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A Fasano & M Primicerio (Editors)

Free boundary problems: theory and applications

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Preface

These volumes contain the proceedings of the interdisciplinary Symposium on "Free Boundary Problems: Theory & Applications" held in Montecatini (Italy) from June 17 to June 26, 1981.

The Scientific Committee was composed of Professors G. Sestini, C. Baiocchi, V. Boffi, G. Capriz, R. Conti, D. Galletto, G. Geymonat, E. Magenes, D. Quilghini, E. Vesentini. The organizers, editors of these volumes, have also had valuable help from a group of four people composing the European Liaison Committee: Professors M. Frémond, K.-H. Hoffmann, J. R. Ockendon, M. Niezgodgka.

In this preface, after a short reference to previous meetings on the same subject, we will explain the aims that the Symposium was pursuing, and thank all the persons and institutions who made it possible.

1. Some history

International conferences on free boundary problems have a recent, but already rich tradition, testifying to the rapidly growing interest in this subject. In 1974 a meeting on "Moving Boundary Problems in Heat Flow and Diffusion" was held at the University of Oxford (see [1]). In that conference a number of problems were presented, in connection with practical applications, exhibiting the common feature that a parabolic equation had to be solved in a region whose boundary was partly unknown, thus requiring additional specification of the data. Particular attention was devoted to phase change problems.

In the workshop held in Gatlinburg (USA) in 1977 on "Moving Boundary Problems" (see [2]) an attempt was made to get pure mathematicians more involved in the debate. Dialogue between pure and applied scientists was one of the principal aims of the meeting in Durham, UK (1978) on "Free and Moving Boundary Problems in Heat Flow and Diffusion" (see [3]), where it appeared that free boundary problems were spreading over a tremendously large field, both from the point of view of applications and of mathematical methods; moreover, attention was brought to bear not only on parabolic problems, but also on elliptic problems.

This trend was clearly confirmed in the intensive "Bimester on Free Boundary Problems", held in Pavia (see [4]), in which a huge number of topics was covered. We would also like to mention the more specialized meeting in Oberwolfach (1980), focussed on numerical methods (see [5]).

2. Scope of the Symposium

On the basis of the experience accumulated in the foregoing Conferences, the meeting in Montecatini was primarily intended as a catalyst for a (hopefully positive) interaction between people working on purely mathematical aspects or on numerical methods, and people from the area of applied research.

With this in mind, we invited people from various branches of engineering, physics, chemistry, etc., in which free boundary problems are commonly encountered, as well as from pure and applied mathematics.

The Conference was attended by 125 participants from 21 countries.

Of course not all of the people invited came, and maybe not all of the people who should have been invited were invited: we apologize for any possible mistakes made at that stage of the organization. Nevertheless, we believe that these volumes bear evidence of the scientific level of the Symposium.

3. Acknowledgements

The Symposium took place at the "Castello La Querceta" in Montecatini, kindly offered by the municipality, whose generosity and cooperation was decisive for the success of the meeting. We would also like to thank many other institutions which supported the Symposium: first of all the Italian C.N.R. and its mathematical branches (particularly the Comitato per la Matematica, the Gruppo Nazionale per la Fisica Matematica and the Istituto di Analisi Numerica), then the Administration of the Regione Toscana and, finally, the Mathematical Institute "Ulisse Dini" of the University of Florence, which bore much of the secretarial work, the European Research Office of the U.S. Army, which mainly supported the publication of the pre-conference literature, and the Azienda di Cura e Soggiorno of Montecatini, which took care of the opening ceremony.

We are also grateful to the members of the European Liaison Committee for their help in setting up the general structure of the conference, and, of course, to the colleagues of the Scientific Committee: in particular we wish to thank Prof. Sestini and Prof. Magenes for their constant interest and valuable suggestions.

4. The general scheme of the Symposium and of the Proceedings

In view of the large number of topics involved in the Conference, we decided to divide most of the speakers into 10 "Discussion Groups", each one consiting of 5-8 rather homogeneous contributions and lead by a "Rapporteur". In addition, 25 not-grouped talks were given. Discussion groups were conceived with the principal aims of stimulating discussion on specific topics and of giving everybody a chance to communicate his own results. We believe that both of these goals were satisfactorily achieved.

These Proceedings are reflecting this same structure. However, the contributors to Discussion Groups had the possibility of choosing between writing a full paper or letting the Rapporteur report briefly their talk in his general introduction.

As a rule, all the authors were asked to provide papers not exceeding ten pages. Of course, the Rapporteurs' papers were allowed to be longer, as well as some survey papers written on request of the editors.

To conclude this Preface, we wish to thank all the participants for their regular attendance (notwithstanding the tremendous amount of talks given!) and everybody who contributed to these volumes.

Firenze
January 1982

Antonio Fasano
Mario Primicerio

M NIEZGODGKA

Stefan-like problems

1. INTRODUCTION

This paper is intended to give a possibly up-to-date survey on the state of the research of Stefan-like problems and its new trends on one hand, and to reflect the activity of the related discussion group on the other. The collaboration of the contributors to that discussion group, A. Bossavit, L. Caffarelli, D. Hilhorst, I. Pawłow and A. Visintin is to be acknowledged (the abstract of the contribution by D. Hilhorst is enclosed in the form of Appendix to this paper).

In view of the size limitations we concentrate ourselves here on complementing those existing publications which review various aspects of the Stefan problems and their derivatives. We refer here to the basic monograph [R6], proceedings [O2,W1,M3,A1] and surveys [F17,F19,F21,K6,P9]. The above works include a large number of references, therefore we shall turn our attention rather towards new and less known issues, directing the readers especially to [P9] as well as to the bibliography [W2] for further information.

Throughout the paper we shall refer to the following general statements of the Stefan-like problems

One-phase problem Determine a pair $\{\theta, S\}$ consisting of a function θ and a moving free surface S so that to satisfy the system

$$c(\theta)\,\theta_t - A(\theta) = f(\theta) \quad \text{in} \quad Q = \bigcup_{t \in (0,T)} \Omega(t) \times \{t\}\,, \tag{1.1}$$

$$\theta(x,0) = \theta_0(x), \quad \text{for } x \in \Omega(0) \quad \text{with } \Omega(0) \text{ given}, \tag{1.2}$$

$B(\theta) = g(\theta)$ on Σ being the fixed part of the lateral boundary of Q, (1.3)

and free boundary conditions at S:

θ is prescribed at S, (1.4)

$$F(\theta,\nabla\theta,x,t,\vec{V}) = 0 \text{ at } S \text{ where } \vec{V} = \{\vec{V}_x, V_t\} \text{ denotes velocity of } S \text{ at the point } (x,t). \tag{1.5}$$

$\Omega(t) \subset R^n (n \geq 1)$, $t \in (0,T)$ are open domains such that the free boundary S is external with respect to Q, i.e. $S \cap Q = \emptyset$, $S \cap \bar{Q} \neq \emptyset$; A is an elliptic operator, e.g. in the quasilinear divergent form

$$A(\theta) = \operatorname{div}(k(\theta)\nabla\theta), \tag{1.6}$$

B denotes a boundary operator assumed to be admissible for A, F takes in particular the standard Stefan form

$$F(\theta,\nabla\theta,x,t,\vec{V}) = k(\theta)\nabla\theta \cdot \vec{V}_x + L\, V_t + h(x,t) \tag{1.7}$$

with f,θ_o,g,c,k,F (L,h, respectively) given as functions of their arguments.

Two-phase problem: Determine a pair $\{\theta,S\}$ satisfying

$$c(\theta)\,\theta_t - A(\theta) = f(\theta) \quad \text{in } Q\backslash S, \quad \text{where } Q = \Omega \times (0,T), \tag{1.8}$$

$\Omega \subset R^n$ $(n \geq 1)$ is a given open domain, free boundary $S = \bigcup_{t \in (0,T)} S(t) \times \{t\}$,

$$\theta(x,0) = \theta_o(x) \quad \text{in } \Omega\backslash S(0), \quad S(0) \text{ - given}, \tag{1.9}$$

$$B(\theta) = g(\theta) \quad \text{on the fixed lateral boundary } \Sigma = \partial\Omega \times (0,T) \tag{1.10}$$

and conditions at the free boundary S:

$$\theta\big|_{S-},\ \theta\big|_{S+} \text{ - prescribed}, \tag{1.11}$$

$$F(\theta\big|_{S-},\theta\big|_{S+},\nabla\theta\big|_{S-},\nabla\theta\big|_{S+},x,t,\vec{V}) = 0 \quad \text{for} \quad (x,t) \in S \tag{1.12}$$

with F in particular taking the standard Stefan form

$$F = \{k(\theta)\nabla\theta\}\Big|_{S-}^{S+} \cdot \vec{V}_x - L\, V_t + h(x,t) \tag{1.13}$$

and the free boundary S internal with respect to Q, i.e. $S \cap Q \neq \emptyset$.

The plan of the paper is as follows. Section 2 reports a variety of applications giving rise to Stefan-like problems. Section 3 offers a review of recent mathematical results on those problems. Certain new aspects of the analysis of the process behaviour in a vicinity of the free interface of a phase-change are reported in Section 4. Section 5 gives some references to new papers devoted to the numerical approaches. In Section 6 a few open areas of the research (being of importance for practical reasons) are exposed. At the end the abstract of the contribution by O. Diekmann and D. Hilhorst is enclosed in the form of an Appendix.

2. APPLICATIVE MOTIVATIONS

Apart from *traditional* examples of the real physical and technological processes giving rise to mathematical models in the form of Stefan-like problems [R6,O2,W1,F19] one can mention a lot of other applications of these problems (see [F19]). The appearance of free boundaries may be caused by phase transitions not only of a thermodynamic or diffusive origin but also those of electric or ferromagnetic types. In particular we would like to specify the following processes:

- freezing by means of cryosurgery techniques [T8],
- heat transfer in large reservoirs [B7],
- heat transfer at high rates including combustion and thermal explosions [S8,T6](see also the papers of Buckmaster and Crowley in this volume),
- processing of coal including gasification and liquefaction, storage of thermal energy [A1,F12,R4],
- diffusion in multi-component systems combined with absorption [P9,R2],
- mass transfer through deformable semi-permeable membranes [R13],
- polymerization [A3,T10],
- heating of oil-saturated porous media by injecting hot water and the associated dissolution of paraffin sediments in thin layers [F9,R7-9],
- induction heating of steel [B14-15],
- heat transfer in electric contacts [K5,M4,P10],
- ferromagnetic phase transitions [B10,K9],
- multi-component alloy solidification and crystal growth [A2,A4-5,D13, L5-6,P1-2,P6,R10-11,S1-3,T5,Y4],
- heat transfer in porous media [R7,M10],
- frost propagation [F13,O3,P3](see also the related papers in this volume),
- diffusion in multi-component systems combined with chemical reactions [K8],

- evolution processes in meteorology and oceanography [T7].

We would like to notice there a few non-typical features of some of the processes mentioned above.

The models of processes of heat transfer in large reservoirs discussed in [B7] stand out by the form of nonlinearities within the coefficient k of equation (1.1) with A defined by (1.6), and the boundary operator B which exhibit an explicit dependence upon $\nabla\theta$ rather than directly upon θ.

A group of the listed processes related to gasification and liquefaction differs from the others by having an irreversible nature.

Certain processes of induction heating are characterized by a non-local dependence of the free boundary evolution upon temperature θ, in particular of an integral form as discussed in [K5].

3. MATHEMATICAL ASPECTS

A broad class of Stefan-like problems goes far beyond the standard explicit frame [R6,P9], comprising not only parabolic problems but also certain problems for pseudo-parabolic, hyperbolic and elliptic equations as well as various degenerate and implicit formulations.

3.1. Parabolic problems

Results available for the Stefan problems and their derivatives have been comprehensively reviewed in [R6,W1,P9]. There are however several directions of the research which have not been sufficiently exposed in those works or are developed only very recently.

3.1.1. Multidimensional situation

We begin with some comments on the existence of *classical solutions* for multi-phase problems [R8]. The first attempts in showing it are due to Budak and Moskal [B19-20] who have exploited potential theory arguments. The question, still regarded as open, has found recently a satisfactory answer in a number of papers by Meirmanov [M5-9]. In these papers multi-dimensional one- and two-phase Stefan problems (as formulated in Introduction) were considered in the case of linear boundary operators B, in general of the first order. The basic result of [M6-7] assures the existence of a classical solution to the problem over a time interval whose length depends only upon bounds on data. The proof is based on:

- introducing a parametrization of the free boundary S:

$$x = x_o + R(x_o,t)\vec{\nu}(x_o), \quad t \in (0,T) \tag{3.1}$$

where $x_o \in S(0)$ - given initial location of the free boundary (assumed to be a regular closed surface), $\vec{\nu}(x_o)$ - vector normal to $S(0)$ at x_o, and R is a function over $S(0) \times (0,T)$ to be determined;

- regularization of the Stefan condition at the free boundary, taking the second order form

$$R_t - \varepsilon\, \Delta_{S(0)} R = \{[k(\theta)\ \nabla\theta\cdot\nabla\theta](\nabla\theta\cdot\vec{\nu}(x_o))^{-1}\}\Big|_{S-}^{S+} \quad \varepsilon > 0 \tag{3.2}$$

with all the terms calculated at $(x,t) \in S$ given by (3.1) and $\Delta_{S(0)}$ being the Laplace-Beltrami operator defined over the surface $S(0)$.

Let us add that in [M10] the multidimensional Stefan problems are studied in the classical framework by applying Lagrange coordinates.

Most of the results on multidimensional Stefan problems were established by considering their *variational formulations*. Such a way of dealing with these problems induces strong limitations as far as the form of problems is concerned [G1,M2,R8].

The variational formulations of the problems take a form of evolution equations expressed either in terms of "temperature" θ, including a multi-valued maximal monotone term $\{\beta(\theta)\}_t$, or with respect to the *enthalpy* w related to θ by the constitutive law $w \in \beta(\theta)$.

This framework has been introduced in [K1,O4], refined in [F14,L1] and next developed by using techniques of nonlinear functional analysis [B17, L2,D1], applicable in the multi-phase case as well as in the one-phase situation. In a number of papers this approach was applied to more general problems, covering:

- nonlinear distributed source terms and coefficients of the equations, also nonlinear conditions at the fixed boundary [B9,C7,C9,N4-6,V1],
- additional source terms in the free boundary conditions [S1,V2],
- a variable in time division of the fixed boundary in the parts where different conditions are imposed [D4](see also [C14] for a physical description),
- nonlinearities of the hysteresis type [V4].

In the above papers the existence of solutions was proved by regularization of the equations. Besides, [N4-6] offer uniqueness, stability and comparison results not only for the problems involving nonlinearities exhibiting linear growth but also those of any arbitrary order.

The problems can be also expressed in terms of the *freezing index* $y = \int_0^t \theta(x,\tau)\, d\tau$. Such formulations have been introduced in [D15] for one-phase and in [F13,D16] for multi-phase problems.

This approach was exploited mostly in the one-phase situation [C1-4, C6,F15-18,J1,K6-7,M2,L3-4,S1]. Apart from existence and uniqueness theorems, in [C1-4,F15-18,J1,K6-7] deep regularity results were established for the one-phase problems.

The two-phase situation was considered in [D11,D16,L4,P4-5,T2], including proofs of existence, uniqueness and stability. In [P4-5] various generalizations of the form of the problem were admitted.

Let us notice a common feature of the variational formulations consisting in only implicit appearance of the free boundaries there, recovered a posteriori as certain level sets in contrast to their explicit presence in the classical framework.

Regularity questions attract much attention in the study of Stefan-like problems. This concerns mostly the multidimensional situation, since practically complete answers have been given to these questions in the case of one space dimension (see the comprehensive survey [P9] and [F9,F15-18, K6-7,M2,W1]). For $n > 1$ the regularity of the free boundary was studied in the one-phase situation [C1-4,F15-18,K6-7] and results up to local infinite differentiability were shown.

Global continuity of weak solutions was discussed in [C1-4,F15] in the case of one-phase problems by using variational inequality techniques (see also [W1]).

The multi-phase problems were studied in this respect in [C5,D8-9] (see also Appendix I, including an abstract of the contribution presented by L. Caffarelli) by exploiting certain modifications of De Giorgi techniques of deriving internal L^∞-estimates of solutions to parabolic equations in the divergence form (see also [L1] for an exposition of the De Giorgi method).

In [C5] the continuity of weak solutions in the interior of the cylinder $Q = \Omega \times (0,T)$ is shown for the equation

$$\{\beta(\theta)\}_t - \Delta\theta \ni 0 \quad \text{in } Q \tag{3.3}$$

with a graph $\beta(\theta)$ maximal monotone in $R \times R$, in particular being multi-valued of the enthalpy type.

More general situation has been considered in [D8-9], covering equations of the form

$$\{\beta(\theta)\}_t - \operatorname{div} \vec{a}(x,t,\theta,\nabla\theta) + b(x,t,\theta,\nabla\theta) \ni 0, \quad (x,t) \in Q \tag{3.4}$$

with $\vec{a} = \{a_1,\ldots,a_n\}$; $a_i b \in C(\bar{Q} \times R^{n+1})$. Every L^∞-solution of (3.4) has been proved there to be continuous not only in the interior of the cylinder Q, but also up to its boundary (with these results valid both for quasilinear first order boundary conditions and Dirichlet conditions).

3.1.2. One-dimensional situation

A comprehensive review of the results concerning one-dimensional Stefan-like problems has been given recently by Primicerio [P9]. We would like to mention here some new advances in this field.

A very general form of nonlinear balance conditions imposed at the free boundary was considered in [F2,F4] for the one-phase problems involving semi-linear equation. Those conditions took either the first order form

$$L(t,s(t),s'(t))s'(t) = F(s(t),t,\theta_x(s(t),t)), \quad t \in (0,T) \tag{3.5}$$

or the higher order one, e.g.

$$L(t,s(t),s'(t))s'(t) = F(s(t),t,\theta_x(s(t),t),\theta_{xx}(s(t),t)), \; t \in (0,T). \tag{3.6}$$

with $x = s(t)$, $t \in (0,T)$ being a parametrization of the free boundary.

For such problems local in time existence of classical solutions was established by constructing convergent successive approximations along the lines of [F3], as well as Lipschitz continuous dependence in C^1-norm of the free boundary upon perturbations of data was shown. Related questions were also studied in [R6,M13,M17,N2,B5].

Differentiability of the mapping from boundary data into the free boundary was analyzed in [B3,J2] (see also the papers by Baumeister and Hoffmann-Niezgódka in this volume), and its monotonicity in [G2,H1].

Problems with Cauchy data on the free boundary and their equivalence to the Stefan problems involving higher order free boundary conditions were considered in [F2,F6,F8].

Critical cases of the non-existence of solutions were analyzed in [F1-2,F5,F8,M8-9,P11] for the problems without compatibility of data.

New results on the analyticity of the free boundary were established in [R12,R14,S3] in the case of one-phase problems with nonlinear first order conditions (3.5) and semi-linear parabolic equation by using Gevrey's analytic extensions of heat potentials onto the complex plane. Some results in this direction are also exposed in [C16].

Curvature of the free boundary was the subject of the study in [F7,F15, P8].

Asymptotic behaviour of the solutions in the case of quasilinear parabolic equations and the possibility of their stabilization were investigated in [B6].

Quasi-steady regimes of heat conduction, resulting as t infinitely grows, were studied by using variational techniques applied to integral functionals with a variable domain of integration [B12-13,D5-6]. The corresponding elliptic quasi-steady Stefan problems were shown to admit unique solutions. We would like to mention here an alternative approach to these problems proposed in [T3-4].

Finally let us note that in [S5] the problems with a discontinuous free boundary were taken into consideration.

3.1.3. Degenerate Problems

In most of the papers on Stefan problems the coefficients c,k of the equations (1.1),(1.8) are assumed to be positive bounded, with the lower bound on c away of zero. There are however certain physical situations (electromachining, partially saturated flows in porous media) where c must be regarded as only non-negative (possibly vanishing in the interior of the domain Q)[C17]. This leads to degenerate formulations involving equations of the mixed parabolic-elliptic type.

The first study of such problems was presented in [C17] where a uniqueness result was proved by an adaptation of the Oleinik's method [O4]. The related existence theorems were proved in [N3,V2-3]. All the results concern the enthalpy formulations of the problems. This kind of formulations was also used in [D11-12] where a broad class of problems, covering not only the degenerate situations but also non-local non-linearities of energetic type, was shown to admit solutions.

There are also results on the existence [H3,P4-5], uniqueness and stability [P4-5] of solutions of the degenerate problems, established by exploiting variational inequality techniques.

3.1.4. Non-conventional formulations

Non-local nonlinearities. All the nonlinearities exposed so far are of local types, depending pointwise on the solution and its derivatives. There are however certain processes characterized by the appearance of global factors like the specification of the total energy of system or prescribing the total flux in it.

One-dimensional one- and multi-phase Stefan problems with the energy specified within one of the phases were studied in [C11-13]. The specification was defined there by the condition

$$\int_0^{s(t)} \theta(x,t)\,dx = E(t), \quad t \in (0,T); \quad E(t) > 0 \text{ - given.}$$

The existence, uniqueness and stability of classical solutions were concluded by means of heat potential arguments employing the Schauder fixed point theorem and a contractive map.

General results covering a broad class of Stefan-like problems with various non-local nonlinearities are due to Di Benedetto and Showalter [D10]. They have considered the evolution equations

$$\frac{dw}{dt} + A(\theta) \ni f, \quad \theta(0) \ni 0_0; \quad w \in \beta(\theta) \tag{3.7}$$

where $\beta, A : V \to V'$ - maximal monotone operators; V - a Hilbert space, V' - its dual; $f \in L^2(0,T;V')$, $\theta_0 \in V'$; moreover the operator $\beta : V \to V'$ is compact.

Abstract results on the existence of solutions and their uniqueness have been proved there. As particular cases the authors of [D10] indicated:

- formulations with specified energy:

$$\beta(\theta) = \begin{cases} \{0\}, & \text{if } E \triangleq \int_\Omega |\theta|^2\,dx < 1 \\ \{\lambda\theta \mid 0 \le \lambda \le 1, & \text{if } E = 1 \\ \{\theta\}, & \text{if } E > 1; \end{cases}$$

- formulations with specified flux:

$$A(\theta) = \vec{b}\cdot\vec{\nabla}\theta + \partial\psi(\theta) \quad \text{where} \quad \partial\psi(\theta) \text{ is the subgradient of } \theta.$$

Concentrated capacity. Problems with a free boundary moving within a concentrated capacity were considered in [B21,F10,R7-9]. One-dimensional heat transfer equations take there the form

$$\theta_{xx} - b(\theta)\theta_x - \theta_t = f(\theta) - a(\theta)\pi^{-1/2}\int_0^t \theta^{**}(x,\tau;s)(t-\tau)^{-1/2}\,d\tau$$

in $Q\backslash S$, where

$$\theta^{**}(x,t;s) = \begin{cases} \theta_t^{*}(x,t;s), & x\neq s(t) \\ 0, & x=s(t) \end{cases} \qquad \theta^{*}(x,t;s) = \begin{cases} \theta_1(x,t), & x \le s(t) \\ \theta_2(x,t), & x > s(t). \end{cases}$$

Problems of such type arise in particular in mathematical modelling of various processes related to the exploitation of oil layers [R7-8].

In [F10,R9] local in time existence of classical solutions was shown and questions concerning their uniqueness were discussed. [B20] offers finite-difference schemes for solving this class of problems.

Time periodicity. There are some technological processes whose mathematical models are characterized by periodicity of all time-dependent parameters (coefficients, source terms and boundary data).

Such problems were analyzed in [B18] in the case of one space dimension and results on the existence and uniqueness of weak solutions of the enthalpy type were established.

Space-periodicity. Multidimensional problems with a free boundary were studied in [B15-16,D2-3,R1] for ε-periodic space structures.

By applying homogenization techniques the existence and asymptotic behaviour of the enthalpy type solutions of the problems as $\varepsilon \downarrow 0$ was discussed in [D2-3,R1].

Applications of those results to electromagnetic composite materials were offered in [B15-16].

Singular perturbations of free boundary problems were also studied by variational techniques in a number of works (see the contribution of Hilhorst, enclosed as Appendix II to this paper, and the paper [N1]).

Hysteresis. Processes of ferro-magnetic phase-changes give rise to Stefan-like problems whose enthalpy formulations (3.7) involve a variety of non-unique nonlinearities of hysteresis type. Such formulations are stated in [V4] where also existence of solutions is established by monotonicity arguments.

3.2. Non-parabolic Stefan-like problems

A number of problems with a free boundary where conditions of the Stefan type are imposed have been stated in other than the parabolic frameworks mentioned above.

Pseudo-parabolic formulations arise in modelling of cooling processes according to the two-temperature heat conduction theory [C15,B2,D10-12,R15]. Such problems differ from the parabolic ones in the form of the governing differential equations, here the following

$$\frac{\partial}{\partial t}(\theta - a\Delta\theta) = \operatorname{div}(k\nabla\theta) \quad \text{in} \quad Q\backslash S. \tag{3.8}$$

In the case of one space dimension the existence of classical solutions to the one-phase Stefan problems for equation (3.8) was shown in [R15]. New results on the existence and uniqueness for multi-dimensional problems were obtained in [D10-12] by analyzing the evolution equation (3.7).

One phase Stefan problems for hyperbolic equation were studied by several authors [B1,D14,M1] who discussed, in the case n=1, basic properties of classical solutions and proposed finite-difference schemes for their determining.

Further generalizations, concerning two-phase analogues of Stefan problems, with the governing equations of the mixed parabolic-hyperbolic type, were considered in [K2-3] with the existence and uniqueness proofs for classical solutions given (n=1).

Stefan problems with convection in the liquid phase were studied in [C8,C10,D10,S1,B9]. The papers [C8,C10] concerned steady state formulations with convection in fluid, governed by either Stokes or Navier-Stokes equations, and included existence results for weak solutions, concluded by a penalty technique combined with compactness arguments. In [B9,S1] the existence of generalized solutions was proved in the evolution case by monotonicity arguments and semi-discretization in t according to Grange-Mignot techniques [G5]. Another variant of the application of the monotonicity techniques was offered by [D10].

Let us note that the problems involving additional convection term in the Stefan conditions were analyzed in [G4,V2,S1].

Coupled systems. In many real processes of phase-change heat transfer is strongly coupled with diffusion. As the most characteristic examples may serve there processes of multi-component diffusion-reaction, frost propagation, as well as alloy solidification and crystal growth.

The corresponding mathematical models take the form of free boundary problems for systems of differential equations (in general of the parabolic or mixed parabolic-elliptic type), involving couplings within coefficients and the conditions imposed at the free boundary [R6-11,W1,M3,A1,A4-5,C18, D10-13](see the related contributions in this volume, too).

The papers [D10,D12-13] contain results on the existence of weak solutions of the coupled problems transferred into variational frames.

4. CHARACTERIZATION OF THE FREE BOUNDARY IN MULTIDIMENSIONAL PROBLEMS

A fundamental information for the study of phase-change processes is usually inherent in determining the evolution of the corresponding free boundaries [R8]. This appears however to be in many cases one of the most difficult questions related to the problems, often lacking any satisfactory answer.

The only situation where the free boundary can be tracked is that of $n=1$ [R6,W1,P9] as well as the case of one-phase problems for $n \geq 1$ [W1,F16], since only there results providing enough global regularity are known.

Before further discussion, let us mention the difficulties associated with giving a satisfactory characterization of the free boundaries in multi-dimensional Stefan-like problems (except of special one-phase formulations). With the only exception for the local in time results on the classical solutions (see Section 3.1.1), the whole study of multi-phase Stefan-like problems for $n > 1$ was carried out within variational weak formulations. The free boundaries were characterized there only implicitly as certain level sets.

Such an approach provides as a rule enough qualitative information neither on the microscopic nor on the macroscopic scale as far as the evolution of the free boundary is concerned, leaving as open such fundamental questions as those concerning its structure, more precisely, its Lebesgue measure, topological properties (closedness, single-connectivity), regularity and stability.

The stability of the free boundaries was discussed in many respects, including both physical considerations [O1,R8,S9] and the mathematical ones (see Section 3.1). For one-phase problems there are two alternative results on continuity of the mappings:

- from boundary Dirichlet data into the characteristic function of the contact set $S \triangleq \{(x,t) \in Q \mid \theta(x,t) = 0\}$, from $H^{3/2,3/4}(\Sigma)$ equipped with the weak topology into $L^2(Q)$ with the strong topology [M16,S1].
- from Dirichlet data and distributed source term into the contact set, from $W^{2-1/p,1-1/2p}(\Sigma) \times L^p(Q)$ $(p > \sup((n+2)/2,2)$ into the space of compacts over Q equipped with the Hausdorff topology for the metric induced by L^2-norm [P7,S1,S6].

In [D7] questions concerning the stability with respect to perturbations of coefficients were analyzed.

There are also certain stability results for general two-phase problems [N4,P5], unfortunately none of them gives any information on the pointwise behaviour of the free boundary. The same remark concerns the results for one-phase problems, too.

In a number of recent publications the derivation of the conditions characterizing free boundaries of phase-transitions has been re-examined on a physical level [A4-5,F11,M15,R10-11,S7,T9,W4](see also [A1,S9]) with the aim of approaching closer to the reality of the processes.

In particular such factors as supercooling, changes of density, surface tension and its local curvature have been taken into account. These considerations lead to new, higher order (usually second) formulations of the conditions specifying evolution of the free boundary, providing good results of numerical simulations [S7](see also the paper of Fix in this volume).

5. NUMERICAL TECHNIQUES

There are many papers and reports devoted to numerical methods of solving the Stefan problems and their manifold derivatives. We would like to refer here only to a number of recent surveys on this subject and mention a few recent advances.

Comprehensive review of numerical methods for Stefan-like problems has been offered in [R6,O2,W1,M3,A1,F19-21,M12,W3,H2] providing large bibliography. The reported numerical techniques concerned both classical and variational statements of the problems.

As far as the recent advances in the field are concerned, we confine ourselves here with giving a few selected references.

Front tracking techniques are developed in several directions, including isotherm migration methods based on interchange of independent and dependent variables (see the Crank's paper in [A1]), application of the method of lines [A1,W1,J3](see also the paper of Meyer in this volume), construction of efficient schemes based on finite element method for multi-phase problems with possibly degenerating phases [B11].

Fixed domain methods are applied both for enthalpy [S1,Z1] and freezing index variational statements of the problems [E1-2,I1,K4]. Another variant of fixed domain techniques, based on the use of alternating phase truncation algorithms is developed in [B8,R5] with a rate of convergence evaluated.

We would like to mention separately the possibility of the use of efficient techniques relying upon solving certain optimization problems (of the optimum design type) instead of the original ones. In this respect we refer to [B2-4] and to the paper of Hoffmann and Niezgódka in this volume.

6. A FEW OPEN PROBLEMS

There are still many open areas of the research on Stefan-like problems. We would like to indicate here some important directions for that research, by no means pretending to any completeness.

One can considerably extend the below list by referring to other papers within this volume as well as to the surveys enclosed in [M3,O2,W1,F19,P9, H2].

(i) A mathematical study of the multidimensional problems involving first order conditions at the free boundary more general than the standard Stefan conditions.

(ii) A mathematical study of the problems involving higher order free boundary conditions, including re-examination of the physical considerations leading to particular formulations.

(iii) A mathematical study of the Stefan-like problems for coupled systems of partial differential equations.

(iv) A development of the study on problems with discontinuous and/or non-unique nonlinearities like hysteresis (especially a discussion of the concepts of the uniqueness of solutions).

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APPENDIX

Variational analysis of a class of perturbed free boundary problems by O. Diekmann and D. Hilhorst.

We study the limiting behaviour as $\varepsilon \downarrow 0$ of the solution u_ε of the problem

$$\begin{cases} -\Delta u + h(\frac{u}{\varepsilon}) = f \quad \text{in } \Omega \\ \int_\Omega h(\frac{u}{\varepsilon}) = C \quad h(-\infty) < C/|\Omega| < h(+\infty) \\ u\big|_{\partial\Omega} = \text{constant (unknown)} \end{cases}$$

Here the function h is continuous, strictly increasing and such that $D(h) = \mathbb{R}$ and $h(0) = 0$ and f is given in $H^{-1}(\Omega)$. This problem occurs in the physics of ionized gases in the case that $h(x)=e^x-1$.

By means of a variational method we prove existence and uniqueness of the solution u_ε of this problem in the direct sum of $H_0^1(\Omega)$ and the constant functions on Ω. We show that, as $\varepsilon \downarrow 0$, u_ε converges in that space to a limit u_0 which is characterised as the unique solution of a minimization problem. It turns out that u_0 depends only on $h(\pm\infty)$, f and C.

If $f \in L^\infty(\Omega)$, we prove that u_ε and u_0 belong to $W^{2,p}(\Omega)$ for all $p \geq 1$ and that u_ε converges weakly to u_0 in $W^{2,p}_{loc}(\Omega)$ [1]. Then either one has convergence in $W^{2,p}(\Omega)$ itself or a boundary layer develops as $\varepsilon \downarrow 0$.

Related work has been done by Brauner and Nicolaenko [2]: they approximate free boundary problems by nonlinear boundary value problems where ε occurs in the argument of an homographic function (see also [3]). Also related is the work of Frank and Van Groesen [4].

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A BOSSAVIT

Stefan models for eddy currents in steel

We present here a variety of Stefan-like models encountered in studies on induction heating. Let us first recall the physical background.

There are electric and thermic phenomena. The former are described by four vector fields: b and h ("induction" and "magnetic field" respectively), j and e ("current density" and "electric field"), subject to the following relations

$$\mathrm{curl}\ h = j \qquad \text{(Ampère's "theorem"; implies div } j = 0) \qquad (1)$$

$$\partial b/\partial j + \mathrm{curl}\ e = 0 \qquad \text{(Faraday's law)} \qquad (2)$$

$$j = \sigma e \qquad \text{(Ohm's law)} \qquad (3)$$

where $\sigma \geq 0$ is the conductivity. The set $\Omega = \{x \in R^3 | \sigma > 0\}$ is known as "the conductors" or "the circuit". Remark that j.n (normal flux) must be zero on $\partial\Omega$ and that b is divergence-free.

Fields b and h are linked by

$$b(x, t) = B(x, h(x, t)) \qquad (4)$$

where $B(x,.)$ is some cyclically monotone operator. We shall deal with a restricted family of such relations. Let $\beta : R \to R$ be a function of the family displayed on Figure 1, parameterized by the temperature $\theta(x)$ at point x. Then

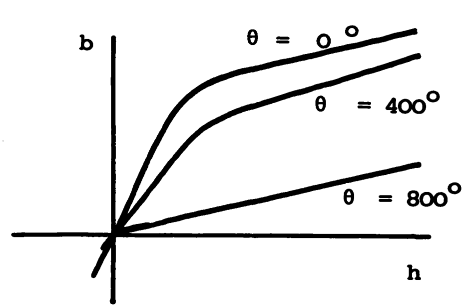

Figure 1.

$$B(x, h) = \beta_{\theta(x)}(|h|)\ h/|h|. \tag{5}$$

So b and h are colinear vectors and the ratio $\mu = |b|/|h|$ depends on the magnitude of h *and* of the temperature at the point considered. This last dependence is only important for steel and other so-called "ferro-magnetic materials". For other metals and insulators, we have the simpler linear relationship

$$b = \mu h \tag{6}$$

where μ is generally the MKSA constant $\mu_0 = 4\pi\ 10^{-7}$. Even for steel,(6) is valid when θ rises above the Curie point (around 760°). (The transition from non-linear to linear behaviour when the temperature reaches this point is quite rapid; this is a source of numerical difficulties.)

It is by no means clear that (1)(2)(3)(4) form a well-posed problem, if only because the source of the field is not apparent. This difficulty will be avoided in the present paper, suitable boundary conditions being derived case by case. For a more thorough treatment, see [7].

As for thermic phenomena, they are described by the classical heat equation with the Joule losses $\rho|j|^2$ in the right-hand side ($\rho = \sigma^{-1}$ is the resistivity).

So the eddy-currents, created in the work-piece by some induction coil fed by the AC mains, heat the metal up, which changes its magnetic characteristics, and this in turn affects the distribution of eddy-currents. The problem is a couple one.

Similar problems are encountered in other parts of electrical engineering, where eddy-currents heating is more often a matter of concern than something sought for. Caution is generally taken to avoid magnetic fields of great magnitude and large temperature ranges, so one works in a limited part of the characteristics domain, near the origin, where a linear model like (6) is right (if one is willing to neglect hysteresis). In induction heating, on the opposite, strong currents are welcome. This means high fields, so that b-h lies most of the time in the right part of the diagram of Figure 1. The following simplification of (5) is thus permitted

$$\beta(u) = b_0 \mathrm{sgn}(u) + \mu_0 u \tag{7}$$

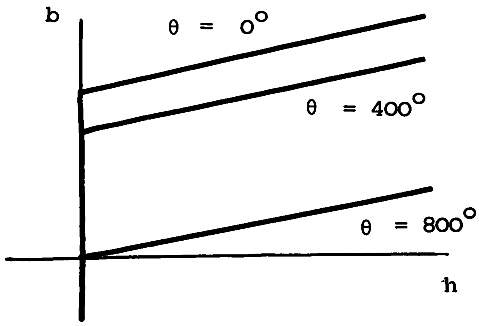

Figure 2. Idealized characteristics (valid for great h)

where only b_0 depends on θ. Such characteristics are displayed on Figure 2.

As will be seen below, the discontinuity at 0 in (7) is responsible for the appearance of free boundaries.

Another point worth mentioning is the generally great magnitude, at low temperatures, of the "Stefan's number"

$$\varepsilon^{-1} = b_0/\mu_0 H \tag{8}$$

where H is a reference for h, e.g. its maximum value on the boundary. It is tempting to dismiss ε completely, i.e. to adopt the characteristics of Figure 3. Models based on this are unphysical, but can be mathematically interesting. Sometimes they can be solved by analytical methods, so giving insight as to the behaviour of the actual solutions.

We shall now examine a few specific models, beginning with the simplest situation (dimension 1, first part), then proceeding to the three possible "scalar" formulations in two dimensions (part 2) and to the full three-

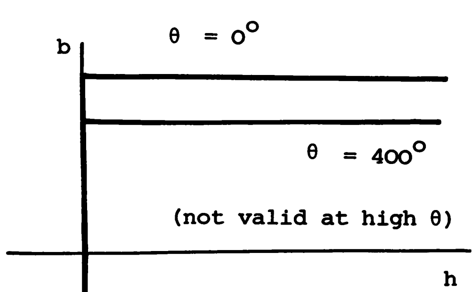

Figure 3. The limit case $\varepsilon = 0$

dimensional case (part 3), where the unknown is a vector field. A new kind of Stefan problem (a "vector" one) is there introduced.

1. DIMENSION 1

1.1. Induction heating of a cylindrical billet

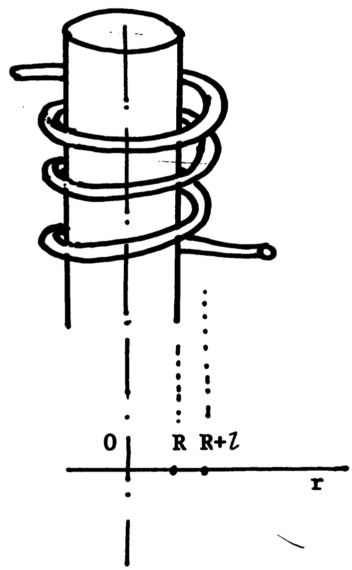

Figure 4.

The situation is as on Figure 4. The billet is surrounded by a coaxial solenoid plugged to the mains which generates a field of about 10^5 A/m. Replacing the coil by a thin sheet of horizontal currents, and assuming invariance in θ and z, we may adopt the following model, where h(r,t) denotes the intensity of the magnetic field (which is vertical)

$$\frac{\partial}{\partial t} q(\theta(r,t)) - \frac{1}{r}\frac{\partial}{\partial r}\left(\kappa(\theta) r\frac{\partial\theta}{\partial r}\right) = \rho\left|\frac{\partial h}{\partial r}\right|^2 \tag{9}$$

$$\kappa r\frac{\partial\theta}{\partial r} + \phi(\theta) = 0 \quad \text{at} \quad r = R \text{ (radiation)} \tag{10}$$

$$\frac{\partial}{\partial t} \beta(\theta(r,t),h) - \frac{1}{r}\frac{\partial}{\partial r}\left(\rho(\theta(r,t)) r\frac{\partial h}{\partial r}\right) = 0 \tag{11}$$

and

$$\rho \frac{\partial h}{\partial r}\bigg|_{r=R} + \underbrace{\mu_0 \ell (1 + \frac{\ell}{2R})}_{\lambda} \frac{d}{dt} h(R, t) = V \sin \omega t \tag{12}$$

No problem here to derive the electric equations: (11) is just the translation of (1)(2)(3) after elimination of e, and (2) is obtained by equating the applied emf V sin ωt (by meter in the z direction) to the derivative of the induction flux and neglecting the resistance and inductance of the coil. Figure 5 shows the variation of the coefficients κ and ρ with the temperature.

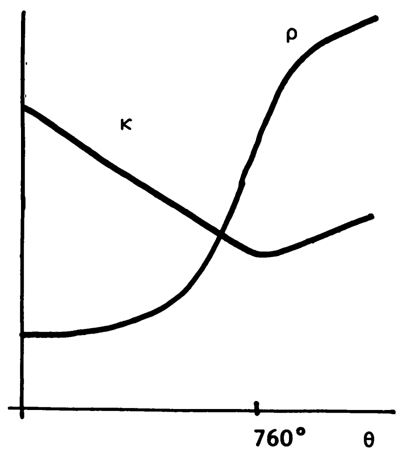

Figure 5. Variations of κ and ρ in θ

Here we have a system of two coupled non-linear parabolic diffusion equations, i.e. two coupled Stefan-like problems ((11)(12) is Stefan *stricto sensu* if one adopts the simplified behaviour law (7)). A first simplification occurs if one notices that the time constant in (9)(10) is far greater than the one in (11)(12): the heating process lasts about ten minutes, to be compared with the mains period (.02 seconds). Consequently, (9)(12) can be replaced by

$$\text{l.h.s. of (9)} = \frac{1}{2\pi} \int_0^{2\pi} \rho \left| \frac{\partial h}{\partial r}(r,s) \right|^2 ds \tag{9'}$$

same as (10) (10')

$$\omega \frac{\partial}{\partial s} \beta(\theta(r,t),h(r,s)) - \frac{1}{r}\frac{\partial}{\partial r} (\rho(\theta(r,t))\, r \frac{\partial h}{\partial r} (r,s)) = 0 \qquad (11')$$

$$\rho \frac{\partial h}{\partial r}(R,s) + \lambda \frac{d}{ds} h(R,s) = V \sin s \qquad (12')$$

In (11')(12'), θ is frozen at its value at time t (the t of (9')(10')) and one looks for the *periodic* solution, the mean value of which is fed back into (9'). Problem (9')...(12') is a *decoupled* approximation of (9)(10). Physically, this is quite justified, but detailed studies of the convergence could be interesting. (For more details, see [4] and [5].)

1.2. The electric equation: a periodic Stefan problem

Let us now concentrate on (11')(12'). One can eliminate non-essential features by ignoring curvature, replacing the boundary condition by a Dirichlet one and using (7) as magnetization characteristics. After adimensionalization, one has:

$$\frac{\partial}{\partial t}(\varepsilon h + \mathrm{sgn}(h)) - \frac{\partial^2 h}{\partial x^2} = 0 \text{ for } x \in [0,\infty) \qquad (13)$$

$$h(0, t) = \sin \omega t \qquad (14)$$

What can be said about *periodic* solutions of (13)(14)?

The coefficient ε is the one which appeared in (8). If $\varepsilon = 0$ (case of Figure 3), (13)(14) can be solved analytically [1]. The shape of the solution is shown in Figure 6. Notice the existence of a "penetration depth". There are three different phases, where h = 0, -1 and 1 respectively.

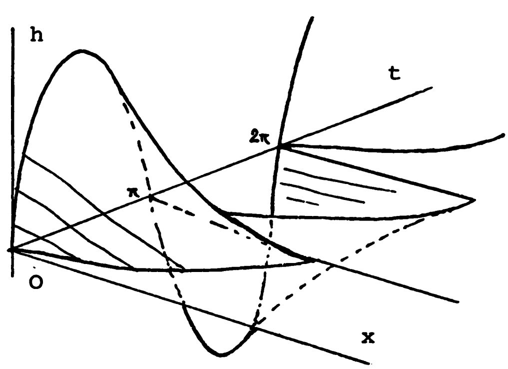

Figure 6.

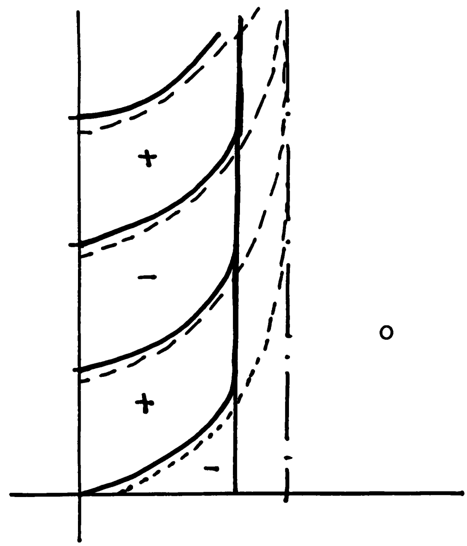

Figure 7.

Figure 7 shows the phase boundaries. When ε is positive, though small enough, one can expect a slight deformation of the phase boundaries, something like the dotted lines on Figure 7. But is there really an accumulation line, as assumed here? This conjecture is supported by numerical simulations (but such evidence is not very conclusive) and by asymptotic estimates due to A. Crowley [12].

1.3. Numerical approach

Solving (9)...(12) brings up interesting information about the induction heating process. Three successive phases can be distinguished. In phase I (cold steel), non-linear, there is a definite Stefan-like behaviour. In phase III, $b = \mu_0 h$ (beware that the non-dimensionalization which led to (13)(14) is not valid there), and one can observe exp-sine like curves, typical of the periodic solutions of the linear diffusion equation. Phase II is more complex: near the edge, the steel is hot and behaves linearly, but a non-linear region subsists below a certain depth. This results in the humped power profile one can see on Figure 8, unpredictable before the numerical computation of the *full* coupled system (9)...(12) was achieved.

As for methods, I relied on linear finite elements and the classical Crank-Nicolson scheme. Equations (9')(10') are solved by Crank-Nicolson, using a Newton procedure to solve the system of non-linear equations at each

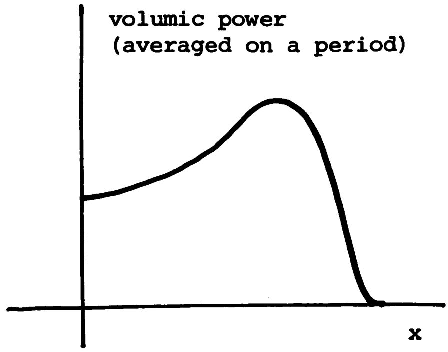

Figure 8. Power profile, i.e. $\overline{|\partial h/\partial x|^2}$, in the intermediate phase.

time-step. To obtain the r.h.s. of (9'), one has to solve (11')(12'), again by Crank-Nicolson, running on a few periods until the periodic solution is established. There, each time-step (of the "small" time-scale, s) requires accurate solution of a strongly non-linear system of algebraic equations. This is done using a method inspired by a suggestion of C. Elliott (see [13]). (The idea is: use SSOR, but reduce the overrelaxation parameter ω *locally*, at points where using its standard value would result in an increase of the convex functional.)

This is not fully satisfactory, for the free boundaries are not well tracked (they jump from point to point in the grid). Better procedures do exist: adaptive grids [20], time-space elements [2], front tracking [15]. All were first rejected on the grounds that there exist (theoretically) an infinity of free boundaries. It is now clear that no more than two or three free boundaries need to be considered simultaneously, so the initial choice of method was rather unwise. (But it is too late: the code is now too big and too old to stand drastic surgery.)

Another decision, the choice of low-degree elements, was perhaps wrong (the rationale was that approximation results depend on global regularity, which is lost in non-linear problems). See [22] for the other side of the argument.

2. DIMENSION 2

2.1. Models using h

If the above cylinder is replaced by a prism of rectangular section, the problem becomes bi-dimensional, but nothing really new emerges. Equations (11)(12) should be replaced by

$$\frac{\partial b}{\partial t} - \mathrm{div}(\rho \ \mathrm{grad}\ h) = 0 \quad \text{in the section } \Omega \tag{16}$$

$$\int_\Gamma \rho \frac{\partial h}{\partial n} + \frac{d}{dt} J(t) = V \sin \omega t \tag{17}$$

$$h_{|\Gamma} = J(t) \tag{18}$$

Notice the integral boundary condition (h is constant on the boundary, for curl h = 0 out of the conductors means here h = Cte); J(t) is the intensity of the current sheet.

In the case of the discontinuous b-h curve (7), it is again likely that free boundaries cluster near an inner fixed boundary. Below this, the field stays 0. Figure 9 shows a numerical simulation.

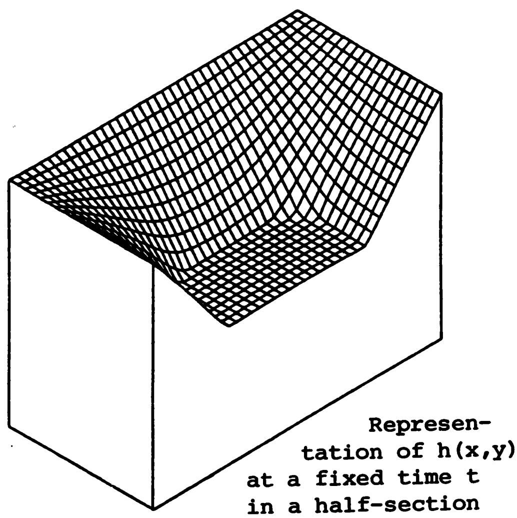

Figure 9.

2.2. Models using a

Figure 10 refers to a situation where another choice of unknown is more appropriate. All currents are parallel to the z direction, so b and h are parallel to the x-y plane. As div b = 0, it is convenient to introduce a, scalar, such that $b = \{\partial a/\partial y, -\partial a/\partial x\}$. Now (1)(2)(3) yield

$$\sigma\frac{\partial a}{\partial t} - \mathrm{div}(\nu(|\mathrm{grad}\ a|)\mathrm{grad}\ a) = \sigma v(t) \tag{19}$$

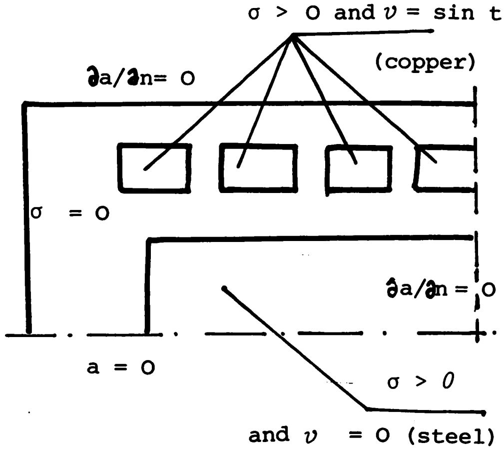

Figure 10. Typical a problem (induction furnace)

where $\nu(s)$ stands for $s/\beta(s)$. Here v is constant in x and y on connected parts of the conductor domain. This constant is interpreted as an applied tension. For justifications, see [6].

Equation (19) holds in the *whole* plane, not only in Ω. It degenerates into an elliptic problem outside of $\Omega(\sigma = 0)$. There, $\nu = 1/\mu_0$. The right functional space for a is $BL(R^3)$. The difficulty can be turned around if the device under study is enclosed within highly permeable metallic walls ($\mu >> \mu_0$). The corresponding boundary condition is $\partial a/\partial n = 0$ and the right space is $H^1(O)/R$ (O denotes the domain within the walls).

Computations in situations similar to Figure 10 are common (see [14, 18, 19]). When the physical setting does not allow consideration of a bounded domain like here, one can avoid discretization of too large regions by a "boundary operator" approach. See [16]. Figure (11) shows results from [16].

As for free boundaries, let us again adopt (7) as b-h relation. Now, using (8),

$$\nu(s) = (s/\varepsilon(s-1))^{+}. \tag{20}$$

The author is not aware of work on periodic solutions of (19)(20) (but see [17]). Interesting also would be the limit case $\varepsilon = 0$. For instance, what happens in the case of Figure 12 (AC current parallel to a conducting

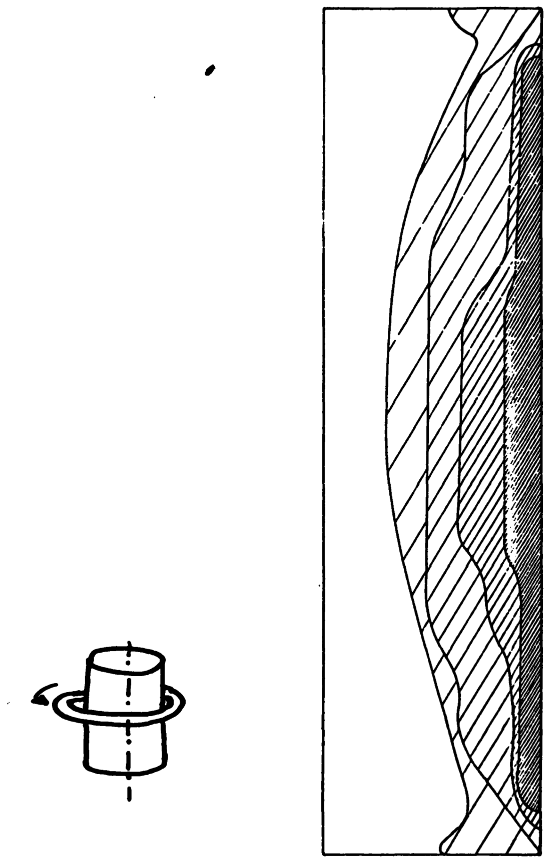

Figure 11. A case of infinite domain. Power density.

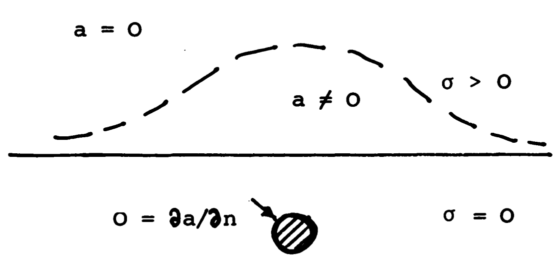

Figure 12. Problem (21).

half-space)? It seems that a free boundary would move periodically up and down below the surface. A model is

$$\left|\begin{array}{l} \sigma\dfrac{\partial a}{\partial t} - \text{div } h = 0 \quad \text{in } \Omega \\ \\ \text{grad } a = h/|h| \\ \\ |\text{grad } a| \le 1 \quad \text{if } h = 0 \\ \\ \partial a/\partial n = \sin t \quad \text{on } \Gamma \end{array}\right. \tag{21}$$

2.3. Models using j

In (19), the current density is

$$j = -\,\text{div}(\nu \text{ grad } a) \tag{22}$$

Using the non-linear Green operator $a = G(j)$ associated with (22), one obtains an alternative formulation:

$$\frac{d}{dt}\,G(j) + \rho j = v(t) \tag{23}$$

This has the advantage of being posed in a bounded domain (Ω, where $\sigma > 0$), and the drawback of being "non-local". See [6] and [7]. This model is equivalent to (19) and does not bring anything new as far as free boundaries are concerned.

2.4. Numerical issues

What was said above for the 1-D case remains valid. Methods are the same, but the computing costs are rather deterring if one wants to treat the full coupled problem.

3. DIMENSION 3

The choice of j, h, or a in two dimensions was dictated by the opportunity of using a scalar unknown rather than a vector one when possible. No such simplifications happen in 3-D situations, where the h-formulation is generally preferable.

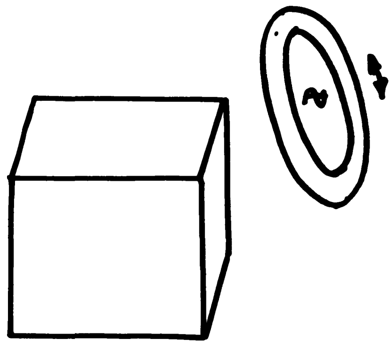

Figure 13.

3.1. A model problem

An induction loop generates eddy-currents in a massive steel workpiece (Figure 13). What happens? This depends on the shape of the magnetization characteristics. In the case of relation (7)(Figure 2), one can conjecture that an infinity of free boundaries (surfaces where h vanishes) exist and cluster around an "inner core", never disturbed by the field.

The mathematical model is as follows. Let Ω be a bounded simply connected open set of R^3, Ω its boundary, $\omega = R^3 - \bar{\Omega}$. Define

$$H = \{h \text{ and curl } h \in [L^2(R^3)]^3 \,|\, \text{curl } h = 0 \text{ on } \omega,\ \text{div } h = 0 \text{ on } \omega\} \tag{24}$$

a Hilbert space for the scalar product

$$((h,\ k)) = \int_{R^3} h.k + \int_\Omega \text{curl } h.\ \text{curl } k \tag{25}$$

and let $h_O(t)$ be a given divergence-free field of L^2-curl, with curl $h_O = 0$ on Ω. (Here Ω is the work-piece and h_O represents the applied field due to the coil.) One can prove [11] that existence and uniqueness hold for the following variational problem

$$\frac{d}{dt}\int_{R^3} B(h+h_O).k + \int_\Omega \rho \text{ curl } h.\text{curl } k = 0 \quad \forall k \in H \tag{26}$$

(where B is as in (4)(5)), with Cauchy or periodicity conditions.

Equation (23) is quite similar to the enthalpy formulation of the nonlinear heat equation. The new feature is that h has three components, so this is a "vector Stefan problem".

The integral on R^3 can be rewritten as

$$\int_{R^3} = \int_{\Omega} B(h+h_o).k + \mu_o \int_{\omega} (h+h_o).k = \ldots + \int_{\Gamma} (R\phi, \psi)$$

where ϕ and ψ are suitably defined potentials and R a stiffness boundary operator. See [9] for details. This is important from the numerical point of view, for only a discretization of the metallic parts is so required. The numerical procedure, which makes use of tetrahedral elements where the degrees of freedom are the circulations of h along the edges, is described in [8]. The computing costs are really huge, even for linear behaviour laws, and no non-linear simulations have been made up to now.

3.2. Homogenization

In application to transformers, alternators, one encounters composite media such as stacks of steel sheets separated by insulating layers. Homogenization [3] is a technique to derive behaviour laws of "equivalent" homogeneous materials. For instance, let us consider (16) again

$$\frac{\partial}{\partial t}[b_o \operatorname{sgn}(h) + \mu_o h] - \operatorname{div}(\rho \operatorname{grad} h) = 0 \tag{27}$$

In a composite medium, b_o and ρ will vary very rapidly in space. Periodicity in two directions may be assumed. When the size of the periodicity cell tends to zero, the solution of (27) tends to the solution of a similar problem, but with constant b_o and ρ (ρ becomes a tensor, so isotropy is lost). Homogenization is the technique by which these limit coefficients are obtained. For its application to (27), see [10].

The extension of this approach to (26) has been done recently by Damlamian. One found that starting from characteristics like (5) with b and h colinear, and isotropy, the "homogenized" operator B is cyclically monotone, but B(h) is no more parallel to h. Therefore, work on the Stefan problem associated with (26) and incorporating such general b-h laws could be of real interest and find applications in electrical engineering.

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A new homographic approximation to multiphase Stefan problems

INTRODUCTION

In a recent paper ([6]), two of the authors have presented a new approximation, both of theoretical and numerical interest, to free boundary problems characterized by variational inequalities. This approximation, based on the properties of the "homographic" mapping

$$\phi(t) = \frac{t}{1 + |t|} \tag{0.1}$$

is directly inspired from the singular limits of reaction diffusion equations modelling heterogeneous chemical catalyst and enzyme kinetics with absorption ([4],[5],[6]). Such singular limits are characterized by free boundaries enclosing zones of frozen reactions. In biochemistry, the homographic function is referred to as the Michaelis-Menten reaction law [3].

In this paper, we extend the method to the parabolic variational inequality formulation of the multiphase Stefan problem ([1][10][11], [9]). It yields a specially robust scheme for multidimensional Stefan problems with evolving clouds. A cloud, or mushy region, is a subdomain where the two phases coexist at the phase change temperature. They do typically occur when an internal source of heat is present or when time-dependent non homogeneous boundary conditions are involved.

We first present the spirit of the method on a model Elliptic Variational Inequality (EVI) : Let Ω be a bounded regular domain in $\mathbb{R}^n$, $n \geq 1$. Let $j(v) = 2\int_\Omega v^- \, dx$ (or $\tilde{j}(v) = 2\int_\Omega v^+ \, dx$), $v^+ = \sup(v, 0)$, $v^- = \sup(-v, 0)$, $v = v^+ - v^-$; j is a convex, continuous, nondifferentiable functional. Let $f \in L^2(\Omega)$. An EVI of the 2^d kind is a problem of the form

$$\int_\Omega \nabla u . \nabla (v-u) \, dx + 2\int_\Omega v^- dx - 2\int_\Omega u^- \, dx \geq \int_\Omega f(v-u) dx \quad \forall v \in H_0^1(\Omega), \tag{0.2}$$

and we look for $u \in H_0^1(\Omega)$.

It is well known that (0.2) is equivalent to the following multiphase free boundary problem (as $u \in W^{2,p}(\Omega)$ if $f \in L^p(\Omega)$, $p \geq 2$):

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega_+ = \{x \in \Omega,\ u(x) > 0\} \\ -\Delta u &= f + 2 \quad \text{in} \quad \Omega_- = \{x \in \Omega,\ u(x) < 0\} \\ f \le -\Delta u &\le f + 2 \quad \text{in} \quad \Omega_o = \{x \in \Omega,\ u(x) = 0\} \end{aligned}\right\} \tag{0.3}$$

It is convenient to rewrite (0.2) with the help of the subdifferential ∂j of $j(v)$. Recall that ∂j is the multivalued operator defined by $\partial j(u) = \{g \in L^2(\Omega),\ j(v) - j(u) \ge \int_\Omega g(v-u)\,dx,\ \forall v \in L^2(\Omega)\}$. Then (0.2) is equivalent to

$$-\Delta u + \partial j(u) \ni f, \quad u \in H_o^1(\Omega). \tag{0.4}$$

Our approximation corresponds to replacing (0.4) by

$$\left.\begin{aligned} &-\Delta u + \phi\left(\frac{u_\varepsilon}{\varepsilon}\right) - 1 = f, \quad u_{\varepsilon/\partial\Omega} = 0, \text{ where} \\ &\phi\left(\frac{t}{\varepsilon}\right) = \frac{t}{\varepsilon + |t|} \end{aligned}\right\} \tag{0.5}$$

This is equivalent to the smoothing of the maximal monotone graph associated to the subdifferential of $2t^-$,

$$2\,\partial t^- = \text{sign } t - 1 \tag{0.6}$$

by $\phi\left(\frac{t}{\varepsilon}\right) - 1 = \dfrac{t}{\varepsilon + |t|} - 1$ (see Figure 1)

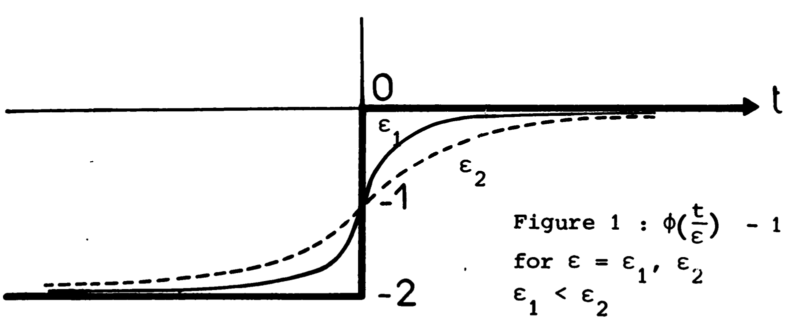

Figure 1. $\phi\left(\frac{t}{\varepsilon}\right) - 1$ for $\varepsilon = \varepsilon_1, \varepsilon_2$ $\varepsilon_1 < \varepsilon_2$

In [7], we have shown that u_ε converges to u in $H^2(\Omega)$ weak, the rate of convergence being

$$||u_\varepsilon - u||_{H_0^1(\Omega)} \leq C \sqrt{\varepsilon}.$$

In a further paper ([8]), we show that we can in fact take very general types of function ϕ, provided they are continuous, monotone with appropriate behaviour as t goes to $\pm\infty$.

1. THE MULTIPHASE STEFAN PROBLEM

We suppose that the temperature $\theta(x, t)$ of a structure Ω varies around a phase change temperature which is renormalized at the origin of the temperature scale; Ω is an open set in $\mathbb{R}^n$. We define as $\mu(x, t)$ the mass proportion of the first phase (unfrozen material for instance); $1 - \mu$ is the mass proportion of the second phase (frozen material). At any time t, Ω is divided into three parts (Figure 2) which we assume to be open:

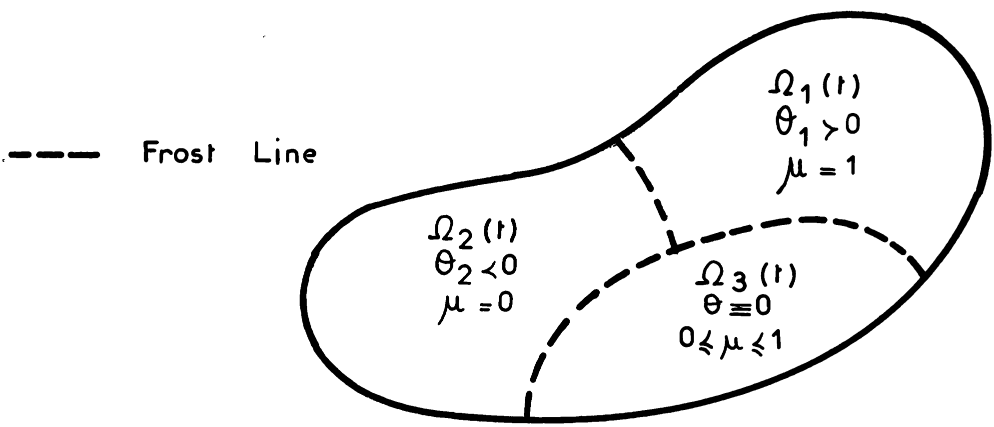

Figure 2. The frost lines depend on the time t in some *a priori* unknown fashion.

- $\Omega_1(t)$ the unfrozen part, where $\theta > 0$ and $\mu = 1$;
- $\Omega_2(t)$ the frozen part, where $\theta < 0$ and $\mu = 0$;
- $\Omega_3(t)$ the intermediate part or the *cloud*, where the two phases coexist at $\theta \equiv 0$, and $0 \leq \mu \leq 1$.

The three parts are separated by the frost lines. In the sequel, the

subscript i refers to the quantities defined in $\Omega_i(t)$, $1 \le i \le 3$.
At $\theta = 0$, the change from one physical state to another requires a quantity of energy equal to the specific phase change latent heat ℓ. The internal energy e has a jump equal to ℓ at $\theta = 0$. Specifically, we define e as

$$e = \theta \, C\,(\theta, x) + \mu \, \ell \tag{1.1}$$

where the specific heat capacities by unit volume are taken piecewise constant

$$C = C_i \text{ in } \Omega_i, \; i = 1,2.$$

Let θ_i be the restriction of θ to Ω_i, $i = 1,2$: The energy conservation law gives then

$$\rho \, C_1 \, \frac{\partial \theta_1}{\partial t} - \lambda_1 \, \Delta\theta_1 = r \quad \text{in} \quad \Omega_1(t) \tag{1.2)a}$$

$$\rho \, C_2 \, \frac{\partial \theta_2}{\partial t} - \lambda_2 \, \Delta\theta_2 = r \quad \text{in} \quad \Omega_2(t), \tag{1.2)b}$$

where λ_i are the thermal conductivities; ρ = mass by unit volume; and $r(x, t)$ is the rate of heat production (e.g. internal electrical heating or chemical reaction in young concrete).

In the cloud, we know that $\theta \equiv 0$, hence $\rho \frac{de}{dt} = r$, and using (1.1)

$$\rho \, \ell \, \frac{\partial \mu}{\partial t} = r \tag{1.3}$$

We give the boundary conditions. We suppose that the boundary Γ of Ω is divided into 5 given parts Γ_i, $0 \le i \le 4$, where

$$\left.\begin{array}{l}
\text{- on } \Gamma_o,\ \lambda(\theta)\ \theta = k_o(x, t)\ \ (\lambda(\theta) = \lambda_i \text{ if } \theta > 0,\ \lambda(\theta) = \lambda_2 \text{ if} \\
\quad \theta < 0) \text{ and } k_o \text{ changes sign;} \\
\text{- on } \Gamma_1,\ \lambda_1\theta_1 = k_1(x, t) > 0\ (\Gamma_1 \text{ is unfrozen})\ ; \\
\text{- on } \Gamma_2,\ \lambda_2\theta_2 = k_2(x, t) < 0\ (\Gamma_2 \text{ is frozen}); \\
\text{- on } \Gamma_3,\ \lambda(\theta)\ \dfrac{\partial \theta}{\partial n} + \ \beta\ [\lambda(\theta)\,\theta - k_3(x, t)] = 0; \\
\text{- on } \Gamma_4,\ \lambda(\theta)\ \dfrac{\partial \theta}{\partial n} = k_4(x, t);
\end{array}\right\} \quad 1.4)$$

where $k_i(x, t)$, $0 \le 1 \le 4$ are taken to be sufficiently regular (see [1][10][11] for more details). The initial conditions are

$$\theta(x, 0) = \theta_0(x), \; \mu(x, 0) = \mu_0(x), \tag{1.5}$$

with appropriate compatibility conditions ([11]).

To solve the multiphase Stefan problem (1.2)...(1.5), a classical formulation ([1][10][11], [9]) uses the freezing index

$$I(x, t) = \int_0^t [\lambda_1 \theta^+(x, s) - \lambda_2 \theta^-(x, s)] \, ds; \tag{1.6}$$

Indeed many engineers who have studied the frost action on civil engineering structures (roads, etc ...) have remarked that many results could be described in terms of the above index. This transformation is analogous to that introduced by C. Baiocchi for the famous free boundary dam problem. It is easy to verify that $I(x, t)$ and grad $I(x, t)$ are continuous across frost lines.

2. A VARIATIONAL FORMULATION FOR THE FREEZING INDEX

Classically (see the above references), the freezing index verifies a Variational Inequality (V.I.). To simplify the notations, we shall suppose, from now on, that all the physical constants, except ℓ, are equal to 1 ($\rho C_i = \lambda_i = 1$). We have then the fundamental relation

$$\frac{\partial I}{\partial t}(x, t) = \theta(x, t). \tag{2.1}$$

We now introduce some mathematical notations. The usual $L^2(\Omega)$ inner product and the associated norm are denoted by

$$(u, v) = \int_\Omega u \, v \, dx, \; |v| = (v, v)^{\frac{1}{2}} \tag{2.2}$$

We introduce the Hilbert space

$$V = \{v \in H^1(\Omega), \; v = 0 \text{ on } \Gamma_i, \; i = 0, 1, 2\}$$

with the usual $H^1(\Omega)$ norm $\|v\|^2 = |u|^2 + |\text{grad}u|^2$. Let a be the following bilinear continuous, coercive form on $V \times V$

$$a(u, v) = \int_\Omega \text{grad}u.\text{grad}v \, dx + \beta \int_{\Gamma_3} uv \, d\Gamma \tag{2.3}$$

(coercive means that $\exists\, \alpha > 0$ such that $a(v, v) \geq \alpha \|v\|^2, \ \forall v \in V$)

We define the continuous linear functional L on V by

$$L(t, v) = \beta \int_{\Gamma_3} K_3(t) \; v \, d\Gamma + \int_{\Gamma_4} K_4(t) \; v \, d\Gamma + \int_\Omega R(t) \; v \, dx \tag{2.4}$$

where $K_j(t) = \int_0^t k_j(s) \, ds$, $0 \leq j \leq 4$, $R(t) = \int_0^t r(s) \, ds$.

Finally, we define the continuous, convex, non differentiable functional

$$\Phi(v) = \ell \int_\Omega \mu_o(x) \; v^- \, dx + \ell \int_\Omega (1 - \mu_o(x)) \; v^+ \, dx \tag{2.5}$$

Following Frémond [1][10][11] (see also Duvaut [9]), the appropriate variational formulation for the multiphase Stefan problem is: find a function I, $\frac{\partial I}{\partial t} = \theta$, such that

$$\frac{\partial I}{\partial t} \in L^\infty(0, T; H^1(\Omega)), \ I \in C^0([0, T]; H^1(\Omega)), \tag{2.6}$$

$$\left.\begin{array}{l}
\text{a.e. } t \in \,]0, T[, \\
\left(\frac{\partial I}{\partial t}, w - \frac{\partial I}{\partial t} \right) + a\left(I, w - \frac{\partial I}{\partial t}\right) + \Phi(w) - \Phi\left(\frac{\partial I}{\partial t}\right) \\
\qquad \geq L\left(t, w - \frac{\partial I}{\partial t}\right) + \int_\Omega \theta_o\left(w - \frac{\partial I}{\partial t}\right) dx \ \ \forall w \in L^\infty(0, T; H^1(\Omega)), \\
\qquad w(x, t) = k_j(t), \ x \in \Gamma_j, \ j = 0, 1, 2. \\
I(x, t) = K_j(t), \ x \in \Gamma_j, \ j = 0, 1, 2; \\
I(x, 0) = 0.
\end{array}\right\} \tag{2.7}$$

For the equivalence of the non-standard V.I. (2.7) with the Stefan problem (1.2)...(1.5), we refer to [1][10][11].

<u>Remark 1</u> $w - \frac{\partial I}{\partial t}$ belongs to $L^\infty(0, T; V)$.

Under sufficient regularity conditions on the data, we have (see [11], Th. 1).

Theorem 1 There exist a unique $I = I(x, t)$ solution of (2.7) such that

$$\left.\begin{aligned} &\frac{d^2 I}{dt^2} \in L^2(0, T; L^2(\Omega)), \ \frac{dI}{dt} \in L^\infty(0, T; H^1(\Omega)), \\ &I \in C^0([0, T]; H^1(\Omega)) \end{aligned}\right\} \tag{2.8}$$

3. HOMOGRAPHIC APPROXIMATION OF THE STEFAN PROBLEM

3.1. Formulation of the approximation

Using the principle described in the introduction we will introduce a convex C^2 regularization of the non-differentiable functional Φ defined in (2.5). Let us rewrite Φ as

$$\begin{aligned} \Phi(v) &= \ell \int_\Omega \mu_o v^- \, dx + \ell \int_\Omega (1 - \mu_o)(v + v^-) \, dx = \\ &= \ell \int_\Omega v^- \, dx + \ell \int_\Omega (1 - \mu_o) \, v \, dx \end{aligned} \tag{3.1}$$

Let us split Φ as

$$\Phi(v) = j(v) + \ell \int_\Omega (1 - \mu_o) \, v \, dx \tag{3.2}$$

where

$$j(v) = \ell \int_\Omega v^- \, dx \tag{3.3}$$

We regularize $j(v)$ by $j_\varepsilon(v)$, $\varepsilon > 0$,

$$j_\varepsilon(v) = j(v) - \frac{\ell \varepsilon}{2} \int_\Omega \text{Log}\,(\varepsilon + |v|) \, dx \tag{3.4}$$

whose differential is

$$j'_\varepsilon(v) = \frac{\ell}{2} \left(\frac{v}{\varepsilon + |v|} - 1 \right). \tag{3.5}$$

Again, this corresponds to smoothing the maximal monotone graph associated to the subdifferential of $2t^-$ (i.e. sign t - 1) by $\left(\frac{t}{\varepsilon + |t|} - 1\right)$, $t \in]-\infty, +\infty[$.

Since $j_\varepsilon(v)$ is differentiable, the approximation of (2.7) is equivalent to the following variational equation for I_ε.

$$\left.\begin{aligned}
&\left(\frac{\partial I_\varepsilon}{\partial t}, v\right) + a(I_\varepsilon, v) + \frac{\ell}{2}\int_\Omega \left[\frac{\frac{\partial I_\varepsilon}{\partial t}}{\varepsilon + \left|\frac{\partial I_\varepsilon}{\partial t}\right|} - 1\right] v\, dx = \\
&= L(t, v) + \int_\Omega \theta_o v\, dx - \ell \int_\Omega (1 - \mu_o)\, v\, dx, \forall\, v \in V \\
&I_\varepsilon(x, t) = K_j(t),\ x \in \Gamma_j,\ j = 0, 1, 2, \\
&I_\varepsilon(x, 0) = 0
\end{aligned}\right\} \quad (3.6)$$

<u>Theorem 2</u> There exist a unique I_ε solution of (3.6), with the same regularity properties as I in (2.8).

The proof is in fact based on the study of the equations verified by I_ε, $\theta_\varepsilon = \frac{\partial I_\varepsilon}{\partial t}$ and $\frac{\partial \theta_\varepsilon}{\partial t}$.

$$\left.\begin{aligned}
&\frac{\partial I_\varepsilon}{\partial t} + \frac{\ell}{2}\left[\frac{\frac{\partial I_\varepsilon}{\partial t}}{\varepsilon + \left|\frac{\partial I_\varepsilon}{\partial t}\right|} - 1\right] - \Delta I_\varepsilon = R(x, t) + \theta_o(x) - \ell(1 - \mu_o(x)) \\
&I_\varepsilon(x, 0) = 0 \\
&\text{B. C. obtained by } I_\varepsilon(x, t) = \int_o^t \theta_\varepsilon(x, s)\, ds,\ \theta_\varepsilon \text{ satisfying the} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{B.C.(1.4).}
\end{aligned}\right\} \quad (3.7)$$

θ_ε formally satisfies the equation

$$\left.\begin{aligned}
&\frac{\partial \theta_\varepsilon}{\partial t} + \frac{\ell}{2}\, \frac{\varepsilon}{(\varepsilon + |\theta_\varepsilon|^2)}\, \frac{\partial \theta_\varepsilon}{\partial t} - \Delta\theta_\varepsilon = r(x, t) \\
&\theta_\varepsilon(x, 0) = \theta_o(x) + \text{B.C.(1.4)}.
\end{aligned}\right\} \quad (3.8)$$

3.2. <u>Error estimate</u>

We will estimate the rate of convergence of I_ε to I and θ_ε to θ.

In the Variational Inequality (2.7), let us choose w as $\frac{\partial I_\varepsilon}{\partial t}$ (again $\theta = \frac{\partial I}{\partial t}$, $\theta_\varepsilon = \frac{\partial I_\varepsilon}{\partial t}$):

$$\left.\begin{aligned}&(\theta, \theta_\varepsilon - \theta) + a(I, \frac{\partial I_\varepsilon}{\partial t} - \frac{\partial I}{\partial t}) + \ell \int_\Omega (\theta_\varepsilon^- - \theta^-)\, dx \geq L(t, \theta_\varepsilon - \theta) + \\ &+ (\theta_o, \theta_\varepsilon - \theta) + \ell(1 - \mu_o, \theta_\varepsilon - \theta).\end{aligned}\right\} \quad (3.9)$$

In the Variational Equality (3.6) verified by I_ε, let us take $v = \theta - \theta_\varepsilon$:

$$\left.\begin{aligned}&(\theta_\varepsilon, \theta - \theta_\varepsilon) + a(I_\varepsilon, \frac{\partial I}{\partial t} - \frac{\partial I_\varepsilon}{\partial t}) + \frac{\ell}{2} \left(\frac{\theta_\varepsilon}{\varepsilon + |\theta_\varepsilon|} - 1, \theta - \theta_\varepsilon\right) = \\ &= L(t, \theta - \theta_\varepsilon) + (\theta_o, \theta - \theta_\varepsilon) + \ell(1 - \mu_o, \theta - \theta_\varepsilon)\end{aligned}\right\} \quad (3.10)$$

Let us add (3.9) and (3.10):

$$(\theta_\varepsilon - \theta, \theta - \theta_\varepsilon) + a\,(I_\varepsilon - I, \frac{\partial I}{\partial t} - \frac{\partial I_\varepsilon}{\partial t}) + \ell \int_\Omega (\theta_\varepsilon^- - \theta^-)\, dx +$$

$$+ \frac{\ell}{2} \left(\frac{\theta_\varepsilon}{\varepsilon + |\theta_\varepsilon|} - 1, \theta - \theta_\varepsilon\right) \geq 0$$

Hence

$$\left.\begin{aligned}&|\theta(t) - \theta_\varepsilon(t)|^2 + \frac{1}{2}\frac{d}{dt} a\,(I - I_\varepsilon, I - I_\varepsilon) \leq \ell \int_\Omega (\theta_\varepsilon^- - \theta^-)\, dx + \\ &+ \frac{\ell}{2} \left(\frac{\theta_\varepsilon}{\varepsilon + |\theta_\varepsilon|} - 1, \theta - \theta_\varepsilon\right)\end{aligned}\right\} \quad (3.11)$$

We now use algebraic lemma:

<u>Lemma 1</u> Let a and b $\in \mathbb{R}$, $\varepsilon > 0$:

$$2(a^- - b^-) + \left(\frac{a}{\varepsilon + |a|} - 1\right)(b - a) \leq 2\,\varepsilon \quad (3.12)$$

<u>Proof</u> $2(a^- - b^-) + \left(\frac{a}{\varepsilon + |a|} - 1\right)(b^+ - a^+ + a^- - b^-) =$

$$= (a^- - b^-)\left(1 + \frac{a}{\varepsilon + |a|}\right) + (a^+ - b^+)\left(1 - \frac{a}{\varepsilon + |a|}\right)$$

$$\leq a^- \left(1 + \frac{a}{\varepsilon + |a|}\right) + a^+ \left(1 - \frac{a}{\varepsilon + |a|}\right)$$

$$\leq a^- \left(1 - \frac{a^-}{\varepsilon + a^-}\right) + a^+ \left(1 - \frac{a^+}{\varepsilon + a^+}\right) \leq \varepsilon \frac{a^-}{\varepsilon + a^-} + \varepsilon \frac{a^+}{\varepsilon + a^+} \leq 2\,\varepsilon$$

Applying the lemma to (3.11)

$$|\theta(t) - \theta_\varepsilon(t)|^2 + \frac{1}{2}\frac{d}{dt}\, a\,(I - I_\varepsilon,\ I - I_\varepsilon) \leq \varepsilon\, \ell \ \text{meas}\ \Omega \tag{3.13}$$

Now integrating (3.13) from O to t, $O < t \leq T$, and using $I(O) = I_\varepsilon(O) = O$:

$$\int_0^t |\theta(s) - \theta_\varepsilon(s)|^2\, ds + \frac{1}{2}\, a(I(t) - I_\varepsilon(t),\ I(t) - I_\varepsilon(t)) \leq \varepsilon\, T\, \ell \ \text{meas}\ \Omega \tag{3.14}$$

Hence using the coercivity of the bilinear form a

<u>Theorem 3</u> $\|I - I_\varepsilon\|_{L^\infty(O,T;V)} \leq \sqrt{\varepsilon}\sqrt{\frac{2}{\alpha}\, T\ell\ \text{meas}\ \Omega}$

$$\|\theta - \theta_\varepsilon\|_{L^2(O,T;L^2(\Omega))} \leq \sqrt{\varepsilon}\ \sqrt{T\ell\ \text{meas}\ \Omega}$$

<u>Remark 2</u> The detailed proof of the existence theorem 2 shows in fact

$$\theta_\varepsilon - \theta \rightharpoonup O \text{ in } L^\infty(O,T;V) \text{ weak*}$$
$$\frac{\partial\theta_\varepsilon}{\partial t} - \frac{\partial\theta}{\partial t} \rightharpoonup O \text{ in } L^2(O,T;L^2(\Omega)) \text{ weak.}$$

But of course there are no estimates for the rate of convergence in these spaces.

4. NUMERICAL APPLICATIONS

There are already several numerical resolution methods for Stefan problems in 2 or 3 dimensions. Some approximations are based on detailed front tracking, and thus are highly specialized ([16][15]); others use a specific heat capacity strongly time-dependent ([13],[14]); still others use the freezing index and proceed with a direct numerical solution of the Variational Inequality (2.7) ([17]); finally, a truncation method can be used [12].

Our method is based on the freezing index and the regularization of the V.I. (2.7). The homographic approximations (3.7) and (3.8) are characterized by a very simple formulation which yields simple and efficient numerical schemes for multiphase Stefan problems in 2 and 3 dimensions, especially where clouds are non trivially present.

The first results which we present here have been obtained with the use of the most straightforward explicit method. Of course, the difficulties generic to the Stefan problem subsist; for instance if a Dirichlet boundary condition is imposed on the surface, the time of transition from temperature $0^\circ + \varepsilon$ and $0^\circ - \varepsilon$ must include several time steps.

There is of course a stability condition coupling ε, Δt and Δx. Beyond the simple example presented below, the numerical program is currently used for the engineering design of cold "gazoducs" (gas pipe-lines) which will be built in permafrost.

In this specific example, we consider $\Omega \subset \mathbb{R}^2$, $0 \le x, y \le 0.2$ m. The physical constants are in the M.K.S.A. units

$$\rho = 10^3 \text{ kg m}^{-3},\ \ell = 3.34\times10^4 \text{ J kg}^{-1},$$

$$\lambda = 2.5 \text{ J s}^{-1} \text{ m}^{-1}\ {}^\circ\text{K}^{-1},\ C = 2.09\times10^3 \text{ J kg}^{-1}\ {}^\circ\text{K}^{-1}.$$

The initial conditions are $\theta_o = 0$, $\mu_o = \frac{1}{2}$, and the boundary conditions are

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } x = 0 \text{ and } x = 0.2$$

$$\theta = -\ (x - 0.1)\ \frac{t}{720} \text{ on } y = 0$$

$$\theta = (x - 0.1)\ \frac{t}{1440} \text{ on } y = 0.2$$

Finally, ε is taken to be 10^{-2}, and $\Delta x = \Delta y = 10^{-2}$, $t = 1$ hour.

The follo

The following figures represent the evolution of the cloud at t = 10, 20, 30 and 40 hours.

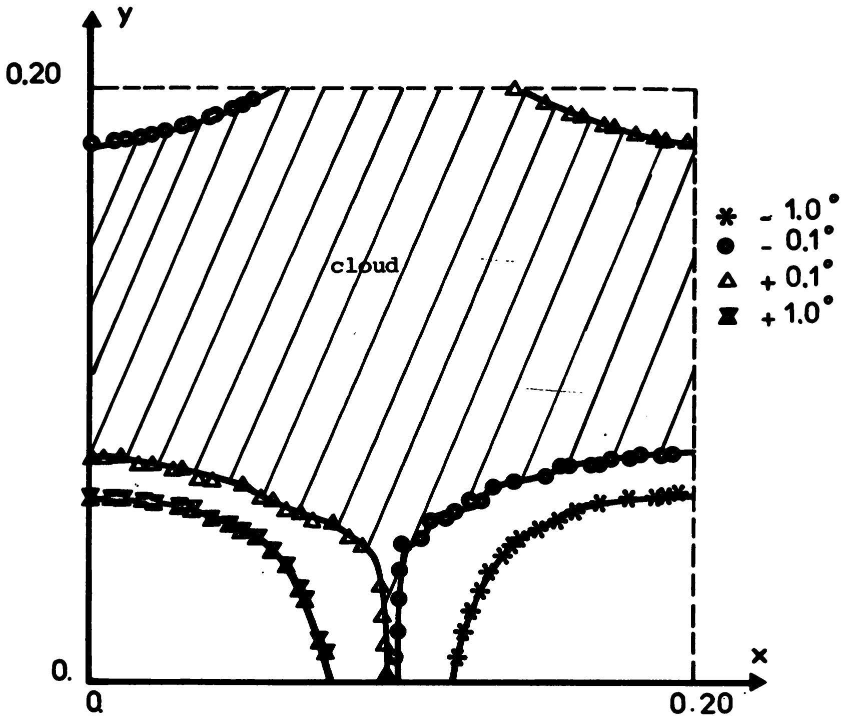

Figure 3

10 heures

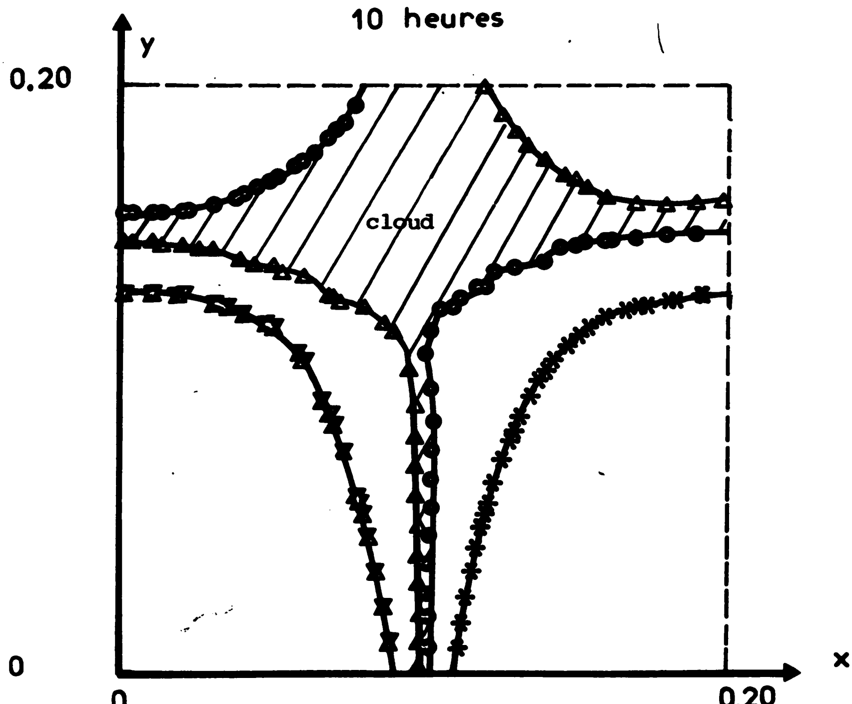

Figure 4

20 heures

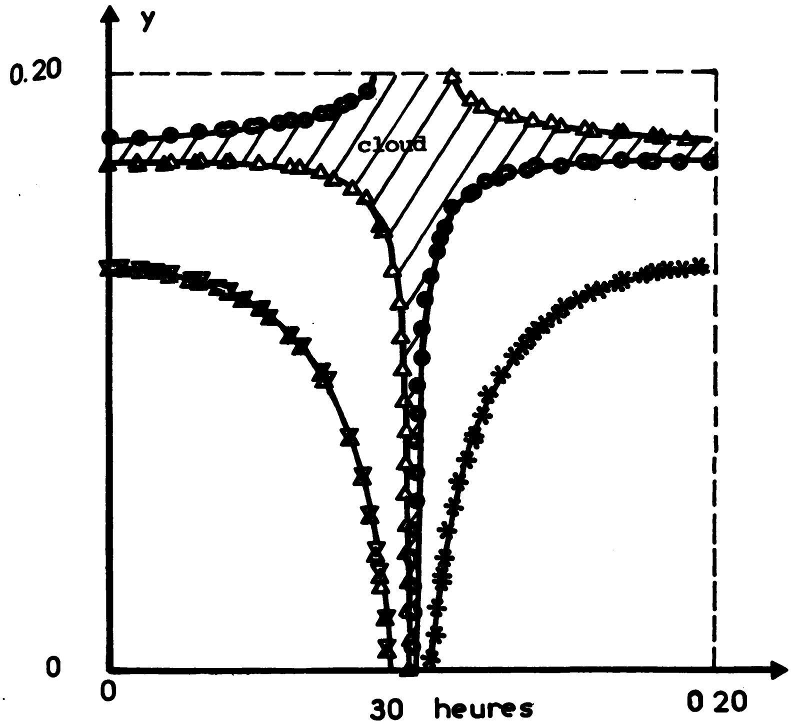

Figure 5

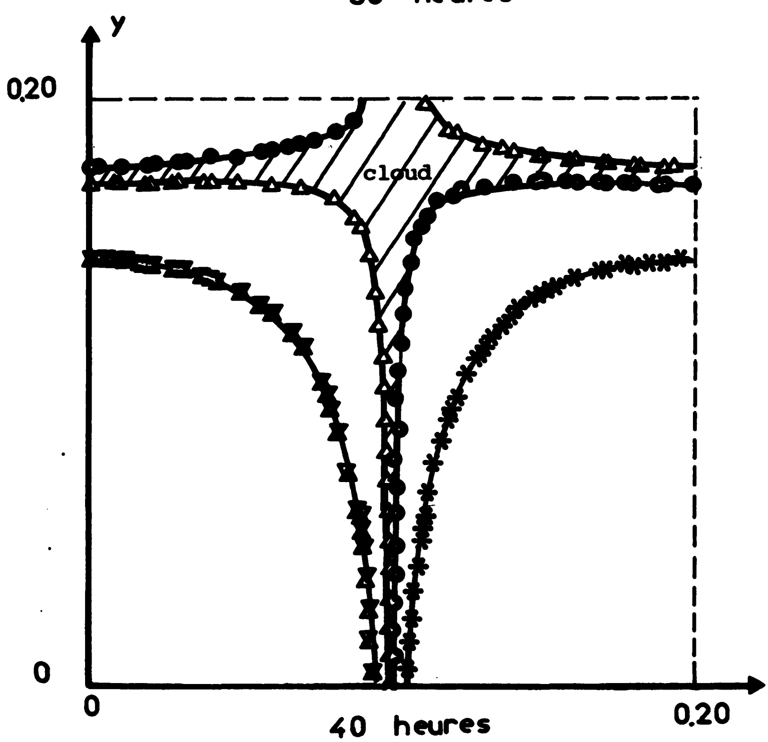

Figure 6

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L A CAFFARELLI & L C EVANS

Continuity of the temperature in two-phase Stefan problems

The classical two-phase Stefan problem is an idealized mathematical model describing the flow of heat within a substance (say water) which changes phase (melts or freezes) at a prescribed temperature. After appropriate transformations and normalizations, the problem becomes the singular parabolic p.d.e.

$$\beta(u(x,t))_t = \Delta u(x,t) \qquad (x,t) \in Q \subset \mathbb{R}^{n+1}, \qquad (*)$$

plus appropriate initial and boundary conditions. Here $\beta(\cdot)$ is multi-valued mapping

$$\beta(x) \equiv \begin{cases} ax - 1 & x < 0 \\ [-1,1] & x = 0 \\ bx + 1 & x > 0, \end{cases}$$

where a and b denote the thermal conductivities in the ice and in the water regions, and the jump in $\beta(\cdot)$ at zero corresponds to the latent heat of fusion. The temperature in the original physical problem is

$$\theta \equiv \begin{cases} \frac{u}{a} & u < 0 \\ 0 & u = 0 \\ \frac{u}{b} & u > 0. \end{cases}$$

It is not particularly difficult to prove that (*) (plus side conditions) has a unique bounded solution u solving the problem in an appropriate weak sense: see, for example, Ladyženskaja et al, Linear and Quasilinear Equations of Parabolic Type, 496-503. In addition, $u_t, Du \in L^2$.

The principal assertion of our work is this:

<u>Theorem</u> The weak solution u of (*) is continuous. In particular, the temperature in the n-dimensional, two-phase Stefan problem is continuous.

Outline of Proof

For purposes of explanation, let us suppose u is in fact a continuous solution of (*) and estimate *a priori* its modulus of continuity. Clearly the major difficulties will occur at points where u vanishes and therefore $\beta(u)$ has a jump; and so we turn our attention to this situation. Let us therefore assume

$$u(0,0) = 0. \tag{1}$$

Consider the cylinders

$$C(R) \equiv \{(x,t) \mid |x| \leq R, -R^2 \leq t \leq 0\},$$

and set

$$M(R) \equiv \max_{C(R)} u \qquad (R > 0).$$

We may assume $M(R) > 0$ for all $R > 0$.

We propose to estimate the rate at which $M(R) \to 0$ as $R \to 0$.

The first lemma asserts that should u be very close on the average over $C(R)$ to its positive maximum $M(R)$, then u would be strictly positive on $C(\frac{R}{2})$ (a contradiction to (1)).

<u>Lemma 1</u> There exists a constant $\alpha > 0$ such that

$$⨍\!\!\!\!⨍_{C(R)} (M(R)-u)\,dxdt \leq \alpha M(R)^{n+4} \tag{2}$$

implies

$$\inf_{C(\frac{R}{2})} u \geq \frac{1}{2} M(R) > 0. \tag{3}$$

(Here "$⨍\!\!\!\!⨍$" denotes the average.)

This lemma is proved by applying De Giorgi's method. The new observation here is that the rapid geometric convergence of these iterative estimates "overwhelms" the problems caused by the singularity in β' at zero.

In view of (1), (3) is impossible; hence (2) fails and so u is strictly less than its maximum by a "little bit" over an "appreciable" subset of $C(R)$:

Lemma 2 There exist constant $\beta,\gamma > 0$ such that

$$\text{meas}\{(x,t) \in C(R) \,|\, u(x,t) \leq M(R) - \gamma M(R)^{n+4}\} \geq \beta M(R)^{n+4} \text{meas } C(R). \tag{4}$$

Next we observe that since $bu_t = \Delta u$ on the set $\{u > 0\}$, $v \equiv u^+$ is a nonnegative *sub*solution of the same linear p.d.e.:

$$bv_t \leq \Delta v, \qquad \text{in } C(R) \tag{5}$$

Thus v satisfies (5) in C(R), $\max_{C(R)} v = M(R)$, but, according to Lemma 2, v is strictly less than M(R) on a set of positive relative measure. In consequence we can use a kind of Green's function representation of the solution to prove

$$M(\theta R) = \max_{C(\theta R)} v \leq M(R) - \sigma(M(R)) \tag{6}$$

for some $0 < \theta < 1$, and some appropriate mapping $\sigma : [0,\infty) \to [0,\infty)$, $\sigma(r) > 0$ for $r > 0$.

Thus the maximum of u^+ over $C(\theta R)$ has diminished by an amount we can estimate in terms of known quantities. This and a similar argument applied to u^- provide an *a priori* estimate on the modulus of u from below, at any point where $u = 0$. A modulus of continuity from above is now easy to obtain by standard techniques.

Owing to this estimate where $u = 0$, we now know that if $u \neq 0$ at some point, then also $u \neq 0$ in some neighborhood of known size. Hence standard linear theory implies u is smooth in this neighbourhood.

Full details of the argument are contained in our paper, which will (someday) appear in Archive for Rational Mechanics and Analysis. Some extensions of our techniques have been discovered independently by E. DiBenedetto, Wm. Ziemer and P. Sacks.

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E DI BENEDETTO

Interior and boundary regularity for a class of free boundary problems

1. INTRODUCTION

Consider parabolic differential inclusions with principal part in divergence form of the type

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div}\vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_2 u) \ni 0 \tag{1.1}$$

in the sense of distributions over a domain $Q \subset \mathbb{R}^{N+1}$.

Here β represents a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that $0 \in \beta(0)$, $\vec{a}$ is a map from $\mathbb{R}^{2N+2}$ into $\mathbb{R}^N$ and b maps $\mathbb{R}^{2N+2}$ into $\mathbb{R}^1$. A particular case of (1.1) is

$$\frac{\partial}{\partial t}\beta(u) - \Delta u + b(x,t,u,\nabla_x u) \ni 0. \tag{1.2}$$

Beside their intrinsic mathematical interest, inclusions such as (1.1), (1.2) arise as models to a variety of diffusion processes. In particular they comprehend in a unifying formulation free-boundary problems of different nature such as Stefan problem, porous media filtration, fast chemical reactions, diffusion in porous media of a partially saturated gas.

The purpose of this note is to review and collect some facts about the regularity of weak solutions of (1.1), (1.2), as well as give statement and a sketch of the proof of new results concerning regularity at the boundary. The theory in question has developed over the past two or three years and therefore this note is far from being unifying and syntetic.

We will consider graphs $\beta(\cdot)$ of the following types.

A. Graphs of Stefan type $\beta(\cdot)$ is given by

$$\beta(r) = \begin{cases} \beta_1(r) & , \quad r > 0 \\ [-\nu, 0] & , \quad r = 0 \\ \beta_2(r) & , \quad r < 0 \end{cases}$$

where $\nu > 0$ is a given constant and β_i $i = 1,2$ are monotone increasing functions in their respective domain of definition, a.e. differentiable and

$$0 < \alpha_0 \le \beta_i'(r) \le \alpha_1 \ , \ i = 1, 2$$

for two positive constants α_0, α_1. Moreover $\beta_1(0) = \beta_2(0) = 0$.

B. Graphs of Porous Media type β is a continuous monotone increasing function in $\mathbb{R}$ such that $\beta(0) = 0$. With $\beta'(r)$ we denote the Dini numbers (whenever they exist)

$$\beta'(r) = \begin{cases} \limsup\limits_{h \downarrow 0} \dfrac{\beta(r) - \beta(r-h)}{h}, & r > 0 \\ \\ \limsup\limits_{h \downarrow 0} \dfrac{\beta(r+h) - \beta(r)}{h}, & r < 0 \end{cases}$$

and on $r \to \beta'(r)$ assume the following

(i) $0 < \alpha_0 \le \beta'(r)$, $\forall r \in \mathbb{R}\backslash\{0\}$, where α_0 is a given constant

(ii) $\liminf\limits_{|r| \to 0} \beta'(r) = +\infty$

(iii) There exists an interval $[-\delta_0, \delta_0]$ around the origin such that $\beta'(s) \le \beta'(r)$ for $s \in \mathbb{R}\backslash[-\delta_0, \delta_0]$ and $r \in [-\delta_0, \delta_0]\backslash\{0\}$, and $\beta'(\cdot)$ is decreasing over $(0, \delta_0]$ and increasing over $[-\delta_0, 0)$.

The model example for such a β is

$$\beta(r) = |r|^{\frac{1}{m}} \operatorname{sign} r, \ m > 1$$

which occurs in filtration of gases in porous media when the flow obeys a polytropic law. Note that no symmetry asuumptions are made on β in a neighbourhood of the origin.

C. Fast diffusion β is such that the graph $\gamma = \beta^{-1}$ has the structure described in B. Graphs of this type occur in the porous-media-like equation

$$u_t = \Delta(|u|^m \operatorname{sign} u), \quad 0 < m < 1$$

which arises as a model for certain problems in plasma physics or for spread of biological populations [21,30].

D. Graphs of partially saturated Porous media type β is continuous monotone increasing such that

(i) $0 \le \beta'(r) \quad \forall r$

(ii) $\limsup_{r \downarrow 0} \beta'(r) \ge 0$

(iii) $\exists p \in (0, \frac{N}{(N-4)^+}) : \beta(r) \ge r^p$ for r sufficiently large.

This last technical assumption arises in the proof of local boundedness of the weak solutions [14].

Graphs such as the above include

(a) $\beta(r) = r^+$

(b) $$\beta(r) = \begin{cases} r^m & , r > 0,\ m > 1 \\ 0 & , r \le 0. \end{cases}$$

Throughout we will assume the following (the notation of [18] is adopted).

Ω is an open set in $\mathbb{R}^N$, $0 < T < \infty$ and $\Omega_T = \Omega \times (0,T]$, $S_T = \partial\Omega \times (0,T]$

$[A_1]$ a_i, b are measurable over Ω_T $\mathbb{R}^{N+1}$, $i = 1,2,\ldots,N$.

$[A_2]$ $\vec{a}(x,t,u,\vec{p}) \cdot \vec{p} \ge C_0(|u|)|\vec{p}|^2 - \phi_0(x,t)$

$|a_i(x,t,u,\vec{p})| \le \mu_0(|u|)|\vec{p}| + \phi_1(x,t)$

$|b(x,t,u,\vec{p})| \le \mu_1(|u|)|\vec{p}|^2 + \phi_2(x,t)$, $i = 1,2,\ldots,N$

where $C_0(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, decreasing and strictly positive; $\mu_i(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and increasing $i = 0, 1$; and ϕ_i, $i = 0,1,2$ are nonnegative and satisfy

$$\|\phi_0, \phi_2\|_{\hat{q},\hat{r},\Omega_T} ; \ \|\phi_1\|_{2\hat{q},2\hat{r},\Omega_T} \le \mu_2 .$$

Here μ_2 is a given constant and $\hat{q}$, $\hat{r}$ are positive numbers linked by

$$\frac{1}{\hat{r}} + \frac{N}{2\hat{q}} = 1 - \kappa_1$$

$$\hat{q} \in [\frac{N}{2(1-\kappa_1)}, \infty]; \ \hat{r} \in [\frac{1}{1-\kappa_1}, \infty], \ 0 < \kappa_1 < 1, \ N \geq 2$$

$$\hat{q} \in (1,\infty), \ \hat{r} \in [\frac{1}{1-\kappa_1}, \frac{1}{1-2\kappa_1}], \ 0 < \kappa_1 < \tfrac{1}{2}, \ N = 1.$$

Definition By a local weak solution of (1.1) in Ω_T we mean a function $u \in W_2^{1,0}(\Omega_T)$ defined by $u \equiv \beta^{-1}(w)$, where $w \in L^\infty(0,T; L_2(\Omega))$ is such that $w \subset \beta(u)$, the inclusion being intended in the sense of the graphs, and w and u satisfy

$$\int_\Omega w(x,\tau)\phi(x,\tau)\,dx \Big|_{t_0}^{t} + \int_{t_0}^{t}\int_\Omega \{-w(x,\tau)\phi_t(x,\tau)$$
$$+ \vec{a}(x,\tau,u,\nabla_x u)\nabla_x\phi(x,\tau) + b(x,\tau,u,\nabla_x u)\phi(x,\tau)\}\,dxd\tau = 0 \tag{1.2}$$

for all $\phi \in \overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, and almost all intervals $[t_0, t] \subset (0,T]$. Informations on local regularity of weak solutions of (1.1) are particularly relevant in compactness arguments. In order to stress this aspect we will assume the following.

$[A_3]$ Let β be a graph of type A and let u be a weak solution of (1.1). We assume u can be constructed as the local weak limit in $V_2(\Omega_T)$ of local solutions of the regularized family of equations

$$\frac{\partial}{\partial t}\beta(u_\varepsilon) - \mathrm{div}\{\vec{a}(x,t,u_\varepsilon,\nabla_x u_\varepsilon) + \varepsilon\nabla_x\beta(u_\varepsilon)\} + b(x,t,u_\varepsilon,\nabla_x u_\varepsilon) \ni 0 \tag{1.3}$$

in the sense of the integral identity (1.2), and each $u_\varepsilon \in W_2^{1,1}(\Omega_T)$. Such a regularization is of the type of Hopf vanishing viscosity.

$[A_4]$ If β is of type B, C, D we assume that a local weak solution of (1.1) can be constructed as the $V_2(\Omega_T)$ weak limit of local solutions of the equations.

$$\frac{\partial}{\partial t}\beta_\varepsilon(u_\varepsilon) - \mathrm{div}\,\vec{a}(x,t,u_\varepsilon,\nabla_x u_\varepsilon) + b(x,t,u_\varepsilon,\nabla_x u_\varepsilon) = 0 \tag{1.4}$$

where $\beta_\varepsilon(\cdot)$ are smooth functions such that $\beta_\varepsilon \to \beta$ uniformly on compacts of $\mathbb{R}\setminus\{0\}$, and $\forall\varepsilon > 0$ $\beta_\varepsilon(u_\varepsilon) \in V_2^{1,0}(\Omega_T)$. Moreover $-\beta'_\varepsilon(r) \le \beta'(r)$ $\forall r \in \mathbb{R}\setminus\{0\}$, $\forall\varepsilon > 0$, if β is of type B and $\beta'_\varepsilon(r) \ge \beta'(r)$ $\forall r \in \mathbb{R}\setminus\{0\}$ if β is of type C or D. Assumptions $[A_3]$, $[A_4]$ are not restrictive in view of the available existence theory (see references in [11]).

2. INTERIOR REGULARITY

We will state facts about interior continuity in terms of equiboundedness and equicontinuity of the family $\{u_\varepsilon\}$ introduced in (1.3)-(1.4).

Graphs of type A and B

Theorem 1 Let β be of type A or B and let $\{u_\varepsilon\}$ be the family of local solutions of either (1.3) or (1.4). Then

(i) u_ε is locally essentially bounded in Ω_T, uniformly in ε.

(ii) u_ε is continuous in Ω_T with a modulus of continuity $\omega_K(\cdot)$ over a compact $K \subset \Omega_T$ which is independent of ε.

Corollary Let u be a local weak solution of (1.1) with β of type A or B and satisfying $[A_3]$ - $[A_4]$. Then

(i) $u \in L_{\infty,loc}(\Omega_T)$

(ii) $u \in C(\Omega_T)$

(iii) The modulus of continuity $\omega_K(\cdot)$ of u over a compact $K \subset \Omega_T$ can be explicitly constructed.

Graphs of type C and D

Theorem 2 Let β be of type C or D and let u be a local weak solution of (1.2) satisfying $[A_4]$. Then for the family $\{\beta_\varepsilon(u_\varepsilon)\}$ satisfying (1.4) we have

(i) $\beta_\varepsilon(u_\varepsilon)$ is locally essentially bounded in Ω_T uniformly in ε.

(ii) $\beta_\varepsilon(u_\varepsilon)$ is continuous in Ω_T with a modulus of continuity $\omega_K(\cdot)$ over a compact $K \subset \Omega_T$ which is independent of ε.

Corollary Let u be a local weak solution of (1.2) with β of type C, D and satisfying $[A_4]$. Then

(i) $\beta(u) \in L_{\infty,loc}(\Omega_T)$

(ii) If $u \in L_\infty^{loc}(\Omega_T)$, then $\beta(u) \in C(\Omega_T)$

(iii) The modulus of continuity of $\omega_K(\cdot)$ of u over a compact $K \subset \Omega_T$ can be explicitly constructed.

Remarks (i) If β is of type C or D, then we assume u is solution of the equation (1.2) of restricted structure on the principal part. It is not known whether Theorem 2 holds for equation with general structure (1.1). Some partial results in this direction can be found in [14].

(ii) If β is of type D, then $(x,t) \to u(x,t)$ need not be continuous as shown by an example due to Alt [1].

The proof of these theorems is in [11,12,14]. Here we give the main idea only in connection with (1.1) and β of type A. The function $(x,t) \to u_\varepsilon(x,t)$ can be modified in a set of measure zero to yield a continuous representative out of the equivalence class $u_\varepsilon \in W_2^{1,1}(\Omega_T)$ if for every $(x_0, t_0) \in \Omega_T$ there exists a family of nested and shrinking cylinders $Q_n(x_0, t_0)$ around (x_0, t_0), such that the essential oscillation ω_n of u_ε in $Q_n(x_0, t_0)$, tends to zero as $n \to \infty$ in a way determined by the operator in (1.1) and the data. Technically the proof is based on the following proposition.

We let $(x_0, t_0) \in \Omega_T$, $t_0 > 0$ and for $R > 0$, Q_R will denote the cylinder

$$Q_R \equiv \{|x - x_0| < R\} \times [t_0 - R^2, t_0] \quad .$$

Let R_0 be so small that $Q_{2R_0} \subset \Omega_T$ and denote with any real number such that

$$2 \operatorname*{ess\,sup}_{Q_{2R_0}} u_\varepsilon \ge \omega \ge \operatorname*{ess\,osc}_{Q_{2R_0}} u_\varepsilon$$

Proposition There exists constants ξ, h and a decreasing function $\pi: \mathbb{R}^+ \to \mathbb{R}^+$ independent of ε such that

$$\operatorname*{ess\,osc}_{Q_{R_*}} u_\varepsilon \le \omega(1 - 2^{\pi(\omega)}), \quad \forall \varepsilon > 0$$

where $R_* = \xi(R_0)^h$, $h > 1$, $\xi > 1$. The number ξ, h, and the function π depend uniquely upon the data and not upon ω, R_0 nor ε.

Repeated application of this proposition, permits to construct two sequences of real numbers $\{R_n\}$, $\{M_n\}$ such that $R_n, M_n \to 0$ and

$$\operatorname*{ess\,osc}_{Q_{R_n}} u_\varepsilon \le M_n ,$$

thereby proving the continuity of u_ε in view of the arbitrarity of $(x_0, t_0) \in \Omega_T$.

For equations with the restricted structure

$$\frac{\partial}{\partial t} \beta(u) - \Delta u \ni 0 \tag{2.1}$$

and β either as in A or B, theorems analogous to theorem 1 have been proved in [6], under the additional restriction of non-negativity of the solutions. In particular if $\beta(u) = u^m$, $m > 1$ and u is a non-negative solution of the Cauchy problem in $\mathbb{R}^N$ associated with (2.1) in [7] it is proved the Hölder continuity of u.

The signum restriction is relaxed in [8] but still the structure (2.1) is retained.

For β as in B and the inhomogeneous equation

$$\frac{\partial}{\partial t}\beta(u) - \Delta u \ni f$$

some continuity results appear in [20]. In this work u is constructed as the solution of a Cauchy problem in all $\mathbb{R}^N$ and on f are assumed more stringent conditions than the ones in $[A_2]$.

The integrability conditions we have assumed in $[A_1]$ - $[A_2]$ are the best possible in the following sense. If β is the identity graph (and therefore (1.1) is the classical parabolic equation with divergence sturcture), then weaker summability conditions might yield unbounded solutions as shown by Kruzkov [22,23]. The methods employed here could handle graphs β of different nature such as for example $\beta(r) = \ell n\ |r|$.

We conclude this section by mentioning that Ziemer has recently proved the continuity of weak solutions of (1.1) for graphs of type A or B [34]. However, the methods employed do not give a modulus of continuity.

3. BOUNDARY REGULARITY

We consider the case when (1.1) or (1.2) are associated with initial-boundary conditions on the parabolic boundary of Ω_T.

(3.i) Variational boundary data

Consider formally the problem

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\ \beta(u) - \operatorname{div} \vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) \ni 0 \text{ in } D'(\Omega_T)\\ &\vec{a}(x,t,u,\nabla_x u) \cdot \vec{n}_{S_T}(x,t) = g(x,t,u) \text{ on } S_T\\ &\beta(u(x,0)) = w_0(x),\ u_0 \in L_2(\Omega).\end{aligned}\right\} \tag{3.1}$$

where $\vec{n}_{S_T} = (n_{x_1}, n_{x_2}, \dots, n_{x_N})$ denotes the outer unit normal to S_T. On the boundary data $g(x,t,u)$ we assume that

[G] g is continuous over $S_T \times \mathbb{R}$ and admits an extension $\tilde{g}(x,t,u)$ over Ω_T such that

$$\left\| \frac{\partial}{\partial u}\ \tilde{g}(x,t,u),\ \frac{\partial}{\partial x}\ \tilde{g}(x,t,u) \right\|_{\infty,\Omega_T} \le C < \infty$$

for some positive constant C.

By a weak solution of (3.1) we mean a function $u \in W_2^{1,0}(\Omega_T)$ defined by $u \equiv \beta^{-1}(w)$, where $w \in L^\infty(0,T;\ L_2(\Omega))$, $w \subset \beta(u)$ and u and w satisfy

$$\begin{aligned}&\int_\Omega w(x,t)\phi(x,t)\,dx + \int_0^t \int_\Omega \{-w(x,\tau)\phi_t(x,\tau)\\ &\quad + \vec{a}(x,\tau,u,\nabla_x u) \cdot \nabla_x \phi + b(x,\tau,u,\nabla_x u)\phi(x,\tau)\}dx\,d\tau \\ &\qquad = \int_{S_T} g(x,\tau,u)\phi(x,\tau)\,ds + \int_\Omega w_0(x)\phi(x,0)\,dx\end{aligned} \tag{3.2}$$

for all $\phi \in W_2^{1,1}(\Omega_T)$, and a.e. $t \in [0,T]$.

We assume that a weak solution u of (3.1) can be constructed as the $V_2(\Omega_T)$ weak limit of weak solutions $\{u_\varepsilon\}$ of regularized problems.

Namely we assume

[A_5] If β is of type A, then a weak solution of (3.1) is the $V_2(\Omega_T)$ - weak limit of the solutions of the problems

$$\begin{cases} \frac{\partial}{\partial t}\beta(u_\varepsilon) - \operatorname{div}\{\vec{a}(x,t,u_\varepsilon,\nabla_x u_\varepsilon) + \varepsilon\nabla_x\beta(u_\varepsilon)\} + b(x,t,u_\varepsilon,\nabla_x u_\varepsilon) \ni 0 \\ [\vec{a}(x,t,u_\varepsilon,\nabla_x u_\varepsilon) + \varepsilon\nabla_x\beta(u_\varepsilon)] \cdot \vec{n}_{S_T} = g(x,t,u_\varepsilon) \text{ on } S_T \\ \beta(u_\varepsilon(x,0)) = w_0(x),\ w_0(x) \in L_2(\Omega) \end{cases}$$

where $u_\varepsilon \in W_2^{1,1}(\Omega_T)$, $\forall\varepsilon > 0$.

$[A_6]$ If β is of type B, C, D then a weak solution of (3.1) can be constructed on the $V_2(\Omega_T)$ weak limit of the solutions $\{u_\varepsilon\}$ of (3.1) with β replaced by β_ε. Here $\beta_\varepsilon(\cdot)$ are smooth functions such that $\beta_\varepsilon \to \beta$ uniformly on compacts of $\mathbb{R}\backslash\{0\}$, and $\forall\varepsilon > 0, \beta_\varepsilon(u_\varepsilon) \in V_2^{1,0}(\Omega_T)$. Moreover $\beta'(r) \le \beta'(r)$ $\forall r \in \mathbb{R}\backslash\{0\}$, $\forall\varepsilon > 0$ if β is of type B and $\beta_\varepsilon'(r) \ge \beta'(r)$, $\forall' r \in \mathbb{R}\backslash\{0\}$ otherwise.

<u>Theorem 3</u> Assume that $\partial\Omega$ is a C^1 manifold in $\mathbb{R}^{N-1}$, and that [G] holds. Let β be of type A or B and let u be a weak solution of (3.1). If for the family $\{u_\varepsilon\}$ in $[A_5]$ - $[A_6]$ we assume

> u_ε are essentially bounded uniformly in ε over $\Omega \times [\sigma,T]$ for every $\sigma > 0$

then

> There exist a continuous non-decreasing function $\omega_\sigma(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_\sigma(0) = 0$ such that

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \le \omega_\sigma(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $\sigma \in (0,T]$, all $\varepsilon > 0$, and all pairs $(x_i, t_i) \in \bar{\Omega} \times [\sigma,T]$, $i = 1,2$.

Moreover if $u_0 = \beta^{-1}(w_0)$ is continuous over all $\bar{\Omega}$, then there exists $s \to \omega_0(s): \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_0(0) = 0$ continuous and nondecreasing such that

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \le \omega_0(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $(x_i, t_i) \in \bar{\Omega}_T$, $i = 1,2$.

The functions $\omega_\sigma(\cdot)$ can be determined in dependence of the data and the positive number σ, and $\omega_0(\cdot)$ can be determined only in terms of the data and the modulus of continuity of u_0.

Theorem 4 Let u be a weak solution of (3.1) where

(a) β is either of type C or D

(b) the equation in (3.1) has the restricted structure (1.2).

Let u satisfy $[A_6]$. Then for the family $\{\beta_\varepsilon(u_\varepsilon)\}$ we have

(i) $\beta_\varepsilon(u_\varepsilon)$ are essentially bounded uniformly in ε over $\bar{\Omega} \times [\sigma,T]$, $\forall\sigma > 0$.

(ii) If $u_\varepsilon \in L_\infty^{loc}(\bar{\Omega} \times [\sigma,T])$, uniformly in ε, then there exists a continuous non decreasing function $\omega_\sigma(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_\sigma(0) = 0$ such that

$$|\beta_\varepsilon(u_\varepsilon(x_1, t_1)) - \beta_\varepsilon(u_\varepsilon(x_2, t_2))| \le \omega_0(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $\sigma \in (0,T]$, all $\varepsilon > 0$ and all pairs $(x_i, t_i) \in \bar{\Omega} \times [\sigma,T]$, $i = 1, 2$.

Moreover if w_0 is continuous over all $\bar{\Omega}$, and $u_\varepsilon \in L_\infty(\Omega_T)$ uniformly in ε, then there exists $\omega_0(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_0(0) = 0$ continuous and non decreasing such that

$$|\beta_\varepsilon(u_\varepsilon(x_1, t_1)) - \beta_\varepsilon(u_\varepsilon(x_2, t_2))| \le \omega_0(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}})$$

for all $(x_i, t_i) \in \bar{\Omega}_T$, $i = 1, 2$, $\forall\varepsilon > 0$.

The function $\omega_\sigma(\cdot)$ can be determined in terms of the data and the positive number σ, whereas $\omega_0(\cdot)$ can be determined only in terms of the data and the modulus of continuity of w_0.

(3-ii) Dirichlet boundary data

Consider the problem: Find $u \in W_2^{1,0}(\Omega_T)$ such that

$$\begin{cases} \dfrac{\partial}{\partial t}\beta(u) - \operatorname{div}\vec{a}(x,t,u,\nabla_x u) + b(x,t,u,\nabla_x u) \ni 0 \text{ in } \mathcal{D}'(\Omega_T) \\ u\Big|_{S_T} = g(x,t) \quad \text{in the sense of the traces on } S_T \\ \beta(u(x,0)) \ni w_0(x) \text{ in } \Omega \end{cases}$$

We leave to the reader the task of defining a weak solution of (3.3) modelled after the previous definitions (see also [18]).

Here we assume that

$[A_7]$ $g \in C(\bar{S}_T)$ and that $\beta^{-1}(w_O) = u_O \in C(\bar{\Omega})$.

$[A_8]$ $\partial\Omega$ is regular for "$-\Delta$" in the sense of the Wiener criterion [34].

Theorem 5 (Ziemer [34]) Let u be a weak solution of (3.3) with β either of type A or B. Assume moreover that $[A_7]$ - $[A_8]$ hold and that

$$u_t \in L_2(\Omega_T)$$

Then

$$u \in C(\bar{\Omega} \times [\sigma,T]), \quad \forall\sigma > 0.$$

This result is very elegant and supplies continuity of the solution under the weakest assumptions on the structure of $\partial\Omega$. The drawback is the lack of a modulus of continuity which does not permit to formulate the result in terms of equicontinuous nets of approximating solutions. This is of importance because the requirement $u_t \in L_2(\Omega_T)$ is not natural for this kind of problem [see 29], whereas it can be verified by the approximations u_ε.

In order to accomplish this we impose homogeneous boundary data and stronger assumptions on $\partial\Omega$. Namely we assume

(P) $\exists\alpha > 0$, $R_O > 0$ such that $\forall x_O \in \partial\Omega$ and every ball $B(R)$ centered at x_O, $R \le R_O$,

$$\text{meas } [\Omega \cap B(R)] < (1-\alpha)\text{meas } B(R).$$

Theorem 6 Let u be a weak solution of (3.3), and let $[A_7]$ and (P) hold. Assume moreover that

(a) If β is of type A, $u_t \in L_2(\Omega_T)$

(b) If β is of type B, $\beta(u) \in C[0,T; L_2(\Omega)]$

(c) $g(x,t) \equiv 0 \quad (x,t) \in \bar{S}_T$

Then

(i) If β is either of type A or B then $u \in C(\bar{\Omega}_T)$ with an explicit modulus of continuity $\omega(\cdot)$ which depends upon the data, the number α in (P) and the modulus of continuity of g over S_T and u_O over $\bar{\Omega}$.

(ii) If β is of type A, then the modulus of continuity $\omega_\sigma(\cdot)$ is of Hölder type near $\partial\Omega \times [\sigma,T]$, $\sigma > 0$.

Remark In view of the precise computation of a modulus of continuity, the theorem could be stated in terms of equicontinuity of an approximating net. In view of this the assumption $u_t \in L_2(\Omega)$ in (a) is not restrictive.

The proof of theorem 6 can be found in [11,12]. Analogous facts hold for β of type C (see [12]) and D (see [14]).

It is not known whether Theorem 6 holds for non-homogeneous data even under the assumption (P) on $\partial\Omega$.

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D T NGUYEN

On a free boundary problem for the heat equation

INTRODUCTION

We deal with the following free boundary problem.

Problem 1 To find the functions $u(x,t), s(t)$ such that $s(o) = b$, $s(t) > 0\ \forall t \in [0,T]$,

$$u_t = u_{xx} \text{ in } \Omega = \{(x,t): 0 < x < s(t),\ 0 < t < T\}, \tag{1}$$

$$u(x,o) = \Psi(x) \text{ for } 0 \le x \le b, \tag{2}$$

$$u(0,t) = f_1(t) \text{ for } 0 \le t \le T, \tag{3}$$

$$u(s(t),t) = f_2(s(t)) \text{ for } 0 \le t \le T, \tag{4}$$

$$u_x(s(t),t) = g(s(t)) \text{ for } 0 \le t \le T, \tag{5}$$

b, T being given positive numbers, $\Psi(x)$, $f_1(t)$, $f_2(x)$, $g(x)$ being given functions.

This problem arises in the study of filtration [1]. In [1] it is proved that the filtration holds if and only if u_x is greater than some critical value g. Therefore the free boundary $x = s(t)$ is the line of distinction between the domain where there is the filtration and the other domain. The analogous problem where f_2, g depends on t is studied by T. D. Ventcel [2].

Here we sketch some of the results of [3] and then we reduced problem 1 to a quasi-variational inequality and sketch the result of [4].

1. UNICITY AND EXISTENCE OF SOLUTION OF PROBLEM 1

The solution of problem 1, by definition, is the pair of functions $u(x,t), s(t)$, such that $s(t)$ is differentiable and positive $\forall t \in [0,T], s(0)=b$, the functions u, u_x, u_{xx}, u_t are Lipschitz in $\bar{\Omega}$ and (1) - (5) are satisfied.

Theorem 1 Under the following assumptions: $f_1'(t)$, $f_2'(x)$, $g(x)$, $g'(x)$, $\Psi''(x)$ are continuous, $f_1'(t) \le 0$, $f_2'(x) \le 0$, $g(x) \ge 0$, $g'(x) \ge 0$, $\Psi''(x) \le 0$, problem 1 cannot have more than one solution such that $\frac{ds}{dt} \ge 0$.

Proof The function $q = u_t$ satisfies the heat equation in Ω and

$$q(x,0) = \Psi''(x) \le 0$$

$$q(0,t) = f_1'(t) \le 0$$

$$q(s(t),t) = [f_2'(s(t))-g(s(t))] \frac{ds}{dt} \le 0.$$

Hence $q = u_t = u_{xx} \le 0$ in $\bar{\Omega}$. Because u_x decreases with respect to x, $u_x(s(t),t) = g(s(t)) \ge 0$, we have $u_x \ge 0$ in $\bar{\Omega}$.

Suppose now that there exist two solutions of problem 1: $u_1(x,t), s_1(t)$ and $u_2(x,t)$, $s_2(t)$. The function $v = u_1 - u_2$ satisfies the heat equation in the domain $\{0 < x < y(t),\ 0 < t < T\}$, where $y(t) = \min(s_1(t),s_2(t))$. Therefore $|v|$ reaches the maximal value only in the line $x = y(t)$, because $v(x,0) = 0$, $v(0,t) = 0$. If $y(t) = s_1(t)$, we have

$$v(y(t),t) = f_2(s_1(t))-u_2(s_1(t),t) \ge f_2(s_2(t))-u_2(s_1(t),t) =$$

$$= u_2(s_2(t),t)-u_2(s_1(t),t) = [s_2(t)-s_1(t)]\frac{\partial u_2}{\partial x}(\xi,t) \ge 0,$$

$$\frac{\partial v}{\partial x}(y(t),t) = g(s_1(t)) - \frac{\partial u_2}{\partial x}(s_1(t),t) \le g(s_2(t)) - \frac{\partial u_2}{\partial x}(s_1(t),t) =$$

$$= \frac{\partial u_2}{\partial x}(s_2(t),t) - \frac{\partial u_2}{\partial x}(s_1(t),t) = [s_2(t)-s_1(t)] \frac{\partial^2 u_2}{\partial x^2}(\xi',t) \le 0,$$

where $s_1(t) \le \xi, \xi' \le s_2(t)$. Similarly, if $y(t) = s_2(t)$, we have $v(y(t),t) \le 0$, $v_x(y(t),t) \ge 0$.

If v reaches a positive maximal value on $x = y(t)$, then $v_x \le 0$, this is impossible. Similarly, v cannot reach a negative minimal value on $x = y(t)$. Therefore $v(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$, $s_1(t) \equiv s_2(t)$.

Theorem 2 We make the following assumptions

(i) $f_2(x),\ g(x) \in C^2\ b,+\infty), f_1(t) \in C^2[0,T],\ \Psi(x) \in C^4[0,b],$

$f_1'(t) < 0,\ f_2'(x) \le 0,\ \Psi''(x) \le 0,\ g(x) > 0,\ g'(x) > 0$

(ii) $f_1(0) = \Psi(0),\ f_2(b) = \Psi(b),\ \Psi''(0) = f_1'(0),\ \Psi'(b) = g(b)$

Then there exists a solution of problem 1 such that $\frac{ds}{dt} \geq 0$. Furthermore we have

$$s(t) \leq X \qquad \forall t \in [0,T], \tag{6}$$

where X is the solution of the equation

$$xg(x) - f_2(x) + f_1(T) = 0 \tag{7}$$

Sketch of the proof

1) Proof of the estimation (6)

Because $u_x > 0$, $u_{xx} \leq 0$, we have for fixed t

$$s(t) \leq \frac{f_2(s(t))-f_1(t)}{g(s(t))} \leq \frac{f_2(s(t))-f_1(T)}{g(s(t))}$$

$$\Rightarrow s(t)g(s(t)) - f_2(s(t)) + f_1(T) \leq 0$$

$$\Rightarrow s(t) \leq X .$$

2) Approximation of problem 1

We denote $u_n(x) = u(x,n\Delta t)$, $f_1(n) = f_1(n\Delta t)$, $s_n=s(n\Delta t)$. The semidiscretized form of problem 1 is the following: to find the functions $u_n(x)$ and the numbers s_n such that $s_o = b$ and

$$u_n'' - \frac{u_n}{\Delta t} = - \frac{u_{n-1}}{\Delta t} \tag{8}$$

$$u_o(x) = \Psi(x) \tag{9}$$

$$u_n(0) = f_1(n) \tag{10}$$

$$u_n(s_n) = f_2(s_n) \tag{11}$$

$$u_n'(s_n) = g(s_n) \tag{12}$$

The general solution of the equation (8) is

$$u_n(x) = C_1 \text{ sh } \frac{x}{\sqrt{\Delta t}} + C_2 \text{ ch } \frac{x}{\sqrt{\Delta t}} - \int_0^x \frac{u_{n-1}(\xi)}{\sqrt{\Delta t}} \text{ sh } \frac{x-\xi}{\sqrt{\Delta t}} d\xi$$

$$\Rightarrow \quad C_2 = f_1(n) \tag{10}$$

$$\Rightarrow \quad f_2(s_n) = C_1 \ \mathrm{sh}\frac{s_n}{\sqrt{\Delta t}} + f_1(n)\ \mathrm{ch}\frac{s_n}{\sqrt{\Delta t}} - \int_0^{s_n} \frac{u_{n-1}(\xi)}{\sqrt{\Delta t}}\ \mathrm{sh}\frac{s_n-\xi}{\sqrt{\Delta t}}\, d\xi \tag{11}$$

$$\Rightarrow \quad g(s_n)\ \sqrt{\Delta t} = C_1\ \mathrm{ch}\frac{s_n}{\sqrt{\Delta t}} + f_1(n)\ \mathrm{sh}\frac{s_n}{\sqrt{\Delta t}} - \int_0^{s_n} \frac{u_{n-1}(\xi)}{\sqrt{\Delta t}}\ \mathrm{ch}\frac{s_n-\xi}{\sqrt{\Delta t}}\, d\xi. \tag{12}$$

By eliminating C_1 from these two equations, we get

$$f_2(s_n)\,\mathrm{ch}\frac{s_n}{\sqrt{\Delta t}} + g(s_n)\ \sqrt{\Delta t}\ \mathrm{sh}\frac{s_n}{\sqrt{\Delta t}} = f_1(n) + \int_0^{s_n} \frac{u_{n-1}(\xi)}{\sqrt{\Delta t}}\ \mathrm{sh}\frac{\xi}{\sqrt{\Delta t}}\, d\xi .$$

So, we have to prove the existence of the solution of the equations

$$\varphi_n(s) = f_2(s)\,\mathrm{ch}\frac{s}{\sqrt{\Delta t}} - g(s)\sqrt{\Delta t}\ \mathrm{sh}\frac{s}{\sqrt{\Delta t}} - \int_0^{s} \frac{u_{n-1}(\xi)}{\sqrt{\Delta t}}\ \mathrm{sh}\frac{\xi}{\sqrt{\Delta t}}\, d\xi - f_1(n) = 0 \tag{13}$$

It is possible to check that $\forall n$

$$\varphi_n(s_{n-1}) > 0,\ \varphi_n'(s) < 0 \text{ for } s \geq s_{n-1},\ \varphi_n''(s) < 0 \text{ for } s \geq s_{n-1}.$$

Hence there exists one solution of the equation (13) such that $s_n > s_{n-1}$.

3) A priori estimations

We can prove by the maximum principle that there exists a positive constant M which does not depend on $n \in \mathbb{N}$ and $x \in [0,s_n]$ such that

$$|s_n| \leq M,\quad \left|\frac{s_n-s_{n-1}}{\Delta t}\right| \leq M,\quad \left|\frac{s_n-2s_{n-1}+s_{n-2}}{\Delta t^2}\right| \leq M,$$

$$|u_n(x)| \leq M,\ |u_n'(x)| \leq M,\ |u_n''(x)| = \left|\frac{u_n(x)-u_{n-1}(x)}{\Delta t}\right| \leq M,$$

$$|u_n'''(x)| \leq M,\ \left|\frac{u_n(x)-2u_{n-1}(x)+u_{n-2}(x)}{\Delta t^2}\right| \leq M.$$

4) Passage to the limit

We denote for $(n-1)\Delta t \leq t \leq n\Delta t$

$$u^{\Delta t}(x,t) = \frac{t-(n-1)\Delta t}{\Delta t}\, u_n(x) + \frac{n\Delta t-t}{\Delta t}\, u_{n-1}(x),$$

$$s^{\Delta t}(t) = \frac{t-(n-1)\Delta t}{\Delta t} s_n + \frac{n\Delta t - t}{\Delta t} s_{n-1}.$$

Because of the above estimates, the set $\{s^{\Delta t}(t)\}$ is compact in $C([0,T])$, the sets $\{u^{\Delta t}(x,t)\}$, $\{\frac{\partial^2 u^{\Delta t}(x,t)}{\partial x^2}\}$, $\{\frac{\partial u^{\Delta t}(x,t)}{\partial t}\}$, are compact in $C(\bar{\Omega}_{\Delta t})$, where $\Omega_{\Delta t} = \{(x,t): 0 < x < s_{\Delta t}, 0 < t < T\}$. Therefore there exists a sequence $\Delta t_i \to 0$ such that $s^{\Delta t_i}(t) \to s(t)$ uniformly with respect to $t \in [0,T]$, $u^{\Delta t_i}(x,t) \to u(x,t)$, $\frac{\partial^2 u^{\Delta t_i}(x,t)}{\partial x^2} \to \frac{\partial^2 u(x,t)}{\partial x^2}$, $\frac{\partial u^{\Delta t_i}(x,t)}{\partial t} \to \frac{\partial u(x,t)}{\partial t}$ uniformly with respect to $(x,t) \in C(\bar{\Omega})$, where $\Omega = \{(x,t): 0 < x < s(t), 0 < t < T\}$. The functions $u(x,t), s(t)$ satisfy the equation (1) and the conditions (2)-(4). Because $\left|\frac{s_n - s_{n-1}}{\Delta t}\right|$ and $\left|\frac{s_n - 2s_{n-1} + s_{n-2}}{\Delta t^2}\right|$ are uniformly bounded, $s_n > s_{n-1}$, the function $s(t)$ is differentiable and $\frac{ds}{dt} \geq 0$.

2. REDUCTION TO A QUASI-VARIATIONAL INEQUALITY

We set $v(x,t) = u(x,t) - f_2(x)$. Because $N_x^-(x,t) = u_x(x,t) - f_2'(x) > 0$, $v(s(t),t) = 0$, we have $v(x,t) < 0$ for $x < s(t)$. We denote

$$\tilde{v}(x,t) = \begin{cases} v(x,t) & \text{for } 0 \leq x \leq s(t) \\ 0 & \text{for } s(t) < x \leq X \end{cases}$$

and we make the transformation

$$w(x,t) = \int_x^X \tilde{v}(\xi,t) \, d\xi. \tag{14}$$

The functions $w(x,t)$, $s(t)$ satisfy the following equation and conditions

$$w_t - w_{xx} = \begin{cases} g(s(t)) - f_2'(x) & \text{for } 0 \leq x \leq s(t) \\ 0 & \text{for } s(t) < x \leq X \end{cases} \tag{15}$$

$$w(x,0) = \int_x^b [\Psi(\xi) - f_2(\xi)] \, d\xi \tag{16}$$

$$w_x(0,t) = f_2(0) - f_1(t) \tag{17}$$

$$w(X,t) = 0. \tag{18}$$

Because $\tilde{v}(x,t) < 0$ for $0 < x < s(t)$, $\tilde{v}(x,t) = 0$ for $s(t) \le x \le X$, we have

$$w(x,t) < 0 \text{ for } 0 \le x < s(t),\ w(x,t) = 0 \text{ for } s(t) \le x \le X \tag{19}$$

We set

$$K = \{z \in L^2(0,X) : z \le 0 \text{ a.e}\}.$$

We have

$$\int_0^{s(t)} (w_t - w_{xx})(z-w)\,dx = \int_0^{s(t)} [g(s(t)-f_2'(x)](z-w)\ dx \quad \forall z \in K,$$

$$\int_0^{s(t)} (w_t - w_{xx})(z-w)\,dx = \int_0^{X} (w_t - w_{xx})(z-w)\,dx,$$

$$\int_0^{s(t)} [g(s(t))-f_2'(x)](z-w)\,dx = \int_0^{X} [g(s(t))-f_2'(x)](z-w)\ dx -$$

$$-\int_{s(t)}^{X} [g(s(t))-f_2'(x)](z-w)\,dx \ge \int_0^{X} [g(s(t))-f_2'(x)](z-w)\,dx$$

because $w = 0$ for $s(t) \le x \le X$, $z \le 0$ a.e. Thus we get

$$(w_t - w_{xx}, z-w) \ge (g(s(t))-f_2'(x), z-w) \quad \forall z \in K \tag{20}$$

whereby $(.,.)$ we denote the scalar product in $L^2(0,X)$. Therefore if $u(x,t)$, $s(t)$ is the solution of problem 1, the functions $w(x,t)$, $s(t)$ satisfy the inequality (20) and the conditions (16)-(19).

<u>Problem 2</u> To find the functions $w(x,t)$, $s(t)$, such that $s(t) > 0$ $\forall t \in [0,T]$, $s(0) = b$

$$w(x,t) \in K \text{ for a.e. } t \in [0,T], \tag{21}$$

$$(w_t - w_{xx}, z-w) \ge (g(s(t))-f_2'(x), z-w) \quad \forall z \in K \tag{22}$$

$$w(x,0) = \int_x^b [\Psi(\xi)-f_2(\xi)]\, d\xi \quad 0 \le x \le b, \tag{23}$$

$$w_x(0,t) = f_2(0) - f_1(t) \quad 0 \le t \le T, \tag{24}$$

$$w(X,t) = 0 \quad 0 \le t \le T, \tag{25}$$

$$s(t) = \inf \{x: w(x,t) = 0\}. \tag{26}$$

If $s(t)$ is a known function, then (22) is a variational inequality with respect to w and the line $x = \tilde{s}(t)$, where

$$\tilde{s}(t) = \inf \{x: w(x,t) = 0\}$$

is the associated free boundary. If $\tilde{s}(t)$ coincide with $s(t)$, the pair $w(x,t)$, $s(t)$ form a solution of the problem 2. Hence (22) is a quasi variational inequality.

The solution of problem 2, by definition, is a pair of functions $w(x,t)$, $s(t)$ such that $s(t)$ is continuous and positive $\forall t \in [0,T]$, $s(0) = b$, $w(x,t)$ is continuous in $\bar{\varphi}$ where $\varphi = \{(x,t): 0 < x < X,\ 0 < t < T\}$, w_{xx}, w_t are in $L^2(\varphi)$ and the relations (21)-(26) are satisfied.

3. EXISTENCE OF THE SOLUTION OF PROBLEM 2

Theorem 3 Under the following assumptions $f_2(x)$, $g(x) \in H^2(0,+\infty)$, $f_1(t) \in H^2(0,T)$, $\Psi(x) \in H^3(0,b)$, $f_1'(t) < 0$, $f_2'(x) \le 0$, $f_2''(x) \le 0$, $\Psi''(x) \le 0$, $g(x) > 0$, $g'(x) > 0$ a.e., there exists a solution of the problem 2 such that $w_x(x,t) \ge 0$, $w_{xx}(x,t) \le 0$, $w_t(x,t) \le 0$ a.e. in Ω, $s'(t) \ge 0$ for a.e. $t \in [0,T]$.

Sketch of the proof

1) Approximation of the problem 2

The semidiscretized form of problem 2 is the following:
To find the functions $w_n(x)$ and the numbers s_n such that $s_0=b$ and

$$w_n(x) \in K, \tag{27}$$

$$\left(\frac{w_n}{\Delta t} - w_n'',\ z-w_n\right) \ge \left(g(s_n)-f_2'(x) + \frac{w_{n-1}}{\Delta t},\ z-w_n\right) \quad \forall z \in K, \tag{28}$$

$$w_o(x) = \int_x^b [\Psi(\xi)-f_2(\xi)]\, d\xi, \tag{29}$$

$$w_n'(0) = f_2(0)-f_1(n), \tag{30}$$

$$w_n(X) = 0, \tag{31}$$

$$s_n = \inf \{x : w_n(x) = 0\}. \tag{32}$$

We have to prove that if $w_{n-1}(x)$ is known, then the problem (27)-(32) has a unique solution. Thus we consider the following problem: to find $w(x)$ and s such that

$$w(x) \in K, \tag{33}$$

$$\left(\frac{w(x)}{\Delta t} - w''(x), z-w(x)\right) \geq \left(g(s)-f_2'(x) + \frac{k(x)}{\Delta t}, z-w(x)\right) \quad \forall z \in K, \tag{34}$$

$$w'(0) = f(0) - c, \tag{35}$$

$$w(X) = 0, \tag{36}$$

$$s = \inf \{x : w(x) = 0\}, \tag{37}$$

where $k(x) = w_{n-1}(x)$, $c = f_1(n)$.

Lemma 1 The problem (33)-(37) has a unique solution.

For the proof, we apply the method of A. Friedman used in [5]. We take $\bar{s} \in [0,X]$ and suppose that $\bar{w}(x)$ is the solution of the following problem:

$$\bar{w}(x) \in K, \tag{38}$$

$$\left(\frac{\bar{w}(x)}{\Delta t} - \bar{w}''(x), z-\bar{w}(x)\right) \geq \left(g(\bar{s})-f_2'(x) + \frac{k(x)}{\Delta t}, z-\bar{w}(x)\right) \quad \forall z \in K, \tag{39}$$

$$\bar{w}'(0) = f_2(0) - c \tag{40}$$

$$\bar{w}(X) = 0 \tag{41}$$

We set $\tilde{s} = \inf \{x : \bar{w}(x) = 0\}$ and denote by A the application $\bar{s} \to \tilde{s}$ from $[0,X]$ into itself.

If $\bar{s}$ increases, then $g(\bar{s})$ increases because $g'(s) > 0$, hence $\bar{w}(x)$ increases, thus $\tilde{s}$ decreases. It is possible to prove that if $s^n \searrow s^*$ then $As^n \nearrow As^*$, and if $s^n \nearrow s^*$ then $As^n \searrow As^*$. So, the application A from $[0,X]$ into itself is continuous. By the Brouwer theorem, A has a fixed point. This proves that the problem (33)-(37) has a solution. It is easy to check that this solution is unique.

Thus the problem (27)-(32) has a unique solution. Furthermore,

$$w_n(x) \in W^{2,p}(0,X) \quad \forall p > 1.$$

2) <u>A priori estimations</u>

<u>Lemma 2</u> We have $\forall n \in \mathbb{N}$, $\forall x \in [0,s_n]$

$$s_n \geq s_{n-1}$$

$$w_n'(x) \geq 0$$

$$w_n''(x) \leq 0$$

$$\frac{w_n(x)-w_{n-1}(x)}{\Delta t} \leq 0$$

<u>Lemma 3</u> There exists a positive constant M which does not depend on $n \in \mathbb{N}$, $x \in [0,s_n]$ such that

$$|s_n| \leq M, \ \left|\frac{s_n-s_{n-1}}{\Delta t}\right| \leq M,$$

$$|w_n(x)| \leq M, \ |w_n'(x)| \leq M, \ |w_n''(x)| \leq M,$$

$$\left|\frac{w_n(x)-w_{n-1}(x)}{\Delta t}\right| \leq M.$$

The proof of the lemmas 2, 3 are based on the maximum principle for parabolic equations.

3) <u>Passage to the limit</u>

We define the functions $w^{\Delta t}(x,t)$, $s^{\Delta t}(t)$ as in §1. By the above estimates, the set $\{s^{\Delta t}(t)\}$ is compact in $C([0,T])$, the set $\{w^{\Delta t}(x,t)\}$ is compact in

$C(\bar{Q})$, the sets $\{\frac{\partial^2 w^{\Delta t}(x,t)}{\partial x^2}\}$, $\{\frac{\partial w^{\Delta t}(x,t)}{\partial t}\}$ are weak compact in $L^2(Q)$. Therefore there exists a sequence $\Delta t_i \to 0$ such that

$$s^{\Delta t_i}(t) \to s(t) \text{ in } C([0,T]) \text{ strong,}$$

$$w^{\Delta t_i}(x,t) \to w(x,t) \text{ in } C(\bar{\varphi}) \text{ strong,}$$

$$\frac{\partial^2 w^{\Delta t_i}(x,t)}{\partial x^2} \to \frac{\partial^2 w(x,t)}{\partial x^2} \text{ in } L^2(\varphi) \text{ weak,}$$

$$\frac{\partial w^{\Delta t_i}(x,t)}{\partial t} \to \frac{\partial w(x,t)}{\partial t} \text{ in } L^2(\varphi) \text{ weak.}$$

The functions $w(x,t)$, $s(t)$ form a solution of problem 2. It is easy to check that $w_x \geq 0$, $w_{xx} \leq 0$, $w_t \leq 0$, $s'(t) \geq 0$.

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I PAWŁOW

Generalized Stefan-type problems involving additional nonlinearities

1. INTRODUCTION

The purpose of this paper is to apply techniques of variational inequalities to the study of a generalized two-phase Stefan problem in several space variables, covering existence of the corresponding solutions and their uniqueness. The problem considered involves quasilinear parabolic or mixed type, parabolic-elliptic, equations, nonlinear conditions at lateral (fixed) boundary, and free boundary conditions either of the Stefan or fast chemical reaction type. The class of problems under consideration assemble various physical processes, including in particular heat conduction with phase transitions of the first (positive latent heat) and of the second kind (latent heat equal to zero). Besides, space-time and temperature dependent thermal properties of the medium and internal heat sources, as well as flux conditions, specific heat coefficient possibly vanishing and space-time dependent latent heat are admitted. This class includes also fast chemical reaction problems, electrochemical machining and saturated/unsaturated flows through porous media.

The variational inequality method, whose general idea is due to Baiocchi [1], has been applied first to the classical Stefan problem in the one-phase case by Duvaut [6], and in the two-phase case by Frémond [10] and Duvaut [7]. This approach was also used in [5,9,11,12,14,15,19] to the study of various Stefan problems.

2. PROBLEM STATEMENT

2.1. Classical formulation

Let Ω be a regular bounded domain in R^n, $n \geq 1$ with boundary Γ admitting representation $\Gamma = \Gamma' \cup \Gamma''$, $\Gamma' \cap \Gamma'' = \emptyset$. For a given $0 < T < \infty$ denote $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\Sigma' = \Gamma' \times (0,T)$, $\Sigma'' = \Gamma'' \times (0,T)$.

Let $S \subset Q$ denote *free boundary*, $S \equiv \bigcup_{t \in (0,T)} S(t)$; and $Q \equiv Q_1 \cup S \cup Q_2$, $Q_1 \cap Q_2 = \emptyset$.

In the sequel the notations ∇u, Δu will refer to the space variables $u' \triangleq \partial u/\partial t$; as a rule we shall use the shortened notations $f(u,\nabla u)$ for

$f(x,t,u(x,t),\partial u/\partial x_1(x,t),\dots,\partial u/\partial x_n(x,t))$ and $f(u)$ for $f(x,t,u(x,t))$; then $[f(u)]' = \partial f/\partial t + \partial f/\partial u\ \partial u/\partial t$.

The problem consists in determining a function $\theta : Q \to R$ and free boundary $S \subset Q$, satisfying

- differential equations (of the parabolic or mixed, parabolic-elliptic, type) in

$$Q_i \triangleq \{(x,t) \in Q \mid (-1)^i\ \Theta(x,t) > 0\},\quad i = 1,2,\ :$$

$$[\tilde{\gamma}(\Theta)]' - \Delta\Theta + \lambda_o\Theta = \begin{cases} \lambda(\Theta,\nabla\Theta) & \text{in } Q_1 \\ \lambda(\Theta,\nabla\Theta) + \lambda & \text{in } Q_2 \end{cases} \tag{1}$$

where $\tilde{\gamma}(x,t,r) \triangleq \int_o^r \rho(x,t,\xi)\ d\xi$;

- conditions on the free boundary $S \triangleq \{(x,t) \in Q \mid \Theta(x,t) = 0\}$:

$$\Theta = 0,\ [\nabla\Theta|_2 - \nabla\Theta|_1]\cdot\vec{N}_x = L\ N_t \tag{2}$$

where $\vec{N} = (\vec{N}_x, N_t)$ is the unit vector normal to S, directed towards Q_1, $\vec{N}_x$ denotes its projection onto R^n, $\nabla\Theta|_i$ indicates the limit of $\nabla\Theta$ on S when approached from Q_i;

- conditions at the lateral boundary Σ:

$$\Theta = f_1 \text{ on } \Sigma',\quad \frac{\partial\Theta}{\partial\nu} + g_o\Theta = g(\Theta) \text{ on } \Sigma'' \tag{3}$$

where $\vec{\nu}$ denotes the unit outward vector normal to Γ;

- initial condition

$$\Theta(0) = \Theta_o \text{ in } \Omega. \tag{4}$$

Here $\rho : Q \times R \to R$, $\lambda_o : \Omega \to R$, $\lambda : Q \times R \times R^n \to R$, $\lambda : Q \to R$, $f_1 : \Sigma' \to R$, $g_o : \Gamma'' \to R$, $g : \Sigma'' \times R \to R$, $L : Q \to R$, $\Theta_o : \Omega \to R$ are given as functions of their arguments; moreover $\rho \geq 0$, $\lambda_o \geq 0$, $g_o \geq 0$, $L \geq 0$ and ρ may possibly have a finite jump over S.

<u>Remark 1</u> Problem (1)-(4) is a reduced formulation expressed in terms of the "flow temperature" $\Theta = K(\upsilon)$ where $[K(\upsilon)](x,t) \triangleq \int_{\upsilon^*}^{\upsilon} k(x,t,\xi)\ d\xi$, of the Stefan problem formulated with respect to temperature υ, with the specific

heat $c(x,t,\upsilon)$, heat conductivity $k(x,t,\upsilon)$, phase transition temperature $\upsilon^*(x,t)$ and latent heat of the phase change $L(x,t)$. Then, in particular $\rho(x,t,\Theta) = c(x,t,K^{-1}(\Theta))/k(x,t,K^{-1}(\Theta))$ where K^{-1} denotes the inverse Kirchhoff transformation (see [16] for details).

Later on we shall distinguish the following two cases:

(C1) *parabolic*: $0 < \bar{\rho} \leq \rho(x,t,r)$ for $(x,t,r) \in Q \times R$,

(C2) mixed type *parabolic-elliptic*: $0 \leq \rho(x,t,r)$ for $(x,t,r) \in Q \times R$.

2.2. Variational formulation

Applying Baiocchi's techniques, introducing the characteristic function of the set Q_2 in Q:

$$\chi(\Theta) \triangleq \begin{cases} 0 & \text{in } Q\setminus\bar{Q}_2 \\ 1 & \text{in } Q_2 \end{cases}$$

the "internal" part (1)-(2) of the problem can be shown to be formally equivalent to

$$[\gamma(\Theta)]' - \Delta\Theta + \lambda_o\Theta = \lambda(\Theta,\nabla\theta) + \lambda\chi(\Theta) - L[\chi(\Theta)]' \quad \text{in } \mathcal{D}'(Q). \tag{5}$$

Let $\mathrm{sign}^+(r)$ denote the Heaviside's graph

$$\mathrm{sign}^+(r) \triangleq \begin{cases} 0 & \text{if } r < 0 \\ [0,1] & \text{if } r = 0 \\ 1 & \text{if } r > 0, \end{cases}$$

and introduce the multi-valued mapping γ representing enthalpy:

$$\gamma(x,t,r) \triangleq \tilde{\gamma}(x,t,r) + L(x,t)\,\mathrm{sign}^+(r), \quad (x,t,r) \in Q \times R. \tag{6}$$

Obviously,

$$\chi(\Theta) \in \mathrm{sign}^+(\Theta) \quad \text{a.e. in } Q. \tag{7}$$

Using the above notations we can rewrite equation (5) in the form

$$h' - \Delta\Theta + \lambda_o\Theta = \lambda(\Theta,\nabla\Theta) + (\lambda + \dot{L}')\,\chi(\Theta) \quad \text{in } \mathcal{D}'(Q) \tag{8}$$

where

$$h(x,t) \in \gamma(x,t,\Theta(x,t)) \quad \text{for a.a.} \quad (x,t) \in Q. \tag{9}$$

Apart from Θ_o we shall assume to know also the initial enthalpy

$$h(0) = h_o \quad \text{in} \quad \Omega, \tag{10}$$

compatible with Θ_o, i.e. $h_o(x) \in \gamma(x,0,\Theta_o(x))$ for a.a. $x \in \Omega$.

Let us remark that in the case of ρ strictly positive the inverse γ^{-1} to the enthalpy mapping is well-defined single-valued function, therefore h_o uniquely determines $\Theta_o = \gamma^{-1}(h_o)$. On the contrary, if ρ is allowed to vanish then γ^{-1} is no more single-valued and the knowledge of the initial pair $\{h_o,\Theta_o\}$ is required.

Obviously, equation (8) has to be satisfied together with boundary conditions (3) and initial conditions (4),(10).

Remark 2 System (8),(9),(3),(4),(10) is in fact so-called enthalpy weak formulation of problem (1)-(4) and suggests the use as a weak solution either a pair $\{h,\Theta\}$ satisfying an integral identity corresponding to (8) or, in the case (C1), a function h satisfying the identity with Θ replaced by $\gamma^{-1}(h)$.

An alternative weak formulation of the problem employs so-called *freezing index* function

$$u(x,t) \triangleq \int_0^t \Theta(x,\tau)\,d\tau, \quad (x,t) \in Q$$

as a new dependent variable. Integrating (8) with respect to t, and taking into account that $u' = \Theta$ in Q, we get

$$h - h_o - \Delta u + \lambda_o u = \Lambda(u',\nabla u') + X(u') \tag{11}$$

where

$$[\Lambda(u',\nabla u')](x,t) \triangleq \int_0^t \lambda(x,\tau,u'(x,\tau),\nabla u'(x,\tau))\,d\tau\,,$$

$$[X(u')](x,t) \triangleq \int_0^t (\lambda(x,\tau) + L'(x,\tau))\chi(u'(x,\tau))\,d\tau.$$

Due to (6), the relationship (9) may be rewritten in the form

$$h(x,t) - \tilde{\gamma}(x,t,u'(x,t)) \in L(x,t)\ \partial\psi_o(u'(x,t)) \text{ for a.a. } (x,t) \in Q \tag{12}$$

where $\psi_o(r) \triangleq r^+$ and $\partial\psi_o$ is the subgradient of ψ_o.

By (11),(12),

$$\begin{aligned} &-\tilde{\gamma}(x,t,u'(x,t)) + [\Lambda(u',\nabla u')](x,t) + [X(u')](x,t) + \Delta u(x,t) - \\ &- \lambda_o(x)\ u(x,t) + h_o(x) \in L(x,t)\ \partial\psi_o(u'(x,t)) \text{ for a.a. } (x,t) \in Q. \end{aligned} \tag{13}$$

In terms of u boundary conditions (3) take the form

$$u = F_1 \text{ on } \Sigma', \quad \frac{\partial u}{\partial \nu} + g_o\ u = G(u') \text{ on } \Sigma'' \tag{14}$$

where

$$F_1(x,t) \triangleq \int_0^t f_1(x,\tau)\ d\tau, \quad [G(u')](x,t) \triangleq \int_0^t g(x,\tau,u'(x,\tau))\ d\tau .$$

By definition of u,

$$u(0) = 0 \text{ in } \Omega. \tag{15}$$

In the sequel we shall use the notations: $V = H^1(\Omega)$, V' - its dual, $\langle .,. \rangle$ - dual pairing between V' and V; $(.,.)$, $(.,.)_{\Gamma''}$ - scalar products in $L^2(\Omega)$ and $L^2(\Gamma'')$, respectively; $V_o \triangleq \{v \mid v \in V,\ v|_{\Gamma'} = 0\}$; $a(v,w) \triangleq (\lambda_o\ v,\ w) + (\nabla v,\ \nabla w) + (g_o\ v,\ w)_{\Gamma''}$; $\Psi(t,v) \triangleq \int_\Omega L(x,t)\ \psi_o(v(x))\ dx$; $\mathcal{B}(.,.): L^2(Q) \times L^2(Q)^n \to L^2(0,T;V')$, defined by

$$\begin{aligned} &\langle [\mathcal{B}(v,\nabla v)](t), w \rangle \triangleq (\tilde{\gamma}(t,v(t)) - [\Lambda(v,\nabla v)](t) - [X(v)](t),\ w) - \\ &\quad - ([G(v)](t),\ w)_{\Gamma''} \quad \text{for all } w \in V, \text{ a.a. } t \in [0,T]. \end{aligned} \tag{16}$$

We shall use also the shifted variable $y \triangleq u - F$ where $F(x,t) \triangleq \int_0^t f(x,\tau)d\tau$, with f being an extension of the function f_1 included in the Dirichlet boundary condition (3), such that $f \in L^\infty(0,T;V)$, $f|_{\Sigma'} = f_1$.

Then problem (13)-(15) may be given the following formulation [18]:

Find a function y such that:

$$
\text{(VI)}\quad\begin{cases}
y \in L^{\infty}(0,T;V_o),\ y' \in L^{\infty}(0,T;V_o);\ \chi(y'+F') \in \operatorname{sign}^{+}(y'+F') \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. in } Q,\ \text{measurable;} \\
<[\mathcal{B}(y'+F',\nabla(y'+F'))](t),z-y'(t)> + a(y(t)+F(t),z-y'(t)) - \\
\quad - (h_o,\ z-y'(t)) + \Psi(t,z+F'(t)) - \Psi(t,y'(t)+F'(t)) \geq 0 \\
\qquad\qquad\qquad\qquad\qquad \text{for all } z \in V_o,\ \text{a.a. } t \in [0,T], \\
y(0) = 0 \text{ in } \Omega.
\end{cases}
$$

On the basis of this formulation we may introduce

Definition By *weak solution of problem* (1)-(4) we mean a function y satisfying (VI).

3. EXISTENCE OF WEAK SOLUTION

Now we list the hypotheses imposed on the data of problem (1)-(4).

In the case (C1) we assume:

(A1) $f' \in H^1(0,T;L^2(\Omega)) \cap L^{\infty}(0,T;V)$;

(A2) there exist constants $\bar{\rho},\bar{\bar{\rho}},c_\rho$ such that

$$0 < \bar{\rho} \leq \rho(x,t,r) \leq \bar{\bar{\rho}} < \infty,\ \left|\frac{\partial\rho}{\partial t}(x,t,r)\right| \leq c_\rho < \infty \text{ for } (x,t,r) \in Q \times R;$$

(A3) $\lambda_o \in L^{\infty}(\Omega)$, $\lambda_o \geq 0$ a.e. in Ω; $\lambda \in L^2(Q)$;

$$\lambda(x,t,r,p) \equiv \lambda_{o1}(x)\ r + \lambda_1(x,t,r) + \sum_{i=1}^{n} \lambda_{2i}(x)\ p_i, \text{ where } \lambda_{o1} \in L^{\infty}(\Omega),$$

$\lambda_{2i} \in L^{\infty}(\Omega)$, $i=1,\ldots,n$; $\lambda_1(x,t,r)$ satisfies the Carathéodory conditions in $Q \times R$, moreover there exists a nonnegative function $\bar{\lambda} \in L^2(Q)$ such that

$$|\lambda_1(x,t,r)| \leq \bar{\lambda}(x,t) \quad \text{for} \quad (x,t,r) \in Q \times R;$$

(A4) $g_o \in L^{\infty}(\Gamma'')$, $g_o \geq 0$ a.e. on Γ''; $g(x,t,r)$ satisfies Carathéodory conditions in $\Sigma'' \times R$, there exist nonnegative functions $\bar{g}_o \in L^{\infty}(\Sigma'')$, $\bar{g} \in L^{\infty}(\Sigma'')$ such that

$$|g(x,0,r)| \leq \bar{g}_o(x) \quad \text{for} \quad (x,r) \in \Gamma'' \times R,$$

$$\left|\frac{\partial g}{\partial t}(x,t,r)\right| \leq \bar{g}(x,t) \quad \text{for} \quad (x,t,r) \in \Sigma'' \times R,$$

or, if $g \equiv g_1(x,t)$, then $g_1 \in H^1(0,T;L^2(\Gamma''))$;

(A5) $h_o \in L^2(\Omega)$, $\Theta_o \in V \cap L^\infty(\Omega)$, $\Theta_o|_{\Gamma'} = f_1(0)$;

(A6) $L \in H^1(0,T;L^2(\Omega))$, $L \geq 0$ a.e. in Q;

(A7) either $\lambda_o > 0$ a.e. in Ω or the set $\Gamma' \cup \{x \in \Gamma'' \mid g_o(x) > 0\}$ has positive measure in Γ.

In the case (C2) we shall keep unchanged the assumptions (A4),(A5),(A7), whereas the assumptions (A1)-(A3),(A6) are to be replaced by the following:

(A1)' assumption (A1) and moreover $f'' \in L^\infty(Q)$;

(A2)' $\rho \equiv \rho(x,r)$ for $(x,r) \in \Omega \times R$; there exists a constant $\bar{\bar{\rho}}$ such that $0 \leq \rho(x,r) \leq \bar{\bar{\rho}} < \infty$ for $(x,r) \in \Omega \times R$;

(A3)' $\lambda_o \in L^\infty(\Omega)$, $\lambda_o \geq 0$ a.e. in Ω; $\lambda' \in L^\infty(Q)$;

$\lambda \equiv \lambda(x,t,r)$ for $(x,t,r) \in Q \times R$; function $\lambda(x,t,r)$ satisfies Carathéodory conditions in $Q \times R$, there exist nonnegative functions $\bar{\lambda}_o \in L^\infty(\Omega)$, $\bar{\lambda} \in L^\infty(Q)$ such that

$$|\lambda(x,0,r)| \leq \bar{\lambda}_o(x) \quad \text{for } (x,r) \in \Omega \times R,$$

$$\left|\frac{\partial\lambda}{\partial t}(x,t,r)\right| \leq \bar{\lambda}(x,t) \quad \text{for} \quad (x,t,r) \in Q \times R,$$

or, if $\lambda \equiv \lambda_1(x,t)$, then $\lambda_1 \in H^1(0,T;L^2(\Omega))$;

(A6)' assumption (A6) and moreover $L'' \in L^\infty(Q)$.

In order to formulate an existence theorem for (VI), let us introduce a family of the corresponding regularized problems. For $\varepsilon > 0$ denote by y_ε a function satisfying:

$$(\text{VI}) \quad \begin{cases} y_\varepsilon \in L^\infty(0,T;V_o),\ y'_\varepsilon \in L^\infty(0,T;V_o), \\ \quad < [B_\varepsilon(y'+F',\nabla(y'_\varepsilon+F'))](t), z-y'_\varepsilon(t) > + a(y_\varepsilon(t)+F(t), z-y'_\varepsilon(t)) - \\ - (h_{o\varepsilon}, z-y'_\varepsilon(t)) + \Psi_\varepsilon(t,z+F'(t)) - \Psi_\varepsilon(t,y'(t)+F'(t)) \geq 0 \\ \qquad\qquad \text{for all } z \in V_o,\ \text{a.a. } t \in [0,T], \\ y_\varepsilon(0) = 0 \text{ in } \Omega \end{cases}$$

where the functionals $\Psi_\varepsilon(t,v) \triangleq \int_\Omega L(x,t)\,\psi_{o\varepsilon}(v(x))\,dx$ are bounded, convex and Gateaux differentiable on V, and they approximate Ψ in the following sense

(i) for every $v \in L^2(0,T;V)$ $\int_0^T \Psi_\varepsilon(t,v(t))\,dt \to \int_0^T \Psi(t,v(t))dt$ as $\varepsilon \to 0$,

(ii) if $v_\varepsilon \to v$ weakly in $L^2(0,T;V)$ then

$$\liminf_{\varepsilon \to 0} \int_0^T \Psi_\varepsilon(t,v_\varepsilon(t))dt \geq \int_0^T \Psi(t,v(t))\,dt \ ;$$

$$h_{0\varepsilon}(x) \triangleq \tilde{\gamma}(x,0,\Theta_0(x)) + L(x,0)\ \psi'_{0\varepsilon}(\Theta_0(x)) \text{ for a.a. } x \in \Omega;$$

$$[X_\varepsilon(v)](x,t) \triangleq \int_0^t (\lambda(x,\tau) + L'(x,\tau))\ \psi'_{0\varepsilon}(v(x,\tau))\,d\tau \ ;$$

B_ε is defined by (16) including X_ε instead of X.

<u>Theorem 1</u>

Case (C1): Assume that (A1)-(A7) hold. Then (VI) has at least one solution y. This solution can be constructed as a limit of solutions y_ε of the regularized problems $(VI)_\varepsilon$ as $\varepsilon \to 0$ (possibly after taking a subsequence):

$$y_\varepsilon \to y,\ \ y'_\varepsilon \to y' \text{ weakly-star in } L^\infty(0,T;V_0), \tag{17}$$

$$y''_\varepsilon \to y'' \text{ weakly in } L^2(Q). \tag{17'}$$

Besides, $y'(0) = \Theta_0 - F'(0)$, and the following a priori estimates hold:

$$\|y\|_{L^\infty(0,T;V_0)} + \|y'\|_{L^\infty(0,T;V_0)} \leq C, \tag{18}$$

$$\|y''\|_{L^2(Q)} \leq C \tag{18'}$$

with the finite constants C dependent only upon the bounds on the data.

Case (C2): Assume that (A1)',(A2)',(A3)',(A4),(A5),(A6)',(A7) hold. Then problem (VI) has at least one solution y that can be constructed as a limit of solutions y_ε of problems $(VI)_\varepsilon$ in the sense of (17). Besides, this solution satisfies a priori estimate (18).

The proof, presented in details in [18] adopts standard techniques exposed in particular in [8]:

1° Galerkin approximation $(VI)_{\varepsilon m}$ to $(VI)_\varepsilon$ is employed,
2° the existence of a solution to each problem $(VI)_{\varepsilon m}$ is obtained by an analysis of a system $(VI)_{\varepsilon,m,\nu}$ involving a singular perturbation of $(VI)_{\varepsilon m}$,
3° appropriate a priori estimates independent of ε,m,ν are established, allowing to pass to the limit with $\nu \to 0$, $m \to \infty$ and $\varepsilon \to 0$, by monotonicity and compactness arguments.

The proof proceeds in the same way in both cases considered with only technical differences in deriving a priori estimates due to the fact that in (C1) $\tilde{\gamma}(t,v)$ is strictly monotone in $L^2(\Omega)$ for almost all $t \in [0,T]$, i.e.

$$(\tilde{\gamma}(t,v) - \tilde{\gamma}(t,w),\ v-w) \geq C\|v-w\|^2_{L^2(\Omega)} \qquad \text{for all} \quad v,w \in L^2(\Omega),$$

a.a. $t \in [0,T]$, with $C \equiv \bar{\rho}$, whereas in the case (C2) $C \equiv 0$.

Remark 3 It follows from Theorem 1 in the case (C1) that $\theta \in H^1(Q) \cap L^\infty(0,T;V)$, whereas in the case (C2) that $\Theta \in L^\infty(0,T;V)$.

Remark 4 The existence questions for Stefan type problems involving additional nonlinearities were considered also in [4,13,16,20] (see also [2,3] for abstract results applicable to some nonlinear situations).

In [16]

$$\lambda(x,t,r,p) \equiv \lambda_1(x,t,r) + \sum_{i=1}^{n} \lambda_{2i}(x,t)\ p_i, \tag{19}$$

where λ_1 is assumed to satisfy Carathéodory conditions in $Q \times R$ and have linear growth with respect to $r \in R$. In [4] similar assumptions on λ_1 are imposed but λ may depend upon p in a more general way than given by (19). The paper [13] deals with Lipschitz continuous source term $\lambda \equiv \lambda(r)$.

A number of existence theorems using another type of hypothesis on nonlinear flux term $g(x,t,r)$ are proved in [20]. In particular g is assumed there to be nonincreasing with respect to r, at the same time satisfying conditions concerning its sign which yield maximum principle or, alternatively, to have growth in r of a bounded order $0 < p < \infty$ (see the Theorems 1,2). In the case when function $r \to -g(x,t,r)$ is not monotone, certain regularity of $\partial g/\partial t$ is required (see the Theorems 4,5 of [20]). Results similar to the Theorem 5 of [20] have been obtained by different techniques in [16] where g is postulated to satisfy Carathéodory conditions in $\Sigma'' \times R$, be of linear growth in r and Lipschitz continuous with respect to t. In [4] the existence is proved under the assumptions of continuity of g with respect its arguments and of the linear growth in r.

For a more comprehensive survey of the results and techniques used in the case of Stefan type problems with nonlinear flux condition at the fixed boundary we refer to [17].

4. UNIQUESNESS OF WEAK SOLUTION

The uniqueness of the solution of (VI) can be proved immediately under the following hypothesis on the operator :

$$\text{(A8)}\quad \int_0^t < [\mathcal{B}(v,\nabla v)](\tau) - [\mathcal{B}(w,\nabla w)](\tau),\ v(\tau) - w(\tau) > d\tau \geq 0$$
$$\text{for all } v,w \in L^2(0,T;V),\ \text{a.a. } t \in [0,T].$$

Clearly, (A8) is satisfied in the affine case

$$\lambda \equiv -\lambda_0(x)\ r + \lambda(x,t),\ g \equiv -g_0(x)\ r + g(x,t) \quad \text{with} \quad \lambda_0, g_0 \geq 0,$$

and provided that $\lambda = -L'$. Then the standard arguments may be employed. Assuming y and $\tilde{y}$ to be two possible solutions of (VI), adding by sides the corresponding inequalities (for y with $z \equiv \tilde{y}'(t)$ whereas that for $\tilde{y}$ with $z \equiv y'(t)$), we can assert

Theorem 2 [18] Let the assumptions of Theorem 1 be satisfied and moreover (A8) holds. Then the solution of (VI) is unique.

Remark 5 Extending the arguments of the proof of the uniqueness one can obtain results on the continuous dependence of the solutions of (VI) upon perturbations of ρ, $\lambda_0, \lambda, f_1, g_0, g, L, h_0$ (see [18] for details).

Remark 6 The uniquesness and stability of the weak solutions of Stefan type problems involving nonlinearities in λ and g were proved in [16] by using special test functions in the integral identities corresponding to the enthalpy.

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A VISINTIN

The Stefan problem for a class of degenerate parabolic equations

I INTRODUCTION

Let D be an open bounded subset of $\mathbb{R}^N$ $(N \geq 1)$, having a "smooth" boundary $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $T > 0$, set $Q \equiv D \times]0,T[$, $\Sigma_i \equiv \Gamma_i \times]0,T[$ $(i = 1,2)$.

Let $c,k : \mathbb{R} \to \mathbb{R}^+$, $f : Q \to \mathbb{R}$ be "smooth" functions, possibly with a jump in 0 for c and k.

We introduce informally the interior conditions of a two-phase free boundary problem:

Find $\tilde{u} : Q \to \mathbb{R}$ such that - setting $\Omega \equiv \{(x,t) \in Q | \tilde{u}(x,t) > 0\}$ and $\mathcal{F} \equiv \partial\Omega \cap Q$ (the free boundary) -

$$c(\tilde{u}) \frac{\partial \tilde{u}}{\partial t} - \bar{\nabla}.[k(\tilde{u})\bar{\nabla}\tilde{u}] = f \qquad \text{in } Q\backslash\mathcal{F} \tag{I.1}$$

(where $\bar{\nabla} \equiv (\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n})$, $\bar{\nabla}. \equiv \text{div}$)

$$\left.\begin{aligned} &\tilde{u} = 0 && \text{(I.2)} \\ &[\![k(\tilde{u})\bar{\nabla}\tilde{u}]\!] . \nu_{\bar{x}} = \nu_t && \text{(I.3)} \end{aligned}\right\} \text{ on } \mathcal{F} \text{ (assumed to be "smooth")}$$

(with $(\nu_t,\nu_{\bar{x}}) \equiv (\nu_t,\nu_{x_1},\ldots,\nu_{x_n})$: vector normal to $\mathcal{F}$ in $\mathbb{R}^{n+1}$)

where $[\![\cdot]\!]$ denotes the jump from $Q\backslash\Omega$ to Ω.

Boundary and initial conditions on Σ and on $D \times \{0\}$ are required.

In most of the works concerning the Stefan problem, (I.1) is assumed to be uniformly parabolic, i.e. with c and k bounded away from zero.

The degenerate case of possibly vanishing $c(\tilde{u})$ arises in some physical problems. An example is the study of the flow of an incompressible fluid in a Hele-Shaw cell; in this case c is identically zero. This problem has been numerically studied by Crowley, who moreover has proved uniqueness also in presence of a nonlinear internal source (see [2]).

Niezgódka has proved existence, uniqueness and stability of the solution and has studied some related control problems in the case of $c \geq 0$ (see [3]). In [4] Pawłow obtains similar results using Duvaut's method of transformation by time integration.

In [5] the Author dealt with a jump condition more general then (I.3) and with $c(\tilde{u})$ possibly vanishing, proving some existence results for the weak formulation of the two-phase problem in several space dimension.

Here we take into account the case of both $c(\tilde{u})$ and $k(\tilde{u})$ possibly vanishing; we prove existence and uniqueness results for a weak formulation and given an error estimate for the time-discretization approximation.

II WEAK FORMULATION

Let χ denote the characteristic function of Ω, i.e.

$$\chi \equiv \begin{cases} 1 & \text{in } \Omega \\ 0 & \text{in } Q \backslash \Omega \end{cases} \tag{II.1}$$

By a technique due to Baiocchi (see [1]), one can check that (I.1),...(I.3) are equivalent to

$$c(\tilde{u}) \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \chi}{\partial t} - \bar{\nabla} \cdot [\bar{k}(\tilde{u}) \bar{\nabla} \tilde{u}] = f \qquad \text{in } \mathcal{D}'(Q) \; . \tag{II.2}$$

Setting

$$C(\xi) \equiv \int_0^\xi c(\eta)\, d\eta, \; K(\xi) \equiv \int_0^\xi k(\eta)\, d\eta \;, \quad \forall \xi \in \mathbb{R} \tag{II.3}$$

(II.2) becomes

$$\frac{\partial}{\partial t}[C(\tilde{u}) + \chi] - \Delta K(\tilde{u}) = f \qquad \text{in } \mathcal{D}'(Q) \; . \tag{II.4}$$

Denote by H the graph of Heaviside:

$$H(\xi) \equiv \begin{cases} \{0\} & \text{if } \xi < 0 \\ [0,1] & \text{if } \xi = 0 \\ \{1\} & \text{if } \xi > 0 \end{cases} \quad ; \tag{II.5}$$

by (II.1) we have

$$\chi \in H(\tilde{u}) \quad \text{a.e. in } Q. \tag{II.6}$$

It is natural to consider an initial condition of the type

$$[C(\tilde{u}) + \chi]_{t=0} = z^0 \quad \text{in } D \times \{0\} \tag{II.7}$$

with z^0 datum, and as boundary conditions

$$k(\tilde{u}) \frac{\partial \tilde{u}}{\partial \bar{\nu}} = -g \qquad \text{on } \Sigma_1 \tag{II.8}$$

($\frac{\partial}{\partial \nu}$: outward normal derivative) and

$$\tilde{u} = \tilde{w} \qquad \text{on } \Sigma_2 \tag{II.9}$$

with $g, \tilde{w}$ known functions; we may assume $\tilde{w}$ to be defined in the whole Q.

As $k \geq 0$, in general K^{-1} (cf. (II.3)) is not a well-defined function, but only a (maximal monotone) graph. Therefore also $CoK^{-1}+H$ is a maximal monotone graph (possibly non-increasing), so that there exists a proper, l.s.c., convex function $\Phi : \mathbb{R} \to \mathbb{R}$ such that

$$CoK^{-1} + H = \partial\phi \qquad \text{in } \mathbb{R} \tag{II.10}$$

we set $\mathrm{Dom}(\Phi) \equiv \{\xi\, \mathbb{R} \,|\, \Phi(\xi) < +\infty\}$.

We transform the unknown function, setting

$$u \equiv K(\tilde{u}) \quad \text{in } Q. \tag{II.11}$$

Set $w \equiv K(\tilde{w})$ a.e. in Q.

Let Γ_1, Γ_2 be "smooth", with Γ_2 of non-vanishing measure.

Set $V \equiv \{v \in H^1(D) \,|\, \gamma_0 v = 0 \text{ on } \Gamma_2\}$ (γ_0 : trace operator), Hilbert space with the norm $\|v\|_V \equiv (\int_D |\bar{\nabla} v|^2 dx)^{\frac{1}{2}}$. Assume that

$$f \in L^2(0,T;V'), g \in L^2(0,T;H^{\frac{1}{2}}_{00}(\Gamma_1))'), \; w \in L^2(0,T;H^1(D)), z^0 \in L^2(D). \tag{II.12}$$

Set

$$A : V \to V' : {}_{V'}\langle Au,v \rangle_V \equiv \int_D \bar{\nabla} u . \bar{\nabla} u dx, \quad \forall u,v \in V \tag{II.13}$$

$$F \in L^2(0,T;V') : {}_{V'}\langle F(t),v \rangle_V \equiv {}_{V'}\langle f(t),v \rangle_V + {}_{(H^{\frac{1}{2}}_{00}(\Gamma_1))'}\langle g(t), \gamma_0 v \rangle_{H^{\frac{1}{2}}_{00}(\Gamma_1)}$$

$$\forall v \in V, \quad \text{a.e. in }]0,T[\tag{II.14}$$

We introduce a first variational problem

(P1) Find (u,z) such that

$$u-w' \in L^2(0,T;V), \quad z \in H^1(0,T;V') \tag{II.15}$$

$$z \text{ is measurable in } Q, \quad z \in \partial\Phi(u) \quad \text{a.e. in } Q \tag{II.16}$$

$$\frac{\partial z}{\partial t} + Au = F \quad \text{in } V', \text{ a.e. in }]0,T[\tag{II.17}$$

$$z(0)=z^0 \qquad \text{in } V'. \tag{II.18}$$

III A FIRST EXISTENCE RESULT

The inverse of the maximal monotone graph $\partial\Phi$ is still a maximal monotone graph; therefore there exists a proper, l.s.c., convex function $\Psi : \mathbb{R} \to \mathbb{R}$ such that

$$\partial\Psi = (\partial\Phi)^{-1} \quad \text{in } \mathbb{R}. \tag{III.1}$$

Theorem 1 Assume that (II.12),(II.14) hold and that

$$\left.\begin{array}{l} \text{there exist two positive constants } X, \Lambda_1 \text{ such that} \\ \forall\xi \in \mathbb{R} \text{ with } |\xi| \geq X_1, \ \forall\zeta \in \partial\Phi(\xi), \quad |\zeta| \leq \Lambda_1 |\xi| \end{array}\right\} \tag{III.2}$$

(this implies $Dom(\Phi) = \mathbb{R}$)

$$\left.\begin{array}{l} \text{there exist two positive constants } X_2, \Lambda_2 \text{ such that} \\ \forall\xi \in \mathbb{R} \text{ with } |\xi| \geq X_2, \ \forall\zeta \in \partial\Phi(\xi), \quad |\zeta| \geq \Lambda_2 |\xi| \end{array}\right\} \tag{III.3}$$

(this implies $Dom(\Psi) = \mathbb{R}$

$$w \in W^{1,1}(0,T;L^2(D)) \cap L^2(0,T;H^1(D)) \tag{III.4}$$

$$\Psi(z^0) \in L^1(D). \tag{III.5}$$

Then problem (P1) has at least one solution such that moreover

$$z \in L^\infty(0,T;L^2(D)). \tag{III.6}$$

Proof 1) Approximation (by time discretization)

Let $m \in N$, $k \equiv \frac{T}{m}$. Set $w_m^n(x) \equiv w(x,nk)$ a.e. in D, $F_m^n \equiv \frac{1}{k}\int_{(n-1)k}^{nk} F(\tau)d\tau$ in V', for $n = 0,\dots,m$.

$\underline{(P1)}_m$ Find $\{(u_m^n, z_m^n)\}_{n=1,\dots,m}$ such that for $n = 1,\dots,m$

$$u_m^n - w_m^n \in V, \quad z_m^n \in L^2(D) \tag{III.7}$$

$$z_m^n \in \partial\Phi(u_m^n) \quad \text{a.e. in } D \tag{III.8}$$

$$\frac{z_m^n - z_m^{n-1}}{k} + Au_m^n = F_m^u \quad \text{in } V' (z_m^o \equiv z^o). \tag{III.9}$$

$\forall m$, this problem has one and only one solution, which can be constructed step by step. With this aim we introduce recursively the following strictly convex, l.s.c., coercive functional

$$J_m^n : V \to \mathbb{R} : v \to \int_D \Phi(v+w_m^n)\,dx - {}_{V'}\langle z_m^{n-1}, v\rangle_V + \frac{k}{2}\|v+w_m^n\|_V^2 - {}_{V'}\langle F_m^n, v\rangle_V \tag{III.10}$$

where z_m^{n-1} is assumed to be known by the preceding step.

Let $\hat{u}_m$ be the minimizing argument of J_m^n (existing and unique); hence

$$0 \in \partial\Phi(\hat{u}_m^n + w_m^n) - z_m^{n-1} + kA(\hat{u}_m^n + w_m^n) - F_m^n \quad \text{in } V' \tag{III.11}$$

Set

$$\left.\begin{aligned} u_m^n &\equiv \hat{u}_m^n + w_m^n && \text{a.e. in } D \\ z_m^n &\equiv z_m^{n-1} - kAu_m^n + F_m^n && \text{in } V' \;; \end{aligned}\right\} \tag{III.12}$$

(III.11) and (III.12) yield

$$z_m^n \in \partial\Phi(u_m^n) \qquad \text{in } V', \tag{III.13}$$

whence, by (III.2),

$$z_m^n \in L^2(Q). \tag{III.14}$$

2) A priori estimates

Let $\ell \in \{1,\dots,m\}$. Multiply (III.9) against $k(u_m^n - w_m^n) \in V$ and sum for $n=1,\dots,\ell$. Notice that

$$\sum_{1}^{\ell}{}_n \int_D (z_m^n - z_m^{n-1})(u_m^n - w_m^n)\,dx \geq \text{(by (III.8) and (III.1))}$$

$$\geq \sum_{1}^{\ell}{}_n \int_D [\Psi(z_m^n) - \Psi(z_m^{n-1})]\,dx + \int_D (-z_m^\ell w_m^\ell + z^o w_m^1)\,dx +$$

$$+ \sum_{2}^{\ell}{}_n \int_D z_m^{n-1}(w_m^n - w_m^{n-1})\,dx \geq \int_D [\Psi(z_m^n) - \Psi(z^o)]\,dx - \| z_m^\ell \|_{L^2(D)} \cdot \| w_m^\ell \|_{L^2(D)} - \tag{III.15}$$

$$- \| z^o \|_{L^2(D)} \| w_m^1 \|_{L^2(D)} - \max_{n=1,\dots\ell-1} \| z_m^n \|_{L^2(D)} \cdot \| w \|_{W^{1,1}(O,T;L^2(D))}$$

$$k \sum_{1}^{\ell}{}_n \int_D \bar{\nabla} u_m^n . \bar{\nabla}(u_m^n - w_m^n)\,dx \geq k \sum_{1}^{\ell}{}_n \| \bar{\nabla} u_m^n \|^2_{L^2(D)^N} -$$

$$- (k \sum_{1}^{\ell}{}_n \| \bar{\nabla} u_m^n \|^2_{L^2(D)^N})^{\frac{1}{2}} . (k \sum_{1}^{\ell}{}_n \| \bar{\nabla} w_m^n \|^2_{L^2(D)^N})^{\frac{1}{2}} \tag{III.16}$$

$$k \sum_{1}^{\ell}{}_n \langle_{V'} F_m^n, u_m^n - w_m^n \rangle_V \leq \| F \|_{L^2(O,T;V')} (k \sum_{1}^{\ell}{}_n \| u_m^n - w_m^n \|^2_V)^{\frac{1}{2}}. \tag{III.17}$$

(III.3) entails that there exist two positive constants α, β such that $\Psi(\xi) \geq \alpha|\xi|^2 - \beta$, $\forall\, \xi \in \mathbb{R}$. Then, using also Gronwall's lemma, by (III.15),..., (III.17) we get (denoting by C various positive constants independent of m)

$$\frac{T}{m} \sum_{1}^{m}{}_n \| u_m^n \|^2_{H^1(D)} \leq C, \tag{III.18}$$

$$\max_{n=0,\dots m} \| z_m^n \|_{L^2(D)} \leq C; \tag{III.19}$$

by (III.18) and (III.9) we have

$$\frac{T}{m}\sum_{1}^{m}{}_{n}\left\|\frac{z_m^n-z_m^{n-1}}{k}\right\|_{V'}\leq C. \tag{III.20}$$

Now denote by z_m the function obtained interpolating

$\{z_m(x,\frac{n}{m}T)\equiv z_m^n(x)\}_{n=0,\ldots,m}$ linearly in $[0,T]$, a.e. in D.

Set $u_m(x,t)\equiv u_m^n(w)$, $w_m(x,t)\equiv w_m^n(x)$ a.e. in D and $F_m(t)\equiv F_m^n$ in V' if $\frac{n-1}{m}T<t\leq\frac{n}{m}T$, for $n=1,\ldots,m$.

Then (III.9) becomes

$$\frac{\partial z_m}{\partial t}+Au_m=F_m \quad \text{in } V', \quad \text{a.e. in }]0,T[\tag{III.21}$$

and (III.18),...,(III.20) yield

$$\|z_m\|_{L^\infty(0,T;L^2(D))\,\cap\, H^1(0,T;V')}\leq C \tag{III.22}$$

$$\|u_m\|_{L^2(0,T;H^1(D))}\leq C. \tag{III.23}$$

3) <u>Limit</u>

There exist z,u such that - possibly taking subsequences -

$$z_m\to z \quad \text{in } L^\infty(0,T;L^2(D)) \text{ weak star, in } H^1(0,T;V') \text{ weak} \tag{III.24}$$

$$u_m\to u \quad \text{in } L^2(0,T;H^1(D)) \text{ weak} \tag{III.25}$$

Taking $m\to\infty$ (i.e. $k\equiv\frac{T}{m}\to 0$) in (III.21), we get (II.17).

Integrating (III.21) in t, we have

$$z_m(t)-z^0+A\int_0^t u_m(\tau)d\tau=\int_0^t F_m(\tau)d\tau \quad \text{in } V', \text{ in }]0,T[; \tag{III.26}$$

multiply this against u_m-w_m and integrate in $]0,T[$; setting

$U_m(x,t)\equiv\int_0^t u_m(x,\tau)d\tau$ a.e. in Q, we get

$$\iint_Q (z_m - z^o).(u_m - w_m)\,dxdt + \iint_Q \bar{\nabla}U_m.\bar{\nabla}(\frac{\partial U_m}{\partial t} - w_m)\,dxdt = \qquad (III.27)$$

$$= \int_o^t {}_{V'}< \int_o^t F_m(\tau)\,d\tau,\ u_m - w_m >_V dt.$$

Analogously, integrating (II.17) w.v.t.t we have

$$z(t) - z^o + A\int_o^t u(\tau)\,d\tau = \int_o^t F(\tau)\,d\tau \quad \text{in } V', \text{ in }]0,T[\ ; \qquad (III.28)$$

multiply this against u-w and integrate in]0,T[; setting $U(x,t) \equiv \int_o^t u(x,\tau)\,d\tau$ a.e. in Q, we get

$$\iint_Q (z - z^o).(u-w)\,dxdt + \iint_Q \bar{\nabla}U.\bar{\nabla}(\frac{\partial U}{\partial t} - w)\,dxdt = \qquad (III.29)$$

$$\int_o^T {}_{V'} < \int_o^t F(\tau)\,d\tau,\ u-w>_V dt.$$

Notice that

$$\lim_{m\to\infty} \iint_Q \bar{\nabla}U_m.\bar{\nabla}\frac{\partial U_m}{\partial t}\,dxdt = \lim_{m\to\infty} \tfrac{1}{2}\,\|U_m(T)\|_V^2 \geq \text{(by III.25)} \qquad (III.30)$$

$$\geq \tfrac{1}{2}\,\|U(T)\|_V^2 = \iint_Q \bar{\nabla}U.\bar{\nabla}\frac{\partial U}{\partial t}\,dxdt.$$

Take the inferior limit as $m\to\infty$ in (III.27); using (III.29) and (III.30) we get

$$\lim_{m\to\infty} \iint_Q z_m.u_m\,dxdt \leq \iint_Q zu\,dxdt. \qquad (III.31)$$

$\forall v \in C^o([0,T];L^2(D))$ such that $\Phi(v) \in L^1(Q)$, setting $v_m^n(x) \equiv v(x,\frac{n}{m}T)$ a.e. in D for $m \in \mathbb{N}$, n=1,...,m, we have

$$\iint_Q [\Phi(u)-\Phi(v)]dxdt \le \text{(by (III.25) and as } \Phi \text{ is l.s.c.)} \tag{III.32}$$

$$\le \varliminf_{m\to\infty} \iint_Q [\Phi(u_m)-\Phi(v)]dxdt = \text{(as } u_m \text{ is constant in}$$

$]\frac{n-1}{m}T, \frac{n}{m}T]$ for $n=1,\ldots,m$ and as v is continuous w.r.t. t)

$$= \varliminf_{m\to\infty} \frac{T}{m} \sum_{1}^{m} {}_n \int_D [\Phi(u_m^n)-\Phi(v_m^n)]dx \le \text{(by (III.8))}$$

$$\le \varliminf_{m\to\infty} \frac{T}{m} \sum_{1}^{m} {}_n \int_D (z_m^n - v_m^n)dx = \varliminf_{m\to\infty} \iint_Q z_m \cdot (u_m - v_m)dxdt \le$$

$$\text{(by (III.31))} \iint_Q z(u-v)dxdt,$$

whence (III.16). □

<u>Theorem 2</u> Assume that (II.12),(II.14),(III.2),(III.5) and

$$\inf \Psi = 0 \tag{III.33}$$

$$w \equiv 0 \quad \text{a.e. in } Q. \tag{III.34}$$

Then problem (P1) has at least one solution such that

$$z \in L^2(Q). \tag{III.35}$$

<u>Proof</u> is similar to that of theorem 2. (III.15) is replaced by

$$\sum_{1}^{\ell} {}_n \int_D (z_m^n - z_m^{n-1})u_m^n dx \ge \int_D [\Psi(z_m^n)-\Psi(z^0)]dx \ge -C, \tag{III.36}$$

therefore here we get only (III.18) and not (III.19);(III.18) and (III.2) yield

$$\frac{T}{m} \sum_{1}^{m} {}_n \|z_m^n\|^2_{L^2(D)} \le C. \ \square \tag{III.37}$$

IV ANOTHER EXISTENCE RESULT

Due to (III.2), theorem 2 cannot be applied if $\partial\Phi$ has a vertical asymptote, i.e. $\mathrm{Dom}(\Phi) \neq \mathbb{R}$. Now we deal with this situation introducing the simplifying assumption that $w \equiv 0$ a.e. in Q.

At first we reformulate the problem in a weaker form

(P2) Find (u,z) such that

$$u \in L^2(0,T;V), \quad z \in H^1(0,T;V') \tag{IV.1}$$

$$z \in \partial\Phi(u) \quad \text{in } V' \tag{IV.2}$$

in the sense that

$$\left.\begin{aligned} &\forall v \in V \text{ such that } v(x) \in \mathrm{Dom}(\Phi) \text{ a.e. in } D \text{ and } \Phi(v) \in L^1(D), \\ &\int_D [\Phi(u)-\Phi(v)]dx \le {}_{V'}\langle z,u-v\rangle_V, \quad \text{a.e. in }]0,T[\end{aligned}\right\} \tag{IV.3}$$

$$\frac{\partial z}{\partial t} + Au = F \quad \text{in } V', \text{ a.e. in }]0,T[\tag{IV.4}$$

$$z(0) = z^0 \quad \text{in } V'. \tag{IV.5}$$

Theorem 3 Assume that (II.14),(II.16),(III.3),(III.5) hold and that

$$\mathrm{Dom}(\Phi) \text{ is bounded} \tag{IV.6}$$

$$\inf \psi > -\infty. \tag{IV.7}$$

Then problem (P2) has at least one solution.

Proof is similar to that of theorem 1; of course, with $z^n_m \in V'$ and $z^n_m \in \partial\Phi(u^n_m)$ in V' in the time-discretized problem.□

V UNIQUENESS

Theorem 4 Problem (P1) has at most one solution.

Proof Let (u_i,z_i) be two solutions. Take the difference between (II.17) written for (u_i,z_i) $(i=1,2)$, integrate w.v.t. t, multiply against u_1-u_2 and integrate w.v.t. t once more; setting

$U_i(x,t) \equiv \int_0^t u_i(x,\tau)\,d\tau$ $(i=1,2)$ we get

$$\int_0^t d\tau \int_D (z_1-z_2)(u_1-u_2)\,dx + \int_0^t d\tau \int_D \bar{\nabla}(U_1-U_2)\cdot\bar{\nabla}\frac{\partial}{\partial t}(U_1-U_2)\,dx = 0, \text{ for } 0<t<T. \quad \text{(V.1)}$$

By the monotonicity of $\partial\Phi$, the first addendum is non-negative; the second one is equal to $\frac{1}{2}\|U_1(t)-U_2(t)\|_V^2 \geq 0$. Therefore $U_1 = U_2$ a.e. in Q, whence $u_1 = u_2$ a.e. in Q; then by (II.17) $\frac{\partial z_1}{\partial t} = \frac{\partial z_2}{\partial t}$ in V' a.e. in $]0,T[$, whence $z_1 = z_2$ a.e. in Q. □

Uniqueness can be proved also for (P2) in the same way.

VI ORDER OF CONVERGENCE ESTIMATE

We present an estimate of the order of convergence of the solutions of the time-discretized problems.

Theorem 5 Assume that the assumptions of theorem 1 hold (so that (P1) has a unique solution (u,z)). Let $Aw=0$ in V', a.e. in $]0,T[$ (this last is not restrictive, as only the trace of w has relevance) and

$$w \in H^1(0,T;L^2(D)) \cap W^{1,1}(0,T;H^1(D)) \quad \text{(VI.1)}$$

$$F \in W^{1,1}(0,T;V'). \quad \text{(VI.2)}$$

$\forall m \in \mathbb{N}$, let (u_m,z_m) be defined as in the proof of theorem 1 and let $Aw_m=0$ in V' a.e. in $]0,T[$. Then

$$\|z-z_m\|_{L^\infty(0,T;V')} = \|\bar{\nabla}\int_0^t (u-u_m)(x,\tau)\,d\tau\|_{L^\infty(0,T;L^2(D)^N)} = o(k^{\frac{1}{2}}) (k \equiv \frac{T}{m}). \quad \text{(VI.3)}$$

If moreover there exists a constant $L > 0$ such that

$$\forall \xi_i \in \mathbb{R},\ \forall \zeta_i \in \partial\Phi(\xi_i)\ (i=1,2),\ (\zeta_1-\zeta_2)\cdot(\xi_1-\xi_2) \geq L|\xi_1-\xi_2|^2 \quad \text{(VI.4)}$$

then

$$\|u-u\|_{L^2(Q)} = o((\tfrac{k}{L})^{\frac{1}{2}}). \square \quad \text{(VI.5)}$$

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Control of parabolic systems involving free boundaries

1. INTRODUCTION

The purpose of this paper is to offer a possibly up-to-date information concerning recent advances in the methods of analysis and solving various control problems for parabolic systems involving free boundaries. In particular the authors have taken advantage of the activities of the special discussion group arranged during the conference in Montecatini, including the contributions of Baumeister, Jochum, Matzeu and Saguez. Such contributions (with the exception of Baumeister's paper appearing in this same volume) are reported as appendixes to this paper.

The scope of this paper has been restricted to the class of Stefan type processes arising in mathematical models of phase-change phenomena, as well as their generalizations. The content of the paper will cover two principal approaches within the deterministic framework, based on the use of:

- techniques of optimal control theory,
- inverse formulations of the boundary value problems.

2. PHYSICAL AND TECHNOLOGICAL APPLICATIONS

The origin of many control problems for phase-change processes is clear in view of the discussion of the practical aspects in particular given in [W1,R1,B10-11,F2,G5,H1,L1,L7,M2,O1,S6,W3]. We shall confine ourselves to mentioning two characteristic situations, related to technological processes of continuous casting [B10,F2,S6-8] and electro-chemical machining (see the papers of McGeough in [A1] and in this volume).

The problem of casting consists of two principal stages:

1° primary cooling: molten metal is poured into a mould to be cooled so that to form a desired shape. This stage lasts until forming a solid shell capable of protecting taken on shape of the cast;

2° secondary cooling: the partially solidified cast undergoes further cooling until achieving the complete freezing.

The following technological objectives are of principal importance for this problem:

- to achieve a required thickness of the slab shell within a given time interval (stage 1^o),
- to achieve the complete solidification in a minimal time provided an imposed velocity of extracting, and next to optimize this velocity (stage 2^o),
- to assure a desired dynamics of the heat transfer within the solid phase in order to provide required mechanical properties of the resulting product (both stages).

Control actions in both stages are realized at the fixed boundary of the solid phase (via the mould, for the first stage).

Mathematical models of the processes of heat transfer at both stages take the form of Stefan type problems. For stage 1^o the model involves one phase, corresponding to the solid, whereas temperature of the liquid is usually assumed to be equal to that of the phase transition [B11,S6-8]. The last assumption is no more justified for stage 2^o in view of the occurrence of large temperature gradients in the liquid [F2,S6-8], therefore the corresponding model involves two phases.

The typical form of modelling the boundary control action consists in using Newton type conditions

$$\frac{\partial\theta}{\partial\nu} + h\,(\theta - u) = 0 \quad \text{at a given subset of the lateral boundary}$$

where either the external temperature u(x,t) (specified by means of the acting heating power) [B10-11] or the heat exchange coefficient h(t) [S6-8] as the control variables.

For some more details concerning optimal control of continuous casting processes we refer to the abstract of the contribution of Saguez (Appendix 1).

We should underline that everything said up to now is related to single-component alloys; in the case of multi-component alloys not only control but also modelling of the processes become incomparably more complicated (see the papers on alloy solidification in the same volume and the paper of Fix in [W3]).

There are some situations when the process of casting does not produce satisfactory results, e.g. providing not enough homogeneous structure or smooth surface of the final product. Then one of the possible solutions consists in applying processes of electrochemical machining (see the related papers in the same volume as well as McGeough's paper in [A1]).

In the last processes a final shape of anode is formed according to the given shape of cathode. As the control variable one usually takes the potential of the anode, treated as a function of time. Typical objectives of control can be expressed in the form of:

- assuring a desired evolution of the surface of the anode in time,
- assuring a satisfactory reconstruction of the shape of the cathode by that of the anode in a minimal time.

There is also an important optimum design problem related to electrochemical machining, consisting in determining such shape of the cathode which makes it possible to produce accurately the desired shape of the anode (of special importance in more complicated geometrical configurations). Problems of such type were studied by many authors [C4,B2].

In applications the performance indexes of controlled systems usually include the requirement of choosing a solution of minimal energy [B9-10].

3. MATHEMATICAL MODELS OF THE CONTROLLED PROCESSES

The processes under consideration give rise to various mathematical models involving free (moving) boundaries. For definiteness, we shall confine ourselves further to the models in the form of Stefan type problems involving one or more phases. In order to have a reference point, we recall first typical classical formulations of the problems in terms of the dependent variable θ corresponding as a rule either to temperature or to concentration of one of the components [R1].

One-phase Stefan problem:

$$c(\theta)\ \theta' - \nabla(k(\theta)\nabla\theta) = f(\theta,\nabla\theta) \quad \text{in} \quad Q^-, \tag{1}$$

$$\theta\big|_{\mathcal{S}} = \theta^* \quad \text{for} \quad t \in (0,T], \tag{2}$$

$$k(\theta)\ \nabla\theta\big|_{\mathcal{S}} \circ \vec{N}_x = - L\, N_t \quad \text{for} \quad t \in (0,T], \tag{3}$$

$$\Omega^-(0) \quad \text{given,} \quad \theta(0) = \theta_o \quad \text{in} \quad \Omega^-(0), \tag{4}$$

$$B(\theta) = C(u) \quad \text{on} \quad \Sigma = \Gamma \times (0,T)\text{-fixed part of the lateral boundary.} \tag{5}$$

Two-phase Stefan problem:

$$c(\theta)\ \theta' - \nabla(k(\theta)\nabla\theta) = f(\theta,\nabla\theta) \quad \text{in} \quad Q^- \cup Q^+, \tag{6}$$

$$\theta\big|_{\mathcal{S}^-} = \theta\big|_{\mathcal{S}^+} = \theta^* \quad \text{for} \quad t \in (0,T] \tag{7}$$

$$[k(\theta)\nabla\theta]\Big|_{\mathcal{S}^-}^{\mathcal{S}^+} \cdot \vec{N}_x = L\, N_t \quad \text{for} \quad t \in (0,T] \tag{8}$$

$$\theta(0) = \theta_o \quad \text{in} \quad \Omega^-(0) \cup \Omega^+(0), \text{ with } \Omega^-(0), \Omega^+(0) \text{ given,} \tag{9}$$

$$B(\theta) = C(u) \quad \text{on } \Sigma = \Gamma \times (0,T) \text{ - fixed lateral boundary.} \tag{10}$$

In the above formulations Q^- and Q^+ correspond to the domains occupied by solid and liquid, respectively; $\Omega^-(0)$, $\Omega^+(0)$ are the corresponding domains at t=0; $\mathcal{S}$ denotes the free boundary, $\vec{N} = (\vec{N}_x, N_t)$ is vector normal to $\mathcal{S}$. We omit here hypotheses imposed on the data $c, k, f, \theta^*, L, \theta_o$, referring to special papers devoted to the analysis of the Stefan problems [C2,F1,N3,P1] and to [R1,O1,W3] for details.

The nonlinear boundary operators $B(\theta)$ are assumed to be admissible for equations (1),(6) in the sense of [L2,L6], in particular taking the mixed form

$$B(\theta) = \begin{cases} h(\theta)\dfrac{\partial\theta}{\partial\nu} + g(\theta) \quad \text{on} \quad \Sigma_N \quad \text{(Newton or Neumann)} \\ \\ \theta \quad \text{on} \quad \Sigma_D = \Sigma \setminus \Sigma_N \qquad \text{Dirichlet} \end{cases} \tag{11}$$

where $\vec{\nu}$ is unit vector normal to boundary of the geometric domain.

The parts Σ_N and Σ_D of the fixed lateral boundary are usually considered as cylindrical sets (invariant in time, i.e. admitting representations $\Sigma_N = \Gamma_N \times (0,T)$, $\Sigma_D = \Gamma_D \times (0,T)$. There are however various processes of practical importance where these sets are time-dependent [D1,B13] what corresponds in particular to so-called moving control actions.

The control variable u enclosed in conditions (5),(10) corresponds to prescribing either flux at the boundary or heating power acting there [B10-13]. Let us also note that there are some situations when the exchange coefficient h in the condition (11) is most suitable as control variable [S6-8].

4. APPLICATION OF INVERSE FORMULATIONS OF BOUNDARY VALUE PROBLEMS

There are many physical processes for which the dependence of characteristic parameters upon space and time variables, and especially upon the state of the process cannot be determined in an experimental way. For example, this type of difficulty often arises in high-speed thermal processing of metals [F2].

One of the possible approaches to the study of such processes consists in determining the unknown characteristics by solving certain inverse problems for differential equations.

Having the parameters undergoing determination in view, one can distinguish the problems involving unknown:

(i) coefficients of the parabolic differential operators (c,k),

(ii) distributed source terms of the process equations (f),

(iii) initial data (θ_o,S(O)),

(iv) boundary data (h,g,C(u),Σ_N,Σ_D).

As an additional information one often assumes to know the evolution of the free boundary. Due to the conditions imposed at this boundary, such an assumption results in over-determination of the problem that can lead to the non-existence of solutions [I1,T1]. Problems of such type arise often in control of systems with the purpose of approaching a prescribed target, in particular in connection with structural questions such as controllability and reachability [S9].

4.1. Basic mathematical properties

Inverse Stefan type problems were studied mostly in the case of one space dimension, with only a few generalizations onto the multidimensional situations [G4,M4].

First let us recall briefly basic features of the one-dimensional framework:

- possibility of exploiting the classical formulations of the direct Stefan type problems and in particular results on regularity of the free boundary [C2,F1,R1],
- availability of explicit expressions defining the motion of the free boundary in terms of the data [R1],
- easy way to introducing an equivalent integral representation of the problem in the form of a system of Volterra integral equations with weakly singular kernels involving appropriately Green or Neumann functions [R1,W2,N1],
- applicability of the approach based on preliminary immobilization of the domain by using Landau transformation [R1,A1] provided $\mathcal{S}$ is a priori globally separated from the fixed part of the lateral boundary; this approach is

especially useful in numerical solving the problems for its efficiency; but one can use also any fixed domain formulation resulting from introducing a weak solution and perhaps eventually regularizing the problem by smoothing irregular coefficients [L2,R1,N3,S6].

Obviously each of the approaches mentioned above has a limited range of applicability, depending on the form of the conditions imposed at the free boundary and that of the governing equations [R1,F1].

The above remarks concern the direct formulations. As far as the inverse formulations are concerned, there are two basic common features:

- these formulations are linear, provided linear form of the problem in each of the phases separately and assuming evolution of the free boundary to be a priori known [R1,N1],
- non-correctness of these formulation in the Hadamard's sense, i.e. non-existence, non-uniqueness or eventually instability of the solutions [T1,N1].

The last feature is especially well-seen when considering the corresponding integral representations which in the case of problems (ii)÷(iv) take form of systems of Volterra equations of the first kind, in view of the properties of the kernels non-transformable into any equations of the second kind [R1,N1,T1]. In this connection one can easily see the importance of various regularization techniques like the methods of Tichonov and Ivanov type [T1,I1], those of approximation theory [C1,W2,H1-3,J1-5] or the quasi-reversibility [L3]. These techniques assure stabilization of the problems and provide constructive procedures of determining the corresponding solutions (understood in appropriate generalized senses). Let us also underline the applicability of the techniques in the situations of only approximate knowledge of data (resulting for example from measurements or discretization).

4.2. Problems involving unknown boundary data

Within this class we can distinguish two types of problems according to the localization of the data to be determined:

(i) at the fixed part of the lateral boundary,

(ii) at the free boundary.

In what follows the evolution of the free boundary is assumed to be a priori prescribed, this hypothesis resulting in the non-correctness of the problems in the Hadamard's sense.

Problems of the first group were studied both in the one- [A1,B3,B5,B8, C1,G2-4,H1-2,J1-5,L3,M4,W1-2] and multi-phase [B4,N1,K4] cases. We are going to present here a number of most typical formulations in the one-dimensional situation, first exploiting various regularization techniques in the spirit of Ivanov [I1] and Tichonov [T1], next using methods of approximation theory.

Problem I G3 (one-phase): Assume that $x = s(t)$, $t \in (0,T)$ is a parametrization of the free boundary (assumed to be prescribed) and the problem is considered after applying the Landau transformation $x/s(t) \to x$, i.e. over fixed domain $[0,1]$ instead of $[0,s(t)]$. The problem consists in finding a pair $\{u(t),\theta(x,t)\}$ satisfying

$$c(\theta)\ \theta' - L_s(\theta) = f(\theta) \quad \text{in} \quad (0,1) \times (0,T) \tag{12}$$

$$\theta(x,0) = \theta_o(x) \qquad \text{in} \quad (0,1) \tag{13}$$

$$\theta(0,t) = u(t), \quad \left[l_s(\theta)\frac{\partial\theta}{\partial x} + \phi(\theta)\right]\Big|_{x=1} = 0 \quad \text{for } t \in (0,T), \tag{14}$$

$$\theta(1,t) = g(t) \quad \text{for} \quad t \in (0,T) \tag{15}$$

with a quasilinear uniformly elliptic operator L_s corresponding to s.

This over-determined problem is formally equivalent to the operator equation

$$P(u) = g, \quad u \in U \triangleq \{u \in H^2(0,T)\ u(0) = \theta_o(0),\ u'(0) = \quad [L_s(\theta_o)](0,0) \tag{16}$$

with P being the nonlinear operator mapping u into the trace at $x = 1$ of the corresponding solution of the system (12)-(14).

Since this problem is non-well posed, one needs to generalize the concept of solution. One can do it by introducing notion of quasi-solution understood as a pair $\{u_R,\theta_R\}$ consisting of any

$$u_R \in \arg\{\inf J_g(u) \mid u \in U_R\} \quad \text{where} \tag{17}$$

$$J_g(u) \triangleq \|P(u) - g\|_{L^2(0,T)}, \quad U_R \triangleq \{u \in U \mid \|u\|_{H^2(0,T)} \le R\},\ R=\text{const.} \tag{18}$$

and θ_R being the corresponding solution of problem (12)-(14).

In order to face more real situations, the author of [G3] admits only approximate knowledge of g,P,U (with an evaluation of the accuracy). The correctness of the problem (17)-(18) is shown for every R, including a detailed study of the stability aspects.

Let us underline that the results of [G3] have been next extended in [G4] onto n-dimensional problems. In the last case Sobolev space $H^2(0,T)$ is replaced in the definitions of the sets U, U_R by $H^{m,m/2}(\Sigma_o)$, $m = 3+(n-1)/2$ where Σ_o is the part of lateral boundary corresponding to u.

Problem II [B8,G2,H1-2] (one-phase): Determine a pair $\{u(t),\theta(x,t)\}$:

$$\theta' - L\,\theta = f \quad \text{in } (0,s(t)) \times (0,T), \tag{19}$$

$$\theta(x,0) = \theta_o(x) \quad \text{in } [0,s(0)], \tag{20}$$

$$\theta(s(t),t) = \theta^*(s(t),t) \quad \text{for } t \in (0,T] \tag{21}$$

$$l(s(t),t)\,\frac{\partial o}{\partial x}\,(s(t),t) = -\varkappa(s(t),t)s'(t) + g(s(t),t),\ t \in (0,T] \tag{22}$$

$$h(t)\,\frac{\partial\theta}{\partial x}\,(0,t) = u(t) \quad \text{for} \quad t \in (0,T] \tag{23}$$

with a prescribed $s(t)$, $t \in [0,T]$ and a linear uniformly elliptic operator L.

The quasi-solution to this problem is defined as a set of elements minimizing functional

$$J(u) \triangleq \|\bar\theta - \bar{\bar\theta}\|_{L^2(Q)}, \quad Q = \bigcup_{t\in[0,T]} [0,s(t)] \times \{t\} \tag{24}$$

over the sets $U_R \triangleq \{u \in H^1[0,T] \mid u(0)=h(0)\theta_o(0),\ \|u\|_{H^1} \le R\}$, $0 < R < \infty$, where $\bar\theta$ is the solution of problem (19)-(21),(23) and $\bar{\bar\theta}$ - that of problem (19),(20),(22),(23).

In [B8] the functional J(u) is shown to be continuous in $H^1[0,T]$ and F-differentiable in $L^2[0,T]$, with the gradient being Lipschitz continuous in $L^2[0,T]$, with the gradient being Lipschitz continuous in $L^2[0,T]$. The problems associated to the construction of quasi-solutions are discussed both in the case of accurate and inaccurate data. These results are extended in [G2] onto quasilinear situation.

In [H1-2] the related numerical aspects are studied with using gradient algorithms and computational results are presented.

Problem III [N1] (two-phase) : Consider system (6)-(10) in one-dimensional case, with a specified free boundary $x = s(t)$, $t \in [0,T]$ or terminal state

$\theta(x,T)$, $x \in [0,1]$, alternatively. The boundary condition (10), with operator B in general of mixed Dirichlet/Neumann form, is splitted as follows:

$$B(\theta)\Big|_{x=0} = u_o \quad \text{in} \quad (0,T], \tag{25}$$

$$B(\theta)\Big|_{x=1} = u_1 \quad \text{in} \quad (0,T] \tag{26}$$

with either u_o or u_1 unknown.

The problem requires determining $\{u_i(t),\theta(x,t)\}$ satisfying the system (6)-(9),(25),(26) including as the additional information either s(t) or $\theta(x,T)$ prescribed. The problem is given an equivalent integral representation in the form of a system of Volterra or respectively Fredholm equations of the first kind with respect to u_i. In view of the non-correctness of the problem the use of Tichonov's regularization procedure is suggested leading to certain minimization problems whose solutions (quasi-solutions to the original problem) exist and are stable with respect to perturbations of the data.

Problem IV [C1,W2,J1-5,B4,H3] (one-phase) : Find u(t) (and $\Theta(x,t)$ corresponding to it) assuring a desired evolution $x = s(t)$, $t \in (0,T]$ of the free boundary, such that

$$\theta' - \theta_{xx} = 0 \quad \text{in} \quad (0,s(t)) \quad \text{for} \quad t \in (0,T) \tag{27}$$

$$\theta(x,0) = {}_o(x) \quad \text{in } [0,s(0)] \tag{28}$$

$$\theta(s(t),t) = 0, \quad \theta_x(s(t),t) = -s'(t) \quad \text{for} \quad t \in (0,T] \tag{29}$$

$$\theta_x(0,\ t) = u(t) \quad \text{for} \quad t \in (0,T] \tag{30}$$

For overcoming the non-correctness, the problem is reformulated as follows:

Find $\hat{u} \in \arg\inf \ \{\|S(u) - s\| \ | \ u \in U_N^-\}$

where $S(\cdot)$ denotes the operator transforming u into the actual free boundary S(u) corresponding to it (unique), $\|\cdot\|$ is C- or L^p-norm $(1 < p \le \infty)$ introduced in the space of functions continuous over [0,T], U_N^- is the non-positive cone in an N-dimensional subspace of C[0,T].

The existence of such best approximation solution is shown in [J1-2] for

C-norms by using integral representation of system (27)-(30) in the form

$$\tilde{s}(t) = [S(u)](t) \triangleq s(0) - \int_0^t u(\tau)\,d\tau - \int_0^{\tilde{s}(t)} \theta(x,t)\,dx + \int_0^{s(0)} \theta_0(x)\,dx.$$

The operator $S(\cdot): A \to B$; $B \triangleq \langle C[0,T], \|\cdot\|\rangle$, $A = B^-$ is shown to be Lipschitz continuously F-differentiable.

The best approximation is constructed as an element of N-dimensional Haar spaces (unique when existing). Numerical results obtained by using Gauss-Newton procedures are presented.

The related two-phase problems were studied in [B4,H3,K4], where u_o, u_1 at x=0,1 were determined so that to assure a best approximation to a desired behaviour of the free boundary [H3] or to a specified state at the lateral boundary x=0,1 [B4], in the sense of minimizing the appropriate L^2-norms. Main results of [B4,K4] concerned F-differentiability in C-spaces of the resolution operators from boundary fluxes into the state in each of two phases and the associated free boundary.

In [H3] a Gauss-Newton proceedure was applied for numerical solving the problem, proving its efficiency.

In [J5] an analogous idea is suggested for the study of two-dimensional one-phase problems.

Let us refer also to the papers [C1,W2] where techniques of mathematical programming were applied for the first time to the analysis and numerical solving one-dimensional one-phase inverse Stefan problems of this type, and to [B3], using there Pontriagin maximum principle.

An alternative approach to solving such problems is proposed in the book [L3]. The basic idea of so-called quasi-reversibility techniques developed there consists in introducing singular perturbations of the parabolic problems taking either elliptic or hyperbolic form. Theoretical results are given and results of computational experiments are presented there.

4.3. Other inverse problems

One-dimensional one-phase inverse Stefan problems of recovering unknown initial state were considered in [L3] by means of the quasi-reversibility techniques. Finite-difference approximations were constructed there and results of computations presented.

Another group of inverse Stefan problems is related to the determining of unknown internal characteristics of the process, including distributed source terms and coefficients of parabolic operators.

In [B7] a well-posed inverse problem of determining the coefficient $k \equiv k(t)$ in equation (1) was considered in one-dimensional case. More precisely the problem consisted in finding $\{k(t), s(t), \theta(x,t)\}$ satisfying one-phase problem (1)-(5).

After applying the Landau transformation, convergent successive approximations were constructed proving the existence of a classical solution to the problem. This solution was shown to be an element of $C^1[0,T] \times C^2[0,T] \times (C^{2,1}(Q) \cap C(\bar{Q}))$. Moreover, by introducing an integral representation of the problem in the form of a system of second kind Volterra equations and making use of their known properties [M1], the uniqueness of the solution and its continuous dependence upon perturbations of initial and boundary data (including source terms at the free boundary) were shown and order of accuracy of the successive approximations was evaluated.

The uniqueness of the solution to one-dimensional one-phase problems of simultaneous determining the coefficients $c \equiv c(\theta)$ and $k \equiv k(\theta)$ was proved in [M4].

Let us refer also to [C5] where a problem of identification of the coefficient $k \equiv k(\theta)$ from state measurements was considered by techniques of optimal control theory and solved numerically by gradient methods.

A problem of the identification of the source term appearing in the Stefan condition was discussed in [S6,S8] both from theoretical and numerical points of view.

Let us finish this section by referring to [K5] for a comprehensive survey of results on the identification for distributed parameter systems.

4.4. Comments

(i) In many papers on inverse Stefan problems the evolution of the free boundary is postulated to be a priori prescribed. At the same time, however, the non-correctness of the problems forces to introduce generalized concepts of solutions, understood as a rule as elements minimizing certain functionals. With except of [B3,J1-5] those functionals only indirectly depend upon the accuracy of the reproduction of the desired evolution of the free boundary.

In one-dimensional case a certain approximation is assured due to known stability results [C2,F1,R1,B4,H3,K4]. The question arises what will happen in the case of more space dimensions.

(ii) Since the study of inverse Stefan problems is motivated mostly by identification and control purposes, the chosen notions of generalized solutions (quasi-solutions) should reflect the main objectives for the particular situations considered. In this connection we refer also to the lectures [L5] devoted in general to non-well posed control problems.

5. OPTIMAL CONTROL PROBLEMS

Problems of optimal control for Stefan type processes were studied by several authors. Most of the results available concern existence aspects, making use of the Weierstrass theorem. Necessary conditions of optimality have been constructed by now only for one-phase processes.

There is a division between the one- and multi-dimensional problems as far as the applicable mathematical tools are concerned. This division results from well-known difference of the regularity properties in each of these cases.

One-dimensional problems: In this case one can choose between classical and variational formulations of the governing Stefan type problems (provided the last being applicable).

The first line of study is represented by the papers [V1,B6,D2-3,G1, K1-3,Y1-2,B3] and also appropriate parts of the books [B11,L1,L7]. All of them concern the case of boundary controls, enclosed either within the conditions (5),(10) at the fixed part of the lateral boundary [V1,B3,K1-3, Y1-2] or the corresponding conditions at the free boundary [B6,G1].

Problem V [V1]:

$$\min J(u) \triangleq \alpha \int_0^{s(T)} [\theta(x,T)-\theta_d(X)]^2 dx + \beta \int_0^T [s(t) - s_d(t)]^2 dt + \gamma[s(T) - s_T]^2 \; ; \; \alpha,\beta,\gamma \geq 0, \quad \alpha^2+\beta^2+\gamma^2 > 0 \tag{31}$$

with respect to $u \in L^\infty[0,T]$, $0 \leq \bar{u} \leq u(t) \leq \bar{\bar{u}}$ a.e. in $[0,T]$, where $\{\theta(x,t),s(t)\}$ is the unique solution of the problem

$$\theta' - \theta_{xx} = 0 \text{ in } (0,s(t)) \quad \text{for} \quad t \in (0,T) \tag{32}$$

$$s(0) = s_o > 0 \text{ - given}, \quad \theta(x,0) = \theta_o(x) \quad \text{in } [0,s_o] \tag{33}$$

$$\theta(s(t),t) = 0, \quad s'(t) = -\theta_x(s(t),t) + g(s(t),t), \quad t \in (0,T] \tag{34}$$

$$\theta_x(0,t) = h(\theta(0,t),t)[\theta(0,t) - u(t)], \quad t \in (0,T) \tag{35}$$

with (32)-(34) understood in the classical sense and (35) interpreted in the sense of L^2-traces.

In [V1] the existence of optimal controls was proved by showing L^2-convergence of minimizing sequences $\{u_m\} \subset C[0,T]$, and using Bernstein estimates [L2]. The results of [V1] were next extended in [K1] onto the two-phase situation.

The existence of optimal controls was also proved in the case of controls acting at the free boundary. Such problems were considered in [B6,G1] in quasilinear statement, involving controls $u=u(t)$ assumed to be Lipschitz continuous together with first derivatives bounded in the norm of $C^1[0,T]$, enclosed in the condition

$$\theta(s(t),t) = \theta^*(s(t),t)\phi(u(t)), \quad t \in (0,T)$$

and performance index defined by the functional

$$J(u) \equiv \int_0^T \int_0^{s(t)} F_o(\theta,\theta_x,s,s',u,u')\,dx\,dt + \int_0^T F_1(\theta|_s,\theta_x|_s,s,s',u,u')\,dt$$

with continuous non-negative functions F_o,F_1.

The proofs of the existence of optimal controls given in [B6,G1] were based on applying the method of lines [B7] and exploiting the Bernstein's estimates [L2]. The results of [G1] can be easily extended onto the multi-phase problems.

Using the above results, the authors of [K2-3,Y1-2] derived necessary conditions of optimality. In [K2-3] these conditions, covering one-phase situation (and also relating to two-phase problems provided strict monotonicity of the free boundary), took the form of Pontriagin's maximum principle. Another approach using an integral representation is given in [B3].

In the papers [Y1-2] the problem (31)-(35) was treated in the case $\gamma = 0$ for admissible controls

$$u \in U_{ad} = \{u \in H^2[0,T] \mid 0 \le \bar{u} \le u(t) \le \bar{\bar{u}},\ t \in [0,T];$$

After applying the Landau transformation $x/s(t) \to x$, system (32)-(35) with $h = \text{const.}$ took the form

$$\theta' = \frac{1}{s^2}\,\theta_{xx} + x\,\frac{s'}{s}\,\theta_x \quad \text{in} \quad (0,1) \times (0,T] \tag{36}$$

$$\theta(1,t) = 0, \quad s'(t) = -\frac{1}{s}\,\theta_x(1,t) \quad \text{for} \quad t \in (0,T] \tag{37}$$

$$\theta(x,0) = \theta_o(x\,s_o) \quad \text{in} \quad [0,1], \quad s(0) = s_o > 0, \tag{38}$$

$$\theta_x(0,t) = h\,s(t)\,[\theta(0,t) - u(t)] \text{ for } t \in (0,T]\,. \tag{39}$$

The following problem, adjoint to (36)-(39), was also introduced with the unknown pair $\{\eta(x,t),\Phi(t)\}$:

$$\eta' = -\frac{1}{s^2}\,\eta_{xx} + x\,\frac{s'}{s}\,\eta_x \quad \text{in} \quad (0,1) \times (0,T) \tag{40}$$

$$\eta(1,t) = \int_0^1 x\,\theta_x(x,t)\,\eta(x,t)\,dx + \Phi(t) \quad \text{for} \quad t \in (0,T) \tag{41}$$

$$\eta(x,T) = 2\alpha[\theta(x,T) - \theta_d(x\,s(T))] \quad \text{in } [0,1] \tag{42}$$

$$\Phi'(t) = -2\beta[s(t) - s_d(t)] + \frac{s'}{s}\,\Phi(t) + \frac{\sigma}{s}\,[z(0,t) - u(t)]\eta(0,t) + $$
$$+ 2\int_0^1 \theta'(x,t)\eta(x,t)\,dx \quad \text{for} \quad t \in (0,T) \tag{43}$$

$$\Phi(T) = \alpha \int_0^1 \{[\theta(x,T)-\theta_d(x\,s(T))]^2 - $$
$$- 2\,x\,\theta_d'(x\,s(T))\,[\theta(x,T) - \theta_d(x\,s(T))]\}\,dx \tag{44}$$

$$\eta_x(0,t) = h\,s(t)\,\eta(0,t) \quad \text{for} \quad t \in [0,T)\,. \tag{45}$$

Making use of the problems (36)-(39) and (40)-(45), the author of [Y1-2] has proved that the functional (31) is F-differentiable in $H^1(0,T)$ with $J'(u) = w(\cdot)$, where $w(t)$ denotes solution of the two-point boundary value problem

$$w''(t) - w(t) = -h\,\eta(0,t) \quad \text{for} \quad t \in (0,T)\,; \quad w'(0) = w'(T) = 0.$$

Moreover the F-derivative of $J(u)$ is shown to be Lipschitz continuous as an operator $H^1(0,T) \to H^1(0,T)$, what implies the convergence of the corresponding gradient procedures.

A finite-difference discretization of the problem (31),(36)-(39) is also constructed. This discretization assures the convergence of the minimal values of the approximate functionals to that for $J(u)$. Since, as it has been observed, the problem is non-well posed in the strong topology of $H^2(0,T)$, a Tichonov's regularization method is applied which yields the desired convergence of optimal controls for the discrete problems (strong in $H^2(0,T)$) and allows to evaluate the accuracy of approaching the minimal value of $J(u)$.

A different approach to the study of optimal control problems for one-dimensional processes of the Stefan type was proposed by Saguez [S1-2,S6]. In those papers the problems possibly admitting convection in liquid were reformulated as variational inequalities. The problems studied there were motivated by metallurgical processes of continuous casting of steel. Control variable $h(t)$ appeared in the source term at the free boundary

$$\theta_x(s(t),t) = - L\, s'(t) - h(t)\ \delta T(t), \quad t \in (0,T)$$

where δT corresponded to superheating and h to convection.

The aim of control consisted in assuring maximal velocity of rolling $\hat{V}$ under metallurgical and technological constraints, in particular under the requirement of complete freezing by a given time moment t_f. For such problems, involving typical state observations in L^2-norms, the existence of optimal controls was proved by semi-discretization in time and necessary conditions of optimality were established.

Multi-dimensional problems: In multi-dimensional case all the known results concern the situations when variational formulations of the Stefan problems are applicable (only then global in time results on the existence and uniqueness of solutions are known [L2,L4,N3,S6,W3]).

Usually the control variables are enclosed in the conditions imposed at the fixed parts of the lateral boundary and are taken as elements of certain closed, bounded and convex subsets of L^p-spaces $(1 \le p \le \infty)$.

Typical performance criteria are expressed in the form of integral functionals corresponding to distributed, boundary or terminal state observations (in the norms of L^p-spaces for $1 < p < \infty$), alternatively to time-optimal realizations of the process.

The existence of optimal controls was proved in the one-phase case in [M3,S3,S6] by penalization of variational inequality formulations, and in the two-phase case by regularization [N3] or regularization combined with semi-discretization in t [S6], applied to the enthalpy type formulations.

In a number of works Saguez established necessary conditions of optimality for various problems, including:

- one-phase problems with distributed L^2-observation of the state, regularized by adding quadratic term dependent on control [S5-6,S8],
- one-phase time-optimal problems [S6],
- two-phase problems with distributed L^2-observation of the state [S6,S8],
- extensions of the results onto the problems admitting convection in liquid [S6].

In two first cases the optimality conditions have been obtained by using combined penalization and regularization techniques for variational inequalities, and showing the possibility of an appropriate passing to the limit. In the third case those conditions have been constructed for the problems regularized and semi-discretized in time (by an implicit scheme).

In the paper of Barbu [B2] necessary conditions of optimality were proved for problems with integral objective functionals and processed governed by semilinear parabolic variational inequalities. Those conditions were expressed in the language of Clarke's generalized gradients. Their derivation was based on the analysis of certain auxiliary regularized problems. The results of [B2] are applicable in the case of processes governed by one-phase Stefan problems.

6. FEEDBACK CONTROL

The real-time control of dynamical processes requires constructing a feedback system. In the case of the Stefan type processes, in view of their structural nonlinearity, there are no results concerning construction of a priori feedback in contrast to the linear-quadratic case [B5].

One of the possible ways of overcoming this difficulty relies upon constructing stochastic feedback law with the state estimation based on measurements discrete in time. This idea was implemented in [G5] for one-dimensional one-phase Stefan type problem with source at the free boundary. In that paper an optimal feedback controller was constructed, showing satisfactory coincidence with experimental data for a laboratory casting process.

Let us also mention here that there is a correspondence between free boundary problems and the problems of optimal stopping arising in decision theory [V3]. As an example of the study of such problems can serve the communication of Capuzzo Dolcetta and Matzeu (see Appendix 3).

7. COMMENTS

(i) In most of the real applications the basic objective of control consists in assuring a desired evolution of the free boundary, as a factor strongly influencing on the physical properties of the final product (frozen solid).

Unfortunately, in contrast to one space dimension, in the multi-dimensional case observation of the evolution of the free boundary can be performed, for multi-phase problems, only implicitly. One can observe there the characteristic function of the set where $\theta(x,t) < \theta^*(x,t)$ (or respectively $\theta(x,t) > \theta^*(x,t)$), using L^2-observations [M3,S3,S4,S6-8]. However such an observation is often not satisfactory for applications due to the appearance of "mushy" regions (see the paper of Rubinstein in this volume). At this point we face the well-known difficulties associated with giving any adequate characterization of the free boundary in multi-dimensional case.

(ii) There are many applications where so-called moving control action (i.e. controls having supports varying with time) is required [B12-13]. Such type of actions seems to be especially suitable for the control of processes involving moving (free) boundaries.

Up to now the only results in this direction concern the existence of solutions to Stefan problems with a prescribed but varying in time division of the lateral boundary onto the parts where appropriately Dirichlet and Neumann conditions are imposed [D1].

The study of related control problems is an open area.

(iii) One of the possible approaches to solving Stefan type problems relies upon treating the functions describing evolution of the free boundary as control variables and defining functionals dependent upon the accuracy of satisfying the free boundary conditions, to be minimized.

Such an approach takes its origin from so-called optimum design (or, more precisely, optimum domain) problems [C4,L4,P2].

In special cases this approach can give rise to interesting game theory formulations [B2].

8. SOME FURTHER OPEN QUESTIONS

(i) The study of multi-phase inverse Stefan problems in multi-dimensional cases.

(ii) The study of inverse Stefan problems with possibly degenerating phases (e.g. in the case of disappearance of some phases).

(iii) The study of multi-phase problems of determining the coefficients and distributed source terms, discontinuous at the free boundary.

(iv) Derivation of necessary conditions of optimality for:

- multi-phase parabolic Stefan problems,
- degenerate mixed type (parabolic-elliptic) Stefan problems,
- implicit free boundary problems like those of fast chemical reaction type.

(v) The study of the optimal control problems with more realistic performance indexes (see also Appendix 1).

(vi) Analysis of the structural aspects of control like controllability, observability and stabilizability (see also [W1]) both in the case of continuous and discrete measurements and control actions.

(vii) Construction of feedback control laws.

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Appendix I

OPTIMAL CONTROL PROBLEMS OF A CONTINUOUS CASTING PROCESS by C. Saguez

The physical problem

To find the best water-spray system such that the following metallurgical constraints are verified:

- to avoid break-out below the mould,
- to assure a complete solidification before the cutting torch,
- to obtain an admissible temperature at the unbending point,
- to avoid the formation of cracks,
- to limit the screep.

Taking into account the limitation of each spray-nozzle and the bound of the global quantity of water.

The mathematical formulation

The problem is formulated as an optimal control problem.

- The state is the temperature and the enthalpy of the system, solution of a two-phase Stefan problem.
- The control variable is the exchange coefficient between water and steel, appearing in the boundary conditions as follows:

$$\lambda(T)\ \overrightarrow{\text{grad}}\ T.\vec{n} = -h(T-T_e)$$

  h is constant on each zone of the spray-system

- The functional is a sum of penalization function associated with each metallurgical constraint.
- The admissible set of controls is the set of exchange coefficients such that the technical constraints are verified.

  Then we have an optimal control problem of a free-boundary problem.

Numerical methods

The problem is solved by a gradient method with projection. The state is computed by an iterative algorithm using the Yosida approximation. With this method we obtain an efficient algorithm, which is necessary due to the important dimensions of the problems. Two softwares have been written, when the speed V of extraction is constant in one (resp. two) dimensional space. We are writing new software, in one dimensional case when the speed varies with the time. These programs are used by IRSID, to study the water-spray system of several installations.

Other problems

- The programs can be used:
  - to optimize the speed of extraction,
  - to compute the optimal structure of the continuous casting (length, number of zones,...)
- Two open problems:
  - to define closed loop algorithms
  - to minimize the domain where the metallurgical constraints are not verified, when the speed of extraction varies.

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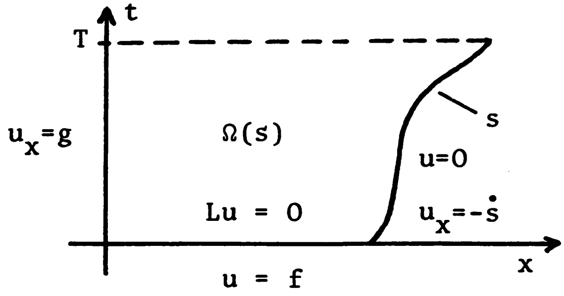
APPENDIX 2

TO THE NUMERICAL SOLUTION OF THE INVERSE STEFAN PROBLEM by P. Jochum

One of the major problems appearing in the mathematical treatment of continuous casting of metal is to find a "cooling function" which produces a prescribed thickness of the solidified part of the metal string at each moment. This question leads to one kind of the so-called "Inverse Stefan Problem" (ISP) which will be treated by Gauss-Newton's method.

Let us first point out the essential properties in a one-dimensional one-phase problem: The temperature distribution u in the liquid phase and the free boundary s satisfy the following systems:

$$\begin{aligned}
&Lu = u_{xx} - u_t = 0 \quad \text{in} \quad \Omega(s) = \{0 < x < s(t),\ 0 < t \le T\},\\
&u(x,0) = f(x), \quad 0 \le x \le b = s(0),\\
&u_x(0,t) = g(t), \quad 0 < t \le T, \qquad\qquad (1)\\
&\left.\begin{aligned} &u(s(t),t) = 0,\\ &u_x(s(t),t) = -\dot{s}(t) \end{aligned}\right\} \quad 0 < t \le T.
\end{aligned}$$



For f and b fixed the solution operator

$$S : A \to C[0,T]\ ,\ g \mapsto s,\quad A = \{g \in C[0,T] \,|\, g \le 0\}\ ,$$

is well known to be continuous and monotone [1]. The inverse Stefan Problem is to find to given f, b and $\tilde{s}$ the temperature distribution u satisfying (1) with $s = \tilde{s}$.

As in general, the ISP has no solution for arbitrary continuous $\tilde{s}$ [3], it is reformulated as a problem of optimal control

$$\| S(g^*) - \tilde{s}\|_p = \inf_{g\in U_{ad}} \| S(g) - \tilde{s}\|_p \ . \tag{2}$$

It has been recently shown by the author [2] and by H. Baumeister by a different method (which is demonstrated in this session) that S is even Frechet differentiable. Thus Gauss-Newton's procedure is suggested for the solution of (2)

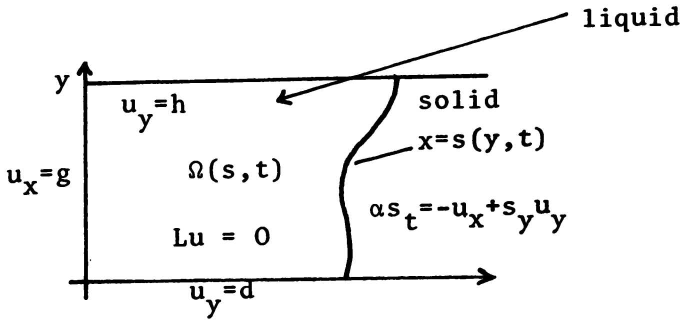
$$\begin{aligned} &\| \tilde{s} - S(g_i) - S'_{g_i} \Delta g_i \|_p = \inf_{g_i+\Delta g \in U_{ad}} \| \tilde{s} - S(g_i) - S'_{g_i} \Delta g \|_p \\ &g_{i+1} = g_i + \Delta g_i \ . \end{aligned} \tag{3}$$

Under some assumptions on the admissible set U_{ad} this algorithm is proved to be locally quadratically convergent for $p = \infty$, i.e.

$$\bigvee_{C>0} \bigwedge_{g_0 \in U_{ad}} \|g_{i+1} - g^*\| \le C\| g_i - g^*\|^2 \ .$$

This idea is now carried over to two space dimensions. Consider the following problem:

Two dimensional Stefan-Problem



Free boundary at the moment t

$$Lu := \Delta u - u_t = 0 \quad \text{in} \quad \bigcup_{t>0} \Omega(s,t),$$

$$\Omega(s,t) = \{(x,y)\,|\,0<x<s(y,t),\ 0<y<Y\}.$$

$u(x,y,0) = f(x,y), \quad (x,y) \in \Omega(s,0),$

$u_x(0,y,t) = g(y,t), \quad 0 < y < Y,\ 0 < t \leq T$

$u_y(x,0,t) = d(x,t) \quad \text{or} \quad u(x,0,t) = D(x,t),\ 0 < x < s(0,t)$

$u_y(x,Y,t) = h(x,t) \quad \text{or} \quad u(x,Y,t) = H(x,t),\ 0 < x < s(Y,t)$

$u(s(y,t),t) = 0,\ 0 < y < Y,\ 0 < t \leq T.$

Free Boundary Condition:

$$\alpha s_t(y,t) = -u_x(s(y,t),y,t) + s_y(y,t)u_y(s(y,t),t).$$

The solution Operator S is now defined by

$$S : B \supset A \to B,\ g \mapsto s,\ B = C([0,Y] \times [0,T]),\ A = \{g \in B\,|\,g \leq 0\}.$$

Proceeding in the same way as in one space dimension (c.f. [2]) one formally arrives at the following representation of the Frechet derivative of S [4]:

$S'_g : B \to B, \quad \delta g \to \delta s$, where

$$Lp = 0 \text{ in } \bigcup_{t>0} \Omega(s,t), \; s = S(g),$$

$$p(x,y,0) = 0, \; p_y(x,0,t) = p_y(x,Y,t) = 0,$$

$$p_x(0,y,t) = \int_0^t \delta g(\tau)\, d\tau ,$$

$$p(s(y,t),y,t) = 0,$$

$$\delta s(y,t) = - p_x(s(y,t),y,t) + s_y(y,t) p_y(s(y,t),y,t).$$

The mathematical proof of Frechet differentiability of S and the numerical test of algorithm (3) in the two-dimensional case is in the phase of investigation.

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APPENDIX 3

ON A FREE BOUNDARY VALUE PROBLEM RELATED TO THE OPTIMAL STOPPING OF A DETERMINISTIC SYSTEM by I. Capuzzo Dolcetta and M. Matzeu.

Let us consider the following first order free boundary value problem:

$$(P)\quad \begin{cases} u \in C^{0,1}(\bar{\mathcal{O}}),\ u = 0 \text{ on } \partial\mathcal{O} \\ \\ u \le \psi,\ -g\cdot\nabla u + \alpha u \le f,\quad (u-\psi)(-g\cdot\nabla u + \alpha u - f) = 0 \text{ in } \mathcal{O} \end{cases}$$

where

- $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^N$, with a smooth boundary $\partial\mathcal{O}$
- $f,\ \psi : \bar{\mathcal{O}} \to \mathbb{R},\ \psi = 0$ on $\partial\mathcal{O}$
- g is a real Lipschitz continuous function on $\mathbb{R}^N$, such that $g(x)\cdot n(x) > 0\ \forall x \in \partial\mathcal{O}$, where $n(x)$ is the outward normal vector to $\partial\mathcal{O}$ at x
- α is a positive constant.

Problem (P) is related to the optimal stopping time problem for the deterministic system, which is to find, for every $x \in \mathcal{O}$, an instant t^*_x minimising the cost function

$$J_x(t) = \int_0^{t\wedge\tau_x} f(y_x(s))e^{-\alpha s}ds + 1_{t<\tau_x}\psi(y_x(t))e^{-\alpha t} \qquad \forall t \ge 0,$$

where

$$y_x'(t) = g(y_x(t)) \quad \forall t > 0$$

$$y_x(0) = x$$

and

$$\tau_x = \inf\{t \ge 0 \,|\, y_x(t) \notin \bar{\mathcal{O}}\}.$$

It is known that, when the Hamiton-Jacobi function

$$\hat{u}(x) = \inf\{J_x(t)\,|\,t \ge 0\}$$

belongs to $C^{0,1}(\bar{\mathcal{O}})$, then it is the unique solution of (P) and that

$$\hat{t}_x = \begin{cases} \inf \{t \in [0,\tau_x] \mid \hat{u}(y_x(t)) = \psi(y_x(t))\} \\ \tau_x \quad \text{if } \hat{u}(y_x(t)) < \psi(y_x(t)) \quad \forall t \in [0,\tau_x] \end{cases}$$

is optimal, that is $\hat{u}(x) = J_x(\hat{t}_x)$.

In a recent paper by Menaldi [3] it is proved that, when f, ψ belongs to $C^{0,1}(\bar{\mathcal{O}})$, α is greater than $|g|_{0,1}$ and some suitable hypotheses on $\partial\mathcal{O}$ are satisfied, then $\hat{u}$ belongs to $C^{0,1}(\bar{\mathcal{O}})$, so it is the unique solution of (P). The result is obtained by using a penalization method: the approximated solutions u_ε satisfy a Bellman-type equation and are therefore interpreted as the value functions of some continuous control problem.

In [1] (case $\mathcal{O} = \mathbb{R}^N$), and [2] (present case, $\mathcal{O}$ bounded), we propose a constructive approach to the solution of (P). We start from the semi-group $\Phi(s)$ generated by the Operator $-g\cdot\nabla+\alpha I$ defined as

$$\Phi(s)v(x) = v(y_x(s\wedge\tau_x))e^{-(s\wedge\tau_x)} \quad \forall s \geq 0,\ v : \mathbb{R}^N \to \mathbb{R}$$

and approximate Φ by the discrete parameter semi-group $\Phi^{\Delta t}(k)$ ($\Delta t > 0$, $k \in \mathbb{N} \cup \{0\}$), associated with an Euler approximation with step size Δt of $y_x(s)$ and $e^{-\alpha s}$, that is

$$\Phi^{\Delta t}(k)v(x) = v(y_x^{\Delta t}(k\Delta t)\Delta t)(1-\alpha\Delta t)^k$$

where

$$\begin{cases} y_x^{\Delta t}((k+1)\Delta t) = y_x^{\Delta t}(k\Delta t) + g(y_x^{\Delta t}(k\Delta t))\Delta t, \quad k = 0, 1, 2, \ldots \\ y_x^{\Delta t}(0) = x \end{cases}$$

and

$$\nu_x^{\Delta t} = \mathrm{Min}\,\{k \geq 0 \mid y_x^{\Delta t}(k\Delta t) \notin \bar{\mathcal{O}}\}$$

Then we consider a sequence of approximate problems of the type

$$(P^{\Delta t}) \quad \begin{cases} u^{\Delta t} \in C^{0,1}(\mathbb{R}^N),\ u^{\Delta t} = 0 \text{ in } \mathbb{R}^N / \bar{\mathcal{O}} \\ u^{\Delta t} = \mathrm{Min}\,\{\psi \mid \Delta t\, f\chi + \Phi^{\Delta t}(1)u^{\Delta t}\} \quad \text{in } \bar{\mathcal{O}} \end{cases}$$

where χ is the characteristic function of $\mathcal{O}$, and prove the following

Theorem Under the above assumptions, for every $\Delta t > 0$ there exists a unique solution $u^{\Delta t}$ of $(P^{\Delta t})$. Furthermore $\{u^{\Delta t}\}$ converges uniformly, as $\Delta t \to 0^+$, to the unique solution u of (P).

Sketch of the proof First we show that the unique solution of $(P^{\Delta t})$ is the maximum solution of the system

$$\left.\begin{array}{l} u^{\Delta t} \in C^{0,1}(\mathbb{R}^N),\ u^{\Delta t} = 0 \text{ in } \mathbb{R}^N/\bar{\mathcal{O}} \\ u^{\Delta t} \leq \Delta t \sum\limits_{k=0}^{n-1} \Phi^{\Delta t}(k)(f\chi) + \Phi^{\Delta t}(n)u^{\Delta t} \text{ in } \bar{\mathcal{O}},\quad n = 0,1,2,\ldots \end{array}\right\} \tag{1}$$

and coincides with the infimum of the cost function

$$J_x^{\Delta t}(n) = \Delta t \sum_{k=0}^{n\wedge\nu_x^{\Delta t}-1} f(y_x^{\Delta t}(k\Delta t))(1-\alpha\Delta t)^k + 1_{n<\nu_x^{\Delta t}}\psi(y_x^{\Delta t}(n\Delta t))(1-\alpha\Delta_t)^n \tag{2}$$

Then we prove that, if $f, \psi \in C^{0,1}(\bar{\mathcal{O}})$ and $\alpha > |g|_{0,1}$, then

$$\|u^{\Delta t}\|_\infty \leq c \quad \text{and} \quad |u^{\Delta t}|_{0,1} \leq c.$$

(it is a consequence of the inequality

$$|y_{x_1}^{\Delta t}(n\Delta t) - y_{x_2}^{\Delta t}(n\Delta t)| \leq |x_1 - x_2|(1 - |g|_{0,1}\Delta t)^{-n}$$

which gives the Lipschitz continuity of $J_x^{\Delta t}(n)$ in the x variable, uniformly with respect to Δt and n).

It enables us to pass to the limit as $\Delta t \to 0^+$ and find a function $\tilde{u} = \lim\limits_{\Delta t\to 0^+} u^{\Delta t}$, which is the maximum solution of the system

$$\left.\begin{array}{l} \tilde{u} \in C^{0,1}(\bar{\mathcal{O}}),\ \tilde{u} = 0 \text{ on } \partial\mathcal{O} \\ \tilde{u} \leq \psi,\ \tilde{u} \leq \int_0^t \Phi(s)(f\chi)\,ds + \Phi(t)\tilde{u} \text{ in } \mathcal{O},\quad t \geq 0. \end{array}\right\} \tag{3}$$

The properties in (3) and the maximality are obtained by using (1), the fact that $\nu_x^{\Delta t}\Delta t \to \tau_x$ as $\Delta t \to 0^+$ and the estimates

$$|y_x^{\Delta t}(k\Delta t) - y_x(s)| \le c_t \Delta t \quad \forall s \in (k\Delta t, (k+1)\Delta t),\ k = 0,1,\dots,[t/\Delta t] - 1$$

$$|\Phi^{\Delta t}(k)(v\chi)(x) - \Phi(s)(v\chi)(x)| \le c_t \Delta t(|v|_{0,1} + \|v\|_\infty) \quad \forall v \in C^{0,1}(\bar{\mathcal{O}}),$$

$x \in \bar{\mathcal{O}}$, $k = 0,1,\dots,\nu_x^{\Delta t} \wedge [t \wedge \tau_x/\Delta t] - 1$, $s \in (k\Delta t, (k+1)\Delta t)$, also for

$k = \nu_x^{\Delta t} \wedge [t \wedge \tau_x/\Delta t]$ and $s \in ((\nu_x^{\Delta t} \wedge [t \wedge \tau_x/\Delta t])\Delta t, \tau_x)$ if $v = 0$ on $\partial\mathcal{O}$.

At this point we can show that u verifies the complementarity system

$$\tilde{u} \in C^{0,1}(\bar{\mathcal{O}}),\ \tilde{u} = 0 \text{ on } \partial\mathcal{O}$$

$$\tilde{u}(x) = \text{Min } \{\psi(x) | \int_0^\sigma \Phi(s) f(x) ds + \Phi(\sigma) u(x)\} \quad \forall \sigma \le t_x.$$

Hence, dividing by the relation in (4) and passing to the limit as $t \to 0^+$, we obtain that $\tilde{u}$ is a solution of (P). The uniquess follows from the inequality

$$\int_0 (-g \cdot \nabla v) v^2 + \alpha v^2) dx \ge (\alpha - |g|_{0,1}/2) \int_{\mathcal{O}} v^2 \, dx \quad \forall v \in C^{0,1}(\bar{\mathcal{O}})$$

which is a consequence of Green's theorem.

The results can be also extended to the case where $\Gamma_0 = \{x \in \partial \ \text{ s.t. } g(x) \cdot n(x) > 0\}$ is a proper subset of $\partial\mathcal{O}$, provided some regularity assumptions are made on the relative boundary of Γ_0.

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On the differentiable dependence upon the data of the free boundary in a two-phase Stefan problem

1. INTRODUCTION

In this note we consider the two-phase Stefan problem with flux boundary conditions. The mathematical problem consists in determining three functions u,v,s such that (u,v,s) satisfy

$$\left.\begin{aligned} & u_t - \alpha u_{xx} = 0 && , \text{ in } \Omega_T^-(s) \\ & u_x(0,t) - h(t)u(o,t) = f(t), && t \in (0,T] \\ & u(s(t),t) = 0, && t \in (0,T] \\ & u(x,o) = \varphi(x), && x \in [0,b] , \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} & v_t - \beta v_{xx} = 0, && \text{in } \Omega_T^+(s) \\ & v_x(1,t) = g(t), && t \in (0,T] \\ & v(s(t),t) = 0, && t \in (0,T] \\ & v(x,o) = \varphi(x), && \varphi x \in [b,1] \end{aligned}\right\} \tag{1.2}$$

and

$$\left.\begin{aligned} & s'(t) = -\gamma u_x(s(t),t) + \delta v_x(s(t),t), \quad t \in (0,T] \\ & s(o) = b \end{aligned}\right\} \tag{1.3}$$

where $T,\alpha,\beta,\gamma,\delta$ are positive constants, $b \in (0,1)$, and

$$\Omega_T^{\pm}(s) := \{(x,t) \in (0,1) \times (0,T] \mid \pm (x - s(t)) > 0\} .$$

The assumptions on the data h,f,g,φ are discussed later.

The boundary conditions at the fixed boundary are chosen in such a way that one-dimensional model for the process of continuous casting of steel is included (see section 4).

Let S be the map which assigns to given data $h,f,g \in C[0,T]$ the free boundary s. Then we show that under suitable assumptions the map S possesses a Frechet-derivative. This result fills a gap in [6]. In the last section we make some remarks on the applicability of this result.

By a solution (u,v,s) in $[0,T]$ of the Stefan problem (1.1)-(1.3) we mean that

$$\left.\begin{array}{ll} \text{i)} & s \in C[0,T] \cap C^1(0,T];\ 0 < s(t) < 1,\ t \in [0,T]\ ; \\ \text{ii)} & u \in C(\overline{\Omega_T^-(s)}) \cap C^{2,1}(\Omega_T^-(s)),\ u_x(s(.),.) \in C(0,T]\ ; \\ \text{iii)} & v \in C(\overline{\Omega_T^+(s)}) \cap C^{2,1}(\Omega_T^+(s)),\ v_x(s(.),.) \in C(0,T]\ ; \\ \text{iv)} & u,v,s \text{ satisfy } (1.1)-(1.3). \end{array}\right\} \qquad (1.4)$$

We use the following notations:

$$Y := C_b^0(0,T] := \{m \in C(0,T] \mid \sup_{t \in (0,T]} |m(t)| < \infty\},$$

$$C_b^1(0,T] := \{m \in C[0,T] \cap C^1(0,T] \mid \sup_{t \in (0,T]} |m'(t)| < \infty\},$$

$$X := \{m \in C_b^1(0,T] \mid m(0) = 0\}.$$

The spaces Y,X are Banach spaces with the following norms:

$$\|m\|_Y := \sup_{t \in (0,T]} |m(t)|, \quad m \in Y,$$

$$\|m\|_X := \sup_{t \in (0,T]} |m'(t)|, \quad m \in X.$$

Concerning the initial data φ we assume in the following (without further mention):

$$\varphi \in C[0,1],\ \sup_{x \in [0,1]} |\varphi(x)(b - x)^{-1}| < \infty.$$

In the following "derivative" means always "Frechet-derivative".

2. A BOUNDARY VALUE PROBLEM WITH FIXED BOUNDARY

Let

$$U := \{\sigma \in X \mid \sigma(t) > -b,\ t \in [0,T]\}.$$

For $\sigma \in U$, $s := \sigma + b$, we consider the boundary value problem (1.1). The aim of this section is to prove that the solution of (1.1) depends differentiable upon the data h,f. This generalizes a result of H. Krüger [9].

It is well known that the boundary value problem (1.1) has for $h, f \in C[0,T]$ a unique solution u in the sense of (1.4)ii) (see for instance [3]).

Theorem 1 Let $h,f \in C[0,T]$. Then we have

i) The map

$$G : U \ni \sigma \mapsto \int_0^{\cdot} u_x(s(\tau),\tau)\,d\tau \in X$$

where $s := \sigma + b$, u solves (1.1), is differentiable.

ii) The derivative of G has the following representation

$$\langle G'(\sigma),\delta\sigma\rangle(t) = \frac{1}{\alpha}\int_0^{s(t)} w(x,t)\,dx + \int_0^t h(\tau)w(0,\tau)\,d\tau,\ t \in [0,T],$$

where $s := \sigma + b$ and w solves

$$\left.\begin{array}{ll} w_t - \alpha w_{xx} = 0, & \text{in } \Omega_T^-(s) \\ w_x(0,t) - h(t)w(0,t) = 0, & t \in (0,T] \\ w(s(t),t) = -u_x(s(t),t)\,\delta\sigma(t), & t \in (0,T] \\ w(x,0) = 0, & x \in [0,b]\ ; \end{array}\right\} \quad (2.1)$$

here u is the solution of (1.1).

Proof The result is proved in [9] for $h \equiv \theta$.

(1) Let us consider the map

$$M : C[0,T] \times U \ni (f,\sigma) \mapsto \int_0^{\cdot} u_x(s(\tau),\tau)\,d\tau \in X$$

where $s := \sigma + b$ and u solves (1.1) with $h \equiv \theta$.

It is easy to see with the result in 9 that the map M is differentiable and that the derivative has the following representation:

$$\langle M'(f,\sigma),(\delta f,\delta\sigma)\rangle(t) = \frac{1}{\alpha}\int_0^{s(t)} w(x,t)\,dx + \int_0^t \delta f(\tau)\,d\tau,\ t \in [0,T]\ ,$$

where w solves

$$\left.\begin{array}{ll} w_t - \alpha w_{xx} = 0, & \text{in } \Omega_T^-(s) \\ w(x,0) = 0 \quad , & x \in [0,b] \end{array}\right\} \tag{2.2}$$

$$\left.\begin{array}{ll} w_x(0,t) = \delta f(t), & t \in (0,T] \\ w(s(t),t) = - u_x(s(t),t)\delta\sigma(t), & t \in (0,T]; \end{array}\right\} \tag{2.3}$$

here u solves (1.1) with $h \equiv \theta$.

(2) Now, consider the map

$$P : C[0,T] \times U \ni (\tilde{f};\sigma) \mapsto hu(o,.) + f - \tilde{f} \in C[0,T]$$

where u solves

$$\left.\begin{array}{ll} u_t - \alpha u_{xx} = 0, & \text{in } \Omega_T^-(s) \\ u(x,0) = \varphi(x), & x \in [0,b] \\ u(s(t),t) = 0, & t \in (0,T] \end{array}\right\} \tag{2.4}$$

$$u_x(0,t) = \tilde{f}(t), \quad t \in (0,T] \; ; \tag{2.5}$$

here $s := \sigma + b$.

Due to the result in [9] P is differentiable and the derivative has the following representation:

$$\langle P'(\tilde{f},\sigma),(\delta\tilde{f},\delta\sigma)\rangle(t) = h(t)w(0,t) - \delta\tilde{f}(t), \quad t \in [0,T],$$

where w solves (2.2) and

$$\left.\begin{array}{ll} w_x(0,t) = \delta\tilde{f}(t) & t \in (0,T] \\ w(s(t)t) = - u_x(s(t),t)\delta\sigma(t), & t \in (0,T]; \end{array}\right\} \tag{2.6}$$

here: $s := \sigma + b$, u solves (2.4),(2.5).
If u is a solution of (1.1) which exists by results in [3] and if $\tilde{f}$ is defined by

$$\tilde{f}(t) := h(t)u(0,t) + f(t), \quad t \in [0,T],$$

we have

$$P(\tilde{f},\sigma) = 0.$$

It is easy to see that the partial derivatives $\frac{\partial P}{\partial \tilde{f}}$, $\frac{\partial P}{\partial \sigma}$ exist, are continuous and that $\frac{\partial P}{\partial \tilde{f}}$ is bijective. By the implicit function theorem there exists for each $\sigma \in U$ an open set $U' \subset X$ with $\sigma \in U'$ and a differentiable map

$$T : U' \to C[0,T]$$

such that

$$P(T(\sigma),\sigma) = 0, \quad \sigma \in U'.$$

This implies

$$T(\sigma) = hu(0,.) + f.$$

(3) Let $\sigma \in U$. Since

$$G(\sigma) = M(T(\sigma),\sigma)$$

we have by (1) and (2) that G is differentiable; the representation of $G'(\sigma)$ follows from the representations of P' and M'. □

3. THE DIFFERENTIABLE DEPENDENCE OF THE FREE BOUNDARY

Let $\bar{h},\bar{f},\bar{g} \in C[0,T]$ such that the problem (1.1)-(1.3) has a (unique) solution $(\bar{u},\bar{v},\bar{s})$ in the sense of (1.4); see [4].

We assume

$$\bar{s} \in C^2[0,T]. \tag{3.1}$$

Sufficient for (3.1) are the following compatibility conditions on the initial function φ:

$$\varphi \in C^4[0,1], \quad \varphi^{(j)}(b) = 0, \quad j = 0,\dots,4. \tag{3.2}$$

By the continuous dependence of the solution of (1.1)-(1.3) upon the data h,f,g (see [4]) there exists a neighborhood V of $(\bar{h},\bar{f},\bar{g})$ in $C[0,T]^3$ such that the problem (1.1)-(1.3) has a solution (u,v,s) for a given $(h,f,g) \in V$.

We denote the solution (u,v,s) for a given $(h,f,g) \in V$ by

$$u = S^-(h,f,g), \quad v = S^+(h,f,g), \quad s = \tilde{S}(h,f,g)$$

and set $\sigma := S(h,f,g) := \tilde{S}(h,f,g) - b$.

Now, we are ready to prove that the map

$$S : V \to X$$

is differentiable. The main tool is the implicit function theorem.

<u>Theorem 2</u> Let the assumption (3.1) hold. Then the map

$$S : V \to X$$

is differentiable in $(\bar{h},\bar{f},\bar{g})$ and the derivative has the following representation:

$$\begin{aligned}\delta\sigma(t) &:= \langle S'(\bar{h},\bar{f},\bar{g}), (\delta h, \delta f, \delta g)\rangle(t) \\ &= -\gamma \int_0^t \{h(\tau)w^1(0,\tau) + \delta h(\tau)\,\bar{u}(0,\tau) + \delta f(\tau)\}d\tau \\ &\quad - \frac{\gamma}{\alpha} \int_0^{\bar{s}(t)} w^1(x,t)\,dx + \delta\int_0^t \delta g(\tau)\,d\tau - \frac{\delta}{\beta} \int_{\bar{s}(t)}^1 w^2(x,t)\,dx\end{aligned}$$

where w^1 solves

$$\left.\begin{aligned} &\alpha w_t - w_{xx} = 0, && \text{in } \Omega_T^-(\bar{s}) \\ &w(x,0) = 0, && x \in [0,b] \\ &w_x(0,t) - h(t)w(0,t) = \delta h(t)u(0,t) + \delta f(t), && t \in (0,T] \\ &w(s(t),t) = -\bar{u}_x(\bar{s}(t),t)\,\delta\sigma(t), && t \in (0,T] \end{aligned}\right\} \quad (3.3)$$

and w^2 solves

$$\left.\begin{aligned} &w_t - \beta w_{xx} = 0, && \text{in } \Omega_T^+(s) \\ &w(x,0) = 0, && x \in [b,1] \\ &w_x(1,t) = \delta g(t), && t \in (0,T] \\ &w(\bar{s}(t),t) = -\bar{v}_x(\bar{s}(t),t)\,\delta\sigma(t), && t \in (0,T] \end{aligned}\right\} \quad (3.4)$$

Proof We define the following map

$$K : V \times U \to X$$

$$K(h,f,g;\sigma) := \sigma + \gamma\int_0^{\cdot} u_x(s(\tau),\tau)\,d\tau - \delta\int_0^{\cdot} v_x(s(\tau),\tau)\,d\tau$$

where $s := \sigma + b$, u solves (1.1), v solves (1.2).

We know

$$K(h,f,g;S(h,f,g)) = 0 \quad \text{for all} \quad (h,f,g) \in V. \tag{*}$$

From the results in section 2 follows that K is differentiable. If we can show that the partial derivative

$$P := \frac{\partial K}{\partial \sigma}(\bar{h},\bar{f},\bar{g};\bar{\sigma}) \quad (\text{where } \bar{\sigma} := \bar{s} - b)$$

is an isomorphism from X onto X the differentiability of S follows from (*) by the implicit function theorem.

Using the results in section 2 we see that the map P has the following representation

$$\begin{aligned}\langle P,\delta\sigma\rangle(t) = \delta\sigma(t) &+ \gamma\int_0^t h(\tau)w^1(0,\tau)\,d\tau + \frac{\gamma}{\alpha}\int_0^{\bar{s}(t)} w^1(x,t)\,dx \\ &+ \frac{\delta}{\beta}\int_{\bar{s}(t)}^1 w^2(x,t)\,dx\end{aligned} \tag{3.5}$$

where w^1 solves (3.3) with $\delta h \equiv \delta\tilde{f} \equiv \theta$ and w^2 solves (3.4) with $\delta g \equiv \theta$.

We obtain

$$\begin{aligned}\langle P,\delta\sigma\rangle'(t) = \delta\sigma'(t) &+ \{-\frac{\gamma}{\alpha}\bar{u}_x(\bar{s}(t),t) + \frac{\delta}{\beta}\bar{v}_x(\bar{s}(t),t)\}\delta\sigma(t) \\ &+ \gamma w_x^1(\bar{s}(t),t) - \delta\bar{w}_x^2(\bar{s}(t),t), \quad t \in (0,T].\end{aligned}$$

In order to prove that P is an isomorphism from X onto X it is enough to show that to a given function $q \in Y$ there exists a unique $\delta\sigma \in X$ such that

$$\langle P,\delta\sigma\rangle' = q. \tag{**}$$

From estimates in [3] (notice that we have $\bar{s} \in C^2[0,T]$) follows that the map

$$\Gamma : X \to Y$$

$$\Gamma(\delta\sigma)(t) := -\{\frac{\gamma}{\alpha}\bar{u}_x(\bar{s}(t),t) + \frac{\delta}{\beta}\bar{v}_x(\bar{s}(t),t)\}\delta\sigma(t)$$
$$- \gamma w_x^1(\bar{s}(t),t) + \delta w_x^2(\bar{s}(t),t), \quad t \in (0,T]$$

where w^1, w^2 are the same functions as in (3.5), satisfies for each $0 < T' \le T$:

$$|\Gamma(\delta\sigma)(t)| \le c(T') \sup_{t \in (0,T]} |\delta\sigma'(t)| , \quad t \in (0,T']$$

where $c(T') = O(t')$.

This shows that the equation (**) considered as a fixed point equation for $\delta\sigma'$ has for a sufficiently small time $T' > 0$ a unique solution $\delta\sigma$ in $[0,T']$. This solution can be continued up to T since the Lipschitz-constant $c(T')$ does not depend upon the initial data of the boundary value problems for w^1, w^2.

Now, we may use the implicit function theorem and we obtain that there exists a neighborhood $V' \subset V$ of $(\bar{h},\bar{f},\bar{g})$ such that

i) $P(h,f,g;S(h,f,g)) = 0, \quad (h,f,g) \in V'$

ii) S is differentiable in $(\bar{h},\bar{f},\bar{g})$.

The representation of $S'(\bar{h},\bar{f},\bar{g})$ follows from the representation of K' which is described in Section 2. □

Remark The differentiable dependence of the free boundary upon the data was first proved in the one-phase case by P. Jochum [7]. Theorem 2 generalizes this result to the case of the boundary condition

$$u_x(0,t) - h(t)u(0,t) = f(t), \quad t \in (0,T].$$

4. CONCLUDING REMARKS

A mathematical model for the continuous casting of steel is of the form (1.1)-(1.3) with $g \equiv \theta$ (symmetry condition); see for instance [5]. An interesting problem is to control this process. Results in developing a control theory for this class of problems can be found in [1],[6],[8],[10]. By theorem 2 the results in [1],[8] for the one-phase problem can be generalized to the two-phase problem.

Another interesting question is to estimate the position of the free boundary from measurements. A problem of this kind was considered in [5]. By the result of theorem 2 it is possible to apply the usual identification techniques (see for instance [2]).

Results for the control problems and the identification problem will be developed elsewhere.

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J A McGEOUGH

Free and moving boundary problems in electrochemical machining and flame fronts

In this section, three papers were presented on electrochemical machining and another two on flame fronts and melting problems.

The first paper described the bases of electrochemical machining and the formulation of theoretical models for this process. Then two papers in which ECM problems involving respectively variational inequalities and axially symmetric three dimensional cases were reported. The numerical solution of flame front problems and the modelling of melting in reactor safety studies were discussed. A summary of these four papers is given at the end of this section.

McGeough firstly described the bases of electrochemical machining:

ASPECTS OF MOVING BOUNDARY PROBLEMS IN ELECTROCHEMICAL MACHINING - McGeough, J. A.

The need for electrochemical machining (ECM) has stemmed from the recent development of high-strength, heat-resistant alloys which are difficult to machine by conventional techniques.

With the alternative ECM procedure, a workpiece and tool are made the anode and cathode respectively of an electrolytic cell, the two electrodes being separated by a small gap, typically 0.5 mm. An appropriate electrolyte, such as aqueous sodium chloride solution, is chosen such that when a small potential difference of about 10 V is applied between the electrodes, metal is dissolved electrolytically from the anode, and gas generation occurs at the cathode. By turbulent flow of the electrolyte through the inter-electrode gap, the solid and gaseous products of machining are rapidly removed.

As electrolytic dissolution proceeds, the cathode-tool can be fed mechanically towards the anode-workpiece in order to maintain the machining action. Under these conditions, the inter-electrode gap width gradually tends to a steady-state value, and a shape, complementary to that of the cathode-tool, is reproduced on the anode-workpiece (Figure 1).

Since the anode-workpiece is shaped electrolytically, the rate of machining does not depend on the hardness of that material. Moreover, if a

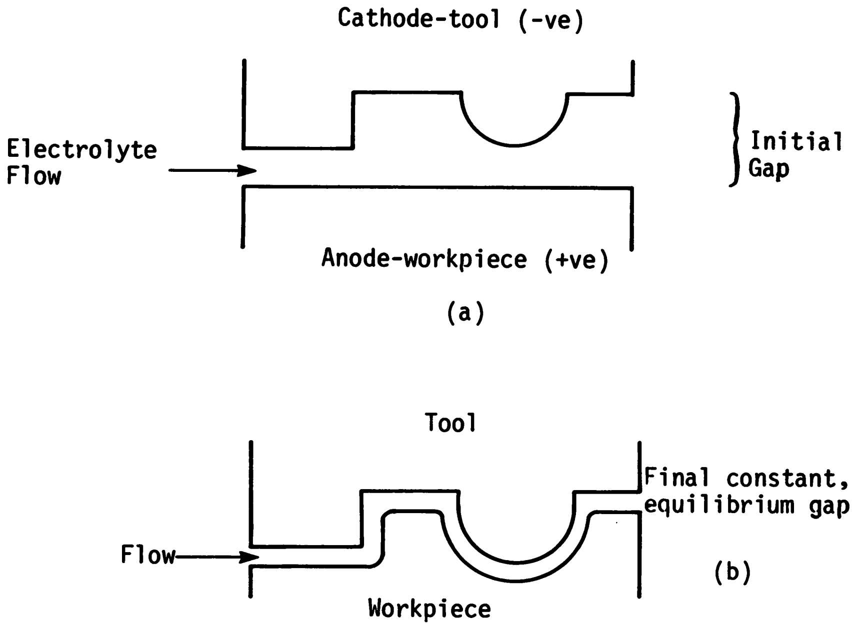


Figure 1. Electrode configuration for ECM (a) initial, (b) final

complicated shape has to be produced in a hard metal, its image form can first be made in a softer metal. Since only gas evolution occurs at the cathode in ECM, there is no tool wear; and the softer metal is made the cathode-tool for the electrochemical shaping of the harder metal.

Most theoretical treatments of ECM deal with a quasi-steady model based on the irrotationality of the electric field between the electrodes (McGeough and Rasmussen 1974). Three basic equations are used in its formulation:

(i) Laplace's equation

$$\nabla^2\phi = 0 \tag{1}$$

the solution of which will give the potential ϕ at any point in the electrolyte, and in particular at the surfaces of the electrodes.

(ii) An equation based on Ohm's law

$$\underset{\sim}{J} = -K_e \nabla\phi \tag{2}$$

where J is the current density, found from the potential now known from equation (1), K_e being the specific conductivity of the electrolyte. In most theoretical treatments of the shaping problem in ECM, K_e is assumed to be maintained constant by the high rate of flow of the electrolyte through the inter-electrode gap.

(iii) An expression incorporating Faraday's law:

$$\dot{\underset{\sim}{r}}_a = \frac{A\underset{\sim}{J}}{ZF\rho_a} \tag{3}$$

is used to describe the recession rate $\dot{\underset{\sim}{r}}_a$ of the anode-workpiece, due to its electrolytic dissolution; here A is the atomic weight, Z the valency and ${}_a$ the density of the anode metal; and F is Faraday's constant. The use of equation (3) in this form implies that all the metal is removed at 100 per cent current efficiency. This quantity is defined as the ratio of the observed amount of metal dissolved to the theoretical amount predicted from Faraday's law, for the same specified conditions of electrochemical equivalent, current density, etc. Often, by judicious choice of the electrolyte solution, such as sodium nitrate or sodium chlorate solution a useful variation of current efficiency with current density can be obtained such that the metal removal is largely confined to specified regions. This condition is particularly useful in hole-drilling by ECM, in which procedure metal machining is desired at the front, leading edge of the cathode-drill (high current densities and current efficiencies), whilst along the side-walls of the anode metal removal has to be reduced (low current efficiency) in order that a dimensionally accurate hole can be formed.

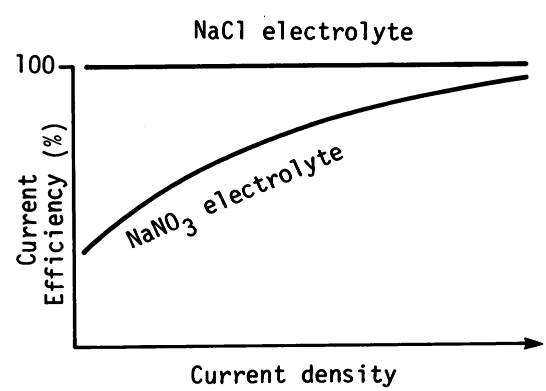
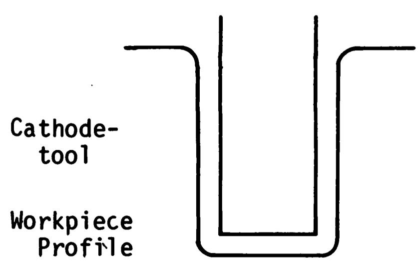
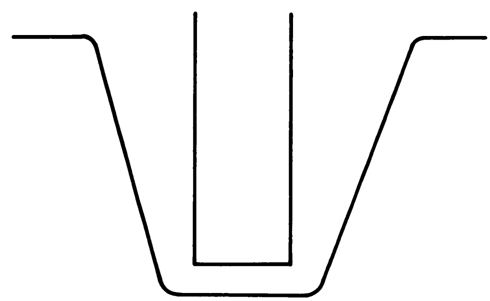


Figure 2. Variation of current efficiency with current density

Figure 2 shows typical variations in current efficiency with increasing current density for $NaNO_3$ and NaCl solutions. For the former electrolyte the current efficiency rises with increasing current density, and this behaviour has the effect of maintaining closely parallel side-walls for the drilled hole (Figure 3). On the other hand, the current efficiency for NaCl is unaffected by current density. The result for this case is a widely tapered hole, as shown in Figure 3(b). In order to achieve a drilled hole of better dimensional accuracy, the walls of the cathode-electrode are sometimes coated with an insulating material. The provision of this layer prevents stray currents causing additional metal removal along the side-walls of the workpiece. It may be noted that this device is used as well as sodium nitrate electrolyte, in order to maintain acceptable dimensional accuracy.



(a) $NaNO_3$ electrolyte



(b) NaCl electrolyte

Figure 3. Hole-drilling by ECM

If the surfaces of both the cathode and anode electrodes are assumed to be equipotentials then the following boundary conditions apply:

$$\phi = 0, \text{ at the cathode-tool} \tag{4}$$

$$\phi = V, \text{ at the anode workpiece} \tag{5}$$

where V is the applied potential difference.

However, the electrochemical reactions at both electrodes can cause "overpotentials" to occur. It is well known that when a metal is placed in an electrolyte solution, an equilibrium potential difference usually becomes established between the metal and solution, the magnitude of which may be estimated from the Nernst equation (see, for example, Gurney 1962). In ECM metal removal is achieved by the application of an external potential difference which gives rise to corresponding current. The larger the current, the greater becomes the difference between the equilibrium and "working" potentials, or overpotential.

The principal overpotentials in any electrochemical process are activation, concentration and resistance. Activation overpotential is known to occur in ECM conditions. The likelihood of the other two overpotentials arising in ECM has been discussed by McGeough (1974).

As an example, therefore, the conditions for activation overpotential are considered. Figure 4(a) shows the free energy-distance diagram for a metal (anode) in an electrolyte. When a potential difference is applied across the cell, the anode ionises at a greater rate than that of discharge of its ions.

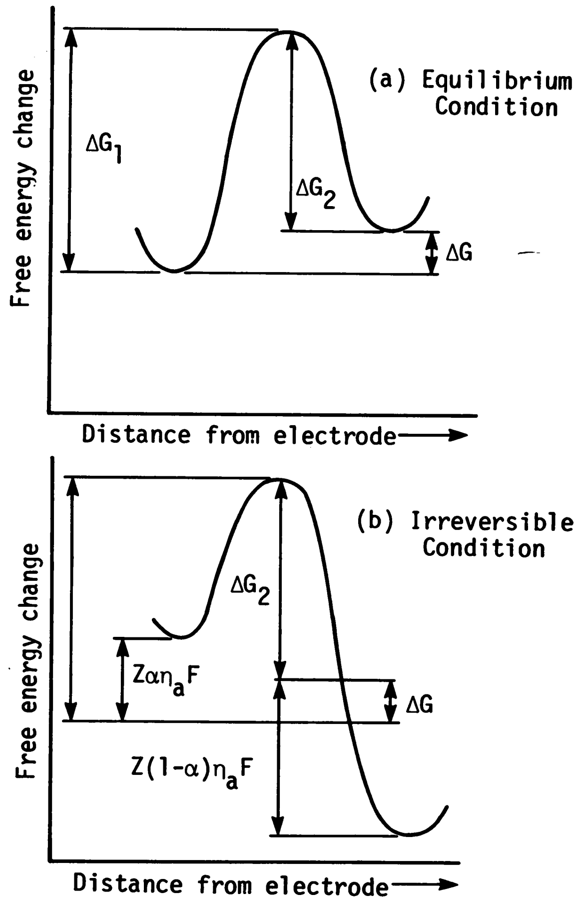


Figure 4. Free energy-distance diagram

The electrode potential is accordingly altered from its equilibrium value by an amount, η_a, the activation overpotential. The equilibrium conditions shown in Figure 4(b) are now replaced by those for anodic dissolution. The free energy for dissolution is reduced from ΔG_1 (the minimum free energy which ions must possess to ionise to the solution) to $(\Delta G_1 - Z\alpha\eta_a F)$. The energy for discharge is increased from ΔG_2 (the minimum free energy for ions to discharge from the solution) to $(\Delta G_2 + Z(1-\alpha)\eta_a F)$. Here Z is the valency of the ions, α is that fraction of overpotential associated with dissolution, and F is Faraday's constant.

Whereas for equilibrium conditions, the equilibrium or exchange current density J_o is given as

$$J_o = \Omega_1 \exp(-\Delta G_1/RT) = \Omega_2 \exp(-\Delta G_2/RT) \qquad (6)$$

where Ω_1 and Ω_2 are characteristic parameters for the dissolution and deposition reactions for forced dissolution and R is the Gas Constant and T is the temperature, the actual current density can be expressed as

$$J = \Omega_1 \exp - \frac{(\Delta G_1 - Z\alpha\eta_a F)}{RT} - \Omega_2 \exp - \frac{(\Delta G_2 + Z(1-\alpha)\eta_a F)}{RT}$$

in which α is that fraction of the overpotential contributing towards dissolution.

$$= J_o\{\exp \frac{Z\alpha\eta_a F}{RF} - \exp \frac{Z(1-\alpha)\eta_a F}{RT}\} \qquad (7)$$

Now ECM is a highly irreversible process. That is the rate of reaction in the direction opposed by the overpotential may be assumed to be negligible. The second term on the right hand side of equation (7) may then be ignored.

Equation (7) then becomes

$$\eta_a = a + b \log J \qquad (8)$$

where $a = - \frac{2.303\ RT}{Z\alpha F} \log J_o$

and $b = \frac{2.303\ RT}{Z\alpha F}$

Equation (8) is known as the Tafel equation, and a and b the Tafel constants. For example, $RT/Z\alpha \simeq 0.025$ V, $\alpha \simeq 0.5$, $J_o \simeq 10^{-5}$ A/cm^2, $J \simeq 100$ A/cm^2. Thus a ~ 0.58 V and 0.12 V. The activation overpotential $\eta_a \simeq 0.82$ V; c.f. a typical applied potential difference in ECM of 10 V.

A similar Tafel expression can be derived for the conditions of gas evolution at the cathode-tool.

However, the high current densities involved in ECM and the steep concentration gradients may induce a more complex form for the activation overpotentials. Thus in theoretical work on moving boundary problems in ECM it is often more appropriate to select arbitrary functions for the overpotentials, e.g. f(J) at the cathode, and g(J) at the anode.

The presence of overpotentials therefore alters the boundary conditions (4) and (5) to

$$\phi = f(J) \text{ at the cathode-tool} \tag{9}$$

and $$\phi = V - g(J) \text{ at the anode-workpiece.} \tag{10}$$

Theoretical problems on moving boundaries in ECM may be divided into three types.

The first problem concerns anodic smoothing. This technique is used widely in industry: burrs or rags produced by previous manufacturing operations are removed by ECM. Figure 5 shows a typical configuration in which a plane-faced cathode-tool is used in the electrochemical dissolution of an initially irregular anode surface. (Figure 5) As ECM proceeds, the surface of the latter electrode gradually becomes smooth, and therefore resembles

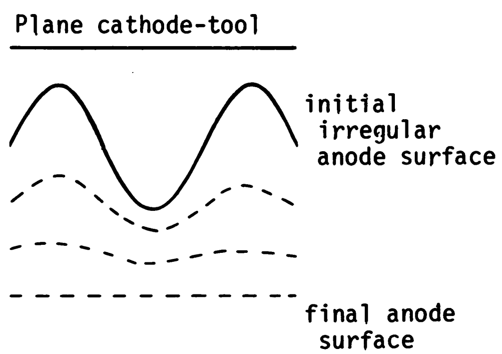


Figure 5. Anodic smoothing

that of the cathode-tool. Anodic smoothing is the only case in ECM in which an exact impression of the cathode-tool is obtained on the anode-workpiece. (Any other cathode-tool is only approximately reproduced on the anode). The problem of anodic smoothing is one in which the role of overpotentials can readily be included in order to assess their effects, (Fitz-Gerald and McGeough 1969)).

The prediction of the resultant anode shape for a fixed cathode profile is a major problem in ECM. The two main problems that have to be investigated are the variation with machining time of the shape of the anode workpiece, and the resultant equilibrium shape of that electrode. (As indicated earlier, the shape of the anode workpiece conforms eventually to that of the tool-electrode). Of practical interest is the time of machining needed for that equilibrium anode shape to achieved.

Hole-drilling by ECM is a useful example of this problem. As discussed above a cathode-electrode is used to drill electrochemically a hole in a workpiece.

When a particular anode shape is required, the corresponding cathode shape has to be designed. This theoretical problem is by far the most difficult, and is generally regarded as the fundamental problem of ECM (Figure 6).

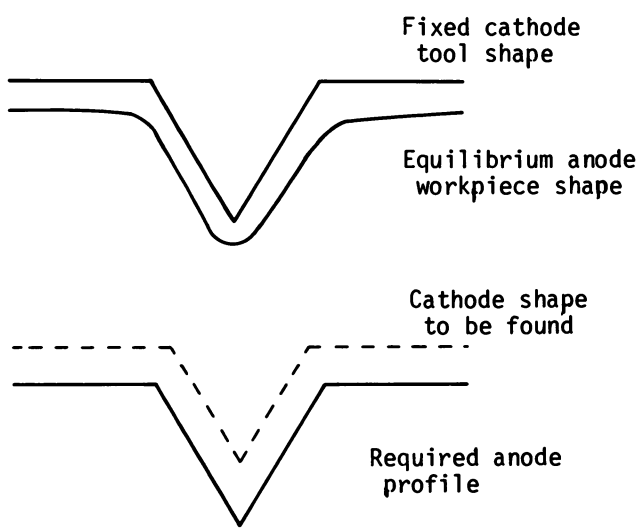


Figure 6. Cathode shape design

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ELECTROCHEMICAL MACHINING AND VARIATIONAL INEQUALITIES - C. Elliot

In this paper, a Baiocchi-type transformation of the dependent variables was shown to lead to variational inequality formulations. A quasi-steady model in which the cathode-tool is fixed was then formulated as an elliptic inequality in which time enters as a parameter. For a case in which the cathode-tool is moved at a constant velocity towards the anode-workpiece, the steady-state equilibrium surface of the latter electrode may be found by solving an elliptic inequality. Efficient numerical methods based on discretisations of the inequalities by which the anode surface may be obtained were described. Solution of the free boundary problem in a mathematical sense by use of the inequality formulations was also discussed.

A COMPARISON OF COMPUTATIONS AND MODELLING IN ELECTROCHEMICAL MACHINING - E. B. Hansen

The author pointed out that most theoretical work on ECM has been based on the assumption that the electrolyte is a linear and homogeneous conductor of electricity, and that Dirichlet boundary conditions apply at the surfaces of the electrodes (that is, given potential values). He explained that little literature had been published in which theoretical findings based on this model had been compared with experimental results mainly because the former had been concerned with comparatively simple two-dimensional cases, whereas more complicated electrode shapes are normally found in practice.

As a contribution towards overcoming this deficiency in ECM work, a computer program has been prepared which can handle axially symmetric three-dimensional problems, the program proceeding in time-steps, so that a series of static potential problems are solved. The computational results were compared with those from experiment.

An attractive feature of this work is the inclusion of conditions in which the electrodes are partly protected by insulating material (This procedure is used in hole-drilling by ECM: the side walls of the drill-electrode are usually insulated to reduce stray ECM action in those regions, and so a greater tolerance is achieved for the drilled hole). The mixed boundary value problems thereby obtained can be solved by an integral equation method. In the presentation, the methods used were discussed in detail, particular attention being paid to the means for handling the singularities between the insulated and bare regions of the electrodes.

The final two papers in this section dealt with flame front free boundary problems (by A. B. Crowley) and the modelling of some melting problems which arise in reactor safety studies (R. S. Peckover).

NUMERICAL SOLUTION OF FLAME FRONT FREE BOUNDARY PROBLEMS - A. B. Crowley

The application of weak solution techniques to several flame front models was outlined (c.f. the "enthalpy method"). The models were then recast in conservation form so that the governing equations can then be solved numerically without the need to apply explicitly the conditions at the flame front. Amongst results described, a model for a flame tip which accounted for convection effects was discussed.

THE MODELLING OF SOME MELTING PROBLEMS - R. S. Peckover

From a review of melting problems that arise in reactor safety studies, two were selected for a more detailed discussion.

The first dealt with the movement of hot particles in low melting point solids. The essential characteristic of their descent through this medium is that it involves the melting of a transient liquid cavity in which it falls under gravity. The calculation of the fall velocity was shown to require the determination of the position of the moving boundary of the cavity. A model appropriate for the low Reynolds number was discussed, and a method for calculating the cavity shape by placing appropriate heat sources and sinks on the melting front was examined.

The ablation of a slab of finite thickness was the second problem discussed. Attention was drawn to the availability of accurate numerical solutions. The time for melt-through was shown to depend not only on the applied heat flux at the front face of the slab but also on the heat transfer coefficient on the near face. Approximate methods which clarify the functional dependence of the melt-through time on the characteristic dimensional parameters were presented.

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J D BUCKMASTER

Free boundary problems in combustion

The discussion of flame propagation as a free-boundary problem (FBP) has a long history (see Burke & Schumann (1928), Darrieus (1938), Landau (1944)). Recent developments add a whole new dimension to the subject, and this paper is, for the most part, an introduction to these new ideas.

In order to show how combustion can be treated as an FBP we must settle on a reasonable set of governing equations. This question is discussed in Williams (1965), and for our purposes it is sufficient to examine the following equations for the temperature T and mass fractions Y_i,

$$\rho \frac{D}{Dt}(T,Y_i) = \Delta(T, \frac{1}{L_i} Y_i) + (\alpha_T,\alpha_i)\omega , \quad i = 1,\ldots,N. \tag{1}$$

To these may be adjoined equations governing the fluid mechanics,

$$\frac{\partial\rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho v) = 0, \ \rho T = \text{Constant}, \ \rho\frac{Dv}{Dt} = -\underset{\sim}{\nabla}p + \Pr[\Delta\underset{\sim}{v} + \frac{1}{3}\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v})] \tag{2}$$

Alternatively, the constant density model can be adopted in which case (2) is replaced by

$$\rho = 1, \ \underset{\sim}{v} \text{ specified } (\underset{\sim}{\nabla}\cdot\underset{\sim}{v}=0). \tag{3}$$

This should be regarded in the spirit of an Oseen approximation.

The chemical reaction rate, a source of heat and a sink of reactants, is denoted by ω and the system (1), (2) or (1), (3) reduces to an FBP if ω is zero everywhere except in thin sheets, these playing the characterizing role. There are four ways in which this reduction can arise: (i) irrational approximation (in the sense of Van Dyke, 1975); (ii) the hydrodynamic limit; (iii) Damkohler number asymptotics; (iv) activation energy asymptotics. Other than noting the existence of some recent analysis of type (i) masquerading as type (iv), we shall say no more about them.

The Hydrodynamic Limit

When the reactants are supplied as a homogeneous mixture, the flame they support is said to be premixed. The mixture can be modelled by a single reactant (N=1), and a possible choice for ω is

$$\omega = DYe^{-\theta/T}H(T-T_c) \tag{4}$$

where θ is the activation energy and the Heaviside function is introduced so that the fresh cold mixture at a temperature $T_f(<T_c)$ does not react (T_c is a cut-off temperature). At high temperatures, reaction only ceases when Y is completely consumed, and for an adiabatic system the burnt gas temperature T_b can be calculated from overall energy considerations. If such a system is examined on a large enough scale (formally this is achieved by rescaling distances using a characteristic length ℓ and then taking the limit $\ell \to \infty$) the combustion field, in general, divides into two distinct regions, one consisting of unburnt gas ($Y=Y_f, T=T_f$), the other burnt ($Y=0$, $T=T_b$), both of which are governed by Euler's equations. These regions are separated by a thin sheet across which the mass and momentum fluxes are conserved, and these conditions, together with the specification of the velocity of the sheet relative to the gas flow, lead to a purely hydrodynamical free boundary problem (e.g. Landau 1944).

Damkohler Number Asymptotics

In the case of diffusion flames, the reactants (oxidant Y_O and fueld Y_F) are initially separated and then an appropriate choice for ω is

$$\omega = DY_OY_Fe^{-\theta/T} \tag{5}$$

where D is the Damkohler number. In the limit $D \to \infty$ a large reaction rate can only be balanced by other terms (derivatives) in thin sheets or for small time intervals, so that in most places and for most times we are led to the (singular) limit

$$Y_O = 0 \text{ or } Y_F = 0 \tag{6}$$

and a thin reaction zone separates a reaction-free region where there is no fuel from one where there is no oxidant. Examination of the singular limit shows that ω has the character of a Dirac δ-function so that T, Y_O and Y_F are continuous across the sheet, but their normal derivatives jump. These jumps are related by $\frac{1}{\alpha_i L_i}\,\delta(\frac{\partial Y_i}{\partial n}) = \frac{1}{\alpha_T}\,\delta(\frac{\partial T}{\partial n})$. All of these conditions are enough to specify the location of the sheet when the field equations are solved. Burke & Schumann (1928) were the first to consider this limit, but formal treatments in the modern style are much more recent (e.g. Kassoy & Williams 1968).

Activation Energy Asymptotics

So far we have examined the classical elements of FBP in combustion. We now turn to developments that are, in essence, less than 10 years old. They pass under the rubric of activation energy asymptotics which is extensively discussed in a soon-to-be-published monograph by Buckmaster & Ludford (1982). The approach can be usefully applied to both diffusion and premixed flames, but we shall confine our discussion to the latter, with the reaction rate (4).

Consider the limit $\theta \to \infty$. By itself this is irrelevant to combustion since ω then vanishes, but if we consider a distinguished limit $D \to \infty$, $\theta \to \infty$ with $D = e^{\theta/T_*}$ where T_* characterizes the magnitude of D, then profound simplifications occur. For in regions where $T < T_*$ ω vanishes in the limit; where $T > T_*$ we are led to $Y = 0$ (for the same reasons that lead to (6)) and again ω vanishes but now because there is no reactant; and for temperatures within $O(\theta^{-1})$ of T_* there is finite reaction and this is usually confined to a thin sheet. This reaction or flame sheet is the free boundary.

In order to formulate a complete description of the FBP it is necessary to consider special classes of problems, and it is important to understand why this is so. Premixed flames have a wavelike character, in that they propagate at a well-defined speed relative to the unburnt gas. Indeed there exists a one-dimensional progressive wave solution of the equations (called a deflagration wave) in which the temperature increases monotonically from the cold temperature T_f to the burnt temperature T_b. A fundamental problem is the calculation of the propagation speed Q_f (for use in the hydrodynamic limit for example), and activation energy asymptotics provides an analytical solution to this problem (Bush & Fendell, 1970) which reveals that $Q_f \propto \exp -[\frac{\theta}{2T_b}]$. It follows that small changes in T_b generate large changes in the flame speed, which implies that if we hope to provide a mathematical structure which will allow us to follow a fluctuating flame through time and space, there will be fundamental difficulties unless, during the passage, the temperature behind the flame changes but a little. There are two known ways to achieve this. Slowly Varying Flames (Sivashinsky 1976, Buckmaster 1977) are characterized by evolution on time and distance scales that are θ multiples of the natural scales for the flame. Near Equidiffusion Flames have the characteristics $L - 1 = \frac{\lambda}{\theta}$, $T + Y = T_f + Y_f + \frac{1}{\theta}\phi$, $\lambda, \phi = O(1)$, are the more important of the two, and are the focus of the remainder of our discussion. Their definition is implicit in the stability analysis of Sivashinsky (1977) and it receives formal statement in Matkowsky & Sivashinsky (1979).

The FBP is defined by chemistry free equations on each side of the flame sheet whose structure defines appropriate jump conditions. Thus for the system (1), (3) it is necessary to solve the field equation

$$\frac{\partial T}{\partial t} + (\underset{\sim}{v}.\underset{\sim}{\nabla})T = \nabla T, \quad \frac{\partial \phi}{\partial t} + (\underset{\sim}{v}.\underset{\sim}{\nabla})\phi = \Delta\phi + \lambda\Delta T \tag{7}$$

where T, ϕ now stand for leading terms in the asymptotic development for large θ, with the additional simplification that $T = T_f + Y_f$ on the hot side of the sheet. Jump conditions are

$$\delta(T) = \delta(\phi) = 0; \quad \delta\left(\frac{\partial T}{\partial n}\right) = -\frac{1}{\lambda}\,\delta\left(\frac{\partial \phi}{\partial n}\right) = -Y_f \exp\frac{\phi}{2(T_f+Y_f)^2} \tag{8}$$

where n is measured normal to the sheet towards the hot gas and δ denotes hot conditions minus cold conditions.

One-Dimensional Deflagration and Perturbation

In the case of a uniform steady flow ($\underset{\sim}{v} = (U,0)$) the system (7), (8) has a one-dimensional steady solution provided $U > 1$. The flame sheet slopes down from left to right at an angle $\arcsin(U^{-1})$ to the flow, so that the non-dimensional flame speed (the normal component of the gas speed) is unity. This solution is

$$\underline{n < 0}\ T = T_f + Y_f e^n, \ \phi = -\lambda Y_f n e^n; \ \underline{n > 0}\ T = T_f + Y_f, \ \phi = 0. \tag{9}$$

Any nonuniform disturbances introduced into the combustion field will change the flame speed in a nonuniform fashion and cause the flame sheet to curve. An example is a noncatalytic wall at $y = 0$ through which $O(\theta^{-1})$ heat losses are permitted. Corresponding boundary conditions might be

$$\underline{y = 0}\ \frac{\partial T}{\partial y} = 0, \quad \frac{\partial \phi}{\partial y} = \alpha(T-T_f) \tag{10}$$

where the coefficient α is a measure of the heat loss. When α is zero the wall is adiabatic and so is equivalent to a line of symmetry; the configuration is then equivalent to that of a flame tip, the plane analogue of the conical tip commonly seen at the mouth of a Bunsen burner. For any α the system (7), (8), (10) with the upstream farfield defined by (9) describes an elliptic free boundary problem which has yet to be solved.

Reduction to a Parabolic System

The elliptic problem can be reduced to a more tractable parabolic problem if $U \gg 1$. Formally it is only necessary to write $x = U\chi$ and take the limit $U \to \infty$ with χ fixed. If at the same time we set $\lambda = 0$, ϕ then satisfies the heat equation everywhere (including the flame sheet) and in view of (10b) may be written

$$\phi = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\chi} ds \, \frac{[T_f - T(s,0)]}{\sqrt{\chi - s}} \exp\left[-\frac{1}{4}\frac{y^2}{\chi - s}\right]. \tag{11}$$

Then it remains to solve the heat equation for T on the cold side of the flame sheet (which is located at $y = h(\chi)$, say) subject to the boundary condition (10a), together with

$$\underline{y = h} \quad T = T_f + Y_f, \; \frac{\partial T}{\partial y} = Y_f \exp\left[\frac{\phi}{2(T_f + Y_f)^2}\right]. \tag{12}$$

Initial conditions are

$$\underline{\chi \to -\infty} \quad h \sim -\chi + o(1), \; T \sim T_f + Y_f e^{y+\chi}. \tag{13}$$

This is a classical problem of Stefan type, readily amenable to numerical analysis (Buckmaster 1979a), and solutions have been constructed for different values of α. These won't be reproduced here. For modest values of α the flame curves towards the wall, ultimately intersecting it at right angles. A local similarity solution can be constructed valid in the neighborhood of the intersection point. For larger values of α the flame has a tendency to curve away from the wall before finally curving in and intersecting. Moreover, there is unpublished evidence that if α is larger than some critical value the flame ultimately moves away from the wall ($h \propto \chi^{\frac{1}{4}}$) without ever intersecting. Thus the final state ie either intersection or everincreasing divergence from the wall, depending on the heat loss. It is commonly accepted that heat loss to a Bunsen burner rim sufficient to prevent the flame from penetrating too close to the surface is an essential mechanism for preventing flashback through the low speed flow near the surface. Thus the mathematical dichotomy is very suggestive.

Flame tips

In the discussion of flame tips (α=O) the variations with Lewis number (λ) are important. The large U approximation now leads to coupled (but still linear) parabolic equations which must be solved on both sides of the flame sheet. Numerical calculations are straightforward and have been carried out by Buckmaster (1979b) using Meyer's (1977) method of lines. Here there can be no question but that the flame always intersects the wall, but if λ is negative and of sufficient magnitude the flame assumes a bulbous shape (Figure 1). In its entirety this is unlike anything that is seen experimentally, but if the solution is terminated at the point Q it is strikingly similar to tips observed experimentally in the combusion of lean hydrogen mixtures (Lewis and von Elbe, 1961), consistent with these values of λ. Unfortunately, although the flame speed at Q is zero (and negative immediately downstream) there is no sound mathematical reason for such a termination and the comparison is merely suggestive. At best the mathematical solutions display significantly decreased reaction rate near the tip (ϕ at the flame sheet assumes large negative values) but there is no termination.

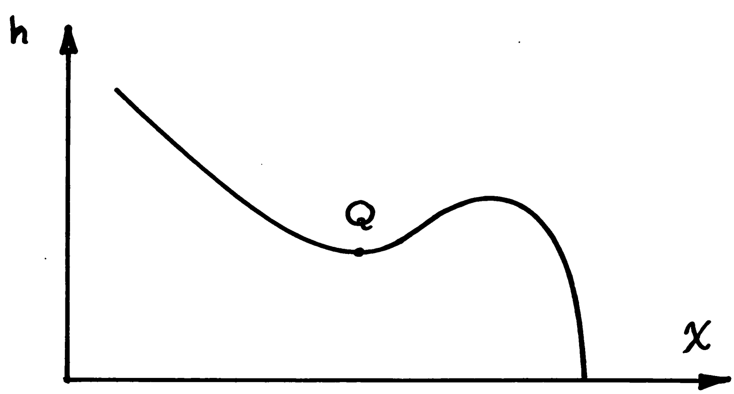

Figure 1. Bulbous Flame Tip

Effects of Shear Flow

Treatments of this type are not restricted to uniform flow. We can, for example, remove the wall and impose a parallel shear flow $\underset{\sim}{v} = U(1-ky)(1,0)$ in the half-space $y < 0$, maintaining a uniform flow in the upper half plane. Now the flame will begin to curve as the fringes of the preheat zone (the region of nonuniform temperature ahead of the flame sheet) penetrate the shear. This is no more difficult to treat than a uniform flow and Buckmaster (1979a) has presented solutions for different values of the velocity gradient k and Lewis number. An interesting limit is $k \to \infty$, for then the solution is once again confined to the region $y > 0$, with boundary conditions $T = T_f$, $\phi = 0$ at $y = 0$. There is then a final solution in which $h \to \exp[\lambda Y_f/2(T_f+Y_f)^2]$ as $\chi \to \infty$, and the field quantities depend only on y. This final state is readily amenable to a stability analysis, a rare opportunity which is presently being exploited.

Straining Flows

A parabolic formulation is possible for straining flows as well as for shear flows. Let $v_s(s)$ be the velocity on a streamline, with s arclength along it. In the limit $v_s \to \infty$, $v_s' = O(1)$, stream line curvature vanishes and the component of velocity normal to the streamline is $v_n = -nv_s'$. Then the equation for T beyond the flame sheet becomes

$$v_s \frac{\partial T}{\partial s} - nv_s' \frac{\partial T}{\partial n} = \frac{\partial^2 T}{\partial n^2} \tag{14}$$

with a corresponding equation for ϕ, and these can be integrated for different choices of v_s. Buckmaster (1979c) has done such a calculation for

$$\tilde{v}_s = 2\beta \sin(\tilde{s}), \; \tilde{v}_s = \epsilon v_s, \; \tilde{s} = \epsilon s, \; \epsilon \to 0 \tag{15}$$

corresponding to potential flow round a large circular cylinder. At the front stagnation point ($\tilde{s}\to 0$) the description is one-dimensional and an analytical description is possible (loc. cit.). This shows that there is a maximum straining rate β that the flame can tolerate. For some values of λ the maximum corresponds to impingement of the flame-sheet on the surface; for others the flame sheet is detached from the surface when β is a maximum. This dichotomy is observed experimentally (Ishizuka, Miyasaka & Law, 1981) and is in qualitative agreement with the theoretical predictions.

With the exception of the stagnation point analysis the discussion so far has been concerned only with numerical computations, the main focus of all efforts to date. An additional exception, which therefore must be mentioned, concerns the response of a flame to a straining flow of the form

$$\tilde{v}_s = 1 + \delta^{\frac{1}{2}} f(\tilde{s}/\delta), \quad \delta \to 0 \tag{16}$$

where f is a smooth function that vanishes far upstream ($f \to 0$ as $\tilde{s} \to -\infty$). In the limit an analysis is possible along the lines of triple-deck theory in the study of boundary layers and leads to an explicit description of the position of the flame sheet (Buckmaster, 1981). The corresponding flame speed is

$$Q_f = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\tilde{s}/\delta} \frac{dt\ f'(t)}{\sqrt{\tilde{s}/\delta - t}} . \tag{17}$$

A sufficiently large positive straining rate can reduce the flame speed to negative values.

Nonlinear Field Equations

The FBP's discussed so far are characterized by linear field equations. This is simply a consequence of the constant density approximation, and incorporation of the fluid mechanics eliminates that simplification. To the jump conditions (8) must now be added

$$\delta(\tilde{v}) = 0, \quad \delta(p) = \frac{4}{3} \frac{P_r v_n}{(T_f + Y_f)} \delta\left(\frac{\partial T}{\partial n}\right) .$$

No calculations including the full fluid mechanics have yet been undertaken, but in the case of flame tips a parabolic approximation ($U \gg 1$) leads to the rational approximation $\tilde{v} \simeq (U, 0)$ so that only the equation of state (2b) needs to be appended to (1) to obtain a closed system. A. Crowley has done some calculations for this system, and there is a preliminary report in these proceedings.

Future Prospects

The elliptic problem, three-dimensional problems, and full incorporation of the Navier-Stokes equations await future study. Let me conclude by describing just one three dimensional problem that is likely to be interesting.

Sivashinsky (1977) has shown that if $|\lambda|$ is large enough, the plane deflagration wave is unstable. For $\lambda < 0$ this instability leads to cellular flames and is associated with the growth of two-dimensional (not plane) disturbances. It follows, for appropriate choices of λ, that if we formulate an axisymmetric flame tip problem, as in the earlier plane case, and impose an asimuthal perturbation to the far-field structure at the initial station, this perturbation will grow with the time-like variable χ and will surely lead to a finite nonaxisymmetric structure rather than a nominally conical flame. Polyhedral flame tips, which have this character, are seen experimentally (Lewis & von Elbe, 1961).

Acknowledgement

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A B CROWLEY

Numerical solution of flame front free boundary problems

1. INTRODUCTION

This paper describes the application of weak solution techniques to the numerical solution of some models of flame fronts as free boundary problems. The advantage of the weak solution approach is that the governing differential equations are reformulated in conservation form, so that no boundary conditions need be explicitly applied at the flame front. This method is based on the so-called enthalpy method for the classical Stefan problem [1,2], and is motivated by the success of this approach in tackling a wide range of moving boundary problems such as electro-chemical machining [3], saturated-unsaturated flow in porous media [3], solidification of a binary alloy [4]. The flame front models considered here are those derived by Buckmaster [5,6,7] using asymptotic analysis in the limit of large activation energy, from the reaction-diffusion equation for an exothermic reaction coupled to the heat conduction equation. The case treated here is that of a slender tip in a laminar premixed flame. The weak solution approach has also produced good results [8] for the model of flame fronts in solid fuel described by Matkowsky and Sivashinsky [9].

First the governing equations for two mathematical models of flame tips, one making the constant density approximation, the other without, are stated. In the next section the equations are recast into conservation form, so that the jump conditions at the flame front are automatically satisfied. This yields a pair of coupled parabolic equations in the first case. In the second model this reformulation yields one parabolic equation and one equation of mixed type, which is parabolic in one region and hyperbolic in the other. The systems of equations may then be solved numerically throughout a fixed domain as described in the third section. In the last section some results for both models are given. It should be noted that although the equations for planar flame tips will be considered in each case for simplicity, the reformulation is equally applicable to the axisymmetric flame tips described in [6].

The first model considered is that for slender flame tips in the constant density approximation and the large activation energy limit derived by Buckmaster [5]. In this approximation it is assumed that all density changes in the gaseous fuel, which may arise from the temperature variations, are negligible. The configuration is shown schematically in Figure 1, where cartesian axes are chosen aligned with the inlet gas velocity, and the x direction is scaled so that this velocity is unity.

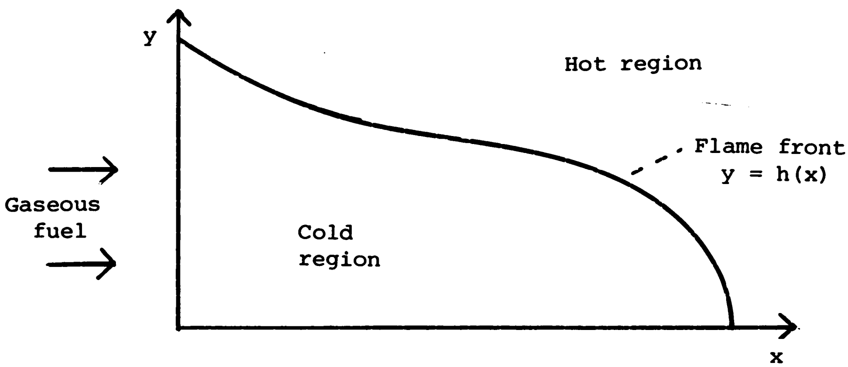

Figure 1. Flame tip configuration

The governing equations in $0 < y < h(x)$ are

$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \tag{1.1}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} + \lambda \frac{\partial^2 T}{\partial y^2}, \tag{1.2}$$

and in $y > h(x)$,

$$T = 1 + T_\infty, \tag{1.3}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y^2}. \tag{1.4}$$

Here T denotes temperature and ϕ is a measure of the total energy (= heat + energy stored in fuel).

$$\lambda = (1 - \frac{1}{L})\theta, \ \lambda \sim O(1),$$

where $\theta \gg 1$ is the activation energy and $L \sim 1$ is the Lewis number. At the flame front $y = h(x)$

$$T = 1 + T_\infty, [\phi]^* = 0, \tag{1.5, 6}$$

$$\left[\frac{\partial \phi}{\partial y}\right] = -\lambda\left[\frac{\partial T}{\partial y}\right] = \lambda \exp\left(\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right). \tag{1.7, 8}$$

The initial and boundary conditions are

$$T(0, y) = \begin{cases} T_\infty + \exp(y - h(0)) & y < h(0) \\ \\ 1 + T_\infty & y > h(0), \end{cases} \tag{1.9}$$

$$\phi(0, y) = \begin{cases} -\lambda(y - h(0)) \exp(y - h(0)) & y < h(0) \\ \\ 0 & y > h(0) \end{cases} \tag{1.10}$$

On $y = 0$,

$$\frac{\partial T}{\partial y} = 0 = \frac{\partial \phi}{\partial y}, \tag{1.11}$$

$$\phi \to 0 \text{ as } y \to \infty. \tag{1.12}$$

The second model with the same configuration is again derived in the large activation energy limit by Buckmaster [7]. This model allows for density changes in the gas, assuming that the pressure remains constant. Consequently there is outward flow of the gas as it is heated. This will henceforward be referred to as the model with fluid mechanics. Here the governing equations are, in $y < h(x)$,

$$\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial y}(\rho q) = 0, \tag{1.13}$$

$$\rho \frac{\partial T}{\partial x} + \rho q \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2}, \tag{1.14}$$

$$\rho \frac{\partial \phi}{\partial x} + \rho q \frac{\partial \phi}{\partial y} = \frac{\partial^2 \phi}{\partial y^2} + \lambda \frac{\partial^2 T}{\partial y^2} \tag{1.15}$$

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* $[f] \equiv f|_{y=h_+} - f|_{y=h_-}$ is the jump in f across the curve $y = h(x)$.

together with Charles' law

$$\rho T = T_\infty.$$

In addition to the variables already defined ρ denotes the gas density and q the gas velocity in the y direction. In $y > h(x)$ we have again $T = 1 + T_\infty$ and the constant temperature forms of (1.13-1.15). The jump conditions are as before with $[q] = 0$ on the flame front. The boundary and initial conditions are also similar with

$$q = \begin{cases} \dfrac{1}{T_\infty} \exp\,(y - h(0)) & y < h(0) \\ \\ \dfrac{1}{T_\infty} & y > h(0), \end{cases} \tag{1.16}$$

and

$$q = 0,\ \frac{\partial \rho}{\partial y} = 0 \text{ on } y = 0. \tag{1.17}$$

2. WEAK FORMULATIONS

The model equations are next rewritten in conservation form, so that all the conditions at the flame front are automatically satisfied.

i) Constant density model

Comparison of (1.1, 1.2) and (1.3, 1.4) shows immediately that (1.1, 1.2) may be considered as holding in both regions $y < h(x)$, $y > h(x)$. Thus

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y^2} + \lambda \frac{\partial^2 T}{\partial y^2} \tag{2.1}$$

in both regions. The corresponding conservation condition at the front is

$$[\phi] \frac{dh}{dx} = - \left[\frac{\partial \phi}{\partial y}\right] - \lambda \left[\frac{\partial T}{\partial y}\right],$$

which yields

$$[\phi] = 0,$$

$$\left[\frac{\partial \phi}{\partial y}\right] = - \lambda \left[\frac{\partial T}{\partial y}\right],$$

which are precisely the desired jump conditions (1.6, 1.7).

The temperature equation (1.1) leads to the jump condition

$$[T] \frac{dh}{dx} = - \left[\frac{\partial T}{\partial y}\right] .$$

This is not the required jump condition, and hence (1.1) must be modified by the addition of a term which vanishes in both regions, but contributes at the flame front, that is, a delta function. A suitable amendment is

$$\frac{\partial T}{\partial x} + \frac{\partial}{\partial y} \left\{ H(1 + T_\infty - T) \exp\left(\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right)\right\} = \frac{\partial^2 T}{\partial y^2} \tag{2.2}$$

where H(u) is the Heaviside function defined by

$$H(U) = \begin{cases} 0 & u \le 0 \\ 1 & u > 0 \end{cases} ,$$

yielding the jump condition

$$[T] \frac{dh}{dx} = \left[\frac{\partial T}{\partial y}\right] - \left[H(1 + T_\infty - T) \exp\left(\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right)\right] ,$$

which simplifies to

$$[T] = 0, \; T = 1 + T_\infty,$$

$$\left[\frac{\partial T}{\partial y}\right] = \left[H(1 + T_\infty - T) \exp\left(\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right)\right]$$

$$= - \exp\left(\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right) .$$

These are the required jump conditions (1.5, 1.8), and hence (2.1), (2.2) is the desired reformulation of the governing equations and jump conditions.

ii) Fluid mechanics model.

(1.13-1.15) are first rearranged in conservation form. As in the constant density model, a δ-function term must be added to the temperature equation to yield the correct jump conditions at the flame front. It is also useful, for reasons which will be made clear later, to introduce a temperature dependent thermal conductivity $\kappa(T)$ which is unity in the cold region, and which may be chosen freely in the hot region $y > h(x)$. (1.13-1.15) may then be written as

$$\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial y}(\rho q) = 0, \tag{2.3}$$

$$\frac{\partial}{\partial x}(\rho T) + \frac{\partial}{\partial y}(\rho q T) + \frac{\partial}{\partial y}\left\{ H(1 + T_\infty - T) \exp\left[\frac{\phi(x, h(x))}{2(1 + T_\infty)}\right]\right\}$$
$$= \frac{\partial}{\partial y}\left(\kappa(T) \frac{\partial T}{\partial y}\right), \tag{2.4}$$

$$\frac{\partial}{\partial x}(\rho\phi) + \frac{\partial}{\partial y}(\rho q \phi) = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial}{\partial y}\left[\kappa(T) \frac{\partial T}{\partial y}\right], \tag{2.5}$$

with $\rho T = T_\infty$, $\kappa(T) \equiv 1$ for $T < 1 + T_\infty$. (2.6)

Using Charles' law and the boundary condition (1.17), (2.5) may be integrated once to give

$$q = \frac{1}{T_\infty}\left\{\kappa(T) \frac{\partial T}{\partial y} + (1 - H(1 + T_\infty - T)) \exp\left[\frac{\phi(x, h(x))}{2(1 + T_\infty)^2}\right]\right\}.$$

Using Charles' law again, substituting for q, and writing $\rho_\infty = T_\infty/(1 + T_\infty)$, the density equation (2.3) may be rewritten as

$$\frac{\partial \rho}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\kappa(\rho)}{\rho} \frac{\partial \rho}{\partial y} - \rho G(\rho)\right], \tag{2.7}$$

where

$$\kappa(\rho) = 1, \ \rho > \rho_\infty,$$

$$G(\rho) = \frac{1}{T_\infty}(1 - H(\rho - \rho_\infty)) \exp\frac{\phi(x, h(x))}{2(1 + T_\infty)^2},$$

and q is now given by

$$q = \left\{-\frac{\kappa(\rho)}{\rho^2} \frac{\partial \rho}{\partial y} + G(\rho)\right\}. \tag{2.8}$$

Thus the problem is fully described by the differential equations (2.5) and (2.7) together with the constitutive relations (2.6) and (2.8). At the flame front (2.5) and (2.7) yield the required jump conditions independently of the choice of $\kappa(\rho)$ for $\rho \leq \rho_\infty$.

In $y > h(x)$ the model imposes the condition $T \equiv 1 + T_\infty$, and consequently $\frac{\partial \rho}{\partial y}$ vanishes in this region. This condition is not explicitly imposed in the weak formulation, but should result in numerical solution with suitable initial conditions such as (1.9, 1.10, 1.16). The front is located as the smallest value of y for which $T = 1 + T_\infty$. However it is found from numerical experiment that the use of (2.8) with $\kappa(\rho) \equiv 1$ everywhere leads to a peak in the velocity q in $y > h(x)$ near the front, resulting from contributions to q from estimates of $\frac{\partial \rho}{\partial y}$. Rather then devise special numerical schemes near the front, it seems more appropriate to the spirit of the weak solution approach simply to define $\kappa(\rho) = 0$ for $\rho \leq \rho_\infty$ (the hot region) so that the velocity q is correct even if the density and temperature vary slightly in numerical computation. This choice leads to smaller variations of ρ and T in the hot region, and correspondingly to smoother curves for the flame front position, and is therefore felt to be vindicated. With this choice of $\kappa(\rho)$, (2.7) is of mixed type being parabolic in the cold region and hyperbolic in the hot region downstream of the flame front.

3. NUMERICAL SOLUTION

In each case finite different methods are used to solve the conservation form of the governing equations. The temperature or density equation is solved explicitly, and then the equation for ϕ is solved implicitly using a Crank-Nicolson difference scheme together with successive over-relaxation. The position of the flame front is located by quadratic extrapolation on the values of temperature from the cold region $y < h(x)$. The value of ϕ at the front is determined as the mean of the values obtained by quadratic extrapolation from $y < h(x)$ and $y > h(x)$.

The details of the finite difference scheme for the constant density model of the flame tips are relatively straightforward, and have been published previously in [8], to which the interested reader is referred. Here the scheme for the model with fluid mechanics is described.

Let f_n^i denote the value of f at the mesh point $(i\,\delta x, n\,\delta y)$. Then (2.7) becomes

$$\rho_n^{i+1} = \rho_n^i + \frac{\delta x}{\delta y} \left[(\rho q)_{n+\frac{1}{2}}^i - (\rho q)_{n-\frac{1}{2}}^i \right] , \tag{3.1}$$

where

$$\rho q^{i}_{n+\frac{1}{2}} = \frac{\kappa(\rho^{i}_{n+\frac{1}{2}})}{\rho^{i}_{n+\frac{1}{2}}} \frac{\rho^{i}_{n+1} - \rho^{i}_{n}}{\delta y} - \rho^{i}_{n+\frac{1}{2}}\, G(\rho^{i}_{n+\frac{1}{2}}), \tag{3.2}$$

$$\rho^{i}_{n+\frac{1}{2}} = (\rho^{i}_{n} + \rho^{i}_{n+1})/2, \tag{3.3}$$

and

$$\kappa(\rho) = \begin{cases} 1 & \rho > \rho_\infty, \\ 0 & \rho \le \rho_\infty, \end{cases} \tag{3.4}$$

and

$$G(\rho) = \begin{cases} 0 & \rho > \rho_\infty \\ \frac{1}{T_\infty} \exp\left(\frac{\phi(x,\ h(x))}{2(1 + T_\infty)^2}\right) & \rho < \rho_\infty \end{cases} \tag{3.5}$$

The finite difference replacement of (2.5) is

$$\begin{aligned} \rho^{i+1}_{n}\, \phi^{i+1}_{n} = \rho^{i}_{n}\, \phi^{i}_{n} &- \frac{\delta x}{2\delta y}\left(\rho q^{i+1}_{n+\frac{1}{2}}\, \phi^{i+1}_{n+\frac{1}{2}} - \rho q^{i+1}_{n-\frac{1}{2}}\, \phi^{i+1}_{n-\frac{1}{2}}\right) \\ &- \frac{\delta x}{2\delta y}\left(\rho q^{i}_{n+\frac{1}{2}}\, \phi^{i}_{n+\frac{1}{2}} - \rho q^{i}_{n-\frac{1}{2}}\, \phi^{i}_{n-\frac{1}{2}}\right) \\ &+ \frac{1}{2}\frac{\delta x}{\delta y^2}\Bigg\{\left(\phi^{i+1}_{n+1} - 2\phi^{i+1}_{n} + \phi^{i+1}_{n-1}\right) + \left(\phi^{i}_{n+1} - 2\phi^{i}_{n} + \phi^{i}_{n-1}\right) \\ &+ \lambda\left(\kappa\left(\rho^{i+1}_{n+\frac{1}{2}}\right)\left(T^{i+1}_{n+1} - T^{i+1}_{n}\right) - \kappa\left(\rho^{i+1}_{n-\frac{1}{2}}\right)\left(T^{i+1}_{n} - T^{i+1}_{n-1}\right)\right) \\ &+ \lambda\left(\kappa\left(\rho^{i}_{n+\frac{1}{2}}\right)\left(T^{i}_{n+1} - T^{i}_{n}\right) - \kappa\left(\rho^{i}_{n-\frac{1}{2}}\right)\left(T^{i}_{n} - T^{i}_{n-1}\right)\right)\Bigg\}, \end{aligned} \tag{3.6}$$

where

$$\rho^{i}_{n}\, T^{i}_{n} = T_\infty. \tag{3.7}$$

The solution procedure is then as follows. Knowing all variables at i δx:

i) Use (3.1) to calculate the values of ρ at $(i+1)\delta x$.

ii) Calculate new values of T using Charles' law (3.7).

iii) Locate the flame front by extrapolation on T from $T < 1 + T_\infty$.

iv) Solve (3.6) iteratively for ϕ^{i+1}_{n}.

v) Calculate new value of $\phi(x,\ h(x))$ using results of (iii) and (iv), and hence obtain improved values of $\rho q_{n+\frac{1}{2}}^{i+1}$ using the new values of $G\left(\rho_{n+\frac{1}{2}}^{i+1}\right)$.

vi) Solve (3.6) using improved values of $\rho q_{n+\frac{1}{2}}^{i+1}$.

Iterate on (v), (vi) until convergence is obtained.

Both models are solved subject to the initial and boundary conditions (1.9-1.11) with, in addition, (1.16) for the model with fluid mechanics. For numerical computation additional boundary conditions are imposed on some boundary $y = \ell > h(0)$ to make the domain finite. For the constant density model these are

$$T = 1 + T_\infty, \ \phi = 0 \text{ on } y = \ell.$$

In the second model, since the density equation is hyperbolic in $y > h(x)$, only the condition on ϕ is required at $y = \ell$.

4. RESULTS AND DISCUSSIONS

All the results described in this section are calculated using $h(0) = 5$, $\ell = 6$, $T_\infty = 0.2$ and $\delta y = 0.05$. For the first model, $\delta x = 0.001$, while in the model with fluid mechanics δx is taken close the stability limit for the explicit representation of the density equation being, in this case, $\delta x = 0.0002$.

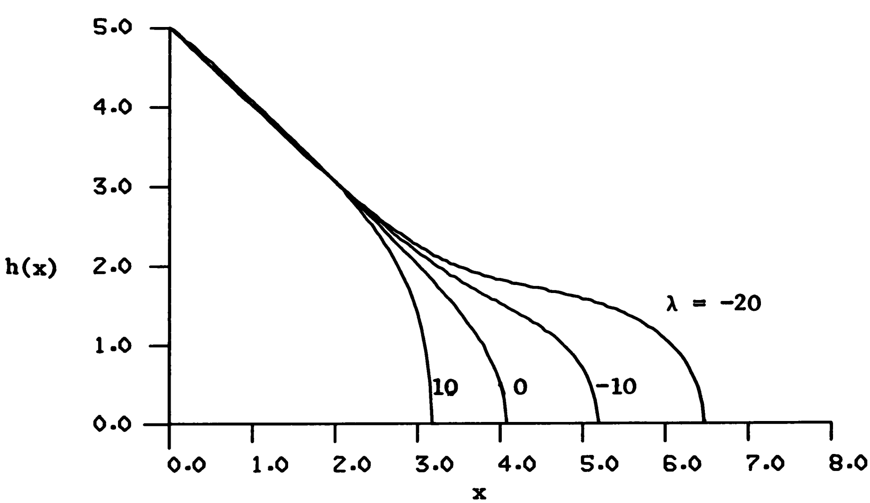

Figure 2. Flame tip shapes for various λ in the constant density approximation

In Figure 2 the flame tip shapes computed for various values of λ in the constant density approximation are shown.

Figure 3 shows the flame tip shapes for the same parameters using the model with fluid mechanics. The effect of the gas velocity in the y direction is to increase the change of shape with variation of λ, the activation energy parameter. In the case of $\lambda = -20$, the front $y = h(x)$

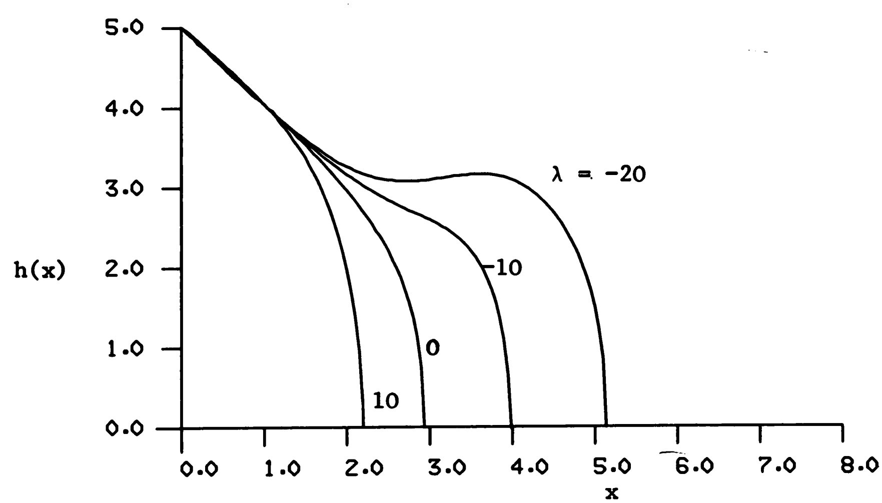

Figure 3. Flame tip shapes for various λ from the model with fluid mechanics

becomes non-monotonic in the model with fluid mechanics, but this causes no difficulty in the weak solution approach to the numerical computation described here. In the model with fluid mechanics, the stream-function for the gas flow is given by

$$\frac{\partial\psi}{\partial x} = -\frac{\rho q}{\rho_\infty}, \quad \frac{\partial\psi}{\partial y} = \frac{\rho}{\rho_\infty} .$$

In Figure 4 the streamlines ψ = constant are plotted for the representative case $\lambda = 0$.

These results, together with those described in [8] for the problem of flame fronts in solid fuel discussed by Matkowsky and Sivashinsky [9] indicate that this approach can be used satisfactorily for the numerical solution of flame front problems as free boundary problems. The advantages

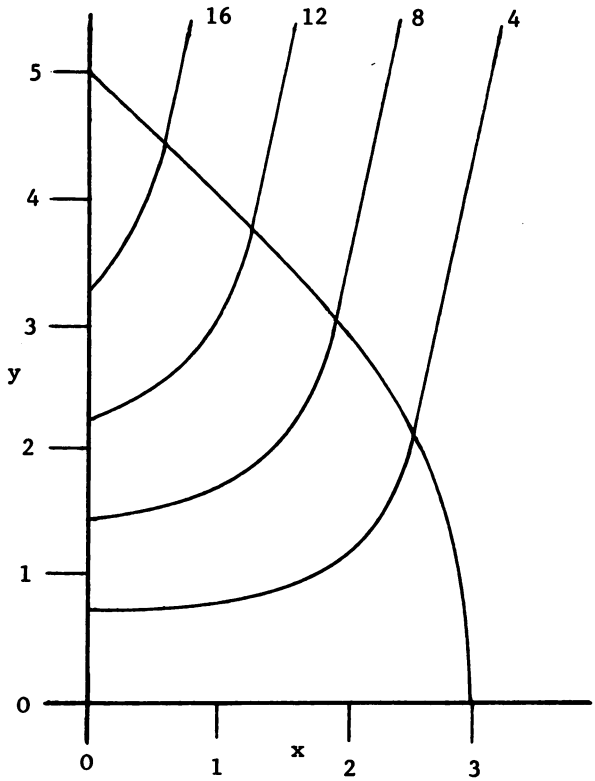

Figure 4. Streamlines ψ = constant for the case $\lambda = 0$ calculated from the model with fluid mechanics

of the weak solution technique are clear. The problem is formulated and solved on a fixed domain, and at the flame front in the interior no jump conditions are explicitly applied, these having been incorporated as the Rankine-Hugoniot conditions for the conservation form of the equations. However, in these flame problems the front must be located as part of the solution in order to calculate the term exp $(\phi(x, h(x))/(1 + T_\infty)^2)$. This is in contrast to the situation in the classical Stefan problem, where the position of the phase change interface need not be determined.

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C M ELLIOTT

A variational inequality formulation of a steady state electrochemical machining free boundary problem

Electrochemical machining [10] concerns the machining of a metal workpiece (the anode) by a metal tool piece (the cathode). A potential difference is applied across the gap which separates the electrodes and which is occupied by a suitable electrolyte causing the electrochemical erosion of the anode surface. Let $\phi(\underset{\sim}{x},t)$ at time $t > 0$, $\underset{\sim}{x} \in D$, denote the potential in the electrolyte $\Omega(t)$ and be identically zero in the anode $D\backslash\Omega(t)$. If the cathode is fed towards the anode with a constant velocity, then in a coordinate system for which the cathode is stationary, the quasi-steady model of [10] leads to the nonlinear evolution equation:

$$\frac{\partial H}{\partial t} - \gamma \frac{\partial H}{\partial y} = \nabla^2\phi \quad \cdot \quad \underset{\sim}{x} \in D, \quad t > 0 \tag{1}$$

where

$$H \in \beta(\phi) = \begin{cases} 0 & \phi > 0 \\ [-1,0] & \phi = 0 \\ -1 & \phi < 0 \end{cases} \tag{2}$$

and $\phi = -1$ on the fixed cathode surface C. (3)

The solution has the property that $\phi = 0$ in the anode $D\backslash\Omega(t)$, $\phi < 0$ and $\nabla^2\phi = 0$ in $\Omega(t)$ and on the unknown moving anode surface $S(\underset{\sim}{x},t) = 0$

$$[H]\frac{\partial S}{\partial t} = [\nabla\phi + H\gamma \underset{\sim}{e}_y]\cdot\nabla S \tag{4}$$

where $[\cdot]$ denotes a jump of the quantity inside the brackets across the surface $S(\underset{\sim}{x},t) = 0$ and $\underset{\sim}{e}_y$ is the unit vector in the y-direction which we have taken to be the feed direction of the cathode.

We note that (1) and (2) also model the time dependent dam problem [7] and are related to the one phase Stefan problem with zero specific heat. By integrating in the cylinder $D \times (0,T)$ along the direction $\underset{\sim}{V} = (-\gamma \underset{\sim}{e}_y, 1)$ it is possible to obtain a linear complementarity system for the transformed variable [3,6,12,13]

$$u = \int_0^{s^*} \phi((\underset{\sim}{x}_0, 0) + s\underset{\sim}{V})\, ds \qquad (5)$$

The case $\gamma = 0$ was considered in [4,5]. The time dependent dam problem was studied by means of the transformation (5) in [13]. The steady state rectangular dam problem was solved in [1] by the original use of (5) and it is this work we follow here. A planar ECM problem, [8], is considered in which the cathode is given by the surface $y = c(x)$ which is symmetric about the line $x = 0$. It is supposed that the asymptotic solution as $x \to \infty$ is known so that the problem is consdered on the finite domain

$$D = \{(x,y): 0 < x < L,\ c(x) < y < Y_2\}\ .$$

Let $c(x) \in C^2[0,L]$ satisfy

$$c(0) = 0,\ c(L) = Y_1,\ c'(0) = 0,\ c''(x) > 0 \qquad (6)$$

and $\Phi(y) \in C^1[Y_1,Y_2]$ satisfy

$$\Phi(Y_1) = -1,\ \Phi(Y_2) = 0,\ \Phi(y) \le 0,\quad 0 \le \Phi'(y) \le \gamma. \qquad (7)$$

The mathematical model leads to the free boundary problem:-

(P1) Find a curve Γ, the anode surface, defined by $y = a(x)$ $x \in (0,L)$ such that

$$a(L) = Y_2,\ a(x) > c(x)\quad x \in [0,L],\quad a(x) \in C[0,L] \cap C^\infty(0,L) \qquad (8)$$

and a function $\phi(x,y) \in H^1(\Omega) \cap C(\bar{\Omega})$ where

$$\Omega \equiv \{(x,y): 0 < x < L,\quad c(x) < y < a(x)\} \qquad (9)$$

such that

$$\nabla^2\phi = 0 \quad \text{in } \Omega \qquad (10)$$

$$\phi(x,c(x)) = -1 \quad x \in [0,L],\quad \phi(L,y) = \Phi(y) \quad y \in [Y_1,Y_2] \qquad (11)$$

and $\quad \phi_x(0,y) = 0 \quad y \in (0,a(0))$

$$\phi = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial\nu} = \gamma/(1+a'(x)^2))^{\frac{1}{2}} \quad \text{on } \Gamma \qquad (12)$$

where ν is the outward pointing unit normal to the surface Γ of Ω. We define also

$$\Gamma_0 \equiv \{(x,y): 0 < x < L, \quad y = c(x)\}, \quad \Gamma_1 \equiv \{(0,y): 0 < y < Y_2\}$$

$$\Gamma_2 \equiv \{(L,y): Y_1 < y < Y_2\}, \qquad \Gamma_3 \equiv \{(x,Y_2): 0 < x < L\} .$$

From the maximum principle we have that

$$-1 < \phi(x,y) < 0 \quad \text{in } \Omega \tag{13}$$

and with the extension $\phi \equiv 0$ in $D\backslash\Omega$, we have from (1)

$$-\gamma \frac{\partial H}{\partial y} = \nabla^2\phi \quad \text{in} \quad D \tag{14}$$

in the sense of distributions, where $H = -1$ in Ω and $H = 0$ in $D\backslash\Omega$. The transformation (5) becomes

$$u(x,y) = -\int_y^{Y_2} \phi(x,\tau)\,d\tau \quad \text{in} \quad D \tag{15}$$

and a straightforward calculation gives the problem for u to be :-

(P2) Find $u \in H^2(\Omega) \cap C^1(\bar{D})$ such that

$$\nabla^2 u = \gamma(x,y) \in \Omega \equiv \{(x,y): u(x,y) > 0\} \tag{16}$$

$$u_y(x,c(x)) = -1 \; x \in (0,L), \; u(L,y) = -\int_y^{Y_2} \Phi(\tau)\,d\tau \equiv U \; y \in [Y_1,Y_2] \tag{17}$$

$$u_x(0,y) = 0 \quad y \in (0,Y_2) \text{ and } u(x,Y_2) = 0 \text{ on } \Gamma_3.$$

$$u = u_x = u_y = 0 \text{ on } \Gamma, \; u \equiv 0 \quad \bar{D}\backslash\Omega . \tag{18}$$

The first equation of (17) is an oblique derivative boundary condition which can be written as

$$\frac{\partial u}{\partial \nu} + c'(x)\,\frac{\partial u}{\partial \sigma} = -\,(1+c'(x)^2)^{\frac{1}{2}} \quad \text{on} \quad \Gamma_0 \tag{19}$$

where ν and σ are, respectively, the unit inward pointing normal and anti-clockwise tangential vectors to the curve Γ_0 at (x,y). Recalling the

Green's formula

$$\int_D -\nabla^2 uv\ dxdy = \int_D u_x(v_x+kv_y+k_yv) + u_y(v_y-kv_x-k_xv)\,dxdy + \qquad (20)$$

$$+ \int_{\partial D} \left(\frac{\partial u}{\partial \nu} + k\ \frac{\partial u}{\partial \sigma}\right) v\ d\sigma$$

which holds for differentiable k, $v \in H^1(D)$ and $y \in H^2(D)$, we introduce the bilinear form

$$a(u,v) = \int_D u_x(v_x+c'(x)v_y) + u_y(v_y-c'(x) \qquad (21a)$$

and the inner products

$$(u,v) = \int_D uv\ dxdy,\ \langle u,v\rangle = \int_0^L (1+c'(x)^2)^{\frac{1}{2}}\ u(x,c(x))\ v(x,c(x))\,dx. \qquad (21b)$$

For $v \in C^\infty(D)$ such that $v = 0\ \Gamma_2 \cup \Gamma_3$ we have that

$$a(v,v) = \int_D |\nabla v|^2\ dxdy + \int_{\Gamma_0} c''(x)\ \frac{v^2}{2}\ dx.$$

Hence

$$a(v,v) \geq |v|_1^{\ 2} \equiv \int_D |\nabla v|^2\ dxdy \quad \text{for } v \in \{H^1(D) : v = 0\ \Gamma_2 \cup \Gamma_3\}\ . \qquad (22)$$

<u>Lemma 1</u>

If a solution to (P2) exists then it is the unique solution of the variational inequality

$$u \quad K:\ a(u,v-u)\ (-\gamma,v-u) + \langle 1,v-u\rangle \quad \text{for all} \quad v \in K \qquad (23)$$

where

$$K = \{H^1(D) : v = U \text{ on } \Gamma_2,\ v = 0 \text{ on } \Gamma_3 \text{ and } v \geq 0 \quad \text{a.e.} D\}\quad .$$

<u>Proof</u>

It is a straightforward calculation to show that any solution of (P2) solves (23). That (23) has a unique solution follows from (22) and the theory of of variational inequalities [9].

Lemma 2

Let ω solve:-

$$\nabla^2\omega = h \text{ in } D,\ \omega = 0 \text{ on } \Gamma_3,\ \omega = U \text{ on } \Gamma_3,\ \omega_y = -1 \text{ on } \Gamma_0$$

where $h \in L^\infty(D)$ and $h = \gamma$ in a neighbourhood of $\Gamma_0 \cup \Gamma_2$. Then

$$\omega \in W^{2,p}(D) \text{ for any } 1 \le p < \infty.$$

Proof

The only difficulties arise at the corners of D where ∂D lacks smoothness. We now follow the arguments in [1,2]. Regularity at $(0,0)$ and $(0,Y_2)$ follows from symmetry and the smoothness of $c(x)$. Regularity at (L,Y_2) follows by considering $\hat{\omega}(x,y) \equiv \omega(x,y)$ in D and $\hat{\omega}(x,y) = -\omega(x,2Y_2-y)$ for $y > Y_2$, see [9,p.235]. Regularity at (L,Y_1) follows from the observation that $\nabla^2(\omega_y) = 0$ in a neighbourhood of (L,Y_1) and that $\omega_y = -1$ on Γ_0 and $\omega_y = U_y$ on Γ_2. So that regularity results for Laplace's equation in a convex Lipschitz domain imply that $\omega_y \in W^{2,2}$ in a neighbourhood of (L,Y_1) which now implies the result of the lemma by Sobolev's imbedding theorem.

Lemma 3

The solution to (23) satisfies:-

(i) $u \in W^{2,p}(D) \cap C^{1,\lambda}(\bar{D})$ for all $1 \le p < \infty$ and $\lambda \in (0,1)$

(ii) $\nabla^2 u = \gamma$ in Ω; $u = u_x = u_y = 0$ in $D\backslash\Omega$

(iii) $u_x = 0$ on Γ_1, $u_y = -1$ on Γ_0, $u = U$ on Γ_2, $u = 0\ \Gamma_3$.

Proof

For each $\varepsilon > 0$, define u_ε by the penalised problem:-

$$-\nabla^2 u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon^- = -\gamma \text{ in } D,\ u_\varepsilon = U \text{ on } \Gamma_2,\ u_\varepsilon = 0 \text{ on } \Gamma_3,$$

$$(u_\varepsilon)_x = 0 \text{ on } \Gamma_1 \text{ and } (u_\varepsilon)_y = -1 \text{ on } \Gamma_0, \text{ where } u_\varepsilon^- = \min(0,u_\varepsilon).$$

The unique solution to this problem converges to u in $H^1(D)$. (See [9] for example). Further, $\|u_\varepsilon^-\|_\infty \le \gamma\varepsilon$ and so u_ε satisfies the conditions of the Lemma 2 since $u_\varepsilon^- = 0$ in a neighbourhood of $\Gamma_0 \cup \Gamma_2$. From which we deduce that u_ε is bounded independently of ε in $W^{2,p}(D)$ and we obtain (i). (ii) and (iii) then follow from the regularity of u.

Lemma 4

The solution u to (23) satisfies

(i) $-1 \le u_y \le 0$ in D

(ii) The second derivatives of u are continuous in a neighbourhood of $\Gamma_0 \cup \Gamma_2$ and

$$u_{xx} = \gamma - \Phi_y \text{ on } \Gamma_2 \quad \text{and} \quad u_{xy} < 0 \text{ on } \Gamma_0.$$

(iii) $u_x \ge 0$ in D

(iv) $u_y = 0$ on Γ_3.

Proof

(i) We have from Lemma 3 that $u_y \in C^{0,\lambda}(\bar{D})$ and that $u_y = 0$ on $D\backslash\Omega$. The maximum/minimum principle for Laplace's equation $\nabla^2(u_y) = 0$ implies the result since $(u_y)_x = 0$ on Γ_1 and $u_y = \Phi$ on Γ_2.

(ii) $u \ge 0$ in D, $u > 0$ on Γ_2 and $u_y = -1$ on Γ_0 imply that there is a neighbourhood in Ω of $\Gamma_2 \cup \Gamma_0$. The smoothness of the boundaries Γ_2 and Γ_0 then implies the continuity of the second derivatives. Since $u_{yy} = \Phi_y$ on Γ_2 we obtain $u_{xx} = \gamma - \Phi_y$ on Γ_2. u_y attains its minimum value on Γ_0, hence by the Hopf principle $(u_y)_x$ (an outward pointing derivative since $c'(x) > 0$ on Γ_0) is negative and so $u_{xy} < 0$ on Γ_0.

(iii) We have that $u_x \in C(\bar{D})$, $\nabla^2(u_x) = 0$ in Ω, $u_x = 0$ on $\Gamma \cup \Gamma_1$, $u_{xx} = \gamma - \Phi_y \ge 0$ on Γ_2 and $(u_x)_y < 0$ on Γ_0. Thus by the Hopf principle, the minimum of u_x cannot lie on $\Gamma_2 \cup \Gamma_0$ and so $u_x \ge 0$ in D.

(iv) The argument for the same result in the rectangular dam situation may be applied without change [9, p.239].

Again following the argument in [9] we obtain

Lemma 5

(i) For $P = (x,y) \in D$ let $Q_p^+ = \{(x',y') \in D\colon x' < x,\ y' > y\}$ and $Q_p^- = \{(x',y') \in D\colon x' > x,\ y' < y\}$. If $P \in D\backslash\Omega$ then $Q_p^+ \subset \overline{D}\backslash\overline{\Omega}$ and if $P \in D \cap \overline{\Omega}$ then $Q_p^- \subset \overline{\Omega}$.

(ii) $\partial\Omega \cap \Gamma_3 = \{\phi\}$, $\partial\Omega \cap D \neq \{\phi\}$.

We can now define

$$a(x) = \inf\{y : (x,y) \in D\backslash\Omega\} \qquad 0 < x < L$$

$$a(0) = \lim_{x \to 0_+} a(x) \qquad a(L) = \lim_{x \to L_-} (x) \ .$$

Since $(x, a(x)) \in \partial\Omega$ Lemma 5(i) implies that $a(x)$ is a non-decreasing function so the above limits exist. Then

$$\Omega = \{(x,y) : 0 < x < L, \quad c(x) < y < a(x)\},$$

and since we may deduce from the Hopf principle that $\partial\Omega$ does not contain segments parallel to the x or y axis we have that $a(x)$ is a strictly increasing continuous function on $[0,L]$. We may now deduce from Theorem 1.1 on p.151 of [9] that $\Gamma = \partial\Omega \cap D \equiv \partial\Omega\backslash(\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ is an analytic curve and so (P1) has been solved in the following sense.

Theorem

(i) There exists a unique solution to (P2) which is the unique solution to (23).

(ii) There exists a unique solution to (P1) $\phi \in C^{0,\lambda}(\bar{\Omega})$ where $\phi = u_y$ and u is the solution to the variation inequality (23).

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Ignition and flame propagation in a strain field

1. INTRODUCTION

Though it has long been recognised that fluid motion plays an essential role in most combustion devices, little is known about the physical mechanisms involved. For flames that are thin compared with the smallest scalar of the fluid motion the dominant effect of the motion must result from its straining (velocity gradients). In this paper we study the influence of a strain field on ignition and flame propagation in a premixed gas. We consider a one-dimensional problem in which a uniform strain rate is applied in a direction transverse to the direction of propagation. Though the model equations are somewhat simplified, they retain all the essential features of combustion: diffusion of reactant and heat, convection and chemical reaction. The non-dimensional conservation equations for the limiting reactant and heat are

$$\frac{\partial Y}{\partial t} - \varepsilon\psi \frac{\partial Y}{\partial \psi} = \frac{\partial}{\partial \psi} (\bar{\rho}\bar{\lambda} \frac{\partial Y}{\partial \psi}) - CY \exp \beta (1 - \frac{1}{\theta + \alpha}) \tag{1}$$

and

$$\frac{\partial \theta}{\partial t} - \varepsilon\psi \frac{\partial \theta}{\partial \psi} = \frac{\partial}{\partial \psi} (\bar{\rho}\bar{\lambda} \frac{\partial \theta}{\partial \psi}) + CY \exp \beta (1 - \frac{1}{\theta + \alpha}) + H(\psi,t) \tag{2}$$

where Y is the concentration of the limiting reactant, θ is temperature, $\bar{\rho}$ and $\bar{\lambda}$ are temperature-dependent density and conductivity, ε is the strain rate, β is the activation energy, L is the Lewis number and H is the heat input from an ignition source such as a spark discharge; C and α are given constants. The continuity equation has been eliminated, and the conservation equations simplified by introducing ψ in place of the space co-ordinate x, using

$$\frac{\partial \psi}{\partial x} = \bar{\rho} \quad \text{and} \quad \frac{\partial \psi}{\partial t} = -\bar{\rho}U - \varepsilon\psi ,$$

where U is the velocity.

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Large activation energies, β, result in very thin flame front regions and their propagation can then be treated as a moving boundary problem, see Crowley (1981). Approximate, analytical solutions are available only for steady-state, large activation energy problems. We wish to model the unsteady problem of the onset of ignition and the subsequent flame propagation over a large range of β values, allowing for the possibility of no ignition/flame front.

2. NUMERICAL SOLUTION

We consider a numerical solution similar to that of Dwyer and Sanders (1981). The governing equations are discretised in space using a non-uniform mesh and corresponding finite-difference approximations, to give a system of ordinary differential equations in time for θ and Y (of order 2N, where N is the number of space points) to be solved by a standard Gear-type library routine. The non-uniform space mesh is generated by the co-ordinate transformations:

$$X(\psi,t) = \frac{\int_0^{\Psi} (1 + bG(\Psi,t))\, d\Psi}{\int_0^{\psi_N} (1 + bG(\psi,t))\, d\psi},$$

where $G(\psi,t)$, the monitor function, is some measure of space derivatives of θ and Y, and b is the transformation stretching parameter. For a given uniform X mesh $X_i = (i-1)\,\Delta X$, $i = 1, \ldots, N$ with $(N-1)\,\Delta X \equiv 1$, constant for all time, the above transformation distorts the spacing of the corresponding ψ_i points proportional to the gradients of θ and Y. The mesh is updated at time intervals inversely proportional to the flame speed and the values of θ and Y are interpolated from the old to the new mesh using spline interpolation. (We note that the moving finite element method of Gelinas and Doss (1981) is a useful alternative since it avoids this interpolation.)

In the simpler problem treated by Dwyer and Sanders, $G(\psi,t)$ is taken as just $|\partial\theta/\partial\psi|$. For our problem, we use the fact that in the high temperature region of the flame, the second derivative (diffusion) terms are large owing to the large, exponential chemical reaction rates, and so, $G(\psi,t)$ is chosen to be the L_2 norm of the second derivatives

$$G(\psi,t) = \sqrt{\left\{\left[\frac{1}{L}\frac{\partial}{\partial\psi}\left(\bar{\rho}\bar{\lambda}\frac{\partial Y}{\partial\psi}\right)\right]^2 + \left[\frac{\partial}{\partial\psi}\left(\bar{\rho}\bar{\lambda}\frac{\partial\theta}{\partial\psi}\right)\right]^2\right\}}$$

The advantages of this non-uniform mesh over a uniform mesh for the travelling flame are:

(i) greater accuracy and resolution of the flame front, θ, Y and U values;

(ii) computational efficiency - since the cost of the computer solution is $O\left[(2N)^2\right]$ and a uniform mesh would require a far larger value of N.

Our numerical results show encouraging agreement with available, asymptotic approximations. The main conclusions of the present analysis are that strain fields increase the energy required for ignition and reduce flame propagation velocity; high Lewis number flames are the most sensitive to straining.

It is suggested that this moving, space-mesh technique could be advantageously used for moving boundary problems when steep gradients exist near the moving boundary.

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E B HANSEN

A comparison between theory and experiment in electrochemical machining

1. INTRODUCTION

Electrochemical machining (ECM) is a technological process in which a workpiece is placed as the anode in an electrolytic cell with a properly shaped cathodic tool so that a desired shape of the anodic work piece is obtained by the electrochemical process. A detailed account of the physical theory of the method, its practical implementation and examples of its widespready applications in industry has been given in [1].

In the simplest and most commonly used mathematical model of ECM the electrodes are assumed to be equipotential surfaces, and the electrolyte which occupies the space between them is treated as a homogeneous and isotropic electrical conductor with constant conductivity. In accordance with Faraday's law and the assumption about constant electrolyte conductivity, the rate of removal of material at a point $\underline{r}$ on the anode surface is assumed to be proportional to the normal derivative of the electric potential at $\underline{r}$. Therefore, if $\eta(\underline{r},t) = 0$ is the equation of the anode surface with time t entering as a parameter, and $\varphi = \varphi(\underline{r},t)$ is the potential, the following boundary condition holds on the anode:

$$\nabla\eta \cdot \nabla\varphi + c\,\frac{\partial\varphi}{\partial t} = 0 \; . \qquad (1.1)$$

Here c is a constant. Because of the other assumptions, φ is harmonic in the region between the electrodes and satisfies Dirichlet boundary conditions

$$\varphi = \varphi_A \;,\; \varphi = \varphi_C \;, \qquad (1.2)$$

where φ_A and φ_C are known constants, on the anode and the cathode, respectively. In some cases parts of the anodic work piece or the cathodic tool are covered by insulating material. In such cases, the boundary conditions in (1.2) only hold on those parts of the electrodes which are exposed to the electrolyte and the boundary condition on the insulator surfaces is the homogeneous Neumann condition

$$\frac{\partial \varphi}{\partial N} = 0 \ . \tag{1.3}$$

The model of the ECM process described here is sometimes referred to as the electrostatic model. We note, that although the electrolyte is usually pumped through the cell with very high fluid velocities in some parts of the electrode gap, a possible influence of the fluid flow is completely neglected in this model.

Laplace's equation and the boundary conditions in (1.1), (1.2) and (1.3) constitute a moving boundary problem from which the potential φ and the anode surface given by the function η may be found as functions of time. It is convenient to solve this problem by proceeding in time steps so that at each step the classical mixed boundary value problem given by Laplace's equation and the relevant boundary conditions from (1.2) and (1.3) is solved with the anode treated as a fixed surface. Thereafter (1.1) is used to determine a new anode surface which is used in the next time step. If, as is sometimes the case, the cathode is moved with a known speed towards the anode during the process, the cathode position is of course also changed from one timestep to the next one. Since the cations form gases when recombining with the electrolyte, the cathode shape remains unchanged during the process.

During recent years several methods for predicting the anode profile corresponding to a given cathode have been developed using the electrostatic model. Thus, Christiansen and Rasmussen [2] developed a method in which the potential problem was formulated as an integral equation of the first kind. They only considered the case of a plane, annular geometry and Dirichlet conditions on both electrodes. Later Hansen and Holm [3] showed that the same situation can be handled using an integral equation of the second kind. For the same geometry Meyer [4] developed a method based on the method of lines and Elliot [5] one using variational inequalities. Forsyth [6] and Forsyth and Rasmussen [7] studied a plane geometry with an infinity long electrode gap in which the cathode moves towards the anode with a constant speed so that the anode profile approaches a stationary shape. Allowing for Neumann as well as Dirichlet type boundary conditions they compared several analytical and numerical methods for computing the time development of the anode surface. For a large number of different cylindrical cathodes Hougaard [8] determined the stationary anode profile using complex variable methods.

Surprisingly enough, in spite of the considerable effort which has thus been made in order to devise computational methods for predicting ECM processes, the published literature on comparisons between this model and experiments seems to be very scarce. Indeed, the only published comparisons known to this author is one by Lawrence [9], which is quoted in [1], and one by Hümbs [10]. In both these references only the stationary anode profile is given and the results which are presented seem only to be those for the upstream side of the electrode gap although, as will appear later, results for the downstream side would have been very interesting. In order that one of the methods mentioned above can be used in a comparison with experiments, it will have to be generalized to a geometry less idealized than those considered so far. In this paper we describe a method capable of dealing with a realistic geometry and use it to compare with experimental results. Our method may be described as an extension of Christiansen's and Rasmussen's integral equation method [2], modified to deal with an axial symmetric geometry and generalized to handle mixed boundary value problems. The next section contains a brief description of the method including the numerical solution procedure and in section 3 we apply the method and compare the results with experiments.

2. THE METHOD

As mentioned in the introduction we compute the anode profile as a function of time by proceeding in time steps where, in each time step, we solve a mixed boundary value potential problem. Since a detailed description of the integral equation method used for the solution of the potential problem has already been presented in [11], we only give a brief description of the method here.

The electrode surfaces are assumed to be axial symmetric (with common axis), the cathode being finite and the anode infinite in extension. The surfaces may have a finite number of circular edges, where the tangent planes meet at an exterior angle $\beta \in]0,2\pi[$. Each electrode surface may consist of ring shaped regions where the boundary condition is either Dirichlet's or Neumann's as in (1.2) or (1.3). In the following we denote the unions of the generatrices of those regions on the cathode where the boundary condition is $\varphi = \varphi_C$ by C_{CD}, and those with boundary condition $\partial\varphi/\partial N = 0$ by C_{CN}. The corresponding unions of generatrices on the anode are denoted by C_{AD} and C_{AN}, respectively. As the fundamental solution we use the function

$$G(\underline{r},\underline{r}') = (\pi R_+)^{-1} K(4rr'/R_+) \ , \tag{2.1}$$

where K is the elliptic integral of the first kind, r and z are the coordinates of $\underline{r}$, r' and z' those of $\underline{r}'$, and $R_+ = ((r+r')^2 + (z-z')^2)^{\frac{1}{2}}$. Using (2.1) and proceeding in a straightforward manner from Green's second identity we derive the following integral equation:

$$\int_C B(s',s)y(s)ds + v(s')y(s') = u(s') \ . \tag{2.2}$$

Here $C = C_{CD} \cup C_{CN} \cup C_{AD} \cup C_{AN}$ and s is the arc length parameter along C. The functions in (2.2) are defined as follows:

$$B(s',s) = \begin{cases} G(\underline{r}',\underline{r}) & \text{if } \underline{r} \in \overline{C}_{CD} \cup \overline{C}_{AD} \\ -\dfrac{\partial G}{\partial N}(\underline{r}',\underline{r}) & \text{if } \underline{r} \in C_{CN} \cup C_{AN} \ , \end{cases} \tag{2.3}$$

where $\underline{r}$ is the position vector $\underline{r}(s)$, $\underline{r}' = \underline{r}(s')$, $\overline{C}_{CD}$ and $\overline{C}_{AD}$ are the closures of C_{CD} and C_{AD} respectively, and $\partial/\partial N$ denotes differentiation in the direction of the normal pointing into the surface,

$$y(s) = \begin{cases} \dfrac{\partial\varphi}{\partial N}(\underline{r}) & \text{if } \underline{r} \in C_{CD} \cup C_{AD} \\ \varphi(\underline{r}) - 1 & \text{if } \underline{r} \in C_{AN} \\ \varphi(\underline{r}) & \text{if } \underline{r} \in C_{CN} \end{cases} \tag{2.4}$$

$$v(s') = \begin{cases} 0 & \text{if } \underline{r}' \in \overline{C}_{CD} \cup \overline{C}_{AD} \\ \frac{1}{2}(\beta/\pi) - 1 & \text{if } \underline{r}' \in C_{CN} \cup C_{AN} \end{cases} \tag{2.5}$$

and

$$u(s') = \begin{cases} 0 & \text{if } \underline{r}' \in C_{AD} \cup C_{AN} \\ -1 & \text{if } \underline{r}' \in C_{CD} \cup C_{CN} \ . \end{cases} \tag{2.6}$$

On a smooth part of $C_{CN} \cup C_{AN}$ the exterior angle between the tangents is of course $\beta = \pi$, so that $v = -\frac{1}{2}$.

At a point on C where intervals with different boundary conditions meet, and at points corresponding to edges, y is singular. For example, near an edge point on C with exterior opening angle β between the tangents, and boundary condition $\varphi = 0$ on the one side $(\theta = 0)$ and $\partial\varphi/\partial N = 0$ on the other $(\theta = \beta)$, φ behaves as

$$\varphi = \sum_{n=1}^{\infty} a_n \rho^{(n-\frac{1}{2})(\pi/\beta)} \sin((n - \tfrac{1}{2})\pi\frac{\theta}{\beta}) , \tag{2.7}$$

where a_1, a_2, ... are constants and ρ is the distance from the edge. In order that a sufficient accuracy could be obtained, the solution procedure had to be particularly adapted to take this kind of singularities into account.

The integral equation is solved by collocation, i.e. we require it to be satisfied only at a finite number of points. With one exception to be mentioned later the collocation points are chosen to include all the singularities which were referred to above. Let $s = s_-$ be one of these singularities and $s = s_+$ the next one along C and let $s_1, s_2, \ldots, s_M$ be the collocation points in $[s_-,s_+]$ with $s_1 = s_-$ and $s_M = s_+$. We then express y in $[s_-,s_+]$ as

$$y(s) = (s_+ - s)^{\alpha_+}(s - s_-)^{\alpha_-} f(s) \tag{2.8}$$

where the constants α_+ and α_- are such that f is finite and non-zero at s_- and s_+. Because of the different physical interpretations which y has above and below a singularity between intervals with different boundary conditions, and because of the factors $(s_+ - s)^{\alpha_+}$ and $(s - s_-)^{\alpha_-}$ in (2.8), f has jump discontinuities at the singularities, but the ratio between the limiting values from above and below can be determined from (2.7) and (2.8). We approximate f by a function which is discontinuous at the singularities with the correct ratios between the limiting values and which is otherwise continuous. In each interval $[s_n,s_{n+1}]$ with $2 \le n \le M - 2$ we approximate f by a first degree polynomial and in the intervals $[s_-,s_2]$ and $[s_{M-1},s_+]$ we use the function determined from the first two terms of (2.7) or the corresponding expression which is valid at the singularity in question. Since, in the applications we have made, all the angles β are rational fractions of π, it is possible, by use of a power function substitution $s = \sigma^m$, to transform the integrand in each subinterval $[s_n, s_{n+1}]$ into one with $y(s)$ being replaced by an analytical function $y_1(\sigma)$. For s' not equal to s_n or s_{n+1} the integral is then evaluated by means of Simpson's rule. For $s = s'$ the kernel $B(s',s)$ has a logarithmic singularity. In that case the integral of the logarithmic term is evaluated by means of a quadrature formula which represents the integral of a function $P(s)\log|s|$ by its exact value, if P is a second degree polynomial. In the integration around a point s_n, C is substituted by the parabola passing through s_{n-1}, s_n, and s_{n+1}.

Since the anode surface extends to infinity, the interval outwards from the last collocation point on the anode is infinite. In that interval f is substituted by a function determined from an asymptotic solution to Laplace's equation.

At an edge point P with opening angle β which belongs to the interior of $C_{AN} \cup C_{CN}$, $v = \frac{1}{2}(\beta/\pi) - 1$, while $v = -\frac{1}{2}$ in a neighbourhood of P. As could be expected, such a discontinuous behavior leads to severe numerical difficulties. It was found that the simplest, satisfactory way to repair this is to omit such a point P from the set of collocation points and to put f at P equal to the value found by interpolation between the f-values at the two nearest collocation points using (2.8) and the first two terms of the formula corresponding to (2.7).

The system of linear algebraic equations for the f-values at the collocation points which is obtained in the manner described above, is solved by Gauss elimination and the value of $\partial\varphi/\partial N$ at the collocation points on C_{AD} is computed. From these values the shift $(\Delta r, \Delta z)$ of each of these collocation points is finally evaluated as $\Delta r = c^{-1}\varphi_r \Delta t$ and $\Delta z = c^{-1}\varphi_z \Delta t$ and used to find the anode surface to be applied in the next time step.

3. RESULTS AND DISCUSSION

The method described in the preceding section was applied to a geometry modelling one used in a series of ECM experiments which has been carried out at the Department of Mechanical Technology, The Technical University of Denmark. The experimental set-up is shown in Figure 1. The cathodic tool was a tube with the upper part of the outer surface covered by insulating material. The electrolyte was a NaCl solution with conductivity 130 mS/cm at 20°C leading to a theoretical metal removal rate of $2.87 \cdot 10^{-2}$ mm/Vmin.

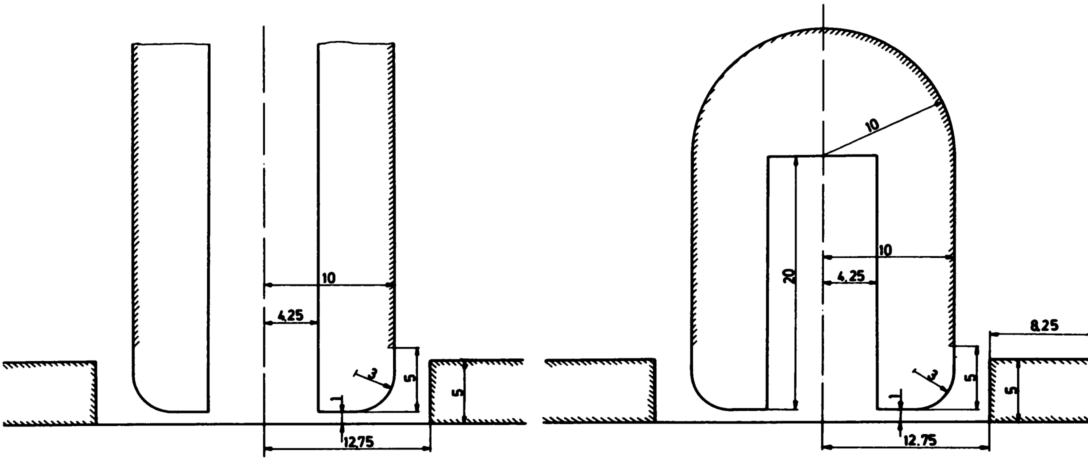

Figure 1. The experimental set-up is shown to the left. The cathode is a cylindrical tube with the upper part of the outer surface insulated. The anode at the bottom of the figure is initially plane. It is covered by a circular insulating ring coaxial with the cathode tube. The mathematical model is shown to the right. The cathode is closed upwards by a semi-spherical cap. Insulated surfaces are hatched. Measures are in mm.

The voltage difference between the electrodes (reduced for a polarization voltage drop of 2.3V) was 17.7V. During the process the cathode was moved towards the anode with a feed rate of 1mm/min. The electrolyte was pumped through the cell in the direction towards the axis and leaving through the cathode tube. In each experiment the anode was initially plane and the cathode moved downwards from a starting position 1mm above the anode as shown in the figure. After some time the experiment was stopped, the anode removed and its surface profile was measured.

The figure also shows the mathematical model of the set-up. In order to have a finite cathode in the computations, the cathode was closed upwards by a semispherical cap which is insulated like the rest of the upper part of the outer surface.

In the computations 102 collocation points were used with 38 on the cathode, 42 on the inner uncovered part of the anode, and 22 on the insulating ring and the uncovered part of the anode outside the ring. The

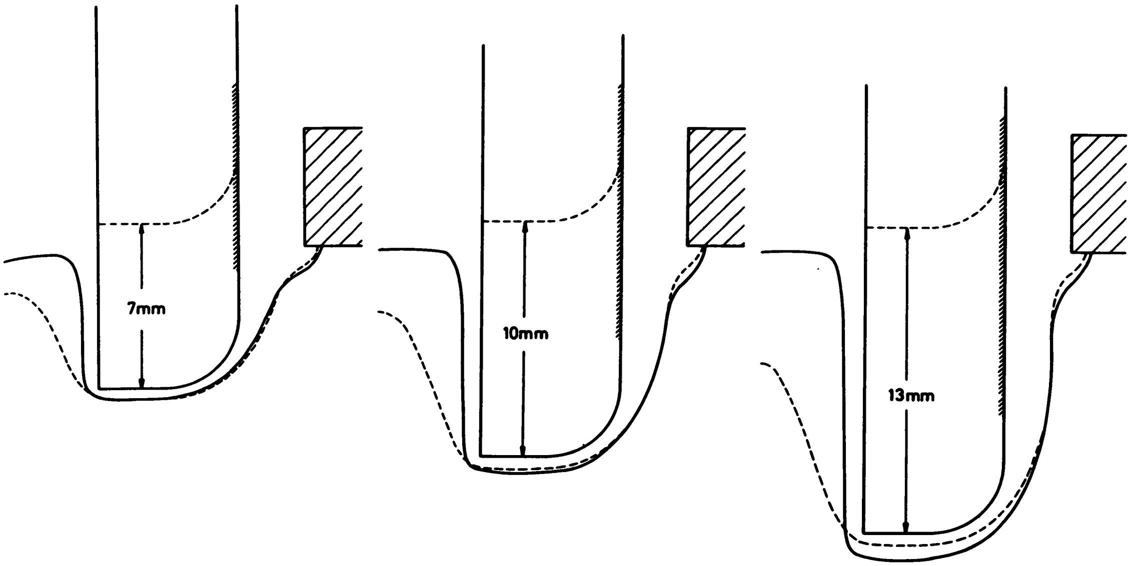

Figure 2. Cross sections through the electrode system showing the cathode position and the anode profile at times 7, 10 and 13 min. after the start of the experiment. In each figure the left edge is on the axis of the electrode system. The full curves are the measured anode profiles, the broken ones are computed.

number of time steps was 130 with 0.1 min. between consecutive steps. The numerical computations were carried out on the IBM 3033 computer at the NEUCC, Lyngby, Denmark.

Experimental and theoretical results are shown in Figure 2. On the outer, vertical side of the anode hole the agreement is seen to be very good at all the three instances presented in the figure. The computed bottom level of the hole is also found correctly after 7 min. run, but after 10 min. a discrepancy has started to develop and it grows further during the rest of the experiment. The stub on the anode, which is formed in the cathode tube, is found theoretically to be considerably smaller at all instances than the one which is actually observed.

The reason why the predictions of the electrostatic model are much poorer on the inside than on the outside of the cathode is undoubtedly that the electrolyte flow is not at all taken into consideration in this model. Since the cations react with the electrolyte to form gases when they recombine, a flow of gas bubbles is created along the cathode and, because of the flow, these bubbles move downstream, that is towards the axis and up through the cathode tube, while they are growing in number and size. When the density of bubbles increases they tend to form an insulating screen in the electrolyte, and this is probably what happens in the inside of the cathode.

It seems to be a good approximation to assume that the electric current density is constant along the gap formed by the parallel horizontal bottom surfaces of the electrodes. This assumption is confirmed by the computed current densities. If the assumption is true, the growth of the bottom gap width with time, which is observed during the process, can only be caused by an increase of the effective conductivity of the electrolyte, which results from the temperature rise due to Joule heating during the process. This explanation agrees with the fact that the increase in conductivity does not occur in the first part of the gap. Further downstream, in the cathode tube, this effect is overruled by the insulating effect of the bubbles.

In conclusion we find, that in the upstream end of the electrode gap the electrostatic model predicts the observed anode profile reasonably well while, in the middle part and in the downstream end, the fact that variations in conductivity and the formation of bubbles are neglected leads to considerable deviations between theory and observations.

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G DUVAUT
Problems in continuum mechanics

I. INTRODUCTION

Most of the contributions presented in this discussion group appear in this volume as separate papers and they need not be summarized here. Therefore, only a brief report on a result by G. Aronsson about an integral inequality related to plasticity problems will be given in the Appendix at the end of this paper.

Here we study a linear elasticity problem with linear viscous friction boundary condition, i.e. the tangential force is proportional to the tangential velocity. The problem is of degenerate parabolic type. We show, by splitting in an elliptic problem and a parabolic one that it possesses a unique solution. The method is explained in a completely linear case but it can be extended to a nonlinear friction law and passing to the limit to a law of the type:

$$|\sigma_T| < k \Longrightarrow \frac{\partial u_T}{\partial t} = 0$$

$$|\sigma_T| = k \Longrightarrow \exists\, \lambda \geq 0, \frac{\partial u_T}{\partial t} = -\lambda\sigma_T$$

which involves a free boundary.

II. VISCOUS FRICTION LAW

Let Ω be the bounded open region contained in $\mathbb{R}^3$, of boundary Γ occupied by the elastic body. A part Γ_0 of Γ is clamped, a part Γ_1 is subjected to given surface forces, and a part Γ_2, $(\Gamma_2 \cap \Gamma_0 = \emptyset)$ is in contact with friction with a rigid support. Equations and boundary conditions are the following:

$$\frac{\partial\sigma_{ij}}{\partial x_j} + f_i = 0 \text{ in } \Omega \qquad (1)$$

$$\sigma_{ij} = a_{ijkh}\, \varepsilon_{kh}(u), \quad \varepsilon_{ij}(v) = 1/2 \quad \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (2)$$

$$u = 0 \text{ on } \Gamma_0 \tag{3}$$

$$\sigma_{ij}\, n_j = F_i \text{ on } \Gamma_1 \tag{4}$$

$$u_N = 0 \text{ on } \Gamma_2 \tag{5}$$

$$\sigma_T = - \mu\, \dot{u}_T \text{ on } \Gamma_2 \qquad (\dot{x} = \frac{\partial x}{\partial t}) \tag{6}$$

$$u_T(x,0) = u_0(x) \tag{7}$$

In these relations (f_i) and (F_i) represent densities of forces respectively in Ω and on Γ_1. Coefficients a_{ijkh} are elasticity coefficients. They may depend on $x \in \Omega$. The displacement field is $u = (u_i)$ and

$$u_N = n \cdot u \ , \quad u_T = u - u_N \cdot n \ ,$$

where n stands for the unit outer normal to Γ. Analogously we split $(\sigma_{ij}\, n_j)$ in a normal part σ_N and a tangential vector σ_T. The relation (6) is the viscous friction law in a linear case and μ is a viscosity coefficient. A nonlinear friction law would be:

$$\sigma_T = - K\, (\dot{u}_T)^{k-1}\, \dot{u}_T \quad , \quad K \text{ and } k > 0 \tag{8}$$

which gives at the limit when k tends to zero:

$$\left[\begin{array}{l} |\sigma_T| < K \Longrightarrow \dot{u}_T = 0 \\ \\ |\sigma_T| = K \Longrightarrow \exists \lambda \geq 0, \ \dot{u}_T = -\lambda \sigma_T. \end{array}\right. \tag{9}$$

The relation (7) is an initial boundary condition on Γ_2 only and $u_0(x)$ a vector field given on Γ_2 with $u_{0N} = 0$.

III. VARIATIONAL FORMULATION

We introduce the notations:

$$a(u,v) = \int_\Omega a_{ijkh}\, \varepsilon_{kh}\, (u)\, \varepsilon_{ij}\, (v)\, dx$$

$$L(v) = \int_\Omega f_i v_i dx + \int_{\Gamma_1} F_i v_i dS$$

If $u = u\ (x,t)$ is a solution of (1) - (6) it satisfies:

$$\begin{cases} u\ (x,t) = 0 \text{ on } \Gamma_0\ ,\ u_N\ (x,t) = 0 \text{ on } \Gamma_2 \\ a\ (u,v) + \mu \int_{\Gamma 2} \dot{u}_T\ v_T\ dS = L\ (v),\ \forall v,\ v = 0 \text{ on } \Gamma_0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad v_N = 0 \text{ on } \Gamma_1 \\ u_T\ (x,0) = u_0\ (x) \end{cases} \tag{10}$$

The relations (9) correspond to the principle of virtual work and are obtained from (1) - (6) by integration by parts.

IV <u>RESULTS</u>

We make the following hypothesis on the data:

$$\begin{cases} f_i \in L^2\ [(0,t_1) \times \Omega]\ ,\ F_i \in L^2\ [(0,t_1) \times \Gamma_1]\ ,\ t_1 > 0 \\ a_{ijkh} \in L^\infty\ (\Omega)\quad ,\quad a_{ijkh} = a_{jikh} = a_{khij} \\ \exists\ \alpha_0 > 0,\quad a_{ijkh}\ \tau_{kh}\ \tau_{ij} \geq \alpha_0\ \tau_{ij}\ \tau_{ij}\ ,\quad \forall\ \tau_{ij} = \tau_{ji} \\ \mu > 0 \end{cases} \tag{11}$$

<u>Theorem</u>

Under hypothesis (11) there exists a unique $u = u(x,t)$ satisfying:

$$u \in L^2\ [(0,t_1),V] \cap L^\infty\ [(0,t_1),L^2(\Omega)] \tag{12}$$

where

$$V = \{v | v \in (H^1\ (\Omega))^3\quad ,\quad v = 0 \text{ on } \Gamma_0,\ v_N = 0 \text{ on } \Gamma_1\}. \tag{13}$$

and satisfying (10) for almost every $t \in (0,t_1)$.

V. <u>PROOF OF THE THEOREM</u>

i) <u>Uniqueness</u>

The problem stated being linear it is sufficient to prove that the corresponding homogeneous problem has the unique zero solution. For that, we choose $v = u(t)$ in the variational formulation and integrate with respect to t.

ii) <u>Splitting of the problem</u>

Let us introduce spaces V_0 and V_1 by

$$\begin{cases} V_0 = \{v \mid v \in V, \quad v = 0 \text{ on } \Gamma_2\} \\ V_1 = \{v \mid v \in V, \quad a(v,\phi) = 0, \quad \forall \phi \in V_0\} . \end{cases} \tag{13}$$

These spaces are orthogonal if we fit V of the scalar product $a(u,v)$. We can get a solution by splitting:

$$u = u^{(0)} + u^{(1)} \text{ , where} \tag{14}$$

$$u^{(0)} \in V_0, \quad a(u^{(0)},v) = L\ (v) \ , \quad \forall v \in V_0 \tag{15}$$

$$\begin{cases} u^{(1)} \in V_1 \ , \ a(u^{(1)},v) + \mu\int_{\Gamma_2} u^{(1)}{}_T\, v_T\, dS = L\ (v), \ \forall v \in V_1 \\ u^{(1)}{}_T\ (x,0) = u_0\ (x) \text{ on } \Gamma_2 \end{cases} \tag{16}$$

iii) <u>Existence</u>

For almost every $t \in (0,t_1)$, $u^{(0)}$ is solution of an elliptic problem by application of Lax-Milgram lemma. We show, using Galerkin method for example, that (16) is a parabolic problem which possesses a unique solution in the class (12).

<u>Remarks</u>

1°) If we introduce

$$\sigma_{ij}{}^{(0)} = a_{ijkh}\, \varepsilon_{kh}\, (u^{(0)})$$

it results from (15) that:

$$\frac{\partial}{\partial x_j} \sigma_{ij}{}^{(0)} + f_i = 0 \text{ in } \Omega \ , \ \sigma_{ij}{}^{(0)}\, n_j = F_i \text{ on } \Gamma_1 \tag{17}$$

Consequently, if

$$\sigma_{ij}^{(1)} = a_{ijkh}\, \varepsilon_{kh}\, (u^{(1)})$$

we get:

$$\frac{\partial}{\partial x_j} \sigma_{ij}^{(1)} = 0 \text{ on } \Omega, \quad \sigma_{ij}^{(1)} n_j = 0 \text{ on } \Gamma_1 \tag{18}$$

2°) Using (17) to express L(v) in (16), we get:

$$\int_{\Gamma_2} (\mu\, \dot{u}^{(1)}_T + \sigma_T^{(1)} + \sigma_T^{(0)})\, v_T\, dS = 0, \quad \forall v \in V_1 \tag{19}$$

which is nothing else than (6) with the introduced notations.

3°) The space V_1 can be identified to the space $\mathscr{T}$ of traces on Γ_2 of functions of V. This identification can be done by solving the elliptic problem (of Dirichlet type):

$$w \in V_1 \;;\; a\,(w,\phi) = 0, \quad \forall\phi \in V_0 \;;\; w = \psi \text{ on } \Gamma_2$$

where $\psi = \Psi|_{\Gamma_2}$, for any $\Psi \in V$. We get a linear application

$$\psi \to w_\psi \quad \text{from } \mathscr{T} \text{ on } V_1.$$

4°) If $\mathscr{T}'$ is the dual space of $\mathscr{T}$, we can define a bilinear form $\mathscr{A}(\psi,\phi)$ on $\mathscr{T}$ by:

$$\mathscr{A}(\psi,\phi) = \langle S(\psi), \phi \rangle \tag{20}$$

where

$$S(\psi) = \sigma_T^{(1)} (w_\psi) \in \mathscr{T}'.$$

Introducing the norm:

$$||\psi|| = \operatorname{Inf} ||\Psi||_V, \quad \Psi \in V, \; \Psi = \psi \text{ on } \Gamma_2 \tag{21}$$

we show easily that $\mathscr{A}$ is symmetric continuous and coercitive on $\mathscr{T}$.

These remarks and relation (19) reduce the research of $u_T^{(1)}$ on Γ_2 to:

$$
\begin{cases}
u^{(1)}_T \in \mathscr{V} \\
\mu \int_{\Gamma_2} \dot{u}^{(1)}_T \, v_T \, dS + \mathscr{A}(u^{(1)}_T, v_T) = -\int_{\Gamma_2} T^{(0)} v_T \, ds, \quad \forall v_T \in \mathscr{V} \\
u^{(1)}_T (x,0) = u_0 (x) \text{ on } \Gamma_2
\end{cases}
\tag{22}
$$

which is a parabolic problem on Γ_2 and gives again solutions of $u^{(1)} \in V_1$.

5°) Some other linear or non linear friction law of viscous type can be considered and solved in a similar way. The case with law (9) lead to a *free boundary problem.*

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APPENDIX A new integral inequality (by G. Aronsson of Minnesota University)

In the paper [1] the author proved the following theorem:
Let $\Omega \subset R^2$ be a bounded domain. Let $f(x_1,x_2)$ be a Lipschitz function on $\bar{\Omega}$ and let $f = 0$ on $\partial\Omega$.

Put $F(\theta) = m\{x | x \in \Omega, \|\text{grad } f(x)\| \leq \theta\}$ and $M = \|\text{grad } f\|_{L^\infty}$.

Then

$$
\left| \iint_\Omega f \, dx_1 dx_2 \right| \leq \frac{1}{3\sqrt{\pi}} \int_0^M ((m\Omega)^{3/2} - F(\theta)^{3/2}) d\theta .
$$

Equality holds if Ω is an open circle and f or $-f$ is a dome function. (A dome function is a concave, non-negative, non-increasing function of the distance to some fixed point).

A similar inequality holds for functions in R^n, but we gave the proof for functions in the plane only.

As an application we solved an optimum design problem in plasticity: Given p different materials with different plastic yield limits and in given quantities, find the shape of a rod of given length which can withstand the largest twisting moment. The solution is a circular cross-section with the different materials in concentric annuli and one in the remaining circle, the strongest material outermost and the others following in order inwards. This generalizes results by L. E. Payne [4] and Leavitt & Ungar [3] for the cases $p = 2$ and $p = 1$, respectively.

The proof of the theorem was based on functions $g(x)$, such that $\|\operatorname{grad} g(x)\|$ is piece-wise constant. Further, the isoperimetric inequality was used. Later, the inequality was proved in [2] in a more general form, namely for functions in the Sobolev space $W^{1,1}$ having a support of finite measure. We also gave a complete determination of the extremal functions, i.e. those for which the case of equality occurs. These functions were also found in [1], but under some additional conditions.

The main tools in the proof are: an integral formula by Fleming and Rishel, the isoperimetric inequality in a form due to De Giorgi, and a classical result by Hardy and Littlewood on rearrangements.

The more general theorem has the following form:

Theorem Let u be a real-valued function defined on euclidean space R^n. Suppose that u is integrable on R^n together with its first order (weak) derivatives, and suppose that the support of u has finite measure. Let E be this support and set $F(t) = m\{x \in E: |Du(x)| \le t\}$ (here m and D stand for Lebesgue measure and the gradient operator, respectively). Then

$$\left| \int_{R^n} u(x)\,dx \right| \le \frac{\Gamma(1+n/2)^{1/n}}{n+1\ \sqrt{\pi}} \int^{\infty} \left[(mE)^{\frac{n+1}{n}} - F(t)^{\frac{n+1}{n}} \right] dt \,. \tag{1}$$

Equality holds in (1) if and only if either u or $-u$ is (equivalent to) a dome function (i.e. a nonnegative continuous function with spherical support, whose restriction to that support is spherically symmetric and concave).

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L A CAFFARELLI

Variational problems with geometrical constraints

The purpose of this survey is to discuss several variational problems with geometric constraints and some of the techniques that have been recently developed to treat them.

For simplicity we will only discuss some classical problems in potential theory and hydrostatics.

The problems under consideration are the following.

Optimal Conductor Design

A perfect conductor ∂D (in R^3) must be surrounded by another conductor $\partial\Omega$ ($\Omega \supset D$) such that a) The space between them $(\Omega \sim D)$ is prescribed or b) The area of the external conductor (Area $(\partial\Omega)$) is prescribed.

We want to find the design that minimizes the mutual capacity, in this case

$$\operatorname{cap}_{(\partial\Omega,\partial D)} = \min \int (\nabla u)^2 d\,x = \min D(u)$$

the minimum taken among those functions u that satisfy

$$u = \begin{cases} 1 \text{ on } \partial D \\ \\ 0 \text{ on } \partial\Omega \end{cases}$$

The hydrostatic problem that we want to consider is that of the shape of a drop in equilibrium on an inhomogeneous plane:

c) The volume of the drop, V_O, is prescribed.

We denote by ϕ_A, the characteristic function of the set $A \subset R_+^{n+1}$, by $(x,y) = X$ a point in R_+^{n+1}.

Then, among sets A of measure V_O, we want to minimize

$$F(A) = \int |D\phi_A| + \int y\phi_A dx - \int \beta(\chi)\phi(\chi,0)\,d\chi.$$

Where the first term denotes the perimeter of A in the geometric measure theory sense, the second corresponds to the potential energy and the third to the contact energy.

Here we assume that $|\beta(\chi)| < 1$ and that β attains its infimum at infinity that is, given R_0, R_1 such that

$$\inf_{B_{R_0}} \beta \geq \sup_{B_{R_1}} \beta \, .$$

It is not difficult to predict, heuristically, what are the (free) boundary conditions that one expects at the free boundary, if we recall the classical variational formulas.

For instance, in problem a), assume that we choose two points x_1, x_2 on the (supposedly analytic) free boundary $\partial\Omega$ and we make small C^∞ volume perturbations, outwards near one point inwards near the other so as to keep $|\{\Omega \sim D\}|$ constant.

Then, the change in Dirichlet integral is known to be

$$\delta D = \; u_\gamma^2(x_2) - u_\nu^2(x_1) \; \delta V + (|\delta V|)$$

(where the sign of δV is taken positive for inward variations).

That is, we expect, for our minimizer, to satisfy the free boundary condition

$$u_\gamma^2(x) = \text{constant}$$

on $\partial\Omega$.

A similar argument, tells us that in problem b) $u_\nu^2 = C\kappa$.
(κ is the mean curvature of $\partial\Omega$ with respect to the inward normal, if we recall that $\delta\text{Area} = -\kappa\delta V$.

Finally, in problem c), we expect again by a similar argument $u + y = C$.

Problem a) (and related free boundary problems) have been treated in the two dimensional case by conformal mapping techniques and variational methods, but only in the case of very restricted geometries (See [A-C,1] for references).

Alt and Caffarelli treated recently ([A-C,1]) the general n-dimensional free boundary problem $u_\nu^2 = C$.

We started there by considering minimizers of the functional

$$j(u) = D(u) + \lambda^2 |\{u>0\}|,$$

and showed that minimizers are weak solutions of the free boundary problem $u_\nu^2 = \lambda^2$, in the sense that, $\partial\{u>0\}$ has finite n-1 dimensional Hausdorff measure, and as a distribution

$$\Delta u = \lambda H^{n-1} (\partial\{u>0\}),$$

and finally, that u grows "linearly" away from the free boundary.

It was then shown that the free boundary of a weak solution is analytic except for a closed set of n-1 dimensional Hausdorff measure zero.

Unfortunately, one cannot expect to obtain, when varying the parameter λ, all values of $|\{u_\lambda>0\}|$, as the interior problem for a circle shows (See [A-C,1]).

Therefore, we resort to a penalization technique.

We consider

$$j_\varepsilon(u) = D(u) + f_\varepsilon(|\{u>0\}|$$

where

$$f_\varepsilon(s) = \begin{cases} \dfrac{s-V_0}{\varepsilon} & \text{for } s > V_0 \\ \varepsilon(s-V_0) & \text{for } s < V_0 \end{cases}$$

It is not hard to prove that a minimizer u_ε of j_ε, is a weak solution, in the sense of [A-C], of $(u_\varepsilon)_\nu^2 = C(\varepsilon)$ and the regularity theorems apply.

We are left with the problem of passing to the limit, and the interesting feature of the technique is that we actually do not have to pass to the limit, that is, for ε small enough $V_\varepsilon = V_0$.

To see this, we prove that $C(\varepsilon)$ remains bounded above and below uniformly in ε ($\frac{1}{\lambda} < C(\varepsilon) > \lambda$).

We then notice that if $V_\varepsilon > V_0$, an inward volume perturbation will increase $D(u)$ by $C(\varepsilon)\delta V \le \lambda\delta V$, and decrease f_ε by $\frac{1}{\varepsilon}\delta V$, a contradiction. The same reasoning applies to the case $V_\varepsilon < V_0$ ([A-A-C]).

A similar idea is applied to problem c) (there is a vast literature on problem c), and related problems, see Giusti [G]).

Again we penalize the volume constraint by considering

$$g(A) = F(A) + f_{\varepsilon}(|A|).$$

The technical interest of such a penalization rests on the fact that we may consider variations that change the volume.

By a very rudimentary variation (cutting or adding pieces of balls) we show that, for any ε, there exists a minimizer, A_{ε}, that has the property that both A and $\complement A$ have uniform positive density at any free boundary point (∂A).

In particular ∂A has finite Hausdorff measure, we can apply the regularity theory (as in Massari [M]), and show that $u + y = C(\varepsilon)$.

As before, it is possible to show that $C(\varepsilon)$ is bounded away from zero and infinity, and hence, we do not have to pass to the limit (that is $V_{\varepsilon} = V_0$. ([C-S]).

Problem b) is somewhat different since it corresponds, as a free boundary problem, to a smaller order of differentiability: that is, if we compare the obstacle problem, where we minimize, among $u > 0$.

$$D(u) + \int u$$

the volume constraintproblem minimizing

$$D(u) + \int \phi_{\{u>0\}}$$

and problem b) associated to minimizing a functional of the type

$$D(u) + \int |D\phi_{\{u>0\}}|$$

we realize that the order of differentiability of the functional decreases in each step. We could ask, therefore, what is the minimum regularity that we need on u to expect a regular free boundary. It is not hard to convince oneself that the critical regularity would be $u \in C^{\frac{1}{2}}$, since in that case $|\nabla u|^2$ will "blow up" at most like $\frac{1}{d}$ when approaching the free boundary, giving changes in the Dirichlet integral of the same homogeneity as those of area ($\frac{1}{f}\,\delta V$).

In fact, u_{ε} is much more regular than that: The free boundary $\partial\{u_{\varepsilon} > 0\}$, is a surface of positive (inwards) mean curvature since any outward perturbation, reducing D(u), must increase perimeter.

The distance function is, therefore, superharmonic in Ω, providing a barrier for u.

Hence, u is Lipschitz, and the regularity theorems of Massari [M] apply.

In short, the Dirichlet integral becomes now the regular term of the functional [A-C,2].

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D KINDERLEHRER

A note on the variational method in contact problems

In a conference whose accent rests on evolution problems and fluids, we offer here an interlude in solid mechanics. Consider an elastic body occupying a region Ω of space which is partially supported by a frictionless rigid planar surface $\Gamma \subset \partial\Omega$, the boundary of Ω. Subjected to assigned volume and surface forces and constrained to remain on or above Γ, the body experiences a displacement and a state of stress. Assuming $\Omega \subset \mathbb{R}^n$ and the planar surface $\Gamma \subset \{x_n = 0\}$, a material point x of the body has new coordinates

$$y = x + u(x), \; x \in \bar{\Omega}$$

obeying the condition

$$u^n(x) \geq 0 \text{ for } x \in \Gamma.$$

The displacement vector $u(x) = (u^1(x),\ldots,u^n(x))$ and its stress distribution are the unknowns of the problem.

Questions of this nature are called contact or indentation problems. Although they have been studied since the eighteenth century, it was only in 1881 that Hertz [3], cf. [9], approached the general problem of two elastic bodies pressed against each other. Approximating the bodies by infinite quadratic solids, he found their stress distributions and the contact area explicitly. The foundations of the contemporary theory were proposed by A. Signorini [13], [14] and draw from the calculus of variations. The problem of contact may be conveniently formulated in terms of a variational inequality. This theory was begun by G. Stampacchia [15]. Our theme here is the use of the complementarity conditions, or natural boundary conditions, to examine the behavior of the solution. By adopting a formulation of this problem where only forces and no values of the displacement are given we are able to recover some qualitative information, even though coerciveness is not present. This includes estimates for the set of contact and the stability of the solution. Since Hertz's method was to solve exactly an approximate problem, perhaps it is of some interest to establish stability properties. Our discussion is based on [5] and [6].

There have been many interesting recent contributions to this theory and its applications. Since the author is not an expert in this subject, he refers to J. Kalker [4] or P. Villaggio [16]. In addition, there is an extensive theory of contact due to Muskhelishvili and his school [10], [11] based on the study of singular integral equations, restricted however to two dimensional problems.

By way of arranging our notations we review some of the apparatus of linear elasticity. Given $\Omega \subset \mathbb{R}^n$, a bounded open region, assume that $\partial\Omega$, the boundary of Ω, is smooth and contains two open smooth (finitely connected) $n - 1$ manifolds Γ and Γ' satisfying

$$\Gamma \subset \{x_n = 0\} \cap \partial\Omega \text{ and } \Gamma' = \partial\Omega - \bar{\Gamma} .$$

Suppose that Ω lies to one side of Γ and $-e_n = (0,\dots,0,-1)$ is the exterior normal to Ω on Γ.

By $H^{m,s}(\Omega)$ we denote the usual Sobolev space of functions whose derivatives through order m are in $L^2(\Omega)$. The space $H^m(\Omega) = H^{m,2}(\Omega)$ and $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in H^1-norm, cf., e.g. [7]. Abusing notation, we also let $H^{m,s}(\Omega)$ stand for the n-fold product $H^{m,s}(\Omega)^n$.

Let $a_{ijhk} \in C^\infty(\bar{\Omega})$ and $\alpha_0 > 0$ satisfy for $x \in \bar{\Omega}$

$$a_{ijhk}\xi_{hk}\xi_{ij} \geq \alpha_0|\xi|^2 \quad \text{for} \quad \xi \in \mathbb{R}^{n^2} \quad \text{with } \xi_{ij} = \xi_{ji} \tag{1.1}$$

and

$$a_{ijhk} = a_{jihk} = a_{hkij} . \tag{1.2}$$

The (linear) strain and stress tensors of $u = (u^1,\dots,u^n)$ are given by

$$\left.\begin{aligned} \varepsilon_{ij} &= \varepsilon_{ij}(u) = \tfrac{1}{2}(u^i_{x_j} + u^j_{x_i}) \\ \sigma_{ij} &= \sigma_{ij}(u) = a_{ijhk}\varepsilon_{hk}(u) \end{aligned}\right\} , \ u \in H^1(\Omega) , \ 1 \leq i, j \leq n . \tag{1.3}$$

From (1.2), (σ_{ij}) is a symmetric matrix and

$$\sigma_{ij} = a_{ijhk} u^h_{x_k} .$$

Define the bilinear form

$$\begin{aligned} a(u,\zeta) &= \int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(\zeta)\,dx \\ &= \int_\Omega \sigma_{ij}(u)\zeta^i_{x_j}\,dx \end{aligned} \quad , \ u,\zeta \in H^1(\Omega) . \tag{1.4}$$

Given functions $f_1,\ldots,f_n$ defined on Ω and $g_1,\ldots,g_n$ defined on Γ', all of which may be taken to be smooth for our purposes, we set

$$(T,\zeta) = \int_\Omega f_i\zeta^i dx + \int_{\Gamma'} g_i\zeta^i dS, \tag{1.5}$$

the distribution of active volume and surface forces, where dS denotes the surface measure on $\partial\Omega$.

The statically admissible displacements for our problem are the functions

$$\mathbb{K} = \{u = (u^1,\ldots,u^n) \in H^1(\Omega) : u^n \geq 0 \text{ on } \Gamma\} \tag{1.6}$$

2. VARIATIONAL INEQUALITY AND COMPLEMENTARITY CONDITIONS

Application of the forces T to the body occupying Ω gives rise to an equilibrium configuration whose displacement $u(x)$ renders a minimum the energy

$$\varepsilon(v) = \frac{1}{2}\,a(v,v) - \langle T,v\rangle \tag{2.1}$$

among all admissible displacements. Thus

$$\varepsilon(u) = \inf_{v\in\mathbb{K}} \varepsilon(v). \tag{2.2}$$

For any $v \in \mathbb{K}$, the convex combination $u + t(v-u) \in \mathbb{K}$, $0 \leq t \leq 1$, so $\lambda(t) = \varepsilon(u+t(v-u))$, $0 \leq t \leq 1$, has a minimum at $t = 0$, whence $\lambda'(0) \geq 0$. This provides us with the variational inequality

<u>Problem</u> (*) To find $u \in \mathbb{K}$: $a(u,v-u) \geq \langle T,v-u\rangle$ for all $v \in \mathbb{K}$.
By convexity of $\lambda(t)$, it is clear that a solution of Problem (*) also satisfies (2.2).

To derive the complementarity conditions, or natural boundary conditions, pertaining to the variational inequality suppose that $u \in H^2(\Omega_\delta)$, $\Omega_\delta = \{x \in \Omega: \text{dist.}(x, \partial\Gamma > \delta\}$, for every $\delta > 0$. Integrating by parts in the variational inequality gives that for $v \in \mathbb{K}$,

$$a(u,v-u) = -\int_\Omega \sigma_{ij}(u)_{x_i}(v^i-u^i)\,dx + \int_{\Gamma'} \sigma_{ij}(u)\nu_j(v^i-u^i)\,dS$$
$$- \int_\Gamma \sigma_{in}(u)(v^i-u^i)\,dx' ,$$

where $\nu = (\nu_1,\ldots,\nu_n)$ denotes the exterior normal to Ω and $dx' = dx_1\ldots dx_{n-1}$. By choosing $v = u + \zeta$ with $\zeta \in H_0^1(\Omega)$ and then with $\zeta = 0$ on Γ, we deduce in the usual fashion that

$$\left.\begin{aligned} (Au)_i &= -\sigma_{ij}(u)x_j = f_i \quad \text{in } \Omega \\ \sigma_{ij}(u)\nu_j &= g_i \qquad\qquad \text{on } \Gamma' \end{aligned}\right\} , \; i = 1,\ldots,n.$$

Thus from the statement of Problem (*),

$$-\int_\Gamma \sigma_{in}(u)(v^i-u^i)\,dx' \geq 0 \quad \text{for} \quad v \in \mathbb{K}.$$

There is no constraint on v^μ, $1 \leq \mu \leq n-1$, for $x \in \Gamma$ so choosing $v = u + (\zeta^1,\ldots,\zeta^{n-1},0)$, $\zeta^1,\ldots,\zeta^{n-1}$ arbitrary, gives that $\sigma_{\mu n}(u) = 0$, on Γ, $\mu, = 1,\ldots,n-1$, whence

$$-\int_\Gamma \sigma_{nn}(u)(v^n-u^n)\,dx' \geq 0 \quad \text{for} \quad v \in \mathbb{K}.$$

Since $v = u+(0,\ldots,0,\zeta^n) \in \mathbb{K}$ for $\zeta^n \geq 0$ on Γ, we learn that $-\sigma_{nn}(u) \geq 0$ on Γ. Choosing $v = 0$,

$$\int_\Gamma \sigma_{nn}(u)u^n\,dx' \geq 0.$$

Because $\sigma_{nn}(u) \leq 0$ and $u^n \geq 0$ we infer that the integrand vanishes identically, namely, $\sigma_{nn}(u)u^n = 0$ on Γ. The argument above has formal elements because $\sigma_{nn} \in L^2_{loc}(\Gamma)$ only, but it may be set right with only a little fuss.

Summarizing the conditions,

$$Au = f \text{ in } \Omega \text{ or}$$

$$(Au)_i = -\sigma_{ij}(u)_{x_j} = f_i \text{ in } \Omega,\ i = 1,\dots,n. \tag{2.3}$$

$$\sigma_{ij}(u)\nu_j = g_i \text{ on } \Gamma',\ i = 1,\dots,n, \tag{2.4}$$

and

$$\begin{aligned} &-\sigma_{nn}(u) \ge 0,\ u^n \ge 0 \\ &\sigma_{nn}(u)u^n = 0 \qquad \text{on } \Gamma,\ \mu = 1,\dots,n-1 \\ &\sigma_{\mu n}(u) = 0 \end{aligned} \tag{2.5}$$

First observations.

I. The system $Au = f$ is elliptic.

II. It is not difficult to show, [5] or [12], that

$$u \in H^2(\Omega_\delta) \text{ for any } \delta > 0 \tag{2.6}$$

where $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x,\partial\Gamma) > \delta\}$. Thus $u \in H^1_{loc}(\Gamma \cup \Gamma')$ and (2.4),(2.5) are valid a.e. Since $u^n_{x_\mu} = 0$ a.e. on the set $\{x \in \Gamma\colon u^n(x) = 0\}$, (2.5) implies that

$$u^n_{x_\mu}\sigma_{nn}(u) = 0 \quad \text{on } \Gamma,\ \mu = 1,\dots,n-1. \tag{2.7}$$

This is instrumental in the proof of regularity of the solution.

III. If the material occupying Ω is homogeneous and isotropic, after suitable normalization

$$\sigma_{ij}(u) = (\alpha - 1)\delta_{ij}\operatorname{div} u + 2\varepsilon_{ij}(u),\ 1 \le i,j \le n \quad \text{and}$$

and (2.8)

$$-(Au)_i = \Delta u^i + \alpha(\operatorname{div} u)_{x_i}$$

We now state a few of the analytical properties we have been able to prove. They are local in character.

<u>Theorem 2.1</u>. Let u be a solution of Problem (*) and $\delta > 0$. Then there exists an $s > 2$ such that

$$u \in H^{2,s}(\Omega_\delta), \; \Omega_\delta = \{x \in \Omega: \mathrm{dist}(x, \partial\Gamma) > \delta\} .$$

In particular, $u \in C^{1,\lambda}(\bar{\Omega}_\delta)$ if $n=2$ and $u \in C^{0,\lambda}(\bar{\Omega}_\delta)$ if $n \le 4$ for some $\lambda > 0$.

Given a solution u of Problem (*), we set

$$I = I(u) = \{x \in \Gamma: u^n(x) = 0\} , \tag{2.9}$$

its set of coincidence. Despite the fact that u need not be unique, I is.

Theorem 2.2. Let $n = 2$. Suppose the material is homogeneous and isotropic.

$$\bar{\Gamma} \subset \tilde{\Gamma} \subset \{x_2 = 0\} \cap \partial\Omega, \; \tilde{\Gamma} \text{ open},$$

and that T vanishes identically in a neighborhood of Γ. Then I is the union of finitely many intervals and isolated points.

The characterization given above does not exclude the possibility that I may be empty. We shall address this question and explain the role of compatible forces. Let $\mathcal{U} \subset H^1(\Omega)$ denote the $n + \frac{1}{2}n(n-1)$ dimensional subspace of affine rigid motions

$$\zeta(x) = c + Bx, \; B = (b_{ij}), \; c \in \mathbb{R}^n, \; b_{ij} + b_{ji} = 0.$$

A force distribution T defined by (1.5) is compatible provided

$$\langle T, \zeta \rangle < 0 \text{ whenever } \zeta \in \mathbb{K} \cap \mathcal{U} \text{ and } -\zeta \in \mathbb{K} \tag{2.10}$$

Our formulation of Problem (*) brings into relief the question of existence and clarifies some qualitative aspects of the behavior of the solution. The theorems of Fichera [2] and Lions and Stampacchia [8] assert the existence of a solution if T is compatible in our sense, cf. also [1], [7]. On the other hand, if T is not compatible the energy (2.1) fails to be bounded below and Problem (*) has no solution [2].

The conditions embodied in (2.10) have an elementary mechanical interpretation. First, it is easy to check that

$$\zeta \in \mathbb{K} \cap \mathcal{U} \text{ with } -\zeta \in \mathbb{K} \text{ if and only if } \zeta^n(x) \equiv 0$$

and moreover if T is compatible,

$\langle T,\zeta\rangle = 0$ for $\zeta \in \mathbb{K} \cap \mathcal{U}$ with $-\zeta \in \mathbb{K}$.

Assume now that Γ is convex, to simplify matters, and let

$$F_i = \int_\Omega f_i dx + \int_{\Gamma'} g_i dS,\ i = 1,\dots,n,$$

be the resultant forces in the various directions. Consideration of the various elements of $\mathcal{U}$

$$e_i,\ x_i e_j - x_j e_i,\quad 1 \le i,\ j \le n,$$

leads us to the conclusions

$$F_n < 0 \text{ and } F_\mu = 0,\quad \mu = 1,\dots,n-1, \tag{2.11}$$

and

$$\int_\Omega \begin{vmatrix} x_\lambda & x_\mu \\ f_\lambda & f_\lambda \end{vmatrix} dx + \int_{\Gamma'} \begin{vmatrix} x_\lambda & x_\mu \\ g_\lambda & g_\mu \end{vmatrix} dS = 0,\quad 1 \le \lambda,\mu \le n-1.$$

We find in this manner that when T is compatible

$$\langle T,\zeta\rangle = F_n \zeta^n(\bar{x}) \text{ for } \zeta \in \mathcal{U},$$

$$\bar{x} = F_n^{-1}(M_1,\dots,M_{n-1},0), \tag{2.12}$$

$$M_\mu = \langle T, x_\mu e_n - x_n e_\mu\rangle$$

From this formula and (2.10) it is easy to infer that $\bar{x} \in \Gamma$. Hence, for a convex Γ, T is compatible if and only if it is statically equivalent to a vertical force applied to an axis $x = \bar{x}$ with $\bar{x} \in \Gamma$. In particular the distance $d = \text{dist.}\,(\bar{x},\partial\Gamma)$ is positive.

3. CAPACITY AND STABILITY

Let us fix a compatible force T and a solution u of Problem (*). By an elementary argument, $\sigma_{nn}(u) \in L^1(\Gamma)$ and

$$a(u,\zeta) = \langle T,\zeta\rangle - \int_\Gamma \sigma_{nn}(u)\zeta^n dx'\ ,\ \zeta \in C^1(\bar{\Omega}) \tag{3.1}$$

Applying (3.1) to $\zeta \in \mathcal{U}$ and noting that supp $\sigma_{nn}(u) \subset I$, by the complementarity conditions, we obtain the balance of forces which may be understood in the classical sense,

$$\int_I \sigma_{nn}(u)\zeta^n dx' - F_n\zeta^n(\bar{x}) = 0 \text{ for } \zeta \in \mathcal{U} . \tag{3.2}$$

If $\zeta = e_n$,

$$-\int_I \sigma_{nn}(u)\,dx' = -F_n > 0$$

so $\text{meas}_{n-1} I > 0$. We have no efficient machinery to appraise this measure, but we are able to estimate the capacity of I. To explain this, let $K \subset \Omega$ be a fixed closed ball and for $E \subset \Gamma$ define

$$\tilde{\text{cap}}_K E = \inf \{\tfrac{1}{2}\int_\Omega |\zeta_x|^2 dx \colon \zeta \in H^1(\Omega),\ \zeta \geq 1 \text{ on } E,\ \zeta = 0 \text{ on } K\} . \tag{3.3}$$

It is not difficult to verify that if $B \supset \bar{\Omega} \supset \Omega$ is an (open) ball and $\text{cap}_B E$ denotes the ordinary capacity of $E \subset B$, namely,

$$\text{cap}_B E = \inf \{\tfrac{1}{2}\int_\Omega |\zeta_x|^2 dx \colon \zeta \in H^1_0(B),\ \zeta \geq 1 \text{ on } E\} ,$$

then

$$c_1 \text{cap}_B E \leq \text{cap}_K E \leq c_2 \text{cap}_B E.$$

The expression (3.3) is more convenient.

<u>Theorem 3.1</u>. Let Γ be convex and let u be a solution of Problem (*) for a compatible force T. There is a function $C_0(d) \geq 1$, $d = \text{dist.}(\bar{x}, \partial\Gamma) > 0$, and a constant $C_K > 0$ such that

$$a(u,u) \leq C_0(d) \|T\|^2$$

and

$$\frac{F_n^2}{C_k C_0(d)\|T\|^2} \leq \tilde{\text{cap}}_K I,$$

$$\|T\|^2 = \int_\Omega f_i^2 dx + \int_{\Gamma'} g_i^2 ds .$$

We have also found a stability property of the solution.

Theorem 3.2. Let Γ be convex and let u and u* be solutions of Problem (*) for compatible forces T and T*. Then

$$a(u-u^*,\ u-u^*) \le \eta \|T-T^*\|,$$

$$\eta = \eta(F_n, F_n^*,\ \|T\|,\ \|T^*\|,\ d, d^*),$$

where F_n, F_n^* are the resultant vertical forces and d,d* are the distances of the central axes to $\partial\Gamma$ of T,T* respectively.

Observations

I. The solution of Problem (*) is unique within addition of a $\zeta \in \mathcal{U}$ with $\zeta^n \equiv 0$. Restricting our attention to solutions u satisfying, for example,

$$\int_\Omega (u^\lambda_{x_\mu} - u^\mu_{x_\lambda})\,dx = 0,\ 1 \le \lambda,\ \mu \le n-1,$$

we may conclude that

$$\|u-u^*\|^2_{H^1(\Omega)} \le \eta_0 \|T-T^*\|$$

for a suitable function η_0.

II. Other measures of capacity may be more appropriate to elastostatics. Set

$$\mathbb{M} = \{v \in H^1(\Omega) : \int_\Omega v^i dx = \int_\Omega (v^i_{x_j} - v^j_{x_i})\,dx = 0,\ 1 \le i,\ j \le n\}$$

and for $E \subset \Gamma$ define

$$\tilde{\operatorname{cap}}_{\mathbb{M}} E = \inf \{\tfrac{1}{2}\, a(v,v) :\ v \in \mathbb{M},\ v^n \ge 1 \text{ on } E\}\ .$$

Using this capacity we have given some estimates in the case of an infinite circular cylinder, in plain strain, initially at contact along a generator with a horizontal rigid plane [6]. Such a capacity was also considered independently by Villaggio.

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B LADANYI

Shear-induced stresses in the pore ice in frozen particulate materials

1. MECHANICAL BEHAVIOUR OF FROZEN SOILS

From the point of view of the science of materials, frozen soil is a natural composite material composed of four different components: solid grains (mineral or organic), ice, unfrozen water, and gases. Its most important characteristic by which it differs from other apparently similar materials, such as unfrozen soils and most particulate composites, is the fact that it contains ice, usually at such a high homologous temperature, that the ice-water ratio in its pores is not a constant but a function of external factors such as temperature and interstitial stresses.

There is only one type of ice in soil pores, i.e., normal hexagonal ice (Type 1 h), but the unfrozen water exists in two state: strongly bound and weakly bound water. The former is the water film surrounding the mineral particle and held to it by high intermolecular forces that suppress its freezing, even at very low temperatures. The rest of the pore water is weakly bound to the particles and can be frozen more easily. (Anderson and Morgenstern, 1973). The amount of unfrozen water present in the frozen soil at a certain temperature can be related to the specific surface area of the mineral particles (Anderson and Tice, 1972), and is also strongly affected by the electrolyte content (Yong and Warkentin, 1975).

Since under natural conditions the temperature of frozen soils in permafrost areas is usually not very low, and many permafrost soils are fine grained, the unfrozen water will nearly always be present in their pores together with the ice. In the terminology used in this Symposium, one can say, therefore, that most engineering problems in permafrost fall in the "mushy region" where two phases are present simultaneously. In addition, the ratio of the two phases changes continuously with changes in temperature and load application.

Nevertheless, the mechanical behaviour of frozen soils depends in a large measure on that of the pore ice which binds the grains together and fills most of the pore space. The pore ice is usually of a polycrystalline type with a random crystal orientation. Its deformation under stress results mainly from the motion of dislocations and can be represented by a power-law creep of the Norton type.

The yielding and failure of polycrystalline ice under a triaxial state of stress differs from most other materials in that, under a high hydrostatic pressure, the ice first weakens and then eventually melts. On the other hand, at low hydrostatic pressures and ordinary temperatures, it shows a ductile yielding at low strain rates, but becomes more and more brittle as the strain rate increases (Mellor, 1979).

Observations made on sand-ice mixtures show that for sand concentrations lower than about 40%, the mechanical behaviour of the mixture is mainly governed by that of ice. However, when the sand fraction exceeds that amount, the behaviour becomes a function of the strength of both the ice cement and the soil skeleton. (Goughnour and Andersland, 1968). When a hydrostatic confining pressure is applied to a frozen granular mass, pressure melting of ice occurs locally, and there is some water migration towards lower stress regions. Theoretically, the freezing point of Ice 1 decreases by about 0.074°C per 1 MPa of pressure increase over the atmospheric pressure (Williams, 1968). This would produce only a negligible melting under ordinary pressures, were it not for stress concentrations at ice-particle contacts which can increase the pressures by a factor of 10 to over 100 (Parameswaran, 1980). In addition, if confining pressures are very high, so that grain fracture occurs, high pressure melting will take place even in a dense sand (Chamberlain et al., 1972).

The type of failure behaviour of frozen sand observed by Chamberlain et al. (1972) in triaxial compression tests under a constant strain rate and temperature but at different confining pressures, can be represented schematically by the Mohr plot in Figure 1 (Ladanyi, 1981). The plot is seen to be composed of three failure lines: Line I is for ice cement and covers the region between the tensile strength of ice and the pressure melting point, Line II represents the drained failure envelope for the soil skeleton, while Line III is the undrained failure envelope for unfrozen sand. The whole failure behaviour of frozen sand under such conditions is represented by the full line, which is a combination of the three basic lines and which, with a transition zone, forms four distinct regions, A to D. Nevertheless, it should be noted that most engineering problems in permafrost are limited to the regions A and B of the plot.

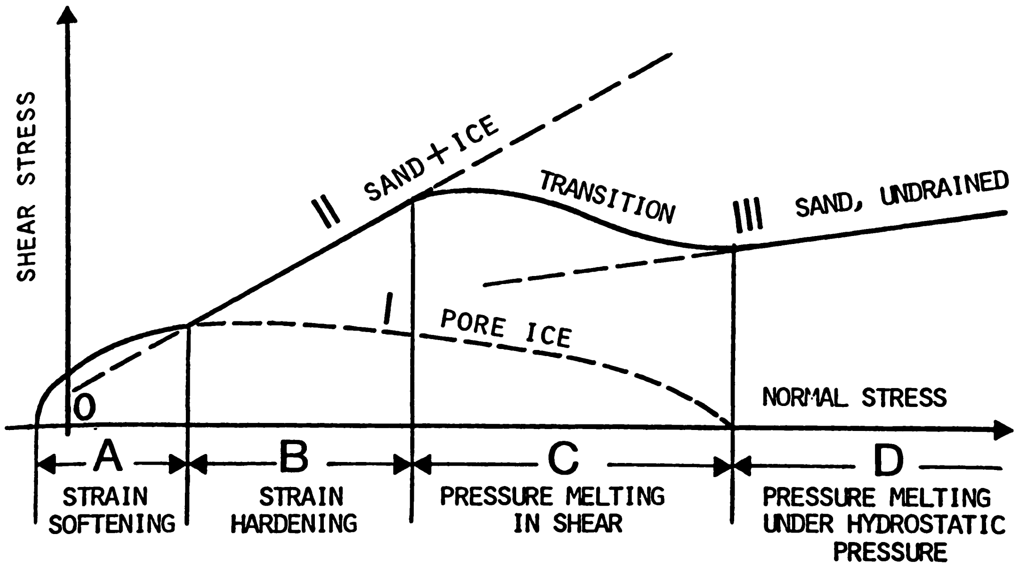

Figure 1. Schematic representation of the whole failure envelope for frozen sand (From Ladanyi, 1981, after Chamberlain et al., 1972).

2. INTERGRANULAR STRESSES IN FROZEN SOIL

As mentioned in the foregoing, at higher grain concentrations, the behaviour of frozen soil is affected not only by ice and unfrozen water, but also by friction, dilatancy and mineral cohesion in the soil skeleton. A proper understanding of that behaviour and its eventual prediction has been considered difficult until now, because of the impossibility to measure directly, or infer indirectly, the value and variation of intergranular stresses in frozen soil, produced by external loading.

In the following it will be shown how the stresses in the pore ice of a frozen soil, during a constant volume triaxial compression test can be indirectly determined from the knowledge of the stress-strain behaviour of the soil skeleton and the pore ice, respectively.

In that, no separate consideration will be given to the stresses in the unfrozen water, because, according to Williams (1968) and other investigators, while the variations of outside pressure affect the quantity of unfrozen water in frozen soil, they affect only very little the pressure difference between ice and water. That pressure difference can be determined from the Clausius-Clapeyron equation (Williams, 1968)

$$p_i - p_w \approx (1 - T/T_o)L/V_\ell \tag{1}$$

where p_i and p_w are pressures in ice and in water, respectively, T is the absolute temperature, T_o is the normal freezing point, depending on pressure according to

$$T_o(^oK) \approx 273.15 - 0.074\ p_i(MPa) \tag{2}$$

L is the latent heat of fusion, and V_ℓ the molar volume of water.

When proper values are substituted for L and V_ℓ, Eq. (1) becomes

$$p_i - p_w \approx 320(1 - T/T_o) \tag{3}$$

with stresses in MPa and temperatures in ^oK.

It follows from the foregoing that the pressure in the pore water can be determined from thermodynamic considerations once the stresses in the ice are known, so that it is not necessary to consider it as a separate phenomenon. The following discussion is therefore limited to the interaction between the soil grains and the pore ice when a particulate frozen soil is submitted to shear strains in a constant volume triaxial compression test. The idea was first expressed by the author in 1974, and was further developed more recently (Ladanyi, 1981).

In unfrozen soil mechanics, it is now reasonably well known how intergranular (effective) stresses change under a deviatoric stress increment and how they are affected by the rate of strain. It is also known that during an undrained (volume constant) shear, the induced pore water pressure, which is a second order phenomenon, depends mainly on the accumulated shear strain and is practically unaffected by the rate of strain (Lo, 1969; Akai et al., 1975).

The response of an unfrozen water-saturated soil to a stress change under cylindrical symmetry conditions, such as provided by an ordinary triaxial test, can be studied very conveniently in the so-called Rendulic plot (Figure 2), which represents a section of the principal stress space containing the σ_1 axis and the diagonal, $\sigma_2 = \sigma_3$ (Rendulic, 1937). In such a plot, with σ_1 and σ_3 as reference axes, each stress point plotted in terms of principal stresses, appears at the same time also in terms of octahedral stresses, if the space diagonal and the normal to it through the origin are

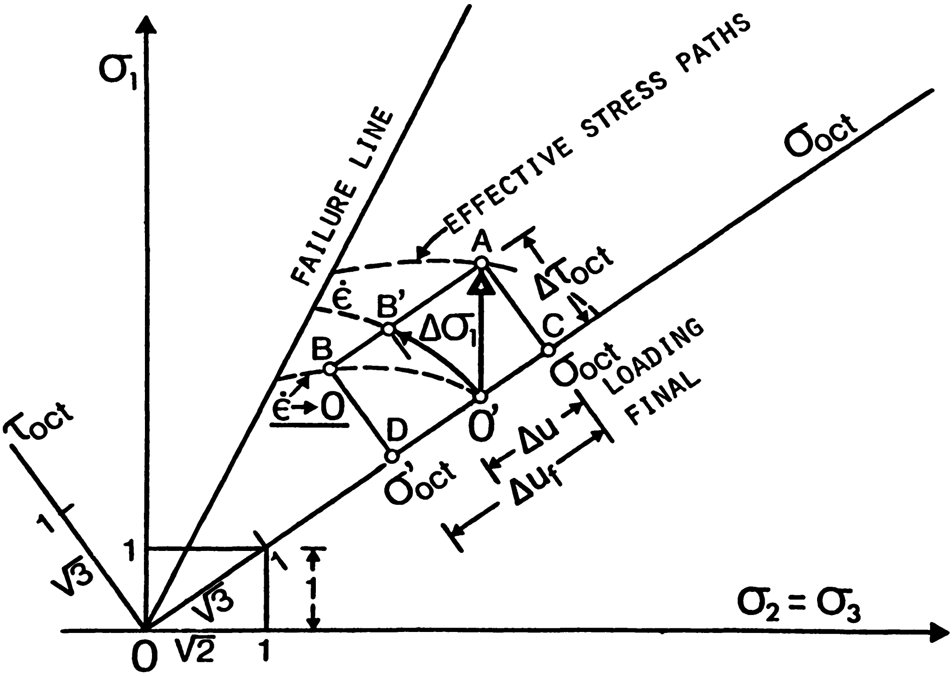

Figure 2. Pore water pressure generation in undrained compression of an unfrozen normally consolidated soil (After Ladanyi, 1981).

used as the alternate axes of reference. It can be shown that, in terms of the units on the σ_1 axis, the lengths along the space diagonal represent $\sqrt{3}\ \sigma_{oct}$ and those normally to it $\sqrt{3}\ \tau_{oct}$. This property of the plot enables any given stress increment to be easily separated into its hydrostatic and deviatoric components.

Using this type of plot for plotting the water content contours from drained tests and effective stress paths from undrained tests, performed with a saturated clay, Rendulic (1937) has shown that, for a given overconsolidation ratio, these two types of contours coincide, and he concluded that the effective stress path is a unique property of a given soil strained at constant volume. The latter property enabled Rendulic to explain the manner in which pore pressure is generated in a saturated soil, and how the part of the total stress that is carried by the soil skeleton can be separated from that which must be carried by the pore water. Rendulic's results were later confirmed and extended by Henkel (1960).

Figure 2 shows schematically how the separation of applied stresses into effective stresses and pore pressure can be performed according to this principle, in the case of a normally consolidated clay under triaxial compression test conditions with $\sigma_1 > \sigma_2 = \sigma_3$. In the figure it is seen that, when a total stress increment $\Delta\sigma_1$(O'A) is applied, the soil skeleton is stressed along the effective stress path O'B, which implies an increase in the octahedral shear stress, $\Delta\tau_{oct}$, (DB), and a decrease in the effective octahedral normal stress, $\Delta\sigma'_{oct}$ (O'D). Since the pore water can support only hydrostatic stresses, the change in pore water pressure is equal to

$$\Delta u = \sigma_{oct} - \sigma'_{oct} \tag{4}$$

and is given by the length BA in units of σ_{oct}. In other words, the total stress change $\Delta\sigma_1$ has been separated in such a manner into its deviatoric part, $\Delta\tau_{oct}$, which is taken over completely by the soil skeleton, and its hydrostatic part $\Delta\sigma_{oct}$, which is taken over partially by soil skeleton ($\Delta\sigma'_{oct}$) and partially by the pore water (Δu). Although it was shown later that the effective stress path in undrained shear is not unique but is affected by the rate of strain (e.g., Akai et al., 1975), the described method remains still valid for unfrozen soils, but the effective stress point will move during loading along a stress path O'B', corresponding to a given strain rate.

If a similar analysis of internal stress distribution is made for a frozen soil, a similar but not quite the same result is obtained. Let us assume that we have in a triaxial cell a specimen of a frozen normally consolidated saturated clay or a frozen, loose sand, both of which have effective stress paths similar to that in Figure 3. If a total stress increment $\Delta\sigma_1$ is applied to such a specimen and the specimen is then deformed under undrained conditions (closed system), the stresses will again be shared between the mineral phase and the matrix as before, but with a very essential difference: while the unfrozen water can take only hydrostatic stresses, the ice can support at least temporarily, also a certain portion of applied shear stresses. If the stress increment $\Delta\sigma_1$ does not reach beyond the long-term strength line ($t = \infty$), the stress separation between the two phases will be similar to that shown in Figure 3.

Because the ice bonds carry a portion of the applied shear stress $\Delta\tau_{oct}$, the soil skeleton is now less strained than it would be in unfrozen soil, and

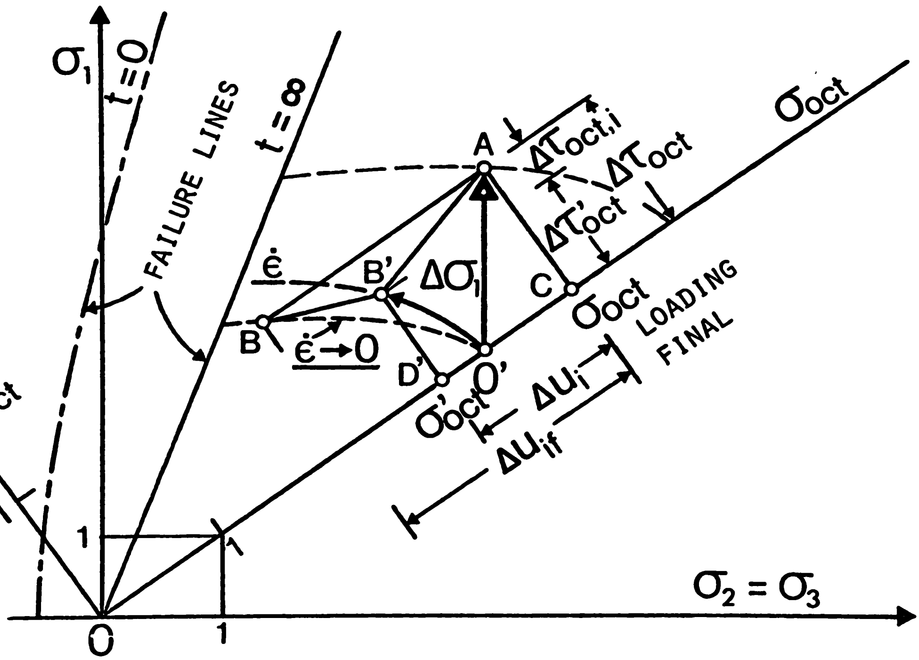

Figure 3. Pore ice stress development in undrained compression of a frozen normally consolidated soil (After Ladanyi, 1981).

its effective strength will be mobilized initially only to the point B' on the effective stress line O'B' which corresponds to the applied strain rate. The hydrostatic pore pressure change in ice Δu_i generated by this straining, is still given by Eq. (4), but it is smaller than Δu in the same soil when unfrozen.

On the other hand, since the shear stresses are now also temporarily shared between the soil skeleton and the ice, in a manner quite similar to the sharing of the hydrostatic pressure in an unfrozen soil, one can write, by analogy, an equation similar to Eq. (4), but in terms of the shear stresses:

$$\Delta\tau_{oct,i} = \Delta\tau_{oct} = -\Delta\tau'_{oct} \tag{5}$$

where $\Delta\tau_{oct}$ denotes the total applied shear stress increment, $\Delta\tau_{oct,i}$ the shear stress in ice bonds, and $\Delta\tau'_{oct}$ the effective shear stress than can be taken over by the soil skeleton at a given strain and rate of strain.

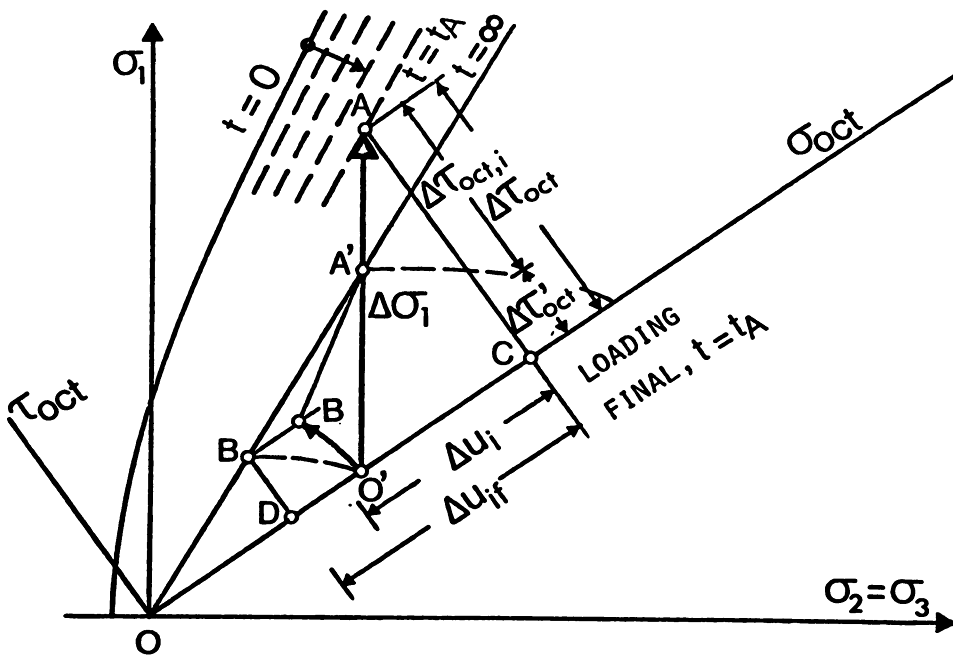

Figure 4. Pore ice stresses in a normally consolidated frozen soil, stressed beyond its long-term strength (After Ladanyi, 1981).

If the frozen soil is allowed to creep under closed system conditions with stress at A, the soil skeleton will take over an increasing portion of the applied shear stress, and the point B' in Figure 3 will move towards B. Eventually, after a sufficiently long time interval, $\Delta u_i \rightarrow \Delta u$ and $\Delta\tau_{oct,i} \rightarrow 0$, as in an unfrozen soil.

Another important difference between an unfrozen and a frozen soil is that the latter can temporarily support also stress increments reaching far outside the long-term strength line. Such a large stress increment applied to a normally consolidated frozen clay or a loose frozen sand is shown schematically in Figure 4.

Initially, if the stresses are rapidly increased from O' to A, and the system is closed, the effective stress point will move only to B'. With time, the soil will continue to creep at a steady or decreasing rate. This will have two simultaneous effects: (1) It will mobilize slowly the whole strength of the soil skeleton, i.e., B' will move towards B along a curved stress path, and (2) the creep will bring the ice bonds closer and closer to

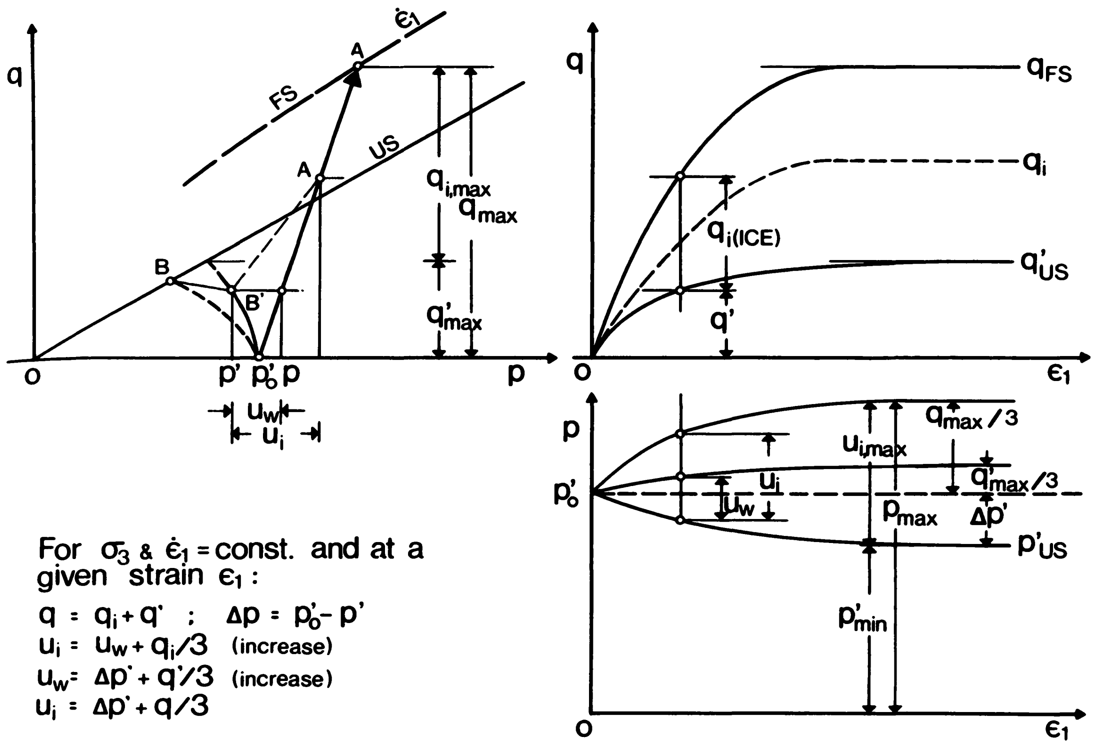

Figure 5. Determination of pore ice stresses in a normally consolidated frozen soil.

failure. Since, in such a creep test under constant stress, the stress A remains fixed, the loss of strength with time can be visualized as a continuous shrinking of the delayed failure surface from its initial position which corresponds ideally to t = O (instantaneous strength), until it reaches A at $t = t_A$, and creep failure occurs. In other words, for any stress increment reaching beyond the longterm strength line, failure will inevitably occur, given sufficient time.

At failure ($t = t_A$), the stresses in such a closed system will be: $\Delta\sigma'_{oct}$ and $\Delta\tau'_{oct}$ in the soil skeleton, Δu_{if} in the ice and $\Delta\tau_{oct,i}$ in the ice bonds. It can be expected that such a specimen under high stress without consolidation will undergo essentially a creep of a stationary type but not necessarily steady state.

Figures 2 to 4 show qualitatively how the intergranular stresses in frozen soil are likely to vary with strain and time. In order to determine their numerical values, the following data should be available:

(a) The exact shape of the relevant effective stress path of the soil and its corresponding stress-strain curve, and

(b) The stress-strain curve of the pore ice, both of them for the same strain rate and temperature.

If one assumes, in addition, as Ruedrich and Perkins (1974), that average shear strains in the soil skeleton and the pore ice are the same throughout the test, the pressure change in the pore ice can be determined as shown in Figure 5 which is similar to Figure 4, but using the coordinate system $p \equiv \sigma_{oct}$ and $q = \sigma_1 - \sigma_3$, and in which the σ_3 = const. loading path appears as a straight line with the slope 3:1. It is implied in that Figure that when the soil is loaded along the stress path O'B corresponding to a test with σ_3 = const. and undergoes a distortion ε_B, this mobilizes the ice strength component q_i and the soil strength $q' = q - q_i$. At the same time, p'_o decreases to p'. The resulting ice pressure change is then equal to:

$$u_i = p - p' = \frac{1}{3} q - (p' - p'_o) \tag{6}$$

For the same soil when unfrozen, the pore pressure at the same strain would be only:

$$u_w = \frac{1}{3} q' - (p' - p'_o) \tag{7}$$

From Equations (6) and (7), and knowing that u_w is practically strain rate independent, one can write:

$$u_i(\varepsilon, \dot{\varepsilon}) = u_w(\varepsilon) + \frac{1}{3} [q(\varepsilon,\dot{\varepsilon}) - q'(\varepsilon,\dot{\varepsilon})] \text{ , or} \tag{8}$$

$$u_i(\varepsilon,\dot{\varepsilon}) = u_w(\varepsilon) + \frac{1}{3} q_i(\varepsilon,\dot{\varepsilon}) \tag{9}$$

In other words, if u_w and q' have been obtained as a function of strain in an undrained compression test carried out on an unfrozen soil sample at the rate $\dot{\varepsilon}$, while the same sample when frozen shows the stress-strain curve $q = f(\varepsilon,\dot{\varepsilon})$, the pore ice pressure u_i in a compression test with σ_3 = const. can be calculated from Eq. (8). If the loading to q is followed by a creep at q = const., ε increases with time, so that both u_w and u_i will also increase with time.

In order to get a theoretical solution for u_i, one should have analytical expressions for both the undrained stress-strain curve of soil and the relevant stress-strain curve for ice. For the former, theoretical equations have been developed for normally consolidated clays by Roscoe et al. (1963) while the effect of strain rate was studied by Sekiguchi (1977), and the overconsolidation effect by Pender (1978). Use in frozen soil mechanics of such solutions, based on the critical state theory, seems very promising, but will need further careful study before it can be generally applied.

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E SANCHEZ-PALENCIA & P SUQUET

Friction and homogenization of a boundary

1° INTRODUCTION

The equilibrium problem for an elastic body on a rigid support with dry friction (Coulomb's law) seems to be an open free boundary problem. The main difficulty is the lack of a variational principle associated to the problem and the consequent failure of the convex analysis technique. In fact, with standard notations (which will be given in the sequel) the *formal* variational formulation of the problem is (cf. [1] or [2] sect. 5.4.4)

$$\left.\begin{aligned} & u \in K \\ & \int_\Omega a_{ijkh}\, e_{kh}(u)\, e_{ij}(v-u)\, dx + \int_{\Gamma_2} k|\sigma_N|(|v_T| - |u_T|)\, ds \geq \\ & \qquad \geq \int_\Omega f(v-u)\, dx \quad \forall v \in K \\ & K = \{v \mid v_N \leq 0 \text{ on } \Gamma_2\} \end{aligned}\right\} \tag{1.1}$$

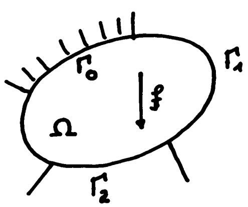
The term containing $|\sigma_N|$ is not defined for $\sigma \cdot n \in H^{-1/2}(\Gamma_2)^3$ and is not the subgradient of a functional. Several mathematical attempts have been made in order to overcome this difficulty: non local friction [1], fixed point techniques (quasi-variational inequalities [3][4]). The former introduces a modification of the law while the latter involves a relation between the friction coefficient and the elasticity coefficients.

In fact, friction seems to be a surface phenomenon associated with roughness. In the present work we apply the *homogenization of boundaries* to the classical (without friction) Signorini's problem on a boundary having small undulations. A small parameter ε is associated with the size of the corrugations. In fact the limit problem (homogenized) is not a dry friction problem. It is a new well posed (variational) problem for which the stress vector on the boundary is contained in the conjugate cone (instead of a halfspace). This law was already proposed in [5].

In fact our result is not very surprising for two reasons. First the hypothesis of small displacements is not probably fitted for the physical problem. Second the Signorini's problem is of standard type (minimization of some energy) and this property is preserved by homogenization of the boundary. As a result, our study is an example of homogenization of a boundary, but it does not furnish a justification of the Coulomb's dry friction law. This justification has to be done.

2° SETTING OF THE PROBLEM

The classical Signorini's problem (without friction) is the following (see for instance (2)). Let Ω be a bounded connected problem in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ formed by three disjoint surfaces $\Gamma_0, \Gamma_1, \Gamma_2$. The solid body fills Ω, is clamped on Γ_0 and free on Γ_1. The surface Γ_2 is such that the body may either lie or part on a rigid support.



$$\left.\begin{array}{l} \dfrac{\partial\sigma_{ij}}{\partial xj} + f_i = 0 \text{ in } \Omega \\[2ex] u_i = 0 \text{ or } \Gamma_0,\ \sigma_{ij}n_j = 0 \text{ on } \Gamma_1 \end{array}\right\} \tag{2.1}$$

$$u_N \leq 0,\ \sigma_N \leq 0,\ \sigma_T = 0,\ u_N\sigma_N = 0 \text{ on } \Gamma_2 \tag{2.2}$$

$$\sigma_{ij} = a_{ijkh}\, e_{khx}\,(u),\ e_{khx}\,(u) = 1/2\,\left(\frac{\partial u_h}{\partial x_k} + \frac{\partial u_k}{\partial x_h}\right)$$

Classical notations are used: in particular a_{ijkh} are the elastic coefficients. We consider them to be constant and satisfying the standard conditions of symmetry and ellipticity.

The variational formulation of (2.1)-(2.3) is as follows. We define the Hilbert space V, the closed convex set K and the bilinear and linear forms a and L by

$$V = \{v \mid v = (v_i),\ v_i \in H^1(\Omega),\ v_i = 0 \text{ on } \Gamma_0\ i = 1, 2, 3\}$$

$$K = \{v \mid v \in V,\ v_N \le 0 \text{ on } \Gamma_2\}$$

$$a(u,v) = \int_\Omega a_{ijkh}\, e_{khx}(u)\, e_{ijx}(v)\, dx$$

$$L(v) = \int_\Omega fv dx$$

Then the problem amounts to:

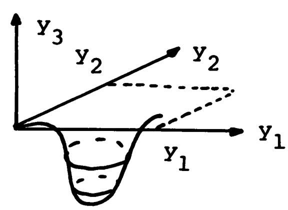
$$\left.\begin{array}{l} \text{find } u \in K \text{ such that:} \\ a(u, v-u) \ge L(v-u) \quad \forall v \in K \end{array}\right\} \tag{2.4}$$

This problem has a unique solution, it is equivalent to the minimization problem:

$$\min_{v \in K} \ \phi(v) = 1/2\ a(v,v) - L(v)$$

We now consider a special case of this situation for domains $\Omega = \Omega_\varepsilon$ depending a parameter $\varepsilon \to 0$ as follows. We consider the plane (Oy_1y_2) shared into periods $Y =]0,Y_1[\times]0,Y_2[$. We give a positive Y-periodic function $F(y_1y_2)$ of class C^∞ taking value zero in a neighbourhood of the boundary of the period. We then consider the surface Σ defined by:

$$y_3 = -F(y_1,y_2)$$



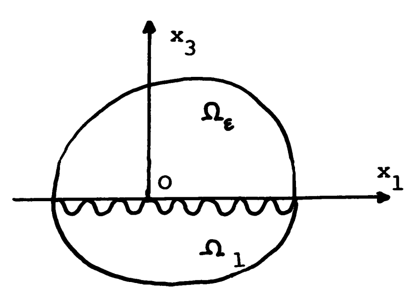
and let Σ_ε be its homothetic with ratio ε.

$$x_3 = -\varepsilon\ F\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)$$

We consider an open connected domain Ω_1 having a non empty intersection with the plane $x_3 = 0$. The domain Ω_ε and the limit domain Ω_0 are defined by

$$\Omega_\varepsilon = \Omega_1 \cap \{x \mid x_3 > -\varepsilon F(\tfrac{x_1}{\varepsilon}, \tfrac{x_2}{\varepsilon})\}$$

$$\Omega_0 = \Omega_1 \cap \{u \mid x_3 > 0\}$$



The undulated boundary is:

$$\Gamma_2^\varepsilon = \Omega_1 \cap \Sigma_\varepsilon$$

The corresponding Signorini's problem amounts to search for $u^{\varepsilon} \in V^{\varepsilon}$ satisfying the analogous of (2.4) with:

$$V^{\varepsilon} = \{v \mid v = (v_i),\ v_i \in H^1(\Omega^{\varepsilon}),\ v_i = 0 \text{ on } \Gamma_0\}$$

$$K^{\varepsilon} = \{v \mid v \in V^{\varepsilon},\ v.n \le 0 \text{ on } \Gamma_2^{\varepsilon}\}$$

3° ASYMPTOTIC EXPANSION AND CONSEQUENCES

Following the classical process of boundary homogenization ([6] sect. 5.7) we define the domain B.

$$B = \{y \mid y_i \in]0,Y_i[,\ i = 1,2,\ \ y_3 > -F(y_1,y_2)\}$$

Then we expand the stress and displacement fields:

$$\sigma^{\varepsilon}(x) = \sigma^0(x,y) + \varepsilon\sigma^1(x,y) + \ldots + \varepsilon^i\sigma_i(x,y) + \ldots \qquad y = \frac{x}{\varepsilon} \tag{3.1}$$

$$u^{\varepsilon}(x) = u^0(x) + \varepsilon u^1(x,y) + \ldots + \varepsilon^i u_i(x,y) + \ldots \qquad y = \frac{x}{\varepsilon} \tag{3.2}$$

with σ^i, u^i B-periodic. Moreover u^1 must satisfy the boundary layer condition:

$$\lim_{y \to +\infty} \operatorname{grad}_y u^1 = 0 \iff \lim_{y \to +\infty} e_y(u^1) = 0 \tag{3.3}$$

The expansions of (2.1), (2.3) give at order ε^{-1} and ε^0:

$$\left.\begin{aligned} &\frac{\partial \sigma_{ij}}{\partial y_j} = 0 \quad \text{in B} \\ &\sigma_{ij} = a_{ijkh}\left[e_{khx}(u^0) + e_{khy}(u^1)\right] \qquad \text{in B} \end{aligned}\right\} \tag{3.4}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \text{ in } \Omega^0 \tag{3.5}$$

in order to expand the boundary condition (2.2) we define two conjugate convex cones of $\mathbb{R}^3$:

$$\Gamma = \{\beta | \beta.n(y) \leq 0 \quad \forall y \in \Sigma\}$$

$$\Gamma^* = \{\tau | \ (\tau,\beta) \leq 0 \quad \forall \beta \in \Gamma\}$$

(2.2) at order ε^0 gives

$$u^0 \in \Gamma. \tag{3.6}$$

The tangential components of $\sigma.n$ are zero and

$$u^0(x)n(y) < 0 \implies \sigma_{ij}(x,y)n_i(y)n_j(y) = 0 \tag{3.7}$$

$$u^0(x)n(y) = 0 \implies \sigma_{ij}(x,y)n_i(y)n_j(y) \leq 0 \tag{3.8}$$

At order ^{1}, if $\underline{u^0 \in \mathrm{Int}\Gamma}$ (interior of Γ) we are in the situation (3.7) and (2.2) gives no new condition on u^1. On the other hand, if $u^0 \in b\Gamma$ (boundary of Γ) there is a subset of Σ, denoted Σ' where (3.8) holds and we have:

$$u^1(x,y)n(y) \leq 0 \qquad \forall y \in \Sigma' \tag{3.9}$$

For sake of simplicity we admit that Σ' is either Σ(if $u^0 = 0$) or a set with zero measure. This happens in particular if ε does not contain any plane portion (apart from $y_3 = 0$).

Thus, the so called *local problem* in B (cf. [6]) (x is a parameter) amounts to find a B-periodic function $u^1(y)$ satisfying (3.3)(3.4)(3.6)(3.7)(3.8)(3.9) where u^0 is given (in fact $u^0 \in \mathbb{R}^3$ and $e_{ijx}(u^0) \in \mathbb{R}^6$ are given). We shall see later that the local problem has a solution if u satisfies some compatibility conditions. These conditions (see later) constitute boundary conditions to be satisfied by $u^0(x)$ on $\{x_3 = 0\}$ in order to define a boundary value problem for u^0 in Ω_0. This limit problem will be explicitly given in Section 5.

In order to obtain the compatibility conditions we multiply (3.4) by any

$$\begin{aligned} 0 &= \int_B \frac{\partial\sigma_{ij}}{\partial y_j} \beta i \ dy = \int_{\partial B} \sigma^0{}_{ij} n_j \beta_i ds \\ &= a_{ijkh}\, e_{khx}\, (u^0)\ \beta_i \int_{\partial B} n_j ds + \int_{\partial B} a_{ijkh}\, e_{khy}\, (u^1) n_j \beta_i ds \end{aligned} \tag{3.10}$$

But $\int_{\partial B} n_j ds = 0$ and $\int_{\Sigma} n_j ds = - \delta_j Y_1 Y_2$

bearing in mind the periodicity of u^1, (3.10) gives:

$$a_{i3kh}\, e_{khx}(u^0) \mid x_3 = 0 \quad \beta_i = \int_{\Sigma} \sigma_N^0\, \beta_N ds \tag{3.11}$$

and by means of the notation:

$$\sigma_{ij}(u^0) = a_{ijkh}\, e_{khx}(u^0)$$

(3.11) becomes, by virtue of (3.6) (3.7) (3.8)

$$\sigma_{i3}(u^0)\beta_i \geq 0 \quad \forall \beta \in \Gamma \quad \sigma_{i3}(u^0)\, u^0_i = 0 \text{ on } x_3 = 0 \tag{3.12}$$

standard properties of convex analysis show that this is equivalent to either (3.13) or (3.14) where I_Γ denotes the indicative function of Γ and ∂I_Γ its subdifferential (see [7]):

$$\left.\begin{array}{l} u^0 \in \Gamma \\ -\sigma(u^0).n \in \partial I_\Gamma(u^0) \end{array}\right\} \quad \text{on } x_3 = 0 \tag{3.13}$$

$$\left.\begin{array}{l} -\sigma(u^0).n\; x_3 = 0 \in \Gamma^* \\ u^0 \in \partial I_{\Gamma^*}(-\sigma(u^0).n) \end{array}\right\} \quad \text{on } x_3 = 0 \tag{3.14}$$

4° <u>INDICATIONS ABOUT EXISTENCE AND UNIQUESNESS OF THE LOCAL PROBLEM</u>

We sum up the local problem. Let $u^0 \in \mathbb{R}^3$, $e_{ijx}(u^0) \in \mathbb{R}^6$ be given, satisfying the compatibility conditions ((3.12),(3.13) or (3.14)): find a B-periodic vector u^1 satisfying:

$$\left.\begin{array}{l} \dfrac{\partial\sigma^0_{ij}}{\sigma Y_j} = 0,\; \sigma^0_{ij} = a_{ijkh}\,[e_{khx}(u^0) + e_{ijy}(u^1)] \quad \text{in B} \\ \lim\limits_{y_z \to +\infty} e_y(u^1) = 0, \text{ tangential components of } \sigma_{ij}n_j \text{ zero on} \\ u^1 n(y) \leq 0 \text{ and } \sigma_{ij}n_jn_i \leq 0 \quad \forall y \in \Sigma',\; \sigma_{ij}n_jn_i = 0 \;\forall y \in \Sigma - \Sigma' \end{array}\right\} \tag{4.1}$$

In order to give a variational formulation of this problem, we define a space and a convex set K as follows. Let B_R be the domain defined by: $B_R = \{y \in B,\ y_3 < R\}$. Let $\mathcal{E}$ be the set of the B-periodic vector functions of class C^∞ which are constant for sufficiently large y_3. Then V is the completed space of $\mathcal{E}$ for the norm associated with the scalar product:

$$(u,v) = \int_B e_{ijy}(u)\, e_{ijy}(v)\, dx + \int_{B_R} u_i v_i dy \tag{4.2}$$

$$K = \{v | v \in V \quad v.n|_{\Sigma'} \le 0\}$$

The problem (4.1) is then equivalent to the following variational problem:

$$\text{Find } u^1 \in K \text{ such that } \forall v \in K$$
$$\int_B a_{ijkh}\, e_{khy}(u^1)\, e_{ijy}(v-u^1)\, dy + \int_\Sigma a_{ijkh}\, e_{khx}(u^0)\,|_{x_3=0}\, n_j (v_i - u_i^1)\, ds \ge 0 \tag{4.3}$$

or equivalently the minimization on K of the functional:

$$\Phi(v) = 1/2 \int_B a_{ijkh}\, e_{khy}(v)\, e_{ijy}(v)\, dy + \int_\Sigma a_{ijkh}\, e_{khx}(u^0)\Big|_{x_3=0} n_j v_i ds \tag{4.4}$$

It is to be noticed that in the case $u^0 \in b\Gamma$, $u^0 \ne 0$, which is a very special case, Σ' is a part of Σ with zero measure. Thus it is not obvious that K is closed for the strong topology of V (and it is probably not: think to the dense embedding of $H^{1/2}_{00}$ into $H^{1/2}$ [8]). Physically the small deformations hypothesis is probably violated and the problem should be formulated in another framework. This point deserves a deeper study.

<u>Case $u^0 = 0$</u>: In this case $\Sigma' = \Sigma$. We admit that the compatibility condition satisfied in such a way that:

$$\sigma_{i3}(u^0) \in \text{Int } \Gamma \quad \text{on } x_3 = 0 \tag{4.5}$$

Then a solution u^1 exists because:

$$\lim_{\substack{v \to +\infty \\ v \in K}} \Phi(v) = +\infty \tag{4.6}$$

Case $u^O \in$ Int: $u^1 = O$ is a solution of the problem.

Case $u^O \in b\Gamma$, $u^O \neq O$: We admit that the compatibility condition is satisfied in such a way that:

$$\left.\begin{array}{l} \sigma_{i3}\,(u^O)\big|_{x_3=O}\ \ \gamma_i = O \\ \\ \gamma_i\, n_i\,(y) \leq O \quad \forall\, y \in \Sigma' \end{array}\right\} \Rightarrow \exists\,\lambda \quad \gamma = \lambda\, u^O$$

In this case we define a space $\dot{V}$ as the quotient space of V by the straight line $\{\lambda u^O,\ \lambda \in \mathbb{R}\}$. We note that $\Phi(v)$ take the same for all the elements of an equivalence class and consequently is a functional $\dot{\Phi}$ on $\dot{V}$. The same thing holds for K from which we get a convex set $\dot{K}$ of $\dot{V}$. The existence of a solution then follows from a property analogous to (4.6) in $\dot{V}$, $\dot{K}$.

5° HOMOGENIZED BOUNDARY CONDITION AND COMPLEMENTS

5.1. The limit problem

According to the considerations of Section 3, the homogenized problem for u^O in Ω_O is:

$$\frac{\partial\sigma_{ij}(u^O)}{\partial x_j} + f_i = O,\ \sigma_{ij} = a_{ijkh}\ \ e_{khx}(u^O)\ \text{ in } \Omega_O \tag{5.1}$$

$$u^O = O \text{ on } \Gamma_O,\ \sigma_{ij}(u^O)n_j = O \text{ on } \Gamma_1 \tag{5.2}$$

$$u^O \in \Gamma,\ \ -\sigma\,(u^O).n \in \partial I_\Gamma(u^O)\ \text{ on } \partial\Omega_O \cap \{x_3 = O\} \tag{5.3}$$

This problem has one and only one solution. Indeed, if K^O is defined by (5.4) the problem (5.1)-(5.3) is equivalent to (5.5).

$$K^O = \{v \,|\, v \in (H^1(\Omega^O))^3,\ u|_{x_3=O} \in \Gamma,\ u = O \text{ on } \Gamma_O\} \tag{5.4}$$

$$\begin{array}{l} \text{Find } u^O \in K^O \text{ such that } \forall v \in K^O \\ \displaystyle\int_\Omega a_{ijkh}\ e_{khx}(u^O)\ \ e_{ijx}(v-u^O)\,dx \geq \int_\Omega f_i\ \ (v_i-u_i^O)\,dx \end{array} \tag{5.5}$$

5.2. On the structure of the friction laws [9]/[10]:

The main difficulty of the Coulomb's law is that it is a *non standard* one (this will be precised later on) and therefore cannot be handled by classical reasonings of Convex Analysis.

We note that for an elastic body the displacement field on Γ_2 is a *global state variable*. Indeed if this displacement field, now denoted α, is known the displacement in the whole body is given by the variational principle:

$$W(\alpha) = \mathrm{Min} \qquad W(u) = 1/2\int_\Omega a_{ijkh}\, e_{khx}(u)\; e_{ijx}(u)\,du - \int_\Omega fudx \tag{5.6}$$

$$u = \alpha \text{ on } \Gamma_2$$

$$u = 0 \text{ on } \Gamma_0$$

The thermodynamical force A associated with α is:

$$A = -\frac{\partial W}{\partial \alpha} = -\sigma.n \text{ on } \Gamma_2 \ .$$

The two laws of thermodynamics show that the dissipated power is:

$$\mathcal{D} = \int_{\Gamma_2} A\dot{\alpha} = -\int_{\Gamma_2} \sigma.n\ \dot{u}ds \geq 0 \quad . \tag{5.7}$$

We shall say that the dissipative process is *standard* if there exists a convex, l.s.c. function φ such that:

$$\dot{\alpha} = \frac{\partial\varphi}{\partial A}(A) \qquad \varphi(A) \geq \varphi(0) = 0 \tag{5.8}$$

The friction law (5.8) is an evolution law which accounts for time effects. Dry (or static) friction laws are built on the same model:

$$\alpha = \frac{\partial\varphi}{\partial A}(A) \tag{5.9}$$

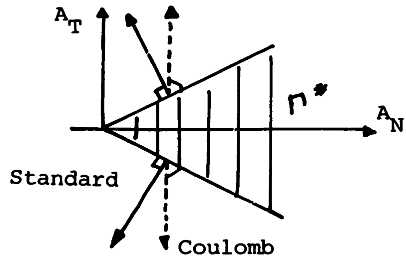
Examples

1. Viscous friction

$$\varphi(A) = 1/2\ k|_T|^2 \qquad \dot{u}_T = -k\sigma_T,\ \dot{u}_N = 0 \quad \text{(evolution)}$$

or $u_T = -k\sigma_T$, $u_N = 0$ (dry) .

2. $\varphi(A) = I_{\Gamma^*}(A)$ where $\Gamma^* = \{A \mid A_N \geq 0,\ |A_T| \leq k|A_N|\}$



The dry friction law (5.9) is: $-\sigma.n \in \Gamma^*$, $u \in \partial I_\Gamma(-\sigma,n)$ on Γ_2 which is exactly the law (5.3) in its form (3.14).

As it can be seen on the figure the Coulomb's law is *not standard.*

6° CONCLUSIONS

The present work illustrates the technique of boundary homogenization. It proposes a law of friction where sliding is allowed only after separation. This law is a standard one and differs from the Coulomb's law. Coulomb's law is still to be justified by more accurate models.

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L COLLATZ

Numerical methods for free boundary problems

1. GENERAL REMARKS

One can classify numerical methods from different points of view.

A. In various cases one is transforming the original problem into another problem:

Aa) To avoid the difficulties arising from having an unknown domain for the differential equation, one transforms the problem into a problem with fixed boundaries; examples for this are the wellknown Landau transform, which is used in the following paper of J. Nitsche, or the Baiocchi transform for the dam problem, which was extensively experienced by Baiocchi [1] for several problems. Then one can use the wellknown numerous numerical subroutines for boundary value problems with fixed domains; the price for this procedure is a more complicated partial differential equation.

Ab) One uses other transforms of the problem (not necessarily into a problem with fixed boundary). The so-called variable-interchange-method interchanges the dependent variable with one of the independent variables, as it is done in the classical theory of partial differential equations by using Legendre's transform. This change of variables is used in the following abstract by H. Kawarada for a one-dimensional Stefan problem with a change of phase. For the obstacle problem Duvaut's transform can be performed. Other methods consist in introducing auxiliary functions as for instance the enthalpy in the so-called H-method. The following paper by Fix is in this spirit, but a piecewise linear enthalpy H is introduced to get a system which is easier to solve than a fully nonlinear diffusion problem.

Ac) One uses formulations with integrals, either dealing with "weak formulations" and "weak solutions" (as in the following papers by Fix and Nitsche), or transforming the problem into an integral equation for which many numerical procedures are available. Hoffmann [9] obtained by this method excellent numerical results, for instance for the Stefan problem with 1 or 2 phases. One can use also immediately variational principles and variational inequalities. In many cases it is easier to formulate the problem in the variational form, starting with considerations about the energy than to formulate the differential equations and the boundary conditions. For variational problems many numerical procedures have been developed.

Ad) One follows the direct approach to the original differential equations, without going to equivalent other formulations as in Aa), Ab), Ac); then a lot of numerical procedures are available.

Various modifications and crossing between the described types are possible.

Another point of view gives different numerical procedures which one can use in each of the types Aa), Ab), Ac), Ad):

Ba) Discretization method, finite differences, finite elements, boundary element methods, multigrid methods, etc.

Bb) Parametric methods, approximation of the wanted solution by functions of prescribed type with a number of parameters which one has to determine, Galerkin's method, etc.

Bc) Various other types of methods, perturbation methods, series expansions, etc.

The following paper of G. H. Meyer gives a "1981-Survey on numerical methods for free boundary problems" with a general description of various numerical methods for 1. One dimensional problems, 2. Partial differential equations, 3. Some open problems. This paper described for different methods advantages and disadvantages in comparing the methods, and also the difficulties with ill posed problems, with computational expense. Special open problems are for instance in the domain of hyperbolic equations and of inverse problems. At the end of his paper G. H. Meyer gives a detailed list of 48 references most of them of the years 1979-1981.

It is impossible to describe all variants of methods which have been published (see for instance Magenes [11], Ockendon [13], Wilson-Solomon-Boggs [16], Kinderlehrer-Stampaccia [10] and many others) and it looks preferable to describe a further method which has been successfully applied to some simple cases.

2. THE FREE BOUNDARY INCLUSION METHOD WITH THE AID OF MONOTONICITY

Sometimes one is interested to get inclusions for free boundaries. This is usually difficult for most of the above described numerical methods. But there exist simple free-boundary problems for which it is possible to get inclusions provided that monotonicity properties can be proved (see Hoffmann [9]). These monotonicity principles (see Collatz [3], [4], Bohl [2], Schröder [14] etc.) often reduce to the physical fact that any increase of the input causes a corresponding increase of the output; of course this is not true in the whole generality and it is often not easy to prove monotonicity mathematically (see Collatz [4], Meyn-Werner [12], Schröder [14]). If there exists a mathematical proof, lower and upper bounds for the free boundary can be computed using approximation and optimization methods. If no proof of monotonicity is available the above numerical methods can be applied as well, but without guarantee for the calculated bounds. So, it seems an important mathematical research to enlarge the field of applicability of such monotonicity methods.

The procedure can be illustrated on a special class of Stefan-type free boundary problems. We consider problems in the x-t-plane. In the domain

$$B = \{(x,t) \quad 0 < x < s(t), \quad 0 < t < T\}, \tag{1}$$

described in Figure 1, we have the parabolic differential equation for a function $u(x,t)$

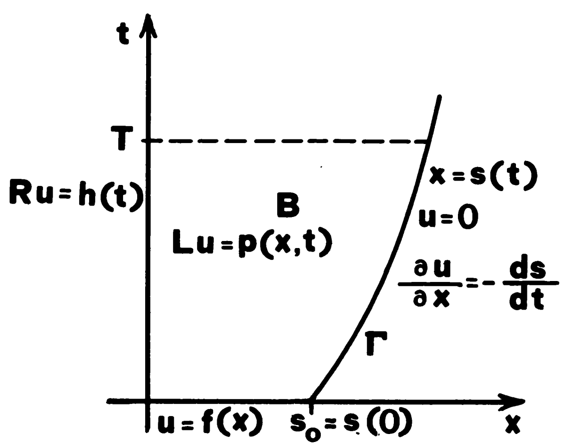

Figure 1

$$Lu \equiv \frac{\partial u}{\partial t} - g(x,t,u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}) = p(x,t) \quad (2)$$

and the boundary conditions

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le s_o = s(0) \quad (3)$$

$$Ru \equiv u(0,t) - c_1 \frac{\partial u}{\partial x}(0,t) = h(t) \quad \text{for} \quad 0 \le t \le T \quad (4)$$

and, on the free boundary Γ

$$\Gamma = \{(x,t), \quad x = s(t)\} \quad (5)$$

$$u = 0, \quad \frac{\partial u}{\partial x} = -\frac{ds}{dt} \quad \text{for} \quad x = s(t). \quad (6)$$

Here the functions $f(x)$, $h(t)$, $p(x,t)$, $g(x,t,...)$ and the real constants c_1,T are supposed to be given, and the functions $s(t)$ and $u(x,t)$ are unknown.

We compare this problem P with another problem P* with corresponding functions $u^*(x,t)$, $p^*(x,t)$, $s^*(t)$, $f^*(x)$, $h^*(t)$, domain B^* and free boundary Γ^*, but with c_1, s_o and g unchanged.

We suppose that the free boundary condition is satisfied for the comparison problem

$$u^* = 0, \quad \frac{\partial u^*}{\partial x} = -\frac{ds^*}{dt} \quad \text{for} \quad x = s^*(t). \quad (7)$$

We assume, that the monotonicity principle holds (see Glashoff-Werner [7], Collatz [5]):

<u>Principle</u>

$$p^* \le p, \; f^* \le f, \; h^* \le h \text{ implies } s^*(t) \le s(t), \quad (8)$$

that means, the domain B^* is contained in B.

3. NUMERICAL EXAMPLE

We describe the method on a Stefan problem with a convective boundary condition considered by Solomon-Wilson-Alexiades [15].

Differential equation:

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{in} \quad B : 0 < x < s(t), \quad 0 < t < T, \quad T = 1. \tag{9}$$

Boundary conditions:

$$Ru = u - \frac{\partial u}{\partial x} = 1 \quad \text{for} \quad x = 0, \tag{10}$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} = -\frac{ds}{dt} \quad \text{for} \quad x = s(t), \quad s_o = s(0) = 0. \tag{11}$$

We take as approximate solutions w:

$$w_j(x,t) = (z(t) - x)\frac{dz}{dt} - c_j[z(t) - x]^2 \qquad (j = 1,2) \tag{12}$$

satisfying the free boundary conditions (11) for every real constant c_j and every functions $z(t) \in C^1$:

$$w = 0, \quad \frac{\partial w}{\partial x} = -\frac{dz}{dt} \quad \text{for } x = z(t) .$$

We consider $x = z(t)$ as approximation for the wanted free boundary Γ and we look for a lower bound $z = z_1(t)$ and an upper bound $z = z_2(t)$. We choose the simple specification

$$z_j(t) = a_j t - b_j t^2 \tag{13}$$

and determine the parameters a_j, b_j, c_j in such a way that

$$\left.\begin{array}{l} Lw_1 \le 0 \le Lw_2 \quad \text{for} \quad 0 < x < z_j(t) \\ \left.\begin{array}{l} Rw_1 \le 1 \le Rw_2 \\ z_1 \le z_2 \end{array}\right\} \text{for} \quad 0 \le t \le T \\ z_1(T) = \text{Max}, \quad z_2(T) = \text{Min} \end{array}\right\} \tag{14}$$

Then the inclusion holds

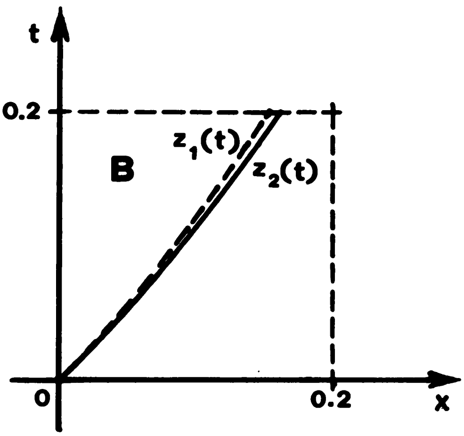

Figure 2.

$$z_1(t) \le s(t) \le z_2(t) \quad \text{for} \quad 0 \le t \le T. \tag{15}$$

With

$a_1 = 0.96664$ $\qquad$ $a_2 = 1$

$b_1 = 0.65752$ $\qquad$ $b_2 = 0.56574$

$c_1 = 0.46720$ $\qquad$ $c_2 = 0.23061$

one gets the inclusion (for instance for $T = 0.2$)

$$|s(0.2) - 0.1725| \le 0.0055$$

Figure 2 gives the graphs for the functions $z_j(t)$; the free boundary is in between.

Of course one can easily improve the accuracy by introducing more free parameters a_ν, b_ν, c_ν in (12), (13).

I thank Mr. Uwe Grothkopf [8] for numerical calculations on a computer.

The numerical method of free boundaries inclusion with the aid of monotonicity was treated also in many other problems:

Stefan problem in more dimensions

Obstacle problem

Biological diffusion-problem, Collatz [6]

etc.

APPENDIX

Numerical Analysis of Stefan Problem by H. Kawarada, Tokyo University.

Free boundary problems appear in various fields of engineering and applied sciences: for example, the problem in mechanics of continuous media, the equilibrium of plasma, the pollution of air and water and others. Here we restrict our interest to systems which produce a change of phase (solidification, liquefaction, sublimation...).

We consider a problem of Stefan type $(Pr)_o$ in one dimensional space. For this problem, we introduce successively i) the variational inequality (V.I) by changing the depending variable of $(Pr)_o$; ii) a first penalized problem $(Pr)_i$, which is directly introduced from $(Pr)_o$ by means of the method of integrated penalty; iii) a second penalized problem $(Pr)_2$ by changing the depending variable of $(Pr)_1$, which arises as a penalized problem associated with (V.I)..

Results on convergence are given and numerical implementation is presented.

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G FIX

Phase field models for free boundary problems

1. THE H-METHOD FOR STEFAN PROBLEMS

The standard description of a solidification process is captured in the classical Stefan problem [1]. In this context $T = T(\underline{x},t)$ denotes a temperature field with T_* denoting the phase transition temperature. In particular, points $\underline{x}$ in the material Ω are in the liquid phase when $T > T_*$, and conversely, they are in the solid phase when $T < T_*$. At points $\underline{x} \in \Omega$ where there is no phase transition, i.e., $T(\underline{x},t) \neq T_*$, the following diffusion equation is valid:

$$\frac{\partial T}{\partial t} = \operatorname{div}(D\, \overrightarrow{\operatorname{grad}}\, T) \quad \underline{x} \in \Omega, \quad t > 0, \quad T(\underline{x},t) \neq T_* . \tag{1.1}$$

The transition region is defined by

$$\Gamma(t) = \{\underline{x} \in \Omega : T(\underline{x},t) = T_*\} , \tag{1.2}$$

and for points in this region

$$\lambda V + [D\cdot\overrightarrow{\operatorname{grad}}\, T\cdot\underline{\nu}]_-^+ = 0, \quad \underline{x} \in \Gamma(t) . \tag{1.3}$$

Here λ is the latent heat, V the normal velocity of $\Gamma(t)$, $\underline{\nu}$ the normal to $\Gamma(t)$, and $[\cdot]_-^+$ denotes the jump across $\Gamma(t)$. To complete the specification of the problem we specify initial conditions, e.g.

$$T(x,0) = T_0(x) \quad \underline{x} \in \Omega , \tag{1.4}$$

for a given initial temperature field T_0, and boundary conditions. For simplicity we use Dirichlet type conditions, namely

$$T(x,t) = T_1(x), \quad \underline{x} \in \partial\Omega, \quad t > 0, \tag{1.5}$$

where T_1 is a given temperature field defined on the boundary $\partial\Omega$ of Ω.

The H-method [2] is a reformulation of (1.1), (1.3) in terms of a single partial differential equation. To do this we introduce an "enthalpy" as follows:

$$H(u) = \begin{cases} u + \lambda/2 & u > 0 \\ & \\ u - \lambda/2 & u < 0 \end{cases} , \tag{1.6}$$

where

$$u = T - T_* . \tag{1.7}$$

Then (1.1), (1.3) are formally equivalent to

$$\frac{\partial H}{\partial t} = \mathrm{div}(D\ \vec{\mathrm{grad}}\ u) . \tag{1.8}$$

Since H is discontinuous across the free boundary $\Gamma(t)$, (1.8) must be interpreted in the weak sense. Perhaps the most easily understood "weak version" of this equation is as a balance law which expresses the conservation of heat. Indeed, let C be any closed curve in $\Omega \times [0,\infty)$. Then the balance of heat in C gives

$$\int_C (H\nu_t + D\ \vec{\mathrm{grad}}\ u \cdot \underline{\nu}_{\underline{x}})\, dC = 0 , \tag{1.9}$$

where $\underline{\nu} = (\nu_t, \underline{\nu}_{\underline{x}})$ is the outer normal to the space-time curve C.

By shrinking C to a point $(\underline{x},t)$ one can formally derive (1.7). Alternately, letting C shrink to a point $(\underline{x},t)$ where $\underline{x} \notin \Gamma(t)$, one obtains (1.1). If on the other hand $\underline{x} \in \Gamma(t)$, we obtain

$$v[H]_-^+ + [D\ \vec{\mathrm{grad}}\ u \cdot \underline{\nu}_{\underline{x}}]_-^+ = 0 ,$$

which is (1.3).

The most common finite element and finite difference approximations to (1.1), (1.3) can be derived directly from (1.9). For example, consider the case where Ω is an interval $[a,b]$, and $[a,b] \times [0,\infty)$ is subdivided into rectangles with nodes at (x_j,t_n). Let Δx denote the spacing in x and Δt denote the spacing in t with u_j^n denoting the approximation to $u(x_j,t_n)$. Then instead of requiring that (1.8) hold for all closed curves C, we require that it hold only for the rectangular paths shown in Figure 1.1. This along with the use of midpoint quadrature to evaluate the integrals gives the following difference scheme:

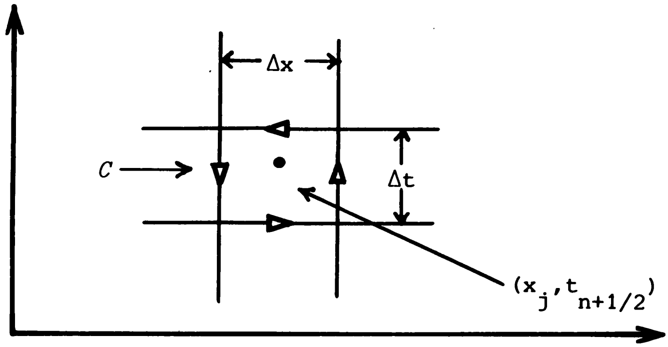

Figure 1.1. Test Curves C

$$\frac{H(u_j^{n+1}) - H(u_j^n)}{\Delta t} = D\,\frac{[u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}]}{\Delta x^2} \tag{1.10}$$

The scheme (1.10) (plus boundary and initial conditions) represents a set of nonlinear equations for the discrete temperature field $\{u_j^n\}$. Since H is piecewise linear in u this scheme is in fact a "piecewise linear" system in the temperature field $\{u_j^n\}$ and as such is easier to solve than a fully nonlinear diffusion problem. This scheme and its multidimensional analogs have proven to be quite effective in practice [3]. Interestingly most of the schemes proposed for solving the Stefan problem either directly reduce to (1.9) or to this scheme with minor modifications.

2. A GENERALIZED STEFAN PROBLEM - SUPERCOOLING AND SURFACE TENSION

In many applications - most notably in crystal growth and the fusion and joining of materials [4] - there are important effects not captured in the classical Stefan problem. One is the effect of surface tension. As solidification takes place the melting temperture T_* itself will change as the curvature K of the free surface $\Gamma(t)$ changes. This can be expressed mathematically by a formula that goes back to Gibbs [5], and which takes the following form:

$$T_* = \bar{T}_*(1 - \ell) \ . \tag{2.1}$$

Here $\bar{T}_*$ is a mean transition temperature, and ℓ is a capillary length.

While (2.1) introduces a full nonlinearity (into the otherwise "piecewise linear" system (1.10), (1.3)), it is in fact a benign nonlinearity which attempts to stabilize perturbations introduced in the system. For example, if we introduce a perturbation in a planar free surface with a positive curvature $K > 0$ as in Figure 2.1, then the transition temperature is lowered, and the perturbation tends to liquidify and disappear.

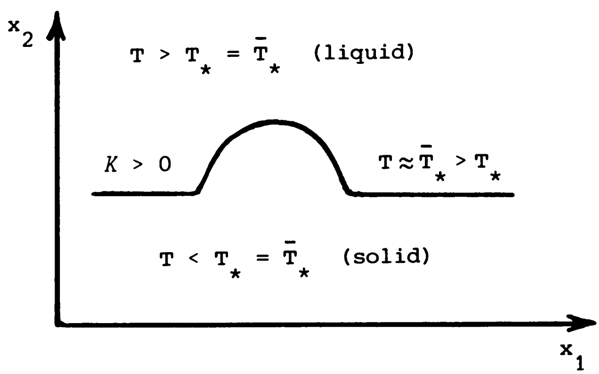

Figure 2.1. Effects of surface tension

The second effect of importance is supercooling. It is possible, for example, for a material to be in the liquid phase with its temperature below the transition temperature, or conversely, in the solid phase with its temperature above the transition temperature. This situation can be captured mathematically by permitting the enthalpy $H = H(u)$ to be multivalued (as in Figure 2.2).

This clearly is an unstable force in solidification. That is, supercooling tends to amplify any perturbation introduced in the system such as shown in Figure 2.1. Without surface tension the influence of supercooling would lead to a totally unstable system. In real physical systems, however, where supercooling is present, local instabilities - often called dendrites - can occur but they are counter balanced by the nonlinear stabilizing effects of surface tension.

One can still use the H-method for problems where surface tension is present (i.e., T_* is given by (2.1)), although it is far more tedious to apply than with standard Stefan problems. The most significant problem is the need to accurately approximate the free surface $\Gamma(t)$ so that reasonable

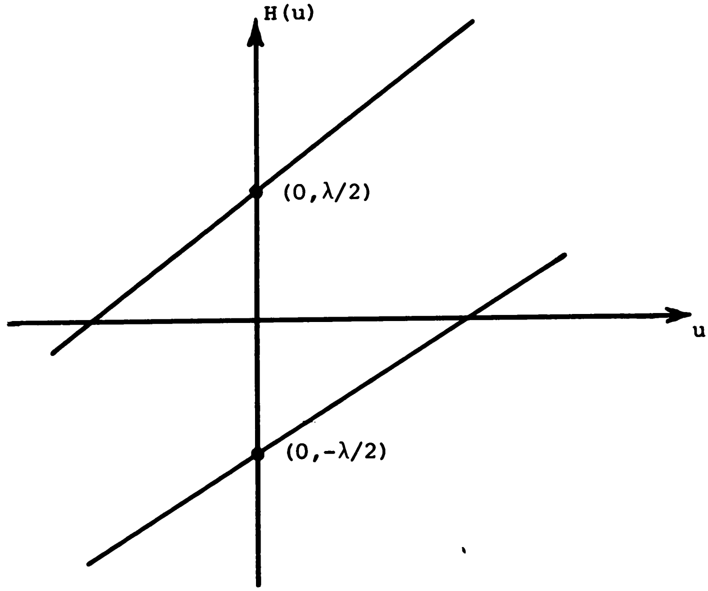

Figure 2.2

approximations to the curvature K of $\Gamma(t)$ can be obtained. This unfortunately works against one of the most attractive features of the H-method when applied to standard Stefan problems; i.e. the ability to get reasonable temperature fields without having to sharply resolve the free boundary [6].

The effects of supercooling, on the other hand, cannot directly be introduced into the H-method. Additional information is needed to resolve the ambiguity created by a multivalued enthalpy $H(u)$. Smith [7] has offered one method for doing this by introducing a local criteria for determining which branch of $H(u)$ should be used. In particular, Smith subdivded space-time into cells and used (1.9) to derive a finite difference approximation. He let a spatial cell change phase only when a "majority" of its neighbors were in the opposite phase. Smith's numerical results seem quite realistic, and from the point of view of statistical mechanics, his local criteria seems intuitive. However, a number of important questions can be raised. First, his local criteria is tied to the numerical discretization. How does one interpret his condition as the mesh spacing goes to zero? Secondly, if there is a "limit condition" is it independent of the grids used in the numerical approximation? In short, can one view Smith's scheme as the approximation to an appropriate continuum model?

In the next section we introduce an alternate model which possibly may be of value in answering these questions. In addition, this model itself can also be used for numerical approximation when supercooling and surface tension is present.

3. PHASE FIELD MODELS

In this model we introduce a phase variable $\phi = \phi(\underline{x},t)$ which is to be determined by an appropriate field equation. Ideally we should have $\phi = \phi_+$ in the liquid region and $\phi = \phi_-$ in the solid region. Thus the enthalpy is given by

$$H = u + \frac{1}{2}\phi \quad . \tag{3.1}$$

As above a balance of heat gives

$$\frac{\partial H}{\partial t} = \mathrm{div}[D\ \vec{\mathrm{grad}}\ u], \quad \underline{x} \in \Omega, \quad t > 0, \tag{3.2}$$

and we recall that this equation is equivalent to (1.1) and (1.3).

The phase field ϕ, on the other hand, satisfies

$$\tau \frac{\partial\phi}{t} = \xi^2 \Delta\ \phi + \frac{1}{2}(\phi - \phi^3) + \lambda u \tag{3.3}$$

for appropriate constants $\tau > 0$, $\lambda > 0$, and ξ. This equation can be derived by introducing a Helmholtz free energy $F(\phi)$, and requiring that ϕ relax in time $\tau > 0$ to a critical point of this functional; i.e.,

$$\tau \frac{\partial\phi}{\partial t} = \frac{\delta F}{\delta\phi} \tag{3.4}$$

(see for example [8]). Here we shall choose a different way of justifying (3.3). In particular, we show that in a suitable sense it reduces to the surface tension relation (2.1) which we rewrite as

$$u = -\bar{T}_* \ell K \quad \text{on} \quad \Gamma(t). \tag{3.5}$$

To do this we view τ as a small relaxation time, i.e.,

$$\tau << 1 \ . \tag{3.6}$$

The diffusion scale ξ is also small, but not as small as τ:

$$\xi \ll 1, \quad \tau/\xi \ll 1; \tag{3.7}$$

i.e., the relaxation to equilibrium takes place at faster rate than the diffusion of the phase.

To study (3.3) under the conditions (3.6)-(3.7) we use the method of matched asymptotic expansions. To take a concrete case consider the situation illustrated in Figure 3.1. where

$$y = \zeta(x,t)$$

describes the free surface

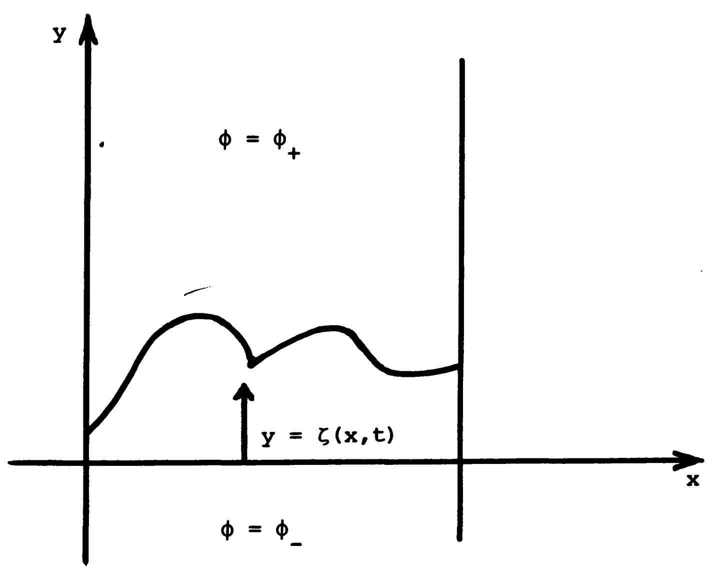

Figure 3.1. The free surface $z = \zeta$

The outer solution (to first order) is obtained by setting $\xi = \tau = 0$. This gives

$$\frac{1}{2}(\phi - \phi^3) - \lambda u = 0. \tag{3.8}$$

For small u this has three real solutions

$$\phi = \phi_+ \approx 1, \qquad \phi \approx 0, \qquad \phi = \phi_- \approx -1 \tag{3.9}$$

It is easy to check that only the first and third are stable orbits of

$$\tau \frac{\partial\phi}{\partial t} = \frac{1}{2} (\phi - \phi^3).$$

Thus away from the free surface we have

$$\phi = \phi_+ \text{ or } \phi = \phi_- .$$

Near the free surface we obtain a boundary layer of order $O(\xi)$. To get the first order contribution we set $\tau = 0$ and obtain the following balance

$$0 = \xi^2\Delta + \frac{1}{2} (\phi - \phi^3) + \lambda u . \tag{3.10}$$

The appropriate boundary layer variables are

$$x^\nabla = x \tag{3.11}$$

$$y^\nabla = [y - \zeta(x,t)]/\xi . \tag{3.12}$$

This gives

$$\xi^2 \frac{\partial^2\phi}{\partial x^{\nabla 2}} - 2\xi \frac{\partial\zeta}{\partial x^\nabla} \frac{\partial^2\phi}{\partial x^\nabla \partial y^\nabla} - \xi \frac{\partial^2\zeta}{\partial x^{\nabla 2}} \frac{\partial\phi}{\partial y^\nabla} + \left(\frac{\partial\zeta}{\partial x^\nabla}\right)^2 \frac{\partial^2\phi}{\partial y^{\nabla 2}} \tag{3.13}$$

$$+ \frac{1}{2} (\phi - \phi^3) + \lambda u = 0$$

Noting that u is small near the free surface $y^\nabla = 0$, the $O(1)$ balance is

$$\left(\frac{\partial\zeta}{\partial x^\nabla}\right)^2 \frac{\partial^2\phi}{\partial y^{\nabla 2}} + \frac{1}{2} (\phi - \phi^3) = 0 . \tag{3.14}$$

Solutions of this equation are approximately

$$\tanh (y^\nabla) = \tanh ([y - \zeta(x,t)]/\xi).$$

The next balance gives the desired Gibbs-Thompson relation, namely

$$u = (\xi/\lambda) \left(\frac{\partial\phi}{\partial y^\nabla}\right) \left(\frac{\partial^2\zeta}{\partial x^2}\right) \quad \text{on} \quad \Gamma(t) . \tag{3.15}$$

Noting that the curvature K of $\Gamma(t)$ is given by

$$K = -\frac{\partial^2 \zeta}{\partial x^2},$$

we conclude from (3.15) that u is proportional to curvature K of the free surface on $\Gamma(t)$: i.e., (3.5) holds.

It is interesting to note that this analysis predicts for positive $\xi > 0$ and $\tau > 0$ a boundary layer of thickness $O(\xi)$ where ϕ rapidly changes from ϕ_- to ϕ_+. Thus it is important when this model is being used to keep ξ sufficiently small so that the free surface is not smeared. In addition, the relaxation time τ must also be small so that the model is describing phenomena near a thermodynamic equilibrium.

4. NUMERICAL APPROXIMATIONS

The use of the phase field model described in the previous section for numerical approximations has certain important advantages. First, supercooling can be directly taken into account in this model. Indeed, this will be the case for example when u and ϕ have apposite signs. Secondly, the jump conditions across the free surface are natural in the sense they are implicit in the model. Thus we do not have to explicitly track the free surface nor compute its curvature.

There are, however, some serious disadvantages. First, as mentioned at the end of previous section ξ and τ are very small. For example, if the diffusion coefficient D in (3.2) is normalized to unity, then ξ typically can be no larger than 10^{-2} and τ can typically be no larger than 10^{-3}. (The actual numbers depend on the material). This means that (3.3) is an extremely stiff equation, and thus care must be used in its numerical approximation.

A few numerical examples for problems in two spatial dimensions are reported in [9]. In this work (3.2) was treated by the analog of (1.10). The phase equation (3.3), on the other hand, was integrated on a very fine grid in space and time by an explicit finite difference scheme. The values of ξ and τ are such that the stability limit on $\xi^2\Delta t/\tau\Delta x^2$ was not very restrictive.

For planar problems reasonable results were obtained with acceptable computing time. However, the number of unknowns required by this approach for three dimensional problems is prohibitively large. Thus work in the future will concentrate on developing adaptively refined moving grids for the phase.

In addition, attention will be given to schemes for using analytical approximations to the phase ϕ (by the matched asymptotic expansion procedure outlined in the previous section) along with numerical approximations to (3.1).

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Numerical methods for free boundary problems: 1981 survey

Free boundary problems occur frequently and are solved routinely with the computer. Numerous methods have been devised for this task, and new or modified methods are appearing regularly. A survey and critical discussion of methods in common use before 1975 for diffusive problems may be found in [21]. Updated surveys of such numerical methods may be found in [22] and [29]. This note is meant to summarize where, in the author's view, we stand today, in what direction research is proceeding, and what some of the open problems are.

1. ONE DIMENSIONAL PROBLEMS

Theory and practice in the numerical solution of fixed domain boundary value problems is nearing the same state of completion as the numerical treatment of initial value problems for ordinary differential equations (see the conference proceedings [10]). It is therefore natural to apply standard solution methods also to free boundary problems. This is readily achieved with the Landau transformation which maps the free interval into the unit interval. The dimension of the system is increased by one by adjoining the boundary equation $s' = 0$. Standard finite difference, finite element or (multiple) shooting codes now apply to this problem. Two-sided transformations are used for problems with a single interface.

The main advantage of this approach to free boundary problems is the applicability of efficient adaptive solvers with automatic error control. The main disadvantage is the highly nonlinear structure of the transformed differential equations, particularly for non-autonomous systems. Since the solution of the finite dimensional nonlinear system of equations resulting from the discretization of the differential equations is one of the critical phases in the application of standard boundary codes, the transformation method may require a good initial guess for the solution of the problem. For small systems the basically non-iterative invariant imbedding method applied to the original free boundary problem may provide a suitable initial guess for high order iterative refinements [28].

Relatively little attention is paid at this time to methods for one dimensional free boundary problems. Yet they deserve attention because they may become essential building blocks for solving multi dimensional problems by locally or sequentially one dimensional approximations [31].

2. PARTIAL DIFFERENTIAL EQUATIONS

Most of the current research into the numerical solution of free boundary problems can be conveniently grouped into three categories. These are A) Problems with a real (physical) free boundary and solution methods which specifically involve the free boundary; B) problems with a real free boundary and solution methods which do not take into account the geometry of the free boundary; and C) problems without a well defined free boundary but which are treated as free boundary problems. We shall briefly discuss each category.

A) Front tracking, boundary fixing (with the Landau transformation) and the isotherm migration method belong under this heading. All three of these methods have been discussed repeatedly (see [22],[29] and the references given there) and their strengths and weaknesses are by now well known. All three methods continue to be developed.

Front tracking for free boundary problems refers to methods where the free boundary is expressly determined from the prescribed Cauchy data. These methods are popular because they apply to the primary variables in the physical domain, leave the differential equations unchanged and can handle complicated free boundary conditions. Front tracking is particularly well developed for transient problems in one space dimension where it can readily be coupled with finite element and finite different methods. Recent research in this area has concentrated on solving complex physical problems such as combined heat and mass transfer [30], on adaptive mesh generation for continuous time Galerkin methods [34],[35], and on high order space-time finite element approximations for the Stefan problem [5],[6].

Multi dimensional front tracking has also long been used. In particular, predictor-corrector type methods are popular in the technical literature. By predicting the location of the free boundary the problem is reduced to a fixed boundary problem, although in general the resulting computational domain is irregular so that considerable interpolation is necessary when discretizing the differential equations. For some problems such as the Stefan problem a correction of the predicted free boundary is not necessary because its position will correct iself from one time step to the next [25]. For other problems, especially in hydrostatics and hydrodynamics the correction is critical and may, in fact, lead to divergence when the prescribed Cauchy data on the free boundary are applied in the wrong order [16].

An alternative more implicit way of front tracking is provided by the method of lines where the multi dimensional equations are replaced through partial discretization by a free multi point system of ordinary differential equations. This system is then solved iteratively as a sequence of one dimensional free boundary problems. The method of lines has been used repeatedly for elliptic and parabolic equations [31]. A recent application to nonlinear water waves is given in [38]. Of particular interest here is the use of a fast Poisson solver over a portion of the domain which is not bounded by the free surface. It is to be expected that any competitive implementation of multi dimensional front tracking will use such tracking only near the free surface and alternative faster methods away from it.

Another important area of front tracking with an extensive literature in its own right but little presence is the diffusion dominated standard free boundary literature concerns the calculation of shocks in transonic flow. Under the heading of "shock fitting" various strategies are described for moving shock points through the computational mesh. Shock fitting based on the Hugoniot Rankine jump conditions is based on tracking the shock along characteristic directions (see, e.g. [48]). A related front tracking approach is the method of random choice for hyperbolic conservation laws which combines local analytic solutions with a sampling process to advance time dependent solutions from one time level to the next. This method is essentially one dimensional and is combined with a fractional step splitting to treat multi dimensional problems (for a recent description of the method see [27]).

A final comment on front tracking. It is apparent from the published literature that this approach is backed by extensive numerical experiments. For several problems in one space dimension there also exist convergence proofs for the numerical methods (see, for example, the discussion of Huber's method for the Stefan problem in [19]). However, for multi dimensional problems the theoretical questions are just now beginning to be considered. For certain elliptic problems with a variational structure a convergence proof for the method of lines is given in [32]. But even for the classical Stefan problem the convergence of front tracking in several space dimensions remains an open mathematical question.

Boundary fixing through the Landau transformation makes specific use of the description of the boundary. On the other hand one is led to a regular computational domain with the attendant formal simplification in discretizing the problem. This method applies under essentially the same conditions as direct front tracking and is very popular in the technical literature. However, great care must be exercised whenever the free boundary is expected to changed rapidly or have corners [12],[40]. For one dimensional one phase Stefan problems the transformed equations can be solved and analyzed with a continuous time Galerkin method [36]. For a two phase problem an analysis of a finite difference discretization is announced in [45]. Boundary fixing is also used successfully for shock fitting in duct flow [43]. However, the theoretical aspects of boundary fixing in several space dimensions remain unexplored.

A third method which specifically involves the free boundary is the socalled isotherm migration method, or more generally, the variable interchange method. Here the dependent variable is interchanged with one of the independent variables, and the differential equations are reformulated for the new dependent variable. The computation then proceeds on a fixed mesh over a regular domain, but the position of the free boundary enters into the discretized equations. A recent application of this method to a travelling wave problem may be found in [15], while the three dimensional dam problem is treated in [11]. It is apparent that this method shares with the Landau transformation the property that originally simple differential equations on unknown, possibly complicated domains are transformed into inherently nonlinear differential equations on simple computational domains. Invariably, interpolation is required in the numerical solution and it becomes a matter of preference whether to interpolate over the physical domain as in front tracking or in the discretization of the nonlinear equations. A mathematical analysis of the variable interchange method has not yet appeared.

Overall then, numerical methods which specifically involve the free boundary are available for elliptic, parabolic and hyperbolic problems. They are supported by convincing numerical evidence but their theoretical foundation is usually weak.

B) The best known numerical methods for free boundary problems with a physical, but without a computational free boundary arise when the classical Stefan problem is solved in its enthalpy formulation. A detailed analysis of the model leading to the enthalpy equation and a discussion of commonly applied numerical methods may be found in [42]. Of particular interest here is an assessment of the observed error introduced by numerical smoothing methods. Additional comments on the implementation of the enthalpy method with a view toward minimizing these errors may be found in [46]. An extension of this model to combined heat and mass transfer in the theory of binary alloy solidification is discussed in [20],[13] and [4] where the socalled chemical activity is introduced as an enthalpy like function of concentration. For this particular model reservations were raised about the performance of the usual fixed domain numerical methods when realistic data reflecting the vastly different time constants of the problem are used [47].

In the same spirit as the enthalpy formulation is the model given in [3] for the classical dam problem. Here the problem is written as a variational inequality for the pressure itself (instead of its Baiocchi transform) and a second function, which for the classical square homogeneous dam problem corresponds to the characteristic function of the wet region. Some impressive numerical results obtained with a finite element method are presented in [3] which indicate that this model is the physically more realistic description than the classical dam problem with a well defined free drainage boundary (compare also with the experimental paper [26]).

The importance of variational inequalities for the numerical solution of free boundary problems is, of course, well recognized. Obstacle problems in their primary formulation, time dependent problems after the Duvaut transformation and the dam problem after the Baiocchi transformation are solvable with finite element methods and analyzable with the tools of functional analysis. The numerical aspects are treated in [24] and new applications are published regularly (see, e.g. [7],[18]).

Methods which do not specifically use the free boundary are also used routinely in transonic flow where they are known as "shock capturing methods". Here the hyperbolic flow equations are regularized through the addition of an artificial small viscosity term [9] which makes the equations elliptic with large convection terms. An analogous approach is used and analyzed in [17] for the Buckley-Leverett equation of two phase flow in a porous medium.

The dominant numerical problem with this approach is the smearing of the shock over several computational cells due to numerical dispersion. In general the computation of elliptic and parabolic equations with strong convective terms is an active area of research. Both the discretization of the equations and mesh refinements are exploited to avoid dispersion. A promising approach here is the moving finite element method [23] where both the Galerkin expansion coefficients and the location of the nodes are unknown and to be determined. An application of this method to the Stefan problem is outlined in [2].

In summary it may be said that most fixed domain models have a realistic physical interpretation. Moreover, the numerical solution methods devised for them apply to regularized equations which are analyzable. As a result these methods have proved to be tractable when examining their convergence properties (see, e.g. [4]).

C) The process described in section B) can also be reversed. Thus, problems without a well defined actual free boundary are on occasion treated as a free boundary problem strictly for computational reasons. For example, the reaction-diffusion equations for laminar premixed flames [8] have strong nonlinear reaction terms near the flame tip. An asymptotic analysis is applied which shrinks the reaction zone to a sheet. The resulting moving boundary problem for a (weakly) coupled one dimensional parabolic system then is solved with front tracking.

The analysis of [3] is justified by the physics of the reaction process and cannot immediately be transferred to other problems where steep local gradients occur or where the coefficients or source terms change only over a small range of the solution. The conversion of such problems to approximating free boundary or interface problems may well be a difficult step. On the other hand a new formulation may be gained in which local changes will not be ignored because they are absorbed into the boundary conditions which strongly drive the solution of the differential equations and as long as the differential equations away from the free boundary are tractable current numerical front tracking methods should solve the problem. In contrast, nonlinear source terms may be undervalued or unduly averaged over the computational cell in standard solution methods for nonlinear partial differential equations. This problem is particularly acute when the computational domain is large compared to the critical zone so that mesh refinements are not economically feasible.

3. SOME OPEN PROBLEMS

Open problems abound in the numerical solution of free boundary problems. There are the usual difficulties associated with the solution of multi-dimensional nonlinear partial differential equations caused by the large size of the discrete approximation, by the multivariate interpolation necessary for the discretization and by the computational expense. Advances in the state of the art of solving partial differential equations efficiently, whether by direct or (multi grid) accelerated iterative techniques should always be adaptable to free boundary problems. An outline of several acceleration procedures for the solution of linear complementarity problems (discrete variational inequalities) is given in [14].

Conduction Stefan problems are reasonably well in hand and further studies of such models should include at least nonlinear material parameters and radiative effects instead of phase-constant properties and pure conduction. Moreover, problems involving phase change and convection in the liquid are only rarely treated. Yet they dominate in practice. Some initial studies are reported in the technical literature [44], but many common methods for the Stefan problem are not suitable for convective problems (e.g. variational inequalities, isotherm migration). It seems unlikely that a solution of the Navier Stokes equations can be avoided when considering the Stefan problem with convection.

It also is apparent that ill-posed problems such as the Hele-Shaw suction problem [37], the Muskat problem with an unfavorable mobility ratio [41] or the astrophysical plasma cloud problem [39] describe important applications which are not routinely solvable with current methods. Some problems of this type have analytic solutions where an initially smooth free boundary eventually forms reentrant cusps. Such solutions provide good but difficult test cases against which numerical results can be compared (see [33]).

In the area of hyperbolic equations the state of the art is even more unsettled. So far the impression is conveyed in the transonic flow literature that the numerical results are not convincing enough for design purposes. Therefore scaled aircraft frames and wings are still subjected to wind tunnel tests to verify the numerical results. Once shock fitting and shock capturing give consistent numerical results for realistic geometries this step should become unnecessary. Compared to this task elliptic and parabolic problems seem manageable.

And once all the forward problems are solved there are always the inverse problems. Such problems are not well understood for fixed domains and even less for free domains. Some initial results for the one phase one dimensional Stefan problem may be found in several papers in [1].

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J A NITSCHE

Finite element approximation to the one-phase Stefan problem

Based on the method of straightening the moving boundary a finite element procedure is proposed.

1. The standard one-phase Stefan problem - the model of moving boundary problems - in one-phase dimension is:

A pair $\{U,S \mid U = U(y,\tau),\ S = S(\tau)\}$ of functions is sought such that U solves the heat equation

$$U_\tau - U_{yy} = 0 \tag{1.1}$$

in the domain

$$\Omega = \{(y,\tau) \mid \tau > 0 \wedge 0 < y < S(\tau)\} \tag{1.2}$$

subject to the initial-boundary conditions

$$\begin{aligned} &\text{(i)} && U(y,0) = f(y) && \text{for } 0 < y < 1\ , \\ &\text{(ii)} && U_y(0,\tau) = 0 && \text{for } \tau > 0\ , \\ &\text{(iii)} && U(S(\tau),\tau) = 0 && \text{for } \tau > 0\ . \end{aligned} \tag{1.3}$$

The 'moving' of the free boundary $y = S(\tau)$ is governed by

$$S_\tau + U_y(S(\tau),\tau) = 0 \qquad \text{for } \tau > 0 \tag{1.4}$$

with $S(0) = 1$.

The Landau-transformation $(y,\tau) \to (x,\tau)$ defined by

$$y = S(\tau)\ x \tag{1.5}$$

transforms the Stefan problem to one with fixed boundary. It is advantageous to introduce in addition a new time scale $\tau \to t$ defined by

$$t = t(\tau) = \int_0^\tau S^{-2}(\tau)\ d\tau\ . \tag{1.6}$$

By straightforward calculation we find: Let $\{U,S\}$ be the solution of the Stefan problem. The function $u(x,t) = U(y,\tau)$ solves

$$\begin{aligned} &u_t - u_{xx} = -x u_x(1,t) u_x && \text{in } Q = \{(x,t) \mid 0 < x < 1 \wedge t > 0\}, \\ &u(x,0) = f(x) && \text{for } x \in I = (0,1) \\ &u_x(0,t) = u(1,t) = 0 && \text{for } t > 0 . \end{aligned} \tag{1.7}$$

The function $s(t) = S(\tau)$ is the solution of

$$\frac{ds}{dt} = -u_x(1,t)s \qquad \text{for } t > 0 \tag{1.8}$$

with $s(0) = 1$.
The dependence $t \to \tau$ is given by

$$\frac{d\tau}{dt} = s^2(t) \qquad \text{for } t > 0 \tag{1.9}$$

with $\tau(0) = 0$.
(1.7) is a nonlinear parabolic initial boundary problem - with fixed boundary - for the function u. Once u is known then s and τ are defined by the ordinary differential equations (1.8) and (1.9), in this way also $U(y,\tau)$ can be computed.

2. In a series of papers - see [1] and the literature given there - we analyzed a finite element procedure. The basis is the weak formulation of problem (1.7):

Find $u \in H_2(I)$ with $u_x(0,t) = u(1,t) = 0$ such that

$$(\dot{u}',w') + (u'',w'') = u'(1)(xu',w'') \quad \text{for} \quad t > 0 \tag{2.1}$$

is valid for $w \in H_2(I)$ fulfilling the same boundary conditions, with the initial condition

$$u(x,0) = f(x) \quad \text{for} \quad x \in I. \tag{2.2}$$

Here "." resp. "'" denote differentiation with respect to time and space, $u'(1) = u'(1,t)$ and $(.,.)$ is the $L_2(I)$ inner product.

The semi-discrete finite element resp. Galerkin procedure is: Let $\overset{\circ}{S}_h$ be an approximation space of $H_2(I)$ according to the boundary conditions

$v_x(0) = v(1) = 0$. Then $u_h = u_h(.,t) \in \overset{\circ}{S}_h$ is defined by (2.1) with $w = w_h$ now restricted to $\overset{\circ}{S}_h$ and the initial condition

$$u_h(.,0) = P_h f \tag{2.3}$$

with P_h being an appropriate projection of $L_2(I)$ onto $\overset{\circ}{S}_h$.
Essential for the method proposed is:
Assume that the projection P_h is bounded in the norm of $H_1(I)$. Then there is a $T_0 > 0$ independent of the choice of the approximation space $\overset{\circ}{S}_h$ such that the finite element solution u_h is uniquely defined (and computable).

The proof is quite simple. Let $\overset{\circ}{S}_h$ be finite dimensional. Then the finite element procedure leads to a finite system of ordinary differential equations with quadratic right hand sides. Thus the existence of a solution for some time $0 \le t \le T_h$ is obvious. In order to verify that T_h will be independent of h resp. $\overset{\circ}{S}_h$ we choose in (2.1) with $u: = u_h$ also $w: = w_h = u_h$. This gives

$$\frac{1}{2}(\|u_h'\|^2)^{\cdot} + \|u_h''\|^2 = u_h'(1)(xu_h', u_h'') \le |u_h'(1)| \, \|u_h'\| \, \|u_h''\| \, . \tag{2.4}$$

Because of $u_h'(0) = 0$ we get

$$u_h'(1)^2 = 2\int_0^1 u_h' \, u_h'' \, dx \le 2 \, \|u_h'\| \, \|u_h''\| \, . \tag{2.5}$$

In this way we come to

$$(\|u_h'\|^2)^{\cdot} + 2\|u_h''\|^2 \le 2^{3/2}\|u_h'\|^{3/2}\|u_h''\|^{3/2} \le \|u_h''\|^2 + c\|u_h'\|^6 . \tag{2.6}$$

Therefore $\lambda : = \|u_h'\|^2$ obeys the differential inequality

$$\dot\lambda \le c\,\lambda^3 \quad \text{for} \quad t > 0 \, ,$$

$$\lambda(0) = \|(P_h f)'\| \le c\|f'\|^2 .$$

Gronwall's lemma then gives the above assertion.

3. The variational formulation (2.1) was won by taking the inner product of the differential equation (1.7_1) with the second derivative of a function w and partial integration of the first term. The corresponding finite element procedure is called an H_2-method since the second derivatives enter.

The corresponding H_1-method seems to be more natural. Then (1.7_1) is multiplied with w itself. This results in the weak formulation:

Find $u = u(.,t) \in H_1(I)$ with $u_x(0,t) = u(1,t) = 0$ such that

$$(\dot{u},w) + (u',w') = -u_1'(xu',w) \text{ for } t > 0 \tag{3.1}$$

is valid for $w \in H_1(I)$ fulfilling the same boundary conditions, with the initial condition

$$u(x,0) = f(x) \quad \text{for} \quad x \in I . \tag{3.2}$$

Obviously this formulation has to be modified since the point value u_1' in general will not exist for $u \in H_1(I)$. Correspondingly a priori estimates of the type (2.6) will not be valid.

In the case of finite dimensional approximation spaces S_h the corresponding finite element procedure still can be applied. In former times various numerical experiments based on the formulation (3.1) were done by Dr. R. Link. The results had been striking: Using splines for S_h the (numerical) convergence order is optimal. The rest of this section is devoted to the explanation of these numerical results.

Let S_h be a spline space subordinate to a uniform (!) subdivision of a fixed degree. Further let $\overset{\circ}{S}_h$ be the subspace of functions with $u'(0) = u(1) = 0$. We extend any $\phi \in \overset{\circ}{S}_h$ which is defined in I to $\tilde{\phi} := E\,\phi$ defined on $\mathbb{R}^1$ by

$$\begin{aligned}
&\text{(i)} & \tilde{\phi}(x) &= \phi(x) & &\text{for} \quad x \in I , \\
&\text{(ii)} & \tilde{\phi}(x) &= \tilde{\phi}(-x) & &\text{for} \quad x < 0 , \\
&\text{(iii)} & \tilde{\phi}(x) &= -\phi(2-x) & &\text{for} \quad x > 1 .
\end{aligned} \tag{3.3}$$

Let $\tilde{S}_h = E\,\overset{\circ}{S}_h$ denote the spline space constructed in this way. To any $\tilde{\phi} \in S_h$ we define the difference operator $\tilde{\nabla}^2$ defined by

$$(\tilde{\nabla}^2\tilde{\phi})(x) = h^{-2}(\tilde{\phi}(x+h) - 2\tilde{\phi}(x) + \tilde{\phi}(h-x)) . \tag{3.4}$$

Because of the uniform subdivision $\tilde{\nabla}^2$ is a mapping of $\tilde{S}_h$ into itself. Correspondingly the operator $\nabla^2 = (\tilde{\nabla}^2 E)_{|I}$ maps $\overset{\circ}{S}_h$ into itself, actually the mapping is onto.

Now let $u_h \in \overset{\circ}{S}_h$ be the solution of the H_1 finite element procedure according to (3.1) and put $w = w_h = \nabla^2 u_h$. By partial summation applied to the terms on the left hand side we come to - ∇ is the first difference oeprator

$$\frac{1}{2}(\|\nabla u_h\|^2)^{\bullet} + \|\nabla u_h'\|^2 = u_{h\ 1}'(xu_h', \nabla^2 u_h) \ . \tag{3.5}$$

For the sake of simplicity let us assume that the elements of $\overset{\circ}{S}_h$ are continuously differentiable (and therefore of degree 3 at least). By direct calculation it can be shown:

In $\overset{\circ}{S}_h$ the norms

$$\|\nabla(.)\| \ , \qquad \|(.)'\| \tag{3.6}$$

and

$$\|\nabla^2(.)\| \ , \quad \|\nabla\{(.)'\}\| \ , \quad \|(.)''\| \tag{3.7}$$

are uniformly equivalent in h, i.e. there is a c independent of h such that for $\phi \in \overset{\circ}{S}_h$

$$c^{-1}\|\nabla\phi\| \le \|\phi'\| \le c\,\|\nabla\phi\| \ , \tag{3.6'}$$

$$c^{-2}\|\nabla^2\phi\| \le c^{-1}\|\nabla\phi'\| \le \|\phi''\| \le c\|\nabla\phi'\| \le c^2\|\nabla^2\phi\| \tag{3.7'}$$

is valid.

In view of these properties (3.5) is the exact counterpart of (2.4). Then it can be shown that the finite element approximation u_h exists in a time interval [O,T] independent of h. The corresponding error analysis is routine and omitted here.

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G ASTARITA

A class of free boundary problems arising in the analysis of transport phenomena in polymers

INTRODUCTION

The mathematical modeling of heat and mass transfer phenomena accompanied by phase transitions in polymeric materials leads to a new class of free boundary problems, which have received much less attention than the classical ones arising from the modeling of the same phenomena in ordinary low molecular weight material. The difference is related to the fact that, in polymer materials, relaxation phenomena take place on a time scale which is comparable to that of the experiments, in contrast with what happens in ordinary materials where relaxation phenomena are essentially instantaneous.

Consider a one-dimensional process of heat or mass transfer in a system which may undergo a phase transition. The following equations can be written in general:

Diffusion equations in the two phases, separated by a moving boundary located at $x = s(t)$:

$$u_{xx} - u_t = 0, \qquad 0 < x < s(t) \tag{1}$$

$$\gamma U_{xx} - U_t = 0, \qquad s(t) < x < X \tag{2}$$

Initial conditions:

$$u(x,0) = h(x), \qquad 0 < x < s(0) \tag{3}$$

$$U(x,0) = H(x), \qquad s(0) < x < X \tag{4}$$

Spatial boundary conditions:

$$u(0,t) = f(t) \tag{5}$$

$$U_x(X,t) = 0 \tag{6}$$

(sometimes equation 6 needs to take a different form, say $U_x(X,t)$ is proportional to $U(X,t)$. This is irrelevant to the present discussion). Heat or mass balance at the phase boundary:

$$-u_x(s(t),t) + \beta U_x(s(t),t) = = \{\alpha + u(s(t),t) - \frac{\beta}{\gamma} U(s(t),t)\} s'(t) \quad . \tag{7}$$

The following conditions are imposed on the functions f, h and H:

$$f > 0 \; ; \; h > 0 \; ; \; H \le 0 \; . \tag{8}$$

The solution to these equations is sought for $0 < t < T$, where $s(T) = X$. The constants γ and β are positive, α is non negative. The system 1-7 is not sufficient to determine the functions $u(x,t)$, $U(x,t)$ and $s(t)$, and additional conditions need to be written.

In the classical case, the assumption is made that the value of the potential (temperature or fugacity) at the *moving* phase transition boundary has the same value of zero it would have at an *equilibrium* phase transition boundary, say:

$$u(s(t),t) = U(s(t),t) = 0 \; . \tag{9}$$

With this assumption, since α is a positive quantity which without loss of generality can be set equal to unity, the right hand side of equation 7 reduces to $s'(t)$.

In the case of polymeric materials, equation 9 has been shown experimentally to be untenable [1,2,3]:Relaxation is too slow for the equilibrium value of the potential to be attained instantaneously at the phase boundary as it moves. A constitutive equation for the rate of displacement of the boundary, $s'(t)$, needs to be written which describes the relaxation phenomenon. This gives rise to a new class of free boundary problems, some elementary aspects of which are discussed in the remainder of this paper.

Another point of interest is that the constant α is, in the case of heat transfer, proportional to the latent heat. In polymeric materials, second order phase transitions may occur for which $\alpha = 0$. In this case, equation 9 would lead at least in one case to a paradox; the fact that this case is a special one does not eliminate the physical paradox and therefore the untenability of equation 9 for polymeric materials. In ordinary materials, α is always positive.

ONE-PHASE PROBLEMS

Turning back attention to the classical problem described by equations 1-9, a considerable simplification arises if the function $H(x)$ has a constant zero value. In this case the solution of the system (2-4-6-9) is, trivially, $U(x,t) = 0$, and the problem collapses to the classical Stefan problem. The Stefan problem, in turn, is considerably simplified if $s(0) = 0$ and $f = 1$, in which case $h(x)$ has a zero domain and the Stefan problem can be solved analytically to yield:

$$u = 1 - \frac{\operatorname{erf}(x/2\sqrt{t})}{\operatorname{erf} K} \tag{10}$$

$$S = 2\,K\sqrt{t} \tag{11}$$

$$T = X^2/4K^2 \tag{12}$$

where K is the root of the following equation:

$$\sqrt{\pi}\,Ke^{K^2}\operatorname{erf} K = \frac{1}{\alpha} \tag{13}$$

Equation 13 shows immediately the paradox arising with second-order phase transitions, where $\alpha = 0$: the solution becomes $K \to \infty$, i.e., the boundary moves at an infinite velocity, and the phase transition takes place instantaneously throughout the spatial domain $0 < x < X$. The conclusion is of more general character, since when $U(x,t) = 0$, if equation 9 is accepted, equation 7 would imply that either $u_x(s(t),t)$ is zero or $s'(t)$ is infinity. The first possibility is ruled out by the fact that no solution of equation 1 can have both $u_x(s(t),t)$ and $u(s(t),t)$ zero, so that only $s'(t) \to \infty$ is left.

Apart from the $\alpha = 0$ paradox, it is important to point out that equation 11 predicts that $s'(t)$ is infinity at $t = 0$, so that there is no neighborhood of $t = 0$ where $s(t)$ is linear in t. Experimental evidence, on the contrary, shows conclusively that such a neighborhood does exist.

Astarita and Sarti [4] have analyzed the one-phase problem for polymeric materials under the simplifying assumptions $s(0) = 0$, $f = 1$ which in the classical case lead to equations 10-13. However, instead of writing equation 9 they have written the following constitutive equation:

$$s'(t) = \frac{|u(s(t),t)|^n}{\Theta} \tag{14}$$

where Θ is a relaxation time of the polymeric material and n is a positive constant (equation 14 is a typical n-order kinetics constitutive equation, with an analytical form commonly used in the description phenomena such as the kinetics of a chemical reaction).

We were unable to obtain a general analytical solution to the problem as formulated above. However, the asymptotic behavior of the solution can be calculated explicitly, and numerical techniques can be used successfully to find complete solutions for different values of the two parameters α and n (X is taken to be infinity in this analysis).

First of all, it may be noted that s'(O) is finite:

$$s'(0) = \frac{1}{\Theta} \tag{15}$$

and that it does tend to infinity when the relaxation time Θ tends to zero. Since s'(O) is finite, a linearized solution in the neighborhood of t = O is obtained:

$$s = \frac{t}{\Theta} + o\left(\frac{t^2}{\Theta^2}\right) \tag{16}$$

$$u = 1 - \frac{1 + \alpha}{\Theta} x + o\left(\frac{t^2}{\Theta^2}\right) \tag{17}$$

which shows (see equation 17) that no paradox arises when $\alpha = 0$.

A solution can also be obtained for $t \to \infty$, by expanding to $O(\varepsilon)$ where ε is the value of u(s(t),t), the solution obtained is again given by equations 10-13, and therefore a (milder) paradox would seem to emerge again when $\alpha = 0$. However, this is only apparent, since the asymptotic solution given by equations 10-13 can be shown to be approached at times $t > t_{crit}$ where t_{crit} is given, when $\alpha \to 0$, by:

$$t_{crit} = 4 \ln(1/\alpha\sqrt{\pi}) \tag{18}$$

and therefore paradox is disposed of. This is confirmed by the numerical solution of the problem, which shows that, when $\alpha = 0$, d ln s/d ln t never attains the value ½, while it always does for any finite value of α. The transition from the $t \to 0$ to the $t \to \infty$ asymptote is extremely smooth, and may cover as much as five order of magnitude of t, particularly at low value of α, since t_{crit} is very much larger than Θ. Of course, when $\alpha = 0$ there is no upper asymptote and therefore the transition region is infinitely long (see Figure 2 of [4]).

EFFECT OF SAMPLE SIZE

In mass transfer experiments, where a slab of polymer is exposed to an atmosphere of a swelling solvent, two quantities are measurable. The first one (which can be measured only in relatively thick samples), is the position of the boundary itself. The other one (which can *always* be measured) is the total weight of solvent sorbed as a function of time: this corresponds to the function $\omega(t)$ defined by:

$$\omega(t) = \int_{0}^{s(t)} (u + \alpha)\ dx + \int_{s(t)}^{X} U\ dx + \delta X \tag{19}$$

(where δ is defined below in equation 20). Possibly the most surprising result is that the *shape* of the $\omega(t)$ function turns out, for a given polymer-solvent pair, to depend on the thickness of the sample X. The experiments are typically carried out under conditions where:

$$s(0) = 0\ ;\ f(t) = 1\ ;\ H(x) = -\ \delta \tag{20}$$

with δ a positive constant.

Astarita and Joshi [5] have presented an approximate analytical solution to the system of equations 1-7, 20, with the constitutive equation for $s'(t)$ written in the form of equation 14, and with the additional condition:

$$U(s(t),t) = 0 \tag{21}$$

notice that, since $u(s(t),t) \neq 0$, there is a discontinuity across the boundary: problems where a discontinuity exists have been discussed by other authors [6],[7], though with different constitutive equations.

The analysis in [5] is based on an evaluation of the order of magnitude of the parameters of the problem, i.e., α, β, γ, Θ and δ. For mass transfer problems, $\beta = \gamma$ (both are the ratio of diffusivities on the two sides of the boundary), though this is not true for heat transfer (where β is the ratio of conductivities and γ the ratio of diffusivities). Therefore, for mass transfer problems there are only four parameters to consider, namely α, γ, Θ and δ. Of these, the first one can be set at will by the experimentalist, and will in general be of order unity; γ and δ are properties of the polymer itself, and they are invariably very small (typically of order 10^{-3}). It is therefore possible to define three *ranges* of sample size, namely:

$$\text{LARGE} \qquad X \gg \Theta \tag{22}$$

$$\text{MEDIUM SAMPLES:} \qquad \Theta \gg X \gg \Theta \frac{\gamma\delta}{1 + \alpha + 2\delta} \tag{23}$$

$$\text{SMALL SAMPLES:} \qquad \Theta \frac{\gamma\delta}{1 + \alpha + 2\delta} \gg X \tag{24}$$

Of course, it is concernable to have samples where X is of the same order of magnitude as either Θ or $\Theta\gamma\delta/(1 + \alpha + 2\delta)$; for these no analytical solution, even approximate, is available. However, for the three ranges defined in equations 22-24 an approximate analytical solution can be found, and the corresponding shape of the $\omega(t)$ curves is presented in Figure 1.

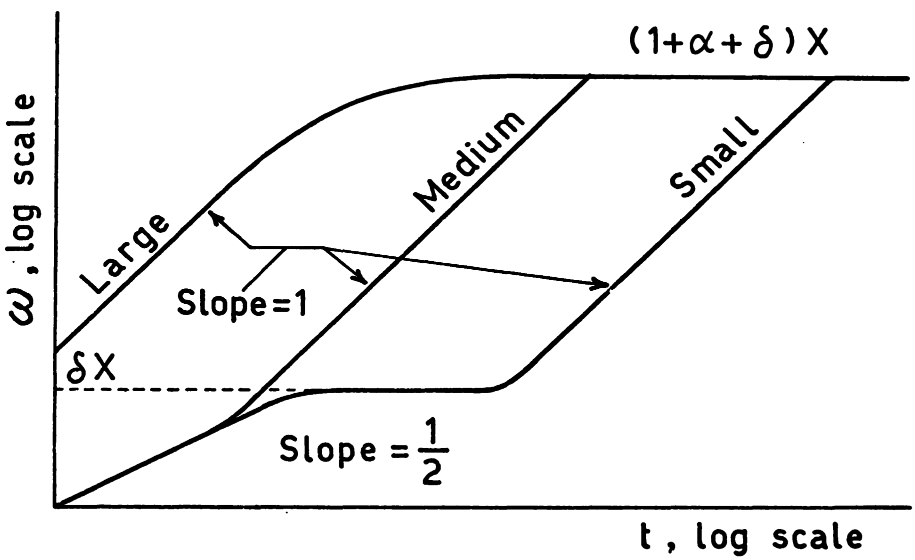

Figure 1. Shape of the $\omega(t)$ curve for different sample sizes.

One may notice that the Figure predicts that different shapes will be observed, at a given polymer-solvent pair, according to the size of the sample. Also notice that, in the case of small samples (such as membranes), a "false equilibrium" (i.e. a region where $d\omega/dt \simeq 0$) is predicted: such false equilibrium have indeed been observed experimentally in membranes.

The "structure" of the approximate solution is as follows. For small samples, the second integral on the right hand side of equation 19 reaches its asymptotic value of zero at a time t where s(t) is so small that the value of the first integral is negligible; correspondingly, a "false equilibrium" at $\omega = \delta x$ is observed. Furthermore, s(t) reaches the value X at times where the $O(t^2/\theta^2)$ term in equations 16 and 17 is still negligible, so that the first integral is simple $(1 + \alpha)t/\theta$.

For medium samples, the first integral becomes the leading term at a time when the second integral is still growing at the low-time asymptotic rate which is proportional to $\sqrt{t}$; it is however still true that up to s(t) = x the first integral is simply $(1 + \alpha)t/\theta$.

Finally, for large samples the contribution of the second integral is at all times so small as to be undetectable by measurement; furthermore, the higher order terms in equations 16 and 17 become non-negligible at times prior to the time when s(t) = X.

Experiments of heat transfer accompanied by phase transition in polymers have been reported where again the classical Stefan problem approach is incapable of correlating the data [8]. Joshi and Astarita [9] have presented an analysis of these data based on equations 14 and 21, which again is able to predict the experimentally observed effect of sample size.

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G CAPRIZ & G CIMATTI

Free boundary problems in the theory of hydrodynamic lubrication: a survey

INTRODUCTION

The theory of lubrication (more precisely the dynamics of thin viscous fluid films confined between solid boundaries) suggests many free-boundary problems, whose solution can be proved to satisfy an appropriate variational inequality (see [1], [5], [24]) which can be treated with the techniques developed in the last twenty years [20]; thus a number of special results, intuitively derived by engineers, can be given a precise mathematical status and new properties can be deduced (see [6] and [9]-[13]). In fact the theory of lubrication can be quoted as a standard field of application of variational techniques ([20], Ch. VII). From a dynamic point of view the flow within a cavitation film can be described as a (rather degenerate type of) two-phase flow; such a point of view has not been sufficiently exploited so far in our opinion; it may suggest further developments in the theory.

In this paper Sections 2,3,4 contain a compact review of the main results of Ref. [5], [9] and [12] and also some new results; in particular we give a complete description of the geometrical properties of the region of cavitation for the case of the journal bearing of small eccentricity ratio. In Sections 5 and 6 we justify from a rigorous mathematical point of view, and in a more general setting than that considered in [6], the classical solution of Sommerfeld (for the very long journal bearings) and Ocvirk (for the very short journal bearing). Finally in Section 7 we discuss the controversial problem of the conditions which must be imposed on the boundary of the region of cavitation, we report on the results of Ref. [3], and mention briefly possible future developments.

1. REYNOLDS' EQUATION

The central problem in the theory of lubrication is the determination of the pressure P in a thin film of lubricant (a viscous fluid of constant density and of given viscosity η) contained in the narrow clearance between two surfaces Σ_1, Σ_2 in relative motion (the surfaces of the "bearing" and of the "journal"). For the purpose direct reference to the Navier-Stokes equations is possible, but not indispensable if certain approximations are directly accepted rather than justified through asymptotic expansions or approximation of the boundary layer type [4], [23].

Because the interest lies in conditions where the clearance H between the surfaces is much smaller than their radii of curvature and also small with respect to a typical dimension R of the portion of the surfaces "wetted" by the film, the film can be locally thought to be delimited by two planes and the flow rate to be that pertaining to a Poiseuille-Couette flow. Thus inertia effects are automatically disregarded and any one of the surfaces, say Σ_1, can be considered as fixed and the other, Σ_2, as moving. Locally one can introduce an orthogonal system of reference (x_1, x_2, x_3) with the plane (x_1, x_2) replacing in approximation Σ_1. If the components of the speed of ε_2 are V_1, V_2, V_3 and the flow rate in the direction of x_1 (x_2) per unit width counted along x_2 (x_1) is called Q_1 (Q_2), then, by the approximation accepted above, one has

$$Q_i = \frac{H}{2} V_i - \frac{H^3}{12\eta} \frac{\partial P}{\partial x_i}, \quad i = 1,2. \tag{1.1}$$

where H is the local thickness of the flim. On the other hand, if v_1, v_2, v_3 are the components of the speed of the fluid (so that $Q_i = \int_0^H v_i dX_3$, $i = 1,2$) the equation of continuity

$$\sum_{1}^{3} {}_i \frac{\partial v_i}{\partial x_i} = 0 ,$$

when integrated over $(0,H)$, yields

$$\sum_{1}^{2} {}_i \left(\frac{\partial Q_i}{\partial x_i} - V_i \frac{\partial H}{\partial x_i}\right) + V_3 = 0,$$

leading, with the use of (1.1), to Reynolds' equation

$$\frac{\partial}{\partial x_1}\left(\frac{H^3}{\eta}\frac{\partial P}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{H^3}{\eta}\frac{\partial P}{\partial x_2}\right) = 12V_3 + 6H^2 \frac{\partial}{\partial x_1}\left(\frac{V_1}{H}\right) + 6H^2 \frac{\partial}{\partial x_2}\left(\frac{V_2}{H}\right). \tag{1.2}$$

The interest lies here exclusively with the case where η is constant, H is a function of x_1 only, $V_2=0$ and either $V_3=0$, $V = -V$ (Section 3) or

$V_3 = V\frac{\partial H}{\partial x_1}$, $V_1=V$, (Sections 4,5,6), V is a positive constant; hence with the following simplified version of (1.2)

$$\frac{\partial}{\partial x_1}\left(H^3 \frac{\partial P}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(H^3 \frac{\partial P}{\partial x_2}\right) = 6\eta V \frac{\partial H}{\partial x_1} . \tag{1.3}$$

Boundary and side conditions must be added to (1.3). Notice, first of all, that the value of P must be normalized to avoid indetermination: it is traditional to take the null value as that of atmospheric pressure. Actually, very often, the film borders with a "reservoir" of lubricant at atmospheric pressure so that the more frequent condition is

$$P = 0, \text{ at the boundary,} \tag{1.4}$$

although pressurized condition (P a positive constant at the boundary) are also met in practice.

The problem owes its distinct mathematical interest to these facts; the solution of (1.3), (1.4) need not be everywhere non negative; on the other hand experiments show that the lubricant supports only slight subatmospheric pressures, so that we must require, to a good degree of approximation, that everywhere

$$p \geq 0. \tag{1.5}$$

The phenomena occurring in the region where $P=0$ (the so called region of cavitation) are not simple to describe but in any case cannot be represented by a Couette-Poseuille flow within the clearance space; we will return on the matter in the Section 7. For the moment we accept the fact that *either* $P>0$ and then P satisfies (1.3) *or* $P=0$, with no further condition except that P be smooth everywhere.

A final note: to avoid dimensional quantities in the following sections, besides the typical dimension R of the bearing, we introduce a typical thickness c of the film and use coordinates x,y

$$x = x_1/R, \quad y = x_2/R$$

and the functions

$$h = H/c, \quad p = \frac{Pc^2}{6\eta VR} .$$

2. THE VARIATIONAL INEQUALITY OF THE HYDRODYNAMIC LUBRICATION WITH CAVITATION

Let Ω be the open, bounded set of $\mathbb{R}^2$ assumed to have smooth boundary which represents the bearing surface, and let $h(X)$ be the non-dimensional film thickness: $X=(x,y) \in \mathbb{R}^2$, $h(X) \in C^1(\bar{\Omega})$, $h(X) \geq h_o > 0$. Then we pose the following

Problem A. To find a function $p(X) \in C^1(\bar{\Omega})$ and an open subset $\mathcal{O}$ of Ω such that

$$p = 0 \quad \text{on} \tag{2.1}$$

$$p \geq 0 \text{ and where } p > 0,\ L[p] = \left\{\frac{\partial}{\partial x}\left[h^3(X)\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[h^3(X)\frac{\partial p}{\partial y}\right]\right\} = \frac{\partial h}{\partial x} \tag{2.2}$$

$$p = 0,\ \frac{dp}{dn} = 0 \text{ on } \partial\mathcal{O} \cap \Omega . \tag{2.3}$$

The last requirement $(2.3)_2$ is traditional, but implies the unnecessary prerequisite that $\partial\mathcal{O} \cap \Omega$ be sufficiently regular (in fact C^1); our original request was that on $\partial\mathcal{O} \cap \Omega$ the whole gradient of p be null, not any specific (and hard to specify) directional derivative. However, we can overcome completely these artificial difficulties if we restate the problem and give it the form of a variational inequality [5], [9], [24].

Let $H_o^1(\Omega)$ be the usual Sobolev space obtained as completion of the $C_o^\infty(\Omega)$ functions with respect to the norm

$$\| v \|_{H_o^1} = \left[\int_\Omega |\nabla v|^2 dX \right]^{\frac{1}{2}} ,$$

and K the closed convex subset of $H_o^1(\Omega)$ given by

$$K = \{ v \in H_o^1(\Omega),\ v \geq 0 \} .$$

Then, the variational inequality reads as follows

$$p \in K, \int_\Omega h^3 \nabla p \cdot \nabla (v-p)\, dX \geq - \int_\Omega \frac{\partial h}{\partial x}(v-p)\, dX, \quad \forall v \in K . \tag{2.4}$$

By the results of [21], [25] the solution of (2.4) exists, is unique and of class $C^{1,\lambda}(\bar{\Omega})$, $0 \le \lambda \le 1$. Moreover if we put

$$\mathcal{O} = \{X \in \Omega,\ p(X) > 0\}$$

it is easy to verify that $\{p, \mathcal{O}\}$ gives a solution to Problem A provided $\partial\mathcal{O} \cap \Omega$ is a C^1 curve.

In the cases which we will study below, $h(X)$ has the form

$$h(X) = 1 - \varepsilon d(X),\quad 0 < \varepsilon < 1, \tag{2.5}$$

hence (2.4) becomes

$$p \in K,\quad \int_\Omega h^3 \nabla p \cdot \nabla(v-p)\, dX \ge \varepsilon \int_\Omega d_x (v-p)\, dX, \quad \forall v \in K\ . \tag{2.6}$$

Correspondingly, it is interesting to consider also the simpler variational inequality

$$p' \in K,\quad \int_\Omega \nabla p' \cdot \nabla(v-p')\, dX \ge \int_\Omega d_x (v-p\ )\, dX, \quad \forall v \in K \tag{2.7}$$

because its solution gives a first order approximation in ε to the solution of (1.8), as the following theorem asserts.

<u>Theorem 2.1</u>. If p, p' are the solutions of (2.6), (2.7) respectively, then

$$\|p - \varepsilon p'\|_{H^1} \le C\varepsilon^2\ . \tag{2.8}$$

<u>Proof</u> By putting $v=0$ in (2.6) we get immediately

$$\|\nabla p\| \le C\varepsilon^2\ .^{(1)}$$

Let us multiply (2.7) by ε^2 and put in it $v = p/\varepsilon$, we obtain

$$\int_\Omega \nabla p' \cdot \nabla(p - \varepsilon p')\, dX \ge \varepsilon \int_\Omega d_x (p - \varepsilon p')\, dX\ . \tag{2.9}$$

By choosing $v = \varepsilon p'$ in (2.6) and adding up with (2.9) term by term we have

---

(1) The various C's denote constants, generally different, which do not depend on ε .

$$\int_\Omega \nabla(p-p')\cdot\nabla(\varepsilon p'-p)\,dX + \int_\Omega (3\varepsilon d_x+3\varepsilon^2 d_x^2+\varepsilon^3 d^3)\nabla p\cdot\nabla(p-\varepsilon p')\,dX \geq 0 .$$

Hence

$$\|\nabla(p-p')\|^2_{L^2(\Omega)} \leq C\varepsilon\|\nabla p\|_{L^2(\Omega)}\ \|\nabla(p-\varepsilon p')\|_{L^2(\Omega)} ,$$

from which (2.8) follows. □

3. A PARTICULAR CASE

To show the type of detail that may be desirable to know and can be achieved, we recall in this section some properties of the coincidence set

$$\mathcal{C} = \{(x,y)\in\Omega,\ p(x,y) = 0\} ,$$

for inequality (2.7) in the case when Ω and $h(X)$ are given by

$$\Omega = \{(x,y),\ x^2+y^2=\rho^2<1\},\quad h(X) = 1+\varepsilon\rho^2 . \tag{3.1}$$

These specifications reflect the physical situation in an experiment on the shape and extension of the cavitated region carried out by D. Dowson [17]: a plane slides at constant speed V under stationary spherical cap of radius $\frac{R^2}{2\varepsilon c}$ and maximal chord R, set at a distance c.

When ε is small we can make use of inequality (2.7) which now becomes

$$p\in K,\ \int_\Omega \nabla p\cdot\nabla(v-p)\,dX \geq -\int_\Omega x(v-p)\,dX,\quad \forall v\in K . \tag{3.2}$$

Then the following theorem holds true:

Theorem 3.1. If $p(x,y)$ is the solution of (2.2), then

$$p(x,y) = p(x,-y) , \tag{3.3}$$

$$p_y \leq 0 \text{ in } \Omega_1 , \tag{3.4}$$

$$p_y \geq 0 \text{ in } \Omega_2 , \tag{3.5}$$

$$p_\phi \geq 0 \text{ in } \Omega_1, \; p_\phi \leq 0 \text{ in } \Omega_2 \;,^{(2)} \tag{3.6}$$

$$p_x \leq 0 \text{ in } \Omega_3 \;, \tag{3.7}$$

where

$$\Omega_1=\{(x,y)\in\Omega, y>0\}, \; \Omega_2=\{(x,y)\in\Omega, y<0\}, \; \Omega_3=\{(x,y)\in\Omega, x>0\} \;.$$

Proof Since all data are symmetric with respect to the x-axis and taking into account that the solution of (3.2) is unique, we get (3.3). Let us show (3.4). Define $\Lambda=\{(x,y)\in\Omega_1, p_y(x,y)>0\}$; it is easy to verify that $p>0$ in Λ: if, by contradiction, we have $p(x^*,y^*)=0$, $(x^*,y^*)\in\Lambda$, then in a neighborhood of (x^*,y^*), $p(x^*,y)$ is an increasing function of y and $p(x^*,y)<0$ when $y<y^*$. Hence $\Delta p_y=0$ in Λ and by the maximum principle, the points of maximum $(\bar{x},\bar{y})$ of p_y in $\bar{\Lambda}$ belong to $\partial\Lambda$. Now $(\bar{x},\bar{y}) \notin \partial\Lambda\cap\partial\Omega_1$, since there $p_y\leq 0$, moreover $(\bar{x},\bar{y})$ cannot belong to $\partial\Lambda\cap\Omega_1$ by the continuity of p_y. This contradiction proves (3.4). The proof of (3.5) is analogous. Again with a maximum principle argument we can obtain (3.6); it suffices to take into account that $p_\phi=0$ on $\partial\Omega_1$ and that, where $p>0$, $-\Delta p_\phi=\rho\sin\phi$. Finally, to obtain (3.7) we apply again the maximum principle noting that $p_x\leq 0$ on $\partial\Omega_3$. □

The above theorem has important consequences on the shape of the region of cavitation. First of all we note that $C\neq\emptyset$, because if $C=\emptyset$, then $p(\rho,\phi)=\frac{\rho^3-\rho}{8}\cos\phi$ and $p<0$ when $-\frac{\pi}{2}<\phi<\frac{\pi}{2}$ which contradicts $p\in K$. We remark also that, by a general property [21] of the solutions of variational inequalities of second order, we have $C\subseteq\Omega_3$; by (3.4)-(3.7) it follows that if $(\bar{x},\bar{y})\in C$ then

$$\{(x,y);\; \bar{x}\leq x<\sqrt{1-\bar{y}^2},\; |\bar{y}|\leq|y|\quad \sqrt{1-x^2}\}\subseteq C \;, \tag{3.8}$$

if $(\bar{\rho};\bar{\phi})\in C$ then

$$\{(\rho,\phi);\; \rho=\bar{\rho},\quad -\bar{\phi}\leq\phi\leq\bar{\phi}\}\subseteq C \;. \tag{3.9}$$

Hence C has a connected interior.

---

(2) The angle ϕ is chosen so that $x=\rho\cos\phi$, $y=\rho\sin\phi$.

Lemma 3.1. The set $\partial C \cap \Omega$ cannot comprise segments parallel to the x-axis or to the y-axis.

Proof Let $\Pi=\{(x_o,y);\ x_o \geq 0,\ y_o \leq y \leq y_1\}$ be a part of $\partial C \cap \Omega$ and let us now consider the Cauchy problem

$$\left.\begin{aligned} -\Delta q &= x \\ q = q_x &= 0 \text{ on } \Pi, \end{aligned}\right\} \qquad (3.10)$$

whose solution is given by $q(x,y) = \frac{x^3}{6} - \frac{x_o^2}{2}x + \frac{1}{3}x_o^3$. By the uniqueness of the solution of (3.2) we have p=q which cannot be, since e.g. p=0 for $(x,y) \in \partial\Omega$, $x > 0$ whereas $q \neq 0$ there. In the same way we can prove that sets of the type

$$\Xi = \{(x,y) \in \Omega,\ y=\bar{y},\ x_2 \leq x \leq x_1\}$$

are not contained in $\partial C \cap \Omega$; it suffices to consider the function $q(x,y) = \frac{x}{2}(y-\bar{y})^2$ which solves in the present case the Cauchy problem similar to (3.10). □

In the following theorem we give an almost complete description of the free boundary of (3.2).

Theorem 3.2. There exists an even, continuous function $\sigma(y)$ defined in $[-d,d]$, $1 \geq d > 0$ which is strictly decreasing in $[0,d]$ and such that

$$C = \{(x,y) \in \Omega,\ |y| < d,\ \sqrt{1-y^2} > x \geq \sigma(y)\}\ . \qquad (3.11)$$

Proof Let us consider the function $\sigma(y)$ defined in $[-d,d]$ as follows

$$\sigma(y) = \sup\{x;\ |x| \leq \sqrt{1-y^2},\ p(x,y) > 0\}\ ,$$

$$d = \sup\{y;\ |y| \leq 1,\ \sigma(y) < \sqrt{1-y^2}\}\ ,$$

By (3.3) $\sigma(y)$ is an even function; moreover from (3.8) $\sigma(y)$ is non-increasing in $[0,d]$. Hence, if $\bar{y} > 0$ we have

$$\lim_{y\to\bar{y}+} \sigma(y) = \sigma(\bar{y}+) \leq \sigma(\bar{y}-) = \lim_{y\to\bar{y}-} \sigma(y)\ .$$

On the other hand the possibility $\sigma(\bar{y}+) < \sigma(\bar{y}-)$ is ruled out by Lemma 3.1 and this implies the continuity of $\sigma(y)$. Moreover we conclude, again by Lemma 3.1, that $\sigma(y)$ is strictly decreasing in $[0,d]$. □

Actually one can go beyond and show by use of a result of H. Lewy (see [20] p.151), that $\sigma(y)$ is an analytic function (for details, see [12]).

4. HYDRODYNAMIC LUBRICATION WITH CAVITATION OF JOURNAL BEARINGS

Let us consider the case of great practical interest when: Σ_1, Σ_2 are two circular cylinders of equal length L and radii R, R+c respectively; the axes of the cylinders are parallel and εc is their mutual distance $(0 \le \varepsilon < 1)$; the internal cylinder Σ_2 rotates with constant angular velocity V/R; at the sides, condition (1.4) applies again.

Then Reynolds' equation in non-dimensional form is

$$L[p] = \{\frac{\partial}{\partial x}[h^3(x)\frac{\partial p}{\partial x}] + \frac{\partial}{\partial y}[h^3(x)\frac{\partial p}{\partial y}]\} = -\varepsilon \sin x \ , \tag{4.1}$$

where $h(x)=1+\varepsilon\cos x$. Notice that now the pressure p must be a periodic function of x with period 2π; this fact makes the present problem slightly different from those described in Sections 2 and 3.

Let $\Omega=\{(x,y)\,;\ 0 \le x \le 2\pi,\ |y| < b\}$, $b = \frac{L}{R}$, be the rectangle of periodicity and define H^1_{0*} as the completion, with respect to the usual norm

$$\|v\| = \{\int_\Omega |\nabla v|^2 dX\}^{\frac{1}{2}} \ ,$$

of the space C^∞_{0*} of the C^∞-functions which are 2π-periodic in x and vanish in a neighborhood of $|y| = b$. If we define now

$$K = \{v;\ v \in H^1_{0*},\ v \ge 0\} \ ,$$

the problem of finding the pressure in the lubricating film can be expressed by the variational inequality

$$p \in K, \quad \int_\Omega h^3 \nabla p \cdot \nabla (v-p)\, dX \ge \varepsilon \int_\Omega \sin x (v-p)\, dX, \quad \forall v \in K \ . \tag{4.2}$$

By Theorem 2.1, which remains valid also in the present situation, the variational inequality

$$p \in K, \int_\Omega \nabla p \cdot \nabla (v-p)\,dX \geq \int_\Omega \sin x (v-p)\,dX, \quad \forall v \in K , \tag{4.3}$$

gives the pressure distribution when ε is very small.

The coincidence set

$$C = \{(\theta,y) \in \Omega, \ p(\theta,y) = 0\} \quad ,$$

has, as usual, the physical meaning of region of cavitation in the lubricating film. We note that the condition of periodicity in x does not permit us to apply the usual theory of existence, uniqueness and regularity from ([21]) directly to (4.2), (4.3); this difficulty can be easily overcome by transforming the rectangle Ω in the annulus $O=\{X \in \mathbb{R}^2, \ a < |X| < a+2b\}$ with the transformation

$$\begin{cases} y_1 = (a+b+y)\cos x \\ \\ y_2 = (a+b+y)\sin x \end{cases} , \quad a > 0 .$$

The result of [21] are directly applicable to the variational inequality restated in O and we obtain the following

<u>Theorem 4.1.</u> Problem (4.2), (4.3) admit a unique solution which belongs to $C^{1,\lambda}(\bar{\Omega})$, $0 \leq \lambda < 1$.

In the remaining part of this section we study in detail the properties of the solution of (4.3) and of the related free boundary $\partial C \cap \Omega$. Use shall be made of the notations

$\Omega_1\{(x,y);\ \pi < x \leq 2\pi,\ |y| < b\}$, $\Omega_2=\{(x,y);\ 0 \leq x \leq 2\pi,\ 0 < y < b\}$,
$\Omega_3=\{(x,y);\ 0 \leq x \leq 2\pi,\ -b < y < 0\}$, $\Omega_4=\{(x,y);\ \frac{\pi}{2} < x < \frac{3}{2}\pi,\ |y| < b\}$,
$\Omega_5=\{(x,y);\ 0 \leq x < \frac{\pi}{2},\ \frac{3}{2}\pi < x \leq 2\pi,\ |y| < b\}$, $\Gamma_1=\{(x,y);\ 0 \leq x \leq 2\pi,\ y=0\}$,
$\Gamma_2=\{(x,y);\ 0 \leq x \leq 2\pi,\ y=b\}$, $\Gamma_3=\{(x,y);\ x=\frac{\pi}{2},\ x=\frac{3}{2}\pi,\ |y| < b\}$,
$\Gamma_4=\{(x,y);\ 0 \leq x \leq 2\pi,\ y=-b\}$.

<u>Theorem 4.2.</u> The solution $p(x,y)$ of (4.3) is symmetric with respect to the straight lines $y=0$, $x=\frac{\pi}{2}$, $x=\frac{3}{2}\pi$. Moreover we have

$$p_y \le 0 \text{ if } 0 \le y \le b, \quad p_y \ge 0 \text{ if } -b \le y \le 0 \ , \tag{4.4}$$

$$\text{sign } p_x = \text{sign } \cos x \ . \tag{4.5}$$

<u>Proof</u> The symmetry of all data with respect to the x-axis in (4.3) implies the symmetry of p with respect to that axis. To prove the symmetry with respect to $x = \frac{\pi}{2}$ we note that problem (4.3) can be restated in any rectangle of periodicity e.g. in $-\frac{\pi}{2} \le x \le \frac{3}{2}\pi$. Taking into account the symmetries of sinx and the uniqueness of the solution we get $p(x,y)=p(\pi-x,y)$. Similarly we obtain $p(x,y)=p(3\pi-x,y)$.

Let us prove (4.4). Suppose, by contradiction, $\Lambda=\{(x,y) \in \Omega_2, p_y(x,y) > 0\} \ne 0$ and let $(\bar{x},\bar{y})$ be any point of maximum of p_y in $\bar{\Lambda}$. Since $p > 0$ in Λ we have $\Delta p_y=0$ in Λ and by the maximum principle $(\bar{x},\bar{y}) \in \partial\Lambda$. Now $(\bar{x},\bar{y})$ cannot belong to $\partial\Lambda \cap \Gamma_2$, because on Γ_2, $p_y \le 0$; on the other hand $(\bar{x},\bar{y}) \notin \partial\Lambda \ \Gamma_1$, since $p_y=0$ on Γ_1; finally $(\bar{x},\bar{y}) \notin \partial\Lambda \cap \Omega_2$ by the continuity of p_y which implies $p_y=0$ on $\partial\Lambda \cap \Omega_2$. Hence $(\bar{x},\bar{y}) \notin \partial\Lambda$. To obtain (4.5) we note that $p_x=0$ on Γ_3 (by the symmetries of p) and on $\Gamma_2 \cup \Gamma_4$. Moreover we have $-\Delta p_x=\cos x$ where $p > 0$ and $\cos x < 0$ in Ω_4. By repeating the same maximum principle argument used before, we get $p_x \le 0$ in Ω_4. In the same way we prove $p_x \ge 0$ in Ω_5 and (4.5) follows.

We note that if $(\bar{x},\bar{y})$ belongs to C, then by (4.4), (4.5) we have

$$\{(x,y);\ \bar{x} \le x \le 3\pi - \bar{x},\ \bar{y} \le |y| < b\} \subseteq C \ . \tag{4.6}$$

We prove now the following

<u>Theorem 4.3</u>. If p is the solution of (4.3), then

$$p \le u + \max_{\bar{\Omega}} u, \text{ where } u(x,y) = \left(1 - \frac{\cosh y}{\cosh b}\right) \sin x \ , \tag{4.7}$$

$$C \subseteq \Omega_1 \ , \tag{4.8}$$

C has a connected interior. (4.9)

<u>Proof</u> Let us consider the Dirichlet's problem

$$-\Delta u = \sin x,\ u(x,\pm b) = 0,\ u(x,y) = u(x+2\pi,y) \ ,$$

whose solution is given by

$$u(x,y) = (1 - \frac{\cosh y}{\cosh b}) \sin x .$$

If we define $w=p-u$, $K_1=\{\phi;\phi \in H^1_{o*},\phi \geq -u\}$, then w is solution of the variational inequality

$$w \in K_1, \int \nabla w \cdot \nabla(\phi - w)\,dX \geq 0, \quad \forall \phi \in K_1 .$$

Hence $w \leq \sup(-u) = \sup u$ and this implies (4.7). The inclusion (4.8) follows from a general property of the coincidence set of second order variational inequalities [21]. Finally we show that C has a connected interior. Let us consider the one-dimensional variational inequality

$$V(x) \in \tilde{K}, \int_0^{2\pi} V'(\phi - V)'dx \geq \int_0^{2\pi} \sin x(\phi - V)dx, \quad \forall \phi \in \tilde{K} , \tag{4.10}$$

where

$$\tilde{K} = \{\phi \in H^1(0,2\pi),\ \phi(0) = \phi(2\pi) = k, \phi \geq 0\} ,$$

with $k = 1 - (\cosh b)^{-1}$. The solution of (4.10) is given by

$$V(\phi) = \begin{cases} \sin x - x\cos\xi + k, & \text{if } 0 \leq x \leq \xi , \\ 0, & \text{if } \xi \leq x \leq \eta, \\ \sin x + (2\pi - x)\cos\eta + k, & \text{if } \eta \leq x \leq 2\pi , \end{cases}$$

where ξ is the unique solution of the equation $\sin\xi - \xi\cos\xi + k = 0$ in $]\pi, \frac{3}{2}\pi[$ and η is the unique solution of $\sin\eta + (2\pi - \eta)\cos\eta + k = 0$ in $]\frac{3}{2}\pi, 2\pi[$. We have $\xi < \frac{3}{2}\pi < \eta$ and $\lim_{k\to 1} \xi = \lim_{k\to 1} \eta = \frac{3}{2}\pi$. By (4.7) $p \leq V$ on $\partial\Omega$, hence the inequality $p \leq V$ in Ω follows from a comparison principle between solution of variational inequalities, and

$$\{(x,y);\ \xi \leq x \leq \eta,\ |y| < b\} \subseteq C . \tag{4.11}$$

Recalling (4.6) we conclude that C has a connected interior.

Finally we have the following

<u>Theorem 4.4</u>. There exists an even, continuous function $\gamma_1(y)$ defined in $[-b,b]$ which is strictly decreasing in $[0,b]$ and such that

$$C = \{(x,y) \in \Omega,\ \gamma_1(y) \le x \le \gamma_2(y) = 3\pi - \gamma_1(y)\} . \tag{4.12}$$

Proof Let us define in $[-b,b]$ the function

$$\gamma_1(y) = \sup\{(x; x < \frac{3}{2}\pi,\ p(x,y)>0\} .$$

By (4.6) $\gamma_1(y)$ is non-increasing in $[0,b]$. Suppose $y_o > 0$, we have

$$\lim_{y\to y_o -} \gamma_1(y) = \gamma_1(y_o -) \ge \gamma_1(y_o +) = \lim_{y\to y_o +} \gamma_1(y).$$

Let us define $\Pi = \{(x,y);\ y=y_o,\ \gamma_1(y_o+) < x < \gamma_1(y_o-)\}$, $\Xi = \{(x,y);\ 0<y<y_o,\ \gamma_1(y_o+) < x < \gamma_1(y_o-)\}$; we have $p_x = p_y = 0$ on Π since $\Pi \subset \partial C$. If $\zeta = p_x$, we have $-\Delta\zeta = \cos x < 0$ in Ξ, but $\zeta < 0$ in Ξ, hence $\zeta_y(x,y_o) > 0$ on Π, by the maximum principle in Hopf's form. On the other hand $\zeta_y = p_{xy} = 0$ on Π and this implies $\Pi = \emptyset$ i.e. $\gamma_1(y_o+) = \gamma_1(y_o-)$. It follows that $\gamma_1(y)$ is a continuous function; moreover a similar maximum principle argument can be used for proving that $\gamma_1(y)$ is strictly decreasing in $[0,b]$. □

Taking into account (4.9), we can finally prove that $\gamma_1(y)$, $\gamma_2(y)$ are two analytic curves, as an application of a theorem of H. Lewy (see [20] p.151).

5. ASYMPTOTIC SOLUTIONS. THE INFINITELY LONG JOURNAL BEARINGS.

In this section and in the following one we study the behaviour of the solution of problem (4.1) when the journal bearing becomes either extremely long or extremely short, two asymptotic cases of great relevance in the history of the theory of lubrication. We will obtain our results as a consequence of the following abstract theorem of the theory of singular perturbation ([22],pp.104,124).

Let V_o, V_b be two real Hilbert space with norm $\|\ \|_a$, $\|\ \|_b$ respectively. Suppose that

$$V_a \subset V_b,\ V_a \textit{ dense in } V_b . \tag{5.1}$$

Let $a(u,v)$, $b(u,v)$ be two continuous bilinear forms on V_a, V_b respectively, and suppose

$$a(v,v) \ge 0,\quad \forall v \in V_a , \tag{5.2}$$

$$a(v,v) + \|v\|_b^2 \geq \gamma\|v\|_a^2 , \quad \forall v \in V_a, \ \gamma > 0 , \tag{5.3}$$

$$b(v,v) \geq \upsilon\|v\|_b^2, \quad \forall v \in V_b, \upsilon > 0 . \tag{5.4}$$

Let IK be a closed, convex, nonempty subset of V_a and let f be an element of the dual V_b' of V_b.

If $\alpha > 0$ is sufficiently small there exists a unique solution of the variational inequality

$$u_\alpha \in IK, \ \alpha a(u_\alpha, v-u_\alpha)+b(u_\alpha, v-u_\alpha) \geq \langle f, v-u_\alpha\rangle, \quad \forall v \in IK . \tag{5.5}$$

Define now $\overline{IK}$ as the closure of IK in V_b; then there exists a unique solution of

$$u \in \overline{IK}, \ b(u,v-u) \geq \langle f,v-u\rangle, \quad \forall v \in \overline{IK} . \tag{5.6}$$

Theorem 5.1. Let assumptions (5.1)-(5.6) hold true, then

$$\lim_{\alpha\to 0} u_\alpha = u \text{ in } V_b, \quad \alpha^{\frac{1}{2}}\|u_\alpha\| \leq C .$$

Moreover, if $u \in IK$, then

$$\lim_{\alpha\to 0} u_\alpha = u \text{ in } V_a, \quad \|u_\alpha - u\|_b \leq C\alpha^{\frac{1}{2}} .$$

With the substitution y=bz, the variational inequality (4.1) can be written in the form

$$p^{(\alpha)} \in K, \int_R h^3 p_x^{(\alpha)} (v-p^{(\alpha)})_x dX + \alpha \int_R h^3 p_z^{(\alpha)} (v-p^{(\alpha)})_z dX \geq \varepsilon \int_R \sin x (v-p^{(\alpha)}) dX, \tag{5.7}$$
$$\forall v \in K ,$$

where $\alpha = b^{-2}$, $R=\{(x,z); \ 0 \leq x \leq 2\pi, |z| < 1\}$.

Consider the following one-dimensional problem

$$-\frac{d}{dx}\left[h^3(x)\frac{dp^{(o)}}{dx}\right] = \varepsilon \sin x, \ p^{(o)}(0)=p^{(o)}(2\pi)=0, \tag{5.8}$$

whose solution is given by

$$p^{(o)}(x) = \varepsilon\frac{(2+\varepsilon\cos x)\sin x}{(2+\varepsilon^2)(1+\varepsilon\cos x)^2} .$$

By applying Theorem 5.1 we will prove that

$$\lim_{\alpha\to 0} p^{(\alpha)}=p^{(o)}+\max p^{(o)}, \quad \lim_{\alpha\to 0} p_x^{(\alpha)}=p_x^{(o)} \text{ in } L^2(R) , \tag{5.9}$$

$$\alpha^{\frac{1}{2}} \|p_y^{(\alpha)}\| \le C . \tag{5.10}$$

Lemma 5.1. If p is the solution of (4.1) then

$$p \le p^{(o)} + \max p^{(o)} \text{ in } \bar{\Omega} . \tag{5.11}$$

Proof

Consider the problem

$$L[P] = -\varepsilon\sin x \text{ in } \Omega,\ P(x,\pm b)=0,\ P(x,y)=P(x+2\pi,y) .$$

It is easy to verify that

$$P(0,y)=P(2\pi,y)=P(\pi,y)=0,$$

$$P(x,y)=-P(2\pi-x,y),\ P(x,y)=P(x,-y) .$$

Taking into account (5.8), we get

$$\max_{\bar{\Omega}} P \le \max_{\bar{\Omega}} p^{(o)} .$$

On the other hand $p-P\le \max P$, hence $p\le P+\max p^{(o)}$. It follows $p\le p^{(o)}(x)+$ $+\max p^{(o)}$ on $\partial\Omega$, but $\tilde{p}(x)=p^{(o)}(x)+\max p^{(o)}$ is the solution of the one-dimensional variational inequality

$$\tilde{p}\in K,\ \int_0^{2\pi} h^3\tilde{p}'(v-\tilde{p})'dx \ge \varepsilon\int_0^{2\pi} \sin x(v-\tilde{p})dx, \quad \forall v\in \tilde{K} .$$

where

$$\tilde{K} = \{v\in H^1(0,2\pi), v(0)=v(2\pi)=\max p^{(o)}, v\ge 0\} .$$

Hence (5.11) holds true by a comparison principle between solutions of variational inequalities. □

Since the point of minimum for $p^{(o)}(x)$ in $[0,2\pi]$ is given by $x_o=2\pi-\arccos\frac{-3\varepsilon}{2+\varepsilon^2}$, Lemma 5.1 implies that the set $\{(x_o,z);|z|<1\}$ is contained in C for all α.

We denote by C_*^∞ the space of the functions which are C^∞ in $-\infty<x<+\infty, |z|\le 1$ and periodic in x with period 2π. Let V be the completion of C_*^∞ with respect to the norm

$$\|v\|_V = [\int_R |v|^2 dx + \int_R |v_x|^2 dx]^{\frac{1}{2}} . \tag{5.12}$$

We have the following

<u>Lemma 5.2</u>. The space H^1_{o*} is dense in V.

<u>Proof</u> Let $f_\upsilon(z)$ be a function of class C^2 defined in $[-1,1]$ as follows

$$f_\upsilon(z) = \begin{cases} 0 \text{ if } 1-\upsilon\le|z|\le 1 \\ \\ 1 \text{ if } |z|\le 1-2\upsilon \end{cases}, \quad 0<\upsilon<\tfrac{1}{2} .$$

Clearly $vf_\upsilon \in H^1_{o*}$ for all $v\in V$. Set

$$R_{2\upsilon} = \{(x,z);\ 0\le x\le 2\pi,\ |z|<1-2\upsilon\} ,$$

we have

$$\|vf_\upsilon - v\|_{L^2(R)} \le C_1\upsilon,\ \|(vf_\upsilon)_x - v_x\|_{L^2(R)} \le C_2\upsilon ,$$

where C_1,C_2 do not depend on υ. This proves the Lemma. □

Define C^∞_{*+}, C^∞_{*o+} as the subspaces of C^∞_*, C^∞_{*o} respectively of the functions which vanish in neighborhood of $x=x_o, |z|\le 1$. Note that in C^∞_{*+} a norm which is equivalent to (5.12) is given by

$$\|v\|_b = [\int_R |v_x|^2 dx]^{\frac{1}{2}}, \quad v\in C^\infty_{*+} . \tag{5.13}$$

Let V_b be the completion of C^{∞}_{*+} with respect to (5.13) and V_a the completion of C^{∞}_{*o+} with respect to

$$\| v \|_a = [\int_R |\nabla v|^2 dx]^{\frac{1}{2}} .$$

Define

$$IK = \{v \in V_a v \geq 0\}, \ \overline{IK} = \{v \in V_b v \geq 0\},$$

$$a(u,v) = \int_R h^3 u_z v_z dX, \quad u,v \in V_a, \ b(u,v) = \int_R h^3 u_x v_x dX, \quad u,v \in V_b .$$

By Lemma 4.2 the space V_a is dense in V_b and $\overline{IK}$ is the closure of IK in V_b; $a(u,v)$, $b(u,v)$ are continuous bilinear forms on V_a, V_b respectively. Restate now problem (5.7) in the following form

$$p_\alpha \in IK, \alpha\ a(p_\alpha, v-p_\alpha) + b(p_\alpha, v-p_\alpha) \geq \varepsilon \int_R \sin x (v-p_\alpha) dX, \quad \forall v \in IK . \tag{5.14}$$

Since the solution $p^{(\alpha)}$ of (5.7) belongs to IK for all α by Lemma 5.1, we conclude that $p^{(\alpha)} = p_\alpha$. In the present setting the limit problem ($\alpha=0$) can be written

$$p \in \overline{IK}, \ b(p,v-p) \geq \varepsilon \int_R (v-p) \sin x \ dX, \quad \forall v \in \overline{IK} . \tag{5.15}$$

<u>Lemma 5.3</u>. Problem (5.15) admits one and only one solution given by

$$p = p^{(o)} + \max p^{(o)} . \tag{5.16}$$

<u>Proof</u> The uniqueness of the solution follows from the coerciveness of the bilinear form $b(u,v)$ in V_b. Moreover we can write

$$\int_R h^3 p_x v_x dX = \varepsilon \int_R v \sin x \ dX, \quad \forall v \in H^1_o ; \tag{5.17}$$

but, by Lemma 5.2, (5.17) holds true for all $v \in V_b$. On the other hand $p \in \overline{IK}$, hence

$$\int_R h^3 p_x (v-p)_x dX \geq \varepsilon \int_R (v-p)\sin x \, dX, \quad \forall v \in \overline{IK},$$

by the uniqueness of the solution; this fact completes the proof. □

All the hypotheses of Theorem 5.1 are now verified and we can conclude that

$$p_\alpha \to p \text{ in } V_b \text{ as } \alpha \to 0, \; \alpha^{\frac{1}{2}} \|p_\alpha\|_{V_a} \leq C,$$

i.e. (5.9)-(5.10). We finally remark that the coincidence set reduces to $\{(x_o,z); x_o = 2\pi - \arcsin \frac{-3\varepsilon}{2+\varepsilon^2}, |z| \leq 1\}$, when $\alpha \to 0$; moreover we lose in the limit the boundary conditions on $z=\pm 1$, in fact $p \notin IK$.

6. THE INFINITELY SHORT JOURNAL BEARINGS

To study asymptotic behaviour of pressure when the bearing becomes extremely short we consider the variational inequality

$$p^{(\beta)} \in K, \; \beta \int_R h^3 p_x^{(\beta)} (v-p^{(\beta)})_x dX + \int_R h^3 p_z^{(\beta)} (v-p^{(\beta)})_z dX \geq \varepsilon \int_R (v-p^{(\beta)}) \sin x \, dX, \tag{6.1}$$

$$\forall v \in \overline{K}.$$

where $\beta = \frac{1}{\alpha} = \frac{L^2}{R^2}$ and $p^{(\beta)}$ is $\frac{1}{\beta}$ times the value of p corresponding to a given choice of β. We intend to analyze the limit of $p^{(\beta)}$ when $\beta \to 0^+$. Set

$$p^{(o)}(x,z) = \begin{cases} \dfrac{\varepsilon(1-z^2)\sin x}{2(1+\varepsilon\cos x)^3}, & 0 \leq x \leq \pi \\ 0, & \pi \leq x \leq 2\pi \end{cases};$$

we will prove that

$$\lim_{\beta \to 0^+} p^{(\beta)} = p^{(o)} \text{ in } H^1_{o*}, \tag{6.2}$$

$$\|p^{(\beta)} - p^{(o)}\|_{L^2(R)} \leq C\beta, \quad \|p_z^{(\beta)} - p_z^{(o)}\|_{L^2(R)} \leq C\,\beta. \tag{6.3}$$

Theorem 5.1 will now be used in the following setting. Set $V_a = H^1_{o*}$ and let

V_b be the completion of C^∞_{o*} under the norm

$$\|v\|_b = [\int_R |v_z|^2 dx]^{\frac{1}{2}} .$$

Define

$$a(u,v) = \int_R h^3 u_x v_x dX, \quad u,v \in V_a, \quad b(u,v) = \int_R h^3 u_z v_z dX, \quad u,v \in V_b ,$$

and $\bar{K}=\{v \in V_b, v \geq 0\}$. The limit problem ($\beta$=0) is now

$$p \in \bar{K}, \; b(p,v-p) \geq \varepsilon \int_R (v-p) \sin x \; dX, \quad \forall v \in \bar{K} . \tag{6.4}$$

The following lemma holds true

Lemma 6.1. Problem (6.4) has one and only one solution given by

$$p = p^{(o)} \tag{6.5}$$

Proof The solution of (6.4) is unique by the coerciveness of $b(u,v)$ in V_b. Moreover we have

$$\int_R h^3 p_z^{(o)} (v-p^{(o)})_z dX = \int_0^\pi \int_{-1}^1 h^3 p_z^{(o)} (v-p^{(o)})_z dX =$$

$$= \varepsilon \int_0^\pi \int_{-1}^1 (v-p^{(o)}) \sin x \; dX \geq \varepsilon \int_R (v-p^{(o)}) \sin x \; dX ;$$

hence by the uniqueness of the solution we get (6.5).

It is easy to verify that the limit solution $p^{(o)}$ belongs to the convex set of admissible functions for problem (6.1) i.e. $p^{(o)} \in K$, thus the present case differs considerably from that of Section 5. We can now apply Theorem 5.1 obtaining

$$\lim_{\beta \to 0} p^{(\beta)} = p^{(o)} \text{ in } V_a, \quad \|p^{(\beta)} - p\|_b \leq C \beta^{\frac{1}{2}} ,$$

i.e. (6.2), (6.3).

7. DISCUSSION ON THE CONDITIONS OF TRANSITION

We return here on one assumption introduced in Section 1 and systematically accepted above: the assumption that the pressure be smooth (i.e., C^1) also across the boundary of the region of cavitation. A decision on this matter must derive from a model of flow in that region [14], [15]; the simplest idea is that the film, having there a thickness $\bar{H}$ less than H, is detached from the stationary surface and is conveyed rigidly by the moving surface. If this idea is accepted, then, within the limits envisaged in stating equation (1.3), the flow in the region of cavitation must be wholly in the direction of the first axis:

$$Q_1 = \bar{H}V_1, \quad Q_2 = 0 \ . \tag{7.1}$$

Disregarding a boundary layer where the actual transition occurs between the Couette-Poseuille type of flow and the rigid motion, we can impose continuity of flow at the boundary of the region of cavitation and require the matching of (1.1) and (7.1) [19]:

$$\frac{\partial P}{\partial x} = \frac{12\eta V_1}{H^3}\left(\frac{H}{2} - \bar{H}\right), \quad \frac{\partial P}{\partial x_2} = 0 \ . \tag{7.2}$$

These conditions need not be in contrast with the required smoothness of P: we have restricted $\bar{H}$ only to be less than H; so, there seem to be no objections to the assumptions $\bar{H} = H/2$. Thus our condition of smoothness for P seems vindicated. Such is certainly the case for example of Section 3; on the contrary, the question is not trivial for the cases treated in Sections 4 to 6, as we shall see below.

An alternative model of flow in the cavitating region could derive from the idea that there the fluid forms with air a compressible foam with much lower gross viscosity. However, this model (perhaps more appropriate for the study of non-stationary films) has not been given yet a precise mathematical formulation and cannot be discussed here. We remark only that the model might lead substantially to the same equations under stationary conditions.

To return now to the question left open above, we observe that, in cylindrical bearings, the condition of periodicity excludes the possibility of matching flows and at the same time ensuring the smoothness of the pressure: because the transport of fluid is rigid along the region of cavitation, $\bar{H}$ remains constant there, if there is not an appropriate external supply of fluid. Actually, under conditions found most often in practice, such external supply is provided. The hypothesis of smoothness of P corresponds to the assumption that the supply is exactly what is wanted so as to satisfy the condition $\bar{H}=H/2$ also at the exit of the region of cavitation.

However, such an exact adjustment is hardly to be expected under all circumstances. A more realistic approach corresponds to the acceptance of the extra supply as a datum, forfeiting the condition of smoothness of the pressure and substituting for it the jump condition implicit in (7.2) [2]. In a very recent, interesting paper [3], Bayada and Chambat show that the difficulties of the corresponding problem can be overcome at least to the point of yielding an existence theorem for the solution.

A problem which remains open is that of the pressurized bearing without supply (the simpler case of the unpressurized bearing, P=0 at the sides, does not seem to offer solution); although the very special case (degenerate in the sense of Section 5) of the infinitely long bearing can be treated in all details [7].

Many other problems remain only partly solved or open: we have ourselves quoted in [8] a few which we regard as particularly interesting. Some involve directly the Stokes equations for slow fluid flow, rather than the simpler equation of Reynolds [16], [18].

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P G DRAZIN & D H GRIFFEL

Free boundary problems in climatology

1. THE MODELS

This lecture is an introduction to some free-boundary problems in global climatology, problems which may be new to many of you. It is a partial account, being written from the point of view of Bristol and broaching only a few uses of free-boundary problems. However, the references provide an entry to the wider literature. Also a recent survey by North et al. (1981) may be found useful.

Following some fundamental work by Budyko (1969) and Sellers (1969), several climatologists have been developing *energy-balance models*. The energy budget of the atmosphere is modelled by a simple system of equations in order to understand, and thereby predict, climatic changes over centuries or millennia. One seeks models which are simple to understand and inexpensive to solve numerically, yet which are not gross oversimplifications of the physics of the atmosphere. A newcomer to the field should not accept the current models uncritically. The dynamics of the earth's turbulent atmosphere is certainly more complex than these models. Also the atmosphere interacts with the land and the oceans. However, there is some hope that the most fundamental long-term *average* properties of the atmosphere may be modelled by quite simple equations, even if the *weather* cannot be forecast accurately a week in advance. Simplicity of a model not only cuts the costs of computation but thence enables the solution to be computed over millennia of model time and over wide ranges of the parameters. Also simplicity enables the mathematical properties of the solutions to be analysed, and thence may lead to physical insight.

There are many energy-balance models. We shall discuss only a subclass of them which are based on a single heat equation of the form,

$$R \frac{\partial T}{\partial t} = \operatorname{div}(k \operatorname{grad} T) + QS(1 - \alpha) - I. \tag{1}$$

This is a nonlinear diffusion equation over the surface of a sphere. The dependent variable is the surface temperature T of the earth averaged over a circle of constant latitude θ. (The longitudinal convection of heat by the winds is much more rapid than the meridional, and hence the average over longitude is taken.) The independent variables used are $x = \sin\theta$, for convenience, and t, the time measured in years. Thus we have

$$\mathrm{div}(k \ \mathrm{grad}\ T) = \frac{\partial}{\partial x}\left\{\frac{k}{r^2}(1 - x^2)\frac{\partial T}{\partial x}\right\},$$

where r is the radius of the earth. We also need the boundary conditions,

$$(1 - x^2)^{\frac{1}{2}}\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = \pm 1, \tag{2}$$

to ensure that there is no meridional heat flux at the poles.

The left-hand side of equation (1) represents the local rate of increase of heat of the atmosphere, R being some specific heat. All of the dynamics of the atmosphere is represented by the diffusion term on the right-hand side, k being some heat conductivity. Analysis of observations of the present climate offers some evidence (Lorenz 1979) that diffusion may represent the average meridional transfer of heat by the winds on a planetary scale over a season or longer; but the analysis does not give direct information about the value of k. The rate of input of heat from the sun is QS, where S varies with x and t in a complicated way governed by the configuration of the orbit and the axis of rotation of the earth. We normalize S by convention so that $\int dt \int_0^1 dx\ S(x,t) = 1$, integrated over a year. Thus 4Q is the so-called solar constant. The albedo, or reflectivity, of the earth's surface is denoted by α, so that the middle term of the right-hand side of equation (1) represents the energy of the sun's radiation (mainly short waves) absorbed by the atmosphere. The last term represents the radiation (mainly long waves) from the atmosphere into space. Note that all these quantities except R and k may be found by averaging direct measurements, although the accuracy of the relevant observations varies greatly at present.

In the simplest models α and I are taken as functions of only T. In more elaborate models they may depend on latitude as well, to represent the uneven distribution of land and sea. Also the temperature of the upper atmosphere, cloudiness etc. may be modelled. Newton's law of cooling,

$$I(T) = A + BT, \tag{3}$$

however, gives a simple expression for I which is quite accurate because of the small relative variation of the absolute temperature of our planet's surface.

Free-boundary problems arise from an approximate representation of the albedo due to Budyko (1969) and used by many others. Low temperatures lead to formation of snow and ice, and hence to a high albedo, although once the temperature for the ice formation is reached the abledo gets little higher when the temperature falls further. On the other hand, forests and ice-free seas absorb the sun's rays efficiently, so that their albedo is low. Thus Budyko took the step function,

$$\alpha(T) = \begin{cases} \alpha_i & \text{for} \quad T < T_s \\ \\ \alpha_w & \text{for} \quad T > T_s \end{cases}, \tag{4}$$

where α_i and α_w represent the albedos of 'icy' and 'watery' surfaces respectively. He chose the constants $0 < \alpha_w < \alpha_i < 1$ and T_s to fit some observations. It should be recognized, however, that this step function is only an approximation to the average of a complex situation (cf. Griffel & Drazin 1981). In any event, the nonlinear dependence of the albedo on temperature is a fundamental mechanism of the energy balance.

As a historical aside, one may note that Stefan himself worked originally on the change of phase of water and ice, though in his problem latent heat is important and in this problem the reflection of the sun's radiation is dominant.

2. MULTIPLE EQUILIBRIA AND STABILITY

Some interesting mean annual temperature distributions $T = T_o(x)$ were found by North (1975) on taking $S = S_o(x)$, its annual average. He followed Budyko in taking $T_s = -10^oC$, $\alpha_i = 0.62$ and $\alpha_w = 0.32$. North fitted observations with $Q = 334.5\ W\ m^{-2}$, $A = 201.4\ W\ m^{-2}$ and $B = 1.45\ W\ m^{-2}\ K^{-1}$. He also used the approximate form $S_o(x) = 1 + S_2P_2(x)$, where P_2 is the Legendre polynomial of second degree, taking $S_2 = -0.482$ for present astronomical circumstances.

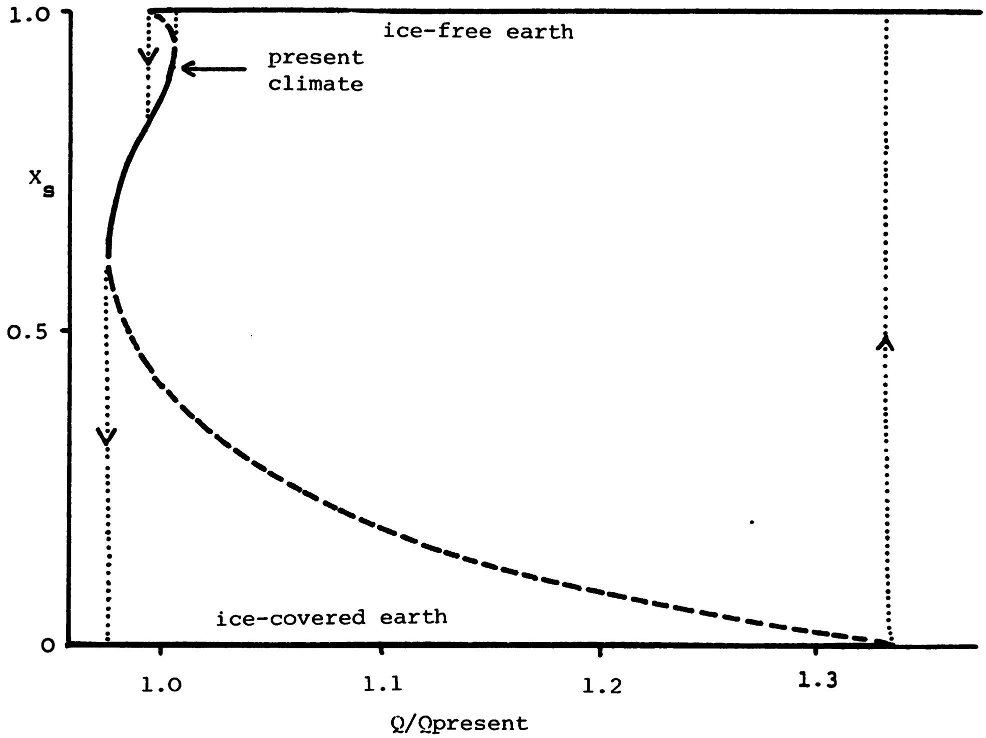

Figure 1. Equilibrium climates as the solar constant Q varies. Note tha $Q_{present} = 334.5$ W m^{-2}, and that x_s is defined as the sine of the latitude of a -10^{o}C isotherm so $T_o(x_s) = T_s$. Continuous curves denote stable and broken curves denote unstable equilibrium solutions. The dotted lines and arrows denote quasi-stationary transitions from one equilibrium solution to another (computed by Griffel & Drazin (1981) from equations (1) and (2)).

He then calculated some steady solutions T_o of equations (1) and (2), and chose the value of k in order to get the present observed value of the mean latitude of the isotherm $T_o = -10^{o}$C. On taking $k/r^2B = 0.31$, $T_o(x)$ agreed with observations quite well for all x.

Thereafter North calculated the equilibrium solutions T_o for a range of Q although there is no firm evidence that the solar constant has varied substantially. Recalculations of North's results by Drazin and Griffel (1977) are shown in Figure 1. It can be seen that there are five equilibria with north-south symmetry possible at present, three stable and two unstable. (In fact there are also three equilibria asymmetric about the equator, one stable and two unstable (Drazin & Griffel 1977). This is an interesting example of bifurcation and symmetry breaking, but may have no climatological significance). Also the solar constant need change by only a small percentage for the steady solution representing the present climate to cease to exist. If, say, Q were to oscillate slowly and periodically, then a climatic cycle might ensue in which the solution would jump from one equilibrium to another in a series of catastrophes. This is suggestive of the cycle of the ice ages.

Of course, Q is not the only parameter which might vary in the model. Griffel and Drazin (1981) computed steady solutions T_o for ranges of the parameters A and B (which can represent climatic changes due to the increase of carbon dioxide in the atmosphere), for a range of S_2 (whose variation is discussed in the next section), and for various albedo functions α.

3. SOME OF THE ASTRONOMICAL BACKGROUND

The most widely accepted theory of the cause of the ice ages is called the *Milankovitch hypothesis*. Last century Adhémar and Croll proposed that ice ages were caused by the variations of the earth's orbital parameters. At the beginning of this century Milankovitch elaborated and substantiated the theory. It has been confirmed in the last decade by statistical analysis of the rapidly growing body of palaeoclimatological observations. The theory shows why there are ice ages rather than how, their mechanism remaining controversial (cf. Imbrie & Imbrie 1979).

There are three important orbital parameters: the obliquity ε, the eccentricity e of the earth's elliptical orbit about the sun, and the longitude Π of the vernal equinox. The obliquity is defined as the angle between the earth's axis of rotation and the normal to the plane of the earth's orbit; thus ε is the latitude of the tropics. The angle Π is that swept out by the line from the sun to the earth while the earth moves from perihelion to the vernal equinox of the northern hemisphere.

Astronomers, since Leverrier in the last century, have used Newton's laws of motion and gravitation to calculate accurately ε, e and Π during the Quaternary Period (last million years) and in the future. Owing to the precession of the equinoxes Π increases steadily; ε and e oscillate nearly sinusoidally. Some details are given in Table 1. To a good approximation, Q is unchanged and the mean annual insolation $S_o(x)=\int_{\text{one year}} S(x,t)\,dt$ depends only on ε.

Table 1

quantity	present value	range	(quasi-)period
Π	perihelion on Jan.3	$0-360^{\circ}$	22 000 yrs
ε	23.45°	$22^{\circ}-24.5^{\circ}$	41 000 yrs
e	0.017	0-0.054	100 000 yrs

The chief climatic effect of the variations of Π and e is the *relative* distribution of insolation over the seasons. For example, now that the winter solstice occurs when the earth is near perihelion, there are short mild winters and long cool summers in the northern hemisphere, but there are short hot summers and long cold winters in the southern hemisphere. This phenomenon is intensified in epochs when e is larger and absent when $e = 0$ (i.e. the earth's orbit is a circle).

4. A NUMERICAL METHOD

To study these climatic problems, we at Bristol have used PDECOL, a program package of Madsen & Sincovec (1979) to integrate equations of diffusion or wave propagation in one space dimension. PDECOL is based on collocation in x with splines, and a choice of finite-difference schemes for t.

Roberts (1981) has adapted PDECOL to deal with free-boundaries of climatological interest. We have $T < T_s$ for $1 \le x < x_s(t)$ and $x_n(t) < x \le 1$, and $T > T_s$ for $x_s(t) < x < x_n(t)$, where x_n and x_s give the boundaries of the northern and southern ice-caps respectively. Roberts made a linear transformation of each of the three sub-intervals of x onto $0 \le y \le 1$, and transformed equation (1) accordingly into three different equations over the same *fixed* interval $0 \le y \le 1$. He joined the solutions for the three new dependent variables by use of appropriate boundary conditions at $x = -1$, x_s, x_n and 1. He then used PDECOL, computing the positions of x_n and x_s at each time step so that $k\partial T/\partial x$ is continuous at $x = x_s, x_n$.

5. SOME SOLUTIONS WITH PERIOD OF ONE YEAR

One can fit the astronomical results at present quite well with the form,

$$S(x,t) = 1 + S_2P_2(x) + S_{11}P_1(x)\cos 2\pi t + S_{22}P_2(x)\cos 4\pi t, \tag{5}$$

where $t = 0$ at a winter solstice in the northern hemisphere. Roberts (1981) showed that $S_2 = -0.48896$, $S_{11} = -0.75622$ and $S_{22} = 0.1470$, and he used the resultant function S together with North's (1975) data for Q, T_s, α_i etc. in computations of unsteady solutions T for various values of R.

The results of Roberts' series of numerical experiments are aptly summarized by his parallel asymptotic analysis. A solution T may be found by regular-perturbation theory if one supposes that, say, S_2 is fixed, $S_{11} = O(\mu)$ and $S_{22} = O(\mu^2)$ as $\mu \to 0$. The first three terms of the expansion of the solution,

$$T(x,t) = T_o(x) + \mu T_1(x,t) + \mu^2 T_2(x,t) + \dots , \tag{6}$$

were then found to agree with direct numerical integration of T to within about 0.1 K, even when μ is as large as 0.5. This is a convincing justification of the use of perturbation theory, even though S_{11} is not smaller in magnitude than S_2.

For example, Roberts (1981) represented S in equation (5) with the 'present' values of all the constants and with $\mu = \frac{1}{2}$, took R = 10B, and found

$$\begin{aligned} x_n(t) &= 0.8887 + 0.0015\ \mu\cos 2\pi(t - 0.2415) \\ &\quad + \mu^2(-\ 0.0532 + 0.0034\ \cos 4\pi t + 0.00054\ \sin 4\pi t) + \dots . \end{aligned} \tag{7}$$

It can be seen that x_n, like S and T, has a period of a year. The term of order μ corresponds to the seasonally forced component, lagging 0.2415 years behind the sun. This is substantially larger than the actual lag of the hottest day after the summer solstice. The rectification of the oscillation, i.e. the cumulative small effect of the oscillation built up over many years, leads to a decrease of the mean latitude of the $-\ 10^{o}C$ isotherm from $\sin^{-1}(0.8887) = 62.7^{o}$ in this model to $\sin^{-1}(0.8887 - 0.0532\mu^2) = 61.1^{o}$, because $\mu = \frac{1}{2}$.

Roberts has also showed that the rectification term in x_n gives different corrections to the mean latitude of the $-\ 10^{o}C$ isotherm at different ages of the Quaternary. In particular, the sign of the rectification term $-\ 0.0532\mu^2$ as well as its magnitude differs according to the age for which S is fitted. The mean latitude of the isotherm seems to shift from about two degree polewards to about two degrees equatorwards of 61.1^{o} according to the values of Π, ε and e.

These preliminary results need some development and refinement. Also palaeo-climatological observations are not complete or accurate enough to warrant quantitative deduction. Nonetheless, the above theoretical results are encouragingly similar to the observations in view of the simplicity of the theory.

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A FASANO & M PRIMICERIO

Classical solutions of general two-phase parabolic free boundary problems in one dimension

1. INTRODUCTION

One phase parabolic free boundary problems are usually classified in two main classes

α) problems in which the velocity of the free boundary enters the free boundary conditions (F.B.C.) explicitly, as e.g. in the Stefan problem,

β) problems in which F.B.C. are of Cauchy type, i.e. the unknown function $u(x_1,\ldots,x_n,t)$ and its normal derivative are prescribed on the free boundary, as e.g. in the oxygen diffusion-consumption problem.

Of course, more general cases could also be imagined (two functional relationships involving u and the free boundary, with suitable independence and compatibility conditions), but classes α) and β) seem to cover the majority of the problems suggested by applications.

Well posedness in a classical sense of a very general problem of class α) in one space dimension was discussed in [1], when F.B.C. involve u and u_x (*first order problems*). In [2] problems of type β) were solved by means of transformations reducing them to problems of type α), but with higher order derivatives of u appearing in F.B.C. (*2nd and 3rd order problems*). It can be recalled that when the parabolic equation is linear, either problems of type β) can be reduced to first order problems of type α), or they are ill-posed (see [3] and [4]).

Concerning two-phase problems, the situation is much more complicated: even if we confine ourselves to those one dimensional problems of the first order (in the sense specified above), which are known to be interesting for applications, we find the following five classes (we denote by u(x,t) and by U(x,t) the unknown functions in the two phases and by x = s(t) the free interface):

A) u and U are given on the free boundary; moreover

A-1) $\dot{s}(t) = f(u_x(s(t),t),\ U_x(s(t),t))$

or

A-2) $u_x = f(U_x)$

Class A-1) contains the two phase Stefan problem; an example of problem of class A-2) is given by the chemical reaction problem (see [5]).

B) u_x and U_x are given on the free boundary; moreover

B-1) $\dot{s} = f(u,U)$

or

B-2) $u = f(U)$.

For an example of class B-2) one can refer to [6] or to some modifications of it.

C) The F.B.C. in this class are of the type

$$u = f(U),$$

$$u_x = g(U_x) \;,$$

$$s = F(u,u_x) \;.$$

A classical example is the problem of the movement of two immiscible liquids in a porous medium (the so-called Muskat problem).

Other examples and an extensive bibliography can be found in [7], [8].

Of course more general F.B.C. can be considered in the same classes, involving dependence upon x and t and functionals instead of functions, as e.g. in [9], or including higher order derivatives of u. As in the case of one phase problems, more general classes could be considered, but for the same reasons we will be concerned only with the five types above; our aim is to discuss their classical well-posedness in a unified theory.

First of all, we recall that classical solvability of problem A-1) is discussed in [1]. Moreover, we note that the definition $\bar{u}(x,t) = u_x(x,t)$, $\bar{U}(x,t) = U_x(x,t)$ allows one to transform problems of type B-1) and B-2) into problems of type A-1) and A-2), at least when the equations to be solved in the two regions are linear. When they are nonlinear, the same goal can be achieved, e.g. by introducing higher order problems (or considering non-linear parabolic equations whose coefficients depend on functionals of the unknown itself).

Thus, it appears that the analysis of problems of type A) and B) reduces basically to the study of problems of type A-2). The remainder of this paper is devoted to proving existence of classical solutions for such a class of free boundary problems. More specifically, we will consider the following problem: to find $T > 0$ and $s(t)$, $u(x,t)$, $U(x,t)$ with the usual regularity properties (see [1]), such that

$$a(u)u_{xx} - u_t = q(u) , \qquad 0 < x < s(t), \quad 0 < t < T, \tag{1.1}$$

$$u_x(0,t) = f(t), \qquad 0 < t < T , \tag{1.2}$$

$$u(x,0) = \phi(x) , \qquad 0 < x < s(0) \equiv b , \tag{1.3}$$

$$A(U)U_{xx} - U_t = Q(U) , \qquad s(t) < x < 1 , \quad 0 < t < T , \tag{1.4}$$

$$U(1,t) = F(t) , \qquad 0 < t < T , \tag{1.5}$$

$$U(x,0) = \Phi(x) , \qquad b < x < 1 , \tag{1.6}$$

with F.B.C.

$$u(s(t),t) = 0 , \qquad 0 < t < T , \tag{1.7}$$

$$U(s(t),t) = 0 , \qquad 0 < t < T , \tag{1.8}$$

$$u_x(s(t),t) = \mathcal{F}_t(s,U,U_x) , \qquad 0 < t < T , \tag{1.9}$$

where for any $t \in [0,T]$ $\mathcal{F}_t$ is a functional acting on the triple of functions $s(\tau), U(x,\tau), U_x(x,\tau)$, $x \in [s(\tau),1]$, $\tau \in [0,t]$.

The argument we will use is to replace (1.9) by

$$u_x(s(t),t) = g(t) , \qquad 0 < t < T . \tag{1.10}$$

where g belongs to a suitable class of functions; then we solve the one-phase free boundary problem (1.1), (1.2), (1.3), (1.7), (1.10) which is of the class β) above. Once s(t) has been found, U(x,t) can be obtained by solving (1.4), (1.5), (1.6), (1.8) (a usual initial boundary value problem) and finally it is possible to use (1.9) to complete the definition of a mapping $h = \mathcal{M}g$.

We will prove that a $T_0 > 0$ exists such that $\mathcal{M}$ has at least one fixed point in $(0,T_0)$. Many of the estimates needed will be derived using classical arguments such e.g. those of [13].

Finally, coming to problems of class C), a parallel theory can be developed based on similar arguments, the only difference being the fact that the free boundary problem to be solved in the first step of the definition of the mapping $\mathcal{M}$ is of class α) instead of class β).

This concludes our introductory discussion about the classical solvability of general two-phase one-dimensional free boundary problems; the question of uniqueness is left open (see Remark 2.2 below) as well as the study of other peculiar aspect (global existence, qualitative properties, etc.), deserving further investigation.

2. NOTATION AND RESULTS

We present our notation for the Banach spaces used throughout the paper (norms are defined as usual). All functions are real.

Let I be a bounded interval on the real line:

$H_\alpha(I)$, $0 < \alpha < 1$, norm $\|\cdot\|_\alpha$: the space of Hölder continuous functions on I, exponent α.

$C_m(I)$, $m \geq 0$, integer, norm $\|\cdot\|_m$: the space of continuously differentiable functions up to the m-th order.

$H_{m+\alpha}(I)$, norm $\|\cdot\|_{m+\alpha}$: the subspace of C_m whose elements have the m-th order derivative in $H_\alpha(I)$.

Let D be a bounded domain in $\mathbb{R}^2$.

$C_\alpha(D)$, $0 < \alpha < 1$: the space of the functions $u(x,t)$ such that

$$\| u \|_\alpha \equiv \sup_{(x',t'),(x",t")\in D} |u(x',t') - u(x",t")|/(|x'-x"|^\alpha + |t'-t"|^{\alpha/2})$$

is finite.

$C_{m,n}(D)$, $m = 0,1,2$, $n = 0,1$, norm $\|\cdot\|_{m,n}$: the space of functions on D which are continuously differentiable m times w.r.t. x and n times w.r.t. t.

$C(D) = C_{o,o}(D)$.

$C_{m+\alpha}(D)$, norm $\|\cdot\|_{m+\alpha}$: the subspace of $C_{m,n}$, $n = [m/2]$, whose elements have the highest order derivatives in $C_\alpha(D)$.

$C_{1,\alpha}(D)$: the subspace of $C_{1,o}(D)$ whose elements are such that the following norm is finite

$$\| u \|_{1,\alpha} \equiv \|u\|_{1,o} + \sup_{(x,t'),(x,t")\in D} |u(x,t') - u(x,t")|/|t'-t"|^\alpha$$

Now we list the hypotheses we shall need in the sequel.

(i) $a(u)$, $q(u)$, $A(u)$, $Q(u)$ are twice continuously differentiable for $u \in (-\infty, +\infty)$, and $a(u)$, $A(u) \geq a_o$ for some constant $a_o > 0$.

(ii) $f \in C_1[0,T]$, $F \in C_2[0,T]$ for any $T > 0$.

(iii) $\phi \in C_3([0,b])$, $\Phi \in C_3([b,1])$

$$\phi'(0) = f(0) \ , \quad \Phi(1) = F(0) \ , \tag{2.1}$$

$$\phi(b) = \Phi(b) = 0 \ , \quad c \equiv \phi'(b) = \Phi'(b) \neq 0 \ , \tag{2.2}$$

$$\phi''(b) = \Phi''(b) \quad , \quad [a(0)\phi''(b) - q(0)][1 + \phi'(b)/c] = 0 \ .$$

(iv) For any $t \in [0,T]$ let $D_t = \{(x,\tau) : s(\tau) < x < 1, \ 0 < \tau < t\}$ and $s \in H_\alpha[0,t])$, $p_1 \in C_{1,\alpha}(\bar{D}_t)$, $p_2 \in C_{1,\alpha}(\bar{D}_t)$, then the function

$$\omega(t) = \mathcal{F}_t(s,p_1,p_2) \tag{2.3}$$

is such that $\omega \in H_\alpha([0,T])$ and

$$\|\omega\|_\alpha \leq \Omega(\|s\|_\alpha, \ \|p_1\|_{1,\alpha}, \ \|p_2\|_{1,\alpha}) \ , \tag{2.4}$$

Ω being a positive, continuous, nondecreasing function of its argument.

(v) This assumption is better stated reducing D_t to the rectangle $(0,1) \times (0,t)$ by means of the transformation $y = (1-x)/(1-s(\tau))$. Setting $\bar{p}_i(y,\tau) = p_i[1-(1-s(\tau))y,\tau]$, $i = 1,2$, we assume that the function (2.3) has the following property: for any choice of pairs $(s^{(1)},s^{(2)})$, $(p_1^{(1)}, p_1^{(2)})$, $(p_2^{(1)}, p_2^{(2)})$ as in (iv), the corresponding functions $\omega^{(1)}(t)$, $\omega^{(2)}(t)$, defined as in (2.3), are such that

$$\|\omega^{(1)} - \omega^{(2)}\|_\alpha \leq K\{\|s^{(1)} - s^{(2)}\|_\alpha + \|\bar{p}_1^{(1)} - \bar{p}_1^{(2)}\|_{1,\alpha} + \|\bar{p}_2^{(1)} - \bar{p}_2^{(2)}\|_{1,\alpha}\} \tag{2.5}$$

for some positive constant K.

For the sake of simplicity we will also assume that (1.4) is satisfied in the corner point $x = 1$, $t = 0$. Of course this assumption is not crucial and also some of the above assumptions can be weakened, but here we are not concerned with such details.

Remark 2.1. Since we are going to prove local existence, no sign specification on the data and no growth conditions for the coefficients are given.

Remark 2.2. The question of uniqueness is by no means trivial: assuming for instance $q = Q = f = F = \phi = \Phi = 0$, the triple $(s, u \equiv 0, U \equiv 0)$ satisfies (1.1)-(1.9) for any continuous boundary $x = s(t)$ such that $s(0) = b$.

The main result of this paper is the following.

Theorem 2.1. Under the assumption (i)-(v), problem (1.1)-(1.9) possesses a classical solution (s,u,U) in some interval $(0,T_0)$; moreover $s \in H_{1+\varepsilon}([0,T_0])$, with $\varepsilon \in (0,\frac{1}{2})$.

3. PROOF OF THEOREM 2.1.

The proof is in several steps: here an outline is presented, omitting most of the details.

(I) Given $T > 0$, $\varepsilon \in (0,\frac{1}{2})$, and two positive constants g_0, g_1 such that $g_0 < c < g_1$, where c is defined in (2.2), let

$$B_g(\varepsilon,g_0,g_1,G_\varepsilon,T) = \{g(t) : g \in H_{\frac{1}{2}+\varepsilon}([0,T]),\ \|g\|_{\frac{1}{2}+\varepsilon} \le G_\varepsilon,$$
$$g(0)=c,\ g_0 \le |g| \le g_1\}.$$

G_ε being a prescribed positive constant.

Consider the following one-phase auxiliary free boundary problem of the Cauchy type.

Problem (C): find (s,u) solving (1.1)-(1.3), (1.7), and

$$u_x(s(t),t) = g(t)\ ,\quad 0 < t < T\ , \tag{3.1}$$

in the classical sense, for some $T > 0$ and $g \in B_g(\varepsilon,g_0,g_1,G_\varepsilon,T)$.

In [2] it is shown that Problem (C) can be reduced to the following

Problem (S II): find (u,s) solving (1.1)-(1.3) and

$$u_x(s(t),t) = g(t)\ ,\quad 0 < t < T\ , \tag{3.2}$$

$$g(t)\ \dot{s}(t) = q(0) - a(0)\ u_{xx}(s(t),t)\ ,\quad 0 < t < T \tag{3.3}$$

in the classical sense, for some $T > 0$.

Well-posedness of problem (S II) is proved in [2] under the assumption $g \in H_{1+\alpha}$. On the other hand, here we chose g in $H_{1/2+\varepsilon}$ (and the motivation for this choice will be clear from steps III and IV of the proof): thus a new proof of the existence of a solution to problem (S II) is needed.

Setting

$$y = x/s(t) \ , \quad v(y,t) = u(s(t)y,t) \ , \tag{3.4}$$

problem (S II) becomes

$$s^{-2}a(v)v_{yy} + s^{-1}\dot{s}yv_{y} - v_{t} = q(v) \ , \quad \text{in } (0,1) \times (0,T) \ , \tag{3.5}$$

$$s(0) = b \ , \tag{3.6}$$

$$v(y,0) = \phi(by) \ , \quad 0 < y < 1 \ , \tag{3.7}$$

$$v_{y}(1.t) = s(t)\ f(t) \ , \ 0 < t < T \ , \tag{3.8}$$

$$v_{y}(1,t) = s(t)\ g(t) \ , \ 0 < t < T \ , \tag{3.9}$$

$$g(t)\ \dot{s}(t) = q(0) - a(0)\ s^{-2}(t)\ v_{yy}(1,t) \ , \quad 0 < t < T \ . \tag{3.10}$$

Thus, the first result we need is the following.

Lemma 3.1. Under the assumptions on a, q, ϕ, f listed in Section 2, there exists $\bar{T} \in (0,T)$ such that for any $g \in B_g(\varepsilon;g_0,g_1,G_\varepsilon,T)$, problem (3.5)-(3.10), i.e. Problem (S II), has one unique classical solution in $(0,\bar{T})$, $\bar{T}$ being estimated in terms of the data and of g_0, G_ε ; moreover $s \in H_{1+\varepsilon}([0,\bar{T}])$.

The proof of this lemma is also lengthy. First of all we regard s in (3.5)-(3.9) as a given function in the set

$$B_s(\varepsilon,b_1,S_0,S_\varepsilon,T) = \{s(t) \ : \ s \in H_{1+\varepsilon}([0,T])\}, \ \ \|s\|_1 \le S_0, \ \ \|s\|_{1+\varepsilon} \le S_\varepsilon$$

$$S(0) = b, \ \dot{s}(0) = c^{-1}(q(0) - a(0)\phi''(b)) \equiv s_0, \ 0 < b_1 \le s(t) \le 1-b_1, \ 0 \le t \le T$$

(of course b_1, S_0, S_ε must be consistent with the data) and we state

Proposition 3.1. There exists a $T_2 \in (0,T)$, depending on the data and on S_ε, such that (3.5)-(3.9) has one unique solution v(y,t) in $(0,T_2)$ for any

$s \in B_s(\varepsilon,b_1,S_o,S_\varepsilon,T)$. Moreover, in $(0,1) \times (0,T_2)$

$$\|v\|_{2+2\varepsilon} \leq N_o + N_1 T_2^{\varepsilon/2} \tag{3.11}$$

$$|v_{yyy}(y,t)| \leq N_2(1-y)^{-1+2\varepsilon} t^{-1/2+\varepsilon}, \tag{3.12}$$

where N_1, N_2 depend on the data and on the parameters entering B_g and B_s, while N_o depends on the data only.

An outline of the proof of Proposition 3.1 is given in the Appendix.

Going on in the proof of Lemma 3.1, we define

$$\dot{r}(t) = [g(0) - a(0)\, v_{yy}(1,t)/s^2(t)]\,/g(t)\ , \quad r(0) = b \tag{3.13}$$

and we consider the operator $\mathcal{T}$ such that

$$r = \mathcal{T}s\ . \tag{3.14}$$

Owing to (3.11)

$$\|\dot{r}\|_o \leq R_1(N_o,b_1,g_o) + R_2(S_o,b_1,g_o,N_1)\ T_2^{\varepsilon}\ . \tag{3.15}$$

Hence we can choose $S_o = S_o'$, depending on the data but not on G_ε, S_ε and $T_3 = T_3(G_\varepsilon,S_\varepsilon)$ such that in $(0,T_3)$

$$\|\dot{r}\|_o \leq S_o'\ .$$

Now we estimate

$$\|\dot{r}\|_\varepsilon \leq R_3\|v\|_{2+2\varepsilon} + R_4\|g\|_\varepsilon$$

with R_3 independent of G_ε, S_ε. Then (3.11) implies

$$\|\dot{r}\|_\varepsilon \leq R_5 + R_6(G_\varepsilon,S_\varepsilon)\ T_3^{\varepsilon/2}\ ,$$

thus defining a value S_ε' independent of G_ε and some $T_4 = T_4(G\ ) \in (0,T_3)$ such that in $(0,T_4)$

$$\|\dot{r}\|_\varepsilon \leq S_\varepsilon'\ .$$

Since T_4 can be reduced, if necessary, to make r fulfill the remaining properties of the functions of $B_1(\varepsilon,b_1,S'_o,S'_\varepsilon,T_4)$, we conclude that such a set is mapped by (3.14) into itself.

Moreover, given s_1, s_2 in such a set and denoting by v_1,v_2 the corresponding solutions of (3.5)-(3.9), from (3.13)-(3.14), we get

$$\|\mathcal{T}s_1 - \mathcal{T}s_2\|_1 \le N_3\|s_1-s_2\|_o + N_4\|v_{1yy}(1,\cdot) - v_{2yy}(1,\cdot)\|_o \ , \tag{3.15}$$

N_3, N_4 being constants of the same kind of N_1, N_2.

The estimate

$$\|v_{1yy}(1,\cdot) - v_{2yy}(1,\cdot)\|_o \le N_5\ T_4^{\varepsilon/2}\|s_1-s_2\|_1 \tag{3.16}$$

can be obtained considering the parabolic boundary value problem satisfied by the function $\zeta(y,t) = v_{1y}(y,t) - v_{2y}(y,t)$ and estimating $|\zeta_y(1,t)|$ by means of classical methods: in this step inequalities (3.11) and (3.12) have to be used. From (3.15) and (3.16) it follows that for some $\bar{T} > 0$ $\mathcal{T}$ is contractive w.r.t. the C_1-norm when acting on $B_s(\varepsilon,b_1,S_o,S'_\varepsilon,\bar{T})$. This concludes the proof of Lemma 3.1.

(II) Now we come back to the proof of Theorem 2.1. By virtue of Lemma 3.1, we can solve Problem (C) for any $g \in B_g(\varepsilon,g_o,g_1,G_\varepsilon,T)$, getting a pair (s,u) in $(0,\bar{T})$. As a consequence, we can find a function $U(x,t)$, $s(t) < x < 1$, $0 < t < \bar{T}$ solving (1.4)-(1.6) and (1.8) with the above specification of s. It will be convenient to use the transformation

$$y = (1-x)/(1-s(t)) \ , \quad V(y,t) = v[1-(1-s(t))y,t] \ , \tag{3.17}$$

to get

$$(1-s)^{-2}A(V)V_{yy} + (1-s)^{-1}\dot{s}yV_y - V_t = Q(V) \ , \quad \text{in } (0,1)\times(0,\bar{T}) \ , \tag{3.18}$$

$$V(y,0) = \Phi(1-(1-b)y) \ , \quad 0 < y < 1 \ , \tag{3.19}$$

$$V(0,t) = F(t) \ , \qquad 0 < t < \bar{T} \ , \tag{3.20}$$

$$V(1,t) = 0 \ , \qquad 0 < t < \bar{T} \ . \tag{3.21}$$

Setting

$$h(t) = \mathcal{F}_f(s,u,U,U_x) \tag{3.22}$$

we consider the operator

$$h = \mathcal{M} g, \tag{3.23}$$

acting on $B_g(\varepsilon, g_o, g_1, G_\varepsilon, \bar{T})$.

For any $\nu \in (0,1)$, using well known results (see e.g. [11], [12]), we obtain the following estimates

$$\|v\|_\nu \le M, \tag{3.24}$$

$$\|v\|_{1+\nu} \le M_2 , \tag{3.25}$$

$$\|v\|_{2+\nu} \le M_3 ; \tag{3.26}$$

here M_1, M_2, M_3 are determined by the data and by ν and $\|S\|_1$, M_3 depends also on S'_ε . Hence all these estimates do not depend on G_ε.

From (3.18), (3.21) we have

$$V_{yy}(1,t) = [1-s(t)]^2/A(0) \; \{Q(0) - (1-s(t))^{-1}\dot{s}(t) \; V_y(1,t)\} . \tag{3.27}$$

Using $V_y(1,t) \in H_{\nu/2}(0,\bar{T})$ and choosing $\nu = 2\varepsilon$, from the above remark and (3.11) it can be seen that (3.27) implies

$$\sup_{0<y<1} \|V_y(y,\cdot)\|_{1/2+\varepsilon} \le M_4 \tag{3.28}$$

for some $M_4 > 0$ independent of G_ε.

To obtain (3.28), one can use the classical heat-potential representation of the function $W = V_y$, which can be regarded as the solution of a linear parabolic equation in $(0,1) \times (0,\bar{T})$, with W_y specified at the boundary according to (3.27)_ this leads easily to estimate $\|V_y(1,\cdot)\|_{1/2+\varepsilon}$ independently of G_ε and estimating likewise $\|V\|_{2,1}$ implies (3.28) immediately.

By virtue of assumption (iv), from (3.21), (3.25) and (3.28) we get

$$\|h\|_{1/2+\varepsilon} \le G'_\varepsilon . \tag{3.29}$$

with G'_ε depending only on the data and on the coefficients.

Therefore, setting $G_\varepsilon = G'_\varepsilon$ we conclude that the operator $\mathfrak{M}$ maps $B_g(\varepsilon,g_o,g_1,G'_\varepsilon,\bar{T})$ into itself, where $\bar{T}$ is determined by the data (also through G'_ε) and ε, g_o, g_1, b_1.

(III) To show that $\mathfrak{M}$ is continuous, first we prove the following continuous dependence result for the solutions of Problem (C).

Lemma 3.2. Let (s_1,u_1), (s_2,u_2) the solutions of Problem (C) corresponding to two data $g^{(1)},g^{(2)} \in B_g(\varepsilon,g_o,g_1,G'_\varepsilon,\bar{T})$. Then

$$\|s_1-s_2\|_{1+\varepsilon} \le K_1(b+S'_o\bar{T})\,\|g^{(1)}-g^{(2)}\|_{1/2+\varepsilon}\ , \tag{3.30}$$

for some positive K_1 depending on the data.

The proof is based on the estimation of the norm $\|v_{1yy}(1,\cdot)-v_{2yy}(1,\cdot)\|_\varepsilon$, taking into account that the most important contribution comes from the $H_{1/2+\varepsilon}$ norm of the boundary value $v_{1y}(1,t)-v_{2y}(1,t) = s_1g^{(1)} - s_2g^{(2)}$, dominated by $g_1\|s_1-s_o\|_{1/2+\varepsilon} + (b+S'_oT)\|g^{(1)} - g^{(2)}\|_{1/2+\varepsilon}$. Details are omitted.

Next, we prove the following inequality.

Proposition 3.2. If V_1, V_2 denote the solutions to (3.18)-(3.21) corresponding to the boundaries $s_1,s_2 \in B_s(\varepsilon,b_1,S'_o,S'_\varepsilon,\bar{T})$, then

$$\sup_{0<y<1}\|V_{1y}(y,\cdot)-V_{2y}(y,\cdot)\|_{1/2+\varepsilon} \le K_2\,\|s_1-s_2\|_{1+\varepsilon}\ , \tag{3.31}$$

for some $K_2 > 0$, depending on the data.

The proof follows the same arguments used to obtain (3.28) (consider the problem satisfied by $V_{1y}-V_{2y}$ and, as a preliminary, note that $\|V_1-V_2\|_{1+\nu}$ is estimated in terms of $\|s_1-s_2\|_1$).

At this point, the continuity of $\mathfrak{M}$ follows from (3.30), (3.31) and (v) Section 2. Thus Schauder's fixed point theorem is applicable, yielding existence of at least one fixed point $g = \mathfrak{M}g$, i.e. of a local solution to (1.1)-(1.9).

APPENDIX

Proof of Proposition 3.1.

For any given $v \in C_{1+\alpha}([0,1] \times [0,T])$ satisfying (3.7), consider the linear problem

$$\left.\begin{aligned} &s^{-2}a(v)w_{yy} - (s^{-2}a'(v)v_y + s^{-1}\dot{s})w_y + s^{-1}\dot{s}w - w_t = q'(v)v_y \ , \\ &w(y,0) = b\phi'(by) \ , \\ &w(0,t) = f(t)\ s(t) \ , \\ &w(1,t) = g(t)\ s(t) \ , \end{aligned}\right\} \tag{A1}$$

obtained by formal differentiation of (3.5) w.r.t. y and replacing v_y by w where appropriate (here s is assumed to belong to $B_s(\varepsilon,b_1,S_o,S_\varepsilon,T)$ and g to the set $B_g(\varepsilon,g_o,g_1,G_\varepsilon,T)$.

By means of classical arguments it is easy to get

$$\|w\|_{(1+\varepsilon)/2} \le c_1 + c_2\ T^{\varepsilon/2} \ , \tag{A2}$$

where c_1 depends on the data and c_2 depends on s, on g and on $\|v\|_{1+\alpha}$.

This means that choosing $\alpha = (1+\varepsilon)/2$, some $T_1 > 0$ and $K > 0$ can be found such that in $[0,1] \times [0,1]$

$$\|w\|_\alpha \le K_1 \quad \text{if} \quad \|v\|_{1+\alpha} \le K \ . \tag{A3}$$

Furthermore the following inequalities can be seen to hold true, utilizing a classical representation for w in terms of heat potentials (all the norms are taken in $[0,1] \times [0,T_1]$).

$$\|w\|_{1+2\varepsilon} \le c_3 + c_4\ T_1^{\varepsilon/2} \ , \tag{A4}$$

$$|w_{yy}(y,t)| \le c_5\ t^{-1/2+\varepsilon}(1-y)^{-1+2\varepsilon} \ , \quad 0 < y < 1, \quad 0 < t \le T_1 \ , \tag{A5}$$

where c_3 depends on the data and c_4, c_5 depends on s, g, K_1.

Now define

$$\tilde{v}(y,t) = \phi(by) + \int_0^t [s^{-2}a(v)\ w_y + y_s^{-1}\dot{s}w - q(v)]d\tau \ , \tag{A6}$$

whence

$$\tilde{v}_y(y,t) = w(y,t) \quad . \tag{A7}$$

As a consequence of (A3), (A4), (A5), for any pair of functions v_1, v_2 satisfying the same conditions as the function v considered above, it is not difficult to estimate the difference w_1-w_2 of the corresponding solutions of (A1).

$$\|w_1 - w_2\|_0 \le c_6 T_1^{\varepsilon/2} \|v_1 - v_2\|_{1,0} \tag{A8}$$

From (A7) and (A8) we deduce that the mapping $v \to \tilde{v}$ on the set $\{v : v \in H_{1+\alpha}, \|v\|_{1+\alpha} \le K_1, \alpha = (1+\varepsilon)/2, v(y,0) = \phi(by)\}$ is contractive in the Banach space $C_{1,0}([0,1] \times [0,T_2])$ for a sufficiently small $T_2 \in (0,T_1)$, thus having one unique fixed point. Recalling once again (A7), (A4), (A5), we complete the proof of Proposition 3.1.

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A FRIEDMAN

Asymptotic estimates for variational inequalities

The regularity theory for solutions of variational inequalities and for their free boundaries is well understood at the present time. Thus attention in this field should be given to the study of qualitative properties for specific types of variational inequalities which arise in physics, engineering, economics, etc. The purpose of this talk is to draw attention to variational inequalities which arise in statistical problems and to questions of asymptotic estimates related to these problems.

We begin with the concept of a stopping time problem (for more details, see [1], [5]). Let w(t) be an n-dimensional Brownian motion and let $f(x,t)$, $\phi(x,t)$, $h(x)$ be given functions. Consider the cost function

$$J_{x,t}(\tau) = E_{x,t}\left\{\int_\tau^{T\wedge\tau} f(w(\lambda),\lambda)\,d\lambda + \phi(w(\tau),\tau)I_{\{\tau<T\}} + h(w(T)I_{\{\tau\geq T\}}\right\} \quad (1)$$

where $E_{x,t}$ denotes the expectation (when $w(t) = x$), I_A = characteristic function of a set A, and τ is a stopping time with respect to the σ-fields $\mathcal{F}_s^w$ generated by $w(\lambda)$, $t \leq \lambda \leq s$, i.e. $\{\tau \leq \mu\} \in \mathcal{F}_\mu^w$ for any $\mu \geq t$.

The problem of finding τ^* such that

$$J_{x,t}(\tau^*) = \min_\tau J_{x,t}(\tau)$$

is called a *stopping time problem*.

Under suitable assumptions on f, ϕ, h there exists a unique solution u of the parabolic variational inequality

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\Delta u + f \geq 0 \\ u \leq \phi \\ \left(\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u + f\right)(u-\phi) = 0 \end{aligned}\right\} \text{ a.e. in } R^n \times (0,T) \quad (2)$$

$$u(x,T) = 0 \text{ if } x \in R^n .$$

By general results [1][5],

$$u(x,t) = \inf_{\tau} J_{x,t}(\tau) = J_{x,t}(\tau^*)$$

where τ^* is the first time $(w(s),s)$ hits the set

$$S = \{(y,s) \in R^n \times [0,T];\ u(y,s) = \phi(y,s)\}.$$

S is called the *stopping set* and its complement in $R^n \times [0,T]$ is called the *continuation set.*

Some problems in economics (stock options, etc.) can be modelled as stopping time problems; see [8] and the references given there.

The above connection between stopping time problems and variational inequalities extends to general diffusion Markov processes $w(t)$. In many instances however τ is not a stopping time with respect to the entire evolving process $w(t)$; it may depend in fact only on observations of one or several components of $w(t)$, or on the observation of $w(t)$ in some random time intervals. In such situations of *partial observation* there is no connection, in general, between the stopping time problem and a variational (or quasi-variational) inequality.

We shall now describe a problem of partial observation arising in sequential analysis which can be reduced to a variational inequality.

The evolving process which we observe is a 1-dimensional process $z(t)$:

$$z(t) = y + \sigma w(t)$$

where $w(t)$ is a 1-dimensional Brownian motion and $\sigma > 0$; y is a random variable with normal distribution $N(\mu_0,\sigma_0^2)$, independent of the Brownian motion. Let $W(y,\delta)$ be a function defined for y real and $\delta = \pm 1$ as follows:

$$W(y,\delta) = \begin{cases} k|y| & \text{if } \delta = 1,\ y < 0 \text{ or if } \delta = 2,\ y > 0, \\ 0 & \text{in all other cases; } k > 0\ . \end{cases}$$

Consider the cost function

$$J(\tau,\delta) = E\left\{ \int_0^{\tau} c\,d\lambda + W(y,\delta(\omega))\right\} \qquad (c > 0)\ .$$

Denote by $\mathcal{F}_\lambda^z$ the σ-field generated by $z(s)$, $s \le \lambda$. We take τ to be a stopping time with respect to $\mathcal{F}_\lambda^z$ and $\delta = \delta(\omega)$ to be a random variable measurable with respect to $\mathcal{F}_\tau^z$ (that is, the σ-field generated by the sets A with the property that $A \cap \{\tau \le \lambda\}$ is in $\mathcal{F}_\lambda^z$, for any $\lambda > 0$).

The cost function $J(\tau,\delta)$ represents the cost incurred when we are faced with the following situation:

We do not have direct knowledge of $y = y(\omega)$; we observe it only indirectly by sampling $z(t) = z(t,\omega)$. Yet we must determine whether $y(\omega) > 0$ (we call this the hypothesis H_1) or whether $y(\omega) < 0$ (the hypothesis H_2). The cost of sampling $z(t)$ is c per unit time and the cost for accepting the wrong hypothesis is given by $W(y,\delta)$ ($\delta = 1$ corresponds to accepting H_1 and $\delta = -1$ corresponds to accepting H_2); W is called the *risk function*.

It is intuitively clear that if $z(\tau(\omega)) > 0$ then we should accept H_1 and if $z(\tau(\omega)) < 0$ then we should accept H_2. Thus the function to be minimized is

$$\tilde{J}(\tau) = E\left[\int_0^\tau c\,dt + \tilde{W}(\tau(\omega))\right]$$

where $\tilde{W}(t)$ can be computed explicitly.

The process $z(t)$ is not a Markov process, but $(z(t),y)$ is a Markov process. Since τ is taken to be any $\mathcal{F}_t^z$-stopping time, we have here an instance of a problem of partial observation of a type mentioned above.

To study such a problem one introduces the *filtered process*

$$\hat{y}(t) = E[y\,|\,\mathcal{F}_\tau^z]$$

obtained by conditioning the unobserved component on the observed σ-fields. The general filtering theory gives some structure for the filter, but this is, in general, insufficient to reduce the stopping time problem to a standard stopping problem related to a variational inequality.

In the present case however, the fact that $z(t)$, y, $w(t)$ are related in a linear fashion enables us to apply the Kalman-Bucy theory which shows that $\hat{y}(t)$ satisfies a simple stochastic differential equation. Furthermore, after making a change of variables

$$s = \frac{1}{t/\sigma^2 + 1/\sigma_0^2},$$

we find that $\tilde{y}(s) = \hat{y}(t)$ is a Brownian motion.

Expressing the term $\tilde{W}$ in $\tilde{J}$ in terms of $\tilde{y}$ we are then led to a standard stopping time problem with respect to the Brownian motion $\tilde{y}(s)$. The corresponding variational inequality is:

$$\left.\begin{aligned} & u_s - \frac{1}{2} u_{xx} + f \geq 0 \\ & u \leq \Psi(x,s) \\ & (u_s - \frac{1}{2} u_{xx} + f)(u - \Psi) = 0 \end{aligned}\right\} \quad \text{a.e. in } R^1 \times (0,\infty) \tag{3}$$

$$u(x,0) = 0 \quad \text{if} \quad x \in R^n,$$

and

$$u(\mu_o,\sigma_o^2) = \inf J(\tau,\delta) \ .$$

Here

$$f = \frac{c\sigma^2}{s^2} ,$$

$$\Psi(x,s) = k \min \{\Psi_1(x,s), \Psi_2(x,s)\} ,$$

where

$$\Psi_1(x,s) = s^{1/2}\Psi(x/s^{1/2}) ,$$

$$\Psi(u) = \phi(u) + u \int_{-\infty}^{u} \phi(\lambda)\, d\lambda \qquad \text{if } u < 0 ,$$

$$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} ,$$

$$\Psi(-u) = \Psi(u) \qquad\qquad \text{if } u < 0 ,$$

and $\Psi_2(x,s) = \Psi_1(x,-s)$, i.e. $\Psi(x,s)$ is even in s.

The main interest here is the shape and location of the free boundary; for this will tell the experimenter when to stop sampling. Since large time t translates into small s, there is a particular interest in the asymptotic behavior of the free boundary as $s \to 0$. In fact, Breakwell and Chernoff [2] have derived an asymptotic expansion for $x = \pm\zeta(s)$ (the two free boundaries):

$$\zeta(s) = \frac{s^2}{4\alpha} - \frac{2s^5}{96\alpha^2} + c_3 \frac{s^8}{\alpha^3} + \ldots$$

(they computed the first five coefficients).

Motivated by this example we now state a general theorem which applies, for instance, in the case of three hypotheses, or two non-symmetric hypotheses. This theorem gives an estimate not only on the free boundaries $x = \zeta_i(s)$, but also on their derivatives. To be specific, let us assume that u satisfies (3) and that $\Psi = \min(\Psi_1, \Psi_2)$; the functions f, Ψ_i satisfy

$$f(x,s) \sim \alpha s^{-p} \qquad \text{as } s \to 0 \ (p > 1),$$

$$(\Psi_1 - \Psi_2)_x = -1 + o(1) \qquad \text{as } s \to 0,$$

$$(\Psi_1 - \Psi_2)_s \sim 0, \quad \Psi_1(0,0) = \Psi_2(0,0) = 0,$$

and, near (0,0),

$$(L\Psi_1 - f)_x \le 0, \quad (L\Psi_2 - f)_x \ge 0, \quad (L\Psi_2 - f)_s \ge 0 \ ;$$

a few other technical but rather general and mild conditions are further assumed. Then we have:

<u>Theorem</u>. The free boundary near $s = 0$ consists of two curves $x = \zeta_i(s)$, and

$$\zeta_i(s) = \frac{s^p}{4\alpha} + (-1)^i \frac{ps^{3p-1}}{96\alpha^2} + O(s^{4p})$$

$$\zeta_1'(s) = \frac{ps^{p-1}}{4\alpha} + O(s^{3p-2}) \ ,$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial s}(\Psi_2(x,s) - u(x,s)) = \frac{p}{2} s^{p-1} + O(s^{3p-2}) \quad \text{if } \zeta_2(s) \le x \le \zeta_1(s).$$

The proof will appear in [7]. The method of proof is applicable to other problems in one space dimension. It uses techniques of variational inequalities as well as the integral equation approach to the Stefan problem.

For other sequential analysis problems related to free boundary problems, see Chernoff [4] and the references given there. Variational inequalities occur also in testing simple hypotheses; here one encounters degenerate elliptic operators; see [3]. Other statistical settings for free boundary problems are mentioned in [6].

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K P HADELER

Free boundary problems in biological models

In many biological models reaction-diffusion equations play an important role ([5],[6]), typically the initial-boundary value problem describes the time evolution of the biological system. However, in cases where diffusion rates or source terms show discontinuities, a description by free boundary value problems may be more appropriate. In the following it is shown how two important problems, the multispecies interaction-diffusion model and the problem of travelling fronts, have their analogues in free boundary value problems.

I. MULTISPECIES SYSTEM

Suppose two species u_1 and u_2 are interacting according to a system of ordinary differential equations

$$\dot{u}^1 = f_1(u^1,u^2), \quad \dot{u}^2 = f_2(u^1,u^2) .$$

The interaction can be competition, a predator-prey or a more general relation. Both species can migrate, possibly with different rates, in a spatial domain. Migration is modeled by diffusion terms,

$$u^1_t = f_1(u^1,u^2) + D_1 u^1_{xx}$$

$$u^2_t = f_2(u^1,u^2) + D_2 u^2_{xx} .$$

Suppose the spatial domain has a fixed boundary, where the populations are either isolated or connected to reservoirs, and a free boundary, where the boundary is determined by the interacting populations themselves. For instance, let u_1 describe the density of plants and u_2 the density of herbivores, and let the free boundary be given by the extent of the plant cover. The plant cover is extended by the spread of plants by seeds and off-shoots, and its boundary is pushed back by the action of the herbivores (modified by climatic factors). It is assumed that the herbivores do not leave the plant covered area. Typical boundary conditions are

$$u^1(0,t) = \bar{u}_1 , \quad u^2(0,t) = \bar{u}_2$$

$$u^1(s(t),t) = 0 , \quad u^2_x(s(t),t) = 0$$

$$\dot{s}(t) = -c_1 u_x^1(s(t),t) - c_2 u^2(s(t),t) .$$

For most applications in biology, chemistry a.s.o., where the interaction terms are given by, say, quadratic polynomials a global linear bound on the source term is not appropriate. It is convenient to infer the theory of invariant rectangles ([11], [3], [1]). A convex set $M \subset R^n$ is called positively invariant with respect to the vector field f, if for any point $u \in \partial M$ and any outer normal p at u the condition $pf(u) \leq 0$ is satisfied. A convex set M is a rectangle if

$$M = \{u: \underline{m}_j \leq u_j \leq \bar{m}_j, \quad j=1,\dots,n\}$$

with some finite numbers $\underline{m}_j < \bar{m}_j$.

We formulate a free boundary value problem for vector-valued functions. In the (x,t)-plane the (unknown) free boundary is described by a function $s = s(t)$, $0 < t \leq T$, starting at

$$s(0) = b > 0 . \tag{1.1}$$

In the domain $\Omega_T = \{(x,t) : 0 < t < T, 0 < x < s(t)\}$ the vector $u = (u^1,\dots,u^n)$ satisfies a semilinear diffusion equation

$$u_t = Du_{xx} + f(x,u) . \tag{1.2}$$

Here D is the positive diagonal matrix of diffusion coefficients. As the fixed boundary $x = 0$ there is a boundary condition

$$u(0,t) = \psi_0 \tag{1.3a}$$

or

$$-u_x(0,t) + B_0 u(0,t) = g_0(u) + B_0\psi_0 . \tag{1.3b}$$

Similarly, at the free boundary there is a boundary condition

$$u(s(t),t) = \psi_1 \tag{1.4a}$$

or

$$u_x(s(t),t) + B_1 u(s(t),t) = g_1(u) + B_1\psi_1 . \tag{1.4b}$$

The matrices B_o, B_1 are positive diagonal matrices.

At this boundary there is an additional condition which relates the displacement of the free boundary to the values of u and u_x,

$$\dot{s}(t) = \varphi(u(s(t),t),\ u_x(s(t),t)) \ . \tag{1.5}$$

There is an initial condition

$$u(x,0) = h(x) \ , \qquad 0 < x < b \ . \tag{1.6}$$

The following invariance properties are required. There is a rectangle M which is invariant with respect to the vector field $f(x,.)$ for all $x \geq 0$, and also with respect to the vector fields g_o, g_1.

The function h assumes its values in M,

$$h : [0,b] \to M \ , \tag{1.7}$$

and

$$\psi_o,\ \psi_1 \in M \ . \tag{1.8}$$

If D=I and if the boundary conditions for the components u^j are of the same type then the rectangle M can be replaced by a general bounded convex set. The vectors ψ_o, ψ_1 can be allowed to depend on the time variable. The symmetric formulation of the differential equation and the boundary conditions follows [7], [10].

With this hypothesis and appropriate smoothness conditions on h, ψ_o, ψ_1 local existence and uniqueness of a classical solution can be shown in much the same way as in the papers by Fasano and Primicerio [4], essentially via the contraction mapping principle. In fact, in the case of Dirichlet boundary data, the proof is verbally the same.

II. TRAVELLING FRONTS

This part of the paper extends the results of [9]. Consider a diffusion equation

$$u_t = \frac{1}{m}(ku_x)_x + f \ . \tag{2.1}$$

The function u can be interpreted as temperature, the concentration of some substance or a population density. The coefficient k is the diffusion rate (or heat conductivity), the term f describes a source, and m is a capacity (product of density and specific heat, respectively).

In the homogeneous case $f = 0$, on an isolated space interval, the equation

$$\frac{d}{dt} \int m\ u\ dx = 0 \tag{2.2}$$

represents the appropriate conservation law.

In the following we assume that k,f and m do not explicitly depend on x and t, but are functions only of u itself. Let $k, f, m \in C^1$ $0,1$,

$$\begin{aligned} &k(u) > 0, \quad m(u) > 0 && \text{for } 0 \le u \le 1 \\ &f(0) = f(1) = 0\ , && f'(0)f'(1) \neq 0 \end{aligned} \tag{2.3}$$

and either (type 1)

$$f(u) > 0 \qquad \text{for } 0 < u < 1, \tag{2.4}$$

or (type 2) for some $\alpha \in (0,1)$

$$f(u) < 0 \qquad \text{for } 0 < u < \alpha, \tag{2.5}$$

$$f(u) > 0 \qquad \text{for } \alpha < u < 1, \tag{2.5}$$

$$f'(\alpha) > 0\ .$$

A travelling front of equation (2.1) is a solution of the form

$$u(x,t) = \phi(x-ct) \tag{2.6}$$

which is subject to the following conditions

$$0 \le \phi(x) \le 1, \tag{2.7}$$

$$\phi(-\infty) = 1\ , \quad \phi(+\infty) = 0\ . \tag{2.8}$$

The function ϕ describes the shape of the front, and the number c is the speed.

For a travelling front the function ϕ satisfies the ordinary differential equation

$$\frac{1}{m(\phi)}\ (k(\phi)\ \phi_x)_x + c\phi_x + f(\phi) = 0 \tag{2.9}$$

If we put $u = \phi$, $v = \phi_x$, and denote the independent variable again by t, then an equivalent first order system obtains,

$$\dot{u} = v$$

$$\dot{v} = -\frac{k'(u)}{k(u)} v^2 - \frac{cm(u)}{k(u)} v - \frac{m(u)}{k(u)} f(u) \quad . \tag{2.10}$$

Define the function $h : [0,1] \to [0,1]$ by

$$h(u) = \frac{1}{\varkappa} \int_0^u m(y)\,dy \tag{2.11}$$

where

$$\varkappa = \int_0^1 m(y)\,dy \ .$$

Let h^{-1}: $[0,1] \to [0,1]$ denote the inverse function. Then define new dependent variables

$$\tilde{u} = h(u)$$

$$\tilde{v} = \frac{1}{\varkappa} k\ (u)\ v \tag{2.12}$$

a new source function

$$\tilde{f}(\tilde{u}) = \frac{1}{\varkappa} f\ (h^{-1}(\tilde{u}))k(h^{-1}(\tilde{u})) \tag{2.13}$$

and a new time variable

$$\tau = \int_0^t \frac{m(u(s))}{k(u(s))}\ ds \quad . \tag{2.14}$$

Then the system (2.10) assumes the form

$$\frac{d\tilde{u}}{d\tau} = \tilde{v}$$

$$\frac{d\tilde{v}}{d\tau} = -\,c\tilde{v} - \tilde{f}(\tilde{u}) \ . \tag{2.15}$$

We have shown that the general system (2.10) can always be transformed in such a way that the system assumes the special form where $k=1$, $m=1$. A related transformation has been used by Aronson [2] for the porous media equation.

The partial differential equation (2.1) does not allow such a simplification. In the following we shall drop the tildes in (2.15).

Earlier we have shown the following results. If the function f in (2.15) is of type 1, then there is a spectrum $[c_o,\infty)$ of speeds of travelling fronts ([8]). The minimal speed c_o can be characterized by two complementary variational principles [9], which bear some similarity with the variational principles for eigenvalues. In the first problem

$$c_o = \inf_g \sup_u \left\{ g'(u) + \frac{f(u)}{g(u)} \right\} \tag{2.16}$$

where $g \in C^1[0,1]$, and $g(u) > 0$ in $(0,1)$, $g(0) = 0$, $g'(0) > 0$. In the second problem

$$c_o = \sup_g \inf \left[\inf_u \left\{ g'(u) + \frac{f(u)}{g(u)} \right\}, 2g'(0) \right] \tag{2.17}$$

where $g \in C^1[0,1]$, and $g(u) > 0$ in $(0,1)$, $g(0) = g(1) = 0$, $g'(0)g'(1) < 0$. The second term in (2.16) is essential.

If the function f in (2.15) is of type 2 then there is a unique speed of a travelling front c_o, and

$$c_o = \left\{ \begin{matrix} \inf & \sup \\ \sup_g & \inf_u \end{matrix} \right\} \left\{ g'(u) + \frac{f(u)}{g(u)} \right\} \tag{2.18}$$

where $g \in C^1[0,1]$, and $g(0) = g(1) = 0$, $g'(0)g'(1) < 0$, $g(u) > 0$ for $0 < u < 1$.

From these variational principles one can derive (see [9]) the following result on the general equation (2.1).

<u>Theorem</u> For functions f of type 1 and of type 2 the speed of c_o depends continuously on the *product* kf with respect to the weighted supremum norm

$$\| p \| = \sup_{0<u<1} \frac{|p(u)|}{u(1-u)} \tag{2.19}$$

The concept of travelling fronts can be carried over to free boundary problems. Let the free boundary be given a function $s = s(t)$. For $x > s(t)$ assume the diffusion equation (2.1), where the functions k and m have the same properties as in (2.3), and the function $f \in C^1[0,1]$ satisfies $f(0) = 0$, and either (type 1)

$$f(u) > 0 \text{ for } 0 < u < 1, \quad f'(0) > 0 \tag{2.20}$$

or (type 2) for some $\alpha \in (0,1)$

$$f(u) < 0 \text{ for } 0 < u < \alpha, \quad f(u) > 0 \text{ for } \alpha < u \le 1, \tag{2.21}$$

$$f'(0) < 0, \qquad f'(\alpha) > 0 .$$

At the free boundary two conditions are given,

$$u(s(t),t) = 1 \tag{2.22}$$

$$u_x(s(t),t) = 0 . \tag{2.23}$$

To make the analogy with the regular problem more obvious, one can extend the fucntion u by

$$u(x,t) = 1 \quad \text{for } x \le s(t) .$$

Then a travelling front is a solution to the free boundary value problem $u(x,t) = \phi(x-ct)$ which satisfies (2.7) and (2.8). Again ϕ is the shape and c the speed.

Thus the shape function ϕ is identically 1 for $x < s(t)$ (simulating instantaneous switch off of the production term at $u = 1$) and it is a solution to (2.1) for $x > s(t)$. These two functions are joined together by the continuity conditions (2.22), (2.23).

One can apply the transformation (2.12) and reduce the problem to the case $k = 1$, $m = 1$. For a source function f of type 1 there is a half-line $[c_o,\infty)$ of speeds, in the case of type 2 there is a single speed c_o. In both cases c_o can be characterized by the variational principles (2.16), (2.17), (2.18), respectively, where the function $g \in C^1[0,1]$ satisfies $g(0) = 0$, $g'(0) > 0$, $g(u) > 0$ for $0 < u < 1$, and furthermore $g(1) = 0$ in the cases of (2.17), (2.18).

From these results one can derive that c_o depends continuously on the product of the functions k and f with respect to the norm

$$\|p\| = \sup \frac{p(u)}{u\sqrt{1-u}} \; .$$

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J MOSSINO & R TEMAM

Free boundary problems in plasma physics: review of results and new developments

INTRODUCTION

In this lecture, we intend to review results concerning some free boundary problems related to plasma fusion, and more precisely to the plasma confinement in a Tokomak machine. The following is a simplified, but typical model: to find a function $u : \Omega \to \mathbb{R}$ (Ω regular open set in $\mathbb{R}^2$ or $\mathbb{R}^N$, $N \geq 2$) u "regular enough" such that

$$\left.\begin{array}{l} - \Delta u + \left\{\begin{array}{ll} 0 & \text{if } u \geq 0 \\ f(x,u) & \text{if } u < 0 \end{array}\right\} = 0 \quad \text{in} \quad \Omega, \\[2ex] u = \gamma \ (\text{unknown constant} > 0) \text{ on } \partial\Omega, \\[2ex] \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n} = I > 0 \text{ given.} \end{array}\right\} \tag{1}$$

The Tokomak machine has the shape of an axisymmetric torus. The plasma is confined away from the boundary of the torus by poloïdal and toroïdal magnetic fields. Here Ω represents the cross section of the Tokomak, u is the flux function. The set $\{u > 0\} = \{y \in \Omega | u(y) > 0\}$ represents the empty region (no plasma), while $\{u < 0\}$ is the region occupied by the plasma, and $\{u = 0\}$ is the free boundary.

Formulation (1) is derived from the Magnetohydrodynamics (M.H.D.) equations ("plasma region") and the Maxwell equations ("empty region"). For a complete derivation of (1), the reader is referred to C. Mercier [17], R. Temam [30] (appendix) and [32].

The term f(x,u) represents the derivative $\frac{dp}{du}$ where p is the pressure inside the plasma, which depends only on u (consequence of the M.H.D. equations). The exact expression of f is unknown, this seems to be a difficult question from the physical point of view. There exists two kinds of models which are proposed:

i) in the first one, f depends on u in a local way

$$f(x,u) = g(x,u(x)) \ , \tag{2}$$

where g is considered as a given function (for example $g(x,t) = \lambda t$ (λ real parameter) which does not depend on x). This problem has now been studied by several authors. The principal aspects are recalled in Section 1.

ii) A more sophisticated model is proposed by H. Grad [11] (see also [32]): f depends on u in a non local way. Assuming that the fluid is adiabatic, a totally different functional dependence of p in term of u is considered which leads to

$$f(x,u) = g(x,u(x),S(u(x)), \frac{du}{ds}(S(u(x))), \frac{d^2u}{ds^2}(S(u(x)))) \tag{3}$$

where S(t) is the area (N = 2) of the level set $\{u < t\}$ (i.e. the distribution of u) and $S \to u(S)$ is an improper notation of the inverse function ($u(S)=t(S)$). Such an f was introduced independently by C. Mercier, sometime after, in the study of the behaviour as $t \to \infty$ of the time dependent magneto-hydrodynamic equations. We describe the two different approaches in Section 2, including recent developments and open problems.

I. THE CASE "f LOCAL"

For this model problem several questions have been solved.

I.1. Existence and uniqueness of solutions

The first case which was considered was $f(x,u) = \lambda u(x)$ (N = 2). Then, for $\lambda > \lambda_1(\Omega)$ = first eigenvalue of the Dirichlet operator in Ω, there exists variational solutions (satisfying $\gamma > 0$), belonging to $W^{3,\alpha}(\Omega)$ and $\mathcal{C}^{2,\eta}(\bar{\Omega})$ for every α, $1 \leq \alpha < \infty$, and every η, $0 \leq \eta < 1$ (see for example R. Temam [31], C. Guillopé [12]). A numerical computation of these solutions is described in [16]. The variational solutions introduced in [31] are also the solutions of a dual optimization problem considered by Berestycki - Brezis (see [4] and [7]).

As for uniqueness, it is shown in [31] that if $\lambda < \lambda_2(\Omega)$ = second eigenvalue of the Dirichlet operator, the solution is unique (for N > 2 see J. P. Puel [23]). This result is almost optimal since D. Schaeffer [24] has shown that there exists Ω("hourglass-shaped") and λ, with $\lambda_2(\Omega) < \lambda < \lambda_3(\Omega)$, for which the problem has at least three solutions. For a complete bifurcation analysis, as well as numerical computations, in the case of the "hourglass-shaped" Ω, see M. Sermange [25] and [26]. An analytical construction (and a computation) of "bifurcated solutions" is also given by J. Sijbrand [27].

These existence and uniqueness results have been extended to more general local nonlinearities g by R. Temam [30], A. Ambrosetti and G. Mancini [2], who used bifurcation theory with $\gamma = u/_{\partial\Omega}$ as a bifurcation parameter, and by G. Keady - J. Norbury [13]. Another extension of the term $\lambda u_-(x)$ (= 0 if $u(x) \geq 0$, $= \lambda u(x)$ if not) which is also valid for some non local f's, has been recently studied by R. Cipolatti [6] who considers more general boundary conditions. Replacing λu_- by $\lambda P_K(u)$, where P_K is the projection mapping on a "general" closed convex set K in $L^2(\Omega)$, he solves the questions of existence and uniqueness, and gives numerical schemes with error estimates.

I.2. Shape and smoothness of configurations, "a priori" estimates (case $f(x,u) = \lambda u(x)$).

Now, we review some recent qualitative results.

When u is a solution of the variational problem defined in [31], the "plasma set" $\{u < 0\}$ was shown to be connected (see L. E. Fraenkel, M. S. Berger [8], and D. Kinderlehrer, J. Spruck [15]). The free boundary $\{u = 0\}$ is an analytic manifold around a regular point ($N \geq 2$) and is globally analytic if $N \geq 2$ (see D. Kinderlehrer, L. Nirenberg, J. Spruck [15] and [14]). If Ω is convex, J. Spruck [28] recently gave a constructive process leading to a solution which has convex level sets $\{u < t\}$. In particular there exists a solution for which the "plasma set" is convex (see also A. Acker [1]).

The asymptotic behaviour ($\lambda \to \infty$) of the configuration of the plasma has been studied by L. A. Caffarelli, A. Friedman [5], L. E. Fraenkel, M. S. Berger [8], G. Keady, J. Norbury [13].

Inequalities of isoperimetric type for the variational solutions were obtained by T. Gallouet [9], C. Bandle and R. Sperb [3]. In particular, it is proved in [3] that, among all domains $\Omega \in \mathbb{R}^2$ of given area, the boundary value of a variational solution u is greatest for the circle, and the area of $\{u < 0\}$ achieves its minimum in that case. Using a maximum principle, I. Stakgold [29] gave a punctual gradient bound (bound for $|\nabla u(x)|$) and derived an information about the location of the minimum of u.

II. THE FUNCTIONAL EQUATION OF GRAD AND MERCIER (A "non local" f)

There exists two different approaches to treat that new kind of non linearities.

II.1. First approach: the operator β

In the first approach, the term S in (3) is considered as a non linear operator which associates to every (measurable) function u defined on Ω a real (measurable) function $\underline{\beta}(u)$ defined on Ω:

$$\begin{aligned}\underline{\beta}(u)(x) &= \text{meas}\{y \in \Omega | u(y) < u(x)\} = \text{meas}\{u < u(x)\} \\ &= |u < u(x)|\end{aligned} \tag{4}$$

This is a (non linear) non local, non continuous and non monotone operator. One may also consider the operator $\overline{\beta}$:

$$\overline{\beta}(u)(x) = |u \le u(x)|, \tag{5}$$

or the multivalued operator β:

$$\beta(u)(x) = [\underline{\beta}(u)(x), \overline{\beta}(u)(x)] . \tag{6}$$

Of course, if u does not possess any flat region, $\underline{\beta}(u)$, $\overline{\beta}(u)$ and $\beta(u)$ coincide. These operators have been studied in [18], where the continuity of the multivalued β is proved, and the problem

$$\left.\begin{aligned}& - \Delta u = g(x,\underline{\beta}(u)(x)) \text{ in } \Omega, \\ & u = 0 \text{ on } \partial\Omega\end{aligned}\right\} \tag{7}$$

is solved, in particular with a fixed point theorem of multivalued analysis. Using some additional arguments, the same method was applied in [10] to solve the free boundary problem (1), with

$$\begin{aligned}f(x,u) &= \lambda g(\delta(u)(x)), \\ \delta(u)(x) &= \text{meas}\{y \in \Omega | u(x) < u(y) < 0\} \\ &= |u(x) < u < 0| ,\end{aligned}$$

g positive, continuous, vanishing (only) at the origin. Then, problem (1) was written

$$\left.\begin{aligned}&- \Delta u + \lambda g(\delta(u)) = 0 \text{ in } \Omega \subset \mathbb{R}^N \ (N \geq 1),\\ &u|_{\partial\Omega} = \gamma \text{ (unknown constant) } > 0,\\ &\int_{\partial\Omega} \frac{\partial u}{\partial n} = I \ ,\end{aligned}\right\} \qquad (1')$$

and the existence result was (see [10]) :

- (a) There exists (at least one) u (in $W^{2,p}(\Omega)$ for any p) if and only if

$$\lambda > \frac{I}{G(|\Omega|)} \text{ , where}$$

$$|\Omega| = \text{meas } \Omega, \ G(s) = \int_0^s g(t)dt \ . \qquad (8)$$

- (b) The measure of the "plasma set" is given by

$$0 < |u < 0| = G^{-1}(\frac{I}{\lambda}) < |\Omega| \ . \qquad (9)$$

Nothing is known about the multiplicity of solutions (which is expected by the physicists) or the smoothness of the free boundary.

II.2. Second approach: S as an independent variable

In a second approach, introduced in [32], S is considered as an independent variable, and the "function $S \to u(S)$" which was considered in the definition (3) is actually u*, which is called (cf. G. Polya - G. Szego [22]) the one dimensional monotone increasing rearrangement of u. The definition of u* is the following one: for u given, the function $t \to \eta_u(t) = |u < t|$, is a well defined increasing function from $\mathbb{R}$ into $[0,|\Omega|]$. Then

$$u^*(S) = \text{Inf}\{\xi \in \mathbb{R} | \eta_u(\xi) > S\} \ . \qquad (10)$$

One can define (at least in the sense of distributions) $\frac{du^*}{dS}$, $\frac{d^2u^*}{dS^2}$, ..., and consider problem (1) with

$$f(x,u) = g(x,u(x),\underline{\beta}(u)(x), \frac{du^*}{dS}(\underline{\beta}(u)(x)), \frac{d^2u^*}{dS^2}(\underline{\beta}(u)(x))) \qquad (11)$$

Up to now, it seems to be very difficult to solve completely this problem, if g depends on the derivatives $\frac{du^*}{ds}$, $\frac{d^2u^*}{ds^2}$. In [32], R. Temam solved heuristically model problem

$$\left.\begin{aligned} & -\ \Delta u(x) - \frac{d^2u^*}{ds^2}(\underline{\beta}(u)(x)) = f(x) \ \text{in } \Omega, \\ & u = 0 \ \text{on } \partial\Omega, \end{aligned}\right\} \tag{12}$$

using a variational method.

This second approach, related to the rearrangement of functions, appeared to be very fruitful. Now we will present in Sections II.2.1 and II.2.2 *two recent developments* in that area.

II.2.1. A priori estimates for problem (1')

The results of this section will appear in [20]. Some of them were announced in [19]. The estimates concern any solution u (in $W^{2,p}(\Omega)$, $\forall$ p) of (1') (not only the variational ones) and any $N \geq 1$. They are of two kinds:

- pointwise estimates on u
- pointwise estimates on ∇u.

The estimates on u are obtained by rearrangement techniques only. They are optimal. In particular, we have

Theorem 1 Among all solutions $u(\Omega)$, for all domains $\Omega \subset \mathbb{R}^N$ $(N > 1)$ of given area, the boundary value and the "energy" $\int_\Omega |\nabla u|^2$ (resp. the minimum value) are greatest (resp. smallest) for the symmetrical solution in the ball.

These optimal bounds (= a priori estimates) are explicitly given in [20] (see also [19]). In fact the symmetrical solution is given explicitly.

Using a maximum principle, and refining some arguments already used by I. Stakgold [29], we get, with Ω convex $(N \geq 1)$, "regular",

$$\forall\, x \quad \bar{\Omega},\ \frac{1}{2}\,|\nabla u(x)|^2 \leq \lambda \int_m^{u(x)} g(\delta(t))dt, \text{ with } \delta(t) = |t<u<0|$$

where $m = \min\limits_{\bar{\Omega}} u$. The equality is achieved in dimension 1. Using the punctual estimates on u, we can majorize the previous integral, and obtain an upper bound for the minimum of the solutions, and lower bounds for the distance from the free boundary to $\partial\Omega$, or to a point where u is minimum.

II.2.2. A new variational approach

The following results will appear in [21]. They are a step in the understanding of the more complicated variational problems presented in [32] (see also [11]). A crucial point in the resolution of these problems appeared to be the computation of the directional derivatives of the rearrangement mapping $u \to u^*$, a result which does not seem to be contained in the literature. We have (see [21])

<u>Theorem 2</u> Let u be a measurable function from Ω into $\mathbb{R}$, and let v be in $L^\infty(\Omega)$. The directional derivative of the rearrangement mapping, at the point u, in the direction v, exists in $L^\infty(0,|\Omega|)$ weak-star. It is given by $\frac{dw}{ds}$, where w is the Lipschitz function

$$w(S) = \int_{u<u^*(S)} v\, dx \quad \text{if } |u = u^*(S)| = 0$$

(If $|u = u^*(S)| \neq 0$, w has a more complicated expression, which is also given in [21]).

This theoretical result is applied to get the Euler equation of a variational problem involving u^*:

$$\operatorname*{Inf}_{u\in H_0^1(\Omega)} \left\{\frac{1}{2}\int_\Omega |\nabla u^2|dx - \int_\Omega f\, u\, dx + \int_0^{|\Omega|} \mu(S,u^*(S))ds\right\} \qquad (\mathcal{P})$$

Doing this, it is proved that, in some sense, the equation

$$\left.\begin{aligned} &- \Delta u + g(\underline{\beta}(u),u) = f \text{ in } \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}\right\} \qquad (13)$$

is variational on the Sobolev space $H_0^1(\Omega)$, i.e. is the Euler equation of a variational problem ($\mathcal{P}$) (take $\mu(S,T) = \int_0^T g(S,t)dt$. Hence the existence of (variational) solutions of (13) is proved, without any assumption of continuity on g with respect to the first variable. This generalizes a result of [18], where continuity was an essential assumption to apply the fixed point method. More precisely (see [21]),

<u>Theorem 3</u> Let $g : (0,|\Omega|) \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function which satisfies

$$- c(1+(t_-)^m) \le g(s,t) \le c(1+(t_+)^m)$$

a.e. $s \in (0,|\Omega|)$, $\forall\ t_1 \in \mathbb{R}$, for some $m \ge 0$, $m < \frac{N+2}{N-2}$ if $N > 2$, $c \in \mathbb{R}^+$. Then for f in $L^{1+\frac{1}{m}}(\Omega)$, there exists (at least) one solution of

$$\begin{cases} -\Delta u + \theta = f \text{ in } \Omega, \\ \theta(x) \in [\operatorname*{EssInf}_{s\in\beta(u)(x)} g(s,u(x)), \operatorname*{EssSup}_{s\in\beta(u)(x)} g(s,u(x))] \\ \qquad\qquad \text{(see (6))} \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

II.2.3. Open problems

There still remain many open problems in this area. We already mentioned the questions of the multiplicity of solutions, or the smoothness of the free boundary, in problem (1'). Moreover, the case

$$f(x,u) = g(x,u, \frac{du^*}{ds}, \frac{d^2u^*}{ds^2})$$

including derivatives, is still an open problem.

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D PHILLIPS

The regularity of solution for a free boundary problem with given homogeneity

0. INTRODUCTION

In this paper we obtain the optimal interior regularity for a function $u(x)$ satisfying

$$J(u) = \int_G \left(\frac{|\nabla u|^2}{2} + u^\gamma \right) dx = \min_{w \in \mathbb{K}} J(w), \; u \in \mathbb{K}, \tag{0.1}$$

where γ is a fixed constant, $0 < \gamma < 1$,

G is an open bounded set in $\mathbb{R}^n$ with a Lipschitz boundary, and

$\mathbb{K} = \{w | w \in H^1(G), w = u_0 \text{ on } \partial G, w \geq 0\}$ with u_0 a fixed element in $H^1(G)$, $u_0 \geq 0$.

We find that for $\overline{H} \subset G$, $u \in C^{1,\beta-1}(H)$ with $\beta = \dfrac{2}{2-\gamma}$. (0.2)

As an example the function:

$$u(x) = \left(\frac{\sqrt{2}}{\beta}\right)^\beta x^\beta \qquad \text{for } x \geq 0$$

$$u(x) = 0 \qquad \text{for } x < 0,$$

will satisfy (0.1) on any subinterval of $\mathbb{R}^1$. This indicates that our result is sharp.

Assuming the regularity assertion, u will satisfy the following free boundary problem (F.B.P.):

$$\Delta u = \gamma u^{\gamma-1} \qquad \text{on} \quad \{u > 0\}$$

$$u = 0, \; \Delta u = 0 \qquad \text{on} \quad \partial\{u > 0\} \backslash \partial G$$

$$u = u_0 \qquad \text{on } \partial G.$$

The preceding F.B.P. is used as a model for chemical heterogeneous catalysts with a pth order reaction term ($p = \gamma - 1$). G represents a porous catalyst pellet, and u the density of a gaseous reactant (see [1], [4]).

The question of regularity for the minimum problem with $\gamma = 1$ has been resolved by H. Brezis and D. Kinderlehrer [3]. The problem with $\gamma = 0$ ($u^0 = \chi_{\{u>0\}}$) has been resolved by H. Alt and L. Caffarelli [2]. A more detailed exposition for γ, $0 < \gamma < 1$ is available in [6].

1. THE LOCAL PROBLEM

Our interest is in interior regularity. Thus we introduce the notion of local minimizers.

__Remark__ We use the notation

$$J_r(w) = \int_{B_r} \left(\frac{|\nabla w|^2}{2} + w^\gamma \right) dx$$

where $B_r \subset G$ and whose center will be understood with each application.

__Definition__ If $w \geq 0$ and in $H^1(G)$, then w is a *local minimizer* with respect to $B_r \subset G$ iff $J_r(w) \leq J_r(v)$ for all v satisfying $w - v \in H_0^1(B_r)$ and $v \geq 0$.

__Remark__ It is clear that our minimizer is a local minimizer with respect to any sub-ball contained in G.

Our regularity result is due primarily to a homogeneity property. Consider w defined on $B_r(0)$ and choose $s > 0$. We define $w_s(x)$ on $B_{r/s}(0)$:

$$w_s(x) = w(sx)/s^\beta \quad \text{with} \quad \beta = \frac{2}{2-\gamma} .$$

We note that

$$J_{r/s}(w_s) = s^{(-n+2-2\beta)} J_r(w) .$$

Of primary importance is that if w is a local minimizer on B_r, w_s is a local minimizer on $B_{r/s}$.

We begin by showing that u is subharmonic. Thus we can define u pointwise as the limit of averages.

__Lemma 1.__ Let $B_r \subset G$ and $h(x)$ be harmonic in B_r with $h = u$ on ∂B_r, then

$u \le h$. Hence we can take $u(x) = \lim_{r \downarrow 0} \fint_{\partial B_r(x)} u$ where $\fint$ denotes the mean value.

Proof Let $v = \min(u,h)$. If we compare $J_r(v)$ and $J_r(u)$, we obtain $J_r(v) \le J_r(u)$ with equality iff $v = u$. A strict inequality would contradict u being a local minimizer. Q.E.D.

In order to prove (0.2) we have two objectives. First near a point p where $u(p) > 0$ we would like to show that u is uniformly bounded away from 0. In such a situation a first variation of $J(\cdot)$ is possible. The regularity of the resulting P.D.E. will imply that u is C^∞ near p.

Second we wish to control the growth of u away from $p \in \partial\{u > 0\}\backslash\partial G$. From the example above we anticipate that

$$u(x) \le c|x - p|^\beta.$$

The weakest such statement of the above behavior can be phrased in terms of the averages of u.

Theorem 1. There exists $\tau(n)$, $c_0(n,\beta) > 0$ such that if u is a local minimizer on $B_r(0)$ and

$$\frac{1}{r^\beta} \fint_{\partial B_r} u \ge c_0$$

then

$$u(x) \ge (\tau \cdot \fint_{\partial B_r} u) \text{ or } B_{r/2} .$$

To prove this, one shows that for c_0 sufficiently large there exists a positive stationary point for $J_r(\cdot)$, $w(x)$. Next a second condition is determined for c_0 that forces $w(x)$ to be the unique local minimizer. Hence $w(x) = u(x)$. Furthermore, $w(x)$ is shown to be near the harmonic function equal to u on ∂B_r. Thus we have the Harnack type conclusion (see [6]).

We have the following corollaries.

Corollary 1. $u \in C^\infty(\{u > 0\})$ and $\Delta u = \gamma u^{\gamma-1}$ on $\{u > 0\}$.

Corollary 2. If $0 \in \partial\{u > 0\}\backslash\partial G$ and $B_{2r}(0) \subset G$ then $u(x) \le C(n,\beta)|x|^\beta$ on B_r.

Thus we have u continuous on G. And since $\beta > 1$, ∇u exists at each point of G. To show the Holder regularity of ∇u we derive estimates of $D_i u(x)$ and $D_{ij}u(x)$ in terms of $u(x)$.

<u>Lemma 2</u>. Let $F_\delta = \{x \in G | d(x,\partial G) \geq \delta\}$ with $0 < \delta < 1$. If $0 \in \{u > 0\} \cap F_\delta$, then:

a) $|D_i u(0)| \leq \theta_1 [u(0)]^{\frac{\beta-1}{\beta}}$

b) $|D_{ij} u(0)| \leq \theta_2 [u(0)]^{\frac{\beta-2}{\beta}}$.

where $\theta_i = \theta_i(n,\beta,\sup\limits_{F_\delta} u,\delta)$.

<u>Proof</u> Take $M = \max[c_0,\sup\limits_{F_\delta} u]$ (c_0 from Theorem 1). Define $\lambda = \left(\frac{u(0)}{M}\right)^{1/\beta}$ $(0 < \lambda \leq 1)$ and consider $u_\lambda(x) = \frac{u(\lambda x)}{\lambda^\beta}$. Note $B_\delta(0)$ is contained in the domain of u. We have

$$\fint_{\partial B_\delta} u_\lambda \geq u_\lambda(0) = M \geq c_0 \delta^\beta \quad .$$

Thus by Theorem 1, $u_\lambda(x) \geq \tau M$ on $B_{\delta/2}$. This allows the following representation on $B_{\delta/2}$:

$$\begin{aligned} u_\lambda(x) &= \int_{B_{\delta/2}} G(x,y)\gamma[u_\lambda(y)]^{\gamma-1}dy + H(x) \\ &= W(x) + H(x). \end{aligned}$$

Above $G(x,y)$ is the Green's function for $B_{\delta/2}$ and $H(x)$ is harmonic with $H = u_\lambda$ on $\partial B_{\delta/2}$.

We will obtain estimates on W, H, their first and second derivatives at $x = 0$ independent of $u(0)$.

Below C will represent different constants depending only on M, n, β, and δ. In particular C is independent of $u(0)$.

With

$$|\Delta W| \le \gamma[\tau \cdot M]^{\gamma-1} ,$$

we have

$$|W|, |D_i W| \le C \qquad \text{on } B_{\delta/2} .$$

With H(x) nonnegative and harmonic,

$$|H|, |D_i H|, |D_{ij} H| \le C \fint_{\partial B_{\delta/2}} H = C \fint_{\partial B_{\delta/2}} u_\lambda \quad \text{on } B_{3\delta/8} .$$

By Theorem 1

$$\fint_{\partial B_{\delta/2}} u_\lambda \le \frac{u_\lambda(0)}{\tau} = \frac{M}{\tau} = C .$$

We notice that

$$|D_i \Delta W| = |\gamma(\gamma-1)[u_\lambda]^{\gamma-2} \cdot (D_i W + D_i H)|$$

$$\le C \quad \text{on } B_{3\delta/8} .$$

Hence by Schauder's estimates,

$$|D_{ij} W(0)| \le C\left[\|\Delta W\|_{C^{0,1}(B_{3\delta/8})} + \|W\|_{C^0(B_{3\delta/8})} \right]$$

$$= C .$$

Thus $|D_i u_\lambda(0)|$, $|D_{ij} u_\lambda(0)| \le C$.

With

$$D_i u_\lambda(0) = [D_i u](0) \cdot \lambda^{1-\beta}$$

and

$$D_{ij} u_\lambda(0) = [D_{ij} u](0) \cdot \lambda^{2-\beta},$$

the lemma follows. Q.E.D.

Theorem 2 There exists a constant $K > 0$ such that if $p_1, p_2 \in F_\delta$ then

$$|D_i u(p_1) - D_i u(p_2)| \le K|p_1 - p_2|^{\beta-1}$$

where

$$K = K(\delta, \sup_{F_{\delta/2}} u, n, \beta).$$

Proof We deal with three separate cases:

1) $u(p_1) > c_0(2|p_1 - p_2|)^\beta$ and $|p_1 - p_2| < \frac{\delta}{4}$

2) $u(p_1) > c_0(2|p_1 - p_2|)^\beta$ and $|p_1 - p_2| \ge \frac{\delta}{4}$

3) $u(p_1), u(p_2) \le c_0(2|p_1 - p_2|)^\beta$.

1) Consider $B_r(p_1)$ with $r = 2|p_1 - p_2|$. We have $B_r(p_1) \subset G$. Furthermore with $u(p_1) \ge c_0 r^\beta$, Theorem 1 implies:

$$u(y) \ge c_0 \tau (2|p_1 - p_2|)^\beta \text{ when } |y - p_1| \le |p_1 - p_2|.$$

Thus $B_{r/2}(p_1) \subset \{u > 0\}$. Hence

$$|D_i u(p_1) - D_i u(p_2)| \le \sup_{y \in \overline{p_1 p_2}} |D_{is} u(y)| |p_1 - p_2|$$

where $\frac{d}{ds}$ is in the direction of $\vec{p}_1 - \vec{p}_2$. By hypothesis $\overline{p_1 p_2} \subset F_{\delta/2}$. From Theorem 2

$$|D_{is} u(y)| \le \bar{\theta}_2 [u(y)]^{\frac{\beta-2}{\beta}} .$$

Using (1.1)

$$|D_{is} u(y)| \le K[|p_1 - p_2|^\beta]^{\frac{\beta-2}{\beta}} .$$

Finally

$$|D_i u(p_1) - D_i u(p_2)| \leq K|p_1 - p_2|^{\beta-2} \cdot |p_1 - p_2| .$$

Case 2) is straight forward and 3) follows from Theorem 2 a). Q.E.D.

The method of proof for Theorem 2 is similar to one developed by L. Caffarelli and D. Kinderlehrer [5].

Acknowledgement

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Free surface flow of heavy oil during in-situ heating

INTRODUCTION

A large part of the oil reserves in Canada consists of heavy crude which is virtually immobile in the reservoir; it is located in the tar sands in Alberta. These oils have so high viscosities under reservoir conditions that they will not flow even through the very porous matrix of unconsolidated sand reservoir. Some deposits are close enough to the surface that strip mining is economical, but most deposits are too deep and in-situ methods must be used. These consist of reducing the viscosity of the oil while it is still in the reservoir.

One method for reducing the viscosity of the oil consists of heating it by steam. In the past cyclic steam stimulation and steam flooding have been used. In cyclic steam stimulation steam is injected through a well into the reservoir, left there for a while, and then the resulting mixture of steam, water and heated oil is pumped out through the same well. In steam flooding steam is injected continuously into the reservoir through an injection well, and it finds it way along natural and artificial paths to the production well where a mixture of steam, water and heated oil is pumped up. Both of these methods have disadvantages which it is believed that gravity drainage will avoid.

The basic concept of gravity drainage of heavy oil during in-situ steam heating is illustrated in Figure 1. Steam is pumped into the reservoir and condenses on the oil particles. This heats up the oil which then becomes mobile and flows due to gravity, together with the condensate, to the bottom where the mixture is continuously removed through a second well. More space is thus formed in the reservoir so more steam can enter, and we get an expanding steam chamber as shown in Figure 1; here we have a vertical section with two horizontal wells placed near the bottom of the reservoir. The steam flows through the porous matrix within the chamber to the oil interface where it condenses. The heat liberated in this way heats up the oil near the interface, and this oil with its reduced viscosity flows due to gravity toward the bottom of the chamber together with the condensate.

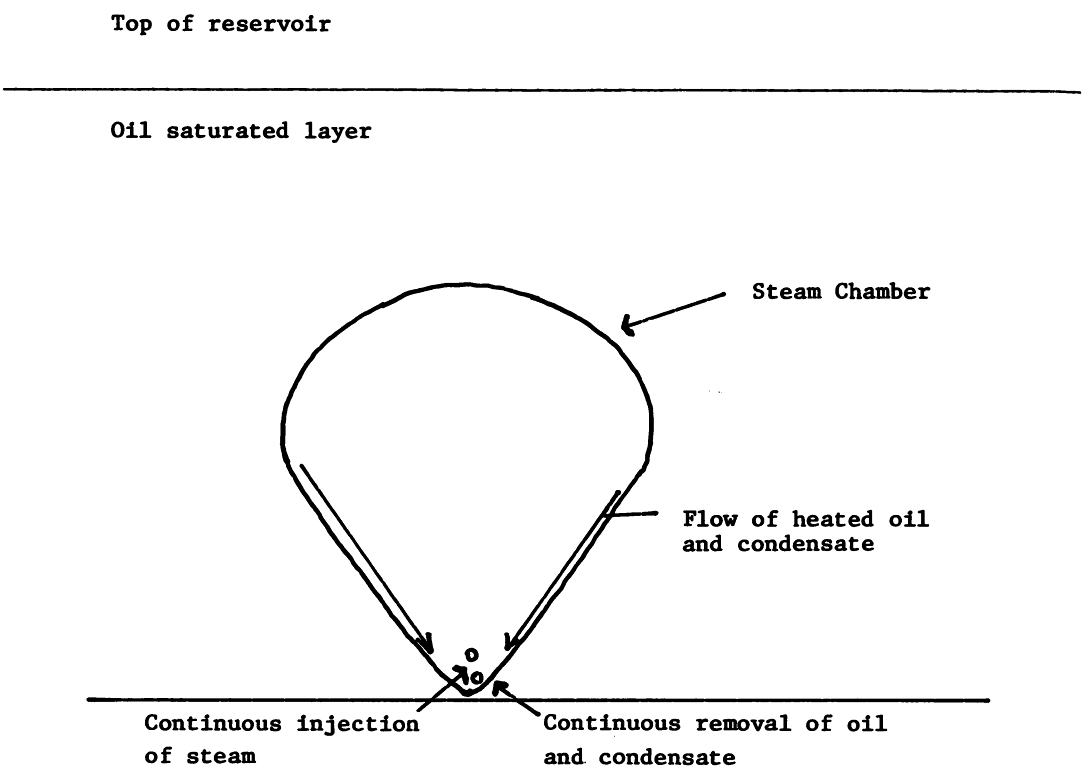

Figure 1. Vertical section of reservoir with steam chamber

It is supposed that initially the rate of growth of the chamber in the vertical direction is higher than in the horizontal direction. After a certain time interval the chamber will reach the top of the reservoir, and the growth will then only take place in the horizontal direction. Eventually the chamber will meet the chambers produced by the neighbouring wells, and finally they will form a single layer of steam above the oil reservoir. This process is illustrated in Figure 2. The above description is mainly based on Butler et al (1979).

It is obvious that the growth of these steam chambers is very complex, and in order to gain some understanding of the basic mechanisms we must look at some simplified key problems. This was first done by Butler et al (1979) and Butler and Stephens (1980) who applied a quasi one-dimensional approach to the movement of a straight interface inclined at an angle to the horizontal.

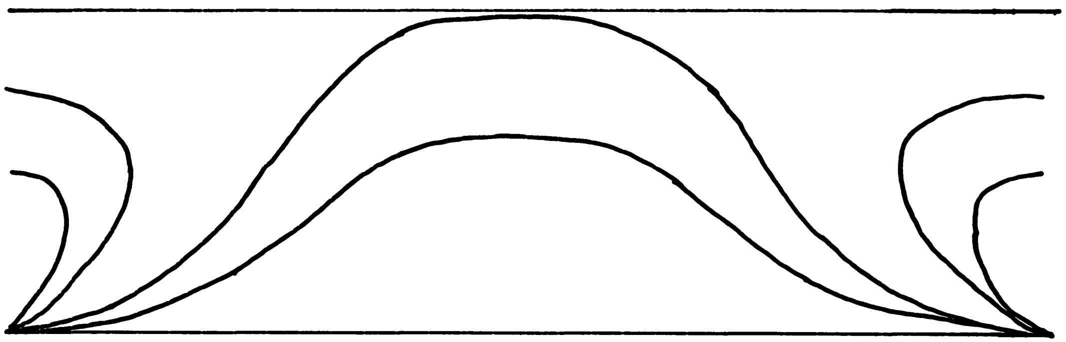

Figure 2. Progression in time of the walls of two adjacent steam chambers

In this paper we look at a different problem. We suppose that at time t=0, we have a layer of oil of thickness H; the temperatures at the oil interface and at some point below the reservoir are T_S and T_R respectively, see Figure 3. This is, of course, much simpler than treating the condensation of steam at the interface. We also suppose that we pump oil through a well located at the bottom of the reservoir. We consider a series of wells located a distance 2L apart; thus we can assume periodicity with period 2L of all dependent variables.

This is, of course, not a realistic model of the complete process, but it is believed that the study of such simplified model problems will lead to a greater understanding of the basic mechanisms appearing in the complete process.

FORMULATION

Let (x,y) be a rectangular coordinate with y=0 corresponding the lower surface of the oil layer and y = h(x,t) indicating the free surface; see Figure 3. Then a fairly complete mathematical model of the oil layer can be written in the form

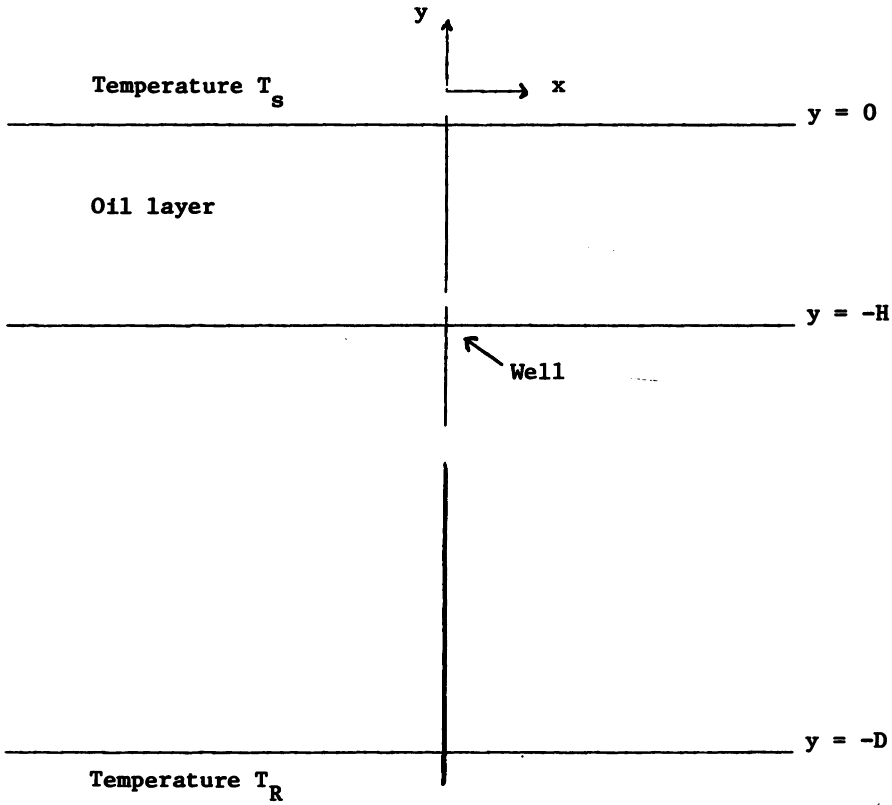

Figure 3. Sketch of the oil layer problem

$$\phi_p \frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{V}) = 0 \tag{1}$$

$$\frac{\rho}{\phi_p} \frac{\partial \underline{V}}{\partial t} + \frac{\rho}{\phi_p^2} (\underline{V} . \nabla) \underline{V} = - \nabla p + \rho \underline{g} - \frac{\nu \rho}{K} \underline{V} \tag{2}$$

$$\nabla . (\lambda \nabla T) - \nabla . [(\rho c)_f \underline{V}\, T] = \frac{\partial}{\partial t} [(\rho c)^* T] \tag{3}$$

$$\rho = \rho_o [1 - \beta (T - T_o)] \tag{4}$$

$$(\rho c)^* = (\rho c)_s (1 - \phi_p) + (\rho c)_f \phi_p \tag{5}$$

where ρ = density of the oil

$(\rho_o$ = reference density)

$\underline{V}$ = velocity of the oil = (V_x, V_y)

p = pressure

$\underline{g}$ = gravity vector

T = temperature (T_o = reference temperature)

λ = thermal conductivity coefficient

β = volumetric thermal expansion coefficient

ν = Kinematic viscosity of the oil

K = permeability

ϕ_p = porosity

$(\rho c)_s$ = heat capacity of the solid phase at constant pressure

$(\rho c)_f$ = heat capacity of the oil;

see Combarnous and Bories (1975).

The boundary and initial conditions are

$$
\begin{aligned}
&x = 0, L \quad h_x = T_x = V_x = 0 \\
&y = -D \quad T = T_R \\
&y = -H \quad V_y \text{ is given} \\
&y = h(x, t) \quad T = T_s, \; V_y - h_x V_x = h_t, \; p=0 \\
&t = 0 \quad h \text{ and } T \text{ are given}
\end{aligned}
\tag{6}
$$

The kinematic viscosity ν of the oil depends on the temperature, and following Butler et al (1979) we shall suppose that this variation can be expressed in the form

$$\frac{1}{\nu} = \frac{1}{\nu_s} \left(\frac{T - T_R}{T_s - T_R}\right)^m \tag{7}$$

where ν_s is the kinematic viscosity of steam and m has a value of 3 to 4 for heavy oil under typical conditions.

The mathematical problem given by equations (1) to (7) is very complicated and in order to simplify it we assume that the following approximations can be applied.

1. Since the thermal expansion coefficient β is very small (between 10^{-3} and

10^{-4}), the variation in the density ρ will be ignored except in the bouyancy term $\rho\underline{g}$. This is usually called the Boussinesq approximation.

2. The quantities λ, $(\rho c)_f$ and $(\rho c)^*$ will be assumed to be constant with the notation $\alpha_1 = (\rho c)_f$ and $\alpha_2 = (\rho c)^*$.

3. The inertial terms in equation (2) will be ignored. With these approximations equations (1) to (4) become

$$\nabla . \underline{v} = 0 \tag{8}$$

$$\underline{v} = - \frac{K}{\mu} [\nabla p + \rho g \hat{j}] \tag{9}$$

$$\lambda \nabla^2 T - \alpha_1 \underline{v} . \nabla T = \alpha_2 \frac{\partial T}{\partial t} \tag{10}$$

$$\rho = \rho_o [1 - \beta (T - T_R)]$$

where $\hat{j}$ is a unit vector in the y direction, and we have used T_R as a reference temperature.

Let us define non-dimensional quantities by

$$\theta = \frac{T - T_R}{\Delta T}, \ \phi = \left(\frac{p}{\rho_o g} + y \right) \frac{1}{H\varepsilon}, \ \underline{v}^* = \frac{\nu_s}{K\rho_o g \varepsilon} \underline{v}$$

$$x^* = \frac{x}{H}, \ y^* = \frac{y}{H}, \ h^* = \frac{h}{\varepsilon H}, \ \nabla^* = H\nabla, \ t^* = \frac{K\rho_o g}{\nu_s H} t$$

where $\Delta T = T_s - T_R$, $\varepsilon = \beta \Delta T$ and H is the average initial depth of the oil layer. Then equations (7) to (10) become

$$\nabla^* . \underline{v}^* = 0 \tag{11}$$

$$\underline{v}^* = \theta^m [- \nabla^* \phi + \theta \hat{j}] \tag{12}$$

$$\nabla^{*2} \theta - \gamma \theta_{t^*} = \gamma \frac{\alpha_1}{\alpha_2} \epsilon \, \underline{v}^* . \nabla^* \theta \tag{13}$$

where $\gamma = \dfrac{\alpha_2 \, K\rho_o \, g \, H}{\lambda \nu_s}$

so $\gamma\varepsilon$ is the Rayleigh number. In our analysis we shall suppose that $\varepsilon = \beta\Delta t << 1$ and that γ is of order unity.

The boundary conditions become

$$
\left.
\begin{array}{ll}
y^* = -\frac{D}{H} = -D^* & \theta = 0 \\
y^* = -1 & V^*_y = f(x^*) \\
y^* = \varepsilon h^* & \theta = 1 \\
 & \phi = h^* \\
 & h^*_{t^*} = \theta^m(- \phi_{y^*} + \theta + \varepsilon h^*_{x^*} \phi_{x^*}) \\
x = 0,\ x = \frac{L}{H} = L^* & h^*_{x^*} = \theta_{x^*} = \phi_{x^*} = 0
\end{array}
\right\} \quad (14)
$$

and the initial conditions are

$$
\left.
\begin{array}{l}
\theta(x^*,y^*,0) = \overline{\theta}(x^*) \\
h^*(x,0) = 0
\end{array}
\right\} \quad (15)
$$

where $f(x^*)$ and $\overline{\theta}(x^*)$ are given functions of x^*.

ANALYSIS

Since ε is small, we can linearize equations (11) to (14); hence we get

$$\nabla . \underline{v} = 0 \tag{16}$$

$$\underline{v} = \theta^m(- \nabla\phi + \theta\hat{j}) \tag{17}$$

$$\nabla^2\theta - \theta_t = 0 \tag{18}$$

with the conditions

$$
\begin{aligned}
&y = -D && \theta = 0 \\
&y = -1 && V_y = f(x) \\
&y = 0 && \theta = 1 \\
& && \phi = h \\
& && h_t = \theta^m(-\phi_y + \theta) \\
&x = 0,\ x = L && h_x = \theta_x = \phi_x = 0
\end{aligned}
\tag{19}
$$

where we have dropped the superscript * and supposed that $\gamma=1$.

Since we wish to model the pumping of oil at $y=0$, we shall suppose that $f(x)$ has the form

$$f(x) = -1 \text{ for } 0 < x < \sigma,\ f(x) = 0 \text{ for } \sigma < x < L$$

which can be written as

$$f(x) = \sum_{n=0}^{\infty} d_n \cos \frac{n\pi x}{L}$$

where $d_o = -\frac{\sigma}{L}$ and $d_n = \frac{-2}{n\pi} \sin \frac{n\pi\sigma}{L}$.

If we suppose that the temperature has reached a steady state before the pumping starts, the solution for θ is

$$\theta = \frac{y+D}{D} \tag{20}$$

Equations (16) and (17) can be written as

$$\theta\, \nabla^2\phi + m\, \nabla\phi . \nabla\phi - (1+m)\theta\theta_y = 0$$

which becomes

$$(y+D)\nabla^2\phi + m\phi_y - (1+m)\, \frac{y+D}{D} = 0 \tag{21}$$

with boundary conditions

$$y = -1\ \frac{\partial\phi}{\partial y} = \frac{1}{c} - c^m d_o - c^m \sum_{n=1}^{\infty} d_n \cos \frac{n\pi x}{L} \tag{22}$$

$$y = 0 \quad \phi = h$$

$$h_t = -\phi_y + 1 \tag{23}$$

where $c = \frac{D}{D-1}$.

The solution for ϕ can be written in the form

$$\phi = \frac{(y+D)^2}{2D} + A(y+D)^{1-m} + B$$

$$+ (y+D)^{\alpha} \sum_{n=1}^{\infty} [C_{1,n} I \left(\frac{n\pi}{L}(y+D)\right)$$

$$+ C_{2,n} K_{\alpha} \left(\frac{n\pi}{L} (y+D)\right)] \cos \frac{n\pi x}{L} . \tag{25}$$

Where A, B, $C_{1,n}$ and $C_{2,n}$ are unknown constants, I_{α} and K_{α} are the modified Bessel functions and $\alpha = \frac{1}{2}(m-1)$.

We also write

$$h(x_1 t) = \sum_{n=0}^{\infty} h_n(t) \cos \frac{n\pi x}{L} .$$

From the boundary conditions (22) and (23) at y = -1 and 0 we then get

$$(1-m) \ A \ (D-1)^{-m} = -c^m d_o ,$$

$$(D-1)^{\alpha} \frac{n\pi}{L} [C_{1,n} I_{\alpha-1} \left(\frac{n\pi}{L} (D-1)\right)$$

$$- C_{2,n} K_{\alpha-1} \left(\frac{n\pi}{L} (D-1)\right)] = - c^m d_n , \tag{26}$$

$$D^{\alpha} [C_{1,n} I \left(\frac{n\pi D}{L}\right) + C_{2,n} K_{\alpha} \left(\frac{n\pi D}{L}\right)] = h_n . \tag{27}$$

The solutions are

$$A = \frac{D^m}{m-1} d_o$$

$$C_{1,n} = V_n h_n + X_n$$

$$C_{2,n} = Y_n h_n + Z_n$$

where

$$V_n = \frac{B}{D^\alpha} K_{\alpha-1}\left(\frac{n\pi}{L}(D-1)\right), \quad X_n = -\frac{c^m BL}{n\pi(D-1)}\alpha d_n K_\alpha\left(\frac{n\pi}{L}D\right)$$

$$Y_n = \frac{B}{D^\alpha} I_{\alpha-1}\left(\frac{n\pi}{L}(D-1)\right), \quad Z_n = \frac{c^m BL}{n\pi(D-1)}\alpha\, d_n I_\alpha\left(\frac{n\pi}{L}D\right)$$

$$B = \left[I_{\alpha-1}\left(\frac{n\pi}{L}(D-1)\right) K_\alpha\left(\frac{n\pi}{L}D\right) + I_\alpha\left(\frac{n\pi}{L}D\right) K_{\alpha-1}\left(\frac{n\pi}{L}(D-1)\right)\right]^{-1}$$

The free surface equations from (23) becomes

$$\frac{dh_n}{dt} = F'_n(0)$$

Thus

$$\frac{dh_o}{dt} = d_o$$

$$\frac{dh_n}{dt} = -\beta_n h_n - \gamma_n$$

where $\beta_n = D^\alpha \frac{n\pi}{L}\left[I_{\alpha-1}\left(\frac{n\pi}{L}D\right)V_n - K_{\alpha-1}\left(\frac{n\pi}{L}D\right)Y_n\right]$ (28)

$$\gamma_n = D^\alpha \frac{n\pi}{L}\left[I_{\alpha-1}\left(\frac{n\pi}{L}D\right)X_n - K_{\alpha-1}\left(\frac{n\pi}{L}D\right)Z_n\right]$$

Hence

$$h_o(t) = d_o t$$

and

$$h_n(t) = \frac{\gamma_n}{\beta_n}\left(e^{-\beta_n t} - 1\right) \tag{29}$$

since $h(0) = 0$. Thus the free surface to the first approximation is given by

$$h(x,t) = d_o t + \sum_{n=1}^{\infty} \frac{\gamma_n}{\beta_n} (e^{-\beta_n t}-1) \cos \frac{n\pi x}{L} \tag{30}$$

It is clear from (30) that the obtained solution is only valid for small times since the term $d_o t$ will eventually become so large that $\varepsilon h = O(1)$, and hence the linearization procedure used in the derivation of equations (16) to (19) is no longer valid.

RESULTS

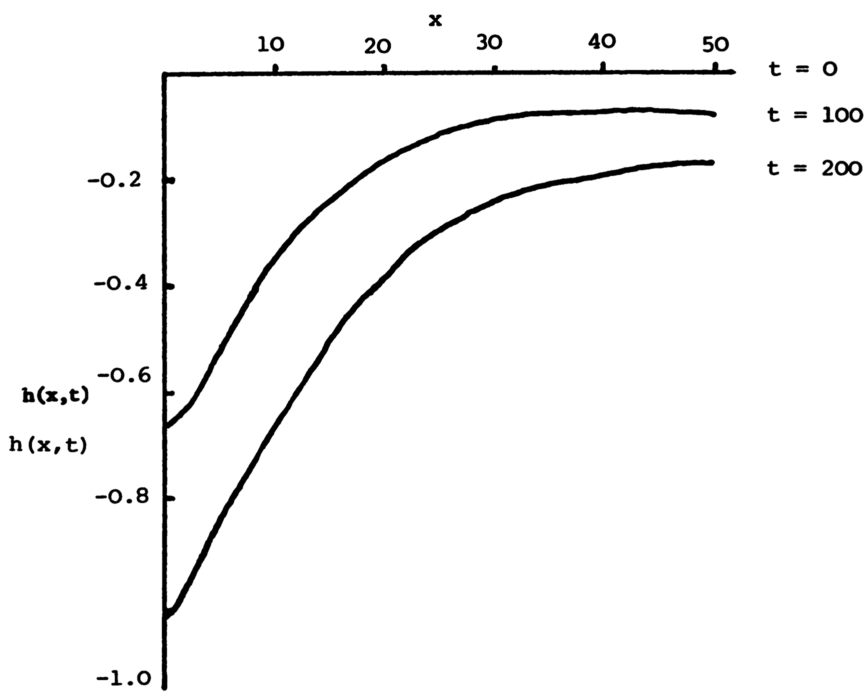

Figure 4. h(x,t) for D=5 and L=50.

We shall now present and discuss some results for the parameter values $\sigma = 0.1$, $L = 50$ and $\gamma = 1$. In Figure 4 we have plotted the free surface $h(x,t)$ as given by (30) for $D = 5$. At $t = 200$ $h(0,t) = 1.0$ so it is doubtful if the perturbation procedure is valid for greater values of t.

It is clear that the value of D is important. In Table 1 we present results for h(O,t) for different values of D. These results indicate the importance of knowing fairly accurately the thermal properties of the material underlying the reservoir. As we can see, an increase of D from 10 to 20 gives a fairly large increase in the draw down due to the pumping.

TABLE 1
h(O,t)

t	D = 2.5	D = 5.0	D = 10.0	D = 20.0
0	0.0	0.0	0.0	0.0
50	-0.264	-0.440	-0.557	-0.623
100	-0.420	-0.665	-0.830	-0.923
150	-0.555	-0.846	-1.044	-1.156
200	-0.679	-1.005	-1.227	-1.354
250	-0.795	-1.148	-1.391	-1.530

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M SCHOENAUER

A monodimensional model for fracturing

INTRODUCTION

Studying a model issued from petroleum research and after a few simplifying hypothesis, we were finally led up to the following Free Boundary Problem:

Find two functions L(t) (the Free Boundary) and u(t,x) satisfying

$$\left.\begin{array}{ll} \dfrac{\partial \Psi(u)}{\partial t} - \dfrac{\partial}{\partial x}\Theta\left(\dfrac{\partial u}{\partial x}\right) = F, \quad 0 \leq t \leq T, \quad 0 \leq x < L(t) & \\ \text{with the boundary conditions} & \\ L(0) = 0 & \\ \Theta\left(\dfrac{\partial u}{\partial x}(t,0)\right) = -1 \qquad 0 \leq t \leq T & \\ \Theta\left(\dfrac{\partial u}{\partial x}(t,L(t)\right) = 0 \qquad 0 \leq t \leq T & \\ u(t,L(t)) = 0 \qquad 0 \leq t \leq T & \end{array}\right\} \qquad (0)$$

where Ψ and Θ are real valued, monotone but both non-linear functions, F being a given function which is integrable, but not square integrable on the domain, with a singularity on the Free Boundary.

This model was introduced in [1] and [5], as a simplification of a two dimensional problem whose study is still open.

But in these papers, the authors only studied the numerical aspects of the problem, and in a particular case where one of the non-linearities vanishes.

We obtain, in the general case, a global existence result, as well as a method to get numerical results we shall not talk about here, for the same reason we shall only give sketches of proofs. All the details, as well as the numerical results, for which there is no room here, can be found in [6].

THE PHYSICAL MODEL

The problem we are dealing with here rises from one of the latest techniques used to improve petroleum extraction, which is called fracturing: while exploiting a well, and in order to increase its production one cracks the

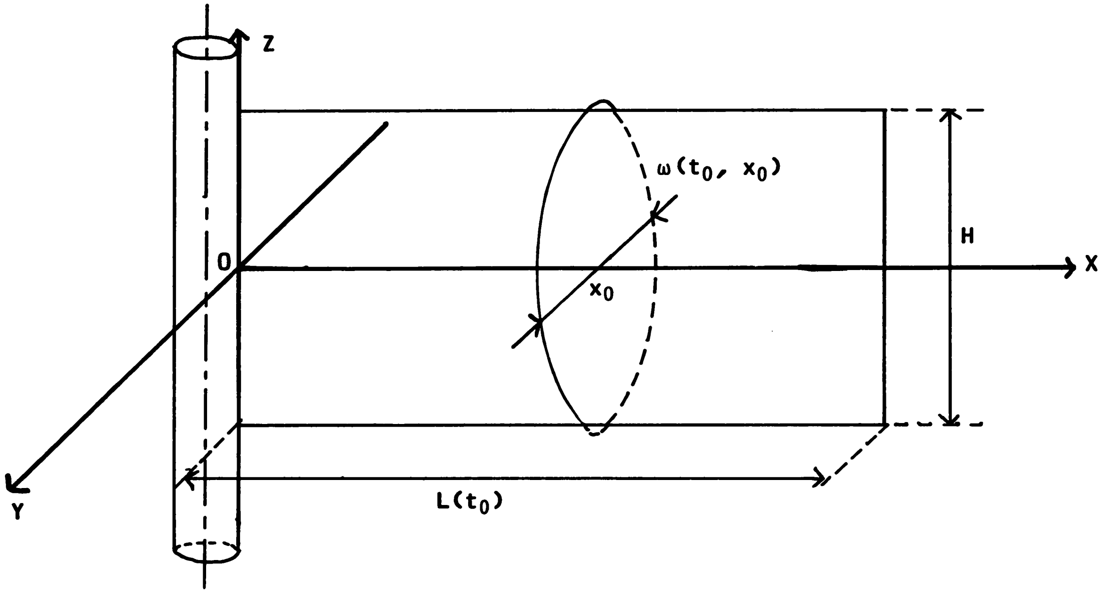

Figure 1. Fracture at time t_o

reservoir rock, creating a fracture in which the oil can flow more easily and more rapidly.

In our model, the fracture is created by an explosion in the well, and is then opened by the injection, from the well, of a liquid at a constant rate. We are interested in finding, at any moment later during the pumping, the length and width of the fracture.

The most important hypothesis we make is that the cracking stays in a plane, propagating in the X-direction, with vertical front parallel to the z-axis, and constant height H (see Figure 1).

We then suppose, following [5], the difference of pressure ΔP between inside and outside the fracture only depends on the x coordinate, and is negligible in front of the total pressure P.

These two assumptions, which are not very realistic, are the reasons why this model is not very accurate. They are discussed in detail in [5].

The third simplification is to say the maximal width $\omega(t,x)$ of the fracture is small when compared to its height H. Experimentally, H is about 10 meters, while $\omega(t,x)$ never exceeds a few centimeters, which makes this hypothesis quite reasonable.

We now deduce from these assumptions and from the classical equations of physics, the equations that will lead us to our free boundary problem.

From the classical elasticity laws for the rock, we get that the fracture is elliptical, and its maximal width is given by

$$\omega(t,x) = \frac{1-\nu}{G} \; H \; \Delta P(t,x),$$

where G and ν are the bulk shear modulus, and the Poisson ratio of the rock.

We then write the conservation of volume for the incompressible fluid flowing in the fracture

$$\frac{\partial Q(t,x)}{\partial x} + \frac{\partial}{\partial t}\left[\frac{\Pi}{4} H \; \omega(t,x)\right] + Q_\ell(t,x) = 0,$$

where $Q(t,x)$ is the volume rate of flow through a cross section $x = \text{cst.}$ of the fracture, and $Q_\ell(t,x)$ is the volume rate of fluid loss per unit of length of the fracture.

Experiments (see [2] for instance) suggest the fluid loss is given by

$$Q_\ell(t,x) = \frac{C_\ell}{\sqrt{t_1 - \tau(x)}}, \text{ for } t > \tau(x),$$

where $\tau(x)$ is the time the loss begins, C_ℓ being a fluid loss coefficient.

Finally, we write the solution (see [3]) for a laminar flow of a viscous incompressible fluid in a tube of elliptical cross section of semi axis $\frac{1}{2}$ H and $\frac{1}{2}\,\omega$, with $\omega \ll H$:

$$\frac{\partial \; \Delta P(t,x)}{\partial x} = - K_n \frac{Q^n(t,x)}{H^n \; \omega(t,x)^{2n+1}},$$

where n only depends on the fluid, the case $n = 1$ corresponding to a Newtonian fluid ($n \in \,]0,1]$), and K_n depends on n and the Lamé coefficient K of the rock.

We now write obvious boundary conditions:

- $t = 0 \quad L(0) = 0 \qquad \omega(0,x) = 0$ for any $x > 0$;

- $x = 0 \quad Q(t,0) = \text{cste}$: the fluid is injected at a constant rate

- $x \geq L(t)$, where $L(t)$ is the length of the fracture at time t:

$$\omega(t,x) = 0$$

$$Q(t,x) = 0.$$

After scaling our equations, we change our unknown function the following way

Let $\Psi(\lambda) = |\lambda|^{\frac{1}{2n+2}} \operatorname{sgn} \lambda$

$$\Theta(\lambda) = |\lambda|^{\frac{1}{n}} \operatorname{sgn} \lambda$$

and define

$$\Psi(u(t,x)) = \omega(t,x).$$

We can now rewrite our problem as problem (O).

EXISTENCE RESULT

Let us first set down our notations and hypotheses

Problem 1 Find two functions L and u,

$$L : [0,T] \to \mathbb{R}, \qquad (T \in \mathbb{R}_+^*)$$

$$u : Q \equiv \{(t,x) \in [0,T] \times \mathbb{R};\ 0 \leq x \leq L(t)\} \to \mathbb{R}$$

s.t.

$$\begin{cases} \dfrac{\partial \Psi(u)}{\partial t} - \dfrac{\partial}{\partial x}\, \Theta\left(\dfrac{\partial u}{\partial x}\right) = -\dfrac{1}{\sqrt{t - \tau(x)}} & \text{a.e. in } Q \\ \Theta\left(\dfrac{\partial u}{\partial x}(t,0)\right) = -1 & \text{for a.e. } t \in [0,T] \\ \Theta\ \dfrac{\partial u}{\partial x}(t,L(t)) = 0 & \text{" " "} \\ u(t,L(t)) = 0 & \text{" " "} \end{cases}$$

with

$$\begin{cases} L(0) = 0, \quad L \text{ non decreasing,} \\ L(\tau(x)) = x \quad \text{for a.e. } x \in [0,L(T)] \end{cases}$$

$$\begin{cases} \Psi(0) = 0, \quad \Psi'(\lambda) > 0 \quad (\forall \lambda \in \mathbb{R}^*), \\ 0 < c_1 \le \dfrac{|\Psi(\lambda)|}{|\lambda|^p} \le c_2 \quad (\forall \lambda \in \mathbb{R}^*), \ (c_1, c_2) \in \mathbb{R}^2 \\ \text{for some } p \in]0,1] \end{cases}$$

$$\begin{cases} \Theta(0) = 0, \ \Theta'(\lambda) > 0 \quad (\forall \lambda \in \mathbb{R}^*) \\ 0 < c_1 \le \dfrac{|\Theta(\lambda)|}{|\lambda|^q} \le c_2 \quad (\forall \lambda \in \mathbb{R}^*) \\ \text{for some } q \in [1,+\infty). \end{cases}$$

Our results are contained in the following

Theorem 1 For any $T > 0$, there exists a solution (L,u) of problem 1 in the sense:

$L \in C^{\frac{1}{2}}([0,T])$, and L is non decreasing on $[0,T]$,

$u \in C^{\frac{1}{3}}(\overline{Q})$, and $u \geq 0$ on $\overline{Q}$

$\frac{\partial \Psi(u)}{\partial t}$ and $\frac{\partial}{\partial x} \Theta\left(\frac{\partial u}{\partial x}\right)$ are in $L'(Q)$ and are non negative a.e. in Q.

The proof of this theorem is based on a semi discretization in the time variable t of the Problem 1: classically, we first construct approximate solutions, then obtain estimates on these solutions and finally prove the convergence of such solutions to a solution of our problem.

We shall only insist here on the points which are not standard. Again, for details, see [6].

Before giving the outline of the proof, we shall state the following a priori estimate on the free boundary L, because this estimate, which is very important in the proof of our results, shows how things work in spite of (or, should we say, because of) the particular form of the right hand side function.

Lemma 1 Suppose (L,u) is solution of Problem 1, with $\frac{\partial \Psi(u)}{\partial t} \geq 0$ (we make no assumptions on the regularity of such solution).

Then

$L \in C^{\frac{1}{2}}([0,T])$.

Proof We simply integrate the equation between instants t_1 and t_2 $(t_1 < t_2)$. It comes:

$$\int_{t_1}^{t_2} \int_0^{L(t)} \frac{\partial\Psi(u)}{\partial t}\, dx\, dt + \int_{t_1}^{t_2} \int_0^{L(t)} \frac{1}{\sqrt{t - \tau(x)}}\, dx\, dt = t_2 - t_1.$$

We then use the hypothesis $\frac{\partial\Psi(u)}{\partial t} \geq 0$, and under estimate the other term of the left hand side by integrating it over a smaller domain (see Figure 2):

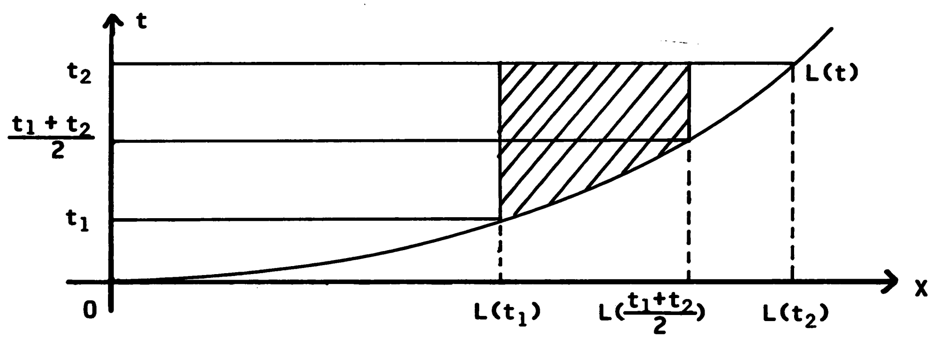

Figure 2

We obtain

$$\int_{L(t_1)}^{L\left(\frac{t_1+t_2}{2}\right)} \int_{\tau(x)}^{t_2} \frac{dt}{\sqrt{t - \tau(x)}}\, dx \leq t_2 - t_1$$

But, in this domain, we have

$$t_2 - \tau(x) \geq \frac{t_2 - t_1}{2}$$

So we get

$$\left[L\left(\frac{t_1 + t_2}{2}\right) - L(t_1)\right] \left(\frac{t_2 - t_1}{2}\right)^{\frac{1}{2}} \leq t_2 - t_1,$$

which is the announced result.

CONSTRUCTION OF THE APPROXIMATE SOLUTION

Fix a time mesh $k > 0$, with $N = \frac{T}{k} \in \mathbb{N}$.

We construct, by induction on $n \in [1,N]$, a solution (L_k^n, u_k^n) of problem (II,n) which is an approximation of Problem 1 at time nk:

$$\left.\begin{aligned}
&\frac{\Psi(u_k^n) - \Psi(u_k^{n-1})}{k} - \frac{\partial}{\partial x}\Theta\left(\frac{\partial u_k^n}{\partial x}\right) = F_k^n \quad \text{a.e. on } [0,L_k^n] \\
&\Theta\left(\frac{\partial u_k^n(0)}{\partial x}\right) = -1 \\
&\Theta\,\frac{\partial u_k^n(L_k^n)}{\partial x} = 0 \\
&u_k^n(L_k^n) = 0\ .
\end{aligned}\right\} \qquad \text{(II-n)}$$

We have to choose an "inverse function" of the approximate free boundary in order to have a right hand side F_k^n as close as possible from the function F of problem 1; we take in this scheme:

$$F_k^n(x) = \frac{1}{k}\int_{(n-1)k}^{nk} - \frac{ds}{\sqrt{s - \tau_k(x)}}\ , \quad 0 \le x \le L_k^n$$

with

$$\tau_k(x) = \rho k \text{ for } x \in [L_k^p, L_k^{p+1}[.$$

Remark 1 The function u_k^{n-1} is a priori defined on $[0, L_k^{n-1}]$. Because of the boundary conditions, it is natural to continue it on $[L_k^{n-1}, +\infty)$ by setting:

$$u_k^{n-1}(x) = 0 \quad \text{if} \quad x \in [L_k^{n-1}, +\infty).$$

We now solve problem (II-n), which is still a free boundary problem, using a shooting method:

Fix $L \ge L_k^{n-1}$, and solve problem (III-L):

$$\left.\begin{aligned}
&\Psi(u_L) - \frac{\partial}{\partial x}\Theta\left(\frac{\partial u_L}{\partial x}\right) = k\, F_k^n + \Psi(u_k^{n-1})\\
&\Theta\left(\frac{\partial u_L}{\partial x}(0)\right) = -1\\
&\Theta\left(\frac{\partial u_L}{\partial x}(L)\right) = 0\ .
\end{aligned}\right\} \qquad \text{(III-L)}$$

By standard method using the monotonicity of Ψ and Θ (see for instance [4]), we prove:

<u>Lemma 2</u> Problem (III-L) has a unique solution u_L in $W^{2,\infty}([0,L])$.

Using a maximum principle for problem III-L, we now study the dependence of $u_L(L)$ upon L, and show:

<u>Lemma 3</u> There exists a unique $L_k^n > L_k^{n-1}$ such that $u_{L_k^n}(L_k^n) = 0$. Let $u_k^n \equiv u_{L_k^n}$. The couple (L_k^n, u_k^n) is then solution of problem (II-n).

Moreover we have:

$$u_k^n > u_k^{n-1} \quad \text{on} \quad [0,L_k^n[$$

$$\frac{\partial}{\partial x}\Theta\left(\frac{\partial u_k^n}{\partial x}\right) > 0.$$

We are now able to construct (L_k, u_k), which are our approximate solutions on the domain $[0,T] \times [0,+\infty]$ of problem I for time step k, by classical linear interpolation of couples $(L_k^n, u_k^n)_{n \in [0,N]}$.

<u>ESTIMATES ON THE APPROXIMATE SOLUTION</u>

Let $\mathcal{O} = [0,T] \times [0,L]$, where

$$L = \sup_{k>0} \{L_k(T)\ ,\ L \in [0, +\infty].$$

Without any surprise in the method we use, we prove the following estimates:

<u>Lemma 4</u> Let C denote various constants independent of k. Then

i) $\|L_k\|_{C^{\frac{1}{2}}[0,T]} \leq C$

ii) $\|\Theta(\frac{\partial u_k}{\partial x}\|_{L^\infty(\mathcal{O})} \leq C$

iii) $\|u_k\|_{L^\infty(\mathcal{O})} \leq C$

iv) $\|\frac{\partial u_k}{\partial t}\|_{L^\infty(\mathcal{O})} \leq C$

v) Let $\Psi_k(t,x) \equiv \dfrac{\Psi(u_k(t+k,x)) - \Psi(u_k(t,x))}{k}$

$\|\Psi_k\|_{L'(\mathcal{O})} \leq C$

vi) $u_k(t,x) \geq \frac{C}{\sqrt{t}}(L_k(t)-x)^{1+\frac{1}{q}}, (t,x) \in \mathcal{O},\ t > k.$

The proof of i) is an exact repetition, in the semi-discrete case, of the proof of Lemma 1.

A consequence of i) is that L is finite, and $\mathcal{O}$ is then a bounded domain.

Inequality ii) is obtained directly everywhere on $\mathcal{O}$, while iii) comes from i) and ii) by convexity. The last three estimates come directly from the equation at step k.

Notice that estimates v) and vi), which are useless to extract a convergent subsequence of (L_k, u_k), are essential to prove the limit of this subsequence is a solution of problem I.

<u>Remark 2</u> We deduce from ii) - iv) that

$\|u_k\|_{C^{\frac{1}{3}}(\mathcal{O})} \leq C.$

<u>CONVERGENCE</u>

After extracting convergent subsequences of (L_k, u_k), using the results of Lemma 4, we pass to the limit in a weak formulation of the problem, by standard methods based upon the monotonicity of Ψ and Θ (see [4] for instance). There are two delicate points to be careful about:

Convergence of Ψ_k to $\frac{\partial\Psi(u)}{\partial t}$:

Define $Q_\varepsilon = \{(t,x) \in \mathcal{O};\ L(t) - x > \varepsilon$ for any $\varepsilon > 0$.
Using estimate vi), we prove that, on Q_ε, we have, for some $\nu_\varepsilon > 0$,

$$u(t,x) > \nu_\varepsilon$$

$$u_k(t,x) > \nu_\varepsilon \text{ for any } k > 0.$$

From this inequality, and from estimates iv), and the hypotheses on Ψ, we deduce

$$\|\Psi_k\|_{L^2(Q_\varepsilon)} \leq C \text{ for any } \varepsilon > 0.$$

We take a sequence ε_n going to 0, and extract a subsequence of Ψ_k such that for any $\varepsilon_n > 0$, $\Psi_k \to \chi$ in $L^2(Q_{\varepsilon_n})$ weak, for some χ.

It is easy to see that $\chi = \frac{\partial\Psi(u)}{\partial t}$, and, by inequality vi), that $\frac{\partial\Psi(u)}{\partial t} \in L^1(\mathcal{O})$.

Convergence of the right hand side: here again, we have to be very precise to derive the fact that F_k, the approximate function to F, converges to F in $L^{2-\alpha}(\mathcal{O})$ weak, for any $\alpha \in]0,1[$. To stay away from the free boundary, where F has a singularity, and because we only know about τ_k converging to τ almost everywhere we have to use Egoroff's theorem.

Apart from these two points, the end of the proof of theorem 1 is just a matter of calculation, without any difficulty.

Remark 3 From point vi) of Lemma 4, and going back to our problem of fracturing, we deduce the shape of the nose of the fracture:

$$\omega(t,x) = \Psi(u(t,x)) \geq C_t (L(t) - x)^{\frac{1}{2}}.$$

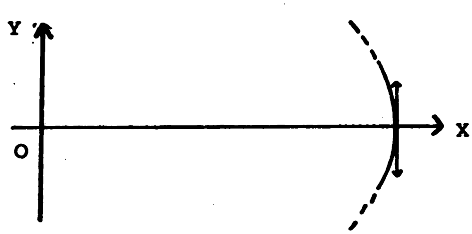

Figure 3. Shape of the fracture

Remark 4 The method used to solve problem I is a constructive method: We only have to discretize in the space variable x the problem (III-L) to solve numerically the problem I. It has been done, and details and numerical results about the width and length of the fracture that can be founded in [6], fit perfectly with the previous computations, in [1] or [5], in the case of a Newtonian fluid ($n = 1$).

CONCLUSION

This problem is not in one of the great classes of free boundary problems interesting every numerical analyst. But, in our opinion, it is interesting because it is an example of a concrete (though a little simplified) situation posed to engineers, where a theoretical mathematical study was not only possible, in spite of the "ugly look" it shows at first sight, but proves to be useful to get the numerical results the engineers wished to obtain.

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