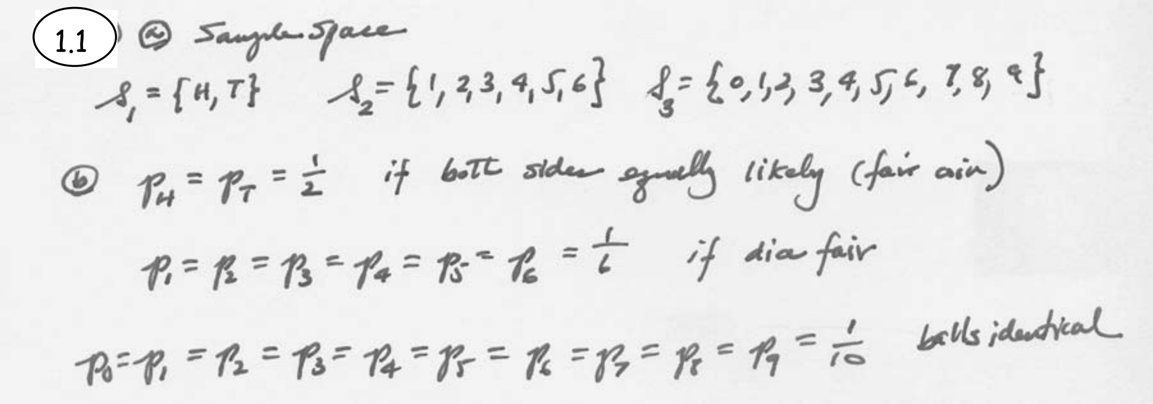
Chapter 1: Probability Models in Electrical and Computer Engineering



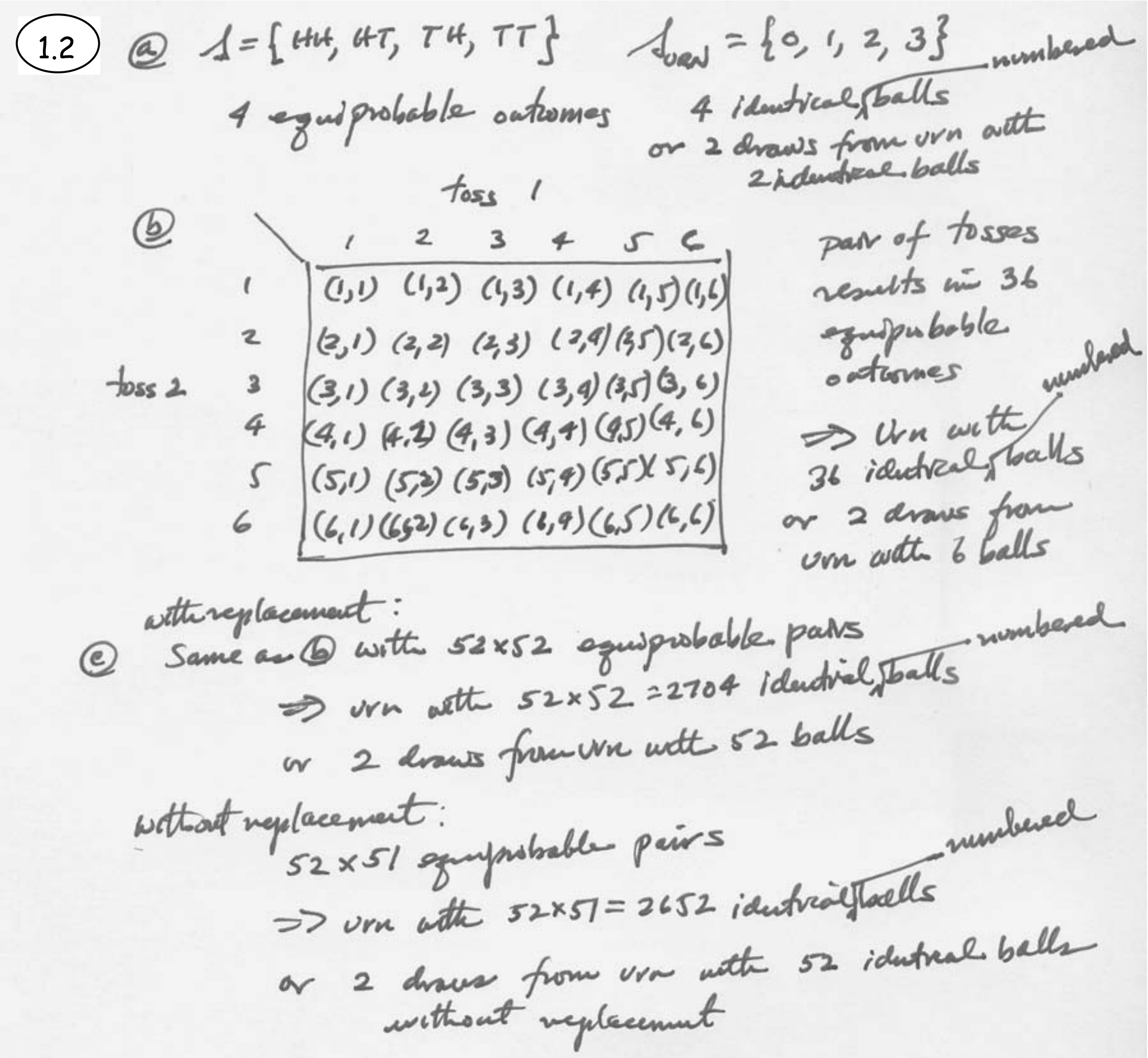
1.1 (a) Sample space

$S_1 = \{H, T\}$ $S_2 = \{1, 2, 3, 4, 5, 6\}$ $S_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

(b) $p_H = p_T = \frac{1}{2}$ if both sides equally likely (fair coin)

$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$ if die fair

$p_0 = p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = \frac{1}{10}$ balls identical



1.2 (a) $S = \{HH, HT, TH, TT\}$ $S_{urn} = \{0, 1, 2, 3\}$

4 equiprobable outcomes 4 identical numbered balls or 2 draws from urn with 2 identical balls

(b)

$$\begin{array}{c|c|cccccc}
 & & & & \text{toss 1} & & & \\
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 1 & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
 & 2 & (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
\text{toss 2} & 3 & (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
 & 4 & (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
 & 5 & (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
 & 6 & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

pair of tosses results in 36 equiprobable outcomes

$\Rightarrow$ Urn with 36 identical numbered balls or 2 draws from urn with 6 balls

with replacement:

(c) Same as (b) with 52×52 equiprobable pairs

$\Rightarrow$ urn with $52 \times 52 = 2704$ identical numbered balls

or 2 draws from urn with 52 balls

without replacement:

52×51 equiprobable pairs

$\Rightarrow$ urn with $52 \times 51 = 2652$ identical numbered balls

or 2 draws from urn with 52 identical balls without replacement

1.3 (a) Equivalent to toss of a biased coin: black = heads; white = tail
white dots much more frequent than black dots so $p_H << 1$.

(b) Binary signal ⇒ 2 outcomes in each transmission
Each outcome can assigned to head or tails
p_H depends on probability of signal outcomes

(c) Device is either working ("heads") or not ("tails")
p_H depends on device and testing schedule

(d) Joe is either online ("heads") or not ("tails")
p_H depends on when observation is made

(e) Received bits equals transmitted bit ("heads") or not ("tails")
p_H depends on properties of transmission channel

1.4 (a) $S_{Lisa} = \{00, 01, 10\}$ $S_{Homer} = \{10, 11\}$ $S_{Bart} = \{00, 10\}$

(b) Lisa: $p_{00} = p_{01} = p_{10} = \frac{1}{3}$

Homer: ball 00 & ball 10 read as 10 $p_{10} = \frac{2}{3}$ $p_{11} = \frac{1}{3}$
ball 01 reads as 11

Bart: ball 00 + ball 01 read as 00 $p_{00} = \frac{2}{3}$ $p_{10} = \frac{1}{3}$
ball 10 reads as 10

1.5 (a) Toss coin: Heads ⇒ "1" Tails ⇒ Do 2nd toss
Heads ⇒ "2" Tails ⇒ Do 3rd toss
Heads ⇒ "3" Tails ⇒ "4"

(b) Urn with 8 identical balls with labels: {1, 1, 1, 1, 2, 2, 3, 4}

(c) Draw Card: if Ace reject outcome and restart experiment
if Not Ace output # assigned to the card
where 24 cards assigned "1" ~ 24/48
12 cards " "2" ~ 12/48
6 " " "3" ~ 6/48
6 " " "4" ~ 6/48

1.6 **a)** In the first draw the outcome can be black (b) or white (w). If the first draw is black, then the second outcome can be b or w. However if the first draw is white, then the run only contains black balls so the second outcome must be b. Therefore $\mathcal{S} = \{bb, bw, wb\}$.

b) In this case all outcomes can be b or w. Therefore $\mathcal{S} = \{bb, bw, wb, ww\}$.

c) In part a) the outcome ww cannot occur so $f_{ww} = 0$. In part b) let N be a larger number of repetitions of the experiment. The number of times the first outcome is w is approximately $N/3$ since the run has one white ball and two black balls. Of these $N/3$ outcomes approximately $1/2$ are also white in the second draw. Thus $N/9$ if the outcome result is ww, and thus $f_{ww} = \frac{1}{9}$.

d) In the first experiment, the outcome of the first draw affects the probability of the outcomes in the second draw. In the second experiment, the outcome of the first draw does not affect the probability of the outcomes in the second draw.

1.7 When the experiment is performed, either A occurs or it doesn't (i.e. B occurs); thus $N_A(n) + N_B(n) = n$ in n repetitions of the experiment, and

$$f_A(n) + f_B(n) = \frac{N_A(n)}{n} + \frac{N_B(n)}{n} = 1.$$

Thus $f_B(n) = 1 - f_A(n)$.

1.8 If A, B, or C occurs, then D occurs. Furthermore since A, B, or C cannot occur simultaneously, in n repetitions of the experiment we have

$$N_D(n) = N_A(n) + N_B(n) + N_C(n)$$

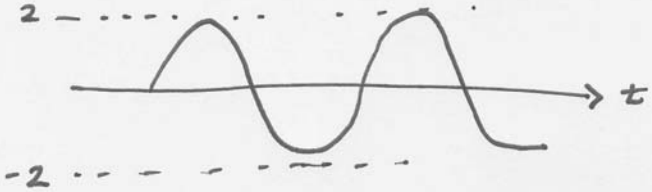
and dividing both sides by n

$$f_D(n) = f_A(n) + f_B(n) + f_C(n).$$

1.9

$$\begin{aligned}
< X >_n &= \frac{1}{n}\sum_{j=1}^{n} X(j) \qquad n > 0 \\
&= \frac{n-1}{n}\frac{1}{n-1}\left\{\sum_{j=1}^{n-1} X(j) + X(n)\right\} \\
&= \left(1 - \frac{1}{n}\right) < X >_{n-1} + \frac{1}{n}X(n) \\
&= < X >_{n-1} + \frac{X(n) - < X >_{n-1}}{n}
\end{aligned}$$

1.10



ⓐ Sample values equally likely to be in positive and negative amplitude regions

Symmetry of function in positive & negative range

$\Rightarrow$ longterm average of samples $= 0$

ⓑ "Voltage positive" $\Leftrightarrow$ half samples $\Leftrightarrow P[>0] = \frac{1}{2}$

"Voltage < -2" does not occur $\Rightarrow P[<-2] = 0$

ⓒ The observed frequencies can change if τ is a rational number times 2π

for example: $\tau = 2\pi$ gives only one observed value

$1 = 2\cos 2\pi = 2\cos 4\pi = 2\cos 6\pi = \ldots$

1.11 By "random" we mean "unpredictable",

but we also mean "occurring as repetition of an identical random experiment."

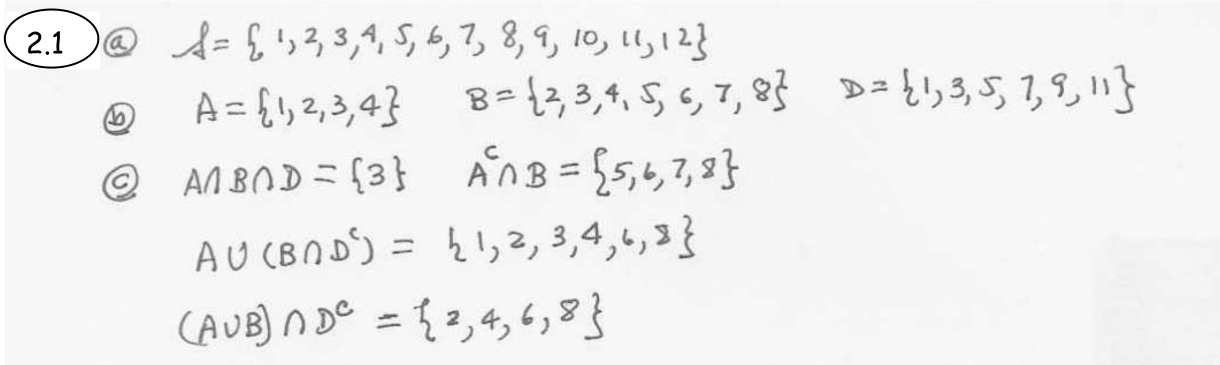
We may also mean "equiprobable outcomes".

If "random" means all 3 of above attributes then we expect long-term relative frequencies of integers to be $\frac{1}{10}$.

We then also expect long term relative frequencies of each possible pair of integers to be $\frac{1}{100}$.

Chapter 2: Basic Concepts of Probability Theory

2.1 Specifying Random Experiments



2.1 ⓐ $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

ⓑ $A = \{1, 2, 3, 4\}$ $B = \{2, 3, 4, 5, 6, 7, 8\}$ $D = \{1, 3, 5, 7, 9, 11\}$

ⓒ $A \cap B \cap D = \{3\}$ $A^c \cap B = \{5, 6, 7, 8\}$

$A \cup (B \cap D^c) = \{1, 2, 3, 4, 6, 8\}$

$(A \cup B) \cap D^c = \{2, 4, 6, 8\}$

2.2 The outcome of this experiment consists of a pair of numbers (x, y) where x = number of dots in first toss and y = number of dots in second toss. Therefore, S = set of ordered pairs (x, y) where $x, y \epsilon \{1, 2, 3, 4, 5, 6\}$ which are listed in the table below:

a) y

x	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

checkmarks indicate elements of events

b) y

x	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6	✓	✓	✓	✓	✓	✓

$A = \{N_1 < N_2\}^c = \{N_1 \geq N_2\}$

c) y

x	1	2	3	4	5	6
1						
2						
3						
4						
5						
6	✓	✓	✓	✓	✓	✓

$B = \{N_1 = 6\}$

d) B is a subset of A so when B occurs then A also occurs, thus B implies A

e) $A \cap B^C = \{N_2 \leq N_1 < 6\}$

x	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6						

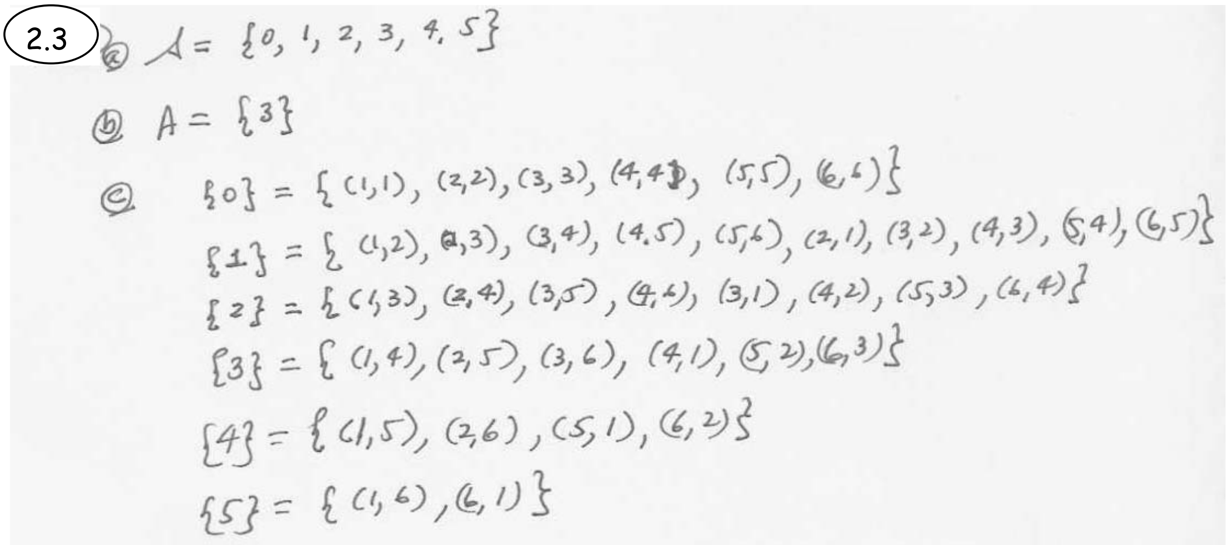
f) C = "number of dots differ by 2"

	1	2	3	4	5	6
1			✓			
2				✓		
3	✓				✓	
4		✓				✓
5			✓			
6				✓		

Comparing the tables for A and C we see

$A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$

2.3



ⓐ $A = \{0, 1, 2, 3, 4, 5\}$

ⓑ $A = \{3\}$

ⓒ $\{0\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\{1\} = \{(1,2), (2,3), (3,4), (4.5), (5,6), (2,1), (3,2), (4,3), (5,4), (6,5)\}$

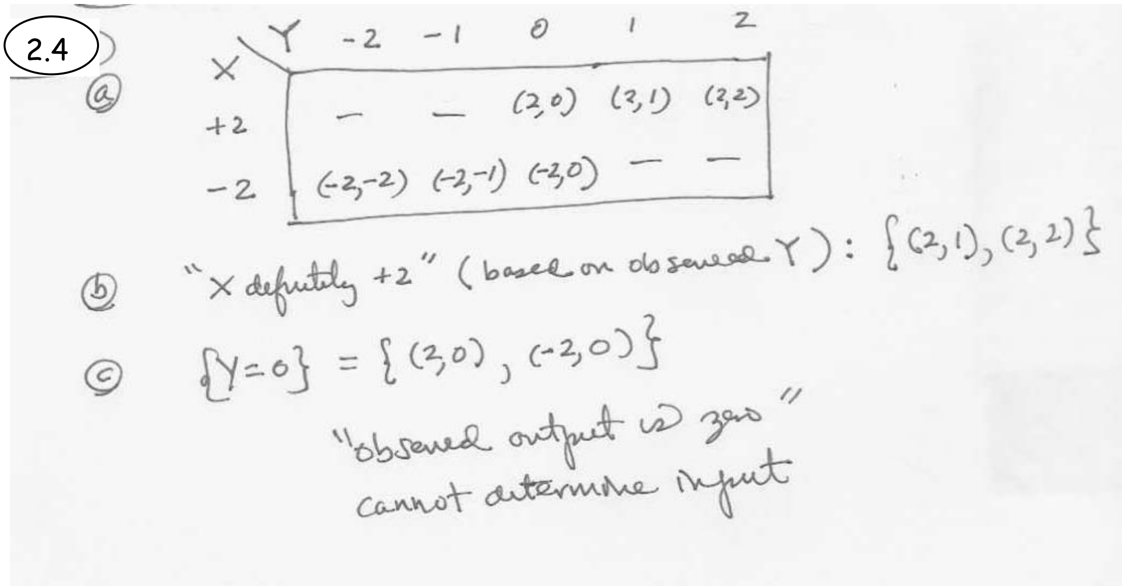
$\{2\} = \{(1,3), (2,4), (3,5), (4,6), (3,1), (4,2), (5,3), (6,4)\}$

$\{3\} = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$

$\{4\} = \{(1,5), (2,6), (5,1), (6,2)\}$

$\{5\} = \{(1,6), (6,1)\}$

2.4



ⓐ

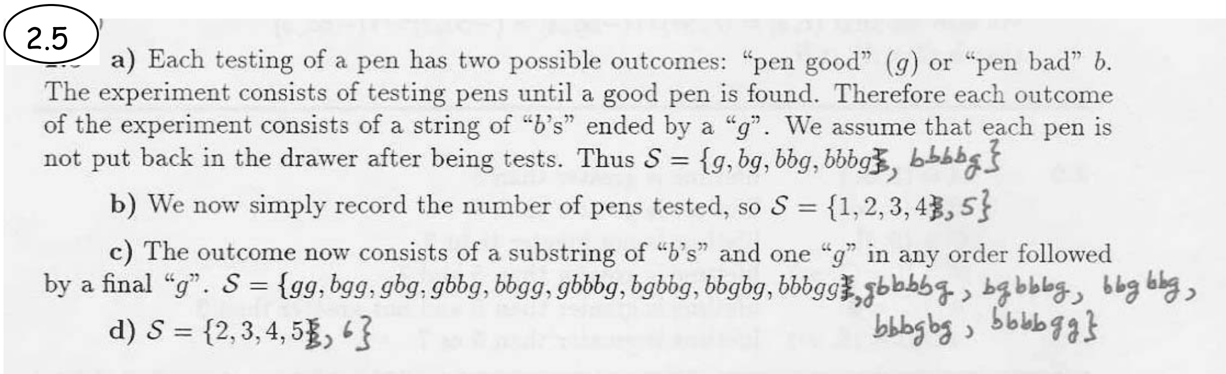
$$\begin{array}{c|ccccc} X \backslash Y & -2 & -1 & 0 & 1 & 2 \\ \hline +2 & - & - & (2,0) & (2,1) & (2,2) \\ -2 & (-2,-2) & (-2,-1) & (-2,0) & - & - \end{array}$$

ⓑ “X definitely +2” (based on observed Y): $\{(2,1), (2,2)\}$

ⓒ $\{Y=0\} = \{(2,0), (-2,0)\}$

“observed output is zero” cannot determine input

2.5



a) Each testing of a pen has two possible outcomes: “pen good” (g) or “pen bad” b. The experiment consists of testing pens until a good pen is found. Therefore each outcome of the experiment consists of a string of “b’s” ended by a “g”. We assume that each pen is not put back in the drawer after being tests. Thus $\mathcal{S} = \{g, bg, bbg, bbbg, bbbbg\}$

b) We now simply record the number of pens tested, so $\mathcal{S} = \{1, 2, 3, 4, 5\}$

c) The outcome now consists of a substring of “b’s” and one “g” in any order followed by a final “g”. $\mathcal{S} = \{gg, bgg, gbg, gbbg, bbgg, gbbbg, bgbbg, bbgbg, bbbgg, gbbbbg, bgbbbg, bbgbbg, bbbgbg, bbbbgg\}$

d) $\mathcal{S} = \{2, 3, 4, 5, 6\}$

2.6

(a) $S = \{abc, cab, bca, acb, bac, cba\}$

(b) $A = \{abc, acb\}$ $B = \{abc, cba\}$ $C = \{abc, bac\}$

(c) $(A \cup B \cup C)^c = \{abc, acb, cba, bac\}^c = \{cab, bca\}$

(d) $A \cap B \cap C = \{abc\}$

(e) $A \cup B \cup C = \{abc, acb, cba, bac\}$.

2.7

(a) $A = \{2, 4, 6, 8, \ldots\}$

(b) $B = \{3, 6, 9, \ldots\}$

(c) $C = \{1, 2, 3, 4, 5, 6\}$

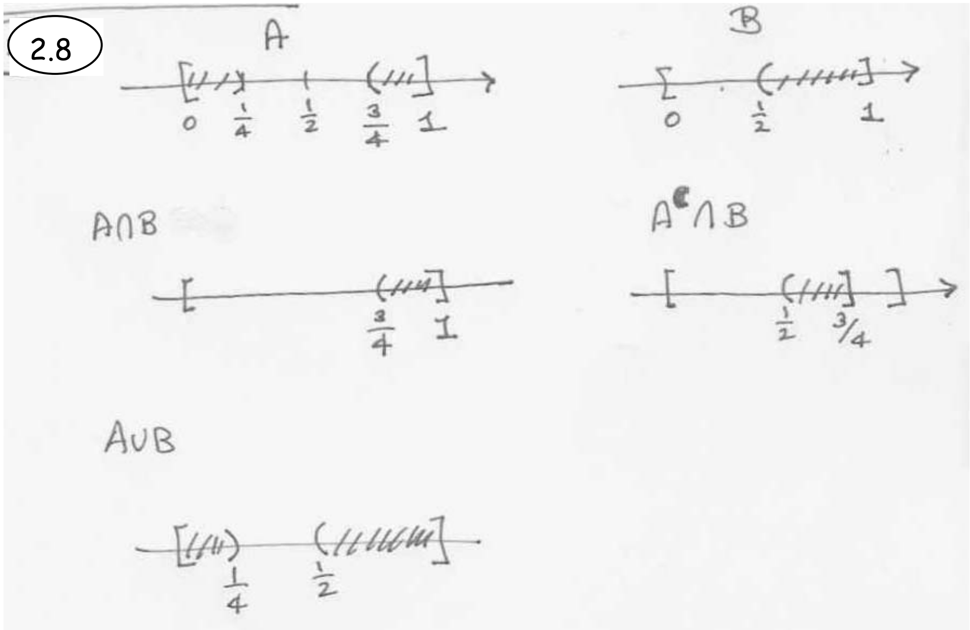
(d) $A \cap B = \{6, 12, 18, \ldots\}$ "multiple of 6"

$A - B =$ "even positive integer and not multiple of 3"

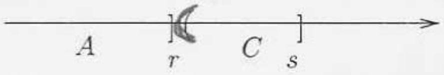
$= \{n = 2m : m$ positive integer, not multiple of $3\}$

$A \cap B \cap C = \{6\}$ "even multiple of 3 less than or equal to 6"

2.8



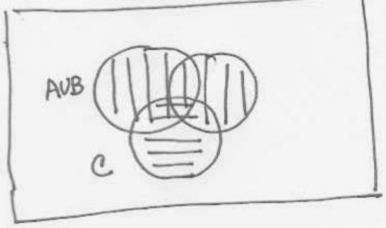
2.9 If we sketch the events A and B we see that $B = A \cup B$. We also see that the intervals corresponding to A and C have no points in common so $A \cap C =$.



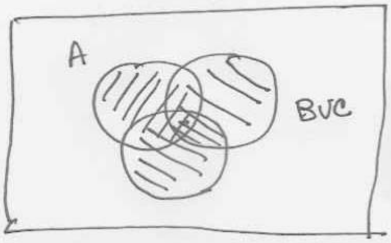
We also see that $(r, s] = (r, \infty) \cap (-\infty, s] = (-\infty, r]^C \cap (-\infty, s]$
that is $C = A^C \cap B$

2.10 ⓐ $A \cup (B \cup C) = (A \cup B) \cup C$

$(A \cup B) \cup C$

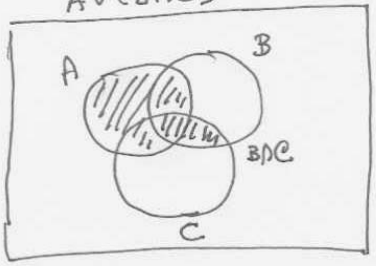


$A \cup (B \cup C)$

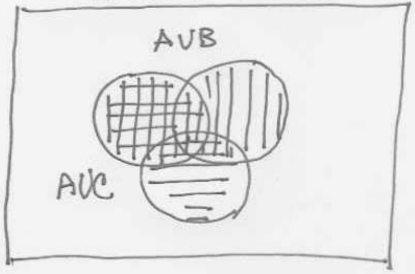


ⓑ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cup (B \cap C)$

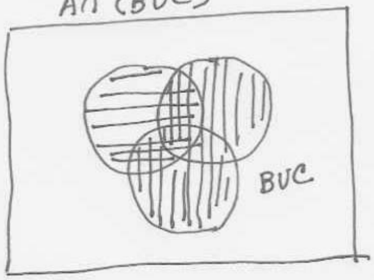


$(A \cup B) \cap (A \cup C)$

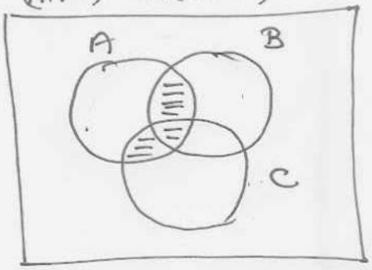


ⓒ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

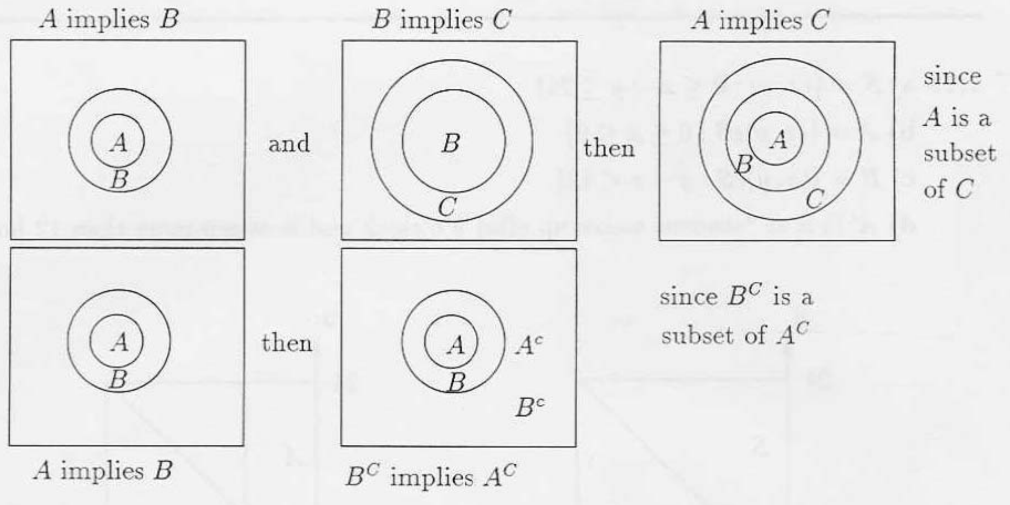
$A \cap (B \cup C)$



$(A \cap B) \cup (A \cap C)$



2.11


A implies B and B implies C then A implies C

since A is a subset of C

A implies B then B^C implies A^C

since B^C is a subset of A^C

2.12

Given $A \cup B = A$ and $A \cap B = A$ claim $A = B$

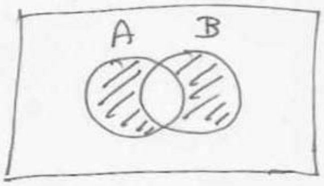
Let $\xi \in A$, then $\xi \in A \cap B \Rightarrow \xi \in B \quad \therefore A \subset B$

Let $\xi \in B$ then $\xi \in A \cup B \Rightarrow \xi \in A \quad \therefore B \subset A$

$\therefore A = B.$

2.13

"Exactly one of A or B occurs" $= (A \cap B^c) \cup (A^c \cap B)$



2.14 **a)** $(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^a)\cup(A^c\cap B^c\cap C)$

b) $(A\cap B\cap C^c)\cup(A\cap B^c\cap C)\cup(A^c\cap B\cap C)$

c) $A\cup B\cup C$

d) $(A\cap B\cap C^c)\cup(A\cap B^c\cap C)\cup(A^c\cap B\cap C)\cup(A\cap B\cap C)$

e) $A^c\cap B^c\cap C^c$

2.15 ⓐ $D = A_1\cap A_2\cap A_3$

ⓑ $D = A_1\cup A_2\cup A_3$

ⓒ $D = (A_1\cap A_2\cap A_3)\cup(A_1^c\cap A_2\cap A_3)\cup(A_1\cap A_2^c\cap A_3)\cup(A_1\cap A_2\cap A_3^c)$

2.16 ⓐ "System j is up" $= A_{j1}\cap A_{j2}$

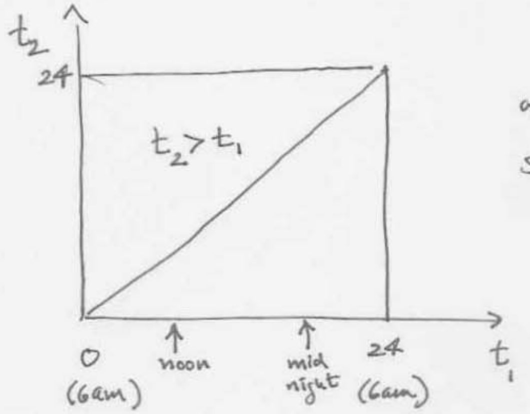
"System is up" $= (A_{11}\cap A_{12})\cup(A_{21}\cap A_{22})\cup(A_{31}\cap A_{32})$

ⓑ "jth level connection active" if $A_{j1}\cap A_{j2}$

"connection active" if any of 3 connections is active

2.17

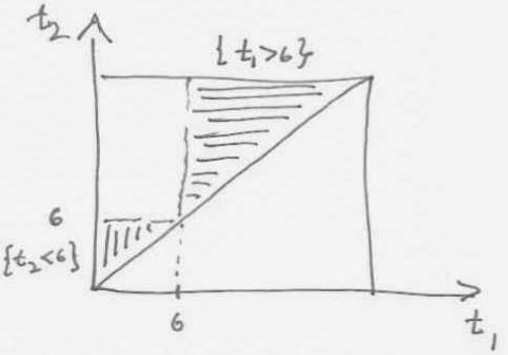
ⓐ

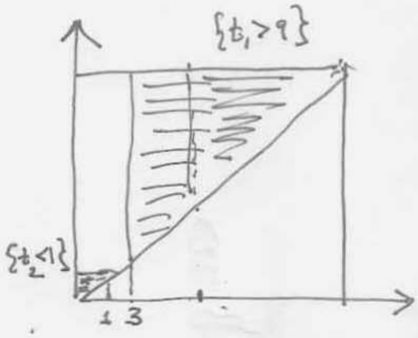


wake at t_1
sleep at $t_2 > t_1$

ⓑ "Asleep at noon" = wake up after noon
or wake up & go to sleep before noon

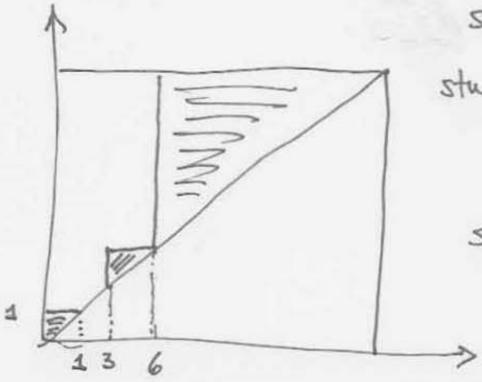
$$= \{t_1 > 6\} \cup \{t_2 < 6\}$$





ⓒ "Student sleeps through breakfast" $= \{t_1 > 3\} \cup \{t_2 < 1\}$

ⓓ $A \cap B$

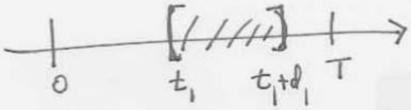


student wakes up after noon
or
student wakes up after 9am
and goes back to sleep
before noon
or
student wakes up and goes
back to sleep before 7am

2.18 (a) $S = \{(t_1, t_2):\ 0<t_1<T,\ 0<t_2<T\}$

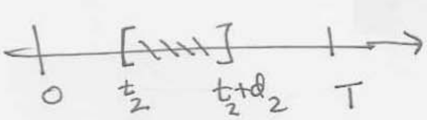
(b) A = train in crossing

$= \{t_1 < t < t_1+d_1\}$

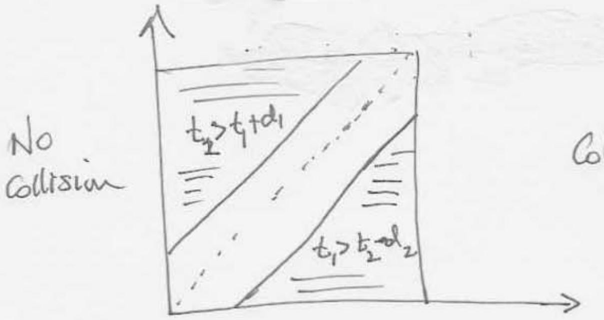


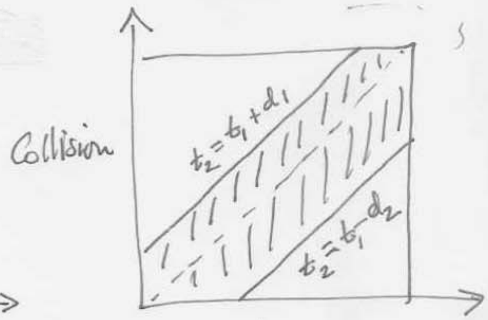
B = car in crossing

$= \{t_2 < t < t_2+d_2\}$



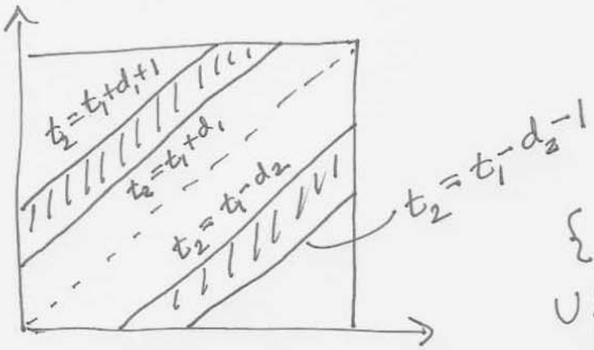
No Collision occurs if $A \cap B$ is empty $\iff$ $t_1 > t_2+d_2$ or $t_2 > t_1+d_1$





(c) Collision missed by 1 second or less.

$= \{\text{No Collision}\} \cap \{\text{within 1 second of collision}\}$



$\{t_1+d_1<t_2<t_1+d_1+1\}$

$\cup\{t_2+d_2<t_1<t_2+d_2+1\}$

2.19 ⓐ ϕ, $\mathcal{S} = \{-1, 0, +1\}$, $\{-1\}$, $\{0\}$, $\{+1\}$, $\{-1, 0\}$, $\{-1, +1\}$, $\{0, +1\}$

ⓑ $\mathcal{S} = \{(-1, 0), (-1, +1), (0, -1), (0, +1), (+1, -1), (+1, 0)\}$

power set has $2^6 = 64$ subsets.

2.20 $\mathcal{S} = \{ HH, HT, TH, TT \}$

ⓐ ϕ, $\mathcal{S}$, $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$, $\{HH, HT\}$, $\{HH, TH\}$, $\{HH, TT\}$,
$\{HT, TH\}$, $\{HT, TT\}$, $\{TH, TT\}$, $\{HH, HT, TH\}$, $\{HH, TH, TT\}$, $\{HH, HT, TT\}$
$\{HT, TH, TT\}$

ⓑ $\mathcal{S}' = \{0, 1, 2\}$

ϕ, $\mathcal{S}$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$

ⓒ $\mathcal{S}$ has 2^{10} elements & its power set has $2^{2^{10}} = 2^{1024}$ subsets

$\mathcal{S}'$ has 11 elements & its power set has 2^{11} subsets

2.2 The Axioms of Probability

2.21) The sample space in tossing a die is $S = \{1,2,3,4,5,6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$\begin{aligned} 1 &= P[S] \\ &= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}] \end{aligned}$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

$$1 = p_1 + p_2 + \ldots + p_6 = 6p \Rightarrow p_i = p = \tfrac{1}{6} \text{ for } i = 1, \ldots, 6$$

ⓑ $P[A] = P[> 3\text{dots}] = P[\{4,5,6\}] = P[\{4\}] + P[\{5\}] + P[\{6\}] = \frac{3}{6}$

$P[B] = P[\text{odd}\#] = P[\{1,3,5\}] = P[\{1\}] + P[\{3\}] + P[\{5\}] = \frac{3}{6}$

ⓒ $P[A \cup B] = P[\{1,3,4,5,6\}] = \frac{5}{6}$

$P[A \cap B] = P[\{5\}] = \frac{1}{6}$

$P[A^c] = 1 - P[A] = \frac{3}{6}$

2.22

ⓐ In first toss, each face occurs with relative frequency 1/6

Each first toss outcome is followed by each possible face 1/6 of the time

∴ Each pair occurs with relative frequency $1/6 \times 1/6 = 1/36$.

ⓑ $P[A] = \frac{21}{36}$ $P[B] = \frac{6}{36}$ $P[C] = \frac{8}{36}$ $P[A \cap B^c] = \frac{15}{36}$ $P[A^c] = \frac{15}{36}$

2.23

$P[\{c,d\}] = p_c + p_d = \frac{3}{8}$ by expressing each event in terms of elementary events

$P[\{b,c\}] = p_b + p_c = \frac{6}{8}$

~~Solving this set of linea~~

$P[\{d\}] = \frac{1}{8}$

$1 = P[S] = p_a + p_b + p_c + p_d = 1$

Solving this set of linear equations gives

$p_a = \frac{1}{8} \quad p_b = \frac{4}{8} \quad p_c = \frac{2}{8} \quad p_d = \frac{1}{8}$

2.24

ⓐ $P[A \cap B^c] = P[A] - P[A \cap B]$

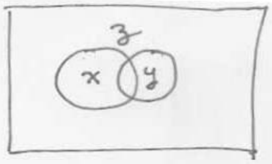
$P[A^c \cap B] = P[B] - P[A \cap B]$

ⓑ $P[A \cap B^c \cup A^c \cap B] = P[A] + P[B] - 2P[A \cap B]$

ⓒ $P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - P[A] - P[B] + P[A \cap B]$

2.25

$z = P[A \cup B] = P[A] + P[B] - P[A \cap B] = x + y - P[A \cap B]$



$P[A \cap B] = x + y - z$

$P[A^c \cap B^c] = 1 - P[(A^c \cap B^c)^c] = 1 - P[A \cup B]$

$= 1 - z$

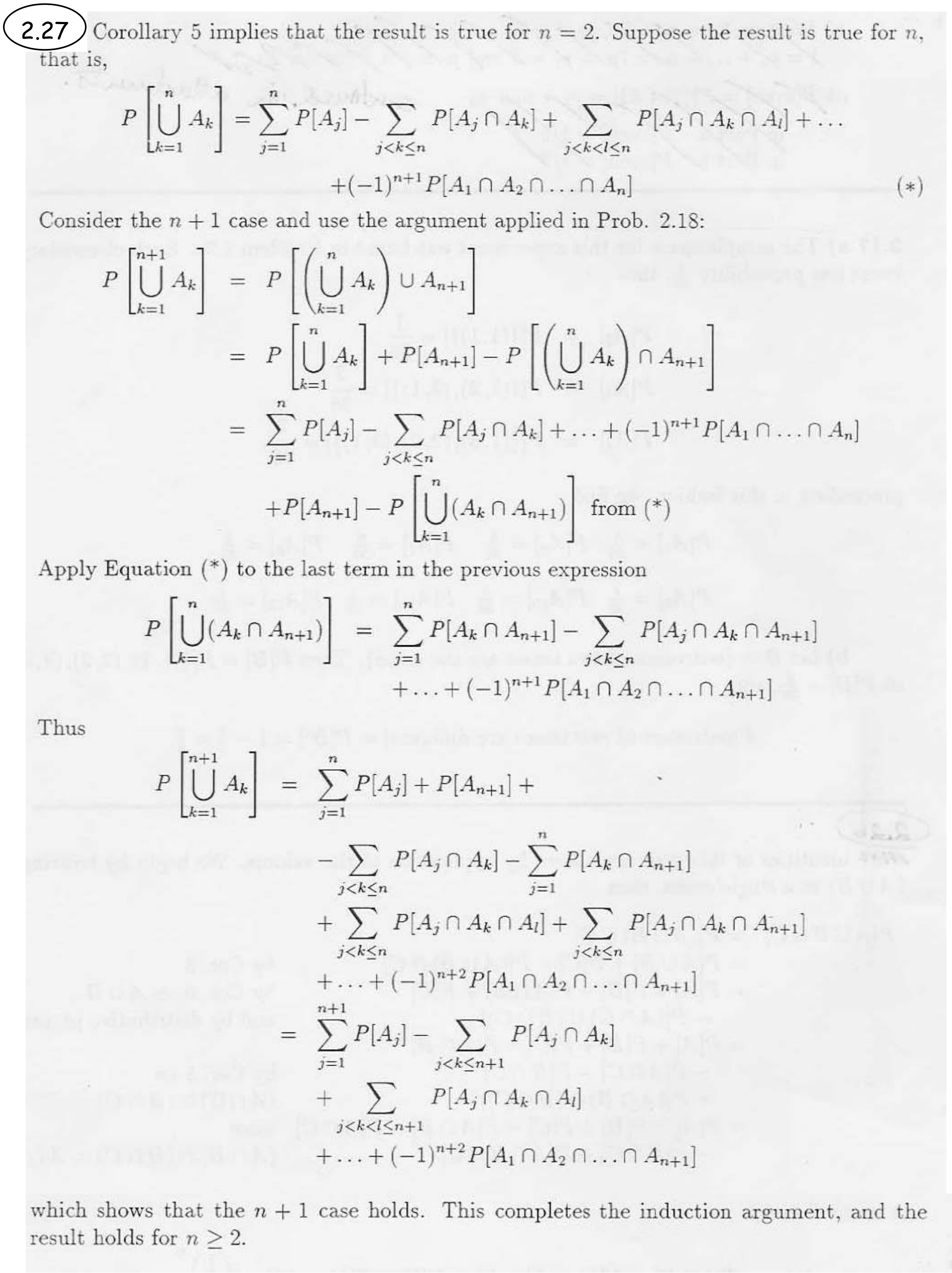
$P[A^c \cup B^c] = 1 - P[(A^c \cup B^c)^c] = 1 - P[A \cap B] = 1 - x - y + z$

$P[A \cap B^c] = P[A] - P[A \cap B] = x - (x + y - z) = z - y$

$P[A^c \cup B] = 1 - P[A \cap B^c] = 1 - z + y$

2.26 Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$$
\begin{aligned}
P[A \cup B \cup C] &= P[(A \cup B) \cup C] \\
&= P[A \cup B] + P[C] - P[(A \cup B) \cap C] && \text{by Cor. 5} \\
&= P[A] + P[B] - P[A \cap B] + P[C] && \text{by Cor. 5 on } A \cup B \\
&\quad - P[(A \cap C) \cup (B \cap C)] && \text{and by distributive property} \\
&= P[A] + P[B] + P[C] - P[A \cap B] \\
&\quad - P[A \cap C] - P[B \cap C] && \text{by Cor. 5 on} \\
&\quad + P[(A \cap B) \cap (B \cap C)] && (A \cap C) \cup (B \cap C) \\
&= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] && \text{since} \\
&\quad - P[B \cap C] + P[A \cap B \cap C]. && (A \cap B) \cap (B \cap C) = A \cap B \cap C
\end{aligned}
$$



2.27 Corollary 5 implies that the result is true for $n = 2$. Suppose the result is true for n, that is,

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k\le n} P[A_j \cap A_k] + \sum_{j<k<l\le n} P[A_j \cap A_k \cap A_l] + \ldots$$
$$+(-1)^{n+1} P[A_1 \cap A_2 \cap \ldots \cap A_n] \qquad (*)$$

Consider the $n+1$ case and use the argument applied in Prob. 2.18:

$$\begin{aligned}
P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] \\
&= P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}] - P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1}\right] \\
&= \sum_{j=1}^{n} P[A_j] - \sum_{j<k\le n} P[A_j \cap A_k] + \ldots + (-1)^{n+1} P[A_1 \cap \ldots \cap A_n] \\
&\quad + P[A_{n+1}] - P\left[\bigcup_{k=1}^{n} (A_k \cap A_{n+1})\right] \text{ from } (*)
\end{aligned}$$

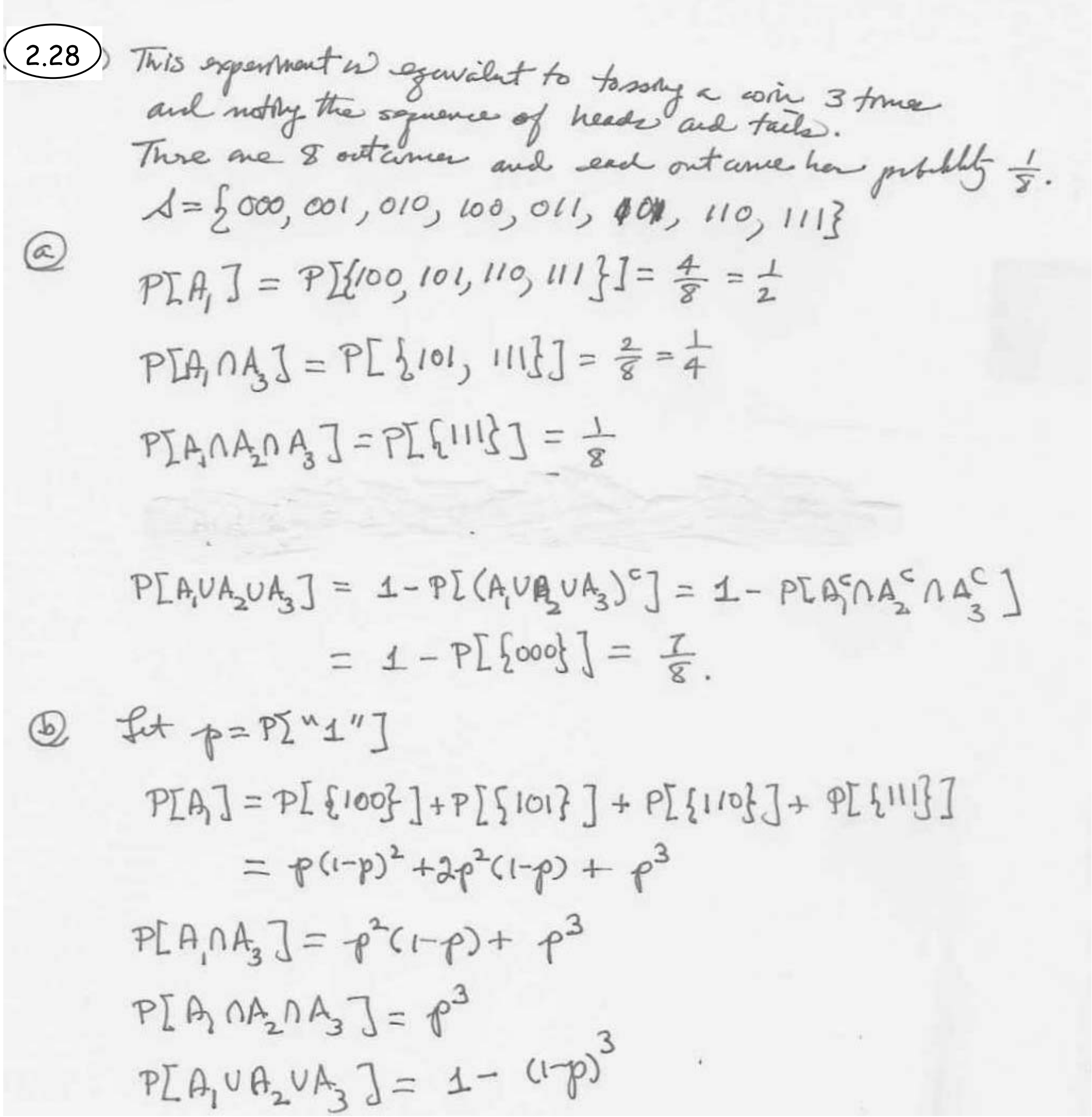
Apply Equation (*) to the last term in the previous expression

$$\begin{aligned}
P\left[\bigcup_{k=1}^{n} (A_k \cap A_{n+1})\right] &= \sum_{j=1}^{n} P[A_k \cap A_{n+1}] - \sum_{j<k\le n} P[A_j \cap A_k \cap A_{n+1}] \\
&\quad + \ldots + (-1)^{n+1} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}]
\end{aligned}$$

Thus

$$\begin{aligned}
P\left[\bigcup_{k=1}^{n+1} A_k\right] &= \sum_{j=1}^{n} P[A_j] + P[A_{n+1}] + \\
&\quad - \sum_{j<k\le n} P[A_j \cap A_k] - \sum_{j=1}^{n} P[A_k \cap A_{n+1}] \\
&\quad + \sum_{j<k\le n} P[A_j \cap A_k \cap A_l] + \sum_{j<k\le n} P[A_j \cap A_k \cap A_{n+1}] \\
&\quad + \ldots + (-1)^{n+2} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}] \\
&= \sum_{j=1}^{n+1} P[A_j] - \sum_{j<k\le n+1} P[A_j \cap A_k] \\
&\quad + \sum_{j<k<l\le n+1} P[A_j \cap A_k \cap A_l] \\
&\quad + \ldots + (-1)^{n+2} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}]
\end{aligned}$$

which shows that the $n+1$ case holds. This completes the induction argument, and the result holds for $n \ge 2$.



2.28

This experiment is equivalent to tossing a coin 3 times and noting the sequence of heads and tails. There are 8 outcomes and each outcome has probability $\frac{1}{8}$.

$S = \{000, 001, 010, 100, 011, 101, 110, 111\}$

ⓐ

$P[A_1] = P[\{100, 101, 110, 111\}] = \frac{4}{8} = \frac{1}{2}$

$P[A_1 \cap A_3] = P[\{101, 111\}] = \frac{2}{8} = \frac{1}{4}$

$P[A_1 \cap A_2 \cap A_3] = P[\{111\}] = \frac{1}{8}$

$$P[A_1 \cup A_2 \cup A_3] = 1 - P[(A_1 \cup A_2 \cup A_3)^c] = 1 - P[A_1^c \cap A_2^c \cap A_3^c]$$
$$= 1 - P[\{000\}] = \frac{7}{8}.$$

ⓑ Let $p = P[\text{"}1\text{"}]$

$$P[A_1] = P[\{100\}] + P[\{101\}] + P[\{110\}] + P[\{111\}]$$
$$= p(1-p)^2 + 2p^2(1-p) + p^3$$

$P[A_1 \cap A_3] = p^2(1-p) + p^3$

$P[A_1 \cap A_2 \cap A_3] = p^3$

$P[A_1 \cup A_2 \cup A_3] = 1 - (1-p)^3$

2.29) Each transmission is equivalent to tossing a fair coin. If the outcome is heads, then the transmission is successful. If tails, then another retransmission is required. As in Example 2.11 the probability that j transmissions are required is:

$$P[j] = \left(\tfrac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\tfrac{1}{4}\right)^{k} = \sum_{k=0}^{\infty} \left(\tfrac{1}{4}\right)^{k} - 1$$

$$= \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}$$

$$P[B] = P[j \text{ multiple of } 3] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{3k} = \frac{1}{1-\frac{1}{8}} - 1 = \frac{1}{7}$$

$$P[C] = \sum_{k=1}^{6} \left(\tfrac{1}{2}\right)^{k} = \frac{1}{2}\sum_{k=0}^{5} \left(\tfrac{1}{2}\right)^{k} = \frac{1}{2}\;\frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = \frac{63}{64}$$

$$P[C^c] = 1 - P[C] = \frac{1}{64}$$

$$P[A \cap B] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{6k} = \frac{1}{1-\frac{1}{64}} - 1 = \frac{1}{63}$$

$$P[A-B] = P[A] - P[A\cap B] = \frac{1}{3} - \frac{1}{63} = \frac{20}{63}$$

$$P[A\cap B \cap C] = \left(\tfrac{1}{2}\right)^6 = \frac{1}{64}$$

2.30 **a)** Corollary 7 implies $P[A \cup B] \leq P[A] + P[B]$. (Eqn. 2.8). Applying this inequality twice, we have

$$P[(A \cup B) \cup C] \leq P[A \cup B] + P[C] \leq P[A] + P[B] + P[C]$$

b) Eqn. 2.8 implies the $n = 2$ case.
Suppose the result is true for n:

$$P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P[A_k] \qquad (*)$$

Then

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] \\ &= \leq P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}] \text{ by Eqn. 2.8} \\ &\leq \sum_{k=1}^{n} P[A_k] + P[A_{n+1}] \text{ by } (*) \\ &= \sum_{k=1}^{n+1} P[A_k] \end{aligned}$$

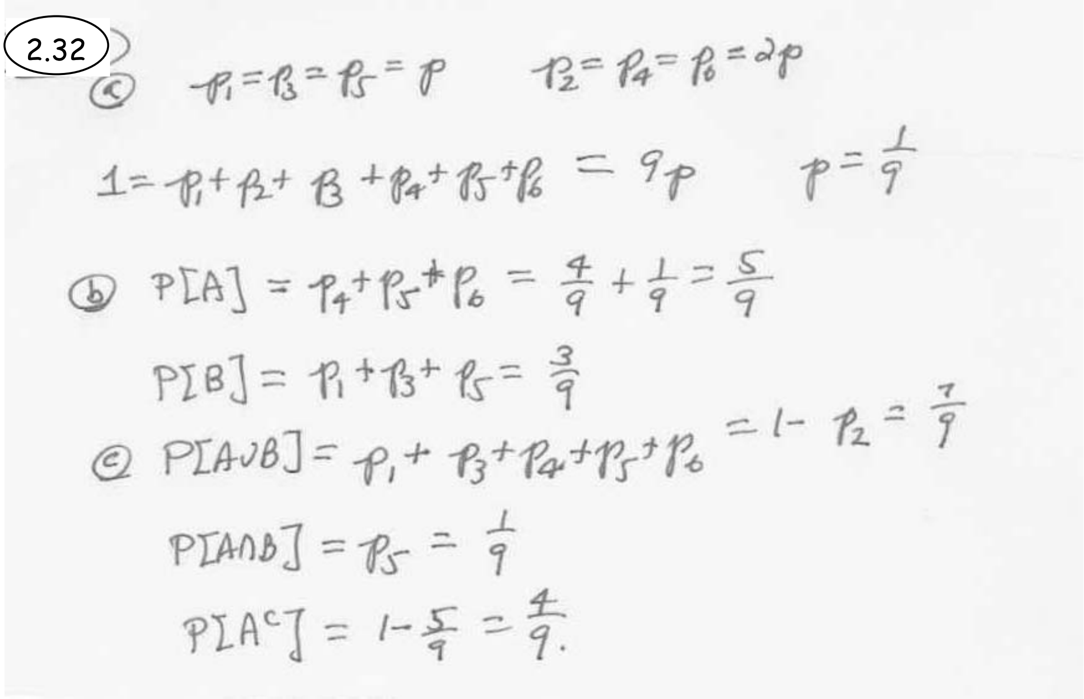
which completes the induction argument.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] = 1 - P\left[\left(\bigcap_{k=1}^{n} A_k\right)^c\right] = 1 - P\left[\bigcup_{k=1}^{n} A_k^c\right]$

$\geq 1 - \sum_{k=1}^{n} P[A_k^c]$ using the result of part b.

2.31 Let $A_i = \{$ith character is in error$\}$

$$P[\text{any error in document}] = P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i] = np$$



2.32

ⓐ $p_1 = p_3 = p_5 = p \qquad p_2 = p_4 = p_6 = 2p$

$1 = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 9p \qquad p = \frac{1}{9}$

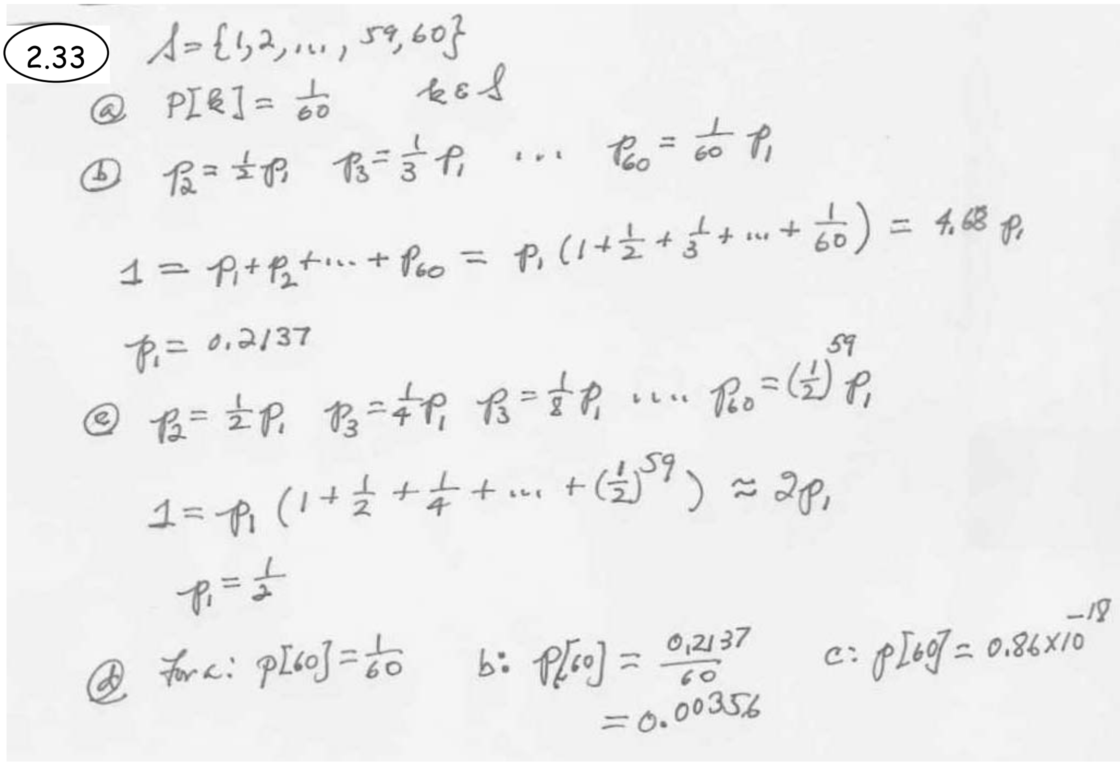
ⓑ $P[A] = p_4 + p_5 + p_6 = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$

$P[B] = p_1 + p_3 + p_5 = \frac{3}{9}$

ⓒ $P[A \cup B] = p_1 + p_3 + p_4 + p_5 + p_6 = 1 - p_2 = \frac{7}{9}$

$P[A \cap B] = p_5 = \frac{1}{9}$

$P[A^c] = 1 - \frac{5}{9} = \frac{4}{9}.$



2.33

$S = \{1, 2, \ldots, 59, 60\}$

ⓐ $P[k] = \frac{1}{60} \qquad k \in S$

ⓑ $p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{3} p_1 \quad \ldots \quad p_{60} = \frac{1}{60} p_1$

$1 = p_1 + p_2 + \cdots + p_{60} = p_1 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{60}\right) = 4.68\, p_1$

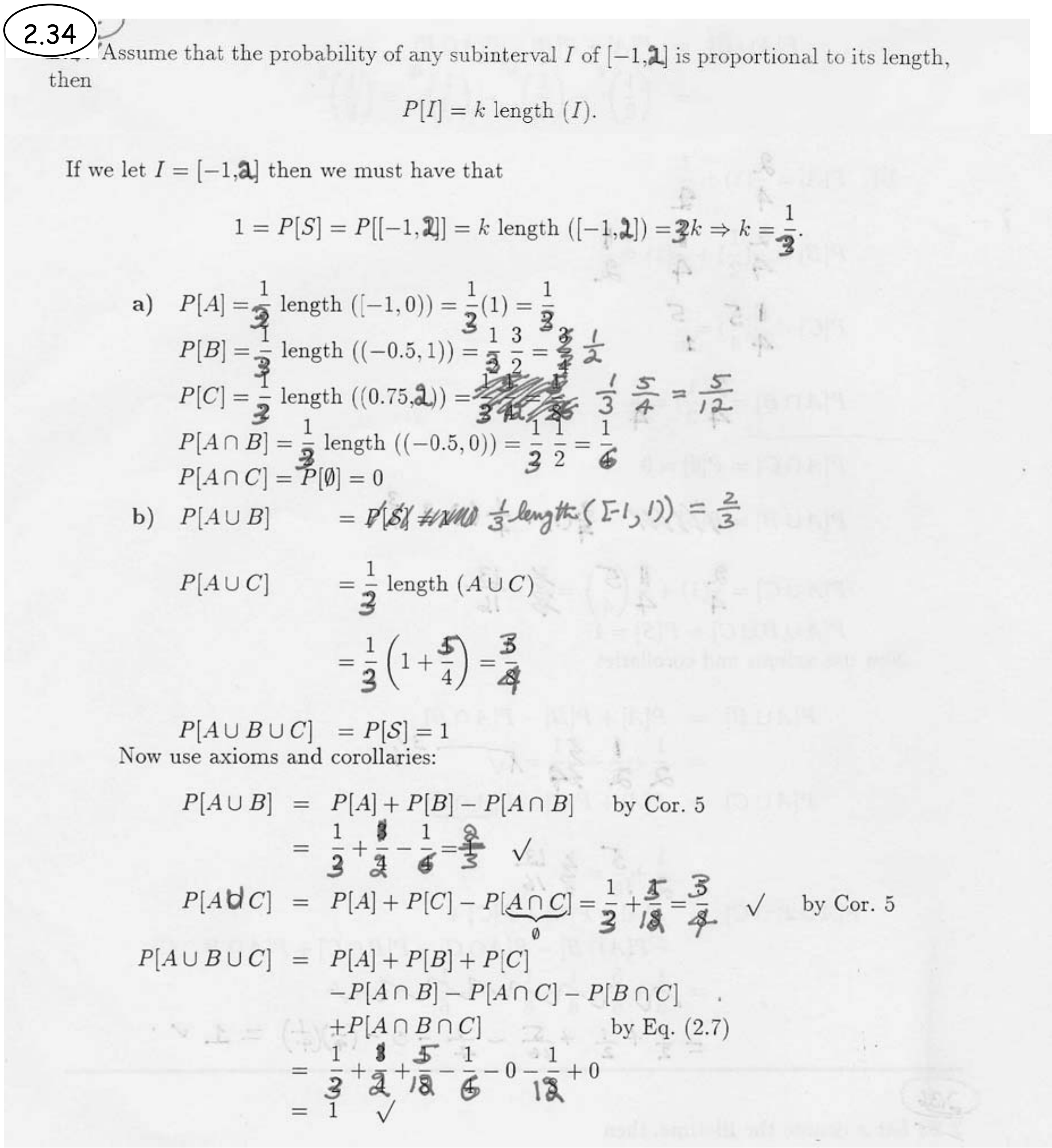
$p_1 = 0.2137$

ⓒ $p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{4} p_1 \quad p_3 = \frac{1}{8} p_1 \quad \ldots \quad p_{60} = \left(\frac{1}{2}\right)^{59} p_1$

$1 = p_1 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{59}\right) \approx 2 p_1$

$p_1 = \frac{1}{2}$

ⓓ for a: $p[60] = \frac{1}{60} \qquad$ b: $p[60] = \frac{0.2137}{60} = 0.00356 \qquad$ c: $p[60] = 0.86 \times 10^{-18}$



2.34 Assume that the probability of any subinterval I of $[-1,2]$ is proportional to its length, then

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1,2]$ then we must have that

$$1 = P[S] = P[[-1,2]] = k \text{ length } ([-1,2]) = 3k \Rightarrow k = \frac{1}{3}.$$

a) $P[A] = \frac{1}{3}$ length $([-1,0)) = \frac{1}{3}(1) = \frac{1}{3}$

$P[B] = \frac{1}{3}$ length $((-0.5,1)) = \frac{1}{3}\frac{3}{2} = \frac{1}{2}$

$P[C] = \frac{1}{3}$ length $((0.75,2)) = \frac{1}{3}\frac{5}{4} = \frac{5}{12}$

$P[A \cap B] = \frac{1}{3}$ length $((-0.5,0)) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}$

$P[A \cap C] = P[\emptyset] = 0$

b) $P[A \cup B] =$ ~~$P[S] = 1$~~ $\frac{1}{3}$ length $([-1,1)) = \frac{2}{3}$

$P[A \cup C] = \frac{1}{3}$ length $(A \cup C)$

$= \frac{1}{3}\left(1 + \frac{5}{4}\right) = \frac{3}{4}$

$P[A \cup B \cup C] = P[S] = 1$

Now use axioms and corollaries:

$$\begin{aligned}
P[A \cup B] &= P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5} \\
&= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \quad \checkmark \\
P[A \cup C] &= P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] = \frac{1}{3} + \frac{5}{12} = \frac{3}{4} \quad \checkmark \quad \text{by Cor. 5} \\
P[A \cup B \cup C] &= P[A] + P[B] + P[C] \\
&\quad - P[A \cap B] - P[A \cap C] - P[B \cap C] \\
&\quad + P[A \cap B \cap C] \quad \text{by Eq. (2.7)} \\
&= \frac{1}{3} + \frac{1}{2} + \frac{5}{12} - \frac{1}{6} - 0 - \frac{1}{12} + 0 \\
&= 1 \quad \checkmark
\end{aligned}$$

2.35 a) Let I be a subinterval of $[-1,2]$ then

$$P[I] = 2k \text{ length } (I \cap [0,2]) + 2k \text{ length } (I \cap [-1,0])$$

Letting $I = [-1,2]$ we have

$$1 = P[[-1,2]] = 2k + 2k = 4k \Rightarrow k = \frac{1}{4}$$

b) $P[A] = \frac{2}{4}(1) = \frac{1}{2}$

$P[B] = \frac{2}{4}(\frac{1}{2}) + \frac{2}{4}(1) = \frac{5}{2}$

$P[C] = \frac{1}{4}(\frac{5}{4}) = \frac{5}{16}$

$P[A \cap B] = \frac{2}{4}(\frac{1}{2}) = \frac{1}{4}$

$P[A \cap C] = P[\emptyset] = 0$

$P[A \cup B] = \cancel{P[S] = 1} \quad \frac{2}{4}(1) + \frac{1}{4}(1) = \frac{3}{4}$

$P[A \cup C] = \frac{2}{4}(1) + \frac{1}{4}\left(\frac{5}{4}\right) = \cancel{\frac{1}{2}} \; \frac{13}{16}$

$P[A \cup B \cup C] = P[S] = 1$

Now use axioms and corollaries

$$\begin{aligned}
P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\
&= \frac{1}{2} + \frac{5}{2} - \frac{2}{34}\frac{1}{4} = \cancel{1}\checkmark \quad 3/4 \\
P[A \cup C] &= P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] \\
&= \frac{1}{2} + \frac{5}{16} = \cancel{\frac{1}{2}} \; \frac{13}{16} \\
P[A \cup B \cup C] &= P[A] + P[B] + P[C] + \\
&\quad -P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C] \\
&= \cancel{\frac{1}{3} + \frac{5}{6} + \frac{1}{6} - \frac{1}{6} - 0 - \frac{1}{6} + 0 = 1}\checkmark \\
&= \frac{1}{2} + \frac{1}{2} + \frac{5}{16} - \frac{1}{4} - 0 - (\tfrac{1}{4})(\tfrac{1}{4}) = 1 \checkmark
\end{aligned}$$

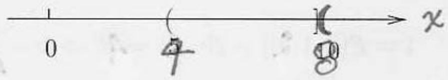
2.36 Let x denote the lifetime, then

$$A = \{x > \overset{4}{\cancel{5}}\} \text{ and } B = \{x > \overset{8}{\cancel{10}}\}$$

a) $P[A \cap B] = P[\{x > \overset{4}{\cancel{5}}\} \cap \{x > \overset{8}{\cancel{10}}\}] = P[\{x > \overset{8}{\cancel{10}}\}] = \frac{1}{8}$

$P[A \cup B] = P[\{x > \underset{4}{\cancel{5}}\} \cup \{x > \underset{8}{\cancel{10}}\}] = P[\{x > \underset{4}{\cancel{5}}\}] = \frac{1}{4}$

b)



$$\begin{aligned}
P[\{x > 5\}] &= P[\{\overset{6}{\cancel{5}} < x \le \overset{12}{\cancel{10}}\} \cup \{x > \overset{12}{\cancel{10}}\}] \\
&= P[\{\overset{6}{\cancel{5}} < x \le \overset{12}{\cancel{10}}\}] + P[\{x > \overset{12}{\cancel{10}}\}] \\
\Rightarrow P[\{\overset{6}{\cancel{5}} < x \le \overset{12}{\cancel{10}}\}] &= P[\{x > \overset{6}{\cancel{5}}\}] - P[\{x > \overset{12}{\cancel{10}}\}] = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}
\end{aligned}$$

2.37 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

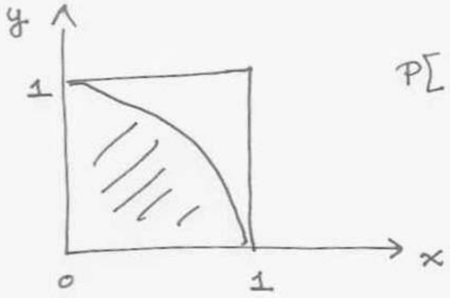
$$P[(-\infty, r]] \le P[(-\infty, s]] \text{ by Corollary 7.}$$

b)

$$\begin{aligned}
P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\
&= P[(-\infty, r]] + P[(r, s]] \\
\Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]]
\end{aligned}$$

2.38

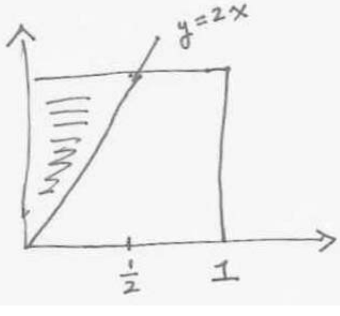
ⓐ



$P[x^2 + y^2 < 1] = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

Area inside circle

ⓑ



$P[y > 2x] = \frac{1}{4}$

Area in right triangle

2.3 *Computing Probabilities Using Counting Methods

2.39 The number of distinct ordered triplets $= 60 \cdot 60 \cdot 60 = 60^3$

2.40 The number of distinct 7-tuples $= 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8(10^6)$

2.41 The number of distinct ordered triplets $= 6 \cdot 2 \cdot 52 = 624$

2.42 #sequences of length $8 = 2^8 = 256$

P[arbitrary sequence = correct sequence] $= \frac{1}{256}$

P[success in two tries] = 1 − P[failure in both tries]

$$= 1 - \frac{255}{256} \cdot \frac{254}{255}$$

2.43 8, 9, or 10 characters long

- at least 1 special character from set of size 24
- numbers from size 10
- upper + lower case letters $26 \times 2 = 52$ } 62 choices

For length n:

- pick position of required special character + pick character
  n positions × 24 characters.
- pick number/letter/special character for remaining n−1 positions
  86^{n-1}

Total # passwords $= n \cdot 24 \cdot 86^{n-1}$

Length 8, 9, or 10 $= 8 \cdot 24 \cdot 86^7 + 9 \cdot 24 \cdot 86^8 + 10 \cdot 24 \cdot 86^9 = 6.24 \times 10^{19}$

Time to try all passwords $= 6.24 \times 10^{13}$ seconds $= 2(10^6)$ years

2.44 $3^{10} = 59049$ possible answers

Assuming each paper selects answers at random

P[two papers are identical] $= \frac{1}{3^{10}} \times \frac{1}{3^{10}} = \frac{1}{3^{20}} = 2.87 \times 10^{-10}$

2.45

(a) # combinations = 5 × 3 = 15

(b) The table below shows the 15 combinations and a schedule that allows all combinations without using the same t-shirt on consecutive days.

jeans \ t-shirts	1	2	3	4	5
1	1	4	7	10	13
2	14	2	5	8	11
3	12	15	3	6	9

2.46

The order in which the 4 toppings are selected does not matter so we have sampling without ordering.

If toppings may not be repeated, Eqn. (2.22) gives

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas}$$

If toppings may be repeated, we have sampling with replacement and without ordering. The number of such arrangements is

$$\binom{14+4}{4} = 3060 \text{ possible deluxe pizzas.}$$

2.47

$$\#\text{student seat selections} = 60 \cdot 59 \cdot 58 \cdot \ldots \cdot 16 = \frac{60!}{15!}$$

2.48

a b b a ⇒ 2 = 2!

a b c cab bca acb bac cba ⇒ 6 = 3!

abcd dabc cdab bcda
acbd dacb bdac cbda
adbc cadb bcad dbca
abdc cabd dcab bdca
acdb bacd dbac cdba
adcb badc cbad dcba

⇒ 24 = 4!

2.49 There are 3! permutations of which only one corresponds to the correct order; assuming equiprobable permutations:

$$P[\text{correct order}] = \frac{1}{3!} = \frac{1}{6}$$

2.50 # ways to cover all buckets $= 5\cdot4\cdot3\cdot2\cdot1 = 5!$
placement of 5 balls w 5 buckets $= 5^5$
probability all buckets covered $= 5!/5^5 = 0.0384$

2.51 Combinations of 2 from 2 objects : ab $\binom{2}{2} = 1$
Combinations of 2 " 3 objects : ab ac bc $\binom{3}{2} = \frac{3!}{2!} = 3$
Combinations of 2 " 4 objects : ab ac ad bc bd cd $\binom{4}{2} = \frac{4!}{2!2!} = 6$

2.52 8! arrangements of people around a table = 40320
Experiment: Select male or female for first spot : 2
Select first spot gender x 4
" 2nd spot gender x+1 4
" 3rd spot gender x 3
⋮ ⋮

$2 \times 4! \times 4! \doteq 1152$

2.53 Number ways of picking one out of 6 $= \binom{6}{1} = 6$

Number ways of picking two out of 6 $= \binom{6}{2} =$ ~~10~~ 15

Number ways of picking none, some or all of ~~5~~ 6 $= \sum_{j=0}^{6} \binom{6}{j} = 2^6 = 64$

2.54a The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M-m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M-m$ nondefectives out of 100 k is $\binom{100-k}{M-m}$.

The number of ways of choosing m defectives out of k
<u>and</u> $M-m$ non-defectives out of $100-k$ is

$$\binom{k}{m}\binom{100-k}{M-m}$$

$$P[m \text{ defectives in } M \text{ samples}] = \frac{\#\text{ outcomes with } k \text{ defective}}{\text{Total }\#\text{ of outcomes}} = \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}}$$

This is called the <u>Hypergeometric</u> distribution.

(b) $P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \dfrac{\binom{100-k}{M}}{\binom{100}{M}} + \dfrac{k\binom{100-k}{M-1}}{\binom{100}{M}}.$

2.55 Number ways of picking 20 raccoons out of $N = \binom{N}{20}$

Number ways of picking $\overset{4}{\cancel{5}}$ tagged raccoons out of $\cancel{10}\ 8$
and $\underset{16}{\cancel{15}}$ untagged raccoons out of $N-\underset{8}{\cancel{10}}$ $= \binom{\overset{8}{\cancel{10}}}{\underset{4}{\cancel{5}}}\binom{N-\overset{8}{\cancel{10}}}{\underset{16}{\cancel{15}}}$

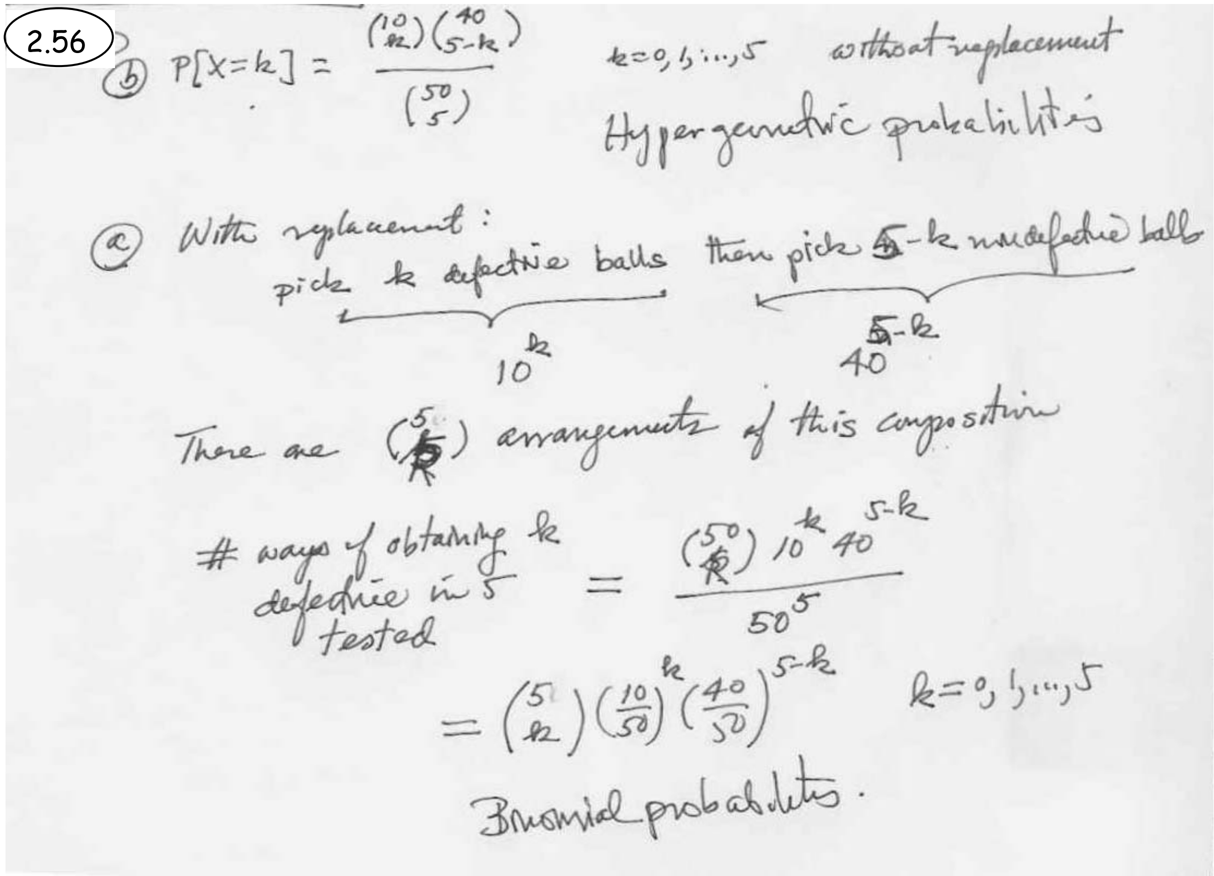
$$P[5 \text{ tagged out of 20 samples}] = \frac{\binom{\overset{8}{\cancel{10}}}{\cancel{5}4}\binom{N-\overset{8}{\cancel{10}}}{\cancel{15}\,16}}{\binom{N}{20}} \triangleq p(N)$$

$p(N)$ increases with N as long as $p(N)/p(N-1) > 1$

$$\frac{p(N)}{p(N-1)} = \frac{\binom{N-\overset{9}{\cancel{10}}}{\cancel{15}\,16}\binom{N-1}{20}}{\binom{N}{20}\binom{N-\cancel{11}9}{\cancel{15}\,16}} = \frac{(N-\overset{8}{\cancel{10}})(N-20)}{(N-\underset{4}{\cancel{25}})N} \geq 1$$

$(N-\overset{8}{\cancel{10}})(N-20) \geq (N-\overset{4}{\cancel{25}})N \Rightarrow 40 \geq N$
$p(40) = p(39) = \underset{0.305}{\cancel{.284}}$ maxima of $p(N)$.

2.56



(b) $P[X=k] = \dfrac{\binom{10}{k}\binom{40}{5-k}}{\binom{50}{5}}$ $k=0,1,\dots,5$ without replacement

Hypergeometric probabilities

(a) With replacement:

$\underbrace{\text{pick } k \text{ defective balls}}_{10^k}$ $\underbrace{\text{then pick } 5-k \text{ nondefective balls}}_{40^{5-k}}$

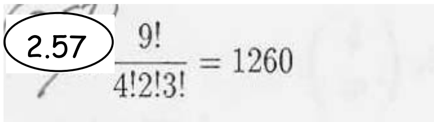
There are $\binom{5}{k}$ arrangements of this composition

ways of obtaining k defective in 5 tested $= \dfrac{\binom{50}{k}10^k 40^{5-k}}{50^5}$

$= \binom{5}{k}\left(\frac{10}{50}\right)^k\left(\frac{40}{50}\right)^{5-k}$ $k=0,1,\dots,5$

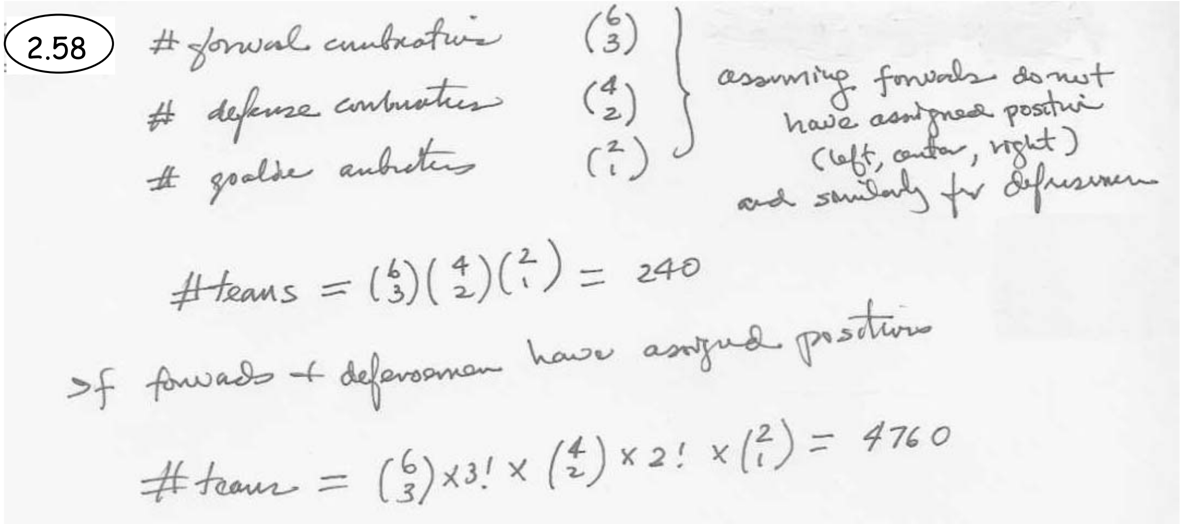
Binomial probabilities.

2.57



$$\frac{9!}{4!2!3!} = 1260$$

2.58



forward combinations $\binom{6}{3}$

defense combinations $\binom{4}{2}$

goalie combinations $\binom{2}{1}$

assuming forwards do not have assigned position (left, center, right) and similarly for defensemen

teams $= \binom{6}{3}\binom{4}{2}\binom{2}{1} = 240$

If forwards + defensemen have assigned positions

teams $= \binom{6}{3}\times 3!\times\binom{4}{2}\times 2!\times\binom{2}{1} = 4760$

2.59

Suppose each student is viewed as selecting one of the 7 days (e.g. placing a ball in one of 7 urns) then there are 7^{28} possible sequences of choices. Of the sequences that have 4 choices for each day there are

$$\frac{28!}{4!4!4!4!4!4!4!}$$ such sequences.

$$\therefore\ P[\text{4 students at each day}] = \frac{28!}{(4!)^7}\frac{1}{7^{28}}$$

2.60

$$\binom{n}{k} = \frac{n!}{k(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$$

2.61

a) Since N_i denotes the number of possible outcomes of the ith subset after $i-1$ subsets have been selected, it can be considered as the number of subpopulations of size k_i from a population of size $n-k_1-k_2-...-k_{i-1}$, hence

$$N_i = \binom{n-k_1-...-k_{i-1}}{k_i} \qquad i = 1,...,J-1$$

Note that after $J-1$ subsets arae selected, the set B_J is determined, i.e. $N_J = 1$.

b) The number of possible outcomes for B_1 is B_1, B_2 is N_2, etc. hence

$$\#\text{ partitions} = N_1N_2...N_{J-1} = \prod_{i=1}^{J-1}\frac{(n-k_1-...-k_{i-1})!}{k_1!(n-k_1-...-k_i)!} = \frac{n!}{k_1!k_2!...k_J!}$$

2.4 Conditional Probability

2.62 $A = \{N_1 \geq N_2\}$ $B = \{N_1 = 6\}$

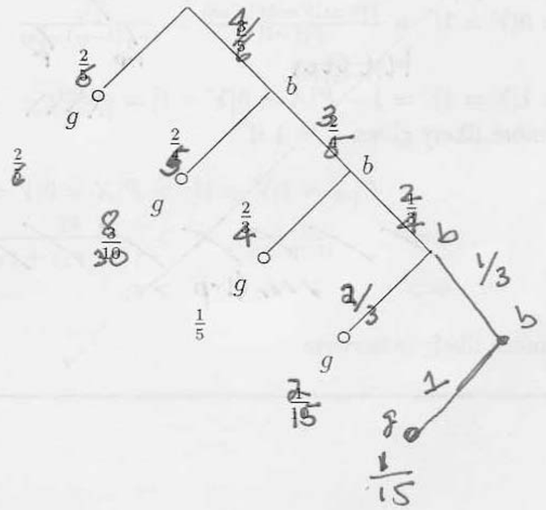
From problem 2.2 we have that $A \supset B$, therefore

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

and

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{6/36}{21/36} = \frac{2}{7}$$

2.63a



$$P[g] = \frac{2}{6}$$

$$P[bg] = P[b]P[g|b] = \cancel{\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}} \quad \frac{4}{6} \cdot \frac{2}{5} = \frac{8}{30}$$

$$P[bbg] = \cancel{\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}} = \frac{4}{6}\,\frac{3}{5}\,\frac{2}{4} = \frac{1}{5}$$

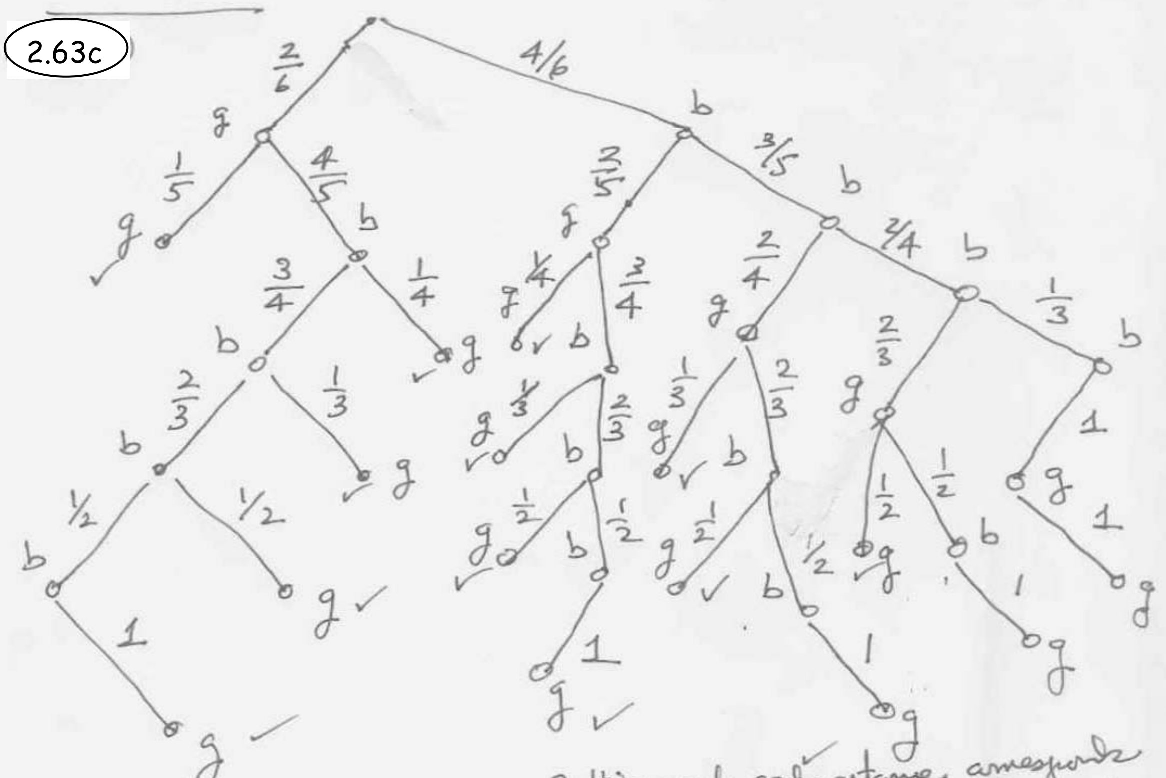
$$P[bbbg] = \cancel{\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}} = \frac{4}{6}\,\frac{3}{5}\,\frac{2}{4}\,\frac{2}{3} = \frac{2}{15}$$

$$P[bbbbg] = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}$$

b) $P[1 \text{ pen tested}] = P[g] \cancel{= \frac{2}{5}}$

$P[2] = P[bg]$ $P[3] = P[bbg]$ $P[4] = P[bbbg]$ $P[5] = P[bbbbg]$

2.63c



In this graph each outcome corresponds to a distinct arrangement of 4 b's and 2 w's. There are $\binom{6}{2}=15$ arrangements.

$$P[2\text{ tests}] = \frac{2}{6}\cdot\frac{1}{5} = \frac{1}{15}$$

$$P[3\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{1}{4}+\frac{4}{6}\frac{2}{5}\frac{1}{4} \quad \frac{1}{15}+\frac{1}{15} = \frac{2}{15}$$

$$P[4\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{1}{3}+\frac{4}{6}\left(\frac{2}{5}\right)\frac{3}{4}\frac{1}{3}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{1}{3} = \frac{3}{15}$$

$$P[5\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{2}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2} = \frac{4}{15}$$

$$P[6\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{2}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{1}{3}$$

$$= \frac{5}{15}$$

2.64

$$P[B\cap C/A] = P[\text{Bob \& Chris pick their names} / \text{Al picked his name}]$$

$$= \frac{P[B\cap C\cap A]}{P[A]} = \frac{P[\{abc\}]}{P[\{abc, acb\}]} = \frac{1/6}{2/6} = \frac{1}{2}.$$

$$P[C|A\cap B] = P[\text{Chris picks his name} \mid \text{Al \& Bob picked their name}]$$

$$= \frac{P[A\cap B\cap C]}{P[A\cap B]} = \frac{P[\{abc\}]}{P[\{abc\}]} = 1.$$

2.65

$$P[B|A] = \frac{P[A\cap B]}{P[A]} = \frac{P[\text{multiple of } 6]}{P[\text{even}]} = \frac{1/63}{1/3} = \frac{1}{21}$$

$$P[A|B] = \frac{P[A\cap B]}{P[A]} = \frac{P[\text{multiple of } 6]}{P[\text{multiple of } 6]} = 1.$$

2.66 from problem 2.8:

$$P[B|A] = \frac{P[A\cap B]}{P[A]} = \frac{P[\frac{3}{4}<U\leq 1]}{P[\,|U-\frac{1}{2}|>\frac{1}{4}]} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P[B|A] = \frac{P[A\cap B]}{P[B]} = \frac{P[\frac{3}{4}<U\leq 1]}{P[\frac{1}{2}<U\leq 1]} = \frac{1/4}{1/2} = \frac{1}{2}.$$

2.67 from problem 2.36

$$P[B|A] = \frac{P[A\cap B]}{P[A]} = \frac{P[x>8]}{P[x>4]} = \frac{1/8}{1/4} = \frac{1}{2}$$

$$P[A|B] = \frac{P[x>8]}{P[x>8]} = 1.$$

2.68

ⓐ $P[A] = P[\text{hand rests in last 10 minutes}]$

$$P[A] = p_{51} + p_{52} + \cdots + p_{60} = \frac{10}{60} = \frac{1}{6}$$

$$P[B] = p_{56} + p_{57} + p_{58} + p_{59} + p_{60} = \frac{5}{60} = \frac{1}{12}$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{1/12}{1/6} = \frac{1}{2}$$

ⓑ

$$P[A] = p_1\left(\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{60}\right)$$

$$P[B] = p_1\left(\frac{1}{56} + \frac{1}{57} + \cdots + \frac{1}{60}\right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\frac{1}{56} + \frac{1}{57} + \cdots + \frac{1}{60}}{\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{60}} = 0.477$$

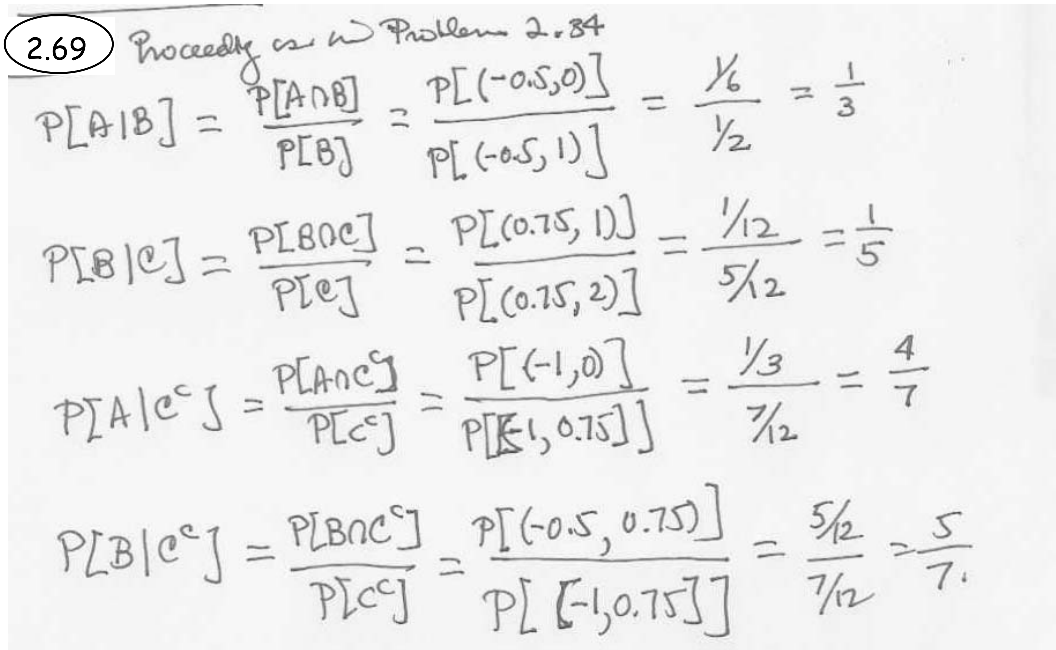
ⓒ

$$P[A] = \frac{1}{2}\left(\left(\frac{1}{2}\right)^{50} + \left(\frac{1}{2}\right)^{56} + \cdots + \left(\frac{1}{2}\right)^{59}\right)$$

$$P[B] = \frac{1}{2}\left(\left(\frac{1}{2}\right)^{55} + \cdots + \left(\frac{1}{2}\right)^{59}\right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\left(\frac{1}{2}\right)^{56} + \cdots \left(\frac{1}{2}\right)^{60}}{\left(\frac{1}{2}\right)^{51} + \cdots + \left(\frac{1}{2}\right)^{60}} = 0.030$$

2.69



Proceeding as in Problem 2.84

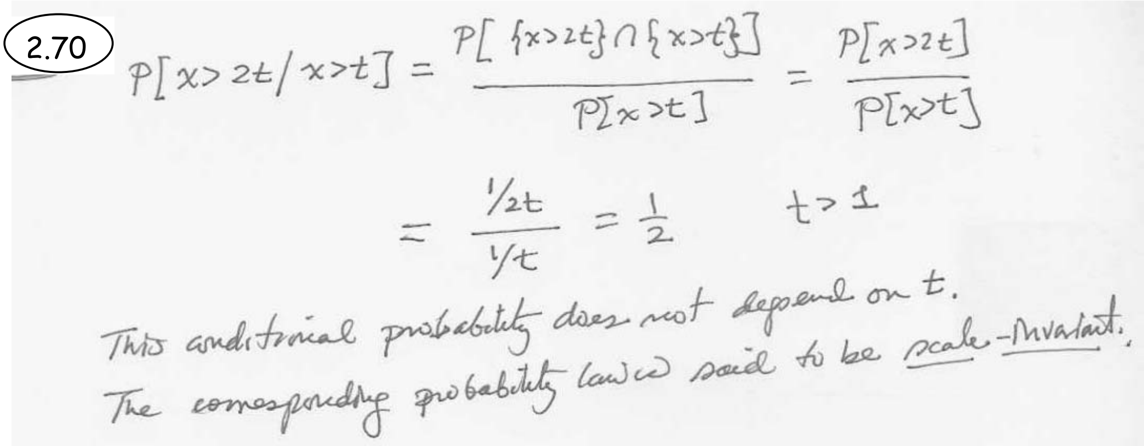
$$P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{P[(-0.5,0)]}{P[(-0.5,1)]} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P[B|C] = \frac{P[B\cap C]}{P[C]} = \frac{P[(0.75,1)]}{P[(0.75,2)]} = \frac{1/12}{5/12} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[A\cap C^c]}{P[C^c]} = \frac{P[(-1,0)]}{P[[-1,0.75]]} = \frac{1/3}{7/12} = \frac{4}{7}$$

$$P[B|C^c] = \frac{P[B\cap C^c]}{P[C^c]} = \frac{P[(-0.5,0.75)]}{P[[-1,0.75]]} = \frac{5/12}{7/12} = \frac{5}{7}.$$

2.70

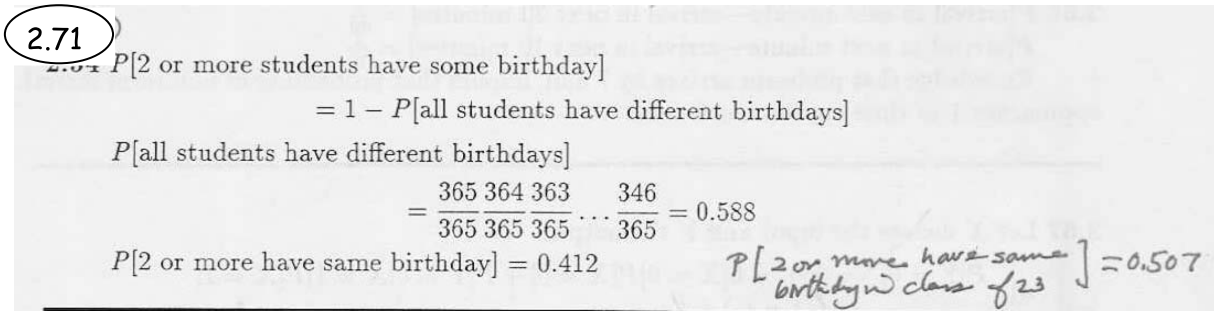


$$P[x>2t/x>t] = \frac{P[\{x>2t\}\cap\{x>t\}]}{P[x>t]} = \frac{P[x>2t]}{P[x>t]}$$

$$= \frac{1/2t}{1/t} = \frac{1}{2} \qquad t>1$$

This conditional probability does not depend on t.
The corresponding probability law is said to be scale-invariant.

2.71



P[2 or more students have some birthday]

$= 1 - P$[all students have different birthdays]

P[all students have different birthdays]

$$= \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{346}{365} = 0.588$$

P[2 or more have same birthday] $= 0.412$

P[2 or more have same birthday in class of 23] $= 0.507$

2.72 # of fingerprints = 2^L $L=64$ or $L=128$

Pick hash at random until we find a repeat.

Same as birthday problem (problem 2.7)

$$P[\text{all hashes different given } N \text{ tries}] = \frac{2^L}{2^L}\,\frac{2^L-1}{2^L}\cdots\frac{2^L-N+1}{2^L}$$

Find N so that

$$\frac{1}{2} = 1-\prod_{j=0}^{N-1}\frac{2^L-j}{2^L} = 1-p(N)$$

$$\ln p(N) = \sum_{j=0}^{N-1}\ln\left(1-\frac{j}{2^L}\right) \approx \sum_{j=0}^{N-1}-\frac{j}{2^L} = -\frac{1}{2^L}\sum_{j=0}^{N-1} j$$

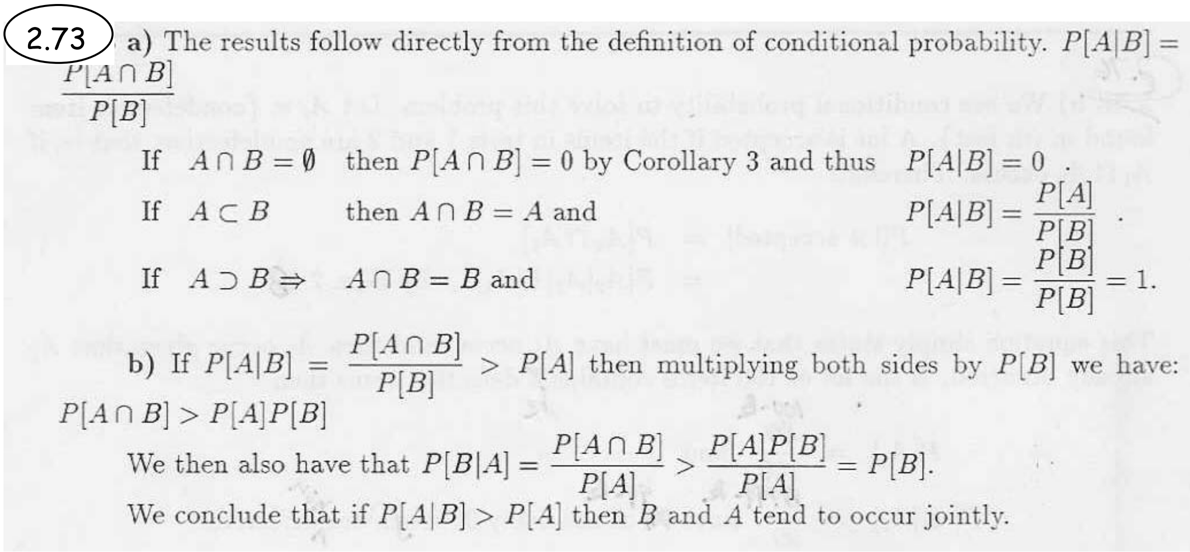
$$\approx -\frac{1}{2^L}\,\frac{N(N-1)}{2}$$

$$p(N) = e^{-\frac{N(N-1)}{2}\frac{1}{2^L}} \approx e^{-\frac{N^2}{2}\frac{1}{2^L}} = \frac{1}{2}$$

$$N \approx \sqrt{(2\ln 2)\,2^L} = 1.17\,2^{L/2}$$

for $L=64$ 2^{32} attempts required
$L=128$ 2^{64} "

2.73



a) The results follow directly from the definition of conditional probability. $P[A|B] = \dfrac{P[A \cap B]}{P[B]}$.

If $A \cap B = \emptyset$ then $P[A \cap B] = 0$ by Corollary 3 and thus $P[A|B] = 0$.

If $A \subset B$ then $A \cap B = A$ and $P[A|B] = \dfrac{P[A]}{P[B]}$.

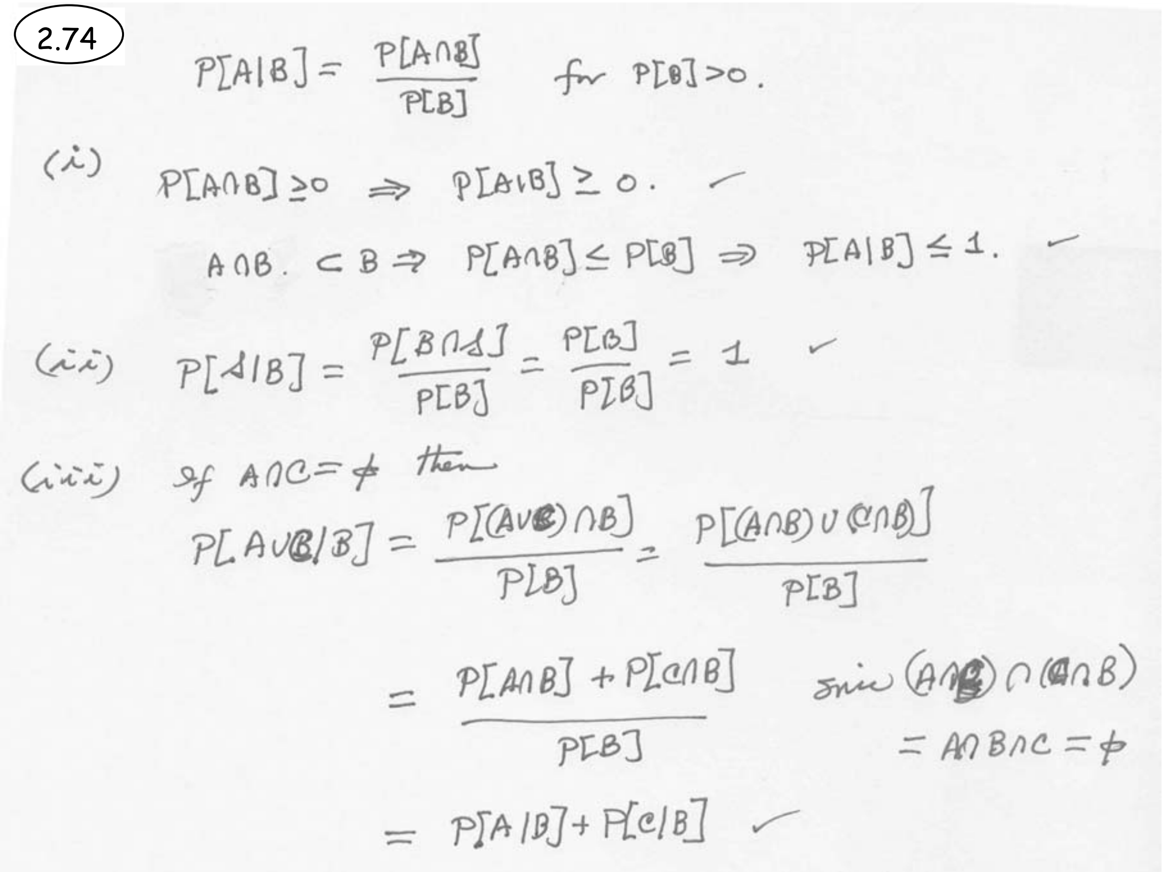
If $A \supset B \Rightarrow A \cap B = B$ and $P[A|B] = \dfrac{P[B]}{P[B]} = 1.$

b) If $P[A|B] = \dfrac{P[A \cap B]}{P[B]} > P[A]$ then multiplying both sides by $P[B]$ we have: $P[A \cap B] > P[A]P[B]$

We then also have that $P[B|A] = \dfrac{P[A \cap B]}{P[A]} > \dfrac{P[A]P[B]}{P[A]} = P[B].$

We conclude that if $P[A|B] > P[A]$ then B and A tend to occur jointly.

2.74



$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0.$$

(i) $P[A \cap B] \geq 0 \Rightarrow P[A|B] \geq 0.$ ✓

$A \cap B \subset B \Rightarrow P[A \cap B] \leq P[B] \Rightarrow P[A|B] \leq 1.$ ✓

(ii) $P[S|B] = \dfrac{P[B \cap S]}{P[B]} = \dfrac{P[B]}{P[B]} = 1$ ✓

(iii) If $A \cap C = \phi$ then

$$P[A \cup B/B] = \frac{P[(A \cup B) \cap B]}{P[B]} = \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}$$

$$= \frac{P[A \cap B] + P[C \cap B]}{P[B]} \quad \text{since } (A \cap B) \cap (C \cap B) = A \cap B \cap C = \phi$$

$$= P[A|B] + P[C|B]$$ ✓

2.75

$$\begin{aligned} P[A\cap B\cap C] &= P[A|B\cap C]P[B\cap C] \\ &= P[A|B\cap C]P[B|C]P[C] \end{aligned}$$

2.76

a) We use conditional probability to solve this problem. Let A_i = {nondefective item found in ith test}. A lot is accepted if the items in tests 1 and 2 are nondefective, that is, if $A_1 \cap A_2$ occurs. Therefore

$$\begin{aligned} P[\text{lot accepted}] &= P[A_2\cap A_1] \\ &= P[A_2|A_1]P[A_1] \quad \text{by Eqn. } 2.2\cancel{5}8 \end{aligned}$$

This equation simply states that we must have A_1 occur, and then A_2 occur given that A_1 already occurred. If the lot of 100 items contains ~~5~~ k defective items then

$$\begin{aligned} P[A_1] &= \frac{\overset{100-k}{\cancel{95}}}{100} \quad \text{and} \\ P[A_2|A_1] &= \frac{\cancel{94}\,99-k}{99} \quad \text{since } \overset{99-k}{\cancel{94}} \text{ of the many 99 itesm are } \overset{\text{non}}{\wedge}\text{defective.} \end{aligned}$$

Thus

$$P[\text{lot accepted}] = \cancel{\frac{94}{99}}\cdot\cancel{\frac{95}{100}}\cdot\frac{99-k}{99}\cdot\frac{100-k}{100}$$

(b) P[1 or more items in m tested are defective] > 99%

⇔ P[no items in m are defective] < 1%

$$P[A_m A_{m-1} \cdots A_1] = \frac{50}{100}\cdot\frac{49}{99}\cdots\frac{50-m+1}{100-m+1} = 0.01$$

for m=6 we have

$$P[A_6A_5A_4A_3A_2A_1] = \frac{50}{100}\cdots\frac{45}{95} = 0.0133$$

2.77 Let X denote the input and Y the output

ⓐ $P[Y=0] = P[Y=0 \mid X=0]\,P[X=0] + P[Y=0 \mid X=1]\,P[X=1]$

$= (1-\varepsilon_1)\,q + \varepsilon_2\,p.$

Similarly

$P[Y=1] = (1-\varepsilon_2)\,p + \varepsilon_1\,q$

ⓑ $P[X=0 \mid Y=1] = \dfrac{P[Y=1 \mid X=0]\,P[X=0]}{P[Y=1]} = \dfrac{\varepsilon_1\,q}{(1-\varepsilon_2)\,p + \varepsilon_1\,q}$

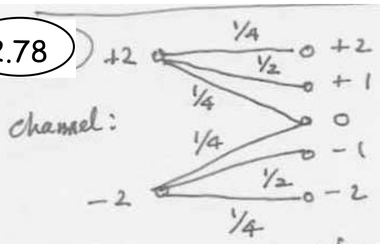
$P[X=1 \mid Y=1] = \dfrac{(1-\varepsilon_2)\,p}{(1-\varepsilon_2)\,p + \varepsilon_1\,q}$

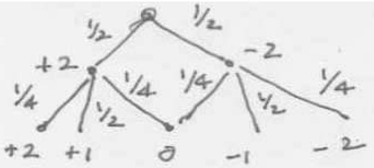
$P[X=1 \mid Y=1] > P[X=0 \mid Y=1]$

$\Leftrightarrow \quad (1-\varepsilon_2)\,p > \varepsilon_1\,q = \varepsilon_1(1-p)$

$\Leftrightarrow \quad p > \dfrac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1}$

2.78





ⓑ $P[X=+2, Y=+2] = P[Y=2 \mid X=2]\,P[X=2]$

$= \tfrac{1}{4}\,\tfrac{1}{2} = \tfrac{1}{8}$

$P[X=2, Y=1] = \tfrac{1}{2}\,\tfrac{1}{2} = \tfrac{1}{4}$

$P[X=2, Y=0] = \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{8}$

$P[X=-2, Y=0] = \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{8}$

$P[X=-2, Y=1] = \tfrac{1}{2}\,\tfrac{1}{2} = \tfrac{1}{4}$

$P[X=-2, Y=-2] = \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{8}$

ⓒ $P[Y=+2] = \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{8} = P[Y=-2]$

$P[Y=+1] = \tfrac{1}{2}\,\tfrac{1}{2} = \tfrac{1}{4} = P[Y=-1]$

$P[Y=0] = 2(\tfrac{1}{2}\cdot\tfrac{1}{4}) = \tfrac{2}{8} = P[Y=0]$

ⓓ $P[X=2 \mid Y=k] = \dfrac{P[Y=k \mid X=2]\,P[X=2]}{P[Y=k]}$

$$= \begin{cases} \tfrac{1}{8}/\tfrac{1}{8} = 1 & k=2 \\ \tfrac{1}{4}/\tfrac{1}{4} = 1 & k=1 \\ \tfrac{1}{8}/\tfrac{1}{4} = \tfrac{1}{2} & k=0 \\ 0 & \text{other } k \end{cases}$$

2.79

ⓐ $P[N=k] = P[N=k \mid \text{coin } 1]\, P[\text{coin } 1] + P[N=k \mid \text{coin } 2]\, P[\text{coin } 2]$

$$= \binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

ⓑ $$P[\text{coin } 1 / N=k] = \frac{P[N=k \mid \text{coin } 1]\, P[\text{coin } 1]}{P[N=k]} \qquad k=0,1,2,3$$

$$= \frac{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2}}{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}}$$

ⓒ Coin 1 is more probable if

$$\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} > \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{3-k} = 2^k \left(\frac{1}{2}\right)^{3-k} = \left(\frac{1}{8}\right) 4^k$$

$$0 > \ln \frac{1}{8} + k \ln 4$$

$$1.5 = \frac{\ln 8}{\ln 4} > k$$

Coin 1 more probable if $N = 0$ or 1
Coin 2 more probable otherwise.

ⓓ In general coin 1 is more probable if

$$\binom{n}{k} p_1^k (1-p_1)^{n-k} \frac{1}{2} > \binom{n}{k} p_2^k (1-p_2)^{n-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{n-k} = \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^n$$

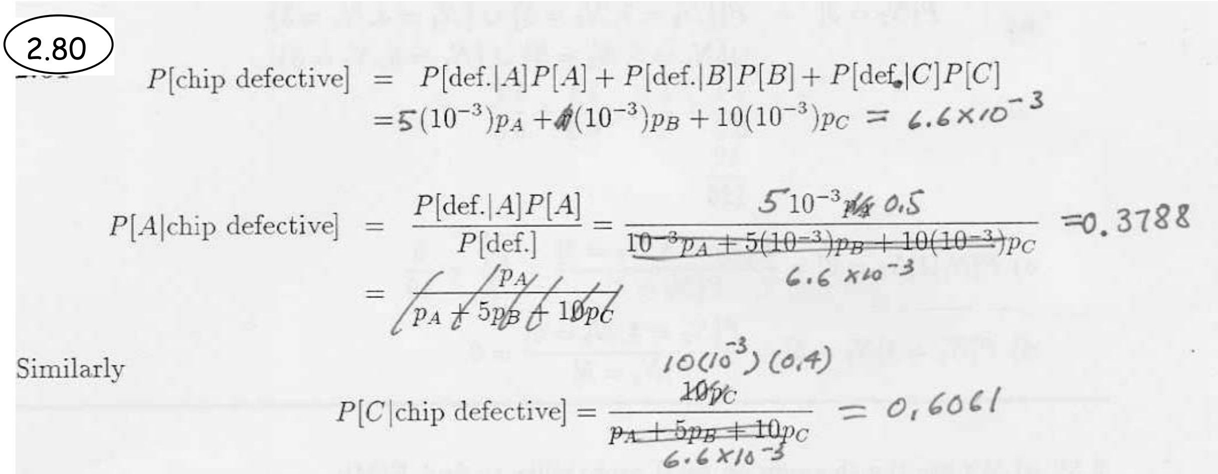
$$T = \frac{n \ln\left(\frac{1-p_1}{1-p_2}\right)}{\ln\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)} > k$$

ⓔ If $p_2 = 1$ then $P[N=k \mid \text{coin } 2] = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{otherwise} \end{cases}$

We cannot determine coin with certainty only if all tosses are heads.

$$P[\text{coin } 1 \mid m \text{ heads}] = (1-p_1)^m / [1 + (1-p_1)^m]$$

2.80



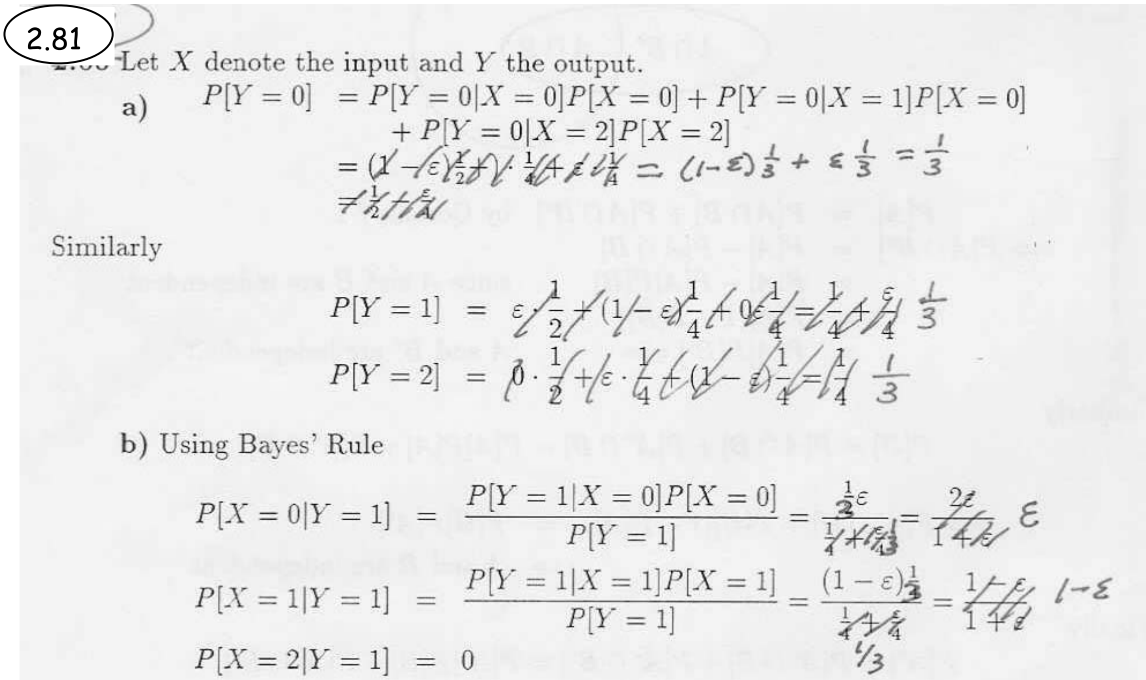
$$P[\text{chip defective}] = P[\text{def.}|A]P[A] + P[\text{def.}|B]P[B] + P[\text{def.}|C]P[C]$$
$$= 5(10^{-3})p_A + \cancel{5}(10^{-3})p_B + 10(10^{-3})p_C = 6.6\times10^{-3}$$

$$P[A|\text{chip defective}] = \frac{P[\text{def.}|A]P[A]}{P[\text{def.}]} = \frac{5\,10^{-3}\cancel{p_A}\ 0.5}{\cancel{10^{-3}p_A + 5(10^{-3})p_B + 10(10^{-3})}p_C \atop 6.6\times10^{-3}} = 0.3788$$
$$= \cancel{\frac{p_A}{p_A + 5p_B + 10p_C}}$$

Similarly

$$P[C|\text{chip defective}] = \frac{\cancel{10p_C} \atop 10(10^{-3})(0.4)}{\cancel{p_A + 5p_B + 10p_C} \atop 6.6\times10^{-3}} = 0.6061$$

2.81



Let X denote the input and Y the output.

a) $$P[Y=0] = P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] + P[Y=0|X=2]P[X=2]$$
$$= \cancel{(1-\varepsilon)\frac{1}{2} + \varepsilon\cdot\frac{1}{4} + \varepsilon\frac{1}{4}} = (1-\varepsilon)\frac{1}{3} + \varepsilon\frac{1}{3} = \frac{1}{3}$$
$$\cancel{= \frac{1}{2} + \frac{\varepsilon}{4}}$$

Similarly

$$P[Y=1] = \cancel{\varepsilon\cdot\frac{1}{2} + (1-\varepsilon)\frac{1}{4} + 0\varepsilon\frac{1}{4} = \frac{1}{4} + \frac{\varepsilon}{4}}\ \frac{1}{3}$$
$$P[Y=2] = \cancel{0\cdot\frac{1}{2} + \varepsilon\cdot\frac{1}{4} + (1-\varepsilon)\frac{1}{4} = \frac{1}{4}}\ \frac{1}{3}$$

b) Using Bayes' Rule

$$P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{\cancel{2}3}\varepsilon}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}}\frac{1}{3}} = \cancel{\frac{2\varepsilon}{1+\varepsilon}}\ \varepsilon$$
$$P[X=1|Y=1] = \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{\cancel{4}3}}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}} \atop 1/3} = \cancel{\frac{1-\varepsilon}{1+\varepsilon}}\ 1-\varepsilon$$
$$P[X=2|Y=1] = 0$$

2.5 Independence of Events

2.82

$P[A\cap B] = P[\{1\}] = \frac{1}{4} = P[A]\,P[B] = \frac{1}{2}\,\frac{1}{2}$ ✓

$P[A\cap C] = P[\{1\}] = \frac{1}{4} = P[A]\,P[C] = \frac{1}{2}\,\frac{1}{2}$ ✓

$P[B\cap C] = P[\{1\}] = \frac{1}{4} = P[B]\,P[C] = \frac{1}{2}\,\frac{1}{2}$ ✓

$P[A\cap B\cap C] = P[\{1\}] = \frac{1}{4} \neq P[A]\,P[B]\,P[C] = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} = \frac{1}{8}$

$\Rightarrow$ Not independent

2.83

$P[A\cap B] = P[\frac{1}{4} < U < \frac{1}{2}] = \frac{1}{4} = P[A]\,P[B] = \frac{1}{2}\,\frac{1}{2}$ ✓ A&B indep

$P[A\cap C] = 0 \neq P[A]\,P[B] = \frac{1}{2}\,\frac{1}{2} \Rightarrow$ A&C Not Indep.

$P[B\cap C] = P[\frac{1}{2} < U < \frac{3}{4}] = \frac{1}{4} = P[B]\,P[C] = \frac{1}{2}\cdot\frac{1}{2}$ ✓ B&C indep.

2.84

Let $A = \{\text{Alice makes shot}\}$ $M = \{\text{Mary makes shot}\}$

We assume that A and M are independent

$P[A] = p_a$

since $AM^c \cap A^cM = \phi$

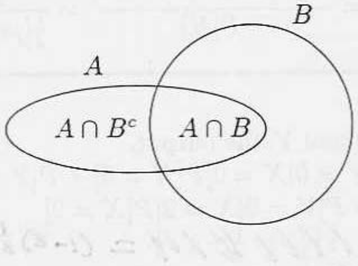
$P[\text{one makes a shot}] = P[AM^c \cup A^cM] = P[AM^c] + P[A^cM]$

$= p_a(1-p_m) + (1-p_a)\,p_m$ by independence

$P[AM] = p_a\,p_m$

$P[A^cM^c] = (1-p_a)(1-p_m).$

2.85 The event A is the union of the mutually exclusive events $A \cap B$ and $A \cap B^c$, thus



$$\begin{aligned}
P[A] &= P[A \cap B] + P[A \cap B^c] && \text{by Corollary 1} \\
\Longrightarrow P[A \cap B^c] &= P[A] - P[A \cap B] \\
&= P[A] - P[A]P[B] && \text{since } A \text{ and } B \text{ are independent} \\
&= P[A](1 - P[B]) \\
&= P[A]P[B^c] \Longrightarrow && A \text{ and } B^c \text{ are independent}
\end{aligned}$$

Similarly

$$P[B] = P[A \cap B] + P[A^c \cap B] = P[A]P[A] + P[A^c \cap B]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B] = P[B](1 - P[A]) &= P[B]P[A^c] \\
&\Rightarrow A \text{ and } B \text{ are independent}
\end{aligned}$$

Finally

$$P[A^c] = P[A^c \cap B] + P[A^c \cap B^c] = P[A^c]P[B] + P[A^c \cap B^c]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B^c] = P[A^c](1 - P[B]) &= P[A^c]P[B^c] \\
&\Rightarrow A^c \text{ and } B^c \text{ are independent}
\end{aligned}$$

2.86

$$P[A|B] = P[A|B^c] \Longrightarrow \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B^c]}{P[B^c]}$$

$$\begin{aligned}
\Longrightarrow P[A \cap B]P[B^c] &= P[A \cap B^c]P[B] \\
&= (P[A] - P[A \cap B])P[B] \quad \text{see Prob. 2.58 solution}
\end{aligned}$$

$$\Longrightarrow P[A \cap B]\underbrace{(P[B^c] + P[B])}_{1} = P[A]P[B]$$

$$\Longrightarrow P[A \cap B] = P[A]P[B]$$

2.87

ⓐ $P[A \cup B] = P[A] + P[B] - P[A \cap B] = p_A + p_B - p_A p_B$

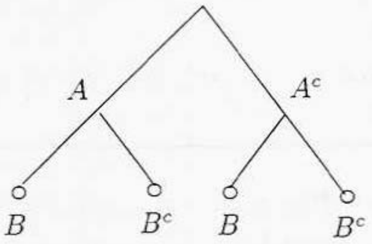
ⓑ $P[A \cup B] = P[A] + P[B] = p_A + p_B$

ⓒ $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$

$= p_A + p_B + p_C - p_A p_B - p_A p_C - p_B p_C + p_A p_B p_C$

ⓓ $P[A \cup B \cup C] = p_A + p_B + p_C$

2.88 We use a tree diagram to show the sequence of events. First we choose an urn, so A or A^c occurs. We then select a ball, so B or B^c occurs:



Now A and B are independent events if

$$P[B|A] = P[B]$$

But

$$P[B|A] = P[B] = P[B|A]P[A] + P[B|A^c]P[A^c]$$

$$\begin{aligned} \Longrightarrow P[B|A](1 - P[A]) &= P[B|A^c]P[A^c] \\ \Longrightarrow P[B|A] = P[B|A^c] &\quad \text{prob. of } B \text{ is the same given } A \text{ or } A^c\text{, that is,} \\ &\quad \text{the probability of } B \text{ is the same for both urns.} \end{aligned}$$

2.89

a) $P[A]P[B^c]P[C^c] + P[A^c]P[B]P[C^c] + P[A^c]P[B^c]P[C]$

b) $P[A]P[B]P[C^c] + P[A^c]P[B]P[C] + P[A]P[B^c]P[C]$

c) $1 - P[A^c]P[B^c]P[C^c]$

d) $P[A]P[B]P[C^c] + P[A]P[B^c]P[C] + P[A^c]P[B]P[C] + P[A]P[B]P[C]$

e) $P[A^c]P[B^c]P[C^c]$

2.90

Series $P[D_a] = P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$

Parallel $P[D_a] = P[A_1 \cup A_2 \cup A_3]$

$$= P[A_1] + P[A_2] + P[A_3] - P[A_1 \cap A_2] - P[A_1 \cap A_3] - P[A_2 \cap A_3] + P[A_1 \cap A_2 \cap A_3]$$

$$= p_{A_1} + p_{A_2} + p_{A_3} - p_{A_1}p_{A_2} - p_{A_1}p_{A_3} - p_{A_2}p_{A_3} + p_{A_1}p_{A_2}p_{A_3}$$

2-of-3 $P[D_a] = P[A_1 \cap A_2 \cap A_3] + P[A_1^c \cap A_2 \cap A_3] + P[A_1 \cap A_2^c \cap A_3] + P[A_1 \cap A_2 \cap A_3^c]$

$$= p_{A_1}p_{A_2}p_{A_3} + (1-p_{A_1})p_{A_2}p_{A_3} + p_{A_1}(1-p_{A_2})p_{A_3} + p_{A_1}p_{A_2}(1-p_{A_3})$$

2.91

$P[\text{System Up}] = P[(A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})]$

$$= P[A_{11} \cap A_{12}] + P[A_{21} \cap A_{22}] + P[A_{31} \cap A_{32}] - P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22}]$$

$$- P[A_{11} \cap A_{12} \cap A_{31} \cap A_{32}] - P[A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$

$$+ P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$

$$= p_{A_{11}}p_{A_{12}} + p_{A_{21}}p_{A_{22}} + p_{A_{31}}p_{A_{32}} - p_{A_{11}}p_{A_{12}}p_{A_{21}}p_{A_{22}}$$

$$- p_{A_{11}}p_{A_{12}}p_{A_{31}}p_{A_{32}} - p_{A_{21}}p_{A_{22}}p_{A_{31}}p_{A_{32}}$$

$$+ p_{A_{11}}p_{A_{12}}p_{A_{21}}p_{A_{22}}p_{A_{31}}p_{A_{32}}$$

2.92 Events A and B are independent iff

$$P[A \cap B] = P[A]P[B]$$

In terms of relative frequencies we expect

$$\underbrace{f_{A\cap B}n}_{\substack{\text{rel. freq. if}\\ \text{joint occurrence}\\ \text{of } A \text{ and } B}} = f_A(n)f_B(n)$$

rel. freq.'s of A and B

2.93 Let the jth bit in the hex character be B_j
To test independence we need:
All pairs ~~of~~ should satisfy $f_{B_j \cap B_k} \approx f_{B_j} f_{B_k}$

All triplets should satisfy $f_{B_j \cap B_k \cap B_l} \approx f_{B_j} f_{B_k} f_{B_l}$

Quadruplets should satisfy $f_{B_1 \cap B_2 \cap B_3 \cap B_4} \approx f_{B_1} f_{B_2} f_{B_3} f_{B_4}$

Note Relative frequencies for different B_j need not be the same.

2.94
$$\begin{aligned} P[\text{System Up}] &= P[\text{at least one controller is working}] \times \\ &\quad\, P[\text{at least two peripherals are working}] \end{aligned}$$

$$\begin{aligned} P[\text{at least one controller working}] &= 1 - P[\text{both not working}] \\ &= 1 - p^2 \end{aligned}$$

$$\therefore P[\text{System Up}] = (1-p^2)\{(1-a)^3 + 3(1-a)^2 a\}$$

2.95)

$P[A_0 \cap B_0] = (1-p)(1-\varepsilon)$

$P[B_0] = (1-p)(1-\varepsilon) + p\varepsilon$

$P[A_0] = (1-p)$

$P[A_0 \cap B_0] = P[B_0]P[A_0]$

$\Leftrightarrow \quad (1-p)(1-\varepsilon)$ [illegible] $= [(1-p)(1-\varepsilon) + p\varepsilon](1-p)$

$\Leftrightarrow \quad (1$ [illegible] $\varepsilon) = (1-p)(1-\varepsilon) + p\varepsilon$

$\Leftrightarrow \quad (1-\varepsilon)p = p\varepsilon$

$\Leftrightarrow \quad \varepsilon = \frac{1}{2}.$

Channel cannot transmit information if output is independent of the input.

2.96) Regardless of the value of ε, we always have

$P[X=2 \mid Y=1] = 0 \neq P[X=2] = \frac{1}{3}$

$\therefore$ the output cannot be independent of the input.

2.6 Sequential Experiments

2.97

(a) $P[0 \text{ or } 1 \text{ errors}] = (1-p)^{100} + 100(1-p)^{99}p \qquad p = 10^{-2}$

$= 0.3660 + .3697$

$= 0.7357$

(b) $p_R = P[\text{Retransmission Required}] = 1 - P[0 \text{ or } 1 \text{ error}] = 0.2642$

$P[M \text{ transmissions in total}] = (1-p_R)\,p_R^{m} \qquad M = 1, 2, \ldots$

$$P[M \text{ or more transmissions required}] = \sum_{j=M}^{\infty}(1-p)\,p_R^{\cancel{M}\,j} = \cancel{p}\,p_R^{M}\sum_{j=0}^{\infty}(1-p)\,p_R^{j}$$

$$= p_R^{m}$$

2.98

(a) $P[N > 1] = 1 - P[N=0 \text{ or } N=1]$

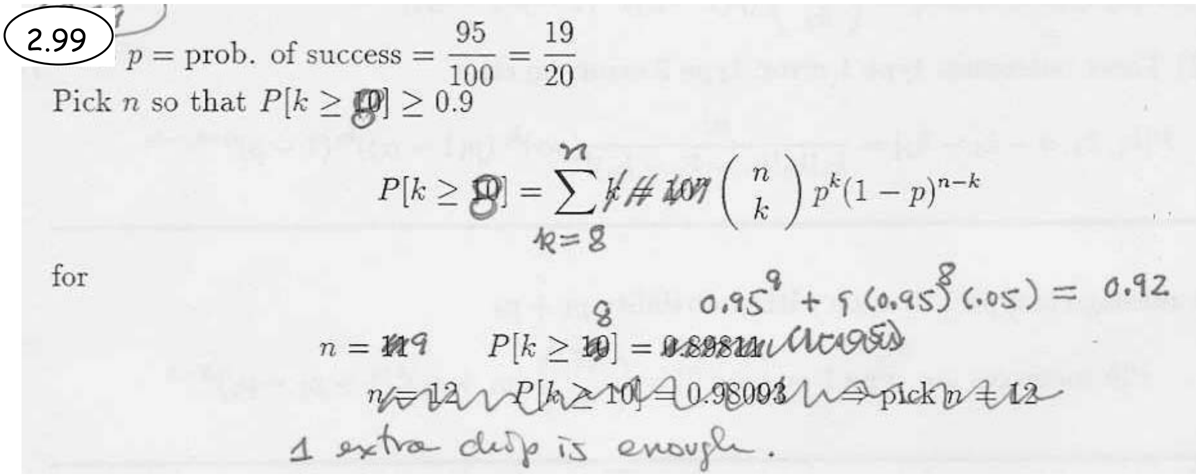
$= 1 - (1-p)^n \mp n(1-p)^{n-1}p.$

(b)

$P[N > 0] = 0.99 = 1 - (1-0.1)^n$

$0.01 = (0.9)^n$

$$n = \frac{\ln 100}{\ln 1/0.9} = 44$$



2.99

$p =$ prob. of success $= \frac{95}{100} = \frac{19}{20}$

Pick n so that $P[k \geq \cancel{10}\, 8] \geq 0.9$

$$P[k \geq \cancel{10}\, 8] = \sum_{k=8}^{n} \cancel{k=10} \binom{n}{k} p^k (1-p)^{n-k}$$

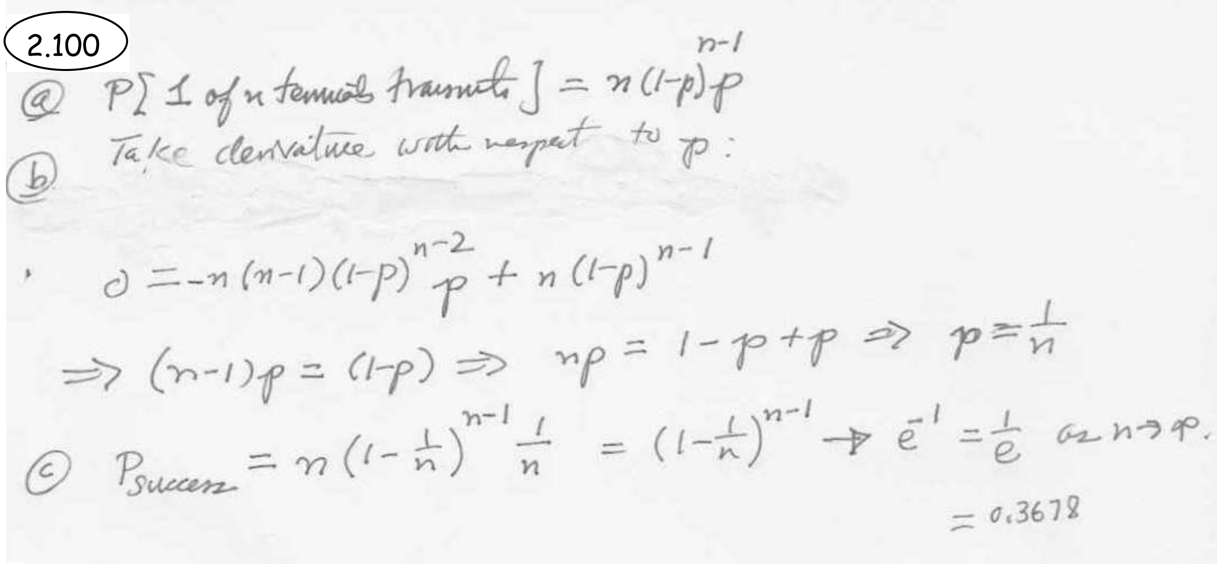
for

$0.95^9 + 9(0.95)^8(.05) = 0.92$

$n = \cancel{11}\, 9 \quad P[k \geq \cancel{10}\, 8] = \cancel{0.89811}$

~~$n = 12 \quad P[k \geq 10] = 0.98093 \Rightarrow$ pick $n = 12$~~

1 extra chip is enough.



2.100

ⓐ $P[$1 of n terminals transmits$] = n(1-p)^{n-1}p$

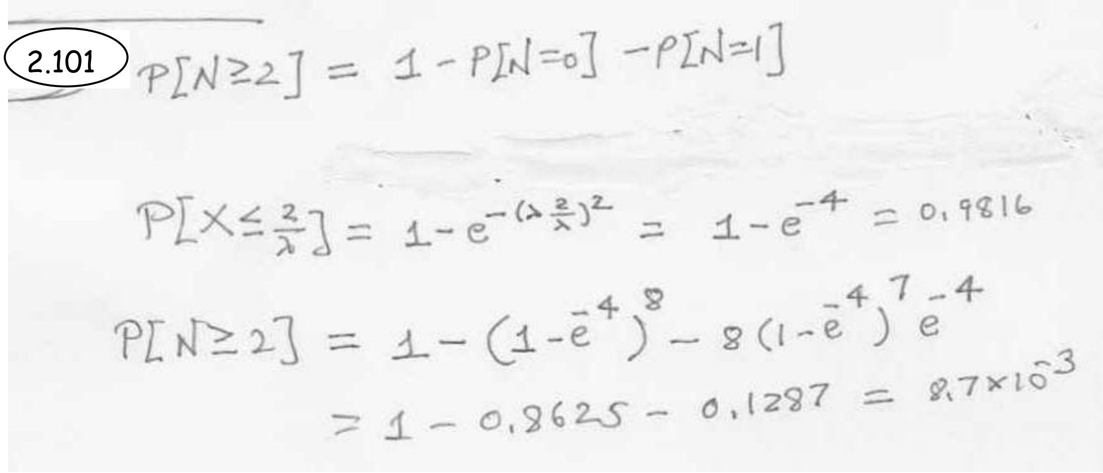
ⓑ Take derivative with respect to p:

$$0 = -n(n-1)(1-p)^{n-2}p + n(1-p)^{n-1}$$

$$\Rightarrow (n-1)p = (1-p) \Rightarrow np = 1-p+p \Rightarrow p = \frac{1}{n}$$

ⓒ $P_{success} = n\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} = \left(1-\frac{1}{n}\right)^{n-1} \to e^{-1} = \frac{1}{e}$ as $n \to \infty$.

$= 0.3678$



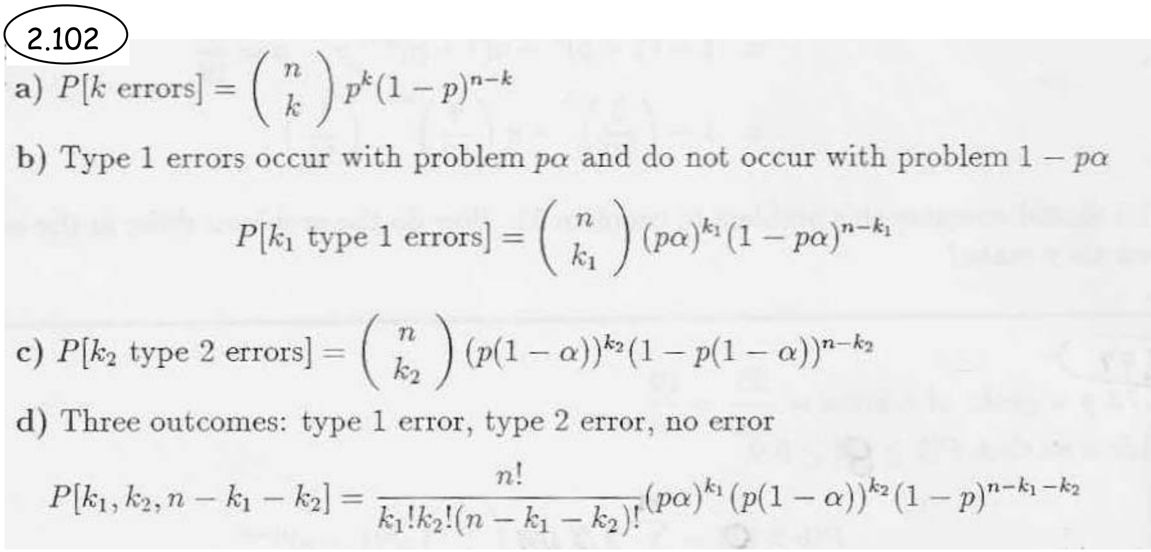
2.101

$$P[N \geq 2] = 1 - P[N=0] - P[N=1]$$

$$P\left[X \leq \tfrac{2}{\lambda}\right] = 1 - e^{-(\lambda \frac{2}{\lambda})^2} = 1 - e^{-4} = 0.9816$$

$$P[N \geq 2] = 1 - (1-e^{-4})^8 - 8(1-e^{-4})^7 e^{-4}$$

$$= 1 - 0.8625 - 0.1287 = 8.7\times10^{-3}$$



2.102

a) $P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$

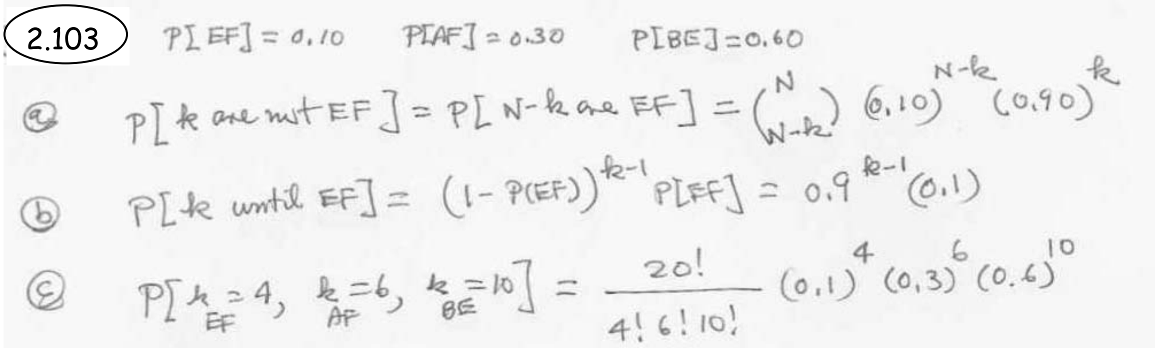
b) Type 1 errors occur with problem $p\alpha$ and do not occur with problem $1 - p\alpha$

$$P[k_1 \text{ type 1 errors}] = \binom{n}{k_1} (p\alpha)^{k_1} (1-p\alpha)^{n-k_1}$$

c) $P[k_2 \text{ type 2 errors}] = \binom{n}{k_2} (p(1-\alpha))^{k_2} (1-p(1-\alpha))^{n-k_2}$

d) Three outcomes: type 1 error, type 2 error, no error

$$P[k_1, k_2, n-k_1-k_2] = \frac{n!}{k_1!k_2!(n-k_1-k_2)!}(p\alpha)^{k_1}(p(1-\alpha))^{k_2}(1-p)^{n-k_1-k_2}$$



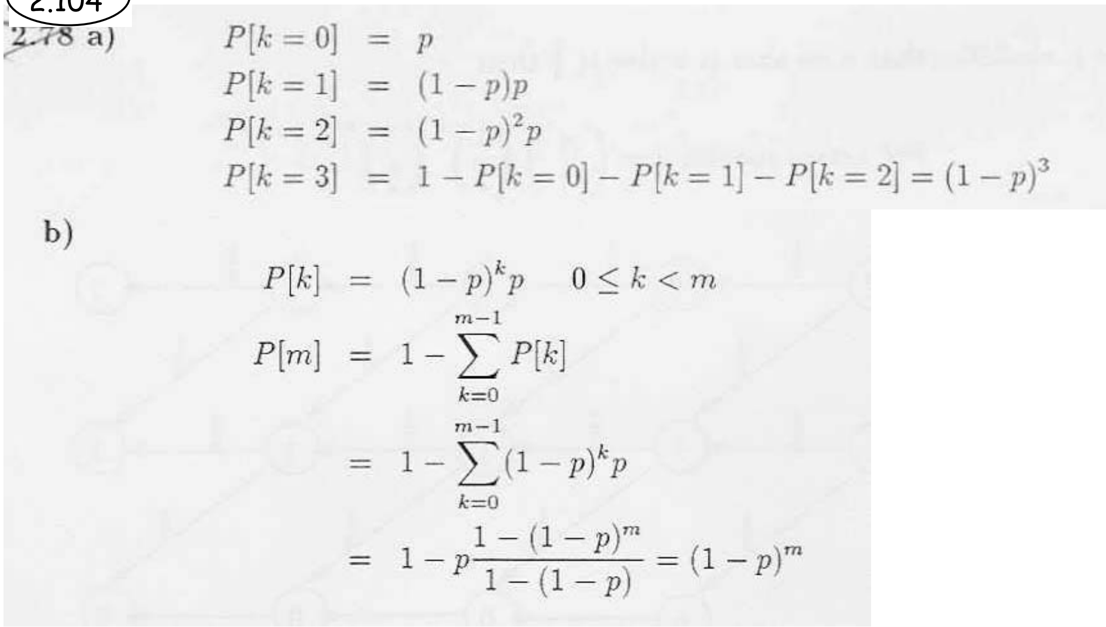
2.103 $P[EF] = 0.10$ $P[AF] = 0.30$ $P[BE] = 0.60$

ⓐ $P[k \text{ are not } EF] = P[N-k \text{ are } EF] = \binom{N}{N-k} (0.10)^{N-k} (0.90)^{k}$

ⓑ $P[k \text{ until } EF] = (1-P(EF))^{k-1} P[EF] = 0.9^{k-1}(0.1)$

ⓒ $P[k_{EF} = 4,\ k_{AF} = 6,\ k_{BE} = 10] = \frac{20!}{4!\,6!\,10!} (0.1)^{4} (0.3)^{6} (0.6)^{10}$

2.104



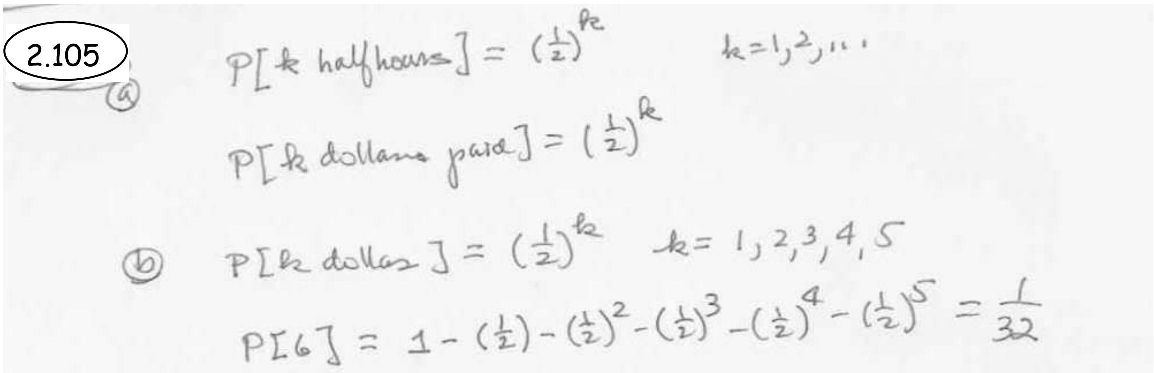
~~2.78~~ a)

$$\begin{aligned} P[k=0] &= p \\ P[k=1] &= (1-p)p \\ P[k=2] &= (1-p)^2 p \\ P[k=3] &= 1 - P[k=0] - P[k=1] - P[k=2] = (1-p)^3 \end{aligned}$$

b)

$$\begin{aligned} P[k] &= (1-p)^k p \quad 0 \le k < m \\ P[m] &= 1 - \sum_{k=0}^{m-1} P[k] \\ &= 1 - \sum_{k=0}^{m-1} (1-p)^k p \\ &= 1 - p\frac{1-(1-p)^m}{1-(1-p)} = (1-p)^m \end{aligned}$$

2.105



ⓐ $P[k \text{ halfhours}] = (\tfrac{1}{2})^k \qquad k = 1, 2, \ldots$

$P[k \text{ dollars paid}] = (\tfrac{1}{2})^k$

ⓑ $P[k \text{ dollars}] = (\tfrac{1}{2})^k \quad k = 1, 2, 3, 4, 5$

$P[6] = 1 - (\tfrac{1}{2}) - (\tfrac{1}{2})^2 - (\tfrac{1}{2})^3 - (\tfrac{1}{2})^4 - (\tfrac{1}{2})^5 = \frac{1}{32}$

2.106

~~2.80~~ $P[k$ tosses required until heads comes up ~~twice~~ three times$] = P[$heads in kth toss ~~|~~ 2 heads in $k-1$ tosses$]P[$2 heads in $k-1$ tosses$] = P[A|B]P[B]$.

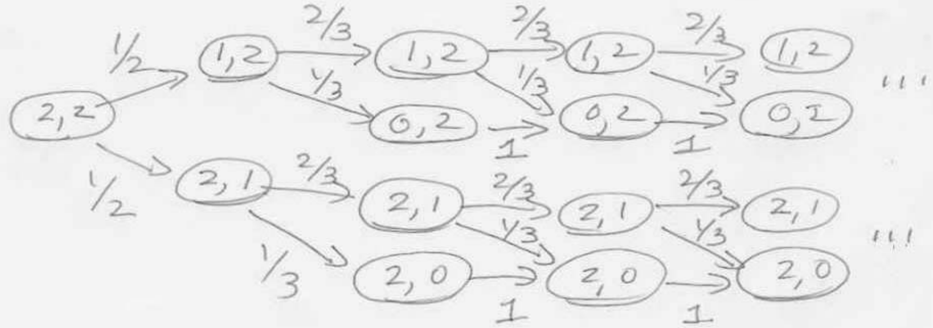
Now $P[A|B] = P[$2 heads in first $k-1$ tosses$] = \binom{k-1}{2} p^2(1-p)^{k-3}$

Thus $P[A|B]P[B] = P[A|B]p = \binom{k-1}{2} p^3(1-p)^{k-3}$ $k=3,4,\ldots$

2.107

The first draw is key since that ball is not put back.

Let (j,k) be a state where $j = \#$ black balls in urn $k = \#$ white balls in urn



ⓑ $P[bb] = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ $P[bw] = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$

$P[ww] = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ $P[wb] = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$

$P[bbw] = \frac{1}{2}\frac{1}{3}1 = \frac{1}{6}$ $P[bww] = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$ $P[bwb] = \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{1}{9}$

$P[wwb] = \frac{1}{2}\frac{1}{3}1 = \frac{1}{6}$ $P[wbb] = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$ $P[wbw] = \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{1}{9}$

ⓒ $P[(0,2)$ after 3 draws$] = P[bbw] + P[bwb] = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

Similarly $P[(2,0)$ after 3 draws$] = \frac{5}{16}$

2.107

ⓓ $P[(2,0)$ after $n] = P[$1st draw is white and at least one white in $n-1]$

$$= \frac{1}{2}\left[1 - \underbrace{\left(\frac{2}{3}\right)^{n-1}}_{R\text{ all blacks}}\right]$$

2.108

a) $p_0(1) = \frac{1}{2} \qquad p_1(1) = \frac{1}{2}$

b) $p_0(n+1) = \frac{2}{3}p_0(n) + \frac{1}{6}p_1(n)$

$p_1(n+1) = \frac{1}{3}p_0(n) + \frac{5}{6}p_1(n)$

In matrix notation, we have

$$[p_0(n+1), p_1(n+1)] = [p_0(n), p_1(n)] \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

or

$$\underline{p}(n+1) = \underline{p}(n)\mathbb{P}$$

c) $\underline{p}(0) = \left[\frac{1}{2}, \frac{1}{2}\right]$

$\underline{p}(1) = \underline{p}(0)\mathbb{P}$

$\underline{p}(2) = \underline{p}(1)\mathbb{P} = \underline{p}(0)\mathbb{P}\mathbb{P} = \underline{p}(0)\mathbb{P}^2$

in general

$$\underline{p}(n) = \underline{p}(0)\mathbb{P}^n$$

To find $\mathbb{P}^n$ we note that if $\mathbb{P}$ has eigenvalues λ_1, λ_2 and eigenvectors $\underline{e}_1$, $\underline{e}_1$ then

$$\mathbb{P} = \mathbb{E}\underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}\mathbb{E}^{-1} \qquad \text{where } \mathbb{E} \text{ has } \underline{e}_1 \text{ and } \underline{e}_2 \text{ as columns}$$

and

$$\begin{aligned} \mathbb{P}^n &= (\mathbb{E}\Lambda E^{-1})(E\Lambda E^{-1})...(E\Lambda E^{-1}) \qquad n \text{ times} \\ &= E\Lambda(E^{-1}E)\Lambda...(E^{-1}E)\Lambda E^{-1} \\ &= E\Lambda^n E^{-1} \end{aligned}$$

Now $\mathbb{P} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$ and eigenvector $\underline{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{e}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Thus

$$\begin{aligned} \mathbb{P}^n &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^n \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n} & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \underline{p}(n) &= \underline{p}(0)\mathbb{P}^n \\ &= \left[\frac{1}{2}, \frac{1}{2}\right]\begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n} & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n} \end{bmatrix} \\ &= \left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}, \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}\right] \end{aligned}$$

c) $\underline{p}(n) \to \left[\frac{1}{3}, \frac{2}{3}\right]$ as $n \to \infty$

2.7 *Synthesizing Randomness: Random Number Generators

2.109

$p_1 = \frac{1}{3}$ $p_2 = \frac{1}{5}$ $p_3 = \frac{1}{4}$ $p_4 = \frac{1}{7}$ $p_5 = 1 - \sum_{i=1}^{4} p_i = 1 - \frac{140+84+105+60}{420}$

$= \frac{31}{420}$

Use an urn with 420 identical balls labeled as follows

140 labeled 1
84 " 2
105 " 3
60 " 4
31 " 5

By finding least common multiple of denominators of rational probabilities we can define an equivalent urn experiment.

2.110

~~2.84~~ Three tosses of a fair coin result in eight equiprobable outcomes:

$$\begin{array}{lcl@{\qquad}lcl}
000 & \rightarrow & 0 & 100 & \rightarrow & 4 \\
001 & \rightarrow & 1 & 101 & \rightarrow & 5 \\
010 & \rightarrow & 2 & 101 & \multirow{2}{*}{$\left.\right\} \rightarrow$} & \text{No output} \\
011 & \rightarrow & 3 & 111 & &
\end{array}$$

a)

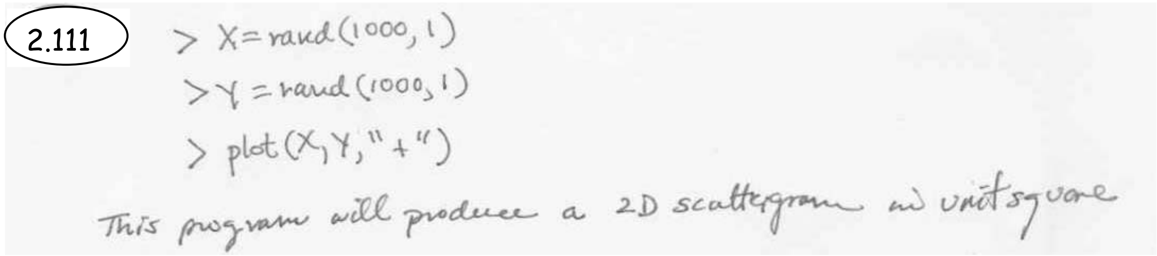
$$\begin{aligned} P[\text{a number is output in step 1}] &= 1 - P[\text{no output}] \\ &= 1 - \frac{2}{8} = \frac{3}{4} \end{aligned}$$

b) Let $A_i = \{\text{output number } i\}$ $\quad i = 0, \ldots, 5$
and $B = \{\text{a number is output in step 1}\}$
then

$$\begin{aligned} P[A_i|B] &= \frac{P[A_i \cap B]}{P[B]} = \frac{P[\text{binary string corresponds to } i]}{\frac{3}{4}} \\ &= \frac{\frac{1}{8}}{\frac{3}{4}} = \frac{1}{6} \end{aligned}$$

c) Suppose we want to an urn experiment with N equiprobable outcomes. Let n be the smallest integer such that $2^n \geq N$. We can simulate the urn experiment by tossing a fair coin n times and outputting a number when the binary string for the numbers $0, \ldots, N-1$ occur and not outputting a number otherwise.

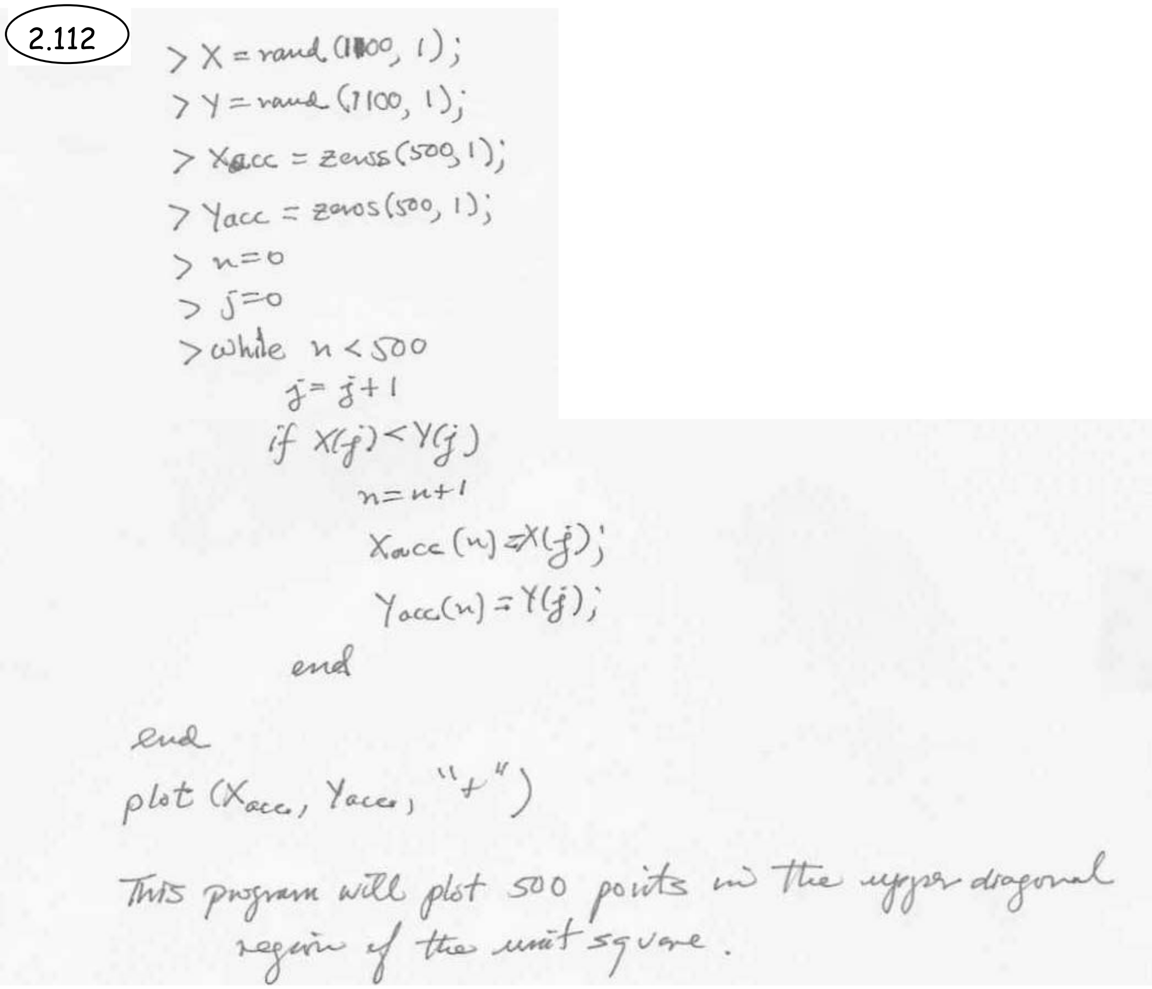
2.111



```
> X=rand(1000,1)
> Y=rand(1000,1)
> plot(X,Y,"+")
```

This program will produce a 2D scattergram in unit square

2.112



```
> X = rand(1100, 1);
> Y = rand(1100, 1);
> Xacc = zeros(500,1);
> Yacc = zeros(500, 1);
> n=0
> j=0
> while n < 500
        j= j+1
        if X(j)<Y(j)
              n=n+1
              Xacc(n)=X(j);
              Yacc(n)=Y(j);
        end
  end
  plot(Xacc, Yacc, "+")
```

This program will plot 500 points in the upper diagonal region of the unit square.

2.113

a) Assume that $X(j)$ assumes values from the sample space $S = \{x_1, x_2, \ldots, x_{,}\}$, and let $N_k(n)$ be the number of tries x_k occurs in n repetitions of the experiment, then

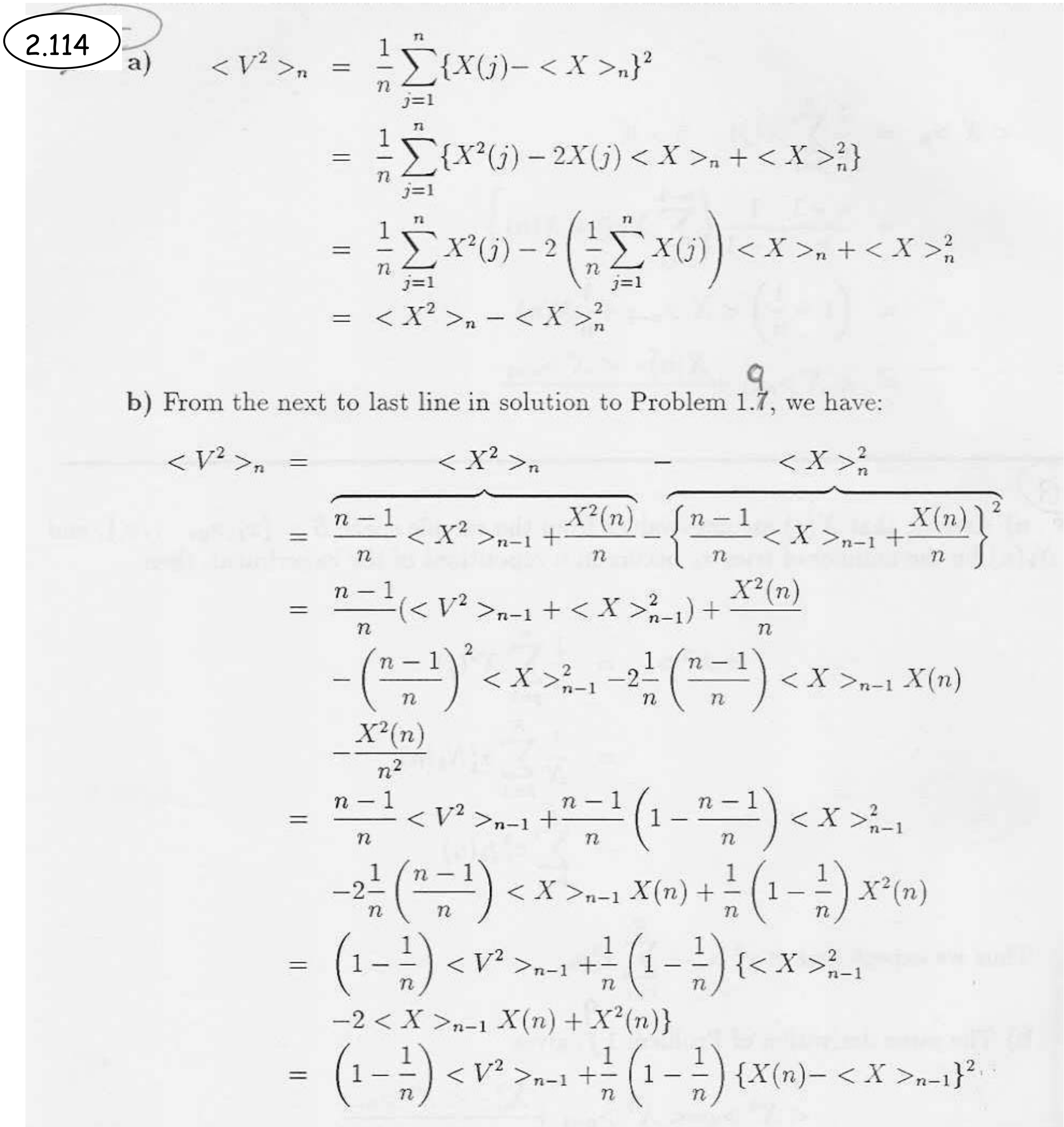
$$\begin{aligned} < X^2 >_n &= \frac{1}{n}\sum_{j=1}^{n} X^2(j) \\ &= \frac{1}{N}\sum_{k=1}^{K} x_k^2 N_k(n) \\ &\rightarrow \sum_{k=1}^{K} x_k^2 f_k(n) \end{aligned}$$

Thus we expect that $< x^2 >_n \rightarrow \sum_{k=1}^{K} x_k^2 p_k$.

b) The same derivation of Problem 1.~~7~~9, gives

$$< X^2 >_n = < X^2 >_{n-1} + \frac{X_n^2 - < X^2 >_{n-1}}{n}$$

2.114



a)

$$\begin{aligned}
<V^2>_n &= \frac{1}{n}\sum_{j=1}^{n}\{X(j)-<X>_n\}^2 \\
&= \frac{1}{n}\sum_{j=1}^{n}\{X^2(j)-2X(j)<X>_n+<X>_n^2\} \\
&= \frac{1}{n}\sum_{j=1}^{n}X^2(j)-2\left(\frac{1}{n}\sum_{j=1}^{n}X(j)\right)<X>_n+<X>_n^2 \\
&= <X^2>_n-<X>_n^2
\end{aligned}$$

b) From the next to last line in solution to Problem 1.$\overset{9}{7}$, we have:

$$\begin{aligned}
<V^2>_n &= \overbrace{\frac{n-1}{n}<X^2>_{n-1}+\frac{X^2(n)}{n}}^{<X^2>_n}-\overbrace{\left\{\frac{n-1}{n}<X>_{n-1}+\frac{X(n)}{n}\right\}^2}^{<X>_n^2} \\
&= \frac{n-1}{n}(<V^2>_{n-1}+<X>_{n-1}^2)+\frac{X^2(n)}{n} \\
&\quad -\left(\frac{n-1}{n}\right)^2<X>_{n-1}^2-2\frac{1}{n}\left(\frac{n-1}{n}\right)<X>_{n-1}X(n) \\
&\quad -\frac{X^2(n)}{n^2} \\
&= \frac{n-1}{n}<V^2>_{n-1}+\frac{n-1}{n}\left(1-\frac{n-1}{n}\right)<X>_{n-1}^2 \\
&\quad -2\frac{1}{n}\left(\frac{n-1}{n}\right)<X>_{n-1}X(n)+\frac{1}{n}\left(1-\frac{1}{n}\right)X^2(n) \\
&= \left(1-\frac{1}{n}\right)<V^2>_{n-1}+\frac{1}{n}\left(1-\frac{1}{n}\right)\{<X>_{n-1}^2 \\
&\quad -2<X>_{n-1}X(n)+X^2(n)\} \\
&= \left(1-\frac{1}{n}\right)<V^2>_{n-1}+\frac{1}{n}\left(1-\frac{1}{n}\right)\{X(n)-<X>_{n-1}\}^2
\end{aligned}$$

2.115 $Y_n = \alpha U_n + \beta$ should map onto $[a, b]$

ⓐ when $U_n = 0$ we want $Y_n = \beta = a$
when $U_n = 1$ we want $Y_n = \alpha + \beta = b$ $\Big\} \Rightarrow \alpha = b - \beta = b - a$

$\alpha = b - a \quad \beta = a$

$\Rightarrow Y_n = (b-a)\,U_n + a$

ⓑ
```
> a=-5
> b=15
> Y = (b-a)* rand(1000,1) + a* ones(1000,1);
> mean(Y)                 % computes sample mean
> cov(Y,Y)                % computes sample variance
```

In a test we obtained

mean$(Y) = 5.2670$ vs $\frac{b-a}{2} = 5$

cov$(Y,Y) = 34.065$ vs $\frac{(b-a)^2}{12} = 33.333$

2.116 ⓐ This problem uses the code in Example 2.47

ⓑ The ~~pdf~~ histogram will change with different values of p.

2.8 *Fine Points: Event Classes

2.117 $f(r)=R \quad f(g)=G \quad f(t)=G$

Homer's events are quite simple:

$\phi, \{R\}, \{G\}, \{R,G\} = S_H$

ⓐ $f^{-1}(\{R\}\cup\{G\}) = f^{-1}(\{R,G\}) = \{r,g,t\}$

and $f^{-1}(\{R\})\cup f^{-1}(\{G\}) = \{r\}\cup\{g,t\} = \{r,g,t\}$ same.

ⓑ $f^{-1}(\{R\}\cap\{R,G\}) = f^{-1}(\{R\}) = \{r\}$

$f^{-1}(\{R\})\cap f^{-1}(\{R,G\}) = \{r\}\cap\{r,t,g\} = \{r\}$ same

ⓒ $f^{-1}(\{\cancel{\xi}G\}^c) = f^{-1}(\{R\}) = \{r\}$

$f^{-1}(\{G\})^c = \{g,t\}^c = \{r\}$ Same

ⓓ (i) $f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$

If $\xi\in f^{-1}(A\cup B)$ then $f(\xi)\in A\cup B \Rightarrow f(\xi)\in A$ ~~and~~/or $f(\xi)\in B$

$\Rightarrow \xi\in f^{-1}(A)$ and/or $\xi\in f^{-1}(B)$

$\Rightarrow \xi\in f^{-1}(A)\cup f^{-1}(B)$ $\quad\therefore\ f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$.

If $\xi\in f^{-1}(A)\cup f^{-1}(B) \Rightarrow \xi\in f^{-1}(A)$ and/or $\xi\in f^{-1}(B)$

$\Rightarrow f(\xi)\in A$ and/or $f(\xi)\in B$

$\Rightarrow f(\xi)\in A\cup B$

$\Rightarrow \xi\in f^{-1}(A\cup B) \Rightarrow f^{-1}(A\cup B)\supset f^{-1}(A)\cup f^{-1}(B)$

$\Rightarrow$ equality.

(d) $f^{-1}(A\cap B) = f^{-1}(A) \cap f^{-1}(B)$

If $\xi \in f^{-1}(A\cap B) \Rightarrow f(\xi) \in A\cap B \Rightarrow f(\xi) \in A$ and $f(\xi) \in B$

$\Rightarrow \xi \in f^{-1}(A)$ and $\xi \in f^{-1}(B) \Rightarrow \xi \in f^{-1}(A) \cap f^{-1}(B)$.

$\Rightarrow f^{-1}(A\cup B) \subset f^{-1}(A) \cap f^{-1}(B)$.

If $\xi \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow \xi \in f^{-1}(A)$ and $\xi \in f^{-1}(B)$

$\Rightarrow f(\xi) \in A$ and $f(\xi) \in \cancel{\xi}B\cancel{\xi} \Rightarrow f(\xi) \in A\cap B$

$\Rightarrow \xi \in f^{-1}(A\cap B)$

$\Rightarrow f^{-1}(A\cup B) \supset f^{-1}(A) \cap f^{-1}(B)$ ✓

$f^{-1}(A^c) = f^{-1}(A)^c$

If $\xi \in f^{-1}(A^c) \Rightarrow f(\xi) \in A^c \Rightarrow f(\xi) \notin A \Rightarrow \xi \notin f^{-1}(A)$

$\Rightarrow \xi \in f^{-1}(A)^c$

$\Rightarrow f^{-1}(A^c) \subset f^{-1}(A)^c$

If $\xi \in f^{-1}(A)^c \Rightarrow \xi \notin f^{-1}(A) \Rightarrow f(\xi) \notin A$

$\Rightarrow f(\xi) \in A^c$

$\Rightarrow \xi \in f^{-1}(A^c)$

$\Rightarrow f^{-1}(A)^c \subset f^{-1}(A^c)$. ✓

2.118

(a) Show that $A_1 \ldots A_n$ forms a partition of S, that is,

$$A_i \cap A_j = \phi \quad i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = S$$

(i) For $i \neq j$ consider $A_i \cap A_j$

$$A_i \cap A_j = \{\xi : \xi \in A_i \text{ and } \xi \in A_j\} = \{\xi : f(\xi) = y_i \text{ and } f(\xi) = y_j\}$$

but if $y_i \neq y_j$ then we cannot have $f(\xi) = y_i$ and $f(\xi) = y_j$.

since each ξ is mapped into a single value

$\therefore A_i \cap A_j = \phi$.

(ii) Now consider $\bigcup_{i=1}^{n} A_i$

Suppose $\xi \in S$, then $f(\xi) \in S' = \{y_1, \ldots, y_n\}$

$\Rightarrow \xi \in A_j$ for some j

$\Rightarrow \xi \in \bigcup_{i=1}^{n} A_i \Rightarrow \bigcup_{i=1}^{n} A_i \supset S$.

But S contains all subsets

$\Rightarrow \bigcup_{i=1}^{n} A_i \subset S$ ✓.

(b) any $B \subset S'$ has form $B = \{y_{i_1}\} \cup \{y_{i_2}\} \ldots \cup \{y_{i_m}\}$

From problem 2.117 (d)

$$f^{-1}(B) = f^{-1}(\{y_{i_1}\} \cup \{y_{i_2}\} \cup \ldots \cup \{y_{i_m}\})$$

$$= f^{-1}(\{y_{i_1}\}) \cup f^{-1}(\{y_{i_2}\}) \cup \ldots \cup f^{-1}(\{y_{i_m}\})$$

$$= A_{i_1} \cup A_{i_2} \ldots \cup A_{i_m}.$$

$\therefore$ Inverse image of B is a union of sets from the partition.

2.119

$\mathcal{F} = \{\phi, A, A^c, S\}$

(i) $\phi \in \mathcal{F}$ ✓

(ii) if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$?

$$A \cup A^c = S \in \mathcal{F}$$

and any other union of events in $\mathcal{F}$ yields an event in $\mathcal{F}$ ✓

(iii) if $B \in \mathcal{F}$ then $B^c \in \mathcal{F}$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$A^c \in \mathcal{F} \Rightarrow A \in \mathcal{F}$$

and similarly for other events in $\mathcal{F}$ ✓

∴ $\mathcal{F}$ is a field.

2.9 *Fine Points: Probabilities of Sequences of Events

2.120

(a) $\bigcup_n A_n = \bigcup_n \left[a+\frac{1}{n}, b-\frac{1}{n}\right] = (a, b)$

(b) $\bigcup_n B_n = \bigcup_n \left(-\infty, b-\frac{1}{n}\right] = (-\infty, b)$

(c) $\bigcup_n C_n = \bigcup_n \left[a+\frac{1}{n}, b\right) = (a, b)$

2.121

(a) $\bigcap_n \left(a-\frac{1}{n}, b+\frac{1}{n}\right) = [a, b]$

(b) $\bigcap_n \left[a, b+\frac{1}{n}\right) = [a, b]$

(c) $\bigcap_n \left[a-\frac{1}{n}, b\right] = [a, b]$

2.122

(a) starting with open sets (a, b)

$(-\infty, b)^c = [b, \infty) \in \mathcal{B}$

then $(-\infty, b) \cap [a, \infty) = [a, b)$ for $a<b$

∴ We can use semi-infinite intervals as in the chapter to show that all elements in the Borel field can be generated.

(b) Closed intervals of the form $[a, b]$ can also be used to generate the Borel field.

2.123

ⓐ $\lim_{n\to\infty} P[A_n] = P[\lim_{n\to\infty} A_n] = P[a < x < b]$

ⓑ $\lim_{n\to\infty} P[B_n] = P[\lim_{n\to\infty} B_n] = P[-\infty < x < b]$

ⓒ $\lim_{n\to\infty} P[C_n] = P[\lim_{n\to\infty} C_n] = P[a < x < b]$

2.124

ⓐ $\lim_{n\to\infty} P[A_n] = P[\lim_{n\to\infty} A_n] = P[a \le x \le b]$

ⓑ $\lim_{n\to\infty} P[B_n] = P[\lim_{n\to\infty} B_n] = P[a \le x \le b]$

ⓒ $\lim_{n\to\infty} P[C_n] = P[\lim_{n\to\infty} C_n] = P[a \le x \le b]$.

Problems Requiring Cumulative Knowledge

2.125

ⓐ

$$P_H[k \text{ defective of 10 tested}] = \begin{cases} \dfrac{\binom{5}{k}\binom{15}{10-k}}{\binom{20}{10}} & k=0,1,2,3,4,5 \\ 0 & k>5 \end{cases}$$

$$P_B[k \text{ defective}] = \binom{10}{k}(.25)^k(.75)^{10-k} \qquad k=0,1,2\ldots,10$$

See Table of values:

Probabilities for hypergeometric and binomial are very different.

k	Hypergeometric	Binomial
0	0.01625	0.18771
1	0.13545	0.28157
2	0.34830	0.25028
3	0.34830	0.14600
4	0.13545	0.058399
5	0.01625	0.016222
6	0	0.003089
7	0	0.00003

ⓑ

$$P_H[k \text{ defective}] = \frac{\binom{250}{k}\binom{750}{10-k}}{\binom{1000}{10}} \qquad k=0,1,\ldots,10$$

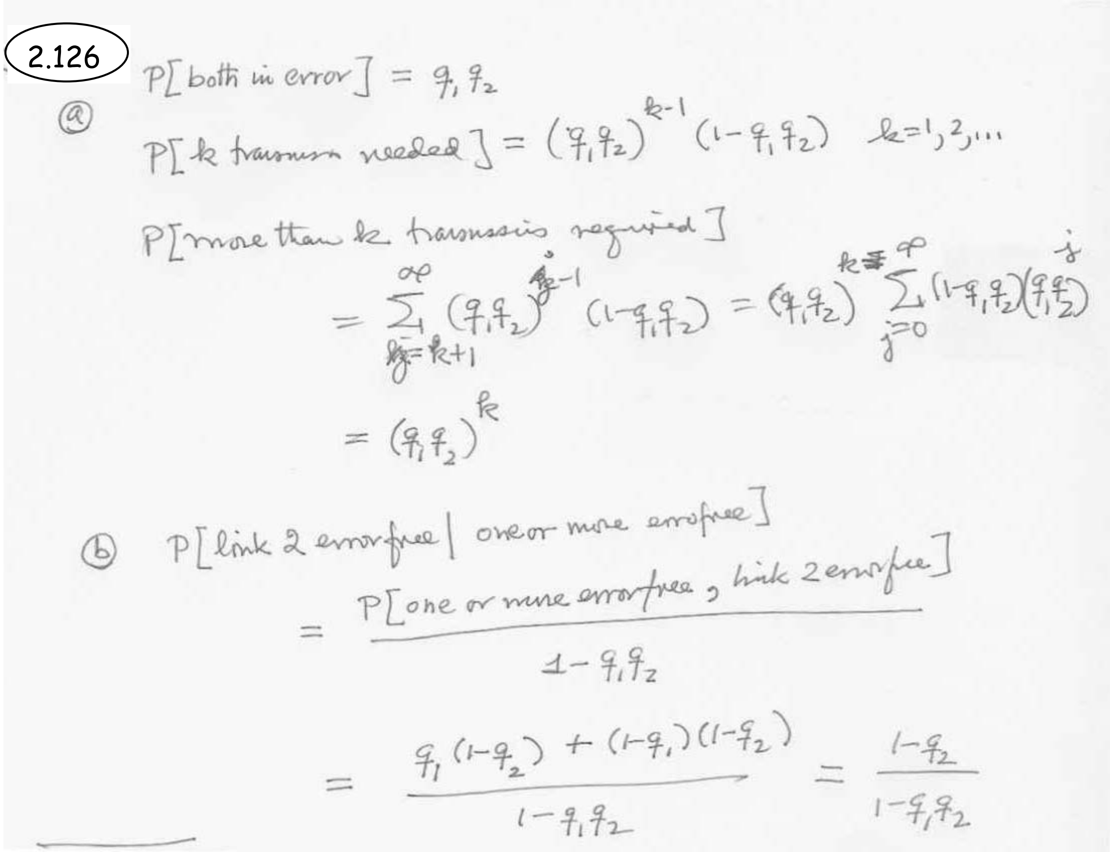
$$P_B[k \text{ defective}] = \binom{10}{k}(.25)^k(.75)^{10-k} \qquad k=0,\ldots,10$$

See Table!

k	Hypergeometric
0	0.18714
1	0.28260
2	0.25154
3	0.14614
4	0.057907
5	0.015848
6	0.0028581
7	0.00036

These are very close to the binomial probabilities

Because of the large population size sampling without replacement is almost the same as sampling with replacement.



2.126

P[both in error] $= q_1 q_2$

ⓐ P[k transmissions needed] $= (q_1 q_2)^{k-1}(1-q_1q_2) \quad k=1,2,\ldots$

P[more than k transmissions required]

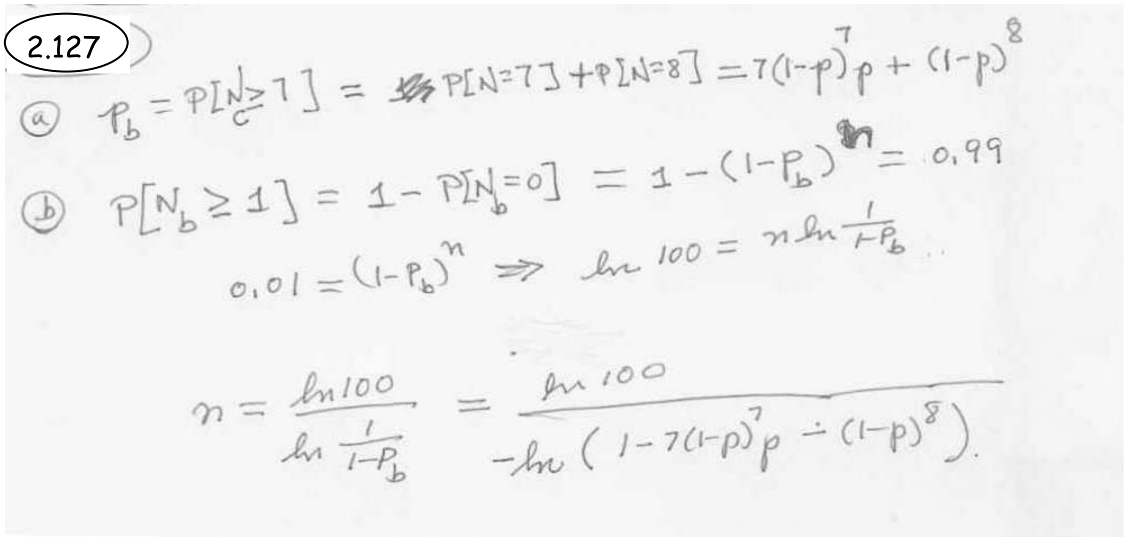
$$= \sum_{j=k+1}^{\infty}(q_1q_2)^{j-1}(1-q_1q_2) = (q_1q_2)^{k}\sum_{j=0}^{\infty}(1-q_1q_2)(q_1q_2)^{j}$$

$$= (q_1q_2)^{k}$$

ⓑ P[link 2 errorfree | one or more errorfree]

$$= \frac{P[\text{one or more errorfree, link 2 errorfree}]}{1-q_1q_2}$$

$$= \frac{q_1(1-q_2)+(1-q_1)(1-q_2)}{1-q_1q_2} = \frac{1-q_2}{1-q_1q_2}$$



2.127

ⓐ $P_b = P[N_c \geq 7] = P[N=7]+P[N=8] = 7(1-p)^7p + (1-p)^8$

ⓑ $P[N_b \geq 1] = 1 - P[N_b=0] = 1-(1-P_b)^n = 0.99$

$$0.01 = (1-P_b)^n \Rightarrow \ln 100 = n \ln\frac{1}{1-P_b}$$

$$n = \frac{\ln 100}{\ln\frac{1}{1-P_b}} = \frac{\ln 100}{-\ln\left(1-7(1-p)^7p-(1-p)^8\right)}$$

2.128

ⓐ $P[\text{ace}] = \frac{4}{52} = \frac{1}{13}$

ⓑ Let A = ace in 1st draw
B = ace in 2nd draw

$P[A] = \frac{1}{13}$ $P[A^c] = \frac{12}{13}$

If we look at 1st draw:

$P[B|A] = \frac{3}{51}$ $P[B|A^c] = \frac{4}{51}$

Suppose we don't look

$$P[B] = P[B|A]\,P[A] + P[B|A^c]\,P[A^c]$$
$$= \frac{3}{51}\frac{1}{13} + \frac{4}{51}\frac{12}{13} = \frac{3+48}{51(13)} = \frac{1}{13}$$

$\Rightarrow$ 2nd Draw has same probability of ace as 1st draw

ⓒ

$$P[\underbrace{3 \text{ aces in } 7 \text{ cards}}_{A}] = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} = 0.00582$$

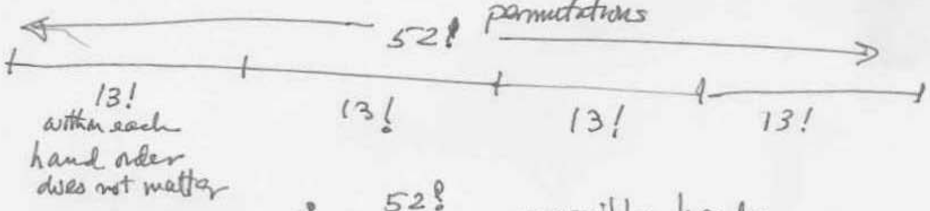
$$P[\underbrace{2 \text{ kings in } 7 \text{ cards}}_{B}] = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}} = 0.07679$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

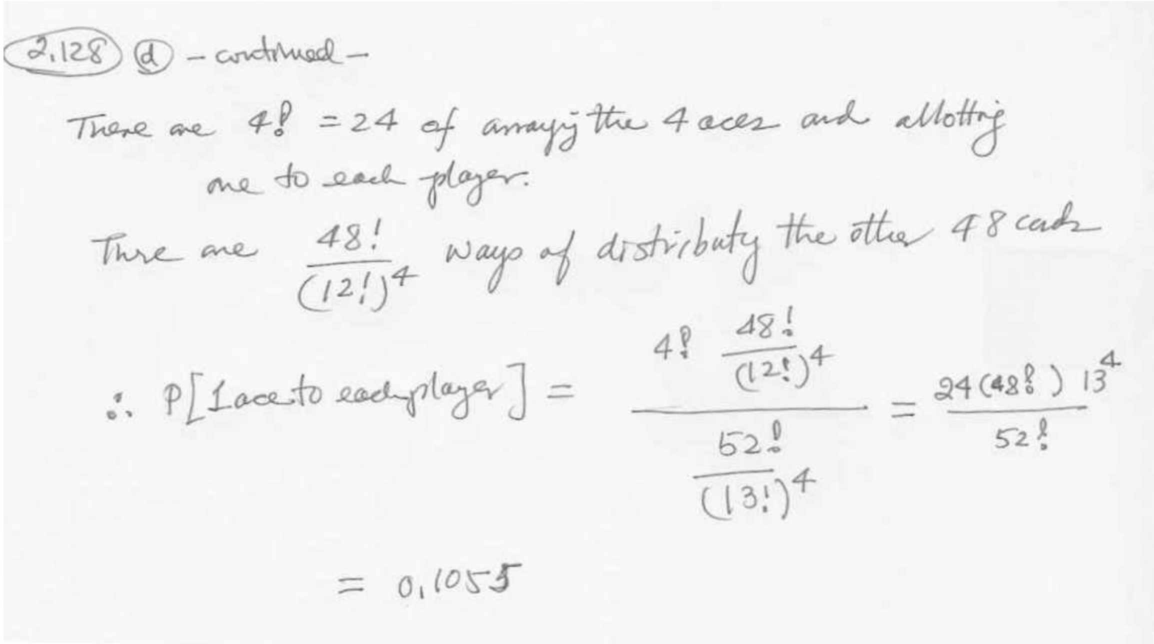
$$P[A \cap B] = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}} = 0.00017$$

$$P[A \cup B] = 0.00582 + 0.07679 - 0.00017 = 0.0824$$

ⓓ



$\therefore \frac{52!}{(13!)^4}$ possible hands



2.128 ⓓ – continued –

There are $4! = 24$ of arranging the 4 aces and allotting one to each player.

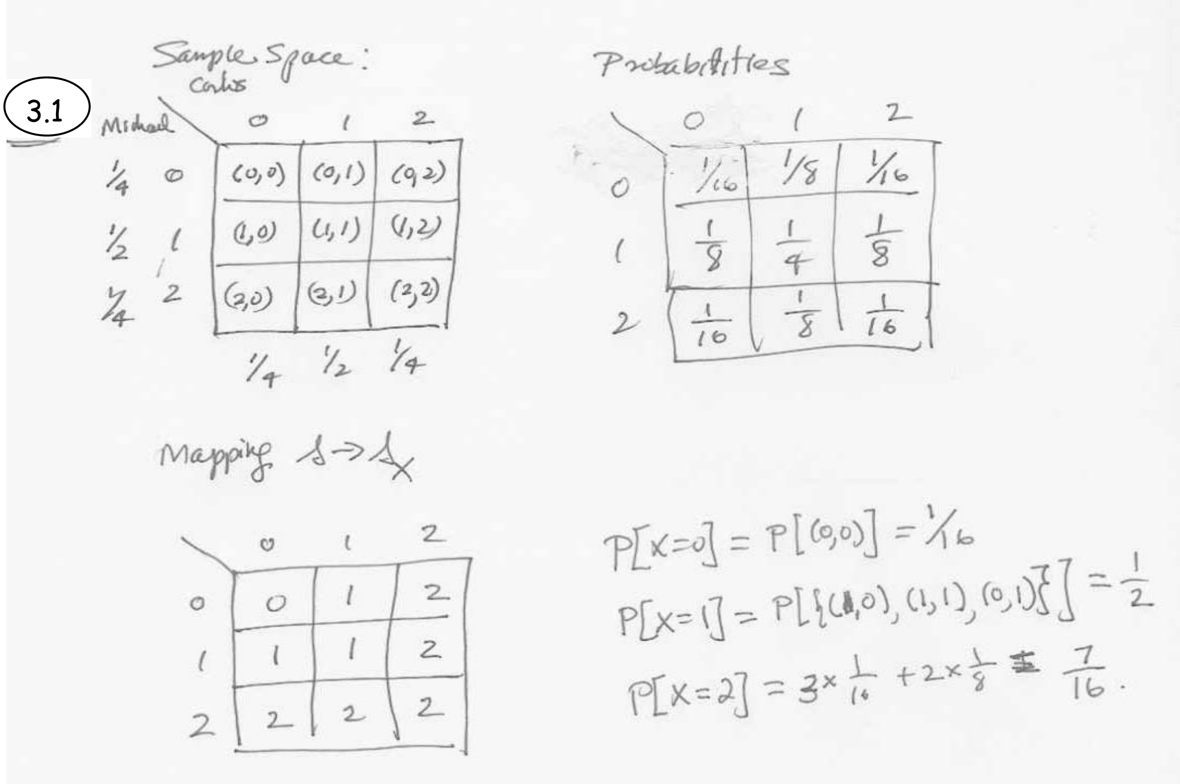
There are $\frac{48!}{(12!)^4}$ ways of distributing the other 48 cards

$$\therefore \; P[\text{1 ace to each player}] = \frac{4! \, \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{24(48!)\, 13^4}{52!}$$

$$= 0.1055$$

Chapter 3: Discrete Random Variables

3.1 The Notion of a Random Variable



3.1

Sample Space:

Michael \ Carlos		0	1	2
1/4	0	(0,0)	(0,1)	(0,2)
1/2	1	(1,0)	(1,1)	(1,2)
1/4	2	(2,0)	(2,1)	(2,2)
		1/4	1/2	1/4

Probabilities

	0	1	2
0	$1/16$	$1/8$	$1/16$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

Mapping $S \to S_X$

	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$$P[X=0] = P[(0,0)] = 1/16$$

$$P[X=1] = P[\{(1,0),(1,1),(0,1)\}] = \frac{1}{2}$$

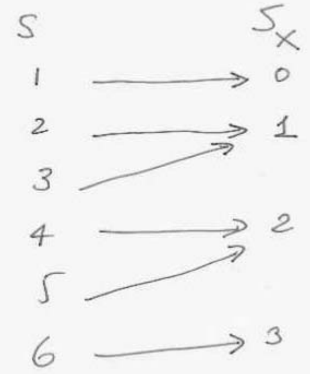
$$P[X=2] = 3 \times \frac{1}{16} + 2 \times \frac{1}{8} = \frac{7}{16}.$$

3.2

ⓐ $S = \{1, 2, 3, 4, 5, 6\}$ $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

where $p_j = P[\{j\}]$

ⓑ



ⓒ $P[X=0] = p_1 = \frac{1}{6}$

$P[X=1] = p_2 + p_3 = \frac{2}{6}$

$P[X=2] = p_4 + p_5 = \frac{2}{6}$

$P[X=6] = p_6 = \frac{1}{6}$

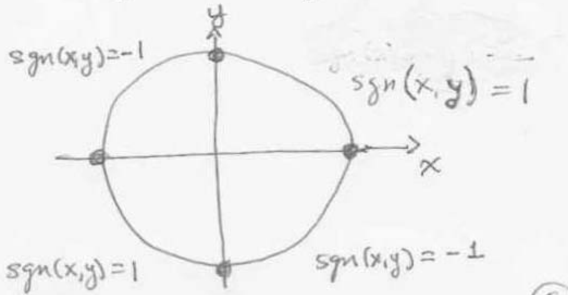
ⓓ $P[\cancel{X}=0] = p_1 + p_2 = \frac{2}{6}$ $P[Y=1] = p_3 + p_4 = \frac{2}{6}$ $P[Y=2] = p_5 + p_6 = \frac{2}{6}$

ⓔ $X=0$ corresponds to $\{1\}$

$Y=0$ corresponds to $\{1, 2\}$

3.3

ⓐ $S = \{(x,y) : x^2 + y^2 = r^2\}$ where r = radius of clock



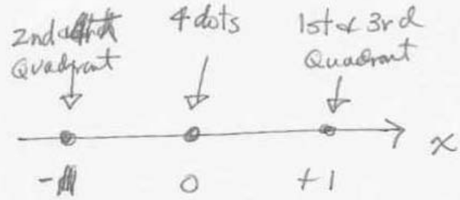
Outcomes from S occur uniformly along the circle.

sgn(xy) = 0 at the dots

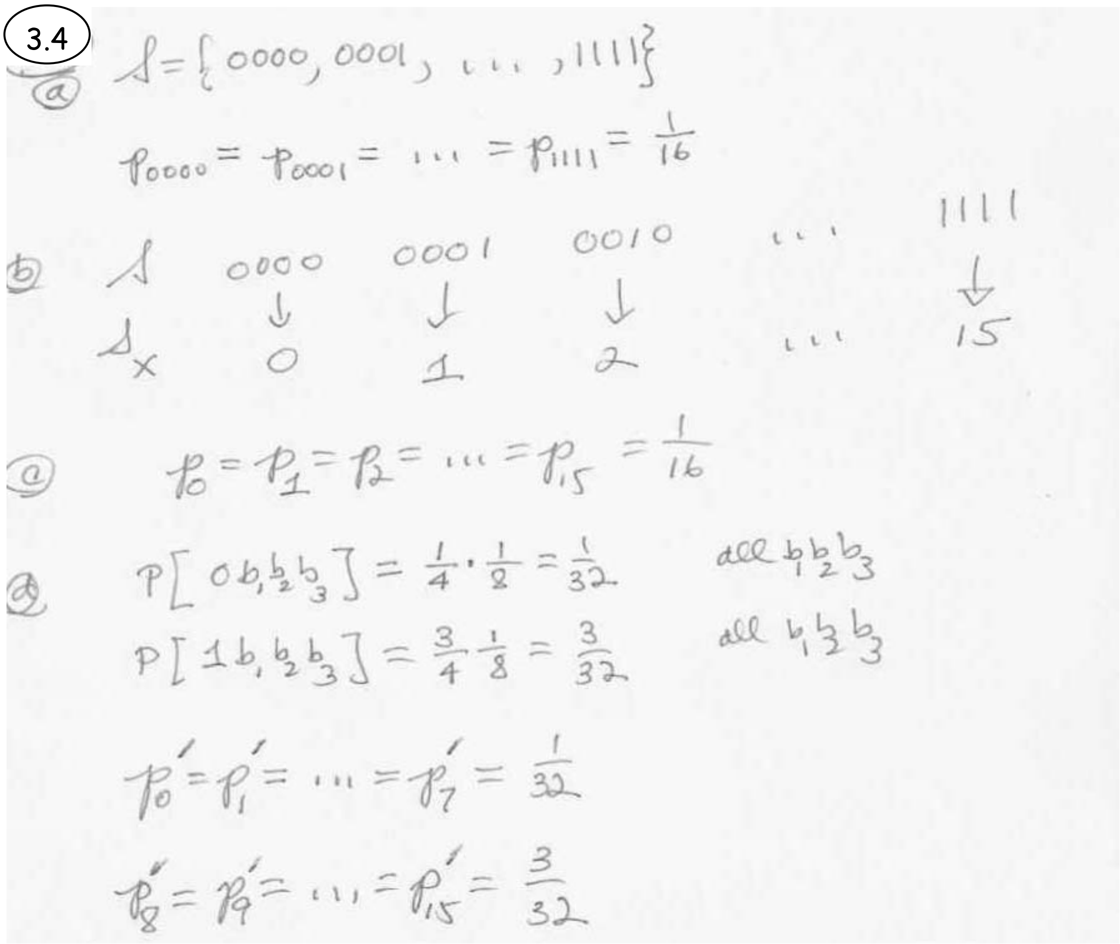
ⓒ $P[X=-1] = P[\text{2nd+3rd Quad}] = \frac{1}{2}$

$P[X=0] = P[\{(r,0), (0,r), (-r,0), (0,-r)\}]$
$= 0$

$P[X=1] = P[\text{1st \& 3rd Quad}] = \frac{1}{2}$



3.4



ⓐ $S = \{0000, 0001, \ldots, 1111\}$

$p_{0000} = p_{0001} = \ldots = p_{1111} = \frac{1}{16}$

ⓑ
$$\begin{array}{cccccc} S & 0000 & 0001 & 0010 & \ldots & 1111 \\ & \downarrow & \downarrow & \downarrow & & \downarrow \\ S_X & 0 & 1 & 2 & \ldots & 15 \end{array}$$

ⓒ $p_0 = p_1 = p_2 = \ldots = p_{15} = \frac{1}{16}$

ⓓ $P[0b_1b_2b_3] = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$ all $b_1b_2b_3$

$P[1b_1b_2b_3] = \frac{3}{4}\frac{1}{8} = \frac{3}{32}$ all $b_1b_2b_3$

$p'_0 = p'_1 = \ldots = p'_7 = \frac{1}{32}$

$p'_8 = p'_9 = \ldots = p'_{15} = \frac{3}{32}$

3.5

Let A_i = Transmitter #1 sends a signal at time slot i

B_i = " #2 "

A signal gets through if $A_iB_i^c \cup A_i^cB_i$ occurs

Each experiment has 4 outcomes

ⓐ Experiment i

	A_i	A_i^c
B_i	1/4	1/4
B_i^c	1/4	1/4

Sample Space consists of a Cartesian product of the outcomes of the basic experiment

$S=(s_1,s_2,\ldots)$ where s_i is an outcome from basic experiment

ⓑ $X(s)=n$

if n is the first occurrence of $A_iB_i^c \cup A_i^cB_i$ in $s_1,s_2,\ldots$

ⓒ $P[A_iB_i^c \cup A_i^cB_i] = P[A_iB_i^c]+P[A_i^cB_i] = \frac{1}{2} = P[\text{success}]$

$P[X=k] = P[(k-1)\text{ failures, } 1 \text{ success}] = \left(\frac{1}{2}\right)^k$

3.6

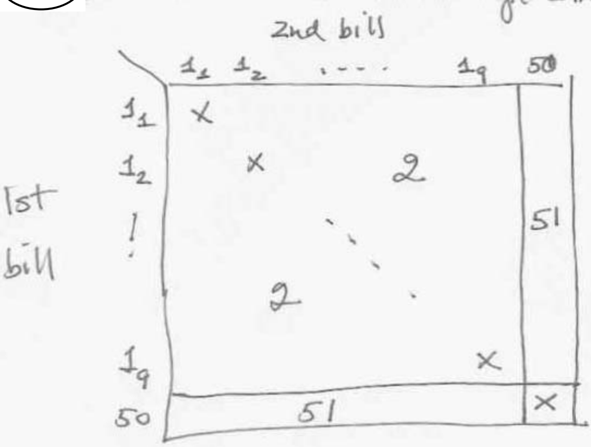
$$\begin{array}{ccccccccc} S = \{ & 000, & 111, & 010, & 101, & 001, & 110, & 100, & 011\} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ X(s): & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 \end{array}$$

$P[X=2] = P[\{000, 111\}] = \frac{1}{2}$

$P[X=3] = P[\{010, 101\}] = \frac{1}{4}$

$P[X=4] = P[\{001, 110, 100, 011\}] = \frac{1}{4}$

3.7 Draw 2 bills without replacement.



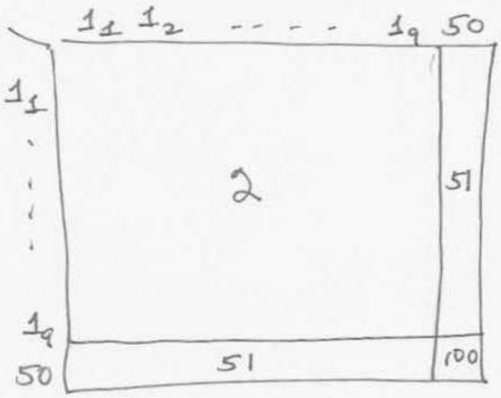
x not allowed

all other outcomes have probability $\frac{1}{9(10)} = \frac{1}{90}$

$$P[X=2] = \frac{9 \cdot 8}{90} = \frac{8}{10} = \frac{4}{5}$$

$$P[X=51] = \frac{9 \cdot 2}{90} = \frac{2}{10} = \frac{1}{5}$$

3.8 Draw 2 bills with replacement.



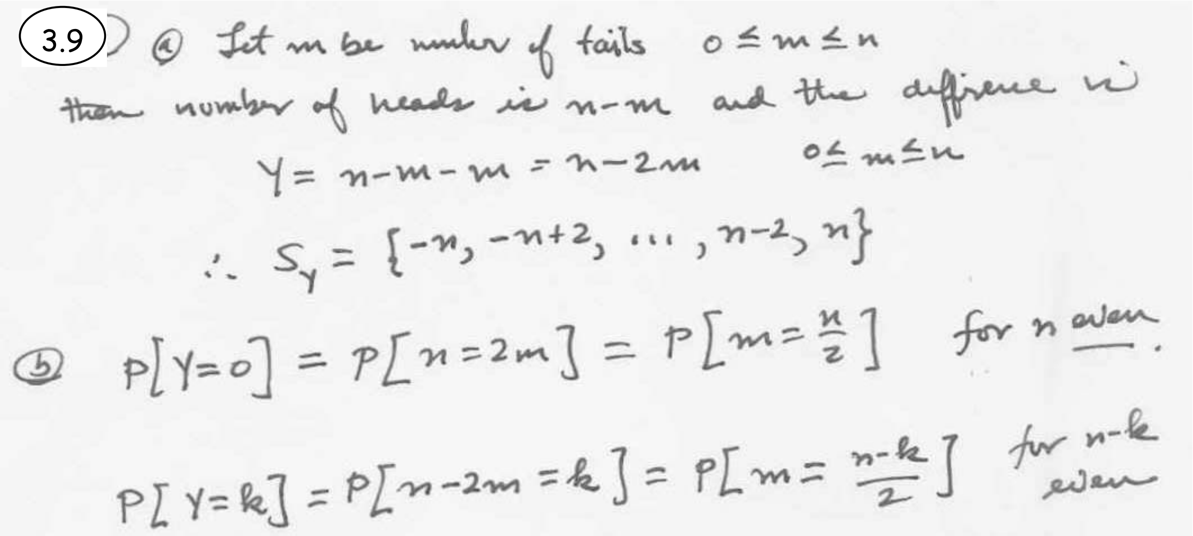
all outcomes have probability $\frac{1}{10(10)} = \frac{1}{100}$

$$P[X=2] = \frac{81}{100}$$

$$P[X=51] = \frac{18}{100}$$

$$P[X=100] = \frac{1}{100}$$

3.9



ⓐ Let m be number of tails $0 \le m \le n$

then number of heads is $n-m$ and the difference is

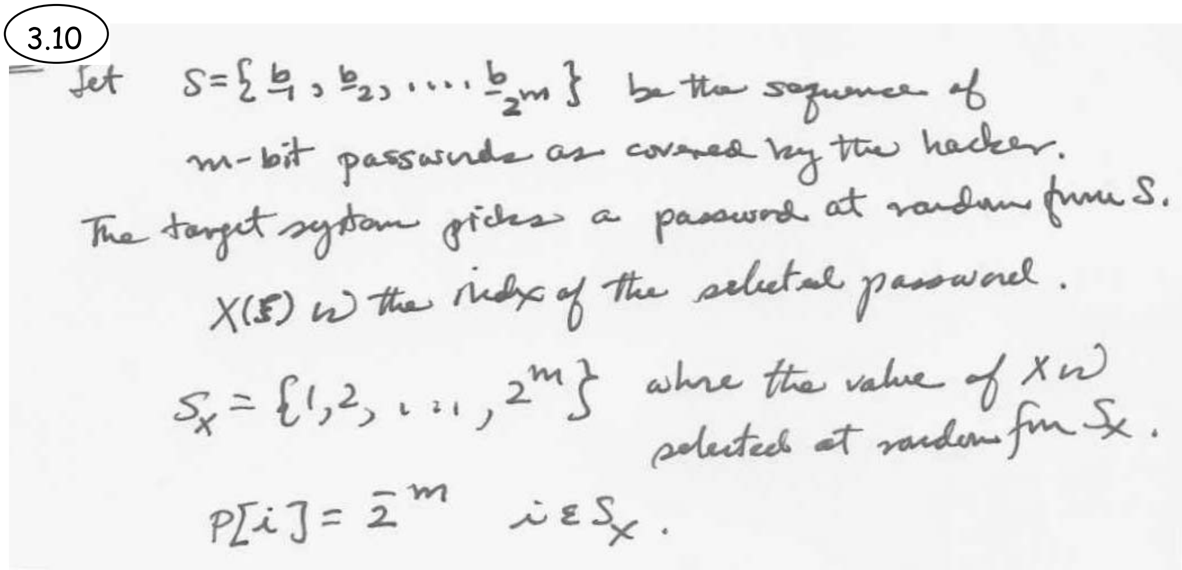
$$Y = n-m-m = n-2m \qquad 0 \le m \le n$$

$$\therefore \; S_Y = \{-n, -n+2, \ldots, n-2, n\}$$

ⓑ $P[Y=0] = P[n=2m] = P[m=\frac{n}{2}]$ for n even.

$P[Y=k] = P[n-2m=k] = P[m=\frac{n-k}{2}]$ for $n-k$ even

3.10



Let $S=\{\underline{b}_1, \underline{b}_2, \ldots \underline{b}_{2^m}\}$ be the sequence of m-bit passwords as covered by the hacker.

The target system picks a password at random from S.

$X(\xi)$ is the index of the selected password.

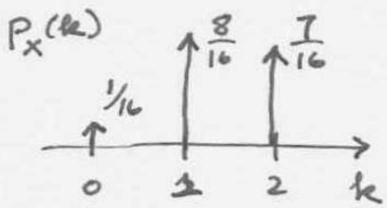
$S_X = \{1, 2, \ldots, 2^m\}$ where the value of X is selected at random from S_X.

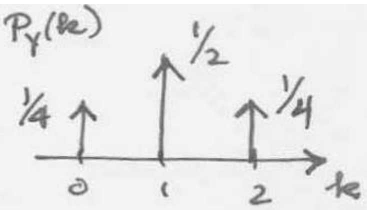
$P[i] = 2^{-m} \quad i \in S_X$.

3.2 Discrete Random Variables And Probability Mass Function

3.11

ⓐ





the max function shifts probability mass to higher values of k

ⓑ If Carlos uses a biased coin:

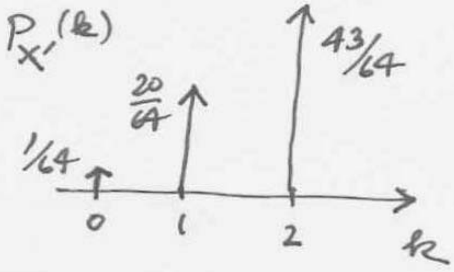
Michel \ Carlos	0	1	2	
0	0	1	2	1/4
1	1	1	2	1/2
2	2	2	2	1/4
	1/16	6/16	9/16	

$$P[X'=0] = \frac{1}{4}\frac{1}{16} = \frac{1}{64}$$

$$P[X'=1] = \frac{1}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{2} + \frac{6}{16}\cdot\frac{1}{4}$$

$$= \frac{20}{64}$$

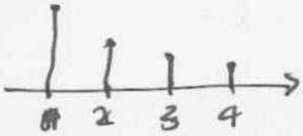
$$P[X'=2] = \frac{43}{64}$$



3.12

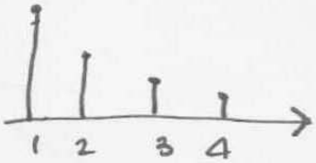
ⓐ $1 = p_1 + p_2 + p_3 + p_4 = p_1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12} p_1 \quad P_1 = \frac{12}{25}$

$p_1 = \frac{12}{25} \quad p_2 = \frac{6}{25} \quad p_3 = \frac{4}{25} \quad p_4 = \frac{3}{25}$



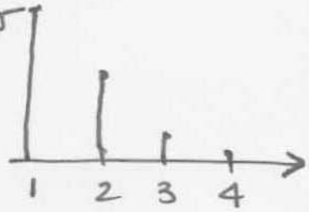
ⓑ $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{15}{8} p_1$

$p_1 = \frac{8}{15} \quad p_2 = \frac{4}{15} \quad p_3 = \frac{2}{15} \quad p_4 = \frac{1}{15}$



ⓒ $1 = p_1\left(1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{64}\right) = \frac{105}{64} p_1$

$p_1 = \frac{64}{105} \quad p_2 = \frac{32}{105} \quad p_3 = \frac{8}{105} \quad p_4 = \frac{1}{105}$



pmf decays more steeply in each example

ⓓ $1 = p_1 \sum_{i=1}^{\infty} \frac{1}{i}$ does not converge so this pmf does not extend to $\{1, 2, \ldots\}$

$1 = p_1 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = p_1 \frac{1}{1 - \frac{1}{2}} \Rightarrow p_1 = \frac{1}{2}.$

this extends to the geometric pmf.

$1 = p_1\left(1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{1+2} + \left(\frac{1}{2}\right)^{1+2+3} + \ldots\right)$

$= p_1 \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j(j-1)/2}$ this is a subseries of the geometric series so it converges

3.13

ⓐ $1 = \sum_{k=1}^{\infty} \frac{c}{k^2} = c \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ is a special case of the zeta function

$= 1.6449 \Rightarrow c = 0.608$

The sum of the first 100 terms gives $1.6349 \Rightarrow c \approx 0.611$

ⓑ $P[X > 4] = 1 - P[X \leq 3] = 1 - c\underbrace{\left[1 + \frac{1}{4} + \frac{1}{9}\right]}_{0.8325}$

$= 0.1675$

ⓒ $P[6 \leq X \leq 8] = c\left[\frac{1}{36} + \frac{1}{49} + \frac{1}{64}\right] = 0.390$

3.14

$P[X \geq 8] = \sum_{k=8}^{15} p_k = \frac{8}{16} = \frac{1}{2}$

$P[Y \geq 8] = \sum_{k=8}^{15} p_k' = 8 \cdot \frac{3}{32} = \frac{24}{32} = \frac{3}{4}$

3.15

		Terminal 2	
		p	$1-p$
Terminal 1	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$
	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$

$P_{success} = \frac{1}{2}q + \frac{1}{2}p = \frac{1}{2}$ same

∴ The pmf of X is unchanged.

$$P[\text{Terminal 2 transmitted} \mid \text{success}] = \frac{P[\text{success and Terminal 2 transmitted}]}{P[\text{success}]}$$

$$= \frac{\frac{1}{2}p}{\frac{1}{2}} = p$$

This suggests that terminal 2 should always transmit (at the expense of terminal 1).

3.16 from problem 3.7b:

(a) $P[X>2] = 1 - P[X=2] = \frac{1}{5}$

$P[X>50] = P[X=51]$

$= \frac{1}{5}$

(b) $P[X>2] = 1 - P[X=2] = \frac{19}{100}$

$P[X>50] = P[X=51] + P[X=100]$

$= \frac{19}{100}$

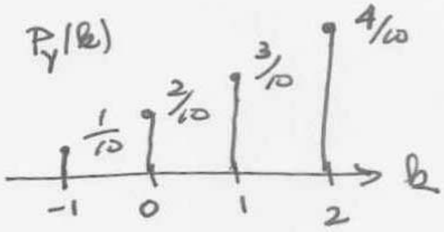
3.17

(a) $Y = 0+2 = 2$ with prob. $4/10$

$Y = -1+2 = 1$ " $3/10$

$Y = -2+2 = 0$ " $2/10$

$Y = -3+2 = -1$ " $1/10$



(b) $P[Y=2] = 4/10$

(c) $P[Y>0] = P[Y=2] + P[Y=1] = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$

3.18 Let ~~K~~ be number of transmissions until success.

$$P[X \le k] = \sum_{j=1}^{k} \left(\tfrac{1}{2}\right)^{j} = \frac{1}{2}\sum_{j=0}^{k-1}\left(\tfrac{1}{2}\right)^{j'} = \frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^{k}}{\frac{1}{2}} = 1-\left(\tfrac{1}{2}\right)^{k}$$

$1-\left(\frac{1}{2}\right)^{k} = 0.99$ $\qquad \left(\frac{1}{2}\right)^{k} = 0.01$

$k = \frac{\ln 100}{\ln 2} = 6.64 \approx 7$

start sending refresh messages

7× 10 seconds before expiry time

3.19

$$P[\text{decoding error}] = P[\text{3 or more bit errors}]$$

$$= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5$$

$$= \frac{5!}{2!\,3!} 10^{-3} (.9)^2 + \frac{5!}{4!\,1!} 10^{-4} (.9) + 10^{-5}$$

$$= (0.81)(10)(10^{-3}) + (0.9)(5)10^{-4} + 10^{-5}$$

$$= 0.00856$$

which is an order of magnitude less than without coding.

3.20

2nd toss

1st toss	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

$P[X=0] = \frac{6}{36}$

$P[X=1] = \frac{5}{36} = P[X=-1]$

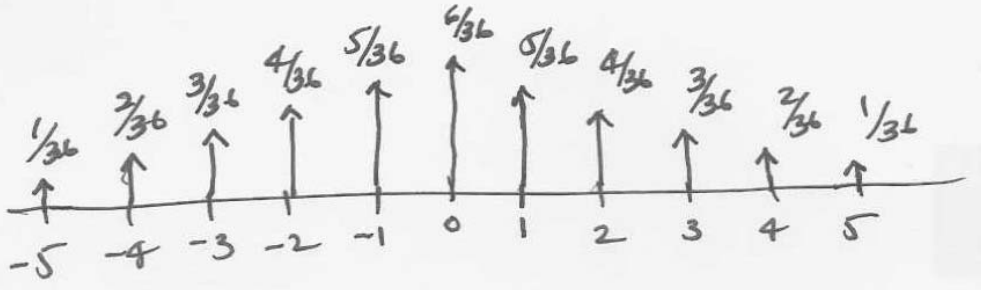
$P[X=2] = \frac{4}{36} = P[X=-2]$

$P[X=3] = \frac{3}{36} = P[X=-3]$

$P[X=4] = \frac{2}{36} = P[X=-4]$

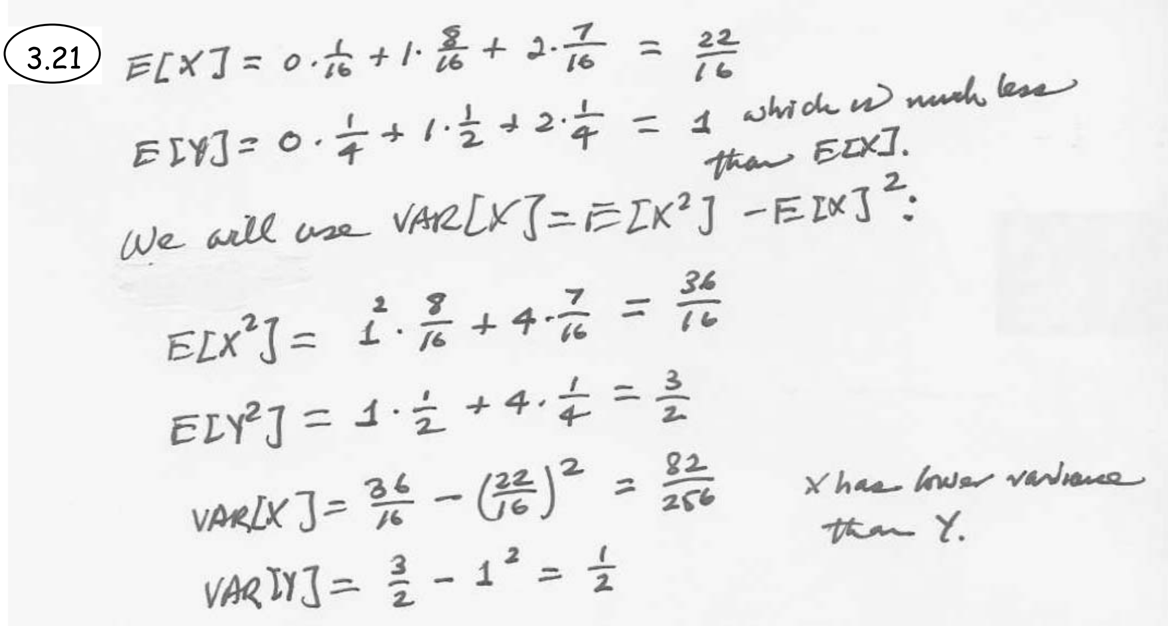
$P[X=5] = \frac{1}{36} = P[X=-5]$

$$P[X=k] = \frac{6-|k|}{36}, \quad |k| \le 5$$



$$P[|X| \le 2] = \frac{6}{36} + 2\frac{5}{36} + 2\cdot\frac{4}{36} = \frac{24}{36}$$

3.3 Expected Value and Moments of Discrete Random Variable



3.21

$E[X] = 0 \cdot \frac{1}{16} + 1 \cdot \frac{8}{16} + 2 \cdot \frac{7}{16} = \frac{22}{16}$

$E[Y] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$ which is much less than $E[X]$.

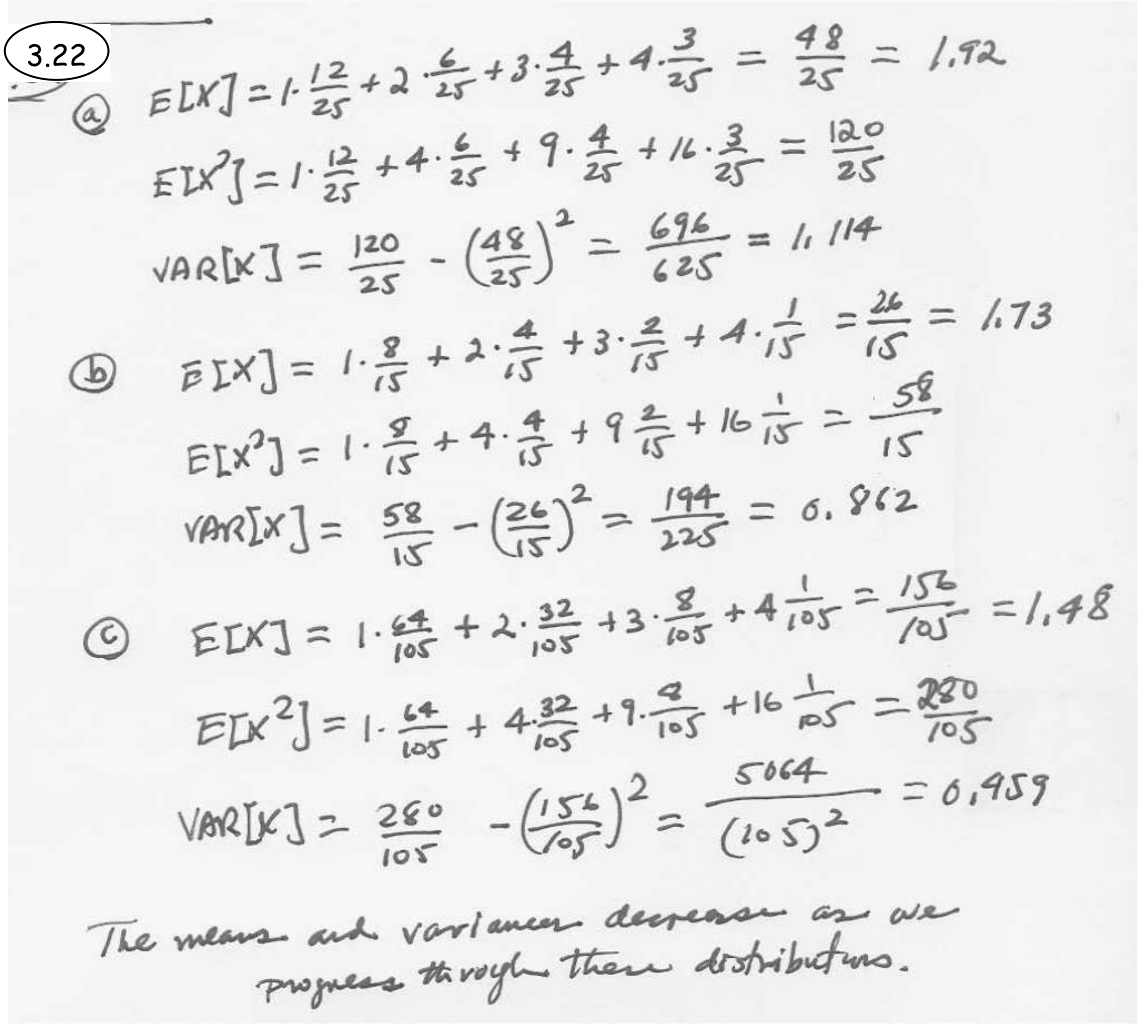
We will use $VAR[X] = E[X^2] - E[X]^2$:

$E[X^2] = 1^2 \cdot \frac{8}{16} + 4 \cdot \frac{7}{16} = \frac{36}{16}$

$E[Y^2] = 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = \frac{3}{2}$

$VAR[X] = \frac{36}{16} - \left(\frac{22}{16}\right)^2 = \frac{82}{256}$ X has lower variance than Y.

$VAR[Y] = \frac{3}{2} - 1^2 = \frac{1}{2}$



3.22

ⓐ $E[X] = 1 \cdot \frac{12}{25} + 2 \cdot \frac{6}{25} + 3 \cdot \frac{4}{25} + 4 \cdot \frac{3}{25} = \frac{48}{25} = 1.92$

$E[X^2] = 1 \cdot \frac{12}{25} + 4 \cdot \frac{6}{25} + 9 \cdot \frac{4}{25} + 16 \cdot \frac{3}{25} = \frac{120}{25}$

$VAR[X] = \frac{120}{25} - \left(\frac{48}{25}\right)^2 = \frac{696}{625} = 1.114$

ⓑ $E[X] = 1 \cdot \frac{8}{15} + 2 \cdot \frac{4}{15} + 3 \cdot \frac{2}{15} + 4 \cdot \frac{1}{15} = \frac{26}{15} = 1.73$

$E[X^2] = 1 \cdot \frac{8}{15} + 4 \cdot \frac{4}{15} + 9 \frac{2}{15} + 16 \frac{1}{15} = \frac{58}{15}$

$VAR[X] = \frac{58}{15} - \left(\frac{26}{15}\right)^2 = \frac{194}{225} = 0.862$

ⓒ $E[X] = 1 \cdot \frac{64}{105} + 2 \cdot \frac{32}{105} + 3 \cdot \frac{8}{105} + 4 \frac{1}{105} = \frac{156}{105} = 1.48$

$E[X^2] = 1 \cdot \frac{64}{105} + 4 \cdot \frac{32}{105} + 9 \cdot \frac{8}{105} + 16 \frac{1}{105} = \frac{280}{105}$

$VAR[X] = \frac{280}{105} - \left(\frac{156}{105}\right)^2 = \frac{5064}{(105)^2} = 0.459$

The means and variances decrease as we progress through these distributions.

3.23

$$E[X] = \sum_{i=0}^{15} i \frac{1}{16} = \frac{1}{16}\sum_{i=0}^{15} i$$

Let $S_k = 1+2+\ldots+k$

$S_k = k+(k-1)+\ldots+1$

$$2S_k = k(k+1) \Rightarrow S_k = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$\therefore E[X] = \frac{1}{16}\sum_{i=1}^{15} i = \frac{1}{16}\,\frac{15(16)}{2} = \frac{15}{2} = 7.5.$$

$$E[X^2] = \frac{1}{16}\sum_{i=1}^{15} i^2 \qquad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$= \frac{1}{16}\,\frac{15(16)(31)}{6} = \frac{155}{2}$$

$$VAR[X] = \frac{155}{2} - \left(\frac{15}{2}\right)^2 = \frac{310-225}{4} = \frac{85}{4}$$

3.24

$$E[X] = 2P[X=2] + 3P[X=3] + 4P[X=4]$$

$$= 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = 2\tfrac{3}{4} \text{ bits/block}$$

Let $X_1, X_2, \ldots,$ be the codeword lengths for a sequence of source outputs. The average codeword length is

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to E[X] \text{ for large } n$$

$\therefore E[X]$ is the long term average number of bits per block.

3.25 Without replacement

$$E[X] = 2 \cdot \frac{4}{5} + 51 \frac{1}{5} = \frac{59}{5} = 11{,}80$$

$$E[X^2] = 4 \cdot \frac{4}{5} + 51^2 \cdot \frac{1}{5} = \frac{2617}{5}$$

$$VAR[X] = \frac{2617}{5} - \left(\frac{59}{5}\right)^2 = \frac{9604}{25} = 384.16$$

with replacement:

$$E[X] = 2 \cdot \frac{81}{100} + 51 \frac{18}{100} + 100 \cdot \frac{1}{100} = \frac{1180}{100} = 11.80$$

$$E[X^2] = 4 \cdot \frac{81}{100} + 51^2 \cdot \frac{18}{100} + 10^4 \cdot \frac{1}{100} = \frac{57142}{100}$$

$$VAR[X] = \frac{57142}{100} - \left(\frac{1180}{100}\right)^2 = \frac{43218}{100} = 432.18$$

Means in both draws is the same!

3.26

$$E[Y] = \sum_{j=-5}^{5} j\, P[Y=j]$$

$$= -5 \cdot \frac{1}{36} - 4 \cdot \frac{2}{36} - 3 \cdot \frac{3}{36} - 2 \cdot \frac{4}{36} - 1 \cdot \frac{5}{36} + 0 \cdot \frac{6}{36}$$

$$+ 1 \cdot \frac{5}{36} + 2 \cdot \frac{4}{36} + 3 \cdot \frac{3}{36} + 4 \cdot \frac{2}{36} + 5 \cdot \frac{1}{36}$$

$$= 0$$

$$VAR[Y] = E[Y^2] = \sum_{j=-5}^{5} j^2 P[Y=j]$$

$$= \sum_{j=1}^{5} j^2 \left[P[X=j] + P[X=-j] \right]$$

$$= 1 \cdot \frac{10}{36} + 4 \cdot \frac{8}{36} + 9 \cdot \frac{6}{36} + 16 \cdot \frac{4}{36} + 25 \cdot \frac{2}{36}$$

$$= \frac{185}{36}$$

3.27

$$E[X] = \sum_{j=1}^{\infty} j\,P[X=j] = \sum_{j=1}^{\infty} j\,\frac{c}{j^2} = c\sum_{j=1}^{\infty}\frac{1}{j} = \infty$$

mean does not exist.

$$E[X^2] = \sum_{j=1}^{\infty} j^2\,\frac{c}{j^2} = c\sum_{j=1}^{\infty} 1 = \infty$$

none of the moments exist

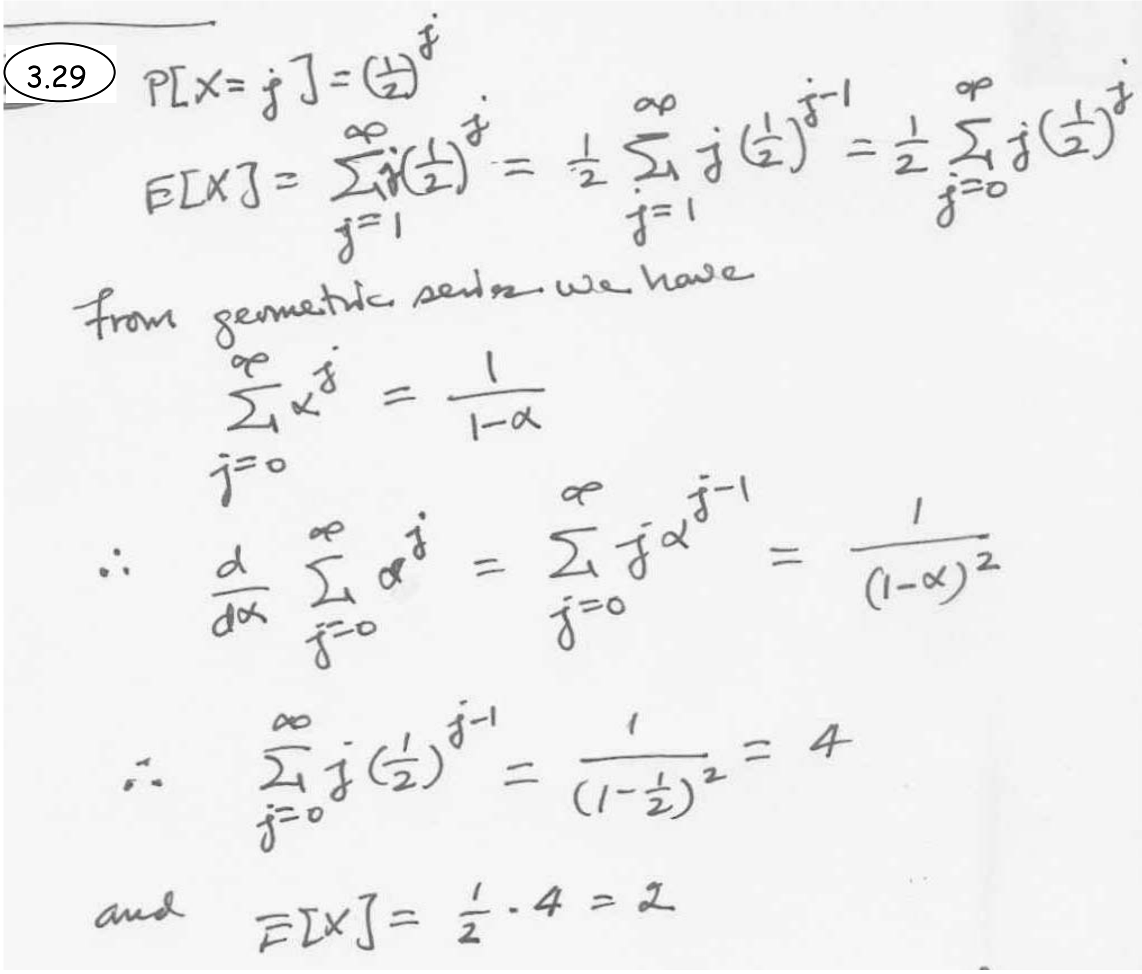
This pmf decays sufficiently fast that probabilities add to 1, but too slowly for moments to exist.

3.28

$$E[Y] = -1\cdot\frac{1}{10} + 0\cdot\frac{2}{10} + 1\cdot\frac{3}{10} + 2\cdot\frac{4}{10} = \frac{10}{10} = 1$$

$$E[Y^2] = 1\cdot\frac{1}{10} + 1\cdot\frac{3}{10} + 4\cdot\frac{4}{10} = \frac{20}{10} = 2$$

$$\text{VAR}[Y] = 2 - 1^2 = 1.$$



3.29

$P[X=j]=\left(\frac{1}{2}\right)^j$

$$E[X]=\sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^j = \frac{1}{2}\sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^{j-1} = \frac{1}{2}\sum_{j=0}^{\infty} j\left(\frac{1}{2}\right)^{j}$$

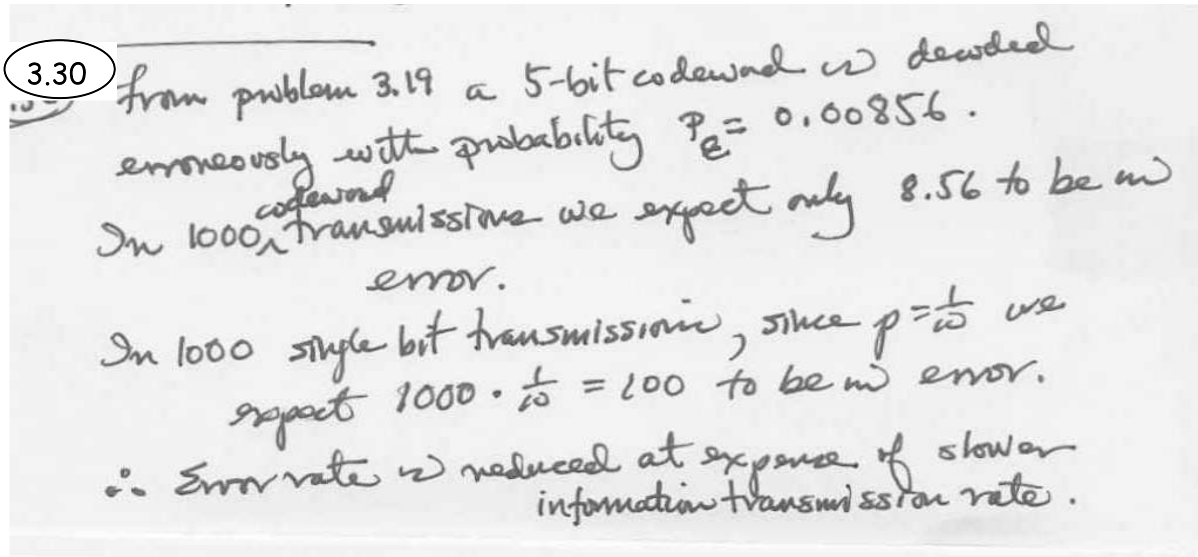
from geometric series we have

$$\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}$$

$$\therefore \quad \frac{d}{d\alpha}\sum_{j=0}^{\infty} \alpha^j = \sum_{j=0}^{\infty} j\alpha^{j-1} = \frac{1}{(1-\alpha)^2}$$

$$\therefore \quad \sum_{j=0}^{\infty} j\left(\frac{1}{2}\right)^{j-1} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

and $E[X]=\frac{1}{2}\cdot 4 = 2$



3.30

from problem 3.19 a 5-bit codeword is decoded erroneously with probability $P_e = 0.00856$.

In 1000 codeword transmissions we expect only 8.56 to be in error.

In 1000 single bit transmissions, since $p=\frac{1}{10}$ we expect $1000\cdot\frac{1}{10}=100$ to be in error.

∴ Error rate is reduced at expense of slower information transmission rate.

3.31

$$P[X=k] = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

$$E[aX^2+bX] = aE[X^2] + bE[X]$$

$$E[X] = \sum_{j=0}^{n} j\binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n \sum_{j=0}^{n} j\,\frac{n!}{j!(n-j)!}$$

$$= \left(\frac{1}{2}\right)^n \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!} \qquad \text{let } j' = j-1$$

$$= \left(\frac{1}{2}\right)^n n \sum_{j'=0}^{n-1} \frac{(n-1)!}{j'!(n-1-j')!} = n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1} \binom{n-1}{j'}$$

$$= n\left(\frac{1}{2}\right)^n 2^{n-1} = \frac{n}{2}.$$

$$E[X^2] = \sum_{j=0}^{n} j^2 \binom{n}{j}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n n \sum_{j=1}^{n} j\,\frac{(n-1)!}{(j-1)!(n-j)!}$$

$$= n\left(\frac{1}{2}\right)^n \sum_{j'=0}^{n-1} (j'+1)\binom{n-1}{j'}$$

$$= n\left(\frac{1}{2}\right)\Bigg[\underbrace{\sum_{j'=0}^{n-1} j'\binom{n-1}{j'}\left(\frac{1}{2}\right)^{n-1}}_{\substack{(n-1)\frac{1}{2} \\ \text{expected value of} \\ \text{binomial}}} + \underbrace{\sum_{j'=0}^{n-1}\binom{n-1}{j'}\left(\frac{1}{2}\right)^{n-1}}_{\substack{1 \\ \text{binomial probs}}}\Bigg]$$

$$= \frac{n}{2}\left[\frac{n}{2}+1\right]$$

$$\therefore E[aX^2+bX] = a\frac{n}{2}\left(\frac{n}{2}+1\right) + b\frac{n}{2}$$ ✓ average reward.

3.31b

$$E[a^X] = \sum_{j=0}^{n} a^j \binom{n}{j}\left(\frac{1}{2}\right)^j = \sum_{j=0}^{n} \binom{n}{j}\left(\frac{a}{2}\right)^j$$

$$= \left(1+\frac{a}{2}\right)^n$$

3.32

ⓐ $E[g(X)] = E[I_A(X)]$ $\quad A = \{X > 10\}$

$$= \sum_{i=1}^{15} I_A(i)\, P[X=i] = \sum_{i=11}^{15} P[X=i]$$

$$= p_1\left[\sum_{i=11}^{15} \frac{1}{i}\right] = \frac{\sum_{i=11}^{15} 1/i}{\sum_{i=1}^{15} 1/i} = 0.1173$$

ⓑ $E[I_A(X)] = p_1 \sum_{i=11}^{15} 1/2^{(i-1)}$

pmf has faster decay than (a)

$$= \frac{\sum_{i=11}^{15} 1/2^{(i-1)}}{\sum_{i=1}^{15} 1/2^{(i-1)}} = .000946$$

ⓒ $E[I_A(X)] = p_1 \sum_{i=11}^{15} \left(\frac{1}{2}\right)^{j(j-1)/2} = \dfrac{\sum_{i=11}^{15}\left(\frac{1}{2}\right)^{j(j-1)/2}}{\sum_{i=1}^{15}\left(\frac{1}{2}\right)^{j(j-1)/2}}$

$$= 1.69 \times 10^{-17}$$

The last pmf decays much faster than the first two.

3.33

ⓐ $E[(X-10)^+] = \sum_{i=1}^{15}(i-10)P[X=i] = p_1\sum_{i=1}^{15}(i-10)\frac{1}{i} = 0.33373$

ⓑ $E[(X-10)^+] = p_1\sum_{i=1}^{15}(i-10)\,2^{-(i-1)} = 0.00174$

ⓒ $E[(X-10)^+] = p_1\sum_{i=1}^{15}(i-10)\,2^{-i(i-1)/2} = 1.69\times10^{-17}$

3.34

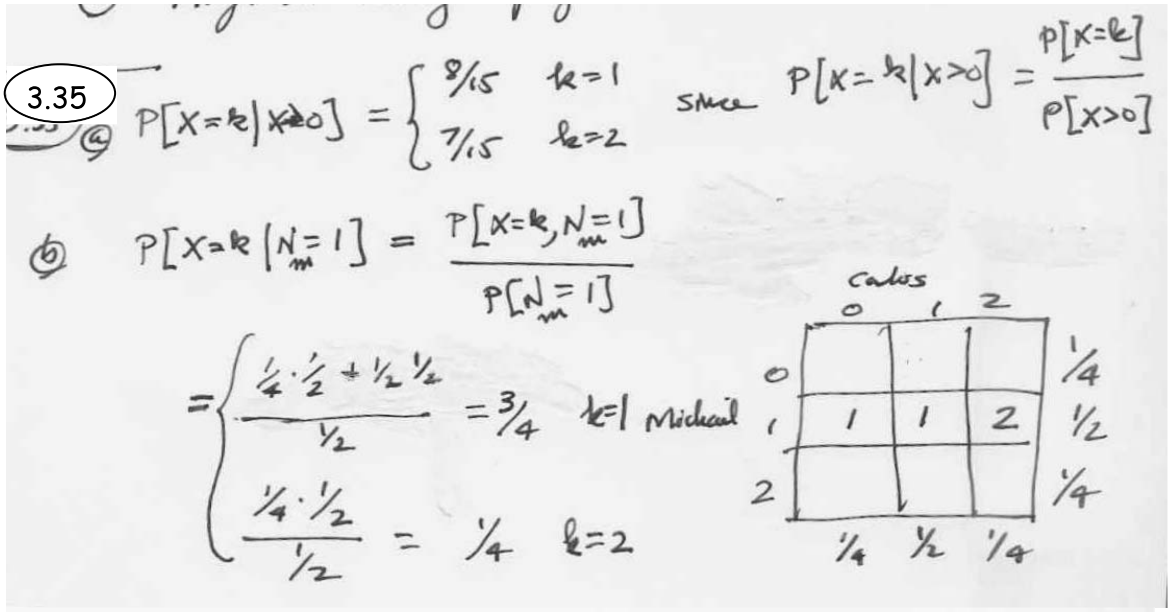
$X_{max} = m$ since casino has 2^m dollars.

ⓐ ∴ casino can only play up to

ⓑ $E[Y] = \sum_{k=1}^{m} 2^k\left(\frac{1}{2}\right)^k = m$

ⓒ Player is willing to pay at most m dollars.

3.4 Conditional Probability Mass Function



3.35

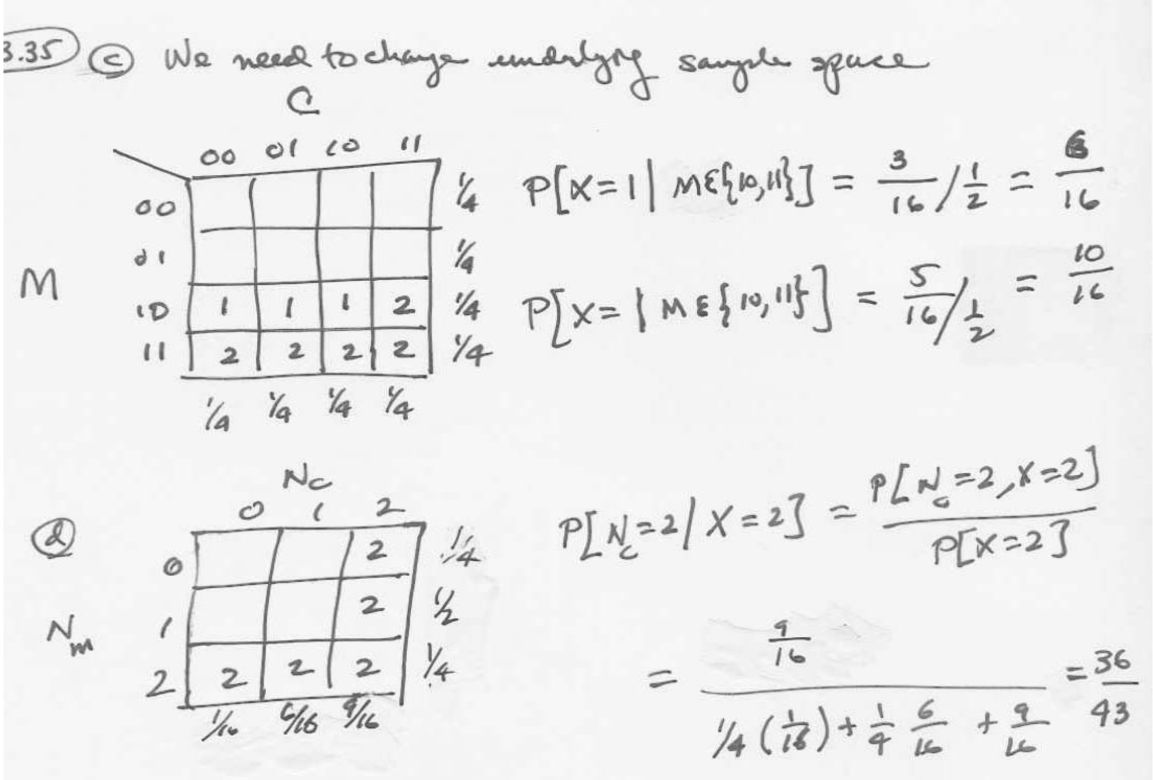
(a) $P[X=k \mid X>0] = \begin{cases} 8/15 & k=1 \\ 7/15 & k=2 \end{cases}$ since $P[X=k \mid X>0] = \dfrac{P[X=k]}{P[X>0]}$

(b) $P[X=k \mid N_m=1] = \dfrac{P[X=k, N_m=1]}{P[N_m=1]}$

$$= \begin{cases} \dfrac{\frac14\cdot\frac12 + \frac12\frac12}{\frac12} = \frac34 & k=1 \text{ Michael} \\[2ex] \dfrac{\frac14\cdot\frac12}{\frac12} = \frac14 & k=2 \end{cases}$$

Carlos

$$\begin{array}{c|ccc|c} & 0 & 1 & 2 & \\ \hline 0 & & & & \frac14 \\ 1 & 1 & 1 & 2 & \frac12 \\ 2 & & & & \frac14 \\ \hline & \frac14 & \frac12 & \frac14 & \end{array}$$



3.35 (c) We need to change underlying sample space

C

$$\begin{array}{cc|cccc|c} & & 00 & 01 & 10 & 11 & \\ \hline & 00 & & & & & \frac14 \\ & 01 & & & & & \frac14 \\ M & 10 & 1 & 1 & 1 & 2 & \frac14 \\ & 11 & 2 & 2 & 2 & 2 & \frac14 \\ \hline & & \frac14 & \frac14 & \frac14 & \frac14 & \end{array}$$

$P[X=1 \mid M\in\{10,11\}] = \frac{3}{16}/\frac12 = \frac{6}{16}$

$P[X= \mid M\in\{10,11\}] = \frac{5}{16}/\frac12 = \frac{10}{16}$

(d)

N_c

$$\begin{array}{cc|ccc|c} & & 0 & 1 & 2 & \\ \hline & 0 & & & 2 & \frac14 \\ N_m & 1 & & & 2 & \frac12 \\ & 2 & 2 & 2 & 2 & \frac14 \\ \hline & & \frac{1}{16} & \frac{6}{16} & \frac{9}{16} & \end{array}$$

$P[N_c=2 \mid X=2] = \dfrac{P[N_c=2, X=2]}{P[X=2]}$

$$= \frac{\frac{9}{16}}{\frac14\left(\frac{1}{16}\right) + \frac14\frac{6}{16} + \frac{9}{16}} = \frac{36}{43}$$

3.36

ⓐ $$P[X=k \mid X<4] = \frac{P[X=k]}{1-P[X=4]} = \begin{cases} \frac{12}{22} & k=1 \\ 6/22 & k=2 \\ 4/22 & k=3 \end{cases}$$

ⓑ $$P[X=k \mid X<4] = \frac{P[X=k]}{104/105} = \begin{cases} \frac{64}{104} & k=1 \\ \frac{32}{104} & k=2 \\ \frac{8}{104} & k=3 \end{cases}$$

ⓒ $$P[X=k \mid X<4] = \frac{P[X=k]}{14/15} = \begin{cases} 8/14 & k=1 \\ 4/14 & k=2 \\ 2/14 & k=3 \end{cases}$$

3.37

ⓐ $$P[X=k \mid X<8] = \frac{P[X=k]}{P[X<8]} = \frac{1/16}{1/2} = \frac{1}{8} \text{ for } k<8$$

ⓑ $P[X=k \mid \text{1st bit is zero}] = P[X=k \mid X<8]$ same as ⓐ.

ⓒ $$P[X=k \mid \text{4th bit is zero}] = P[X=k \mid X \text{ is even}]$$

$$= \frac{P[X=k,\ k \text{ even}]}{P[X \text{ is even}]} = \frac{1/16}{1/2} = 1/8 \text{ for } k \text{ even}$$

3.38) "No message gets through" $\Leftrightarrow X > 1$

ⓐ

$$P[X=k \mid X>1] = \frac{P[X=k]}{P[X>1]} = \frac{(1/2)^k}{1/2} = \left(\frac{1}{2}\right)^{k-1} \text{ for } k>1$$

ⓑ If $\underbrace{\text{1st transmitter transmitted in slot 1}}_{=C}$, then

collision occurs in time slot 1 with prob 1/2 $\Leftrightarrow X>1$

success " " " " " 1/2 $\Leftrightarrow X=1$

$\therefore P[X=1 \mid C] = 1/2$

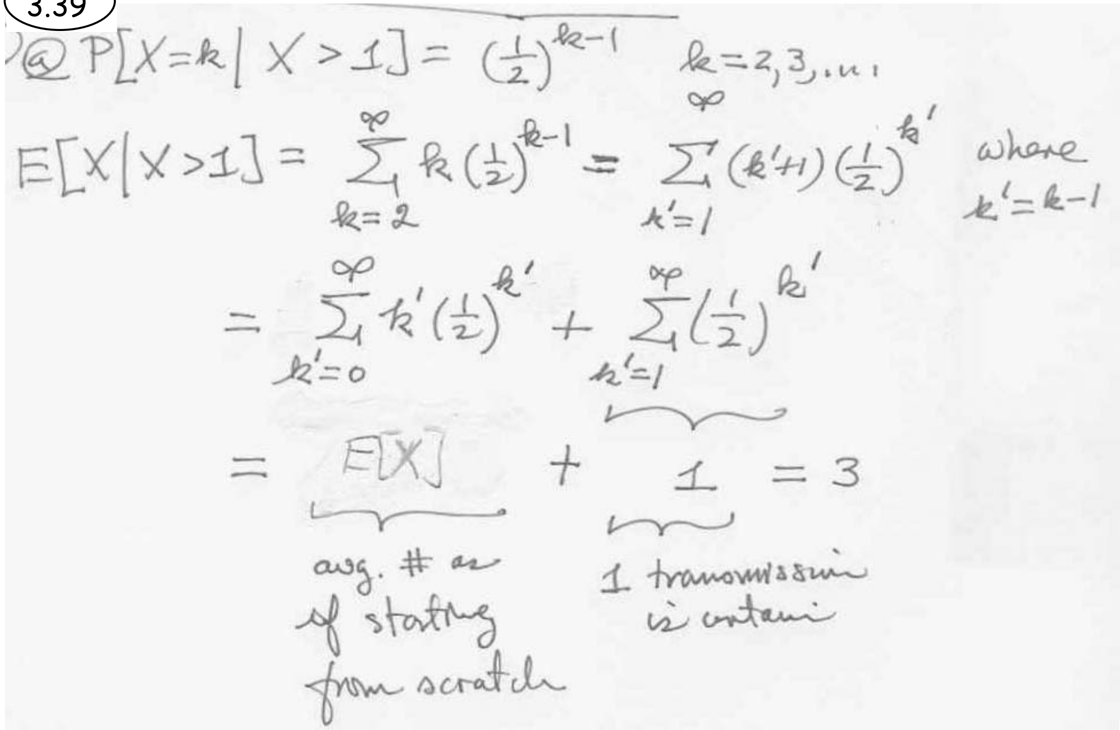
for $k>1$

$$P[X=k \mid C] = P[X=k, X>1] = P[X=k \mid X>1]\,P[X>1]$$

$$= \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^k \quad k>1$$

$\therefore$ knowledge that C occurred does not change the pmf of X.

3.39



(a) $P[X=k \mid X>1] = \left(\frac{1}{2}\right)^{k-1}$ $k=2,3,\ldots$

$$E[X \mid X>1] = \sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty}(k'+1)\left(\frac{1}{2}\right)^{k'}$$ where $k'=k-1$

$$= \sum_{k'=0}^{\infty} k'\left(\frac{1}{2}\right)^{k'} + \underbrace{\sum_{k'=1}^{\infty}\left(\frac{1}{2}\right)^{k'}}$$

$$= \underbrace{E[X]}_{\text{avg. \# as of starting from scratch}} + \underbrace{1}_{\text{1 transmissi is certai}} = 3$$

3.39b "message gets through in 1st time slot" $= X=1$

$$P[X=k \mid X=1] = \begin{cases} 0 & k>1 \\ 1 & k= \text{[illegible]} \end{cases}$$

$$E[X \mid X=1] = 1 \cdot P[X=1] = 1$$

ⓒ Let $A=\{X=1\}$ $B=\{X>1\}$ then A & B form a partition

$$E[X] = E[X|A]P[A] + E[X|B]P[B]$$

$$= 1 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = 2$$

note we can use the result of part a to find $E[X]$:

$$E[X] = 1 \cdot \tfrac{1}{2} + (E[X]+1)\tfrac{1}{2} \Rightarrow E[X]=2.$$

ⓓ
$$E[X^2 \mid X>1] = \sum_{k=2}^{\infty} k^2 \left(\tfrac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)^2 \left(\tfrac{1}{2}\right)^{k'}$$

$$= \sum_{k'=1}^{\infty} k'^2 \left(\tfrac{1}{2}\right)^{k'} + 2\sum_{k'=1}^{\infty} k' \left(\tfrac{1}{2}\right)^{k'} + \sum_{k'=1}^{\infty} \left(\tfrac{1}{2}\right)^{k'}$$

$$= E[X^2] + 2E[X] + 1$$

$$= E[X^2] + 5$$

$$E[X^2 \mid X=1] = 1$$

$$\therefore\ E[X^2] = E[X^2 \mid X=1]\tfrac{1}{2} + E[X^2 \mid X>1]\tfrac{1}{2}$$

$$= \tfrac{1}{2} + [E[X^2]+5]\tfrac{1}{2}$$

$$\Rightarrow E[X^2] = 6$$

$$VAR[X] = E[X^2] - E[X]^2 = 6 - 2^2 = 2$$

3.40

By (3.31b)

$$E[X^2] = \sum_{i=1}^{n} E[X^2/B_i]P[B_i] \quad \text{and} \quad E[X] = \sum_{i=1}^{n} E[X/B_i]P[B_i]$$

$$VAR[X] = E[X^2] - E[X]^2$$

$$= \sum_{i=1}^{n} E[X^2/B_i]P[B_i] - \left(\sum_{i=1}^{n} E[X|B_i]P[B_i]\right)^2$$

$$\neq \sum_{i=1}^{n} \left(E[X^2/B_i] - E[X/B_i]^2\right) P[B_i]$$

3.41

(a) $P[X = k \mid \text{1st draw} = k]$ $k = 1, 50$

$$P[X = j \mid \text{1st draw} = 1] = \begin{cases} \frac{8}{9} & j = 2 \\ \frac{1}{9} & j = 51 \end{cases}$$

$$P[X = j \mid \text{1st draw} = 50] = \begin{cases} 1 & j = 51 \\ 0 & \text{otherwise} \end{cases}$$

(b) $E[X \mid \text{1st draw} = 1] = 2 \cdot \frac{8}{9} + 51 \cdot \frac{1}{9}$

$E[X \mid \text{1st draw} = 50] = 51$

(c) $E[X] = E[X|1] \cdot \frac{9}{10} + E[X|50]\frac{1}{10}$

$$= \frac{67}{9}\,\frac{9}{10} + \frac{51}{10} = \frac{118}{10}$$

(d) $E[X^2|1] = 4 \cdot \frac{8}{9} + (51)^2\frac{1}{9}$ $E[X^2|50] = (51)^2$

$$E[X^2] = \left(4 \cdot \frac{8}{9} + \frac{51^2}{9}\right)\frac{9}{10} + \frac{51^2}{10} = \frac{32}{10} + 2\left(\frac{51^2}{10}\right) = \frac{5234}{10}$$

$$VAR[X] = \frac{32}{10} + 2\left(\frac{51^2}{10}\right) - \left(\frac{118}{10}\right)^2 = 384.16$$

3.42 Assume # of heads is k

then $E[Y|k] = n-2k$

$$\therefore \quad E[Y] = \sum_{k=0}^{n} E[Y|k]P[k] = \sum_{k=0}^{n} (n-2k)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n - 2E[X] = n - 2np$$

$$= n(1-2p)$$

Similarly

$$E[Y^2|k] = (n-2k) = n^2 - 4kn + 4k^2$$

$$E[Y^2] = \sum_{k=0}^{n} (n^2 - 4kn + 4k^2)\binom{n}{k}p^k(1-p)^{n-k}$$

$$= n^2 - 4nE[X] + 4E[X^2]$$

$$= n^2 - 4n^2p + 4(npq + (np)^2)$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

$$= n^2 - 4n^2p + 4npq + 4n^2p^2 - n^2\underbrace{(1-2p)^2}_{1-4p+4p^2}$$

$$= 4npq$$

3.43

ⓐ If password has not been found after k tries then there remain $2^m - k$ possible passwords.

$$P[X=j \mid X>k] = \begin{cases} \dfrac{1}{2^m-k} & j = k+1, \ldots, 2^m \\ 0 & \text{otherwise} \end{cases}$$

ⓑ

$$E[X \mid X>k] = \sum_{j=k+1}^{2^m} j \, \frac{1}{2^m-k} = \frac{1}{2^m-k} \underbrace{\sum_{j=k+1}^{2^m} j}_{\sum_{j=1}^{2^m} j \; - \; \sum_{j'=1}^{k} j'}$$

$$= \frac{1}{2^m-k}\left[\frac{2^m(2^m+1)}{2} - \frac{k(k+1)}{2}\right]$$

$$= \frac{1}{2^m-k}\left[\frac{(2^m-k)(2^m+k+1)}{2}\right] = \frac{2^m+k+1}{2}$$

$$= \underbrace{(k+1)}_{\text{minimum}} + \underbrace{\frac{2^m-(k+1)}{2}}_{\text{average additional number of tries}}$$

3.5 Important Discrete Random Variables

3.44

ⓐ $S = \{1,2,3,4,5\}$ $A = \{\zeta > 3\}$

$P[I_A = 0] = \frac{3}{5}$ $P[I_A = 1] = \frac{2}{5}$

$E[I_A] = 0 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{2}{5}$

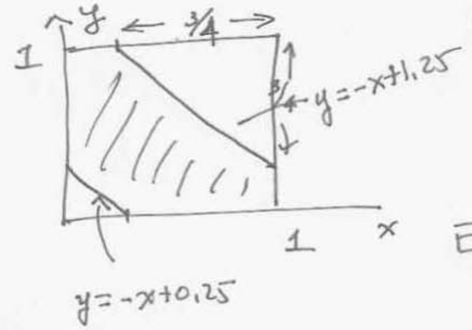
ⓑ $S = [0,1]$ $A = \{0.3 < \zeta \le 0.7\}$

$P[I_A = 0] = P[0 < \zeta \le 0.3] + P[0.7 < \zeta \le 1] = 0.6$

$P[I_A = 1] = P[0.3 < \zeta \le 0.7] = 0.4$

$E[I_A] = 0 \cdot 0.6 + 1 \cdot 0.4 = 0.4$

ⓒ

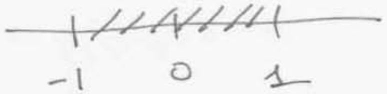


$P[I_A = 0] = \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{10}{16}$

$P[I_A = 1] = 1 - \frac{10}{16} = \frac{6}{16}$

$E[I_A] = 0 \cdot \frac{10}{16} + 1 \cdot \frac{6}{16} = \frac{6}{16}.$

ⓓ $A = \{\zeta > a\}.$

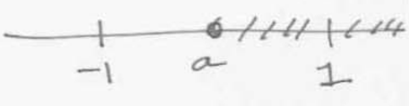


$a < -1$

$P[I_A = 1] = 1$

$E[I_A] = 1$

$-1 < a < 1$



$P[I_A = 1] = \frac{1-a}{2}$

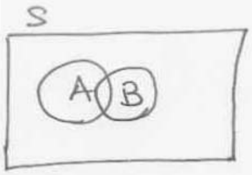
$E[I_A] = \frac{1-a}{2}$

$1 < a$

$P[I_A = 1] = 0$

$E[I_A] = 0$

3.45



ⓐ $I_S = 1$ iff $\xi \in S \Rightarrow I_S = 1$ all ξ

$I_\phi = 1$ iff $\xi \in \phi \Rightarrow I_\phi = 0$ all ξ

ⓑ $I_{A\cap B}(\xi) = 1$ iff $\xi \in A$ and $\xi \cap B \Leftrightarrow I_A(\xi) = 1$ and $I_B(\xi) = 1$

$\Leftrightarrow I_{A\cap B}(\xi) = I_A(\xi) I_B(\xi)$.

$I_{A\cup B}(\xi) = 0$ iff $\xi \notin A \cup B \Leftrightarrow \xi \in A^c \cap B^c \Leftrightarrow I_{A^c}(\xi) I_{B^c}(\xi) = 1$

$\Leftrightarrow (1 - I_A(\xi))(1 - I_B(\xi)) = 1$

$\Leftrightarrow 1 - I_A(\xi) - I_B(\xi) + I_A(\xi) I_B(\xi) = 1$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_A(\xi) I_B(\xi) = 0$

$\Leftrightarrow I_A(\xi) + I_B(\xi) - I_{A\cap B}(\xi) = 0 = I_{A\cup B}(\xi)$

ⓒ $E[I_S] = 1 \cdot P[S] = 1$

$E[I_\phi] = 1 \cdot P[\phi] = 0$

$E[I_{A\cap B}] = 1 \cdot P[A \cap B]$

$E[I_{A\cup B}] = E[I_A] + E[I_B] - E[I_{A\cap B}]$

$= P[A] + P[B] - P[A \cap B]$.

3.46 $n = 8$ $p = 0.25$ Binomial random variable

ⓐ $P[N=0] = \binom{8}{0} p^0 (1-p)^{8-0} = (0.75)^8 = 0.100$

ⓑ $P[N=1] = \binom{n}{1} p (1-p)^7 = 8(0.25)(0.75)^7 = 0.267$

ⓒ $P[N>4] = \sum_{j=5}^{8} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.0273$

ⓓ $P[2<N<6] = \sum_{j=3}^{5} \binom{8}{j} (.25)^j (.75)^{8-j} = 0.3172$

3.47

ⓐ $A_i = \{U_i < 0.25\}$

$A_i^c = \{U_i > 0.25\}$

$$P[\overbrace{A_1 A_2 A_3 A_4 A_5^c A_6^c A_7^c A_8^c}^{\text{intersections}}] = (0.25)^4 (0.75)^4$$

$$= 0.00124$$

ⓑ $P[N=4] = \binom{8}{4}(0.25)^4 (0.75)^4 = 0.0865$

ⓒ $A_i = \{U_i < 0.25\}$

$B_i = \{0.25 < U_i < 0.75\}$

$C_i = \{U_i > 0.75\}$

$$P[A_1 A_2 A_3 B_4 B_5 C_6 C_7 C_8] = (0.25)^3 (0.5)^2 (0.25)^3$$

$$= (0.25)^6 (0.5)^2$$

$$= 6.10 \times 10^{-5}$$

ⓓ $P[N_1 = 3, N_2 = 2, N_3 = 3] = \dfrac{8!}{3!\,2!\,3!}\, 0.25^3\ 0.5^2\ 0.25^3$

multinomial

$$= 0.0342$$

ⓔ $P[A_1 A_2 A_3 A_4 C_5 C_6 C_7 C_8] = (0.25)^4 (0.25)^4 = 1.526 \times 10^{-5}$

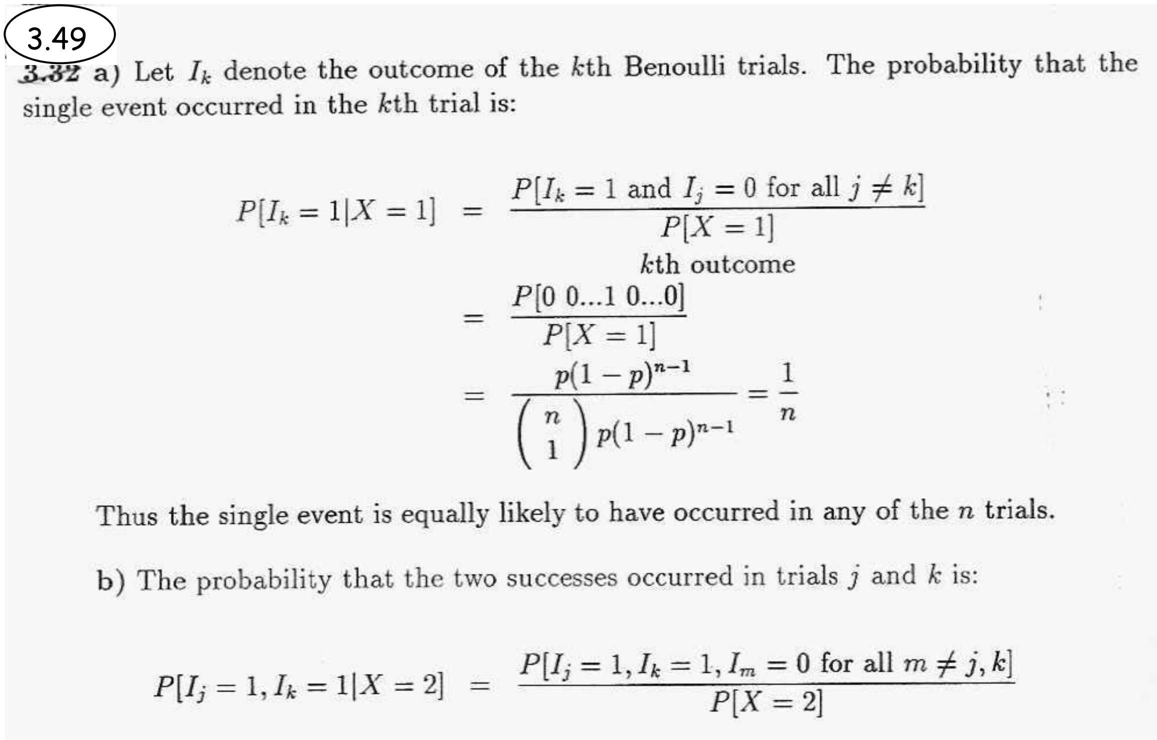
ⓕ $P[N_1 = 4, N_2 = 0, N_3 = 4] = \dfrac{8!}{4!\,0!\,4!} (0.25)^4 (0.5)^0 (0.25)^4$

$$= 0.00107$$

3.48

This Octave program will plot binomial pmf.

```
> n=4;
> x=[0:n];
> p=0.10;
> stem(binomial-pdf(x,n,p))
```



3.49

3.32 a) Let I_k denote the outcome of the kth Benoulli trials. The probability that the single event occurred in the kth trial is:

$$
\begin{aligned}
P[I_k = 1 | X = 1] &= \frac{P[I_k = 1 \text{ and } I_j = 0 \text{ for all } j \neq k]}{P[X = 1]} \\
&= \frac{P[0\;0 \ldots \overset{k\text{th outcome}}{1}\;0 \ldots 0]}{P[X = 1]} \\
&= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} = \frac{1}{n}
\end{aligned}
$$

Thus the single event is equally likely to have occurred in any of the n trials.

b) The probability that the two successes occurred in trials j and k is:

$$
P[I_j = 1, I_k = 1 | X = 2] = \frac{P[I_j = 1, I_k = 1, I_m = 0 \text{ for all } m \neq j, k]}{P[X = 2]}
$$

3.50 a)

$$\frac{p_k}{p_{k-1}} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!}} \frac{p}{q} = \frac{(n-k+1)p}{kq}$$

$$= \frac{(n+1)p - k(1-q)}{kq} = 1 + \frac{(n+1)p - k}{kq}$$

b) First suppose $(n+1)p$ is not an integer, then
for $0 \leq k \leq [(n+1)p] < (n+1)p$

$$(n+1)p - k > 0$$

so

$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p - k}{kq} > 1$$

$\Rightarrow p_k$ increases as k increases from 0 to $[(n+1)p]$
for $k > (n+1)p \geq [(n+1)p]$

$$(n+1)p - k < 0$$

so

$$\frac{p_k}{p_{k-1}} = 1 + \frac{(n+1)p - k}{kq} < 1$$

$\Rightarrow p_k$ decreases as k increases beyond $[(n+1)p]$
$\therefore p_k$ attains its maximum at $k_{MAX} = [(n+1)p]$
If $(n+1)p = k_{MAX}$ then above implies that

$$\frac{p_{k_{MAX}}}{p_{k_{MAX}-1}} = 1 \Rightarrow p_{k_{MAX}} = p_{k_{MAX}} - 1$$

3.51

ⓐ $$(a+b+c)^n = \sum_{k=0}^{n} \binom{n}{k} (a+b)^k c^{n-k}$$

ⓑ $$= \sum_{k=0}^{n} \binom{n}{k} c^{n-k} \sum_{j=0}^{k} \binom{k}{j} a^{j} b^{k-j}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!\, k!}{k!\,(n-k)!\, j!\,(k-j)!} a^{j} b^{k-j} c^{n-k}$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{n!}{j!\,(n-k)!\,(k-j)!} a^{j} b^{k-j} c^{n-k}$$

$$= \sum_{\substack{j_1 \\ j_i \geq 0 \\ j_1+j_2+j_3=n}} \sum_{j_2} \sum_{j_3} \frac{n!}{j_1!\, j_2!\, j_3!} a^{j_1} b^{j_2} c^{j_3}$$

ⓒ $$1 = (p_1+p_2+p_3)^n = \sum_{j_1 j_2 j_3} \frac{n!}{j_1!\, j_2!\, j_3!} p_1^{j_1} p_2^{j_2} p_3^{j_3}$$

3.52 $p = 0.01$ $N = \#$ error free characters until first error

ⓐ $P[N=k] = (1-p)^k p \qquad k = 0, 1, 2, \ldots$

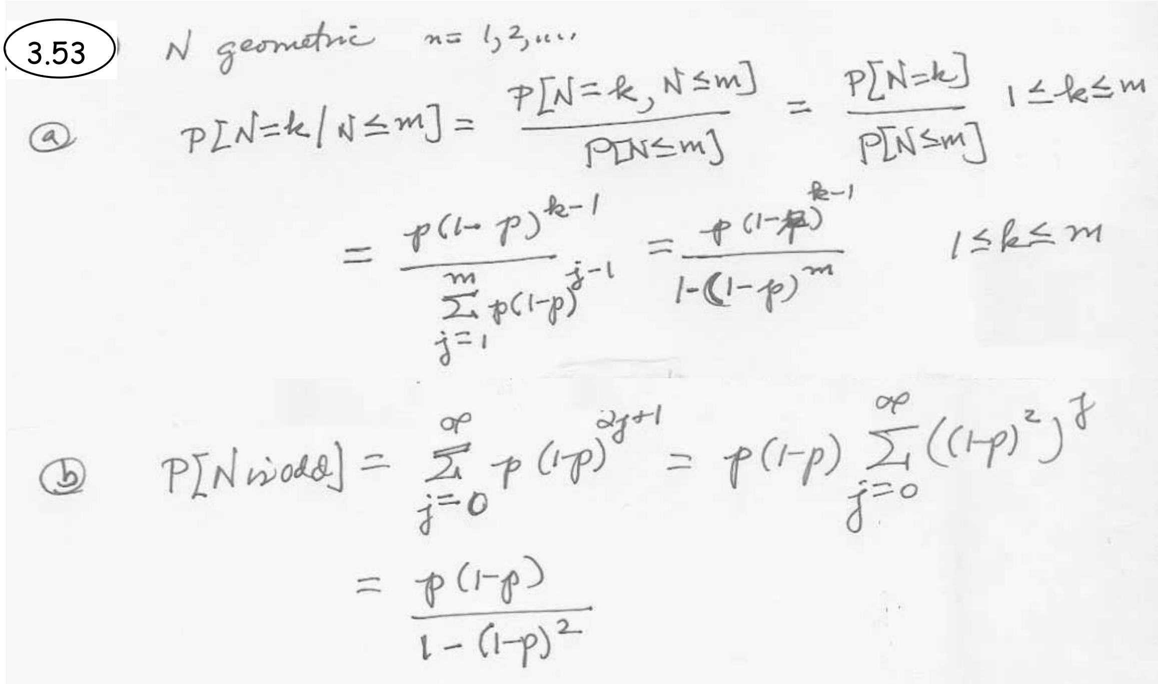
ⓑ $$E[N] = \sum_{k=0}^{\infty} k(1-p)^k p = (1-p)p \sum_{k=0}^{\infty} k(1-p)^{k-1}.$$

$$= (1-p)p \frac{1}{(1-(1-p))^2} = \frac{1-p}{p}$$ by Eqn 3.14

ⓒ $$0.99 = P[N > k_0] = \sum_{k_0+1}^{\infty} (1-p)^k p = p(1-p)^{k_0+1} \sum_{k=0}^{\infty} (1-p)^k$$

$$= (1-p)^{1001} \Rightarrow p = 1 - 0.99^{\frac{1}{1001}}$$

$$= 1.004 \times 10^{-5}$$



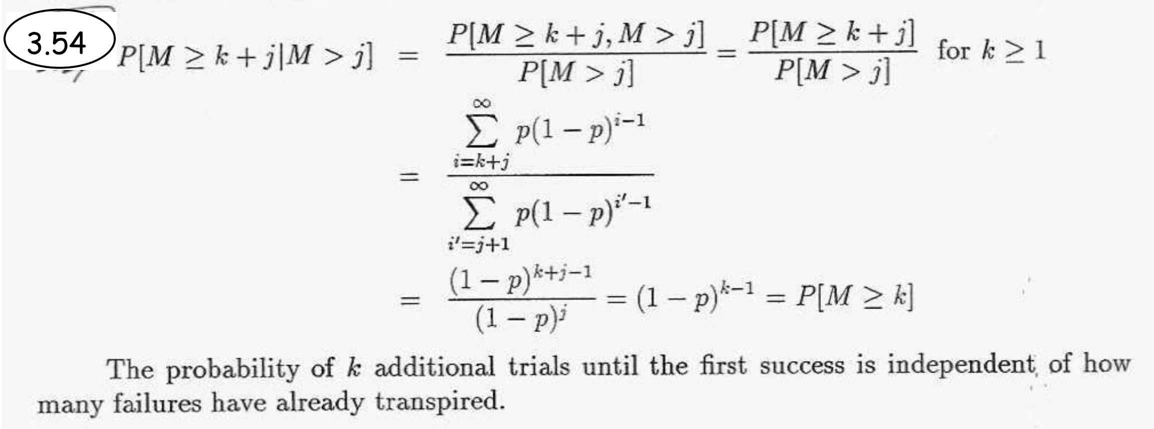
3.53 N geometric $n = 1, 2, \ldots$

ⓐ $$P[N=k \,/\, N \le m] = \frac{P[N=k,\, N \le m]}{P[N \le m]} = \frac{P[N=k]}{P[N \le m]} \quad 1 \le k \le m$$

$$= \frac{p(1-p)^{k-1}}{\sum_{j=1}^{m} p(1-p)^{j-1}} = \frac{p(1-p)^{k-1}}{1-(1-p)^{m}} \quad 1 \le k \le m$$

ⓑ $$P[N \text{ is odd}] = \sum_{j=0}^{\infty} p(1-p)^{2j+1} = p(1-p)\sum_{j=0}^{\infty}\left((1-p)^2\right)^j$$

$$= \frac{p(1-p)}{1-(1-p)^2}$$



3.54
$$P[M \ge k+j \mid M > j] = \frac{P[M \ge k+j,\, M > j]}{P[M > j]} = \frac{P[M \ge k+j]}{P[M > j]} \quad \text{for } k \ge 1$$

$$= \frac{\sum\limits_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum\limits_{i'=j+1}^{\infty} p(1-p)^{i'-1}}$$

$$= \frac{(1-p)^{k+j-1}}{(1-p)^{j}} = (1-p)^{k-1} = P[M \ge k]$$

The probability of k additional trials until the first success is independent of how many failures have already transpired.

3.55

3.36 The memoryless property states that for $j, k \geq 1$.

$$\begin{aligned} P[M \geq k] &= P[M \geq k + j | M > j] \\ &= \frac{P[M \geq k + j]}{P[M > j]} = \frac{P[M \geq k + j]}{P[M \geq j + 1]} \end{aligned}$$

$\Longrightarrow$

$$P[M \geq k + j] = P[M \geq k]P[M \geq j + 1]$$

Let

$$a_k = P[M \geq k],$$

then we have

$$(*) \qquad a_{k+j} = a_k a_{j+1} \qquad j \geq 1, k \geq 1$$

where $a_1 = 1$ and $a_2 = 1 - P[M = 1] = 1 - p$.
Equation (*) with $j = 1$ becomes

$$a_{k+1} = a_2 a_k \qquad k \geq 1$$

$$\Rightarrow a_k = a_2^{k-1} \qquad k \geq 1$$

$$\Rightarrow P[M \geq k] = (1 - p)^{k-1} \qquad k \geq 1$$

$$\begin{aligned} P[M = k] &= P[M \geq k] - P[M \geq k + 1] \\ &= (1 - p)^{k-1} - (1 - p)^k \\ &= (1 - p)^{k-1}(1 - (1 - p)) \\ &= (1 - p)^{k-1} p \end{aligned}$$

3.56

$C_{visit} = \$50$ $\quad p = \frac{1}{12}$ $\quad N =$ # failures in 12 months

$C_{repair} = \$20$ $\quad P[N=k] = \binom{12}{k}\left(\frac{1}{12}\right)^k\left(\frac{11}{12}\right)^{12-k}$ $\quad k = 0, 1, \ldots, 12$

$$0.99 = \sum_{j=k_0}^{12} P[N=j] \Rightarrow k_0 = ?$$

k	p_k	$P[N \leq k]$
0	0.352	0.352
1	0.384	0.736
2	0.192	0.928
3	0.058	0.986

$\Rightarrow k_0 = 3$

$\Rightarrow$ Charge $\$50 + 3 \times \$20 = \$110.$

Avg cost per player $= \$50 + \$20 \underbrace{E[N]}_{12(\frac{1}{12})} = \$70.$

3.57

$\alpha_S = 48 \quad \alpha_R = 24 \quad \alpha_G = 12 \quad$ slice $= 1/12$

ⓐ $P[N_G = 0] = \dfrac{(\alpha_G \frac{1}{12})^0}{0!} e^{-\alpha_G \frac{1}{12}} = e^{-1} = 0.368$

ⓑ $P[N_G = 0, N_R \leq 2] = P[N_G = 0]\, P[N_R \leq 2] = e^{-1} \sum_{k=0}^{2} \dfrac{(\alpha_R \frac{1}{12})^k}{k!} e^{-\alpha_R \frac{1}{12}}$

$$= e^{-1}\left[e^{-2} + \frac{2}{1!}e^{-2} + \frac{4}{2!}e^{-2}\right]$$

$$= e^{-3}[1 + 2 + 2] = 5e^{-3} = 0.249$$

ⓒ $P[N_G = 0, N_R = 0, N_S \geq 5] = P[N_G = 0]\, P[N_R = 0]\, P[N_S \geq 5]$

$$= e^{-1} e^{-2} e^{-4} \sum_{k=0}^{5} \frac{4^k}{k!} = e^{-3}(0.785)$$

$$= 0.039$$

It's hard to avoid the red and green thingies!

3.58

$$P[X > 4] < 0.9 \Leftrightarrow P[X \leq 4] > 0.1$$

$$P[X \leq 4] = \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha} = \sum_{k=0}^{4} \frac{(5/n)^k}{k!} e^{-5/n}$$

Since $\alpha = \frac{\lambda}{n\mu} = \frac{5}{n}$
If $n = 2$ then $P[X \leq 4] = 0.891$. Therefore ~~one~~ two employees ~~is~~ are sufficient.

$$P[X = 0] = e^{-\alpha} = e^{-2.5} = \cancel{0.05} = 0.082.$$

3.59

$\lambda = 6000$ requests/minute. $= 100$ requests/sec

$\alpha = \lambda \frac{1}{10} = 10$ requests/100 ms

ⓐ $P[N=0] = e^{-10} = 4.54 \times 10^{-5}$

ⓑ $P[5 \leq N \leq 10] = \sum_{k=5}^{10} \frac{10^k}{k!} e^{-10} = 0.554$

3.60

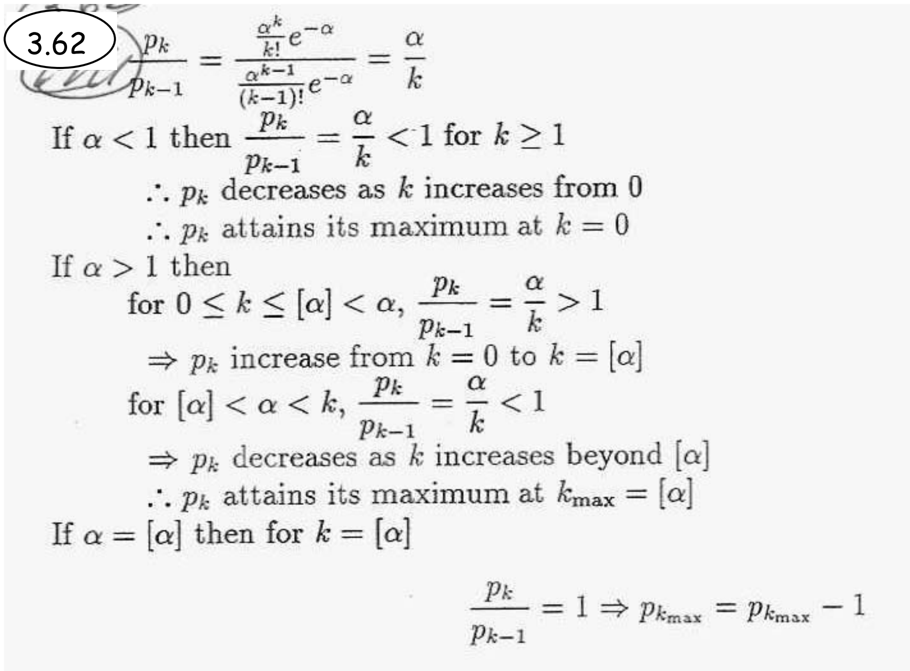
Use octave to plot pmf.

```
> a = 0.1 ;
> j = [0: 20];
> stem (poisson_pdf (j, a))
```

3.61

$$\mathcal{E}[X] = \sum_{k=0}^{\infty} k \frac{\alpha^k}{k!} e^{-\alpha} = \alpha \sum_{k=1}^{\infty} \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha} = \alpha \underbrace{\sum_{k'=0}^{\infty} \frac{\alpha^{k'}}{k'!} e^{-\alpha}}_{1} = \alpha$$

$$\begin{aligned}
\mathcal{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\alpha^k}{k!} e^{-\alpha} = \alpha \sum_{k=1}^{\infty} k \frac{\alpha^{k-1}}{(k-1)!} e^{-\alpha} \\
&= \alpha \sum_{k'=0}^{\infty} (k'+1) \frac{\alpha^{k'}}{k'!} e^{-\alpha} = \alpha\{\alpha + 1\} \\
\sigma_X^2 &= \mathcal{E}[X^2] - \mathcal{E}[X]^2 = \alpha\{\alpha + 1 - \alpha\} \\
&= \alpha
\end{aligned}$$



3.62

$$\frac{p_k}{p_{k-1}} = \frac{\frac{\alpha^k}{k!}e^{-\alpha}}{\frac{\alpha^{k-1}}{(k-1)!}e^{-\alpha}} = \frac{\alpha}{k}$$

If $\alpha < 1$ then $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1$ for $k \geq 1$

$\therefore p_k$ decreases as k increases from 0

$\therefore p_k$ attains its maximum at $k = 0$

If $\alpha > 1$ then

for $0 \leq k \leq [\alpha] < \alpha$, $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} > 1$

$\Rightarrow p_k$ increase from $k = 0$ to $k = [\alpha]$

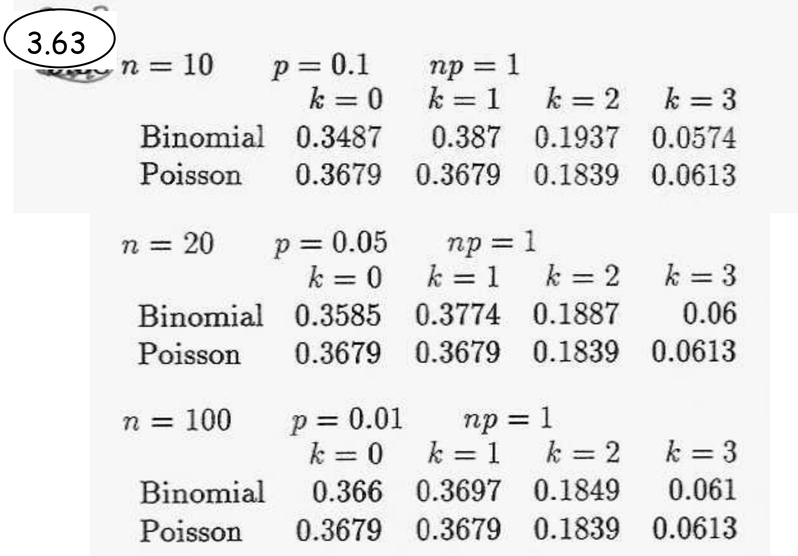
for $[\alpha] < \alpha < k$, $\frac{p_k}{p_{k-1}} = \frac{\alpha}{k} < 1$

$\Rightarrow p_k$ decreases as k increases beyond $[\alpha]$

$\therefore p_k$ attains its maximum at $k_{\max} = [\alpha]$

If $\alpha = [\alpha]$ then for $k = [\alpha]$

$$\frac{p_k}{p_{k-1}} = 1 \Rightarrow p_{k_{\max}} = p_{k_{\max}} - 1$$



3.63

$n = 10$ $p = 0.1$ $np = 1$

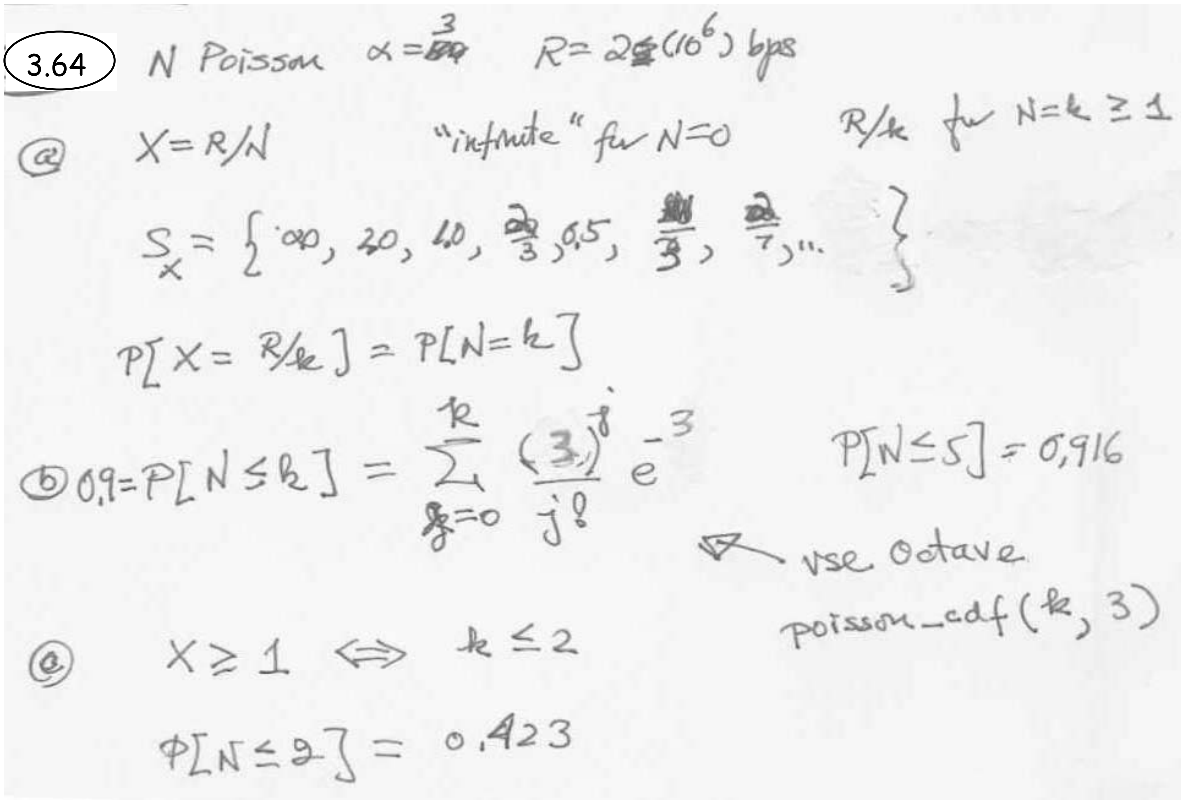
	$k = 0$	$k = 1$	$k = 2$	$k = 3$
Binomial	0.3487	0.387	0.1937	0.0574
Poisson	0.3679	0.3679	0.1839	0.0613

$n = 20$ $p = 0.05$ $np = 1$

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
Binomial	0.3585	0.3774	0.1887	0.06
Poisson	0.3679	0.3679	0.1839	0.0613

$n = 100$ $p = 0.01$ $np = 1$

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
Binomial	0.366	0.3697	0.1849	0.061
Poisson	0.3679	0.3679	0.1839	0.0613



3.64 N Poisson $\alpha = 3$ $R = 2(10^6)$ bps

ⓐ $X = R/N$ "infinite" for $N=0$ R/k for $N=k \geq 1$

$S_X = \{\infty, 2.0, 1.0, \frac{2}{3}, 0.5, [illegible], \frac{2}{7}, \ldots\}$

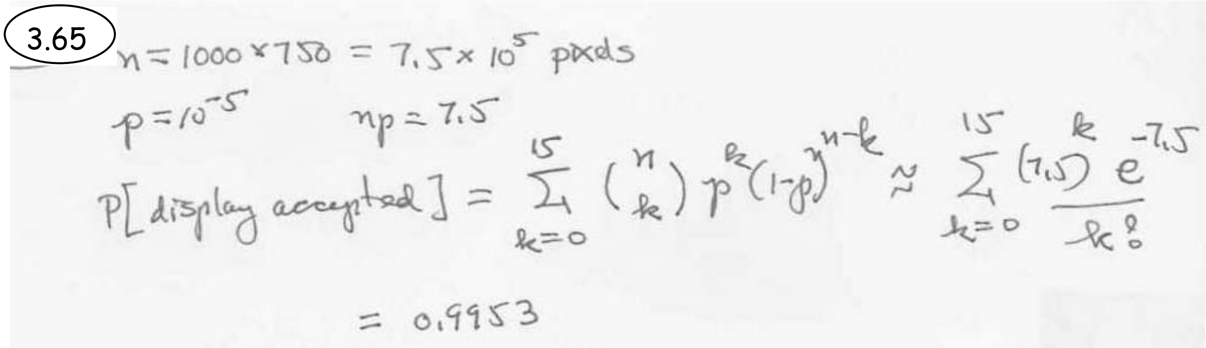
$P[X = R/k] = P[N=k]$

ⓑ $0.9 = P[N \leq k] = \sum_{j=0}^{k} \frac{(3)^j}{j!} e^{-3}$ $P[N \leq 5] = 0.916$

use Octave poisson_cdf(k, 3)

ⓒ $X \geq 1 \Leftrightarrow k \leq 2$

$P[N \leq 2] = 0.423$



3.65 $n = 1000 \times 750 = 7.5 \times 10^5$ pixels

$p = 10^{-5}$ $np = 7.5$

$$P[\text{display accepted}] = \sum_{k=0}^{15} \binom{n}{k} p^k (1-p)^{n-k} \approx \sum_{k=0}^{15} \frac{(7.5)^k e^{-7.5}}{k!}$$

$= 0.9953$

3.66) $n = 10^4$ drives $\quad p = 10^{-3} \quad np = 10^4(10^{-3}) = 10/\text{day}$

ⓐ $P[N=0] \simeq e^{-np} = e^{-10} = 4.54 \times 10^{-5}$

ⓑ Failure rate in 2 days $= 20$

$$P[N \leq 10] = \sum_{j=0}^{10} \frac{(20)^j}{j!} e^{-20} = 1.08 \times 10^{-2}$$

ⓒ

$$0.99 = P[N \leq k] = \sum_{j=0}^{k} \frac{10^j}{j!} e^{-10} \Rightarrow P[N \leq \text{[redacted]}] = 0{,}986$$

3.67 $p = 10^{-6} \quad n = 10^4 \quad np = 10^{-2}$

ⓐ $P[N=0] = e^{-np} = 0{,}990$

$$P[N \leq 3] = \sum_{k=0}^{3} \frac{(0{,}01)^k}{k!} e^{-np} \approx 1$$

ⓑ Find p so that

$$0.99 = P[N \geq 1] = 1 - P[N=0] = 1 - e^{-np}$$

$$0{,}01 = e^{-np}$$

$$\Rightarrow p = \frac{\ln 100}{n} = 4.6 \times 10^{-6}$$

3.68

.65

$$\begin{aligned}
\mathcal{E}[X] &= \sum_{k=1}^{n} kP[X=k] = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n} = \frac{n+1}{2} \\
\sigma_X^2 &= \mathcal{E}[X^2] - \mathcal{E}[X]^2 = \sum_{k=1}^{n} \frac{k^2}{n} - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\
&= \frac{n^2-1}{12}
\end{aligned}$$

3.69

3.69) X uniform in $\{-3,-2,\ldots,3,4\}$ $P[X=j]=\frac{1}{8}$

ⓐ $E[X] = -4 + \frac{8+1}{2} = 0.5$

$VAR(X) = \frac{8^2-1}{12} = \frac{63}{12} = \frac{21}{4}$

ⓑ $E[Y] = E[-2X^2+3] = -2E[X^2]+3$

$= -2[VAR(X)+E[X]^2]+3$

$= -2\left[\frac{21}{4}+(0.5)^2\right]+3 = -8$

$E[Y^2] = E[(-2X^2+3)^2] = E[4X^4-12X^2+9]$

$VAR[Y] = E[Y^2] - E[Y]^2$

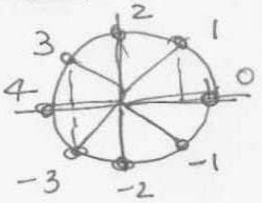
$= 4E[X^4] - 12E[X^2] + 9 - (-8)^2$

$E[x^4] = \frac{1}{8}\left[(-3)^2+(-2)^2+(-1)^2+0^2+1^2+2^2+3^2+4^2\right]$

$= \frac{44}{8} = \frac{11}{2}$

$VAR[Y] = 4(\frac{11}{2}) - 12$ ([illegible]) $+9-64 = 99$

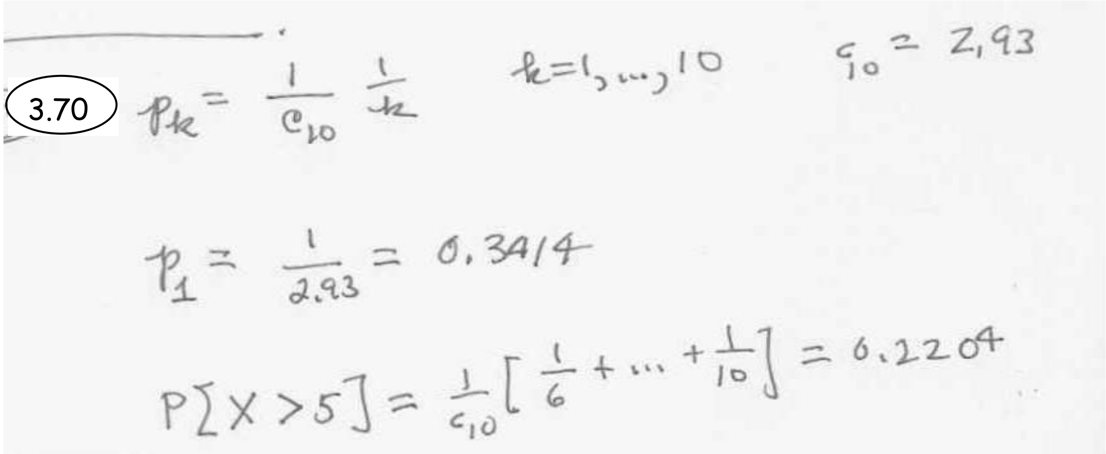
ⓒ $W = \cos(\frac{\pi X}{8})$



x	-3	-2	-1	0	1	2	3	4
w	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1

$P[W=-\frac{1}{\sqrt{2}}] = \frac{2}{8}$ $P[W=0]=\frac{2}{8}$

$P[W=\frac{1}{\sqrt{2}}] = \frac{2}{8}$ $P[W=1]=\frac{1}{8}$ $P[W=-1]=\frac{1}{8}$

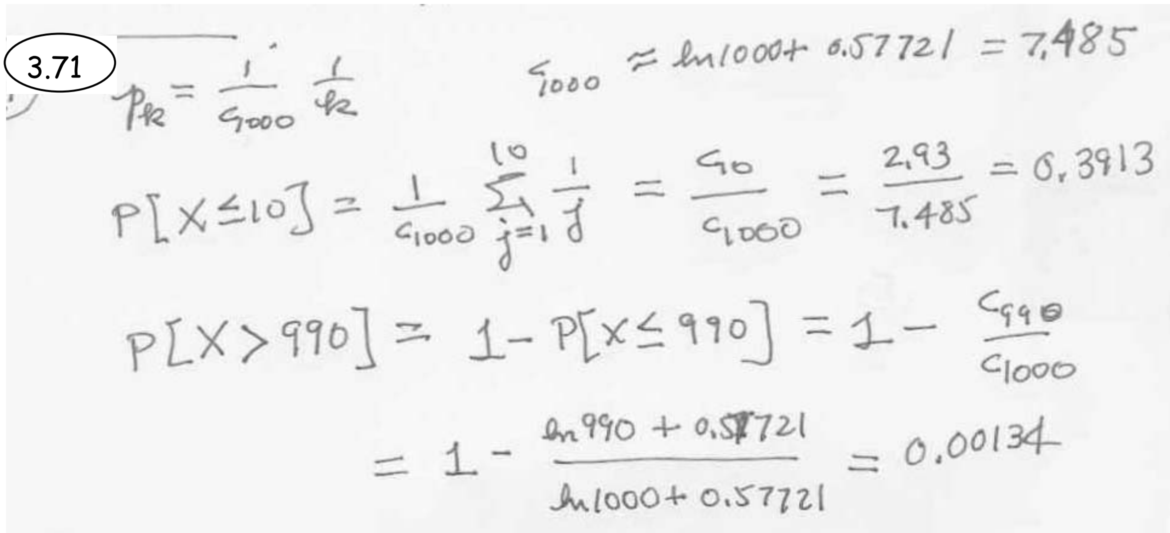


3.70

$p_k = \frac{1}{c_{10}}\frac{1}{k}$ $k=1,\dots,10$ $c_{10} = 2.93$

$p_1 = \frac{1}{2.93} = 0.3414$

$P[X>5] = \frac{1}{c_{10}}\left[\frac{1}{6}+\dots+\frac{1}{10}\right] = 0.2204$



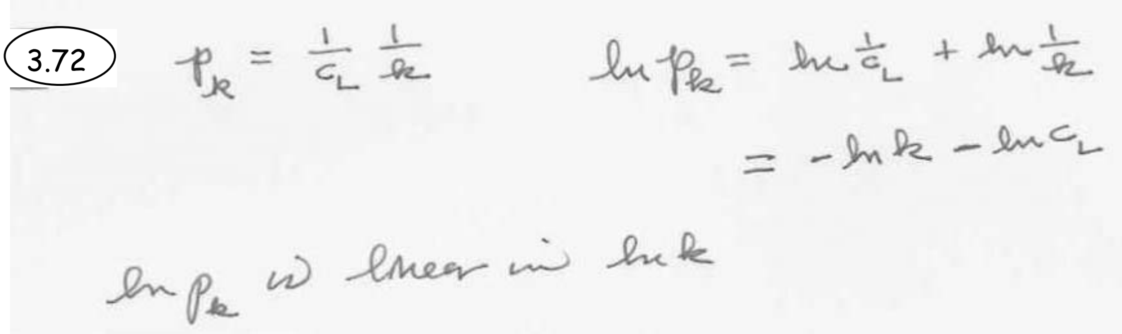
3.71

$p_k = \frac{1}{c_{1000}}\frac{1}{k}$ $c_{1000} \approx \ln 1000 + 0.57721 = 7.485$

$P[X\le 10] = \frac{1}{c_{1000}}\sum_{j=1}^{10}\frac{1}{j} = \frac{c_{10}}{c_{1000}} = \frac{2.93}{7.485} = 0.3913$

$P[X>990] = 1-P[X\le 990] = 1-\frac{c_{990}}{c_{1000}}$

$= 1-\frac{\ln 990 + 0.57721}{\ln 1000 + 0.57721} = 0.00134$



3.72

$p_k = \frac{1}{c_L}\frac{1}{k}$ $\ln p_k = \ln\frac{1}{c_L} + \ln\frac{1}{k}$

$= -\ln k - \ln c_L$

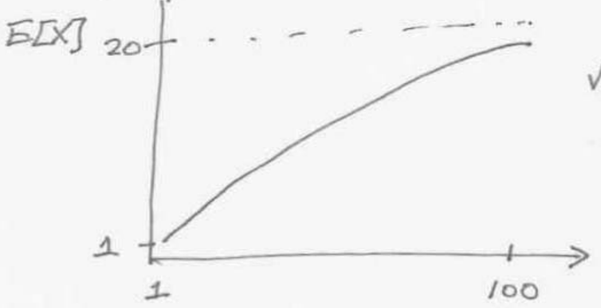
$\ln p_k$ is linear in $\ln k$

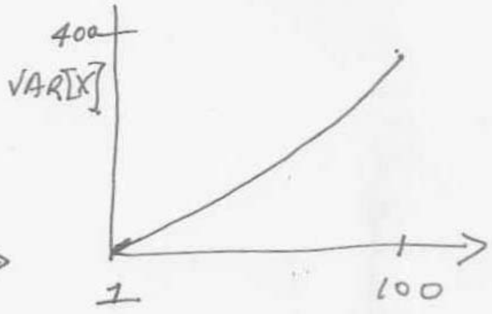
3.73 $E[X] = \frac{L}{c_L} \approx \frac{L}{\ln L + 0.57721}$ for large L

$VAR[X] = L^2/c_L^2 = E[X]^2$

To plot $E[X]$ vs L use Octave

```
>L=[1:100];
>p = L.^(-1);
>cL = cumsum(p);        array of coefficients

>plot(L./cL)            plots means
>plot((L.^2)./(cL.^2))  plots variances
```





3.74 $p_k = \frac{1}{c_L}\frac{1}{k}$ $L = 10^4$ $c_L = \ln 10^4 + 0.57721 = 9.7876$

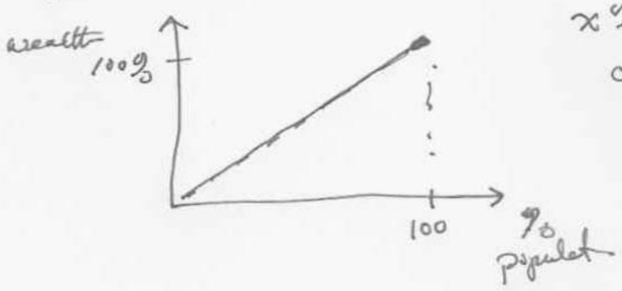
$$0.99 = P[X \leq k_0] = \frac{1}{c_{10000}} \sum_{j=1}^{k_0} \frac{1}{j} = \frac{c_{k_0}}{c_{10000}} \approx \frac{\ln k_0 + 0.57721}{9.7876}$$

$$\ln k_0 \approx 0.99(9.7876) - 0.57721 = \underline{9067}$$

Zipf decays very slowly!

3.75 Equal ownership

If I is total income then every individual has I/L of the wealth.

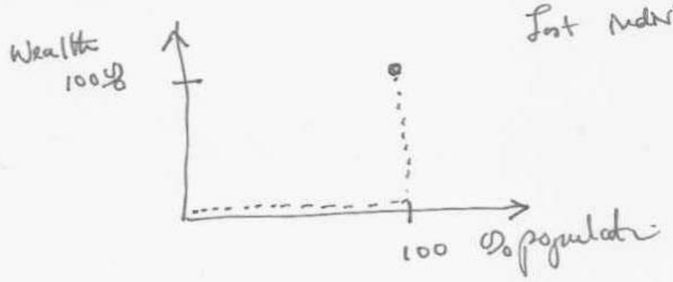


x% of wealth owned by x% of population

Most unequal ownership

0% of wealth owned by x% of population for x<100

Last individual owns 100%



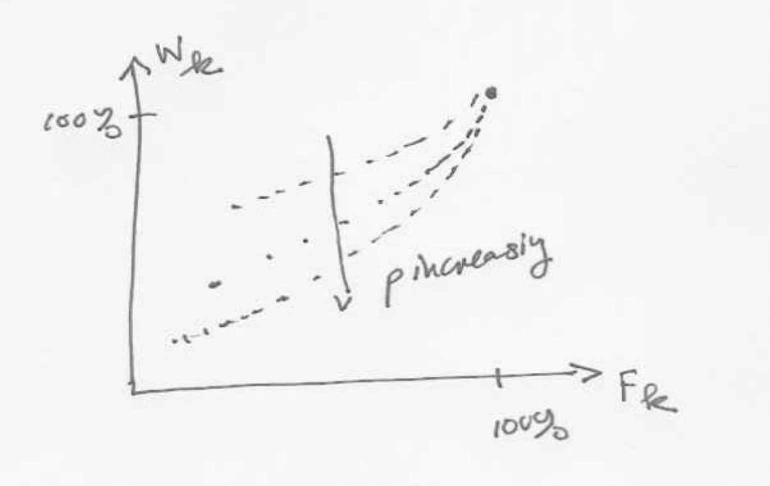
3.76

$$P[X=k] = (1-p)p^{k-1} \qquad k=1,2,\ldots$$

$$F_k = P[X \le k] = c\sum_{j=1}^{k} p^{j-1} = 1-p^k.$$

$$w_k = \frac{\sum_{j=1}^{k} j\,c\,p^{j-1}}{\sum_{j'=1}^{\infty} j'\,c\,p^{j'-1}} = \frac{\sum_{j=1}^{k} j\,p^{j-1}}{\sum_{j'=1}^{\infty} j'\,p^{j'-1}} = (1-p^{k+1})-(k+1)p^k(1-p)$$

$$\sum_{j=0}^{m} j\,p^{j-1} = \frac{d}{dp}\,\frac{1-p^{m+1}}{1-p} = \frac{(1-p^{m+1})-(m+1)p^m(1-p)}{1-p^2}$$



3.77 $X = \frac{1}{z_\alpha}\frac{1}{k^\alpha}$ $k=1,2,\ldots$

ⓐ $P[X \le k] = \frac{1}{z_\alpha}\sum_{j=1}^{k}\frac{1}{j^\alpha} = \frac{z_{\alpha,k}}{z_\alpha}$ where $z_{\alpha,k} = \sum_{j=1}^{k}\frac{1}{j^\alpha}$

ⓑ z_α is given by the zeta function evaluated at α

Using Octave, we have

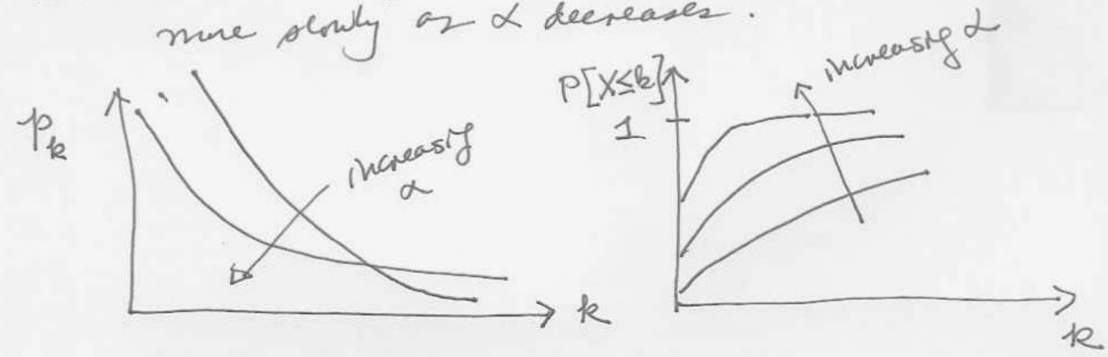
```
> zeta(1.5)
> ans = 2.6124
```

$z(1.5) = 2.6124$
$z(2) = 1.6449$
$z(3) = 1.2021$

If we add the first 100 terms to estimate z_α we have:

$z_{1.5,100} = 2.41$ $z_{2,100} = 1.635$ $z_{3,100} = 1.202$

The series that defines the zeta function decays more slowly as α decreases.



3.6 Generation of Discrete Random Variables

3.78 The following Octave commands will give the requested plots:

(a)

```
x = [0:1:10];
lambda = 0.5;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));

x = [0:1:20];
lambda = 5;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));

x = [0:1:100];
lambda = 50;
figure;
plot(x, poisson_pdf(x, lambda));
figure;
plot(x, poisson_cdf(x, lambda));
figure;
plot(x, 1-poisson_cdf(x, lambda));
```

(b)

```
x = [0:1:15];
figure;
plot(x, binomial_pdf(x, 48, 0.1));
figure;
plot(x, binomial_cdf(x, 48, 0.1));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.1));

x = [0:1:30];
figure;
plot(x, binomial_pdf(x, 48, 0.3));
figure;
plot(x, binomial_cdf(x, 48, 0.3));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.3));

x = [0:1:50];
figure;
```

```
plot(x, binomial_pdf(x, 48, 0.5));
figure;
plot(x, binomial_cdf(x, 48, 0.5));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.5));

x = [20:1:50];
figure;
plot(x, binomial_pdf(x, 48, 0.75));
figure;
plot(x, binomial_cdf(x, 48, 0.75));
figure;
plot(x, 1-binomial_cdf(x, 48, 0.75));
```

(c)

```
x = [0:1:10];
n = 100; p = 0.01;
figure;
plot(x, binomial_pdf(x, n, p), "1");
hold on;
plot(x, poisson_pdf(x, n*p), "3");
hold off;
```

3.79 The following Octave commands produce the request plots:

(a)

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
figure;
plot(k, discrete_pdf(k, k, pk));
figure;
plot(k, discrete_cdf(k, k, pk));
figure;
plot(k, 1-discrete_cdf(k, k, pk));
```

(b)

```
m = 20;
k = [1:1:m];
pk = (1/2).^k;
figure;
semilogy(k, pk);
```

3.80 The following Octave commands will plot the Lorenze curves:

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Wk = k./L;
Fk = discrete_cdf(k, k, pk);
figure;
plot(Fk, Wk);
```

3.81 The following Octave commands will plot the requested curves:

```
figure;
hold on;
n = 100; p = 0.1;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);

n = 100; p = 0.5;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);

n = 100; p = 0.9;
k = [0:1:n];
Wk = zeros(1, n+1);
for i = 0:n,
      v = [0:i];
      Wk(i+1) = sum(v.*binomial_pdf(v, n, p))./(n*p);
end;
Fk = binomial_cdf(k, n, p);
plot(Fk, Wk);
```

3.82

(a)

(b)

(c)

3.83 The following Octave commands will generate the requested samples of the Zipf random variable and the requested plots.

```
L = 10;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);

L = 100;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);

L = 1000;
k = [1:1:L];
cL = sum(1./k);
pk = (1/cL)./k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);
```

3.84 The following Octave commands generate the samples of the St. Peter's Paradox random variable and the requested plots.

```
m = 20;
k = [1:1:m];
pk = (1/2).^k;
Sk = discrete_rnd(200, k, pk);
figure;
plot(Sk);
figure;
hist(Sk, k);
```

(3.85) The following Octave commands generate the requested pairs and plots:

(a)

```
k = [1:10];
pk = ones(1,10)./10;
Sx = discrete_rnd(200, k, pk);
Sy = discrete_rnd(200, k, pk);
figure;
hist(Sx, k);
figure;
hist(Sy, k);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [2:20]);
```

(c)

```
Sw = Sx .* Sy;
figure;
hist(Sw, 10);
```

(d)

```
Sv = Sx ./ Sy;
figure;
hist(Sv, 10);
```

3.86 The following Octave commands generate the requested pairs and plots:

(a)

```
Sx = binomial_rnd(8, 0.5, 1, 200);
Sy = binomial_rnd(4, 0.5, 1, 200);
figure;
hist(Sx, [0:8]);
figure;
hist(Sy, [0:4]);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [0:12]);
```

3.87 The following Octave commands generate the requested pairs and plots:

(a)

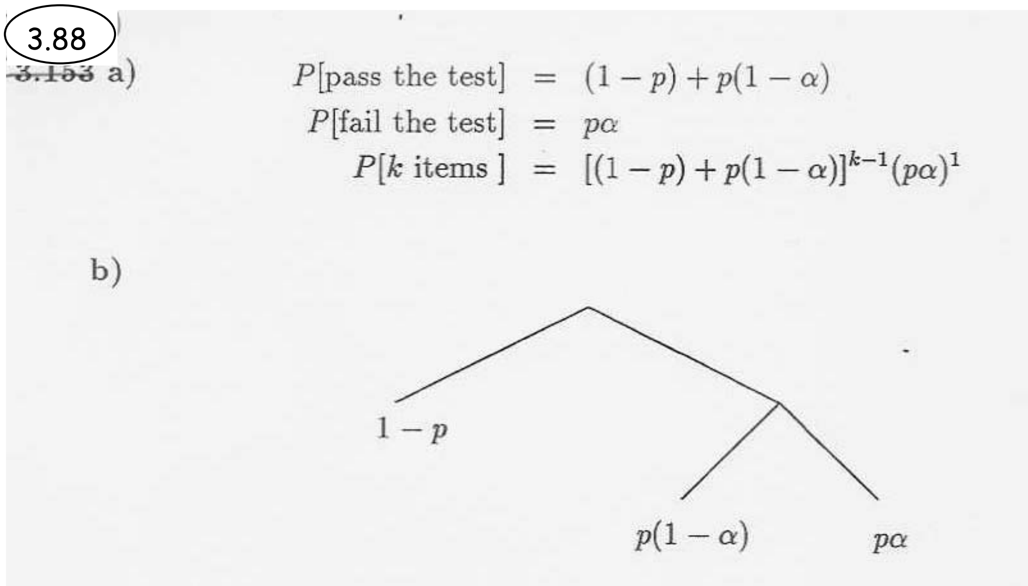
```
Sx = poisson_rnd(5, 1, 200);
Sy = poisson_rnd(10, 1, 200);
figure;
hist(Sx, [0:15]);
figure;
hist(Sy, [0:20]);
```

(b)

```
Sz = Sx + Sy;
figure;
hist(Sz, [0:35]);
```

Problems Requiring Cumulative Knowledge

3.88

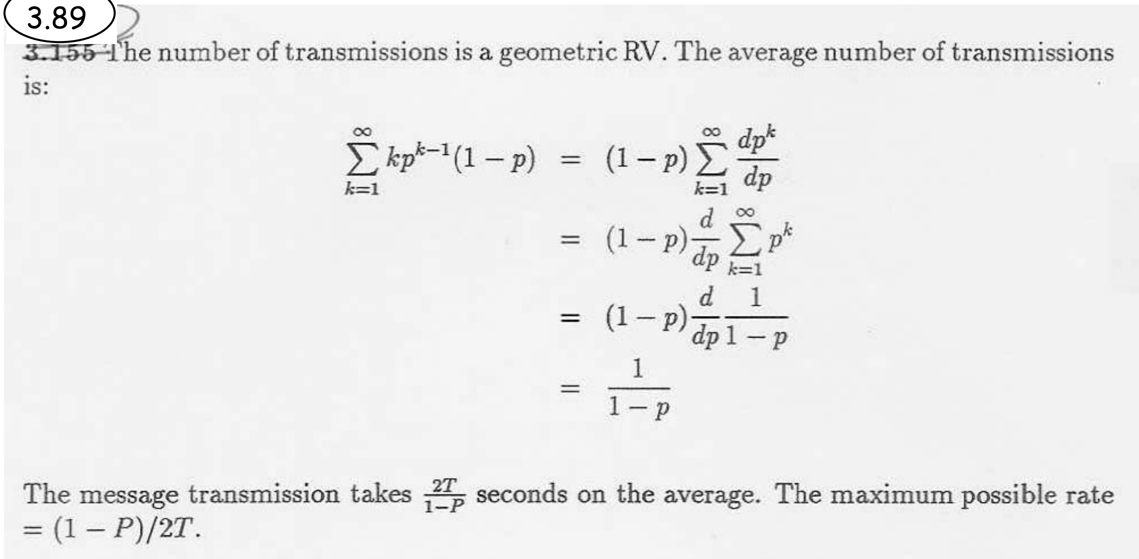


~~3.153~~ a)

$$\begin{aligned} P[\text{pass the test}] &= (1-p)+p(1-\alpha) \\ P[\text{fail the test}] &= p\alpha \\ P[k \text{ items }] &= [(1-p)+p(1-\alpha)]^{k-1}(p\alpha)^1 \end{aligned}$$

b)

3.89



~~3.155~~ The number of transmissions is a geometric RV. The average number of transmissions is:

$$\begin{aligned} \sum_{k=1}^{\infty} kp^{k-1}(1-p) &= (1-p)\sum_{k=1}^{\infty}\frac{dp^k}{dp} \\ &= (1-p)\frac{d}{dp}\sum_{k=1}^{\infty}p^k \\ &= (1-p)\frac{d}{dp}\frac{1}{1-p} \\ &= \frac{1}{1-p} \end{aligned}$$

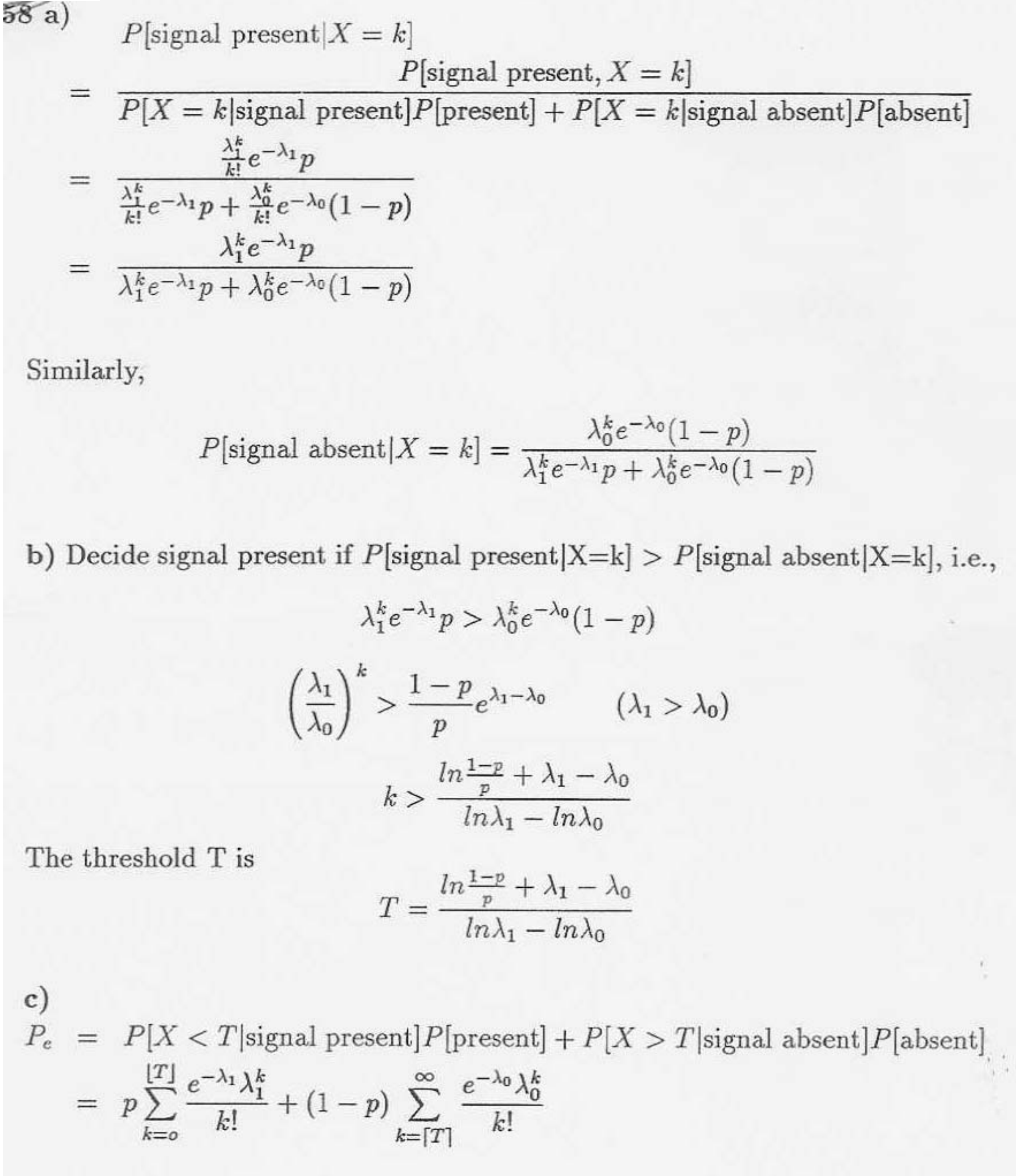
The message transmission takes $\frac{2T}{1-P}$ seconds on the average. The maximum possible rate $= (1-P)/2T$.

3.90 We want to find n so that the nth arrival is after more than 2 minutes 90% of the time:

$$P[N(2) \leq n] = 0.90 = \sum_{k=0}^{n} \frac{2^k}{k!} e^{-2}$$

By trial and error we find $n=5$.

3.91



~~58~~ a)

$$
\begin{aligned}
& P[\text{signal present}|X=k] \\
= & \frac{P[\text{signal present}, X=k]}{P[X=k|\text{signal present}]P[\text{present}] + P[X=k|\text{signal absent}]P[\text{absent}]} \\
= & \frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p}{\frac{\lambda_1^k}{k!}e^{-\lambda_1}p + \frac{\lambda_0^k}{k!}e^{-\lambda_0}(1-p)} \\
= & \frac{\lambda_1^k e^{-\lambda_1}p}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}
\end{aligned}
$$

Similarly,

$$P[\text{signal absent}|X=k] = \frac{\lambda_0^k e^{-\lambda_0}(1-p)}{\lambda_1^k e^{-\lambda_1}p + \lambda_0^k e^{-\lambda_0}(1-p)}$$

b) Decide signal present if P[signal present|X=k] > P[signal absent|X=k], i.e.,

$$\lambda_1^k e^{-\lambda_1}p > \lambda_0^k e^{-\lambda_0}(1-p)$$

$$\left(\frac{\lambda_1}{\lambda_0}\right)^k > \frac{1-p}{p}e^{\lambda_1-\lambda_0} \qquad (\lambda_1 > \lambda_0)$$

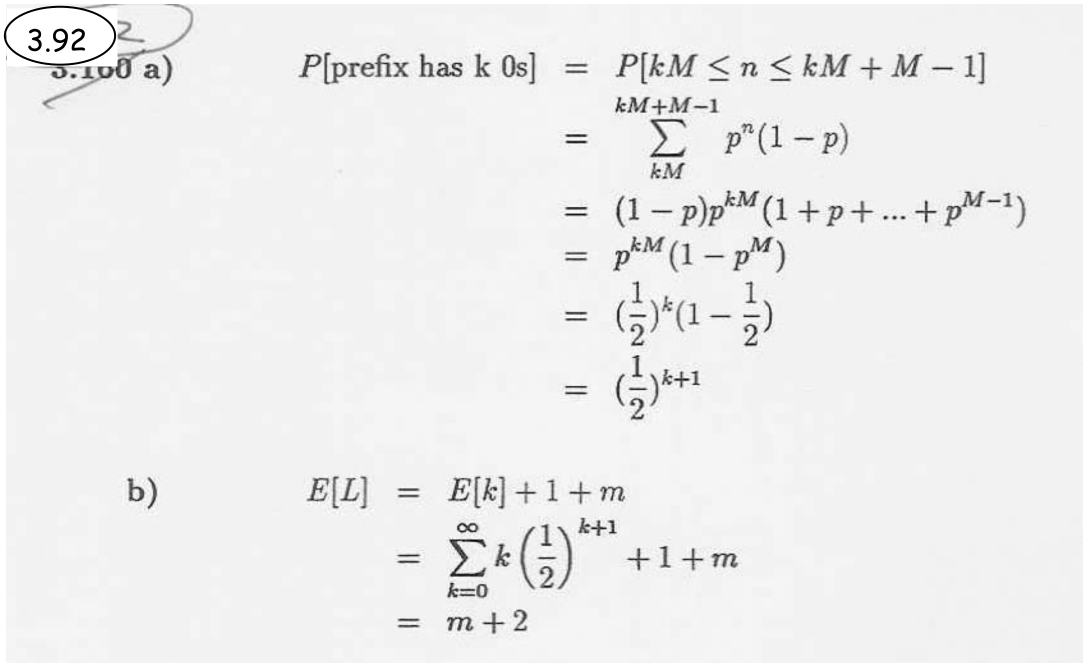
$$k > \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}$$

The threshold T is

$$T = \frac{ln\frac{1-p}{p} + \lambda_1 - \lambda_0}{ln\lambda_1 - ln\lambda_0}$$

c)

$$
\begin{aligned}
P_e = & P[X < T|\text{signal present}]P[\text{present}] + P[X > T|\text{signal absent}]P[\text{absent}] \\
= & p\sum_{k=o}^{\lfloor T \rfloor}\frac{e^{-\lambda_1}\lambda_1^k}{k!} + (1-p)\sum_{k=\lceil T \rceil}^{\infty}\frac{e^{-\lambda_0}\lambda_0^k}{k!}
\end{aligned}
$$



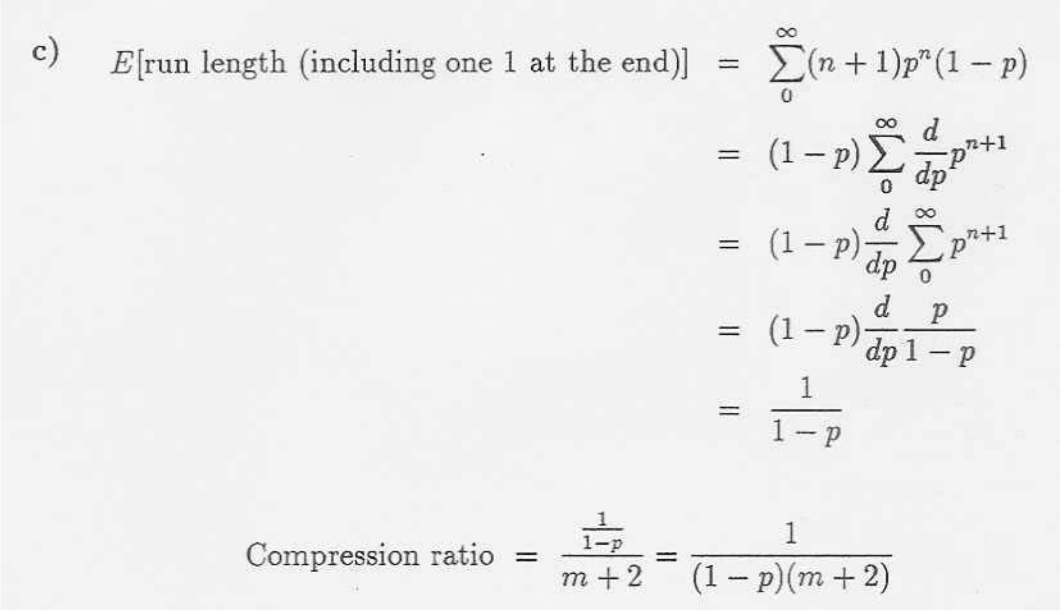
3.92

~~3.100~~ a)

$$\begin{aligned}
P[\text{prefix has k 0s}] &= P[kM \le n \le kM+M-1] \\
&= \sum_{kM}^{kM+M-1} p^n(1-p) \\
&= (1-p)p^{kM}(1+p+\ldots+p^{M-1}) \\
&= p^{kM}(1-p^M) \\
&= (\frac{1}{2})^k(1-\frac{1}{2}) \\
&= (\frac{1}{2})^{k+1}
\end{aligned}$$

b)

$$\begin{aligned}
E[L] &= E[k]+1+m \\
&= \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^{k+1} + 1 + m \\
&= m+2
\end{aligned}$$



c)

$$\begin{aligned}
E[\text{run length (including one 1 at the end)}] &= \sum_0^{\infty}(n+1)p^n(1-p) \\
&= (1-p)\sum_0^{\infty}\frac{d}{dp}p^{n+1} \\
&= (1-p)\frac{d}{dp}\sum_0^{\infty}p^{n+1} \\
&= (1-p)\frac{d}{dp}\frac{p}{1-p} \\
&= \frac{1}{1-p}
\end{aligned}$$

$$\text{Compression ratio } = \frac{\frac{1}{1-p}}{m+2} = \frac{1}{(1-p)(m+2)}$$

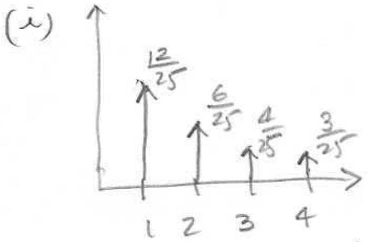
Chapter 4: One Random Variable

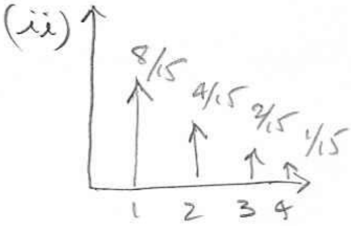
4.1 The Cumulative Distribution Function

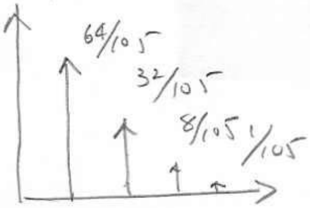
4.1 (i) $1 = p_1\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{25}{12}p_1 \Rightarrow p_1 = \frac{12}{25}$ $\quad p = \left(\frac{12}{25}, \frac{6}{25}, \frac{4}{25}, \frac{3}{25}\right)$

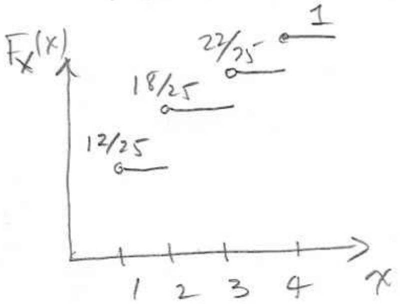
(ii) $1 = p_1\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right) = \frac{15}{8}p_1 \Rightarrow p_1 = \frac{8}{15}$ $\quad p = \left(\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}\right)$

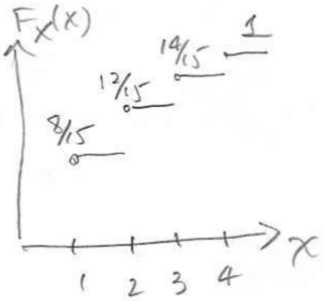
(iii) $1 = p_1\left(1+\frac{1}{2}+\frac{1}{8}+\frac{1}{64}\right) = \frac{105}{64}p_1 \Rightarrow p_1 = \frac{64}{105}$ $\quad p = \left(\frac{64}{105}, \frac{32}{105}, \frac{8}{105}, \frac{1}{105}\right)$

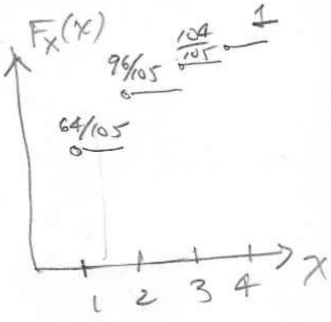












$P[X \le 1] = \frac{12}{25}$

$P[X < 2.5] = \frac{18}{25}$

$P[0.5 < X \le 2] = \frac{6}{25}$

$P[1 < X < 4] = \frac{10}{25}$

$P[X \le 1] = \frac{8}{15}$

$P[X \le 2.5] = \frac{12}{15}$

$P[0.5 < X \le 2] = \frac{4}{15}$

$P[1 < X < 4] = \frac{6}{15}$

$P[X \le 1] = \frac{64}{105}$

$P[X \le 2.5] = \frac{96}{105}$

$P[0.5 < X \le 2] = \frac{32}{105}$

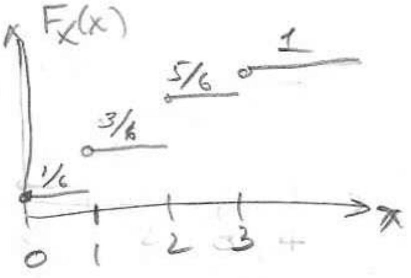
$P[1 < X < 4] = \frac{40}{105}$

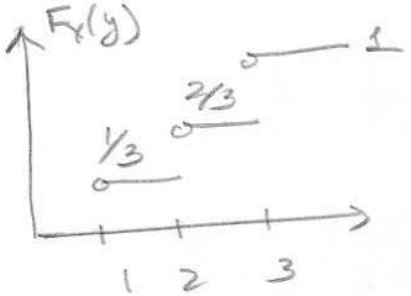
4.2

ξ	1	2	3	4	5	6
$X(\xi)$	0	1	1	2	2	3
$Y(\xi)$	1	1	2	2	3	3

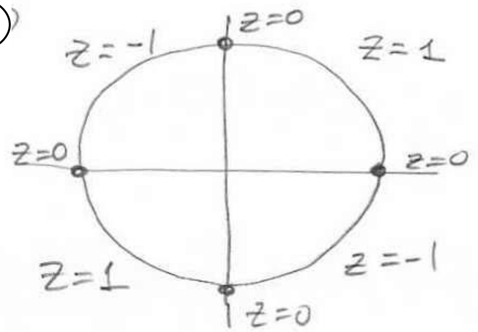
$S_X = \{0, 1, 2, 3\}$ $\quad p = \left(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right)$

$S_Y = \{1, 2, 3\}$ $\quad p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$



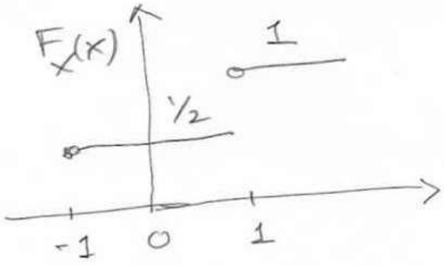


4.3



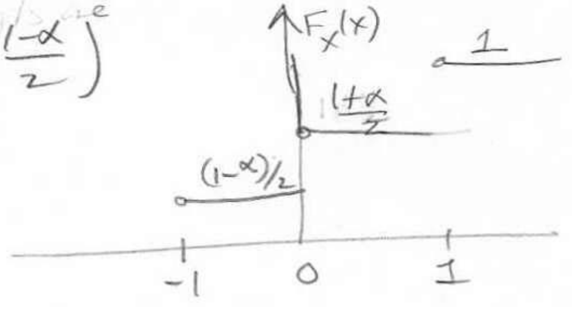
$S_X = \{-1, 0, 1\}$

$p = \left(\frac{1}{4}+\frac{1}{4}, 0, \frac{1}{4}+\frac{1}{4}\right)$
$= \left(\frac{1}{2}, 0, \frac{1}{2}\right)$



If clock has propensity
to stop at 3/6/9,12
α of the time then

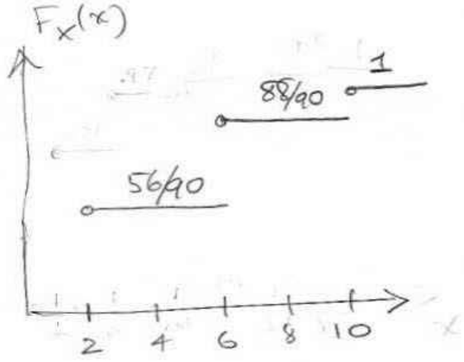
$p = \left(\frac{1-\alpha}{2}, \alpha, \frac{1-\alpha}{2}\right)$



4.4 $8 \times \$1$ $2 \times \$5$

without replacement
$1+1$
$1+5$
$5+5$
$S_X = \{2, 6, 10\}$

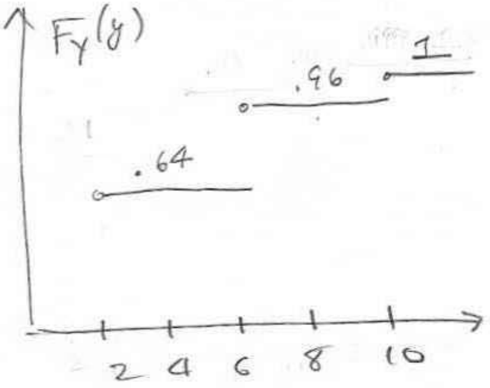
with replacement
$1+1$
$1+5$
$5+5$
$S_Y = \{2, 6, 10\}$



$P[2] = \frac{8}{10} \cdot \frac{7}{9} = \frac{56}{90}$

$P[6] = 2 \cdot \frac{8}{10} \frac{2}{9} = \frac{32}{90}$

$P[10] = \frac{2}{10} \cdot \frac{1}{9} = \frac{2}{90}$



$P[2] = .8^2 = .64$

$P[6] = 2(.8)(.2) = .32$

$P[10] = (.2)^2 = .04$

$P[X=2] = \frac{56}{90}$ $P[Y=2] = \frac{64}{100}$

$P[X<7] = \frac{88}{90}$ $P[Y<7] = \frac{96}{100}$

$P[X \geq 6] = 1 - \frac{56}{90}$ $P[Y \geq 6] = 1 - .64$

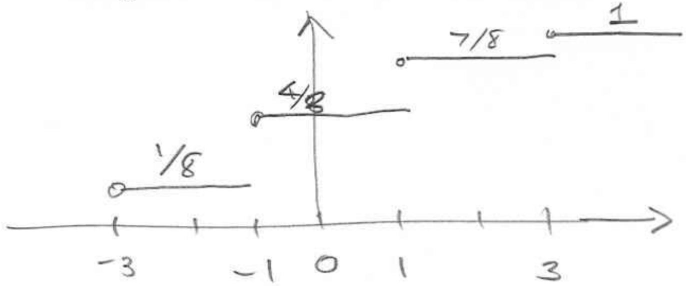
$= \frac{34}{90}$ $= \frac{.36}{100}$

4.5 $Y = N_H - N_T = N_H - (n - N_H) = 2N_H - n$

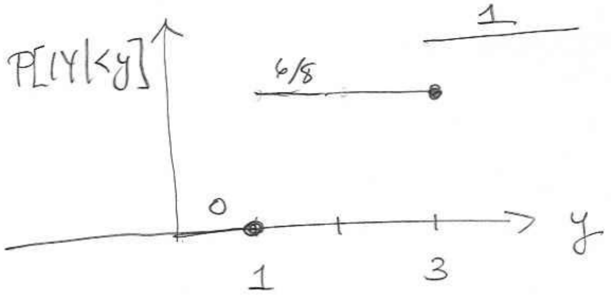
$n = 3$

n_H	0	1	2	3
$y(n_H)$	-3	-1	1	3
$P[y]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$P[N_H = k] = \binom{3}{k}\left(\frac{1}{2}\right)^3$

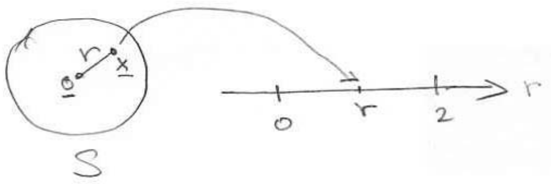


$P[|Y| < y] = P[-y < Y < y] = F_Y(y^-) - F_Y(-y)$

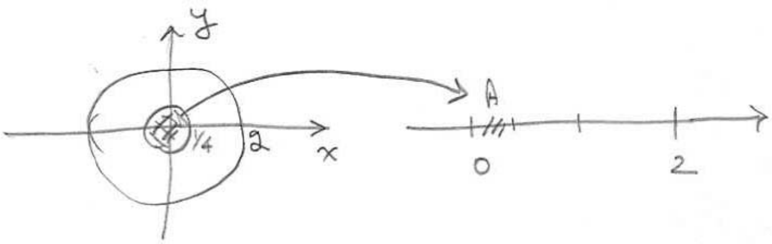


4.6 $S = \{(x,y) : x^2+y^2 \le 4\}$ $\quad S_R = \{r : 0 \le r \le 2\}$

$R = \sqrt{x^2+y^2}$

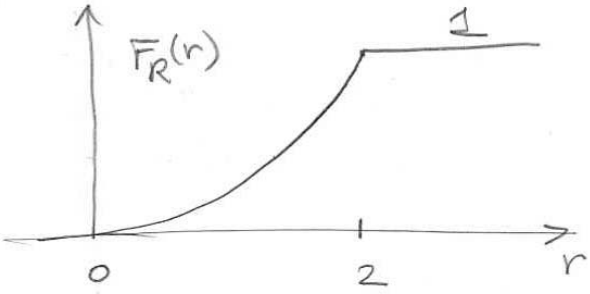


$$P[A] = P[R \le \tfrac{1}{4}] = \frac{\pi(\frac{1}{4})^2}{\pi(2)^2} = \frac{1}{64}$$



for $0 \le r \le 2$

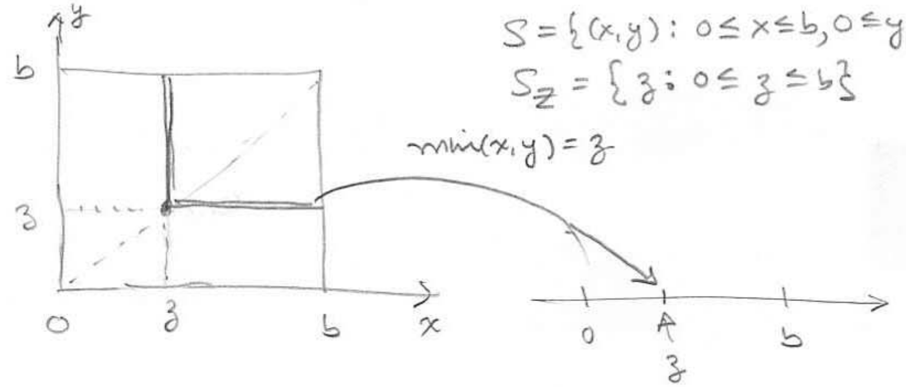
$$F_R(r) = P[R \le r] = \frac{\pi r^2}{\pi 2^2} = \left(\frac{r}{2}\right)^2$$

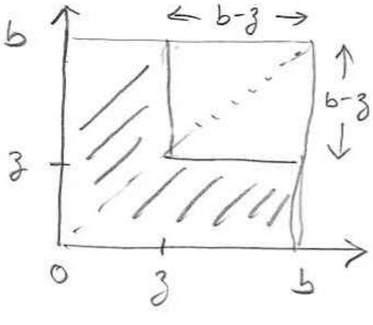


4.7

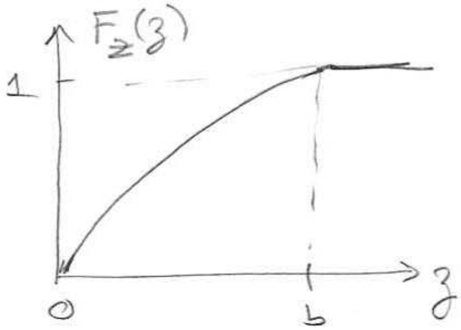
$S = \{(x,y) : 0 \le x \le b, 0 \le y \le b\}$

$S_Z = \{z : 0 \le z \le b\}$





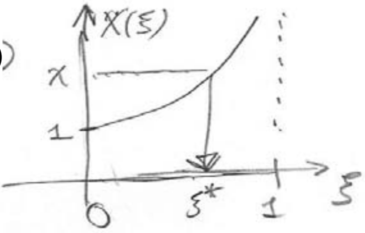
$$P[Z \le z] = 1 - \left(\frac{b-z}{b}\right)^2$$
$$= 1 - \left(1 - 2\frac{z}{b} + \frac{z^2}{b^2}\right)$$
$$= \frac{2z}{b} - \frac{z^2}{b^2}$$
$$= \frac{z}{b}\left(2 - \frac{z}{b}\right)$$



$$P[Z > 0] = 1$$
$$P[Z > b] = 1$$
$$P\left[Z \le \frac{b}{2}\right] = F_Z\left(\frac{b}{2}\right)$$
$$= \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$$

$$P\left[Z > \frac{b}{4}\right] = 1 - F_Z\left(\frac{b}{4}\right)$$
$$= 1 - \frac{1}{4}\left(2 - \frac{1}{4}\right)$$
$$= \frac{9}{16}$$

4.8

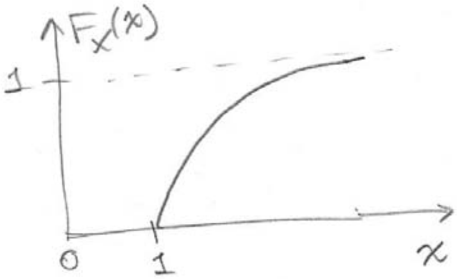


$$X(\xi) = \frac{1}{\sqrt{1-\xi^2}}$$

$$S_X = \{x : 1 \le x < \infty\}$$

$$P[X(\xi) \le x] = P\left[\frac{1}{\sqrt{1-\xi}} \le x\right] = P\left[\frac{1}{1-\xi} \le x^2\right]$$

$$= P\left[\frac{1}{x^2} \le 1-\xi\right] = P\left[\xi \le 1-\frac{1}{x^2}\right] = 1-\frac{1}{x^2}$$

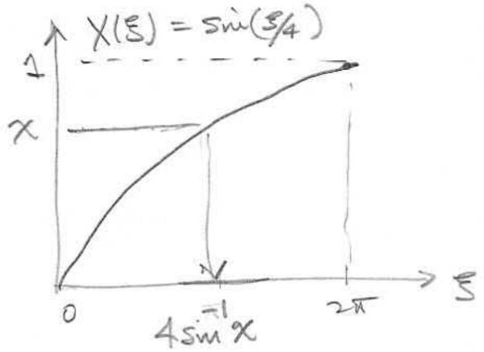


$$P[X > 10] = 1 - F_X(10)$$
$$= 1 - \left(1 - \frac{1}{10^2}\right) = \frac{1}{100}$$

$$P[5 < X < 7] = F_X(7) - F_X(5) = \left(1 - \frac{1}{49}\right) - \left(1 - \frac{1}{25}\right)$$
$$= \frac{1}{25} - \frac{1}{49} = 0.01959$$

$$P[X \le 20] = 1 - \frac{1}{400} = 0.9975$$

4.9

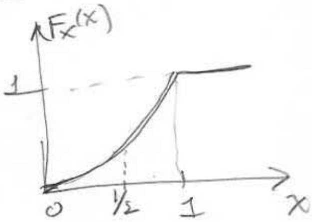


$S = \{\xi : 0 \le \xi < 2\pi\}$

$S_X = \{x : 0 \le x < 1\}$

$$P[X \le x] = P\left[\sin\left(\tfrac{\xi}{4}\right) \le x\right] = P\left[\xi \le 4\sin^{-1} x\right]$$

$$= \frac{4\sin^{-1} x}{2\pi} \qquad 0 \le x \le 1$$



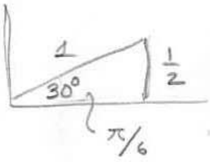
$$P[X > 1] = 1 - F_X(1) = 1 - 1 = 0$$

$$P\left[-\tfrac{1}{2} < X < \tfrac{1}{2}\right] = P\left[0 < X < \tfrac{1}{2}\right]$$

$$= F_X\left(\tfrac{1}{2}\right)$$

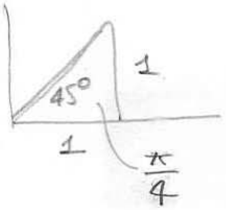
$$= \frac{4\sin^{-1}\frac{1}{2}}{2\pi}$$

$$= \frac{4}{2\pi}\,\frac{\pi}{6} = \frac{1}{3}$$



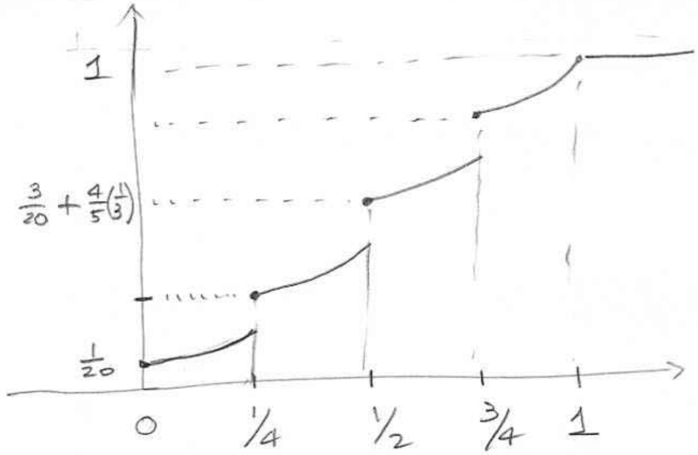
$$P\left[X \le \tfrac{1}{\sqrt{2}}\right] = \frac{4\sin^{-1}\frac{1}{\sqrt{2}}}{2\pi}$$

$$= \frac{4}{2\pi}\,\frac{\pi}{4} = \frac{1}{2}$$



4.10 The probability law over $S = \{\xi : 0 \le x < 2\pi\}$ places probability $\frac{0.2}{4}$ on the points $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and probability 0.8 uniformly in the interval $[0, 2\pi)$.

$X(\xi)$ has probability mass at the points $0, 1/4, 1/2, 3/4$



$$F_X(x) = \frac{1}{20}u(x) + \frac{1}{20}u(x - \tfrac{1}{4}) + \frac{1}{20}u(x - \tfrac{1}{2}) + \frac{1}{20}u(x - \tfrac{3}{4}) + \frac{4}{5}\,\frac{4\sin^{-1}x}{2\pi}$$

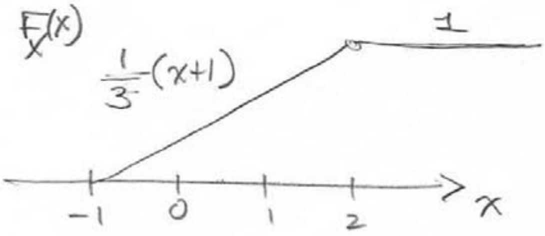
$$P[X > 1] = 1 - F_X(1) = 0$$

$$P[-\tfrac{1}{2} < X < \tfrac{1}{2}] = P[0 < X < \tfrac{1}{2}]$$

$$= \frac{2}{20} + \frac{4}{5}\,\frac{4\sin^{-1}(\frac{1}{2})}{2\pi} = \frac{2}{20} + \frac{4}{5}(\tfrac{1}{3}) = \frac{11}{30}$$

$$P[X \le \tfrac{1}{\sqrt{2}}] = \frac{3}{20} + \frac{4}{5}\,\frac{\sin^{-1}\frac{1}{\sqrt{2}}}{2\pi} = \frac{3}{20} + \frac{4}{5}(\tfrac{1}{2}) = \frac{11}{20}$$

4.11



$$P[X<0] = F_X(0) = \frac{1}{3}$$

$$P\left[\left|X-\tfrac{1}{2}\right|<1\right] = P\left[-1<X-\tfrac{1}{2}<1\right] = P\left[-\tfrac{1}{2}<X<\tfrac{3}{2}\right]$$

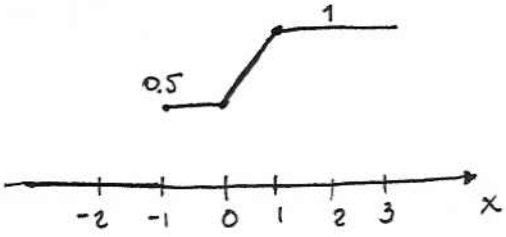
$$= \frac{1}{3}\left(\frac{3}{2}+1\right) - \frac{1}{3}\left(-\frac{1}{2}+1\right)$$

$$= \frac{1}{3}\left(\frac{3}{2}+1+\frac{1}{2}-1\right) = \frac{2}{3}$$

$$P\left[X>-\tfrac{1}{2}\right] = 1-P\left[X\le-\tfrac{1}{2}\right] = 1-\frac{1}{3}\left(-\frac{1}{2}+1\right) = \frac{5}{6}$$

4.12

a)



Mixed type random variable

b) $P[X \leq -1] = 0.5$

$P[X = -1] = 0.5$

$$P[X < 0.5] = P[X \leq 0.5] - P[X = 0.5] = \frac{1+0.5}{2} - 0 = 0.75$$

$$P[-0.5 < X < 0.5] = P[X \leq 0.5] - P[X = 0.5] - P[X \leq -0.5] = \frac{1+0.5}{2} - 0 - 0.5 = 0.75 - 0.5 = 0.25$$

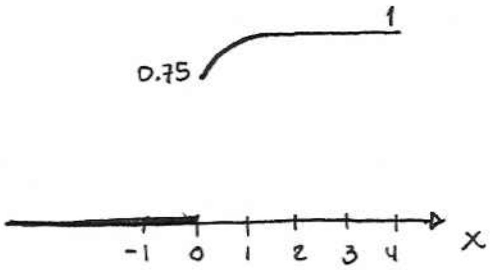
$$P[X > -1] = 1 - P[X \leq -1] = 1 - 0.5 = 0.5$$

$P[X \leq 2] = 1$

$$P[X > 3] = 1 - P[X \leq 3] = 1 - 1 = 0$$

4.13

a)



Mixed type random variable

b) $P[X \le 2] = 1 - \frac{1}{4}e^{-2(2)}$

$= 0.9954$

$P[X=0] = 1 - \frac{1}{4}e^{-2(0)}$

$= 0.75$

$P[X<0] = 0$

$P[2<X<6] = P[X \le 6] - P[X \le 2]$

$= 1 - \frac{1}{4}e^{-2(6)} - 1 + \frac{1}{4}e^{-2(2)}$

$= 0.0046$

$P[X>10] = 1 - P[X \le 10]$

$= 1 - \left(1 - \frac{1}{4}e^{-2(10)}\right)$

$= 5.15 \times 10^{-10}$

4.14

a) Mixed Type

b) $P[X<-1]=0$

$P[X\le -1]=\frac{2}{10}$

$P[-1<X<-0.75]= P[X\le -0.75]-P[X\le -1]$

$=\frac{2}{10}-\frac{2}{10}$

$=0$

$P[-0.5\le X\le 0.5]= P[X\le 0.5]-P[X\le -0.5]$

$=\frac{8}{10}-\frac{2}{10}$

$=\frac{6}{10}$

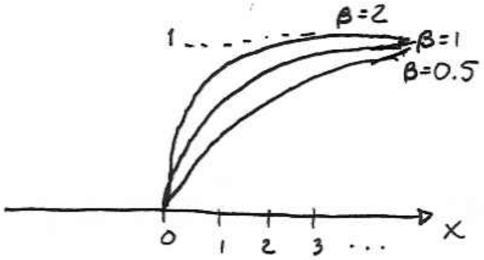
$P[|X-0.5|<0.5]= P[\{X<1\}\cup\{X>0\}] = P[\{0<X<1\}]$

$=1-\frac{6}{10}$

$=\frac{4}{10}$

4.15

a)



b) $P[j\lambda < X < (j+1)\lambda] = P[X \le (j+1)\lambda] - P[X \le j\lambda]$

$$= 1 - e^{-\left((j+1)\frac{\lambda}{\lambda}\right)^{\beta}} - 1 + e^{-\left(j\frac{\lambda}{\lambda}\right)^{\beta}} \qquad j \geq 0$$

$$= e^{-j^{\beta}} - e^{-(j+1)^{\beta}} \qquad j \geq 0$$

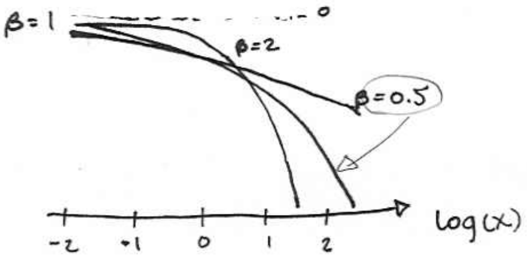
$$= \begin{cases} 0 & j < -1 \\ 1 - e^{-(j+1)^{\beta}} & -1 \le j < 0 \\ e^{-j^{\beta}} - e^{-(j+1)^{\beta}} & j \geq -1 \end{cases}$$

$$P[X > j\lambda] = 1 - P[X \le j\lambda] = 1 - \left(1 - e^{-\left(j\frac{\lambda}{\lambda}\right)^{\beta}}\right) \qquad j \geq 0$$

$$= \begin{cases} 0 & j < 0 \\ e^{-j^{\beta}} & j \geq 0 \end{cases}$$

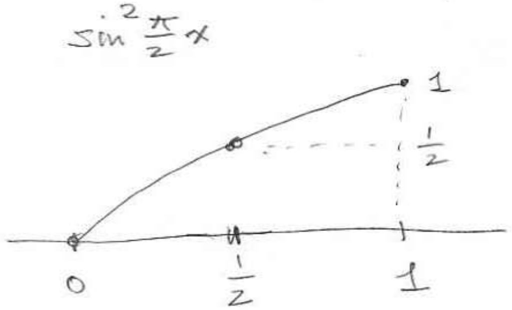
c) $$\log P[X > x] = \begin{cases} \text{Log}(1) & x < 0 \\ \text{Log}\left(e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right) & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ -\left(\frac{x}{\lambda}\right)^{\beta} & x \geq 0 \end{cases}$$



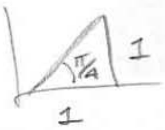
4.16

$$F_X(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} + c\sin^2\frac{\pi}{2}x & 0 \le x \le 1 \\ 1 & x>1 \end{cases}$$



$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

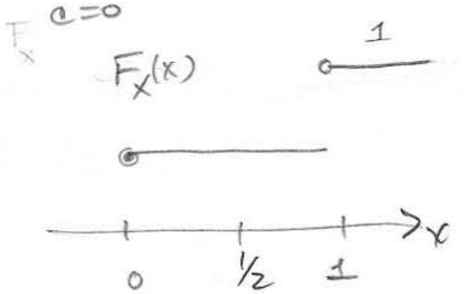
$\sin^2\frac{\pi}{4} = \frac{1}{2}$



The constant c should be selected so properties of the cdf are satisfied: $c>0$ so cdf is increasing

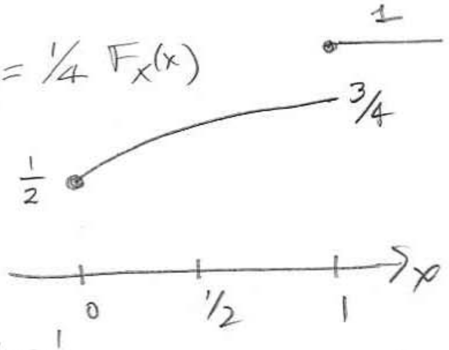
$c<\frac{1}{2}$ so cdf ≤ 1

$c=0$



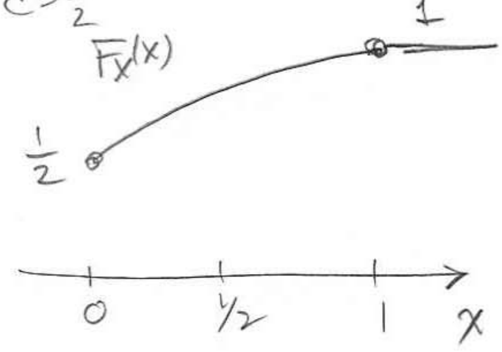
$P[X>0] = 1 - P[X\le 0] = \frac{1}{2}$

$c = 1/4$



$P[X>0] = \frac{1}{2}$

$c=\frac{1}{2}$



$P[X>0] = \frac{1}{2}$

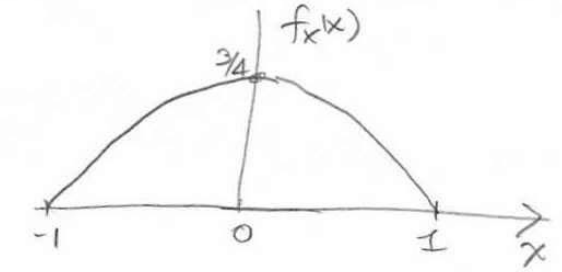
4.2 The Probability Density Function

4.17

$$1 = c\int_{-1}^{1}(1-x^2)\,dx = c\left[x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1}\right] = c\left[2-\frac{1}{3}2\right] = \frac{4}{3}c$$

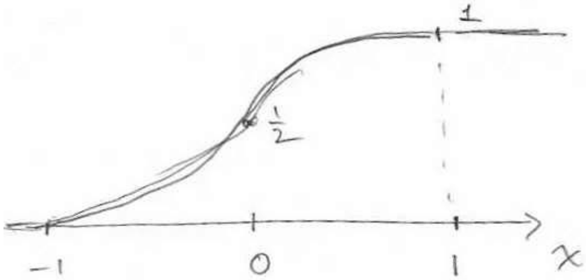
$$\Rightarrow c = \frac{3}{4}$$

$$f_X(x) = \frac{3}{4}(1-x^2) \qquad -1 \le x \le 1$$



$$F_X(x) = \frac{3}{4}\int_{-1}^{x}(1-y^2)\,dy = \frac{3}{4}\left[y\Big|_{-1}^{x} - \frac{y^3}{3}\Big|_{-1}^{x}\right]$$

$$= \frac{3}{4}\left[(x+1) - \frac{1}{3}(x^3+1)\right]$$



$$P[X=0] = F_X(0) = 0$$

$$P[0<X<0.5] =$$
$$= \frac{3}{4}\left[\left(\frac{1}{2}+1\right) - \frac{1}{3}\left(\frac{1}{8}+1\right)\right]$$
$$- \frac{3}{4}\left[1-\frac{1}{3}\right]$$
$$= \frac{11}{32}$$

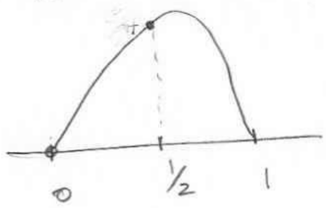
$$P\left[\,|X-\tfrac{1}{2}| < \tfrac{1}{4}\right] = P\left[\tfrac{1}{4} < X < \tfrac{3}{4}\right]$$
$$= \frac{3}{4}\left[\left(\frac{3}{4}+1\right) - \frac{1}{3}\left(\left(\frac{9}{4}\right)^3+1\right)\right] - \frac{3}{4}\left[\left(\frac{1}{4}+1\right) - \frac{1}{3}\left(\left(\frac{1}{4}\right)^3+1\right)\right]$$
$$= 0.2734$$

4.18

$$1 = c\int_0^1 x(1-x^2)\,dx = c\left[\frac{x^2}{2}\Big|_0^1 - \frac{x^4}{4}\Big|_0^1\right] = c\,\underbrace{\left[\frac{1}{2}-\frac{1}{4}\right]}_{\frac{1}{4}} =$$

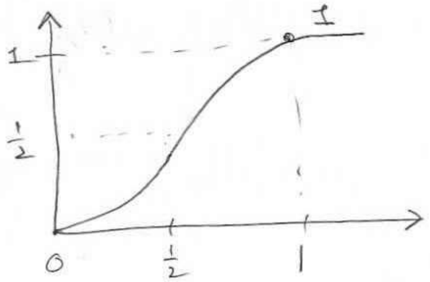
$$\Rightarrow c = 4$$

$f_X(x) = 4x(1-x^2) \quad 0 \le x \le 1$



$$F_X(x) = 4\int_0^x (y - y^3)\,dy$$

$$= 4\left[\frac{x^2}{2} - \frac{x^4}{4}\right]$$



$P[0 < X < \frac{1}{2}] = F_X(\frac{1}{2})$

$= 7/16 = .4375$

$P[X = 1] = 0$

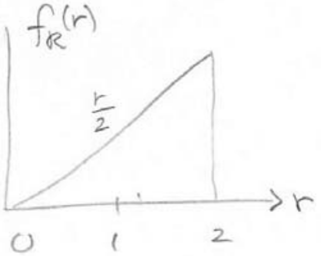
$$P\left[\frac{1}{4} < X < \frac{1}{2}\right] = F_X\left(\tfrac{1}{2}\right) - F_X\left(\tfrac{1}{4}\right)$$

$$= \frac{7}{16} - 4\left[\frac{1}{32} - \frac{1}{1024}\right] = .3164$$

4.19 from 4.6

$F_R(r) = \left(\frac{r}{2}\right)^2 \quad 0 \le r \le 2$

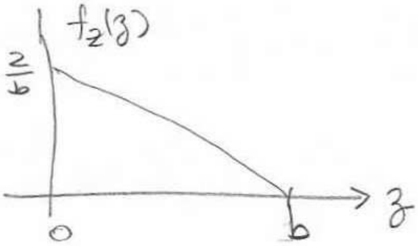
ⓐ $f_R(r) = \frac{d}{dr}F_R(r) = 2\left(\frac{r}{2}\right)\left(\frac{1}{2}\right) = \frac{r}{2} \quad 0 \le r \le 2$



ⓑ $P[R > \frac{1}{4}] = \int_{1/4}^{2} \frac{r}{2}\,dr = \frac{1}{2}\frac{r^2}{2}\Big|_{1/4}^{2} = \frac{1}{4}\left[4 - \frac{1}{16}\right] = \frac{63}{64}$

4.20 $F_Z(z) = 2\frac{z}{b} - \frac{z^2}{b^2}$ from prob. 4.7

ⓐ $f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{2}{b} - \frac{2z}{b^2} = \frac{2}{b}\left(1 - \frac{z}{b}\right)$



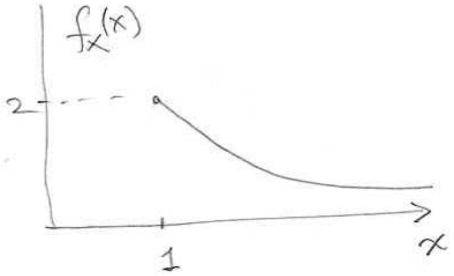
ⓑ $$P\left[Z > \frac{b}{3}\right] = \int_{b/3}^{b} \frac{2}{b}\left(1 - \frac{z}{b}\right)dz = \frac{2}{b}\left[z\Big|_{b/3}^{b} - \frac{1}{b}\frac{z^2}{2}\Big|_{b/3}^{b}\right]$$

$$= \frac{2}{b}\left(b - \frac{b}{3}\right) - \frac{2}{2b^2}\left[b^2 - \frac{b^2}{9}\right]$$

$$= \frac{4}{3} - \frac{8}{9} = \frac{4}{9}$$

4.21 from 4.8 $F_X(x) = 1 - \frac{1}{x^2}$ $x \geq 1$.

$f_X(x) = \frac{d}{dx}F_X(x) = 2\frac{1}{x^3}$ $x \geq 1$



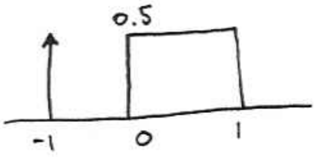
$a > 1$

$$P[X > a] = 2\int_{a}^{\infty} \frac{1}{x^3}dx = \frac{1}{x^2}\Big|_{a}^{\infty} = \frac{1}{a^2}$$

$$P[X > 2a] = \frac{1}{x^2}\Big|_{2a}^{\infty} = \frac{1}{4a^2}$$

4.22

a) $$f_X(x) = \begin{cases} 0.5\delta(x+1) & x \le 0 \\ 0.5 & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$



b) $$P[-1 \le X < 0.25] = \int_{-1^-}^{0.25^-} f_X(x)\,dx$$

$$= \int_{-1^-}^{0} 0.5\,\delta(x+1)\,dx + \int_{0}^{0.25} 0.5\,dx$$

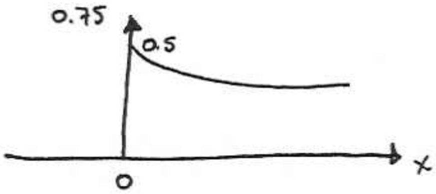
$$= 0.5x\Big|_0^{0.25} + 0.5$$

$$= 0.125 + 0.5$$

$$= 0.625$$

4.23

a) $$f_X(x) = \begin{cases} 0 & x<0 \\ 0.75\,\delta(x) & x=0 \\ 0.5\,e^{-2x} & x>0 \end{cases}$$

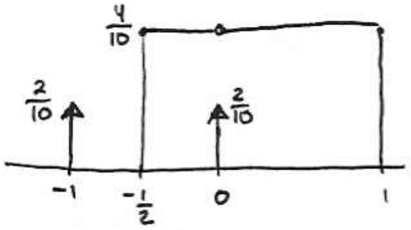


b) $P[X=0] = 0.75$

$$P[X>8] = \int_8^\infty 0.5\,e^{-2x}\,dx$$
$$= -0.25\,e^{-2x}\Big|_8^\infty$$
$$= 0 + 0.25\,e^{-2(8)}$$
$$= 0.25\,e^{-16}$$

4.24

a)



b) $P[X<-1]=0$

$$P[X\le -1]=\int_{-\infty}^{-1}\frac{2}{10}\delta(x+1)\,dx=\frac{2}{10}$$

$$P[-1<X<-0.75]=\int_{-1^+}^{-0.75}0\,dx=0$$

$$P[-0.5\le X<0]=\int_{-0.5}^{0^-}\frac{4}{10}\,dx=\frac{4}{10}\left(\frac{1}{2}\right)=\frac{2}{10}$$

$$P[-0.5\le X\le 0.5]=\int_{-0.5}^{0.5}f_X(x)\,dx=\int_{-0.5}^{0^-}\frac{4}{10}dx+\int_{0^-}^{0^+}\frac{2}{10}\delta(x)\,dx$$

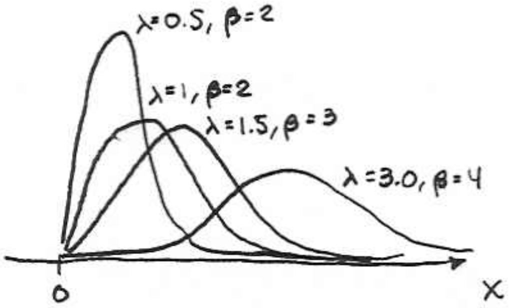
$$+\int_{0^+}^{0.5}\frac{4}{10}dx=\frac{4}{10}(0+0.5)+\frac{2}{10}+\frac{4}{10}(0.5-0)$$

$$=\frac{6}{10}$$

$$P[|X-0.5|<0.5]=P[\{X<1\}\cup\{X>0\}]=\int_{0^+}^{1}\frac{4}{10}dx=\frac{4}{10}$$

4.25

a) $$f_X(x)=\begin{cases}0 & x<0\\ \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}e^{-\left(\frac{x}{\lambda}\right)^{\beta}} & x\ge 0\end{cases}$$



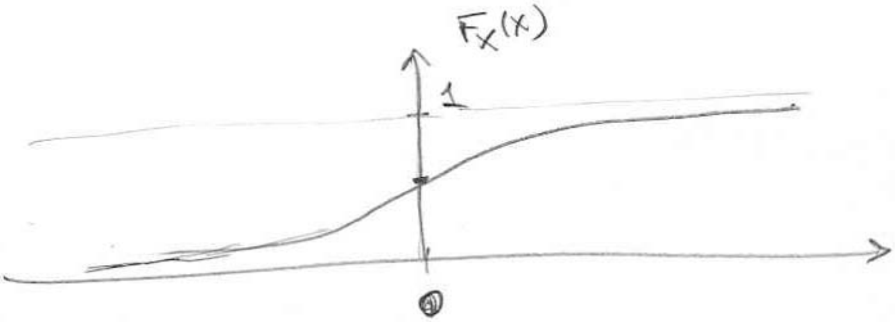
4.26

$$f_X(x) = \frac{\alpha/\pi}{x^2+\alpha^2}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{\alpha/\pi}{t^2+\alpha^2}\,dt = \int_{-\infty}^{x} \frac{1/\pi}{1+\left(\frac{t}{\alpha}\right)^2}\,d\left(\frac{t}{\alpha}\right)$$

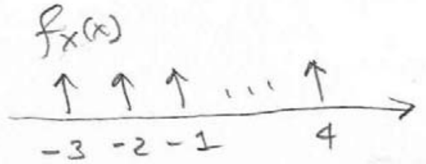
$$= \frac{1}{\pi}\tan^{-1}\left(\frac{t}{\alpha}\right)\Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x}{\alpha}\right)+\frac{\pi}{2}\right] \qquad -\infty < x < \infty$$

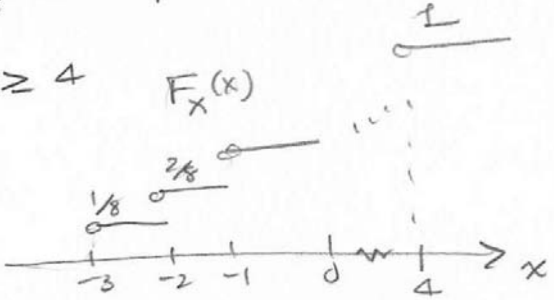


4.27

ⓐ $f_X(x) = \sum_{j=-3}^{4} \frac{1}{8}\delta(x-j)$



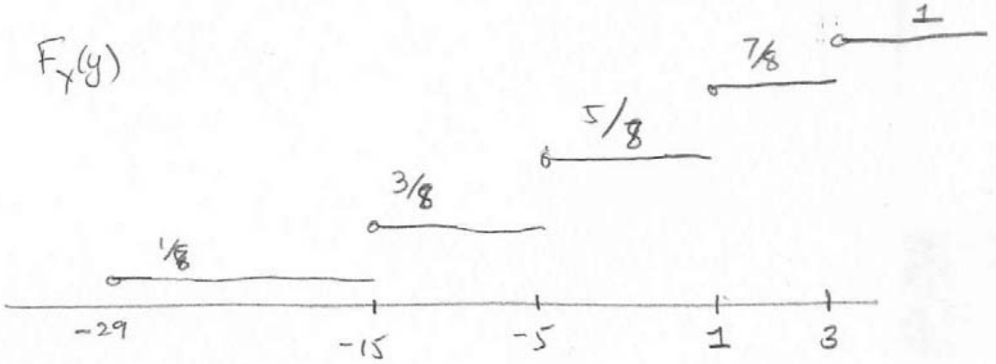
$$F_X(x) = \begin{cases} 0 & x < -3 \\ \frac{j+3}{8} & \frac{j+3}{7} \le x < \frac{j+4}{7} \qquad j = -3, \ldots, 3 \\ 1 & x \ge 4 \end{cases}$$



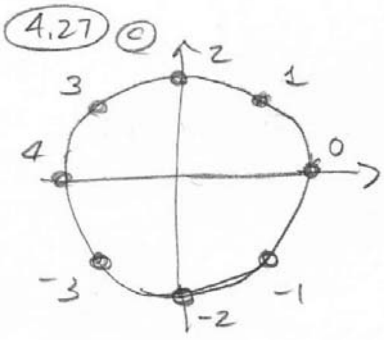
ⓑ

X	-3	-2	-1	0	1	2	3	4
$Y = -2X^2 + 3$	-15	-5	1	3	1	-5	-15	-29

$$f_Y(y) = \frac{1}{8}\delta(y+29) + \frac{2}{8}\delta(y+15) + \frac{2}{8}\delta(y+5) + \frac{2}{8}\delta(y-1) + \frac{1}{8}\delta(y-3)$$

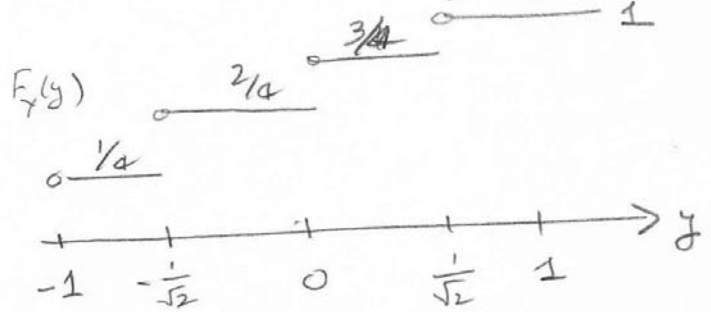


4.27 ⓒ



X	-3	-2	-1	0	1	2	3	4
$\cos\frac{\pi X}{8}$	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1

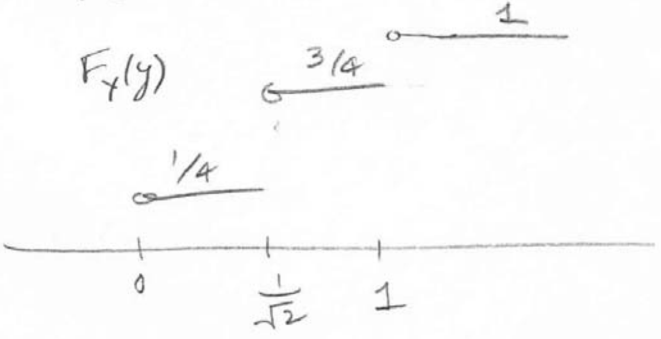
$$f_Y(y) = \frac{2}{8}\delta(y+1) + \frac{2}{8}\delta\left(y+\frac{1}{\sqrt{2}}\right) + \frac{2}{8}\delta\left(y-\frac{1}{\sqrt{2}}\right) + \frac{2}{8}\delta(y-1)$$



4.27 d

$$Y = \cos^2\frac{\pi X}{8} = \begin{cases} \frac{1}{2} & x = -3, -1, 1, 3 \\ 1 & x = 0, 4 \\ 0 & x = 2, -2 \end{cases}$$

$$f_Y(y) = \frac{1}{4}\delta(y) + \frac{2}{4}\delta\left(y-\frac{1}{\sqrt{2}}\right) + \frac{1}{4}\delta(y-1)$$



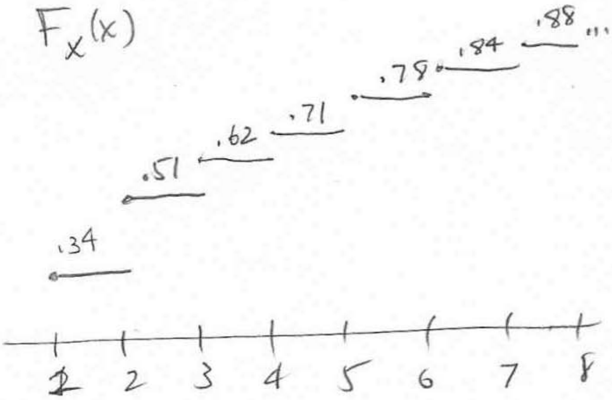
4.28

$$p_X(i) = \frac{1}{c_{10}} \frac{1}{i} \qquad i = 1, 2, \ldots, 10$$

$$c_{10} = \sum_{i=1}^{10} \frac{1}{i} = 2.929$$

	$P[X=i]$
1	0.341
2	0.171
3	0.114
4	0.0853
5	0.0683
6	0.0569
7	0.0488
8	0.0427
9	0.0379
10	0.0341
	1

Note slow decay rate



$$f_X(x) = \sum_{i=1}^{10} p_X(i)\, \delta(x-i)$$

4.29

~~**3.26**~~ $F_X(x|A) = \dfrac{P[\{X \le x\} \cap A]}{P[A]}$ if $P[A] > 0$

i) $P[A] > 0, \;\; P[\{x \le x\} \cap A] \ge 0 \Rightarrow F_x(x|A) \ge 0$
$\{\{X \le x\} \cap A\} \subset A \Rightarrow P[\{X \le x\} \cap A] \le P[A]$

Therefore

$$F_X(x|A) \le 1$$

ii) $\lim\limits_{x\to\infty} F_X(x|A) = \lim\limits_{x\to\infty} \dfrac{P[\{X \le x\} \cap A]}{P[A]} = \dfrac{P[A]}{P[A]} = 1$

iii) $\lim\limits_{x\to-\infty} P[\{X \le x\} \cap A] = P[\Phi] = 0 \Rightarrow \lim\limits_{x\to\infty} F_X(x|A) = 0$

iv)
$$\begin{aligned} a < b \;\; &\Rightarrow \;\; \{\{X \le a\} \cap A\} \subset \{\{X \le b\} \cap A\} \\ &\Rightarrow \;\; P[\{X \le a\} \cap A] \le P[\{x \le b\} \cap A] \\ &\Rightarrow \;\; F_x(a|A) \le F_x(b|A) \end{aligned}$$

v)
$$\begin{aligned} P[\{X \le b\} \cap A] &= \lim_{h\to 0} P[\{X \le b+h\} \cap A] \\ F_X(b|A) &= \lim_{h\to 0} F_X(b+h|A) \end{aligned}$$

vi) $\{\{X \le a\} \cap A\} \cup \{\{a < X \le b\} \cap A\} = \{\{X \le b\} \cap A\}\}$
The two event on LHS are mutually exclusive. Therefore,

$$P[\{X \le a\} \cap A] + P[\{a < X \le b\} \cap A] = P[\{X \le b\} \cap A]$$

$$P[a < X < b|A] = F_X(b|A) - F_X(a|A)$$

vii)
$$\begin{aligned} P[\{X = b\} \cap A] &= P[\{X \le b\} \cap A] - P[\{X < b^-\} \cap A] \\ P[\{X = b\}|A] &= F_X(b|A) - F_X(b^-|A) \end{aligned}$$

4.30

a) $$F_X(x|C) = \frac{P[\{X\le x\}\cap\{X>0\}]}{P[X>0]} = \frac{P[0<X\le x]}{P[X>0]} \qquad x\geq 0$$

$$= \begin{cases} 0 & x\le 0 \\ \dfrac{F_X(x)-F_X(0)}{1-F_X(0)} & x\geq 0 \end{cases}$$

$$= \begin{cases} 0 & x\le 0 \\ \dfrac{-\frac{1}{4}e^{-2x}+\frac{1}{4}}{1-(1-\frac{1}{4})} = 1-e^{-2x} & x\geq 0 \end{cases}$$

b) $$F_X(x|C) = \frac{P[\{X\le x\}\cap\{X=0\}]}{P[X=0]}$$

$$= \begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$$

4.31

$$B = \left\{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\}^c \qquad P[B] = 1-0.2 = 4/5$$

ⓐ

$$F_X(x|B) = \frac{P[X\le x, B]}{P[B]} = \frac{\frac{4}{5}\,\frac{4\sin^{-1}x}{2\pi}}{4/5} = \frac{4\sin^{-1}x}{2\pi}$$

ⓑ $$B^c = \left\{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\}$$

$$F_X(x|B^c) = \frac{P[X\le x, B^c]}{P[B^c]} \qquad P[B^c] = 1/5$$

$$= \frac{1}{1/5}\left[\frac{1}{20}u(x) + \frac{1}{20}u(x-\tfrac{1}{4}) + \frac{1}{20}u(x-\tfrac{1}{2}) + \frac{1}{20}u(x-\tfrac{3}{4})\right]$$

$$= \frac{1}{4}u(x) + \frac{1}{4}u(x-\tfrac{1}{4}) + \frac{1}{4}u(x-\tfrac{1}{2}) + \frac{1}{4}u(x-\tfrac{3}{4})$$

4.32

$$F_X(x|B) = \frac{P[\{X \leq x\} \cap \{X > 0.25\}]}{P[X > 0.25]} = \frac{P[0.25 < X \leq x]}{P[X > 0.25]}$$

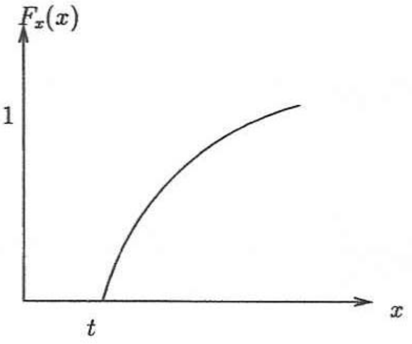
$$= \begin{cases} 0 & x < 0.25 \\ \dfrac{F_X(x) - F_X(0.25)}{1 - F_X(0.25)} & x \geq 0.25 \end{cases}$$

$$= \begin{cases} 0 & x \leq 0.25 \\ \dfrac{-\frac{1}{4}e^{-2x} + \frac{1}{4}e^{-2(\frac{1}{4})}}{1 - (1 - \frac{1}{4}e^{-2(\frac{1}{4})})} = \dfrac{e^{-\frac{1}{2}} - e^{-2x}}{e^{-\frac{1}{2}}} = 1 - e^{-(2x - \frac{1}{2})} & x \geq 0.25 \end{cases}$$

$$f_X(x|B) = \begin{cases} \dfrac{f_X(x)}{1 - F_X(0.25)} & x \geq 0.25 \\ 0 & x < 0.25 \end{cases}$$

$$= \begin{cases} \dfrac{\frac{1}{2}e^{-2x}}{\frac{1}{4}e^{-2(\frac{1}{4})}} = 2e^{-(2x - \frac{1}{2})} & x \geq 0.25 \\ 0 & x \leq 0.25 \end{cases}$$

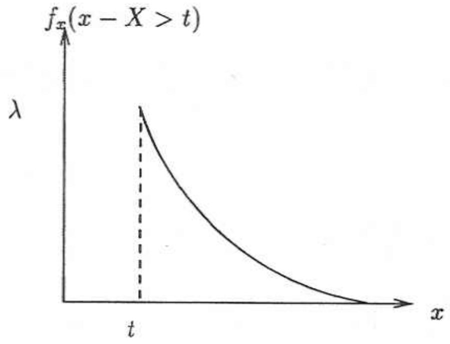
4.33 **a)**



$$\begin{aligned}
F_X(x|X>T) &= \frac{P[\{X\le x\}\cap\{X>t\}]}{P[X>t]}\\
&= \begin{cases} 0 & x<t \\ \frac{F_X(x)-F_X(t)}{1-F_X(t)} & x\ge t \end{cases}\\
&= \begin{cases} 0 & x<t \\ \frac{(1-e^{-\lambda x})-(1-e^{-\lambda t})}{1-(1-e^{-\lambda t})} & x\ge t \end{cases}\\
&= \begin{cases} 0 & x<t \\ \frac{e^{-\lambda x}-e^{-\lambda t}}{e^{-\lambda t}} & x\ge t \end{cases}
\end{aligned}$$

$F_x(x|x>t)$ is delayed version of $F_x(x)$

b) $f_x(x|x>t) = \dfrac{f_x(x)}{1-F_x(t)} = \dfrac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)}, \quad x \ge t$



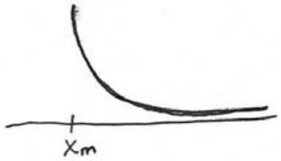
c)

$$\begin{aligned}
P &= [X>t+x|X>t] \qquad x\ge 0\\
&= \frac{P[\{X>t+x\}\cap\{X>t\}]}{P[X>t]}\\
&= \frac{1-F_X(t+x)}{1-F_X(t)}\\
&= \frac{1-(1-e^{-\lambda(t+x)})}{1-(1-e^{-\lambda t})}\\
&= e^{-\lambda x}\\
&= P[X>x]
\end{aligned}$$

Additional waiting time does not depend on time already spent waiting ⇒ "memoryless"

4.34

a) $$f_X(x) = \begin{cases} 0 & x < x_m \\ \dfrac{\alpha x_m^\alpha}{x^{\alpha-1}} & x \geq x_m \end{cases}$$



b) $$F_X(x \mid X > t) = \frac{P[\{X \leq x\} \cap \{X > t\}]}{P[X > t]} = \frac{P[t < X \leq x]}{P[X > t]}$$

$$= \begin{cases} 0 & x \leq t \\ \dfrac{F_X(x) - F_X(t)}{1 - F_X(t)} & x \geq t \end{cases}$$

if $t \geq x_m$ $$F_X(x \mid X > t) = \frac{1 - \frac{x_m^\alpha}{x^\alpha} - 1 + \frac{x_m^\alpha}{t^\alpha}}{1 - \left(1 - \frac{x_m^\alpha}{t^\alpha}\right)} = t^\alpha\left(\frac{1}{t^\alpha} - \frac{1}{x^\alpha}\right) = 1 - \left(\frac{t}{x}\right)^\alpha \quad x \geq t$$

if $t < x_m$ $$F_X(x \mid X > t) = 1 - \left(\frac{x_m}{x}\right)^\alpha \quad x \geq x_m$$

$$f_X(x \mid X > t) = \frac{f_X(x)}{1 - F_X(t)} \quad x \geq t$$

if $t \geq x_m$ $$f_X(x \mid X > t) = \frac{\alpha \frac{x_m^\alpha}{x^{\alpha-1}}}{\frac{x_m^\alpha}{t^\alpha}} = \alpha t \left(\frac{t}{x}\right)^{\alpha-1} \quad x \geq t$$

if $t < x_m$ $$f_X(x \mid X > t) = \alpha \frac{x_m^\alpha}{x^{\alpha-1}} \quad x \geq x_m$$

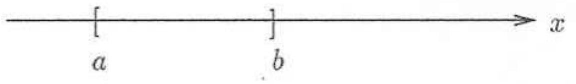
c) $$\frac{P[\{X > t + x\} \cap \{X > t\}]}{P[X > t]} \xrightarrow[t \to \infty]{} 1$$

The longer you wait
the longer you are likely
to wait more!

4.35

3.~~28~~ a) From the definition of conditional probability we have:

$$F_X(x|a \leq X \leq b) = \frac{P[\{X \leq x\} \cap \{a \leq X \leq b\}]}{P[a \leq X \leq b]}$$



From the above figure we see that

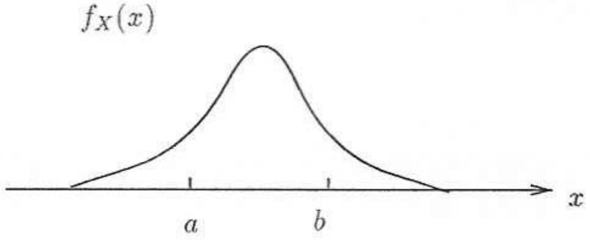
$$\{X \leq x\} \cap \{a \leq X \leq b\} = \begin{cases} \emptyset & \text{for } x < a \\ \{a \leq X \leq x\} & \text{for } a \leq x \leq b \\ \{a \leq X \leq b\} & \text{for } x > b \end{cases}$$

Therefore

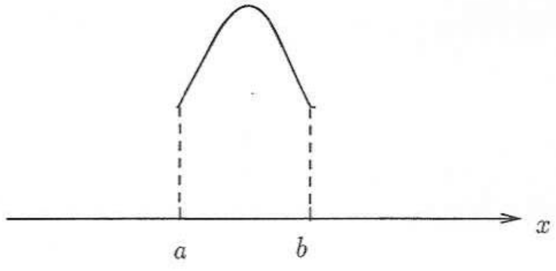
$$F_X(x|a \leq X \leq b) = \begin{cases} \frac{P[\emptyset]}{P[a \leq X \leq b]} = 0 & x < a \\ \frac{P[a \leq X \leq x]}{P[a \leq X \leq b]} = \frac{F_X(x) - F_X(a^{-}1)}{F_X(b) - F_X(a^-)} & a \leq x \leq b \\ \frac{P[a \leq X \leq b]}{P[a \leq X \leq b]} = 1 & x > b \end{cases}$$

b)
$$\begin{aligned} f_X(x|a \leq X \leq b) &= \frac{d}{dx} F_X(x|a \leq X \leq b) \\ &= \begin{cases} 0 & x < a \\ \frac{f_X(x)}{F_X(b) - F_X(a)} & a \leq x \leq b \\ 0 & x > b \end{cases} \end{aligned}$$

Thus if X has pdf:



then $f_X(x|a \leq X \leq b)$ is



4.36

$$F_R(r \mid R>1) = \frac{P[R \le r, R>1]}{P[R>1]} \qquad 1 \le r \le 2$$

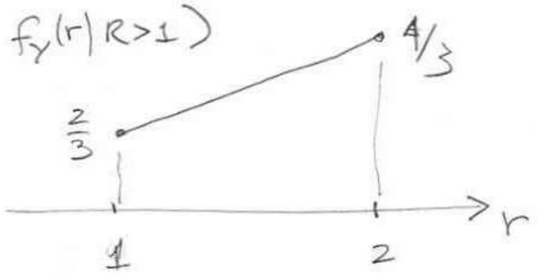
$$= \frac{P[1 \le R \le r]}{P[R>1]} = \frac{\left(\frac{r}{2}\right)^2 - \left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2}$$

$$= \frac{r^2-1}{4-1} = \frac{1}{3}(r^2-1)$$

$$f_Y(r \mid R>1) = \frac{d}{dr}\,\frac{1}{3}(r^2-1) = \frac{2}{3}r \qquad 1 \le r \le 2$$

$$0$$

$$= \frac{2}{3}r$$



4.37

a)
$$F_Z\left(z \,\middle|\, \tfrac{b}{4} \le Z \le \tfrac{b}{2}\right) = \frac{P\left[Z \le z, \tfrac{b}{4} \le Z \le \tfrac{b}{2}\right]}{P\left[\tfrac{b}{4} \le Z \le \tfrac{b}{2}\right]}$$

$$= \frac{P\left[\tfrac{b}{4} \le Z \le z\right]}{P\left[\tfrac{b}{4} \le Z \le \tfrac{b}{2}\right]} \quad \text{for } \frac{b}{4} \le z \le \frac{b}{2}$$

$$= \frac{F_Z(z) - F_Z\left(\tfrac{b}{4}\right)}{F_Z\left(\tfrac{b}{2}\right) - F_Z\left(\tfrac{b}{4}\right)} = \frac{\frac{z}{b}\left(2 - \frac{z}{b}\right) - \frac{1}{4}\left(2 - \frac{1}{4}\right)}{\frac{1}{2}\left(2 - \frac{1}{2}\right) - \frac{1}{4}\left(2 - \frac{1}{4}\right)}$$

$$= \frac{\frac{z}{b}\left(2 - \frac{z}{b}\right) - \frac{7}{16}}{5/16}$$

$$f_Z\left(z \,\middle|\, \tfrac{b}{4} \le Z \le \tfrac{b}{2}\right) = \frac{16}{5}\left[\frac{2}{b} - \frac{2z}{b^2}\right] \quad \frac{b}{4} \le z \le \frac{b}{2}$$

$$= \frac{32}{5b}\left[1 - \frac{z}{b}\right]$$

b)
$$F_Z\left(z \,\middle|\, X > \tfrac{b}{2}\right) = \frac{P\left[\tfrac{b}{2} < X < z\right]}{P\left[X > \tfrac{b}{2}\right]} \quad \frac{b}{2} < z < b$$

$$= \frac{\frac{z}{b}\left(2 - \frac{z}{b}\right) - \frac{3}{4}}{1 - \frac{3}{4}}$$

$$f_Z\left(z \,\middle|\, X > \tfrac{b}{2}\right) = \frac{\frac{2}{b}\left[1 - \frac{z}{b}\right]}{1/4} \quad \frac{b}{2} < z < b$$

4.38

a) $F_Y(x) = F_Y(x|B_0)P[B_0] + F_Y(x|B_1)P[B_1]$

$= P[Y \le x \,|\, X=-1](1-p) + P[Y \le x \,|\, X=1]p$

$= P[X+N \le x \,|\, X=-1](1-p) + P[X+N \le x \,|\, X=1]p$

$= P[N \le x+1](1-p) + P[N \le x-1]p$

$= F_N(x+1)(1-p) + F_N(x-1)p$

$$f_Y(x) = \frac{d}{dx}F_Y(x)$$

$$= (1-p)f_N(x+1) + pf_N(x-1)$$

$$f_Y(x|B_0) = f_N(x+1) = \frac{\alpha}{2}e^{-\alpha|x+1|}$$

$$f_Y(x|B_1) = f_N(x-1) = \frac{\alpha}{2}e^{-\alpha|x-1|}$$

$$f_Y(x) = \frac{1}{2}\left[\frac{\alpha}{2}e^{-\alpha|x+1|} + \frac{\alpha}{2}e^{-\alpha|x-1|}\right] = \frac{1}{4}\alpha\left[e^{-\alpha|x+1|} + e^{-\alpha|x-1|}\right]$$

b) $P[Y<0|B_1] = P[X+N<0|X=1] = P[N<-1]$

$= \frac{\alpha}{2}e^{-\alpha|-1|} = \frac{\alpha}{2}e^{-\alpha}$

$P[Y \geq 0|B_0] = P[X+N \geq 0|X=-1] = P[N \geq 1]$

$= \frac{\alpha}{2}e^{-\alpha}$

c) $P_E = P[Y<0|B_1]P[B_1] + P[Y \geq 0|B_0]P[B_0]$

$= 0.5\frac{\alpha}{2}e^{-\alpha} + 0.5 \quad \frac{\alpha}{2}e^{-\alpha} \quad = \frac{\alpha}{2}e^{-\alpha}$

4.3 The Expected Value of *X*

4.39

$f_X(x) = \frac{3}{4}(1-x^2) \qquad -1 \le x \le 1$

$$E[X] = \frac{3}{4}\int_{-1}^{1} x(1-x^2)\,dx = \frac{3}{4}\left[\left.\frac{x^2}{2}\right|_{-1}^{1} - \left.\frac{x^4}{4}\right|_{-1}^{1} \right] = 0$$

$$E[X^2] = \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)\,dx = \frac{3}{4}\left[\left.\frac{x^3}{3}\right|_{-1}^{1} - \left.\frac{x^5}{5}\right|_{-1}^{1} \right]$$

$$= \frac{3}{4}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{1}{5}$$

$$VAR[X] = E[X^2] - E[X]^2 = \frac{1}{5}$$

4.40

$f_X(x) = 4x(1-x^2) \qquad 0 \le x \le 1$

$$E[X] = 4\int_0^1 x\left(x(1-x^2)\right)dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$= 8/15$$

$$E[X^2] = 4\int_0^1 x^2\left(x(1-x^2)\right)dx = 4\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = \frac{1}{3}$$

$$VAR[X] = E[X^2] - E[X]^2 = \frac{1}{3} - \frac{64}{225} = \frac{11}{225}$$

4.41

$$f_R(r) = \frac{r}{2} \qquad 0 \le r \le 2$$

$$E[R] = \int_0^2 r\left(\frac{r}{2}\right)dr = \frac{1}{2}\int_0^2 r^2\,dr = \frac{1}{2}\left.\frac{r^3}{3}\right|_0^2 = \frac{8}{6} = \frac{4}{3}$$

$$E[R^2] = \int_0^2 r^2\frac{r}{2}dr = \frac{1}{2}\left[\frac{r^4}{4}\right]_0^2 = \frac{16}{8} = 2$$

$$VAR[R] = E[R^2] - E[R]^2 = 2 - \frac{16}{9} = \frac{2}{9}$$

4.42

$$f_Z(z) = \frac{2}{b}\left(1 - \frac{z}{b}\right) \qquad 0 \le z \le b$$

$$E[Z] = \frac{2}{b}\int_0^b z\left(1 - \frac{z}{b}\right)dz = \frac{2}{b}\left[\frac{z^2}{2} - \frac{1}{b}\frac{z^3}{3}\right]_0^b$$

$$= \frac{2}{b}\left[\frac{b^2}{2} - \frac{1}{b}\frac{b^3}{3}\right] = b\left[1 - \frac{2}{3}\right] = \frac{b}{3}$$

$$E[Z^2] = \frac{2}{b}\int_0^b z^2\left(1 - \frac{z}{b}\right)dz = \frac{2}{b}\left[\frac{z^3}{3} - \frac{1}{b}\frac{z^4}{4}\right]_0^b$$

$$= 2b^2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{b^2}{6}$$

$$VAR[Z] = E[Z^2] - E[Z]^2 = \frac{b^2}{6} - \frac{b^2}{9} = \frac{b^2}{18}$$

4.43

$f_X(x) = \frac{2}{x^3} \qquad x \geq 1$

$$E[X] = \int_1^\infty x \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x^2}\,dx = -\frac{2}{x}\Big|_1^\infty = 2$$

$$E[X^2] = \int_1^\infty x^2 \frac{2}{x^3}\,dx = \int_1^\infty \frac{2}{x}\,dx = 2\ln x\Big|_1^\infty = \infty$$

Second moment does not exist

$\Rightarrow$ variance does not exist.

Using Eqn. 4.28

$$E[X] = \int_0^\infty (1-F_X(t))\,dt = \int_0^1 1\,dt + \int_1^\infty \frac{1}{t^2}\,dt$$

$$= 1 + -\frac{1}{t}\Big|_1^\infty = 1+1=2. \checkmark$$

4.44

$$E[X] = \frac{1}{2}\int_{-1^-}^{-1^+} (-1)\,\delta(x+1)\,dx + \int_0^1 x\,\frac{1}{2}\,dx$$

$$= -1\left(\frac{1}{2}\right) + \frac{1}{2}\frac{x^2}{2}\Big|_0^1 = \frac{3}{4}$$

$$E[X^2] = \frac{1}{2}\int_{-1^-}^{-1^+} (-1)^2\,\delta(x+1)\,dx + \int_0^1 \frac{1}{2}x^2\,dx$$

$$= \frac{1}{2} + \frac{1}{2}\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}+\frac{1}{6} = \frac{4}{6}$$

$$VAR[X] = E[X^2] - E[X]^2$$

$$= \frac{4}{6} - \frac{9}{16} = \frac{32-27}{48} = \frac{5}{48}$$

4.45

$$f_X(x) = \frac{3}{4}\delta(x) + \frac{1}{4}2e^{-2x} \qquad x > 0.$$

$$E[X] = \int_0^\infty x f_X(x)\,dx = 0 + \frac{1}{4}\underbrace{\int_0^\infty 2x e^{-2x}\,dx}_{\text{mean of exponential RV}}$$

$$= \frac{1}{4}\frac{1}{2} = \frac{1}{8}$$

$$E[X^2] = \int_0^\infty x^2 f_X(x)\,dx = 0 + \frac{1}{4}\underbrace{\int_0^\infty 2x^2 e^{-2x}\,dx}_{\frac{2}{2^2}}$$

2nd moment of exponential RV

$$= \frac{1}{4}\frac{1}{2}$$

$$= \frac{1}{8}$$

see Problem 4.48 for solution of this integral

$$VAR[X] = E[X^2] - E[X]^2$$

$$= \frac{1}{8} - \left(\frac{1}{8}\right)^2$$

$$= \frac{7}{64}$$

4.46

$$\begin{aligned}
\mathcal{E}[X] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(y+m)e^{-y^2/2\sigma^2}dy \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y e^{-y^2/2\sigma^2}dy + \frac{m}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}dy \\
&= \frac{-\sigma^2}{\sqrt{2\pi}\sigma}\left[e^{-y^2/2\sigma^2}\right]_{-\infty}^{\infty} + m = m \\
\sigma_X^2 &= \int_{-\infty}^{\infty}(x-m)^2 \frac{e^{-(x-m)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^2 e^{-y^2/2\sigma^2}dy \\
&= \frac{1}{\sqrt{2\pi}\sigma}\left\{\left[-\sigma^2 y e^{-y^2/2\sigma^2}\right]_{-\infty}^{\infty} + \sigma^2\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}dy\right\} \\
&= \sigma^2 \quad \text{where we used integration by parts with} \\
&\qquad u = y \quad dv = e^{-y^2/2\sigma^2}
\end{aligned}$$

4.47

$$\begin{aligned}\int_0^x tf_X(t)dt &= tF_X(t)|_0^x - \int_0^x F_X(t)dt \text{ where we let } u = t \; dv = f_X(t)\\ &= xF_X(x) - \int_0^x F_X(t)dt\\ &= x(F_X(x)-1) + \int_0^x (1-F_X(t))dt\end{aligned}$$

If $\mathcal{E}[X] < \infty$ then the first term on the right-hand side approaches zero as $x \to \infty$, so

$$\mathcal{E}[X] = \int_0^t tf_X(t)dt = \int_0^\infty (1-F_X(t))dt$$

For the discrete case, we have

$$\begin{aligned}\sum_{k=0}^{\infty} P[X>k] &= \sum_{k=0}^{\infty}\left(\sum_{j=k+1}^{\infty} P[X=j]\right)\\ &= (P[X=1]+P[X=2]+...)+(P[X=2]+...)+(P[X=3]+...)+...\\ &= P[X=1]+2P[X=2]+3P[X=3]+...\\ &= \sum_{k=0}^{\infty} kP[X=k] \triangleq \mathcal{E}[X]\end{aligned}$$

4.48

$$E[X]^2 = \left(\frac{1}{\lambda}\right)^2$$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$u = x^2$ $dv = \lambda e^{-\lambda x} dx$
$du = 2x dx$ $v = -e^{-\lambda x}$

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}(2x)dx$$

$u = 2x$ $dv = e^{-\lambda x} dx$
$du = 2dx$ $v = -\frac{1}{\lambda} e^{-\lambda x}$

$$= -x^2 e^{-\lambda x}\Big|_0^\infty + \left(-\frac{2x}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty \frac{1}{\lambda} e^{-\lambda x}(2)dx$$

$$= -\frac{2}{\lambda^2} e^{-\lambda x}\Big|_0^\infty = \frac{2}{\lambda^2}$$

$$VAR[X] = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4.49

a) $E[X] = \int_0^\infty x \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^\beta} dx = \int_0^\infty \beta\left(\frac{x}{\lambda}\right)^{\beta} e^{-\left(\frac{x}{\lambda}\right)^\beta} dx$

$y = \left(\frac{x}{\lambda}\right)^\beta \qquad dy = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} dx$

$x = y^{1/\beta}\lambda \qquad dy = \frac{\beta}{\lambda} y^{1-\frac{1}{\beta}} dx$

$= \int_0^\infty \beta \left(\frac{y^{1/\beta}\lambda}{\lambda}\right)^\beta e^{-y} \frac{\lambda}{\beta} y^{-(1-\frac{1}{\beta})} dy$

$= \int_0^\infty \lambda y y^{-1+\frac{1}{\beta}} e^{-y} dy$

$= \int_0^\infty \lambda y^{1+\frac{1}{\beta}-1} e^{-y} dy$

$= \lambda \Gamma\left(1+\frac{1}{\beta}\right)$

b) $E[X^2] = \int_0^\infty x^2 \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^\beta} dx = \int_0^\infty (y^{1/\beta}\lambda)\, \beta \left(\frac{y^{1/\beta}\lambda}{\lambda}\right)^\beta e^{-y} \frac{\lambda}{\beta} y^{\frac{1}{\beta}-1} dy$

$= \int_0^\infty \lambda^2 y^{\frac{1}{\beta}} y\, y^{\frac{1}{\beta}-1} e^{-y} dy$

$= \int_0^\infty \lambda^2 y^{1+\frac{2}{\beta}-1} e^{-y} dy$

$= \lambda^2 \Gamma\left(\frac{2}{\beta}+1\right)$

$VAR[X] = E[X^2] - E[X]^2$

$= \lambda^2 \Gamma\left(1+\frac{2}{\beta}\right) - \lambda^2 \Gamma\left(1+\frac{1}{\beta}\right)^2$

4.50

~~3.72~~ $\mathcal{E}[X] = \int_{-\infty}^{0} x \frac{1}{\pi(1+x^2)} dx + \int_0^\infty \frac{x}{\pi(1+x^2)} dx$

Consider the latter term:

$$\int_0^y \frac{x}{\pi(1+x^2)} dx = \frac{1}{2\pi} \ln(1+x^2)\Big|_0^y == \frac{\ln(1+y)}{2\pi} \to \infty$$

Thus the integrals do not exist ⇒ $\mathcal{E}[X]$ does not exist.

4.51

$$E[X] = \int_0^\infty (1 - F_X(t))\,dt = \int_0^{x_m} dt + \int_{x_m}^\infty \left(\frac{x_m}{x}\right)^\alpha dx$$

$$= x_m - \frac{x_m^\alpha}{(\alpha-1)} \frac{1}{x^{\alpha-1}}\Big|_{x_m}^\infty$$

$$= x_m + \frac{x_m^\alpha}{\alpha-1}\frac{1}{x_m^{\alpha-1}}$$

$$= x_m + \frac{x_m}{\alpha-1}$$

$$= \frac{\alpha x_m}{\alpha-1} \qquad \rightarrow E[X] \nexists \quad \alpha = 1$$

Alternatively, consider

$$E[X] = \int_{x_m}^\infty x \cdot \alpha \frac{x_m^\alpha}{x^{\alpha+1}}\,dx$$

$$= \alpha x_m^\alpha \int_{x_m}^\infty \frac{1}{x^\alpha}\,dx \qquad \text{when } \alpha = 1 \quad \int_{x_m}^\infty \frac{1}{x}\,dx$$

does not exist.

4.52

$$\begin{aligned}
\mathcal{E}[c] &= \int_{-\infty}^\infty c f_X(x)dx = c\int_{-\infty}^\infty f_X(x)dx = c \\
\mathcal{E}[c^2] &= \int_{-\infty}^\infty c^2 f_X(x)dx = c^2 \\
VAR[c] &= \mathcal{E}[c^2] - \mathcal{E}[c]^2 = c^2 - c^2 = 0 && \overset{4.36}{(\cancel{3.68})} \\
VAR[X+c] &= \mathcal{E}[((X+c) - \mathcal{E}[X+c])^2] \\
&= \mathcal{E}[(X + C - \mathcal{E}(X) - C)^2] \\
&= \mathcal{E}[(X - \mathcal{E}(X))^2] = VAR[X] && \overset{4.37}{(\cancel{3.69})} \\
VAR[cX] &= \mathcal{E}[(cX - \mathcal{E}[cX])^2] \\
&= \mathcal{E}[c^2(X - \mathcal{E}[X])^2] \\
&= c^2 VAR[X] && \overset{4.38}{(\cancel{3.70})}
\end{aligned}$$

4.53

$$\begin{aligned}
Y &= A\cos \overset{\omega}{\cancel{W}} t + c \\
E[Y] &= E[A]\cos \overset{\omega}{\cancel{W}} t + c = m\cos \overset{\omega}{\cancel{W}} t + c \\
VAR[Y] &= \cos^2 \overset{\omega}{\cancel{W}} t\, VAR[A] = \sigma^2 \cos^2 \overset{\omega}{\cancel{W}} t
\end{aligned}$$

4.54

~~3.17~~

ⓐ

$$
\begin{aligned}
\mathcal{E}[Y] &= \int_{-\infty}^{\infty} g(x) f_X(x)dx \quad \text{write integral into three parts} \\
&= -a\int_{-\infty}^{-a} f_X(x)dx + \int_{-a}^{a} x f_X(x)dx + a\int_{a}^{\infty} f_X(x)dx \\
&= -aF_X(-a) + \int_{-a}^{a} x f_X(x)dx + a(1 - F_X(a^-)) \\
\mathcal{E}[Y^2] &= a^2 F_X(-a) + \int_{-a}^{a} x^2 f_X(x)dx + a^2(1 - F_X(a^-)) \\
VAR[Y] &= \mathcal{E}[Y^2] - \mathcal{E}[Y]^2
\end{aligned}
$$

4.54 ⓑ

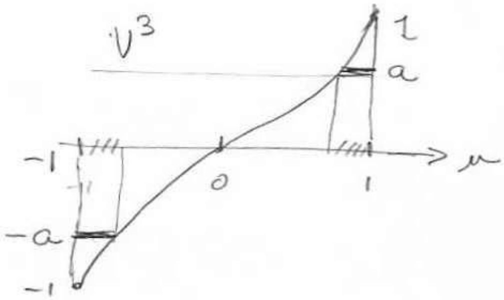
$$E[Y] = \underbrace{-(1)\underbrace{P[Y \le -1]}_{\frac{1}{2}e^{-1}} + (1)\underbrace{P[Y \ge 1]}_{\frac{1}{2}e^{-1}}}_{=0} + \underbrace{\int_{-1}^{1} \underbrace{x}_{\text{odd}} \underbrace{\tfrac{1}{2}e^{-|x|}}_{\text{even}} dx}_{=0} = 0$$

$$
\begin{aligned}
VAR[Y] = E[Y^2] &= (-1)^2 P[Y \le -1] + (1)^2 P[Y \ge 1] \\
&\quad + \int_{-1}^{1} x^2 \tfrac{1}{2} e^{-|x|} dx \\
&= e^{-1} + \underbrace{2 \cdot \tfrac{1}{2} \int_0^1 x^2 e^{-x} dx}_{e^{-x}(x^2+2x+2)\Big|_0^1} \quad \text{from appendix} \\
&= e^{-1} + 5e^{-1} - 2 = 6e^{-1} - 2
\end{aligned}
$$

ⓒ $$E[Y] = \left(-\tfrac{1}{2}\right)\underbrace{P\left[X \le -\tfrac{1}{2}\right]}_{5/32} + \left(\tfrac{1}{2}\right)\underbrace{P\left[X \ge \tfrac{1}{2}\right]}_{5/32} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \underbrace{x}_{\text{odd}} \cdot \underbrace{\tfrac{3}{4}(1-x^2)}_{\text{even}} dx = 0$$

$$
\begin{aligned}
VAR[Y] = E[Y^2] &= \left(-\tfrac{1}{2}\right)^2 P\left[X \le -\tfrac{1}{2}\right] + \left(\tfrac{1}{2}\right)^2 P\left[X \ge \tfrac{1}{2}\right] \\
&\quad + \tfrac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2(1-x^2)\,dx \\
&= 2 \cdot \tfrac{1}{4} \cdot \tfrac{5}{32} + \tfrac{3}{4} \cdot 2 \underbrace{\int_0^{\frac{1}{2}} (x^2 - x^4)\,dx}_{\frac{x^3}{3} - \frac{x^5}{5}\Big|_0^{1/2}} = \frac{1}{24} - \frac{1}{160} \\
&= \frac{5}{64} + \frac{3}{2}\left(\frac{17}{480}\right) \\
&= 0.13125
\end{aligned}
$$

4.54 d



$u_+^3 = \frac{1}{2}$

$u_+ = \left(\frac{1}{2}\right)^{1/3} = .7937$

$u_-^3 = -\frac{1}{2}$

$u_- = \left(-\frac{1}{2}\right)^{1/2} = -.7937$

$$Y = \begin{cases} \frac{1}{2} & u^3 > \frac{1}{2} \\ -\frac{1}{2} & u^3 < -1/2 \\ u^3 & -\frac{1}{2} < u^3 < \frac{1}{2} \end{cases}$$

$$E[Y] = \underbrace{\underbrace{\tfrac{1}{2} P[Y \geq \tfrac{1}{2}]}_{\frac{1-(\frac{1}{2})^{1/3}}{2}} - \underbrace{\tfrac{1}{2} P[Y \leq \tfrac{1}{2}]}_{\frac{(-\frac{1}{2})^{1/3}-(-1)}{2}}}_{0} + \underbrace{\int_{-1/2}^{1/2} u^3 \frac{du}{2}}_{0}$$

$$VAR[Y] = E[Y^2]$$

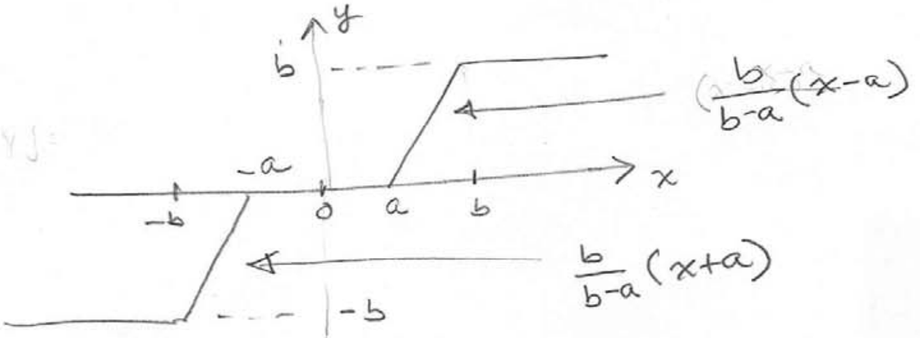
$$= \left(\tfrac{1}{2}\right)^2 P[Y \geq \tfrac{1}{2}] + \left(-\tfrac{1}{2}\right)^2 P[Y \leq \tfrac{1}{2}] + \int_{-1/2}^{1/2} u^6 \frac{du}{2}$$

$$= 2\left(\tfrac{1}{4}\right)\frac{\left(1-\left(\frac{1}{2}\right)^{1/3}\right)}{2} + \frac{2}{7}\left(\tfrac{1}{2}\right)^7$$

$$= \tfrac{1}{4}\left(1-\left(\tfrac{1}{2}\right)^{1/3}\right) + \tfrac{2}{7}\left(\tfrac{1}{2}\right)^7$$

4.55

ⓐ



$$E[Y] = -b\,P[X \le -b] + b\,P[X \ge b] + 0 \cdot P[-a \le X \le a] + \int_{-b}^{a} \frac{b}{b-a}(x+a) f_X(x)\,dx + \int_{a}^{b} \frac{b}{b-a}(x-a) f_X(x)\,dx$$

$$E[Y^2] = b^2\,P[X \le b] + b^2\,P[X \ge b] + \int_{-b}^{-a} \frac{b^2}{(b-a)^2}(x+a)^2 f_X(x)\,dx + \int_{a}^{b} \frac{b^2}{(b-a)^2}(x-a)^2 f_X(x)\,dx$$

$$VAR[Y] = E[Y^2] - E[Y]^2$$

ⓑ $f_X(x) = \frac{1}{2} e^{-|x|}\,dx \quad -\infty < x < \infty, \quad a = 1,\ b = 2$

$$E[Y] = -2\,\underbrace{P[X \le -2]}_{\frac{1}{2}e^{-2}} + 2\,\underbrace{P[X \ge 2]}_{\frac{1}{2}e^{2}} - \underbrace{+ \int_{-2}^{1} 2(x+1)\, e^{x}\,dx + \int_{1}^{2} 2(x-1)\, e^{-x}\,dx}_{=0}$$

$$E[Y^2] = 4 \cdot \frac{1}{2} e^{-2} + 4 \cdot \frac{1}{2} e^{-2} + 4\int_{-2}^{-1} (x+1)^2 e^{x}\,dx + 4\int_{1}^{2} (x-1)^2 e^{-x}\,dx$$

$$= 4e^{-2} + 8\int_{1}^{2} (x-1)^2 e^{-x}\,dx$$

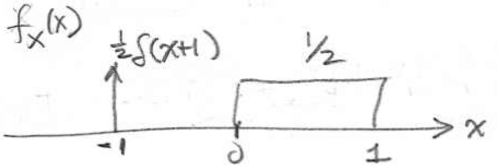
4.55

$$\int_1^2 \underbrace{(x-1)}_{y}^2 e^{-x}\,dx = \int_0^1 y^2 e^{-y-1}\,dy = e^{-1}\left[e^{-y}(x^2+2x+2)\right]_0^1$$

$$= e^{-1}\left[5e^{-1}-2\right] = 5e^{-2}-2e^{-1}$$

$$\therefore\ VAR[Y] = 4e^{-2} + 40e^{-2} - 16e^{-1} = 44e^{-2} - 16e^{-1}$$

ⓒ $a = \tfrac{1}{2}\quad b = \tfrac{3}{2}$ $\qquad P[X \le -\tfrac{3}{2}] = 0 = P[X \ge \tfrac{3}{2}]$



$$E[Y] = \int_{-3/2}^{-1/2} \frac{3}{2}\left(x+\frac{1}{2}\right)\frac{1}{2}\delta(x+1)\,dx + \int_{1/2}^{1} \frac{3}{2}\left(x-\frac{1}{2}\right)\cdot\frac{dx}{2}$$

$$= \frac{3}{4}\left(-1+\frac{1}{2}\right) + \frac{3}{4}\int_0^{1/2} x'\,dx' = -\frac{3}{8} + \frac{3}{4}\frac{1}{2}\left(\frac{1}{2}\right)^2 = -\frac{9}{32}$$

$$E[Y^2] = \int_{-3/2}^{-1/2} \frac{9}{4}\left(x+\frac{1}{2}\right)^2\frac{1}{2}\delta(x+1)\,dx + \frac{9}{4}\int_{1/2}^{1}\left(x-\frac{1}{2}\right)^2\frac{dx}{2}$$

$$= \frac{9}{4}\left(-\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + \frac{9}{8}\int_0^{1/2} x'^2\,dx' = \frac{21}{64}$$

$$VAR(Y) = \frac{21}{64} - \left(\frac{9}{32}\right)^2 = \frac{255}{1024}$$

ⓓ

$$Y = \begin{cases} \frac{1}{2} & \mu^3 > \frac{1}{2} \\ 2\left(\mu^3 - \frac{1}{4}\right) & \frac{1}{4} < \mu^3 < \frac{1}{2} \\ 0 & -\frac{1}{4} < \mu^3 < \frac{1}{4} \\ 2\left(\mu^3 + \frac{1}{4}\right) & -\frac{1}{2} < \mu^3 < -\frac{1}{4} \\ -\frac{1}{2} & \mu^3 < -\frac{1}{2} \end{cases}$$

(4.55d) – continued –

$$E[Y] = \underbrace{\tfrac{1}{2}P[U^3 > \tfrac{1}{2}] - \tfrac{1}{2}P[U^3 < -\tfrac{1}{2}]}_{=0}$$

$$\underbrace{+ \underbrace{\int_{-1/2}^{-1/4} 2(u^3+\tfrac{1}{4})\frac{du}{2}}_{\text{let } u'=-u} + \int_{1/4}^{1/2} 2(u^3-\tfrac{1}{4})\frac{du}{2}}_{=0}$$

$$E[Y]=0$$

$$VAR[Y] = E[Y^2] = (\tfrac{1}{2})^2 P[U^3 > \tfrac{1}{2}] + (-\tfrac{1}{2})^2 P[U^3 < -\tfrac{1}{2}]$$

$$+ \int_{-1/2}^{-1/4} 4(u^3+\tfrac{1}{4})^2\frac{du}{2} + \int_{1/4}^{1/2} 4(u^3-\tfrac{1}{4})^2\frac{du}{2}$$

$$= 2(\tfrac{1}{2})^2 \frac{1-(\frac{1}{2})^{1/3}}{2} + \cancel{2}\cdot 4\int_{1/4}^{1/2} \underbrace{(u^3-\tfrac{1}{4})^2}_{u^6-\frac{1}{2}u^3+\frac{1}{16}}\frac{du}{\cancel{2}}$$

$$= \frac{1-(\frac{1}{2})^{1/3}}{1/4} + 4\underbrace{\int_{1/4}^{1/2} u^6\,du}_{\frac{u^7}{7}\Big|_{1/4}^{1/2}} - 2\underbrace{\int_{1/4}^{1/2} u^3\,du}_{\frac{u^4}{4}\Big|_{1/4}^{1/2}} + \frac{1}{4}\underbrace{\int_{1/4}^{1/2} du}_{1/4}$$

$$= \frac{1-(\frac{1}{2})^{1/3}}{1/4} + \frac{4}{7}\left[(\tfrac{1}{2})^7 - (\tfrac{1}{4})^7\right] - \frac{1}{2}\left[(\tfrac{1}{2})^4 - (\tfrac{1}{4})^4\right] + \frac{1}{16}$$

$$= 4(.7937) + \frac{4}{7}(\tfrac{1}{2})^7\left[1-(\tfrac{1}{2})^7\right] - \frac{1}{2}(\tfrac{1}{2})^4\left[1-(\tfrac{1}{2})^4\right] + \frac{1}{16}$$

$$= 3,212$$

4.56

a) $E[Y] = 3E[X] + 2$

$VAR[Y] = VAR[3X+2] = VAR[3X] = 9\,VAR[X]$

b) Laplacian R.V. $E[X] = 0$

$VAR[X] = \frac{2}{\alpha^2}$

$E[Y] = 2$

$VAR[Y] = 9\left(\frac{2}{\alpha^2}\right) = \frac{18}{\alpha^2}$

c) Caussian R.V. $E[X] = m$

$VAR[X] = \sigma^2$

$E[Y] = 3m + 2$

$VAR[Y] = 9\sigma^2$

d) $E[X] = b\int_0^1 \cos(2\pi u)\,du = -b\sin(2\pi u)\Big|_0^1 = 0$

$$\begin{aligned} VAR[X] &= b^2\int_0^1 \cos^2(2\pi u)\,du \\ &= b^2\int_0^1 \tfrac{1}{2}du + \frac{b^2}{2}\int_0^1 \cos 4\pi u\,du \\ &= b^2\frac{1}{2} + b^2\left(\frac{1}{4\pi}\right)\left(-\sin 4\pi u\right)\Big|_0^1 \\ &= \frac{b^2}{2} \end{aligned}$$

$E[Y] = 2$

$VAR[Y] = \frac{9b^2}{2}$

4.57

$$\begin{aligned} \mathcal{E}[X^n] &= \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \\ \mathcal{E}[Y^n] &= \frac{1}{b-a}\int_a^b y^n dy = \frac{1}{b-a}\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right] \end{aligned}$$

4.58

a)
$$F_X(x \mid d \le x \le 2d) = \begin{cases} 0 & x \le d \\ \dfrac{F_X(x) - F_X(d)}{F_X(2d) - F_X(d)} & d \le x \le 2d \\ 1 & x \ge 2d \end{cases}$$

$$f_X(x \mid d \le x \le 2d) = \frac{f_X(x)}{F_X(2d) - F_X(d)} \qquad d \le x \le 2d$$

$$f_X(x) = \frac{1}{2x_{max}} \qquad -x_{max} \le x \le x_{max}$$

$$F_X(x) = \frac{x + x_{max}}{2x_{max}} \qquad -x_{max} \le x \le x_{max}$$

$$f_X(x \mid d \le x \le 2d) = \frac{1}{2d + x_{max} - d - x_{max}} = \frac{1}{d}$$

b)
$$E[X \mid d \le X \le 2d] = \int_{-\infty}^{\infty} x f_X(x \mid d \le x \le 2d)\,dx = \int_d^{2d} \frac{x}{d}\,dx$$
$$= \frac{1}{d}\frac{x^2}{2}\Big|_d^{2d} = \frac{1}{d}\left(\frac{4d^2}{2} - \frac{d^2}{2}\right) = \frac{3d}{2}$$

$$VAR[X \mid d \le X \le 2d] = \int_{-\infty}^{\infty} x^2 f_X(x \mid d \le x \le 2d)\,dx - E^2[X \mid d \le X \le 2d]$$
$$= \int_d^{2d} \frac{x^2}{d}\,dx - \left(\frac{3d}{2}\right)^2$$
$$= \frac{1}{d}\frac{x^3}{3}\Big|_d^{2d} - \frac{9d^2}{4} = \frac{7d^2}{3} - \frac{9d^2}{4} = \frac{d^2}{12}$$

c)
$$E[(X-c)^2 \mid d < X < 2d] = \int_d^{2d} \frac{x^2}{d}\,dx - 2c\int_d^{2d} \frac{x}{d}\,dx + c^2\int_d^{2d} \frac{1}{d}\,dx$$
$$= \frac{7d^2}{3} - 2c\left(\frac{3d}{2}\right) + c^2\left(\frac{1}{d}\right)d = c^2 - 3cd + \frac{7}{3}d^2$$

d) $2c - 3d + 0 = 0$

$c = \dfrac{3d}{2}$ it is the midpoint of the interval $(d, 2d)$

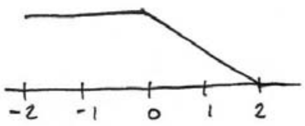
4,4 Important Continuous Random Variables

4.59

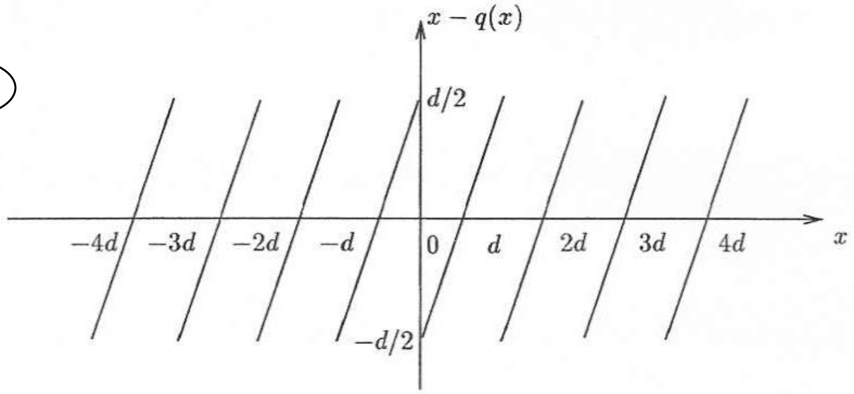
$$\begin{aligned} P[|X|>x] &= P[\{X>x\}\cup\{X<-x\}] \\ &= P[X>x] + P[X<-x] \\ &= 1 - F_X(x) + F_X(-x) \end{aligned}$$

$$F_X(x) = \begin{cases} 0 & x\le -2 \\ \frac{x+2}{4} & -2\le x\le 2 \\ 1 & x\ge 2 \end{cases}$$

$$P[|X|>x] = \begin{cases} 1 & x\le 0 \\ 1-\left(\frac{x+2}{4}\right)+\left(\frac{-x+2}{4}\right) = 1-\frac{x}{2} & 0\le x\le 2 \\ 0 & x\ge 2 \end{cases}$$



4.60



for $-\frac{d}{2} < y < \frac{d}{2}$ the equation $y = x - q(x)$ has 8 roots, thus from Eqn. 3.55:

$$f_Y(y) = \sum_{k=1}^{8} \frac{f_X(x_k)}{\frac{dy}{dx}\big|_{x=x_k}}$$

Since $x - q(x)$ consists of piecewise linear unit-slope segments, we have that $\left.\frac{dy}{dx}\right|_{x=x_k} =$ 1 all x_k.

Thus

$$f_Y(y) = \sum_{k=1}^{9} f_X(x_k) = \sum_{k=1}^{8} \frac{1}{8d} = \frac{1}{d}$$

for $-\frac{d}{2} < y < \frac{d}{2}$ √

4.61

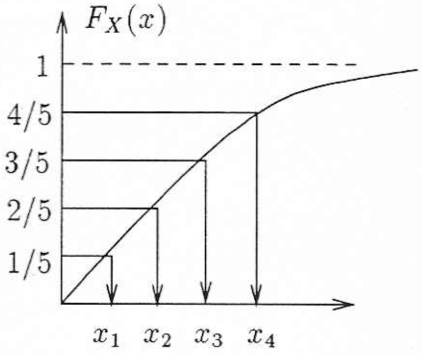
~~3.18~~ **a)** $P[X \le d] = F_X(d) = 1 - e^{-\lambda d} \qquad d > 0$

$$\begin{aligned} P[kd \le X \le (k+1)d] &= F_X((k+1)d) - F_X(kd) \\ &= e^{-\lambda kd} - e^{-\lambda(k+1)d} = e^{-\lambda kd}(1 - e^{-\lambda d}) \end{aligned}$$

$P[X > kd] = 1 - P[X \le kd] = 1 - F_X(kd) = e^{-\lambda kd}$

b) Find x_k, $k = 1, 2, 3, 4$ such that

$$\begin{aligned} F_X(x_k) &= \frac{k}{5} \\ &= 1 - e^{-\lambda x_k} \end{aligned}$$



$$x_1 = \frac{\ln \frac{5}{4}}{\lambda} \qquad x_2 = \frac{\ln \frac{5}{3}}{\lambda} \qquad x_3 = \frac{\ln \frac{5}{2}}{\lambda} \qquad x_4 = \frac{\ln 5}{\lambda}$$

4.62 a)

$$\begin{aligned} P[X \le x] &= 1 - e^{-\lambda x} = \frac{r}{100} \\ 1 - \frac{r}{100} &= e^{-\lambda x} \\ \Rightarrow \pi(r) &= x = -\frac{1}{\lambda}\ln\left(1 - \frac{r}{100}\right) = \frac{1}{\lambda}\ln\left(\frac{100}{100-r}\right) \end{aligned}$$

$$\pi(90) \cong \frac{23}{\lambda} \qquad \pi(95) \approx \frac{3}{\lambda} \qquad \pi(99) \approx \frac{4.6}{\lambda}$$

b) $P[X \le x] = 1 - Q\left(\frac{x}{\sigma}\right)$ Using Tables 4.2 and 4.3:

$$\begin{aligned} 1 - Q\left(\frac{x}{\sigma}\right) &= 0.90 \quad \Rightarrow \tfrac{x}{\sigma} = 1.28 \quad \Rightarrow \pi(90) = 1.28\sigma \\ 1 - Q\left(\frac{x}{\sigma}\right) &= 0.95 \quad \Rightarrow \tfrac{x}{\sigma} \approx 1.5 \quad \Rightarrow \pi(95) \approx 1.5\sigma \\ 1 - Q\left(\frac{x}{\sigma}\right) &= 0.99 \quad \Rightarrow \tfrac{x}{\sigma} \approx 2.33 \quad \Rightarrow \pi(99) \approx 2.33\sigma \end{aligned}$$

4.63

a) $P[X>4] = 1-F_X(4) = 1-\Phi\left(\frac{4-5}{4}\right) = 1-\Phi\left(-\frac{1}{4}\right) = \Phi\left(\frac{1}{4}\right) = 0.598$

$P[X\geq 7] = 1-F_X(7) = 1-\Phi\left(\frac{7-5}{4}\right) = 1-\Phi\left(\frac{1}{2}\right) = 0.308$

$P[6.72<X<10.16] = \Phi\left(\frac{10.16-5}{4}\right)-\Phi\left(\frac{6.72-5}{4}\right) = \Phi(1.29)-\Phi(0.43) = 0.235$

$P[2<X<7] = \Phi\left(\frac{7-5}{4}\right)-\Phi\left(\frac{2-5}{4}\right) = \Phi\left(\frac{1}{2}\right)-\Phi\left(-\frac{3}{4}\right) = 0.465$

$P[6\leq X\leq 8] = \Phi\left(\frac{8-5}{4}\right)-\Phi\left(\frac{6-5}{4}\right) = \Phi\left(\frac{3}{4}\right)-\Phi\left(\frac{1}{4}\right) = 0.175$

b) $P[X<a] = 0.8869$

$\Phi\left(\frac{a-5}{4}\right) = 0.8869 = 1-Q(x)$

$Q(x) = 0.1131 \rightarrow x = 1.2 = \frac{a-5}{4} \rightarrow a = 9.8$

c) $P[X>b] = 1-\Phi\left(\frac{b-5}{4}\right) = 0.11131$

$Q(x) = 0.11131 \rightarrow x = 1.2 = \frac{b-5}{4} \rightarrow b = 9.8$

d) $P[13<X\leq c] = 0.0123$

$\Phi\left(\frac{c-5}{4}\right)-\Phi\left(\frac{13-5}{4}\right) = \Phi\left(\frac{c-5}{4}\right)-\Phi(2) = 0.0123$

$\Phi\left(\frac{c-5}{4}\right) = 0.0123+0.9772 = 0.9895$

$Q\left(\frac{c-5}{4}\right) = 0.0105 \longrightarrow x = 2.3 = \frac{c-5}{4} \longrightarrow c = 14.2$

4.64

$$\begin{aligned}
Q(-x) &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{t^2/2}dt = 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x} e^{-t^2/2}dt \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{\infty}^{x} e^{-t'^2/2}(-dt') \text{ where } t'=-t \\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-t'^2/2}dt' = 1-Q(x)
\end{aligned}$$

4.65) To generate Q values

```
> x = [0: 0.1: 10];
> format short e
> 1-normal_cdf(x)
```

To generate $Q(x_k) = 10^{-k}$

$$10^{-k} = Q(x_k) = 1 - Q(-x) = F_X(-x_k)$$

$$x_k = -F_X^{-1}(10^{-k})$$

```
> k = [1: 1: 10];
> p = ones(1,10)/10;
> p2 = p.^k;
> -normal_inv(p2)
```

4.66

$$P[X < m] = P[X \le m] = \Phi\left(\frac{n-m}{\sigma}\right) = \Phi(0) = \frac{1}{2}$$

$$\begin{aligned} P[|X-m| > k\sigma] &= 1 - P[-k\sigma + m \le X \le m + k\sigma] \\ &= 1 - \left(\Phi\left(\frac{m+k\sigma-m}{\sigma}\right) - \Phi\left(\frac{m-k\sigma-m}{\sigma}\right)\right) \\ &= \underbrace{1-\Phi(k)} + \underbrace{\Phi(-k)} \\ &= \quad Q(k) \quad + \quad Q(k) \quad = 2Q(k) \end{aligned}$$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	from 4.2
$2Q(k)$	0.318	$4.56(10^{-2})$	$4.05(10^{-3})$	$6.34(10^{-5})$	$5.74(10^{-7})$	Table ~~3.3~~

$$P[X > m + k\sigma] = Q\left(\frac{m+k\sigma-m}{\sigma}\right) = Q(k)$$

	$k=1.28$	$k=3.09$	$k=4.26$	$k=5.20$	from 4.3
$Q(k)$	$\approx 10^{-1}$	$\approx 10^{-3}$	$\approx 10^{-5}$	$\approx 10^{-7}$	Table ~~3.4~~

4.67

a) $F_Y(X+N \leq y \mid X=+1) = F_N(y-1)$

$F_Y(X+N \leq y \mid X=-1) = F_N(y+1)$

$f_Y(y \mid X=+1) = f_N(y-1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-1)^2/2\sigma^2}$

$f_Y(y \mid X=-1) = f_N(y+1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y+1)^2/2\sigma^2}$

b) $f_Y(y \mid X=-1) P[X=-1] > f_Y(y \mid X=+1) P[X=+1]$ decide "0"

$\frac{e^{-(y+1)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}(3p_1) > \frac{e^{-(y-1)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} p_1$

$3e^{-(y+1)^2/2\sigma^2} e^{(y-1)^2/2\sigma^2} > 1$

$3e^{-y^2-2y-1+y^2-2y+1/2\sigma^2} > 1$

$3e^{-4y/2\sigma^2} > 1$

$\frac{-4y}{2\sigma^2} > \ln\left(\frac{1}{3}\right)$

$y < -\frac{\sigma^2}{2}\ln\left(\frac{1}{3}\right)$ decide "0" $T = -\frac{\sigma^2}{2}\ln\left(\frac{1}{3}\right)$

c) $P[X+N<T \mid X=+1] = P[N<T-1] = \Phi\left(\frac{T-1}{\sigma^2}\right)$

$P[X+N \geq T \mid X=-1] = P[N \geq T+1] = \left(1-\Phi\left(\frac{T+1}{\sigma^2}\right)\right)$

d) $P[X+N<T \mid X=+1] P[X=+1] + P[X+N \geq T \mid X=-1] P[X=-1] =$

$= p_1 \Phi\left(\frac{T-1}{\sigma^2}\right) + \left(1-\Phi\left(\frac{T+1}{\sigma^2}\right)\right) 3p_1$

4.68

$$\begin{aligned}
P[X_1 > x] &= Q\left(\frac{x-20}{5}\right) \\
P[X_2 > x] &= Q\left(\frac{x-22}{1}\right) \\
P[X_1 > 20] &= Q\left(\frac{20-20}{5}\right) = Q(0) = \frac{1}{2} \\
P[X_2 > 20] &= Q\left(\frac{20-22}{1}\right) = Q(-2) = 1 - Q(2) = 0.9722
\end{aligned}$$

} pick #2

$$\begin{aligned}
P[X_1 > 24] &= Q\left(\frac{24-20}{5}\right) = Q(\overset{0.8}{\cancel{1}}) = 0.\cancel{159}.212 \\
P[X_2 > 24] &= Q\left(\frac{24-22}{1}\right) = Q(2) = 0.023
\end{aligned}$$

} pick #1

Note that in the second case, the chip with the smaller mean (but larger variance) is selected.

4.69

$$\begin{aligned}
P[X > 10] &= 1 - P[X \le 10] \\
&= \sum_{k=0}^{7-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= \sum_{k=0}^{6} \frac{10^k}{k!} e^{-10} \\
&= 0.1301
\end{aligned}$$

(4.70) Gamma RV

ⓐ $$E[X] = \int_0^\infty x \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx = \int_0^\infty \frac{\lambda(\lambda x)^{\alpha} e^{-\lambda x}}{\lambda\,\Gamma(\alpha)} dx$$

$$= \int_0^\infty \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+1)/\Gamma(\alpha)}{\lambda} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda x} dx}_{1}$$

$$= \frac{\alpha}{\lambda}$$

ⓑ $$E[X^2] = \int_0^\infty x^2 \frac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} dx = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \frac{1}{\lambda^2} \underbrace{\int_0^\infty \frac{\lambda(\lambda x)^{\alpha+1}}{\Gamma(\alpha+2)} e^{-\lambda x} dx}_{1}$$

$$= \frac{(\alpha+1)(\alpha)}{\lambda^2}$$

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha)$$

$$VAR[X] = \frac{(\alpha+1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\alpha}{\lambda^2}.$$

ⓒ m-Erlang $\alpha = m$

$$E[X] = \frac{m}{\lambda} \qquad VAR[X] = \frac{m}{\lambda^2}$$

ⓓ chi-square $\alpha = k/2$ $\lambda = \frac{1}{2}$

$$E[X] = \frac{k}{2} \frac{1}{1/2} = k$$

$$VAR[X] = \frac{k}{2} \frac{1}{1/4} = 2k$$

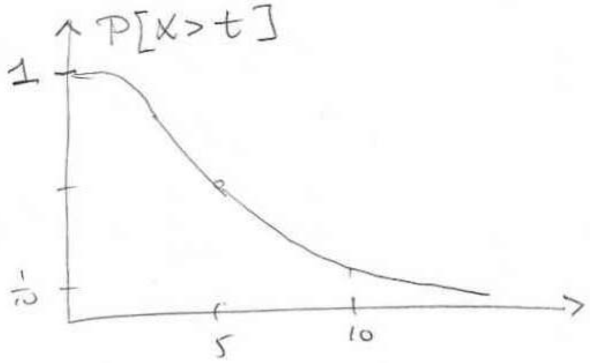
4.71 Gamma RV

$$m = 4 = \frac{\alpha}{\lambda} \qquad \frac{\sigma^2}{m} = 2 = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{2}$$

$$\sigma^2 = 8 = \frac{\alpha}{\lambda^2} \qquad \alpha = m\lambda = 4 \cdot \frac{1}{2} = 2.$$

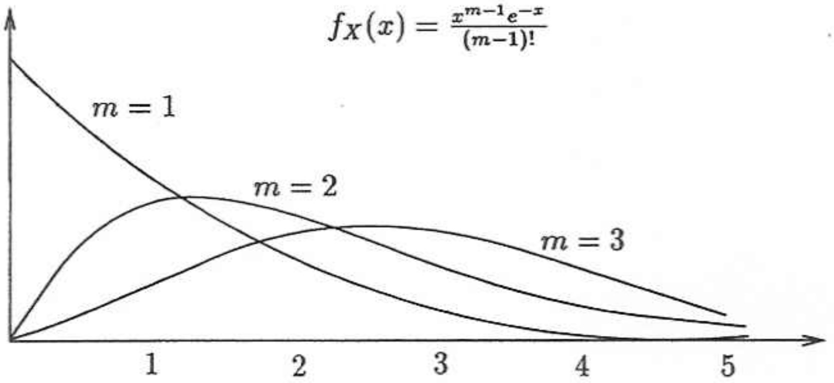
Note: Gamma pdf frequently stated in terms of α and $\beta = 1/\lambda$.
Octave uses this latter notation

```
                                   α
                                      β
> x=[0: 0.1: 5];
> plot (1- gamma_cdf(x, 2, 2))
```

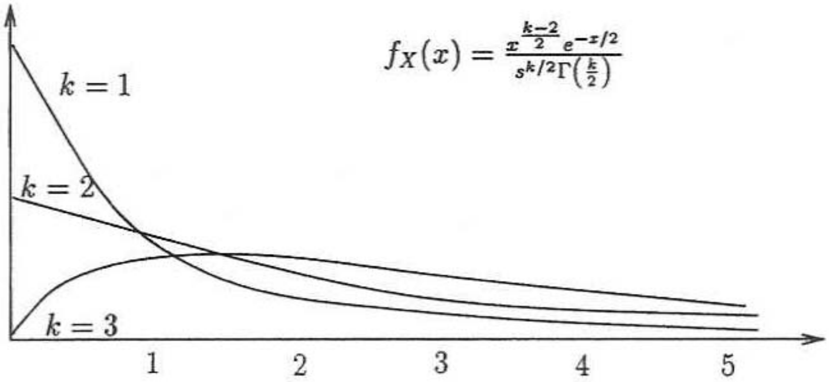


4.72 Use gamma_cdf with appropriate adjustments for m-Erlang and chi-square RV's.

(a)



(b)



4.73

$$P[X \geq 15] = \sum_{k=0}^{4-1} \frac{\left(\frac{1}{3}15\right)^k}{k!} e^{-\frac{15}{3}}$$

$$= \sum_{k=0}^{3} \frac{5^k}{k!} e^{-5}$$

$$= 0.2650$$

4.74

~~3.50~~ a) The cdf is given by the integral

$$F_X(x) = \frac{\lambda^{m-1}}{(m-1)!}\int_0^x y^{m-1}\lambda e^{-\lambda y}dy,$$

since $\Gamma(m)(m-1)!$. Integrate by parts using $u = y^{m-1}$ and $dv = \lambda e^{-\lambda y}dy$ so that $du = (m-1)y^{m-2}dy$ and $v = -e^{-\lambda y}$:

$$\begin{aligned} F_X(x) &= \frac{\lambda^{m-1}}{(m-1)!}\left\{-y^{m-1}e^{-\lambda y}\Big|_0^x - \int_0^x -(m-1)y^{m-2}e^{-\lambda y}dy\right\} \\ &= -\frac{(\lambda x)^{m-1}}{(m-1)!}e^{-\lambda x} + \frac{\lambda^{m-2}}{(m-2)!}\int_0^x y^{m-2}\lambda e^{-\lambda y}dy. \end{aligned}$$

The integral on the righ-hand side is identical to the equation for $F_X(x)$ with $m-1$ replaced by $m-2$. We can therefore repeatedly perform integration by parts to obtain the cdf of X:

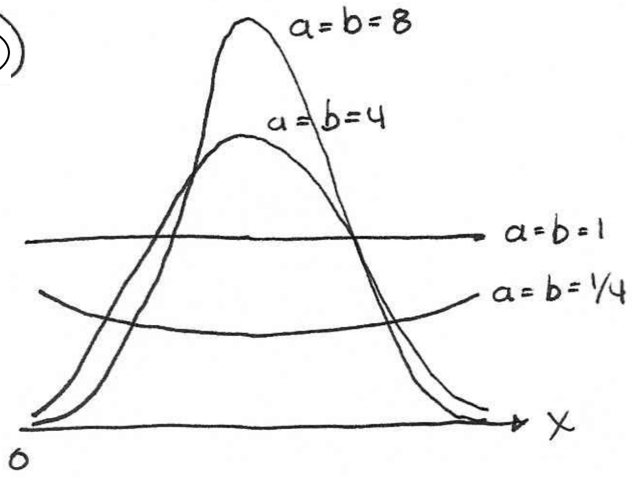
$$\begin{aligned} F_X(x) &= -\frac{(\lambda x)^{m-1}}{(m-1)!}e^{-\lambda x} - \frac{(\lambda x)^{m-2}}{(m-2)!}e^{-\lambda x} - \ldots + \int_0^x e^{-\lambda y}dy \\ &= -\frac{(\lambda x)^{m-1}}{(m-1)!}e^{-\lambda x} - \frac{(\lambda x)^{m-2}}{(m-2)!}e^{-\lambda x}\ldots - e^{-\lambda x} + 1 \\ &= 1 - \sum_{k=0}^{m-1}\frac{(\lambda x)^k}{k!}e^{-\lambda x}. \end{aligned}$$

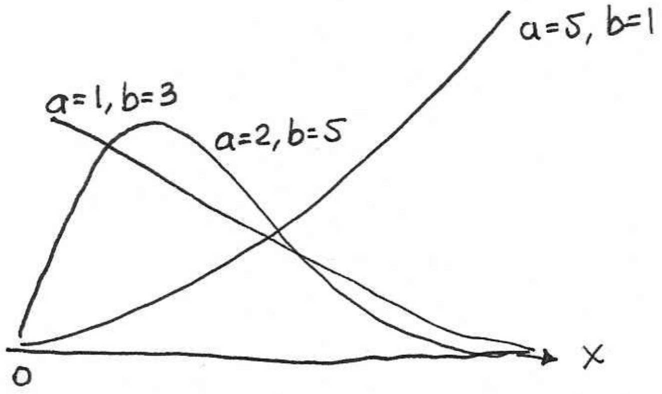
b) From part a)

$$F_X(x) = 1 - \sum_{k=0}^{m-1}\frac{(\lambda x)^k}{k!}e^{-\lambda x} = \frac{\lambda^{m-1}}{(m-1)!}\int_0^x y^{m-1}\lambda e^{-\lambda y}dy$$

$$f_X(x) = \frac{df_X(x)}{dx} = \frac{\lambda(\lambda y)^{m-1}e^{-\lambda y}}{(m-1)!}$$

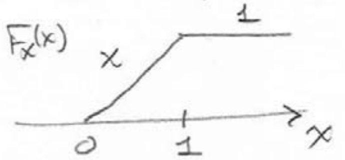
4.75

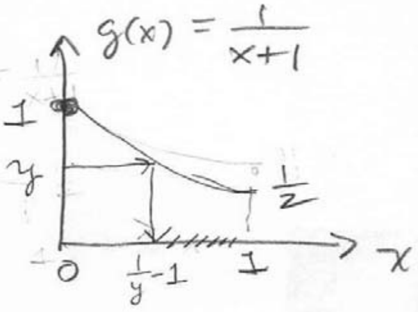




4.5 Functions of a Random Variable

4.76 X unif in $[0,1]$.





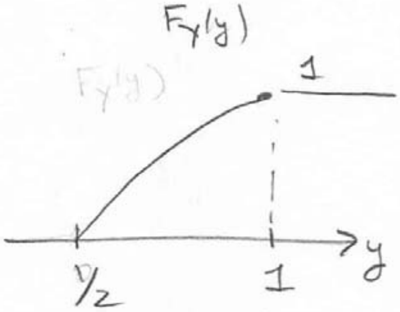
$$P[Y \leq y] = P\left[\frac{1}{X+1} \leq y\right] \qquad \frac{1}{2} < y < 1$$

$$= P\left[\frac{1}{y} \leq X+1\right]$$

$$= P\left[X \geq \frac{1}{y} - 1\right]$$

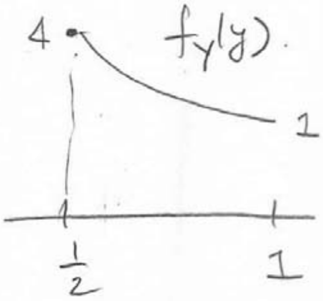
$$= 1 - \left(\frac{1}{y} - 1\right)$$

$$= 2 - \frac{1}{y}$$



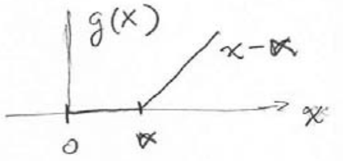
$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= +\frac{1}{y^2} \qquad \frac{1}{2} < y < 1$$

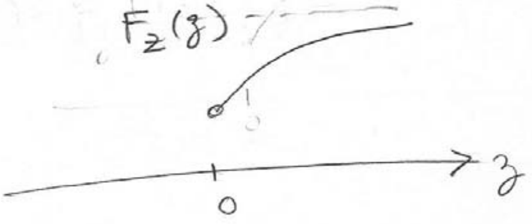


4.77

ⓐ $$z = (X-\alpha)^+ = \begin{cases} 0 & X \le \alpha \\ X-\alpha & X > \alpha \end{cases}$$



$$P[Z \le z] = \begin{cases} 0 & z<0 \\ P[X \le r] = 1 - e^{-\alpha^2/2\alpha^2} = 1-e^{-1/2} & z=0 \\ P[X-\alpha \le z] = P[X \le z+\alpha] & z>0 \\ \quad = 1 - e^{-(z+\alpha)^2/2\alpha^2} & \end{cases}$$



$$f_Z(z) = \frac{d}{dz} F_Z(z) = (1-e^{-1/2})\,\delta(z) + \frac{d}{dz} F_X(z+\alpha)$$

$$= (1-e^{-1/2})\,\delta(z) + f_X(z+\alpha)$$

$$= (1-e^{-1/2})\,\delta(z) + \frac{z+\alpha}{\alpha^2} e^{-(z+\alpha)^2/2\alpha^2}$$

ⓑ $$P[Z \le z] = P[X^2 \le z] = P[X \le \sqrt{z}]$$ since X is nonnegative

$$f_Z(z) = \frac{d}{dx} F_X(\sqrt{z}) = f_X(\sqrt{z})\, \frac{1}{2} z^{-1/2}$$

$$= \frac{f_X(\sqrt{z})}{2\sqrt{z}} = \frac{\sqrt{z}}{2\sqrt{z}} \frac{1}{\alpha^2} e^{-\sqrt{z}^2/2\alpha^2} = \frac{1}{2\alpha^2} e^{-z/2\alpha^2}$$

exponential RV.

4.78

$P[N=0] = F_X(\pi)$

$P[N=1] = F_X(2\pi) - F_X(\pi)$

$\vdots$

$$
\begin{aligned}
P[N=n] &= F_X((n+1)\pi) - F_X(n\pi) \\
&= 1 - e^{-\lambda(n+1)\pi} - (1 - e^{-\lambda n\pi}) \\
&= (e^{-\lambda n\pi} - e^{-\lambda(n+1)\pi}) \\
&= e^{-\lambda n\pi}(1 - e^{-\lambda\pi}) \qquad \lambda = \frac{1}{5\pi} \\
&= e^{-\frac{n\pi}{5\pi}}\left(1 - e^{-\frac{\pi}{5\pi}}\right) \\
&= (1 - e^{-1/5})(e^{-1/5})^n \qquad n = 0, 1, 2, \ldots
\end{aligned}
$$

geometric RV

4.79

a) $P[\{X<-4d\} \cup \{X>4d\}] = 0.01$

$$P[X<-4d] + P[X>4d] = \Phi(-4d) + 1 - \Phi(4d)$$
$$= 1 - (1-Q(4d)) + Q(4d)$$
$$= 2Q(4d)$$

$$Q(4d) = 0.005 \rightarrow 4d = 2.57 \rightarrow d = 0.6425$$

b) $P[0<X<d] = F_X(d) - F_X(0) = Q(0) - Q(0.64) = 0.5 - 0.258$
$= 0.242$

$P[d<X<2d] = F_X(2d) - F_X(d) = Q(0.64) - Q(1.28) = 0.258 - 0.0994$
$= 0.1586$

$P[2d<X<3d] = F_X(3d) - F_X(2d) = Q(1.28) - Q(1.92) = 0.0994 - 0.0273$
$= 0.0721$

$P[3d<X<4d] = Q(3d) - Q(4d) = Q(1.92) - Q(2.57) = 0.0273 - 0.005$
$= 0.0223$

4.80

$Y = 2X + 3$

$$F_Y(y) = P[2X+3 \le y] = P\left[X \le \frac{y-3}{2}\right]$$

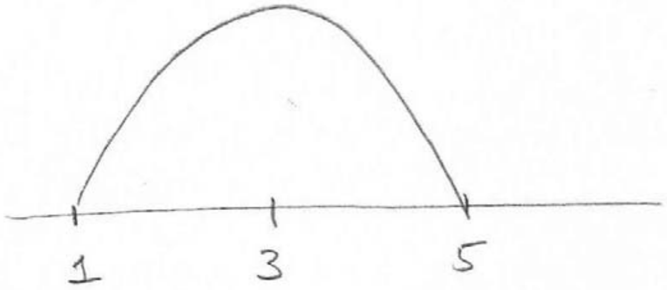
$$= F_X\left(\frac{y-3}{2}\right)$$

$$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-3}{2}\right)$$

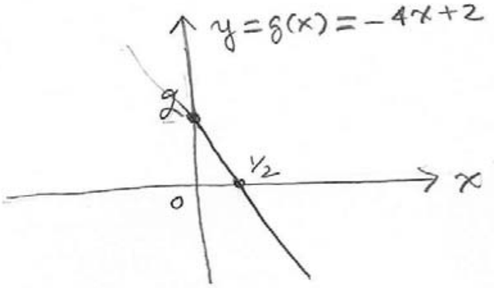
$$= \frac{1}{2} f_X\left(\frac{y-3}{2}\right)$$

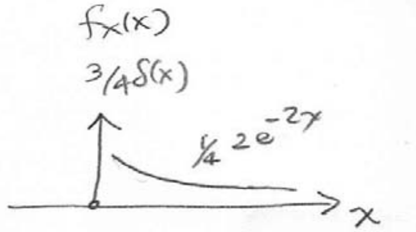
$$f_X(x) = \frac{3}{4}(1-x^2) \qquad -1 < x < 1$$

$$\Rightarrow f_Y(y) = \frac{3}{8}\left(1 - \left(\frac{y-3}{2}\right)^2\right) \qquad 1 < y < 5$$



4.81



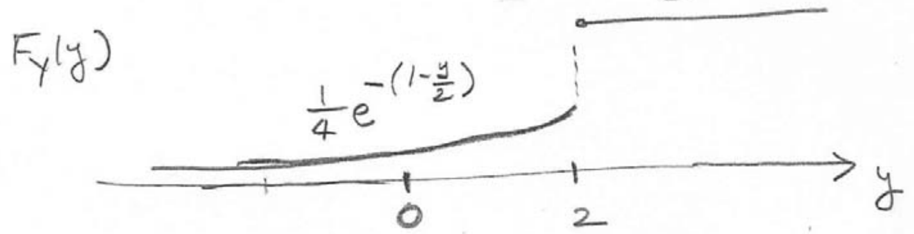


$$F_X(x) = \frac{3}{4} + \frac{1}{4}(1-e^{-2x})$$

$$F_Y(y) = P[-4X+2 \le y] = P[-4X \le y-2]$$

$$= P\left[X \ge \frac{2-y}{4}\right] \qquad -\infty < y < 2$$

$$= \begin{cases} P[X \ge 0] = 1 & y=2 \\ P\left[X \ge \frac{2-y}{4}\right] = \frac{1}{4}e^{-2\left(\frac{2-y}{4}\right)} & y<2 \end{cases}$$

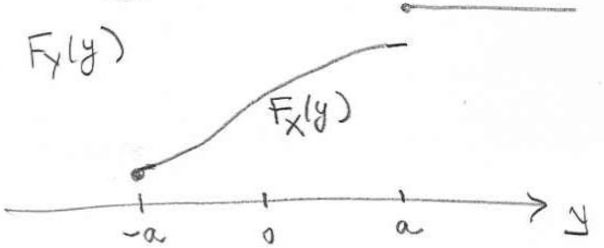


$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{4}e^{-(1-\frac{y}{2})}\left(\frac{1}{2}\right) + \frac{3}{4}\delta(y-2)$$

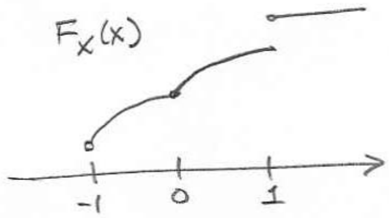
$$= \frac{1}{8}e^{-(1-y/2)} + \frac{3}{4}\delta(y-2)$$

4.82

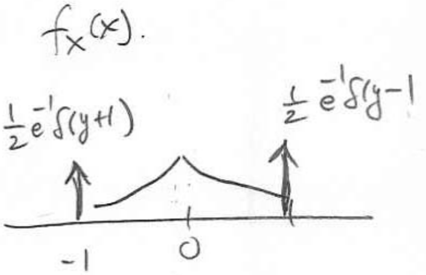
$$F_Y(y) = \begin{cases} 0 & y<-a \\ F_X(-a) = P[X\le -a] & y=-a \\ F_X(y) = P[X\le y] & -a<y<a \\ 1 & y=+a \end{cases}$$



ⓐ X Laplacian $\lambda = a = 1$



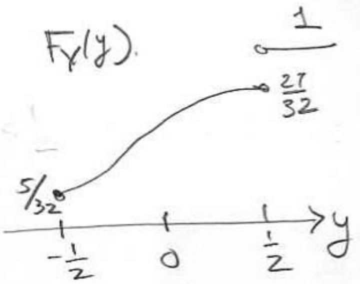
$$F_Y(y) = \begin{cases} 0 & y<-1 \\ \frac{1}{2}e^{-1} & y=-1 \\ \frac{1}{2}e^{x} & -1\le y\le 0 \\ \frac{1}{2}+\frac{1}{2}(1-e^{-x}) & 0\le y\le 1 \\ 1 & y\ge 1 \end{cases}$$



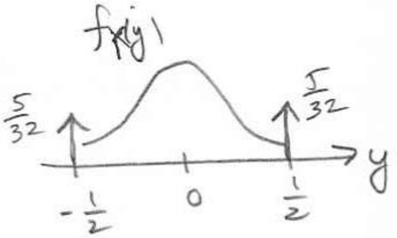
$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-1}\,\delta(y+1) & y=-1 \\ \frac{1}{2}e^{-|x|} & -1<y<1 \\ \frac{1}{2}e^{-1}\,\delta(y-1) & y=+1 \end{cases}$$

(c)

$$F_Y(y) = \begin{cases} 0 & y < -1/2 \\ \frac{5}{32} & y = -\frac{1}{2} \\ \frac{3}{4}\left[(x+1) - \frac{1}{3}(x^3+1)\right] & -\frac{1}{2} < y < \frac{1}{2} \\ 1 & y = 1 \end{cases}$$

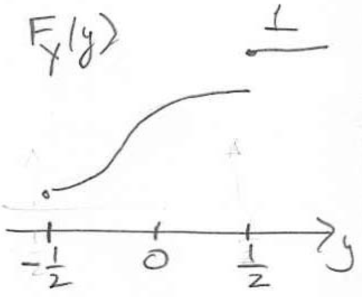


$$f_Y(y) = \begin{cases} \frac{5}{32}\delta(y+\frac{1}{2}) & y = -\frac{1}{2} \\ \frac{3}{4}(1-y^2) & -\frac{1}{2} < y < \frac{1}{2} \\ \frac{5}{32}\delta(y-\frac{1}{2}) & y = \frac{1}{2} \end{cases}$$

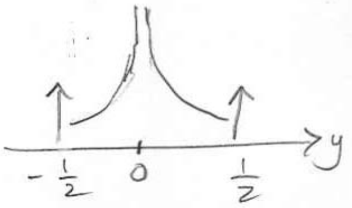


(d)

$$F_Y(y) = \begin{cases} 0 & y < -\frac{1}{2} \\ (1-(\frac{1}{2})^{1/3})/2 & y = -\frac{1}{2} \\ P[U^3 \le y] = \frac{y^{1/3}-(-1)}{2} & -\frac{1}{2} < y < \frac{1}{2} \\ 1 & y = \frac{1}{2} \end{cases}$$

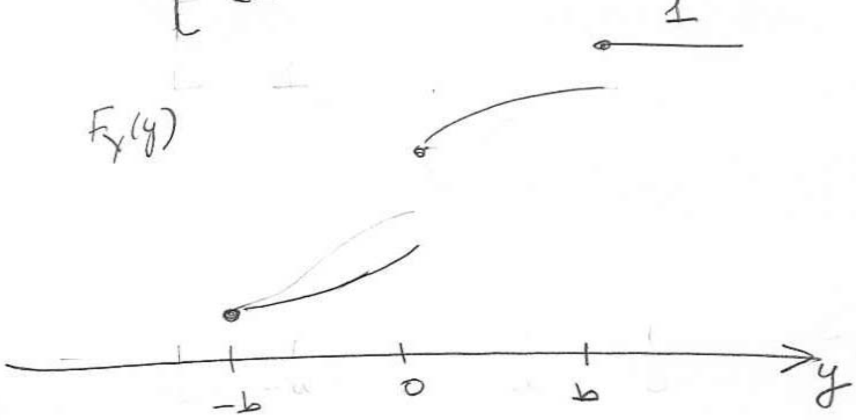


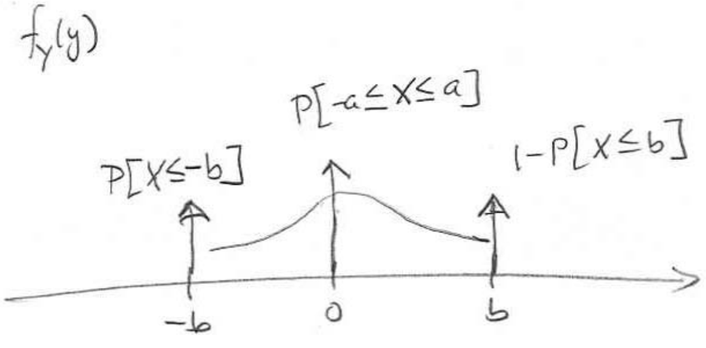
$$f_Y(y) = \begin{cases} (1-(\frac{1}{2})^{1/3})/2\,\delta(y+\frac{1}{2}) & y = -\frac{1}{2} \\ \frac{1}{6y^{2/3}} & -\frac{1}{2} < y < \frac{1}{2} \\ (1-(\frac{1}{2})^{1/3})/2\,\delta(y-\frac{1}{2}) & y = \frac{1}{2} \end{cases}$$



4.83a

$$F_Y(y) = \begin{cases} 0 & y < -b \\ P[X \le -b] & y = -b \\ P\left[\frac{b}{b-a}(X+a) \le y\right] = F_X\left(\frac{b-a}{b}y - a\right) & -b < y < 0 \\ P[X \le a] & y = 0 \\ P\left[\frac{b}{b-a}(X-a) \le y\right] = F_X\left(\frac{b-a}{b}y + a\right) & 0 < y < b \\ 1 & y = b \end{cases}$$





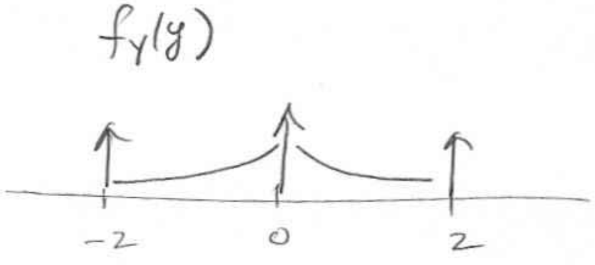
4.83b) $a=1,\ b=2$

$$P[X \le -2] = \frac{1}{2}e^{-2} = P[X \ge 2]$$

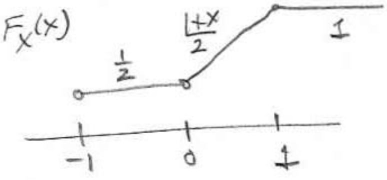
$$P[-1 \le X \le 1] = 2P[0 \le X \le 1] = 2\cdot\frac{1}{2}(1-e^{-1})$$

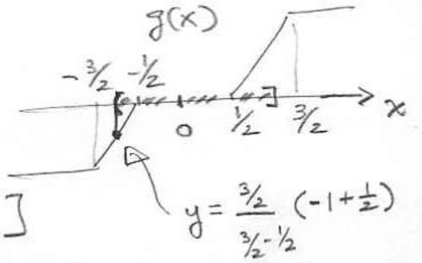
$$F_Y(y) = \begin{cases} 0 & y < -2 \\ \frac{1}{2}e^{-2} & y = -2 \\ \frac{1}{2}e^{\frac{y}{2}-1} & -2 < y < 0 \\ \frac{1}{2} + \frac{1}{2}(1-e^{-1}) & y = 0 \\ \frac{1}{2} + \frac{1}{2}(1-e^{-(\frac{y}{2}+1)}) & 0 < y < 2 \\ 1 & y = 2 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-2}\delta(y+2) & y = 2 \\ \frac{1}{4}e^{\frac{y}{2}-1} & -2 < y < 0 \\ (1-e^{-1})\delta(y) & y = 0 \\ \frac{1}{4}e^{-(\frac{y}{2}+1)} & 0 < y < 2 \\ \frac{1}{2}e^{-2}\delta(y-2) & y = 2 \end{cases}$$



4.83c Recall

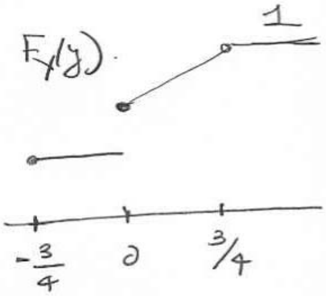




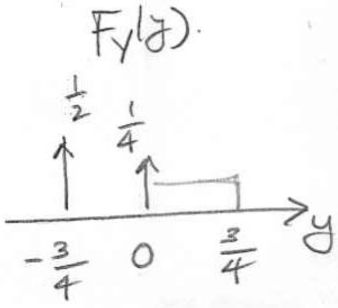
$$P[X \le -b] = P[X \le -\tfrac{3}{2}] = 0 = P[X \ge \tfrac{3}{2}]$$

$$P[-\tfrac{1}{2} \le X \le \tfrac{1}{2}] = P[0 \le X \le \tfrac{1}{2}] = \tfrac{1}{4}$$

$$F_Y(y) = \begin{cases} 0 & y < -\frac{3}{4} \\ \frac{1}{2} & y = -\frac{3}{4} \\ \frac{1}{2} & -\frac{3}{4} \le y < 0 \\ \frac{3}{4} & y = 0 \\ F_X\left(\frac{2}{3}y + \frac{1}{2}\right) = \dfrac{1 + \frac{2}{3}y + \frac{1}{2}}{2} & 0 < y \le \frac{3}{4} \\ \quad = \frac{1}{3}y + \frac{3}{4} & \end{cases}$$



$$f_Y(y) = \begin{cases} \frac{1}{2}\delta(y + \frac{3}{4}) & y = -3/4 \\ \frac{1}{4}\delta(y) & y = 0 \\ \frac{1}{3} & 0 < y \le \frac{3}{4} \\ 0 & \text{elsewhere} \end{cases}$$



4.84

$$F_Y(y) = F_X\left(\frac{y-2}{3}\right)$$

$$f_Y(y) = \frac{1}{3} f_X\left(\frac{y-2}{3}\right)$$

- X is Laplacian

$$F_Y(y) = \begin{cases} \frac{1}{2} e^{\alpha\left(\frac{y-2}{3}\right)} & y \le 2 \\ 1 - \frac{1}{2} e^{-\alpha\left(\frac{y-2}{3}\right)} & y \geq 2 \end{cases} . \qquad f_Y(y) = \frac{1}{3}\frac{\alpha}{2} e^{-\alpha\left|\frac{y-2}{3}\right|}$$

- X is Gaussian

$$F_Y(y) = \Phi\left(\frac{\frac{y-2}{3} - m}{\sigma}\right) = \Phi\left(\frac{y-(2+3m)}{3\sigma}\right)$$

$$f_Y(y) = \frac{1}{3\sigma\sqrt{2\pi}} e^{-\left(\frac{y-2}{3}-m\right)^2/2\sigma^2} = \frac{1}{3\sigma\sqrt{2\pi}} e^{-(y-(2+3m))^2/2(3\sigma)^2}$$

- $X = b\cos(2\pi U)$

$$F_Y(y) = \begin{cases} 0 & y < -3b+2 \\ \frac{1}{\pi}\sin^{-1}\left(\frac{y-2}{3b}\right) + \frac{1}{\pi}\sin^{-1}\left(-\frac{1}{b}\right) & 2-3b \le y \le 3b+2 \\ 1 & y > 3b+2 \end{cases}$$

$$f_Y(y) = \frac{1}{3}\frac{1}{\pi b\sqrt{1-\left(\frac{y-2}{3b}\right)^2}} \qquad 2-3b \le y < 3b+2$$

4.85 X : Gaussian, $Y = aX + b$, a linear combination of X.
Y is also Gaussian

$$E[Y] = aE[X] + b = am + b = m'$$

$$Var[Y] = a^2 Var[X] = a^2\alpha^2 = \alpha'^2$$

$$a = \alpha'/\alpha, \;\; b = m' - am = m' - m\alpha'/\alpha$$

4.86

$$f_X(x) = \sum_k f_U(u) \left| \frac{1}{n} x^{\frac{1}{n}-1} \right| \Big|_{x=x_k}$$

$$\cdot\ f_X(x) = f_U(\sqrt[n]{x}) \left(\frac{1}{n} x^{\frac{1}{n}-1} \right)$$

$$= \frac{1}{n} x^{\frac{1}{n}-1} \qquad 0 \leq x \leq 1$$

$$F_X(x) = \int_0^x \frac{1}{n} x^{\frac{1}{n}-1}\, dx = x^{\frac{1}{n}} \Big|_0^x = x^{\frac{1}{n}} \qquad 0 \leq x \leq 1$$

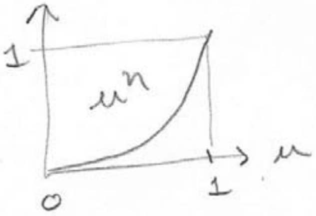
$$= \begin{cases} 0 & x < 0 \\ x^{1/n} & x \geqslant 0 \end{cases}$$

Alternatively we could start with the cdf:

$$F_X(x) = P[U^n \leq x]$$

$$= P[U \leq x^{1/n}]$$

$$= x^{1/n}$$

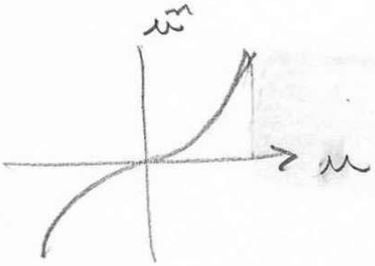


$$0 \leq x \leq 1$$

4.87

$X = U^n$ n odd

$F_X(x) = P[U^n \le x] = P[U \le x^{1/n}]$

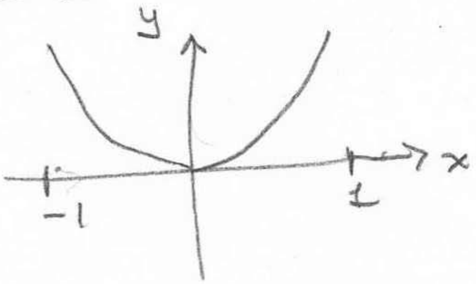


$= \frac{1}{2}(x^{1/n} - (-1))$

$= \frac{1}{2}(x^{1/n} + 1)$

$f_X(x) = \frac{1}{2}\frac{1}{n}x^{\frac{1}{n}-1} \quad -1 \le x \le 1$

$X = U^n$ n even



$F_X(x) = P[U^n \le x]$

$= P[-x^{1/n} \le U \le x^{1/n}]$

$= 2(\frac{1}{2}x^{1/n}) \quad 0 \le x \le 1$

$= x^{1/n}$

$f_X(x) = \frac{1}{n}x^{1/n - 1}$

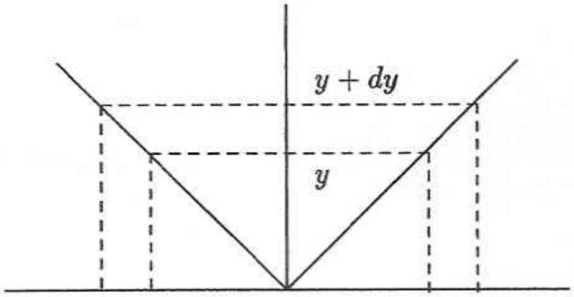
4.88 **a)** The equivalent event for $\{Y \le y\}$ is $\{|X| \le y\}$, therefore:

$$\begin{aligned} F_Y(y) &= P[|X| \le y] = P[-y \le X \le y] \\ &= \begin{cases} 0 & y < 0 \\ F_X(y) - F_X(y^-) & y \ge 0 \end{cases} \end{aligned}$$

Assuming X is a continuous random variable,

$$f_Y(y) = F'_Y(y) = f_X(y) + f_X(-y) \quad \text{for } y > 0 .$$

b) The equivalent event for $\{dy < Y \le y + dy\}$ is shown below:



Therefore

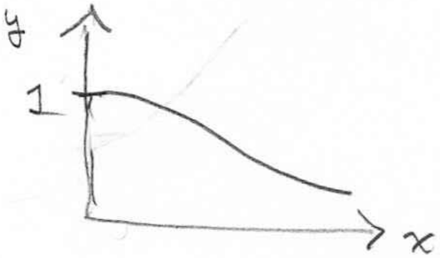
$$\begin{aligned} P[y < Y \le y + dy] &= P[y < X \le y + dy] \\ &\quad + P[-y - dy < X \le -y] \end{aligned}$$

$$\begin{aligned} \Rightarrow f_Y(y)dy &= f_X(y)dy + f_X(-y)|dy| \\ \Rightarrow f_Y(y) &= f_X(y) + f_X(-y) \qquad \text{for } y > 0 . \end{aligned}$$

c) If $f_X(x)$ is an even factor of x, then $f_X(x) = f_X(-x)$ and thus $f_Y(y) = 2f_X(y)$.

4.89

$Y = \frac{1}{(x+1)^2}$

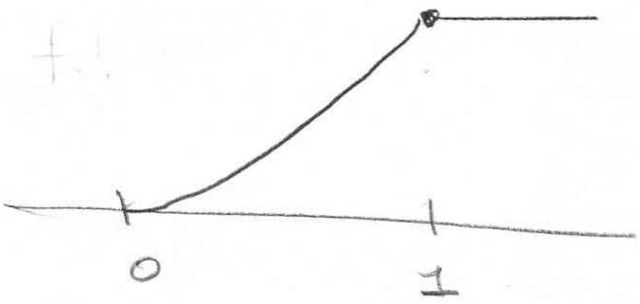


$$P[Y \le y] = P\left[\frac{1}{(x+1)^2} \le y\right]$$

$$= P\left[\frac{1}{y} \le (x+1)^2\right]$$

$$= P\left[\sqrt{\frac{1}{y}} - 1 \le X\right]$$

$$= e^{-\sqrt{1/y} - 1} \qquad 0 < y < 1$$



4.90

$$X = \pm\sqrt{\frac{P}{R}} \qquad \frac{dx}{dp} = \pm\frac{1}{2}\frac{p^{\frac{1}{2}-1}}{\sqrt{R}} = \pm\frac{1}{2\sqrt{RP}}$$

$$f_P(p) = \left[f_X(x) + f_X(-x)\right]\left|\frac{dx}{dP}\right|$$

$$= \left[f_X\left(\sqrt{\frac{P}{R}}\right) + f_X\left(-\sqrt{\frac{P}{R}}\right)\right] \frac{\cdot 1}{2\sqrt{RP}}$$

$$= \left[\frac{1}{2\sqrt{2\pi}}\left(e^{-\left(\sqrt{\frac{P}{R}}-1\right)^2/2(2)} + e^{-\left(-\sqrt{\frac{P}{R}}-1\right)^2/2(2)}\right)\right]\frac{1}{2\sqrt{RP}}$$

$$= \frac{1}{2\sqrt{2\pi}}\,\frac{1}{2\sqrt{RP}}\left(e^{-(\sqrt{P}-\sqrt{R})^2/2(2R)} + e^{-(\sqrt{P}-(-\sqrt{R}))^2/2(2R)}\right)$$

4.91

a) For $y \le 0$ $\quad P[Y \le y] = 0$

For $y > 0$ $\quad P[Y \le y] = P[e^X \le y] = P[X \le \ln y] = F_X(\ln y)$

$$\therefore \quad F_Y(y) = \begin{cases} 0 & y \le 0 \\ F_X(\ln y) & y > 0 \end{cases}$$

For $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = F_X'(\ln y)\frac{d}{dy}\ln y \\ &= \frac{1}{y}f_X(\ln y) \end{aligned}$$

b) If X is a Gaussian random variable, then

$$f_Y(y) = \begin{cases} 0 & y \le 0 \\ \dfrac{e^{-(\ln y - m)^2/2\sigma^2}}{y\sqrt{2\pi}\sigma} & y > 0 \end{cases}$$

4.92 $f_X(x) = 4x(1-x^2) \qquad 0 \le x \le 1$

ⓐ $Y = \pi X^2 \qquad \frac{dy}{dx} = 2\pi x_1 \qquad x_1 = \sqrt{\frac{y}{\pi}}$

$$f_Y(y) = \frac{f_X(x_1)}{|2\pi x_1|}$$

only a positive root since x is non-negative

$$= \frac{4\sqrt{\frac{y}{\pi}}\left(1-\frac{y}{\pi}\right)}{\left|2\pi\sqrt{\frac{y}{\pi}}\right|} = \frac{2}{\pi}\left(1-\frac{y}{\pi}\right) \qquad 0<y<\pi$$

ⓑ $Y = \frac{4}{3}\pi X^3$

$$x_1 = \left(\frac{3}{4\pi}y\right)^{1/3} \qquad \frac{dy}{dx} = 4\pi x^2$$

$$f_Y(y) = \frac{f_X(x_1)}{|4\pi x_1^2|} = \frac{4x_1(1-x_1^2)}{4\pi x_1^2} = \frac{1-x_1^2}{\pi x_1}$$

$$= \frac{1-\left(\frac{3}{4\pi}y\right)^{2/3}}{\pi\left(\frac{3}{4\pi}y\right)^{1/3}} \qquad 0<y<\frac{4}{3}\pi$$

ⓒ $Y = cX^n$

$$x_1 = \left(\frac{y}{c}\right)^{1/n} \qquad \frac{dy}{dx} = ncx^{n-1}$$

$$f_Y(y) = \frac{f_X(x_1)}{|ncx^{n-1}|} = \frac{4x_1(1-x_1^2)}{nc\,x_1^{n-1}} = \frac{4(1-x_1^2)}{nc\,x_1^{n-2}}$$

$$= \frac{4}{nc}\,\frac{\left(1-\left(\frac{y}{c}\right)^{2/n}\right)}{(y/c)^{\frac{n-2}{n}}}$$

4.93) Qzr defined on intervals

$(-\infty, -3d], (-3d, -2d], (-2d, -d], (-d, 0]$

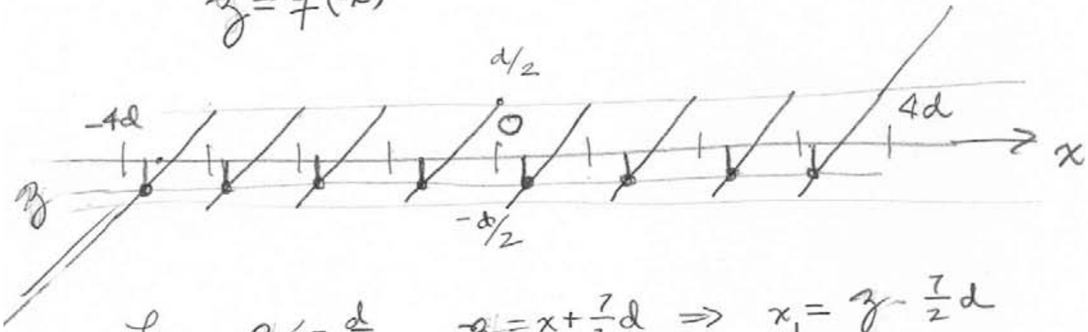
$(0, d], (d, 2d], (2d, 3d], (3d, \infty)$

for $(kd, (k+1)d)$ $\quad Z = X - (k+\frac{1}{2})d \qquad k = -3, \ldots, 2$

$(-\infty, -3d) \quad Z = X + \frac{7}{2}d$

$(3d, \infty) \quad Z = X - \frac{7}{2}d$

$z = q(x)$



for $z < -\frac{d}{2} \quad z = x + \frac{7}{2}d \Rightarrow x_1 = z - \frac{7}{2}d$

$z > \frac{d}{2} \quad z = x - \frac{7}{2}d \Rightarrow x_1 = z + \frac{7}{2}d$.

for $-\frac{d}{2} < z < \frac{d}{2}$ there are 8 solutions to

$z = q(x_i)$

$z = x_1 + \frac{7}{2}d \Rightarrow x_1 = z - \frac{7}{2}d \qquad x_5 = z + \frac{1}{2}d$

$x_2 = z - \frac{5}{2}d \qquad x_6 = z + \frac{3}{2}d$

$x_3 = z - \frac{3}{2}d \qquad x_7 = z - \frac{5}{2}d$

$x_4 = z - \frac{1}{2}d \qquad x_7 = z + \frac{7}{2}d$

also $\frac{dz}{dx} = 1$.

(4.93) continued → $f_X(x) = \frac{1}{2}\frac{2}{d} e^{-2|x/d|}$ $\quad \frac{1}{\lambda} = \frac{d}{2}$

for $-\frac{d}{2} < z < \frac{d}{2}$ the pdf of Z is

$$f_Z(z) = \sum_{i=1}^{8} \frac{f_X(x_i)}{|dz/dx|} = \underbrace{\sum_{i=1}^{4} f_X(x_i)}_{\text{negative roots}} + \underbrace{\sum_{i=5}^{8} f_X(x_i)}_{\text{positive roots}}$$

$$= \frac{1}{d}\left[e^{+\frac{2}{d}(z-\frac{7}{2}d)} + e^{+\frac{2}{d}(z-\frac{5}{2}d)} + e^{+\frac{2}{d}(z-\frac{3}{2}d)} + e^{+\frac{2}{d}(z-\frac{d}{2})} \right]$$

$$+ \frac{1}{d}\left[e^{-\frac{2}{d}(z+\frac{1}{2}d)} + e^{-\frac{2}{d}(z+\frac{3}{2}d)} + e^{-\frac{2}{d}(z+\frac{5}{2}d)} + e^{-\frac{2}{d}(z+\frac{7}{2}d)} \right]$$

$$= \frac{e^{2z}}{d}\left[e^{-7} + e^{-5} + e^{-3} + e^{-1} \right]$$

$$+ \frac{e^{-2z}}{d}\left[e^{-1} + e^{-3} + e^{-5} + e^{-7} \right]$$

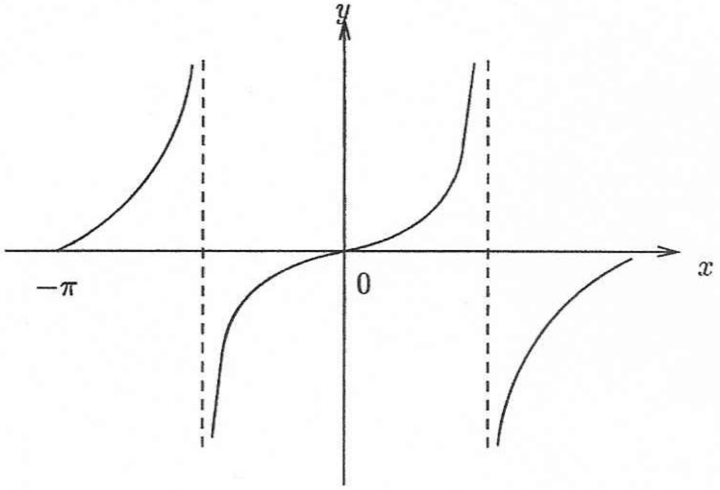
$$= 2\frac{e^{-2|z|}}{d}\left[e^{-1} + e^{-3} + e^{-5} + e^{-7} \right]$$

pdf for $z < -\frac{d}{2}$ and $z > \frac{d}{2}$ involve only one root.

④.94 $Y = a \tan X$.

$$x = \tan^{-1}(y/a), \ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\frac{dx}{dy} = \frac{1}{1 + (y/a)^2}\frac{1}{a} = \frac{a}{y^2 + a^2}$$



$$\begin{aligned} f_X(y) &= \sum_k f_X(x) \left|\frac{dx}{dy}\right| |_{x=x_k} \\ &= 2 \cdot \frac{1}{2\pi}\frac{a}{y^2 + a^2} \\ &= \frac{a/\pi}{y^2 + a^2} \end{aligned}$$

Y is a Cauchy RV.

4.95

$Y = \left(\frac{X}{\lambda}\right)^{\beta}$

$X = \lambda \sqrt[\beta]{Y}$

$$F_Y(y) = P\left[\left(\frac{X}{\lambda}\right)^{\beta} \le y\right] = P\left[X \le \lambda \sqrt[\beta]{y}\right]$$

$$= \begin{cases} 0 & y<0 \\ 1-e^{-\left(\frac{\lambda\sqrt[\beta]{y}}{\lambda}\right)^{\beta}} & y \geq 0 \end{cases}$$

$$= \begin{cases} 0 & y<0 \\ 1-e^{-y} & y \geq 0 \end{cases}$$

$$f_Y(y) = e^{-y} \qquad y \geq 0$$

4.96

$X = -\ln(1-U)$

$e^{-x} = (1-U)$

$U = 1-e^{-x}$

$$f_X(x) = f_U(u)\left|\frac{du}{dx}\right|$$

$$= f_U(1-e^{-x})\,|e^{-x}|$$

$$= e^{-x}$$

4.6 The Markov and Chebyshev Inequalities

4.97

a) $P[X>c] = 1-\frac{c}{b} \qquad 0\le c\le b$

$E[X] = \frac{b}{2}$

$1-\frac{c}{b} \overset{?}{\le} \frac{b}{2c}$

$c=0 \qquad 1\le\infty$ ✓

$c=b \qquad 0\le 1/2$ ✓

b) $P[X>c] = e^{-\lambda c} \qquad c\geq 0$

$E[X] = 1/\lambda$

$e^{-\lambda c} \overset{?}{\le} \frac{1}{\lambda c}$

$c=0 \qquad 1\le\infty$ ✓

$c\to\infty \qquad 0\le 0$ ✓

c) $P[X>c] = \frac{x_m^{\alpha}}{c^{\alpha}} \qquad c\geq x_m$

$E[X] = \frac{\alpha x_m}{\alpha-1} \qquad \alpha>1$

$\frac{x_m^{\alpha}}{c^{\alpha}} \overset{?}{\le} \frac{\alpha x_m}{\alpha-1}\frac{1}{c}$

$c=x_m \qquad 1\le\frac{\alpha}{\alpha-1}$ ✓ $\alpha>1$

$c\to\infty \qquad 0\le 0$ ✓

d) $P[X>c] = e^{-c^2/2\alpha^2} \qquad c\geq 0$

$E[X] = \alpha\sqrt{\pi/2} \qquad \alpha>0$

$e^{-c^2/2\alpha^2} \overset{?}{\le} \frac{\alpha\sqrt{\pi/2}}{c}$

$c=0 \qquad 1\le\infty$ ✓

$c\to\infty \qquad 0\le 0$ ✓

4.98

a) $P[X > c] = 1 - \left(\frac{c-1+1}{L-1+1}\right)$ $\quad c \in \{1, 2, \ldots, L\}$

$E[X] = \frac{L+1}{2}$

$1 - \frac{c}{L} \overset{?}{\leq} \frac{L+1}{2c} - \frac{1}{L}$

$c = 1$

$1 - \frac{1}{L} \leq \frac{L}{2} + \frac{L}{2} - \frac{1}{L}$ ✓ $L \geq 1$

$c = L$

$1 - \frac{L}{L} \leq \frac{L+1}{2L} - \frac{1}{L}$

$0 \leq \frac{1}{2} - \frac{1}{2L}$ ✓

b) $P[X > c] = (1-p)^{c+1}$ $\quad c \in \{0, 1, 2, \ldots\}$

$E[X] = \frac{1-p}{p}$

$(1-p)^{c+1} \overset{?}{\leq} \frac{1-p}{pc} - (1-p)^c p$

$(1-p)^c \leq \frac{1-p}{pc}$

$c = 0$ $\quad 1 \leq \infty$ ✓

$c \to \infty$ $\quad 0 \leq 0$ ✓

$c = 1$ $\quad 1-p \leq \frac{1-p}{p}$ ✓

c) $p_X[k] = \frac{1}{C_L}\frac{1}{k}$ $\quad k = 1, 2, \ldots, L$ $\qquad P[X > k] = 1 - \frac{C_k}{C_L}$

$E[X] = \frac{L}{C_L}$

$L = 10 \quad C_L = 2.929$

$L = 100 \quad C_L = 5.1874$

$1 - \frac{C_k}{C_L} \overset{?}{\leq} \frac{1}{k}\frac{L}{C_L}$

$C_L - C_k \leq \frac{L}{k}$

$L = 10,\ k = 1 \quad 1.929 \leq 10$ ✓

$k = 10 \quad 0 \leq 1$ ✓

$L = 100\ \ k = 1 \quad 4.1874 \leq 100$ ✓

$k = 100 \quad 0 \leq 1$ ✓

… 4.98

d) $P[X>c] = 1 - \sum_{j=0}^{c} \frac{n!}{j!(n-j)!} p^n$

$E[X] = np$

$P[X>c] \leq \frac{np}{c} - P[X=c]$

$1 - \sum_{j=0}^{c-1} \binom{n}{j} p^n \leq \frac{np}{c} \qquad 0 \leq c \leq n$

$n=10, \quad c=0 \qquad 1-(0.5)^{10} \leq \infty$ ✓

$c=1 \qquad 1-(0.5)^{10} \leq 5$ ✓

$c=10 \qquad 1-(0.5)^{10}(1023) \leq 0.5$ ✓

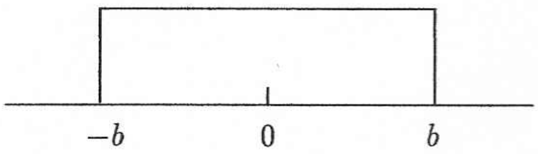
$n=100, \quad c=0 \qquad 1-(0.5)^{100} \leq \infty$ ✓

$c=1 \qquad 1-(0.5)^{100} \leq 50$ ✓

$c=100 \qquad 1-(0.5)^{100}(1.2677\times 10^{30}) \leq 0.5$ ✓

4.99

~~81~~ a) For a uniform random variable in $[-b, b]$ we have



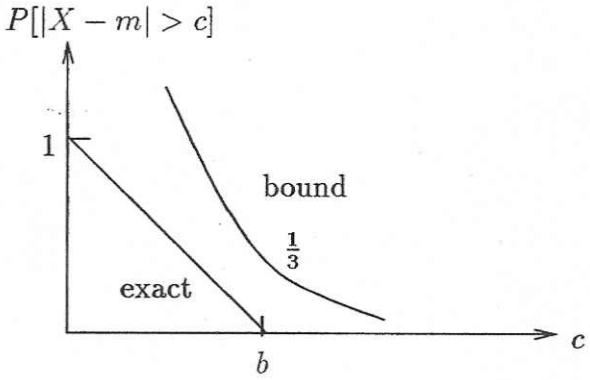
$\mathcal{E}[X] = 0$

Exact:

$$P[|X - m| > c] = \begin{cases} 1 - \frac{c}{b} & 0 \leq c \leq b \\ 0 & c > b \end{cases}$$

Chebyshev Bound gives

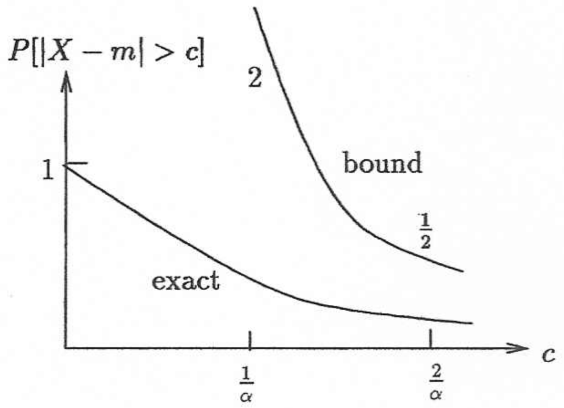
$$P[|X - m| > c] \leq \frac{\sigma_X^2}{c^2} = \frac{b^2}{3c^2}$$



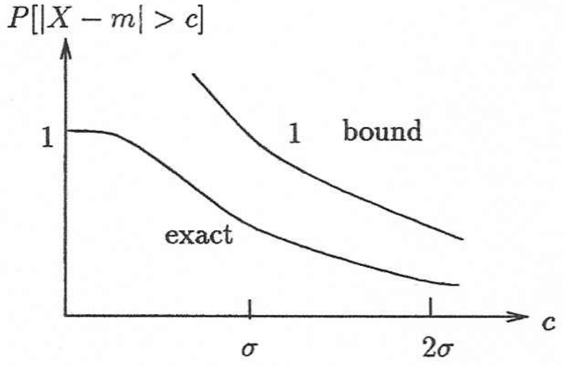
b) For the Laplacian random variable $\mathcal{E}[X] = 0$ and $VAR[X] = 2/\alpha^2$

Exact: $P[|X - m| > c] = P[|X| > c] = e^{-\alpha c}$

Bound: $P[|X - m| > c] \leq \dfrac{2}{\alpha^2 c^2}$



c) For the Gaussian random variable $\mathcal{E}[X]=0$ and $VAR[X]=\sigma^2$
Exact: $P[|X-m|>c]=2Q\left(\frac{c}{\sigma}\right)$
Bound: $P[|X-m|>c]\le\frac{\sigma^2}{c^2}$



ⓓ Binomial $n=10,\ p=\frac{1}{2}$ $n=50\ p=\frac{1}{2}$

$m=np=5\quad \sigma^2=npq=2.5$ $m=np=25\quad \sigma^2=npq=12.5$

$P[|X-5|\ge c]\le\frac{2.5}{c^2}$ $P[|X-25|>c]\le\frac{12.5}{c^2}$

c	$P[\lvert X-5\rvert\ge c]$	$\frac{2.5}{c^2}$
1	0.754	2.5
2	0.344	0.625
3	0.109	0.277
4	0.0215	0.156
5	0.0019	0.1

c	$P[\lvert X-25\rvert>c]$	$12.5/c^2$
5	0.119	0.5
10	0.0026	0.125
15	5.61×10^{-6}	0.0556
20	4.46×10^{-10}	0.03125

4.100

~~.82~~ $Y = X/n$

$$E[Y] = E[X]/n = np/n = p$$

$$VAR[Y] = VAR[X]/n^2 = npq/n^2 = pq/n, \; q = 1 - p$$

$$P[\{|Y - p|\} > a] \leq \frac{\sigma^2}{a^2} = \frac{pq}{na^2}$$

$$\text{as } n \to \infty \;\; P[\{|Y - p|\} > a] \to 0 \text{ for any fixed } a > 0$$

4.101

$$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E[Y] = \frac{1}{n}\sum_i E[X_i] = E[X]$$

$$Var[Y] = \frac{1}{n^2} Var[\sum_i X_i] = \frac{1}{n^2} \cdot \frac{n}{\lambda^2} = \frac{1}{n\lambda^2}$$

$$P[\{|Y - E[X]| > a\}] = P[\{|Y - E[Y]| > a\}]$$

$$\leq \frac{1}{n\lambda^2 a^2}$$

$$\text{as } n \to \infty \;\; P[\{|Y - E[X]| > a\}] \to 0$$

4.7 Transform Methods

4.102

$$\phi_X(w) = \int_{-\infty}^{\infty} f_X(x)e^{jwx}dx$$

$$\underset{-b}{=} \int_{\cancel{a}}^{b} \frac{1}{b-a}e^{jwx}dx$$

$$= \frac{e^{jwb} - e^{-jw\cancel{a}b}}{jw(b-\underset{-b}{\cancel{a}})} = \frac{e^{j\omega b} - e^{-j\omega b}}{j2\omega b}$$

$$E[X] = \frac{1}{j}\frac{d\phi_X(w)}{dw}\Big|_{w=0}$$

$$= -\frac{1}{b+b}\left[-\frac{1}{2}b^2 + \frac{1}{2}b^2\right]$$

$$= \frac{1}{2}(b+b) = 0$$

$$E[X^2] = \frac{1}{j^2}\frac{d^2\phi_x(w)}{dw^2}\Big|_{w=0}$$

$$= -\frac{1}{j(b-a)}\left[-\frac{1}{3}jb^3 + \frac{1}{3}ja^3\right]_{a=-b}$$

$$= \frac{1}{3}(b^2+ab+a^2) = \frac{b^2}{3}$$

$$VAR[X] = E[X^2] - E^2[X]$$

$$\cancel{= \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(b+a)^2}$$

$$\cancel{= \frac{1}{12}(b-a)^2}$$

$$= \frac{4b^2}{12} = \frac{b^2}{3}$$

4.103

$$\begin{aligned}\phi_X(w) &= \int_{-\infty}^{\infty} f_X(x)e^{jwx}dx \\ &= \int_{-\infty}^{0} \frac{\alpha}{2}e^{\alpha x}e^{jwx}dx + \int_{0}^{\infty} \frac{\alpha}{2}e^{-\alpha x}e^{jwx}dx \\ &= \frac{\alpha}{2}\frac{1}{\alpha + jw} + \frac{\alpha}{2}\frac{1}{\alpha - jw} \\ &= \frac{\alpha^2}{\alpha^2 + w^2}\end{aligned}$$

$$\begin{aligned}E[X] &= \frac{1}{j}\frac{d\phi_X(w)}{dw}|_{w=0} \\ &= \frac{1}{j}\cdot\frac{\alpha^2\cdot 2w}{-(\alpha^2+w^2)^2}|_{w=0} \\ &= 0\end{aligned}$$

$$\begin{aligned}E[X^2] &= \frac{1}{j^2}\frac{d\phi_X(w)}{dw^2}|_{w=0} \\ &= -\frac{\alpha^2}{j^2}\frac{2(\alpha^2+w^2)^2 - 2w\cdot 2(\alpha^2+w^2)\cdot 2w}{(\alpha^2+w^2)^4}|_{w=0} \\ &= \frac{2}{\alpha^2} \\ VAR\,[X] &= E[X^2] - E^2[X] = \frac{2}{\alpha^2}\end{aligned}$$

4.104

X exponential

$$\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$$

$$\Phi_X^n(\omega) = \left(\frac{\lambda}{\lambda - j\omega}\right)^n = \frac{1}{\left(1 - \frac{j\omega}{\lambda}\right)^n}$$

Corresponds to a Gamma RV, and specifically, an n-Erlang RV

4.105

$$\begin{aligned}
E[X] &= \frac{1}{j}\frac{d}{d\omega}e^{jm\omega-\sigma^2\omega^2/2}\bigg|_{\omega=0} \\
&= \frac{1}{j}(jm-\sigma^2\omega)e^{jm\omega-\sigma^2\omega^2/2}\bigg|_{\omega=0} \\
&= m \\
E[X^2] &= \frac{1}{j^2}\frac{d^2}{d\omega^2}e^{jm\omega-\sigma^2\omega^2/2}\bigg|_{\omega=0} \\
&= \frac{1}{j^2}\left[-\sigma^2 e^{jm\omega-\sigma^2\omega^2/2}+(jm-\sigma^2\omega)^2 e^{jm\omega-\sigma^2\omega^2/2}\right]_{\omega=0} \\
&= \frac{1}{j^2}[-\sigma^2+j^2m^2]=\sigma^2+m^2 \\
VAR[X] &= E[X^2]-\mathcal{E}[X]^2=\sigma^2
\end{aligned}$$

4.106

$$\begin{aligned}
\Phi_Y(\omega) &= E\left[e^{j\omega Y}\right] = E\left[e^{j\omega(aX+b)}\right] \\
&= E\left[e^{ja\omega X}\right]e^{j\omega b} \\
&= e^{j\omega b}\,\Phi_X(a\omega) \\
&= e^{j\omega b}\; e^{j\omega m-\frac{\sigma^2\omega^2}{2}}\bigg|_{\omega=a\omega} \\
&= \underbrace{e^{j\omega b}\; e^{j\omega a m-\frac{a^2\sigma^2\omega^2}{2}}}_{}
\end{aligned}$$

characteristic fcn for Gaussian RV
with mean $a\omega+b$
and variance $a^2\sigma^2$

4.107

91 We take the inverse transform of $e^{-|\omega|}$ to show that it corresponds to a Cauchy pdf:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\omega|}e^{j-\omega x}d(x) &= \frac{1}{2}\int_{-\infty}^{0} e^{\omega}e^{-j\omega x}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} e^{-\omega}e^{-j\omega}d\omega \\ &= \frac{1}{2\pi}\left[\frac{e^{\omega(1-jx)}}{1-jx}\right]_{-\infty}^{0} + \frac{1}{2\pi}\left[\frac{e^{-\omega(1+jx)}}{-(1+jx)}\right]_{0}^{\infty} \\ &= \frac{1}{2\pi}\left[\frac{1}{1-jx}+\frac{1}{1+jx}\right] = \frac{1}{\pi(1+x^2)} \quad \checkmark\end{aligned}$$

4.108

$$P[X \ge a] \le e^{-sa}E[e^{sX}]$$

$$E[e^{sX}] = \int_0^\infty e^{sx}e^{-x}dx = \int_0^\infty e^{-(1-s)x}dx = \frac{1}{1-s}$$

$$P[X\ge a] \le \min_{s\ge 0} \frac{e^{-sa}}{1-S}$$

$$0 = \frac{d}{ds}\frac{e^{-sa}}{1-s} = \frac{-ae^{-sa}}{1-s} + \frac{e^{-sa}}{(1-s)^2}$$

$$+a(1-s) = 1 \Rightarrow +a \cancel{-} as = 1$$

$$\Rightarrow s = \frac{a-1}{a} \qquad s \ge 0 \Rightarrow a \ge 1.$$

$$\Rightarrow P[X>a] \le \frac{e^{-(a-1)}}{1-\frac{a-1}{a}} = ae^{-(a-1)} = aee^{-a}$$

Exact probability:

$$P[X \ge a] = e^{-a}$$

4.109

$$\begin{aligned}
G_X(z) &= \frac{p}{1-qz} \\
E[X] &= \frac{d}{dz}G_X(z)|_{z=1} = \frac{p}{(1-qz)^2}(q)\Big|_{z=1} = \frac{pq}{(1-q)^2} = \frac{q}{p} \\
E[X^2]-E[X] &= \frac{d^2}{dz^2}G_X(z)\Big|_{z=1} = \frac{pq}{(1-qz)^3}2q\Big|_{z=1} = \frac{2pq^2}{(1-q)^3} = \frac{2q^2}{p^2}
\end{aligned}$$

4.110

$$\begin{aligned}
G_N(z) &= \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}z^k \\
&= \sum_{k=0}^{n}\binom{n}{k}(pz)^k(1-p)^{n-k} \\
&= [pz+(1-p)]^n \quad \text{from Binomial Thm.}
\end{aligned}$$

$E[N]=G_N'(z)\Big|_{z=1} = n(pz+q)^{n-1}p\Big|_{z=1} = np$

$E[N^2]-E[N] = G_N''(z)\Big|_{z=1} = n(n-1)[\cdot]p^2\Big|_{z=1}$

$= (n^2-n)p^2$

$VAR[N] = n^2p^2 - np^2 + np - (np)^2 = np(1-p)$ ✓

4.111

$G_X(z) = (pz+q)^n$

$G_Y(z) = (pz+q)^m$

$G_X(z)G_Y(z) = (pz+q)^n(pz+q)^m = (pz+q)^{n+m}$

corresponds to pgf of Binomial RV with parameters $(n+m)$ and p.

this is a legitimate pgf.

4.112

$G_N(z) = e^{\alpha(z-1)}$ $G_M(z) = e^{\beta(z-1)}$

$G_N(z)\,G_M(z) = e^{(\alpha+\beta)(z-1)}$

corresponds to pgf of a Poisson RV with $(\alpha+\beta)$

this is a legitimate pgf.

4.113

$$P[X=k] = \frac{\alpha^k}{k!}e^{-\alpha}$$

$$E[e^{sX}] = \sum_{k=0}^{\infty} \frac{e^{sk}\alpha^k}{k!}e^{-\alpha} = e^{-\alpha}\sum_{k=0}^{\infty}\frac{(\alpha e^s)^k}{k}$$

$$= e^{-\alpha}e^{(\alpha e^s)}$$

$$P[X \geq a] \leq \min_{s \geq 0} e^{-sa}e^{-\alpha(1-e^s)} =$$

$$= \min_{s \geq 0} e^{-(sa+\alpha(1-e^s))}$$

We minimize the exponential by maximizing the exponent

$$0 = \frac{d}{ds}\left(sa + \alpha(1-e^s)\right) = a - \alpha e^s \Rightarrow a = \alpha e^s$$

$$s = \ln\left(\frac{a}{\alpha}\right) \qquad s \geq 0 \text{ for } a \geq \alpha$$

$$P[X \geq a] \leq e^{-a\ln\frac{a}{\alpha}}e^{-\alpha(1-\frac{a}{\alpha})} = e^{(a-\alpha-a\ln\frac{a}{\alpha})}$$

for $\alpha = 1$, $a = 5$ $\quad P[X \geq 5] \leq e^{(5-1-5\ln 5)}$

$$= e^{4-5\ln 5}$$

$$= .01747$$

Exact Probability

$$P[X \geq 5] = 1 - \sum_{k=0}^{4}\frac{(1)^k}{k!}e^{-1}$$

$$= 0,00366$$

4.114

a) $G_U(z) = E[z^U] = \sum_{k=0}^{\infty} p_U(k) z^k = \sum_{k=a}^{b} \frac{1}{b-a+1} z^k$

$= \frac{z^a}{b-a+1}\left[\sum_{k=0}^{b-a} z^k\right] \qquad = \frac{z^a}{b-a+1}\left[\frac{1-z^{b+1-a}}{1-z}\right]$

$= \frac{1}{b-a+1}\left[\frac{z^a - z^{b+1}}{1-z}\right]$

b) $E[U] = G_U'(1) = \frac{1}{b-a+1} \frac{1}{(1-z)^2}\left[z^a(az^{-1}-a+1) - z^b(b+1-zb)\right]\Big|_{z=1}$

$\lim_{z \to 1} G_U'(z) = \frac{1}{(b-a+1)(2)(1-z)(-1)} \cdot \left[az^{a-2}(-1+a) + az^{a-1}(1-a) + bz^b(1+b) - bz^{b-1}(b+1)\right.$

$\ldots = \frac{a(a-1)z^{a-2} - b(1+b)z^{b-1}}{(b-a+1)(-1)(2)}$

$= \frac{a(a-1) - b(1+b)}{-2(b-a+1)} = \frac{b+a}{2}$

$VAR[U] = G_U''(1) + G_U'(1) - (G_U'(1))^2$

$\lim_{z \to 1} G_U''(1) = \frac{a(a-1)(a-2) - b(b-1)(b+1)}{-3(b-a+1)}$

$VAR[U] = \frac{b(b-1)(b+1) - a(a-1)(a-2)}{3(b-a+1)} + \frac{b+a}{2} - \left(\frac{b+a}{2}\right)^2$

$= \frac{(b-a+1)^2 - 1}{12}$

c) $G_U(z)^2$ corresponds to a pgf

$G_V(z) = G_U(z)^2$

$E[V] = G_V'(1) = 2G_U'(1)G_U(1) = 2G_U'(1)$

$= b + a$

4.115

~~5~~ The negative Binomial random variable has

$$\begin{aligned} G_X(z) &= \left(\frac{pz}{1-qz}\right)^r \\ P[X=r] &= \frac{1}{r!}\frac{d^r}{dz^r}G_X(z)\bigg|_{z=0} = \frac{1}{r!}\frac{d^r}{dz^r}\left(\frac{pz}{1-pz}\right)^r \\ &= \frac{1}{r!}\frac{d^{r-1}}{dz^{r-1}}r\left(\frac{pz}{1-qz}\right)^{r-1}\frac{p}{(1-qz)^2} \end{aligned}$$

First, consider the $r=2$ negative Binomial random variable:

$$\begin{aligned} P[X=2] &= \frac{1}{2!}\frac{d^2}{dz^2}G_X(z)\bigg|_{z=0} = \frac{1}{2!}\frac{d^2}{dz^2}\frac{(pz)^2}{(1-qz)^2}\bigg|_{z=0} \\ &= \frac{p^2}{2!}\frac{d}{dz}\left[\frac{2z}{(1-qz)^2} + \frac{z^2 2q}{(1-qz)^3}\right]_{z=0} \\ &= \frac{p^2}{2!}\left[f2(1-qz)^2 + 2\frac{4zq}{(1-qz)^3} + \frac{z^2(z)(3)q^2}{(1-qz)^4}\right]_{z=0} \\ &= p^2 \end{aligned}$$

In the general case, we have

$$\begin{aligned} P[X=r] &= \frac{1}{r!}\frac{d^r}{dz^r}\frac{(pz)^r}{(1-qz)^r}\bigg|_{z=0} \\ &= \frac{p^r}{r!}\sum_{k=0}^{r}\binom{r}{k}\frac{d^k}{dz^k}(z^r)\frac{d^{r-k}}{dz^{r-k}}\frac{1}{(10qz)^r}\bigg|_{z=0} \end{aligned}$$

$$\frac{d^k}{dz^k}(z^r)\bigg|_{z=0} = \begin{cases} z^r|_{z=0} = 0 & k=0 \\ (r-r1)\ldots(r-k+1)z^{r-k}|_{z=0} = 0 & 0<k<r \\ r(r-1)\ldots(3)(2)(1) = r! & k=r \end{cases}$$

$\therefore$ only the $k=r$ term in the summation is nonzero

$$\begin{aligned} P[X=r] &= \frac{p^r}{r!}\left\{\binom{r}{r}f!\frac{1}{(1-qz)^r}\right\}_{z=0} = p^r \quad \checkmark \\ \mathcal{E}[X] &= \frac{d}{dz}G_N(z)\bigg|_{z=1} = \frac{d}{dz}\left(\frac{pz}{1-qz}\right)^r_{z=1} \\ &= \frac{d}{dz}\frac{p^r}{(z^{-1}-q)^r}\bigg|_{z=1} = p^r(-r)(z^{-1}-q)^{-r-1}(-z^{-2})\Big|_{z=1} \\ &= \frac{rp^r}{(1-q)^{r+1}} = \frac{r}{p} \end{aligned}$$

4.116

$$\begin{aligned}
X^*(s) &= \int_{-\infty}^{\infty} f_X(x)e^{-sx}dx \\
\frac{d^nX^*(s)}{ds^n} &= \int_{-\infty}^{\infty} f_X(x)\cdot(-x)^n e^{-sx}dx \\
&= (-1)^n \int_{-\infty}^{\infty} x^n f_X(x)e^{-sx}dx \\
\Rightarrow E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x)dx \\
&= (-1)^n \frac{d^nX^*(s)}{ds^n}|_{s=0}
\end{aligned}$$

4.117

$$\begin{aligned}
\Phi_X(\omega) &= \int_0^{\infty} \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} e^{j\omega x}dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_{0-}^{\infty} x^{\alpha-1}e^{-(\lambda-j\omega)x}dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}(\lambda-j\omega)^{-\alpha} \underbrace{\int_0^{\infty}((\lambda-j\omega)x)^{\alpha-1}e^{-(\lambda-j\omega)x}\alpha(\lambda-j\omega)x}_{\Gamma(\alpha)} \\
&= \frac{(\lambda^\alpha)}{(\lambda-j\omega)^\alpha} = \frac{1}{(1-j\omega/\lambda)^\alpha} \\
\mathcal{E}[X^n] &= \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_X(\omega)\bigg|_{\omega=0} = \lambda^{-n}\frac{(\alpha+n-1)!}{(\alpha-1)!}\left(1-\frac{j\omega}{\lambda}\right)^{-(\alpha+n)}\bigg|_{\omega=0} \\
&= \lambda^{-n}\frac{(\alpha+n-1)!}{(\alpha-1)!}
\end{aligned}$$

4.118

$$X = \begin{cases} X_1 & \text{with prob } p \\ X_2 & \text{with prob } 1-p \end{cases}$$

$$\begin{aligned}
X^*(s) &= \mathcal{E}[e^{-sX}] = \mathcal{E}[e^{-sX}|X=X_1]p + \mathcal{E}[e^{-sX}|X=X_2](1-p) \\
&= \mathcal{E}[e^{-sX_1}]p + \mathcal{E}[e^{-sX_2}](1-p) \\
&= p\frac{\lambda_1}{s+\lambda_1} + (1-p)\frac{\lambda^2}{s+\lambda_2}
\end{aligned}$$

4.119

$$\begin{aligned}
X^*(s) &= \left(\frac{\alpha}{s+\alpha}\right)\left(\frac{\beta}{s+\beta}\right) \\
&= \frac{\alpha\beta}{\alpha+\beta}\left[\frac{1}{s+\beta}-\frac{1}{s+\alpha}\right] \\
f_X(t) &= \frac{\alpha\beta}{\alpha-\beta}\left[\mathcal{L}^{-1}\left[\frac{1}{s+\beta}\right]-\mathcal{L}^{-1}\left[\frac{1}{s+\alpha}\right]\right] \\
&= \frac{\alpha\beta}{\alpha-\beta}\left[\frac{1}{\beta}e^{-\beta t}-\frac{1}{\alpha}e^{-\alpha t}\right] \\
&= \frac{\alpha}{\alpha-\beta}e^{-\beta t}-\frac{\beta}{\alpha-\beta}e^{-\alpha t}
\end{aligned}$$

4.120

- Gamma$(\lambda; \alpha)$

$$X^*(s) = \frac{\lambda^\alpha}{(\lambda+s)^\alpha}$$

- Gamma$(\lambda, \alpha-1)$

$$\begin{aligned}
Y^*(s) &= \int_0^\infty \frac{\lambda(\lambda y)^{\alpha-2}}{\Gamma(\alpha-1)} e^{-\lambda y} e^{-sy}\, dy \\
&= \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^\infty y^{\alpha-2} e^{-(\lambda+s)y}\, dy \\
&= \frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)} \int_0^\infty \left(\frac{\omega}{\lambda+s}\right)^{\alpha-2} e^{-\omega} \frac{d\omega}{\lambda+s} \\
&= \frac{\lambda^{\alpha-1}}{(\lambda+s)^{\alpha-1}} \cdot \frac{1}{\Gamma(\alpha-1)} \int_0^\infty \omega^{(\alpha-1)-1} e^{-\omega}\, d\omega \\
&= \frac{\lambda^{\alpha-1}}{(\lambda+s)^{\alpha-1}}
\end{aligned}$$

$$X_*(s) = \frac{\lambda}{\lambda+s} Y_*(s)$$

4.121 a)

$$P[X \ge a] \le \min_{s \ge 0} \frac{e^{-sa}}{(1-s/\lambda)^{\alpha}}$$

$$0 = \frac{d}{ds}\frac{e^{-sa}}{(1-\frac{s}{\lambda})^{\alpha}} = \frac{-ae^{-sa}}{(1-\frac{s}{\lambda})^{\alpha}} - \frac{\alpha e^{-sa}}{(1-\frac{s}{\lambda})^{\alpha+1}}\left(-\frac{1}{\lambda}\right)$$

$$a\left(1-\frac{s}{\lambda}\right) = \frac{\alpha}{\lambda} \qquad 1-\frac{s}{\lambda} = \frac{\alpha}{a\lambda}$$

$$s = \lambda\left(1-\frac{\alpha}{a\lambda}\right) = \lambda - \frac{\alpha}{a} \qquad s \ge 0 \Rightarrow \lambda \ge \frac{\alpha}{a}$$

$$a \ge \frac{\alpha}{\lambda}$$

$$P[X \ge a] \le \frac{e^{-sa}}{(1-s/\lambda)^{\alpha}} = \frac{e^{-(a\lambda-\alpha)}}{\left(1-\left(1-\frac{\alpha}{a\lambda}\right)\right)^{\alpha}}$$

$$= \frac{e^{-(a\lambda-\alpha)}}{(\alpha/a\lambda)^{\alpha}}$$

ⓑ $P[X \ge 9]$ $m=3, \lambda=1$ Erlang $\Rightarrow \alpha=3, \lambda=1$

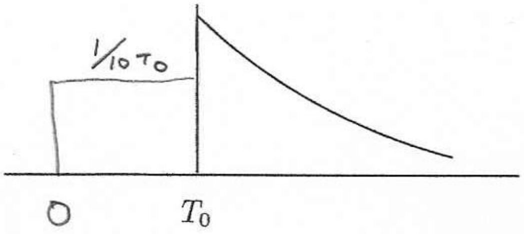
$$P[X \ge 9] \le \frac{e^{-(9-3)}}{\left(\frac{3}{9}\right)^{3}} = 27e^{-6} = 0.067$$

Exact for Gamma

$P[X \ge 9] = 0.00623$

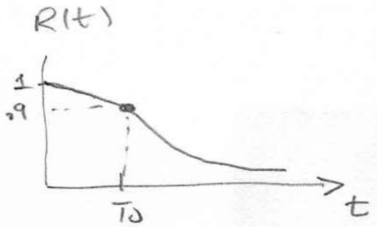
4.8 Basic Reliability Calculations

4.122 a) $$f_T(t) = \begin{cases} \frac{1}{10} & 0 \le t \le T_0 \\ \frac{1}{9}\lambda e^{-\lambda(t-T_0)} & t \ge T_0 \\ 0 & t < 0 \end{cases}$$



$$R(t) = P[T > t] = \begin{cases} 1 - t/10T_0 & 0 < t < T_0 \\ .9\, e^{-\lambda(t-T_0)} & t \ge T_0 \end{cases}$$

$\frac{1}{10} + 0.9\times$



where we used the fact that

$$\cancel{R(t) = P[T > t] =} \int_t^\infty \lambda e^{-(\lambda t' - T_0)} dt' = e^{-\lambda(t-T_0)} \quad t > T_0$$

The $MTTF$ is given by the expected value of X:

$$\begin{aligned} MTTF = E[T] &= \int_0^\infty R(t')dt' \\ &= \cancel{\frac{1}{10}\int_0^{T_0} 1\, dt'} + \int_{T_0}^\infty e^{-\lambda(t'-T_0)} dt' \\ &= .95\, T_0 + \frac{1}{\lambda} \end{aligned}$$

$\int_0^{T_0} \left(1 - \frac{t}{10T_0}\right) dt$

b) $$r(t) = \frac{-R'(t)}{R(t)} = \begin{cases} (10T_0 - t)^{-1} & 0 \le t \le T_0 \\ \lambda & t < T_0 \end{cases}$$

$\frac{1}{10} + .9*$

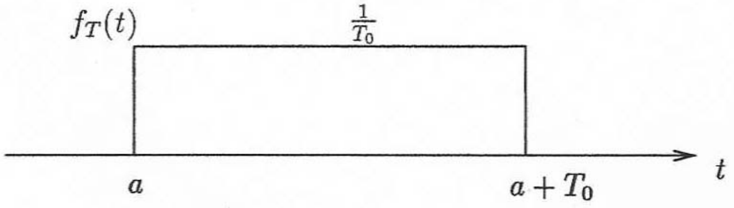
c) $R(t) = e^{-\lambda(t-T_0)} = 0.99 \quad \Rightarrow \quad e^{-\lambda(t-T_0)} = \frac{.89}{.9}$

$$\Rightarrow \lambda(t - T_0) = \ln \frac{1}{0.99} \quad \frac{.9}{.89}$$

$$\Rightarrow t = T_0 + \frac{1}{\lambda}\ln\frac{.9}{0.89} = T_0 + \frac{.011}{\lambda}$$

4.123 **a)**

$$R(t) = P[T > t] = \int_t^\infty f_T(t')dt' = \begin{cases} 1 & t < a \\ 1 - \dfrac{t-a}{T_0} & a < t < a + T_0 \\ 0 & t > T_0 \end{cases}$$



$$MTTF = \int_0^\infty R(t)dt = a + \frac{T_0}{2}$$

b) $r(t) = \dfrac{-R'(t)}{R(t)} = \begin{cases} \dfrac{1}{a + T_0 - t} & a < t < a + T_0 \\ 0 & \text{elsewhere} \end{cases}$

c) $R(t) = 1 - \dfrac{t-a}{T_0} = 0.99 \Rightarrow t = \alpha + 0.01T_0$

4.124

3 a) $R(t) = \int_0^\infty \dfrac{x}{\alpha^2} e^{-s^2/2\alpha^2} dx = -e^{-x^2/2\alpha^2}\Big|_t^\infty = e^{-t^2/2\alpha^2}$

b) $r(t) = \dfrac{f_T(t)}{R(t)} = \dfrac{t}{\alpha^2} \qquad t \geq 0$

(c) $R_{\text{series}}(t) = e^{-t^2/2\alpha^2} \cdot e^{-t^2/2\alpha^2}$
$= e^{-t^2/\alpha^2}$

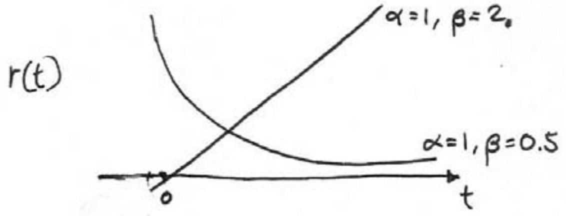
(d) $R_{\text{parallel}} = 1 - (1 - e^{-t^2/2\alpha^2})(1 - e^{-t^2/2\alpha^2})$

4.125

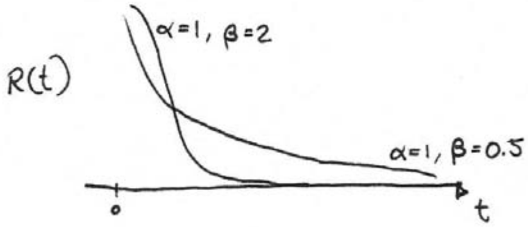
a) $R(t) = 1 - F_T(t) = e^{-(t/\lambda)^{\beta}}$

$R'(t) = -\frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1} e^{-(t/\lambda)^{\beta}}$

$r(t) = \frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1} = \alpha\beta t^{\beta-1}$



b)



c) $R(t) = R_1(t)R_2(t) = \alpha_1\beta_1 t^{\beta_1-1}\alpha_2\beta_2 t^{\beta_2-1} = e^{-(\alpha_1 t^{\beta_1}+\alpha_2 t^{\beta_2})}$

d) $R(t) = 1-(1-R_1(t))(1-R_2(t)) = 1-(1-e^{-\alpha_1 t^{\beta_1}})(1-e^{-\alpha_2 t^{\beta_2}})$

$= \cancel{1}-\cancel{1}+e^{-\alpha_2 t^{\beta_2}} + e^{-\alpha_1 t^{\beta_1}} - e^{-\alpha_1 t^{\beta_1}} e^{-\alpha_2 t^{\beta_2}}$

(4.126) a) $R(t) = P[T>t]$

$$= \sum_{n=0}^{m-1} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad \text{Erlang.}$$

b) $r(t) = \frac{-f_T(t)}{R(t)}$

$$= \frac{-\lambda^m t^{m-1} e^{-\lambda t}}{(m-1)!} \frac{1}{\sum_{n=0}^{m-1} e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

$$= \frac{-\lambda (\lambda t)^{m-1} / (m-1)!}{\sum_{n=0}^{m-1} (\lambda t)^n / n!}$$

(4.127) ~~3.105~~ The failure rate function of the memory chips is obtained as follows:

$$\begin{aligned}
F_T(x|t>t) &= P[T \le x | T > t] \\
&= \begin{cases} 0 & x < t \\ \frac{F_T(x) - F_T(t)}{1P[T>t]} & x \ge t \end{cases} \\
&= \begin{cases} 0 \\ \frac{[1-(1-p)e^{-\alpha x} - pe^{-1000\alpha x}] - [1-(1-p)e^{\alpha t} - pe^{-1000\alpha t}]}{(1-p)e^{-\alpha t} + pe^{-1000\alpha t}} \end{cases} \\
f_T(x|T>t) &= \frac{\alpha(1-p)e^{-\alpha x} + 1000\alpha p e^{-1000\alpha x}}{(1-p)e^{-\alpha t} + pe^{-1000\alpha t}} \\
r(t) &= f_T(t|T>t) = \frac{\alpha(1-p)e^{-\alpha t} + 1000\alpha p e^{-1000\alpha t}}{(1-p)e^{-\alpha t} + pe^{-1000\alpha t}}
\end{aligned}$$

For small t, $r(t)$ is dominated by the second term is the numerator. For large t, $r(t)$ is dominated by the first term.

4.128

ⓐ

$$R(t) = P[T > t]$$

$$= P[T > t \mid s=1]\, p + P[T > t \mid s=2]\,(1-p)$$

$$= (1 - F_{exp}(t))\, p + (1 - F_{pareto}(t))(1-p)$$

$$= \begin{cases} p e^{-t/m} + (1-p)\left(\dfrac{x_m}{t}\right)^{\alpha} & t > x_m \\ p e^{-t/m} + (1-p)\cdot 1 & x_m > t > 0 \end{cases}$$

$$x_m = \frac{m(\alpha-1)}{\alpha}$$

ⓑ

$$r(t) = -\frac{R'(t)}{R(t)}$$

$$= \begin{cases} \dfrac{-\frac{p}{m} e^{-t/m} - \alpha x_m^{\alpha} t^{-\alpha-1}(1-p)}{p e^{-t/m} + (1-p)\, x_m^{\alpha} t^{-\alpha}} & t > x_m \\[2ex] \dfrac{-\frac{p}{m} e^{-t/m}}{p e^{-t/m} + (1-p)} & 0 < t < x_m \end{cases}$$

4.129

$$R(t) = \exp\{-\int_0^t r(t')dt'\}$$

For $0 \le t < 1$:

$$\begin{aligned} R(t) &= \exp\{-\int_0^t [1+9(1-t')]dt'\} \\ &= \exp\{-10t + \frac{9}{2}t^2\} \end{aligned}$$

For $1 \le t < 10$:

$$\begin{aligned} R(t) &= \exp\{-\int_0^1 [1+9(1-t')]dt' - \int_1^t 1dt'\} \\ &= \exp\{-10 + \frac{9}{2} - (t-1)\} \\ &= \exp\{-4,5 - t\} \end{aligned}$$

For $t \ge 10$:

$$\begin{aligned} R(t) &= \exp\{-\int_0^1 [1+9(1-t')]dt' - \int_1^{10} 1dt' - \int_{10}^t [1+10(t'-10)]dt'\} \\ &= \exp\{-4,5 - 10 - [-99(t-10) + 5(t^2 - 10^2)]\} \\ &= \exp\{-5t^2 + 99t - 1504.5\} \end{aligned}$$

$$\begin{aligned} f_T(t) &= -r(t)R(t) \\ &= \begin{cases} -[1+9(1-t)]\exp\{-10t + \frac{9}{2}t^2\} & 0 \le t < 1 \\ 1\exp\{-4.5 - t\} & 1 \le t < 10 \\ 1 + 10(t-10)\exp\{-5t^2 + 99t - 1504.5\} & t > 10 \end{cases} \end{aligned}$$

4.130)

~~3.108~~ Each component has reliability: $R_i(t) = e^{-t}$

a) $R(t) = P[\text{system working at time}] = P[\text{2 or more working at time } t]$
$= \binom{3}{2}(e^{-t})^2(1-e^{-t}) + \binom{3}{3}(e^{-t})^3$
$= 3e^{-2t} - 2e^{-3t}$

$$\begin{aligned} MTTF &= \int_0^\infty R(t')dt' = \int_0^\infty (3e^{-2t'} - 2e^{-3t'})dt' \\ &= \frac{3}{2} - \frac{2}{3} = \frac{5}{6} \end{aligned}$$

b) Now $R_1(t) = R_2(t) = e^{-t}$ and $R_3(t) = e^{-t/2}$. $R(t) = P[\text{2 or more working at time } t]$

$= R_1(t)R_2(t)(1-R_3(t)) + R_1(t)(1-R_2(t))R_3(t)$
$\quad +(1-R_1(t))R_2(t)R_3(t) + R_1(t)R_2(t)R_3(t)$
$= e^{-2t}(1-e^{-t/2}) + 2e^{-t}(1-e^{-t})e^{-t/2} + e^{-2t}e^{-t/2}$
$= e^{-2t} + 2e^{-3t/2} - 2e^{-st/2}$

$$\begin{aligned} MTTF &= \int_0^\infty R(t')dt' = \int_0^\infty (3e^{-2t'} + 2e^{-\frac{3t}{2}} - 2e^{-\frac{5t'}{2}})dt' \\ &= \frac{1}{2} + 2\frac{2}{3} - 2\frac{2}{5} = \frac{31}{30} \end{aligned}$$

4.131 $R(t) = e^{-\alpha t^{3}}$ Weibull $E[X] = 1 = \frac{\Gamma(4/3)}{\alpha^{1/3}}$

ⓐ P[2 or more working at time t]

$$= \binom{3}{2}\left(e^{-\alpha t^\beta}\right)^2\left(1-e^{-\alpha t^\beta}\right) + \binom{3}{3}\left(e^{-\alpha t^\beta}\right)^3$$

$$= 3e^{-2\alpha t^\beta} - 3e^{-3\alpha t^\beta} + 1\, e^{-3\alpha t^\beta}$$

$$= 3e^{-2\alpha t^\beta} - 2e^{-3\alpha t^\beta} = 3e^{-2\alpha t^3} - 2e^{-3\alpha t^3}$$

ⓑ P[2 or more working at time t]

$$= \left(e^{-\alpha t^\beta}\right)^2\left(1-e^{-\alpha' t^{\beta'}}\right) + 2e^{-\alpha t^\beta}\left(1-e^{-\alpha t^\beta}\right)e^{-\alpha' t^{\beta'}}$$

$$+ \left(e^{-\alpha t^\beta}\right)^2 e^{-\alpha' t^{\beta'}}$$

$$= e^{-2\alpha t^3}\left(1-e^{-\alpha' t^3}\right) + 2e^{-\alpha t^3}\left(1-e^{-\alpha t^3}\right)e^{-\alpha' t^3}$$

$$+ e^{-2\alpha t^3}e^{-\alpha' t^3}$$

$$= e^{-2\alpha t^3} \cancel{- e^{-(2\alpha+\alpha')t^3}} + 2e^{-(\alpha+\alpha')t^3} - 2e^{-(2\alpha+\alpha')t^3}$$

$$\cancel{+ e^{-(2\alpha+\alpha')t^3}}$$

$$= e^{-2\alpha t^3} + 2e^{-(\alpha+\alpha')t^3} - 2e^{-(2\alpha+\alpha')t^3}$$

MTTF = $\int_0^\infty R(t)\,dt$

for Weibull $\int_0^\infty e^{-\alpha t^\beta}\,dt = \frac{\Gamma(1+\frac{1}{\beta})}{\alpha^{1/\beta}} \overset{\beta=3}{=} \frac{\Gamma(\frac{4}{3})}{\alpha^{1/3}} = E[X]$

$$\int_0^\infty e^{-2\alpha t^\beta}\,dt = \frac{\Gamma(1+\frac{1}{\beta})}{(2\alpha)^{1/\beta}} = \frac{\Gamma(1+1/\beta)}{2^{1/\beta}(\alpha)^{1/\beta}} = \frac{E[X]}{2^{1/\beta}}$$

$$\int_0^\infty e^{-\alpha' t^\beta}\,dt = 2 = \frac{\Gamma(1+\frac{1}{\beta})}{(\alpha')^{1/\beta}} = \frac{\Gamma(\frac{4}{3})}{(\alpha')^{1/\beta}}$$

$\Rightarrow \Gamma(\frac{4}{3}) = 2(\alpha')^{1/\beta}$ $\qquad \Rightarrow 2(\alpha')^{1/\beta} = \alpha^{1/\beta}$

vs $\Rightarrow \Gamma(\frac{4}{3}) = 1(\alpha)^{1/\beta}$ $\qquad \Rightarrow (\alpha')^{1/\beta} = \frac{1}{2}\alpha^{1/\beta}$

$\Rightarrow \alpha' = \alpha/2^\beta = \frac{\alpha}{8}$

ⓐ MTTF $= \int_0^\infty 3e^{-2\alpha t^3}\,dt - 2\int_0^\infty e^{-3\alpha t^3}\,dt = 3\frac{\Gamma(4/3)}{(2\alpha)^{1/3}} - 2\frac{\Gamma(4/3)}{(3\alpha)^{1/3}}$

$$= \left(\frac{3}{2^{1/3}} - \frac{2}{3^{1/3}}\right)\underbrace{\frac{\Gamma(4/3)}{\alpha^{1/3}}}_{1} = 0.381$$

ⓑ MTTF $= \int_0^\infty e^{-2\alpha t^3}\,dt + 2\int_0^\infty e^{-(\alpha+\alpha')t^3}\,dt - 2\int_0^\infty e^{-(2\alpha+\alpha')t^3}\,dt$

$$= \frac{\Gamma(4/3)}{(2\alpha)^{1/3}} + \frac{2\Gamma(4/3)}{(\alpha+\alpha')^{1/3}} - 2\frac{\Gamma(4/3)}{(2\alpha+\alpha')^{1/3}}$$

$$= \frac{\Gamma(\frac{4}{3})}{\alpha^{1/3}}\left[\frac{1}{(2)^{1/3}} + \frac{2}{(9/8)^{1/3}} - \frac{2}{(17/8)^{1/3}}\right]$$

4.132

.110 **a)** Reliability of processor $R_S(t) = e^{-t/5}$
Reliability of peripheral units $R_P(t) = e^{-t/10}$

$$\begin{aligned} R(t) &= P[\text{1 or more processors functioning at time } t] \\ &\quad \times P[\text{2 or more peripherals functioning at time } t] \\ &= \left[\binom{2}{1} R_S(t)(1-R_S(t)) + R_S^2(t)\right]\left[\binom{3}{2} R_P^2(t)(1-R_P(t)) + \binom{3}{3} R_P^3(t)\right] \\ &= 2e^{-tt/5}(1-e^{-t/5}) + e^{-2t/5}][3e^{-2t/20}(1-e^{-t/10} + e^{-3t/10}] \\ &= e^{-2t/5}[(2-e^{-t/5})(3-2e^{-t/10})] \end{aligned}$$

$$MTTF = \int_0^\infty R(t)dt = 6\left(\frac{5}{2}\right) - 3\left(\frac{5}{3}\right) - 4\left(\frac{10}{5}\right) + 2\left(\frac{10}{7}\right) = \frac{34}{7}$$

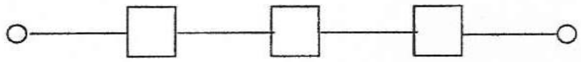
b) If $R_S(t) = e^{-t/10}$ $R_P(t) = e^{-t/5}$

$$\begin{aligned} R(t) &= [2e^{-t/10}(1-e^{-t/10}) + e^{-2t/10}][3e^{-2t/5}(1-e^{-t/5}) + e^{-3t/5}] \\ &= \underbrace{e^{-t/10}e^{-2t/5}}_{e^{-t/2}}[(2-e^{-t/10})(3-2e^{-t/5})] \end{aligned}$$

$$MTTF = \int_0^\infty R(t)dt = 6(2) - 3\left(\frac{10}{6}\right) - 4\left(\frac{10}{7}\right) + 2\left(\frac{10}{8}\right) = \frac{53}{14}$$

4.133

(a)



$$R_{subsystem}(t) = R^3(t) = e^{-3t}$$

$$R_{parallel}(T) = 1 - (1 - e^{-3T})^n = 0.99$$

$$n \ln(1 - e^{-3T}) = \ln \frac{1}{100}$$

$$n = \frac{\ln 100}{\ln \frac{1}{1-e^{-3T}}}$$

(b)

$$1 - (1 - e^{-T^2/2\alpha^2})^n = 0.99$$

$$n \ln(1 - e^{-T^2/2\alpha^2}) = \ln \frac{1}{100}$$

$$n = \frac{\ln 100}{\ln(1 - e^{-T^2/2\alpha^2})}$$

Rayleigh

(c)

$$n = \frac{\ln 100}{\ln(1 - e^{-\alpha T^3})}$$

Weibull

4.9 Computer Methods for Generating Random Variables

4.134 The following Octave commands generate the requested plots:

(a)

```
x = [-4:0.01:4];
y0 = normal_pdf(x, -2, 1);
y1 = normal_pdf(x, 2, 1);
figure;
hold on;
plot(x, y0, "1");
plot(x, y1, "3");
```

(b)

```
x = [-5:0.01:5];
y0 = 1-normal_cdf(x, -2, 1);
y1 = 1-normal_cdf(x, 2, 1);
ey0 = e.^(-(x+2).^2/2);
ey1 = e.^(-(x-2).^2/2);
figure;
hold on;
plot(x, y0, "1");
plot(x, y1, "3");
plot(x, ey0, "1");
plot(x, ey1, "3");
```

4.135 The following Octave commands generate plots of the pdf and cdf of the gamma random variable:

(a)

```
x = [0:0.01:15];
figure;
hold on;
plot(x, gamma_pdf(x, 1, 1), "1");
plot(x, gamma_pdf(x, 2, 1), "2");
plot(x, gamma_pdf(x, 4, 1), "3");
figure;
hold on;
plot(x, gamma_cdf(x, 1, 1), "1");
plot(x, gamma_cdf(x, 2, 1), "2");
plot(x, gamma_cdf(x, 4, 1), "3");
```

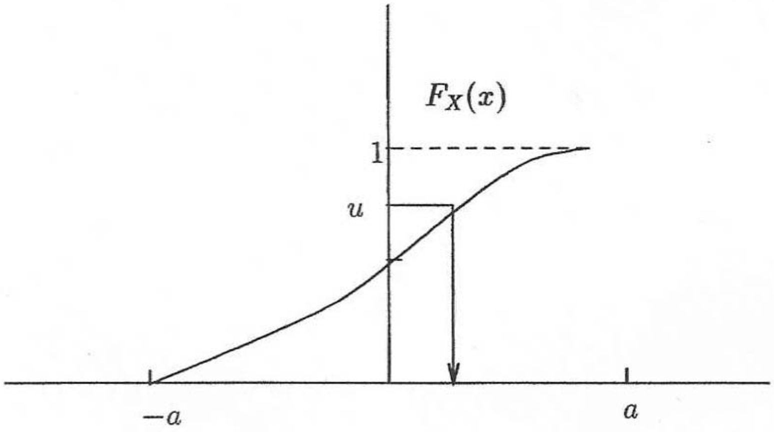
(b)

```
x = [0:0.01:15];
figure;
hold on;
plot(x, gamma_pdf(x, 1/2, 1/2), "1");
plot(x, gamma_pdf(x, 1, 1/2), "2");
plot(x, gamma_pdf(x, 3/2, 1/2), "3");
plot(x, gamma_pdf(x, 5/2, 1/2), "4");
figure;
hold on;
plot(x, gamma_cdf(x, 1/2, 1/2), "1");
plot(x, gamma_cdf(x, 1, 1/2), "2");
plot(x, gamma_cdf(x, 3/2, 1/2), "3");
plot(x, gamma_cdf(x, 5/2, 1/2), "4");
```

4.136

$$f_X(x) = \begin{cases} (a+x)/a^2 & -a \le x \le 0 \\ (a-x)/a^2 & 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < -a \\ \frac{1}{2}\left[\left(\frac{x}{a}\right)^2 + 2\left(\frac{x}{a}\right) + 1\right] & -a \le x \le 0 \\ \frac{1}{2} + \frac{1}{2}\left[2\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^2\right] & 0 \le x \le a \\ 1 & x > a \end{cases}$$



Solving the equation $u = F_X(x)$ for x we obtain

$$= F_X^{-1}(u) = \begin{cases} -a + a\sqrt{2u} & 0 \le u \le \frac{1}{2} \\ a - a\sqrt{2-2u} & \frac{1}{2} \le u \le 1 \end{cases}$$

4.137 The following Octave commands generate the requested samples and plots:

(a)

```
x = [-6:0.01:6];
u = rand(1, 1000);
%Multiply all values by discretely generated -1 or 1
z = -log(u).*discrete_rnd(length(u), [-1 1], [0.5 0.5]);
figure;
hold on;
%Normalize to 2 because bar width is 0.5
hist(z, [-6:0.5:6], 2);
plot(x, laplace_pdf(x), "1");
```

(b)

```
x = [1:0.01:10];
u = rand(1, 1000);
k = 1.5;
z = u.^(-1/k);
figure;
hold on;
hist(z, [1.25:0.5:10], 2);
plot(x, k./x.^(k+1));

x = [1:0.01:10];
u = rand(1, 1000);
k = 2;
z = u.^(-1/k);
figure;
hold on;
hist(z, [1.25:0.5:10], 2);
plot(x, k./x.^(k+1));

x = [1:0.01:10];
u = rand(1, 1000);
k = 2.5;
z = u.^(-1/k);
figure;
hold on;
hist(z, [1.25:0.5:10], 2);
plot(x, k./x.^(k+1));
```

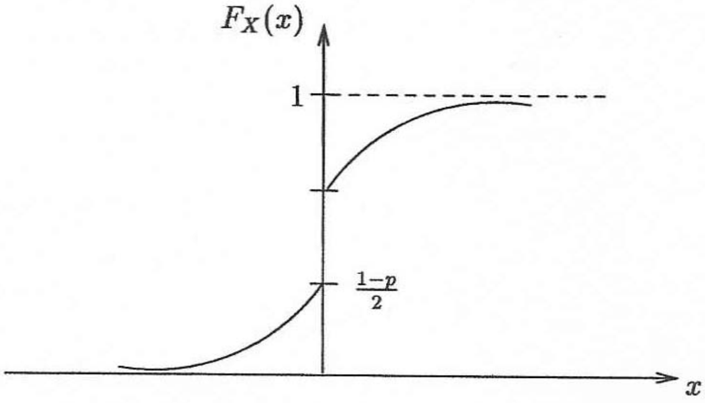
(c)

```
x = [0:0.01:5];
u = rand(1, 1000);
b = 0.5;
z = log(1./u).^(1./b);
figure;
hold on;
hist(z, [0:0.25:5], 4);
plot(x, b.*x.^(b-1).*e.^(-x.^b), "1");

x = [0:0.01:5];
u = rand(1, 1000);
```

```
b = 2;
z = log(1./u).^(1./b);
figure;
hold on;
hist(z, [0:0.125:5], 8);
plot(x, b.*x.^(b-1).*e.^(-x.^b), "1");

x = [0:0.01:5];
u = rand(1, 1000);
b = 3;
z = log(1./u).^(1./b);
figure;
hold on;
hist(z, [0:0.125:5], 8);
plot(x, b.*x.^(b-1).*e.^(-x.^b), "1");
```

4.138

$$F_X(x) = \begin{cases} \frac{1-p}{2}e^{\alpha x} & x < 0 \\ 1 - \frac{1-p}{2}e^{-\alpha x} & x \geq 0 \end{cases}$$



Note that $\mu = F_X(x)$ for

$$\frac{1-p}{2} < \mu < 1 - \frac{1-p}{2} \Rightarrow z = 0$$

$$Z = F_X^{-1}(\mu) = \begin{cases} \frac{1}{\alpha}\ln\frac{2U}{1-p} & 0 \leq \mu \leq \frac{1-p}{2} \\ -\frac{1}{\alpha}\ln\frac{2(1-U)}{1-p} & 1 - \frac{1-p}{2} \leq \mu \leq 1 \\ 0 & \frac{1-p}{2} \leq \mu \leq 1 - \frac{1-p}{2} \end{cases}$$

4.139

$$\begin{array}{rcl} 0 < U < \frac{1}{2} & \Rightarrow & X = 1 \\ \frac{1}{2} < U < \frac{3}{4} & \Rightarrow & X = 2 \\ \frac{3}{4} < U < \frac{7}{8} & \Rightarrow & X = 3 \\ \vdots & & \vdots \\ \mathcal{E}[N] = \mathcal{E}[X] & = & \frac{1}{\frac{1}{2}} = 2 \end{array}$$

4.140

$$\begin{array}{rcl} 0 < U < e^{-\alpha} & \Rightarrow & X = 0 \\ e^{-\alpha} < U < e^{-\alpha}(1+\alpha) & \Rightarrow & X = 1 \\ e^{-\alpha}(1+\alpha) < U < e^{-\alpha}(1+\alpha+\frac{\alpha^2}{2!}) & \Rightarrow & X = 2 \\ \vdots & & \vdots \\ \text{Average number of comparisons} & = & \sum_{k=0}^{\infty}(k+1)\frac{\alpha^k}{k!}e^{-\alpha} = \alpha + 1 \end{array}$$

4.141 The following Octave commands describe the function for performing the rejection method and the code to call the function:

```
function z = gaussian_rejection_method(N)
     z = zeros(1, N);
     k = 1;
     while k <= N
          while true
               u1 = rand;
               u2 = rand;
               x1 = -log(u1);
               if (u2 <= e.^(-((x1-1).^2))/2)
                    z(k) = x1.*discrete_rnd(1,[-1 1],[0.5 0.5]);
                    break;
               end
          end
          k = k + 1;
     end
end

x = [-4:0.01:4];
z = gaussian_rejection_method(10000);
figure;
hold on;
hist(z, [-4:0.125:4], 8);
plot(x, normal_pdf(x, 0, 1), "1");
```

The cdf of X_1 is

$$P[X_1 \le x] = P[-\ln U_1 \le x] = P[U_1 \ge e^{-x}] = 1 - e^{-x}$$

$\therefore X_1$ is exponential with parameter $\lambda = 1$, and $f_{X_1}(x) = e^{-x}$.
If X_1 is accepted, its pdf is given by:

$$P[x \le X_1 < x + dx | X_1 \text{ accepted}] = \frac{P[\{X_1 \text{ accepted}\} \cap \{x \le X_1 < x + dx\}}{P[X_1 \text{ accepted}]}$$

$$\begin{aligned} P[X_1 \text{ accepted}] &= \int_0^\infty P[X_1 \text{ accepted} | X_1 = x] e^{-x} dx \\ &= \int_0^\infty e^{-(x_1-1)^2/2} e^{-x} dx \\ &= e^{-\frac{1}{2}} \int_0^\infty e^{-x^2/2} dx \\ &= \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} P[x \le X_1 < x + dx | X_1 \text{ accepted}] &= \frac{e^{-(x_1-1)^2/2} e^{-x} dx}{\sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}}} = 2 \frac{e^{-x^2/2}}{\sqrt{2\pi}} \\ &= f_Y(x) dx \end{aligned}$$

where $Y = |X|$ and X is a zero-mean, unit-variance random variable.

4.142

$$\begin{aligned} F_Z(x) &= \int_0^\infty \frac{\lambda \alpha^\lambda t^{\lambda-1} dt}{(\alpha^\lambda + \lambda)^2} = \int_0^{x^\lambda} \frac{\alpha^\lambda dy}{(\alpha^\lambda + y)^2} \qquad \begin{array}{l} \text{where we let} \\ y = t^\lambda \\ dy = \lambda t^{\lambda-1} dt \end{array} \\ &= \alpha^\lambda \left[\frac{-1}{(\alpha^\lambda + y)} \right]_0^{x^\lambda} = \alpha^\lambda \left[\frac{1}{\alpha^3} - \frac{1}{\alpha^\lambda + x^\lambda} \right] \\ &= 1 - \frac{\alpha^\lambda}{\alpha^\lambda + x^\lambda} \qquad x > 0 \end{aligned}$$

To generate Z we need to solve

$$\begin{aligned} \mu &= 1 - \frac{\alpha^\lambda}{\alpha^\lambda + x^\lambda} \\ 1 - \mu &= \frac{\alpha^\lambda}{\alpha^\lambda + x^\lambda} \\ \Rightarrow x^\lambda &= \alpha^\lambda \left[\frac{1}{1-\mu} - 1 \right] = \alpha^\lambda \frac{\mu}{1-\mu} \\ \Rightarrow x &= \alpha \left[\frac{\mu}{1-\mu} \right]^{1/\lambda} \\ \therefore \hat{Z} &= \alpha \left[\frac{U}{1-U} \right]^{1/\lambda} \qquad \text{where } U \text{ is uniform in } [0,1] \end{aligned}$$

4.143 a) The key observation is that

$$P[X_1 \text{ is accepted} | X_1 = x] = \frac{f_X(x)}{K f_W(x)}$$

since Y is uniform in $[0, K f_W(x)]$. We then have that

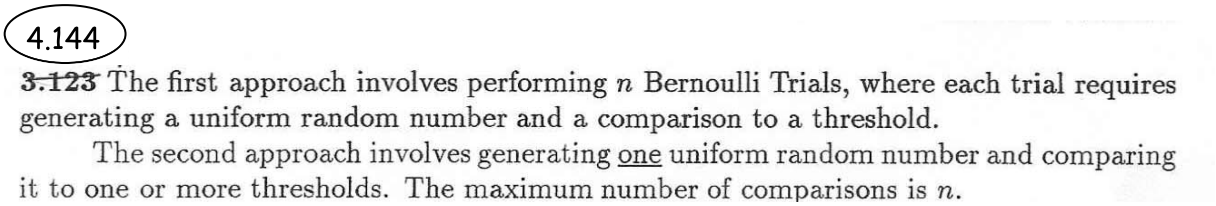
$$\begin{aligned}
P[X_1 \text{ is accepted}] &= \int_{-\infty}^{\infty} P[X_1 \text{ is accepted} | X_1 = x] f_W(x) dx \\
&= \int_{-\infty}^{\infty} \frac{f_X(x)}{K f_W(x)} f_W(x) dx \\
&= \frac{1}{K}
\end{aligned}$$

b)

$$\begin{aligned}
P[x < X_1 < x + dx | X_1 \text{ accepted}] &= \frac{P[\{X_1 \text{ accepted}\} \cap \{x < X_1 < x + dx\}]}{P[X_1 \text{ accepted}]} \\
&= \frac{\frac{f_X(x)}{K f_W(x)} f_W(x) dx}{1/K} \\
&= f_X(x) dx
\end{aligned}$$

$\therefore$ X_1 when accepted as pdf $f_X(x)$ as desired.

4.144



~~3.123~~ The first approach involves performing n Bernoulli Trials, where each trial requires generating a uniform random number and a comparison to a threshold.

The second approach involves generating one uniform random number and comparing it to one or more thresholds. The maximum number of comparisons is n.

The following Octave commands generate the requested samples and plots:

```
function z = binomial_bernoulli_method(N, P)
        z = sum(discrete_rnd(N, [0, 1], [P, 1-P]));
end

function z = binomial_unit_interval_method(N, P)
     u = rand;
     z = 0;
     pos = 0;
     for j = 0:N
          pos = pos + bincoeff(N, j).*P.^j.*(1-P).^(N-j);
          if u < pos
               return;
          end
          z = z + 1;
     end
end

z1 = zeros(1,1000);
z2 = zeros(1,1000);
for i = 1:1000
     z1(i) = binomial_bernoulli_method(5, 0.5);
     z2(i) = binomial_unit_interval_method(5, 0.5);
end
figure;
hist(z1, [0:10], 1);
figure;
hist(z2, [0:10], 1);
```

4.145

~~3.124~~ Let $T_1, T_2, \ldots$ be exponential interarrival times, then

$$S_n = T_1 + T_2 + \ldots + T_n$$

is the time of the nth arrival. Thus

$$N(t) = k \quad \text{iff} \quad S_k \leq t < S_{k+1} \qquad (*)$$

Therefore to generate $N(t)$ we generate interarrival times $S_1, S_2, \ldots$ until the time t is exceeded as in $(*)$. Then $N(t) = k$.

4.146 The following Octave commands create the necessary functions for the requested program:

```
%This generates random numbers from the gamma distribution
%for alpha > 1.
function X = gamma_rejection_method_agtone(alpha, lambda)
     while(true),
          %Step 1: Generate X with pdf fx(x).
          X = cheng_inverse(alpha, lambda);
          %Step 2: Generate Y uniform in [0, Kfx(X)].
          B = cheng_pdf(X, alpha, lambda);
          Y = rand.*B;
          %Step 3: Accept or reject...
          if (Y <= fx_gamma_pdf(X, alpha, lambda)),
               break;
          end
     end
end

%This helper function generates RVs according to Kfz(x) that will bound
%our distribution.
%We will first generate random numbers according to the following pdf:
%fz(x) = (l.a^l).(x^(l-1))/(a^l + x^l)^2
%and with K = (2a-1)^(1/2)
%First we integrate to obtain the cdf:
%Fz(x) = x^l/(x^l + a^l)
%We have u = Kfz(x). Inverting the function by solving for x, we obtain:
%x = ((u.a^l)/(K-u))^(1/l)
function X = cheng_inverse(alpha, lambda)
     u = rand;
     X = ((u.*alpha.^lambda)./(1-u)).^(1./lambda);
end

%We also want to return B as we have to generate uniform variables in
%[0, Kfz(X)]
function B = cheng_pdf(X, alpha, lambda)
     K = (2.*alpha-1).^(1/2);
     B = (K.*lambda.*alpha.^lambda.*X.^(lambda-1))./((alpha.^lambda +
X.^lambda).^2);
end

%pdf of the gamma distribution.
%You could also use the Octave function gamma_pdf(X,A,B).
function y = fx_gamma_pdf(x, alpha, lambda)
     y = (x.^(alpha-1)).*(e.^(-
x./lambda))./(gamma(alpha).*lambda.^alpha);
end

function X = m_erlang_sum_of_m_exponentials(m, lambda)
     X = sum(exponential_rnd(lambda, 1, m));
end
```

4.10 *Entropy

4.147

~~3.126~~ **a)** $H_X = \log 6$

b) $H_{X|A} = \log 3$

$H_X - X_{X|A} = \log 6 - \log 3 = \log 2$

4.148

~~3.127~~ **a)**

$$\begin{aligned} H_X &= \sum_k P_k \log \frac{1}{P_k} \\ &= -p^3 \log p^3 - p^2 q \log p^2 q - pqp \log pqp - pqq \log pqq \\ &\quad -qpp \log qpp - qpq \log qpq - q^2 p \log q^2 p - q^3 \log q^3 \\ &= -p^3 \log p^3 - 3p^2 \log p^2 q - 3pq^2 \log pq^2 - q^3 \log q^3 \end{aligned}$$

b)

$$\begin{aligned} H_X &= -\sum_k P_k \log P_k \\ &= -p^3 \log p^3 - 3p^2 q \log 3p^2 q - 3pq^2 \log 3pq^2 - q^3 \log q^3 \end{aligned}$$

c) The sample space and the number of outcomes are different in the experiments of parts a) and b).

4.149

8 $P[X=n] = q^n p\ ,\ N = 0, 1, 2, \ldots\ ,\ q = 1-p$

a)

$$\begin{aligned}
P[X=n|X \geq k] &= P[X=n]/P[X \geq k] \text{ for } n \geq k \\
&= \frac{q^n p}{\sum_{i=k}^{\infty} q^i p} \\
&= \frac{q^n}{q^k \frac{1}{1-q}} \\
&= pq^{n-k} \\
H_{X|A} &= -\sum_{n=k}^{\infty} pq^{n-k} \log pq^{n-k} \\
&= -\sum_{n=0}^{\infty} pq^n \log pq^n \\
&= \frac{h(P)}{P}
\end{aligned}$$

This is consistent with the memoryless property of the geometric random variable.

b)

$$\begin{aligned}
P[X=n|X \leq k] &= P[X=n]/P[X \leq k] \text{ for } n \leq k \\
&= \frac{q^n p}{\sum_{i=0}^{k} q^i p} \\
&= \frac{q^n}{\frac{1-q^{k+1}}{1-q}} \\
&= \frac{pq^n}{1-q^{k+1}}
\end{aligned}$$

$$H_{X|A} = -\sum_{n=0}^{k} \frac{pq^n}{1-q^{k+1}} = \log \frac{pq^n}{1-q^{k+1}} \log(1-q^{k+1}) - E[X|A] \log pq$$

4.150 a)

$$\begin{aligned} P[\text{Head}] &= P[\text{Head}|A]P(A) + P[\text{Head}|B]P(B) \\ &= \frac{1}{10}\cdot\frac{1}{2} + \frac{9}{10}\cdot\frac{1}{2} \\ &= \frac{1}{2} \\ P[\text{Tail}] &= \frac{1}{2} \\ H_X &= \log 2 = 1 \text{ bit} \end{aligned}$$

b)

$$\begin{aligned} P[HH] &= P[HH|A]P(A) + P(HH|B)P(B) \\ &= \frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{2} + \frac{9}{10}\cdot\frac{9}{10}\cdot\frac{1}{2} \\ &= \frac{41}{100} \\ P[HT] &= P[HT|A]P(A) + P(HT|B)P(B) \\ &= \frac{1}{10}\cdot\frac{9}{10}\cdot\frac{1}{2} + \frac{9}{10}\cdot\frac{1}{10}\cdot\frac{1}{2} \\ &= \frac{9}{100} \\ P[TH] &= \frac{9}{100} \\ P[TT] &= 1 - \frac{41}{100} - \frac{9}{100} - \frac{9}{100} = \frac{41}{100} \\ H_X &= -\frac{41}{50}\log\frac{41}{100} - \frac{9}{50}\log\frac{9}{100} = 1.68 \text{ bits} \end{aligned}$$

4.151

$$\begin{aligned}
P(A|kth\,\text{toss}) &= \frac{P[A, kth\,\text{toss}]}{P[kth\,\text{toss}]} \\
&= \frac{P[A, kth\,\text{toss}]}{P[kth\,\text{toss}\,|A]P(A) + P[kth\,\text{toss}\,|B]P(B)} \\
&= \frac{\left(\frac{9}{10}\right)^{k-1} \cdot \frac{1}{10}}{\left(\frac{9}{10}\right)^{k-1} \cdot \frac{1}{10} + \left(\frac{1}{10}\right)^{k-1} \cdot \frac{9}{10}} \\
&= \frac{9^{k-1}}{9^{k-1} + 9} \\
&= \frac{9^{k-2}}{9^{k-2} + 1}
\end{aligned}$$

$$P(B|kth\,\text{toss}) = \frac{1}{9^{k-2} + 1}$$

$$H_X = -\frac{9^{k-2}}{9^{k-2}+1}\log\frac{9^{k-2}}{9^{k-2}+1} - \frac{1}{9^{k-2}+1}\log\frac{1}{9^{k-2}+1}$$

$$P[A|1] = \frac{1}{10} \;\; P[A|2] = \frac{1}{2} \;\; P[A|3] = 0.9 \;\; P[A|4] = .9878...$$

The entropy peaks at $k = 2$ and approaches 0 as $k \to \infty$ as we become certain that coin A was selected.

4.152

~~3.131~~ a) $H_I = \log 7$

b) $X = 4$, $I = 3$ or 5, $H_I = \log 2$

4.153

~~3.132~~ The entropy is a function of probabilities and it does not depend on the values taken by the RV. Thus $H_Y = H_X$.

4.154

~~3.133~~ a) $H_X = \log 6$

b) $H_{X,Y} = \log 36 = 2 \log 6$

c) $P[\text{every outcome}] = \left(\frac{1}{6}\right)^n$
$H = \log 6^n = n \log 6$
The uncertainty in each toss is $\log 6$. In n independent tosses, the uncertainty increases linearly, $n \log 6$.

4.155

3.134 a)

$$\begin{aligned} P[Y=1] &= \sum_{i=1}^{6} P[Y=1|X=i]P[X=i] \\ &= \left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{6}\right)\cdot\frac{1}{6}=\frac{147}{360} \end{aligned}$$

$$\begin{aligned} P[Y=2] &= \left(\ldots\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{6}\right)\cdot\frac{1}{6}=\frac{87}{360} \\ P[Y=k] &= \left(\frac{1}{k}+\ldots+\frac{1}{6}\right)\cdot\frac{1}{6} \\ H_Y &= -\sum_k P_k \log P_k = 1.51 \end{aligned}$$

b)

$$\begin{aligned} H(X,Y) &= -\sum_j \sum_k P[X=j,Y=k]\log P[X=j,Y=k] \\ &= -\sum_{j=1}^{6}\sum_{k=1}^{j}\frac{1}{6}\frac{1}{j}\log\frac{1}{6}\frac{1}{j} \\ &= \sum_{j=1}^{6}\frac{1}{6}\log 6j \\ &= \log 6+\frac{1}{6}\log 6! \end{aligned}$$

c) $H(Y|X=k)=\log k$, therefore

$$E[H(Y|X)]=\sum_{k=1}^{6}H(Y|X=k)=\sum_{k=1}^{6}\frac{1}{6}\log k=\frac{1}{6}\log 6!$$

d)

$$\begin{aligned} P(x,y) &= P(y|x)P(x) \\ \log P(x,y) &= \log P(y|x)+\log P(x) \end{aligned}$$

Take expectation on both sides.

$$H(X,Y)=H(Y|X)+H(X)$$

The joint uncertainty (X,Y) is equal to the sum of uncertainty in X and uncertainty of Y given X is observed.

4.156

$$\begin{aligned} H_X &= -\sum_{k=1}^{K} P_k \log P_k \\ &= -\sum_{k=1}^{K-1} P_k \log P_k - P_K \log P_K \\ &\quad -(1-P_K)\log(1-P_K) + (1-P_K)\log(1-P_K) \end{aligned}$$

But $(1-P_K) = \sum_{k=1}^{K-1} P_k$. Therefore,

$$\begin{aligned} H_X &= -P_K \log P_K - (1-P_K)\log(1-P_K) - \sum_{K=1}^{K-1} P_k \log P_k + \sum_{k=1}^{K-1} P_k \log(1-P_K) \\ &= -P_K \log P_K - (1-P_K)\log(1-P_K) - \sum_{k=1}^{K-1} P_k \log \frac{P_k}{(1-P_K)} \end{aligned}$$

We finish the proof by noting that

$$H_Y = -\sum_{k=1}^{K-1} P_k \log \frac{P_k}{(1-P_K)} \quad \text{since} \quad P[Y=k|X \neq K] = \frac{P_k}{1-P_K}$$

4.157

$$VAR[X - Q(X)] = \Delta^2/12 = \alpha^2$$

$$\Delta = \sqrt{12}\alpha$$

$$\begin{aligned} H_Q &= -\ln \Delta - \sum_{k=1}^{K} f_X(x_k) \log f_X(x_k)\Delta \\ &= -\ln\sqrt{12}\alpha - \sum_{k=1}^{K} \frac{1}{b-a} \cdot \sqrt{12}\alpha \log \frac{1}{b-a} \\ &= -\ln\sqrt{12}\alpha - \frac{K\cdot\sqrt{12}\alpha}{b-a} \log \frac{1}{b-a} \end{aligned}$$

4.158

$$\begin{aligned} P[\text{Input } 000|\text{Output } 000] &= \frac{P[\text{ Output } 000, \text{ Input } 000]}{P[\text{ Output } 000]} \\ &= \frac{(1-P)^3 \cdot \frac{1}{2}}{(1-P)^3 \cdot \frac{1}{2} + P^3 \cdot \frac{1}{2}} \\ &= \frac{(1-P)^3}{(1-P)^3 + P^3} \end{aligned}$$

$$P[\text{Input } 111|\text{ Output } 000] = \frac{P3}{(1-P)^3 + P^3}$$

$$H_{X|A} = -\frac{(1-P)^3}{(1-P)^3+P^3} \log \frac{(1-P)^3}{(1-p)^3+P^3} - \frac{P^3}{(1-P)^3+P^3} \log \frac{P^3}{(1-P)^3+P^3}$$

If the output is 010,

$$\begin{aligned} H_{X|A} &= -\frac{P(1-P)^2}{P(1-P)^2+P^2(1-P)} \log \frac{P(1-P)^2}{P(1-P)^2+P^2(1-P)} \\ &\quad -\frac{P^2(1-P)}{P(1-P)^2+P^2(1-P)} \log \frac{P(1-P)^2}{P(1-P)^2+P^2(1-P)} \end{aligned}$$

4.159

~~138~~ X is uniform RV in $[-a, a]$, $f_X(x_k) = \frac{1}{2a}$,

$$\begin{aligned} H_Q &= -\log \Delta - \sum_{k=1}^{K} f_X(x_k)\Delta \log(f_X(x_k)) \\ &= -\log \Delta - \log(f_X(x)) \\ &= -\log \Delta - \log \frac{1}{2a} \\ H_{Q|A} &= -\log \Delta - \log(f_{X|A}(x)) \\ &= -\log \Delta - \log \frac{1}{a} \\ H_Q - H_{Q|A} &= \log \frac{1}{a} - \log \frac{1}{2a} = \log 2 \end{aligned}$$

The difference of the differential entropy $\log(a-(-a)) - \log(a-0) = \log 2$

4.160

$$\begin{aligned}
H_Y &= -\int f_Y(y)\log f_Y(y)dy \\
&= -\int \frac{1}{2}f_X(x)\log\frac{1}{2}f_X(x)\cdot 2dx \\
&= -\log\frac{1}{2} + H_X \\
&= \log 2 + H_X
\end{aligned}$$

Note that $f_Y(y)$ is different from $f_X(x)$.

4.161

~~3.140~~

X	P	l(X)
1	1/2	1
2	1/4	2
3	1/8	3
4	1/16	4
5	1/32	5
6	1/64	6
7	1/128	7
8	1/128	7

For this pmf Equation 3.114 gives $E[L] - H_X = 0$ implying that the code is optimum. An intutive way of seeing that this is optimum is to note that the alternatives in each question are always equiprobable for this pmf.

4.162

$$\begin{aligned}
H_X &= -2\cdot\frac{3}{8}\log\frac{3}{8} - \frac{1}{8}\log\frac{1}{8} - \frac{1}{16}\log\frac{1}{16} - 2\cdot\frac{1}{32}\log\frac{1}{32} \\
&= 1.06 + \frac{3}{8} + \frac{4}{16} + \frac{5}{16} \\
&= 2.0\,\text{bits}
\end{aligned}$$

X	P	Codeword
1	3/8	0
2	3/8	10
3	1/8	110
4	1/16	1110
5	1/32	11110
6	1/32	11111

4.163

~~3.142~~ $\log_2 \begin{pmatrix} 52 \\ 7 \end{pmatrix} = 27\,\text{bits}$

4.164

~~3.143~~ $P[X = k] = \left(\frac{1}{2}\right)^k$

$x = 1$, the codeword is 0 .

$x \geq 2$ the codeword begins with $(x-1)$ ones, and ends with 0 .

The outcomes at each toss is equiprobable implying that we use 1 bit to encode the result.

4.165

~~3.144~~ a)

$$\begin{aligned} E[L] &= \frac{6}{10} \cdot 3 + \frac{4}{10} \cdot 4 = 3 \cdot 4\,\text{bits} \\ H_X &= \log_2 10 = 3.3\,\text{bits} \end{aligned}$$

b) Choose the smallest k, s.t. $2^k > 10^n$

In the full binary tree with depth $(k-1)$, $(10^n - 2^{k-1})$ nodes have to be expanded.

$$E[C] = \frac{2^{k-1} - (10^n - 2^{k-1})}{10^n} \cdot (k-1) + \frac{2(10^n - 2^{k-1})}{10^n} \cdot k$$

$$H_X = \log_2 10^n = n \log_2 10.$$

The performance will be better if $2^k - 10^n$ is small.

4.166

~~3.145~~ a) There are $\begin{pmatrix} n \\ k \end{pmatrix}$ equiprobable patterns, so a code with codewords of lengths $\lceil \log \begin{pmatrix} n \\ k \end{pmatrix} \rceil$ and $\lfloor \log_2 \begin{pmatrix} n \\ k \end{pmatrix} \rfloor$ will be optimum.

b) $\log_2(n+1)$ bits are sufficient.

4.167

~~3.146~~ Set

$$\begin{aligned}
P_k &= Ce^{-\lambda k} = C\alpha^k \\
&\begin{cases} 1 = \sum P_k = C\alpha + C\alpha^2 + C\alpha^3 + C\alpha^4 \\ 2 = E[X] = 1 \cdot C\alpha + 2 \cdot C\alpha^2 + 3 \cdot C\alpha^3 + 4 \cdot C\alpha^4 \end{cases} \\
&C = 0.64,\ a = 0.66 \\
&P_1 = 0.42,\ P_2 = 0.28,\ P_3 = 0.18,\ P_4 = 0.12
\end{aligned}$$

4.168

~~.147~~ Set

$$\begin{aligned}
f_X(x) &= Ce^{-\lambda x} \quad x \geq 0 \\
1 &= \int_0^\infty f_X(x)dx = \int_0^\infty Ce^{-\lambda x}dx = C/\lambda \\
C &= \lambda \\
10 &= E[X] = \int_0^\infty xCe^{-\lambda x}dx \\
&= -\int_0^\infty x de^{-\lambda x} \\
&= -xe^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x}dx \\
&= \frac{1}{\lambda} \\
\lambda &= C = \frac{1}{10}
\end{aligned}$$

$$f_X(x) = \frac{1}{10}e^{-\frac{1}{10}x} \text{ for } x \geq 0$$

X is an exponential RV.

4.169

~~148~~ Set

$$\begin{aligned}
f_x(x) &= C_1 e^{-\lambda x^2} \\
1 &= \int_{-\infty}^{\infty} f_x(x)dx = 2C_1 \int_0^{\infty} e^{-\lambda x^2} dx = C_1\sqrt{\frac{\pi}{\lambda}} \\
\lambda &= \pi C_1^2 \\
C &= E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
2C_1 \int_0^{\infty} x^2 e^{-\lambda x^2} dx &= -\frac{1}{\pi C_1}\int_0^{\infty} x de^{-\pi C_1^2 x^2} \\
&= +\frac{1}{\pi C_1}\int_0^{\infty} e^{-\pi C_1^2 x^2} dx \\
&= +\frac{1}{\pi C_1}\frac{\sqrt{\pi}}{2\sqrt{\pi C_1^2}} \\
&= \frac{1}{2\pi C_1^2}
\end{aligned}$$

Therefore,

$$C_1 = \sqrt{\frac{1}{2\pi C}},\ \lambda = \pi C_1^2 = \frac{1}{2c}$$

$$f_X(x) = \sqrt{\frac{1}{2\pi c}} e^{-\frac{x^2}{2c}}$$

4.170

~~3.149~~ a)

$$\int f_X(x)dx = 1$$
$$\int g_1(x)f_X(x)dx = C_1$$
$$\int g_2(x)f_X(x)dx = C_2$$

Using Lagrange Multipliers,

$$-\int f_X(x)\ln f_X(x)dx + \lambda_1[\int g_1(x)f_X dx - C_1] + \lambda_2[\int g_2(x)f_X(x)dx - C_2]$$
$$= -\int f_X(x)\ln\frac{f_X(x)}{Ce^{-\lambda_1 g_1(x)-\lambda_2 g_2(x)}}dx$$

where $C = e^{-\lambda_1 C_2 - \lambda_2 C_2}$ so $f_X(x)$ has the form of $e^{-\lambda_1 g_1(x)-\lambda_2 g_2(x)}$

b)

$$\begin{aligned} h_X(x) &= -\int f_X(x)\ln f_X(x)dx \\ &= -\int f_X(x)[\ln C - \lambda_1 g_1(x) - \lambda_2 g_2(x)]dx \\ &= -\ln C + \lambda_1 C_1 + \lambda_2 C_2 \end{aligned}$$

4.171

~~3.150~~ From the results of Problem 149, $f_X(x) = Ce^{-\lambda_1 x - \lambda_2 x^2}$. It should be in the form of Gaussian RV.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\alpha}exp\left[-\frac{(x-m)^2}{2\alpha^2}\right]$$

Problems Requiring Cumulative Knowledge

4.172

3.151

$$x = \begin{cases} X_1, & \text{exponential RV, } \frac{1}{\lambda_1} = 1, & \text{with} P_1 = 1/2 \\ X_2, & \text{exponential RV, } \frac{1}{\lambda_1} = 10, & \text{with} P_2 = 1/8 \\ X_3, & \text{constant } 2, & \text{with} P_3 = 3/8 \end{cases}$$

$$\begin{aligned} P[X > 15] &= P[X > 15 | X = X_1]P_1 + P[X > 15 | X = X_2]P_2 + P[X > 15 | X = X_3]P_3 \\ &= e^{-\lambda_1 \cdot 15} \cdot \frac{1}{2} + e^{-\lambda_2 \cdot 15} \cdot \frac{1}{8} + 0 \\ &= 0.028 \\ E[X] = E[E[X|\text{type}]] &= P_1 E[X|X = X_1] + P_2 E[X|X = X_2] + P_3 E[X|X = X_3] \\ &= \frac{1}{2} \cdot 1 + \frac{1}{8} \cdot 10 + \frac{3}{8} \cdot 2 \\ &= 2.5 \end{aligned}$$

Markov's inequality $P[X \geq 15] \leq E[X]/15 = \frac{1}{6}$. The bound is loose.

4.173

.152

$$P[X \geq 9 | X \geq 1] = \frac{P[X \geq 9, X \geq 1]}{P[X \geq 1]} = \frac{\frac{1}{1+9}}{\frac{1}{1+1}} = \frac{1}{5}$$

$$P[X < 9 | X \geq 1] = \frac{4}{5}$$

$$P[\text{At least one bulb is working}] = 1 - (\frac{4}{5})^3$$

4.174

~~3.154~~ $Y = \max\{X_1, X_2, ..., X_n\}$.

$$\begin{aligned}
P[Y \le y] &= P[X_1 \le y, X_2 \le y, ..., X_n \le y] \qquad 0 \le y \le a \\
&= P[X \le y]^n \\
&= \left(\frac{y}{a}\right)^n \\
E[Y] &= \int_0^a y f_Y(y) dy = \int_0^1 y \frac{n y^{n-1}}{a^n} dy \\
&= \frac{n}{a^n} \frac{y^{n+1}}{n+1}\bigg|_0^1 = \frac{n}{n+1} a \\
E[Y^2] &= \frac{n}{a^n} \int_0^1 y^2 y^{n-1} dy = \frac{n}{a^n} \frac{y^{n+2}}{n+2}\bigg|_0^1 = \frac{n}{n+2} a^2 \\
VAR(Y) &= E[Y^2] - E[Y]^2 = \frac{n}{n+2} a^2 - \left(\frac{n}{n+1}\right)^2 a^2 \\
&= \left[\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right] a^2
\end{aligned}$$

The value of "a" is by definition larger than any value Y can assume. In addition, when n is large the above results show that Y tends to be close to "a".

4.175

a) $P[X \le -a] = \Phi_X(-a)$
$P[-a < X \le 0] = \Phi_X(0) - \Phi_X(-a)$
$P[0 < X \le a] = \Phi_X(a) - \Phi_X(0)$
$P[X > a] = 1 - \Phi_X(a)$

since $P[X \le -a] = P[-a < X \le 0] = P[0 < X \le a] = P[X > a]$
$\Phi(-a) = 1/4$
$\Phi(0) = 1/2$
$\Phi(a) = 3/4$

$\Phi(-a) = Q(a) = 1/4 \longrightarrow a \simeq 0.7$

b) $\int_0^a 2(x - x_1)(-1) f_X(x)\,dx = 0$
$2x_1 \int_0^a f_X(x)\,dx = 2\int_0^a x f_X(x)\,dx$
$2x_1 P[0 \le X \le a] = 2\int_0^a \frac{x}{\sqrt{2\pi}} e^{-x^2/2}\,dx$
$2x_1 \Phi(-a) = \frac{2}{\sqrt{2\pi}}(1 - e^{-a^2/2})$
$x_1 = \frac{(1 - e^{-a^2/2})}{\sqrt{2\pi}\,\Phi(-a)} = \frac{1 - e^{-0.245}}{\sqrt{2\pi}}(4) = 0.3468$

Similarly
$x_2 = \frac{4}{\sqrt{2\pi}} e^{-0.245} = 1.249$

(x)

$\rightarrow x_{-1} = -x_1$
$x_{-2} = -x_2$

c) $E[(X - q(X))^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx + (x_{-2}^2 + x_{-1}^2 + x_1^2 + x_2^2)\int_{-\infty}^{\infty} f_X(x)\,dx - 2(x_{-2} + x_{-1} + x_1 + x_2)\int_{-\infty}^{\infty} x\, f_X$
$= x_2^2 + x_{-1}^2 + x_1^2 + x_2^2$
$= 2\left(\frac{4}{\sqrt{2\pi}} e^{-0.245}\right)^2 + 2\left(\frac{4}{\sqrt{2\pi}}(1 - e^{-0.245})\right)^2$
$= \frac{16}{\pi}(1 + e^{-0.49} - 2e^{-0.245}) = 0.2405$

4.176

~~3.159~~ **a)**

$$\begin{aligned} P[\text{input is } 1 | y < Y < y+h] &= \frac{P[\text{input is } 1, y < Y < y+h]}{P[y < Y < y+h]} \\ &= \frac{\int_y^{y+h} f_1(t)dt \cdot p}{\int_y^{y+h} f_1(t)dt \cdot p + \int_y^{y+h} f_0(t)dt \cdot (1-p)} \\ &\approx \frac{f_1(t)hp}{f_1(t)ph + f_0(t)h(1-p)} \end{aligned}$$

where we assume $h \ll 1$.

b)

$$\begin{aligned} & P[\text{input is } 1 | y < Y < y+h] &&> P[\text{input is } 0 | y < Y < y+h] \\ \text{iff}\quad & f_1(t)p &&> f_0(t)(1-p) \\ \text{iff}\quad & \tfrac{p}{\sqrt{2\pi}} e^{-(y-1)^2/2} &&> \tfrac{(1-p)}{\sqrt{2\pi}} e^{-y^2/2} \\ \text{iff}\quad & e^{-\frac{1}{2}(y^2-2y+1-y^2)} &&> \tfrac{1-p}{p} \\ \text{iff}\quad & -\tfrac{1}{2}(-2y+1) &&> \ln \tfrac{1-p}{p} \\ \text{iff}\quad & y > \tfrac{1}{2} + \ln \tfrac{1-p}{p} &&= T. \end{aligned}$$

c)

$$\begin{aligned} P_e &= P[Y > T, \text{input } 0](1-p) + P[Y < T, \text{input } 1]p \\ &= (1-p)\int_T^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + p \int_{-\infty}^{T} \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2} dy \\ &= (1-p)Q(T) + pQ(1-T) \end{aligned}$$

Chapter 5: Pairs of Random Variables

5.1 Two Random Variables

5.1

(a) S

Michael \ Carlos	0	1	2	
0	00	01	02	$\frac{1}{4}$
1	10	11	12	$\frac{1}{2}$
2	20	21	22	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

S_{XY}

X = min \ Y = max	0	1	2
0	00	01	02
1		11	12
2			22

(b)

$P[X=0, Y=0] = P[\{00\}] = \frac{1}{16}$

$P[X=0, Y=1] = P[\{01, 10\}] = \frac{1}{4}\frac{1}{2} + \frac{1}{4}\frac{1}{2} = \frac{1}{4}$

$P[X=0, Y=2] = P[\{02, 20\}] = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

$P[X=1, Y=1] = P[\{11\}] = \frac{1}{4}$

$P[X=1, Y=2] = P[\{12, 21\}] = \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4}$

$P[X=2, Y=2] = \frac{1}{16}$

(c) $P[X=Y] = P[(X,Y) \in \{00, 11, 22\}] = \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{3}{8}$

(d)

M \ C	0	1	2	
0				$\frac{1}{4}$
1				$\frac{1}{2}$
2				$\frac{1}{4}$
	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$	

$P[X=0, Y=0] = \frac{1}{64}$

$P[X=0, Y=1] = \frac{6}{64} + \frac{2}{64} = \frac{8}{64}$

$P[X=0, Y=2] = \frac{9}{64} + \frac{1}{64} = \frac{10}{64}$

$P[X=1, Y=1] = \frac{12}{64}$

$P[X=1, Y=2] = \frac{18}{64} + \frac{6}{64} = \frac{24}{64}$

$P[X=2, Y=2] = \frac{9}{64}$

$P[X=Y] = \frac{1}{64} + \frac{12}{64} + \frac{9}{64} = \frac{22}{64}$

5.2

ⓐ S

M \	0	1	2	
0	00	01	02	$\frac{1}{4}$
1	10	11	12	$\frac{1}{2}$
2	20	21	22	$\frac{1}{4}$
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

S_{XY}

X \ Y	0	1	2	3	4
−2			02		
−1		01		12	
0	00		11		22
1		10		21	
2			20		

ⓑ

	fair coin	for biased coin:
$P[X=-2, Y=2] = P[\{02\}]$	$= \frac{1}{16}$	$9/64$
$P[X=-1, Y=1] = P[\{01\}]$	$= \frac{1}{8}$	$6/64$
$P[X=-1, Y=3] = P[\{12\}]$	$= \frac{1}{8}$	$18/64$
$P[X=0, Y=0] = P[\{00\}]$	$= \frac{1}{16}$	$1/64$
$P[X=0, Y=2] = P[\{11\}]$	$= \frac{1}{4}$	$12/64$
$P[X=0, Y=4] = P[\{22\}]$	$= \frac{1}{16}$	$9/64$
$P[X=1, Y=1] = P[\{10\}]$	$= \frac{1}{8}$	$2/64$
$P[X=1, Y=3] = P[\{21\}]$	$= \frac{1}{8}$	$6/64$
$P[X=2, Y=2] = P[\{20\}]$	$= \frac{1}{16}$	$1/64$

ⓒ $P[X+Y=1] = 0$

$$P[X+Y=2] = P[(X,Y) \in \{(-1,3), (0,2), (1,1)\}] = \frac{1}{8} + \frac{1}{4} + \frac{1}{8}$$

$$= \frac{1}{2}$$

5.3

(a) Sample Space: a set of outcomes where each outcome is a pair $\underline{\zeta} = (\zeta_1, \zeta_2)$ where ζ_1 is the input and ζ_2 is the output.

$$S_{XY} = \{(-1,-1), (-1,0), (-1,1), (1,-1), (1,0), (1,1)\}$$

(b) $P[X=1, Y=-1]$

$= P[Y=-1 \mid X=1]\,P[X=1]$

$= \frac{3}{4}p$ //

$P[X=-1, Y=-1]$

$= \frac{1}{4}(1-p-p_e)$ //

$P[X=1, Y=0]$

$= P[Y=0 \mid X=1]\,P[X=1]$

$= \frac{3}{4}p_e$ //

$P[X=-1, Y=0]$

$= \frac{1}{4}p_e$ //

$P[X=1, Y=1]$

$= P[Y=1 \mid X=1]\,P[X=1]$

$= \frac{3}{4}(1-p-p_e)$ //

$P[X=-1, Y=1]$

$= \frac{1}{4}p$ //

(c) $P[X \neq Y]$

$= \frac{1}{4}p_e + \frac{1}{4}p + \frac{3}{4}p_e + \frac{3}{4}p$

$= p_e + p$ //

$P[Y=0]$

$= \frac{3}{4}p_e + \frac{1}{4}p_e$

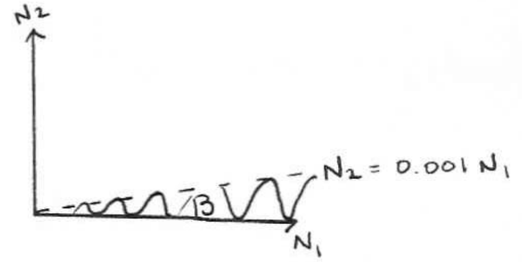
$= p_e$ //

5.4

(a) Let ζ_n be the number of arrivals specified by ζ

$$0 \le N_1 \le \zeta_n \quad \text{and} \quad N_2 = \zeta_n - N_1$$

(b) $B = \{ 0.001 N_1 > N_2 \}$

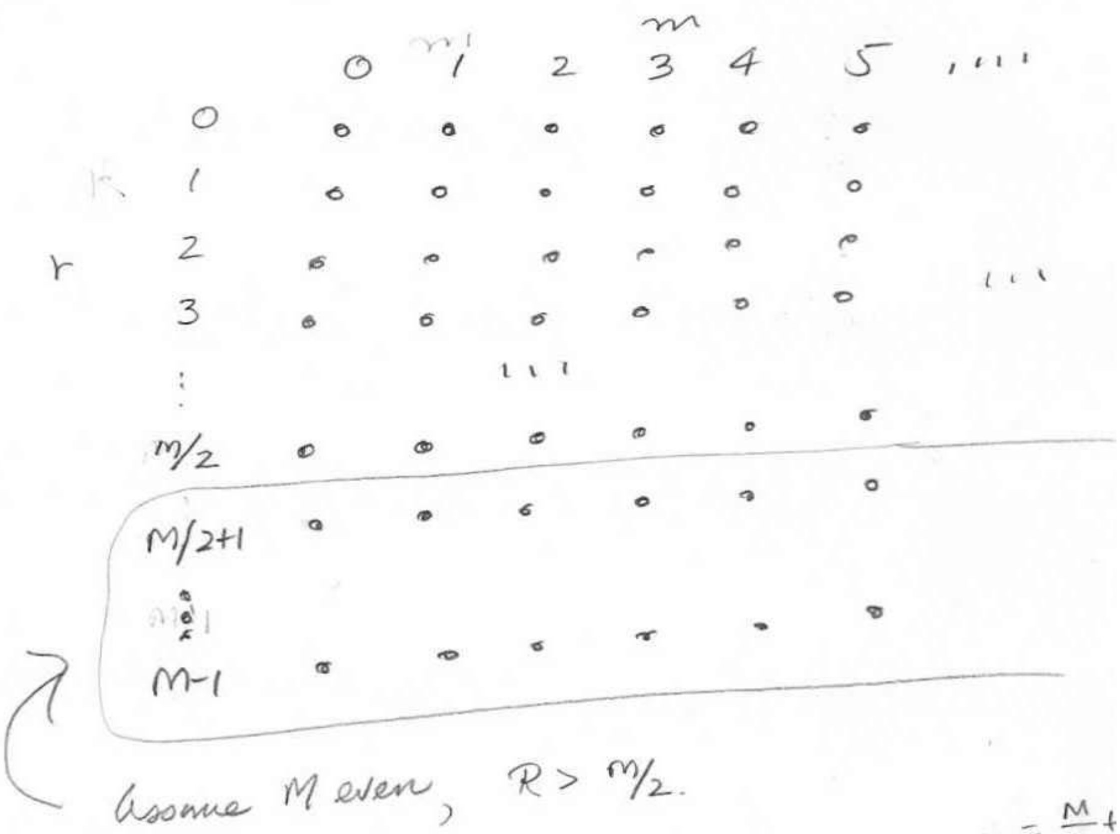


5.5

$\xi \in \{1, 2, \ldots\}$

ⓐ $Q = \left\lfloor \frac{\xi}{M} \right\rfloor$ $\quad Q \in \{0, 1, 2, \ldots\}$

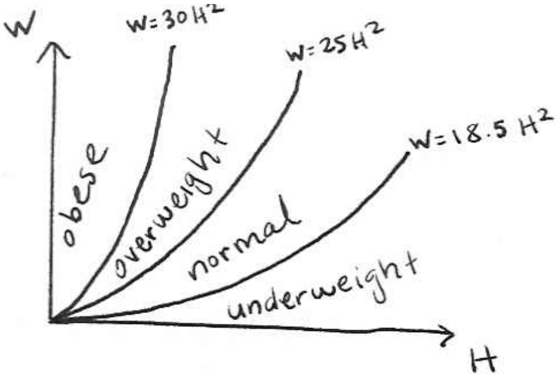
$R = Q - \left\lfloor \frac{\xi}{M} \right\rfloor M$ $\quad R \in \{0, 1, \ldots, M-1\}$



Assume M even, R > M/2.

"Last Packet More than Half full" $= \{(q, r) : q = 0, 1, \ldots$ and $r = \frac{M}{2}+1, \ldots, M-1\}$

5.6

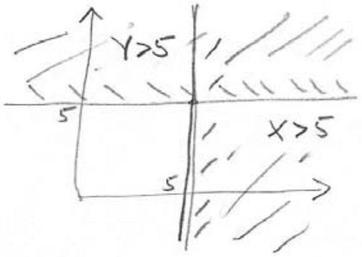


$W/H^2 < c$

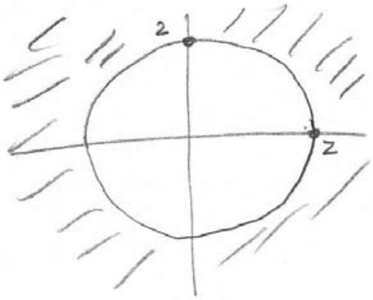
$W < c H^2$

5.7

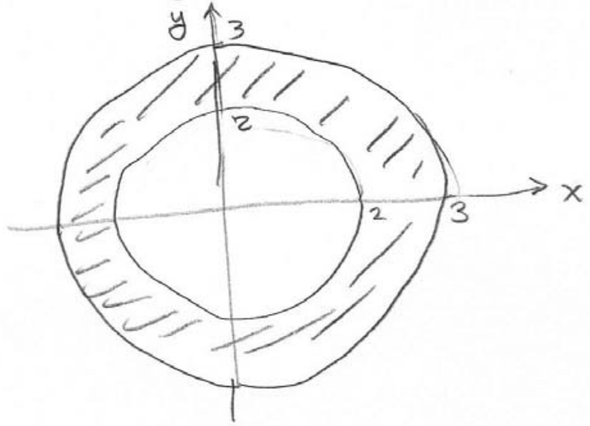
ⓐ $\{\max(X,Y) > 5\} = \{X > 5\} \cup \{Y > 5\}$



ⓑ $\{X^2+Y^2 > 4\}$



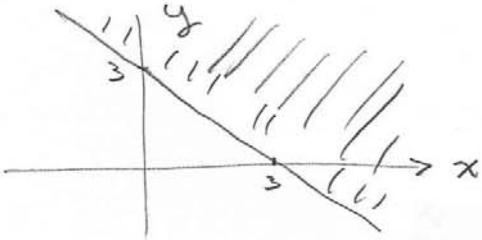
ⓒ $\{4 < X^2+Y^2 < 9\}$



5.8

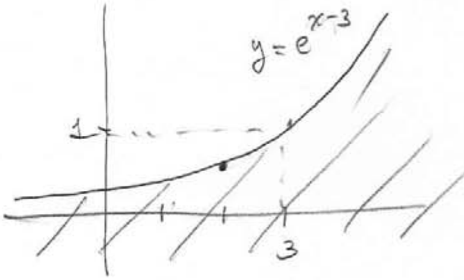
ⓐ $\{X+Y>3\} = \{Y>3-X\}$

Not product form



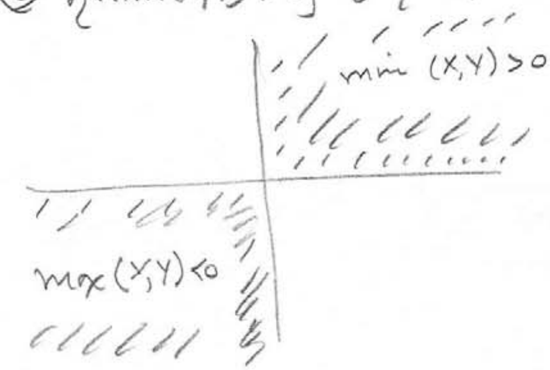
ⓑ $\{e^X > Ye^3\}$

$= \{Y < e^{X-3}$



Not product form

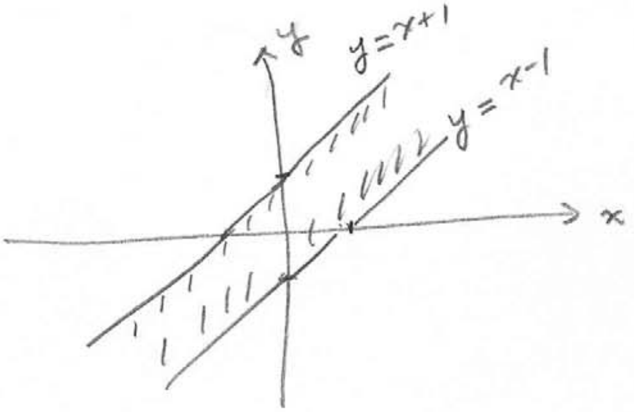
ⓒ $\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}$



Not product form
but union of product form

ⓓ $\{|X-Y| \geq 1\} = \{-1 \leq X-Y \leq 1\}$

$= \{Y-1 \leq X \leq Y+1\}$



Not product form

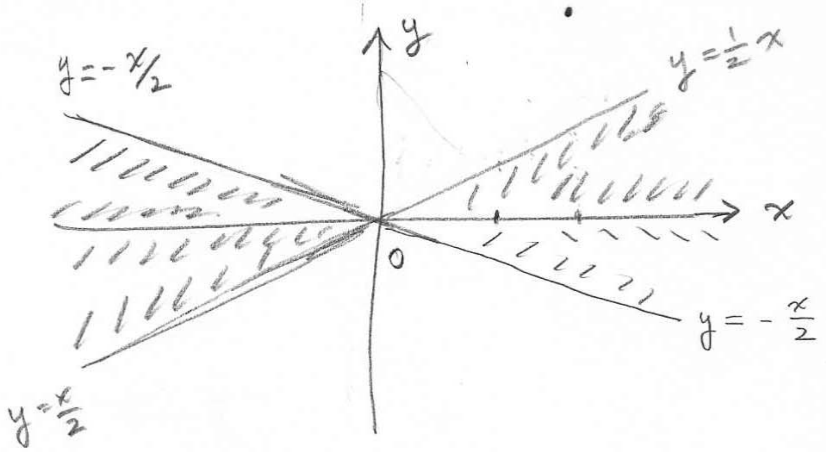
ⓔ $\{|X/Y| > 2\}$

$-\frac{x}{y} > 2 \iff -x/2 > y$

$x/y > 2 \iff \frac{1}{2}x > y$

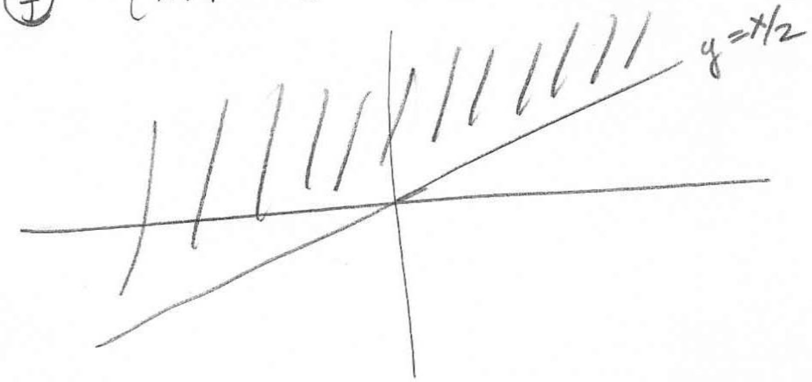
$x/y > 2 \iff \frac{x}{2} < y$

$-x/y > 2 \iff -\frac{x}{2} < y$



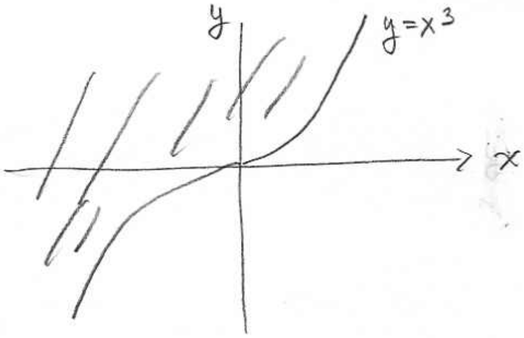
Not product form.

ⓕ $\{X/Y < 2\} = \{X/2 < Y\}$



Not product form.

(g) $\{X^3 > Y\}$

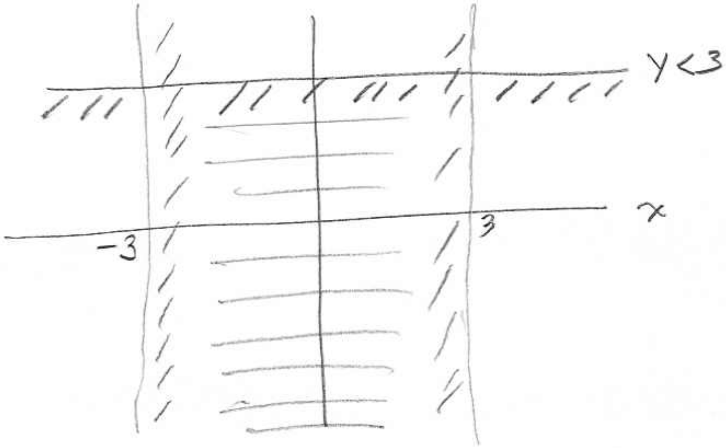


Not product form.

(h) $\{XY<0\} = \{X<0, Y>0\} \cup \{X>0, Y<0\}$

Not product form but union of product form.

(i) $\max\{(|X|, Y) < 3\} = \max\{(-X, X, Y) < 3\}$
$= \{-X<3, X<3, Y<3\}$

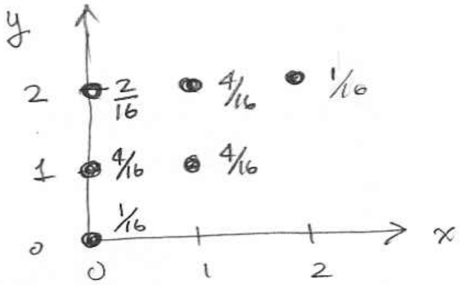


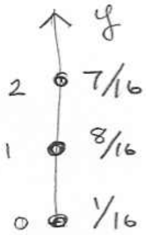
product form

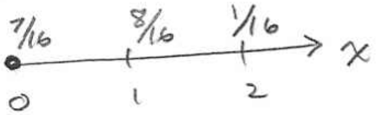
5.2 Pairs of Discrete Random Variables

5.9

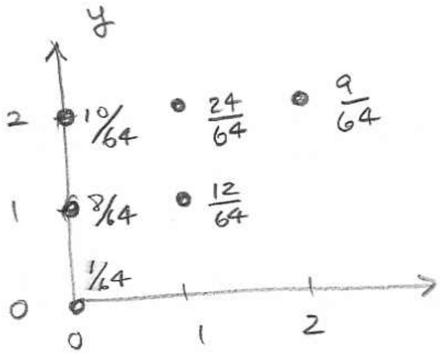
a)

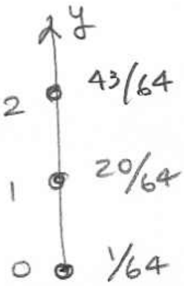




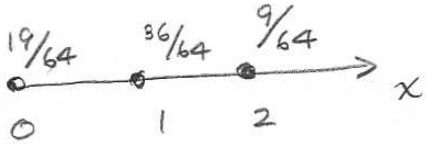


b)





shift towards higher values of y



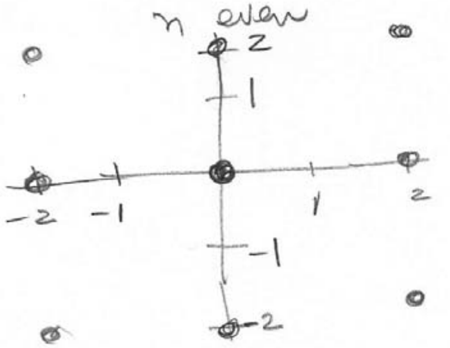
shift to higher values of x.

ⓒ If n is even then X and Y can only take on even values.

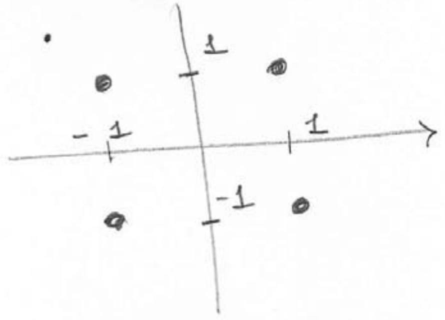
If n is odd then X and Y can only take odd values

$P[(X,Y) \text{ within } \sqrt{2} \text{ of origin}]$

n even



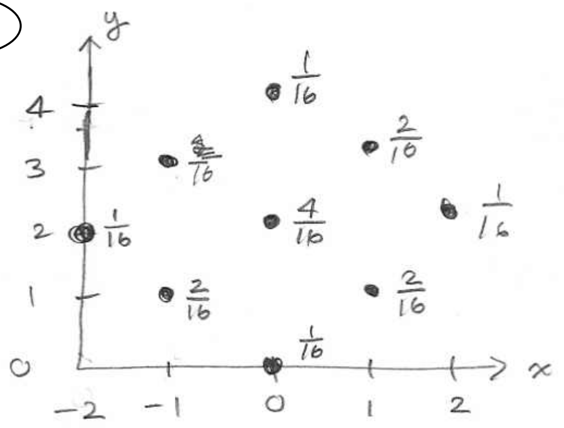
$n = 2l+1$ odd

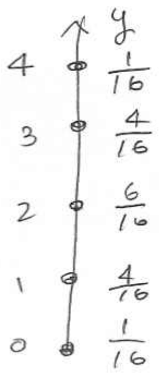


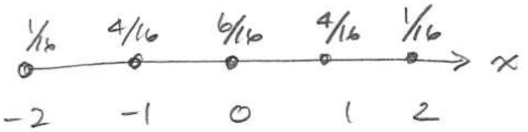
$$P[X=0, Y=0] = P\left[N_1 = \frac{n}{2}, N_2 = \frac{n}{2}\right] = \left(\binom{n}{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}\right)^2$$

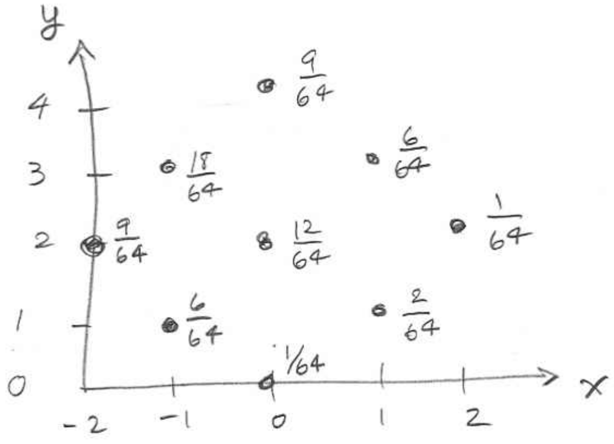
$$\left(P[X=-1] + P[X=1]\right) \times \left(P[Y=-1] + P[Y=1]\right) = \left[\binom{n}{l} p^{l} (1-p)^{n-l} + \binom{n}{l+1} p^{l+1} (1-p)^{n-l-1}\right]^2$$

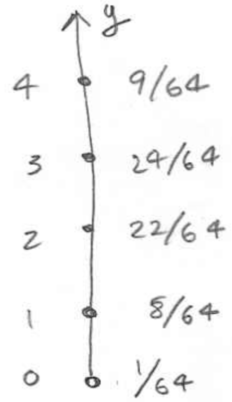
5.10



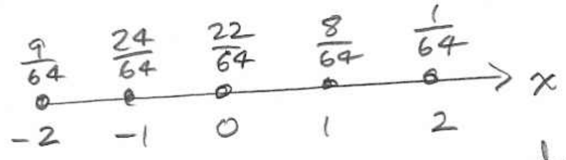








shift to higher values



shift to more negative values

5.11

(i)

X \ Y	-1	0	1	
-1	$1/6$	$1/6$	0	$1/3$
0	0	0	$1/3$	$1/3$
1	$1/6$	$1/6$	0	$1/3$
	$1/3$	$1/3$	$1/3$	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=\frac{1}{2}$

$P[X=-Y]=1/6$

(ii)

X \ Y	-1	0	1	
-1	$1/9$	$1/9$	$1/9$	$1/3$
0	$1/9$	$1/9$	$1/9$	$1/3$
1	$1/9$	$1/9$	$1/9$	$1/3$
	$1/3$	$1/3$	$1/3$	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=\frac{2}{3}$

$P[X=-Y]=\frac{1}{3}$

(iii)

X \ Y	-1	0	1	
-1	$1/3$	0	0	$1/3$
0	0	$1/3$	0	$1/3$
1	0	0	$1/3$	$1/3$
	$1/3$	$1/3$	$1/3$	

$P[X=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[Y=i]=\frac{1}{3} \quad i\in\{-1,0,1\}$

$P[X>0]=\frac{1}{3}$

$P[X\geq Y]=1$

$P[X=-Y]=1/3$

Three different joint pmf's have the same marginal pmf's.

Events that involve joint behavior have different probabilities.

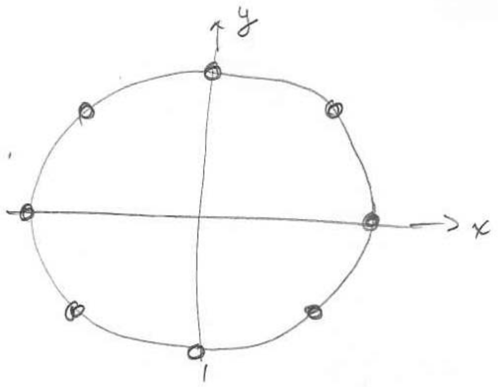
5.12

(a) Mapping S to S_{XY}

Θ	0	1	2	3	4	5	6	7
X, Y	$(r,0)$	$(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$	$(0,r)$	$(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$	$(-r,0)$	$(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$	$(0,-r)$	$(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$

(b) Joint PMF

X \ Y	$-r$	$\frac{-r}{\sqrt{2}}$	0	$\frac{r}{\sqrt{2}}$	r
$-r$	0	0	1/8	0	0
$\frac{-r}{\sqrt{2}}$	0	1/8	0	1/8	0
0	1/8	0	0	0	1/8
$\frac{r}{\sqrt{2}}$	0	1/8	0	1/8	0
r	0	0	1/8	0	0



(c) $p_X(x) = \sum_k p_{XY}(x,k)$

$p_X(r) = 1/8$	$p_Y(r) = 1/8$
$p_X(\frac{r}{\sqrt{2}}) = 1/4$	$p_Y(\frac{r}{\sqrt{2}}) = 1/4$
$p_X(0) = 1/4$	$p_Y(0) = 1/4$
$p_X(\frac{-r}{\sqrt{2}}) = 1/4$	$p_Y(-\frac{r}{\sqrt{2}}) = 1/4$
$p_X(-r) = 1/8$	$p_Y(-r) = 1/8$

(d) $P[A] = p_X(0) = 1/4$

$P[B] = 1 - p_Y(1) = 7/8$

$P[C] = p_{XY}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}) = 1/8$

$P[D] = p_{XY}(-r, 0) = 1/8$

5.13

(a) Sample space: 200 outcomes ζ_i representing zero or one page request for the given 1ms interval

$$S_{XY} = \{(i,j): 0 \le i \le 100,\ 0 \le j \le 100\}$$

(b) $$p_{XY}(x,y) = \binom{100}{x}(0.05)^x(0.95)^{100-x} \cdot \binom{100}{y}(0.05)^y(0.95)^{100-y}$$

(c) $$P[X=x] = \binom{100}{x}(0.05)^x(0.95)^{100-x}$$

$$P[Y=y] = \binom{100}{y}(0.05)^y(0.95)^{100-y}$$

(d) $$P[X \ge Y] = \underbrace{P[X>Y]}_{\frac{1}{2}(1-P[X=Y])} + P[X=Y] = \frac{1}{2} + \frac{1}{2}P[X=Y] = \frac{1}{2} + \frac{1}{2}\underbrace{\sum_{j=0}^{100}(P[X=j])^2}_{.13073}$$

$$= .5654$$

$$P[X=0, Y=0] = \binom{100}{0}\binom{100}{0}(0.05)^0(0.05)^0(0.95)^{100}(0.95)^{100}$$

$$= 0.95^{202}$$

$$\simeq 3.5 \times 10^{-5}$$

$$P[X>5, Y>3]$$

$$= (1-P[X \le 5])(1-P[Y \le 3])$$

$$= \left(1-\sum_{x=0}^{5}\binom{100}{x}(0.05)^x(0.95)^{100-x}\right)\left(1-\sum_{y=0}^{3}\binom{100}{y}(0.05)^y(0.95)^{100-y}\right)$$

$$= 0.285$$

(e) $P[X+Y=10]$ ⇐ P["10 requests in 200 ms"]

$$= \binom{200}{10}(0.05)^{10}(0.95)^{190}$$

$$= 0.128$$

5.14

(a) Sample Space: 200 outcomes representing zero or one page request for the given 1ms interval.

$S_{XY} = \{(i,j) : 0 \le i \le 100,\ i \le j \le 200\}$

(b) $p_{N_1N_2}(n_1,n_2) = P[N_1 = n_1, N_2 = n_2]$

$= \binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}\binom{100}{n_2-n_1}(0.05)^{n_2-n_1}(0.95)^{100-(n_2-n_1)}$

(c) $p_{N_1}(n_1) = \binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}$

$p_{N_2}(n_2) = \binom{200}{n_2}(0.05)^{n_2}(0.95)^{200-n_2}$

(d) $P[A] = P[N_1 < N_2]$

$= 1 - P[N_1 = N_2]$ since $N_1 \le N_2$

$= 1 - P[$"0 requests in second 100ms interval"$]$

$= 1 - \binom{100}{0}(0.05)^0(0.95)^{106}$

$= 0.994$ //

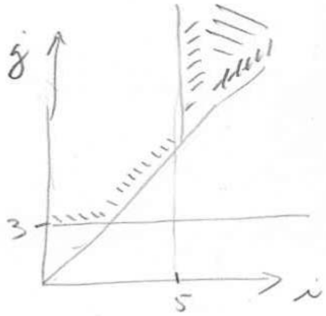
$P[B] = P[N_2 = 0]$

$= \binom{200}{0}(0.05)^0(0.95)^{200}$

$= 0.95^{200}$

$\cong 3.5 \times 10^{-5}$

$P[C] = P[N_1 > 5, N_2 > 3]$ →



$= P[N_1 > 5]$ since $N_1 \le N_2$

$= 1 - P[N_1 \le 5]$

$= 1 - \sum_{n_1=0}^{5}\binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}$

$= 0.384$ //

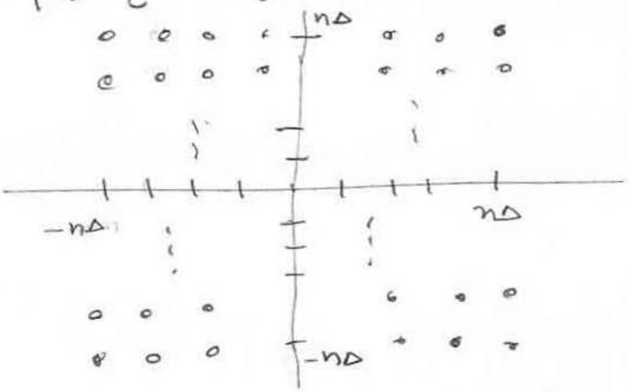
5.15

(a) $S = \{(k_1, k_2) : 0 \le k_1 \le n,\ 0 \le k_2 \le n\}$

where $N_1 = k_1$ = # heads in n tosses of a coin
$N_2 = k_2$ = # heads in n tosses of a coin

$$X = \Delta N_1 - (n - N_1)\Delta = 2N_1\Delta - n\Delta$$
$$Y = \Delta N_2 - (n - N_2)\Delta = 2N_2\Delta - n\Delta$$

$$X \in \{-n\Delta, (-n+2)\Delta, \ldots, n\Delta\}$$
$$Y \in \{-n\Delta, (-n+2)\Delta, \ldots, n\Delta\}$$

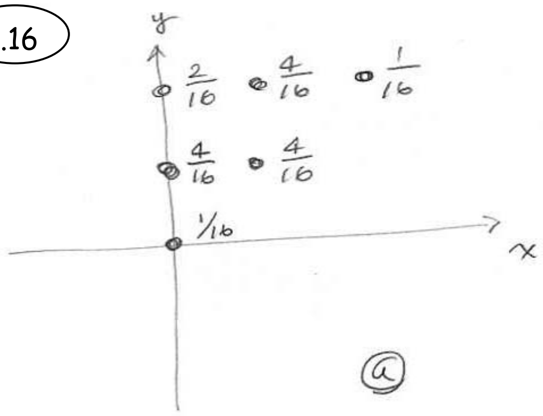


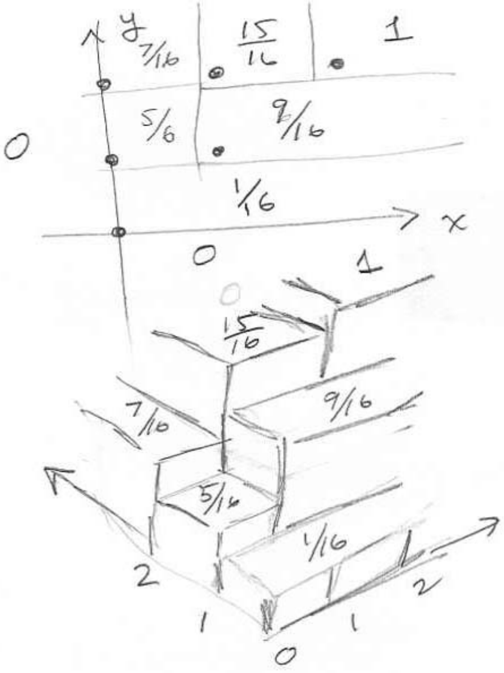
(b)
$$P[X = (2k-n)\Delta] = \binom{n}{k} p^k (1-p)^{n-k}$$
$$P[Y = (2j-n)\Delta] = \binom{n}{k} p^j (1-p)^{n-j}$$
$$P[X = (2k-n)\Delta, Y = (2j-n)\Delta]$$
$$= \binom{n}{k} p^k (1-p)^{n-k} \binom{n}{j} p^j (1-p)^{n-j}$$

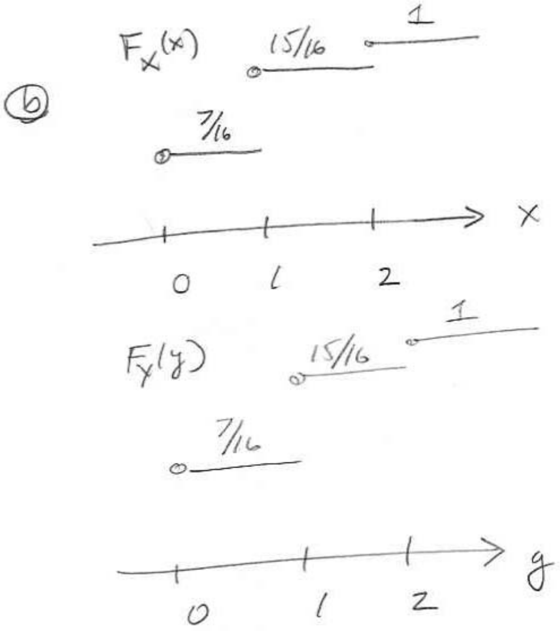
assuming same coin is used for both directions.

5.3 The Joint pdf of *X* and *Y*

5.16

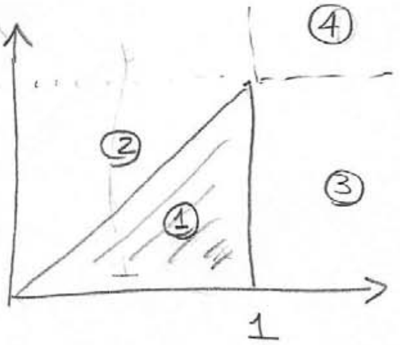






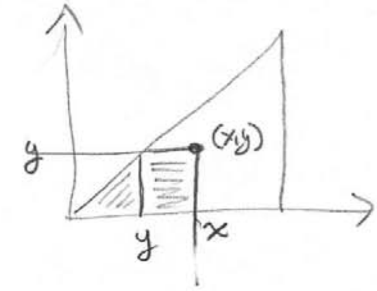
5.17

(a)



Area of Triangle is $1/2$

Region 1

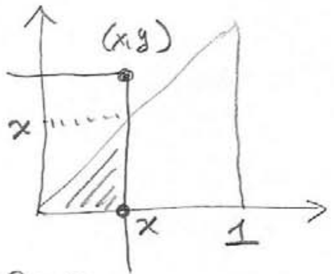


$0 < y < x < 1$.

$$P[X \le x, Y \le y] = \frac{\overbrace{\frac{y^2}{2}}^{\text{triangle}} + \overbrace{y(x-y)}^{\text{rectangle}}}{1/2}$$

$$= 2\left(xy - \frac{y^2}{2}\right)$$

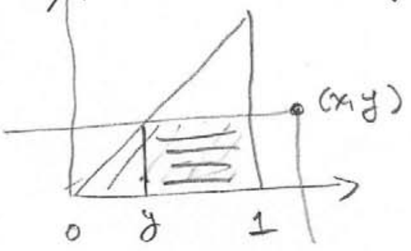
Region 2



$0 < x < y$

$$P[X \le x, Y \le y] = \frac{x^2/2}{1/2} = x^2$$

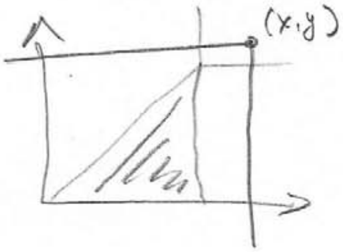
Region 3



$y < x,\ x > 1$

$$P[X \le x, Y \le y] = \frac{\frac{y^2}{2} + y(1-y)}{1/2}$$

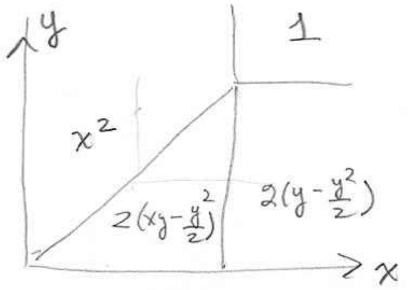
$$= 2\left(y - \frac{y^2}{2}\right)$$



$x > 1,\ y > 1$

$$P[X \le x, Y \le y] = 1$$

5.17



ⓑ $P[X \le x] = F_{XY}(x,\infty) = x^2$

$P[Y \le y] = F_{XY}(\infty, y) = 2\left(y - \frac{y^2}{2}\right)$

ⓒ $P\left[X \le \frac{1}{2}, Y \le \frac{3}{4}\right] = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ since $\left(\frac{1}{2},\frac{3}{4}\right)$ is in Region 2

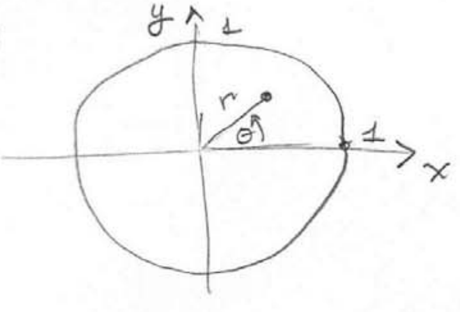
$$P\left[\frac{1}{4} < X \le \frac{3}{4}, \frac{1}{4} < Y \le \frac{3}{4}\right]$$

$$= F_{XY}\left(\frac{3}{4},\frac{3}{4}\right) - F_{XY}\left(\frac{3}{4},\frac{1}{4}\right) - F_{XY}\left(\frac{1}{4},\frac{3}{4}\right) + F_{XY}\left(\frac{1}{4},\frac{1}{4}\right)$$

$$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2\right) - \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2$$

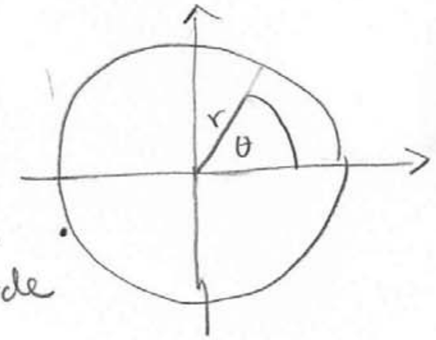
$$= \frac{1}{4}$$

5.18



$0 \le R \le 1$

$0 \le \Theta \le 2\pi$

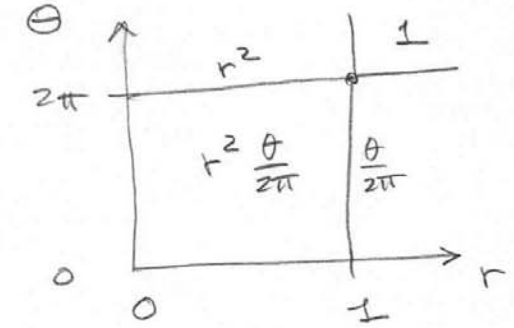


(a) $P[R \le r, \Theta \le \theta]$

$= \frac{\text{area of pie slice}}{\text{area of unit circle}}$

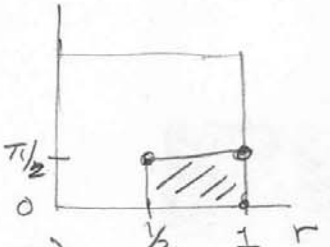
$= \frac{\pi r^2 \cdot \frac{\theta}{2\pi}}{\pi(1)^2} = r^2 \frac{\theta}{2\pi}$ $\quad 0 \le r \le 1$, $\quad 0 \le \theta \le 2\pi$

(b)



(b) $F_R(r) = r^2 \quad 0 \le r \le 1$

$F_\Theta(\theta) = \frac{\theta}{2\pi} \quad 0 \le \theta \le 2\pi$



(c) $P[R > \frac{1}{2}, 0 < \Theta < \frac{\pi}{2}]$

$= F_{R\Theta}(1, \frac{\pi}{2}) - F_{R\Theta}(1, 0) - F_{R\Theta}(\frac{1}{2}, \frac{\pi}{2}) + F_{R\Theta}(\frac{1}{2}, 0)$

$= \frac{1}{4} - 0 - \frac{1}{4} \cdot \frac{1}{4} + 0 = \frac{3}{16}$

5.19

(g)

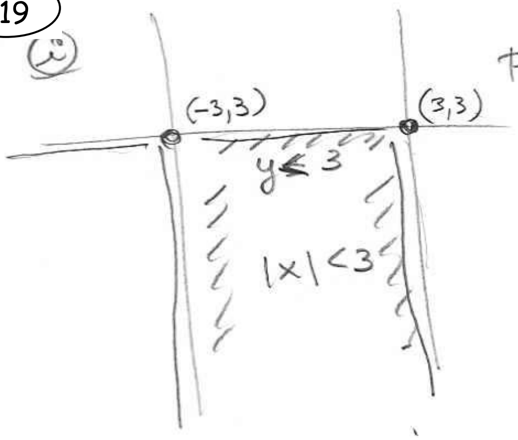
$$\begin{aligned}
&P[\{\min(X,Y)>0\} \cup \{\max(X,Y)<0\}] \\
&= P[\{\min(X,Y)>0\}] + P[\{\max(X,Y)<0\}] \\
&= P[X>0, Y>0] + P[X<0, Y<0] \\
&= F_{XY}(0^-, 0^-) + 1 - P[\{X>0\}^c \cup \{Y>0\}^c] \\
&= F_{XY}(0^-, 0^-) + 1 - P[X\le 0] - P[Y \le 0] \\
&\qquad + P[X \le 0, Y \le 0] \\
&= F_{XY}(0^-, 0^-) + 1 - F_{XY}(0, \infty) \\
&\qquad - F_{XY}(\infty, 0) + F_{XY}(0, 0)
\end{aligned}$$

(h)

$$\begin{aligned}
&P[\{X<0, Y>0\} \cup \{X>0, Y<0\}] \\
&= P[\{X<0, Y>0\}] + P[\{X>0, Y<0\}] \\
&= P[\{X<0\}] - P[X<0, Y\le 0] \\
&\qquad + P[\{Y<0\}] - P[X \le 0, Y<0] \\
&= F_{XY}(0^-, \infty) - F_{XY}(0^-, 0) \\
&\qquad + F_{XY}(\infty, 0^-) - F_{XY}(0, 0^-)
\end{aligned}$$

5.19

(i)



$$P[-3 < X < 3,\ Y < 3]$$

$$= F_{XY}(3^-, 3^-) - F_{XY}(-3, 3^-)$$

5.20 (b) $F_X(x) = F_{XY}(x, \infty)$

$$= \begin{cases} 1 - 1/x^2 & , x > 1 \\ 0 & , \text{otherwise} \end{cases}$$

$F_Y(y) = F_{XY}(\infty, y)$

$$= \begin{cases} 1 - 1/y^2 & , y > 1 \\ 0 & , \text{otherwise} \end{cases}$$

(c) $P\{X < 3, Y \le 5\}$

$= F_{XY}(3, 5)$

$= (1 - \frac{1}{9})(1 - \frac{1}{25})$

$= \frac{64}{75}$ //

$P\{X > 4, Y > 3\}$

$= 1 - F_{XY}(4, \infty) - F_{XY}(\infty, 3) + F_{XY}(4, 3)$

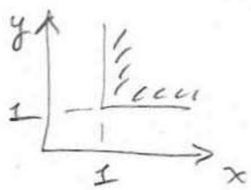
$= 1 - (1 - \frac{1}{16}) - (1 - \frac{1}{9}) + (1 - \frac{1}{16})(1 - \frac{1}{9})$

$= 1 - \frac{15}{16} - \frac{8}{9} + \frac{5}{6}$

$= \frac{1}{144}$ //

5.21

$$F_{XY}(x,y) = \begin{cases} 1-\frac{1}{x^2y^2} & x>1,\ y>1 \\ 0 & \text{elsewhere.} \end{cases}$$



$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = 1 \quad \text{all } x>1$$

$F_X(x)$ cannot be equal to 1 for all x

∴ not a valid cdf

5.22 Properties

(i) $F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$

$F_X(x_1)F_Y(y_1) \le F_X(x_2)F_Y(y_2)$

is true since $0 \le F_X(x_1) \le F_X(x_2)$

and $0 \le F_Y(y_1) \le F_Y(y_2)$

(ii) $F_{XY}(x_1, -\infty) = 0$

$F_X(x_1)F_Y(-\infty) = 0$

is true since $F_Y(-\infty) = 0$

$F_{XY}(-\infty, y_1) = 0$

$F_X(-\infty)F_Y(y_1) = 0$

is true since $F_X(-\infty) = 0$

$F_{XY}(\infty, \infty) = 1$

$F_X(\infty)F_Y(\infty) = 1$

is true since $F_X(\infty) = F_Y(\infty) = 1$

(iii) $F_X(x_1) = F_{XY}(x_1, \infty)$

$= F_X(x_1)F_Y(\infty)$

$= F_X(x_1) \cdot 1$

$= F_X(x_1)$

$F_Y(y_1) = F_{XY}(\infty, y_1)$

$= F_X(\infty)F_Y(y_1)$

$= 1 \cdot F_Y(y_1)$

$= F_Y(y_1)$

(iv) $\lim_{x \to a^+} F_{XY}(x, y)$

$= \lim_{x \to a^+} F_X(x)F_Y(y)$

$F_X(x)$ right continuous

$= F_X(a)F_Y(y)$

$= F_{XY}(a, y)$

$\lim_{y \to b^+} F_{XY}(x, y)$

$= \lim_{y \to b^+} F_X(x)F_Y(y)$

$F_Y(y)$ right continuous

$= F_X(x)F_Y(b)$

$= F_{XY}(x, b)$

ont)

(v) $P[x_1 < X \le x_2,\ y_1 < Y \le y_2]$

$= P[x_1 < X \le x_2]\,P[y_1 < Y \le y_2]$

$= (F_X(x_2) - F_X(x_1))(F_Y(y_2) - F_Y(y_1))$

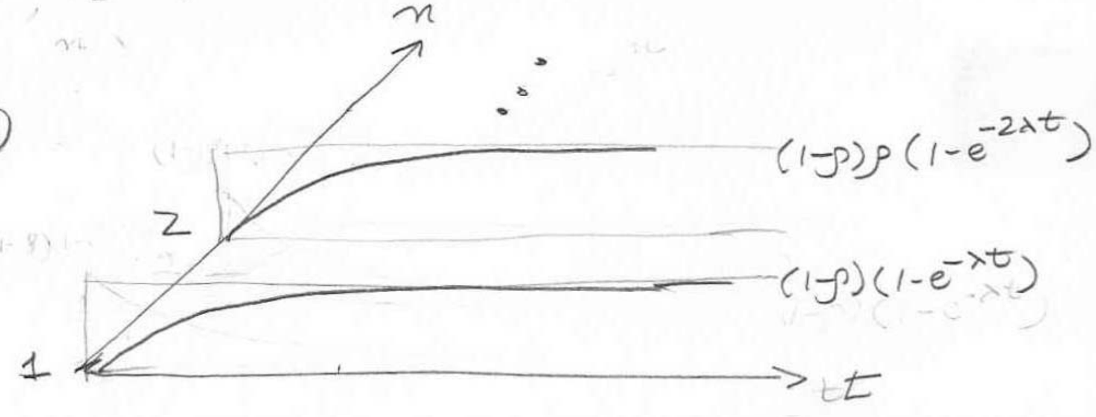
$= F_X(x_2)F_Y(y_2) - F_X(x_1)F_Y(y_2) - F_X(x_2)F_Y(y_1) + F_X(x_1)F_Y(y_1)$

$= F_{XY}(x_2, y_2) - F_X(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$

5.23 The N is discrete and X is continuous

$$P[N=n, X\le t] = (1-\rho)\rho^{n-1}(1-e^{-n\lambda t}) \qquad n=1,2,\ldots \quad t>0$$

ⓐ



ⓑ

$$P[N=n] = \lim_{t\to\infty} P[N=n, X\le t] = (1-\rho)\rho^{n-1} \qquad n=1,2,\ldots$$

geometric pmf

ⓒ

$$P[X\le t] = \sum_{n=1}^{\infty}(1-\rho)\rho^{n-1}(1-e^{-\lambda n t})$$

$$= 1-(1-\rho)e^{-\lambda t}\sum_{n=1}^{\infty}\underbrace{\rho^{n-1}e^{-\lambda(n-1)t}}_{(\rho e^{-\lambda t})^{n-1}}$$

$$= 1-\frac{(1-\rho)e^{-\lambda t}}{1-\rho e^{-\lambda t}} = \frac{1-e^{-\lambda t}}{1-\rho e^{-\lambda t}}$$

ⓓ

$$P\left[N\le 3, X>\frac{3}{\lambda}\right] = P[N\le 3] - P\left[N\le 3, X\le\frac{3}{\lambda}\right]$$

$$= \sum_{n=1}^{3}(1-\rho)\rho^{n-1} - \sum_{n=1}^{3}(1-\rho)\rho^{n-1}(1-e^{-3n})$$

$$= \sum_{n=1}^{3}(1-\rho)\rho^{n-1}e^{-3n}$$

5.24a

$$\sum_{n=1}^{3} (1-p)\, p^{n-1} e^{-3n} = (1-p)\, e^{-3} \sum_{n=1}^{3} (pe^{-3})^{n-1}$$

$$= (1-p)\, e^{-3} \underbrace{\left(1 + pe^{-3} + (pe^{-3})^2\right)}_{\dfrac{1-(pe^{-3})^3}{1-pe^{-3}}}$$

$$= (1-p)\, e^{-3}\, \frac{1-(pe^{-3})^3}{1-pe^{-3}}$$

5.24

$$\begin{aligned}
P[N=k] &= \binom{n}{k} p^k (1-p)^{n-k} \\
P[T \le t \mid N=k] P[N=k] &= (1-e^{-k\alpha t}) \binom{n}{k} p^k (1-p)^{n-k} \\
P[T \le t] &= \sum_{k=0}^{n} (1-e^{-k\alpha t}) \binom{n}{k} p^k (1-p)^{n-k} \\
&= 1 - \sum_{k=0}^{n} \binom{n}{k} (pe^{-\alpha t})^k (1-p)^{n-k} \\
&= 1 - (pe^{-\alpha t} + (1-p))^n
\end{aligned}$$

$$\lim_{t\to\infty} P[T \le t] = 1 - \underbrace{(1-p)^n}_{\substack{\text{Prob. that all} \\ \text{machines \underline{not} working}}}$$

that is, the time to complete an item is infinite when no machines are available to do the work.

5.4 The Joint cdf of Two Continuous Random Variables

5.25

(a) for $x>0,\ y>0$

$$F_{XY}(x,y) = \int_0^x \int_0^y \tfrac{1}{2}e^{-x/2}\,2ye^{-y^2}\,dx\,dy$$

$$= (1-e^{-x/2})(1-e^{-y^2})$$

(b)

$$P[X>Y] = \int_0^\infty \int_0^x 2ye^{-y^2}dy\,\tfrac{1}{2}e^{-x/2}dx$$

$$= \int_0^\infty (1-e^{-x^2})\tfrac{1}{2}e^{-x/2}dx \qquad (x+\tfrac{1}{4})^2 = \tfrac{1}{16}$$

$$= 1 - \tfrac{1}{2}\int_0^\infty e^{-(x^2+x/2)}dx \qquad = x^2+\tfrac{1}{2}x+\tfrac{1}{16}-\tfrac{1}{16}$$

$$= 1-\tfrac{1}{2}e^{1/16}\underbrace{\int_0^\infty e^{-(x+\frac{1}{4})^2}dx}_{\sqrt{2\pi\frac{1}{2}}\,=\,\sqrt{\pi}} \qquad \leftarrow \text{Gaussian pdf mean } 1/4 \text{ variance } 1/2$$

$$= 1-\frac{\sqrt{\pi}\,e^{1/10}}{2}$$

(c)

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = 1-e^{-x/2} \qquad x>0$$

$$f_X(x) = \tfrac{1}{2}e^{-x/2}$$

$$F_Y(y) = 1-e^{-y^2}$$

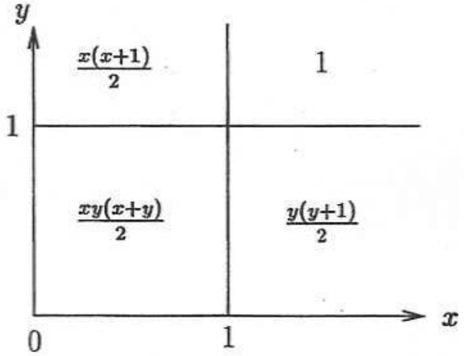
$$f_Y(y) = 2y\,e^{-y^2}$$

5.26

$$\begin{aligned} 1 &= k\int_0^1\int_0^1 (x+y)dxdy = k\int_0^1 \left(\frac{x^2}{2}+xy\right)_0^1 dy \\ &= k\int_0^1 \left(\frac{1}{2}+y\right)dy = k\left[\frac{1}{2}y+\frac{y^2}{2}\right]_0^1 = k \end{aligned}$$

$\therefore k = 1$

b)



$$\begin{aligned} \text{c)}\quad F_X(x) &= \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,1) \qquad 0<x<1 \\ &\Rightarrow\ f_X(x) = \frac{d}{dx}F_X(x) = x+\frac{1}{2} \end{aligned}$$

Similarly

$$f_Y(y) = y+\frac{1}{2}$$

5.27

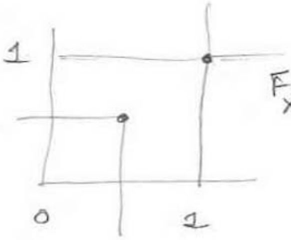
$f_{XY}(x,y) = kx(1-x)y \qquad 0<x<1,\ 0<y<1$

ⓐ

$$1 = k\int_0^1\int_0^1 x(1-x)\,y\,dx\,dy = k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 = k\underbrace{\left[\frac{1}{2}-\frac{1}{3}\right]}_{\frac{3-2}{6}}\left[\frac{1}{2}\right]$$

$\Rightarrow k=12$

ⓑ



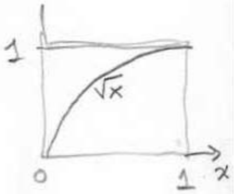
$0<x<1 \quad 0<y<1$

$$F_{XY}(x,y) = 12\int_0^x\int_0^y (x'-x'^2)\,y'\,dx'\,dy'$$
$$= 12\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\left(\frac{y^2}{2}\right)$$
$$= (3x^2-2x^3)\,y^2$$

ⓒ

$$f_X(x) = \int_0^1 f_{XY}(x,y')\,dy' = 12x(1-x)\int_0^1 y\,dy = 6x(1-x)$$

$$f_Y(y) = \int_0^1 f_{XY}(x',y)\,dx' = 12y\int_0^1(x-x^2)\,dx = 12y\left(\frac{x^2}{2}-\frac{x^3}{3}\right)_0^1$$
$$= 2y$$



ⓓ

$$P[Y<\sqrt{X}] = 12\int_0^1 dx\int_0^{\sqrt{x}}(x-x^2)\,y\,dy$$
$$= 12\int_0^1 dx\,(x-x^2)\underbrace{\left[\frac{y^2}{2}\right]_0^{\sqrt{x}}}_{x}$$
$$= 6\int_0^1 dx\,(x^2-x^3) = 6\underbrace{\left[\frac{x^3}{3}-\frac{x^4}{4}\right]}_{\frac{4-3}{12}} = \frac{1}{2}$$

$$P[X<Y] = 12\int_0^1 dy\int_0^y (x-x^2)\,y\,dx$$
$$= 12\int_0^1 y\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^y dy = 12\int_0^1\left(\frac{y^3}{2}-\frac{y^4}{3}\right)dy$$
$$= 12\left[\frac{1}{2}\frac{y^4}{4}-\frac{1}{3}\frac{y^5}{5}\right]_0^1 = 12\left[\underbrace{\frac{1}{2}\frac{1}{4}}_{\frac{1}{8}}-\underbrace{\frac{1}{3}\frac{1}{5}}_{\frac{1}{15}}\right] = \frac{7}{10}$$

5.28

a)

$$k \cdot \pi = 1 \qquad \Rightarrow k = \frac{1}{\pi} \quad \text{(i)}$$

$$k \cdot \sqrt{2} \cdot \sqrt{2} = 1 \quad \Rightarrow k = \frac{1}{2} \quad \text{(ii)}$$

$$k \cdot 1^2/2 = 1 \qquad \Rightarrow k = 2 \quad \text{(iii)}$$

b) (i) $f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \frac{2\sqrt{1-x^2}}{\pi} \qquad -1 < x < 1$

Similarly

$$f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi} \qquad -1 < x < 1$$

(ii) $f_X(x) = \int_{|x|-1}^{1-|x|} \frac{dy}{2} = 1 - |x| \qquad -1 < x < 1$

Similarly

$$f_Y(y) = 1 - |y| \qquad -1 < y < 1$$

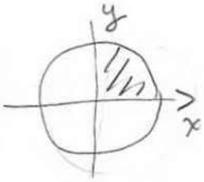
(iii) $f_X(x) = \int_0^{1-x} 2dy = 2(1-x) \qquad 0 < x < 1$

Similarly

$$f_Y(y) = 2(1-y) \qquad 0 < y < 1$$

5.28c

(i) $P[X>0, Y>0] = \int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{1}{\pi}\int_0^1 \sqrt{1-x^2}\, dx$



$$= \frac{1}{\pi}\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_0^1$$

$$= \frac{1}{\pi}\Big[0 + \underbrace{\sin^{-1} 1}_{\pi/4} - 0 + \underbrace{\sin^{-1} 0}_{0}\Big] = \frac{1}{\pi}\frac{\pi}{4} = \frac{1}{4}$$

(ii) $P[X>0, Y>0] = \int_0^1 dx \int_0^{1-x} \frac{1}{2} dy = \frac{1}{2}\int_0^1 (1-x)\, dx = \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1$

$$= \frac{1}{2}\left[1 - \frac{1}{2} - 0 + 0\right] = \frac{1}{4} \checkmark$$

(iii) $P[X>0, Y>0] = \int_0^1 dx \int_0^{1-x} 2dy = 2\left[\frac{1}{2}\right] = 1 \checkmark$

5.29 ~~4.8~~ **a)** For $0 \le y_0 \le x_0$ we integrate along the strip indicated below.

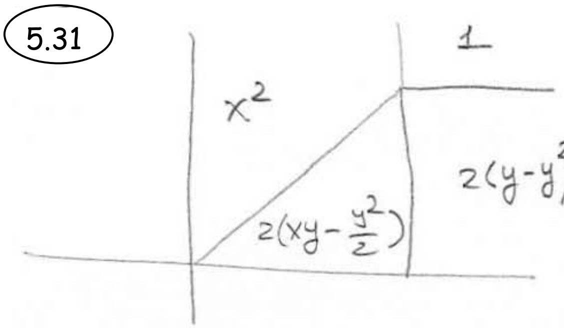
b) The marginal cdf's are obtained by taking the appropriate limits of the joint cdf:

$$\begin{aligned} F_X(x_0) &= \lim_{y_0 \to \infty} F_{XY}(x_0, y_0) = F_{XY}(x_0, x_0) = 1 - 2e^{-x_0} + e^{-2x_0} \\ F_Y(y_0) &= \lim_{x_0 \to \infty} F_{XY}(x_0, y_0) = 1 - e^{-2y_0} \end{aligned}$$

5.30

$$\begin{aligned} f_X(x) &= \int_0^\infty x e^{-x} e^{-xy}\,dy = x e^{-x} \left(\frac{-1}{x} e^{-xy} \right)_0^\infty = e^{-x} \\ f_Y(y) &= \int_0^\infty x e^{-x(1+y)}\,dx = \left. \frac{e^{-x(1+y)}((1+y)x - 1)}{(1+y)^2} \right|_0^\infty \\ &= \frac{1}{(1+y)^2} \end{aligned}$$

5.31

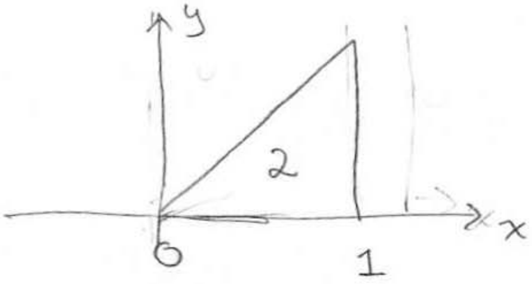


$$f_{XY}(x,y)=\frac{\partial^2}{\partial x\partial y}F_{XY}(x,y)$$

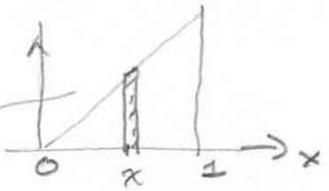
$\Rightarrow$ only the region $0<y<x<1$ has non zero pdf

$0<y<x<1$

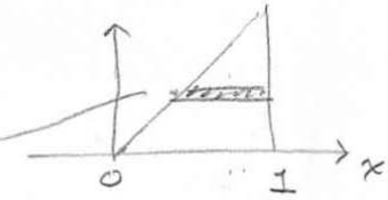
$$f_{XY}(x,y)=\frac{\partial^2}{\partial x\partial y}\left(2xy-\frac{y^2}{2}\right)=\frac{\partial}{\partial x}(2x-2y)=2$$



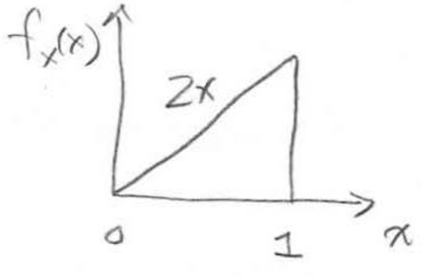
pdf is constant inside the triangle



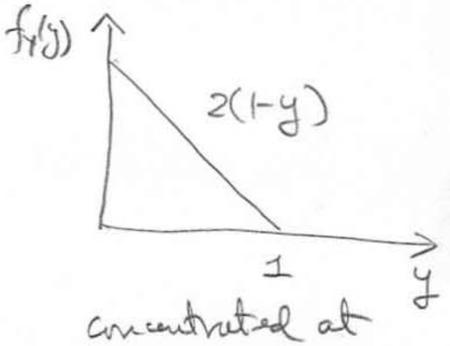
$$f_X(x)=\int_{-\infty}^{\infty}f_{XY}(x,y)\,dy=\int_0^x 2\,dy=2x \qquad 0<x<1$$



$$f_Y(y)=\int_y^1 2\,dx=2(1-y)$$



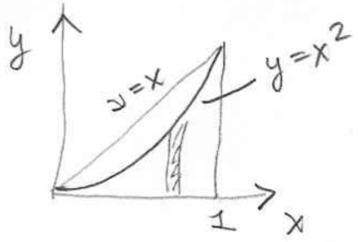
concentrated at higher values



concentrated at lower values

5.31 ⓒ $P[Y < X^2] = 2\int_0^1 dx \int_0^{x^2} dy = 2\int_0^1 x^2\,dx = 2\cdot \frac{x^3}{3}\Big|_0^1$

$= 1/6$



5.32

ⓐ $$f_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r\partial\theta}F_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial r\partial\theta}\, r^2\,\frac{\theta}{2\pi} \qquad \begin{matrix} 0\le r\le 1 \\ 0\le\theta\le 2\pi\end{matrix}$$

$$= \frac{\partial}{\partial r}\, r^2\,\frac{1}{2\pi} = (2r)\left(\frac{1}{2\pi}\right) \qquad \begin{matrix} 0\le r\le 1 \\ 0\le\theta\le 2\pi\end{matrix}$$

ⓑ $$f_R(r) = \int_0^{2\pi} 2r\,\frac{1}{2\pi}\,d\theta = 2r \qquad 0\le r\le 1$$

$$f_\Theta(\theta) = \int_0^1 2r\,\frac{1}{2\pi}\,d\theta = \frac{1}{2\pi}\,r^2\Big|_0^1 = \frac{1}{2\pi}$$

$0\le\theta\le 2\pi$

5.33

$$\begin{aligned}
P[X^2+Y^2<R^2] &= \iint_{x^2+y^2<R^2}\frac{1}{2\pi\sigma^2}e^{-(x^2+y^2)/2\sigma^2}dx\,dy \\
&= \int_0^{2\pi}\int_0^R \frac{1}{2\pi\sigma^2}e^{-r^2/2\sigma^2}r\,dr\,d\theta \\
&\quad \text{where we let } x = r\cos\theta,\ y = r\sin\theta \\
&= \frac{r}{\sigma^2}\int_0^R re^{-r^2/2\sigma^2}dr \\
&= 1-e^{-R^2/2\sigma^2}
\end{aligned}$$

5.34

$$f_X(x) = \int_{-\infty}^{\infty} \frac{\exp\left\{-\frac{\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \left(\frac{y-m_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} dy$$

Complete the square of the argument in the exponent:

$$\begin{aligned}
&\left(\frac{x-m_1}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right) + \rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2 \\
&\quad -\rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2 + \left(\frac{y-m_2}{\sigma_2}\right)^2 \\
&\quad = \left[\left(\frac{y-m_2}{\sigma_2}\right) - \rho\left(\frac{x-m_1}{\sigma_1}\right)\right]^2 + (1-\rho^2)\left(\frac{x-m_1}{\sigma_1}\right)^2
\end{aligned}$$

Thus

$$\begin{aligned}
f_X(x) &= \frac{e^{-(x-m_1)^2/2\sigma_1^2}}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{\infty} \frac{e^{-\left[y-\left(m_2+\rho\sigma_2\left(\frac{x-m_1}{\sigma_1}\right)\right)\right]^2/2\sigma_2^2(1-\rho^2)}}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \\
&= \underbrace{\frac{e^{-(x-m_1)^2/2\sigma_1^2}}{\sqrt{2\pi}\sigma_1}}_{1}
\end{aligned}$$

integral of Gaussian pdf with
mean $m_2 + \rho\sigma_2\left(\frac{x-m_1}{\sigma_1}\right)$
variance $\sigma_2^2(1-\rho^2)$

$f_Y(y)$ has the same form.

5.35

(a) for $j=-1$:

$P[X=j, Y\le y]$

$= P[X=-1, N-1\le y]$

$= P[X=-1, N\le y+1]$

$= (1-p)\cdot \frac{1}{0.5\sqrt{2\pi}} \int_{-\infty}^{y+1} e^{-x^2/2(0.25)}\, dx$

$= (1-p)\sqrt{\frac{2}{\pi}} \int_{-\infty}^{y+1} e^{-2x^2}\, dx$ //

for $j=1$:

$P[X=1, Y\le y]$

$= P[X=1, N+1\le y]$

$= P[X=1, N\le y-1]$

$= p\sqrt{\frac{2}{\pi}} \int_{-\infty}^{y-1} e^{-2x^2}\, dx$ //

(b) $p_X(-1) = 1-p \qquad p_X(1) = p$

$F_Y(y) = P[Y\le y \mid X=-1]P[X=-1] + P[Y\le y\mid X=1]P[X=1]$

$= P[N-1\le y](1-p) + P[N+1\le y](p)$

$$= \int_{-\infty}^{y} \frac{(1-p)e^{-(y'+1)^2/2(0.25)}}{0.5\sqrt{2\pi}}\, dy' + \int_{-\infty}^{y} \frac{pe^{-(y'-1)^2/2(0.25)}}{0.5\sqrt{2\pi}}\, dy'$$

$f_Y(y) = \frac{d}{dy}F_Y(y) =$

$= (1-p)e^{-2(y+1)^2}\cdot\sqrt{\frac{2}{\pi}} + pe^{-2(y-1)^2}\cdot\sqrt{\frac{2}{\pi}}$

(c) test for $X=1$:

$$P[X=1 \mid Y>0]$$

$$= \frac{P[X=1, Y>0]}{P[Y\geq 0]}$$

$$= \frac{P[X=1, N+1>0]}{P[Y>0 \mid X=-1]P[X=-1]+P[Y>0 \mid X=1]P[X=1]}$$

$$= \frac{P[X=1, N>-1]}{(1-p)P[N>1]+p\,P[N>-1]}$$

$$= \frac{p\,Q(\frac{-1}{0.5})}{(1-p)\,Q(\frac{1}{0.5})+p\,Q(-\frac{1}{0.5})}$$

$$= \frac{p\,Q(-2)}{(1-p)\,Q(2)+p\,Q(-2)}$$

$$= \frac{p\,(1-Q(2))}{(1-p)\,Q(2)+p\,(1-Q(2))}$$

from table:

$$= \frac{0.9772p}{0.0228-0.0228p+0.9772p}$$

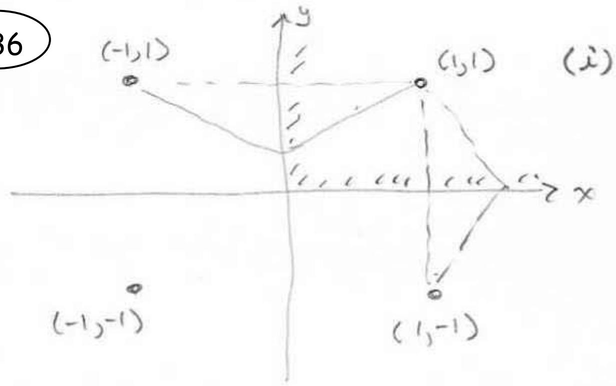
$$= \frac{0.9772p}{0.0228+0.9544p}$$

$P[X=1 \mid Y>0] > \frac{1}{2}$ when $0.0228 < p \leq 1$

and $< \frac{1}{2}$ when $0 \leq p \leq 0.0228$

$\therefore$ $X=1$ more likely when $p \in [0.0228, 1]$

5.36



(i) The y axis is perpendicular to the line between (-1,1) and (1,1). All points on the y axis are equidistant to (-1,1) and to (1,1). All pts to the right of the y axis are closer to (1,1) than to (-1,1).

(ii) Similarly all points above the x-axis are closer to (1,1) than to (1,-1). ∴ the first quadrant is the set of points closer to (1,1) than to the other points.

$$P[\underline{Y} \text{ in 1st quadrant} \mid \underline{X}=(1,1)]$$

$$= \int_0^\infty\int_0^\infty \frac{1}{2\pi\sigma^2} e^{-\{(x_1-1)^2+(x_2-1)^2\}/2\sigma^2} dx_1 dx_2$$

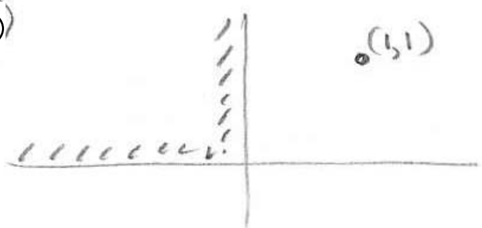
$$= \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_1-1)^2/2\sigma^2} dx_1 \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_2-1)^2/2\sigma^2} dx_2$$

$$= \int_{-\frac{1}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \int_{-\frac{1}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$$= Q(-\tfrac{1}{\sigma})\, Q(-\tfrac{1}{\sigma})$$

$$= (1-Q(\tfrac{1}{\sigma}))^2$$

5.36b



Given $\underline{X} = (1,1)$
Find prob of 2nd quadrant

$$P[\underline{Y} \text{ in 2nd Quad } | \underline{X} = (1,1)]$$

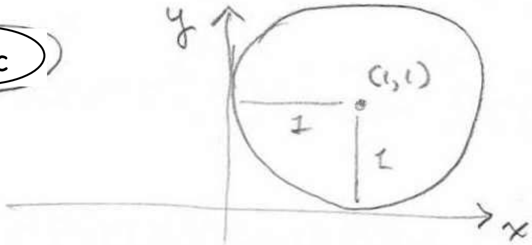
$$= \int_{-\infty}^{0} dx_1 \int_{0}^{\infty} dx_2 \frac{1}{2\pi\sigma^2} e^{-\{(x_1-1)^2+(x_2-1)^2\}/2\sigma^2}$$

$$= \int_{-\infty}^{-\frac{1}{\sigma}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \int_{-1/\sigma}^{\infty} \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv$$

$$= \left(1-Q\left(-\tfrac{1}{\sigma}\right)\right) Q\left(-\tfrac{1}{\sigma}\right)$$

$$= Q\left(\tfrac{1}{\sigma}\right)\left(1-Q\left(\tfrac{1}{\sigma}\right)\right)$$

5.36c



$$\{(x,y): \sqrt{(x-1)^2+(y-1)^2} > 1\} \supset \left[\underbrace{\{x \le 0, y \ge 0\}}_{\text{quad 2}} \cup \underbrace{\{x<0, y<0\}}_{\text{quad 3}} \cup \underbrace{\{x>0, y<0\}}_{\text{quad 4}}\right]$$

$$P\left[\sqrt{(X-1)^2+(Y-1)^2} \le 1\right] = \iint_{\sqrt{(x-1)^2+(y-1)^2} \le 1} \frac{1}{2\pi\sigma^2} e^{-\{(x_1-1)^2+(x_2-1)^2\}/2\sigma^2} dx_1 dx_2$$

$$= \iint_{\sqrt{x^2+y^2} \le 1} \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} dx\, dy$$

5.36c —continued— change to polar coordinates

$$P\left[\sqrt{(X-1)^2+(Y-1)^2} \le 1\right] = \int_0^{2\pi}\int_0^{1} \frac{1}{2\pi\sigma^2}\, e^{-r^2/2\sigma^2}\, r\,dr\,d\theta$$

$$= \int_0^1 \frac{r\, e^{-r^2/2\sigma^2}}{\sigma^2}\, dr = -e^{-r^2/2\sigma^2}\Big|_0^1$$

$$= 1 - e^{-1/2\sigma^2}$$

$$\therefore\ P\left[\sqrt{(X-1)^2+(Y-1)^2} \ge 1\right] = e^{-1/2\sigma^2}$$

5.5 Independence of Two Random Variables

5.37

N	1	2	3	4	5	6	outcome of toss
x	0	1	1	2	2	3	full pairs
y	1	0	1	0	1	0	remainder
$p(x,y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	

$x \backslash y$	0	1	
0		$1/6$	$1/6$
1	$1/6$	$1/6$	$2/6$
2	$1/6$	$1/6$	$2/6$
3	$1/6$		$1/6$
	$\frac{1}{2}$	$\frac{1}{2}$	

$P[Y=0] = \frac{1}{2} = P[Y=1]$

$P[X=0] = \frac{1}{6} = P[X=3]$

$P[X=1] = \frac{2}{6} = P[X=2]$

$p(x,y) \neq p(x)\,p(y)$ all x, y

$\Rightarrow$ X and Y not independent

5.38 from 5.16 b

$0 \le k \le n$

$0 \le j \le n$

$$P[X=(2k-n)\Delta,\ Y=(2j-n)\Delta]$$

$$= P[X=(2k-n)\Delta,\ Y=(2j-n)\Delta]$$

$\Rightarrow$ X and Y are independent RVs

5.39

(a) $p_{XY}(r,r) = 0$

but $p_X(r) = 1/8$
and $p_Y(r) = 1/8$

$\therefore$ Since $p_{XY}(r,r) \neq p_X(r) \cdot p_Y(r)$
X and Y are <u>not</u> independent.

(b) joint pmf:

$X \backslash Y$	$-r$	$-\frac{r}{\sqrt{2}}$	0	$\frac{r}{\sqrt{2}}$	r
$-r$	0	0	1/6	0	0
$-\frac{r}{\sqrt{2}}$	0	1/12	0	1/12	0
0	1/6	0	0	0	1/6
$r/\sqrt{2}$	0	1/12	0	1/12	0
r	0	0	1/6	0	0

marginal pmf:

$p_X(-r) = \frac{1}{6}$ $p_X(-\frac{r}{\sqrt{2}}) = \frac{1}{6}$ $p_X(0) = \frac{1}{3}$ $p_X(\frac{r}{\sqrt{2}}) = \frac{1}{6}$ $p_X(r) = \frac{1}{6}$

$p_Y(-r) = \frac{1}{6}$ $p_Y(-\frac{r}{\sqrt{2}}) = \frac{1}{6}$ $p_Y(0) = \frac{1}{3}$ $p_Y(\frac{r}{\sqrt{2}}) = \frac{1}{6}$ $p_Y(r) = \frac{1}{6}$

$p_{XY}(0,0) = 0$

but
$p_X(0)\, p_Y(0) = 1/9$

$\therefore$ X and Y are <u>not</u> independent.

5.40

Only the joint pmf for 5.11 (ii) corresponds to a pair of independent RV's.

5.41

Let M represent Michael's arrival time (minutes after 7:00)
Let B represent the arrival time of the bus (minutes after 7:00)

M, B uniform RVs

$f_M(m) = \frac{1}{15} \quad , 25 \le m \le 40$

$f_B(b) = \frac{1}{10} \quad , 27 \le b \le 37$

(a) $f_{MB}(m,b) = f_M(m) f_B(b)$ since M, B independent

$= \frac{1}{150} \quad , 25 \le m \le 40, \; 27 \le b \le 37$

$0 \quad$, otherwise

$$P[M+5 < B] = \int_{30}^{37}\int_{25}^{b-5} \left(\frac{1}{150}\right) dm\,db$$

$$= \frac{1}{150}\int_{30}^{37} (b-30)\,db$$

$$= \frac{1}{150}\left[\frac{b^2}{2} - 30b\right]_{30}^{37}$$

$$= 0.163$$

(b) $$P[M>B] = \int_{27}^{37}\int_{b}^{40} \frac{1}{150}\, dm\,db$$

$$= \frac{1}{150}\int_{27}^{37} (40-b)\,db$$

$$= \frac{1}{150}\left[40b - \frac{b^2}{2}\right]_{27}^{37}$$

$$= 8/15$$

$$\cong 0.533$$

5.42 From marginal cdfs in 5.19bc

$$F_{R,\Theta}(r,\theta) = P[R \leq r, \Theta \leq \theta] = F_R(r)\,F_\Theta(\theta) \qquad \text{all } r, \theta$$

$\Rightarrow$ R and Θ are indep RVs.

5.43 $F_{XY}(x,y) = F_X(x)F_Y(y)$ all x, y

$\therefore$ X and Y are independent

5.44

$$F_{XY}(x,y) = (1-e^{-x/2})(1-e^{-y^2})$$

$$= F_X(x)\,F_Y(y) \qquad \text{all } x, y$$

$\Rightarrow$ X and Y are independent

5.45

$$F_{XY}(x,y) = \frac{xy(x+y)}{2} \qquad \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array}$$

$$F_X(x) = x + \tfrac{1}{2} \qquad 0 \leq x \leq 1$$

$$F_Y(y) = y + \tfrac{1}{2} \qquad 0 \leq y \leq 1$$

$$F_{XY}(x,y) \neq F_X(x)\,F_Y(y)$$

$\Rightarrow$ X and Y are not independent.

5.46

$$f_{XY}(x,y) = 12x(1-x)y$$

$$f_X(x) = 6x(1-x)$$

$$f_Y(y) = 2y$$

$\therefore$ X and Y independent since $f_{XY}(x,y) = f_X(x) f_Y(y)$

5.47

~~4.23~~ **a)** $P[a < X \leq b, Y \leq d] = P[a < X \leq b]P[Y \leq d]$
$= (F_X(b) - F_X(a))F_Y(d)$

b) $P[a \leq X \leq b, c \leq Y \leq a] = (F_X(b) - F_X(a^-))(F_Y(d) - F_X(c^-))$

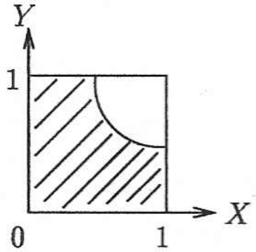
c) $P[|X| > a, c \leq Y \leq d] = (1 - F_X(a) + F_X(a^-))(F_Y(d) - F_X(c^-))$

5.48

~~4.24~~ **a)** $P[X^2 < \frac{1}{2}, |Y$ [redacted] $| < \frac{1}{2}] = P[X^2 < \frac{1}{2}]P[|Y$ [redacted] $| < \frac{1}{2}]$
$= P[X < \frac{1}{\sqrt{2}}]P[Y < \frac{1}{2}] = \frac{1}{2}\frac{1}{\sqrt{2}}$

b) $P[4X < 1, Y < 0] = P[X < \frac{1}{4}]P[Y < 0] = \left(\frac{1}{4}\right) \cdot 0 = 0$

c)



$$f(x,y) = f(x)f(y) = 1$$

$$\begin{aligned} P[XY < \frac{1}{2}] &= \frac{1}{2} + \int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2x}} 1 \cdot dy dx \\ &= \frac{1}{2} + \int_{\frac{1}{2}}^{1} \frac{1}{2x} dx \\ &= \frac{1}{2} + \frac{1}{2}\ln|x| \,\Big|_{\frac{1}{2}}^{1} \\ &= 0.85 \end{aligned}$$

d) $P[\min(X,Y) > \frac{1}{3}] == P[X > \frac{1}{3}]P[Y > \frac{1}{3}] = (\frac{2}{3})^2\ \frac{4}{9}$

5.49

~~4.29~~ **a)**

$$\begin{array}{cc|ccc} & Y & -1 & 0 & 1 \\ X & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \hline -1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{6} & \frac{1}{16} \end{array} \qquad \begin{array}{cc|cc} & Y^2 & -1 & 0 \\ X^2 & & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}$$

b)

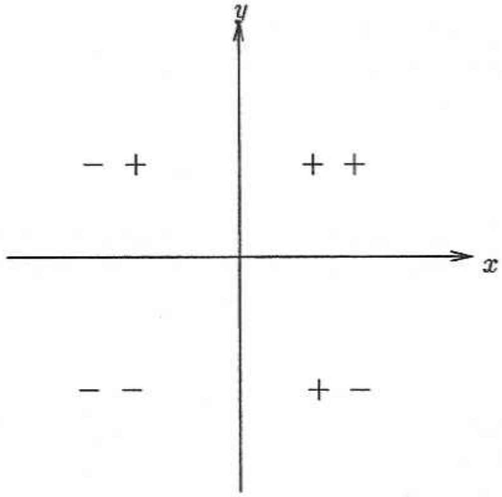
$$\begin{array}{cc|ccc} & Y & -1 & 0 & 1 \\ X & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \hline -1 & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ 1 & \frac{1}{2} & 0 & \frac{1}{8} & 0 \end{array} \qquad \begin{array}{cc|cc} & Y^2 & 0 & 1 \\ X^2 & & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array}$$

5.50

~~4.25~~ a) If $\rho = 0$ in Problem ~~15~~ 35, we have

$$\begin{aligned}
f_{XY}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\left[\left(\frac{x-m_1}{\sigma_1}\right)^2 + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]/2} \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-m_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-m_2)^2}{2\sigma_2^2}} \\
&= f_X(x) f_Y(y) \quad \text{for all } x, y \\
&\Rightarrow X, Y \text{ indep. RV's}
\end{aligned}$$

b) If $\rho = 0$ then



$$\begin{aligned}
P[XY > 0] &= P[X \text{ and } Y \text{ have same sign}] \\
&= \iint_{\substack{++ \\ \text{quadrant}}} f_{XY}(x,y)\,dx\,dy + \iint_{\substack{-- \\ \text{quadrant}}} f_{XY}(x,y)\,dx\,dy \\
&= \int_0^\infty f_X(x)\,dx \int_0^\infty f_Y(y)\,dy + \int_{-\infty}^0 f_X(x)\,dx \int_{-\infty}^0 f_Y(y)\,dy
\end{aligned}$$

but

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(x-m_1)^2/2\sigma_1^2}\,dx = \int_{-\frac{m_1}{\sigma_1}}^\infty \frac{e^{-t^c/2}}{\sqrt{2\pi}}\,dt = Q\left(-\frac{m_1}{\sigma_1}\right)$$

and similarly for other integrals, thus

$$P[XY > 0] = Q\left(-\frac{m_1}{\sigma_1}\right) Q\left(-\frac{m_2}{\sigma_2}\right) + \left(1 - Q\left(-\frac{m_1}{\sigma_1}\right)\right)\left(1 - Q\left(-\frac{m_2}{\sigma_2}\right)\right)$$

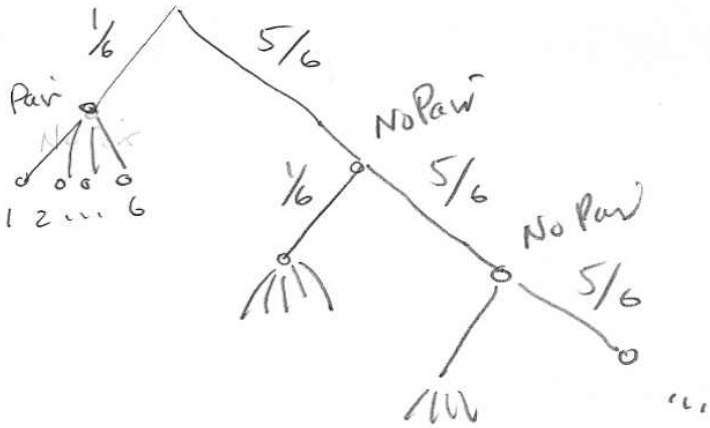
5.51

$p_{KX}(k,x) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) \qquad 1 \le x \le 6,\ k > 0$

$p_K(k) = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right), \quad k > 0$

$p_X(x) = \frac{1}{6} \qquad , 1 \le x \le 6$

$\therefore$ K and X are independent since $p_{KX}(k,x) = p_K(k)\,p_X(x)$



5.52 L is geometric dist p, # devices produced in 1 day

N = # working devices

M = # defective devices

$$P[L=k] = (1-p)^{k} p \qquad k=0,1,\ldots$$

$$P[N=n \mid L=k] = \binom{k}{n} \alpha^{n} (1-\alpha)^{k-n} \qquad \begin{matrix} n=0,\ldots,k \\ 0 \le n \le k \end{matrix}$$

$$P[N=n, M=m \mid L=n+m] = \binom{n+j}{n} \alpha^{n} (1-\alpha)^{m}$$

$$P[N=n, M=m] = \binom{n+j}{n} \alpha^{n} (1-\alpha)^{m} (1-p)^{n+m} p$$

$$P[N=n] = \sum_{k=n}^{\infty} \binom{k}{n} \alpha^{n} (1-\alpha)^{k-n} (1-p)^{k} p$$

R * Note k must be greater than n in order for N=n to be possible.

$$= \alpha^{n} (1-p)^{n} p \underbrace{\sum_{k=n}^{\infty} \binom{n}{k} \left[(1-\alpha)(1-p)\right]^{k-n}}_{\frac{1}{(1-(1-\alpha)(1-p))^{n+1}}}$$

$$= \frac{\alpha^{n}(1-p)^{n} p}{(1-(1-\alpha)(1-p))^{n+1}} = \frac{(\alpha\bar{p})^{n} p}{(1-\bar{\alpha}\bar{p})^{n+1}} = \left(\frac{\alpha\bar{p}}{1-\bar{\alpha}\bar{p}}\right)^{n} \frac{p}{1-\bar{\alpha}\bar{p}}$$

Similarly

$$P[M=m] = \frac{(\bar{\alpha}\bar{p})^{m} P}{(1-\alpha\bar{p})^{n+1}} = \left(\frac{\bar{\alpha}\bar{p}}{1-\alpha\bar{p}}\right)^{n} \left(\frac{P}{1-\alpha\bar{p}}\right)$$

(5.52) – continued –

$$P[N=n, M=m] = \binom{n+j}{n}\alpha^n(1-\alpha)^m(1-p)^{n+m}p$$

only one way for n & m at the same time

$$\neq P[N=n]P[M=m]$$

∴ many ways for n and m to occur separately

⇒ N & M are not independent

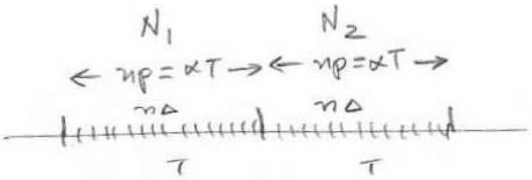
Also

$$P[N=n|L=k] = \binom{k}{n}\alpha^n(1-\alpha)^{k-n}$$

$$\neq P[N=n]$$

∴ L and N are not independent

(5.53)



In problem 13 we found that N_1 and N_2 are independent Binomial RV's

$$P[N_1=k, N_2=j] = \binom{n}{k}p^k(1-p)^{n-k}\binom{n}{j}p^j(1-p)^{n-j}$$

Allow the intervals Δ to decrease while keeping $np=\alpha T$

then each Binomial approaches a Poisson pmf

$$P[N_1=k, N_2=j] = \frac{(\alpha T)^k}{k!}e^{-\alpha T}\frac{(\alpha T)^j}{j!}e^{-\alpha T}.$$

The intervals need not be of the same length T.

5.54

4.27 a) Without loss of generality, let event A, be $x_1 < X \le x_2$, event A_2 be $y_1 < Y \le y_2$

$$\begin{aligned} P[A] &= P[A_1 \cap A_2] \\ &= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_2) \\ &= F_X(x_2)F_Y(y_1) - F_X(x_2)F_Y(y_1) - F_X(x_1)F_Y(y_2) + F_X(x_1)F_Y(y_2) \\ &= [F_X(x_2) - F_X(x_1)][F_Y(y_2) - F_Y(y_1)] \\ &= P[A_1]P[A_2] \end{aligned}$$

b) Let event A be $\{-\infty < X < x,\ -\infty < Y < y\}$, A_1 be $\{-\infty < X < x\}$, A_2 be $\{-\infty < Y < y\}$, then

$$\begin{aligned} P[A] &= F_{X,Y}(x, y) \\ &= P[A_1]P[A_2] \\ &= F_X(x)F_Y(y) \end{aligned}$$

5.55 **a)** (4.20) ⇒ (4.21)

$$\begin{aligned} F_{X,Y}(x, y) &= F_X(x)F_Y(y) \\ \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} &= \frac{\partial^2}{\partial x \partial y}(F_X(x)F_Y(y)) \\ f_{X,Y}(x, y) &= \frac{\partial}{\partial x}(F_X(x)f_Y(y)) \\ &= f_X(x)f_Y(y) \end{aligned}$$

b) (4.21) ⇒ (4.20)

$$\begin{aligned} f_{X,Y}(x', y') &= f_X(x')f_Y(y') \\ \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x', y')dx'dy' &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(x')f_Y(y')dx'dy' \\ \text{i.e.}\quad F_{X,Y}(x, y) &= \int_{-\infty}^{x} f_X(x')F_Y(y)dx' \\ &= F_X(x)F_Y(y) \end{aligned}$$

5.6 Joint Moments and Expected Value of a Function of Two Random Variables

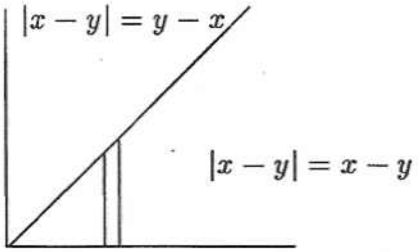
5.56

~~4.59~~ a) $\mathcal{E}[(X+Y)^2] = \mathcal{E}[X^2+2XY+Y^2] = \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2]$

b)
$$\begin{aligned} VAR[X+Y] &= \mathcal{E}[(X+Y)^2] - \mathcal{E}[X+Y]^2 \\ &= \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2] - \mathcal{E}[X]^2 \\ &\quad -2\mathcal{E}[X]\mathcal{E}[Y] - \mathcal{E}[Y]^2 \\ &= VAR[X] + VAR[Y] + 2[\mathcal{E}[XY] - \mathcal{E}[X]\mathcal{E}[Y]] \end{aligned}$$

c) $VAR[X+Y] = VAR[X] + VAR[Y]$ if $\mathcal{E}[XY] = \mathcal{E}[X]\mathcal{E}[Y]$ that is, if X and Y are uncorrelated.

5.57



$$\begin{aligned} \mathcal{E}[|X-Y|] &= \cancel{1}\int_0^\infty \int_0^\infty 2\,|x-y|e^{-\overset{2}{\cancel{(}}x+y)}dxdy \\ &= 2\int_0^\infty \int_0^x 2(x-y)e^{-\overset{2}{\cancel{x}}}e^{-y}dydx \\ &= 2\int_0^\infty 2\,e^{-\overset{2}{\cancel{x}}}\underbrace{[x(1-e^{-x}) - \underbrace{\int_0^x ye^{-y}dy}_{1-(1+x)e^{-x}}}_{x+e^{-x}-1} \\ &= 2\int_0^\infty 2(xe^{-\overset{2}{\cancel{x}}} + e^{-\overset{3}{\cancel{2}}x} - e^{-\overset{2}{\cancel{x}}})dx \\ &= \cancel{2[1+\tfrac{1}{2}-1]}\quad 2\left[\frac{1}{2} + 2\cdot\frac{1}{3} - 1\right] = 1 + \frac{4}{3} - 2 = \frac{1}{3} \end{aligned}$$

5.58

8) $E[X^2e^Y] = E[X^2]E[e^Y] = 1\cdot E[e^Y] = \frac{1}{3}\cdot(e^3-1)$

~~1 $E[X^2Y] = \mathcal{E}[X^2]\mathcal{E}[Y] = 1(1) = 1$~~

$\frac{1}{3}\int_0^3 e^y dy = \frac{1}{3}e^y\Big|_0^3 = \frac{1}{3}(e^3-1)$

5.59 from problem 5.9:

$$E[X] = 0\cdot\frac{7}{16} + 1\cdot\frac{8}{16} + 2\cdot\frac{1}{16} = \frac{10}{16}$$

$$E[Y] = 0\cdot\frac{1}{16} + 1\cdot\frac{8}{16} + 2\cdot\frac{7}{16} = \frac{22}{16}$$

$$E[XY] = 0\cdot\big(\underbrace{0\cdot\tfrac{1}{16}\cdot 1\cdot\tfrac{4}{16}\cdot 2\cdot\tfrac{2}{16}}_{0}\big)$$

$$+ 1\cdot\left(1\cdot\frac{4}{16} + 2\cdot\frac{4}{16}\right) + 2\cdot 2\cdot\frac{1}{16}$$

$$= \frac{4}{16} + \frac{8}{16} + \frac{4}{16} = 1 \quad \Rightarrow \text{not orthogonal}$$

$$\text{COV}(X,Y) = E[XY] - E[X]E[Y]$$

$$= 1 - \frac{10}{16}\cdot\frac{22}{16} = \frac{36}{256} = \frac{9}{64} \Rightarrow \text{not uncorrelated}$$

$$\Rightarrow \text{not independent}$$

5.60 from Problem 5.10

$$E[X] = -2\cdot\frac{1}{16} - 1\cdot\frac{4}{16} + 0\cdot\frac{6}{16} + 1\cdot\frac{4}{16}\cdot 2\cdot\frac{1}{16} = 0$$

$$E[Y] = 0 \quad \text{same pmf}$$

$$E[XY] = -2(2)\cdot\frac{1}{16} - 1\left(1\cdot\frac{2}{16} + 3\cdot\frac{2}{16}\right) + 0(\quad)$$

$$+1\cdot\left(1\cdot\frac{2}{16} + 3\cdot\frac{2}{16}\right) + 2(2)\frac{1}{16}$$

$$= 0 \quad \Rightarrow X \text{ and } Y \text{ are orthogonal}$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0$$

$$\Rightarrow X \text{ and } Y \text{ are uncorrelated}$$

However $P[X=i, Y=j] \neq P[X=i]P[Y=j]$

$$\Rightarrow \text{not independent}$$

5.61) (i) $E[X] = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$

$E[Y] = 0$ same pmf

$E[XY] = (-1)(-1)\frac{1}{6} + (-1)(1)\frac{1}{6} = 0$

$\Rightarrow$ X and Y are orthogonal

$\text{COV}(X,Y) = E[XY] - E[X]E[Y] = 0$

$\Rightarrow$ X and Y are uncorrelated

However X and Y are not independent

Since $P[X=i, Y=j] \neq P[X=i]P[Y=j]$

(ii) $E[X] = E[Y] = 0$ as before

$E[XY] = (-1)(-1)\frac{1}{9} + (-1)(1)\frac{1}{9} + (1)(-1)\frac{1}{9} + (1)(1)\frac{1}{9}$

$= 0$

$\therefore$ X and Y uncorrelated and orthogonal

furthermore $P[X=i, Y=j] = P[X=i]P[Y=j]$ all i, j

$\Rightarrow$ X and Y indep.

(iii) $E[X] = E[Y] = 0$.

$E[XY] = (-1)(-1)\frac{1}{3} + (1)(1)\frac{1}{3} = \frac{2}{3}$

$\Rightarrow$ not orthogonal and not uncorrelated

Also $P[X=i, Y=j] \neq P[X=i]P[Y=j]$

$\Rightarrow$ not indep.

5.62

$$p_{N_1N_2}(n_1,n_2) = \binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}\binom{100}{n_2}(0.05)^{n_2}(0.95)^{100-n_2}$$

correlation

$$E[N_1N_2] = \sum_{n_1=0}^{100}\sum_{n_2=0}^{100}\binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}\binom{100}{n_2}(0.05)^{n_2}(0.95)^{100-n_2}\cdot n_1 n_2$$

$$= \sum_{n_1=0}^{100}\binom{100}{n_1}(0.05)^{n_1}(0.95)^{100-n_1}\cdot n_1 \sum_{n_2=0}^{100}\binom{100}{n_2}(0.05)^{n_2}(0.95)^{100-n_2}\cdot n_2$$

$$= E[N_1]E[N_2]$$

$$= 100(0.65)\cdot 100(0.05)$$

$$= 25$$

covariance

$$E[N_1N_2] - E[N_1]E[N_2]$$

$$= 25-25$$

$$=0$$

$\therefore$ Independent: yes since $\mathrm{COV}(N_1,N_2) = 0$
Orthogonal: no since $E[N_1N_2] \neq 0$
Uncorrelated: yes since N_1 and N_2 independent.

5.63

N_1 = # req in 1st n

N_2 = total # req in 1st n and 2nd n

Let M = # requests in 2nd n, ~~then~~

$N_2 = N_1 + M$ where N_1 & M are indep (from Problem 5.62)

$E[N_1] = np \quad E[N_2] = 2np$

$E[N_1 N_2] = E[N_1(N_1+M)] = E[N_1^2] + E[N_1]E[M]$

$= npq + (np)^2 + (np)(np)$

$= npq + 2(np)^2 \Rightarrow$ not orthogonal

$\text{COV}(N_1, N_2) = npq + 2(np)^2 - np(2np)$

$= npq \Rightarrow$ not uncorrelated

Also N_1 and N_2 are not independent (N_2 always $\geq N_1$)

5.64

$$P[N=n, X\le t] = (1-\rho)\rho^{n-1}(1-e^{-n\lambda t}) \qquad n=1,\ldots \quad t>0$$

$$E[NX] = \sum_{n=1}^{\infty}\int_0^{\infty} n\,P[N=n, X>t]\,dt$$

$$= \sum_{n=1}^{\infty}(1-\rho)\rho^{n-1} n \underbrace{\int_0^{\infty} e^{-n\lambda t}\,dt}_{-\frac{e^{-n\lambda t}}{n\lambda}\Big|_0^{\infty}}$$

$$= \sum_{n=1}^{\infty}(1-\rho)\rho^{n-1} n\cdot\frac{1}{n\lambda}$$

$$= \frac{1}{\lambda} \implies \text{not orthogonal}$$

$$E[N] = \sum_{n=1}^{\infty} n(1-\rho)\rho^{n-1} = \frac{1}{1-\rho}$$

$$E[X] = \int_0^{\infty} P[X>t]\,dt = \int_0^{\infty}\frac{(1-\rho)e^{-\lambda t}}{1-\rho e^{-\lambda t}}\,dt$$

$$= (1-\rho)\left[\frac{1}{\rho}\ln(1-\rho e^{-\lambda t})\right]_0^{\infty}$$

$$= \frac{1-\rho}{\rho}\left[0-\ln(1-\rho)\right]$$

$$= \frac{(1-\rho)}{\rho}\left(\ln\frac{1}{1-\rho}\right)$$

5.64 – continued –

$$\text{COV}(N, X) = E[NX] - E[N]E[X]$$

$$= \frac{1}{\lambda} - \frac{1}{1-\rho}\,\frac{1-\rho}{\rho}\ln\frac{1}{1-\rho}$$

$$= \frac{1}{\lambda} + \frac{\ln(1-\rho)}{\rho}$$

$\Rightarrow$ correlated

$\Rightarrow$ not independent

5.65

$$\begin{aligned}
f(x,y) &= x+y \quad 0<x<1 \quad 0<y<1 \\
f_X(x) &= x+\frac{1}{2} \quad 0<x<1 \quad f_Y(y)=y+\frac{1}{2} \quad 0<y<1 \\
\mathcal{E}[X] &= \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{7}{12} = \mathcal{E}[Y] \\
\mathcal{E}[X^2] &= \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \frac{5}{12} = \mathcal{E}[Y^2] \\
\Rightarrow VAR[X] &= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144} = VAR[Y] \\
\mathcal{E}[XY] &= \int_0^1\int_0^1 xy(x+y)dxdy = 2\int_0^1\int_0^1 x^2y\,dxdy = \frac{1}{3} \\
\rho &= \frac{\frac{1}{3}-\left(\frac{7}{12}\right)^2}{\frac{11}{144}} = -\frac{1}{11}
\end{aligned}$$

5.66 $f_{XY}(x,y) = 12x(1-x)y \quad 0<x<1,\ 0<y<1$

correlation $E[XY] = \int_0^1\int_0^1 12x(1-x)y\cdot x\cdot dy\,dx$

$$= \int_0^1\int_0^1 12x^2(1-x)y^2\,dy\,dx$$

$$= 12\int_0^1 \left[\frac{y^3}{3}x^2(1-x)\right]_0^1 dx$$

$$= \frac{12}{3}\int_0^1 x^2(1-x)\,dx$$

$$= 4\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= \frac{4}{12}$$

$$= \frac{1}{3} \quad //$$

covariance · $E[XY] - E[X]E[Y]$

$$E[X] = \int_0^1 6x(1-x)\cdot x\,dx$$

$$= 6\int_0^1 (x^2 - x^3)\,dx$$

$$= 6\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1$$

$$= \frac{1}{2}$$

$$E[Y] = \int_0^1 2y\cdot y\,dy$$

$$= 2\left[\frac{1}{3}y^3\right]$$

$$= \frac{2}{3}$$

$$\text{cov}(X,Y) = 1/3 - \frac{1}{2}\cdot\frac{2}{3}$$

$$= 0$$

∴ orthogonal : no since $E[XY] \neq 0$
independent : yes since $\text{cov}(X,Y) = 0$
uncorrelated yes since X,Y independent.

5.67 i) $\mathcal{E}[X] = \mathcal{E}[Y] = 0$

$$\mathcal{E}[XY] = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\frac{1}{x}dydx = 0$$

$\Rightarrow \rho = 0$ orthogonal & uncorrelated

ii) $\mathcal{E}[X] = \mathcal{E}[Y] = 0$

$$\mathcal{E}[XY] = \int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|} xydydx = 0$$

$\Rightarrow \rho = 0$ orthogonal & uncorrelated

iii) $\mathcal{E}[X] = \int_0^1 2x(1-x)dx = \frac{1}{3} = \mathcal{E}[Y]$

$$\mathcal{E}[X^2] = \int_0^1 2x^2(1-x)dx = \frac{1}{6}$$

$$VAR[X] = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

$$\mathcal{E}[XY] = \int_0^1\int_0^{1-x} 2xydydx = \int_0^1 x(1-x)^2dx = \frac{1}{12}$$

$$\rho = \frac{\frac{1}{12} - \left(\frac{1}{3}\right)^2}{\frac{1}{18}} = -\frac{1}{2}$$

5.68

$$\begin{aligned}
\rho &= \frac{COV(X,Y)}{\sigma_x\sigma_Y} = \frac{\mathcal{E}[X(aX+b) - \mathcal{E}(X)\mathcal{E}(aX+b)]}{\sqrt{\mathcal{E}[X^2]-\mathcal{E}[X]^2}\sqrt{\mathcal{E}[(aX+b)^2 - \mathcal{E}[aX+b]^2]}} \\
&= \frac{a\mathcal{E}[X^2] + b\mathcal{E}[X] - a\mathcal{E}[X]^2 - b\mathcal{E}[X]}{\sqrt{\mathcal{E}[X^2]-\mathcal{E}[X]^2}\sqrt{a\mathcal{E}[X^2]-a\mathcal{E}[X]^2}} \\
&= \frac{a}{|a|} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}
\end{aligned}$$

5.69 The following expression suggests an approach to estimate the covariance:

$$\text{COV}(X,Y) = E[XY] - E[X]E[Y]$$

Let (X_1,Y_2), (X_2,Y_2), ..., (X_n,Y_n) be n sample pairs

then

$$E[X] \text{ is estimated by } \frac{1}{n}\sum_{i=1}^{n} X_i$$

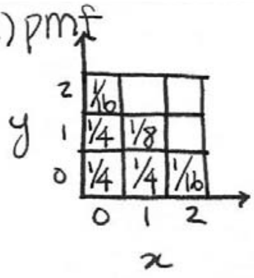
$$E[Y] \quad \text{"} \quad \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$E[XY] \quad \text{"} \quad \frac{1}{n}\sum_{i=1}^{n} X_iY_i$$

COV(X, Y) is estimated by

$$\frac{1}{n}\sum_{i=1}^{n} X_iY_i - \frac{1}{n^2}\sum_{i=1}^{n} X_i \sum_{j=1}^{n} Y_j$$

5.70 (a) pmf



y \ x	0	1	2
2	1/16		
1	1/4	1/8	
0	1/4	1/4	1/16

$$\rho_{XY} = \frac{\text{COV}(X,Y)}{\sigma_X \sigma_Y}$$

$$E[XY] = \sum_{x=0}^{2} \sum_{y=0}^{2} p_{XY}(x,y)\cdot x\cdot y$$

$$= \frac{1}{8}(1)(1)$$

$$= \frac{1}{8}$$

$$E[X] = \frac{6}{16}(1) + \frac{1}{16}(2)$$

$$= \frac{1}{2}$$

$$E[Y] = \frac{6}{16}(1) + \frac{1}{16}(2)$$

$$= \frac{1}{2}$$

$$\text{COV}(X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{1}{8} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= -\frac{1}{8}$$

$$E[X^2] = \frac{6}{16}(1) + \frac{1}{16}(4)$$

$$= 5/8$$

$$\sigma_Y = \sigma_X = (\text{VAR}[X])^{1/2}$$

$$= (E[X^2] - E[X]^2)^{1/2}$$

$$= (5/8 - 1/4)^{1/2}$$

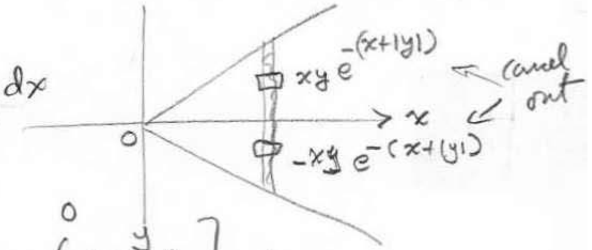
$$= \sqrt{3/8}$$

$$\rho_{XY} = \frac{-\frac{1}{8}}{(\sqrt{3/8})(\sqrt{3/8})}$$

$$= -\frac{1}{3} \; //$$

5.70 (b) $f_{XY}(x,y) = e^{-(x+|y|)}$ $\quad x>0, \; -x<y<x$

$$E[XY] = \int_0^\infty \int_{-x}^{x} xy\, e^{-(x+|y|)}\, dy\, dx$$



$$= \int_0^\infty x e^{-x} \Big[\underbrace{\int_0^x y e^{-y}\, dy + \underbrace{\int_{-x}^0 y e^{y}\, dy}_{-\int_0^x y' e^{-y'}}}_{0} \Big]\, dx$$

$y' = -y$
$dy' = -dy$

$= 0$

X and Y are orthogonal.

$$E[X] = \int_0^\infty x e^{-x} \int_{-x}^{x} e^{-|y|}\, dy\, dx = \int_0^\infty x e^{-x} \Big[2\int_0^x e^{-y}\, dy \Big] dx$$

$$= 2\int_0^\infty x e^{-x}(1-e^{-x})\, dx = 2\int_0^\infty x\,(e^{-x} - e^{-2x})\, dx$$

$$= 2\Big[1 - \tfrac{1}{2}\Big] = 1$$

$E[Y] = 0$ from above

$\mathrm{COV}(X,Y) = E[XY] - E[X]E[Y] = 0$

$\Rightarrow$ X and Y are uncorrelated.

5.71

(a) $$\rho_{XY} = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$

$$COV(X,Y) = E[XY] - E[\cancelto{0}{X}]E[Y]$$
$$= E[XY]$$

$$E[XY]$$
$$= E[X(X+N)]$$
$$= E[X^2] + E[XN]$$

$$E[X^2] = VAR[X] + E[\cancelto{0}{X}]^2$$
$$= \sigma_X{}^2$$

$$E[XN] \overset{\text{independence}}{=} E[X]\underbrace{E[N]}_{0} = 0$$

$$COV(X,Y) = \sigma_X{}^2$$

$$\rho_{XY} = \frac{\sigma_X{}^2}{\sigma_X \sigma_Y}$$

$$= \frac{\sigma_X}{\sigma_Y}, \quad \sigma_Y = \sqrt{\sigma_X{}^2 + \sigma_N{}^2}$$

(b)

$$E[(X-aY)^2]$$

$$= E[X^2 - 2aXY + a^2Y^2]$$

$$= E[X^2] - 2aE[XY] + a^2E[Y^2]$$

$$= \sigma_X^2 - 2a\sigma_X^2 + a^2(\sigma_X^2+\sigma_N^2)$$

Minimize:

$$\frac{d}{da}\,MSE = 0$$

$$-2\sigma_X^2 + 2a\sigma_X^2 + 2a\sigma_N^2 = 0$$

$$\cancel{2}a(\sigma_X^2+\sigma_N^2) = \cancel{2}\sigma_X^2$$

$$a = \frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}$$

//

(c)
$$MSE = \sigma_X^2 - 2\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}\right)\sigma_X^2 + \left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}\right)^2(\sigma_X^2+\sigma_N^2)$$

$$= \sigma_X^2 - \frac{2\sigma_X^4}{\sigma_X^2+\sigma_N^2} + \frac{\sigma_X^4}{\sigma_X^2+\sigma_N^2}$$

$$= \sigma_X^2 - \frac{\sigma_X^4}{\sigma_X^2+\sigma_N^2}$$

5.72

$$E[XY] = \int_0^{2\pi} \cos\frac{\theta}{4} \sin\frac{\theta}{4} \frac{d\theta}{2\pi}$$

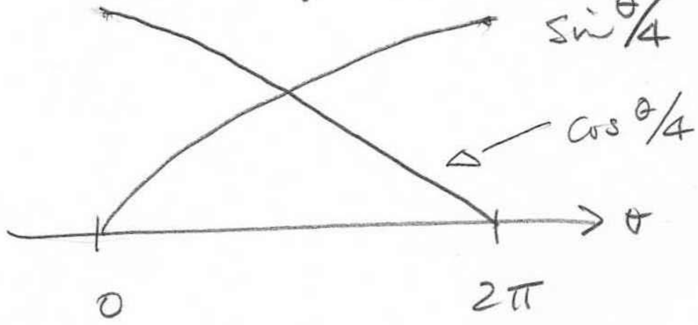
$$= \frac{1}{4\pi}\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = \frac{1}{4\pi}\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= \frac{2}{4\pi}\Big[\underbrace{\cos 0}_{1} - \underbrace{\cos\frac{2\pi}{2}}_{-1}\Big]$$

$$= \frac{4}{4\pi} = \frac{1}{\pi}$$

X and Y are correlated.

This is evident from the following



5.73

ⓐ

$$\mathrm{cov}(X, E[Y|X]) = \underbrace{E[X\,E[Y|X]]}_{\underbrace{E[E[XY|X]]}_{E[XY]}} - E[X]\underbrace{E[E[Y|X]]}_{E[Y]}$$

$$= \mathrm{cov}(X,Y).$$

ⓑ If $E[Y|X=x]=E[Y]$ for all x

$$\Rightarrow E[Y|X] = E[Y]$$

$$\therefore \quad \mathrm{cov}(X, E[Y|X]) = \underbrace{\underbrace{E[X\,E[Y]]}_{E[X]E[Y]} - E[X]E[Y]}_{=0}$$

$\Rightarrow$ uncorrelated

5.74

$$0 \le E[(tX+Y)^2] = t^2E[X^2] + 2tE[XY] + E[Y^2]$$

View as quadratic equation in t, then equation has at most a double real root

$\therefore$ the discriminant is non-positive

$$(2E[XY])^2 - 4E[X^2]E[Y^2] \le 0$$

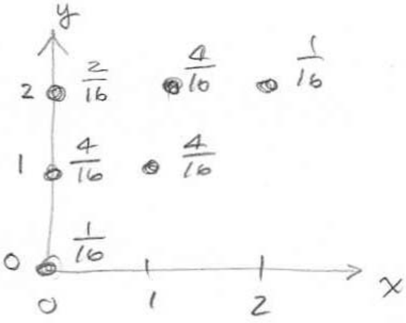
$$\Rightarrow \quad E[XY]^2 \le E[X^2]\,E[Y^2]$$

$$|E[XY]| \le \sqrt{E[X^2]\,E[Y^2]}$$

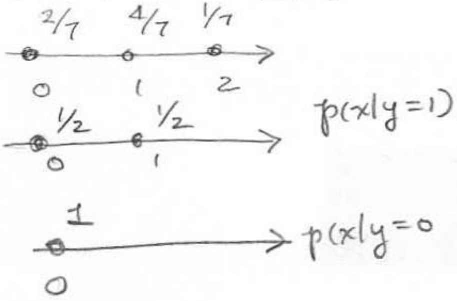
5.7 Conditional Probability and Conditional Expectation

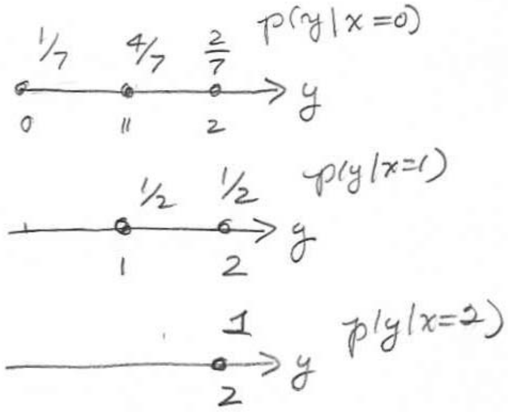
5.75

ⓐ

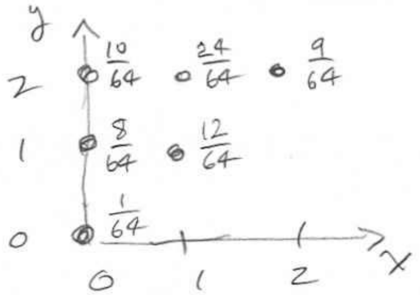


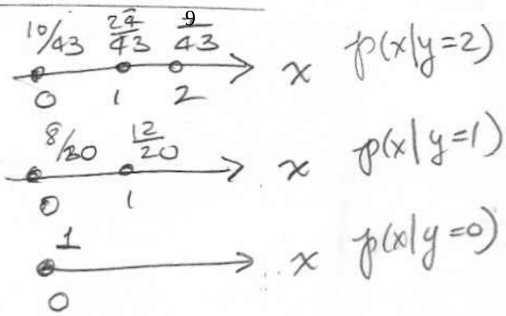
$p(x \mid y=2) = \frac{P(x,2)}{P(Y=2)}$

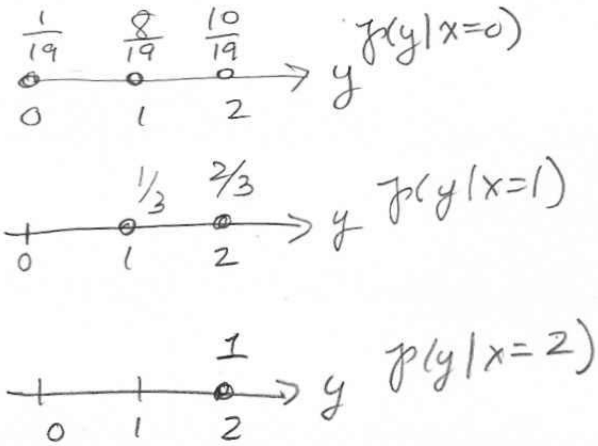




ⓑ







ⓒ Trend towards higher values of X and Y

(5.75)

(a) $E[X|y=2] = 0\cdot\frac{2}{7} + 1\cdot\frac{4}{7} + 2\cdot\frac{1}{7} = \frac{6}{7}$

$E[X|y=1] = 0\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{1}{2}$

$E[X|y=0] = 0$

$E[X] = 0\cdot\frac{1}{16} + \frac{1}{2}\cdot\frac{8}{16} + \frac{6}{7}\,\frac{7}{16} = \frac{5}{8}$

$E[Y|x=0] = 0\cdot\frac{1}{7} + 1\cdot\frac{4}{7} + 2\cdot\frac{2}{7} = \frac{8}{7}$

$E[Y|x=1] = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = \frac{3}{2}$

$E[Y|x=2] = 1$

$E[Y] = \frac{15}{8}\cdot\frac{7}{16} + \frac{3}{2}\,\frac{8}{16} + 1\cdot\frac{1}{16} = \frac{35+96+8}{128}$

$= \frac{139}{128}$

$E[X|y=2] = 1\cdot\frac{24}{43} + 2\cdot\frac{9}{43} = \frac{42}{43}$

$E[X|y=1] = \frac{12}{20}\cdot 1 = \frac{12}{20}$

$E[X|y=0] = 0$

$E[X] = \frac{42}{43}\cdot\frac{43}{64} + \frac{12}{20}\cdot\frac{20}{64} = \frac{54}{64}$

$E[Y|x=0] = 1\cdot\frac{8}{19} + 2\cdot\frac{10}{19} = \frac{28}{19}$

$E[Y|x=1] = 1\cdot\frac{1}{3} + 2\cdot\frac{2}{3} = \frac{5}{3}$

$E[Y|x=2] = 1$

$E[Y] = \frac{28}{19}\cdot\frac{19}{64} + \frac{5}{3}\cdot\frac{36}{64} + 1\cdot\frac{9}{64} = \frac{28+60+9}{64} = \frac{97}{64}$

5.76

(a) $p_X(x|y) = \dfrac{p_{XY}(x,y)}{p_Y(y)}$

$p_X(-1|-1) = \dfrac{\frac{1}{4}(1-p-p_e)}{\frac{1}{4}(1-p-p_e)+\frac{3}{4}p}$ $\qquad p_X(1|-1) = \dfrac{\frac{3}{4}p}{\frac{1}{4}(1-p-p_e)+\frac{3}{4}p}$

$p_X(-1|0) = \dfrac{\frac{1}{4}p_e}{p_e} = \dfrac{1}{4}$ $\qquad p_X(1|0) = \dfrac{\frac{3}{4}p_e}{p_e} = \dfrac{3}{4}$

$p_X(-1|1) = \dfrac{\frac{1}{4}p}{\frac{3}{4}(1-p-p_e)+\frac{1}{4}p}$ $\qquad p_X(1|1) = \dfrac{\frac{3}{4}(1-p-p_e)}{\frac{3}{4}(1-p-p_e)+\frac{1}{4}p}$

(b) for $y=0$, $p_X(x|y)$ maximum for $x=1$

for $y=-1$, $p_X(x|y)$ maximum for $x=1$ assuming $p>\frac{1}{4}-\frac{1}{4}p_e$

for $y=1$, $p_X(x|y)$ maximum for $x=1$ assuming $p<\frac{3}{4}-\frac{3}{4}p_e$

(c) $P_{error} = 1 - p_{XY}(1,1) - p_{XY}(-1,-1)$

$= 1 - \frac{1}{4}(1-p-p_e) - \frac{3}{4}(1-p-p_e)$

$= p + p_e$

5.77

ⓐ $p(y|-1)$ $\frac{1}{2}$ $\frac{1}{2}$ 0

$p(y|0)$ 0 0 $\frac{1}{3}$ $\Leftarrow$ this pmf provides most info about Y

$p(y|1)$ $\frac{1}{2}$ $\frac{1}{2}$ 0

ⓑ The conditional pmf's in (ii) are equiprobable so are most random;

The conditional pmf's in (iii) have one value with probability 1 and hence remove all uncertainty about the outcome

ⓒ (i) $E[Y|-1] = -1\cdot\frac{1}{2} = -\frac{1}{2}$ $E[Y] = \frac{1}{3}\left[-\frac{1}{2} + 1 - \frac{1}{2}\right] = 0$

$E[Y|0] = 1\cdot 1 = 1$

$E[Y|+1] = -1\cdot\frac{1}{2}$

$E[X|-1] = -\frac{1}{2}+\frac{1}{2} = 0$ $E[X] = 0.$

$E[X|0] = -\frac{1}{2}+\frac{1}{2} = 0$

$E[X|1] = 0$

(ii) $E[Y|i] = 0$ all i $E[X] = E[Y] = 0$

$E[X|i] = 0$ all i

(iii) $E[Y|-1] = -1$ $E[Y] = \frac{1}{3}[-1+1] = 0$

$E[Y|0] = 0$

$E[Y|1] = 1$

$E[X|-1] = -1$

$E[X|0] = 0$ $E[X] = 0.$

$E[X|1] = 1$

5.77 d

i) $E[Y^2|-1] = \frac{1}{2}$ $\qquad E[Y^2] = \frac{1}{3}\left[\frac{1}{2}+1+\frac{1}{2}\right] = \frac{2}{3}$

$E[Y^2|0] = 1$ $\qquad VAR[Y] = \frac{2}{3} - 0^2 = \frac{2}{3}$

$E[Y^2|+1] = \frac{1}{2}$

$E[X^2|-1] = \frac{1}{2}+\frac{1}{2} = 1$ $\qquad E[X^2] = \frac{1}{3}[1+1] = \frac{2}{3}$

$E[X^2|0] = \frac{1}{2}+\frac{1}{2} = 1$ $\qquad VAR(X) = \frac{2}{3}$

$E[X^2|1] = 0$

(ii) $E[Y^2|i] = ((-1)^2+0+1^2)\frac{1}{3} = \frac{2}{3}$ all i

$\Rightarrow E[Y^2] = \frac{2}{3} = VAR[Y]$

Similarly $E[X^2|i] = \frac{2}{3} = E[X^2] = VAR[X]$.

(iii) $E[Y^2|-1] = 1$ $\qquad E[Y^2] = \frac{1}{3}(1+1) = \frac{2}{3} = VAR[Y]$

$E[Y^2/0] = 0$

$E[Y^2|+1] = 1$

$E[X^2|-1] = 1$ $\qquad E[X^2] = \frac{2}{3} = VAR[X]$

$E[X^2|0] = 0$

$E[X^2|+1] = 1$

5.78) for $l \geq k$ M ~~=#~~ heads in second n trials

$$P[N_1=k, N_2=l] = P[N_1=k]P[M=l-k]$$

ⓐ
$$P[N_1=k \mid N_2=l] = \frac{P[N_2=l \mid N_1=k]P[N_1=k]}{P[N_2=l]}$$

$$= \frac{\binom{n}{l-k}p^{l-k}(1-p)^{n-l+k}\binom{n}{k}p^{k}(1-p)^{n-k}}{\binom{2n}{l}p^{l}(1-p)^{2n-l}}$$

$$= \frac{\binom{n}{l-k}\binom{n}{k}}{\binom{2n}{l}}$$

this is the hypergeometric probability we saw in chapter 3

ⓑ
$$P[N_1=k \mid N_2=2k] = \frac{\binom{n}{k}\binom{n}{k}}{\binom{2n}{2k}}$$

Use Stirling's formula $n \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \approx \frac{\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}}{\sqrt{2\pi}\, k^{k+\frac{1}{2}} e^{-k} \sqrt{2\pi}\, (n-k)^{n-k+\frac{1}{2}} e^{-(n-k)}}$$

$$= \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}\, k^{k+\frac{1}{2}} (n-k)^{n-k+\frac{1}{2}}}$$

$$\frac{\binom{n}{k}\binom{n}{k}}{\binom{2n}{2k}} \approx \frac{n^{2n+1}}{2\pi k^{2k+1} (n-k)^{2(n-k)+1}} \; \frac{\sqrt{2\pi}\,(2k)^{2k+\frac{1}{2}} (2n-2k)^{2n-2k+\frac{1}{2}}}{(2n)^{2n+1/2}}$$

5.78 — continued —

$$\frac{\binom{n}{k}\binom{n}{k}}{\binom{2n}{2k}} \approx \frac{n^{1/2}\, 2^{2k+\frac{1}{2}}\, 2^{2\cdot(n-k)+\frac{1}{2}}}{\sqrt{2\pi}\; 2^{2n+1/2}\, k^{1/2}\,(n-k)^{1/2}} = \frac{n^{1/2}\, 2^{1/2}}{\sqrt{2\pi}\, k^{1/2}(n-k)^{1/2}}$$

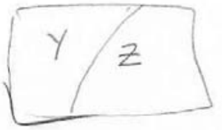
$$= \sqrt{\frac{2n}{2\pi k(n-k)}}$$

$n=100$	$k=5$	$k=10$	$k=20$
	$\sqrt{\frac{200}{2\pi\, 5(95)}}$	$\sqrt{\frac{200}{2\pi\, 10(90)}}$	$\sqrt{\frac{200}{2\pi\, 20(80)}}$
	$=$	$=$	$=$
	0.2589	0.1880	0.1404

ⓒ $$E[N_1 \mid N_2 = l] = \sum_{k=0}^{l} k\, \frac{\binom{n}{l-k}\binom{n}{k}}{\binom{2n}{l}} = \frac{l}{2}$$

$$E[N_1] = E\left[\tfrac{1}{2}N_2\right] = \tfrac{1}{2}E[N_2] = \tfrac{1}{2}\,2np = np$$

5.79



$X = Y + Z$ X is Poisson

$$P[Y=j, Z=l] = P[Y=j \mid X=j+l]\, P[X=j+l]$$

$$= \binom{j+l}{j} p^{j} (1-p)^{k-j} \frac{\alpha^{j+l}}{(j+l)!} e^{-\alpha}$$

$$P[Z=l \mid Y=j] = \frac{P[Y=j, Z=l]}{P[Y=j]}$$

$$= \frac{\binom{j+l}{j} p^{j} (1-p)^{l} \alpha^{j+l} e^{-\alpha}}{(j+l)!\; \frac{(\alpha p)^{j} e^{-\alpha p}}{j!}}$$

$$= \frac{(1-p)^{l} \alpha^{l}}{l!} e^{-\alpha(1-p)}$$

$$= \frac{((1-p)\alpha)^{l}}{l!} e^{-\alpha(1-p)}$$ Poisson and indep of $Y=j$.

Result is "intuitive" if we view defect process as aggregation of multiplicity of Bernoulli trials over different regions of the device

5.80

ⓐ $f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}} \qquad 0<y<1$

ⓑ $P[Y>X|x] = \int_x^1 \frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\int_x^1 (x+y)\,dy$

$= \frac{1}{x+\frac{1}{2}}\left[xy\Big|_x^1 + \frac{y^2}{2}\Big|_x^1\right]$

$= \frac{1}{x+\frac{1}{2}}\left[x(1-x) + \frac{1}{2} - \frac{x^2}{2}\right]$

$= \frac{1}{x+\frac{1}{2}}\left[x + \frac{1}{2} - \frac{3}{2}x^2\right]$

ⓒ $P[Y>X] = \int_0^1 P[Y>X|x]\, f_X(x)\,dx$

$= \int_0^1 \frac{x+\frac{1}{2}-\frac{3}{2}x^2}{x+\frac{1}{2}}\left(x+\frac{1}{2}\right)dx$

$= \int_0^1 \left(x+\frac{1}{2}-\frac{3}{2}x^2\right)dx$

$= \frac{1}{2} + \frac{1}{2} - \frac{3}{2}\,\frac{1}{3} = \frac{1}{2}$ ✓

ⓓ $E[Y|X=x] = \int_0^1 y\,\frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\int_0^1 (xy+y^2)\,dy$

$= \frac{\frac{1}{2}x+\frac{1}{3}}{x+\frac{1}{2}}$

5.81

ⓐ (i) $f_Y(y|x) = \dfrac{f_{XY}(x,y)}{f_X(x)} = \dfrac{\pi}{2\sqrt{1-x^2}}$ $\quad -1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

(ii) $f_Y(y|x) = \dfrac{1}{1-|x|}$ $\quad -1 \le x \le 1$

$-(1-|x|) \le y \le 1-|x|$

(iii) $f_Y(y|x) = \dfrac{1}{2(1-x)}$ $\quad 0 < x < 1$

$0 < y < 1-x$

ⓑ (i) $E[Y|x] = \dfrac{\pi}{\sqrt{1-x^2}} \displaystyle\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\,dy = 0 \Rightarrow E[Y] = 0$

(ii) $E[Y|x] = \dfrac{1}{1-|x|} \displaystyle\int_{-(1-|x|)}^{1-|x|} y\,dy = 0 \Rightarrow E[Y] = 0$

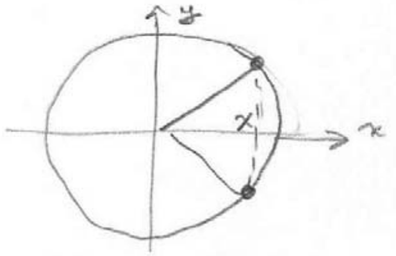
(iii) $E[Y|x] = \dfrac{1}{2(1-x)} \displaystyle\int_0^{1-x} y\,dy = \dfrac{1}{2(1-x)} \times \dfrac{(1-x)^2}{2} = \dfrac{1-x}{2}$

ⓒ $E[Y] = \displaystyle\int_0^1 \frac{1-x}{2} \cdot 2(1-x)\,dx = \int_0^1 (1-x)^2\,dx$

$= \displaystyle\int_0^1 (1-2x+x^2)\,dx$

$= \left[1 - 1 + \frac{1}{3}\right] = \frac{1}{3}.$ ✓

5.82 (a)



$X = \cos\Theta$
$Y = \sin\Theta$
Θ uniform in $(0, 2\pi)$

$X^2 + Y^2 = 1$

$$f_Y(y|x) = \frac{1}{2}\delta\left(y - \sqrt{1-x^2}\right) + \frac{1}{2}\delta\left(y + \sqrt{1-x^2}\right)$$

(a)
$$E[xY|x] = \frac{x}{2}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} y\left\{\delta(y-\sqrt{\ }) + \delta(y+\sqrt{\ })\right\} dy$$
$$= \frac{x}{2}\left\{\sqrt{1+x^2} - \sqrt{1+x^2}\right\}$$
$$= 0$$

(a) $E[XY] = E[E[xY|X]] = 0$

(b)
$$E[Y|x] = \frac{1}{2}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} y\left\{\delta(y-\sqrt{\ }) + \delta(y+\sqrt{\ })\right\} dy$$
$$= \frac{1}{2}\left\{\sqrt{1+x^2} - \sqrt{1+x^2}\right\} = 0$$

(c) $E[Y] = 0$

5.83

$$f_Y(y|x) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{\left(\frac{x-m_1}{\sigma_1}\right)^2-2\rho\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)+\left(\frac{y-m_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-m_1)^2}{2\sigma_1^2}\right\}}$$

$$\begin{aligned}
f_Y(y|x) &= \frac{\exp\left\{-\frac{\left(\frac{y-m_2}{\sigma_2}\right)^2-2\rho\left(\frac{y-m_2}{\sigma_2}\right)\left(\frac{x-m_1}{\sigma_1}\right)+\rho^2\left(\frac{x-m_1}{\sigma_1}\right)^2}{2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \\
&= \frac{\exp\left\{-\frac{\left\{\left(\frac{y-m_2}{\sigma_2}\right)-\rho\left(\frac{x-m_1}{\sigma_1}\right)\right\}^2}{2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \\
&= \frac{\exp\left\{-\frac{(y-m_2-\rho\frac{\sigma_2}{\sigma_1}(x-m_1))^2}{2\sigma_2^2(1-\rho^2)}\right\}}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \\
&= \text{Gaussian pdf with mean } m_2+\rho\tfrac{\sigma_2}{\sigma_1}(x-m_1) \\
&\quad\ \text{and variance } \sigma_2^2(1-\rho^2)
\end{aligned}$$

Similarly $f_X(x|y)$ is a Gaussian pdf with mean $m_1+\rho\frac{\sigma_1}{\sigma_2}(y-m_2)$ and variance $\sigma_1^2(1-\rho^2)$.

5.84

(a) $f_X(x|N=n) = \frac{d}{dx} F_X(x|N)$

$$F_X(x|N) = \frac{P[X \le x, N=n]}{P[N=n]}$$

$$= \frac{\cancel{(1-g)}\cancel{g^{n-1}}(1-e^{-n\lambda x})}{\cancel{(1-g)}\cancel{g^{n-1}}}$$

$$= 1 - e^{-n\lambda x}$$

$$f_X(x|N) = \frac{d}{dx}(1 - e^{-n\lambda x})$$

$$= n\lambda e^{-n\lambda x} \qquad x > 0$$

//

(b) $E[X|N=n]$

$$= \int f_X(x|n)\, x\, dx$$

$$= \int_0^{\infty} n\lambda e^{-n\lambda x} \cdot x\, dx$$

integration by parts...

$$= \left.\frac{-e^{-n\lambda x}}{n\lambda}\right]_0^{\infty}$$

$$= 1/n\lambda$$

//

(5,84) continued

$$P[N=n \mid x<T<t+dt] = \frac{P[N=n,\ t<T<t+dt]}{P[t<T<t+dt]}$$

$$P[N=n,\ t<T<t+dt] = P[N=n,\ T<t+dt] - P[N=n,\ T<t]$$

$$= (1-\rho)\rho^{n-1}\left[e^{-n\lambda t} - e^{-n\lambda(t+dt)}\right]$$

$$= (1-\rho)\rho^{n-1} e^{-n\lambda t}\left[1 - e^{-n\lambda dt}\right]$$

$$\simeq (1-\rho)\rho^{n-1} e^{-n\lambda t}\left[n\lambda dt + o(n\lambda dt)\right]$$

from 5.24 ⓒ

$$P[T\le t] = \frac{1-e^{-\lambda t}}{1-\rho e^{-\lambda t}} \Rightarrow f_T(t) = \frac{\lambda(1-\rho)e^{-\lambda t}}{(1-\rho e^{-\lambda t})^2}$$

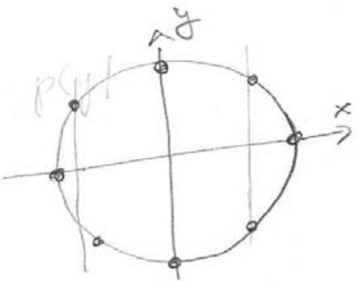
ⓒ $$P[N=n \mid t<T<t+dt] = \frac{(1-\rho)\rho^{n-1}e^{-n\lambda t}\,n\lambda\,dt}{f_T(t)\,dt}$$

$$= A(t)\,\underbrace{(\rho e^{-\lambda t})^{n}}_{}\,n$$

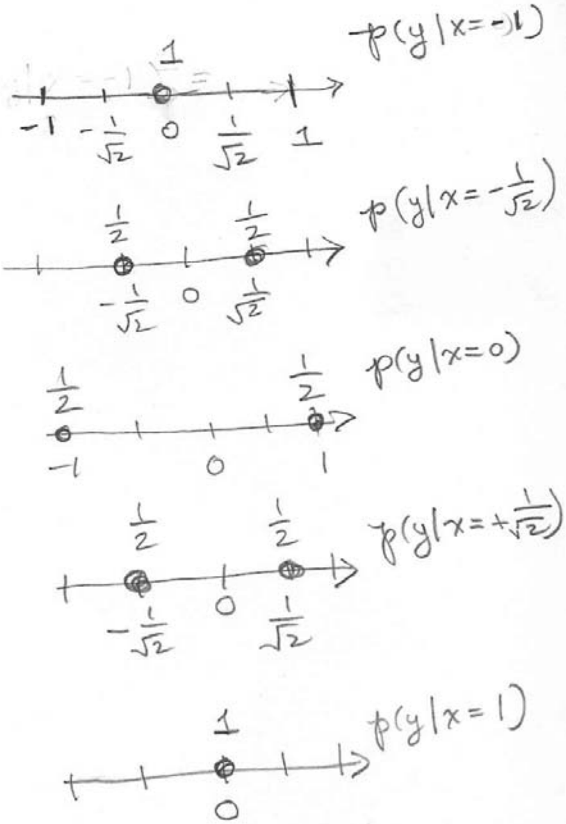
To find the value of n that maximizes this probability we consider the ratio

$$1 = \frac{\beta^{n+1}(n+1)}{\beta^{n}n} = \beta\left(1+\frac{1}{n}\right) \Rightarrow n^{*} \approx \frac{\beta}{\beta-1} = \frac{\rho e^{-\lambda t}}{1-\rho e^{-\lambda t}}$$

5.85



ⓐ



ⓑ $E[Y|-1] = 0$

$E[Y|-\frac{1}{\sqrt{2}}] = 0$

$E[Y|0] = 0$

$E[Y|\frac{1}{\sqrt{2}}] = 0$

$E[Y|1] = 0$

ⓒ $E[xY|x] = \sum_{j} jx \, p_Y(j|x) = x\,E[Y|x]$

$\therefore$ $E[xY|x] = 0$ all x

$E[XY] = E[E[XY|X]] = 0$

$\Rightarrow$ X and Y are uncorrelated.

5.86

$$2 = \frac{\alpha x_m}{\alpha - 1} = \frac{2.5}{1.5} x_m$$

$$f_X(t) = \sum_{k=1}^{3} f_X(t|k)\, p_k$$

$$= \frac{1}{3}\left[\ \delta(t-2) + \frac{1}{2}e^{-t/2} + 2.5\,\frac{\left(\frac{3}{2.5}\right)^{2.5}}{\cancel{t}^{3.5}}\, u\left(t - \frac{3}{2.5}\right)\right]$$

$$E[X] = \sum_{k=1}^{3} \underbrace{E[X|k]}_{2}\, p_k = 2$$

$$VAR[X] = E\left[(X-2)^2\right]$$

$$= \int_0^\infty (t-2)^2 f_X(t)\,dt = \sum_{k=1}^{3} \int_0^\infty (t-2)^2 f_X(t|k)\,dt\, p_k$$

$$= \frac{1}{3}\left\{0 + \left(\frac{1}{1/2}\right)^2 + \frac{(3/2.5)^2\, 2.5}{(1.5)^2(.5)^2}\right\} = \frac{52}{15}$$

5.87

$$P[K=k|N=n] = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P[K=k] = \sum_{n=\max(k,1)}^{\infty} P[K=k|N=n]P[N=n]$$

can have $k>0$ new messages only if $N \geq k$

$$\begin{aligned}
P[K=0] &= \sum_{n=1}^{\infty}(1-p)^n(1-a)a^{n-1} \\
&= (1-p)(1-a)\sum_{n=1}^{\infty}[(1-p)a]^{n-1} \\
&= \frac{(1-p)(1-a)}{1-(1-p)a}
\end{aligned}$$

For $k \geq 1$

$$\begin{aligned}
P[K=k] &= \sum_{n=k}^{\infty}\binom{n}{k}p^k(1-p)^{n-k}(1-a)a^{n-1} \\
&= \frac{(1-a)p^k a^k}{a}\sum_{n=k}^{\infty}\binom{n}{k}[(1-p)a]^{n-k} \\
&= \frac{(1-a)p^k a^k}{a(1-(1-p)a)^{k+1}} \\
&= \frac{(1-a)}{a(1-(1-p)a)}\left(\frac{pa}{1-(1-p)a}\right)^k \qquad k=1,2,\ldots
\end{aligned}$$

K is geometric-like for $k \geq 1$, but the $P[K=0]$ is not consistent with the probability of success.

b)

$$\begin{aligned}
\mathcal{E}[K] &= \mathcal{E}[\mathcal{E}[K|N]] = \sum_{n=1}^{\infty}\mathcal{E}[K|n](1-\alpha)a^{n-1} \\
&= \sum_{n=1}^{\infty} n-p(1-a)a^{n-1} = p\mathcal{E}[N] = \frac{p}{1-a} \\
\mathcal{E}[K^2] &= \sum_{n=1}^{\infty}\mathcal{E}[K^2|n](1-a)a^{n-1} \\
&= \sum_{n=1}^{\infty}(npq+(np)^2)(1-a)a^{n-1} \\
&= pq\mathcal{E}[N]+p^2\mathcal{E}[N^2] \\
VAR[K] &= \mathcal{E}[K^2]-\mathcal{E}[K]^2 \\
&= pq\mathcal{E}[N]+p^2\mathcal{E}[N^2]-p^2\mathcal{E}[N]^2 \\
&= pq\mathcal{E}[N]+p^2VAR[N] \\
&= \frac{pq}{1-a}+\frac{p^2a}{(1-a)^2}
\end{aligned}$$

5.88

$$\begin{aligned}
P[N=k] &= \int_0^\infty P[N=k|R=r]f_R(r)dr \\
&= \int_0^\infty \frac{r^k}{k!}e^{-r}\frac{\lambda(\lambda r)^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda r}dr \\
&= \frac{\lambda^\alpha}{k!\Gamma(\alpha)}\int_0^\infty r^{k+\alpha-1}e^{-(1+\lambda)r}dr \qquad \text{let } t=(1+\lambda)r \\
&= \frac{\lambda^\alpha}{k!\Gamma(\alpha)}\frac{1}{(1+\lambda)^{k+\alpha}}\underbrace{\int_0^\infty t^{k+\alpha-1}e^{-t}dt}_{\Gamma(k+\alpha)} \\
&= \underbrace{\frac{\Gamma(k+\alpha)}{\Gamma(\alpha)k!}}_{\substack{\text{generalization}\\ \text{of Binomial}\\ \text{Coefficient}}} \quad \underbrace{\left(\frac{\lambda}{1+\lambda}\right)^\alpha\left(\frac{1}{1+\lambda}\right)^k}_{\substack{\text{form of}\\ \text{Binomial}\\ \text{dist.}}}
\end{aligned}$$

N is called the generalized Binomial RV.

$$\begin{aligned}
E[N] &= \int_0^\infty \mathcal{E}[N|r]f_R(r)dr = \int_0^\infty rf_R(r)dr = \mathcal{E}[R] = \frac{\alpha}{\lambda} \\
\mathcal{E}[N^2] &= \int_0^\infty \mathcal{E}[N^2|r]f_R(r)dr = \int_0^\infty (r+r^2)f_R(r) = \mathcal{E}[R]+\mathcal{E}[R^2] \\
VAR[N] &= \mathcal{E}[R^2]+\mathcal{E}[R] = \mathcal{E}[R]^2 = VAR[R]+\mathcal{E}[R] \\
&= \frac{\alpha}{\lambda^2}+\frac{\alpha}{\lambda}
\end{aligned}$$

5.89

(a)

$$P[X=+1 \mid y<Y\le y+dy] = \frac{P[X=+1,\ y<Y\le y+dy]}{f_Y(y)\,dy}$$

$$= \frac{f_Y(y|+1)\,p\,dy}{f_Y(y)\,dy} = p\,\frac{f_Y(y|+1)}{f_Y(y)}$$

$\sigma^2 = \frac{1}{4}$

$$= p\,\frac{e^{-(y-1)^2/2\sigma^2}}{p\,e^{-(y-1)^2/2\sigma^2} + (1-p)\,e^{-(y+1)^2/2\sigma^2}}$$

$$= \frac{p}{p + (1-p)\,e^{-2[(y+1)^2-(y-1)^2]}}$$

$$= \frac{p}{p + q\,e^{-8y}}$$

(b) +1 is more probable if

$$\frac{p}{p+q\,e^{-8y}} > \frac{1}{2} \iff 1 + \frac{q}{p}e^{-8y} < 2$$

$$\iff y > -\frac{1}{8}\ln\frac{p}{q} = \gamma$$

(c)

$$P[\text{error}] = P[\text{error}|+1]\,p + P[\text{error}|-1]\,(1-p)$$

$$= p\int_{-\infty}^{\gamma} f_Y(y|+1)\,dy + q\int_{\gamma}^{\infty} f_Y(y|-1)\,dy$$

$$= p\left(1 - Q\left(\frac{\gamma-1}{\sigma}\right)\right) + q\,Q\left(\frac{\gamma+1}{\sigma}\right)$$

5.8 Functions of Two Random Variables

5.90

X_1 exponential $\frac{1}{\lambda} = 100$

X_2 Rayleigh $E[X_2] = \alpha\sqrt{\pi/2} = 100$

$T = \min(X_1, X_2)$

ⓐ $P[T > t] = P[X > t]P[Y > t] = e^{-\lambda t} e^{-\alpha t^2}$

$P[X \le t] = 1 - e^{-\lambda t} e^{-\alpha t^2}$

$t_0 = 100$

ⓑ $P[T - t_0 > t \mid T > t_0] = \dfrac{P[T > t + t_0]}{P[T > t_0]}$ where $T - t_0$ is the additional time

$= P[T > t + t_0 \mid T > t_0] =$

$$= \frac{e^{-\lambda(t+t_0) - \alpha(t+t_0)^2}}{e^{-\lambda t_0 - \alpha t_0^2}} = e^{-\lambda t - \alpha(t+t_0)^2 + \alpha t_0^2} \qquad t > 0$$

ⓒ $P[T > t + t_0 \mid T > t_0]$ given above

where T is the total time

5.91 ⓐ $T = \max(X_1, X_2)$

ⓐ $P[T \le t] = P[X_1 \le t]P[X_2 \le t] = (1 - e^{-\lambda t})(1 - e^{-\alpha t^2})$

$t_0 = 100$

ⓑ $P[T > t + t_0 \mid T > t_0] = \dfrac{P[T > t + t_0]}{P[T > t_0]}$

$$= \frac{1 - (1 - e^{-\lambda(t+t_0)})(1 - e^{-\alpha(t+t_0)^2})}{1 - (1 - e^{-\lambda t_0})(1 - e^{-\alpha t_0^2})}$$

5.92

(a)
$$\begin{aligned} p_M(m) &= P[M=m] \\ &= P[K+N=m] \\ &= \sum_{n=0}^{m} P[N=n]\,P[K=m-n] \\ &= \sum_{n=0}^{m} p_N(n)\,p_K(m-n) \end{aligned}$$

(b)
$$\begin{aligned} p_M(m) &= \sum_{t=0}^{m} p_N(t)\,p_K(m-t) \\ &= \sum_{t=0}^{m} \binom{n}{t} p^t (1-p)^{n-t} \binom{k}{m-t} p^{m-t} (1-p)^{k-m+t} \\ &= \sum_{t=0}^{m} \binom{n}{t}\binom{k}{m-t} p^m (1-p)^{k+n-m} = p^m (1-p)^{k+n-m} \underbrace{\sum_{t=0}^{m} \binom{n}{t}\binom{k}{m-t}}_{\binom{n+k}{m}} \\ &= \binom{n+k}{m} p^m (1-p)^{k+n-m} \end{aligned}$$

Also binomial

(c)
$$\begin{aligned} p_M(m) &= \sum_{t=0}^{m} p_N(t)\,p_K(m-t) \\ &= \sum_{t=0}^{m} \frac{\alpha_1^{\,t}}{t!} e^{-\alpha_1} \cdot \frac{\alpha_2^{\,m-t}}{(m-t)!} e^{-\alpha_2} \\ &= \frac{\alpha_2^m e^{-(\alpha_1+\alpha_2)}}{m!} \underbrace{\sum_{t=0}^{m} \frac{m!}{t!(m-t)!} \left(\frac{\alpha_1}{\alpha_2}\right)^t}_{\left(1+\frac{\alpha_1}{\alpha_2}\right)^m} \\ &= \frac{(\alpha_1+\alpha_2)^m}{m!} e^{-(\alpha_1+\alpha_2)}. \end{aligned}$$

Also Poisson

5.93

$$E[X] = \frac{1-p}{p} = 2 \qquad E[Y] = \frac{1-q}{q} = 4$$

$$p = \frac{1}{3} \qquad q = \frac{1}{5}$$

(a) $p_Z(z) = P[Z = z]$

$= P[X - Y = z]$

for $z \geq 0$:

$$p_Z(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{k-z} \frac{1}{3}\left(\frac{2}{3}\right)^k \cdot \frac{1}{5}\left(\frac{4}{5}\right)^l$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{\infty} \sum_{l=0}^{k-z} \frac{1}{15}\left(\frac{2}{3}\right)^k \left(\frac{4}{5}\right)^l & , z \geq 0 \\ \displaystyle\sum_{k=0}^{\infty} \sum_{l=0}^{k-z} \frac{1}{15}\left(\frac{4}{5}\right)^k \left(\frac{2}{3}\right)^l & , z < 0 \end{cases}$$

(b) P["Bulldogs beat Flames"]

$= P[X > Y]$

$$= \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^k \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^l$$

$$= \sum_{k=1}^{\infty} \frac{1}{15}\left(\frac{2}{3}\right)^k \left(\frac{1 - \frac{4}{5}^k}{1/5}\right)$$

$$= \frac{1}{3}\left(\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k - \sum_{k=1}^{\infty}\left(\frac{8}{15}\right)^k\right)$$

$$= \frac{1}{3}\left(2 - \frac{8}{7}\right)$$

$$= \frac{2}{7}$$

P5.93
cont

$$P[\text{"tie"}]$$
$$= P[X=Y]$$
$$= \sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^k \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^k$$
$$= \frac{1}{15} \sum_{k=0}^{\infty} \left(\frac{8}{15}\right)^k$$
$$= \frac{1}{15}\left(\frac{15}{7}\right)$$
$$= \frac{1}{7}$$

5.94

(a) $p_T(t) = P[T=t]$

$$= \begin{cases} \binom{t}{2} p^2 (1-p)^{t-2} & , t \geq 2 \\ 0 & , \text{otherwise} \end{cases}$$

(b) $p_W(w) = (1-p)^{w-1} p \quad , w \geq 1$

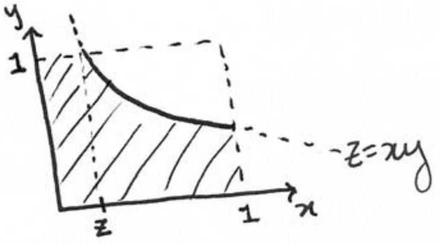
5.95

$Z = XY \quad f_{XY}(x,y) = 1 \quad \begin{matrix} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{matrix}$

$$F_Z(z) = P[Z \leq z]$$

$$= P[XY \leq z]$$

$$= P[Y \leq z/X]$$



$$= f_{XY}(x,y) \times \text{shaded area}$$

$$= z + \int_z^1 \int_0^{z/x} dy\,dx$$

$$= z + \int_z^1 z/x \, dx$$

$$= z + [z \ln x]_z^1$$

$$= z - z \ln z$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

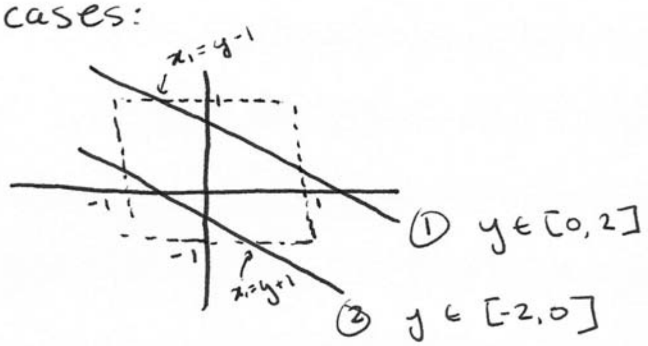
$$= 1 - \ln z - \frac{z}{z}$$

$$= \begin{cases} -\ln z & , 0 \leq z \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

5.96

(a) $f_{X_1,X_2}(x_1,x_2) = \frac{1}{4}$ $\quad -1 \le x_1 \le 1$, $\; -1 \le x_2 \le 1$

2 cases:



case ①:

$$F_Y(y) = P[X_1 + X_2 \le y]$$

$$= P[X_2 \le y - X_1]$$

$$= \frac{1}{4}(2y) + \int_{y-1}^{1} \int_{-1}^{y-x_1} \frac{1}{4}\, dx_2\, dx_1$$

$$= \frac{y}{2} + \frac{1}{4} \int_{y-1}^{1} (y - x_1 + 1)\, dx_1$$

$$= \frac{y}{2} + \frac{1}{4} \left[y x_1 - \frac{1}{2} x_1^2 + x_1 \right]_{y-1}^{1}$$

$$= \frac{y}{2} + \cancel{\frac{y}{4}} + \frac{3}{8} - \frac{y^2}{4} + \frac{y^2}{8} - \cancel{\frac{y}{4}} + \frac{1}{8}$$

$$= -\frac{y^2}{8} + \frac{y}{2} + \frac{1}{2}$$

case ②

$$F_X(y)$$

$$= \frac{1}{4}\int_{-1}^{y+1}\int_{-1}^{y-x_1} dx_2\,dx_1$$

$$= \frac{1}{4}\int_{-1}^{y+1}(y-x_1+1)\,dx_1$$

$$= \frac{1}{4}\left[x_1 y - \frac{1}{2}x_1^2 + x_1\right]_{-1}^{y+1}$$

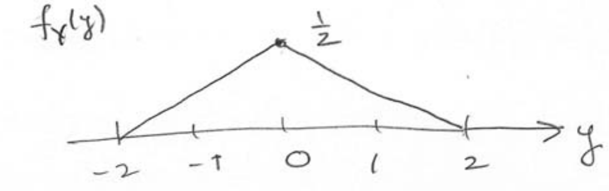
$$= \frac{1}{4}\left(y(y+1) - \frac{1}{2}(y+1)^2 + y + 1 + y + \frac{1}{2} + 1\right)$$

$$= \frac{1}{4}\left(y^2 + y - \frac{1}{2}y^2 - y - \frac{1}{2} + y + 1 + y + \frac{1}{2} + 1\right)$$

$$= \frac{1}{8}y^2 + \frac{1}{2}y + \frac{1}{2}$$

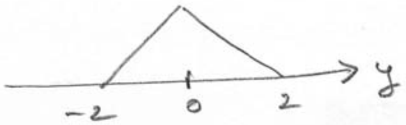
$$\therefore\ F_Y(y) = \begin{cases} -\frac{1}{8}y^2 + \frac{1}{2}y + \frac{1}{2} & , 0 \le y \le 2 \\ \frac{1}{8}y^2 + \frac{1}{2}y + \frac{1}{2} & , -2 \le y < 0 \\ 1 & , y > 2 \\ 0 & , y < -2 \end{cases}$$

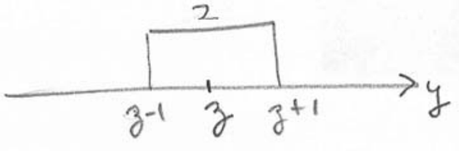
$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} -\frac{1}{4}y + \frac{1}{2} & , 0 \le y \le 2 \\ \frac{1}{4}y + \frac{1}{2} & , -2 \le y < 0 \\ 0 & , \text{otherwise} \end{cases}$$



(b) Use the convolution method

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y)\, f_{X_3}(z-y)\, dy$$





for $z+1<-2$ $\quad f_Z(z)=0$

for $z+1<0$

$$f_Z(z) = \int_{-2}^{z+1} \left(\tfrac{1}{2}+\tfrac{1}{4}y\right) dy = \left[\tfrac{1}{2}y + \tfrac{1}{4}\tfrac{y^2}{2}\right]_{-2}^{z+1}$$

$$= \tfrac{1}{2}(z+1) + 1 + \tfrac{1}{8}(z+1)^2 - \tfrac{1}{2} = \tfrac{1}{8}(z+1)^2 + \tfrac{1}{2}(z+1) + \tfrac{1}{2}$$

for $0 < z+1 < 2$

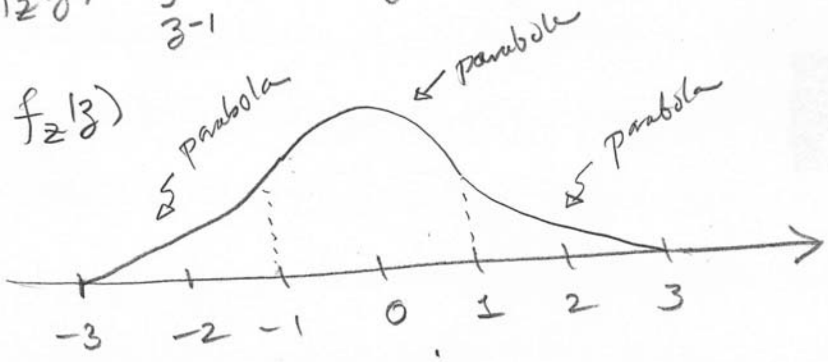
$$f_Z(z) = \int_{z-1}^{0} \left(\tfrac{1}{2}+\tfrac{1}{4}y\right) dy + \int_{0}^{z+1} \left(\tfrac{1}{2}-\tfrac{1}{4}y\right) dy$$

$$= -\tfrac{1}{2}(z-1) - \tfrac{1}{4}\tfrac{(z-1)^2}{2} + \tfrac{1}{2}(z+1) - \tfrac{1}{4}\tfrac{(z+1)^2}{2}$$

$$= 1 - \tfrac{1}{8}(z-1)^2 - \tfrac{1}{8}(z+1)^2 = 1 - \tfrac{1}{4}(1+z^2) = \tfrac{3}{4} - \tfrac{1}{4}z^2$$

for $2 < z+1 < 4$

$$f_Z(z) = \int_{z-1}^{2} \left(\tfrac{1}{2}-\tfrac{1}{4}y\right) dy = \tfrac{1}{2}(2-z+1) - \tfrac{1}{4}\tfrac{1}{2}\left(4-(z-1)^2\right)$$



5.97

$$f_X(x) = \frac{\lambda^{\alpha_1} x^{\alpha_1-1}}{\Gamma(\alpha_1)} e^{-\lambda x} \qquad f_Y(y) = \frac{\lambda^{\alpha_2} x^{\alpha_2-1}}{\Gamma(\alpha_2)} e^{-\lambda x}$$

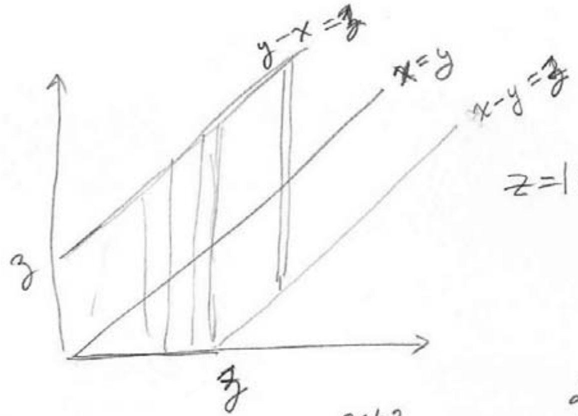
$Z = X+Y$

$$f_Z(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\underbrace{\Gamma(\alpha_1)\Gamma(\alpha_2)}_{c}} \int_0^\infty x^{\alpha_1-1} e^{-\lambda x} (z-x)^{\alpha_2-1} e^{-\lambda(z-x)} \, dx$$

$$= c e^{-\lambda z} \underbrace{\int_0^\infty x^{\alpha_1-1} (z-x)^{\alpha_2-1} \, dx}_{\frac{z^{\alpha_1+\alpha_2-1}\,\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2-1)}} = \frac{\lambda^{\alpha_1+\alpha_2} z^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2-1)}$$

Gamma pdf

5.98



$$Z = |X-Y| = \begin{cases} X-Y & X>Y \\ Y-X & X<Y \end{cases}$$

ⓐ

$$P[Z\le z] = \int_0^z dx\, e^{-x}\int_0^{x+z} e^{-y}\,dy + \int_z^\infty dx\, e^{-x}\int_{x-z}^{x+z} e^{-y}\,dy$$

$$= \int_0^z dx\, \underbrace{e^{-x}\left(1-e^{-(x+z)}\right)}_{e^{-x}-e^{-2x-z}} + \int_z^\infty dx\, \underbrace{e^{-x}\left(e^{-(x-z)}-e^{-(x+z)}\right)}_{e^{-2x+z}-e^{-2x-z}}$$

$$= \int_0^z e^{-x}\,dx - \int_0^\infty e^{-2x-z}\,dx + \int_z^\infty e^{-2x+z}\,dx$$

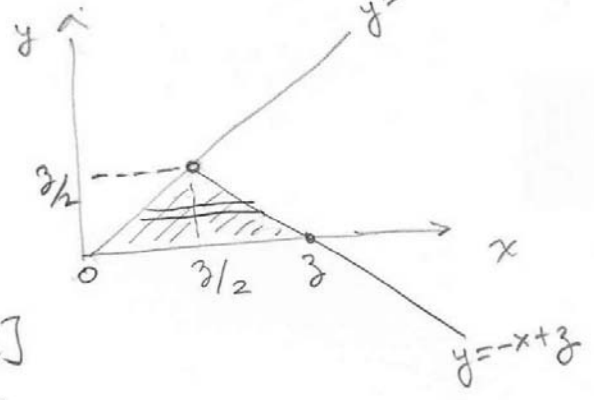
$$= 1-e^{-z} - \frac{1}{2}e^{-z} + \frac{1}{2}e^{z}e^{-2z} = 1-e^{-z}$$

ⓑ

$$E[Z] = \int_0^\infty P[Z>z]\,dz = \int_0^\infty e^{-z}\,dz = 1 \checkmark$$

5.99 $f_{XY}(x,y) = e^{-(x+y)}$ $\quad 0 < y < x < 1$

$Z = X+Y$



$P[Z \le z] = P[X+Y \le z]$

$= P[Y \le -X+z]$

$= 2\int_0^{z/2} dy\, e^{-y} \int_y^{z-y} e^{-x}\, dx = \int_0^{z/2} dy\, e^{-y}\left[-e^{-x}\right]_y^{z-y}$

$= 2\int_0^{z/2} dy\, e^{-y}\left[e^{-y} - e^{y-z}\right]$

$= 2\int_0^{z/2} \left[e^{-2y} - e^{-z}\right] dy$

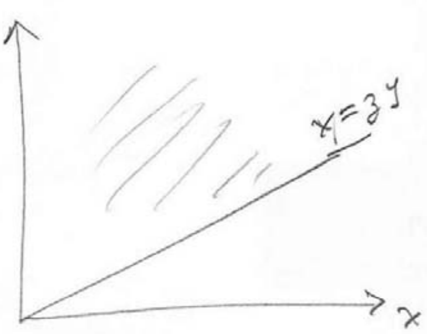
$= 2\left\{-\frac{e^{-2y}}{2}\Big|_0^{z/2} - e^{-z}\left[y\right]_0^{z/2}\right\}$

$= 2\left\{\frac{1}{2} - \frac{1}{2}e^{-z} - e^{-z}\, z/2\right\}$

$= 1 - e^{-z} - ze^{-z}$

$f_Z(z) = e^{-z} - e^{-z} + ze^{-z} = ze^{-z}$ $\quad z > 0$

5.100 X, Y Rayleigh w. $\alpha = \beta = 1$



$Z = X/Y$

for $z > 0$

$$P[Z \le z] = P[X/Y \le z]$$

$$= P[X \le zY]$$

$$P[Z \le z] = \int_0^\infty dx\, x e^{-x^2/2} \int_{x/z}^\infty y e^{-y^2/2} dy$$

$$= \int_0^\infty dx\, x e^{-x^2/2} \left[e^{-(x/z)^2/2} \right]$$

$$= \int_0^\infty dx\, x\, e^{-\frac{1}{2}\left(1 + \frac{1}{z^2}\right)x^2}$$

$$= \frac{1}{\left(1 + \frac{1}{z^2}\right)} = \frac{z^2}{z^2 + 1}$$

$$f_Z(z) = \frac{2z}{(1 + z^2)^2}$$

5.101

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\left(\frac{x^2}{2}+\frac{y^2}{2}\right)}$$

$Z = X/Y$

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f(zy, y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |y| e^{-\left(\frac{y^2}{2}+\frac{z^2y^2}{2}\right)} dy$$

$$= \frac{1}{\pi} \int_{0}^{\infty} y e^{-y^2\left(\frac{1}{2}+\frac{1}{2}z^2\right)} dy$$

but $\int_0^\infty y e^{-ay^2} dy = \left[-\frac{1}{2a} e^{-ay^2}\right]_0^\infty = \frac{1}{2a}(0-(-1)) = \frac{1}{2a}$

So

$$f_Z(z) = \frac{1}{\pi} \cdot \frac{1}{2\left(\frac{1}{2}+\frac{1}{2}z^2\right)}$$

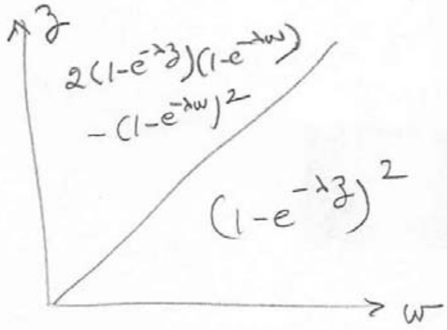
$$= \frac{1}{\pi(1+z^2)}$$

$\therefore$ Z is a Cauchy RV w/ $\alpha = 1$ //

5.103) $F_{XY}(x,y) = (1-e^{-\lambda x})(1-e^{-\lambda y}) \quad x>0 \ y>0$

If $z < w$ then

$$F_{WZ}(w,z) = F_{XY}(z,z)$$
$$= (1-e^{-\lambda z})^2$$



If $z > w$ then from Ex. 5.43

$$F_{WZ}(w,z) = F_{XY}(w,z) + F_{XY}(z,w) - F_{XY}(w,w)$$
$$= 2(1-e^{-\lambda z})(1-e^{-\lambda w}) - (1-e^{-\lambda w})^2$$

Check: find marginal cdf of $W = \min(X,Y)$.

$$\lim_{z\to\infty} F_{WZ}(w,z) = 2(1-e^{-\lambda w}) - (1-e^{-\lambda w})^2$$
$$= 1 - e^{-2\lambda w} \qquad w > 0$$

exponential w. rate 2λ

5.104

$$F_{XY}(x,y) = \left(1-\left(\frac{x_m}{x}\right)^{\alpha}\right)\left(1-\left(\frac{x_m}{y}\right)^{\alpha}\right) \qquad \begin{array}{l} x > x_m \\ y > x_m \end{array}$$

If $z < w$ then

$$F_{WZ}(w,z) = F_{XY}(z,z) = \left(1-\left(\frac{x_m}{z}\right)^{\alpha}\right)^2$$

If $z > w$ then

$$F_{WZ}(w,z) = 2\left(1-\left(\frac{x_m}{z}\right)^{\alpha}\right)\left(1-\left(\frac{x_m}{w}\right)^{\alpha}\right) - \left(1-\left(\frac{x_m}{z}\right)^{\alpha}\right)^2$$

Let's look at marginals of W and Z

$$F_W(w) = \lim_{z\to\infty} F_{WZ}(w,z) = 2\left(1-\left(\frac{x_m}{w}\right)^{\alpha}\right) - 1$$

$$= 1 - \left(\frac{x_m}{w}\right)^{\alpha}$$

the minimum is also Pareto with the same parameter!

The marginal for the max is

$$\lim_{w\to\infty} F_{WZ}(w,z) = \left(1-\left(\frac{x_m}{z}\right)^{\alpha}\right)^2$$

5.105

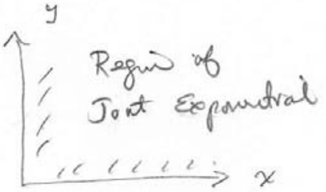
(a) $\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} W \\ Z \end{bmatrix}$$

$$x = \frac{w+z}{2} \qquad y = \frac{w-z}{2}$$

$$\therefore f_{WZ}(w,z) = f_{XY}\left(\frac{w+z}{2}, \frac{w-z}{2}\right)$$



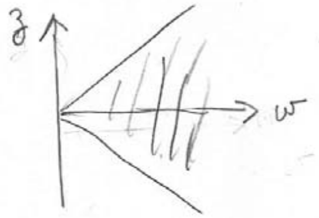
(b) $f_{WZ}(w,z) = f_X\left(\frac{w+z}{2}\right) f_Y\left(\frac{w-z}{2}\right)$

$$= e^{-\frac{(w+z)}{2}} \cdot e^{-\frac{(w-z)}{2}} \rightarrow w > z$$

$$= e^{-w} \rightarrow w > -z$$

//

for $w > 0$ $-w < z < w$



(c) $f_{WZ}(w,z) = f_X\left(\frac{w+z}{2}\right) f_Y\left(\frac{w-z}{2}\right)$

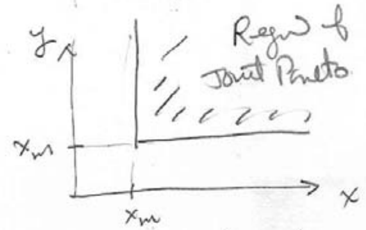
$$= k \frac{x_m^k}{\left(\frac{w+z}{2}\right)^{k+1}} \cdot k \frac{x_m^k}{\left(\frac{w-z}{2}\right)^{k+1}}$$

$$= \frac{k^2 \, x_m^{2k}}{\left(\frac{w^2 - z^2}{4}\right)^{k+1}}$$

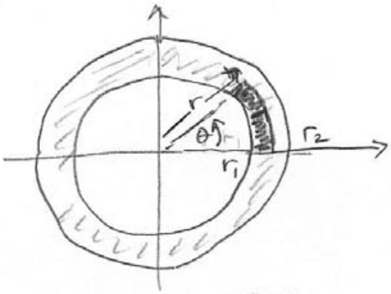
//

$w > x_m$

$w + z > x_m$ $\quad w - z > x_m$



5.106



$r_1 < r < r_2 \quad 0 < \theta < 2\pi$

$$P[R \le r, \Theta \le \theta] = \frac{(\pi r^2 - \pi r_1^2)\frac{\theta}{2\pi}}{\pi r_2^2 - \pi r_1^2}$$

$$\frac{\partial^2}{\partial r \partial \theta} F_{R,\Theta}(r,\theta) = \frac{\partial}{\partial r} \frac{1}{2\pi}\left(\frac{\pi r^2 - \pi r_1^2}{\pi r_2^2 - \pi r_1^2}\right)$$

$$= \frac{1}{2\pi} \frac{2r}{r_2^2 - r_1^2} \qquad \begin{array}{l} r_1 < r < r_2 \\ 0 < \theta < 2\pi \end{array}$$

5.107

(a) $\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

then, $\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{|ae-bc|} \begin{bmatrix} e & -b \\ -c & a \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$

$$f_{VW} = f_{XY}\left(\frac{ev-bw}{|ae-bc|}, \frac{-cv+aw}{|ae-bc|}\right)$$

$$= f_X\left(\frac{ev-bw}{|ae-bc|}\right) f_Y\left(\frac{-cv+aw}{|ae-bc|}\right)$$

where f_X and f_Y Gaussian pdf with $\mu = 0$ and $\sigma = 1$.

ⓑ The matrix A is not invertible if its rows are linearly dependent, that is,

$V = aX + bY$

$W = kaX + kbY$

which implies that $W = kV$

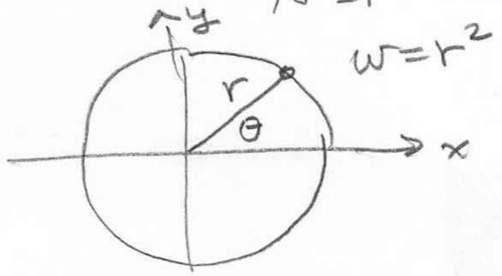
$\therefore$ W is simply a constant multiple of V

5.108

X, Y indep. jointly Gaussian RV's $m=0, \sigma^2=1$

$W = X^2+Y^2$

$\Theta = \tan^{-1} Y/X$



$$P[W \le w, \Theta \le \theta_0] = \iint\limits_{\{(x,y):\, x^2+y^2 \le r_0 \,\&\, \tan^{-1}(\frac{y}{x}) < \theta_0\}} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\, dx\, dy$$

$$= \int_0^{\sqrt{w}} \int_0^{\theta_0} \frac{e^{-r^2/2}}{2\pi}\, r\, dr\, d\theta$$

$$= \frac{\theta_0}{2\pi}\left(1 - e^{-w/2}\right) \qquad 0<\theta_0<2\pi \quad 0<r_0<\infty$$

$$F_W(w) = 1 - e^{-w/2} \qquad w > 0$$

$$F_\Theta(\theta) = \frac{1}{2\pi} \qquad 0 < \theta \le 2\pi$$

W is an exponential RV with mean 2

5.109

$$X = \left(2\ln\frac{2\pi}{\xi_1}\right)^{1/2}\cos\xi_2 \qquad Y = \left(2\ln\frac{2\pi}{\xi_1}\right)^{1/2}\sin\xi_2$$

$$R = \sqrt{X^2+Y^2} \qquad \Theta = \tan^{-1} Y/X$$

Consider Θ first:

$$\Theta = \tan^{-1}\frac{Y}{X} = \tan^{-1}\left(\frac{\sin\xi_2}{\cos\xi_2}\right) = \tan^{-1}(\tan\xi_2) = \xi_2$$

But ξ_2 is uniform in $[0, 2\pi)$ $\Rightarrow$ Θ also unif in $[0, 2\pi)$

Now consider R

$$R = \sqrt{X^2+Y^2} = \sqrt{2\ln\left(\frac{2\pi}{\xi_1}\right)\cos^2\xi_2 + 2\ln\left(\frac{2\pi}{\xi_1}\right)\sin^2\xi_2}$$

$$= \sqrt{2\ln\left(\frac{2\pi}{\xi_1}\right)}$$

ξ_1 is indep of ξ_2 $\Rightarrow$ R and Θ are independent

$$F_R(r) = P[R<r] = P\left[\sqrt{2\ln\frac{2\pi}{\xi_1}} < r\right]$$

$$= P\left[\frac{2\pi}{\xi_1} < e^{r^2/2}\right] = P\left[2\pi e^{-r^2/2} < \xi_1\right]$$

$$= 1 - \frac{2\pi e^{-r^2/2}}{2\pi} = 1 - e^{-r^2/2}$$

R is Rayleigh, Θ is unif in $(0, 2\pi)$ as in Ex. 5.44

$\therefore$ X and Y are indep jointly Gaussian RVs with zero mean + unit variance

5.9 Pairs of Jointly Gaussian Random Variables

5.110

$$f_{XY}(x,y) = \frac{e^{-(2x^2+y^2/2)}}{2\pi c}$$

The sol'n involves matching the coefficients of the polynomial in the exponent of the Gaussian pdf.

coeff. of $x^2 \Rightarrow \frac{1}{2(1-\rho^2)\sigma_1^2} = 2.$

coeff. of $y^2 \Rightarrow \frac{1}{2(1-\rho^2)\sigma_2^2} = \frac{1}{2}$

coeff of $xy \Rightarrow \frac{-2\rho}{2(1-\rho^2)\sigma_1\sigma_2} = 0 \quad \therefore \rho = 0$

$$\sigma_1^2 = \frac{1}{2(1-0^2)\cdot 2} = \frac{1}{4}$$

$$\sigma_2^2 = \frac{1}{2(1-0^2)\cdot\frac{1}{2}} = 1$$

$\therefore \; COV(X,Y) = \rho_{XY}\sigma_1\sigma_2 = 0$

$VAR[X] = \frac{1}{4}$

$VAR[Y] = 1$

5.111

$$x^2 + 4y^2 - 3y(x-1) - 2x + 1 =$$

$$= (x-1)^2 - 3(x-1)y + 4y^2$$

$$= \left(\frac{x-1}{1}\right)^2 - 2\left(\frac{3}{4}\right)\left(\frac{x-1}{1}\right)\left(\frac{y}{1/2}\right) + \left(\frac{y}{1/2}\right)^2$$

$m_1 = 1$ $\rho = \frac{3}{4}$ $m_2 = 0$

$\sigma_1 = 1$ $\sigma_2 = 1/2$

5.112 $m_Y = 0$ $\sigma_1 = 1$ $\sigma_2 = 2$

$$E[X|Y] = Y/4 + 1$$

$$E[X|Y] = Y/4 + 1 = \underbrace{\rho\frac{\sigma_1}{\sigma_2}}_{\frac{1}{2}}(y - \underbrace{m_2}_{0}) + \underbrace{m_1}_{+1}$$

$$\Rightarrow \rho = \frac{1}{2}$$

$$f_{XY}(x,y) = \frac{\exp\left\{\frac{-1}{2(3/4)}\left[\left(\frac{x-1}{1}\right)^2 - 2\,\frac{1}{2}\left(\frac{x-1}{1}\right)\left(\frac{y}{2}\right) + \left(\frac{y}{2}\right)^2\right]\right\}}{4\pi\sqrt{3/4}}$$

5.113 ~~4.78 a)~~

$$\begin{aligned} P[\sqrt{X^2+Y^2} \le r] &= \iint_{x^2+y^2\le r^2} \frac{e^{-(x^2+y^2)/2}}{2\pi} dx\,dy \\ &= \int_0^{2\pi}\int_0^r \frac{e^{-r^2/2}}{2\pi} r\,dr\,d\theta \\ &\quad \text{let } x = r\cos\theta,\ y = r\sin\theta \\ &= 1 - e^{-r^2/2} = \frac{1}{2} \\ &\quad \text{Note: } \sqrt{X^2+Y^2} \text{ has a Rayleigh dist.} \\ \Rightarrow r &= \sqrt{2\ln 2} \end{aligned}$$

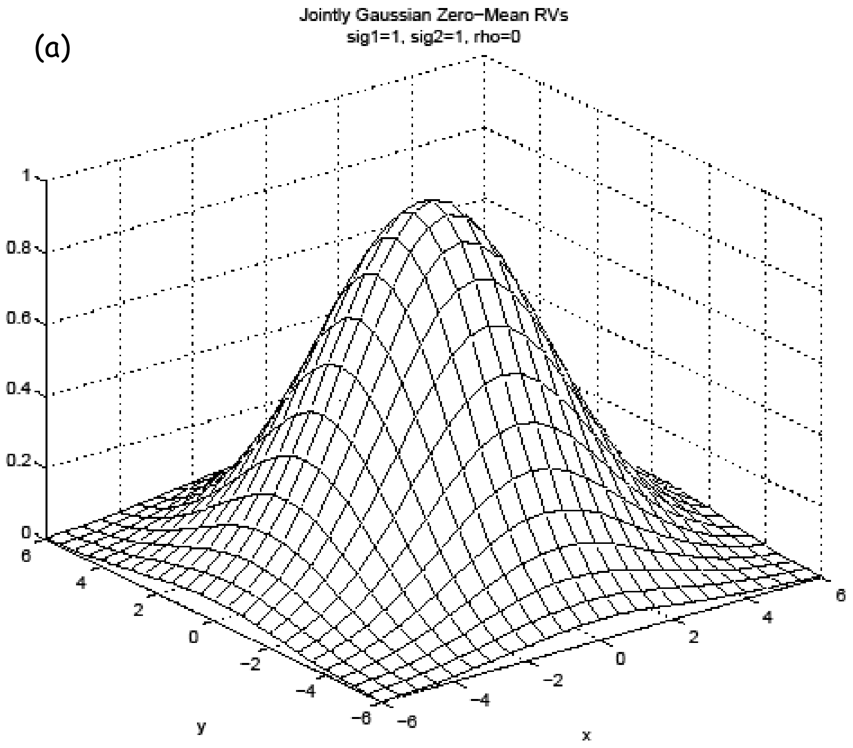
b) $P[\sqrt{X^2+Y^2} > r] = e^{-r^2/2}$ from above.

For (x, y) such that $x^2 + y^2 > r^2$

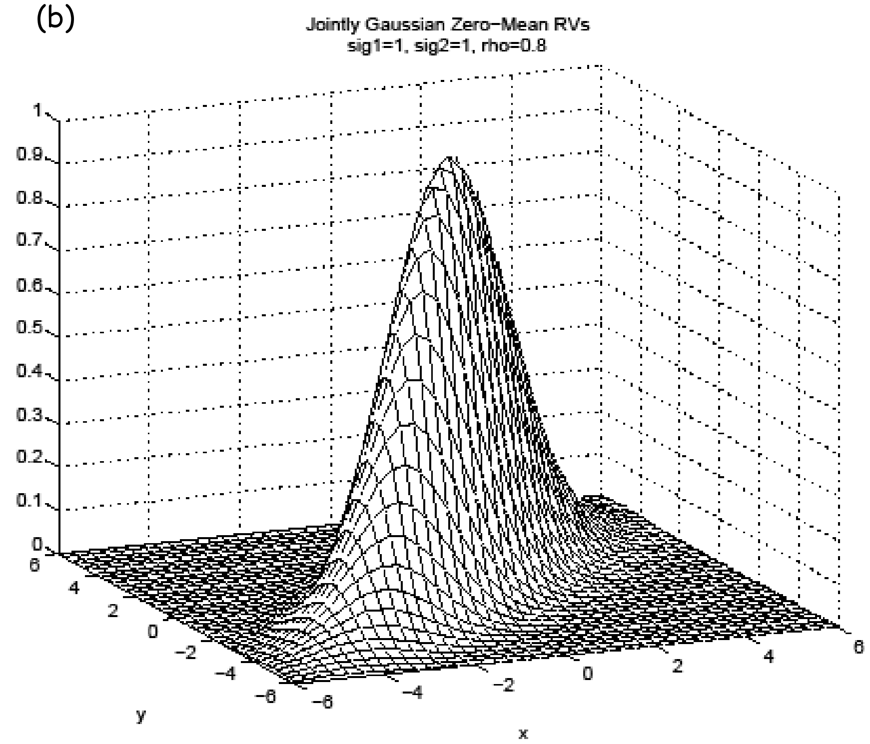
$$\begin{aligned} f_{XY}(x, y|R > r)dx\,dy &= P[x < X \le x + dx, y < Y \le y + dy|R > r] \\ &= \frac{f_{XY}(x,y)dx\,dy}{P[R > r]} \\ \Rightarrow f_{XY}(x, y|R > r) &= \frac{f_{XY}(x,y)}{P[R > r]} \\ &= \frac{e^{-(x^2+y^2-r^2)/2}}{2\pi} \end{aligned}$$

5.114

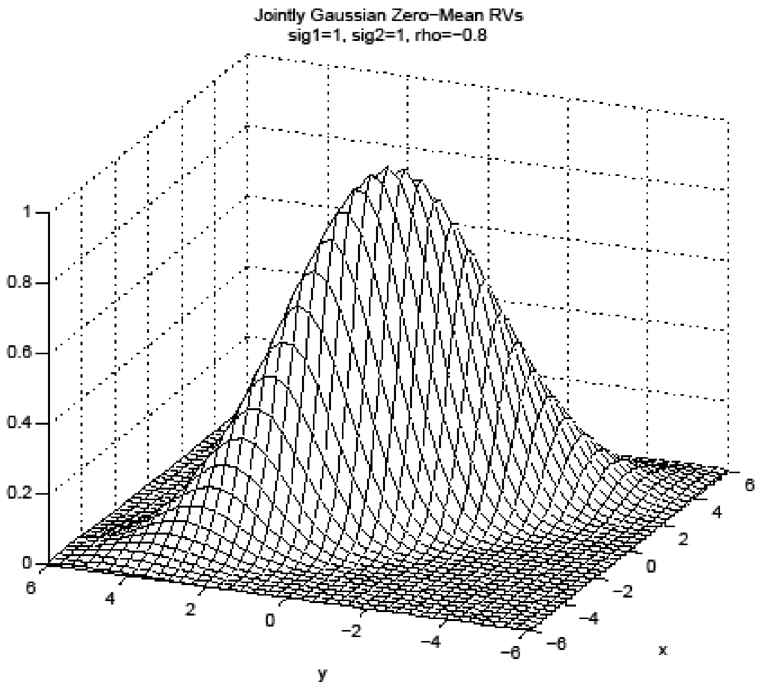
(a)


Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=1, rho=0
1
0.8
0.6
0.4
0.2
0
6
4
2
0
−2
−4
−6
y
−6
−4
−2
0
2
4
6
x

(b)

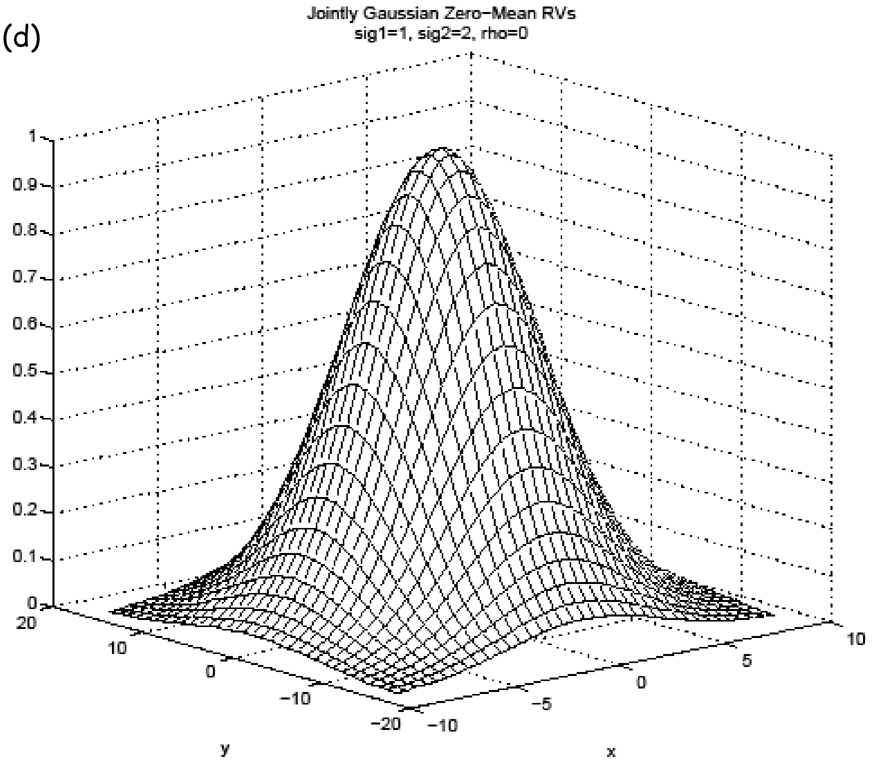

Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=1, rho=0.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
6
4
2
0
−2
−4
−6
y
−6
−4
−2
0
2
4
6
x

(c)



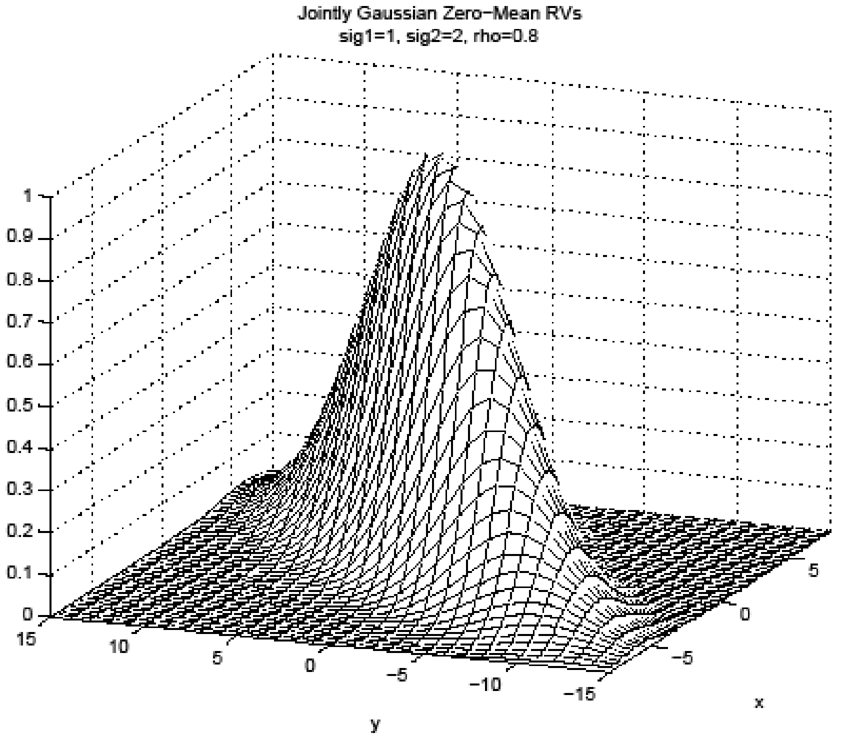
Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=1, rho=−0.8

(d)

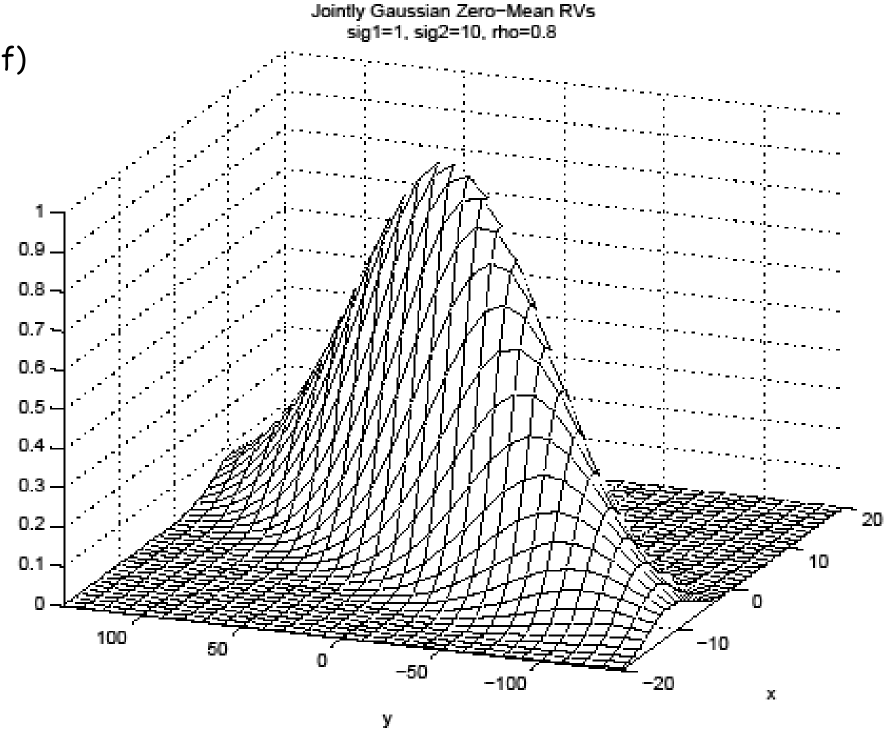


Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=2, rho=0

(e)

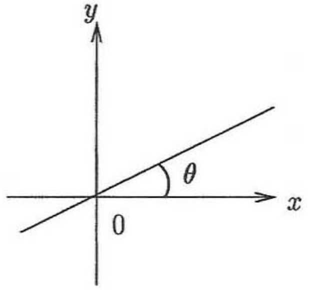

Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=2, rho=0.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
15
10
5
0
−5
−10
−15
y
5
0
−5
x

(f)


Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=10, rho=0.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100
50
0
−50
−100
y
20
10
0
−10
−20
x

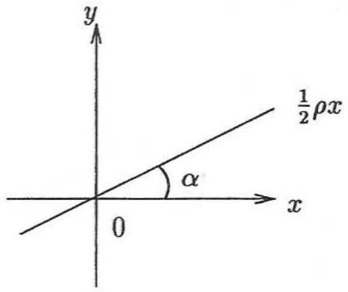
5.115

~~116~~ a)



$$\theta = \frac{1}{2}\arctan\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2} = \frac{1}{2}\arctan\frac{4\rho}{3}$$

b)



$$E[Y|X=x] = m_2 \overset{-}{\cancel{+}} \rho\frac{\sigma_2}{\sigma_1}(x-m_1) = \frac{1}{2}\rho x$$

c) The plots in parts a) and b) are the same only when $\rho = 1$. In this case $E[Y|X = x] = \frac{1}{2}x$, i.e.

$$\tan\alpha = \frac{1}{2}, \qquad \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{4}{3}\ .$$

$$\therefore \quad \alpha = \frac{1}{2}\arctan\frac{4}{3} = \theta$$

5.116

~~4.80~~ $$\begin{vmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{vmatrix} = \sigma_X^2\sigma_Y^2(1-\rho^2) = 0 \Rightarrow \rho = \pm 1$$

$\Rightarrow P[X = \rho Y] = 1 \Rightarrow$ all probl. mass concentrated along $X = \rho Y$ line

Assume $\rho = 1$:

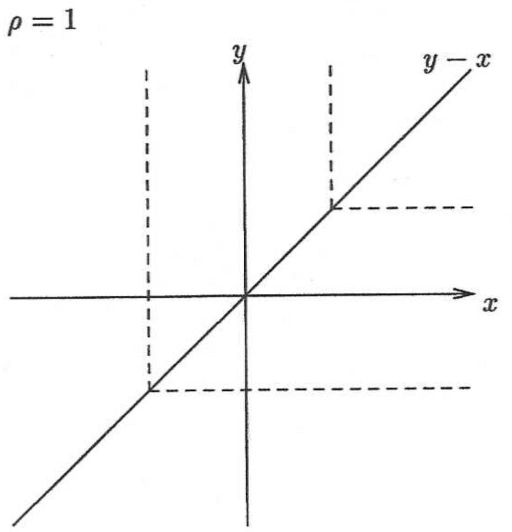
$$\begin{aligned} R_{XY}(x,y) &= P[X \le x, Y \le y] = P[X \le x, X \le y] \\ &= P[\{X \le x\} \cap \{X \le y\}] = P[X \le \min(x,y)] \\ &= \int_{-\infty}^{\min(x,y)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \end{aligned}$$

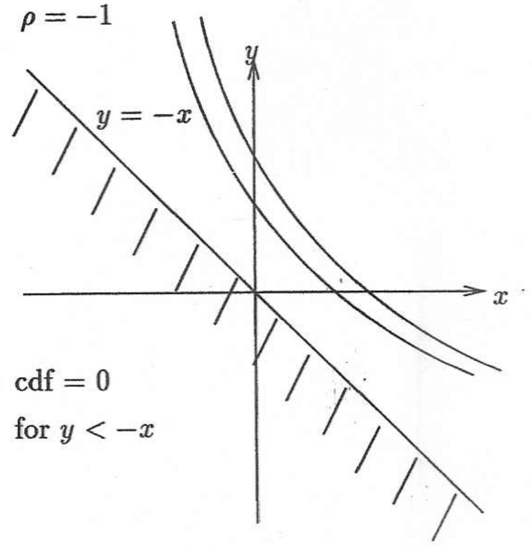
The joint pdf does not exist.

Similarly if $\rho = -1$

$$\begin{aligned} F_{XY}(x,y) &= P[X \le x, -X \le y] = P[X \le x, X \ge -y] \\ &= P[-y \le X \le x] \\ &= \begin{cases} \displaystyle\int_{-y}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt & x \ge -y \Leftrightarrow x + y \ge 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The joint pdf does not exist.





5.117

~~4.81~~ $h(x,y) = \dfrac{e^{-(x^2-2\rho_1 xy+y^2)/2(1-\rho_1^2)}}{2\pi\sqrt{1-\rho_1^2}} \qquad g(x,y) = \dfrac{e^{-(x^2-2\rho_2 xy+y^2)/2(1-\rho_2^2)}}{2\pi\sqrt{1-\rho_2^2}}$

a) $f_X(x) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} h(x,y)dy + \dfrac{1}{2}\int_{-\infty}^{\infty} g(x,y)dy = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

Similarly

$$f_Y(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$\therefore$ X and Y, individually, are Gaussian RV's.

b) However,

$$f_{XY}(x,y) = \frac{\sqrt{1-\rho_2^2}\,e^{-(x^2-2\rho_1 xy+y^2)/2(1-\rho_1^2)} + \sqrt{1-\rho_1^2}\,e^{-(x^2-2\rho_2 xy+y^2)/2(1-\rho_2^2)}}{2\pi\sqrt{1-\rho_1^2}\sqrt{1-\rho_2^2}}$$

does not have the form required for jointly Gaussian RV's.

5.118

~~4.82~~

$$\begin{aligned}
\mathcal{E}[X^2|y] &= VAR[X|y] + \mathcal{E}[X|y]^2 \qquad \text{where we assume } \mathcal{E}[X] = \mathcal{E}[Y] = 0 \\
&= \sigma_X^2(1-\rho^2) + \left(\rho\frac{\sigma_X}{\sigma_Y}y\right)^2 \\
\mathcal{E}[X^2Y^2] &= \mathcal{E}[\mathcal{E}[X^2Y^2|Y]] = \mathcal{E}[Y^2\mathcal{E}[X^2|Y]] \\
&= \mathcal{E}[\sigma_X^2(1-\rho^2)Y^2 + \rho\frac{\sigma_X^2}{\sigma_Y^2}Y^4] \\
&= \sigma_X^2\sigma_Y^2(1-\rho^2) + \rho^2\frac{\sigma_X^2}{\sigma_Y^2}\underbrace{\mathcal{E}[Y^4]}_{3\sigma_Y^4}\} \text{ shown below} \\
&= \sigma_X^2\sigma_Y^2(1+2\rho^2) \\
&= \sigma_X^2\sigma_Y^2 + 2\mathcal{E}[XY] = \mathcal{E}[X^2]\mathcal{E}[Y^2] + 2\mathcal{E}[XY] \\
\mathcal{E}[Y^4] &= \int_{-\infty}^{\infty}\frac{y^4e^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma}dy \qquad t = \frac{y}{\sigma} \\
&= \frac{\sigma^5}{\sqrt{2\pi}\sigma}\underbrace{2\int_0^{\infty}t^4e^{-t^2/2}dt}_{\frac{\Gamma\left(\frac{5}{2}\right)}{2\alpha^5}} \qquad \text{from Table in Appendix A where } \alpha^2 = \tfrac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\Gamma\left(\tfrac{5}{2}\right) &= \tfrac{3}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{3}{2}\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) \\
&= \tfrac{3}{4}\sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma^5}{\sqrt{2\pi}\sigma}\frac{2\frac{3}{4}\sqrt{\pi}}{2\left(\frac{1}{\sqrt{2}}\right)^5} \\
&= 3\sigma^4
\end{aligned}$$

5.119

$$\begin{aligned} f(x_1,x_2,x_3) &= \exp(-x_1^2 - x_2^2 + \sqrt{2}x_1x_2 - \frac{1}{2}x_3^2)/(2\pi\sqrt{\pi}) \\ &= \frac{\exp(-x_1^2 - x_2^2 + \sqrt{2}x_1x_2)}{\sqrt{2\pi}\cdot\sqrt{\pi}}\,\frac{\exp(-\frac{1}{2}x_3^2)}{\sqrt{2\pi}} \\ &= \frac{\exp\left\{-\frac{1}{2(1-\frac{1}{2})}[x_1^2 - 2\frac{1}{\sqrt{2}}x_1x_2 + x_2^2]\right\}}{2\pi\sqrt{1-\frac{1}{2}}}\,\frac{\exp(-\frac{1}{2}x_3^2)}{\sqrt{2\pi}} \end{aligned}$$

We only need to "decorrelate" X_1 and X_2. Try the transform

$$\begin{aligned} A &= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Note that

$$K = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check

$$\begin{aligned} C &= AKA^T \\ &= \begin{bmatrix} 1+\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 1-\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

C is a diagonal matrix

5.120

$$\hat{X} = \frac{1}{1+\sigma_N^2/\sigma_X^2} Y = cY$$, where $Y = X+N$

$$E[(X-cY)^2] = E[(X-c(X+N))^2]$$

$$= E[((1-c)X - cN)^2]$$

$$= (1-c)^2 E[X^2] - 2(1-c)\,c\underbrace{\underbrace{E[XN]}_{E[X]E[N]}}_{0} + c^2 E[N^2]$$

$$= (1-c)^2 \sigma_X^2 + c^2 \sigma_N^2$$

$$= \left(1 - \frac{1}{1+\sigma_N^2/\sigma_X^2}\right)^2 \sigma_X^2 + \left(\frac{1}{1+\frac{\sigma_N^2}{\sigma_X^2}}\right)^2 \sigma_N^2$$

$$= \left(\frac{\sigma_N^2/\sigma_X^2}{1+\sigma_N^2/\sigma_X^2}\right)^2 \sigma_X^2 + \left(\frac{1}{1+\frac{\sigma_N^2}{\sigma_X^2}}\right)^2 \sigma_N^2$$

$$= \frac{\sigma_N^4 \sigma_X^2 + \sigma_X^4 \sigma_N^2}{(\sigma_X^2+\sigma_N^2)^2}$$

$$= \frac{\sigma_X^2 \sigma_N^2}{\sigma_X^2+\sigma_N^2} \longrightarrow \begin{cases} \sigma_N^2 \text{ @ Hi SNR} \\ \sigma_X^2 \text{ at low SNR} \end{cases}$$

5.10 Generating Independent Gaussian Random Variables

5.121 The following Octave code produces the inverse:

```
function z = rayleigh_rnd(s)
     u = rand;
     z = s.*(2.*log(1./(1-u))).^(1/2);
end
```

5.123 The following Octave generates the requested pairs and plot:

```
len = 10000;
X = discrete_rnd([1 -1], [0.5 0.5], 2, len);
N = normal_rnd(0, 1, 2, len);
Y = X + N;
figure;
plot(Y(1,:), Y(2,:), ".");
Xr = sign(Y);
sigerr = (X ~= Xr);
biterr = (sigerr(1,:) | sigerr(2,:));
proberr = sum(biterr)./len
```

5.124 The following Octave generates the requested pairs and plot:

```
X = normal_rnd(0, 2, 1, 1000);
N = normal_rnd(0, 1, 1, 1000);
Y = X + N;
Xr = Y./(1 + 1/2);
err = Xr - X;
figure;
hist(err, [-3:0.25:3], 4);
m = mean(err)
v = var(err)
```

5.125 The following Octave generates the sequence of X_n and Y_n:

```
X = normal_rnd(0, 1, 1, 1000);
Y = (X + [0 X(1:999)])./2;
figure;
plot(Y(1:999), Y(2:1000), ".");
Z = (X - [0 X(1:999)])./2;
figure;
plot(Z(1:999), Z(2:1000), ".");
```

5.126 The following Octave generates the specified jointly Gaussian random variables:

```
function vw = gaussian_correlate(xy, mu1, mu2, var1, var2, rho)
    sig1 = sqrt(var1);
    sig2 = sqrt(var2);
    K = [var1, rho*sig1*sig2; rho*sig1*sig2, var2];
    [evec eval] = eig(K);
    A = evec * sqrt(eval);
    vw = zeros(size(xy));
    for i = 1:length(xy)
        vw(:,i) = A*xy(:,i) + [mu1; mu2];
    end
end
```

5.127 The following Octave generates the specified jointly Gaussian random variables and plot:

```
xy = normal_rnd(0, 1, 2, 1000);
vw = gaussian_correlate(xy, 1, -1, 1, 2, -1/2);
plot(vw(1,:), vw(2,:), ".");
```

5.128 The following Octave generates the specified plot:

```
xy = normal_rnd(0, 1, 2, 1000);
muh = 174;
muv = 4.4;
varh = 42.36;
varv = 0.021;
covhv = 0.458;
sigh = sqrt(varh);
sigv = sqrt(varv);
rhohv = covhv/(sigh*sigv);
xy = normal_rnd(0, 1, 2, 1000);
hv = zeros(size(xy));
hv = gaussian_correlate(xy, muh, muv, varh, varv, rhohv);
hw = zeros(size(hv));
hw(1,:) = hv(1,:);
hw(2,:) = e.^hv(2,:);
bmi = hw(2,:)./(hw(1,:).^2);
hist(bmi);
```

Problems Requiring Cumulative Knowledge

5.129

a)

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\pi/2} c\sin(x+y)dxdy &= 1\\ c\int_0^{\pi/2}[-\cos(x+y)]\Big|_0^{\pi/2}dy &= 1\end{aligned}$$

$$\begin{aligned}1 &= c\int_0^{\pi/2}(\cos y-\cos(\frac{\pi}{2}+y))dy = c\int_0^{\pi/2}(\cos y+\sin y)dy\\ &= 2c\sin y\Big|_0^{\pi/2}\\ &= 2c\\ c &= \frac{1}{2}\end{aligned}$$

b)

$$\begin{aligned}F_{X,Y}(x,y) &= \int_0^y\int_0^x \frac{1}{2}\sin(u+v)dudv\\ &= \int_0^y\left[-\frac{1}{2}\cos(u+v)\,\right]\Big|_0^x dv\\ &= \frac{1}{2}\int_0^y(\cos v-\cos(x+v))dv\\ &= \frac{1}{2}(\sin v-\sin(x+v))\Big|_0^y\\ &= \frac{1}{2}(\sin y-\sin(x+y)+\sin x)\end{aligned}$$

c)

$$\begin{aligned}f_X(x) &= \int_0^{\pi/2}\frac{1}{2}\sin(x+y')dy'\\ &= \frac{1}{2}(-\cos(x+y'))\Big|_0^{\pi/2}\\ &= \frac{1}{2}(\cos x+\sin x)\\ f_Y(y) &= \frac{1}{2}(\cos y+\sin y)\end{aligned}$$

d)

$$
\begin{aligned}
E[X] &= \int_0^{\pi/2} x\frac{1}{2}(\cos x + \sin x)dx \\
&= \frac{1}{2}\int_0^{\pi/2} x d\sin x - \frac{1}{2}\int_0^{\pi/2} x d\cos x \\
&= \frac{1}{2}\left[x\sin x|_0^{\pi/2} - \int_0^{\pi/2}\sin x dx\right] - \frac{1}{2}\left[x\cos x|_0^{\pi/2} - \int_0^{\pi/2}\cos x dx\right] \\
&= \frac{1}{2}\left[\frac{\pi}{2} - 1\right] + \frac{1}{2} \\
&= \pi/4 \\
E[Y] &= \pi/4 \\
E[X^2] &= \int_0^{\pi/2} x^2\frac{1}{2}(\cos x + \sin x)dx \\
&= \frac{1}{2}\int_0^{\pi/2} x^2 d\sin x - \frac{1}{2}\int_0^{\pi/2} x^2 d\cos x \\
&= \frac{1}{2}x^2\sin x|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin x\cdot 2x dx \\
&\quad -\frac{1}{2}x^2\cos x|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos x\cdot 2x dx \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2}\left(\frac{\pi}{2} - 1\right) \\
&= \pi^2/8 + \pi/2 \\
VAR[X] &= E[X^2] - E^2[X] \\
&= \pi^2/16 + \pi/2 \\
VAR[Y] &= \pi^2/16 + \pi/2 \\
E[XY] &= \int\int xy\frac{1}{2}\sin(x+y)dxdy \\
&= \frac{1}{2}\int_0^{\pi/2} ydy\int_0^{\pi/2} -xd\cos(x+y) \\
&= -\frac{1}{2}\int_0^{\pi/2} y[x\cos(x+y)|_0^{\pi/2} - \int_0^{\pi/2}\cos(x+y)dx]dy \\
&= -\frac{1}{2}\int_0^{\pi/2} y\cdot\left[-\frac{pi}{2}\sin y - \sin(x+y)|_0^{\pi/2}\right]dy \\
&= \frac{\pi}{2}\int_0^{\pi/2}\frac{1}{2}y\sin y dy + \frac{1}{2}\int_0^{\pi/2} y(\cos y - \sin y)dy \\
&= \left(\frac{\pi}{2} - 1\right)\in_0^{\pi/2}\frac{1}{2}y\sin y dy + \int_0^{\pi/2}\frac{1}{2}y\cos y dy \\
&= \left(\frac{\pi}{2} - 1\right)\frac{1}{2} + \frac{1}{2}\left(\frac{\pi}{2} - 1\right) \\
&= \pi/2 - 1 \\
COV[X,Y] &= E[XY] - E[X]E[Y] \\
&= \pi/2 - 1 - (\pi/4)^2
\end{aligned}
$$

5.130

a) The number of items between consecutive inspections is a geometric random variable with proof.

$$P[M=m] = p(1-p)^{k-1} \qquad k = 1, 2, \ldots$$

b) The time between inspections is the sum of the M interarrival times:

$$T = \sum_{i=1}^{M} X_i$$

where the X_i are iid exponential random variables with mean 1.

$$f_T(t) = \sum_{j=1}^{\infty} f_T(t|M=j)P[M=j]$$

The sume of j independent exponential random variables is Erlang:

$$f_T(t|M=j) = \frac{\lambda e^{-\lambda x}(\lambda x)^{j-1}}{(j-1)!}$$

Therefore

$$\begin{aligned} f_T(t) &= \sum_{j=1}^{\infty} \frac{\lambda e^{-\lambda x}(\lambda x)^{j-1}}{(j-1)!} p(1-p)^{j-1} \\ &= \lambda p e^{-\lambda x} \sum_{j=1}^{\infty} \frac{(\lambda x(1-p))^{j-1}}{(j-1)!} \\ &= \lambda p e^{-\lambda x} e^{\lambda x(1-p)} \\ &= \lambda p e^{-\lambda p x} \end{aligned}$$

$\therefore$ T is an exponential random variable.

c) Choose p so that

$$\begin{aligned} 0.90 = P[T > t] &= e^{-pt} \\ \Rightarrow \quad p &= \frac{1}{t} \ln \frac{1}{0.90} \end{aligned}$$

5.131 **a)**

$$\begin{aligned} f_{X,R}(x,r) &= f_X(x|r)f_R(r) \\ &= re^{-rx}\frac{\lambda(\lambda r)^{\alpha-1}e^{-\lambda r}}{\Gamma(\alpha)} \end{aligned}$$

b)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,R}(x,r)dr \\ &= \int_0^{\infty} \frac{(\lambda r)^{\alpha}e^{-(\lambda+X)r}}{\Gamma(\alpha)}dr \\ &= \int_0^{\infty} \frac{\lambda^{\alpha}}{(\lambda+x)^{\alpha}}\frac{[(\lambda+X)r]^{\alpha}e^{-(\lambda+X)r}}{\Gamma(\alpha)}dr \\ &= \frac{\lambda^{\alpha}}{(\lambda+X)^{\alpha}\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{\lambda+X} \\ &= \frac{\alpha\lambda^{\alpha}}{(\lambda+x)^{\alpha+1}} \qquad x>0 \end{aligned}$$

c)

$$\begin{aligned} E[X] &= \int_0^{\infty} xf_X(x)dx \\ &= \int_0^{\infty} \frac{\alpha\lambda^{\alpha}x}{(\lambda+x)^{\alpha+1}}dx \\ &= \frac{\lambda^{\alpha}}{-\alpha+1}(\lambda+x)^{-\alpha+1}\Big|_0^{\infty} \\ &= \frac{\lambda}{\alpha-1} \qquad (\alpha>1) \\ E[X^2] &= \int_0^{\infty} x^2 f_X(x)dx \\ &= \int_0^{\infty} \frac{\alpha\lambda^{\alpha}x^2}{(\lambda+x)^{\alpha+1}}dx \\ &= -\lambda^{\alpha}\int_0^{\infty} x^2 d(\lambda+x)^{-\alpha} \\ &= -\lambda^{\alpha}x^2(\lambda+x)^{-\alpha}\Big|_0^{\infty} + \lambda^{\alpha}\int_0^{\infty}(\lambda+x)^{-\alpha}2xdx \\ &= \frac{2\lambda^{\alpha}}{-\alpha+1}\int_0^{\infty} xd(\lambda+x)^{-\alpha+1} \\ &= \frac{2\lambda^{\alpha}}{-\alpha+1}x(\lambda+x)^{-\alpha+1}\Big|_0^{\infty} + \frac{2\lambda^{\alpha}}{\alpha-1}\int_0^{\infty}(\lambda+x)^{-\alpha+1}dx \\ &= \frac{2\lambda^{\alpha}}{-\alpha-1}\cdot\frac{1}{-\alpha+2}(\lambda+x)^{-\alpha+2}\Big|_0^{\infty} \\ &= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} \\ VAR[X] &= E[X^2]-E^2[X] = \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2} \end{aligned}$$

5.132 $R^2 = X^2 + Y^2$

a) When signal 0 is present

$$\frac{R^2}{\sigma_0^2} = \left(\frac{X}{\sigma_0}\right)^2 + \left(\frac{Y}{\sigma_0}\right)^2$$

$\frac{X}{\sigma_0}$, $\frac{Y}{\sigma_0}$ are independent, zero-mean, unit-variance RVs. R^2/σ_0^2 is a chi-square RV with 2 degrees of freedom. The pdf of R^2/σ_0^2 is

$$\frac{u^0 e^{-u/2}}{2' r(1)} = \frac{e^{-u/2}}{2}$$

The pdf of R_1^2, or $\sigma_0^2 \cdot \frac{R^2}{\sigma_0^2}$ is

$$\frac{1}{\sigma_0^2} \cdot \frac{e^{-u/2\sigma_0^2}}{2} = \frac{e^{-T^2/2\sigma_0^2}}{2\sigma_0^2}$$

Similarly, the pdf of R^2 when signal 1 is present

$$f_{R^2}(R^2|1) = \frac{e^{-R^2/2\sigma_1^2}}{2\sigma_0^2}$$

$$\begin{aligned} f_{R^2}(R^2) &= f_{R^2}(R^2|0)p(0) + f_{R^2}(R^2|1)p(1) \\ &= \frac{e^{-R^2/2\sigma_0^2}}{2\sigma_0^2}p + \frac{e^{-R^2/2\sigma_1^2}}{2\sigma_1^2}(1-p) \end{aligned}$$

b)

$$\begin{aligned} p_e &= p[R^2 > T|0]p(0) + p[R^2 < T|]p(1) \\ &= \int_T^\infty \frac{e^{-R^2/2\sigma_0^2}}{2\sigma_0^2} p\,dR^2 + \int_0^T \frac{e^{-R^2/2\sigma-1^2}}{2\sigma_1^2}(1-p)\,dR^2 \end{aligned}$$

c) $\frac{dP_e}{dT} = 0$

$$-p \cdot \frac{e^{-T/2\sigma_0^2}}{2\sigma_0^2} + \frac{e^{-T/2\sigma_1^2}}{2\sigma_1^2}(1-p) = 0$$

$$\ln\frac{p}{2\sigma_0^2} - \frac{T}{2\sigma_0^2} = \ln\frac{1-p}{2\sigma_1^2} - \frac{T}{2\sigma_1^2}$$

$$T = \frac{\ln\frac{p}{2\sigma_0^2} - \ln\frac{1-p}{2\sigma_1^2}}{\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}}$$

5.133 **a)**

$$\begin{aligned} X_n &= \frac{1}{2}(U_n + U_{n-1}) \\ X_{n-1} &= \frac{1}{2}(U_{n-1} + U_{n-2}) \end{aligned}$$

or

$$\begin{bmatrix} X_n \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} U_n \\ U_{n-1} \\ U_{n-2} \end{bmatrix}$$

X_n and X_{n-1} are jointly Gaussian

$$\begin{aligned} E[X_n] &= 0, \qquad E[X_{n-1}] = 0 \\ VAR[X_n] &= \frac{1}{4}(VAR[U_n] + VAR[U_{n-1}]) = \frac{1}{2}, \qquad VAR[X_{n-1}] = \frac{1}{2} \\ E[X_nX_{n-1}] &= \frac{1}{4}(E[U_nU_{n-1}] + E[U_{n-1}U_{n-1}] + E[U_nU_{n-2}] + E[U_{n-1}U_{n-2}]) \\ &= \frac{1}{4} \\ COV[X_nX_{n-1}] &= E[X_nX_{n-1}] - E[X_n]E[X_{n-1}] = \frac{1}{4} \\ \rho &= \frac{COV[X_nX_{n-1}]}{\sigma_n\sigma_{n-1}} = \frac{1}{2} \\ f(X_n, X_{n-1}) &= \frac{1}{2\pi \cdot \frac{1}{2}\sqrt{1-\frac{1}{4}}} \exp\left\{\frac{-1}{2(1-\frac{1}{4})}\left[\frac{x_{n-1}^2}{1/2} - \frac{x_nx_{n-1}}{1/2} + \frac{x_{n-1}^2}{1/2}\right]\right\} \\ &= \frac{1}{\pi\sqrt{3/4}} \exp\left\{-\frac{4}{3}[x^2 - x_nx_{n-1} + x_{n-1}^2]\right\} \\ COV[X_nX_{n+m}] &= 0 \quad \text{for } m > 1 \\ f(X_nX_{n+m}) &= f(X_n)f(X_{n+m}) \\ &= \frac{1}{\pi}\exp[-x_n^2 - x_{n+m}^2], \qquad m > 1 \end{aligned}$$

b) In this case, ρ is negative

$$\begin{aligned} f(y_n, y_{n-1}) &= \frac{1}{\pi\sqrt{3/4}} \exp\left\{-\frac{4}{3}[y_n^2 + y_ny_{n-1} + y_{n-1}^2]\right\} \\ f(y_n, y_{n+m}) &= \frac{1}{\pi}\exp[-y_n^2 - y_{n+m}^2] < \quad , m > 1 \end{aligned}$$

c) $m = n$

$$\begin{aligned} E[X_nY_m] &= \frac{1}{4}E[U_n^2 - U_{n-1}^2] = 0, \qquad \rho = 0 \\ f(x_n, y_m) &= \frac{1}{\pi}\exp[-x_n^2 - y_n^2], \qquad m = n. \end{aligned}$$

$m = n + 1$

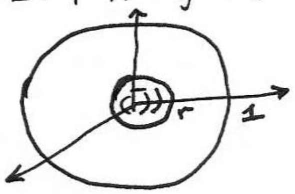
$$E[X_nY_m] = \frac{1}{4}E[(U_n + U_{n-1})(U_{n+1} - U_n)] = -\frac{1}{4} = COV[X_nY_m]$$

Chapter 6: Vector Random Variables

6.1 Vector Random Variables

6.1

a) $P[x^2+y^2+z^2 \le r \mid x^2+y^2+z^2 \le 1] = \frac{4\pi r^3}{3} \frac{3}{4\pi 1^3} = r^3, \; r \le 1$



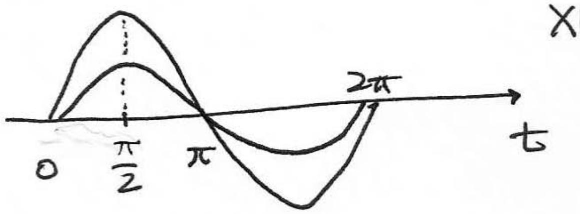
b) $P[|X| \le \frac{1}{\sqrt{3}} \cap |Y| < \frac{1}{\sqrt{3}} \cap |Z| \le \frac{1}{\sqrt{3}} \mid x^2+y^2+z^2 \le 1]$

$$= \frac{\frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}}{\frac{4\pi 1^3}{3}} = \frac{8}{3\sqrt{3}\,4\pi} = \frac{2}{\pi\sqrt{3}} = 0.3675$$

c) $P[X>0 \cap Y>0 \cap Z>0] = \frac{1}{8}$

d) $P[Z \le 0 \mid x^2+Y^2+z^2 \le 1] = \frac{1}{2}$

6.2



$X(t) = A \sin t$

A unif $[0,1]$

ⓐ $X_1 = X(0) = 0 \quad X_2 = X(\frac{\pi}{2}) = A \quad X_3 = X(\pi) = 0$

$$F_{\underline{X}}(x_1, x_2, x_3) = P[X_1 \le x_1, X_2 \le x_2, X_3 \le x_3]$$

$$= \begin{cases} 0 & \text{if } x_1 < 0 \text{ or } x_2 < 0 \text{ or } x_3 < 0 \\ P[A \le x_2] & x_1 > 0 \text{ and } x_3 > 0 \text{ and } x_2 > 0 \end{cases}$$

$X(t_1), X(t_2), X(t_3)$ are independent in this case

ⓑ $X_1 = X(t_1) \quad X_2 = X(t_1 + \frac{\pi}{2}) \quad X_3 = X(t_1 + \pi)$

$= A \sin t_1 \quad = A \sin(t_1 + \frac{\pi}{2}) \quad = A \sin(t_1 + \pi)$

$= \frac{1}{2}A \quad = A \cos t_1 \quad = -A \sin t_1$

$\quad = \sqrt{3}/2 \quad = -\frac{1}{2}A$

$$F_{\underline{X}}(x_1, x_2, x_3) = P[A \sin t_1 \le x_1, A \cos t_1 \le x_2, -A \sin t_1 \le x_3]$$

$$= P[A \le 2x_1, A \le \tfrac{2}{\sqrt{3}} x_2, A \ge 2x_3]$$

This is the prob. of 3 events involving A which is readily found once x_1, x_2 and x_3 are specified.

$X(t_1), X(t_2)$ and $X(t_3)$ are not independent.

6.3

ⓐ $P[|X|<5, Y<4, Z^3>8] = P[|X|<5]P[Y<4]P[Z^3>8]$

$= P[-5<X<5]P[Y<4]P[Z>2]$

$= [F_X(5^-) - F_X(-5)][F_Y(4^-)][1-F_Z(2)]$

ⓑ $P[X=5, Y<0, Z>1] = P[X=5]P[Y<0]P[Z>1]$

$= [F_X(5) - F_X(5^-)]F_Y(0^-)[1-F_Z(1)]$

ⓒ $P[\min(X,Y,Z)<2] = 1-P[\min(X,Y,Z)\geq 2]$

$= 1 - P[X\geq 2]P[Y\geq 2]P[Z\geq 2]$

$= 1-[1-F_X(2^-)][1-F_Y(2^-)][1-F_Z(2^-)].$

ⓓ $P[\max(X,Y,Z)>6] = 1-P[\max(X,Y,Z)\leq 6]$

$= 1-P[X\leq 6]P[Y\leq 6]P[Z\leq 6]$

$= 1-F_X(6)F_Y(6)F_Z(6).$

6.4

a) $F_{\bar{X}}(x_1, x_2, x_3) = P[N_1 \le x_1 - s, N_2 \le x_2 - s, N_3 \le x_3 - s]$

$= F_{N_1}(x_1 - s) F_{N_2}(x_2 - s) F_{N_3}(x_3 - s)$

$f_{\bar{X}}(x_1, x_2, x_3) = f_{N_1}(x_1 - s) f_{N_2}(x_2 - s) f_{N_3}(x_3 - s)$

$$= \frac{e^{-(x_1-s)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_2-s)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_3-s)^2/2}}{\sqrt{2\pi}}$$

b) $P[\min(X_1, X_2, X_3) > 0] = P[X_1 > 0] P[X_2 > 0] P[X_3 > 0]$

$F_Y(y) = 1 - (1 - F_{N_1}(-s))(1 - F_{N_2}(-s))(1 - F_{N_3}(-s))$

$= 1 - (1 - F_N(-s))^3$

$= 1 - (1 - \Phi_N(-s))^3$

c) $P[X_1 > 0, X_2 > 0, X_3 > 0] + P[X_1 > 0, X_2 > 0, X_3 \le 0]$
$+ P[X_1 \le 0, X_2 > 0, X_3 > 0] + P[X_1 > 0, X_2 \le 0, X_3 > 0]$

$= (1 - F_N(-s))^3 + 3F_N(-s)(1 - F_N(-s))^2$

6.5 $I_k = 1$ if kth drawn is black 1B, 2W.

ⓐ $(I_1, I_2, I_3) \in \{000, 010, 001, 100, 011, 101, 110, 111\} = S$

$p_{\underline{I}}(ijk) = \frac{1}{8}$ all $ijk \in S$

$X = I_1 + I_2 + I_3$ $X \in \{0, 1, 2, 3\}$

$Y = \min(I_1, I_2, I_3)$ $Y \in \{0, 1\}$

$Z = \max(I_1, I_2, I_3)$ $Z \in \{0, 1\}$

$\underline{I}$	X	Y	Z
000	0	0	0
001	1	0	1
010	1	0	1
100	1	0	1
011	2	0	1
101	2	0	1
110	2	0	1
111	3	1	1

$p_{\underline{X}}[000] = 1/8$

$p_{\underline{X}}[101] = 3/8$

$p_{\underline{X}}[201] = 3/8$

$p_{\underline{X}}[311] = 1/8$

$P[X=0] = 1/8$ $P[Y=0] = 7/8$ $P[Z=0] = 1/8$

$P[X=1] = 3/8$ $P[Y=1] = 1/8$ $P[Z=1] = 7/8$

$P[X=2] = 3/8$

$P[X=3] = 1/8$

(b)

X and Z not indep since $P[Y=1, Z=1] = \frac{1}{8} \neq \underbrace{P[Y=1]}_{\frac{1}{8}}\underbrace{P[Z=1]}_{\frac{7}{8}}$

X and Y not indep since $P[X=0, Y=0] = \frac{1}{8} \neq P[X=0]P[Y=0]$

X and Z not indep since $P[X=0, Z=0] \neq P[X=0]P[Z=0]$.

6.5 — continued —

(c) $\underline{I} \in \{001, 010, 100\}$

$\underline{I}$	X	Y	Z
001	1	0	1
010	1	0	1
100	1	0	1

$P[101] = 1$
$\underline{X}$

$P[X=1] = 1$
$P[Y=0] = 1$
$P[Z=1] = 1$

$P[X=1, Y=0] = P[X=1]\,P[Y=0] = 1$
$P[X=0, Y=0] = P[X=0]\,P[Y=0] = 0$
etc

X and Y are indep.

Similarly X and Z are independent
Y and Z are independent

$P[X=1, Y=0, Z=1] = P[X=1]\,P[Y=0]\,P[Z=1] = 1$
all other triplets have prob zero

$\therefore$ X, Y, Z are independent.

6.6 Every time instant there are 3 arrivals; of these

(a) $X_1 = i$ go to port 1, $X_2 = j$ to port 2, $X_3 = k$ to port 3
and $3-i-j-k$ to the fictional port #4

$$P[X_1 = i, X_2 = j, X_3 = k] = \frac{3!}{i!\, j!\, k!\, (3-i-j-k)!} \left(\frac{p}{3}\right)^i \left(\frac{p}{3}\right)^j \left(\frac{p}{3}\right)^k (1-p)^{3-i-j}$$

(b) $$P[X_1 = i, X_2 = j] = \sum_{k=0}^{3-i-j} P[X_1 = i, X_2 = j, X_3 = k] = \ldots$$

Alternatively, $N_1 = i$ go to port 1, $N_2 = j$ to port 2
and $3-i-j$ to ports 3 or 4 (prob $1-2\frac{p}{3}$)

$$P[X_1 = i, X_2 = j] = \frac{3!}{i!\, j!\, (3-i-j)!} \left(\frac{p}{3}\right)^i \left(\frac{p}{3}\right)^j \left(1 - 2\frac{p}{3}\right)^{3-i-j}$$

$i > 0$ $j > 0$ $i+j \leq 3$

(c) $$P[X_2 = j] = \sum_{i=0}^{3-j} P[X_1 = i, X_2 = j] = \ldots$$

or reasoning as above

$$P[X_2 = j] = \frac{3!}{j!\, (3-j)!} \left(\frac{p}{3}\right)^j \left(1 - \frac{p}{3}\right)^{3-j}$$

(d) Consider $P[X_1 = 0, X_2 = 0] = \left(1 - \frac{p}{3}\right)^3 \neq$ Not Independent

$$P[X_1 = 0]P[X_2 = 0] = \left(1 - \frac{p}{3}\right)^3 \left(1 - \frac{p}{3}\right)^3$$

(6.6)

(e) $(X_i-1)^+$ = # pkts discarded by port i.

$$E[(X_i-1)^+] = 1\cdot P[X_i=2] + 2P[X_i=3]$$

$$= 1\cdot \frac{3!}{2!1!}\left(\frac{p}{3}\right)^2\left(1-\frac{p}{3}\right)^1 + 2\left(\frac{p}{3}\right)^3$$

$$= 3\left(\frac{p}{3}\right)^2\left(1-\frac{p}{3}\right) + 2\left(\frac{p}{3}\right)^3$$

Average # discarded by all ports is

$$3\cdot E[(X_i-1)^+] = 9\left(\frac{p}{3}\right)^2\left(1-\frac{p}{3}\right) + 2\left(\frac{p}{3}\right)^3$$

(6.7) **a)**

$$\begin{aligned} 1 &= \int_0^1\int_0^1\int_0^1 k(x+y+z)\,dx\,dy\,dz \\ &= k\int_0^1\int_0^1\left(\frac{1}{2}+y+z\right)dy\,dz \\ &= k\int_0^1\left(\left(\frac{1}{2}+z\right)+\frac{1}{2}\right)dz \\ &= k\left(1+\frac{1}{2}\right) \Rightarrow k=\frac{2}{3} \end{aligned}$$

b) $f_{XY}(x,y) = \frac{2}{3}\int_0^1 (x+y+z)dz = \frac{2}{3}\left[x+y+\frac{1}{2}\right]$

$$f_Z(z|x,y) = \frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)} = \frac{x+y+z}{x+y+\frac{1}{2}}$$

c) $f_X(x) = \frac{2}{3}\int_0^1 \left(x+y+\frac{1}{2}\right)dy = \frac{2}{3}\left[xy\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 + \frac{1}{2}y\Big|_0^1\right] = \frac{2}{3}[x+1]$

6.8

~~40~~ a)

$$\begin{aligned} f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z')dz' \\ &= \int_{-(1-x^2-y^2)^{1/2}}^{(1-x^2-y^2)^{1/2}} \frac{3}{4\pi}dz' \\ &= \frac{3}{2\pi}(1-x^2-y^2)^{1/2} \end{aligned}$$

b)

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y',z')dy'dz' \\ &= \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} \int_{(1-x^2-y'^2)^{1/2}}^{(1-x^2-y'^2)^{1/2}} \frac{3}{4\pi}dz'dy' \\ &= \int_{-(1-x^2)^{1/2}}^{(1-x^2)^{1/2}} \frac{3}{2\pi}(1-x^2-y'^2)^{1/2}dy' \end{aligned}$$

Let $a^2 = 1 - x^2$:

$$\begin{aligned} \int_0^a \sqrt{a^2-t^2}\,dt &= \int_0^{\pi/2} a\cos u a \cos u du \qquad (= a\sin u) \\ &= \frac{1}{2}a^2\int_0^{\pi/2}(1+\cos 2u)du \\ &= \frac{1}{2}a^2\left(\frac{\pi}{2}+\frac{1}{2}\sin 2u|_0^{\pi/2}\right) \\ &= \frac{1}{4}\pi a^2 \end{aligned}$$

$$\therefore \quad f_X(x) = \frac{3}{2\pi}\cdot 2\cdot\frac{1}{4}\pi a^2 = \frac{3}{4}(1-x^2)$$

c)

$$\begin{aligned} f(x,y|z) &= f(x,y,z)/f(z) \\ &= \frac{\frac{3}{4\pi}}{\frac{3}{4}(1-z^2)} \\ &= \frac{1}{\pi(1-z^2)} \end{aligned}$$

d) X, Y, Z are not independent RVs.

e) $P[A] = P[R > \frac{1}{2}$, $\underline{X}$ in positive orthant$] = \frac{1}{8}\left(1-\frac{4\pi r^3}{4\pi}\right) = \frac{1}{8}(1-r^3)$

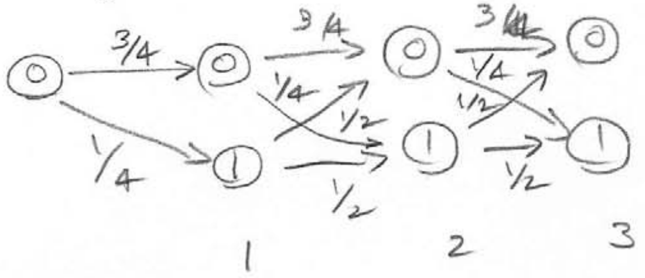
$$f(x,y,z|A] = \frac{3}{32\pi(1-r^3)}$$

6.9

$$p_{X_1X_2X_3}(x_1,x_2,x_3) = p_{X_3}(x_3|x_1,x_2)\,p_{X_1X_2}(x_1,x_2)$$

$$= p_{X_3}(x_3|x_1,x_2)\,p_{X_2}(x_2|x_1)\,p_{X}(x_1). \checkmark$$

6.10 $X_0 = 0$



$$P[X_3 X_2 X_1 = 000] = P[X_3=0|00]\,P[X_2=0|X_1=0]\,P[X_1=0]$$
$$= \frac{3}{4}\,\frac{3}{4}\,\frac{3}{4}$$

$$P[100] = \underbrace{P[X_3=1|00]}_{1/4}\,\underbrace{P[0|0]}_{3/4}\,\underbrace{P[0]}_{3/4}$$

$$P[010] = \underset{1/2}{P[X_3=0|10]}\,\underset{1/4}{P[1|0]}\,\underset{3/4}{P[0]}$$

$$P[110] = \underset{1/2}{P[X_3=1|10]}\,\underset{1/4}{P[1|0]}\,\underset{3/4}{P[0]}$$

$$P[111] = \underbrace{P[X_3=1|11]}_{\frac{1}{2}}\,\underbrace{P[1|1]}_{1/2}\,\underbrace{P[1]}_{1/4}$$

$$P[011] = \underset{1/2}{P[X_3=0|11]}\,\underset{1/2}{P[1|1]}\,\underset{1/4}{P[1]}$$

$$P[101] = \underset{1/4}{P[X_3=1|01]}\,\underset{1/2}{P[0|1]}\,\underset{1/4}{P[1]}$$

$$P[001] = \underset{\frac{3}{4}}{P[X_3=0|01]}\,\underset{\frac{1}{2}}{P[0|1]}\,\underset{1/4}{P[1]}$$

6.10 - continued

X_3	X_2	X_1	Prob
0	0	0	27/64
1	0	0	9/64
0	1	0	6/64
1	1	0	6/64
1	1	1	4/64
0	1	1	4/64
1	0	1	2/64
0	0	1	6/64

$$P[X_3=0] = \frac{27+6+4+6}{64} = \frac{43}{64}$$

$$P[X_3=1] = \frac{21}{64}$$

$$P[X_2=0] = \frac{27+9+2+6}{64} = \frac{44}{64}$$

$$P[X_2=1] = \frac{20}{64}$$

$$P[X_1=0] = \frac{27+9+6+6}{64} = \frac{48}{64}$$

$$P[X_1=1] = \frac{16}{64}$$

6.11

$$\begin{aligned} f_{XYZ}(x,y,z) &= f_Z(z|x,y) f_{XY}(x,y) \\ &= f_Z(z|x,y) f_Y(y|x) f_X(x) \end{aligned}$$

6.12

a)
$$\begin{aligned} f_Y(y|x) &= f_{U_2}(y-x) \\ f_Z(z|x,y) &= f_{U_3}(z-y) \\ f_{X,Y,Z}(x,y,z) &= f_Z(z|x,y)f_Y(y|x)f_X(x) \\ &= f_{U_3}(z-y)f_{U_2}(y-x)f_{U_1}(x) \end{aligned}$$

b)
$$\begin{aligned} f_{YZ}(y,z) &= \int_{-\infty}^{\infty} f_{U_3}(z-y)f_{U_2}(y-x)f_{U_1}(x)dx \\ &= f_{U_3}(z-y)\underbrace{\int_{-\infty}^{\infty} f_{U_2}(y-x)f_{U_1}(x)dx}_{f_Y(y)} \end{aligned}$$

We next find $f_Y(y)$:
For $0 \le y \le 1$

$$\begin{aligned} f_Y(y) &= \int_0^y f_{U_1}(u_1)\cdot f_{U_2}(y-u_1)du_1 \\ &= \int_0^y 1\cdot 1du_1 \\ &= y \end{aligned}$$

For $1 \le y \le 2$

$$\begin{aligned} f_Y(y) &= \int_{y-1}^1 f_{U_1}(u_1)f_{U_2}(y-u_1)du_1 \\ &= 2-y \end{aligned}$$

$$\therefore \quad f_{Y,Z}(y,z) = \begin{cases} y & 0\le y\le 1, \quad y\le z\le y+1 \\ 2-y & 1\le y\le 2, \quad y\le z\le y+1 \\ 0 & \text{elsewhere} \end{cases}$$

The pdf of Z is:

$$f_Z(z) = \int f_{Y,Z}(y', z)dy'$$

For $0 \leq z \leq 1$

$$f_Z(z) = \int_0^z y'dy' = \frac{1}{2}z^2$$

For $1 \leq z \leq 2$

$$\begin{aligned} f_Z(z) &= \int_{z-1}^1 y'dy' + \int_1^z (2-y')dy' \\ &= \frac{1}{2}[1-(z-1)^2] + 2(z-1) - \frac{1}{2}[z^2-1] \\ &= \frac{1}{2}[1-z^2+2z-1] + 2z - 2 - \frac{1}{2}[z^2-1] \\ &= -z^2 + 3z - \frac{3}{2} \end{aligned}$$

For $2 \leq z \leq 3$

$$\begin{aligned} f_Z(z) &= \int_{z-1}^2 (2-y')dy' \\ &= 2[2-(z-1)] - \frac{1}{2}[4-(z-1)^2] \\ &= 6 - 2z - \frac{1}{2}[-z^2+2z+3] \\ &= \frac{1}{2}z^2 - 3z + \frac{9}{2} \\ f_Z(z) &= \begin{cases} \frac{1}{2}z^2 & 0 \leq z \leq 1 \\ -z^2+3z-\frac{3}{2} & 1 \leq z \leq 2 \\ \frac{1}{2}z^2 - 3z + \frac{9}{2} & 2 \leq z \leq 3 \end{cases} \end{aligned}$$

c) From part b) we have

$$\begin{aligned} f_{Y,Z}(y,z) &= f_{U_3}(z-y)f_Y(y) \\ f_{U_3}(z-y) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(z-y)^2}{2}\right) \\ f_Y(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\{(y-x)^2+x^2\}}dx = \frac{1}{\sqrt{2\pi \cdot 2}}e^{-\frac{y^2}{2\cdot 2}} \end{aligned}$$

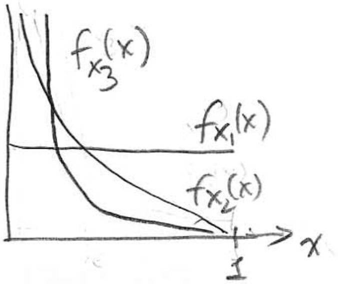
$$\begin{aligned} f_{Y,Z}(y,z) &= \frac{1}{2\pi\sqrt{2}} \exp\left[-\frac{y^2}{4} - \frac{(z-y)^2}{2}\right] \\ f_Z(z) &= \frac{1}{2\pi\sqrt{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\{(z-y)^2 + \frac{y^2}{2}\}}dy \\ &= \frac{1}{\sqrt{2\pi \cdot 3}} \exp\left(-\frac{y^2}{2\cdot 3}\right) \end{aligned}$$

6.13) Using the result from ~~problem 4~~1: Example 6.7

$$\begin{aligned} f_{X_1X_2X_3} &= f_{X_1}(x_1)f_{X_2}(x_2|x_1)f_{X_3}(x_3|x_1,x_2) \\ &= 1\cdot\frac{1}{x_1}\cdot\frac{1}{x_2} \qquad \text{for } 0 < x_3 < x_2 < x_1 < 1. \end{aligned}$$

a) Here we need to carefully determine the limits of the integrals: For a given x_3, x_2 varies from x_3 to 1; for a given x_2, x_1 varies from x_2 to 1. Thus

$$\begin{aligned} f_{X_3}(x_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1X_2X_3}(x_1,x_2,x_3)dx_1dx_2 \\ &= \int_{x_3}^{1} dx_2 \int_{x_2}^{1}\frac{dx_1}{x_1x_2} = \int_{x_3}^{1}\frac{dx_2}{x_2}\ln x_1|_{x_2}^{1} \\ &= \int_{x_3}^{1}(-1)\frac{\ln x_2}{x_2}dx_2 = (-1)\frac{(\ln x_2)^2}{2}\,|_{x_3}^{1} \\ &= \frac{(-1)^2}{2}(\ln x_3)^2 \\ f_{X_2}(x_2) &= \int_0^{x_2} dx_3\int_{x_2}^{1}\frac{dx_1}{x_1x_2} = -\int_0^{x_2}dx_3\frac{\ln x_2}{x_2} = -\ln x_2 \end{aligned}$$



We could also find the marginal pdf of X_2 by noting from the way the experiment is defined that:

$$f_{X_1X_2}(x_1,x_2) = 1\cdot\frac{1}{x_1} \qquad 0 < x_2 < x_1 < 1$$

Thus

$$f_{X_2}(x_2) = \int_{x_2}^{1} f_{X_1X_2}(x_1,x_2)dx_1 = \int_{x_2}^{1}\frac{dx_1}{x_1} = -\ln x_2\ .$$

Clearly X_1 is uniform in [0,1]. Nevertheless we carry out the integral to find $f_{X_1}(x_1)$:

$$\begin{aligned} f_{X_1}(x_1) &= \int_0^{x_1} dx_2\int_0^{x_2}\frac{dx_3}{x_1x_2} \\ &= \int_0^{x_1} dx_2\frac{1}{x_1} = 1 \qquad 0 < x_1 < 1\ . \end{aligned}$$

b)

$$\begin{aligned} f_{X_3}(x_3|x_1) &= \int_{x_3}^{x_1} f_{X_2X_3}(x_2,x_3|x_1)dx_2 \\ &= \int_{x_3}^{x_1}\frac{f(x_1,x_2,x_3)}{f(x_1)}dx_2 \\ &= \int_{x_3}^{x_1}\frac{dx_2}{x_1x_2} \\ &= \frac{1}{x_1}[\ln x_1 - \ln x_3] = \frac{1}{x_1}\ln\frac{x_1}{x_3} \qquad 0 < x_3 < x_1 \end{aligned}$$

3) ⓒ

$$f(x_1, x_2 | x_3) = \frac{1}{x_1 x_2} \frac{2}{(\ln x_3)^3} \qquad 0 < x_3 < x_2 < x_1 < 1$$

$$f(x_1 | x_3) = \int_{x_3}^{x_1} \frac{2}{x_1(\ln x_3)^2} \frac{1}{x_2} dx_2 = \frac{2}{x_1(\ln x_3)^2} \left[\ln x_1 - \ln x_3 \right]$$

$$= \frac{2 \ln \frac{x_1}{x_3}}{x_1 (\ln x_3)^2} \qquad x_3 < x_1 < 1$$

6.14

a) $$p(X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4) = \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!}\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{x_4}$$

$$p(X_1=x_1, X_2=x_2, X_3=x_3) = \frac{n!\,\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{n-x_1-x_2-x_3}}{x_1!\,x_2!\,x_3!\,(n-x_1-x_2-x_3)!}$$

b) $$= p(x_1, x_2) = \sum_{x_3=0}^{n} \frac{n!\,\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{n-x_1-x_2-x_3}}{x_1!\,x_2!\,x_3!\,(n-x_1-x_2-x_3)!}$$

$$= \frac{\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}}{x_1!\,x_2!}\sum_{x_3=0}^{n}\frac{(n-x_1-x_2)!\,\frac{n!}{(n-x_1-x_2)!}}{x_3!\,(n-x_1-x_2-x_3)!}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{n-x_1-x_2-x_3}$$

$$= \frac{\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}}{x_1!\,x_2!}\,\frac{n!}{(n-x_1-x_2)!}\left(\frac{1}{8}+\frac{1}{8}\right)^{n-x_1-x_2}$$

c) $$p(x_1) = \sum_{x_2=0}^{n}\left(\frac{1}{2}\right)^{x_1}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{4}\right)^{n-x_1-x_2}\frac{n!}{x_1!\,x_2!\,(n-x_1-x_2)!}$$

$$= \frac{\left(\frac{1}{2}\right)^{x_1}}{x_1!}\sum_{x_2=0}^{n}\frac{n!}{(n-x_1)!}\cdot\frac{(n-x_1)!}{(n-x_1-x_2)!\,x_2!}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{4}\right)^{n-x_1-x_2}$$

$$= \frac{\left(\frac{1}{2}\right)^{x_1}}{x_1!}\,\frac{n!}{(n-x_1)!}\left(\frac{1}{2}\right)^{n-x_1}$$

$$= \binom{n}{x_1}\left(\frac{1}{2}\right)^{n}$$

d) $$p(x_2, x_3 \mid x_1=m) = \frac{\frac{n!}{m!\,x_2!\,x_3!\,(n-x_2-x_3-m)!}\left(\frac{1}{2}\right)^{m}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{n-m-x_2-x_3}}{\frac{n!}{m!\,(n-m)!}\left(\frac{1}{2}\right)^{n}}$$

$$= \frac{(n-m)!}{x_2!\,x_3!\,(n-m-x_2-x_3)!}\underbrace{\left(\frac{1}{2}\right)^{m}\left(\frac{1}{2}\right)^{-n}\left(\frac{1}{4}\right)^{x_2}\left(\frac{1}{8}\right)^{x_3}\left(\frac{1}{8}\right)^{n-m-x_2-x_3}}_{\left(\frac{2}{4}\right)^{x_2}\left(\frac{2}{8}\right)^{x_3}\left(\frac{2}{8}\right)^{n-m-x_2-x_3}}$$

6.15

$p_N(n) = \frac{\alpha^n}{n!} e^{-\alpha}$ $\quad n = 0, 1, \ldots$

$$p_{\bar{X}}(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3, x_4 \mid n)\, p_N(n)$$

$$= \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!} \left(\frac{1}{2}\right)^{x_1} \left(\frac{1}{4}\right)^{x_2} \left(\frac{1}{8}\right)^{x_3} \left(\frac{1}{8}\right)^{x_4} \frac{\alpha^n e^{-\alpha}}{n!}$$

$n = x_1 + x_2 + x_3 + x_4$

$$= \frac{1}{x_1!\,x_2!\,x_3!\,x_4!} \left(\frac{\alpha}{2}\right)^{x_1} \left(\frac{\alpha}{4}\right)^{x_2} \left(\frac{\alpha}{8}\right)^{x_3} \left(\frac{\alpha}{8}\right)^{x_4} e^{-\alpha}$$

$$= \left(\frac{\left(\frac{\alpha_1}{2}\right)^{x_1}}{x_1!} e^{-\frac{\alpha_1}{2}} \right) \left(\frac{\left(\frac{\alpha}{4}\right)^{x_2}}{x_2!} e^{-\frac{\alpha}{4}} \right) \left(\frac{\left(\frac{\alpha}{8}\right)^{x_3}}{x_3!} e^{-\frac{\alpha}{8}} \right) \left(\frac{\left(\frac{\alpha}{8}\right)^{x_4}}{x_4!} e^{-\frac{\alpha}{8}} \right)$$

independent Poisson RV's!

6.16

~~4.46~~ For $k_j \geq 0$ such that $k_1 + k_2 + k_3 \leq$n

$$p(k_1, k_2, k_3) = \frac{1}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}}$$

Note: $\begin{pmatrix} n+3 \\ 3 \end{pmatrix}$ is the number of ways of distributing n identical balls in 4 cells: See Sampling with Replacement and Without Ordering in Section 2.3.

a) $$p(k_1, k_2) = \sum_{k_3=0}^{n-k_1-k_2} p(k_1, k_2, k_3) = \frac{n - k_1 - k_2 + 1}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}}$$

b) $$p(k_1) = \sum_{k_2=0}^{n-k_1} \frac{n - k_1 - k_2 + 1}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} \qquad j = n - k_1 - k_2 + 1$$

$$= \sum_{j=1}^{n-k_1+1} \frac{j}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} = \frac{(n-k_1+2)(n-k_1+1)}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}}$$

Check

$$\begin{aligned}
\sum_{k_1=0}^{n} p(k_1) &= \frac{1}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} \sum_{k_1=0}^{n} (n-k_1+2)(n-k_1+1) \qquad j = n - k_1 + 1 \\
&= \frac{1}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} \sum_{j=1}^{n+1} j(j+1) \qquad \begin{array}{l} \displaystyle\sum_{j=1}^{n+1} j = \frac{(n+2)(n+1)}{2} \\ \displaystyle\sum_{j=1}^{n+1} j^2 = \frac{(n+1)(n+2)(2n+3)}{6} \end{array} \\
&= \frac{1}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} \underbrace{\left[\frac{(n+2)(n+1)}{2} + \frac{(n+1)(n+2)(2n+3)}{6}\right]}_{} \\
&= \frac{1}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}} \frac{(n+1)(n+2)(n+3)}{3} = 1 \qquad \checkmark
\end{aligned}$$

c) $$p(k_2, k_3 | k_1) = \frac{p(k_1, k_2, k_3)}{p(k_1)} = \frac{\dfrac{(n-k_1+2)(n-k_1+1)}{2\begin{pmatrix} n+3 \\ 3 \end{pmatrix}}}{\begin{pmatrix} n+3 \\ 3 \end{pmatrix}}$$

$$= \frac{(n-K-1+2)(n-k_1+1)}{2}$$

$j \geq 0, k \geq 0, l \geq 0$

6.17

$$P[N_1=j, N_2=k, N_3=l \mid T=t] = \frac{(\lambda_1 t)^j}{j!}\frac{(\lambda_2 t)^k}{k!}\frac{(\lambda_3 t)^l}{l!} e^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

$$P[N_1=j, N_2=k, N_3=l] = \int_0^\infty \frac{\lambda_1^j \lambda_2^k \lambda_3^l}{j!k!l!} t^{j+k+l} e^{-(\lambda_1+\lambda_2+\lambda_3)t} \alpha e^{-\alpha t} dt$$

$$= \frac{\alpha \lambda_1^j \lambda_2^k \lambda_3^l}{j!k!l!} \underbrace{\int_0^\infty t^{j+k+l} e^{-(\alpha+\lambda_1+\lambda_2+\lambda_3)t} dt}_{\frac{\Gamma(j+k+l+1)}{(\alpha+\lambda_1+\lambda_2+\lambda_3)^{j+k+l+1}}}$$

$$= \frac{(j+k+l)!}{j!k!l!}\left(\frac{\alpha}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)\left(\frac{\lambda_1}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^j \left(\frac{\lambda_2}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^k \left(\frac{\lambda_3}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^l$$

Neat Answer!

Imagine a sequence of 4 way races among :

- customer in service : rate α wins w. prob $\frac{\alpha}{\alpha+\lambda_1+\lambda_2}$
- arrival type 1 : rate λ_1 " $\frac{\lambda_1}{\alpha_1+\lambda_1+\lambda_2+}$
- " 2 " λ_2 " $\frac{\lambda_2}{\alpha+\lambda_1+\lambda_2+}$
- " 3 " λ_3 " $\frac{\lambda_3}{\alpha+\lambda_1+\lambda_2+\lambda_3}$

Game ends when customer in service wins.

6.17

b) $$P[N_1 = j] = \int_0^\infty \frac{(\lambda_1 t)^j}{j!} e^{-\lambda t} \alpha e^{-\alpha t}\, dt$$

$$= \frac{\alpha \lambda_1^j}{j!} \underbrace{\int_0^\infty t^j e^{-(\lambda_1+\alpha)t}\, dt}_{\frac{\Gamma(j+1)}{(\alpha+\lambda_1)^{j+1}}}$$

$$= \left(\frac{\alpha}{\alpha+\lambda_1}\right)\left(\frac{\lambda_1}{\alpha+\lambda_1}\right)^j \qquad j = 0, 1, \ldots$$

c)

$$P[N_1 = j, N_2 = k \mid N_3 = l] =$$

$$= \frac{(j+k+l)!}{j!\,k!\,l!} \; \frac{\left(\frac{\alpha}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)\left(\frac{\lambda_1}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^j\left(\frac{\lambda_2}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^k\left(\frac{\lambda_3}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^l}{\left(\frac{\alpha}{\alpha+\lambda_3}\right)\left(\frac{\lambda_3}{\alpha+\lambda_3}\right)^l}$$

$$= \frac{(j+k+l)!}{j!\,k!\,l!} \left(\frac{\alpha+\lambda_3}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)\left(\frac{\alpha+\lambda_3}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^l\left(\frac{\lambda_1}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^j\left(\frac{\lambda_2}{\alpha+\lambda_1+\lambda_2+\lambda_3}\right)^k$$

6.2 Functions of Several Random Variables

6.18

a) $Y = \min(L_1, L_2, \ldots, L_N)$

$$1 - F_Y(y) = P[\min(L_1, L_2, \ldots, L_N) > y]$$

$$F_Y(y) = 1 - (1 - F_L(y))^N$$

$$= 1 - \left(1 - \left(1 - \left(\frac{y_{min}}{y}\right)^{\alpha}\right)\right)^N$$

$$= 1 - \left(\frac{y_{min}}{y}\right)^{\alpha N}$$

b) $F_Y(y) = 1 - \left(1 - \left(1 - e^{-(y/\lambda)^{\beta}}\right)\right)^N$

$$= 1 - \left(e^{-(y/\lambda)^{\beta}}\right)^N$$

6.19

$I_k(t) = 1$ iff item k still working.

$P[I_k(t) = 1] = P[X_k > t] = R(t)$ This is a Bernoulli Trial

R success prob.

$N(t) = I_1(t) + \ldots I_N(t)$ # successes in N Bernoulli trials

$N(t)$ is a Binomial RV

$$P[N(t) = n] = \binom{N}{n} R(t)^n (1 - R(t))^{N-m}$$

$$E[N(t)] = N \cdot R(t)$$

$$VAR[N(t)] = N\, R(t)(1 - R(t))$$

6.20

$$P[\max(X_1^2, X_2^2 \dots X_n^2) \le \gamma]$$

$$= P[X_1^2 \le \gamma] \dots P[X_n^2 \le \gamma]$$

$$P[X^2 \le \gamma] = P[X \le \sqrt{\gamma}] = \int_0^{\sqrt{\gamma}} \frac{r}{\alpha^2} e^{-r^2/2\alpha^2} dr$$

$$= 1 - e^{-\gamma/2\alpha^2}$$

$$\therefore P[\max(\quad) \le \gamma] = \left(1 - e^{-\gamma/2\alpha^2}\right)^n$$

6.21

a) $$P_{Y_1 Y_2 \dots Y_K}(y_1, y_2, \dots y_K) = P[N_1 \le y_1 - \alpha_1 b_1, N_2 \le y_2 - \alpha_2 b_2, \dots N_K \le y_K - \alpha_K b_K]$$

$$= F_{N_1 N_2 \dots N_K}(y_1 - \alpha_1 b_1, y_2 - \alpha_2 b_2 \dots y_K - \alpha_K b_K)$$

$$f_{\bar{Y}}(y_1, y_2, \dots, y_K) = f_{N_1}(y_1 - \alpha_1 b_1) f_{N_2}(y_2 - \alpha_2 b_2) \dots f_{N_K}(y_K - \alpha_K b_K)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^K e^{-(y_1 - \alpha_1 b_1)^2/2} e^{-(y_2 - \alpha_2 b_2)^2/2} \dots e^{-(y_K - \alpha_K b_K)^2/2}$$

b) $$P[Y_1 > 0, Y_2 > 0, Y_3 > 0 \dots Y_K > 0 \mid b_1 = 1, b_2 = 1, \dots b_K = 1]$$

$$= P[N_1 > -\alpha_1 b_1, N_2 > -\alpha_2 b_2, \dots N_K > -\alpha_K b_K \mid b_1 = 1, b = 2 \dots b_K = 1]$$

$$= P[N_1 > -\alpha_1, N_2 > -\alpha_2 \dots N_K > -\alpha_K]$$

$$= (1 - \Phi_{N_1}(-\alpha_1))(1 - \Phi_{N_2}(-\alpha_2)) \dots (1 - \Phi_{N_K}(-\alpha_K))$$

6.22

~~4.53~~ a) $\underline{Z} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \qquad |A| = 1$

$$\begin{aligned} X_1 &= \, = U \\ X_2 &= V - X_1 = V - U \\ X_3 &= W - X_1 - X_2 = W - V \end{aligned}$$

$$f_{\underline{Z}}(u,v,w) = \left.\frac{f_{\underline{X}}(\underline{x})}{|A|}\right|_{\underline{x}=A^{-1}\underline{u}} = f_{\underline{X}}(u, v-u, w-v)$$

b)
$$\begin{aligned} f_{\underline{Z}}(u,v,w) &= \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{u^2}{2}} e^{-(v-u)^2/2} e^{-(w-v)^2/2} \\ &= \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{2}[2u^2+2v^2+w^2-2uv-2vw]} \\ &= \frac{1}{(\sqrt{2\pi})^3} e^{-[u^2+v^2+\frac{1}{2}w^2-uv-vw]} \end{aligned}$$

6.22 ⓒ

$$f(v,w) = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-[u^2-uv]} du}_{} \; e^{-[v^2+\frac{1}{2}w^2-vw]}$$

$$= e^{\frac{1}{4}v^2} \int_{-\infty}^{\infty} \frac{1/2}{\sqrt{2\pi \frac{1}{2}}} e^{-[u-\frac{1}{2}v]^2}$$

$$f(v,\omega) = \frac{1}{4\pi} e^{-[\frac{3}{4}v^2+\frac{1}{2}\omega^2-v\omega]}$$

ⓘ 6.23

4.54

$$\begin{aligned} M &= \frac{1}{2}X_1 + \frac{1}{2}X_2 = \frac{1}{2}(X_1 + X_2) \\ V &= \frac{1}{2}(X_1 - M)^2 + \frac{1}{2}(X_2^2 - M)^2 = \frac{1}{2}\left(\frac{X_1}{2} - \frac{X_2}{2}\right)^2 + \frac{1}{2}\left(\frac{X_2}{2} - \frac{X_1}{2}\right)^2 \\ &= \frac{1}{8}(X_1 - X_2)^2 \end{aligned}$$

ⓐ

$$\left.\begin{array}{l} \sqrt{8V} = X_1 - X_2 \\ 2M = X_1 + X_2 \end{array}\right\} \quad \begin{array}{l} X_1 = M + \sqrt{2V} \\ X_2 = M - \sqrt{2V} \end{array}$$

$$\begin{aligned} \begin{vmatrix} \frac{\partial M}{\partial x_1} & \frac{\partial M}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{vmatrix} &= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4}(x_1 - x_2) & -\frac{1}{4}(x_1 - x_2) \end{vmatrix} = \left| -\frac{1}{8}(x_1 - x_2) - \frac{1}{8}(x_1 - x_2) \right| \\ &= \frac{1}{4}|x_1 - x_2| = \frac{1}{4}|2\sqrt{2V}| = \sqrt{v/2} \end{aligned}$$

$$f_{M,V}(m,v) = \frac{f_{X_1,X_2}(m + \sqrt{2v}, m - \sqrt{2v})}{\sqrt{v/2}}$$

ⓒ

If $f_{X_1X_2}(x,y) = \lambda e^{-\lambda x}\lambda e^{-\lambda y} \qquad x,y > 0$. Then

$$\begin{aligned} f_{M,V}(m,v) &= \frac{\lambda^2 e^{-\lambda(m+\sqrt{2v})}e^{-\lambda(m-\sqrt{2v})}}{\sqrt{v/2}} \qquad \underbrace{\underbrace{m - \sqrt{2v} > 0}_{m > \sqrt{2v} > 0}}_{\frac{m^2}{2} > v > 0} \\ &= \frac{\lambda^2 e^{-2\lambda m}}{\sqrt{v/2}} \qquad 0 < v < \frac{m^2}{2} \end{aligned}$$

As a check, find $f_M(m)$:

$$f_M(m) = \int_0^{m^2/2} \frac{\lambda^2 e^{-2\lambda m}}{\sqrt{v/2}} dv = \lambda^2 e^{-2\lambda m} 2m$$

This is an Erlang RV for the sum of 2 exponential RV's

(6.23) ⓑ

$$f_{X_1 X_2}(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$f_{MV}(m,v) = \frac{1}{2\pi\sqrt{v/2}} e^{-\frac{1}{2}\left[(m+\sqrt{2v})^2 + (m-\sqrt{2v})^2\right]}$$

$$= \frac{e^{-\frac{1}{2}[2m^2+4v]}}{2\pi\sqrt{v/2}} = \underbrace{\frac{e^{-m^2}}{\sqrt{2\pi\frac{1}{2}}}}_{\text{Gaussian}} \underbrace{\frac{e^{-2v}}{\sqrt{4\pi v}}}_{\text{chi-square}}$$

6.24

~~4.55~~ **a)** Let the auxiliary function be $W = Y$ then

$$J(z,w) = \begin{vmatrix} \frac{w}{(1-z)^2} & \frac{z}{1-z} \\ 0 & 1 \end{vmatrix} = \frac{w}{(1-z)^2}$$

and

$$f_{Z,W}(z,w) = f_{XY}\left(\frac{zw}{1-z}, w\right)\frac{|w|}{(1-z)^2}$$

so

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}\left(\frac{zw}{1-z}, w\right)\frac{|w|}{(1-z)^2}dw$$

b)

$$\begin{aligned} f_Z(z) &= \int_0^\infty \alpha e^{-\alpha zw/(1-z)}\alpha e^{-\alpha w}\frac{w}{(1-z)^2}dw \qquad 0 \le z \le 1 \\ &= \frac{\alpha^2}{(1-z)^2}\int_0^\infty we^{-\frac{\alpha}{(1-z)}w} = \frac{\alpha^2}{(1-z)^2}\frac{(1-z)^2}{\alpha^2} \\ &= 1 \end{aligned}$$

That is Z is unif. dist. in [0,1].

ⓒ

$$f_Z(z) = \int_{x_m}^{\infty} \alpha^2 \frac{x_m^{2\alpha}(1-z)^{\alpha+1}}{(zw)^{\alpha+1} w^{\alpha+1}} \frac{w}{(1-z)^2}dw$$

$$= \frac{\alpha^2 x_m^{2\alpha}(1-z)^{\alpha+1}}{z^{\alpha+1}(1-z)^2}\underbrace{\int_{x_m}^{\infty}\frac{w}{w^{2\alpha+1}}dw}_{-\frac{1}{2}w^{-2\alpha}\Big|_{x_m}^{\infty}}$$

$$= \alpha^2\frac{(1-z)^{\alpha-1}}{z^{\alpha+1}}\frac{x_m^{2\alpha}}{2x_m^{2\alpha}} = \frac{\alpha^2}{2}\frac{(1-z)^{\alpha-1}}{z^{\alpha+1}} \qquad x_m < z$$

$$0 < z < 1$$

6.25

$Z = X/Y$

$W = Y$

$$J(x,y) = \begin{vmatrix} 1/y & -x/y^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{y} = \frac{1}{w}$$

$$f_{ZW}(z,w) = f_{XY}(zw, w)\,|w|$$

$$f_Z(z) = \int_0^\infty \alpha e^{-\alpha z w} \alpha e^{-\alpha w} w \, dw$$

$$= \alpha \int_0^\infty w e^{-(\alpha z+1)w} dw$$

$$= \frac{\alpha}{\alpha z+1} \underbrace{\int_0^\infty w (\alpha z+1) e^{-(\alpha z+1)w} dw}_{\text{mean of exponential RV}}$$

$$= \frac{\alpha}{(\alpha z+1)^2} \qquad z > 0$$

6.26

~~4.56~~ $U = X^2$ Four $(\sqrt{u}, {}^4\sqrt{v})$ $(+\sqrt{u}, -{}^4\sqrt{v})$
$V = Y^4$ Roots: $(-\sqrt{u}, {}^4\sqrt{v})$ $(-\sqrt{u}, -{}^4\sqrt{v})$

$$J_{XY} = \begin{vmatrix} 2x & 0 \\ 0 & 4y^3 \end{vmatrix} = 8|xy^3|$$

$$\begin{aligned} f_{UV}(u,v) &= \sum_i \frac{f_{XY}(x_i,y_i)}{8|x_i y_i^3|} \qquad u > 0, v > 0 \\ &= \frac{1}{8\sqrt{u}\,{}^4\sqrt{v^3}} \left[\frac{2e^{-(u-2\rho\sqrt{u}\,{}^4\sqrt{v}+\sqrt{v})/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \right. \\ &\qquad \left. + \frac{2e^{-(u+2\rho\sqrt{u}\,{}^4\sqrt{v}+\sqrt{v})/2(1-\rho^2)}}{2\pi\sqrt{1-\rho^2}} \right] \end{aligned}$$

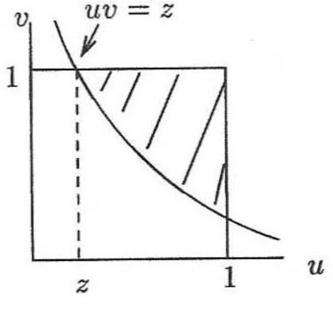
6.27

~~4.57~~ Defining two auxiliary functions $U = X_2$, $V = X_3$

$$J_{X_1X_2X_3} = \begin{vmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |x_2x_3|$$

$$f_{Z_1,U,V}(z,u,v) = \frac{f_{\underline{X}}(z/uv,u,v)}{uv} = \frac{1}{uv} \qquad \begin{array}{l} 0 < \frac{z}{uv} < 1 \\ 0 < u < 1 \\ 0 < v < 1 \end{array}$$

$$= \frac{1}{uv} \qquad \begin{array}{l} 0 < z < uv \\ 0 < u < 1 \\ 0 < v < 1 \end{array}$$



$$\begin{aligned} f_Z(z) &= \int\int_{\text{shaded area}} \frac{du\,dv}{uv} = \int_z^1 \int_{z/u}^1 \frac{1}{uv} dv\,du \\ &= \frac{1}{2}(\ln z)^2 \end{aligned}$$

6.28

~~4.58~~ Use spherical coordinates:

ⓐ $$X = R\cos\Theta\sin\phi \qquad Y = R\sin\Theta\sin\phi \qquad Z = R\cos\phi$$

$$J(r,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -r\sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ r\cos\theta\cos\phi & r\sin\theta\cos\phi & -r\sin\phi \end{vmatrix} = |-r^2\sin\phi|$$

$$\begin{aligned} f_{R,\Theta,\phi}(r,\theta,\phi) &= f_{\underline{X}}(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)r^2\sin\phi \\ &= \frac{e^{-r^2/2}}{\sqrt{2\pi}^3}r^2\sin\phi \end{aligned}$$

$$\begin{aligned} f_R(r) &= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \frac{e^{-r^2/2}}{\sqrt{2\pi}^3}r^2\sin\phi \\ &= \int_0^{\pi} \frac{r^2e^{-r^2/2}}{\sqrt{2\pi}}\sin\phi d\phi \\ &= \sqrt{\frac{2}{\pi}}r^2e^{-r^2/2} \qquad r > 0 \end{aligned}$$

ⓑ Let $W = R^2$ then

$$f_W(w) = \frac{f_R(\sqrt{w})}{2\sqrt{w}} = \sqrt{\frac{2}{\pi}}\,\frac{w\,e^{-w/2}}{2\sqrt{w}}$$

$$= \frac{\sqrt{w}\,e^{-w/2}}{\sqrt{2\pi}}$$

6.29

$Y_1 = X_1$ $\qquad$ $X_1 = Y_1$

$Y_2 = X_1 + X_2$ $\qquad$ $X_2 = -Y_1 + Y_2$

$Y_3 = X_2 + X_3$ $\qquad$ $X_3 = Y_1 - Y_2 + Y_3$

$Y_4 = X_3 + X_4$ $\qquad$ $X_4 = -Y_1 + Y_2 - Y_3 + Y_4$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad |A| = 1.$$

ⓐ
$$f_{\underline{Y}}(\underline{y}) = \frac{f_{\underline{X}}(y_1, -y_1+y_2, y_1-y_2+y_3, -y_1+y_2-y_3+y_4)}{|1|}$$

ⓑ
$$f_{\underline{Y}}(\underline{y}) = \frac{1}{(2\pi)^2} \exp\left\{-\frac{1}{2}\left[y_1^2 + (-y_1+y_2)^2 + (y_1-y_2+y_3)^2 + (-y_1+y_2-y_3+y_4)^2 \right]\right\}$$

6.3 Expected Values of Vector Random Variables

6.30

~~32~~
$$\begin{aligned}\mathcal{E}[M] &= \frac{1}{2}\mathcal{E}[X_1] + \frac{1}{2}\mathcal{E}[X_2] \\ \mathcal{E}[V] &= \mathcal{E}\left[\frac{(X_1-M)^2}{2} + \frac{(X_2-M)^2}{2}\right] = \frac{1}{8}\mathcal{E}[(X_1-X_2)^2] \\ &= \frac{1}{8}[\mathcal{E}[X_1^2] - 2\mathcal{E}[X_1x_2] + \mathcal{E}[X_2^2]].\end{aligned}$$

In 6.30c M and V are independent, so E[MV] = E[M] E[V] = 0.

6.31

~~.63~~ **a)** $\mathcal{E}[Z] = \int_0^1 z\frac{1}{2}(\ln z)^2 dz = \frac{1}{4}\left[z^2(\ln z)^2 - z^2\ln z + \frac{z^2}{2}\right]_0^1 = \frac{1}{8}$

b)

$$\begin{aligned}\mathcal{E}[X_1X_2X_3] &= \int_0^1\int_0^1\int_0^1 x_1x_2x_3dx_1dx_2dx_3 \\ &= \int_0^1 x_1dx_1\int_0^1 x_2dx_2\int_0^1 x_3dx_3 = \left(\frac{1}{2}\right)^3\end{aligned}$$

6.32)

$$m_x = E[\bar{X}] = \begin{bmatrix} s \\ s \\ s \end{bmatrix}$$

$$VAR[X_k] = VAR[s + N_k] = 1$$

$$K_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.33 $E[X(t)] = E[A \sin t] = E[A] \sin t = \frac{1}{2} \sin t$

$E[X(t_1) X(t_2)] = E[A \sin t_1 \, A \sin t_2]$ $\quad E[A^2] = \int_0^1 x^2 dx = \frac{1}{3}$

$= E[A^2] \sin t_1 \sin t_2$

$= \frac{1}{3} \sin t_1 \sin t_2$

$\text{COV}(X(t_1), X(t_2)) = \frac{1}{3} \sin t_1 \sin t_2 - \frac{1}{4} \sin t_1 \sin t_2$

$= \frac{1}{12} \sin t_1 \sin t_2.$

$$\underline{m}_{\underline{X}} = \begin{bmatrix} \frac{1}{2} \sin t_1 \\ \frac{1}{2} \sin t_2 \\ \frac{1}{2} \sin t_3 \end{bmatrix}$$

$$K_{\underline{X}} = \frac{1}{12} \begin{bmatrix} \sin^2 t_1 & \sin t_1 \sin t_2 & \sin t_1 \sin t_3 \\ \sin t_1 \sin t_2 & \sin^2 t_2 & \sin t_2 \sin t_3 \\ \sin t_1 \sin t_3 & \sin t_2 \sin t_3 & \sin^2 t_3 \end{bmatrix}$$

6.34

(a)

$E[X] = 3\cdot\frac{1}{2} = \frac{3}{2}$

$E[Y] = \frac{1}{8}$ $\quad\quad m_{\underline{X}} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{8} \\ \frac{7}{8} \end{bmatrix}$

$E[Z] = \frac{7}{8}$

$E[XY] = 3\cdot 1\cdot \frac{1}{8} = \frac{3}{8}$

$E[XZ] = 1\cdot 1\cdot\frac{3}{8} + 2\cdot 1\cdot\frac{3}{8} + 3\cdot 1\cdot\frac{1}{8} = \frac{12}{8} = \frac{3}{2}$

$E[YZ] = 1\cdot 1\cdot\frac{1}{8} = \frac{1}{8}$

$E[X^2] = 3\cdot\frac{1}{2}\cdot\frac{1}{2} + \left(3\cdot\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{9}{4} = 3$

$E[Y^2] = \frac{1}{8}$

$E[Z^2] = \frac{7}{8}$

$$K_{\underline{X}} = \begin{bmatrix} \frac{3}{2} & \frac{3}{8} - \frac{3}{2}\frac{1}{8} & \frac{3}{2} - \frac{3}{2}\frac{7}{8} \\ — & \frac{1}{8} - \left(\frac{1}{8}\right)^2 & \frac{1}{8} - \frac{1}{8}\frac{7}{8} \\ — & — & \frac{7}{8} - \left(\frac{7}{8}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & \frac{3}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{7}{64} & \frac{1}{64} \\ \frac{3}{16} & \frac{1}{64} & \frac{7}{64} \end{bmatrix}$$

(b)

$E[X] = 1 \quad E[Y] = 0 \quad E[Z] = 1$

$E[X^2] = 1 \quad E[Y^2] = 0 \quad E[Z^2] = 1$

$$m_X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$K_X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6.35

Note symmetry in x, y, and z so number of calculations can be reduced drastically.

$$E[X] = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{2}{3}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = \frac{5}{9}$$

$$= E[Y] = E[Z]$$

$$E[X^2] = \frac{2}{3}\int_0^1 x^2(x+1)\,dx = \frac{2}{3}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1 = \frac{7}{18}$$

$$VAR[X] = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$E[XY] = \frac{2}{3}\int_0^1\int_0^1 xy\left(x+y+\frac{1}{2}\right)dx\,dy$$

$$= \frac{2}{3}\int_0^1 x\,dx\left(\underbrace{\int_0^1\left(yx+y^2+\frac{1}{2}y\right)dy}_{\frac{1}{2}x+\frac{1}{3}+\frac{1}{4}}\right)$$

$$= \frac{2}{3}\int_0^1\left(\frac{1}{2}x^2+\frac{7}{12}x\right)dx$$

$$= \frac{2}{3}\left[\frac{1}{2}\,\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{2}\right] = \frac{11}{36}$$

$$E[XY]-E[X]E[Y] = \frac{11}{36}-\left(\frac{5}{9}\right)^2 = -\frac{1}{81}$$

$$\underline{m}_{\underline{X}} = \begin{bmatrix}\frac{5}{9}\\ \frac{5}{9}\\ \frac{5}{9}\end{bmatrix} \qquad K_{\underline{X}} = \begin{bmatrix}\frac{13}{162} & -\frac{2}{162} & -\frac{2}{162}\\ -\frac{2}{162} & \frac{13}{162} & -\frac{2}{162}\\ -\frac{2}{162} & -\frac{2}{162} & \frac{13}{162}\end{bmatrix}$$

6.36

$$E[X] = \int_{-1}^{1} x \frac{3}{4}(1-x^2)\,dx = \frac{3x^2}{8} - \frac{3x^4}{16}\Big|_{-1}^{1} = 0$$

$E[Y] = 0$

$E[Z] = 0$

$$E[X^2] = \int_{-1}^{1} x^2 \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\cdot\frac{x^3}{3} - \frac{3}{4}\cdot\frac{x^5}{5}\Big|_{-1}^{1} = \frac{1}{5}$$

$E[Y^2] = \frac{1}{5}$

$E[Z^2] = \frac{1}{5}$

$$E[XY] = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy\, \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{2\pi}\, y\, \sqrt{1-y^2}\,\sin u\, \sqrt{1-y^2}\cos u\, \sqrt{1-y^2}\cos u\, du\, dy$$

$$= \int_{-1}^{1} \frac{3}{2\pi}\, y\, (1-y^2)^{3/2} \left(-\frac{\cos^3 u}{3}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dy = 0$$

$$m = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad K = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

6.37

(a) X_2 is Binomial with parameter 3 and $\frac{p}{3}$

$$E[X_2] = 3 \cdot \frac{p}{3} = p$$

(b)

$$E[X_1X_2] = \sum_{i=0}^{3}\sum_{j=0}^{3-i} ij \frac{3!\left(\frac{p}{3}\right)^i\left(\frac{p}{3}\right)^j}{i!\,j!\,(3-i-j)!}\left(1-2\frac{p}{3}\right)^{3-i-j}$$

$$= \sum_{i=0}^{3} i \frac{3!\left(\frac{p}{3}\right)^i\left(\frac{p}{3}\right)}{i!} \underbrace{\sum_{j=1}^{3-i}\frac{\left(\frac{p}{3}\right)^{j-1}}{(j-1)!}\frac{\left(1-2\frac{p}{3}\right)^{3-i-j}}{(3-i-j)!}}_{\underbrace{\sum_{j'=0}^{3-i-1}\frac{\left(\frac{p}{3}\right)^{j'}}{j'!}\frac{\left(1-2\frac{p}{3}\right)^{3-i-j'-1}}{(3-i-j'-1)!}}_{\frac{1}{(3-i-1)!}\underbrace{\sum_{j'=0}^{3-i-1}\binom{3-i-1}{j'}\left(\frac{p}{3}\right)^{j'}\left(1-2\frac{p}{3}\right)^{3-i-j'-1}}_{\left(\frac{p}{3}+1-2\frac{p}{3}\right)^{3-i-1}}}}$$

$$E[X_1X_2] = \sum_{i=1}^{3}\frac{3!\left(\frac{p}{3}\right)^i\left(\frac{p}{3}\right)}{(i-1)!}\frac{\left(1-\frac{p}{3}\right)^{3-i-1}}{(3-i-1)!}$$

$$= 3!\left(\frac{p}{3}\right)\sum_{i'=0}^{2}\frac{\left(\frac{p}{3}\right)^{i'+1}}{i'!}\frac{\left(1-\frac{p}{3}\right)^{3-i'-2}}{(3-i'-2)!}$$

6.38

$$E\left[E[N_1N_2|T]\right] = E\left[(\lambda_1 T)(\lambda_2 T)\right]$$
$$= \lambda_1\lambda_2 E[T^2]$$

$$\text{COV}(N_1, N_2) = \lambda_1\lambda_2 E[T^2] - \lambda_1 E[T]\lambda_2 E[T]$$
$$= \lambda_1\lambda_2\left[\frac{2}{\alpha^2}\right] - \lambda_1\lambda_2\frac{1}{\alpha^2}$$
$$= \lambda_1\lambda_2\frac{1}{\alpha^2}$$

$$E\left[E[N_1^2|T]\right] = E\left[\lambda_1 T + (\lambda_1 T)^2\right]$$
$$= \lambda_1 E[T] + \lambda_1^2 E[T^2]$$
$$= \frac{\lambda_1}{\alpha} + \lambda_1^2\frac{2}{\alpha^2}$$

$$\text{VAR}[N_1] = \frac{\lambda_1}{\alpha} + \lambda_1^2\frac{2}{\alpha^2} - \left(\frac{\lambda_1}{\alpha}\right)^2$$
$$= \frac{\lambda_1^2}{\alpha^2} + \frac{\lambda_1}{\alpha}$$

$$\underline{m}_{\underline{N}} = \begin{bmatrix}\lambda_1/\alpha \\ \lambda_2/\alpha \\ \lambda_3/\alpha\end{bmatrix} \quad \underline{K}_{\underline{N}} = \begin{bmatrix}\frac{\lambda_1^2}{\alpha^2}+\frac{\lambda_1}{\alpha} & \frac{\lambda_1\lambda_2}{\alpha^2} & \frac{\lambda_1\lambda_3}{\alpha^2} \\ \frac{\lambda_2\lambda_1}{\alpha^2} & \frac{\lambda_2^2}{\alpha^2}+\frac{\lambda_2}{\alpha} & \frac{\lambda_2\lambda_3}{\alpha^2} \\ \frac{\lambda_1\lambda_3}{\alpha^2} & \frac{\lambda_2\lambda_3}{\alpha^2} & \frac{\lambda_3^2}{\alpha^2}+\frac{\lambda_3}{\alpha}\end{bmatrix}$$

6.39

$U = X_1$

$V = X_1 + X_2$

$W = X_1 + X_2 + X_3$

X_i $\underline{X}$ iid Gaussian

$\underline{m}_{\underline{X}} = \underline{0}$ $K_{\underline{X}} = I$

ⓐ $$\underline{Y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \underline{X} \qquad \underline{m}_Y = A\,\underline{m}_{\underline{X}} = \underline{0}$$

$$K_{\underline{Y}} = A K_{\underline{X}} A^T = A A^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

ⓑ $$K_{\underline{X}\underline{Y}} = K_{\underline{X}} A^T = A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

6.40

a) & b)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad m_Y = \begin{bmatrix} E[X_1] \\ E[X_1] + E[X_2] \\ E[X_2] + E[X_3] \\ E[X_3] + E[X_4] \end{bmatrix}$$

$$K_{XY} = K_X A^T = \begin{bmatrix} k_{11}+k_{21} & k_{11}+k_{12} & k_{12}+k_{13} & k_{13}+k_{14} \\ k_{21} & k_{21}+k_{22} & k_{22}+k_{23} & k_{23}+k_{24} \\ k_{31} & k_{31}+k_{32} & k_{32}+k_{33} & k_{33}+k_{34} \\ k_{41} & k_{41}+k_{42} & k_{42}+k_{43} & k_{43}+k_{44} \end{bmatrix}$$

$$K_Y = \begin{bmatrix} k_{11} & k_{11}+k_{12} & k_{12}+k_{13} & k_{13}+k_{14} \\ k_{11}+k_{12} & k_{11}+k_{12}+k_{21}+k_{22} & k_{12}+k_{13}+k_{22}+k_{23} & k_{13}+k_{14}+k_{23}+k_{24} \\ k_{21}+k_{31} & k_{31}+k_{32}+k_{32}+k_{33} & k_{22}+k_{23}+k_{32}+k_{33} & k_{23}+k_{24}+k_{33}+4_{34} \\ k_{31}+k_{41} & k_{31}+k_{32}+k_{41}+k_{42} & k_{32}+k_{33}+k_{42}+k_{43} & k_{33}+k_{34}+k_{43}+k_{44} \end{bmatrix}$$

$k_{11} = VAR[X_1]$ $\quad k_{22} = VAR(X_2)$ $\quad k_{33} = VAR(X_3)$ $\quad k_{44} = VAR(X_4)$

$k_{12} = k_{21} = COV(X_1, X_2)$ $\quad k_{13} = k_{31} = COV(X_1, X_3)$ ...

c) $k_{12} = k_{21} = k_{13} = k_{31} = \ldots = 0$

d)

$$m_Y = \begin{bmatrix} E[X_1] \\ E[X_1] + E[X_2] \\ \vdots \\ E[X_{n-1}] + E[X_n] \end{bmatrix} \qquad K_Y = \begin{bmatrix} k_{11} & k_{11}+k_{12} & & & 0 \\ k_{11}+k_{12} & k_{11}+k_{12}+k_{21}+k_{22} & & & 0 \\ 0 & & & \ddots & \vdots \\ \vdots & 0 & 0 \cdots & & 2(k_{n-1\,n-1}+k_{n-1\,n})+1 \end{bmatrix}$$

$k_{nn}+k_{n-1}k_{n-1}+2k_{n-1}k_n$

6.41

a) $$m_Y = Am_X = \begin{bmatrix} m+\frac{1}{2}m+\frac{1}{4}m+\frac{1}{8}m \\ m+\frac{1}{2}m+\frac{1}{4}m \\ m+\frac{1}{2}m \\ m \end{bmatrix} = \begin{bmatrix} 15/8\,m \\ 7/4\,m \\ 3/2\,m \\ m \end{bmatrix}$$

$$K_{XY} = K_X A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 \\ 1/8 & 1/4 & 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 \\ 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}$$

$$K_Y = AK_{XY} = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/4 & 1/2 & 1 & 0 \\ 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}$$

$$K_Y = \begin{bmatrix} 1.32812 & 0.65625 & 0.3125 & 0.125 \\ 0.65625 & 1.3125 & 0.625 & 0.25 \\ 0.3125 & 0.625 & 1.25 & 0.5 \\ 0.125 & 0.25 & 0.5 & 1 \end{bmatrix}$$

b) $$m_Y = Am_X = \begin{bmatrix} 4m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$K_{XY} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$K_Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

6.42

~~4.70~~ **a)**

$$\begin{aligned}\phi_V(w) &= E[e^{jwV}] = E[e^{jw(zX+bY+c)}] \\ &= e^{jwc}\phi_{X,Y}(aw, bw)\end{aligned}$$

b)

$$\begin{aligned}\phi_V(w) &= e^{jwc}\phi_{X,Y}(aw, bw) \\ &= e^{jwc}\phi_U(w_1 + 2w_1)\phi_V(w_1 + w_2)|_{w_1=aw, w_2=bw} \\ &= e^{jwc}\phi_U((a+2b)w)\phi_V((a+b)w) \\ &= e^{jwc}\exp\left[-\frac{1}{2}(aw+2bw)^2\right]\exp\left[-\frac{1}{2}(aw+bw)^2\right] \\ &= e^{jwc}\exp\left[-\frac{1}{2}w^2(a^2+4ab+4b^2+a^2+2ab+b^2)\right] \\ &= e^{jwc}\exp\left[-\frac{1}{2}w^2(2a^2+6ab+5b^2)\right] \\ f_V(v) &= \frac{1}{\sqrt{2\pi(2a^2+6ab+5b^2)}}\exp\left[-\frac{(v-c)^2}{2(a^2+6ab+5b^2)}\right]\end{aligned}$$

6.43

~~4.71~~ a)

$$
\begin{aligned}
\phi_{X,Y}(w_1,w_2) &= E[e^{jw_1X+jw_1Y}] \\
&= E\left[\exp\left[jw_1\left(\frac{1}{\sqrt{2}}V-\frac{1}{\sqrt{2}}W\right)+jw_2\left(\frac{1}{\sqrt{2}}V+\frac{1}{\sqrt{2}}W\right)\right]\right] \\
&= E\left[\exp\left[\left(j\frac{1}{\sqrt{2}}w_1+j\frac{1}{\sqrt{2}}w_2\right)V+\left(-j\frac{1}{\sqrt{2}}w_1+j\frac{1}{\sqrt{2}}w_2\right)W\right]\right] \\
&= \phi_V\left(\frac{1}{\sqrt{2}}w_1+\frac{1}{\sqrt{2}}w_2\right)\phi_W\left(-\frac{1}{\sqrt{2}}w_1+\frac{1}{\sqrt{2}}w_2\right) \\
&= \exp\left[-\frac{1}{2}(1+\rho\left(\frac{1}{\sqrt{2}}w_1+\frac{1}{\sqrt{2}}\right)^2\right] \\
&\quad \exp\left[-\frac{1}{2}(1-\rho)\left(-\frac{1}{\sqrt{2}}w_1+\frac{1}{\sqrt{2}}w_2\right)^2\right] \\
&= \exp\left[-\frac{1}{4}(1+\rho(w_1+w_2)^2-\frac{1}{4}(1-\rho(-w_1+w_2)^2\right] \\
&= \exp\left[-\frac{1}{2}(w_1^2+2w_1w_2\rho+w_2^2)\right]
\end{aligned}
$$

b)

$$
\begin{aligned}
\frac{\partial\phi_{X,Y}(w_1,w_2)}{\partial w_1^2\partial w_2} &= \frac{\partial}{\partial w_1^2}\left[(-w_1\rho-w_2)\exp\left(-\frac{1}{2}(w_1^2+2w_1w_2\rho+w_2^2)\right)\right] \\
&= \frac{\partial}{\partial w_1}\left\{-\rho\exp\left[-\frac{1}{2}(w_1^2+w_1w_2\rho+w_2^2)\right]\right. \\
&\quad \left.+(w_1\rho+w_2)(w_1+w_2\rho)\cdot\exp\left[-\frac{1}{2}(w_1^2+2w_1w_2\rho+w_2^2)\right]\right\} \\
&= p(w_1+w_2\rho)\exp\left[-\frac{1}{2}(w_1^2+2w_1w_2\rho)+w_2^2)\right] \\
&\quad +p(w_1+w_2\rho)\exp\left[-\frac{1}{2}(w_1^2+w_1w_2\rho+w_2^2)\right] \\
&\quad +(w_1\rho+w_1)\exp\left[-\frac{1}{2}(w_1^2+2w_1w_2\rho)+w_2^2\right] \\
&\quad -(w_1\rho+w_2)(w_1+w_2\rho)^2\exp\left[-\frac{1}{2}(w_1^2+w_1w_2\rho+w_2^2)\right] \\
E[X^2Y] &= \frac{1}{j^3}\frac{\partial\phi_{X,Y}(w_1,w_2)}{\partial w_1^2\partial w_2}\bigg|_{w_1=0,w_2=0} = 0
\end{aligned}
$$

c)

$$
\begin{aligned}
\phi_{X',Y'}(w_1,w_2) &= E[e^{jw_1X'+jw_2Y'}] \\
&= E[e^{jw_1(X+a)+jw_2(Y+b)}] \\
&= e^{jw_1a+jw_2b}\phi_{X,Y}(w_1,w_2) \\
&= \exp[jw_1a+jw_2b]\exp\left[-\frac{1}{2}(w_1^2+2w_1w_2\rho+w_2^2)\right]
\end{aligned}
$$

6.44

$$\begin{aligned}\begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} \\ \begin{pmatrix} U \\ V \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\end{aligned}$$

a)

$$\begin{aligned}\phi_{X,Y}(w_1,w_2) &= E[e^{jw_1X+jw_2Y}] \\ &= E[\exp[jw_1(aU+bV)+jw_2(cU+dV)]] \\ &= E[\exp[(jw_1a+jw_2c)U+(jw_1b+jw_2d)V]] \\ &= \phi_{U,V}(aw_1+cw_2, bw_1+dw_2)\end{aligned}$$

b)

$$\begin{aligned}\frac{\partial^2}{\partial w_1 \partial w_2}\phi_{X,Y}(w_1,w_2) &= \frac{\partial}{\partial w_1}\left[\frac{\partial \phi}{\partial u}c + \frac{\partial \phi}{\partial v}d\right] \\ &= c\left[\frac{\partial^2 \phi}{\partial u^2}a + \frac{\partial^2 \phi}{\partial u \partial v}b\right] + d\left[\frac{\partial \phi}{\partial u \partial v}a + \frac{\partial \phi}{\partial v^2}b\right]\end{aligned}$$

$$\begin{aligned}E[XY] &= \left.\frac{\partial^2}{\partial w_1 \partial w_2}\phi_{X,Y}(w_1,w_2)\right|_{w_1=w_2=0} \\ &= acE[U^2]+(bc+ad)E[UV]+bdE[V^2]\ .\end{aligned}$$

We check this result by direct calculation:

$$\begin{aligned}E[XY] &= E[(aU+bV)(cU+dV)] \\ &= acE[U^2]+(bc+ad)E[UV]+bdE[V^2]\end{aligned}$$

6.45

~~4.73~~ a) Poisson RV X_1 and X_2 with rates σ_1 and σ_2

$$G_1(z) = e^{+\sigma_1(z-1)}, \qquad G_2(z) = e^{+\sigma_2(z-1)}$$

X_1 and X_2 are independent

$$\begin{aligned}G_{X_1,X_2}(z_1,z_2) &= E[z_1^{X_1}z_2^{X_2}] = E[z_1^{X_1}]E[z_2^{X_2}] \\ &= G_1(z_1)G_2(z_2) \\ &= e^{+\sigma_1(z_1-1)+\sigma_2(z_2-1)}\end{aligned}$$

b) $G_1(z) = (q+ps)^n$, $G_2(z) = (q+pz)^m$

$$G_{X_1,X_2}(z_1,z_2) = G_1(z_1)G_2(z_2) = (q+pz_1)^n(q+pz_1)^m$$

6.46

~~4.74~~ a)

$$
\begin{aligned}
G_X(z) &= G_{X,Y}(z_1,z_2)\big|_{z_1=z,z_2=1} = e^{(\sigma_1+\beta)(z-1)} \\
G_Y(z) &= e^{(\sigma_2+\beta)(z-1)}
\end{aligned}
$$

$\therefore$ X and Y are Poisson RVs.

b) $G_Z(z) = E[z^{X+Y}] = G_{X,Y}(z_1,z_2)\big|_{z_1=z_2=z} = e^{(\sigma_2+\sigma_2)(z_1)+\beta(z^2-1)}$
Z is not a Poisson RV.

6.47

~~4.75~~ a)

$$
\begin{aligned}
G_{X,Y}(z_1,z_2) &= E[z_1^X z_2^Y] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} z_1^j z_2^k P[X=j,Y=k] \qquad j+k \le n \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} z_1^j z_2^k \frac{n!}{j!k!(n-j-k)!} p_1^j p_2^k (1-p_1-p_2)^{n-j-k} \\
&= \sum_{j=0}^{n} \frac{z_1^j n! p_1^j}{j!(n-j)!} \sum_{k=0}^{n-j} \frac{(n-j)! z_2^k}{k!(n-j-k)!} p_2^k (1-p_1-p_2)^{n-j-k} \\
&= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (p_1 z_1)^j (p_2 z_2 + 1 - p_1 - p_2)^{n-j} \\
&= (p_1 z_1 + p_2 z_2 + 1 - p_1 - p_2)^n
\end{aligned}
$$

b)

$$
\begin{aligned}
E[XY] &= \left.\frac{\partial G(z_1,z_2)}{\partial z_1 \partial z_2}\right|_{z_1=z_2=1} \\
&= \left.\frac{\partial}{\partial z_1} n p_2 (p_1 z_1 + p_2 z_2 + 1 - p_1 - p_2)^{n-1}\right|_{z_1=z_2=1} \\
&= p_1 p_2 n(n-1)
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \left.\frac{\partial G(z_1,1)}{\partial z_1}\right|_{z_1=1} \\
&= \left.\frac{\partial}{\partial z_1}(p_1 z_1 + 1 - p_1)^n\right|_{z_1=1} \\
&= n p_1
\end{aligned}
$$

$$
E[Y] = n p_2
$$

$$
\begin{aligned}
COV[X,Y] &= E[XY] - E[X]E[Y] \\
&= p_1 p_2 n^2 - p_1 p_2 n - n^2 p_1 p_2 \\
&= -n p_1 p_2
\end{aligned}
$$

6.48

$$G_{XY}(z_1,z_2) = e^{\alpha_1(z_1-1)+\alpha_2(z_2-1)+\beta(z_1z_2-1)}$$

$$\frac{\partial}{\partial z_1} G_{XY}(z_1,z_2) = e^{(\ )}\left[\alpha_1+\beta z_2\right]$$

$$\frac{\partial^2}{\partial z_2 \partial z_1} G_{XY}(z_1\, z_2)\Big|_{z_1=z_2=1} = e^{(\ )}\left[\alpha_1+\beta z_2\right]\left[\alpha_2+\beta z_1\right] + e^{(\ )}(\beta)$$

$$= (\alpha_1+\beta)(\alpha_2+\beta)+\beta$$

$$\frac{\partial^2}{\partial z_2 \partial z_1} G_{XY}(z_1,z_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} j z_1^{k-1}\, k z_2^{k-1}\, P[X=j,Y=k]$$

$$= E[XY]$$

$$\Rightarrow E[XY] = (\alpha_1+\beta)(\alpha_2+\beta)+\beta$$

$$COV(X,Y) = \alpha_1\alpha_2+\alpha_1\beta+\alpha_2\beta+\beta-\alpha_1\alpha_2$$

from 6.46 we know: X and Y are Poisson RV's

$E[X] = \alpha_1$ $VAR[X] = \alpha_1$

$E[Y] = \alpha_2$ $VAR[Y] = \alpha_2$

$$m_{\underline{X}} = \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \qquad K_{\underline{X}} = \begin{bmatrix}\alpha_1 & \beta(\alpha_1+\alpha_2+1)\\ \beta(\alpha_1+\alpha_2+1) & \alpha_2\end{bmatrix}$$

6.49 from 6.47

$E[X] = np_1 \qquad COV(X,Y) = -np_1p_2$

$E[Y] = np_2 \qquad VAR(X) = np_1(1-p_1)$

$VAR(Y) = np_2(1-p_2).$

$$m_{\underline{X}} = \begin{bmatrix} np_1 \\ np_2 \end{bmatrix} \qquad K_{\underline{X}} = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 \\ -np_1p_2 & np_2(1-p_2) \end{bmatrix}$$

6.50

a) $\det\begin{bmatrix} 1-\lambda & 1/4 \\ 1/4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1/16 = \lambda^2 - 2\lambda + 15/16 = \left(\lambda - \frac{5}{4}\right)\left(\lambda - \frac{3}{4}\right)$

$\lambda_1 = \frac{5}{4}$

$\lambda_2 = \frac{3}{4}$

$\begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{5}{4}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

$e_1 + \frac{1}{4}e_2 = \frac{5}{4}e_1 \qquad e_1 + \frac{1}{4}e_2 = \frac{3}{4}e_1$

$-\frac{1}{4}e_1 + \frac{1}{4}e_2 = 0 \qquad \frac{1}{4}e_1 + \frac{1}{4}e_2 = 0$

$e_1 = [1,1]^T$ - form $\qquad e_2 = [1,-1]^T$ - form

$e_1 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T \qquad e_2 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$

b) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad P^TP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$P^TK_kP = \left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5/4 & 0 \\ 0 & 3/4 \end{bmatrix}$

c) $X = PY = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} y_1+y_2 \\ y_1-y_2 \end{bmatrix}$

6.51

a) $\det K_X = -\lambda(\lambda - \frac{3}{2})^2$

$\lambda_1 = 0 \qquad e_1 = \frac{1}{\sqrt{3}}[1 \ 1 \ 1]^T$

$\lambda_2 = \frac{3}{2} \qquad e_2 = \frac{1}{\sqrt{6}}[1 \ -1 \ 2]^T$

$\lambda_3 = \frac{3}{2} \qquad e_3 = \frac{1}{\sqrt{2}}[1 \ -1 \ 0]^T$

b) $$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \qquad P^TP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T K_X P = P^T \begin{bmatrix} 0 & -3/2\sqrt{6} & 3/2\sqrt{2} \\ 0 & -3/2\sqrt{6} & -3/2\sqrt{2} \\ 0 & 3/\sqrt{6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

c) $$X = PY = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3 \\ \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 - \frac{1}{\sqrt{2}}y_3 \\ \frac{1}{\sqrt{3}}y_1 + \frac{2}{\sqrt{6}}y_2 \end{bmatrix}$$

6.52

$$K_X = E[(\bar{X} - \bar{m}_X)(\bar{X} - \bar{m}_X)^T]$$

$$\begin{aligned} \bar{a}^T K_X \bar{a} &= a^T E[(\bar{X} - \bar{m}_X)(\bar{X} - \bar{m}_X)^T]\, a \\ &= E[\bar{a}^T(\bar{X} - \bar{m}_X)(\bar{X} - \bar{m}_X)^T \bar{a}] \\ &= E[(\bar{a}^T(X - \bar{m}_X))^2] \geq 0 \end{aligned}$$

6.53

$\underline{b} = P^T \underline{a}$ where $K_{\underline{x}} = P\Lambda P^T$

(a)

$$\underline{b}^T K_{\underline{x}} \underline{b} = (P^T\underline{a})^T K_{x}(P\underline{a}) = \underline{a}^T P K_{x} P\underline{a}$$

$$= \underline{a}^T \Lambda \underline{a}$$

$$= \sum_{i=0}^{n} a_i^2 \lambda_i > 0 \quad \text{if } \underset{\text{all } i}{\lambda_i > 0} \text{ and at } \underset{\text{some } i}{a_i > 0}$$

$\Rightarrow K_{\underline{x}}$ is positive definite

(b) If K_x is positive definite, then

$a^T K_{\underline{x}} \underline{a} > 0$ for any nonzero vector

in particular if $\underline{a}$ is an eigenvector, then

$$0 < a^T K_{\underline{x}} \underline{a} = a^T \lambda_i \underline{a} = \lambda_i \underbrace{a^T a}_{>0}$$

$$\Rightarrow \lambda_i > 0$$

∴ all evalues of K are positive.

6.4 Jointly Gaussian Random Vectors

6.54

a) $$f_{\bar{X}}(\bar{x}) = \frac{e^{-\frac{1}{2}(\bar{x}-\bar{m}_x)^T K_x^{-1}(\bar{x}-\bar{m}_x)}}{2\pi\sqrt{2}}$$

$$\det[K] = 2$$

$$K_x^{-1} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \qquad \bar{m}_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) $$[x_1-1 \quad x_2]\begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}\begin{bmatrix} x_1-1 \\ x_2 \end{bmatrix} = [x_1-1 \quad x_2]\begin{bmatrix} 3(x_1-1)+x_2 \\ (x_1-1)+3x_2 \end{bmatrix}\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}\left[3(x_1-1)^2 + x_2(x_1-1) + x_2(x_1-1) + 3x_2^2\right]$$

$$= \frac{3}{4}\left[(x_1-1)^2 + x_2^2 + \frac{2}{3}x_2(x_1-1)\right]$$

$$f_{\bar{X}}(x) = \frac{e^{-\frac{1}{2}\left(\frac{3}{4}\right)\left[(x_1-1)^2 + x_2^2 + \frac{2}{3}x_2(x_1-1)\right]}}{2\pi\sqrt{2}}$$

c) $$f_{X_1}(x_1) = \frac{e^{-\frac{1}{2}(x_1-1)^2/(3/2)}}{\sqrt{2\pi}\sqrt{3/2}}$$

$$f_{X_2}(x_2) = \frac{e^{-\frac{1}{2}(x_2)^2/(3/2)}}{\sqrt{2\pi}\sqrt{3/2}}$$

d) $$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{for } K_x$$

$$A = P^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \qquad K_Y = \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad m_Y = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

e) $$\det[K_Y] = 2 \qquad K_Y^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$f_Y(y) = \frac{e^{-\frac{1}{2}\left[(y_1+\frac{1}{\sqrt{2}})^2 + (y_2+\frac{1}{\sqrt{2}})^2/2\right]}}{2\pi\sqrt{2}}$$

6.55

a) $\det[K_X] = \frac{3}{2}$

$$K^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \qquad m_X = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$f_{\bar{X}}(\bar{x}) = \frac{e^{-\frac{1}{2}(\bar{x}-\bar{m})^T K^{-1}(\bar{x}-\bar{m})}}{(2\pi)^{3/2}\sqrt{3/2}}$$

b) $$K^{-1}(\bar{x}-\bar{m}) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1-1 \\ x_2 \\ x_3-2 \end{bmatrix} = \begin{bmatrix} x_1-1-(x_3-2) \\ x_2 \\ -\frac{1}{3}(x_1-1)+(x_3-2) \end{bmatrix}$$

$$(\bar{x}-\bar{m})^T K^{-1}(\bar{x}-\bar{m}) = [x_1-1 \quad x_2 \quad x_3-2] \begin{bmatrix} x_1-1-(x_3-2) \\ x_2 \\ -\frac{1}{3}(x_1-1)+(x_3-2) \end{bmatrix}$$

$$= (x_1-1)^2 - (x_1-1)(x_3-2) + x_2^2 - \frac{1}{3}(x_1-1)(x_3-2) + (x_3-2)^2$$

$$f_{\bar{X}}(x) = \frac{e^{-\frac{1}{2}[(x_1-1)^2 + x_2^2 + (x_3-2)^2 - 4/3(x_1-1)(x_3-2)]}}{(2\pi)^{3/2}\sqrt{3/2}}$$

c) $$f_{X_1}(x_1) = \frac{e^{-\frac{1}{2}(x_1-1)^2/(3/2)}}{\sqrt{2\pi}\sqrt{3/2}} \qquad f_{X_2}(x_2) = \frac{e^{-\frac{1}{2}x_2^2}}{\sqrt{2\pi}} \qquad f_{X_3}(x_3) = \frac{e^{-\frac{1}{2}(x_3-2)^2/(3/2)}}{\sqrt{2\pi}\sqrt{3/2}}$$

d) $$P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$K_Y = \Lambda$
$A = P^T$

e) $\det(K_Y) = 2 \qquad m_Y = [0 \quad -1/\sqrt{2} \quad 3/\sqrt{2}]^T$

$$f_{\bar{Y}}(\bar{y}) = \frac{e^{-\frac{1}{2}[y_1^2 + (y_2+1/\sqrt{2})^2 + (y_3-3/\sqrt{2})^2/2]}}{(2\pi)^{3/2}\sqrt{2}}$$

6.56

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \qquad K_U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad m_U = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$K_Y = AK_{\cancel{u}}A^T = AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

b) $\det(K_Y) = 1$

$$K_Y^{-1} = \begin{bmatrix} +2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\cancel{u}^T K_Y^{-1} u^T = 2x^2 + 2y^2 + z^2 - 2xy - 2yz$$

$$f_Y(\bar{y}) = \frac{e^{-\frac{1}{2}[2x^2+2y^2+z^2-2xy-2yz]}}{(2\pi)^{3/2}}$$

c) $$f(y,z|x) = \frac{f(x,y,z)}{f_X(x)} = \frac{e^{-\frac{1}{2}[x^2+2y^2+z^2-2xy-2yz]}}{2\pi}$$

$$f_X(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}$$

d) $$f(z|x,y) = \frac{f(x,y,z)}{f(x,y)} = \frac{e^{-\frac{1}{2}[y^2+z^2-2yz]}}{\sqrt{2\pi}}$$

$$f(x,y) = \frac{e^{-\frac{1}{2}[2x^2+y^2-2xy]}}{2\pi} \qquad K' = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad (K')^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

6.57

a) $K_X = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $\qquad A = \begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$

$K_Y = AA^T = \begin{bmatrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&1\end{bmatrix}$ $\qquad K_Y(y_1,y_2,y_3) = \begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}$

b) $\det(K_Y) = 4$

$K_Y^{-1} = \frac{1}{4}\begin{bmatrix}3&-2&1\\-2&4&-2\\1&-2&3\end{bmatrix}$

$$f_Y(y_1,y_2,y_3) = \frac{e^{-\frac{1}{2}(\frac{1}{4})[3x_1^2+4x_2^2+3x_3^2-4x_1x_2-4x_2x_3+2x_1x_3]}}{(2\pi)^{3/2}\,(2)}$$

c) $$f(y_1,y_2) = \frac{e^{-\frac{1}{2}[2y_1^2+2y_2^2-2y_1y_2](\frac{1}{3})}}{2\pi\sqrt{3}}$$

$K' = \begin{bmatrix}2&1\\1&2\end{bmatrix}$ $\qquad \det(K') = 3$ $\qquad K'^{-1} = \frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$

$$f(y_1,y_3) = \frac{e^{-\frac{1}{2}(\frac{1}{2})[x^2+z^2)}}{2\pi\,(2)}$$

$K' = \begin{bmatrix}2&0\\0&2\end{bmatrix}$ $\qquad \det(K') = 4$

d) $P = \begin{bmatrix}1/2&-1/\sqrt{2}&1/2\\-1/\sqrt{2}&0&1/\sqrt{2}\\1/2&1/\sqrt{2}&1/2\end{bmatrix}$ $\qquad \Lambda = \begin{bmatrix}2-\sqrt{2}&0&0\\0&2&0\\0&0&2+\sqrt{2}\end{bmatrix}$ $\qquad \Lambda = K_Z$

$A = P^T$

6.58

$\underline{Y} = AR\underline{b} + \underline{N}$ $\quad A = \text{diag}[a_k] \quad a_k > 0$

$R = R^T$

$\underline{N}$ Gaussian $\underline{m}_{\underline{N}} = \underline{0} \quad K_{\underline{N}} = I$

ⓐ

$$\underline{m}_{\underline{Y}} = E[\underline{Y}] = E[AR\underline{b} + \underline{N}] = AR\underline{b} + E[\underline{N}] = AR\underline{b}$$

$$K_{\underline{Y}} = E[(\underline{Y} - \underline{m}_{\underline{Y}})(\underline{Y} - \underline{m}_{\underline{Y}})^T] = E[\underline{N}\,\underline{N}^T] = I$$

$$f_{\underline{Y}}(\underline{y}) = \frac{\exp\{-\frac{1}{2}(\underline{y} - AR\underline{b})^T(\underline{y} - AR\underline{b})\}}{(2\pi)^{k/2}}$$

ⓑ

$$\underline{Z} = (AR)^{-1}\underline{Y} = (AR)^{-1}((AR)\underline{b} + \underline{N})$$

$$= \underline{b} + (AR)^{-1}\underline{N}$$

$$\underline{m}_{\underline{Z}} = E[\underline{Z}] = \underline{b} + (AR)^{-1}\underbrace{E[\underline{N}]}_{\underline{0}} = \underline{b}$$

$$K_{\underline{Z}} = E[(\underline{Z} - \underline{m}_{\underline{Z}})(\underline{Z} - \underline{m}_{\underline{Z}})^T]$$

$$= E[(AR)^{-1}\underline{N}((AR)^{-1}\underline{N})^T]$$

$$= E[(AR)^{-1}\underline{N}\,\underline{N}^T(AR)^{-1T}]$$

$$= (AR)^{-1}\underbrace{E[\underline{N}\,\underline{N}^T]}_{I}(AR)^{-1T}$$

$$= R^{-1}A^{-1}(R^{-1}A^{-1})^T = (R^{-1}A^{-1})(\underbrace{A^{-1T}}_{\text{diagonal}})(\underbrace{R^{-1T}}_{\text{symmetric}}) = R^{-1}A^{-2}R^{-1}$$

$$K_{\underline{Z}}^{-1} = (R^{-1}A^{-2}R^{-1})^{-1} = RA^2R$$

$$f_{\underline{Z}}(\underline{z}) = \frac{\exp\{-\frac{1}{2}(\underline{z} - \underline{b})RA^2R(\underline{z} - \underline{b})^T\}}{(2\pi)^{k/2}\,|RA^2R|}$$

6.59

a) $K_2 = \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix}$ $Q_2 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(K_2) = \frac{3}{2}$

$K_3 = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}$ $Q_3 = \begin{bmatrix} 3/4 & 0 & -1/4 \\ 0 & 1 & 0 \\ -1/4 & 0 & 3/4 \end{bmatrix}$ $\det(K_3) = 2$

b) $f(x_3 | x_1, x_2) = \dfrac{f(x_1, x_2, x_3)}{f(x_1, x_2)}$

$$f(x_1, x_2, x_3) = \frac{e^{-\frac{1}{2}(\frac{1}{4})[3x_1'^2 + 4x_2^2 + 3x_3'^2 - 2(x_1')(x_3')]}}{\sqrt{2}\,(2\pi)^{3/2}}$$

$x_1' = (x_1 - 1)$
$x_3' = (x_3 - 2)$

$$f(x_1, x_2) = \frac{e^{-\frac{1}{2}[\frac{2}{3}x_1'^2 + x_2'^2]}}{(2\pi)\sqrt{3/2}}$$

$$f(x_3 | x_1, x_2) = \frac{e^{-\frac{1}{2}[\frac{1}{12}x_1'^2 + \frac{3}{4}x_3'^2 - \frac{1}{2}x_1'x_3']}}{\sqrt{2\pi}\,\sqrt{4/3}}$$

6.60

If we write out the quadratic form in the exponent we obtain:

$$\sum_{j=1}^{n}\sum_{k=1}^{n}Q_{jk}(x_j-m_j)(x_k-m_k)-\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}Q_{jk}(x_j-m_j)(x_k-m_k)$$

$$=Q_{nn}(x_n-m_n)^2+2(x_n-m_n)\sum_{j=1}^{n-1}Q_{jk}(x_j-m_j)$$

$$=Q_{nn}\left\{(x_n-m_n)^2+2(x_n-m_n)\frac{1}{Q_{nn}}\sum_{j=1}^{n-1}Q_{jk}(x_j-m_j)\right\}$$

$$=Q_{nn}\left\{(x_n-m_n)^2+2(x_n-m_n)B+B^2\right\}-Q_{nn}B^2$$

$$=Q_{nn}\left\{(x_n-m_n)+B\right\}^2-Q_{nn}B^2$$

where $B=\frac{1}{Q_{nn}}\sum_{j=1}^{n-1}Q_{jk}(x_j-m_j)$

In the first line, all the terms in the second summation are contained in the first summation. We then completed the square to obtain an expression involving x_k, its mean $m_k - B$, and its variance $1/Q_{kk}$. The term $Q_{kk}B^2$ is part of the normalization constant. We therefore conclude that:

$$f_{X_n}(x_n\,|\,x_1,\ldots,x_{n-1})=\frac{\exp\left\{-\frac{1}{2Q_{nn}}\left(x-m_n+\frac{1}{Q_{nn}}\sum_{j=1}^{n-1}Q_{jk}(x_j-m_j)\right)^2\right\}}{\sqrt{2\pi Q_{nn}}}$$

We see that the conditional mean of x_n is a linear function of the observations x_1, x_2,…, x_{n-1}.

6.61

a) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $m_Z = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $K_Z = AK_XA^T = \begin{bmatrix} 5 & 1 & 3/2 \\ 1 & 1 & 0 \\ 2 & 0 & 3/2 \end{bmatrix}$

$$f_Z(z) = \frac{e^{-\frac{1}{2}(z-3)^2/5}}{\sqrt{2\pi}\sqrt{5}}$$

b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $m_Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $K_Z = AK_YA^T = \begin{bmatrix} 14 & 5 & 6 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

$$f_Z(z) = \frac{e^{-\frac{1}{2}z^2/14}}{\sqrt{2\pi}\sqrt{14}}$$

c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $m_Z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $VAR[Z] = 10 = \sum_i \sum_j COV(Y_i, Y_j)$

$$f_Z(z) = \frac{e^{-\frac{1}{2}z^2/10}}{\sqrt{2\pi}\sqrt{10}}$$

6.62

$$\Phi_X(\omega) = e^{j\omega^T m - \frac{1}{2}\omega^T K \omega}$$

$$j[\omega_1 \ \ \omega_2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = j\omega_1$$

$$-\frac{1}{2}[\omega_1 \ \ \omega_2]\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = -\frac{1}{2}[\omega_1 \ \ \omega_2]\begin{bmatrix} 3/2\,\omega_1 - 1/2\,\omega_2 \\ -1/2\,\omega_1 + 3/2\,\omega_2 \end{bmatrix}$$

$$= -\frac{1}{2}\left[\frac{3}{2}\omega_1^2 - \omega_1\omega_2 + \frac{3}{2}\omega_2^2\right]$$

$$\Phi_X(\omega) = e^{j\omega_1 - \frac{1}{4}[3\omega_1^2 + 3\omega_2^2 - 2\omega_1\omega_2]}$$

6.63

a) $$[\omega_1 \ \omega_2 \ \omega_3]\begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix}$$

$$= \omega_1^2+\omega_2^2+\omega_3^2 - \omega_1\omega_2 - \omega_2\omega_3 - \omega_1\omega_3$$

$$\Phi_X(\omega) = e^{-\frac{1}{2}[\omega_1^2+\omega_2^2+\omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3]}$$

b) NO

6.64

~~4.85~~
$$\begin{aligned}
\phi_{X,Y}(w_1,w_2) &= E[e^{jw_1X}e^{jw_2Y}] \\
&= E[E[e^{jw_1X}e^{jw_2Y}|Y]] \\
&= E[e^{jw_2}E[e^{jw_1X}|Y]] \\
&= E[e^{jw_2Y}e^{jw_1[m_1+\rho\frac{\sigma_1}{\sigma_2}(Y-m_2)]-\frac{1}{2}w_2^2\sigma_1^2(1-\rho^2)}] \\
&= \exp(jw_1m_1 - jw_1\rho\frac{\sigma_1}{\sigma_2}m_2 - \frac{1}{2}w_1^2\sigma_1^2(1-\rho^2))E[e^{jw_2Y+jw_1\rho\frac{\sigma_1}{\sigma_2}Y}] \\
&= \exp(jw_1m_2 - jw_1\rho\frac{\sigma_1}{\sigma_2}m_2 - \frac{1}{2}w_1^2\sigma_1^2(1-\rho^2)) \\
&\quad \cdot \exp\left(j\left(w_2+w_1\rho\frac{\sigma_1}{\sigma_2}\right)m_2 - \frac{1}{2}\sigma_2^2\left(w_2+w_1\rho\frac{\sigma_1}{\sigma_2}\right)^2\right) \\
&= \exp(jw_1m_1 + jw_2m_2 - \frac{1}{2}w_1^2\sigma_1^2 - w_1w_2\rho\sigma_1\sigma_2 - \frac{1}{2}w_2^2\sigma_2^2)
\end{aligned}$$

6.65

$$\begin{aligned}
\phi_{X_1,\ldots,X_n}(w_1,\ldots,w_n) &= E[e^{j\mathbf{w}^TX}] \qquad \mathbf{w} = (w_1,w_2,\ldots,w_n)^T \\
&= \int \exp(j\mathbf{w}^TX)\frac{1}{(2\pi)^{n/2}|K^{1/2}}\exp\{-\frac{1}{2}(X-m)^TK^{-1}(X-m)\}dX \\
&= \frac{1}{(2\pi)^{n/2}|K|^{1/2}}\int \exp\{-\frac{1}{2}[X^TK^{-1}X - m^TK^{-1}X - jw^TX - X^TK^{-1}m \\
&\quad -X^TK^{-1}Kjw + (m+jkw)^TK^{-1}(m+jKw) \\
&\quad -(m+jKw)^TK^{-1}(m+jKw) + m^TK^{-1}m\}dX \\
&\qquad (w^TK = X^Tw = w_1x_1 + w_2x_2 + \ldots + w_nx_n) \\
&= \frac{1}{(2\pi)^{n/1}|K|^{1/2}}\int \exp\{-\frac{1}{2}(x-m-jw)^TK^{-1}(x-m-jw) \\
&\quad -jw^TK^TK^{-1}m - jm^TK^{-1}Kw + w^TK^TK^{-1}Kw\}dX \\
&= \exp[jw^Tm - \frac{1}{2}w^TKw] \\
&\qquad (w^Tm = m^Tw, \quad K = K^T)
\end{aligned}$$

6.66

~~1.87~~

$$
\begin{aligned}
\phi_{X,Y}(w_1,w_2) &= \exp\left(-\frac{1}{2}w_1^2\sigma_1^2-\frac{1}{2}w_1^2\sigma_2^2-w_1w_2\sigma_1\sigma_2\right)\\
E[X^2Y^2] &= \left.\frac{\partial^4}{\partial w_1^2\partial w_2^2}\phi_{X,Y}(w_1,w_2)\right|_{w_1=w_2=0}\\
\frac{\partial\phi}{\partial w_1} &= \phi\cdot(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)\\
\frac{\partial^2\phi}{\partial w_1^2} &= \phi(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)^2+\phi(-\sigma_1^2)\\
\frac{\partial^3\phi}{\partial w_1^2\partial w_2} &= \phi(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)^2(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)\\
&\quad+\phi\cdot 2(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)(-\sigma_1\sigma_2p)\\
&\quad+\phi(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)(-\sigma_1^2)\\
\frac{\partial^4\phi}{\partial w_1^2\partial w_2^2} &= \phi(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)^3(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)\\
&\quad+\phi\cdot 2(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)(-\sigma_1\sigma_2p)(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)\\
&\quad+\phi(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)(-\sigma_2^2)\\
&\quad+\phi\cdot(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)\cdot 2(-\sigma_1^2w_1-w_2\sigma_1\sigma_2p)(-\sigma_1\sigma_2p)\\
&\quad+\phi\cdot 2(-\sigma_1\sigma_2p)(-\sigma_1\sigma_2p)\\
&\quad+\phi(-\sigma_2^2w_2-w_1\sigma_1\sigma_2p)^2(-\sigma_1^2)+\phi(-\sigma_2^2)(-\sigma_1^2)\\
E[X^2Y^2] &= \left.\frac{\partial^4\phi(w_1,w_2)}{\partial w_1^2\partial w_2^2)}\right|_{w_1=w_2=0}=2(\sigma_1\sigma_2p)^2+\sigma_1^2\sigma_2^2\\
&= E[X^2]E[Y^2]+2E^2[XY]
\end{aligned}
$$

6.67

~~4.88~~ The joint characteristic function for (X_1, X_2, X_3, X_4) is:

$$\Phi_{\underline{X}}(\underline{w}) = e^{-\frac{1}{2}\underline{w}^T K \underline{w}}$$

where

$$\underline{w}^T K \underline{w} = (w_1, w_2, w_3, w_4)\,[E[X_i X_j]] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \sum_{i=1}^{4}\sum_{j=1}^{4} E[X_i X_j] w_i w_j$$

Expanding the exponential in a power series:

$$\Phi_{\underline{X}}(\underline{w}) = 1 - \frac{1}{2}\underline{w}^T K \underline{w} + \frac{1}{8}(\underline{w}^T K \underline{w})^2 + \ldots$$

From the moment theorem we know that $E[X_1 X_2 X_3 X_4]$ is the coefficient of $w_1 w_2 w_3 w_4$ in the above series. This coefficient appears in the third term:

$$\begin{aligned} \frac{1}{8}(\underline{w}^T K \underline{w})^2 &= \frac{1}{8}\left(\sum_{ij} E[X_i X_j] x_i x_j\right)\left(\sum_{i'j'} E[X_{i'} X_{j'}] w_{i'} w_{j'}\right) \\ &= \frac{1}{8}\sum_{ij}\sum_{i'j'} E[X_i X_j] E[X_{i'} X_{j'}] w_i w_j w_{i'} w_{j'} \end{aligned}$$

By grouping all terms that give $w_1 w_2 w_3 w_4$ we find

$$\begin{aligned} E[X_1 X_2 X_3 X_4] &= \frac{1}{8}[8E[X_1 X_2]E[X_3 X_4] + 8E[X_1 X_3]E[X_2 X_4] + 8E[X_1 X_4]E[X_2 X_3]] \\ &= E[X_1 X_2]E[X_3 X_4] + E[X_1 X_3]E[X_2 X_4] + E[X_1 X_4]E[X_2 X_3]\ . \end{aligned}$$

6.5 Estimation of Random Variables

6.68

(i)
$$\begin{array}{c|ccc} X \backslash Y & -1 & 0 & 1 \\ \hline -1 & 1/6 & 1/6 & \\ 0 & & & 1/3 \\ 1 & 1/6 & 1/6 & \end{array}$$

(ii)
$$\begin{array}{c|ccc} X \backslash Y & -1 & 0 & 1 \\ \hline -1 & 1/9 & 1/9 & 1/9 \\ 0 & 1/9 & 1/9 & 1/9 \\ 1 & 1/9 & 1/9 & 1/9 \end{array}$$

(iii)
$$\begin{array}{c|ccc} X \backslash Y & -1 & 0 & 1 \\ \hline -1 & 1/3 & & \\ 0 & & 1/3 & \\ 1 & & & 1/3 \end{array}$$

from Problem 5.61:

(i)
$\text{COV}(X,Y)=0$

$\text{VAR}(X) = (-1)^2\frac{1}{3}+(1)^2\frac{1}{3}$
$= \frac{2}{3}$

$\text{VAR}(Y) = \frac{2}{3}$

$\rho_{XY}=0$

(ii)
$\text{COV}(X,Y)=0$

$\text{VAR}(X)=\frac{2}{3}$

$\text{VAR}(Y)=\frac{2}{3}$

$\rho_{XY}=0$

(iii)
$\text{COV}(X,Y)=\frac{2}{3}$

$\text{VAR}(X)=\frac{2}{3}$

$\text{VAR}(Y)=\frac{2}{3}$

$\rho_{XY}=\dfrac{\frac{2}{3}}{\sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}}=1.$

ⓐ $\hat{Y}=\rho_{XY}\sigma_Y\dfrac{X-m_X}{\sigma_X}+m_X$
$=0$

$\hat{Y}=0$

$\hat{Y}=X$

ⓑ $\hat{Y}(-1)=E[Y|-1]=-\frac{1}{2}$
$\hat{Y}(0)=E[Y|0]=1$
$\hat{Y}(+1)=E[Y|+1]=-\frac{1}{2}$

$\hat{Y}(-1)=0$
$\hat{Y}(0)=0$
$\hat{Y}(+1)=0$

$\hat{Y}(-1)=-1$
$\hat{Y}(0)=0$
$\hat{Y}(+1)=+1$

ⓒ $P_Y(j|X=k)=\dfrac{P[X=k|Y=j]P[Y=j]}{P[X=k]}=P_X(k|Y=j]$

Since $P[Y=j]=1/3=P[X=k]$
$\Rightarrow$ ML and MAP estimates the same

$\hat{Y}(-1)=0$ or 1
$\hat{Y}(0)=1$
$\hat{Y}(+1)=0$ or 1

$\hat{Y}(-1)=0,1,$ or -1
$\hat{Y}(0)=$ "
$\hat{Y}(+1)=$ "

$\hat{Y}(-1)=-1$
$\hat{Y}(0)=0$
$\hat{Y}(+1)=+1$

6.68

ⓓ Mean Square Errors

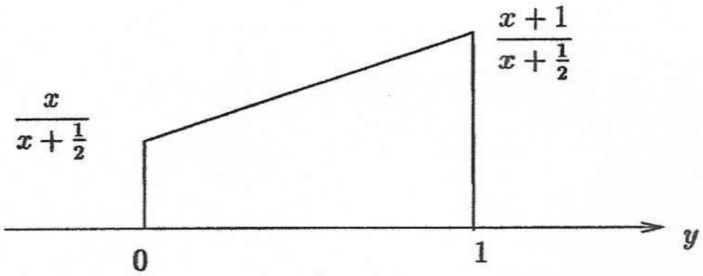
	(i)	(ii)	(iii)
Lin MSE	$(-1)^2\frac{1}{6}+(1)^2\frac{1}{6}=\frac{1}{3}$	$\left((-1)^2\cdot\frac{1}{9}+(1)^2\frac{1}{9}\right)2$ $=\frac{4}{9}$	0
MMSE	$\left(\frac{1}{2}\right)^2\frac{1}{6}\cdot 2\cdot 2$ $=\frac{1}{6}$	$\frac{4}{9}$	0
ML/MAP	$\frac{1}{3}$	$\frac{4}{9}$	0

6.69

~~4.90~~ a) $\hat{Y} = -\frac{1}{11}\left(x - \frac{7}{12}\right) + \frac{7}{12}$

$$\mathcal{E}[(Y-\hat{Y})^2] = VAR[Y](1-\rho^2) = \frac{11}{144}\left(1 - \left(\frac{1}{11}\right)^2\right) = \frac{55}{726} = .075$$

b)



$$f(y|x) = \frac{x+y}{x+\frac{1}{2}} \qquad 0 \le y \le 1$$

$$\max\ y = 1 \Rightarrow \hat{Y} = 1$$

$$\therefore\ \mathcal{E}[(Y-\hat{Y})^2] = \mathcal{E}[(Y-1)^2] = \mathcal{E}[Y^2] - 2\mathcal{E}[Y] + 1 = 0.25$$

c)
$$\begin{aligned}
\hat{Y} = \mathcal{E}[Y|x] &= \int_0^1 \frac{x+y}{x+\frac{1}{2}} y\,dy = \frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}} \\
\mathcal{E}[(Y-\hat{Y})^2] &= \int_0^1 dx \int_0^1 \left(y - \frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}\right)^2 (x+y)dy \\
&= \int_0^1 dx \int_0^1 \left[y^2 - 2\frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}} + \left(\frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}\right)^2\right](x+y)dy \\
\mathcal{E}[(Y-\hat{Y})^2] &= \int_0^1 \left[\left(\frac{x}{3}+\frac{1}{4}\right) - \frac{\frac{2}{2}+\frac{1}{3}}{x+\frac{1}{2}}\left(x+\frac{2}{3}\right) + \left(\frac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}\right)^2\left(x+\frac{1}{2}\right)\right] dx \\
&= \int_0^1 \left[\left(\frac{x}{3}+\frac{1}{4}\right) + \frac{-\left(\frac{x}{2}+\frac{1}{3}\right)\left(x+\frac{2}{3}\right) + \left(\frac{x}{2}+\frac{1}{3}\right)^2}{x+\frac{1}{2}}\right] dx \\
&= \int_0^1 \left(\left(\frac{x}{3}+\frac{1}{4}\right) - \frac{\left(\frac{x}{2}+\frac{1}{3}\right)^2}{x+\frac{1}{2}}\right] dx \\
&= \int_0^1 \left(\frac{x}{3}+\frac{1}{4}-\frac{x}{4}-\frac{5}{24}-\frac{\frac{1}{144}}{x+\frac{1}{2}}\right) dx \\
&= \int_0^1 \left(\frac{x}{12}+\frac{1}{24}-\frac{1}{144}\frac{1}{x+\frac{1}{2}}\right) dx \\
&= \frac{1}{24}+\frac{1}{24}-\frac{1}{144}\ln\left(x+\frac{1}{2}\right)\Big|_0^1 \\
&= \frac{1}{12}-\frac{1}{144}\ln 3 \\
&= 0.07570
\end{aligned}$$

This is slightly better than the linear predictor.

6.70) $X_i = s + N_i \quad i = 1, 2, 3$ $\quad N_i$ indep, zero mean unit variance Gaussian RV's

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(\sqrt{2\pi})^3} e^{-\frac{1}{2}\sum_{i=1}^{3}(x_i - s)^2}$$

To find maximum with respect to s, minimize the argument in the exponent:

$$0 = \frac{d}{ds}\sum_{i=1}^{3}(x_i - s)^2 = \sum_{i=1}^{3} 2(x_i - s)$$

$$\sum_{i=1}^{3} x_i = 3s$$

$$s = \frac{1}{3}\sum_{i=1}^{3} x_i$$

This is the sample mean of the three received signals.

6.71 from Problem 5.63

$E[N_1] = np \qquad E[N_2] = 2np.$

$VAR[N_1] = npq \qquad VAR[N_2] = 2npq$

$COV(N_1, N_2) = npq \qquad \rho_{N_1 N_2} = \frac{npq}{\sqrt{npq}\sqrt{2npq}} = \frac{1}{\sqrt{2}}$

ⓐ $$\hat{N}_2 = \rho_{N_1 N_2} \frac{\sigma_{N_2}}{\sigma_{N_1}}(N_1 - m_1) + m_2 = \frac{1}{\sqrt{2}} \frac{\sqrt{2npq}}{\sqrt{npq}}(N_1 - np) + 2np$$

$$\hat{N}_2 = N_1 + np$$ expected value of additional arrivals

ⓑ $$\hat{N}_2 = E[N_2 / N_1] = E[(N_2 - N_1) + N_1 / N_1]$$
$$= np + N_1$$ same as ⓐ

ⓒ $$P[N_2 = j+k \mid N_1 = j] = \frac{\binom{n}{k} p^k (1-p)^{n-k} \binom{n}{j} p^j (1-p)^{n-j}}{\binom{n}{j} p^j (1-p)^{n-j}}$$

$$= \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{\text{max at } k \doteq np} = P[N_2 = k]$$

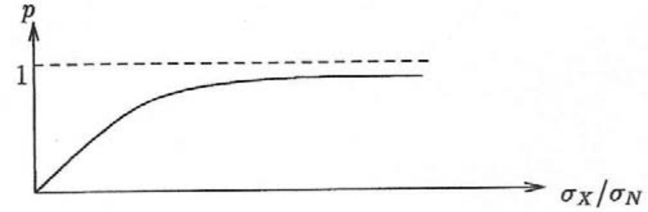
$$\hat{N}_2 = np + N_1$$

ⓓ $$\hat{N}_1 = \frac{1}{\sqrt{2}} \frac{\sqrt{npq}}{\sqrt{2npq}}(N_2 - 2np) + np = \frac{1}{2} N_2$$ Linear Est

$$p_{N_1}(j \mid N_2 = k) = \frac{\binom{n}{k-j} p^{k-j} (1-p)^{n-k+j} \binom{n}{j} p^j (1-p)^{n-j}}{\binom{2n}{k} p^k (1-p)^{2n-k}}$$

6.72 **a)** $COV[XY] = E[XY] = E[X(X+N)] = VAR[X] = \sigma_X^2$

$$\rho = \frac{COV[XY]}{\sigma_X \sigma_Y} = \frac{\sigma_X^2}{\sigma_X(\sigma_X^2 + \sigma_N^2)^{1/2}} = \left(\frac{1}{1 + \sigma_N^2/\sigma_X^2}\right)^{1/2}$$



b)

$$\begin{aligned}
\hat{X} &= \rho\frac{\sigma_X}{\sigma_Y}Y = \rho^2 Y \\
&= \frac{COV[XY]}{\sigma_Y^2}Y \\
&= \frac{\sigma_X^2}{\sigma_Y^2}Y \\
MSE &= E[(X-\hat{X})^2] \\
&= E[(X-\rho^2 Y)^2] \\
&= VAR[X^2] + \rho^4 VAR[Y] - 2\rho^2 COV[X,Y] \\
&= \sigma_X^2 + \rho^2\sigma_X^2 - 2\rho^2\sigma_X^2 \\
&= \sigma_X^2(1-\rho^2)
\end{aligned}$$

— continued —

6.72 From Example 6.26
the MAP estimator is the same as the MMSE estimator

On the other hand, the ML receiver is given by

$$\hat{X}_{ML} = \frac{\sigma_X}{\rho\sigma_Y}(Y - m_Y) + m_X = \frac{\sigma_X}{\rho\sigma_Y} Y$$

$$= \frac{\sigma_X\sqrt{1+\sigma_N^2/\sigma_X^2}}{\sqrt{\sigma_X^2+\sigma_N^2}} Y = Y.$$

Thus the ML estimator gives a different estimate.

The MSE for the ML estimator is

$$MSE_{ML} = E[(X - \hat{X}_{ML})^2]$$

$$= E[(X-Y)^2]$$

$$= E[N^2]$$

$$= \sigma_N^2$$

In comparison to the MAP estimator MSE we have

$$MSE_{MAP} = \sigma_X^2(1-\rho^2) = \sigma_X^2\left(1 - \frac{1}{1+\frac{\sigma_N^2}{\sigma_X^2}}\right) = \sigma_N^2 \underbrace{\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2}}_{<1}$$

$\therefore \quad MSE_{MAP} < MSE_{ML}$

6.73

$$f(x,y,z) = \frac{2}{3}(x+y+z) \qquad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$$

$$f(x,y) = \frac{2}{3}\left[x+y+\frac{1}{2}\right] \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

$$f(x) = \frac{2}{3}[x+1] \qquad 0 \le x \le 1$$

(a)

$$\Rightarrow E[X] = \frac{2}{3}\int_0^1 x(x+1)\,dx = \frac{2}{3}\left[\frac{1}{3}+\frac{1}{2}\right] = \frac{2}{3}\,\frac{5}{6} = \frac{5}{9} = E[Y] = E[Z]$$

$$E[X^2] = \frac{2}{3}\int_0^1 x^2(x+1)\,dx = \frac{2}{3}\left[\frac{1}{4}+\frac{1}{3}\right] = \frac{2}{3}\,\frac{7}{12} = \frac{7}{18}$$

$$VAR[X] = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{7}{18} - \frac{25}{81} = \frac{63-50}{162} = \frac{13}{162} = VAR[Y] = VAR[Z]$$

$$E[XY] = \frac{2}{3}\int_0^1\!\!\int_0^1 xy\left(x+y+\frac{1}{2}\right)dx\,dy = \frac{2}{3}\left[\int_0^1\!\!\int_0^1\left(x^2y + xy^2 + \frac{1}{2}xy\right)dx\,dy\right]$$

$$= \frac{2}{3}\int_0^1 dx\left[x^2\frac{1}{2} + x\,\frac{1}{3} + \frac{1}{2}x\,\frac{1}{2}\right] = \frac{2}{3}\underbrace{\left[\frac{1}{2}\,\frac{1}{3} + \frac{1}{3}\,\frac{1}{2} + \frac{1}{4}\,\frac{1}{2}\right]}_{\frac{4+4+3}{24}}$$

$$= \frac{11}{36}$$

$$COV(X,Y) = \frac{11}{36} - \left(\frac{5}{9}\right)^2 = \frac{99-100}{324} = \frac{-1}{324}$$ almost uncorrelated

$$= COV(X,Z) = COV(Y,Z)$$

The optimum linear estimator for Y given X and Z is:

$$\hat{X} = (a_1, a_2)\begin{bmatrix} X-m_X \\ Z-m_Z \end{bmatrix} + m_Y$$

where from Eqn 6.63a

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} VAR(X) & COV(X,Z) \\ COV(X,Z) & VAR(Z) \end{bmatrix}^{-1}}_{K_{XZ}^{-1}} \underbrace{\begin{bmatrix} COV(Y,X) \\ COV(Y,Z) \end{bmatrix}}_{E\left\{(Y-m_Y)\begin{bmatrix} X-m_X \\ Z-m_Z \end{bmatrix}\right\}}$$

6.73

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \left(\frac{1}{324} \begin{bmatrix} 26 & -1 \\ -1 & 26 \end{bmatrix} \right) \begin{bmatrix} -1/324 \\ -1/324 \end{bmatrix} = \frac{324}{705} \begin{bmatrix} 26 & 1 \\ 1 & 26 \end{bmatrix} \begin{bmatrix} -\frac{1}{324} \\ -\frac{1}{324} \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{27}{705} \\ \frac{27}{705} \end{bmatrix}$$

$$\bar{X}_{LinMSE} = -\frac{27}{705}[1, 1] \begin{bmatrix} X - \frac{5}{9} \\ Z - \frac{5}{9} \end{bmatrix} + \frac{5}{9}$$

$$= -\frac{27}{705}\left(X - \frac{5}{9}\right) - \frac{27}{705}\left(Z - \frac{5}{9}\right) + \frac{5}{9}$$

$$= -\frac{27}{705}(X+Z) + \frac{5}{9}\left(\frac{651}{705}\right)$$

fm Eqn 6.63b

$$MSE_{LMSE} = \sigma_X^2 - a^T \begin{bmatrix} \operatorname{cov}(YX) \\ \operatorname{cov}(YZ) \end{bmatrix} = \frac{13}{162} - \frac{27}{705}(1,1)\begin{bmatrix} -\frac{1}{324} \\ -\frac{1}{324} \end{bmatrix}$$

$$= \frac{13}{162} - \frac{54}{324(705)} = \frac{13}{162} - \frac{1}{235(6)} = 0.0795$$

ⓑ

$$f(y \mid x, z) = \frac{\frac{2}{3}(x+y+z)}{\frac{2}{3}\left(x+z+\frac{1}{2}\right)} \qquad 0 < y < 1$$

$$E[Y \mid x, z] = \frac{1}{x+z+\frac{1}{2}} \int_0^1 y(x+y+z)\,dy = \frac{x\frac{1}{2} + \frac{1}{3} + z\frac{1}{2}}{x+z+\frac{1}{2}} = \frac{\frac{1}{2}(x+z) + \frac{1}{3}}{x+z+\frac{1}{2}}$$

$$\hat{Y}_{MMSE} = \frac{\frac{1}{2}(X+Z) + \frac{1}{3}}{X+Z+\frac{1}{2}}$$

(b) continued —

6.73 from equations following Eqn 6.59

$$MSE_{MMSE} = \int_0^1\int_0^1 dx\,dz\; E[(Y-\hat{Y})^2 \mid x,z)\, f_{XZ}(x,z)$$

$$E[(Y-\hat{Y})^2 \mid x,z] = \int_0^1 (y - E[Y\mid x,z])^2\, \frac{\frac{2}{3}(x+y+z)}{x+z+\frac{1}{2}}\, dy$$

$$= E[Y^2\mid x,z] - 2E^2[Y\mid x,z] + E[Y\mid x,z]^2$$

$$E[Y^2\mid x,z] = \int_0^1 y^2\, \frac{x+y+z}{x+z+\frac{1}{2}}\, dy = \frac{\frac{1}{3}x + \frac{1}{4} + \frac{1}{3}z}{x+z+\frac{1}{2}}$$

$$E[(Y-\hat{Y})^2 \mid x,z] = \frac{\frac{1}{3}(x+z) + \frac{1}{4}}{x+z+\frac{1}{2}} - \left(\frac{\frac{1}{2}(x+z)+\frac{1}{3}}{x+z+\frac{1}{2}}\right)^2$$

$$= \frac{\frac{1}{12}\left[(x+z)^2 + (x+z) + \frac{1}{6}\right]}{\left((x+z)+\frac{1}{2}\right)^2}$$

$$E[(Y-\hat{Y})^2] = \frac{1}{12}\int_0^1\int_0^1 \frac{(x+z)^2 + (x+z) + \frac{1}{6}}{\left((x+z)+\frac{1}{2}\right)^2}\, \frac{2}{3}\left((x+z)+\frac{1}{2}\right) dx\,dz$$

$$= \frac{1}{18}\int_0^1\int_0^1 \frac{(x+z)^2 + (x+z) + \frac{1}{6}}{(x+z)+\frac{1}{2}}\, dx\,dz$$

$$= \frac{1}{18}\int_0^1\int_0^1 \left((x+z) + \frac{1}{2} - \frac{1/12}{(x+z)+\frac{1}{2}}\right) dx\,dz$$

$$= \frac{1}{18}\int_0^1 dx\left[x + \frac{1}{2} + \frac{1}{2} - \frac{1}{12}\underbrace{\ln\left(z+x+\tfrac{1}{2}\right)\Big|_0^1}_{\ln(x+\frac{3}{2}) - \ln(x+\frac{1}{2})}\right]$$

(b) · continued –

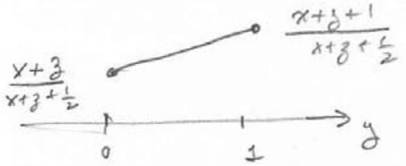
6.73

$$MSE_{MMSE} = \frac{1}{18}\left[\frac{1}{2} + 1 - \frac{1}{12}\left\{\underbrace{\left[\left(x+\frac{3}{2}\right)\ln\left(x+\frac{3}{2}\right) - x\right]_0^1}_{\left(\frac{5}{2}\ln\frac{5}{2} - 1 - \frac{3}{2}\ln\frac{3}{2}\right)} - \underbrace{\left[\left(x+\frac{1}{2}\right)\ln\left(x+\frac{1}{2}\right) - x\right]_0^1}_{\left(\frac{3}{2}\ln\frac{3}{2} - 1 - \frac{1}{2}\ln\frac{1}{2}\right)}\right\}\right.$$

$$\underbrace{\left(\frac{5}{2}\ln\frac{5}{2} - 1 - \frac{3}{2}\ln\frac{3}{2}\right) - \left(\frac{3}{2}\ln\frac{3}{2} - 1 - \frac{1}{2}\ln\frac{1}{2}\right)}_{\frac{5}{2}\ln\frac{5}{2} - 2\frac{3}{2}\ln\frac{3}{2} + \frac{1}{2}\ln\frac{1}{2}}$$

$$= \frac{1}{18}\left[\frac{3}{2} - \frac{1}{12}\left(\frac{5}{2}\ln\frac{5}{2} - 3\ln\frac{3}{2} + \frac{1}{2}\ln\frac{1}{2}\right)\right]$$

$$= 0.08187$$ larger than LW MSE (need to recheck both).



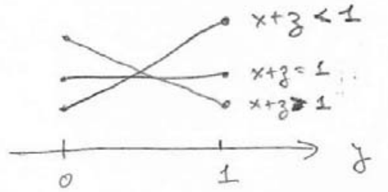
ⓒ MAP Estimator

$$f(y|x,z) = \frac{(x+y+z)}{(x+z+\frac{1}{2})} \qquad 0 < y < 1$$

$$\Rightarrow \hat{Y}_{MAP} = 1 \qquad MSE_{MAP} = E[(Y-1)^2] = E[Y^2] - 2E[Y] + 1$$

$$= \frac{7}{18} - 2\left(\frac{5}{9}\right) + 1 = .277$$

ML Estimator



$$f(x,z|y) = \frac{\frac{2}{3}(x+y+z)}{\frac{2}{3}(y+1)}$$

$$\therefore \hat{Y}_{ML} = \begin{cases} 1 & x+z<1 \\ 0 & x+z>1 \end{cases}$$

(6.73)

(d) $$MSE_{ML} = \iiint_{x+z<1} (y-1)^2 f(x,y,z)\,dx\,dy\,dz + \iiint_{x+z>1} y^2 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 dx \int_0^{1-x} dz \int_0^1 (y-1)^2 \tfrac{2}{3}(x+y+z)\,dy$$

$$+ \int_0^1 dx \int_{1-x}^1 dz \left(\int_0^1 y^2 \tfrac{2}{3}(x+y+z)\right) dy$$

$$= \frac{2}{3}\int_0^1 (y-1)^2 dy \int_0^1 dx \underbrace{\int_0^{1-x} dz\,(x+y+z)}_{x(1-x)+y(1-x)+(1-x)^2}$$

$$+ \frac{2}{3}\int_0^1 y^2 dy \int_0^1 dx \underbrace{\int_{1-x}^1 dz\,(x+y+z)}_{x(x)+yx+\frac{1}{2}\left(1-(1-x)^2\right)}$$

$$= \frac{2}{3}\int_0^1 dy\,(y-1)^2 \left(1+y-y\tfrac{1}{2}-\tfrac{1}{2}\right)$$

$$+ \frac{2}{3}\int_0^1 dy\, y^2 \left(\tfrac{2}{3}+\tfrac{1}{2}y\right)$$

$$= \frac{5}{36} + \frac{17}{54} = \frac{15+34}{108} = \frac{49}{108} = .453$$

6.74

$$E[X_1] = \int_0^1 x_1\,dx_1 = \frac{1}{2} \qquad E[X_1^2] = \int_0^1 x_1^2\,dx_1 = \frac{1}{3} \qquad VAR[X_1] = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$E[X_2] = E[\underbrace{E[X_2|X_1]}_{\frac{1}{2}X_1}] = \frac{1}{2}E[X_1] = \frac{1}{4}$$

given X_1, X_2 is unif in $[0, X_1]$

$$E[X_3] = E[\underbrace{E[X_3|X_2]}_{\frac{1}{2}X_2}] = \frac{1}{2}E[X_2] = \frac{1}{8}$$

$$E[X_2^2] = E[\underbrace{E[X_2^2|X_1]}_{\frac{X_1^2}{3}}] = \frac{1}{3}E[X_1^2] = \frac{1}{9} \qquad VAR[X_2] = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

$$E[X_3^2] = E[\tfrac{1}{3}X_2^2] = \frac{1}{3}E[X_2^2] = \frac{1}{27} \qquad VAR[X_3] = \frac{1}{27} - \frac{1}{64} = \frac{37}{\underbrace{27(64)}_{1728}}$$

$$E[X_1X_2] = E[X_1\underbrace{E[X_2|X_1]}_{\frac{1}{2}X_1}] = \frac{1}{2}E[X_1^2] = \frac{1}{6}$$

$$E[X_1X_3] = E[X_1\underbrace{E[\underbrace{E[X_3|X_2X_1]}_{\frac{1}{2}X_2}|X_1]}_{\frac{1}{4}X_1}] = \frac{1}{4}E[X_1^2] = \frac{1}{12}$$

$$E[X_2X_3] = E[X_2\underbrace{E[X_3|X_2]}_{\frac{1}{2}X_2}] = \frac{1}{2}E[X_2^2] = \frac{1}{18}$$

(a) Best Linear MSE Estimate

$$\boxed{\hat{X}_{2\,LMSE} = (a_1, a_2)\begin{bmatrix} X_1 - m_1 \\ X_3 - m_3 \end{bmatrix} + m_2 = -1.696\left(X_1 - \frac{1}{2}\right) + 2.786\left(X_3 - \frac{1}{8}\right) + \frac{1}{4}}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} - \frac{1}{2}\frac{1}{8} \\ \frac{1}{48} & \frac{37}{1728} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{6} - \frac{1}{2}\frac{1}{4} \\ \frac{1}{18} - \frac{1}{4}\frac{1}{8} \end{bmatrix} \doteq \begin{bmatrix} -1.696 \\ 2.786 \end{bmatrix}$$

6.74

$$f(x_1, x_2, x_3) = \frac{1}{x_1 x_2} \qquad 0 < x_3 < x_2 < x_1 < 1$$

$$f(x_1, x_3) = \int_{x_3}^{x_1} dx_2 \, \frac{1}{x_1 x_2} = \frac{1}{x_1}\left[\ln x_1 - \ln x_3\right] = \frac{1}{x_1} \ln \frac{x_1}{x_3}$$

$$f(x_2 | x_1, x_3) = \frac{\frac{1}{x_1 x_2}}{\frac{1}{x_1} \ln \frac{x_1}{x_3}} = \frac{1/x_2}{\ln x_1/x_3} \qquad x_3 < x_2 < x_1$$

decreases with increasing x_2

$$E[X_2 | x_1, x_3] = \frac{1}{\ln \frac{x_1}{x_3}} \int_{x_3}^{x_1} x_2 \frac{1}{x_2} \, dx_2 = \frac{x_1 - x_3}{\ln(x_1/x_3)}$$

(b)

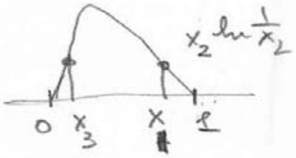
$$\boxed{\hat{X}_{2\,MMSE} = \frac{X_1 - X_3}{\ln(X_1/X_3)}} \qquad \boxed{\hat{X}_{2\,MAP} = \frac{1/x_3}{\ln X_1/X_3}}$$

from Prob. 6.13 : $f(x_2) = -\ln x_2$

(c)

$$0 < x_3 < x_2 < x_1 < 1$$

$$f(x_1, x_3 | x_2) = \frac{\frac{1}{x_1 x_2}}{-\ln x_2} = \frac{1/x_1}{x_2 \ln \frac{1}{x_2}}$$



$f(x_1, x_3 | x_2)$ is maximized at either x_3 or x_1 according to whether $x_3 \ln \frac{1}{x_3} \overset{x_3}{\underset{x_1}{\lessgtr}} x_1 \ln \frac{1}{x_1}$

$$\boxed{\hat{X}_{2\,ML} = \begin{cases} X_3 & \text{if } x_3 \ln \frac{1}{x_3} < x_1 \ln \frac{1}{x_1} \\ X_1 & \text{if } x_3 \ln \frac{1}{x_3} > x_1 \ln \frac{1}{x_1} \end{cases}}$$

ii

6.74 d

$$\hat{X}_3 = (a_1, a_2)\begin{bmatrix} X_1 - m_1 \\ X_2 - m_2 \end{bmatrix} + m_3$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{6} - \frac{1}{2}\frac{1}{4} \\ \frac{1}{24} & \frac{7}{144} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{12} - \frac{1}{2}\frac{1}{8} \\ \frac{1}{18} - \frac{1}{4}\frac{1}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\therefore \hat{X}_{3_{LMSE}} = \frac{1}{2}\left(X_2 - \frac{1}{4}\right) + \frac{1}{8} = \frac{1}{2}X_2$$

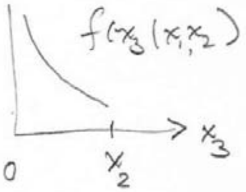
Correlation structure captures fact that X_3 does not depend on X_1 when X_2 is known!

$$f(x_3 \mid x_1 x_2) = \frac{\frac{1}{x_1 x_2}}{\frac{1}{x_1}} = \frac{1}{x_2} \qquad 0 < x_3 < x_2$$

$$E[X_3 \mid x_1, x_2] = \int_0^{x_2} x_3 \cdot \frac{1}{x_2}\, dx_2 = \frac{1}{x_2}\frac{x_2^2}{2} = \frac{x_2}{2}$$

$$\hat{X}_{3_{MMSE}} = \frac{1}{2}X_2 \quad \text{same as linear estimator}$$

$$\hat{X}_{3_{MAP}} = 0$$



$$f(x_1, x_2 \mid x_3) = \frac{\frac{1}{x_1 x_2}}{\frac{1}{2}(\ln x_3)^2} \qquad 0 < x_3 < x_2 < x_1 < 1$$

the pdf is maximized wrt x_3 when the denominator is minimized. This occurs at $X_3 = X_2$

$$\hat{X}_{3_{ML}} = X_2$$

6.75

ⓐ $f(\underline{y}|\underline{b}) = \frac{1}{\sqrt{2\pi}^k} e^{-\frac{1}{2}\sum_{j=1}^{k}(y_i-\alpha_i b_i)^2}$

$P[\underline{b}] = \left(\frac{1}{2}\right)^k$

$f(\underline{y}) = \sum_{\underline{b}} f(\underline{y}|\underline{b})\,P[\underline{b}] = \frac{1}{2}^k \sum_{\underline{b}} f(\underline{y}|\underline{b})$

ML Estimator maximizes

$$\max_{\underline{b}} f(\underline{y}|\underline{b}) = \max_{\underline{b}} \frac{1}{\sqrt{2\pi}^k} e^{-\frac{1}{2}\sum_{j=1}^{K}(y_i-\alpha_i b_i)^2}$$

which is equivalent to minimizing the exponent

$$\min_{\underline{b}} \sum_{j=1}^{K}(y_i-\alpha_i b_i)^2 \Rightarrow$$ find $A\underline{b}$ that is closest to $\underline{Y}$ in Euclidean distance.

MAP Estimator is same as ML estimator since $\underline{b}$ are equally probable.

ⓑ This problem involves finding a estimator for a vector of random variables $\underline{b}$ based on a vector of observations $\underline{Y}$. For linear MSE estimation we generalize Eqs 6.63ab, for $\underline{X}=\underline{b}$:

$$\hat{X} = A^T(\underline{Y}-\underline{m}_Y) + \underline{m}_{\underline{X}}$$

where

$A = K_Y^{-1}K_{XY}$ where $K_{XY} = E[(\underline{X}-m_{\underline{X}})(\underline{Y}-m_{\underline{Y}})^T]$

6.75 ii

$$K_{\underline{bY}} = E\left[(\underline{b}-\underline{m}_b)(\underline{Y}-\underline{m}_Y)^T\right]$$

$\underline{m}_b = \underline{0}$ since b_i equally likely to be ± 1.

$$E\left[\underline{b}(\underline{Y}-\underline{m}_Y)^T\right] = \left[\; E[B_i(Y_j - m_j)] \;\right]$$

Recall $Y_j = \alpha_j B_j + N_j$

$$\therefore \quad E[B_i(Y_j - m_j)] = \begin{cases} E[B_i(Y_i - m_i)] & i = j \\ E[B_i]E[(Y_j - m_j)] = 0 & i \neq j \end{cases}$$

$$E[B_i(Y_i - m_i)] = E[B_i(Y_i - m_i) \mid B_i = +1]\tfrac{1}{2} + E[B_i(Y_i - m_i) \mid B_i = -1]\tfrac{1}{2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}(y - m_i)\, e^{-(y-\alpha_i)^2/2}\, dy - \frac{1}{2}\int_{-\infty}^{\infty}(y_i - m_i)\, e^{-(y+\alpha_i)^2/2}\, dy$$

$$= \frac{1}{2}\left[\alpha_i - m_i\right] - \frac{1}{2}\left[-\alpha_i - m_i\right]$$

$$= \alpha_i$$

$$\therefore \quad K_{\underline{bY}} = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_K \end{bmatrix} = A$$

$$K_{\underline{Y}} = E\left[(\underline{Y}-\underline{m}_{\underline{Y}})(\underline{Y}-\underline{m}_{\underline{Y}})^T\right]$$

$$= E\left[\ E[(y_i-m_i)(y_j-m_j)]\ \right]$$

$$E[(y_i-m_o)(y_j-m_j)] = \begin{cases} E[(y_i-m_i)^2] & i=j \\ E[y_i-m_i]E[(y_j-m_j)]=0 & i\neq j \end{cases}$$

$$m_i = E[Y_i] = \underbrace{E[Y_i \mid B_i=+1]}_{+\alpha_i}\frac{1}{2} + \underbrace{E[Y_i \mid B_i=-1]}_{-\alpha_i}\frac{1}{2} = 0$$

$$E[(Y_i-m_i)^2] = E[Y_i^2]$$

$$= E[Y_i^2 \mid B_i=+1]\frac{1}{2} + E[Y_i^2 \mid B_i=-1]\frac{1}{2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} y^2 e^{-(y-\alpha_i)^2/2}\,dy + \frac{1}{2}\int_{-\infty}^{\infty} y^2 e^{-(y+\alpha_0)^2/2}$$

$$= \frac{1}{2}\left(1+\alpha_i^2\right) + \frac{1}{2}\left(1+(-\alpha_i)^2\right)$$

$$= 1+\alpha_i^2$$

$$K_Y = \begin{bmatrix} 1+\alpha_1^2 & & 0 \\ & 1+\alpha_2^2 & \\ 0 & & \ddots \ 1+\alpha_k^2 \end{bmatrix}$$

$$\hat{b}_{LMSE} = \begin{bmatrix} \frac{1}{1+\alpha_1^2} & & 0 \\ & \frac{1}{1+\alpha_2^2} & \\ 0 & & \ddots \ \frac{1}{1+\alpha_k^2} \end{bmatrix} \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_k \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_k \end{bmatrix}$$

$$\hat{b}_{MLSE} = \begin{bmatrix} \frac{\alpha_1}{1+\alpha_1} Y_1 \\ \vdots \\ \frac{\alpha_K}{1+\alpha_K} Y_K \end{bmatrix}$$

The decision/estimate for each component is based solely on the corresponding observation. There is no "coupling" between different channel components.

6.76 From problem 6.58

$$E[\underline{Y} \mid \underline{b}] = AR\underline{b} \Rightarrow E[\underline{Y}] = E[AR\underline{b}] = ARE[\underline{b}] = \underline{0}$$

Also

$$f(\underline{y} \mid \underline{b}) = \frac{\exp\left\{-\frac{1}{2}(\underline{y} - AR\underline{b})^T(\underline{y} - AR\underline{b})\right\}}{(2\pi)^{K/2}} = \exp\left\{-\frac{1}{2}\right.$$

$$= \frac{\exp\left\{-\frac{1}{2}(\underline{y} - \underline{b}')^T(\underline{y} - \underline{b})\right\}}{(2\pi)^{K/2}} \quad \text{where } \underline{b}' = AR\underline{b}$$

$$= \frac{\exp\left\{-\frac{1}{2}\sum_{j=1}^{K}(y_j - b_j')^2\right\}}{(2\pi)^{K/2}}$$

This has the same form as the pdf in problem 6.75

The ML estimator minimizes the exponent by finding

$$\min_{\underline{b}'} \sum_{j=1}^{K} (y_i - b_i') \Rightarrow \text{find } \underline{b}' \text{ that is closest to observation } \underline{Y} \text{ in Euclidean distance.}$$

MAP estimator is same as ML estimator.

The estimate for b is obtained from

$$\hat{b}_i = (AR)^{-1}\underline{b}_i'.$$

6.76

$$K_{\underline{b}\underline{Y}} = E\left[(\underline{b} - \underbrace{\underline{m}_{\underline{b}}}_{0})(\underline{Y} - m_{\underline{Y}})^T\right] = E[\underline{b}\,\underline{Y}^T]$$

$$= E\left[\underline{b}(AR\underline{b} + \underline{N})^T\right]$$

$$= E\left[\underline{b}(\underline{b}^T R^T A^T + \underline{N}^T)\right]$$

$$= \underbrace{E[\underline{b}\,\underline{b}^T]}_{I} R^T A^T + \underbrace{E[\underline{b}]E[\underline{N}^T]}_{0}$$

$$= R^T A^T$$

$$K_{\underline{Y}} = E[\underline{Y}\,\underline{Y}^T] = E\left[(AR\underline{b} + \underline{N})(AR\underline{b} + \underline{N})^T\right]$$

$$= AR\,E[\underline{b}\,\underline{b}^T] R^T A^T + \underbrace{E[\underline{N}\,\underline{N}^T]}_{I}$$

$$= ARR^T A^T + I$$

$$= AR^2 A + I$$ since R is symmetric

$$\hat{b}_{MSE} = (AR^2A + I)^{-1} R^T A^T \underline{Y}$$

This expression generalizes the result for estimating a signal from a noisy observation in Prob. 6.72,

6.77

From Eqns. 4.84, letting D correspond to X_1 and B to X_2, the best coefficients are:

a)

$$\begin{aligned} a &= \frac{\sigma_B^2 COV(D,E) - COV(B,D)COV(B,E)}{\sigma_D^2\sigma_B^2 - COV(B,D)^2} \\ &= \frac{\sigma^2(\sigma^2\rho) - \sigma^2\rho^2\sigma^2\rho}{\sigma^4 - \sigma^4\rho^4} = \frac{\rho - \rho^3}{1-\rho^4} = \frac{\rho}{1+\rho^2} \end{aligned}$$

and

$$\begin{aligned} b &= \frac{\sigma_D^2 COV(B,E) - COV(B,D)COV(D,E)}{\sigma_D^2\sigma_B^2 - COV(B,D)^2} \\ &= \frac{\sigma^2(\sigma^2\rho) - \sigma^2\rho^2(\sigma^2\rho)}{\sigma^4 - \sigma^4\rho^4} = \frac{\rho}{1+\rho^2} \end{aligned}$$

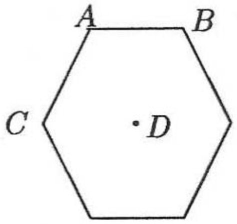
b)

$$\begin{aligned} \mathcal{E}[(E - \underbrace{(aD+bB)}_{\hat{E}})^2] &= \mathcal{E}[(E-\hat{E})(E-aD-bB)] \\ &= \mathcal{E}[(E-\hat{E})E] - a\,\underbrace{E[(E-\hat{E})D]}_{0} - b\,\underbrace{\mathcal{E}[(E-\hat{E})B]}_{0} \\ &\qquad \text{since error and observations are orthogonal} \\ &= \mathcal{E}[(E-aD-bB)E] \\ &= \mathcal{E}[E^2] - a\mathcal{E}[DE] - b\mathcal{E}[BE] \\ &= \sigma^2 - \frac{\rho}{1+\rho^2}\rho\sigma^2 - \frac{\rho}{1+\rho^2}\rho\sigma^2 \\ &= \sigma^2 - \frac{2\rho^2}{1+\rho^2}\sigma^2 = \sigma^2\left\{1 - \frac{2\rho^2}{1+\rho^2}\right\} \end{aligned}$$

6.78

$$\begin{aligned} \mathcal{E}[(X_3 - aX_1 - bX_2)^2] &= \mathcal{E}[X_3(X_3 - aX_1 - bX_2)] \\ &\quad - a\,\underbrace{\mathcal{E}[X_1(X_3 - aX_1 - bX_2)]}_{0} - b\,\underbrace{\mathcal{E}[X_2(X_3 0 aX_1 - bX_2)]}_{0} \\ &\qquad \text{since error and observations are orthogonal} \\ &= \mathcal{E}[X_3^2] - a\mathcal{E}[X_1X_3] - b\mathcal{E}[XX_3X_2] \\ &= \sigma^2 - \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}\rho_2\sigma^2 - \frac{\rho_1(1-\rho_2)}{1-\rho_1^2}\rho_1\sigma^2 \\ &= \sigma^2 - \frac{\rho_2^2 - \rho_1^3\rho_2 + \rho_1^2 - \rho_1^3\rho_2}{1-\rho_1^2}\sigma^2 \\ &= \sigma^2 - \frac{\rho_2^2 - 2\rho_1^2\rho_2 + \rho_1^4 + \rho_1^2 - \rho_1^4}{1-\rho_1^2}\sigma^2 \\ &= \sigma^2\left\{1 - \rho_1^2 - \frac{(\rho_1^2 - \rho_2)^2}{1-\rho_1^2}\right\} \quad \checkmark \end{aligned}$$

6.79



The performance of using $\{A, C\}$ or $\{A, B\}$ is the same.

a) Using $\{A, B\}$ assuming $E[X_i] = 0$

$\cdot x_1$ $\cdot x_3$

$\cdot x_3$

$$\begin{aligned}
a &= \frac{1 \cdot p - p \cdot p}{1 \cdot 1 - p^2} = \frac{p}{1+p} \\
b &= \frac{p}{1+p} \\
c &= E[(X_3 - aX_1 - bX_2)^2] \\
&= E[X_3^2] + a^2E[X_1^2] + b^2E[X_2^2] - 2aE[X_1X_3] + 2abE[X_1X_2] \\
&= 1 + a^2 + b^2 - 2ap - 2bp + 2abp \\
&= \frac{1 - p - 2p^2}{1+p}
\end{aligned}$$

b) Using $\{B, C\}$, assuming $e[X_i] = 0$

$\cdot x_1$

$\cdot x_1$ $\cdot x_3$

$$\begin{aligned}
a &= \frac{1 \cdot p - p^{\sqrt{e}} \cdot p}{1 \cdot 1 - p^{2\sqrt{3}}} = \frac{p}{1+p^{\sqrt{3}}} \\
b &= \frac{p}{1+p^{\sqrt{3}}} \\
e &= E[(X_3 - aX_1 - bX_2)^2] \\
&= E[X_3^2] + a^2E[X_1^2] + b^2E[X_2^2] - 2aE[X_1X_3] - 2abe[X_2X_3] + 2abe[X_2X - 3] \\
&= 1 + a^2 + b^2 - 2ap - 2bp + 2abp^{\sqrt{3}} \\
&= \frac{1 - p^{\sqrt{3}} - 2p^2}{1+p^{\sqrt{3}}}
\end{aligned}$$

$$\frac{1 - p - 2p^2}{1+p} > \frac{1 - p^{\sqrt{3}} - 2p^2}{1+p^{\sqrt{3}}} \qquad \text{for} \quad 0 < p < 1$$

We should use samples B and C to give a smaller predicttion error.

6.6 Generating Correlated Vector Random Variables

6.80

4.96 $K = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

$$\begin{aligned} \det|K - \lambda I| &= \lambda^2 - 6\lambda + 7 \\ \lambda_1, \lambda_2 &= 3 \pm \sqrt{2} \quad \text{eigenvalues} \end{aligned}$$

The orthonormal eigenvectors are:

$$\underline{e}_1 = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix} \qquad \underline{e}_2 = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{bmatrix} 1 \\ 1-\sqrt{2} \end{bmatrix}$$

$$P = [\underline{e}_1, \underline{e}_2] = \begin{bmatrix} .38268 & .92388 \\ .92388 & -.38268 \end{bmatrix}$$

$$\begin{aligned} A &= PD^{1/2} = \begin{bmatrix} .80401 & 1.16342 \\ 1.94107 & -.48190 \end{bmatrix} \\ A &= \begin{bmatrix} .80401 & 1.16342 \\ 1.94107 & -.48190 \end{bmatrix} \end{aligned}$$

Check

$$AA^{+} = \begin{bmatrix} .80401 & 1.16342 \\ 1.94107 & -.48190 \end{bmatrix}\begin{bmatrix} .80401 & 1.94107 \\ 1.16342 & -.48190 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \quad \checkmark$$

6.81

$K = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

Using $[P, D] = \text{eig}(K)$ we obtain

$$P = \begin{bmatrix} -0.92388 & 0.38268 \\ 0.38268 & 0.92388 \end{bmatrix} \quad D = \begin{bmatrix} 1.58579 & 0 \\ 0 & 4.41421 \end{bmatrix}$$

then

$$A = (P * \text{sqrt}(D))' = \begin{bmatrix} -1.16342 & 0.48191 \\ 0.80402 & 1.94107 \end{bmatrix}$$

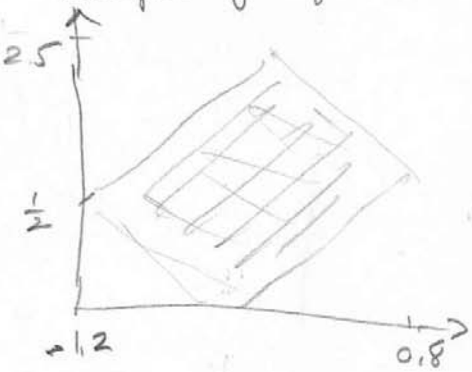
The desired transformation is then

$$\underline{Y} = A'\underline{X}$$

(a) To generate pairs with covariance K where $\underline{X}$ are iid uniform pairs from [0, 1]:

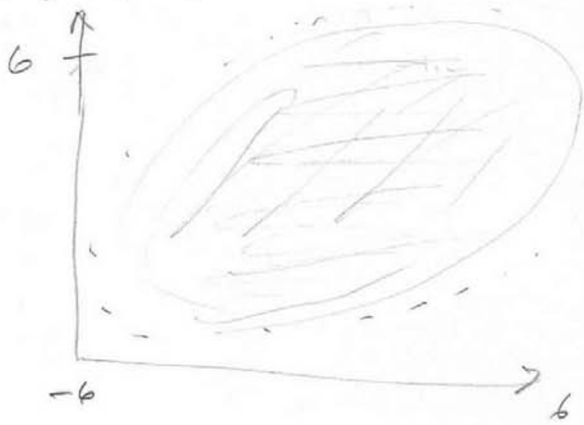
```
x=uniform_rnd(0,1,2,1000);
z= A'*x
plot( z(1,:), z(2,:),'+' )
```

1000 pairs of uniform iid RVs



(b)

```
x=normal_rnd(0,1,2,100)
z=A'*x
plot (z(1,:), z(2,:), '+')
```



jointly Gaussian RVs

6.82

$$\underline{m}_X = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad K_Y = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}$$

$[P, D] = \text{eig}(K)$ gives $P = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

First we need to generate $\underline{X}$: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$A = (P * \text{sqrt}(D))' =$

$$A' = \begin{bmatrix} 0 & 1/\sqrt{2} & 1 \\ -1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1 \end{bmatrix}$$

$\underline{X}$ = normal_rnd(0,1,3,1000)

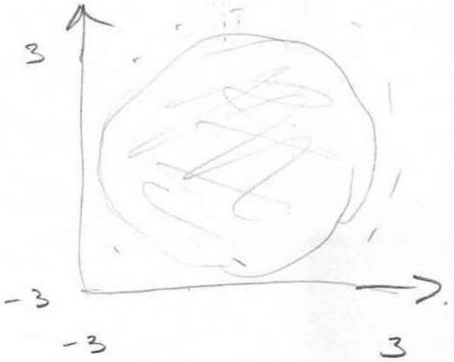
$\underline{Z} = A' * X$

plot(z(1,:), z(3,:), '+') ← plots x_1, x_3 scattergram

Next we need to obtain iid zero-mean unit variance Gaussian pairs:

$\underline{W} = A\underline{Z}$

plot(w(1,:), w(2,:), '+')



This is somewhat artificial since $AA^T = I$; but we cannot avoid the need to generate $\underline{X}$.

6.83 Let A be such that

$$A^T K_X A = \Lambda \Rightarrow K_X = A \Delta A^T$$

Consider

$$\underline{Y} = A^{-1}(\underline{x} - \underline{m}_x)$$

$$E[\underline{Y}] = A^{-1} E[\underline{x} - \underline{m}_x] = \underline{0}$$

$$K_Y = E[\underline{Y}\,\underline{Y}^T]$$

$$= E[A^{-1}(\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^T A^{-1T}]$$

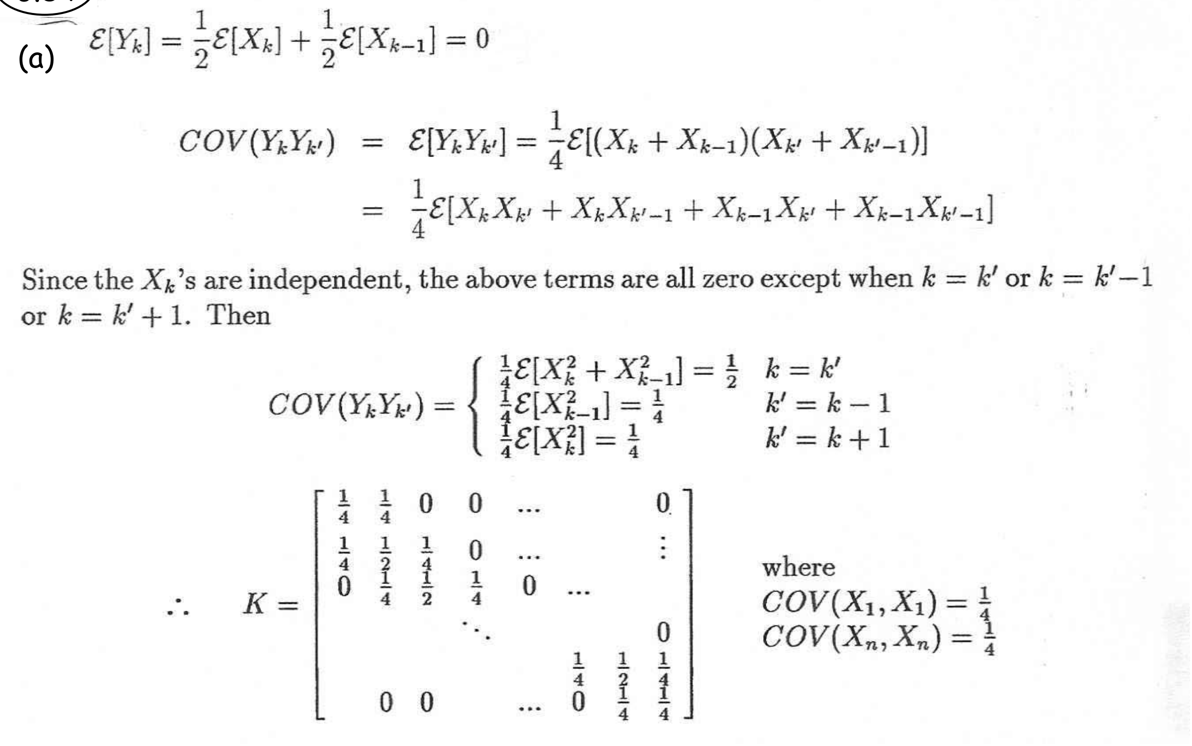
$$= A^{-1} K_X A^{-1T}$$

$$= A^{-1} \cancel{A} \Lambda A^T (A^T)^{-1}$$

$$= \Lambda = \mathrm{diag}[\lambda_i]$$

$$\therefore \quad f_Y(\underline{y}) = \frac{1}{(\sqrt{2\pi})^n} \exp\left\{ -\frac{1}{2} \sum_{i=1}^{n} y_i^2 / 2\lambda_i \right\}$$

6.84



(a) $\mathcal{E}[Y_k] = \frac{1}{2}\mathcal{E}[X_k] + \frac{1}{2}\mathcal{E}[X_{k-1}] = 0$

$$\begin{aligned} COV(Y_kY_{k'}) &= \mathcal{E}[Y_kY_{k'}] = \frac{1}{4}\mathcal{E}[(X_k + X_{k-1})(X_{k'} + X_{k'-1})] \\ &= \frac{1}{4}\mathcal{E}[X_kX_{k'} + X_kX_{k'-1} + X_{k-1}X_{k'} + X_{k-1}X_{k'-1}] \end{aligned}$$

Since the X_k's are independent, the above terms are all zero except when $k = k'$ or $k = k'-1$ or $k = k'+1$. Then

$$COV(Y_kY_{k'}) = \begin{cases} \frac{1}{4}\mathcal{E}[X_k^2 + X_{k-1}^2] = \frac{1}{2} & k = k' \\ \frac{1}{4}\mathcal{E}[X_{k-1}^2] = \frac{1}{4} & k' = k-1 \\ \frac{1}{4}\mathcal{E}[X_k^2] = \frac{1}{4} & k' = k+1 \end{cases}$$

$$\therefore \quad K = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 & \cdots & & & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & & & \vdots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & & \\ & & & \ddots & & & & 0 \\ & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & 0 & 0 & & \cdots & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

where
$COV(X_1, X_1) = \frac{1}{4}$
$COV(X_n, X_n) = \frac{1}{4}$

(b) The following Octave code generates a sequence of 1000 samples:

```
X = normal_rnd(0, 1, 1, 1000);
Y = (X + [0 X(1:length(X)-1)])./2;
cov(Y);
```

6.85

~~4.98~~

$$\begin{aligned}
\mathcal{E}[Y-K] &= \mathcal{E}[X_k - X_{k-1}] = 0 \\
COV(Y_k, Y_{k'}) &= \mathcal{E}[(X_k - X_{k-1})(X_{k'} - X_{k'-1})] \\
&= \mathcal{E}[X_k X_{k'}] - \mathcal{E}[X_k X_{k'-1}] - \mathcal{E}[X_{k-1} X_{k'}] + \mathcal{E}[X_{k-1} X_{k'-1}] \\
COV(Y_k Y_{k'}) &= 0 \text{ except when } k' = k, k-1, k+1 \text{ then} \\
COV(Y_k, Y_{k'}) &= \begin{cases} \mathcal{E}[X_k^2] + \mathcal{E}[X_{k-1}^2] = 2 & k = k' \\ -\mathcal{E}[X_k^2] = -1 & k' = k+1 \\ -\mathcal{E}[X_{k-1}^2] = -1 & k' = k-1 \end{cases} \\
COV(X_1, X_1) &= 1 = COV(X_n, X_n)
\end{aligned}$$

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & & \cdots & & \\ -1 & 2 & -1 & 0 & 0 & & \cdots & & \vdots \\ 0 & -1 & 2 & -1 & 0 & & \cdots & & \\ & & \ddots & & \ddots & & & & 0 \\ & & & & & 0 & -1 & 2 & -1 \\ 0 & 0 & & \cdots & & 0 & 0 & -1 & 1 \end{bmatrix}$$

6.86

Given $AK_xA^T = \Lambda$

Let $B = UA$ where U is an orthogonal matrix $UU^T = I$

$$\begin{aligned}
BK_xB^T &= UAK_x(UA)^T \\
&= UAK_xA^TU^T \\
&= U\Lambda U^T \\
&= \Lambda UU^T \\
&= \Lambda
\end{aligned}$$

6.87

$X = U_1$

$Y = U_1 + U_2$

$Z = U_1 + U_2 + U_3$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$E[\underline{U}] = \underline{0} \qquad K_{\underline{U}} = I$

$$K_{\underline{X}} = A K_{\underline{U}} A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Let $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.30798 & 0 & 0 \\ 0 & 0.64310 & 0 \\ 0 & 0 & 5.04892 \end{bmatrix}$

$$K_{\underline{X}} = A \Lambda^{-\frac{1}{2}} \Lambda \Lambda^{-\frac{1}{2}} A^T$$

$$= (A \Lambda^{-\frac{1}{2}}) \Lambda (\Lambda^{-\frac{1}{2}} A^T)$$

$$\underbrace{\Lambda^{\frac{1}{2}} A^{-1}} K_X \underbrace{A^{-1T} \Lambda^{\frac{1}{2}}} = \Lambda$$

$$\Lambda^{\frac{1}{2}} A^{-1} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ -\sqrt{\lambda_2} & \sqrt{\lambda_2} & 0 \\ 0 & -\sqrt{\lambda_3} & \sqrt{\lambda_3} \end{bmatrix}}_{\text{causal}}$$

6.88

$$K_X = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

(a)

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} \frac{3}{2}a & \frac{3}{2}b - \frac{1}{2}c \\ -\frac{1}{2}a & -\frac{1}{2}b + \frac{3}{2}c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\frac{3}{2}a^2 = 1 \qquad a = \sqrt{\frac{2}{3}}$$

$$\frac{3}{2}ab - \frac{1}{2}ac = 0 \implies 3b = c$$

$$\frac{3}{2}b^2 - \frac{1}{2}bc - \frac{1}{2}bc + \frac{3}{2}c^2 = 2$$

$$\frac{3}{2}b^2 - b(3b) + \frac{3}{2}(9b^2) = 2$$

$$12b^2 = 2 \implies b = \sqrt{\frac{1}{6}}$$

$$\implies c = 3\sqrt{\frac{1}{6}} = \sqrt{\frac{9}{6}} = \sqrt{\frac{3}{2}}$$

$$A = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{3}{2}} \end{bmatrix}$$

then $AK_XA^T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ✓

6.88

b

$$K_X = \begin{bmatrix} 3/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{bmatrix}$$

Consider the reduced matrix $\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$

Proceeding as in 6.88a

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} \frac{3}{2}a & \frac{3}{2}b+\frac{1}{2}c \\ \frac{1}{2}a & \frac{1}{2}b+\frac{3}{2}c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\frac{3}{2}a^2 = 1 \qquad a = \sqrt{\frac{2}{3}}$$

$$\frac{3}{2}ab + \frac{1}{2}ac = 0 \Rightarrow -3b = c$$

$$\frac{3}{2}b^2 - b(3b) + \frac{3}{2}(9b^2) = 2$$

$$12b^2 = 2 \Rightarrow b = \sqrt{\frac{2}{12}} = \sqrt{\frac{1}{6}}$$

$$\Rightarrow c = -3\sqrt{\frac{1}{6}} = -\sqrt{\frac{9}{6}} = -\sqrt{\frac{3}{2}}$$

$$A = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{3}{2}} \end{bmatrix}$$

then $A K_X A' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Problems Requiring Cumulative Knowledge

6.89)

a)

$$X_{n-1} = \tfrac{1}{2}U_{n-2} + \tfrac{1}{2}U_{n-1}$$
$$X_n = \tfrac{1}{2}U_{n-1} + \tfrac{1}{2}U_n$$
$$X_{n+1} = \tfrac{1}{2}U_n + \tfrac{1}{2}U_{n+1}$$
$$Z = U_{n+1}$$

U_n are iid zero-mean unit-variance Gauss RV

$$\underline{X}_a = \begin{bmatrix} X_{n-1} \\ X_n \\ X_{n+1} \\ Z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n-2} \\ U_{n-1} \\ U_n \\ U_{n+1} \end{bmatrix}$$

$$E[\underline{X}_a] = A\,E[\underline{U}] = \underline{0}$$

$$K_{\underline{X}_a} = A K_{\underline{U}} A^T = AA^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

We are only interested in

$$\underline{X} = \begin{bmatrix} X_{n-1} \\ X_n \\ X_{n+1} \end{bmatrix}$$ which are jointly Gaussian with $E[\underline{X}] = \underline{0}$

and $$K_{\underline{X}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

6.89 — continued —

$X_{n} = \frac{1}{2}U_{n-1} + \frac{1}{2}U_{n}$

$X_{n+m} = \frac{1}{2}U_{n+m-1} + \frac{1}{2}U_{n+m}$

$X_{n+2m} = \frac{1}{2}U_{n+2m-1} + \frac{1}{2}U_{n+2m}$

$E[X_{n+km}] = 0$

$VAR[X_{n+km}] = \frac{1}{2}VAR(U_{n-1} + U_n) = \frac{1}{2}\underbrace{2VAR(U)}_{1} = 1$

$\therefore$ X_n, X_{n+m}, X_{n+2m} are iid zero-mean, unit variance Gaussian RVs.

(b)

$$\underline{X}_a = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{n-2} \\ U_{n-1} \\ U_n \\ U_{n+1} \end{bmatrix}$$

$E[\underline{X}_a] = \underline{0}$

$$K_{\underline{X}_a} = AK_UA^T = AA^T = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ are jointly Gaussian with $m_{\underline{X}} = \underline{0}$ and

$$K_X = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

X_n, X_{n+m}, X_{n+2m} have the same joint pdf as in part a.

6.89

ⓒ

$$X_n = \frac{1}{2}U_{n-1} + \frac{1}{2}U_n$$
$$Y_n = -\frac{1}{2}U_{n-1} + \frac{1}{2}U_n$$

Assume $m > n+1$

$$X_m = \frac{1}{2}U_{m-1} + \frac{1}{2}U_m$$
$$Y_m = -\frac{1}{2}U_{m-1} + \frac{1}{2}U_m$$

$$\begin{bmatrix} X_n \\ Y_n \\ X_m \\ Y_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} U_{n-1} \\ U_n \\ U_{m-1} \\ U_m \end{bmatrix}$$

(X_n, Y_n, X_m, Y_m) are jointly Gaussian with zero mean

and
$$K_{\underline{X}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$
that is, they are $\underline{\underline{iid}}$.

ⓓ

$$\Phi_{\underline{X}}(\omega) = e^{j\underline{\omega}^T \underline{m} - \frac{1}{2}\underline{\omega}^T K \underline{\omega}}$$

a) $$\underline{\omega}^T K \underline{\omega} = (\omega_1, \omega_2, \omega_3) \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_1\omega_2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_2\omega_3 + \frac{1}{2}\omega_3^2$$

6.89

(b) $$\omega^T K \omega = (\omega_1, \omega_2, \omega_3) \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$= \frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_1\omega_2 + \frac{1}{2}\omega_2^2 - \frac{1}{2}\omega_2\omega_3 + \frac{1}{2}\omega_3^2$$

(c) $$\omega^T K \omega = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_3^2 + \frac{1}{2}\omega_4^2$$

6.90 a) $\hat{X}_2 = aX_1 = bX_3$.

We use the orthogonality principle

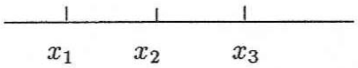
$$\begin{aligned} E[(X_2 - aX_2 - bX_3)X_1] &= 0 \\ E[(X_2 - aX_2 - bX_3)X_3] &= 0 \end{aligned}$$

i.e.

$$\begin{bmatrix} E[X_1^2] & E[X_1X_3] \\ E[X_1X_3] & E[X_3^2] \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E[X_1X_2] \\ E[X_2X_3] \end{bmatrix}$$

$$\begin{aligned} a &= \frac{VAR[X_3]COV(X_1,X_2) - COV(X_1,X_3)COV(X_2,X_3)}{VAR[X_1]VAR[X_3] - COV(X_1,X_3)^2} \\ b &= \frac{VAR[X_1]COV(X_2,X_3) - COV(X_1,X_2)COV(X_1,X_3)}{VAR[X_1]VAR[X_3] - COV(X_1,X_3)^2} \end{aligned}$$

b)



$$\begin{aligned} a &= \frac{\sigma^2\cdot\rho_1\sigma^2 - \rho^2\sigma^2\cdot\rho_1\sigma^2}{\sigma^2\cdot\sigma^2 - (\rho_2\sigma^2)^2} = \frac{\rho_1 - \rho_1\rho_2}{1-\rho_2^2} \\ b &= \frac{\sigma^2\cdot\rho_1\sigma^2 - \rho^1\sigma^2\cdot\rho_1\sigma^2}{\sigma^2\cdot\sigma^2 - (\rho_2\sigma^2)^2} = \frac{\rho_1 - \rho_1\rho_2}{1-\rho_2^2} \\ MSE &= E[(X_2 - aX_1 - bX_3)^2] \\ &= E[(X_2 - a(X_1+X_3))^2] \\ &= E[X_2^2] - 2aE[X_2(X_1+X_3)] + a^2E[(X_1+X_3)^2] \\ &= \sigma^2 - 2a(\rho_1\sigma^2 + \rho_1\sigma^2) + a^2(\sigma^2 + 2\rho_2\sigma^2 + \sigma^2) \\ &= \sigma^2[1 - 4a\rho_1 + 2a^2 + 2a^2\rho_2] \\ &= \sigma^2\frac{1-2\rho_1^2+\rho_2}{1+\rho_2} \\ &= \sigma^2\left(1 - \frac{2\rho_1^2}{1+\rho_2}\right) . \end{aligned}$$

The interpolation results in smaller MSE.

c) $e = X_2 - aX_1 - bX_3$, e is Gaussian

$$\begin{aligned} E[e] &= 0, \qquad VAR[e] = MSE = \sigma^2(1 - 2\rho_1^2/(1+\rho_2)) \\ f(e) &= \frac{1}{\sqrt{2\pi VAR[e]}}\exp\left[1\frac{e^2}{2\;VAR[e]}\right] \end{aligned}$$

6.91 $K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$

a) If we assume the signals are zero-mean, then the components of $\underline{X}$ correspond to the jointly Gaussian random variables in Ex. 4.13 which are transferred into an independent pair $\underline{Y}$ by the inner transformation given in Ex. 4.36:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

b) Consider how two consecutive blocks $\underline{X}_1$ are $\underline{X}_2$ are transformed into $\underline{Y}_1$ and $\underline{Y}_2$:

$$\begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} = \begin{bmatrix} A\underline{X}_1 \\ A\underline{X}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

which expanded gives:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \frac{1}{\sqrt{2}} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{A'} \begin{bmatrix} X_1 \\ X_2 \\ X-3 \\ X_4 \end{bmatrix}$$

The covariance matrix for $\underline{Y}$ is:

$$AKA^T = \begin{bmatrix} \sigma^2 + \rho\sigma^2 & 0 & \frac{\rho\sigma^2}{2} & -\frac{\rho\sigma^2}{2} \\ 0 & \sigma^2 - \rho\sigma^2 & \frac{\rho\sigma^2}{2} & -\frac{\rho\sigma^2}{2} \\ \frac{\rho\sigma^2}{2} & \frac{\rho\sigma^2}{2} & \sigma^2 + \rho\sigma^2 & 0 \\ -\frac{\rho\sigma^2}{2} & -\frac{\rho\sigma^2}{2} & 0 & \sigma^2 - \rho\sigma^2 \end{bmatrix}$$

It can be seen that the components of $\underline{Y}$ are not independent.

6.92 $X = X_1 + X_2 + \ldots + X_N$

$$\begin{aligned} m_a &= E[X] = Nm \\ \sigma_N^2 &= VAR[X] = N\ VAR[X_i] = N\sigma^2 \\ f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left[-\frac{(x-m_a)^2}{2\sigma_N^2}\right] \end{aligned}$$

a)

$$\begin{aligned} p_{Loss} &= \int_T^\infty f_X(x)dx \\ &= \int_{m_a + t\sigma_N}^\infty \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{(x-m_a)^2}{2\sigma_N^2}\right) dx \\ &= \int_{t\sigma_N}^\infty \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{x^2}{2\sigma_N^2}\right) dx \\ &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= Q(t) \end{aligned}$$

b)
$$\begin{aligned}
E[X_{Loss}] &= \int_T^\infty (x-T)f(x)dx \\
&= \int_T^\infty x\cdot f(x)dx - TQ(t) \\
&= \int_{m_a+t\sigma_N}^\infty x\frac{1}{\sqrt{2\pi}\sigma_N}\exp\left(-\frac{(x-m_a)^2}{2\sigma_N^2}\right)dx - TQ(t) \\
&= \int_{t\sigma_N}^\infty (y+m_a)\frac{1}{\sqrt{2\pi}\sigma_N}\exp\left(-\frac{y^2}{2\sigma_N^2}\right)dy - (m_a+t\sigma_N)Q(t) \\
&= \int_{t\sigma_N}^\infty y\frac{1}{\sqrt{2\pi}\sigma_N}\exp\left(-\frac{y^2}{2\sigma_N^2}\right)dy - t\sigma_N Q(t) \\
&= \int_{\sigma_N}^\infty \sigma_N u\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du - t\sigma_N Q(t) \\
&= \sigma_N\int_{\sigma_N}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)d\left(\frac{u^2}{2}\right) - t\sigma_N Q(t) \\
&= \frac{\sigma_N}{\sqrt{2\pi}}e^{-\frac{\sigma_N^2}{2}} - t\sigma_N Q(t)
\end{aligned}$$

c) fraction of bits lost $= \dfrac{\text{bits lost}/33\text{ ms}}{\text{bits produced}/33\text{ ms}} = \dfrac{E[X_{Loss}]}{m_a}$.

d)
$$\begin{aligned}
\text{Avg. \# bits allocated per source} &= \frac{m_a+t\sigma_N}{N} \\
&= \frac{Nm+t\sqrt{N}\sigma}{N} = m+t\frac{\sigma}{\sqrt{N}} \\
\text{Avg. \# bits lost per source} &= \frac{\frac{\sigma_N}{\sqrt{2\pi}}e^{-\frac{\sigma_N^2}{2}}+t\sigma_N Q(t)}{N} \\
&= \frac{\sigma e^{-\sqrt{N}\sigma}}{\sqrt{2\pi N}}+\frac{tQ(t)\sigma}{\sqrt{N}}
\end{aligned}$$

Both quantities decrease with N.

e) We need to keep
$$c = \text{constant} = \frac{E[X_{Loss}]}{m_a} = \frac{\frac{\sqrt{N}\sigma e^{-N\sigma^2/2}}{\sqrt{2\pi}} - t\sqrt{N}\sigma Q(t)}{Nm}$$

$$\begin{aligned}
tQ(t) &= \frac{1}{\sqrt{N}\sigma}\left[\frac{\sqrt{N}\sigma e^{-N\sigma^2/2}}{\sqrt{2\pi}} - Nmc\right] \\
&= \frac{e^{-N\sigma^2/2}}{\sqrt{2\pi}} - \sqrt{N}e\left(\frac{m}{\sigma}\right)
\end{aligned}$$

Solve this equation for t.

f) If $COV(X_i, X_j) = \rho$ then

$$\begin{aligned}
m_a &= Nm \\
\sigma_N^2 &= N\sigma^2 + N(N-1)\rho\sigma^2
\end{aligned}$$

The expressions in terms of m_a and σ_N^2 still hold. However, σ_N^2 now has a stronger dependence on N.

6.93

ⓐ

$$P[N_1 = j \mid T = t] = \frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t}$$

$$P[N_1 = j] = \int_0^\infty \frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t} \alpha e^{-\alpha t} dt$$

$$= \frac{\alpha \lambda_1^j}{j!} \underbrace{\int_0^\infty t^j e^{-(\alpha+\lambda_1)t} dt}_{\frac{j!}{(\alpha+\lambda_1)^j}}$$

$$= \left(\frac{\alpha}{\alpha+\lambda_1}\right)\left(\frac{\lambda_1}{\alpha+\lambda}\right)^j$$

geometric rv
$j = 0, 1, ..$

$$f_T[t \mid N = j] = \frac{\frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t} \alpha e^{-\alpha t}}{\left(\frac{\alpha}{\alpha+\lambda_1}\right)\left(\frac{\lambda_1}{\alpha+\lambda}\right)^j}$$

$$= \frac{(\alpha+\lambda_1)^{j+1}}{j!} t^j e^{-(\alpha+\lambda_1)t}$$

ML $$0 = \frac{d}{dt} P[N_1 = j \mid T = t] = \frac{j(\lambda_1 t)^{j-1}\lambda_1}{j!} e^{-\lambda t} + \frac{(\lambda_1 t)^j}{j!}\left(-\lambda e^{-\lambda t}\right)$$

$$1 = \frac{\lambda t}{j} \qquad \hat{t} = \frac{j}{\lambda}$$

MAP

$$0 = \frac{d}{dt} f[T = t \mid N_1 = j] = \cancel{c}\, \frac{j t^{j-1}}{j!} e^{-(\alpha+\lambda_1)t} + \frac{c\, t^j}{j!}(-\lambda_1 - \alpha)e$$

$$\Rightarrow \hat{t} = \frac{j}{\alpha+\lambda_1}$$

6.93 b for linear estimator need correlation

$$E[NT] = E\left[T\,E[N|T]\right] = E[\lambda_1 T^2]$$

$$= \lambda_1 E[T^2] = \lambda_1\left[\frac{1}{\alpha^2} + \left(\frac{1}{\alpha}\right)^2\right] = \lambda_1 \frac{2}{\alpha^2}$$

$$E[T] = \frac{1}{\alpha} \qquad E[N] = \frac{\frac{\lambda_1}{\alpha+\lambda}}{\frac{\alpha}{\alpha+\lambda}} = \frac{\lambda_1}{\alpha}$$

$$VAR[T] = \frac{1}{\alpha^2} \qquad VAR[N] = \frac{\frac{\lambda_1}{\alpha+\lambda_1}}{\left(\frac{\alpha}{\alpha+\lambda_1}\right)^2} = \frac{\lambda_1(\alpha+\lambda_1)}{\alpha^2}$$

$$COV(N,T) = \frac{2\lambda_1}{\alpha^2} - \frac{1}{\alpha}\frac{\lambda_1}{\alpha} = \frac{\lambda_1}{\alpha^2}$$

Min MSE Linear Estimator

$$\hat{T} = \frac{COV(N,T)}{VAR[N]}\left(N + E[N]\right) + E[T]$$

$$= \frac{\lambda_1/\alpha^2}{\lambda_1(\alpha+\lambda_1)/\alpha^2}\left(N - \frac{\lambda_1}{\alpha}\right) + \frac{1}{\alpha}$$

$$= \frac{1}{\alpha+\lambda_1} N - \frac{\lambda_1/\alpha}{\alpha+\lambda_1} + \frac{1}{\alpha}$$

(6.93) ⓒ

$$P[N_1=j, N_2=k] = \int_0^t \frac{(\lambda_1 t)^j}{j!}\frac{(\lambda_2 t)^k}{k!} e^{-(\lambda_1+\lambda_2)t} \alpha e^{-\alpha t} dt$$

$$= \frac{\alpha \lambda_1^j \lambda_2^k}{j!k!} \underbrace{\int_0^\infty t^{j+k} e^{-(\alpha+\lambda_1+\lambda_2)t} dt}_{\frac{(j+k)!}{(\alpha+\lambda_1+\lambda_2)^{j+k+1}}}$$

$$= \binom{j+k}{j}\left(\frac{\alpha}{\alpha+\lambda_1+\lambda_2}\right)\left(\frac{\lambda_1}{\alpha+\lambda_1+\lambda_2}\right)^j \left(\frac{\lambda_2}{\alpha+\lambda_1+\lambda_2}\right)^k$$

$j=0,1,\ldots$
$k=0,1,\ldots$

$$f_T(t|N_1=j,N_2=k) = \frac{\frac{(\lambda_1 t)^j}{j!}\frac{(\lambda_2 t)^k}{k!} e^{-(\lambda_1+\lambda_2)t} \alpha e^{-\alpha t}}{\binom{j+k}{j}\frac{\alpha \lambda_1^j \lambda_2^k}{(\alpha+\lambda_1+\lambda_2)^{j+k+1}}}$$

$$= \frac{(\alpha+\lambda_1+\lambda_2)^{j+k+1} t^{j+k} e^{-(\alpha+\lambda_1+\lambda_2)t}}{(j+k!)}$$

Same basic form as part ⓐ.
Following same procedure we find

$$\hat{T}_{ML} = \frac{j+k}{(\lambda_1+\lambda_2)} \qquad \hat{T}_{MAP} = \frac{j+k}{\alpha+\lambda_1+\lambda_2}$$

Chapter 7: Sums of Random Variables and Long-Term Averages

7.1 Sums of Random Variables

7.1 $\mathcal{E}[X+Y+Z] = \mathcal{E}[X] + \mathcal{E}[Y] + \mathcal{E}[Z] = 0$

a) From Eqn. 5.3 we have

$$\begin{aligned} VAR(X+Y+Z) &= VAR(X) + VAR(Y) + VAR(Z) \\ &\quad +2COV(X,Y) + 2COV(X,Z) + 2COV(Y,Z) \\ &= 1+1+1+2\left(\frac{1}{4}\right) + 2(0) + 2\left(-\frac{1}{4}\right) = 3 \end{aligned}$$

b) From Eqn. 5.3 we have

$$VAR(X+Y+Z) = VAR(X) + VAR(Y) + VAR(Z) = 3$$

7.2 $\mathcal{E}[S_n] = \mathcal{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathcal{E}[X_i] = n\mu$.

$$VAR(S_n) = \underbrace{\sum_{k=1}^{n} VAR(X_k)}_{\substack{\text{sum of diag.} \\ \text{elements of} \\ \text{covariance matrix } K}} + \underbrace{\sum_{j=1}^{n}\sum_{\substack{k=1 \\ j\neq k}}^{n} COV(X_j, X_k)}_{\substack{\text{sum of off-diag.} \\ \text{element of } K}}$$

$$K = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & 0 & & \dots & 0 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & 0 & \dots & 0 \\ & & & & & 0 \\ & \ddots & \ddots & \ddots & & \rho\sigma^2 \\ & & & \rho\sigma^2 & & \sigma^2 \end{bmatrix}$$

$\therefore VAR(S_n) = n\sigma^2 + 2(n-1)\rho\sigma^2$

7.3 Proceeding as in previous problem:

$$\begin{aligned}
\mathcal{E}[S_n] &= n\mu \\
K &= \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho^2\sigma^2 & \ldots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \ldots & \rho^{n-2}\sigma^2 \\ \vdots & & & & \\ \rho^{n-1}\sigma^2 & & \ldots & & \sigma^2 \end{bmatrix} \\
VAR(S_n) &= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1}\sum_{k=0}^{j-1} \rho^k \\
&= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1} \frac{1-\rho^j}{1-\rho} \\
&= n\sigma^2 + 2\rho\sigma^2 \left[\frac{n-1}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)\frac{1-\rho^{n-1}}{1-\rho}\right]
\end{aligned}$$

7.4 **a)** By Eqn. 7.7 we have

$$\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega) = e^{-\alpha|\omega|}e^{-\beta|\omega|} = e^{-(\alpha+\beta)|\omega|}$$

b) Taking the inverse transform:

$$f_Z(z) = \Phi_Z^{-1}(\omega) = \frac{1}{\lambda}\frac{\alpha+\beta}{(\alpha+\beta)^2+z^2} \Rightarrow Z$$

is also Cauchy

7.5 $\Phi_{S_k}(\omega) = \left(\frac{1}{1-2j\omega}\right)^{\frac{n_1}{2}} \left(\frac{1}{1-2j\omega}\right)^{\frac{n_2}{2}} \ldots \left(\frac{1}{1-2j\omega}\right)^{\frac{n_k}{2}} = \left(\frac{1}{1-2j\omega}\right)^{\frac{n_1+n_2+\ldots+n_k}{2}}$
$\Rightarrow S_k$ is chi-square RV with $n = n_1 + n_2 + \ldots + n_k$.

7.6

a) From Ex.4.34 X_i^2 is chi-square with one degree of freedom. From Prob. 7.5, S_n is then chi-square with n degrees of freedom

b)
$$\begin{aligned} T_n &= \sqrt{S_n} \\ \Rightarrow f_{T_n}(x) &= \frac{f_{S_n}(x^2)}{\frac{1}{2}|(x^2)^{-\frac{1}{2}}|} = 2x f_{X_n}(x^2) \\ &\text{Now use fact that } S_n \text{ is chi-square:} \\ &= \frac{2x(x^2)^{\frac{n-2}{2}}e^{-x^2/2}}{2^{n/2}\Gamma(n/2)} = \frac{x^{n-1}e^{-x^2/2}}{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)} \qquad x > 0 \end{aligned}$$

c) $f_{T_2}(x) = xe^{-x^2/2}$ $x > 0$

d) $f_{T_3}(x) = \dfrac{x^2e^{-x^2/2}}{2^{1/2}\Gamma\left(\frac{3}{2}\right)} = \dfrac{x^2e^{-x^2/2}}{\sqrt{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \sqrt{\dfrac{2}{\pi}}x^2e^{-x^2/2}$ $x > 0$

7.7

a) $\Phi_Z(\omega) = \left(\dfrac{\alpha}{\alpha - j\omega}\right)\left(\dfrac{\beta}{\beta - j\omega}\right)$

b) $\Phi_Z(\omega) = \dfrac{a}{\alpha - j\omega} + \dfrac{b}{\beta - j\omega}$ is partial fraction expansion, where $b = \dfrac{\alpha - \beta}{\alpha\beta}$, $a = \dfrac{\beta - \alpha}{\alpha\beta}$

$$\begin{aligned} \Phi_Z(\omega) &= \frac{\beta - \alpha}{\alpha^2\beta}\left(\frac{\alpha}{\alpha - j\omega}\right) - \frac{\beta - \alpha}{\alpha\beta^2}\left(\frac{\beta}{\beta - j\omega}\right) \\ \Rightarrow f_Z(t) &= \mathcal{F}^{-1}[\Phi_Z(\omega)] = \frac{\beta - \alpha}{\alpha\beta}e^{-\alpha t} - \frac{\beta - \alpha}{\alpha\beta}e^{-\beta t} \qquad t > 0 \end{aligned}$$

7.8

a) $\Phi_Z 9\omega) = \mathcal{E}[e^{j(aX+bY)}] = \mathcal{E}[e^{jaX}]\mathcal{E}[e^{-jbY}] = \Phi_X(a\omega)\Phi_Y(b\omega)$

b)
$$\begin{aligned} \mathcal{E}[Z] &= \frac{1}{j}\Phi'_Z(\omega)|_{\omega=0} = \frac{1}{j}\Phi'_X(a\omega)a\Phi_Y(b\omega)|_{\omega=0} + \frac{1}{j}\Phi'_Y(b\omega)b\Phi_X(a\omega|_{\omega=0} \\ &= a\mathcal{E}[X] + b\mathcal{E}[Y] \\ \mathcal{E}[Z^2] &= -\Phi''_Z(\omega) \\ &= -[\Phi''_X(a\omega)a^2\Phi_Y(b\omega) + 2\Phi'_X(a\omega)a\Phi'_Y(b\omega)b + \Phi_X(a\omega)\Phi_Y(b\omega)b^2]_{\omega=0} \\ &= a^2\mathcal{E}[X^2] + b^2\mathcal{E}[Y^2] - 2ab\mathcal{E}[X]\mathcal{E}[Y] \\ VAR[Z] &= \mathcal{E}[Z^2] - \mathcal{E}[Z]^2 = a^2VAR[X] + b^2\varepsilon[Y] \end{aligned}$$

7.9 ~~5.9~~

$$\begin{aligned}\Phi_{M_n}(\omega) &= \mathcal{E}[e^{j\omega M_n}] = \mathcal{E}[e^{j\omega\frac{1}{n}\sum_{j=1}^n X_j}]\\ &= \prod_{j=1}^{n}\mathcal{E}[e^{j\frac{\omega}{n}X_j}]\\ &= \left(\Phi_X\left(\frac{\omega}{n}\right)\right)^n\end{aligned}$$

7.10

$$\begin{aligned}G_{S_k}(z) &= \mathcal{E}[z^{X_1+\ldots+X_k}] = \mathcal{E}[z^{X_1}]\ldots\mathcal{E}[z^{X_k}] = G_{X_1}(z)\ldots G_{X_k}(z)\\ &= [pz+q]^{n_1}[pz+q]^{n_2}\ldots[pz+q]^{n_k}\\ &= [pz+q]^{n_1+\ldots+n_k}\end{aligned}$$

where the second equality follows from the independence of the X_i's. The result states that S_k is Binomial with parameters $n_1+\ldots+n_k$ and p. This is obvious since S_k is the number of heads in $n_1+\ldots+n_k$ tosses.

7.11

$$\begin{aligned}G_{S_k}(z) &= G_{X_1}(z)\ldots G_{X_k}(z) = e^{\alpha_1(z-1)}e^{\alpha_2(z-1)}\ldots e^{\alpha_k(z-1)}\\ &= e^{(\alpha_1+\ldots+\alpha_k)(z-1)}\end{aligned}$$

$\Rightarrow X_k$ Poisson with rate $\alpha_1+\ldots+\alpha_k$.

7.12) Note first that

$$\mathcal{E}[S/N=n]=\mathcal{E}\left[\sum_{k=1}^{n}X_k\right]=nE[X]\ ,$$

thus

$$\begin{aligned}\mathcal{E}[S] &= \mathcal{E}[\mathcal{E}[S/N]]=\mathcal{E}[N\mathcal{E}[X]]=\mathcal{E}[N]\mathcal{E}[X]\ .\\ \mathcal{E}[S^2] &= \mathcal{E}[\mathcal{E}[S^2/N]]\end{aligned}$$

which requires that we find

$$\begin{aligned}\mathcal{E}[S^2|N=n] &= \mathcal{E}\left[\sum_{i=1}^{n}X_i\sum_{j=1}^{n}X_j\right]=\sum_{i=1}^{n}\sum_{j=1}^{n}\mathcal{E}[X_iX_j]\\ &= n\mathcal{E}[X^2]+n(n-1)\mathcal{E}[X]^2\end{aligned}$$

since $E[X_iX_j]=\mathcal{E}[X^2]$ if $i=j$ and $E[X_iX_j]=\mathcal{E}[X]^2$ if $i\neq j$. Thus

$$\begin{aligned}\mathcal{E}[S^2] &= \mathcal{E}[N\mathcal{E}[X^2]+N(N-1)\mathcal{E}[X]^2]\\ &= \mathcal{E}[N]\mathcal{E}[X^2]+\mathcal{E}[N^2]\mathcal{E}[X^2]-\mathcal{E}[N]\mathcal{E}[X]^2\end{aligned}$$

Then

$$\begin{aligned}VAR(S) &= \mathcal{E}[S^2]-\mathcal{E}[S]^2\\ &= \mathcal{E}[N]\mathcal{E}[X^2]+\mathcal{E}[N^2]\mathcal{E}[X^2]-\mathcal{E}[N]\mathcal{E}[X]^2-\mathcal{E}[N]^2\mathcal{E}[X]^2\\ &= \mathcal{E}[N]VAR[X]+VAR[N]\mathcal{E}[X]^2\end{aligned}$$

b) First note that

$$\mathcal{E}[z^S/N=n]=\mathcal{E}[z^{\sum_{i=1}^{n}X_i}]=\mathcal{E}[z^{X_1}]...\mathcal{E}[z^{X_n}]=G_X(z)^n$$

Then

$$\begin{aligned}\mathcal{E}[z^S] &= \mathcal{E}[\mathcal{E}[z^S|N]\\ &= \mathcal{E}[G_X^N(z)]\\ &= \mathcal{E}[\omega^N]_{\omega=G_X(z)}\\ &= G_N(G_X(z))\end{aligned}$$

7.13)

$X_j = \begin{cases} 500 & 1/2 \\ 1000 & 1/2 \end{cases}$ $E[X] = 750$, $E[X^2] = 625000$

a) $R = \sum_{j=1}^{N} X_j$, from problem 7.12, we have : $E[R] = E[N]\,E[X]$

So, $E[R] = 750L$,

Also $VAR(R) = E[N]\,VAR[X] + VAR[N]\,E[X]^2$. Since $E[N] = VAR[N] = L$

then $VAR(R) = L\left[VAR[X] + E^2[X]\right] = L\,E[X^2] = 625000L$

b) $G_R = G_N(G_X(z))$ (P.7.12) , $G_X(z) = \frac{1}{2}(z^{500} + z^{1000})$, $G_N(z) = e^{L(z-1)}$

Therefore: $G_R(z) = e^{L(\frac{1}{2}z^{500} + \frac{1}{2}z^{1000} - 1)}$

7.14

N # of widgets tested in 1-hour $N \sim \text{Binomial}(600, p)$

$X_i \sim \text{Bernoulli}(a)$

$S = \sum_{i=1}^{N} X_i$

a) from P 7.12: $E[S] = E[N]\;E[X] = np \times a = 600pa$

$VAR(S) = E[N]\,VAR[X] + VAR[N]\,E[X]^2 = 600p(1-a)a + 600p(1-p)a^2$

b) from P.7.12

$G_S(z) = G_N(G_X(z))$, $G_N(z) = (1-p+pz)^{600}$, $G_X(z) = (1-a+az)$

therefore $G_S(z) = (1-p+p(1-a+az))^{600} = (1-pa+paz)^{600}$

7.2 The Sample Mean and the Laws of Large Numbers

7.15

$$\begin{aligned} P\left[\left|\frac{N(t)}{t}-\lambda\right| \geq \varepsilon\right] &= P[|N(t)-\lambda t| \geq \varepsilon t] \\ &\leq \frac{VAR[N(t)]}{(\varepsilon t)^2} \quad \text{by Chebyshev Inq.} \\ &= \frac{\lambda t}{\varepsilon^2 t^2} = \frac{\lambda}{\varepsilon^2 t} \end{aligned}$$

7.16

~~5.16~~ $p = \frac{\overset{2}{\cancel{1}}}{10}$

$$P[|f_A(n)-p| < \varepsilon] \geq 1 - \frac{p(1-p)}{n\varepsilon^2} = 0.95$$

letting $p = \frac{\overset{2}{\cancel{1}}}{10}$, $\varepsilon = \frac{1}{50} \Rightarrow n =$ 8000

7.17

$$\begin{aligned} M_{\underset{20}{\cancel{100}}} &= \frac{1}{20\cancel{100}}(X_1 + \dots + X_{100}) = \frac{1}{20\cancel{100}} S_{\cancel{100}\,20} \\ \mu &= \mathcal{E}[X] = \frac{1+2+\dots+6}{6} = 3.5 \\ \sigma_X^2 &= \frac{1}{\overset{}{\cancel{7}}6}(1+s^2+3^2+4^2+5^2+6^2)-(3.5)^2 = 2.91667 \end{aligned}$$

$$\begin{aligned} P[\overset{60}{\cancel{300}} < S_{100} < \overset{80}{\cancel{400}}] &= \left[3 < \overset{M_{100}}{\frac{S_{\cancel{100}}{}^{20}}{\cancel{100}\,20}} < 4\right] \\ &= P[-.5 < M_{\overset{20}{\cancel{100}}} - 3.5 < .5] \\ &= P[|M_{\overset{20}{\cancel{100}}} - 35| < .5] \\ &\geq 1 - \frac{2.92}{20\cancel{100}\left(\frac{1}{2}\right)^2} = 0.416 \end{aligned}$$

7.18 For $n = 16$, Eqn. 7.20 gives $\sigma_x^2 = 1$

$$P[|M_{16} - 0| < \varepsilon] \geq 1 - \frac{1^2}{16\varepsilon^2} = 1 - \frac{1}{16}\frac{1}{\varepsilon^2}$$

Since M_{16} is Gaussian with mean 0 and variance $\frac{1}{16}$

$$\begin{aligned} P[|M_{16} - 0| < \varepsilon] &= P[-\varepsilon < M_{16} < \varepsilon] = 1 - 2Q(\sqrt{16}\varepsilon) \\ &= 1 - 2Q(\underset{4}{\cancel{3.16}}\varepsilon) \end{aligned}$$

Similarly for $n = \overset{81}{1\cancel{0}0}$ we obtain

$$\begin{aligned} P[|M_{81} - 0] < \varepsilon] &\geq 1 - \frac{1}{800}\frac{1}{\varepsilon^2} \\ P[|M_{81} - 0] < \varepsilon] &= 1 - 2Q(9\varepsilon) \end{aligned}$$

For example if $\varepsilon = \frac{1}{2}$

$$\begin{aligned} P[|M_{16} < \frac{1}{2}] &\geq 1 - \frac{1}{16/4} = .75 \\ P[|M_{16}| < \frac{1}{2}] &= 1 - 2Q(2.0) = 1 - 2(\overset{2.28 \times 10^{-2}}{\cancel{5.44(10^{-2})}})) = \cancel{.89}\,0.954 \\ P[|M_{81}| < \frac{1}{2}] &\geq 1 - \frac{1}{81/4} = .951 \\ P[|M_{81}| < \frac{1}{2}] &= 1 - 2Q(\underset{4.5}{\cancel{5}}) = 1 - 2(\overset{3.4}{\cancel{2.87}})10^{-6} \end{aligned}$$

Note the significant discrepancies between the bounds and the exact values.

7.19

$$\begin{aligned} P\left[\left|\frac{1}{n}S_n - \mu\right| > \varepsilon\right] &\leq \frac{VAR(\frac{1}{n}S_n)}{\varepsilon^2} = \frac{VAR(S_n)}{n^2\varepsilon^2} \\ &= \frac{n\sigma^2 + 2(n-1)\rho\sigma^2}{n^2\varepsilon^2} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

$\Rightarrow$ Week Law of Large Numbers holds.

7.20

$$\begin{aligned}
P\left[\left|\frac{1}{n}S_n-\mu\right|>\varepsilon\right] &\le \frac{VAR(S_n)}{n^2\varepsilon^2}\\
&= \frac{1}{n^2\varepsilon^2}\left[n\sigma^2+2\rho\sigma^2\left(\frac{n-1}{1-\rho}-\frac{\rho}{1-\rho}\frac{1-\rho^{n-1}}{1-\rho}\right)\right]\\
&= \frac{\sigma^2}{n\varepsilon^2}+\frac{2\rho\sigma^2}{\varepsilon^2}\left(\frac{n-\frac{1}{n}}{n(1-\rho)}-\frac{1}{n^2}\frac{\rho(1-\rho^{n-1}}{(1-\rho)^2}\right)\\
&\to 0 \text{ as } n\to\infty \text{ (assuming } \rho<1)
\end{aligned}$$

$\Rightarrow$ Weak Law of Large Numbers holds.

7.21 a)

$$\begin{aligned}
LHS &= \sum_{j=1}^{n}(X_j^2-2\mu X_j+\mu^2)=\sum_{j=1}^{n}X_j^2-2\mu(nM_n)+n\mu^2\\
RHS &= \sum_{j=1}^{n}(X_j^2-2M_nX_j+M_n^2)+n(M_n-\mu)^2\\
&= \sum_{j=1}^{n}X_j^2-2M_n(nM_n)+nM_n^2+nM_n^2-2n\mu M_n+n\mu^2\\
&= \sum_{j=1}^{n}X_j^2-2n\mu M_n+n\mu^2=LHS \qquad \checkmark
\end{aligned}$$

b)

$$\begin{aligned}
\mathcal{E}\left[k\sum_{j=1}^{n}(X_j-M_n)^2\right] &= k\mathcal{E}\left[\underbrace{\sum_{j=1}^{n}(X_j-\mu)^2-n(M_n-\mu)^2}_{\text{from part a}}\right]\\
&= k\sum_{j=1}^{n}\mathcal{E}[(X_j-\mu)^2]-kn\mathcal{E}[(M_n-\mu)^2]\\
&= kn\sigma^2-kn\frac{\sigma^2}{n}\\
&= k(n-1)\sigma^2 \qquad \text{since } VAR[M_n]=\tfrac{\sigma^2}{n}.
\end{aligned}$$

c) If $k=\frac{1}{n-1}$ then $\mathcal{E}[V_n^2]=\sigma^2$

d) if $k=\frac{1}{n}$ then

$$\mathcal{E}\left[\frac{1}{n}\sum_{j=1}^{n}(X_j-M_n)^2\right]=\left(1-\frac{1}{n}\right)\sigma^2=\sigma^2-\underbrace{\frac{1}{n}\sigma^2}_{\text{bias}}$$

7.3 The Central Limit Theorm

7.22 The relevant parameters are $n = 100$, $m = np = 50$, $\sigma^2 = npq = 25$. The Central Limit Theorem then gives:

$$P[40 \leq N \leq 60] = P\left[\underbrace{\frac{40-50}{\sqrt{25}}}_{-2} \leq \frac{N-m}{\sigma} \leq \underbrace{\frac{60-50}{\sqrt{25}}}_{2}\right]$$

$$\approx Q(\overset{-2}{\cancel{-6.324}}) - Q(\overset{2}{\cancel{6.324}}) = 1 - 2Q(\overset{2}{\cancel{6.324}}) =$$

$$= 1 - \cancel{2.54(10^{-10})})\ 2(2.28)\times 10^{-2} = 0.9544$$

$$P[50 \leq N \leq 55] \approx Q(0) - Q(\underset{1}{\cancel{3.162}}) = \frac{1}{2} - \underbrace{\cancel{7.3(10^{-4})}}_{.159}) = 0.441$$

7.23 The relative frequency $f_A(n)$ has mean $\frac{2}{10}$ and variance $\frac{1}{n}p(1-p) = \frac{\overset{0.16}{\cancel{0.09}}}{n}$

$$\begin{aligned} P[|f_Z(n) - 0.2| < 0.02] &= P[0.18 < f_A(n) < 0.22] \\ &= P\left[\frac{0.18 - 0.20}{\sqrt{\frac{0.16}{n}}} < \frac{f_A(n) - 0.1}{\sqrt{\frac{0.16}{n}}} < \frac{0.22 - 0.20}{\sqrt{\frac{0.16}{n}}}\right] \\ &\approx 1 - 2Q\left(\frac{0.02}{\sqrt{\frac{0.16}{n}}}\right) = 0.95 \end{aligned}$$

$$\Rightarrow Q\left(\frac{\sqrt{n}}{20}\right) = 0.025 \Rightarrow \frac{\sqrt{n}}{20} = 1.95 \Rightarrow n = \cancel{866}\ 1521$$

7.24

$$S = \sum_{i=1}^{20} X_i \Rightarrow E[S] = 20 \cdot E[X] = 20 \times 3.5 = 70$$

$$VAR[S] = 20\ VAR[X] = 20 \times 2.92 = 58.4$$

Using CLT we have: $S \sim N(70, \sqrt{58.4})$

$$P\left\{60 < S < 80\right\} = P\left\{\frac{60-70}{7.64} < \frac{S-70}{7.64} < \frac{80-70}{7.64}\right\}$$

$$= 1 - 2Q(1.3089) = 0.8094$$

7.25

~~**5.25**~~ $\mathcal{E}[X_i] = \frac{1}{\lambda} = 36 \qquad VAR(X_i) = \frac{1}{\lambda^2} = 36^2$

$$S = X_1 + \ldots + X_{16} \qquad \mathcal{E}[S] = 16(36) \qquad VAR(S) = 16(36)^2$$

$$\begin{aligned} P[S < 600] &= P\left[\frac{S - 16(36)}{4(36)} < \frac{600 - 16(36)}{4(36)}\right] \\ &\cong 1 - Q\left(\frac{1}{6}\right) = 0.5692 \end{aligned}$$

7.26

$1/\lambda = 1$

$$\begin{aligned} \mathcal{E}[S_n] &= n\mathcal{E}[X_i] = n \cdot 1 = n \\ VAR[S_n] &= n\sigma_{x_i}^2 = n \cdot 1^2 = n \end{aligned}$$

Assuming S_n approximately Gaussian:

$$P[S_n > 15] = P\left[\frac{S_n - n}{\sqrt{n}} > \frac{15 - n}{\sqrt{n}}\right] \approx Q\left(\frac{15 - n}{\sqrt{n}}\right) = 0.99$$

From Table 3.4

$$\frac{15 - n}{\sqrt{n}} = -2.3263$$

$\Rightarrow n - 2.3263\sqrt{n} - 15 = 0 \Rightarrow n = 27.04$

$\Rightarrow$ by 28 pens

7.27 $n = 80 \qquad \lambda = \frac{1}{4}$

$$\mathcal{E}[S_n] = n\lambda = 20 \qquad VAR(S_n) = 20$$

$$\begin{aligned} P[S_n = k] &\approx \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{e^{-(x-20)^2/2(20)}}{\sqrt{2\pi(20)}}dx \quad \text{as per Eqn. 7.29} \\ &\approx \frac{e^{-(k-20)^2/40}}{\sqrt{40\pi}} \quad \text{as per Eqn. 7.30} \end{aligned}$$

A comparison of the exact value of $P[S_n = k]$ and the above approximation is given below:

k	Poisson	approx.	k	poisson	approx
0	0.000000	0.000004	21	0.084605	0.087003
1	0.000000	0.000010	22	0.076913	0.080717
3	0.000002	0.000064	23	0.066881	0.071232
4	0.000013	0.000148	24	0.055734	0.059796
5	0.000054	0.000321	25	0.044587	0.047748
6	0.000183	0.000664	26	0.034298	0.036268
7	0.000523	0.001304	27	0.025406	0.026205
8	0.001308	0.002437	28	0.018147	0.018010
9	0.002908	0.004331	29	0.012515	0.011774
10	0.005816	0.007322	30	0.008343	0.007322
11	0.010575	0.011774	31	0.005382	0.004331
12	0.017625	0.018010	32	0.003364	0.002437
13	0.027115	0.026205	33	0.002038	0.001304
14	0.038736	0.036268	34	0.001199	0.000664
15	0.051648	0.047748	35	0.000685	0.000321
16	0.064561	0.059796	36	0.000380	0.000148
17	0.075954	0.071232	37	0.000205	0.000064
18	0.084393	0.080717	38	0.000108	0.000027
19	0.088835	0.087003	39	0.000055	0.000010
20	0.088835	0.089206	40	0.000027	0.000004

7.28

~~5.28~~ Using the fact that the time between arrivals is an exponential RV with mean $\frac{1}{15}$ and variance $\left(\frac{1}{15}\right)^2$; the time of the nth arrival is:

$$S_n = X_1 + \ldots + X_n$$

where

$$\begin{aligned} \mathcal{E}[S_n] &= n\mathcal{E}[X_i] = n/15 \\ VAR[S_n] &= nVAR[X_i] = n/225 \end{aligned}$$

$E[S_{950}] = 63.33$

$VAR[S_{950}] = 4.22$

$$\begin{aligned} P\begin{bmatrix}\text{more than } 950 \\ \text{messages in} \\ \text{60 seconds}\end{bmatrix} &= P[S_{950} < 60] \\ &= P\left[\frac{S_{950} - \overset{63.33}{\cancel{65}}}{\sqrt{\cancel{6.5}\,4.22}} < \frac{60 - \overset{63.33}{\cancel{65}}}{\sqrt{\cancel{6.5}\,4.22}}\right] \\ &\approx 1 - Q(\underset{-1.62}{\cancel{-1.96}}) = Q(\underset{1.62}{\cancel{1.96}}) = \overset{5.4}{\cancel{2.49}}(10^{-2}) \\ \\ &\approx Q(\overset{-2}{\cancel{-6.324}}) - Q(\overset{2}{\cancel{6.324}}) = 1 - 2Q(\overset{2}{\cancel{6.324}}) = \\ &= 1 - \cancel{2.54(10^{-10})}\, 2(2.28)\times 10^{-2} = 0.9544 \\ P[50 \le N \le 55] &\approx Q(0) - Q(\underset{1}{\cancel{3.162}}) = \frac{1}{2} - \underbrace{\cancel{7.3(10^{-4})}}_{.159} = 0.441 \end{aligned}$$

7.29

~~5.29~~ The total number of errors S_{100} is the sum of iid Bernoulli random variables

$$\begin{aligned} S_{100} &= X_1 + \ldots + X_{100} \\ \mathcal{E}[S_{100}] &= 100p = 15 \\ VAR[S_{100}] &= 100pq = 12.75 \end{aligned}$$

The Central Limit Theorem gives:

$$\begin{aligned} P[S_{100} \le 20] &= 1 - P[S_{100} > 20] \\ &= 1 - P\left[\frac{S_{100} - 15}{\sqrt{12.75}} > \frac{20 - 15}{\sqrt{12.75}}\right] \\ &\approx 1 - Q(1.4) = 0.92 \end{aligned}$$

7.30

~~5.30~~ Total error is

$$S_{\cancel{100}\,64} = X_1 + X_2 + \dots + X_{\cancel{100}\,64}$$

where X_i uniform is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\mathcal{E}[X_i] = 0 \qquad VAR[X_i] = \frac{1}{12}$$

$$P[S_{\cancel{100}\,64} > \cancel{6}\,4] = P\left[\frac{S_{100}}{\sqrt{\frac{\cancel{100}\,64}{12}}} > \frac{4}{\sqrt{\frac{\cancel{100}\,64}{12}}}\right] \approx Q(\underset{1.7321}{\cancel{2.078}}) = \overset{4.16}{\cancel{1.79}}(10^{-2})$$

7.31 1) $X_i \sim \text{Bernoulli}\,(1/2)$ $\quad x_i = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$

$$S_{100} = \sum_{i=1}^{100} X_i \qquad \mathcal{E}[S_{100}] = 50\,, \quad VAR[S_{100}] = 25$$

a) $$P\{S_{100} \geq 91\} \leq \left(\frac{\frac{1}{2}}{\left(\frac{91}{100}\right)^{\frac{91}{100}} \times \left(\frac{9}{100}\right)^{\frac{9}{100}}}\right)^{100} = 1.0886 \times 10^{-17}$$

using CLT: $$P\{S_{100} > 91\} = Q\left(\frac{91-50}{5}\right) = Q(8.2) = 1.11 \times 10^{-16}$$

b) $$P\{S_{1000} \geq 651\} \leq \left(\frac{\frac{1}{2}}{\left(\frac{651}{1000}\right)^{\frac{651}{1000}} \left(\frac{349}{1000}\right)^{\frac{349}{1000}}}\right)^{1000} = 7.6332 \times 10^{-21}$$

using CLT: $$P\{S_{1000} > 651\} = Q\left(\frac{651-500}{\sqrt{250}}\right) = Q(9.5501) = 6.47 \times 10^{-22}$$

7.32)

Total error $E_{100} = \sum_{i=1}^{100} X_i$, X_i represents an error in ith transmission

$$P\left[E_{100} \geq 4\right] \leq \left(\frac{0.01^{0.04} \times 0.99^{0.96}}{0.04^{0.04} \times 0.96^{0.96}} \right)^{100} = 0.0749$$

using CLT : $P\{E_{100} \geq 4\} \simeq Q\left(\frac{4-1}{\sqrt{0.99}}\right) = Q(3.0151) = 0.0013$

7.33

$$W_{20} = \sum_{i=1}^{20} x_i \qquad x_i = \begin{cases} 1 & p = \frac{2}{5} \\ 0 & p = 3/5 \end{cases}$$

a) $$P\{W_{20} \geq 11\} \leq \left(\frac{\left(\frac{2}{5}\right)^{11/20} \times \left(\frac{3}{5}\right)^{9/20}}{\left(\frac{11}{20}\right)^{11/20} \times \left(\frac{9}{20}\right)^{9/20}} \right)^{20} = 0.4010$$

b) $$P\{W_{100} \geq 51\} \leq \left(\frac{\left(\frac{2}{5}\right)^{\frac{51}{100}} \times \left(\frac{3}{5}\right)^{49/100}}{\left(\frac{51}{100}\right)^{0.51} \times \left(\frac{49}{100}\right)^{0.49}} \right)^{100} = 0.0849$$

c) $$P\left\{W_n \geq \frac{n}{2}+1\right\} \leq \left(\frac{\left(\frac{2}{5}\right)^{\frac{1}{2}+\frac{1}{n}} \times \left(\frac{3}{5}\right)^{\frac{1}{2}-\frac{1}{n}}}{\left(\frac{1}{2}+\frac{1}{n}\right)^{\left(\frac{1}{2}+\frac{1}{n}\right)} \left(\frac{1}{2}-\frac{1}{n}\right)^{\left(\frac{1}{2}-\frac{1}{n}\right)}} \right)^{n} = P$$

for $n = 92$, $P = 0.0998$

7.34 a

$$P[X \geqslant a] \leqslant e^{-sa} E[e^{sX}] \quad , s > 0$$

$$G_X(z) = e^{(z-1)}$$

$$E[(e^s)^X] = e^{\alpha(e^s-1)}$$

therefore $P[X \geqslant a] \leqslant e^{-sa+\alpha e^s-\alpha}$, $s > 0$

To minimize bound we have to find the root of power.

So: $f(s) = -sa+\alpha e^s-\alpha$, $f'(s) = -a+\alpha e^s = 0 \Rightarrow e^s = \frac{a}{\alpha} \Rightarrow s = \ln\frac{a}{\alpha}$

if $\ln a/\alpha \geqslant 0$ or $a \geqslant \alpha$

Therefore: $P[X \geqslant a] \leqslant e^{-a\ln\frac{a}{\alpha}+a-\alpha}$

7.35

number of faulty pens in the duration of 15 weeks is Poisson RV with mean 15. So: $S \sim \text{Poisson}(15)$

according to P 7.34 we have

$$P[S > a] \leqslant e^{-a\ln\left(\frac{a}{15}\right)+a-15} \quad \text{for } a > 15$$

Therefore: $e^{-a\ln\left(\frac{a}{15}\right)+a-15} = 0.01$

$$\Rightarrow -a\ln\left(\frac{a}{15}\right)+a-15 = \ln 0.01$$

by trial & error we find: $a = 28$

So, the student should buy 28 pens.

7.36)

$$P[X \geq a] \leq e^{-sa} E[e^{sX}] \quad , s > 0$$

$X \sim N(\mu, \sigma)$

$$E[e^{sX}] = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

Therefore $P[X \geq a] \leq e^{-sa + \mu s + \frac{\sigma^2 s^2}{2}}$

$$f(s) = -sa + \mu s + \frac{\sigma^2 s^2}{2} \Longrightarrow f'(s) = -a + \mu + \sigma^2 s = 0 \Longrightarrow s^* = \frac{a-\mu}{\sigma^2} \quad \text{if } a > \mu$$

$$\Rightarrow P[X \geq a] \leq e^{-a\left(\frac{a-\mu}{\sigma^2}\right) + \mu\left(\frac{a-\mu}{\sigma^2}\right) + \frac{1}{2\sigma^2}(a-\mu)^2}$$

$$= e^{-\frac{(a-\mu)^2}{2\sigma^2}} \quad , a > \mu$$

7.37)

$$P[X \geq a] = Q\left(\frac{a-\mu}{\sigma}\right) \approx \frac{1}{\left(1-\frac{1}{\pi}\right)\left(\frac{a-\mu}{\sigma}\right) + \frac{1}{\pi}\sqrt{\left(\frac{a-\mu}{\sigma}\right)^2 + 2\pi}} \, e^{-\frac{(a-\mu)^2}{2\sigma^2}} \quad \text{for } a > \mu$$

if you compare this value with the bound in (P 7.36) you can see that the difference is coefficient $\dfrac{1}{\left(1-\frac{1}{\pi}\right)\left(\frac{a-\mu}{\sigma}\right) + \frac{1}{\pi}\sqrt{\left(\frac{a-\mu}{\sigma}\right)^2 + 2\pi}}$

7.38

a) $P[X \geq a] \leq e^{-sa} E[e^{sX}]$, $s>0$ $X \sim \exp(\lambda)$, $E[e^{sX}] = \frac{\lambda}{\lambda - s}$

Therefore:

$$P[X \geq a] \leq e^{-sa} \times \frac{\lambda}{\lambda - s}$$

$f(s) = e^{-sa} \times \frac{\lambda}{\lambda - s}$, $f'(s) = \frac{-ae^{-sa}\lambda(\lambda - s) + \lambda e^{-sa}}{(\lambda - s)^2}$

$f'(s) = 0 \Longrightarrow s^* = \lambda - \frac{1}{a}$, Since $s^* > 0$ then $\lambda > 1/a$ or $a > 1/\lambda$

Therefore $P[X \geq a] \leq a\lambda e^{-(\lambda a - 1)}$ for $a > 1/\lambda$

b) we know $P[X \geq \frac{k}{\lambda}] = e^{-\lambda \times \frac{k}{\lambda}} = e^{-k}$

if $a = \frac{k}{\lambda}$, chernoff's bound gives us: $P[X \geq \frac{k}{\lambda}] \leq ke^{-(k-1)}$, $k>1$

$ke^{-k+1} = ke\, e^{-k}$, Therefore chernoff bound is always greater than actual probability, with a factor of ke

(7.39) $X \sim \Gamma(\lambda, \alpha)$, $E[e^{sX}] = \frac{1}{(1-\frac{s}{\lambda})^{\alpha}}$

a)

$$P[X \geq a] \leq e^{-sa} \times \frac{1}{(1-\frac{s}{\lambda})^{\alpha}}$$

$$f(s) = e^{-sa} \times \frac{1}{(1-\frac{s}{\lambda})^{\alpha}} , \quad f'(s) = \frac{-a e^{-sa}(1-\frac{s}{\lambda})^{\alpha} + e^{-sa}\alpha\frac{1}{\lambda}(1-\frac{s}{\lambda})^{\alpha-1}}{(1-\frac{s}{\lambda})^{2\alpha}}$$

$f'(s) = 0 \Longrightarrow s^* = \lambda - \frac{\alpha}{a}$, since $s > 0 \Rightarrow a > \frac{\alpha}{\lambda}$

Therefore: $P[X \geq a] \leq e^{-(\lambda a - \alpha)} \frac{\lambda^{\alpha}}{(a\alpha)^{\alpha}}$ for $a > \frac{\alpha}{\lambda}$

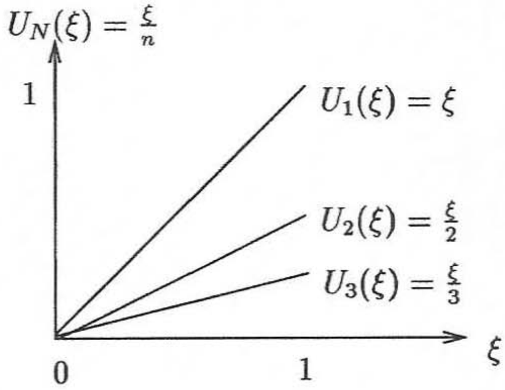
b) chi-square with k degree of freedom, $X \sim$ chi-square$(k) \sim \Gamma(\frac{1}{2}, \frac{k}{2})$

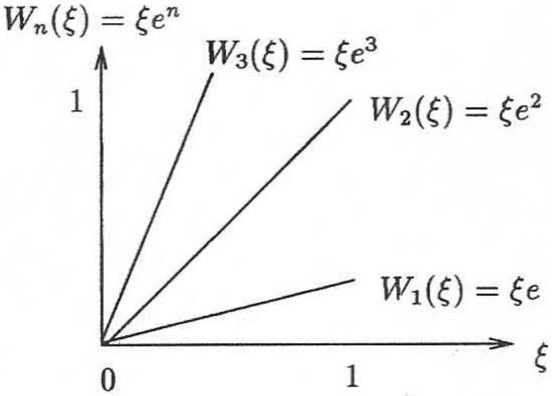
$\alpha = \frac{k}{2}$, $\lambda = \frac{1}{2}$ from part a: $P[X \geq a] \leq e^{-\frac{1}{2}(a-k)} (a\,k)^{-k/2}$ for $a > k$

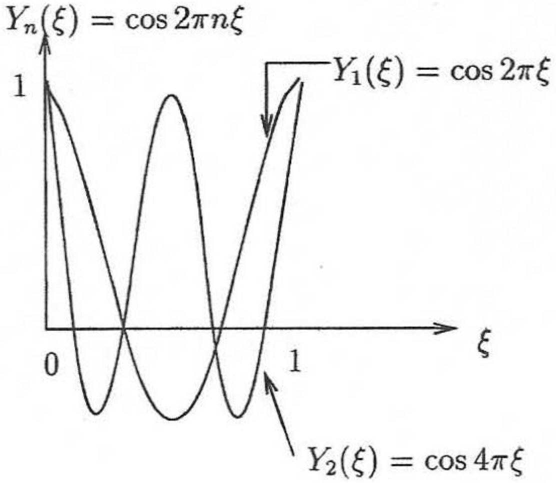
*7.4 Convergence of Sequences of Random Variables

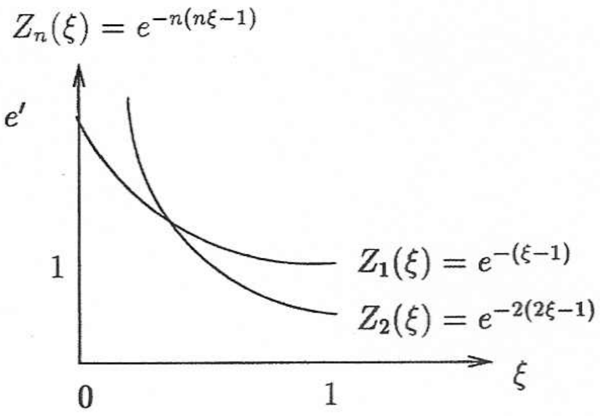
7.40

~~5~~.40 a)

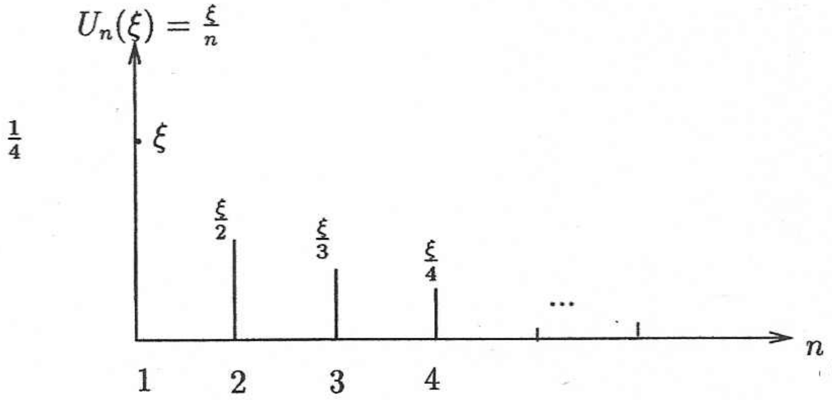


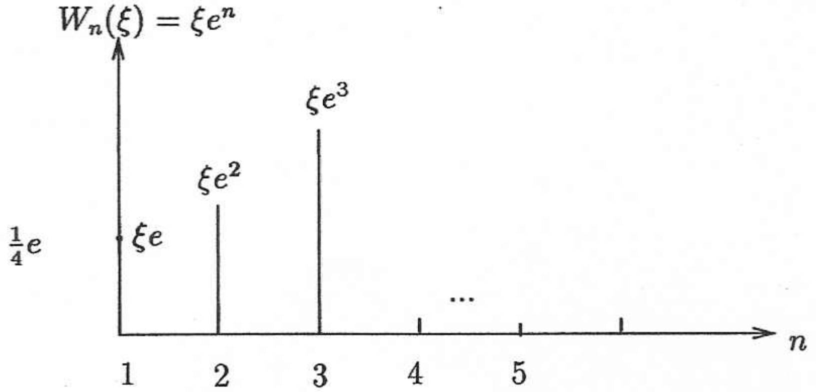


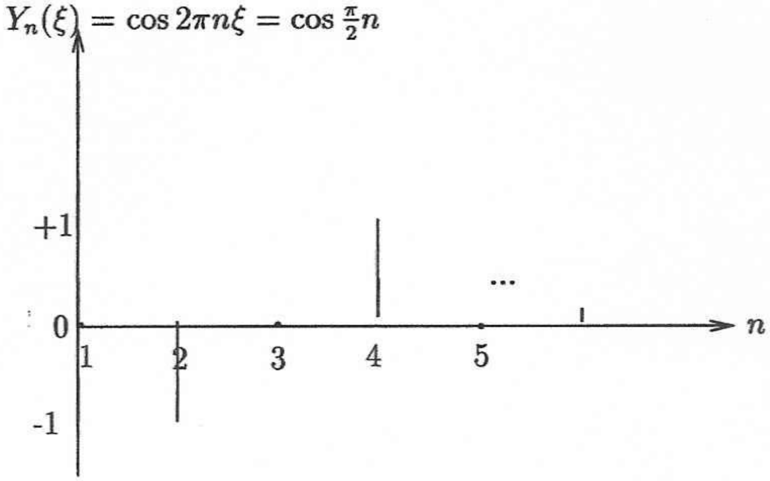


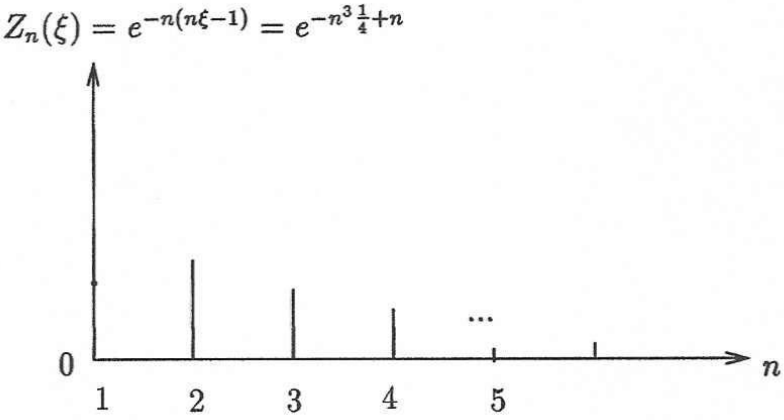

$Z_n(\xi) = e^{-n(n\xi-1)}$
e'
1
$Z_1(\xi) = e^{-(\xi-1)}$
$Z_2(\xi) = e^{-2(2\xi-1)}$
ξ
0
1

b)


$U_n(\xi) = \frac{\xi}{n}$
$\frac{1}{4}$
ξ
$\frac{\xi}{2}$
$\frac{\xi}{3}$
$\frac{\xi}{4}$
...
n
1
2
3
4


$W_n(\xi) = \xi e^n$
ξe^3
ξe^2
$\frac{1}{4}e$
ξe
...
n
1
2
3
4
5





7.41 $X_n(\xi) = \xi^n \to 0$ for all $0 \le \xi < 1$

$X_n(\xi)$ converges to 0 almost surely

$Y_n(\xi) = \cos^2 2\pi\xi$ (seems to be $\cos^2 2\pi n\xi$)

$\cos^2 2\pi\xi$ doesn't depend on n

$Y_n(\xi) = \cos^2 2\pi\xi \to Y(\xi) = \cos^2 2\pi\xi$ for all $\xi \in S$

$Z_n(\xi) = \cos^n 2\pi\xi \to 0$ for $0 \le \xi \le 1$ except $\xi = 0, \frac{1}{2}, 1$

$Z_n(\xi)$ converges to 0 almost surely.

7.42

$$
\begin{aligned}
\xi &= \sum_{i=1}^{\infty} b_i 2^{-i} \\
P[b_1 = 0] &= P[b_1 = 1] = \frac{1}{2} \\
\therefore\ P[\xi < 0.5] &= P[\xi > 0.5] = \frac{1}{2} \\
P[b_1 b_2 = 00] &= P[b_1 b_2 = 01] = P[b_1 b_2 = .10] = P[b_1 b_2 = 11] = \frac{1}{4} \\
P[\xi < 0.25] &= P[0.25 < \xi < 0.5] = P[0.5 < \xi < 0.75] \\
&= P[\xi > 0.75] = \frac{1}{4}
\end{aligned}
$$

and so on.

So ξ is uniformly distributed in [0,1].

b) If $b_1 = 0$ then we suppose a black ball was selected in first draw; Thereafter we interpret $b_n = 1$ as "remove black ball, if any, from urn."

Similarly if $b_1 = 1$ then we suppose a white ball was selected in first draw; Thereafter we interpret $b_n = 1$ as "remove white ball, if any, from urn."

7.43

~~5.43~~ a) $Y_n = 2^n X_1 X_2 ... X_n = \begin{cases} 2^n & X_1 = X_2 = ... = X_n = 1 \\ 0 & \text{otherwise} \end{cases}$

$\therefore Y_n \to 0$ almost surely.

b)
$$\begin{aligned} E[Y_n] &= 2^n P[X_1 = X_2 = ... = X_n = 1] \\ &\quad +0 \cdot (1 - P[X_1 = X_2 = ... = X_n]) \\ &= 2^n \left(\frac{1}{2}\right)^n \\ &= 1 \end{aligned}$$

Furthermore
$$E[Y_n^2] = (2^n)^2 \left(\frac{1}{2}\right)^n = 2^n \to \infty .$$

Thus Y_n does not converge to 0 in the m.s. sense.

7.44

$$\begin{aligned} E[(M_n - m)^2] &= E\left[\left(\frac{1}{n}\sum_i (X_i - m)\right)^2\right] \\ &= \frac{1}{n^2} E\left[\sum_i (X_i - m)^2\right] \\ &= \frac{1}{n^2} \cdot n\sigma^2 \end{aligned}$$

$$\therefore \quad \lim_{n\to\infty} E[(M_n - m)^2] = 0$$

7.45

~~5.45~~ We are given that $X_n \to X$ ms and $Y_n \to Y$ ms. Consider

$$\begin{aligned} \mathcal{E}[((X_n+Y_n)-(X+Y))^2] &= E[((X_n-X)+(Y_n-Y))^2] \\ &= E[(X_n-X)^2]+E[(Y_n-Y)^2] \\ &\quad +2E[(X_n-X)(Y_n-Y)] \end{aligned}$$

The first two terms approach zero since $X_n \to X$ and $Y_n \to Y$ in mean square sense. We need to show that the last term also goes to zero. This requires the Schwarz Inequality:

$$E[ZW] \leq \sqrt{E[Z^2]}\sqrt{E[W^2]}\ .$$

When the inequality is applied to the third term we have:

$$\begin{aligned} E[((X_n+Y_n)-(X+Y))^2] &\leq E[(X_n-X)^2]+E[(Y_n-Y)^2] \\ &\quad +2\sqrt{E[(X_n-X)^2]}\sqrt{E[(Y_n-Y)^2]} \\ &= \left(\sqrt{E[(X_n-X)^2]}+\sqrt{E[(Y_n-Y)^2]}\right)^2 \\ &\to 0 \quad \text{as } n \to \infty\ . \end{aligned}$$

To prove the Schwarz Inequality we take

$$0 \leq E[(Z+aW)^2]$$

and minimize with respect to a:

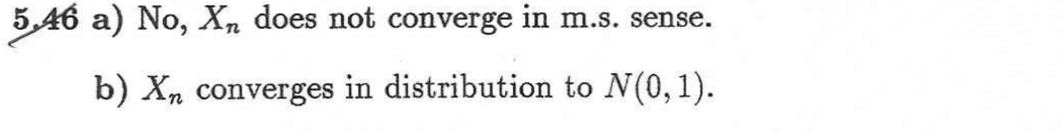
$$\begin{aligned} \frac{d}{da}(E[Z^2]+2aE[ZW]+a^2E[W^2]) &= 0 \\ 2E[ZW]+2aE[W^2] &= 0 \end{aligned}$$

$\Rightarrow$ minimum attained by $a^* = -\dfrac{E[ZW]}{E[W^2]}$. Thus

$$0 \leq E[(Z+a^*W)^2] = E[Z^2] = 2\frac{E[ZW]^2}{E[W^2]} + \frac{E[ZW]^2}{E[W^2]}$$

$$\begin{aligned} \Rightarrow \quad \frac{E[ZW]^2}{E[W^2]} &\leq E[Z^2] \\ \Rightarrow \quad E[ZW] &\leq \sqrt{E[Z^2]}\sqrt{E[W^2]} \qquad \text{as required} \end{aligned}$$

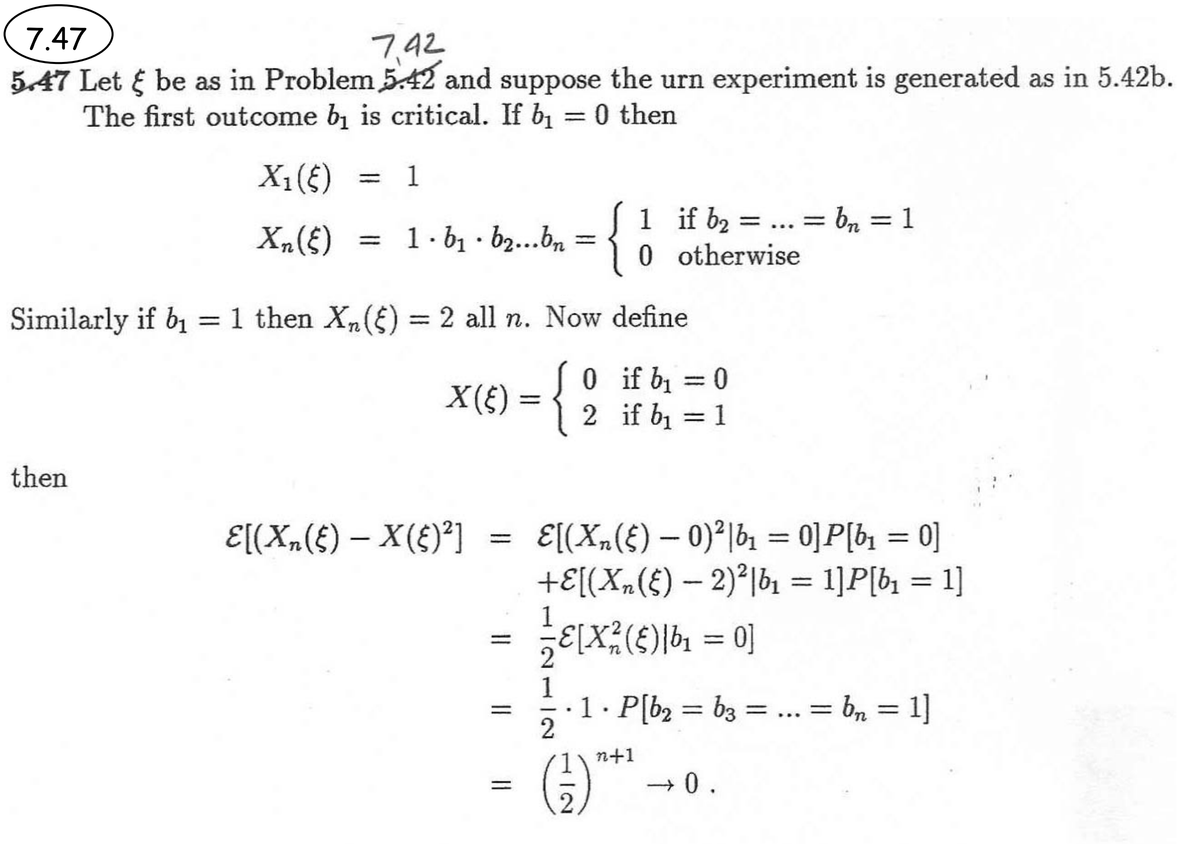
7.46



~~5.46~~ a) No, X_n does not converge in m.s. sense.

b) X_n converges in distribution to $N(0,1)$.

7.47



~~5.47~~ Let ξ be as in Problem ~~5.42~~ 7.42 and suppose the urn experiment is generated as in 5.42b. The first outcome b_1 is critical. If $b_1 = 0$ then

$$\begin{aligned} X_1(\xi) &= 1 \\ X_n(\xi) &= 1 \cdot b_1 \cdot b_2 \ldots b_n = \begin{cases} 1 & \text{if } b_2 = \ldots = b_n = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Similarly if $b_1 = 1$ then $X_n(\xi) = 2$ all n. Now define

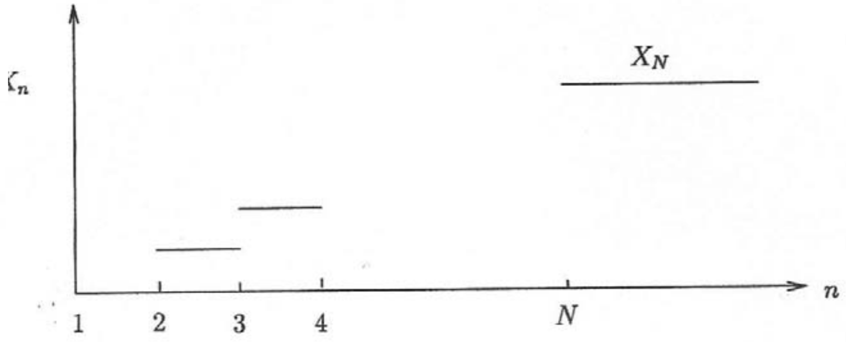
$$X(\xi) = \begin{cases} 0 & \text{if } b_1 = 0 \\ 2 & \text{if } b_1 = 1 \end{cases}$$

then

$$\begin{aligned} \mathcal{E}[(X_n(\xi) - X(\xi)^2] &= \mathcal{E}[(X_n(\xi) - 0)^2 | b_1 = 0] P[b_1 = 0] \\ &\quad + \mathcal{E}[(X_n(\xi) - 2)^2 | b_1 = 1] P[b_1 = 1] \\ &= \frac{1}{2} \mathcal{E}[X_n^2(\xi) | b_1 = 0] \\ &= \frac{1}{2} \cdot 1 \cdot P[b_2 = b_3 = \ldots = b_n = 1] \\ &= \left(\frac{1}{2}\right)^{n+1} \to 0 \,. \end{aligned}$$

7.48

a)



b) Let $A_1, A_2, \ldots$ be a sequence of iid Bernoulli trials with parameter p corresponding to potential customer arrivals. Let $B_1, B_2, \ldots$ be a sequence of iid Bernoulli trials with probability of success $\frac{99}{100}$, i.e. $P[B_i = 1] = \frac{99}{100} = \alpha$.

The sequence X_n is then given by

$$X_n = A_1B_1 + A_2B_1B_2 + \ldots + A_nB_1B_2\ldots B_n + \ldots$$

Let N be the largest value of n for which $B_1 \ldots B_N = 1$, then

$$X_n = A_1 + \ldots + A_N \qquad \text{for } n \geq N.$$

Now define the limiting random variable X by

$$X = A_1 + \ldots + A_N$$

where N is determined by the sequence of B_i's, then

$$X_n \to X \qquad \text{as } n \to \infty \text{ almost surely.}$$

c)

$$\begin{aligned} \mathcal{E}[(X_n - X)^2] &= \mathcal{E}[\mathcal{E}[(X_n - X)^2|N]] \\ \mathcal{E}[(X_n - X)^2|N = k] &= \begin{cases} 0 & n \geq k \\ \mathcal{E}\left[\left(\sum_{i=n+1}^{k} A_i\right)^2\right] & n < k \end{cases} \end{aligned}$$

The summation above defines a Binomial random variable with parameters $k-n-1$ and p, thus the second moment is:

$$\mathcal{E}\left[\left(\sum_{i=n+1}^{k} A_i\right)^2\right] = (k-n-1)pq + (k-n-1)^2p^2 \ .$$

Finally we have

$$\begin{aligned} \mathcal{E}[(X_n - X)^2] &= \sum_{k=1}^{\infty} \mathcal{E}[(X_n - X)^2|N = k]P[N = k] \\ &= \sum_{k=n+1}^{\infty} [(k-n-1)pq + (k-n-1)^2p^2]\alpha^k(1-\alpha) \\ &= \alpha^{n+1} \sum_{l=0}^{\infty} (lpq + l^2p^2)\alpha^l(1-\alpha) \\ &= \alpha^{n+1}\left[pq\frac{\alpha}{1-\alpha} + p^2\frac{\alpha+\alpha^2}{(1-\alpha)^2}\right] \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

Therefore X_n converges in mean square sense.

7.49

~~5.~~49 $Y_n = \cos 2\pi n\xi$. ξ: uniform RV in [0,1].

Note that there are $2n$ solutions to the equation

$$y = \cos 2\pi n x$$

for each fixed y in [-1,1]

$$\begin{aligned} f_{Y_n}(y_n) &= \sum_k \frac{f_X(x)}{|dy/dx|}\Big|_{x=x_k} \\ &= 2n \cdot \frac{1}{2\pi n\sqrt{1-y_n^2}} \\ &= \frac{1}{\pi\sqrt{1-y_n^2}} \end{aligned}$$

Define

$$\begin{aligned} F_Y(y) &= \int_{-1}^{y} \frac{1}{\pi\sqrt{1-y'^2}}dy' \\ F_n(y_n) &\to F_Y(y) \quad \text{as } n \to \infty. \end{aligned}$$

7.50

~~5.~~50 Laplacian RV

$$\begin{aligned} f_{X_n}(x_n) &= \frac{n}{2}\exp(-n|x_n|) \\ E[X_n] &= 0, \qquad VAR[X_n] = \frac{2}{n^2} \to 0 \quad \text{as } n \to \infty \end{aligned}$$

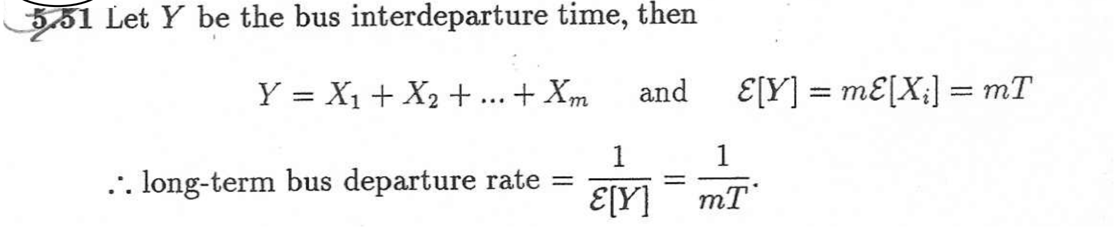
From Chebyshev inequality

$$P[|X_n - 0| \geq e] \leq \frac{\sigma^2}{e^2} \to 0 \quad \text{as } n \to \infty$$

The sequence converges in probability, and hence in distribution.

*7.5 Long-Term Arrival Rates and Associated Events

7.51

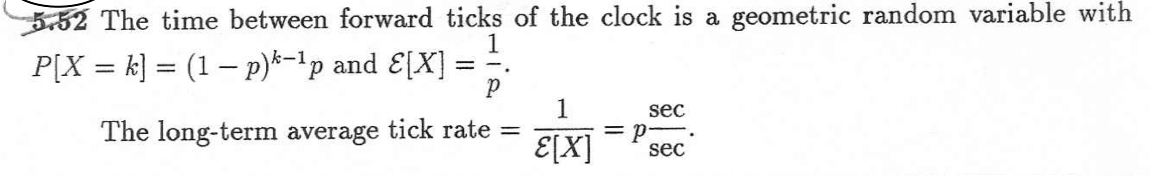


~~5.51~~ Let Y be the bus interdeparture time, then

$$Y = X_1 + X_2 + ... + X_m \quad \text{and} \quad \mathcal{E}[Y] = m\mathcal{E}[X_i] = mT$$

$$\therefore \text{ long-term bus departure rate} = \frac{1}{\mathcal{E}[Y]} = \frac{1}{mT}.$$

7.52



~~5.52~~ The time between forward ticks of the clock is a geometric random variable with $P[X = k] = (1-p)^{k-1}p$ and $\mathcal{E}[X] = \frac{1}{p}$.

$$\text{The long-term average tick rate} = \frac{1}{\mathcal{E}[X]} = p\frac{\text{sec}}{\text{sec}}.$$

7.53

~~5.53~~ Show $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$.

a) We first show that $\{N(t) \geq n\} \Rightarrow \{S_n \leq t\}$.

$$\begin{aligned}
\text{If } \{N(t) \geq n\} \Rightarrow t \geq S_{N(t)} &= X_1 + X_2 + \dots + X_{N(t)} \\
&\geq X_1 + \dots + X_n = S_n \\
\Rightarrow \{S_n \leq t\}
\end{aligned}$$

Next we show that $\{S_n \leq t\} \Rightarrow \{N(t) \geq n\}$. If $\{S_n \leq t\}$ then nth event occurs before time t

$$\begin{aligned}
&\Rightarrow \quad N(t) \text{ is at least } n \\
&\Rightarrow \quad \{N(t) \geq n\} \qquad \checkmark
\end{aligned}$$

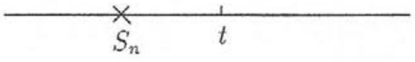
b)

$$\begin{aligned}
P[N(t) \leq n] &= 1 - P[N(t) \geq n+1] \\
&= 1 - P[S_{n+1} \leq t] \\
&= 1 - \left(1 - \sum_{k=0}^{n} \frac{(\alpha t)^k}{k!} e^{-\alpha t}\right) \\
&= \sum_{k=0}^{n} \frac{(\alpha t)^k}{k!} e^{-\alpha t}
\end{aligned}$$

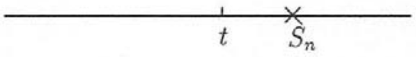
but S_{n+1} is an Erlang RV so by Ex. 3.16 we have that $N(t)$ is a Poisson random variable.

7.54

~~5.54~~ a) $\{N(t) = n\} \Rightarrow \{S_n \leq t\}$

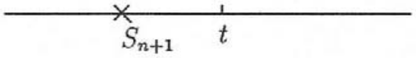


$\{S_n \geq t\}$

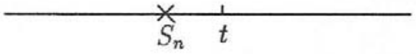


$\therefore$ $\{N(t) = n\}$ and $\{S_n > t\}$ are mutually exclusive
$\therefore$ $\{N(t) \leq n\}$ and $\{S_n \geq t\}$ cannot be equivalent

b) $\{N(t) > n\} \Rightarrow \{N(t) \geq n+1\} \Rightarrow \{S_{n+1} < t\}$



$\{S_n < t\}$



If $S_n < t < S_{n+1}$ then event $\{S_n > t\}$ occurs but event $\{N(t) > n\}$ doesn't; thus the two events cannot be equivalent.

7.55

~~5.55~~ Define

$$\begin{aligned} \text{cycle} &= \text{error free period} + \text{error period} \\ \text{“cost” per cycle} &= \text{duration of error free period} \end{aligned}$$

Then

$$\begin{array}{c}\text{long-term proportion}\\ \text{of time when channel}\\ \text{is error free}\end{array} = \frac{\text{avg. cost for cycle}}{\text{avg. cycle length}} = \frac{m_1}{m_1+m_2}$$

7.56

~~5.56~~ cycle $\triangleq \begin{array}{c}\text{time boss}\\ \text{present}\end{array} + \begin{array}{c}\text{time boss}\\ \text{absent}\end{array} = X_i + Y_i$

$$\begin{aligned}\text{“cost”/cycle} &\triangleq \text{work done} = r_1 \times \begin{array}{c}\text{time boss}\\ \text{present}\end{array} + r_2 \begin{array}{c}\text{time boss}\\ \text{absent}\end{array}\\ &= r_1 X_i + r_2 Y_i\end{aligned}$$

$$\begin{aligned}\begin{array}{c}\text{long term avg. rate at}\\ \text{which worker does work}\end{array} &= \frac{\mathcal{E}[r_1X_i + r_2Y_i]}{\mathcal{E}[X_i+Y_i]} = \frac{r_1m_1 + r_2m_2}{m_1+m_2}\\ &= r_1\frac{m_1}{m_1+m_2} + r_2\frac{m_2}{m_1+m_2}\end{aligned}$$

7.57

~~5.57~~ a) $\lim\limits_{t\to\infty} \dfrac{N(t)}{t} = \dfrac{1}{\mathcal{E}[X_1+X_2+X_3]} = \dfrac{1}{\mathcal{E}[X_1]+\mathcal{E}[X_2]+\mathcal{E}[X_3]}$

b) Let $c_i(t) = \sum\limits_{j=1}^{N(t)} X_{ij}$, $i = 1, 2, 3$. Then long term proportion of time spent servicing task i is

$$\lim_{t\to\infty} \frac{c_i(t)}{t} = \frac{\mathcal{E}[X_i]}{\mathcal{E}[X_1]+\mathcal{E}[X_2]+\mathcal{E}[X_3]}$$

c) Replace X_i by $Y_i = X_i + W_i$ then

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathcal{E}[X_1]+\mathcal{E}[X_2]+E[X_3]+3\mathcal{E}[W]}$$

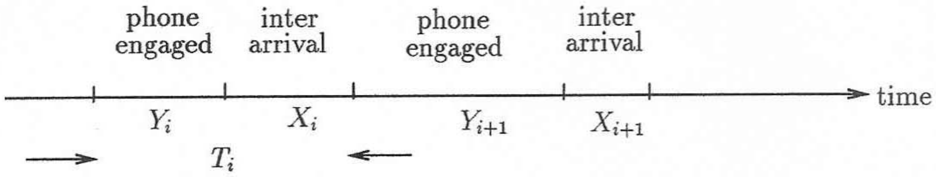
and

$$\lim_{t\to\infty} \frac{c_i(t)}{t} = \frac{\mathcal{E}[X_i]}{\mathcal{E}[X_1]+\mathcal{E}[X_2]+\mathcal{E}[X_3]+3\mathcal{E}[W]}$$

7.58

~~5.58~~ **a)** Let T_i be the phone interseizure time, then

$$T_i = Y_i + X_i$$



$$\therefore \text{ long-term seizure rate} = \frac{1}{\mathcal{E}[T]} = \frac{1}{\mathcal{E}[Y] + \mathcal{E}[X]}$$

In order for the X_i to be iid, we need the memoryless property of the exponential RV.

b)

$$\begin{aligned}
\text{Define a "cycle"} &= \#\text{ of customer arrivals in } T_i \\
&= 1 + \#\text{ arrivals in } Y_i \triangleq 1 + N \\
\text{Define a "cost"} &= \#\text{ customers that leave without using phone} \\
&= \#\text{ arrivals in } Y_i \triangleq N \\
\text{Then long-term "cost" rate} &= \text{long-term proportion of customers that leave without using phone} \\
&= \frac{\mathcal{E}[N]}{1+\mathcal{E}[N]} \qquad \text{From Ex. 4.26 } \mathcal{E}[N] = \lambda\mathcal{E}[Y] \\
&= \frac{\lambda\mathcal{E}[Y]}{1+\lambda\mathcal{E}[Y]} = \frac{\mathcal{E}[Y]}{\frac{1}{\lambda}+\mathcal{E}[Y]} = \text{proportion of time phone is engaged}
\end{aligned}$$

7.59

~~5.59~~ The interreplacement time is

$$\tilde{X}_i = \begin{cases} X_i & \text{if } X_i < 3T \quad \text{that is item breaks down before } 3T \\ 3T & \text{if } X_i \geq 3T \quad \text{that is item is replaced at time } 3T \end{cases}$$

where the X_i are iid exponential random variables with mean $\mathcal{E}[X_i] = T$.

The mean of $\tilde{X}_i$ is:

$$\mathcal{E}[\tilde{X}_i] = \int_0^{3T} x\frac{1}{T}e^{-x/T}dx + 3TP[X > 3T] = T(1-e^{-3})$$

a) Therefore the

$$\begin{array}{c}\text{long-term}\\ \text{replacement}\\ \text{rate}\end{array} = \frac{1}{\mathcal{E}[\tilde{X}]} = \frac{1}{T(1-e^{-3})}$$

b) Let

$$c_i = \begin{cases} 1 & X_i \geq 3T \\ 0 & X_i < 3T \end{cases} \qquad \text{i.e. a good item is replaced}$$

Then

$$\mathcal{E}[C] = P[X_i \geq 3T] = e^{-3}$$

$\therefore$ long term rate at which working components are replaced is

$$\lim_{t\to\infty} \frac{\sum_{i=0}^{N(t)} C_i}{t} = \frac{\mathcal{E}[C]}{\mathcal{E}[\tilde{X}]} = \frac{e^{-3}}{T(1-e^{-3})}$$

7.60

~~5.60~~ a) A codeword is produced each time a pattern is completed. The average pattern length X is

$$\mathcal{E}[X] = 1(0.1) + 2(0.9) + 3(0.81) + 4(0.0729 + .6561) = 3.439$$

$$\therefore \text{codeword production rate} = \frac{1}{3.439}\,\frac{\text{codewords}}{\text{ms}}$$

b)

Define "cycle length" $\triangleq$ pattern length for one codeword
"cost" $\triangleq$ codeword length

$$\begin{array}{c}\text{long term ratio}\\ \text{of encoded bits to}\\ \text{information bits}\end{array} = \frac{\mathcal{E}[C]}{\mathcal{E}[X]} = \frac{1(.6561)+3(1-.6561)}{3.439} = 0.49$$

7.61

~~5.61~~ **a)** $\mathcal{E}[X] = T \qquad F_X(y) = \frac{y}{2T} \qquad 0 < y < 2T$

$$\begin{array}{l}\text{prop. of time}\\ r(t) \text{ exceed } c\end{array} = \frac{1}{T}\int_C^{2T}\left(1 - \frac{y}{2T}\right)dy = 1 - \frac{c}{T}\left(1 - \frac{c}{2T}\right)$$

b) $\mathcal{E}[X] = T \qquad F_X(y) = 1 - e^{-y/T}$

$$\begin{array}{l}\text{prop. of time}\\ r(t) \text{ exceed } c\end{array} = \frac{1}{T}\int_C^{\infty}(1 - (1 - e^{-y/T}))dy = e^{-c/T}$$

c) $\mathcal{E}[X] = T \qquad F_X(y) = 1 - e^{-\pi y^2/4T^2}$

$$\begin{array}{rcl}\begin{array}{l}\text{prop. of time}\\ r(t) \text{ exceeds } c\end{array} & = & \frac{1}{T}\int_c^{\infty}(1 - (1 - e^{-\pi y^2/4T^2}))dy \\ & = & \frac{1}{T}\int_c^{\infty} e^{-\pi y^2/4^2}dy = 2Q\left(\sqrt{\frac{\pi}{2}}\frac{c}{T}\right)\end{array}$$

d) Let R be the residual time at a randomly-selected time instant. Then parts a)-c) involved finding $P[R > C] = 1 - P[R \leq C]$

X uniform:

$$\mathcal{E}[R] = \int_0^{\infty} P[R > x]dx = \int_0^{2T}\left(1 - \frac{x}{T} - \frac{x^2}{4T^2}\right)dx = \frac{2}{3}T$$

X exponential:

$$\mathcal{E}[R] = \int_0^{\infty} P[R > x]dx = \int_0^{2T} e^{-x/T}dt = T$$

X Rayleigh, the pdf of R is

$$\begin{array}{rcl} f_R(x) & = & -\frac{d}{dx}P[R > x] = \frac{1}{T}e^{-\pi x^2/4T^2} \\ \therefore \mathcal{E}[R] & = & \int_0^{\infty} x f_R(x)dx = \int_0^{\infty}\frac{x}{T}e^{-\pi x^2/4T^2}dx = \frac{2}{\pi}T\end{array}$$

7.62

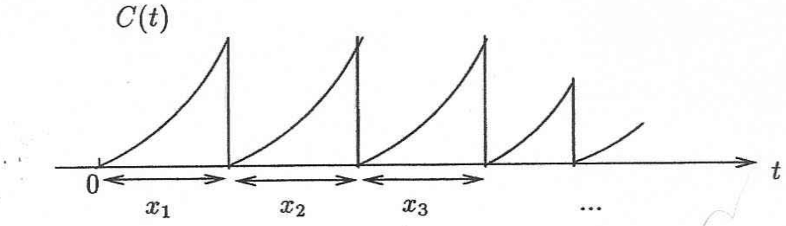
~~5.62~~ The amount of time that the age exceeds C in a cycle of length X is $(X - C)^+$. The long-term proportion of time that $a(t)$ exceeds C can be found by defining the cost per cycle by $C_j = (X_j - C)^+$. This is the same cost as considered in Example ~~5.23~~ 7,21. Therefore the long-term proportion of time $a(t)$ exceeds C is given by Eqn. ~~5.50~~ 7.39.

7.63

~~5.63~~ **a)** Since the age $a(t)$ is the time that has elapsed from the last arrival up to time t, then

$$C_j = \int_0^{X_j} a(t')dt' = \int_0^{X_j} t'dt' = \frac{X_j^2}{2}$$

The figure below shows the relation between $a(t)$ and the C_j's.



b) $\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathcal{E}[C]}{\mathcal{E}[X]} = \frac{\mathcal{E}[X^2]}{2\mathcal{E}[X]}$

c) From the above figure:

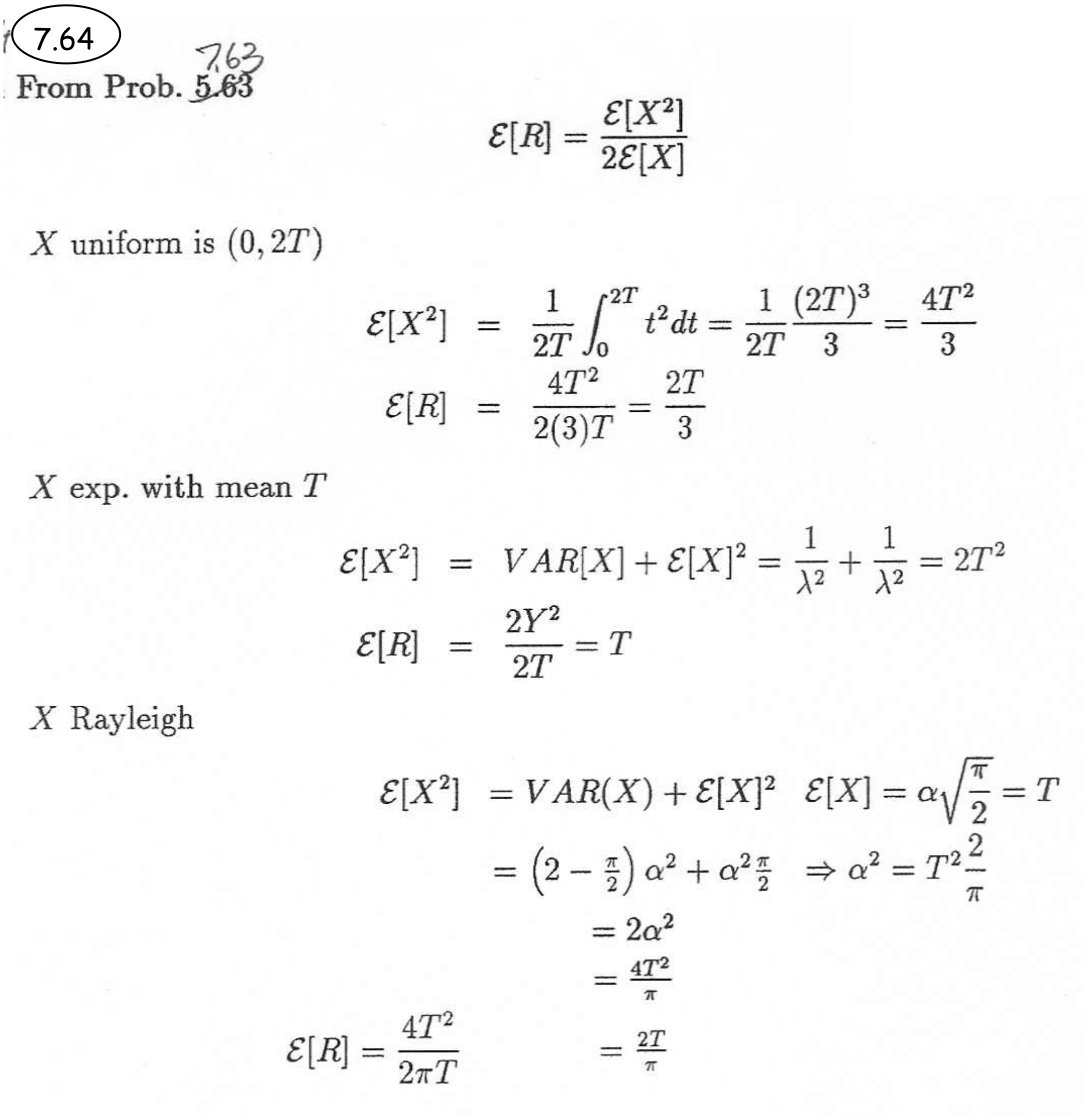
$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\int_0^t a(t')dt' &= \lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{N(t)} \int_0^{X_j} a(t')dt' \\
&= \lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{N(t)} C_j \\
&= \frac{\mathcal{E}[X^2]}{2\mathcal{E}[X]} \quad \text{from part b)}
\end{aligned}$$

d) For the residual life in a cycle

$$C_j' = \int_0^{X_j} r(t')dt' = \int_0^{X_j} (X_j - t')dt' = \frac{X_j^2}{2} = C_j$$

$\Rightarrow$ same cost as for age of a cycle

7.64



From Prob. ~~5.63~~ 7.63

$$\mathcal{E}[R] = \frac{\mathcal{E}[X^2]}{2\mathcal{E}[X]}$$

X uniform is $(0, 2T)$

$$\begin{aligned}\mathcal{E}[X^2] &= \frac{1}{2T}\int_0^{2T} t^2 dt = \frac{1}{2T}\frac{(2T)^3}{3} = \frac{4T^2}{3} \\ \mathcal{E}[R] &= \frac{4T^2}{2(3)T} = \frac{2T}{3}\end{aligned}$$

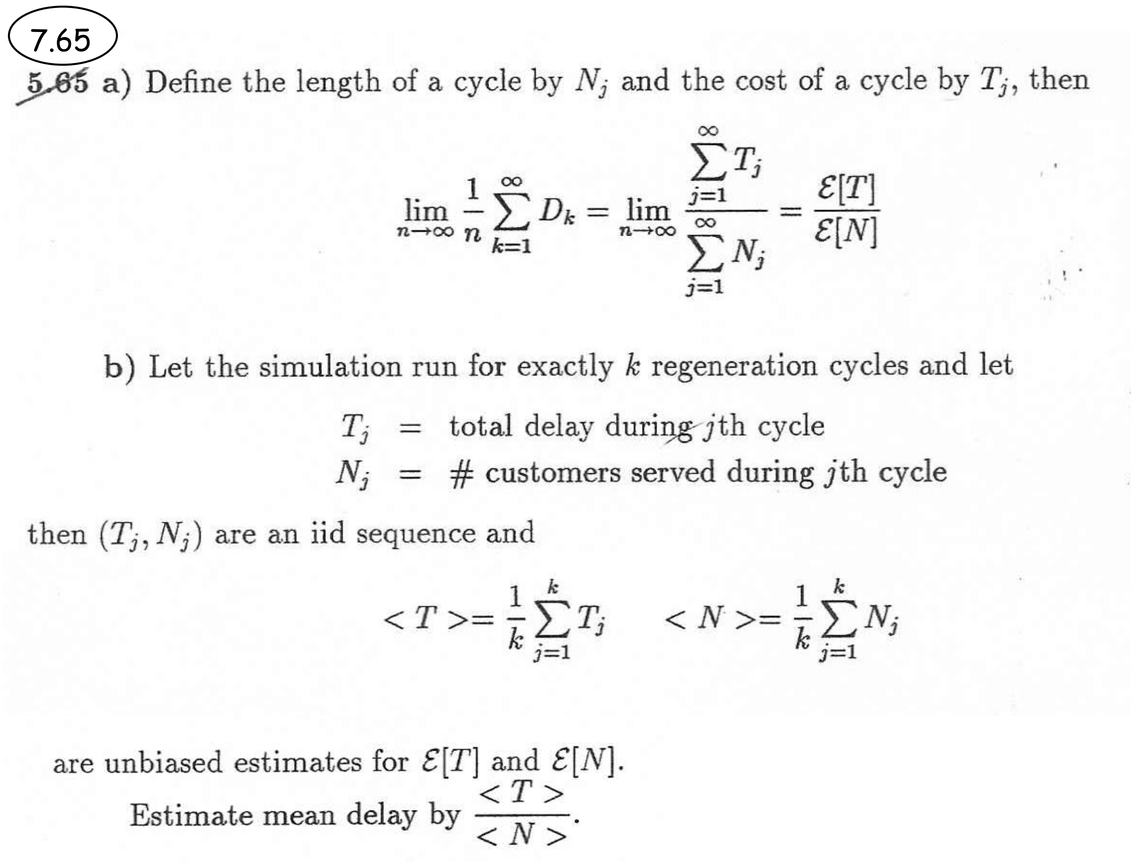
X exp. with mean T

$$\begin{aligned}\mathcal{E}[X^2] &= VAR[X] + \mathcal{E}[X]^2 = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 2T^2 \\ \mathcal{E}[R] &= \frac{2Y^2}{2T} = T\end{aligned}$$

X Rayleigh

$$\begin{aligned}\mathcal{E}[X^2] &= VAR(X) + \mathcal{E}[X]^2 \quad \mathcal{E}[X] = \alpha\sqrt{\frac{\pi}{2}} = T \\ &= \left(2 - \tfrac{\pi}{2}\right)\alpha^2 + \alpha^2\tfrac{\pi}{2} \quad \Rightarrow \alpha^2 = T^2\frac{2}{\pi} \\ &= 2\alpha^2 \\ &= \tfrac{4T^2}{\pi}\end{aligned}$$

$$\mathcal{E}[R] = \frac{4T^2}{2\pi T} \qquad = \tfrac{2T}{\pi}$$



7.65

~~5.65~~ a) Define the length of a cycle by N_j and the cost of a cycle by T_j, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty}D_k = \lim_{n\to\infty}\frac{\sum_{j=1}^{\infty}T_j}{\sum_{j=1}^{\infty}N_j} = \frac{\mathcal{E}[T]}{\mathcal{E}[N]}$$

b) Let the simulation run for exactly k regeneration cycles and let

T_j = total delay during jth cycle
N_j = # customers served during jth cycle

then (T_j, N_j) are an iid sequence and

$$< T >= \frac{1}{k}\sum_{j=1}^{k}T_j \qquad < N >= \frac{1}{k}\sum_{j=1}^{k}N_j$$

are unbiased estimates for $\mathcal{E}[T]$ and $\mathcal{E}[N]$.

Estimate mean delay by $\dfrac{< T >}{< N >}$.

*7.6 Calculating Distributions Using the Discrete Fourier Transform

7.66

~~5.66~~ **a)**

$$\begin{aligned} c_0 &= \frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1 \\ c_1 &= \frac{1}{3}+\frac{1}{3}e^{j\frac{2\pi}{3}}+\frac{1}{3}e^{j\frac{2\pi}{3}}=0 \\ c_2 &= \frac{1}{3}+\frac{1}{3}e^{j\frac{4\pi}{3}}+\frac{1}{3}e^{j\frac{8\pi}{3}}=0 \end{aligned}$$

b) $P[X=1]=\frac{1}{3}\left[1+0\cdot e^{-j\frac{2}{3}\pi}+0\cdot e^{-j\frac{4}{3}\pi}\right]=\frac{1}{3}$

7.67

$$\begin{aligned} \Phi_Z(\omega) &= \Phi_X(\omega)^2\left(\frac{1}{3}+\frac{1}{3}e^{j\omega}+\frac{1}{3}e^{j2\omega}\right)^2 \\ c_m &= \Phi_Z\left(\frac{2\pi m}{5}\right)=\left(\frac{1}{3}+\frac{1}{3}e^{j\frac{2\pi m}{5}}+\frac{1}{3}e^{j\frac{4\pi m}{5}}\right)^2 \qquad m=0,\ldots,4 \end{aligned}$$

a)

$$\begin{aligned} \Rightarrow c_0 &= 1 \\ c_1 &= \frac{1}{9}(1+e^{j2\pi/5}+e^{j4\pi/5})^2=-0.235+j(0.171) \\ c_2 &= 0.0131+j(-0.04) \\ c_3 &= 0.0131+j(0.04) \\ c_4 &= -0.235-j(0.171) \end{aligned}$$

b)

$$\begin{aligned} P[S=2] &= \frac{1}{5}\sum_{m=0}^{4}c_m e^{-j4\pi m/5} \\ &= \frac{1}{5}[1+(-.235+j(.171))e^{-j4\pi/5} \\ &\quad +(.0131-j(.04))e^{-j2\pi/5} \\ &\quad +(.0131+j(.04))e^{-j12\pi/5} \\ &\quad +(-.235-j(.171))e^{-j16\pi/5}] \\ &= \frac{1}{(1.666)}=\frac{1}{3} \end{aligned}$$

7.68 The following Octave code produces the FFTs:

```
N = 8;
P = 1/2;
n = [0:N-1];
cms = fft(binomial_pdf(n, N, P), 16);
%You can also evaluate the characteristic function directly...
%w = 2.*pi.*n./N;
%cms = (1-P+P.*e.^(j.*w)).^N;
pmf = ifft(cms.*cms);
figure;
stem([1:16], pmf, "b");
```

7.69 The following Octave code produces the FFTs:

```
N = 5;
P = ones(1,10)./10;
pmf = ifft(fft(P, 9.*N).^N);
stem([0:9.*N-1], pmf);

N = 10;
P = ones(1,10)./10;
pmf = ifft(fft(P, 9.*N).^N);
stem([0:9.*N-1], pmf);
```

7.70 The following Octave code produces the FFT for evaluating Eq. (7.55):

```
N = 8;
P = 1/2;
n = [0:N-1];
w = 2.*pi.*n./N;
cms = P.*e.^(j.*w)./(1-(1-P).*e.^(j.*w));
pmf = ifft(cms);
figure;
stem([N-1:-1:0], pmf);
ek = (1-P).*P.^n.*(P.^N./(1-P.^N));

N = 16;
P = 1/2;
n = [0:N-1];
w = 2.*pi.*n./N;
cms = P.*e.^(j.*w)./(1-(1-P).*e.^(j.*w));
pmf = ifft(cms);
figure;
stem([N-1:-1:0], pmf);
ek = (1-P).*P.^n.*(P.^N./(1-P.^N));
```

7.71 The following Octave code produces the FFTs:

```
%N=16 achieves an error percent less than 0.01. This can be found
%by simple trial and error using different values for N.
N = 16;
L = 5;
n = [0:N-1];
w = 2.*pi.*n./N;
cms = e.^(L.*(e.^(j.*w)-1));
pmf = ifft(cms);
figure;
stem([N-1:-1:0], pmf);

N = 5;
L = 5;
n = [0:19];
P = poisson_pdf(n, L);
pmf = ifft(fft(P, 65).^N);
stem([0:64], pmf);
```

7.74 The following Octave code produces the FFT to obtain the pdf of *Z*:

```
function fx = ift(phix, n, N)
    phixs = [phix((N/2+1):N) phix(1:(N/2))];
    fxs = fft(phixs)./(2.*pi);
    fx = fftshift(fxs);
end

N = 512;
n = [-(N/2):(N/2-1)];
d = 2.*pi.*n./N;
alphaX = 1;
alphaY = 2;
phiX = 1./(1 + alphaX.^2.*n.^2);
phiY = 1./(1 + alphaY.^2.*n.^2);
phiZ = phiX.*phiY;
pdf = ift(phiZ, n, N);
figure;
plot(d, pdf, "b");
hold on;
plot(d, ift(phiX, n, N), "g");
plot(d, ift(phiY, n, N), "r");
```

7.75 The following Octave code produces the FFT to obtain the pdf of a Gaussian random variable:

```
function fx = ift(phix, n, N)
    phixs = [phix((N/2+1):N) phix(1:(N/2))];
    fxs = fft(phixs)./(2.*pi);
    fx = fftshift(fxs);
end

N = 512;
n = [-(N/2):(N/2-1)];
d = 2.*pi.*n./N;
phiX = e.^(-n.^2./2);
pdf = ift(phiX, n, N);
figure;
plot(d, pdf, "b");
hold on;
plot(d, normal_pdf(d, 0, 1), "r");
```

7.76 The following Octave code produces Figs. 7.2 through 7.4:

```
function stepplot(x, y)
    xn = zeros(1, 2*length(x));
    yn = zeros(1, 2*length(x));
    for k = 1:length(x)
        xn(2*k-1) = x(k);
        xn(2*k) = x(k) + 1;
        yn(2*k-1) = y(k);
        yn(2*k) = y(k);
    end
    plot(xn, yn);
end

%Figure 7.2a
N = 5;
P = ones(1,2)./2;
pmf = ifft(fft(P, N+1).^N);
cdf = zeros(size(pmf));
for i = 1:N+1
    cdf(i) = sum(pmf(1:i));
end
figure;
stepplot([0:N], cdf);
hold on;
x = [0:0.01:N+1];
plot(x, normal_cdf(x, 2.5, 1.25), "r");

%Figure 7.2b
N = 25;
P = ones(1,2)./2;
pmf = ifft(fft(P, N+1).^N);
cdf = zeros(size(pmf));
for i = 1:N+1
    cdf(i) = sum(pmf(1:i));
end
figure;
stepplot([0:N], cdf);
hold on;
x = [0:0.01:N+1];
plot(x, normal_cdf(x, 12.5, 6.25), "r");

%Figure 7.3
N = 5;
P = ones(1,10)./10;
pmf = ifft(fft(P, 9.*N+1).^N);
cdf = zeros(size(pmf));
for k = 1:9.*N+1
    cdf(k) = sum(pmf(1:k));
end
figure;
stepplot([0:9.*N], cdf);
hold on;
x = [0:0.01:9.*N+1];
plot(x, normal_cdf(x, 22.5, 41.3), "r");
```

Problems Requiring Cumulative Knowledge

7.77

~~5.77~~ **a)** $S = X + Y$

Assume X and Y are independent

$$\begin{aligned} G_S(z) &= G_X(z)G_Y(z) \\ &= (pz + 1 - p)^n(rz + 1 - r)^m \end{aligned}$$

b) $G_S(z) = E[z^S] = \sum_{k=0}^{n+m} z^k P[S = k]$

$P[S = k]$ is the coefficient of z^k in $G_S(z)$.

$$\begin{aligned} G_S(z) &= (pj + 1 - p)^n(rj + 1 - r)^m \\ P[S = k] &= \sum_{j=0}^{k} \binom{n}{j} p^j(1 - p)^{n-j} \binom{m}{k-j} r^{k-j}(1 - r)^{m-(k-j)} \end{aligned}$$

c) Obtain $G_S(z)$ at $z = \exp(j2\pi k/(n + m))$. Use IDFT to obtain p_k.

7.78

~~5.78~~ $X_n = \frac{1}{2}U_n + \left(\frac{1}{2}\right)^2 U_{n-1} + \ldots + \left(\frac{1}{2}\right)^n U_1 \quad n \geq 1$

This "low-pass filter" weighs recent samples more heavily than older samples. Note that we can also write X_n as follows:

$$X_n = \frac{1}{2}X_{n-1} + \frac{1}{2}U_n \qquad X_0 = 0, \quad n \geq 1$$

We will see in Chapter 6 that X_n is an autoregressive random process.

a)

$$\begin{aligned}
E[X_n] &= E\left[\frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^j U_{n-j}\right] = \frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^j E[U] \\
&= \frac{1}{2}E[U]\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = E[U]\left(1-\left(\frac{1}{2}\right)^n\right) \\
&= 0 \qquad \text{since } E[U] = 0. \\
E[X_n^2] &= E\left[\frac{1}{2}\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^n U_{n-j}\ \frac{1}{2}\sum_{j'=0}^{n-1}\left(\frac{1}{2}\right)^{j'} U_{n-j'}\right] \\
&= \frac{1}{4}\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1}\left(\frac{1}{2}\right)^{j+j'} E[U_{n-j}U_{n-j'}] \\
&= \frac{1}{4}\sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^{2j} E[U^2] \quad \text{since the } U_j \text{ are iid} \\
&= \frac{\sigma^2}{3}\left(1-\left(\frac{1}{4}\right)^n\right) \quad \text{where } E[U^2] = \sigma^2 \\
VAR(X_n) &= E[X_n^2] - E[X_n]^2 = \frac{\sigma^2}{3}\left(1-\left(\frac{1}{4}\right)^n\right) \ .
\end{aligned}$$

Thus X_n is a zero-mean Gaussian random variable with the variance found in part a).

As $n \to \infty$

$$\Phi_{X_n}(\omega) \to e^{-\frac{1}{2}\frac{\sigma^2}{3}\omega^2}$$

so X_n approaches a zero-mean Gaussian random variable with variance $\sigma^2/3$.

c) The result in Part b) shows that X_n converges in distribution to a Gaussian random variable X with zero mean and variance $\sigma^2/3$.

To determine whether X_n converges in mean-square sense consider the Cauchy Criterion in Eq. 5.50. Consider X_n and X_{n+m}:

$$\begin{aligned}
\mathcal{E}[(X_{n+m} - X_n)^2] &= \mathcal{E}\left[\left(\frac{1}{2}\sum_{j=0}^{n+m-1}\left(\frac{1}{2}\right)^n U_{n-j} - \frac{1}{2}\sum_{j'=0}^{n+m-1}\left(\frac{1}{2}\right)^{j'} U_{n-j'}\right)^2\right] \\
&= \frac{1}{4}\mathcal{E}\left[\left(\sum_{j=n}^{n+m-1}\left(\frac{1}{2}\right)^j U_{n-j}\right)^2\right] \\
&= \frac{1}{4}\sum_{j=n}^{n+m-1}\sum_{j'=n}^{n+m-1}\left(\frac{1}{2}\right)^{j+j'} E[U_{n-j}U_{n-j'}] \\
&= \frac{\sigma^2}{4}\sum_{j=n}^{n+m-1}\left(\frac{1}{2}\right)^{2j} \\
&= \frac{\sigma^2}{4}\left(\frac{1}{4}\right)^n\left(\frac{1-(\frac{1}{4})^m}{1-\frac{1}{4}}\right) \\
&\to 0 \quad \text{as } n, m \to \infty
\end{aligned}$$

Therefore X_n converges in mean-square sense.

To determine almost-sure convergence of X_n would take us beyond the scope of the text. See Gray and Davisson, page 183 for a discussion on how this is done.

7.79

a) $S_n = X_1 + X_2 + \ldots + X_n$, S_n is Gaussian

$$\begin{aligned}
E[S_n] &= n\mu \\
VAR[S_n] &= E[[(X_1 - m) + \ldots + (X_n - m)]^2] \\
&= (\sigma^2 + p\sigma^2) + (p\sigma^2 + \sigma^2 + p\sigma^2) + (p\sigma^2 + \sigma^2 + p\sigma^2) \\
&\quad + \ldots + (\sigma^2 + \sigma^2 + p\sigma^2) + (p\sigma^2 + \sigma^2) \\
&= n\sigma^2 + [1 + 2(n-2) + 1]p\sigma^2 \\
&= (n + (2n-2)p)\sigma^2 \\
\phi_{S_n}(\omega) &= e^{jn\mu\omega - [n+2np-2p]\sigma^2\omega^2/2}
\end{aligned}$$

b) Suppose $n \geq m$.

$$\begin{aligned}
S_n - S_m &= X_{m+1} + X_{m+2} + \ldots + X_n \quad \text{also Gaussian} \\
E[S_n - S_m] &= (n-m)\mu \\
VAR[S_n - S_m] &= [(n-m) + 2(n-m)p - 2p]\sigma^2
\end{aligned}$$

c) Assume $n < m$.

$$\begin{aligned}
\phi_{S_m,S_n}(\omega_1,\omega_2) &= E[e^{j\omega_1 S_m + j\omega_2 S_n}] \\
&= E[E[e^{j\omega_1 S_m + j\omega_2 S_n}|S_m]] \\
&= E[e^{j\omega_1 S_m} E[e^{j\omega_2 S_n}|S_m]] \\
&= E\left[e^{j\omega_1 S_m}\left\{\exp\left[j\omega_2 n\mu - \frac{\omega_2^2\sigma^2}{2}[(n - m + 2(n-m)p - 2p]\right]\right\}\right] \\
&= \exp\left\{j\omega_1 m\mu - \frac{\omega_1^2\sigma^2}{2}[m + (2m-2)p]|\cdot\right. \\
&\quad \left.\exp\left\{j\omega_2 n\mu - \frac{\omega_2^2\sigma^2}{2}[(n-m) + 2(n-m)p - 2p]\right\}\ \right\}
\end{aligned}$$

d) No.

7.80)

$$S_n = \sum_{i=1}^{n} X_i$$

a) $$E[S_n] = \sum_{i=1}^{n} E[X_i] = n\mu$$

$$VAR[S_n] = E\left[\left[(X_1-\mu_1) + \dots + (X_n-\mu_n)\right]^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} COV(X_i, X_j)$$

$$= n\sigma^2 + (\rho\sigma^2) + (\underline{\rho\sigma^2 + \rho\sigma^2}) + \cdots + (\rho\sigma^2 + \rho\sigma^2) + \rho\sigma^2$$

$$= n\sigma^2 + 2\rho\sigma^2 + 2(n-2)\rho\sigma^2 = n\sigma^2 + (2n-2)\rho\sigma^2 = (n + (2n-2)\rho)\sigma^2$$

S is a Gaussian RV, Therefore:

$$\Phi_{S_n}(\omega) = e^{\{j\omega n\mu) - \frac{1}{2}(n+(2n-2)\rho)\sigma^2\omega^2\}}$$

b) Suppose $n > m$

$S_n - S_m = X_{m+1} + \dots + X_n$ is also Gaussian

$$E[S_n - S_m] = E[S_n] - E[S_m] = (n-m)\mu$$

$$VAR[S_n - S_m] = \sum_{i=m+1}^{n}\sum_{j=m+1}^{n} cov(X_i, X_j) = [(n-m) + 2(n-m)\rho - 2\rho]\sigma^2$$

P7.80)

c) S_m & $S_n - S_m$ are indepent if $m < n$

we have:

$$\varphi_{S_n - S_m}(\omega) = e^{\{j\omega \quad) - \frac{\sigma^2\omega^2}{2}((n-m) + 2(n-m)\rho - 2\rho]\}}$$

$$\varphi_{S_m}(\omega) = e^{\{j\omega \quad) - \frac{1}{2}(m + (2m-2)\rho)\sigma^2\omega^2\}}$$

Therefore: $\varphi_{S_m, S_n}(\omega_1, \omega_2) = \quad = E[e^{j\omega_1 S_m + j\omega_2 S_n}]$

$$= E\left[E\left[e^{j\omega_1 S_m + j\omega_2 S_n} \middle| S_m\right]\right]$$

$$= E\left[e^{j\omega_1 S_m} \; E\left[e^{j\omega_2 S_n} \middle| S_m\right]\right]$$

$$= E\left[e^{j\omega_1 S_m}\right] E\left[e^{j\omega_2 S_n} \middle| S_m\right]$$

$$= E\left[e^{j\omega_1 S_m}\right] E\left[e^{j\omega_2 (S_n - S_m)}\right]$$

$$= \varphi_{S_m}(\omega_1)\, \varphi_{S_n - S_m}(\omega_2)$$

d) Using Cauchy Criterion, it can be seen that it doesn't converge.

7.81

7,81

~~5.80~~ Z_n does not converge in mean square sense.

7.82

~~5.81~~ a)

$$\begin{aligned}
E[Y_n] &= -\Sigma P[X_n]\log_2 P[X_n] = H(X) \\
&= \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \frac{1}{16}\cdot 4 \\
&= 1\frac{7}{8} \\
VAR[Y_n] &= E[(Y_n - 1\frac{7}{8})^2] \\
&= \frac{1}{2}(1-1\frac{7}{8})^2 + \frac{1}{4}(2-1\frac{7}{8})^2 + \frac{1}{8}(3-1\frac{7}{8})^2 + 2\frac{1}{16}(4+1\frac{7}{8})^2 \\
&= 1.11
\end{aligned}$$

b) Functions of independent RVs are also independent, so $\{Y_n\}$ is an iid sequence. The weak law and strong of large numbers apply.

c) The long term bit rate $S_n = \frac{1}{n}\sum_{k=1}^{n} L(x_k) = \frac{1}{n}\sum_{k=1}^{n} Y_k$ will approach the entropy H_x of the source with probability 1.

Chapter 8: Statistics

8.1 Samples and Sampling Distributions

8.1 $\mu = 10 \quad \sigma_X^2 = 4 \quad n = 9$

ⓐ $P[\bar{X}_9 < 9] = P\left[\frac{\bar{X}_9 - \mu}{\sigma_X/\sqrt{n}} < \frac{9-\mu}{\sigma_X/\sqrt{n}}\right] = P\left[\frac{\bar{X}_9 - 10}{2/\sqrt{9}} < \frac{9-10}{2/3}\right]$

$= 1 - Q\left(-\frac{3}{2}\right) = 0.0668$

ⓑ $P[\min(X_1, \ldots X_9) > 8] = P[X_1 > 8]\, P[X_2 > 8] \cdots P[X_9 > 8]$

$= P[X_1 > 8]^9$

$= Q\left(\frac{8-10}{2}\right)^9 = Q(-1)^9$

$= 0.2112$

ⓒ $P[\max(X_1, \ldots, X_9) < 12] = P[X_1 < 12] \cdots P[X_9 < 12]$

$= \left(1 - Q\left(\frac{12-10}{2}\right)\right)^9 = (1 - Q(1))^9$

$= 0.2112$

ⓓ $0.95 = P[|\bar{X}_n - 10| < 1] = P\left[\left|\frac{\bar{X}_n - 10}{2/\sqrt{n}}\right| < \frac{1}{2/\sqrt{n}}\right]$

$= P\left[-\frac{\sqrt{n}}{2} < \frac{\bar{X}_n - 10}{2/\sqrt{n}} < \frac{\sqrt{n}}{2}\right]$

$= P\left[-1.96 < \frac{\bar{X}_n - 10}{2\sqrt{n}} < 1.96\right]$

$\Rightarrow \sqrt{n} = 2(1.96) \qquad n = 4(1.96)^2 = 15.366 = 16$

8.1 – continued –

Octave command to generate 100 samples of groups of 9

```
> normal_rnd (10, 4, 9, 100)
```

To find the sample mean of each group of 9:

```
> mean (normal_rnd (10, 4, 9, 100)
```

From sample of 100 we found:

$0.07 = \frac{7}{100}$ had values less than 9 vs. 0.0668 theory.

$0.19 = \frac{19}{100}$ had max of group < 12 vs. 0.2112

$0.18 = \frac{18}{100}$ had min of group > 8 vs. 0.2112

Max & MIN obtained using:

```
> max (normal_rnd (10, 4, 9, 100))
> min (normal_rnd (10, 4, 9, 100))
```

8.2 X exponential $\mu=50$ $n=25$ $\sigma^2 = \frac{1}{\lambda^2} = \mu^2 = 50^2$

ⓐ $P[|\bar{X}_{25}-50|<1] = P\left[\left|\frac{\bar{X}_{25}-50}{50/\sqrt{25}}\right| < \frac{1}{50/\sqrt{25}}\right]$

$= P\left[-\frac{1}{10} < \left|\frac{\bar{X}_{25}-50}{10}\right| < \frac{1}{10}\right]$

$= 0{,}07966$

ⓑ $P[\max(X_1,\ldots,X_{25}) > 100] = 1-P[\max(\quad) < 100]$

$= 1 - P[X_1<100]\,P[X_2<100]\cdots P[X_{25}<100]$

$= 1-(1-e^{-100/50})^{25} = 1-(1-e^{-2})^{25}$

$= 0{,}9736$

ⓒ $P[\min(X_1\ldots X_{25}) < 25] = 1-P[\min(X_1\ldots X_{25}) > 25]$

$= 1-P[X_1>25]^{25} = 1-(e^{-25/50})^{25}$

$= 1-e^{-25/2} = 1-3.73\times10^{-6}$

ⓓ $0.90 = P[|\bar{X}_n-50|<5] = P\left[\left|\frac{\bar{X}_n-50}{50/\sqrt{n}}\right| < \underbrace{\frac{5}{50/\sqrt{n}}}_{1{,}64}\right]$

$\frac{\sqrt{n}}{10} = 1{,}64$

$\sqrt{n} = 16.4$ $n = 269$

ⓔ Using approach in problem 8.1 (but generating exponential samples)

$0.08 = \frac{8}{100}$ samples were between 49 & 50 vs. 0.07966

$0.97 = \frac{97}{100}$ samples of max >100 all samples of min <25

8.3 X uniform [-3,3] $n=50$ $\mu=0$ $\sigma^2 = \frac{6^2}{12} = 3$

ⓐ $P[|X_{50}| > 0.5] = P\left[\left|\frac{X_{50}}{\sqrt{3}/\sqrt{50}}\right| > \underbrace{\frac{0.5}{\sqrt{3}/\sqrt{50}}}_{2.041}\right] = 0.0206$

ⓑ $P[\max(X_1, \ldots X_{50}) < 2.5] = P[X_1 < 2.5]^{50} = \left(\frac{5.5}{6}\right)^{50} = 0.0129$

ⓒ $E[Y] = E[X^2] = \frac{1}{6}\int_{-3}^{3} x^2\,dx = 3$

$E[Y^2] = E[X^4] = \frac{1}{6}\int_{-3}^{3} x^4\,dx = \frac{81}{5}$

$VAR[Y] = E[Y^2] - E[Y]^2 = \frac{81}{5} - 9 = \frac{36}{5}$

$P[\bar{Y}_{50} > 3] = P\left[\frac{\bar{Y}_{50} - 3}{6/\sqrt{5}\cdot\sqrt{50}} > \frac{3-3}{6/\sqrt{5}\sqrt{50}}\right] = Q($[illegible]$)$

$= 1/2$

ⓓ In 100 samples of sample means in groups of 50

0.0[illegible] $= \frac{\text{[illegible]}}{100}$ were > 0.5 vs. 0.0206

none of the max's were < 2.5 vs. 0.0129

$0.50 = \frac{50}{100}$ of the sample mean of $Y^2 < 3$ vs 0.50.

8.4 $X \quad \mu = 2 \quad \sigma^2 = 2 \quad n = 16$

ⓐ $$P[\bar{X}_{16} > 2.5] = P\left[\frac{\bar{X}_{16} - 2}{\sqrt{2}/4} > \frac{2.5-2}{\sqrt{2}/4}\right] =$$

$$= Q(\sqrt{2}) = 0.0786$$

ⓑ $$P[|\bar{X}_{16} - 2| > 0.5] = P\left[\frac{|\bar{X}_{16} - 2|}{\sqrt{2}/4} > \frac{0.5}{\sqrt{2}/4}\right]$$

$$= 2Q(\sqrt{2}) = 0.1572$$

ⓒ $$0.95 = P[|\bar{X}_n - 2| > 0.5] = P\left[\frac{|\bar{X}_n - 2|}{\sqrt{2}/\sqrt{n}} > \underbrace{\frac{0.5}{\sqrt{2}/\sqrt{n}}}_{1.96}\right]$$

$$\frac{0.5\sqrt{n}}{\sqrt{2}} = 1.96 \qquad n = \left((1.96)(2)\sqrt{2}\right)^2$$

$$= 30.73$$

$$= 31$$

ⓓ Use method in Problem 8.1

8.5 X exponential $\frac{1}{\lambda}=\frac{1}{4}$ $n=9$ $\sigma^2=\frac{1}{\lambda^2}=\frac{1}{16}$

ⓐ
$$P[|\hat{\lambda}_1-4|>1]=P\left[\left|\frac{1}{\bar{X}_9}-4\right|>1\right]=1-P\left[\left|\frac{1}{\bar{X}_9}-4\right|<1\right]$$
$$=1-P\left[-1<\frac{1}{\bar{X}_9}-4<1\right]$$
$$=1-P\left[3<\frac{1}{\bar{X}_9}<5\right]=1-P[3\bar{X}_9<1<5\bar{X}_9]$$
$$=1-P\left[\frac{1}{5}<\bar{X}_9<\frac{1}{3}\right]$$
$$=1-P\left[\frac{\frac{1}{5}-\frac{1}{4}}{1/\sqrt{16}\sqrt{3}}<\bar{X}_9<\frac{\frac{1}{3}-\frac{1}{4}}{1/\sqrt{16}\sqrt{3}}\right]$$
$$=1-P\left[-\frac{\sqrt{3}}{5}<\bar{X}_9<\frac{1}{\sqrt{3}}\right]$$
$$\approx 1-0{,}3536=0{,}6463$$

(b)
$$\hat{\lambda}_2=\frac{1}{9\min(X_1,\ldots,X_9)}$$

$$P[|\hat{\lambda}_2-4|>1]=P[\hat{\lambda}_2<3 \text{ or } \hat{\lambda}_2>5]$$
$$=P\left[\frac{1}{9\min()}<3\right]+P\left[\frac{1}{9\min()}>5\right]$$
$$=P\left[\frac{1}{27}<\min()\right]+P\left[\frac{1}{45}>\min()\right]$$
$$=P\left[X>\frac{1}{27}\right]^9+1-P\left[X>\frac{1}{45}\right]^9$$
$$=(e^{-4/27})^9+1-(e^{-4/45})^9$$
$$=e^{-4/3}+1-e^{-4/5}=0{,}814$$

ⓒ Out of 100 samples of minimum of group of 9

$\frac{80}{100}$ had values <3 or >5.

8.6 ⓐ X uniform in $[0, \theta]$ $E[X] = \frac{\theta}{2}$

$$\hat{m}_1 = \frac{1}{n}\sum_{j=1}^{n} X_j = \frac{\theta}{2}$$

$$\Rightarrow \hat{\theta} = 2\hat{m}_1$$

ⓑ

$$E[\hat{\theta}] = E\left[2\frac{1}{n}\sum_{j=1}^{n} X_j\right] = 2\frac{1}{n}\sum_{j=1}^{n} E[X] = 2E[X].$$

$$VAR[\hat{\theta}] = E\left[(\hat{\theta} - 2E[X])^2\right] = E\left[\left(\frac{2}{n}\sum_{j=1}^{n} X_j - 2E[X]\right)^2\right]$$

$$= \frac{4}{n^2} E\left[\left(\sum_{j=1}^{n}(X_j - E[X])\right)^2\right]$$

$$= \frac{4}{n^2} E\left[\sum_{j=1}^{n}\sum_{i=1}^{n}(X_j - E[X])(X_i - E[X])\right]$$

$$= \frac{4}{n^2} \sum_{j=1}^{n} E[(X_j - E[X])^2] + \sum_{\substack{i\ j \\ i\neq j}}\sum \underbrace{E[X_j - E[X]]\,E[X_i - E[X_i]]}_{0}$$

$$= \frac{4}{n^2} n VAR[X]$$

$$= \frac{4}{n} VAR[X].$$

8.7 X Gamma α and $\beta = 1/\lambda$.

ⓐ $$\hat{m}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \approx \frac{\alpha}{\lambda} \Rightarrow \alpha = \lambda \hat{m}_1$$

$$\hat{m}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \approx VAR[X] + E[X]^2 = \frac{\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda^2}$$

$$\Rightarrow \hat{m}_2 = \frac{\lambda \hat{m}_1}{\lambda^2} + \frac{\lambda^2 \hat{m}_1^2}{\lambda^2} = \frac{\hat{m}_1}{\lambda} + \hat{m}_1^2$$

$$\hat{m}_2 - \hat{m}_1^2 = \frac{\hat{m}_1}{\lambda} \qquad \lambda = \frac{\hat{m}_1}{\hat{m}_2 - \hat{m}_1^2} \quad //$$

$$\alpha = \lambda \hat{m}_1 = \frac{\hat{m}_1^2}{\hat{m}_2 - \hat{m}_1^2} \quad //$$

ⓑ As n becomes large $\hat{m}_1$ and $\hat{m}_2$ approach $E[X]$ and $E[X^2]$ and so the estimators approach their true values.

Note: Means are known.

(8.8)

ⓐ

$$E[\hat{C}_{XY}] = E\left[\frac{1}{n}\sum_{j=1}^{n}(X_j-\mu_1)(Y_j-\mu_2)\right]$$

$$= \frac{1}{n}\sum_{j=1}^{n}\underbrace{E[(X_j-\mu_1)(Y_j-\mu_2)]}_{\mathrm{COV}(X,Y)}$$

$$= \frac{1}{n}\, n\, \mathrm{COV}(X,Y) = \mathrm{COV}(X,Y).$$

$$E[(\hat{C}_{XY}-\mathrm{COV}(X,Y))^2] = E\left[\left(\frac{1}{n}\sum(X_j-\mu_1)(Y_j-\mu_2)-\mathrm{COV}(X,Y)\right)^2\right]$$

$$= E\left[\frac{1}{n^2}\sum_j\sum_i\{(X_j-\mu_1)(Y_j-\mu_2)-\mathrm{COV}(X,Y)\}\{(X_i-\mu_1)(Y_i-\mu_2)-\mathrm{COV}(X,Y)\}\right]$$

$i \neq j$ cross product terms have zero expected value

$$= \frac{1}{n^2}\sum_j E\left[\left((X_j-\mu_1)(Y_j-\mu_2)-\mathrm{COV}(X,Y)\right)^2\right]$$

$$= \frac{1}{n^2}\sum_j \Big\{E[(X_j-\mu_1)^2(Y_j-\mu_2)^2] - 2E[(X_j-\mu_1)(Y_j-\mu_2)]\,\mathrm{COV}(X,Y) + \mathrm{COV}^2(X,Y)\Big\}$$

$$= \frac{1}{n}\left[E[(X-\mu_1)^2(Y-\mu_2)^2] - \mathrm{COV}^2(X,Y)\right]$$

ⓑ If $E[(X-\mu_1)^2(Y-\mu_2)^2]$ is bounded then variance of estimator approaches zero as $n\to\infty$.

8.9 Means unknown.

ⓐ
$$\hat{K}_{XY} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j-\bar{X}_n)(Y_j-\bar{Y}_n)$$

$$= \frac{1}{n-1}\sum_{j=1}^{n}(X_j-\mu_1+\mu_1-\bar{X}_n)(Y_j-\mu_2+\mu_2-\bar{Y}_n)$$

$$= \frac{1}{n-1}\sum_{j}\Big[(X_j-\mu_1)(Y_j-\mu_2)+(\mu_1-\bar{X}_n)(Y_j-\mu_2)$$
$$+(X_j-\mu_1)(\mu_2-\bar{Y}_n)+(\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$= \frac{1}{n-1}\Big\{\sum_{j}(X_j-\mu_1)(Y_j-\mu_2)+(\mu_1-\bar{X}_n)\sum_{j}(Y_j-\mu_2)$$
$$+(\mu_2-\bar{Y}_n)\sum_{j}(X_j-\mu_1)+(\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$= \frac{1}{n-1}\sum_{j}\Big\{(X_j-\mu_1)(Y_j-\mu_2)-(\mu_1-\bar{X}_n)(\mu_2-\bar{Y}_n)\Big]$$

$$E[\hat{K}_{XY}] = \frac{1}{n-1}\sum_{j}\Big\{COV(X,Y)-\frac{1}{n^2}\underbrace{\sum_{i}(X_i-\mu_1)\sum_{j}(Y_j-\mu_2)}_{n\,COV(X,Y)}\Big\}$$

$$= \frac{n}{n-1}\Big\{1-\frac{1}{n}\Big\}COV(X,Y)$$

$$= COV(X,Y)$$

ⓑ as n becomes large $\bar{X}_n \to E[X]$ and $\bar{Y}_n \to E[Y]$

so the estimator approaches the estimator in problem 8.8.

8.10) $W = \min(X_1 \ldots X_n) \qquad Z = \max(X_1 \ldots X_n)$

ⓐ

$$P[\min(X_1 \ldots X_n) > x] = P[X_1 > x, X_2 > x, \ldots, X_n > x]$$
$$= P[X > x]^n$$

$$\Rightarrow \quad F_W(x) = 1 - P[X > x]^n$$
$$f_W(x) = n F_X(x)^{n-1} f_X(x)$$

ⓑ

$$P[\max(X_1 \ldots X_n) \le x] = P[X_1 \le x, \ldots, X_n \le x]$$
$$= P[X < x]^n = F_X^n(x)$$

$$\Rightarrow \quad F_Z(x) = F_X^n(x)$$
$$f_Z(x) = n F_X^{n-1}(x) f_X(x).$$

8.2 Parameter Estimation

8.11

$$E[(\hat{\theta}-\theta)^2] = E[(\hat{\theta} - E[\hat{\theta}] + E[\hat{\theta}] - \theta)^2]$$

$$= E\left[(\hat{\theta} - E[\hat{\theta}])^2 + 2(\hat{\theta} - E[\hat{\theta}])(E[\hat{\theta}] - \theta) + (E[\hat{\theta}] - \theta)^2\right]$$

$$= VAR[\hat{\theta}] + 2\underbrace{E[\hat{\theta} - E[\hat{\theta}]]}_{0}(E[\hat{\theta}] - \theta) + (\underbrace{E[\hat{\theta}] - \theta}_{B(\hat{\theta})})^2$$

$$= VAR[\hat{\theta}] + B(\hat{\theta})^2$$

8.12 X_i Poisson, $\alpha = 4$

ⓐ $E[\hat{\alpha}_1] = E\left[\frac{X_1+X_2}{2}\right] = \frac{1}{2}E[X_1] + \frac{1}{2}E[X_2] = \alpha$ unbiased

$VAR[\hat{\alpha}_1] = VAR\left[\frac{X_1+X_2}{2}\right] = \frac{VAR[X]}{2} = \frac{\alpha}{2}$

ⓑ $E[\hat{\alpha}_2] = E\left[\frac{X_3+X_4}{2}\right] = \alpha$ unbiased

$VAR[\hat{\alpha}_2] = VAR\left[\frac{X_3+X_4}{2}\right] = \frac{VAR[X]}{2} = \frac{\alpha}{2}$

ⓒ $E[\hat{\alpha}_3] = E\left[\frac{X_1+2X_2}{3}\right] = \frac{1}{3}E[X_1] + \frac{2}{3}E[X_2] = \alpha$ unbiased

$VAR[\hat{\alpha}_3] = E\left[\left(\frac{1}{3}X_1 + \frac{2}{3}X_2 - \alpha\right)^2\right] = E\left[\left(\frac{1}{3}X_1 - \frac{1}{3}\alpha + \frac{2}{3}X_2 - \frac{2}{3}\alpha\right)^2\right]$

$= \frac{1}{9}E[(X_1-\alpha)^2] + \frac{4}{9}E[(X_2-\alpha)^2] + 0$

$= \frac{1}{9}\alpha + \frac{4}{9}\alpha = \frac{5}{9}\alpha$

ⓓ $E[\hat{\alpha}_4] = E\left[\frac{X_1+X_2+X_3+X_4}{4}\right] = E[X] = \alpha$ unbiased

$VAR[\hat{\alpha}_4] = \frac{VAR[X]}{4} = \frac{\alpha}{4}$

8.13

ⓐ $E[\hat{\theta}] = E[p\hat{\theta}_1 + (1-p)\hat{\theta}_2] = pE[\hat{\theta}_1] + (1-p)E[\hat{\theta}_2] = p\theta + (1-p)\theta = \theta$

$\Rightarrow$ unbiased.

ⓑ $$\frac{d}{dp}\left[E\left[\left(p\hat{\theta}_1 + (1-p)\hat{\theta}_2 - \theta\right)^2\right]\right] = E\left[2\left(p\hat{\theta}_1 + (1-p)\hat{\theta}_2\right)\left(\hat{\theta}_1 - \hat{\theta}_2\right)\right] = 0$$

$$0 = pE[\hat{\theta}_1(\hat{\theta}_1 - \hat{\theta}_2)] + (1-p)E[\hat{\theta}_2(\hat{\theta}_1 - \hat{\theta}_2)]$$

$$= pE[\hat{\theta}_1(\hat{\theta}_1 - \hat{\theta}_2)] + E[\hat{\theta}_2(\hat{\theta}_1 - \hat{\theta}_2)] - pE[\hat{\theta}_2(\hat{\theta}_1 - \hat{\theta}_2)]$$

$$\Rightarrow \quad p = \frac{E[\hat{\theta}_2(\hat{\theta}_1 - \hat{\theta}_2)]}{E[\hat{\theta}_2(\hat{\theta}_1 - \hat{\theta}_2)] - E[\hat{\theta}_1(\hat{\theta}_1 - \hat{\theta}_2)]}$$

ⓒ $\hat{\theta}_1 = \frac{X_1 + X_2}{2} \quad \hat{\theta}_2 = \frac{X_3 + X_4}{2}$ Note that $E[\hat{\theta}_1\hat{\theta}_2] = E[\hat{\theta}_1]E[\hat{\theta}_2]$ since X_1, X_2 are indep of X_3, X_4

$$p = \frac{E[\hat{\theta}_2]E[\hat{\theta}_1] - E[\hat{\theta}_2^2]}{E[\hat{\theta}_2]E[\hat{\theta}_1] - E[\hat{\theta}_2^2] - E[\hat{\theta}_1^2] + E[\hat{\theta}_1]E[\hat{\theta}_2]}$$

$$= \frac{\alpha^2 - (\alpha + \alpha^2)}{\alpha^2 - (\alpha + \alpha^2) - (\alpha + \alpha^2) + \alpha^2}$$

$$= \frac{-\alpha}{-2\alpha} = \frac{1}{2}$$

ⓓ $$E[\hat{\theta}_1\hat{\theta}_4] = E\left[\frac{1}{2}(X_1 + X_2) \bullet \frac{1}{4}(X_1 + X_2 + X_3 + X_4)\right]$$

$$= \frac{1}{8}\underbrace{E[(X_1 + X_2)^2]}_{2\alpha + 4\alpha^2} + \frac{1}{8}\underbrace{E[(X_1 + X_2)]}_{2\alpha}\underbrace{E[(X_3 + X_4)]}_{2\alpha}$$

$$= \frac{1}{4}\alpha + \alpha^2$$

8.13d — continued — from part ⓑ

$$\rho = \frac{E[\hat{\theta}_4 \hat{\theta}_1] - E[\hat{\theta}_4^2]}{E[\hat{\theta}_4 \hat{\theta}_1] - E[\hat{\theta}_4^2] - E[\hat{\theta}_1^2] + E[\hat{\theta}_4 \hat{\theta}_1]}$$

$$E[\hat{\theta}_4^2] = VAR[\hat{\theta}_4] + E[\hat{\theta}_4]^2 = \frac{\alpha}{4} + \alpha^2$$

$\therefore \ \rho = 0 \Rightarrow$ Estimator 1 is not used in combined estimator because its terms already accounted for.

ⓔ $E[\hat{\theta}_1] = E[X] \qquad E[\hat{\theta}_2] = E[X^2]$

$$VAR[X] = E[X^2] - E[X]^2$$

An obvious choice to estimate VAR[X] is

$$\hat{\theta}_3 = \hat{\theta}_2 - \hat{\theta}_1^2$$

then

$$E[\hat{\theta}_3] = E[\hat{\theta}_2] - E[\hat{\theta}_1^2] = E[X^2] - (VAR[\hat{\theta}_1^2] + E[\hat{\theta}_1]^2)$$

$$= E[X^2] - E[X]^2 - \underbrace{VAR[\hat{\theta}_1^2]}_{\text{bias}}$$

8.14 $Y = \theta + N$ N unif in $[0,2]$ $Y_i = \theta + N_i$ $i = 1, \ldots, n$
N_i iid.

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

$$E[\bar{Y}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}(\theta + N_i)\right] = \theta + E[N_i]$$

$$= \theta + 1 \Rightarrow \text{biased estimator}$$

$$E[(\bar{Y}_n - \theta)^2] = VAR[\bar{Y}_n] + B(\bar{Y}_n)^2 = \frac{1}{n}VAR[Y] + 1$$

$$= \frac{1}{n}\frac{2^2}{12} + 1 = \frac{1}{3n} + 1.$$

8.15 X Poisson $\alpha = 2$ req/min

ⓐ $\hat{p}_0 = e^{-\alpha}$ $\hat{\alpha} = \ln 1/\hat{p}_0$

\# of zero-arrival intervals.
Binomial with parameter n + $p_0 = e^{-\alpha}$

ⓑ $$E[\hat{\alpha}] = -E[\ln \hat{p}_0] = -E\left[\ln \frac{k_0}{n}\right]$$

$$= -\sum_{j=0}^{n}\binom{n}{j} p_0^{\,j}(1-p_0)^{n-j}\ln\frac{j}{n}$$

$$= -\sum_{j=0}^{n}\binom{n}{j}(e^{-2})^{j}(1-e^{-2})^{n-j}\ln\frac{j}{n}$$

$$= \underbrace{-\ln\frac{0}{n}}_{\infty}(1-e^{-2})^{n} + \text{other terms}$$

if no zero-arrival intervals occur estimator decides arrival rate is infinite.

ⓒ $MSE[\hat{\alpha}] = VAR[\hat{\alpha}] + B(\hat{\alpha})^2 =$ infinity.

ⓓ As $n \to \infty$ $\hat{p}_0 \to p_0 \Rightarrow \hat{\alpha} = \ln\frac{1}{\hat{p}_0} \to \alpha$

$\Rightarrow \hat{\alpha}$ is consistent.

Sample Mean Estimator

8.16

```
> x = poisson_rnd(2, 20, 100);      generate 100 groups of 10 poisson samples
> mean(mean(x));                    sample mean of group means
ans = 2.0585
> std(mean(x))^2;                   Sample variance of group means
ans = 0.098488
```

∴ sample mean estimator works well.

Zero-Count Estimator

Let x be poisson samples from above

```
> y = min(x, 1);                    indicator function for non-zero arrival
> sum(y);                           # of nonzero arrivals
> 20.- sum(y);                      # of zero counts in 20 minutes
```

Histogram for 100 samples are

k_0	# occurences	estimate value
0	10	∞
1	22	$-\ln 1/20 = +3$
2	24	$-\ln 2/20 = 2.3$
3	15	$-\ln 3/20 = 1.90$
4	19	$-\ln 4/20 = 1.61$
5	6	$-\ln 5/20 = 1.39$
6	3	$-\ln 6/20 = 1.20$
7	1	$-\ln 7/20 = 1.05$

$\Bigg\}$ sample mean of 90 nonzero estimates 2.1465

Excluding the intervals where no arrivals occur leads to a viable estimator.

8.17 $\hat{p} = \frac{k}{n}$ $\quad \hat{\sigma}_n^2 = \hat{p}(1-\hat{p}) = \frac{k}{n}\left(1-\frac{k}{n}\right)$

ⓐ $E[\sigma_n^2] = E\left[\frac{k}{n}\left(1-\frac{k}{n}\right)\right] = E\left[\frac{k}{n} - \frac{k^2}{n^2}\right] = \frac{1}{n}E[k] - \frac{1}{n^2}E[k^2]$

$$= \frac{1}{n}np - \frac{1}{n^2}[npq + (np)^2]$$

$$= p - \frac{pq}{n} - p^2 = \underbrace{(p - p^2)}_{\substack{p(1-p) \\ \text{variance of} \\ \text{Bernoulli}}} - \underbrace{\frac{pq}{n}}_{\text{bias.}}$$

ⓑ as $n \to \infty$ $\hat{p} = \frac{k}{n} \to p$

$\therefore \hat{\sigma}_n^2 \to p(1-p)$ as well

$\therefore \hat{\sigma}_n^2$ is consistent.

ⓒ $E[\sigma_n^2] = p(1-p) - \frac{p(1-p)}{n} = p(1-p)\underbrace{\left(1-\frac{1}{n}\right)}_{\frac{n-1}{n}}$

$\Rightarrow c = \frac{n}{n-1}$

ⓓ $\text{MSE}[\hat{\sigma}_n^2] = E\left[\left(\hat{\sigma}_n^2 - p(1-p)\right)^2\right]$

$$= E[(\hat{\sigma}_n^2)^2] - 2\underbrace{E[\hat{\sigma}_n^2]}_{\frac{p(1-p)(n-1)}{n}}p(1-p) + p^2(1-p)^2$$

$$E[(\hat{\sigma}_n^2)^2] = E\left[\left(\frac{k^2}{n}\right)\left(1-\frac{k}{n}\right)^2\right] = E\left[\frac{k^2}{n^2}\left(1 - \frac{2k}{n} + \frac{k^2}{n^2}\right)\right]$$

$$= \frac{1}{n^2}E[k^2] - \frac{2}{n^3}E[k^3] + \underbrace{\frac{1}{n^4}E[k^4]}_{\text{Need first 4 moments!}}$$

– These moments can be found from the generating function discussed in chapter 4.

8.18 $\hat{\Theta} = \max(X_1, \ldots, X_n)$ X_i uniform in $[0, \theta]$

ⓐ $f_{\hat{\theta}}(x) = n F_X(x)^n f_X(x) = n \left(\frac{x}{\theta}\right)^n \frac{1}{\theta} \quad 0 < x < \theta$

ⓑ $E[\hat{\Theta}] = \int_0^\theta x f_{\hat{\theta}}(x)\,dx = \frac{n}{\theta^{n+1}} \int_0^\theta x^{n+1}\,dx = \frac{n}{\theta^{n+1}} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2}\theta$

$B[\hat{\theta}] = E[\hat{\theta}] - \theta = -\frac{2}{n+2}\theta$

ⓒ $E[\hat{\theta}^2] = \frac{n}{\theta^{n+1}} \int_0^\theta x^{n+2}\,dx = \frac{n}{\theta^{n+1}} \frac{\theta^{n+3}}{n+3} = \frac{n}{n+3}\theta^2$

$VAR[\hat{\theta}] = \frac{n}{n+3}\theta^2 - \left(\frac{n}{n+2}\theta\right)^2 = \theta^2\left[\frac{n}{n+3} - \frac{n^2}{(n+2)^2}\right]$

$MSE[\hat{\theta}] = VAR[\hat{\theta}] + \frac{4\theta^2}{(n+2)^2} \to 0$ as $n \to \infty$ so estimator is consistent.

ⓓ $\hat{\theta}' = \frac{n+2}{n} \max(X_1, \ldots, X_n)$ is unbiased.

ⓔ $\hat{\Theta}$ in 100 trials of 20 observations had a sample mean of 4.7923 vs theoretical of 4.5455

$\hat{\Theta}'$ in same 100 trials had sample mean of 5.2716

8.19

$1-F_X(x)=\frac{\theta^k}{x^k} \quad x \geq \theta$ Pareto.

$\therefore \hat{\theta}=\min(X_1,\ldots,X_n) \qquad f_{\hat{\theta}}(x)=n[1-F_X(x)]^{n-1}f_X(x).$

$$= n\left(\frac{\theta^k}{x^k}\right)^{n-1} k \frac{\theta^k}{x^{k+1}}$$

ⓐ $E[\hat{\theta}]=nk\int_\theta^\infty x \frac{\theta^{kn}}{x^{kn}}\frac{1}{x}\,dx = nk\int_\theta^\infty \frac{\theta^{kn}}{x^{kn}}dx$

$$= nk\theta^{kn}\left.\frac{x^{-kn+1}}{-kn+1}\right|_\theta^\infty = -\frac{nk\theta^{kn}}{(1-kn)\theta^{kn-1}} = \left(\frac{nk}{nk-1}\right)\theta$$

$$= \left(1+\frac{1}{nk-1}\right)\theta = \theta + \underbrace{\frac{1}{nk-1}\theta}_{\text{bias.}}$$

ⓑ $E[\hat{\theta}^2] = nk\int_\theta^\infty x^2\frac{\theta^{kn}}{x^{kn}}\frac{1}{x}\,dx = nk\theta^{kn}\left.\frac{x^{-kn+2}}{-kn+2}\right|_\theta^\infty = \frac{nk\theta^2}{nk-2}$

$$MSE[\hat{\theta}] = VAR[\hat{\theta}]+B(\hat{\theta})^2$$

$$= \underbrace{\frac{nk\theta^2}{nk-2}}_{E[\hat{\theta}^2]} + \underbrace{\left(\frac{\theta}{nk-1}\right)^2}_{B(\hat{\theta})^2} \underbrace{- \left(\frac{nk}{nk-1}\right)^2\theta^2}_{-E[\hat{\theta}]^2}$$

$$= \frac{nk\theta^2}{nk-2} + \frac{\theta^2-(nk)^2\theta^2}{(nk-1)^2} \rightarrow \theta^2-\theta^2=0.$$

ⓒ

$\therefore \hat{\theta}$ is consistant.

8.20

Sample Mean Over 100 Samples

Group Size	Unbiased	Biased
5	1,0163	0.84690
10	0,98573	0.86157
20	0,97089	0.92235

8.21 We are interested in the variance of the sample variance estimator

$$\hat{\sigma}_N^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

Note that the sample variance of X_i and $X_i - E[X]$ will have the same distribution. For this reason we can assume that X_i is zero mean, $m_X = 0$.

Consider:

$$s = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \sum_{i=1}^{n}\left(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2\right)$$

$$= \sum_{i=1}^{n} X_i^2 - 2n\bar{X}_n^2 + n\bar{X}_n^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

$$E[s] = E\left[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\right] = \sum_{i=1}^{n} E[X_i^2] - nE[\bar{X}^2]$$

$$= n\underbrace{E[X^2]}_{\sigma_X^2} - n\underbrace{E[\bar{X}^2]}_{\frac{\sigma_X^2}{n}}$$ ← variance of sample mean (if $m_X = 0$)

$$= (n-1)\sigma_X^2$$ which we already knew from Eqn 8.16

Now consider the second moment

$$(*) \quad E[s^2] = E\left[\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right)^2\right]$$

$$= E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2\right] - 2nE\left[\bar{X}^2\sum_{i=1}^{n} X_i^2\right] + n^2E[\bar{X}^4]$$

Take each of these terms separately

(8.21) 1st term:

$$E\left[\left(\sum_{i=1}^{n} X_i^2\right)^2\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} E[X_i^2 X_j^2]$$

$$= \sum_{i=1}^{n} E[X_i^4] + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} E[X_i^2]E[X_j^2]$$

$$= n E[X^4] + n(n-1) E[X^2]^2$$

2nd term:

$$E\left[\bar{X}_n^2 \sum_{i=1}^{n} X_i^2\right] = E\left[\frac{1}{n^2}\sum_i X_i \sum_j X_j \sum_k X_k^2\right]$$

$$= \frac{1}{n^2}\sum_k E[X_k^4] + \frac{1}{n^2}\sum_{k=1}^{n} E[X_k^2] \sum_{i=j\neq k}\sum E[X_i^2] + 0$$

$$= \frac{1}{n}E[X^4] + \frac{n(n-1)E[X^2]^2}{n^2}$$

3rd term:

$$E[\bar{X}_n^4] = \frac{1}{n^4} E\left[\sum_i X_i \sum_j X_j \sum_k X_k \sum_l X_l\right]$$

$$= \frac{1}{n^4}\Bigg[\sum_{i=j=k=l} E[X_i^4] + \sum_{\substack{i=1 \\ j=i}}^{n} E[X_i^2] \sum_{\substack{k=l \\ k\neq i}}^{n} E[X_k^2]$$

$$+ \sum_{\substack{i=1 \\ k=l}}^{n} E[X_i^2] \sum_{\substack{j=l \\ j\neq i}} E[X_j^2]$$

$$+ \sum_{\substack{i=1 \\ l=i}}^{n} E[X_i^2] \sum_{\substack{j=k \\ j\neq i}} E[X_j^2]\Bigg]$$

$$= \frac{1}{n^4}\left[n E[X^4] + 3n(n-1)E[X^2]^2\right]$$

8.21 substitute 3 terms back into (*)

$$E[s^2] = nE[X^4] + n(n-1)E[X^2]^2$$
$$-2n\left\{ \frac{E[X^4]}{n} + \frac{n(n-1)E[X^2]^2}{n^2} \right\}$$
$$+ \frac{n^2}{n^4}\left\{ nE[X^4] + 3(n)(n-1)E[X^2]^2 \right\}$$

$$= E[X^4]\left\{ n - 2 + \frac{1}{n} \right\}$$
$$+E[X^2]^2\left\{ n(n-1) - 2(n-1) + \frac{3(n-1)}{n} \right\}$$

$$VAR[s] = E[s^2] - E[s]^2 \qquad E[s]^2 = (n-1)^2 E[X^2]^2$$

$$= E[X^4]\left\{ \frac{n^2-2n+1}{n} \right\}$$
$$+E[X^2]^2\left\{ \underbrace{\frac{n^2(n-1) - 2n(n-1) + 3(n-1) - n(n-1)^2}{n}}_{-\frac{(n-1)(n-3)}{n}} \right\}$$

$$= E[X^4]\frac{(n-1)^2}{n} - E[X^2]^2\frac{(n-1)(n-3)}{n}$$

$$VAR\left[\frac{1}{n-1}\sum_i (X_i - \bar{X}_n)^2\right] = VAR\left[\frac{1}{n-1} s\right] = \frac{1}{(n-1)^2} VAR(s]$$
$$= \frac{1}{n}E[X^4] - E[X^2]^2\frac{(n-3)}{n(n-1)}$$
$$= \frac{1}{n}\left[E[X^4] - \frac{n-3}{n-1}E[X^2]^2 \right]$$
$$= \frac{1}{n}\left[E[(X-m_X)^4] - \frac{n-3}{n-1}E[(X-m_X)^2]^2 \right]$$
$$= \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma_X^4 \right]$$

8.22

```
x= normal_rnd (0,1,2,2000)
y= A' * x
cxy1 = y(1,:).*y(2,:)
```

$= \{X_iY_i\}$

```
Z=reshape (cxy1,20,100)
hist (mean (Z))
mean(mean(Z)') = 0.50009
```

$\frac{1}{20}\sum_{i=1}^{20} X_iY_i$, known mean

```
% for unknown means and variances
for i=1:100
mx(i) = mean ( y(1, i:i+20);
my(i) = mean ( y(2, i: i+20);
xy(i) = sum( y(1, i: i+20) .* y(2, i: i+20));
cxy2(i) = (xy(i)-20*mx(i) my(i))/19
end
hist(cxy2)
mean( cxy2') = 0.51493
```

8.23

```
x= normal_rnd(0, 1, 2, 2000);
y= A' *x
for i=1:100
mx(i) = mean( y(1, i: i+20))
my(i)= mean ( y(2, i: i+20))
vx(i) = sum((y(1, i: i+20)-mx(i)).^2)
vy(i) = sum((y(2, i: i+20) -my(i)).^2)
rhoxy(i) = ( xy(i) -20* mx(i) *my(i))/ sqrt(vx(i)*vy(i))
end
hist (rhoxy)
mean (rhoxy') = 0.49916
```

8.3 Maximum Likelihood Estimation

8.24

ⓐ $f(x) = \frac{1}{\theta} e^{-x/\theta} \quad x \ge 0$

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \frac{1}{\theta^n} e^{-\sum_{i=1}^{n} x_i/\theta}$$

$$\ln f(x_1\ldots x_n|\theta) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$0 = \frac{d}{d\theta}\ln f = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_i x_i \Rightarrow n\theta = \sum_i x_i$$

$$\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

ⓑ By invariance property

$$\hat{\lambda}_{ML} = \frac{1}{\hat{\theta}_{ML}} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} X_i}$$

Try direct approach anyway:

$$f(x_1\ldots x_n|\lambda) = \lambda^n \prod_{i=1}^{n} e^{-\lambda\sum_i x_i}$$

$$0 = \frac{d}{d\lambda}\ln f = \frac{d}{d\lambda}\left[n\ln\lambda - \lambda\sum_i X_i\right] = \frac{n}{\lambda} - \sum_i X_i$$

$$\Rightarrow \hat{\lambda} = \frac{n}{\sum_i X_i} \quad ✓$$

ⓒ $\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ scaled version of n Erlang RV

$$\hat{\lambda}_{ML} = \frac{1}{\hat{\theta}_{ML}} \Rightarrow f_\lambda(y) = \frac{f_\theta(1/y\,|}{y^2} \qquad y > 0$$

where f_θ is n-Erlang.

ⓓ $\hat{\theta}_{ML}$ is unbiased and consistent because it is a sample mean

8.25

ⓐ $f_X(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x-\theta-1)^2/2}$

$$0 = \frac{d}{d\theta} \ln f = \frac{d}{d\theta} \sum_{i=1}^{n} \left(\ln \frac{1}{\sqrt{2\pi}} - \frac{1}{2}(x_i - \theta - 1)^2 \right)$$

$$= +\sum_{i=1}^{n}(x_i - \theta - 1) = \sum_{i=1}^{n} x_i - n\theta - n$$

$$\hat{\theta}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i - 1$$

ⓑ $\hat{\theta}_{ML}$ is Gaussian with mean

$$E[\hat{\theta}_{ML}] = \frac{1}{n} \sum_{i=1}^{n} E[X_i] - 1 = \frac{1}{n} n [\theta + 1] - 1 = \theta$$

$$VAR[\hat{\theta}_{ML}] = \frac{1}{n} VAR[X] = \frac{1}{n}$$

ⓒ From ⓑ $\hat{\theta}_{ML}$ is unbiased

$VAR[\hat{\theta}_{ML}] \to 0$ so $\hat{\theta}_{ML}$ is consistent.

8.26

$f(x_1, \ldots, x_n \mid \theta) = \frac{1}{\theta^n} \qquad 0 \le x_i \le \theta$

f is increases as θ decreases

∴ f is maximized when $\theta = \max(X_1, \ldots, X_n)$.

8.27

ⓐ $f(x_1,\ldots,x_n|\alpha) = \prod_{i=1}^{n} \alpha \frac{x_m^{\alpha}}{x_i^{\alpha+1}} = \qquad x_i \ge x_m$

$$0 = \frac{d}{d\alpha} f = \frac{d}{d\alpha} \sum_{i=1}^{n} \left(\ln\alpha + \alpha \ln x_m - (\alpha+1)\ln x_i \right)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{\alpha} + \ln x_m - \ln x_i \right) = \frac{n}{\alpha} + n \ln x_m - \sum_{i=1}^{n} \ln x_i$$

$$\frac{n}{\alpha} = -n \ln x_m + \sum_{i=1}^{n} \ln x_i$$

$$\alpha = \frac{n}{-n\ln x_m + \sum_{i=1}^{n} \ln x_i} = \frac{n}{\sum_{i=1}^{n} \ln(x_i/x_m)}$$

ⓑ If x_m is unknown we have additional condition

$$f(x_1,\ldots,x_n|\alpha, x_m) = \frac{\alpha^n x_m^{n\alpha}}{x_i^{\alpha+1}} \qquad x_i \ge x_m$$

f is an increasing function of x_m, so f is maximized by letting $\underline{\underline{\hat{x}_m}} = \min(x_1,\ldots,x_n)$.

Then

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(x_i/\hat{x}_m)}$$

8.27 (c) Consider $1/\alpha$ first:

$$\frac{1}{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \ln x_i/x_m$$

- x_i/x_m is a normalized Pareto, which has long tail
- $\ln x_i/x_m$ compacts the tail closer to the origin
- the estimator takes the arithmetic average

Let $Y = X/x_m$

$$P[Y > y] = P\left[\ln \frac{X}{x_m} > y\right] = P[X > x_m e^{y}]$$

$$= \frac{x_m^{\alpha}}{(x_m e^{y})^{\alpha}} = e^{-\alpha y}$$

exponential RV with mean $1/\alpha$

∴ above estimator is sample mean for the transformed random variable

∴ the estimator is consistent.

$\hat{x}_m = \min(x_1, \ldots, x_m)$ has pdf given by $\quad x \geq x_m$

$$f_{\hat{X}_m}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x) = n\left(\frac{x_m^{\alpha}}{x^{\alpha}}\right)^{n-1} \alpha \frac{x_m}{x^{\alpha+1}}$$

$$= n\alpha \frac{x_m^{\alpha n}}{x^{\alpha n}} \frac{1}{x}$$

which is Pareto with parameter αn

with

$$E[\hat{X}_m] = \frac{\alpha n x_m}{\alpha n - 1} \to x_m \quad \Rightarrow \hat{X}_m \text{ is consistent.}$$

$$VAR[\hat{X}_m] = \frac{\alpha n x_m^2}{(\alpha n - 2)(\alpha n - 1)} \to 0$$

8.28

$\theta = \alpha^2$

$$f_X(x) = \frac{x}{\theta} e^{-x^2/2\theta^2} \qquad x \geq 0.$$

(a) $$f_X(x_1 \ldots x_n | \theta) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{-x_i^2/2\theta}$$

$$0 = \frac{d}{d\theta} \ln f = \frac{d}{d\theta} \sum_{i=1}^{n} \left(\ln x_i - \ln\theta - x_i^2/2\theta \right)$$

$$0 = \sum_{i=1}^{n} \left(-\frac{1}{\theta} + \frac{x_i^2}{2\theta^2} \right) \Rightarrow 0 = -n\theta + \frac{1}{2} \sum_{i=1}^{n} x_i^2$$

$$\hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} X_i^2$$

X_i^2 is an exponential RV with $\lambda = 1/2\alpha^2$

$$\therefore E[\hat{\theta}] = \frac{1}{2n} \sum_{i=1}^{n} E[X_i^2] = \alpha^2 \quad \text{unbiased.}$$

$$VAR[\hat{\theta}] = \frac{1}{4n} VAR[X_i] = \frac{1}{4n} (4\alpha^2)^2 = \frac{\alpha^4}{n} \to 0$$

$\therefore$ $\hat{\theta}$ is consistent

8.29

$$f_X(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} x^{\alpha-1} = \frac{1}{\alpha} x^{\alpha-1} \quad \text{since } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$0 < x < 1$$

ⓐ

$$f_X(x_1, \ldots x_n) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{\alpha}$$

$$0 = \frac{d}{d\alpha} \ln f = \frac{d}{d\alpha} \sum_{i=1}^{n} \left(-\ln\alpha + (\alpha-1)\ln x_i\right)$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{\alpha} + \ln x_i\right) = -\frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_i$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

8.30

$$f_X(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}} \qquad t > 0$$

$$f_X(x_1, \ldots, x_n) = (\alpha\beta)^n \prod_{i=1}^{n} x_i^{\beta-1} e^{-\alpha x_i^{\beta}}$$

$$0 = \frac{d}{d\alpha} \ln f = \frac{d}{d\alpha}\left[n \ln \alpha\beta + \sum_{i=1}^{n} \left((\beta-1)\ln x_i - \alpha x_i^{\beta}\right)\right]$$

$$= \frac{n}{\alpha} + \sum_{i=1}^{n} - x_i^{\beta}$$

$$\alpha = \frac{n}{\sum_{i=1}^{n} x_i^{\beta}}$$

8.31 $P_f = (1 - e^{-T/\tau})$ where $\tau = 1/\lambda$ is the mean lifetime

$$P[N_f = k] = \binom{n}{k} P_f^k (1-P_f)^{n-k}$$

the ML estimate for P_f is

$$\hat{P}_f = \frac{k}{n}$$

We are interested in the following function of P_f

$$e^{-T/\tau} = 1 - P_f$$

$$-T/\tau = \ln(1-P_f)$$

$$\tau = -\frac{T}{\ln(1-P_f)}$$

By invariance property

$$\hat{\tau}_{ML} = -\frac{T}{\ln(1-\frac{k}{n})}$$

(8.32) $f_X(x) = \dfrac{\lambda(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}$ $\quad x > 0 \quad \alpha > 0 \; \lambda > 0$

(a) $f_X(x_1, \dots, x_n | \lambda) = \prod_{i=1}^{n} \dfrac{\lambda(\lambda x_i)^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x_i}$

$$0 = \frac{d}{d\lambda} \ln f = \frac{d}{d\lambda} \Big[\sum_{i=1}^{n} \Big(\alpha \ln \lambda + (\alpha-1) \ln x_i - \lambda x_i - \ln \Gamma(\alpha) \Big) \Big)$$

$$0 = \sum_{i=1}^{m} \left(\frac{\alpha}{\lambda} - x_i \right) = \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i$$

$\hat{\lambda} = \dfrac{n\alpha}{\sum_{i=1}^{n} x_i}$ reciprocal of sample mean scaled by α.

(b) also require

$$0 = \frac{d}{d\alpha} \ln f = \sum_{i=1}^{n} \left[\ln \lambda + \ln x_i - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \right]$$

$$= n \ln \lambda + \sum_{i=1}^{n} \ln x_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

We replace λ by $\hat{\lambda}$ from part (a)

$$0 = n \ln \left(\frac{\alpha}{\overline{X}_n} \right) + \sum_{i=1}^{n} \ln x_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

$$\underbrace{\ln \alpha - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}}_{\text{must be solved for } \alpha.} = \ln \overline{X}_m - \frac{1}{m} \sum_{i=1}^{m} \ln x_i$$

8.33 First assume known $m_X=0$, $m_Y=0$ $\sigma_X^2=1$, $\sigma_Y^2=1$

ⓐ Find ML estimate for ρ:

$$f(x,y)=\frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)\right\}}{2\pi(1-\rho^2)^{1/2}}$$

$$\ln\prod_{i=1}^{n}f(x_i,y_i)=\sum_{i=1}^{n}\left(-\ln 2\pi-\frac{1}{2}\log(1-\rho^2)-\frac{x_i^2-2\rho x_iy_i+y_i^2}{2(1-\rho^2)}\right)$$

$$=-n\ln 2\pi-\frac{n}{2}\ln(1-\rho^2)-\frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}\left(x_i^2-2\rho x_iy_i+y_i^2\right)$$

$$0=\frac{\partial}{\partial\rho}(\quad)=\frac{n\rho}{1-\rho^2}-\frac{\cancel{2}\rho}{\cancel{2}(1-\rho^2)^2}\sum_i(\quad)+\frac{2}{2(1-\rho^2)}\sum_i x_iy_i$$

multiply by $(1-\rho^2)^2$

$$0=n\rho(1-\rho^2)-\rho\sum_i\left(x_i^2-2\rho x_iy_i+y_i^2\right)+(1-\rho^2)\sum x_iy_i$$

$$=n\rho-n\rho^3-\rho\sum_i x_i^2+2\rho^2\sum_i x_iy_i-\rho\sum_i y_i^2+(1-\rho^2)\sum x_iy_i$$

$$0=\rho-\rho^3+(1+\rho^2)\frac{1}{n}\sum_i x_iy_i-\rho\left(\frac{1}{n}\sum_i x_i^2+\frac{1}{n}\sum_i y_i^2\right)$$

cubic eqn in ρ.

There is always at least one root in $-1<\rho<1$

If more than one root pick the root that gives the maximum likelihood.

8.33 — contnd —

$$0=\frac{\partial \ln f}{\partial \rho}=\frac{1}{(1-\rho^2)}\left\{n\rho-\frac{1}{1-\rho^2}\left[\rho\,\frac{\sum_i(x_i-m_x)^2}{\sigma_x^2}+\rho\,\frac{\sum_i(y_i-m_y)^2}{\sigma_y^2}\right.\right.$$

$$\left.\left.-(1+\rho^2)\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}\right]\right\} \qquad (v)$$

We have 5 equations for the 5 unknown parameters.

(iii) and (iv) become

$$(iii') \qquad n(1-\rho^2)=\frac{\sum_i(x_i-m_x)^2}{\sigma_x^2}-\rho\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}$$

$$(iv') \qquad n(1-\rho^2)=\frac{\sum_i(y_i-m_y)^2}{\sigma_y^2}-\rho\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}$$

(v) becomes

$$n(1+\rho^2)=\frac{\sum_i(x_i-m_x)^2}{\sigma_x^2}+\frac{\sum_i(y_i-m_y)^2}{\sigma_y^2}-\frac{1+\rho^2}{\rho}\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}$$

subtract (iii) + (iv) from (v):

$$-n(1-\rho^2)=-\underbrace{\frac{1+\rho^2-\rho^2-\rho^2}{\rho}}_{-\frac{1-\rho^2}{\rho}}\,\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}$$

$$\boxed{\Rightarrow\ \hat{\rho}=\frac{\frac{1}{n}\sum_i(x_i-m_x)(y-m_y)}{\sigma_x\sigma_y}}$$

8.33

ⓑ $$f(x,y) = \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-m_x}{\sigma_x}\right)^2 + 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y-m_y}{\sigma_y}\right)^2\right\}\right\}}{2\pi\sqrt{\sigma_x^2\sigma_y^2(1-\rho^2)}}$$

$$\ln \prod_i f(x_i,y_i) = -n\ln 2\pi - \frac{n}{2}\left(\ln\sigma_x^2 + \ln\sigma_y^2 + \ln(1-\rho^2)\right)$$
$$-\frac{1}{2(1-\rho^2)}\sum_{i=1}^n\left\{\left(\frac{x_i-m_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right) + \left(\frac{y_i-m_y}{\sigma_y}\right)^2\right\}$$

We take derivatives wrt to m_x, m_y, σ_x^2, σ_y^2, and ρ

$$0 = \frac{\partial \ln\prod f}{\partial m_x} = \frac{-1}{2(1-\rho^2)}\sum_{i=1}^n\left(-2\left(\frac{x_i-m_x}{\sigma_x^2}\right) + \frac{2\rho}{\sigma_x}\left(\frac{y_i-m_y}{\sigma_y}\right)\right)$$

$$= \frac{n}{\sigma_x(1-\rho^2)}\left[\frac{\frac{1}{n}\sum_i x_i - m_x}{\sigma_x} - \rho\,\frac{\frac{1}{n}\sum_i y_i - m_y}{\sigma_y}\right] \qquad (i)$$

for m_y we have

$$0 = \frac{n}{\sigma_y(1-\rho^2)}\left[\frac{\frac{1}{n}\sum_i y_i - m_y}{\sigma_y} - \rho\,\frac{\frac{1}{n}\sum_i x_i - m_x}{\sigma_x}\right] \qquad (ii)$$

$$0 = \frac{\partial \ln\prod f}{\partial \sigma_x^2} = -\frac{n/2}{\sigma_x^2} - \frac{1}{2(1-\rho^2)}\sum_{i=1}^n\left(-\frac{(x_i-m_x)^2}{\sigma_x^4} - 2\rho\frac{(x_i-m_x)}{\sigma_x^3}\left(\frac{y_i-m_y}{\sigma_y}\right)\left(-\frac{1}{2}\right)\right)$$

$$(iii) \qquad = -\frac{1}{2\sigma_x^2(1-\rho^2)}\left[n(1-\rho^2) - \frac{\sum_i(x_i-m_x)^2}{\sigma_x^2} + \rho\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}\right]$$

Similarly

$$(iv) \qquad 0 = -\frac{1}{2\sigma_y^2(1-\rho^2)}\left[n(1-\rho^2) - \frac{\sum_i(y_i-m_y)^2}{\sigma_y^2} + \rho\frac{\sum_i(x_i-m_x)(y_i-m_y)}{\sigma_x\sigma_y}\right]$$

8.33 substitute ρ into (iii')

$$n(1-\rho^2) = \frac{\sum_i (x_i - m_x)^2}{\sigma_X^2} - n\rho^2$$

$$\Rightarrow \boxed{\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - m_x)^2}$$

Similarly we obtain

$$\boxed{\hat{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - m_Y)^2}$$

Finally we obtain the estimates for m_X and m_Y from (i) and (ii)

$$\text{(i')} \qquad \frac{\bar{X}_n - m_X}{\sigma_X} = \rho\,\frac{\bar{Y}_n - m_Y}{\sigma_Y}$$

$$\text{(ii')} \qquad \frac{\bar{Y}_n - m_Y}{\sigma_Y} = \rho\,\frac{\bar{X}_n - m_X}{\sigma_X} = \rho^2\,\frac{\bar{Y}_n - m_Y}{\sigma_Y^2}$$

$$\Rightarrow \bar{Y}_n - m_Y = 0 \Rightarrow \boxed{\hat{m}_Y = \frac{1}{n}\sum_i Y_i}$$

Similarly $\boxed{\hat{m}_X = \frac{1}{n}\sum X_i}$.

8.34 Invariance Property

ML estimator for $h(\theta)$ finds h^* such that

$$f(x_1,\ldots,x_n|h^*) = \max f(x_1\ldots x_n|h^*)$$

ML estimator for θ finds θ^* such that

$$f(x_1\ldots x_n|\theta^*) = \max f(x_1\ldots x_n|\theta^*)$$

Let $\theta_0 = h^{-1}(h^*)$ the inverse image of the optimum h^* and suppose that $\theta_0 \neq \theta^*$ the optimal MLE for θ, then

$$f(x_1\ldots x_n|\theta^*) = f(x_1\ldots x_n|h(\theta^*))$$
$$\leq f(x_1\ldots x_n|h^*) = f(x_1\ldots x_n|\theta_0)$$

contradicting the optimality of θ^*.

8.35 From (8.35) we have

$$0 = E\left[\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right] = \int\cdots\int_{\underline{x}_n}\left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right) f_X(\underline{x}|\theta)\,d\theta$$

Take another derivative wrt θ

$$0 = E\left[\frac{\partial^2}{\partial\theta^2}\ln f_X(\underline{x}|\theta)\right]$$

$$= \iint_{\underline{x}_n}\left\{\frac{\partial^2}{\partial\theta^2}\ln f_X(\underline{x}|\theta)\, f_X(\underline{x}|\theta) + \underbrace{\left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right)\frac{\partial}{\partial\theta} f_X(\underline{x}|\theta)}\right\} d\underline{x}$$

Note that $\dfrac{\partial f(\underline{x}|\theta)}{\partial\theta} = \left(\dfrac{\partial}{\partial\theta}\ln f(\underline{x}|\theta)\right) f(\underline{x}|\theta)$, so

$$0 = E\left[\frac{\partial^2}{\partial\theta^2}\ln f(\underline{x}|\theta)\right] + \iint_{\underline{x}_n}\left(\frac{\partial}{\partial\theta}\ln f_X(\underline{x}|\theta)\right)^2 f(\underline{x}|\theta)\,d\underline{x}$$

$$\Rightarrow \quad E\left[\frac{\partial^2}{\partial\theta^2}\ln f(\underline{x}|\theta)\right] = -E\left[\left(\frac{\partial}{\partial\theta}\ln f(\underline{x}|\theta)\right)^2\right]$$

$$= -I_n(\theta).$$

8.36

ⓐ Binomial

$$\ln l(\underline{X}|p) = \sum_{i=1}^{n}\left(\ln\binom{n}{k_i} + k_i \ln p + (n-k_i)\ln(1-p)\right)$$

$$\frac{\partial}{\partial p}\ln l(\underline{X}|p) = \sum_{i=1}^{n}\left(\frac{k_i}{p} - \frac{n-k_i}{1-p}\right)$$

$$\frac{\partial^2}{\partial p}\ln l(\underline{X}|p) = \sum_{i=1}^{n}\left(-\frac{k_i}{p^2} - \frac{n-k_i}{(1-p)^2}\right)$$

$$-E\left[\frac{\partial^2}{\partial p}\ln l(\underline{X}|p)\right] = +\sum_{i=1}^{n}\left(\frac{E[k_i]}{p^2} + \sum_{i=1}^{n}\frac{n-E[k_i]}{(1-p)^2}\right.$$

$$= \frac{n^2}{p} + \frac{n^2}{1-p} = \frac{n^2}{p(1-p)}$$

8.36b Gaussian: known σ^2 unknown mean:

$$\ln l(\underline{X}|\mu) = \sum_{i=1}^{n}\left(\ln\sqrt{2\pi}\,\sigma^2 \mp (x_i-\mu)^2/2\sigma^2\right)$$

$$\frac{\partial}{\partial \mu}\ln l = \sum_{i=1}^{n} + 2(x_i-\mu)/2\sigma^2$$

$$\frac{\partial^2}{\partial \mu}\ln l = \sum_{i=1}^{n} -2/2\sigma^2 = -\frac{n}{\sigma^2}$$

$$I_n(\mu) = -E\left[\frac{\partial^2}{\partial \mu}\ln l\right] = \frac{n}{\sigma^2}.$$

8.36 c Gaussian unknown variance, known mean

$$\ln \ell(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(-\ln\sqrt{2\pi\sigma^2} - (x_i-\mu)^2/2\sigma^2\right)$$

$$\frac{\partial}{\partial\sigma^2}\ln \ell(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(\frac{-1}{2\sigma^2} + \frac{(x_i-\mu)^2}{2(\sigma^2)^2}\right)$$

$$\frac{\partial^2}{\partial{\sigma^2}^2}\ln \ell(\underline{X}|\sigma^2) = \sum_{i=1}^{n}\left(\frac{+1}{2(\sigma^2)^2} + \frac{(x_i-\mu)^2}{2(\sigma^2)^3}(-2)\right)$$

$$I_n(\sigma^2) = \frac{-n}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\underbrace{\sum_{i=1}^{n} E[(X_i-\mu)^2]}_{n\sigma^2}$$

$$= \frac{-n}{2\sigma^4} + \frac{n}{\sigma^4} = \frac{n}{2\sigma^4}$$

If the mean μ is unknown, the above computation does not change.

from Ex 8.8 the variance of the unbiased sample variance estimator is

$$VAR[\hat{\sigma}_n^2] = \frac{1}{n}\left[\mu_4 - \frac{n-3}{n-1}\sigma^4\right] = \frac{1}{n}\left[\overset{\text{Gaussian}}{3\sigma^4} - \frac{n-3}{n-1}\sigma^4\right]$$

$$= \frac{2\sigma^4}{n-1} > \frac{2\sigma^4}{n} = \text{CramerRao LB.}$$

8.36d $$f_X(x|\beta) = \frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} e^{-x/\beta}$$

ⓐ $$\ln l(\underline{x}|\beta) = \sum_{i=1}^{n}\left(-\alpha\ln\beta - \ln\Gamma(\alpha) + (\alpha-1)\ln x_i - \frac{x_i}{\beta}\right)$$

$$\frac{\partial}{\partial\beta}\ln l = \sum_{i=1}^{n}\left(-\frac{\alpha}{\beta} + \frac{x_i}{\beta^2}\right)$$

$$\frac{\partial^2}{\partial\beta}\ln l = \sum_{i}\left(\frac{\alpha}{\beta^2} - \frac{2x_i}{\beta^3}\right)$$

$$I(\beta) = -\left(\frac{n\alpha}{\beta^2} - \frac{2}{\beta^3}\underbrace{\sum_i E[X]}_{n\alpha\beta}\right) = -\left(\frac{n\alpha}{\beta^2} - \frac{2n\alpha}{\beta^2}\right) = +\frac{n\alpha}{\beta^2}$$

8.36e

$$\ln l = \sum_i\left(k_i\ln\alpha - \ln k_i! - \alpha\right)$$

$$\frac{\partial}{\partial\alpha}\ln l = \sum_i \frac{k_i}{\alpha} - 1$$

$$\frac{\partial^2}{\partial\alpha^2}\ln l = \sum_i -\frac{k_i}{\alpha^2}$$

$$I_n(\alpha) = +E\left[\sum_i \frac{k_i}{\alpha^2}\right] = \frac{1}{\alpha^2}n\alpha = \frac{n}{\alpha}.$$

8.37 $\hat{\theta}_{ML}$ estimator for $1/\lambda$, the mean of an exponential RV

from 8.24a $\quad \hat{\theta}_{ML} = \frac{1}{n}\sum_i X_i$

from invariance property $\hat{\theta}^2_{ML}$ is the ML estimator for $1/\lambda^2$

We are interested in

$$P\left[-\frac{1}{20\lambda^2} < \hat{\theta}^2_{ML} - \frac{1}{\lambda^2} < \frac{1}{20\lambda^2}\right]$$

$$= P\left[\frac{19}{20\lambda^2} < \hat{\theta}^2_{ML} < \frac{21}{20\lambda^2}\right]$$

$$= P\left[\sqrt{\frac{19}{20\lambda^2}} < \hat{\theta}_{ML} < \sqrt{\frac{21}{20\lambda^2}}\right]$$

$\hat{\theta}_{ML}$ has mean $\frac{1}{\lambda}$ and variance $\frac{1}{n\lambda^2}$ and is approx. Gaussian

$$\simeq \int_{\sqrt{\frac{19}{20\lambda^2}}}^{\sqrt{\frac{21}{20\lambda^2}}} \frac{1}{\sqrt{2\pi\left(\frac{1}{n\lambda^2}\right)}}\, e^{-\left(x-\frac{1}{\lambda}\right)/2\left(\frac{1}{n\lambda^2}\right)}\, dx$$

$$\simeq Q\left(\frac{\sqrt{\frac{19}{20\lambda^2}} + \frac{1}{\lambda}}{\frac{1}{\sqrt{n\lambda^2}}}\right) - Q\left(\frac{\sqrt{\frac{21}{20\lambda^2}} + 1}{\frac{1}{\sqrt{n\lambda^2}}}\right)$$

$$= Q\left(\frac{\sqrt{\frac{19}{20}}+1}{1/\sqrt{n}}\right) - Q\left(\frac{\sqrt{\frac{21}{20}}+1}{1/\sqrt{n}}\right).$$

8.38 $\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ estimator for α, Poisson

Estimate $\hat{h}(\theta) = e^{-\hat{\theta}_{ML}}$ estimator for $P[N=0]$

$\hat{h}(\theta)$ is ML est for $P[N=0]$ by invariance property

We are interested in

$$P\left[-\left|e^{-\hat{\theta}} - e^{-\alpha}\right| < \frac{1}{10}e^{-\alpha}\right]$$

$$= P\left[-\frac{1}{10}e^{-\alpha} < e^{-\hat{\theta}} - e^{-\alpha} < \frac{1}{10}e^{-\alpha}\right]$$

$$= P\left[\frac{9}{10}e^{-\alpha} < e^{-\hat{\theta}} < \frac{11}{10}e^{-\alpha}\right]$$

$$= P\left[\alpha + \ln\frac{10}{11} < \hat{\theta} < \alpha + \ln\frac{10}{9}\right]$$

[$\hat{\theta}_{ML}$ has mean α and variance $\frac{\alpha}{n}$ and is approx Gaussian]

$$\approx \int_{\alpha+\ln\frac{10}{11}}^{\alpha+\ln\frac{10}{9}} \frac{1}{\sqrt{2\pi\alpha/n}}\, e^{-(x-\alpha)/2(\alpha/n)}\, dx$$

$$= Q\left(\frac{\ln\frac{10}{11}}{\alpha/n}\right) - Q\left(\frac{\ln\frac{10}{9}}{\alpha/n}\right)$$

we have dependence on the actual value of α

8.4 Confidence Intervals

8.39

The ith measurement is $X_i = m + N_i$ where $\mathcal{E}[N_i] = 0$ and $VAR[N_i] = 10$. The sample mean is $M_{100} = 100$ and the variance is $\sigma = \sqrt{10}$.

Eqn. ~~5.37~~ 8.52 with $z_{\alpha/2} = 1.96$ gives

$$\left(100 - \frac{1.96\sqrt{10}}{\sqrt{30}}, 100 + \frac{1.96\sqrt{10}}{\sqrt{30}}\right) = (98.9, 101.1)$$

8.40

5.32 The width of the confidence interval given by Eqn. ~~5.37~~ 8.52 is

$$\left(M_n + \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) - \left(M_n - \frac{z_{\alpha/2}\sigma}{\sqrt{n}}\right) = \frac{2z_{\alpha/2}\sigma}{\sqrt{n}}$$

a) For 95% confidence intervals $z_{\alpha/2} = 1.96$, so $(\sigma = 1)$

$$\text{width of interval} = \frac{2(1.96)}{\sqrt{n}} = \begin{cases} 1.96 & n = 4 \\ 0.98 & n = 16 \\ 0.29 & n = 100 \end{cases}$$

b) For 99% confidence intervals $z_{\alpha/2} = 2.576$ so

$$\text{width of interval} = \frac{2(2.576)}{\sqrt{n}} = \begin{cases} 2.576 & n = 4 \\ 1.288 & n = 16 \\ 0.515 & n = 100 \end{cases}$$

8.41

5.33 $M_n = 223 \qquad V_N^2 = 100 \qquad n = 225$

$\Rightarrow V_n = 10$

ⓐ Assuming that individual lifetimes are Gaussian RV's, Eqn. ~~5.43~~ 8.58 with $n = \infty$

$$\begin{aligned}\left(M_n - \frac{z_{\alpha/2,\infty}V_n}{\sqrt{n}}, \frac{M_n + z_{\alpha/2,\infty}V_n}{\sqrt{n}}\right) &= \left(223 - \frac{1.96(10)}{\sqrt{225}}, 223 + \frac{1.96(10)}{\sqrt{225}}\right) \\ &= (222, 224)\end{aligned}$$

41 ⓑ

from Eqn 8.59 the confidence interval for the sample variance is

$$\left[\frac{224\,(100)}{\chi^2_{.025,\,224}}, \frac{224(100)}{\chi^2_{0.975,\,224}}\right] = \left[\frac{224\,(100)}{267.35}, \frac{224\,(100)}{184.44}\right]$$

$$= [83.785, 121.45].$$

8.42 **~~5.34~~** $M_n = \frac{1}{n}\sum_{j=1}^{10} X_j = \frac{350}{10} = 35$

$$\begin{aligned}
\sum_{j=1}^{n}(X_j - M_n)^2 &= \sum_{j=1}^{n} X_j^2 - 2M_n\sum_{j=1}^{n} X_j + nM_n^2 \\
&= \sum_{j=1}^{n} X_j^2 - nM_n^2 \\
V_n^2 &= \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n)^2 = \frac{1}{n-1}\sum_{j=1}^{n} X_j^2 - \frac{n}{n-1}M_n^2 \\
&= \frac{1}{9}(12645) - \frac{10}{9}(35)^2 = 43.88 \\
\Rightarrow V_n &= 6.624
\end{aligned}$$

For 90% confidence interval

$$z_{\alpha/2,9} = 1.833$$

So Eqn. ~~5.43~~ 8.58 gives

$$\left(35 - \frac{1.833(6.624)}{\sqrt{10}}, 35 + \frac{1.833(6.624)}{\sqrt{10}}\right) = (31.16, 38.84)$$

8.42 ⓑ

from Eq. 8.59:

$$\left[\frac{9(43,88)}{\underbrace{\chi^2_{0.05,9}}_{16,92}}, \frac{9(43.88)}{\underbrace{\chi^2_{0.95,9}}_{3,325}}\right] = [23,34, 118,77]$$

8.43

~~5.35~~ a) $M_n = 57.3 \qquad V_n^2 = 23.2 \qquad n = 10$

$$\left(M_n - \frac{1.833V_n}{\sqrt{10}}, M_n + \frac{1.833V_n}{\sqrt{10}}\right) = (54.5, 60.1) \qquad 90\%$$

$$\left(M_n - \frac{2.262V_n}{\sqrt{10}}, M_n + \frac{2.262V_n}{\sqrt{10}}\right) = (53.85, 60.75) \qquad \cancel{90\%}\ 95\%$$

$$\left(M_n - \frac{3.25V_n}{\sqrt{10}}, M_n + \frac{3.25V_n}{\sqrt{10}}\right) = (52.35, 62.25) \qquad 99\%$$

b) $M_n = 57.3 \qquad V_n^2 = 23.2 \qquad n = 20$

$$\left(M_n - \frac{1.725V_n}{\sqrt{20}}, M_n + \frac{1.725V_n}{\sqrt{20}}\right) = (55.44, 59.16) \qquad 90\%$$

$$\left(M_n - \frac{2.086V_n}{\sqrt{20}}, M_n + \frac{2.086V_n}{\sqrt{20}}\right) = (55.05, 59.55) \qquad \cancel{90\%}\ 95\%$$

$$\left(M_n - \frac{2.895V_n}{\sqrt{20}}, M_n + \frac{2.895V_n}{\sqrt{20}}\right) = (54.24, 60.36) \qquad 99\%$$

Note: the entry for $z_{\alpha/2,20}$ was used instead of $z_{\alpha/2,19}$.

8.43 ⓒ $\left[\dfrac{9(23.2)}{\chi^2_{\alpha/2,9}}, \dfrac{9(23.2)}{\chi^2_{1-\alpha/2,9}}\right]$

$n = 10$ measurements

$[12.34, 62.78]$ 90%

$[10.98, 77.33]$ 95%

$[8.85, 120.69]$ 99%.

$\left[\dfrac{19(23.2)}{\chi^2_{\alpha/2,19}}, \dfrac{19(23.2)}{\chi^2_{1-\alpha/2,19}}\right]$.

$[14.62, 43.56]$ 90%

$[13.42, 49.47]$ 95%

$[11.43, 64.44]$ 99%.

8.44

$$M_{15} = -1.154$$
$$V_{15}^2 = 3.711$$

From Table ~~5.2~~ 8.2 with $1-\alpha = 90\%$ and $n-1 = 14$, we have

$$z_{\alpha/2,14} \approx z_{\alpha/2,15} = 1.753, \qquad \text{so}$$

$$\left(M_{15} - \frac{z_{\alpha/2,15}V_n}{\sqrt{15}}, M_{15} + \frac{z_{\alpha/2,15}V_n}{\sqrt{15}}\right) = (-2.026, -0.282)$$

8.45

~~5.37~~ The sample mean and variance of the batch sample means are $M_{10} = 24.9$ and $V_{10}^2 = 3.42$. The mean number of heads in a batch is $\mu = \mathcal{E}[M_{10}] = \mathcal{E}[X] = 50p$.
From Table ~~5~~8.2, with $1-\alpha = 95\%$ and $n-1 = 9$ we have

$$z_{\alpha/2,9} = 2.262$$

The confidence interval for μ is

$$\left(M_{10} - \frac{z_{\alpha/2,9}V_{10}}{\sqrt{10}}, M_{10} + \frac{z_{\alpha/2}V_{10}}{\sqrt{10}}\right) = (23.58, 26.22)$$

The confidence interval for $p = M_{10}/50$ is then

$$\left(\frac{23.58}{50}, \frac{26.22}{50}\right) = (0.4716, 0.5244)$$

8.46

$Z = 1.645 \quad n = 10 \qquad \sigma^2 = 1.$

ⓐ

$$\left(M_n - \frac{1.645\sigma}{\sqrt{10}}, M_n + \frac{1.645\sigma}{\sqrt{10}}\right) = (M_n - 0.5202, M_n + 0.5202)$$

8.47 90% confidence intervals

ⓐ $n=4$ batches $\left(M_n - \frac{2{,}353 V_n}{\sqrt{4}}, M_n + \frac{2{,}353 V_n}{\sqrt{4}}\right)$

$n=8$ $\left(M_n - \frac{1{,}895 V_n}{\sqrt{8}}, M_n + \frac{1{,}895 V_n}{\sqrt{8}}\right)$

$n=16$ $\left(M_n - \frac{1{,}753 V_n}{\sqrt{16}}, M_n + \frac{1{,}753 V_n}{\sqrt{16}}\right)$

$n=32$ $\left(M_n - \frac{1{,}697 V_n}{\sqrt{31}}, M_n + \frac{1{,}697 V_n}{\sqrt{31}}\right).$

8.48 $\mu=25$ $\sigma^2=36$

Gaussian mean and variance unknown.

$$\left(M_n - z_{\alpha/2,n-1}\frac{\hat{\sigma}_n}{\sqrt{n}}, M_n + z_{\alpha/2,n-1}\frac{\hat{\sigma}_n}{\sqrt{n}}\right)$$

$\alpha = 0{,}10$
$n = 50, 100, \ldots$

$$\left(\frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)\hat{\sigma}_n^2}{\chi^2_{1-\alpha/2,n-1}}\right)$$

$\alpha = 0{,}10$
$n = 50, 100, \ldots$

8.5 Hypothesis Testing

8.49 $H_0 : \alpha = 30$ $\quad$ $n = 8$ measurements

$H_1 : \alpha > 30$ $\quad$ $\bar{X}_8 = 32 \Rightarrow \sum_{i=1}^{8} N_i = 256$

The experiments involves n measurements of a Poisson random variable. We take the sum of the total number of orders $N = \sum_{i=1}^{8} N_i$ (equivalent to taking the sample mean)

Accept H_0 if $N_T < T$ $\quad$ N Poisson with mean $n\alpha = 8\alpha$

Reject H_0 if $N_T \geq T$.

$$\alpha = 5\% = P[\text{Reject } H_0 \mid H_0] = P[N_T \geq T \mid H_0]$$

$$= \sum_{k=T}^{\infty} \frac{240^k}{k!} e^{-240} \qquad \bar{X}_8 = \frac{1}{8} N$$

$$\approx P\left[\frac{\bar{X}_8 - 30}{\sqrt{30}/\sqrt{8}} > \frac{T-30}{\sqrt{30}/\sqrt{8}}\right] = Q(1.64)$$

$$\Rightarrow T - 30 = \frac{1.64\sqrt{8}}{\sqrt{30}} + 30 = 30.847$$

$$\bar{X}_8 = 32 > 30.847 \Rightarrow \text{Reject } H_0$$

$\alpha = 1\%$ $\quad$ $1\% = Q(2.326)$

$$\Rightarrow T = 30 + \frac{2.326\sqrt{8}}{\sqrt{30}} = 31.201$$

$$\bar{X}_8 = 32 > 31.2 \Rightarrow \text{Reject } H_0$$

8.50

Carlos

		T	H
Michael	T	tie	C wins
	H	M wins	tie

In a fair game

$\frac{1}{2}$ games are ties

$\frac{1}{4}$ carlos wins

$\frac{1}{4}$ michael wins

ⓐ If we count how many times Carlos wins N_c we are testing

$H_0: p = 1/4$ so Accept H_0 if $N_c < T$

$H_1: p > 1/4$ Reject H_0 if $N_c \geq T$

$$\alpha = 10\% = P[N_c \geq T / H_0] = \sum_{k=T}^{6} \binom{6}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{6-k}$$

for $T=3$ $P[N_c \geq 3 | H_0] = 0.169$

$T=4$ $P[N_c \geq 4 | H_0] = 0.0375$ Use $T=4$.
& much more stringent than 10%

for $n=12$

$T=5$ $P[N_c \geq 5 / H_0] = 0.157$

$T=6$ $P[N_c \geq 6 | H_0] = 0.054$ Use $T=6$

ⓑ If we count how many times Carlos' toss is heads, N_c'

$H_0: p = \frac{1}{2}$ $\alpha = 10\% = \sum_{k=T}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n$

$H_0: p > \frac{1}{2}$

for $n=6$ $P[N_c' \geq 5 | H_0] = 0.109$

$n=12$ $P[N_c' \geq 9 | H_0] = 0.073$

Counting heads rather than wins is more effective because it uses more information about the expt.

8.50 ⓒ $P_{\text{Detection}} = P[H_1 \mid H_1]$

1/4	3/4	
1/8	3/8	1/2
1/8	3/8	1/2

for $n = 6$, $p = 0.75$, counting wins

$$P_D = \sum_{k=4}^{6} \binom{6}{k} \left(\frac{3}{8}\right)^{k} \left(\frac{5}{8}\right)^{6-k} = 0.146 \approx 15\%$$

$n=6$ $p=0.55$ counting wins $P_D = .052 \approx 5\%$

$n=12$ $p=0.75$ counting wins $P_D = .271 \approx 21\%$

$n=12$ $p=0.55$ " $P_D = 0.082 \approx 8\%$

$\Rightarrow$ Difficult to detect by counting wins

$n=6$ $p=0.75$ counting heads $P_D = 534 \approx 53\%$

$p=0.55$ " $P_D = 0.164 \approx 16\%$

$n=12$ $p=0.75$ " $P_D = 0.649 \approx 65\%$

$p=0.55$ " $P_D = 0.134 \approx 13\%$

These results confirm that counting heads is more effective.

8.51 Gaussian $m=0$ $\sigma^2=4$

(a) $H_0: m=0$ Accept if $-c<\overline{X}_n<c$

$H_1: m\neq 0$ Reject otherwise

$$\alpha=0.01=P[|\overline{X}_n|>c|H_0]=P\left[\left|\frac{\overline{X}_n}{2/\sqrt{n}}\right|>\frac{c}{2/\sqrt{n}}\right]=2Q(2.576)$$

$$c=2.576(2)/\sqrt{n}=5.152/\sqrt{n}=1.692$$

(b) $\overline{X}_m=-0.75$ $n=10$ $c=1.692$

$$|\overline{X}_{10}|=0.75<1.692\Rightarrow \text{Accept } H_0$$

(c) $P[\text{Type II}]=P[\text{Accept } H_0|H_1]=P[|\overline{X}_m|<c|H_1]$

$$=\frac{1}{\sqrt{2\pi 4/n}}\int_{-c}^{c}e^{-(x-\nu)^2/2(4/n)}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\frac{-c-\nu}{2\sqrt{n}}}^{\frac{(c-\nu)}{2/\sqrt{n}}}e^{-x'^2/2}\,dx'=Q\left(\frac{-c-\nu}{2\sqrt{n}}\right)-Q\left(\frac{c-\nu}{2/\sqrt{n}}\right)$$

for $\nu=1$, $n=10$, $c=1.692$

$$P[\text{Type II}]=Q\left(\frac{-2.692}{2/\sqrt{10}}\right)-Q\left(\frac{0.692}{2/\sqrt{10}}\right)=.863$$

for $\nu=0.01$ $n=10$ $c=1.692$

$$P[\text{Type II}]=Q\left(\frac{-1.702}{2/\sqrt{10}}\right)-Q\left(\frac{1.682}{2/\sqrt{10}}\right)=0.993$$

Type II errors are very high because most samples fall in acceptance region when $\nu=1$ and $\nu=0.01$.

8.52 (a) α is used to determine the most extreme value that defines the boundary of the acceptance region. α is also the probability of a sample falling in the acceptance region given H_0

The p-value is the prob. of observing a sample as extreme or more extreme than the given observation. When the p-value equals α then the observation is at the boundary of the acceptance region.

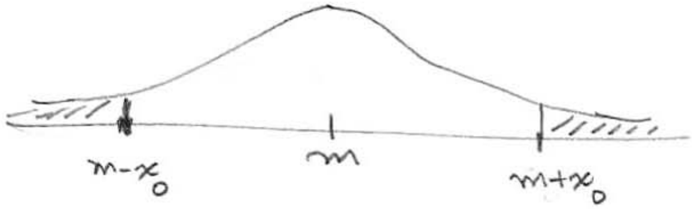
(b) The hypothesis test gives a binary answer as to whether the sample is in the acceptance region or not.

The p-value gives an indication of the whether the observation would be accepted at different significance levels.

(c) $$P[\bar{X}_n > x_0 \mid H_0] = Q\left(\frac{x_0 - m}{\sigma/\sqrt{n}}\right)$$ where x_0 is observed value

(d) $$P\left[\,|\bar{X}_n - m| > |x_0 - m| \;\middle|\; H_0\right]$$

$$= 2Q\left(\frac{|x_0 - m|}{\sigma/\sqrt{n}}\right)$$



8.53

$\beta > \alpha$ Poisson rate

$H_0: \alpha = 2$ $\qquad P[\cancel{X} = k | H_0] = \frac{\alpha^k}{k!} e^{-\alpha}$

$H_1: \beta = 6$

ⓐ $\qquad P[X = k | H_1] = \frac{\beta^k}{k!} e^{-\beta}$

$$R^c = \left\{ \underline{x} : \ln \frac{p(\underline{x} | H_1)}{p(\underline{x} | H_0)} > t \right\}$$

$$\ln \frac{p(\underline{x} | H_1)}{p(\underline{x} | H_0)} = k \ln \beta - \beta - \ln k! - k \ln \alpha + \alpha + \ln k!$$

$$= k \ln \beta/\alpha - \beta - \alpha > t$$

$$k > \frac{t + \alpha + \beta}{\ln \beta/\alpha} = t'$$

$$0.05 = \alpha = P[k > t | H_0] = \sum_{k=t+1}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} = \sum_{k=t+1}^{\infty} \frac{2^k}{k!} e^{-2}$$

$$\Rightarrow t = 4$$

ⓑ $\quad P_D = P[k > 4 | H_1] = \sum_{k=5}^{\infty} \frac{6^k}{k!} e^{-6} = 0.714$

ⓒ If we take n measurements the test becomes

$H_0: \alpha = 2n$ $\qquad 0.05 = \sum_{k=t+1}^{\infty} \frac{(2n)^k}{k!} e^{-2n}$

$H_1: \beta = 6n$

Try $n = 2 \Rightarrow t = 7$

$$P_D = [k > 7 | H_1] = \sum_{k=8}^{\infty} \frac{12^k}{k!} e^{-6} = 0.91$$

8.54) Assume $\bar{X}_n$ is used in test

H_0 : Gaussian $m=8$ $\sigma^2 = 1/n$

H_1 : Gaussian $m=9$ $\sigma^2 = 1/n$

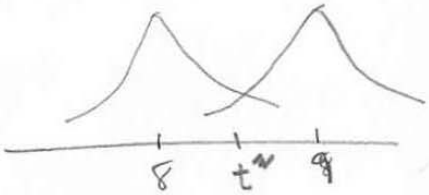
Apply Neyman-Pearson criterion:

$$\ln \Lambda(x) = -\frac{1}{2}\ln 2\pi/n - \frac{1}{2}\frac{(x-9)^2}{1/n} + \frac{1}{2}\ln\frac{2\pi}{n} + \frac{1}{2}\frac{(x-8)^2}{1/n} \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$-(x-9)^2 + (x-8)^2 \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$-x^2+18x-81+x^2-16x+64 \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} t''$$



$$\alpha = 0.01 = P[X > t'' \mid H_0]$$

$$= \int_{t''}^{\infty} \frac{1}{\sqrt{2\pi/n}} e^{-(x-8)^2/2/n}\, dx = \underbrace{Q\left(\frac{t''-8}{1/\sqrt{n}}\right)}_{2.326}$$

$$\Rightarrow (t''-8)\sqrt{n} = 2.326$$

$$P_D = 0.99 = P[X > t'' \mid H_1] = \underbrace{Q\left(\frac{t''-9}{1/\sqrt{n}}\right)}_{-2.326}$$

$$\left.\begin{aligned} \frac{t''-8}{\sqrt{n}} &= 2.326 \\ \frac{t''-9}{\sqrt{n}} &= -2.326 \end{aligned}\right\} \Rightarrow n = 21.64 \text{ — use } n = 22$$

then $t'' = 8 + \frac{2.326}{\sqrt{22}}$

$= 8.4959$

8.55

H_0: exponential $m=2$ $\qquad f_X(x)=\lambda e^{-\lambda x}$

H_1: exponential $m=4$

Neyman Pearson:

$$\ln\Lambda(x) = \ln 1/4 - x/4 - \ln 1/2 + x/2 \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

ⓐ $\alpha = 0.05 = P[X > t' \mid H_0] = \int_{t'}^{\infty} \frac{1}{2} e^{-x/2}\,dx = e^{-t'/2}$

$\Rightarrow \ln 0.05 = -t'/2$

$\Rightarrow t' = -2\ln 0.05 = 5.9915$

ⓑ $P_D = P[X > t' \mid H_1] = \int_{t'}^{\infty} \frac{1}{4} e^{-x/4}\,dx = e^{-5.9915/4} = 0.22$

Difficult to identify heavy users without misidentifying light users.

8.56

H_0: Pareto $m=3$ $a=3$ $x_m = \frac{m(a-1)}{a} = 2$ for both distributions

H_1: Pareto $m=16$ $a=8/7$

$$f_X(x) = a\frac{x_m^{a}}{x^{a+1}} \quad x \geq 2$$

ⓐ Neyman-Pearson

$$\ln\Lambda(x) = \ln\frac{8}{7} + \frac{8}{7}\ln x_m - \frac{15}{7}\ln x - \ln 3 + 3\ln x_m + 4\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$-\frac{15}{7}\ln x + 4\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$\frac{13}{7}\ln x \underset{H_0}{\overset{H_1}{\gtrless}} t''$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\alpha = 0.01 = P[X>\gamma \mid H_0] = \frac{2^3}{\gamma^3}$$

$$\Rightarrow \gamma = \left(\frac{8}{.01}\right)^{1/3} = 9.283$$

ⓑ

$$P_D = P[X>\gamma \mid H_1] = \frac{2^{8/7}}{\gamma^{8/7}} = 0.173$$

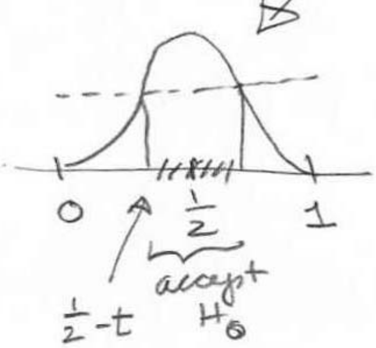
8.57

H_0: Beta $a=b=10$ $f(x|H_0)=c\,x^9(1-x)^9$ $0<x<1$

H_1: Beta $a=b=5$ $f(x|H_0)=c'x^5(1-x)^5$

$$\Lambda(x)=\frac{f(x|H_1)}{f(x|H_0)}=\frac{c'x^5(1-x)^5}{c\,x^9(1-x)^9}=\frac{c'}{c\,x^4(1-x)^4}\underset{H_0}{\overset{H_1}{\gtrless}}t$$

$$\left|x-\tfrac{1}{2}\right|\underset{H_0}{\overset{H_1}{\gtrless}}t' \quad \Leftarrow$$



$$\alpha=0.05=P\left[\left|X-\tfrac{1}{2}\right|>t\,\middle|\,H_0\right]$$

$$=2\int_0^{\frac{1}{2}-t}c\,x^9(1-x)^9\,dx$$

Use Octave
beta_inv(0.025, 10, 10)

$$\Rightarrow \tfrac{1}{2}-t=0.28864$$

$$\Rightarrow t=.31136$$

$$P_D=P\left[\left|X-\tfrac{1}{2}\right|>t\,\middle|\,H_1\right]=2\int_0^{\frac{1}{2}-t}c'x^5(1-x)^5\,dx$$

$$=0.17098$$

Use octave
beta_cdf(0.28864, 5, 5)

8.58 $m_0 = 2$ $m_1 = 1.75$ $\sigma^2 = 0.04$ $n = 10$ $\bar{x}_{10} = 1.82$

ⓐ H_0: Gaussian $m_0 = 2$ $\sigma^2 = 0.04$

H_1: Gaussian $m_1 = 1.75$ $\sigma^2 = 0.04$

Simple Binary Hypothesis Test as in Example 8.25

$$\ln \Lambda(x) = (m_1 - m_0)\bar{X}_n + c \underset{H_0}{\overset{H_1}{\gtrless}} t$$

$$\bar{X}_n \underset{H_0}{\overset{H_1}{\lessgtr}} \gamma$$

$$0.05 = \alpha = P[\bar{X}_n < \gamma \mid H_0] = 1 - Q\Big(\underbrace{\sqrt{n}\,\frac{\gamma - 2}{\sigma_X}}\Big)$$ Reject H_0

$$\frac{\sqrt{10}(\gamma - 2)}{0.2} = -1.644$$

$\Rightarrow \gamma = 1.896$ $\bar{X}_n = 1.82 \Rightarrow$ Reject H_0

ⓑ

$$P_D = P[\bar{X}_n < \gamma \mid H_1] = \int_{-\infty}^{\gamma} \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-(x-1.75)^2/2\sigma_X^2}\,dx = 1 - Q\left(\sqrt{10}\cdot\frac{\gamma - 1.75}{.2}\right)$$

$$= 0.9895$$

ⓒ $\bar{X}_n = 1.82$ $p = P[\bar{X}_n - 2 < 1.82] = 1 - Q\left(\frac{\sqrt{10}\,(1.82 - 2)}{0.2}\right)$

$$= 0.00221$$

The p-value is lower than $\alpha = 0.05$ or even $\alpha = 0.01$

8.59

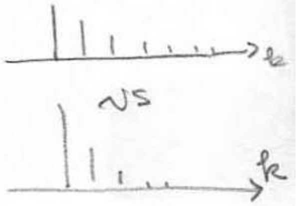
$H_0: p = 1/2$

$H_1: p = 3/4$

$$P[X=k \mid p] = p(1-p)^{k-1} \quad k = 1, 2, \ldots$$

ⓐ

$$\Lambda(k) = \frac{\frac{3}{4}\left(\frac{1}{4}\right)^{k-1}}{\left(\frac{1}{2}\right)^{k}} = \frac{3}{4}\left(\frac{4}{1}\right)\left(\frac{1}{2}\right)^{k} \underset{H_0}{\overset{H_1}{\gtrless}} t$$



$$\Leftrightarrow \quad k \ln\frac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$\Leftrightarrow \quad k \underset{H_0}{\overset{H_1}{\lessgtr}} \gamma \qquad \text{since } \ln\frac{1}{2} < 0$$

$\Rightarrow$ Reject if $k < \gamma$

$$0.05 = \alpha = P[X < \gamma \mid H_0] = \sum_{k=1}^{\gamma-1}\left(\frac{1}{2}\right)^{k} = \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{\gamma-1}$$

Unless we let the rejection region be $\{k < 1\}$ = all values of k we cannot find a region that satisfies the above equation.

ⓑ for $H_0 = p = 1/2$ $\quad k \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$

$H_1 = p = 1/4$

$$0.05 = \alpha = P[X \geq \gamma \mid H_0] = \sum_{k=\gamma}^{\infty}\left(\frac{1}{2}\right)^{k} = \left(\frac{1}{2}\right)^{\gamma-1}$$

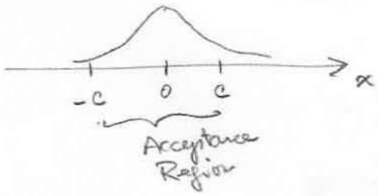
$$\ln 0.05 = (\gamma - 1)\ln\frac{1}{2}$$

$$\gamma = 1 + \frac{\ln 0.05}{\ln 1/2} = 5.3219 \Rightarrow \gamma = 6$$

$$P_D = P[X \geq \gamma \mid H_0] = \sum_{k=\gamma}^{\infty}\left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^{\gamma-1} = \left(\frac{3}{4}\right)^{5} = 0.237$$

$\uparrow$ low

$\boxed{8.60}$ H_0 : Gaussian $m=0$ $\sigma^2 = 1/n$
H_1 : Gauss $m \neq 0$ $\sigma^2 = 1/n$



ⓐ Proceeding as in Ex. 8.28

$$0.10 = \alpha = P[\bar{X}_n > c \mid H_0] = 2Q(c\sqrt{n})$$

$$c = z_{\alpha/2}/\sqrt{n} = 1.644/\sqrt{n}$$

ⓑ

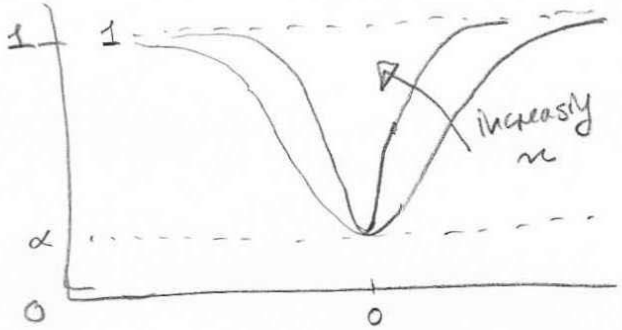
$$P[\text{Type II error}] = P[\,|\bar{X}_n| < c \mid m = \mu \neq 0\,]$$
$$= Q(-z_{\alpha/2} - \sqrt{n}\mu) - Q(z_{\alpha/2} - \sqrt{n}\mu)$$
$$= Q(-1.644 - \sqrt{n}\mu) - Q(1.644 - \sqrt{n}\mu)$$
$$= \beta(\mu)$$

Power of test $= 1 - \beta(\mu)$

ⓒ The following Octave commands plot the power curve for $n = 64$

```
> mu = [-10: 0.10 : 10]
> plot (mu, 1.-(-normal_cdf (-1.6449.-8mu)
              +normal_cdf (1.6449.-8mu)))
```

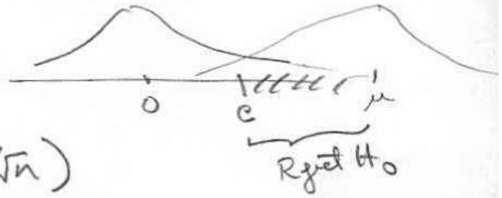
We obtain



8.61

H_0: Gaussian $m=0$ $\sigma^2=1/n$

H_1: Gaussian $m>0$ $\sigma^2=1/n$



ⓐ $0.10 = \alpha = P[\bar{X}_n > c \mid H_0] = Q(c/\sqrt{n})$

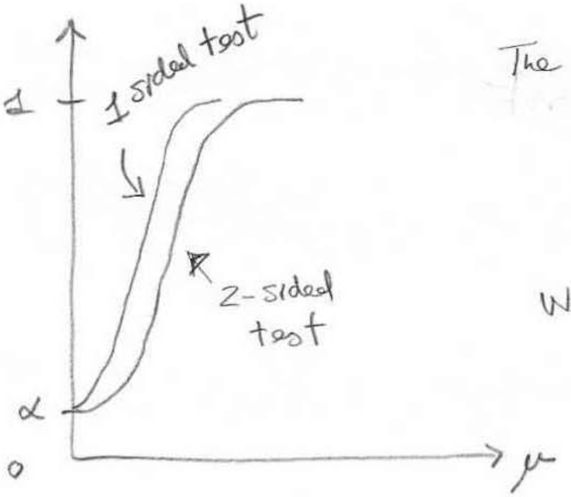
$\Rightarrow c = z_\alpha/\sqrt{n} = 1.2816/\sqrt{n}$

ⓑ $P[\text{Type II error}] = P[\bar{X}_n < c \mid H_1] = 1 - Q(1.2816 - \sqrt{n}\mu)$

ⓒ The following commands plot the power curve

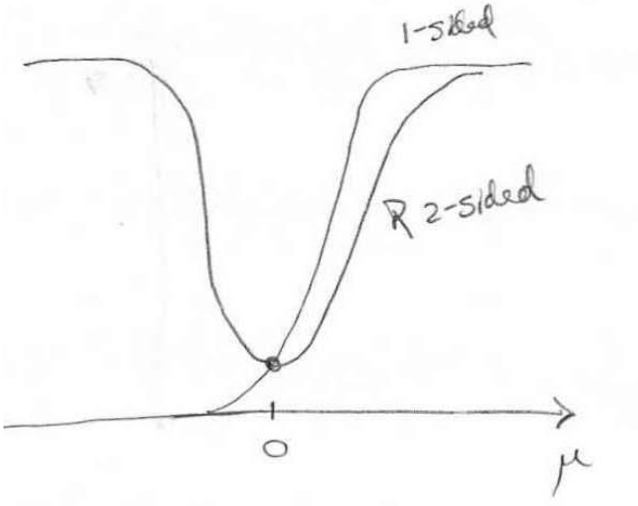
```
> mu=[0:0.1:10]
> plot(mu, 1.- normal_cdf(1.2816.- 2*mu))
```

8.62 Using the Octave commands in 8.60 and 8.61 we obtain the following



The 1-sided test is more powerful in the region μ>0, than the 2-sided test

We already saw that the 1-sided test is UMP for μ>0



If we compare the power of the 1-sided test for μ>0 vs the 2-sided test for values μ<0 we obtain the curve on the left

The 1-sided test is useless for values μ<0

∴ The 2-sided test strikes a compromise to perform well for all $\mu \neq 0$, and consequently it is unable to outperform the more specialized 1-sided test in the latter's design region.

8.65

$H_0: p = \frac{1}{2}$ $n = 100$ tosses

$H_1: p \neq \frac{1}{2}$ let $k = \#$ heads.

a(i) $$\Lambda(k) = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k}\left(\frac{1}{2}\right)^n} \underset{H_0}{\overset{H_1}{\gtrless}} t \Leftrightarrow \left(\frac{p}{1-p}\right)^k \underset{H_0}{\overset{H_1}{\gtrless}} t'$$

$$\Leftrightarrow k \ln \frac{p}{1-p} \underset{H_0}{\overset{H_1}{\gtrless}} t' \Rightarrow \begin{matrix} k \underset{H_0}{\overset{H_1}{\gtrless}} t'' & p > \frac{1}{2} \\ k \underset{H_0}{\overset{H_1}{\lessgtr}} & p < \frac{1}{2} \end{matrix}$$

$$\Leftrightarrow \left|k - \underbrace{\frac{n}{2}}_{50}\right| \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$$

$$\alpha = 0.01 = P\left[|k-50| > \gamma \mid H_0\right] \approx P\left[\left|\frac{X-50}{\sqrt{25}}\right| > \underbrace{\frac{\gamma}{\sqrt{25}}}_{2.5758}\right]$$

$$\Rightarrow \gamma = 2.5758\sqrt{25} = 12.88$$

$$\Rightarrow \text{Use } \gamma = 13$$

b(i) $$P[\text{Type II error}] = P\left[\,|k-50| < 13 \mid H_1\right]$$

$$\approx \int_{37}^{63} \frac{1}{\sqrt{2\pi np(1-p)}} e^{-(x-np)/2np(1-p)} dx$$

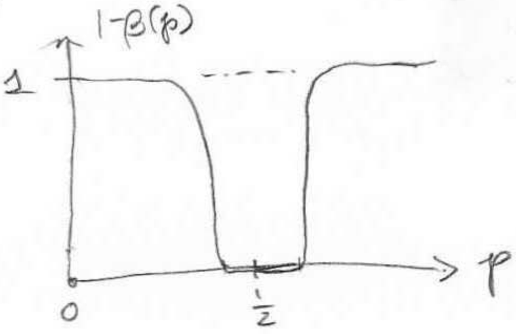
$$\approx Q\left(\frac{37-100p}{10\sqrt{p(1-p)}}\right) - Q\left(\frac{63-100p}{10\sqrt{p(1-p)}}\right)$$

$$\approx \beta(p)$$

Power $= 1-\beta(p)$

(ii) & (iii) done similarly

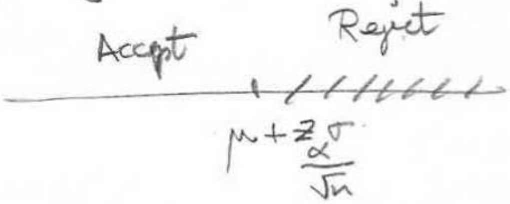
Conclusions of comparison as in Prob. 8.62



8.67 H_0: X Gaussian $m \le \mu$ σ_X^2 known $\equiv$ composite hypotheses

H_1: X Gaussian $m > \mu$ σ_X^2 known

Use the following decision regions



$$P[\text{Type I error}] = P\left[X < \mu + \frac{z_\alpha \sigma}{\sqrt{n}} \,\middle|\, H_0\right]$$

$$= \int_{-\infty}^{\mu + \frac{z_\alpha \sigma}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu')/2\sigma^2} dx \qquad \mu' < \mu$$

$$= 1 - Q\left(\frac{\mu - \mu' + \frac{z_\alpha \sigma}{\sqrt{n}}}{\sigma/\sqrt{n}}\right)$$

$$= 1 - Q\left(\underbrace{\frac{\mu - \mu'}{\sigma/\sqrt{n}}}_{>0} + z_\alpha\right)$$

$$\le 1 - Q(z_\alpha) = \alpha \quad ✓$$

8.68 $m = 2 \quad n=10 \quad \bar{X}_{10} = 2.2 \quad \hat{\sigma}^2_{10} = 0.04$

Proceeding as in Ex. 8.29

H_0: $m=2$ σ^2 unknown
H_1: $m \neq 2$ σ^2 unknown

$$T = \frac{\bar{X}_n - m}{\hat{\sigma}_n / \sqrt{n}} = \sqrt{10}\,\frac{\bar{X}_n - 2}{(.2)}$$

@ $\alpha = 0.05$, degree $10-1=9$ $t_{0.025,9} = 2.2622$

$$\therefore \text{ Accept } H_0 \text{ if } \left| \frac{x-2}{\hat{\sigma}_n/\sqrt{n}} \right| \leq 2.2622$$

In the example we have $T_0 = \sqrt{10}\,\frac{2.2-2}{.2} = 3.1623 > 2.2622$

$\Rightarrow$ Reject H_0

$$p = P\left[\left| \frac{x-2}{\hat{\sigma}_n/\sqrt{n}} \right| > 3.1623 \right] = 2F_9(3.1623)$$

$= 0.011508$

which is lower than

$\alpha = 0.05$

8.69 a)

H_0: mean 50 variance unknown $\quad n=8$

H_1: mean 55 variance unknown $\quad \bar{X}_8 = 52.5$

We assume that X has a Gaussian distribution $\quad \hat{\sigma}_8 = 3$
and use the Student-t statistic

$$T = \frac{\bar{X} - m}{\sigma/\sqrt{n}}$$

Accept H_0 if $\bar{X} < \gamma$

$$\alpha = P[\bar{X} > \gamma \mid H_0] = P\left[\frac{\bar{X} - 50}{\hat{\sigma}_n/\sqrt{n}} > \frac{\gamma - 50}{\hat{\sigma}_n/\sqrt{n}}\right]$$

$$= 1 - F\Bigg(\underbrace{\frac{\gamma - 50}{\hat{\sigma}_n/\sqrt{n}}}_{t_{\alpha, n-1}}\Bigg)$$

$\alpha = 0.01 \Rightarrow t_{0.01, 7} = 2.998$

$\alpha = 0.05 \Rightarrow t_{0.05, 7} = 1.8946$

$$\alpha = 0.01 \qquad \gamma = 50 + \frac{\sqrt{n}\, t_{\alpha, n-1}}{\hat{\sigma}_n} = 50 + \frac{\sqrt{8}\,(2.998)}{3} = 52.824$$

$\Rightarrow$ Accept $\bar{X}_8 = 52.5$

$$\alpha = 0.05 \qquad \gamma = 50 + \frac{\sqrt{8}(1.8946)}{3} = 51.7862$$

$\Rightarrow$ Reject $\bar{X}_8 = 52.5$

ⓑ $$p = P\left[\frac{\bar{X} - 50}{\hat{\sigma}_n/\sqrt{n}} > \frac{52.5 - 50}{3/\sqrt{8}}\right] = 0.0252$$

less than 0.05
but greater than 0.01

8.70) $H_0: m=4$ $\quad \bar{X}_n = 3{,}3$ $\quad n = 100$

$H_1: m<4$ $\quad \hat{\sigma}_n = \frac{1}{2}$

(a) Assume $\bar{X}_n$ Gaussian since n is large

This is a one-sided test:

Accept H_0 if $\bar{X}_n > \gamma$

Reject H_0 if $\bar{X}_n < \gamma$

$z_{0{,}01} = 2{,}3263$

$$\gamma = 4 - \frac{\sigma}{\sqrt{n}} z_\alpha = 4 - \frac{1}{2(10)} z_\alpha, \qquad z_{0.05} = 1{,}6449$$

$$\gamma = \begin{cases} 3{,}8837 & \alpha = 0.01 \\ 3{,}9178 & \alpha = 0{,}05 \end{cases}$$

Both tests reject H_0 for $\bar{X} = 3{,}3$

Frosh rule!

(b)
$$p = P[\bar{X}_n < 3{,}3 \mid H_0] = Q\left(\frac{3{,}3-4}{(\frac{1}{2})/10}\right) = Q(.7(20))$$

$$= Q(14) \sim e^{-(14)^2/2} = 0$$

8.75

H_0: Gaussian $m=0$ $\sigma^2=4$

H_1: Gauss $m=0$ $\sigma^2<4$

ⓐ Accept H_0 if $\hat{\sigma}_n^2 > \gamma$

Reject H_0 if $\hat{\sigma}_n^2 < \gamma$

$$\alpha = P[\hat{\sigma}_n^2 < \gamma \mid H_0] = P\left[\frac{(n-1)\hat{\sigma}_n^2}{\sigma_0^2} < \frac{(n-1)\gamma}{\sigma_0^2}\right] = 1-\chi^2$$

$$= 1 - P\left[\chi^2 > \frac{(n-1)\gamma}{\sigma_0^2}\right]$$

$$1-\alpha = P\left[\chi^2 > \underbrace{\frac{(n-1)\gamma}{\sigma_0^2}}_{\chi^2_{1-\alpha,n-1}}\right] = \chi^2_{.99,n-1}$$

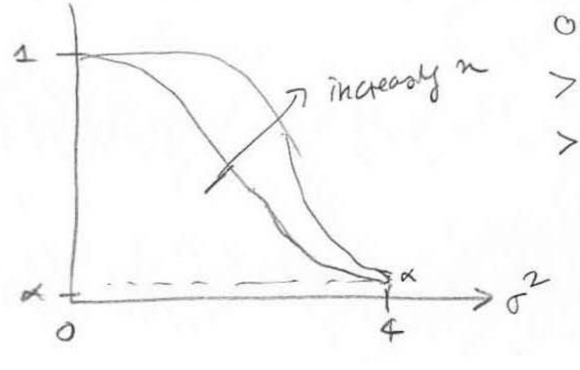
$$\gamma = \frac{\chi^2_{.99,n-1}}{n-1}\sigma_0^2$$

n	8	64	256
$\chi^2_{.99,n-1}$	1.2390	39.85	205.4
γ	0.708	2.53	3.22

ⓑ Power $= P[\hat{\sigma}_n^2 < \gamma \mid H_1]$

$$= P\left[\frac{(n-1)\sigma_n^2}{\sigma^2} < \frac{(n-1)\gamma}{\sigma^2}\right] = P\left[\chi^2 < \frac{(n-1)\gamma}{\sigma^2}\right]$$

↑ for σ^2 not σ_0^2



octave:

```
> sig2 = [0:0.1:4];
> plot(sig2, chisquare_cdf
        (63*2.53./sig2, 63));
```

8.76

H_0: Gaussian $m=0$ $\sigma^2=4$

H_1: Gauss $m=0$ $\sigma^2>4$

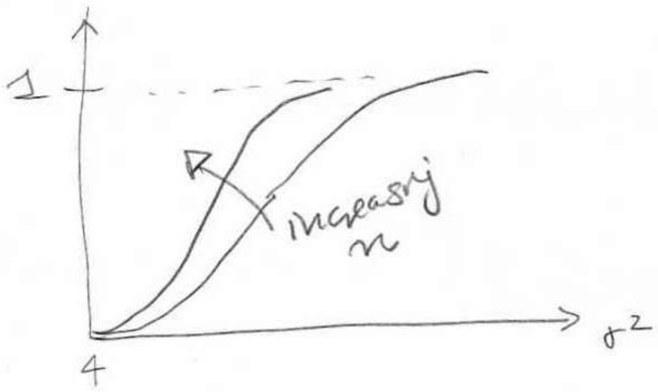
ⓐ Accept H_0 if $\sigma_n^2 < \gamma$

$$\alpha = P[\sigma_n^2 > \gamma \mid H_0] = P\Big[\chi^2 > \underbrace{\frac{(n-1)\gamma}{\sigma_0^2}}_{\chi^2_{\alpha,n-1}}\Big]$$

n	8	64	256
$\chi^2_{.01,n-1}$	18.47	92.01	310.46
γ	10.56	5.84	4.87

ⓑ $\text{Power} = P[\hat{\sigma}_n^2 > \gamma \mid H_1] = P\Big[\chi^2 > \frac{(n-1)\gamma}{\sigma^2}\Big]$

↑ for σ^2



```
> sig2 [4: 0.1: 8]
> plot ( sig2, 1-chisquare_cdf (63*5.84./sig2,63))
```

8.77

$H_0: \sigma^2 = \mu_0 = 7$

$H_1: \sigma^2 = \mu \neq \mu_0$

(a) Accept H_0 if $a < \hat{\sigma}_n^2 < b$

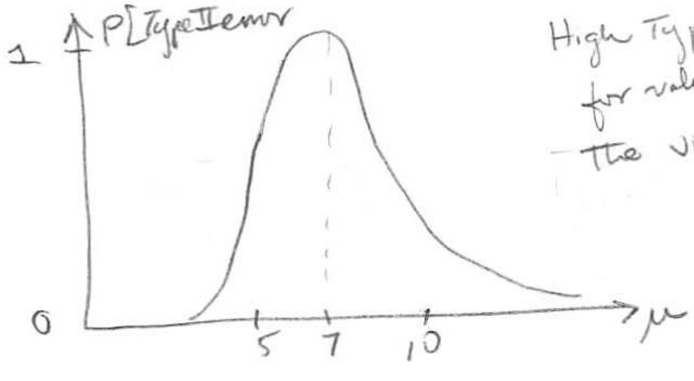
$$\Leftrightarrow \frac{(n-1)a}{\mu_0} < \frac{(n-1)\hat{\sigma}_n^2}{\mu_0} < \frac{(n-1)b}{\mu_0}$$

$$\alpha = 1 - P[a < \hat{\sigma}_n^2 < b \mid H_0] = 1 - P\Big[\underbrace{\underbrace{\frac{(n-1)a}{7}}_{\chi^2_{1-\alpha/2,\,n-1}}}_{\chi^2_{.995,70}} < \chi^2 < \underbrace{\underbrace{\frac{(n-1)b}{7}}_{\chi^2_{\alpha/2,\,n-1}}}_{\chi^2_{.005,70}}\Big]$$

$$a = \frac{\mu_0 \chi^2_{.995,n-1}}{(n-1)} = 4.3275 \qquad b = \frac{\mu_0 \chi^2_{.005,n-1}}{n-1} = 10.42$$

(b) $P[\text{Type II error}] = P[a < \hat{\sigma}_n^2 < b \mid H_1]$

$$= P\left[\frac{70(4.3275)}{\mu} < \frac{(n-1)\sigma_n^2}{} < \frac{70(10.42)}{\mu}\right]$$



High Type II error rates for values of μ in the vicinity of 7

8.6 Bayesian Decision Methods

8.81 H_0: Exponential $m = \frac{1}{2}$ $\quad p_0 = \frac{1}{10}$

H_1: Exponential $m = 5$ $\quad 1 - p_0 = \frac{9}{10}$

$C_{00} = 0 \qquad C_{10} = 3 \qquad\qquad C_{01} - C_{00} = 5$

$C_{01} = 5 \qquad C_{11} = 0 \qquad\qquad C_{10} - C_{11} = 3$

cost of short life sold as long

$$\text{Accept } H_0 \qquad \underbrace{\frac{f(x|H_1)}{f(x|H_0)}} < \frac{\frac{1}{10}\cdot 5}{\frac{9}{10}\cdot 3} = \frac{5}{27}$$

cost of long life sold as short

$$\frac{\frac{1}{5}e^{-x/5}}{2e^{-2x}} < \frac{5}{27}$$

$$-x/5 + 2x < \ln\frac{50}{27}$$

$$\frac{9}{5}x < \ln\frac{50}{27}$$

$$x < \frac{5}{9}\ln\frac{50}{27} = 0.3423$$

8.82 2) ⓐ Maximum Likelihood Decision Rule

$x=0$ $p(0|H_1) < p(0|H_0) \Rightarrow$ decide H_0

$x=1$ $p(1|H_1) > p(1|H_0) \Rightarrow$ decide H_1

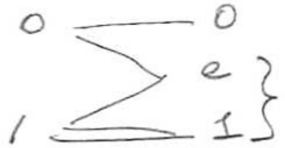
$x=e$ $p(0|H_1) = p(1|H_0) \Rightarrow$ no clear decision

The cost does not enter in the ML decision rule

We can use knowledge of cost to break the tie when $x=e$

If $C_{10} > C_{01}$ then cost of mistaking 1 for 0 is higher so decide H_1 when $x=e$

This gives



if $C_{10} > C_{01}$

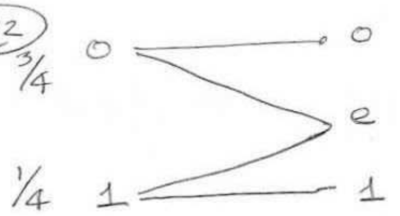
These are the rules obtained in the Bayes' case

Similarly

if $C_{10} < C_{01}$

If $C_{10} = C_{01}$ then we have no basis for deciding one way or another

8.82



$P[0|H_1] = 1/2$

$P[e|H_1] = 1/2$ $P[e|H_0] = 1/2$

$P[1|H_1] = 1/2$

$H_0 : \oplus = 0$

$H_1 : \oplus = 1$

Accept H_0 if $\dfrac{p(x|H_1)}{p(x|H_0)} < \dfrac{P_0 \, b \, C_{10}}{P_1 \, C_{10}} = 3b$

ⓑ

Bayes' Decision Rule:

$\dfrac{p(0|H_1)}{p(0|H_0)} = \dfrac{0}{1/2} = 0 \Rightarrow$ decide H_0

$\dfrac{p(1|H_1)}{p(1|H_0)} = \dfrac{1/2}{0} = \infty \Rightarrow$ decide H_1

$\dfrac{p(e|H_1)}{p(e|H_0)} = 1 \overset{?}{<} 3b$

→ No $b = 1/6$ $\Rightarrow$ decide H_1

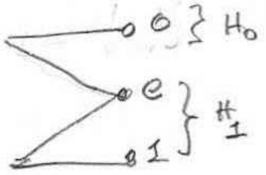
→ Yes $b = 6$ $\Rightarrow$ decide H_0

→ Yes $b = 1$ decide H_0

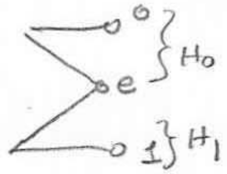
Average cost is:

$C = C_{01} P[\text{decide } H_1 | H_0] P_0 + C_{10} P[\text{decide } H_0 | H_1] P_1$

$b = 1/6$



$b=1$
$b=6$

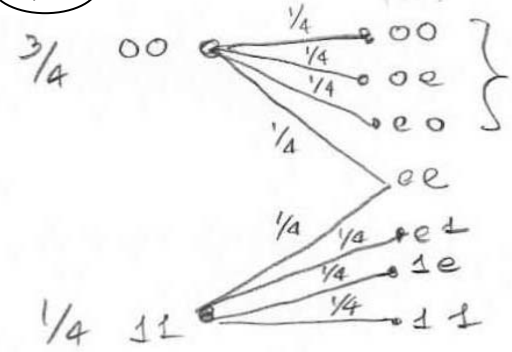


$C = C_{01} \frac{1}{2} \frac{3}{4} = \frac{1}{6} C_{10} \frac{3}{8}$

$= \frac{1}{16} C_{10}$

$C = C_{10} \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} C_{10}$

8.83



$p(\underline{x}|H_0) > 0$ and $p(\underline{x}|H_1) = 0$
$\Rightarrow$ decide H_0

$p(\underline{x}|H_1) > 0$ and $p(\underline{x}|H_0) = 0$
$\Rightarrow$ decide H_1

$$1 = \frac{p(ee|H_1)}{p(ee|H_0)} \overset{?}{<} \frac{p_0 b C_{10}}{p_1 c_{10}} = 3b = \begin{cases} 1/2 & b = 1/6 \\ 3 & b = 1 \\ 18 & b = 6 \end{cases}$$

for $b = \frac{1}{6}$ $\quad \underline{x} = ee \Rightarrow$ decide H_1

$b = 1, 6$ $\quad \underline{x} = ee \Rightarrow$ decide H_0

Average cost w:

$b = \frac{1}{6}$ $\quad C = C_{01} \cdot \frac{1}{4} \cdot \frac{3}{4}$
$= \frac{1}{6} \frac{3}{16} c_{10}$
$= \frac{1}{32} c_{10}$

$b = 1, 6$ $\quad C = c_{10}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$
$= \frac{c_{10}}{16}$

ML Rules give same decision rules as above if cost used to break ties (in this case only)

8.84

Bob H_0: 2D Gaussian mean zero, variance 1

Rick H_1: 2D Gauss mean zero, variance 4

$P_0 = 1/2$ assume

$P_1 = 1/2$ equal turns

$c_{01} = 1$

$c_{10} = 1$

$$C = c_{01} \cdot P[1|0]\,\tfrac{1}{2} + c_{10}\, P[0|1]\,\tfrac{1}{2}$$

$$= P[1|0]\,\tfrac{1}{2} + P[0|1]\,\tfrac{1}{2}$$

Min Cost Rule:

Accept H_0 if $$\underbrace{\frac{f(\underline{x}|H_1)}{f(\underline{x}|H_0)}}_{\dfrac{2\pi(1)\, e^{-(x^2+y^2)/8}}{2\pi(8)\, e^{-(x^2+y^2)/2}}} < \frac{P_0 c_{01}}{P_1 c_{10}} = 1$$

$$e^{\frac{3}{8}(x^2+y^2)} < 8$$

$$x^2+y^2 < \frac{8}{3}\ln 8 = 5.5452 = R^2$$

$$\Rightarrow R = 2.3548$$

If $c_{01} = 2$, $c_{10} = 1$

$$x^2+y^2 \overset{H_0}{<} \frac{8}{3}\ln 16 = 7.3936$$

$$\Rightarrow R = 2.7191$$

Bob expands his radius to reduce cost.

8.85

H_0: Binomial n, $p = 10^{-3}$ $\qquad P[H_0] = 1-\alpha = 4/5$

H_1: Binomial n, $1-p = 1-10^{-3}$ $\qquad P[H_1] = \alpha = 1/5$

ⓐ ML Rule

$$\frac{P[N=k \mid H_1]}{P[N=k \mid H_0]} = \frac{\binom{n}{k}(1-p)^k p^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \left(\frac{1-p}{p}\right)^{2k}\left(\frac{p}{1-p}\right)^n$$

$$= \left(\frac{p}{1-p}\right)^{n-2k} \overset{H_1}{\underset{H_0}{\gtrless}} 1$$

$$\underbrace{(n-2k)\ln\underbrace{\left(\frac{p}{1-p}\right)}_{<1}}_{<0} \overset{H_1}{\underset{H_0}{\gtrless}} 0$$

$$n \overset{H_1}{\underset{H_0}{\lessgtr}} 2k \qquad \boxed{\frac{n}{2} \overset{H_1}{\underset{H_0}{\lessgtr}} k}$$

Majority Rule

ⓑ
$$\frac{P[N=k \mid H_1]}{P[N=k \mid H_0]} = \left(\frac{1-p}{p}\right)^{2k}\left(\frac{p}{1-p}\right)^n \gtrless \frac{(1-\alpha)\cdot 1}{\alpha \cdot 1}$$

$$(n-2k)\ln\frac{p}{1-p} \overset{H_1}{\underset{H_0}{\gtrless}} \ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$n-2k \overset{H_1}{\underset{H_0}{\lessgtr}} \frac{\ln(1-\alpha)/\alpha}{\ln p/(1-p)} = \frac{1.3863}{-6.91} = -.2006$$

$$\frac{1}{2}(8.2) \overset{H_0}{\underset{H_1}{\lessgtr}} k \qquad \begin{array}{l} \{0,1,\ldots,4\} \Rightarrow H_0 \\ \{5,6,7,8\} \Rightarrow H_1 \end{array}$$

ⓒ
$$P[\text{Type I error}] = \sum_{k=5}^{8}\binom{8}{k}(10^{-3})^k(1-10^{-3})^{8-k} \approx \binom{8}{5}10^{-15}$$

$$P[\text{Type II error}] = \sum_{k=0}^{4}\binom{8}{k}(1-10^{-3})(10^{-3})^{8-k} \approx \binom{8}{4}(10^{-12})$$

$$P_e = \binom{8}{5}10^{-15}\cdot\frac{4}{5} + \binom{8}{4}10^{-12}\frac{1}{5} \approx \binom{8}{4}10^{-12}\frac{1}{5}$$

8.86

H_0: Gaussian $m=-1$ σ^2/n $1-\alpha$

H_1: Gauss $m=+1$ σ^2/n α

ⓐ ML Decision rule

$$\frac{f_X(x|H_1)}{f_X(x|H_0)} = \frac{\frac{e^{-(x-1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}}{\frac{e^{-(x+1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

take ln of both sides

$$-(x-1)^2 + (x+1)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 0 \iff \boxed{x \underset{H_0}{\overset{H_1}{\gtrless}} 0}$$

Decide based on sign of x

ⓑ Bayes' Decision Rule

$$-\frac{(x-1)^2}{2\sigma^2/n} + \frac{(x+1)^2}{2\sigma^2/n} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$4x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2\sigma^2}{n}\ln\left(\frac{1-\alpha}{\alpha}\right)$$

$$x \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{2n}\ln\left(\frac{1-\alpha}{\alpha}\right) \triangleq \gamma$$

$$P[\text{Type I}] = P[X>\gamma \mid H_0] = \int_\gamma^\infty \frac{e^{-(x+1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}\,dx = Q\left(\frac{\gamma+1}{\sigma/\sqrt{n}}\right)$$

$$P[\text{Type II}] = P[X<\gamma \mid H_1] = \int_{-\infty}^{\gamma} \frac{e^{-(x-1)^2/2\sigma^2/n}}{\sqrt{2\pi\sigma^2/n}}\,dx = Q\left(\frac{1-\gamma}{\sigma/\sqrt{n}}\right)$$

$$P_e = Q\left(\frac{\gamma+1}{\sigma/\sqrt{n}}\right)(1-\alpha) + Q\left(\frac{1-\gamma}{\sigma/\sqrt{n}}\right)\alpha$$

for ML Rule we have $\gamma=0$: $P_e^{ML} = Q\left(\frac{\sqrt{n}}{\sigma}\right)$

ⓒ $P[N>1] = Q\left(\frac{1}{\sigma}\right) = 10^{-3}$

$$\Rightarrow \frac{1}{\sigma} = 3,090$$

Consider the ML Rule:

$$P_e = Q\Big(\underbrace{\frac{\sqrt{n}}{\sigma}}_{5,9978}\Big) = 10^{-9}$$

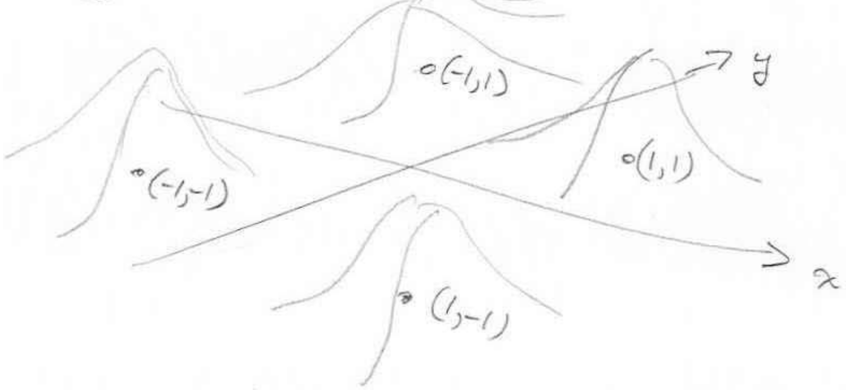
$$\Rightarrow \frac{\sqrt{n}}{3,09} = 5,9978$$

$$n = \left(\frac{5,9978}{3.09}\right)^2 = 3.7676 \Rightarrow \text{Use } \underline{\underline{n=4}}$$

8.87

$$f(x,y|\theta_1,\theta_2) = \frac{1}{2\pi\sigma^2} e^{-[(x-\theta_1)^2 + (y-\theta_2)^2]/2\sigma^2}$$

ⓐ



ⓑ

$$C_{ij} = \begin{cases} 1 & i \neq j \\ 0 & i \neq j \end{cases}$$

$i = \{1,2,3,4\}$

Quadrat
$1 \sim (1,1)$
$2 \sim (-1,1)$
$3 \sim (-1,-1)$
$4 \sim (1,-1)$

$$C = \sum_{i=1}^{4} \sum_{j=1}^{4} C_{ij}\, P[\text{Decide } j \,|\, H_i]\, P[H_i]$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{4} \sum_{j=1}^{4} P[\text{Decide } j \,|\, H_i] \underbrace{P[H_i]}_{p_i}$$

$$P[\text{Decide } j \,|\, H_i] = \int_{R_j} f(\underline{x}|H_i)\, d\underline{x}$$

R_j = Region where we decide j

$$C = \sum_{j=1}^{4} \sum_{\substack{i=1 \\ i \neq j}}^{4} \int_{R_j} f(\underline{x}|H_i) P[H_i]\, d\underline{x}$$

$$= \sum_{j=1}^{4} \underbrace{\int_{R_j} \left(\sum_{\substack{i=1 \\ i \neq j}}^{4} f(\underline{x}|H_i) P[H_i] \right) d\underline{x}}$$

contribution to cost (prob of error)
by H_i not selected in region R_j.

(8.87) – continued –

C is minimized by selecting for each $\underline{x}$ the index j that maximizes

$$P(H_i|\underline{x}) = \frac{f(\underline{x}|H_i)P[H_i]}{f(\underline{x})}$$

ⓓ if $P[H_i] = 1/4$ all i, then

$$P[H_i|\underline{x}] = \frac{f(\underline{x}|H_i)\,1/4}{f(x)}$$

so maximizing $P[H_i|\underline{x}]$ is same as maximizing $\underbrace{f(\underline{x}|H_i)}_{\text{max likelihood}}$

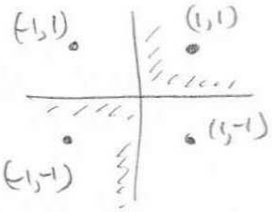
Compare 2 pdf's

1st quad $f(\underline{x}|11) = \frac{1}{2\pi\sigma^2} e^{-((x-1)^2+(y-1)^2)/2\sigma^2}$

4th quad $f(\underline{x}|1-1) = \frac{1}{2\pi\sigma^2} e^{-((x-1)^2+(y+1)^2)/2\sigma^2}$

For a point (x,y) $f(\underline{x}|11)$ is larger than $f(\underline{x}|1-1)$ if its exponent former's exponent is smaller than the latter's, that is, if

$$\underbrace{(x-1)^2+(y-1)^2}_{\substack{\text{distance from}\\(1,1)\text{ to }(x,y)}} < \underbrace{(x-1)^2+(y+1)^2}_{\substack{\text{distance from}\\(1,-1)\text{ to }(x,y)}}$$



∴ the ML decision rule picks the pt $(\pm 1, \pm 1)$ closest to (x,y). The resulting 4 decision regions are the 4 quadrants associated with the 4 signal points.

8.88 $C(g(\underline{x}),\Theta) = |g(\underline{x}) - \Theta|$

estimator

$$E[C(g(\underline{x}),\Theta)] = \int_{\underline{x}} \int_{T} |\theta - g(\underline{x})| \, f_\Theta(\theta|\underline{x}) \, f_{\underline{X}}(\underline{x}) \, d\theta \, d\underline{x}$$

$$= \int_{\underline{x}} \underbrace{\left[\int_{-\infty}^{g(\underline{x})} (g(\underline{x})-\theta) \, f_\Theta(\theta|\underline{x}) \, d\theta + \int_{g(\underline{x})}^{\infty} (\theta - g(\underline{x})) \, f_\Theta(\theta|\underline{x}) \, d\theta \right]}_{\text{need to minimize this expression for each } \underline{x}} f_{\underline{X}}(\underline{x}) \, d\underline{x}$$

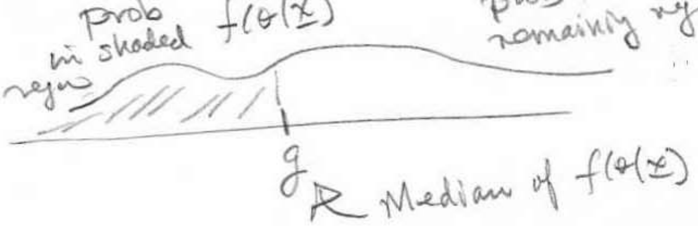
for each $\underline{x}$ we need to minimize

$$\int_{-\infty}^{g} (g-\theta) f(\theta|\underline{x}) d\theta + \int_{g}^{\infty} (\theta - g) f(\theta|\underline{x}) d\theta$$

Take derivative with respect to g

$$0 = \int_{-\infty}^{g} f(\theta|\underline{x}) d\theta - \int_{g}^{\infty} f(\theta|\underline{x}) d\theta$$

$$\Rightarrow \underbrace{\int_{-\infty}^{g} f(\theta|\underline{x}) d\theta}_{\text{prob in shaded region}} = \underbrace{\int_{g}^{\infty} f(\theta|\underline{x}) d\theta}_{\text{prob in remaining region}}$$



$\therefore$ $g(\underline{x})$ is median of $f(\theta|\underline{x})$.

8.89

$$C(g(\underline{x}),\Theta) = \begin{cases} 1 & \text{if } |g(\underline{x})-\theta|>\delta \\ 0 & \text{if } |g(x)-\theta|<\delta \end{cases}$$

$$E[C(g(\underline{x}),\Theta] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_{\{|g(\underline{x})-\theta|>\delta\}}\, f(\theta|\underline{x})\, f(\underline{x})\, d\theta\, d\underline{x}$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[1 - \underbrace{\int_{g(\underline{x})-\delta}^{g(\underline{x})+\delta} f(\theta/\underline{x})\, d\theta}_{\text{maximize this}} \right]}_{\text{then minimize this}} f(\underline{x})\, d\underline{x}$$

$\Rightarrow$ $g(\underline{x})$ selects θ so that $f(\theta|\underline{x})$ is <u>max</u>

This is the <u>maximum</u> a <u>posteriori</u> estimate

8.90 $X_1 \ldots X_n$ iid $E[X] = \Theta$ $\sigma_X^2 = 1$

Θ Gaussian mean 0 and unit variance

$$f_{\underline{X}}(\underline{x}|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i-\theta)^2/2} = \frac{1}{\sqrt{2\pi}^n} e^{-\sum_{i=1}^{n}(x_i-\theta)^2/2}$$

$$f(\underline{x},\theta) = \frac{1}{\sqrt{2\pi}^n} e^{-\sum_{i=1}^{n}(x_i-\theta)^2/2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\theta^2/2}$$

$$= c\, e^{-\left[\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2 + \frac{1}{2}\theta^2\right]}$$

$$= c\, e^{-\frac{1}{2}\left[\sum_{i=1}^{n}(x_i^2 - 2\theta x_i + \theta^2) + \theta^2\right]}$$

$$= c\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}\, e^{-\frac{1}{2}\left[2\theta^2 - 2\theta\sum_{i=1}^{n} x_i\right]}$$

$$= c\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}\, e^{-\left[\theta - \frac{1}{2}\sum_{i=1}^{n} x_i\right]^2}\, e^{\frac{1}{4}\left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= c'\, e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2}\, e^{\frac{1}{4}\left(\sum_1 x_i\right)^2}\, \frac{e^{-\left[\theta - \frac{1}{2}\sum_{i=1}^{n} x_i\right]^2/2\cdot\frac{1}{2}}}{\sqrt{2\pi\left(\frac{1}{2}\right)}}$$

$$f(\underline{x}) = c'\, e^{-\frac{1}{2}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{2}\left(\sum_1 x_i\right)^2\right]} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\left(\theta - \frac{1}{2}\sum_1\right)^2/2\cdot\frac{1}{2}}}{\sqrt{2\pi\left(\frac{1}{2}\right)}}\, d\theta}_{1}$$

$$\Rightarrow f(\theta|\underline{x}) = \frac{c\, e^{-\frac{1}{2}\left[\left(\sum_1 x_i^2 - 2\theta x_i + \theta^2\right) + \theta^2\right]}}{c'\, e^{-\frac{1}{2}\left[\sum_1 x_i^2 - \frac{1}{2}\left(\sum_1 x_i\right)^2\right]}}$$

$$= \frac{c}{c'}\, e^{-\frac{1}{2}\left[-\frac{1}{2}\left(\sum_1 x_i\right)^2 - 2\theta\sum_1 x_i + (n+1)\theta^2\right]}$$

8.90 – continued –

$$f(\theta|\underline{x}) = \frac{c}{c'}\, e^{-\frac{1}{2}(n+1)\left[\theta^2 - 2\frac{1}{n+1}\theta\sum x_i + \frac{1}{n+1}\left(\sum x_i^2\right)^2\right]}$$

$$= \frac{c}{c'}\, c''(\underline{x})\, e^{-\frac{1}{2}(n+1)\left[\theta - \frac{1}{n+1}\sum x_i\right]^2}$$

$$E[\theta|\underline{x}] = \frac{1}{n+1}\sum_{i=1}^{n} x_i = \frac{1}{1+\frac{1}{n}}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)}_{\bar{X}_n}$$

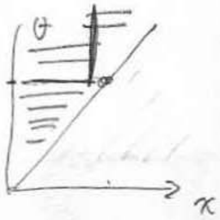
for Gaussian median = mean

∴ the minimum absolute error estimator $= E[\theta|\underline{x}]$

The maximum of the Gaussian occurs at the mean

∴ MAP estimator is also $E[\theta|\underline{x}]$

8.91 X unifm in $[0, \Theta]$. $f_\Theta(\theta) = \theta e^{-\theta}$



ⓐ $f(x, \theta) = f(x|\theta) f(\theta)$

$$= \frac{1}{\theta} \theta e^{-\theta} \qquad 0 < x < \theta$$

$$\theta > 0$$

$$= e^{-\theta}$$

$$f(x) = \int_x^\infty e^{-\theta} \, d\theta = e^{-x}$$

$$f(\theta|x) = \frac{f(x,\theta)}{f(x)} \qquad 0 < x < \theta$$

$$= \frac{e^{-\theta}}{e^{-x}} = e^{-(\theta - x)} \qquad \theta \geq x \geq 0$$

$$\frac{1}{2} = \int_x^{g(x)} e^{-(\theta - x)} \, d\theta = \int_0^{g(x)-x} e^{-y} \, dy = 1 - e^{-g(x)+x}$$

$$\frac{1}{2} = e^{-g(x)+x} \qquad -g(x) + x = \ln \frac{1}{2}$$

$$\Longrightarrow \boxed{\begin{aligned} g(x) &= x - \ln \tfrac{1}{2} \\ &= x + \ln 2 \end{aligned}}$$

$$E[\theta|X] = \int_x^\infty \theta e^{-(\theta - x)} \, dx$$

$$= \int_0^\infty (y + x) e^{-y} \, dy = x + 1$$

8.92 X Binomial n, Θ $\quad \Theta \sim$ Beta α, β

ⓐ

$$P[k, \Theta] = \binom{n}{k} \theta^{k} (1-\theta)^{n-k} \, c \, \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$P[k] = c\binom{n}{k} \underbrace{\int_0^1 \theta^{k+\alpha-1} (1-\theta)^{n-k+\beta-1} \, d\theta}_{B(k+\alpha-1,\, n-k+\beta-1)}$$

$$f(\Theta \mid k) = \frac{c\binom{n}{k} \theta^{k+\alpha-1} (1-\theta)^{n-k+\beta-1}}{c\binom{n}{k} B(k+\alpha-1,\, n-k+\beta-1)}$$

$$= \frac{\theta^{k+\alpha-1} (1-\theta)^{n-k+\beta-1}}{B(k+\alpha-1,\, n-k+\beta-1)} \quad \text{Beta pdf.}$$

$$E[\Theta \mid k] = \frac{k+\alpha-1}{k+\alpha+n-k+\beta} = \frac{k+\alpha}{\alpha+n+\beta} \quad ✓$$

8.93

$P[k|\theta] = \binom{n}{k}\theta^{k}(1-\theta)^{n-k}$ $\quad f_\theta(\theta)=1 \quad 0<\theta<1$

$$C(g(x),\theta) = \frac{(\theta-g(x))^2}{\theta(1-\theta)}$$

$$E[C(g(x),\theta)] = \int_0^1 \sum_{k=0}^{n} \frac{(\theta-g(k))^2}{\theta(1-\theta)}\binom{n}{k}\theta^{k}(1-\theta)^{n-k}\,d\theta$$

$$= \sum_{k=0}^{n} \int_0^1 \underbrace{(\theta-g(k))^2}_{\text{Sqd Error}} \underbrace{\binom{n}{k}\theta^{k-1}(1-\theta)^{n-k-1}}_{\text{Beta Dist.}}\,d\theta$$

Minimized by letting

$$g(k) = E[\theta|k] = \frac{k}{k+n-k} = \frac{k}{n}$$

8.7 Testing the Fit of a Distribution to Data

8.94

	Obs.	Expected	$(0-\varepsilon)^2/\varepsilon$
0	0	10.5	10.5
1	0	10.5	10.5
2	24	10.5	17.36
3	2	10.5	6.88
4	25	10.5	20.02
5	3	10.5	5.36
6	32	10.5	44.02
7	15	10.5	1.93
8	2	10.5	6.88
9	2	10.5	6.88
	105		$D^2 = 130.33$

\# degrees of freedom = 9
1% significance level ⇒ 21.7
$D^2 > 21.7$
⇒ Reject hypothesis
that #'s are
unif. dist. in $\{0, 1, \ldots, 9\}$

	Obs.	Expected	$(0-\varepsilon)^2/\varepsilon$
2	24	105/8	9.01
3	2	105/8	9.43
4	25	105/8	10.74
5	3	105/8	7.81
6	32	105/8	77.41
7	15	105/8	0.27
8	2	105/8	9.43
9	2	105/8	9.93
	105		83.26

\# degrees of freedom = 9
1% significance level ⇒ 21.7
$D^2 > 21.7$
⇒ Reject hypothesis
that #'s are
unif. dist. in $\{0, 1, \ldots, 9\}$

8.95

k	Observed N_k	Expected m_k	$(N_k - m_k)^2/m_k$
1	25	16	81/16
2	6	16	100/16
3	19	16	9/16
4	16	16	0/16
5	10	16	36/16
6	20	16	16/16

$$D^2 = \sum_k (N_k - m_k)^2/m_k = 242/16 = 15.125 > 11.07$$

⇒ Reject hypothesis.

8.96 Suppose N pairs of numbers are generated.

1. Partition the unit square into K disjoint subregions of equal area such that

$$\frac{N}{K} \geq 5 \Rightarrow N \geq 5K$$

2. Apply the Chi-Square test:

$$D^2 = \sum_{j=1}^{K} \frac{(N_j - N/K)^2}{N/K} \lesssim t_\alpha$$

where N_j is the number of pairs that fall in the jth region, and t_α is the threshold value determined by the significance level and the degrees of freedom $K-1$.

Chapter 9: Random Processes – Part II

9.6 Stationary Random Processes

9.61 a)

$$\begin{aligned} X(t) &= A\cos 2\pi t \\ m_X(t) &= \mathcal{E}[A]\cos 2\pi t = 0 \\ C_X(t_1,t_2) &= VAR[A]\cos 2\pi t_1 \cos 2\pi t_2 \qquad \text{from Example } \cancel{6.46}\ 9.9 \\ &= \frac{1}{3}\cos 2\pi t_1 \cos 2\pi t_2 \end{aligned}$$

Autocovariance does not depend only on $t_1 - t_2$

$\Rightarrow X(t)$ not stationary, not wide sense stationary

b) $X(t) = \cos(\omega t + \Theta)$

From Example ~~6.47~~ 9.10 $m_X(t) = 0$, $C_X(t_1,t_2) = \frac{1}{2}\cos\omega(t_1 - t_2)$

$\Rightarrow X(t)$ is wide sense stationary

In order to determine whether $X(t)$ is stationary, consider the third-order joint pdf:

$$\begin{aligned} & f_{X(t_1)X(t_2)X(t_3)}(x_1,x_2,x_3)dx_1dx_2dx_3 \\ &\quad = P[x_1 < \cos(\omega t_1 + \Theta) \le x_1 + dx_1,\ \ x_2 < \cos(\omega t_2 + \Theta) \le x_2 + dx_2, \\ &\qquad x_3 < \cos(\omega t_3 + \Theta) \le x_3 + dx_3] \\ &= P[A_1 \cap A_2 \cap A_3] \end{aligned}$$

where

$$A_i = \left\{ \cos^{-1} x_i - \omega t_i < \Theta \le \cos^{-1} x_i - \omega t_i + \frac{dx_i}{\sqrt{1-x_i^2}} \right\}$$

see Example ~~3.28~~ 4.36 and Figure ~~3.10~~ 4.14

$$\begin{aligned} & f_{X(t_1+\tau)X(t_2+\tau)X(t_3+\tau)}(x_1, x_2, x_3) \\ & \quad P[x_1 < \cos(\omega t_1 + \omega\tau + \Theta) \le x_1 + dx_1, \\ & \quad\; x_2 < \cos(\omega t_2 + \omega\tau + \Theta) \le x_2 + dx_2, \\ & \quad\;\; x_3 < \cos(\omega t_3 + \omega\tau + \Theta) \le x_3 + dx_3] \\ & \quad = P[A_1' \cap A_2' \cap A_3'] \end{aligned}$$

where

$$A_i' = \left\{ \cos^{-1} x_i - \omega t_i - \omega\tau < \Theta \le \cos^{-1} x_i - \omega t_i - \omega\tau + \frac{dx_i}{\sqrt{1-x_i^2}} \right\}$$

Since Θ is uniformly distributed, $P[A_i] = P[A_i']$.

In addition

$$P[A_1 \cap A_2 \cap A_3] = P[A_1' \cap A_2' \cap A_3']$$

since the intersection depends only on the relative values of t_1, t_2 and t_3. The same procedure can be used for nth order pdf's.

$\therefore X(t)$ is a stationary random process.

9.62

Head : X_n: … 1 1 1 ….

Tail: X_n: … -1 -1 -1 …

a) $E\{X_n\} = P\{\text{Head}\}\times 1 + (-1)\times P\{\text{Tail}\} = 0$

$$C_X(n_1,n_2) = E[X_{n_1} X_{n_2}] - E[X_{n_1}]E[X_{n_2}] = E[X_{n_1} X_{n_2}]$$

$$= 1\times 1\times P\{\text{Head}\} + (-1)\times(-1)\times P\{\text{Tail}\} = \frac{1}{2} + \frac{1}{2} = 1$$

therefore X_n is WSS

b) $VAR\{X_n\} = E[X_n^2] - E[X_n]^2 = 1 - 0 = 1$

$R_X(n_1,n_2) = 1$

for third order pmf we have

$$P\{X_{n_1} = k_1, X_{n_2} = k_2, X_{n_3} = k_3\} = \begin{cases} \frac{1}{2} & k_1 = k_2 = k_3 = 1 \\ \frac{1}{2} & k_1 = k_2 = k_3 = -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

it is not dependent on sample difference. and placement of origin

True to the nth order,

then X_n is stationary.

c) X_n stationary $\Rightarrow$ X_n cyclostationary with period $T=1$

9.63)

a) Head: X_n : . . . . $-1 \quad 1 \quad -1 \quad 1 \dots \qquad (-1)^n$

Tail: X_n : . . . . . $1 \quad -1 \quad 1 \quad -1 \dots \qquad (-1)^{n+1}$

$$E[X_n] = (-1)^n \times \frac{1}{2} + (-1)^{n+1}\left(\frac{1}{2}\right) = 0$$

$$R_X(n_1, n_2) = E[X_{n_1} X_{n_2}] = \begin{cases} -1 & \text{if } n_2 - n_1 \text{ odd} \\ 1 & \text{if } n_2 - n_1 \text{ even} \end{cases}$$

$$C_X(n_1, n_2) = R_X(n_1, n_2)$$

Therefore:
X_n is WSS

b) $\text{VAR}[X_n] = E[X_n^2] - E^2[X_n] = 1 \times \frac{1}{2} + 1 \times \frac{1}{2} - 0 \times 0 = 1$

We have to show that:

$$P\{X_{n_1} = k_1, X_{n_2} = k_2, X_{n_3} = k_3\} = P\{X_{n_1+m} = k_1, X_{n_2+m} = k_2, X_{n_3+m} = k_3\}$$

if m is even The above equality holds trivially.

if m is odd The above equality ~~again~~ is also ~~would be~~ true.

This can be shown for nth order pmf this is true as well.

then X_n is stationary.

c) X_n is cyclostationary with period $T = 1$.

9.64

$$
\begin{aligned}
& f_{X(t_1)X(t_2)\dots X(t_n)}(x_1, x_2, \dots, x_k) \\
&\quad = f_{X(t_1)\dots X(t_k)}(x_2, \dots, x_k | x_1) \underbrace{f_{X(t_1)}(x_1)}_{\text{1 unif. dist.}} \\
&\quad = \prod_{i=2}^{k} \delta(g^{-1}(x_i) - (t_i - \tilde{t})) \cdot 1 \\
&\quad \prod_{i=2}^{k} \delta(g^{-1}(x_i) - (t_i - t_1 + g^{-1}(x_1)))
\end{aligned}
$$

where

$$\tilde{t} = t_1 - g^{-1}(x_1)$$

once x_1 is known the phase shift is determined unambiguously.

$$f_{X(t_1)\dots X(t_k)}(x_1, \dots, x_k) = \prod_{i=2}^{k} \delta(g^{-1}(x_i) - g^{-1}(x_1) - (t_2 - t_1))$$

$\therefore X(t)$ is a stationary random process.

9.65

~~6.53~~ $X(t) = A\cos\omega t + B\sin\omega t$

a)

$$\begin{aligned}
\mathcal{E}[X(t)] &= \mathcal{E}[A\cos\omega t + B\sin\omega t] \\
&= \mathcal{E}[A]\cos\omega t + \mathcal{E}[B]\sin\omega t = 0 \\
C_X(t_1,t_2) &= \mathcal{E}[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\
&= \mathcal{E}[A^2]\cos\omega t_1\cos\omega t_2 + \mathcal{E}[B^2]\sin\omega t_1\sin\omega t_2 \\
&\quad + \mathcal{E}[A]\mathcal{E}[B]\cos\omega t_1\sin\omega t_2 + \mathcal{E}[A]\mathcal{E}[B]\sin\omega t_1\cos\omega t_2 \\
&= \mathcal{E}[A^2]\cos\omega t_1\cos\omega t_2 + \mathcal{E}[B^2]\sin\omega t_1\sin\omega t_2 \\
&= \mathcal{E}[A^2]\underbrace{(\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2)}_{\frac{1}{2}\cos\omega(t_1-t_2)} \\
&\quad \text{where we assumed } \mathcal{E}[A^2] = \mathcal{E}[B^2] \\
&= \frac{1}{2}\mathcal{E}[A^2]\cos\omega(t_1 - t_2)
\end{aligned}$$

$\therefore X(t)$ is WSS.

b)

$$\begin{aligned}
\mathcal{E}[X^3(t)] &= \mathcal{E}[(A\cos\omega t + B\sin\omega t)^3] \\
&= \mathcal{E}[A^3\cos^3\omega t + 3A^2B\cos^2\omega t\sin\omega t + 3AB^2\cos\omega t\sin^2\omega t \\
&\quad + B^3\sin^{-3}t] \\
&= \mathcal{E}[A^3]\cos^3\omega t + \mathcal{E}[B^3]\sin^3\omega t \\
&= \mathcal{E}[A^3](\cos^3\omega t + \sin^3\omega t) \\
&= \frac{\mathcal{E}[A^3]}{4}\{3\underbrace{(\cos\omega t + \sin\omega t) + (\cos 3\omega t - \sin 3\omega t)}_{\text{neither of these are constant}}\}
\end{aligned}$$

⇒ third ~~through~~ moment of $X(t)$ depends on time
⇒ $X(t)$ is <u>not</u> stationary
Since $\mathcal{E}[A^2B] = \mathcal{E}[A^2]\mathcal{E}[B] = \tau$ and $\mathcal{E}[AB^2] = 0$

$$\begin{aligned}
\cos^3\omega t &= \frac{1}{4}\cos 3\omega t + \frac{3}{4}\cos\omega t \\
\sin^3\omega t &= \frac{3}{4}\sin\omega t - \frac{1}{4}\sin 3\omega t
\end{aligned}$$

9.66

~~**6.54**~~ Assume X_n is discrete-valued, for simplicity, so that we can work with pmf's. Consider the third-order joint pmf of Y_n: for $n_1 < n_2 < n_3$ we need to show that for all $\tau > 0$

$$(\star)\quad P[Y_{n_1} = y_1, Y_{n_2} = y_2, Y_{n_3} = y_3] = P[Y_{n_1+\tau} = y_1, Y_{n_2+\tau} = y_2, Y_{n_3+\tau} = y_3]$$

Express the above probabilities in terms of the X_n's:

$$\begin{aligned} &P[Y_{n_1} = y_1, Y_{n_2} = y_2, Y_{n_3} = y_3] \\ &\quad = P\left[\frac{1}{2}(X_{n_1} + X_{n_1-1}) = y_1, \frac{1}{2}(X_{n_2} + X_{n_2-1}) = y_2, \frac{1}{2}(X_{n_3} + X_{n_3-1}) = y_3\right] \\ &\quad = P\left[\frac{1}{2}(X_2 + X_1) = y_1, \frac{1}{2}(X_{n_2-n_1+2} + X_{n_2-n_1+1}) = y_2,\right. \\ &\qquad \left.\frac{1}{2}(X_{n_3-n_1+2} + X_{n_3-n_1+1}) = y_3\right] \end{aligned}$$

Since the joint pdf of $(X_{n_1-1}, X_{n_1}, X_{n_2-1}, X_{n_2}, X_{n_3-1}, X_{n_3})$ is identical to that of $(X_1, X_2, X_{n_2-n_1+1}, X_{n_2-n_1+2}, \ldots, X_{n_3-n_1+2})$ if X_n is a stationary process.

Similarly we have that

$$\begin{aligned} &P[Y_{n_1+\tau} = y_1, Y_{n_2+\tau} = y_2, Y_{n_3+\tau} = y_3] \\ &\quad = P\left[\frac{1}{2}(X_{n_1+\tau} + X_{n_1+\tau-1}) = y_1, \ldots \frac{1}{2}(X_{n_3+\tau} + X_{n_3+\tau-1}) = y_3\right] \\ &\quad = P\left[\frac{1}{2}(X_2 + X_1) = y_1, \frac{1}{2}(X_{n_2-n_1+2} + X_{n_2-n_1+1}) = y_2, \frac{1}{2}(X_{n_3-n_1+2} + X_{n_3-n_1+1}) = y_3\right] \end{aligned}$$

$\therefore$ $(\star)$ holds if X_n is a stationary random process and in particular if X_n is an iid process. (a)

(b)

— continued —

ⓒ

In order for X_n and Y_n to be jointly stationary, their joint distributions should be invariant to shifts in the origin.

In parts ⓐ and ⓑ we express the joint pmf of Y_n's in terms of the X_n's. Therefore we can also express joint pmf's of X_n's and Y_n's in terms of joint pmf's of X_n's only. The stationarity of X_n then implies shift invariance for the joint pmf's of X_n and Y_n.

$\therefore$ X_n and Y_n are jointly stationary if X_n is iid and if X_n is stationary.

9.67

~~**6.55**~~ $Z_n = \frac{3}{4} Z_{n-1} + X_n \qquad Z_0 = 0$

a) $(\star)\ Z_n = \sum_{i=1}^{n} \left(\frac{3}{4}\right)^{n-i} X_i \qquad \mathcal{E}[Z_n] = 0$

$m < n$

$$\begin{aligned}
C_Z(m,n) &= \mathcal{E}[Z_m Z_n] = \mathcal{E}\left[\sum_{i=1}^{m}\left(\frac{3}{4}\right)^{m-i} X_i \sum_{j=1}^{n}\left(\frac{3}{4}\right)^{n-j} X_j\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{3}{4}\right)^{m+n-i-j}\mathcal{E}[X_i X_j] \\
&= \sum_{i=1}^{m}\left(\frac{3}{4}\right)^{m+n-2i}\mathcal{E}[X^2] \\
&\quad \text{since } \mathcal{E}[X_i X_j] = \begin{cases} 0 & i \neq j \\ \mathcal{E}[X^2] & i = j \end{cases} \\
&= \left(\frac{3}{4}\right)^{m+n}\mathcal{E}[X^2]\sum_{i=1}^{m}\left(\frac{16}{9}\right)^{i} \\
&= \left(\frac{3}{4}\right)^{m+n}\mathcal{E}[X^2]\frac{1-\left(\frac{16}{9}\right)^{m}}{1-\frac{16}{9}} \\
&= \frac{9}{7}\mathcal{E}[X^2]\left[\underbrace{\left(\frac{3}{4}\right)^{n-m}}_{\substack{\text{dependence}\\ \text{on } n-m}} - \underbrace{\left(\frac{3}{4}\right)^{m+n}}_{\substack{\text{dependence}\\ \text{on } m,n}}\right]
\end{aligned}$$

$\Rightarrow Z_n$ is <u>not</u> WSS.

b) For τ a fixed time shift, $m, n = m + \tau$ as $m \to \infty$

$$C_Z(m, m+\tau) = \frac{9}{7}\mathcal{E}[X^2]\left(\frac{3}{4}\right)^{\tau}$$

$\therefore Z_n$ is asymptotically WSS.

Indeed as $m \to \infty$, we can suppose that the process stated at $t = -\infty$, then

$$Z_n = X_n + \frac{3}{4}X_{n-1} + \left(\frac{3}{4}\right)^2 X_{n-2} + \ldots$$

If X_n is a stationary process (and hence its joint pmf's are invariant with respect to time shifts) then Z_n will also be <u>stationary</u>.

c) From part a), $\mathcal{E}[Z_n] = 0$, and from part b)

$$C_Z(m, m+\tau) = \frac{9}{7}(1)\left(\frac{3}{4}\right)^{\tau} \qquad \text{as } m \to \infty.$$

Z_n is then a discrete-time, zero-mean, Gaussian random process with the above covariance function.

9.68

$Y(t) = X(t+s) - \beta X(t) \qquad E[X(t)] = m_X, \quad C_X(t_1, t_2) = C_X(t_1 - t_2) = C_X(\tau)$

a) $E[Y(t)] = E[X(t+s)] - \beta E[X(t)] = m_X - \beta m_X = m_X(1-\beta)$

$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[X(t_1+s)X_2(t_2+s)] - \beta^2 E[X(t_1)X(t_2)]$$

$$- \beta E[X(t_1+s)X(t_2)] - \beta E[X(t_2+s)X(t_1)]$$

$$= R_X(t_1 - t_2) - \beta^2 R_X(t_2 - t_2) - \beta R_X(t_1+s-t_2) - \beta R_X(t_2+s-t_1)$$

$$= (1-\beta^2) R_X(\tau) - \beta\left(R_X(\tau+s) + R_X(s-\tau)\right) \qquad \tau = t_1 - t_2$$

if s is fixed then $R_Y(t_1, t_2)$ only depends on τ.

Also, $C_Y(t_1, t_2) = R_Y(t_1, t_2) - m_Y(t_1) m_Y(t_2)$ only depends on $t_1 - t_2$

Therefore Y is WSS

b) $$C_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] - E[X(t_1)]E[Y(t_2)]$$

$$= E[X(t_1)X(t_2+s)] - \beta E[X(t_1)X(t_2)] - m_X^2(1-\beta)$$

$$= R_X(t_1 - t_2 - s) - \beta R_X(t_1 - t_2) - m_X^2(1-\beta)$$

again for fixed S, X & Y are jointly WSS

c) & d) $Y(t)$ has jointly Gaussian pdf's with mean m_X and covariance $C_X(\tau)$.

e) $X(t)$ and $Y(t)$ have jointly Gaussian pdf's with means m_X, m_Y, and covariances $C_X(\tau)$, $C_Y(\tau)$ and $C_{XY}(\tau)$.

9.69 $Z(t) = 3X(t) - 5Y(t)$

X & Y are WSS then $m_X = m_Y = 0$ & $\sigma_X^2 = \sigma_Y^2 = C_X(0)$

a) $$m_Z(t) = 3m_X(t) - 5m_Y(t) = 0$$

$$C_Z(t_1,t_2) = R_Z(t_1,t_2) - 0 = R_Z(t_1,t_2) =$$

$$= E\{(3X(t_1) - 5Y(t_1))(3X(t_2) - 5Y(t_2))\}$$

$$= 9\,C_X(t_1,t_2) + 25C_Y(t_1,t_2) - 15C_{XY}(t_1,t_2) - 15C_{YX}(t_1,t_2)$$

X&Y independt $= 9C_X(\tau) + 25C_Y(\tau)$

Then Z is WSS

b) $Z(t)$ would be a Gaussian RV. with zero mean

and $\sigma^2_{Z(t)} = 34\sigma^2_{X(t)}$ $\qquad \sigma^2_{Z(t)} = 34\sigma^2_{X(t)} = 34\,C_X(0) = 34 \times 4 = 132$

$$f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi}\,\sigma_{Z(t)}}\, e^{-\frac{(z)^2}{2\sigma^2_{Z(t)}}}$$

c) $Z(t_1) = 3X(t_1) - 5Y(t_1)$ & $Z(t_2) = 3X(t_2) - 5Y(t_2)$

$Z(t_1)$ & $Z(t_2)$ are both Gaussian RVs. with mean $m_{Z(t_1)}$ & $m_{Z(t_2)}$ and variance $\sigma^2_{Z(t_1)}$ & $\sigma^2_{Z(t_2)}$, Also $C_Z(t_1,t_2) = C_Z(t_1 - t_2) = 132\,e^{-|t_1 - t_2|}$

So $f_{Z(t_1)Z(t_2)}(z_1,z_2)$ can be obtained as in P9.51

9.70

~~6.58~~ **a)** From Problem ~~6.18~~ 9.55:

$$\begin{aligned}\mathcal{E}[Z(t)] &= 0\\ C_Z(t_1,t_2) &= C_X(t_2-t_1)\cos\omega(t_2-t_1)\end{aligned}$$

$\Rightarrow Z(t)$ is WSS

b) $Z(t)$ is a Gaussian RV with mean zero and variance $C_X(0)$.

ⓒ $Z(t)$ is WSS Gaussian RP with zero mean and covariance

$$C_X(t_2-t_1)\cos\omega(t_2-t_1)$$

$\therefore$ $(Z(t_1), Z(t_2))$ ~~has~~ jointly Gaussian pdf with

$$\underline{m}_{\underline{X}} = \begin{bmatrix}0\\0\end{bmatrix} \qquad \underline{K}_{\underline{X}} = \begin{bmatrix} C_X(0) & C_X(t_2-t_1)\cos\omega(t_2-t_1)\\ C_X(t_2-t_1)\cos\omega(t_2-t_1) & C_X(0)\end{bmatrix}$$

ⓓ Since process are zero mean

$$\begin{aligned}COV(Z(t_1)X(t_2)) &= E[Z(t_1)X(t_2)]\\ &= [(X(t_1)\cos\omega t_1 + Y(t_1)\sin\omega t_1)X(t_2)]\\ &= E[X(t_1)X(t_2)]\cos\omega t_1\\ &= R_X(t_2-t_1)\underbrace{\cos\omega t_1}_{\text{dependence on } t_1}\end{aligned}$$

Use auxiliary variables

ⓔ $Z(t_1) = X(t_1)\cos\omega t_1 + Y(t_1)\sin\omega t_1$

$W(t_2) = X(t_2)$

$V(t_1) = Y(t_1)$

$$\begin{bmatrix} Z \\ W \\ V \end{bmatrix} = \begin{bmatrix} \cos\omega t_1 & 0 & \sin\omega t_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y_1 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos} & 0 & -\frac{\sin}{\cos} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ W \\ V \end{bmatrix}$$

The joint pdf of $X(t)$ and $Z(t)$ have parameters

$m_X(t) = 0$, $C_X(t_1 - t_2)$

$m_Z(t) = 0$, $C_Z(t_1, t_2) = C_X(t_2 - t_1)\cos\omega(t_2 - t_1)$

and $\text{COV}(Z(t_1), X(t_2)) = R_X(t_2 - t_1)\cos\omega t_1$

$$\underline{m}_{XZ} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad K_{X,Z} = \begin{bmatrix} C_X(t_2 - t_1) & R_X(t_2 - t_1)\cos\omega t_1 \\ R_X(t_2 - t_1)\cos\omega t_1 & C_X(t_2 - t_1)\cos\omega(t_2 - t_1) \end{bmatrix}$$

9.71

$$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = E[X^2(t_1)X^2(t_2)] =$$

$X(t_1)$ & $X(t_2)$ are zero mean jointly Gaussian RV

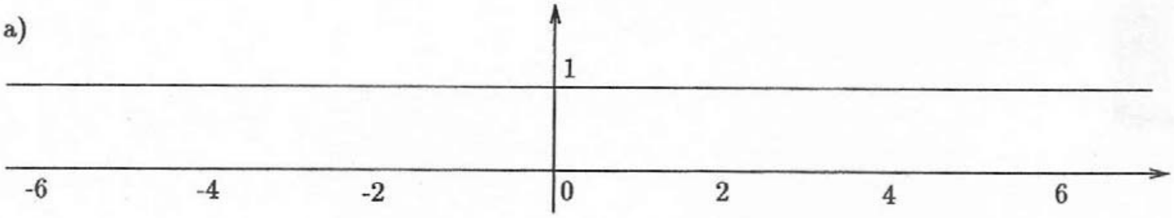
Hint: $$R_Y(t_1,t_2) = E[X^2(t_1)]E[X^2(t_2)] + 2E[X(t_1)X(t_2)]^2$$

$$= R_X(0)R_X(0) + 2R_X^2(\tau) \quad , \tau = t_1 - t_2$$

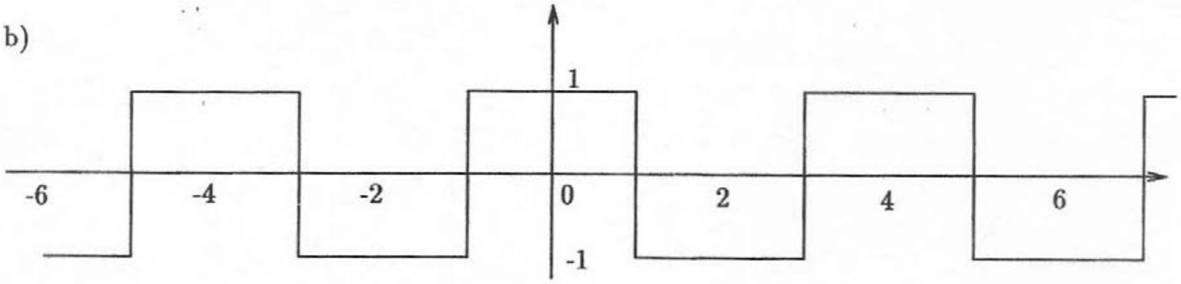
$$\Rightarrow \quad R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$$

9.72

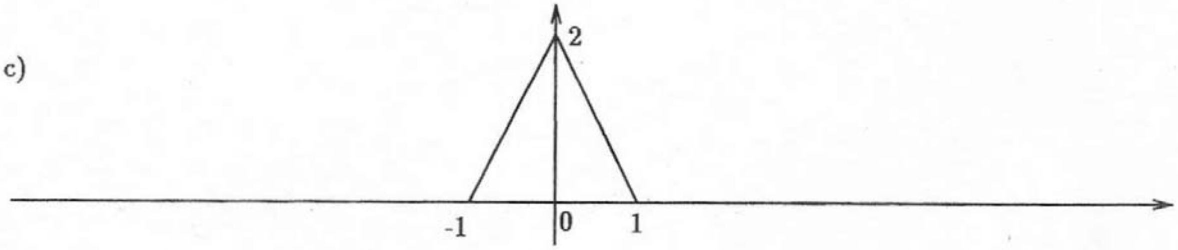
a)



b)



c)



9.73) The sequence U_n and the sequences X_n and Y_n are related as shown below:

$$\begin{array}{c|cc|cc|cc|cc|c} \cdots & U_{-2} & U_{-1} & U_0 & U_1 & U_2 & U_3 & U_4 & U_5 & \cdots \\ \cdots & X_{-1} & Y_{-1} & X_0 & Y_0 & X_1 & Y_1 & X_2 & Y_2 & \cdots \end{array}$$

If the sequence U_n is shifted by $2k$, then the subsequences X_n and Y_n are shifted by k

If the X_n (Y_n) are stationary then their joint pmf's are shift invariant

ⓑ ∴ the joint pmf of U_n is also shift invariant and U_n is cyclostationary.

ⓐ If X_n and Y_n are iid then they are stationary
$\Rightarrow U_n$ is cyclostationary.

ⓒ If X_n and Y_n are WSS then $m_X =$ constant $m_Y =$ constant
$R_X(\tau)$ and $R_Y(\tau)$ depend on $t_2 - t_1$ only.

Consider a shift of $2k$, then

$$m_U(t+2k) = m_U(t) = \begin{cases} m_X & t \text{ even} \\ m_Y & t \text{ odd} \end{cases}$$

also

$$C_U(t_1+2k, t_2+2k) = C_U(t_1,t_2) = \begin{cases} C_X(t_1,t_2) & t_1 \,\&\, t_2 \text{ even} \\ C_Y(t_1,t_2) & t_1 \,\&\, t_2 \text{ odd} \\ C_{XY}(t_1,t_2) & t_1 \text{ even}, t_2 \text{ odd} \\ C_{YX}(t_1,t_2) & t_1 \text{ odd}, t_1 \text{ even} \end{cases}$$

∴ U_n is wide sense cyclostationary

In general U_n is NOT WSS since X and Y may have different mean and covariance functions.

(d) $m_U = \frac{1}{2} m_X + \frac{1}{2} m_Y$

$$C_U(t_1,t_2) = \frac{1}{2} C_X(t_1,t_2) + \frac{1}{2} C_Y(t_1,t_2)$$
$$= \frac{1}{2} C_X(t_2-t_1) + \frac{1}{2} C_Y(t_2-t_1)$$

9.74

~~6.62~~ a) If n is even

$P[B_n = 1] = 1/3$ $P[B_{n+1} = 0] = 1/3$

$$\begin{aligned} P[B_n = 0, B_{n+1} = 0] &= \frac{1}{3} \\ P[B_n = 0, B_{n+1} = 1] &= \frac{1}{3} \\ P[B_n = 1, B_{n+1} = 0] &= \frac{1}{3} \\ P[B_n = 1, B_{n+1} = 1] &= 0 \end{aligned}$$

$B_n B_{n+1} = 0$ always

If n is odd

$$\begin{aligned} P[B_n = 0, B_{n+1} = 0] &= \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \\ P[B_n = 0, B_{n+1} = 1] &= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \\ P[B_n = 1, B_{n+1} = 0] &= \frac{2}{9} \\ P[B_n = 1, B_{n+1} = 1] &= \frac{1}{9} \end{aligned}$$

$B_n B_{n+1} = 1$ $\frac{1}{9}$ of time

B_n is not stationary, but is cyclostationary with period 2.

n even

$\left|\, \underline{n}\ \ \underline{n+1} \,\middle|\, n+2\ \ n+3 \,\right|$

b) $E[B_n] = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}$.

If n is even

$$\begin{aligned} R_B(n, n+j) &= E[B_n B_{n+j}] = P[B_n = 1, B_{n+j} = 1] \\ &= \begin{cases} 0 & \text{if } j = 1 \\ \frac{1}{3} & \text{if } j = 0 \\ \frac{1}{9} & \text{otherwise} \end{cases} \end{aligned}$$

n odd

$\left|\, n-1\ \ n \,\middle|\, n+1\ \ n+2 \,\right|$

if n is odd,

$$\begin{aligned} R_B(n, n+j) &= E[B_n B_{n+j}] = P[B_n = 1, B_{n+j} = 1] \\ &= \begin{cases} 0 & \text{if } j = -1 \\ \frac{1}{3} & \text{if } j = 0 \\ \frac{1}{9} & \text{otherwise} \end{cases} \end{aligned}$$

so B_n is not wide-sense stationary, but wide-sense cyclostationary.

c) After introducing random phase, we obtain B_n^s.

$$P[B_n^s = 0] = \frac{2}{3}, \qquad P[B_n^s = 1] = \frac{1}{3}$$

$$\begin{aligned} P[B_n^s = 0, B_{n+1}^s = 0] &= P[B_n = 0, B_{n+1} = 0 | n \text{ even}]\frac{1}{2} + P[B_n = 0, B_{n+1} = 0 | n \text{ odd}]\frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{4}{9} = \frac{7}{18} \end{aligned}$$

Similarly

$$\begin{aligned} P[B_n^s = 0, B_{n+1}^s = 1] &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{9} = \frac{5}{18} \\ P[B_n^s = 1, B_{n+1}^s = 0] &= \frac{5}{18} \\ P[B_n^s = 1, B_{n+1} = 1] &= \frac{1}{2}\frac{1}{9} = \frac{1}{18} \end{aligned}$$

$$E[B_n^s] = \frac{1}{3}$$

$$\begin{aligned} R_{B^s}(n, n+j) &= \begin{cases} \frac{1}{3} & j = 0 \\ \frac{1}{18} & j = \pm 1 \\ \frac{1}{9} & \text{otherwise} \end{cases} \\ C_{B_s}(n, n+j) &= \begin{cases} \frac{2}{9} & j = 0 \\ -\frac{1}{18} & j = \pm 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

9.75

~~6.63~~ a)

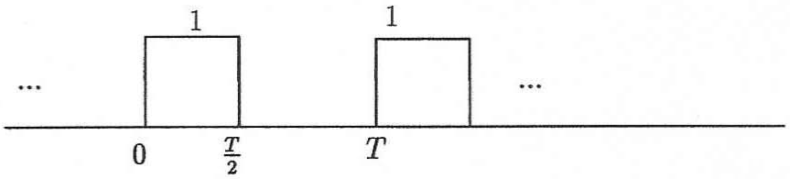
$$\begin{aligned}
E[X(t)] &= E[As(t)] = E[A]s(t) \\
R_X(t_1,t_2) &= E[As(t_1)As(t_2)] = E[A^2]s(t_1)s(t_2) \\
C_X(t_1,t_2) &= R_X(t_1,t_2) - E[X(t_1)]E[X(t_2)] = VAR[A]s(t_1)s(t_2)
\end{aligned}$$

X(t) is not WSS and hence not mean-square periodic.

b)

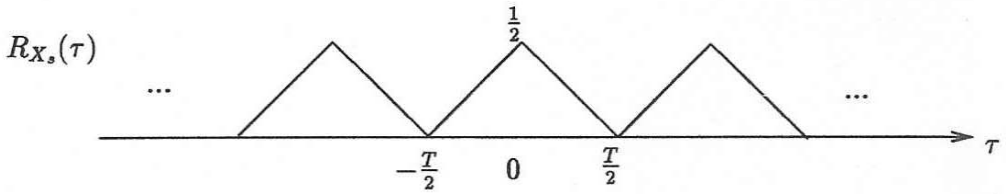
$$\begin{aligned}
X_s(t) &= X(t+\theta) \\
E[X_s(t)] &= \frac{1}{T}\int_0^T m_X(t)dt \\
&= \frac{1}{T}\int_0^T E[A]s(t)dt = E[A]\frac{1}{T}\int_0^T s(t)dt \\
R_{X_s}(\tau) &= \frac{1}{T}\int_0^T R_X(t+\tau,t)dt \\
&= \frac{1}{T}\int_0^T E[A^2]s(t+\tau)s(t)st \\
&= E[A^2]\frac{1}{T}\int_0^T s(t+\tau)s(t)dt
\end{aligned}$$

Thus the mean and autocorrelation of $X_s(t)$ are determined by time averages of $s(t)$. If $s(t)$ is as below



then

$$\frac{1}{T}\int_0^T s(t)dt = \frac{1}{2} \quad \text{and} \quad \frac{1}{T}\int_0^T s(t)s(t-\tau)dt = \frac{1}{2} - \frac{\tau}{T} \qquad |\tau| < \frac{T}{2}$$



9.76

a) if $A=1$, $X(t_1)=s(t_1)$

if $A=-1$, $X(t_1)=-s(t_1)$

$$P\{X(t_1)=1\}=\frac{1}{2}\times P\{s(t_1)=1\}+\frac{1}{2}\times P\{s(t_1)=-1\}=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$$

Also, similarly $P\{X(t_1)=-1\}=\frac{1}{2}$, $P\{Y(t_2)=1\}=\frac{1}{2}$, $P\{Y(t_2)=-1\}=\frac{1}{2}$

$$P\{X(t_1)=\pm1, Y(t_2)=\pm1\}=P\{X(t_1)=\pm1\}P\{Y(t_2)=\pm1\}=\frac{1}{4}$$

note that since A & B are independent, identically distributed RVs, then X & Y become independent for all t.

I{·} is indicator function

$$P\{X(t_1)=1, Y(t_2)=1\}=P\{A=1,B=1\}I\{s(t_1)=1, s(t_2)=1\}$$
$$+P\{A=1,B=-1\}I\{s(t_1)=+1, s(t_2)=-1\}$$
$$+P\{A=-1,B=1\}I\{s(t_1)=-1, s(t_2)=1\}$$
$$+P\{A=-1,B=-1\}I\{s(t_1)=-1, s(t_2)=-1\}$$
$$=\frac{1}{4}=P\{X(t_1)=1\}\times P\{Y(t_2)=1\} \quad \forall t_1,t_2$$

since one and only one of the indicator functions is 1.

This can be shown for any value of $X(t_1)$ & $Y(t_2)$, and for n-th order pmf

b) $C_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]-E[X(t_1)]E[Y(t_2)]$

$E[X(t_1)]=E[Y(t_2)]=0$

therefore $C_{XY}(t_1,t_2)=E[X(t_1)Y(t_2)]=\frac{1}{4}\times1+\frac{1}{4}\times-1+\frac{1}{4}\times-1+\frac{1}{4}\times1=0 \quad \forall t_1,t_2$

which means X & Y are uncorrelated.

c) $C_{XY}(t_1+mT, t_2+mT)=\emptyset=C_{XY}(t_1,t_2) \Rightarrow$ jointly WS cyclostationary; ~~and they~~ X_n and Y_n are independent so ~~they~~ their n-th order pmf is also jointly cyclostationary.

9.77

~~6.64~~ Recall the application of the Schwarz Inequality (Eqn. ~~6.60~~ Eqn 9.67) during the discussion ~~after~~ where the mean square periodic process ~~was defined on page 360.~~ We had:

$$E[(X(t+\tau+d)-X(t+\tau)X(t))]^2 \leq E[(X(t+\tau+d)-X(t+\tau))^2]E[X^2(t)]$$

If $X(t)$ is mean-square periodic, then

$$E[(X(t+\tau+d)-X(t+d))^2]=0 \quad .$$

Thus

$$\begin{aligned} & E[(X(t+\tau+d)-X(t+\tau))X(t)]^2=0 \\ \Rightarrow\ & (E[X(t+\tau+d)X(t)]-E[X(t+\tau)X(t)])^2=0 \\ \Rightarrow\ & E[X(t+\tau+d)X(t)]=E[X(t+\tau)X(t)] \\ \Rightarrow\ & R_X(t_1+d,t_2)=R_X(t_1,t_2) \end{aligned}$$

Repeated applications of this argument to t_1 and t_2 implies

$$R_X(t_1+md,t_2+nd)=R_X(t_1,t_2) \quad \text{for every integer } m,n\ .$$

The special case $m=n$ implies (~~6.62b~~ 9.70b) and hence that $X(t)$ is wide-sense cyclostationary.

9.78

~~6.65~~ Since the data sequence is iid, each T-second interval can be viewed as an independent trial. Therefore, time shifts of the process by integer multiples of T leave the joint distributions unchanged. Thus the process is cyclostationary.

9.79

~~6.66~~

$$\begin{aligned} m_X(t) &= E[A]\cos\frac{2\pi t}{T} \\ E[X_S(t)] &= \frac{1}{T}\int_0^T E[A]\cos\frac{2\pi t}{T}dt=0 \\ R_X(t,t+\tau) &= E\left[A\cos\frac{2\pi t}{T}A\cos\frac{2\pi(t+\tau)}{T}\right] \\ &= E[A^2]\frac{1}{2}\left(\cos\frac{2\pi\tau}{T}+\cos\frac{4\pi t+2\pi\tau}{T}\right) \\ R_{X_S}(\tau) &= \frac{1}{T}\int_0^T R_X(t,t+\tau)dt \\ &= \frac{1}{2}\cos\frac{2\pi\tau}{T}E[A^2] \end{aligned}$$

9.80

~~6.67~~ Note that $\int_{-\tau}^{T-\tau} f(t)dt = \int_0^T f(t)dt$ if $f(t+mT) = f(t)$

$$
\begin{aligned}
&P[X_S(t_1+\tau) \le x_1, \ldots, X_S(t_k+\tau) \le x_k] \\
&\quad = P[X(t_1+\tau+\Theta) \le x_1, \ldots, X(t_k+\tau+\Theta) \le x_k] \\
&\quad = \int_0^T P[X(t_1+\tau+\Theta) \le x_1, \ldots, X(t_k+\tau+\Theta) \le x_k | \Theta = 0] d\Theta \\
&\quad \frac{1}{T}\int_{-\tau}^{T-\tau} P[X(t_1+\tau+\Theta) \le x_1, \ldots, X(t_k+\tau+\Theta) \le x_k] d\Theta \\
&\quad = \frac{1}{T}\int_0^T P[X(t_1+\Theta) \le x_1, \ldots, X(t_k+\Theta) \le x_k] d\Theta \\
&= P[X_S(t_1) \le x_1, \ldots, X_S(t_k) \le x_k] \\
&\quad \therefore\ X_S(t) \text{ is stationary}
\end{aligned}
$$

9.81

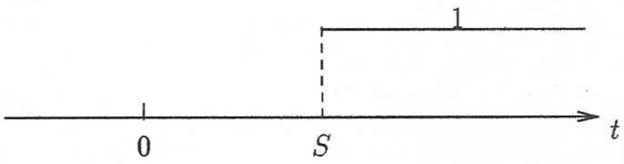
~~6.68~~ $X_S(t) = X(t+\Theta)$

$$
\begin{aligned}
E[X_S(t)] &= E[X(t+\Theta)] \\
&= E[E[X(t+\Theta)|\Theta]] \\
&= E[m_X(t+\Theta)] \\
&= \frac{1}{T}\int_0^T m_X(t+\Theta)d\Theta \\
&= \frac{1}{T}\int_{+t}^{T+t} m_X(u)du \\
&= \frac{1}{T}\int_0^T m_X(u)du \\
R_{X_S}(\tau) &= E[X(t_1+\Theta)X(t_1+\tau+\Theta)] \\
&= E[E[X(t_1+\Theta)X(t_1+\tau+\Theta)|\Theta]] \\
&= E[R_X(t_1+\Theta, t_1+\tau+\Theta)] \\
&= \frac{1}{T}\int_0^T R_X(t_1+\Theta, t_1+\tau+\Theta)d\Theta \\
&= \frac{1}{T}\int_{+t_1}^{T+t_1} R_X(u, u+\tau)du \\
&= \frac{1}{T}\int_0^T F_X(u, u+\tau)du
\end{aligned}
$$

$\therefore$ mean of $X_S(t)$ is constant, and autocorrelation depends only on τ.

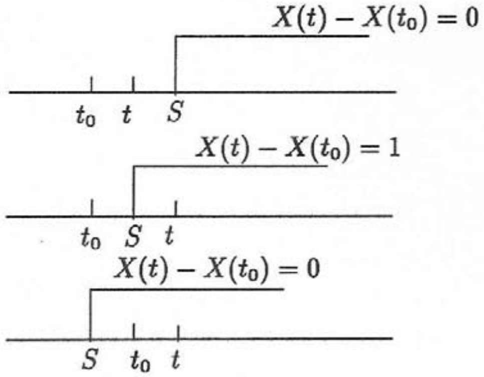
9.7 Continuity, Derivatives, and Integrals of Random Processes

9.82



a) $P[X(t) \text{ discontinuous at } t_0] = P[s = t_0] = 0$

b)



$$\begin{aligned}\lim_{t_0 \to 0} E[X(t) - X(t_0)^2] &= 1 \cdot P[t_0 < S < t] \\ &= e^{-\lambda t} - e^{-\lambda t_0} \\ &\to 0 \quad \Rightarrow \quad X(t) \text{ is M.S. continuous}\end{aligned}$$

We can also determine continuity from the autocorrelation function:

$$\begin{aligned}E[X(t_1)X(t_2)] &= P[X > \max(t_1, t_2)] \\ &= e^{-\lambda \max(t_1, t_2)}\end{aligned}$$

Next we determine if $R_X(t_1, t_2)$ is continuous at (t_0, t_0):

$$\begin{aligned}R_X(t_0 + \varepsilon_1, t_0 + \varepsilon_2) - R_X(t_0, t_0) &= e^{-\lambda \max(t_0+\varepsilon_2, t_0+\varepsilon_2)} - e^{-\lambda t_0} \\ &= e^{-\lambda(t_0 + \max(\varepsilon_1, \varepsilon_2))} - e^{-\lambda t_0} \\ &= e^{-\lambda t_0}[e^{-\lambda \max(\varepsilon_1, \varepsilon_2)} - 1] \\ &\to 0 \quad \text{as } \varepsilon_1 \text{ and } \varepsilon_2 \to 0\,. \\ &\qquad \Rightarrow \quad X(t) \text{ is M.S. continuous}\end{aligned}$$

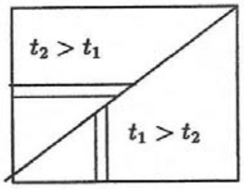
c) We expect that the mean square derivative is zero (if it exists). We thus consider the limit:

$$E\left[\left(\frac{X(t+\tau) - X(t)}{\varepsilon}\right)^2\right] = \frac{1}{\varepsilon^2}(e^{-\lambda(t+\varepsilon)} - e^{-\lambda t})$$

$$
\begin{aligned}
&= e^{-\lambda t} \underbrace{\left(\frac{e^{-\lambda\varepsilon}-1}{\varepsilon^2}\right)}_{\frac{\lambda\varepsilon - \frac{\lambda^2\varepsilon^2}{2}+\ldots}{\varepsilon^2}} \\
&= e^{-\lambda t}\frac{1}{\varepsilon} \to \infty \ .
\end{aligned}
$$

Thus the M.S. derivative does not exist.

d) $X(t)$ is M.S. integrable if the following integral exists:



Two regions of integration

$$
\begin{aligned}
&\int_0^t \int_0^t e^{-\lambda \max(t_1,t_2)} dt_1 dt_2 \\
&= \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-\lambda t_1} + \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-\lambda t_2} \\
&= \int_0^t dt_1 t_1 e^{-\lambda t_1} + \int_0^{t_2} dt_2 t_2 e^{-\lambda t_2} \\
&= \left.\frac{e^{-\lambda t_1}(-\lambda t_1 - 1)}{\lambda^2}\right|_0^t + \left.\frac{e^{-\lambda t_2}(-\lambda t_2 - 1)}{\lambda^2}\right|_0^t \\
&= \frac{e^{-\lambda t}(-\lambda t - 1) - (-1)}{\lambda^2} + \frac{e^{-\lambda t}(-\lambda t - 1) - (-1)}{\lambda^2} \\
&= \frac{2}{\lambda^2}[1 - e^{-\lambda t}(\lambda t + 1)] \\
&\Rightarrow \quad X(t) \text{ is M.S. integrable}
\end{aligned}
$$

Let

$$Y(t) = \int_0^t X(\lambda) d\lambda \ .$$

Then from Eqn. 9.91

$$
\begin{aligned}
m_Y(t) &= \int_0^t m_X(u) du \\
m_X(t) &= E[X(t)] = 1 \cdot P[S < t] = 1 - e^{-\lambda t} \\
m_Y(t) &= \int_0^t (1 - e^{-\lambda u}) du = t - \left(\frac{e^{-\lambda u}}{-\lambda}\right)_0^t \\
&= t + \frac{1}{\lambda}[e^{-\lambda t} - 1]
\end{aligned}
$$

From Eqn. 9.92 we have:

$$
\begin{aligned}
R_Y(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} R_X(u,v) du dv \\
&= \frac{1 - e^{-\lambda t_1}(\lambda t_1 + 1)}{\lambda^2} + \frac{1 - e^{-\lambda t_2}(\lambda t_2 + 1)}{\lambda^2}
\end{aligned}
$$

9.83

~~6.70~~ **a)** $R_X(t_1, t_2) = e^{-2\alpha|t_1+t_2|}$

$R_X(t_1, t_2)$ is continuous in both t_1 and t_2.

$X(t)$ is M.S. continuous.

b)

$$\begin{aligned} R_X(t_1, t_2) &= \begin{cases} e^{-2\alpha(t_1-t_2)} & t_2 < t_1 \\ e^{-2\alpha(t_2-t_1)} & t_2 \geq t_1 \end{cases} \\ \frac{\partial R_X(t_1, t_2)}{\partial t_2} &= \begin{cases} 2\alpha e^{-2\alpha(t_1-t_2)} & t_2 < t_1 \\ -2\alpha e^{-2\alpha(t_2-t_1)} & t_2 \geq t_1 \end{cases} \\ &= sgn(t_1 - t_2) 2\alpha e^{-2\alpha|t_1-t_2|} \end{aligned}$$

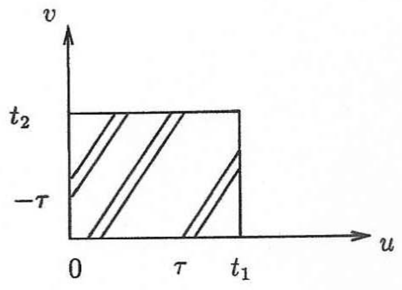
The second derivative does not exist at discontinuity points. $X(t)$ does not have a mean square derivative.

$$\frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2} = \delta(t_1 - t_2) 4\alpha e^{-2\alpha(t_1-t_2)} + 4\alpha^2 e^{-2\alpha(t_1-t_2)}$$

The transitions from $+1$ to -1 or from -1 to 1 give rise to delta function in $R_{X'}(t_1, t_2)$.

c) $X(t)$ has a M.S. integral since it is M.S. continuous. $m_X(t) = 0$, so

$$E\left[\int_0^t x(t')dt'\right] = \int_0^t E[x(t')]dt' = 0$$



Consider $Y(t) = \int_0^t X(t')dt$

$$\begin{aligned}
R_Y(t_1,t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u,v)dudv \qquad (t_2, \le t_1) \\
&= \int_0^{t_1}\int_0^{t_2} R_X(u-v)dudv \\
&= \int_{-t_2}^{0} (t_2+\tau)R_X(\tau)d\tau + \int_0^{t_1-t_2} t_2 R_X(\tau)d\tau \\
&\quad + \int_{t_1-t_2}^{t_1} (t_1-\tau)R_X(\tau)d\tau \\
&= \int_{-t_2}^{0} e^{-2\alpha(-\tau)}(t_2+\tau)d\tau + \int_0^{t_1-t_2} t_2 e^{-2\alpha\tau}d\tau \\
&\quad + \int_{t_1-t_2}^{t_1} (t_1-\tau)e^{-2\alpha\tau}d\tau \\
&= \frac{t_2}{2\alpha}(1-e^{-2\alpha t_2}) + \frac{1}{2\alpha}\int_{-t_2}^{0} \tau de^{2\alpha\tau} + \frac{t_2}{2\alpha}(1-e^{-2\alpha(t_1-t_2)}) \\
&\quad + \frac{t_1}{2\alpha}(e^{-2\alpha(t_1-t_2)} - e^{-2\alpha t_1}) + \frac{1}{2\alpha}\int_{t_1-t_2}^{t_1} \tau de^{-2\alpha\tau} \\
&= \frac{t_2}{2\alpha}(1-e^{-2\alpha t_2}) + \frac{t_2}{2\alpha}e^{-2t_2} - \frac{1}{2\alpha}\int_{-t_2}^{0} e^{2\alpha\tau}d\tau + \frac{t_2}{2\alpha}(1-e^{-2\alpha(t_1-t_2)}) \\
&\quad + \frac{t_1}{2\alpha}(e^{-2\alpha(t_1-t_2)} - e^{-2\alpha t_1}) + \frac{1}{2\alpha}(t_1e^{-2\alpha t_1} - (t_1-t_2)e^{-2\alpha(t_1-t_2)}) \\
&\quad - \frac{1}{2\alpha}\int_{t_1-t_2}^{t_1} e^{-2\alpha\tau}d\tau \\
&= \frac{t_2}{2\alpha}(1-e^{-2\alpha t_2}) + \frac{t_2}{2\alpha} - \frac{1}{4\alpha^2}(1-e^{-2\alpha t_2}) + \frac{t_2}{2\alpha}(1-e^{-2\alpha(t_1-t_2)}) \\
&\quad + \frac{t_1}{2\alpha}(e^{-2\alpha(t_1-t_2)} - e^{-2\alpha t_1}) + \frac{1}{2\alpha}(t_1e^{-2\alpha t_1} - (t_1-t_2)e^{-2\alpha(t_1-t_2)}) \\
&\quad = \frac{1}{4\alpha^2}[e^{-2\alpha t_1} - e^{-2\alpha(t_1-t_2)}] \\
&= \frac{t_2}{2\alpha} - \frac{1}{4\alpha^2}(1-e^{-2\alpha t_2}) + \frac{t_2}{2\alpha}(1-2e^{-2\alpha(t_1-t_2)}) + \frac{1}{4\alpha^2}[e^{-2\alpha t_1} - e^{-2\alpha(t_1-t_2)}]
\end{aligned}$$

In general

$$\begin{aligned}
R_Y(t_1,t_2) &= \frac{\min(t_1,t_2)}{\alpha}(1-e^{-2\alpha(t_1-t_2)}) \\
&\quad + \frac{1}{4\alpha^2}(e^{-2\alpha t_1} + e^{-2\alpha t_2} - 1 - e^{-2\alpha(t_1-t_2)})
\end{aligned}$$

9.84

~~6.71~~ $R_X(\tau) = \sigma^2 e^{-\alpha\tau^2}, \quad R_X(t_1, t_2) = \sigma^2 e^{-\alpha(t_1-t_2)^2}$

a) Yes since $R_X(\tau)$ is continuous at τ.

b) Yes since $R_X(\tau)$ has derivatives of all orders at $\tau = 0$.

$$\begin{aligned} E\left[\frac{d}{dt}X(t)\right] &= \frac{d}{dt}[E[X(t)]] = 0, \qquad \text{because } E[X(t)] = \text{constant} \\ R_{X'}(\tau) &= -\frac{d}{d\tau^2}R_X(\tau) \\ &= 2\alpha\sigma^2 e^{-\alpha\tau^2}(1 - 2\alpha\tau^2) \end{aligned}$$

c) Yes since $R_X(\tau)$ is M.S. continuous.
Consider $Y(t) = \int_0^t X(t)dt$.

$$\begin{aligned} E[Y(t)] &= \int_0^t E[X(t)]dt = m_X t \\ R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u, v)dudv \end{aligned}$$

If $t_2 \le t_1$

$$\begin{aligned} R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u - v)dudv \\ &= \int_{t_2}^{0}(t_2 + \tau)\sigma^2 e^{-\alpha\tau^2}d\tau + \int_0^{t_1-t_2} t_2\sigma^2 e^{-\alpha\tau^2}d\tau \\ &\quad + \int_{t_1-t_2}^{t_1}(t_1 - \tau)\sigma^2 e^{-v\tau^2}d\tau \\ &= \sigma^2 t_2\int_{-t_2}^{0} e^{-\alpha\tau^2}d\tau + \frac{\sigma^2}{2\alpha}\int_{-t_2}^{0} e^{-\alpha\tau^2}d\alpha\tau^2 + \sigma^2 t_2\int_0^{t_1-t_2} e^{-\alpha\tau^2}d\tau \\ &\quad +\sigma^2 t_1\int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2}d\tau - \frac{\sigma^2}{2\alpha}\int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2}d\sigma\tau^2 \\ &= \sigma^2 t_2\int_{-t_2}^{t_1-t_2} e^{-\alpha\tau^2}d\tau - \frac{\sigma^2}{2\alpha}(t - e^{-\alpha t_2^2}) \\ &\quad +\sigma^2 + 1\int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2}d\tau + \frac{\sigma^2}{2\alpha}(e^{-\alpha t_1^2} - e^{-\alpha(t_1-t_2)^2}) \end{aligned}$$

If $t_2 > t_1$

$$\begin{aligned} R_Y(t_1, t_2) &= \sigma^2 t_1\int_{-t_1}^{t_2-t_1} e^{-\alpha\tau^2}d\tau + \sigma^2 t_2\int_{t_2-t_1}^{t_2} e^{-\alpha\tau^2}d\tau \\ &\quad +\frac{\sigma^2}{2\alpha}(-1 + e^{-\alpha t_1^2} + e^{-\alpha t_2^2} - e^{-\alpha(t_1-t_2)^2}) \end{aligned}$$

d) The fact that the autocorrelation has the shape of a Gaussian pdf does not imply that the process is Gaussian.

9.85

~~6.72~~ The independent increments property implies that

$$E[(N(t)-N(t_0))^2] = E[(N^2(t-t_0))] = \lambda(t-t_0)+\lambda^2(t-t_0)^2$$

So

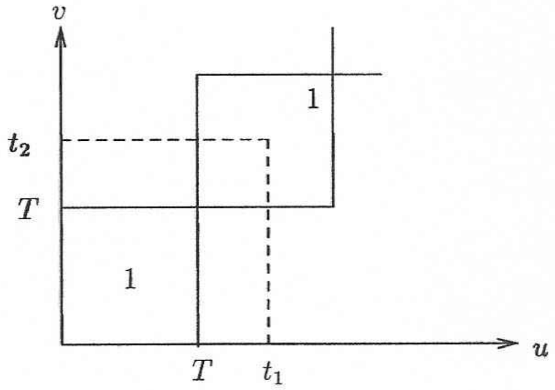
$$\lim_{t\to t_0} E[(N(t)-N(t_0))^2] = 0$$

and $N(t)$ is M.S. continuous.

9.86

~~6.73~~ $X(t)$ is not M.S. continuous so we don't know whether it is integrable. Consider $Y(t)=\int_0^t X(t')dt'$, then

$$\begin{aligned} E[Y(t)] &= 0 \\ \text{Now consider} \qquad R_Y(t_1,t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u,v)\,du\,dv \end{aligned}$$



$R_Y(t_1,t_2)$ is the area of the non-zero region within the rectangle defined by (t_1,t_2). For fixed values t_1, t_2 this area is well-defined. Therefore $X(t)$ is M.S. integrable.

9.87

~~6.74~~ From ordinary calculus we know that the integral in Eqn. 6.83 exists if its argument $R_X(u,v)$ is a continuous function of u and v. Therefore, it suffices for us to show that if $X(t)$ is M.S. continuous, then $R_X(u,v)$ is continuous. Therefore consider

$$\begin{aligned} R_X(t_1,t_2) - R_X(t_0,t_0) &= E[X(t_1)X(t_2) - X(t_0)X(t_0)] \\ &= E[X(t_1)X(t_2) - X(t_0)X(t_2) + X(t_0)X(t_2) - X(t_0)X(t_0)] \\ &= E[(X(t_1) - X(t_0))X(t_2)] + E[X(t_0)(X(t_2) - X(t_0))] \end{aligned}$$

From the Schwarz Inequality (Eqn. 6.60)

$$|E[(X(t_1) - X(t_0))X(t_2)]|^2 \le E[(X(t_1) - X(t_0))^2]E[X^2(t_2)]$$

and

$$|E[(X(t_0))(X(t_2) - X(t_0))]|^2 \le E[X^2(t_0)]E[(X(t_2) - X(t_0))^2]$$

If $X(t)$ is M.S. continuous then the right-hand side approaches zero as $t_2 \to t_0$ and $t_1 \to t_0$. Therefore

$$R_X(t_1,t_2) \to R_X(t_0,t_0)$$

and hence $R_X(u,v)$ is continuous as required.

9.88

~~6.75~~ $Y(t) = \int_0^t X(u)du$

$$\begin{aligned} E\left[\left(\frac{d}{dt}Y(t) - X(t)\right)^2\right] &= E\left[\frac{d}{dt}Y(t)\frac{d}{dt}Y(t)\right] - 2E\left[X(t)\frac{d}{dt}Y(t)\right] \\ &\quad + E[X(t)X(t)] \\ &= \frac{\partial^2}{\partial t_1 \partial t_2} R_Y(t_1,t_2)\bigg|_{t_1=t_2=t} - 2\,\frac{\partial R_{XY}(t_1,t_2)}{\partial t_2}\bigg|_{t_1=t_2=t} + R_X(t,t) \\ R_Y(t_1,t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u,v)dudv \\ \frac{\partial^2 R_Y(t_1,t_2)}{\partial t_1 \partial t_2} &= R_X(t_1,t_2) \\ R_{XY}(t_1,t_2) &= \int_0^{t_2} R_X(t_1,u)du \\ \frac{\partial R_{XY}(t_1,t_2)}{\partial t_2} &= R_X(t_1,t_2) \end{aligned}$$

$$\therefore \quad E\left[\left(\frac{d}{dt}Y(t) - X(t)\right)^2\right] = R_X(t,t) - 2R_X(t,t) + R_X(t,t) = 0$$

9.89

$$\begin{aligned} R_{XX'}(t_1,t_2) &= \frac{\partial}{\partial t_2} R_X(t_1,t_2) \\ &= \frac{\partial}{\partial t_2} R_X(t_1-t_2) \\ &= -\frac{dR_X(\tau)}{d\tau}, \qquad \tau = t_1 - t_2 \end{aligned}$$

$$R_X(\tau) \le R_X(0), \qquad R_X(0) \text{ is the max of } R_X(\tau)$$

$$\therefore \left.\frac{dR_X(\tau)}{d\tau}\right|_{\tau=0} = 0$$

$$\text{i.e.,} \qquad R_{XX'}(t,t) = 0$$

9.90 $m_Z(t) = 0 \Rightarrow m_X(t) = 0$.
Proceeding as in Ex. 6.41:

$$R_{ZX}(t_1,t_2) = \int_0^{t_2} e^{-\alpha(t_2-\tau)} R_Z(t_1,\tau)d\tau = \int_0^{t_2} e^{-\alpha(t_2-\tau)}\sigma^2 e^{-\beta|t_1-\tau|}d\tau$$

We note that:

$$e^{-\beta|t_1-\tau|} = \begin{cases} e^{-\beta(t_1-\tau)} & \text{for } t_1 \ge \tau \\ e^{\beta(t_1-\tau)} & \text{for } t_1 \le \tau \end{cases}$$

We now suppose that $t_1 \le t_2$ so the above integral becomes:

$$\begin{aligned} R_{ZX}(t_1,t_2) &= \sigma^2 e^{-\alpha t_2}\left[\int_0^{t_1} e^{\alpha\tau}e^{-\beta t_1}e^{\beta\tau}d\tau + \int_{t_1}^{t_2} e^{\alpha\tau}e^{\beta t_1}e^{-\beta\tau}d\tau\right] \\ &= \sigma^2 e^{-\alpha t_2}\left[\frac{e^{-\beta t_1}(e^{(\alpha+\beta)t_1}-1)}{\alpha+\beta} + \frac{e^{\beta t_1}(e^{(\alpha-\beta)t_2}-e^{(\alpha-\beta)t_1})}{\alpha-\beta}\right] \end{aligned}$$

The autocorrelation of $X(t)$ is then

$$\begin{aligned} R_X(t_1,t_2) &= \int_0^{t_1} e^{-\alpha(t_1-\tau)} R_{ZX}(\tau,t_2)d\tau \\ &= \sigma^2 e^{-\alpha t_1}e^{-\alpha t_2}\left[\int_0^{t_1} e^{\alpha\tau}\frac{e^{-\beta\tau}(e^{(\alpha+\beta)\tau}-1)}{\alpha+\beta}d\tau\right. \\ &\quad \left. + \int_0^{t_1} e^{\alpha\tau}\frac{e^{\beta\tau}(e^{(\alpha-\beta)t_2}-e^{(\alpha-\beta)\tau})}{\alpha-\beta}d\tau\right] \\ &= \sigma^2 e^{-\alpha t_1}e^{-\alpha t_2}\left\{\frac{1}{\alpha+\beta}\left[\frac{1}{2\alpha}(e^{2\alpha t_1}-1) - \frac{1}{\alpha-\beta}(e^{(\alpha-\beta)t_1}-1)\right]\right. \\ &\quad \left. + + \frac{1}{\alpha-\beta}\left[\frac{e^{(\alpha-\beta)t_2}}{\alpha+\beta}(e^{(\alpha+\beta)t_1}-1) - \frac{1}{2\alpha}(e^{2\alpha t_1}-1)\right]\right\} \\ &= \frac{\sigma^2}{\alpha^2-\beta^2}\left[e^{-\beta(t_1-t_2)} - \frac{\beta}{\alpha}e^{-\alpha(t_2-t_1)} + \frac{\beta+\alpha}{\alpha}e^{-\alpha t_1-\alpha t_2}\right. \\ &\quad \left. - e^{-\alpha t_2-\beta t_1} - e^{-\beta t_2-\alpha t_1}.\right] \end{aligned}$$

Letting $t_2 = t_1 + \tau$, we see that as $t_1 \to \infty$ transient effects die out and the autocorrelation becomes

$$R_Z(t_1,t_2) \to \frac{\sigma^2}{\alpha-\beta^2}\left[e^{-\beta(t_2-t_1)} - \frac{\beta}{\alpha}e^{-\alpha(t_2-t_1)}\right]$$

9.8 Time Averages of Random Processes and Ergodic Theorems

9.91

~~6.78~~ $\mathcal{E}[X(t)] = \mathcal{E}[A] = 0 \qquad \mathcal{E}[X(t_1)X(t_2)] = \mathcal{E}[A^2] = 1$

$$\begin{aligned} VAR[< X(t) >_T] &= \frac{1}{2T}\int_{-2T} 2T\left(1 - \frac{|u|}{2T}\right) C_X(u)du \\ &= \frac{2}{2T}\int_0^{2T}\left(1 - \frac{u}{2T}\right) du \\ &= 1 \end{aligned}$$

$\Rightarrow$ process is not mean-ergodic.

9.92

a) First we have to check for WSS

$E[X_n] = \frac{21}{6}$, $C_X(n, n+k) = 2.9167$

Therefore X_n is WSS

we have to find:

$$VAR[\langle X_n \rangle_T] = \frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_X(k)$$

$$= \frac{2.9167}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) = 2.9167\left(1 - \frac{\sum |k|}{(2T+1)^2}\right)$$

it can be seen that $\lim_{T \to \infty} [\langle x_n \rangle_T] = \lim_{T \to \infty} 2.9167\left(1 - \frac{2T(2T+1)}{(2T+1)^2}\right)$

$= 0$

Therefore X_n is mean ergodic.

b) $E[X_n] = 0$, $C_X(n, n+k) = \begin{cases} 1 & , k \text{ even} \\ -1 & , k \text{ odd} \end{cases} = (-1)^k$

Therefore X_n is WSS

$$VAR[\langle X_n \rangle_T] = \frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_X(k)$$

$$= \frac{1}{2T+1}\left(1 - \sum_{k=-2T}^{2T} \frac{|k|(-1)^k}{2T+1}\right) = \frac{1}{2T+1}\left(1 - \frac{2\sum_{1}^{2T} k(-1)^k}{2T+1}\right)$$

$$= \frac{1}{2T+1}\left(1 - \frac{2T}{2T+1}\right)$$, which $\to 0$ if $T \to \infty$ then X_n is mean ergodic

P9.92)

c) $X_n = s^n \quad n \geq 0 \quad s \sim U(0,1)$

$$E[X_n] = E[s^n] = \int_0^1 s^n ds = \frac{1}{n+1}$$

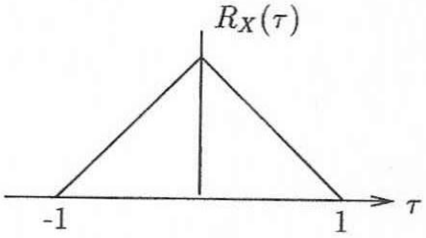
$$C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] = E[s^{2n+k}] - \frac{1}{(n+1)} \times \frac{1}{(n+k+1)}$$

$$= \frac{1}{2n+k+1} - \frac{1}{(n+1)(n+k+1)}$$

which shows X_n is not WSS, then we can't check for mean ergodicity using Eqn. (9.108)

9.93)

~~6.79~~ $R_X(\tau) = A(1 - |\tau|), \qquad |\tau| \leq 1$



$$\begin{aligned}
VAR[<X(t)>_T] &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right) R_X(u)du \\
&< \frac{1}{2T}\int_{-2T}^{2T} R_X(u)du \\
&= \frac{1}{2T}\int_{-2T}^{2T} A(1-|u|)du \\
&= \frac{1}{2T}\left(\frac{A}{2}\right) \qquad \text{for } T > 1 \\
&\to 0 \qquad \text{as } T \to \infty
\end{aligned}$$

$\Rightarrow X(t)$ is mean-ergodic.

9.94

6.80 a) $\mathcal{E}[X(t)] = \mathcal{E}[A]\cos 2\pi ft = m\cos 2\pi ft$

$$\begin{aligned} <X(t)>_T &= \frac{1}{2T}\int_{-T}^{T} X(t)dt = \frac{1}{2T}\int_{-T}^{T} A\cos 2\pi ft dt \\ &= \frac{1}{2T}\frac{2A\sin 2\pi fT}{2\pi f} \to 0 \text{ as } T\to\infty \end{aligned}$$

b)

$$\begin{aligned} R_X(t,t+\tau) &= \mathcal{E}[A^2]\cos 2\pi ft\cos 2\pi f(t+\tau) \\ <X(t)X(t+\tau)>_T &= \frac{1}{2T}\int_{-T}^{T} A^2\cos 2\pi f(t+\tau)\cos 2\pi ft dt \\ &= \frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\left[\cos 2\pi f\tau + \frac{A^2}{2}\cos 2\pi f(2t+\tau)\right]dt \\ &\to \frac{A^2}{2}\cos 2\pi f\tau \end{aligned}$$

9.95

.81 a)

$$\begin{aligned} \mathcal{E}[X(t)] &= \mathcal{E}[A]\mathcal{E}[\cos(2\pi ft+\Theta)] = 0 = m_X(t) \\ <X(t)>_T &= \frac{1}{2T}\int_{-T}^{T} A\cos(2\pi ft+\Theta) = \frac{A\sin(2\pi fT+\Theta)}{2\pi fT} \to 0 = m_X(t) \end{aligned}$$

b)

$$\begin{aligned} \mathcal{E}[X(t)X(t+\tau)] &= \mathcal{E}[A^2]\mathcal{E}[\cos(2\pi ft+\Theta)\cos(2\pi f(t+\tau)+\Theta)] \\ &= \frac{\mathcal{E}[A^2]}{2}\cos 2\pi f\tau = R_X(\tau) \\ <X(t)X(t+\tau)>_T &= \frac{1}{2T}\int_{-T}^{T} A^2\cos(2\pi f\tau+\Theta)\cos(2\pi f(t+\tau)+\Theta)dt \\ &= \frac{1}{2T}\int_{-T}^{T} A^2[\cos 2\pi f\tau + \cos(2\pi f(2t+\tau)+\Theta)]dt \\ &\to \frac{A^2}{2}\cos 2\pi f\tau \neq R_X(\tau) \end{aligned}$$

9.96

$$\begin{aligned} VAR[<X(t)>_T] &= \frac{1}{T}\int_0^{2T}\left(1-\frac{u}{2T}\right)e^{-2\alpha u}du \\ &= \frac{1}{T}\int_0^{2T} e^{-2\alpha u}du - \frac{1}{2T^2}\int_0^{2T} ue^{-2\alpha u}du \\ &= \frac{4\alpha T + e^{-4\alpha T} - 1}{8\alpha^2T^2} \quad \to 0 \end{aligned}$$

9.97

$$\begin{aligned} VAR[<X_n>_T] &= \frac{1}{2T+1}\sum_{k=-2T}^{2T}\left(1-\frac{|k|}{2T+1}\right)C_X(k) < \frac{1}{2T+1}\sum_{k=-2T}^{2T}\left(\frac{1}{2}\right)^{|k|} < \frac{2}{2T+1} \\ &\to 0 \text{ as } T\to\infty \\ &\Rightarrow X_n \text{ is mean-ergodic} \end{aligned}$$

9.98

a) $Y_n = \frac{1}{2}(X_n + X_{n-1})$, $X_0 = 0$ & X_n is Bernoulli(p)

all the computations are for $n > 2$

$$E[X_n] = p, \quad R_X(k) = E[X_n X_{n+k}] = \begin{cases} p^2 & k \neq 0 \\ p & k = 0 \end{cases} \qquad C_Y(k) = \begin{cases} p(1-p) & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$R_Y(k) = E[Y_n Y_{n+k}] = \frac{1}{4}\left[R_X(k+1) + 2R_X(k) + R_X(k-1)\right]$$

$$E[Y_n] = p, \quad C_Y(k) = R_Y(k) - p^2$$

Y_n is WSS

$$C_Y(k) = \begin{cases} \frac{1}{4}[2p^2 + 2p] - p^2 = \frac{1}{2}p - \frac{1}{2}p^2 & k = 0 \\ \frac{1}{4}(p + 3p^2) - p^2 = \frac{1}{4}p - \frac{1}{4}p^2 & k = \pm 1 \\ \frac{1}{4}[4p^2] - p^2 = 0 & |k| > 1 \end{cases}$$

It can be easily shown that the following limit is zero

$$\lim_{T \to \infty} VAR[\langle Y_n \rangle_T] = \lim_{T \to \infty} \frac{1}{2T+1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T+1}\right) C_Y(k) = 0$$

Then Y_n is mean ergodic.

ⓑ Since $E[Z_n] = m_X \dfrac{1 - \left(\frac{3}{4}\right)^{n-1}}{1 - \frac{3}{4}}$ depends on n, Z_n is <u>not</u> WSS.

However if we suppose the process was started at time $-\infty$, then

$$E[Z_n] = m_X$$

and

$$COV(Z_n, Z_{n+k}) = \frac{\left(\frac{3}{4}\right)^{|k|}}{1 - \left(\frac{3}{4}\right)^2} \sigma_X^2 \quad \text{(see problem 9.32).}$$

$$= \beta \alpha^{|k|}$$

We then have

$$VAR[\langle X_n \rangle_T] < \frac{2}{2T+1} \beta \sum_{k=0}^{2T} \alpha^{|k|} = \frac{2\beta}{2T+1} \frac{1}{1-\alpha} \to 0$$

as $T \to \infty$

$\therefore$ the steady state process is mean-ergodic

9.99 ⓐ

~~6.84~~ $\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)du < \frac{1}{2T}\int_{-2T}^{2T}|C_X(u)|du$

$\therefore$ If $\int_{-\infty}^{\infty}|C_X(u)|du = M < \infty$ then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)C_X(u)du < \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|C_X(u)|du = 0$$

$\Rightarrow X(t)$ is mean-ergodic.

ⓑ

$$\lim_{T\to\infty}\frac{1}{2T+1}\sum_{k=-2T}^{2T}\left(1-\frac{|k|}{2T+1}\right)C_X(k) < \lim_{T\to\infty}\frac{1}{2T+1}\sum_{k=-2T}^{2T}|C_X(k)| <$$

$$< \lim_{T\to\infty}\frac{M}{2T+1} = 0.$$

$\Rightarrow X_n$ is mean ergodic

9.100

~~6.85~~ In order for $< X^2(t) >_T$ to be a valid estimate for $\mathcal{E}[X^2(t)]$, the process $Y(t) = X^2(t)$ must be mean-ergodic. Since $\mathcal{E}[X^2(t)] = C_X(0)$, a constant, $X^2(t)$ is mean-ergodic iff $C_{X^2}(t_1, t_2)$ is such that

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C_{X^2}(t_1, t_2) dt_1 dt_2 \\ &= \lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} [\mathcal{E}[X^2(t_1)X^2(t_2)] - C_X^2(0)] dt_1 dt_2 = 0\end{aligned}$$

Note that, in general, $X(t)$ WSS does not necessarily imply that $X^2(t)$ is WSS.

ⓑ

$X(t) = A\cos(2\pi t + \Theta)$ $\quad \Theta$ uniform $(0, 2\pi)$

$X^2(t) = A^2\cos^2(2\pi t + \Theta) = \frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi t + 2\Theta)$

$E[X^2(t)] = \frac{A^2}{2}$

$$\begin{aligned}C_{X^2}(t_1, t_2) &= E\left[\left(X^2(t_1) - \frac{A^2}{2}\right)\left(X^2(t_2) - \frac{A^2}{2}\right)\right] \\ &= E\left[\frac{A^2}{2}\cos(4\pi t_1 + 2\Theta)\frac{A^2}{2}\cos(4\pi t_2 + 2\Theta)\right] \\ &= \frac{A^4}{8}E\left[\cos 4\pi(t_2 - t_1) + \cos(4\pi(t_1 + t_2) + 4\Theta)\right] \\ &= \frac{A^4}{8}\cos 4\pi(t_2 - t_1).\end{aligned}$$

Applying Theorem for mean ergodicity:

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|\tau|}{2T}\right)\frac{A^4}{8}\cos 4\pi\tau\, d\tau$$

$$< \frac{1}{2T}\frac{A^4}{8}\underbrace{\int_{-2T}^{2T}\cos 4\pi\tau\, d\tau}_{\underbrace{\frac{1}{4\pi}\sin 4\pi\tau\Big|_{-2T}^{2T}}_{\frac{2}{4\pi}\sin 8\pi T}}$$

$$< \frac{A^4}{32\pi}\frac{1}{T} \to 0 \quad \Rightarrow \quad X^2(t) \text{ is mean ergodic}$$

$$\langle X^2(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T}\left(\frac{A}{2} + \frac{A}{2}\cos(4\pi t + 2\Theta)\right)dt \to \frac{A^2}{2}.$$

9.101a In order for $< X(t)X(t+\tau) >_T$ to be a valid estimate for $R_X(\tau)$, $Y(t) = X(t)X(t+\tau)$ must be mean-ergodic.

Note that

$$\mathcal{E}[< X(t)X(t+\tau) >_T] = \frac{1}{2T}\int_{-T}^{T} \mathcal{E}[X(t)X(t+\tau)]dt = R_X(\tau)$$

does not depend on t. Thus $X(t)X(t+\tau)$ is mean ergodic iff $C_{X(t)X(t+\tau)}(t_1, t_2)$ is such that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{4T^2}\int_{-T}^{T}\int_{-}^{T} C_{X(t)X(t+\tau)}(t_1, t_2)dt_1dt_2 \\
&= \lim_{T\to\infty} \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}[\mathcal{E}[X(t_1)X(t_1+\tau)X(t_2)X(t_2+\tau)] - R_X^2(\tau)]dt_1dt_2 \\
&= 0
\end{aligned}$$

(b) $X(t) = A\cos(2\pi t + \Theta)$ $\quad \Theta$ uniform in $(0, 2\pi)$

$$X(t)X(t+\tau) = A^2\cos(2\pi t + \Theta)\cos(2\pi(t+\tau)+\Theta)$$

$$= \frac{A^2}{2}\left[\cos 2\pi\tau + \cos(2\pi(2t+\tau)+2\Theta)\right]$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos 2\pi\tau.$$

$$C_{X(t)X(t+\tau)}(t_1,t_2) = E\left[\frac{A^2}{2}\cos(2\pi(2t_1+\tau)+2\Theta)\frac{A^2}{2}\cos(2\pi(2t_2+\tau)+2\Theta)\right]$$

$$= \frac{A^4}{8}E\left[\cos 4\pi(t_2-t_1) + \cos(2\pi(2t_1+2t_2+2\tau)+4\Theta))\right]$$

$$= \frac{A^4}{8}\cos 4\pi(t_2-t_1)$$

Applying the theorem on ergodicity:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)\frac{A^4}{8}\cos 4\pi\tau\, d\tau$$

$$< \frac{1}{2T}\,\frac{A^4}{8}\,\frac{2}{4\pi}\sin 8\pi T$$

$$< \frac{A^4}{32\pi}\,\frac{1}{T} \to 0 \quad \Rightarrow X(t)X(t+\tau) \text{ is mean ergodic}$$

$$\langle X(t)X(t+\tau)\rangle_T = \frac{1}{2T}\int_{-T}^{T}\frac{A^2}{2}\cos 2\pi\tau + \frac{A^2}{2}\cos(2\pi(2t+\tau)+2\Theta)\,dt$$

$$\to \frac{A^2}{2}\cos 2\pi\tau. \quad ✓$$

9.101

a) $< Y(T) >_T = \frac{1}{2T}\int_{-T}^{T} Y(t)dt = \frac{1}{2T}$ [total time $X(t) \in (a, b]$ during $t \in [-T, T]$]

b)
$$\begin{aligned}\mathcal{E}[< Y(t) >_T] &= \frac{1}{2T}\int_{-T}^{T}\mathcal{E}[Y(t)]dt \\ &= \frac{1}{2T}\int_{-T}^{T}P[a < X(t) \leq b]dt\end{aligned}$$

If $P[a < X(t) \leq b]$ does not depend on t, then

$$\mathcal{E}[< Y(t) >_T] = P[a < X(t) \leq b] \triangleq P[a < X \leq b]$$

(c) $< Y(t) >_T \to P[a < X \leq b]$ if $Y(t)$ is mean ergodic. That is, if

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{T}\int_{T}^{T}C_Y(t_1, t_2)dt_1dt_2 \\ &= \lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\{\mathcal{E}[Y(t_1)Y(t_2)] - P[a < X \leq b]^2\}dt_1dt_2 \\ &= \lim_{T\to\infty}\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T}\{P[X(t_1) \in (a, b], X(t_2) \in (a, b]] - P^2[a < X \leq b]\}dt_1dt_2 \\ &= 0\end{aligned}$$

d) Let $a = -\infty$

ⓔ for the random telegraph signal. let $a = -\infty$, so we seek $P[X(t) \leq 0]$. Assume $X(t)$ ω

$$Y(t) = I_{\{X(t)<0\}}$$

$$E[Y(t)] = E[I_{\{X(t)<0\}}] = P[X(t)<0] = \frac{1}{2}$$

$$\begin{aligned}C_Y(t_1, t_2) &= E[I_{\{X(t_1)<0\}}I_{\{X(t_2)<0\}}] - \left(\tfrac{1}{2}\right)^2 \\ &= P[\text{even \# of transitions in } (t_1, t_2) \mid X(t_1)<0]\,\frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{2}\left(1 + e^{-2\alpha|t_2-t_1|}\right)\frac{1}{2} - \frac{1}{4} = \frac{1}{2}e^{-2\alpha|t_2-t_1|}\end{aligned}$$

Applying theorem on mean ergodicity

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right)\frac{1}{2}e^{-2\alpha|\tau|}d\tau$$

$$< \lim_{T\to\infty}\frac{2}{4T}\int_{0}^{T}e^{-2\alpha\tau}d\tau$$

$$= \lim_{T\to\infty}\frac{1}{\alpha T}\left(1 - e^{-2\alpha T}\right) \to 0$$

$\Rightarrow$ mean ergodic

9.103

~~6.88~~ **a)** $< Y_n >_T = \frac{1}{2T+1}\sum_{n=-T}^{T} Y_n = \frac{1}{2T+1}$ [# occurrences of $\{a < X_n \leq b\}$ during $t \in \{-T, T\}$]

b)
$$\begin{aligned} \mathcal{E}[< Y_n >_T] &= \frac{1}{2T+1}\sum_{n=-T}^{T}\mathcal{E}[Y_n] = \frac{1}{2T+1}\sum_{n=-T}^{T} P[a < X_n \leq b] \\ &= P[a < X \leq b] \quad \text{if } P[a < X_n \leq b] \text{ does not depend on } n \end{aligned}$$

c) Y_n is mean-ergodic iff

$$\lim_{T\to\infty}\frac{1}{(2T+1)^2}\sum_{k=-T}^{T}\sum_{j=-T}^{T}\{P[X_k \in (a,b], X_j \in (a,b]] - P^2[a < X \leq b]\} = 0$$

d) $a = -\infty$, $b = x$.

9.104

~~6.89~~ **a)** Here we suppose that we observe X_n only for $n \leq 1$

$$\begin{aligned} Z_n &= u(a - X_n) \\ \frac{1}{n}\sum_{k=1}^{n} Z_n &= \frac{1}{n} \underbrace{\sum_{k=1}^{n} u(a - X_n)}_{\substack{\text{counting process for} \\ \text{event } \{X_n \leq a\}}} \end{aligned}$$

b) If Z_n is mean-ergodic, then

$$\frac{1}{n}\sum_{k=1}^{n} Z_n \to \mathcal{E}[Z_n] = \mathcal{E}[u(a - X_n)] = P[X_n \leq a] = F_X(a)$$

9.105

$$C_X(k) = \frac{\sigma^2}{2}\left\{ |k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right\}$$

$$VAR[\langle X_n \rangle_T] = \frac{1}{(2T+1)^2}\left\{ (2T+1)C_X(0) + 2\times 2T\, C_X(1) + 2\times(2T-1)\, C_X(2) + \cdots + 2C_X(2T) \right\}$$

if you replace $C_X(k)$s in the VAR equation you can see:

$$(2T+1)\, C_X(0) = (2T+1)\frac{\sigma^2}{2}\left\{ 1^{2H} - 2\times|0|^{2H} + 1^{2H} \right\}$$

$$2\times 2T \quad C_X(1) = 2(2T)\frac{\sigma^2}{2}\left\{ 2^{2H} - 2\times 1^{2H} + 0^{2H} \right\}$$

$$2\times(2T-1) \quad C_X(2) = 2(2T-1)\frac{\sigma^2}{2}\left\{ 3^{2H} - 2\times 2^{2H} + 1^{2H} \right\}$$

$$\vdots$$

$$2\times 2 \quad C_X(2T-1) = 2\times 2\times\frac{\sigma^2}{2}\left\{ (2T)^{2H} - 2\times(2T-1)^{2H} + (2T-2)^{2H} \right\}$$

$$2\times 1 \quad C_X(2T) = 2\times 1\times\frac{\sigma^2}{2}\left\{ (2T+1)^{2H} - 2\times(2T)^{2H} + (2T-1)^{2H} \right\}$$

$$VAR[\langle X_n \rangle_T] = \frac{1}{(2T+1)^2}\left((2T+1)^{2H} \right)\times 2\times 1\times\frac{\sigma^2}{2} = \sigma^2 (2T+1)^{2H-2}$$

9.106

```
T=1:5:100;
%P.9.106
H2=2*0.5;Cx1=0.5*((T+1).^H2-2*T.^H2+(T-1).^H2);
H2=2*0.6;Cx2=0.5*((T+1).^H2-2*T.^H2+(T-1).^H2);
H2=2*0.75;Cx3=0.5*((T+1).^H2-2*T.^H2+(T-1).^H2);
H2=2*0.99;Cx4=0.5*((T+1).^H2-2*T.^H2+(T-1).^H2);
plot(T, Cx1, '-', T, Cx2, '-*', T,Cx3, '-^', T,Cx4, '--');
legend('H=0.5', 'H=0.6', 'H=0.75', 'H=0.99');
title('Problem 9.106');
%as it can be seen long range dependence increases with H
```

9.107

(a)

```
T=1:5:100;
%P.9.107a
H=0.5;var1=(2*T+1).^(2*H-2);
H=0.6;var2=(2*T+1).^(2*H-2);
H=0.75;var3=(2*T+1).^(2*H-2);
H=0.99;var4=(2*T+1).^(2*H-2);
figure(2);
plot(T, var1, '-', T, var2, '-*', T,var3, '-^', T,var4, '--');
legend('H=0.5', 'H=0.6', 'H=0.75', 'H=0.99');
title('Problem 9.107a');
```

(b)

```
T=1:5:100;
%P.9.107b
H=0.5;var1=(2*T+1).^(2*H-1);
H=0.6;var2=(2*T+1).^(2*H-1);
H=0.75;var3=(2*T+1).^(2*H-1);
H=0.99;var4=(2*T+1).^(2*H-1);
figure(3);
plot(T, var1, '-', T, var2, '-*', T,var3, '-^', T,var4, '--');
legend('H=0.5', 'H=0.6', 'H=0.75', 'H=0.99');
title('Problem 9.107b');
```

9.108

(a)

```
T=1:5:100;
%P.9.106
H=0.5;var1=(2*T+1).^(2*H-2);
H=0.6;var2=(2*T+1).^(2*H-2);
H=0.75;var3=(2*T+1).^(2*H-2);
H=0.99;var4=(2*T+1).^(2*H-2);
figure(4);
loglog(T, var1, '-', T, var2, '-*', T,var3, '-^', T,var4, '--');
legend('H=0.5', 'H=0.6', 'H=0.75', 'H=0.99');
title('Problem 9.106');
%answer is H=0.75;
```

9.109

(9.109)

$$C_X(k) = \sigma^2 H(2H-1)\, k^{2H-2}$$

$$\sum_{k=-\infty}^{+\infty} |C_X(k)| < \infty$$ is the sufficient condition.

$$\sum_{k=-\infty}^{+\infty} |C_X(k)| = \sigma^2 H(2H-1) \sum_{k=-\infty}^{+\infty} |k|^{2H-2} = 2\sigma^2 H(2H-1) \sum_{k=1}^{\infty} k^{2H-2}$$

$$= 2\sigma^2 H(2H-1) \sum_{k=1}^{\infty} k^{-\alpha} \qquad \alpha = 2-2H,\ \tfrac{1}{2} < H < 1 \Rightarrow 0 < \alpha < 1$$

This sum does not converge and is not bounded.

Although the sufficient condition is not met, ~~that~~ the process is still mean ergodic as discussed in Example 9.50

*9.9 Fourier Series and Karhunen-Loeve Expansion

9.110

a) $X(t) = Xe^{j\omega t}$

$$R_X(t_1, t_2) = E[X(t_1)X^*(t_2)] = E[Xe^{j\omega t_1} X^* e^{-j\omega t_2}]$$

$$= E[XX^*]e^{j\omega(t_1 - t_2)} = E[|X|^2]\, e^{j\omega(t_1 - t_2)}$$

b) $E[X(t)] = E[X]e^{j\omega t}$

$R_X(\tau) = E[|X|^2]e^{j\omega\tau}$

if $E[X] = 0$, then $X(t)$ is a WSS random process.

9.111) $X(t) = X_1 e^{j\omega_1 t} + X_2 e^{j\omega_2 t} \quad , \omega_1 \neq \omega_2$

a) $E[X(t_1)X^*(t_2)] = E[(X_1 e^{j\omega_1 t_1} + X_2 e^{j\omega_2 t_2})(X_1^* e^{-j\omega_1 t_2} + X_2^* e^{-j\omega_2 t_2})]$

$$= E[|X_1|^2] e^{j\omega_1(t_1-t_2)} + E[|X_2|^2] e^{j\omega_2(t_1-t_2)}$$

$$+ E[X_1 X_2^*] e^{j(\omega_1 t_1 - \omega_2 t_2)} + E[X_2 X_1^*] e^{j(\omega_2 t_1 - \omega_1 t_2)}$$

$$E[X(t)] = E[X_1] e^{j\omega_1 t} + E[X_2] e^{j\omega_2 t}$$

$$C_X(t_1, t_2) = E[X(t_1) X_2^*(t)] - E[X(t_1)] E[X^*(t_2)]$$

$$E[X(t_1)] E[X^*(t_2)] = E[X_1] E[X_1^*] e^{j\omega_1(t_1-t_2)} + E[X_2] E[X_2^*] e^{j\omega_2(t_1-t_2)}$$

$$+ E[X_1] E[X_2^*] e^{j(\omega_1 t_1 - \omega_2 t_2)} + E[X_2] E[X_1^*] e^{j(\omega_2 t_1 - \omega_1 t_2)}$$

Therefore:

$$C_X(t_1, t_2) = VAR[X_1] e^{j\omega_1(t_1-t_2)} + VAR[X_2] e^{j\omega_2(t_1-t_2)}$$

$$+ COV(X_1, X_2) e^{j(\omega_1 t_1 - \omega_2 t_2)} + COV(X_2, X_1) e^{j(\omega_2 t_1 - \omega_1 t_2)}$$

b) $E[X(t)] = E[X_1] e^{j\omega_1 t} + E[X_2] e^{j\omega_2 t} \quad , \omega_1 \neq \omega_2$

first condition : $E[X_1] = E[X_2] = 0$

Since $\omega_1 \neq \omega_2$

another condition is : X_1 & X_2 should be uncorrelated.

so that $C_X(t_1, t_2)$ could be function of $t_1 - t_2$ only.

c) $X(t) = \frac{(U - jV)}{2} e^{j\omega t} + \frac{(U + jV)}{2} e^{-j\omega t}$, $\omega_1 = -\omega_2 = \omega$

$$= \frac{U}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) - j\cdot\frac{V}{2}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$= U\cos\omega t + V\sin\omega t$, $X(t)$ is real.

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[(U\cos\omega t_1 + V\sin\omega t_1)(U\cos\omega t_2 + V\sin\omega t_2)]$$

$$= E[U^2]\cos\omega t_1\cos\omega t_2 + E[V^2]\sin\omega t_1\sin\omega t_2 + E[UV]\sin(\omega t_1 + \omega t_2)$$

d) $E[X(t)] = E[U]\cos\omega t + E[V]\sin\omega t$

first condition $E[U] = E[V] = 0$, and therefore $E[X(t)] = 0 \ \forall t$

Also:

$C_X(t_1,t_2) = R_X(t_1,t_2) - E[X(t_1)]E[X(t_2)] = R_X(t_1,t_2)$

second condition: $E[U^2] = E[V^2]$, and U & V should be orthogonal

e) $X(t) = U\cos\omega t + V\sin\omega t$

$X(t_1)$ is a linear combination of two jointly Gaussian Random Variables.

Then $X(t_1)$ is also a Gaussian Random Variable.

(chapter 6)

Similarly, it can be easily shown that $X(t_1)$ & $X(t_2)$ .... & $X(t_k)$ are jointly Gaussian Random variables.

Therefore $X(t)$ is a Gaussian Random process.

9.112

~~6.90~~ **a** The correlation between Fourier coefficients is:

$$\begin{aligned} E[X_k X_m^*] &= E\left[\frac{1}{T}\int_0^T X(t')e^{-j2\pi kt'/T}dt'\frac{1}{T}\int_0^T X(t'')e^{j2\pi mt''/T}dt''\right] \\ &= \frac{1}{T^2}\int_0^T\int_0^T R_X(t'-t'')e^{-j2\pi kt'/T}e^{j2\pi mt''/T}dt'dt'' \end{aligned}$$

This is Eqn.9.118..

b) Now suppose $X(t)$ is M.S. periodic:

$$\begin{aligned} E[X_k X_m^*] &= \frac{1}{T^2}\int_0^T e^{j2\pi nt''/T}dt''\int_0^T R_X(t't-'')e^{-j2\pi k't/T}dt' \\ &= \frac{1}{T^2}\int_0^T e^{j2\pi mt''/T}dt''\int_{-t''}^{T-t''} R_X(u)e^{-j2\pi k(u+t'')}du \\ &= \frac{1}{T}\int_0^T e^{j2\pi(m-k)t''/T}dt''\frac{1}{T}\int_{-t''}^{T-t''} R_X(u)e^{-j2\pi ku}du \end{aligned}$$

If $X(t)$ is M.S. periodic then $R_X(u)$ is periodic and the inner integral is a_k, thus

$$\begin{aligned} E[X_k X_m^*] &= a_k\frac{1}{T}\int_0^T e^{j2\pi(m-k)t''/T}dt'' \\ &= a_k\delta_k m \qquad \checkmark \end{aligned}$$

9.113 X(t) WSS Gaussian RP, $R_X(\tau) = e^{-|\tau|}$

X(t) is not mean square periodic, but we expa it in the interval $[0, T]$

$$X(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T} \qquad 0 \le t \le T$$

where

$$X_k = \frac{1}{T}\int_0^T X(t') e^{-j2\pi kt' T} dt'$$

X_k is defined by a linear transformation of a jointly Gaussian RP so it is also a Gaussian RV

mean
$$E[X_k] = \frac{1}{T}\int_0^T E[X(t')] e^{-j2\pi kt' T} dt'$$

$$= \frac{1}{T}\int_0^T \underbrace{m_X(t)}_{=0} e^{-j2\pi kt'/T} dt' \quad \text{assume zero mean WSS}$$

$$= 0$$

By Eqn 9.118

$$E[X_k X_k^*] = \frac{1}{T^2}\int_0^T\int_0^T e^{-|t-u|} e^{-j2\pi \frac{kt}{T}} e^{j2\pi \frac{ku}{T}} dt\,du$$

Variance ↗

$$= \frac{1}{T^2}\int_{-T}^{T} e^{-|\tau|} e^{-j\frac{2\pi k}{T}\tau} (T-|\tau|)\,d\tau$$

Note we can also write

$$X(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos\frac{2\pi kt}{T} + B_k \sin\frac{2\pi kt}{T}$$

where $A_k = X_k + X_{-k}$

$B_k = j(X_k - X_{-k})$

$$E[X_k X_k^*] = \frac{1}{T}\int_{-T}^{T} e^{-|\tau|}\cos\frac{2\pi k\tau}{T}\left(1-\frac{|\tau|}{T}\right)d\tau \qquad b=\frac{2\pi k}{T}$$

$$= \frac{2}{T}\int_0^T e^{-\tau}\cos b\tau\, d\tau - \frac{2}{T}\int_0^T \frac{\tau}{T} e^{-\tau}\cos b\tau\, d\tau$$

$$\int_0^T e^{-\tau}\cos b\tau\, d\tau = \frac{e^{-\tau}}{1+b^2}\Big\{-\cos b\tau + b\sin b\tau\Big\}_0^T = \frac{1-e^{-T}}{1+b^2}$$

$$\int_0^T \tau e^{-\tau}\cos b\tau\, d\tau = \left[\frac{\tau e^{-\tau}}{1+b^2}\{-\cos b\tau + b\sin b\tau\} - \frac{1}{1+b^2}\int e^{-\tau}(-\cos b\tau + b\sin b\tau)\,d\tau\right]_0^T$$

$$\int_0^T e^{-\tau}\sin b\tau\, d\tau = \frac{e^{-\tau}}{1+b^2}\Big\{-\sin b\tau - b\cos b\tau\Big\}_0^T = \frac{b}{1+b^2}(1-e^{-T})$$

$$\Rightarrow \int_0^T \tau e^{-\tau}\cos b\tau\, d\tau = \frac{-Te^{-T}}{1+b^2} + \frac{1-e^{-T}}{(1+b^2)^2} - \frac{b^2}{(1+b^2)^2}(1-e^{-T})$$

Finally
$$E[X_k X_k^*] = \frac{2}{T}\,\frac{1-e^{-T}}{1+b^2} - \frac{2}{T^2}\left\{\frac{-Te^{-T}(1+b^2)+(1-e^{-T})-b^2(1-e^{-T})}{(1+b^2)^2}\right.$$

Note that
$$E[X_k X_m^*] \neq 0$$

9.114

~~6.91~~ Assume $X(t)$ is a WSS mean square periodic process

$$K_X(t_1, t_2) = K_X(t_2 - t_1)$$

Eigenvalue equation:

$$\int_0^T K_X(t_1 - t_2)\phi_k(t_2)dt_2 = \lambda_k \phi_k(t_1)$$

Try

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt}$$

$\{\phi_k(t)\}$ is orthonormal set and

$$\begin{aligned}
\int_0^T K_X(t_2 - t_1)\frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_2} dt_2 &= \int_{-t_1}^{T-t_1} K_X(u)\frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}ku} e^{+j\frac{2\pi}{T}kt_1} du \\
&= \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_1} \int_0^T K_X(u) e^{j\frac{2\pi}{T}ku} du \\
&= \frac{1}{\sqrt{T}} e^{j\frac{2\pi}{T}kt_1} \cdot \lambda_k \\
&= \lambda_k \phi_k(t_1)
\end{aligned}$$

where the eigenvalue is given by

$$\lambda_k = \int_0^T K_X(u) e^{j\frac{2\pi}{T}ku} du \ .$$

Therefore the KL Expansion is:

$$X(t) = \sum_{k=-\infty} X_k \phi_k(t)$$

where

$$\begin{aligned}
X_k &= \int_0^T X(t)\phi_k^*(t)dt \\
&= \int_0^T X(t)\frac{1}{\sqrt{T}} e^{-j\frac{2\pi}{T}kt} dt
\end{aligned}$$

Thus

$$\frac{X_K}{\sqrt{T}} = \frac{1}{T}\int_0^T X(t) e^{-j\frac{2\pi}{T}kt} dt, \quad \text{the Fourier coefficients}$$

Therefore $K - L$ expansion of $X(t)$ yields the Fourier series.

9.115

~~**6.92**~~ For white Gaussian noise process

$$K_X(t_1,t_2) = \alpha\delta(t_1 - t_2)$$

Take any set of orthonormal functions $\{\phi_k(t)\}$

$$\begin{aligned}
\int_0^T K_X(t_1,t_2)\phi_k(t_2)dt_2 &= \int_0^T \alpha\delta(t_1 - t_2)\phi_k(t_2)dt_2 \\
&= \alpha\phi_k(t_1)
\end{aligned}$$

The eigenvalue equation is satisfied and $\lambda_k = \alpha$.

9.116

$$\begin{aligned}
R_{XW}(t_1,t_2) &= 0 \\
K_W(t_1,t_2) &= \alpha\delta(t_1 - t_2) \\
\int_0^T K_X(t_1,t_2)\phi_n(t_2)dt_2 &= \lambda_n\phi_n(t_1) \\
K_Y(t_1,t_2) &= E[(X(t_1) + W(t_1))(X(t_2) + W(t_2))] \\
&= K_X(t_1,t_2) + K_W(t_1,t_2) \\
\int_0^T K_Y(t_1,t_2)\phi_n(t_2)dt_2 &= \int_0^T [K_X(t_1,t_2) + \alpha\delta(t_1 - t_2)]\phi_n(t_2)dt_2 \\
&= \lambda_n\phi(t_1) + \alpha\phi(t_1) \\
&= (\lambda_n + \alpha)\phi(t_1)
\end{aligned}$$

So $\phi_n(t)$ is also an eigenfunction for $K_Y(t_1,t_2)$ with the eigenvalue $\lambda_n + \alpha$.

9.117

~~6.94~~ **a)**

$$\int_{-T}^{T} R_X(t_1,t_2)\phi(t_2)dt_2 = \lambda\phi(t_1)$$

or

$$\begin{aligned}\lambda\phi(t_1) &= \int_{-T}^{T}\sigma^2 e^{-\alpha|t_1-t_2|}\phi(t_2)dt_2\\ &= \int_{-T}^{t_1}\sigma^2 e^{-\alpha(t_1-t_2)}\phi(t_2)dt_2 + \int_{t_1}^{T}\sigma^2 e^{\alpha(t_1-t_2)}\phi(t_2)dt_2\end{aligned}$$

b)

$$\begin{aligned}\lambda\frac{d\phi(t_1)}{dt_1} &= \sigma^2\phi(t_1) - \int_{-T}^{t_1}\sigma^2\alpha e^{-\alpha(t_1-t_2)}\phi(t_2)dt_2\\ &\quad -\sigma^2\phi(t_1) + \int_{t_1}^{T}\sigma^2\alpha e^{\alpha(t_1-t_2)\phi(t_2)}dt_2\\ &= -\alpha\int_{-T}^{t_1}\sigma^2 e^{-\alpha(t_1-t_2)}\phi(t_2)dt_2 + \alpha\int_{t_2}^{T}\sigma^2 e^{\alpha(t_1-t_2)}\phi(t_2)dt_2\\ \lambda\frac{d^2\phi(t_1)}{dt_1^2} &= -\alpha\sigma^2\phi(t_1) + \alpha^2\int_{-T}^{t_1}\sigma^2 e^{-\alpha(t_1-t_2)}\phi(t_2)dt_2\\ &\quad -\alpha\sigma^2\phi(t_1) + \alpha^2\int_{t_1}^{T}\sigma^2 e^{-\alpha(t_1-t_2)}\phi(t_2)dt_2\\ &= -2\alpha\sigma^2\phi(t_1) + \alpha^2\int_{-T}^{T}R_X(t_1,t_2)\phi(t_2)dt_2\\ &= (-2\alpha\sigma^2 + \lambda\alpha^2)\phi(t_1)\\ \frac{d^2\phi(t_1)}{dt_1^2} &= \frac{\alpha^2(\lambda - 2\frac{\sigma^2}{\alpha})}{\lambda}\phi(t_1)\end{aligned}$$

c) $\dfrac{d^2\phi(t_1)}{dt_1^2} = \dfrac{\alpha^2(2\frac{\sigma^2}{\alpha} - \lambda)}{\lambda}\phi(t_1) = 0$

$$\phi(t_1) = A\sin bt + B\cos bt$$

where

$$b^2 = \frac{\alpha^2(2\frac{\sigma^2}{\alpha} - \lambda)}{\lambda}$$

In order to satisfy orthogonal condition of $\phi(t)$, $\phi(t)$ should be in the form of $A\sin bt$ or $B\cos bt$.

d) $\phi(t) = A\cos bt$. Substitute the $\phi(t)$ into the integral condition.

$$\begin{aligned}\lambda A\cos bt_1 &= \int_{-T}^{T}\sigma^2 e^{-\alpha|t_1-t_2|}A\cos bt_2 dt_2\\ &= A\sigma^2\int_{-T}^{t_1} e^{-\alpha(t_1-t_2)}\cos bt_2 dt_2\\ &\quad + A\sigma^2\int_{t_1}^{T} e^{\alpha(t_1-t_2)}\cos bt_2 dt_2\\ &= A\sigma^2 e^{-\alpha t_1}\int_{-T}^{t_1} e^{\alpha t_2}\cos bt_2 dt_2\\ &\quad + A\sigma^2 e^{\alpha t_1}\int_{t_1}^{T} e^{-\alpha t_2}\cos bt_2 dt_2\end{aligned}$$

Cancel A on both sides and let $t_1 \to 0$.

$$\begin{aligned}
\frac{\lambda}{\sigma^2} &= \int_{-T}^{0} e^{\alpha t_2}\cos bt_2 dt_2 + \int_0^T e^{-\alpha t_2}\cos bt_2 dt_2 \\
&= \frac{e^{\alpha t_2}(\alpha\cos bt_2 + b\sin bt_2)|_{-T}^{0}}{\alpha^2+b^2} + \frac{e^{-\alpha t_2}(-\alpha\cos bt_2 + b\sin bt_2)|_0^T}{\alpha^2+b^2} \\
\alpha^2 + b^2 &= \alpha^2 + 2\sigma^2\alpha/\lambda - \alpha^2 = 2\sigma^2\alpha/\lambda \\
\therefore\ 1 &= \frac{\alpha + e^{-\alpha T}(b\sin bT - \alpha\cos bT)}{\alpha} \\
\tan bT &= \alpha/b
\end{aligned}$$

Substitute $\phi(t) = B\sin bt$ into the integral equation, cancel B on both sides,

$$\lambda\sin bt_1 = \int_{-T}^{t_1}\sigma^2 e^{-\alpha(t_1-t_2)}\sin bt_2 dt_2 + \int_{t_1}^{T}\sigma^2 e^{\alpha(t_1-t_2)}\sin bt_2 dt_2$$

Let $t_1 \to T$

$$\begin{aligned}
\frac{\lambda\sin bT}{\sigma^2} &= \int_{-T}^{T} e^{-\alpha T}e^{\alpha t_2}\sin bt_2 dt_2 \\
e^{\alpha T}\frac{\lambda\sin bT}{\sigma^2} &= \frac{e^{\alpha t_2}(+\alpha\sin bt_2 - b\cos bt_2)|_{-T}^{T}}{\alpha^2+b^2} \\
&= \frac{e^{\alpha T}(\alpha\sin bT - b\cos bT) + e^{-\alpha T}(\alpha\sin bT + b\cos bT)}{2\sigma^2\alpha/\lambda} \\
e^{\alpha T}\tan bT &= \frac{e^{\alpha T}(\alpha\tan bT - b)Ee^{-\alpha T}(\alpha\tan bT + b)}{2\alpha}
\end{aligned}$$

Try

$$\tan bT = -b/\alpha$$

$$RHS = \frac{e^{\alpha T}(-b-b)+0}{2\alpha} = e^{\alpha T}\tan bT = LHS$$

So b is the root of $\tan bT = -b/\sigma$.

e)

$$\begin{aligned}
\int_{-T}^{T}\phi(t)\phi^*(t)dt &= 1 \\
\int_{-T}^{T} A\cos btA\cos btdt &= 1 \\
\frac{1}{2}A^2\int_{-T}^{T}(1+\cos 2bt)dt &= 1 \\
\frac{1}{2}A^2\cdot 2T - \frac{A^2}{4b}\sin 2bt\Big|_{-T}^{T} &= 1 \\
A^2\left[T - \frac{1}{2b}\sin 2bT\right] &= 1
\end{aligned}$$

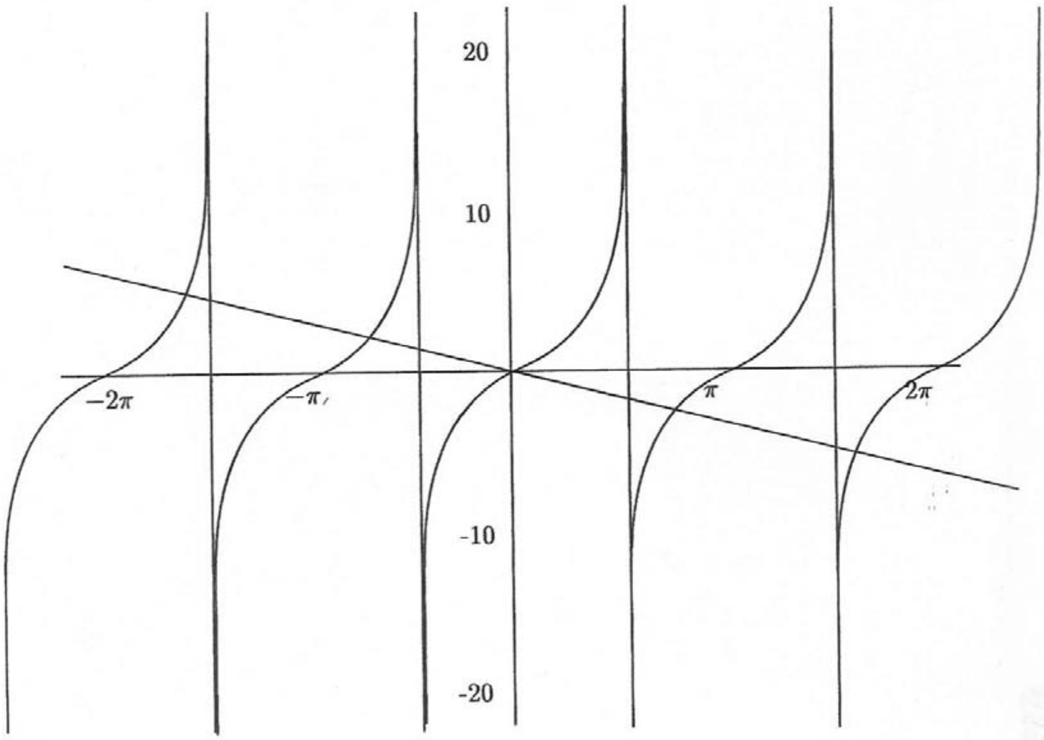
$$A^2 = \frac{1}{T - \frac{1}{2b}\sin 2bT}$$

$$\begin{aligned} \int_{-T}^{T} B \sin btB \sin b + dt &= 1 \\ \frac{A^2}{2}\int_{-T}^{T}(1 - \cos 2bt)dt &= 1 \end{aligned}$$

So

$$B^2 = \frac{1}{T + \frac{1}{2b}\sin 2bT}$$

f)



*9.10 Generating Random Processes

9.118

(a)

```
%P9.118
%part a
clear all;
close all;
s=zeros(200,10,3);
%s dimensions are: (n, realization, p)
p=[0.25 0.5 0.75];
for sample=1:1:10
    for i=1:1:3
        if (rand < p(i))
            s(1,sample,i) = 1;
        end
        for n=2:1:200
            s(n,sample,i)=s(n-1,sample,i);
            if (rand < p(i))
                s(n,sample,i)=s(n-1,sample,i)+1;
            end
        end
    end

    figure(sample);
    plot(1:200,s(:,sample,1),'--',1:200,s(:,sample,2),'-*',
       1:200, s(:,sample,3), '-o');
    legend('p=0.25', 'p=0.5', 'p=0.75');
    xlabel('n')
    ylabel('Sn, random process')
    title('Problem 9.118a');

end
```

(b)

```
%P9.118
%part b
clear all;
close all;
s(1:200,1:50) = 0;
p=0.5;
for sample=1:1:50
    if (rand < p)
        s(1,sample) = 1;
    end
    for n=2:1:200
        s(n,sample)=s(n-1,sample);
        if (rand < p)
            s(n,sample)=s(n-1,sample)+1;
        end
    end
```

```
end
m=mean(s');
v=var(s');
plot(1:200, m(1:200), '--', 1:200, v(1:200), '-o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
      title('Problem 9.118b');
```

(c)

(d)

Octave code for parts c and d:

```
%P9.118
%parts c & d
clear all;
close all;
s(1:200,1:50) = 0;
inc(1:4,1:50) = 0;
p=0.5;
for sample=1:1:50
    if (rand < p)
        s(1,sample) = 1;
    end
    for n=2:1:200
        %{
        %for the distortion case at the end of part d uncomment this part
        %if (n<50)
        %    p=0.9;
        %else
        %    p=0.5;
        %end
        %}
        s(n,sample)=s(n-1,sample);
        if (rand < p)
            s(n,sample)=s(n-1,sample)+1;
        end
    end
    inc(1,sample)=s(50,sample)-s(1,sample);
    inc(2,sample)=s(100,sample)-s(51,sample);
    inc(3,sample)=s(150,sample)-s(101,sample);
    inc(4,sample)=s(200,sample)-s(151,sample);
end

figure(1);
subplot(2,2,1);
hist(inc(1,:),5);
```

```
xlabel('increments [1-50]');
ylabel('number of samples');
subplot(2,2,2);
hist(inc(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
%hist(inc',5);

figure(2);
plot(inc(1,:),inc(2,:),'*');
xlabel('inc in [1,50]');
ylabel('inc in [51,100]');
axis([1 50 1 50]);
title('Problem 9.118d');

%for test we can distort inc in one range and see if it affects increments
%in the other range, for example we can modify the parameter p for range
%(1,50) and change it back to the original value for the range (51,100)
```

9.119

(a)

```
clear all;
close all;
s=zeros(3,200,10);
p=[0.25 0.5 0.75];
for sample=1:1:10
    for i=1:1:3
        if (rand < p(i))
            s(i,1,sample) = 1;
        else
            s(i,1,sample) = -1;
        end
        for n=2:1:200
            s(i,n,sample)=s(i,n-1,sample);
            if (rand < p(i))
                s(i,n,sample)=s(i,n-1,sample)+1;
            else
                s(i,n,sample)=s(i,n-1,sample)-1;
            end
        end
    end

    figure(sample);
    plot(1:200, s(1,:,sample), '--', 1:200,s(2,:,sample), '*',
      1:200,s(3,:,sample), 'o');
    legend('p=0.25', 'p=0.5', 'p=0.75');
    xlabel('n')
    ylabel('Sn, random process')
    title('Problem 9.119a');

end
```

(b)

```
clear all;
close all;
s(1:200,1:500) = 0;
p=0.5;
for sample=1:1:50
    if (rand < p)
        s(1,sample) = 1;
    else
        s(1,sample) = -1;
    end
    for n=2:1:200
        s(n,sample)=s(n-1,sample);
        if (rand < p)
            s(n,sample)=s(n-1,sample)+1;
        else
            s(n,sample)=s(n-1,sample)-1;
        end
    end
```

```
end
m=mean(s');
v=var(s');
plot(1:1:200, m, '--', 1:1:200, v, '-o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
title('Problem 9.119b');
```

(c)

(d)

Octave code for parts c and d:

```
clear all;
close all;
s(1:200,1:50) = 0;
inc(1:4,1:50) = 0;
p=0.5;
for sample=1:1:50
    if (rand < p)
        s(1,sample) = 1;
    else
        s(1,sample) = -1;
    end
    for n=2:1:200
        %{
        %for the distortion case at the end of part d uncomment this part
        %if (n<50)
        %    p=0.9;
        %else
        %    p=0.5;
        %end
        %}
        s(n,sample)=s(n-1,sample);
        if (rand < p)
            s(n,sample)=s(n-1,sample)+1;
        else
            s(n,sample)=s(n-1,sample)-1;
        end
    end
    %increment values:
    inc(1,sample)=s(50,sample)-s(1,sample);
    inc(2,sample)=s(100,sample)-s(51,sample);
    inc(3,sample)=s(150,sample)-s(101,sample);
    inc(4,sample)=s(200,sample)-s(151,sample);
end
close all;
figure(1);
subplot(2,2,1);
hist(inc(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
```

```
subplot(2,2,2);
hist(inc(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');

%hist(inc',5);
figure(2);
plot(inc(1,:),inc(2,:),'*');
xlabel('inc in [1,50]');
ylabel('inc in [51,100]');
axis([1 50 1 50]);
title('Problem 9.119d');

%for test we can distort inc in one range and see if it affects increments
%in the other range, for example we can modify the parameter p for range
%(1,50) and change it back to the original value for the range (51,100)
```

9.120

(a)

```
clear all;
close all;
s=zeros(2,200,10);
p=[0 0.5];
for sample=1:1:10
    for i=1:1:2
        s(i,1,sample) = 0;
        for n=2:1:200
            s(i,n,sample)=s(i,n-1,sample)+randn+p(i);
        end
    end

    figure(sample);
    plot(1:1:200, s(1,:,sample), '--', 1:1:200, s(2,:,sample), '-*');
    legend('m=0', 'm=0.5');
    xlabel('n')
    ylabel('Sn, random process')
    title('Problem 9.120a');

end
```

(b)

```
clear all;
close all;
s(1:200,1:50) = 0;
p=0.5;
for sample=1:1:50
    s(1,sample) = 0;
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+randn+p;
    end
end
m=mean(s');
v=var(s');
plot(1:200, m, '--', 1:200, v, '-o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
title('Problem 9.120b');
```

(c)

(d)

Octave code for parts c and d:

```
%for the test in part d, we can distort inc in one range and see if it
affects increments
```

```
%in the other range, for example we can modify the parameter p for range
%(1,50) and change it back to the original value for the range (51,100)

clear all;
close all;
s(1:200,1:50) = 0;
inc(1:4,1:50) = 0;
p=0.5;
for sample=1:1:50
    s(1,sample) = 0;
    for n=2:1:200
        %{
        %for the distortion case at the end of part d uncomment this part
        %if (n<50)
        %    p=0.9;
        %else
        %    p=0.5;
        %end
        %}
        s(n,sample)=s(n-1,sample)+randn+p;
    end
    %increment values in each realization:
    inc(1,sample)=s(50,sample)-s(1,sample);
    inc(2,sample)=s(100,sample)-s(51,sample);
    inc(3,sample)=s(150,sample)-s(101,sample);
    inc(4,sample)=s(200,sample)-s(151,sample);
end
figure(1);
subplot(2,2,1);
hist(inc(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
subplot(2,2,2);
hist(inc(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
%hist(inc',5);


figure(2);
plot(inc(1,:),inc(2,:),'*');
title('Problem 9.120d');
xlabel('inc in [1,50]');
ylabel('inc in [51,100]');
axis([1 100 1 100]);
```

9.121

(a)

```
clear all;
close all;
s(1:200,1:10) = 0;
for sample=1:1:10
    s(1,sample) = poisson_inv(rand,1);
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+poisson_inv(rand,1);
    end

    figure(sample);
    plot(1:200, s(:,sample));
    xlabel('n')
    ylabel('Sn, random process')
    title('Problem 9.121a');

end
```

(b)

```
clear all;
close all;
s(1:200,1:50) = 0;
for sample=1:1:50
    s(1,sample) = poisson_inv(rand,1);
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+poisson_inv(rand,1);
    end
end
m=mean(s');
v=var(s');
plot(1:200, m, '--', 1:200, v, '-o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
title('Problem 9.121b');
```

(c)

(d)

Octave code for parts c and d:

```
clear all;
close all;
s(1:200,1:50) = 0;
inc(1:4,1:50) = 0;
p=1;
for sample=1:1:50
    s(1,sample) = poisson_inv(rand,p);%poissinv(rand, p);
    for n=2:1:200
```

```
        %{
        %for the distortion case at the end of part d uncomment this part
        %if (n<50)
        %    p=0.9;
        %else
        %    p=0.5;
        %end
        %}
        s(n,sample)=s(n-1,sample)+poisson_inv(rand,p);%poissinv(rand, p);
    end
    inc(1,sample)=s(50,sample)-s(1,sample);
    inc(2,sample)=s(100,sample)-s(51,sample);
    inc(3,sample)=s(150,sample)-s(101,sample);
    inc(4,sample)=s(200,sample)-s(151,sample);
end

%hist(inc',5);
figure(1);
subplot(2,2,1);
hist(inc(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
subplot(2,2,2);
hist(inc(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.121c');

figure(2);
plot(inc(1,:),inc(2,:), '*');
xlabel('inc in [1,50]');
ylabel('inc in [51,100]');
axis([1 50 1 50]);
title('Problem 9.121d');

%for test we can distort inc in one range and see if it affects increments
%in the other range, for example we can modify the parameter p for range
%(1,50) and change it back to the original value for the range (51,100)
```

9.122

(a)

```
%Cauchy random variable with parameter 1 has CDF of arctan (x)/pi+0.5
clear;
s(1:200,1:10) = 0;
for sample=1:1:10
    s(1,sample) = tan(pi*(rand-0.5));
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+tan(pi*(rand-0.5));
    end

    figure(sample);
    plot(1:1:200, s(:,sample));
    xlabel('n')
    ylabel('Sn, random process')
    title('Problem 9.122a');
end
```

(b)

```
clear all;
close all;
s(1:200,1:50) = 0;
for sample=1:1:50
    s(1,sample) = tan(pi*(rand-0.5));
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+tan(pi*(rand-0.5));
    end
end
m=mean(s');
v=var(s');
plot(1:1:200, m, '--', 1:1:200, v, 'o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
title('Problem 9.122b');
```

(c)

(d)

Octave code for parts c and d:

```
%for test in part d, we can distort inc in one range and see if it affects
increments
%in the other range, for example we can modify the parameter p for range
%(1,50) and change it back to the original value for the range (51,100)

clear all;
close all;
```

```
s(1:200,1:50) = 0;
inc(1:4,1:50) = 0;
for sample=1:1:50
    s(1,sample) = tan(pi*(rand-0.5));
    for n=2:1:200
        s(n,sample)=s(n-1,sample)+tan(pi*(rand-0.5));
        %{
        %for the distortion case at the end of part d uncomment this part
        %if (n<50)
        %    s(n,sample)=s(n-1,sample)+tan(pi*(rand/2-0.5));
        %end
        %}
    end
    inc(1,sample)=s(50,sample)-s(1,sample);
    inc(2,sample)=s(100,sample)-s(51,sample);
    inc(3,sample)=s(150,sample)-s(101,sample);
    inc(4,sample)=s(200,sample)-s(151,sample);
end

%hist(inc',5);
figure(1);
subplot(2,2,1);
hist(inc(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
subplot(2,2,2);
hist(inc(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.122c');

figure(2);
plot(inc(1,:),inc(2,:), '*');
xlabel('inc in [1,50]');
ylabel('inc in [51,100]');
axis([1 100 1 100]);
title('Problem 9.122d');
```

9.123

(a)

```
clear all;
close all;
y=zeros(5,200,3,2);
%dimensions in y are:  (realization, n, alpha, p)
alpha=[0.25 0.5 0.9];
step=0;
p=[0.5 0.25];
for sample=1:1:5
    for i=1:1:3
        for j=1:1:2
            rn=rand;
            step = -1*(rn <= p(j))+1*(rn > p(j));
            y(sample,1,i,j)=step;
            for n=2:1:200
                rn=rand;
                step = -1*(rn <= p(j))+1*(rn > p(j));
                y(sample,n,i,j)=alpha(i)*y(sample,n-1,i,j)+step;
            end
        end

        figure(sample*4+i);
        plot(1:200, y(sample,1:200,i,1), '--', 1:200, y(sample,1:200,i,2));
        legend('p=0.5', 'p=0.25');
        xlabel('n')
        ylabel('Yn, random process')
        str=sprintf('Problem 9.123a, alpha=%1.1f',alpha(i));
        title(str);

    end
end
m=mean(y);
v=var(y);
%plotting mean and variance
for i=1:1:3
    figure(200+i);
    subplot(2,1,1);
    plot(1:1:200, m(1,1:200,i,1), '--', 1:1:200, m(1,1:200,i,2));
    legend('p=0.5', 'p=0.25');
    xlabel('n')
    ylabel('mean of Yn')
    str=sprintf('Problem 9.123a, alpha=%1.1f',alpha(i));
    title(str);
    subplot(2,1,2);
    plot(1:200, v(1,1:200,i,1), '--', 1:200, v(1,1:200,i,2));
    legend('p=0.5', 'p=0.25');
    xlabel('n')
    ylabel('variance of Yn')
    title(str);
end
```

```
%histogram
for sample=1:1:5
    for i=1:1:3
        figure(300+sample*4+i);
        for j=1:1:2
            subplot(2,1,j);
            hist(y(sample,:,i,j));
            xlabel('Yn, output')
            ylabel('Histogram count')
            str=sprintf('Problem 9.121a, histogram for alpha=%1.1f,
               p=%1.1f,sample#%d',alpha(i),p(j),sample);
            title(str);
        end
    end
end
```

(b)

```
clear all;
close all;
y=zeros(50,200,2);
%dimensions are: (realization, n, p)
alpha=0.5;
step=0;
p=[0.5 0.25];
for sample=1:1:50
    for j=1:1:2
        rn = rand;
        step = -1*(rn <= p(j))+1*(rn > p(j));
        y(sample,1,j)=step;
        for n=2:1:200
            rn = rand;
            step = -1*(rn <= p(j))+1*(rn > p(j));
            y(sample,n,j)=alpha*y(sample,n-1,j)+step;
        end
    end
end

m=mean(y);
v=var(y);
%plotting mean and variance:
figure(100);
subplot(2,1,1);
plot(1:1:200, m(1,1:200,1), '--', 1:1:200, m(1,1:200,2));
legend('p=0.5', 'p=0.25');
xlabel('n')
ylabel('mean of Yn')
str=sprintf('Problem 9.123b, alpha=%f',alpha);
title(str);
subplot(2,1,2);
plot(1:1:200, v(1,1:200,1), '--', 1:1:200, v(1,1:200,2));
legend('p=0.5', 'p=0.25');
xlabel('n')
ylabel('variance of Yn')
title(str);
figure;
```

```
%histogram
figure(200);
for j=1:1:2
    subplot(2,3,(j-1)*3+1);
    hist(y(:,5,j));
    xlabel('outcome')
    str=sprintf('p=%f',p(j));
    ylabel(str)
    str=sprintf('P.9.123b, n=5');
    title(str);
    subplot(2,3,(j-1)*3+2);
    hist(y(:,50,j));
    str=sprintf('P.9.123b, n=50');
    title(str);
    subplot(2,3,(j-1)*3+3);
    hist(y(:,200,j));
    str=sprintf('P.9.123b, n=200');
    title(str);
end
```

(c)

```
clear all;
close all;
y=zeros(50,200,2);
%y dimensions: (realization, n, p)
inc1(1:4,1:50)=0;
inc2(1:4,1:50)=0;
alpha=0.5;
step=0;
p=[0.5 0.25];
for sample=1:1:50
    for j=1:1:2
        rn=rand;
        step = -1*(rn <= p(j))+1*(rn > p(j));
        y(sample,1,j)=step;
        for n=2:1:200
            rn = rand;
            step = -1*(rn <= p(j))+1*(rn > p(j));
            y(sample,n,j)=alpha*y(sample,n-1,j)+step;
        end
    end
    inc1(1,sample)=y(sample,50,1)-y(sample,1,1);
    inc1(2,sample)=y(sample,100,1)-y(sample,51,1);
    inc1(3,sample)=y(sample,150,1)-y(sample,101,1);
    inc1(4,sample)=y(sample,200,1)-y(sample,151,1);
    inc2(1,sample)=y(sample,50,2)-y(sample,1,2);
    inc2(2,sample)=y(sample,100,2)-y(sample,51,2);
    inc2(3,sample)=y(sample,150,2)-y(sample,101,2);
    inc2(4,sample)=y(sample,200,2)-y(sample,151,2);
end

%hist(inc1',5);
figure(1);
subplot(2,2,1);
```

```
hist(inc1(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
title('Problem 9.123c, p=0.5');
subplot(2,2,2);
hist(inc1(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
title('Problem 9-121-c, p=0.5');
subplot(2,2,3);
hist(inc1(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
title('Problem 9.123c, p=0.5');
subplot(2,2,4);
hist(inc1(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.123c, p=0.5');
replot;

figure(2);
%hist(inc2',5);
subplot(2,2,1);
hist(inc2(1,:),5);
title('Problem 9.123c, p=0.25');
xlabel('increments [1-50]');
ylabel('number of samples');
subplot(2,2,2);
hist(inc2(2,:),5);
title('Problem 9.123c, p=0.25');
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc2(3,:),5);
title('Problem 9.123c, p=0.25');
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc2(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.123c, p=0.25');
replot
```

9.124

(a)

```
clear all;
close all;
y=zeros(5,200,2);
%y dimensions: (realization, n, gaussian mean)
step=0;
p=[0 0.5];
for sample=1:1:5
    for j=1:1:2
        step = randn+p(j);
        y(sample,1,j)=step;
        for n=2:1:200
            step = randn+p(j);
            y(sample,n,j)=y(sample,n-1,j)+step;
        end
    end

    figure(sample);
    plot(1:200, y(sample,1:200,1), 1:200, y(sample,1:200,2));
    legend('m=0', 'm=0.5');
    xlabel('n')
    ylabel('Yn, random process')
    title('Problem 9.124a');
end
m=mean(y);
v=var(y);
figure(100);
subplot(2,1,1);
plot(1:200, m(1,1:200,1), '--', 1:200, m(1,1:200,2));
legend('m=0', 'm=0.5');
xlabel('n')
ylabel('mean of Yn')
title('Problem 9.124a');
subplot(2,1,2);
plot(1:1:200, v(1,1:200,1), '--', 1:1:200, v(1,1:200,2));
legend('m=0.5', 'm=0.25');
xlabel('n')
ylabel('variance of Yn')
title('Problem 9.124a');

%histogram
for sample=1:1:5
    figure(200+sample);
    for j=1:1:2
        subplot(2,1,j);
        hist(y(sample,1:200,j));
        xlabel('n, number of trials')
        ylabel('Histogram count')
        str=sprintf('Problem 9.124a, histogram for
m=%1.1f,sample#%d',p(j),sample);
        title(str);
    end
end
```

(b)

```
clear all;
close all;
y=zeros(50,200,2);
step=0;
p=[0 0.5];
for sample=1:1:50
    for j=1:1:2
        step = randn+p(j);
        y(sample,1,j)=step;
        for n=2:1:200
            step = randn+p(j);
            y(sample,n,j)=y(sample,n-1,j)+step;
        end

    end
end
m=mean(y);
v=var(y);
figure(1);
subplot(2,1,1);
plot(1:200, m(1,1:200,1), '--', 1:200, m(1,1:200,2));
legend('m=0', 'm=0.5');
xlabel('n')
ylabel('mean of Yn')
title('Problem 9-122-b');
subplot(2,1,2);
plot(1:1:200, v(1,1:200,1), '--', 1:1:200, v(1,1:200,2));
legend('m=0', 'm=0.5');
xlabel('n')
ylabel('variance of Yn')
title('Problem 9.124b');


%histogram

figure(100);
for j=1:1:2
    subplot(2,3,(j-1)*3+1);
    hist(y(:,5,j));
    xlabel('outcome')
    str=sprintf('m=%1.1f',p(j));
    ylabel(str)
    str=sprintf('P9.124b, n=5');
    title(str);
    subplot(2,3,(j-1)*3+2);
    hist(y(:,50,j));
    str=sprintf('P9.124b, n=50');
    title(str);
    subplot(2,3,(j-1)*3+3);
    hist(y(:,200,j));
    str=sprintf('P9.124b, n=200');
    title(str);
end
```

(c)

```
clear;
close all;
y=zeros(50,200,2);
step=0;
p=[0 0.5];
for sample=1:1:50
    for j=1:1:2
        step = randn+p(j);
        y(sample,1,j)=step;
        for n=2:1:200
            step = randn+p(j);
            y(sample,n,j)=y(sample,n-1,j)+step;
        end
    end
    inc1(1,sample)=y(sample,50,1)-y(sample,1,1);
    inc1(2,sample)=y(sample,100,1)-y(sample,51,1);
    inc1(3,sample)=y(sample,150,1)-y(sample,101,1);
    inc1(4,sample)=y(sample,200,1)-y(sample,151,1);
    inc2(1,sample)=y(sample,50,2)-y(sample,1,2);
    inc2(2,sample)=y(sample,100,2)-y(sample,51,2);
    inc2(3,sample)=y(sample,150,2)-y(sample,101,2);
    inc2(4,sample)=y(sample,200,2)-y(sample,151,2);
end

figure(1);
subplot(2,2,1);
hist(inc1(1,:),5);
xlabel('increments [1-50]');
ylabel('number of samples');
title('Problem 9.124c, m=0');
subplot(2,2,2);
hist(inc1(2,:),5);
xlabel('increments [51-100]');
ylabel('number of samples');
title('Problem 9.124c, m=0');
subplot(2,2,3);
hist(inc1(3,:),5);
xlabel('increments [101-150]');
ylabel('number of samples');
title('Problem 9.124c, m=0');
subplot(2,2,4);
hist(inc1(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.124c, m=0');
replot;

figure(2);
%hist(inc2',5);
subplot(2,2,1);
hist(inc2(1,:),5);
title('Problem 9.124c, m=0.5');
xlabel('increments [1-50]');
ylabel('number of samples');
```

```
subplot(2,2,2);
hist(inc2(2,:),5);
title('Problem 9.124c, m=0.5');
xlabel('increments [51-100]');
ylabel('number of samples');
subplot(2,2,3);
hist(inc2(3,:),5);
title('Problem 9.124c, m=0.5');
xlabel('increments [101-150]');
ylabel('number of samples');
subplot(2,2,4);
hist(inc2(4,:),5);
xlabel('increments [151-200]');
ylabel('number of samples');
title('Problem 9.124c, m=0.5');
replot
```

9.125

Octave code for parts a, b and c:

```
clear all;
close all;
s(1:200,1:50) = 0;
p=0.5;
for sample=1:1:50
    if (rand < p)
        s(1,sample) = 1;
    end
    for n=2:1:200
        s(n,sample)=s(n-1,sample);
        if (rand < p)
            s(n,sample)=s(n-1,sample)+1;
        end
    end
end
m=mean(s');
v=var(s');

%For auto correlation correlation:
%acor=s*s'/number of samples
%for autocovariance: acov=acor-m*m';

acor=s*s'/50;
acov=acor-m'*m;

%for WSS, mean and variance should not depend on (n)
%so for P9.125c since variance depends on (n) it is not WSS.

plot(1:200, m, '--', 1:200, v, 'o');
legend('mean','variance');
xlabel('n')
ylabel('mean,variance')
title('Problem 9.123c');
```

(d)

Octave code for parts a, b and d:

```
clear all;
close all;
numberofsamples=1000;
y1=zeros(numberofsamples,200);
alpha=0.5;
step=0;
p=0.5;
for sample=1:1:numberofsamples
    rn=rand;
    step = -1*(rn <= p)+1*(rn > p);
    y1(sample,1)=step;
    for n=2:1:200
        rn = rand;
```

```
        step = -1*(rn <= p)+1*(rn > p);
        y1(sample,n)=alpha*y1(sample,n-1)+step;
    end
end
m1=mean(y1);
v1=var(y1);


%For auto correlation correlation:
%acor=y*y'/number of samples
%for autocovariance: acov=acor-m'*m;


acor1=y1'*y1/numberofsamples;
acov1=acor1-m1'*m1;

%for WSS, mean and variance should not depend on (n), and acor should
%depend only on (n1-n2)
for i=1:1:200
    v12(i)=acov1(i,i);
end

figure (1);
plot(1:1:200, m1, '--', 1:1:200, v1, '-', 1:200, v12, '*');
legend('mean', 'variance', 'var from acov');
xlabel('n')
ylabel('mean of Yn, p=0.5')
title('Problem 9-123-d');

y2=zeros(numberofsamples,200);
step=0;
p=0.25;
for sample=1:1:numberofsamples
    rn = rand;
    step = -1*(rn <= p)+1*(rn > p);
    y2(sample,1)=step;
    for n=2:1:200
        rn = rand;
        step = -1*(rn <= p)+1*(rn > p);
        y2(sample,n)=alpha*y2(sample,n-1)+step;
    end
end
m2=mean(y2);
v2=var(y2);

%For auto correlation correlation:
%acor=y*y'/number of samples
%for autocovariance: acov=acor-m'*m;

acor2=y2'*y2/numberofsamples;
acov2=acor2-m2'*m2;

%for WSS, mean and variance should not depend on (n), and acor should
%depend only on (n1-n2)
for i=1:1:200
    v22(i)=acov2(i,i);
end
```

```
figure (2);
plot(1:200, m2, '--', 1:200, v2, '-', 1:200, v22, '*');
legend('mean', 'var', 'var from acov');
xlabel('n')
ylabel('mean of Yn, p=0.25')
title('Problem 9.125d');

figure (3);
mesh(acov1);
figure (4);
mesh(acov2);

%you can see that v22 & v12 are the variances computed from autocovariance
%functions, now if you draw the autocovariance, you can see that it is
%almost zero for n1-n2 <> 0
%also it can be seen that mean and variance are independent of n
```

9.126

```
%lambda*100=n*p, if lambda=1 then n*p should be 100
clear all;
close all;
%as n grows N would be a better approximation of a Poisson process.
%probably n=10*100 would be a good pick
%you can draw N for n large than 10*t and you can see that the result would
%not change significantly

n=1000;
p=100/n;
N(1:n)=0;
N(1)=0;
for i=2:1:n
    if (rand < p)
        N(i)=N(i-1)+1;
    else
        N(i)=N(i-1);
    end
end

plot(N);
```

9.127

Octave code for parts a and b:

```
clear all;
close all;
%lambda is 1/4 per second
%lamda*t=n*p=60*1/4=15
n=1200;
p=15/n;
rep=100;
cnt = 0;
dur(1:rep)=0;
for r=1:1:rep
    N(1:1:n)=0;
    N(1)=0;
    arr = 0;
    for i=2:1:n
        if (rand < p)
            arr = arr + 1;
            if (arr == 1)
                previ=i;
            end
            if (arr == 2)
                dur(r) = i-previ;
            end
            N(i)=N(i-1)+1;
        else
            N(i)=N(i-1);
        end
    end
    if ((N(10*n/60) == 3) && ((N(60*n/60)-N(45*n/60)) == 2))
        cnt = cnt + 1;
    end
end
disp('relative frequency');
cnt/rep
disp('theoretical prob');
(10/4)^3*(15/4)^2*exp(-10/4-15/4)/12
figure(1);
hist(dur,1:n/60:40*n/60);
axis([0 40*n/60]);
figure(2);
z=hist(dur,1:n/60:40*n/60);
%pdf is exponential with rate lambda
t=1:1:40;
f=0.25*exp(-0.25*t);
plot(t,f, '-', t,z/rep, '--');
legend('theoretical', 'simulation');
xlabel('seconds');
title('pdf');
figure(3);
cf=1-exp(-0.25*t);
plot(t,cf, '-', t,cumsum(z)/rep, '--');
legend('theoretical', 'simulation');
xlabel('seconds');
title('cdf');
```

9.128

(a)

```
%lambda*10=n*p, if lambda=1 then n*p should be 10
clear all;
close all;
n=100;
p=10/n;
maxrep = 100;
N(1:maxrep,1:n)=0;
N1(1:maxrep,1:n)=0;
N2(1:maxrep,1:n)=0;
for r=1:1:maxrep
    N(r,1)=0;
    N1(r,1)=0;
    N2(r,1)=0;
    for i=2:1:n
        if (rand < p)
            N(r,i)=N(r,i-1)+1;
            if (rand < 0.25)
                N1(r,i)=N1(r,i-1)+1;
                N2(r,i)=N2(r,i-1);
            else
                N1(r,i)=N1(r,i-1);
                N2(r,i)=N2(r,i-1)+1;
            end
        else
            N(r,i)=N(r,i-1);
            N1(r,i)=N1(r,i-1);
            N2(r,i)=N2(r,i-1);
        end
    end
end

y=hist(N(:,n),1:21);
y1=hist(N1(:,n),1:21);
y2=hist(N2(:,n),1:21);

%we know that theoretical pmf for N1 and N2 are Poisson random variables
%pmf with rate lambda*p*t, and lambda*(1-p)*t
expected=poisson_pdf(1:21,10);
expected1=poisson_pdf(1:21,10*0.25);
expected2=poisson_pdf(1:21,10*0.75);
figure(1);
plot(1:21,y/maxrep, '-*', 1:21,expected, '-o');
legend('histogram','Poisson');
title('N(t)');
figure(2);
plot(1:21,y1/maxrep, '-*', 1:21,expected1, '-o');
legend('histogram','Poisson');
title('N1(t)');
figure(3);
plot(1:21,y2/maxrep, '-*', 1:21,expected2, '-o');
legend('histogram','Poisson');
title('N2(t)');
```

```
%chi-square goodness-of-fit test: sum(y-expected)^2/expected

p=sum((y/maxrep-expected).^2./expected)
p1=sum((y1/maxrep-expected1).^2./expected1)
p2=sum((y2/maxrep-expected2).^2./expected2)



%as n grows N would be a better approximation of a Poisson process.
```

(b)

```
%lambda*10=n*p, if lambda=1 then n*p should be 10
clear all;
close all;
n=100;
p=10/n;
maxrep = 100;
N(1:maxrep,1:1:n)=0;
N1(1:maxrep,1:1:n)=0;
N2(1:maxrep,1:1:n)=0;
for r=1:1:maxrep
    N(r,1)=0;
    N1(r,1)=0;
    N2(r,1)=0;
    for i=2:1:n
        if (rand < p)
            N(r,i)=N(r,i-1)+1;
            if (rand < 0.25)
                N1(r,i)=N1(r,i-1)+1;
                N2(r,i)=N2(r,i-1);
            else
                N1(r,i)=N1(r,i-1);
                N2(r,i)=N2(r,i-1)+1;
            end
        else
            N(r,i)=N(r,i-1);
            N1(r,i)=N1(r,i-1);
            N2(r,i)=N2(r,i-1);
        end
    end
end

y=hist(N(:,n),0:21);
y1=hist(N1(:,n),0:21);
y2=hist(N2(:,n),0:21);

%we know that theoretical pmf for N1 and N2 are Poisson random variables
%pmf with rate lambda*p*t, and lambda*(1-p)*t
expected=poisson_pdf(0:21,10);
expected1=poisson_pdf(0:21,10*0.25);
expected2=poisson_pdf(0:21,10*0.75);
figure(1);
plot(0:21,y/maxrep, '-*', 0:21,expected, '-o');
legend('histogram','poisson');
title('N(t)');
```

```
figure(2);
plot(0:21,y1/maxrep, '-*', 0:21,expected1, '-o');
legend('histogram','poisson');
title('N1(t)');
figure(3);
plot(0:21,y2/maxrep, '-*', 0:21,expected2, '-o');
legend('histogram','poisson');
title('N2(t)');

%chi-square goodness-of-fit test: sum((y-expected)^2/expected)

p=sum((y/maxrep-expected).^2./expected)
p1=sum((y1/maxrep-expected1).^2./expected1)
p2=sum((y2/maxrep-expected2).^2./expected2)

%y/maxrep,y1/maxrep,y2/maxrep are pmf of N,N1, and N2

expec(1:20,1:20)=0;
Freq(1:20,1:20)=0;
for r=1:1:maxrep
    i = N1(r,n)+1;
    j = N2(r,n)+1;
    if (i>20)
        i = 20;
    end
    if (j>15)
        j = 20;
    end
    Freq(i,j)=Freq(i,j)+1;
end
chi = 0;
for i=1:1:19
    for j=1:1:19
       expec(i,j)=expected1(i)*expected2(j);
       chi=chi+(Freq(i,j)/maxrep-expec(i,j))^2/(expec(i,j));
    end
end
chi
figure(10);
subplot(2,1,1);
mesh(expec);
ylabel('theoretical prob');
subplot(2,1,2);
mesh(Freq/maxrep);
ylabel('simulation');
```

9.129

(a)

```
%lambda*t=n*p, if lambda=1 then n*p should be t
clear all;
close all;
n=2000;
p=200/n;
maxrep = 1;
N(1:maxrep,1:1:n)=0;
A(1:maxrep,1:1:n)=0;
D(1:maxrep,1:1:n)=0;
L(1:maxrep,1:1:n)=0; % # of customers leaving at time t in (1,n)
for r=1:1:maxrep
    %assuming we will have maximum of 3000 customer and their service time
distribution is exponential.
    T(1:1:3000) = floor(exponential_inv(rand(1,3000), 5*n/200));
    N(r,1)=0;
    A(r,1)=0;
    D(r,1)=0;
    cust=1;
    for i=2:1:n
        if (rand < p)
            A(r,i)=A(r,i-1)+1;
            N(r,i)=N(r,i-1)+1;
            if (i+T(cust) < n)
                L(r,i+T(cust))=L(r,i+T(cust))+1;
            end
            cust = cust+1;
        else
            A(r,i)=A(r,i-1);
            N(r,i)=N(r,i-1);
        end
        D(r,i)=D(r,i-1)+L(r,i);
        N(r,i)=N(r,i)-L(r,i);
    end
end

plot(1:1:n,A(1,:),1:1:n,N(1,:),1:1:n,D(1,:));
legend('Arrival','Number in system','Departure');
```

(b)

(c)

First part of code is the same as part a.

```
%lambda*t=n*p, if lambda=1 then n*p should be 200
clear all;
close all;
n=2000;
p=200/n;
t=n/200; % one second is t steps
```

```
maxrep = 100;
N(1:maxrep,1:1:n)=0;
A(1:maxrep,1:1:n)=0;
D(1:maxrep,1:1:n)=0;
L(1:maxrep,1:1:n)=0; % # of customers leaving at time t in (1,n)
for r=1:1:maxrep
    %assuming we will have maximum of 3000 customer and their service time
distribution is exponential.
    T(1:1:3000) = floor(exponential_inv(rand(1,3000), 5*t));
    N(r,1)=0;
    A(r,1)=0;
    D(r,1)=0;
    cust=1;
    for i=2:1:n
        if (rand < p)
            A(r,i)=A(r,i-1)+1;
            N(r,i)=N(r,i-1)+1;
            if (i+T(cust) < n)
                L(r,i+T(cust))=L(r,i+T(cust))+1;
            end
            cust = cust+1;
        else
            A(r,i)=A(r,i-1);
            N(r,i)=N(r,i-1);
        end
        D(r,i)=D(r,i-1)+L(r,i);
        N(r,i)=N(r,i)-L(r,i);
    end
end

figure(1);
plot(1:1:n,A(1,:),1:1:n,N(1,:),1:1:n,D(1,:));
legend('Arrival','Number in system','Departure');

d50=hist(D(:,50*t),1:max(D(:,50*t)))/maxrep;
d100=hist(D(:,100*t),1:max(D(:,100*t)))/maxrep;
d150=hist(D(:,150*t),1:max(D(:,150*t)))/maxrep;
d200=hist(D(:,200*t),1:max(D(:,200*t)))/maxrep;

%we know that theoretical pmf for D(t) is poisson random variables
%pmf with rate lambda*(t-5)

expected50=poisson_pdf(1:max(D(:,50*t)), 45);
expected100=poisson_pdf(1:max(D(:,100*t)), 95);
expected150=poisson_pdf(1:max(D(:,150*t)), 145);
expected200=poisson_pdf(1:max(D(:,200*t)), 195);

figure(2);
plot(1:max(D(:,50*t)),d50, '-*', 1:max(D(:,50*t)),expected50, '-o');
legend('histogram','poisson');
title('D(50)');

figure(3);
plot(1:max(D(:,100*t)),d100, '-*', 1:max(D(:,100*t)),expected100, '-o');
legend('histogram','poisson');
title('D(100)');
figure(4);
```

```
plot(1:max(D(:,150*t)),d150, '-*', 1:max(D(:,150*t)),expected150, '-o');
legend('histogram','poisson');
title('D(150)');
figure(5);
plot(1:max(D(:,200*t)),d200, '-*', 1:max(D(:,200*t)),expected200, '-o');
legend('histogram','poisson');
title('D(200)');


%chi-square goodness-of-fit test: sum(y-expected)^2/expected

p50=sum((d50-expected50).^2./expected50)
p100=sum((d100-expected100).^2./expected100)
p150=sum((d150-expected150).^2./expected150)
p200=sum((d200-expected200).^2./expected200)
```

(d)

```
%lambda*t=n*p, if lambda=1 then n*p should be 200 seconds
clear all;
close all;
n=2000;
p=200/n;
t=n/200; % one second is t steps
maxrep = 100;
N(1:maxrep,1:1:n)=0;
A(1:maxrep,1:1:n)=0;
D(1:maxrep,1:1:n)=0;
L(1:maxrep,1:1:n)=0; % # of customers leaving at time t in (1,n)
for r=1:1:maxrep
    N(r,1)=0;
    A(r,1)=0;
    D(r,1)=0;
    cust=1;
    for i=2:1:n
        if (rand < p)
            A(r,i)=A(r,i-1)+1;
            N(r,i)=N(r,i-1)+1;
            if (i+5*t < n)
                L(r,i+5*t)=L(r,i+5*t)+1;
            end
            cust = cust+1;
        else
            A(r,i)=A(r,i-1);
            N(r,i)=N(r,i-1);
        end
        D(r,i)=D(r,i-1)+L(r,i);
        N(r,i)=N(r,i)-L(r,i);
    end
end

figure(1);
plot(1:1:n,A(1,:),1:1:n,N(1,:),1:1:n,D(1,:));
legend('Arrival','Number in system','Departure');

d50=hist(D(:,50*t),1:max(D(:,50*t)))/maxrep;
```

```
d100=hist(D(:,100*t),1:max(D(:,100*t)))/maxrep;
d150=hist(D(:,150*t),1:max(D(:,150*t)))/maxrep;
d200=hist(D(:,200*t),1:max(D(:,200*t)))/maxrep;

%we know that theoretical pmf for D(t) is poisson random variables
%pmf with rate lambda*(t-5)

expected50=poisson_pdf(1:max(D(:,50*t)), 45);
expected100=poisson_pdf(1:max(D(:,100*t)), 95);
expected150=poisson_pdf(1:max(D(:,150*t)), 145);
expected200=poisson_pdf(1:max(D(:,200*t)), 195);

figure(2);
plot(1:max(D(:,50*t)),d50, '-*', 1:max(D(:,50*t)),expected50, '-o');
legend('histogram','poisson');
title('D(50)');
figure(3);
plot(1:max(D(:,100*t)),d100, '-*', 1:max(D(:,100*t)),expected100, '-o');
legend('histogram','poisson');
title('D(100)');
figure(4);
plot(1:max(D(:,150*t)),d150, '-*', 1:max(D(:,150*t)),expected150, '-o');
legend('histogram','poisson');
title('D(150)');
figure(5);
plot(1:max(D(:,200*t)),d200, '-*', 1:max(D(:,200*t)),expected200, '-o');
legend('histogram','poisson');
title('D(200)');

%chi-square goodness-of-fit test: sum(y-expected)^2/expected

p50=sum((d50-expected50).^2./expected50)
p100=sum((d100-expected100).^2./expected100)
p150=sum((d150-expected150).^2./expected150)
p200=sum((d200-expected200).^2./expected200)
```

9.130

(a)

```
clear all;
close all;
s=100; %each second divided to 100 steps
%alpha=1, so h=sqrt(alpha/s)=0.1
alpha=1;
h=sqrt(alpha/s);
n=s*3.5;
maxrep=1000;
x(1:maxrep,1:n)=0;
inc(1:maxrep,3)=0;
for r=1:1:maxrep
    x(r,1)=0;
    for i=2:1:n
        if (rand < 0.5)
            x(r,i)=x(r,i-1)+h;
        else
            x(r,i)=x(r,i-1)-h;
        end
    end
    inc(r,1)=x(r,0.5*s)-x(r,1);
    inc(r,2)=x(r,1.5*s)-x(r,0.5*s+1);
    inc(r,3)=x(r,3.5*s)-x(r,1.5*s+1);
end

[y b]=hist(inc(:,1),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(0.5));
sno=sum(no);
figure(1);
plot(-6:0.4:6,no/sno,b,y/maxrep)
legend('normal pdf','simulation');
title('inc (0,0.5)');

[y b]=hist(inc(:,2),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(1.5-0.5));
sno=sum(no);
figure(2);
plot(-6:0.4:6,no/sno,b,y/maxrep)
legend('normal pdf','simulation');
title('inc (0.5,1.5)');

[y b]=hist(inc(:,3),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-8:0.4:8,0,sqrt(3.5-1.5));
sno=sum(no);
figure(3);
plot(-8:0.4:8,no/sno,b,y/maxrep)
legend('normal pdf','simulation');
title('inc (3.5,1.5)');
```

(b)

```
clear all;
close all;
s=100; %each second divided to 100 steps
%alpha=1, so h=sqrt(alpha/s)=0.1
alpha=1;
h=sqrt(alpha/s);
n=s*3.5;
maxrep=100;
x(1:maxrep,1:n)=0;
inc(1:maxrep,3)=0;
for r=1:1:maxrep
    x(r,1)=0;
    for i=2:1:n
        if (rand < 0.5)
            x(r,i)=x(r,i-1)+h;
        else
            x(r,i)=x(r,i-1)-h;
        end
    end
    inc(r,1)=x(r,0.5*s)-x(r,1);
    inc(r,2)=x(r,1.5*s)-x(r,0.5*s+1);
    inc(r,3)=x(r,3.5*s)-x(r,1.5*s+1);
end

[y b]=hist(inc(:,1),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(0.5));
sno=sum(no);
figure(1);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(0,0.5)');
pdf1=no/sno;

[y b]=hist(inc(:,2),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(1.5-0.5));
sno=sum(no);
figure(2);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(0.5,1.5)');
pdf2=no/sno;

[y b]=hist(inc(:,3),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(3.5-1.5));
sno=sum(no);
pdf3=no/sno;
figure(3);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(1.5,3.5)');
```

```
value=-6:0.4:6;
vallen=length(value);
%number of values is 12*2.5=30

chi=0;
p0(1:1:vallen,1:1:vallen)=0;
freq(1:1:vallen,1:1:vallen)=0;
for i=2:1:vallen-1
    for j=2:1:vallen-1
        p0(i,j)=pdf1(i)*pdf2(j);
        for r=1:1:maxrep
            if (inc(r,1) <= ((value(i)+value(i+1))/2) && inc(r,1) >
((value(i-1)+value(i))/2))
                if (inc(r,2) <= ((value(j)+value(j+1))/2) && inc(r,2) >
((value(j-1)+value(j))/2))
                    freq(i,j) = freq(i,j)+1;
                end
            end
        end
        freq(i,j)=freq(i,j)/maxrep;
        chi=chi+((freq(i,j)-p0(i,j))^2/p0(i,j));
    end
end
chi %(chi for 10000 sim is 0.02), (chi for 1000 sim is 0.35)
figure(4);
subplot(2,1,1);
mesh(value,value,p0);
axis([-5 5 -5 5]);
zlabel('theoretical prob');
subplot(2,1,2);
mesh(value,value,freq);
axis([-5 5 -5 5]);
zlabel('simulation');
title('Dependency check, inc (0,0.5) & inc(0.5,1.5)');
```

9.131

(a)

```
clear all;
close all;
s=100; %each second divided to 100 steps
%each sample is a gaussian random variable with zero mean, and variance
%alpha*t. Since alpha=1, then just generate randn(0,step)+prevvalue
n=3.5*s;
alpha=1;
maxrep=100;
x(1:maxrep,1:n)=0;
inc(1:maxrep,3)=0;
for r=1:1:maxrep
    x(r,1)=0;
    for i=2:1:n
        x(r,i)=x(r,i-1)+sqrt(1/s)*randn;
    end
    inc(r,1)=x(r,0.5*s)-x(r,1);
    inc(r,2)=x(r,1.5*s)-x(r,0.5*s+1);
    inc(r,3)=x(r,3.5*s)-x(r,1.5*s+1);
end

[y b]=hist(inc(:,1),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(0.5));
sno=sum(no);
figure(1);
plot(-6:0.4:6,no/sno,b,y/maxrep)
legend('normal pdf','simulation');

[y b]=hist(inc(:,2),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(1.5-0.5));
sno=sum(no);
figure(2);
plot(-6:0.4:6,no/sno,b,y/maxrep)
legend('normal pdf','simulation');

[y b]=hist(inc(:,3),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-8:0.4:8,0,sqrt(3.5-1.5));
sno=sum(no);
figure(3);
plot(-8:0.4:8,no/sno,b,y/maxrep)
legend('normal pdf','simulation');
```

(b)

```
clear all;
close all;
s=100; %each second divided to 100 steps
%each sample is a gaussian random variable with zero mean, and variance
%alpha*t. Since alpha=1, then just generate randn(0,step)+prevvalue
```

```
n=3.5*s;
alpha=1;
maxrep=100;
x(1:maxrep,1:n)=0;
inc(1:maxrep,3)=0;
for r=1:1:maxrep
    x(r,1)=0;
    for i=2:1:n
        x(r,i)=x(r,i-1)+sqrt(1/s)*randn;
    end
    inc(r,1)=x(r,0.5*s)-x(r,1);
    inc(r,2)=x(r,1.5*s)-x(r,0.5*s+1);
    inc(r,3)=x(r,3.5*s)-x(r,1.5*s+1);
end

[y b]=hist(inc(:,1),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(0.5));
sno=sum(no);
figure(1);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(0,0.5)');
pdf1=no/sno;

[y b]=hist(inc(:,2),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(1.5-0.5));
sno=sum(no);
figure(2);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(0.5,1.5)');
pdf2=no/sno;

[y b]=hist(inc(:,3),-6:0.4:6);
[sum(y/maxrep) mean(y/maxrep) var(y/maxrep)]
no=normal_pdf(-6:0.4:6,0,sqrt(3.5-1.5));
sno=sum(no);
pdf3=no/sno;
figure(3);
plot(-6:0.4:6,no/sno, '--',b,y/maxrep,'-o')
legend('normal','histogram');
title('inc(1.5,3.5)');

value=-6:0.4:6;
vallen=length(value);
%number of values is 12*2.5=30

chi=0;
p0(1:1:vallen,1:1:vallen)=0;
freq(1:1:vallen,1:1:vallen)=0;
for i=2:1:vallen-1
    for j=2:1:vallen-1
        p0(i,j)=pdf1(i)*pdf2(j);
        for r=1:1:maxrep
```

```
            if (inc(r,1) <= ((value(i)+value(i+1))/2) && inc(r,1) >
((value(i-1)+value(i))/2))
                if (inc(r,2) <= ((value(j)+value(j+1))/2) && inc(r,2) >
((value(j-1)+value(j))/2))
                    freq(i,j) = freq(i,j)+1;
                end
            end
        end
        freq(i,j)=freq(i,j)/maxrep;
        chi=chi+((freq(i,j)-p0(i,j))^2/p0(i,j));
    end
end
chi %(chi for 10000 sim is 0.02), (chi for 1000 sim is 0.35)
figure(4);
subplot(2,1,1);
mesh(value,value,p0);
axis([-5 5 -5 5]);
zlabel('theoretical prob');
subplot(2,1,2);
mesh(value,value,freq);
axis([-5 5 -5 5]);
zlabel('simulation');
title('Dependency check, inc (0,0.5) & inc(0.5,1.5)');
```

Problems Requiring Cumulative Knowledge

9.132

~~6.95~~ The increment of $X(t)$ in the interval $(t_1, t_2]$ has pdf:

$$f_{X(t_2)-X(t_1)}(x) = \frac{\lambda^{t_2-t_1}}{\Gamma(t_2-t_1)} x^{t_2-t_1-1} e^{-\lambda x}$$

a) We assume that $X(0) = 0$, then

$$\begin{aligned} f_{X(t_1)X(t_2)}(x,y) &= f_{X(t_1)}(x) f_{X(t_2)-X(t_1)}(y-x) \quad \text{by indep. increment property} \\ &= \frac{\lambda^{t_1}}{\Gamma(t_1)} x^{t_1} e^{-\lambda x} \frac{\lambda^{t_2-t_1}}{\Gamma(t_2-t_1)} (y-x)^{t_2-t_1-1} e^{-\lambda(y-x)} \\ &= \frac{\lambda^{t_2}}{\Gamma(t_1)\Gamma(t_2-t_1)} x^{t_1} (y-x)^{t_2-t_1-1} e^{-\lambda y} \end{aligned}$$

b)

$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \qquad \text{assume } t_2 \geq t_1 \\ &= E[X(t_1)(X(t_2) - X(t_1) + X(t_1))] \\ &= E[X(t_1)^2] + E[X(t_1)]E[\underbrace{X(t_2) - X(t_1)}_{\text{increment}}] \end{aligned}$$

From Table 4.1

$$\begin{aligned} E[X(t_1)] &= \frac{\alpha}{\lambda} = \frac{t_1}{\lambda} \\ E[X^2(t_1)] &= VAR[X(t_1)] + E[X(t_1)]^2 \\ &= \frac{t_1}{\lambda^2} + \frac{t_1^2}{\lambda^2} \\ R_X(t_1,t_2) &= \frac{t_1}{\lambda^2} + \frac{t_1^2}{\lambda^2} + \frac{t_1}{\lambda}\left(\frac{t_2-t_1}{\lambda}\right) = \frac{t_1}{\lambda^2} + \frac{t_1 t_2}{\lambda^2} \\ &= \frac{t_1(1+t_2)}{\lambda^2} \qquad t_2 \geq t_1 \end{aligned}$$

If $t_1 \leq t_2$, then

$$R_X(t_1,t_2) = \frac{t_2(1+t_1)}{\lambda^2}$$

Note the similarities to the Wiener Process ~~discussed in Ex. 6.38~~.

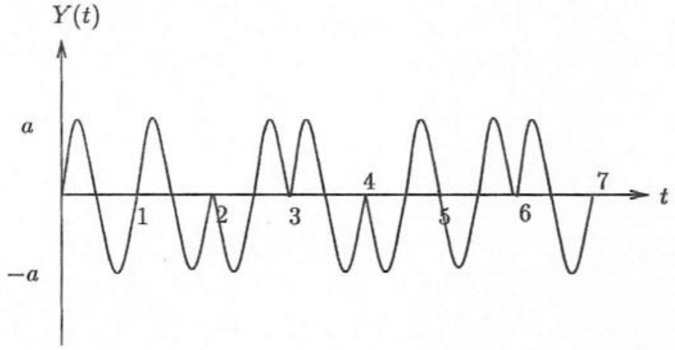
c) $R_X(t_1,t_2)$ is continuous at the point $t_2 = t_2 = t$ so $X(t)$ is M.S. continuous.

d)

$$
\begin{aligned}
R_X(t_1,t_2) &= \begin{cases} \dfrac{t_2(1+t_1)}{\lambda^2} & t_1 < t_2 \\ \dfrac{t_2(1+t_2)}{\lambda^2} & g_2 \geq t_1 \end{cases} \\
\frac{\partial R_X(t_1,t_2)}{\partial t_2} &= \begin{cases} \dfrac{1+t_1}{\lambda^2} & t_1 \leq t_2 \quad \Rightarrow \; X(t) \text{ is } \underline{\text{NOT}} \\ \dfrac{t_1}{\lambda^2} & t_2 \geq t_1 \qquad \text{M.S. differentiable} \end{cases} \\
&= \frac{t_1}{\lambda^2} + \frac{1}{\lambda^2} u(t_1 - t_2) \\
R_{X'}(t_1,t_2) &= \frac{\partial^2 R_X(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\delta(t_1 - t_2)
\end{aligned}
$$

This suggests that $X'(t)$ has this autocorrelation function if we generalize the notion of derivative of a random process.

9.133 a)



b) $f_{Y_1Y_2}(y_1,y_2)$

$$= \begin{cases} \frac{1}{2}\delta(y_1 - a\cos(2\pi t_1 + \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_2 + \frac{\pi}{2})) & \\ +\frac{1}{2}\delta(y_1 - a\cos(2\pi t_1 - \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_2 - \frac{\pi}{2})), & nT \le t_1, t_2 < (n+1)T \\ \frac{1}{4}\delta(y_1 - a\cos(2\pi t_1 + \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_2 + \frac{\pi}{2})) & \\ +\frac{1}{4}\delta(y_1 - a\cos(2\pi t_1 - \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_2 + \frac{\pi}{2})) & \\ +\frac{1}{4}\delta(y_1 - a\cos(2\pi t_1 - \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_2 + \frac{\pi}{2})) & \\ +\frac{1}{4}\delta(y_1 - a\cos(2\pi t_1 - \frac{\pi}{2}))\delta(y_2 - a\cos(2\pi t_1 - \frac{\pi}{2})), & \begin{array}{l} nT \le t_1 < (n+1)T \\ mT \le t_2 < (m+1)T, \quad m \ne n \end{array} \end{cases}$$

c)
$$\begin{aligned} E[Y(t)] &= \frac{1}{2}a\cos\left(2\pi t + \frac{\pi}{2}\right) + \frac{1}{2}a\cos\left(2\pi t - \frac{\pi}{2}\right) \\ &= -\frac{a}{2}\sin 2\pi t + \frac{a}{2}\sin 2\pi t \\ &= 0 \end{aligned}$$

$$\begin{aligned} & E[Y(t_1)Y(t_2)] \\ &= \frac{1}{2}a\cos\left(2\pi t_1 + \frac{\pi}{2}\right) a\cos\left(2\pi t_2 + \frac{\pi}{2}\right) + \frac{1}{2}a\cos\left(2\pi t_1 - \frac{\pi}{2}\right)\cos\left(2\pi t_1 - \frac{\pi}{2}\right) \\ &= \frac{a}{2}\sin(2\pi t_1)\sin(2\pi t_2) + \frac{a}{2}\sin(2\pi t_1)\sin(2\pi t_2) \\ &= a\sin(2\pi t_1)\sin(2\pi t_2), \qquad nT \le t_1, t_2 < (n+1)T \end{aligned}$$

$$E[Y(t_1)Y(t_2)] = E[Y(t_1)]E[Y(t_2)] = 0 \quad \text{otherwise}$$

d) $Y(t)$ is cyclostationary.

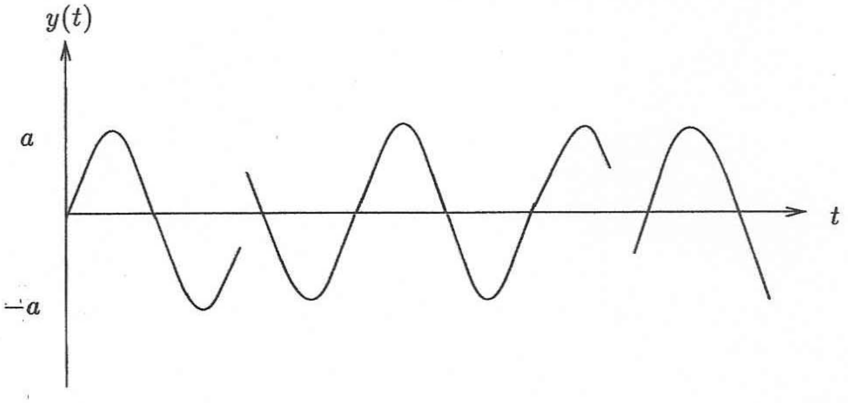
e) Yes.

f) $X(t)$ is differentiable at all points $t \ne nT$.

$$\begin{aligned} E\left[\frac{dY(t)}{dt}\right] &= \frac{d}{dt}E[Y(t)] = 0 \\ R_{Y'}(t_1,t_2) &= \frac{\partial^2 R_Y(t_1,t_2)}{\partial t_1 \partial t_2} \\ &= \begin{cases} 4\pi^2 a\cos 2\pi t_1 \cos 2\pi t_2, & nT \le t_1, t_2 < (n+1)T \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

9.134

~~6.97~~ **a)** Each event in the Poisson process causes a sign reversal in $Y(t)$.



b) At time t_1, $Y(t)$ assumes the values $\cos 2\pi f t_1$ or $-\cos 2\pi f t_1$ depending on whether $N(t_1)$ is even or odd. Similarly $Y(t_2)$ is $\cos 2\pi f t_2$ or $-\cos 2\pi f t_2$ depending on whether an even or odd number of events occur in the interval $(t_1, t_2]$. Thus from Example 6.22 we know that

$$\begin{aligned} P[\text{even \# events in } t \text{ seconds}] &= \frac{1}{2}\{1+e^{-2\alpha t}\} \triangleq \pi_e(t) \\ P[\text{odd \# events in } t \text{ seconds}] &= \frac{1}{2}\{1-e^{-2\alpha t}\} \triangleq \pi_o(t) \end{aligned}$$

and

$$\begin{aligned} f_{Y(t_1)Y(t_2)}(y_1,y_2) &= \pi_e(t_1)\delta(y_1-\cos 2\pi f t_1)[\pi_e(t_2-t_1)\delta(y_2-\cos 2\pi f t_2) \\ &\qquad +\pi_o(t_2-t_1)\delta(y_2+\cos 2\pi f t_2)] \\ &\quad +\pi_o(t_1)\delta(y_1+\cos 2\pi f t_1)[\pi_e(t_2-t_1)\delta(y_2+\cos 2\pi f t_2)] \\ &\qquad +\pi_o(t_2-t_1)\delta(y_2-\cos 2\pi f t_2)\ . \end{aligned}$$

c)

$$\begin{aligned} E[Y(t)] &= \cos 2\pi f t \pi_e(t) - \cos 2\pi f t \pi_o(t) \\ &= (\cos 2\pi f t)e^{-2\alpha t} \\ E[Y(t_1)Y(t_2)] &= \Sigma\Sigma y_1 y_2 P[y_1,y_2] \\ &= \cos 2\pi f t_1 \cos 2\pi f t_2 \pi_e(t_1)\pi_e(t_2-t_1) \\ &\quad - \cos 2\pi f t_1 \cos 2\pi f t_2 \pi_e(t_1)\pi_o(t_2-t_1) \\ &\quad - \cos 2\pi f t_1 \cos 2\pi f t_2 \pi_o(t_1)\pi_o(t_2-t_1) \\ &\quad + \cos 2\pi f t_1 \cos 2\pi f t_2 \pi_o(t_1)\pi_e(t_2-t_1) \end{aligned}$$

$$\begin{aligned}
&= \cos 2\pi f t_1 \cos 2\pi f t_2 (\pi_e(t_2 - t_1) - \pi_o(t_2 - t_1)) \\
&= \cos 2\pi f t_1 \cos 2\pi f t_2 e^{-2\alpha(t_2 - t_1)} \\
&= \frac{1}{2}\cos 2\pi f(t_2 - t_1) e^{-2\alpha(t_2 - t_1)} + \frac{1}{2}\cos 2\pi f(t_2 + t_1) e^{-2\alpha(t_2 - t_1)}
\end{aligned}$$

If $t_1 \geq t_2$ we have

$$E[Y(t_1)Y(t_2)] = \cos 2\pi f t_1 \cos 2\pi f t_2 e^{-2\alpha(t_1 - t_2)}.$$

Thus

$$E[Y(t_1)Y(t_2)] = \cos 2\pi f t_1 \cos 2\pi f t_2 e^{-2\alpha|t_2 - t_1|}.$$

d) $E[Y(t)]$ varies with time so $Y(t)$ is not stationary. $Y(t)$ does not depend solely on $t_2 - t_1$ so it is not WSS. If we let $t_2 = t_1 + \tau$ and let $t_1 \to \infty$, $E[Y(t_1)Y(t_2)]$ still does not depend solely on $t_2 - t_1$ so it is not asymptotically WSS.

If we consider $t_1 + mT$ and $t_2 + mT$ we have

$$\begin{aligned}
E[Y(t_1 + mT)Y(t_2 + mT)] &= \cos 2\pi f(t_1 + mT) \cos 2\pi f(t_2 + mT) e^{-2\alpha|t_2 - t_1|} \\
&= \cos 2\pi f t_1 \cos 2\pi f t_2 e^{-2\alpha|t_2 - t_1|} \\
&= E[Y(t_1)Y(t_2)]
\end{aligned}$$

However the mean is:

$$\begin{aligned}
E[Y(t_1 + mT)] &= \cos 2\pi f(t_1 + mT) e^{-2\alpha(t_1 + mT)} \\
&\neq E[Y(t_1)]\ .
\end{aligned}$$

As $t_1 \to \infty$ both of these mean terms approach zero. We conclude that $Y(t)$ is asymptotically wide sense cyclostationary.

e) $R_Y(t_1, t_2)$ is continuous in t_1 and t_2 so $Y(t)$ is M.S. continuous.

f)

$$R_Y(t_1, t_2) = \begin{cases} \cos 2\pi f t_1 \ \cos 2\pi f t_2 e^{-2\alpha(t_2 - t_1)} & t_2 \geq t_1 \\ \cos 2\pi f t_1 \ \cos 2\pi f t_2 e^{2\alpha(t_2 - t_1)} & t_1 < t_2 \end{cases}$$

We see that $R_Y(t_1, t_2)$ has a cusp at $t_1 = t_2$ so $Y(t)$ is not M.S. differentiable.

9.135

~~6.98~~ Assume $h(t) = 0$ for $t < 0$.

a) We condition on the number of occurrence, $N(t)$, up to t

$$\begin{aligned}
E[X(t)] &= E[E[X(t)|N(t)]] \\
E[X(t)|N(t)=k] &= E\left[\sum_{j=1}^{k} A_j h(t-s_j)\right] \\
&= \sum_{j=1}^{k} E[A_j]E[h(t-s_j)] \\
E[h(t-s_j)] &= \int_0^t h(t-s)\cdot\frac{1}{t}ds \\
&= \int_0^t h(u)\frac{1}{t}du \\
E[X(t)|N(t)=k] &= E[A_j]\cdot k\cdot\int_0^t h(u)\frac{1}{t}du \\
E[X(t)] &= E[E[X(t)|N(t)]] \\
&= E\left[E[A_j]\frac{N(t)}{t}\int_0^t h(u)du\right] \\
&= E[A_j]\lambda\int_0^t h(u)du
\end{aligned}$$

For the autocorrelation function, we condition on the number of occurrences at t_1 and t_2

$$E[X(t_1)X(t_2)] = E[E[X(t_1)X(t_2)|N(t_1)=k,\ N(t_2)=n]],\quad t_1 \le t_2$$

$$\begin{aligned}
E[X(t_1)X(t_2)|N(t_1)=k, N(t_2)=k+n] &= E\left\{\sum_{j=1}^{k} A_j h(t_1-s_j)\sum_{l=1}^{k+n} A_l h(t_2-s_l)\right\} \\
&= \sum_{j=1}^{k}\sum_{l=1}^{k+n} E[A_jA_l]E[h(t_1-s_j)h(t_2-s_l)] \\
E[A_jA_l] &= \begin{cases} E[A_j^2] & j=l \\ E[A_j]E[A_l] & j\neq l \end{cases} \\
E[h(t_1-s_j)h(t_2-s_l)] &= \begin{cases} E[h(t_1-s_j)h(t_2-s_j)] & j=l \\ E[h(t_1-s_j)]E[h(t_2-s_l)] & j\neq l \end{cases}
\end{aligned}$$

When $j = l$, s_j occurs in $(0, t_1)$:

$$E[h(t_1-s_j)h(t_2-s_j)] = \int_0^{t_1} h(t_1-s)h(t_2-s)\frac{ds}{t_1}$$

Substitution into first equation gives:

$$\begin{aligned}
E[X(t_1)X(t_2)|N(t_1)=k, N(t_2)=k+n] &= kE[A^2]\int_0^{t_1} h(t_1-s)h(t_2-s)\frac{ds}{t_1} \\
&\quad +k(k-1)E[A]^2\int_0^{t_1} h(t_1-s)\frac{ds}{t_1}\int_0^{t_1} h(t_2-s)\frac{ds}{t_1} \\
&\quad +k\cdot nE[A]^2\int_0^{t_1} h(t_2-s)\frac{ds}{t_1}\int_{t_1}^{t_2} h(t_2-s)\frac{ds}{t_2-t_1}
\end{aligned}$$

Now

$$\begin{aligned}
E[N(t_1)] &= \lambda t_1 \qquad E[N(t_1)(N(t_1)-1] = \lambda^2 t_1^2 \\
E[N(t_2)-N(t_1)] &= \lambda(t_2-t_1)
\end{aligned}$$

Thus

$$\begin{aligned}
E[X(t_1)X(t_2)] &= \lambda\int_0^{t_1} h(t_1-s)h(t_2-s)ds \\
&\quad +\lambda^2 E[A]^2\int_0^{t_1} h(t_1-s)ds\int_0^{t_1} h(t_2-s)ds \\
&\quad +\lambda^2 E[A]^2\int_0^{t_1} h(t_2-s)ds\int_{t_1}^{t_2} h(t_2-s)ds \\
&= \lambda\int_0^{t_1} h(t_1-s)h(t_2-s)ds \\
&\quad +\lambda^2 E[A]^2\int_0^{t_1} h(t_1-s)ds\int_0^{t_2} h(t_2-s)ds
\end{aligned}$$

b) $h(t) = u(t)$

$$E[X(t)] = E[A]\lambda\int_0^t h(u)du - E[A]\lambda t$$

This is consistent with the expected value of Poisson RV when $A = 1$

$$\begin{aligned}
E[X(t_1)X(t_2)] &= \lambda t_1 E[A^2]\frac{1}{t_1}\int_0^{t_1} |\cdot|du \\
&\quad +\lambda^2 t_1^2 E[A]^2\frac{1}{t_1^2}\int_0^{t_1} du\int_0^{t_1} du \\
&\quad +\lambda(t_2-t_1)\lambda t_1 E[A^2]\frac{1}{t_1}\int_0^{t_1} du\frac{1}{t_2-t_1}\int_0^{t_2-t_1} du \\
&= \lambda t_1 E[A^2] + \lambda^2 t_1^2 E[A^2] + \lambda(t_2-t_1)\lambda t_1 E[A^2] \\
&= \lambda t_1 E[A^2] + (\lambda^2 t_1 t_2)E[A^2] \qquad (t_1 \le t_2)
\end{aligned}$$

This is consistent again with the autocorrelation function of Poisson RV when $A = 1$.

c) $h(t) = p(t)$, a rectangular pulse of duration T

$$\begin{aligned}
E[X(t)] &= E[A]\lambda\int_0^t p(u)du \\
&= \begin{cases} E[A]\lambda T & \text{if } t \ge T \\ E[A]\lambda t & \text{if } t < T \end{cases} \\
&= E[A]\min(t,T) \\
E[X(t_1)X(t_2)] &= \lambda t_1 E[A^2]\frac{1}{t_1}\int_0^{t_1} p(u)h(t_2-t_1+u)du \\
&\quad +\lambda^2 t_1^2 E[A]^2 + \frac{1}{t_1^2}\min(t_1,T)\int_0^{t_1} h(t_2-t_1+u)du \\
&\quad +\lambda(t_2-t_1)\lambda t_1 E[A]^2\frac{1}{t_1}\min(t_1,T)\frac{1}{t_2-t_1}\min(t_2-t_1,T) \\
&\qquad \text{for } t_1 \le t_2
\end{aligned}$$

9.136

~~6.99~~ **a)**

$$
\begin{aligned}
E[X(t)] &= E[A_1\cos(\omega_0 t+\theta_1)] + E[A_2\cos(\sqrt{2}\omega_2 t+\theta_2)] \\
&= E[A_1]E[\cos(\omega_0 t+\theta_1)] + E[A_2]E[\cos(\sqrt{2}\omega_0 t+\theta_2)] \\
&= 0 \\
E[X(t_1)X(t_2)] &= E[\{A_1\cos(\omega_0 t_1+\theta_1) + A_2\cos(\omega_0 t_1+\theta_2)\}]\cdot \\
&\quad \{A_1\cos(\sqrt{2}\omega_0 t_2+\theta_1) + A_2\cos(\sqrt{2}\omega_0 t_2+\theta_2)\} \\
&= E[A_1^2\cos(\omega_0 t_1+\theta_1)\cos(\sqrt{2}\omega_0 t_2+\theta_1)] \\
&\quad +E[A_2A_1\cos(\omega_0 t_1+\theta_2)\cos(\sqrt{2}\omega_0 t_2+\theta_1)] \\
&\quad +E[A_1A_2\cos(\omega_0 t_1+\theta_1)\cos(\sqrt{2}\omega_0 t_2+\theta_2)] \\
&\quad +E[A_2^2\cos(\omega_0 t_1+\theta_2)\cos(\sqrt{2}\omega_0 t_2+\theta_2)] \\
&= \frac{1}{2}E[A_1^2]\cos(\omega_0 t_1-\sqrt{2}\omega_0 t_2) + \frac{1}{2}E[A_2^2]\cos(\omega_0 t_1-\sqrt{2}\omega_0 t_2) \\
&= \frac{1}{2}\{E[A_1^2]+E[A_2^2]\}\cos\omega_0(t_1-\sqrt{2}t_2)
\end{aligned}
$$

b) If $X(t)$ were mean-square periodic then its autocorrelation would depend only on $\tau = t_2 - t_1$ and it would be periodic in τ. The $E[X(t_1)X(t_2)]$ above does not satisfy these properties, so $X(t)$ is not mean square periodic.

c) If we condition on Θ_1 and Θ_2 then $X(t_1)$ and $X(t_2)$ are defined by a linear transformation on A_1 and A_2:

$$\begin{bmatrix} X(t_1) \\ X(t_2) \end{bmatrix} = \begin{bmatrix} \cos(\omega_0 t_1+\theta_1) & \cos(\sqrt{2}\omega_0 t_1+\theta_2) \\ \cos(\omega_0 t_2+\theta_1) & \cos(\sqrt{2}\omega_0 t_2+\theta_2) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \Gamma(t_1,t_2,\theta_1,\theta_2)\underline{A}$$

Since A_1 and A_2 are jointly Gaussian, the conditional joint pdf of $X(t_1)$ and $X(t_2)$ are also jointly Gaussian with correlation matrix

$$K_X(t_1,t_2,\theta_1,\theta_2) = \Gamma(t_1,t_2,\theta_1,\theta_2)K_A\Gamma^*(t_1,t_2,\theta_1,\theta_2)$$

where K_A is the covariance matrix of $\underline{A}$. The joint pdf of $X(t_1)$ and $X(t_2)$ is then found by averaging over the random phases Θ_1 and Θ_2:

$$f_{X(t_1)X(t_2)}(x_1,x_2) = \int_0^{2\pi}\int_0^{2\pi} \frac{e^{-\frac{1}{2}\underline{x}^+(\Gamma K_A\Gamma^+)^{-1}\underline{x}}}{2\pi|\Gamma K_A\Gamma^+|^{1/2}}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}$$

where we assumed that $\underline{A}$ (and hence $\underline{X}$) is zero mean. A slightly simpler expression is obtained if we write the joint characteristic function

$$\Phi_{X(t_1)X(t_2)}(\omega_1,\omega_2) = \int_0^{2\pi}\int_0^{2\pi} e^{-\frac{1}{2}\underline{\omega}^+\Gamma K_A\Gamma^+\underline{\omega}}\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}.$$

9.137

6.100 a) Since $X(t)$ is Markov for $X(t_1) = x_1$, $X(t_2) = x_2$, $X(t_3) = x_3$, we have

$$f(x_3, x_1|x_2)\frac{f(x_1, x_2)f(x_3|x_2, x_1)}{f(x_2)} = f(x_3|x_2)f(x_1|x_3)$$

$$\begin{aligned}
C_X(t_3, t_1) &= E[X_3X_1] - m_3m_1 \\
&= E[E[X_3X_1|X_2]] - m_3m_1 \\
&= E[E[X_3|X_2]E[X_1|X_2]] - m_3m_1 \\
&= E\left[\left\{m_3 + \rho_{2,3}\frac{\sigma_3}{\sigma_2}(X_2 - m_2)\right\}\left\{m_1 + \rho_{1,2}\frac{\sigma_1}{\sigma_2}(X_2 - m_2)\right\}\right] - m_3m_1 \\
&= E\left[\rho_{2,3}\frac{\sigma_3}{\sigma_2}(X_2 - m_2)\rho_{1,2}\frac{\sigma_1}{\sigma_2}(X_2 - m_2)\right] \\
&= \frac{\rho_{2,3}\sigma_3\sigma_2\rho_{1,2}\sigma_1\sigma_2}{\sigma_2\sigma_2} \\
&= \frac{C_X(t_3, t_2)C_X(t_2, t_1)}{C_X(t_2, t_2)} \qquad (t_1 \le t_2 \le t_3)
\end{aligned}$$

b) Wiener Process

$$\begin{aligned}
C_X(t_3, t_2) &= \alpha t_2, \qquad C_X(t_2, t_1) = \alpha t_1, \qquad C_X(t_3, t_1) = \alpha t_1 \\
C_X(t_3, t_2) &= \frac{C_X(t_3, t_2)C_X(t_2, t_1)}{C_X(t_2, t_2)}
\end{aligned}$$

So Wiener process is Gauss-Markov.

For Ornstein-Uhlenbeck process

$$\begin{aligned}
\frac{C_X(t_3, t_2)C_X(t_2, t_1)}{C_X(t_2, t_2)} &= \frac{\sigma^2}{2\alpha}\frac{(e^{-\alpha(t_3-t_2)} - e^{-\alpha(t_3+t_2)})(e^{-\alpha(t_2-t_1)} - e^{-\alpha(t_2+t_1)})}{e^{-\alpha(t_2-t_2)} - e^{-\alpha(t_2+t_2)}} \\
&= \frac{\sigma^2}{2\alpha}\frac{e^{-\alpha(t_3-t_1)} - e^{-\alpha(t_3+2t_2-t_1)} - e^{-\alpha(t_3+t_1)} + e^{-\alpha(t_3+2t_2+t_1)}}{1 - e^{-2\alpha t_2}} \\
&= \frac{\sigma^2}{2\alpha}(e^{-\alpha(t_3-t_1)} - e^{-\alpha(t_3+t_1)})
\end{aligned}$$

So Ornstein-Uhlenbeck process is also Gauss-Markov.

9.138

~~6.101~~ **a)** $Y_{4n+1} = A_{2n+1}$, $Y_{4n+2} = A_{2n+2}$, $Y_{4n+1} = B_{2n+1}$, $Y_{4n+4} = B_{2n+2}$, $n = 0, 1, \ldots$

$$\begin{aligned}
E[Y_{4j+k}Y_{4m+n}] &= E[A_{2j+k}A_{2m+n}] = \sigma_1^2\rho_1^{|2m+n-2j-k|} \qquad 1 \le k,\ n \le 2 \\
E[Y_{4j+k}Y_{4m+n}] &= E[B_{2j+k}B_{2m+n-2}] = \sigma_1^2\rho_1^{|2m+n-2j-k|} \qquad 3 \le k,\ n \le 4 \\
E[Y_{4j+k}Y_{4m+n}] &= 0 \quad \text{otherwise}
\end{aligned}$$

b) Y_m is not WS stationary, but is cyclostationary.

c) $m = 4n + 1$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{A_{2n+1}A_{2n+2}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_1^2, \sigma_1^2, \rho_1\sigma_1^2)$$

$m = 4n + 3$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{B_{2n+1}B_{2n+2}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_2^2, \sigma_2^2, \rho_2\sigma_2^2)$$

$m = 4n + 2$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{A_{2n+2}B_{2n+1}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_1^2, \sigma_2^2, 0)$$

$m = 4n + 4$

$$f_{Y_mY_{m+1}}(y_m, y_{m+1}) = f_{B_{2n+2}A_{2n+3}}(y_m, y_{m+1}) \sim N(0, 0, \sigma_2^2, \sigma_1^2, 0)$$

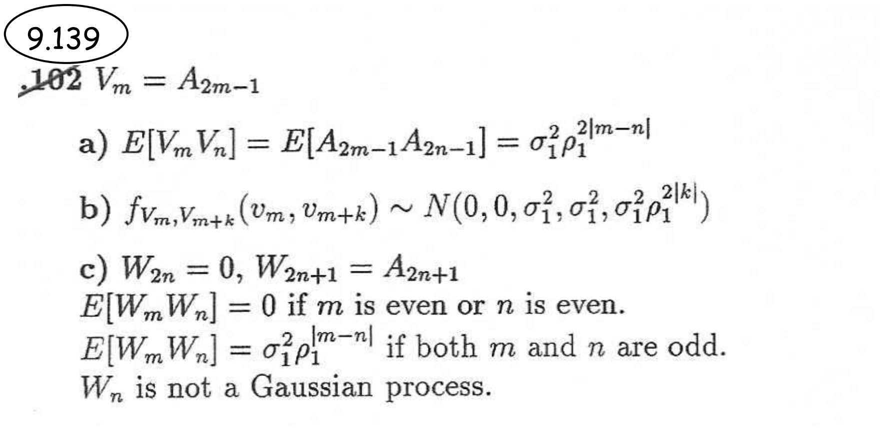
d) $Z_m = Y_{m+T}$, will "stationarize" Y_m.

$$\begin{aligned}
E[Z_mZ_n] &= E[Y_{m+T}T_{n+T}] \\
&= \sum_{T=0}^{3} E[Y_{m+T}Y_{n+T}] \cdot \frac{1}{4}
\end{aligned}$$

Z_m is stationary.

$$f_{Z_mZ_{m+1}}(z_m, z_{m+1}) = \sum_{T=0}^{3} f_{Y_{m+T}Y_{m+1+T}}(z_m, z_{m+1}) \cdot \frac{1}{4}$$

9.139



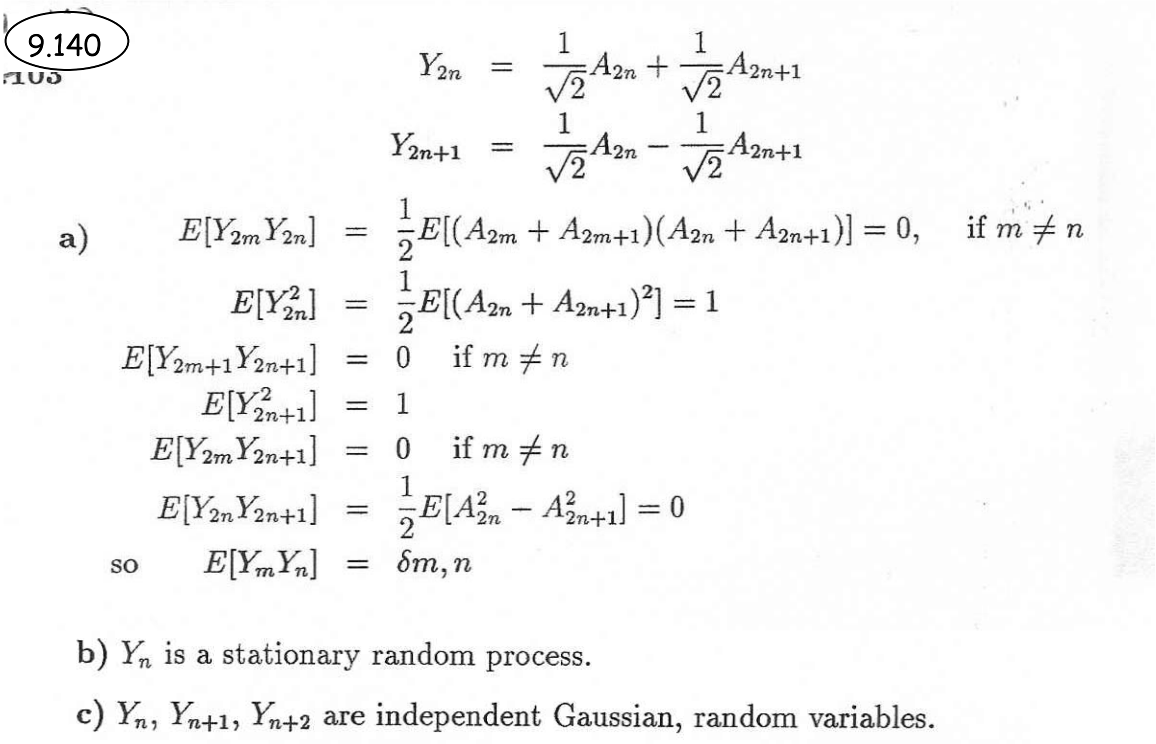
~~102~~ $V_m = A_{2m-1}$

a) $E[V_m V_n] = E[A_{2m-1}A_{2n-1}] = \sigma_1^2 \rho_1^{2|m-n|}$

b) $f_{V_m,V_{m+k}}(v_m, v_{m+k}) \sim N(0, 0, \sigma_1^2, \sigma_1^2, \sigma_1^2\rho_1^{2|k|})$

c) $W_{2n} = 0$, $W_{2n+1} = A_{2n+1}$
$E[W_m W_n] = 0$ if m is even or n is even.
$E[W_m W_n] = \sigma_1^2 \rho_1^{|m-n|}$ if both m and n are odd.
W_n is not a Gaussian process.

9.140



103

$$\begin{aligned} Y_{2n} &= \frac{1}{\sqrt{2}}A_{2n} + \frac{1}{\sqrt{2}}A_{2n+1} \\ Y_{2n+1} &= \frac{1}{\sqrt{2}}A_{2n} - \frac{1}{\sqrt{2}}A_{2n+1} \end{aligned}$$

a)

$$\begin{aligned} E[Y_{2m}Y_{2n}] &= \frac{1}{2}E[(A_{2m} + A_{2m+1})(A_{2n} + A_{2n+1})] = 0, \quad \text{if } m \neq n \\ E[Y_{2n}^2] &= \frac{1}{2}E[(A_{2n} + A_{2n+1})^2] = 1 \\ E[Y_{2m+1}Y_{2n+1}] &= 0 \quad \text{if } m \neq n \\ E[Y_{2n+1}^2] &= 1 \\ E[Y_{2m}Y_{2n+1}] &= 0 \quad \text{if } m \neq n \\ E[Y_{2n}Y_{2n+1}] &= \frac{1}{2}E[A_{2n}^2 - A_{2n+1}^2] = 0 \\ \text{so} \quad E[Y_m Y_n] &= \delta m, n \end{aligned}$$

b) Y_n is a stationary random process.

c) Y_n, Y_{n+1}, Y_{n+2} are independent Gaussian, random variables.

9.141

~~9.104~~ $N = X_1 + X_2 + \ldots + X_n$

$$P[X_i = 1] = p, \qquad P[X_i = 0] = 1 - p$$

$$S_n = Y_1 + Y_2 + \ldots + Y_n$$

Y_i is exponential if $X_i = 1$, or with probability p.
Y_i is zero if $X_i = 0$, or with probability $1 - p$.

a)

$$\begin{aligned} E[Y] &= E[E[Y|X]] \\ E[Y|X=1] &= \frac{1}{\lambda}, \qquad E[Y|X=0] = 0 \\ E[Y] &= p \cdot \frac{1}{\lambda} + (1-p)0 = p \cdot \frac{1}{\lambda} \end{aligned}$$

Similarly

$$\begin{aligned} E[Y^2] &= p \cdot \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\right) = p \cdot \frac{2}{\lambda^2} \\ E[S_n] &= nE[Y] = np\frac{1}{\lambda} \end{aligned}$$

Assume $m \leq n$

$$\begin{aligned} E[S_m S_n] &= E[(Y_1 + \ldots + Y_m)(Y_1 + \ldots + Y_m + \ldots + Y_n)] \\ &= mE[Y^2] + m(m-1)E[Y_i Y_j] + (n-m) \cdot mE[Y_k Y_l] \quad (i \neq j, \quad k \neq l) \\ &= m \cdot p\frac{1}{\lambda^2} + m(m-1) \cdot \left(p\frac{1}{\lambda^2}\right)^2 + (n-m)m\left(p\frac{1}{\lambda}\right)^2 \\ &= mp\frac{1}{\lambda^2} + m(n-1)p^2\frac{1}{\lambda^2} \end{aligned}$$

b) No, the mean square linearly with n.

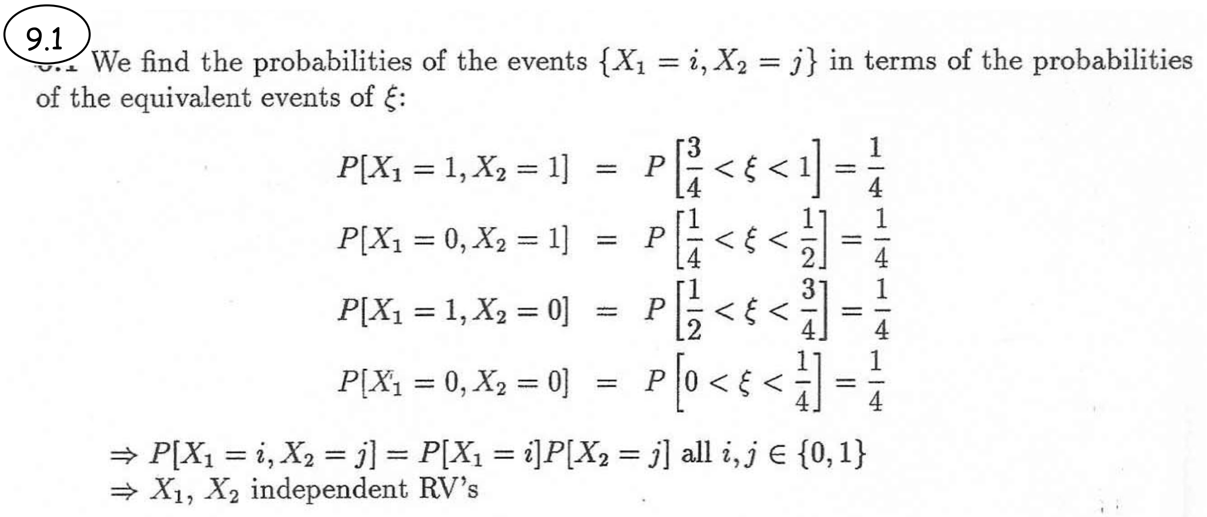
c) Yes, the process has independent increments and is therefore Markov.

d) $f_{S_n S_{n+m}}(x, y) = f_{S_n}(x) f_{S_m}(y - x)$ where

$$\begin{aligned} f_{S_n}(x) &= \sum_{k=0}^{n} f_{S_n}(x|N=k)P[N=k] \\ &= \sum_{k=0}^{n} \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!} \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Chapter 9: Random Processes – Part I

9.1 & 9.2 Definition and Specification of a Stochastic Process



9.1 We find the probabilities of the events $\{X_1 = i, X_2 = j\}$ in terms of the probabilities of the equivalent events of ξ:

$$\begin{aligned}
P[X_1 = 1, X_2 = 1] &= P\left[\frac{3}{4} < \xi < 1\right] = \frac{1}{4} \\
P[X_1 = 0, X_2 = 1] &= P\left[\frac{1}{4} < \xi < \frac{1}{2}\right] = \frac{1}{4} \\
P[X_1 = 1, X_2 = 0] &= P\left[\frac{1}{2} < \xi < \frac{3}{4}\right] = \frac{1}{4} \\
P[X_1 = 0, X_2 = 0] &= P\left[0 < \xi < \frac{1}{4}\right] = \frac{1}{4}
\end{aligned}$$

$\Rightarrow P[X_1 = i, X_2 = j] = P[X_1 = i]P[X_2 = j]$ all $i, j \in \{0, 1\}$
$\Rightarrow X_1$, X_2 independent RV’s

9.2

a) ξ is the output of fair die

if $\xi=1$: X_n: $\cdots 1111 \cdots$

$\xi=2$: X_n: $\cdots 2222 \cdots$

$\vdots$

$\xi=6$: X_n : $\cdots 6666 \cdots$

b) $P[X_n=1] = P\{k=1\} = 1/6$

$P[X_n=2] = P\{k=2\} = 1/6$

$\vdots$

$P[X_n=6] = P\{k=6\} = 1/6$

$P[X_n=k] = \emptyset \quad \forall k \notin \{1,2,3,4,5,6\}$

c) $P[X_n=1, X_{n+k}=1] = \frac{1}{6}$

$P[X_n=2, X_{n+k}=2] = 1/6$

$\vdots$

$P[X_n=6, X_{n+k}=6] = 1/6$

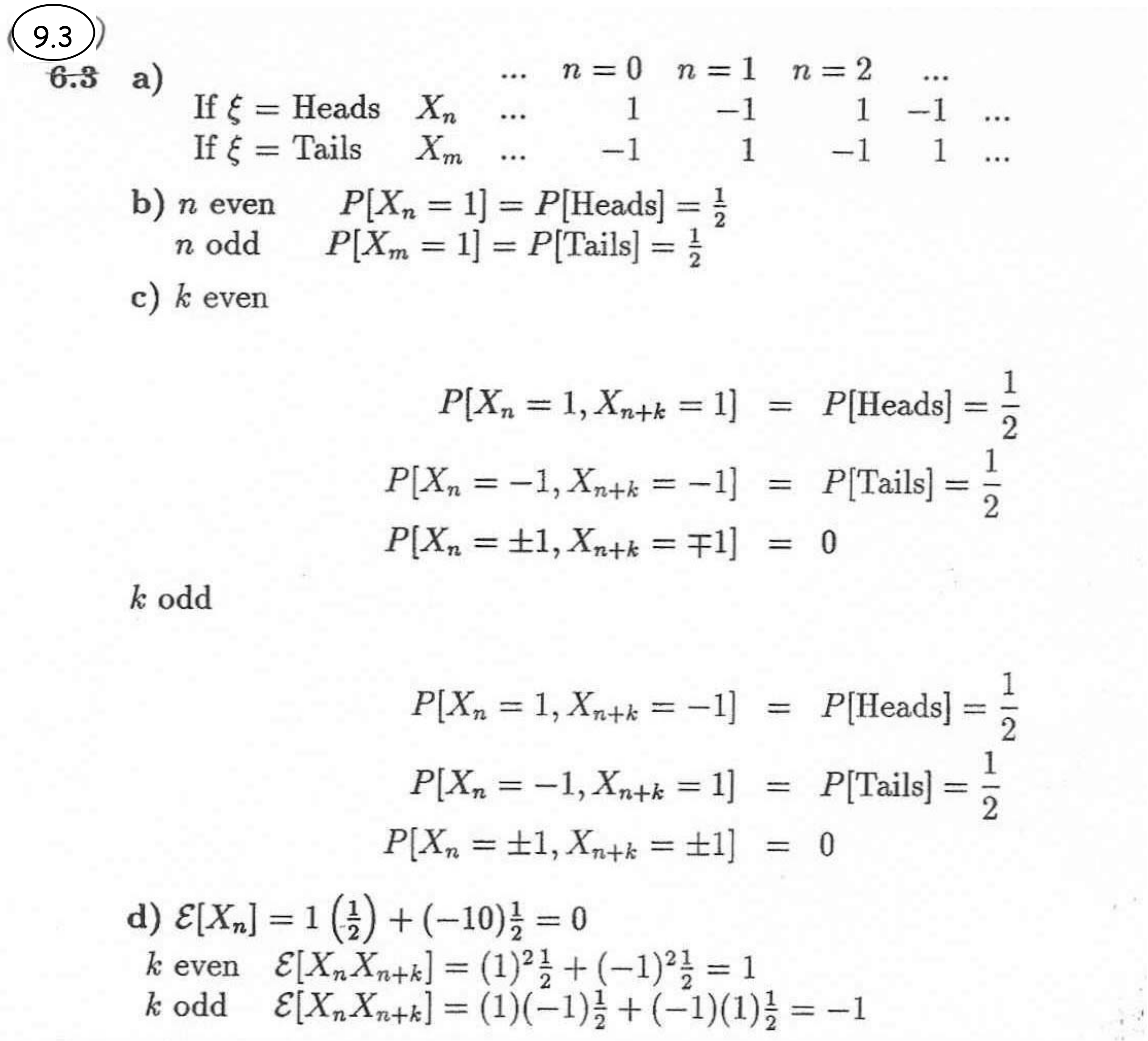
$P[X_n=k_1, X_{n+k}=k_2] = 0 \quad \forall k_1 \neq k_2$

$P[X_n=k_1, X_{n+k}=k_2] = 0 \quad \forall k_1 \notin \{1,2,3,4,5,6\}$

d) $E[X_n] = 1 \times P\{X_n=1\} + 2 \times P\{X_n=2\} + 3 \times P\{X_n=3\} + \cdots + 6 \times P\{X_n=6\} = \frac{21}{6}$

$E[X_n X_{n+k}] = 1 \times 1 \times P\{X_n=1, X_{n+k}=1\} + 2 \times 2 \times P\{X_n=2, X_{n+k}=2\} + \cdots + 6 \times 6 \times P\{X_n=6, X_{n+k}=6\} = \frac{91}{6}$

$C_X(n, n+k) = E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] = \frac{91}{6} - \frac{21}{6} \times \frac{21}{6} = 2.9167$



9.3

~~6.3~~ **a)**

$$\begin{array}{lllllll} & & \ldots & n=0 & n=1 & n=2 & \ldots \\ \text{If } \xi = \text{Heads} & X_n & \ldots & 1 & -1 & 1 & -1 \quad \ldots \\ \text{If } \xi = \text{Tails} & X_m & \ldots & -1 & 1 & -1 & 1 \quad \ldots \end{array}$$

b) n even $P[X_n = 1] = P[\text{Heads}] = \frac{1}{2}$
n odd $P[X_m = 1] = P[\text{Tails}] = \frac{1}{2}$

c) k even

$$\begin{aligned} P[X_n = 1, X_{n+k} = 1] &= P[\text{Heads}] = \frac{1}{2} \\ P[X_n = -1, X_{n+k} = -1] &= P[\text{Tails}] = \frac{1}{2} \\ P[X_n = \pm 1, X_{n+k} = \mp 1] &= 0 \end{aligned}$$

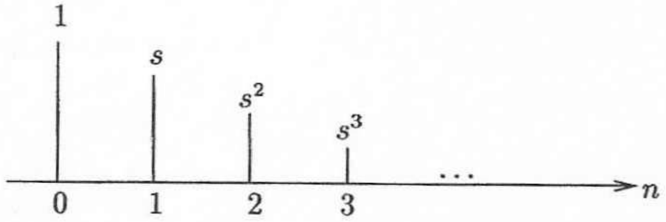
k odd

$$\begin{aligned} P[X_n = 1, X_{n+k} = -1] &= P[\text{Heads}] = \frac{1}{2} \\ P[X_n = -1, X_{n+k} = 1] &= P[\text{Tails}] = \frac{1}{2} \\ P[X_n = \pm 1, X_{n+k} = \pm 1] &= 0 \end{aligned}$$

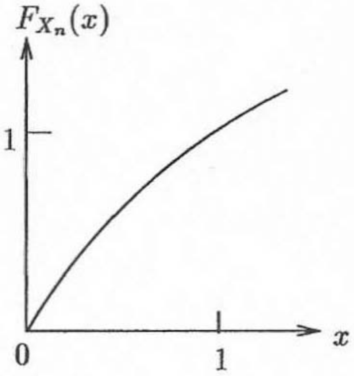
d) $\mathcal{E}[X_n] = 1\left(\frac{1}{2}\right) + (-10)\frac{1}{2} = 0$
k even $\mathcal{E}[X_n X_{n+k}] = (1)^2\frac{1}{2} + (-1)^2\frac{1}{2} = 1$
k odd $\mathcal{E}[X_n X_{n+k}] = (1)(-1)\frac{1}{2} + (-1)(1)\frac{1}{2} = -1$

9.4

~~6~~.4 a) $X_n = s^n \qquad 0 < s < 1$



b) $P[X_n \leq x] = P[s^n \leq x] = P[s \leq x^{1/n}] = x^{1/n} \qquad 0 \leq x \leq 1$



c) For $0 < x,\ y < 1$

$$\begin{aligned} P[X_n \leq x, X_{n+1} \leq y] &= P[s^n \leq x, s^{n+1} \leq y] \\ &= P[s \leq x^{1/n}, s \leq y^{1/n+1}] \\ &= P[s \leq \min(x^{1/n}, y^{1/n+1})] \\ &= \min(x^{1/n}, y^{1/n+1}) \end{aligned}$$

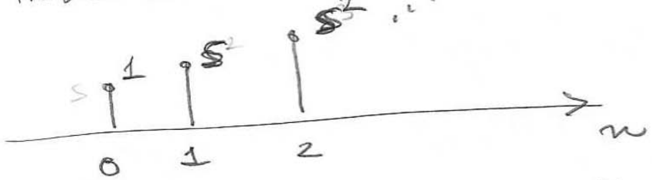
d) $\mathcal{E}[X_n] = \mathcal{E}[s^n] = \int_0^1 s^n ds = \dfrac{1}{n+1}$

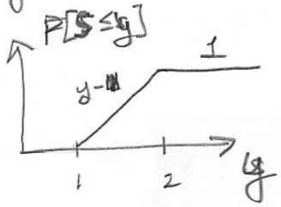
$\mathcal{E}[X_n X_{n+k}] = \mathcal{E}[s^n s^{n+k}] = \mathcal{E}[s^{2n+k}] = \frac{1}{2n+k+1}$

$C_X(n, n+k) = \frac{1}{2n+k+1} - \left(\frac{1}{n+1}\right)\left(\frac{1}{n+k+1}\right)$

— continued —

9.4 ⓔ If s is uniform in (1,2) then the sample paths increase with n





$$P[X_n \le x] = P[s^n \le x] = P[s \le x^{1/n}] \qquad 1 < x^{1/n} < 2$$

$$= x^{1/n} - 1$$

$$P[X_n \le x, X_{n+1} \le y] = P[s^n \le x, s^{n+1} \le y]$$

$$= P[s \le \min(x^{1/n}, y^{1/(n+1)})]$$

$$= \min(x^{1/n}, y^{\frac{1}{n+1}}) - 1$$

$$E[X_n] = E[s^n] = \int_1^2 s^n ds = \frac{2^{n+1} - 1}{n+1}$$

$$E[X_n X_{n+k}] = E[s^{2n+k}] = \frac{2^{2n+k+1} - 1}{2n+k+1}$$

$$C_X(n, n+k) = \frac{2^{2n+k+1} - 1}{2n+k+1} - \left(\frac{2^{n+1}-1}{n+1}\right)\left(\frac{2^{n+k+1}-1}{n+k+1}\right)$$

9.5

~~6~~.5 a) Since $g(t)$ is zero outside the interval [0,1]:

$$P[X(t) = 0] = 1 \text{ for } t \notin [0,1]$$

For $t \in [0,1]$, we have

$$P[X(t) = 1] = P[X(t) = -1] = \frac{1}{2}$$

b) $m_X(t) = \begin{cases} 1 \cdot P[X(t) = 1] + (-1)P[X(t) = -1] = 0 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$

c) For $t \in [0,1]$, $t+d \in [0,1]$, $X(t)$ must be the same value, thus:

$$\begin{aligned} P[X(t) = \pm 1, X(t+d) = \pm 1] &= \frac{1}{2} \\ P[X(t) = \pm 1, X(t+d) = \mp 1] &= 0 \end{aligned}$$

For $t \in [0,1]$, $t+d \notin [0,1]$:

$$P[X(t) = \pm 1, X(t+d) = 0] = \frac{1}{2}$$

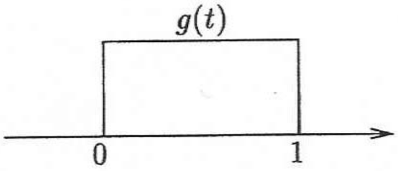
For $t \notin [0,1]$, $t+d \notin [0,1]$:

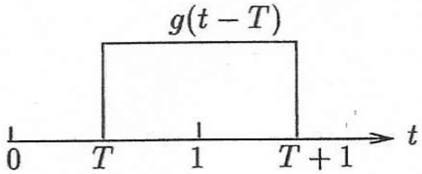
$$P[X(t) = 0, X(t+d) = 0] = 1$$

d)

$$\begin{aligned} C_X(t, t+d) &= \mathcal{E}[X(t)X(t+d)] - m_X(t)m_X(t+d) \\ &= \mathcal{E}[X(t)X(t+d)] \\ &= \begin{cases} 1 & t \in [0,1] \text{ and } t+d \in [0,1] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

9.6





a)

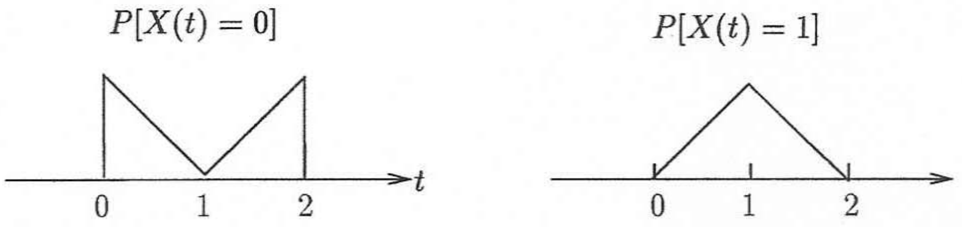
$$P[Y(t)=1] = P[g(t-T)=1]$$

For $0 < t < 1$

$$P[Y(t)=0] = P[t < T] = 1 - t = 1 - P[Y(t)=1]$$

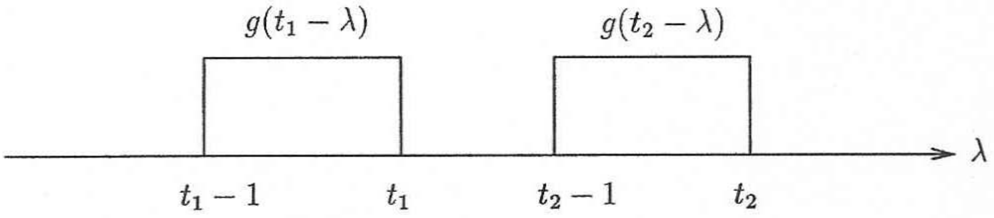
For $1 < t < 2$

$$\begin{aligned} P[Y(t)=1] &= P[t < T+1] = P[T > t-1] = 1-(t-1) \\ &= 2-t \\ P[Y(t)=0] &= 1 - P[Y(t)=1] = t-1 \end{aligned}$$



b) $\mathcal{E}[Y(t)] = 1 \cdot P[Y(t)=1] = P[Y(t)=1]$

$$\begin{aligned} E[Y(t_1)Y(t_2)] &= \int_0^1 E[g(t_1-T)g(t_2-T)|T=\lambda]f_T(\lambda)d\lambda \\ &= \int_0^1 g(t_1-\lambda)g(t_2-\lambda)d\lambda \end{aligned}$$



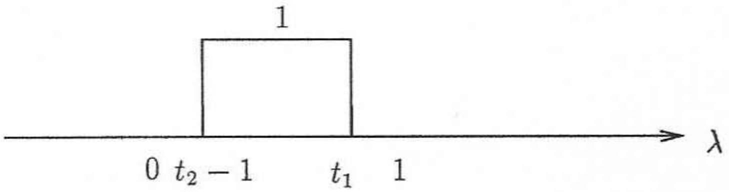
$$g(t_1-\lambda)g(t_2-\lambda) = 0 \quad \text{for} \quad t_1 < t_2 - 1$$

$$\Rightarrow \; R_Y(t_1,t_2) = 0 \quad \text{for} \quad t_2 - t_1 > 1$$

If $t_2 - 1 < t$, then

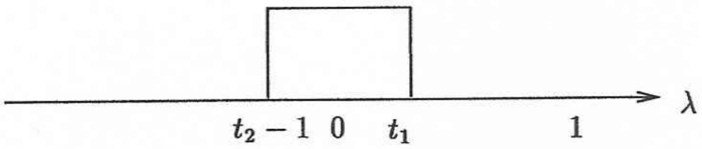
$$g(t_1-\lambda)g(t_2-\lambda) = \begin{cases} 1 & t_2 - 1 < \lambda < t_1 \\ 0 & \text{elsewhere} \end{cases}$$

Case 1



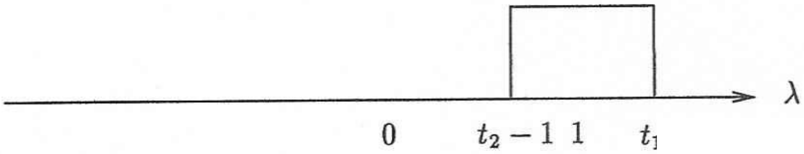
$R_Y(t_1,t_2) = t_1 - (t_2 - 1) = 1 - (t_2 - t_1) \qquad t_1 < 1, \;\; 0 < t_2 - 1, \;\; t_2 - t_1 < 1$

Case 2



$$R_Y(t_1,t_2)=t_1 \qquad t_1<1,\ \ t_2<1,\ \ t_2-t_1<1$$

Case 3



$$R_Y(t_1,t_2)=2-t_2 \qquad t_1>1,\ \ t_2<2$$

9.7 a) We will use conditional probability:

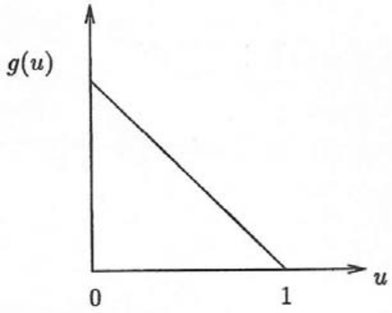
$$\begin{aligned} P[X(t) \le x] &= P[g(t-T) \le x] \\ &= \int_0^1 P[g(t-T) \le x | T = \lambda] f_T(\lambda) d\lambda \\ &= \int_0^1 P[g(t-\lambda) \le x] d\lambda \quad \text{since } f_T(\lambda) = 1 \\ &= \int_{t-1}^{t} P[g(u) \le x] du \quad \text{after letting } u = t - \lambda \end{aligned}$$

$g(u)$ (and hence $P[g(u) \le x]$) is a periodic function of u with period, so we can change the limits of the above integral to any full period. Thus

$$P[X(t) \le x] = \int_0^1 P[g(u) \le x] du$$

Note that $g(u)$ is deterministic, so

$$P[g(u) \le x] = \begin{cases} 1 & u : g(u) \le x \\ 0 & u : g(u) > x \end{cases}$$



So finally

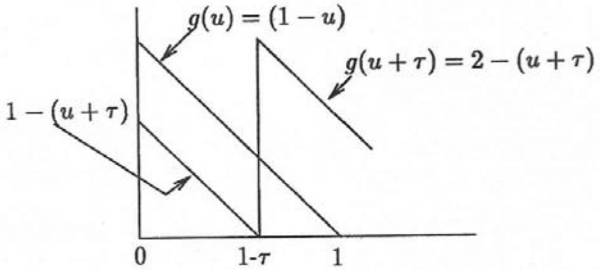
$$P[X(t) \le x] = \int_{u:g(u)\le x} 1\ du = \int_{1-X}^{1} 1\ du = x\ .$$

b) $m_X(t) = E[X(t)] = \int_0^1 x\ dx = \frac{1}{2}$.
The correlation is again found using conditioning on T:

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_0^1 E[g(t-T)g(t+\tau-T)|T=\lambda]f_T(\lambda)d\lambda \\ &= \int_0^1 g(t-\lambda)g(t+\tau-\lambda)d\lambda \\ &= \int_{t-1}^{t} g(u)g(u+\tau)du \end{aligned}$$

$g(u)g(u+\tau)$ is a periodic function in u so we can change the limits to (0,1):

$$E[X(t)X(t+\tau)] = \int_0^1 g(u)g(u+\tau)du$$



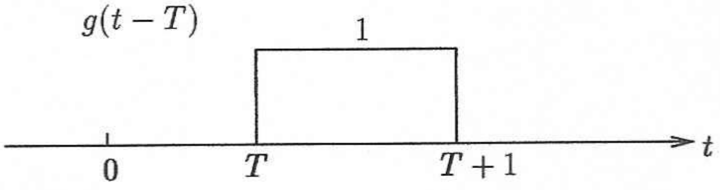
here we assume $0 < \tau < 1$ since $E[X(t)X(t+\tau)]$ is periodic in τ.

$$\begin{aligned} E[X(t)X(t+\tau)] &= \int_0^{1-\tau}(1-u)(1-u-\tau)du + \int_{1-\tau}^{1}(1-u)(2-u-\tau)du \\ &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^3}{6} + \frac{\tau^2}{2} - \frac{\tau^3}{6} \\ &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^2}{2}\ . \end{aligned}$$

Thus

$$\begin{aligned} C_X(t,t+\tau) &= \frac{1}{3} - \frac{\tau}{2} + \frac{\tau^2}{2} - \frac{1}{4} \\ &= \frac{1}{12} - \frac{\tau}{2} + \frac{\tau^2}{2} \end{aligned}$$

9.8

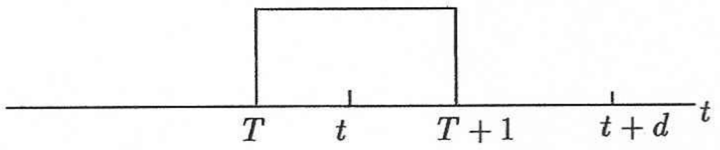


a)

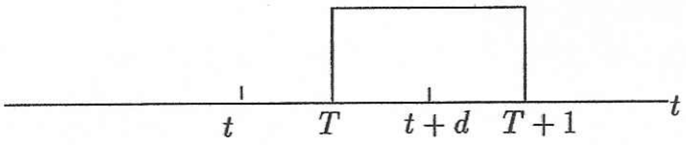
$$\begin{aligned} P[Y(t)=1] &= P[T<t<T+1] \\ &= P[t-1<T<t] = e^{-\alpha(t-1)} - e^{-\alpha t} \\ P[Y(t)=0] &= 1 - P[Y(t)=1] \end{aligned}$$

b) Case 1 $d>1$

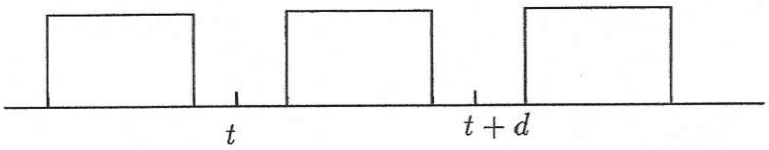
$$P[Y(t)=1, Y(t+d)=1] = 0$$



$$\begin{aligned}
&P[Y(t) = 1, Y(t+d) = 0] \\
&\quad = P[\{T < t < T+1\} \cap \{T+1 < t+d\}] \\
&\quad = P[\{t-1 < T < t\} \cap \{T < t + (d-1)\}] \\
&\quad = P[\{t-1 < T < t\}] = e^{-\alpha(t-1)} e^{-\alpha t}
\end{aligned}$$



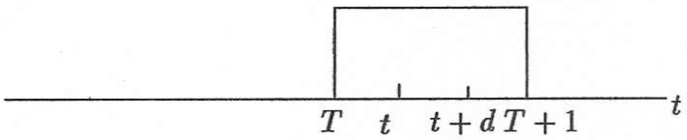
$$\begin{aligned}
&P[Y(t) = 0, Y(t+d) = 1] \\
&\quad = P[\{T > t\} \cap \{T < t+d < T+1\}] \\
&\quad = P[\{T > t\} \cap \{t+d-1 < T < t+d\}] \\
&\quad = P[t+d-1 < T < t+d] \\
&\quad = e^{-\alpha(t+d-1)} - e^{-\alpha(t+d)}
\end{aligned}$$



$$\begin{aligned}
&P[Y(t) = 0, Y(t+d) = 0] \\
&\quad = P[\{T+1 < t\} \cup \{T > t+d\}] \\
&\qquad \cup \{\{t < T\} \cap \{T+1 < t+d\}\}
\end{aligned}$$

$$\begin{aligned}
&P[Y(t) = 0, Y(t+d) = 0] \\
&\quad = P[T \le t-1] + P[T \ge t+d] + P[t \le T \le t+d-1] \\
&\quad = 1 - e^{-\alpha(t-1)} + e^{-\alpha(t+d)} \\
&\qquad + e^{-\alpha t} - e^{-\alpha(t+d-1)}
\end{aligned}$$

Case 2: $0 < d < 1$



$$\begin{aligned}&P[Y(t)=1, Y(t+d)=1]\\ &\quad = P[\{T<t\}\cap\{t+d<T+1\}]\\ &\quad = P[\{T<t\}\cap\{t+d-1<T\}] = P[t+d-1<T<t]\\ &\quad = e^{-\alpha(t+d-1)} - e^{-\alpha t}\end{aligned}$$

$$\begin{aligned}&P[Y(t)=1, Y(t+d)=0]\\ &\quad = P[\{t-1<T<t\}\cap\{T<t+(d-1)\}] \text{ as in Case 1}\\ &\quad = P[\{t-1<T<t+d-1\}]\\ &\quad = e^{-\alpha(t-1)} - e^{-\alpha(t+d-1)}\end{aligned}$$

$$\begin{aligned}&P[Y(t)=0, Y(t+d)=1]\\ &\quad = P[\{T>t\}\cap\{t+d-1<T<t+d\}] \text{ as in Case 1}\\ &\quad = P[t<T<t+d]\\ &\quad = e^{-\alpha t} - e^{-\alpha(t+d)}\end{aligned}$$

$$\begin{aligned}&P[Y(t)=0, Y(t+d)=0]\\ &\quad = P[\{T+1<t\}\cup\{T>t+d\}]\\ &\quad = P[T<t-1] + P[T>t+d]\\ &\quad = 1 - e^{-\alpha(t-1)} + e^{-\alpha(t+d)}\end{aligned}$$

c)

$$\begin{aligned}m_Y(t) &= 1\cdot P[Y(t)=1] + 0P[Y(t)=0]\\ &= e^{-\alpha(t-1)} - e^{-\alpha t}\end{aligned}$$

$$\begin{aligned}\mathcal{E}[Y(t)Y(t+d)] &= 1\cdot 1\cdot P[Y(t)=1, Y(t+d)=1] \\ &= \begin{cases} e^{-\alpha(t+d-1)} - e^{-\alpha t} & 0<d<1 \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

$$\begin{aligned}C_Y(t,t+d) &= \mathcal{E}[Y(t)Y(t+d)] - m_Y(t)m_Y(t+d) \\ &= e^{-\alpha(t+d-1)} - e^{-\alpha t} \\ &\quad -(e^{-\alpha(t-1)} - e^{-\alpha t})(e^{-\alpha(t+d-1)} - e^{-\alpha(t+d)})\end{aligned}$$

9.9 $Z(t) = At^3 + B \qquad Z(t) = B \text{ if } t = 0$

a)

$$\begin{aligned}f_{Z(t)}(z)dz &= P[z < Z(t) \le z + dz] \text{ assume } t \neq 0 \\ &= P[z < At^3 + B \le z + dz] \\ &= \int_{-\infty}^{\infty} P[z < At^3 + b \le z + dz] f_B(b)db \\ &= \int_{-\infty}^{\infty} P\left[\frac{z-b}{t^3} < A \le \frac{z-b}{t^3} + \frac{dz}{t^3}\right] f_B(b)db \\ &= \int_{-\infty}^{\infty} f_A\left(\frac{z-b}{t^3}\right)\frac{dz}{|t^3|} f_B(b)db \\ \therefore f_{Z(t)}(z) &= \int_{-\infty}^{\infty} \frac{1}{|t^3|} f_A\left(\frac{z-b}{t^3}\right) f_B(b)db\end{aligned}$$

This is the convolution of the pdf of At^3 and B.

b)

$$\begin{aligned}\mathcal{E}[Z(t)] &= \mathcal{E}[At^3 + B] \\ &= \mathcal{E}[A]t^3 + \mathcal{E}[B]\end{aligned}$$

$$\mathcal{E}[Z(t_1)Z(t_2)] = \mathcal{E}[(At_1^3 + B)(At_2^3 + B)] = \mathcal{E}[A^2]t_1^3t_2^3 + \mathcal{E}[AB](t_1^3 + t_2^3) + \mathcal{E}[B^2]$$

9.10

$$\begin{aligned}\mathcal{E}[|X_{t_2} - X_{t_1}|^2] &= \mathcal{E}[(X_{t_2} - X_{t_1})^2] \\ &= \mathcal{E}[X_{t_2}^2 - 2X_{t_2}X_{t_1} + X_{t_1}^2] \\ &= \mathcal{E}[X_{t_2}^2] - 2\mathcal{E}[X_{t_2}X_{t_2}] + \mathcal{E}[X_{t_1}^2] \\ &= R_X(t_2,t_2) - 2R_X(t_2,t_1) + R_X(t_1,t_1)\end{aligned}$$

9.11

~~6.10~~ a) $P[H(t)=1] = P[X(t) \geq 0] = P[\xi \cos 2\pi t \geq 0] = \frac{1}{2} = P[H(t)=-1]$

$$\begin{aligned}
\mathcal{E}[H(t)] &= 1 \cdot P[H(t)=1] + (-1)P[H(t)=-1] = 0 \\
C_H(t,t+\tau) &= \mathcal{E}[H(t)H(t+\tau)] \\
&= 1 \cdot P[\underbrace{H(t)H(t+\tau)=1}_{\substack{H(t)\ \&\ H(t+\tau) \\ \text{same sign}}}] + (-1)P[\underbrace{H(t)H(t+\tau)=-1}_{\substack{H(t)\ \&\ H(t+\tau) \\ \text{opposite sign}}}]
\end{aligned}$$

$$\begin{aligned}
H(t)H(t+\tau) &= 1 \Leftrightarrow \cos 2\pi t \text{ and } \cos 2\pi(t+\tau) \text{ have same sign} \\
H(t)H(t+\tau) &= -1 \Leftrightarrow \cos 2\pi t \text{ and } \cos 2\pi(t+\tau) \text{ have different sign}
\end{aligned}$$

$$\therefore C_H(t,t+\tau) = \begin{cases} 1 & \text{for } t,\tau \text{ such that } \cos 2\pi t \cos 2\pi(t+\tau) = 1 \\ -1 & \text{for } t,\tau \text{ such that } \cos 2\pi t \cos 2\pi(t+\tau) = -1 \end{cases}$$

b) $P[H(t)=1] = P[X(t) \geq 0] = P[\cos(\omega t + \Theta) \geq 0] = \frac{1}{2} = P[H(t)=-1]$

$$\begin{aligned}
\mathcal{E}[H(t)] &= 1\left(\frac{1}{2}\right) + (-1)\frac{1}{2} = 0 \\
\mathcal{E}[H(t)H(t+\tau)] &= 1 \cdot \underbrace{P[X(t)X(t+\tau)>0]}_{1-P[X(t)X(t+\tau)<0]} + (-1)P[X(t)X(t+\tau)<0] \\
&= 1 - 2P[X(t)X(t+\tau)<0]
\end{aligned}$$

$$\begin{aligned}
P[X(t)X(t+\tau)<0] &= P[\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta) < 0] \\
&= \left[\frac{1}{2}\cos\omega\tau + \frac{1}{2}\cos(2\omega t + \omega\tau + 2\Theta) < 0\right] \\
&= P[\cos(2\omega t + \omega\tau + 2\Theta) < \cos\omega\tau] \\
&= 1 - \frac{\text{shaded region in figure}}{2\pi}
\end{aligned}$$

c) $P[H(t)=1] = P[X(t) \geq 0] = 1 - F_{X(t)}(0^-) = 1 - P[H(t)=-1]$

$$\begin{aligned}
\mathcal{E}[H(t)] &= 1 \cdot P[H(t)=1] + (-1)P[H(t)=-1] \\
&= 1 - F_{X(t)}(0^-) - F_{X(t)}(0^-) \\
&= 1 - 2F_{X(t)}(0^-)
\end{aligned}$$

9.12

ⓐ X(t) and Y(t) independent implies X(t) and Y(t) uncorrelated

Since $\text{cov}(X(t_1), Y(t_2)) = E[X(t_1)Y(t_2)] - E[X(t_1)]E[Y(t_2)]$

$= E[X(t_1)]E[Y(t_2)] - E[X(t_1)]E[Y(t_2)]$

$= 0$

ⓓ If X(t) and Y(t) are uncorrelated then

$E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)]$

X(t) and Y(t) will be orthogonal if X(t) and/or Y(t) is zero mean.

ⓑ If X(t) and Y(t) are orthogonal, then

$E[X(t_1)Y(t_2)] = 0$, so

$\text{cov}(X(t_1), Y(t_2)) = E[X(t_1)]E[Y(t_2)]$

then X(t) and Y(t) will be uncorrelated if either X(t) and/or Y(t) are zero mean.

ⓒ Uncorrelated random variables are not necessarily independent and so uncorrelated random processes are not necessarily independent.

9.13

$$E[Z(t)] = E[2Xt - Y] = 2E[X]t - E[Y]$$

$$= 2tm_X - m_Y$$

$$\triangleq m_Z(t)$$

$$C_Z(t_1,t_2) = E[(2Xt_1 - Y)(2Xt_2 - Y)] - m_Z(t_1)m_Z(t_2)$$

$$= 4t_1t_2E[X^2] - 2(t_1+t_2)E[XY] + E[Y^2]$$

$$- 4t_1t_2m_X^2 + 2(t_1+t_2)m_Xm_Y - m_Y^2$$

$$= 4t_1t_2\sigma_X^2 - 2(t_1+t_2)\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2$$

$$\sigma_{Z(t)}^2 = C_Z(t,t) = 4t^2\sigma_X^2 - 4t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2$$

$$f_{Z(t)}(z) = \frac{\exp\left\{-\dfrac{(z - 2tm_X + m_Y)^2}{2(4t^2\sigma_X - 4t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2)}\right\}}{\sqrt{2\pi(4t^2\sigma_X - 4t\sigma_X\sigma_Y\rho_{XY} + \sigma_Y^2}}$$

9.14

a) $E[H(t)X(t)] = E[|X(t)|]$ since $H(t)X(t) = \begin{cases} X(t), & X(t) \geq 0 \\ -X(t), & X(t) < 0 \end{cases}$

$E[|X(t)|] = E[|\xi \cos(2\pi t)|] = E[|\xi||\cos(2\pi t)|] = |\cos 2\pi t| E[|\xi|] = \frac{1}{2}|\cos 2\pi t|$

My cos is cos, My sin is sin ← Note!

$E[H(t)X(t)] - E[H(t)]E[X(t)] = E[|X(t)|] - 0 \times E[X(t)] = E[|X(t)|] = \frac{1}{2}|\cos 2\pi t|$

Not Uncorrelated, Not Orthogonal

b)

Again: $E[H(t)X(t)] = E[|X(t)|]$

$E[H(t)] = +1\,P\{X(t) > 0\} - 1 \times P\{X(t) < 0\} = 0$

$C_V(X,H) = E[X(t)H(t)] - E[X(t)]E[H(t)] = E[|X(t)|]$

$E[|X(t)|] = E[|\cos(2\pi t + \xi)|] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\cos(2\pi t + \xi)|\, d\xi = \frac{2}{\pi}$

So $C_V(X,H) = \frac{2}{\pi}$

Not Uncorrelated, Not Orthogonal

9.15

$Y_n = X_n + g(n)$

a) $E[Y_n] = E[X_n] + g(n)$

$VAR[Y_n] = VAR[X_n + g(n)] = VAR[X_n]$

b) $F_{Y_n}(x) = P[Y_n < x] = P[(X_n + g(n)) < x] = P[X_n < x - g(n)] = F_{X_n}(x - g(n))$

$F_{Y_n, Y_{n+1}}(x_1, x_2) = P[Y_n < x_1, Y_{n+1} < x_2] = P[X_n < x_1 - g(n), X_{n+1} < x_2 - g(n+1)]$

$= F_{X_n, X_{n+1}}(x_1 - g(n), x_2 - g(n+1))$

c) $R_Y(n_1, n_2) = E[Y_{n_1} Y_{n_2}] = E[(X_{n_1} + g(n_1))(X_{n_2} + g(n_2)]$

$= E[X_{n_1} X_{n_2}] + g(n_1) E[X_{n_2}] + g(n_2) E[X_{n_1}] + g(n_1) g(n_2)$

$C_Y(n_1, n_2) = R_Y(n_1, n_2) - E[Y_{n_1}] E[Y_{n_2}] = E[X_{n_1} X_{n_2}] - E[X_{n_1}] E[X_{n_2}]$

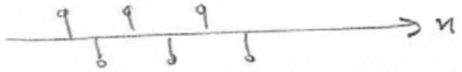
$= C_X(n_1, n_2)$

d) Based on X_n, Y_n can be easily plotted

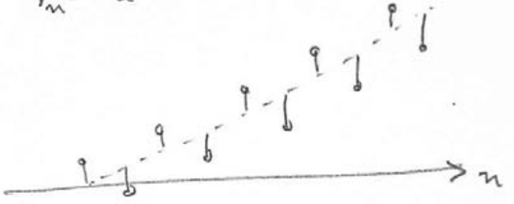
— continued —

9.15 d

X_n

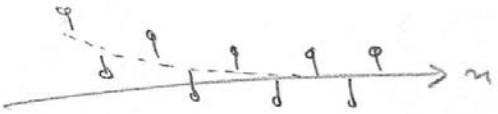


$Y_n = X_n + n$



$Y_n = X_n + 1/n^2$

$Y = X_n + \frac{1}{n}$
$\Rightarrow$ similar



9.16

a) $Y_n = c(n) X_n$

$$E[Y_n] = c(n)\, E[X_n]$$

$$VAR[Y_n] = VAR[c(n)X_n] = E\left[c^2(n)\left(E[X_n^2] - E^2[X_n]\right)\right] = c^2(n)\, VAR(X_n]$$

b)
$$F_{Y_n}(x) = P\{Y_n < x\} = P\{c(n)X_n < x\} = \begin{cases} P\{X_n < \frac{x}{c(n)}\} = F_{X_n}\left(\frac{x}{c(n)}\right) & \text{if } c(n) > 0 \\ P\{X_n > \frac{x}{c(n)}\} = 1 - F_{X_n}\left(\frac{x}{c(n)}\right) & \text{if } c(n) < 0 \\ u(x) & \text{if } c(n) = 0 \end{cases}$$

$$F_{Y_n,Y_{n+1}}(x_1, x_2) = P\{Y_n < x_1, Y_{n+1} < x_2\} = P\{c(n)X_n < x_1, c(n+1)X_{n+1} < x_2\}$$

$$= \begin{cases} F_{X_n,X_{n+1}}\left(\frac{x_1}{c(n)}, \frac{x_2}{c(n+1)}\right) & \text{if } c(n), c(n+1) > 0 \\ P\{X_n > \frac{x_1}{c(n)}, X_{n+1} < \frac{x_2}{c(n+1)}\} = F_{X_{n+1}}\left(\frac{x_2}{c(n+1)}\right) - F_{X_n,X_{n+1}}\left(\frac{x_1}{c(n)}, \frac{x_2}{c(n+1)}\right) & \text{if } c(n+1) > 0 \;\&\; c(n) < 0 \\ P\{X_n < \frac{x_1}{c(n)}, X_{n+1} > \frac{x_2}{c(n+1)}\} = F_{X_n}\left(\frac{x_1}{c(n)}\right) - F_{X_n,X_{n+1}}\left(\frac{x_1}{c(n)}, \frac{x_2}{c(n+1)}\right) & \text{if } c(n+1) < 0,\ c(n) > 0 \\ P\{X_n > \frac{x_1}{c(n)}, X_{n+1} > \frac{x_2}{c(n+1)}\} = 1 - F_{X_n}\left(\frac{x_1}{c(n)}\right) - F_{X_{n+1}}\left(\frac{x_2}{c(n+2)}\right) + F_{X_n,X_{n+1}}\left(\frac{x_1}{c(n)}, \frac{x_2}{c(n+2)}\right) & \text{if } c(n) \;\&\; c(n+1) < 0 \end{cases}$$

P9.16)

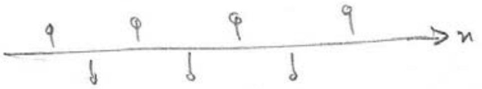
c) $R_Y(n_1, n_2) = E[Y_{n_1} Y_{n_2}] = c(n_1)c(n_2)\,E[X_{n_1} X_{n_2}] = c(n_1)c(n_2)R_X(n_1, n_2)$

$$C_Y(n_1, n_2) = c(n_1)c(n_2)R_X(n_1, n_2) - c(n_1)c(n_2)E[X_{n_1}]E[X_{n_2}]$$

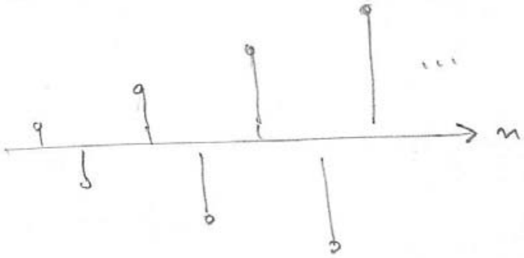
$$= c(n_1)c(n_2)C_X(n_1, n_2)$$

d) Based on X_n, we can plot Y_n

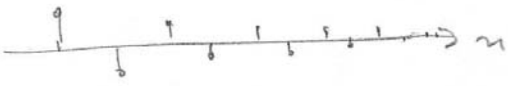
X_n



$Y_n = n X_n$



$Y_n = \frac{1}{n^2} X_n$ $\qquad$ $Y_n = \frac{1}{n} X_n$ similar



9.17

a) $R_{X,Y}(n_1, n_2) = E[X_{n_1} Y_{n_2}] = E[X_{n_1}(X_{n_2} + g(n_2))] = E[X_{n_1} X_{n_2}] + g(n_2) E[X_{n_1}$

$$= R_X(n_1, n_2) + g(n_2) E[X_{n_1}]$$

$$C_{X,Y}(n_1, n_2) = R_{X,Y}(n_1, n_2) - E[X_{n_1}]E[Y_{n_2}] = R_X(n_1, n_2) + g(n_2) E[X_{n_1}]$$

$$- E[X_{n_1}]E[X_{n_2}] - g(n_2) E[X_{n_1}]$$

$$= R_X(n_1, n_2) - E[X_{n_1}]E[X_{n_2}] = C_X(n_1, n_2)$$

b) $P\{X_n < x, Y_{n+1} < y\} = P\{X_n < x, X_{n+1} < y - g(n+1)\}$

$$= F_{X_n, X_{n+1}}(x, y - g(n+1))$$

---

c). X_n and Y_n are not independent since $Y_n = X_n + g(n)$.

X_n and Y_n are orthogonal if $R_{XY}(t_1, t_2) = 0$ all t_1, t_2

but $R_{XY}(n,n) = \underbrace{R_X(n,n)}_{\geq 0} + g(n) E[X_n]$

so X_n and Y_n are not orthogonal (except in very contrived cases).

Similarly X_n and Y_n are not uncorrelated.

9.18

a) $R_{X,Y}(n_1,n_2) = E[X_{n_1}Y_{n_2}] = c(n_2)R_X(n_1,n_2)$

$C_{X,Y}(n_1,n_2) = c(n_2)C_X(n_1,n_2)$

b) $$P[X_n < x, Y_{n+1} < y] = \begin{cases} F_{X_n,X_{n+1}}\left(x, \frac{y}{c(n+1)}\right) & \text{if } c(n+1) > 0 \\ F_{X_n}(x) - F_{X_n,X_{n+1}}\left(x, \frac{y}{c(n+1)}\right) & \text{if } c(n+1) < 0 \end{cases}$$

c). X_n and Y_n are orthogonal if

$R_{X,Y}(n_1,n_2) = 0$ for all n_1, n_2

$\Leftrightarrow$ $c(n_2)R_X(n_1,n_2) = 0$ all n_1, n_2

However in general $R_X(n,n) > 0$ and $c(n) > 0$ for at least some n $\Rightarrow$ X_n and Y_n are not orthogonal

Similarly X_n and Y_n are not uncorrelated

To test independence consider

$$\begin{aligned} P[X_n < x, Y_n < y] &= P[X_n < x, c(n)X_n < y] \quad \text{assuming } c(n) > 0 \\ &= P\left[X_n < x, X_n < \frac{y}{c(n)}\right] \\ &= P\left[X_n < \min\left(x, \frac{y}{c(n)}\right)\right] \\ &\neq P[X_n < x]P[Y_n < y]. \end{aligned}$$

X_n and Y_n are not independent.

9.19 $U(t) = X(t) - Y(t)$, $V(t) = X(t) + Y(t)$

a) $C_{UX}(t_1,t_2) = E[U(t_1)X(t_2)] - E[U(t_1)]E[X(t_2)]$

$= E[X(t_1)X(t_2) - Y(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)] + E[Y(t_1)]E[X(t_2)]$

$= C_X(t_1,t_2) - C_{YX}(t_1,t_2) = C_X(t_1,t_2)$ (X&Y are ind.)

$C_{UY}(t_1,t_2) = E[U(t_1)Y(t_2)] - E[U(t_1)]E[Y(t_2)]$

$= E[X(t_1)Y(t_2) - Y(t_1)Y(t_2)] - E[X(t_1)]E[Y(t_2)] + E[Y(t_1)]E[Y(t_2)]$

$= C_{XY}(t_1,t_2) - C_Y(t_1,t_2) = -C_Y(t_1,t_2)$

$C_{UV}(t_1,t_2) = E[(X(t_1)-Y(t_1))(X(t_2)+Y(t_2))] - E[X(t_1)-Y(t_1)]E[X(t_2)+Y(t_2)]$

$= C_X(t_1,t_2) + C_{XY}(t_1,t_2) - C_{YX}(t_1,t_2) - C_Y(t_1,t_2)$

$= C_X(t_1,t_2) - C_Y(t_1,t_2)$

P9.19)

b) for $f_{U(t_1)X(t_2)}(u,x)$ we define two aux. variables W & Z:

$$U(t_1) = X(t_1) - Y(t_1)$$
$$W(t_2) = X(t_2)$$
$$Z(t_1) = Y(t_1)$$

$$, \begin{bmatrix} U(t_1) \\ W(t_2) \\ Z(t_1) \end{bmatrix} = A \begin{bmatrix} X(t_1) \\ Y(t_1) \\ X(t_2) \end{bmatrix}, A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \det(A) = -1$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$f_{U(t_1)W(t_2)Z(t_1)}(u,w,z) = f_{X(t_1)Y(t_1)X(t_2)}(u+z, z, w) = f_{Y(t_1)}(z)\, f_{X(t_1)X(t_2)}(u+z, w$$

$$f_{U(t_1)X(t_2)}(u,x) = \int_{-\infty}^{+\infty} f_{U(t_1)W(t_2)Z(t_1)}(u, \cancel{x}, z)\,dz = \int_{-\infty}^{+\infty} f_{Y(t_1)}(z)\, f_{X(t_1)X(t_2)}(u+z, \cancel{x})\,dz$$

for $f_{U(t_1)V(t_2)}(u,v)$ we need to define two aux. variables as well:

$$U(t_1) = X(t_1) - Y(t_1)$$
$$V(t_2) = X(t_2) + Y(t_2)$$
$$W(t_1) = Y(t_1)$$
$$Z(t_2) = Y(t_2)$$

$$, \text{Trans. Matrix, } A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\det(A) = -1$$

Therefore $f_{U(t_1)V(t_2)}(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{UVWZ}(u,v,w,z)\,dw\,dz$

$$= \iint_{-\infty}^{+\infty} f_{X(t_1)Y(t_1)X(t_2)Y(t_2)}(u+w, w, v-z, z)\,dw\,dz$$

$$= \iint_{-\infty}^{+\infty} f_{X(t_1)X(t_2)}(u+w, v-z)\, f_{Y(t_1)Y(t_2)}(w,z)\,dw\,dz$$

9.20

a) $$C_{UX}(n_1,n_2) = E\left[U(n_1)X(n_2)\right] - E\left[U(n_1)\right]E\left[X(n_2)\right]$$

$$= C_X(n_1,n_2) - C_{YX}(n_1,n_2) = C_X(n_1,n_2) = \begin{cases} \sigma_X^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$$

$$C_{UY}(n_1,n_2) = E\left[U(n_1)Y(n_2)\right] - E\left[U(n_1)\right]E\left[Y(n_2)\right]$$

$$= C_{XY}(n_1,n_2) - C_Y(n_1,n_2) = -C_Y(n_1,n_2) = \begin{cases} -\sigma_Y^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$$

$$C_{UV}(n_1,n_2) = E\left[U(n_1)V(n_2)\right] - E\left[U(n_1)\right]E\left[V(n_2)\right]$$

$$= C_X(n_1,n_2) + C_{XY}(n_1,n_2) - C_{YX}(n_1,n_2) - C_Y(n_1,n_2)$$

$$= C_X(n_1,n_2) - C_Y(n_1,n_2)$$

since X & Y are different iid random processes:

$$C_{UV}(n_1,n_2) = \begin{cases} \sigma_X^2 - \sigma_Y^2 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$$

b) $$f_{U(t_1)X(t_2)}(u,x) = \int_{-\infty}^{+\infty} f_Y(z)\, f_X(u+z)\, f_X(x)\, dz =$$

$$= f_X(x) \cdot f_X(-u) * f_Y(u) \quad \text{, if } t_1 \neq t_2$$

$$f_{U(t_1)V(t_2)}(u,v) = \iint_{-\infty}^{+\infty} f_X(u+w)\, f_X(v-z)\, f_Y(w)\, f_Y(z)\, dw\, dz$$

$$= f_X(v) * f_Y(v) \cdot f_X(-u) * f_Y(u) \qquad \text{if } t_1 \neq t_2$$

if $t_1 = t_2$, $$f_{U(t_1)X(t_1)}(u,x) = f_{XY}(x, x+u) = f_X(x)\, f_Y(x+u)$$

$$f_{U(t_1)X(t_2)}(u,v) = f_{XY}\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(v-u)\right) = f_X\left(\tfrac{1}{2}(u+v)\right) f_Y\left(\tfrac{1}{2}(v-u)\right)$$

9.3 Sum Process, Binomial Counting Process, and Random Walk

9.21 ~~6.20~~ ⓐ

$$\begin{aligned} P[Y_n = 1] &= P[I_n \text{ is not erased } | I_n = 1] P[I_n = 1] \\ &= (1-\alpha)p \text{ where } I_n \text{ is Bernoulli process} \end{aligned}$$

The Y_n are then a Bernoulli process with success probability

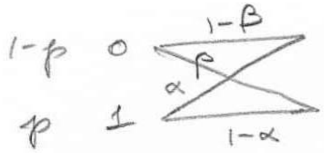
$$(1-\alpha)p \triangleq p'.$$

S'_n is then the binomial count~~er~~ing process with

$$P[S'_n = k] = \binom{n}{k} p'^k (1-p')^{n-k}$$

S'_n has independent and stationary increments.

ⓑ



$$(1-p)(1-\beta) + p\alpha$$

$$\beta(1-p) + p(1-\alpha) \triangleq p''$$

S''_n is a Binomial counting process with p''

S''_n has stationary and independent increments.

9.22

~~**6.21**~~ **a)** Assume $n' > n$, $i \geq j$

$$\begin{aligned} P[S_n = j, S_{n'} = i] &= P[S_n = j, \overbrace{S_{n'-n} = i - j}^{\text{increment}}] \\ &= P[S_n = j] P[S_{n'-n} = i - j] \\ &\qquad \text{by indep. increment property} \end{aligned}$$

In general

$$\begin{aligned} P[S_{n'} = i] &\neq P[S_{n'-n} = i - j] \\ \therefore P[S_n = j, S_{n'} = i] &\neq P[S_n = j] P[S_{n'} = i]. \end{aligned}$$

b)

$$\begin{aligned} P[S_{n_2} = j | S_{n_1} = i] &= \frac{P[S_{n_2} = j, S_{n_1} = i]}{P[S_{n_1} = i]} \\ &= \frac{P\overbrace{[S_{n_2} - S_{n_1} = j - i]}^{\text{increment}} P[S_{n_1} = i]}{P[S_{n_1} = i]} \\ &= P[S_{n_2} - S_{n_1} = j - i] = \begin{pmatrix} n_2 - n_1 \\ j - 0 \end{pmatrix} p^{j-i} (1-p)^{n_2 - n_1 - j + i} \end{aligned}$$

$$\begin{aligned} P[S_{n_2} | S_{n_1} = i, S_{n_0} = k] &= \frac{P[S_{n_2} = j, S_{n_1} = i, S_{n_0} = k]}{P[S_{n_1} = i, S_{n_0} = k]} \\ &= \frac{P[S_{n_0} = k, S_{n_1} - S_{n_0} = i - k, S_{n_2} - S_{n_1} = j - i]}{P[S_{n_0} = k, S_{n_1} - S_{n_0} = i]} \\ &= \frac{P[S_{n_0} = k] P[S_{n_1} - S_{n_0} = i - k] P[S_{n_2} - S_{n_1} = j - i]}{P[S_{n_0} = k] P[S_{n_1} - S_{n_0} = i]} \\ &= P[S_{n_2} - S_{n_1} = j - i] \\ &= P[S_{n_2} = j | S_{n_1} = i]. \end{aligned}$$

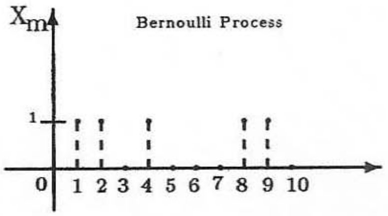
9.23

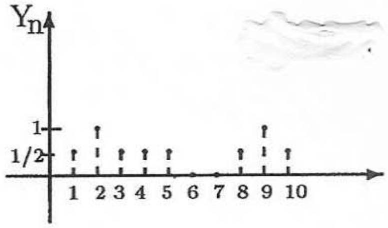
~~6.22~~

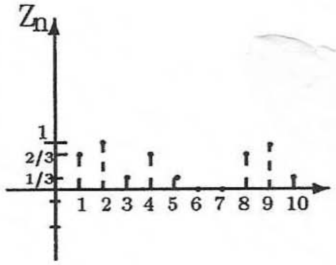
$$\begin{aligned} S_n = 0 \quad &\Leftrightarrow \quad \#\ 1\text{'s} = \#\ 0\text{'s} \\ &\Leftrightarrow \quad n \ \text{ is even and } \#\ 1\text{'s} = \#\ 0\text{'s} = \frac{n}{2} \end{aligned}$$

$$\therefore P[X_n = 0] = \begin{cases} \begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

9.24







$$
\begin{aligned}
\mathcal{E}[Y_n] &= \frac{1}{2}\mathcal{E}[X_n] + \frac{1}{2}\mathcal{E}[X_{n-1}] = \frac{1}{2}p + \frac{1}{2}p = p \\
\mathcal{E}[Y_n^2] &= \mathcal{E}\left[\frac{1}{4}X_n^2 + \frac{2}{4}X_nX_{n-1} + \frac{1}{4}X_{n-1}^2\right] \\
&= \frac{1}{4}p\,\underbrace{\mathcal{E}[X_n^2]}_{p} + 2\,\underbrace{\mathcal{E}[X_n]}_{p}\underbrace{\mathcal{E}[X_{n-1}]}_{p} + \underbrace{\mathcal{E}[X_{n-1}^2]}_{p} \\
&= \frac{1}{2}p(1+p) \\
\mathcal{E}[Y_nY_{n+1}] &= \frac{1}{4}\mathcal{E}[X_nX_{n+1} + X_n^2 + X_{n-1}X_{n+1} + X_{n-1}X_n] \\
&= \frac{1}{4}[p + 3p^2] \\
\mathcal{E}[Y_nY_{n+1}] &= \mathcal{E}\left[\frac{1}{4}(X_n + X_{n-1})(X_{n+1} + X_n)\right] = \frac{1}{4} + \frac{3}{4}(2p-1)^2
\end{aligned}
$$

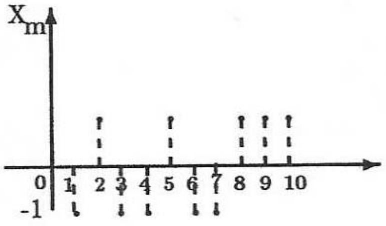
For $k > 1 \quad \mathcal{E}[Y_nY_{n+k}] = \mathcal{E}[Y_n]\mathcal{E}[Y_{n+k}] = p^2$

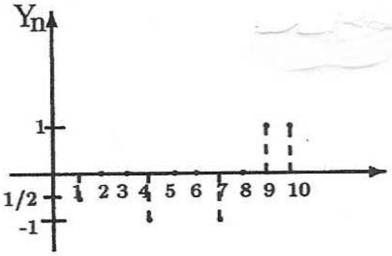
$$\therefore C_Y(n, n+k) = \begin{cases} \frac{p}{2} - \frac{p^2}{2} & k = 0 \\ \frac{p}{4} - \frac{p^2}{4} & k = 1 \\ 0 & k > 1 \end{cases}$$

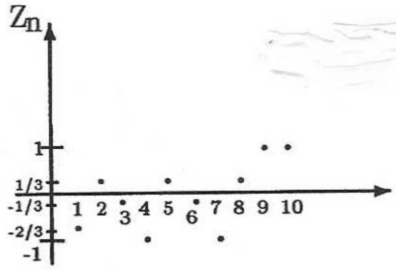
$$\begin{aligned}
\mathcal{E}[Z_n] &= \frac{2}{3}\mathcal{E}[X_n] + \frac{1}{3}\mathcal{E}[X_{n-1}] = p \\
\mathcal{E}[Z_n^2] &= \frac{1}{9}\mathcal{E}[(4X_n^2 + 4X_nX_{n-1} + X_{n-1}^2)] = \frac{5}{9}p + \frac{4}{9}p^2 \\
\mathcal{E}[Z_nZ_{n+1}] &= \frac{1}{9}\mathcal{E}[4X_nX_{n+1} + 2X_n^2 + 2X_{n+1}X_{n-1} + X_nX_{n-1}] = \frac{7}{9}p^2 + \frac{2}{9}p \\
\mathcal{E}[Z_nZ_{n+k}] &= \mathcal{E}[Z_n]\mathcal{E}[Z_{n+k}] = p^2 \text{ for } k > 1
\end{aligned}$$

$$\therefore C_Z(n, n+k) = \begin{cases} \frac{5}{9}p - \frac{5}{9}p^2 & k = 0 \\ \frac{2}{4}p - \frac{2}{9}p^2 & k = 1 \\ 0 & k > 1 \end{cases}$$

Random Step Process







$$
\begin{aligned}
\mathcal{E}[Y_n] &= \frac{1}{2}\mathcal{E}[X_n] + \frac{1}{2}\mathcal{E}[X_{n-1}] = 2p-1 \\
\mathcal{E}[Z_n] &= \frac{2}{3}\mathcal{E}[X_n] + \frac{1}{3}\mathcal{E}[X_{n-1}] = 2p-1 \\
\mathcal{E}[Y_n^2] &= \mathcal{E}\left[\frac{1}{4}(X_n^2 + 2X_nX_{n-1} + X_{n-1}^2)\right] \\
&= \frac{1}{4}\{1 + 2(2p-1)^2 + 1\} = \frac{1}{2} + \frac{(2p-1)^2}{2} \\
\mathcal{E}[Z_n^2] &= \frac{1}{9}\mathcal{E}[4X_n^2 + 4X_nX_{n-1} + X_{n-1}^2] \\
&= \frac{1}{9}[5 + 4(2p-1)^2] = \frac{5}{9} + \frac{4}{9}(2p-1)^2
\end{aligned}
$$

For $k > 1$

$$
\begin{aligned}
\mathcal{E}[Y_nY_{n+k}] &= \mathcal{E}[Y_n]\mathcal{E}[Y_{n+k}] = (2p-1)^2 \\
\mathcal{E}[Z_nZ_{n+1}] &= \mathcal{E}\left[\frac{1}{9}(2X_n + X_{n-1})(2X_{n+1} + X_n)\right] = \frac{2}{9} + \frac{7}{9}(2p-1)^2
\end{aligned}
$$

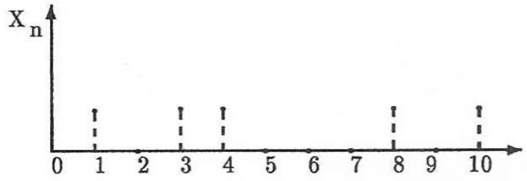
For $k > 1$

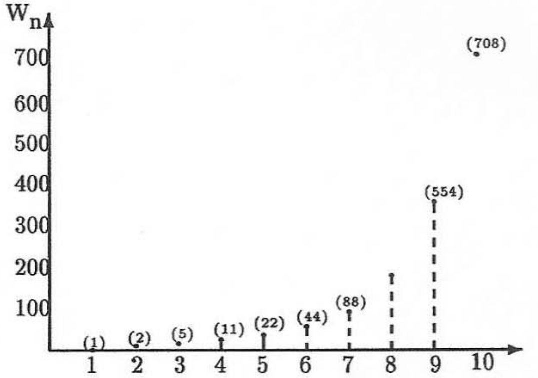
$$\mathcal{E}[Z_nZ_{n+1}] = \mathcal{E}[Z_n]\mathcal{E}[Z_{n+1}] = (2p-1)^2$$

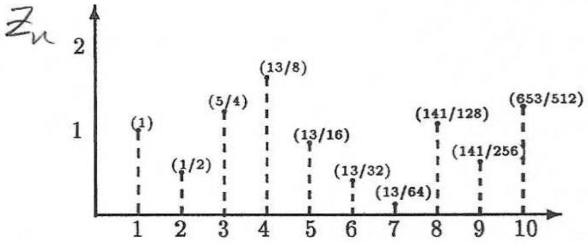
$$
\begin{aligned}
\therefore C_Y(n, n+k) &= \begin{cases} \frac{1}{2} - \frac{1}{2}(2p-1)^2 & k = 0 \\ \frac{1}{4} - \frac{1}{4}(2p-1)^2 & k = 1 \\ 0 & k > 1 \end{cases} \\
C_Z(n, n+k) &= \begin{cases} \frac{5}{9} - \frac{5}{9}(2p-1)^2 & k = 0 \\ \frac{2}{9} - \frac{2}{9}(2p-1)^2 & k = 1 \\ 0 & k > 1 \end{cases}
\end{aligned}
$$

In all cases, the sample functions are close to the mean of the processes.

9.25 **a)**







$(Z_n = \frac{3}{4} Z_{n-1} + \frac{1}{4} X_n)$
in this graph

W_n is exponentially increasing without bound as $n \to \infty$ and has meaningless sample mean.

Z_n is exponentially decreasing unless X_n is 1 and hence, the sample mean is about twice of the sample mean of X_n.

b)

$$\begin{aligned}
W_n &= 2W_{n-1} + X_n \qquad n > 1 \\
&= 2(W_{n-2} + S_{n-1}) + X_n \\
&= X_n + 2X_{n-1} + 4X_{n-2} + \ldots + 2^{n-1}X_1 \\
\mathcal{E}[W_n] &= \mathcal{E}[X]\{1 + 2 + \ldots + 2^{n-1}\} = \mathcal{E}[X]\frac{1-2^n}{1-2} = (2^n - 1)\mathcal{E}[X] \\
Z_n &= \frac{3}{4}Z_{n-1} + X_n = \frac{3}{4}\left(\frac{3}{4}|Z_{n-2} + X_{n-1}\right) + X_n \\
&= X_n + \frac{3}{4}X_{n-1} + \ldots + \left(\frac{3}{4}\right)^{n-1} X_1 \\
\mathcal{E}[Z_n] &= \mathcal{E}[X]\left\{1 + \frac{3}{4} + \ldots + \left(\frac{3}{4}\right)^n\right\} = \mathcal{E}[X]\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\frac{3}{4}}
\end{aligned}$$

c) Since $W_n - W_{n-1} = W_{n-1} + X_n$, W_n does not have independent increments.
Since $Z_n - Z_{n-1} = X_n - \frac{1}{2}Z_{n-1}$, Z_n does not have independent increments.

9.26

~~**6.25**~~ **a)** $\mathcal{E}[M_n] = \frac{1}{n}\mathcal{E}[X_1 + X_2 + \ldots + X_n] = \frac{n\mathcal{E}[X]}{n} = \mathcal{E}[X]$

$$\begin{aligned}
C_M(n,k) &= \mathcal{E}[(M_n - \mathcal{E}(X))(M_k - \mathcal{E}(X))] \\
&= \mathcal{E}\left[\frac{1}{n}[S_n - n\mathcal{E}(X)]\frac{1}{k}[S_k - k\mathcal{E}[X]]\right] \\
&= \frac{1}{nk}\mathcal{E}[(S_n - \mathcal{E}[S_n])(S_k - \mathcal{E}[S_k])] \\
&= \frac{1}{nk}C_S(n,k) = \frac{1}{nk}\min(n,k)\sigma_X^2 \\
VAR(M_n) &= C_M(n,n) = \frac{1}{n}\sigma_X^2
\end{aligned}$$

b) Since $M_{n+1} - M_n = \frac{1}{n+1}X_n - \frac{1}{n+1}M_n$, M_n does not have indep. increments.

9.27

Y_n and Z_n are Gaussian random processes with mean

$$\begin{aligned}
\mathcal{E}[Y_n] &= \frac{1}{2}\mathcal{E}[X_n] + \frac{1}{2}\mathcal{E}[X_{n-1}] = 0 \\
\mathcal{E}[Z_n] &= \frac{2}{3}\mathcal{E}[X_n] + \frac{1}{3}\mathcal{E}[X_{n-1}] = 0
\end{aligned}$$

and variance

$$\begin{aligned}
\mathcal{E}[Y_n^2] &= \mathcal{E}\left[\frac{1}{4}(X_n + X_{n-1}^2)\right] = \frac{1}{4}(1+1) = \frac{1}{2} \\
\mathcal{E}[Z_n^2] &= \mathcal{E}\left[\left(\frac{2}{3}X_n + \frac{1}{3}X_{n-1}\right)^2\right] = \frac{5}{9}
\end{aligned}$$

$$\therefore f_{Y_n}(y) = \frac{e^{-y^2}}{\sqrt{\pi}} \qquad f_{Z_n}(z) = \frac{3}{\sqrt{10\pi}}e^{-9z^2/10}$$

9.28

~~6.27~~ **a)** $\Phi_{S_n}(\omega) = \Phi_X(\omega)^n = (e^{-\alpha|\omega|})^n = e^{-n\alpha|\omega|}$

$\Rightarrow f_{S_n}(x) = \dfrac{n\alpha/\pi}{x^2 + n^2\alpha^2}$

b) Since S_n has independent and stat. increments

$$\begin{aligned} f_{S_n,S_{n+k}}(y_1, y_2) &= f_{S_n}(y_1) f_{S_{n+k-n}}(y_2 - y_1) \\ &= \frac{n\alpha/\pi}{(y_1^2 + n^2\alpha^2)} \frac{k\alpha/\pi}{((y_2 - y_1)^2 + k^2\alpha^2)} \end{aligned}$$

9.29

~~6.28~~ **a)** $G_{S_n}(z) = G_X(z)^n = (e^{\alpha(z-1)})^n = e^{n\alpha(z-1)}$

$\Rightarrow P[S_n = k] = \dfrac{(n\alpha)^k}{k!} e^{-n\alpha}$

b) $P[S_n = i, S_{n+k} = j] = P[S_n = i] P[S_{n+k-n} = j - i]$

$= \dfrac{(n\alpha)^i}{i!} e^{-n\alpha} \dfrac{(k\alpha)^{j-1}}{(j-i)!} e^{-k\alpha}$ for $j \geq i$, $k > 0$

9.30

~~6.29~~ **a)**
$$\begin{aligned} \Phi_{M_n}(\omega) &= \mathcal{E}[e^{j\omega(X_1+\ldots+S_n)/n}] = \Phi_X^n\left(\frac{\omega}{n}\right) = e^{-n\left(\frac{\omega}{n}\right)^2/2} \\ &= e^{-\omega^2/2n} \end{aligned}$$

$\Rightarrow M_n$ is Gaussian with mean zero and variance $\frac{1}{n}$

$$F_{M_n}(x) = \sqrt{\frac{n}{2\pi}} e^{-nx^2/2}$$

b) M_n and M_{n+k} are related to S_n and S_{n+k} by

$$\begin{aligned} M_n &= \frac{1}{n} S_n \\ M_{n+k} &= \frac{1}{n+k} S_{n+k} \\ J(S_n, S_{n+k}) &= \begin{vmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n+k} \end{vmatrix} = \frac{1}{n(n+k)} \end{aligned}$$

$$\Rightarrow f_{M_n M_{n+k}}(x, y) = n(n+k) f_{S_n S_{n+k}}(nx, (n+k)y)$$

$$\begin{aligned} f_{M_n M_{n+k}}(x, y) &= f_{S_n}(nx) f_{S_k}((n+k)y - nx)(n)(n+k) \\ &= n(n+k) \frac{e^{-n^2x^2/2n}}{\sqrt{2\pi n}} \frac{e^{-[(n+k)y - nx]^2/2k}}{\sqrt{2\pi k}} \end{aligned}$$

since S_n is Gaussian with mean zero and variance n.

9.31 $X_n = \frac{1}{2}(Y_n + Y_{n-1})$ $\quad Y_n$ iid.

$$E[X_n] = \frac{1}{2}\left[E[Y_n] + E[Y_{n-1}]\right] = E[Y] \triangleq m$$

$$\text{COV}(X_n, X_{n+k}) = E[(X_n - m)(X_{n+k} - m)] = E[X_n X_{n+k}] - m^2$$

$$= \begin{cases} \frac{1}{4}E[(Y_j + Y_{j-1})^2] - m^2 = \frac{1}{2}E[Y^2] - \frac{1}{2}m^2 = \frac{1}{2}\text{VAR}(Y) & \text{for } k=0 \\ \frac{1}{4}E[(Y_j + Y_{j-1})(Y_{j-1} + Y_j)] \\ \quad = \frac{1}{4}E[Y^2] + \frac{3}{4}m^2 - m^2 = \frac{1}{4}\text{VAR}(Y) & k = \pm 1 \\ \frac{1}{4}E[(Y_j + Y_{j-1})]E[(Y_{j+k} + Y_{j+k+1})] = 0 & \text{otherwise.} \end{cases}$$

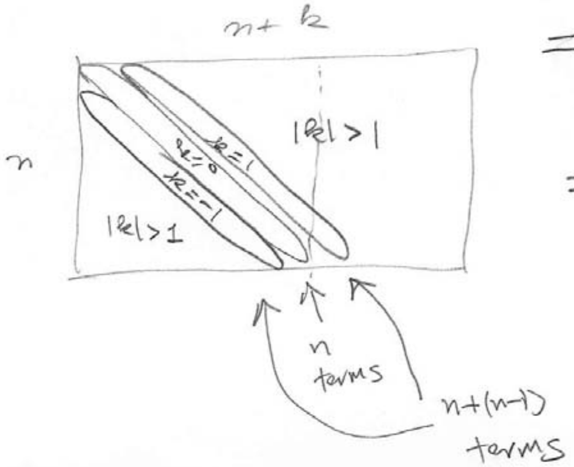
$$M_n = \frac{1}{n}\sum_{j=1}^{n} X_j \qquad E[M_n] = \frac{1}{n}\sum_{j=1}^{n} E[X_j] = m$$

for $k > 1$

$$\text{COV}(M_n, M_{n+k}) = E[(M_n - m)(M_{n+k} - m)]$$

$$= E\left[\frac{1}{n}\sum_{j=1}^{n}(X_j - m)\,\frac{1}{n+k}\sum_{l=1}^{n+k}(X_l - m)\right]$$

$$= \frac{1}{n(n+k)}\sum_{j=1}^{n}\sum_{l=1}^{n+k}\underbrace{E[(X_j - m)(X_l - m)]}_{\text{COV}(X_j, X_l)}$$



$$= \frac{1}{n(n+k)}\left\{n\,\frac{1}{2}\sigma_Y^2 + (2n-1)\frac{\sigma_Y^2}{4}\right\}$$

$$= \frac{4\sigma_Y^2(4n-1)}{n(n+k)} \qquad k \geq 1$$

$$\text{VAR}(M_n) = \frac{4\sigma_Y^2(4n-2)}{n(n+k)}$$

$\to 0$
as $n \to \infty$.

9.32

$$X_n = \alpha X_{n-1} + Y_n \qquad \alpha = 3/4$$

$$= Y_n + \alpha(\alpha X_{n-2} + Y_{n-1})$$

$$= Y_n + \alpha^{2} Y_{n-1} + \alpha^{2} Y_{n-2} + \dots + \alpha^{n-1} Y_0$$

To simplify derivation assume process started at time $-\infty$ then,

$$X_n = Y_n + \alpha^{2} Y_{n-1} + \alpha^{2} Y_{n-1} + \dots$$

$$= \sum_{l=0}^{\infty} \alpha^{l} Y_{n-l}$$

$$E[X_n] = \sum_{l=0}^{\infty} \alpha^{l} \underbrace{E[Y_{n-l}]}_{m_Y} = m_Y \frac{1}{1-\alpha} \triangleq m_X$$

$$\mathrm{COV}(X_n, X_{n+k}) = E[(X_n - m_X)(X_{n+k} - m_X)$$

$$= E\left[\sum_{l=0}^{\infty} \alpha^{l} Y_{n-l} \sum_{l'=0}^{\infty} \alpha^{l'} Y_{n-l'+k}\right] - m_X^2$$

$$= \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty} \alpha^{l+l'} \underbrace{E[Y_{n-l} Y_{n-l'+k}]}_{E[Y^2]\,\delta_{l-l'-k}} - m_X^2$$

$$= \alpha^{k} \sum_{l=0}^{\infty} \alpha^{2l} E[Y^2] + \sum_{\substack{l=0 \\ l \neq l'+k}}^{\infty}\sum_{l'=0}^{\infty} \alpha^{l+l'} m_Y^2 - m_X^2$$

$$= \frac{\alpha^{k}}{1-\alpha^2} E[Y^2] + \left(\left(\frac{1}{1-\alpha}\right)^2 - \frac{\alpha^{k}}{1-\alpha^2}\right) m_Y^2 - \underbrace{m_X^2}_{\frac{m_Y^2}{(1-\alpha)^2}}$$

$$= \frac{\alpha^{k}}{1-\alpha^2} \sigma_Y^2$$

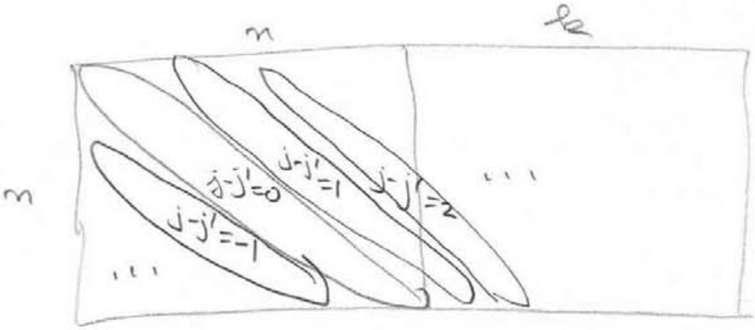
9.32 — Now consider M_n —

$$E[M_n] = \frac{1}{n}\sum_{j=1}^{n} E[X_j] = m_x$$

$$\text{COV}(M_n, M_{n+k}) = E[(M_n - m_x)(M_{n+k} - m_x)]$$

$$= \frac{1}{n(n+k)} E\left[\sum_{j=1}^{n}(X_j - m_x)\sum_{j'=1}^{n+k}(X_{j'} - m_x)\right]$$

$$= \frac{1}{n(n+k)}\sum_{j=1}^{n}\sum_{j'=1}^{n+k}\underbrace{\text{cov}(X_j, X_{j'})}_{\frac{\alpha^{|j-j'|}}{1-\alpha^2}\sigma_Y^2}$$



$$\text{COV}(M_n, M_{n+k}) =$$

$$= \frac{\sigma_Y^2/(1-\alpha^2)}{n(n+k)}\left(n\cdot\alpha^0 + (n+(n-1))\alpha^1 + (n+(n-2))\alpha^2 + \ldots\right)$$

$$\text{VAR}(M_n) = \frac{\sigma_Y^2/(1-\alpha^2)}{n^2}\sum_{k=-(n-1)}^{n-1}(n-|k|)\,\alpha^{|k|}$$

$$= \frac{\sigma_Y^2/(1-\alpha^2)}{n}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\alpha^{|k|}$$

$$\leq \frac{\sigma_Y^2/(1-\alpha^2)}{n}\,2\,\frac{1}{1-\alpha} \to 0 \text{ as } n \to \infty.$$

9.33

~~6.30~~ a)

$$\begin{aligned}
\underline{S}(n) &= (S_0(n), S_1(n), S_2(n)) \\
&= (X_0(1) + \ldots + S_0(n), S_1(1) + \ldots + X_1(n), X_2(n) + \ldots + X_2(n))
\end{aligned}$$

$\Rightarrow S_i(n) =$ number of occurences of i in n trials

$S_0(n) + S_1(n) + S_2(n) = n$

$$\Rightarrow P[S_0(n) = i, S_1(n) = j, S_2(n) = n - j - i] = \frac{n!}{i!j!(n-i-j)!} p_0^i p_1^j p_2^{n-i-j}$$

$\Rightarrow \underline{S}(n)$ Has a multinomial dist.

b)

$$\left.\begin{aligned}
\underline{S}_{n_1} - \underline{S}_{n_0} &= \underline{X}_{n_0+1} + \ldots + \underline{X}_{n_1} \\
\underline{S}_{n_3} - \underline{S}_{n_2} &= \underline{X}_{n_2+1} + \ldots + \underline{X}_{n_3}
\end{aligned}\right\}$$

if $n_2 > n_1$ then $\underline{S}_{n_1} - \underline{S}_{n_0}$ and $\underline{S}_{n_3} - \underline{S}_{n_2}$ depends on different sets of X's and hence they are independent

$$\begin{aligned}
P[\underline{S}(n) = \underline{S}, \underline{S}(n+k) = \underline{S}'] &= P[\underline{S}(n) = \underline{S}]P[\underline{S}(k) = \underline{S}' - \underline{S}] \\
&= \frac{n!}{s_0!s_1!s_2!} \frac{k!}{(s_0' - s_0)!(s_1' - s_1)!(s_2' - s_2)!} p_0^{s_0'} p_1^{s_1'} p_2^{s_2'}
\end{aligned}$$

c) $S_j(n) = X_j(1) + \ldots + X_j(n)$ where $X_j(i)$ are Bernoulli RV's
$\Rightarrow S_j(n)$ is a binomial counting process.

9.4 Poisson and Associated Random Processes

9.34

$\lambda = 10$ queries/minute

$T = 20$ sec $= 1/3$ minute

$$P[N(\tfrac{1}{3}) = 0] = \frac{(10/3)^0}{0!} e^{-10/3} = 0.0356$$

9.35

$$P[N_i(t) = k \mid N(t) = n] = \binom{n}{k} p^k (1-p)^{n-k}$$

items dispensed

$$P[N_i(t) = k] = \sum_{l=k}^{\infty} \binom{l}{k} p^k (1-p)^{l-k} \frac{(\lambda t)^l}{l!} e^{-\lambda t}$$

$$P[N_i(t) = k] = \frac{e^{-\lambda t} p^k (\lambda t)^k}{k!} \underbrace{\sum_{l=k}^{\infty} \frac{((1-p)\lambda t)^{l-k}}{(l-k)!}}_{\sum_{l'=0}^{\infty} \frac{((1-p)\lambda t)^{l'}}{l'!} = e^{(1-p)\lambda t}}$$

$$= \frac{(\lambda t p)^k}{k!} e^{-\lambda t p}$$

9.36

(a) $$P[N(t) = 0] = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

(b) $$P[N(t) > 2] = 1 - P[N(t) \le 2]$$

$$= 1 - e^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2}\right)$$

9.37

a)

$$P\{T_1<T_2\} = \int_0^\infty P[T_2>t \mid T_1=t]\, f(t)\,dt = \int_0^\infty e^{-\lambda_2 t}\,\lambda_1 e^{-\lambda_1 t}\,dt = \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{2}{3}$$

b)

$$f_1(t)=\lambda_1 e^{-\lambda_1 t},\quad f_2(t)=\lambda_2 e^{-\lambda_2 t},\quad F_1(t)=1-e^{-\lambda_1 t},\quad F_2(t)=1-e^{-\lambda_2 t}$$

$$\begin{aligned} f_{min}(t) &= f_1(t)+f_2(t)-f_1(t)F_2(t)-F_1(t)f_2(t) \\ &= \lambda_1 e^{-\lambda_1 t}+\lambda_2 e^{-\lambda_2 t}-\lambda_1 e^{-\lambda_1 t}-\lambda_2 e^{-\lambda_2 t}+\lambda_1 e^{-(\lambda_1+\lambda_2)t}+\lambda_2 e^{-(\lambda_1+\lambda_2)t} \\ &= (\lambda_1+\lambda_2)\,e^{-(\lambda_1+\lambda_2)t} = 3e^{-3t} \end{aligned}$$

$$1-F_{min}(t) = P\{\min(T_1,T_2)>t\} = P[T_1>t,\ T_2>t] = P[T_1>t]\,P[T_2>t] = e^{-\lambda_1 t}\cdot e^{-\lambda_2 t} = e^{-(\lambda_1+\lambda_2)t} = e^{-3t}$$

c) $N(t)$ is a poisson random variable with rate $(\lambda_1+\lambda_2)t$

Hence, $P\{N(t)=k\} = e^{-(\lambda_1+\lambda_2)t}\,\dfrac{(\lambda_1+\lambda_2)^k t^k}{k!} = e^{-3t}\,\dfrac{(3t)^k}{k!}$

Note that sum of independent poisson RVs is a poisson RV (P7.11)

d) at time t, $N_i(t)$ is a Poisson RV with rate $\lambda_i t$.

so, if we define $N(t)=\sum_{i=1}^{k} N_i(t)$, $N(t)$ is sum of K independent random variable with rat $\sum_{i=1}^{k} \lambda_i t$. if we define $\lambda := \sum \lambda_i(t)$

we have:

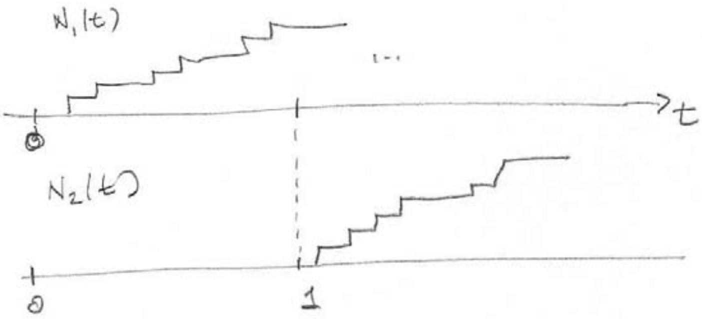
$$P\{N(t)=m\} = e^{-\lambda t}\,\frac{(\lambda t)^m}{m!}$$

it can be shown that $N(t)$ is a Poisson process with rate λ

9.38

$$\begin{aligned}
P[N(t-d)=j \mid N(t)=k] &= \frac{P[N(t-d)=j, N(t)=k]}{P[N(t)=k]} \\
&= \frac{P[N(t-d)=j]P[N(t)-N(t-d)=k-j]}{P[N(t)=k]} \\
&= \frac{\frac{\lambda^j (t-d)^j}{j!} e^{-\lambda(t-d)} \frac{\lambda^{k-j} d^{k-j}}{(k-j)!} e^{-\lambda d}}{\frac{\lambda^k t^k}{k!} e^{-\lambda t}} \\
&= \binom{k}{j} \left(\frac{t-d}{t}\right)^j \left(\frac{d}{t}\right)^{k-j} \quad \text{binomial}
\end{aligned}$$

9.39



$$N(t) = \begin{cases} N_1(t) & 0 \le t < 1 \\ N_1(t) + N_2(t) & 1 \le t \end{cases} \qquad m(t) = \begin{cases} \lambda_1 t & 0 \le t < 1 \\ (\lambda_1+\lambda_2)t & 1 \le t \end{cases}$$

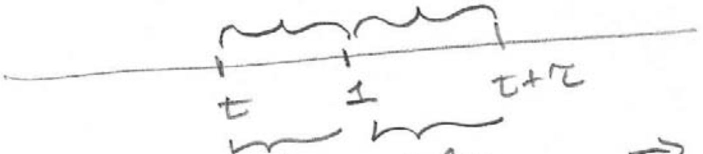
Since $N_1(t)$ and $N_2(t)$ are independent random processes $N_1(t)+N_2(t)$ has a Poisson pmf.

$$P[N(t)=k] = P[N_1(t)+N_2(t)=k] = \frac{((\lambda_1+\lambda_2)t)^k}{k!} e^{-(\lambda_1+\lambda_2)t}$$

for $t \ge 1$.

$N(t)$ has indep increments for intervals in $t<1$, and for $t>1$

$N(t)$ also has indep increments for intervals that span $t=1$:



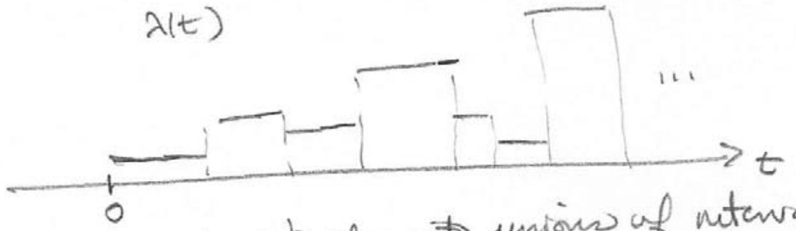
indep increment for $(t,1]$ $m(1)-m(t)$

indep incr. for $[1, t+\tau]$ $m(t+\tau)-m(1)$

$\Rightarrow$ indep incr. for $(t, t+\tau]$

$$\therefore \quad P[N(t+k)-N(t)=k] = \frac{(m(t+\tau)-m(t))^k}{k!} e^{-(m(t+\tau)-m(t))}$$

ⓑ The above 2-interval arrival rate process is generalized to a piecewise constant process



By decomposing intervals into unions of intervals where $\lambda(t)$ is constant, we can show that pmf in part (a) holds.

For the interval that begins at $t=0$, we have:

$$P[N(t)=k] = \frac{m(t)^k}{k!} e^{-m(t)} \quad \text{where } m(t) = \int_0^t \lambda(t')\,dt'$$

ⓒ The above pmf holds for $\lambda(t)$ sufficiently smooth that they can be approximated by a sequence of piecewise constant functions.

9.40

$$\begin{aligned}
P[N=k] &= \int_0^\infty P[N=k|T=t] f_T(t)dt \\
&= \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} \beta e^{-\beta t} dt \\
&= \frac{\beta}{k!} \int_0^\infty (\lambda t)^k e^{-(\lambda+\beta)t} dt \qquad t' = (\lambda+\beta)t \\
&= \frac{\beta}{k!} \frac{\lambda^k}{(\lambda+\beta)^{k+1}} \underbrace{\int_0^\infty t'^k e^{-t'} dt'}_{\Gamma(k+1)=k!} \\
&= \left(\frac{\beta}{\lambda+\beta}\right)\left(\frac{\lambda}{\lambda+\beta}\right)^k \qquad k=0,1,\ldots
\end{aligned}$$

9.41) ⓐ

~~**6.37**~~ $D(t) = N_1(t) - N_2(t)$ can assume negative values
$\Rightarrow D(t)$ cannot be a Poisson process.

ⓑ

The times between events in this process is the sum of two exponential random variables (Erlang). Thus the process is no longer Poisson.

9.42)

~~**6.38**~~ **a)**

$$\begin{aligned} P[N_1(t) = j, N_2(t) = k \mid N(t) = k+j] &= P[j \text{ heads in } k+j \text{ tosses}] \\ &= \binom{j+k}{j} p^j (1-p)^k \end{aligned}$$

b)

$$\begin{aligned} P[N_1(t) = j, N_2(t) = k] &= P[N_1(t) = j, N_2(t) = k, N(t) = k+j] \\ &= P[N_1(t) = j, N_2(t) = k \mid N(t) = k+j] P[N(t) = k+j] \\ &= \binom{j+k}{j} p^j (1-p)^k \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \\ &= \frac{(p\lambda)^j}{j!} e^{-p\lambda} \frac{((1-p)\lambda)^k}{k!} e^{-(1-p)\lambda} \end{aligned}$$

9.43

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

$$P[X(t)=j] = \sum_{n=0}^{\infty} P[X(t)=j \mid N(t)=n]\, \underbrace{P[N(t)=n]}_{\frac{(\lambda t)^n}{n!} e^{-\lambda t}}$$

$=$

ⓐ X_i is Bernoulli, then

$$P[X(t)=j \mid N(t)=n] = \binom{n}{j} p^j (1-p)^{n-j}$$

Proceeding as in prob. 9.35,

$$P[X(t)=j] = \frac{(\lambda t p)^j}{j!} e^{-\lambda t p}$$

ⓑ $X_i \in \{0, 5\}$ with $p_0 = \frac{5}{6}$ $p_1 = \frac{1}{6}$

same as ⓐ:

$$P[X(t)=5j] = \frac{(\lambda t p)^j}{j!} e^{-\lambda t p} \qquad p = \frac{1}{6}.$$

ⓒ X_i is Poisson with $\alpha = 1$, $X(t)$ is sum of n iid Poisson w. mean n

$$P[X(t)=j \mid N(t)=n] = \frac{n^j}{j!} e^{-n}$$

$$P[X(t)=j] = \sum_{n=0}^{\infty} \frac{n^j}{j!} e^{-n} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

$$= \frac{e^{-\lambda t + \frac{\lambda t}{e}}}{j!} \underbrace{\sum_{n=0}^{\infty} n^j \frac{\left(\frac{\lambda t}{e}\right)^n}{n!} e^{-\frac{\lambda t}{e}}}_{E[N^j] \text{ for Poisson w rate } \lambda t/e}$$

ⓓ

$$P[\underbrace{X(t)=j}\mid \underbrace{N(t)=n}]$$

there have been n arrivals

# of consecutive "tails" since last "heads" up to maximum of n trials

$\Rightarrow$

$$P[X(t)=j \mid N(t)=n] = \frac{(1-p)p^j}{(1-p^{n})} \qquad j=1,2,\ldots,n$$

$$P[X(t)=j] = \sum_{n=j}^{\infty} \frac{(1-p)p^j}{1-p^n}\,\frac{(\lambda t)^n}{n!}\,e^{-\lambda t}$$

after system has been in operation a long time (i.e. $t\to\infty$, λt is large and p^n becomes negligible)

then

$$P[X(t)=j] \to (1-p)p^j \qquad \text{geometric RV}$$

9.44 $Y(t)$ is a random telegraph process with transition rate $p\alpha$, since if T = time till next transition, then

$$T = \tau_1 + \ldots + \tau_N \qquad \text{where } N \text{ is geometric RV}$$

$$\begin{aligned}
\phi_T(N) &= \mathcal{E}[\mathcal{E}[e^{j\omega t}|N]] = \mathcal{E}\left[\left(\frac{\alpha}{\alpha-j\omega}\right)^N\right] = \sum_{k=1}^{\infty}\left(\frac{\alpha}{\alpha-j\omega}\right)^k p(1-p)^{k-1} \\
&= \frac{\alpha p}{\alpha-j\omega}\,\frac{1}{1-\frac{\alpha(1-p)}{\alpha-j\omega}} = \frac{\alpha p}{\alpha p-j\omega} \Rightarrow T \text{ exp. rate } \alpha p
\end{aligned}$$

$$\therefore P[Y(t)=+1] = \frac{1}{2} = P[Y(t)=-1] \text{ if } P[Y(0)] = \frac{1}{2}$$

and

$$C_Y(t_1,t_2) = e^{-2\alpha t|t_2-t_1|}$$

9.45

~~6~~.41 Let $X(t)$ be the random telegraph process, then

$$\begin{aligned}
P[Y(t)=1] &= P[X(t)=1]=\frac{1}{2}\\
P[Y(t)=0] &= P[X(t)=-1]=\frac{1}{2}\\
m_Y(t) &= 1\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)=\frac{1}{2}\\
C_Y(t_1,t_2) &= \mathcal{E}[Y(t_1)Y(t_2)]-\left(\frac{1}{2}\right)^2\\
&= P[Y(t_1)=1,Y(t_2)=1]-\frac{1}{4}\\
&= P[Y(t_1)=1]P[\text{even }\#\text{ transitions in }t_2-t_1]-\frac{1}{4}\\
&= \frac{1}{2}(1+e^{-2\alpha|t_2-t_1|})\frac{1}{2}-\frac{1}{4}\\
&= \frac{1}{4}e^{-2\alpha|t_2-t_1|}
\end{aligned}$$

9.46

.42 a) $P[Z(t)=0|Z(0)=0]=$

$$\begin{aligned}
P[\text{even }\#\text{ transitions in }[0,t]] &= \sum_{j=0}^{\infty}\frac{1}{1+\alpha t}\left(\frac{\alpha t}{1+\alpha t}\right)^{2j}\\
&= \frac{1}{1+\alpha t}\frac{1}{1-\left(\frac{\alpha t}{1+\alpha t}\right)^2}=\frac{1+\alpha t}{1+2\alpha t}
\end{aligned}$$

$P[Z(t)=0|Z(0)=1]=$

$$P[\text{odd }\#\text{ transitions in }[0,t]]=\sum_{j=0}^{\infty}\frac{1}{1+\alpha t}\left(\frac{\alpha t}{1+\alpha t}\right)^{2j+1}=\frac{\alpha t}{1+2\alpha t}$$

$$\begin{aligned}
P[Z(t)=0] &= P[Z(t)=0|Z(t)=0]P[Z(0)=0]+P[Z(t)=0|Z(0)=1]P[Z(0)=1]\\
&= \frac{1+\alpha t}{1+2\alpha t}\frac{1}{2}+\frac{\alpha t}{1+2\alpha t}\frac{1}{2}=\frac{1}{2}\\
&\quad \text{where we assume } P[Z(0)=0]=\tfrac{1}{2}\\
P[Z(t)=1] &= 1-P[Z(t)=0]=\frac{1}{2}
\end{aligned}$$

b) $m_Z(t)=1\cdot P[Z(t)=1]=\dfrac{1}{2}$

9.47 ~~6.43~~ $h(t) = u(t) - u(t-T)$

a) $$X(t) = \sum_{i=1}^{\infty} h(t-S_i) = \underbrace{\underbrace{\sum_{i=1}^{\infty} u(t-S_i)}_{\substack{\#\text{ arrivals to}\\ \text{time } t\\ N(t)}} - \underbrace{\sum_{i=1}^{\infty} u(t-T-S_i)}_{\substack{\#\text{ arrivals to}\\ \text{time } t-T\\ N(t-T)}}}_{\text{increments in } (t-T,t)}$$

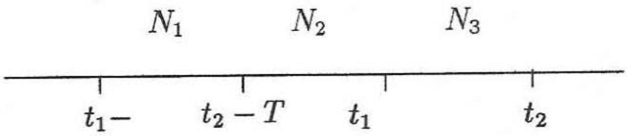
b) $\mathcal{E}[X(t)] = \lambda T$

Assume $t_1 < t_2$

Case 1: If $t_1 < t_2 - T$

Then $\mathcal{E}[X(t_1)X(t_2)] = \mathcal{E}[X(t_1)]\mathcal{E}[X(t_2)] = \lambda^2 T^2$

Case 2: If $t_2 - T < t_1 < t_2$



Then

$$\begin{aligned}
\mathcal{E}[X(t_1)X(t_2)] &= \mathcal{E}[(N_1+N_2)(N_2+N_3)] \qquad \begin{array}{l} X(t_1) = N_1 + N_2 \\ X(t_2) = N_2 + N_3 \end{array}\\
&= \mathcal{E}[N_1]\mathcal{E}[N_2] + \mathcal{E}[N_2^2] + \mathcal{E}[N_1]\mathcal{E}[N_3] + \mathcal{E}[N_2]\mathcal{E}[N_3]\\
\mathcal{E}[N_1] &= \lambda(t_2 - t_1) = \mathcal{E}[N_3]\\
\mathcal{E}[N_2] &= \lambda(t_1 - t_2 + T)\\
\mathcal{E}[N_2^2] &= VAR[N_2] + \mathcal{E}[N_2]^2\\
&= \lambda(t_1 - t_2 + T) + \lambda^2(t_1 - t_2 + T)^2
\end{aligned}$$

$$\begin{aligned}
\therefore \mathcal{E}[X(t_1)X(t_2)] &= \lambda^2(t_1 - t_2 + T)^2 + \lambda(t_1 - t_2 + T)\\
&\quad + 2\lambda^2(t_2 - t_1)(t_1 - t_2 + T) + \lambda^2(t_2 - t_1)^2\\
&= \lambda^2 T^2 + \lambda(t_1 - t_2 + T)
\end{aligned}$$

$$\Rightarrow C_X(t_1, t_2) = \begin{cases} \lambda(T - |t_2 - t_2) & |t_2 - t_1| < T \\ 0 & \text{otherwise} \end{cases}$$

9.48

~~6.44~~ **a)**

$$\begin{aligned}
\mathcal{E}[X^2(t)] &= \mathcal{E}[\mathcal{E}[X^2(t)/N(t)]] \\
\mathcal{E}[X^2(t)|N(t)=k] &= \mathcal{E}\left[\sum_{j=1}^{k}\sum_{i=1}^{k} h(t-S_j)h(t-S_i)\right] \\
&= k\mathcal{E}[h^2(t-s)] + k(k-1)\mathcal{E}[h(t-s)]^2 \\
\mathcal{E}[X^2(t)] &= \mathcal{E}[N(t)]\mathcal{E}[h^2(t-s)] + \mathcal{E}[N(t)(N(t)-1)]\mathcal{E}[h(t-s)]^2 \\
VAR[X(t)] &= \mathcal{E}[X^2(t)] - \mathcal{E}[X(t)]^2 \\
VAR[X(t)] &= \mathcal{E}[N(t)]\mathcal{E}[h^2(t-s)] + (\mathcal{E}[N^2]-\mathcal{E}[N])\mathcal{E}[h(t-s)]^2 \\
&\quad -\mathcal{E}[N]^2\mathcal{E}[h(t-s)]^2 \\
&= \underbrace{\mathcal{E}[N]}_{\lambda t}\,\mathcal{E}[h^2(t-s)] \\
&\quad + \underbrace{(\underbrace{VAR[N]}_{\lambda t} - \underbrace{\mathcal{E}[N]}_{\lambda t})}_{0}\,\mathcal{E}[h(t-s)]^2 \\
&= \lambda t \int_0^t h^2(t-s)\frac{ds}{t} \\
&= \lambda \int_0^t h^2(u)du
\end{aligned}$$

b)

$$\begin{aligned}
\mathcal{E}[X(t)] &= \lambda\int_0^t e^{-\beta u}du = \frac{\lambda}{\beta}(1-e^{-\beta t}) \to \frac{\lambda}{\beta} \\
VAR[X(t)] &= \lambda\int_0^t e^{-2\beta u}du = \frac{\lambda}{2\beta}(1-e^{-2\beta t}) \to \frac{\lambda}{2\beta}
\end{aligned}$$

9.49

Suppose $N=k$, the # of arrivals in 1 hr, and let $S_1, S_2, ..., S_k$ be the arrival times.

Then customer i waits $60 - S_i$ minutes

$$\mathcal{E}[W|N=k] = \mathcal{E}\left[\sum_{i=1}^{k}(60-S_i)\right] = \sum_{j=1}^{k}\int_0^{60}(60-s)\frac{ds}{60} = 30k$$

Since arrival times are iid unif. dist. in [9.60]

$$\therefore \mathcal{E}[W] = \mathcal{E}[\mathcal{E}[W?N=k]] = \mathcal{E}[30N] = 30\mathcal{E}[N] = 30\lambda$$

9.5 Gaussian Random Processes, Wiener Process and Brownian Motion

9.50

$X(t)$ and $Y(t)$ independent

$\Rightarrow X(t)$ and $Y(t)$ uncorrelated (whether Gaussian or not)

$X(t)$ and $Y(t)$ jointly Gaussian and uncorrelated

$\Rightarrow X(t)$ and $Y(t)$ independent

(since joint pdf factors into product of marginal pdf's).

$X(t)$ and $Y(t)$ uncorrelated then

$X(t)$ and $Y(t)$ orthogonal iff $\underset{\text{all } t}{m_X(t) = 0}$ or $\underset{\text{all } t}{m_Y(t) = 0}$.

$X(t)$ and $Y(t)$ orthogonal then

$X(t)$ and $Y(t)$ uncorrelated if $\underset{\text{all } t}{m_X(t) = 0}$ or $\underset{\text{all } t}{m_Y(t) = 0}$.

For jointly Gaussian RP's:

$X(t)$ & $Y(t)$ independent

$\Updownarrow$

$X(t)$ & $Y(t)$ uncorrelated

$\Updownarrow$ and $m_X(t)$ or $m_Y(t) = 0$.

$X(t)$ & $Y(t)$ orthogonal and $m_X(t) = 0$ or $m_Y(t) = 0$.

9.51)

$C_X(t_1,t_2) = 4e^{-2|t_1-t_2|}$ $X(t)$ & $X(t+s)$ are jointly Gaussian RVs with $COV(X(t),X(t+s)) = C_X(t,t+s) = 4e^{-2|s|}$

then $f_{X(t),X(t+s)}(x_1,x_2)$ can be achieved using Eq. 9.47 as follows:

$$f_{X(t),X(t+s)}(x_1,x_2) = \frac{e^{-\frac{1}{2}(x-m)^T K^{-1}(x-m)}}{2\pi\,|K|^{1/2}}$$

in which $m=0$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $K = \begin{bmatrix} 4 & 4e^{-2|s|} \\ 4e^{-2|s|} & 4 \end{bmatrix}$

and $K^{-1} = \dfrac{1}{16(1-e^{-4|s|})}\begin{bmatrix} 4 & -4e^{-2|s|} \\ -4e^{-2|s|} & 4 \end{bmatrix}$ $|K| = 16(1-e^{-4|s|})$

So

$$f_{X(t),X(t+s)}(x_1,x_2) = \frac{e^{\left\{-\frac{1}{32(1-e^{-4|s|})}\left(4x_1^2+4x_2^2-8e^{-4|s|}x_1x_2\right)\right\}}}{8\pi\sqrt{1-e^{-4|s|}}}$$

9.52)

$Z = 2tX - Y$ X & Y are jointly Gaussian, then $Z(t)$ would be a Gaussian RV too. (Example 6.24)

$m_{Z(t)} = E[Z(t)] = 2t\,E[X] - E[Y] = 2t\,m_X - m_Y$

$\sigma^2_{Z(t)} = VAR[Z(t)] = 4t^2\sigma_X^2 + \sigma_Y^2 + 4t\,\sigma_X\sigma_Y\rho_{XY}$

$$f_{Z(t)}(z) = \frac{e^{-\frac{(z-m_{Z(t)})^2}{2\sigma^2_{Z(t)}}}}{\sqrt{2\pi}\,\sigma_{Z(t)}}$$

9.53

(a) $E[Y(t)] = E[X(t+d)] - E[X(t)] = m_X(t+d) - m_X(t) = m_Y(t)$

$$E[Y(t_1)Y(t_2)] = E\Big[\big(X(t_1+d) - X(t_1)\big)\big(X(t_2+d) - X(t_2)\big)\Big]$$

$$= E[X(t_1+d)X(t_2+d)] - E[X(t_1)X(t_2+d)] - E[X(t_2)X(t_1+d)] + E[X(t_1)X(t_2)]$$

$$C_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] - m_Y(t_1)\,m_Y(t_2)$$

$$= C_X(t_1+d,t_2+d) - C_X(t_1,t_2+d) - C_X(t_2,t_1+d) + C_X(t_1,t_2)$$

(b) Since $X(t)$ and $X(t+d)$ are jointly Gaussian RV's, then $Y(t)$ is also Gaussian with mean $m_Y(t)$ and

$$\sigma_Y^2(t) = C_Y(t,t) = C_X(t+d,t+d) - 2C_X(t,t+d) + C_X(t,t)$$

(c) $Y(t)$ and $Y(t+s)$ are jointly Gaussian RVs defined by the linear transformation of jointly Gaussian RV's

$$Y(t) = X(t+d) - X(t)$$
$$Y(t+s) = X(t+s+d) - X(t+s)$$

∴ $Y(t)$ and $Y(t+s)$ have a jointly Gaussian pdf with mean $m_Y(t)$, variance, $\sigma_Y^2(t)$ and covariance $C_Y(t_1,t_2)$.

(d) Any n time samples of $Y(t)$ are defined by a linear transformation of jointly Gaussian random variables. Thus the samples have a jointly Gaussian pdf.

9.54

~~6.16~~ **a)** $\mathcal{E}[X(t)] = \mathcal{E}[A\cos\omega t + B\sin\omega t] = \mathcal{E}[A]\cos\omega t + \mathcal{E}[B]\sin\omega t = 0$

$$\begin{aligned}
C_X(t_1,t_2) &= \mathcal{E}[X(t_1)X(t_2)] - m_X(t_1)m_X(t_2) \\
&= \mathcal{E}[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\
&= \mathcal{E}[A^2]\cos\omega t_1\cos\omega t_2 + \mathcal{E}[A]\mathcal{E}[B]\cos\omega t_1\sin\omega t_2 \\
&\quad +\mathcal{E}[A]\mathcal{E}[B]\cos\omega t_2\sin\omega t_1 + \mathcal{E}[B^2]\sin\omega t_2\sin\omega t_2 \\
&= \sigma^2(\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2) \\
&= \sigma^2\cos\omega(t_1-t_2)
\end{aligned}$$

b) Because A and B are jointly Gaussian RV's, $X(t) = A\cos\omega t + B\sin\omega t$ and $X(t+d) = A\cos\omega(t+s) + B\sin\omega(t+d)$ are also jointly Gaussian, with zero means and covariance matrix

$$K = \begin{bmatrix} \sigma^2 & \sigma^2\cos\omega d \\ \sigma^2\cos\omega d & \sigma^2 \end{bmatrix} \qquad |K|^{1/2} = \sigma^2|1-\cos^2\omega d| = \sigma^2|\sin\omega d|$$

$$K^{-1} = \frac{1}{\sigma^4\sin^2\omega d}\begin{bmatrix} \sigma^2 & -\sigma^2\cos\omega d \\ -\sigma^2\cos\omega d & \sigma^2 \end{bmatrix}$$

$$\begin{aligned}
f_{X(t)X(t+d)}(x_1,x_2) &= \frac{\exp\left\{-\frac{1}{2}\underline{x}'K^{-1}\underline{x}\right\}}{2\pi\sigma^2|\sin\omega s|} \\
&= \frac{\exp\left\{-\frac{x_1^2-2\cos\omega d\, x_1x_2+x_2^2}{2\sigma^2\sin^2\omega d}\right\}}{2\pi\sigma^2|\sin\omega d|}
\end{aligned}$$

9.55

~~6.18~~ $Z(t) = X(t)\cos\omega t + Y(t)\sin\omega t$

a) $\mathcal{E}[Z(t)] = m_X(t)\cos\omega t + m_Y(t)\sin\omega t = 0$

$$\begin{aligned}
C_Z(t_1,t_2) &= \mathcal{E}[(X(t_1)\cos\omega t_1 + Y(t_1)\sin\omega t_1) \\
&\qquad (X(t_2)\cos\omega t_2 + Y(t_2)\sin\omega t_2)] \\
&= \mathcal{E}[X(t_1)X(t_2)]\cos\omega t_1\cos\omega t_2 \\
&\quad + \underbrace{\mathcal{E}[X(t_1)Y(t_2)]}_{0}\cos\omega t_1\sin\omega t_2 \\
&\quad + \underbrace{\mathcal{E}[Y(t_1)X(t_2)]}_{0}\sin\omega t_1\cos\omega t_2 + \mathcal{E}[Y(t_1)Y(t_2)]\sin\omega t_1\sin\omega t_2 \\
&= C(t_1,t_2)\cos\omega t_1\cos\omega t_2 + C(t_1,t_2)\sin\omega t_1\sin\omega t_2 \\
&= C(t_1,t_2)\cos\omega(t_1-t_2)
\end{aligned}$$

b) $f_{Z(t)}(z) = \dfrac{1}{\sqrt{2\pi C(t,t)}}e^{-z^4/2C(t,t)}$

9.56

~~6.19~~ $\mathcal{E}[Y(t)] = \mathcal{E}[X^2(t)] = C_X(t,t)$

$\mathcal{E}[Y(t_1)Y(t_2)] = \mathcal{E}[X^2(t_1)X^2(t_2)]$

To proceed further we need the result from Problem 4.69

$$\begin{aligned}\mathcal{E}[X^2(t_1)X^2(t_2)] &= \mathcal{E}[X^2(t_1)]\mathcal{E}[X^2(t_2)] + 2\mathcal{E}[X(t_1)X(t_2)]^2 \\ \Rightarrow \mathcal{E}[Y(t_1)Y(t_2)] &= C_X(t_1,t_1)C_X(t_2,t_2) + 2C_X^2(t_1,t_2)\end{aligned}$$

Since arrival times are iid unif. dist. in [9.60]

$$\therefore \mathcal{E}[W] = \mathcal{E}[\mathcal{E}[W?N=k]] = \mathcal{E}[30N] = 30\mathcal{E}[N] = 30\lambda$$

9.57

~~6.46~~ a)

$$\begin{aligned}F_{Y(t)}(y) &= P[Y(t) \le y] = P[X(t) + \mu t \le y] = P[X(t) \le y - \mu t] \\ &= F_{X(t)}(y - \mu t) \\ \Rightarrow \quad f_{Y(t)}(y) = F_X(y - \mu t) &= \frac{1}{\sqrt{2\pi\alpha t}} e^{-(y-\mu t)^2/2\alpha t}\end{aligned}$$

b)

$$\begin{aligned}F_{Y(t)Y(t+s)}(y_1,y_2) &= P[X(t) + \mu t \le y_1, X(t+s) + \mu(t+s) \le y_2] \\ &= F_{X(t),X(t+s)}(y_1 - \mu t, y_2 - \mu(t+s)) \\ \Rightarrow f_{Y(t)Y(t+s)}(y_1,y_2) &= f_{X(t),X(t+s)}(y_1 - \mu t, y_2 - \mu(t+s)) \\ &= f_{X(t)}(y_1 - \mu t) f_{X(s)}(y_2 - y_1 - \mu s) \\ &= \frac{3^{-(y_1-\mu t)^2/2\alpha t}}{\sqrt{2\pi\alpha t}} \frac{e^{-(y_2-y_1-\mu s)^2/2\alpha s}}{\sqrt{2\pi\alpha s}}\end{aligned}$$

9.58

~~**6.47**~~ **a)** $Y(t) = X^2(t)$

$$\Rightarrow f_{Y(t)}(y) = \frac{1}{\sqrt{2\pi\alpha t y}} e^{-y/2\alpha t}$$ from Example ~~3.26~~ 4.33

b) $$f_{Y(t_2)}(y_2|Y(t_1) = y_1) = \frac{f_{Y(t_2),Y(t_1)}(y_2, y_1)}{f_{Y(t_1)}(y_1)}$$

$$\begin{aligned} Y(t_1) &= X^2(t_1) \\ Y(t_2) &= X^2(t_2) \end{aligned}$$

$$\begin{aligned} J(x(t_1), x(t_2)) &= \left| \begin{array}{cc} 2x(t_1) & 0 \\ 0 & 2x(t_2) \end{array} \right| = 4|x(t_1)x(t_2)| \\ &= 4\sqrt{y(t_1)y(t_2)} \end{aligned}$$

$$\begin{aligned} f_{Y(t_1)Y(t_2)}(y_1, y_2) &= \frac{f_{X_1X_2}(\sqrt{y_1}, \sqrt{y_2})}{4\sqrt{y_1y_2}} + \frac{f_{X_1X_2}(\sqrt{y_1}, -\sqrt{y_2})}{4\sqrt{y_1y_2}} \\ &\quad + \frac{f_{X_1X_2}(-\sqrt{y_1}, \sqrt{y_2})}{4\sqrt{y_1y_2}} + \frac{f_{X_1X_2}(-\sqrt{y_1}, -\sqrt{y_2})}{4\sqrt{y_1y_2}} \\ &= \frac{e^{-\frac{y_1}{2\alpha t_1}} \left\{ e^{-\frac{y_2+y_1+2\sqrt{y_1y_2}}{2\alpha(t_2-t_1)}} + e^{-\frac{y_2+y_1+2\sqrt{y_1y_2}}{2\alpha(t_2-t_1)}} \right\}}{4\pi\alpha\sqrt{t_1(t_2-t_1)y_1y_2}} \end{aligned}$$

$$\Rightarrow f_{Y(t_2)}(y_2|Y(t_1) = y_1) = \frac{e^{-\frac{y_2+y_1-2\sqrt{y_1y_2}}{2\alpha(t_2-t_1)}} + e^{-\frac{y_2+y_1+2\sqrt{y_1y_2}}{2\alpha(t_2-t_1)}}}{2\sqrt{2\pi\alpha(t_2-t_1)y_2}}$$

9.59

~~6.48~~ **a)** From Example ~~4.32~~ 6.24 we know that $Z(t) = X(t) - \alpha X(t-s)$ is a Gaussian RV since $X(t)$ and $X(t-s)$ are jointly Gaussian. Therefore we need only find $m_Z(t)$ and $VAR[Z(t)]$

$$\begin{aligned}
m_Z(t) &= \mathcal{E}[X(t)] - a\mathcal{E}[X(t-s)] = 0 \\
VAR[Z(t)] &= \mathcal{E}[(X(t) - aX(t-s))^2] \\
&= \mathcal{E}[X^2(t)] - 2a\mathcal{E}[X(t)X(t-s)] + a^2\mathcal{E}[X^2(t-s)] \\
VAR[Z(t)] &= \alpha t - 2a(\alpha(t-s)) + a^2\alpha(t-s) \\
&= \alpha t(1 - 2a + a^2) + 2a\alpha s - a^2\alpha s \\
&= \alpha t(a-1)^2 - a\alpha s(a-2) \\
f_{Z(t)}(z) &= \frac{\exp\left\{-\frac{z^2}{2VAR[Z(t)]}\right\}}{\sqrt{2\pi VAR[Z(t)]}}
\end{aligned}$$

b)

$$\begin{aligned}
m_Z(t) &= E[X(t) - aX(t-s)] = 0 \\
C_Z(t_1, t_2) &= E[\{Z(t_1) - m(t_2)\}\{Z(t_2) - m(t_2)\}] \\
&= E[\{X(t_2) - aX(t_1 - s)\}\{X(t_2) - aX(t_2 - s)\}] \\
&= \alpha \min(t_1, t_2) - a\alpha \min(t_1 - s, t_2) \\
&\quad - a\alpha \min(t_1, t_2 - s) + a^2\alpha \min(t_1, t_2)
\end{aligned}$$

(9.60)) $\alpha = 1$ & $0 < t < 1$

a) Using Eq. 9.52 we have:

$$f_{X(t),X(1)}(x_1, x_2) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t} + \frac{(x_2 - x_1)^2}{(1-t)}\right]\right\}}{2\pi\sqrt{t(1-t)}}$$

b) $$f_{X(t)}(x \mid X(0) = X(1) = 0) = f_{X(t),X(1)}(x_1, 0) = \frac{\exp\left\{-\frac{1}{2}\frac{x_1^2}{t(1-t)}\right\}}{2\pi\sqrt{t(1-t)}}$$

c) $$f_{X(t)}(x \mid X(t_1) = a, X(t_2) = b) = f_{Y(t)}(y \mid Y(t_2 - t_1) = b - a)$$

$$= f_{Y(t),Y(t_2-t_1)}(y, b-a)$$

$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{y^2}{t} + \frac{(b-a-y)^2}{(t_2 - t_1 - t)}\right]\right\}}{2\pi\sqrt{t(t_2 - t_1 - t)}}$$

End of Section: END OF PART I.

Chapter 10: Analysis and Processing of Random Signals

10.1 Power Spectral Density

10.1 ~~1.1~~ a) $S_X(f) = \mathcal{F}\left[g\left(\frac{\tau}{T}\right)\right] = AT\left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}\right)^2$

Table in Appendix B.

b) $S_X(f) = g\left(\frac{f}{W}\right)$.

$$R_X(\tau) = AW\left(\frac{\sin\frac{W\tau}{2}}{\frac{W\tau}{2}}\right)^2$$

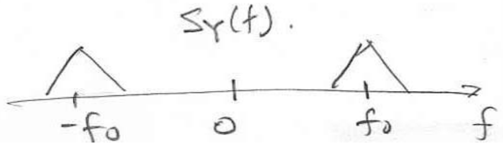
10.2 ~~1.2~~ $\mathcal{F}\left[p\left(\frac{\tau}{T}\right)\right] = 2AT\frac{\sin 2\pi fT}{2\pi fT}$ which is negative for some values of f. Since power spectral densities are always non-negative, $p\left(\frac{\tau}{T}\right)$ is <u>not</u> a valid autocorrelation function.

10.3

$$\begin{aligned}
S_Y(f) &= \mathcal{F}[R_X(\tau)\cos 2\pi f_0\tau] \\
&= \mathcal{F}\left[R_X(\tau)\left[\frac{e^{j2\pi f_0\tau} + e^{-j2\pi f_0\tau}}{2}\right]\right] \\
&= \frac{1}{2}\mathcal{F}[R_X(\tau)e^{j2\pi f_0\tau}] + \frac{1}{2}\mathcal{F}[R_X(\tau)e^{-j2\pi f_0\tau}] \\
&= \frac{1}{2}S_X(f-f_0) + \frac{1}{2}S_X(t+f_0)
\end{aligned}$$

where

$$S_X(f) = \mathcal{F}[R_X(\tau)]$$



10.4

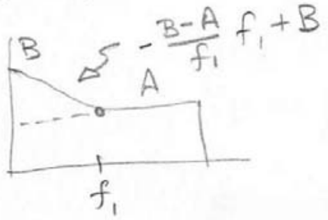
.4 $S_X(f) = Ap\left(\frac{f}{f_2}\right) + (B-A)g\left(\frac{f}{f_1}\right)$ where $p(x)$ is as in Fig. P10.2 and $g(x)$ as in Fig. P10.1.
Then from Table C in Appendix B:

$$R_X(\tau) = 2Af_2\frac{\sin 2\pi f_2\tau}{2\pi f_2\tau} + (B-A)f_1\left(\frac{\sin 2\pi f_1\tau/2}{2\pi f_1\tau/2}\right)^2$$

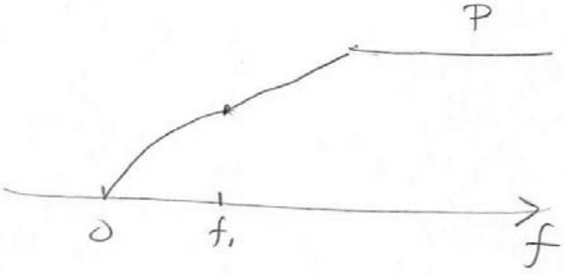
– continued –

ⓑ $$P = \int_{-\infty}^{\infty} S_X(f)\,df = A(2f_2) + (B-A)(2f_1)\frac{1}{2}$$

$$= 2Af_2 + (B-A)f_1$$



ⓒ $$2\int_0^{f_0} S_X(f)\,df = \begin{cases} -2\left(\frac{B-A}{f_1}\right)\frac{f_0^2}{2} + Bf_0 & 0<f_0<f_1 \\ 2\left[\frac{B-A}{2}f_1 + A(f_0-f_1)\right] & f_1<f_0<f_2 \end{cases}$$



10.5

$R_X(\tau) = \delta_X^2 e^{-2\alpha^2\tau^2}$,

a) From Fourier transform table we have: $e^{-\frac{t^2}{2\beta^2}} \xleftrightarrow{\mathcal{F}} \beta\sqrt{2\pi}\, e^{-\frac{\beta^2(2\pi f)^2}{2}}$

in our case $\beta^2 = \frac{1}{4\alpha^2}$ & $\beta = \frac{1}{2\alpha}$

Therefore: $S_X(f) = \delta_X^2 \times \frac{\sqrt{2\pi}}{2\alpha} e^{-\frac{f^2(2\pi)^2}{2(4\alpha^2)}} = \delta_X^2 \frac{1}{\frac{\alpha}{\pi}\sqrt{2\pi}} e^{-\frac{f^2}{2(\frac{\alpha}{\pi})^2}}$

as it can be seen $S_X(f)$ is δ_X^2 times of a Gaussian function with mean zero and variance $\frac{\alpha^2}{\pi^2}$.

b) $\int_{-\infty}^{-k\alpha} S_X(f)\,df + \int_{k\alpha}^{+\infty} S_X(f)\,df = 2 \times \int_{k\alpha}^{\infty} S_X(f)\,df = 2\delta_X^2\, Q\left(\frac{k\alpha}{\frac{\alpha}{\pi}}\right) = 2\delta_X^2 Q(k\pi)$

in which Q is the Q-function.

Therefor: $k = 1$, $2\delta_X^2 Q(\pi) = 2\delta_X^2 \times 8.4\times10^{-4}$

$k = 2$, $2\delta_X^2 (Q(2\pi)) = 2\delta_X^2 \times 1.65\times10^{-10}$

$k = 3$, $2\delta_X^2 Q(3\pi) \approx 2\delta_X^2 \times Q(9.5) = 2\delta_X^2 \times 1.05\times10^{-21}$

10.6

~~7.5~~ From Example 10.4

$$\begin{aligned} R_Z(\tau) &= R_X(\tau) + R_{YX}(\tau) + R_{XY}(\tau) + R_Y(\tau) \\ \Rightarrow S_Z(f) &= S_X(f) + S_{YX}(f) + S_{XY}(f) + S_Y(f) \end{aligned}$$

If $X(t)$ and $Y(t)$ are orthogonal processes, then

$$R_{XY}(\tau) = R_{XY}(\tau) = 0 \Rightarrow S_Z(f) = S_X(f) + S_Y(f)$$

10.7

~~7.6~~ a)

$$\begin{aligned} R_{XY}(\tau) &= \mathcal{E}[X(t+\tau)Y(t)] \\ &= \mathcal{E}[Y(t)X(t+\tau)] \\ &= R_{YX}(-\tau) \end{aligned}$$

b)

$$\begin{aligned} S_{XY}(t) &= \mathcal{F}[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} R_{YX}(-\tau)e^{-j2\pi f\tau}d\tau \\ &= \int_{-\infty}^{\infty} R_{YX}(\tau')e^{+j2\pi f\tau'}d\tau' \\ &= S_{YX}^*(f) \end{aligned}$$

10.8

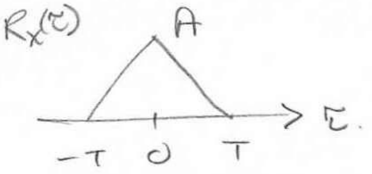
~~7.7~~ a)

$$\begin{aligned} R_{XY}(\tau) &= \mathcal{E}[X(t+\tau)[X(t) - X(t-d)]] \\ &= R_X(\tau) - R_X(\tau+d) \\ S_{XY}(f) &= \mathcal{F}[R_X(\tau)] - \mathcal{F}[R_X(\tau+d)] = S_X(f) - S_X(f)e^{j2\pi fd} \end{aligned}$$

b)

$$\begin{aligned} R_Y(\tau) &= \mathcal{E}[(X(t+\tau) - X(t+\tau-d))(X(t) - X(t-d))] \\ &= R_X(\tau) - R_X(\tau+d) - R_X(\tau-d) + R_X(\tau) \\ &= 2R_X(\tau) - R_X(\tau+d) - R_X(\tau-d) \\ S_Y(f) &= 2S_X(f) - S_X(f)e^{j2\pi fd} - S_X(f)e^{-j2\pi fd} \\ &= 2S_X(f)[1 - \cos 2\pi fd] \end{aligned}$$

10.9



$g(\frac{\tau}{T})$

ⓐ

$$R_{XY}(\tau) = g\left(\frac{\tau}{T}\right) - g\left(\frac{\tau+d}{T}\right)$$

$$S_{XY}(f) = S_X(f) - S_X(f)e^{j2\pi d}$$

$$= AT\left(\frac{\sin \pi fT}{\pi fT}\right)^2\left(1 - e^{-j2\pi fd}\right)$$

ⓑ

$$R_Y(\tau) = 2g\left(\frac{\tau}{T}\right) - g\left(\frac{\tau+d}{T}\right) - g\left(\frac{\tau-d}{T}\right)$$

$$S_Y(f) = 2AT\left(\frac{\sin \pi fT}{\pi fT}\right)^2\left(1 - \cos 2\pi fd\right)$$

10.10

$$\begin{aligned}
\mathcal{E}[Z(t)] &= \mathcal{E}[X(t)]\mathcal{E}[Y(t)] = m_X m_Y \\
R_Z(t,t+\tau) &= \mathcal{E}[X(t)Y(t)X(t+\tau)Y(t+\tau)] \\
&= \mathcal{E}[X(t)X(t+\tau)]\mathcal{E}[Y(t)Y(t+\tau)] \\
&= R_X(\tau)R_Y(\tau) \\
S_Z(f) &= \mathcal{F}[R_X(\tau)R_Y(\tau)] = S_X(f) \star S_Y(f)
\end{aligned}$$

10.11

$X(t) = a\cos(2\pi f_0 t + \theta)$, $\Theta \sim U(0, 2\pi]$

from P.10.10 we have : $S_Z(f) = S_X(f) * S_Y(f)$

$R_Z(\tau) = R_X(\tau) . R_Y(\tau)$

$$R_X(\tau) = E[X(t)X(t+\tau)] = \frac{a^2}{2\pi}\int_0^{2\pi} \cos(2\pi f_0 t + \theta)\cos(2\pi f_0 t + 2\pi f\tau + \theta)\,d\theta$$

$$= \frac{a^2}{2\pi} \times \frac{1}{2}\left[\int_0^{2\pi} \cos(2\pi f_0 \tau)\,d\theta + \int_0^{2\pi} \cos(4\pi f_0 t + 2\pi f_0 \tau + 2\theta)\,d\theta\right]$$

$$= \frac{a^2}{2\pi} \times \frac{1}{2} \times 2\pi \times \cos(2\pi f_0 \tau) + 0 = \frac{a^2}{2}\cos(2\pi f_0 \tau)$$

$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \frac{1}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$$

Therfore

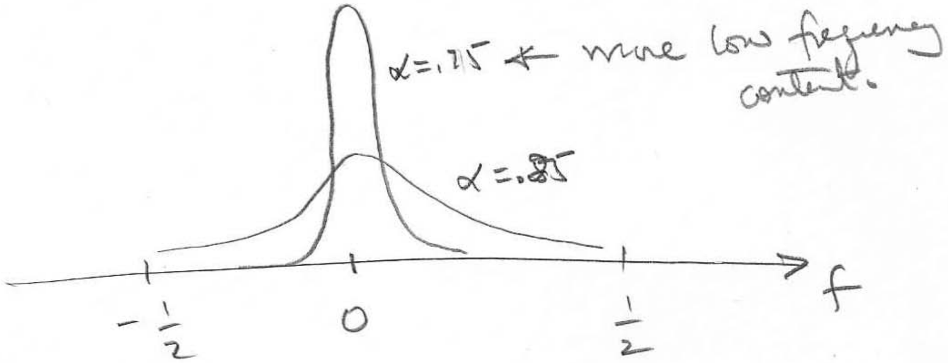
$$S_Z(f) = \frac{1}{2}\left[S_Y(f - f_0) + S_Y(f + f_0)\right]$$

and

$$R_Z(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau)\,R_Y(\tau)$$

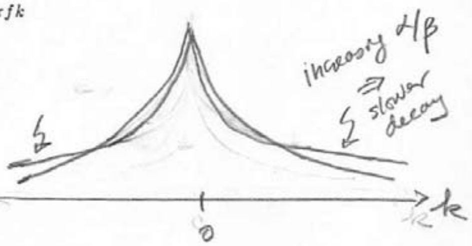
10.12 $R_X(k) = 4\alpha^{|k|} \quad |\alpha| < 1.$

ⓐ
$$S_X(f) = 4\sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi fk}$$
$$= 4 + 4\sum_{k=1}^{\infty} \alpha^k e^{-j2\pi fk} + 4\sum_{k=-\infty}^{-1} \left(\frac{1}{\alpha}\right)^k e^{-j2\pi fk}$$
$$= 4 + \frac{\alpha e^{-j2\pi f}}{1-\alpha e^{-j2\pi f}} + \frac{4\alpha e^{j2\pi f}}{1-\alpha e^{j2\pi f}}$$
$$= \frac{4(1-\alpha^2)}{1+\alpha^2-2\alpha\cos 2\pi f}$$



10.13

$$\sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-j2\pi fk} = 1 + \sum_{k=1}^{\infty} \alpha^k e^{-j2\pi fk} + \sum_{k=-\infty}^{-1} \left(\frac{1}{\alpha}\right)^k e^{-j2\pi fk}$$
$$= 1 + \frac{\alpha e^{-j2\pi f}}{1-\alpha e^{-j2\pi f}} + \frac{\alpha e^{j2\pi f}}{1-\alpha e^{j2\pi f}}$$
$$= \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f}$$
$$S_X(f) = \mathcal{F}\left[4(\alpha)^{|k|} + 16(\beta)^{|k|}\right]$$
$$= 4\frac{1-(\alpha)^2}{1+(\alpha)^2-2(\alpha)\cos 2\pi f} + 16\frac{1-(\beta)^2}{1+(\beta)^2-2(\beta)\cos 2\pi f}$$



10.14

$$\begin{aligned}
S_X(f) &= \sum_{k=-N}^{N}\left(1-\frac{|k|}{N}\right)e^{-j2\pi fk} \\
&= 1+\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)e^{-j2\pi fk}+\sum_{k=-(N-1)}^{-1}\left(1+\frac{k}{N}\right)e^{-j2\pi fk} \qquad k'=-k \\
&= 1+\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)e^{-j2\pi fk}+\sum_{k'=1}^{N-1}\left(1-\frac{k'}{N}\right)e^{j2\pi fk'}
\end{aligned}$$

We need the following summations

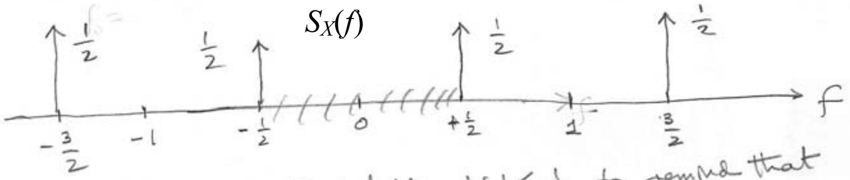
$$\sum_{k=1}^{N-1}\alpha^k=\alpha\sum_{k'=0}^{N-2}\alpha^{k'}=\frac{\alpha-\alpha^N}{1-\alpha}$$

Taking derivatives with respect to α:

$$\begin{aligned}
\sum_{k=1}^{N-1}k\alpha^{k-1} &= \frac{(1-N\alpha^{N-1})(1-\alpha)+(\alpha-\alpha^N)}{(1-\alpha)^2} \\
\Rightarrow\sum_{k=1}^{N-1}k\alpha^{k} &= \frac{(\alpha-N\alpha^{N})(1-\alpha)+\alpha(\alpha-\alpha^N)}{(1-\alpha)^2} \\
\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\alpha^k &= \sum_{k=1}^{N-1}\alpha^k-\frac{1}{N}\sum_{k=1}^{N-1}k\alpha^k \\
&= \frac{\alpha-\alpha^N}{1-\alpha}-\frac{1}{N}\frac{(\alpha-N\alpha^{N})(1-\alpha)+\alpha(\alpha-\alpha^N)}{(1-\alpha)^2} \\
&= \left(1-\frac{1}{N}\right)\frac{\alpha}{1-\alpha}-\frac{1}{N}\frac{\alpha^2(1-\alpha^{N-1})}{(1-\alpha)^2} \\
S_X(f) &= 1+\left(1-\frac{1}{N}\right)\left\{\frac{e^{-j2\pi f}}{1-e^{-j2\pi f}}+\frac{e^{j2\pi f}}{1-e^{j2\pi f}}\right\} \\
&\quad -\frac{1}{N}\left\{\left(\frac{e^{-j2\pi f}}{1-e^{-j2\pi f}}\right)^2(1-e^{-j2\pi f(N-1)})+\left(\frac{e^{j2\pi f}}{1-e^{j2\pi f}}\right)^2(1-e^{j2\pi f(N-1)})\right\} \\
&= 1+\left(1-\frac{1}{N}\right)\left\{\frac{e^{-j\pi f}}{e^{j\pi f}-e^{-j\pi f}}+\frac{e^{j\pi f}}{e^{-j\pi f}-e^{+j\pi f}}\right\} \\
&\quad -\frac{1}{N}\left\{\left(\frac{e^{-j\pi f}}{e^{j\pi f}-e^{-j\pi f}}\right)^2 e^{-j\pi f(N-1)}(e^{j\pi f(N-1)}-e^{-j\pi f(N-1)})\right. \\
&\quad \left.+\left(\frac{e^{j\pi f}}{e^{-j\pi f}-e^{j\pi f}}\right)^2 e^{j\pi f(N-1)}(e^{-j\pi f(N-1)}-e^{+j\pi f(N-1)})\right\} \\
&= 1+\left(1-\frac{1}{N}\right)\left\{\frac{e^{-j\pi f}-e^{j\pi f}}{2j\sin\pi f}\right\} \\
&\quad -\frac{1}{N}\left\{e^{-j\pi f(N+1)}\frac{2j\sin\pi f(N-1)}{-4\sin^2\pi f}+e^{j\pi f(N+1)}\frac{(-2j)\sin\pi f(N-1)}{-4\sin^2\pi f}\right\} \\
&= \frac{1}{N}\left\{1-\frac{\sin\pi f(N-1)}{4\sin^2\pi f}\sin\pi f(N+1)\right\} \\
&= \frac{\sin^2\pi Nf}{4N\sin^2\pi f}
\end{aligned}$$

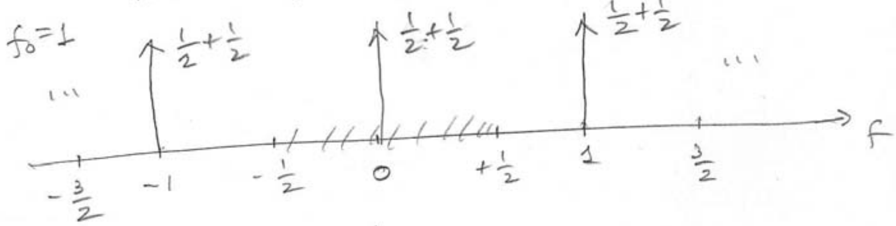
10.15 $X_n = \cos(2\pi f_0 n + \Theta)$

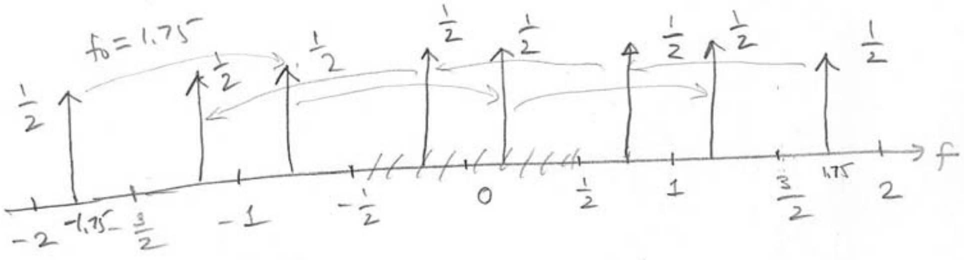
$$\begin{aligned}
R_X(n, n+k) &= \mathcal{E}[X_n X_{n+k}] = \mathcal{E}[\cos(2\pi f_0 n + \Theta)\cos(2\pi f_0(n+k) + \Theta)] \\
&= \mathcal{E}\left[\frac{1}{2}\cos 2\pi f_0 k + \frac{1}{2}\cos\left(4\pi f_0 n + 2\pi f_0 k + 2\Theta\right)\right] \\
&= \frac{1}{2}\cos 2\pi f_0 k \\
S_X(f) &= \sum_{k=-\infty}^{\infty} \frac{1}{2}\cos 2\pi f_0 k e^{-j2\pi f k} \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} (e^{j2\pi f_0 k} + e^{-j2\pi f_0 k})e^{-j2\pi f k} \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty} \left\{e^{j2\pi(f+f_0)k} + e^{-j2\pi(f-f_0)k}\right\} \\
&= \frac{1}{2}\delta(f+f_0) + \frac{1}{2}\delta(f-f_0)
\end{aligned}$$

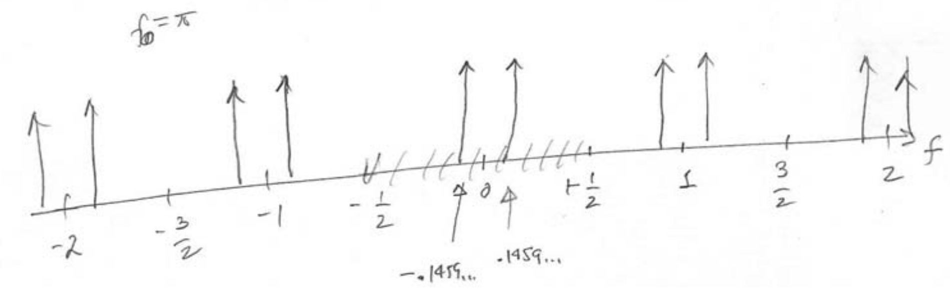


We show frequencies outside $|f| \le \frac{1}{2}$ to remind that the power spectral density of discrete-time processes is periodic.

Thus sinusoids with frequencies outside $|f| \le \frac{1}{2}$ reflect back into the range $|f| \le \frac{1}{2}$.



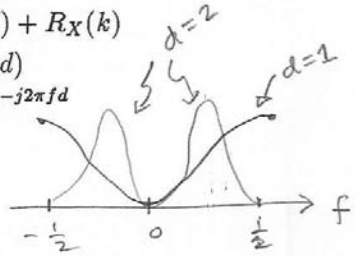




10.16

~~7.12~~ a) $\mathcal{E}[D_n] = \mathcal{E}[X_n] - \mathcal{E}[X_{n-d}] = 0$

$$\begin{aligned}
R_D(n, n+k) &= \mathcal{E}[(X_n - X_{n-d})(X_{n+k} - X_{n+k-d})] \\
&= R_X(k) - R_X(d+k) - R_X(k-d) + R_X(k) \\
&= 2R_X(k) - R_X(k+d) - R_X(k-d) \\
S_D(f) &= 2S_X(f) - S_X(f)e^{j2\pi fd} - S_X(f)e^{-j2\pi fd} \\
&= 2S_X(f)\underbrace{(1 - \cos 2\pi fd)}_{}
\end{aligned}$$



b) $\mathcal{E}[D_n^2] = R_D(0) = 2R_X(0) - 2R_X(d)$

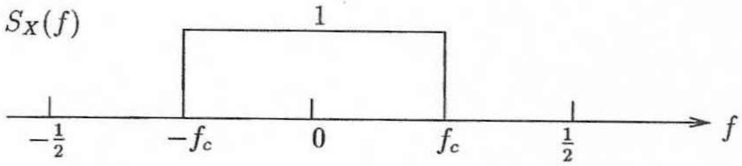
10.17

~~7.~~13 The moving average process with $\alpha = 1$ has

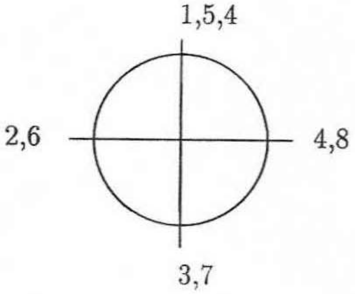
$$R_X(k) = \begin{cases} 2\sigma_X^2 & k = 0 \\ \sigma_X^2 & k = \pm 1 \\ 0 & k > 1 \end{cases}$$

$$\begin{aligned}
R_D(k) &= 2R_X(k) - R_X(k+d) - R_X(k-d) \\
S_D(f) &= 2S_X(f)(1 - \cos 2\pi fd) \\
&= 4\sigma_X^2(1 + \cos 2\pi f)(1 - \cos 2\pi fd) \\
&= 4\sigma_X^2(1 - \cos 2\pi fd + \cos 2\pi f - \frac{1}{2}\cos 2\pi f(d-1) \\
&\quad -\frac{1}{2}\cos 2\pi f(d+1))
\end{aligned}$$

10.18



$$\begin{aligned} R_X(k) &= \int_{-f_c}^{f_c} e^{j2\pi fk} df = \left.\frac{e^{j2\pi fk}}{j2\pi k}\right|_{-f_c}^{f_c} = \frac{e^{j2\pi f_c k} - e^{-j2\pi f_c k}}{j2\pi k} \\ &= \frac{\sin 2\pi f_c k}{\pi k} \end{aligned}$$



If $f_c = \frac{1}{4}$

$$R_X(k) = \frac{\sin k\frac{\pi}{2}}{\pi k} = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{\pi k} & k = 1, 5, 9, ... \\ -\frac{1}{\pi k} & k = 3, 7, 11, ... \\ 0 & k \text{ even} \end{cases}$$

10.19

~~7.15~~ a) $\mathcal{E}[Y_n] = \mathcal{E}[W_n]\mathcal{E}[X_n] = 0$

$$\begin{aligned} \mathcal{E}[Y_n, Y_{n+k}] &= \mathcal{E}[W_n W_{n+k} X_n X_{n+k}] \\ &= \mathcal{E}[W_n W_{n+k}]\mathcal{E}[X_n X_{n+k}] \\ &= \delta_{k,0}\mathcal{E}[X_n^2] \Rightarrow Y_n \text{ is a white noise sequence} \end{aligned}$$

$$\sigma_{Y_n}^2 = R_Y(0) = \mathcal{E}[X_n^2]$$

b)

$$\begin{aligned} \mathcal{E}[Y_n] &= 0 \\ R_Y(k) &= \begin{cases} \left(\frac{1}{2}\right)^0 = 1 & k = 0 \\ 0 & ew \end{cases} \end{aligned}$$

$$\begin{aligned} f_{Y_{n_1} Y_{n_2} \ldots Y_{n_k}}(y_1, \ldots, y_k) &= f_{Y_{n_1}}(y_1) \ldots f_{Y_{n_k}}(y_k) \\ &= \frac{e^{-\frac{1}{2}(y_1^2 + \ldots + y_k^2)}}{\sqrt{2\pi}^k} \end{aligned}$$

10.20

~~7.16~~

$$\begin{aligned}
\tilde{x}_T(f) &= \int_0^T a\cos(2\pi f_0 t' + \Theta)e^{-j2\pi f t'}dt' \\
&= \frac{a}{2}\int_0^T e^{j\Theta}e^{j2\pi f_0 t'}e^{-j2\pi f t'}dt' + \frac{a}{2}\int_0^T e^{-j\Theta}e^{-j2\pi f_0 t'}e^{-j2\pi f t'}dt' \\
&= \frac{a}{2}e^{j\Theta}\left[\frac{e^{j2\pi(f_0-f)T}-1}{j2\pi(f_0-f)}\right] + \frac{a}{2}e^{-j\Theta}\left[\frac{1-e^{-j2\pi(f_0+f)T}}{j2\pi(f_0+f)}\right] \\
&= \frac{a}{2}e^{j\Theta}e^{j\pi(f_0-f)T}\left[\frac{e^{j\pi(f_0-f)T}-e^{-j\pi(f_0-f)T}}{j2\pi(f_0-f)}\right] \\
&\quad + \frac{a}{2}e^{-j\Theta}e^{-j\pi(f_0+f)}\left[\frac{e^{j\pi(f_0+f)T}-e^{-j\pi(f_0+f)T}}{j2\pi(f_0+f)}\right] \\
&= \frac{a}{2}e^{j\Theta}e^{j\pi(f_0-f)}\frac{\sin\pi(f_0-f)T}{\pi(f_0-f)} \\
&\quad + \frac{a}{2}e^{-j\Theta}e^{-j\pi(f_0+f)}\frac{\sin\pi(f_0+f)T}{\pi(f_0+f)}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_T(f) &= \frac{1}{T}|\tilde{x}_T(f)|^2 \\
&= \frac{a^2}{4T}\frac{\sin^2\pi(f_0-f)T}{\pi(f_0-f)^2} + \frac{a^2}{4T}\frac{\sin^2\pi(f_0+f)T}{\pi^2(f_0+f)^2}
\end{aligned}$$

$$\begin{aligned}
&+\frac{a^2}{4T}e^{j2\Theta}e^{j\pi(2f_0)}\frac{\sin\pi(f_0-f)T}{\pi(f_0-f)}\frac{\sin\pi(f_0+f)T}{\pi(f_0+f)} \\
&+\frac{a^2}{4T}e^{-j2\Theta}e^{-j\pi 2f_0}\frac{\sin\pi(f_0-f)T}{\pi(f_0-f)}\frac{\sin\pi(f_0+f)T}{\pi(f_0+f)}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}_T(f) &= \frac{a^2}{4T}\frac{\sin^2\pi(f_0-f)T}{\pi^2(f_0-f)^2} + \frac{a^2}{4T}\frac{\sin^2\pi(f_0+f)T}{\pi^2(f_0+f)^2} \\
&\quad + \frac{a^2}{4T}\cos(2\pi f_0 + 2\Theta)\frac{\sin\pi(f_0-f)T}{\pi(f_0-f)}\frac{\sin\pi(f_0+f)T}{\pi(f_0+f)}
\end{aligned}$$

NOTE

$$\begin{aligned}
\tilde{p}_T(f_0) &= \frac{a^2T}{4} + \frac{a^2}{4T}\frac{\sin^2\pi 2f_0T}{\pi^2 4f_0^2} + \frac{a^2\cos(2\pi f_0 + 2\Theta)}{4}\frac{\sin\pi(2f_0T)}{2\pi f_0} \\
\tilde{p}_T(-f_0) &= \frac{a^2T}{4} + \frac{a^2}{4T}\frac{\sin^2\pi 2f_0T}{\pi^2 4f_0^2} + \frac{a^2}{4}\cos(2\pi f_0 + 2\Theta)\frac{\sin 2\pi f_0T}{2\pi f_0}
\end{aligned}$$

$\therefore$ as $T \to \infty$

$$\tilde{p}_T(f) \to \frac{a^2}{4}\delta(f-f_0) + \frac{a^2}{4}\delta(f+f_0)$$

10.21

```
%P.10.21
clear all;
close all;
samples = 50;
len=128;
x=randn(len,samples);
xf=fft(x);
pf=sqrt(xf'.*conj(xf'))/len;
sf=mean(pf);
sf1=mean(pf(1:10,:));
sf2=mean(pf(1:20,:));
sf3=mean(pf(1:30,:));
sf4=mean(pf(1:40,:));
subplot(2,3,1);
plot(1:len,sf);
axis([1 len 0.06 0.11]);
title('All (50) realizations');
subplot(2,3,2);
plot(1:len,sf1);
axis([1 len 0.06 0.11]);
title('10 realizations');
subplot(2,3,3);
plot(1:len,sf2);
axis([1 len 0.06 0.11]);
title('20 realizations');
subplot(2,3,4);
plot(1:len,sf3);
axis([1 len 0.06 0.11]);
title('30 realizations');
subplot(2,3,5);
plot(1:len,sf4);
axis([1 len 0.06 0.11]);
title('40 realizations');
```

10.2 Response of Linear Systems to Random Signals

10.22 ~~7.17~~ **a)** $S_Y(f) = |H(f)|^2 S_X(f) = 4\pi^2 f^2 S_X(f)$

b) $R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = -\frac{d^2}{d\tau^2} R_X(\tau)$

10.23 ~~7.18~~ **a)** $S_Y(f) = 4\pi^2 f^2 \frac{N_0}{2} \qquad \mathcal{F} < W$

$$\begin{aligned}
R_Y(\tau) &= \int_{-W}^{W} 2\pi^2 N_0 f^2 e^{j2\pi f\tau} df \\
&= 2\pi^2 N_0 \left[e^{j2\pi f\tau} \frac{(-4\pi^2 f^2\tau^2 - 2j2\pi f\tau + 2)}{(j2\pi\tau)^3} \right]_{-W}^{W} \\
&= 2\pi^2 N_0 \left[e^{j2\pi W\tau} \frac{(2 - 4\pi^2 W^2\tau^2 - 4j\pi W\tau)}{-j8\pi^3\tau^3} \right. \\
&\qquad \left. - e^{-j2\pi W\tau} \frac{(2 - 4\pi^2 W^2\tau^2 + 4j\pi W\tau)}{-j8\pi^3\tau^3} \right] \\
&= \frac{4\pi^2 N_0}{8\pi^3\tau^3} [-(2 - 4\pi^2 W^2\tau^2) \sin 2\pi W\tau + 4\pi W\tau \cos 2\pi W\tau]
\end{aligned}$$

b) $R_Y(0) = \int_{-W}^{W} S_Y(f) df = \int_{-W}^{W} 4\pi^2 f^2 \frac{N_0}{2} = \frac{4\pi^2 N_0 W^3}{3}$

10.24

$$S_X(f) = \beta^2 e^{-\pi f^2} \Rightarrow R_X(\tau) = \beta^2 e^{-\pi\tau^2}$$

$$S_Y(f) = 4\pi^2 f^2 \beta^2 e^{-f^2/2}$$

$$R_Y(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau)$$

$$= -\frac{d}{d\tau}\left[\beta^2 e^{-\pi\tau^2}(-2\pi\tau)\right]$$

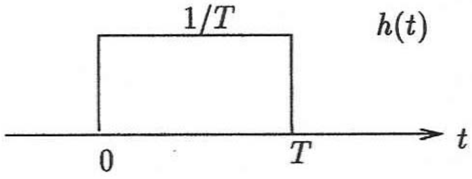
$$= -\beta^2\left[e^{-\pi\tau^2}(-2\pi\tau) - e^{-\pi\tau^2}(-2\pi)\right]$$

$$= 2\pi\beta^2 e^{-\pi\tau^2}\left[\tau - 1\right]$$

10.25

~~7.19~~ **a)** The impulse response is

$$\begin{aligned} h(t) &= \frac{1}{T}\int_{t-T}^{t} \delta(t')dt' = \frac{1}{T}\int_{-\infty}^{t} \delta(t')dt' - \frac{1}{T}\int_{-\infty}^{t-T} \delta(t')dt' \\ &= \frac{1}{T}[u(t) - u(t-T)] \end{aligned}$$



$$\begin{aligned} H(f) &= \frac{1}{T}\int_0^T e^{-j2\pi ft}dt = \frac{1}{T}\frac{1 - e^{-j2\pi ft}}{j2\pi f} = \frac{1}{T}\frac{e^{j2\pi f\frac{T}{2}} - e^{-j2\pi f\frac{T}{2}}}{j2\pi f}e^{-j2\pi f\frac{T}{2}} \\ &= \frac{1}{T}\frac{\sin \pi fT}{\pi f}e^{-j\pi fT} \end{aligned}$$

b)

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) \\ &= \frac{\sin^2 \pi fT}{T^2\pi^2 f^2} S_X(f) \end{aligned}$$

10.26

$$R_X(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| < T \\ 0 & ew \end{cases}$$

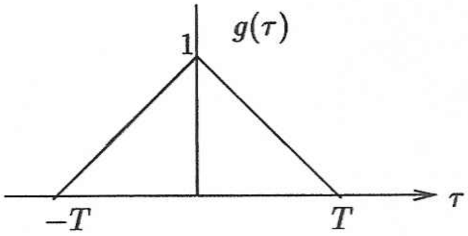
$$\begin{aligned} S_X(f) &= T\left(\frac{\sin \pi f T}{\pi f T}\right)^2 = \frac{1}{T}\left(\frac{\sin \pi f T}{\pi f}\right)^2 \\ S_Y(f) &= \frac{1}{T^3}\left(\frac{\sin \pi f T}{\pi f}\right)^4 \\ \mathcal{E}[Y^2(t)] &= R_Y(0) \end{aligned}$$

$$S_Y(f) = TG(f)G(f)$$

where

$$G(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

and



$$\begin{aligned} \therefore R_Y(\tau) &= Tg(\tau) \star g(\tau) \\ &= T\int_{-\infty}^{\infty} g(\lambda)g(\tau - \lambda)d\lambda \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}[Y^2(t)] &= R_Y(0) = T\int_{-\infty}^{\infty} g(\lambda)g(-\lambda)d\lambda \\ &= T\int_{-T}^{T} g^2(\lambda)d\lambda \quad \text{since } g(\lambda) = g(-\lambda) \\ &= 2T\int_0^T \left(1 - \frac{\lambda}{T}\right)^2 d\lambda \\ &= \frac{2T^2}{3} \end{aligned}$$

10.27

~~7.21~~ **a)** $S_{YX}(f) = H(f)S_X(f) = \dfrac{N_0/2}{1+j2\pi f}$

$$R_{YX} = \mathcal{F}^{-1}[S_{YX}(f)] = \frac{N_0}{2}e^{-\tau} \qquad \tau > 0$$

b) $S_Y(f) = |H(f)|^2 S_X(f) = \dfrac{N_0/2}{1+4\pi^2 f^2}$

$$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \frac{N_0}{4}e^{-|\tau|}$$

c) $R_Y(0) = \dfrac{N_0}{4}$

10.28)

a)

$$S_{YX}(f) = H(f)S_X(f) = (1+j2\pi f)S_X(f) = S_X(f) + j2\pi f\, S_X(f)$$

$$R_{YX}(\tau) = \mathcal{F}^{-1}\{S_{YX}(f)\} = R_X(\tau) + R_X'(\tau)$$

b)

$$S_Y(f) = |H(f)|^2 S_X(f) = (1+(2\pi f)^2)S_X(f) = S_X(f) + (2\pi f)^2 S_X(f)$$

$$R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\} = R_X(\tau) - R_X''(\tau)$$

c)

average power: $R_Y(0) = R_X(0) - R_X''(0)$

10.29

~~7.22~~

$$\begin{aligned}
Y(t) &= \int_0^{\infty} h(t-s)X(s)ds = \int_{-\infty}^{t} h(u)X(t-u)du \\
\mathcal{E}[Y(t)] &= \int_{-\infty}^{t} h(u)\mathcal{E}[X(t-u)]du = m_X \int_{-\infty}^{t} h(u)du \\
\mathcal{E}[Y(t)Y(t+\tau)] &= \mathcal{E}\left[\int_{-\infty}^{t}\int_{-\infty}^{t+\tau} h(u)h(v)X(t-u)X(t+\tau-v)dudv\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} h(u)h(v)R_X(\tau+u-v)dudv
\end{aligned}$$

depends on t and $t+\tau$. As $t \to \infty$ this expression approaches the expression in Eqn 10.42 and the process becomes WSS.

10.30

~~7.23~~ $X(t) \to \boxed{h(t)} \to Y(t)$

From Eqn. 10.45a:

$$\begin{aligned}
S_{YX}(f) &= H(f)S_X(f) = H(f)\frac{N_0}{2} \\
&\qquad\qquad\qquad \text{input is white noise} \\
\Rightarrow R_{YX}(\tau) &= \mathcal{F}^{-1}[S_{YX}(f)] = \frac{N_0}{2}h(t) \qquad \checkmark
\end{aligned}$$

10.31
~~7.24~~

$$\begin{aligned} Y(t_1) &= \int_{-\infty}^{\infty} h_1(u)X(t_1-u)du \approx \sum_i h_i(u_i)X(t_1-x_i)\Delta u \\ W(t_2) &= \int_{-\infty}^{\infty} h_2(v)X(t_2-v)dv \approx \sum_j h_2(v_j)X(t_2-v_j)\Delta v \end{aligned}$$

$\therefore Y(t_1)$ and $W(t_2)$ are jointly Gaussian

$$\begin{aligned} \mathcal{E}[Y(t_1)W(t_2)] &= \mathcal{E}\left[\int_{-\infty}^{\infty} h_1(u)X(t_1-u)du \int_{-\infty}^{\infty} h_2(v)X(t_2-v)dv\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)R_X(\underbrace{t_2-t_1}_{\tau}-v+u)dudv \end{aligned}$$

$\therefore Y(t_1)$ and $W(t_2)$ are jointly WSS.

From Eqn. 10.42, $Y(t)$ has variance function

$$\sigma_Y^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(s)h_1(r)R_X(s-r)dsdr - m_X^2 H_1(0)^2$$

and $W(t)$ has

$$\sigma_W^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(s)h_2(r)R_X(s-r)dsdr - m_X^2 H_2(0)^2$$

If $X(t)$ is white noise then

$$R_X(\tau) = \sigma_x^2 \delta(\tau)$$

$$E[Y(t_1)W(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)\, h_2(v)\, \delta_X(\tau - v + u)\, du\, dv$$

$$= \underbrace{\int_{-\infty}^{\infty} h_1(u)\, h_2(u+\tau)\, du}_{\text{correlation of impulse responses}}$$

$$\sigma_Y^2(t) = \iint_{-\infty}^{\infty} h_1(s)\, h_1(r)\, \delta_X(s-r)\, ds\, dr - m_x^2 H_1(0)^2$$

$$= \int_{-\infty}^{\infty} h_1^2(s)\, ds - m_x^2 H_1^2(0)$$

10.32

from 10.31:

$$E[Y(t_1)W(t_2)] = \iint_{-\infty}^{\infty} h_1(u)\, h_2(v)\, R_X(t_2 - t_1 - v + u)\, du\, dv$$

$$S_{YW}(f) = \iiint_{-\infty}^{\infty} h_1(u)\, h_2(v)\, R_X(\tau - v + u)\, du\, dv\, e^{-j2\pi f \tau}\, d\tau$$

$$\tau' = \tau - v + u$$

$$= \iiint_{-\infty}^{\infty} h_1(u)\, h_2(v)\, R_X(\tau')\, du\, dv\, e^{-j2\pi f(\tau' + v - u)}\, d\tau'$$

$$= \int_{-\infty}^{\infty} h_1(u)\, e^{j2\pi f u}\, du \int_{-\infty}^{\infty} h_2(v)\, e^{-j2\pi f v}\, dv \int_{-\infty}^{\infty} R_X(\tau')\, e^{-j2\pi f \tau'}\, d\tau'$$

$$= H_1^*(f)\, H_2(f)\, S_X(f)$$

$$= 0 \quad \text{if } H_1(f) \text{ \& } H_2(f) \text{ are non-overlapping.}$$

$\Rightarrow R_{YW}(t_1, t_2) = 0$

$\Rightarrow$ $Y(t_1)$ and $W(t_2)$ are uncorrelated.

$\Rightarrow$ $Y(t_1)$ and $W(t_2)$ are independent since they are jointly Gaussian

10.33

~~7.25~~ $Z(t) = X(t) - Y(t) \qquad Y(t) = h(t) \star X(t)$

a)

$$\begin{aligned}
E[Z(t)Z(t+\tau)] &= \mathcal{E}[(X(t) - Y(t))(X(t+\tau) - Y(t+\tau))] \\
&= R_X(\tau) + R_Y(\tau) - R_{YX}(-\tau) - R_{XY}(-\tau) \\
&= R_X(\tau) + R_Y(\tau) - R_{XY}(\tau) - R_{XY}(-\tau) \\
S_Z(f) &= S_X(f) + S_Y(f) - S_{XY}(f) - S^*_{XY}(f) \\
&= S_X(f) + |H(f)|^2 S_X(f) - H^*(f)S_X(f) - H(f)S_X(f) \\
&= \left\{ 1 + |H(f)|^2 - \underbrace{(H^*(f) + H(f))}_{2\text{Re}[H(f)]} S_X(f) \right\} \\
&= |\{1 - H(f)\}|^2 S_X(f)
\end{aligned}$$

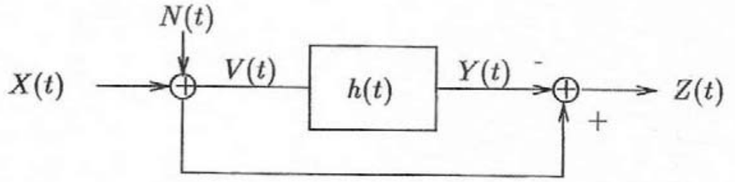
b)

$$\begin{aligned}
\mathcal{E}[Z(t)^2] &= R_X(0) + R_Y(0) - 2R_{XY}(0) \\
&= \mathcal{E}[X^2(t)] + \int\int_{-\infty}^{\infty} h(s)h(r)R_X(s-r)\,ds\,dr \\
&\quad -2\int_{-\infty}^{\infty} h(r)R_X(-r)\,dr
\end{aligned}$$

Also

$$\mathcal{E}[Z^2(t)] = \int_{-\infty}^{\infty} |1 - H(f)|^2 S_X(f)\,df$$

10.34
~~7.26~~



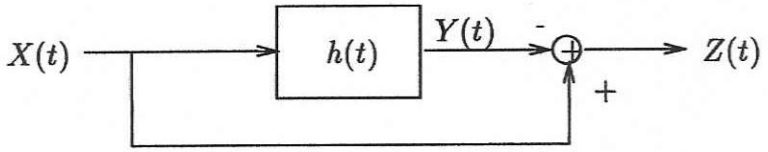
$$\begin{aligned}
R_Z(\tau) &= \mathcal{E}[Z(t+\tau)Z(t)] = \mathcal{E}[(X(t+\tau)-Y(t+\tau))(X(t)-Y(t))] \\
&= R_X(\tau)+R_Y(\tau)-R_{XY}(\tau)-R_{YX}(\tau) \\
R_{XY}(\tau) &= \mathcal{E}[X(t+\tau)Y(t)] = \mathcal{E}\left[X(t+\tau)\int_{-\infty}^{\infty} h(\lambda)V(t-\lambda)d\lambda\right] \\
&= \int_{-\infty}^{\infty} h(\lambda)R_{XV}(\tau+\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} h(\lambda)R_X(\tau+\lambda)d\lambda \quad \text{since} \quad \begin{aligned} R_{XV}(\tau) &= \mathcal{E}[X(t+\tau)(X(t)+N(t))] \\ &= R_X(\tau) \end{aligned} \\
&= h(-\tau)\star R_X(\tau) \\
S_Z(f) &= S_X(f)+S_Y(f)-S_{XY}(f)-S_{YX}(f) \\
&= S_X(f)+|H(f)|^2(S_X(f)+S_N(f))-H(f)S_X(f)-H^*(f)S_X(f) \\
S_Z(f) &= |1-H(f)|^2S_X(f)+|H(f)|^2S_N(f) \qquad (*)
\end{aligned}$$

Comments: If we view $Y(t)$ as our estimate for $X(t)$, then $S_Z(f)$ is the power spectral density of the error signal $Z(t)=Y(t)-X(t)$. Equation (*) suggests the following:

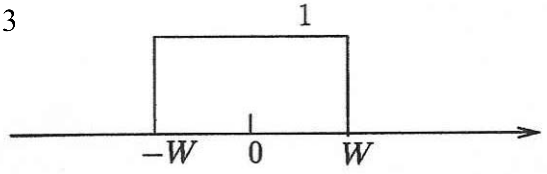
if $S_X(f) >> S_N(f)$ let $H(f)\approx 1$
if $S_X(f) << S_N(f)$ let $H(f)\approx 0$

i.e. select $H(f)$ to "pass" the signal and reject the noise.

10.35



From Problem 10.33



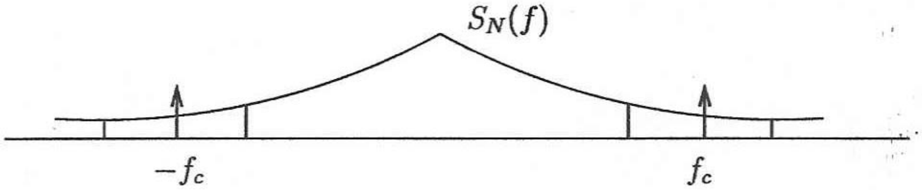
$$
\begin{aligned}
S_Z(f) &= |1 - H(f)|^2 S_X(f) \\
&= \begin{cases} 0 & |f| < W \\ \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} & |f| > W \end{cases} \\
\mathcal{E}[Z^2(t)] &= 2\int_W^{\infty} \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} df \qquad \begin{array}{l} x = 2\pi f \\ dx = 2\pi df \end{array} \\
&= 8\alpha \int_{2\pi W}^{\infty} \frac{1}{4\alpha^2 + x^2} \frac{dx}{2\pi} \\
&= \frac{4\alpha}{\pi} \frac{1}{2\alpha} \tan^{-1} \left. \frac{x}{2\alpha} \right|_{2\pi W}^{\infty} \\
&= \frac{2}{\pi} \left[\frac{\pi}{2} - \tan^{-1} \frac{\pi W}{\alpha} \right] \\
&= 1 - \frac{2}{\pi} \tan^{-1} \frac{\pi W}{\alpha}
\end{aligned}
$$

10.36

~~7.28~~ $Y(t) = a\cos(\omega_c(t) + \Theta) + N(t)$

We will assume that Θ and $N(t)$ are statistically independent:

$$\begin{aligned} \therefore R_Y(\tau) &= a\cos\omega_c\tau + R_N(\tau) \\ S_Y(f) &= \frac{a^2}{4}\delta(f - f_c) + \frac{a^2}{4}\delta(f + f_c) + S_N(f) \end{aligned}$$



$$\begin{aligned} \text{Signal Power} &= \int_{-f_c-W}^{-f_c+W}\left(\frac{a^2}{4}\delta(f - f_c) + \frac{a^2}{4}\delta(f + f_c)\right)df \\ &\quad + \int_{f_c-W}^{f_c+W}\left(\frac{a^4}{4}\delta(f - f_c) + \frac{a^2}{4}\delta(f + f_c)\right)df \\ &= \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2} \end{aligned}$$

$$\begin{aligned} \text{Noise Power} &= \int_{-f_c-W}^{-f_c+W} S_N(f)df + \int_{f_c-W}^{f_c+W} S_N(f)df \\ &= 2\int_{f_c-W}^{f_c+W} S_N(f)df \\ \text{SNR} &= \frac{a^2/2}{2\int_{f_c-W}^{f_c+W} S_N(f)df} \end{aligned}$$

10.37

~~7.31~~ Find the impulse response:
If input is $\delta_n = 1$, $n = 0$ and $\delta_n = 0$ elsewhere

$$\begin{aligned}
h_n &= \begin{cases} \frac{1}{3} & n = -1, 0, 1 \\ 0 & \text{elsewhere} \end{cases} \\
\Rightarrow H(f) &= \frac{1}{3}[e^{-j2\pi f} + 1 + e^{j2\pi f}] \\
&= \frac{1}{3}[1 + 2\cos 2\pi f] \\
S_Y(f) &= |H(f)|^2 S_X(f) = \frac{1}{9}[1 + 2\cos 2\pi f]^2 S_X(f) \\
&= \frac{1}{9}[1 + e^{j2\pi f} + e^{-j2\pi f}]^2 S_X(f) \\
&= \frac{1}{9}[1 + e^{j2\pi f} + e^{-j2\pi f} + e^{j2\pi f} + e^{-j2\pi f} \\
&\quad +1 + e^{-j2\pi f} + 1 + e^{-j4\pi f}]S_X(f) \\
&= \frac{1}{9}[3 + 2e^{j2\pi f} + 2e^{-j2\pi f} + e^{j4\pi f} + e^{-j4\pi f}]S_X(f) \\
\therefore R_Y(k) &= \mathcal{F}^{-1}[S_Y(f)] = \frac{1}{3}R_X(k) + \frac{2}{9}R_X(k+1) + \frac{2}{9}R_X(k-1) \\
&\quad + \frac{1}{9}R_X(k+2) + \frac{1}{9}R_X(k-2) \\
\mathcal{E}[Y_n^2] &= \frac{1}{3}R_X(0) + \frac{4}{9}R_X(1) + \frac{2}{9}R_X(2)
\end{aligned}$$

10.38

~~7.32~~ $R_X(n) = \begin{cases} \sigma^2 & n = 0 \\ 0 & n \neq 0 \end{cases}$ where we assumed $\mathcal{E}[X_n] = 0$

$$\Rightarrow R_Y(k) = \begin{cases} \frac{1}{3}\sigma^2 & k = 0 \\ \frac{2}{9}\sigma^2 & k = \pm 1 \\ \frac{1}{9}\sigma^2 & k = \pm 2 \\ 0 & ew \end{cases}$$

$\therefore (Y_n, Y_{n+1}, Y_{n+2})$ is jointly Gaussian with zero mean vector and covariance matrix

$$K = \sigma^2 \begin{bmatrix} \frac{1}{3} & \frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{1}{3} \end{bmatrix}$$

10.39

~~7.30~~ a)

$$\begin{aligned}
R_{YX}(k) &= \mathcal{E}[Y_{n+k}X_n] = \mathcal{E}[(X_{n+k} + \beta X_{n+k-1})X_n] \\
&= R_X(k) + \beta R_X(k-1) \\
S_{YX}(f) &= \mathcal{F}[R_{YX}(k)] = S_X(f) + \beta S_X(f)e^{-j2\pi f} \\
&= \frac{(1+\beta e^{-j2\pi f})(1-\alpha^2)}{1+\alpha^2-2\alpha\cos 2\pi f}\sigma^2
\end{aligned}$$

since

$$\mathcal{F}[\alpha^{|k|}] = \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f}$$

See Problem ~~7.9~~. 10.13.

b)

$$\begin{aligned}
R_Y(k) &= \mathcal{E}[(X_{n+k} + \beta X_{n+k-1})(X_n + \beta X_{n-1})] \\
&= (1+\beta^2)R_X(k) + \beta R_X(k+1) + \beta R_X(k-1) \\
S_Y(f) &= (1+\beta^2)S_X(f) + \beta S_X(f)e^{j2\pi f} + \beta S_X(f)e^{-j2\pi f} \\
&= [(1+\beta^2) + 2\beta\cos 2\pi f]S_X(f) \\
&= \frac{1+\beta^2+2\beta\cos 2\pi f}{1+\alpha^2-2\alpha\cos 2\pi f}(1-\alpha^2)\sigma^2 \\
\mathcal{E}[Y_n^2] &= R_Y(0) = (1+\beta^2)R_X(0) + \beta R_X(1) + \beta R_X(-1) \\
&= (1+\beta^2)\sigma^2 + 2\beta\sigma^2\alpha
\end{aligned}$$

c) if $\beta = -\alpha$ then $S_Y(f) = (1-\alpha^2)\sigma^2$ and $\mathcal{E}[Y_n^2] = (1-\alpha^2)\sigma^2$.

10.40

~~.29~~ **a)**

$$\begin{aligned}
H(f) &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j2\pi fn} = \frac{1}{1-\frac{1}{2}e^{-j2\pi f}} \\
G(f) &= \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j2\pi fn} = \frac{1}{1-\frac{1}{4}e^{-j2\pi f}} \\
S_Y(f) &= |H(f)|^2\frac{N_0}{2} = \frac{N_0/2}{\frac{5}{4}-\cos 2\pi f} \\
S_Z(f) &= |G(f)|^2 S_Y(f) = \frac{N_0/2}{\left(\frac{5}{4}-\cos 2\pi f\right)\left(\frac{17}{16}-\frac{1}{2}\cos 2\pi f\right)}
\end{aligned}$$

b)

$$\begin{aligned}
S_{WY}(f) &= H(f)S_X(f) = \frac{N_0/2}{1-\frac{1}{2}e^{-j2\pi f}} \\
\Rightarrow R_{WY}(k) &= \frac{N_0}{2}\left(\frac{1}{2}\right)^k u(k) \\
S_{WZ}(f) &= H(f)G(f)S_X(f) = \frac{N_0/2}{(1-\frac{1}{2}e^{-j2\pi f})(1-\frac{1}{4}e^{-j2\pi f})} \\
&= \frac{N_0}{1-\frac{1}{2}e^{-j2\pi f}} - \frac{N_0/2}{1-\frac{1}{4}e^{-j2\pi f}} \\
R_{WZ}(k) &= \left(N_0\left(\frac{1}{2}\right)^k - \frac{N_0}{2}\left(\frac{1}{4}\right)^k\right)u(k)
\end{aligned}$$

c) $S_Z(f) = \dfrac{N_0/2}{\left(\frac{5}{4}-\cos 2\pi f\right)\left(\frac{17}{16}-\frac{1}{2}\cos 2\pi f\right)} = \dfrac{\frac{8}{7}N_0}{\frac{5}{4}-\cos 2\pi f} - \dfrac{\frac{4}{7}N_0}{\frac{17}{16}-\frac{1}{2}\cos 2\pi f}$

From Problem 10.13 we know that

$$\mathcal{F}[|\alpha|^k] = \sum_{k=-\infty}^{\infty}|\alpha|^k e^{-j2\pi fk} = \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos 2\pi f}$$

10.41 a)

$$\begin{aligned}\mathcal{E}[X_nX_{n+k}] &= \mathcal{E}\left[\left(\sum_{i=0}^{p}\alpha_iW_{n-i}\right)\left(\sum_{j=0}^{p}\alpha_jW_{n+k-j}\right)\right]\\ &= \sum_{i=0}^{p}\sum_{j=0}^{p}\alpha_i\alpha_j\underbrace{\mathcal{E}[W_{n-i}W_{n+k-j}]}_{R_W(k-j+i)}\end{aligned}$$

$$\begin{aligned}k>p &\Rightarrow k-j+i>0\\ &\Rightarrow R_W(k-j+i)=0 \quad \text{since } W \text{ is white}\end{aligned}$$

Similarly

$$k<p \Rightarrow R_W(k-j+i)=0$$

$$\therefore \mathcal{E}[X_nX_{n+k}]=0 \quad \text{for} \quad |k|>p$$

For $|k|\le p$

$$\begin{aligned}R_X(k) &= \sigma^2\sum_{i=0}^{p}\sum_{l=0}^{p}\alpha_i\alpha_l\overbrace{\delta_{k-l+i}}^{R_W(k-l+i)}\\ &= \sigma^2\sum_{l=0}^{p}\alpha_l\alpha_{l-k} \quad \text{since } \delta_{k-l+i}=1\cdot\sigma^2 \Leftrightarrow i=l-k\\ &= \sigma^2\sum_{l=-\infty}^{\infty}\alpha_l\alpha_{l-k} \quad \text{where we define } \alpha_l=0,\ l<0 \text{ and } l>p\\ S_X(f) &= \sum_{k=-\infty}^{\infty}R_X(k)e^{-j2\pi fk}\\ &= \sigma^2\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\alpha_l\alpha_{l-k}e^{-j2\pi fk} \quad m=l-k\\ &= \sum_{l=-\infty}^{\infty}\alpha_le^{-j2\pi fl}\sum_{m=-\infty}^{\infty}\alpha_me^{j2\pi fm}\\ &= H(f)H^*(f)\sigma^2\end{aligned}$$

where $H(f)=\sum_{l=-\infty}^{\infty}\alpha_le^{-j2\pi fl}$ is the impulse response of the system.

10.42

~~7.34~~ **a)** If input is δ_n then $Y_0 = 1$ and $Y_1 = \frac{3}{4}$. We seek a solution to

$$Y_n = \frac{3}{4}Y_{n-1} - \frac{1}{8}Y_{n-2}$$

of the form

$$Y_n = c_1 z_1^n + c_2 z_2^n$$

that satisfies the above boundary conditions. The z_i must satisfy

$$\begin{aligned}
cz^n &= \frac{3}{4}cz^{n-1} - \frac{1}{8}cz^{n-2} \Rightarrow z^2 - \frac{3}{4}z + \frac{1}{8} = 0 \\
&\Rightarrow z_1 = \frac{1}{2} \qquad z_2 = \frac{1}{4}
\end{aligned}$$

$$\begin{array}{c} \text{Boundary} \\ \text{Conditions} \end{array} \Rightarrow \left.\begin{array}{l} Y_0 = 1 = c_1 + c_2 \\ Y_1 = \frac{3}{4} = \frac{c_1}{2} + \frac{c_2}{4} \end{array}\right\} \qquad \begin{array}{l} c_1 = 2 \\ c_2 = -1 \end{array}$$

$$\Rightarrow Y_n = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \qquad n \geq 0$$

b)

$$\begin{aligned}
H(f) &= 2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j2\pi fn} - \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j2\pi fn} \\
&= 2\frac{1}{1-\frac{1}{2}e^{-j2\pi f}} - \frac{1}{1-\frac{1}{4}e^{-j2\pi f}} \\
&= \frac{\frac{1}{2}e^{-j2\pi f}}{(1-\frac{1}{2}e^{-j2\pi f})(1-\frac{1}{4}e^{-j2\pi f})}
\end{aligned}$$

c)

$$\begin{aligned}
S_Y(f) &= |H(f)|^2\sigma_W^2 \\
&= \frac{\sigma_W^2/4}{(\frac{3}{4} - \cos 2\pi f)(\frac{17}{16} - \frac{1}{2}\cos 2\pi f)} \\
S_Y(f) &= \frac{4}{7}\left(\frac{4}{3}\right)\frac{\frac{3}{4}\sigma_X^2}{\frac{5}{4} - \cos 2\pi f} - \frac{2}{7}\frac{16}{15}\frac{\frac{15}{16}\sigma_X^2}{\frac{17}{16} - \frac{1}{2}\cos 2\pi f}
\end{aligned}$$

$$\Rightarrow R_Y(k) = \frac{16}{21}\left(\frac{1}{2}\right)^{|k|} - \frac{32}{105}\left(\frac{1}{4}\right)^{|k|}$$

See Problem 10.13 solution.

10.43

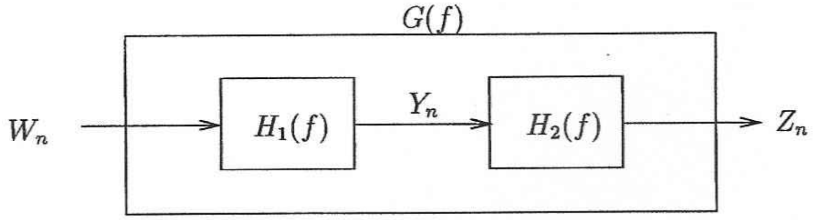
~~7.35~~ $Z_n = Y_n - \frac{1}{4}Y_{n-1}$

a) The impulse response for this system is

$$h_0 = 1 \qquad h_1 = -\frac{1}{4} \qquad h_n = 0 \qquad n \neq 0 \text{ or } 1$$

$$\therefore H_2(f) \;=\; 1 - \frac{1}{4}e^{-j2\pi f}$$

Let W_n be the input to the system in Problem 10.42, then



where

$$H_1(f) = \frac{\frac{1}{2}e^{-j2\pi f}}{(1-\frac{1}{2}e^{-j2\pi f})(1-\frac{1}{4}e^{-j2\pi f})}$$

The power spectral density of Z_n is

$$\begin{aligned} S_Z(f) &= |G(f)|^2 S_W(f) = |H_1(f)H_2(f)|^2\sigma_W^2 \\ &= \frac{\sigma_W^2/4}{\frac{5}{4} - \cos 2\pi f} \end{aligned}$$

where $G(f)$ is the transfer function that defines a first-order autoregressive process.

$$R_Z(k) = \frac{\sigma_W^2}{4}\frac{4}{3}\mathcal{F}^{-1}\left[\frac{\frac{3}{4}}{\frac{5}{4} - \cos 2\pi f}\right] = \frac{\sigma_W^2}{3}\left(\frac{1}{2}\right)^{|k|}$$

b) $G(f) = H_1(f)H_2(f) = \dfrac{\frac{1}{2}e^{-j2\pi f}}{1-\frac{1}{2}e^{-j2\pi f}} = \dfrac{1}{2}e^{-j2\pi f}\displaystyle\sum_{l=0}^{\infty}\left(\frac{1}{4}e^{-j2\pi f}\right)$ which corresponds to a first-order autoregressive process.

c) If we let $H_3(f) = 1 - \frac{1}{2}e^{-j2\pi f}$, then

$$|H_1(f)H_2(f)H_3(f)|^2 = \frac{1}{4}$$

and

$$|H_3(f)|^2 S_Z(f) = \frac{\sigma_W^2}{4}$$

10.44

~~7.36~~ **a)**

$$\begin{aligned}\mathcal{E}[Y_n^2] &= \mathcal{E}\left[Y_n\left(\sum_{i=1}^{q}\alpha_i Y_{n-i}+W_n\right)\right]\\ &= \sum_{i=1}^{q}\alpha_i R_Y(i)+R_{YW}(0)\end{aligned}$$

$$\begin{aligned}R_{YW}(0) &= \mathcal{E}\left[\left(\sum_{i=1}^{q}\alpha_i Y_{n-i}+W_n\right)W_n\right]\\ &= \sum_{i=1}^{q}\alpha_i \underbrace{\mathcal{E}[Y_{n-i}W_n]}_{0}+R_W(0)=R_W(0)\\ \therefore\quad R_Y(0) &= \sum_{i=1}^{q}\alpha_i R_Y(i)+R_W(0)\\ R_Y(k) &= \mathcal{E}\left[Y_{n-k}\left(\sum_{i=1}^{q}\alpha_i Y_{n-i}+W_n\right)\right]\\ &= \sum_{i=1}^{q}\alpha_i R_Y(k-i)+\underbrace{\underbrace{\mathcal{E}[Y_{n-k}W_n]}_{\mathcal{E}[Y_{n-k}]\mathcal{E}[W_n]}}_{0}\\ &= \sum_{i=1}^{q}\alpha_i R_Y(k-i)\end{aligned}$$

b)

$$\begin{aligned}Y_n &= rY_{n-1}+W_n\\ R_Y(0) &= rR_Y(1)+R_W(0)\\ R_Y(k) &= rR_Y(k-1)\Rightarrow R_Y(1)=rR_Y(0)\\ \Rightarrow R_Y(0) &= r^2R_Y(0)+R_W(0)\Rightarrow R_Y(0)=\frac{R_W(0)}{1-r^2}\\ \Rightarrow R_Y(k) &= \begin{cases}\frac{r^kR_W(0)}{1-r^2}=\underbrace{\left(\frac{R_W(0)}{1-r^2}\right)}_{\sigma_Y^2}r^k & k>0\\ R_Y(-k)=\sigma_Y^2 r^{|k|} & k<0\end{cases}\end{aligned}$$

10.3 Bandlimited Random Processes

10.45)

a) Suppose you have a signal $x(t)$, which is bandlimited and has a fourier transform $X(f)$, $|f| < W$. if you sample it with rate T_s you have

$$S(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

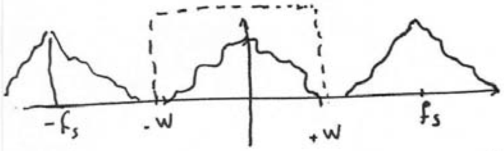
$$\hat{x}(t) = S(t)x(t) = x(t) \sum_{-\infty}^{+\infty} \delta(t - nT_s)$$

$$\hat{X}(f) = X(f) * \mathcal{F}\left\{ \sum_{-\infty}^{+\infty} \delta(t - nT_s) \right\}$$

it can be easily shown that fourier of a δ train is a δ train, since δ train in t, is a periodic signal. so we have:

$$\hat{X}(f) = X(f) * \left[\frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \delta(f - nf_s) \right] = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X(f - nf_s)$$

Since $X(f)$ is bandlimited to $|f| < W$, $X(f)$ can be recovered from $\hat{X}(f)$ if we use an Ideal LP filter with bandwidth W, and if $\underline{f_s > 2W}$



This sampling rate is called Nyquist rate.

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} e^{jk\omega_0 T} \quad \text{(Fourier Series)}$$

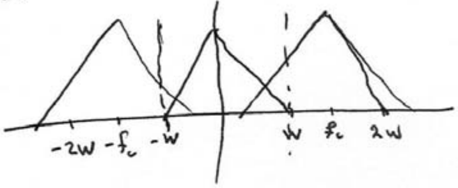
Therefore:

$$S(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \delta(f - nf_s)$$

P10.45)

b) Assume $W < f_S < 2W$

Therefore, from figure 10.10(b)



Therefore the error part would be:

$$e(f) = X(f - f_S)\left[1 - u(f - W)\right] + X(f + f_S)\left[u(f + W)\right]$$

and:

$$\underset{\text{recovered}}{\hat{X}(f)} = f_S\left[X(f) + e(f)\right]$$

for $0 < f_S < 2W$, $e(f) = \left[\sum_{K=1}^{\infty} X(f - Kf_S)\right]\left[1 - u(f - W)\right] + u(f + W)\sum_{K=1}^{\infty} X(f + f_S)$

P10.45)

c) $\hat{X}(t) = X(t)\, s(t)$

$$E\left[\hat{X}(t)\hat{X}(t+\tau)\right] = E\left[X(t)X(t+\tau)\, s(t)s(t+\tau)\right] = s(t)s(t+\tau)R_X(\tau)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_S)$$

$$s(t+\tau) = \sum_{m=-\infty}^{+\infty} \delta(t + \tau - nT_S)$$

$$s(t) \times s(t+\tau) = \sum_{K=-\infty}^{+\infty} \delta(\tau - KT_S)$$

Therefore: $R_{\hat{X}}(\tau) = R_X(\tau)\sum_{K=-\infty}^{+\infty} \delta(\tau - KT_S)$

And: $S_{\hat{X}}(f) = S_X(f) * f_S\sum_{n=-\infty}^{+\infty} \delta(f - nf_S) = f_S\sum_{n=-\infty}^{+\infty} S_X(f - nf_S)$

P10.45)

d) just like part b:

Assume $W < f_s < 2W$

Therefore $S_{X_{\text{recovered}}}(f) = f_s \left[S_X(f) + e(f) \right]$

in which $e(f) = S_X(f - f_s)\left[1 - u(f - W)\right] + S_X(f + f_s)\left[u(f + W)\right]$

for any $f_s < 2W$: $e(f) = \left[\sum_{k=1}^{\infty} S_X(f - kf_s)\right]\left[1 - u(f - W)\right] + \sum_{k=1}^{\infty} S_X(f + kf_s)\, u(f + W)$

e)

$$S_X(f) = \begin{cases} A + \frac{A}{W} f, & -W \leq f < 0 \\ A - \frac{A}{W} f, & 0 \leq f < W \end{cases}$$

if $W < f_s < 2W$

$\Rightarrow$ $e(f) = S_X(f - f_s)\left[1 - u(f - W)\right] + S_X(f + f_s)\left[u(f + W)\right]$

$$\int_{-\infty}^{+\infty} e(f)\, df = \int_{-W}^{-f_s + W} \left(A - \frac{A}{W}(f + f_s)\right) df + \int_{f_s - W}^{W} \left(A + \frac{A}{W}(f - f_s)\right) df$$

$$= \frac{1}{2} \times \frac{W}{A}\left(A - \frac{A}{W}(f_s - W)\right)^2 + \frac{1}{2} \times \frac{W}{A}\left(A + \frac{A}{W}(W - f_s)\right)^2 = \frac{W}{A}\left(2A - \frac{Af_s}{W}\right)^2 = WA\left(2 - \frac{f_s}{W}\right)^2$$

(10.46))

$$H(f) = \begin{cases} 1 & |f|<f_c<1/2 \\ 0 & f_c<|f|<1/2 \end{cases}$$

a)

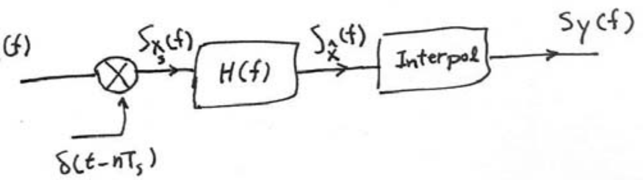
inverse fourier transform:

$$h(n) = \int_{-f_c}^{f_c} e^{j2\pi fn}\,df = \int_{-f_c}^{f_c} (\cos 2\pi fn + j\sin 2\pi fn)\,df$$

$$= \frac{2}{2\pi n}\sin 2\pi f_c n = \frac{\sin 2\pi f_c n}{\pi n}$$

Approach 1:

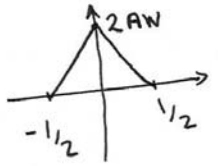
b) $S_Y(f)$



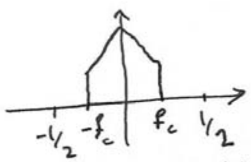
After sampling with Nyquist rate, the discrete time process has
periodic
a shape, with base period like the original signal

Therefore:

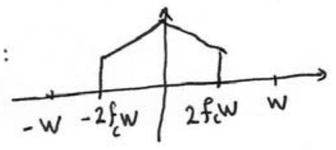
$S_{X_s}(f)$:



Therefore $S_{\hat{X}}(f) = |H(f)|^2 S_{X_s}(f)$:



and interpolutor achieves the equivalent continuous time prosess with following $S_Y(f)$:



P10.46)

b) Approach 2: , let $X_s(n) = X(nT_s)$, $R_{X_s}(k) = R_X(kT_s)$

$$R_{\hat{X}}(k) = h(n) * h(-n) * R_{X_s}(k)$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} h(m)h(l) R_{X_s}(k-l+m)$$

$$R_Y(\tau) = \sum_{k=-\infty}^{+\infty} R_{\hat{X}}(k)\, p(t-kT_s) = \sum_{m=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} h(m)h(l) \sum_{k=-\infty}^{+\infty} R_{X_s}(k-l+m)p(t-kT_s)$$

Therefore $R_Y(\tau) = \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} h(m)h(l) R_X(\tau-(l-m)T_s)$

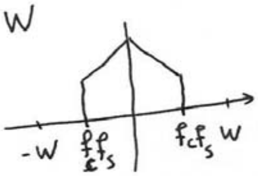
So, we have

$$S_Y(f) = \sum_{m=-\infty}^{+\infty} h(m)\, e^{-j2\pi T_s m f} \sum_{l=-\infty}^{+\infty} h(l)\, e^{j2\pi T_s l f} \, S_X(f)$$

$$= \left| H(fT_s) \right|^2 S_X(f) = \left| H\left(\frac{f}{f_s}\right) \right|^2 S_X(f) \quad , f_s = \frac{1}{2W}$$

$$\text{if} \quad H\left(\frac{f}{f_s}\right) = \begin{cases} 1 & |f| < |f_c| f_s < W \\ 0 & f_c f_s < |f| < W \end{cases} \qquad f_s = \frac{1}{2W}$$

Then: $S_Y(f)$ would be :



10.47

a) $H(f) = \frac{j2\pi f}{T} \quad |f| < \frac{1}{2}$

$$h(n) = \int_{-1/2}^{1/2} H(f) e^{j2\pi f n} df = \frac{1}{T}\int_{-1/2}^{1/2} j2\pi f e^{j2\pi f n} df$$

$$= \frac{j2\pi}{T}\left(\frac{1}{j2\pi n} f e^{j2\pi f n} - \frac{1}{j2\pi n} \times \frac{1}{j2\pi n} e^{j2\pi f n} \right]_{-\frac{1}{2}}^{\frac{1}{2}} \quad , n \neq 0$$

$$= \frac{j2\pi}{T}\left[\frac{1}{j4\pi n}\left(e^{j\pi n} + e^{-j\pi n}\right) + \frac{1}{(2\pi n)^2}\left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{1}{T}\left[\frac{\cos \pi n}{n} - \frac{\sin \pi n}{\pi n^2}\right] = \frac{\pi n \cos \pi n - \sin \pi n}{T\pi n^2} = \frac{(-1)^n}{nT}, \; n \neq 0$$

and $h(n) = 0 \, , \; n = 0$

P10.47)

b)

$$X(t) = a\cos(2\pi f_o t + \theta), \quad \frac{1}{T} = 4f_o \Rightarrow X(n) = X(nT) = a\cos\left(\frac{\pi}{2}n + \theta\right)$$

Therefore: $X_o(n) = X(n) * h(n)$ or $X_o(f) = X(f)H(f)$

$$X(f) = \frac{a}{2}\,\delta\left(f - \frac{1}{4}\right)\left[\cos\theta + j\sin\theta\right] + \frac{a}{2}\,\delta\left(f + \frac{1}{4}\right)\left[\cos\theta - j\sin\theta\right]$$

$$= \frac{a}{2}\left[\delta\left(f - \frac{1}{4}\right)e^{j\theta} + \delta\left(f + \frac{1}{4}\right)e^{-j\theta}\right]$$

$$H(f) = j\,\frac{2\pi f}{T}$$

Therefore:

$$X_o(f) = X(f)\,H(f) = \frac{j2\pi f}{T} \times \frac{a}{2}\left[\delta\left(f - \frac{1}{4}\right)e^{j\theta} + \delta\left(f + \frac{1}{4}\right)e^{-j\theta}\right]$$

$$= \frac{j\pi a}{4T}\left[\delta\left(f - \frac{1}{4}\right)e^{j\theta} - \delta\left(f + \frac{1}{4}\right)e^{-j\theta}\right]$$

Therefore $$x_o(n) = \frac{j\pi a}{4T}\left[e^{j\frac{\pi}{2}n}e^{j\theta} - e^{-j\frac{\pi}{2}n - j\theta}\right]$$

$$= \frac{j\pi a}{4T}\left[+2j\sin\left(\frac{\pi}{2}n + \theta\right)\right] = \frac{-\pi a}{2T}\sin\left(\frac{\pi}{2}n + \theta\right)$$

$$X_o(nT) = x_o(n) = \frac{-\pi a}{2T}\sin(2\pi n f_o T + \theta)$$

after interpolation $$Y(t) = -\sum_{n=-\infty}^{+\infty} \frac{\pi a}{2T}\sin(2\pi f_o nT + \theta)\,p(t - nT)$$

since $$f_o = \frac{1}{4T} < \frac{1}{T}, \text{ then } Y(t) = \frac{-\pi a}{2T}\sin(2\pi f_o T + \theta)$$

10.48

we have

$$R_X(\tau) = \sum R_X(nT)\,p(\tau - nT)$$

define $R'_X(\tau) = R_X(\tau - \alpha)$, $R'_X(\tau)$ has the same ~~BW~~ spectrum as $R_X(\tau)$

So: $R'_X(\tau) = \sum R'_X(nT)\,p(\tau - nT)$

or

$$R_X(\tau - \alpha) = \sum R_X(nT - \alpha)\,p(\tau - nT) \qquad ①$$

first we define $y(t) = E\left[(X(t_0) - \hat{X}(t_0))X(t)\right]$, where t_0 is a fixed ti

$$y(mT) = E\left[(X(t_0) - \hat{X}(t_0))\,X(mT)\right] = R_X(t_0 - mT) - \sum_n R_X(mT - nT)\,p(t_0 - nT)$$

From ①: $\sum_n R_X(mT - nT)\,p(t_0 - nT) = R_X(t_0 - mT)$

Therefore $y(mT) = 0$

now for general t we have:

$$y(t) = R_X(t_0 - t) - \sum_n R_X(t - nT)\,p(t_0 - nT)$$

Since $BW < \frac{1}{T} \Longrightarrow y(t) = R_X(t_0 - t) - R_X(t_0 - t) = 0$

Therefore $y(t) = 0$, $\forall t$

Now, if you let $t = t_0$, $y(t_0) = E\left\{(X(t_0) - \hat{X}(t_0))X(t_0)\right\} = 0$, $\forall t_0$

therefore $E\left[(X(t_0) - \hat{X}(t_0))\,X(t_0)\right] = 0$, $\forall t_0$

P10.48) — continued —

Also: $E\left[(X(t)-\hat{X}(t))\hat{X}(t)\right] = 2(t)$

we showed that $E[X(t)\hat{X}(t)] = E[X(t)X(t)] = R_X(0)$

Now we show that $E[\hat{X}(t)\hat{X}(t)] = R_X(0)$

To do so:

$$E\{\hat{X}(t)\hat{X}(t)\} = E\Big\{\sum_n \sum_m X(mT)X(nT)\,p(t-nT)p(t-mT)\Big]$$

$$= \sum_n \sum_m R_X(mT-nT)\,p(t-nT)p(t-mT)$$

using ① $= \sum_n R_X(t-nT)p(t-nT) = R_X(0)$

Therefore $E[(X(t)-\hat{X}(t))\hat{X}(t)] = R_X(0) - R_X(0) = 0$

And $\hat{X}(t) = X(t)$ in Mean Square Sense.

10.49

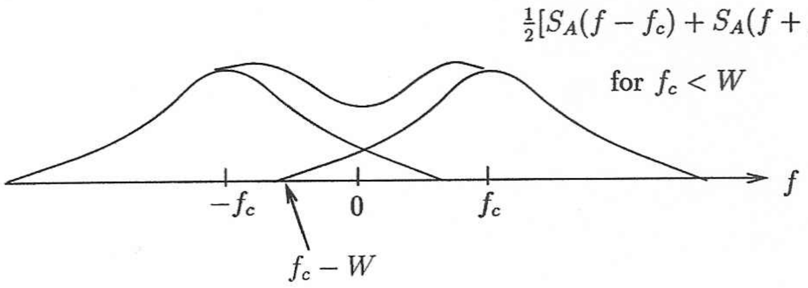
~~7.37~~ $Y(t) = \underbrace{A(t)\cos(2\pi f_c t + \Theta)}_{X(t)} + N(t)$

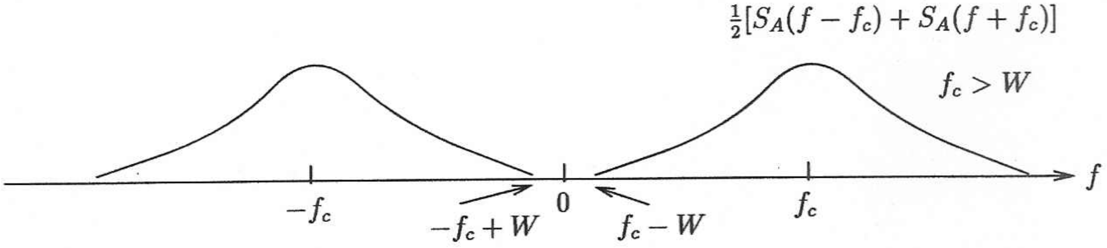
Assuming $X(t)$ and $N(t)$ are independent random processes:

$$R_Y(\tau) = R_X(\tau) + R_N(\tau) \ ,$$

from Example10.4 and the fact that $\mathcal{E}[X(t)] = 0$.

$$\begin{aligned} S_Y(f) &= S_X(f) + S_N(f) \\ &= \frac{1}{2}S_A(f - f_c) + \frac{1}{2}S_A(f + f_c) + S_N(f) \end{aligned}$$



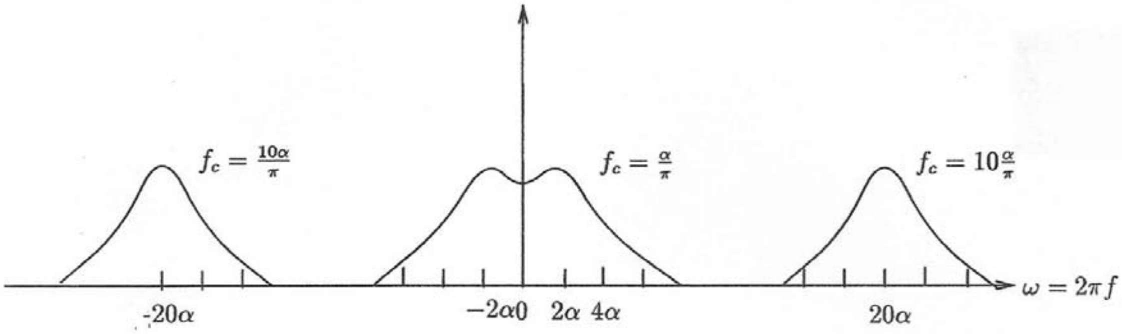


where we assumed that $S_A(f)$ is bandlimited to $|f| < W$.

10.50

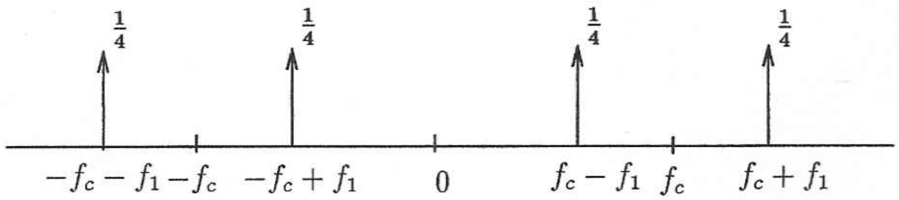
~~7.38~~ For the random telegraph signal:

$$
\begin{aligned}
S_X(f) &= \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} \\
S_Y(f) &= \frac{1}{2}S_X(f+f_c) + \frac{1}{2}S_X(f-f_c) \\
&= \frac{2\alpha}{4\alpha^2 + 4\pi^2(f+f_c)^2} + \frac{2\alpha}{4\alpha^2 + 4\pi^2(f-f_c)^2} \\
&= \frac{2\alpha}{4\alpha^2(\omega+2\pi f_c)^2} + \frac{2\alpha}{4\alpha^2 + (\omega-2\pi f_c)^2}
\end{aligned}
$$



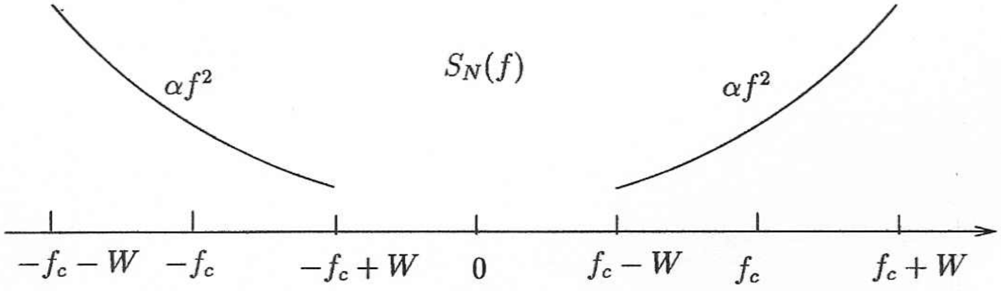
10.51

$$
\begin{aligned}
A(t) &= 2\cos(2\pi f_1 + \Phi) \\
R_A(\tau) &= \cos 2\pi f_1 \tau \qquad S_A(f) = \frac{1}{2}\delta(f+f_1) + \frac{1}{2}\delta(f-f_1) \\
S_X(f) &= \frac{1}{2}S_A(f+f_c) + \frac{1}{2}S_A(f-f_c) \\
&= \frac{1}{4}[\delta(f+f_c+f_1) + \delta(f+f_c-f_1) \\
&\quad +\delta(f-f_c+f_1) + \delta(f-f_c-f_1)]
\end{aligned}
$$

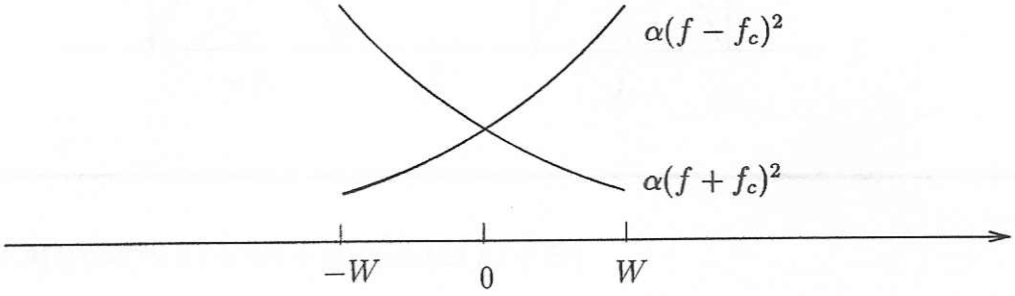


10.52

7.40



$$S_{N_c}(f) = [S_N(f - f_c) + S_N(f + f_c)]_2 = \alpha(f - f_c)^2 + \alpha(f + f_c)^2 \qquad |f| < W$$

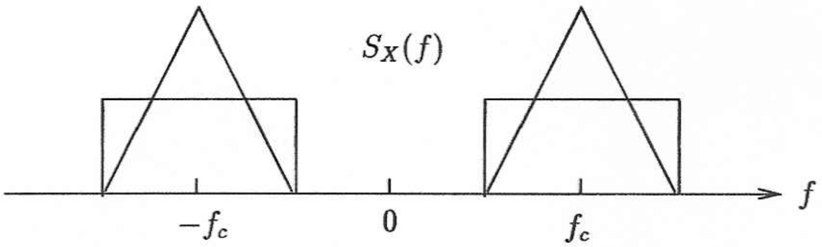


$$\begin{aligned}
\sigma^2_{N_c} &= \int_{-W}^{W} \alpha(f - f_c)^2 df + \int_{-W}^{W} \alpha(f + f_c)^2 df \\
&= \frac{2}{3}[(f_c + W)^3 - (f_c - W)^3] \\
&= \frac{2}{3}W^3 + r f_c^2 W \\
\text{SNR} &= \frac{\sigma_X^2}{4 f_c^2 W + \frac{2}{3}W^3}
\end{aligned}$$

10.53

$X(t) = A(t)\cos(2\pi f_c t + \Theta) + B(t)\sin(2\pi f_c t + \Theta)$

$$\begin{aligned}
\mathcal{E}[X(t)X(t+\tau)] &= \mathcal{E}[A(t)A(t+\tau)\cos(2\pi f_c t+\Theta)\cos(2\pi f_c t + 2\pi f_c\tau + \Theta)] \\
&\quad +\mathcal{E}[A(t)B(t+\tau)\cos(2\pi f_c t+\Theta)\sin(2\pi f_c t + 2\pi f_c\tau + \Theta)] \\
&\quad +\mathcal{E}[A(t+\tau)B(t)\cos(2\pi f_c t + 2\pi f_c\tau+\Theta)\sin(2\pi f_c t + \Theta)] \\
&\quad +\mathcal{E}[B(t)B(t+\tau)\sin(2\pi f_c t+\Theta)\sin(2\pi f_c t + 2\pi f_c\tau + \Theta)] \\
&= \frac{1}{2}R_A(\tau)\cos 2\pi f_c\tau + \frac{1}{2}R_B(\tau)\cos 2\pi f_c\tau \\
&= \frac{1}{2}(R_A(\tau)+R_B(\tau))\cos 2\pi f_c\tau \\
S_X(f) &= \frac{1}{4}[S_A(f+f_c)+S_B(f+f_c)+S_A(f-f_c)+S_A(f+f_c)]
\end{aligned}$$

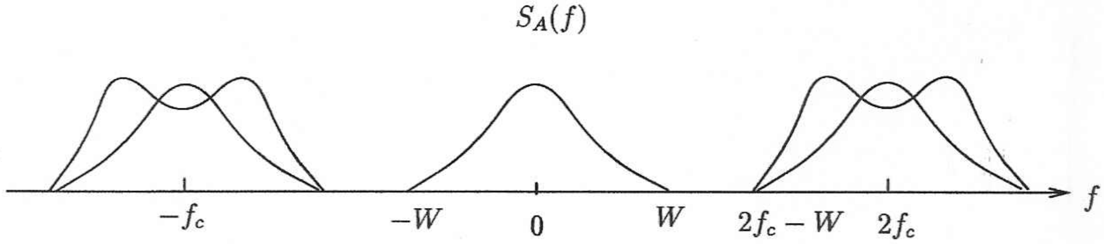


10.54

~~7.42~~ $X(t) = A(t)\cos(2\pi f_c t + \Theta) + B(t)\sin(2\pi f_c t + \Theta)$

$$\begin{aligned}
2X(t)\cos(\omega_c t + \Theta) &= 2A(t)\cos^2(\omega_c t + \Theta) \\
&\quad +2B(t)\cos(\omega_c t + \Theta)\sin(\omega_c t + \Theta) \\
&= 2A(t)\left(\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t + 2\Theta)\right) \\
&\quad +2B(t)\frac{1}{2}\sin(2\omega_c t + 2\Theta) \\
&= A(t) + A(t)\cos(2\omega_c t + 2\Theta) \\
&\quad +B(t)\sin(2\omega_c t + 2\Theta)
\end{aligned}$$

The power spectral density of this signal is shown below:



As long as $2f_c - W > W$, then the output of the ideal LPF is $A(t)$.
The same condition guarantees the recovery of $B(t)$.

10.55

$$\begin{aligned} R_{BA}(\tau) &= \mathcal{E}[B(t+\tau)A(t)] \\ &= \mathcal{E}[A(t)B(t+\tau)] \\ &= R_{AB}(-\tau) \end{aligned}$$

Eqn. 10.67b implies

$$R_{AB}(-\tau) = R_{BA}(\tau) = -R_{AB}(\tau)$$

$\Rightarrow R_{AB}(\tau)$ is an odd function of τ. Then

$$\begin{aligned} S_{AB}(f) &= \mathcal{F}[R_{AB}(\tau)] \\ &= \int_{-\infty}^{\infty} R_{AB}(\tau) e^{-j2\pi f\tau} d\tau \\ &= \underbrace{\int_{-\infty}^{\infty} R_{AB}(\tau) \cos 2\pi f\tau d\tau}_{0} - j\int_{-\infty}^{\infty} R_{AB}(\tau) \sin 2\pi f\tau d\tau \\ &= -j\int_{-\infty}^{\infty} R_{AB}(\tau) \sin 2\pi f\tau d\tau \end{aligned}$$

$\therefore S_{AB}(f)$ is a purely imaginary, odd form of f.
From Prob 10.7 we also have

$$S_{BA}(f) = S^*_{AB}(f) = j\int_{-\infty}^{\infty} \alpha_{AB}(\tau) \sin 2\pi f\tau d\tau$$

10.4 Optimum Linear Systems

10.56

~~7.44~~ $X_\alpha = Z_\alpha + N_\alpha$ where $R_Z(k) = \sigma_Z^2 r^{|k|}$

$$\begin{bmatrix} 1+\Gamma & r \\ r & 1+\Gamma \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 1 \\ r \end{bmatrix}$$

where $\Gamma = \dfrac{\sigma_N^2}{\sigma_Z^2}$.

$$\begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \frac{1}{(1+\Gamma)^2 - r^2} \begin{bmatrix} 1+\Gamma & -r \\ -r & 1+\Gamma \end{bmatrix} \begin{bmatrix} 1 \\ r \end{bmatrix} = \frac{1}{(1+\gamma)^2 - r^2} \begin{bmatrix} 1+\Gamma - r^2 \\ \Gamma r \end{bmatrix}$$

$$\begin{aligned} \mathcal{E}[(Z_t - Y_t)^2] &= R_Z(0) - \sum_{\beta=0}^{1} h_\beta R_{ZX}(\beta) = R_Z(0) - \sum_{\beta=0}^{1} h_\beta \sigma_Z^2 r^{|\beta|} \\ &= \sigma_Z^2 \left[1 - \frac{1 - \Gamma - r^2}{(1+\Gamma)^2 - r^2} - \frac{\Gamma r}{(1+\Gamma)^2 - r^2} r \right] \\ &= \sigma_Z^2 \left[1 - \frac{(1+\Gamma)(1-r^2)}{(1+\Gamma)^2 - r^2} \right] = \frac{29}{64} \sigma_Z^2 \end{aligned}$$

since $\Gamma = \frac{1}{4}$ $r = \frac{3}{4}$.

10.57

~~7.45~~ $R_Z(k) = \sigma_Z^2 r_1^{|k|} \qquad R_N(k) = \sigma_N^2 r_2^{|k|}$

a) Eqn. 10.83 implies

$$\sigma_Z^2 r_1^{|m|} = \sum_{\beta=0}^{p} h_\beta \left\{ \sigma_Z^2 r_1^{|m-\beta|} + \sigma_N^2 r_2^{|m-\beta|} \right\}$$

b)
$$\begin{bmatrix} \sigma_Z^2+\sigma_N^2 & \sigma_Z^2 r_1 + \sigma_N^2 r_2 & \dots & \sigma_Z^2 r_1^p + \sigma_N^2 r_2^p \\ \sigma_Z^2 r_1 + \sigma_N^2 r_2 & \sigma_Z^2+\sigma_N^2 & \dots & \\ \vdots & & \ddots & \\ \sigma_Z^2 r_1 + \sigma_N^2 r_2 & & \dots & \sigma_Z^2+\sigma_N^2 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_p \end{bmatrix} = \sigma_Z^2 \begin{bmatrix} 1 \\ r_1 \\ r_1^2 \\ \vdots \\ r_1^p \end{bmatrix}$$

c) Let $p=2$, $\underbrace{\sigma_Z^2 = 9 \qquad r_1 = \frac{2}{3}}_{\substack{\text{decays}\\\text{slowly}\\\text{with } |k|}} \qquad \underbrace{\sigma_N^2 = 1 \qquad r_2 = \frac{1}{3}}_{\substack{\text{decays}\\\text{quickly}\\\text{with } |k|}}$

$$\begin{bmatrix} 10 & 9(\frac{2}{3}) + 1(\frac{1}{3}) & 9(\frac{4}{9}) + 1(\frac{1}{9}) \\ 6.3 & 10 & 6.3 \\ 4.11 & 6.3 & 10 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix}$$

$\Rightarrow h_0 = 0.867 \;\; h_1 = 0.0383 \;\; h_2 = 0.0189$
This is essentially a low pass filter.

d)
$$\begin{aligned} \mathcal{E}[e_t^2] &= R_Z(0) - \sum_{\beta=0}^{p} h_\beta R_Z(\beta) \\ &= 9 - 8.1165 \\ &= 0.8835 \end{aligned}$$

10.58 ~~7.46~~ **a)** $R_Z(k) = \begin{cases} (1+\alpha^2)\sigma^2 \triangleq \sigma_Z^2 & k=0 \\ \alpha\sigma^2 & k=\pm 1 \\ 0 & ew \end{cases}$

Eqn. 10.83 is then $R_Z(m) = \sum_{\beta=0}^{p} h_\beta \left\{ R_Z(m-\beta) + \sigma_N^2 \delta_{m-\beta} \right\}$

$$\begin{bmatrix} \sigma_Z^2+\sigma_N^2 & \alpha\sigma^2 & 0 & 0 & \dots & 0 \\ \alpha\sigma^2 & \sigma_Z^2+\sigma_N^2 & \alpha\sigma^2 & 0 & & \vdots \\ 0 & \alpha\sigma^2 & \sigma_Z^2+\sigma_N^2 & & & 0 \\ \vdots & & & & \ddots & \alpha\sigma^2 \\ 0 & & \dots & 0 & \alpha\sigma^2 & \sigma_Z^2+\sigma_N^2 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ \\ h_p \end{bmatrix} = \begin{bmatrix} \sigma_Z^2 \\ \alpha\sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

b) Let $p = 2$ in previous matrix equation, and divide both sides by σ^2

$$\begin{bmatrix} 1+\alpha^2+\Gamma & & \\ \alpha & 1+\alpha^2+\Gamma & \alpha \\ 0 & \alpha & 1+\alpha^2+\Gamma \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 \\ \alpha \\ 0 \end{bmatrix}$$

where $\Gamma = \sigma_N^2/\sigma^2$. The last equation implies

$$h_2 = \frac{-\alpha}{1+\alpha^2+\Gamma} h_1$$

Then substituting for h_2 we obtain

$$\begin{bmatrix} 1+\alpha^2+\Gamma & \alpha \\ \alpha & \frac{(1+\alpha^2+\Gamma)^2-\alpha^2}{1+\alpha^2+\Gamma} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 \\ \alpha \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} &= \frac{1}{(1+\alpha^2+\Gamma)^2 - 2\alpha^2} \begin{bmatrix} \frac{(1-\alpha^2+\Gamma)^2-\alpha^2}{1+\alpha^2+\Gamma} & -\alpha \\ -\alpha & 1+\alpha^2+\Gamma \end{bmatrix} \begin{bmatrix} 1+\alpha^2 \\ \alpha \end{bmatrix} \\ h_0 &= \frac{\frac{(1+\alpha^2+\Gamma)^2(1+\alpha^2)-\alpha^2(1+\alpha^2)}{1+\alpha^2+\Gamma} - \alpha^2}{(1+\alpha^2+\Gamma)^2-2\alpha^2} \\ h_1 &= \frac{\alpha\Gamma}{(1+\alpha^2+\Gamma)^2-2\alpha^2} \\ h_2 &= \frac{-\frac{\alpha^2\Gamma}{1+\alpha^2+\Gamma}}{(1+\alpha^2+\Gamma)^2-2\alpha^2} \end{aligned}$$

c)

$$\begin{aligned} \mathcal{E}[e_t^2] &= R_Z(0) - \sum_{\beta=0}^{2} h_\beta R_Z(\beta) \\ &= R_Z(0) - h_0 R_Z(0) - h_1 R_Z(1) - \underbrace{h_2 R_Z(z)}_{0} \\ &= \sigma^2\{(1+\alpha^2)(1-h_0) - \alpha h_1\} \end{aligned}$$

Check: If $\Gamma = \sigma_N^2/\sigma^2 = 0$, i.e. no noise. Then $h_0 = 1$, $h_1 = 0$, $h_2 = 0$, i.e. no filtering and $\mathcal{E}[e_t^2] = 0$, i.e. no error.

10.59 $X_\alpha = Z_\alpha + N_\alpha$

a)

$$\begin{aligned} Y_t &= \sum_{\beta=-p}^{p} h_\beta X_{t-\beta} \\ R_{ZX}(m) &= \sum_{\beta=-p}^{p} h_\beta R_X(m-\beta) \qquad |m| \le p \\ R_{ZX}(m) &= \mathcal{E}[Z_n X_{n-m}] = \mathcal{E}[Z_n(Z_{n-m} + N_{n-m})] \\ &= R_Z(m) \end{aligned}$$

where we assume noise and desired signal are independent and noise is zero mean

$$R_X(m-\beta) = R_Z(m-\beta) + R_N(m-\beta)$$

$\therefore$ the optimum filter must satisfy

$$R_Z(m) = \sum_{\beta=-p}^{p} h_\beta \{R_Z(m-\beta) + R_N(m-\beta)\} \qquad |m| \le p$$

b)

$$\begin{bmatrix} R_Z(0)+R_N(0) & R_Z(1)+R_N(1) & \dots & R_Z(2p)+R_N(2p) \\ R_Z(1)+R_N(1) & R_Z(0)+R_N(0) & \dots & R_Z(2p-1)+R_N(2p-1) \\ \vdots & & & \vdots \\ R_Z(2p)+R_N(2p) & & \dots & R_Z(0)+R_N(0) \end{bmatrix} \begin{bmatrix} h_{-p} \\ h_{-p+1} \\ \vdots \\ h_0 \\ \vdots \\ h_p \end{bmatrix}$$

$$= \begin{bmatrix} R_Z(-p) \\ \vdots \\ R_Z(0) \\ \vdots \\ R_Z(p) \end{bmatrix}$$

c)

$$\begin{bmatrix} 1+\Gamma & r & r^2 \\ r & 1+\Gamma & r \\ r^2 & r & 1+\Gamma \end{bmatrix} \begin{bmatrix} h_{-1} \\ h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} r \\ 1 \\ r \end{bmatrix}$$

$$r = \frac{3}{4}, \qquad \Gamma = \frac{\sigma_N^2}{\sigma^2} = \frac{1}{4}$$

$$\begin{bmatrix} \frac{5}{4} & \frac{3}{4} & \frac{9}{16} \\ \frac{3}{4} & \frac{5}{4} & \frac{3}{4} \\ \frac{9}{16} & \frac{3}{4} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} h_{-1} \\ h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ 1 \\ \frac{3}{4} \end{bmatrix}$$

$$h_{-1} = .16438 \quad h_0 = .60274 \quad h_1 = .16438$$

d)

$$\begin{aligned} \mathcal{E}[e_t^2] &= R_Z(0) - \sum_{\beta=-1}^{1} h_\beta R_Z(\beta) \\ &= R_Z(0) - h_{-1}R_Z(-1) - h_0 R_Z(0) - h_1 R_Z(1) = 1.6828 \end{aligned}$$

10.60

~~7.48~~ **a)** $9\left(\frac{1}{3}\right)^m = 9\sum_{\beta=1}^{p} h_\beta \left(\frac{1}{3}\right)^{|m-\beta|} \qquad m \in \{1, 2, \ldots, p\}$

b) $p = 2$:

$$\begin{bmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{9} \end{bmatrix}$$

$$\Rightarrow h_1 = \frac{1}{3} \qquad h_2 = 0$$

c) $\mathcal{E}[e_g^2] = 9 - 9\left(\frac{1}{3}\right) = 6$

10.61

~~7.49~~ $\hat{X}(t) = aX(t_1) + bX(t_2)$

a)

$$\begin{aligned} e(t) &= \hat{X}(t) - X(t) \\ &= aX(t_1) + bX(t_2) - X(t) \end{aligned}$$

Orthogonality condition implies that

$$\begin{aligned} \mathcal{E}[(aX(t_1) + bX(t_2) - X(t))X(t_1)] &= 0 \\ \mathcal{E}[(aX(t_1) + bX(t_2) - X(t))X(t_2)] &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad aR_X(0) + bR_X(t_2 - t_1) &= R_X(t - t_1) \\ aR_X(t_1 - t_2) + bR_X(0) &= R_X(t - t_2) \end{aligned}$$

$$\Rightarrow \quad \begin{bmatrix} R_X(0) & R_X(t_2 - t_1) \\ R_X(t_2 - t_1) & R_X(0) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} R_X(t - t_1) \\ R_X(t - t_2) \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{R_X^2(0) - R_X^2(t_2 - t_1)} \begin{bmatrix} R_X(0)R_X(t - t_1) - R_X(t_2 - t_1)R_X(t - t_2) \\ R_X(0)R_X(t - t_2) - R_X(t_2 - t_1)R_X(t - t_1) \end{bmatrix}$$

b)

$$\begin{aligned} \mathcal{E}[e^2(t)] &= \mathcal{E}[e(t)[aX(t_1) + bX(t_2) - X(t)]] \\ &= \underbrace{a\mathcal{E}[e(t)X(t_1)]}_{0} + \underbrace{b\mathcal{E}[e(t)X(t_2)]}_{0} - \mathcal{E}[e(t)X(t)] \\ &= -a\mathcal{E}[X(t_1)X(t)] - b\mathcal{E}[X(t_2)X(t)] + \mathcal{E}[X(t)X(t)] \end{aligned}$$

$$\begin{aligned} \mathcal{E}[e^2(t)] = R_X(0) \quad &- \quad \frac{R_X(0)R_X(t - t_1) - R_X(t_2 - t_1)R_X(t - t_2)}{R_X^2(0) - R_X^2(t_2 - t_1)} R_X(t - t_1) \\ &- \quad \frac{R_X(0)R_X(t - t_2) - R_X(t_2 - t_1)R_X(t - t_1)}{R_X^2(0) - R_X^2(t_2 - t_1)} R_X(t - t_2) \end{aligned}$$

Check: If $t = t_1$ then $a = 1$, $b = 0$ and $\mathcal{E}[e^2(t)] = 0$.

10.62

~~7.50~~ $t_1 = t - d \qquad t_2 = t + d \qquad t_2 - t_1 = 2d$

$$\begin{aligned}
a &= \frac{R_X(0)R_X(d) - R_X(2d)R_X(d)}{R_X^2(0) - R_X^2(2d)} = \frac{R_X(0) - R_X(2d)}{R_X^2(0) - R_X^2(2d)} R_X(d) \\
&= \frac{R_X(d)}{R_X(0) + R_X(2d)} \\
b &= \frac{R_X(0)R_X(d) - R_X(2d)R_X(d)}{R_X^2(0) + R_X(2d)} = \frac{R_X(d)}{R_X(0) + R_X(2d)} = a \\
\mathcal{E}[e^2(t)] &= R_X(0) - a[R_X(d) + R_X(d)] \\
&= R_X(0) - \frac{2R_X^2(d)}{R_X(0) + R_X(2d)}
\end{aligned}$$

10.63

~~7.51~~ $t_1 = t - d \qquad t_2 = t - 2d \qquad t_1 - t_2 = d \qquad t - t_1 = d \qquad t - t_2 = 2d$

$$\begin{aligned}
a &= \frac{R_X(0)R_X(d) - R_X(d)R_X(2d)}{R_X^2(0) - R_X^2(d)} = \frac{e^{-\alpha d} - e^{-\alpha 3d}}{1 - e^{-\alpha 2d}} = e^{-\alpha d} \\
b &= \frac{R_X(0)R_X(2d) - R_X(d)R_X(d)}{R_X^2(0) - R_X^2(d)} = \frac{e^{-\alpha 2d} - e^{-\alpha 2d}}{1 - e^{-\alpha 2d}} = 0 \\
\mathcal{E}[e^2(t)] &= R_X(0) - aR_X(d) = 1 - e^{-\alpha 2d}
\end{aligned}$$

The optimum predictor of the form $\hat{X}(t) = aX(t-d)$ must satisfy

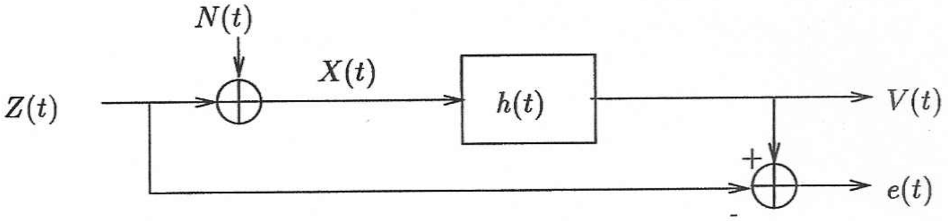
$$\begin{aligned}
\mathcal{E}[e(t)X(t-d)] &= \mathcal{E}[(aX(t-d) - X(t))X(t-d)] = 0 \\
aR_X(0) &= R_X(d) \\
a &= e^{-\alpha d}
\end{aligned}$$

This is the same predictor as obtained above.

Thus in this example (i.e. when $R_X(\tau) = e^{-\alpha|\tau|}$), the optimum predictor uses only the most recent observations.

10.64

~~7.52~~



In Problem 10.30 we considered the above system. After making adjustments for the difference in notation, we have

$$S_e(f) = |1 - H(f)|^2 S_Z(f) + |H(f)|^2 S_N(f)$$

Equation 10.92 implies that

$$\begin{aligned}
|1 - H(f)|^2 &= \left(\frac{S_N(f)}{S_Z(f) + S_N(f)}\right)^2 \\
\therefore \quad S_e(f) &= \frac{S_N^2(f) S_Z(f)}{(S_Z(f) + S_N(f))^2} + \frac{S_N(f) S_Z^2(f)}{(S_Z(f) + S_N(f))^2} \\
&= \frac{S_N(f) S_Z(f)}{S_Z(f) + S_N(f)} \\
\mathcal{E}[e^2(t)] &= R_e(0) = \int_{-\infty}^{\infty} \frac{S_N(f) S_Z(f)}{S_Z(f) + S_N(f)} df.
\end{aligned}$$

10.65

~~7.53~~ If $Z(t)$ is the random telegraph signal and $N(t)$ is white noise, then

$$\begin{aligned}
S_Z(f) &= \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} \qquad S_N(f) = \frac{N_0}{2} \\
H(f) &= \frac{S_Z(f)}{S_Z(f) + S_N(f)} = \frac{\frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}}{\frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} + \frac{N_0}{2}} \\
&= \frac{4\alpha}{4\alpha + \frac{N_0}{2}(4\alpha^2 + 4\pi^2 f^2)} \\
&= \frac{4\alpha}{(4\alpha + 2N_0\alpha^2) + \frac{N_0}{2} 4\pi^2 f^2}
\end{aligned}$$

From Problem 10.64 ~~7.52~~

$$\begin{aligned}
\mathcal{E}[e^2(t)] &= \int_{-\infty}^{\infty} \frac{\frac{N_0}{2} 4\alpha}{(4\alpha + 2N_0\alpha^2) + \frac{N_0}{2} 4\pi^2 f^2} df \qquad x = \sqrt{2N_0}\pi f \\
&= \frac{2N_0\alpha}{\sqrt{2N_0}\pi} \int_{-\infty}^{\infty} \frac{1}{(4\alpha + 2N_0\alpha^2) + x^2} dx \qquad a^2 = 4\alpha + 2N_0\alpha^2 \\
&= \frac{\sqrt{2N_0}\alpha}{\pi\sqrt{4\alpha + 2N_0\alpha^2}} \underbrace{\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} dx}_{\pi} = \sqrt{\frac{2N_0\alpha}{4 + 2N_0\alpha}}
\end{aligned}$$

10.66

~~7.54~~ $S_Z(f) = \frac{N_1}{2} \qquad |f| < W \qquad S_N(f) = \frac{N_0}{2}$

$$H(f) = \frac{S_Z(f)}{S_Z(f) + S_N(f)} = \frac{\frac{N_1}{2}}{\frac{N_1}{2} + \frac{N_0}{2}} = \frac{N_1}{N_1 + N_0} \qquad |f| < w$$

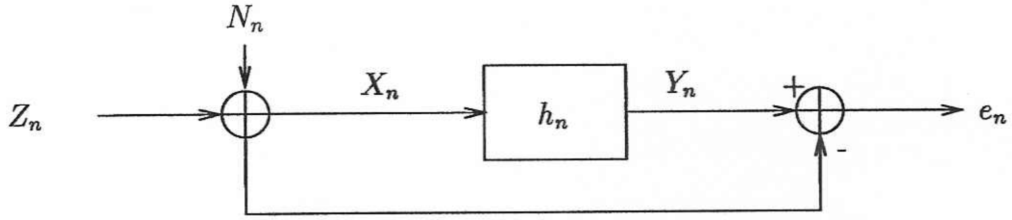
$$\mathcal{E}[e^2(t)] = \int_{-\infty}^{\infty} \frac{S_Z(f) S_N(f)}{S_Z(f) + S_N(f)} df = \frac{\frac{N_1 N_2}{2}}{N_1 + N_0} 2W = \frac{N_1 N_0}{N_1 + N_0} W$$

10.67

~~7.55~~ $S_Z(f) = \frac{2}{1 + 4\pi^2 f^2} \qquad S_N(f) = 1$

$$\begin{aligned} H(f) &= \frac{\frac{2}{1+4\pi^2 f^2}}{\frac{2}{1+4\pi^2 f^2} + 1} = \frac{2}{2 + 1 + 4\pi^2 f^2} = \frac{2}{3 + 4\pi^2 f^2} \\ \mathcal{E}[e^2(t)] &= \int_{-\infty}^{\infty} \frac{S_Z(f) S_N(f)}{S_Z(f) + S_N(f)} df = \int_{-\infty}^{\infty} \frac{2}{3 + 4\pi^2 f^2} df \qquad x = 2\pi f \\ &= \frac{1}{\pi} \frac{1}{\sqrt{3}} \int_{-\infty}^{\infty} \frac{\sqrt{3}}{3 + x^2} dx = \frac{1}{\sqrt{3}} = 0.577 \end{aligned}$$

10.68



Equation 10.89 states that the optimum filter is given by

$$\begin{aligned} H(f) &= \frac{S_{ZX}(f)}{S_X(f)} \\ &= \frac{S_Z(f)}{S_Z(f)+S_N(f)} \\ R_{ZX}(k) &= \mathcal{E}[Z_{n+k}(Z_n+X_n)] \\ &= R_Z(k) \\ \Rightarrow S_{ZX}(f) &= S_Z(f) \end{aligned}$$

From Example 10.14

$$\begin{aligned} S_Z(f) &= \frac{\sigma^2}{(1-\alpha e^{-j2\pi f})(1-\alpha e^{j2\pi f})} \\ S_Z(f)+S_N(f) &= \frac{\sigma^2}{(1-\alpha e^{-j2\pi f})(1-\alpha e^{j2\pi f})}+\sigma_N^2 \\ &= \frac{\sigma^2+\sigma_N^2(1-\alpha e^{-j2\pi f})(1-\alpha e^{j2\pi f})}{(1-\alpha e^{-j2\pi f})(1-\alpha e^{j2\pi f})} \\ \Rightarrow H(f) &= \frac{\sigma^2}{\sigma^2+\sigma_N^2(1-\alpha e^{j2\pi f})(1-\alpha e^{j2\pi f})} \\ \mathcal{E}[e_n^2] &= R_e(0) \end{aligned}$$

Proceeding as in Problem 10.64 we can show that

$$\begin{aligned} S_e(f) &= \frac{S_N(f)S_Z(f)}{S_N(f)+S_Z(f)} = S_N(f)H(f) \\ &= \frac{\sigma^2}{\frac{\sigma^2}{\sigma_N^2}+(1-\alpha e^{-j2\pi f})(1-\alpha e^{j2\pi f})} \end{aligned}$$

We need to find $R_e(k) = \mathcal{F}^{-1}[S_e(f)]$

Consider the denominator of $S_e(f)$; let $Z = e^{j2\pi f}$ and $\Gamma = \dfrac{\frac{\sigma^2}{\sigma_N^2}+1+\alpha^2}{\alpha}$

$$\frac{\sigma^2}{\sigma_N^2}+1+\alpha^2-\alpha e^{-j2\pi f}-\alpha e^{j2\pi f} = -\alpha Z^{-1}[Z^2-\Gamma Z+1]$$

$$
\begin{aligned}
&= -\alpha Z^{-1}[Z - Z_1][Z - Z_2] \text{ where } Z_i = \frac{\Gamma \pm \sqrt{\Gamma^2 - 4}}{2} \\
&= -\alpha[1 - Z_1 Z^{-1}[Z - Z_2] \\
&= \alpha Z_2[1 - Z_1 Z^{-1}][1 - \frac{1}{Z_2} Z] \\
&= \alpha Z_2[1 - Z_1 Z^{-1}][1 - Z_1 Z] \text{ since } Z_1 = \frac{1}{Z_2} \\
&= \alpha Z_2[1 - Z_1 e^{-j2\pi f}][1 - Z_1 e^{j2\pi f}] \\
\therefore S_e(f) &= \frac{\sigma^2}{\alpha Z_2[1 - Z_1 e^{-j2\pi f}][1 - Z_1 e^{j2\pi f}]} \\
\therefore R_e(k) &= \mathcal{F}^{-1}[S_e(f)] \\
&= \frac{\sigma^2 Z_1}{\alpha(1 - Z_1^2)} \mathcal{F}^{-1}\left[\frac{1 - Z_1^2}{(1 - Z_1 e^{-j2\pi f})(1 - Z_1 e^{j2\pi f})}\right] \\
&= \frac{\sigma^2 Z_1}{\alpha(1 - Z_1^2)} Z_1^{|k|}
\end{aligned}
$$

$\Gamma = 2\left(\frac{9}{4}\right) + 2 + \frac{1}{2} = 7$

$Z = \frac{7 \pm \sqrt{49 - 4}}{2}$

$= .195$

Finally we find that

$$\mathcal{E}[e_n^2] = R_e(0) = \frac{\sigma^2 Z_1}{\alpha(1 - Z_1^2)} = \frac{9}{1/2}\left(\frac{.195}{(1-(.19)^2}\right)$$

10.69

~~7.57~~ $S_Z(f) = \dfrac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$

$$\begin{aligned} S_X(f) &= S_Z(f) + S_N(f) \\ &= \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2} + \frac{N_0}{2} = \frac{N_0}{2}\frac{4\pi^2 f^2 + 4\alpha^2 + \frac{8\alpha}{N_0}}{4\pi^2 f^2 + 4\alpha^2} \\ &= \frac{N_0}{2}\frac{(j2\pi f + 2\beta)(-j2\pi f + 2\beta)}{(j2\pi f + 2\alpha)(j2\pi f + 2\alpha)} \qquad \beta = \alpha\sqrt{1 + \frac{2}{\alpha N_0}} \end{aligned}$$

$\frac{2}{N_0} = 3$

Let

$$G(f) = \sqrt{N_0}2\frac{j2\pi f + 2\beta}{j2\pi f + 2\alpha}$$

Equation 10.99 is then

$$\begin{aligned} S_{ZX'}(f) &= \frac{S_{ZX}(f)}{G^*(f)} = \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}\frac{2\alpha - j2\pi f}{2\beta - j2\pi f} \\ &= \frac{4\alpha}{(2\alpha + j2\pi f)(2\beta - j2\pi f)} \end{aligned}$$

$$= \frac{2\alpha}{\alpha + \beta}\left[\frac{1}{2\alpha + j2\pi f} + \frac{1}{2\beta - j2\pi f}\right]$$

The inverse transform is

$$\begin{aligned} R_{ZX'}(\tau) &= \begin{cases} \frac{2\alpha}{\alpha+\beta}e^{-2\alpha\tau} & \tau > 0 \\ \frac{2\alpha}{\alpha+\beta}e^{2\beta\tau} & \tau < 0 \end{cases} \\ \therefore \quad H_2(f) &= \mathcal{F}\left[\frac{2\alpha}{\alpha + \beta}e^{-2\alpha\tau}\mu(\tau)\right] = \frac{2\alpha/(\alpha + \beta)}{2\alpha + j2\pi f} \end{aligned}$$

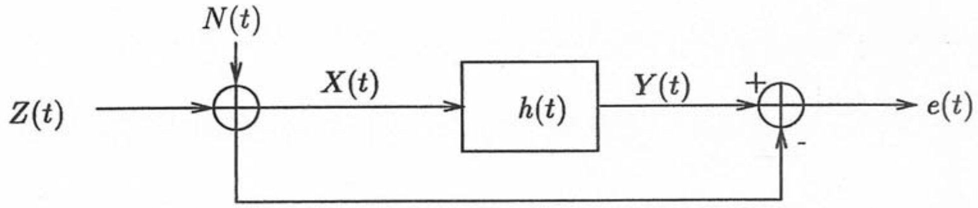
and

$$\begin{aligned} H(f) &= \frac{H_2(f)}{G(f)} = \sqrt{\frac{2}{N_0}}\frac{2\alpha}{\alpha + \beta}\frac{1}{j2\pi f + 2\beta} \\ h(t) &= \sqrt{\frac{2}{N_0}}\frac{2\alpha}{\alpha + \beta}e^{-\beta t} \qquad t \geq 0 \end{aligned}$$

where

$$\beta = \alpha\sqrt{1 + \frac{2}{\alpha N_0}}$$

10.70



In Problem 10.30 we considered the above system. After making adjustments for notation, we have

$$S_e(f) = |1 - H(f)|^2 S_Z(f) + |H(f)|^2 \underbrace{S_N(f)}_{1}$$

The Wiener filter in Example 10.25 is

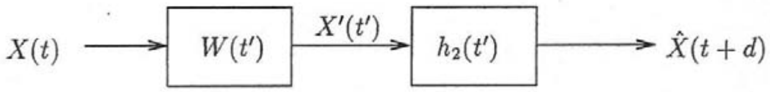
$$\begin{aligned}
H(f) &= \frac{c}{\sqrt{3} + j2\pi f} \text{ where } c = \frac{2}{1+\sqrt{3}} \\
1 - H(f) &= \frac{\sqrt{3} + j\omega - \frac{2}{1+\sqrt{3}}}{\sqrt{3} + j\omega} = \frac{1 + j\omega}{\sqrt{3} + j\omega} \\
S_e(f) &= \left|\frac{1 + j\omega}{\sqrt{3} + j\omega}\right|^2 \left|\frac{\sqrt{2}}{1 + j\omega}\right|^2 + \left|\frac{c}{\sqrt{3} + j\omega}\right|^2 \\
&= \frac{2 + c^2}{|\sqrt{3} + j\omega|^2} \\
R_e(\tau) &= \frac{1}{2\sqrt{3}}(2 + c^2)e^{-\sqrt{3}|t|} \\
R_e(0) &= \frac{2 + c^2}{2\sqrt{3}} = .732
\end{aligned}$$

As expected this is a larger error than the smoothing filter which uses the entire observation of $Z(\alpha)$.

10.71 First we find the causal filter that whitens the observation process

$$\begin{aligned} S_X(f) &= \mathcal{F}[e^{-\alpha|\tau|}] = \frac{2\alpha}{\alpha^2+4\pi^2f^2} = \frac{\sqrt{2\alpha}}{\alpha+j2\pi f}\frac{\sqrt{2\alpha}}{\alpha-j2\pi f} \\ \Rightarrow W(f) &= \frac{\alpha+j2\pi f}{\sqrt{2\alpha}} \end{aligned}$$

Next we seek the optimum estimate for $X(t+d)$ in terms of the whitened process $X'(t)$



The estimator we seek is

$$\hat{X}(t+d) = \int_0^\infty h_2(\lambda_1)X'(t-\lambda)d\lambda$$

The orthogonality condition requires that

$$\begin{aligned} &\mathcal{E}[(\hat{X}(t+d)-X(t+d))X'(t')] = 0 \qquad \text{for } t'<t \\ &\Rightarrow \mathcal{E}[\hat{X}(t+d)X'(t')] = \mathcal{E}[X(t+d)X'(t')] \qquad (*) \end{aligned}$$

The right hand side of the above equation is:

$$\begin{aligned} \mathcal{E}[X(t+d)X'(t')] &= \mathcal{E}\left[X(t+d)\int_0^\infty \omega(\lambda)X(t'-\lambda)d\lambda\right] \\ &= \underbrace{\int_0^\infty \omega(\lambda)R_X(t'-t-d-\lambda)d\lambda}_{\text{convolution of } \omega(\lambda)\star R_X(\lambda)'} \end{aligned}$$

$$\begin{aligned} \mathcal{F}[\omega(\lambda)\star R_X(\lambda)] &= W(f)S_X(f) = \frac{\alpha+j2\pi f}{\sqrt{2\alpha}}\frac{2\alpha}{\alpha^2+4\pi^2f^2} = \frac{\sqrt{2\alpha}}{\alpha-j2\pi f} \\ \Rightarrow \omega(\tau)\star R_X(\tau) &= \sqrt{2\alpha}e^{\alpha\tau} \qquad \tau<0 \\ \Rightarrow \int_0^\infty \omega(\lambda)R_X(t'-t-d-\lambda)d\lambda &= \sqrt{2\alpha}e^{\alpha(t'-t-d)} \qquad t'-t-d<0 \end{aligned}$$

Now consider the left hand side of (*)

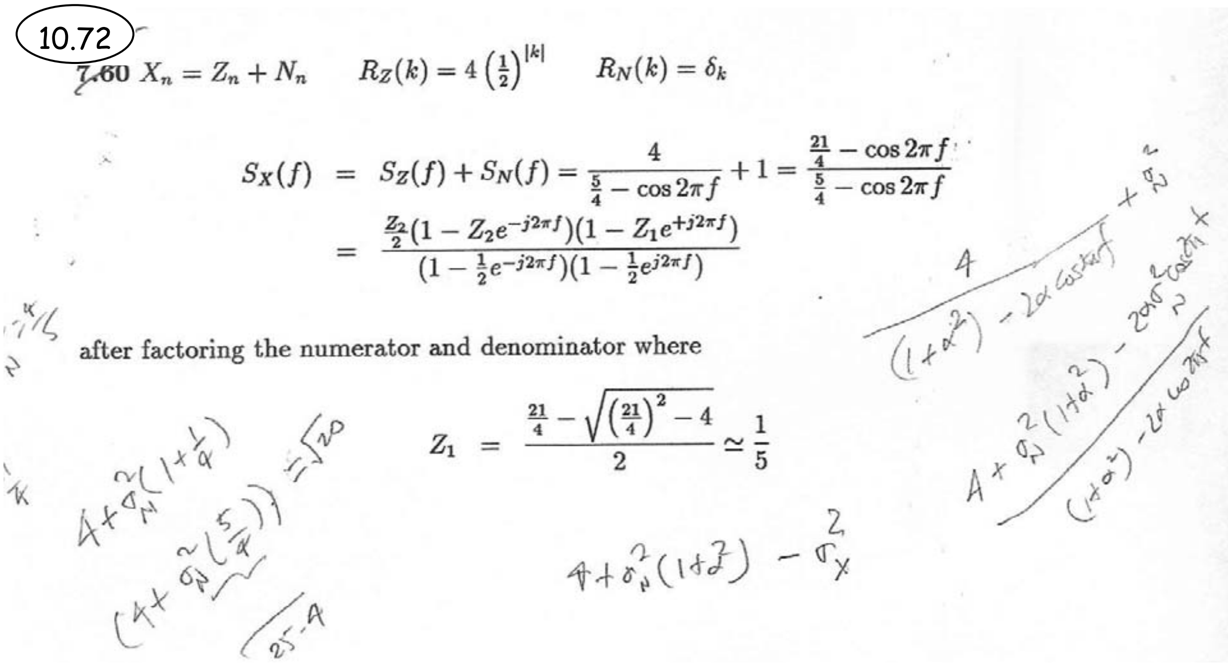
$$\begin{aligned} \mathcal{E}\left[\int_0^\infty h_2(\lambda)X'(t-\lambda)d\lambda X'(t')\right] &= \int_0^\infty h_2(\lambda)\underbrace{R_{X'}(t-t'-\lambda)}_{S_{X'}(t-t'-\lambda)}d\lambda \qquad t'<t \\ &= h_2(t'-t) \qquad t'<t \end{aligned}$$

Let $t''=t-t'$

$$\begin{aligned} \therefore h_2(t'') &= \sqrt{2}e^{-\alpha d}e^{-\alpha t''} \qquad t''>0 \\ \Rightarrow H_2(f) &= \frac{\sqrt{2}e^{-\alpha d}}{\alpha+j2\pi f} \end{aligned}$$

Finally the optimum filter is given by

$$\begin{aligned} H(f) &= W(f)H_2(f) = e^{-\alpha d} \\ \Rightarrow h(t) &= e^{-\alpha d}\delta(t) \\ \Rightarrow \hat{X}(t+d) &= e^{-\alpha d}X(t) \end{aligned}$$



10.72

7.60 $X_n = Z_n + N_n \qquad R_Z(k) = 4\left(\frac{1}{2}\right)^{|k|} \qquad R_N(k) = \delta_k$

$$\begin{aligned} S_X(f) &= S_Z(f) + S_N(f) = \frac{4}{\frac{5}{4} - \cos 2\pi f} + 1 = \frac{\frac{21}{4} - \cos 2\pi f}{\frac{5}{4} - \cos 2\pi f} \\ &= \frac{\frac{Z_2}{2}(1 - Z_2 e^{-j2\pi f})(1 - Z_1 e^{+j2\pi f})}{(1 - \frac{1}{2}e^{-j2\pi f})(1 - \frac{1}{2}e^{j2\pi f})} \end{aligned}$$

$$\frac{4}{(1+\alpha^2) - 2\alpha\cos 2\pi f} + \sigma_N^2$$

$$\frac{4 + \sigma_N^2(1+\alpha^2) - 2\alpha\sigma_N^2\cos 2\pi f}{(1+\alpha^2) - 2\alpha\cos 2\pi f}$$

after factoring the numerator and denominator where

$$Z_1 = \frac{\frac{21}{4} - \sqrt{\left(\frac{21}{4}\right)^2 - 4}}{2} \simeq \frac{1}{5}$$

$4 + \sigma_N^2\left(1 + \frac{1}{\alpha}\right)$

$\left(4 + \sigma_N^2\left(\frac{5}{4}\right)\right) = \sqrt{20}$

$25 - 4$

$4 + \sigma_N^2(1+\alpha^2) - \sigma_X^2$

$$\begin{aligned} Z_2 &= \frac{\frac{21}{4}+\sqrt{\left(\frac{21}{4}\right)^2-4}}{2} = \frac{1}{Z_1} \\ \Rightarrow G(f) &= \sqrt{\frac{Z_2}{2}}\frac{1-Z_1e^{-j2\pi f}}{1-\frac{1}{2}e^{-j2\pi f}} = \frac{1}{W(f)} \end{aligned}$$

Next consider:

$$\begin{aligned} R_{ZX}(k) &= \mathcal{E}[Z_{n+n}(Z_n+N_n)] = R_Z(k) \\ &\Rightarrow \; S_{ZX}(f) = S_Z(f) \end{aligned}$$

Thus

$$\begin{aligned} S_{ZX'}(f) &= W^*(f)S_{ZX}(f) \\ &= \sqrt{\frac{2}{Z_2}}\frac{1-\frac{1}{2}e^{+j2\pi f}}{1-Z_1e^{+j2\pi f}}S_Z(f) \\ &= \sqrt{\frac{2}{Z_2}}\left(\frac{1}{1-Z_1e^{+j2\pi f}}\right)\left(\frac{4}{1-\frac{1}{2}e^{-j2\pi f}}\right) \\ &= \frac{4\sqrt{\frac{2}{Z_2}}}{1-\frac{1}{2}Z_1}\Bigg[\frac{1}{1-Z_1e^{j2\pi f}} + \underbrace{\frac{1}{1-\frac{1}{2}e^{-j2\pi f}}}_{\substack{\text{this yields the positive} \\ \text{time component}}} \quad \Bigg] \text{ after partial fraction expansion} \end{aligned}$$

$$\therefore H_2(f) = \frac{4\sqrt{\frac{2}{Z_2}}}{1-\frac{1}{2}Z_1}\frac{1}{1-\frac{1}{2}e^{-j2\pi f}}$$

and finally

$$\begin{aligned} H(f) &= W(f)H_2(f) = \sqrt{\frac{2}{Z_2}}\frac{1-\frac{1}{2}e^{-j2\pi f}}{1-Z_1e^{-j2\pi f}}\left(\frac{4\sqrt{\frac{2}{Z_2}}}{1-\frac{1}{2}Z_1}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j2\pi f}}\right) \\ &= \frac{\frac{8}{Z_2(1-\frac{1}{2}Z_1)}}{1-Z_1e^{-j2\pi f}} \\ &= \left(\frac{8}{Z_2-\frac{1}{2}}\right)\frac{1}{1-Z_1e^{-j2\pi f}} \\ h_n &= \left(\frac{8}{Z_2-\frac{1}{2}}\right)Z_1^k \qquad k \geq 0 \end{aligned}$$

is the impulse response of the optimum filter.

*10.5 The Kalman Filter

10.73

~~7.63~~ $Z_n = \alpha_{n-1}Z_{n-1} + W_{n-1}$

$$\begin{aligned} P[Z_n|Z_{n-1}, Z_{n-2}, ...] &= P[W_{n-1}|Z_{n-1}, Z_{n-1}, ...] \\ &= P[W_{n-1}] \\ &= P[Z_n|Z_{n-1}] \end{aligned}$$

So Z_n is Markovian.

$$\begin{aligned} X_n &= Z_n + N_n \\ &= \alpha_{n-1}Z_{n-1} + W_{n-1} + N_n \\ &= \alpha_{n-1}(X_{n-1} - N_{n-1}) + W_{n-1} + N_n \\ &= \alpha_{n-1}X_{n-1} - \alpha N_{n-1} + N_n + W_{n-1} \end{aligned}$$

Similarly, X_n is Markovian.

10.74

~~7.64~~ From the equation just above (~~7.114~~ 10.119), we have

$$k_n(a_n - k_n)E[\mathcal{E}_n^2] = k_n^2 E[N_n^2]$$

$$\begin{aligned} E[\mathcal{E}_{n+1}^2] &= (a_n - k_n)^2 E[\mathcal{E}_n^2] + E[W_n^2] + k_n^2 E[N_n^2] \\ &= (a_n - k_n)^2 E[\mathcal{E}_n^2] + E[W_n^2] + k_n(a_n - k_n)E[\mathcal{E}_n^2] \\ &= a_n(a_n - k_n)^2 E[\mathcal{E}_n^2] + E[W_n^2] \end{aligned}$$

10.75

~~7.65~~ Initialization: $Y_0 = 0$, $E[\mathcal{E}_0^2] = E[Z_0^2] = 0$

$$\begin{aligned}
k_n &= \frac{aE[\mathcal{E}_n^2]}{E[\mathcal{E}_n^2] + E[N - n^2]} = \frac{aE[\mathcal{E}_n^2]}{E[\mathcal{E}_n^2] + 1} \\
Y_{n+1} &= a_n Y_n + k_n (X_n - Y_n) \\
E[\mathcal{E}_{n+1}^2] &= a_n(a_n - k_n)E[\mathcal{E}_n^2] + E[W_n^2] \\
&= a\frac{a}{1 + E[\mathcal{E}_n^2]}E[\mathcal{E}_n^2] + 0.36
\end{aligned}$$

We then have

$$e_\infty = a\frac{a}{1 + e_\infty}e_\infty + 0.36$$

For $a = 0.5$, $e_\infty = 0.44$, $k_\infty = 0.15$

$$Y_{n+1} = 0.8Y_n + 0.15(X_n - Y_n)$$

For $a = 2$, $e_\infty = 3.46$, $k_\infty = 1.55$

$$Y_{n+1} = 0.8Y_n + 1.55(X_n - Y_n)$$

10.76

~~7.66~~ Define the innovation as

$$I_n = X_n - bY_n$$

We have

$$\begin{aligned}
\mathcal{E}_{n+1} &= (a_n - k_n b_n)\mathcal{E}_n + W_n - k_n N_n \\
E[\mathcal{E}_{n+1}^2] &= (a_n - k_n b_n)^2 E[\mathcal{E}_n^2] + E[W_n^2] + k_n^2 E[N_n^2] \\
k_n &= \frac{a_n b_n E[\mathcal{E}_n^2]}{b_n^2 E[\mathcal{E}_n^2] + E[N - n^2]}
\end{aligned}$$

And

$$\begin{aligned}
Y_{n+1} &= aY_n + k_n(X_n - bY_n) \\
E[\mathcal{E}_{n+1}^2] &= a_n(a_n - k_n b_n)E[\mathcal{E}_n^2] + E[W_n^2]
\end{aligned}$$

*10.6 Estimating the Power Spectral Density

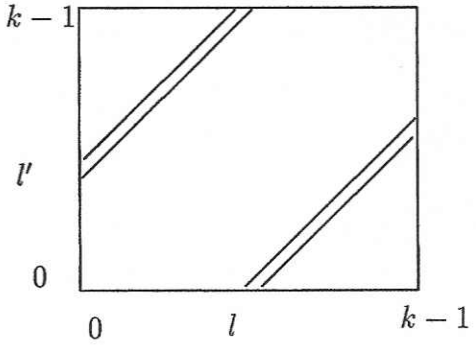
10.77

$$\begin{aligned}
\tilde{p}_k(f) &= \frac{1}{k}\left|\sum_{l=0}^{k-1} X_l e^{-j2\pi fl}\right|^2 \\
&= \frac{1}{k}\sum_{l=0}^{k-1}\sum_{l'=0}^{k-1} X_l X_{l'} e^{-j2\pi fl} e^{+j2\pi fl'} \\
&= \frac{1}{k}\sum_{l=0}^{k-1}\sum_{l'=0}^{k-1} X_l X_{l'} e^{-j2\pi f(l-l')}
\end{aligned}$$

$$m = l - l' \Leftrightarrow l' = l - m$$

where $(k-1) \leq m \leq k-1$

We change the order of summation as indicated by the figure below:

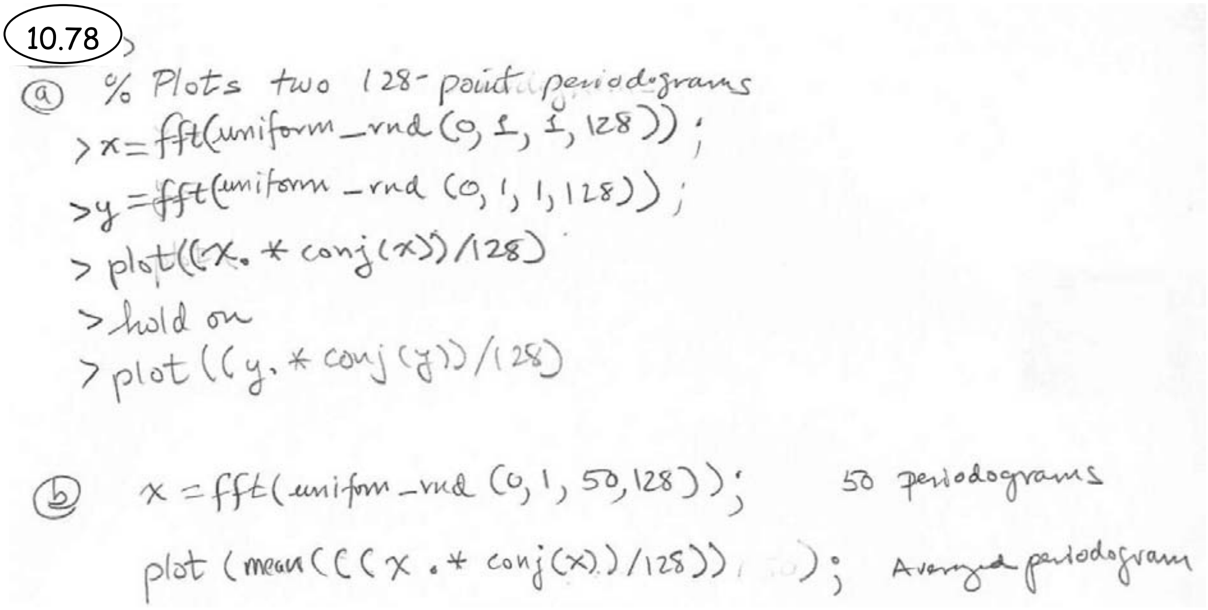


We then obtain:

$$= \sum_{m=-(k-1)}^{k-1}\left\{\underbrace{\frac{1}{k}\sum_{l=0}^{k-|m|-1} X_l X_{l+n}}_{\hat{r}_k(m)}\right\} e^{-j2\pi fm}$$

$$= \sum_{m=-(k-1)}^{k-1} \hat{r}_k(m) e^{-j2\pi fm} \qquad \checkmark$$

10.78



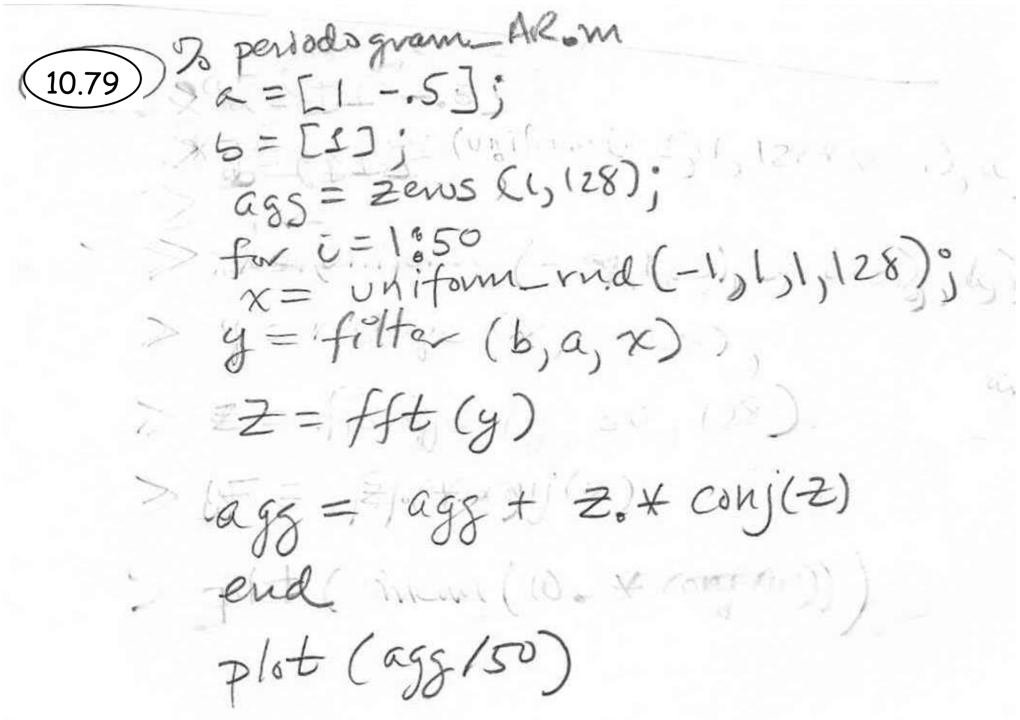
ⓐ

```
% Plots two 128-point periodograms
>x=fft(uniform_rnd(0,1,1,128));
>y=fft(uniform_rnd(0,1,1,128));
>plot((x.*conj(x))/128)
>hold on
>plot((y.*conj(y))/128)
```

ⓑ

```
x = fft(uniform_rnd(0,1,50,128));        50 periodograms
plot(mean(((x .* conj(x))/128)));        Averaged periodogram
```

10.79



```
% periodogram_AR.m
a = [1 -.5];
b = [1];
agg = zeros(1,128);
for i=1:50
x= uniform_rnd(-1,1,1,128);
y = filter(b,a,x)
z = fft(y)
agg = agg + z.*conj(z)
end
plot(agg/50)
```

10.80

$$\mathcal{E}\left[\sum_{m=-(k-1)}^{k-1}\left(\frac{1}{k-|m|}\sum_{n=0}^{k-|m|-1}X_nX_{n+m}\right)e^{-j2\pi fm}\right]$$

$$= \sum_{m=(k-1)}^{k-1}\frac{1}{k-|m|}\left(\underbrace{\sum_{n=0}^{k-|m|-1}R_X(m)}_{(k-|m|)R_X(m)}\right)e^{-j2\pi fm}$$

$$= \sum_{m=-(k-1)}^{k-1}R_X(m)e^{-j2\pi fm}$$

The estimate is biased because the limits of the summation are finite.

10.7 Numerical Techniques for Processing Random Signals

10.81)

$$f_s = 2KW_0$$

a) $$e(f) = \left[\sum_{n=1}^{\infty} S_X(f - nf_s)\right](1 - u(f - KW_0)) + u(f + KW_0)\sum_{n=1}^{\infty} S_X(f + nf_s)$$

$$\int_{-\infty}^{+\infty} e(f) = \sum_{n=1}^{\infty} \int_{-KW_0}^{+KW_0} S_X(f + nf_s)\,df + \sum_{n=1}^{\infty} \int_{-KW_0}^{KW_0} S_X(f - nf_s)\,df$$

Since $S_X(f)$ is a like a Gaussian pdf function with mean 0 & variance W_0

we have

$$\int_{-\infty}^{+\infty} e(f)\,df = \beta^2 W_0 \sum_{n=1}^{\infty} \left(Q(nf_s - KW_0) - Q(nf_s + KW_0) \right)$$

$$+ \beta^2 W_0 \sum_{n=1}^{\infty} \left(Q(nf_s - KW_0) - Q(nf_s + KW_0) \right) =$$

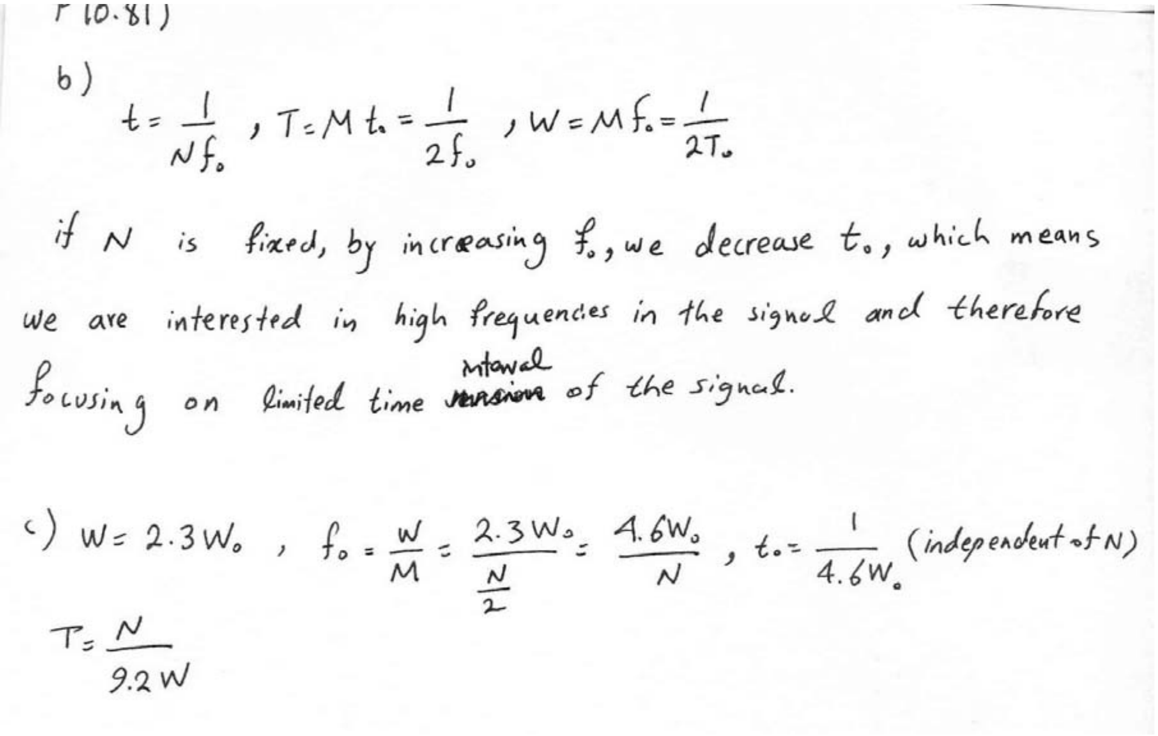
$$= 2\beta^2 W_0 \sum_{n=1}^{\infty} \left(Q(nf_s - KW_0) - Q(nf_s + KW_0) \right)$$

in order to make this error less than 1%, we neglect effects of

$n > 1$, and therefore we have

$$Q(f_s - KW_0) - Q(f_s + KW_0) \approx Q(f_s - KW_0) = Q(2KW_0 - KW_0) = Q(KW_0) = 0.01$$

as a result $K > 2.3$



10.81)

b)

$t = \frac{1}{Nf_0}$, $T = M t_0 = \frac{1}{2f_0}$, $W = M f_0 = \frac{1}{2T_0}$

if N is fixed, by increasing f_0, we decrease t_0, which means we are interested in high frequencies in the signal and therefore focusing on limited time ~~version~~ interval of the signal.

c) $W = 2.3W_0$, $f_0 = \frac{W}{M} = \frac{2.3W_0}{\frac{N}{2}} = \frac{4.6W_0}{N}$, $t_0 = \frac{1}{4.6W_0}$ (independent of N)

$T = \frac{N}{9.2W}$

(d) The Octave code for part d:

```
%P10.81
%part d
clear all;
close all;
N=1024;
M=N/2;
W0=10;
t0=1/(4.6*W0);
f0=4.6*W0/N;
T=N/(9.2*W0);
m=-M:1:M-1;
Sx=exp(-1*(m.^2*f0^2)/(2*W0^2));
k=-M:1:M-1;
Rfex=fft(Sx);
Rf=Rfex.*conj(Rfex)/N;
Rex=Sx*exp(-j*2*pi*m'*k/N);
Rex2=f0*sqrt(Rex.*conj(Rex));
%Rx=sum(Sx*exp(-j*2*pi*m*k'/N));
Rx=exp(-1*(m.^2*t0^2)*(2*pi^2*W0^2))*(sqrt(2*pi)*W0);
subplot(2,1,1)
plot(m,Rx,m,Rex2);
legend('Actual','FFT');
subplot(2,1,2)
plot(Rx,Rex2);
title('Actual vs. FFT');
```

10.82

```
%P10.82
%Part a
clear all;
close all;
N=256;
M=N/2;
k=-M:1:M-1;
alpha=0.75;

%part a
Rx=4*alpha.^abs(k);
Sx=fft(Rx);
figure(1);
plot((k+M)/N,sqrt(Sx.*conj(Sx)));
title('P10.82a');


%part b
Rx=4*0.5.^abs(k)+16*0.25.^abs(k);
Sx=fft(Rx);
figure(2);
plot((k+M)/N,sqrt(Sx.*conj(Sx)));
title('P10.82b');


%part c
Rx=0.5*cos(2*pi*0.1*k);
Sx=fft(Rx);
figure(3);
plot((k+M)/N,sqrt(Sx.*conj(Sx)));
title('P10.82c');
```

10.83

```
%P10.83
%Part a
clear all;
close all;
N=256;
M=N/2;
k=-M:1:M-1;
fc=1/8;
Sx(1:1:N)= 1;
for i=1:1:N
    if (((i-M-1)/N) >= -fc)
        if (((i-M-1)/N) <= fc)
            Sx(i)= 0;
        end
    end
end
Rx=fft(Sx);
Rx2=sqrt(Rx.*conj(Rx))/N;
%note that Rx would be periodic with period N
figure(1);
plot(k+M,Rx2);
title('Problem 10.83a');


%Part b
Sx= 0.5+0.5*cos(2*pi*k/N);
Rx=fft(Sx);
Rx2=sqrt(Rx.*conj(Rx))/N;
%note that Rx would be periodic with period N
figure(2);
plot(k+M,Rx2);
title('Problem 10.83b');
```

10.84

```
%P10.84
%Part a,b,c
clear all;
close all;
N=256;
M=N/2;
k=-M:1:M-1;
fc=1/8;
alpha=0.25;
Rx=4*alpha.^abs(k)
Sx=fft(Rx);
Sx1=sqrt(Sx.*conj(Sx));
Sx2=Sx1;
fc = 1/4;
for i=1:1:N
    if (((i-M-1)/N) >= -fc)
        if (((i-M-1)/N) <= fc)
            Sx2(i) = 0;
        end
    end
end
figure(1);
plot((k+M)/N,Sx2);
title('Problem 10.84a');

%Part b
%Assume f0=0.1
Rx=0.5*cos(2*pi*0.1*k);
Sx=fft(Rx);
Sx1=sqrt(Sx.*conj(Sx));
Sx2=Sx1;
h2=-4*pi^2*(k/N).^2;
Sx2=Sx2.*h2;
figure(2);
plot((k+M)/N,Sx2);
title('Problem 10.84b');

%Part c
Rx=9*(1-abs(k)/3);
Rx(1:M-3)=0;
Rx(M+4:2*M)=0;
Sx=fft(Rx);
Sx1=sqrt(Sx.*conj(Sx));
figure(3)
plot((k+M)/N,Sx1);
title('Problem 10.84c');
```

10.85)

a)

$$R_X(\tau) = \int_{-\infty}^{+\infty} S_x(f)\, e^{-j2\pi\tau}\, df = \int_{-\infty}^{0} S_x(f)\, e^{-j2\pi f\tau}\, df + \int_{0}^{+\infty} S_x(f)\, e^{-j2\pi f}\, df$$

$$= \int_{0}^{+\infty} S_X(-f)\, e^{j2\pi f\tau}\, df + \int_{0}^{+\infty} S_X(f)\, e^{-j2\pi f\tau}\, df$$

$$= \int_{0}^{+\infty} \left[\left(S_X(f)\, e^{-j2\pi f\tau}\right)^* + \left(S_X(f)\, e^{-j2\pi f\tau}\right)\right] df$$

$$= 2\,\mathrm{Real}\left\{\int_{0}^{+\infty} S_X(f)\, e^{-j2\pi f\tau}\, df\right\}$$

we used the fact that if X is a real random process:

we have: $S_X(-f) = S_X^*(f)$

P10.85)

b)

$$R_X(\tau) = 2\mathrm{Re}\left\{ \int_0^\infty S_X(f)\, e^{-j2\pi f\tau}\, df \right\}$$

define $y(\tau) := \int_0^\infty S_X(f)\, e^{-j2\pi f\tau}\, df \approx \int_0^W S_X(f)\, e^{-j2\pi f\tau}\, df$

divide the range $(0,W)$ to N steps, therefore:

$$y(\tau) \approx \sum_{m=0}^{N-1} S_X(mf_0)\, e^{-j2\pi m f_0 \tau} f_0$$

and dividing τ to N steps:

$$y(kt_0) \approx f_0 \sum_{m=0}^{N-1} S_X(mf_0)\, e^{-j2\pi m k f_0 t_0}$$

in order to make the above equation as FFT, we should have:

$t_0 f_0 = \frac{1}{N}$.

Therefore $R_X(kt_0) = 2 f_0\, \mathrm{Re}\{ y(kt_0) \}$ $\quad 0 \leq k < N-1$

also f_0 $\quad -N < k < 0 : R_X(kt_0) = R_X(-kt_0)$, Assuming X is real

c) This approach gives a better resolution to $R_X(\tau)$ with the same amount of FFT computations.

10.86

a)

```
%P10.86
close all;
clear all;
n=0:1023;
X=normal_rnd(0,1,1,1024);
h=exp(-2*n);
Y=conv(X,h);
z=autocov(Y,30);
%theoretical autocovariance w
w=exp(-2*n)/(1-exp(-4));
plot(1:30,w(1:30),1:30,z(1:30));
title('Problem 10.86');
```

b)

$$R_X(K) = \begin{cases} \sigma_X^2 = 1 & K=0 \\ 0 & K\neq 0 \end{cases}$$

$$R_Y(K) = R_X(K) * h(K) * h(-K) = \sigma_X^2 \cdot h(K) * h(-K)$$

$$h(K) = e^{-2K},\ K \geq 0 \Rightarrow h(K) * h(-K) = \sum_{n=0}^{\infty} h(k)\,h(n+K) = e^{-2K} \sum_{n=0}^{\infty} e^{-4k} = \frac{e^{-2K}}{1-e^{-4}}$$

$$E[Y] = m_X(0)\,H(0) = 0$$

Therefore $C_Y(K) = R_Y(K) = \dfrac{e^{-2K}}{1-e^{-4}}$

c) The output is also Gaussian, with the above autocovariance.

10.87

```
%p 10.87
close all;
clear all;
n=0:0.01:10.23;
r=exp(-2.*n);
%r=(-0.5).^n;
K=toeplitz(r);
[U,D,V]=svd(K);
X=normal_rnd(0,1,1,1024);
y=V*(D^0.5)*X';
plot(y);
z=autocov(y,200);
plot(1:200,r(1:200),1:200,z(1:200));
title('Problem 10.87');
```

Problems Requiring Cumulative Knowledge

10.88

~~7.71~~ In Example 9.38, $R_X(t_1, t_2) \neq R_X(t_1 - t_2)$ so the process is not WSS. However, the process is WS cyclostationary so after stationarizing $X(t)$,

$$R_{X_S}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \leq T \\ 0 & |\tau| > T \end{cases}$$

and

$$S_X(f) = T\left(\frac{\sin \pi f T}{\pi f T}\right)^2$$

10.89

~~7.72~~ We perform Wiener filtering for the signal of Example 10.26 to compare its operation and performance with Kalman filtering

$$\begin{aligned}
Z_n &= aZ_{n-1} + W_n, \qquad a = 0.8, \quad \sigma_N^2 = 0.36 \\
S_Z(f) &= |H_Z(f)|^2\sigma_N^2 = \left|\frac{1}{1 - ae^{-j2\pi f}}\right|^2 \sigma_N^2 = \frac{0.36}{1.64 - 1.6\cos 2\pi f} = \frac{0.225}{1.025 - \cos 2\pi f} \\
R_Z(k) &= \frac{\sigma_N^2}{1 - a^2}a^k = \frac{0.36}{1 - 0.64}a^k = a^k = (0.8)^k \\
X_n &= Z_n + N_n, \qquad E[N_n^2] = 1 \\
S_X(f) &= S_Z(f) + S_N(f) = S_Z(f) + 1 = \frac{1.25 - \cos 2\pi f}{1.025 - \cos 2\pi f}
\end{aligned}$$

Since

$$c - \cos 2\pi f = \frac{1}{2b}(1 - be^{-j2\pi f})(1 - be^{j2\pi f}), \qquad b = c \pm \sqrt{c^2 - 1}$$

then

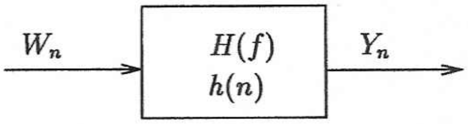
$$S_X(f) = \frac{(1 - 0.5e^{-j2\pi f})(1 - 0.5e^{j2\pi f})}{\frac{1}{1.6}(1 - 0.8e^{-j2\pi f})(1 - 0.8e^{j2\pi f})}$$

$$S_X(f) = G(f) \cdot G^*(f) \Rightarrow G(f) = \frac{1}{W(f)} = \sqrt{1.6}\frac{1 - 0.5e^{-j2\pi f}}{1 - 0.8e^{-j2\pi f}}$$

$$S_{ZX'}(f) = W^*(f) \cdot S_{ZX}(f), \qquad S_{ZX}(f) = S_Z(f)$$

$$\begin{aligned}
\Rightarrow S_{ZX'}(f) &= \frac{1}{\sqrt{1.6}}\frac{1.08e^{j2\pi f}}{1 - 0.5e^{j2\pi f}} \cdot \frac{0.225}{\frac{1}{1.6}(1 - 0.8e^{-j2\pi f})(1 - 0.8e^{j2\pi f})} \\
S_{ZX'}(f) &= 0.225 \times \sqrt{1.6}\frac{1}{1 - 0.5e^{j2\pi f}} \cdot \frac{1}{1 - 0.8e^{-j2\pi f}} \\
&= 0.225 \times \sqrt{1.6} \times \frac{1}{1 - 0.5 \times 0.8} \times \left[\frac{e^{j2\pi f}}{1 - 0.5e^{j2\pi f}} + \frac{1}{1 - 0.8e^{-j2\pi f}}\right] \\
\Rightarrow H_2(f) &= \frac{0.225\sqrt{1.6}}{1 - 0.4}\frac{1}{1 - 0.8e^{-j2\pi f}} \\
\Rightarrow H(f) &= W(f) \cdot H_2(f) = \frac{1}{\sqrt{1.6}}\frac{1 - 0.8e^{-j2\pi f}}{1 - 0.5e^{-j2\pi f}} \cdot \frac{0.225\sqrt{1.6}}{0.6}\frac{1}{1 - 0.8e^{-j2\pi f}} = \frac{0.375}{1 - 0.5e^{-j2\pi f}}
\end{aligned}$$

10.90 a) Let's first find the impulse response of the linear system



By taking the Fourier transform of both sides of the Y_n equation, we have

$$\begin{aligned}
\mathcal{F}[Y_n] &= -\sum_{i=1}^{q} \alpha_i \mathcal{F}[Y_{n-i}] + \sum_{i'=0}^{p} \beta_{i'} \mathcal{F}[W_{n-i'}] \\
Y(f) &= -\sum_{i=1}^{q} \alpha_i e^{-j2\pi i} Y(f) + \sum_{i'=0}^{p} \beta_{i'} e^{-j2\pi i'} W(f) \\
\Rightarrow H(f) &= \frac{Y(f)}{W(f)} = \frac{\sum_{i'=0}^{p} \beta_{i'} e^{-j2\pi i'}}{1 + \sum_{i=1}^{q} \alpha_i e^{-j2\pi i}}
\end{aligned}$$

For a linear system, we have $S_Y(f) = |H(f)|^2 \cdot S_W(f)$. Since W_n is white noise with variance σ_N^2, $S_W(f) = \sigma_W^2$

$$\Rightarrow S_Y(f) = \left| \frac{\Sigma \beta_{i'} e^{-j2\pi i'}}{1 + \Sigma \alpha_i e^{-j2\pi i}} \right|^2 \cdot \sigma_W^2$$

b) $Y_n W_{n-k} = -\Sigma \alpha_i Y_{n-i} W_{n-k} + \Sigma \beta_{i'} W_{n-i'} W_{n-k}$
Taking the expectations of both sides, we have:

$$\begin{aligned}
E[Y_n W_{n-k}] &= -\Sigma \alpha_i E[Y_{n-i} W_{n-k}] + \Sigma \beta_i E[W_{n-i'} W_{n-k}] \\
R_{YW}(-K) &= -\sum_i \alpha_i R_{YW}(i-k) + \sum_{i'} \beta_{i'} R_W(i'-k)
\end{aligned}$$

Let's take the Fourier transform of both sides with respect to k

$$\begin{aligned}
S_{YW}^*(f) &= -\sum_i \alpha_i e^{j2\pi i} S_{YW}^*(f) + \sum_{i'} \beta_{i'} e^{j2\pi i'} S_W^*(f) \\
\Rightarrow S_{YW}(f) &= \frac{\sum_{i'} \beta_{i'} e^{-j2\pi i'}}{1 + \Sigma \alpha_i e^{-j2\pi i}} S_W(f)
\end{aligned}$$

We assumed β_i and α_i's are real parameters

$$\begin{aligned}
Y_n Y_{n-k} &= -\Sigma \alpha_i Y_{n-i} Y_{n-k} + \Sigma \beta_{i'} W_{n-i'} Y_{n-k} \\
R_Y(-K) &= -\Sigma \alpha_i E[Y_{n-i} Y_{n-k} + \Sigma \beta_i R_{WY}(i'-k)] \\
S_Y^*(f) &= -\Sigma \alpha_i e^{j2\pi i} S_Y^*(f) + \Sigma \beta_{i'} e^{j2\pi i'} S_{WY}^*(f) \\
S_Y(f) &= \frac{\sum_{i'=0}^{p} \beta_{i'} e^{-j2\pi i'}}{1 + \sum_{i=1}^{q} \alpha_i e^{-j2\pi i}} S_{WY}(f), \qquad S_{WY}(f) = S_{YW}^*(f) \\
\Rightarrow S_Y(f) &= \underbrace{\frac{\sum_{i'=0}^{p} \beta_{i'} e^{-j2\pi i'}}{1 + \sum_{i=1}^{q} \alpha_i e^{-j2\pi i}}}_{H(f)} \cdot \underbrace{\frac{\sum_{i'=0}^{p} \beta_{i'} e^{j2\pi i'}}{1 + \sum_{i=1}^{q} \alpha_i e^{j2\pi i}}}_{H^*(f)} S_W(f) \\
\Rightarrow S_Y(f) &= H(f) \cdot H^*(f) \cdot S_W(f) = |H(f)|^2 \cdot \sigma_W^2
\end{aligned}$$

10.91

~~7.75~~ **a)**

$$\begin{aligned} R_{Y_1Y_2}(\tau) &= E[Y_1(t+\tau)Y_2(t)] \\ &= E\left[\int_{-\infty}^{\infty} h_1(u)x_1(t+\tau-u)du \int_{-\infty}^{\infty} h_2(v)x_2(t-v)dv\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)R_{X_1X_2}(\tau-u+v)dudv \\ &= h_1(\tau)\star h_2(-\tau)R_{X_1X_2}(\tau) \\ S_{Y_1Y_2}(f) &= H_1(f)H_2^*(f)S_{X_1X_2}(f) \end{aligned}$$

b) From Section 7.2, we know that $Y_1(t)$ and $Y_2(t)$ are WSS. From part (a), the cross-correlation only depends on $t_1 - t_2 = \tau$, so $Y_1(t)$ and $Y_2(t)$ are jointly WSS.

$Y_1(t)$ and $Y_2(t)$ are jointly Gaussian since $X_1(t)$ and $X_2(t)$ are jointly Gaussian, and the operation of convolution is linear.

c)

$$\begin{aligned} S_{Y_1Y_2}(f) &= H_1(f)H_2^*(f)S_{X_1X_2}(f) = 0 \\ R_{Y_1Y_2}(\tau) &= 0 \\ C_{Y_1Y_2}(\tau) &= 0 \end{aligned}$$

Y_1 and Y_2 are uncorrelated and therefore independent because they are jointly Gaussian.

d) $Y_1(t)$ and $Y_2(t)$ are still jointly Gaussian.

10.92

a) $Y(t) = X(t) + N(t)$

$Y(t)$ should be sampled at points $t = mT$

$$X(t) = \sum_{K=-\infty}^{+\infty} a_K \, p(t - KT), \quad p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = KT \end{cases}$$

Therefore the best point to look for a_K is at time KT.
and the sampling rate should be $\frac{1}{T}$

b) $Y(t) = X(t) + N(t) = \sum a_K \, p(t - KT) + N(t)$

$$Y(KT) = a_K + N(KT), \text{ and } a_K = \begin{cases} 1 & \text{if data is 1} \\ 0 & \text{if data is 0} \end{cases}$$

So, as ~~it~~ can be seen, $N(KT)$ can ~~drift~~ change the sampled value from the actual value of a_K.

Therefore, we define a threshold as V, $(0 < V < 1)$ so that if $Y(KT) > V$, we assume the transmitted bit was 1, and we assome it was 0, otherwise.

P10.92)

c)
Probability of error:

$$P_{e_1} = P\{\text{we detect } 0 \text{ while bit was } 1\} = P\{Y_K < V \mid a_K=1\} = P\{N_K < V - a_K \mid a_K=1\}$$

$$= P\{N_K < V-1 \mid a_K=1\} = \int_{-\infty}^{V-1} P_N(y|a_{K=1})\,dy$$

in which P_N is pdf of noise $(N(t))$

$$P_{e0} = P\{\text{we detect } 1 \text{ while signal was } 0\} = P\{Y_K > V \mid a_K=0\} = \int_{V}^{\infty} P_N(y|a_{K=0})\,dy$$

Total error: $P_e = P_{e_1} \times P_{a_K=1} + P_{e0} \times P_{a_{K=0}}$

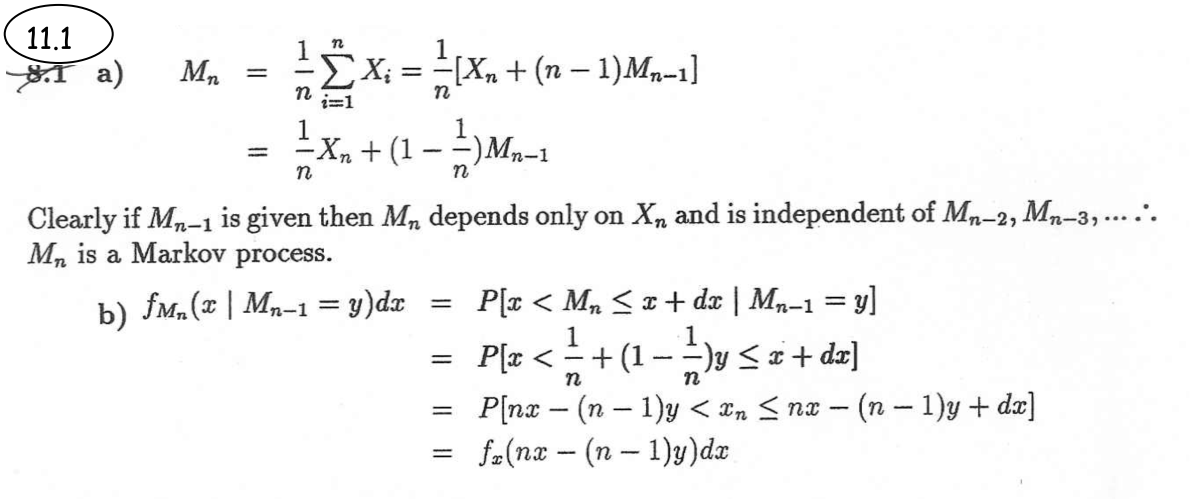
if N is independent of X and $P_{a_K=1} = P_{a_K=0} = \frac{1}{2}$

The best Threshold would be $1/2$ and:

$$P_e = \frac{1}{2}\int_{-\infty}^{-1/2} P_N(y)\,dy + \frac{1}{2}\int_{\frac{1}{2}}^{\infty} P_N(y)\,dy = \frac{1}{2}\left[1 - \int_{-1/2}^{1/2} P_N(y)\,dy\right]$$

Chapter 11: Markov Chains

11.1 Markov Processes



11.1 ~~8.1~~ **a)**

$$\begin{aligned} M_n &= \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}[X_n + (n-1)M_{n-1}] \\ &= \frac{1}{n}X_n + (1-\frac{1}{n})M_{n-1} \end{aligned}$$

Clearly if M_{n-1} is given then M_n depends only on X_n and is independent of $M_{n-2}, M_{n-3}, \ldots \therefore$ M_n is a Markov process.

b)

$$\begin{aligned} f_{M_n}(x \mid M_{n-1} = y)dx &= P[x < M_n \leq x + dx \mid M_{n-1} = y] \\ &= P[x < \frac{1}{n} + (1-\frac{1}{n})y \leq x + dx] \\ &= P[nx - (n-1)y < x_n \leq nx - (n-1)y + dx] \\ &= f_x(nx - (n-1)y)dx \end{aligned}$$

11.2

~~8.2~~ **a)** The number X_n of black balls in the urn completely specifies the probability of outcomes of a trial; therefore X_n is independent of its past values and X_n is a Markov process.

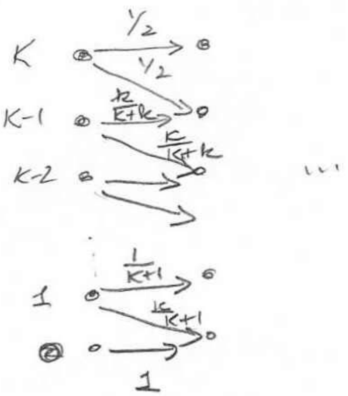
$$\begin{aligned}
P[X_n = 4|X_{n-1} = 5] &= \frac{5}{10} = 1 - P[X_n = 5|X_{n-1} = 5] \\
P[X_n = 3|X_{n-1} = 4] &= \frac{4}{9} = 1 - P[X_n = 4|X_{n-1} = 4] \\
P[X_n = 2|X_{n-1} = 3] &= \frac{3}{8} = 1 - P[X_n = 3|X_{n-1} = 3] \\
P[X_n = 1|X_{n-1} = 2] &= \frac{2}{7} = 1 - P[X_n = 2|X_{n-1} = 2] \\
P[X_n = 0|X_{n-1} = 1] &= \frac{1}{6} = 1 - P[X_n = 1|X_{n-1} = 1] \\
P[X_n = 0|X_{n-1} = 0] &= 1
\end{aligned}$$

All transition probability are independent of time.

11.2 ⓒ Let X_n = # black balls in urn

$$P[X_n = k-1 \mid X_n = k] = 1 - \frac{k}{K+k}$$

$$P[X_n = k \mid X_n = k] = \frac{k}{K+k}$$

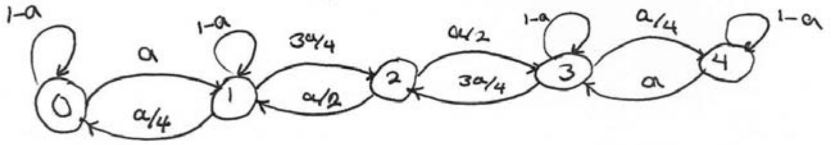


$$\begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
\frac{K}{K+1} & \frac{1}{K+1} & 0 & \cdots & 0 \\
0 & \frac{K}{K+2} & \frac{2}{K+2} & 0 \cdots & 0 \\
 & & \ddots & \ddots & 0 \\
 & 0 & & & \\
 & & & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}$$

11.3 a) X_n is Markov since $x_n \in \underbrace{\{0,1,2,3,4\}}_{X}$ and for $\forall x_n \in X$

$$x_{n+1} = x_n \quad \text{or} \quad x_{n+1} = x_n + 1 \quad \text{or} \quad x_{n+1} = x_n - 1$$

$$\Rightarrow P(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots) = P(X_{n+1} = x_{n+1} \mid X_n = x_n)$$



$$P = \begin{pmatrix} 1-a & a & 0 & 0 & 0 \\ a/4 & 1-a & \frac{3a}{4} & 0 & 0 \\ 0 & a/2 & 1-a & a/2 & 0 \\ 0 & 0 & \frac{3a}{4} & 1-a & a/4 \\ 0 & 0 & 0 & a & 1-a \end{pmatrix}$$

(b) Transition probabilities are independent from n as P shows.

(c) If $a=1$ then in all events a change in the color of the selected ball takes place and there is no self-loop. The transition matrix would be:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(d) $P(X_{n+1} = m+1 \mid X_n = m) = \underbrace{P(m+1 \mid m)}_{\text{simplified notation}} = \frac{m}{K} a \qquad \forall\ 1 < m < K, \ m \in N$

$P(X_{n+1} = m-1 \mid X_n = m) = P(m-1 \mid m) = \frac{K-m}{m} a \qquad \forall\ 1 < m < K, \ m \in N$

$P(m \mid m) = 1 - a$

obviously X_n is Markov.

$P(m+1 \mid m) = a \qquad m = 1$

$P(m-1 \mid m) = a \qquad m = K$

11.3 d (cont.)

$$P = \begin{pmatrix} 1-a & 0 & 0 & \cdots & & 0 \\ a/K & 1-a & \frac{K-1}{K}a & \cdots & & 0 \\ 0 & 2a/K & 1-a & \frac{K-2}{K}a & \cdots & 0 \\ \vdots & & \frac{3a}{K} & \ddots & \ddots & \ddots \\ \vdots & & & \ddots & a & 1-a \end{pmatrix}$$

independent of n

in case of $a=1$, there is always a change in colors

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{K} & 0 & \frac{K-1}{K} & 0 & \cdots & 0 \\ 0 & \frac{2}{K} & 0 & \frac{K-2}{K} & \cdots & 0 \\ \vdots & & & & & \\ \vdots & \cdots & & & 1 & 0 \end{pmatrix}$$

11.4 (a) X_n is Markov since
$$\begin{cases} x_{n+1} = x_n \pm 1 & \text{for } x_n \in \{1,2,3\} \\ x_{n+1} = x_n + 1 & \text{if } x_n = 0 \\ x_{n+1} = x_n - 1 & \text{if } x_n = 4 \end{cases}$$

we show:

$P(X_{n+1} = a \mid X_n = b) = P(a|b)$

$P(1|0) = 1$

$P(0|1) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

$P(1|1) = \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} = \frac{6}{16}$

$P(1|2) = \frac{4}{16}$

$P(2|2) = \frac{2}{4} \times \frac{2}{4} + \frac{2}{4} \times \frac{2}{4} = \frac{8}{16}$

$P(3|2) = \frac{2}{4} \times \frac{2}{4} = \frac{4}{16}$

$P(3|3) = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{6}{16}$

$P(4|3) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$

$P(2|3) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{6}{16} & \frac{9}{16} & 0 & 0 \\ 0 & \frac{4}{16} & \frac{8}{16} & \frac{4}{16} & 0 \\ 0 & 0 & \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

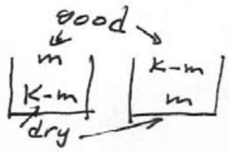
11.4 (b) Transition probabilies do not depend on n.

(C) $P(X_{n+1} = a \mid X_n = b) = P(a|b)$ ← notation

$P(m|m) = \frac{m}{K} \times \frac{K-m}{K} + \frac{K-m}{K} \times \frac{m}{K} = \frac{2m(K-m)}{K^2}$

$P(m-1|m) = \frac{m}{K} \times \frac{m}{K} = \frac{m^2}{K^2}$

$P(m+1|m) = 1 - \frac{m^2}{K^2} - \frac{2m(K-m)}{K^2} = \frac{(K-m)^2}{K^2}$



$\forall\ 1 \leq m < K$

$P(1|0) = 1$

$P(K-1|K) = 1$

11.5

~~8.3~~ If $X(t)$ has independent increments, then

$$f_{X(t_{k+1})X(t_k)\dots X(t_1)}(x_{k+1,\dots,x_1}) = f_{X(t_1)}(x_1)f_{X(t_2)-X(t_1)}(x_2-x_1)\dots f_{X(t_{k+1})-X(t_k)}(x_{k+1}-x_k)$$

$$\begin{aligned}
\Rightarrow f_{X(t_{k+1})}(x_k|x(t_k) &= x_k,\dots,x(t_1)=x_1) \\
&= \frac{f_{X(t_{k+1})\dots x(t_1)}(x_{k+1},\dots,x_1)}{f_{X(t_k)\dots x(t_1)}(x_k,\dots,x_1)} \\
&= \frac{f_{X(t_1)}(x_1)\dots f_{X(t_k)-x(t_{k-1})}(x_k-x_{k-1})f_{X(t_{k+1})-x(t_k)}(x_{k+1}-x_k)}{f_{X(t_1)}(x_1)\dots f_{X(t_k)-x(t_{k-1})}(x_k-x_{k-1})} \\
&= f_{X(t_{k+1})-x(t_k)}(x_{k+1}-x_k) \\
&= f_{X(t_{k+1})}(x_{k+1}|x(t_k)=x_k)
\end{aligned}$$

$\Rightarrow X(t)$ is a Markov process.

However if $X(t)$ is Markov, then $X(t)$ need not have independent increments. To see why consider the Markov chain in Problem ~~8~~.2 11.2. Since X_n is Markov we have:

$$P[X_i=i,X_2=j]=P[X_1=i]P[X_2=j|X_1=i]$$

If X_n had independent increments, we would have

$$P[X_1=i,X_2=j]=P[X_1=i]P[X_2-X_1=j-i]$$

so we require that

$$P[X_2=j|X_1=i]=P[X_2-X_1=j-i]\ .$$

Let $j_1=5$, $i_1=4$ then

$$P[X_2=4|X_1=5]=\frac{1}{2}\overset{?}{=}P[X_2-X_1=-1]$$

but if $j_2=4$, $i_2=3$ then

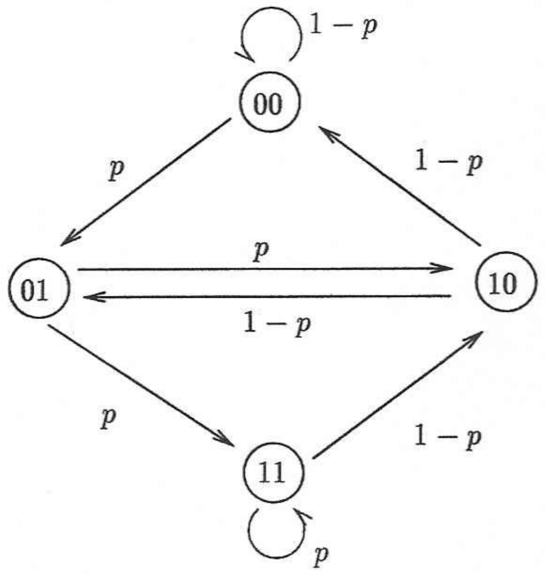
$$P[X_2=3|X_1=4]=\frac{4}{9}\overset{?}{=}P[X_2-X_1=-1]\ .$$

Thus X_n cannot have independent increments.

11.6 ~~8.5~~ $\underline{Z}_n = (x_n, x_{n-1})$

$$\begin{aligned}
P[\underline{Z}_{n+1} &= (x_{n+1}, x_n) | \underbrace{\underline{Z}_n = (x_n, x_{n-1}), \underline{Z}_{n-1} = (x_{n-1}, x_{n-2}) \ldots]}_{\text{all past vectors}} \\
&= P[\underline{Z}_{n+1} = (x_{n+1}, x_n) | \quad \underbrace{X_n = x_n, X_{n-1} \ldots]}_{\text{all past Bernoulli trials}} \\
&= P[\overbrace{X_{n+1} = x_{n+1}}^{\text{next trial}}] \\
&= P[\underline{Z}_{n+1} = (x_{n+1}, x_n) | \underline{Z}_n = (x_n, x_{n-1})]
\end{aligned}$$

$\therefore \underline{Z}_n$ is a Markov process.



where $p = P[x = 1]$

11.7 $Y_n = rY_{n-1} + X_n \qquad Y_0 = 0 \qquad Y_n - rY_{n-1} = X_n$

$$\begin{aligned} f_{Y_n}(y|y_{n-1} = y_1 \;\; Y_{n-2} = y_2 \ldots) &= f_{X_n}(y - ry_1) \\ &= f_{Y_n}(y|Y_{n-1} = y_1) \end{aligned}$$

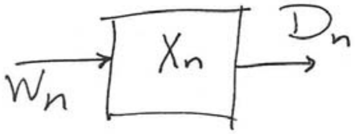
$\Rightarrow Y_n$ is a Markov process.

ⓑ

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2}$$

$$\begin{aligned} f_{Y_{n+1}}(y_{n+1}|Y_n = y_n)\,dy &= P[y_{n+1} < Y_{n+1} \le y_{n+1} + dy \,|\, Y_n = y_n] \\ &= P[y_{n+1} < ry_n + X_{n+1} \le y_{n+1} + dy] \\ &= P[y_{n+1} - ry_n < X_{n+1} \le y_{n+1} - ry_n + dy] \\ &= f_{X_{n+1}}(y_{n+1} - ry_n)\,dy \end{aligned}$$

11.8



(a)
$$\begin{aligned} X_2 &= X_1 + (W_1 - D_1) \\ X_3 &= X_2 + (W_2 - D_2) \\ &\vdots \\ X_{n+1} &= X_n + (W_n - D_n) \end{aligned}$$

(b) According to the above recursive equations, X_{n+1} depends only on X_n plus $(W_n - D_n)$. The X_n is Markov only if W_n and D_n are independent random variables.

11.2 Discrete-Time Markov Chains

11.9

~~8.7~~ Assume X_n is discrete, then

$$\begin{aligned} P[X_n = x | X_{n-1} = x_1, X_{n-2} = x_2, \ldots] &= P[X_n = x] \quad \text{since } X_n \text{ is iid} \\ &= P[X_n = x | X_{n-1} = x_1] \end{aligned}$$

$\therefore X_n$ is a Markov process with transition probabilities:

$$P[X_n = x | X_{n-1} = x_1] = P[X_n = x] \quad \text{all } x_1$$

11.10 (a)

$$\begin{aligned} P(O=0) &= P(O=0|I=0)\,P(I=0) + P(O=0|I=1)\,P(I=1) \\ &= (1-\varepsilon)a + \varepsilon(1-a) = a + \varepsilon - 2a\varepsilon \end{aligned}$$

$$\begin{aligned} P(O=1) &= P(O=1|I=0)|P(I=0) + P(O=1|I=1)\,P(I=1) \\ &= \varepsilon a + (1-\varepsilon)(1-a) = 1 - a - \varepsilon + 2a\varepsilon \end{aligned}$$

(b) $P = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}$ in a system of K independent symmetric channels:

$$\begin{aligned} &P(O_{n+1} = o_{n+1} | O_n = o_n, \ldots) \\ &= P(O_{n+1} = o_{n+1} | O_n = o_n) \quad \text{Markov} \end{aligned}$$

since if we know O_n, the O_{n+1} could be found (channels are independent)

(C) $P_K = (P)^K = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{pmatrix}^K \Rightarrow \begin{array}{l} P_K(0) = (1-\varepsilon)P_{K-1}(0) + \varepsilon P_{K-1}(1) \\ P_K(1) = \varepsilon P_{K-1}(0) + (1-\varepsilon)P_{K-1}(1) \end{array}$

$(P_K(0), P_K(1))^T$ is the output probability of channel K

To find a close form for P^K we can use diagonalization method

$$|\lambda I - P| = 0 \Rightarrow (\lambda+\varepsilon-1)^2 - \varepsilon^2 = 0 \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 1-2\varepsilon \end{cases}$$

$\lambda = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda = 1-2\varepsilon \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $U = (v_1 \vdots v_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$P = UDU^{-1}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 1-2\varepsilon \end{pmatrix}$

$\Longrightarrow$

$$P = UDU^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-2\varepsilon \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$P^K = (UDU^{-1})^K = UD^KU^{-1}$$
$$= -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (1-2\varepsilon)^K \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow P^K = -\frac{1}{2}\begin{pmatrix} -1-(1-2\varepsilon)^n & -1+(1-2\varepsilon)^n \\ -1+(1-2\varepsilon)^n & -1-(1-2\varepsilon)^n \end{pmatrix}$$

$$P^K = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(1-2\varepsilon)^n & \frac{1}{2}+\frac{1}{2}(1-2\varepsilon)^n \\ \frac{1}{2}-\frac{1}{2}(1-2\varepsilon)^n & \frac{1}{2}+(1-2\varepsilon)^n \end{pmatrix}$$

If $K \to \infty$ for any $|1-2\varepsilon| < 1$ $(1-2\varepsilon)^K \to 0$

$$\Rightarrow P^\infty \to \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

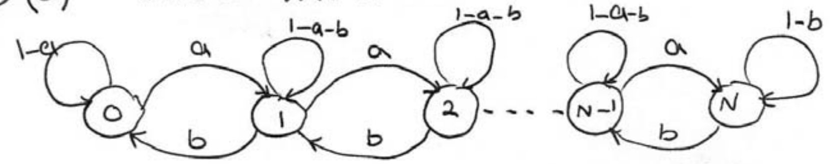
and the steady state probability vector is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

(d) When $K \to \infty$ $\begin{cases} P_K(0) = P_{K-1}(0) \\ P_K(1) = P_{K-1}(1) \end{cases} \Rightarrow \begin{cases} P_K(0) = (1-\varepsilon)P_K(0) + \varepsilon P_K(1) \\ P_K(1) = \varepsilon P_K(0) + (1-\varepsilon)P_K(1) \end{cases}$

$\left.\begin{array}{l} \Rightarrow P_K(0) = P_K(1) \\ \text{we know } P_K(0) + P_K(1) = 1 \end{array}\right\} \Rightarrow P_K(0) = P_K(1) = 0.5$ $\underset{K\to\infty}{\text{output}} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

11.11 Version 1: Arrival and Departure are mutually exclusive.

(a) There are $N+1$ states



Clearly every state x_{n+1} be found either from x_n or $x_n \pm 1 \Rightarrow$ Markov

(b)

$$\underset{(N+1)\times(N+1)}{P} = \begin{pmatrix} 1-a & a & 0 & 0 & \cdots & 0 \\ b & 1-a-b & a & 0 & \cdots & 0 \\ 0 & b & 1-a-b & a & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & b & 1-a-b & a \\ 0 & 0 & \cdots & 0 & 1-b & b \end{pmatrix}$$

(c) $xP = x \Rightarrow (1-a)x_0 + bx_1 = x_0 \Rightarrow x_1 = \frac{a}{b}x_0$

$ax_0 + (1-a-b)x_1 + bx_2 = x_1 \Rightarrow (a+b)\frac{a}{b}x_0 - ax_0 = bx_2 \Rightarrow x_2 = \frac{a^2}{b^2}x_0$

We claim $x_k = \frac{a^k}{b^k}x_0$

for $k=1,2$ it's proved now if it is true for $k-1, k$ we show it's valid for $k+1$ as well: $ax_{k-1} + (1-a-b)x_k + bx_{k+1} = x_k$

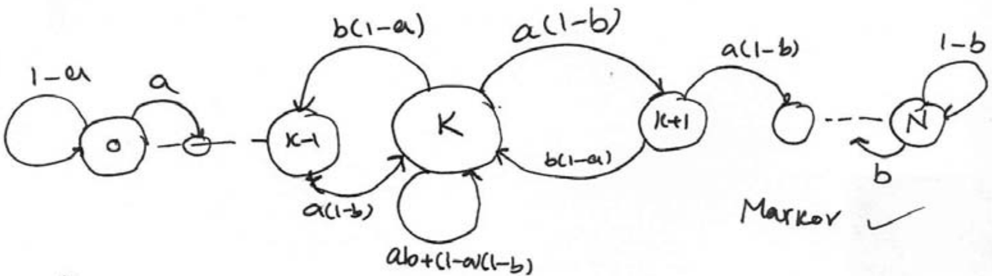
$\Rightarrow a \times \frac{a^{k-1}}{b^{k-1}}x_0 + bx_{k+1} = (a+b)\frac{a^k}{b^k}x_0 \Rightarrow bx_{k+1} = \frac{a^k}{b^{k-1}}\left(1+\frac{a}{b}-1\right)x_0$

$\Rightarrow x_{k+1} = \frac{a^{k+1}}{b^{k+1}}x_0$

Finally $x_0 + x_1 + \cdots + x_N = 1 \Rightarrow x_0\left(1+\frac{a}{b}+\frac{a^2}{b^2}+\cdots+\frac{a^N}{b^N}\right) = 1$

$$\Rightarrow x_0 = \frac{1}{\sum_{i=0}^{N}\left(\frac{a}{b}\right)^i} \Rightarrow \boxed{\begin{array}{l} x = (x_0, x_1, \ldots, x_N) \\ x_i = \dfrac{\left(\frac{a}{b}\right)^i}{\sum_{j=0}^{N}\left(\frac{a}{b}\right)^j} \end{array}}$$

Version 2: Arrivals and departures occur simultaneously



Markov ✓

$$P_2 \begin{pmatrix} 1-a & a & 0 & 0 & 0 & \cdots & 0 \\ b(1-a) & ab+(1-a)(1-b) & a(1-b) & 0 & \cdots & \cdots & 0 \\ 0 & b(1-a) & ab+(1-a)(1-b) & a(1-b) & \cdots & \cdots & \cdots \\ \vdots & & & & & & \\ \cdots & \cdots & 0 & 0 & b(1-a) & ab+(1-a)(1-b) & a(1-b) \\ 0 & \cdots & & 0 & 0 & b & 1-b \end{pmatrix}$$

$$\pi = \pi P \Rightarrow \left\{ \begin{aligned} \pi_0 &= (1-a)\pi_0 + b(1-a)\pi_1 \\ \pi_1 &= a\pi_0 + [ab+(1-a)(1-b)]\pi_1 + b(1-a)\pi_2 \\ &\vdots \\ \pi_i &= a(1-b)\pi_{i-1} + [ab+(1-a)(1-b)]\pi_i + b(1-a)\pi_{i+1} \\ \pi_{N-1} &= a(1-b)\pi_{N-2} + [ab+(1-a)(1-b)]\pi_{N-1} + b\pi_N \\ \pi_N &= a(1-b)\pi_{N-1} + (1-b)\pi_N \end{aligned} \right.$$

$2 < i < N$

$$a\pi_0 = b(1-a)\pi_1 \qquad \pi_1 = \frac{a}{b(1-a)}\pi_0 = \frac{a}{b(1-a)} \propto \frac{1}{A}$$

$$\pi_2 = \frac{a(1-b)}{b(1-a)}\pi_1 \Rightarrow \pi_2 = \frac{ab(1-b)}{b^2(1-a)^2}\pi_0 = \frac{ab(1-b)}{b^2(1-a)^2} \wedge \frac{1}{A}$$

$$\vdots$$

$$\sum_{i=0}^{N} \pi_i = 1 \qquad \Big[\underbrace{1 + \frac{a}{b(1-a)} + \frac{ab(1-b)}{b^2(1-a)^2} + \cdots}_{A}\Big]\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{A}$$

11.12

a) $$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{5}{10} \end{bmatrix}$$

b) $$\mathbb{P}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{36} & \frac{25}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{65}{144} & \frac{25}{49} & 0 & 0 & 0 \\ 0 & \frac{3}{28} & \frac{225}{448} & \frac{25}{64} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{95}{192} & \frac{25}{81} & 0 \\ 0 & 0 & 0 & \frac{2}{9} & \frac{19}{36} & \frac{1}{4} \end{bmatrix}$$

$$\begin{aligned} p_{54}(2) &= \frac{19}{36} \quad \text{from } \mathbb{P}^2 \\ p_{54}(2) &= p_{55}^{\text{no change}}(1)p_{54}^{\text{change}}(1) + p_{54}^{\text{change}}(1)p_{44}^{\text{no change}}(1) \\ &= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{5}{9} = \frac{19}{36} \quad \checkmark \end{aligned}$$

c) As $n \to \infty$ eventually all black balls are removed. Thus

$$\mathbb{P}^n \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

11.13 (a) $a = \frac{1}{10} \Rightarrow P = \begin{pmatrix} 0.9 & 0.1 & 0 & 0 & 0 \\ 0.025 & 0.9 & 0.075 & 0 & 0 \\ 0 & 0.05 & 0.9 & 0.05 & 0 \\ 0 & 0 & 0.075 & 0.9 & 0.025 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{pmatrix}$

(b)

11.13.b

P =

```
    0.9000    0.1000         0         0         0
    0.0250    0.9000    0.0750         0         0
         0    0.0500    0.9000    0.0500         0
         0         0    0.0750    0.9000    0.0250
         0         0         0    0.1000    0.9000
```

P^2 =

```
    0.8125    0.1800    0.0075         0         0
    0.0450    0.8163    0.1350    0.0037         0
    0.0013    0.0900    0.8175    0.0900    0.0013
         0    0.0037    0.1350    0.8163    0.0450
         0         0    0.0075    0.1800    0.8125
```

P^4 =

```
    0.6683    0.2939    0.0365    0.0014    0.0000
    0.0735    0.6865    0.2214    0.0183    0.0003
    0.0061    0.1476    0.6926    0.1476    0.0061
    0.0003    0.0183    0.2214    0.6865    0.0735
    0.0000    0.0014    0.0365    0.2939    0.6683
```

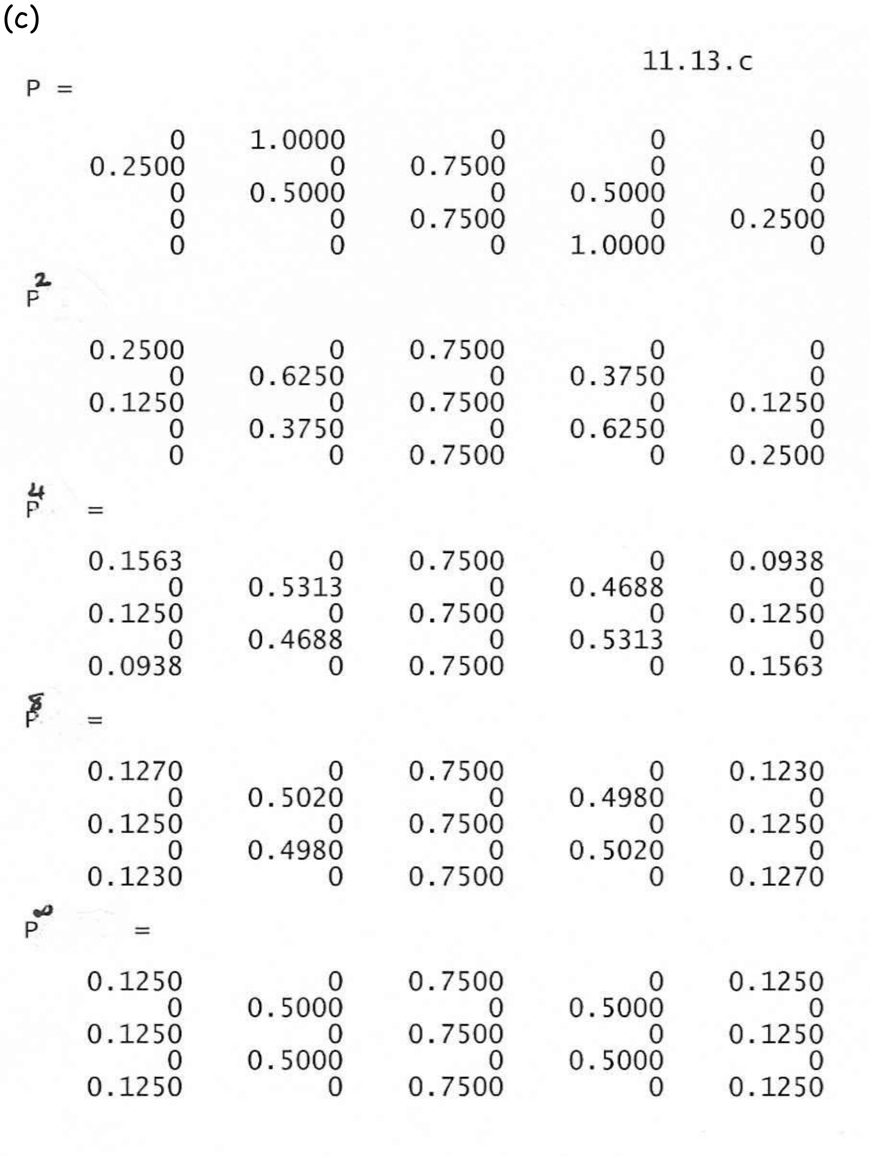
P^8 =

```
    0.4684    0.4035    0.1151    0.0126    0.0004
    0.1009    0.5259    0.3121    0.0580    0.0031
    0.0192    0.2081    0.5455    0.2081    0.0192
    0.0031    0.0580    0.3121    0.5259    0.1009
    0.0004    0.0126    0.1151    0.4035    0.4684
```

P^∞ =

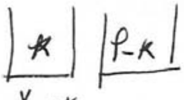
```
    0.0625    0.2500    0.3750    0.2500    0.0625
    0.0625    0.2500    0.3750    0.2500    0.0625
    0.0625    0.2500    0.3750    0.2500    0.0625
    0.0625    0.2500    0.3750    0.2500    0.0625
    0.0625    0.2500    0.3750    0.2500    0.0625
```

(c)

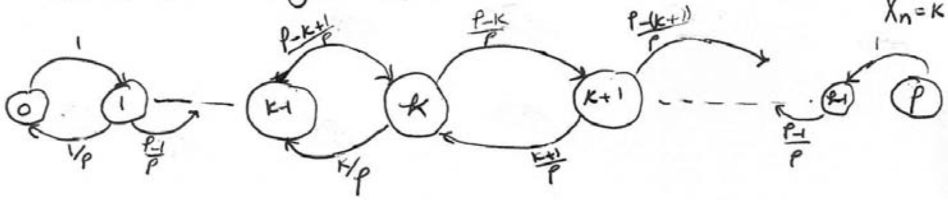


```
                                             11.13.c
P =

         0    1.0000         0         0         0
    0.2500         0    0.7500         0         0
         0    0.5000         0    0.5000         0
         0         0    0.7500         0    0.2500
         0         0         0    1.0000         0
P^2

    0.2500         0    0.7500         0         0
         0    0.6250         0    0.3750         0
    0.1250         0    0.7500         0    0.1250
         0    0.3750         0    0.6250         0
         0         0    0.7500         0    0.2500
P^4  =

    0.1563         0    0.7500         0    0.0938
         0    0.5313         0    0.4688         0
    0.1250         0    0.7500         0    0.1250
         0    0.4688         0    0.5313         0
    0.0938         0    0.7500         0    0.1563
P^8  =

    0.1270         0    0.7500         0    0.1230
         0    0.5020         0    0.4980         0
    0.1250         0    0.7500         0    0.1250
         0    0.4980         0    0.5020         0
    0.1230         0    0.7500         0    0.1270
P^∞      =

    0.1250         0    0.7500         0    0.1250
         0    0.5000         0    0.5000         0
    0.1250         0    0.7500         0    0.1250
         0    0.5000         0    0.5000         0
    0.1250         0    0.7500         0    0.1250
```

11.14 (a) There are $\rho+1$ states (0 to ρ).



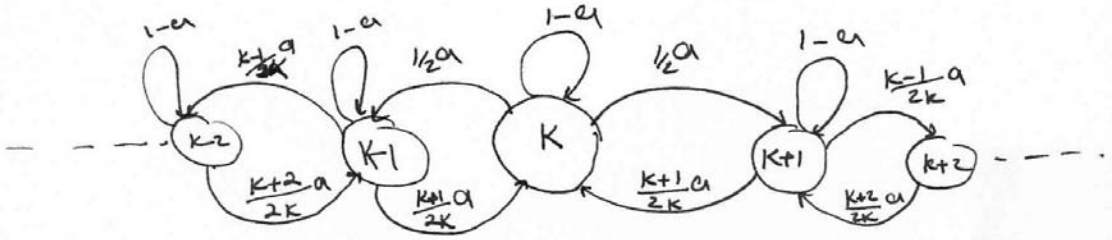
The state diagram is:



This model is the same as 11.3-d when $a=1$ and $K=\rho$

* Another way to interpret this problem is to assume that we have one container but to distinguish between particles of container 2 (from 1) we use color for particles, white for container 1 and black for container 2 particles. This converts the problem to 11.3.d with $a=1$ (because when a particle is chosen it will definitely converts its color)

(b) "Central force" $\rho=2K$



This diagram shows that when the system is not in the central state (K), the probability of moving toward state K is increasing as the system state goes to the left (right).

for example

$$\begin{cases} P_{K-1,K} = \frac{K+1}{2K}a, \quad P_{K-2,K-1} = \frac{K+2}{2K}a \cdots \\ P_{K,K+1} = \frac{1}{2}a, \quad P_{K+1,K+2} = \frac{K-1}{2K}a, \quad P_{K+2,K+3} = \frac{K-2}{2K}a \cdots \end{cases}$$

This shows that the system exhibits a force towards central state.

* to unify notation with Problem 11.3.d we set $\rho = K$

(c) $x = (x_0, x_1, \dots, x_K)$ is pmf (stationary)

$$xP = x \Rightarrow \frac{1}{K}x_1 = x_0 \Rightarrow x_1 = Kx_0 = \binom{K}{1}x_0$$

$$x_0 + \frac{2}{K}x_2 = x_1 \Rightarrow x_2 = \frac{K(K-1)}{2}x_0 = \binom{K}{2}x_0$$

$$\frac{K-1}{K}x_1 + \frac{3}{K}x_3 = x_2$$

$$\vdots$$

$$\frac{K-i}{K}x_i + \frac{i+2}{K}x_{i+2} = x_{i+1}$$

$$x_3 = K\left(\frac{K(K-1)}{2} - \frac{K-1}{K}\times K\right)x_0 \Rightarrow x_3 = \frac{K(K-1)(K-2)}{3!}x_0 = \binom{K}{3}x_0$$

we claim $x_i = \binom{K}{i}x_0 = \frac{K!}{i!(K-i)!}x_0$

by induction we prove our claim. for $i = 1, 2, 3$ it is already shown

we assume that it is valid for i & $(i+1)$ then:

$$\frac{K-i}{K}x_i + \frac{i+2}{K}x_{i+2} = x_{i+1}$$

$$\frac{i+2}{K}x_{i+2} = \frac{K!}{(i+1)!(K-i-1)!}x_0 - \frac{K-i}{K}\times\frac{K!}{i!(K-i)!}x_0$$

$$\frac{i+2}{K}x_{i+2} = \frac{(K-1)!}{i!(K-i-1)!}\left(\frac{K}{i+1} - 1\right)x_0 = \frac{(K-1)!}{(i+1)!(K-i-2)!}x_0$$

$$\Rightarrow x_{i+2} = \frac{K!}{(i+2)!(K-i-2)!}x_0 = \binom{K}{i+2}x_0$$

on the other hand $\sum_{i=0}^{K} x_i = 1 \Rightarrow \sum_{i=0}^{K}\binom{K}{i}x_0 = 1$

$$\Rightarrow x_0 = \frac{1}{\sum_{i=0}^{K}\binom{K}{i}} = \frac{1}{2^K} \qquad (K = \rho)$$

$$\Rightarrow x = \frac{1}{2^\rho}\left(\binom{\rho}{0}, \binom{\rho}{1}, \binom{\rho}{2}, \dots, \binom{\rho}{\rho-1}, \binom{\rho}{\rho}\right)$$

11.15 (a)

$$P=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{16} & \frac{6}{16} & \frac{9}{16} & 0 & 0 \\ 0 & \frac{4}{16} & \frac{8}{16} & \frac{4}{16} & 0 \\ 0 & 0 & \frac{4}{16} & \frac{6}{16} & \frac{1}{16} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

11.15 – b, c

```
P =

         0    1.0000         0         0         0
    0.0625    0.3750    0.5625         0         0
         0    0.2500    0.5000    0.2500         0
         0         0    0.5625    0.3750    0.0625
         0         0         0    1.0000         0

P^2 =

    0.0625    0.3750    0.5625         0         0
    0.0234    0.3438    0.4922    0.1406         0
    0.0156    0.2188    0.5313    0.2188    0.0156
         0    0.1406    0.4922    0.3438    0.0234
         0         0    0.5625    0.3750    0.0625

P^4 =

    0.0215    0.2754    0.5186    0.1758    0.0088
    0.0172    0.2544    0.5131    0.2043    0.0110
    0.0144    0.2280    0.5151    0.2280    0.0144
    0.0110    0.2043    0.5131    0.2544    0.0172
    0.0088    0.1758    0.5186    0.2754    0.0215

P^8 =

    0.0147    0.2317    0.5143    0.2254    0.0139
    0.0145    0.2301    0.5143    0.2270    0.0141
    0.0143    0.2286    0.5143    0.2286    0.0143
    0.0141    0.2270    0.5143    0.2301    0.0145
    0.0139    0.2254    0.5143    0.2317    0.0147

P^∞ =

    0.0143    0.2286    0.5143    0.2286    0.0143
    0.0143    0.2286    0.5143    0.2286    0.0143
    0.0143    0.2286    0.5143    0.2286    0.0143
    0.0143    0.2286    0.5143    0.2286    0.0143
    0.0143    0.2286    0.5143    0.2286    0.0143
```

11.16 (a) It is easy to see that this problem is equivalent to 11.4.c in fact

$$P(X_{n+1}=m / x_n=m) = \frac{m}{\rho} \times \frac{\rho-m}{\rho} + \frac{\rho-m}{\rho} \times \frac{m}{\rho} = \frac{2m(\rho-m)}{\rho^2}$$

$$P(X_{n+1}=m-1 \mid x_n=m) = \frac{m}{\rho} \times \frac{m}{\rho} = \frac{m^2}{\rho^2}$$

$$P(X_{n+1}=m+1 \mid x_n=m) = 1 - \frac{m^2}{\rho^2} - \frac{2m(\rho-m)}{\rho^2} = \frac{(\rho-m)^2}{\rho^2}$$

The same as in 11.4.c

(b)

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & & & 0 \\ \frac{1}{\rho^2} & \frac{2(\rho-1)}{\rho^2} & \frac{(\rho-1)^2}{\rho^2} & \cdots & & & 0 \\ 0 & \frac{4}{\rho^2} & \frac{4(\rho-2)}{\rho^2} & \frac{(\rho-2)^2}{\rho^2} & \cdots & & 0 \\ \vdots & & \ddots & & \ddots & & \\ \vdots & & & & \frac{(\rho-1)^2}{\rho^2} & \frac{2(\rho-1)}{\rho^2} & \frac{1}{\rho^2} \\ 0 & \cdots & & & 0 & 1 & 0 \end{pmatrix}$$

$x = (x_0, x_1, \ldots, x_\rho)$

$$xP = x \Rightarrow \frac{x_1}{\rho^2} = x_0 \Rightarrow x_1 = \rho^2 x_0 = \binom{\rho}{1}^2 x_0$$

$$x_0 + \frac{2(\rho-1)}{\rho^2} x_1 + \frac{4}{\rho^2} x_2 = x_1 \Rightarrow x_2 = \frac{1}{4}(\rho-1)^2 x_1 = \frac{1}{4}\rho^2(\rho-1)^2 x_0 = \binom{\rho}{2}^2 x_0$$

$$\frac{(\rho-1)^2}{\rho^2} x_1 + 4\frac{(\rho-2)}{\rho^2} x_2 + \frac{9}{\rho^2} x_3 = x_2 \Rightarrow x_3 = \frac{1}{9}(\rho-2)^2 x_2 = \frac{1}{9} \times \frac{1}{9} \rho^2(\rho-1)^2(\rho-2)^2 x_0$$

$$\Rightarrow x_3 = \binom{\rho}{3}^2 x_0$$

Although we can obtain the general form $x_m = \binom{\rho}{m}^2 x_0$ by induction, we take another approach. As the Markov chain here is reversible: $x_i P_{ij} = x_j P_{ji} \Rightarrow x_i P_{i,i+1} = x_{i+1} P_{i+1,i}$

$$\Rightarrow x_i \frac{(\rho-i)^2}{\rho^2} = x_{i+1} \frac{(i+1)^2}{\rho^2} \Rightarrow x_{i+1} = \frac{(\rho-i)^2}{(i+1)^2} x_i$$

$$x_{i+1} = \frac{(\rho-i)^2(\rho-i+1)^2(\rho-i+2)^2 \cdots \rho^2}{(i+1)^2 i^2 (i-1)^2 \cdots 1} x_0 = \left(\frac{(\rho-i)\times(\rho-i+1)\times\cdots\times\rho}{(i+1)!}\right)^2 x_0 = \left(\frac{\rho!}{(i+1)!(\rho-i-1)!}\right)^2 = \binom{\rho}{i+1}^2$$

11.16 (b) (Cont.)

$$x_{i+1} = \binom{\rho}{i+1}^2 x_0 \qquad \text{we know} \quad \sum_{i=0}^{\rho} x_i = 1$$

$$\Rightarrow x_0 = \frac{1}{\sum_{i=0}^{\rho} \binom{\rho}{i}^2} = \frac{1}{\binom{2\rho}{\rho}}$$

$$\Rightarrow x_i = \frac{\binom{\rho}{i}^2}{\binom{2\rho}{\rho}}$$

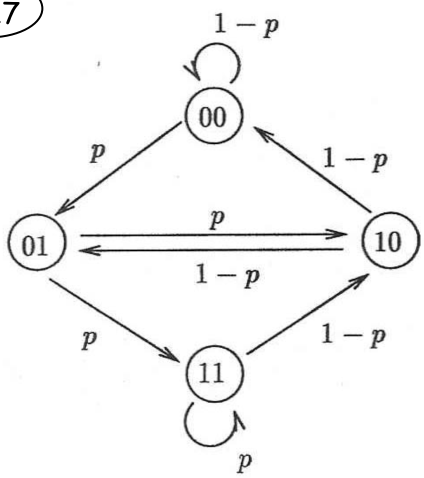
We have used the fact that $\sum_{i=0}^{\rho} \binom{\rho}{i}^2 = \binom{2\rho}{\rho}$

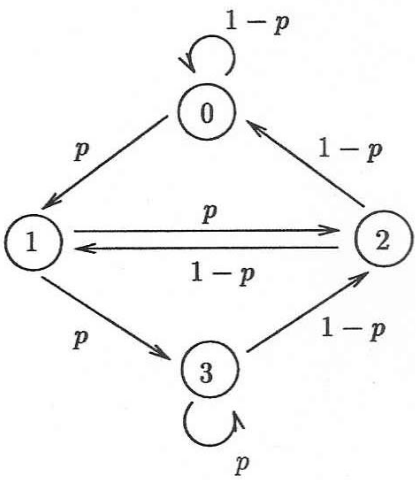
to see this, we can use $\binom{\rho}{i} = \binom{\rho}{\rho-i}$

$$\Rightarrow \sum_{i=0}^{\rho} \binom{\rho}{i}^2 = \sum_{i=0}^{\rho} \binom{\rho}{i}\binom{\rho}{\rho-i} = \binom{2\rho}{\rho}$$

which follows if we note that both sides show the number of subgroups of size ρ one can select from a set of n white balls and n black balls.

11.17





a) $\mathbb{P} = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 0 & q & p \end{bmatrix}$

b) $\mathbb{P}^2 = \begin{bmatrix} q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \end{bmatrix}$

(01) to (01) in 2 steps corresponds to $p_{11}(2) = pq$, i.e.

$$p(2|1)p(1|2) = qp \qquad \surd$$

c) Let A be a matrix with rows $\underline{r}_1, \underline{r}_2, \underline{r}_3, \underline{r}_4$, then

$$\mathbb{P}A = \begin{bmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 0 & q & p \end{bmatrix} \begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \underline{r}_3 \\ \underline{r}_4 \end{bmatrix} = \begin{bmatrix} q\underline{r}_1 + p\underline{r}_2 \\ q\underline{r}_3 + p\underline{r}_4 \\ q\underline{r}_1 + p\underline{r}_2 \\ q\underline{r}_3 + p\underline{r}_4 \end{bmatrix}$$

But all rows of $\mathbb{P}^2$ are equal, thus

$$\mathbb{P}\mathbb{P}^2 = \begin{bmatrix} q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \\ q^2 & qp & qp & p^2 \end{bmatrix} = \mathbb{P}^2$$

and if we assume $\mathbb{P}^{n-1} = \mathbb{P}^2$. Then $\mathbb{P}^n = \mathbb{P}\mathbb{P}^{n-1} = \mathbb{P}\mathbb{P}^2 = \mathbb{P}^2$

$$\therefore \mathbb{P}^n = \mathbb{P}^2 \quad n \geq 2$$

After two steps, the process is independent of the post.

d) Let $\underline{p}(0)$ be the initial state post, then

$$\underline{p}(0)\mathbb{P}^n = (q^2, qp, qp, p^2) \ \text{ for } n \geq 2$$

11.18 (a) If $X_n = i$ then X_{n+1} could be either $i+1$, or 0

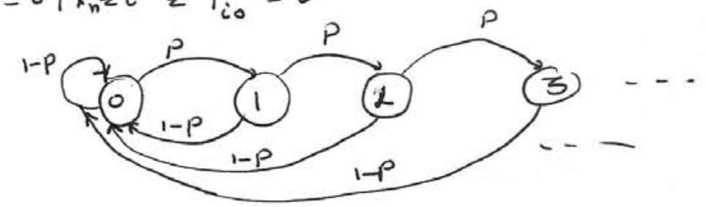
SFSSFFSSSS|S| $\rightarrow$ $X_n=4$ $X_{n+1}=5$

SFSSFFSSSS|F| $\rightarrow X_{n+1}=0$ $X_n=4$

This is independent of the events/states before $X_{n-1} \Rightarrow$ Markov

(b) $P(X_{n+1}=i+1 / X_n=i) = P_{i,i+1} = p \rightarrow$ probability of success

$P(X_{n+1}=0 \mid X_n=i = P_{i0} = 0$



(c)

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 1-p & 0 & 0 & p & & \\ 1-p & 0 & 0 & 0 & p & \\ \vdots & \vdots & \vdots & & & \ddots \end{bmatrix} \qquad \pi P = \pi$$

$$\Rightarrow \quad \pi_0 = (1-p)(\pi_0 + \pi_1 + \pi_2 + \cdots)$$

$\pi_1 = p\pi_0$

$\pi_2 = p\pi_1 = p^2\pi_0$

$\pi_3 = p\pi_2 = p^3\pi_0$

$\vdots$

$\pi_n = p^n\pi_0$

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow (1+p+p^2+\cdots)\pi_0 = 1$$

$$p<1 \Rightarrow \frac{1}{1-p} \times \pi_0 = 1 \Rightarrow \pi_0 = 1-p$$

$$\pi = \begin{pmatrix} 1-p & p(1-p) & p^2(1-p) & \cdots \end{pmatrix}$$

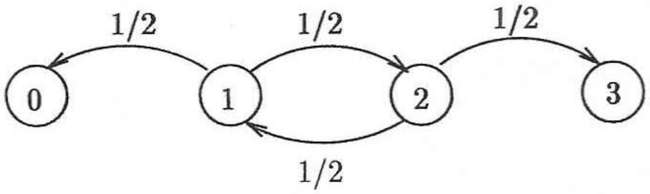
$$\pi_n = p^n(1-p)$$

11.19

~~8.11~~ $X_n \epsilon 0, 1, 2, 3$

a) $P[X_n = k | X_{n-1} = j, \ldots] = P[X_n = k | X_{n-1} = j]$
since $X_n = X_{n-1} \pm 1$ for $X_{n-1} \epsilon \{1, 2\}$
and $X_n = X_{n-1}$ if $X_{n-1} \epsilon \{0, 3\}$

b)



$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) for $n = 2k$, $i \epsilon \{1, 2\}$

$$\begin{aligned} p_{ii}(n) &= P[\overbrace{HT \;\; HT \;\; HT \;\; \ldots \;\; HT}^{2k}] = \left(\frac{1}{2}\right)^n \\ p_{10}(n) &= \sum_{j=0}^{k-1} P[j \; 1 \to 2 \text{ cycles and then go to } 0] \\ &= \sum_{j=0}^{k-1} \left(\frac{1}{2}\right)^{2j} \frac{1}{2} \\ &= \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) \\ p_{23}(n) &= p_{10}(n) \qquad \text{by symmetry.} \end{aligned}$$

d) $$\mathbb{P}(n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3}(1 - (\frac{1}{4})^k) & (\frac{1}{2})^n & 0 & \frac{1}{3}(1 - (\frac{1}{4})^k \\ \frac{1}{3}(1 - (\frac{1}{4})^k) & 0 & (\frac{1}{2})^n & \frac{2}{3}(1 - (\frac{1}{4})^k) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

e) $\mathbf{P}(n) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

f)

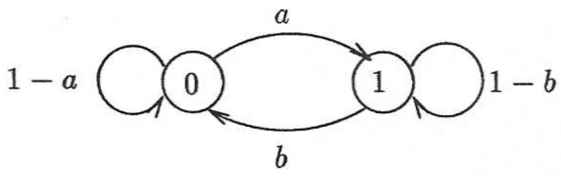
$$\begin{aligned} \underline{p}(n) &= [0100]\mathbb{P}(n) \\ &= \left[\frac{2}{3}\left(1-\left(\frac{1}{4}\right)^{k}\right), \left(\frac{1}{4}\right)^{k}, 0, \frac{1}{3}\left(1-\left(\frac{1}{4}\right)^{k}\right)\right] \\ &\to \left[\frac{2}{3}, 0, 0, \frac{1}{3}\right] \end{aligned}$$

$P[\text{player } A \text{ wins}] = \frac{1}{3}$.

11.20

~~8.12~~ $X_n \epsilon \{0,1\}$ where $0 =$ working, $1 =$ not working

a) $\mathbb{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$



b) To find the eigenvalues, consider

$$\begin{aligned} |\mathbb{P} - \lambda I| &= (1-b-\lambda)(1-a-\lambda) - ab = 0 \\ &\Rightarrow \lambda_1 = 1 \qquad \lambda_2 = 1-a-b \end{aligned}$$

Then the eigenvectors are $\underline{e}_1 = [1, \frac{b}{a}]$, $\underline{e}_2 = [1, -1]$, so

$$E = \begin{bmatrix} \underline{e}_1 \\ \underline{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 1 & -1 \end{bmatrix}$$

and

$$E^{-1} = \frac{1}{a+b} \begin{bmatrix} a & b \\ a & -a \end{bmatrix}$$

and thus

$$\begin{aligned} \mathbb{P}^n &= E^{-1} \begin{bmatrix} 1 & 0 \\ 0 & (1-a-b)^n \end{bmatrix} E \\ &= \frac{1}{a+b} \begin{bmatrix} a + b(1-a-b)^n & b - b(1-a-b)^n \\ a - a(1-a-b)^n & b + c(1-a-b)^n \end{bmatrix} \end{aligned}$$

c) $0 < a + b < 2$ since $0 < a < 1$ and $0 < b < 1$
$\Rightarrow -1 < 1 - a - b < 1 \Rightarrow (1-a-b)^n \to 0$

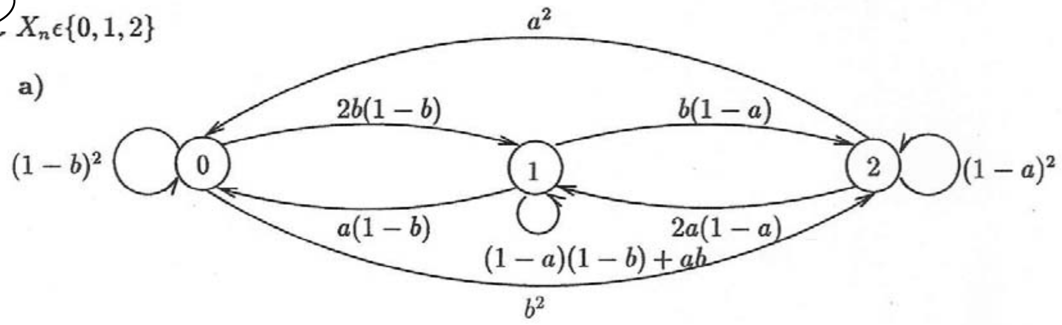
$$\therefore \mathbb{P}^n \to \begin{bmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{a}{a+b} & \frac{b}{a+b} \end{bmatrix}$$

and

$$\underline{p}(n) \to [\frac{a}{a+b}, \frac{b}{a+b}]$$

11.21 $X_n \epsilon \{0,1,2\}$

a)



$$\mathbf{P} = \begin{bmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & (1-a)(1-b)+ab & b(1-a) \\ a^2 & 2a(1-a) & (1-a)^2 \end{bmatrix}$$

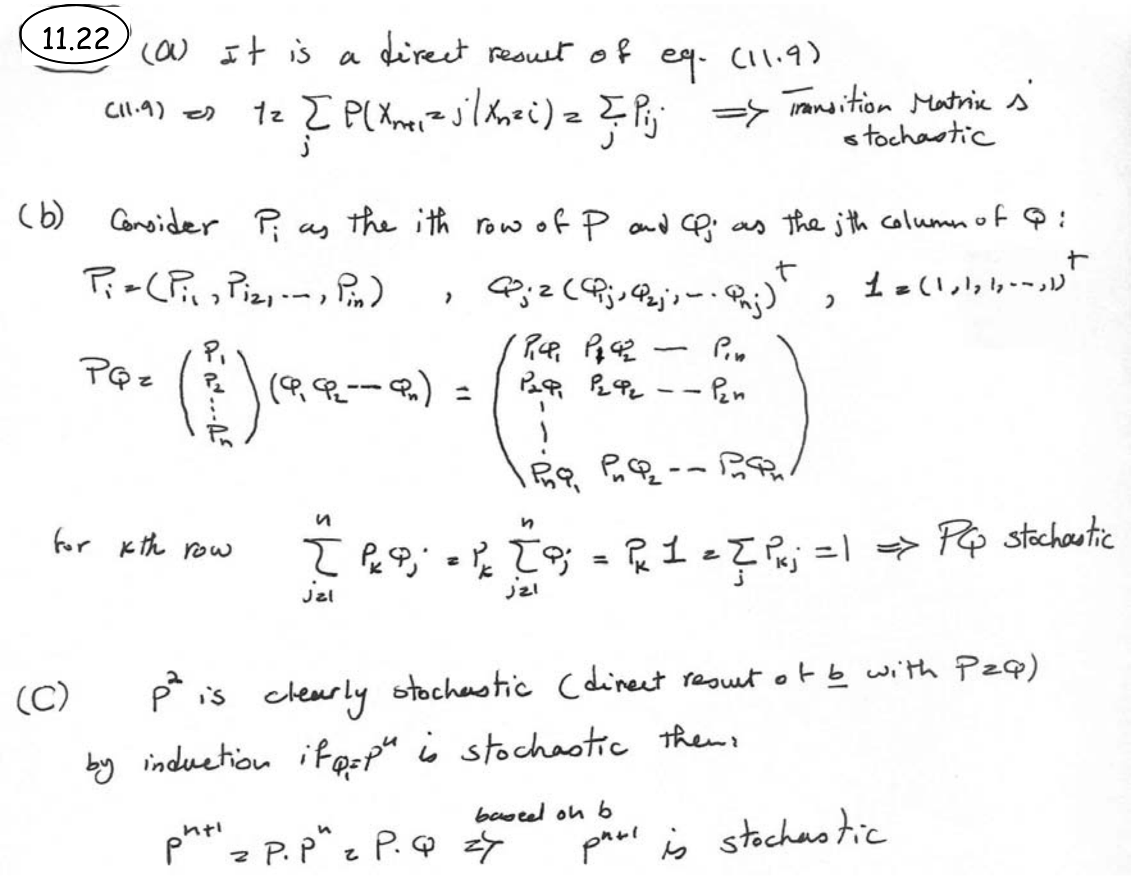
b) Claim: the steady state pmf is

$$\begin{aligned} \underline{p} &= \left(\left(\frac{a}{a+b}\right)^2, 2\left(\frac{b}{a+b}\right)\left(\frac{a}{a+b}\right), \left(\frac{b}{a+b}\right)^2 \right) \\ &= \frac{1}{(a+b)^2}(a^2, 2ab, b^2) \end{aligned}$$

$$\begin{aligned} \underline{p}\mathbf{P} &= \frac{1}{(a+b)^2}(a^2, 2ab, b^2) \begin{bmatrix} (1-b)^2 & 2b(1-b) & b^2 \\ a(1-b) & (1-a)(1-b)+ab & b(1-a) \\ a^2 & 2a(1-a) & (1-a)^2 \end{bmatrix} \\ &= \frac{1}{(a+b)^2} \begin{bmatrix} a^2(1-b)^2 + 2a^2b(1-b) + a^2b^2 \\ 2a^2b(1-b) + 2ab(1-a)(1-b) + 2a^2b^2 + 2ab^2(1-a) \\ a^2b^2 + 2ab^2(1-a) + b^2(1-a)^2 \end{bmatrix}^{+} \\ &= \frac{1}{(a+b)^2}(a^2, 2ab, b^2) = \underline{p} \quad \checkmark \end{aligned}$$

c) In the general case with n machines, the steady state part is given by

$$P[X_n = k] = \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k} \qquad 0 \le k \le n$$



11.22 (a) It is a direct result of eq. (11.9)

(11.9) ⟹ $1 = \sum_j P(X_{n+1} = j \mid X_n = i) = \sum_j P_{ij}$ ⟹ Transition Matrix is stochastic

(b) Consider P_i as the ith row of P and φ_j as the jth column of φ:

$P_i = (P_{i1}, P_{i2}, \dots, P_{in})$, $\varphi_j = (\varphi_{1j}, \varphi_{2j}, \dots \varphi_{nj})^T$, $\mathbb{1} = (1, 1, 1, \dots, 1)^T$

$$P\varphi = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix} (\varphi_1 \, \varphi_2 \cdots \varphi_n) = \begin{pmatrix} P_1\varphi_1 & P_1\varphi_2 & \cdots & P_{1n} \\ P_2\varphi_1 & P_2\varphi_2 & \cdots & P_{2n} \\ \vdots & & & \\ P_n\varphi_1 & P_n\varphi_2 & \cdots & P_n\varphi_n \end{pmatrix}$$

for kth row $\sum_{j=1}^{n} P_k \varphi_j = P_k \sum_{j=1}^{n} \varphi_j = P_k \mathbb{1} = \sum_j P_{kj} = 1$ ⟹ $P\varphi$ stochastic

(c) P^2 is clearly stochastic (direct result of b with $P = \varphi$)

by induction if $\varphi = P^n$ is stochastic then:

$$P^{n+1} = P \cdot P^n = P \cdot \varphi \overset{\text{based on b}}{\Longrightarrow} P^{n+1} \text{ is stochastic}$$

11.23

~~8.10~~ If $\mathbf{P}^k = \begin{bmatrix} \underline{r} \\ \underline{r} \\ \vdots \\ \underline{r} \end{bmatrix}$, that is, $\mathbf{P}^k$ has identical rows

then the j^{th} row of $\mathbf{P}^{k+1}$ is

$$\begin{aligned} \mathbf{P}\mathbf{P}^k &= \begin{bmatrix} & & \cdots & \\ p_{j1} & p_{j2} & \cdots & p_{jn} \\ & & \cdots & \end{bmatrix} \begin{bmatrix} \underline{r} \\ \underline{r} \\ \vdots \\ \underline{r} \end{bmatrix} \\ &= \begin{bmatrix} \cdots \\ p_{j1}\underline{r} + p_{j2}\underline{r} + \ldots + p_{jn}\underline{r} \\ \cdots \end{bmatrix} \\ &= \begin{bmatrix} \cdots \\ \underline{r} \\ \cdots \end{bmatrix} = \mathbf{P}^k \end{aligned}$$

11.24

$$P(n+m) = P(n)P(m)$$

$$\Rightarrow P(2) = P(1) \times P(1) = P^2$$

if $P(n) = P^n$ by induction

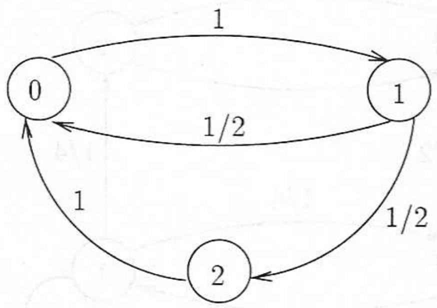
then $P(n+1) = P(n) \times P(1) = P^n \times P = P^{n+1}$

11.3 Classes of States, Recurrence Properties, and Limiting Probabilities

11.25

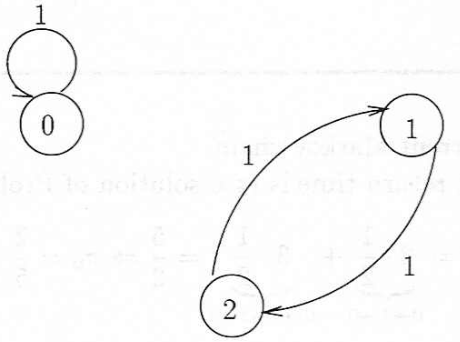
(a)

(i)



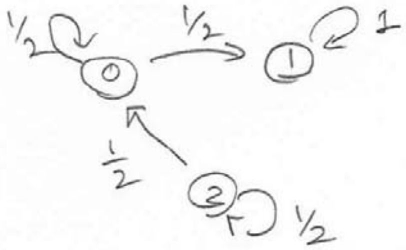
one class {1,2,3}
recurrent

(ii)



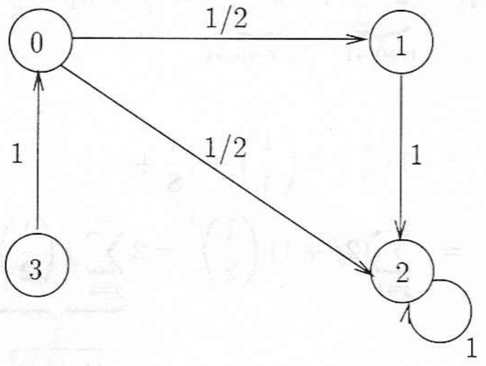
{0} recurrent
{2,3} recurrent

(iii)



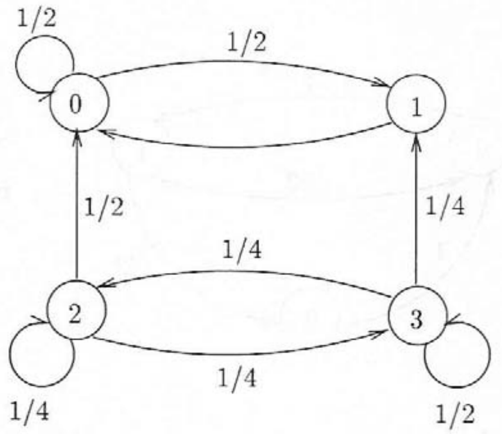
{2}, {0} transient
{1} recurrent

(iv)



{0} transient
{1} transient
{2} recurrent
{3} transient

(v)



{0,1} recurrent

{2,3} transient

(c) (i)

$$A^2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{bmatrix} \quad A^{50} = \begin{bmatrix} .4 & .4 & .2 \\ .4 & .4 & .2 \\ .4 & .4 & .2 \end{bmatrix}$$

(ii)

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^3 = A \quad A^4 = I \quad A^5 = A \;\ldots$$

alternates

(iii)

$$A^2 = \begin{bmatrix} .25 & .75 & 0 \\ 0 & 1 & 0 \\ .5 & .25 & .25 \end{bmatrix} \quad A^4 = \begin{bmatrix} .0625 & .9375 & 0 \\ 0 & 1 & 0 \\ .25 & 0.6875 & 0.0625 \end{bmatrix} \quad A^{50} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(iv)

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

converges in 3 steps.

(v)

$$A^2 = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/8 & \frac{5}{16} & 1/8 & \frac{3}{16} \\ 3/8 & 1/8 & \frac{3}{16} & \frac{5}{16} \end{bmatrix} \quad A^8 = \begin{bmatrix} .668 & .332 & 0 & 0 \\ .664 & .336 & 0 & 0 \\ .648 & .327 & .009 & .015 \\ .639 & .321 & .015 & .0244 \end{bmatrix} \quad A^{50} = \begin{bmatrix} 2/3 & 1/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \end{bmatrix}$$

11.26

~~8.21~~ **a)** This is a positive recurrent Markov chain.
For state 0 the mean first return time is (see solution of Problem 11.25)

$$\mathcal{E}[T_0] = \underbrace{2\cdot\frac{1}{2}}_{0\to1\to0} + \underbrace{3\cdot\frac{1}{2}}_{0\to1\to2\to0} = \frac{5}{2} \Rightarrow \pi_0 = \frac{2}{5}$$

For state 1

$$\mathcal{E}[T_1] = \underbrace{2\cdot\frac{1}{2}}_{1\to0\to1} + \underbrace{3\cdot\frac{1}{2}}_{1\to2\to0\to1} = \frac{5}{2} \Rightarrow \pi_1 = \frac{2}{5}$$

For state 2

$$\begin{aligned}\mathcal{E}[T_2] &= 3\cdot\frac{1}{2} + 5\left(\frac{1}{4}\right) + 7\frac{1}{8} + \ldots\\ &= \sum_{j=1}^{\infty}(2j+1)\left(\frac{1}{2}\right)^j = 2\underbrace{\sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^j}_{\frac{\frac{1}{2}}{\left(1-\left(\frac{1}{2}\right)\right)^2}}\\ &\Rightarrow \pi_2 = \frac{1}{5}\end{aligned}$$

It can easily be shown that above are solutions to equations for the stationary proof.

b) If system starts in state 0 then $\mathcal{E}[T_0] = 1 \Rightarrow \pi_0 = 1$
If system starts in state 1 or 2 then $\mathcal{E}[T_1] = \mathcal{E}[T_2] = 2$

$$\Rightarrow \pi_1 = \pi_2 = \tfrac{1}{2}$$

c) Only state 2 is recurrent and $\mathcal{E}[T_2] = 1 \Rightarrow \pi_2 = 1$

d) States 0 and 1 are recurrent and

$$\begin{aligned}\mathcal{E}[T_0] &= 1\cdots\frac{1}{2} + 2\cdot\frac{1}{2} = \frac{3}{2}\\ \Rightarrow \pi_0 &= \frac{2}{3}\\ \Rightarrow \pi_1 &= \frac{1}{3}\end{aligned}$$

11.26

(i) This is an irreducible positive recurrent MC.

$$(p_0, p_1, p_2) = (p_0, p_1, p_2)\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$p_0 = \frac{1}{2}p_1 + p_2$

$p_1 = p_0$

$p_2 = \frac{1}{2} p_1$

$1 = p_0 + p_1 + p_2 = p_0 + p_0 + \frac{1}{2}p_0 = \frac{5}{2}p_0$

$p_0 = \frac{2}{5} = p_1$

$p_2 = \frac{1}{5}$

(ii) This is a multiclass MC

Class 1 consists of a single recurrent state

Class 2 is has 2 recurrent states and has period 2

Suppose we calculate the stationary pmf:

$p_0 = p_0$ $p_0 = p_0$ $\underbrace{p_0 + 2p_1 = 1}_{\text{No unique solution}}$

$p_1 = p_2$ $p_1 = p_2$

$p_2 = p_1$

Let $p_0 = \alpha$

then $p_1 = p_2 = \frac{1-\alpha}{2}$

$$(p_0, p_1, p_2) = \left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right)$$

$$= \alpha \underbrace{(1,0,0)}_{\substack{\text{stationary pmf} \\ \text{for class 1} \\ \text{in isolation}}} + (1-\alpha)\underbrace{\left(0, \frac{1}{2}, \frac{1}{2}\right)}_{\substack{\text{stationary pmf} \\ \text{for class 2} \\ \text{in isolation}}}$$

(iii)

$$p_0 = \tfrac{1}{2}p_0 + p_2$$
$$p_1 = \tfrac{1}{2}p_0 + p_1 \Rightarrow p_0 = 0 \qquad p_0 + p_1 + p_2 = 1$$
$$p_2 = \tfrac{1}{2}p_0 + \tfrac{1}{2}p_2 \Rightarrow p_2 = 0 \qquad \Rightarrow p_1 = 1$$

This is a multiclass MC with 2 transient classes and one recurrent single state class, {1}.

(iv)

$$p_0 = p_3$$
$$p_1 = \tfrac{1}{2}p_0$$
$$p_2 = \tfrac{1}{2}p_0 + p_1 + p_2$$
$$p_3 = 0 \Rightarrow p_0 = 0 \Rightarrow p_1 = 0 \Rightarrow p_2 = 1$$

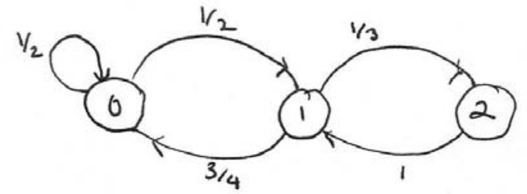
This MC has 3 transient classes and one recurrent, single state class.

(v)

$$p_0 = \tfrac{1}{2}p_0 + p_1 + \tfrac{1}{2}p_2 \qquad \tfrac{1}{2}p_0 = p_1 \Rightarrow p_0 = \tfrac{2}{3}, \quad p_1 = \tfrac{1}{3}$$
$$p_1 = \tfrac{1}{2}p_0 + \tfrac{1}{4}p_3$$
$$p_2 = \tfrac{1}{4}p_2 + \tfrac{1}{4}p_3 \Rightarrow p_3 = 3p_2$$
$$p_3 = \tfrac{1}{4}p_2 + \tfrac{1}{2}p_3 \Rightarrow p_3 = \tfrac{1}{2}p_2 \quad \Big\} \Rightarrow p_2 = p_3 = 0$$

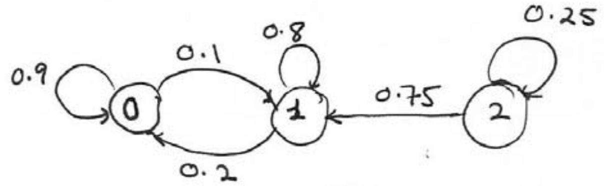
This MC has a transient class and a two-state, aperiodic, positive recurrent class.

11.27 (a)



irreducible

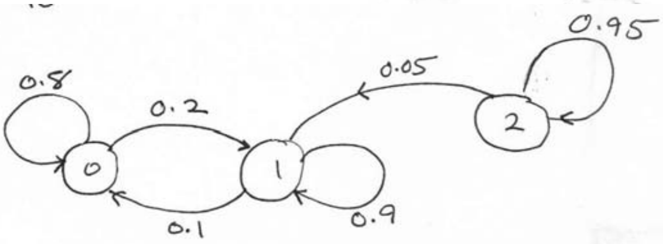
(b)



{2} transient

{0,1} recurrent

(C)



11.28 (a)

i)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

ii)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

iii)

$$P = \begin{array}{cccc} 0 & 1 & 3 & 4 \\ \hline \end{array}$$

$$P = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{pmatrix} \end{array}$$

iv

$$P = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \end{pmatrix} \end{array}$$

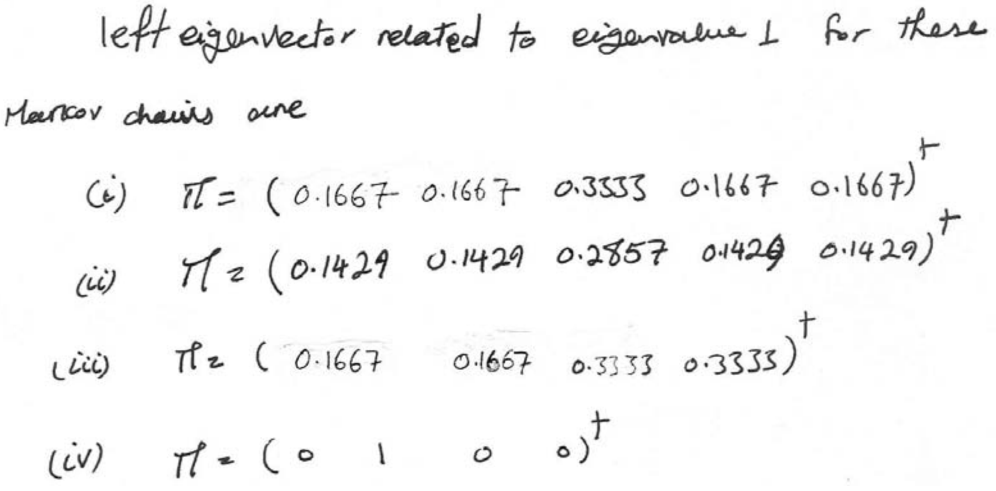
(b) (i) Recurrent, Periodic

(ii) Recurrent, Periodic

(iii) Recurrent, Periodic

(iv) Transient, aperiodic

(c)



left eigenvector related to eigenvalue 1 for these Markov chains are

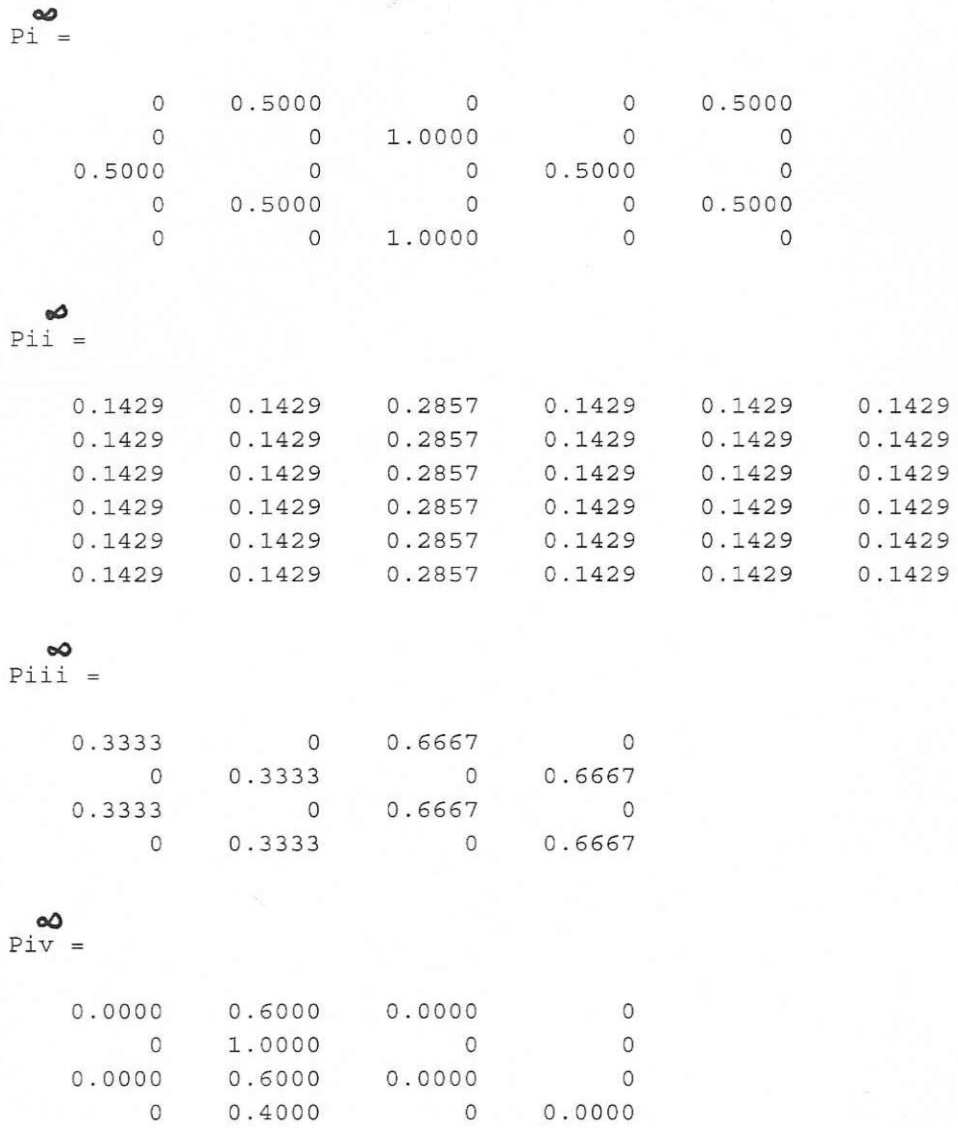
(i) $\pi = (0.1667 \quad 0.1667 \quad 0.3333 \quad 0.1667 \quad 0.1667)^t$

(ii) $\pi = (0.1429 \quad 0.1429 \quad 0.2857 \quad 0.1429 \quad 0.1429)^t$

(iii) $\pi = (0.1667 \quad 0.1667 \quad 0.3333 \quad 0.3333)^t$

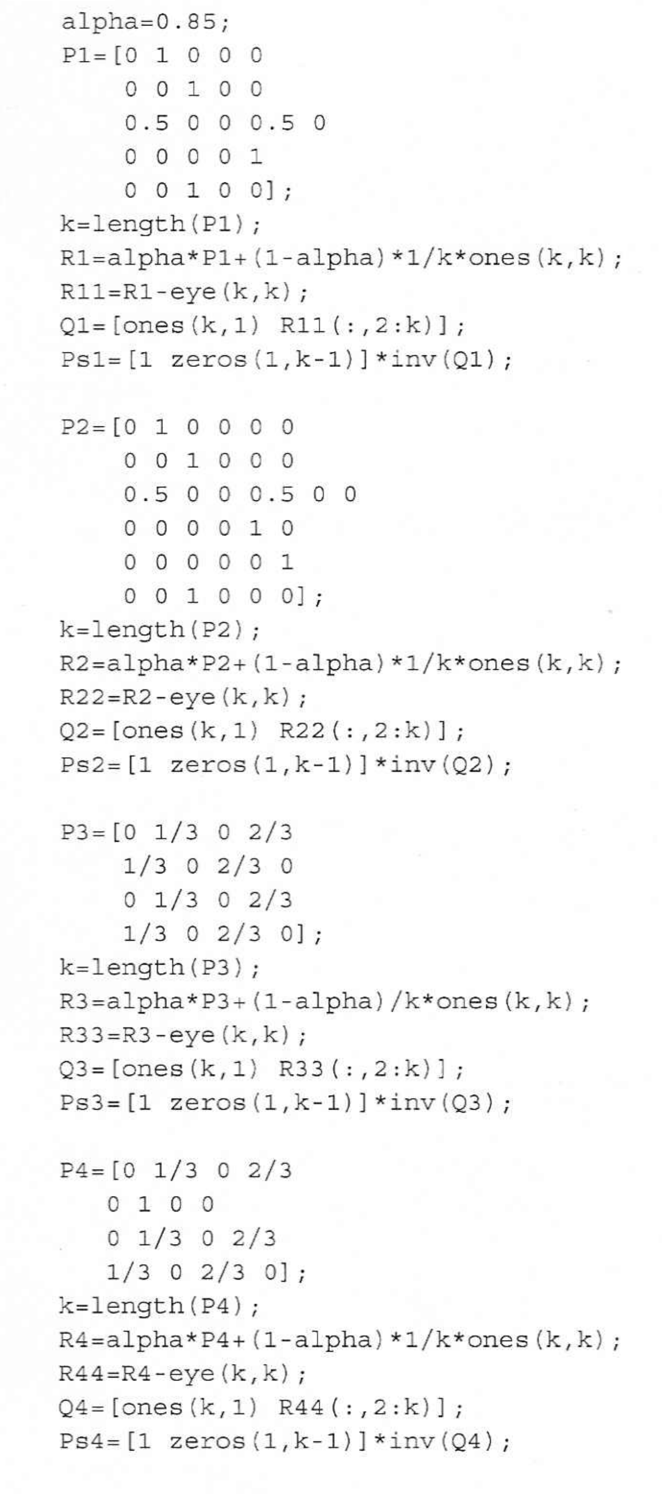
(iv) $\pi = (0 \quad 1 \quad 0 \quad 0)^t$

(d)

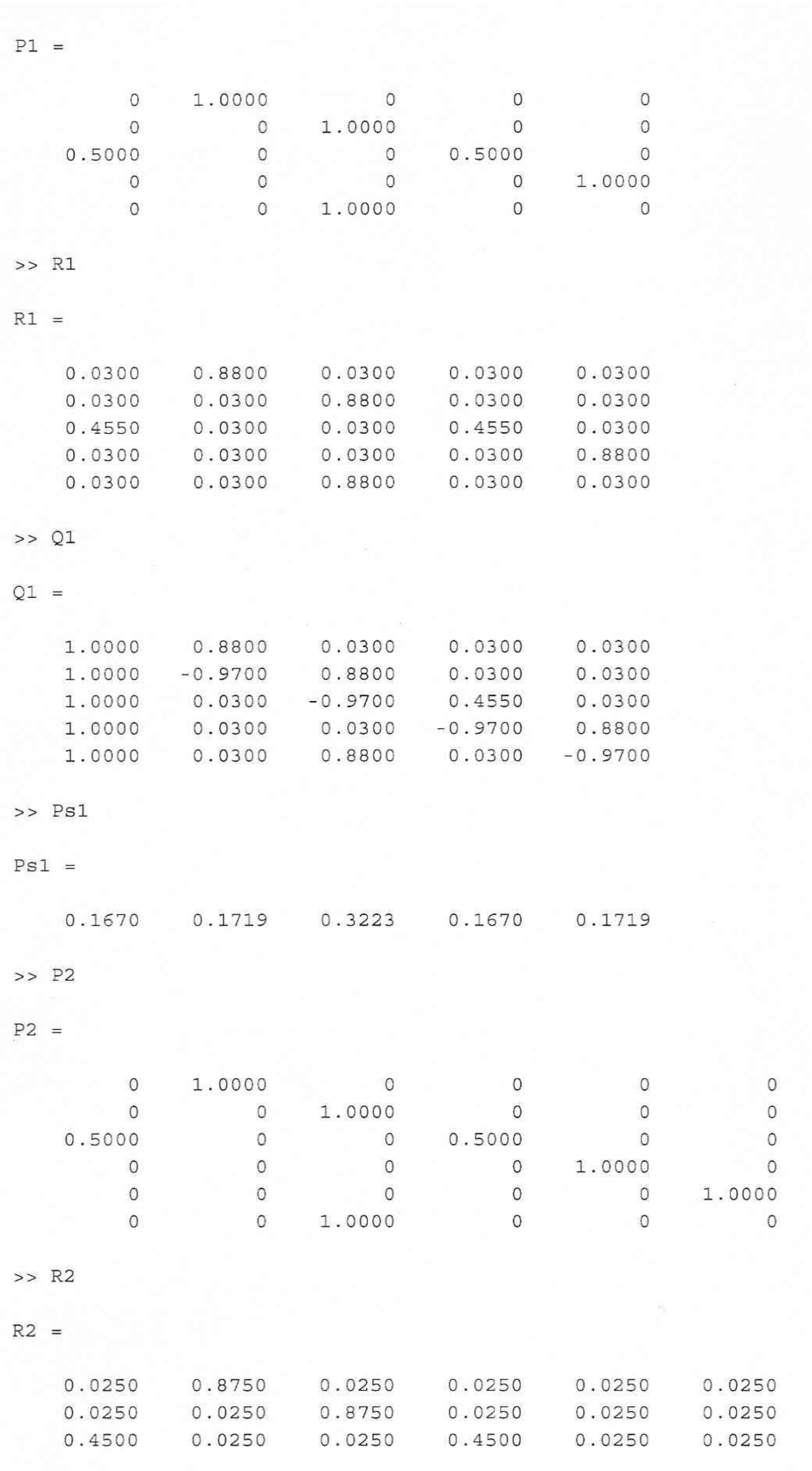


```
 ∞
Pi  =

         0    0.5000         0         0    0.5000
         0         0    1.0000         0         0
    0.5000         0         0    0.5000         0
         0    0.5000         0         0    0.5000
         0         0    1.0000         0         0


  ∞
Pii  =

    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429
    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429
    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429
    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429
    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429
    0.1429    0.1429    0.2857    0.1429    0.1429    0.1429


  ∞
Piii  =

    0.3333         0    0.6667         0
         0    0.3333         0    0.6667
    0.3333         0    0.6667         0
         0    0.3333         0    0.6667


 ∞
Piv  =

    0.0000    0.6000    0.0000         0
         0    1.0000         0         0
    0.0000    0.6000    0.0000         0
         0    0.4000         0    0.0000
```

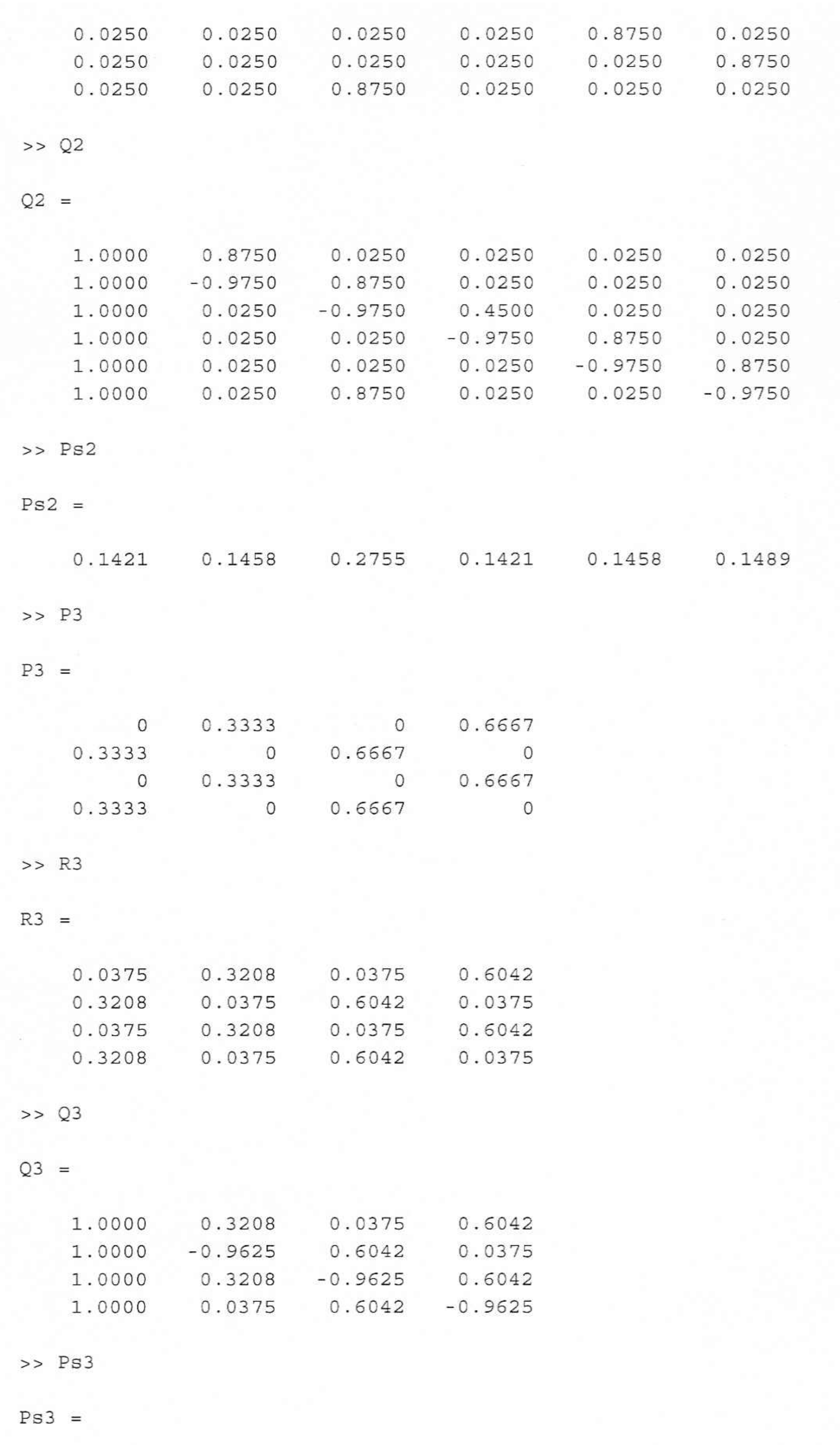
11.29



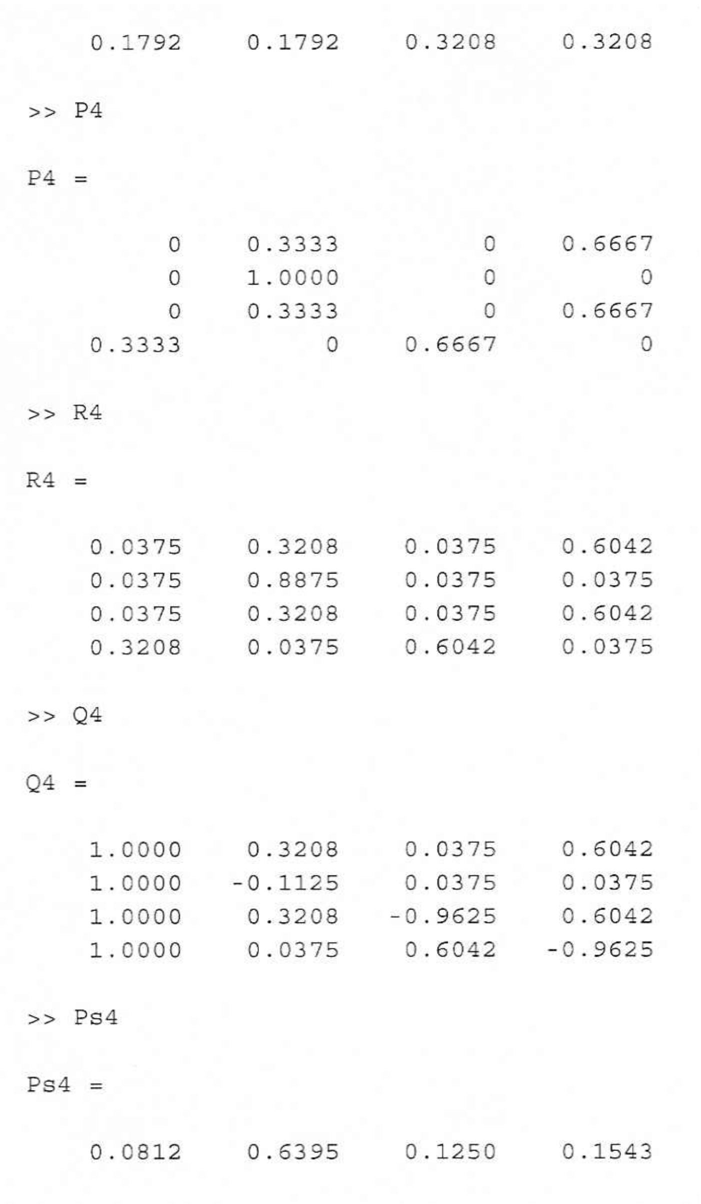
```
alpha=0.85;
P1=[0 1 0 0 0
    0 0 1 0 0
    0.5 0 0 0.5 0
    0 0 0 0 1
    0 0 1 0 0];
k=length(P1);
R1=alpha*P1+(1-alpha)*1/k*ones(k,k);
R11=R1-eye(k,k);
Q1=[ones(k,1) R11(:,2:k)];
Ps1=[1 zeros(1,k-1)]*inv(Q1);

P2=[0 1 0 0 0 0
    0 0 1 0 0 0
    0.5 0 0 0.5 0 0
    0 0 0 0 1 0
    0 0 0 0 0 1
    0 0 1 0 0 0];
k=length(P2);
R2=alpha*P2+(1-alpha)*1/k*ones(k,k);
R22=R2-eye(k,k);
Q2=[ones(k,1) R22(:,2:k)];
Ps2=[1 zeros(1,k-1)]*inv(Q2);

P3=[0 1/3 0 2/3
    1/3 0 2/3 0
    0 1/3 0 2/3
    1/3 0 2/3 0];
k=length(P3);
R3=alpha*P3+(1-alpha)/k*ones(k,k);
R33=R3-eye(k,k);
Q3=[ones(k,1) R33(:,2:k)];
Ps3=[1 zeros(1,k-1)]*inv(Q3);

P4=[0 1/3 0 2/3
   0 1 0 0
   0 1/3 0 2/3
   1/3 0 2/3 0];
k=length(P4);
R4=alpha*P4+(1-alpha)*1/k*ones(k,k);
R44=R4-eye(k,k);
Q4=[ones(k,1) R44(:,2:k)];
Ps4=[1 zeros(1,k-1)]*inv(Q4);
```



```
P1 =

         0    1.0000         0         0         0
         0         0    1.0000         0         0
    0.5000         0         0    0.5000         0
         0         0         0         0    1.0000
         0         0    1.0000         0         0

>> R1

R1 =

    0.0300    0.8800    0.0300    0.0300    0.0300
    0.0300    0.0300    0.8800    0.0300    0.0300
    0.4550    0.0300    0.0300    0.4550    0.0300
    0.0300    0.0300    0.0300    0.0300    0.8800
    0.0300    0.0300    0.8800    0.0300    0.0300

>> Q1

Q1 =

    1.0000    0.8800    0.0300    0.0300    0.0300
    1.0000   -0.9700    0.8800    0.0300    0.0300
    1.0000    0.0300   -0.9700    0.4550    0.0300
    1.0000    0.0300    0.0300   -0.9700    0.8800
    1.0000    0.0300    0.8800    0.0300   -0.9700

>> Ps1

Ps1 =

    0.1670    0.1719    0.3223    0.1670    0.1719

>> P2

P2 =

         0    1.0000         0         0         0         0
         0         0    1.0000         0         0         0
    0.5000         0         0    0.5000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0         0    1.0000         0         0         0

>> R2

R2 =

    0.0250    0.8750    0.0250    0.0250    0.0250    0.0250
    0.0250    0.0250    0.8750    0.0250    0.0250    0.0250
    0.4500    0.0250    0.0250    0.4500    0.0250    0.0250
```

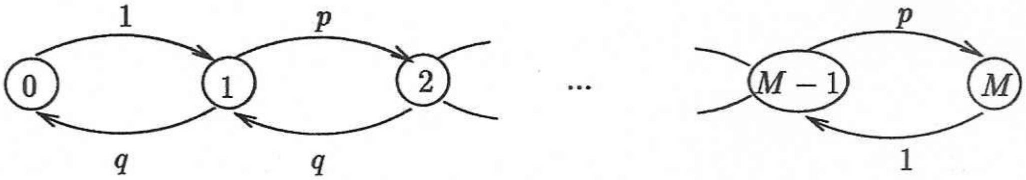


```
    0.0250    0.0250    0.0250    0.0250    0.8750    0.0250
    0.0250    0.0250    0.0250    0.0250    0.0250    0.8750
    0.0250    0.0250    0.8750    0.0250    0.0250    0.0250

>> Q2

Q2 =

    1.0000    0.8750    0.0250    0.0250    0.0250    0.0250
    1.0000   -0.9750    0.8750    0.0250    0.0250    0.0250
    1.0000    0.0250   -0.9750    0.4500    0.0250    0.0250
    1.0000    0.0250    0.0250   -0.9750    0.8750    0.0250
    1.0000    0.0250    0.0250    0.0250   -0.9750    0.8750
    1.0000    0.0250    0.8750    0.0250    0.0250   -0.9750

>> Ps2

Ps2 =

    0.1421    0.1458    0.2755    0.1421    0.1458    0.1489

>> P3

P3 =

         0    0.3333         0    0.6667
    0.3333         0    0.6667         0
         0    0.3333         0    0.6667
    0.3333         0    0.6667         0

>> R3

R3 =

    0.0375    0.3208    0.0375    0.6042
    0.3208    0.0375    0.6042    0.0375
    0.0375    0.3208    0.0375    0.6042
    0.3208    0.0375    0.6042    0.0375

>> Q3

Q3 =

    1.0000    0.3208    0.0375    0.6042
    1.0000   -0.9625    0.6042    0.0375
    1.0000    0.3208   -0.9625    0.6042
    1.0000    0.0375    0.6042   -0.9625

>> Ps3

Ps3 =
```



```
    0.1792    0.1792    0.3208    0.3208

>> P4

P4 =

         0    0.3333         0    0.6667
         0    1.0000         0         0
         0    0.3333         0    0.6667
    0.3333         0    0.6667         0

>> R4

R4 =

    0.0375    0.3208    0.0375    0.6042
    0.0375    0.8875    0.0375    0.0375
    0.0375    0.3208    0.0375    0.6042
    0.3208    0.0375    0.6042    0.0375

>> Q4

Q4 =

    1.0000    0.3208    0.0375    0.6042
    1.0000   -0.1125    0.0375    0.0375
    1.0000    0.3208   -0.9625    0.6042
    1.0000    0.0375    0.6042   -0.9625

>> Ps4

Ps4 =

    0.0812    0.6395    0.1250    0.1543
```

11.30



This Markov chain has period 2 $\Rightarrow p_{ii}(2n) = 2\pi_i$ as $n \to \infty$

$$\begin{aligned}
\pi_0 = q\pi_1 \quad &\Rightarrow \quad \pi_1 = \frac{1}{q}\pi_0 \\
\pi_1 = \pi_0 + q\pi_2 \quad &\Rightarrow \quad \pi_2 = \frac{1}{q}\left(\frac{1}{q}-1\right)\pi_0 = \frac{p}{q^2}\pi_0 \\
\pi_2 = p\pi_1 + q\pi_3 \quad &\Rightarrow \quad \pi_3 = \frac{1}{q}(\pi_2 - p\pi_1) = \frac{1}{q}\left(\frac{p}{q^2}-\frac{p}{q}\right)\pi_0 \\
\vdots \qquad\quad & \qquad\quad = \frac{p}{q^2}\left(\frac{1}{q}-1\right) = \frac{p^2}{q^3}\pi_0 \\
\pi_{M-1} = p\pi_{M-2} + q\pi_M \quad & \qquad\quad \pi_4 = \frac{1}{q}(\pi_3 - p\pi_2) = \frac{1}{q}\left(\frac{p^2}{q^3}-\frac{p^2}{q^2}\right)\pi_0 \\
\pi_M = p\pi_{M-1} \quad & \qquad\quad = \frac{1}{q}\left(\frac{p^2}{q^2}\right)\left(\frac{1}{q}-1\right)\pi_0 = \frac{p^3}{q^4}\pi_0 \\
& \qquad\quad \vdots \\
\text{and} \quad & \qquad\quad \pi_M = \pi_{M-1} - p\pi_{M-2} \\
& \qquad\quad = \left(\frac{p^{M-2}}{q^{M-1}} - p\frac{p^{M-3}}{q^{M-2}}\right)\pi_0
\end{aligned}$$

$$\begin{aligned} &= \frac{p^{M-2}}{q^{M-2}}\left(\frac{1}{q}-1\right)\pi_0 \\ &\frac{p^{M-1}}{q^{M-1}}\pi_0 \end{aligned}$$

To find π_0 we note that

$$\begin{aligned} 1 &= \pi_0\left(1+\frac{1}{q}+\frac{p}{q^2}+\frac{p^2}{q^3}+\dots+\frac{p^{M-2}}{q^{M-1}}+\frac{p^{M-1}}{q^{M-1}}\right) \\ &= \pi_0\left(1+\frac{1}{q}\left(\underbrace{\left(\frac{p}{q}\right)^0+\left(\frac{p}{q}\right)^1+\dots+\left(\frac{p}{q}\right)^{M-2}}_{\frac{1-\left(\frac{p}{q}\right)^{M-1}}{1-\frac{p}{q}}}\right)+\frac{p^{M-1}}{q^{M-1}}\right) \end{aligned}$$

$$\therefore \qquad \pi_0 = \frac{1-2p}{2\left(1-p\left(1+\left(\frac{p}{q}\right)^{M-1}\right)\right)}$$

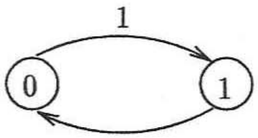
and

$$\begin{aligned} \pi_i &= \frac{1}{q}\left(\frac{p}{q}\right)^{i-1}\pi_0 \qquad 1 \le i \le M-1 \\ \pi_M &= \left(\frac{p}{q}\right)^{M-1}\pi_0 \end{aligned}$$

and finally

$$\lim_{n\to\infty} p_{ii}(2n) = 2\pi_i$$

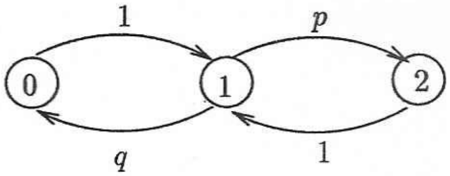
Some special cases:
$M = 1$



$$\pi_0 = \frac{1}{2} \quad \Rightarrow \quad \begin{array}{l} p_{00}(2n) = 1 \\ p_{11}(2n) = 1 \end{array}$$

that is, an even number of steps implies certain return to the same state.

$M = 2$



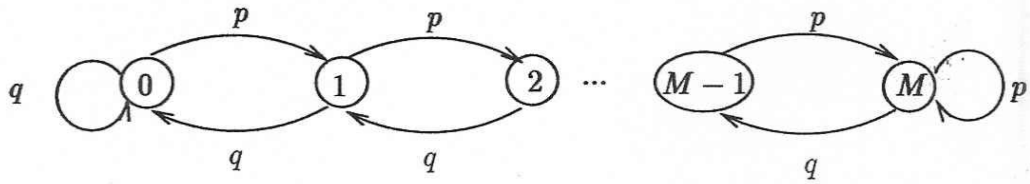
$$\pi_0 = \frac{q}{2} \qquad \pi_1 = \frac{1}{2} \qquad \pi_2 = \frac{p}{2}$$

Note $p_{11}(2n) = 1$, that is, every other step involves a return to state 1.

If $p = q = \frac{1}{2}$

$$\begin{aligned} 1 &= \pi_0(1 + \underbrace{2 + 2 + \ldots + 2}_{M-1} + 1) = 2M\pi_0 \\ &\Rightarrow \pi_0 = \pi_M = \frac{1}{2M} \qquad \pi_i = \frac{1}{M} \qquad 2 \le i \le M - 1 \end{aligned}$$

11.31



This is an aperiodic Markov chain $\Rightarrow p_{ii}(n) \to \pi_i$ as $n \to \infty$

$$\begin{array}{ll}
\pi_0 = q\pi_0 + q\pi_1 \quad \Rightarrow & \pi_1 = \dfrac{p}{q}\pi_0 \\
\pi_1 = p\pi_0 + q\pi_2 & \pi_2 = \dfrac{1}{q}(\pi_1 - p\pi_0) = \dfrac{1}{q}\left(\dfrac{p}{q} - p\right)\pi_0 = \left(\dfrac{p}{q}\right)^2 \pi_0
\end{array}$$

$$\begin{array}{ll}
\pi_2 = p\pi_1 + q\pi_3 & \pi_3 = \dfrac{1}{q}(\pi_2 - p\pi_1) = \dfrac{1}{q}\left(\dfrac{p^2}{q^2} - \dfrac{p^2}{q}\right)\pi_0 \\
\vdots & \quad \dfrac{1}{q}\dfrac{p^2}{q}\left(\dfrac{1}{q} - 1\right) = \dfrac{p^3}{q^3}\pi_0 \\
\pi_{M-1} = p\pi_{M-1} + q\pi_M & \quad \vdots \\
\pi_M = p\pi_M + p\pi_{M-1} & \pi_M = \left(\dfrac{p}{q}\right)^M \pi_0
\end{array}$$

$$\begin{aligned}
1 &= \pi_0\left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \ldots + \left(\frac{p}{q}\right)^M\right) = \pi_0 \frac{1 - \left(\frac{p}{q}\right)^{M+1}}{1 - \frac{p}{q}} \\
\pi_0 &= \frac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^{M+1}} \\
\pi_i &= \frac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^{M+1}} \left(\frac{p}{q}\right)^i
\end{aligned}$$

11.32 If an irreducible class has n states, then every state can be reached from every other state in n or fewer steps. If a state has zero probability then all states that lead to it must also have zero probability from the transition probability equations. But this implies all the states must have zero probability, which is not possible for a finite state MC.

11.34 suppose i is a positive recurrent state.

If state i communicates with j, then there exist n, m such that $P_{ij}(n) > 0$, $P_{ji}(m) > 0$.

For any k one can see $P_{jj}(m+n+k) \geq P_{ji}(m)\, P_{ii}(n)\, P_{ij}(k)$

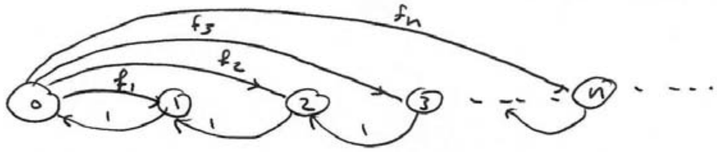
Since $P_{jj}(m+n+k)$ is the probability of going from j to j in $(m+n+k)$ steps, while $P_{ji}(m)\, P_{ii}(n)\, P_{ij}(k)$ is the probability of going from j to i in m steps and then from i to i in (n) steps and finally from i to j in k steps.

$$\Rightarrow \sum_{n\geq 1}^{\infty} P_{jj}(m+n+k) \geq P_{ji}(m)\, P_{ij}(k) \underbrace{\sum_{n\geq 1}^{\infty} P_{ii}}_{\infty\ (i \text{ is pos. recurrent})} = \infty$$

$\Rightarrow j$ is also positive recurrent

11.36

(a)



(b) There is a single class in this Markov chain ⇒ irreducible

(c) state zero is recurrent

(d) $X = (x_0, x_1, \ldots x_n \ldots)$

$$P = \begin{pmatrix} 0 & f_1 & f_2 & f_3 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & & \ddots & & \\ \vdots & & & 1 & 0 \\ \vdots & & & & \ddots \end{pmatrix}$$

$\Rightarrow \quad xP = x \quad \Rightarrow$

$x_1 = x_0$

$f_1 x_0 + x_2 = x_1 \Rightarrow x_2 = (1-f_1)x_0$

$f_2 x_0 + x_3 = x_2 \Rightarrow x_3 = (1-f_1-f_2)x_0$

$$\Rightarrow \boxed{x_n = (1-f_1-f_2-\cdots-f_{n-1})x_0}$$

$$\sum_{n=0}^{\infty} x_n = 1 \quad \Rightarrow \quad \begin{array}{l} 1 \\ 1-f_1 \\ 1-f_1-f_2 \\ 1-f_1-f_2-f_3 \\ \vdots \end{array}$$

$$\sum_{i=0}^{n-1} x_i = 1x_0 + (1-f_1)x_0 + (1-f_1-f_2)x_0 + (1-f_1-f_2-f_3)x_0 + \cdots (1-f_1-\cdots-f_{n-1})x_0$$

$$= \left[n - (n-1)f_1 - (n-2)f_2 - \cdots - f_{n-1}\right]x_0$$

$$= \underbrace{\left[n(1-f_1-f_2-\cdots-f_{n-1}) + f_1 + 2f_2 + \cdots + (n-1)f_{n-1}\right]}_{K} x_0$$

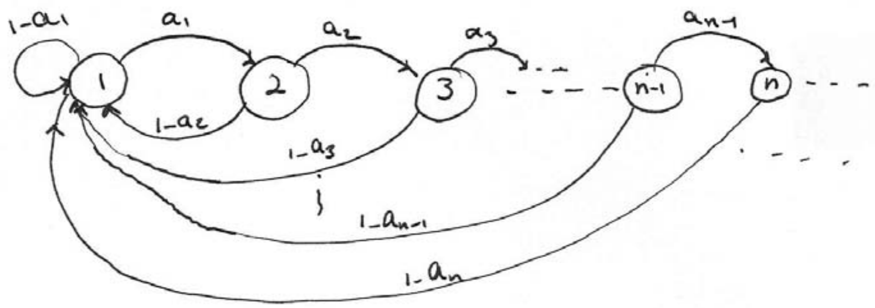
$$\Rightarrow x_0 = \frac{1}{K} \qquad x_n = \frac{(1-f_1-f_2-\cdots-f_{n-1})}{K}$$

It means that there is no stationary Pmf.

11.37 - $S =$ state space $= \{1, 2, 3, \dots\}$

$\forall j \in S \quad P_{j,j+1} = a_j \; ; \; P_{j1} = 1 - a_j \quad 0 < a_j < 1$

(a)



(b) All the states are reachable and construct one class
$\Rightarrow$ irreducible

(c) state 1 is positive recurrent.

(d)

$$P = \begin{bmatrix} 1-a_1 & a_1 & 0 & 0 & 0 & \cdots \\ 1-a_2 & 0 & a_2 & & & \\ 1-a_3 & 0 & 0 & a_3 & & \\ \vdots & & & & \ddots & \\ \vdots & & & & & \end{bmatrix}$$

$\pi P = \pi$

$(1-a_1)\pi_1 + (1-a_2)\pi_2 + \cdots = \pi_1$

$a_1\pi_1 = \pi_2 \qquad \pi_2 = a_1\pi_1$

$a_2\pi_2 = \pi_3 \Rightarrow \pi_3 = a_1a_2\pi_1$

$\vdots \qquad \pi_{n+1} = a_1a_2 \cdots a_n\pi_1$

$$\sum_{i=1}^{\infty} \pi_i = 1 \Rightarrow \pi_1 + a_1\pi_1 + a_2a_1\pi_1 + \cdots = 1$$

$$\pi_1 = \frac{1}{\underbrace{1 + a_1 + a_1a_2 + a_1a_2a_3 + \cdots}_{\text{assume } A}} = \frac{1}{A}$$

$$\begin{cases} \pi_1 = \frac{1}{A} \\ \pi_2 = \frac{a_1}{A} \\ \pi_3 = \frac{a_2a_1}{A} \\ \vdots \end{cases}$$

(e) i) $A = 1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots = \frac{1}{1-\frac{1}{2}} = 2 \Rightarrow \pi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$

11.37 (d) (ii) $a_j = \frac{j-1}{j} \Rightarrow a_1 = 0 \Rightarrow \begin{cases} \pi_1 = 1 \\ \pi_i = 0 \quad i > 1 \end{cases}$ always in state 1

(iii) $a_j = \frac{1}{j}$ $\pi_0 = 1 + 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{2} \times \frac{1}{3} + \cdots = e$

$\Rightarrow \pi_1 = \frac{1}{e}$ $\pi_j = \frac{a_j}{A} = \frac{1}{j \times e}$

(iv) $a_j = (1/2)^j \Rightarrow A = 1.64$

$\pi_1 = \frac{1}{1.64} = 0.6098$ $\pi_2 = (\frac{1}{2}) \times 0.6098 = 0.3046 \ldots$ $\pi_{n+1} = (\frac{1}{2})^n \times 0.6098$

(V) $a_j = 1 - (\frac{1}{2})^j \Rightarrow A =$

11.38

(a) suppose event A is $(X_{n+1}=j \mid X_n=i, X_{n-1}=i, \ldots)$

Then $P(A) = P(A|H)P(H) + P(A|T)P(T) = \frac{1}{2}P(A|H) + \frac{1}{2}P(A|T)$

but $\begin{cases} P(A|H) = (P_1)_{ij} \\ P(A|T) = (P_2)_{ij} \end{cases}$ $\Rightarrow P(A) = \frac{1}{2}(P_1)_{ij} + \frac{1}{2}(P_2)_{ij}$

$\Rightarrow P = \frac{1}{2}(P_1 + P_2) \Rightarrow$ Markov ergodic it has pmf

(b) This process is not Markov since at each time instant one needs to know the previous steps in order to be able to find the probability of X_{n+1}.

(C) In any time instant, if it is an even number P_2 is used and in case of an even number P_1. In this case in fact in any time constant The transition-Probability matrix alternates between P_1 & P_2.

If π_0 be the initial state, then the other states can be calculated :

$$\pi_1 = \pi_0 P_1$$
$$\pi_2 = \pi_1 P_2 = \pi_0 (P_1 P_2)$$
$$\pi_3 = \pi_0 (P_1 P_2 P_1)$$
$$\pi_4 = \pi_0 (P_1 P_2)^2$$
$$\vdots$$
$$\begin{cases} \pi_{2n} = \pi_0 (P_1 P_2)^n \\ \pi_{2n+1} = \pi_0 (P_1 P_2)^n P_1 \end{cases} \Rightarrow$$ we have two recurrent classes (two pmf exists)

11.4 Continuous-Time Markov Chains

11.41

~~**8.14**~~ From Ex.11.36 we have

$$\begin{aligned} p_0(t) &= \frac{\beta}{\alpha+\beta} + \left(p_0(0) - \frac{\beta}{\alpha+\beta}\right) e^{-(\alpha+\beta)t} \\ p_1(t) &= \frac{\alpha}{\alpha+\beta} + \left(p_1(0) - \frac{\alpha}{\alpha+\beta}\right) e^{-(\alpha+\beta)t} \end{aligned}$$

a) Now suppose we know the initial state is 0, then $p_0(0) = 1 \Rightarrow$

$$\begin{aligned} p_{00}(t) &= \frac{\beta}{\alpha+\beta} + \left(1 - \frac{\beta}{\alpha+\beta}\right) e^{-(\alpha+\beta)t} = \frac{\beta + \alpha e^{-(\alpha+\beta)t}}{\alpha+\beta} \\ p_{01}(t) &= 1 - p_{00}(t) = \frac{\alpha(1 - e^{-(\alpha+\beta)t})}{\alpha+\beta} \end{aligned}$$

If the initial state is 1, then $p_1(0) = 1 \Rightarrow$

$$\begin{aligned} p_{11}(t) &= \frac{\alpha}{\alpha+\beta} + \left(1 + \frac{\alpha}{\alpha+\beta}\right) e^{-(\alpha+\beta)t} = \frac{\alpha + \beta e^{-(\alpha+\beta)t}}{\alpha+\beta} \\ p_{10}(t) &= 1 - p_{11}(t) = \frac{\beta(1 - e^{-(\alpha+\beta)t}}{\alpha+\beta} \\ \therefore \mathbb{P}(t) &= \frac{1}{\alpha+\beta} \begin{bmatrix} \beta + \alpha e^{-(\alpha+\beta)t} & \alpha(1 - e^{-(\alpha+\beta)t}) \\ \beta(q - e^{-(\alpha+\beta)t}) & \alpha + \beta e^{-(\alpha+\beta)t} \end{bmatrix} \end{aligned}$$

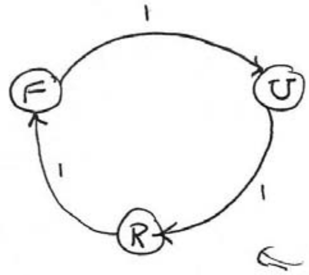
b) $P[X(1.5) = 1, X(3) = 1/X(0) = 0]$

$$\begin{aligned} &= P[X(3) = 1/X(1.5) = 1, X(0) = 0] P[X(1.5) = 1/X(0) = 0] \\ &= P[X(3) = 1/X(1.5) = 1] P[X(1.5) = 1/X(0) = 0] \\ &= p_{11}(1.5) p_{01}(1.5) \end{aligned}$$

$$\begin{aligned} P[X(1.5) = 1, X(3) = 1] &= P[X(3) = 1/X(1.5) = 1] P[X(1.5) = 1] \\ &= p_{11}(1.5) \left[\frac{\alpha}{\alpha+\beta} + \left(p_1(0) - \frac{\alpha}{\alpha+\beta}\right) e^{-(\alpha+\beta)1.5}\right] \end{aligned}$$

11.42

(a)



From Full-charged (F) one can go to "in use" (U) state and from in use to "Recharge" (R). Only from "R" one can go to "F"

⇐ embedded Markov chain

(b) $E(T_i) = \frac{1}{\nu_i}$ $\Rightarrow \begin{cases} \nu_F = \lambda \\ \nu_U = 1 \\ \nu_R = \frac{1}{3} \end{cases}$

$i = F, U, R$

$\pi_F = \pi_U = \pi_R = \frac{1}{3}$

(stationary probability of embedded Markov chain)

$$P_i = \frac{\frac{\pi_i}{\nu_i}}{\sum_j \frac{\pi_j}{\nu_j}}$$

$$P_F = \frac{\frac{1}{\lambda} \times \frac{1}{3}}{\frac{1}{3}(\frac{1}{\lambda}+1+3)} = \frac{1}{4\lambda+1}$$

$$P_U = \frac{1 \times \frac{1}{3}}{\frac{1}{3}(\frac{1}{\lambda}+1+3)} = \frac{\lambda}{4\lambda+1}$$

$$P_R = \frac{3 \times \frac{1}{3}}{\frac{1}{3}(\frac{1}{\lambda}+1+3)} = \frac{3\lambda}{4\lambda+1}$$

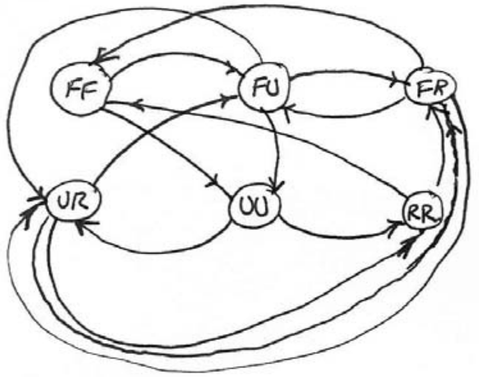
for $\lambda \to 0^+$ $P_F \to 1$ and $\begin{cases} P_U \to 0 \\ P_R \to 0 \end{cases}$ which is expected since $\lambda \to 0^+$ means that the occupancy time for state "F" is going to ∞.

For $\lambda \to +\infty$ $P_F \to 0$ and $\begin{cases} P \to \frac{1}{4} \\ P \to \frac{3}{4} \end{cases}$

Again, when $\lambda \to \infty$ the occupancy time of state "F" is negligible and the whole time is shared between "U" and "R" states

11.43

(a)



	FF	FU	FR	UR	UU	RR
FF	0	$2/3$	0	0	$1/3$	0
FU	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
FR	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0
UR	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{3}$
UU	0	0	0	$2/3$	0	$1/3$
RR	$\frac{1}{3}$	0	$\frac{2}{3}$	0	0	0

$\Rightarrow \pi = \left(\frac{1}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{1}{9} \ \frac{1}{9}\right)$

steady state of embedded Markov chain

(b) To find P_i we notice that:

$$P(T_m > t \ \& \ T_n > t) = P(T_m > t) \times P(T_n > t) = e^{-(\nu_m + \nu_n)t}$$

$$\Rightarrow \quad \nu_{mn} = \nu_m + \nu_n \Rightarrow$$

$\nu_{FF} = 2\lambda$; $\nu_{RR} = \frac{2}{3}$ $\nu_{UU} = 2$

$\nu_{FR} = \lambda + \frac{1}{3}$; $\nu_{FU} = \lambda + 1$ $\nu_{UR} = 1 + \frac{1}{3} = 4/3$

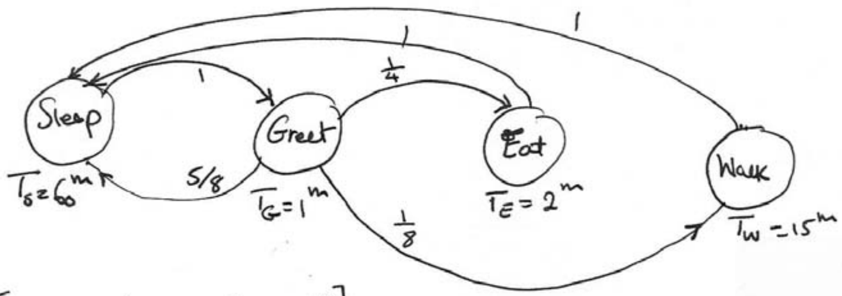
$$P_i = \frac{\frac{\pi_i}{\nu_i}}{\sum \frac{\pi_i}{\nu_i}} \Rightarrow P_{FF} = \frac{\frac{1}{9} \times \frac{1}{2\lambda}}{\frac{1}{2\lambda} + \frac{3}{3\lambda+1} + \frac{1}{\lambda+1} + \frac{11}{4}}$$

The rest can be obtained in a similar way.

When $\lambda \to \infty$ $P_{FF} \to 0$ and when $\lambda \to 0^+$ $P_{FF} \to \frac{1}{9}$

11.44

(a)



$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{5}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\pi M = \pi \Rightarrow$

$$\begin{cases} \frac{5}{8}\pi_1 + \pi_2 + \pi_3 = \pi_0 & \\ \pi_0 = \pi_1 & \pi_1 = \pi_0 \\ \frac{1}{4}\pi_1 = \pi_2 & \pi_2 = \frac{1}{4}\pi_0 \\ \frac{1}{8}\pi_1 = \pi_3 & \pi_3 = \frac{1}{8}\pi_0 \end{cases}$$

$\Rightarrow \pi = \pi_0 \left(1, 1, \frac{1}{4}, \frac{1}{8}\right)$ $\pi = \left(\frac{8}{19}, \frac{8}{19}, \frac{2}{19}, \frac{1}{19}\right)$

↳Normalization

$\nu_S = \frac{1}{60}$, $\nu_G = \frac{1}{1}$, $\nu_E = \frac{1}{2}$ $\nu_W = \frac{1}{15}$

$$P_i = \frac{\frac{\pi_i}{\nu_i}}{\sum \frac{\pi_i}{\nu_i}} \Rightarrow P_S = \frac{\frac{8}{19} \times 60}{\frac{8}{19}\times 60 + \frac{8}{19}\times 1 + \frac{2}{19}\times 2 + \frac{1}{19}\times 15} = \frac{480}{507} = \frac{160}{169}$$

$P_G = \frac{8}{507}$ $P_E = \frac{4}{507}$ $P_W = \frac{15}{507}$

11.44

~~8.15~~ Let $N(t) = \#$ of spares at time t

$N(t)$ decreases by one each time a part breaks down, and the time between breakdowns is independent exponential RV's with rate α.

a)

$$\begin{aligned}
p_{ij}(t) &= P[N(s+t) = j \mid N(s) = i] \qquad 1 \le j \le i \le n \\
&= P[i-j \text{ breakdowns in time } t] \\
&= \frac{(\alpha t)^{i-j}}{(i-j)!} e^{-\alpha t} \\
p_{i0}(t) &= P[i \text{ or more ``breakdowns'' in time } t] \\
&= 1 - \sum_{k=0}^{i-1} \frac{(\alpha t)^k}{k!} e^{-\alpha t}
\end{aligned}$$

b) $$\mathbf{P} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 - e^{-\alpha t} & e^{-\alpha t} & 0 & 0 & \cdots & \\
1 - \sum_{k=0}^{1} \frac{(\alpha t)^k}{k!} e^{-\alpha t} & \alpha t e^{-\alpha t} & e^{-\alpha t} & 0 & & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
1 - \sum_{k=0}^{n-1} \frac{(\alpha t)^k}{k!} e^{-\alpha t} & \frac{(\alpha t)^{n-1}}{(n-1)!} e^{-\alpha t} & & \cdots & \alpha t e^{-\alpha t} & e^{-\alpha t}
\end{bmatrix}$$

c) $p_n(0) = 1$

$$\begin{aligned}
\Rightarrow p_j(t) &= p_{nj}(t) \qquad 1 \le j \le n \\
p_0(t) &= 1 - \sum_{j=0}^{n-1} p_j(t)
\end{aligned}$$

11.45

~~8.16~~ **a)** For $j \leq i \leq n$

$$\begin{aligned} p_{ij}(t) &= P[N(s+t) = j \mid N(s) = i] \\ &= P[i-j \text{ machines out of } i \text{ machines break down} \\ &\qquad \text{in } t \text{ seconds}] \\ &= \binom{i}{i-j} p^{i-j}(1-p)^j \end{aligned}$$

where

$$\begin{aligned} p &= P[\text{a particular machine breaks down by } t \text{ seconds}] \\ &= 1 - e^{-\alpha t} \end{aligned}$$

$$\therefore p_{ij}(t) = \binom{i}{i-j} (1 - e^{-\alpha t})^{i-j} (e^{-\alpha t})^j$$

b) $$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 - e^{-\alpha t} & e^{-\alpha t} & 0 & \ldots & 0 \\ (1 - e^{-\alpha t})^2 & 2(1 - e^{-\alpha t})e^{-\alpha t} & e^{-2\alpha t} & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ (1 - e^{-\alpha t})^n & n(1 - e^{-\alpha t})^{n-1} e^{-\alpha t} & \ldots & & e^{-n\alpha t} \end{bmatrix}$$

c) $p_n(0) = 1$

$$\Rightarrow p_j(t) = p_{nj}(t) \qquad 0 \leq j \leq n$$

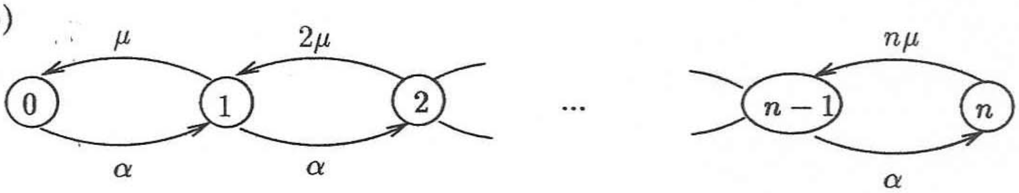
11.46

~~8.17~~ Let $\tau_i \triangleq$ time till next breakdown of machine i

a)

$$\begin{aligned}
T &\triangleq \text{time till next breakdown of any machine} \\
\Rightarrow T &= \min(\tau_1, \tau_2, ..., \tau_k) \\
P[T > t] &= P[\min(\tau_1, ..., \tau_k) > t] \\
&= P[\tau_1 > t, \tau_2 > t, ..., \tau_k > t] \\
&= P[\tau_1 > t]P[\tau_2 > t]...P[\tau_k > t] \\
&= (e^{-\mu t})^k = e^{-k\mu t}
\end{aligned}$$

b)



$$[\gamma_{ij}] = \begin{bmatrix} -\alpha & \alpha & 0 & 0 & \cdots & & \\ \mu & -(\alpha+\mu) & \alpha & 0 & \cdots & & \\ \vdots & & & \ddots & & & \\ & & & & (n-1)\mu & -(\alpha+(n-1))\mu & \alpha \\ 0 & 0 & & \cdots & 0 & n\mu & -n\mu \end{bmatrix}$$

c)

$$\begin{aligned}
(\alpha + j\mu)P_j &= \alpha P_{j-1} + (n+1)\mu P_{j+1} \qquad 0 < j \le n-1 \\
\alpha P_0 &= \mu P_1
\end{aligned}$$

Following M/M/1 example,

$$\begin{aligned}
j\mu P_j &= \alpha P_{j-1} \\
\Rightarrow P_j &= \frac{\alpha}{j\mu} P_{j-1} \\
\Rightarrow P_j &= \frac{(\alpha/\mu)^j}{j!} P_0
\end{aligned}$$

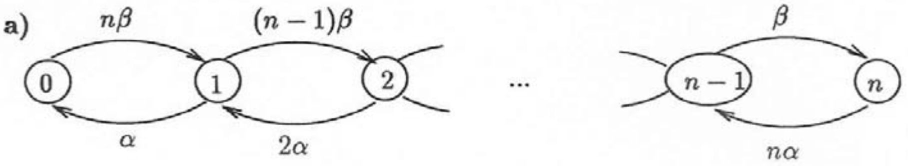
where

$$1 = \sum_{j=0}^{n} \frac{(\alpha/\mu)^j}{j!} P_0 \qquad P_0 = \frac{1}{\sum_{j=0}^{n} \frac{(\alpha/\mu)^j}{j!}}$$

11.47

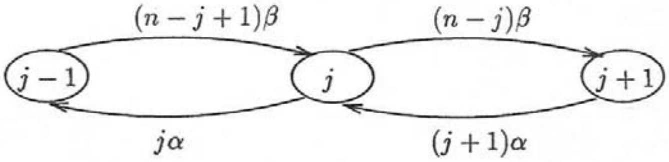
~~8.18~~ Suppose $N(t) = k$ speakers are active, then from Problem 11.46a the time until the next speaker goes silent is an exponential RV with rate $k\alpha$ and the transition rate from state k to $k-1$ is $k\alpha$. When $N(t) = k$, $n-k$ speakers are silent, and the time till the next speaker goes active is an exponential RV with rate $(n-k)\beta$; thus the transition rate from k to $k+1$ is $(n-k)\beta$.

a)



$$[\gamma_{ij}] = \begin{bmatrix} -n\beta & n\beta & 0 & 0 & \cdots & & 0 \\ \alpha & -(\alpha+(n-1)\beta) & (n-1)\beta & 0 & \cdots & & 0 \\ \alpha & -(\alpha+(n-1)\beta) & (n-1)\beta & 0 & \cdots & & 0 \\ 0 & 2\alpha & -(2\alpha+(n-2)\beta) & (n-2)\beta & \cdots & & 0 \\ \vdots & & & \ddots & \ddots & & \\ & & & & (n-1)\alpha & -(n-1)\alpha-\beta & \beta \\ 0 & & \cdots & & 0 & n\alpha & -n\alpha \end{bmatrix}$$

b)



$$\begin{aligned} [j\alpha + (n-j)\beta]P_j &= (n-j+1)\beta P_{j-1} + (j+1)\alpha P_{j+1} \qquad 0 < j < n-1 \\ n\beta P_0 &= \alpha P_1 \end{aligned}$$

Proceeding as in M/M/1 example, we can show that the above are equivalent to

$$\begin{aligned} (n-j)\beta P_j &= (j+1)\alpha P_{j+1} \\ \Rightarrow P_{j+1} &= \frac{n-j}{j+1}\left(\frac{\beta}{\alpha}\right)P_j \qquad (\star)\ \ j = 0, ..., n-1 \end{aligned}$$

Claim: $P_j = \binom{n}{j}\left(\frac{\beta}{\alpha}\right)^j P_0$

Proof: for $j = 0$ $\quad P_0 = P_0 \surd$
Assume P_j as above, then $(\star)$ implies

$$\begin{aligned}
P_{j+1} &= \frac{n-j}{j+1}\left(\frac{\beta}{\alpha}\right)\frac{n!}{(n-j)!j!}\left(\frac{\beta}{\alpha}\right)^j P_0 \\
&= \frac{n!}{(n-j-1)!(j+1)!}\left(\frac{\beta}{\alpha}\right)^{j+1} P_0 \\
&= \binom{n}{j+1}\left(\frac{\beta}{\alpha}\right)^{j+1} P_0 \quad \surd \\
1 &= P_0\sum_{j=1}^{n}\binom{n}{j}\left(\frac{\beta}{\alpha}\right)^j = P_0\left(\frac{\beta}{\alpha}+1\right)^n \\
&\Rightarrow P_0 = \left(\frac{\alpha}{\alpha+\beta}\right)^n \\
\Rightarrow P_j &= \binom{n}{j}\left(\frac{\beta}{\alpha}\right)^j\left(\frac{\alpha}{\alpha+\beta}\right)^n = \binom{n}{j}\left(\frac{\beta}{\alpha+\beta}\right)^j\left(\frac{\alpha}{\alpha+\beta}\right)^{n-j} \qquad 0 \le j \le n
\end{aligned}$$

This is a binomial distribution for n independent speakers where each speaker is active with probability $\dfrac{\beta}{\alpha+\beta}$.

11.48 (a) Assuming $\delta << 1$ we can recognize three situations in each state. One arrival can come with the probability of $\lambda\delta$. One departure can occure with Prob. $\mu\delta$, and no-arrival or departure with prob. $1-(\mu+\lambda)\delta$.

The Transition prob. matrix can be written as:

$P_{j,j+1} = \lambda\delta$ $\quad P_{j,j-1} = \delta\mu$ $\quad P_{j,j} = 1-(\lambda+\mu)\delta$ $\quad P_{00} = (1-\lambda)\delta$, $P_{01} = \lambda\delta$

$$P = \begin{pmatrix} (1-\lambda)\delta & \delta\lambda & 0 & 0 & 0 & 0 & \cdots \\ \delta\mu & 1-(\lambda+\mu)\delta & \delta\lambda & 0 & 0 & 0 \\ 0 & \delta\mu & 1-(\lambda+\mu)\delta & \delta\lambda & 0 & \cdot \\ 0 & 0 & \delta\mu & 1-(\lambda+\mu)\delta & \delta\lambda & \\ \vdots & \vdots & \vdots & & \cdots & \\ \vdots & \vdots & \vdots & & \cdots & \end{pmatrix}$$

(b) $\pi = \pi P$ $\qquad \pi = (\pi_0, \pi_1, \cdots)$

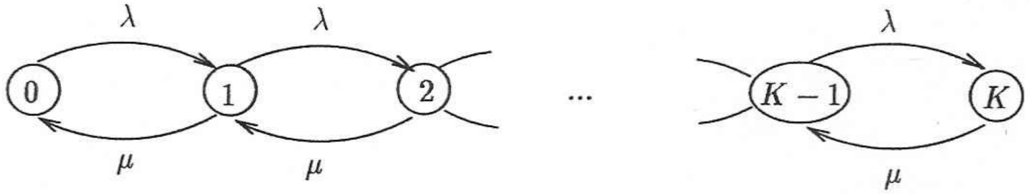
$$\Rightarrow \begin{cases} (1-\lambda)\delta\,\pi_0 + \delta\mu\,\pi_1 = \pi_0 & \Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0 \\ \delta\lambda\,\pi_0 + [1-(\lambda+\mu)\delta]\pi_1 + \delta\mu\pi_2 = \pi_1 \\ \vdots \\ \delta\lambda\,\pi_{i-1} + [1-(\lambda+\mu)\delta]\pi_i + \delta\mu\,\pi_{i+1} = \pi_i \Rightarrow \end{cases}$$

$$\Rightarrow \delta\lambda\,\pi_{i-1} + \delta\mu\,\pi_{i+1} = (\lambda+\mu)\delta\,\pi_i$$

$$\begin{cases} \pi_1 = \frac{\lambda}{\mu}\pi_0 = \rho\pi_0 \\ \lambda\pi_{i-1} + \mu\,\pi_{i+1} = (\lambda+\mu)\pi_i \end{cases} \Rightarrow \boxed{\pi_i = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i = (1-\rho)(\rho)^i}$$

11.49

~~**8.19**~~ The transition rate diagram is:



Eqn.11.42 applies here, so we have

$$P_{j+1} = \left(\frac{\lambda}{\mu}\right) P_j = \left(\frac{\lambda}{\mu}\right)^{j+1} P_0$$

To find P_0 consider

$$\begin{aligned} 1 &= P_0 \sum_{j=0}^{K} \left(\frac{\lambda}{\mu}\right)^j = P_0 \frac{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\frac{\lambda}{\mu}} \\ \Rightarrow P_j &= \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} \left(\frac{\lambda}{\mu}\right)^j \qquad 0 \le j \le K \end{aligned}$$

11.50

~~**8.24**~~ The transition rate diagram for the continuous-time process is:

Let

T_+ be exponential RV with parameter λ
T_- be exponential RV with parameter μ

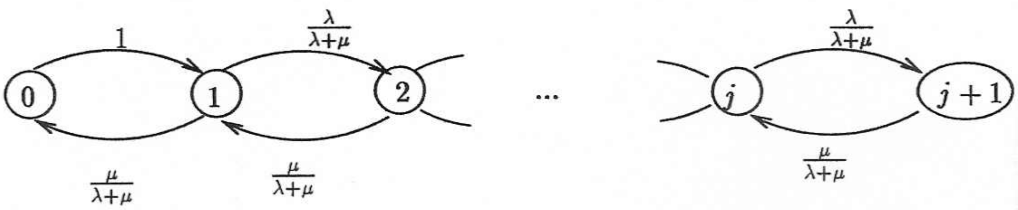
Then

$$P_{i,i+1} = P[T_+ < T_-] = 1 - P_{i,i-1}$$

From the solution of Problem 9.37 we know that

$$P[T_+ < T_-] = \frac{\lambda}{\lambda + \mu}$$

Thus transition probability diagram for the embedded process is:



This discrete-time Markov chain is the one discussed in Problem 11.30 with $M = \infty$, thus letting $p = \dfrac{\lambda}{\lambda+\mu} < \dfrac{\lambda}{\lambda+\lambda} = \dfrac{1}{2}$

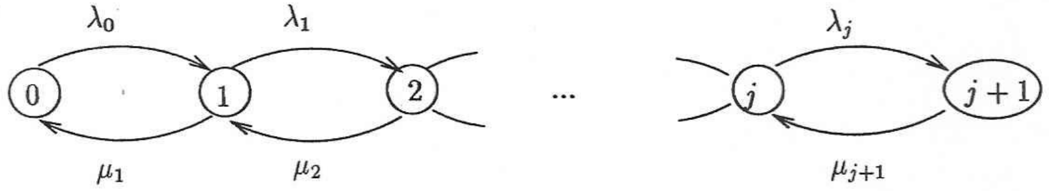
$$\pi_i = \frac{p^{i-1}}{q^i}\pi_0$$

and

$$\begin{aligned}
1 &= \pi_0\left[1 + \frac{1}{q}\underbrace{(1 + \frac{p}{q} + \frac{p}{q^2} + \ldots)}_{\frac{1}{1-\frac{p}{q}}}\right] \\
1 &= \pi_0\left[1 + \frac{1}{q-p}\right] = \pi_0\left[1 + \frac{1}{1-2p}\right] \\
&= \pi_0\frac{1-2p+1}{1-2p} \\
\pi_0 &= \frac{1-2p}{2(1-p)} = \frac{1-\frac{2\lambda}{\lambda+\mu}}{2\frac{\mu}{\lambda+\mu}} = \frac{\lambda+\mu-2\lambda}{2\mu} = \frac{\mu-3}{2\mu} \\
&= \frac{1-p}{2} \\
\pi_i &= \frac{1}{q}\left(\frac{p}{q}\right)^{i-1}\pi_0 = \frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^{i-1}\frac{\mu-\lambda}{2\mu} \\
P_0 &= c\frac{\mu-\lambda}{2\mu}\frac{1}{\lambda} = c\frac{(1-\rho)}{2\lambda} \\
P_i &= c\frac{\lambda+\mu}{\mu}\left(\frac{\lambda}{\mu}\right)^{i-1}\left(\frac{1-\rho}{2\mu}\right)\frac{1}{\lambda+\mu} \\
&= c(1-\rho)\frac{1}{2\mu}\left(\frac{\lambda}{\mu}\right)^{i-1} \\
1 &= \frac{c(1-\rho)}{2}\underbrace{\left[\frac{1}{\lambda} + \frac{1}{\lambda}\underbrace{[1+\rho+\rho^2+\ldots]}_{\frac{1}{1-\rho}=\frac{\mu}{\mu-\lambda}}\right]}_{\frac{\mu-\lambda+\lambda}{\lambda(\mu-\lambda)}=\frac{\mu}{\lambda(\mu-\lambda)}=\frac{1}{\lambda}\left(\frac{1}{1-\rho}\right)} \\
1 &= \frac{c}{2\lambda} \Rightarrow c = 2\lambda \\
&\Rightarrow P_i = (1-\rho)\rho^i \qquad \surd
\end{aligned}$$

11.51

~~8.25~~



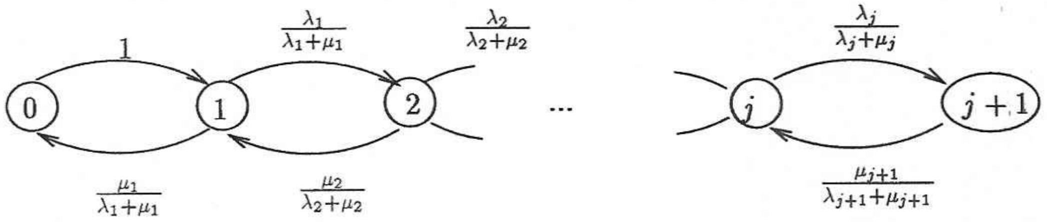
Let

$T_+(j)$ be exp RV with rate λ_j
$T_-(j)$ be exp RV with rate μ_j

then

$$P_{j,j+1} = P[T_+(j) < T_-(j)] = \frac{\lambda_j}{\lambda_j + \mu_j} = 1 - P_{j,j-1}$$

The transition probability diagram for the embedded draw is:



$$\begin{aligned}
\pi_0 &= \frac{\mu_1}{\lambda_1+\mu_1}\pi_1 \\
\pi_1 &= \pi_0 + \frac{\mu_2}{\lambda_2+\mu_2}\pi_2 \\
\pi_j &= \frac{\lambda_{j-1}}{\lambda_{j-1}+\mu_{j-1}}\pi_{j-1} + \frac{\mu_{j+1}}{\lambda_{j+1}+\mu_{j+1}}\pi_{j+1}
\end{aligned}$$

Substitute

$$\pi_j = \frac{P_j\nu_j}{c} = \frac{P_j(\lambda_j+\mu_j)}{c} \qquad j > 0 \qquad \pi_0 = \frac{P_0\lambda_0}{c}$$

$$\left.\begin{aligned}
\lambda_0 P_0 &= \mu_1 P_1 \\
(\lambda_j+\mu_j)P_j &= \lambda_{j-1}P_{j-1} + \mu_{j+1}P_{j+1}
\end{aligned}\right\} \Rightarrow \text{global balance eqns.}$$

$\therefore$ solution to π_j eqns. consistent with P_j's found in Ex. 11.40.

(11.52) a) Occ. time is exp. with $\mu = 1$

$$\Rightarrow \nu_i = \frac{1}{E(T_i)} = 1$$

$$P_i = \frac{\pi_i/\nu_i}{\sum_j \frac{\pi_j}{\nu_j}} \quad \text{where from 11.36} \quad \pi_i = \frac{1-f_1-f_2-\cdots-f_{i-1}}{A}$$

$$A = 1 + 1 + (1-f_1) + (1-f_1-f_2) + (1-f_1-f_2-f_3) + \cdots)$$

$$\Rightarrow P_i = \frac{\pi_i}{\sum_j \pi_j} = \pi_i = \frac{1-\sum_{k=1}^{i-1} f_k}{A} = \frac{1-g_i}{A}$$

On the other hand f_i is the pmf of a geometric dist.

$$\text{Then: } g_i = \sum_{k=1}^{i-1} f_k = \sum_{k=1}^{i-1} p(1-p)^{k-1} = p \times \frac{1-q^k}{1-q} = 1-q^k$$

$$\Rightarrow 1-g_i = q^k,$$

$$\text{and } A = 2 + \sum_{i=1}^{\infty}(1-g_i) = 2 + \sum_{i=1}^{\infty} q^i = 2 + \frac{q}{1-q} = \frac{2-q}{1-q}$$

$$\Rightarrow \boxed{P_i = \frac{1-g_i}{A} = q^i\,\frac{1-q}{2-q}}$$

(b) In this part $\nu_i = \frac{1}{E(T_i)} = \frac{1}{i}$

$$P_i = \frac{\frac{\pi_i}{i}}{\sum_j \pi_j/j} = \frac{\frac{1}{i}q^i}{\sum_{j=1}^{\infty}\frac{1}{j}q^j} = \boxed{\frac{1}{i}\,\frac{q^i}{-\ln(1-q)} = \frac{q^i}{i\ln(p)}}$$

$$\text{Since } \frac{1}{1-x} = 1+x+x^2+\cdots \Rightarrow \int\frac{dx}{1-x} = \left(x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\cdots\right)$$

$$= -\ln(1-x)$$

$$\Rightarrow \sum_{j=1}^{\infty}\frac{1}{j}q^j = -\ln(1-q) = -\ln p$$

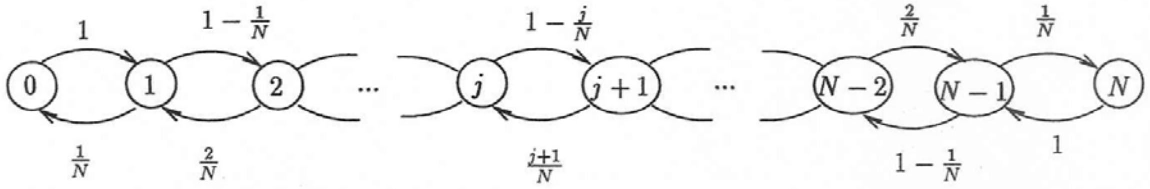
11.52 (c) $V_i = \frac{1}{E(T_i)} = 2^{-i}$

$$P_i = \frac{\frac{\pi_i}{v_i}}{\sum_j \frac{\pi_j}{v_j}} = \frac{2^{-i} q^i}{\sum_{j=1}^{\infty} 2^{-j} q^j} = \frac{2^{-i} q^i}{\sum_{j=1}^{\infty} (q/2)^j} = \frac{2^{-i} q^i}{q/2 \frac{1}{1-q/2}}$$

$$\Rightarrow \boxed{P_i = \frac{2^{-i} q^{i-1}}{2-q}}$$

*11.5 Time-Reversed Markov Chains

11.53 **a)** The state transition diagram is:



This is a birth-death process, so by Example 11.44 it is reversible. If process is time reversible then

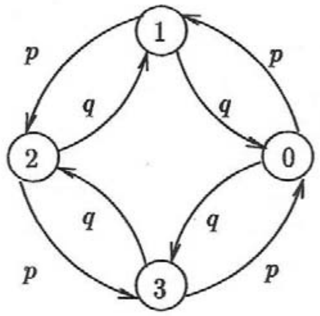
$$\begin{aligned}\pi_i p_{ij} &= \pi_j p_{ji}\\ \Rightarrow \pi_i p_{i,i+1} &= \pi_{i+1} p_{i+1,i}\\ \pi_i\left(1-\frac{1}{N}\right) &= \pi_{i+1}\frac{i+1}{N}\\ \pi_{i+1} &= \frac{N-i}{i+1}\pi_i\end{aligned}$$

It is then easy to show that

$$\pi_j = \begin{pmatrix} N \\ j \end{pmatrix}\pi_0$$

$$\begin{aligned}1 = \pi_0\sum_{j=0}^{N}\begin{pmatrix} N \\ j \end{pmatrix} &= \pi_0 2^N\\ \Rightarrow \pi_j &= \begin{pmatrix} N \\ j \end{pmatrix}\left(\frac{1}{2}\right)^N\end{aligned}$$

11.54



a)
$$\left.\begin{aligned}\pi_0 &= q\pi_1 + p\pi_3\\ \pi_1 &= q\pi_2 + p\pi_0\\ \pi_2 &= q\pi_3 + p\pi_1\\ \pi_3 &= q\pi_0 + p\pi_2\end{aligned}\right\} \Rightarrow \pi_0 = \pi_1 = \pi_2 = \pi_3 = \frac{1}{4}$$

b) Clearly if $p > q$ then point tends to circulate counter clockwise whereas if $p < q$ it tends to circulate clockwise. Thus we expect that the process is time reversible only when $p = q = \frac{1}{2}$.

Indeed the condition for time reversibility requires that:

$$\pi_0 p_{01} = \pi_1 p_{10} \Leftrightarrow \underbrace{\pi_0}_{\frac{1}{4}} p = \underbrace{\pi_1}_{\frac{1}{4}} q \Leftrightarrow p = q$$

and similarly for the other states.

11.55

$$\begin{aligned} q_{ij} &= \frac{\pi_i p_{ji}}{\pi_i} = \frac{\left(\frac{p}{q}\right)^j p_{ji}}{\left(\frac{p}{q}\right)^i}\\ \Rightarrow q_{i,i+1} &= \frac{\left(\frac{p}{q}\right)^{i+1}}{\left(\frac{p}{q}\right)^i} p_{i+1,i} = \frac{p}{q} q = p = p_{i,i+1}\\ q_{i,i-1} &= \frac{\left(\frac{p}{q}\right)^{i-1}}{\left(\frac{p}{q}\right)^i} p_{i-1,i} = \frac{q}{p} p = q = p_{i,i-1}\\ q_{00} &= q = p_{00}\end{aligned}$$

$\Rightarrow$ Yes, process is time reversible.

11.56

From the solution of problem 11.16 we have

$$\pi_i = \frac{\binom{\rho}{i}^2}{\binom{2\rho}{\rho}} \quad \text{and} \quad \begin{cases} P_{i,i+1} = \dfrac{(\rho-i)^2}{\rho^2} \\ P_{i+1,i} = \dfrac{(i+1)^2}{\rho^2} \end{cases} \quad \text{and } P_{ij} = 0 \text{ for all } j \neq i \ \& \ j \neq {}^{i+1}_{i-1}$$

one can see: $\pi_i P_{i,i+1} = \underset{j=i+1}{\pi_i P_{ij}} = \dfrac{\binom{\rho}{i}^2}{\binom{2\rho}{\rho}} \dfrac{(\rho-i)^2}{\rho^2}$

and $\pi_j P_{ji} = \pi_{i+1} P_{i+1,i}$

$$= \frac{\binom{\rho}{i+1}^2}{\binom{2\rho}{\rho}} \times \frac{(i+1)^2}{\rho^2}$$

$$= \frac{\dfrac{\rho!^2}{(i+1)!^2\,(\rho-i-1)!^2}}{\binom{2\rho}{\rho}} \times \frac{(i+1)^2}{\rho^2}$$

$$= \frac{1}{\binom{2\rho}{\rho}} \times \frac{\rho!^2}{i!^2\,(\rho-i)!^2} \times \frac{(\rho-i)^2}{\cancel{(i+1)}^2} \times \frac{\cancel{(i+1)}^2}{\rho^2}$$

$$= \frac{\binom{\rho}{i}^2}{\binom{2\rho}{\rho}} \times \frac{(\rho-i)^2}{\rho^2}$$

$$= \pi_i \times P_{ij}$$

$\Rightarrow \pi_j P_{ji} = \pi_i P_{ij} \rightarrow$ time-reversible ✓

11.57 For the Markov chain of problem 11.17 we have:

$$P = \begin{pmatrix} q & p & 0 & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \\ 0 & 0 & q & p \end{pmatrix} \Rightarrow \pi = (q^2, qp, qp, p^2)$$

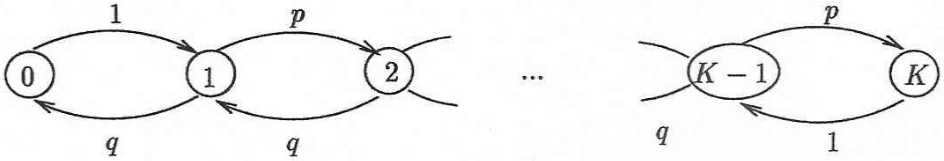
for these (i,j) pairs P_{ij} is non-zero $\begin{cases}(0,1)\\(0,0)\end{cases}, \begin{cases}(2,3)\\(2,2)\end{cases}, \begin{cases}(3,1)\\(3,2)\end{cases}, \begin{cases}(3,2)\\(3,3)\end{cases}$

where $\begin{cases} P_{01} = p \\ P_{10} = 0 \end{cases}$ $\quad \left.\begin{array}{l} \pi_0 P_{01} = q^2 \times p = q^2 p \\ \pi_1 P_{10} = qp \times 0 = 0 \end{array}\right\} \Rightarrow \pi_0 P_{01} \neq P_1 P_{10}$

$\Rightarrow$ it is not time-reversible

11.58

~~8.29~~ **a)** The transition probability diagram for the embedded chain of the process in Problem 49 is:



where $p = \dfrac{\lambda}{\lambda+\mu}$, $q = \dfrac{\mu}{\lambda+\mu}$ (see Problem 11.50a).

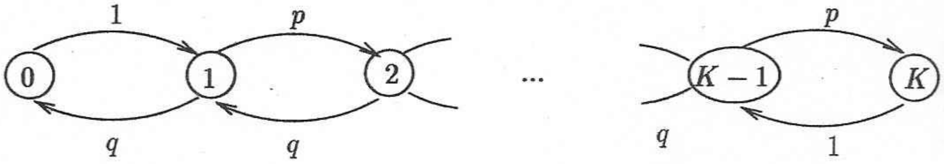
In Problem 11.30 we showed that

$$
\begin{aligned}
x_0 &= \frac{1-2p}{2\left(1-p\left(1+\left(\frac{p}{q}\right)^{K-1}\right)\right)} \\
\pi_i &= \frac{1}{q}\left(\frac{p}{q}\right)^{i-1}\pi_0 \qquad 1 \le i \le K-1 \\
\pi_K &= \left(\frac{p}{q}\right)^{K-1}\pi_0
\end{aligned}
$$

$\therefore$ the transition probabilities for the time reversed process are:

$$
\begin{aligned}
q_{01} &= \frac{\pi_1}{\pi_0}\tilde{q}_{10} = \frac{1}{q}q = 1 \\
q_{10} &= \frac{\pi_0}{\pi_1}\tilde{q}_{01} = q \cdot 1 = q \\
q_{i,i+1} &= \frac{\pi_{i+1}}{\pi_i}\tilde{q}_{i+1,i} = \frac{p}{q}q = p \\
q_{i,i-1} &= \frac{\pi_{i-1}}{\pi_1}\tilde{q}_{i-1,i} = \frac{q}{p}p = q \\
q_{K-1,K} &= \frac{\pi_K}{\pi_{K-1}}\tilde{q}_{K,K-1} = p \cdot 1 = p \\
q_{K,K-1} &= \frac{\pi_{K-1}}{\pi_K}\tilde{q}_{K-1,K} = \frac{1}{p}p = 1
\end{aligned}
$$

$\Rightarrow$ transition probability diagram for reverse process is:



This is identical to forward process ⇒ process is reversible.

b) Eq. (11.67) implies

$$\begin{aligned} p_i \gamma_{i,i+1} &= p_{i+1}\gamma_{i+1,i} \\ \Rightarrow \lambda p_i &= \mu p_{i+1} \\ \Rightarrow p_{i+1} &= \frac{\lambda}{\mu} p_i \qquad i \geq 0 \end{aligned}$$

which implies that $p_{i+1} = \left(\frac{\lambda}{\mu}\right)^{i+1} p_0$ and

$$p_i = \frac{1 = \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} \left(\frac{\lambda}{\mu}\right)^i$$

as in Problem 11.49.

(11.59) The Markov process in 11.37i is a time-reversed version of Ex 11.42 (Figure 11.15a)
where $a_j = 1 - b_j$
From Ex 11.49 we have $\pi_j = \dfrac{P(X \geq i)}{E(X)}$

$$\pi_i P_{ij} = \pi'_j q_{ji} \Rightarrow \pi_i P_{i,i+1} = \pi'_{i+1} q_{i+1,i}$$

$$\left.\begin{array}{l} \dfrac{P(X \geq i)}{E(X)} \times (1 - b_i) = \pi'_{i+1} \times a_i \\ \qquad 1 - b_i = a_i \end{array}\right\} \Rightarrow \pi'_{i+1} = \frac{P(X \geq i)}{E(X)}$$

$$\text{or} \quad \pi'_i = \frac{P(X \geq i-1)}{E(X)}$$

11.60

From Ex. 11.36 we have $\begin{cases} P_0 = \dfrac{\beta}{\alpha+\beta} & \gamma_{01} = \alpha \\ P_1 = \dfrac{\alpha}{\alpha+\beta} & \gamma_{10} = \beta \end{cases}$

$$\Rightarrow P_0 \gamma_{01} = \frac{\beta}{\alpha+\beta} \times \alpha = \frac{\alpha}{\alpha+\beta} \times \beta = P_1 \gamma_{10} \Rightarrow \text{reversible}$$

11.61

From problem 11.46 $\begin{cases} \gamma_{i,i+1} = \alpha \\ \gamma_{i+1,i} = (i+1)\mu \end{cases}$

$$P_i = \frac{(\alpha/\mu)^i}{i!} P_0$$

$$\Rightarrow P_i \gamma_{i,i+1} = \frac{\alpha \times \left(\frac{\alpha}{\mu}\right)^i}{i!} P_0$$

$$P_{i+1} \gamma_{i+1,i} = \frac{(\alpha/\mu)^{i+1}}{(i+1)!} P_0 \times \cancel{(i+1)}\mu = \frac{\alpha\left(\frac{\alpha}{\mu}\right)^i}{i!} P_0 = P_i P_{i,i+1}$$

$\Rightarrow$ reversible

11.62

From problem 11.47 $\begin{cases} \gamma_{i,i+1} = (n-i)\beta \\ \gamma_{i+1,i} = (i+1)\alpha \end{cases}$ & $P_i = \binom{n}{i}\left(\frac{\beta}{\alpha}\right)^i P_0$

$i \geq 3$

$$\Rightarrow P_i \gamma_{i,i+1} = \binom{n}{i}\left(\frac{\beta}{\alpha}\right)^i P_0 \times (n-i)\beta$$

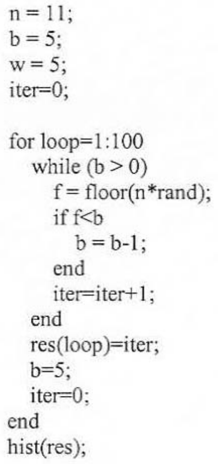
$$P_{i+1} \gamma_{i+1,i} = \binom{n}{i+1}\left(\frac{\beta}{\alpha}\right)^{i+1} P_0 \times (i+1)\alpha = \frac{n!}{(i+1)!\,(n-i-1)!}\left(\frac{\beta}{\alpha}\right)^i P_0\,(i+1)\beta$$

$$= \frac{n!\,(n-i)}{i!\,(n-i)!}\left(\frac{\beta}{\alpha}\right)^i P_0 \beta = \binom{n}{i}\left(\frac{\beta}{\alpha}\right)^i P_0 (n-i)\beta$$

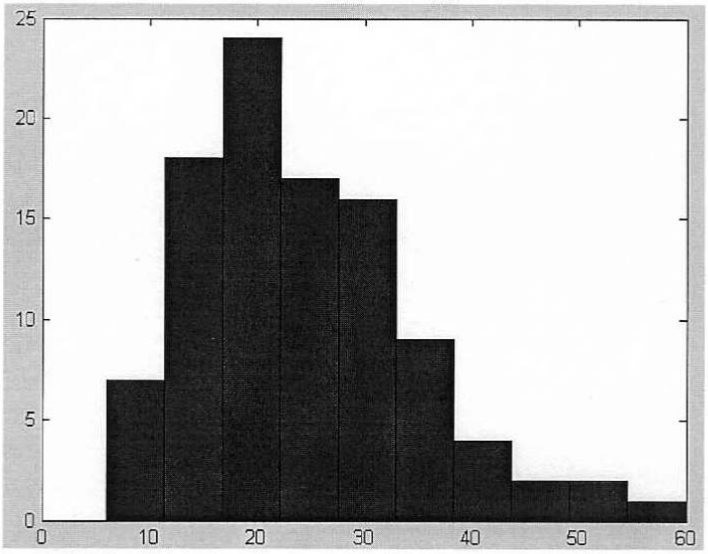
$$= P_i \gamma_{i,i+1}$$ reversible

11.6 Numerical Techniques for Markov Chains

11.63

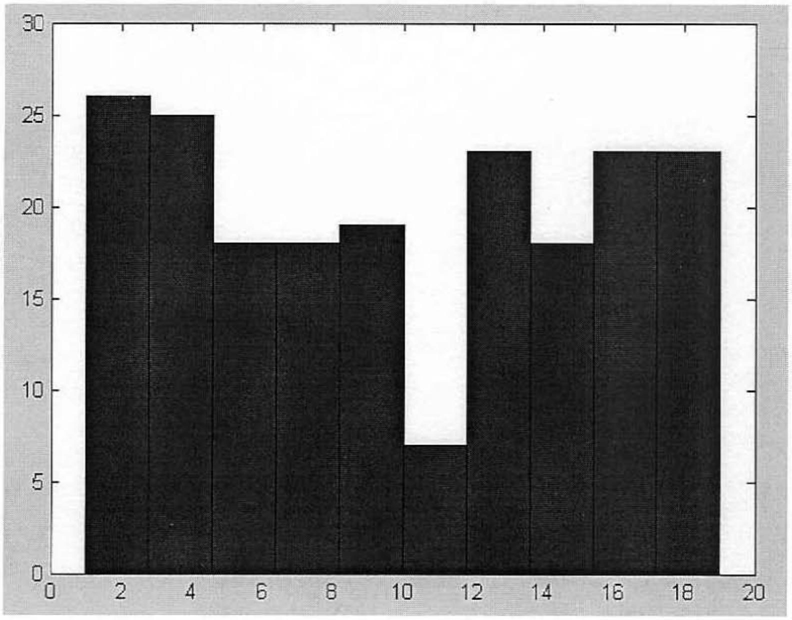


```
n = 11;
b = 5;
w = 5;
iter=0;

for loop=1:100
   while (b > 0)
      f = floor(n*rand);
      if f<b
         b = b-1;
      end
      iter=iter+1;
   end
   res(loop)=iter;
   b=5;
   iter=0;
end
hist(res);
```



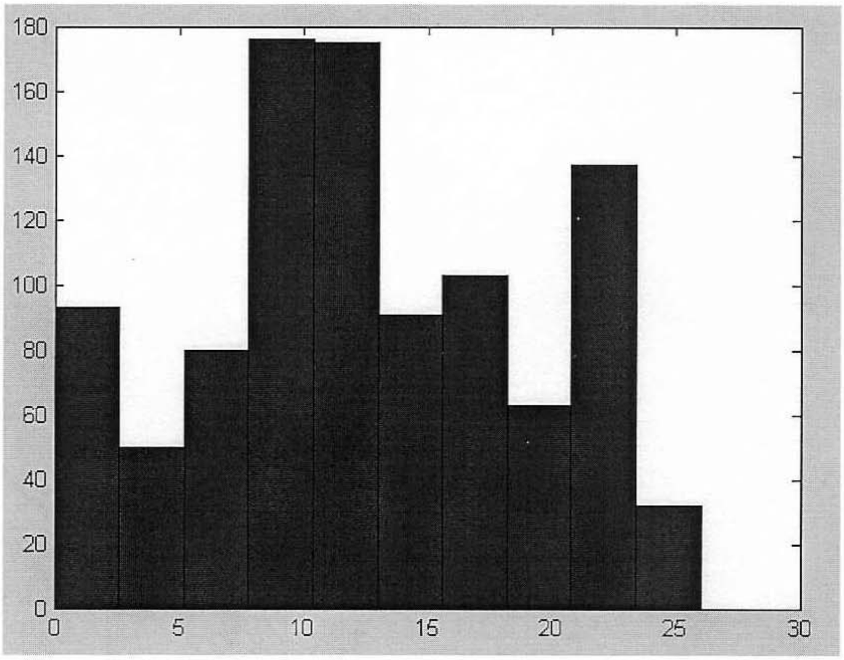
11.64

```
ro=20;

for i=1:200
    c1_w=floor((ro+1)*rand);
    c1_b=20-c1_w;
    c2_w=20-c1_w;
    c2_b=20-c2_w;

    f1=floor((ro+1)*rand);
    f2=floor((ro+1)*rand);

    if f1<=c1_w
        c2_w=c2_w+1;
        c1_w=c1_w-1;
    else
        c2_b=c2_b+1;
        c1_b=c1_b-1;
    end

    if f2<=c2_w
        c1_w=c1_w+1;
        c2_w=c2_w-1;
    else
        c1_b=c1_b+1;
        c2_b=c2_b-1;
    end

    res(i)=c1_w;
end

hist(res);
```



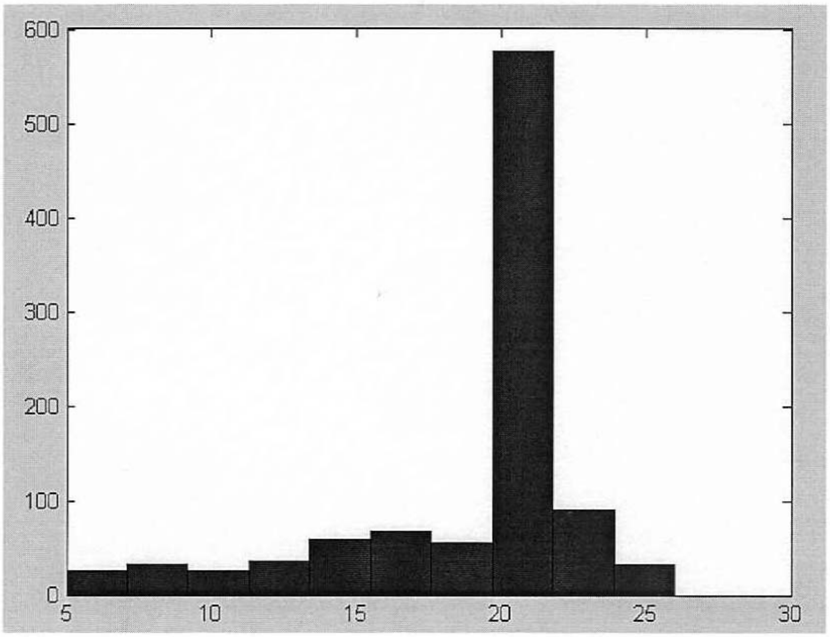
11.65

```
N=100;
b=0.5;
a=0.5;
iter=0;
X=0;

for i=1:1000
   p1=rand;
   if p1<a && X<N
      X=X+1;
   end

   p2=rand;
   if p2<b && X>0
      X=X-1;
   end

   res(i)=X;
end

hist(res);
```



11.66

```
a=10;
b=5;

for i=1:500
while a>0 && b>0
      f=rand;
      if f<0.5
         a=a+1;
         b=b-1;
      else
         b=b+1;
         a=a-1;
      end
      iter=iter+1;
   end
   res(i)=iter;
end
hist(res);
```

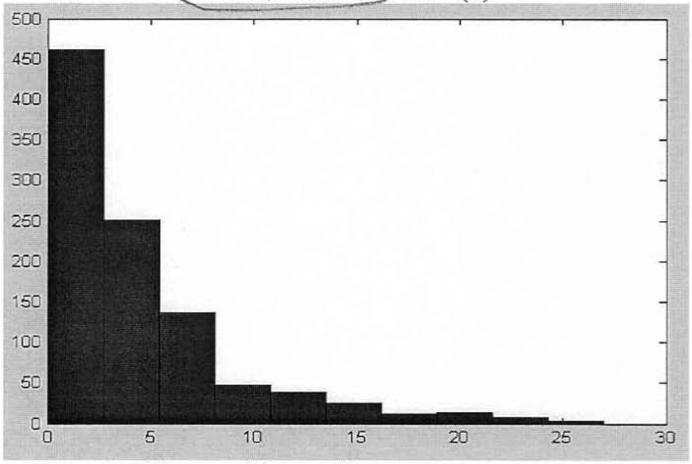


11.67

```
%For Geometric toggle=0, p=1/5. For Zipf toggle=1, p=5.
s=0;
N=1000;
toggle=0;
nxtS=zeros(N+1,1);
p=1/5;
for i=2:N+1
   if s==0
      if toggle==0
         s=randraw('geom',p,1);
      else
        s=randraw('zipf',p,1);
      end
      nxtS(i)=s;
   else
      s=nxtS(i-1)-1;
      nxtS(i)=s;
   end
end
hist(nxtS);
```

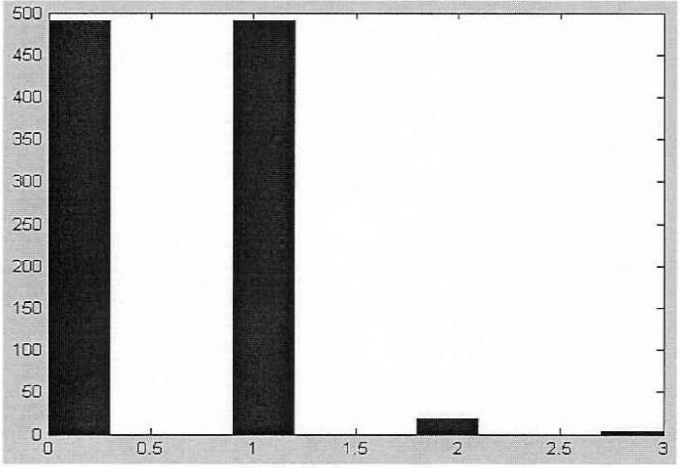
```
ac=cov(nxtS, nxtS);
```

11.67-(a)



Geometric with mean 5.

11.67-(b)

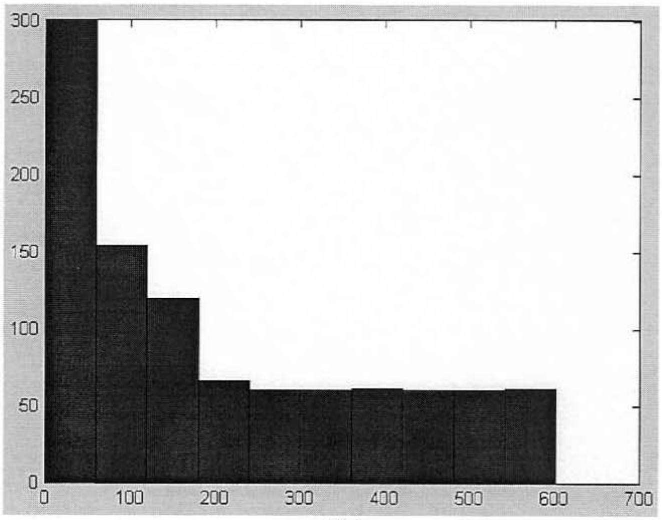


Zipf with parameter 5.

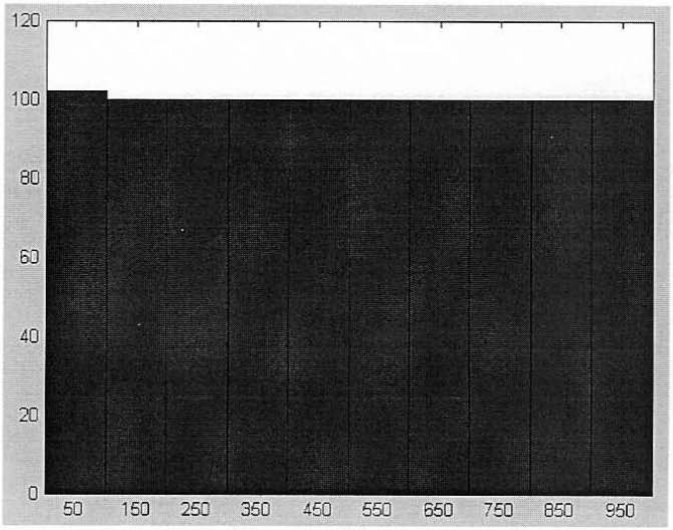
```
ac(geom) =[ 18.5959   18.5959          ac(zipf) =[  0.2759    0.2759
            18.5959   18.5959  ]                     0.2759    0.2759 ];
```

11.68

```
%For part (a) toggle=0. For part (2) toggle=1.
s=0;
a=0;
toggle=1;

for i=1:N+1
   if toggle==0
      a(i)=(i-1)/i;
   else
      a(i)=1-(1/2)^i;
   end
   if rand < a(i)
      s=s+1;
      nxtS(i+1)=s;
   else
      s=1;
      nxtS(i+1)=s;
   end
end
hist(nxtS);
```

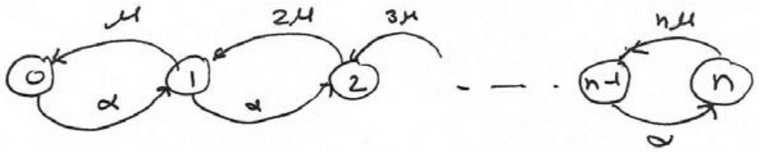


part (a)



part (b)

11.70



$$\Gamma = \begin{pmatrix} -\alpha & \alpha & 0 & 0 & - & - \\ \mu & -(\alpha+\mu) & \alpha & 0 & - & - \\ 0 & 2\mu & -(\alpha+2\mu) & \alpha & & \\ \vdots & \vdots & \vdots & \ddots & & \\ - & 0 & 0 & (n-1)\mu & & \alpha \\ 0 & 0 & 0 & 0 & n\mu & -n\mu \end{pmatrix}$$

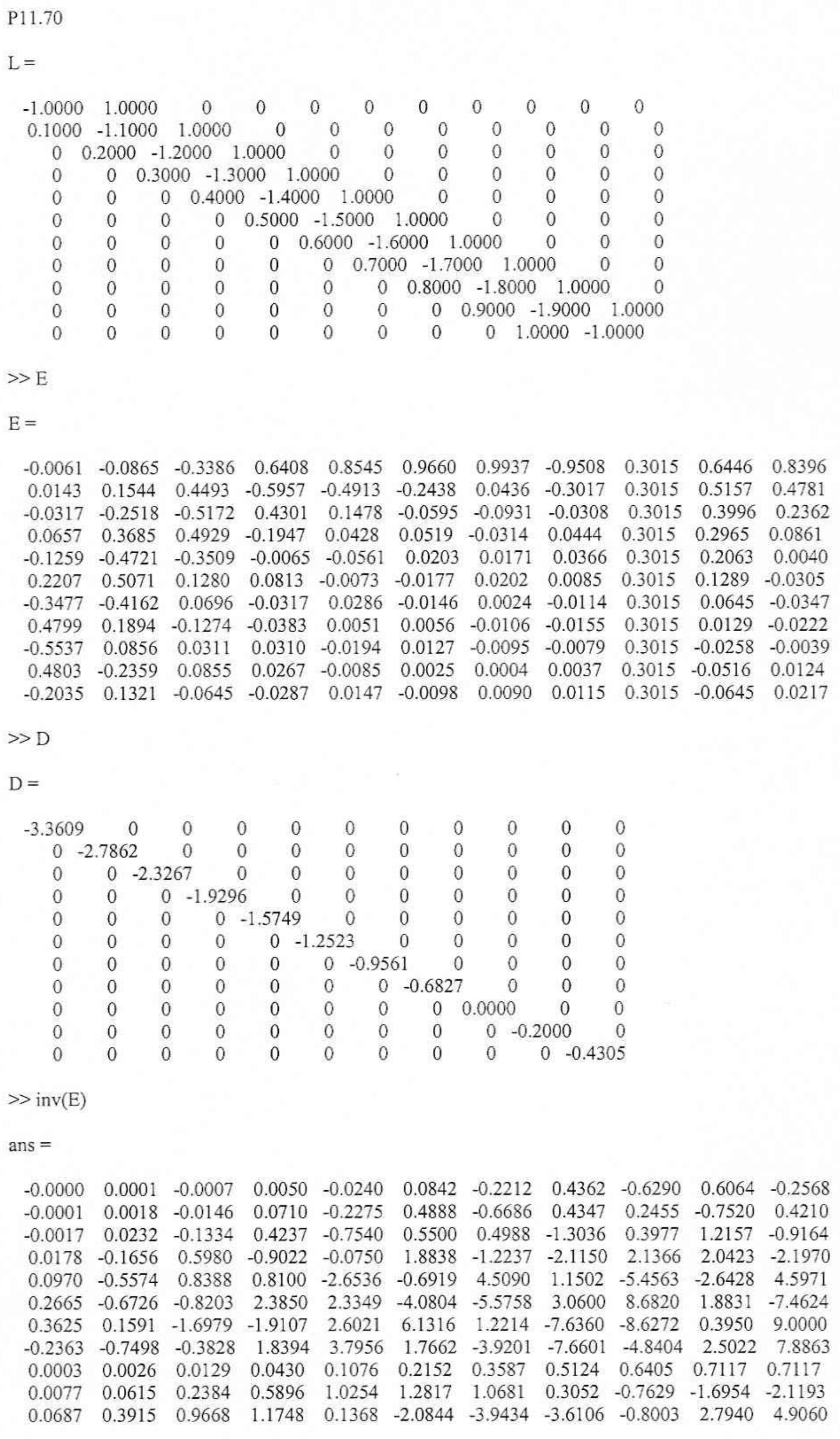
$n=10$
$\mu=0.1$
$\alpha=1$

$$\Rightarrow \Gamma = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & - & - & 0 \\ 0.1 & -1.1 & 1.0 & & & & & \\ 0 & 0.2 & -1.2 & 1.0 & 0 & 0 & - & \\ & & \ddots & \ddots & \ddots & & & \vdots \\ 0 & & & & 0 & 0.9 & -1.9 & 1 \\ 0 & & - & - & - & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{cases} (\alpha + i\mu) P_i = (n+1)\mu P_{i+1} + \alpha P_{i-1} & 0 \le i \le n-1 \\ \alpha P_0 = \mu P_1 \end{cases}$$

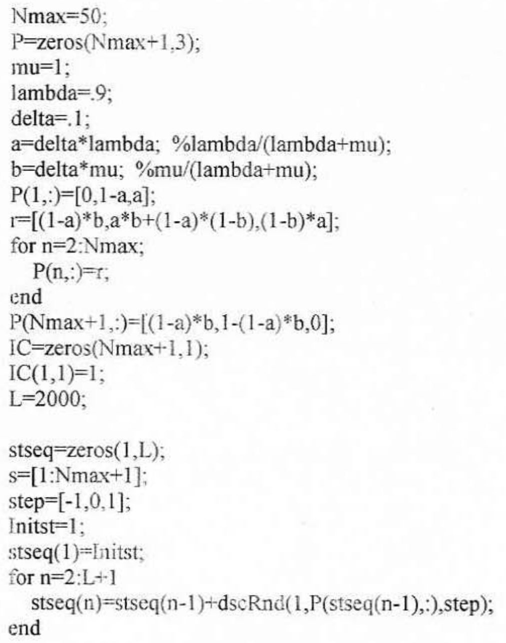
$\hookrightarrow$ recursive equations to program P_i

```
%P 1170
alpha=1;
mu=1/10;
n=10;

v=(alpha+mu);

L=[-alpha alpha 0 0 0 0 0 0 0 0 0
    mu -(alpha+mu) alpha 0 0 0 0 0 0 0 0
    0 2*mu -(alpha+2*mu) alpha 0 0 0 0 0 0 0
    0 0 3*mu -(alpha+3*mu) alpha 0 0 0 0 0 0
    0 0 0 4*mu -(alpha+4*mu) alpha 0 0 0 0 0
    0 0 0 0 5*mu -(alpha+5*mu) alpha 0 0 0 0
    0 0 0 0 0 6*mu -(alpha+6*mu) alpha 0 0 0
    0 0 0  0 0 0 7*mu -(alpha+7*mu) alpha 0 0
    0 0 0 0 0 0 0 8*mu -(alpha+8*mu) alpha 0
    0 0 0 0 0 0 0 0 9*mu -(alpha+9*mu) alpha
    0 0 0 0 0 0 0 0 0 10*mu -10*mu];

[E D]=eig(L);
t=sym('t');

N0=1;
p0=zeros(1,n+1);
p0(N0)=1;
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(E,t,D);
% Example: p0*f(D,E,2) will compute the amount of p(2)
```

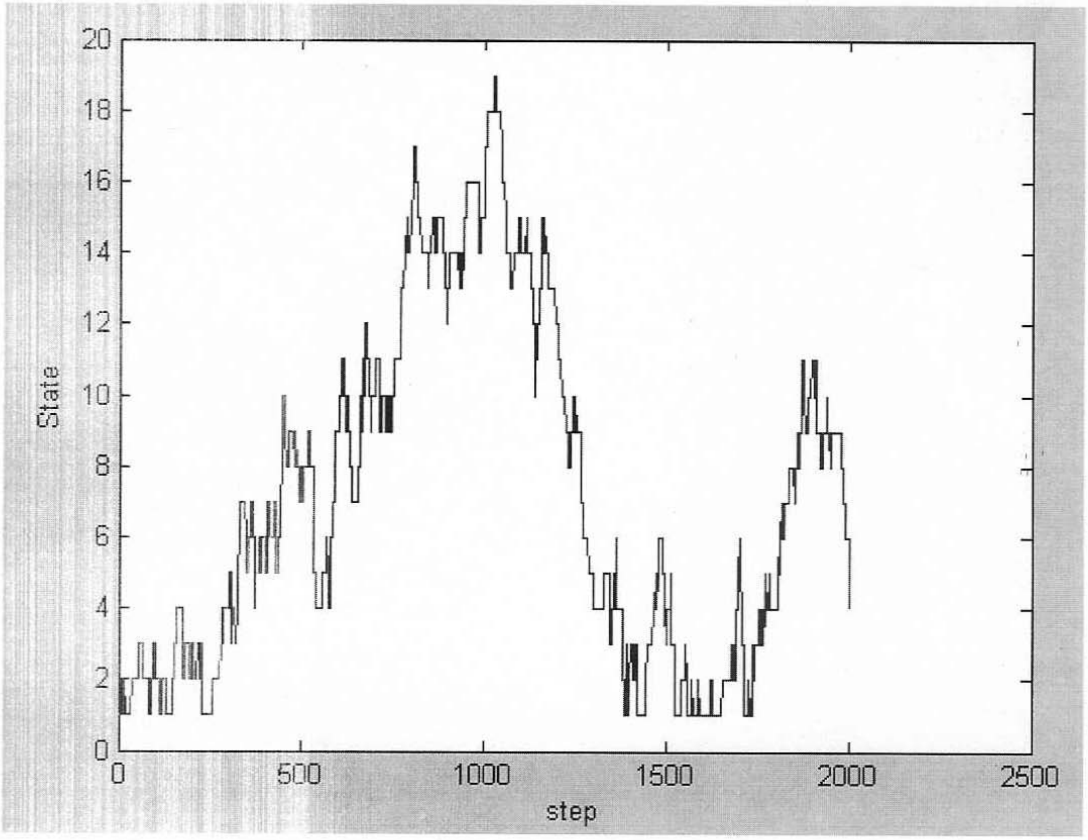


```
P11.70

L =

  -1.0000   1.0000        0        0        0        0        0        0        0        0        0
   0.1000  -1.1000   1.0000        0        0        0        0        0        0        0        0
        0   0.2000  -1.2000   1.0000        0        0        0        0        0        0        0
        0        0   0.3000  -1.3000   1.0000        0        0        0        0        0        0
        0        0        0   0.4000  -1.4000   1.0000        0        0        0        0        0
        0        0        0        0   0.5000  -1.5000   1.0000        0        0        0        0
        0        0        0        0        0   0.6000  -1.6000   1.0000        0        0        0
        0        0        0        0        0        0   0.7000  -1.7000   1.0000        0        0
        0        0        0        0        0        0        0   0.8000  -1.8000   1.0000        0
        0        0        0        0        0        0        0        0   0.9000  -1.9000   1.0000
        0        0        0        0        0        0        0        0        0   1.0000  -1.0000

>> E

E =

  -0.0061  -0.0865  -0.3386   0.6408   0.8545   0.9660   0.9937  -0.9508   0.3015   0.6446   0.8396
   0.0143   0.1544   0.4493  -0.5957  -0.4913  -0.2438   0.0436  -0.3017   0.3015   0.5157   0.4781
  -0.0317  -0.2518  -0.5172   0.4301   0.1478  -0.0595  -0.0931  -0.0308   0.3015   0.3996   0.2362
   0.0657   0.3685   0.4929  -0.1947   0.0428   0.0519  -0.0314   0.0444   0.3015   0.2965   0.0861
  -0.1259  -0.4721  -0.3509  -0.0065  -0.0561   0.0203   0.0171   0.0366   0.3015   0.2063   0.0040
   0.2207   0.5071   0.1280   0.0813  -0.0073  -0.0177   0.0202   0.0085   0.3015   0.1289  -0.0305
  -0.3477  -0.4162   0.0696  -0.0317   0.0286  -0.0146   0.0024  -0.0114   0.3015   0.0645  -0.0347
   0.4799   0.1894  -0.1274  -0.0383   0.0051   0.0056  -0.0106  -0.0155   0.3015   0.0129  -0.0222
  -0.5537   0.0856   0.0311   0.0310  -0.0194   0.0127  -0.0095  -0.0079   0.3015  -0.0258  -0.0039
   0.4803  -0.2359   0.0855   0.0267  -0.0085   0.0025   0.0004   0.0037   0.3015  -0.0516   0.0124
  -0.2035   0.1321  -0.0645  -0.0287   0.0147  -0.0098   0.0090   0.0115   0.3015  -0.0645   0.0217

>> D

D =

  -3.3609        0        0        0        0        0        0        0        0        0        0
        0  -2.7862        0        0        0        0        0        0        0        0        0
        0        0  -2.3267        0        0        0        0        0        0        0        0
        0        0        0  -1.9296        0        0        0        0        0        0        0
        0        0        0        0  -1.5749        0        0        0        0        0        0
        0        0        0        0        0  -1.2523        0        0        0        0        0
        0        0        0        0        0        0  -0.9561        0        0        0        0
        0        0        0        0        0        0        0  -0.6827        0        0        0
        0        0        0        0        0        0        0        0   0.0000        0        0
        0        0        0        0        0        0        0        0        0  -0.2000        0
        0        0        0        0        0        0        0        0        0        0  -0.4305

>> inv(E)

ans =

  -0.0000   0.0001  -0.0007   0.0050  -0.0240   0.0842  -0.2212   0.4362  -0.6290   0.6064  -0.2568
  -0.0001   0.0018  -0.0146   0.0710  -0.2275   0.4888  -0.6686   0.4347   0.2455  -0.7520   0.4210
  -0.0017   0.0232  -0.1334   0.4237  -0.7540   0.5500   0.4988  -1.3036   0.3977   1.2157  -0.9164
   0.0178  -0.1656   0.5980  -0.9022  -0.0750   1.8838  -1.2237  -2.1150   2.1366   2.0423  -2.1970
   0.0970  -0.5574   0.8388   0.8100  -2.6536  -0.6919   4.5090   1.1502  -5.4563  -2.6428   4.5971
   0.2665  -0.6726  -0.8203   2.3850   2.3349  -4.0804  -5.5758   3.0600   8.6820   1.8831  -7.4624
   0.3625   0.1591  -1.6979  -1.9107   2.6021   6.1316   1.2214  -7.6360  -8.6272   0.3950   9.0000
  -0.2363  -0.7498  -0.3828   1.8394   3.7956   1.7662  -3.9201  -7.6601  -4.8404   2.5022   7.8863
   0.0003   0.0026   0.0129   0.0430   0.1076   0.2152   0.3587   0.5124   0.6405   0.7117   0.7117
   0.0077   0.0615   0.2384   0.5896   1.0254   1.2817   1.0681   0.3052  -0.7629  -1.6954  -2.1193
   0.0687   0.3915   0.9668   1.1748   0.1368  -2.0844  -3.9434  -3.6106  -0.8003   2.7940   4.9060
```

11.71

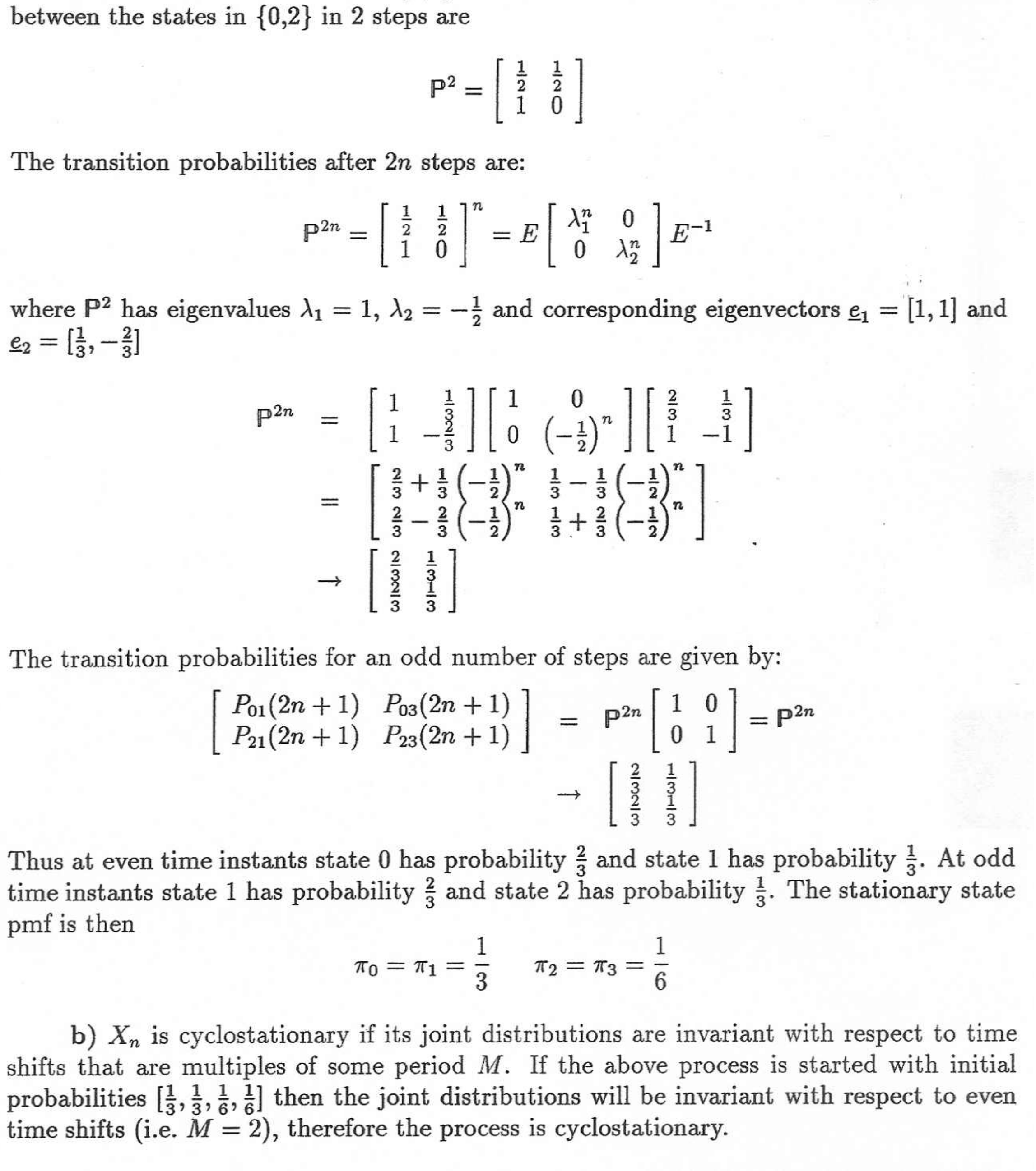


```
Nmax=50;
P=zeros(Nmax+1,3);
mu=1;
lambda=.9;
delta=.1;
a=delta*lambda;  %lambda/(lambda+mu);
b=delta*mu;  %mu/(lambda+mu);
P(1,:)=[0,1-a,a];
r=[(1-a)*b,a*b+(1-a)*(1-b),(1-b)*a];
for n=2:Nmax;
   P(n,:)=r;
end
P(Nmax+1,:)=[(1-a)*b,1-(1-a)*b,0];
IC=zeros(Nmax+1,1);
IC(1,1)=1;
L=2000;

stseq=zeros(1,L);
s=[1:Nmax+1];
step=[-1,0,1];
Initst=1;
stseq(1)=Initst;
for n=2:L+1
   stseq(n)=stseq(n-1)+dscRnd(1,P(stseq(n-1),:),step);
end
```



Problems Requiring Cumulative Knowledge

11.72



8.30 a) If the process starts in state 0 at times 0, then the process will be in states {0,2} at even time instants and in states {1,3} at odd time instants. The transition probabilities between the states in {0,2} in 2 steps are

$$\mathbb{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

The transition probabilities after $2n$ steps are:

$$\mathbb{P}^{2n} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}^n = E \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} E^{-1}$$

where $\mathbb{P}^2$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{2}$ and corresponding eigenvectors $\underline{e}_1 = [1,1]$ and $\underline{e}_2 = [\frac{1}{3}, -\frac{2}{3}]$

$$\begin{aligned} \mathbb{P}^{2n} &= \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n \\ \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n & \frac{1}{3} + \frac{2}{3}\left(-\frac{1}{2}\right)^n \end{bmatrix} \\ &\to \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

The transition probabilities for an odd number of steps are given by:

$$\begin{aligned} \begin{bmatrix} P_{01}(2n+1) & P_{03}(2n+1) \\ P_{21}(2n+1) & P_{23}(2n+1) \end{bmatrix} &= \mathbb{P}^{2n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{P}^{2n} \\ &\to \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

Thus at even time instants state 0 has probability $\frac{2}{3}$ and state 1 has probability $\frac{1}{3}$. At odd time instants state 1 has probability $\frac{2}{3}$ and state 2 has probability $\frac{1}{3}$. The stationary state pmf is then

$$\pi_0 = \pi_1 = \frac{1}{3} \qquad \pi_2 = \pi_3 = \frac{1}{6}$$

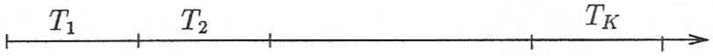
b) X_n is cyclostationary if its joint distributions are invariant with respect to time shifts that are multiples of some period M. If the above process is started with initial probabilities $[\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}]$ then the joint distributions will be invariant with respect to even time shifts (i.e. $M = 2$), therefore the process is cyclostationary.

11.73

~~8.31~~ X_n is an ergodic Markov chain $\Rightarrow$ positive recurrent, aperiodic, irreducible

$$\text{time average of } I_j(n) = \overline{I} = \frac{\sum_T I_j(n)}{T}$$

Suppose the process has returned to state 'i' times as shown below:



Then

$$T = \sum_{i=1}^{K} T_j(i)$$

where $T_j(i)$ where $T_j(i)$ are interarrival times to state 'j'.

Therefore,

$$\overline{I} = \frac{K}{\sum_{i=1}^{K} T_j(i)}$$

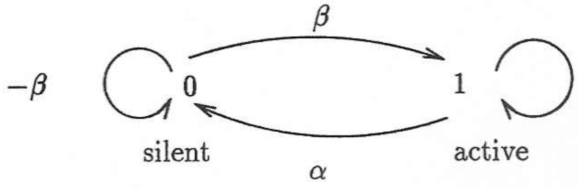
As

$$T \to \infty \Rightarrow \overline{I} \to \frac{1}{E[T_j]} = \pi_j$$

Therefore, the limiting value of time average of $I_j(n)$ is equal to long-term proportion of time spent in state 'j'. This result is an ergodic theorem.

11.74

.32 a) $X(t)$ has transition rate diagram:



b) The associated Chapman Kolmogorov equations are:

$$P_j'(t) = \sum_i \gamma_{ij} P_i(t)$$

$$\Rightarrow \begin{cases} P_0'(t) = -\beta P_0(t) + \alpha P_1(t) \\ P_1'(t) = t\alpha P_0(t) - \beta P_1(t) \end{cases}$$

$P_0(t)$ and $P_1(t)$ are obtained by solving this set of differential equations:

$$P_0(t) = P_0 + (P_0(0) - P_0)e^{-(\alpha+\beta)t}$$

$$P_1(t) = P_1 + (P_1(0) - P_1)e^{-(\alpha+\beta)t}$$

where $P_0 = \frac{\alpha}{\alpha+\beta}$ and $P_1 = \frac{\beta}{\alpha+\beta}$

c)
$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)\cdot X(t_2)] \\ &= \sum_{i=0}^{1}\sum_{j=0}^{1} a_i a_j \underline{P}[X(t_1) = a_i, X(t_2) = a_j] \end{aligned}$$

where a_i = state of the system = ϕ or 1

$$\begin{aligned} \Rightarrow R_X(t_1,t_2) &= P(X(t_1) = 1, X(t_2) = 1) \\ &= P(X(t_2) = 1/X(t_1)) \cdot P(X(t_1) = 1) \end{aligned}$$

Without loss of generality, let's assume $t_2 \geq t_1$.
Let's suppose $P_0 = 1, P_1(0) = 0$ then

$$P(X(t_1) = 1) = P_1 \cdot (1 - e^{-(\alpha+\beta)t_1})$$

$P(X(t_2) = 1/X(t_1) = 1)$ can be also obtained using the results of part (b) assuming that $\tau = t_2 - t_1$ and $P_0(0) = 0$, $P_1(0) = 1$:

$$\begin{aligned} P(X(t_2) = 1/X(t_1) = 1) &= P_1 + (1 - P_1)e^{-(\alpha+\beta)(t_2-t_1)} \\ &= P_1 + P_0 e^{-(\alpha+\beta)\tau} \end{aligned}$$

$$\Rightarrow R_X(t_1,t_2) = P_1^2 + P_1 P_0 e^{-(\alpha+\beta)\tau} + P_1^2 e^{-(\alpha+\beta)t_1} + P_1 P_0 e^{-(\alpha+\beta)(t_1+\tau)}$$

d) From the equation of $R_X(t_1,t_2)$, it is seen that $X(t)$ is not WSS. However, if $t_1 \to \infty$, the last two terms will die out and we have:

$$R_X(t_1,t_2) \to R_X(\tau) = P_1^2 + P_0 P_1 e^{-(\alpha+\beta)\tau}$$

and

$$\eta_X^{(t)} = E[X(T)] = P_1(T)\eta_X = P_1 \text{ as } t \to \infty$$

So, $X(t)$ is asymptotically WSS.
The power spectral density of $X(t)$ is:

$$S_X(f) = \mathcal{F}[R_X(\tau)]$$

$$\Rightarrow \quad S_X(W) = p_1^2\delta(f) + p_1 p_0 \frac{2(\alpha+\beta)}{(\alpha+\beta)^2 + 4\pi^2 f^2}$$

e) If we have 'n' independent speakers with $X_i(t) = 0$ or 1 , $i = 1, ..., n$, then

$$\Rightarrow \quad N(t) = X_1(t) + X_2(t) + ... + X_n(t) = \sum_{i=1}^{n} X_i(t)$$

$$\begin{aligned} R_N(t_1, t_2) &= E[N(t_1) \cdot N(t_2)] = E\left[\sum_{i=1}^{n} X_i(t) \cdot \sum_{j=1}^{n} X_j(t)\right] \\ &= \sum_{i=1}^{n} E[X_i(t_1) X_i(t_2)] + \sum_{i=1}^{n} \sum_{j \neq i} E[X_i(t_1) X_j(t_2)] \\ &= n \cdot R_X(t_1, t_2) + \sum \sum E[X_i(t_1)] \cdot E[X_j(t_2)] \end{aligned}$$

$$R_N(t_1, t_2) = n \cdot R_X(t_1, t_2) + n(n-1)\eta_X(t_1) \cdot \eta_X(t_2)$$

Suppose $t_2 = t_1 + \tau$ and $\tau \geq 0$
If $t_1 \to \infty$

$$\Rightarrow R_N(t_1, t_2) = R_N(\tau) = n \cdot R_X(\tau) + n(n-1)\eta_X^2$$

$$\begin{aligned} S_N(f) &= \mathcal{F}[R_N(\tau)] = n \cdot S_X(f) + n(n-1)\eta_X^2 \cdot 2\delta(f) \\ S_N(f) &= n^2 p_1^2 \delta(f) + n p_1 p_o \frac{2(\alpha + \beta)}{(\alpha + \beta)^2 + 4\pi^2 f^2} \end{aligned}$$

11.75

~~8.33~~ X_n : continuous-valued discrete-time Markov process.

(a) The joint pdf of $(n+1)$ values of the process X_n is denoted by

$$f_{X_n \ldots X_0}(x_n, x_{n-1}, \ldots, x_1, x_0)$$

According to the theorem of conditional pdf we can write

$$f_{X_n,\ldots,X_0}(x_n, \ldots, x_0) = f_{X_n/X_{n-1},\ldots,X_0}(x_n/x_{n-1}, \ldots, x_0) \cdot f_{X_{n-1}\ldots X_0}(x_{n-1}, \ldots, x_0)$$

However, for a first-order Markov process, we have:

$$f_{X_n/X_{n-1},\ldots,X_0}(x_n/x_{n-1}, \ldots, x_0) = f_{X_n/X_{n-1}}(x_n/x_{n-1})$$

Continuing, this process for $f_{X_{n-1},\ldots,X_0}(x_{n-1}, \ldots, x_0)$, the following can be easily obtained:

$$f_{X_n,\ldots,X_0}(x_n, x_{n-1}, \ldots, x_0) = f_{X_n/X_{n-1}}(x_n/x_{n-1}) \cdot f_{X_{n-1}/X_{n-2}}(x_{n-1}/x_{n-2}), \ldots, f(x_0)$$

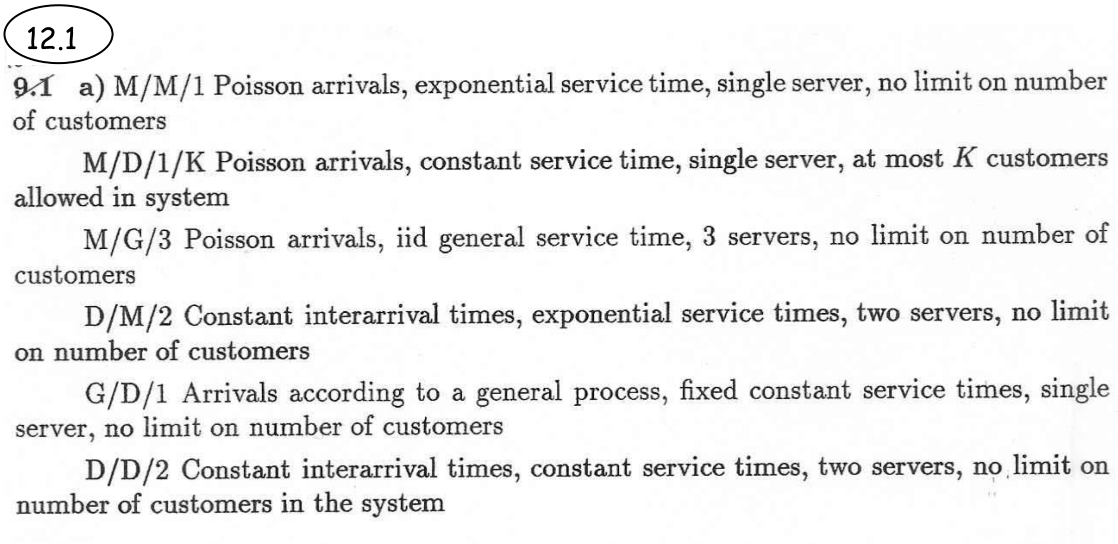
b) Consider the two-step transition pdf: $f_{X_{n+2}/X_n}(x_{n+2}/x_n)$

$$\begin{aligned}
f_{X_{n+2}/X_n}(x_{n+2}/x_n) &= \frac{f_{X_{n+2}\cdot X_n}(x_{n+2}, x_n)}{f_{X_n}(x_n)} \\
&= \frac{\int f_{X_{n+2}X_{n+1}+X_n}(x_{n+2}, x_{n+1}.x_n)dx_{n+1}}{f_{X_n}(x_n)} \\
&= \int f_{X_{n+2}/X_{n+1}}(x_{n+2}/x_{n+1}) \cdot f_{X_{n+1}/X_n}(x_{n+1}/x_n) \cdot dx_{n+1} \\
&\Rightarrow f_{X_{n+2}/X_n}(x_{n+2}/x_n) = \int f_{X_{n+2}/X_{n+1}}(x_{n+2}/x_{n+1}) \\
&\quad \cdot f_{X_{n+1}/X_n}(x_{n+1}/x_n)dx_{n+1}
\end{aligned}$$

Chapter 12: Introduction to Queueing Theory

12.1 & 12.2 The Elements of a Queueing Network and Little's Formula

12.1



~~9.1~~ **a)** M/M/1 Poisson arrivals, exponential service time, single server, no limit on number of customers

M/D/1/K Poisson arrivals, constant service time, single server, at most K customers allowed in system

M/G/3 Poisson arrivals, iid general service time, 3 servers, no limit on number of customers

D/M/2 Constant interarrival times, exponential service times, two servers, no limit on number of customers

G/D/1 Arrivals according to a general process, fixed constant service times, single server, no limit on number of customers

D/D/2 Constant interarrival times, constant service times, two servers, no limit on number of customers in the system

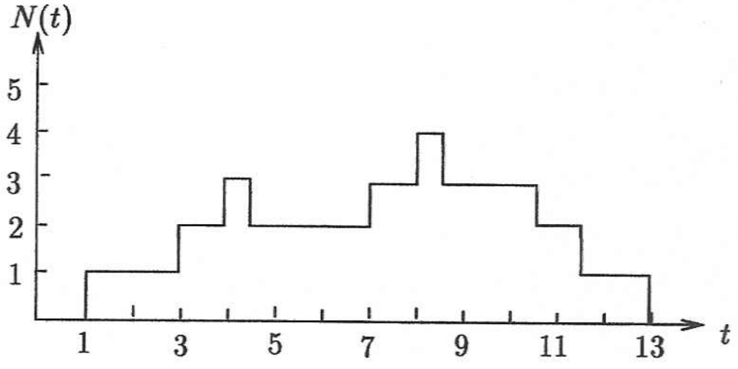
12.2 $\{S_i\} = \{1, 3, 4, 7, 8, 15\}$

$\{\tau_i\} = \{3.5, 4, 2, 1, 1.5, 4\}$

a) FCFS

i	S_i	τ_i	D_i	W_i	T_i
1	1	3.5	4.5	0	3.5
2	3	4	8.5	1.5	5.5
3	4	2	10.5	4.5	6.5
4	7	1	11.5	3.5	4.5
5	8	1.5	13.0	3.5	5.0
6	15	4			

where $W_i = D_{i-1} - S_i = T_i - \tau_i$ and $T_i = D_i - S_i = W_i + \tau_i$



$$< N >_{13} = \frac{1}{13}\sum_{i=1}^{A_{13}} T_i = \frac{25}{13}$$

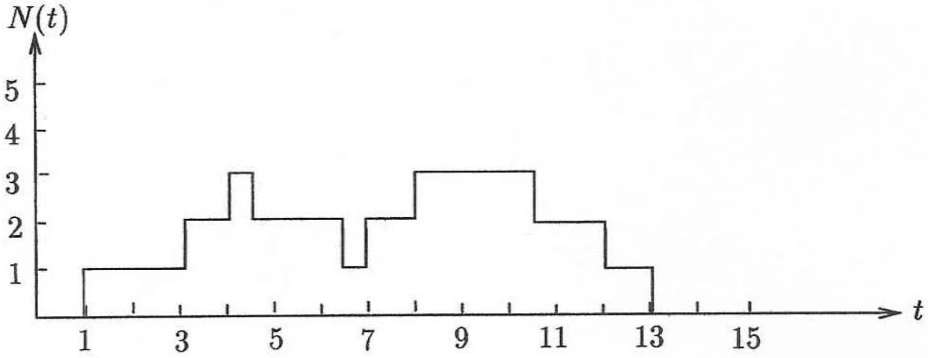
$$< \lambda >_{13} = \frac{A_{13}}{13} = \frac{5}{13}$$

$$< T >_{13} = \frac{1}{A_{13}}\sum_{i=1}^{A_{13}} T_i = \frac{25}{5}$$

$$< N >_{13} = \frac{25}{13} = < \lambda >_{13} < T >_{13} = \frac{5}{13}\frac{25}{5} \quad \checkmark$$

b) LCFS

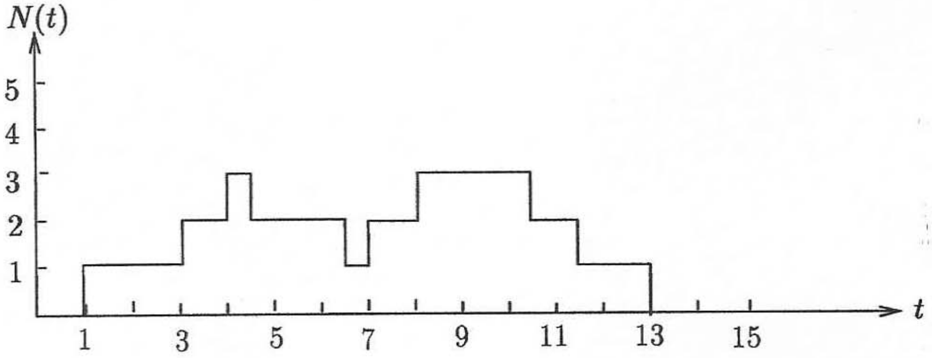
i	S_i	τ_i	D_i	$W_i = T_i - \tau_i$	$T_i = D_i - S_i$
1	1	3.5	4.5	0	3.5
2	3	4	10.5	3.5	7.5
3	4	2	6.5	0.5	2.5
4	7	1	13.0	5.0	6.0
5	8	1.5	12.0	2.5	4.0



$$< N >_{13} = \frac{23.5}{13} \qquad < \lambda >_{13} = \frac{5}{13} \qquad < T >_{13} = \frac{23.5}{5}$$
$$< N >_{13} = < \lambda >_{13} < T >_{13}$$

c) Shortest Job First:

i	S_i	τ_i	D_i	$W_i = T_i - \tau_i$	$T_i = D_i - S_i$
1	1	3.5	4.5	0	3.5
2	3	4	10.5	3.5	7.5
3	4	2	6.5	0.5	2.5
4	7	1	11.5	3.5	4.5
5	8	1.5	13.0	3.5	5.0



$$< N >_{13} = \frac{23}{13} \qquad < \lambda >_{13} = \frac{5}{13} \qquad < T >_{13} = \frac{23}{5}$$
$$< N >_{13} = < \lambda >_{13} < T >_{13}$$

12.3

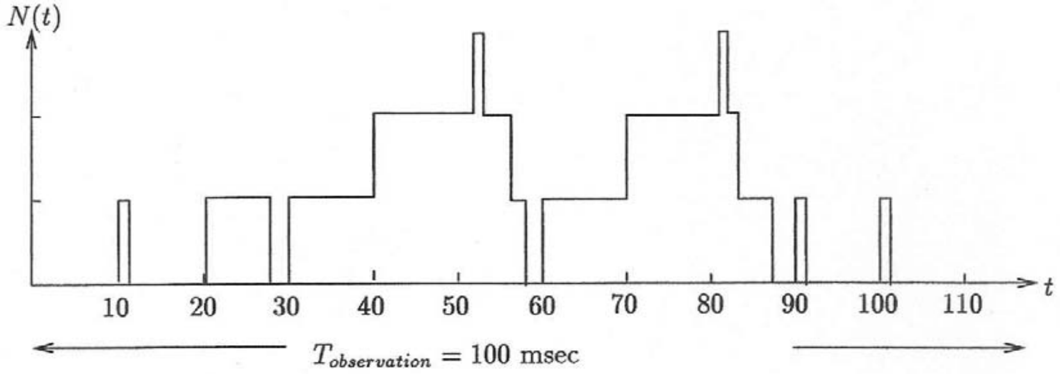
1) Interarrivals are constant with interarrival times = 10 μsec

2) Service time

if 0 error 1 μsec
1 error 1+5 μsec
>1 error 1+20 μsec

arrival time	10	20	30	40	50	60	70	80	90	100
errors	0	1	3	1	0	4	0	1	0	0
service time	1	6	21	6	1	21	1	6	1	1
dep. time	11	26	51	57	58	81	82	88	91	101

a)



b)
$$\begin{aligned}
&\frac{1}{100}\int_{10}^{110} N(t)dt \\
&= \frac{1}{100}[1+6+10+20+3+12+1+10+20+3+2+6+1+1] \\
&= 0.96
\end{aligned}$$

c) Server is working during 65 msec = Σ service times

$$\text{proportion idle time} = 1 - \frac{65}{100} = 0.35$$

12.4

~~9.4~~ a) One customer ⇒ no waiting

$$\Rightarrow \mathcal{E}[T] = m_1 + m_2 + m_3 \Rightarrow \lambda = \frac{1}{m_1 + m_2 + m_3}$$

Little's formula ⇒

$$\begin{aligned} \mathcal{E}[N_i] &= \lambda m_i = \frac{m_i}{m_1 + m_2 + m_3} = \% \text{ time customer in queue } i \\ \sum_{i=1}^{3} \mathcal{E}[N_i] &= \sum_{i=1}^{3} \frac{m_i}{m_1 + m_2 + m_3} = 1 \Rightarrow \text{one customer in system} \end{aligned}$$

b) Let $\mathcal{E}[T]$ = mean cycle time per customer, then

$$\begin{matrix}\text{total \#} \\ \text{in system}\end{matrix} = N = \lambda \mathcal{E}[T] \quad \text{by Little's formula}$$

12.5

~~9.5~~ a) $\lambda T = 5$

b) $\lambda m = 2$

c) $T = \frac{5}{\lambda} = 5\left(\frac{m}{2}\right) = \frac{5}{2} m$

12.6

~~9.6~~ Let τ be the service time, then

$$\begin{aligned} P[\tau = 1] &= p_0 \qquad [\tau = 1 + 5] = p_1 \qquad P[\tau = 1 + 20] = p_2 \\ \mathcal{E}[\tau] &= 1 \cdot p_0 + 6p_1 + 21p_2 \;\; 10^{-6} \text{ sec} \\ \lambda &= 1 \text{ arrival every } 10\ \mu\text{s} = \frac{1}{10^{-5}} = 10^5 \\ \mathcal{E}[N_d] &= \lambda \mathcal{E}[\tau] = \frac{p_o + 6p_1 + 21p_2}{10} \end{aligned}$$

12.7 **a)** $\mathcal{E}[T_i] = \dfrac{1}{\lambda_i}\mathcal{E}[N_i]$

b) $\mathcal{E}[T] = \dfrac{1}{\lambda}\mathcal{E}[N] = \dfrac{1}{\lambda}\sum_i \mathcal{E}[N_i]$

c) $\mathcal{E}[T] = \dfrac{1}{\lambda}\sum_i \lambda_i \mathcal{E}[T_i] = \sum_i \dfrac{\lambda_i}{\lambda}\mathcal{E}[T_i]$

Let

$$\begin{aligned}
A_i(t) &= \#\text{ type } i \text{ arrivals during } [0,t] \\
A(t) &= \sum_i A_i(t) = \text{total } \# \text{ arrivals} \\
<T> &= \frac{1}{A(t)}\sum_i^{A(t)} T_i \qquad \text{average time in system} \\
&= \frac{1}{A(t)}\left[\sum_{i_1}^{A_1(t)} T_{i_1} + \sum_{i_2}^{A_2(t)} T_{i_2} + \ldots + \sum_{i_n}^{A_n(t)} T_{i_n}\right] \\
&= \frac{1}{A(t)}\left[\frac{A_1(t)}{A_1(t)}\sum_{i_1}^{A_1(t)} T_{i_1} + \ldots + \frac{A_n(t)}{A_n(t)}\sum_{i_n}^{A_n(t)} T_{i_n}\right] \\
&= \underbrace{\frac{A_1(t)}{A(t)}}_{\frac{\lambda_1}{\lambda}}\left(\underbrace{\frac{1}{A_1(t)}\sum_{i_1}^{A_1(t)} T_{i_1}}_{\mathcal{E}[T_1]}\right) + \ldots + \underbrace{\frac{A_n(t)}{A(t)}}_{\frac{\lambda_n}{\lambda}}\left(\underbrace{\frac{1}{A_n(t)}\sum_{i_n}^{A_n(t)} T_{i_n}}_{\mathcal{E}[T_n]}\right)
\end{aligned}$$

as $t \to \infty$

$\Rightarrow$ same result as above.

12.3 The M/M/1 Queue

12.8 ~~9.8~~ **a)** $P[N \geq n] = (1-\rho)\sum_{j=n}^{\infty} \rho^j = (1-\rho)\dfrac{\rho^n}{1-\rho} = \rho^n$

b) $P[N \geq 10] = \rho^{10} = 10^{-3} \Rightarrow \rho = 10^{-0.3} \approx \dfrac{1}{2}$

$$\Rightarrow \lambda \approx \frac{1}{2}\mu$$

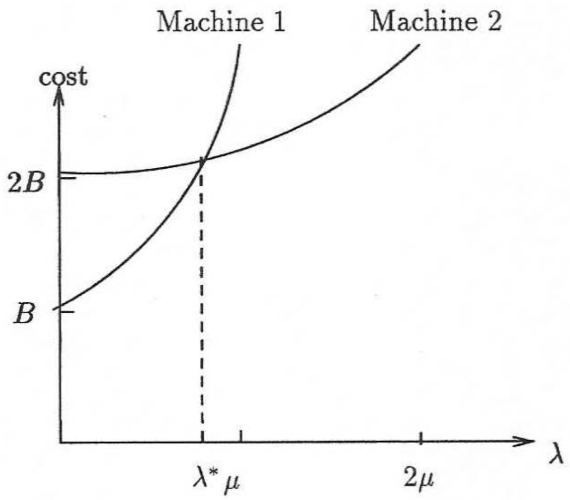
12.9

a) Machine 1 μ transactions/hr B \$/hr operating cost
Machine 2 $2\ \mu$ transactions/hr $2B$ \$/hr operating cost

We assume "operating" cost is incurred regardless of whether machine is idle.

$$\begin{aligned}\text{Cost for }\#1 &= B + \lambda\frac{\text{cost}}{\text{hr}}\cdot\bar{W}\frac{\text{waiting hrs.}}{\text{cust}}\cdot A\frac{\$}{\text{hr}} \\ &= B + \lambda\left(\frac{\rho}{1-\rho}\frac{1}{\mu}\right)A \qquad \text{where } \rho = \frac{\lambda}{\mu} \\ &= B + A\frac{\rho^2}{1-\rho}\end{aligned}$$

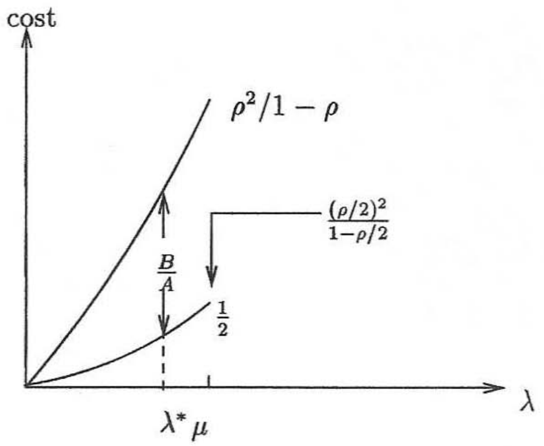
$$\text{Cost for }\#2 = 2B + \lambda\left(\frac{\frac{\rho}{2}}{1-\frac{\rho}{2}}\right)\left(\frac{1}{2\mu}\right)A = 2B + A\frac{\left(\frac{\rho}{2}\right)^2}{1-\frac{\rho}{2}}$$



for $\lambda < \lambda^*$ machine 1 is less costly. Let $\rho^* = \lambda^*/\mu$

$$B + A\frac{{\rho^*}^2}{1-{\rho^*}^2} = 2B + A\frac{\left(\frac{\rho^*}{2}\right)^2}{1-\frac{\rho^*}{2}}$$

$$\Leftrightarrow \frac{{\rho^*}^2}{1-\rho^*} - \frac{\left(\frac{\rho^*}{2}\right)^2}{1-\frac{\rho^*}{2}} = \frac{B}{A}$$



This requires finally the root of a cubic polynomial but we can estimate the root from the figure.

For $\dfrac{B}{A} >> 1$

$$\frac{\left(\frac{\rho^*}{2}\right)^2}{1-\frac{\rho^*}{2}} \approx \frac{1}{2}$$

so we solve the quadratic equation associated with

$$\frac{\rho^{*2}}{1-\rho^*} - \frac{1}{2} = \frac{B}{A}$$

In particular if $\dfrac{B}{A} = 10$ then $\rho^* \approx 0.91$.

For $\dfrac{B}{A} << 1$

$$\frac{\left(\frac{\rho^*}{2}\right)^2}{1-\frac{\rho^*}{2}} \approx 0$$

so we solve

$$\frac{\rho^{*2}}{1-\rho^*} = \frac{B}{A}$$

In particular if $\dfrac{B}{A} = \dfrac{1}{10}$ then $\rho^* \approx 0.27$.

NOTE: If "operating" cost is incurred only when a machine is in use then:

$$\text{Cost of Machine 1} = \underbrace{B\rho}_{\substack{\text{prop. of time}\\ \text{machine 1 in use}}} + \frac{\rho^2}{1-\rho}$$

$$\text{Cost of Machine 2} = 2B\left(\frac{\rho}{2}\right) + \frac{\left(\frac{\rho}{2}\right)^2}{1-\frac{\rho}{2}} = B\rho + \frac{\left(\frac{\rho}{2}\right)^2}{1-\frac{\rho}{2}}$$

In this case machine 2 is less costly than machine 1 for all ρ.

12.10

~~**9.10**~~ A net profit is made if

$$5 > \mathcal{E}[T] = \frac{\frac{1}{\mu}}{1-\rho}$$

$$\begin{aligned}
\Rightarrow &\quad 1 - \frac{\lambda}{\mu} > \frac{1}{5\mu} \Rightarrow 5\mu - 5\lambda > 1 \\
\Rightarrow &\quad 0 < \lambda < \mu - \frac{1}{5}
\end{aligned}$$

12.11

9.11 a) $\mathcal{E}[N_q] = \dfrac{\rho^2}{1-\rho} = 5 \Rightarrow \rho^2 - 5\rho - 5 = 0$

$$\Rightarrow \rho = \frac{-5 + \sqrt{25 - 4(-5)}}{2} = \frac{\sqrt{45} - 5}{2} = 0.854$$

b)

$$\begin{aligned}
P[N_q = j | N_q > 0] &= \frac{P[N_1 = j]}{P[N_q > 0]} \qquad j > 0 \\
&= \frac{(1-\rho)\rho^j}{\rho} = (1-\rho)\rho^{j-1} \\
\mathcal{E}[N_q | N_q > 0] &= \sum_{j=1}^{\infty} j(1-\rho)\rho^{j-1} = \frac{1}{1-\rho} = 5 \\
&\Rightarrow \rho = \frac{4}{5} = 0.8
\end{aligned}$$

c) It depends on whether one is concerned with average queue over all time (Part a) or on the average queue when one forms (Part b).

12.12

$$\begin{aligned}
p &= P[W \le x] = \int_0^x ((1-\rho)\delta(t') + \lambda(1-\rho)e^{-\mu(1-\rho)t'})dt' \\
&= (1-\rho) + \left| -\frac{\lambda}{\mu} e^{-\mu(1-\rho)t'} \right|_0^x \\
&= 1 - \rho e^{-\mu(1-\rho)x} \\
\Rightarrow \frac{1-p}{\rho} &= e^{-\mu(1-\rho)x} \\
\Rightarrow x &= \frac{1}{\mu(1-\rho)} \ln \frac{\rho}{1-p} \qquad \checkmark
\end{aligned}$$

12.13 $\frac{1}{\mu} = \frac{1}{2}$

a) From Example 12.5

$$\begin{aligned} x &= \frac{1}{\mu-\lambda}\ln\frac{1}{1-p} \\ \Rightarrow \mu-\lambda &= \frac{1}{x}\ln\frac{1}{1-p} \\ \lambda &= \mu - \frac{1}{x}\ln\frac{1}{1-p} = 2 - \frac{1}{3}\ln\frac{1}{1-.9} = 1.232 \end{aligned}$$

b) From Problem 12.12

$$\begin{aligned} x &= \frac{\frac{1}{\mu}}{1-\rho}\ln\frac{\rho}{1-\rho} &&= \frac{1}{\mu-\lambda}\ln\frac{\lambda}{\mu(1-p)} \\ 2 &= \frac{1}{2-\lambda}\ln 5\lambda && \Rightarrow \lambda = 2 - \frac{1}{2}\ln 5\lambda \\ & && \Rightarrow \lambda = 1.13 \end{aligned}$$

12.14

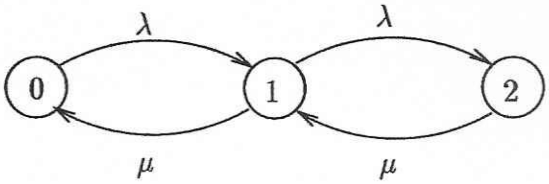
~~9.14~~ From Example 11.40 we have

$$\begin{aligned} P_j &= \frac{\lambda}{\mu}P_{j-1} = \left(\frac{\lambda}{\mu}\right)^j P_0 \qquad 1 \le j \le K \\ 1 &= \sum_{j=0}^{K}\left(\frac{\lambda}{\mu}\right)^j P_0 \Rightarrow P_0 = \frac{1}{\sum_{j=0}^{K}\left(\frac{\lambda}{\mu}\right)^j} = \frac{1-\rho}{1-\rho^{K+1}} \end{aligned}$$

where $\rho = \frac{\lambda}{\mu}$

$$\therefore P_j = \frac{(1-\rho)}{1-\rho^{K+1}}\rho^j \qquad 0 \le j \le K$$

12.15



$$\begin{aligned} P_0 &= \frac{1-\rho}{1-\rho^3} = \frac{1}{1+\rho+\rho^2} \\ P_1 &= \frac{\rho}{1+\rho+\rho^2} \qquad P_2 = \frac{\rho^2}{1+\rho+\rho^2} \end{aligned}$$

In a very long time interval of length T

$$\begin{aligned} \text{Profit} &= \#\text{ accepted into system} \times \$5 - \#\text{ blocked} \times 1 \\ &= \lambda T \times (1-P_B) \times 5 - \lambda T \times P_B \times 1 \\ \text{Profit} &= 0 \quad \text{if} \quad 5\lambda T(1-P_B) = \lambda T P_B \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow\ 5 = 6P_B \\ &\Leftrightarrow\ P_B = \frac{\rho^2}{1+\rho+\rho^2} = \frac{5}{6} \\ &\Leftrightarrow\ \rho^2 - 5\rho - 5 = 6 \\ &\Rightarrow\ \rho = \frac{5+\sqrt{25+20}}{2} = 5.854 \\ &\Rightarrow\ \lambda = 5.854\mu \end{aligned}$$

12.16 $P[N=k|N<K] = \dfrac{P[N=k, N<K]}{P[N<K]} = \begin{cases} 0 & k \geq K \\ \frac{P[N=k]}{P[N<K]} & 0 \leq k < K \end{cases}$

$\therefore$ for $0 \leq k < K$

$$P[N=k|N<K] = \frac{P[N=k]}{P[N<K]} = \frac{P[N=k]}{1-P[N=K]}$$

Arriving customers are allowed into the system only when $N < K$.

$\therefore$ $P[N=k|N<K]$ represents the proportion of time when there are k in the system and arriving customers are allowed in. Since Poisson arrivals pick their arrival times at random, then $P[N=k|N<K]$ is the proportion of customers that see k in system upon being admitted in.

12.17 a) M/M/1/5 ⇒ 6 states

(i)

$$\Gamma = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 & 0 & \mu & -\mu \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & -1.5 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & -1.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & -1.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & -1.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$P(t) = e^{\Gamma t} = E\,[e^{\lambda_i t}]\,E^{-1}$$

$$E = \begin{pmatrix} -0.2644 & -0.4487 & 0.6172 & 0.8661 & -0.7487 & -0.8318 \\ 0.5882 & 0.7661 & -0.6172 & 0.4331 & 0.2193 & -0.1870 \\ -0.5882 & -0.3173 & -0.3086 & 0.2165 & 0.5294 & 0.1870 \\ 0.4263 & -0.1587 & 0.3086 & 0.1083 & 0.2647 & 0.3225 \\ -0.2280 & 0.2708 & 0.1543 & 0.0541 & -0.0775 & 0.3014 \\ 0.0661 & -0.1122 & -0.1543 & 0.0271 & -0.1872 & 0.2080 \end{pmatrix}$$

$$[e^{\lambda_i t}] = \begin{pmatrix} e^{-2.72t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-2.21t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-1.50t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-0.79t} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-0.27t} \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} -0.0578 & 0.2573 & -0.5147 & 0.7460 & -0.7980 & 0.4627 \\ -0.1262 & 0.4309 & -0.3570 & -0.3570 & 1.2188 & -1.0097 \\ 0.1800 & -0.3600 & -0.3600 & -0.7201 & 0.7201 & -1.4402 \\ 0.5864 & 0.5864 & 0.5864 & 0.5864 & 0.5864 & 0.5864 \\ -0.2106 & 0.1233 & 0.5956 & 0.5956 & -0.3489 & -1.6846 \\ -0.1820 & -0.0818 & 0.5643 & 0.5643 & 1.0551 & 1.4558 \end{pmatrix}$$

$$E[e^{\lambda t}]E^{-1} = X \quad \Rightarrow \quad P(t) = P(0)X$$

$N(0)=0 \Rightarrow P(0)=(1,0,0,0,0,0) \Rightarrow P(t) =$ first row of X

$$\Rightarrow P(t) = \begin{pmatrix} 0.0153e^{-2.72t} + 0.0556e^{-2.21t} + 0.1111e^{-1.5t} + 0.5091 + 0.1577e^{-.79t} + 0.1514e^{-0.27t}, \\ -0.3040e^{-2.72t} + (-0.0967)e^{-2.21t} - 0.1111e^{-1.5t} + 0.2340 + (-0.0462)e^{-0.79t} + 0.0340e^{-0.27t}, \\ 0.3040e^{-2.72t} + 0.04e^{-2.21t} - 0.0555e^{-1.5t} + 0.1270 - 0.1115e^{-0.79t} - 0.0390e^{-0.27t}, \\ -0.0246e^{-2.72t} + 0.02e^{-2.21t} + 0.0555e^{-1.5t} + 0.0635 - 0.0557e^{-0.79t} - 0.0587e^{-0.27t}, \\ 0.0132e^{-2.72t} - 0.0342e^{-2.21t} + 0.0278e^{-1.5t} + 0.0317 + 0.0163e^{-0.79t} - 0.0549e^{-0.27t}, \\ -0.0582e^{-2.72t} + 0.0142e^{-2.21t} - 0.0278e^{-1.5t} + 0.0159 + 0.0394e^{-0.79t} - 0.0379e^{-0.27t} \end{pmatrix}^t$$

for other initial conditions also we can obtain $P(t)$ in the same way:

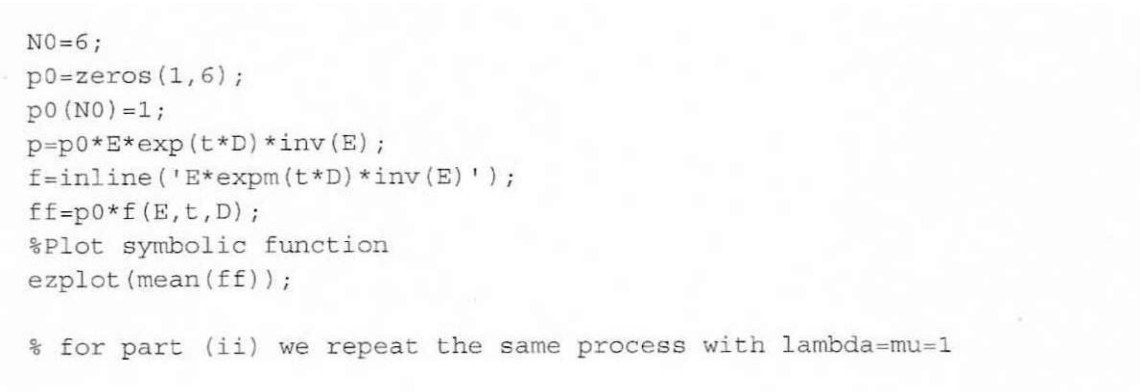
$N(0)=2 \Rightarrow P(0)=(0,0,1,0,0,0)$, $P(t) = P(0)E[e^{\lambda_i t}]E^{-1} \Rightarrow$ third row of X

$N(0)=5 \Rightarrow P(0)=(0,0,0,0,0,1)$, $P(t) = P(0)E[e^{\lambda_i t}]E^{-1} \Rightarrow$ last row of X

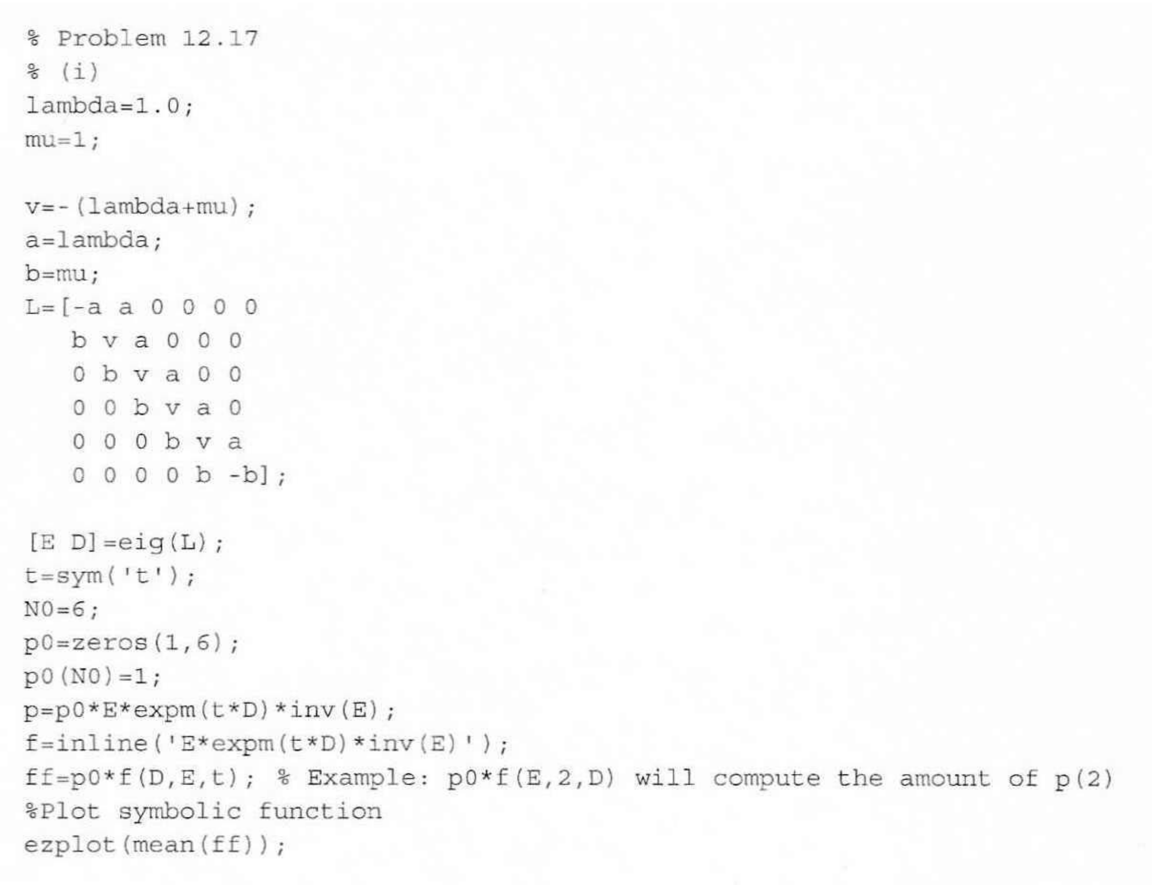
(ii) if $\rho = 1 \Rightarrow$

$$\Gamma = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ +1 & -2 & 1 & 0 & 0 & 0 \\ 0 & +1 & -2 & 1 & 0 & 0 \\ 0 & 0 & +1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

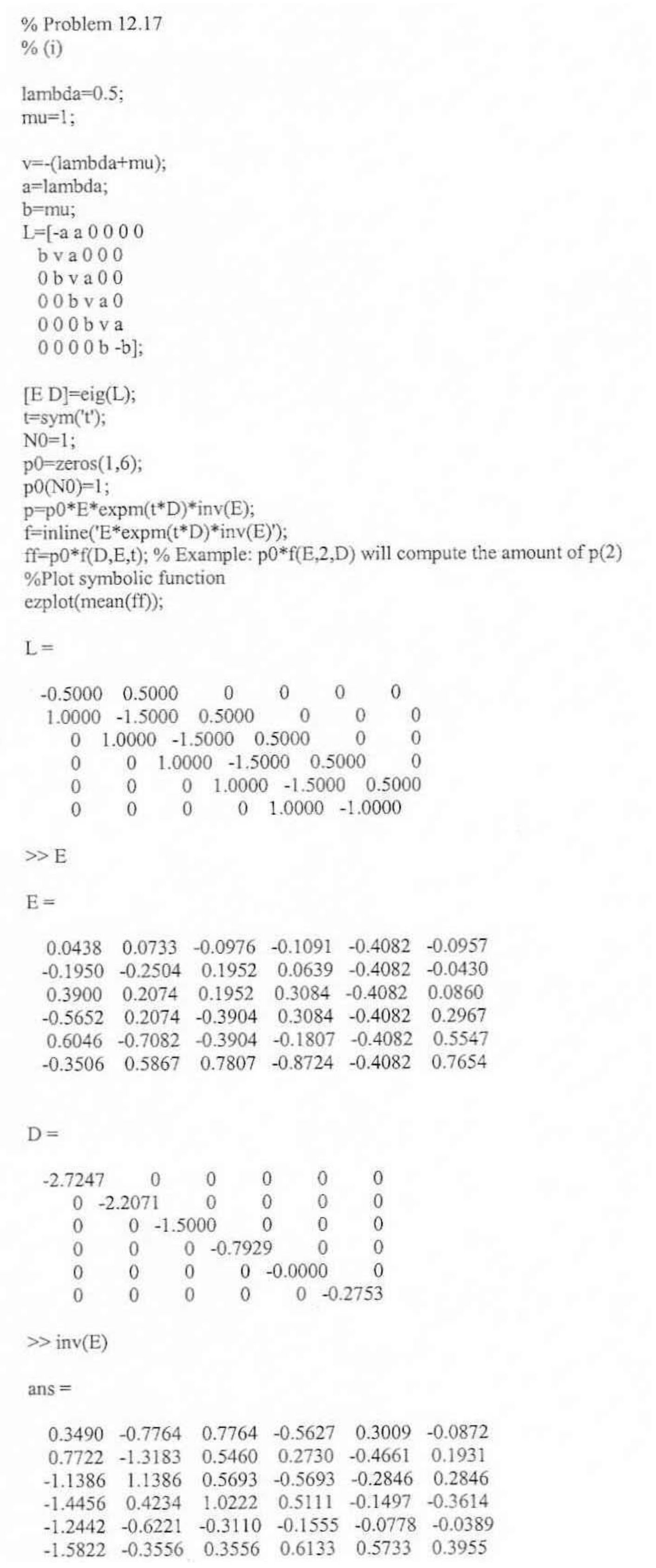
```
% Problem 12.17
% (i)
lambda=0.5;
mu=1;

v=-(lambda+mu);
a=lambda;
b=mu;
L=[-a a 0 0 0 0
   b v a 0 0 0
   0 b v a 0 0
   0 0 b v a 0
   0 0 0 b v a
   0 0 0 0 b -b];

[E D]=eig(L);
t=sym('t');
N0=1;
p0=zeros(1,6);
p0(N0)=1;
p=p0*E*expm(t*D)*inv(E);
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(E,t,D); % Example: p0*f(E,2,D) will compute the amount of p(2)
%Plot symbolic function
ezplot(mean(ff));

N0=3;
p0=zeros(1,6);
p0(N0)=1;
p=p0*E*expm(t*D)*inv(E);
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(E,t,D);
%Plot symbolic function
ezplot(mean(ff));
```



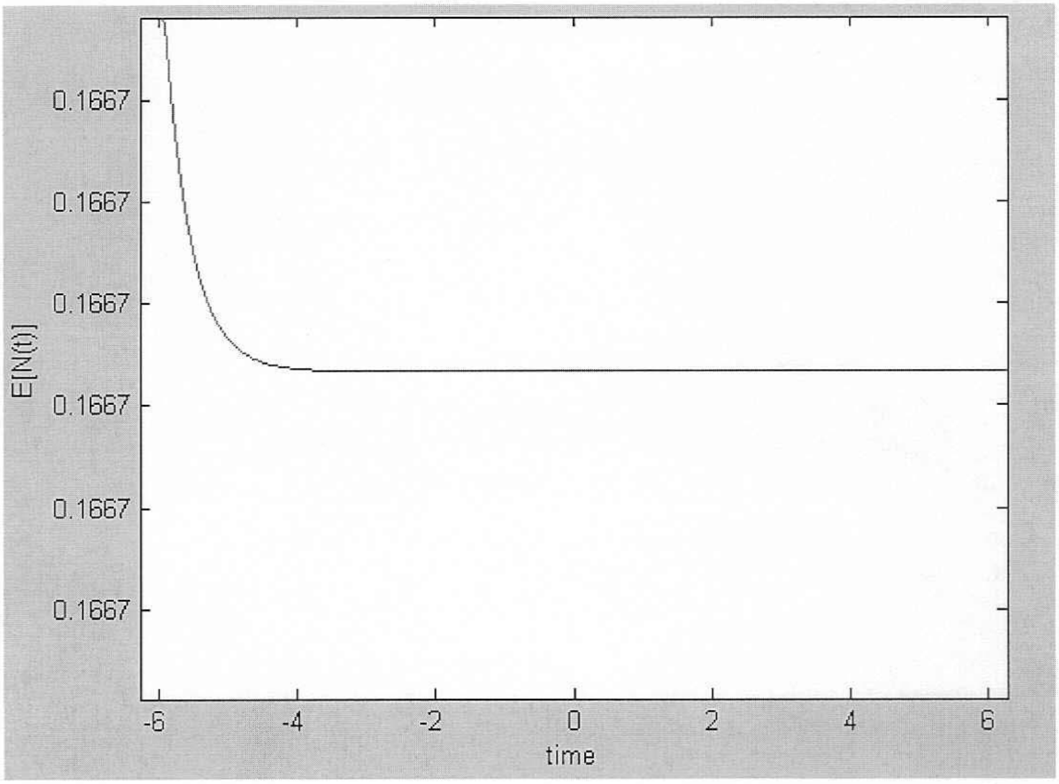
```
N0=6;
p0=zeros(1,6);
p0(N0)=1;
p=p0*E*exp(t*D)*inv(E);
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(E,t,D);
%Plot symbolic function
ezplot(mean(ff));

% for part (ii) we repeat the same process with lambda=mu=1
```



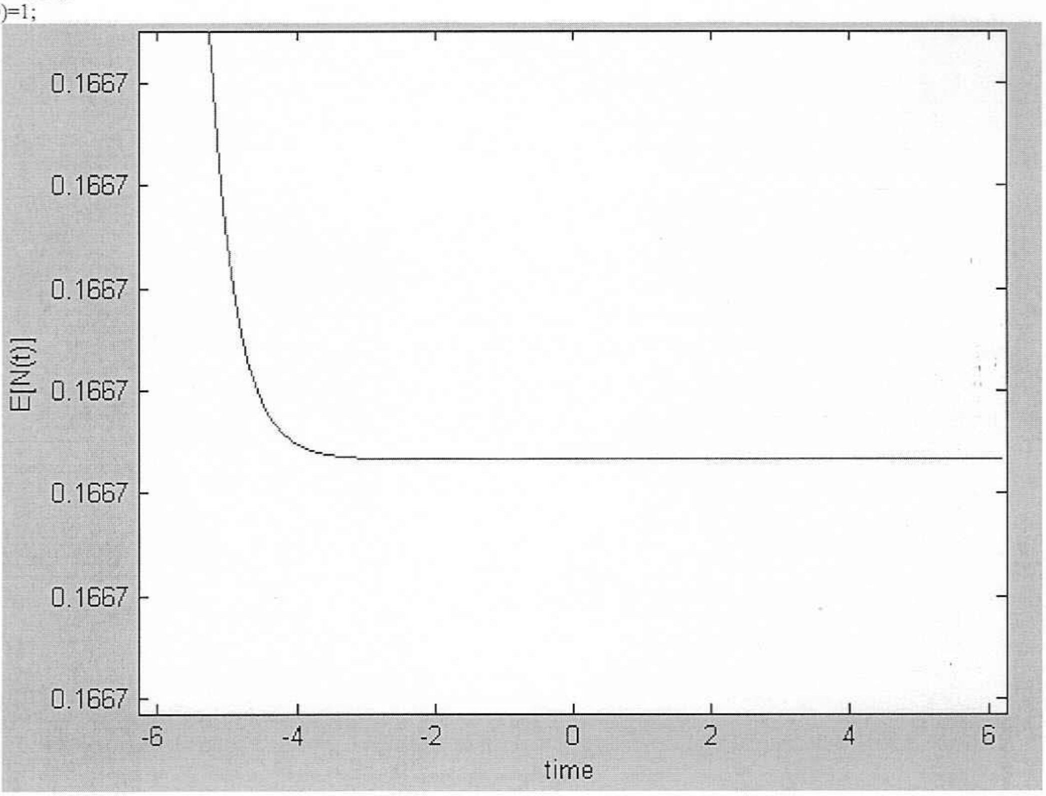
```
% Problem 12.17
% (i)
lambda=1.0;
mu=1;

v=-(lambda+mu);
a=lambda;
b=mu;
L=[-a a 0 0 0 0
   b v a 0 0 0
   0 b v a 0 0
   0 0 b v a 0
   0 0 0 b v a
   0 0 0 0 b -b];

[E D]=eig(L);
t=sym('t');
N0=6;
p0=zeros(1,6);
p0(N0)=1;
p=p0*E*expm(t*D)*inv(E);
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(D,E,t); % Example: p0*f(E,2,D) will compute the amount of p(2)
%Plot symbolic function
ezplot(mean(ff));
```



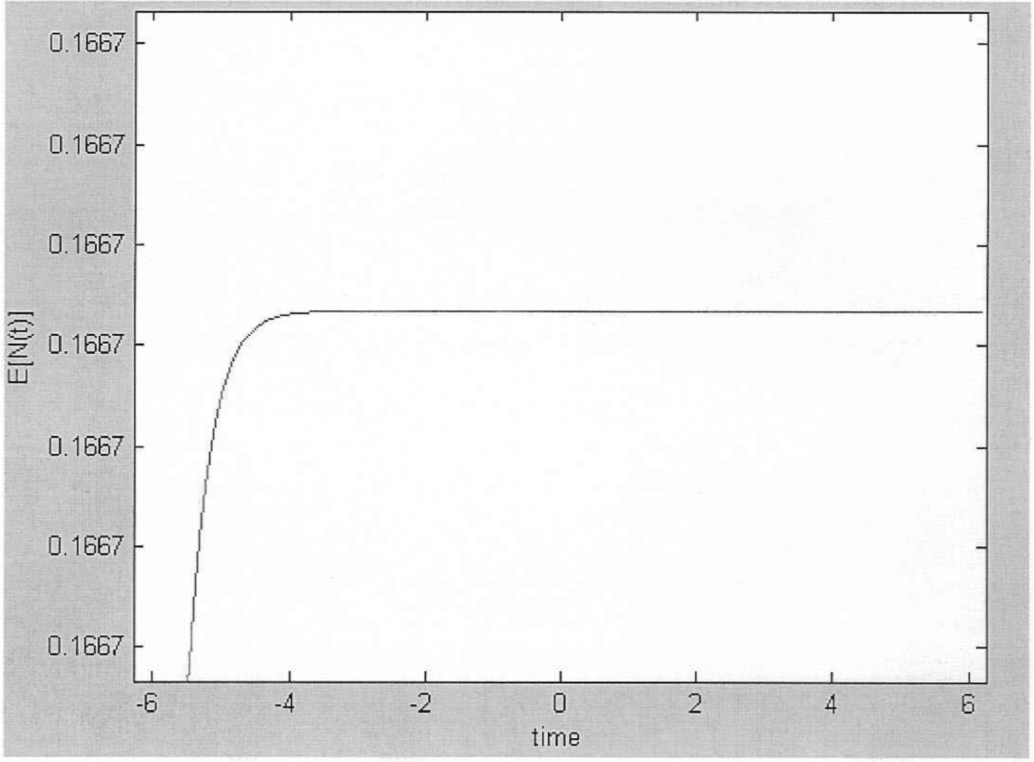
```
% Problem 12.17
% (i)

lambda=0.5;
mu=1;

v=-(lambda+mu);
a=lambda;
b=mu;
L=[-a a 0 0 0 0
  b v a 0 0 0
  0 b v a 0 0
  0 0 b v a 0
  0 0 0 b v a
  0 0 0 0 b -b];

[E D]=eig(L);
t=sym('t');
N0=1;
p0=zeros(1,6);
p0(N0)=1;
p=p0*E*expm(t*D)*inv(E);
f=inline('E*expm(t*D)*inv(E)');
ff=p0*f(D,E,t); % Example: p0*f(E,2,D) will compute the amount of p(2)
%Plot symbolic function
ezplot(mean(ff));

L =

  -0.5000   0.5000        0        0        0        0
   1.0000  -1.5000   0.5000        0        0        0
        0   1.0000  -1.5000   0.5000        0        0
        0        0   1.0000  -1.5000   0.5000        0
        0        0        0   1.0000  -1.5000   0.5000
        0        0        0        0   1.0000  -1.0000

>> E

E =

   0.0438   0.0733  -0.0976  -0.1091  -0.4082  -0.0957
  -0.1950  -0.2504   0.1952   0.0639  -0.4082  -0.0430
   0.3900   0.2074   0.1952   0.3084  -0.4082   0.0860
  -0.5652   0.2074  -0.3904   0.3084  -0.4082   0.2967
   0.6046  -0.7082  -0.3904  -0.1807  -0.4082   0.5547
  -0.3506   0.5867   0.7807  -0.8724  -0.4082   0.7654


D =

  -2.7247        0        0        0        0        0
        0  -2.2071        0        0        0        0
        0        0  -1.5000        0        0        0
        0        0        0  -0.7929        0        0
        0        0        0        0  -0.0000        0
        0        0        0        0        0  -0.2753

>> inv(E)

ans =

   0.3490  -0.7764   0.7764  -0.5627   0.3009  -0.0872
   0.7722  -1.3183   0.5460   0.2730  -0.4661   0.1931
  -1.1386   1.1386   0.5693  -0.5693  -0.2846   0.2846
  -1.4456   0.4234   1.0222   0.5111  -0.1497  -0.3614
  -1.2442  -0.6221  -0.3110  -0.1555  -0.0778  -0.0389
  -1.5822  -0.3556   0.3556   0.6133   0.5733   0.3955
```



```
N0=3;
p0=zeros(1,6);
p0(N0)=1;
```

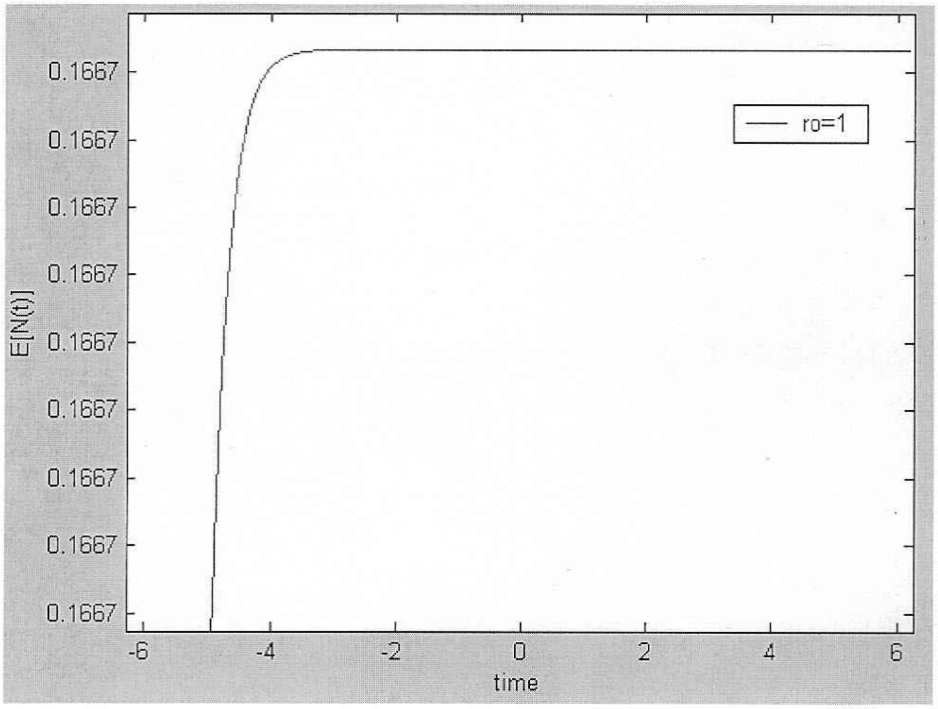


```
N0=6;
p0=zeros(1,6);
p0(N0)=1;
```

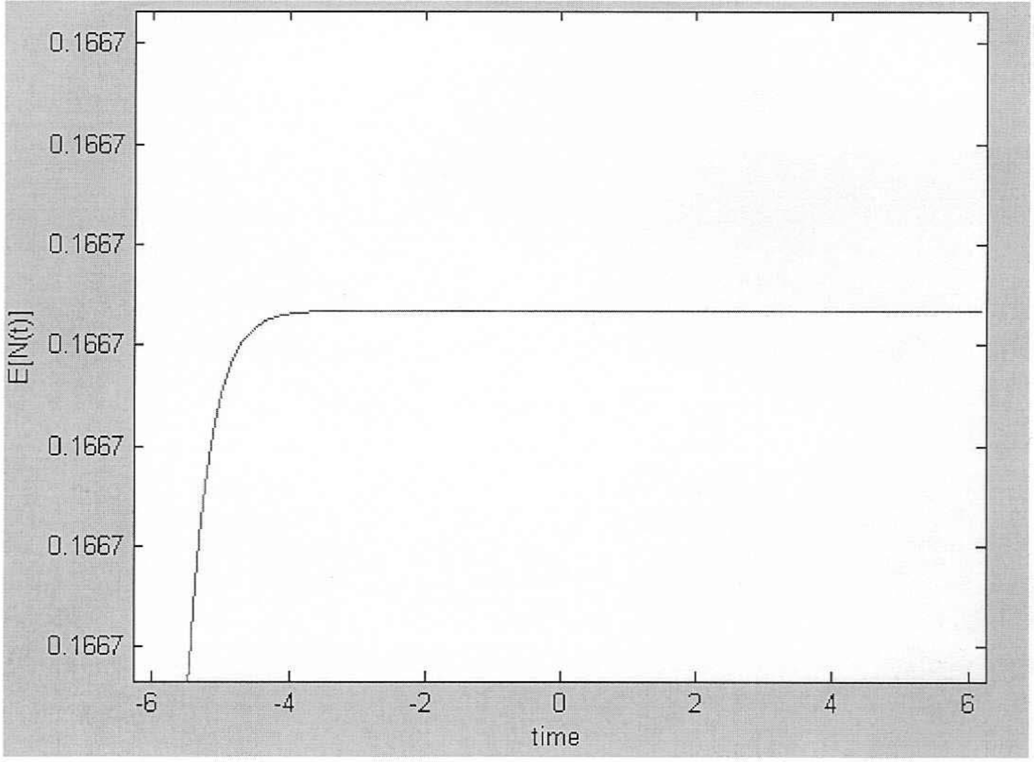


Part (ii)

```
Lambda=1;
N0=1;
p0=zeros(1,6);
p0(N0)=1;
```

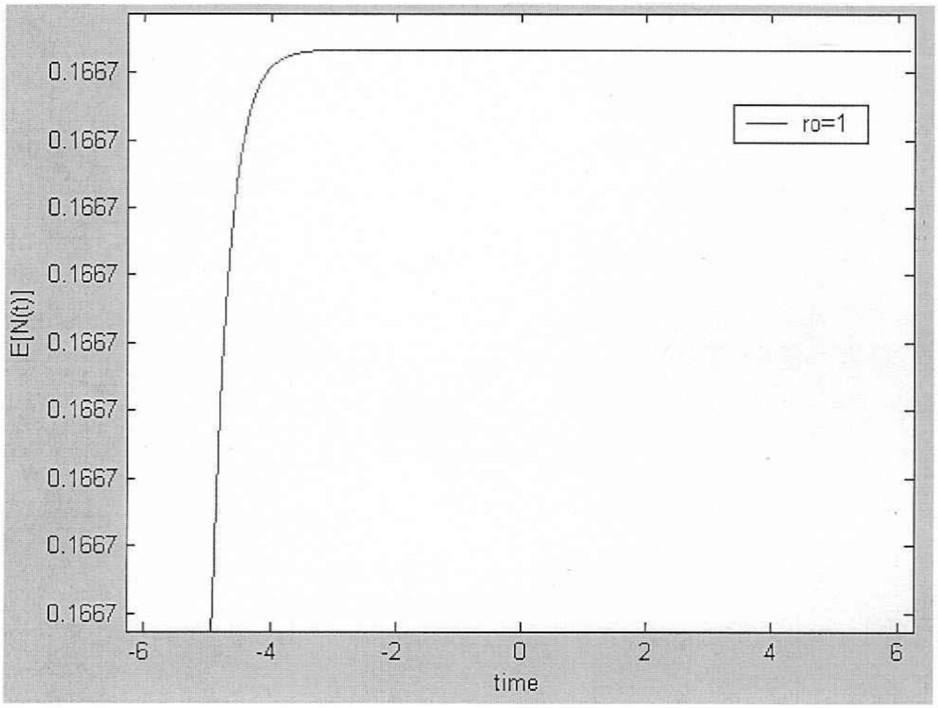


```
N0=6;
p0=zeros(1,6);
p0(N0)=1;
```

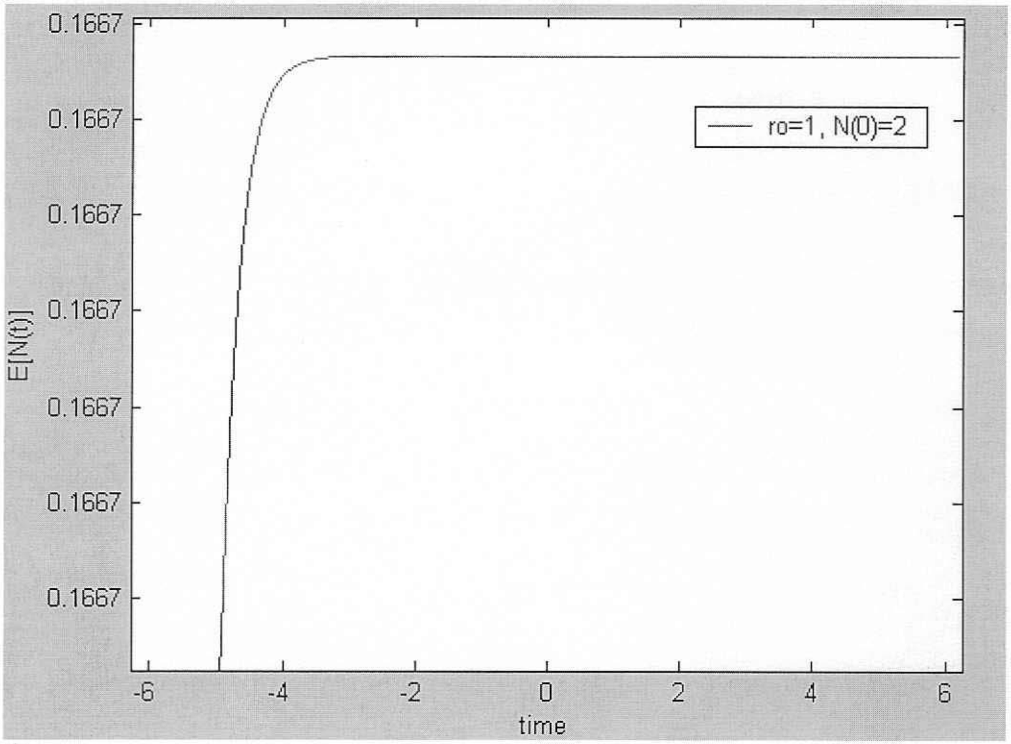


Part (ii)

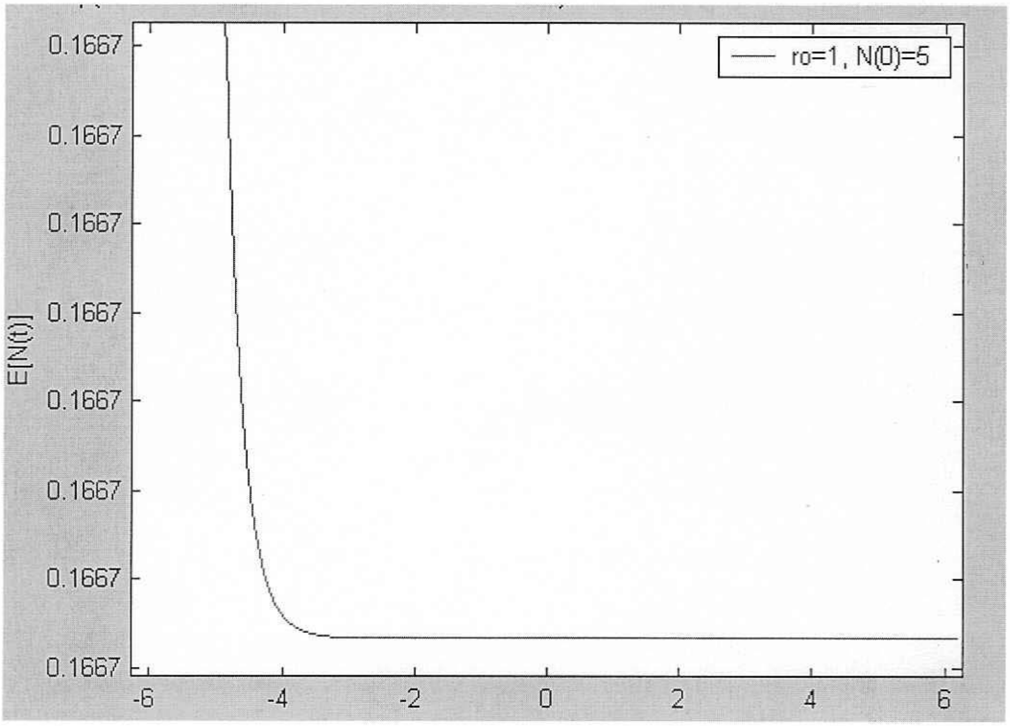
```
Lambda=1;
N0=1;
p0=zeros(1,6);
p0(N0)=1;
```



```
Lambda=1;
N0=1;
p0=zeros(1,6);
p0(N0)=3
```

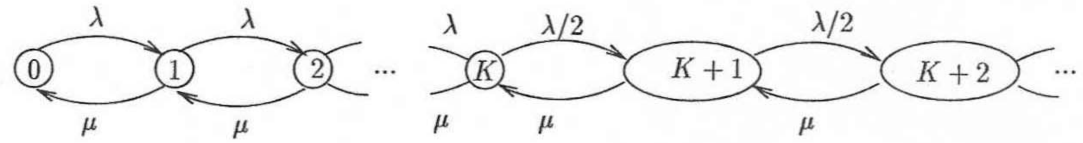


```
Lambda=1;
N0=1;
p0=zeros(1,6);
p0(N0)=6
```



12.18

~~9.17~~ If $N < K$ arrival rate is λ
If $N \geq K$ arrival rate is reduced to $\frac{\lambda}{2}$



For $0 \leq j \leq K$

$$P_j = \frac{\lambda}{\mu} P_{j-1} = \left(\frac{\lambda}{\mu}\right)^j P_0$$

For $K < j$

$$P_j = \frac{\lambda}{2\mu} P_{j-1} = \left(\frac{\lambda}{2\mu}\right)^{j-K} P_K = \left(\frac{\lambda}{2\mu}\right)^{j-K} \left(\frac{\lambda}{\mu}\right)^K P_0$$

$$\begin{aligned} 1 &= \sum_{j=0}^{\infty} P_j = \underbrace{P_0 \sum_{j=0}^{K-1} \left(\frac{\lambda}{\mu}\right)^j}_{\frac{1-\left(\frac{\lambda}{\mu}\right)^K}{1-\frac{\lambda}{\mu}}} + \underbrace{P_0 \sum_{j=K}^{\infty} \left(\frac{\lambda}{2\mu}\right)^{j-K} \left(\frac{\lambda}{\mu}\right)^K}_{\frac{\left(\frac{\lambda}{\mu}\right)^K}{1-\frac{\lambda}{2\mu}}} \\ P_0 &= \left[\frac{1-\left(\frac{\lambda}{\mu}\right)^K}{1-\frac{\lambda}{\mu}} + \frac{\left(\frac{\lambda}{\mu}\right)^K}{1-\frac{\lambda}{2\mu}}\right]^{-1} \end{aligned}$$

12.4 Multi-Server Systems: M/M/*c*, M/M/*c*/*c*, and M/M/∞

12.20

~~9.18~~

$$P[N \geq c+k] = \sum_{j=c+k}^{\infty} \frac{\rho^{j-c}}{c!} a^c P_0 = \frac{a^c}{c!} P_0 \rho^k \sum_{j'=0}^{\infty} \rho^j$$

$$= \frac{\frac{a^c}{c!} P_0 \rho^k}{1-\rho} = \frac{p_c \rho^k}{1-\rho}$$

12.21

~~9.19~~ $\lambda = 12 \qquad \frac{1}{\mu} = \frac{5}{60} \qquad c = 2$

$$\Rightarrow a = \frac{\lambda}{\mu} = 1 \qquad \rho = \frac{a}{2} = \frac{1}{2}$$

a)

$$P[N \geq c] = \frac{\frac{a^c}{c!} P_0}{1-\rho} - C(c,a)$$

$$P_0 = \left\{ \sum_{j=0}^{1} \frac{a^j}{j!} + \frac{a^2}{2!} \sum_{j=0}^{\infty} \rho^j \right\}^{-1} = \left\{ 1 + 1 + \frac{1}{2} \frac{1}{1-\frac{1}{2}} \right\}^{-1} = \frac{1}{3}$$

$$\Rightarrow P[N \geq 2] = \frac{\frac{1}{2!}\frac{1}{3}}{1-\frac{1}{2}} = \frac{1}{3} = C(2,1)$$

b)

$$\mathcal{E}[N] = \mathcal{E}[N_q] + a = \frac{\rho}{1-\rho} C(c,a) + a = \frac{\frac{1}{2}}{1-\frac{1}{2}} \frac{1}{3} + 1 = \frac{4}{3}$$

$$\mathcal{E}[T] = \frac{1}{\lambda} \mathcal{E}[N] = \frac{1}{9}$$

c) $P[N > 4] = P[N_q > 2] = \sum_{j=3}^{\infty} \rho^{j-2} P_2 = \frac{P_2 \rho}{1-\rho} = \frac{1}{6}$

12.22
~~9.20~~

$$\begin{aligned}\mathcal{E}[N_s] &= \sum_{j=0}^{c} jP_j + \sum_{j=c+1}^{\infty} cP_j \\ &= \sum_{j=0}^{c} j\frac{a^j}{j!}P_0 + c\sum_{j=c+1}^{\infty} \rho^{j-c}\frac{a^c}{c!}P_0\end{aligned}$$

$$= \left[a\sum_{j=1}^{c}\frac{a^{j-1}}{(j-1)!} + \frac{ca^c}{c!}\frac{\rho}{1-\rho}\right]P_0 \qquad \text{but } c\rho = a$$

$$= a\underbrace{\left[\sum_{j=0}^{c-1}\frac{a^j}{j!} + \frac{\frac{a^c}{c!}}{1-\rho}\right]}_{P_0^{-1}}P_0 = a = \lambda\mathcal{E}[\tau] \qquad \checkmark$$

12.23
~~9.21~~ $\lambda = 10 \qquad \frac{1}{\mu} = \frac{1}{2} \qquad a = \frac{\lambda}{\mu} = 5$

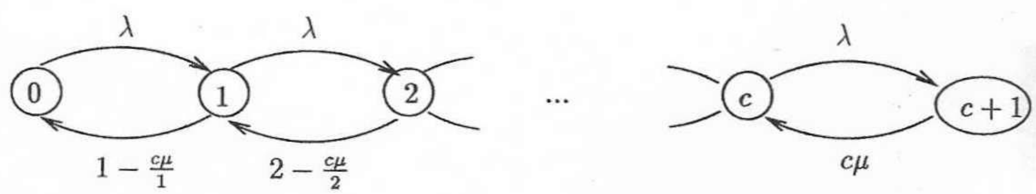
a) $$\begin{aligned}\mathcal{E}[T] &= \mathcal{E}[W] + \frac{1}{\mu} \le 4 \Rightarrow \mathcal{E}[W] \le 4 - \frac{1}{\mu} = 2 \\ \mathcal{E}[W] &= \frac{\frac{1}{\mu}}{c(1-\rho)}C(c,a) \le 2 \Rightarrow C(c,a) \le c(1-\rho) \\ C(c,a) &\le c - c\rho = c - a = c - 5 \\ &\Rightarrow c \ge 5 + C(c,a) \\ &\Rightarrow \text{try } c = 6 \text{ and check waiting time requirement}\end{aligned}$$

$$C(6,5) = \frac{p_0}{1-\rho} = 0.5875 \text{ where } p_0 = 0.004512$$

$$\begin{aligned}\therefore P[W \le 8] &= 1 - c(6,5)e^{-6\left(\frac{1}{2}\right)\left(1-\frac{5}{6}\right)8} = 0.9892 \\ &\Rightarrow \quad c = 6 \quad \text{OK} \\ p_0 &= \left\{\sum_{j=0}^{5}\frac{5^j}{j!} + \frac{\frac{5^6}{6!}}{1-\frac{5}{6}}\right\}^{-1} = 0.004512\end{aligned}$$

$$P[\text{all servers busy}] = P[N \ge c] = C(c,a) = 0.5875$$

12.24 **a)**



b)

$$\begin{aligned}
P_j &= \frac{\lambda}{c\mu}P_{j-1} \qquad j \geq 1 \\
&= \left(\frac{\lambda}{c\mu}\right)^j P_0 \\
\Rightarrow P_j &= (1-\rho)\rho^j \qquad j \geq 0, \qquad \rho = \frac{\lambda}{c\mu}
\end{aligned}$$

c)

$$\begin{aligned}
\mathcal{E}[N] &= \frac{\rho}{1-\rho} \qquad \mathcal{E}[T] = \frac{1}{\lambda}\mathcal{E}[N] = \frac{\frac{1}{c\mu}}{1-\rho} \\
\mathcal{E}[N_q] &= \sum_{k=c}^{\infty}(k-c)P_k = \sum_{k=c}^{\infty}(k-c)(1-\rho)\rho^k \\
&= \rho^c(1-\rho)\sum_{k'=0}^{\infty}k'\rho^{k'} = \frac{\rho^{c+1}}{1-\rho} \\
\mathcal{E}[W] &= \frac{1}{\lambda}\mathcal{E}[N_q] = \frac{\rho^c}{1-\rho}\frac{1}{c\mu}
\end{aligned}$$

d) M/M/1

$$\begin{aligned}
\mathcal{E}[T] &= \frac{\frac{1}{c\mu}}{1-\rho} \qquad \text{same as above system} \\
\mathcal{E}[W] &= \frac{\rho}{1-\rho}\frac{1}{c\mu}
\end{aligned}$$

M/M/2

$$\begin{aligned}
C(2,a) &= \frac{a^2/2}{1-\rho}\left[1+a+\frac{a^2/a}{1-\rho}\right]^{-1} = \frac{2\rho^2}{1+\rho} \qquad \text{since } a = 2\rho \\
\mathcal{E}[W] &= \frac{\frac{1}{\mu}}{c(1-\rho)}C(c,a) = \frac{2\rho^2(1+\rho)}{2\mu(1-\rho)} \\
\mathcal{E}[T] &= \mathcal{E}[W] + \frac{1}{\mu} = \frac{2/(1+\rho)}{2\mu(1-\rho)}
\end{aligned}$$

Comparison:

	M/M/1	M/M/2	New System
$\mathcal{E}[T]$	$\frac{1}{2\mu(1-\rho)}$	$\frac{2/(1+\rho)}{2\mu(1-\rho)}$	$\frac{1}{2\mu(1-\rho)}$
$\mathcal{E}[W]$	$\frac{\rho}{2\mu(1-\rho)}$	$\frac{2\rho^2/(1+\rho)}{2\mu(1-\rho)}$	$\frac{1}{2\mu(1-\rho)}$

for $\mathcal{E}[T]$ New = M/M/1 < M/M/2

for $\mathcal{E}[W]$ New < M/M/2 < M/M/1

12.25 for the queue of P. 12.24 we have:

$$P_i = \frac{\lambda}{c\mu} P_{i-1} \Rightarrow P_i = \left(\frac{\lambda}{c\mu}\right)^i P_0 = (1-\rho)\rho^i$$

$$\sum_{j=0}^{\infty} P_j = 1 \Rightarrow P_0 + \sum_{j=1}^{\infty} P_j = 1 \Rightarrow \frac{1}{1-\frac{\lambda}{c\mu}} P_0 = 1 \Rightarrow P_0 = 1-\frac{\lambda}{c\mu} = 1-\rho$$

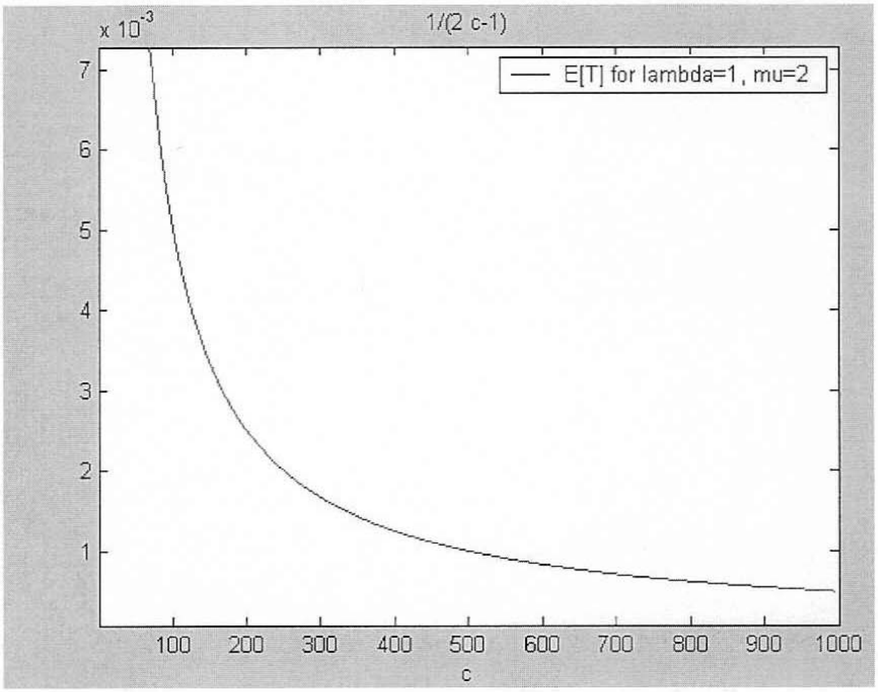
$$E(N) = \frac{\rho}{1-\rho}, \quad E(T) = \frac{E(N)}{\lambda} = \frac{\frac{1}{c\mu}}{1-\frac{\lambda}{c\mu}} = \boxed{\frac{1}{c\mu-\lambda}}$$

$$E(N_q) = \sum_{i=c}^{\infty}(i-c)P_i = \sum_{i=c}^{\infty}(i-c)(1-\rho)\rho^i = \rho^c(1-\rho)\sum_{j=0}^{\infty} j\rho^j$$

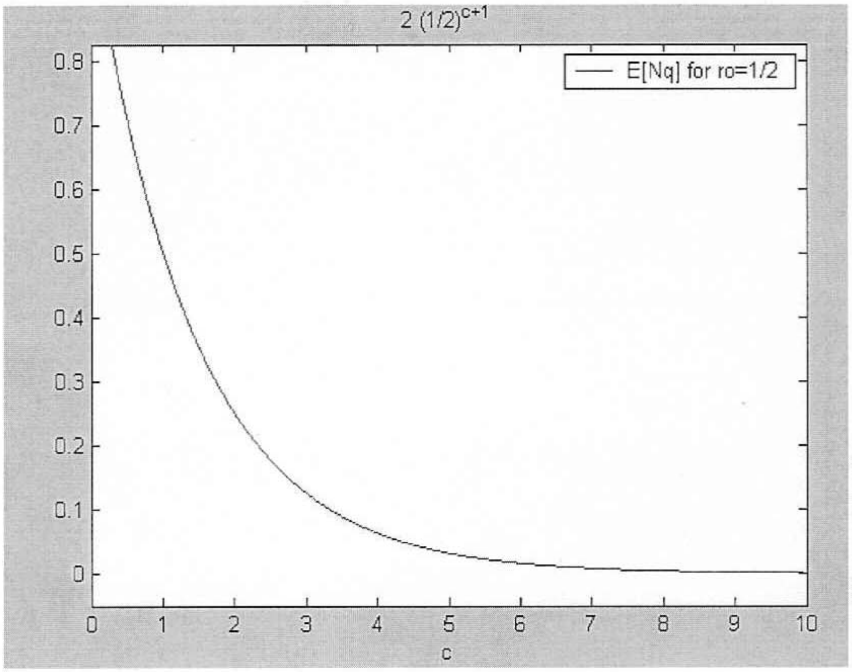
$$= \rho^c(1-\rho) \times \frac{\rho}{(1-\rho)^2} = \boxed{\frac{\rho^{c+1}}{1-\rho}}$$

we write a simple code to plot $E[T]$ & $E(N_q)$

```
c=sym('c');
>> f=inline('1/(c*mu-lambda)');
>> f

f =

     Inline function:
     f(c,lambda,mu) = 1/(c*mu-lambda)

>> ezplot(f(c,1,1))
```



```
f=inline('ro^(c+1)/(1-ro)');
ezplot(f(c,1/2))
```



12.26

$$B(c,a) = \frac{\frac{a^c}{c!}}{\sum_{j=0}^{c}\frac{a^j}{j!}} = \frac{\frac{a^c}{c!}}{\frac{a^c}{c!}+\sum_{j=0}^{c-1}\frac{a^j}{j!}} = \frac{\frac{a^c}{c!}/\sum_{j=0}^{c-1}\frac{a^j}{j!}}{1+\frac{a^c}{c!}/\sum_{j=0}^{c-1}\frac{a^j}{j!}}$$

$$= \frac{\frac{a}{c}B(c-1,a)}{1+\frac{a}{c}B(c-1,a)} = \frac{aB(c-1,a)}{c+aB(c-1,a)}$$

12.27

~~9.24~~ $\lambda = 10 \qquad \frac{1}{\mu} = \frac{1}{2} \qquad \frac{\lambda}{\mu} = 5 = a$

$$\begin{aligned}
B(0,5) &= 1\\
B(1,5) &= \frac{5\cdot 1}{1+5\cdot 1} = \frac{5}{6}\\
B(2,5) &= \frac{5\left(\frac{5}{6}\right)}{2+5\left(\frac{5}{6}\right)} = \frac{25}{37}\\
B(3,5) &= \frac{5\left(\frac{25}{37}\right)}{3+5\left(\frac{25}{37}\right)} = \frac{125}{236}\\
B(4,5) &= \frac{5\left(\frac{125}{236}\right)}{4+5\left(\frac{125}{236}\right)} = \frac{625}{1569}\\
\rightarrow B(5,5) &= \frac{5\left(\frac{625}{1569}\right)}{5+5\left(\frac{625}{1569}\right)} = \frac{625}{2194} \approx 28.5\%\\
&\vdots\\
B(8,5) &= 0.070 \qquad \text{need 3 more servers}
\end{aligned}$$

12.28

~~9.25~~ **a)** $\lambda = \frac{1}{2} \qquad \frac{1}{\mu} = 2 \qquad a = 1$

$$B(4,1) = \frac{1}{65} = 1.54\%$$

b) $\mathcal{E}[N] = a(1-B(4,1)) = \frac{64}{65} = 0.985$

c) $B(3,1) = \frac{1}{16} = 6.25\%$ an increase of 4.7%

12.29

~~9.26~~ For $a < c$

$$\textbf{a)}\quad C(c,a) = \frac{\frac{a^c}{c!}\frac{1}{1-\rho}}{\sum_{j=0}^{c-1}\frac{a^j}{j!}+\frac{a^c}{c!}\left(\frac{1}{1-\rho}\right)} = \frac{\frac{a^c}{c!}}{(1-\rho)\sum_{j=0}^{c-1}\frac{a^j}{j!}+\frac{a^c}{c!}} \qquad \text{but } \rho = \frac{a}{c}$$

$$= \frac{\frac{a^c}{c!}}{\left(1-\frac{a}{c}\right)\sum_{j=0}^{c-1}\frac{a^j}{j!}+\frac{a^c}{c!}} = \frac{a\frac{a^{c-1}}{(c-1)!}}{(c-a)\sum_{j=0}^{c-1}\frac{a^j}{j!}+a\frac{a^{c-1}}{(c-1)!}}$$

$$= \frac{aB(c-1,a)}{(c-a)+1B(c-1,a)} \qquad \text{since} \quad \frac{\frac{a^{c-1}}{(c-1)!}}{\sum_{j=0}^{c-1}\frac{a^j}{j!}} = B(c-1,a)$$

Problem 12.26 $\Rightarrow aB(c-1,a) = \dfrac{cB(c,a)}{1-B(c,a)}$

$$\therefore C(c,a) = \frac{cB(c,a)}{(c-a)(1-B(c,a))+cB(c,a)} = \frac{cB(c,a)}{c-a(1-B(c,a))}$$

b) $C(c,a) = \dfrac{B(c,a)}{1-\frac{a}{c}(1-B(c,a))} > B(c,a)$ since

$$\frac{1}{1-\frac{a}{c}\underbrace{(1-B(c,a))}_{<1}} > \frac{1}{1-\underbrace{\frac{a}{c}}_{<1}} > 1$$

12.30

~~9.27~~ $\lambda = 1 \qquad \dfrac{1}{\mu} = 2 \qquad a = \dfrac{\lambda}{\mu} = 2$

a) $P[\text{redirected}] = B(3,2) = \dfrac{4}{19} = 21.1\%$

b) $P[\text{redirected}] = B(6,4) = \dfrac{256}{2185} = 11.7\%$

$\lambda' = 2 \qquad \dfrac{1}{\mu} = 2$

12.31

M/M/10

λ cust/sec $\frac{\lambda}{2}$ Type 1 $\overline{X}_1 = 1$ $\overline{X} = \frac{1}{2}\overline{X}_1 + \frac{1}{2}\overline{X}_2 = \frac{1}{2}(1+3) = 2$

$\frac{\lambda}{2}$ Type 2 $\overline{X}_2 = 3$

$P_B = B(10, a)$ $a = \lambda\overline{X} = 2\lambda$

alternative: 2 M/M/5 systems

	$\lambda=1$	$\lambda=2$	$\lambda=3$
$P_{B_1} = B(5, \frac{\lambda}{2})$	0%	0%	1%
$P_{B_2} = B(5, \frac{3\lambda}{2})$	1%	11%	24%
$P_B = B(10, 2\lambda)$	0%	1%	4%

Combined system does better for overall blocking;
short service customers affected only at higher load

M/M/100

	$\lambda=46$	$\lambda=48$	$\lambda=50$
$P_{B1} = B(50, \frac{\lambda}{2})$	0%	0%	0%
$P_{B2} = B(50, \frac{3\lambda}{2})$	31%	33%	35%
$P_B = B(100, 2\lambda)$	3%	5%	8%

Combined system hugely beneficial to customers with
longer service time.

12.32

$$P[N=c] = \frac{a^c}{c!}e^{-a}$$

$$B(c,a) = \frac{\frac{a^c}{c!}}{\sum_{j=0}^{c}\frac{a^j}{j!}} < \frac{\frac{a^c}{c!}}{\sum_{j=0}^{\infty}\frac{a^j}{j!}} = \frac{a^c}{c!}e^{-a}$$

$\Rightarrow P[N=c]$ estimate is conservative

12.33 This system can be modeled with an M/M/∞ queue. $\lambda = 10$ $\frac{1}{\mu} = 3600 \Rightarrow \mu = \frac{1}{3600}$

$a = \frac{\lambda}{\mu} = 36{,}000$

$$\Rightarrow P_j = \frac{a^j}{j!} e^{-a}$$ a poisson dist.

when a is large enough, the poisson dist. can be approximated with a Gaussian, which is the case in this problem

12.5 Finite-Source Queueing Systems

12.34

~~9.29~~ a)

$$\begin{aligned}\rho &= 1 - p_0 = 1 - \frac{1}{\sum_{k=0}^{K} \frac{K!}{(K-k)!}\left(\frac{\alpha}{\mu}\right)^k} \qquad \text{let } j = K-k \\ &= 1 - \frac{\left(\frac{\mu}{\alpha}\right)^K / K!}{\sum_{j=0}^{K} \frac{(\mu\alpha)^j}{j!}} \\ &= 1 - \underbrace{B\left(K, \frac{\mu}{\alpha}\right)}_{\text{Erlang } B}\end{aligned}$$

$$K = 15 \qquad \frac{1}{\mu} = 2 \qquad \frac{1}{\alpha} = 30 \qquad \frac{\mu}{\alpha} = 15$$

$$B(15,15) = 0.18$$

$$\begin{aligned}\rho &= 1 - B\left(K, \frac{\mu}{\alpha}\right) = 11 - 0.18 = 0.82 \\ \lambda &= \mu\rho = \frac{1}{2}0.82 = 0.41 \\ \mathcal{E}[T] &= \frac{K}{\lambda} - \frac{1}{\alpha} = \frac{15}{0.41} - 30 = 6.6\end{aligned}$$

b) $K^* = \dfrac{\frac{1}{\mu} + \frac{1}{\alpha}}{\frac{1}{\mu}} = \dfrac{32}{2} = 16$

c) If we add 5 users we exceed K^* so

$$\begin{aligned}\mathcal{E}[T] &\approx \frac{K}{\mu} - \frac{1}{\alpha} = 20(2) - 30 = 10 \\ \lambda &\approx \mu = 2\end{aligned}$$

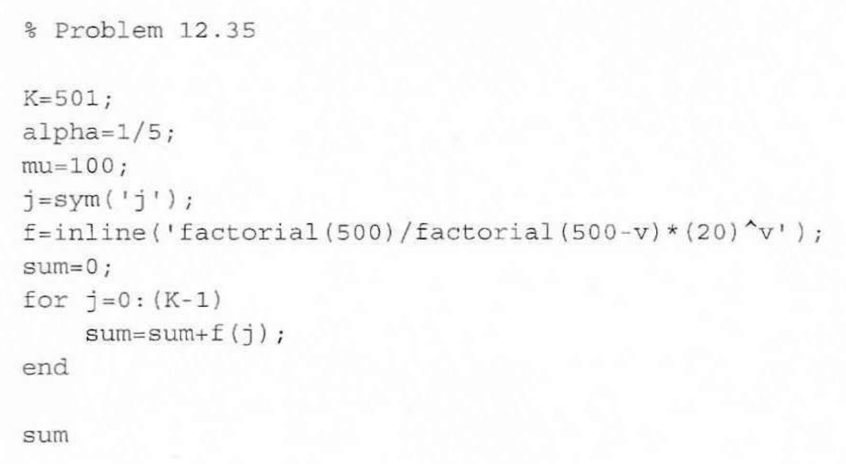
12.35

(a) $\frac{1}{\mu} = 0.01$, $\frac{1}{\alpha} = 5$ $\qquad \frac{\alpha}{\mu} = \frac{1/5}{\frac{1}{0.01}} = 20$

$$K^* \simeq \frac{\frac{1}{\mu}+\frac{1}{\alpha}}{\frac{1}{\mu}} = \frac{0.01+5}{0.01} = 501$$

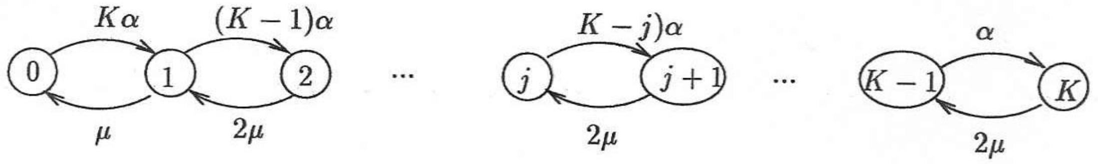
(b) The pmf of the number of request can be obtained using eq 12.84 $P[N_a = k] = \frac{\left[(K_r-1)!/(K_r-k-1)!\right](\alpha/\mu)^k}{\sum_{j=0}^{K_r-1}\left[(K_r-1)!/(K_r-j-1)!\right](\alpha/\mu)^j}$.

$$\Rightarrow P[N_a = k] = \frac{\frac{500!}{(500-k)!}(20)^k}{\sum_{j=0}^{500}\frac{500!}{(500-j)!}(20)^j}$$



```
% Problem 12.35

K=501;
alpha=1/5;
mu=100;
j=sym('j');
f=inline('factorial(500)/factorial(500-v)*(20)^v');
sum=0;
for j=0:(K-1)
    sum=sum+f(j);
end

sum
```

12.36



$$\begin{aligned}
p_1 &= \frac{K\alpha}{\mu}p_0 \\
p_{j+1} &= \frac{(K-j)\alpha}{2\mu}p_j \qquad 1 < j \leq K-1 \\
\Rightarrow p_j &= \frac{K(K-1)\ldots(K-j+1)}{2^{j-1}}\left(\frac{\alpha}{\mu}\right)^j p_0 = 2\frac{K!}{(K-j)!}\left(\frac{\alpha}{2\mu}\right)^j p_0 \\
p_0 &= \left[1 + 2\sum_{j=1}^{K}\frac{K!}{(K-j)!}\left(\frac{\alpha}{2\mu}\right)^j\right]^{-1}
\end{aligned}$$

12.37

$$P[N_a = k] = \frac{\frac{(K-1)!(\alpha/\mu)^k}{(K-1-k)!}}{\sum\limits_{k'=0}^{K-1} \frac{(K-1)!(\alpha/\mu)^{k'}}{(K-1-k')!}} = \frac{\frac{(\alpha/\mu)^k}{(K-1-k)!}}{\sum_{k'=0}^{K-1} \frac{(\alpha/\mu)^{k'}}{(K-1-k')!}}$$

$$\begin{aligned}
\mathcal{E}[T] &= \frac{1}{\mu}\sum_{k=0}^{K-1}(k+1)P[N_a = k] \\
&= \frac{1}{\mu}\sum_{k=0}^{K-1}(k+1)\frac{\frac{(\alpha/\mu)^k}{(K-1-k)!}}{\sum\limits_{k'=0}^{K-1}\frac{(\alpha/\mu)^{k'}}{(K-1-k')!}} \qquad \text{Let } j = K-1-k,\ j' = K-1-k' \\
&= \frac{1}{\mu}\sum_{j=0}^{K-1}(K-j)\frac{\frac{(\mu/\alpha)^j}{j!}}{\underbrace{\sum\limits_{j'=0}^{K-1}\frac{(\mu/\alpha)^{j'}}{j'!}}_{\text{probs of M/M/K-1/K-1}}} \\
&= \frac{1}{\mu}\left[K - \underbrace{\frac{\mu}{\alpha}\left(1 - B(K-1, \frac{\mu}{\alpha}\right)}_{\text{mean \# in M/M/K-1/K-1}}\right] \\
&= \frac{K}{\mu} - \frac{1}{\alpha}\left(1 - B\left(K-1, \frac{\mu}{\alpha}\right)\right)
\end{aligned}$$

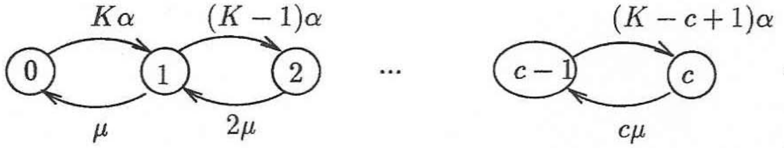
From Problem 12.26

$$\begin{aligned}
B\left(K-1, \frac{\mu}{\alpha}\right) &= \frac{\frac{\alpha K}{\mu} B\left(K, \frac{\mu}{\alpha}\right)}{1 - B\left(K, \frac{\mu}{\alpha}\right)} \\
\mathcal{E}[T] &= \frac{K}{\mu} - \frac{1}{\alpha} + \frac{\frac{K}{\mu} B\left(K, \frac{\mu}{\alpha}\right)}{1 - B\left(K, \frac{\mu}{\alpha}\right)} \\
\mathcal{E}[T] &= \frac{K}{\mu}\left[1 + \frac{B\left(K, \frac{\mu}{\alpha}\right)}{1 - B\left(K, \frac{\mu}{\alpha}\right)}\right] - \frac{1}{\alpha} \\
&= \frac{K}{\mu}\frac{1}{1 - B\left(K, \frac{\mu}{\alpha}\right)} - \frac{1}{\alpha}
\end{aligned}$$

But for Problem 12.34 solution

$$\begin{aligned}
\rho &= \frac{\lambda}{\mu} = 1 - B\left(K, \frac{\mu}{\alpha}\right) \\
\Rightarrow \mathcal{E}[T] &= \frac{K}{\lambda} - \frac{1}{\alpha} \qquad \text{as desired} \qquad \checkmark
\end{aligned}$$

12.38



where $c \le K$

$$K\alpha P_0 = \mu P_1 \quad\Rightarrow\quad P_1 = \frac{K\alpha}{\mu}P_0$$

$$(K-j+1)\alpha P_{j-1} + j\mu P_j \quad\Rightarrow\quad P_j = \frac{(K-j+1)\alpha}{j\mu}P_{j-1}$$

$$\Rightarrow P_j \;=\; \frac{K\ldots(K-j+1)}{j!}\left(\frac{\alpha}{\mu}\right)^j P_0 = \underbrace{\frac{K!}{j!(K-j)!}}_{\binom{K}{j}}\left(\frac{\alpha}{\mu}\right)^j P_0$$

$$\Rightarrow P_0 \;=\; \left[\sum_{j=0}^{c}\binom{K}{j}\left(\frac{\alpha}{\mu}\right)^j\right]^{-1}$$

$$\therefore P_j \;=\; \frac{\binom{K}{j}\left(\frac{\alpha}{\mu}\right)^j}{\sum_{j'=0}^{c}\binom{K}{j'}\left(\frac{\alpha}{\mu}\right)^{j'}} = \frac{\binom{K}{j}\left(\frac{\alpha}{\alpha+\mu}\right)^j\left(\frac{\mu}{\alpha+\mu}\right)^{K-j}}{\sum_{j'=0}^{c}\binom{K}{j'}\left(\frac{\alpha}{\alpha+\mu}\right)^{j}\left(\frac{\mu}{\alpha+\mu}\right)^{K-j}}$$

$$= \frac{\binom{K}{j}p^j(1-p)^{K-j}}{\sum_{j'=0}^{c}\binom{K}{j'}p^{j'}(1-p)^{K-j}}$$

b) $P[\text{all servers busy}] = P_c = \dfrac{\binom{K}{c}p^c(1-p)^{K-c}}{\sum_{j'=0}^{c}\binom{K}{j'}p^{j'}(1-p)^{K-j'}}$

c) Since an arriving customer sees the steady state of the system with one fewer server:

$P[\text{an arriving customer sees } c \text{ customers in system}]$

$$= P_{K-1}[N=c]$$

$$= \frac{\binom{K-1}{c}p^c(1-p)^{K-1-c}}{\sum_{j'=0}^{c}\binom{K-1}{j'}p^{j'}(1-p)^{K-1-j'}}$$

12.39 The probability that all the servers are busy in Engset (P. 12.38) is:

$$P_{busy} = \frac{\binom{K}{c} p^c (1-p)^{K-c}}{\sum_{i=0}^{c} \binom{K}{i} p^i (1-p)^{K-i}} \quad \text{where } p = \frac{\alpha}{\alpha+\mu}$$

$\mu = \frac{1}{0.1} = 10 \Rightarrow p = \frac{1}{11}$ $\quad c = 10$

$\alpha = 1$

we want $P_{busy} = 10\% = 0.1$

$$\Rightarrow \frac{\binom{K}{10} \left(\frac{1}{11}\right)^{10} \left(\frac{10}{11}\right)^{K-10}}{\sum_{i=0}^{10} \binom{K}{i} \left(\frac{1}{11}\right)^{i} \left(\frac{10}{11}\right)^{K-i}} = 0.1$$

12.6 M/G/1 Queueing Systems

12.40

~~**9.33**~~ A k-Erlang RV X with parameter k and λ has

$$\mathcal{E}[X] = \frac{k}{\lambda} \qquad VAR[X] = \frac{k}{\lambda^2}$$

Since $\mathcal{E}[X] = \dfrac{1}{\mu}$ we have that $\lambda = k\mu$ and

$$\begin{aligned}
VAR[X] &= \frac{k}{k^2\mu^2} = \frac{1}{k\mu^2} \\
\Rightarrow C_X^2 &= \frac{VAR[X]}{\mathcal{E}[X]^2} = \frac{1}{k} \\
\therefore \mathcal{E}[W]_{M/E_k/1} &= \frac{\rho(1+C_X^2)}{2(1-\rho)}\mathcal{E}[\tau] = \frac{\rho\left(1+\frac{1}{k}\right)}{2(1-\rho)}\mathcal{E}[\tau]
\end{aligned}$$

For M/M/1 we let $k = 1$ and obtain

$$\mathcal{E}[W]_{M/M/1} = \frac{2\rho}{2(1-\rho)}\mathcal{E}[\tau]$$

For M/D/1 $C_X^0 = 0$ so

$$\mathcal{E}[W]_{M/D/1} = \frac{\rho}{2(1-\rho)}\mathcal{E}[\tau]$$

$$\therefore \mathcal{E}[W]_{M/D/1} < \mathcal{E}[W]_{M/E_k/1} \leq \mathcal{E}[W]_{M/M/1}$$

Since $\mathcal{E}[T] = \mathcal{E}[W] + \mathcal{E}[\tau]$ the same ordering applies for total delay.

12.41

$$\begin{aligned}
\mathcal{E}[\tau] &= \mathcal{E}[\tau|1]p + \mathcal{E}[\tau|2](1-p) = \frac{1}{\mu_1}p + \frac{1}{\mu_2}(1-p) \\
\mathcal{E}[\tau^2] &= \mathcal{E}[\tau^2|1]p + \mathcal{E}[\tau^2|2](1-p) = \frac{2}{\mu_1^2}p + \frac{2}{\mu_2^2}(1-p) \\
\mathcal{E}[W] &= \frac{\lambda\mathcal{E}[\tau^2]}{2(1-\rho)} = \frac{\lambda/2}{1-\rho}\left[\frac{2}{\mu_1^2}p + \frac{2}{\mu_2^2}(1-p)\right] \\
\mathcal{E}[T] &= \mathcal{E}[W] + \mathcal{E}[\tau]
\end{aligned}$$

where $\rho = \lambda\mathcal{E}[\tau]$.

12.42

$$\begin{aligned}
\mathcal{E}[\tau] &= \mathcal{E}[\tau|1]\alpha + \mathcal{E}[\tau|2](1-\alpha) = d\alpha + \frac{1}{\mu}(1-\alpha) \\
\mathcal{E}[\tau^2] &= \mathcal{E}[\tau^2|1]\alpha + \mathcal{E}[\tau^2|2](1-\alpha) = d^2\alpha + \frac{2}{\mu_2^2}(1-\alpha) \\
\mathcal{E}[W] &= \frac{\lambda\mathcal{E}[\tau^2]}{2(1-\rho)} = \frac{\lambda/2}{1-\rho}\left[\alpha d^2 + (1-\alpha)\frac{2}{\mu^2}\right] \\
\mathcal{E}[T] &= \mathcal{E}[W] + \mathcal{E}[\tau]
\end{aligned}$$

where $\rho = \lambda\mathcal{E}[\tau]$.

12.43

$$\begin{aligned}
\tau &= d + \tau_1 \\
\mathcal{E}[\tau] &= d + \mathcal{E}[\tau_1] = d + \frac{1}{\mu} \\
\mathcal{E}[\tau^2] &= d^2 + 2d\mathcal{E}[\tau_1] + \mathcal{E}[\tau_1^2] \\
&= d^2 + \frac{2d}{\mu} + \frac{2}{\mu^2} \\
\mathcal{E}[W] &= \frac{\lambda/2}{1-\rho}\mathcal{E}[\tau^2] = \frac{\lambda/2}{1-\rho}\left[d^2 + \frac{2d}{\mu} + \frac{2}{\mu^2}\right] \\
\mathcal{E}[T] &= \mathcal{E}[W] + \mathcal{E}[\tau]
\end{aligned}$$

12.44

9.37 A message is transmitted until a successful acknowledgement is received:

a) $P[\tau = kd] = (1-p)p^{k-1} \qquad k =, 1, 2, \ldots$

$$\mathcal{E}[\tau] = \frac{d}{1-p} \qquad VAR[\tau] = \frac{d^2 p}{(1-p)^2}$$

b) $C_\tau^2 = \frac{VAR[\tau]}{\mathcal{E}[\tau]^2} = \frac{d^2 p}{(1-p)^2}\frac{(1-p)^2}{d^2} = p$

$$\begin{aligned}
\mathcal{E}[T] &= \mathcal{E}[\tau] + \mathcal{E}[\tau]\frac{\rho}{2(1-\rho)}(1 + C_\tau^2) \qquad \text{where } \rho = \frac{\lambda d}{1-p} \\
&= \frac{2-\lambda d}{2(1-p-\lambda d)}d
\end{aligned}$$

12.47

~~9.38~~ **a)** Let τ = total job time, X = service time, $N(X)$ = # breakdowns during X, R_i = repair times

$$\tau = X + \sum_{i=1}^{N(X)} R_i$$

where $N(X)$ is the total number of times the machine breaks down. To find $C[\tau]$ we use conditional expectation:

$$\begin{aligned}
\mathcal{E}[\tau] &= \mathcal{E}[\mathcal{E}[\tau|X]] \\
\mathcal{E}[\tau|X=t] &= t + \mathcal{E}\left[\sum_{i=1}^{N(t)} R_i\right] = t + \alpha t\mathcal{E}[R] \qquad \text{from Eq. 5.13} \\
\Rightarrow \mathcal{E}[\tau] &= \mathcal{E}[X + \alpha X\mathcal{E}[R]] = \underbrace{\mathcal{E}[X] + \alpha\mathcal{E}[X]\mathcal{E}[R]}_{\mathcal{E}[X](1+\alpha\mathcal{E}[R])} = \frac{1}{\mu}\left[1 + \frac{\alpha}{\beta}\right]
\end{aligned}$$

We also use conditional expectation to find $E[\tau^2]$:

$$\begin{aligned}
\mathcal{E}[\tau^2] &= \mathcal{E}[\mathcal{E}[\tau^2|X]] \\
\mathcal{E}[\tau^2|X=t] &= \mathcal{E}\left[\left(t + \sum_{i=1}^{N(t)} R_i\right)^2\right] \\
&= t^2 + 2t\mathcal{E}\left[\sum_{i=1}^{N(t)} R_i\right] + \mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2\right] \\
&= t^2 + 2t(\alpha t + \mathcal{E}[R]) + \mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2\right] \\
\mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2\right] &= \mathcal{E}\left[\mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2 |N(t)\right]\right] \\
\mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2 |N(t)=k\right] &= \mathcal{E}\left[\sum_{i=1}^{k}\sum_{j=1}^{k} R_iR_j\right] \\
&= k\mathcal{E}[R^2] + (k^2-k)\mathcal{E}[R]^2 \\
\therefore \mathcal{E}\left[\left(\sum_{i=1}^{N(t)} R_i\right)^2\right] &= \mathcal{E}[N(t)\mathcal{E}[R^2] + [N^2(t) - N(t)]\mathcal{E}[R]^2] \\
&= \mathcal{E}[N(t)]\mathcal{E}[R^2] + (\mathcal{E}[N^2(t)] - \mathcal{E}[N(t)])\mathcal{E}[R]^2
\end{aligned}$$

$$\begin{aligned}
&= \alpha t\mathcal{E}[R^2] + (\alpha t + (\alpha t)^2 - \alpha t)\mathcal{E}[R]^2 \\
&= \alpha t\mathcal{E}[R^2] + \alpha^2 t^2\mathcal{E}[R]^2 \\
\therefore \mathcal{E}[\tau^2|X = t] &= t^2 + 2\alpha t^2\mathcal{E}[R] + \alpha t\mathcal{E}[R^2] + \alpha^2 t^2\mathcal{E}[R]^2
\end{aligned}$$

finally

$$\begin{aligned}
\mathcal{E}[\tau^2] &= \mathcal{E}[X^2 + 2\alpha X^2\mathcal{E}[R] + \alpha X\mathcal{E}[R^2] + \alpha^2 X^2\mathcal{E}[R]^2] \\
&= \mathcal{E}[X^2]\underbrace{[1 + 2\alpha\mathcal{E}[R] + \alpha^2\mathcal{E}[R]^2]}_{(1+\alpha\mathcal{E}[R])^2} + \mathcal{E}[X]\alpha\mathcal{E}[R^2]
\end{aligned}$$

$$\begin{aligned}
VAR[\tau] &= \mathcal{E}[\tau^2] - \mathcal{E}[\tau]^2 \\
&= \mathcal{E}[X^2](1 + \alpha\mathcal{E}[R])^2 + \mathcal{E}[X]\alpha\mathcal{E}[R^2] - \mathcal{E}[X]^2(1 + \alpha\mathcal{E}[R])^2 \\
&= VAR[X](1 + \alpha\mathcal{E}[R])^2 + \mathcal{E}[X]\alpha\mathcal{E}[R^2] \\
&= \frac{1}{\mu^2}\left(1 + \frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\mu}\frac{2}{\beta^2}
\end{aligned}$$

b) The coefficient of variation of τ is:

$$C_\tau^2 = \frac{VAR[\tau]}{\mathcal{E}[\tau]^2} = \frac{\frac{1}{\mu^2}\left(1 + \frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\mu}\frac{2}{\beta^2}}{\frac{1}{\mu^2}\left(1 + \frac{\alpha}{\beta}\right)^2} = 1 + \frac{2\alpha}{(\alpha + \beta)^2}$$

Thus the mean delay in the system is

$$\begin{aligned}
\mathcal{E}[T] &= \mathcal{E}[\tau] + \mathcal{E}[\tau]\frac{\rho}{2(1-\rho)}(1 + C_\tau^2) \\
&= \mathcal{E}[\tau]\left[1 + \frac{\rho}{(1-\rho)}\left(1 + \frac{\alpha}{(\alpha+\beta)^2}\right)\right]
\end{aligned}$$

where

$$\rho = \lambda\mathcal{E}[\tau] = \frac{\lambda}{\mu}\left[1 + \frac{\alpha}{\beta}\right]$$

12.48

9.39 a) The proportion of time that the server works on low priority jobs is

$$\begin{aligned}\rho_2' &= 1-\rho_1 = \lambda_2'\mathcal{E}[\tau_2] \\ \Rightarrow \lambda_2' &= \frac{1-\rho_1}{\mathcal{E}[\tau_2]} = \frac{1-\lambda_1\mathcal{E}[\tau_1]}{\mathcal{E}[\tau_2]}\end{aligned}$$

b) From Eq. (12.105)

$$\begin{aligned}\mathcal{E}[W_1] &= \frac{\lambda_1\mathcal{E}[\tau_1^2]+\lambda_2'\mathcal{E}[\tau_2^2]}{2(1-\rho_1)} \\ &= \frac{\lambda_1\mathcal{E}[\tau_1^2]}{2(1-\rho_1)} + \frac{\lambda_2'\mathcal{E}[\tau_2^2]}{2(1-\rho_1)} \\ &= \frac{\frac{\lambda_1}{2}\mathcal{E}[\tau_1^2]}{1-\lambda_1\mathcal{E}[\tau_1]} + \frac{\mathcal{E}[\tau_2^2]}{2\mathcal{E}[\tau_2]}\end{aligned}$$

since $\rho_1 = \lambda_1\mathcal{E}[\tau_1]$, $1-\rho_1 = \lambda_2'\mathcal{E}[\tau_2]$

12.49

9.40 The server vacations can be viewed as the servicing of a low priority class of fictitious customers whose service times are a vacation time and whose arrival rate saturates the system. The result of Problem 9.39b then implies

$$\mathcal{E}[W] = \frac{\frac{1}{2}\lambda\mathcal{E}[\tau^2]}{1-\lambda\mathcal{E}[\tau]} + \frac{\mathcal{E}[V^2]}{2\mathcal{E}[V]}$$

where W and τ correspond to the real customers.

12.50

9.41 If we suppose that the server in Problem 12.49 takes vacations of fixed duration d, then we have the system described in the problem. Thus

$$\mathcal{E}[W] = \frac{\frac{1}{2}\lambda d^2}{1-\lambda d} + \frac{d^2}{2d} = \frac{\frac{1}{2}\lambda d^2}{1-\lambda d} + \frac{d}{2}$$

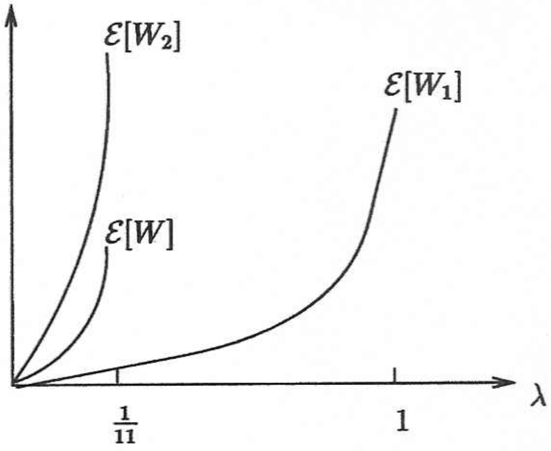
12.51

$$\begin{aligned}\mathcal{E}[\tau^2] &= \mathcal{E}[\tau^2|\text{type 1}]P[\text{type 1}] + \mathcal{E}[\tau^2|\text{type 2}]P[\text{type 2}] \\ &= \frac{2}{\mu_1^2}\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{2}{\mu_2^2}\frac{\lambda_2}{\lambda_1+\lambda_2} \\ &= 1 + 100 = 101\end{aligned}$$

where $\lambda = \lambda_1 = \lambda_2$ and $\mu_1 = 1$, $\mu_2 = \frac{1}{10}$.

$$\mathcal{E}[W_1] = \frac{\mathcal{E}[R'']}{1-\rho_1} = \frac{\frac{2\lambda}{2}\mathcal{E}[\tau^2]}{1-\rho_1} = \frac{101\lambda}{1-\lambda} \qquad \text{since } \rho_1 = \lambda/\mu_1 = \lambda$$

$$\begin{aligned}\mathcal{E}[W_2] &= \frac{101\lambda}{(1-\lambda)(1-\lambda-10\lambda)} \qquad \text{since } \rho_2 = \lambda/\mu_2 = 10\lambda \\ &= \frac{101\lambda}{(1-\lambda)(1-11\lambda)} \\ \mathcal{E}[W] &= \frac{\lambda_1}{\lambda_1+\lambda_2}\mathcal{E}[W_1] + \frac{\lambda_1}{\lambda_1+\lambda_2}\mathcal{E}[W_2] \\ &= \frac{1}{2}\frac{101\lambda}{1-\lambda}\left[1 + \frac{1}{1-11\lambda}\right]\end{aligned}$$



12.52

~~9.43~~ **a)** The low priority customers are "invisible" to the high priority customers. Thus the mean waiting time and delay of high priority customers is that of a single-class M/G/1 system:

$$\begin{aligned}\mathcal{E}[W_1] &= \frac{\lambda_1 \mathcal{E}[\tau_1^2]}{2(1-\rho_1)}\\ \mathcal{E}[T_1] &= \mathcal{E}[W_1] + \mathcal{E}[\tau]\end{aligned}$$

b) The time required to service all customers found by a low priority arrival is the time required to service all such customers in an ordinary M/G/1 system in which both classes are combined and neither receives priority. The reason for this is that the priority mechanism alters the order in which customers are served but not the rate at which the backlog is reduced. The mean waiting time in such a system is

$$\frac{\lambda \mathcal{E}[\tau^2]}{2(1-\rho)} = \frac{\frac{1}{2}\sum_{j=1}^{2} \lambda_j \mathcal{E}[\tau_j^2]}{(1-\rho_1-\rho_2)}$$

since $\lambda = \lambda_1 + \lambda_2$, $\rho = \rho_1 + \rho_2$

c) The mean time required to service all the high priority customers that arrive while a low priority customer is in the system is

$$\begin{aligned}\mathcal{E}\left[\sum_{i=1}^{N_1(T_2)} \tau_{i_1}\right] &= \mathcal{E}[N_1(T_2)]\mathcal{E}[\tau_1] = \lambda_1 \mathcal{E}[T_2]\mathcal{E}[\tau_1]\\ &= \rho_1 \mathcal{E}[T_2]\end{aligned}$$

d)

$$\begin{aligned}\mathcal{E}[T_2] &= \frac{R_2}{1-\rho_1-\rho_2} + \rho_1\mathcal{E}[T_2] + \frac{1}{\mu_2}\\ \mathcal{E}[T_2] &= \frac{R_2}{(1-\rho_1)(1-\rho_1-\rho_2)} + \frac{\frac{1}{\mu}}{1-\rho_1}\\ &= \frac{\frac{1}{\mu}(1-\rho_1-\rho_2) + R_2}{(1-\rho_1)(1-\rho_1-\rho_2)}\end{aligned}$$

12.53

~~9.44~~ $\lambda = \lambda_1 = \lambda_2 \qquad \mu_1 = 1 \qquad \mu_2 = \frac{1}{10}$

$$\begin{aligned}\mathcal{E}[W_1] &= \frac{\lambda_1 \mathcal{E}[\tau_1^2]}{2(1-\rho_1)} = \frac{\lambda}{1-\lambda}\\ \mathcal{E}[T_1] &= \frac{\lambda}{1-\lambda} + 1 = \frac{1}{1-\lambda}\\ R_2 &= \frac{1}{2}\lambda_1\frac{2}{\mu_1^2} + \frac{1}{2}\lambda_2\frac{2}{\mu_2^2} = \lambda + 100\lambda = 101\lambda\\ \mathcal{E}[T_2] &= \frac{101\lambda}{(1-\lambda)(1-11\lambda)} + \frac{10}{1-\lambda} = \frac{10-9\lambda}{(1-\lambda)(1-11\lambda)}\end{aligned}$$

The mean waiting time and delay of class 1 is reduced greatly while those of class 2 are not significantly affected relative to the corresponding values for a non-preemptive priority system.

12.7 M/G/1 Analysis Using Embedded Markov Chains

12.54 ~~9.45~~ $\rho = \frac{\lambda}{\mu} = \left(\frac{\mu}{2}\right)/\mu = \frac{1}{2}$

a) For an M/G/1 system we have:

$$G_N(z) = \frac{(1-\rho)(z-1)\hat{\tau}(\lambda(1-z))}{z - \hat{\tau}(\lambda(1-z))}$$

where

$$\begin{aligned}
\hat{\tau}(\lambda(1-z)) &= \frac{4\mu^2}{(s+2\mu)^2}\Big|_{s=\lambda(1-z)} = \frac{4\mu^2}{(\lambda - \lambda z + 2\mu)^2} \\
\Rightarrow G_N(z) &= \frac{\left(1-\frac{1}{2}\right)(z-1)4\mu^2}{z(\lambda - \lambda z + 2\mu) - 4\mu^2} = \frac{8}{z^2 - 9z + 16} \\
& \quad \text{where we used the fact that } \frac{\lambda}{\mu} = \frac{1}{2} \\
&= \frac{8}{(z-z_1)(z-z_2)} \qquad z_1 = \frac{9+\sqrt{17}}{2} \qquad z_2 = \frac{9-\sqrt{17}}{2} \\
&= \frac{8/z_1 z_2}{\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)} = \frac{\frac{1}{2}}{\left(1-\frac{1}{z_1}z\right)\left(1-\frac{1}{z_2}z\right)} \\
&= \frac{A}{1-\frac{1}{z_1}z} + \frac{B}{1-\frac{1}{z_2}z} \Rightarrow \begin{array}{l} A = \frac{-z_2/2}{z_1 - z_2} \\ B = \frac{z_1/2}{z_1 - z_2} \end{array} \quad \text{partial fraction expansion} \\
&= \frac{z_1/2}{(z_1 - z_2)\left(1-\frac{1}{z_2}z\right)} = \frac{z_2/2}{(z_1 - z_2)\left(1-\frac{z}{z_1}\right)} \\
&= \frac{1}{2(z_1 - z_2)}\left[z_1 \sum_{j=0}^{\infty}\left(\frac{z}{z_2}\right)^j - z_2 \sum_{j=0}^{\infty}\left(\frac{z}{z_1}\right)^j\right] \\
\therefore P[N=j] &= \frac{z_1}{2(z_1-z_2)}\left(\frac{1}{z_2}\right)^j - \frac{z_2}{2(z_1-z_2)}\left(\frac{1}{z_1}\right)^j \quad \text{coefficient of } Z^j \\
P[N=j] &= \frac{9+\sqrt{17}}{4\sqrt{17}}\left(\frac{2}{9-\sqrt{17}}\right)^j - \frac{9-\sqrt{17}}{4\sqrt{17}}\left(\frac{2}{9+\sqrt{17}}\right)^j \\
&= \frac{8}{\sqrt{17}}\left(\frac{2}{9-\sqrt{17}}\right)^{j+1} - \frac{8}{\sqrt{17}}\left(\frac{2}{9+\sqrt{17}}\right)^j \qquad j = 0, 1, \ldots
\end{aligned}$$

b) The Laplace Transform of the waiting time is:

$$\begin{aligned}
\hat{W}(s) &= \frac{(1-\rho)s}{s - \lambda + \lambda\hat{\tau}(s)} = \frac{\frac{1}{2}s}{s - \lambda + \frac{\lambda 4\mu^2}{(s+2\mu)^2}} = \frac{1}{2}\left[\frac{s^2 + 8\lambda s + 16\lambda^2}{s^2 + 7\lambda s + 8\lambda^2}\right] \\
&= \frac{1}{2}\left[1 + \frac{\left(\frac{\sqrt{17}+9}{2\sqrt{17}}\right)\lambda}{s + \left(\frac{7-\sqrt{17}}{2}\right)\lambda} + \frac{\left(\frac{\sqrt{17}-9}{2\sqrt{17}}\right)\lambda}{s + \left(\frac{7+\sqrt{17}}{2}\right)\lambda}\right]
\end{aligned}$$

$$\begin{aligned} f_W(t) &= \mathcal{L}^{-1}[\hat{W}(s)] \\ &= \frac{1}{2}\delta(t) + \frac{1}{2}\left(\frac{\sqrt{17}+9}{2\sqrt{17}}\right)\lambda e^{-\left(\frac{7-\sqrt{17}}{2}\right)\lambda t}u(t) \\ &\quad +\frac{1}{2}\left(\frac{\sqrt{17}-9}{2\sqrt{17}}\right)\lambda e^{-\left(\frac{7+\sqrt{17}}{2}\right)\lambda t}u(t) \end{aligned}$$

The total delay transform is:

$$\begin{aligned} \hat{T}(s) &= \frac{(1-\rho)s\hat{\tau}(s)}{s-\lambda+\lambda\hat{\tau}(s)} = \frac{\frac{1}{2}s\frac{4\mu^2}{(s+2\mu)^2}}{s-\lambda+\lambda\frac{4\mu^2}{(s+2\mu)^2}} \\ &= \frac{8\lambda^2}{s^2+7\lambda s+8\lambda^2} \\ \hat{T}(s) &= \frac{8\lambda}{\sqrt{17}}\left[\frac{1}{s+\left(\frac{7-\sqrt{17}}{2}\right)\lambda} - \frac{1}{s+\left(\frac{7+\sqrt{17}}{2}\right)\lambda}\right] \\ f_T(t) &= \mathcal{L}^{-1}[\hat{T}(s)] = \frac{8\lambda}{\sqrt{17}}\left[e^{-\left(\frac{7-\sqrt{17}}{2}\right)\lambda t} - e^{-\left(\frac{7+\sqrt{17}}{2}\right)\lambda t}\right]u(t) \end{aligned}$$

12.55
~~9.46~~ **a)** $\tau = X + \sum_{i=1}^{N(X)} R_i$ (see solution to 12.47)

$$\begin{aligned} \hat{\tau}(s) &= \mathcal{E}[e^{-s\tau}] = \mathcal{E}[\mathcal{E}[e^{-s\tau}|X]] \\ \mathcal{E}\left[e^{-s\left(X+\sum_{i=1}^{N(x)} R_i\right)}|X=t\right] &= e^{-st}\mathcal{E}\left[e^{-s\sum_{i=1}^{N(t)} R_i}\right] \\ \mathcal{E}\left[e^{-s\sum_{i=1}^{N(t)} R_i}\right] &= \mathcal{E}\left[\mathcal{E}\left[e^{-s\sum_{i=1}^{N} R_i}|N\right]\right] \\ &= \mathcal{E}[\underbrace{\mathcal{E}[e^{-sR}]^N}] \\ &= G_N[\hat{R}(s)] \quad \text{but} \quad G_N(z) = e^{\alpha t(z-1)} \\ &= e^{\alpha t(\hat{R}(s)-1)} \end{aligned}$$

$$\begin{aligned}
\hat{\tau}(s) &= \mathcal{E}[e^{-sX} e^{\alpha X(\hat{R}(s)-1)}] = \mathcal{E}[e^{-X(s-\alpha(\hat{R}(s)-1))}] \\
&= \hat{X}(s-\alpha(\hat{R}(s)-1))
\end{aligned}$$

But $\hat{R} = \dfrac{\beta}{s+\beta}$ and $\hat{X}(s) = \dfrac{\mu}{s+\mu}$

$$\begin{aligned}
\therefore \hat{\tau}(s) &= \frac{\mu}{s-\alpha(\hat{\alpha}(s)-1)+\mu} = \frac{\mu}{s-\alpha\frac{-s}{s+\beta}+\mu} \\
&= \frac{\mu(s+\beta)}{(s+\mu)(s+\beta)+\alpha s} \qquad \text{as required.}
\end{aligned}$$

b)

$$\begin{aligned}
\hat{W}(s) &= \frac{(1-\rho)s}{s-\lambda+\lambda\hat{\tau}(s)} = \frac{(1-\rho)s[(s+\mu)(s+\beta)+\alpha s]}{(s-\lambda)[(s+\mu)(s+\beta)+\alpha s]+\lambda\mu(s+\beta)} \\
&= (1-\rho)s[s^2+(\alpha+\beta+\mu)s+\mu\beta] \\
&= \frac{(1-\rho)(s^2+(\alpha+\beta+\mu)s+\mu\beta)}{s^2+(\alpha+\beta+\mu-\lambda)s-(\alpha+\beta+\mu-\mu\beta-\lambda\mu)} \\
&= (1-\rho)\left[1+\frac{\lambda s+(\alpha+\beta+\mu-\lambda\mu)}{s^2+(\alpha+\beta+\mu-\lambda)s-(\alpha+\beta+\mu-\mu\beta-\lambda\mu)}\right] \\
&\quad \text{where } \lambda_1 \text{ and } \lambda_2 \text{ are roots of denominator} \\
&= (1-\rho)\left[1+\frac{A}{s+\lambda_1}+\frac{B}{s+\lambda_2}\right] \\
f_W(t) &= (1-\rho)\delta(t)+(Ae^{-\lambda_1 t}+Be^{-\lambda_2 t})\mu(t)
\end{aligned}$$

where A, B are obtained from a partial fraction expansion

$$\begin{aligned}
\hat{T}(s) &= \frac{(1-\rho)s\hat{\tau}(s)}{s-\lambda+\lambda\hat{\tau}(s)} = \frac{(1-\rho)s\mu(s+\beta)}{(s-\lambda)[(s+\mu)(s+\beta)+\alpha s]+\lambda\mu(s+\beta)} \\
&= \frac{(1-\rho)\mu(s+\beta)}{s^2+(\alpha+\beta+\mu-\lambda)s-(\alpha+\beta+\mu-\mu\beta-\lambda\mu)} \\
&= \frac{A'}{s+\lambda_1}+\frac{B'}{s+\lambda_2} \\
f_T(t) &= (A'e^{-\lambda_1 t}+B'e^{-\lambda_2 t})\mu(t)
\end{aligned}$$

(12.56)

~~9.47~~ a) $N_j = N_{j-1} - U(N_{j-1}) + M_j = \begin{cases} N_j - 1 + M_j & N_{j-1} \geq 1 \quad (9.110\text{a}) \\ M_j & N_{j-1} = 0 \quad (9.110\text{b}) \end{cases}$ $\surd$

b)

$$\begin{aligned} \mathcal{E}[N_j] &= \mathcal{E}[N_j] - \mathcal{E}[U(N_{j-1})] + \mathcal{E}[M_j] \\ \Rightarrow \mathcal{E}[M_j] &= \mathcal{E}[U(N_{j-1})] = P[N_{j-1} > 0] \\ \Rightarrow P[N > 0] &= \mathcal{E}[M] = \lambda \mathcal{E}[\tau] \end{aligned}$$

c)

$$\begin{aligned} N_j^2 &= N_{j-1}^2 - 2N_{j-1}U(N_{j-1}) + U(N_{j-1})^2 + M_j^2 \\ &\quad +2(N_{j-1} - U(N_{j-1}))M_j \end{aligned}$$

$N_{j-1}U(N_{j-1}) = N_{j-1}$ and $U(N_{j-1})^2 = U(N_{j-1})$

$$\begin{aligned} 0 = -2\mathcal{E}[N_{j-1}] &+ \mathcal{E}[U(N_{j-1})] + \mathcal{E}[M_j^2] \\ &+ 2\mathcal{E}[N_{j-1}]\mathcal{E}[M_j] - \mathcal{E}[U(N_{j-1})]\mathcal{E}[M_j] \end{aligned}$$

$$\begin{aligned} \mathcal{E}[N_{j-1}] &= \frac{\mathcal{E}[U(N_{j-1})](1 - \mathcal{E}[M_j]) + \mathcal{E}[M_j^2]}{2(1 - \mathcal{E}[M_j])} \\ &= \mathcal{E}[U(N_{j-1})] + \frac{\mathcal{E}[M_j^2]}{2(1 - \mathcal{E}[M_j])} \end{aligned}$$

From part b) $\mathcal{E}(U[N_j]) = \mathcal{E}[M] = \lambda \mathcal{E}[\tau]$

$$\begin{aligned} \mathcal{E}[M^2] &= \mathcal{E}[\mathcal{E}[M^2|\tau]] = \mathcal{E}[X\tau + \lambda^2\tau^2] \\ &= \lambda \mathcal{E}[\tau] + \lambda^2 \mathcal{E}[\tau^2] \end{aligned}$$

Finally

$$\begin{aligned} \mathcal{E}[N] &= \lambda\mathcal{E}[\tau] + \frac{\lambda\mathcal{E}[\tau] + \lambda^2\mathcal{E}[\tau^2]}{2(1 - \lambda\mathcal{E}[\tau])} \qquad C_\tau^2 = \frac{\lambda^2\mathcal{E}[\tau^2]}{\lambda^2\mathcal{E}[\tau]^2} \\ &= \lambda\mathcal{E}[\tau] + \lambda\mathcal{E}[\tau]\frac{(1 + C_\tau^2)}{2(1 - \lambda\mathcal{E}[\tau])} \\ &= \lambda\mathcal{E}[T] \leftarrow \text{as given by 9.94} \qquad \surd \end{aligned}$$

12.57

~~9.48~~ a)

$$\begin{aligned} G_N(z) &= \frac{(1-\rho)(z-1)\hat{\tau}(\lambda(1-z))}{z-\hat{\tau}(\lambda(1-z))} \qquad \hat{\tau}(s) = e^{-sd} \\ &= \frac{(1-\rho)(z-1)e^{-\lambda d(t-z)}}{z-e^{-\lambda d(1-z)}} = \frac{(1-\rho)(1-z)}{1-ze^{\rho(1-z)}} \qquad \text{where } \rho = \lambda d \end{aligned}$$

b)

$$\begin{aligned} \frac{1}{1-e^{\rho(1-z)}z} &= \sum_{k=0}^{\infty} e^{\rho(1-z)k} z^k = \sum_{k=0}^{\infty} e^{-\rho z k} e^{\rho k} z^k \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\rho k z)^l}{l!} e^{\rho k} z^k \\ \therefore G_N(z) &= (1-\rho)(1-z) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\rho k)^l}{l!} e^{\rho k} z^{l+k} \\ &= (1-\rho) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-\rho k)^l}{l!} e^{\rho k} (z^{l+k} - z^{l+k+1}) \\ &= \sum_{k'=0}^{\infty} P[N=k'] z^{k'} \end{aligned}$$

where

$$\begin{aligned} P[N=k'] &= (1-\rho) \left\{ \sum_{l,k:l+k=k'} \frac{(-\rho k)^l e^{\rho k}}{l!} - \sum_{l,k:l+k+1=k'} \frac{(-\rho k)^l e^{\rho k}}{l!} \right\} \\ P[N=k'] &= (1-\rho) \sum_{j=0}^{k'} \left[\frac{(-j\rho)^{k'-j} e^{j\rho}}{(k'-j)!} - \frac{(-j\rho)^{k'-j-1} e^{j\rho}}{(k'-j-1)!} \right] \\ &= (1-\rho) \sum_{j=0}^{k'} \frac{(-j\rho)^{k'-j-1}(-j\rho - k' + j)}{(k'-j)!} e^{j\rho} \qquad \checkmark \end{aligned}$$

12.58

9.49

$$\begin{aligned}\hat{W}(s) &= \frac{(1-\rho)s}{s-\lambda+\lambda\hat{\tau}(s)} = \frac{1-\rho}{1-\lambda\frac{1-\hat{\tau}(s)}{s}} = \frac{1-\rho}{1-\rho\frac{1-\hat{\tau}(s)}{s\mathcal{E}[\tau]}} \\ &= \frac{1-\rho}{1-\rho\hat{R}(s)} \quad \text{where } \hat{R}(s) = \frac{1-\hat{\tau}(s)}{s\mathcal{E}[\tau]}\end{aligned}$$

But

$$f_R(t) = \mathcal{L}^{-1}\left[\frac{1-\hat{\tau}(s)}{s\mathcal{E}(\tau)}\right] = \frac{1}{\mathcal{E}[\tau]}[1-F_\tau(x)]$$

which is pdf of residual service time as given by Eqn. 12.87

$$\begin{aligned}\therefore \hat{W}(s) &= (1-\rho)\sum_{k=0}^{\infty}(\rho\hat{R}(s))^k \\ &= \sum_{k=0}^{\infty}(1-\rho)\rho^k\hat{R}^k(s)\end{aligned}$$

and

$$f_W(t) = \sum_{k=0}^{\infty}(1-\rho)\rho^k f^{(k)}(x)$$

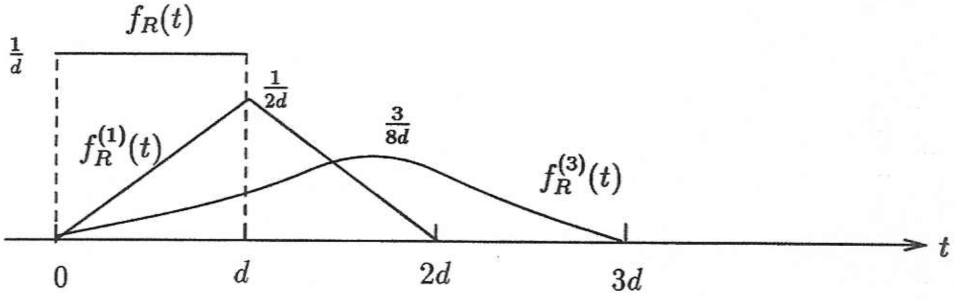
where

$$f^{(k)}(x) = \mathcal{L}^{-1}[\hat{R}^k(s)]$$

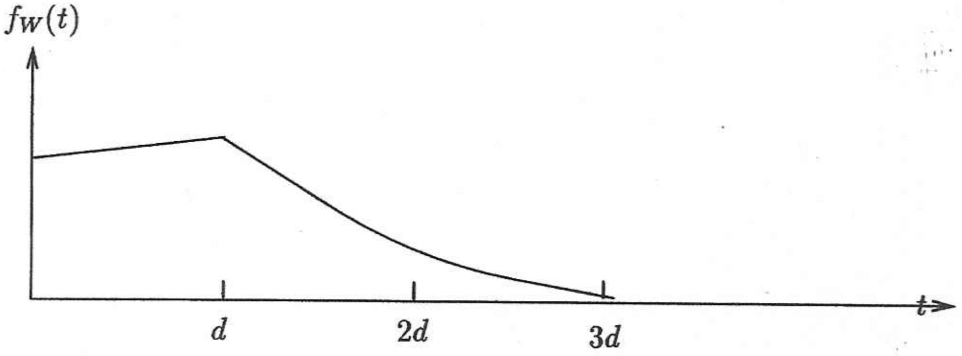
12.59

~~9.50~~ For M/D/1 $\hat{\tau}(s) = e^{-sd}$ and

$$\begin{aligned}\hat{R}(s) &= \frac{1-e^{-sd}}{sd} \\ \Rightarrow f_R(t) &= \frac{1-\mu(t-d)}{d} = \begin{cases} 1 & 0 \le t \le d \\ 0 & ew \end{cases}\end{aligned}$$



$$\begin{aligned}\therefore f_W(t) &\approx (1-\rho)f_R(t) + (1-\rho)\rho f_R^{(1)}(t) + (1-\rho)\rho^2 f_R^{(2)}(t) \\ &= \frac{1}{2}f_R(t) + \frac{1}{4}f_R^{(1)}(t) + \frac{1}{8}f_R^{(2)}(t)\end{aligned}$$



12.8 Burke's Theorem: Departures From M/M/*c* Systems

12.60

~~9.51~~ **a)** If a departure leaves the system nonempty, then another customer commences service immediately. Thus the time until the next departure is an exponential random variable with mean $1/\mu$.

b) If a departure leaves the system empty, then the time until the next departure is equal to the sum of an exponential interarrival time (of mean $1/\lambda$) followed by an exponential service time (of mean $1/\mu$).

c) The Laplace transform of the interdeparture time is

$$\frac{\mu}{s+\mu} \quad \text{when a departure leaves system nonempty}$$

$$\frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu} \quad \text{when a departure leaves system empty}$$

$$\begin{aligned}
\therefore \mathcal{E}[e^{-sT_d}] &= \frac{\mu}{s+\mu}\underbrace{\rho}_{\substack{\text{prob. system}\\ \text{left nonempty}}} + \frac{\lambda}{s+\lambda}\frac{\mu}{s+\mu}\underbrace{(1-\rho)}_{\substack{\text{prob. system}\\ \text{left empty}}} \\
&= \frac{\lambda}{s+\mu} + \frac{\lambda(\mu-\lambda)}{(s+\lambda)(s+\mu)} = \frac{\lambda(s+\lambda)+\lambda\mu-\lambda^2}{(s+\lambda)(s+\mu)} \\
&= \frac{\lambda}{s+\lambda} \Rightarrow T_d \text{ exponential with mean } 1/\lambda
\end{aligned}$$

12.61

~~9.52~~ Claim:

$$P[N_1 = n, N_2 = m] = (1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^m \qquad \begin{array}{l} n, m \geq 0 \\ \rho_i = \lambda/\mu_i \end{array}$$

Eq. 9.135a

$$\begin{aligned} \lambda P[N_1 = 0, N_2 = 0] &= \lambda(1-\rho_1)(1-\rho_2) \\ &= \mu_2(1-\rho_1)(1-\rho_2)\rho_2 = \mu_2 P[N_1 = 0, N_2 = 1] \quad \checkmark \end{aligned}$$

Eq. 9.135b

$$\begin{aligned} (\lambda+\mu_1)P[N_1 = 0, N_2 = 0] &= (\lambda+\mu_1)(1-\rho_1)\rho_1^n(1-\rho_2) \\ &= \lambda(1-\rho_1)\rho_1^n(1-\rho_2) + \mu_1(1-\rho_1)\rho_1^n(1-\rho_2) \\ &= \mu_2\rho_2(1-\rho_1)\rho_1^n(1-\rho_2) + \lambda(1-\rho_1)\rho_1^{n-1}(1-\rho_2) \\ &= \mu_2 P[N_1 = n, N_2 = 1] + \lambda P[N_1 = n-1, N_2 = 0] \quad \checkmark \end{aligned}$$

Eqn. 9.135c

$$\begin{aligned} (\lambda+\mu_2)P[N_1 = 0, N_2 = m] &= (\lambda+\mu_2)(1-\rho_1)(1-\rho_2)\rho_2^m \\ &= \mu_2(1-\rho_1)(1-\rho_2)\rho_2^{m+1} + \mu_1(1-\rho_1)\rho_1(1-\rho_2)\rho_2^{m-1} \\ &= \mu_2 P[N_1 = 0, N_2 = m+1] + \mu_1 P[N_1 = 1, N_2 = m-1] \quad \checkmark \end{aligned}$$

Eqn. 9.135d

$$\begin{aligned} (\lambda+\mu_1+\mu_2)P[N_1 = n, N_2 = m] &= \lambda(1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^m + \mu_1(1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^m \\ &\quad +\mu_2(1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^m \\ &= \mu_2(1-\rho_1)\rho_1^n(1-\rho_2)\rho_2^{m+1} + \lambda(1-\rho_1)\rho_1^{n-1}(1-\rho_2)\rho_2^m \\ &\quad +\mu_1(1-\rho_1)\rho_1^{n+1}(1-\rho_2)\rho_2^{m-1} \\ &= \mu_2 P[N_1 = n, N_2 = m+1] + \lambda P[N_1 = n-1, N_2 = m] \\ &\quad +\mu_1 P[N_1 = n+1, N_2 = m-1] \quad \checkmark \end{aligned}$$

12.62

~~9.53~~ The arrival process at queue #3 is the merge of two independent Poisson processes with combined rate

$$\lambda_1 + \frac{1}{2}\lambda_2$$

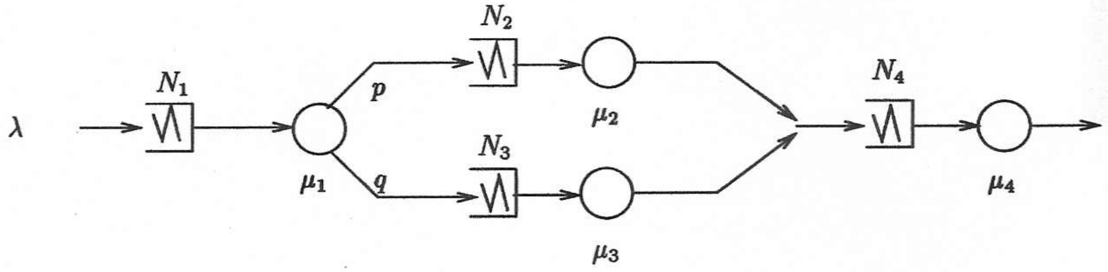
The state of queue #3 at time t is independent of those at queues #1 and #2 at time t:

$$P[N_1(t) = i, N_2(t) = j, N_3(t) = k] = (1-\rho_1)\rho_1^2(1-\rho_2)\rho_2^j(1-\rho_3)\rho_3^k$$

where

$$\begin{aligned} \rho_1 &= \frac{\lambda_1}{\mu_1} < 1 \\ \rho_2 &= \frac{\lambda_2}{\mu_2} < 1 \\ \text{and } \rho_3 &= \frac{(\lambda_1 + \frac{1}{2}\lambda_2)}{\mu_3} < 1 \end{aligned}$$

12.63



Claim:

$$\begin{aligned} P[N_1 = k, N_2 = l, N_3 = m, N_4 = n] &= (1-\rho)\rho_1^k(1-\rho_2)\rho_2^l(1-\rho_3)\rho_3^m(1-\rho_4)\rho_4^n \\ &\triangleq A\rho_1^k\rho_2^l\rho_3^m\rho_4^n \end{aligned}$$

where

$$\rho_1 = \frac{\lambda}{\mu_1} \qquad \rho_2 = \frac{p\lambda}{\mu_2} \qquad \rho_3 = \frac{q\lambda}{\mu_3} \qquad \rho_4 = \frac{\lambda}{\mu_4}$$

Let $\underline{e}_i$ be the ith unit vector.
Let $\underline{s} = (klmn)$ where $k, l, m, n > 0$. The balance equation for this state is

$$\begin{aligned} (\lambda + \mu_1 + \mu_2 + \mu_3 + \mu_4)P[\underline{s}] &= \lambda P[\underline{s} - \underline{e}_1] + p\mu_1 P[\underline{s} + \underline{e}_1 - \underline{e}_2] \\ &\quad + q\mu_1 P[\underline{s} + \underline{e}_1 - \underline{e}_2] + \mu P[\underline{s} + \underline{e}_2 - \underline{e}_4] \\ &\quad + \mu_3 P[\underline{s} + \underline{e}_3 - \underline{e}_4] + \mu_4 P[\underline{s} + \underline{e}_4] \\ &= \lambda P[\underline{s}]\rho_1^{-1} + p\mu_1 P[\underline{s}]\rho_1\rho_2^{-1} + q\mu_1 P[\underline{s}]\rho_1\rho_3^{-1} \\ &\quad + \mu_2 P[\underline{s}]\rho_2\rho_4^{-1} + \mu_3 P[\underline{s}]\rho_3\rho_4^{-1} | \mu_4 P[\underline{s}]\rho_4 \\ &= \mu_1 P[\underline{s}] + \mu_2 P[\underline{s}] + \mu_3 P[\underline{s}] + \mu_4 P[\underline{s}] \\ &\quad + \lambda P[\underline{s}] \quad \checkmark \end{aligned}$$

$\therefore P[\underline{s}] = A\rho_1^k\rho_2^l\rho_3^m\rho_4^n$ satisfies this balance equation.

There are 15 other special cases of boundary balance equations. These are shown to be satisfied by $P[\underline{s}]$ in similar fashion.

12.9 Networks of Queues: Jackson's Theorem

12.64

~~9.55~~ $\lambda_1 = \lambda \qquad \lambda_2 = \frac{1}{2}\lambda + \frac{1}{2}\lambda_3 \qquad \lambda_3 = \lambda_2 + \frac{1}{2}\lambda$

$$\Rightarrow \quad \lambda_1 = \lambda \qquad \lambda_2 = \frac{3}{2}\lambda \qquad \lambda_3 = 2\lambda$$

$$\Rightarrow \quad \rho_1 = \frac{\lambda}{\mu_1} \qquad \rho_2 = \frac{3\lambda}{2\mu_2} \qquad \rho_3 = \frac{2\lambda}{\mu_3}$$

Assuming $\rho_i < 1$, $i = 1, 2, 3$, then

$$P[N_1 = k, N_2 = l, N_3 = m] = (1-\rho_1)\rho_1^k(1-\rho_2)\rho_2^l(1-\rho_3)\rho_3^m \qquad k, l, m \geq 0$$

12.65

~~9.56~~ $I = 3$

$$\left.\begin{array}{rcl} \pi_0 & = & p\pi_0 + \pi_1 + \pi_2 \\ \pi_1 & = & \frac{1}{2}(1-p)\pi_0 \\ \pi_2 & = & \frac{1}{2}(1-p)\pi_0 \end{array}\right\} \begin{array}{l} \pi_0 = \frac{1}{2-p} \\ \pi_1 = \pi_2 = \frac{1-p}{2(2-p)} \end{array}$$

a) Then

$$\begin{array}{ll} \lambda_0 = \lambda(3)\pi_0 = \dfrac{\lambda(3)}{2-p} & \rho_0 = \dfrac{\lambda_0}{\mu} \\ \lambda_1 = \lambda(3)\pi_1 = \dfrac{\lambda(3)(1-p)}{2(2-p)} & \rho_1 = \dfrac{\lambda_1}{\mu_1} \\ \lambda_2 = \lambda_1 & \rho_2 = \dfrac{\lambda_1}{\mu_2} \end{array}$$

$$\begin{aligned} S(3) &= (1-\rho_0)(1-\rho_1)(1-\rho_2)[\rho_0^3 + \rho_1^3 + \rho_2^3 + \rho_0\rho_1^2 + \rho_0\rho_2^2 \\ &\quad + \rho_1\rho_2^2 + \rho_1\rho_0^2 + \rho_2\rho_0^2 + \rho_2\rho_1^2 + \rho_0\rho_1\rho_2] \\ &= (1-\rho_0)(1-\rho_1)(1-\rho_2)[(\rho_0^2 + \rho_1^2 + \rho_2^2)(\rho_0 + \rho_1 + \rho_2) + \rho_0\rho_1\rho_2] \end{aligned}$$

$$\therefore P[N_0 = i, N_1 = j, N_2 = 3 - i - j] = \frac{\rho_0^i\rho_1^j\rho_2^{3-i-j}}{(\rho_0^2 + \rho_1^2 + \rho_2^2)(\rho_0 + \rho_1 + \rho_2) + \rho_1\rho_2\rho_3}$$

$$0 \leq i, j \quad \text{and} \quad i + j \leq 3$$

b) The program completion rate is

$$p\mu[1 - P[N_0 = 0]] = p\mu\frac{\rho_0^3 + \rho_0^2\rho_1 + \rho_0^2\rho_2 + \rho_0\rho_2^2 + \rho_0\rho_1^2 + \rho_0\rho_1\rho_2}{(\rho_0^2 + \rho_1^2 + \rho_2^2)(\rho_0 + \rho_1 + \rho_2) + \rho_1\rho_2\rho_3}$$

12.10 Simulation and Data Analysis of Queueing Systems

12.66

~~9.5~~ From Eq. (12.56) we have:

$$\pi_0 = \frac{1}{2-p} \qquad \pi_1 = \pi_2 = \frac{1-p}{2(2-p)}$$

We need to find $p\lambda_0(3) = p\pi_0\lambda(3)$

$I = 1$

$$\mathcal{E}[T_0(1)] = \frac{1}{\mu} \qquad \mathcal{E}[T_1(1)] = \frac{1}{\mu_1} \qquad \mathcal{E}[T_2(1)] = \frac{1}{\mu_2}$$

$$\begin{aligned}
\lambda(1) &= 1\left[\frac{\frac{1}{\mu}}{2-p} + \left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\frac{1-p}{2(2-p)}\right]^{-1} \\
\mathcal{E}[N_0(1)] &= \frac{\frac{1}{\mu}}{\frac{1}{\mu}+\frac{1-p}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)} \triangleq \frac{a}{a+b+c} \\
\mathcal{E}[N_1(1)] &= \frac{\frac{1-p}{2}\frac{1}{\mu_1}}{\frac{1}{\mu}+\frac{1-p}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)} \triangleq \frac{b}{a+b+c} \\
\mathcal{E}[N_2(1)] &= \frac{\frac{1-p}{2}\frac{1}{\mu_2}}{\frac{1}{\mu}+\frac{1-p}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)} \triangleq \frac{c}{a+b+c}
\end{aligned}$$

where

$$a \triangleq \frac{1}{\mu} \qquad b = \frac{1-p}{2}\frac{1}{\mu_1} \qquad c = \frac{1-p}{2}\frac{1}{\mu_2}$$

$I = 2$

$$\begin{aligned}
\mathcal{E}[T_0(2)] &= \frac{1}{\mu}\left[\frac{2a+b+c}{a+b+c}\right] \\
\mathcal{E}[T_1(2)] &= \frac{1}{\mu_1}\left[\frac{a+2b+c}{a+b+c}\right] \\
\mathcal{E}[T_2(2)] &= \frac{1}{\mu_2}\left[\frac{a+b+2c}{a+b+c}\right]
\end{aligned}$$

$$\lambda(2) = 2\left[\frac{1}{\frac{1}{2-p}\mathcal{E}[T_0(2)] + \frac{1-p}{2}\frac{1}{2-p}\mathcal{E}[T_1(2)] + \frac{1-p}{2}\frac{1}{2-p}\mathcal{E}[T_2(2)]}\right]$$

$$\begin{aligned}
&= 2\left[\frac{(2-p)(a+b+c)}{\underbrace{\frac{1}{\mu}}_{a}(2a+b+c)+\underbrace{\frac{1-p}{2}\frac{1}{\mu_1}}_{b}(a+2b+c)+\underbrace{\frac{1-p}{2}\frac{1}{\mu_2}}_{c}(a+b+2c)}\right]\\
&= 2\frac{(2-p)(a+b+c)}{2a^2+2b^2+2c^2+2ab+2ac+2bc}\\
&= \frac{(2-p)(a+b+c)}{a^2+b^2+c^2+ab+ac+bc}
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}[N_0(2)] &= \lambda(2)\pi_0\mathcal{E}[T_0(2)] = \lambda(2)\frac{1}{2-p}\frac{1}{\mu}\left[\frac{2a+b+c}{a+b+c}\right]\\
&= \frac{1}{\mu}\frac{2a+b+c}{a^2+b^2+c^2+ab+ac+bc} = \frac{2a^2+ab+ac}{a^2+b^2+c^2+ab+ac+bc}\\
\mathcal{E}[N_1(2)] &= \frac{\overbrace{\frac{1}{\mu_1}\frac{1-p}{2}}^{b}(a+2b+c)}{a^2+b^2+c^2+ab+ac+bc} = \frac{ab+2b^2+bc}{a^2+b^2+c^2+ab+ac+bc}\\
\mathcal{E}[N_2(2)] &= \frac{c(a+b+2c)}{a^2+b^2+c^2+ab+ac+bc} = \frac{ac+bc+2c^2}{a^2+b^2+c^2+ab+ac+bc}
\end{aligned}$$

$I = 3$

$$\begin{aligned}
\mathcal{E}[T_0(3)] &= \frac{1}{\mu}[1+\mathcal{E}[N_0(2)]] = \frac{3a^2+b^2+c^2+2ab+2ac+bc}{a^2+b^2+c^2+ab+ac+bc}\\
\mathcal{E}[T_1(3)] &= \frac{1}{\mu_1}\left[\frac{2ab+3b^2+2bc+a^2+c^2+ac}{a^2+b^2+c^2+ab+ac+bc}\right]\\
\mathcal{E}[T_2(3)] &= \frac{1}{\mu_2}\left[\frac{2ac+2bc+3c^2+a^2+b^2+c^2+ab}{a^2+b^2+c^2+ab+ac+bc}\right]\\
\lambda(3) &= 3\frac{1}{\frac{1}{2-p}\mathcal{E}[T_0(3)]+\frac{1-p}{2}\frac{1}{2-p}\mathcal{E}[T_1(3)]+\frac{1-p}{2}\frac{1}{2-p}\mathcal{E}[T_2(3)]}
\end{aligned}$$

Program completion rate is

$$\begin{aligned}
p\lambda_0 &= px_0\lambda(3)\\
&= \frac{p}{2-p}\lambda(3)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3p}{\mathcal{E}[T_0(3)] + \frac{1-p}{2}\mathcal{E}[T_1(3)] + \frac{1-p}{2}\mathcal{E}[T_2(3)]} \\
&= [3p(a^2+b^2+c^2+ab+ac+bc)]/[a(3a^2+b^2+c^2+2ab+2ac+bc) \\
&\quad +b(2ab+3b^2+2bc+a^2+c^2+ac)+c(2ac+2bc+3c^2+a^2+b^2+ab)] \\
&= \frac{p\mu\overbrace{\left(\frac{1}{\mu}\right)}^{a}(a^2+b^2+c^2+ab+ac+bc)}{a^3+b^3+c^3+ab^2+ac^2+a^2b+a^2c+b^2c+bc^2+abc}
\end{aligned}
$$

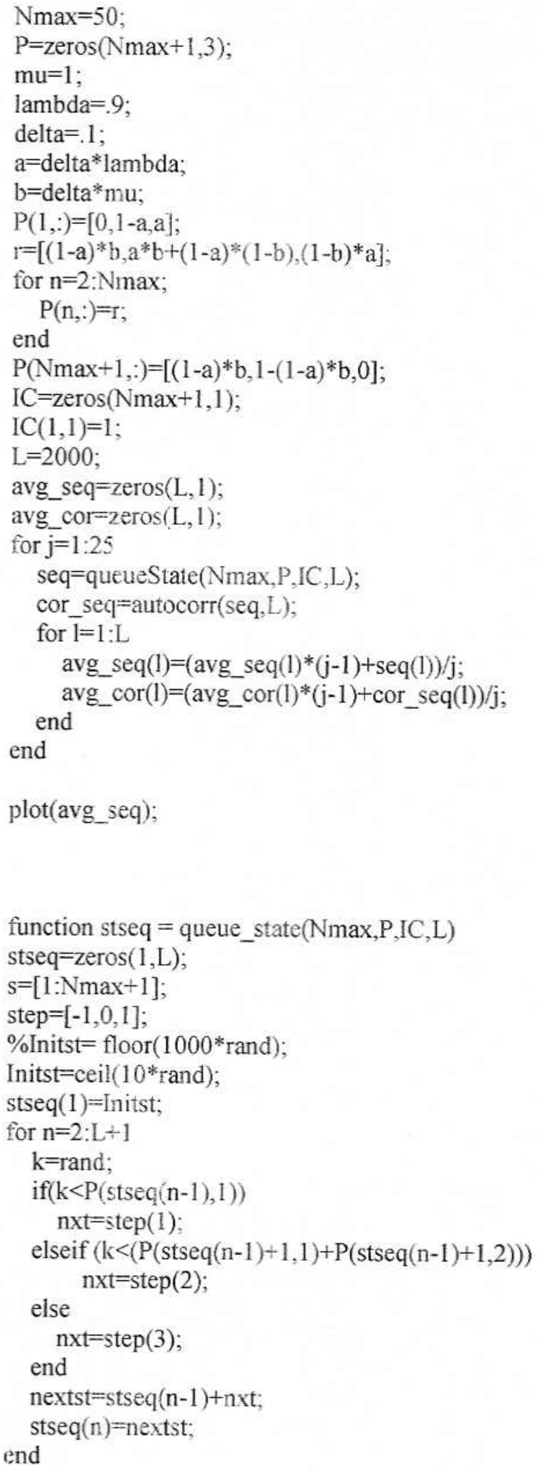
We will multiply the numerator and denominator above by $\left(\frac{\lambda(3)}{2-p}\right)^3$ but first note that

$$
\begin{aligned}
\frac{\lambda(3)a}{2-p} &= \frac{\lambda(3)}{\mu(2-p)} = \lambda_0(3)\frac{1}{\mu} = \rho_0 \\
\frac{\lambda(3)b}{2-p} &= \frac{\lambda(3)(1-p)}{2(2-p)\mu_1} = \rho_1 \qquad \frac{\lambda(3)c}{2-p} = \rho_2
\end{aligned}
$$

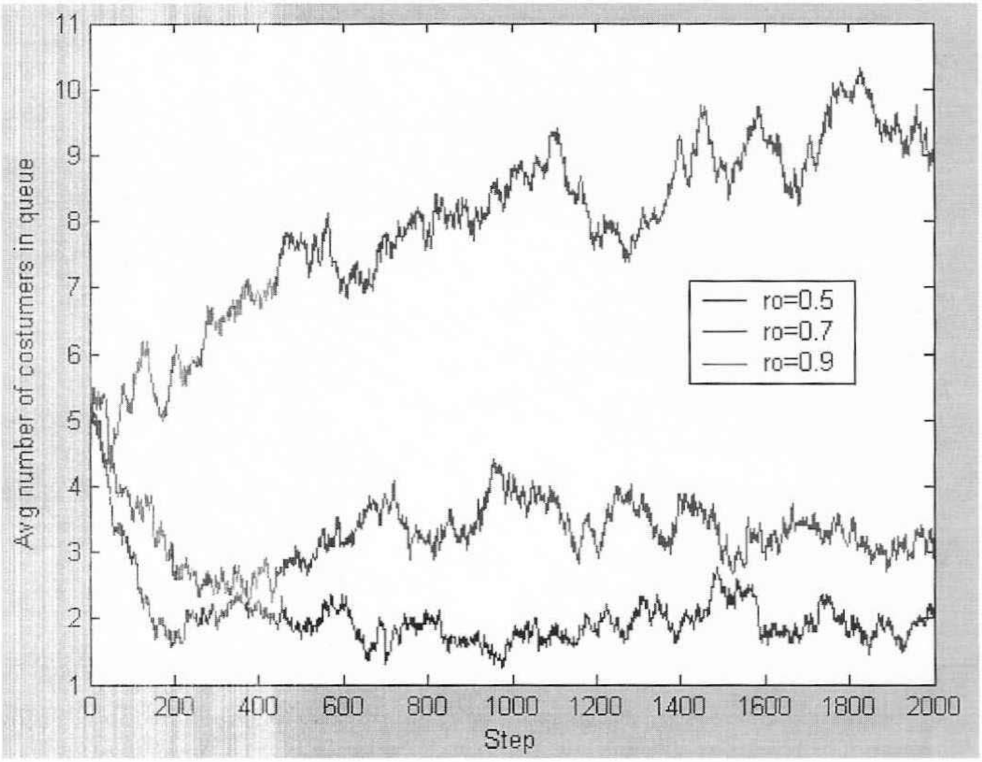
$\therefore$ Completion Rate

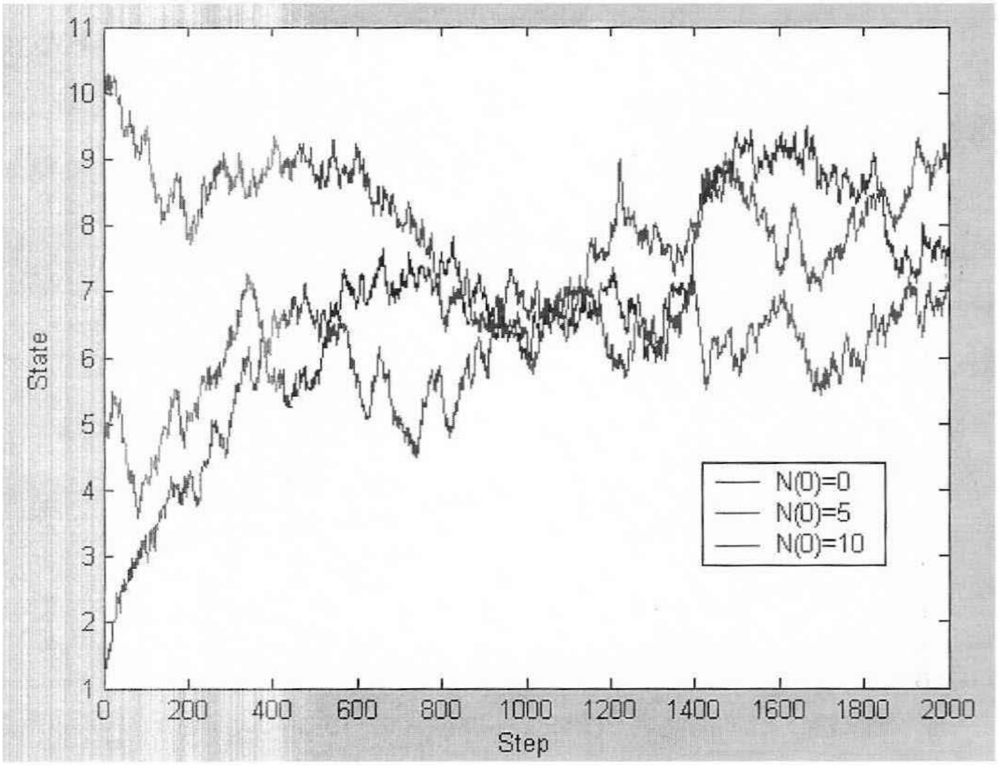
$$
= p\mu\frac{\rho_0^3+\rho_0^2\rho_1+\rho_0^2\rho_2+\rho_0\rho_1^2+\rho_0\rho_1\rho_2}{(\rho_0^2+\rho_1^2+\rho_2^2)(\rho_0+\rho_1+\rho_2)+\rho_1\rho_2\rho_3} \qquad \checkmark
$$

12.67

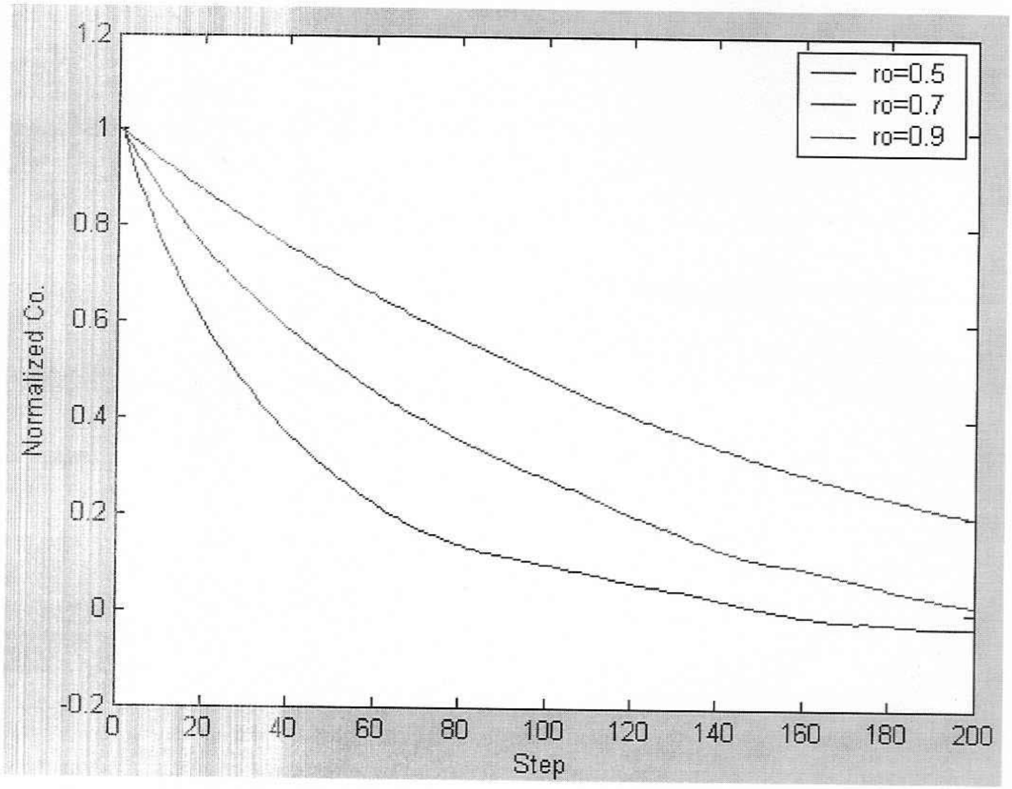


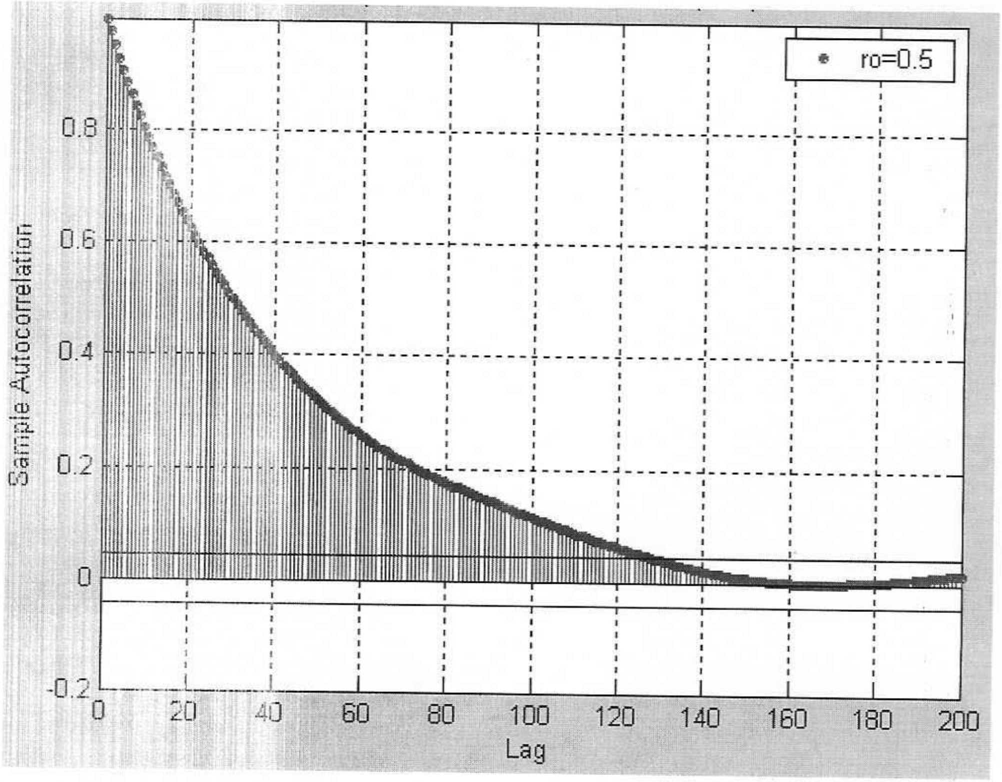
```
Nmax=50;
P=zeros(Nmax+1,3);
mu=1;
lambda=.9;
delta=.1;
a=delta*lambda;
b=delta*mu;
P(1,:)=[0,1-a,a];
r=[(1-a)*b,a*b+(1-a)*(1-b),(1-b)*a];
for n=2:Nmax;
  P(n,:)=r;
end
P(Nmax+1,:)=[(1-a)*b,1-(1-a)*b,0];
IC=zeros(Nmax+1,1);
IC(1,1)=1;
L=2000;
avg_seq=zeros(L,1);
avg_cor=zeros(L,1);
for j=1:25
  seq=queueState(Nmax,P,IC,L);
  cor_seq=autocorr(seq,L);
  for l=1:L
    avg_seq(l)=(avg_seq(l)*(j-1)+seq(l))/j;
    avg_cor(l)=(avg_cor(l)*(j-1)+cor_seq(l))/j;
  end
end

plot(avg_seq);



function stseq = queue_state(Nmax,P,IC,L)
stseq=zeros(1,L);
s=[1:Nmax+1];
step=[-1,0,1];
%Initst= floor(1000*rand);
Initst=ceil(10*rand);
stseq(1)=Initst;
for n=2:L+1
  k=rand;
  if(k<P(stseq(n-1),1))
    nxt=step(1);
  elseif (k<(P(stseq(n-1)+1,1)+P(stseq(n-1)+1,2)))
      nxt=step(2);
  else
    nxt=step(3);
  end
  nextst=stseq(n-1)+nxt;
  stseq(n)=nextst;
end
```

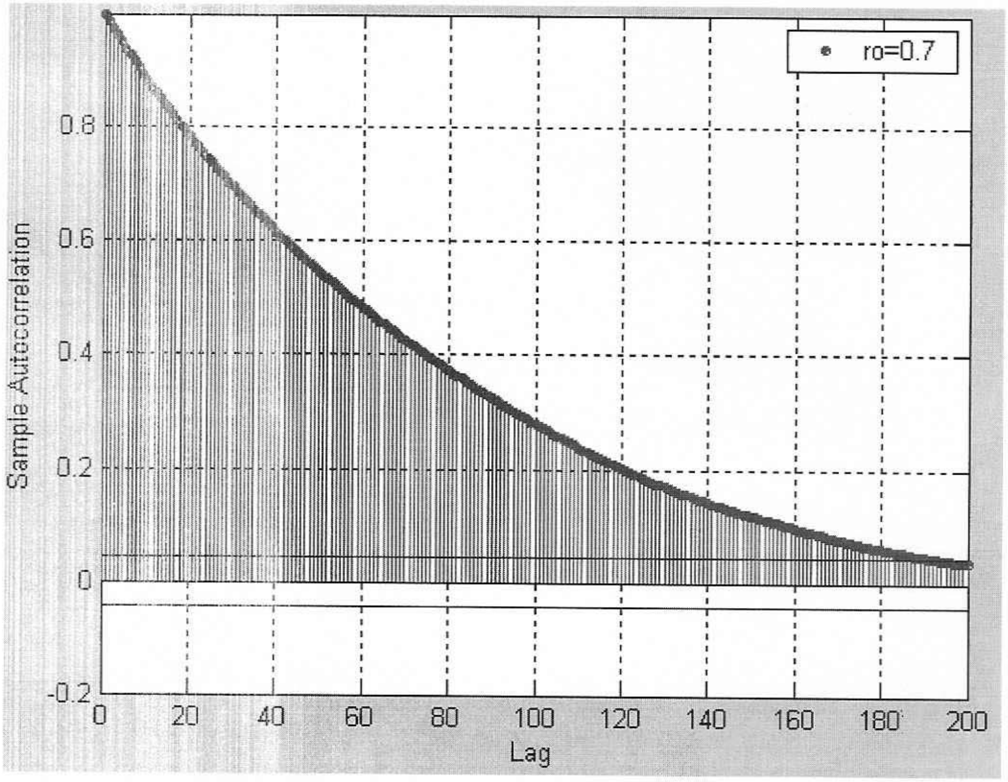

Avg number of costumers in queue
Step
ro=0.5
ro=0.7
ro=0.9
1
2
3
4
5
6
7
8
9
10
11
0
200
400
600
800
1000
1200
1400
1600
1800
2000

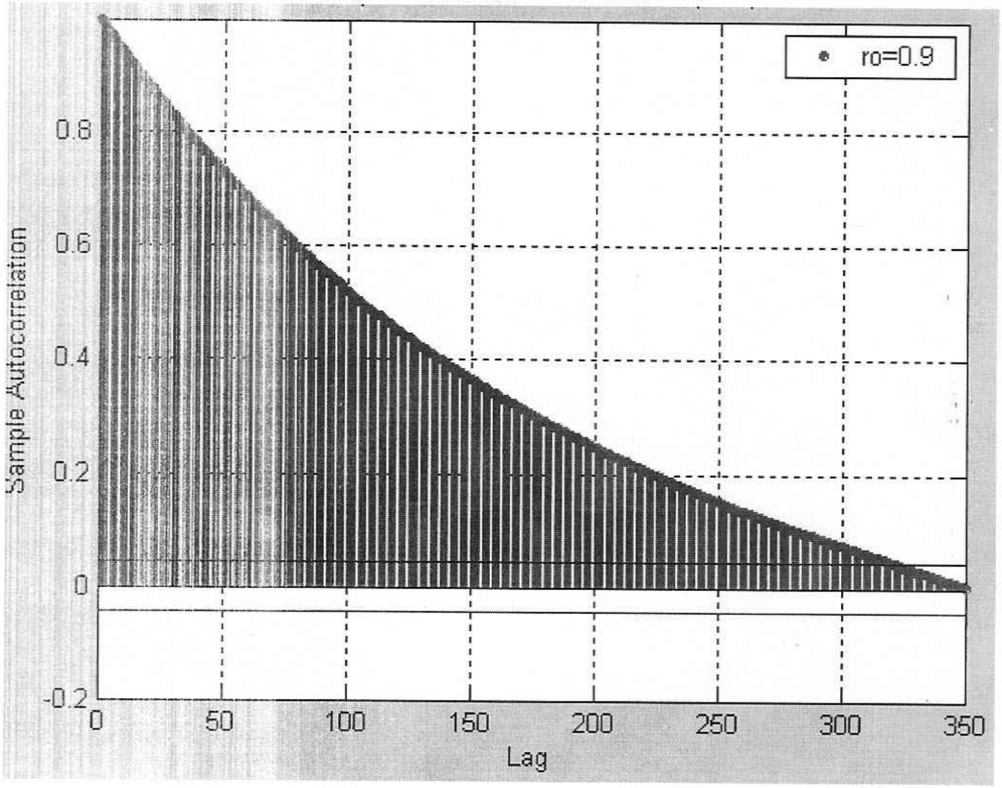

State
Step
N(0)=0
N(0)=5
N(0)=10
1
2
3
4
5
6
7
8
9
10
11
0
200
400
600
800
1000
1200
1400
1600
1800
2000

12.68

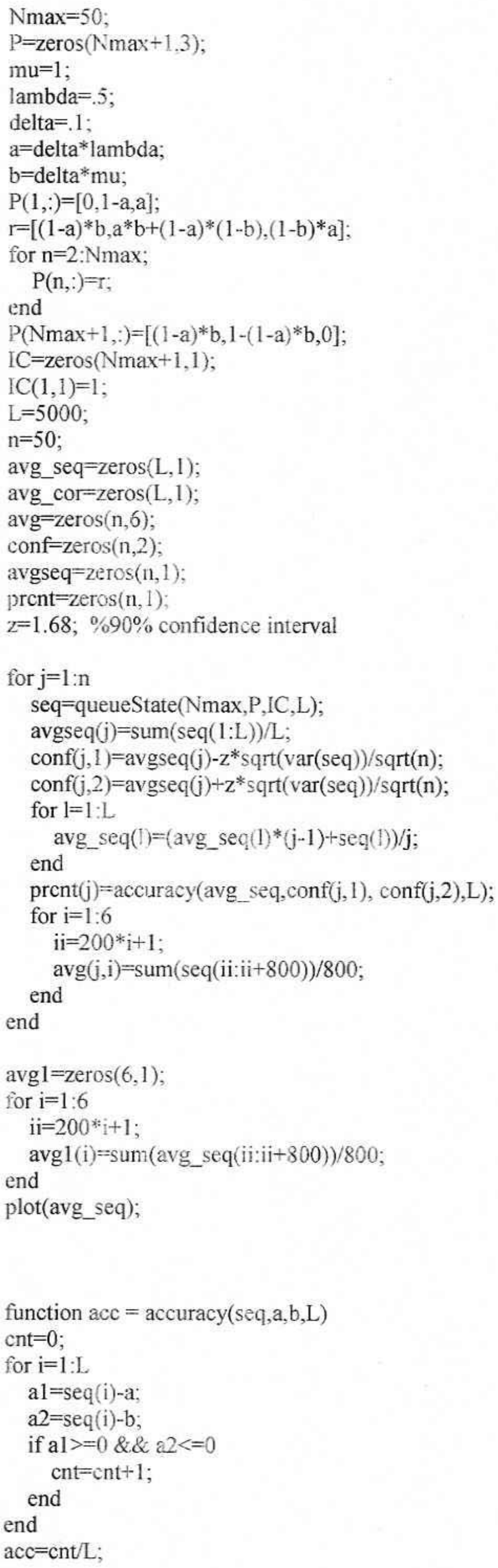

Normalized Co.
Step
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
20
40
60
80
100
120
140
160
180
200
ro=0.5
ro=0.7
ro=0.9


Sample Autocorrelation
Lag
0.8
0.6
0.4
0.2
0
-0.2
0
20
40
60
80
100
120
140
160
180
200
ro=0.5

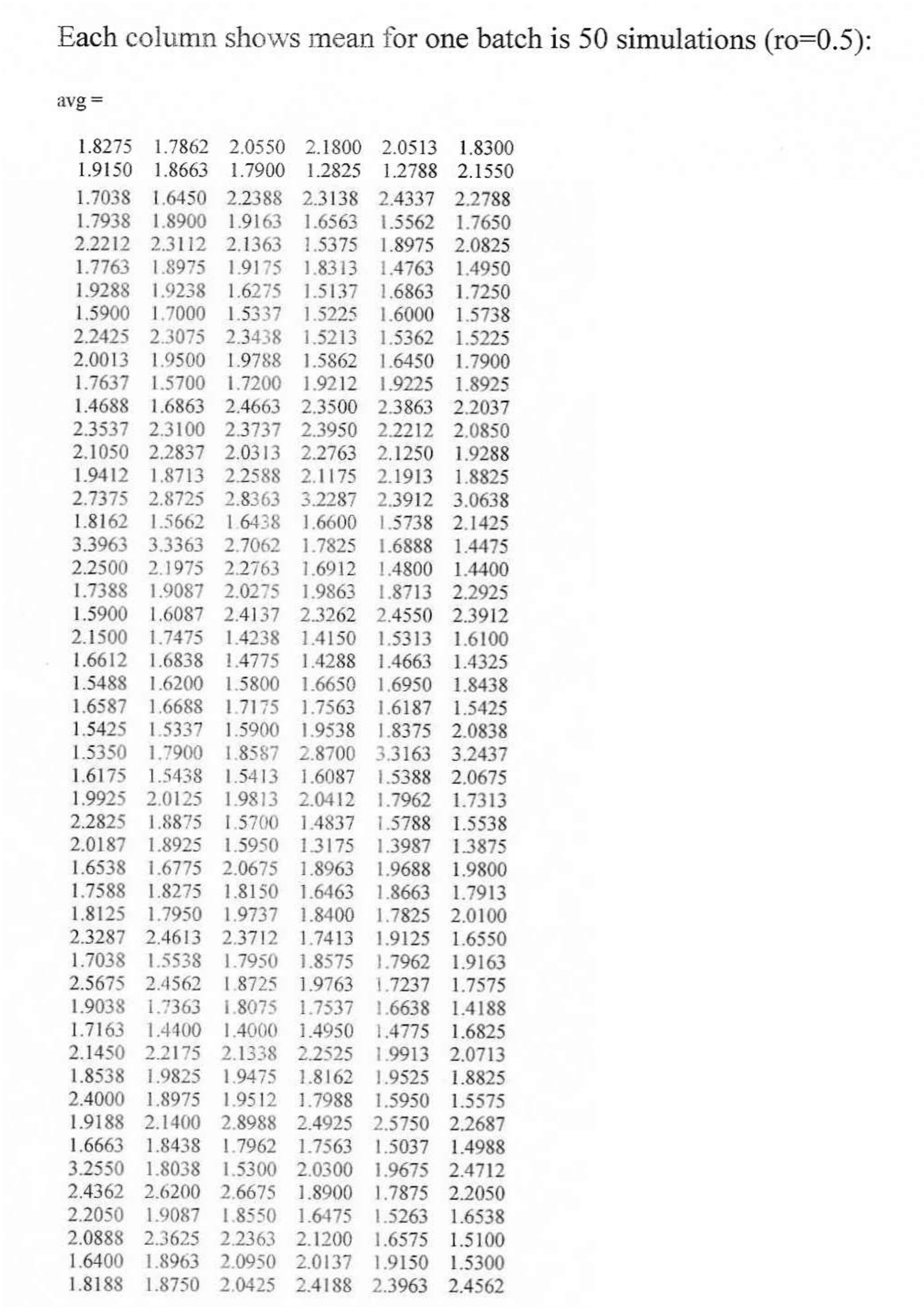

ro=0.7
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
20
40
60
80
100
120
140
160
180
200
Lag


ro=0.9
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
50
100
150
200
250
300
350
Lag

12.70

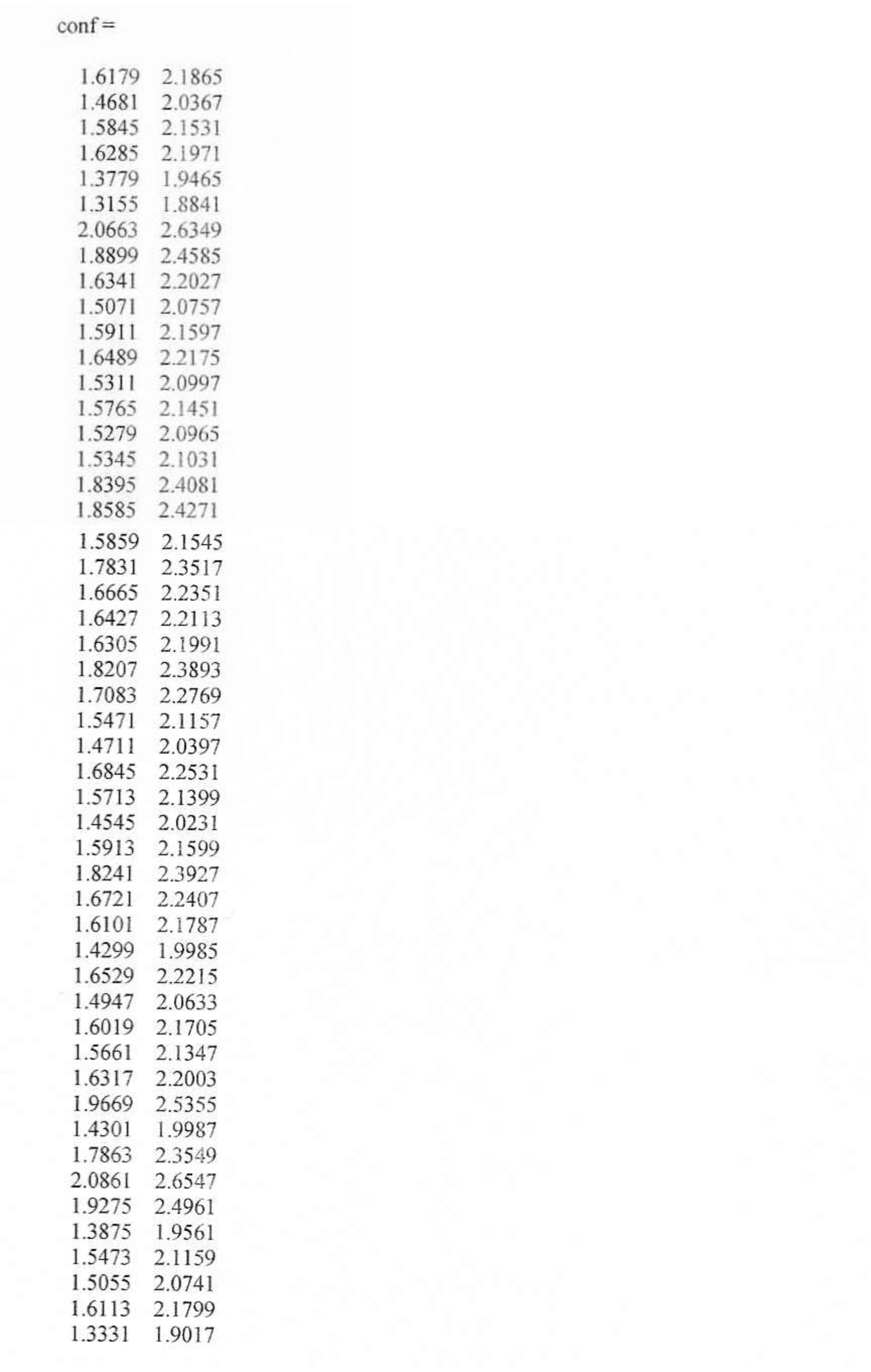


```
Nmax=50;
P=zeros(Nmax+1,3);
mu=1;
lambda=.5;
delta=.1;
a=delta*lambda;
b=delta*mu;
P(1,:)=[0,1-a,a];
r=[(1-a)*b,a*b+(1-a)*(1-b),(1-b)*a];
for n=2:Nmax;
   P(n,:)=r;
end
P(Nmax+1,:)=[(1-a)*b,1-(1-a)*b,0];
IC=zeros(Nmax+1,1);
IC(1,1)=1;
L=5000;
n=50;
avg_seq=zeros(L,1);
avg_cor=zeros(L,1);
avg=zeros(n,6);
conf=zeros(n,2);
avgseq=zeros(n,1);
prcnt=zeros(n,1);
z=1.68;  %90% confidence interval

for j=1:n
   seq=queueState(Nmax,P,IC,L);
   avgseq(j)=sum(seq(1:L))/L;
   conf(j,1)=avgseq(j)-z*sqrt(var(seq))/sqrt(n);
   conf(j,2)=avgseq(j)+z*sqrt(var(seq))/sqrt(n);
   for l=1:L
      avg_seq(l)=(avg_seq(l)*(j-1)+seq(l))/j;
   end
   prcnt(j)=accuracy(avg_seq,conf(j,1), conf(j,2),L);
   for i=1:6
      ii=200*i+1;
      avg(j,i)=sum(seq(ii:ii+800))/800;
   end
end

avg1=zeros(6,1);
for i=1:6
   ii=200*i+1;
   avg1(i)=sum(avg_seq(ii:ii+800))/800;
end
plot(avg_seq);



function acc = accuracy(seq,a,b,L)
cnt=0;
for i=1:L
   a1=seq(i)-a;
   a2=seq(i)-b;
   if a1>=0 && a2<=0
      cnt=cnt+1;
   end
end
acc=cnt/L;
```



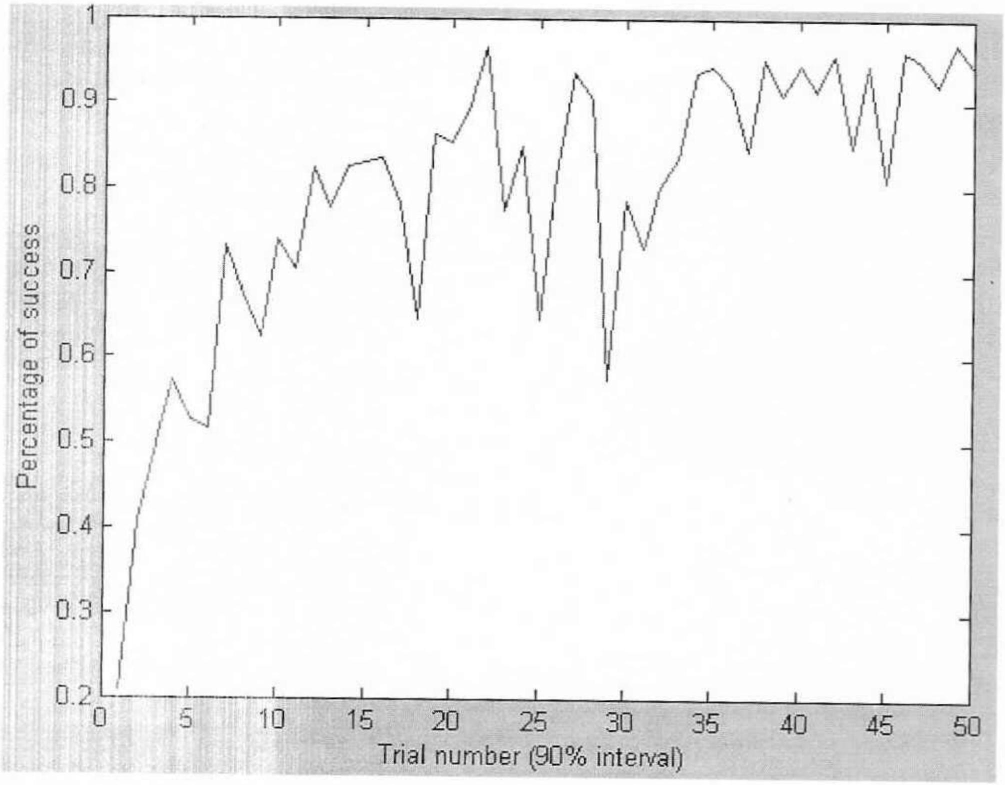
Each column shows mean for one batch is 50 simulations (ro=0.5):

```
avg =

    1.8275    1.7862    2.0550    2.1800    2.0513    1.8300
    1.9150    1.8663    1.7900    1.2825    1.2788    2.1550
    1.7038    1.6450    2.2388    2.3138    2.4337    2.2788
    1.7938    1.8900    1.9163    1.6563    1.5562    1.7650
    2.2212    2.3112    2.1363    1.5375    1.8975    2.0825
    1.7763    1.8975    1.9175    1.8313    1.4763    1.4950
    1.9288    1.9238    1.6275    1.5137    1.6863    1.7250
    1.5900    1.7000    1.5337    1.5225    1.6000    1.5738
    2.2425    2.3075    2.3438    1.5213    1.5362    1.5225
    2.0013    1.9500    1.9788    1.5862    1.6450    1.7900
    1.7637    1.5700    1.7200    1.9212    1.9225    1.8925
    1.4688    1.6863    2.4663    2.3500    2.3863    2.2037
    2.3537    2.3100    2.3737    2.3950    2.2212    2.0850
    2.1050    2.2837    2.0313    2.2763    2.1250    1.9288
    1.9412    1.8713    2.2588    2.1175    2.1913    1.8825
    2.7375    2.8725    2.8363    3.2287    2.3912    3.0638
    1.8162    1.5662    1.6438    1.6600    1.5738    2.1425
    3.3963    3.3363    2.7062    1.7825    1.6888    1.4475
    2.2500    2.1975    2.2763    1.6912    1.4800    1.4400
    1.7388    1.9087    2.0275    1.9863    1.8713    2.2925
    1.5900    1.6087    2.4137    2.3262    2.4550    2.3912
    2.1500    1.7475    1.4238    1.4150    1.5313    1.6100
    1.6612    1.6838    1.4775    1.4288    1.4663    1.4325
    1.5488    1.6200    1.5800    1.6650    1.6950    1.8438
    1.6587    1.6688    1.7175    1.7563    1.6187    1.5425
    1.5425    1.5337    1.5900    1.9538    1.8375    2.0838
    1.5350    1.7900    1.8587    2.8700    3.3163    3.2437
    1.6175    1.5438    1.5413    1.6087    1.5388    2.0675
    1.9925    2.0125    1.9813    2.0412    1.7962    1.7313
    2.2825    1.8875    1.5700    1.4837    1.5788    1.5538
    2.0187    1.8925    1.5950    1.3175    1.3987    1.3875
    1.6538    1.6775    2.0675    1.8963    1.9688    1.9800
    1.7588    1.8275    1.8150    1.6463    1.8663    1.7913
    1.8125    1.7950    1.9737    1.8400    1.7825    2.0100
    2.3287    2.4613    2.3712    1.7413    1.9125    1.6550
    1.7038    1.5538    1.7950    1.8575    1.7962    1.9163
    2.5675    2.4562    1.8725    1.9763    1.7237    1.7575
    1.9038    1.7363    1.8075    1.7537    1.6638    1.4188
    1.7163    1.4400    1.4000    1.4950    1.4775    1.6825
    2.1450    2.2175    2.1338    2.2525    1.9913    2.0713
    1.8538    1.9825    1.9475    1.8162    1.9525    1.8825
    2.4000    1.8975    1.9512    1.7988    1.5950    1.5575
    1.9188    2.1400    2.8988    2.4925    2.5750    2.2687
    1.6663    1.8438    1.7962    1.7563    1.5037    1.4988
    3.2550    1.8038    1.5300    2.0300    1.9675    2.4712
    2.4362    2.6200    2.6675    1.8900    1.7875    2.2050
    2.2050    1.9087    1.8550    1.6475    1.5263    1.6538
    2.0888    2.3625    2.2363    2.1200    1.6575    1.5100
    1.6400    1.8963    2.0950    2.0137    1.9150    1.5300
    1.8188    1.8750    2.0425    2.4188    2.3963    2.4562
```

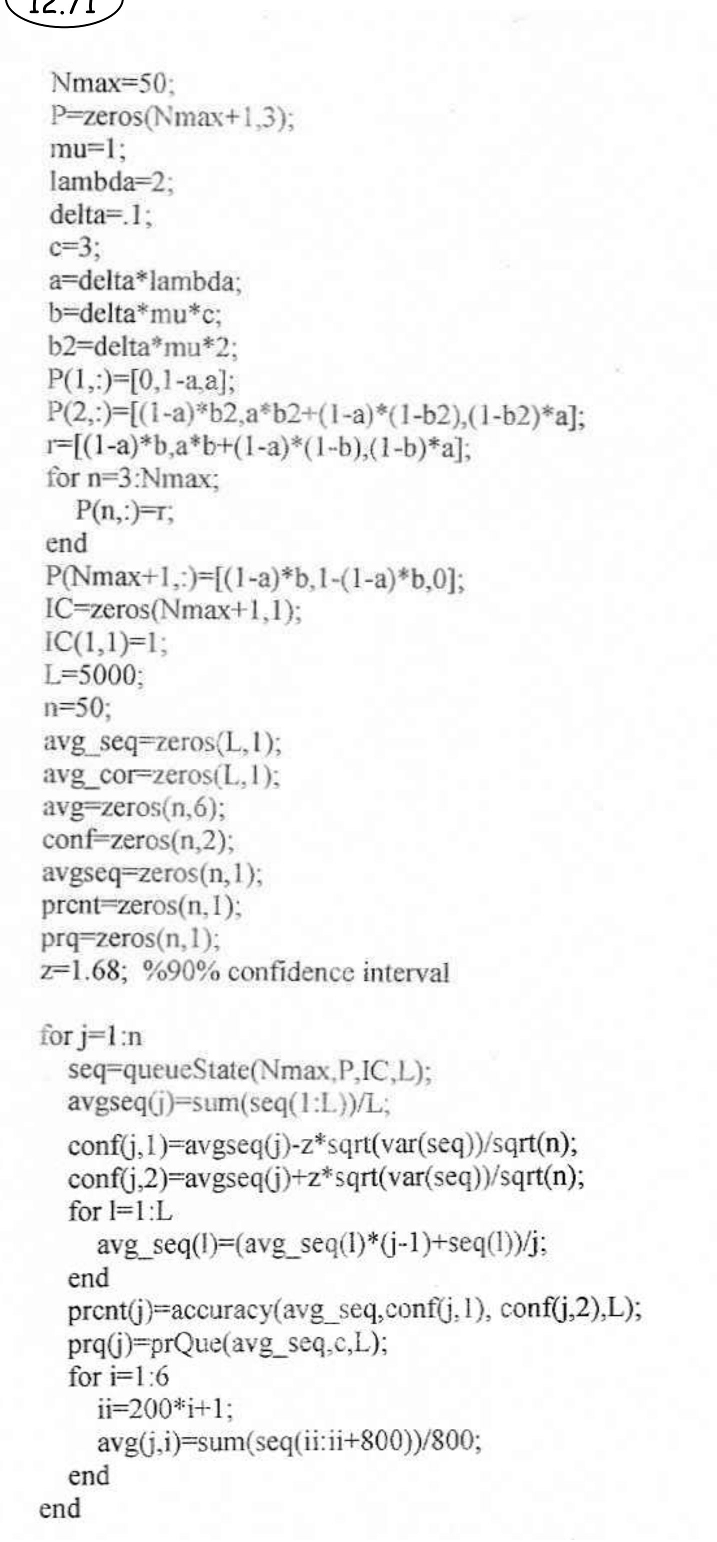


```
conf =

    1.6179    2.1865
    1.4681    2.0367
    1.5845    2.1531
    1.6285    2.1971
    1.3779    1.9465
    1.3155    1.8841
    2.0663    2.6349
    1.8899    2.4585
    1.6341    2.2027
    1.5071    2.0757
    1.5911    2.1597
    1.6489    2.2175
    1.5311    2.0997
    1.5765    2.1451
    1.5279    2.0965
    1.5345    2.1031
    1.8395    2.4081
    1.8585    2.4271
    1.5859    2.1545
    1.7831    2.3517
    1.6665    2.2351
    1.6427    2.2113
    1.6305    2.1991
    1.8207    2.3893
    1.7083    2.2769
    1.5471    2.1157
    1.4711    2.0397
    1.6845    2.2531
    1.5713    2.1399
    1.4545    2.0231
    1.5913    2.1599
    1.8241    2.3927
    1.6721    2.2407
    1.6101    2.1787
    1.4299    1.9985
    1.6529    2.2215
    1.4947    2.0633
    1.6019    2.1705
    1.5661    2.1347
    1.6317    2.2003
    1.9669    2.5355
    1.4301    1.9987
    1.7863    2.3549
    2.0861    2.6547
    1.9275    2.4961
    1.3875    1.9561
    1.5473    2.1159
    1.5055    2.0741
    1.6113    2.1799
    1.3331    1.9017
```

The percentage of times that the real value of the state is in the confidence interval is shown below:



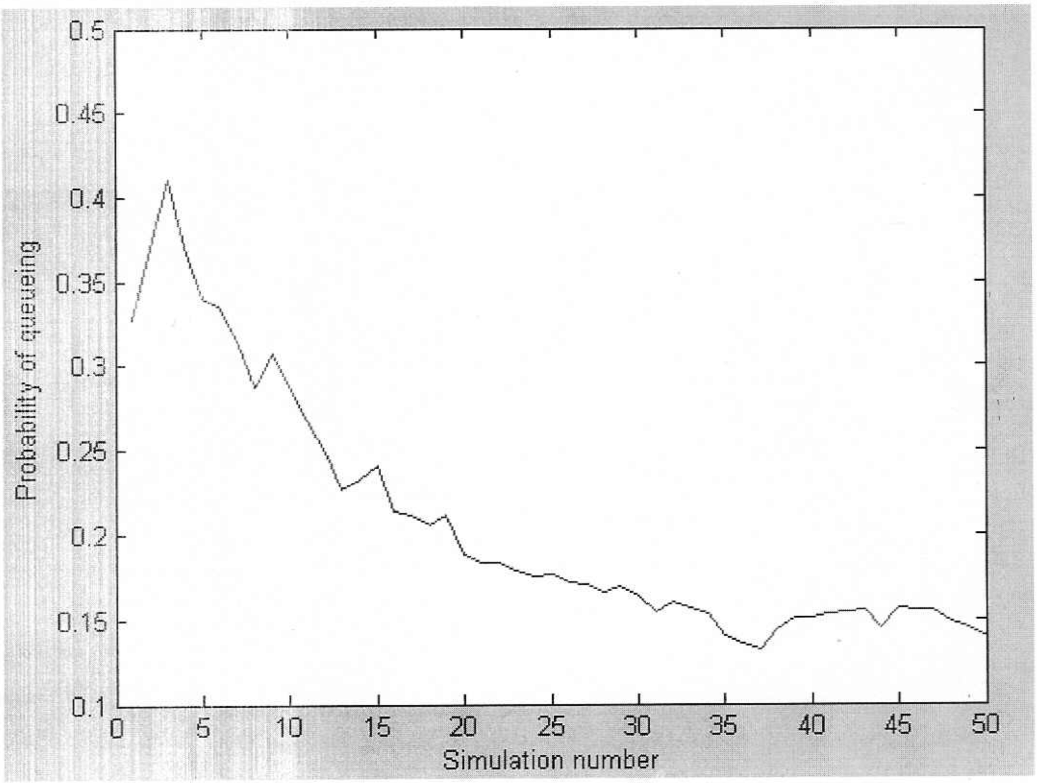
12.71



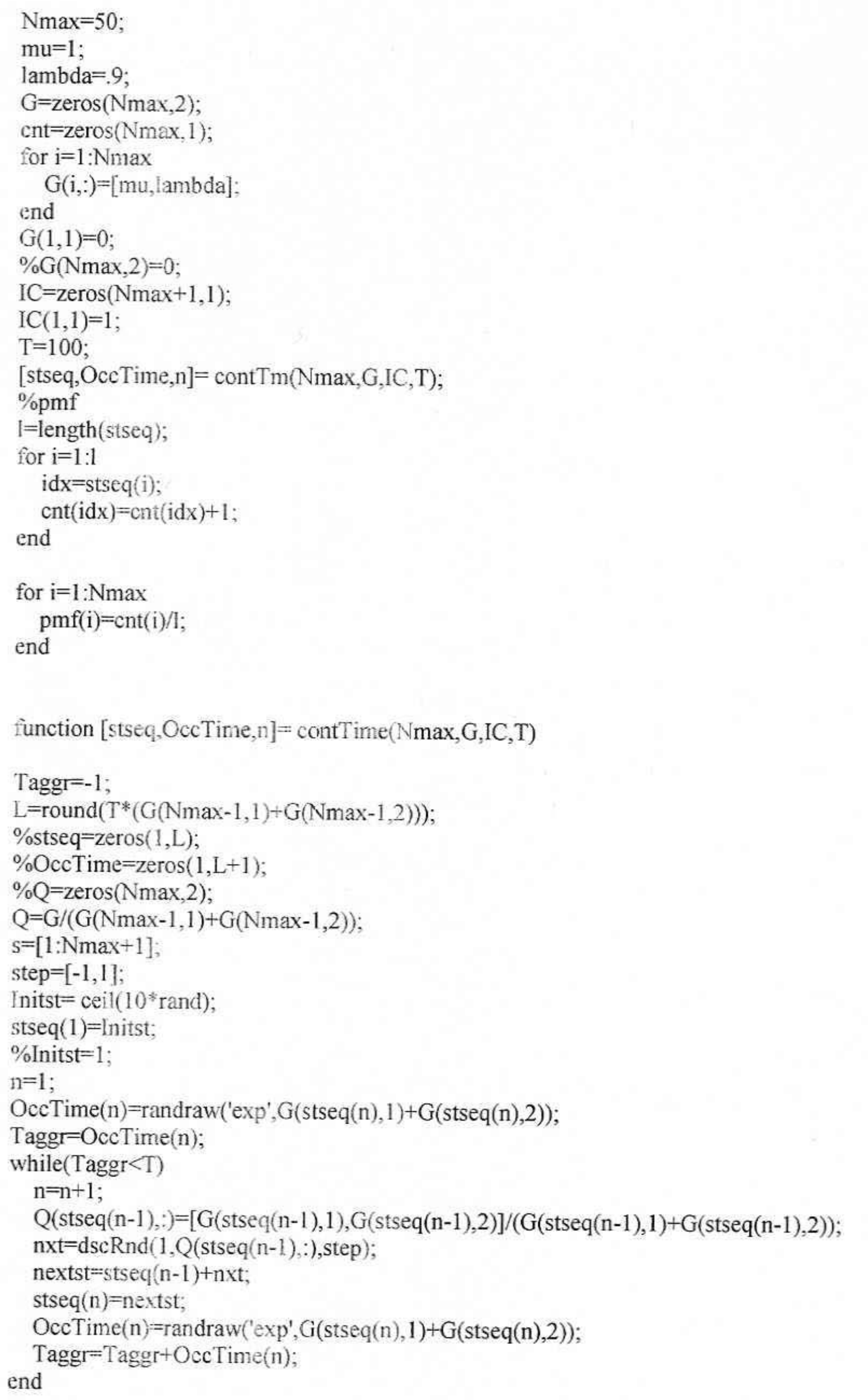
```
Nmax=50;
P=zeros(Nmax+1,3);
mu=1;
lambda=2;
delta=.1;
c=3;
a=delta*lambda;
b=delta*mu*c;
b2=delta*mu*2;
P(1,:)=[0,1-a,a];
P(2,:)=[(1-a)*b2,a*b2+(1-a)*(1-b2),(1-b2)*a];
r=[(1-a)*b,a*b+(1-a)*(1-b),(1-b)*a];
for n=3:Nmax;
   P(n,:)=r;
end
P(Nmax+1,:)=[(1-a)*b,1-(1-a)*b,0];
IC=zeros(Nmax+1,1);
IC(1,1)=1;
L=5000;
n=50;
avg_seq=zeros(L,1);
avg_cor=zeros(L,1);
avg=zeros(n,6);
conf=zeros(n,2);
avgseq=zeros(n,1);
prcnt=zeros(n,1);
prq=zeros(n,1);
z=1.68;  %90% confidence interval

for j=1:n
   seq=queueState(Nmax,P,IC,L);
   avgseq(j)=sum(seq(1:L))/L;
   conf(j,1)=avgseq(j)-z*sqrt(var(seq))/sqrt(n);
   conf(j,2)=avgseq(j)+z*sqrt(var(seq))/sqrt(n);
   for l=1:L
      avg_seq(l)=(avg_seq(l)*(j-1)+seq(l))/j;
   end
   prcnt(j)=accuracy(avg_seq,conf(j,1), conf(j,2),L);
   prq(j)=prQue(avg_seq,c,L);
   for i=1:6
      ii=200*i+1;
      avg(j,i)=sum(seq(ii:ii+800))/800;
   end
end
```

```
avg1=zeros(6,1);
for i=1:6
   ii=200*i+1;
   avg1(i)=sum(avg_seq(ii:ii+800))/800;
end
plot(seq);



function acc = prQue(seq,c,L)
cnt=0;
for i=1:L
   if seq(i)>c
      cnt=cnt+1;
   end
end
acc=cnt/L;
```

Probability of being queued before getting the service:

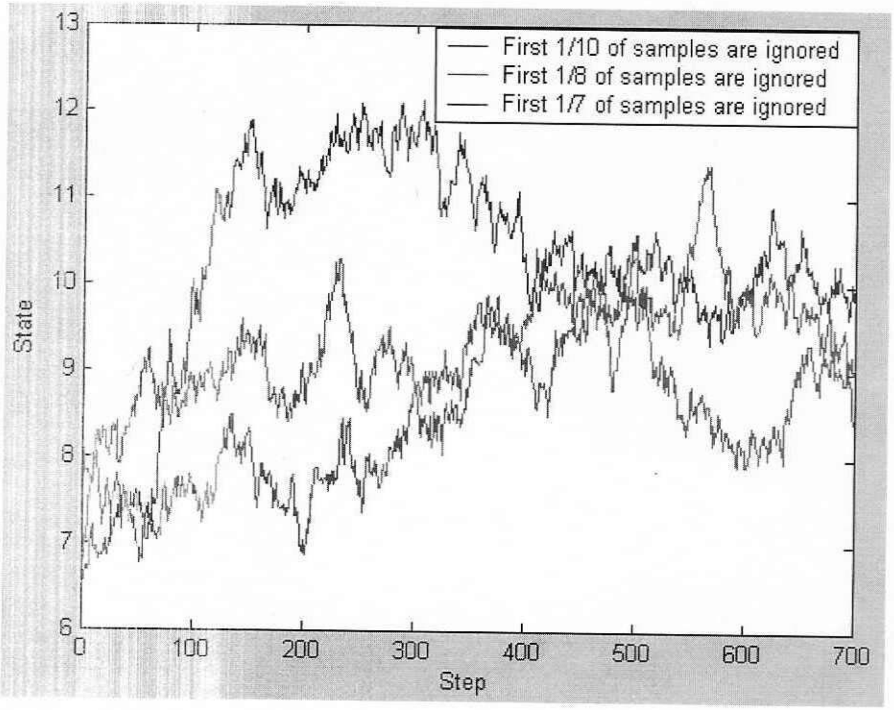


12.72

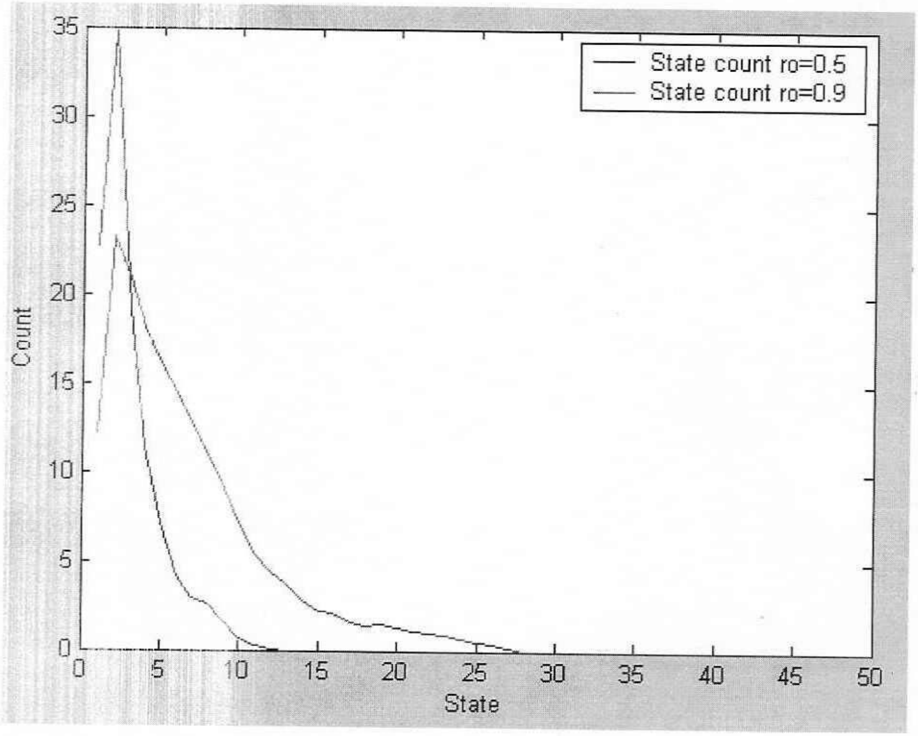


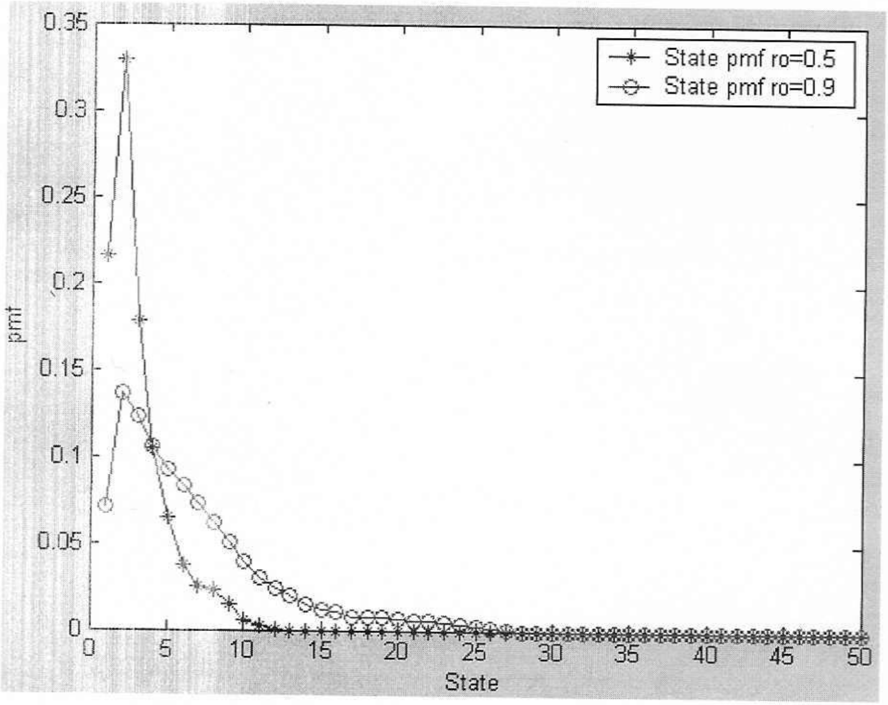
```
Nmax=50;
mu=1;
lambda=.9;
G=zeros(Nmax,2);
cnt=zeros(Nmax,1);
for i=1:Nmax
   G(i,:)=[mu,lambda];
end
G(1,1)=0;
%G(Nmax,2)=0;
IC=zeros(Nmax+1,1);
IC(1,1)=1;
T=100;
[stseq,OccTime,n]= contTm(Nmax,G,IC,T);
%pmf
l=length(stseq);
for i=1:l
   idx=stseq(i);
   cnt(idx)=cnt(idx)+1;
end

for i=1:Nmax
   pmf(i)=cnt(i)/l;
end


function [stseq,OccTime,n]= contTime(Nmax,G,IC,T)

Taggr=-1;
L=round(T*(G(Nmax-1,1)+G(Nmax-1,2)));
%stseq=zeros(1,L);
%OccTime=zeros(1,L+1);
%Q=zeros(Nmax,2);
Q=G/(G(Nmax-1,1)+G(Nmax-1,2));
s=[1:Nmax+1];
step=[-1,1];
Initst= ceil(10*rand);
stseq(1)=Initst;
%Initst=1;
n=1;
OccTime(n)=randraw('exp',G(stseq(n),1)+G(stseq(n),2));
Taggr=OccTime(n);
while(Taggr<T)
   n=n+1;
   Q(stseq(n-1),:)=[G(stseq(n-1),1),G(stseq(n-1),2)]/(G(stseq(n-1),1)+G(stseq(n-1),2));
   nxt=dscRnd(1,Q(stseq(n-1),:),step);
   nextst=stseq(n-1)+nxt;
   stseq(n)=nextst;
   OccTime(n)=randraw('exp',G(stseq(n),1)+G(stseq(n),2));
   Taggr=Taggr+OccTime(n);
end
```

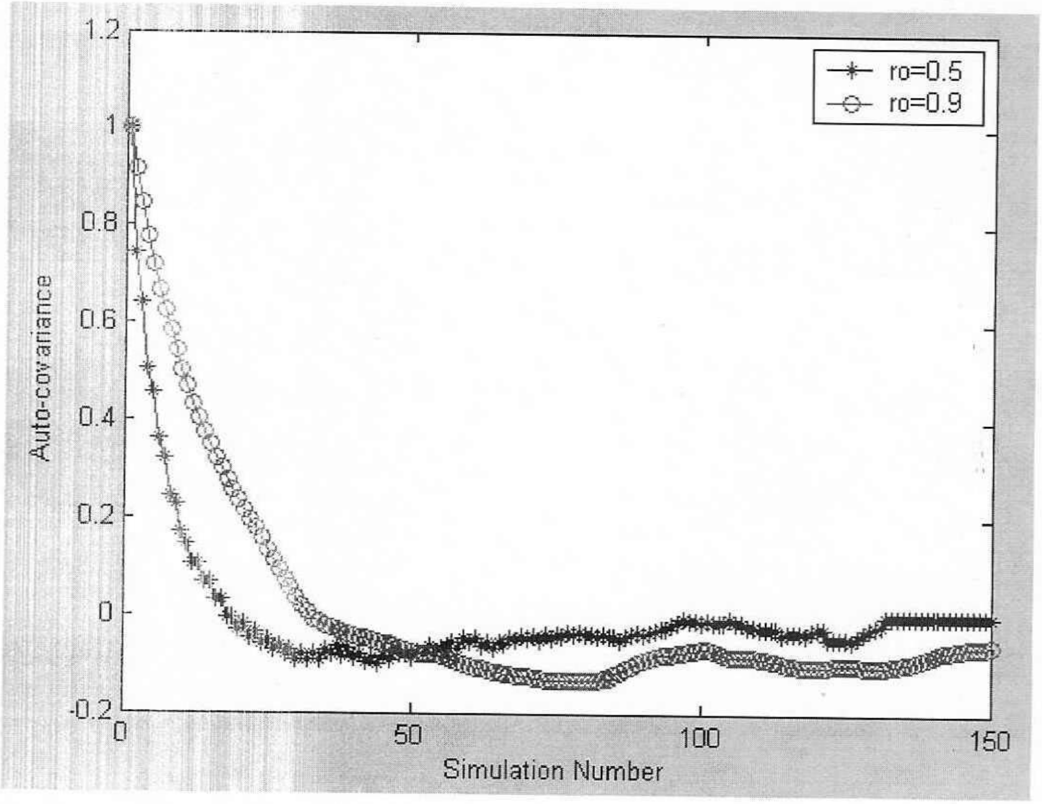
This diagram compares three different warm-up periods:



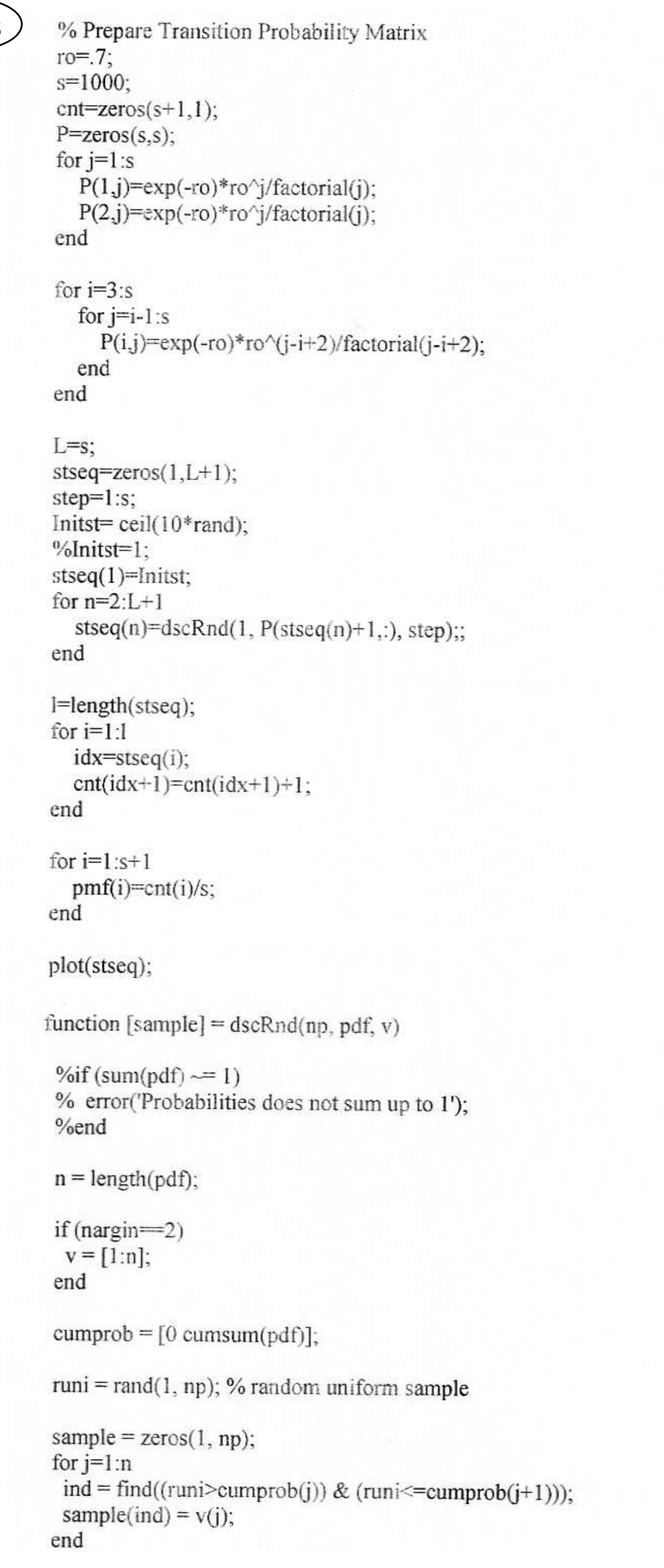
The diagram of stay in each one of the states and the pmf's:



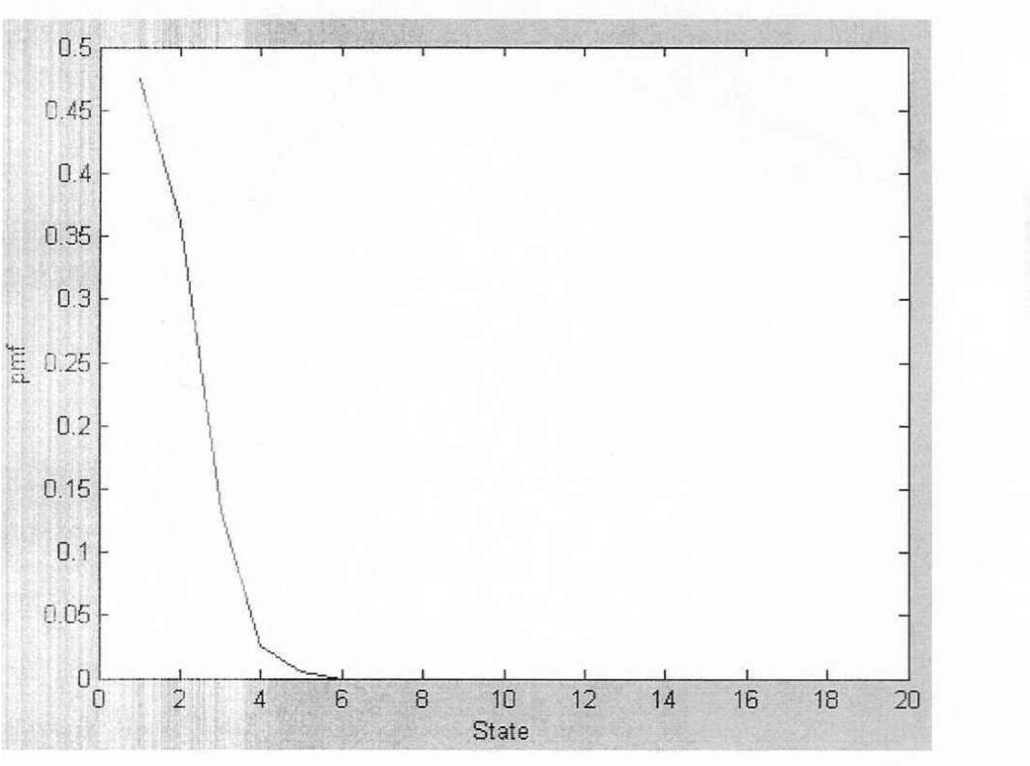

State pmf ro=0.5
State pmf ro=0.9
pmf
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
35
40
45
50
State


ro=0.5
ro=0.9
Auto-covariance
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
50
100
150
Simulation Number

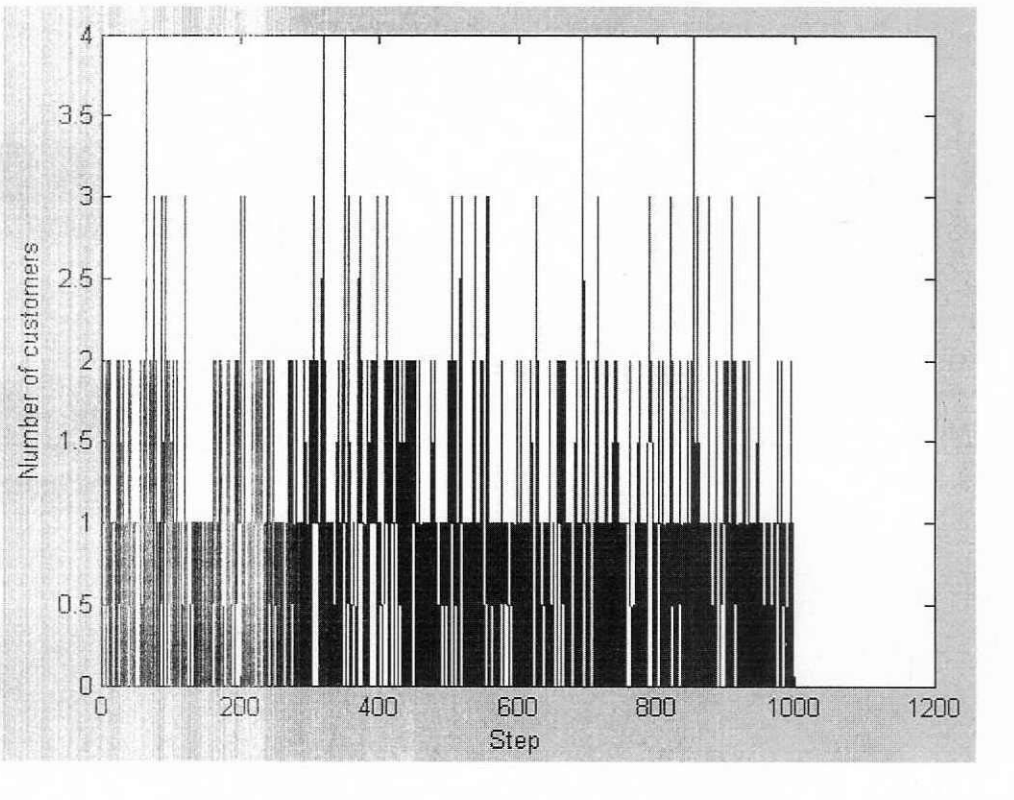
12.73



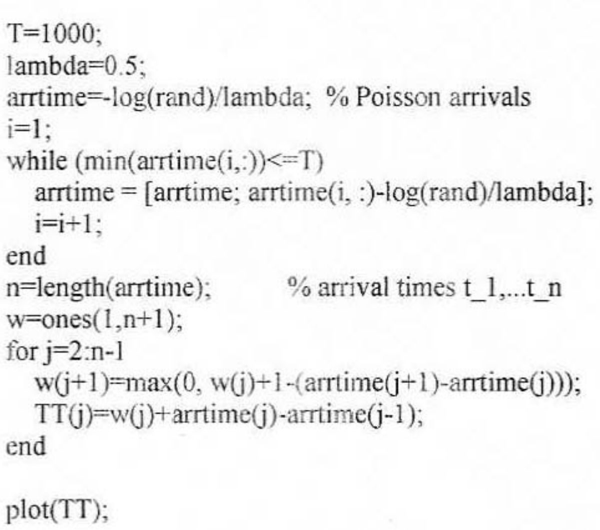
```
% Prepare Transition Probability Matrix
ro=.7;
s=1000;
cnt=zeros(s+1,1);
P=zeros(s,s);
for j=1:s
   P(1,j)=exp(-ro)*ro^j/factorial(j);
   P(2,j)=exp(-ro)*ro^j/factorial(j);
end

for i=3:s
   for j=i-1:s
      P(i,j)=exp(-ro)*ro^(j-i+2)/factorial(j-i+2);
   end
end

L=s;
stseq=zeros(1,L+1);
step=1:s;
Initst= ceil(10*rand);
%Initst=1;
stseq(1)=Initst;
for n=2:L+1
   stseq(n)=dscRnd(1, P(stseq(n)+1,:), step);;
end

l=length(stseq);
for i=1:l
   idx=stseq(i);
   cnt(idx+1)=cnt(idx+1)+1;
end

for i=1:s+1
   pmf(i)=cnt(i)/s;
end

plot(stseq);

function [sample] = dscRnd(np, pdf, v)

 %if (sum(pdf) ~= 1)
 %  error('Probabilities does not sum up to 1');
 %end

 n = length(pdf);

 if (nargin==2)
   v = [1:n];
 end

 cumprob = [0 cumsum(pdf)];

 runi = rand(1, np); % random uniform sample

 sample = zeros(1, np);
 for j=1:n
   ind = find((runi>cumprob(j)) & (runi<=cumprob(j+1)));
   sample(ind) = v(j);
 end
```



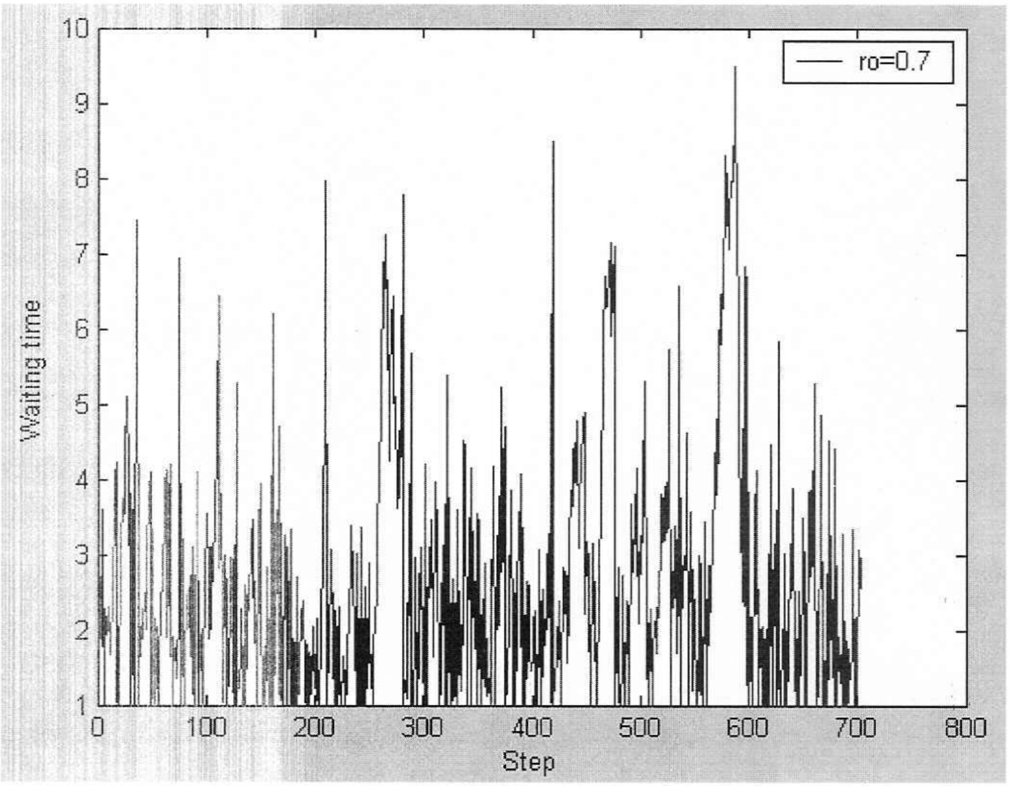
Number of customers in each step:

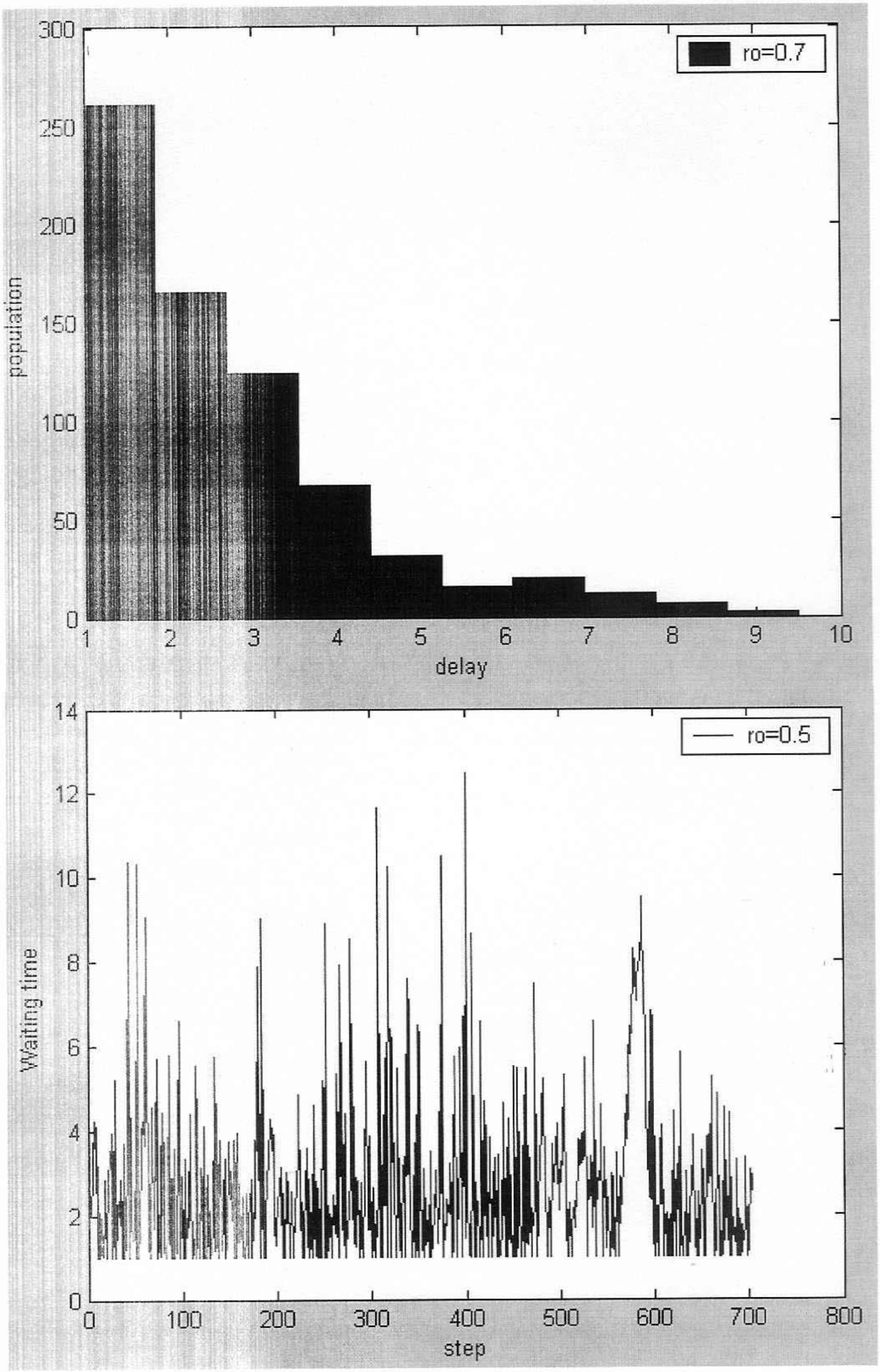


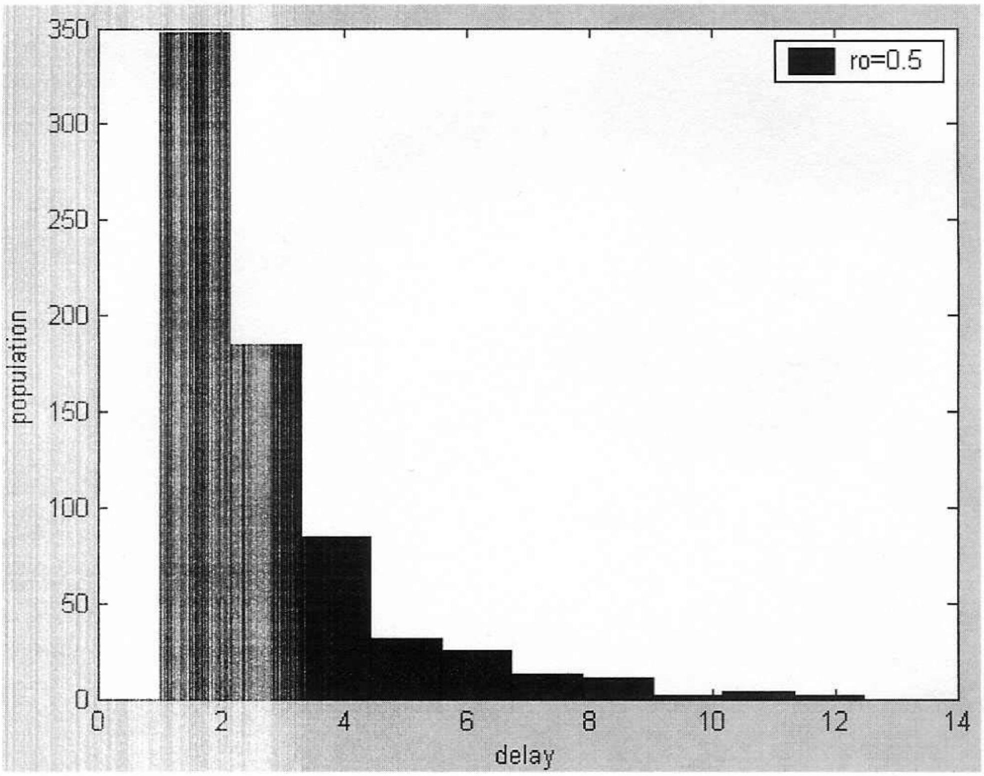
12.74



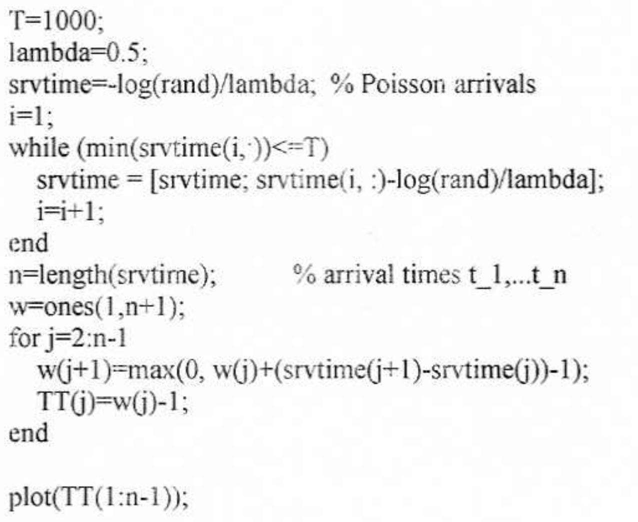
```
T=1000;
lambda=0.5;
arrtime=-log(rand)/lambda;  % Poisson arrivals
i=1;
while (min(arrtime(i,:))<=T)
   arrtime = [arrtime; arrtime(i, :)-log(rand)/lambda];
   i=i+1;
end
n=length(arrtime);          % arrival times t_1,...t_n
w=ones(1,n+1);
for j=2:n-1
   w(j+1)=max(0, w(j)+1-(arrtime(j+1)-arrtime(j)));
   TT(j)=w(j)+arrtime(j)-arrtime(j-1);
end

plot(TT);
```



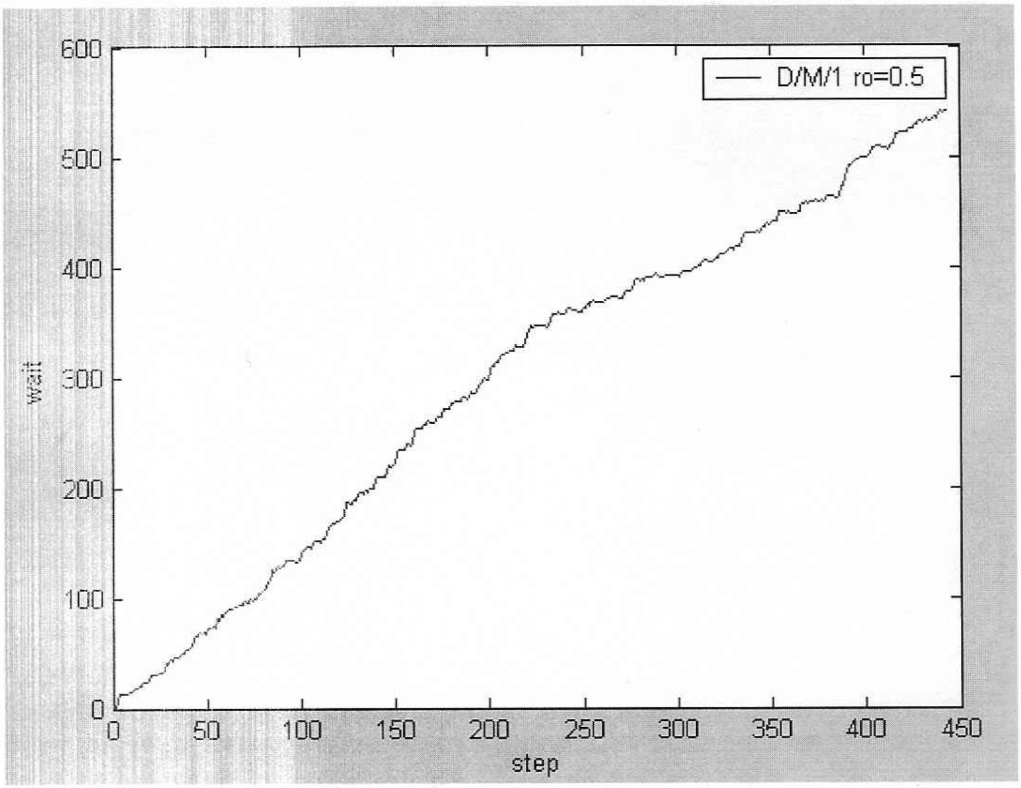

ro=0.7
population
delay
0
50
100
150
200
250
300
1
2
3
4
5
6
7
8
9
10
ro=0.5
Waiting time
step
0
2
4
6
8
10
12
14
0
100
200
300
400
500
600
700
800

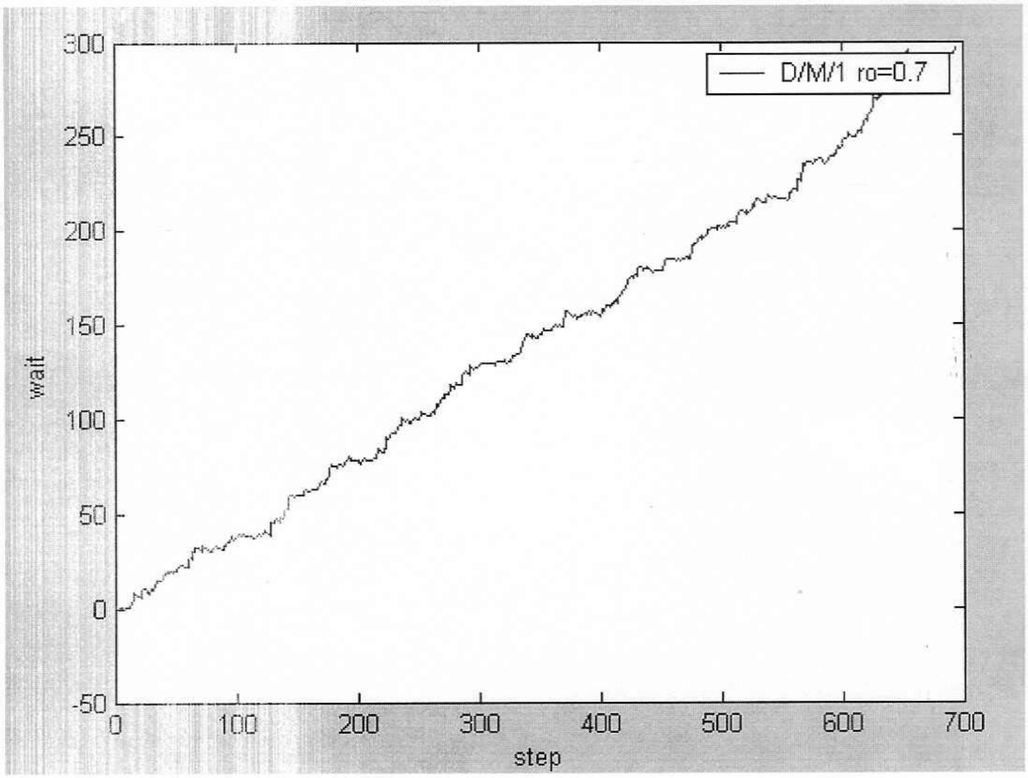


12.75

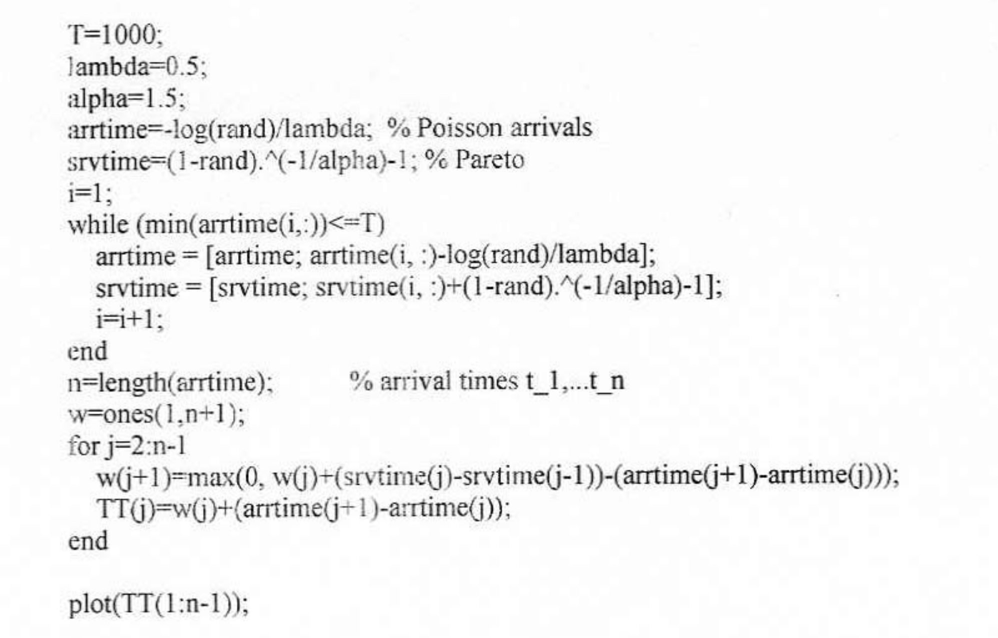


```
T=1000;
lambda=0.5;
srvtime=-log(rand)/lambda;  % Poisson arrivals
i=1;
while (min(srvtime(i,:))<=T)
   srvtime = [srvtime; srvtime(i, :)-log(rand)/lambda];
   i=i+1;
end
n=length(srvtime);          % arrival times t_1,...t_n
w=ones(1,n+1);
for j=2:n-1
   w(j+1)=max(0, w(j)+(srvtime(j+1)-srvtime(j))-1);
   TT(j)=w(j)-1;
end

plot(TT(1:n-1));
```

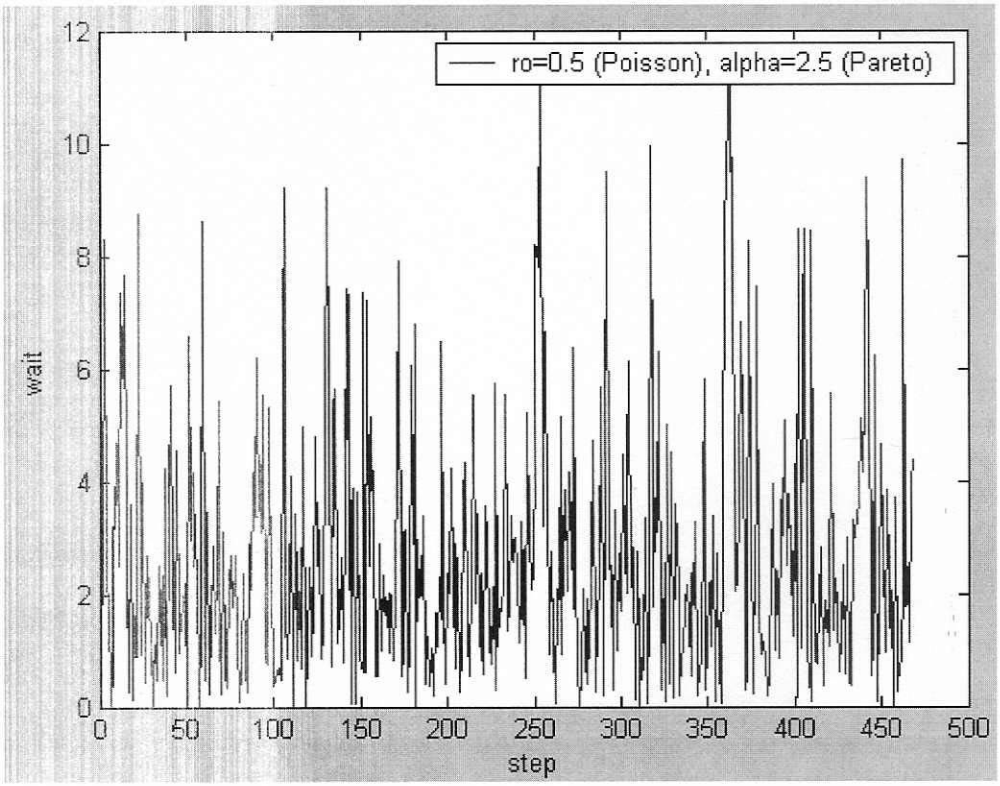

D/M/1 ro=0.5
wait
step
0
100
200
300
400
500
600
0
50
100
150
200
250
300
350
400
450

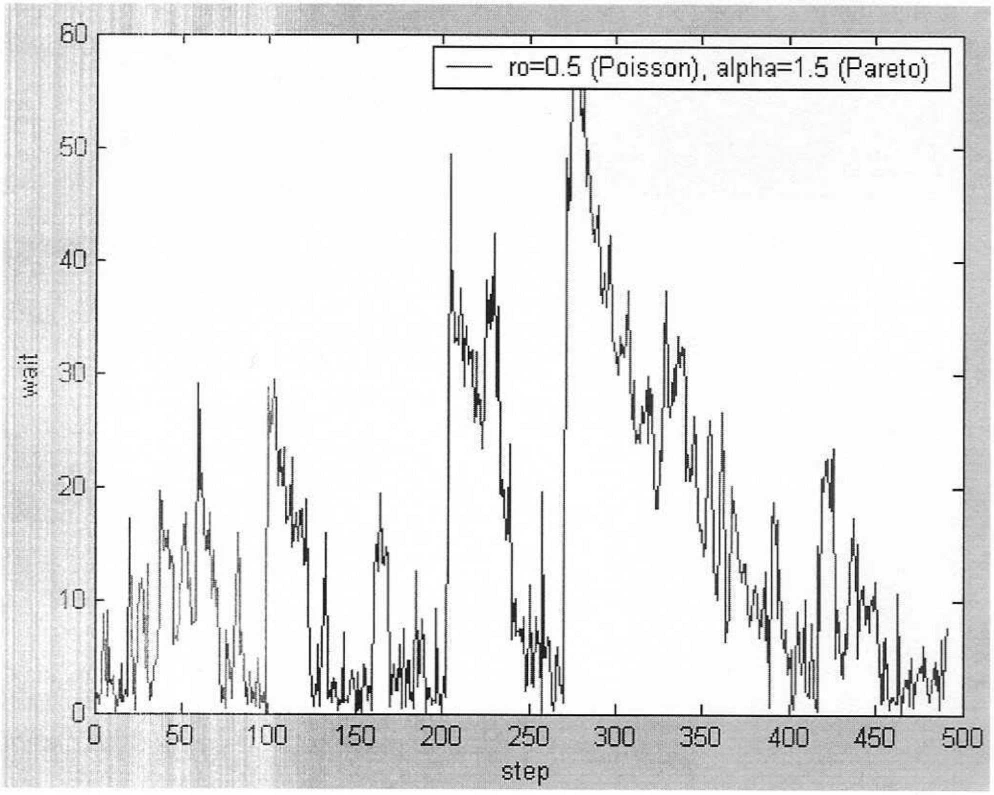

D/M/1 ro=0.7
wait
step
-50
0
50
100
150
200
250
300
0
100
200
300
400
500
600
700

12.76



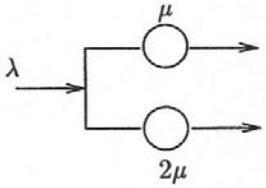
```
T=1000;
lambda=0.5;
alpha=1.5;
arrtime=-log(rand)/lambda;  % Poisson arrivals
srvtime=(1-rand).^(-1/alpha)-1; % Pareto
i=1;
while (min(arrtime(i,:))<=T)
   arrtime = [arrtime; arrtime(i, :)-log(rand)/lambda];
   srvtime = [srvtime; srvtime(i, :)+(1-rand).^(-1/alpha)-1];
   i=i+1;
end
n=length(arrtime);          % arrival times t_1,...t_n
w=ones(1,n+1);
for j=2:n-1
   w(j+1)=max(0, w(j)+(srvtime(j)-srvtime(j-1))-(arrtime(j+1)-arrtime(j)));
   TT(j)=w(j)+(arrtime(j+1)-arrtime(j));
end

plot(TT(1:n-1));
```




ro=0.5 (Poisson), alpha=1.5 (Pareto)
wait
step
0
10
20
30
40
50
60
0
50
100
150
200
250
300
350
400
450
500

Problems Requiring Cumulative Knowledge

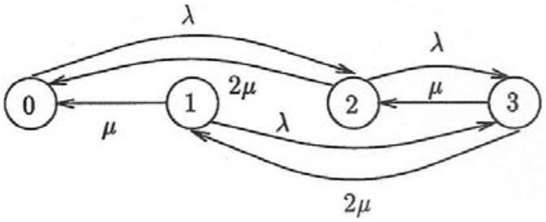
12.80



a) There are four possible cases: both servers are idle, the slower server is idle, the faster server is idle or both servers are busy. If we define each of these cases as a state, then $X(t)$ will be a four-state continuous-time Markov chain.

b) In this case, the transition rate diagram will be as follows:

State 1: Slower server is busy & faster server is idle.
State 2: Faster server is busy & slower server is idle.



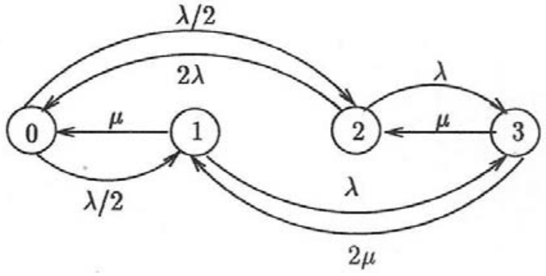
Writing global balance equations, we will have:

$$\begin{aligned} P_0\lambda &= \mu P_1 + 2\mu P_2 \\ 2\mu P_3 &= (\mu+\lambda)P_1 \\ (\lambda+2\mu)P_2 &= \mu P_3 + \lambda P_0 \\ 2\mu P_3 &= \lambda P_2 + \lambda P_1 \end{aligned}$$

Incorporating the fact that $\sum_i P_i = 1$ and having $\rho = \frac{\lambda}{\mu}$, the following is obtained:

$$\begin{aligned} P_0 &= \frac{2\rho+3}{\rho^3+2\rho^2+3.5\rho+3} \\ P_1 &= \frac{\rho^2}{\rho^3+2\rho^2+3.5\rho+3} \\ P_2 &= \frac{0.5\rho^2+1.5\rho}{\rho^3+2\rho^2+3.5\rho+3} \\ P_3 &= \frac{\rho^3+0.5\rho^2}{\rho^3+2\rho^2+3.5\rho+3} \end{aligned}$$

c) The Markov chain is as follows:



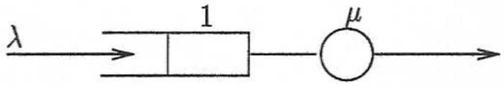
$$\begin{aligned}
\rho P_0 &= \mu P_1 + 2\mu P_2 \\
2\mu P_3 + \frac{\lambda}{2} P_0 &= (\lambda + \mu) P_1 \\
(\lambda + 2\mu) P_2 &= \mu P_3 + \frac{\lambda}{2} P_0 \\
3\mu P_3 &= \lambda P_2 + \lambda P_1
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad P_0 &= \frac{4}{\rho^2 + 3\rho + 4} \\
P_1 &= \frac{2\rho}{\rho^2 + 3\rho + 4} \\
P_2 &= \frac{\rho}{\rho^2 + 3\rho + 4} \\
P_3 &= \frac{\rho^2}{\rho^2 + 3\rho + 4}
\end{aligned}$$

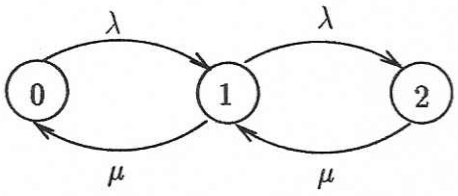
It is expected that the average waiting time in the case of part (b) will be less than that of part (c).

12.81

~~9.59~~ M/M/1/2



This is a three-state continuous-time Markov chain:



The associated Chapman-Kolmogorov: $P_i' = \sum_j \Pi_{ji} P_j$

$$\Rightarrow \quad \begin{aligned} P_0' &= -\lambda P_0 + \mu P_1 \\ P_1' &= \lambda P_0 + \mu P_2 - (\mu + \lambda) P_1 \\ P_2' &= \lambda P_1 - \mu P_2 \end{aligned}$$

This set of differential equations can be solved much easier using Laplace transform:

$$\begin{cases} sP_0(s) - P_0(0) = -\lambda P_0(s) + \mu P_1(s) \\ sP_1(s) - P_0(0) = \lambda P_0(s) + \mu P_2(s) - (\mu + \lambda) P_1(s) \\ sP_2(s) - P_2(0) = \lambda P_1(s) - \mu P_2(s) \end{cases}$$

$P_0(s)$, $P_1(s)$, and $P_2(s)$ are obtained by solving this set of linear equations from which the transient pmfs, $P_i(t) = P[N(t) = i]$, are obtained.

a) Suppose that the system is empty at $t = 0 \Rightarrow P_0(0) = 1, P_1(0) = P_2(0) = 0$

$$\Rightarrow \quad \begin{aligned} P_1(s) &= \frac{\lambda(s+\mu)}{s(s^2 + 2(\mu+\lambda)s + \mu\lambda + \lambda^2 + \mu^2)} = \frac{\lambda(s+\mu)}{F(s)} \\ P_0(0) &= \frac{\mu\lambda(s+\mu)}{s(s+\mu)(s^2 + 2(\mu+\lambda)s + \mu\lambda + \lambda^2 + \mu^2)} + \frac{1}{s+\lambda} \\ P_2(s) &= \frac{\lambda^2}{s(s^2 + 2(\mu+\lambda)s + \mu\lambda + \lambda^2 + \mu^2)} \end{aligned}$$

$\Rightarrow$ By partial fraction expansion, we will have:

$$P_i(s) = \frac{A_i}{s} + \frac{B_i}{s+\lambda} + \frac{C_i}{s+a_1} + \frac{D_i}{s+a_2}$$

where $a_1, a_2 = -(\mu + \lambda) \pm \sqrt{\mu\lambda}$.
Applying inverse Laplace transform, we obtain:

$$P_i(t) = A_i + B_i e^{-\lambda t} + C_i e^{-\alpha_1 t} + D_i e^{-\alpha_2 t}$$

Finding the factors $A_i - D_i$ is very easy because $P_i(s)$ have only simple 1st order roots.

b) If the system is full at $t = 0 \Rightarrow P_2(0) = 1, P_0(0) = P_1(0) = 0$

$$\Rightarrow \quad \begin{aligned} P_1(s) &= \frac{\mu(s+\lambda)}{F(s)} \\ P_0(s) &= \frac{\mu^2}{F(s)} \\ P_2(s) &= \frac{\mu\lambda(s+\lambda)}{(s+\mu)F(s)} + \frac{1}{s+\mu} \end{aligned}$$

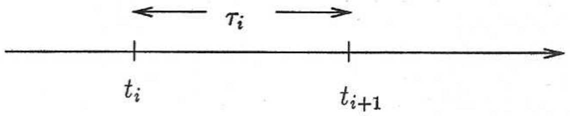
The expansion of these fractions will be as follows:

$$P_i(s) = \frac{A_i}{s} + \frac{B_i}{s+\mu} + \frac{C_i}{s+a_1} + \frac{P_i}{s+a_2}$$

$$\Rightarrow \quad P_i(t) = A_i + B_i e^{-\mu t} + C_i e^{-a_i t} + D_i e^{-a_2 t}$$

12.82

~~9.60~~ **a)** Let's divide the time axis into time intervals or cycles τ_i which are the times when customers arrive to an empty system.



We have $t_j = \tau_0 + \tau_1 + \tau_2 + ... + \tau_i$.

If the τ_i's are identically distributed independent random variables, then they form a renewal process. In an M/G/1, each time a customer arrives to an empty system restarts a process which is independent of all past events. In addition, all cycles have the same statistics, i.e. they constitute a repeated trial.

b) Let $N(t)$ be the number of customers in the system.

$$P[N(t) = j] = \lim_{t\to\infty} \frac{\text{Total time that there are 'j' customers in the system}}{t}$$

To find this, let's associate a cost, C_i, to each renewal interval τ_i. C_i is equal to amount of time that there are 'j' customers in the system in time τ_i. Also, $C(t) = \sum_i C_i$.

$$P[N(t) = j] = \lim_{t\to\infty} \frac{C(t)}{t}$$

According to the renewal theorem

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E(C)}{E[\tau]}$$

Therefore,

$$P[N(t) = j] = \frac{E(C)}{E(\tau)} \approx \frac{\frac{1}{n}\sum_{j=1}^{n} C_j}{\frac{1}{n}\sum_{j=1}^{n} \tau_j}$$

c) We can find confidence intervals for the numerator and denominator of the above expression using the methods from section 8.4.

12.83

~~9.61~~ **a)**
$$\begin{aligned} P[N_1(t) = i, N_2(t) = j] &= P[N_1(t) = i, N(t) = i + j] \\ &= P[N_1(t) = i/N(t) = i + j]P[N(t) = i + j] \end{aligned}$$

If we know that $N(t) = i + j$, then each of the $i + j$ arrivals picks its arrival time according to a uniform distribution in the interval $(0, t)$ independently of the other arrivals

Therefore the probability that an arrival for $N(t)$ is also an arrival for $N_1(t)$ is

$$\int_0^t Pr[\text{arrival for } N_1(t)/\text{arrival time is } t]\frac{1}{t}dt = \frac{1}{t}\int_0^t p(t)dt \triangleq p$$

Since the $i + j$ arrivals are independent, we have

$$P[N_1(t) = i|N(t) = i + j] = \binom{i+j}{i} p^i(1-p)^j$$

Finally, we have

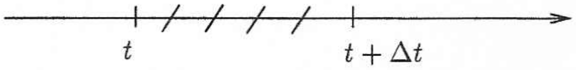
$$\begin{aligned} P[N_1(t)=i, N_2(t)=j] &= \binom{i+j}{i} p^i(1-p)^j \frac{(\lambda t)^{i+j}}{(i+j)!} e^{-\lambda t} \\ &= \frac{p^i(1-p)^j}{i!j!}(\lambda t)^{i+j} e^{-\lambda t} \\ &= \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^j}{j!} e^{-\lambda t} e^{-\lambda(1-p)t} \\ &= P[N_1(t)=i] P[N_2(t)=j]\ . \end{aligned}$$

b) In order for $N_1(t)$ and $N_2(t)$ to be independent random processes we require that

$$P[N_1(t_1)=i, N_2(t_2)=j] = P[N_1(t_1)=i] P[N_2(t_2)=j]$$

for any i, j and any choices of times t_1 and t_2.

Consider a very small time interval at time t of duration Δt:



Consider the evolution of the joint process $(N_1(t), N_2(t))$

$$\begin{aligned} P[N_1(t+\Delta t)=i, N_2(t+\Delta t)=j] &= P[N_1(t)=i-1,\ \text{a type 1 arrival in } \Delta t] \\ &\quad +P[N_2(t)=j-1,\ \text{a type 2 arrival in } \Delta t] \\ &\quad +P[N_1(t)=i, N_2(t)=j,\ \text{no arrival in } \Delta t] \\ &\quad +O(\Delta t) \\ &= P[N_1(t)=i-1]\lambda p(t)\Delta t \\ &\quad +P[N_2(t)=j-1]\lambda(1-p(t))\Delta t \\ &\quad +P[N_1(t)=i, N_2(t)=j](1-\lambda\Delta t) \\ &\quad +O(\Delta t) \end{aligned}$$

Now consider the individual evolution of the two processes:

$$\begin{aligned} P[N_1(t+\Delta t)=i] &= P[N_1(t)=i-1]\lambda p(t)\Delta t \\ &\quad +P[N_1(t)=i](1-\lambda p(t)\Delta t)+O(\Delta t) \\ P[N_2(t+\Delta t)=j] &= P[N_2(t)=j-1]\lambda(1-p(t))\Delta t \\ &\quad +P[N_2(t)=j](1-\lambda(1-p(t))\Delta t)+O(\Delta t) \end{aligned}$$

If we multiply these two equations we obtain:

$$\begin{aligned} P[N_1(t+\Delta t)=i]P[N_2(t+\Delta t)=j] &= P[N_1(t)=i-1]P[N_2(t)=j]\lambda p(t)\Delta t \\ &\quad +P[N_1(t)=i]P[N_2(t)=j-1]\lambda(1-p(t))\Delta t \\ &\quad +P[N_1(t)=i]P[N_2(t)=j](1-\lambda\Delta t)+O(\Delta t) \end{aligned}$$

By comparing this equation to that of the joint process, we see that we will end up with independent random processes.

12.84

~~9.62~~ a) Suppose a customer arrived at time $t_1 < t$, then the customer has completed its service time by time t with probability

$$P[X < t - t_1] = F_X(t - t_1) \triangleq p(t_1)$$

Therefore customers that arrived in the interval $(0, t)$ have probability of completing service:

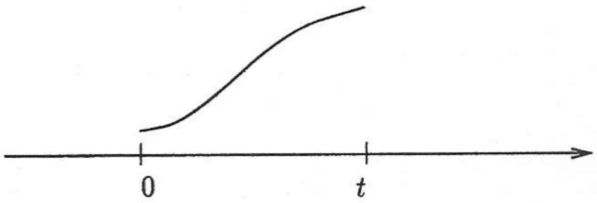
$$p = \frac{1}{t}\int_0^t F_X(t - t_1)dt_1$$

Thus

$$P[N_1(t) = i, N_2(t) = j] = \frac{(\lambda tp)^i}{i!}e^{-\lambda tp}\frac{(\lambda t(1-p))^j}{j!}e^{-\lambda t(1-p)}$$

Note that

$$\begin{aligned}\lambda tp &= \lambda \int_0^t F_X(t - t_1)dt_1 \\ \lambda t(1-p) &= \lambda t\left[\frac{1}{t}\int_0^t (1 - F_X(t - t_1))dt_1\right] \\ &= \lambda \int_0^t (1 - F_X(t - t_1))dt_1\end{aligned}$$



b) As $t \to \infty$

$$\lambda t(1-p) \to \lambda \int_0^t (1 - F_X(t))dt = \lambda E[X]$$

$$\therefore P[N_2(t) = j] = \frac{(\lambda E[X])^j}{j!}e^{-\lambda F[X]} \quad \text{Poisson RV!}$$

c) Little's formula

$$E[N] = \lambda E[X] \qquad \checkmark$$