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A Fasano & M Primicerio (Editors)

Free boundary problems: theory and applications

VOLUME I

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University of Florence

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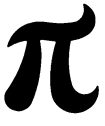

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Preface

These volumes contain the proceedings of the interdisciplinary Symposium on "Free Boundary Problems: Theory & Applications" held in Montecatini (Italy) from June 17 to June 26, 1981.

The Scientific Committee was composed of Professors G. Sestini, C. Baiocchi, V. Boffi, G. Capriz, R. Conti, D. Galletto, G. Geymonat, E. Magenes, D. Quilghini, E. Vesentini. The organizers, editors of these volumes, have also had valuable help from a group of four people composing the European Liaison Committee: Professors M. Frémond, K.-H. Hoffmann, J. R. Ockendon, M. Niezgodgka.

In this preface, after a short reference to previous meetings on the same subject, we will explain the aims that the Symposium was pursuing, and thank all the persons and institutions who made it possible.

1. Some history

International conferences on free boundary problems have a recent, but already rich tradition, testifying to the rapidly growing interest in this subject. In 1974 a meeting on "Moving Boundary Problems in Heat Flow and Diffusion" was held at the University of Oxford (see [1]). In that conference a number of problems were presented, in connection with practical applications, exhibiting the common feature that a parabolic equation had to be solved in a region whose boundary was partly unknown, thus requiring additional specification of the data. Particular attention was devoted to phase change problems.

In the workshop held in Gatlinburg (USA) in 1977 on "Moving Boundary Problems" (see [2]) an attempt was made to get pure mathematicians more involved in the debate. Dialogue between pure and applied scientists was one of the principal aims of the meeting in Durham, UK (1978) on "Free and Moving Boundary Problems in Heat Flow and Diffusion" (see [3]), where it appeared that free boundary problems were spreading over a tremendously large field, both from the point of view of applications and of mathematical methods; moreover, attention was brought to bear not only on parabolic

problems, but also on elliptic problems.

This trend was clearly confirmed in the intensive "Bimester on Free Boundary Problems", held in Pavia (see [4]), in which a huge number of topics was covered. We would also like to mention the more specialized meeting in Oberwolfach (1980), focussed on numerical methods (see [5]).

2. Scope of the Symposium

On the basis of the experience accumulated in the foregoing Conferences, the meeting in Montecatini was primarily intended as a catalyst for a (hopefully positive) interaction between people working on purely mathematical aspects or on numerical methods, and people from the area of applied research.

With this in mind, we invited people from various branches of engineering, physics, chemistry, etc., in which free boundary problems are commonly encountered, as well as from pure and applied mathematics.

The Conference was attended by 125 participants from 21 countries.

Of course not all of the people invited came, and maybe not all of the people who should have been invited were invited: we apologize for any possible mistakes made at that stage of the organization. Nevertheless, we believe that these volumes bear evidence of the scientific level of the Symposium.

3. Acknowledgements

The Symposium took place at the "Castello La Querceta" in Montecatini, kindly offered by the municipality, whose generosity and cooperation was decisive for the success of the meeting. We would also like to thank many other institutions which supported the Symposium: first of all the Italian C.N.R. and its mathematical branches (particularly the Comitato per la Matematica, the Gruppo Nazionale per la Fisica Matematica and the Istituto di Analisi Numerica), then the Administration of the Regione Toscana and, finally, the Mathematical Institute "Ulisse Dini" of the University of Florence, which bore much of the secretarial work, the European Research Office of the U.S. Army, which mainly supported the publication of the pre-conference literature, and the Azienda di Cura e Soggiorno of Montecatini, which took care of the opening ceremony.

We are also grateful to the members of the European Liaison Committee for their help in setting up the general structure of the conference, and, of course, to the colleagues of the Scientific Committee: in particular we wish to thank Prof. Sestini and Prof. Magenes for their constant interest and valuable suggestions.

4. The general scheme of the Symposium and of the Proceedings

In view of the large number of topics involved in the Conference, we decided to divide most of the speakers into 10 "Discussion Groups", each one consiting of 5-8 rather homogeneous contributions and lead by a "Rapporteur". In addition, 25 not-grouped talks were given. Discussion groups were conceived with the principal aims of stimulating discussion on specific topics and of giving everybody a chance to communicate his own results. We believe that both of these goals were satisfactorily achieved.

These Proceedings are reflecting this same structure. However, the contributors to Discussion Groups had the possibility of choosing between writing a full paper or letting the Rapporteur report briefly their talk in his general introduction.

As a rule, all the authors were asked to provide papers not exceeding ten pages. Of course, the Rapporteurs' papers were allowed to be longer, as well as some survey papers written on request of the editors.

To conclude this Preface, we wish to thank all the participants for their regular attendance (notwithstanding the tremendous amount of talks given!) and everybody who contributed to these volumes.

Firenze
January 1982

Antonio Fasano
Mario Primicerio

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J NORBURY

Free boundary problems of fluid mechanics

The first session of the conference focussed on free boundary flow problems from fluid mechanics. Classical ideal liquid flow problems (we discussed inviscid flows at first) can be separated, broadly, into two categories. The first includes those problems where the nonlinearities are mainly from the boundary conditions (usually Bernoulli's equation on the free streamline boundary); such problems are the steady propagation of gravity water waves over running streams and stationary oceans, or the steady flow of a jet of liquid from an orifice. The second type consists of flow problems where the nonlinearity is principally in the form of a body force, such as in the problem of determining the shape of steady vortex rings which move through an ideal liquid.

In R_2 the basic equations for the steady flow of an ideal liquid (no viscosity, constant density normalised to be unity) are that the x and y velocity components are given in terms of the streamfunction ψ by (respectively) $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, where the streamfunction is given by, for a given corticity function $\omega(t)$, $t \in R$,

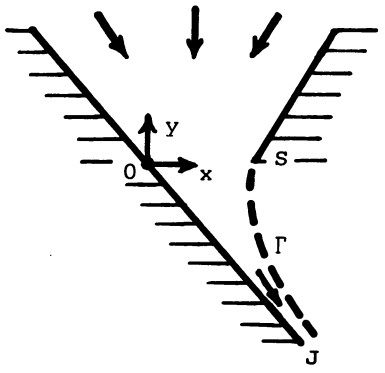
$$-\Delta\psi \equiv -\partial^2\psi/\partial x^2 - \partial^2\psi/\partial y^2 = \omega(\psi) \quad \text{for } (x,y) \text{ in the flow domain } D. \tag{1}$$

On the boundary of D either ψ or the normal derivative of ψ is prescribed. However when part of the boundary of D is unknown, say Γ, another boundary condition has to be prescribed. Typically on Γ we have ψ = constant (Γ is a free streamline) and a general Bernoulli's equation holds. When p is the pressure, g is gravity acting in the -y direction, and k is the surface tension force, we have

$$p + \frac{1}{2}(u^2+v^2) + gy + k = \text{constant on } \Gamma. \tag{2}$$

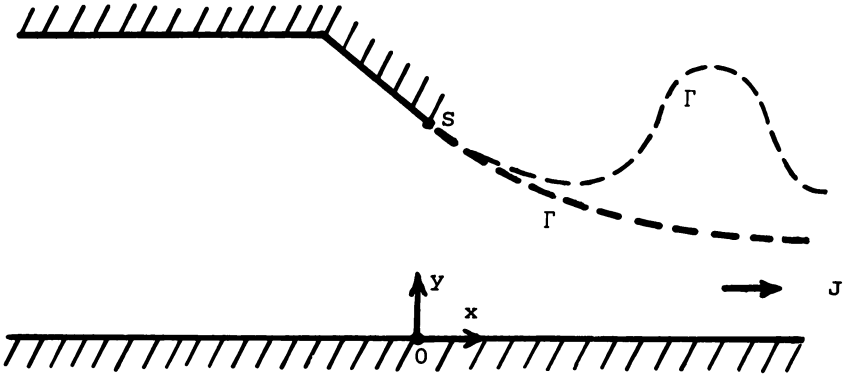
When $\omega(\psi) \equiv 0 \equiv k$ we are led to the classical free surface flow problems where the effect of gravity is significant. The simplest situation in R_2 is when, for example, a steady jet of liquid issues from an orifice down an

an inclined plane. The fluid accelerates to infinite speed at downstream

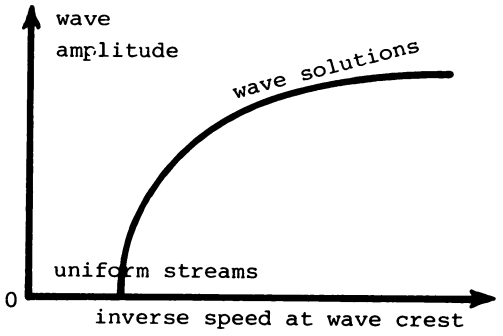


infinity J. There is only one such flow and it exists. Global existence using contraction mapping arguments and uniqueness using a priori bounds are proved in [1], and these results are typical of such geometries.

However, when the supporting plane is horizontal at J (y = constant as $x \to \infty$) then a new sort of flow with waves is also possible, and nonuniqueness can occur.



A global existence theory for gravity water waves on a uniform stream gives the simplest proof of nonuniqueness for these free boundary problems; see [2] where the problem is treated as one in bifurcation theory for a positive operator in a cone. Local existence theories dating to the 1920's are referred to in [3], and the importance of the global existence theory of Krasovskii in 1960-1 is discussed. Nonconstructive existence theories for



jet flows under gravity were announced from the floor by Friedman [4].

A priori bounds for these free boundary flows can often be found using Comparison Theorems [5], and these results may lead to uniqueness independent of existence. The following contribution extends the Lavrentiev-Gilbarg-Serrin Comparison Theorems to include those flows with vorticity. Numerical solution of the above problems was discussed along the lines of the variational principle finite element approach outlined in the following contribution from Aitchison.

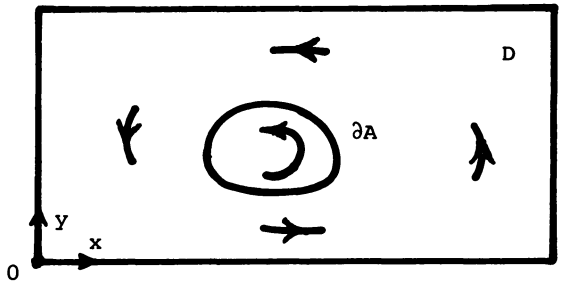
Do and Guevel looked at a linearised version of the water wave problem, applicable when the disturbing effect of a slender submerged obstacle on a uniform stream is small. Their aim is to get closer to explicit approximate formulae for drag corrections etc., which will be useful in applications.

If $\omega(\psi) = 0$ and the liquid is stationary (or is in steady rotation in the absence of gravity) we are led to the problems involving capillarity. Emmer, Gonzalez and Tamanini demonstrated the existence and regularity of the solutions to certain capillarity problems by means of a new variational approach, and pooled their efforts to give a concise introduction to this area.

In the above problems the difficulties are associated with the boundary condition (2) on the free boundary. A different sort of unknown boundary problem arises if Γ is absent but $\omega(\psi)$ is non-zero for a certain range of ψ. Typically we have a flow domain D with a subset $A \subset D$ in which vorticity is present (that is, $\omega(\psi)$ is non-zero) and the free boundary ∂A is contained in D.

These classical vortex ring problems have received much interest recently due to the analogy with the plasma problem from high energy physics.

Turkington proved existence and established asymptotic estimates for such a problem in his contribution.



Again the existence may follow from a variational principle (this nonconstructive approach often yields information on connectedness of the set A, symmetry, regularity of ∂A and on the asymptotic behaviour of ∂A); see Turkington and, for different variational approaches [6, 7, 8]. Existence may also follow from a contraction mapping argument [9] which yields, in addition, local uniqueness and explicit approximate formulae. When there is non-uniqueness the type of solution favoured by the variational principle is often unknown [7]. For another example, do the wavy or the non-wavy flows of the earlier (second) figure maximise the energy functional?

Finally the effect of viscosity must be considered, and the session concluded with Boudourides giving us a look at a problem where viscosity affects the stability of a thin film on a rotating cylinder.

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J M AITCHISON

The numerical solution of free surface flow problems using variable finite elements

In this paper we will briefly discuss the numerical solution of steady state free surface flow problems for both hydrodynamic and viscous fluids using the finite element method. The techniques will be described with respect to two model problems, one for each type of fluid. They are deliberately chosen to be very similar geometrically, so that the differences in solution technique will be due to differences in the flow models.

Hydrodynamic flows - flow through an orifice

We take as our example the case of two-dimensional flow through an orifice, shown in figure 1, and aim to find the stream function $\psi(x,y)$ and the position of the free surface CD. Since the flow is symmetric about the centre line we need only consider the region Ω bounded by ABCDEF.

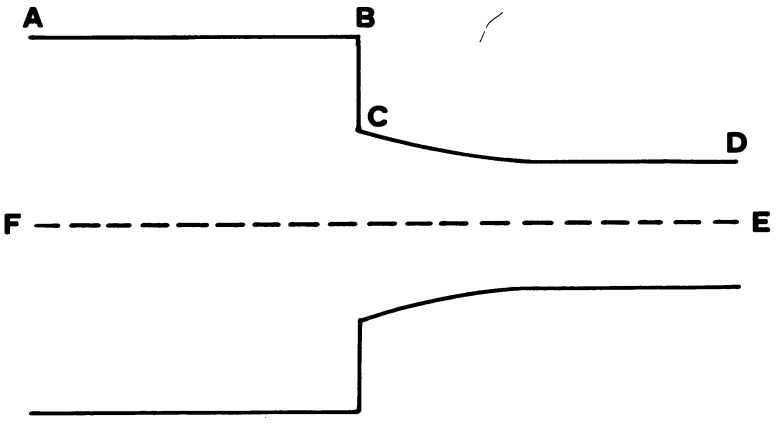

Figure 1

The governing differential equation and boundary conditions are

$$\nabla^2\psi = 0 \qquad \text{on } \Omega$$

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on AF, DE}$$

$$\psi = 0 \qquad \text{on FE}$$

$$\psi = 1 \qquad \text{on AB, BC, CD}$$

and $$\frac{\partial\psi}{\partial n} = q \qquad \text{on CD ,}$$

where q is an unknown constant to be determined as part of the solution. If the half-width of the pipe, AF, is unity then q is related to the pressure far upstream, p_∞, by

$$p_\infty = \frac{1}{2}(q^2 - 1) \ . \tag{1}$$

It is easily shown that for a given value of q the solution of equation (1) is equivalent to finding a stationary value of the functional

$$E[\psi,\Omega] = \iint_\Omega \{\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}q^2\}dxdy \tag{2}$$

where ψ is constrained to satisfy

$$\begin{aligned} \psi &= 0 \qquad \text{on FE} \\ \psi &= 1 \qquad \text{on AB, BC, CD} \end{aligned} \tag{3}$$

and ψ is allowed to vary by moving the line CD (the point C being fixed).

The method of variable finite elements

The finite element method for the solution of variational problems on fixed domain is well known (see, for example, Strang and Fix [1] or Zienkiewicz [2]). However many problems in fluid dynamics can be expressed in terms of a variational principle over a variable domain, typified by the example given here. The numerical solution of such problems can be obtained by the method of *variable* finite elements.

The region Ω is covered by a net of triangular elements and the function $\psi(x,y)$ is approximated, as usual, by piecewise polynomials. The region Ω is varied by distorting the geometry of the elements, by allowing the

coordinates of the vertices of the triangles to vary. We only allow a limited number of degrees of freedom in this geometric variation - otherwise we would be solving not only for the position of CD but also for the optimal position of the nodes within Ω. For this particular problem all the x-coordinates are kept constant and only a few y-coordinates are allowed to vary independently. The linkage structure between the elements remains fixed.

The initial triangulation is performed on a region where the line CD has an arbitrary, simple form (see figure 2).

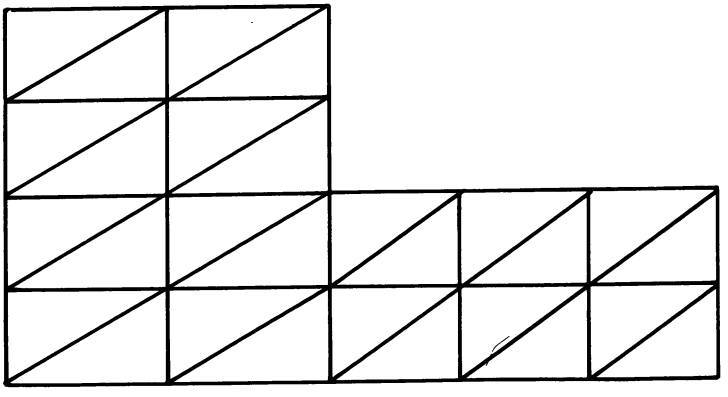

Figure 2

Each subsequent position of the free boundary is specified by a set of heights $\{h_i\}_{i=1}^{n}$ and the initial triangulation scaled in the y direction as shown in figure 3.

An expression for the discrete approximation to the functional $E[\psi,\Omega]$ can now be written down in terms of the nodal values for ψ and the geometric variables $\{h_i\}$. The discrete functional E_d is a quadratic function of the ψ-values but a more complicated function of the parameters $\{h_i\}$. The solution of the discrete stationary value problem is described in detail in Aitchison [3] in connection with the determination of a free surface cavity flow.

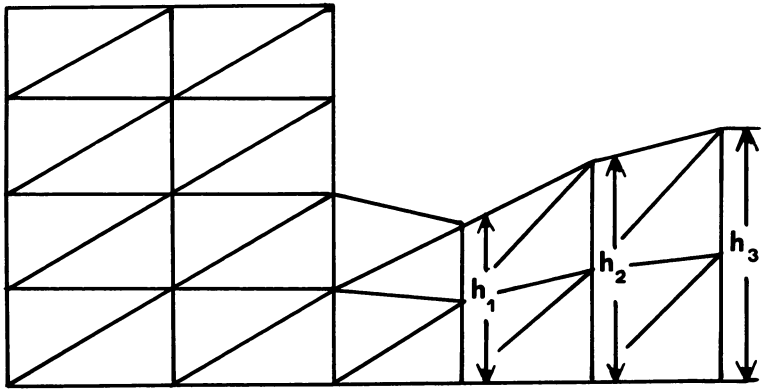

Figure 3

The method described above can apparently be used to find a solution for $\psi(x,y)$ and the position of CD for any value of the constant q. However many of these solutions are unrealistic. The pressure, p, at any point inside the fluid is given by

$$p + \frac{1}{2} (\nabla\psi)^2 = \frac{1}{2} q^2 \ .$$

For a given value of q, and a corresponding calculated value of $\psi(x,y)$, we can calculate the value of p in each element. If the value of q is taken to be too low then there are areas of fluid where the pressure is negative. This is obviously wrong. Therefore we increase q (equivalent to increasing the upstream pressure) until all such negative pressures disappear.

The application of the variable finite element method to a flow under gravity (critical flow over a weir) is described in Aitchison [4].

Viscous flows - the drawing problem

We now consider a similar problem for a viscous flow, which describes the drawing of a viscous fluid from a pipe, and is shown in figure 4. Again we have an axis of symmetry and consider the flow region Ω bounded by ABCDE. We solve for the primitive variables u(x,y), v(x,y) and p(x,y) (horizontal and vertical components of velocity and pressure), and the position of the

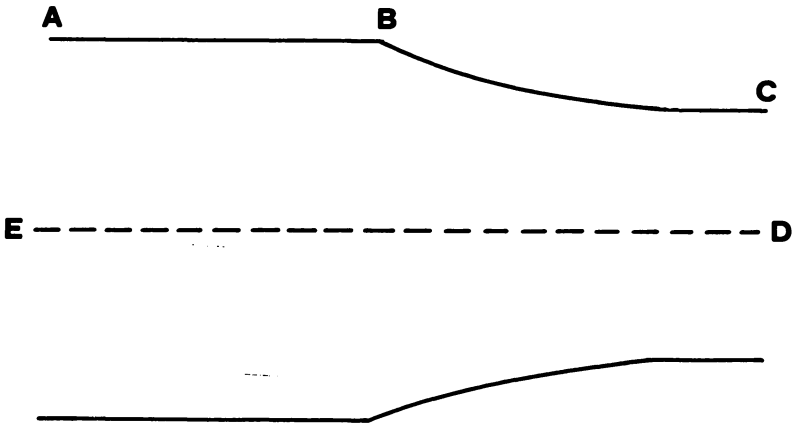

Figure 4

free surface BC. The governing differential equations are

$$\rho\{u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\} = - \frac{\partial p}{\partial x} + \mu \nabla^2 u$$

$$\{u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\} = - g - \frac{\partial p}{\partial y} + \mu \nabla^2 v$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 ,$$

where ρ is the density of the fluid, μ is the viscosity and g is the acceleration due to gravity. The boundary conditions are

$u = u_o(1 - y^2)$, $v = 0$ on AE

$\frac{\partial u}{\partial y} = 0$, $v = 0$ on ED

$u = u_1$, $v = 0$ on CD

$u = v = 0$ on AB

$T_{nn} = T_{ns} = 0$, $u \frac{\partial x}{\partial n} + v \frac{\partial y}{\partial n} = 0$ on BC ,

where u_o and u_1 are given constants, and T_{nn}, T_{ns} are respectively the normal and shear stresses.

Although we cannot derive a variational principle for this problem, we can nevertheless use the method of variable finite elements as follows. An initial position for the free surface is specified using height parameters as before. The flow region is covered by a net of triangular elements and the differential equation solved approximately using the Galerkin technique. This solution satisfies the stress conditions on BC as natural boundary conditions. We then move the free surface BC iteratively in an attempt to satisfy the streamline condition

$$u \frac{\partial x}{\partial n} + v \frac{\partial y}{\partial n} = 0 \qquad \text{on} \quad BC \ .$$

As the boundary is moved the finite element grid is distorted as before. Further details of the application of this technique to various viscous flows is found in [5].

Conclusion

The variable finite element technique can be used for both those problems which have a related variational principle and those which do not. In either case the free surface is represented by a set of ordinates $\{h_i\}$ and the triangulation for a given region found by a distortion of a basic grid. This avoids much expensive repetition in the calculation since the linkage structure between the nodes and the elements need only be set once. Furthermore it is frequently possible to avoid complete recalculation of the various matrices.

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M A BOUDOURIDES

Stability of a free boundary hydrodynamical flow

We consider the flow of a liquid film sitting on the outer surface of a rotating solid cylinder and having an interface that separates it from the ambient air. The liquid is assumed to be an incompressible, Newtonian fluid and the interface is assumed to exhibit constant surface tension with the air. Under these circumstances, such a flow can be described by the solution of the Navier-Stokes equations in a free boundary domain.

There are several industrial applications of these flows. For example we mention the carrying of paper pulp in a Fourdrinier machine, in processes of coating pipes and tubes, in rotational moulding, in spin casting of plastics, etc.

The initial free boundary value problem is going to be written in cylindrical coordinates (r,θ,z), where the solid cylinder rotates with constant angular velocity around the z-axis. The data are the non-negative numbers ν and σ, standing for a (dimensionless) viscosity and a (dimensionless) surface tension coefficient, respectively. The dependent variables are the velocity field, $(u,v,w)(r,\theta,z,t)$, the pressure field, $p(r,\theta,z,t)$, and the free boundary, $r = s(\theta,z,t)$, which is denoted by $\Gamma(t)$ (a surface in $\mathbb{R}^3$). The fixed boundary $r = \rho$ (we assume $\rho < 1$) is denoted by Σ and the domain between $\Gamma(t)$ and Σ by $\Omega(t)$. Thus one is asked to solve the following problem:

$$\left.\begin{aligned}
&D_t u - \frac{1}{r}v^2 = -p_r + \nu\{\Delta u - \frac{1}{r^2}u - \frac{2}{r^2}v_\theta\}\\
&D_t v + \frac{1}{r}uv = -\frac{1}{r}p_\theta + \nu\{\Delta v + \frac{2}{r^2}u_\theta - \frac{1}{r^2}v\}\\
&D_t w = -p_z + \nu\Delta w\\
&(ru)_r + v_\theta + (rw)_z = 0
\end{aligned}\right\} \quad \text{in } \Omega(t),\ t > 0$$

$$u = w = 0, \quad v = 1 \qquad \text{on } \Sigma,\ t > 0$$

$$D_t(r - s) = 0 \qquad \text{on } \Gamma(t),\ t > 0$$

$$\{p - \nu D\}n = \sigma Hn \qquad \text{on } \Gamma(t),\ t > 0$$

$$u,\ v,\ w,\ p,\ s \quad \text{given at} \quad t = 0\ ,$$

where D is the deformation tensor for (u,v,w), H, n are the mean curvature and an outward normal vector on the free boundary $\Gamma(t)$ and

$$D_t\phi = \phi_t + u\phi_r + \frac{1}{r}v\phi_\theta + w\phi_z$$

$$\Delta\phi = \frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz}\ .$$

The above problem admits the following steady axisymmetric solution (i.e. independent of t and θ), called swirl flow in cylindral free boundary,

$$\hat{u} = \hat{w} = 0$$

$$\hat{v} = \begin{cases} \hat{v}(r), & \text{if } \nu = 0 \\ r\ , & \text{if } \nu > 0 \end{cases}$$

$$\hat{p} = \begin{cases} \sigma - \displaystyle\int_r^1 \frac{1}{\xi}\hat{v}^2(\xi)\, d\xi, & \text{if } \nu = 0 \\ \sigma + \frac{1}{2}(r^2 - 1), & \text{if } \nu > 0 \end{cases}$$

$$\hat{s} = 1\ ,$$

where $\hat{v}$ is an arbitrary twice continuously differentiable function on $[\rho,1]$.

To determine the stability of the swirl flow in cylindrical free boundary under infinitesimal axisymmetric disturbances, one should substitute in the equations $u = \mathrm{Re}\{u'(r)\exp(\lambda t + ikz)\}$, $v = \hat{v} + \mathrm{Re}\{v'(r)\exp(\lambda t + ikz)\}$, $p = \hat{p} + \mathrm{Re}\{p'(r)\exp(\lambda t + ikz)\}$, $s = 1 + \mathrm{Re}\{s'\exp(\lambda t + ikz)\}$ and solve for the growth rate of the disturbances $\lambda = \lambda(k,\nu,\sigma)$. If $\mathrm{Re}\,\lambda \le 0$ ($\mathrm{Re}\,\lambda > 0$), then the above solution is stable (unstable).

Thus in the inviscid case, $\nu = 0$, we obtain the following result.

Theorem 1 The swirl flow $(0,\hat{v},0)$ in cylindrical free boundary is stable if and only if

$$\Phi(r) = \frac{1}{r^3}((r\hat{v})^2)_r \geq 0, \quad r \in [\rho,1]$$

and

$$\Psi = \sigma(k^2 - 1) - \hat{v}^2(1) \geq 0 .$$

If either (or both) Φ is negative somewhere in $[\rho,1]$ or Ψ is negative, then the swirl flow $(0,\hat{v},0)$ in cylindrical free boundary is unstable.

Indication of proof. The equations for the disturbances u',v',w',p',s' can be reduced to the following eigenvalue problem for $x(r) = ru'(r)$

$$\frac{1}{r}x_{rr} - \frac{1}{r^2}x_r - \frac{k^2}{r}\left(1 + \frac{1}{\lambda^2}\Phi(r)\right)x = 0 , \quad r \in [\rho,1]$$

$$x = 0, \quad r = \rho$$

$$x_r - \frac{k^2}{\lambda^2}\Psi x = 0, \quad r = 1 .$$

Note that the eigenvalue appears also in the boundary condition. The conclusions follow as in [1] and [2].

We remark that Theorem 1 generalizes the well known Rayleigh criterion for the stability of the swirl flow $(0,\hat{v},0)$ between two concentric cylinders ([3],[4]), according to which a necessary and sufficient condition for stability is that $\Phi(r) \geq 0$, $r \in [\rho,1]$. Now, in the case of the swirl flow in cylindrical free boundary, a centrifugally stable profile (i.e. one for which $\Phi \geq 0$) can be destabilized because of capillarity effects on the free boundary (i.e. when $\sigma < \hat{v}^2(1)/(k^2 - 1)$).

In the viscous case, $\nu > 0$, the swirl flow $(0,\hat{v},0)$ being the rigid body rotation $(0,r,0)$ in cylindrical free boundary, we obtain the next result.

Theorem 2 A sufficient condition for the stability of the rigid body rotation $(0,r,0)$ in cylindrical free boundary is that

$$\sigma(k^2 - 1) \geq 1 .$$

Indication of proof. The equations for the disturbances u',v',w',p',s' can be reduced to a system involving only u',v' and s'. Then using

integration by parts and the boundary conditions of the reduced equations we get the following (integral) relation

$$\lambda\nu I_0 + \lambda^* k^2 \nu\{I_1 + I_3[\sigma(k^2 - 1) - 1]\} = -I_2 ,$$

where I_0, I_1, I_2, I_3 are non-negative functionals (integrals) of u',v' and s' and the asterisk denotes the complex conjugate. The conclusion of the theorem follows immediately from the above relation. Moreover it results that a sufficient condition for instability is that $\mathrm{Im}\,\lambda = 0$, i.e. the principle of exchange of stabilities holds (cf. [6]).

Finally we remark that Theorem 2 is a generalization of Synge criterion for the stability of the rigid body rotation (0,r,0) between two concentric rotating cylinders ([5]). Now, the rigid body rotation (0,r,0) in cylindrical free boundary is stable whenever the sufficient condition for stability of Theorem 1 is satisfied. In other words, the addition of viscous effects does not destabilize the rigid body rotation (0,r,0) in cylindrical free boundary.

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C DO & P GUEVEL

Waves on a uniform flow in a channel of constant depth

INTRODUCTION

Since Michell's [1] and Havelock's [2] fundamental work, many hydrodynamics research teams have tried to build up a method to calculate the wave resistance of surface ships in steady motion in still water. Notwithstanding these tremendous efforts, no really satisfying solution has been obtained yet.

Indeed, very interesting results have already been obtained such as [3], [4], [5], [6], [7], [8], but none of the aforementioned methods has such a universal character so that total confidence can be had whatever the hull shape and forward velocity.

The main difficulty lies in the highly non linear behaviour of the free surface conditions near the ship's bow. We thus easily conceive that the results heretofore obtained are the more satisfying, the more slender the ship's hull shape, and still more the submerged ships.

In the following, we shall restrain ourselves to the description of the two dimensional problem; from the above remark, it would be illusory to look for a mathematical solution when the hull pierces the free surface. The following results have been forecast in the note [15]. §1, 2 and 3 describe the physical problem; mathematical analysis is given in §4, 5.

1. THE TWO DIMENSIONAL PROBLEM

A cylindrical streamlined profile of infinite span is moving in uniform horizontal motion with velocity C in a dense fluid bounded by a free surface and a horizontal bottom at depth h.

An irrotational motion of a perfect and incompressible fluid is assumed. In these conditions, the velocity vector field $\vec{V}$ stems from a harmonic potential Φ. We assume furthermore that the motion is established; the flow is thus steady in a coordinate system where the profile is at rest.

Owing to symmetry, the flow is determined in the (x O y) plane, Ox being horizontal in the direction of C and Oy vertically upward with its origin at the bottom. For an observer in the moving frame with velocity C the free surface has the aspect depicted in figure 1, the wave field is limited to

the downstream half space.

Figure 1

2. FREE-SURFACE AND BOUNDARY CONDITIONS

The a priori unknown free surface, of equation $y = h + \eta(x)$, derives from two independent conditions. In the present case, its trace (Γ) on ($x\,O\,y$) is a streamline of the relative flow field of velocity $\vec{V}_R = \vec{V} - \vec{C}$, $\vec{V}$ being the absolute velocity, i.e.

$$\vec{\text{grad}}\ \Phi.\vec{n} = \vec{C}.\vec{n} \qquad \text{on } (\Gamma) \qquad \vec{n} \text{ being normal to } (\Gamma) \tag{2.1}$$

Secondly, we write the constancy of the pressure on (Γ); using Bernoulli's theorem and taking account of the stagnation in the upstream far field, we get

$$g\,\eta + \frac{V_R^2}{2} - \frac{C^2}{2} = 0 \qquad \text{on } (\Gamma) \tag{2.2}$$

We can derive tangentially the above condition; taking $\vec{s}$ as the unit streamwise tangent vector to (Γ) and putting $\vec{V}_R = V_R\,\vec{s}$, we get

$$g\,\vec{s}.\vec{y} + V_R \frac{\partial V_R}{\partial s} = 0$$

Explicitly, we have

$$\vec{s}.\vec{y} = \frac{\vec{V}_R}{V_R}.\vec{y} = \frac{\vec{V}}{V_R}.\vec{Y} = \frac{1}{V_R}\frac{\partial \Phi}{\partial y}$$

and

$$\frac{\partial V_R}{\partial s} = \frac{\partial}{\partial s}(\vec{V}_R.\vec{s}) = \frac{\partial}{\partial s}(\vec{V}.\vec{s}) - \frac{\partial}{\partial s}(\vec{C}.\vec{s}) = \frac{\partial^2 \Phi}{\partial s^2} - C\frac{\partial}{\partial s}(\vec{x}.\vec{s})$$

thus finally giving a new condition by substitution in (2.1) and (2.2)

$$\frac{\partial^2 \Phi}{\partial s^2} + \frac{g}{V_R^2}\frac{\partial \Phi}{\partial y} - C\frac{\partial}{\partial s}(\vec{x}.\vec{s}) = 0 \qquad (2.3)$$

In practice, we prefer to retain the following couple of equations

$$\left.\begin{aligned} &\frac{\partial^2 \Phi}{\partial s^2} + \frac{g}{V_R^2}\frac{\partial \Phi}{\partial y} - C\frac{\partial}{\partial s}(\vec{x}.\vec{s})\Big|_{y=h+\eta} = 0 \\ &g\eta + \left(\frac{V^2}{2} - C\frac{\partial \Phi}{\partial x}\right)\Big|_{y=h+\eta} = 0 \end{aligned}\right\} \qquad (2.4)$$

The boundary conditions on the profile contour (c) and on the bottom are obtained from the usual zero normal velocity condition, viz.

$$\frac{\partial \Phi}{\partial n} = \vec{C}.\vec{n} \quad \text{on (C)} \; ; \quad \frac{\partial \Phi}{\partial y} = 0, \; y = 0 \qquad (2.5)$$

Finally, we have to solve a boundary problem given by the following system

$$\left.\begin{aligned} &\text{(a)} \quad \Delta\Phi = 0 \\ &\text{(b)} \quad \left\{\begin{aligned} &\frac{\partial^2 \Phi}{\partial s^2} + \frac{g}{V_R^2}\frac{\partial \Phi}{\partial y} - C\frac{\partial}{\partial s}(\vec{x}.\vec{s}) = 0 \\ &g\eta + \frac{V^2}{2} - C\frac{\partial \Phi}{\partial x} = 0 \end{aligned}\right. \quad y = h + \eta \\ &\text{(c)} \quad \frac{\partial \Phi}{\partial n} = 0 \qquad y = 0 \\ &\text{(d)} \quad \frac{\partial \Phi}{\partial n} = \vec{C}.\vec{n} \qquad \text{on (C)} \end{aligned}\right\} \qquad (2.7)^*$$

*In the case of a lift, it will also be necessary to enforce a Kutta-Jukowsky condition at the trailing edge F, by adjusting the circulation around the profile so that the velocity remains finite at F.

The free surface conditions (b) are obviously too intricate to leave some hope of directly calculating a solution. However, several numerical studies using an iterative procedure have been proposed [9], [10]. Before going to the task of using one of those, it is often advantageous to solve the simplified Neumann-Kelvin problem [11], [12] which can be thought of as first step of an iterative scheme. In this case, we first admit that the absolute velocity V of the fluid is small compared to the profile velocity C. It is furthermore assumed that the curvature of the free surface is small enough to allow to assimilate the unit vectors $\vec{s}$ and $-\vec{x}$; finally, if the amplitude of the free surface displacement is small enough, the conditions (b) can be taken on $y = h$ instead of $y = h + \eta$.

The free surface conditions are thus linearized and simplified to

$$(b')\begin{cases} \dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial\Phi}{\partial y} = 0 & y = h \\ \eta = -\dfrac{C}{g}\dfrac{\partial\Phi}{\partial x} & \nu = \dfrac{g}{C^2} \end{cases}$$

These two equations are now independent: the first unambiguously characterizes the potential for the linearized boundary problem and the second gives the free surface elevation once the potential is determined.

3. KELVIN SOURCE

The Neumann Kelvin problem can be solved by either a finite element method or a singularity method. Whatever the adopted procedure, one is first led to solve, via potential theory, the following elementary problem, pertaining to a point mass source of intensity q at location P(a,b)

$$\left.\begin{array}{ll} \Delta\Phi = \dfrac{q}{2\pi}\,\delta(x-a)\,\delta(y-b) & \\ \dfrac{\partial^2\Phi}{\partial x^2} + \nu\dfrac{\partial\Phi}{\partial y} = 0\,, & y = h \\ \dfrac{\partial\Phi}{\partial n} = 0\,, & y = 0 \end{array}\right\} \qquad (3.1)$$

4. THE MATHEMATICAL PROBLEM

The previous considerations lead to study the flow generated by a source of constant intensity q, moving with a uniform motion in a channel of constant depth. We use the stream function ψ, the conjugate of the potential Φ. The notations are as following. Now, the relative variables (x,y) are dimensionless. The fluid fills the domain $\mathcal{B} = \{(x,y)/0 < y < 1\}$. The source is set at the point P = (0,b), $0 < b < 1$. The equations depend on the parameter $\nu = g\,h\,C^{-2}$ (1), without dimension. (4.5) is the slip condition on the bottom. The condition at the down-stream (resp. upstream) takes the form (4.10) (resp. (4.11)). In addition, we observe the stream function ψ cannot be uniform in $\mathcal{B}$: if γ is a simple curve surrounding the point P, we have

$$\int_{\gamma} \frac{\partial \psi}{\partial s}\, ds = q \tag{4.1}$$

Then, we have to introduce a cutting Γ in $\mathcal{B}$: we choose $\Gamma = \{(0,y)/b \leq y < 1\}$ and we put $\Omega = \mathcal{B} - \Gamma$. From (4.1), we have $\psi^{+} - \psi^{-} = q$ on Γ, and (the velocity being uniform in $\mathcal{B}$) $\frac{\partial \psi^{+}}{\partial x} = \frac{\partial \psi^{-}}{\partial x}$ on Γ (2). Using the function h defined by

$$h(y) = 0 \text{ if } 0 < y < b; \quad h(y) = q \text{ if } b < y < 1$$

and indicating by $[\![\ ]\!]$ the discontinuity through $x = 0$, these conditions take the form (4.8), (4.9). The condition on the free boundary can be written

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \nu \frac{\partial \psi}{\partial x} = 0 \qquad \text{on } y = 1 \ ; \tag{4.3}$$

this relation can be separately integrated on the two half straight lines $x > 0$ and $x < 0$ (because the cutting Γ meets $y = 1$) and gives $\frac{\partial \psi}{\partial y} - \nu \psi = C^{\pm}$; from (4.5) and (4.10), first we have $C^{+} = 0$, and, after, $C^{-} = \nu q$, because (4.8) and (4.9). Then (4.3) is equivalent to the two conditions (4.6), (4.7). Thus we have the

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(1) with the notations of the previous sections

(2) in the following, the indice + (resp. -) indicate a quantity relating to $x > 0$ (resp. $x < 0$).

Problem 4.1. To find a function ψ defined on Ω, such that

$$\Delta\psi = 0 \qquad \text{in } \Omega \tag{4.4}$$

$$\psi(x,0) = 0 \tag{4.5}$$

$$\frac{\partial\psi}{\partial y} - \nu\psi = \nu q \qquad \text{on } y = 1,\ x < 0 \tag{4.6}$$

$$\frac{\partial\psi}{\partial y} - \nu q = 0 \qquad \text{on } y = 1,\ x > 0 \tag{4.7}$$

$$[\![\psi]\!] = h \tag{4.8}$$

$$[\![\frac{\partial\psi}{\partial x}]\!] = 0 \tag{4.9}$$

ψ is bounded, uniformly with respect to y, if $x \to -\infty$ (4.10)

$\psi \to 0$, uniformly with respect to y, if $x \to +\infty$ (4.11)

5. EXISTENCE, UNIQUENESS, AND REPRESENTATION FORMULA

The space variables absolute (X,Y) and relative (x,y) are related by $X = x + Ct$ $Y = y$; then x is (more or less) the time, and the idea is to use the Fourier method, in considering the conditions (4.8), (4.9) like initial conditions. This method consists in looking for the solution by means of its expansion on the special basis $(s_n,\ n \geq 0)$ defined in the lemma 5.1 later.

The first part of this section describes the formal method and the uniqueness theorem. After, the results are given for the most interesting case $\nu > 1$ (similar formulas would be easy written for $\nu \leq 1$). The third part is devoted to studying the convergence problem and gives the existence theorem.

Lemma 5.1. The system (5.1) below has countable eigenelements $(\lambda_n,\ s_n,\ n \geq 0)$, and the corresponding sequence of eigenfunctions is a complete orthonormal system in $L^2(0,1)$.

$$s'' + \lambda s = 0,\ 0 < y < 1\ ;\quad s(0) = 0;\quad s'(1) = \nu s(1) \quad (1) \tag{5.1}$$

---

(1) In the following, we use the notations: $\frac{\partial f}{\partial y} = f'$; $\frac{\partial f}{\partial x} = \dot{f}$

The proof consists to show that the Garding inequality (cf. [13]) is satisfied by the system (5.1).

Remark 5.2. Except for the first values of n (for which the result depends on ν) we have $\lambda_n = \omega_n^2$; in addition $\omega_n/n \to \pi$ if $n \to \infty$.

Principle of the method. Formal study and uniqueness. A solution ψ of the problem can be expanded on the basis (s_n), with coefficients ψ_n functions of x, in the form (5.2). The same thing can be done for the function h defined in (4.2).

$$\psi = \sum_{n\geq 0} \psi_n s_n \quad (\text{or } \psi(x,y) = \sum_{n\geq 0} \psi_n(x)\, s_n(y)); \quad \psi_n(x) = \int_0^1 \psi(x,y)\, s_n(y)\, dy \tag{5.2}$$

$$h = \sum_{n\geq 0} \gamma_n s_n; \quad \gamma_n = \int_0^1 h(y)\, s_n(y)\, dy \tag{5.3}$$

Formally at least, the expansion (5.2) verify the boundary conditions (4.3), (4.5). For each n, we compute ψ_n with (5.2), taking into account (4.4), and (after an integration by parts) (4.6), (4.7); from this, we obtain the differential equations (5.4), (5.5) for ψ_n on the domains $x < 0$ and $x > 0$. The relations of discontinuity (4.8), (4.9), the conditions at infinity (4.10), (4.11) give respectively (5.6),...(5.9).

$$\ddot{\psi}_n = \lambda_n \psi_n - \nu q\, s_n(1) \qquad x < 0 \tag{5.4}$$

$$\ddot{\psi}_n = \lambda_n \psi_n \qquad x > 0 \tag{5.5}$$

$$[\![\psi_n]\!] = \gamma_n \tag{5.6}$$

$$[\![\dot{\psi}_n]\!] = 0 \tag{5.7}$$

$$\psi_n \text{ is bounded if } x \to -\infty \tag{5.8}$$

$$\psi_n \to 0 \text{ if } x \to +\infty \tag{5.9}$$

From that, it is easy to obtain the uniqueness: the linear problem 4.1 becomes homogeneous when $q = 0$; in this case, the coefficients γ_n are zero

and $\psi_n = 0$ is the unique solution of the system (5.4), ... (5.9). Then, we have the

Theorem 5.3. The problem 4.1 has at most one solution.

The formal construction of the solution is easy; integration of the differential equations (5.4), (5.5) gives

$$\psi_n^- = F_n^- + \nu q \frac{s_n(1)}{\lambda_n} , \quad \psi_n^+ = F_n^+$$

where $F_n^{\pm}$ are (in general) linear combinations of trigono exponential functions, according to the sign of λ_n, each one depending on two numerical coefficients. Conditions (5.6), ..., (5.9) take the form (5.10), (5.11) and permit to compute all these coefficients

$$F_n^- \text{ is bounded if } x \to -\infty ; \quad F_n^+ \to 0 \text{ if } x \to +\infty ; \tag{5.10}$$

$$[\![F_n]\!] = \gamma_n + \nu q \frac{s_n(1)}{\lambda_n} = \alpha_n ; \quad [\![\dot{F}_n]\!] = 0 \tag{5.11}$$

Remark 5.4. Taking (5.1) into account, we have

$$\int_0^1 y\, s_n(y)\, dy = (1 - \nu) \frac{s_n(1)}{\lambda_n} ;$$

we deduce:

$$\sum_{n \geq 0} \nu q \frac{s_n(1)}{\lambda_n} s_n(y) = \frac{\nu q}{1-\nu} y ;$$

then, the coefficients α_n can be defined by the expansion

$$h(y) + \frac{\nu q}{1-\nu} y = \sum_{n \geq 0} \alpha_n s_n(y) \tag{5.12}$$

Representation formulas. They are only given for $\nu > 1$; in this case, the eigenelements of (5.1) are given by (5.13); the coefficients V_n are chosen to assure that the norm of s_n in $L^2(0,1)$ is the unity. Then we obtain the formal solution (5.14), where the coefficients $\beta_n = V_n \alpha_n$ are given by (5.15).

$$\lambda_o = -\omega_o^2 \ , \ \omega_o = \nu \text{ th } \omega_o \ , \ s_o(y) = V_o \text{ sh } \omega_o y$$

$$\lambda_n = \omega_n^2 \ , \quad \omega_n = \nu \text{ tg } \omega_n \ , \ s_n(y) = V_n \sin \omega_n y \ , \ n \geq 1 \tag{5.13}$$

$$\psi^+(x,y) = \frac{1}{2} \sum_{n \geq 0} \beta_n \, e^{-\omega_n x} \sin \omega_n y \tag{5.14}$$

$$\psi^-(x,y) = \beta_o \cos \omega_o x \text{ sh } \omega_o y + \frac{q\nu}{1-\nu} y - \frac{1}{2} \sum_{n \geq 0} \beta_n \, e^{\omega_n x} \sin \omega_n y$$

$$\beta_o = 4q \frac{\text{ch } \omega_o b}{\text{sh } 2\omega_o - 2\omega_o} \ ; \quad \beta_n = 4q \frac{\cos \omega_n b}{2\omega_n - \sin 2\omega_n} \tag{5.15}$$

Existence theorem We have essentially to give a meaning to the series (5.14), i.e. to state precisely the convergences. Lemma 5.7 and 5.8 constitute the proof of theorem 5.5.

Theorem 5.5. The problem 4.1 has a (unique) solution ψ, which is given by (5.14), (5.15). This solution is very regular for $x \neq 0$, and verifies, for all finite $X > 0$ and $0 < s < 1/2$:

$$\psi \in L^2(0,X; H^s(0,1)) \tag{5.16}$$

Remark 5.6. From (5.16), we deduce $\psi'' \in L^2(0,X; H^{s-2}(0,1))$; then because (4.4), ψ has the same property. In using the theorem of intermediate derivatives and the trace theorem (cf. [14]), we deduce that (4.8) and (4.9) have a meaning.

Lemma 5.7. (i) For all $X > 0$, the series (5.14) and all their derivatives are uniformly convergent in the domains $|x| > X$.

(ii) For all finite $X > 0$, and $0 < s < 1/2$, the series (5.14) are convergent in the space $L^2(0,X; H^s(0,1))$.

Lemma 5.8. The function ψ defined by (5.14) is solution of the problem 4.1.

Proof of lemma 5.7. From the remark 5.2, and (5.15), we obtain

$\omega_n/n \le C_1$, $n\,\beta_n \le C_2$ [(1)], for $n \to \infty$. Then (i) is obvious.

On the other hand, we have:

$$\|\psi_n\|^2_{L^2(0,X)} = c\int_0^X \beta_n^2\, e^{-2\omega_n x}\, dx \le \frac{C_1}{\omega_n^3}$$

$$\|s_n\|^2_{H^2(0,1)} = \iint_{(0,1)\times(0,1)} \frac{|s_n(y) - s_n(\eta)|^2}{|y-\eta|^{1+2s}}\, dy\, d\eta \qquad \text{(cf. [13])}$$

$$= v_n^2 \iint_{(0,1)\times(0,1)} \frac{|\sin \omega_n y - \sin \omega_n \eta|^2}{|y-\eta|^{1+2s}}\, dy\, d\eta$$

$$\le c\,\omega_n^{2-2\sigma} \iint_{(0,1)\times(0,1)} \frac{dy\, d\eta}{|y-\eta|^{2s+2\sigma-1}}$$

in using $|\sin u| \le |u|^{1-\sigma}$ with $0 \le \sigma \le 1$. Then, if $2s + 2\sigma - 1 < 1$, $\|s_n\|^2_{H^s(0,1)} \le C_2\,\omega_n^{1-\sigma}$ and

$$\|\psi_n\, s_n\|_{L^2(0,X;\, H^s(0,1))} \le \frac{c}{\omega_n^{\sigma+1/2}}$$

Then, the series (5.14) are convergent if there exists σ, $0 \le \sigma \le 1$, such that $2s + 2\sigma - 1 < 1$ and $\sigma + 1/2 > 1$, i.e. if $0 < s < \frac{1}{2}$.

Proof of lemma 5.8. From lemma 5.7. (i), we deduce (4.4), ..., (4.7), (4.10), (4.11), (for $x \neq 0$) in a classical meaning. Now, we call S_N the function defined by its restrictions S_N^- and S_N^+ to Ω^- and Ω^+, obtained in truncating the series (5.14) at N. Because the part (ii) of the lemma 5.7, the continuity of the trace operator (cf. [14]), we obtain

$$\begin{aligned} &[\![S_N]\!] \to [\![\psi]\!] \quad (\text{and } [\![S_N]\!] \to h) \quad \text{in } H^{s-1/2}(0,1) \\ &[\![\dot{S}_N]\!] \to \dot{\psi} \quad (\text{and } [\![\dot{S}_N]\!] \to 0) \quad \text{in } H^{s-3/2}(0,1)\ , \end{aligned} \tag{5.17}$$

from which we conclude (4.8), (4.9).

---

(1) In the following, C, C_1, C_2, ... are some positive constants

Now, let $\varphi \in \mathcal{D}(\Omega)$. We compute $(\Delta S_N, \varphi) = (S_N, \Delta\varphi)$ in using an integration by parts, which introduces the discontinuities of S_N and $\dot{S}_N$ on $x = 0$:

$$(S^N, \Delta\varphi) = \int_{x=0} \left\{ [\![S_N]\!] \frac{\partial \varphi}{\partial x} + [\![\dot{S}_N]\!] \varphi \right\} dy \to 0$$

because the support of h does not meet Ω, and (5.17); we conclude (4.4), in the distribution sense.

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Sets of finite perimeter and capillarity phenomena

1. THE THEORY OF PERIMETERS

For simplicity in the exposition, we shall consider only subsets of the euclidean 3-dimensional space $\mathbb{R}^3$, this being enough for the subsequent physical applications. The general theory can be found e.g. in [2] and [7]. We begin with the following

Definition Let Ω be an open subset of $\mathbb{R}^3$; then, for any $E \subset \mathbb{R}^3$, E measurable, we can define:

$$\text{area } (\partial E \cap \Omega) = \sup \{\int_E \text{div}\phi(x,y,z)\,dxdydz : \phi \in C_o^1(\Omega; \mathbb{R}^3), |\phi| \leq 1\}$$

a quantity which is simply called the *perimeter of* E *in* Ω.

A straightforward application of the Gauss-Green theorem shows, that in the case when $\partial E \cap \Omega$ is the graph of a Lipschitz-continuous function $f : A \to \mathbb{R}$, A open in $\mathbb{R}^2$, then it holds

$$\text{area } (\partial E \cap \Omega) = \int_A \sqrt{1 + |Df(x,y)|^2}\,dxdy .$$

Moreover, the Riesz representation theorem shows that if area $(\partial E \cap \Omega) < +\infty$, then the (distributional) gradient $D\phi_E$ of the characteristic function ϕ_E of the set E is a Radon vector measure on Ω; in this case,

$$\text{area } (\partial E \cap \Omega) = \int_\Omega |D\phi_E|$$

where the integral on the right denotes the total variation of the measure $D\phi_E$, evaluated on Ω. We can also prove the following results:

Compactness theorem Let $\{E_h\}$ be a sequence of sets satisfying, for every $K \subset\subset \Omega$: $\int_K |D\phi_{E_h}| < c(K)\ \forall h$; then there exist a subsequence $\{E_{h(k)}\}$ of $\{E_h\}$ and a set E such that $\forall K \subset\subset \Omega$ there holds: $\lim_{h\to+\infty} \int_K |\phi_{E_{h(k)}} - \phi_E|\,dxdydz = 0$,

i.e. $E_{h(k)}$ converges, locally on Ω, to the set E.

Semicontinuity theorem The application $E \to \int |D\phi_E|$ is lower semi-continuous with respect to the local convergence on Ω.

The existence of solutions to the problems we shall consider later, usually follows from the results just stated; this approach, of course, opens the question of their regularity. For sets of *minimal boundary*, the next theorem was proved by E. De Giorgi in 1960:

Regularity theorem Let $E \subset \mathbb{R}^3$ be such that $\int_\Omega |D\phi_E| \leq \int_\Omega |D\phi_F|$, $\forall F : E \Delta F \subset\subset \Omega$; then $\partial E \cap \Omega$ is locally an analytic surface.

The method of proof of the preceding theorem, which involves a careful examination of the behaviour of the *excess*: $\Lambda(\rho) = \rho^{-2}\left(\int_{B\rho(x)} |D\phi_E| - \left|\int_{B\rho(x)} D\phi_E\right|\right)$ on balls $B_\rho(x)$ with decreasing radius, can be carried out for "almost minimal boundaries" as well. Precisely, it can be proved that if for some ε, $0 < \varepsilon < 2$, and $\forall F : F \Delta E \subset\subset B\rho(x)$ and $\forall B\rho(x) \subset\subset \Omega$ it holds: $\int_{B\rho(x)} |D\phi_E| \leq \int_{B\rho(x)} |D\phi_F| + c\rho^{2+\varepsilon}$, then $\partial E \cap \Omega$ is smooth (of class $C^{1,\varepsilon/2}$) in Ω. See [14, 16].

In conclusion, we recall that for every set E of finite perimeter in Ω (i.e. $\int_\Omega |D\phi_E| < +\infty$) we can define the *trace* of E on $\partial\Omega$ (still denoted by ϕ_E), at least when $\partial\Omega$ is Lipschitz-continuous. This will be especially useful when considering the energy of contact between a fluid E and the solid walls of a container Ω.

2. THE CAPILLARY TUBE

We will show in the sequel how the previous ideas apply to the study of some typical problems of capillarity. We consider at first a capillary tube of section A (A open and bounded in $\mathbb{R}^2$), partially immersed into a liquid, and we assume that a certain configuration of the liquid in the tube is described by the graph S of a function $f : A \to [0, +\infty)$; its energy will have the following expression

$$\text{energy of } S = \mathcal{F}(f) = \int_A \sqrt{1 + |Df|^2} - \nu \int_{\partial A} f + \frac{K}{2} \int_A f^2$$

where the integrals on the right correspond respectively to surface energy, energy of contact liquid-walls of the tube, and gravitational energy.

Here, κ is a constant ≥ 0 (physically, $\kappa = \rho g \sigma^{-1}$, where ρ = density difference across S, g = gravitational acceleration and σ = surface tension coefficient), while $\nu \in (0,1]$ (in fact, for $\nu \leq 0$ the minimum of $\mathcal{J}(\cdot)$ is clearly $f \equiv 0$, while $\inf \mathcal{J}(\cdot) = -\infty$ if $\nu > 1$); physically, ν represents the cosine of the angle γ between S and the walls of the tube.

The solutions to the problem $\mathcal{J}(\cdot) \to \min$ will be found among the L^1-functions $f : A \to [0, +\infty)$, whose subgraph $E_f = \{(x,y,z) : (x,y) \in A \text{ and } z < f(x,y)\}$ has finite perimeter in $A \times \mathbb{R}$; for such a function f we have by definition

$$\int_A \sqrt{1 + |Df|^2} = \text{area } (\partial E_f \cap (A \times \mathbb{R}))$$

while $\int_A f$ corresponds to the integral of the trace of E_f on $\partial A \times [0, +\infty)$; here, we are assuming ∂A Lipschitz-continuous, with Lipschitz constant $L > 0$.

The existence theorem relies on the following "trace estimate" (see [3]):

$$\int_{\partial A} f \leq \sqrt{1 + L^2} \int_A |Df| + c \int_A f^2 \tag{2.1}$$

where the constant c depends on the geometry of A. From (2.1) we get easily a lower bound for the energy, when $0 < \nu \leq \dfrac{1}{\sqrt{1 + L^2}}$

$$\inf \mathcal{J}(\cdot) \geq -\frac{\nu^2 \cdot c^2 \cdot |A|}{\kappa} \qquad (|A| \equiv \text{meas } A) \ . \tag{2.2}$$

Moreover, if $0 < \nu < \dfrac{1}{\sqrt{1 + L^2}}$ and $\mathcal{J}(f_j) \leq \text{const.}\ \forall j$, then we obtain

$$\int_A \sqrt{1 + |Df_j|^2} \leq |A| + \frac{\text{const'.}}{1 - \nu\sqrt{1 + L^2}} \quad \text{and} \quad \int_A f_j^2 \leq \text{const"}.$$

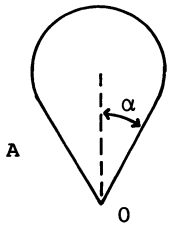
In conclusion, from the compactness and semicontinuity theorems of section 1, we derive the existence of a solution f, which is in fact unique (since $\mathcal{J}(\cdot)$ is strictly convex) and bounded on every ball B cc A, because the graph S of f, a minimum of "area + gravity" in B, must lie below the (analytic, hence locally bounded) graph of least area in B, and coinciding with S outside B.

The regularity of the solution f can be proved without appealing to De Giorgi theorem in section 1. For, let u_ε be the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} Tu = \kappa u_\varepsilon & \text{in } B_R \\ u_\varepsilon = f_\varepsilon & \text{on } \partial B_R \end{cases}$$

with f_ε = ε-regularized of f and B_R a ball $\subset\subset$ A (with sufficiently small radius R). Then, known maximum principles and a priori bounds for solutions of quasilinear elliptic equations (see [12, 13] imply that $u_\varepsilon \in C^2(\bar{B}_R)$ and $\sup_{B_{R/2}} (|u_\varepsilon| + |Du_\varepsilon|) < \text{const.}$ (independent of $\varepsilon > 0$). Defining $v_\varepsilon = u_\varepsilon$ in B_R, $v_\varepsilon = f_\varepsilon$ in $A - B_R$, it follows that f is the limit of $\{v_\varepsilon\}$ as $\varepsilon \to 0^+$, from which we conclude that $f \in \operatorname{Lip}(B_{R/2})$ and then, in view of general results for elliptic equations, f is analytic in A.

Hence, we can solve our problem under the hypothesis $0 < \nu < \dfrac{1}{\sqrt{1 + L^2}}$, which turns out to be an "almost necessary" assumption. In fact easy examples show that when the reverse inequality holds, the problem may admit no solution; this is the case for the domain A in figure



here, if $\alpha + \gamma < \frac{\pi}{2}$, that is, if $\nu > \dfrac{1}{\sqrt{1 + L^2}}$, then $\inf \mathcal{J}(\cdot) = -\infty$, as one can easily check by means of the function $f(r) = (r \lg r)^{-1}$, truncated near the origin. Such a behaviour agrees with physical experiments, which show a discontinuous transition at $\alpha = \frac{\pi}{2} - \gamma$ (cf. [5], section 1.6). Nevertheless the condition $0 < \nu < \dfrac{1}{\sqrt{1 + L^2}}$ may really be too severe; a more manageable assumption, which applies to a wide class of cases of physical interest, is worked out in [15].

As a final remark, we note that the preceding techniques do not work when $\kappa = 0$, i.e. in the absence of gravity, as we may suspect looking at (2.2). The study of the behaviour of the solutions f_κ, corresponding to values $\kappa \to 0^+$, has been undertaken by the first author in [4], where the uniform convergence to $+\infty$ of $\{f_\kappa\}$ is proved.

3. LIQUID DROPS

Let us now consider the problem of a liquid drop, sitting on the horizontal plane $z = 0$; as in section 2, we can write the global energy of a certain configuration $E \subset \{z > 0\}$ in the following way

$$\mathcal{J}(E) = \int_{z>0} |D\phi_E| + \nu \int_{z=0} \phi_E \, dxdy + \kappa \int_{z>0} z\phi_E(x,y,z)\, dxdydz \tag{3.1}$$

The constant κ is again ≥ 0, while $\nu \in [-1,1]$. The admissible configurations are represented by the sets $E \subset \{z > 0\}$, having a given volume: meas $E = v > 0$.

This way, we are led to the study of a constrained non-cartesian problem. It is exactly the volume constraint which makes the proof of the existence of solutions a non trivial one, since it is not preserved (in general) by the L^1_{loc}-convergence on the half-space $\{z > 0\}$. However, from the trivial inequality: $\int_{z>0} |D\phi_E| > \int_{z=0} \phi_E \, dxdy$, we derive immediately the relation $\mathcal{J}(E) \geq \frac{1+\nu}{2} \int |D\phi_E|$, and then a uniform bound for the perimeters of any minimizing sequence E_j, satisfying $\mathcal{J}(E_j) \to \inf \mathcal{J}(\cdot)$

$$\int |D\phi_{E_j}| < \frac{\text{const.}}{1+\nu} \quad , \text{ if } \nu > -1 \ . \tag{3.2}$$

In order to prevent the spreading of the sets E_j over $\{z > 0\}$, we use a symmetrization procedure, which consists in replacing each horizontal section of E_j by a disc (with the same measure), centered on the z-axis. This operation decreases the perimeter of E_j, unless E_j is already symmetric (see [9]); as a consequence of (3.2), the elements of a symmetrized minimizing sequence are in fact all confined in a (fixed) cylinder around the z-axis; on the other hand, gravity pulls them toward the plane $z = 0$, so that, in conclusion, a subsequence of $\{E_j\}$ converges to some set E in $L^1(\{z > 0\})$ (and not only locally, as we would have derived from (3.2)).

Thus

Existence theorem for sitting drops For any $v > 0$, $\kappa > 0$, $\nu \in (-1,1]$, the functional (3.1) admits a minimum in the class $\{E \subset \{z > 0\} : \text{meas } E = v\}$. This minimum is unique and symmetric with respect to the z-axis. (see [9] and [6], where the uniqueness is proved).

The same functional (3.1), with $\kappa < 0$, can be chosen to represent the energy of a pendent liquid drop. However, for better understanding, we rewrite it as follows

$$\mathcal{J}(E) = \int_{z<0} |D\phi_E| + \nu \int_{z=0} \phi_E \, dxdy + \kappa \int_{z<0} z\phi_E(x,y,z)\,dxdydz \qquad (3.3)$$

with $E \subset \{z < 0\}$ satisfying meas $E = v > 0$, $\nu \in [-1, 1]$ and $\kappa > 0$.

The new difficulty arises from the unboundedness of $\mathcal{J}(\cdot)$: this is consistent with the physical meaning, since a pendent drop corresponds to a relative minimum of the energy. The existence of such a *relative minimum* can be proved as follows: we firstly minimize $\mathcal{J}(\cdot)$ among the admissible configurations $E \subset \{T < z < 0\}$ (this corresponds to insert a "ground floor" - the plane $\{z = T\}$ with $T < 0$ - below the "ceiling" $\{z = 0\}$); in doing this we proceed as we did above, when studying sitting drops. Next, we examine the behaviour of the solutions E_κ (for fixed values of $\nu \in (-1,1)$, $v > 0$ and $T < 0$), as $\kappa \to 0^+$. It is an easy calculation to show that $E_\kappa \to E_0$ in $\{T < z < 0\}$, and that E_0 is a solution of the same problem *in the absence of gravity* (i.e. E_0 is a minimum, in $\{T < z < 0\}$, of the functional (3.3) with $\kappa = 0$); moreover, the convergence $\partial E_\kappa \to \partial E_0$ is *uniform* in $\{T < z < 0\}$. On the other hand, if $|T|$ is large enough, then E_0 is a portion of a suitable ball (completely determined by the volume constraint: meas $E_0 = v$, and by the requirement on the contact angle γ between ∂E_0 and the plane $\{z = 0\}$: $\cos \gamma = \nu$), which lies strictly above the plane $\{z = T\}$; finally, from the uniform convergence $\partial E_\kappa \to \partial E_0$, we obtain that for a sufficiently small κ the solution E_κ does not touch the ground $\{z = T\}$: therefore, E_κ represents a pendent drop.

To sum up, we have the following result (see [11] and [8]):

Existence theorem for pendent drops For every $\nu \in (-1,1)$ and $v > 0$ there exist $T < 0$ and $\kappa_0 > 0$ such that for every $\kappa \in (0,\kappa_0)$, the solution E_κ of $\mathcal{J}(E) \to \min$ ($\mathcal{J}(\cdot)$ given by (3.3)) in the class $\{E \subset \{T < z < 0\}, \text{meas } E = v\}$

satisfies $E_K \subset \{T + \delta < z < 0\}$ for some $\delta > 0$.

The regularity (analyticity) of the solution can be proved again in a straightforward way. For, the free surface of the drop, being rotationally symmetric, can locally be described as the graph of a suitable function f, which turns out to minimize a functional of the type

$$\int \sqrt{1 + |Df|^2} + \text{gravity} + \lambda \int f$$

in which the integrals have to be evaluated on a two-dimensional domain, while λ is a suitable "Lagrange multiplier" which takes into account the volume constraint. From this, the regularity result follows arguing as in section 2 (see [10, 11]).

4. ROTATING FLUIDS

As we have just seen, the symmetry of the solutions plays the chief role in deriving both the existence and regularity of pendent or sitting drops. When symmetry fails, deeper methods must be used. As a meaningful example in this direction, we shall consider in this section the problem of a rotating drop, i.e. of a quantity of liquid in rotation around a fixed axis through its center of mass, held together by surface tension.

Denoting by E the drop which rotates with constant angular velocity around the z-axis, and assuming its center of mass coinciding with the origin of the co-ordinate system, we may express its energy by

$$\mathcal{F}_\omega(E) = \int_{\mathbb{R}^3} |D\phi_E| - \omega \int_E (x^2+y^2)\,dxdydz \tag{4.1}$$

where ω is a non-negative coefficient.

The admissible configurations are the sets $E \subset \mathbb{R}^3$, satisfying meas E = 1 and $\int_E x\,dxdydz = \int_E y\,dxdydz = \int_E z\,dxdydz = 0$. Again, we have to find a relative minimum of the problem $\mathcal{F}_\omega(\cdot) \to \min$, since $\inf \mathcal{F}_\omega(\cdot) = -\infty$ if $\omega > 0$. To this aim, we define E to be a *local minimum* of the energy functional (4.1) if there exists $R > R_o = \left(\frac{3}{4\pi}\right)^{1/3}$ such that $E \subset\subset C_R$ and $\mathcal{F}_\omega(E) \le \mathcal{F}_\omega(F)$, for every admissible $F \subset\subset C_R$, where C_R denotes the cylindrical container

$$C_R = \{(x,y,z) : x^2 + y^2 < R^2 \text{ and } |z| < R\}\ .$$

As a first immediate result we have that for every $\omega \geq 0$ and every $R > R_o$ there is a solution E_ω to the problem $\mathcal{F}_\omega(E) \to \min$, *restricted* to the admissible configurations $E \subset C_R$. Such a solution, however, will not generally satisfy the condition $E_\omega \subset\subset C_R$ for a local minimum. Next, we can prove without difficulties, that for a fixed $R > R_o$ the solutions E_ω converge, as $\omega \to o^+$, to the ball $E_o = \{(x,y,z) : x^2 + y^2 + z^2 < R_o^2\} \subset\subset C_R$; unfortunately, the L^1-convergence $E_\omega \to E_o$ is too weak to conclude that $E_\omega \subset\subset C_R$ for ω small enough. In order to do this, we will use the following result:

Fundamental lemma For sufficiently small $\omega > 0$ there exists a value $r \in (R_o, R)$ such that $\int_{\partial C_r} \phi_{E_\omega} = 0$.

Its proof (rather long and technically complicated) can be found in [1]. From the lemma, the fact that $E_\omega \subset\subset C_R$ for ω small can be proved as follows. Define

$$G = E_\omega \cap C_r \quad \text{and} \quad \alpha = \text{meas}\,(E_\omega - C_r)$$

and choose $F = (1-\alpha)^{-1/3} G$ (dilatation of G with coefficient $(1-\alpha)^{-1/3} > 1$), so that meas F = 1. Then perform a suitable translation $F \to \tilde{F}$ in such a way that the center of mass of $\tilde{F}$ is 0. Since $\alpha \to 0$ as $\omega \to 0$, it is not difficult to prove, that for ω small $\tilde{F}$ is an admissible configuration $\subset C_R$; on the other hand, the difference $\mathcal{F}_\omega(E_\omega) - \mathcal{F}_\omega(\tilde{F})$ can easily be estimated from below by the quantity

$$\left[1 - (1 - \alpha)^{-2/3}\right] \cdot \int_{\mathbb{R}^3} |D\phi_G| + \text{const.}\alpha + \text{const!}\ \alpha^{2/3} . \tag{4.2}$$

Now, from the Taylor expansion of the coefficient of $\int_{\mathbb{R}^3} |D\phi_G|$, we conclude that $\mathcal{F}_\omega(E_\omega) - \mathcal{F}_\omega(\tilde{F}) > 0$ if α is positive and small, which would contradict the minimality of E_ω. Therefore, for ω small enough, α must be 0, so that the corresponding solution E_ω is contained in $C_r \subset\subset C_R$, thus concluding the proof of the existence of local minima for the energy (4.1).

The preceding method can also be used to prove the regularity of the solution E. In fact, arguing as above, we obtain in particular the existence of two balls B_1, $B_2 \subset C_R$, with radius $\rho > 0$ and satisfying:

$\text{mis}(E \cap B_1) = \text{mis}\, B_1$, $\text{mis}(E \cap B_2) = 0$. For $x \in \partial E \cap C_R$ and $0 < t < \rho$ choose F such that $F \Delta E \subset\subset B_t(x)$; if $\beta \equiv \text{mis}(E \cap B_t(x)) - \text{mis}(F \cap B_t(x))$ is positive, we can translate B_1 until it reaches a new position B_3 such that $\text{mis}(B_3 - E) = \beta$. Then, defining $F_t = (F \cap B_t(x)) \cup (E - B_t(x)) \cup B_3$ and adjusting the position of the center of mass with a translation $F_t \to \tilde{F}_t$, we find, as a consequence of the minimality of E

$$\int_{B_t(x)} |D\phi_E| + \int_{B_3} |D\phi_E| \leq \int_{B_t(X)} |D\phi_{F_t}| + \int_{\partial B_3} \phi_{F_t} + c \cdot t^3 \tag{4.3}$$

From the inequality $\int_{\partial B_3} \phi_L \leq \int_{B_3} |D\phi_L| + \frac{3}{\rho} \text{meas}\, L$, which holds $\forall L \subset B_3$, see [15], it follows

$$\int_{\partial B_3} \phi_{F_t} + c \cdot t^3 \leq \int_{B_3} |D\phi_E| + \left(\frac{c'}{\rho} + c\right) \cdot t^3 \tag{4.4}$$

Combining (4.3) and (4.4) we find $\int_{B_t(x)} |D\phi_E| \leq \int_{B_t(x)} |D\phi_F| + \left(\frac{c'}{\rho} + c\right) \cdot t^3$

which holds if $\beta \leq 0$ as well. The regularity of ∂E then follows from the discussion following the regularity theorem in section 1.

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Comparison theorems for flows with vorticity

1. INTRODUCTION

The comparison theorems of Lavrentiev, Gilbarg and Serrin for steady ideal fluid flows without vorticity [1-3] have long been useful in establishing a priori global bounds for various nonlinear free boundary problems of ideal fluid mechanics [4-6]. The comparison theorems are the Inclusion Theorem, in which flow domain D_2 lies inside flow domain D_1, and the Under-Over Theorem, in which the upper boundary Γ_1 of D_1 crosses smoothly over the corresponding boundary Γ_2 of D_2.

When D_1 is changed to D_2 (so that $D_1 \supset D_2$) by moving the upper flow boundary Γ_1 then the flow speed (i) increases on the (other stationary) boundary of $D_1 \cap D_2$, and (ii) decreases on the unchanged portions of Γ_1. Further, the streamlines are all moved closer to the stationary boundary of $D_1 \cap D_2$. When the upper boundaries of the flow domains D_1 and D_2 cross over by having only the arc MN in common and such that Γ_2 starts above Γ_1 then the ratio of the flow speeds q_1/q_2 increases from M to N.

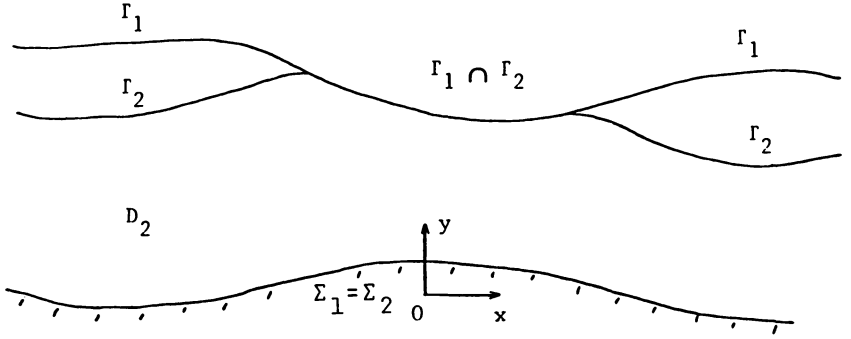

Fig. 1. The flow geometries for the Inclusion Theorem

Since all real fluid flows possess vorticity, this work generalises the above theorems to include those steady ideal fluid flows possessing the vorticity function $\omega(\psi)$ which is constant along the streamlines of the flow. The generalisations require either that $\omega'(\psi) < \lambda_1$ (the first eigenfunction for the domain $D_1 \cup D_2$), or that the flows compared be forward (positive solutions of the elliptic boundary value problem).

In the next section we give an outline of the general results, which are then applied to fluid flows in section three. In section four we extend the previous work to axisymmetric flows in R_3. A more detailed version of the R_2 work is given in [7], although the Vorticity Comparison Theorem is new. An application to rotational wave flows which obtains a priori bounds on waves on a stream with vorticity appears in [8].

2. GENERAL RESULTS

For simplicity we consider only domains D which are simply-connected in the upper half plane R_2^+ extending from $x = -\infty$ to $x = +\infty$. We suppose that D has as boundaries the smooth non-selfintersecting arcs Γ (upper) and Σ (lower) such that

$$\inf \{y_s \mid (x,y_s) \in \Sigma\} = 0, \quad \sup \{y_g \mid (x,y_g) \in \Gamma\} = h < \infty.$$

The flux Q is a fixed positive constant. Given a function $\omega \in C^1(R)$ we say that the bounded function $\psi \in C(\bar{D})\, C^2(D)$ solves problem (I) if

$$\begin{aligned} &\Delta\psi (=\nabla^2\psi = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = -\omega(\psi) \quad \text{for } (x,y) \in D, \\ &\psi = 0 \text{ for } (x,y) \in \Sigma \text{ and } \psi = Q \text{ for } (x,y) \in \Gamma. \end{aligned} \tag{I}$$

By a *flow* in D (with flux Q and vorticity function ω) we mean a streamfunction ψ solving (I) in D.

We now give the general form of the comparison theorems, on which the fluid mechanical applications depend.

Inclusion Theorem Let ω' be bounded on R. Let flows (with the same $\omega(\psi)$) be defined in domains D_1 and D_2 with $\Sigma_1 = \Sigma_2$. If $D_1 \supseteq D_2$ and $\psi_1 < \psi_2$ in D_2 then either $\psi_1 \equiv \psi_2$ in D_2 or
(i) for any outward normal n from D_2,

$$\frac{\partial \psi_1}{\partial n} > \frac{\partial \psi_2}{\partial n} \quad \text{for } P \in \Sigma_1 = \Sigma_2 \text{ and for } P \in \Gamma_1 \cap \Gamma_2 \ , \tag{2.1}$$

and (ii) the sets (for $j = 1,2$ and $\tilde{\psi}$ a constant)

$$D_1(\tilde{\psi}) = \{(x,y) \in D_j \,|\, \psi_j(x,y) < \tilde{\psi}\}$$

satisfy

$$D_1(\tilde{\psi}) \supseteq D_2(\tilde{\psi}) \ , \tag{2.2}$$

with strict inclusion whenever $D_2(\tilde{\psi}) \subset D_2$.

Proof Define $\psi \equiv \psi_1 - \psi_2$ in D_2. Assume $\psi \not\equiv 0$ in D_2, since otherwise there is nothing to prove. Then ψ satisfies

$$\Delta\psi - c(x,y)\psi = 0 \quad \text{in } D_2 \ , \tag{2.3}$$

with

$$c(x,y) = \begin{cases} \omega(\psi_1) - \omega(\psi_2) \ (\psi_1 - \psi_2)^{-1} & \text{if } \psi_1(x,y) \neq \psi_2(x,y) \\ \omega'(\psi_1(x,y)) & \text{if } \psi_1(x,y) = \psi_2(x,y) \end{cases} \tag{2.4}$$

Since $\psi < 0$ in D_2 the maximum principle [11] implies that $\psi < 0$ in D_2, and hence the Hopf Lemma [11], see also [7], gives us (2.1). If for some constant $\tilde{\psi} > 0$, we assume that $D_1(\tilde{\psi}) = D_2(\tilde{\psi})$, then we have $\psi = 0$ on $\partial D_2(\tilde{\psi})$, and this contradiction on $\psi < 0$ in D_2 establishes (2.2).

Corollary If in the Inclusion Theorem $\psi_1 > 0$ and $\psi_2 < 0$ in D_2 then either $\psi_1 \equiv \psi_2$ in D_2 or the flow speeds $q_i \equiv |\nabla\psi_i|$ satisfy

$$q_1(P) > q_2(P) \text{ for } P \in \Gamma_1 \ \Gamma_2 \text{ and } q_1(P) < q_2(P) \text{ for } P \in \Sigma_1 = \Sigma_2.$$

Proof Since $\psi_1 = \psi_2$ on $\Gamma_1 \cap \Gamma_2$ and $\Sigma_1 = \Sigma_2$ we have $q_i \equiv |\partial\psi_i/\partial_n|$ there ($\partial/\partial n$ is the outward normal derivative to the smooth boundary); and the Corollary follows by considering the signs of $\partial\psi/\partial n$ on $\Gamma_1 \cap \Gamma_2$ and $\Sigma_1 = \Sigma_2$.

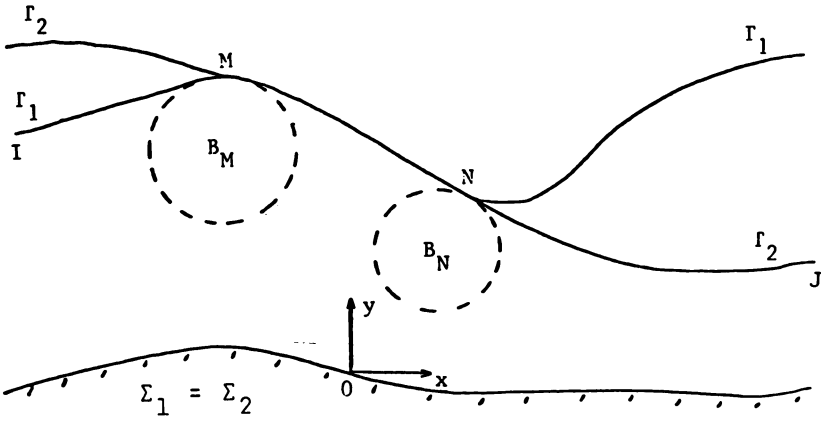

Fig. 2. The flow geometries for the Under-Over Theorem

Under-Over Theorem Let ω' be bounded on R. Let flows (with the same $\omega(\psi)$) be defined in domains D_1 and D_2 with $\Sigma_1 = \Sigma_2$. Suppose Γ_1 and Γ_2 have a common arc MN which forms part of the boundary of $D = D_1 \cap D_2$. Let the remaining boundary of D consist of an arc NJ of Γ_2 (J at $x = +\infty$), an arc IM of Γ_1 (I at $x = -\infty$) and $\Sigma_1 = \Sigma_2$. Suppose that there exist in D balls B_M and B_N containing respectively M and N as boundary points and in which respectively $\psi_1 > \psi_2$ and $\psi_1 < \psi_2$. Then either the flows are identical in D or

$$\psi_1 > \psi_2 \text{ in } B_M, \quad \frac{\partial \psi_1}{\partial n}(M) < \frac{\partial \psi_2}{\partial n}(M), \tag{2.6}$$

$$\psi_1 < \psi_2 \text{ in } B_N \text{ and } \frac{\partial \psi_1}{\partial n}(N) > \frac{\partial \psi_2}{\partial n}(N) . \tag{2.7}$$

Corollary If in the Under-Over Theorem $\psi_1 < Q$ in B_M then $q_1(M) < q_2(M)$, and if $\psi_2 < Q$ in B_N then $q_1(N) > q_2(N)$.

Proofs See [7].

Remark Since $\partial\psi/\partial n$ is continuous on MN there is at least one point

$P \in MN$ where $\partial\psi/\partial n$ vanishes, so that $q_1(P) = q_2(P)$. This condition is necessary for the Under-Over Theorems of sections 2 and 3. The above results justify the second paragraph of the Introduction.

Vorticity Comparison Theorem Suppose that the vorticity functions $\omega_1(\psi)$ and $\omega_2(\psi)$ satisfy condition (3.1). If $\omega_1(t) > \omega_2(t)$ then the flows ψ_1 and ψ_2, determined by ω_1 and ω_2 respectively, with the same flux Q through the same domain D satisfy the results (2.1) and (2.2) (strictly).

Proof As in the proof of the Domain Comparison Theorem we define $\psi = \psi_1 - \psi_2$ in D; then we have

$$-\Delta\psi = \omega_1(\psi_1) - \omega_2(\psi_2) = \omega_1(\psi_1) - \omega_1(\psi_2) + \omega_1(\psi_2) - \omega_2(\psi_2) ,$$

$$-\Delta\psi - \omega_1'(t)\psi = \omega_1(\psi_2) - \omega_2(\psi_2) > 0 ,$$

for some $t(x,y)$ lying between $\psi_1(x,y)$ and $\psi_2(x,y)$ in D. Thus the maximum principle may be used, as in the aforementioned proof, to establish the results of the theorem.

Remark If (say) $\omega_1(\psi) \equiv 0$, then we may compare the rotational flow determined by $\omega_2(\psi)$ with the potential flow through the same geometry.

3. APPLICATIONS TO FLUID FLOWS

The simplest situations for using the general theorems of section two are those for which there is only one possible flow through the domain D. Suppose that the widths h_i of flows in D_i (i=1,2) satisfy $h_i < h$.

Our first general restriction (which guarantees uniqueness of flow i(=1,2) in domain D_i) is that the vorticity function be continuously differentiable and satisfy

$$\sup_t \omega'(t) < \pi^2/h^2 . \tag{3.1}$$

Then, in equations (2.3,4) we have, on using the mean value theorem,

$$\sup_{D_2} c(x,y) = \sup_t \omega'(t) < \pi^2/h^2 = \lambda_1 , \tag{3.2}$$

where $t(x,y)$ lies between $\psi_1(x,y)$ and $\psi_2(x,y)$. Since $\lambda_1 = \pi^2/h^2$ is the first eigenvalue for the domain $D_h = \{(x,y) \mid 0 < y < h, -\infty < x < \infty\}$, we have a positive eigenfunction in D_h which can be used in the application of the generalised maximum principle [11] in domains $D_i \subseteq D_h$ when condition (3.2) holds. For flows satisfying condition (3.1) we have uniqueness of each flow in D_i; if, as well, $\psi_i < Q$ in D_i then when $D_1 > D_2$ the Inclusion Theorem holds. If, in addition to (3.1) and $\psi_i < Q$ in D_i, we have the flow geometry of the Under-Over Theorem such that there is a point $P \in MN$ at which $q_1(P) = q_2(P)$, then the Under-Over Theorem implies that either

$$\psi_1 \equiv \psi_2 \quad \text{in } D_1 \cap D_2$$

or

$$q_1(M)/q_2(M) < 1 < q_1(N)/q_2(N). \tag{3.3}$$

Precise statements of the theorems and proofs are given in Keady and Norbury [7]. Note that the conditions $\psi_i < Q$ in D_i still have to be established in a particular situation. A general result is

$$\omega(\psi) < 0 \text{ implies } \psi_i < Q \text{ in } D_i,$$

which follows directly from the maximum principle; a more careful argument (by contradiction, after assuming $\psi_i > Q$ in D_i somewhere) shows that

Theorem 3.1. If $\omega(t) < 0$ when $t > Q$ then $\psi_i < Q$ in D_i.

Our second general restriction is to *forward* flows, which we define as follows. A flow is forward if

$$\Sigma = \{y = \sigma(x) > 0 \text{ for } -\infty < x < \infty\}, \quad \Gamma = \{y = \gamma(x) > \sigma(x), -\infty < x < \infty\} \tag{3.4}$$

and

$$u \equiv \partial\psi/\partial y > 0 \text{ for } (x,y) \in \overline{D}. \tag{3.5}$$

A forward flow is the basic solution of the nonlinear elliptic boundary value problem. Other flows may exist with the same geometry and vorticity function, but there is only one forward flow (unless no flows exist). Provided that we restrict attention to these forward flows then Inclusion

and Under-Over Theorems may be established with less difficulty than in the previous application.

For virtually arbitrary $\omega(\psi)$ we have comparison theorems for forward flows since the restriction to forward flows implies that ψ increases monotonically with y, and hence $0 < \psi < Q$ in D. The only extra condition needed is for the Under-Over Theorem, when the existence of the point $P \in MN$ with $q_1(P) = q_2(P)$ is necessary (see the remark in section two).

Examples are given in Keady and Norbury [7] to show that all the previous restrictions are best possible. Forward flows may satisfy the condition (3.1), but most do not; and flows which satisfy condition (3.1) may be forward, but most are not.

4. AXISYMMETRIC EXTENSIONS

We now consider axisymmetric flows of an ideal liquid in R_3, the flows having no swirl. The governing differential equation for the streamfunction ψ is now, in the domain D in the plane of symmetry with $y^{-1}\nabla\psi \in C(\bar{D})$,

$$-\Delta\psi + y^{-1}\partial\psi/\partial y = y^2\,\omega(\psi) \quad \text{in D}, \tag{4.1}$$

where

$$u = y^{-1}\partial\psi/\partial y \quad \text{and} \quad v = -\,y^{-1}\,\partial\psi/\partial x, \tag{4.2}$$

and D is as before except that

$$\inf_x \{y_s \,|\, (x,y) \in \Sigma\} = h_o > 0. \tag{4.3}$$

When $h_o = 0$ we have a modification of the maximum principle results when the boundary point P lies on the axis of the symmetry $y = 0$. We now have $\partial\psi/\partial y = 0 = v$ at P and in the Hopf Theorem the outward normal derivative becomes (see §3. Boundary point lemmas [10])

$$\lim_{y\downarrow 0} -\frac{1}{y}\,\frac{\partial\psi}{\partial y} = -\,u(P). \tag{4.4}$$

In condition (3.1) we take λ_1 to be the first eigenvalue of the strip $h_o < y < h$. The proofs of the uniqueness of the flow in D for each $\omega(\psi)$ satisfying condition (3.1), and of the comparison theorems, now goes through with only the minor modification of multiplying $\partial\psi/\partial y$ terms by y^{-1} and $\omega(\psi)$ terms by y^2.

As (4.4) implies we must now replace $\partial\psi/\partial n$ by $-u(P)$ when P is on the axis $y = 0$ in the Inclusion and Under-Over Theorems.

We treat forward flows just as we did in §3. Again we restrict the domain D as in (3.4), and we ask that $u > 0$ in $\bar{D}$. Then for any $\omega(\psi) > -\pi^2(h_1 + h_2)^{-1}$ for which a forward flow exists (continuous $\omega(\psi)$ is sufficient) we have uniqueness in the class of forward flows, and comparison theorems for axisymmetric forward flows follow exactly as in the plane case. The proofs use an ingenious transformation of dependent and independent variables, that is, from $\psi(x,y)$ to $y(x,\psi)$, and D is transformed to the strip $E = \{-\infty < x < \infty,\ 0 < \psi < Q\}$. The governing differential equation is, for the linear uniformly elliptic differential operator

$$Q(y)u = y_\psi^{-3}\{y_\psi^2 u_{xx} - 2y_x y_\psi u_{x\psi} + (1+y_x^2)u_{\psi\psi}\},$$

$$Q(y)y + y_\psi^{-1}y^{-1} = y^2\omega(\psi) \quad \text{in } E. \tag{4.5}$$

Note that $-Q(y)y \equiv \operatorname{div} Ay \equiv (\partial/\partial x, \partial/\partial\psi)\cdot(-y_x y_\psi^{-1}, \tfrac{1}{2}(1+y_x^2)y_\psi^{-2})$ and $|\partial A_i|\partial p_j| = y_\psi^{-4} > 0$.

We now have

$$y(x,Q) = \gamma(x) > y(x,0) = \sigma(x) \geq 0, \tag{4.6}$$

while

$$u = y^{-1}y_\psi^{-1} \quad \text{and} \quad v = y^{-1}y_x y_\psi^{-1}. \tag{4.7}$$

When we compare domains note that at the points where the speed is evaluated we always have $y_1(x,\psi) = y_2(x,\psi)$ and $\partial y_2/\partial x = \partial y_1/\partial x$ so that $u_1 > u_2$ and $v_1 > v_2$ when $\partial y_1/\partial\psi < \partial y_2/\partial\psi$; and note also that, if $D_1 \supset D_2$ strictly, then $y_1 > y_2$ in E. Thus $y_1 > y_2$ in E implies that $u_1 > u_2$, $v_1 > v_2$ on $\psi = Q$ while $u_1 < u_2$, $v_1 < v_2$ on $\psi = 0$.

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Steady flow with vorticity

In this note we discuss a variational method used to prove both existence and certain asymptotic properties for a class of steady ideal fluid flows possessing vorticity. The approach requires that the flows be either planar or axisymmetric, that is, we deal with either vortex pairs of vortex rings and their generalizations.

We consider first the fluid domain $D = \{x = (x_1,x_2) \in R^2;\ x_2 > 0\}$. We seek a velocity field $u(x) = (u_1(x),u_2(x))$ satisfying the steady Euler equations

$$\nabla \cdot u = 0$$

$$u \cdot \nabla u = -\nabla p \quad \text{in} \quad D\ ,$$

along with the boundary conditions

$$u_2 = 0 \quad \text{on} \quad \partial D$$

$$u \to U(1,0) \quad \text{as} \quad |x| \to \infty\ .$$

Introducing the streamfunction $\psi(x)$ so that $u = (\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1})$, and the vorticity $\omega(x)$ defined by $\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = -\Delta\psi$, we find that the Euler equations are satisfied whenever

$$\omega = f(\psi) \tag{1}$$

where $f(t)$ is an arbitrary function. Further, in view of the boundary conditions, we seek ψ in the form

$$\psi(x) = \int_D g(x,x')\omega(x')dx' - Ux_2 - \mu \tag{2}$$

where $g(x,x')$ is the Green's function for $-\Delta$ on D (with zero Dirichlet data)

and U (stream speed) and μ (flux) are constants. We now state a variational problem whose solutions yield a triple $\omega(x)$, U, μ which satisfies (1,2) for a given f(t).

Let the kinetic energy functional

$$E(\omega) = \frac{1}{2}\int_D \int_D g(x,x')\omega(x)\omega(x')\,dxdx' \tag{3}$$

be maximized over all measurable functions $\omega(x_1,x_2) = \omega(-x_1,x_2) \geq 0$ on D subject to the constraints

$$1 = \int_D x_2\omega(x)\,dx, \tag{4}$$

$$\int_D \omega(x)\,dx \leq 1, \tag{5}$$

$$\operatorname*{ess\,sup}_{x\in D} \omega(x) \leq \lambda; \tag{6}$$

λ is a positive parameter. Our basic existence result is the following

Theorem There exists a solution ω_λ of the problem (3,4,5,6) having the form

$$\omega_\lambda(x) = \lambda I_{\{\psi(x)>0\}}$$

for ψ defined by (2) with Lagrange multipliers $U_\lambda > 0$ and $\mu_\lambda \geq 0$. Also, the free boundary $\partial\{\psi > 0\}$ is a smooth graph $x_1 = \pm\phi(x_2)$. In this result $f(t) = \lambda I_{\{t>0\}}$.

Proof The proof is obtained by first solving a penalized problem in which $f(t) = \lambda(t^+)^\beta$ and then taking the limit $\beta \to 0^+$; this permits the rigorous justification of the computation of the first variation. The penalized functional

$$E_\beta(\omega) = E(\omega) - \int_D \frac{1}{p}\left[\omega(x)/\lambda\right]^p dx \qquad (p = 1 + 1/\beta)$$

is maximized over the constraints without (6). The proof is somewhat

complicated by the fact that the domain D is unbounded and hence a priori bounds on the support of ω must be found; otherwise the proof is quite direct.

We note that the constraints are imposed on quantities with the classical physical meanings: (4) fluid impulse, (5) circulation, (6) vortex strength. Except for the constraints (5,6) the form of the variational principle is idential with that suggested by Benjamin [1], who in turn refers to Arnold and Kelvin.

We may now consider the limiting behaviour of ω_λ as $\lambda \to \infty$. First it can be shown that $\mu_\lambda \to \infty$ as $\lambda \to \infty$, and hence (5) is an equality for large λ (since $\mu_\lambda \neq 0$). Then in view of (4) we except that $\omega_\lambda \sim \delta_{(0,1)}$, the delta measure at $x = (0,1)$. This is confirmed by the estimate

$$\text{diam supp } \omega_\lambda = O(\lambda^{-1/2}) \quad \text{as } x \to \infty$$

which constitutes the main technical result in the asymptotic analysis; the fact that ω_λ is a maximizer and hence that $E(\omega_\lambda) = \frac{1}{4\pi} \log \lambda^{-1/2} + O(1)$ is crucial here.

We now turn our attention to the case when D is a subdomain of the halfplane; in particular, when $\{x_2 > 0\} \backslash D$ is a smooth and bounded profile. The variational principle is modified so that $g(x,x')$ is the Green's function for D, and constraint (4) is replaced by

$$P = \int_D \eta(x)\omega(x)dx \qquad (P > 0 \text{ given}) \tag{4'}$$

when $\eta(x)$ is the streamfunction for potential flow in D with unit speed at infinity; that is, $\Delta\eta = 0$ in D, $\eta|_{\partial D} = 0$, and $\eta = x_2 + O(|x|^{-1})$ as $|x| \to \infty$. Also, the symmetry of $\omega(x)$ in x, is lost. The existence of solutions follows in fundamentally the same way as before (the term Ux_2 in (2) is replaced by $U\eta$). However, since D is no longer homogeneous to scaling, we have a *two*-parameter family of solutions $\omega_{\lambda,P}$.

The physical interpretation is now rather less standard: the kinetic energy is $E + \frac{1}{2} MU^2$ and the fluid impulse is $P + MU$ where

$$M = \int_D |\Delta(\eta - x_2)|^2 dx \qquad \text{(induced mass)}.$$

In the asymptotic analysis as $\lambda \to \infty$ we find as before that $\omega_{\lambda,p} \sim \delta_{x_p}$, the delta measure at $X_p \in D$; however, now the point X_p is to be determined. We assert that X_p is a solution of the minimization problem

$$\min H(x) \quad \text{subject to} \quad \eta(x) = p \tag{7}$$

where $H(x) = \frac{1}{2}h(x,x)$ and $h(x,x')$ is the regular part of Green's function:

$$g(x,x') = \frac{1}{2\pi} \log|x - x'|^{-1} - h(x,x') \ .$$

It can be shown that (7) always possesses a solution and that all solutions satisfy $|x| \leq C_p$. In specific situations the variational conditions $\nabla(H + U\eta) = 0$ can be used to determine the point X_p explicitly; for example, in the case $D = \{x_2>0, |x|>1\}$ this was done by Föppl, cf. Lamb [3,p.223].

Extension of the above to axisymmetric flow (vortex rings) may be achieved with the use of analogous variational principles; cf. Friedman and Turkington [2].

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H W ALT

The dam problem

For the construction of a dam it is important to know how water will seep through the different layers of the earth dam. The mathematical treatment of this problem gave rise to many contributions during the last decade. Here I will describe the physical data, the mathematical formulation of the problem, and the main results obtained in the last years. In the stationary case I will give a detailed existence proof.

THE STATIONARY DAM PROBLEM

The porous medium is an open bounded connected set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary ($n = 2$ or 3 in physical situations). The boundary of Ω is divided into three measurable sets S_{res}, S_{air} and S_{imp}, which denote respectively the boundary to reservoirs, to atmosphere and the impervious part of the boundary (Fig. 1). To be sure that nontrivial solutions exist we assume that $\mathcal{H}^{n-1}(S_{res}) > 0$. The water flow through the porous medium is described by its pressure u. On S_{res} and S_{air} the pressure is given by the boundary values of a function $u^o \in H^{1,2}(\Omega) \cap L^\infty(\Omega)$. For simplicity we assume that the atmospheric pressure is constant, that is $u^o = \text{const} \geq 0$ on S_{air}, and that the water pressure u^o on S_{res} is nonnegative, which means that the pressure is normalized such that zero is the capillary pressure. This is the pressure below which saturated flow fails to exist. The capillary pressure depends on the fluid and on the porous medium and is usually less than the atmospheric pressure, but sometimes the difference can be neglected.

The law by which fluid flow through porous media can be described was found by Darcy experimentally and reads

$$\vec{v} = -K(\nabla u + e).$$

This says that the velocity $\vec{v}$ is proportional to the gradient of the piezometric head, which is the sum of the pressure u and the height. Here the e term comes in because of the gravity, that is, e is the vertical unit vector. The conductivity K in general is a matrix. Since water can be considered to be incompressible, the equation of continuity is

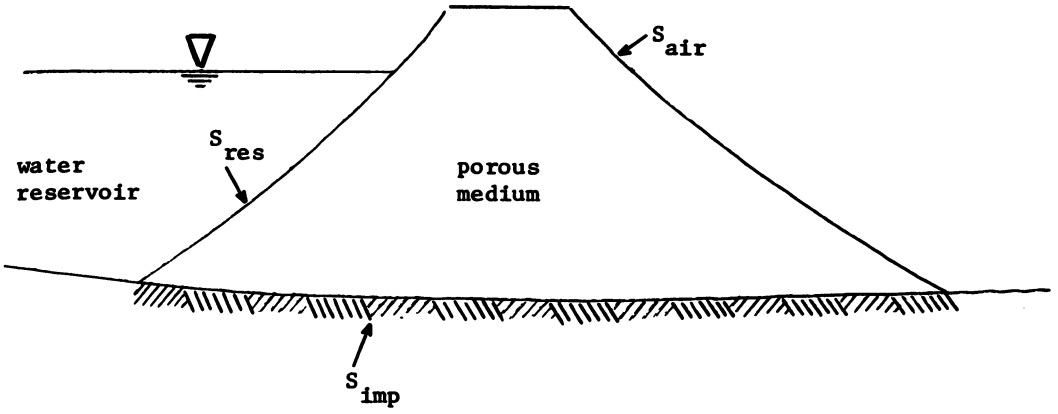

Figure 1

$$\nabla \cdot \vec{v} = 0.$$

The boundary conditions on the three boundary sets are

$$u = u^0 \quad \text{on } S_{res},$$

$$\vec{v} \cdot \nu = 0 \quad \text{on } S_{imp},$$

$$u \le u^0, \quad \vec{v} \cdot \nu \ge 0,$$

$$(u-u^0)\vec{v} \cdot \nu = 0 \quad \text{on } S_{air}.$$

The condition S_{imp} may be replaced by a nonzero flux condition (see [4]). The condition on S_{air} says that because of the capillary force the pressure there may be less than the outside pressure. In this case there is no flow across this part of the boundary. On the other hand the pressure on S_{air} cannot exceed the atmospheric pressure; instead of this fluid is seeping out of the porous medium. Therefore we call this also overflow condition.

The conductivity K depends on x and u. For simplicity we may assume that

$$K(x,y) = k(u)a(x),$$

where k is a nonnegative bounded function, and where a is a measurable bounded uniformly elliptic matrix. We consider two different kinds of functions k since we deal with two different mathematical models. The mathematical problem related to the gross model is called the free boundary problem for fluid flow through porous media. For this we take a function k as in Figure 2 (more general situations were considered in [3]). The

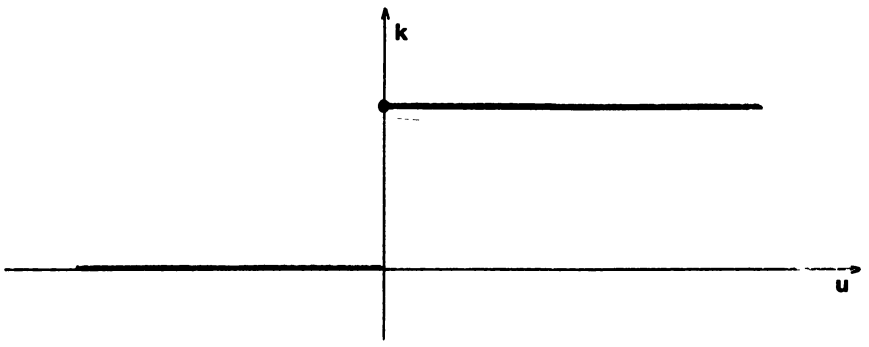

Figure 2

problem then can be formulated in the convex set

$$M(u^o) := \{v \in H^{1,2}(\Omega)/v = u^o \text{ on } S_{res} \text{ and } v \le u^o \text{ on } S_{air}\}.$$

In fact, in a weak sense the above properties are equivalent to the variational inequality

$$\int_\Omega \nabla(v-u)(k(u)a)(\nabla u+e) \ge 0 \qquad \text{for all } v \in M(u^o)$$

with u also in $M(u^o)$. Since $u^o \ge 0$ and $k(s) = 0$ for $s < 0$ we may assume that $u \ge 0$ (and then $k(u) = 1$ in the above integral). Thus the domain Ω is divided into the two regions $\{u > 0\}$ and $\{u = 0\}$ and formally u satisfies the free boundary conditions

$$u = 0 \text{ and } (\nabla u+e)\cdot\nu = 0 \quad \text{on } \partial\{u > 0\}.$$

Although first one always might expect a solution in this sense, it turns

out that the following more general notion of a solution must be introduced.

1. Stationary problem A pair u,γ is called a solution of the stationary dam problem, if

$$u \in M(u^o) \quad \text{with } u \geq 0,$$

$$\gamma \in L^\infty(\Omega) \quad \text{with } \chi_{\{u>0\}} \leq \gamma \leq 1,$$

such that

$$\int_\Omega \nabla(v-u)\, a(\nabla u+\gamma e) \geq 0 \qquad \text{for every } v \in M(u^o).$$

This means that in the saturated region $\{u > 0\}$ the pressure satisfies an elliptic equation, whereas in $\Omega\backslash\overline{\{u > 0\}}$ the function γ is the solution of a first order equation describing the flow in unsaturated regions. $\{\gamma = 0\}$ is the dry region. An example for which the unique solution has an unsaturated region, that is, the set $\{0 < \gamma < 1\}$ has positive Lebesgue measure, was discussed in [3] 4.6, see Figure 3 here. On the other hand the set $\{u > 0\}$ is a subgraph, if $\nabla\cdot(ae) \geq 0$ (in Figure 3 not satisfied), which was proved in [3] 4.1-4.4. Under additional technical assumptions this implies that $\gamma = \chi_{\{u>0\}}$, which in a particular case was proved in [5] 4.

It is known that

2. Theorem There is a solution u,γ of the stationary dam problem. u is Hölder continuous.

The proof of this theorem was first done in the very special case of a rectangular dam in two dimensions based on the now called Baiocchi transformation, see [7]. Later this was generalized in [17] to the three dimensional case. The transformed function satisfies a variational inequality as described in the following. Theorem 2 will then be proved below.

3. Lemma Suppose Ω is the cylinder $\Omega^* \times]0,h[$ and that a is the identity. Suppose further that

$$\Omega^* \times \{0\} \subset S_{imp}, \quad \Omega^* \times \{h\} \subset S_{air},$$

and that each line segment $\{x\} \times]0,h[$ on the shell either belongs to S_{imp}

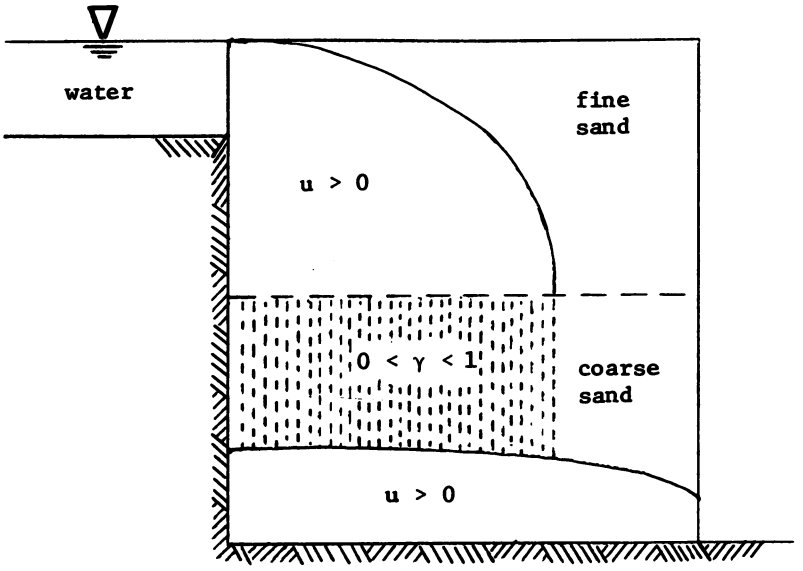

Figure 3

or to $S_{res} \cup S_{air}$, such that ∂S_{imp} is removable for $H^{1,2}$ (n-2 dimensional), and such that S_{res} does not reach the top of Ω with $u^{o} = 0$ on S_{air}. Then, if $u, \chi_{\{u>0\}}$ is a solution of the stationary dam problem, the function

$$w(x^*, x_n) := \int_{x_n}^{h} u(x^*, s)\, ds$$

lies in the convex set (w^{o} defined similar)

$$K(w^{o}) := \{v \in H^{1,2}(\Omega) / v \geq 0 \text{ and } v = w^{o} \text{ on } S_{res} \cup S_{air}$$

$$\text{and } v = w^* \text{ on } \Omega^* \times \{0\}\}$$

and is a solution of the variational inequality

$$\int_{\Omega} (\nabla(v-w)\nabla w + (v-w)w) \geq 0 \quad \text{for all } v \in K(w^{o}).$$

Here w^* is in

$$K^*(w^o) := \{v \in H^{1,2}(\Omega^*)/v = w^o \text{ on } \partial\Omega^* \cap \overline{S_{res} \cup S_{air}}\}$$

and is a solution of

$$\int_\Omega \nabla(v-w^*)\nabla w^* = 0 \qquad \text{for all } v \in K^*(w^o)$$

proof First we notice that $\{u > 0\}$ is a subgraph (see [3] 4.4). This property is essential for the transformation since we conclude that

$$\{u > 0\} = \{w > 0\}.$$

The second observation is, that $u = 0$ in a neighbourhood of the top of Ω. This follows from the comparison lemma [5] 3.2. Then we consider the test function

$$\zeta(x^*,x_n) := \eta(x^*) + \int_0^{x_n} \phi(x^*,s)ds$$

where $\eta \in C_o^\infty(\Omega^* \cup S^*)$ and $\phi \in C_o^\infty(\Omega \cup S^* \times]0,h[)$. Here S^* is the set of points $x \in \partial\Omega$ such that $\{x\} \times]0,h[$ belongs to S_{imp}. Since $u = 0$ in a neighbourhood of the top, ζ is an admissible test function, that is

$$0 = \int_\Omega \nabla\zeta(\nabla u+\chi_{\{u>0\}}e)$$

$$= \int_\Omega \nabla\eta\nabla w + \int_\Omega (\nabla\phi\nabla w+\phi\chi_{\{w>0\}}),$$

which proves the lemma.

It is well known that variational inequalities as in Lemma 3 admit a unique solution. Although this is a very nice method to solve the dam problem, it turns out that in general Baiocchi type transformations do not lead to variational inequalities. Therefore other methods for proving the existence had to be developed. The first result in this direction was proved by the author in [1] with the overflow condition proved later in [2]. In [1] supersolutions of the problem were considered. It was possible to prove that a smallest supersolution exists, which then must be a solution of the problem. For the last conclusion a local Baiocchi transformation was used.

Also this method requires that $\{u > 0\}$ is a subgraph, which is not true in general for inhomogeneous porous media (Figure 2). Therefore the author gave another existence proof by approximating the free boundary problem by nonlinear elliptic equations, see [3]. In a special case this was done independently in [10] (which was written after [3] was submitted). Contrary to [10] the author used in [3] approximating problems which themselves describe physical situations, namely saturated-unsaturated fluid flow through porous media (see e.g. [8]). From the physical point of view it is even reversed. The free boundary problem is an approximation for saturated-unsaturated flow, and for porous media up to a size of several meters the latter description is more satisfactory.

Let us now give the mathematical formulation for saturated-unsaturated flow, which is the same as before including the boundary conditions except that we have to use positive continuous functions k as in Figure 4. Curves like this can be

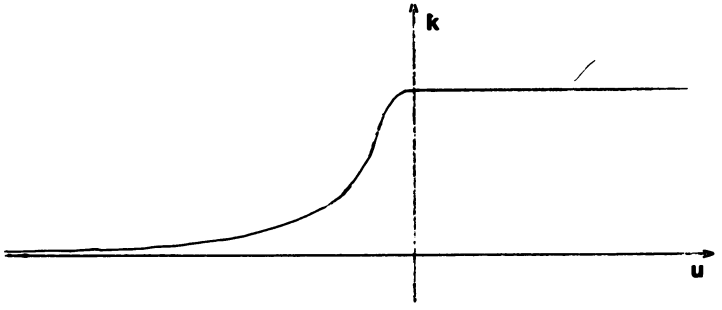

Figure 4

obtained by measurements and depend on the porous medium as well as on the fluid. We then see that

4. Lemma If k is Lipschitz continuous, positive, $k(s) = 1$ for $s \geq 0$, and $\int_{-\infty}^{0} k(s)ds = +\infty$, then there is a unique function $u \in M(u^0) \cap L^\infty(\Omega)$ such that

$$\int_\Omega \nabla(v-u)\,(k(u)a)\,(\nabla u+e) \geq 0 \qquad \text{for every } v \in M(u^0).$$

proof (In [3] this was proved under more general assumptions on k.) Using the Kirchhoff transformation

$$\bar{u}(x) := f_k(u(x)) := \int_0^{u(x)} k(s)ds$$

we see that we have to find a function $\bar{u} \in M(\bar{u}^o)$ (with $\bar{u}^o := f_k(u^o)$) such that

$$\int_\Omega \nabla(v-\bar{u})a(\nabla\bar{u}+k\circ f_k^{-1})(\bar{u})e) \geq 0 \quad \text{for every } v \in M(\bar{u}^o) .$$

Since this is a uniform elliptic variational inequality, it has a solution, and since the nonlinearity is Lipschitz continuous this solution is unique. Using [9] one sees that the solution is bounded, therefore

$$u := f_k^{-1}(\bar{u}) \in H^{1,2}(\Omega) \cap L^\infty(\Omega)$$

and solves the original variational inequality.

Approximating the free boundary problem by equations as in Lemma 4 is justified by the following consideration. Assume a sequence of problems is given with k as in Lemma 4 and boundary conditions depending on a parameter ε, say, large domains and large boundary values given by

$$\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega, \quad S^\varepsilon_{res} = \ldots, \quad u^o_\varepsilon(x) = \frac{1}{\varepsilon}u^o(\varepsilon x).$$

Let u_ε be the corresponding solutions, and consider the transformed functions

$$\tilde{u}_\varepsilon(x) := \varepsilon u_\varepsilon\left(\frac{x}{\varepsilon}\right) .$$

Then $\tilde{u}_\varepsilon$ is a solution with respect to Ω, S_{res}, ..., u^o and

$$\tilde{k}_\varepsilon(s) := k\left(\frac{s}{\varepsilon}\right) .$$

We see that the functions $\tilde{k}_\varepsilon$ converge to a step function. Therefore the next proof implies that for large porous media separating reservoirs the free boundary problem leads to a good approximating of the physical quantities.

5. Proof of Theorem 2 (See [3] Satz 2.6, 2.8.) Let us consider a sequence of Lipschitz continuous positive functions k_ε satisfying

$$k_\varepsilon(s) = 1 \quad \text{for } s \geq 0,$$

$$k_\varepsilon(s) \leq 1 \quad \text{for } s \leq 0,$$

$$k_\varepsilon \downarrow 0 \text{ uniformly in }]-\infty,-\delta[\quad \text{for all } \delta,$$

$$\int_{-\infty}^{0} k_\varepsilon(s)\,ds = +\infty.$$

Then the corresponding solution $u_\varepsilon \in M(u^0)$ satisfies

$$\int_\Omega \nabla(v-u_\varepsilon) k_\varepsilon(u_\varepsilon) a(\nabla u_\varepsilon + e) \geq 0 \quad \text{for every } v \in M(u^0),$$

and the Kirchhoff transformation $\bar{u}_\varepsilon := f_{k_\varepsilon}(u_\varepsilon)$ satisfies

$$\int_\Omega \nabla(v-\bar{u}_\varepsilon) a(\nabla \bar{u}_\varepsilon + \bar{k}_\varepsilon(\bar{u}_\varepsilon) e) \geq 0 \quad \text{for every } v \in M(u^0),$$

where $\bar{k}_\varepsilon := k_\varepsilon \circ f^{-1}_{k_\varepsilon}$. (Here $\bar{u}^0 = u^0$ since $u^0 \geq 0$ and $k_\varepsilon(s) = 1$ for $s \geq 0$.) Setting $v = \bar{u}^0$ we obtain that $\bar{u}_\varepsilon$ are bounded in $H^{1,2}(\Omega)$. Therefore we can choose a subsequence such that

$$\bar{u}_\varepsilon \to \bar{u} \quad \text{weakly in } H^{1,2}(\Omega),$$

$$\bar{u}_\varepsilon \to \bar{u} \quad \text{almost everywhere in } \Omega.$$

Moreover, since k_ε are bounded there is a function $\gamma \in L^\infty(\Omega)$ with

$$\bar{k}_\varepsilon(\bar{u}_\varepsilon) \to \gamma \quad \text{weakly star in } L^\infty(\Omega).$$

Then

$$\int_\Omega \nabla v a(\nabla \bar{u}_\varepsilon + \bar{k}_\varepsilon(\bar{u}_\varepsilon) e) \geq \int_\Omega \nabla \bar{u}_\varepsilon a(\nabla \bar{u}_\varepsilon + \bar{k}_\varepsilon(\bar{u}_\varepsilon) e)$$

$$= \int_\Omega \nabla \max(\bar{u}_\varepsilon, 0) a(\nabla \max(\bar{u}_\varepsilon, 0) + e) + \int_{\{\bar{u}_\varepsilon < 0\}} \nabla \bar{u}_\varepsilon a(\nabla \bar{u}_\varepsilon + \bar{k}_\varepsilon(\bar{u}_\varepsilon) e),$$

and in the limit $\varepsilon \to 0$ we obtain

$$\int_\Omega \nabla v a(\nabla \bar{u}+\gamma e) \geq \int_\Omega \nabla \max(\bar{u},0)\, a(\nabla \max(\bar{u},0)+e) + \limsup_{\varepsilon \to 0} \int_{\{\bar{u}_\varepsilon<0\}} \nabla \bar{u}_\varepsilon a(\nabla \bar{u}_\varepsilon+\bar{k}_\varepsilon(\bar{u}_\varepsilon)e),$$

or

$$\int_\Omega \nabla(v-\bar{u})\, a(\nabla \bar{u}+\gamma e) \geq \int_\Omega \nabla \max(\bar{u},0)\, ae(1-\gamma) - \int_\Omega \nabla \min(\bar{u},0)\, a(\nabla \bar{u}+\gamma e) + \limsup_{\varepsilon \to 0} \int_{\{\bar{u}_\varepsilon<0\}} \nabla \bar{u}_\varepsilon a(\nabla \bar{u}_\varepsilon+\bar{k}_\varepsilon(\bar{u}_\varepsilon)e).$$

Thus we have to show that the three terms on the right vanish. The first term is zero since

$$\bar{k}_\varepsilon(\bar{u}_\varepsilon) \to \bar{k}(\bar{u}) = 1 \quad \text{almost everywhere in } \{\bar{u} > 0\},$$

in other words

$$\chi_{\{\bar{u}>0\}} \leq \gamma \leq 1 .$$

Next consider the functions $v = \bar{u}_\varepsilon + \delta \min(\bar{u}_\varepsilon,0)$ which are in $M(u^0)$ for $\delta \geq -1$. Therefore we can use them as test functions for $\bar{u}_\varepsilon$ and obtain that the third term in the above inequality vanishes for each ε. It remains to prove that $\bar{u} \geq 0$. For this let $\delta > 0$ and consider the solution $v_\varepsilon \in H^{1,2}(\Omega)$ of the problem

$$v_\varepsilon = 0 \text{ on } S_{res}$$

$$\int_\Omega \nabla \zeta a(\nabla v_\varepsilon+\bar{k}_\varepsilon(\min(\bar{u}_\varepsilon,-\delta))e) = 0$$

for every $\zeta \in H^{1,2}(\Omega)$ vanishing on S_{res}. Then by the well known L^∞ estimate (see [9]) we have

$$\|v\|_{L^\infty(\Omega)} \leq C \sup_{s \leq -\delta} \bar{k}_\varepsilon(s) =: \delta_\varepsilon .$$

Then $v := \max(v_\varepsilon-\delta_\varepsilon-\delta,\bar{u}_\varepsilon)$ is in $M(u^0)$, which yields

$$\int_\Omega \nabla \max(v_\varepsilon-\delta_\varepsilon-\delta-\bar{u}_\varepsilon,0)\, a(\nabla \bar{u}_\varepsilon+\bar{k}_\varepsilon(\bar{u}_\varepsilon)e) \geq 0,$$

that is

$$c \int_{\Omega} |\nabla \max(v_\varepsilon - \delta_\varepsilon - \delta - \bar{u}_\varepsilon, 0)|^2$$

$$\leq \int_{\Omega} \nabla \max(v_\varepsilon - \delta_\varepsilon - \delta - \bar{u}_\varepsilon, 0) a(\nabla v_\varepsilon + \bar{k}_\varepsilon(\min(\bar{u}_\varepsilon, -\delta))e) = 0,$$

hence

$$\bar{u}_\varepsilon \geq v_\varepsilon - \delta_\varepsilon - \delta \geq -2\delta_\varepsilon - \delta.$$

Since $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ and since δ was arbitrarily chosen this shows that $\bar{u} \geq 0$, which ends the proof of Theorem 2.

The general formulation of the stationary problem in 1. is not only of theoretical interest, it can also be used for numerical approximations. Using a suitable discretization of 1. the author developed an algorithm, which can be used for inhomogeneous isotropic media of general shape, even in three dimensions, see [4]. The only fixed domain method available before was for the variational inequality of the Baiocchi transformation. In the contribution by J. C. Bruch, J. Remar and J. M. Sloss [11] Baiocchi transformations are used in the special case of a nonhomogeneous axisymmetric porous medium for numerical computations.

The regularity was studied in [2] and [3]. In [3] the Lipschitz continuity of the pressure was proved in regions where the conductivity a is Hölder continuous. In [2] the regularity of the free boundary was shown for homogeneous media. After this the behaviour of the free boundary was studied in [5] covering the two dimensional case. This paper contains also a comparison lemma and a uniqueness theorem. Independently another uniqueness theorem. Independently another uniqueness theorem was proved by J. Carillo-Menendez and M. Chipot, see [12].

A fundamental question is to justify Darcy's law by homogenization. In this volume the contribution of M. Codegone and J. F. Rodrigues [13] deals with the simpler question, whether Darcy's law is closed under the homogenization procedure, and of course the answer is yes.

THE NON-STEADY DAM PROBLEM

For the time dependent dam problem we have again to combine Darcy's law

$$\vec{v} = -K(\nabla u+e)$$

with the equation of continuity, which now reads

$$\partial_t \theta + \nabla\cdot\vec{v} = 0,$$

where θ is the water content. Here let us first describe saturated-unsaturated flow. In this case θ is a continuous monotone non-decreasing function of u as in Figure 5, in particular $\theta(s) = \theta(0) =: \theta_o$ for $s \geq 0$ and $\theta(s) > 0$ for all s. The conductivity K is similar as before

$$K(x,u) = k(\theta(u))a(x)$$

where a is again a bounded uniformly elliptic matrix, but k is a continuous

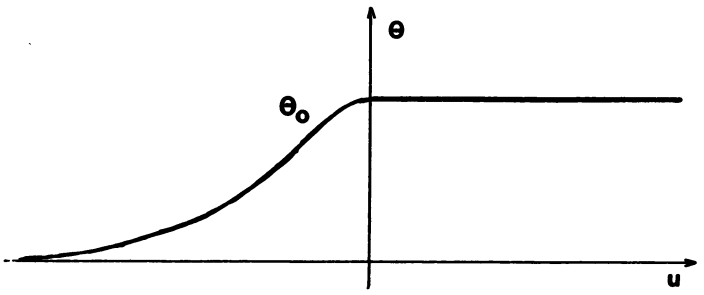

Figure 5

positive function with $k(0) = 0$ as in Figure 6. Both functions, θ and k, are the functions obtained originally by measurements, see [8]. The differential equation

$$\partial_t \theta(u) - \nabla\cdot(k(\theta(u))a(\nabla u+e)) = 0$$

is therefore an elliptic-parabolic differential equation with degenerate elliptic part. Using the Kirchhoff transformation

$$\bar{u}(x) := \int_0^{u(x)} k(\theta(s))ds$$

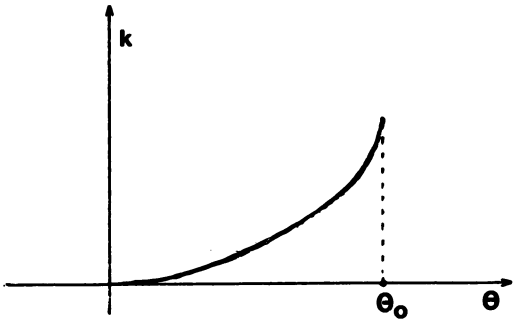

Figure 6

this equation is transformed into

$$\partial_t \bar{\theta}(\bar{u}) - \nabla\cdot(a(\nabla u + k(\bar{\theta}(\bar{u}))e)) = 0,$$

where $\bar{\theta}$ is a function with similar properties as θ.

Next we have to describe the initial and boundary conditions. We consider the time interval $[0,T]$. The initial condition is

$$\bar{\theta}(\bar{u}(0)) = \bar{\theta}(\bar{u}^1)$$

for some measurable function $\bar{u}^1$. As before, $]0,T[\times \Omega$ consists of three measurable sets S_{res}, S_{air} and S_{imp}, and we assume again that

$$\mathcal{H}^{n-1}(S_{res}(t)) \geq c > 0$$

for all t, where

$$S_{res}(t) := \{x \in \partial\Omega / (t,x) \in S_{res}\}.$$

The class of admissible functions is analogously

$$M_T(\bar{u}^o) := \{\bar{u} \in L^2(0,T; H^{1,2}(\Omega)) / \bar{u} = \bar{u}^o \text{ on } S_{res} \text{ and } \bar{u} \leq \bar{u}^o \text{ on } S_{air}\}.$$

This means that we, for example, allow the level of the reservoirs to change

with time. Of course we need some assumptions on the time dependence of the data. For example, it is enough to assume that $\bar{u}^o \in C^{o,1}([0,T] \times \bar{\Omega})$, where as before $u^o = \text{const} \geq 0$ on S_{air} and $u^o \geq 0$ on S_{res}, and that the sets $S_{res}(t)$ and $S_{res}(t) \cup S_{air}(t)$ do not change too much in the sense that the speed of progression is bounded. Then one can prove the following

6. Theorem There is a function $\bar{u} \in M(\bar{u}^o)$ such that $\partial_t \bar{\theta}(\bar{u}) \in L^2(0,T;H^{1,2}(\Omega)^*)$ with initial values $\bar{\theta}(\bar{u}^1)$, and such that

$$\int_0^T \left\langle \partial_t \bar{\theta}(\bar{u}), v-\bar{u} \right\rangle + \int_0^T \int_\Omega \nabla(v-\bar{u}) a(\nabla\bar{u}+k(\bar{\theta}(\bar{u}))e) \geq 0$$

for every $v \in M_T(\bar{u}^o)$.

This can be proved by approximating the boundary conditions

$$\bar{u} \leq \bar{u}^o \quad \text{on} \quad S_{res} \cup S_{air},$$

$$\bar{u} \geq \bar{u}^o \quad \text{on} \quad S_{res}$$

by corresponding inequalities in the entire domain Ω. For such an obstacle problem the existence of a solution was proved in [6].

Now we will go to the non-steady free boundary problem. As in the stationary case we justify considering a sequence of functions θ_ε converging to a step function with jump θ_o at 0. Then formally the solutions of Theorem 6 converge to a solution of the time dependent problem described below. This is the same procedure as in the stationary case, but up to now there is no rigorous proof for this.

7. The non-steady problem A pair u,β is a solution of the time dependent dam problem, if

$$u \quad M_T(u^o) \qquad \text{with } u \geq 0,$$

$$\beta \quad L^\infty(]0,T[\times \Omega) \qquad \text{with } \chi_{\{u>0\}} \leq \beta \leq \theta_o,$$

such that $\partial_t \beta \in L^2(0,T;H^{1,2}(\Omega)^*)$ with initial values $\theta(u^1)$, and such that

$$\int_0^T \langle \partial_t \beta, v-u \rangle + \int_0^T \int_\Omega \nabla(v-u)\, a(\nabla u + k(\beta) e) \geq 0$$

for every $v \in M_T(u^0)$.

The problem in proving the existence of such a solution is that one must show the strong convergence of $\theta_\varepsilon(u_\varepsilon)$ to β because one has to conclude that $k(\theta_\varepsilon(u_\varepsilon)) \to k(\beta)$. In the case of a linear function k, which is not true for the physical data, one needs only that β is a weak limit, and indeed, in this case the existence of a solution has already been proved in [14]. In this paper it was done by an elliptic regularization, but the proof also applies to parabolic regularizations. If one can show that the solution in [14] satisfies the identity

$$\beta = \chi_{\{u>0\}},$$

it follows that it is also a solution of the time dependent problem formulated above, and that for all (!) functions k with the same values at 0 and θ_0.

Before this the existence of a solution has been proved by a time dependent version of Baiocchi's transformation introduced in [18]. This transformation is

$$w(t,x^*,x_n) = \int_0^t u(\tau,x^*,x_n+t-\tau)\,d\tau.$$

A similar computation as in Lemma 3 shows that solutions u of 7. are transformed into solutions w of a parabolic variational inequality.

Using this transformation in the case of a rectangular dam J. F. Rodrigues derives in [16] continuity properties for the free boundary and the convergence for $t \to \infty$.

We should remark that here we considered only incompressible fluid. For gas flow through porous media the function θ usually is taken to be

$$\theta(u) = u^{1/m} \quad \text{with } m > 1,$$

where only values $u \geq 0$ are allowed. In [15] the case $m = 1$ is considered.

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H W ALT & G GILARDI

The free boundary in the dam problem

1. INTRODUCTION

Let Ω be an open bounded connected set in $\mathbb{R}^2$ with Lipschitz boundary which denotes a homogeneous isotropic porous medium. On $\partial\Omega$ three relatively open sets are given, S_{res}, S_{air}, S_{imp}, which denote respectively the parts to reservoirs, to atmosphere and the impervious part of the boundary.

We assume that they are disjoint and the union of their closures is $\partial\Omega$. A function $u^o \in C^{0,1}(\mathbb{R}^2)$ is given such that $u^o > 0$ on S_{res} (hydrostatic pressure) and $u^o = 0$ on S_{air} (atmospheric pressure).

Define $e = (0,1)$ and

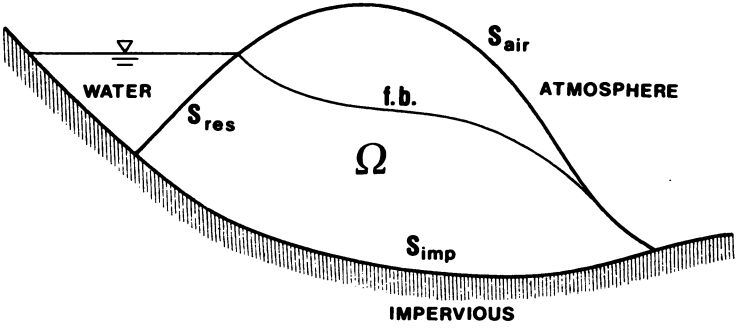
$$M(u^o) := \{v \in H^{1,2}(\mathbb{R}^2) : v = u^o \text{ on } S_{res} \text{ and } v \le 0 \text{ on } S_{air}\}$$

According to [3], a solution of the dam problem is a pair u,γ with

$$u \in M(u^o),\ u \ge 0,\ \gamma \in L^\infty(\Omega),\ \chi_{\{u>0\}} \le \gamma \le 1\ ; \tag{1.1}$$

$$\text{for every } v \in M(u^o) \tag{1.2}$$

$$\int_\Omega \nabla(v-u)\cdot(\nabla u+\gamma e) \ge 0\ .$$



The physical meaning is the following:
u is the pressure of the fluid and $\vec{v} = -(\nabla u + \gamma e)$ is the velocity, by Darcy's law.

Formally, the following relations are satisfied

$$\nabla \cdot \vec{v} = 0 \qquad \text{in } \Omega,$$

$$u = u^0 \qquad \text{on } S_{res} \cup S_{air},$$

$$\vec{v} \cdot \nu = 0 \qquad \text{on } S_{imp},$$

$$\vec{v} \cdot \nu \geq 0 \qquad \text{on } S_{air}.$$

Existence and Lipschitz continuity of u and smoothness of the free boundary $\Omega \cap \partial\{u>0\}$ are known [2,3]; we are interested in the behaviour of u and its free boundary near the fixed boundary $\partial\Omega$. Whenever $\partial\Omega$ is $C^{1,1}$ near the detachment point of $\Omega \cap \partial\{u > 0\}$, we prove the existence of the tangent of the free boundary and compute its slope.

To do this we use only local arguments and assume that there is some dry neighbourhood (i.e. $\gamma = 0$) above the point of interest, whenever Ω lies below $\partial\Omega$; otherwise local counterexamples can be given. However, under suitable global conditions on the geometry of Ω, we have $\gamma = \chi_{\{u>0\}}$ so that the above assumption is verified.

As a consequence of our local results we prove that the solution is unique up to ground water lakes. Therefore, whenever ground water lakes are not allowed to form because of the geometry of Ω, the solution is unique. This implies that all other approaches to the problem (see [1,4,6,15] and for further references [7]) gave the same solution.

For all proofs we refer to [5].

2. LOCAL RESULTS

2.1. Local solutions If B is a ball, a local solution in $\Omega \cap B$ is a pair u,γ such that

$$u \in H^{1,2}(\Omega \cap B),\ u \geq 0,\ \gamma \in L^{\infty}(\Omega \cap B),\ \chi_{\{u>0\}} \leq \gamma \leq 1;$$

$$u = u^0 \quad \text{on} \quad (S_{res} \cup S_{air}) \cap B$$

and satisfying the inequality

$$\int_{\Omega} \nabla\zeta\cdot(\nabla u + \gamma e) \geq 0$$

for all $\zeta \in H_o^{1,2}(B)$ with $\zeta = 0$ on $S_{res} \cap B$ and $\zeta \leq 0$ on $S_{air} \cap B$.

2.2. Assumptions

1) u,γ is a local solution in $\Omega \cap B_R$, $0 \in \partial\Omega$, $e^{i\sigma_o}$ is tangent to $\partial\Omega$ at 0, and $\Omega \cap B_R$ is a $C^{1,1}$ perturbation of the half ball $B_R \cap \{\sigma_o - \pi < \theta < \sigma_o\}$.

We consider the following three cases:

2) $0 < \sigma_o < \pi$, $\partial\Omega \cap B_R \cap \{h < 0\} \subset S_{res}$, $\partial\Omega \cap B_R \cap \{h > 0\} \subset S_{air}$,

and $u(y,h) = \max\{-h,0\}$ on $\partial\Omega$;

3) $0 \leq \sigma_o \leq \pi$ and $\partial\Omega \cap B_R \subset S_{air}$;

4) $0 \leq \sigma_o \leq \pi$ and $\partial\Omega \cap B_R \subset S_{imp}$.

Moreover we assume

5) if $\sigma_o \geq \pi/2$, $\gamma = 0$ near some point of $\Omega \cap B_R$ above 0;

6) 0 is a limit point of the free boundary.

Our local results are given in the following three theorems.

2.3. Theorem Assume 2.2.1), 2), 5) and 6). Define

1) $\sigma_- := \min(0,\sigma_o - \pi)$ and $\sigma_+ := \max(0,\sigma_o - \pi)$.

Then the free boundary is tangent at 0 to $\{\theta = \sigma_-\}$ or to $\{\theta = \sigma_+\}$, that is, it is horizontal or perpendicular to $\partial\Omega$.

2.4. Theorem Assume 2.2.1), 3), 5) and 6). Then either $\sigma_o \leq \pi/2$, the free boundary is tangent at 0 to $\partial\Omega$, and the tangent cone of $\Omega \cap \{u > 0\}$ at 0 is the half plane $\{\sigma_o - \pi < \theta < \sigma_o\}$, or $\sigma_o > \pi/2$ and the free boundary starts vertically at 0 (the case of a cusp is included).

2.5. Theorem Assume 2.2.1), 4), 5) and 6). Then either $\sigma_o < \pi/2$ and the free boundary is tangential to $\partial\Omega$ or horizontal at 0, or $\sigma_o \geq \pi/2$ and the

free boundary is horizontal at O.

2.6. Remark The only interesting remaining case are separation points between atmosphere and impervious boundary. In this case we allow the fixed boundary to have two different tangents, that is $\Omega \cap B_R$ is a $C^{1,1}$ perturbation of a sector. In this case we can prove that the tangent cone of the free boundary consists of a finite number of rays and compute all possible slopes.

2.7. Blow-up limits To prove our local results we use a blow-up technique combined with super- and subsolution arguments. The corresponding definitions and basic comparison results are given in 2.9-2.11. Now we deal with blow-up limits.

If u,γ is a local solution and z_r is any sequence in $\overline{\Omega} \cap \{u=0\}$, we define the blow-up sequence with respect to balls $B_r(z_r)$ by

$$u_r(x) = \frac{1}{r}\, u(z_r + r\,x) \quad \text{and} \quad \gamma_r(x) = \gamma(z_r + r\,x).$$

The pair u_r, γ_r is a solution in an appropriate domain.

The main remark is that, if u is Lipschitz continuous near the set of points z_r, then every weak limit u_*, γ_* of u_r, γ_r is a solution in a suitable domain Ω_*.

Therefore it is important to prove the Lipschitz continuity of u up to the boundary, provided no compatibility condition is violated.

2.8. Theorem Assume 2.2.1), 6) and either 2) or 3) or 4) hold. Then

$$|\nabla u| \le C \qquad \text{in } \Omega \cap B_r$$

where C is a universal constant and r depends only on R and on the curvature of $\partial\Omega$.

2.9. Super- and subsolutions Let $u \in H^{1,2}(\mathbb{R}^2)$ and $\gamma \in L^\infty(\Omega)$ with $u \ge 0$ and $\chi_{\{u > 0\}} \le \gamma \le 1$.

We call u,γ a:

1) Supersolution, if $u \ge u^0$ on S_{res}, and

$$\int_{\Omega} \nabla\zeta\cdot(\nabla u + \gamma e) \le 0$$

for every non-negative $\zeta \in H^{1,2}(\Omega)$ vanishing on $S_{res} \cup S_{air}$;

2) Subsolution, if $u \le u^{o}$ on $S_{res} \cup S_{air}$, and

$$\int_{\Omega} \nabla\zeta\cdot(\nabla u + \gamma e) \le 0$$

for every non-negative $\zeta \in H^{1,2}(\Omega)$ vanishing on S_{res}.

Our technique is based on the following comparison results

2.10. Comparison lemma from above Suppose u,γ is a solution and $v, \chi_{\{v \ge 0\}}$ is a supersolution such that $\overline{\Omega \cap \partial\{v > 0\}}$ consists of Lipschitz graphs in vertical direction locally in Ω and a set $E \subset \partial\Omega$ with

$$\int_{\Omega \cap B_{\rho}(E)} (1 + |\nabla u| + |\nabla v|) = o(\rho) \qquad \text{for } \rho \to 0$$

Then $u \le v$ in connected components of $\Omega \cap \{v > 0\}$ touching S_{res} and $\gamma = 0$ above these components.

2.11. Comparison lemma from below Suppose u,γ is a solution and $v,\chi_{\{v > 0\}}$ is a subsolution with $\nabla v\cdot(\nabla v+e) \ge 0$ in $\Omega \cap \{v > 0\}$. Then $u \ge v$ in every connected component of $\Omega \cap \{u > 0\}$ touching S_{res}.

2.12. To give an idea of our results, consider for instance the theorem 2.3, assuming that $\partial\Omega$ is linear near 0.

For $\sigma_o - \pi < \sigma < \sigma_o$ define $u^{\sigma} \in H^{1,2}(\Omega \cap B_R)$ by $u^{\sigma} = 0$ for $\theta > \sigma$, u^{σ} is linear for $\theta < \sigma$; $u^{\sigma}(y,h) = \max(-h, 0)$ on $\partial\Omega \cap B_R$.

One sees that u^{σ} is a supersolution or a subsolution according to $\sigma \in [\sigma_-,\sigma_+]$ or $\sigma \notin]\sigma_-,\sigma_+[$. Therefore, assuming that everything is smooth up to the boundary and blowing up with respect to balls $B_r(0)$, one gets a solution u_* which is linear where positive. Thus $u_* = u^{\sigma_-}$ or $u_* = u^{\sigma_+}$. This fact suggest the statement 2.3.

In the correct proof we use 2.10 and 2.11 to estimate from above and from below the solution u by the functions $u^{\sigma\pm}$ and get the result.

3. GLOBAL RESULTS

3.1. Assumptions We assume that $\partial\Omega$ consists of a finite number of $C^{1,1}$ curves contained in S_{res}, S_{air}, or S_{imp}, and that $\partial\Omega$ is globally Lipschitz. Moreover the number of points and segments of $\{h: (y,h) \in \partial\Omega\}$ is supposed to be bounded uniformly in y. We assume that $u^o > 0$ at points of $\overline{S_{res} \cap S_{imp}}$ and that elsewhere on $\partial\Omega$ the assumptions 2.2 or 2.6 hold.

Under assumptions 3.1 the following is true.

3.2. Theorem If u,γ is a solution, then $\gamma = \chi_{\{u > o\}}$ and there is a finite number of functions $g_i \in C^o(\mathbb{R}, \mathbb{R})$ such that

$$\Omega \cap \partial\{u > o\} = \{(y,h) \in \Omega : h = g_i(y) \quad \text{for some } i\} \ .$$

3.3. Remark Our aim is to give a uniqueness theorem, applying 2.10 to solutions u and v. But this cannot be done directly. Indeed, even if the free boundary of v consists of analytic graphs in the interior [2], it is not known whether it is globally Lipschitz continuous or not. Our local results only give the differentiability at end-points. However, using this and a suitable modification of the proof of 2.10, we can give the following:

3.4. Uniqueness theorem Under the assumptions 3.1 the solution of the dam problem is unique up to ground water lakes, that is, there exists a solution u of the dam problem such that any other solution u has the form

$$\tilde{u}(y,h) = u(y,h) + \sum_j \max(h_j - h, o)\ \chi_{D_j}(y,h)$$

with certain real numbers h_j (may be infinitely many) and connected components D_j of $\Omega \cap \{h < h_j\}$.

3.5. Remark By 3.4, to be sure that the solution is unique, it is sufficient to give assumptions on the geometry which exclude ground water lakes.

If we restrict ourselves to situations for which the solution is unique, it is easy to give stability results in the interior with respect to small perturbations of the data, for instance the local uniform convergence of the free boundaries. However, the control of the behaviour of the free boundary at its end-points is more complicated. Nevertheless, we can prove

the following results.

3.6. Stability of overflow Suppose that Ω^λ, S^λ_{res}, etc. for small λ is a $C^{1,1}$ perturbation of Ω, S_{res}, etc., and that u^o_λ is an $H^{1,2}$ perturbation of u^o satisfying the assumption 3.1. Denote by u^λ and u the corresponding unique solutions. If $u > 0$ near a point $x_o \in \overline{S_{res}} \cap \overline{S_{air}}$, then the same is true for u^λ for small λ.

3.7. Stability of the right angle property Assume the same as in 3.6. If at a point $x_o \in S_{res} \cap S_{air}$ with $\sigma_o < \pi/2$ the free boundary of u has tangent direction $e^{i\sigma-}$ (see 2.3.1.), then the same is true for u^λ for small λ.

Using 3.6, 3.7, sect. 2 and [9], one can describe satisfactorily some general global situations and show in particular that all cases in section 2 are actually possible.

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H BREZIS

The dam problem revisited

1. INTRODUCTION

We recall the problem of determining a flow through a porous dam. The dam is represented by a given domain $\Omega \subset \mathbb{R}^{2(1)}$ whose boundary $\partial\Omega$ is divided into 3 parts

$$\partial\Omega = S_1 \cup S_2 \cup S_3 \qquad \text{(see Figure 1)}$$

where

S_1 is the bottom of the dam, which is in contact with rock and is impermeable,

S_2 is the part of $\partial\Omega$ which is in contact with open air,

S_3 is the part of $\partial\Omega$ which is in contact with water reservoirs.

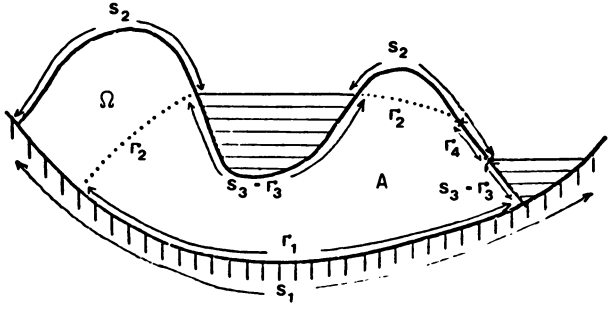

Figure 1

We denote by n the outward normal to Ω. We have to find a region $A \subset \Omega$ (A is the wet part of the dam) and a function p defined on A (p is the internal pressure) such that

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(1) We assume for simplicity that $\Omega \subset R^2$ but all the results we shall describe here extend to dimension N.

(i) the boundary ∂A of A consists of 4 parts,

$$A = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \quad \text{with}$$

$\Gamma_1 \subset S_1$, Γ_1 is the part of ∂A in contact with the rock,

$\Gamma_2 \subset \Omega$, Γ_2 is the free boundary inside the dam,

$\Gamma_3 = S_3$, Γ_3 is the part of ∂A in contact with the water reservoirs,

$\Gamma_4 \subset S_2$, Γ_4 is the seepage surface, i.e. the part of ∂A in contact with open air.

(ii) the function p satisfies

$$\Delta p = 0 \qquad \text{in } A \tag{1}$$

$$p = 0 \qquad \text{on } \Gamma_2 \cup \Gamma_4 \tag{2}$$

$$p = \phi \qquad \text{on } \Gamma_3 \tag{3}$$

where ϕ is a given function; $\phi(x,y)$ is the height of water in the reservoir, above the point $(x,y) \in \Gamma_3$

$$\frac{\partial}{\partial \nu}(p+y) = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \tag{4}$$

i.e. no water filtrates through $\Gamma_1 \cup \Gamma_2$

$$\frac{\partial}{\partial \nu}(p+y) \leq 0 \qquad \text{on } \Gamma_4 \tag{5}$$

i.e. there is an outgoing filtration of water through Γ_4. Here ν denotes the outward normal to A. Problem (1) - (5) has been beautifully solved (i.e. existence *and* uniqueness) using variational inequalities, by Baiocchi, Magenes and the Pavia school, in case Ω is a rectangle (Figure 2); see [2],[5].

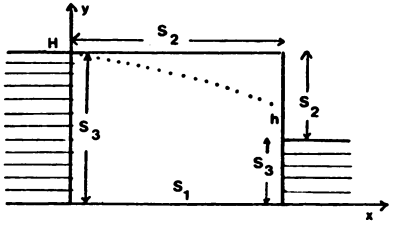

Figure 2

Existence has been proved by Baiocchi [3], [4] in the case of a general dam with flat bottom and two reservoirs, using quasi-variational inequalities. The question of uniqueness was left open but the importance of condition (5) was emphasized by Baiocchi. Later Alt [1] proved the existence of a solution for (1) - (4) in general domains Ω. He obtains p as the minimum of a family of supersolutions (à la Perron). Caffarelli and Riviere [7] announced that Alt's solution satisfies also [5]; they hinted that the solution of (1) - (5) is unique. I will present a simple and constructive proof of existence for general domains, due to Kinderlehrer-Stampacchia and myself [6]. I will also discuss recent results of Carillo and Chipot dealing with the question of uniqueness and non uniqueness.

2. A WEAK FORMULATION OF PROBLEM (1) - (5)

Assume

$$y \cdot n \leq 0 \qquad \text{on } S_1. \tag{6}$$

Suppose (A,p) is a solution of (1) - (5). Then

$$p > 0 \qquad \text{on } A. \tag{7}$$

This follows from the maximum principle since $p \geq 0$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ (recall that $\phi \geq 0$ on Γ_3) and $\frac{\partial p}{\partial \nu} = -\frac{\partial y}{\partial \nu} = -\, y \cdot \nu = -y \cdot n \geq 0$ on Γ_1. We extend p to Ω by setting $p = 0$ on $\Omega \backslash A$ and we still denote the extension by p. On physical grounds it is "reasonable" to expect that $p \in H^1(\Omega)$.

Let $H_0(t)$ denote the Heaviside function

$$H_0(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Let

$$K = \{\zeta \in H^1(\Omega);\ \zeta = 0 \text{ on } S_3,\ \zeta \geq 0 \text{ on } S_2\}.$$

Lemma 1 We have

$$\int_\Omega \nabla p \nabla \zeta + \int_\Omega H_0(p)\, \frac{\partial \zeta}{\partial y} \leq 0 \qquad \forall \zeta \in K \tag{8}$$

Proof Clearly

$$\int_\Omega \nabla p \nabla \zeta = \int_A \nabla p \nabla \zeta = - \int_A \Delta p \zeta + \int_{\partial A} \frac{\partial p}{\partial \nu} \zeta = \int_{\partial A} \frac{\partial p}{\partial \nu} \zeta \ .$$

and

$$\int_\Omega H_0(p) \frac{\partial \zeta}{\partial y} = \int_A \frac{\partial \zeta}{\partial y} = \int_{\partial A} (y \cdot \nu) \zeta = \int_{\partial A} \frac{\partial y}{\partial \nu} \zeta \ .$$

Thus

$$\int_\Omega \nabla p \nabla \zeta + \int_\Omega H_0(p) \frac{\partial \zeta}{\partial y} = \int_{\partial A} \frac{\partial}{\partial \nu} (p+y) \zeta$$

Using (4), (5) and the fact that $\zeta = 0$ on Γ_3, $\zeta \geq 0$ on $\Gamma_4 \subset S_2$ we obtain (8). Conversely if a function $p \in H^1(\Omega)$, $p \geq 0$ on Ω, satisfies (8) and

$$p = 0 \qquad \text{on } S_2 \tag{9}$$

$$p = \phi \qquad \text{on } S_3, \tag{10}$$

then p is a solution of (1) - (5) with $A = \{p > 0\}$. Since it is not convenient to work with the Heaviside function H_0 we consider the multivalued Heaviside *graph* H defined by

$$H(t) = \begin{cases} 1 & \text{if } t > 0 \\ [0,1] & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

and we introduce a new formulation which is slightly weaker than the previous one.

Problem (*) Find $p \in H^1(\Omega)$, $p > 0$ on Ω satisfying (9), (10) and

$$\int_\Omega \nabla p \nabla \zeta + \int_\Omega H(p) \frac{\partial \zeta}{\partial y} \leq 0 \qquad \forall \zeta \in K \ . \tag{11}$$

More precisely, by (11) we mean that there is a function $g \in L^\infty(\Omega)$ such that $g \in H(p)$ a.e. and

$$\int_\Omega \nabla p \nabla \zeta + \int_\Omega g \frac{\partial \zeta}{\partial y} \leq 0 \qquad \forall \zeta \in K \ . \tag{12}$$

3. EXISTENCE

We assume that there is a function $\Phi \in H^1(\Omega)$ such that

$$\Phi = \begin{cases} 0 & \text{on } S_2 \\ \\ \phi & \text{on } S_3 \end{cases}$$

Let

$$M = \{\zeta \in H^1(\Omega);\ \zeta = 0 \text{ on } S_2 \cup S_3\}.$$

Our main existence result is the following

Theorem 1 There is at least one solution of Problem (*). Moreover $p \in W^{1,q}(\Omega)$ for every $q < \infty$, and in particular $p \in C^{0,\alpha}(\Omega)$ for every $\alpha < 1$.

Proof For each $\varepsilon > 0$ set

$$H_\varepsilon(t) = \begin{cases} t & \text{if } t > \varepsilon \\ \\ \frac{1}{\varepsilon} t & \text{if } 0 \le t \le \varepsilon \\ \\ 0 & \text{if } t < 0 \end{cases}$$

Consider the Problem $(*)_\varepsilon$:

Find $p_\varepsilon \in H^1(\Omega)$, $p_\varepsilon > 0$ in Ω such that

$$p_\varepsilon = \Phi \qquad \text{on } S_2 \cup S_3 \tag{13}$$

$$\int_\Omega \nabla p_\varepsilon \nabla \zeta + \int_\Omega H_\varepsilon(p_\varepsilon)\, \frac{\partial \zeta}{\partial y} = 0 \qquad \forall \zeta \in M \tag{14}$$

Lemma 2 For each $\varepsilon > 0$, Problem $(*)_\varepsilon$ has a unique solution.

Proof of Lemma 2

a) Existence We rely on Schauder fixed point Theorem. Given $f \in L^2(\Omega)$ there is a unique solution $u \in H^1(\Omega)$ of the linear (non-homogeneous) problem

$$u = \Phi \quad \text{on } S_2 \cup S_3$$

$$\int_\Omega \nabla u \nabla \zeta + \int_\Omega H_\varepsilon(f) \frac{\partial \zeta}{\partial y} = 0 \qquad \forall \zeta \in M. \tag{15}$$

(this follows from the Lax-Milgram Theorem). Moreover

$$\| u \|_{H^1} \leq C, \tag{16}$$

where C is a constant independent of f (choose $\zeta = u - \Phi$ in (15) and note that $|H_\varepsilon(f)| \leq 1$). Clearly the mapping $T : f \to u$ is continuous (even Lipschitz continuous) from $L^2(\Omega)$ into $H^1(\Omega)$. Therefore T has a fixed point p_ε. Choosing $\zeta = (p_\varepsilon)^-$ in (14) we see that $p_\varepsilon \geq 0$ on Ω.

b) Uniqueness Assume p_ε and $\overline{p_\varepsilon}$ are two solutions of Problem$_\varepsilon$(*). Set $u = p_\varepsilon - p_\varepsilon$. We have $u \in M$ and

$$\int_\Omega \nabla u \nabla \zeta + \int_\Omega [H_\varepsilon(p_\varepsilon) - H_\varepsilon(\overline{p_\varepsilon})] \frac{\partial \zeta}{\partial y} = 0 \qquad \forall \zeta \in M.$$

Therefore

$$\left| \int_\Omega \nabla u \nabla \zeta \right| \leq L \int_\Omega |u| \left| \frac{\partial \zeta}{\partial y} \right| \qquad \forall \zeta \in M, \text{ where } L = \frac{1}{\varepsilon}.$$

Given $\delta > 0$ we choose[1] in (17)

$$\zeta = G(u) \qquad \text{where} \qquad G(t) = \frac{t}{(|t|+\delta)}$$

Note that $\zeta \in M$ and $\nabla \zeta = \dfrac{\delta \nabla u}{(|u|+\delta)^2}$. Thus we find

$$\int_\Omega \frac{|\nabla u|^2}{(|u|+\delta)^2} \leq L \int_\Omega \frac{|u|}{(|u|+\delta)} \leq L \int_\Omega \frac{\left|\frac{\partial u}{\partial y}\right|}{|u|+\delta},$$

and by Cauchy-Schwarz,

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[1] Such testing functions have been introduced by Trudinger, see [11].

$$\int_\Omega \frac{|\nabla u|^2}{(|u|+\delta)^2} \le L^2|\Omega|,$$

Hence $\int_\Omega |\nabla \log(\frac{|u|}{\delta} + 1)|^2 \le C$ where C is independent of δ, and by Poincare's inequality.

$$\| \log(\frac{|u|}{\delta} + 1) \|_{L^2} \le C.$$

Passing to the limit as $\delta \to 0$ we see that $u \equiv 0$.

Proof of Theorem 1 concluded First we claim that

$$\int_\Omega \nabla p_\varepsilon \nabla \zeta + \int_\Omega H_\varepsilon(p_\varepsilon) \frac{\partial \zeta}{\partial y} \le 0 \qquad \forall \zeta \in K. \tag{18}$$

(Note that (18) is a stronger assertion than (14). Indeed if $\zeta \in M$, then $\pm\zeta \in M$, and thus (18) $\Rightarrow$ (14)). From (14) we deduce that

$$-\Delta p_\varepsilon - \frac{\partial}{\partial y} H_\varepsilon(p_\varepsilon) = 0 \qquad \text{in } \Omega$$

$$\frac{\partial p_\varepsilon}{\partial n} + H_\varepsilon(p_\varepsilon)(y \cdot n) = 0 \qquad \text{on } S_1$$

For $\zeta \in K$ we have

$$\int_\Omega \nabla p_\varepsilon \nabla \zeta + \int_\Omega H_\varepsilon(p_\varepsilon) \frac{\partial \zeta}{\partial y} = \int_\Omega \left[-\Delta p_\varepsilon - \frac{\partial}{\partial y} H_\varepsilon(p_\varepsilon)\right]\zeta$$

$$+ \int_{\partial\Omega} \left[\frac{\partial p_\varepsilon}{\partial n} + H_\varepsilon(p_\varepsilon)(y \cdot n)\right]\zeta = \int_{S_2} \left[\frac{\partial p_\varepsilon}{\partial n} + H_\varepsilon(p_\varepsilon)(y \cdot n)\right]\zeta$$

$$= \int_{S_2} \frac{\partial p_\varepsilon}{\partial n} \zeta \le 0$$

Indeed $p_\varepsilon = 0$ on S_2, $p_\varepsilon > 0$ in Ω and therefore $\frac{\partial p_\varepsilon}{\partial n} \le 0$ on S_2. Next, we pass to the limit as $\varepsilon \to 0$. We have $\| p_\varepsilon \|_{H^1} \le C$ where C is independent of ε.

Indeed, choose $\zeta = p_\varepsilon - \Phi$ in (14) and recall that $|H_\varepsilon(p_\varepsilon)| < 1$. Therefore we may extract a subsequence $\varepsilon_n \to 0$ such that $p_{\varepsilon_n} \to p$ weakly in $H^1(\Omega)$, $p_{\varepsilon_n} \to p$ strongly in $L^2(\Omega)$, $p_{\varepsilon_n} \to p$ a.e. on Ω, $H_{\varepsilon_n}(p_{\varepsilon_n}) \to g$ weakly in $L^2(\Omega)$. Since $0 \le H_\varepsilon \le 1$ we also have $0 \le g \le 1$ a.e. on Ω. On the other hand $H_{\varepsilon_n}(p_{\varepsilon_n}) \to 1$ a.e. on $[p > 0]$. Therefore $g = 1$ a.e. on $[p > 0]$ and so $g \in H(p)$ a.e. Passing to the limit in (18) we obtain a solution of Problem (*).

Finally we observe that

$$-\Delta p - \frac{\partial}{\partial y} g = 0 \qquad \text{in } \mathcal{D}'(\Omega)$$

with $g \in L^\infty(\Omega)$; we conclude from the elliptic regularity theory that $p \in W^{1,q}_{loc}(\Omega)\ \forall q < \infty$.

Remark 1 It is of course superfluous to prove that p_ε is unique since anyway we pass to the limit as $\varepsilon \to 0$. However we want to emphasize that our approximation method is constructive; instead of using Schauder fixed point Theorem it is presumably possible to obtain p through a convergent iteration scheme. Note that Problem (*) is of the form

$$-\Delta p - \frac{\partial}{\partial y} F(p) = 0 \qquad \text{in } \Omega \tag{19}$$

$$+ \text{ boundary condition}$$

Very little seems to be known about the uniqueness of the solution of problem (19) [except when F is Lipschitz continuous - see the proof above - or when F is the Heaviside graph - see Theorem 2 below]. It would be of interest to consider the question of uniqueness (or non-uniqueness) for example when $F(p) = |p|^{m-1}p$ with $0 < m < 1$.

4. UNIQUENESS

I will describe the main result obtained by Carillo and Chipot (for more details see [8] [9] [10] [11]). First observe that the solution of (1) - (5) is *not always unique*. Indeed consider a dam $\tilde{\Omega}$ having the following shape. Let Ω be the dam of Figure 1. Dig a little hole G in the rock at the bottom of Ω in the dry region. Then fill G with sand. Define the new

dam $\tilde{\Omega} = \Omega \cup G$ (see Figure 3). For the dam $\tilde{\Omega}$ we have many solutions of (1) - (5). Namely:

(i) The solution p for the dam Ω is still a solution for the dam $\tilde{\Omega}$

(ii) All solutions $\tilde{p}$ obtained by (partially) filling G with water and adding it to p (see Figure 3).

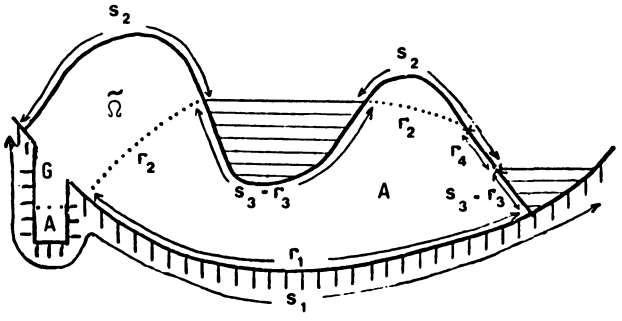

Figure 3

In other words, for some configurations Ω there are solutions involving *underground pools* in addition to the natural filtration of water connecting the reservoirs [Imagine some water left by God when He created the world - see the first verses in the book of Genesis!]. In order to classify all solutions, Carillo and Chipot have introduced the following

Definition A solution p of Problem (*) is S_3-connected if each component C of the set $[p > 0]$ has the property that $\bar{C} \cap S_3 \neq 0$. In other words, a solution p is S_3-connected if every drop of water inside the dam is connected to the reservoirs. The main result of Carillo-Chipot is the following

Theorem 2

a) There is a unique S_3-connected solution of Problem (*).

b) Every solution p of Problem (*) can be written as the sum of the S_3-connected solution plus underground pools (i.e. functions locally of the form $(h-y)^+$ for some constant h)

c) The function g occurring in (12) equals $\begin{cases} 1 & \text{on } [p > 0] \\ 0 & \text{on } [p = 0] \end{cases}$.

Remark 2 The S_3-connected solution coincides of course with the minimal solution of Alt [1].

Remark 3 For some geometries, [for example when Ω is a rectangle or if Ω has a convex bottom] then underground pools can not appear.

Remark 4 It would be of interest to know whether the approximation scheme p_ε used in the proof of Theorem 1 converges to the S_3-connected solution (or to another solution?). From the point of view of numerical analysis it is also important to estimate $\| p_\varepsilon - p \|_{L^\infty}$.

Remark 5 Suppose Ω is a rectangle (see Figure 2). The connection between our approach and Baiocchi's approach (through variational inequalities) is the following. Let p be a solution of Problem (*). Set $u = \int_y^H p(x,s)ds$ (Baiocchi's transform), so that $p = -\frac{\partial u}{\partial y}$. Since we have

$$-\Delta p - \frac{\partial}{\partial y} H(p) = 0 \qquad \text{in } \Omega^{(1)}$$

It follows that

$$-\Delta \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} H(p) = 0 \qquad \text{in } \Omega$$

and thus

$$-\Delta u + H(p) = f(x) \qquad \text{in } \Omega. \tag{20}$$

Choosing $y = H$ in (20) we see that $f \equiv 0$ and therefore

$$-\Delta u + H(p) = 0 \qquad \text{in } \Omega. \tag{21}$$

Finally one checks that

$$[u > 0] = [p > 0] \text{ and } [u = 0] = [p = 0]$$

and consequently we have

$$-\Delta u + H(u) = 0 \tag{22}$$

---

(1) The letter H denotes both the Heaviside graph and the height of water in the upper reservoir, but no confusion is possible.

which is exactly the variational inequality formulation used by Baiocchi. Thus our approximation scheme $(*)_\varepsilon$ corresponds to the penalty method in variational inequalities.

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J CARRILLO-MENENDEZ & M CHIPOT

On the uniqueness of the solution of the dam problem

1. STATEMENT OF THE PROBLEM

(For more details one can see Brezis Lecture in this issue and [1] to [10].

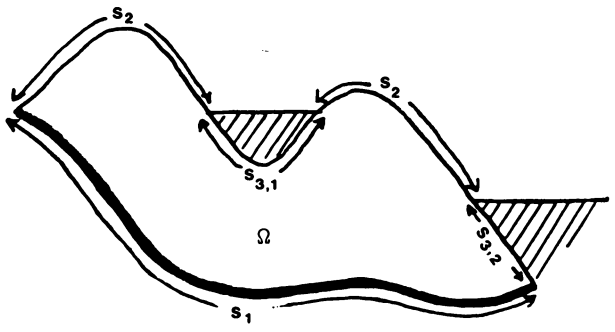

Figure 1

Ω will be a bounded open set of $\mathbb{R}^2$ with a Lipschitz boundary S

S_1 - a closed sub-set of S - denotes the impervious part of the dam

S_3 - a non-empty open set in the complement of S_1 in S - will be the bottoms of the reservoirs

S_2 - the remainder of S - is the part in contact with the air

If p is the number of reservoirs we will denote by $S_{3,i}$ the different connected components of S_3 - i.e. $S_3 = S_{3,1} \cup S_{3,2} \cup \ldots \cup S_{3,p}$. Moreover, π_x being here the projection of $\mathbb{R}^2$ on the x-axis and if we set for $x \in \pi_x(\bar{\Omega})$

$$S^-(x) = \mathrm{Sup}\{y \mid (x,y) \in S_1\} \qquad S^+(x) = \mathrm{Inf}\{y \mid (x,y) \in \overline{S_2 \cup S_3}\}$$

we shall assume all along that

S^- (Resp S^+) is continuous except on a finite set $\mathcal{S}^-$ (Resp $\mathcal{S}^+$) of $\pi_x(\bar{\Omega})$

$$\Omega = \{(x,y) \mid x \in \pi_x(\Omega),\ S^-(x) < y < S^+(x)\}.$$

In other words we shall assume that S_1 and $\overline{S_2 \cup S_3}$ are defined by two continuous functions except for a finite number of points where vertical walls can occur.

Let us consider now the problem (see Brezis Lecture for an existence proof of a solution, or [6]).

$$(*)\quad \begin{cases} \text{Find } (p,g) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that} \\ \text{(i) } p \geq 0 \text{ a.e., } p = \varphi \text{ on } S_2 \cup S_3 \\ \text{(ii) } 0 \leq g \leq 1, \quad g = 1 \text{ a.e. on } [p > 0] \\ \text{(iii) } \int_\Omega \nabla p \cdot \nabla \zeta + g \cdot \zeta_y \leq 0 \quad \forall \zeta \in H^1(\Omega),\ \zeta = 0 \text{ on } S_3,\ \zeta \geq 0 \text{ on } S_2 \end{cases}$$

where φ is the function defined by

$$\varphi(xy) = \begin{cases} h_i - y \text{ on } S_{3,i} \\ 0 \text{ on } S_2 \end{cases}$$

h_i being the level of the water in the reservoir of bottom $S_{3,i}$.

More precisely, consider the problem of the uniqueness of the solution of (*).

2. TWO USEFUL LEMMAS

Lemma 1 Let (p,g) be a solution of (*) then

(i) $\Delta p + g_y = 0$ in Ω

(ii) $\Delta p \leq 0$, $g_y \leq 0$ in Ω

Proof For (i) take $\pm\zeta \in \mathcal{D}(\Omega)$ in (*) (iii). For (ii) if $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ take (see [1]) $\zeta = \min(\frac{p}{\varepsilon}, \xi)$ where $\varepsilon > 0$. By (*) this leads to

$$0 \geq \int_\Omega \nabla p \cdot \nabla \min(\tfrac{p}{\varepsilon}, \xi) + \int_\Omega g [\min(\tfrac{p}{\varepsilon}, \xi)]_y .$$

Since $g = 1$ a.e. on $[p > 0]$ and $[\min(\frac{p}{\varepsilon}, \xi)]_y = 0$ a.e. on $p = 0$ (for a function $u \in H^1(\Omega)$, $\nabla u = 0$ a.e. on $[u=0]$ - see [9]), we deduce

$$0 \geq \int_\Omega \nabla p \cdot \nabla \min\left(\frac{p}{\varepsilon}, \varepsilon\right) + \int_\Omega \left[\min\left(\frac{p}{\varepsilon}, \varepsilon\right)\right]_y = \int_\Omega \nabla p \cdot \nabla \min\left(\frac{p}{\varepsilon}, \xi\right)$$

which leads to

$$0 \geq \frac{1}{\varepsilon} \cdot \int_{[p \leq \varepsilon\xi]} |\nabla p|^2 + \int_{[p > \varepsilon\xi]} \nabla p \cdot \nabla \xi .$$

Thus the first integral being positive we have

$$\int_{[p > \varepsilon\xi]} \nabla p \cdot \nabla \xi \leq 0$$

and the result follows by letting ε go to zero.

Remark 1 It results from (i) that $p \in W^{1,s}_{loc}(\Omega)$ for all $s \geq 1$, thus p is continuous in Ω and $[p > 0]$ is open. Moreover, since $\Delta p = -g_y = -1_y = 0$ in $[p > 0]$, p is analytic in this set.

Remark 2 By (ii) g is a non-increasing function of y and it is easy to check using the maximum principle that if $p(x_0, y_0) > 0$ then p is positive on a small cylinder below a ball centered at (x_0, y_0). In particular p is positive near and below S_3. This is very clear from a physical point of view

Lemma 2 Let (p,g) be a solution of (*) and x_0, x_1 two reals such that $]x_0, x_1[\subset \pi_x(S_2)$. If h is real, let us denote by Z_h the set defined by

$$Z_h = \Omega \cap]x_0, x_1[\times]h, +\infty[.$$

If in Ω we have

$$p(x_0, y) = 0 \quad \forall y > h \quad \text{and} \quad p(x_1, y) = 0 \quad \forall y > h$$

(where we understand that our assumptions are fulfilled if one or both (x_i, h) belongs to $\bar{S}_2$) then

$$\int_{Z_h} g \cdot p_y + g^2 \leq \int_{Z_h} p_y + g \leq 0.$$

Proof The first inequality is easy to check (note that $g^2 \leq g$). For the second one (formally and if $\chi(\ )$ denotes here as in the sequel the characteristic function of the set in brackets) $\zeta = \chi(Z_h) \cdot (y-h)$ is a test function for (*) and the result follows. (To correct the proof one has only to choose in (*) in place of the above ζ, $\alpha_\varepsilon \zeta$ where α_ε is for ε small enough a smooth function of x only, which lies between 0 and 1, has a support in $[x_0, x_1]$ and takes the value 1 on $[x_0 + \varepsilon, x_1 - \varepsilon]$ and to let ε go to 0 after some easy computations (see [7]).)

3. S_3-CONNECTED SOLUTIONS

A main point is now

Theorem 1 Let (p,g) be a solution of (*) and C a connected component of $[p > 0]$ such that $\bar{C} \cap S_3 = \emptyset$ where $\bar{C}$ denotes the closure of C in $\mathbb{R}^2$, then, C is a connected component of the set $[y < h] = \{(x,y) \in \Omega | y < h\}$ (for some h) and on the strip $\pi_x(C) \times \mathbb{R} \cap \Omega$ we have

$$(p,g) = ((h-y)^+, \chi(C)).$$

Proof From Remark 2 and the definition of C it is easy to see, if we put $\pi_x(C) =]x_0, x_1[$, $Z =]x_0, x_1[\times \mathbb{R} \cap \Omega$, that $]x_0, x_1[\subset \pi_x(S_2)$ and $p(x_i, y) = 0$ $\forall (x_i, y) \in \Omega (i=0,1)$. Thus $\pm \chi(Z) p$ is a test function for (*) and we have

$$\int_Z |\nabla p|^2 + g p_y = 0.$$

But now applying the lemma 2 to Z with $h = \min \{y | (x,y) \in Z\}$ we obtain

$$\int_Z g p_y + g^2 \leq 0.$$

Adding both leads to

$$\int_Z p_x^2 + (p_y + g)^2 \leq 0 \quad (=) \qquad p_x = 0 \; p_y = -g \quad \text{in} \quad Z.$$

But on $[p > 0]$ we have $g = 1$ whereas on $p = 0$, since $g = -p_y$ we have $g = 0$ a.e. hence $g = \chi(C)$ in Z. Moreover, since $\nabla p = (0,-1)$ we have in C $p = h - y$ with obviously $h = \text{Sup}\{y | (x,y) \in C\}$ and the proof is complete.

This leads to the following definitions

Definition 1 We shall say that a solution (p,g) of (*) is S_3-connected if for all connected component C of $[p > 0]$ we have $\bar{C} \cap S_3 \neq \emptyset$.

Definition 2 If C is a connected component of $[y < h]$ we call a pool a pair (P,G) of functions on Ω which vanishes in Ω except on the strip $\pi_x(C) \times \mathbb{R}$ where we have $(P,G) = ((h-y)^+, \chi(C))$.

Thus with these definitions the theorem 1 leads to

Theorem 2 All (p,g) solutions of (*) can be written as the sum of a S_3-connected solution (p',g') and pools.

Proof If $(C_i)_{i \in I}$ denote the different connected components of $[p > 0]$ satisfying $\bar{C}_i \cap S_3 = \emptyset$ (I could, of course, be empty), it is easy to check that

$$(p',g') = (p,g) - \sum_{i \in I} (\chi(C_i)p, \chi(C_i))$$

is a S_3-connected solution of (*). Indeed by Theorem 1 we have for all $\zeta \in H^1(\Omega)$

$$\int \nabla(p-p') \cdot \nabla\zeta + (g-g')\zeta_y = \int_{\cup_I C_i} -\zeta_y + \zeta_y = 0$$

and the result follows.

Now we shall prove that there is only one S_3-connected solution of (*). Of course this solution will be also the minimal solution of (*) (see [1]) and it is clear from theorem 2 that to obtain all other solutions it is enough to add eventually some pools to this S_3-connected solution.

4. UNIQUENESS OF S_3-CONNECTED SOLUTION

Let (p_1,g_1), (p_2,g_2) be two solutions of (*) and set for i = 1,2

$$\Phi_i(x) = \begin{cases} \sup \{y \mid (x,y) \in [p_i > 0]\} & \text{if this set is not empty} \\ & \text{(see remark 2)} \\ s^-(x) & \text{otherwise.} \end{cases}$$

Moreover set

$$P_o = \min(p_1,p_2) \quad g_o = \min(g_1,g_2) \quad \Phi_o = \min(\Phi_1,\Phi_2) \quad A_o = [p_o > o].$$

First we need the following lemma.

Lemma 3 For all $\xi \in H^1(\Omega) \cap C(\bar{\Omega})$, $\xi \geq o$ we have for $i = 1,2$

$$\int \nabla(p_i - p_o)\cdot\nabla\xi + (g_i - g_o)\xi_y \leq \int_{D_i} \xi(x,\Phi_i(x))dx$$

where D_i is the set defined by $D_i = \{x \in \pi_x(\Omega) \mid \Phi_o(x) < \Phi_i(x)\}$.

Proof Note first that it is easy to check that the functions Φ_i are lower semi-continuous except perhaps on $\mathcal{S}$ (use remark 2). Thus the Φ_i are measurable and so the second integral above makes sense. Now for $\varepsilon > o$ consider

$$\zeta = \min [(p_i - p_o)/\varepsilon,\xi] .$$

Clearly $\zeta = o$ on $S_2 \cup S_3$ and $\pm\zeta$ is a test function for (*). So we have for $j \neq i$, $j = 1,2$

$$\int_\Omega \nabla(p_i - p_j)\cdot\nabla\zeta + (g_i - g_j)\zeta_y = o .$$

But in this integral it is enough to integrate on the set $p_i - p_o > o$ where $p_j = p_o$ and so

$$\int_\Omega \nabla(p_i - p_o)\cdot\nabla\zeta + (g_i - g_o)\zeta_y = o$$

$$\Longleftrightarrow \frac{1}{\varepsilon}\int_{[p_i - p_o \leq \varepsilon\xi]} |\nabla(p_i - p_o)|^2 + \int_{[p_i - p_o > \varepsilon\xi]} \nabla(p_i - p_o)\cdot\nabla\xi + \int_\Omega (g_i - g_o)\zeta_y = o$$

The first integral being positive this leads to

$$\int_{[p_i - p_o > \varepsilon\xi]} \nabla(p_i - p_o)\cdot\nabla\xi + \int_\Omega (g_i - g_o)\xi_y \leq \int_\Omega (g_i - g_o)(\xi - \min [(p_i - p_o)/\varepsilon,\xi])_y .$$

Let us assume here that $g_i = \chi([p_i > 0])$ for $i = 1,2$ (for a proof see [7]) then the last integral can be written (since on $[p_0 > 0]$ we have $p_i > 0$ and thus $g_i = g_0 = 1$)

$$\int_\Omega (g_i - g_0)\left[\xi - \frac{(p_i - p_0)}{\varepsilon}\right]_y^+ = \int_{[\Phi_0(x) \le y \le \Phi_i(x)]} \left(\xi - \frac{p_i}{\quad}\right)_y^+$$

where $[\Phi_0(x) \le y < \Phi_i(x)] = \{(x,y) \in \Omega \mid \Phi_0(x) \le y < \Phi_i(x)\}$.

Now integrating in the y direction leads to

$$\int_{[p_i - p_0 > \varepsilon\xi]} \nabla(p_i - p_0)\cdot\nabla\xi + \int_\Omega (g_i - g_0)\,\xi_y \le \int_{D_i} \xi(x, \Phi_i(x))\,dx$$

and the result follows by letting ε go to zero. (Note that $p_i - p_0 \ge 0$ and on the set $p_i - p_0 = 0$ we have $\nabla(p_i - p_0) = 0$ a.e.)

Now we can prove our main result

Theorem 3 There is one and only one S_3-connected solution of problem (*).

Proof Existence is clear from Theorem 2. Now let B_r be an open ball of radius r centered on $S_{3,j}$ (for some j) and such that $B_r \subset [p_i > 0]$, $i = 1,2$ (see remark 2). Let Γ_1 be an open connected part of ∂B_r (the boundary of B_r) outside Ω, and Γ_0 the complement of Γ_1 in ∂B_r (see the figure below).

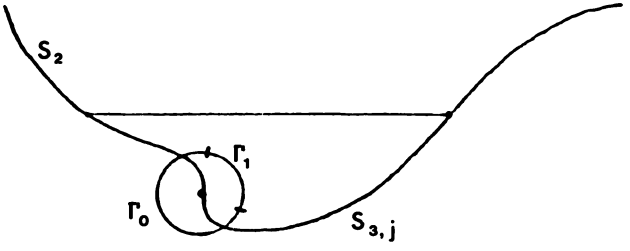

Figure 2

Moreover, let us denote by σ a smooth function such that

$$\Delta\sigma = 0 \text{ in } B_r,\ \sigma = 1 \text{ on } \Gamma_0,\ 0 \le \sigma < 1 \text{ on } \Gamma_1. \tag{0}$$

By Green's formula we have for i = 1 or 2

$$\int_{\Gamma_0\cap\Omega} (p_i-p_0)\cdot\frac{\partial\sigma}{\partial\nu} = \int_{\Omega\cap B_r} \nabla(p_i-p_0)\cdot\nabla\sigma + \int_{\Omega\cap B_r} (p_i-p_0)\cdot\Delta\sigma = \int_{\Omega\cap B_r} \nabla(p_i-p_0)\cdot\nabla\sigma.$$

If we denote also by σ the function (clearly in $H^1(\Omega) \cap C(\bar{\Omega})$ which agrees with σ in B_r and is equal to 1 outside since $g_i = g_0 = 1$ on B_r this gives

$$\int_{\Gamma_0\cap\Omega} (p_i-p_0)\cdot\frac{\partial\sigma}{\partial\nu} = \int_{\Omega} \nabla(p_i-p_0)\cdot\nabla\sigma + (g_i-g_0)\sigma_y\ .$$

Now for $\varepsilon > 0$ let α_ε be a smooth function in $\mathbb{R}^2$ satisfying $0 \le \alpha_\varepsilon \le 1$, $\alpha_\varepsilon = 1$ on A_0, $\alpha_\varepsilon(X) = 0$ when $d(X,A_0) > \varepsilon$ (for instance $(1 - \frac{d(X,A_0)}{\varepsilon})^+$ would be suitable). By maximum principle from (0) we deduce that $0 \le \sigma \le 1$. So $(1-\alpha_\varepsilon)\cdot\sigma$ is a test function for (*). This leads to

$$\int_\Omega \nabla p_i\cdot\nabla[(1-\alpha_\varepsilon)\sigma] + g_i\cdot[(1-\alpha_\varepsilon)\sigma]_y \le 0.$$

Since $(1-\alpha_\varepsilon)\sigma = 0$ on A_0 and also $p_0 = 0 = g_0$ outside A_0

$$\int_\Omega \nabla p_0\cdot\nabla[(1-\alpha_\varepsilon)\sigma] + g_0\cdot[(1-\alpha_\varepsilon)\sigma_y] = 0.$$

By subtracting from the above inequality we obtain

$$\int_\Omega \nabla(p_i-p_0)\cdot\nabla\sigma + (g_i-g_0)\cdot\sigma_y \le \int \nabla(p_i-p_0)\cdot\nabla(\alpha_\varepsilon\sigma) + (g_i-g_0)\cdot(\alpha_\varepsilon\sigma)_y$$

Now combining this with an above equality and lemma 3 gives us

$$\int_{\Gamma_0\cap\Omega} (p_i-p_0)\cdot\frac{\partial\sigma}{\partial\nu} \le \int_{D_i} (\alpha_\varepsilon\sigma)(x,\Phi_i(x))\,dx$$

and moreover after letting ε go to 0 (by Lebesque's theorem and since $\alpha_\varepsilon(x,\Phi_i(x)) \to 0$ on D_i) we have

$$\int_{\Gamma_0 \cap \Omega} (p_i - p_0) \cdot \frac{\partial \sigma}{\partial \nu} \leq 0$$

But by maximal principle it is easy to see that $\frac{\partial \sigma}{\partial \nu} > 0$ on Γ_0 thus the above inequality implies that $p_i = p_0$ on $\Gamma_0 \cap \Omega$ for $i = 1,2$ and consequently

$$p_1 = p_2 \text{ on } \Gamma_0 \cap \Omega.$$

But if we denote by C_j^i the connected component of $[p_i > 0]$ which contains $\Omega \cap B_r$, by analytic continuation (see remark 1) we have $p_1 = p_2$ on $C_j^1 \cap C_j^2$ since clearly this set is connected (see remark 2). This implies that $C_j^1 = C_j^2$ and the result follows since the above arguments are valid for all $j = 1,\ldots,p$.

Remark 3 It is now easy to see that the uniqueness of the solution of (*) depends on the shape of Ω. For instance for a dam with a horizontal bottom, where no pools can appear, the solution of (*) is unique and so agrees with the solution introduced by Baiocchi in [3],[4].

Of course one can also deduce the uniqueness of the solution of (*) in many other cases but also construct very simple examples where the uniqueness fails (see [7]).

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Homogenization of the dam problem with layer-structure

Our aim is to illustrate how the theory of the homogenization of elliptic variational inequalities with obstacles may be applied to a typical free boundary problem, the well known model dam problem. We shall consider the simple case of a rectangular dam with several horizontal layers, considered in [3]. In the limit we obtain the classical homogeneous problem of Baiocchi [2] and we prove the convergence in mean of the free boundary.

1. THE DAM PROBLEM WITH HORIZONTAL LAYERS

Consider a non homogeneous porous dam occupying a rectangle

$$D = \{(x,y) : o < x < a , \quad o < y < b\}$$

with two vertical walls separating two reservoirs of water at fixed levels y_o and y_a, $O < y_a < y_o < b$, and with an impervious bottom. The permeability of the material of the dam is given by a positive function $K^\varepsilon = K^\varepsilon(y)$.

We denote by p^ε the pressure in the liquid and we consider p^ε extended into the dry part of the dam by O. Setting

$$u^\varepsilon(x,y) = \int_y^b K^\varepsilon(t)\ p^\varepsilon(x,t)\,dt \tag{1}$$

and assuming that $u^\varepsilon > O$ in the wet part of the dam, denoted by Ω^ε, and that $D\backslash\bar{\Omega}^\varepsilon$ is connected one has (see [3]) that u^ε satisfies the following variational inequality

$$u^\varepsilon \in H^1(D), \quad u^\varepsilon \geq O, \quad u^\varepsilon = g^\varepsilon \quad \text{on } \partial D \tag{2}$$

$$\int_D \frac{1}{k^\varepsilon} \nabla u^\varepsilon \cdot \nabla(v-u^\varepsilon)\,dxdy \geq -\int_D (v-u^\varepsilon)\,dxdy, \ \forall v \geq o, \ v^\varepsilon = g^\varepsilon \text{ on } \partial D \tag{3}$$

where g^ε is defined by

$$g^{\varepsilon}(x,y) = \begin{cases} \displaystyle\int_{y}^{y_i} K^{\varepsilon}(t)[y_i - t]\,dt \ , \ \text{if } o \le y \le y_i, \ x = i \ (i = o,a) \\ \left[g^{\varepsilon}(a,o)\ x + g^{\varepsilon}(o,o)(a-x)\right]/a \ \text{if } o \le x \le a, \ y = o \\ \qquad o \ \text{elsewhere on } \partial D \end{cases} \tag{4}$$

When $K^{\varepsilon}(y)$ is piecewise constant (each constant value of K^{ε} corresponds to a different layer) it is known (see [3]) that the unique solution u^{ε} of (2) (3) is such that

$$u^{\varepsilon} \in C^{o}(\bar{D}), \ u_x^{\varepsilon} \in C^{o}(\bar{D}) \cap H^{1}(D) \ \text{and} \ \frac{u_y^{\varepsilon}}{K^{\varepsilon}} \in C^{o}(\bar{D}) \cap H^{1}(D).$$

Moreover Baiocchi and Friedman have shown in [4] that the set

$$\Omega^{\varepsilon} = \{(x,y) \in D : u^{\varepsilon}(x,y) > o\} \tag{5}$$

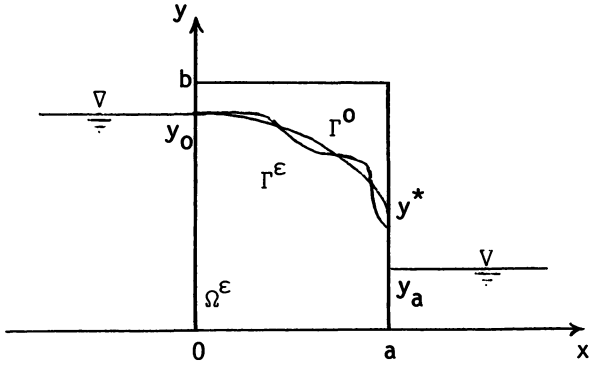
is bounded by $x = o$, $y = o$, and by a continuous curve

$$\hat{\Gamma}^{\varepsilon} : x = \hat{\psi}^{\varepsilon}(y) = \sup\{x : (x,y) \in \Omega^{\varepsilon}\},$$

being the restriction of $\hat{\psi}^{\varepsilon}$ to $o < x < a$,

$$\Gamma^{\varepsilon} = \hat{\Gamma}^{\varepsilon} \cap D : x = \psi^{\varepsilon}(y), \tag{6}$$

the free boundary which separates the wet part of the dam from the dry one.



By the results of Caffarelli and Friedman (see [6]) one knows that, if $\hat{\Gamma}^\varepsilon \cap \partial D$ is connected and the dam has several layers, the function $\psi^\varepsilon(y)$ has at most one local maximum and at most one local minimum in each layer. Furthermore in that paper they have studied the asymptotic behaviour of the free boundary when one permeability coefficient increases to ∞ whereas the width of the corresponding layer shrinks to 0.

Here we are interested in a different problem corresponding to the case in which the widths of an increasing number of layers shrink asymptotically to zero but the permeability coefficients, for instance, oscillate very rapidly (with "period" $\varepsilon \downarrow 0$). More generally we assume that

$$K^\varepsilon(y) \rightharpoonup K^o \qquad \text{in } L^2(0,b)\text{-weak} \tag{7}$$

$$1/K^\varepsilon(y) \rightharpoonup q_1 \qquad \text{in } L^2(0,b)\text{-weak} \tag{8}$$

where K^o and q_1 are positive constants and K^ε is a family of one real parameter $\varepsilon \downarrow o$ of functions satisfying

$$0 < a \leq K^\varepsilon(y) \leq M, \text{ ae. in } (o,b), \ \forall \varepsilon > 0.$$

In particular, if $K^\varepsilon(y) = K(\frac{y}{\varepsilon})$ with $K(t)$ a periodic function on $[0,1]$, one has $K^o = \int_o^1 K(t)\,dt$ and $q_1 = \int_o^1 \frac{dt}{K(t)}$. For more details on homogenization theory we refer to [5].

2. HOMOGENIZATION OF THE VARIATIONAL INEQUALITY

If we consider the function g^ε defined by (4) with K^ε satisfying (7) it is easy to see that, for $i = 0,a$, one has

$$g^\varepsilon(i,y) \rightharpoonup g^o(i,y) \text{ in } H^1(o,b)\text{-weak, as } \varepsilon \downarrow o, \tag{9}$$

where g^o is defined on ∂D by

$$g^o(x,y) = \begin{cases} K^o(y-y_i)^2/2, \text{ if } o < y < y_i, \ x = i, \ i = o,a \\ K^o\left[y_a^2 x + y_o^2(a-x)\right]/2a, \text{ if } o < x < a, \ y = o \\ o \text{ elsewhere on } \partial D. \end{cases} \tag{10}$$

By Rellich-Kondrachoff compactness imbeddings, one has from (9) and the definitions that

$$g^{\varepsilon} \to g^{o} \quad \text{on} \quad C^{o,\lambda}(\partial D) \text{ - strong, for some } \lambda > o. \tag{11}$$

Consider now the homogenized variational inequality

$$u^{o} \in H^{1}(D),\ u^{o} \geq o,\ u^{o} = g^{o} \quad \text{on} \quad \partial D \tag{12}$$

$$\int_{D} (Q\nabla u^{o}) \cdot \nabla (v-u^{o})\, dxdy \geq - \int_{D} (v-u^{o})\, dxdy,\ \forall v \geq o\ v = g^{o} \quad \text{on} \quad \partial D. \tag{13}$$

Here Q is the matrix of the homogenized coefficients, which by the classical theory of multilayers homogenization, is given by

$$Q = \begin{pmatrix} q_1 & o \\ o & q_2 \end{pmatrix}$$

where q_1 is given by (8) and $q_2 = 1/K^{o}$, K^{o} given by (7).

We note that, in general, one has $q_1 \neq q_2$.

Theorem 1 Under assumptions (7) and (8) one has

$$u^{\varepsilon} \rightharpoonup u^{o} \qquad \text{in } H^{1}(D)\text{-weak} \tag{14}$$

$$\frac{1}{K^{\varepsilon}} \nabla u^{\varepsilon} \rightharpoonup Q\nabla u^{o} \qquad \text{in } [L^{2}(D)]^{2}\text{-weak} \tag{15}$$

where u^{ε} and u^{o} are the solutions of (2) (3) and (12) (13) respectively.

Proof The solutions G^{ε} of the following Dirichlet problem, for $\varepsilon > o$,

$$\nabla \cdot \left(\frac{1}{K^{\varepsilon}} \nabla G^{\varepsilon}\right) = o \quad \text{in } D, \text{ and } G^{\varepsilon} = g^{\varepsilon} \quad \text{on} \quad \partial D,$$

are non-negative functions (by the maximum principle and $g^{\varepsilon} \geq o$), which belongs to a bounded set of $C^{o,\mu}(\bar{D})$ for some $o < \mu < \lambda$ (by (11) and the DeGiorgi-Nash-Moser estimates). On the other hand, by the homogenization theory (see [5]), one deduces that

$G^\varepsilon \to G^o$ in $H^1(D)$-weak and in $C^o(\bar{D})$-strong, where G^o is the solution of the problem

$$\nabla\cdot(Q\nabla G^o) = o \quad \text{in } D, \text{ and } G^o = g^o \quad \text{on } \partial D.$$

Now we apply Theorem 4 of [8] to the translated problem for $w^\varepsilon = u^\varepsilon - G^\varepsilon$ (with $\zeta = o$)

$$w^\varepsilon \in |K(-G^\varepsilon) \equiv \{v \in H_o^1(D) : v \geq -G^\varepsilon \text{ a.e. in } D\}$$

$$\int_D \frac{1}{k^\varepsilon} \nabla w^\varepsilon \cdot \nabla(v-w^\varepsilon)\,dxdy \geq -\int_D (v-w^\varepsilon)\,dxdy, \quad \forall v \in |K(-G^\varepsilon),$$

to conclude that $w^\varepsilon \longrightarrow w^o$ in $H_o^1(D)$-weak, where $w^o \in |K(-G^o)$ is the solution of the variational inequality (13) for any $v \in |K(-G^o)$. Since $u^o = w^o + G^o$, (14) is proved. Finally (15) follows by Theorem 2 of [8].

Remark 1 Problem (12) (13) corresponds to an anisotropic homogeneous dam problem. If we make the change of coordinates

$$x = \sqrt{q_1}\,\xi \quad \text{and} \quad y = \sqrt{q_2}\,\eta$$

one can easily see that we get exactly the classical dam problem of Baiocchi [2] in the homotetic rectangle $\tilde{D} = (o,a/\sqrt{q_1})x(o,b/\sqrt{q_2})$ and with levels $y_o/\sqrt{q_2}$ and $y_a/\sqrt{q_2}$ respectively.

Remark 2 One can also consider the more general case where the permeability is in the form $K(x,y) = K_1(x)K_2(y)$ which still reduces to the homogenization of a variational inequality (see [7]). For the general non-homogeneous dam problem we refer to Alt [1]. The homogenization of the general case is an open problem.

3. CONVERGENCE OF THE FREE BOUNDARY

It is well known that Baiocchi's problem has a very smooth free boundary, given by a continuous and strictly monotone decreasing function $\tilde{\phi}$

$$\eta = \tilde{\phi}(\xi), \quad o \leq \xi \leq a/\sqrt{q_1}, \quad \text{(see Remark 1)}.$$

Therefore the free boundary corresponding to the homogenized problem is given by

$$x = \psi^o(y) = \sqrt{q_1}\ \tilde{\phi}^{-1}\ (y/\sqrt{q_2}),\ \text{for } y^* \le y \le y_o,$$
$$\text{where } y^* = \sqrt{q_2}\phi(a/\sqrt{q_1}). \tag{16}$$

We shall consider ψ^o extended by a for $o \le y \le y^*$ and by O for $y_o \le y \le b$.

On the other hand, if we assume that K^ε are piecewise constants functions we have already said that the free boundary of (2) (3) is given by a continuous curve in the form (6).

Theorem 2 Under assumptions of the Theorem 1 and assuming that K^ε are piecewise constants functions, the free boundary ψ^ε converges in mean to the homogeneous free boundary ψ^o.

Proof Since mes $(\partial\Omega^\varepsilon) = O$, the solution u^ε of the variational inequality (3) satisfies

$$\nabla\cdot\left(\frac{1}{K^\varepsilon}\nabla u^\varepsilon\right) = \chi^\varepsilon \text{ a.e. in } D, \tag{17}$$

where χ^ε is the characteristic function of the set $\Omega^\varepsilon = \{u^\varepsilon > o\}$. Using (15) and passing to the limit in (17) one finds that $\chi^\varepsilon \longrightarrow \chi^o$ in $L^p(D)$-weak, $\forall p < \infty$, since u^o satisfies the limit equation. But since if characteristic functions converge weakly in L^p also converge strongly, one has

$$\int_o^b |\psi^\varepsilon(y) - \psi^o(y)|\,dy = \int_D |\chi^\varepsilon - \chi^o|\ dxdy \to O, \text{ as } \varepsilon\downarrow o$$

and the theorem is proved.

Remark 3 With an additional assumption on K^ε, for instance, if $\frac{dK^\varepsilon}{dy} \ge o$ (this strong condition may be relaxed - see [4]), the free boundary may be represented in the form

$$\Gamma^\varepsilon : y = \phi^\varepsilon(x),\ o < x < a,$$

where ϕ^ε is a continuous and monotone decreasing function. In that case, one

can show the uniform convergence of the graphs of the free boundaries (see [7]). This is the case of vertical layers.

Remark 4 The free boundary convergence in the homogenization of the one phase Stefan problem may be studied by similar techniques extended to the parabolic case (see [9]).

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S HUANG & Y WANG

A free boundary flow well problem for an unbounded domain

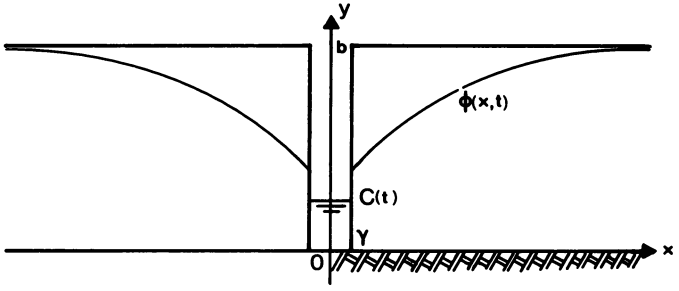
We study a free boundary problem related to an unsteady flow of compressible water moving towards a well in an unbounded porous medium; a natural generalization of the bounded case (see [1],[2]). If the discharge of the pumping well is not very large or the pumping time is not very long, the influence of the natural boundary is negligible and the problem can be treated as an unbounded problem.

1. CLASSICAL FORMULATION OF THE PROBLEM

We consider an axisymmetric fully penetrating well with radius r and depth b, the horizontal bottom of the aquifer is impervious. The aquifer is homogeneous and its permeability is assumed to be one. Denote $u(x,y,t)$ the the hydraulic head, which is b at $t=0$, $C(t)$ is the water level in the well, which satisfies

$$C(0) = b, \quad 0 < C(t) \le b; \quad C(t) \in C^1([0,T]), \quad C'(t) > -1 . \tag{1.1}$$

is the equation of seepage surface (free boundary)



Problem 1.1. Find a triplet $\{\phi,\Omega,u\}$ such that

(i) $\phi(x,t)$ is a regular function defined on $[r,+\infty)\times[0,T]$ satisfying

$$0 < \phi(x,t) \le b, \ (x,t) \in [r,+\infty)\times[0,T]; \quad \phi(r,t) \ge C(t), \ t \in [0,T]; \tag{1.2}$$

$$\phi(x,t) \to b \ (x \to +\infty) \quad \text{uniformly for } t \in [0,T].$$

(ii) Ω is defined by the relation

$$\Omega = \{(x,y,t);\ r < x < +\infty,\ 0 < t < T,\ 0 < y < \phi(x,t)\}. \tag{1.3}$$

(iii) u is a regular function defined on $\bar{\Omega}$ such that

$$Eu \equiv (x u_x)_x + u_{yy} - x u_t = 0, \text{ in } \Omega \tag{1.4}$$

$$u(r,y,t) = C(t),\ 0 \le y \le C(t),\ 0 < t \le T;$$

$$u(r,y,t) = y,\ C(t) < y \le \phi(r,t),\ 0 < t \le T; \tag{1.5}$$

$$u_y(x,0,t) = 0,\ r < x < +\infty,\ 0 < t \le T.$$

There exist two constants M and μ_o such that

$$|u(x,y,t)| \le Me^{\mu_o x^2},\quad |u_x(x,y,t)| \le Me^{\mu_o x},$$
$$|u_y(x,y,t)| \le Me^{\mu_o x},\quad (x,y,t) \in \bar{\Omega}. \tag{1.6}$$

On the free boundary $\Sigma = \{x,y,t);\ (x,t) \in (r,+\infty) \times (0,T),\ y = \phi(x,t)\}$ we have

$$u(x,y,t) = y;\ u_x^2 + u_y^2 - u_y = u_t\ ; \tag{1.7}$$

$$u(x,y,0) = b,\ r \le x < +\infty,\ 0 \le y \le b. \tag{1.8}$$

A similar technique in the chapter 10 in [3] may be applied here to conclude that

$$u(x,y,t) \le b, \text{ in } \bar{\Omega};\ u(x,y,t) \to b\ (x \to +\infty)$$
$$\text{uniformly for } t \in [0,T],\ 0 < y < \phi. \tag{1.9}$$

2. FORMULATION OF THE VARIATIONAL INEQUALITY AND UNIQUENESS

Let

$$D = \{(x,y);r<x<+\infty,0<y<b\};\ \Gamma_n = \{(x,y);r<x<+\infty,y=0\};\ \Gamma_d = \partial D \backslash \Gamma_n;$$

$$Q = DX(0,T);\quad Q_1 = \{(x,y,t);\ r<x<+\infty,\ 0<t<T,\ 0<y<b+T-t\}.$$

We introduce the transform of u used by Torelli in [1],[2]

$$\tilde{u}(x,y,t) = \begin{cases} u(x,y,t), & (x,y,t) \in \bar{\Omega} \\ y, & (x,y,t) \in \bar{Q}_1 \setminus \bar{\Omega} \end{cases}$$

$$\tilde{w}(x,y,t) = \int_0^t \{\tilde{u}(x,y+t-\tau,\tau) - (y+t-\tau)\} d\tau, \quad (x,y,t) \in \bar{Q} \tag{2.1}$$

Set

$$Z(\lambda) = \frac{1}{2}[(b-\lambda)^+]^2, \ \lambda \in \mathbb{R}$$

$$g_1(y,t) = \int_0^t [C(\tau) - (y+t-\tau)]^+ d\tau + Z(y+t);$$

$$g_2(y,t) = \int_0^t [b - (y+t-\tau)]^+ d\tau + Z(y+t); \tag{2.2}$$

$$G(x,y,t) = \begin{cases} g_1 - \dfrac{(x-r)(x-2R_o+r)}{(R_o-r)^2}(g_2-g_1), & (x \leq R_o) \\ g_2, & (x > R_o) \end{cases}$$

$$w(x,y,t) = \tilde{w}(x,y,t) + Z(y+t); \ S(x,y,t) = W - G; \ h(x,y,t) = \int_0^t S(x,y,\tau)\, dt.$$

where $R_o > r$ is arbitrarily given.

We introduce the Sobolev spaces with the weight $\sigma = \sqrt{x}e^{-\mu x}$, $\mu > \mu_o$

$$L^2_\sigma(D) = \{v; \ \sigma v \in L^2(D)\}; \ H^1(D;\sigma) = \{v; \ D^\alpha v \in L^2_\sigma(D), \ \forall |\alpha| \leq 1\};$$

$$H^1_0(\Gamma_n;\sigma) = \{v; \ v \in H^1(\Gamma_n;\sigma), \ v(0) = 0\};$$

$$W_\sigma = \{v \in H^1(D;\sigma); \ v|\Gamma_d = 0\}; \quad V_\sigma = \{v \in W_\sigma; \ r_o v \in H^1_0(\Gamma_n;\sigma)\}; \tag{2.3}$$

$$\|v\|_{W_\sigma} = \|v\|_{H^1(D;\sigma)} = \{\sum_{|\alpha| \leq 1} \int_D |\sigma D^\alpha v|^2 dxdy\}^{\frac{1}{2}};$$

$$\|v\|_{V_\sigma} = \{\|v\|^2_{W_\sigma} + \|r_o v\|^2_{H^1(\Gamma_n;\sigma)}\}^{\frac{1}{2}}.$$

where $r_o v$ is the trace of v on Γ_n.

We define the following bilinear forms and the linear form

$$a_1(u,v) = \int_D \sigma^2(u_x v_x + u_y v_y)\,dxdy + \int_{\Gamma_n} \sigma^2 r_o u \cdot r_o v\,dx;$$

$$a_2(u,v) = -\int_D 2\mu\sigma^2 u_x v\,dxdy; \quad b_1(u,v) = \int_{\Gamma_n} \sigma^2 D_x r_o u \cdot D_x r_o v\,dx; \tag{2.4}$$

$$b_2(u,v) = -\int_{\Gamma_n} 2\mu\sigma^2 D x r_o u \cdot r_o v\,dx; \quad (u,v)_\sigma = \int_D \sigma^2 u \cdot v\,dxdy;$$

$$\langle F_1(t),v\rangle_\sigma = -(G_t,v)_\sigma - a_1(G,v) - a_2(G,v) - b_1\Big(\int_o^t G(\tau)\,d\tau, v\Big)$$

$$- b_2\Big(\int_o^t G(\tau)\,d\tau, v\Big) - \int_{\Gamma_n} x \cdot q \cdot r_o v\,dx, \quad q = Z'(O) - Z(O). \tag{2.5}$$

Theorem 2.1. If $\{\phi,\Omega,u\}$ is a solution of Problem 1.1, h is given by (2.2), then h is a solution of the following problem of the variational equality

Problem 2.1. Find a function h(x,y,t) such that

$$h,\ h'_t \in L^\infty(O,T;V_\sigma);\quad h''_t \in L^\infty(O,T;\ L^2_\sigma(D))\ ; \tag{2.6}$$

$$(h'',v-h')_\sigma + a_1(h',v-h') + a_2(h',v-h') + b_1(h,v-h') + b_2(h,v-h')$$
$$+ (1,\ (v+G)^+ - (h'+G)^+_\sigma) \geq \langle F_1(t),v-h'\rangle_\sigma\ ,\ \forall v \in V_\sigma,\ \text{a.e. } t \in [O,T]; \tag{2.7}$$

$$h(x,y,O) = h'(x,y,O) = O \tag{2.8}$$

Theorem 2.2. There exists at most one solution of Problem 2.1.

Proof If h_1 and h_2 are two solutions of problem 2.1, taking $v = h'_2$ (resp. h'_1) in the inequality (2.7) for h_1 (resp. h_2) and adding the two inequalities, setting $h = h_1 - h_2$ we obtain

$$(h'',h')_\sigma + a_1(h',h') + a_2(h',h') + b_1(h,h') + b_2(h,h') \leq O. \tag{2.9}$$

then by integration on [O,t] noting h(O) = h'(O) = O we have

$$\tfrac{1}{2}\|h'\|^2_{L^2_\sigma(D)} + \int_o^t a_1(h',h')d\tau + \tfrac{1}{2}b_1(h,h) \le -\int_o^t a_2(h',h')d\tau - \int_o^t b_2(h,h')d\tau \tag{2.10}$$

by the inequalities

$$\int_o^t a_1(h',h')d\tau \ge \alpha \int_o^t \|h'\|^2_{W_\sigma} d\tau; \quad \tfrac{1}{2}b_1(h,h) \ge \alpha \|r_o h\|^2_{H^1_o(\Gamma_n;\sigma)};$$

$$\left| \int_o^t a_2(h',h')d\tau \right| \le \frac{\alpha}{3} \int_o^t \|h'\|^2_{W_\sigma} d\tau + C \int_o^t \|h'\|^2_{L^2_\sigma(D)} d\tau;$$

$$\left| \int_o^t b_2(h,h')d\tau \right| \le \frac{\alpha}{3} \int_o^t \|h'\|^2_{W_\sigma} d\tau + C \int_o^t \|r_o h\|^2_{H^1_o(\Gamma_n;\sigma)} d\tau$$

we obtain

$$\{ \|h'\|^2_{L^2_\sigma(D)} + \|r_o h\|^2_{H^1_o(\Gamma_n;\sigma)} \} + \int_o^t \|h'\|^2_{W_\sigma} d\tau$$

$$\le C \int_o^t \{ \|h'\|^2_{L^2_\sigma(D)} + \|r_o h\|^2_{H^1_o(\Gamma_n;\sigma)} \} d\tau$$

using the Gronwall inequality, it easily follows that

$$\|h'\|^2_{L^2_\sigma(D)} = 0$$

hence $h' = 0$ since $h(0) = 0$ we have $h(t) = 0$, then $h_1 = h_2$.

3. EXISTENCE OF THE SOLUTION OF PROBLEM 2.1.

Problem 3.1. Find a function h_{Rm} such that

$$h_{Rm},\ h_{Rm}' \in L^{\infty}(0,T;\ V_{\sigma R});\quad h_{Rm}'' \in L^{2}(0,T;W_{\sigma R}) \cap L^{\infty}(0,T;L^{2}_{\sigma}(D));\tag{3.1}$$

$$h_{Rm}(x,y,0) = h_{Rm}'(x,y,0) = 0;\tag{3.2}$$

$$\begin{aligned}&(h_{Rm}'',v)_{\sigma} + a_1(h_{Rm}',v) + a_2(h_{Rm}',v) + b_1(h_{Rm},v) + b_2(h_{Rm},v)\\ &+ (1,\ H_m(h_{Rm}' + G)\cdot v)_{\sigma} = \langle F_1(t),v\rangle_{\sigma},\quad \forall v \in V_{\sigma R},\ \text{a.e. } t \in [0,T].\end{aligned}\tag{3.3}$$

where

$$W_{\sigma R} = \{v \in W_{\sigma};\ v=0,\ x \geq R\};\quad V_{\sigma R} = \{v \in V_{\sigma};\ v=0,\ x \geq R\};\ R > r.\tag{3.4}$$

$$H_m(\lambda) = \begin{cases} 0, & \lambda \leq 0\\ m\lambda, & 0 < \lambda \leq \frac{1}{m}\\ 1, & \lambda > \frac{1}{m}\end{cases}\tag{3.5}$$

By the Galerkin method, we solve the problem 3.1, and then derived a priori estimates on the solutions h_{Rm} :

$$\|h_{Rm}\|_{L^{\infty}(0,T;V_{\sigma})} + \|h_{Rm}'\|_{L^{\infty}(0,T;V_{\sigma})}$$

$$+ \|h_{Rm}''\|_{L^{2}(0,T;W_{\sigma})} + \|h_{Rm}''\|_{L^{\infty}(0,T;L^{2}_{\sigma})} \leq C.$$

where C is a constant independent from R and m, firstly taking $m \to \infty$ then $R \to \infty$ passing to the limit, we conclude

Theorem 3.1. There exists a solution of the problem 2.2. verifying

$$h'' \in L^{2}(0,T;W_{\sigma}).\tag{3.6}$$

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Seepage from a pond: fixed domain formulation

1. INTRODUCTION

The physical problem of steady-state free surface seepage from axisymmetric ponds through homogeneous or nonhomogeneous porous media to a drained layer at a finite depth will be investigated using several alternative formulations. The Baiocchi transformation will have a central role in these formulations which will all be derived on fixed solution domains which simplifies the numerical computational aspects of these free surface problems.

Several authors have used the Baiocchi transformation and method on axisymmetric type seepage flows. Shao-yun Huang and Yao-dong Wang from the Department of Mathematics, Peking University, have done considerable theoretical work on the well flow problem using this approach. These are listed here but unfortunately the full citations for the references are unknown to us. Some of these results were presented at the Symposium in Montecatini, Italy. Huang and Wang [4] have studied the free boundary problem related to an unsteady flow of compressible fluid toward a well in an unbounded porous medium. Huang [2, 3], Wang [10], and Huang and Wang [5] have also investigated the bounded case of such a flow.

Cryer and Fetter [1] obtained a solution and some theoretical properties of the solution for the steady-state, free boundary problem dealing with a fully penetrating well.

Remar et al. [8] solve the problem of axisymmetric free surface seepage from a pond through homogeneous porous media to a drained layer at a finite depth. In Remar et al. [9] they solve a similar problem, but in a nonhomogeneous medium where the permeability is given as a linear function of the depth.

2. DIFFERENTIAL EQUATION FORMULATION AND EXTENSION

The physical problem investigated is the steady-state, axisymmetric seepage from a pond through a nonhomogeneous, isotropic porous medium underlain by a drain at a finite depth. Capillarity and evaporation will be neglected. Figure 1 shows the relevant information. The bottom curve $z = \alpha(r)$, consists of a small segment being horizontal and the remainder being an arbitary

convex shaped curve.

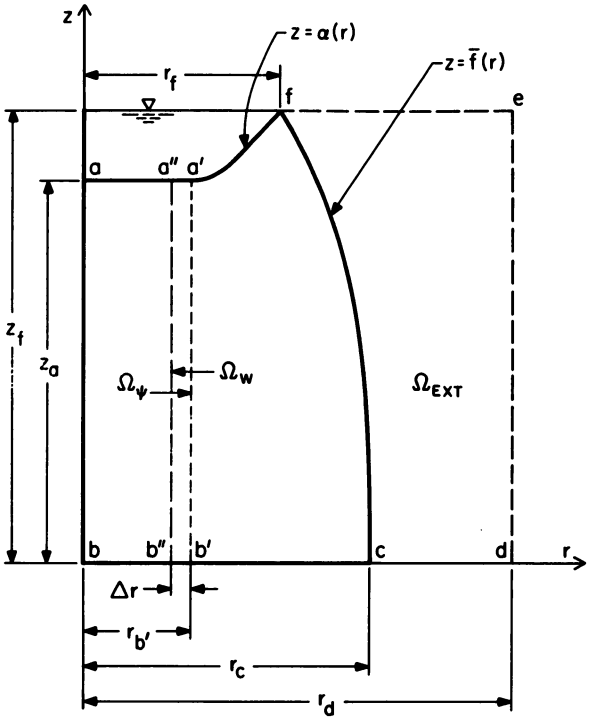

Figure 1. Mathematical configuration for first solution approach formulation of seepage from a circular pond.

Problem 1 Find $\{\bar{f}(r), \Omega, \phi, \psi, q\}$ such that

(i) Ω (seepage region) is an open subset of D;

$\Omega = \{(r,z) \mid 0 < r \leq r_f,\ 0 < z < \alpha(r);\ r_f < r < r_c,\ 0 < z < \bar{f}(r)\}$ and

$D = \{(r,z) \mid 0 < r \leq r_f,\ 0 < z < \alpha(r);\ r_f < r < r_d,\ 0 < z < z_f\}$;

(ii) $\bar{f}(r)$ (the boundary between the wet region and the dry region) $\in C^0[r_f, r_c]$ with r_c to be determined, $r_f < r_c < r_d$, $\bar{f}(r_c) = 0$, $\bar{f}(r_f) = z_f$, and $\bar{f}(r)$ is strictly decreasing in $[r_f, r_c]$; graph $\bar{f}(r)$ is given by the set $\partial\Omega - \partial D$, $r_f < r < r_c$;

(iii) $\phi(r,z)$ (force potential function) and $\psi(r,z)$ (Stokes' stream function) $\in H^1(\Omega) \cap C^0(\bar{\Omega})$, where $\phi(r,z) = \frac{p(r,z)}{\rho g} + z$ in which p is the pressure at a point (r,z) in the flowfield, ρ is the density of the fluid, g is the acceleration constant, and z is the

vertical coordinate (positive upwards);

$$\phi_r - \frac{1}{rk(r,z)}\psi_z = 0, \quad \phi_z + \frac{1}{rk(r,z)}\psi_r = 0 \text{ in } \Omega, \tag{1}$$

where $k(r,z)$ is the permeability function and will be chosen as a constant or as a function of the depth only, i.e., $k(r,z) = \mu(z)$;

(iv) $q = \frac{Q}{2\pi} \in \mathbb{R}$ (Q is the flowrate or discharge);

(v) Boundary conditions

$$\begin{aligned} \phi(r,z) &= z_F \text{ on } \widehat{af}, \\ \phi(r,0) &= 0 \text{ on } [bc], \\ \psi(0,z) &= q \text{ on } [ab], \\ \phi(r,z) &= z \text{ on } \widehat{cf}, \\ \text{and } \psi(r,z) &= 0 \text{ on } \widehat{cf}. \end{aligned} \tag{2}$$

For application of the following solution approaches, the domain will be extended beyond the free surface $\widehat{cf}$ to the vertical line de. The functions ϕ and ψ are now extended into Ω_{ext} by

$$\tilde{\phi}(r,z) = \phi(r,z), \quad \tilde{\psi}(r,z) = \psi(r,z) \text{ in } \Omega; \tag{3}$$

$$\tilde{\phi}(r,z) = z, \qquad \tilde{\psi}(r,z) = 0 \text{ in } D - \Omega, \tag{4}$$

where $\Omega_{ext} = \{(r,z) \mid r_f < r \le r_c,\ \bar{f}(r) < z < z_f;\ r_c < r < r_d,\ 0 < z < z_f\}$. The extended functions $\tilde{\phi}$ and $\tilde{\psi}$ then satisfy

$$\tilde{\phi}_r - \frac{1}{r\mu(z)}\tilde{\psi}_z = 0 \text{ and } \tilde{\phi}_z + \frac{1}{r\mu(z)}\tilde{\psi}_r = \chi_{D-\bar{\Omega}} \tag{5}$$

in the sense of distributions defined on the fixed domain D where $\chi_{D-\bar{\Omega}} = 1$ in $D-\bar{\Omega}$ and $\chi_{D-\bar{\Omega}} = 0$ in Ω.

3. FIRST SOLUTION APPROACH FORMULATION

The following approach is that used by Remar et al. [8, 9] for homogeneous, as well as nonhomogeneous, axisymmetric free surface seepage problems.

The Baiocchi transformation that will be used is

$$w(P) = \int_{\widehat{fP}} -\frac{\tilde{\psi}}{r}\,dr + \mu(z)(z - \tilde{\phi})\,dz, \tag{6}$$

where $\widehat{fP}$ is a smooth path in D joining f to P in D. Note this transformation is independent of the path and is undefined at r = 0. This latter difficulty will be handled later in this section. The function w satisfies the following differential equation in D

$$w_{rr} + w_{zz} + \frac{1}{r} w_r - \frac{d\mu(z)}{dz} \frac{w_z}{\mu(z)} = \mu(z)\ \chi_\Omega \tag{7}$$

and for the special case when $\mu(z)$ is a linear function, i.e. $\mu(z) = \xi + \zeta z$, Eq. (7) becomes

$$w_{rr} + w_{zz} + \frac{1}{r} w_r - \frac{\zeta}{\xi + \zeta z} w_z = (\xi + \zeta z)\chi_\Omega\ , \tag{8}$$

where ξ and ζ are constants. For a seepage problem in homogeneous media $\zeta = 0$ and ξ equals a constant.

The boundary conditions in terms of the function w are

$$w_z = (\xi + \zeta z)(z - z_f) \quad \text{on } \widehat{af}, \tag{9}$$

$$w_z = 0 \quad \text{on }]bd[, \tag{10}$$

$$w \text{ on } [ab] \text{ is undefined}, \tag{11}$$

$$w = 0 \quad \text{on } [fe], \tag{12}$$

$$\text{and } w = 0 \quad \text{on } [de]. \tag{13}$$

The obvious difficulty arising in Eq. (11) for r = 0 will be avoided by dividing the domain Ω into two overlapping subdomains Ω_ψ and Ω_w (see Figure 1), and formulating the problem in terms of the function ψ in Ω_ψ. (See Remark et al. [9]). Here $\Omega_\psi = \{(r,z) \mid 0 < r < r_{b'}, 0 < z < z_a\}$ and

$$\Omega_w = \{(r,z) \mid r_{b''} < r \le r_f,\ 0 < z < \alpha(r);\ r_f < r < r_c,\ 0 < z < \bar{f}(r)\}.$$

The differential equation used in Ω_ψ will be

$$\psi_{rr} + \psi_{zz} - \frac{1}{r}\psi_r - \frac{\mu'}{\mu}\psi_z = 0 \tag{14}$$

and for linear variation of μ this becomes

$$\psi_{rr} + \psi_{zz} - \frac{1}{r}\psi_r - \frac{\zeta}{\xi + \zeta z}\psi_z = 0 \ . \tag{15}$$

Equation (14) is obtained by eliminating the function ϕ from Eqs. (1).

The boundary conditions used for the domain Ω_ψ will be

$$\psi = q \qquad \text{on } [ab], \tag{16}$$

$$\psi_z = 0 \qquad \text{on }]aa'[, \tag{17}$$

$$\psi_z = 0 \qquad \text{on }]bb'[, \tag{18}$$

and $$\psi = -r_{b'} w_r \text{ on } [a'b']. \tag{19}$$

The coupling takes place on the overlap a"b"b'a'. The free boundary problem for w in the region $\tilde{D} = \Omega_w + \gamma_{cf} + \Omega_{ext}$ has the following boundary conditions

$$w_z = (\xi + \zeta z)(z - z_f) \text{ on } \widehat{a"f}, \tag{20}$$

$$w(r_{b"},z) = w(r_{b'},z)\big|_{\gamma_{a'b'}} + \int_{r_{b'}}^{r_{b"}} -\frac{\psi}{r}\, dr \text{ on } [a"b"], \tag{21}$$

$$w_z = 0 \quad \text{on }]b"d[, \tag{22}$$

$$w = 0 \quad \text{on } [fe], \tag{23}$$

and $$w = 0 \quad \text{on } [de], \tag{24}$$

where $\gamma_{cf} = \{(r,z)\,|\,r_f < r_c < r,\ z = \bar{f}(r)\}$ and $\gamma_{a'b'} = \{(r,z)\,|\,r = r_{b'},\ 0 < z < z_a\}$. Since $w \geq 0$ in D and $w > 0$ in Ω, the following "complementarity system" can be written for the w-formulation

$$\left.\begin{aligned} & w \geq 0 \\ & [(\xi + \zeta z) - (w_{rr} + \frac{1}{r} w_r + w_{zz} - \frac{\zeta}{\xi+\zeta z} w_z)] \geq 0, \\ \text{and} \quad & [(\xi + \zeta z) - (w_{rr} + \frac{1}{r} w_r + w_{zz} - \frac{\zeta}{\xi+\zeta z} w_z)]w = 0. \end{aligned}\right\} \quad \text{in } \tilde{D} \tag{25}$$

If a $\bar{w}$ and a ψ are found satisfying the above boundary value problems found in Equations (15)-(25), then $\{\bar{f}, \Omega, \phi, \psi, q\}$ is obtained as a solution of Problem 1 by setting

$$\Omega_w = \{(r,z) \,|\, w(r,z) > 0\}; \tag{26}$$

$$\Omega = \Omega_\psi + \Omega_w; \tag{27}$$

$$\bar{f}(r) = \sup \{z \,|\, (r,z) \in \Omega,\ r > r_f\}; \tag{28}$$

$$\bar{\psi} = -rw_r \text{ in } \Omega_w; \tag{29}$$

and $q = \psi(0,z)$ on $[ab]$. (30)

Note since ψ and $\bar{\psi}$ satisfy the same differential equation and boundary conditions on the overlap, $\psi = \bar{\psi}$ on the overlap and permits the definition of ψ in Ω and thus ϕ can also be obtained from Equation (1).

4. SECOND SOLUTION APPROACH FORMULATION

In setting up this problem formulation, line a"b" and a'b' will be moved to fb" and curve $\widehat{fb'}$, respectively, as seen in Figure 2. In this formulation there is now no need for a partially flat bottom to the pond. The problem now given for the region Ω_ψ will be formulated in terms of ϕ. Thus $\Omega_\psi = \Omega_\phi$. The differential equation holding in Ω_ϕ is

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \frac{\mu'}{\mu}\phi_z = 0 \tag{31}$$

and for linear variation of μ this becomes

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \frac{\zeta}{\xi+\zeta z}\phi_z = 0. \tag{32}$$

The boundary conditions for the domain Ω_ϕ are

$$\phi = z_f \quad \text{on } \widehat{af}, \tag{33}$$

$$\phi = 0 \quad \text{on } [bb'], \tag{34}$$

$$\phi_r = 0 \quad \text{on }]ab[, \tag{35}$$

and $$\phi = z - \frac{1}{\mu(z)} w_z \quad \text{on } \widehat{fb'}. \tag{36}$$

The coupling again will take place on an overlap. This time it is on fb"b'. It is essential in the iterative solution of this problem to assume that fb"b' remains an overlap region under the free surface for each iteration. Equations (22)-(25) will still hold in the corresponding $\tilde{D}$. However, Equation (21) becomes

$$w(r_{b"},z) = \int_{z_f}^{z} \mu(\bar{z})(\bar{z} - \phi)d\bar{z} \text{ on } [fb"] \tag{37}$$

and Equation (20) is irrelevant since $\widehat{a"f}$ is outside the new $\tilde{D}$. A similar procedure to obtain $\{\bar{f}, \Omega, \phi, \psi, q\}$ as a solution to Problem 1 is used here as in Equations (26)-(28) with Equation (29) becoming $\bar{\phi} = z - \frac{1}{\mu(z)} w_z$ in Ω_w and q obtained from Equation (30) after solving for ψ from Equation (1).

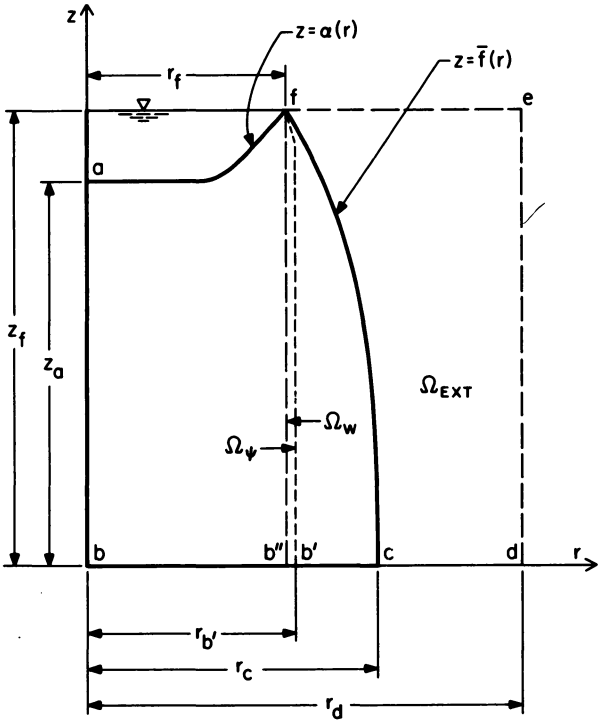

Figure 2. Mathematical configuration for second solution approach formulation.

5. THIRD SOLUTION APPROACH FORMULATION

The solution domain $\tilde{D}$ for this formulation will be equal to D. The transformation will now be

$$w(P) = \int_{\widehat{fP}} - \frac{(\tilde{\psi} - q)}{r}\, dr + \mu(z)(z - \tilde{\phi})\, dz. \tag{38}$$

The partial derivatives of this function are

$$w_r = - \frac{\tilde{\psi} - q}{r} \quad \text{and} \quad w_z = \mu(z)(z - \tilde{\phi}). \tag{39}$$

It can be shown that the function w satisfies the differential equation, Equation (7), in the region D in the sense of distributions.

The boundary conditions given by Equations (2) and (22) also apply in this case. However, a major difference arises on the axis, where w in Equation (11) was undefined. Here not only r = 0, but also $(\tilde{\psi} - q) = 0$. Applying l'Hospital's rule to w_r gives

$$w_r = - \frac{\tilde{\psi}_r}{1}\bigg|_{r=0}$$

and because $\tilde{\psi}$ is symmetric with respect to r, the boundary condition

$$w_r = 0 \text{ on }]ab[\tag{40}$$

is obtained.

The situation also changes significantly in the region Ω_{ext}. On the free surface $\widehat{cf}$, the boundary conditions are

$$\phi(r,z) = z \text{ on } \widehat{cf} \tag{41}$$

and $\psi(r,z) = 0$ on $\widehat{cf}$.

Therefore, the transformation Equation (38) gives along this line

$$w(P) = \int_{\widehat{fP}} \frac{q}{r}\, dr = q \ln r \,\Big|_{r_f}^{r} = q \ln \frac{r}{r_f}. \tag{42}$$

This equation is also valid for any point in the extended region Ω_{ext}. Therefore, the remaining boundary conditions will be

$$w = q \ln \frac{r}{r_f} \text{ on } [fe] \tag{43}$$

and $$w = q \ln \frac{r_d}{r_f} \text{ on } [de]. \tag{44}$$

It is obvious that the function w must again satisfy a constraint. Using the transformation Equation (5), $w \geq 0$ in $\tilde{D}$. If the alternative transformation given by Equation (35) is used, the function w must now satisfy the relationship $w \geq q \ln \frac{r}{r_f}$ in $\tilde{D} = D$. The line singularity at $r = 0$ now does not cause any difficulties for the solution of the problem. In this case the values of w can be negative. The following *complementarity system* can be written for this w-formulation

$$\left.\begin{aligned} &(w - q \ln \frac{r}{r_f}) \geq 0, \\ &[(\xi + \zeta z) - (w_{rr} + \frac{1}{r} w_r + w_{zz} - \frac{\zeta}{\xi+\zeta z} w_z)] \geq 0, \\ \text{and} \quad &(w - q \ln \frac{r}{r_f})[(\xi + \zeta z) - (w_{rr} + \frac{1}{r} w_r + w_{zz} - \frac{\zeta}{\xi+\zeta z} w_z)] = 0. \end{aligned}\right\} \text{ in } \tilde{D} \tag{45}$$

If a w is found satisfying the above boundary value problem, then $\{\bar{f}, \Omega, \phi, \psi, q\}$ is obtained as a solution to Problem 1 by setting

$$\Omega = \{(r,z) \,|\, w(r,z) \geq \ln \frac{r}{r_f}\} ; \tag{46}$$

$$\bar{f}(r) = \sup \{z \,|\, (r,z) \in \Omega, \; r > r_f\}; \tag{47}$$

$$\psi = q - rw_r \text{ and } \phi = z - \frac{1}{\mu(z)} w_z \text{ in } \Omega; \tag{48}$$

and $$q = rw_r \text{ at } (r_f, z_f). \tag{49}$$

6. RESULTS

The first and third formulation will now be used on an example of a pond with seepage in homogeneous media taken from Jeppson [6, 7]. A finite difference successive over-relaxation approach with projection was used for these two

formulations to solve the w parts of the problem. For the first formulation the ψ-problem was solved by a finite difference successive over-relaxation technique. The discretized forms of the necessary equations are found in Remar et al. [8] for a fixed q value since an outer iteration was necessary to solve for this unknown parameter. After obtaining solutions for two guessed q's the secant method was used to determine a new q and the iterations proceed until a compatibility condition, Equation (52), is satisfied. Stopping error criteria for the computations consisted of the following for the first formulation

$$\max_{i,j} \left| w^{n+1}_{q^{(k)},i,j} - w^{n}_{q^{(k)},i,j} \right| \leq \varepsilon_1, \tag{50}$$

$$\max_{i,j} \left| \psi^{n+1}_{q^{(k)},i,j} - \psi^{n}_{q^{(k)},i,j} \right| \leq \varepsilon_1, \tag{51}$$

and $$\left| f_h(q^k) \right| \leq \varepsilon_2, \tag{52}$$

where ε_1 and ε_2 are given small positive constants,

$$f_h(q^k) = (w_q) \Big|_{r_f, z_f - \Delta z} - \frac{\Delta z^2}{2}$$

is a compatibility condition at point f requiring that both Dirichlet and Neumann boundary conditions, Equations (20) and (23) for the first formulation and Equations (20) and (43) for the third, be satisfied there; and Equations (50) and (52) for the third formulation.

Jeppson [6, 7] used an inverse transformation method in conjunction with finite differences to obtain his results. However, in his approach the shape of the bottom is a result of the calculations, and he presents no examples with a flat part of the bottom. Nevertheless, it is possible to approximate his profiles closely enough so that the results should be comparable. The problems treated herein were normalized with respect to k, which was set equal to 1 ft/s. Therefore, the discharge Q is given in ft^2.

Jeppson's approach requires starting values within 10 to 30% of the correct values. An initialization of all zeros was used in the present work. The calculations were carried out on an Itel AS/6 computer (which is similar to an IBM/370 Model 168) with a program written in single precision.

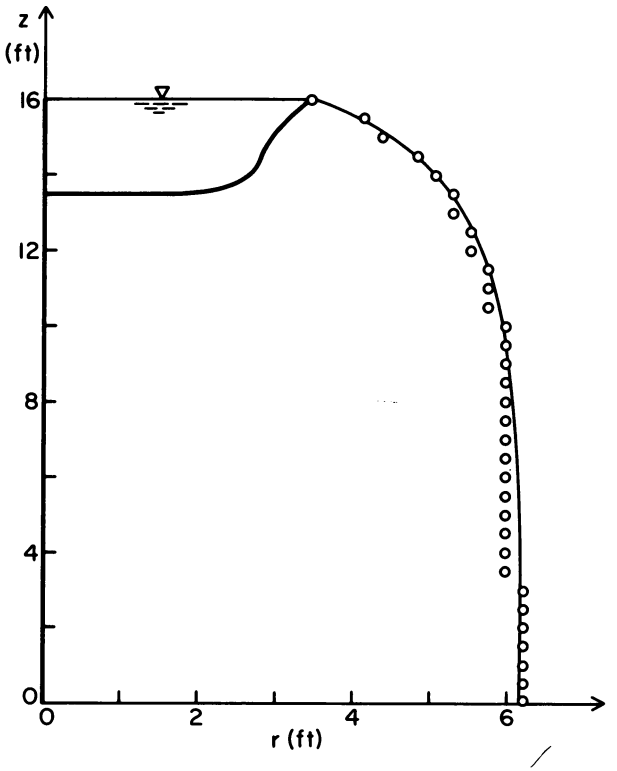

Figure 3. Comparison between:- Jeppson's ([6], [7]) results and o - the first and third (except for two points) solution approach results.

This example analyzed is Figure 4 of Jeppson [6] and Figure 3 of Jeppson [7] and Figure 5 of Remar et al [8]. The mesh size used was $\Delta r = 0.23$ ft, $\Delta z = 0.50$ ft, while the iteration criterion was $\varepsilon_1 = \varepsilon_2 = 0.0002$. The flat part of the bottom extends to r = 1.84 ft. The results using the first solution approach formulation are presented in Table 1. The resulting discharge rate is only 0.67% more than the corresponding Jeppson's result, which is $Q = 120.0$ ft^2. The iteration was stopped because $f_h(q^k)$ became negative and once this happens the iterations do not converge. Using somewhat different starting values $q^{(0)}$ and $q^{(1)}$ would prevent such a situation. As can be seen in Figure 3, the free surface is in very good agreement. In order to plot the free surface, the first point with the value zero that occurred in each line as the line is scanned

in the increasing r-direction was taken as a point on the free surface. The average value of the optimum relaxation parameter resulting from computation was ω_{opt} = 1.809. The computer time (the GO step only) was 8.12 sec.

The results obtained using the third solution approach formulation are given in Table 2. All input parameters of the problem were exactly the same, except the relaxation parameter, which was set ω = 1.80 and kept constant. The free surface was almost identical (two points moved by Δr). The discharge rate is now 1.95% below Q = 120.0 ft^2. The computer time was 17.40 sec, more than twice the time required by the first solution approach.

Table 1. Iteration Information (ψ/w method - first solution approach formulation)

k	number of iterations	$q^{(k)}$	$Q^{(k)}=2\pi q^{(k)}$	$f_h(q^{(k)})$
0	156	21.0000	131.947	0.515026 E+00
1	116	22.0000	138.230	0.841132 E+00
2	119	19.4207	122.024	0.505741 E-01
3	70	19.2557	120.987	0.649863 E-02
4	52	19.2313	120.834	0.114232 E-02
5	32	19.2261	120.801	-0.135076 E-02

Table 2. Iteration Information (third solution approach formulation)

k	number of iterations	$q^{(k)}$	$Q^{(k)}=2\pi q^{(k)}$	$f_h(q^{(k)})$
0	134	22.0000	138.230	0.104739 E+01
1	125	21.0000	131.947	0.701232 E+00
2	128	18.9742	119.218	0.682269 E-01
3	123	18.7559	117.847	0.794148 E-02
4	115	18.7271	117.666	-0.493526 E-04

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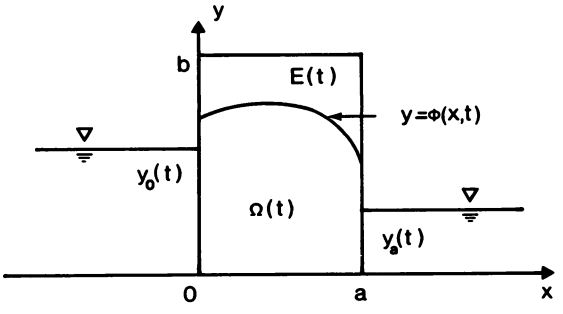
On the free boundary of the evolution dam problem

1. INTRODUCTION

The present work extends some results of Friedman and Torelli [6] on the free boundary in the evolution problem representing a non-steady flow of an incompressible fluid moving across a homogeneous porous dam. One shows the continuity in time of the free boundary and one gives sufficient conditions in order to obtain its stability as $t \to +\infty$ with respect to the stationary dam problem of Baiocchi [1]. This is done after the introduction of the main results of [6] on the evolution dam problem and using the continuous variation, with respect to the Hausdorff distance, of the coincidence sets of a family of elliptic variational inequalities depending on a parameter (the time) together with the uniform regularity in space variables of the corresponding free boundaries.

2. PRELIMINARIES

Consider a dam $D = \{(x,y) / 0 < x < a,\ 0 < y < b\}$, with walls $x = 0$, $x = a$, which separates two reservoirs at levels $y_o(t)$ and $y_a(t)$ respectively.



The piezometric head (the velocity potential) $u = u(x,y,t) = y+p(x,y,t)$ (where p is the pressure of the liquid in the dam) satisfies

$$\left.\begin{aligned} &\Delta u = u_{xx} + u_{yy} = 0 \quad \text{if} \quad 0 < y < \phi(x,t), \quad 0 < x < a, \quad t > 0 \\ &u(i,y,t) = y_1(t) \quad \text{if} \quad 0 < y < y_i(t), \quad u(i,y,t) = y \\ &\qquad\qquad \text{if} \quad y_i(t) < y < \phi(i,t), \quad (i=0,a) \\ &u_y(x,0,t) = 0 \quad \text{if } 0 < x < a, \quad t > 0, \end{aligned}\right\} \tag{2.1}$$

where $y = \phi(x,t)$ is the seepage line, or the free boundary separating the wet part of the dam from the dry one; on this line one has

$$u = y, \quad u_t = u_x^2 + u_y^2 - u_y \quad \text{for} \quad y = \phi(x,t), \quad 0 < x < a, \quad t > 0; \tag{2.2}$$

finally, the initial condition is given by

$$\phi(x,0) = \phi_0(x), \quad 0 \le x \le a, \tag{2.3}$$

where $\phi_0 \in C^0[0,a]$, $\phi_0(x) > 0$, $\phi_0(0) \ge y_0(0)$, $\phi_0(a) \ge y_a(0)$.

In this paper we assume as in [6], that the levels $y_i(t)$ verify

$$y_1 \in C^1[0,T], \quad 0 < y_i(t) < b \quad \text{and} \quad y_i(t) + 1 > 0, \quad 0 \le t \le T, \quad (i=0,a). \tag{2.4}$$

Following Baiocchi's method, Torelli has introduced the function

$$w(x,y,t) = \int_0^t [u(x,y+t-\tau,\tau) - (y+t-\tau)]\, d\tau, \tag{2.5}$$

where $u(x,y,t) = y$ if $y > \phi(x,t)$, to which he has shown an existence and uniqueness result (cf. [9]).

Now we state the main results of [6] which we will need and which show, in particular, that the functions w and u have various properties that one may expect from the physical point of view

$$w \in L^2(0,T;H^2(D)) \cap C^0([0,T];\, H^{5/4}(D)); \tag{2.6}$$

$w_t - w_y \geq 0$, this means that the internal pressure of the liquid is non negative; (2.7)

$w_y \leq 0$, this implies that the wet part of the dam is a subgraph i.e. there is a function ϕ, such that, (2.8)

$$\Omega(t) \equiv \{(x,y)/w(x,y,t) > 0\} = \{(x,y)/0 < y < \phi(x,t),\ 0 < x < a\},\ t > 0;$$

For almost all $t \geq t_o \equiv \sup_x \phi_o(x)$, the function w(t) is a solution of the elliptic variational inequality

$$w(t) \geq 0 \text{ a.e. } \int_D (-\Delta w(t) + 1)(v - w(t))\, dxdy \geq 0 \quad \forall v \in L^2(D),\ v \geq 0 \text{ a.e.}, \quad (2.9)$$

verifying $w(t) = G(t)$ on $\partial D \setminus \{y=o\}$, where G is any function in $H^2(Dx(0,T))$, which coincides with $\int_o^t y_i(\tau) - (y+t-\tau)^+ d$ on $x=i\ (i=o,a)$ and vanish on $y=b$.

For almost all $t \geq t_o \equiv \sup \phi_o(x)$, the free boundaries verify

$$x \to \phi(x,t) \text{ are continuous in } x \in [0,a], \quad (2.10)$$

(moreover they are also continuously differentiable in x when $0 < x < a$, $0 < \phi(x,t) < b$, for a.a. $t \geq t_o$, by the results of Caffarelli [3]).

If the data $\phi_o(x)$ is *quasi-stationary* (q.s.), i.e. if it is the free boundary at time t=0 of a "regular" solution u(x,y,t) of the dam problem (see [9]), which exists for $-t_o \leq t \leq 0$, where $t_o = \sup \phi_o(x)$, then we can extend (2.10) for a.a. $t \geq 0$. Indeed, in that case, from [6], we have $\Omega(t) = \{(x,y) \mid \lambda(x,y,t) > 0\}$ for some function λ satisfying the variational inequality (2.9) for a.a.$t > 0$ to which we can apply the same argument used to obtain the conclusion (2.10) with the function w.

Let also mention that under the restriction that the initial data are quasi-stationary, Friedman and Torelli have proved [6] comparison theorems, a monotonicity result which asserts that if $y_o(t)$ and $y_a(t)$ are decreasing then $t \to \phi(x,t)$ is decreasing, and with additional conditions on the derivatives of $y_i(t)$ the seepages lines $\phi(x,t)$ remains bounded by a fixed number b_o, independent of b. Moreover if one assumes that the initial data $\phi_o(x)$ is q.s. and bounded above by some constant c_o, and the levels $y_i(t)$

verify $0 < c_1 \le y_i(t) \le c_2$, $\forall 0 \le t < \infty$, then Theorem 8.6 of [6] states that there exist constants c_3 and c_4 (depending only on c_o, c_1, c_2) such that

$$0 < c_3 \le \phi(x,t) \le c_4 \quad \text{for all} \quad 0 \le x \le a,\ 0 \le t \le \infty.$$

3. CONTINUITY OF THE FREE BOUNDARY

In this paper we assume that the free boundary is such that

$$0 < \gamma \le \phi(x,t) < b, \text{ for all } o \le x \le a \text{ and } t \ge t_o, \text{ where} \tag{3.1}$$

$$t_o = \sup_x \phi_o(x), \text{ or } t_o = 0 \text{ if the initial data } \phi_o \text{ is quasi-stationary.}$$

Consider the family of compact subset of $\bar{D}$ given by

$$E(t) = \{(x,y) \in \bar{D} \,|\, x(x,y,t) = 0\}, \quad t \ge t_o. \tag{3.2}$$

Lemma 3.1. $\Delta w \in C^o([t_o, T]; L^p(D)), \forall p < \infty$.

Proof By (2.9) we have, a.a. $t \ge t_o$, $w(t) = 0$ and $\Delta w(t) = 0$ a.e. in $E(t)$; $w(t) > 0$ and $\Delta w(t) = 1$ a.e. in $D \setminus E(t)$, say

$$\Delta w(t) = \chi(t) \text{ in } L^p(D), \tag{3.3}$$

where $\chi(t)$ is the characteristic function of the set $D \setminus E(t)$.

Let $t_n \to t$ in $[t_o, T]$. $\{\chi(t_n)\}$ being bounded, one has for a subsequence $\chi(t_n) \to q$ in $L^p(D)$ - weakly. But from (2.6), $w \in C^o([0,T]; H^{-1}(D))$, and by (3.3) we find $q = \chi(t)$. Since if characteristic functions converge weakly in $L^p(D)$ to a characteristic function they converge also strongly, we find $x \in C^o([t_o, T]; L^p(D))$, and the lemma is proved.

Consider, for each fixed ε, $0 < \varepsilon < \gamma$, $D_\varepsilon = (\varepsilon, a-\varepsilon) x (\varepsilon, b)$, and

$$E_\varepsilon(t) = E(t) \cap \bar{D}_\varepsilon, \tag{3.4}$$

where $E(t)$ is given by (3.2).

Denote by $\mathcal{H}(\bar{D})$ the space of the compact subsets of $\bar{D}$ with the Hausdorff metric. We recall that the Hausdorff distance δ between two sets E_1 and E_2 is defined by

$$\delta(E_1,E_2) = \sup\left[\sup_{x_1 \in E,} d(x_1,E_2),\ \sup_{x_2 \in E_2} d(x_2,E_1)\right],$$

where $d(x_1,E_2) = \inf_{x_2 \in E_2} [x_1 - x_2]$. It is well known that $\mathcal{H}(K)$ is a compact metric space for δ if K is a compact of R^n (see, for instance [5, p.42]).

PROPOSITION 3.2. For each $\varepsilon > 0$, the application

$$[t_o,T] \quad t \to E_\varepsilon(t) \in \mathcal{H}(\bar{D}) \tag{3.5}$$

is continuous for all T, $t_o < T < \infty$.

Proof Let $t_n \to t$ in $[t_o,T]$ and denote, $E_\varepsilon(t_n) = E^n$ and $E_\varepsilon(t) = E$. Since E^n is a family of compact subsets of $\bar{D}_\varepsilon$ we can extract a subsequence (also denoted by $\{E^n\}$) converging in the Hausdorff metric for some compact subset E^* of $\bar{D}_\varepsilon$. If we show that $E^* = E$, the proof of the proposition will be complete.

Step 1 Since $\delta(E^n, E^*) \to 0$, for every $(x,y) \in E^*$ there exists $(x_n,y_n) \in E^n$, such that $(x_n,y_n) \overset{n}{\to} (x,y)$. Using

$$|w_n(x_n,y_n)-w(x,y)| \le |w_n(x_n,y_n)-w(x_n,y_n)| + |w(x_n,y_n)-w(x,y)|$$

(where $w_n = w(t_n)$ and $w = w(t)$) and remarking that by (2.6) and Sobolev imbeddings, one has $w \in C^o([0,T]; C^o(\bar{D}))$, we find that $w(x,y) = 0$. By (3.4), this means that $(x,y) \in E$ and therefore $E^* \subset E$.

Step 2 $E = E^* \cup N$, with int $(N) = \emptyset$. Suppose, to the contrary, that there exists a ball $B \subset$ int $(E \backslash E^*)$. Then we have

$$\Delta w = 1 \quad \text{in} \quad B,\ w = 0 \text{ on } \partial B. \tag{3.6}$$

In fact, by one hand, $\Delta w^n = 1$ a.e. in $D_\varepsilon \backslash E^n$. Since $E^n \to E^*$ (Hausdorff), by the other hand, for every $\phi \in D(D_\varepsilon \backslash E^*)$ there exists n_o, such that supp $\phi \subset D_\varepsilon \backslash E^n$, $\forall n \ge n_o$. Therefore, using Lemma 3.1, one has

$$\int_D (\Delta w - 1)\tilde{\phi}\, dxdy = \lim_n \int_D (\Delta w^n - 1)\tilde{\phi}\, dxdy = 0,$$

where $\tilde{\phi}$ denotes the extension in D by zero. From (3.6), the maximum

principle implies $w < o$ in B, which is impossible. Therefore we conclude $\text{int}\,(E \backslash E^*) = \emptyset$.

Step 3 Finally, the other inclusion follows easily

$$E = \overline{\text{int}(E)} = \overline{\text{int}(E^*)} \subset E^* ,$$

since from step 2, one has $\text{int}(E) = \text{int}\,(E^*)$.

Consider the free boundary (see (2.8))

$$\phi(t)(x) = \phi(x,t) = \sup \{y \mid w(x,y,t) > o \text{ in } D\} .$$

Lemma 3.3 For each $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$, independent of $t \in [t_o,T]$, such that

$$|\phi_x(x,t)| \leq C_\varepsilon, \quad \forall x \in [\varepsilon, a-\varepsilon,], \forall t \in [t_o,T]. \tag{3.7}$$

Proof By a classical localisation argument, using Lemma 3.1. and observing that w is a solution of (2.9), one easily finds that

$$w \in L^\infty(t_o,T;C^{1,1}(\bar{D}_\varepsilon)) \quad \text{(see [2])}.$$

Then (3.7) is a consequence of the regularity result on ϕ from [3].

Denote, for $0 < \varepsilon < \gamma$,

$$\Gamma_\varepsilon(t) = \Gamma(t) \cap \bar{D}_\varepsilon = \{(x,y) : \varepsilon \leq x \leq a-\varepsilon, \ y = \phi(x,t)\}.$$

Now we can state the following result, which extends Theorem 6.1. of [6].

Theorem 3.4. For each $0 < \varepsilon < \gamma$, the maps

$$[t_o,T] \quad t \to \Gamma_\varepsilon(t) \in \mathcal{H}(\bar{D}) \tag{3.8}$$

$$[t_o,T] \quad t \to \phi(t) \in C^o[\varepsilon,a-\varepsilon] \tag{3.9}$$

are continuous, where $t_o = \sup_x \phi_o(x)$. Particularly, $\phi \in C^o((o,a)x[t_o,T])$.

Proof By Lemma 3.1, (2.10) is valid for all $t \geq t_o$ and then (3.9) is an easy consequence of (3.8). To prove this we use the Proposition 3.2 and the regularity of $\Gamma_\varepsilon(t)$.

Let $t_n \to t$ in $[t_o,T]$ and denote $E^n = E(t_n)$, $\Gamma^n = \Gamma_\varepsilon(t_n)$ and $Q^n = \bar{\Omega}_\varepsilon(t_n)$, where

$$\Omega_\varepsilon(t) = \{(x,y) : \varepsilon < x < a-\varepsilon,\ \varepsilon < y < \phi(x,t)\} \quad (0 < \varepsilon < \gamma).$$

Consider a subsequence (denoted in the same way) such that, in the Hausdorff sense,

$$E_n \to E^*,\ Q^n \to Q^* \text{ and } \Gamma^n \to \Gamma^*,$$

where E^*, Q^* and Γ^* are compact subsets of $\bar{D}_\varepsilon$.

From Proposition 3.2, one has $E^* = E$.

Since $\Gamma^n = E^n \cap Q^n$, it is obvious that $\Gamma^* \subset E \cap Q^*$. Conversely, if $(x,y) \in E \cap Q^*$, (x,y) is a limit point of $(x_n,y_n) \in E^n$ and $(\xi_n,n_n) \in Q^n$, and we have $(x,y) \in \Gamma^*$. So $\Gamma^* = E \cap Q^*$. Since by definition $\Gamma = E \cap Q$, to prove that $\Gamma = \Gamma^*$ it is sufficient to establish that $Q = Q^*$.

From $\bar{D}_\varepsilon \backslash E = Q \backslash \Gamma \subset Q^*$, it is clear that $Q \subset Q^*$. To prove the other inclusion we use a uniform cone property of the sets $\Omega_\varepsilon(t)$, with respect to t.

Assume to the contrary that $(x,y) \in Q^* \backslash Q = Q^* \cap (E \backslash \Gamma)$, and let $(x_n,y_n) \to (x,y)$, with $(x_n,y_n) \in Q^n$. As a consequence of Lemma 3.3, there exists an open cone K_n with vertex (x_n,y_n) (with angle and height independents of n) such that

$$K_n \subset Q^n \quad \text{and} \quad K_n \cap E^n = \emptyset, \text{ for every } n. \tag{3.10}$$

By the other hand, for all $n \geq n_1$, one has $(x_n,y_n) \in E \backslash \Gamma$ and $K_n \cap (E \backslash \Gamma) \neq \emptyset$. Since each point of E is a limit point of a sequence of points in E_n, for $(\xi_n,\eta_n) \in (E \backslash \Gamma) \cap K_n$ there exists $(\bar{\xi}_n,\bar{\eta}_n) \in E_n$ such that

$$|\xi_n - \bar{\xi}_n| + |\eta_n - \bar{\eta}_n| \leq 1/n.$$

Then, for $n \geq n_2$, one has $(\bar{\xi}_n,\bar{\xi}_n) \in K_n$, which is a contradiction with (3.10). Therefore $Q^* = Q$ and the proof is complete.

Corollary 3.5. If the initial data is quasi-stationary then (3.8) and (3.9) are valid for $t \geq 0$.

Remark 3.6. The continuous variation with respect to the Hausdorff metric of

the coincidence set of a variational inequality was earlier used in [7] for control problems and in [8] to discuss the stability of regular free boundaries. However the continuity in time of the evolution dam problem can also be obtained as a consequence of the paper [4], where the stability of coincidence sets is considered by a different and independent method.

4. ASYMPTOTIC BEHAVIOUR OF THE FREE BOUNDARY

Assuming

$$|y_i(t) - y_i| \leq c/t^{\delta}, \text{ with } \delta > 1 \text{ and } i = 0,a, \tag{4.1}$$

Friedman and Torelli have shown the following stability result

$$\int_{t_1}^{\infty} \| w(t) - w_{\infty} \|^2_{H^1(D)} \, dt < \infty, \tag{4.2}$$

where w_{∞} is the solution of the stationary dam problem (see [1]) with levels y_o and y_a $(0 < y_a < y_o < b)$.

With further assumptions on the stabilization of the levels of the reservoirs, in particular if the levels became constants in finite time, one can prove

Theorem 4.1. If $y_i(t)$ $(i=0,a)$ verify (4.1) and

$$y_1' \in L^1(0,\infty) \cap L^2(0,\infty) \text{ and } 1+y_i'(t) \geq C_* > 0, \; 0 < t < \infty, \tag{4.3}$$

then one has $w \in L^{\infty}(t_o,\infty;W^{2,\infty}(D_{\varepsilon}))$, $(\varepsilon > 0$ arbitrary), and

$$w(t) \underset{t\to\infty}{\to} w_{\infty} \text{ in } H^1(D) \;\; C^{1,\alpha}(\bar{D}_{\varepsilon}), \; 0 < \alpha < 1. \tag{4.4}$$

Proof The condition (4.3) is sufficient to establish

$$\int_{0}^{\infty} \| w'(t) \|^2_{H^1(D)} \, dt < \infty. \tag{4.5}$$

To prove (4.5) use Lemma 10.2 of [6] and inspect carefully the second estimate in the proof of Theorem 4.1 of [9] - formulae (4.26-38). Now (4.2) and (4.5) imply the asymptotic convergence (4.4) in

$H^1(D)$ - strong (*) and by a classic argument on the local regularity for the solution w(t) of (2.9) (see [2]) it follows that $w \in L^\infty(t_o,\infty;W^{2,\infty}(D_\varepsilon))$. Finally the Rellich-Kondratchoff compactness imbeddings give the second part of (4.4).

It is well known that in the stationary dam problem (see [1]) the free boundary is given by a continuous and strictly monotone decreasing function $y = \phi_\infty(x)$, $o \le x \le a$.

Then we can use the arguments of the proof of the Theorem 3.4. together with the results of Theorem 4.1. in order to get the asymptotic convergence of the free boundaries:

Theorem 4.2. Under assumptions (4.1) and (4.3), one has $\phi(x,t) \underset{t\to\infty}{\to} \phi_\infty(x)$ uniformly in $\varepsilon \le x \le a-\varepsilon$, for any $\varepsilon > o$.

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(*) Recall that, if $f(t) \in H^1(t_1,\infty)$, then $\lim_{t\to\infty} f(t) = o$.

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D G ARONSON

Nonlinear diffusion problems

This Discussion Group was devoted to various diffusion, diffusion-convection, and diffusion-interaction problems involving nonlinear diffusion mechanisms. In this essay I shall sketch some of the background of the problems involved and attempt to place the various discussion papers within that background. They are summarized at the end of this report. The scope of my survey has been determined by the nature of the discussion session. Thus, although I will cover a somewhat broader range of topics, I have not attempted to be comprehensive. Peletier [32] has written an excellent survey which covers several points that have been omitted here.

Nonlinearity in the diffusion mechanism often leads to the occurrence of interfaces or free boundaries. Here I shall be concerned only with problems in which the interface occurs as an a posteriori property of the solution rather than as part of the formulation of the problem. Other types of problems which may be of considerable theoretical or practical interest are beyond the scope of this discussion.

One of the simplest example of a nonlinear diffusion mechanism occurs in modeling isentropic flow of an ideal gas through a homogeneous porous medium. If $u = u(x,t)$ denotes the density of the gas then its evolution is governed by the equation

$$\frac{\partial u}{\partial t} = \Delta(u^m) \tag{1}$$

where Δ is the Laplace operator on $\mathbb{R}^d$ and m is a constant which is greater than 1. More information about the derivation of equation (1) can be found in reference [5]. Although equation (1) is usually called the porous medium equation it also occurs in other contexts, for example, in plasma physics [9] and in population dynamics [19].

Since u in equation (1) represents a density it is natural to restrict attention to solutions of (1) which satisfy $u \geq 0$. If one writes equation (1) in the standard Fickian diffusion form one sees that the diffusivity is mu^{m-1}. In particular, since $m > 1$ it follows that the diffusivity vanishes

with the density. The most striking consequence of this nonlinear degeneracy in the diffusion is the finite speed of propagation of disturbances from rest. This is in stark contrast to the classical linear diffusion case, $m = 1$, where disturbances are propagated with infinite speed.

To be more specific about the finite speed of propagation, consider the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta(u^m) \text{ in } \mathbb{R}^d \times \mathbb{R}^+ \\ &u(\cdot,0) = u_0 \text{ in } \mathbb{R}^d \end{aligned} \tag{2}$$

where u_0 is a given nonnegative function on $\mathbb{R}^d$. For simplicity, assume that

$$\text{supp } u_0 \text{ is homeomorphic to } \bar{B}_1(0) \tag{3}$$

where

$$B_r(\xi) = \{x \in \mathbb{R}^d : |x - \xi| < r\}.$$

Although problem (2) does not have a classical solution in this case, appropriate classes of weak solutions with the associated existence, uniqueness, and regularity theory are provided by Oleinik, Kalashnikov and Chzou [30] for $d = 1$ and Sabinina [34] for $d > 1$. Further regularity results can be found in [1], [2], [12], [14], [18] and [26]. The essential fact to note here is that, under the hypothesis (3) on supp u_0, the weak solution u of (2) exists and

$$\text{supp } u(\cdot,t) \text{ is homeomorphic to } \bar{B}_1(0)$$

for all $t \in \mathbb{R}^+$. I shall refer to the set

$$\{(x,t) \in \mathbb{R}^d \times \mathbb{R}^+ : x \in \partial \text{ supp } u(\cdot,t)\}$$

as the interface.

Very little is known about the nature of the interface in $\mathbb{R}^d$ for $d > 1$. Indeed the only result which I am aware of is the Caffarelli-Friedman proof that the interface is Hölder continuous [13]. On the other hand, there has been a considerable amount of work on the interface for $d = 1$ and I shall

discuss some of it here.

Consider the initial value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(u^m) \qquad \text{in } \mathbb{R} \times \mathbb{R}^+ \\ &u(\cdot,0) = u_0 \qquad \text{in } \mathbb{R}, \end{aligned} \tag{4}$$

where

$$u_0 \begin{cases} > 0 \text{ on } J \equiv (a_1,a_2) \\ \\ = 0 \text{ on } \mathbb{R}\backslash J \end{cases} \tag{5}$$

for $-\infty < a_1 < a_2 < +\infty$. In this case it is known that the interface consists of two Lipschitz continuous curves $x = \zeta_i(t)$ for $i = 1$ and 2, where $\zeta_i(0) = a_i$, ζ_1 is nonincreasing, and ζ_2 is nondecreasing. Moreover, there exist numbers $t_i^* \in [0,+\infty)$ such that

$$\zeta_i \text{ is strictly monotone and } C^1 \text{ on } (t_i^*,+\infty)$$

and

$$\zeta_i = a_i \quad \text{on} \quad [0,t_i^*] .$$

For example, the case $t_i^* > 0$ occurs if

$$\{u_0(x)\}^{m-1} \le c(x - a_i)^2$$

in a neighbourhood of $x = a_i$ for some constant $c \in \mathbb{R}^+$. These results are proved in [3], [11], and [25]. They are described in more detail in [5]. When the waiting time t_i^* is positive the corresponding interface is stationary for t_i^* units of time. After the waiting time it begins to move and never again stops. In reference [6] Aronson, Caffarelli and Kamin estimate t_i^* in terms of the behavior of u_0^{m-1} and show that under certain circumstances it can be computed a priori. In addition, they derive criteria which ensure that $\zeta_i \in C^1(\mathbb{R}^+)$. For further information see reference [5] in this volume.

In reference [22], Kamin has shown that if u is the solution of problem (4) with u_0 satisfying (5) then

$$\lim_{t \uparrow \infty} t^{1/(m+1)} |u(x,t) - \bar{u}(x,t)| = 0$$

where $\bar{u}$ is an appropriately normalized Barrenblatt-Pattle similarity solution of the porous medium equation. This result shows that the large time behavior of solution of problem (4) for data with compact support is independent of the form of u_0. In his paper in this volume [35], Vazquez considerably sharpens Kamin's result by obtaining a more precise characterization of the similarity solution $\bar{u}$ using not only the total mass carried by u_0 but also its center of mass. This allows him to obtain, among other results, the leading terms in the asymptotic development of the interface for large t.

In addition to the initial value problem (2), various initial-boundary value problems for the porous medium equation are also of interest. For example, Aronson and Peletier [8] study the large time behavior of solutions of the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta(u^m) && \text{in } \Omega \times \mathbb{R}^+ \\ &u = 0 && \text{on } \partial\Omega \times \mathbb{R}^+ \\ &u(\cdot,0) = u_0 && \text{in } \Omega \end{aligned} \tag{6}$$

where $\Omega \subset \mathbb{R}^d$ is a smoothly bounded domain with compact closure. Roughly speaking their result is that the large time behavior of the solution of (1) is independent of the detailed behavior of u_0 and is essentially determined by the unique positive solution of the nonlinear elliptic problem

$$\begin{aligned} &\Delta(v^m) + \frac{1}{m-1} v = 0 \quad \text{in} \quad \Omega \\ &v = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

In various applications the porous medium equation (1) is replaced by the more general equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) \tag{7}$$

where it is often assumed that

$$\varphi(0) = \varphi'(0) = 0, \ \varphi > 0 \text{ on } \mathbb{R}^+, \text{ and } \varphi' > 0 \text{ on } \mathbb{R}^+. \tag{8}$$

Existence, uniqueness and some properties of weak solutions for the initial value problem and certain initial-boundary value problems are given by Oleinik et al. in reference [30] while Knerr [25] provides a great deal of detailed information about the interface when $d = 1$. In a different direction, Kalashnikov [20] and Peletier [31] have shown independently that for $d = 1$ the necessary and sufficient condition for a finite speed of propagation (and hence the occurrence of interfaces) is the convergence of the integral

$$\int_{0+} \frac{\varphi'(u)}{u}\, du.$$

VanDuyn [15] has studied equation (7) with $\varphi(0) = \varphi(1) = 0$ in connection with his work on flow of immiscible fluids. The equation

$$\rho(x)\, \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\, \varphi(u)$$

arises in the work of Rosenau and Kamin [33] on nonlinear heat conduction in nonhomogeneous media. Here $\rho(x)$ is the density of the medium and $\int_{\mathbb{R}} \rho$ its total mass. They prove existence and uniqueness of weak solutions of the initial value problem and derive the large time behavior of solutions in both the finite and infinite mass cases. An account of their results can be found in Kamin's paper in this volume [23].

Gurtin and MacCamy [19] have derived the equation

$$\frac{\partial u}{\partial t} = \Delta\varphi(u) + f(u) \tag{9}$$

for the density of a dispersing population. Here f describes the intrinsic growth characteristics of the population while φ describes the dispersal mechanism. If $\varphi(u) = u$ one has simply linear (random) dispersal. However, Gurtin and MacCamy observed that if the population avoids crowds then a nonlinear dispersal mechanism satisfying (8) is more appropriate. For the most part they work with the model equation

$$\frac{\partial u}{\partial t} = \Delta(u^m) + \mu u, \tag{10}$$

where $m > 1$ and $\mu \in \mathbb{R}$. Equation (10) can be transformed to the porous medium equation for $U(x,\tau)$ where

$$U(x,\tau) = e^{-\mu t} u(x,t) \text{ and } \tau = \frac{\gamma}{\mu}\left(e^{-\mu t/\gamma} - 1\right)$$

with $\gamma = 1/(m-1)$. Using this observation, Aronson and Peletier [8] have obtained results on the large time behavior of solutions to the initial-boundary value problem for equation (10). For other work on the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(u^m) + f(u)$$

for various choices of the growth function f see references [4] and [7].

Equations of the form (9) also occur in study of nonlinear heat conduction with absorption. In this application the growth term f in (9) satisfies $f \le 0$. Existence and uniqueness of weak solutions of the initial value problem are proved by Kalashnikov [21], while Kersner [24] and Bertsch [10] treat the initial-boundary value problem. Kalashnikov and Kersner consider only $d = 1$ while Bertsch works in $\mathbb{R}^d$ for any $d \ge 1$. Much of this work has been focused on the model equation

$$\frac{\partial u}{\partial t} = \Delta(u^m) - \lambda u^p$$

where $\lambda > 0$, $m > 1$, and $p \ge 1$. Let

$$\Gamma = B_r(0) \times (0,T]$$

for some $r \in \mathbb{R}^+$ and $T \in \mathbb{R}^+$ and define the parabolic boundary of Γ to be

$$\partial_p \Gamma = \bar{\Gamma} \backslash \Gamma.$$

Consider the initial-boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta(u^m) - \lambda u^p \text{ in } \Gamma \\ &u = g \text{ on } \partial_p \Gamma. \end{aligned} \tag{11}$$

Kersner [24] observed that under certain circumstances it is possible to have $u = 0$ at interior points of Γ even though g is strictly positive on $\partial_p\Gamma$. Bertsch [10] has made this observation rigorous. These interior interfaces provide interesting objects for further study. The paper of Phillips in this session is concerned with equilibrium problems which are related to this phenomenon.

Nambu's paper in this discussion group is also concerned with problem (11). He considers the case in which $T = +\infty$ and $g = 0$ on $\partial B_r(0) \times \mathbb{R}^+$. His results concern the large time behavior of solutions to this problem and are an extension of the results obtained for the porous medium equation in reference [8]. It should be noted that for problem (11) the nature of the large time behavior depends in a crucial way on the relative sizes of m and p.

So far I have only considered diffusion mechanisms which are of the porous medium type. Another type of nonlinear diffusion mechanism occurs in the diffusion-convection equation which describes the flow of ground water. Let u denote the hydrostatic potential, $c = c(u)$ the volumetric water content, and $K = K(c)$ the hydraulic conductivity of the soil. The functions c and K are experimentally determined. Usually K is assumed to be a smooth positive function of c, while c is a continuous piecewise smooth function which is increasing for $u < 0$ and is constant at its saturation value $\bar{c}$ for $u \geq 0$. Let x measure vertical distance downward from ground level and let y denote the remaining spatial coordinates. The equation governing the flow is then

$$\frac{\partial}{\partial t} c(u) = \operatorname{div}\{K(c)\operatorname{grad} u\} + \frac{\partial}{\partial x} K(c) \tag{12}$$

where div and grad are with respect to the coordinates (x,y) ([17]). Note that equation (12) is parabolic on the unsaturated set

$$\{(x,y,t): u(x,y,t) < 0\}$$

and elliptic on the saturated set

$$\{(x,y,t): u(x,y,t) > 0\}.$$

The interface is the set

$$\{(x,y,t): u(x,y,t) = 0\}.$$

In a typical boundary value problem for equation (12) with $y \in \mathbb{R}^d$ for $d = 1$ or 2 one assumes that there is a source of water (for example, rain) at $x = 0$ so that $u(0,y,t)$ is given in $\mathbb{R}^d \times \mathbb{R}^+$. Moreover, at some level $x = x_o > 0$ there is either impervious bedrock

$$u_x(x_o,y,t) = 0,$$

or else $u(x_o,y,t)$ is given. In general these problems do not admit a classical solution and it is necessary to define an appropriate class of weak solutions.

So far there has not been a great deal of work on problems of ground water type. vanDuyn and Peletier [17] and vanDuyn [16] have studied the one-dimensional problem with Dirichlet boundary conditions in the case $K(c) \equiv$ constant, that is for the equation

$$\frac{\partial}{\partial t} c(u) = \frac{\partial^2 u}{\partial x^2} .$$

In his paper in this session, Bertsch extends their results to the more general equation

$$\frac{\partial}{\partial t} c(u) = \frac{\partial}{\partial x} \{K(c) (\frac{\partial u}{\partial x} + 1)\}$$

for positive smooth K. Specifically his results include existence and uniqueness of weak solutions, asymptotic behavior, and, with appropriate assumptions, Hölder continuity of the interface.

I shall conclude this brief survey with a few remarks about open problems and directions for further research. In practically all of the work cited above in which there is an interface the principal results have been obtained only in one space dimension. One of the most important and challenging open problems is the problem of obtaining detailed information about the interface in various models in $\mathbb{R}^d$ for $d > 1$. Another important direction for future work is the study of systems of diffusion-interaction equations involving nonlinear diffusion mechanisms. Problems of this type arise in a natural way in various ecological contexts in which some of the species involved will disperse in response to population pressures from their own or other species. Okubo's monograph [29] is an excellent source of examples of these phenomena.

Some initial work in this area can be found in references [4], [27] and [28].

The second part of this report is a short summary of two contributions presented within the discussion group.

A. NONSTATIONARY FILTRATION IN PARTIALLY SATURATED POROUS MEDIA

In his talk, Michiel Bertsch (University of Leiden, Leiden, The Netherlands) considered the nonstationary filtration of a fluid in a homogeneous, isotropic and rigid porous medium.

Let u denote the hydrostatic potential due to capillary suction, c the volumetric moisture content, and K the hydraulic conductivity. Assume that c is a function of u, and that K is a function of c, both given by experiment. Moreover, assume that c is close to its saturation value $\bar{c} > 0$, and that

$$c \in C^1([-a,0)) \cap C([-a,\infty)) \text{ for some } a \geq +1;\ c(u) = \bar{c} \text{ for } u \geq 0; \text{ and}$$
$$0 < c' \leq C_0 \text{ on } [-a,0) \text{ for some } C_0 > 0. \qquad \text{H1}$$

K is assumed to satisfy

$$K \in C^1([c(-a),\bar{c}]);\ K > 0 \text{ and } K' \text{ is Lipschitz continuous on } [c(-a),1]. \qquad \text{H2}$$

Using Darcy's law and the continuity equation, the liquid flow can be described by

$$\frac{\partial}{\partial t}(c(u)) = \operatorname{div}(K(c)\operatorname{grad} u) + \frac{\partial}{\partial x}(K(c)),$$

where the gravity works along the x-axis. Thus u satisfies a parabolic equation in the unsaturated region ($u < 0$) and in the saturated region it satisfies Laplace's Equation $\Delta u = 0$. Across the interface between these regions require c and the macroscopic velocity of the fluid $q = -K(c)\operatorname{grad} u$ to be continuous. Consider the initial-boundary value problem for the one-dimensional vertical flow

$$\begin{aligned} (c(u))_t &= \{K(c)(u_x + 1)\}_x && \text{in } Q_T = (0,1) \times (0,T], \\ u(0,t) &= -1, u(1,t) = +1 && \text{for } t \in [0,T], \\ c(u(x,0)) &= v_0(x) && \text{in } [0,1]. \end{aligned} \qquad \text{(I)}$$

Assume that there exists a function $u_o : [0,1] \to [-a,\infty)$ such that $c(u_o(x)) = v_o(x)$ for $x \in [0,1]$ and such that

u_o is Lipschitz continuous on $[0,1]$; $u_o(0) = -1$ and $u_o(1) = +1$. H3

Problem (I) is studed by vanDuyn and Peletier in the case that K is constant ([16],[17]).

Definition A function $u(x,t)$, defined a.e. in $\bar{Q}_T$, is said to be a weak solution of Problem (I), if (i) $c(u) \in C(\bar{Q}_T)$, u possibly redefined on a set of measure zero, (ii) $u - g \in L^2(0,T:H_o^1(\Omega))$, where $g(x) = 2x - 1$ and (iii) u satisfies the integral identity.

$$\iint_{Q_T} \{\varphi_x K(c)(u_x + 1) - \varphi_t c(u)\} dxdt = \int_0^1 \varphi(x,0) v_o(x) dx$$

for all test functions $\varphi \in C^1(\bar{Q}_T)$ which vanish for $x = 0,1$ and $t = T$.

The main results are the following theorems.

Theorem 1 Let H1-2 be satisfied. Then there exists one and only one stationary solution $\bar{u} \in C^2([0,1])$ related with Problem (I).

Theorem 2 Let H1-2-3 be satisfied. Then Problem (I) possesses a unique weak solution u. Moreover $u \in L^2(0,T:H^2(0,1))$.

The existence of a weak solution is proved by approximating c, u_o and K by sequences of smooth functions $\{c_n\}$, $\{u_{on}\}$ and $\{K_n\}$ such that $c_n' \geq 1/n$. Then we get a classical problem (I_n) with a unique smooth solution u_n. It is shown that a subsequence of $\{u_n\}$ converges weakly in $L^2(0,T:H^1(0,1))$ to an element u, which possesses all the properties of a weak solution.

To describe the properties of the interface, we define for $t \in [0,T]$

$$\bar{z}(t) = \sup \{x \in (0,1) : c(u(\zeta,t)) < \bar{c} \ \forall \zeta \in [0,x)\},$$

$$z^+(t) = \inf \{x \in (0,1) : c(u(\zeta,t)) = \bar{c} \ \forall \zeta \in (x,1]\},$$

where $c(u) \in C(\bar{Q}_T)$.

Theorem 3 Let H1-2-3 be satisfied. If $z^-(0) = z^+(0)$ then $z^-(t) = z^+(t)$ for all $t \in [0,T]$.

In what follows we shall write $Z(t) = Z^{\pm}(t)$. In order to derive a regularity result for the interface $Z(t)$ we need another hypothesis.

H4. $u_0 \in C^{2+\alpha}([0,1])$ $(\alpha \in (0,1))$; $(K(c)(u_0'+1))' = 0$ at the points $x = 0,1$; $(K(c)(u_0'+1))' \geq -C_1 c'(u_0)$ for all $x \in (0,1)$ where $u_0'(x) \neq 0$ and for some $C_1 > 0$; the restriction of c to $[-a,0]$ belongs to $C^2([-a,0])$; $C'' \leq 0$ on $[-a,0)$; $K' \geq 0$ and $\liminf_{u \uparrow 0} c'(u) > 0$.

Theorem 4 Let H1-2-3-4 be fulfilled. Then the interface $Z(t)$ is Hölder continuous on $[0,T]$.

An important tool in the proof of Theorem 4 is a lower bound for u_t.

Theorem 5 Let H1-2-3 be satisfied, and let u be the weak solution of Problem (I). Then

$$|c(u(x,t)) - c(\bar{u}(x))| \leq C_2 e^{-C_3 t} \quad \text{in } [0,1] \times [0,\infty),$$

where $C_3 > 0$ does not depend on v_0.

B. DECAY OF SOLUTIONS OF A DEGENERATE NONLINEAR DIFFUSION EQUATION

In his contribution, T. Nanbu (Kyushu University, Fukuoka, Japan) illustrated some results obtained jointly with M. Bertsch and L. A. Peletier.

Consider the large time behaviour of the non-negative (weak) solution of the initial-boundary value problem

$$u_t = \Delta u^m - \lambda u^p \text{ in } Q \equiv \Omega \times R^+, \tag{1.1}$$

(I) $$u(x,t) = 0 \quad \text{on } S \equiv \partial\Omega \times R^+, \tag{1.2}$$

$$u(x,0) = u_0(x), \tag{1.3}$$

where $m > 1$, $\lambda > 0$ and $p \geq 1$ are constants, $u_0(x)$ is a given non-negative function ($u_0(x) \not\equiv 0$ in Ω), Ω is a bounded domain in R^N ($N > 1$) with smooth boundary $\partial\Omega$ and $R^+ = (0,\infty)$.

For $\lambda = 0$, when equation (1.1) is called the Porous Medium Equation, Aronson and Peletier [8] have recently shown that

$$u(x,t) \leq K_1 f_0(x) \quad (1+t)^{-\gamma} \text{ in } \bar{Q} \tag{1.4}$$

and

$$\left|(1+t)^{\gamma}u(x,t)-f_0(x)\right| \le K_2 f_0(x)(1+t)^{-1} \quad \text{in} \quad \bar{Q} \tag{1.5}$$

where $\gamma = 1/(m-1)$, K_1 and K_2 are positive constants depending only on m, u_0 and Ω, and $f_0(x)$ is the unique positive solution of the problem

$$\Delta f^m + \gamma f = 0 \text{ in } \Omega,$$
$$f = 0 \text{ on } \partial\Omega.$$

This work deals with the extension of these results to the case $\lambda > 0$. The main results depend on m, p, and λ. In particular, the cases $p \ge m$, $1 < p < m$ and $p = 1$ are quite different. The case $p = 1$ is fully discussed in [8].

The methods are closely related to [8].

I. Case $p \ge m$.

Theorem 1 Let u be the solution of problem (I). Then (1.4) is still valid and K_1 does not depend λ. Let $\beta = (p-m)/(m-1)$. Then we have that

(i) if $\beta > 0$ $(p > m)$, then in $\bar{Q}$

$$\left|(1+t)^{\gamma}u(x,t) - f_0(x)\right| \le K f_0(x) \begin{cases} (1+t)^{-1} & \text{if } \beta > 1 \\ (1+t)^{-1}\{1+\log(t+1)\} & \text{if } \beta = 1 \\ (1+t)^{-\beta} & \text{if } \beta < 1, \end{cases}$$

where K is a positive constant, which depends on the data: m, p, λ, N, u_0 and Ω, and

(ii) if $\beta = 0 (p = m)$, then

$$\left|(1+t)^{\gamma}u(x,t) - f_\lambda(x)\right| \le K f_\lambda(x)(1+t)^{-1} \quad \text{in} \quad \bar{Q},$$

where K is a positive constant, which depends on the data, and $f_\lambda(x)$ is the unique positive solution of the problem

$$\Delta f^m + \gamma f - \lambda f^m = 0 \quad \text{in} \quad \Omega,$$
$$f = 0 \quad \text{on} \quad \partial\Omega.$$

II. Case $1 < p < m$.

In the case $1 < p < m$, if $u_0(x)$ has a compact support in Ω, then it may happen that the set

$$p(t) = \{x \in \Omega;\ u(x,t) > 0\}$$

remains a proper subset of Ω for all time $t \geq 0$. Now, we set

$$p(\infty) = \bigcup_{t \geq 0} P(t).$$

Theorem 2 Let u be the solution of problem (I). Then

$$u(x,t) \leq K(1+t)^{-1/(p-1)} \quad \text{in } \bar{Q},$$

where the constant K depends on the data, and at every point $x \in \Omega$, as $t \to \infty$,

$$(1+t)^{1/(p-1)} u(x,t) \to \bar{C}\chi_{p(\infty)}(x),$$

where $\bar{C} \equiv \{\lambda(p-1)\}^{-1/(p-1)}$ and $\chi_{p(\infty)}(x)$ is the characteristic function of the set $p(\infty)$.

In addition, let $x_0 \in P(\infty)$.
Then there exists a positive constant K such that for all $t \geq 0$,

$$\left|(1+t)^{\frac{1}{p-1}} u(x_0,t) - \bar{C}\right| \leq K \begin{cases} (1+t)^{-1} & \text{if } \delta > 1 \\ (1+t)^{-1}\{1+\log(t+1)\} & \text{if } \delta = 1 \\ (1+t)^{-\delta} & \text{if } \delta < 1, \end{cases}$$

where $\delta \equiv (m-p)\ (p-1)$ and K depends on x_0 and the data.

The proofs of Theorems 1 and 2 are based on a comparison principle.

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D G ARONSON

Some properties of the interface for a gas flow in porous media

1. THE POROUS MEDIUM EQUATION

Consider an ideal gas flowing in a homogeneous porous medium. If the flow is isentropic the equation of state is

$$\gamma = \gamma_o p^{\alpha} ,$$

where $\gamma = \gamma(x,t)$ is the density, $p = p(x,t)$ is the pressure, and $\gamma_o \in \mathbb{R}^+$ and $\alpha \in (0,1]$ are constants. The dynamics of the flow are described by the conservation of mass

$$\operatorname{div} (\gamma \underset{\sim}{v}) = -f \frac{\partial \gamma}{\partial t}$$

and Darcy's Law

$$\underset{\sim}{v} = - \frac{\varkappa}{\mu} \operatorname{grad} p,$$

where $\underset{\sim}{v} = \underset{\sim}{v}(x,t)$ is the velocity vector, $f \in \mathbb{R}^+$ is the porosity of the medium, $\varkappa \in \mathbb{R}^+$ is the permeability of the medium, and $\mu \in \mathbb{R}^+$ is the viscosity of the gas. Darcy's Law is an empirical relationship which characterizes porous medium flow.

If one eliminates p and $\underset{\sim}{v}$ from the above equations one finds

$$\frac{\partial \gamma}{\partial t} = \frac{\varkappa}{\mu \gamma_o^{1/\alpha} (1+\alpha) f} \Delta(\gamma^m) ,$$

where

$$m = 1 + \frac{1}{\alpha} .$$

By a suitable change of time or distance scale one can reduce the constant to unity and thus obtain the *porous medium equation*

$$\frac{\partial u}{\partial t} = \Delta(u^m). \tag{1.1}$$

Since u is a (scaled) density it is natural to assume that

$$u \geq 0.$$

Moreover, $\alpha \in (0,1]$ implies that $m \in [2,+\infty)$. However, for the problems I will discuss here this restriction is irrelevant and I will only assume that

$$m > 1.$$

Equation (1.1) arises in contexts other than porous medium flow, for example, in population dynamics and plasma physics. Further references can be found in [3] and [6].

If one computes the Laplacian,

$$\Delta(u^m) = \operatorname{div}(m\, u^{m-1} \operatorname{grad} u) = m\, u^{m-1}\Delta u + m(m-1)u^{m-2}|\operatorname{grad} u|^2 ,$$

one sees that the porous medium equation (1.1) is uniformly parabolic in any region where u is bounded away from zero, but that it is degenerate in the neighborhood of any point where $u = 0$. In terms of Fickian diffusion, the diffusivity mu^{m-1} vanishes with the density u. The most striking manifestation of this nonlinear degeneracy is that in porous medium flow there is a finite speed of propagation of disturbances from rest. This should be contrasted with the classical heat conduction case, $m = 1$, where this is an infinite speed of propagation. In this lecture I want to explore some consequences of the finite speed of propagation in porous medium flow.

2. THE INITIAL VALUE PROBLEM IN ONE SPACE DIMENSION

Consider the initial value problem for gas flow in a one dimensional porous medium

$$\begin{aligned} u_t &= (u^m)_{xx} \quad \text{in } \mathbb{R} \times \mathbb{R}^+ \\ u(\cdot,0) &= u_o \quad \text{in } \mathbb{R} , \end{aligned} \tag{2.1}$$

where

$$u_o(x)\begin{cases} >0 & \text{for } x \in J \equiv (a_1,a_2) \\ \\ =0 & \text{for } x \in \mathbb{R}\backslash J \end{cases} \tag{2.2}$$

with

$$-\infty < a_1 < a_2 < +\infty.$$

The assumption (2.2) is for simplicity and is by no means essential. Problem (2.1) does not, in general, have a classical solution and it is necessary to deal with some sort of generalized solution.

Definition A function $u = u(x,t)$ is said to be a *weak solution* of problem (2.1) if

(i) u is a nonnegative and continuous in $\mathbb{R} \times \mathbb{R}^+$,

(ii) $(u^m)_x$ exists in the sense of distributions and is bounded in $\mathbb{R} \times \mathbb{R}^+$,

(iii) u satisfies the integral identity

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \{\frac{\partial\varphi}{\partial x}\frac{\partial u^m}{\partial x} - \frac{\partial\varphi}{\partial t}u\}dx\,dt = \int_{\mathbb{R}} \varphi(x,0)u_o dx$$

for all smooth functions φ which vanish for $|x|$ and t large.

The basic existence and uniqueness theory for weak solutions of problem (2.1) is provided by the following result due to Oleinik, Kalashnikov and Chzou [12].

Theorem 1 ([12]) If u_o satisfies (2.2) and $u_o^m \in \mathrm{Lip}(\mathbb{R})$ then there exists a unique weak solution $u(x,t)$ of problem (2.1). Moreover, $u(\cdot,t)$ has compact support for each $t \in \mathbb{R}^+$.

Let

$$P[u] \equiv \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : u(x,t) > 0\}$$

I shall be concerned with the nature of the set $\partial P[u]\backslash J$, that is, the *interface* between the part of medium which contains gas and the part which

does not. The next result shows that each time-section of $P[u]$ is an interval and that these intervals, in general, expand as time increases.

Theorem 2 ([2],[9]) Let u be the weak solution of problem (2.1) with u_o satisfying (2.2). Then

$$\partial P[u] = J \cup \{(\zeta_1(t),t): t \in \mathbb{R}^+\} \cup \{(\zeta_2(t),t): t \in \mathbb{R}^+\},$$

where $\zeta_i \in \mathrm{Lip}(\mathbb{R}^+)$, $\zeta_i(0) = a_i$, ζ_i is nonincreasing, and ζ_2 is nondecreasing.

Naturally, one expects the interface to move with the local velocity of the gas. Set

$$v = \frac{m}{m-1} u^{m-1}.$$

Then, in view of the equation of state, v is essentially the pressure, and, by Darcy's Law, one expects

$$\dot{\zeta}_i(t) = - \lim_{x \to \zeta_i(t)^{\pm}} v_x(x,t) \tag{2.3}$$

where $x \to \zeta_i(t)^{\pm}$ means that the interface is approached from within $P[u]$. The next result shows that (2.3) is almost true!

Theorem 3 ([2],[11]) The

$$\lim_{x \to \zeta_i(t)^{\pm}} v_x(x,t)$$

exists and

$$D^+\zeta_i(t) = -v_x(\zeta_i(t),t) \tag{2.4}$$

for all $t \in \mathbb{R}^+$. Here D^+ denotes right hand derivative.

More detailed information about the behavior of the interface is provided by the following result proved by Knerr [11].

Theorem 4 ([11]) There exist numbers $t_i^* \in [0,+\infty)$ for $i = 1,2$ such that

ζ_i is strictly monotone for $t \in (t_i^*,+\infty)$

and

$$\zeta_i(t) \equiv a_i \text{ for } t \in [0, t_i^*].$$

If $t_i^* > 0$ then the interface $x = \zeta_i(t)$ remains stationary (vertical) for t_i^* units of time after which it begins to move without further stops. I shall call t_i^* the *waiting time*. The case $t_i^* > 0$ actually occurs. Let

$$v_o = \frac{m}{m-1} u_o^{m-1} .$$

If

$$v_o(x) \geq c|x - a_i|^q \text{ for some } c \in \mathbb{R}^+ \text{ and } q \in (0,2) \text{ then } t_i^* = 0,$$

but if

$$v_o(x) \leq c(x - a_i)^2 \text{ for some } c \in \mathbb{R}^+ \text{ then } t_i^* > 0,$$

([11]). For example, if m=2 and

$$u_o(x) = \begin{cases} \cos^2 x & \text{for } |x| \leq \pi/2 \\ 0 & \text{for } |x| > \pi/2 \end{cases}$$

then $t_1^* = t_2^* = 1/12$, ([1]).

Since ζ_i is monotone, $\dot{\zeta}_i$ exists and is equal to $D^+\zeta_i$ almost everywhere in $\mathbb{R}^+$. Thus, in particular, (2.3) holds for almost all $t \in \mathbb{R}^+$. The next result due to Caffarelli and Friedman [7], shows that (2.3) is valid whenever the interface is in motion.

Theorem 5 ([7]) For i = 1 and 2, $\zeta_i \in C^1(t_i^*, +\infty)$ and

$$\dot{\zeta}_i = -v_x(\zeta_i, t) \quad \text{on} \quad (t_i^*, +\infty) .$$

The Caffarelli-Friedman proof starts from the estimate

$$v_{xx}(x,t) \geq -\frac{k}{t}$$

which is proved for $x \in \mathbb{R}$ in [2] and for $x \in \mathbb{R}^d$ in [4]. Using this they derive the differential equation

$$\ddot{\zeta}_i + K_i \dot{\zeta}_i = \mu_i \tag{2.5}$$

where $K_i \in \mathbb{R}^+$ is a constant, μ_i is a positive measure, and the equation is to be understood in the sense of distributions. Two important consequences of (2.5) are the inequality

$$e^{Kt_2} D^{\pm}\zeta_i(t_2) \geq e^{Kt_1} D^{\pm}\zeta_i(t_1) \tag{2.6}$$

for $0 < t_1 < t_2$ and the representation formula

$$\zeta_i(t) = \eta_i(t) + \xi_i(t)$$

where η_i is Lipschitz continuously differentiable and ξ_i is convex. It follows easily from (2.6) that $D^+\zeta_i > 0$ on $(t_i^*, +\infty)$. The rest of the Caffarelli-Friedman result follows from the representation formula.

In the light of all these results there still remain two problems of interest:

(1) Evaluate or estimate the waiting time t_i^*.

(2) When does one have $t_i^* > 0$ and $\zeta_i \in C^1(\mathbb{R}^+)$, that is when does the interface begin to move smoothly?

Assume that $a_2 = 0$. In what follows I will consider only the right hand interface $x = \zeta_2(t)$ and to simplify the notation I will drop the subscript 2 throughout. The following results recently obtained in joint work with Caffarelli and Kamin [5] give partial solutions to problems (1) and (2). Define

$$t_m = 1/2(m + 1).$$

Theorem 6 ([5]) Let u be the weak solution of problem (2.1) and let $v_0 = \frac{m}{m-1} u_0^{m-1}$. If

$$v_0(x) = \alpha x^2 + o(x^2) \text{ as } x \uparrow 0 \tag{2.7}$$

and

$$v_o(x) \in \beta x^2 \text{ in } \mathbb{R}^- \tag{2.8}$$

for some constants $\alpha \in [0,+\infty)$ and $\beta \in \mathbb{R}^+$ then

$$t_m/\beta \leq t^* \leq t_m/\alpha.$$

In particular, if $\alpha = \beta$ then

$$t^* = t_m/\alpha.$$

Suppose that

$$v_o(x) = \begin{cases} (1-\theta)\sin^2 x + \theta \sin^4 x & \text{for} \quad x \in [-\pi,0] \\ 0 & \text{for} \quad x \notin [-\pi,0] \end{cases} \tag{2.9}$$

where $\theta \in [0,1]$. It is easy to verify that for $\theta \in [0,1/4]$ both (2.7) and (2.8) hold with $\alpha = \beta = 1 - \theta$. Therefore, if $\theta \in [0,1/4]$ then

$$t^* = t_m/(1-\theta).$$

On the other hand, for $\theta \in (1/4,1]$, (2.7) holds with $\alpha = 1 - \theta$ while the smallest β for which (2.8) holds is obtained by solving the nonlinear equations

$$\beta\xi^2 = v_o(\xi), \quad 2\beta\xi = v_o'(\xi).$$

Since $\beta \neq 1 - \theta$ for $\theta > 1/4$, Theorem 6 gives only the estimate

$$t_m/\beta \leq t^* \leq t_m/(1-\theta).$$

For example, if $\theta = 1$ then

$$1.904538\ldots t_m \leq t^* < +\infty.$$

Kath and Cohen [10] have done a formal asymptotic analysis for $m - 1$ small which leads to an estimate for the waiting time up to terms which are $o(m-1)$. It would be very useful to have more examples in which the waiting time was known accurately. In particular, good numerical schemes for the

determination of the waiting time are sorely needed.

The next result gives a sufficient condition for $\zeta \in C^1(\mathbb{R}^+)$. When this condition is satisfied $D^+\zeta(t^*) = D^-\zeta(t^*)$ and (2.3) holds for all $t \in \mathbb{R}^+$.

Theorem 7 ([5]) Suppose that v_o satisfies (2.7) and (2.8) for some $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$. If

$$t^* = t_m/\alpha$$

and v_{xx} is a nondecreasing function of x in $(-\delta,0) \times (0,t_m/\alpha)$ for some $\delta \in \mathbb{R}^+$ then

$$\zeta \in C^1(\mathbb{R}^+) .$$

The requirement that v_{xx} be monotone near the vertical interface is very awkward. It can be circumvented when v_o is such that it is possible to construct a dominating comparison function which satisfies all of the hypotheses of Theorem 7. This observation leads to the following more practical criterion for smoothness.

Corollary ([5]) If v_o satisfies (2.2) with $a_2 = 0$,

$$v_o(x) = \alpha x^2 + o(x^2) \text{ as } x \uparrow 0,$$

and

$$v_o(x) \leq \alpha x^2 \text{ in } \mathbb{R}^-$$

for some $\alpha \in \mathbb{R}^+$, then $t^* = t_m/\alpha$ and $\zeta \in C^1(\mathbb{R}^+)$.

The Corollary to Theorem 7 can be applied to example (2.9) when $\theta \in [0,1/4]$ to show that $\zeta \in C^1(\mathbb{R}^+)$. On the other hand, neither Theorem 7 or its Corollary give any information about the smoothness of ζ if $\theta \in (1/4,1]$. Further work in this direction is needed.

The proofs of Theorem 6 and 7 are rather technical. They rely heavily on similarity transformations and comparison with certain explicit solutions of the pressure equation

$$v_t = (m-1)vv_{xx} + v_x^2 \tag{2.10}$$

For example, under the hypotheses of Theorem 6

$$v(x,t) = \frac{t_m x^2}{\frac{1}{\alpha} t_m - t} + o(x^2) \quad \text{as} \quad x \uparrow 0$$

uniformly on $(0,t^*)$. Thus near a vertical interface the pressure behaves like the quadratic solution

$$V(x,t) = \frac{t_m x^2}{\frac{1}{\alpha} t_m - t} .$$

In the neighbourhood of a point where the interface is not vertical the pressure behaves like a linear solution of (2.10). Specifically, if $x_o = \zeta(t_o)$ and

$$\lim_{x \uparrow x_o} - v_x(x,t_o) = \gamma > 0$$

then

$$v(x,t) = L_\gamma(x - x_o, t - t_o) + o(|x - x_o| + |t - t_o|),$$

where

$$L_\gamma(x,t) = \gamma(\gamma t - x)^+.$$

3. PROBLEMS IN MORE THAN ONE SPACE DIMENSION

In the study of the interface for the one dimensional porous medium equation there still remain some interesting and challenging open problems. Nevertheless, the theory in the one dimensional case is almost complete. In contrast, very little is known about the interface for the porous medium equation with $x \in \mathbb{R}^d$ for $d > 1$. Caffarelli and Friedman [8] have proved Hölder continuity. However, aside from qualitative properties such as preservation of bounded support, there is as yet very little specific information about the behavior of the interface. I hope that this situation will change in the near future.

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S KAMIN & P ROSENAU

Propagation of thermal waves in inhomogeneous media

In this work we are concerned with the solutions of the heat equation in an inhomogeneous medium

$$\rho(x)u_t = [A(u)]_{xx} \quad \text{in} \quad R^1 \times R^+, \tag{1}$$

$$u(x,0) = u_0(x). \tag{2}$$

We shall assume that $A(0) = 0$, $A'(o) \geq 0$, $A'(u) > 0$ if $u > 0$ and that $\rho(x) > 0$, $A(u)$ and $\rho(x)$ are smooth functions.

The particular case of (1) is the equation

$$u_t = (u^n)_{xx} \tag{3}$$

which is often cited in the mathematical literature as the porous medium equation ([1]). We refer to (1) as the equation of the thermal evolution of a heated plasma, where u is the temperature and where $\rho(x)$ is the particle density; thus we are concerned with thermal phenomena in an unbounded medium.

Thermal phenomena in inhomogeneous ambience are of interest in a variety of laboratory situations e.g. inertial and magnetic confinement, or extraterrestrial situations where the response to an impulsively initiated blast, takes initially the form of a supersonic wave while the medium is quiescent [2]. It is only in a later stage when the wave slows down to the sonic range that the gas is set into motion. It will be assumed that the density is stationary but, unlike in previous studies, inhomogeneous. In fact, it is the study of the impact of inhomogeneity that distinguishes this work from previous studies.

We consider the *asymptotic behavior* for large t of the solutions of (1), (2).

As is known the initial value problem (1), (2) does not, in general, have a solution in the classical sense (see [3]), and the definition of weak solution is necessary. A function u(x,t) is said to be a weak solution of

problem (1) and (2), if

(i) $u(x,t)$ is bounded, continuous, and nonnegative in $R^1 \times R^+$,

(ii) $$\int_0^T\int_{-\infty}^{\infty} [\rho(x)uf_t + A(u)f_{xx}]dxdt + \int_{-\infty}^{\infty} \rho(x)u_0(x)f(x,0)dx =$$

$$= \int_{-\infty}^{\infty} \rho(x)u(x,T)f(x,T)dx$$

for all $f(x,t) \in C^2(R^1 \times R^+)$ which vanish for large $|x|$.

We prove in [4], [5] the existence and uniqueness of the weak solution for the Cauchy problem (1), (2). In order to study the asymptotic behavior we distinguish two cases

Case a) The total mass of the medium is finite, i.e.

$$\int_{-\infty}^{\infty} \rho dx = m < \infty. \tag{4}$$

This is the case of diffusion in a finite mass medium. Our main result is

Theorem 1 Assume that $0 \le u_0(x) \le M$, the total mass m is finite and $u(x,t)$ is the solution of (1) and (2). Then

$$u(x,t) \to \bar{u} = \frac{E}{M} \quad \text{as} \quad t \to \infty \tag{5}$$

where $E = \int_{-\infty}^{\infty} \rho u_0(x)dx$. The convergence in (5) is uniform in every bounded interval in x.

The proof of this theorem based on the conservation law

$$\int_{-\infty}^{\infty} \rho(x)u(x,t)dx = E$$

and integral estimates. For the detailed proof see [5]. Simply stated, the main result, contained in Theorem 1, ensures the isothermalization of the

medium to a positive average temperature given by equation (5). Such a result would be natural in a finite domain with homogeneous Neumann conditions. Here it is derived for a Cauchy problem. It is of course drastically different from diffusion in a homogeneous medium or any infinite inhomogeneous mass medium, where the average temeperature is zero. In a standard diffusion problem the first non-vanishing term describes the decay to "average" zero of the thermal pulse. On the other hand, in our case the calculation of how this average is approached constitutes the second term in an appropriate asymptotic expansion. We plan to report on this in the near future.

One should, however, distinguish between the approach to the average temperature $\bar{u}$ at a given point, as guaranteed by Theorem 1, and the behavior at infinity. Whether the isothermalization of the whole space takes a finite or infinite time still remains to be answered. Note that the possibility that arbitrarily far particles have a finite temeperature is physically plausible. Because there are so few of them all energy contained in the tail is negligible.

Case b) The total mass of the medium is infinite. This includes the homogeneous case $\rho(x) = \text{const}$. We consider now the case $A(u) = u^n$ and initially we assume $\rho(x) = \frac{1}{|x|^\ell}$, $0 \le \ell < 1$. Thus the equation (1) takes the form

$$\frac{1}{|x|^\ell} \frac{\partial u}{\partial t} = \frac{\partial^2 u^n}{\partial x^2} \tag{6}$$

We find the self-similar solution of (6) which is of the form

$$w(x,t) = \frac{1}{t^{(1-\ell)N}} \psi\left(\frac{x}{t^N}\right) \tag{7}$$

where $N = \frac{1}{1+n-n\ell}$ and

$$\psi(\xi) = C(\ell,n) \left[\xi_0^{2-\ell} - \xi^{2-\ell}\right]^{\frac{1}{n-1}} \tag{8}$$

In (8), $C(\ell,n)$ is a constant that depends on ℓ and n, and ξ_0 is the constant of integration. We choose ξ_0 in such a way, that

$$\int_{-\infty}^{\infty} \frac{1}{|x|^{\ell}} w(x,t)\,dx = E \tag{9}$$

and therefore

$$w(x,0) = E\delta(x) \tag{10}$$

where $\delta(x)$ is the Dirac measure. For the homogeneous case ($\ell=0$) the self-similar solution is well known ([6]), but to the best of our knowledge the problem of a non-linear propagation in an inhomogeneous medium has not been hitherto treated. Our solutions are a generalization of solutions known as Barenblatt-Pattle solutions.

Next we consider the equation

$$\rho(x)\,\frac{\partial u}{\partial t} = \frac{\partial^2 u^n}{\partial x^2} \tag{11}$$

We prove that the self-similar solution (7) describes the later phase of evolution for a wide class of initial data and density, provided that $\int_{-\infty}^{\infty} \rho(x)\,u_0(x)\,dx = E > 0$ and that $\rho(x) \sim |x|^{-\ell}$ for $|x| \to \infty$. (Precise formulation is given below.) Thus asymptotically in time the solution is completely characterised in terms of two "numbers", ℓ, and the initial energy E. It is important and interesting that it is the distribution of the density far from the origin that has a decisive influence on the propagation while the distribution of the density at a finite distance inasmuch as E is preserved, is irrelevant.

Theorem 2 Assume $u(x,t)$ to be a solution of the Cauchy problem (11), (2). Suppose that $u_0(x)$ has a compact support and $\int_{-\infty}^{\infty} \rho(x)\,u_0(x)\,dx = E$. Let $\rho_1(1 + |x|^{\ell})^{-1} \le \rho(x) \le \rho_2(1 + |x|^{\ell})^{-1}$, $\forall x \in R^1$ where $\rho_1 > 0$, $\rho_2 > 0$ are some constants. Let $w_E(x,t)$ be a similarity solution (7) which satisfies (10). Then if $t \to \infty$ $t^{(-\ell)/(1+n-n\ell)}\,|\,u(x,t) - w_E(x,t)\,| \to 0$. The convergence is uniform inside $|x| \le \alpha t^{1/(1+n-n\ell)}$ for every α=const.

The proof of this theorem follows the proof for the homogeneous medium

given in [7]. See [4].

Returning now to case a) we notice that the infinite spread of a finite mass suggests that it may be natural to consider our problem in mass coordinate. Thus we define

$$z = \int_0^x \rho(x)\,dx \tag{12}$$

and the equation (1) transforms to

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\,\bar{\rho}(z)\,\frac{\partial A}{\partial z}\,,\quad a < z < b$$

where $a = \int_0^\infty \rho\,dx$, $b = \int_0^\infty \rho\,dx$.

We exploit the inverse transformation in order to analyze the following class of boundary value problems, often encountered in plasma physics

$$\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\,C(x)\,\frac{\partial A(u)}{\partial x}\,,\quad x \in (-1,1)\,, \tag{13}$$

where $\rho(x)$ and $C(x)$ are some smooth functions that may vanish at $|x| = 1$. Here we shall assume this actually to occur, i.e. $C(\pm 1) = 0$, $\rho(\pm 1) = 0$. Also

$$u(x,0) = u_0(x)\,,\quad x \in (-1,1) \tag{14}$$

The relevant boundary conditions will be discussed later.

For the homogeneous case i.e., $\rho(x) = \text{const.}$, $C(x) = \text{const.}$, and $A(u) = u^n$, $n > 1$, Berryman and Holland [8], Aronson and Peletier [9] (for several dimensions) have shown that for large times the asymptotic solution of (13), (14) with the homogeneous b.c.,

$$u(-1,t) = u(1,t) = 0 \tag{15}$$

is given by

$$u \sim \Phi(t)X(x) \tag{16}$$

where $\Phi(t) = (1 + \lambda t)^{1/(1-n)}$ and $X(x)$ is the solution of the nonlinear boundary value problem

$$\frac{d^2X^n}{dx^2} + \frac{\lambda}{n-1} X = 0 \tag{17}$$

satisfying $X(\pm 1) = 0$ and $X(0) = 1$.

In the singular case when $C(x)$ (and $\rho(x)$) not only varies but also vanishes at $|x| = 1$ there are two separate cases to distinguish according to the behavior of

$$y = F(x) = \int_0^x \frac{dx}{C(x)} \tag{18}$$

as $|x| \to 1$. If F becomes unbounded i.e., C^{-1} has nonintegrable singularities, then viewing y as a new coordinate transformation, $y:(-1,1) \to (-\infty,\infty)$, (18) yields

$$\bar{\rho}(y) \frac{\partial u}{\partial t} = \frac{\partial^2 A(u)}{\partial y^2}, \quad y \in R^1 \tag{19}$$

$$u(y,0) = u_1(y) \tag{20}$$

$$\bar{\rho}(y) = C[x(y)]\, \rho[x(y)], \quad u_1(y) = u_0[x(y)].$$

Since $\int_{-\infty}^{\infty} \bar{\rho}(y)\, dy = \int_{-1}^{1} \rho(x)\, dx < \infty$ problem (19), (20) corresponds exactly to case a). Using Theorem 1 we are ensured that $u \to \bar{u} = \text{const.} > 0$ where

$$\bar{u} = \frac{\int_{-\infty}^{\infty} \bar{\rho}(y)\, u_1(y)\, dy}{\int_{-\infty}^{\infty} \bar{\rho}(y)\, dy} = \frac{\int_{-1}^{+1} \rho(x)\, u_0(x)\, dx}{\int_{-1}^{+1} \rho(x)\, dx}$$

Note that in the present case, boundness of the solution is the only boundary condition that can be imposed. This has a simple physical interpretation: $C(x)$ which is part of a product composing the thermal conductivity, decreases at the edge to the extent that it thermally isolates the medium from outside. Heat cannot escape and hence isothermalization occurs. On the other hand, when $C^{-1}(x)$ has integrable singularities $x \to y$ maps $(-1,1)$ into a finite

interval in y. Looking into solutions of the form (16) in either x or y units, one obtains a problem similar to (17). Here one naturally imposes (15) and expects u to decay with time to zero. We prove in [10] that as in the homogeneous case, the asymptotic solution of (13), (15) with $A(u) = u^n$ has the separable form (16).

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J L VAZQUEZ

Large time behaviour of the solutions of the one-dimensional porous media equation

Suppose that a compressible gas flows through a porous medium in $\mathbb{R}^3$. Let ρ be its density, p its pressure and $\vec{v}$ its velocity. The conservation of mass implies the following equation of continuity (t is time)

$$K \frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \vec{v}) = 0 \tag{0.1}$$

where K depends on the properties of the porous medium. Also the gas obeys to a law of state of the form

$$\frac{p}{p_o} = (\frac{\rho}{\rho_o})^{\gamma} \tag{0.2}$$

with a constant $\gamma \geq 1$ ($\gamma = 1$ in isothermal flow, $\gamma > 1$ in adiabatic flow). Finally we assert Darcy's law and neglect gravity so that

$$\vec{v} = - c \nabla p \tag{0.3}$$

for a constant $c > 0$. If we combine (0.1),(0.2),(0.3) and rescale to eliminate the remaining constant we obtain

$$u_t = \Delta u^m \tag{0.4}$$

with $m = 1 + \gamma$, u being the density but for a constant. Because of this application (0.4) is often called the porous medium equation.

In this paper we supplement (0.4) with initial conditions at $t = 0$ and restrict the space dimension to one (by assuming uniformity in y,z). Thus we consider the following Cauchy problem

$$(P) \begin{cases} (E) \quad u_t = (u^m)_{xx} \quad \text{for} \quad x \in \mathbb{R},\ t > 0 \\ (IC) \quad u(x,0) = u_o(x) \quad \text{for} \quad x \in \mathbb{R} \end{cases}$$

we allow m to be any number greater than 1, $m > 1$ and impose in u_o the following conditions

$$u_o \in L^1(\mathbb{R}) \ , \ u_o \geq 0 \tag{C 1}$$

u_o vanishes a.e. outside a bounded interval $[a_1, a_2] \subset \mathbb{R}$ (C 2)

Sticking to the precedent application we refer to u as a density, to $v = \frac{m}{m-1} u^{m-1}$ as a pressure and to $V = - v_x$ as a local velocity.
A large number of other possible applications have been suggested for equation (0.4). We refer to [14] for a survey. A particular application is interesting with respect to our result on the conservation of the center of mass: The spread of a thin drop of viscous fluid over a horizontal plane under gravity. Under the assumptions of the lubrication theory the profile of the drop $\eta(x,t)$ satisfies

$$\frac{\partial \eta}{\partial t} = \frac{1}{3} \frac{\partial}{\partial x} (\eta^3 \frac{\partial \eta}{\partial x}) \tag{0.5}$$

(Prof. Ockendon has drawn our attention to this example, see [11]. See also [9]). In the paper we prove that the mass and the center of mass are time - invariants for our problem and we introduce a new comparison Principle by means of which we study the behaviour of the solutions for large time.

1. MODEL SOLUTIONS

We are interested in describing the properties and behaviour of the solutions of (P) in terms of what we know on u_o. The leading idea is that there is a *model* behaviour to which we shall always refer.

In fact Barenblatt [5], Pattle [13] and others have introduced the following class of self-similar solutions, i.e. solutions of the form

$$u(x,t) = t^{\alpha} f \left(\frac{x}{t^{\beta}} \right) \tag{1.1}$$

for some numbers α, β and a real function f, see [9]. They are

$$\bar{u}(x,t;M) = \frac{[r(t)^2-x^2]_+^{1/m-1}}{b_m t^{1/m-1}} \tag{1.2}$$

where $r(t) = c_m(M^{m-1}t)^{1/m+1}$, $b_m = (\frac{2m(m+1)}{m-1})^{1/m-1}$, $c_m = b_m^{\frac{m-1}{m+1}} B(\frac{m}{m-1}, \frac{1}{2})$,

$B(\cdot,\cdot)$ is Euler's Beta Function and for $s \in \mathbb{R}$, $[s]_+ = \max(s,0)$.
The functions (1.2) solve equation (E) and satisfy an initial condition

$$\bar{u}(x,0;M) = M\,\delta(x) \tag{1.3}$$

where δ is Dirac's delta.

For $x_o \in \mathbb{R}$ we define the translates

$$\bar{u}(x,t;M,x_o) = \bar{u}(x-x_o,t;M) \tag{1.4}$$

These solutions exhibit free-boundaries: $\bar{\zeta}_i(t) = x_o + (-1)^i r(t)$, $i = 1,2,$. They are C^∞, strictly monotone curves such that $\bar{\zeta}_i''(t) + \frac{m}{(m+1)t}\bar{\zeta}_i'(t) = 0$.

2. GENERAL SOLUTIONS

The equation is parabolic on the set

$$\Omega = \Omega[u] = \{(x,t) \in \mathbb{R} \times (0,\infty) : u(x,t) > 0\} \tag{2.1}$$

but degenerates at the points where $u = 0$. (1.2) shows that the solutions need not be smooth at these points. Hence generalized solutions must be defined for (P). Oleinik, Kalashnikov and Yui-Lin [12] introduced a class of unique weak solutions. Using Semigroup Theory we associate to every u_o satisfying (C_1) a unique solution $u(x,t)$ of (P) satisfying

$$\begin{aligned} &\text{i)} \quad u \in c([0,\infty), L^1(\mathbb{R})) \cap L^\infty(\mathbb{R}\times[\delta,\infty)) \text{ for every } \delta > 0 \\ &\text{ii)} \quad u(0) = u_o \text{ a.e.} \\ &\text{iii)} \quad u_t = (u^m)_{xx} \text{ in the sense of distributions} \end{aligned} \tag{2.2}$$

(see [14]). Since the construction will play a role in our proofs we outline it (cfr. [6]): The operator A defined in $D(A) = \{u \in L^1(\mathbb{R}) : u^m \in W^{1,\infty}(\mathbb{R})$ and $(u^m)_{xx} \in L^1(\mathbb{R})$ by $Au = (u^m)_{xx}$ is m-dissipative in $L^1(\mathbb{R})$,

i.e. for every $\lambda > 0$ $(1 - \lambda A)^{-1}$ is a well defined contraction on $L^1(\mathbb{R})$. Also $D(A)$ is dense in $L^1(\mathbb{R})$. Hence it generates a semigroup of contractions in $L^1(\mathbb{R})$ by the formula

$$S_t u_o = \lim (I - \frac{t}{n} A)^{-n} u_o \tag{2.3}$$

$u(x,t) = S_t u_o(x)$, the semigroup solution of (P) has the above properties and is continuous in $\mathbb{R} \times (0,\infty)$. Also if $u_o^1 \leq u_o^2$ (a.e.) then for every $t > 0$ $S_t u_o^1 \leq S_t u_o^2$, a pointwise comparison principle.

If $m > 1$ 'finite propagation' occurs: if u_o has compact support the $u(\cdot,t)$ has also compact support for every $t > 0$. We define the *external interfaces or free boundaries* $\zeta_1(t) = \inf \Omega(t)$ and $\zeta_2(t) = \sup \Omega(t)$, where $\Omega(t) = \{x \in \mathbb{R} : u(x,t) > 0\}$. By comparison to the model situation one can remark that $\Omega(t)$ is open, $(-1)^i \zeta_i(t)$ is nondecreasing for $i = 1,2$ and ζ_i is Lipschitz-continuous for $t \geq \tau > 0$. A first difference is that a *waiting-time* can occur: a time $T_i > 0$ such that $\zeta_i(t) = a_i$ if $0 \leq t < T_i$ [4], [10], [11]. Once the free boundary starts to move we know that ζ_i is C^1 and $(-1)^i \zeta_1' > 0$, [9].

Another main difference is the possible occurrence of *internal interfaces*, i.e. parts of the boundary of Ω lying in $(a_1,a_2) \times (0,\infty)$. Then $\Omega(t)$ is not always an interval. This does not occur if for instance $u_o \in C(\mathbb{R})$ and $u_o > 0$ in (a_1,a_2) (use the inequality $u_t \geq \frac{u}{(m+1)t}$, [3]).

3. TWO INVARIANTS

Our aim is to show to what extent the solution for a general u_o satisfying (C_1), (C_2) behaves like a suitable model solution. To choose this particular one we first observe that the motion described by (E) presents two invariants, the *total mass* and the *center of mass*. The first is a consequence of the equations of continuity (0.1) in our typical application: the second is more involved and comes from the fact that the total external force is zero (cfr. the application to the spread of a drop).

The total mass at time $t \geq 0$ is given by

$$M(t) = \int_{\mathbb{R}} u(x,t)dx \ , \ M(0) = M \tag{3.1}$$

and the center of mass by

$$x_c(t) = \frac{1}{M(t)} \int_{\mathbb{R}} Xu(x,t)\,dx, \quad x_c(0) = x_c. \tag{3.2}$$

We prove

Proposition 1 For every $t > 0$ and every solution u of (P), $M(t) = M$ and $x_c(t) = x_c$.

Idea of the proof. A formal computation shows that

$$\frac{d}{dt} M(t) = \int u_t \, dx = \int (u^m)_{xx} dx = [(u^m)_x]_{-\infty}^{\infty} = 0$$

$$M \frac{d}{dt} x_c(t) = \int x \, u_t \, dx = \int x (u^m)_{xx} dx = [x u_x^m - u^m]_{-\infty}^{+\infty} = 0$$

These proofs can be justified but it is easier to discretize (E) in time, i.e. use formula (2.3) and prove the corresponding elliptic analogue. This is a trick that we shall use repeatedly.

Remark The proof applied to problem (P) on the sole assumptions $m > 0$ and $u_0 \in L^1(R)$.

In view of this result the candidate model solution is the one having the same mass and center of mass, i.e.

$$\bar{u}(x,t) = \bar{u}(x-x_c, t; M) \tag{3.3}$$

In that direction Kamin [9] proved that u is asymptotic to a model solution $\bar{u}$ with the same mass and

$$\lim_{t\to\infty} t^{1/m+1} \left| u(x,t) - \bar{u}(x,t) \right| = 0 \tag{3.4}$$

uniformly in x. Observe that (3.4) determines M but not x_c; in fact (3.4) remains valid with $\bar{u} = \bar{u}(x-a,t;M)$ for every $a \in \mathbb{R}$.

4. COMPARISON BY SHIFTING

In order to improve (3.4) we introduce a new Comparison Principle based in

the evaluation of masses rather than that of densities. The idea is that if the initial mass of one problem is *shifted* to the right (or the left) with respect to that of another problem, it will remain so for every $t > 0$. We precise the meaning of shifted mass by means of the *distribution function*: for a solution u of (P) we define

$$U(x,t) = \int_{-\infty}^{x} u(x,t)\,dx = \text{mass lying in } (-\infty,x) \tag{4.1}$$

and say that the mass of u_1 is shifted to the right with respect to that of u_2 at time t if $U_1(x,t) \le U_2(x,t)$ for every $x \in \mathbb{R}$. We prove the following

Shifting Comparison Principle: Let u_o^i, $i = 1,2$ be integrable and let u_o^1 be shifted to the right respect to u_o^2 (i.e. $U_o^1(x) \le U_o^2(x)$ in $\mathbb{R}$), then the same holds for the corresponding solutions $u_1(\cdot,t)$ and $u_2(\cdot,t)$ for every $t > 0$ (i.e. $U_1(x,t) \le U_2(x,t)$ for $x \in \mathbb{R}$, $t > 0$).

Idea of the proof. Using formula (2.3), i.e. discretizing in time, we are reduced to prove the elliptic analogue: let $f_i \in L^1(\mathbb{R})$, $i = 1,2$ and let u_i solve $-\lambda A u_i + u_i = f_i$, for some $\lambda > 0$. Let $F_i(x) = \int_{-\infty}^{x} f_i(s)\,ds$ and $U_i(x) = \int_{-\infty}^{x} u_i(s)\,ds$ be their distribution functions. We have to prove that $F_1 \le F_2$ implies $U_1 \le U_2$.

Assume that the conclusion is not true and let $I = (a,b)$ be a maximal interval in the nonvoid open set $\{x \in \mathbb{R} : U_1(x) > U_2(x)\}$. Integration of the equation gives on I:

$$u_1'(x) = U_1(x) - F_1(x) > U_2(x) - F_2(x) = u_2'(x)$$

hence $(u_1 - u_2)$ is strictly increasing on I. Since necessarily $u_1(a) \ge u_2(a)$ this implies that $U_1 - U_2$ is strictly increasing on I, therefore $b = \infty$ and $\lim_{x\to\infty} (U_1 - U_2) = \infty$. But both U_1 and U_2 are finite.

Remark The proof applies to problem (P) under the sole assumptions $m > 0$, $u_o^i \in L^1(\mathbb{R})$.

We use the Shifting Comparison Principle (S.C.P.) to obtain a first result on the asymptotic behaviour of the free-boundaries $\zeta_i(t)$, $i = 1,2$.

Corollary For every $t > 0$

$$a_1 - r(t) \le \zeta_1(t) \le a_2 - r(t) \tag{4.2}$$

$$a_2 - r(t) \le \zeta_2(t) \le a_2 + r(t) \tag{4.3}$$

hence

$$\lim_{t\to\infty} \zeta_i(t) t^{-1/m+1} = (-1)^i \, c_m M^{\frac{m-1}{m+1}} \tag{4.4}$$

The proof consists simply of comparing the solution u with the solutions corresponding to initial data $u_o^1(x) = M\,\delta(x - a_1)$ (shifted to the left with respect to u_o) and $u_o^2(x) = M\,\delta(x - a_2)$ (shifted to the right).

5. THE LARGE-TIME BEHAVIOUR OF THE SOLUTIONS

5.1. We have already remarked that a general solution of problem (P) under hypotheses $m > 1$, (C_1), (C_2) can exhibit two peculiarities not present at the model solutions: the occurrence of a 'waiting-time' and the existence of 'internal interfaces'. Both depend on the particular shape of u_o. In particular a 'waiting time' exists at $x = a_1$ if $u_o(x) = O(|x - a_i|^{2/m-1})$ in a neighbourhood of $x = a_i$, this exponent being best possible, [10]. For recent work see [4], [11]. Our interest is not to describe the conditions under which these phenomena occur but to estimate the time at which they *cease* to occur ('normalization time') in terms of simplest possible information on u_o. We take this information to be the initial mass M and the length of the initial dispersion $l = a_2 - a_1$.

Using the S.C.P. we are able to prove

Theorem 1 There exists a time $T \ge 0$ such that for $t > T$ $\Omega(t)$ is an open interval with strictly expanding borders $\zeta_i(t)$ and

$$T < T_* = (1/c_m)^{m+1} \, M^{l-m}$$

the estimate being sharp in terms of M and l.

5.2. Next we center our attention at the asymptotic behaviour of u, v, V and ζ_i as $t \to \infty$. The conclusion we arrive at is that in first approximation they behave like the model solution with same total mass, M, and center of mass, x_c. In other words: once we know these two data on the initial situation we can describe in first approximation the behaviour of all the above quantities,

i.e. the *dispersion* of the initial distribution plays a secondary role as $t \to \infty$. We prove the following result.

Theorem 2 Let $\bar{u} = \bar{u}(x - x_c, t; M)$ as in (1.2) and analogously define $\bar{v}, \bar{V}$ and $\bar{\zeta}_i$, $i = 1,2$. As $t \to \infty$ we have

$$(-1)^i \, (\zeta_i(t) - \bar{\zeta}_i(t)) \downarrow 0 \tag{5.1}$$

$$(-1)^i \, t(\zeta_i'(t) - \bar{\zeta}_i'(t)) \uparrow 0 \tag{5.2}$$

$$t \, | V(x,t) - \bar{V}(x,t) | \to 0 \qquad \text{uniformly in } x: |x - x_c| \le r(t) \tag{5.3}$$

$$t^{m/m+1} \, | v(x,t) - \bar{v}(x,t) | \to 0 \qquad " \qquad " \qquad " \tag{5.4}$$

$$t^{\alpha/m+1} \, | u(x,t) - \bar{u}(x,t) | \to 0 \qquad " \qquad " \qquad " \tag{5.5}$$

with $\alpha = \min \left(\frac{m}{m-1}, 2\right)$.

Thus we can write for the free boundaries the asymptotic expansion

$$\zeta_i(t) = x_c + (-1)^i \, c_m \, M^{\frac{m-1}{m+1}} \, t^{\frac{m}{m+1}} + o(1) \quad (*) \tag{5.6}$$

$$\zeta_i'(t) = (-1)^i \, \frac{c_m}{m+1} \, M^{\frac{m-1}{m+1}} \, t^{-\frac{m}{m+1}} + o\left(\frac{1}{t}\right) \tag{5.7}$$

Idea of the proof: we begin by focusing our attention on the interfaces ζ_i. We know that for $t \ge T_i \ge 0$ they are C^1 curves and that $(-1)^i \, \zeta_i'(t) > 0$. For brevity we consider only the right interface $\zeta(t) = \zeta_2(t)$. We can assume $M = 1$. Caffarelli and Friedman [8] have proved that not only $\zeta \in C^1(T_2, \infty))$ but also that ζ'' is a bounded measure and there exists a constant $c > 0$ such that

$$\zeta''(t) + c\zeta'(t) \ge 0 \tag{5.8}$$

We show that c in (5.8) can be replaced by $\frac{m}{(m+1)t}$ (thus we obtain a sharp inequality: for the model solutions it is exact equality). From this we conclude that $\zeta'(t) t^{m/m+1}$ is nondecreasing. But we know that $\zeta(t) t^{-1/m} \to c_m$,

---

(*) $o(1)$, $o(t^\alpha)$, $O(1)$, $O(t^\alpha)$ are the usual notations of Landau for $t \to \infty$

(4.4), so that $\zeta'(t)t^{m/m+1} \uparrow \frac{c_m}{m+1}$.

By repeated manipulation of (5.8) we arrive at the following result

Lemma 1 Let $\eta(t) = \zeta(t) - r(t)$. Then $a_1 < \eta(t) < a_2$, $\eta'(t) \le 0$, $t^{m/m+1}\eta'(t)$ is nondecreasing and $t\eta'(t) \to 0$ as $t \to \infty$.

Let now $l(t) = \zeta_2(t) - \zeta_1(t)$ be the length of $\Omega(t)$ and $d(t) = l(t) - 2r(t)$ the difference with respect to the model situation. Lemma 1 implies that $d'(t) \le 0$. We next prove

Lemma 2 $\lim_{t\to\infty} d(t) = 0$, i.e. there exists $x_o \in (a_1,a_2)$ such that

$$\zeta_i(t) = x_o + (-1)^i r(t) + o(1).$$

Lemma 2 is proved as a consequence of the conservation of mass and the estimates on u, v, V that are derived from Lemma 1: we start from $v_x(\zeta(t),t) = -\zeta'(t) = -\frac{r(t)}{(m+1)t} + o(\frac{1}{t})$ and use $v_{xx} \ge -\frac{1}{(m+1)t}$, [3], to conclude that if the limit is not zero, $\int u(x,t)dx$ and $\int \bar{u}(x-x_o,t;M)dx$ cannot be equal for large enough t. In this process we derive (5.1)-(5.5) with $\bar{u} = \bar{u}(x-x_o,t;M)$.

The final step is to show that $x_c = x_o$. Here we use the invariance of the center of mass. For $m \le 2$ the proof is simple

$$M|x_c - x_o| \le \int x|u(x,t) - \bar{u}(x - x_o,t;M)|dx =$$

$$= O(r(t)^2) \cdot o(t^{-\frac{2}{m+1}}) = o(1)$$

now let $t \to \infty$ to get $x_c = x_o$.
If $m \ge 2$ the estimate (5.5) for u is not good and we have to calculate $M|x_c - x_o|$ in terms of $|v - \bar{v}|$.

5.3. Optimal rates of convergence it is reasonable to expect in the convergences of Theorem 2 the following exponents

$$|\zeta(t) - \bar{\zeta}(t)|\, t^{m/m+1} \le K_1, \quad |\zeta'(t) - \bar{\zeta}'(t)|\, t^{\frac{2m+1}{m+1}} \le K_2,$$

$$|v - \bar{v}|\, t^{\frac{2m+1}{m+1}} \le K_3, \quad t^{\frac{2m+1}{m+1}} |v - \bar{v}| \le K_4.$$

We are able to prove these estimates with K_i depending only on m,M and 1 for the particular case where $u_o(x)$ is *symmetric* (i.e. $u_o(-x) = u_o(x)$) by means of a new comparison theorem (where the masses $\int_{-x}^{x} u(x,t)dx$ are involved). Thus in this case we have instead of (5.6),(5.7)

$$\zeta_i(t) = x_c + (-1)^i c_m M^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}} + O(t^{-\frac{m}{m+1}}) \tag{5.9}$$

$$\zeta_i'(t) = (-1)^i \frac{c_m}{m+1} M^{\frac{m-1}{m+1}} t^{-\frac{m}{m+1}} + O(t^{-\frac{2m+1}{m+1}}). \tag{5.10}$$

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S XIAO & C ZHOU

Numerical models for infiltration into unsaturated–saturated porous media

This paper describes a rather general one-dimensional unsteady watertable behaviour related to rainfall infiltration. A moving boundary model for unsaturated-saturated flow in soil, both for homogeneous and layered media, is established and solved by numerical method. For a given rainfall process, the relation between watertable fluctuation and other factors are discussed.

1. MATHEMATICAL MODEL

For a one-dimensional unsaturated-saturated unsteady flow of an incompressible fluid, the governing equation can be expressed as follows [1]

$$
(A)\quad \begin{cases}
C(\psi)\dfrac{\partial\psi}{\partial t} = \dfrac{\partial}{\partial x}\left(K(\psi)\dfrac{\partial\psi}{\partial x} - K(\psi)\right), & (\psi < \psi_c,\ t > 0,\ 0 < x < H),\\[2ex]
\dfrac{\partial^2\psi}{\partial x^2} = 0, & (\psi \geq \psi_c,\ t > 0,\ 0 < x < H).\\[2ex]
\dfrac{\partial\psi}{\partial x} = 1, & (x = H,\ t \geq 0),\\[2ex]
K(\psi)\left(\dfrac{\partial\psi}{\partial x} - 1\right) = R(t), & (x = 0,\ t \geq 0,\ \psi(0,t) < 0),\\[2ex]
K(\psi)\left(\dfrac{\partial\psi}{\psi x} - 1\right) = -R(t) - \dfrac{d}{dt}\psi(0,t), & (x = 0,\ t \geq 0,\ \psi(0,t) \geq 0),\\[2ex]
\psi = \psi_c, & (x = S(t),\ t \geq 0),\\[2ex]
\dfrac{d\psi}{dx} = 1, & (x = S(t),\ t \geq 0),\\[2ex]
\psi = \psi_0(x), & (t = 0,\ 0 \leq x \leq H).
\end{cases}
$$

$$
S(t) = x + \psi_c - \psi(x,t), \qquad (S(t) \leq x \leq H,\ \psi \geq \psi_c).
$$

where x is positive downward, x = 0 is the groundsurface, and

$\theta(x,t)$: volumetric moisture content; (θ_s: Saturated value)

$\psi(x,t)$: soil tension head or pressure head;

ψ_c : air entry pressure head;

$K(\psi)$: hydraulic conductivity;

K_s : hydraulic conductivity of a saturated soil;

$C(\psi)$: specific moisture capacity; $(\frac{d\theta}{d\psi})$;

$R(t)$: intensity of rainfall;

$S(t)$; watertable.

For a layered soil, the conditions on the interfaces are [2] (Figure 2)

$$K_i(\psi)\left(\frac{\partial\psi}{\partial x} - 1\right)_{x=H_i-0} = K_{i+1}(\psi)\left(\frac{\partial\psi}{\partial x} - 1\right)_{x=H_i+0}, \tag{1}$$

$$\psi\big|_{x=H_i-0} = \psi\big|_{x=H_i+0}, \tag{2}$$

$$(i = 1,2,\ldots,p).$$

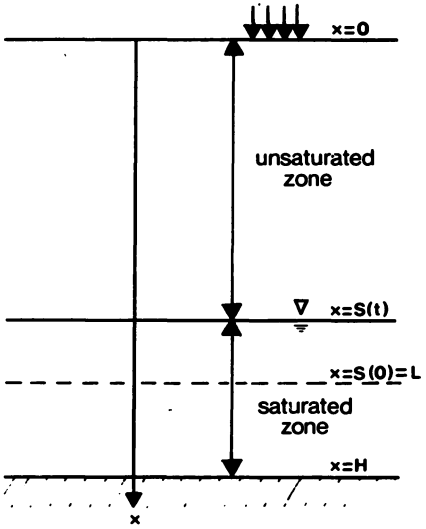

Figure 1

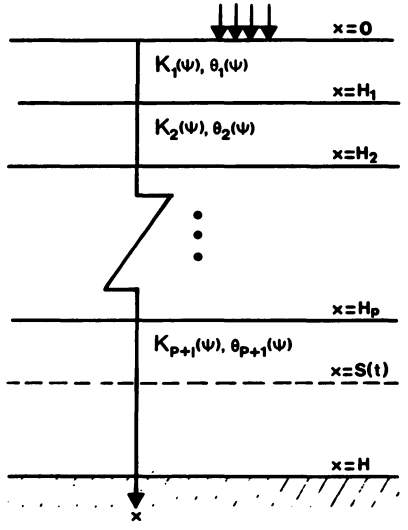

Figure 2

2. NUMERICAL SCHEME

It is convenient to introduce the following dimensionless variables

$$\varphi = \psi/L,$$

$$\zeta = x/L,$$

$$\tau = t\, K_{sl}/\theta_{sl} L,$$

$$\rho(\tau) = R(t)/K_{sl},$$

$$\gamma_i(\varphi,\zeta) = L\, C_i(\psi,x)/\theta_{sl},$$

$$x_i(\varphi,\zeta) = K_i(\psi)/K_{sl},$$

$$(i = 1,2,\ldots,p+1)$$

where

$$L = s(0).$$

(A) is transformed into

$$(A)^* \quad \begin{cases} \gamma_i \dfrac{\partial \varphi}{\partial \tau} = \dfrac{\partial}{\partial \zeta}\left(x_i(\varphi)\dfrac{\partial \varphi}{\partial \zeta} - x_i(\varphi)\right), & (\zeta_{i-1} < \zeta < \zeta_i,\ \varphi < \psi_{c_i}/L), \\ \dfrac{\partial^2 \varphi}{\partial \zeta^2} = 0, & (\zeta_{i-1} < \zeta < \zeta_i,\ \varphi \geq \psi_{c_i}/L), \\ \qquad (i = 1,2,\ldots,p+1), & \\ x_i(\varphi)\left(\dfrac{\partial \varphi}{\partial \zeta} - 1\right)_{\zeta = \zeta_i - 0} = x_{i+1}(\varphi)\left(\dfrac{\partial \varphi}{\partial \zeta} - 1\right)_{\zeta = \zeta_i + 0}, & \\ \varphi|_{\zeta = \zeta_i - 0} = \varphi|_{\zeta = \zeta_i + 0}, & \\ \qquad (i = 1,2,\ldots,p), & \\ \dfrac{\partial \varphi}{\partial \zeta} = 1, & (\zeta = H/L), \\ x_1(\varphi)\left(\dfrac{\partial \varphi}{\partial \zeta} - 1\right) = -\rho(\tau), & (\zeta = 0,\ \varphi(0,\tau) < 0), \\ x_1(\varphi)\left(\dfrac{\partial \varphi}{\partial \zeta} - 1\right) - \dfrac{1}{\theta_{sl}}\dfrac{d}{d\tau}\varphi(0,\tau) = -\rho(\tau), & (\zeta = 0, \varphi(0,\tau) \geq 0), \\ \varphi = \psi_0/L, & (\tau = 0). \end{cases}$$

Crank-Nicolson's implicit scheme gives

$$-A_j \varphi_{j+1}^{n+1} + B_j \varphi_j^{n+1} - C_j \varphi_{j-1}^{n+1} = D_j, \qquad (\varphi_j^n < \psi_{si}/L),$$

$$\varphi_{j+1}^{n+1} - 2\varphi_j^{n+1} + \varphi_{j-1}^{n+1} = 0, \qquad (\varphi_j^n \geq \psi_{si}/L), \tag{3}$$

$$(j = 2,3,\ldots,N-1;\ n = 1,2,\ldots;\ (j-1)\Delta\zeta \neq \zeta_i).$$

For $j = 1$,

$$\varphi_2^{n+1} - \varphi_1^{n+1} = \left(1 - \frac{\rho^{n+1}}{x_1(\varphi_1^{n+\frac{1}{2}})}\right)\Delta\zeta, \qquad (\varphi_1^n < 0),$$

$$\varphi_2^{n+1} - \left(1 + \frac{\Delta\zeta}{\theta_{s1}\Delta\tau}\right)\varphi_1^{n+1} = (1 - \rho^{n+1})\Delta\zeta - \frac{\Delta\zeta}{\theta_{s1}\Delta\tau}\varphi_1^n, \qquad (\varphi_1^n \geq 0), \tag{4}$$

if $\varphi_1^n < \psi_{s1}/L$, $\varphi_1^{n+\frac{1}{2}}$ in (4) is determined by

$$\varphi_2^n - \varphi_1^{n+\frac{1}{2}} = \left(1 - \frac{\rho^{n+1}}{x_1(\varphi_1^{n+\frac{1}{2}})}\right)\Delta\zeta. \tag{4}'$$

For $j = N$,

$$\varphi_N^{n+1} - \varphi_{N-1}^{n+1} = \Delta\zeta. \tag{5}$$

At the interface,

$$-x_{i+1}(\varphi_j^{n+\frac{1}{2}})\,\varphi_{j_i+1}^{n+1} + \left(x_i(\varphi_{j_i}^{n+\frac{1}{2}}) + x_{i+1}(\varphi_{j_i}^{n+\frac{1}{2}})\right)\varphi_{j_i}^{n+1}$$

$$- x_i(\varphi_{j_i}^{n+\frac{1}{2}})\,\varphi_{j_i-1}^{n+1} = \Delta\zeta\left(x_i(\varphi_{j_i}^{n+\frac{1}{2}}) - x_{i+1}(\varphi_{j_i}^{n+\frac{1}{2}})\right), \tag{6}$$

$$((j_i-1)\Delta\zeta = \zeta_i;\ i = 1,2,\ldots p).$$

As θ is increasing rapidly near the interface, x_i changes abruptly, so significant errors may be induced for the calculation of $\varphi_{j_i}^{n+\frac{1}{2}}$.

In order to overcome this, we solve $\varphi_{j_i}^{n+\frac{1}{2}}$ in (6) from the following equation

$$x_i\left(\begin{smallmatrix} n+\frac{1}{2} \\ j_i \end{smallmatrix}\right)\left(\frac{1}{\Delta\zeta}(\varphi^{n+\frac{1}{2}}_{j_i} - \varphi^{n}_{j_i-1}) - 1\right)$$

$$= x_{i+1}\left(\varphi^{n+\frac{1}{2}}_{j_i}\right)\left(\frac{1}{\Delta\zeta}(\varphi^{n}_{j_i+1} - \varphi^{n+\frac{1}{2}}_{j_i}) - 1\right), \tag{7}$$

which can be solved by Newton's method.

3. RESULTS AND DISCUSSION

For illustration we take 8 soils, among which a fitted empirical equation is given for S1, and by changing parameters we get the others.

Conductivity-pressure head relation:

$$K(\psi) = \begin{cases} \dfrac{K_s A}{A + (\psi - \psi_c)^2}, & (\psi < \psi_c), \\ K_s, & (\psi \geq \psi_c). \end{cases} \tag{8}$$

Moisture content-pressure head relation:

$$\theta(\psi) = \begin{cases} -\dfrac{\sqrt{3}}{2\pi}\theta_s\left\{\dfrac{1}{2}\ln\dfrac{1 + (\psi_c - \psi)^2 - (\psi_c - \psi)}{1 + (\psi_c - \psi) + 2(\psi_c - \psi)}\right. \\ \qquad \left. + \sqrt{3}\ \text{arc tg}\dfrac{2(\psi_c - \psi) - 1}{\sqrt{3}}\right\} + \dfrac{3}{4}\theta_s, & (\psi < \psi_c), \\ \theta_s, & (\psi \leq \psi_c). \end{cases} \tag{9}$$

$$\tilde{\theta}(\psi) = \begin{cases} -\dfrac{\sqrt{3}}{\pi}\theta_s\left\{\dfrac{1}{2}\ln\dfrac{1 + (\psi_c - \psi)^2 - (\psi_c - \psi)}{1 + (\psi_c - \psi)^2 + 2(\psi_c - \psi)}\right. \\ \qquad \left. + \sqrt{3}\ \text{arc tg}\dfrac{2(\psi_c - \psi) - 1}{\sqrt{3}}\right\} + \left(\dfrac{3}{2} - e^{-\frac{1}{2}(\psi_c - \psi)}\right)\theta_s, & (\psi < \psi_c), \\ \theta_s, & (\psi \geq \psi_c). \end{cases} \tag{10}$$

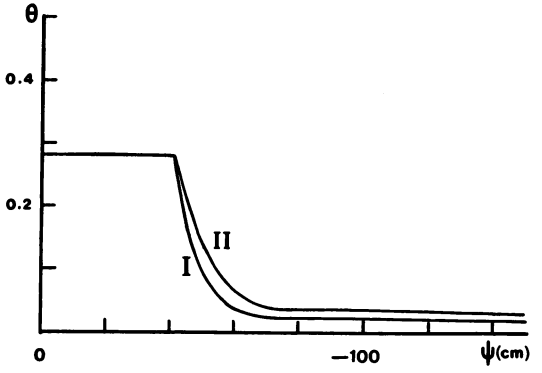

Figure 3. $\theta = \theta(\psi)$
I. (9) $\theta_s = 0.28, \psi_c = -20$
II. (10) $\theta_s = 0.28, \psi_c = -20$

All parameters are shown in the following:

	K_s	θ_s	ψ_c	$\theta(\psi)$	A
S1	0.048	0.28	-20	(9)	16
S2	0.042	0.28	-20	(9)	16
S3	0.048	0.28	-20	(10)	16
S4	0.042	0.28	-20	(10)	16
S5	0.054	0.28	-20	(10)	16
S6	0.048	0.28	-12	(9)	16
S7	0.048	0.28	-20	(9)	4
S8	0.080	0.38	-12	(9)	4

We take static pressure distribution to be the initial state in all cases, i.e.

$$\psi_o(x) = \psi_c + x - L. \tag{11}$$

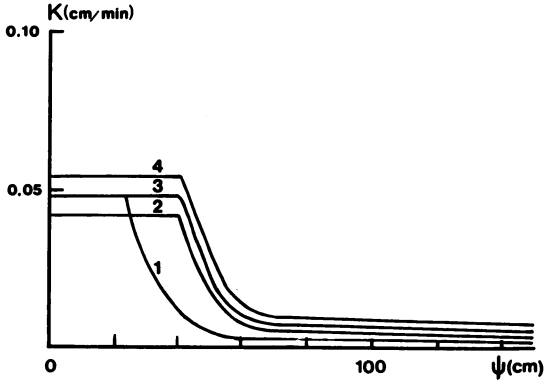

Figure 4. $K = K(\psi)$

1. $K_s = 0.048$, $\psi_c = -12$
2. $K_s = 0.042$, $\psi_c = -20$
3. $K_s = 0.048$, $\psi_c = -20$
4. $K_s = 0.054$, $\psi_c = -20$

1. Asymptotic behaviour of redistribution in the unsaturated region

Figure 5 shows the pressure head profiles for large t, with different initial depth L and different rainfall rate. It can be seen that the

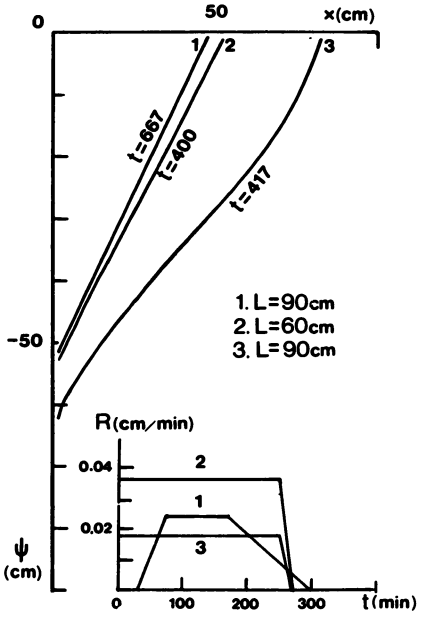

Figure 5

redistribution of soil moisture following a cessation of rainfall should tend to the static pressure distribution, regardless of the initial water-table positions and the rainfall intensities, and the process is longer as L is larger.

2. Effect of Soil type

Figure 6 illustrates the relation between the rise of watertable and soil type. Under the same initial condition, rainfall intensity (R = 0.036 cm/min), and initial watertable depth (L = 90 cm), we can see that

(i) The rate of the rise of watertable s(t) is constant.

(ii) For sufficiently large t, the rate of s(t) is determined by K_s, the rate is increasing for increasing K_s.

(iii) For soil with larger $\theta(\psi)$ the time of arrival t_s of the infiltrating wet front to the watertable should be later.

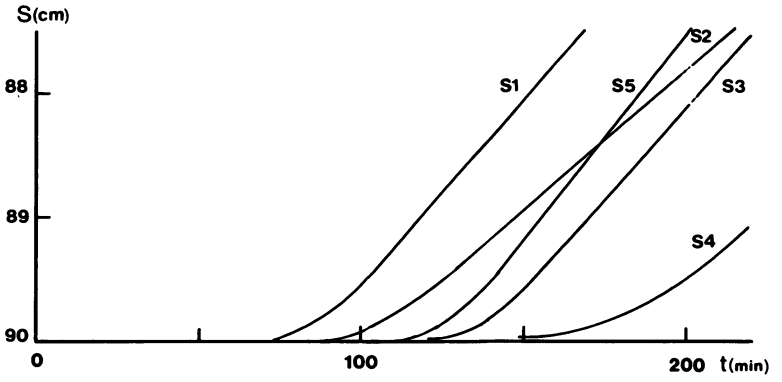

Figure 6 S = S(t)
R = 0.036 cm min
L = 90 cm
θ_s = 0.28
ψ_c = −20

3. The velocity of infiltrating wet front

In Figure 7, the position of θ = 0.02 (wet front) of S7 is plotted as a function of t, under different rainfall rates. We see that the velocity of such a wet front is nearly invariant even after ponding time t_p. Also, it can be seen that higher rainfall rate induces higher velocity, as one expects.

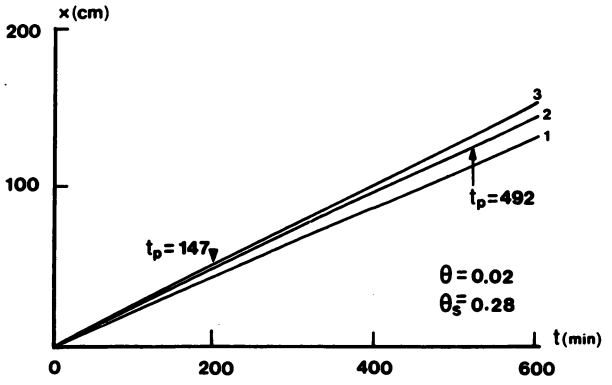

Figure 7 1. R = 0.024
2. R = 0.060
3. R = 0.096

Figure 8 shows the relation between the time of watertable rise and the initial depth L = s(0). We see again that the velocity of the wet front is constant.

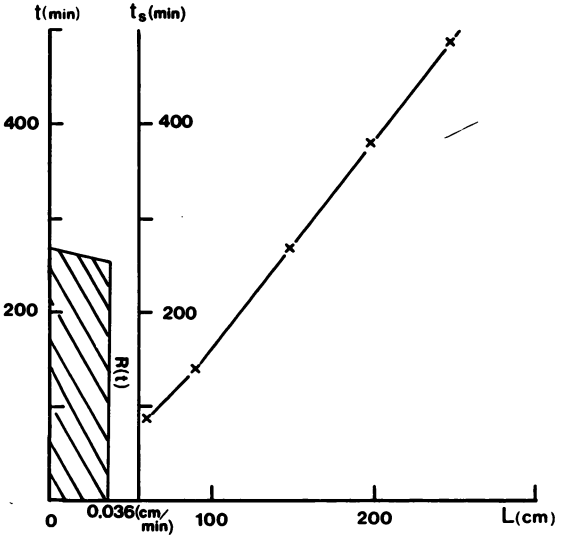

Figure 8

4. Effect of Ponding

As soon as ponding takes place on the ground surface, the surface infiltration rate declines, and then it tends gradually to a constant rate, which depends on the rainfall rate and is always larger than K_s, as shown

in Figure 9. This result is somewhat different from that in [1] because no surface runoff takes place in our case.

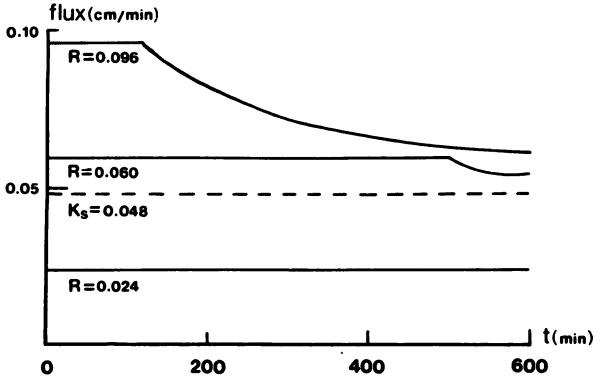

Figure 9

5. Effect of Rainfall Intensity

Figures 10 and 11 show the moisture profiles under different rainfall intensities for the same soil S6 and equal amount of water (20cm). It can be seen that watertable rise is earlier and faster for higher rainfall intensity.

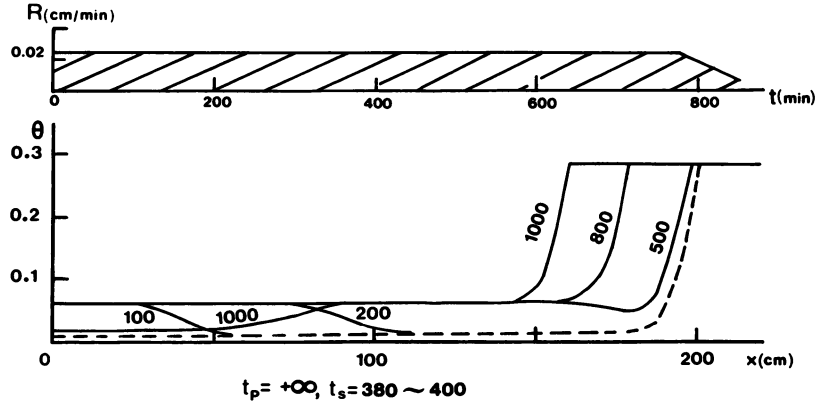

Figure 10

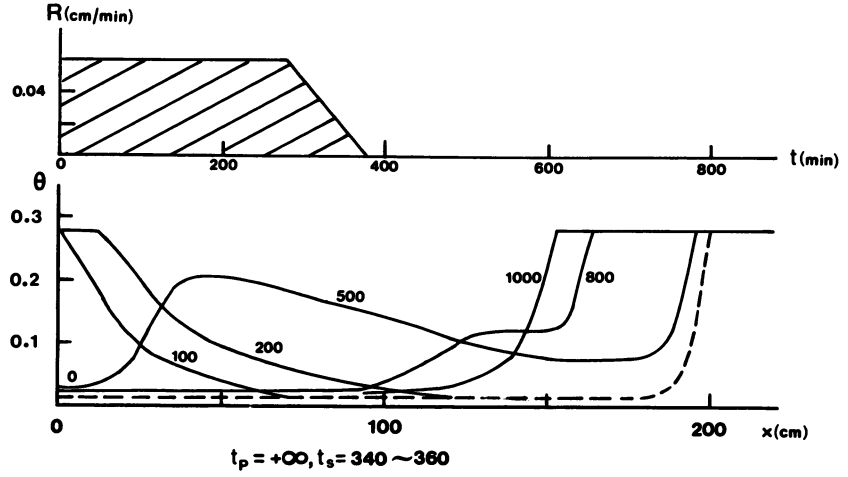

Figure 11

6. Effect of the Discontinuity of Soil Texture

Figure 12 shows the hold-up effect of the wet front when a fine layer is over a coarse layer [2]. Figure 13 shows the case when a coarse layer is over a fine layer.

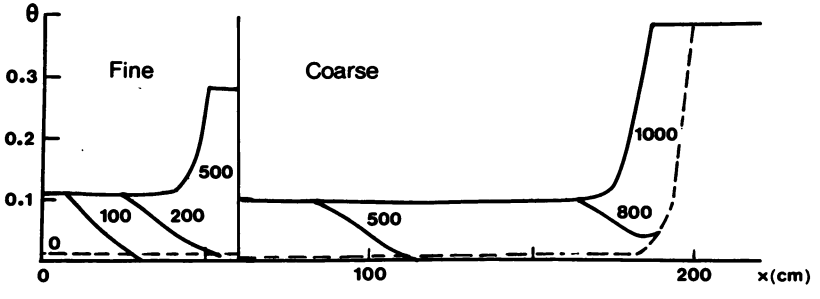

Figure 12 R=0.024, L=200
$t_p = +\infty$, $t_s = 650 \sim 700$

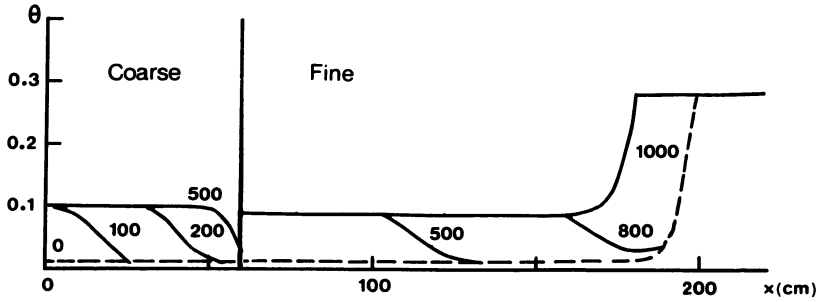

Figure 13 R=0.024, L=200
$t_p = +\infty$, $t_s = 650 \sim 700$

7. Effect of thin layer

An effect of a thin layer is reduction of the velocity of the infiltrating wet front, regardless of whether it is fine or coarse, this fact can be shown in Figure 14. Another result is that the velocity of wet front decreases as the depth of the thin layer decreases.

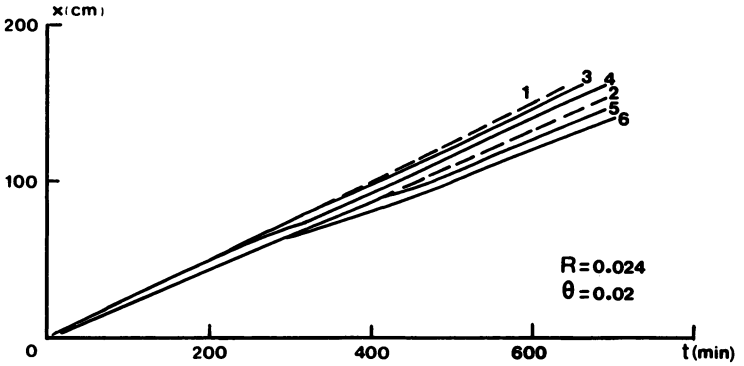

Figure 14

1. S8
2. S7
3. S8—S7—S8, depth of thin layer: 165 cm
4. S8—S7—S8, " 60 cm
5. S7—S8—S7, " 165 cm
6. S7—S8—S7, " 60 cm

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[2] Rubin, J., Numerical Analysis of Ponded Rainfall Infiltration, Water in the Unsaturated Zone. IASH-AIHS Proceedings of the Wageningen Symposium, Vol. 1 (1969), 440-451.

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M FRÉMOND

Frost action in soils

The session on the frost action in soils comprised four lectures on applications (J. Aguirre-Puente, M. Frémond and Xu Xiaozu) and theory (G. Aronson and M. Frémond, H. Ghidouche, N. Point).

After a brief description of the physical problems (paragraph 1) and applications (paragraph 2), we concentrate on the descriptions of the models (paragraph 3) and mathematical results (paragraph 4).

1. THE PHYSICAL PROBLEMS

The soils contain water which freezes when the temperature is below $0^{\circ}C$ [23], [24], [25], [26]. The thermal, mechanical and hydraulic properties of the soils are changed by freezing. The frozen soils are more heat conductive and more rigid than unfrozen soils. They are practically impermeable. The freezing of soils has not only such obvious effects but others, not very well known, which modify more profoundly their structure. Most importantly, movements of water are induced by freezing. These movements of water cause vertical displacements of the soils up to many meters (as shown by pictures of Xu Xiaozu). These movements can be explained by microscopic physics and chemistry [4] or by the macroscopic theory of continuum media [6], [12], [15], [24].

Experiments and theory indicate then, that it is necessary to take into account both thermal and hydraulic aspects to have a realistic predictive model.

2. THE APPLICATIONS

The developments of engineering in northern regions, the need to construct all-weather highways in areas of seasonal frost are important reasons for developing studies of soil and road freezing [5], [7], [24].

These concerns apply not only to natural freezing but also to artificial freezing (e.g. underground or semi-underground storage of L.N.G.; freezing of soil for civil engineering [26]).

In the session two types of problems were discussed: determination of

permafrost distribution and thickness; frost propagation in structures.

3. THE MODELS

They are based on the mass and energy conservation laws, the constitutive laws (Fourier and Darcy laws, specific energy as a function of temperature and water content) and certain assumptions to simplify the equations.

3.1. ASSUMPTIONS FOR ALL THE MODELS

We assume that the wet soil is a water saturated porous medium, i.e. there are only three phases liquid water, solid water (ice) and the skeleton of the porous medium. The classical theory of porous media allows us to define at each point x of the domain Ω, which is occupied by the soil, and at each time t of the time interval [O,T], average quantities: temperature $\theta(x,t)$, water content $\mu(x,t)$, heat of water $h(x,t)$... .

We assume that the heat exchanges between the different phases are instantaneous so that a single temperature applies to all the phases. In the same way we assume that the heat flux vector is the same in each phase.

We neglect the mechanical aspects (stresses). It is then possible to assume that the porosity ε is constant.

Since the displacements we spoke of (i.e. frost heaving) are principally induced by accumulation of ice due to cryogenic suction at the frost line (isotherm $0^{\circ}C$) [21], [24], [25], [26], we can neglect the volume expansion of water upon freezing.

Without being necessary for the models, we assume that all the densities are constant.

Heat is transferred in the soil by conduction, by diffusion of water and by convection of the skeleton, in decreasing order of importance.

Models exist which do not make all these assumptions, such as those which assume that the soil is not saturated [17].

3.2. THE CONSERVATION LAWS

3.2.1. Conservation of the mass of solid and liquid water

Let us define ρ^w the mass by unit volume of water, μ the mass liquid water content ($\mu = 1$ for pure water, $\mu = 0$ for pure ice and $0 < \mu < 1$ for a mixture of ice and water). We have for any domain $\mathcal{D}$

$$\frac{d}{dt}\int_{\mathfrak{D}} \varepsilon\rho^w \mu \, d\mathfrak{D} + \frac{d}{dt}\int_{\mathfrak{D}} \varepsilon\rho^w(1-\mu)\, d\mathfrak{D} = 0, \tag{1}$$

where the derivatives are particular derivatives [14].

Let us define $\vec{W}$ the velocity of the frost line with $\vec{N}$ a unit normal vector, $\vec{U}^w$, $\vec{U}^i$ the absolute velocities of the liquid and solid water, $\vec{V}^w = \vec{U}^w - \vec{W}$, $\vec{V}^i = \vec{U}^i - \vec{W}$ the relative velocities. It follows from (1) that, if the quantities are continuous

$$\mathrm{div}\varepsilon\rho^w\mu\vec{U}^w + \mathrm{div}\varepsilon\rho^w(1-\mu)\vec{U}^i = 0. \tag{2}$$

and if they are not continuous

$$[\rho^w\varepsilon\mu\vec{V}^w + \rho^w(1-\mu)\vec{V}^i]\cdot\vec{N} = 0, \tag{3}$$

where $[\mathcal{A}]$ is the discontinuity of the quantity $\mathcal{A}$.

3.2.2. Conservation of the mass of the skeleton

Let us define ρ^s the mass by unit volume of the skeleton. We have for any domain $\mathfrak{D}$

$$\frac{d}{dt}\int_{\mathfrak{D}} (1-\varepsilon)\rho^s \, d\mathfrak{D} = 0, \tag{4}$$

where the derivative is the particular derivative. The absolute velocity of the skeleton being $\vec{U}^s$, it follows from (4) that if the quantities are continuous

$$\mathrm{div}(1-\varepsilon)\rho^s\vec{U}^s = 0, \tag{5}$$

and if they are not continuous,

$$[(1-\varepsilon)\rho^s\vec{V}^s]\cdot\vec{N} = 0, \tag{6}$$

where $\vec{V}^s = \vec{U}^s - \vec{W}$ is the relative velocity.

3.2.3. Conservation of the energy of the water and the skeleton

Let us define e^w, e^i, e^s the specific energies of the liquid water, the solid

water and the skeleton, q the heat flow vector in the porous medium, r the rate of heat production by unit volume. We have for any domain $\mathfrak{D}$

$$\frac{d}{dt}\int_{\mathfrak{D}} \varepsilon\rho^w e^w \, d\mathfrak{D} + \frac{d}{dt}\int_{\mathfrak{D}} \varepsilon\rho^w(1-\mu)e^i \, d\mathfrak{D} + \frac{d}{dt}\int_{\mathfrak{D}} (1-\varepsilon)\rho^e e^s \, d\mathfrak{D} =$$

$$= -\int_{\Gamma} \vec{q}\cdot\vec{n} \, d\Gamma + \int_{\mathfrak{D}} r \, d\mathfrak{D}, \tag{7}$$

where Γ is the boundary of $\mathfrak{D}$, $\vec{n}$ an outwards normal to Γ unit vector. The derivatives in (7) are particular derivatives. It follows from (7) that if the quantities are continuous

$$\rho\frac{\partial e}{\partial t} + \operatorname{div}(\varepsilon\rho^w\mu e^w\vec{U}^w + \varepsilon\rho^w(1-\mu)e^i\vec{U}^i + (1-\varepsilon)\rho^s e^s\vec{U}^s) + \operatorname{div}\vec{q} = r, \tag{8}$$

with

$$\rho = \varepsilon\rho^w + (1-\varepsilon)\rho^s, \quad \rho e = \varepsilon\rho^w(\mu e^w + (1-\mu)e^i) + (1-\varepsilon)\rho^s e^s,$$

and if the quantities are discontinuous

$$[\varepsilon\rho(\mu e^w\vec{V}^w + (1-\mu)e^i\vec{V}^i) + (1-\varepsilon)\rho^s e^s\vec{V}^s]\cdot\vec{N} + [\vec{q}]\cdot\vec{N} = 0. \tag{9}$$

3.3. THE CONSTITUTIVE LAWS

We assume Fourier's law

$$\vec{q} = -\lambda \operatorname{grad}\theta, \tag{10}$$

where λ is the thermal conductivity which can depend on θ. For simplicity we assume that it depends only on the frozen or unfrozen state.

We also assume Darcy's law

$$\vec{U}^w - \vec{U}^s = -\frac{m}{\varepsilon\mu}\operatorname{grad}h \tag{11}$$

where m is the mobility of the water in the soil.

We assume that the specific energy e depends only on θ and μ. Let us define $e(\theta,\mu)$ for $\theta = 0$, the phase change temperature. The energies e^w and e^i are constant. The quantity $e^w - e^i$ is the specific latent heat of the water ℓ^w. One can assume that there is no phase change in the skeleton and

that e^s is constant. The specific energy is then

$$e(0,\mu) = \frac{1}{\rho} \varepsilon \rho^w \ell^w (\mu - 1) = \ell(\mu - 1),$$

where ℓ is the specific latent heat of the wet soil. When θ is not 0, we have the classical definition of the energy

$$e(\theta,\mu) = C\theta + \ell(\mu - 1), \tag{12}$$

with $\mu = 1$ if $\theta > 0$, $\mu = 0$ if $\theta < 0$ and $0 \leq \mu \leq 1$ if $\theta = 0$.

The heat capacity of the soil C can depend on the temperature θ. For simplicity we assume that it depends only on the frozen or unfrozen state.

3.4. THE CLOUDS (zones where the temperature is 0°)

It follows from (12) that there can exist intermediate states ($0 < \mu < 1$) between pure water ($\mu = 1$) and pure ice ($\mu = 0$) which are called the clouds. They are mixtures of water and ice at the temperature 0°. In contrast with metallurgy [27], clouds are not well known in soil freezing, because they can appear when $r \neq 0$ or when they exist at the time $t = 0$ [3], [12]. This situation is not common in soil freezing. The rate of heat production is in practice almost always equal to 0, because there are no chemical reactions or electrical currents within the soils.

3.5. THE THERMAL MODEL

We define the unfrozen part $Q_1(\theta(x,t) > 0)$, the frozen part $Q_2(\theta(x,t) < 0)$ and the cloud $Q_3(\theta(x,t) = 0)$ of $Q = \Omega \times]0,T[$. In the following the subscript i refers to quantities defined in Q_i (figure 1).

The thermal model neglects all other form of heat transfer except conduction. This is equivalent to assuming that all the velocities are zero. The mass conservation equations are automatically verified. There remain equations (8) and (9) which can be combined with the constitutive laws to obtain the classical Stefan problem

$$\rho C_i \frac{\partial \theta_i}{\partial t} - \lambda_i \Delta \theta_i = r \text{ in } Q_i \quad (i = 1 \text{ or } 2), \tag{12}$$

$$\rho \ell \frac{\partial \mu}{\partial t} = r \text{ in } Q_3,$$

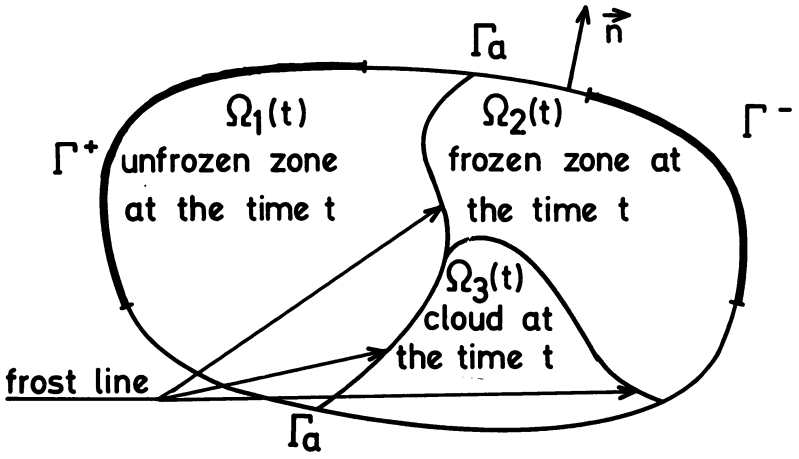

Figure 1. The domain Ω at the time $(\Omega_i(t) = Q_i \cap \{\Omega \times \{t\}\})$.

$$\mu = 1 \text{ in } Q_1,\ \mu = 0 \text{ in } Q_2,\ 0 \le \mu \le 1 \text{ in } Q_3 \tag{13}$$

At the frost line the temperature is zero and the conservation of energy gives

$$[-\lambda\ \mathrm{grad}\theta]\cdot\vec{N} = [\rho e]\vec{W}\cdot\vec{N} = \rho\ell[\mu]\vec{W}\cdot\vec{N}, \tag{14}$$

These equations are equivalent to

$$\frac{\partial e}{\partial t} - \lambda\ \Delta\theta = r$$

in the sense of distributions.

The classical Stefan problem involves the clouds. But, as we have said, they are not common in soil freezing. The equations are completed by classical boundary and initial conditions.

3.5.1. The quasi static thermal model

Close to the phase change temperature $0^\circ C$, the energy can be approximated by

$$e = \ell(\mu - 1),$$

with $\mu = 1$ if $\theta > 0$, $\mu = 0$ if $\theta < 0$, $0 \le \mu \le 1$ if $\theta = 0$.

The approximation is best when ℓ is large. This can be the case for a wet soil even if not very porous because the latent heat of the water ℓ^w is very large. We have then the quasi-static thermal model [11], [16]. The equations are (12), (13) and (14) with $C_i = 0$ (i = 1 and 2).

Another assumption can further simplify the model. Let us assume that the heat flux $\vec{q} = 0$ in the unfrozen zone Q_1. The temperature is then 0 in Q_1. We obtain the one phase quasi-static thermal model. The equations are (12), (13) and (14) with $\theta_1 = 0$ and $C_2 = 0$.

These simple models are adequate in many practical applications as described by Aronson.

In one dimension this model gives the well known Bergren (or Stefan) formula which gives the frost penetration s(t) as a function of the freezing index [12] $I(t) = \int_0^t -\theta(\tau)d\tau$ at the surface of the soil when freezing begins at t = 0:

$$s(t) = \sqrt{\frac{\lambda}{\rho\ell} I(t)}.$$

This formula is used with correction factors by Xu Xiaozu to calculate the maximum thaw penetration in permafrost [9], [22].

3.6. THE HYDRAULIC AND THERMAL MODEL

This model takes into account both conduction and heat transferred by water diffusion. In the frozen and unfrozen zones we consider only the conduction. In the clouds we consider the heat transferred by water diffusion (the conduction being zero). At the frost line we consider both conduction and heat transferred by water diffusion since the latent heat of the diffused water is important. One knows that this is a characteristic aspect of the freezing of wet porous media.

In the zones Q_1 and Q_2 we have then the heat equation

$$\rho C_i \frac{\partial \theta_i}{\partial t} - \lambda_i \Delta\theta_i = r \text{ in } Q_i, \text{ (i = 1 and 2)}, \tag{15}$$

$\mu = 1$ in Q_1 and $\mu = 0$ in Q_2.

In the cloud Q_3, by multiplying equation (2) by e^i and subtracting from (8) we obtain by using equation (5)

$$\ell^w \frac{\partial}{\partial t} \varepsilon\rho^w\mu + \ell^w \mathrm{div}\rho^w\mu\vec{U}^w = r.$$

The heat transferred by conduction being zero, we consider, as we have said, the heat transferred by water diffusion and neglect the heat transferred by convection of the skeleton. We obtain then by using Darcy's law (1)

$$\ell^w\varepsilon\rho^w \frac{\partial\mu}{\partial t} - \rho^w\ell^w \Delta h = r. \tag{16}$$

Note To neglect the heat transferred by convection of the skeleton in the clouds is to neglect the mass of liquid water transferred by convection of the skeleton. The equation (16) is also in the cloud the equation of conservation of the mass of liquid water. This is natural because in the cloud the energy and the mass of liquid water are proportional, ℓ^w being the coefficient of proportionality.

In the unfrozen zone the equations (2) and (5) give

$$- m \Delta h = 0. \tag{17}$$

At the frost line where the temperature is zero, by multiplying equation (3) by e^i and subtracting from (9) we obtain by using (6)

$$\varepsilon\rho^w\ell^w[\mu\vec{U}^w]\cdot\vec{N} - \varepsilon\rho^w\ell^w[\mu]\vec{W}\cdot\vec{N} + [\vec{q}]\cdot\vec{N} = 0.$$

We use Darcy's law and neglect the convective term $\varepsilon\rho^w\ell^w[\mu\vec{W}^s]$ following our assumptions. We obtain

$$[-\lambda\ \mathrm{grad}\theta - \rho^w\ell^w m\ \mathrm{grad}h]\cdot\vec{N} = \varepsilon\rho^w\ell^w[\mu]\vec{W}\cdot\vec{N}. \tag{18}$$

We let $\mathrm{grad}h_2 = 0$ when this term appears in the preceding formulae because we assume that there is no movement of liquid water within the frozen zone Ω_2. This assumption is consistent with the description of the cryogenic suction which follows.

It is necessary to add a boundary condition at the frost line to describe the cryogenic suction. We assume that the cryogenic suction is known at the frost line between the frozen part Ω_2 and the other part Ω_3 and Ω_1 [1], [12]. Its value is $- d$ $(d > 0)$. In fact we define the function h in Ω_2 by

$$h = -d \text{ in } \Omega_2, \tag{19}$$

and we assume that the head of water h, like the temperature θ, is continuous at the frost lines.

The equations are completed by boundary and initial conditions. Many situations are possible. Let us take two examples. We assume that the boundary Γ of Ω is divided into three parts Γ^+ (unfrozen), Γ^- (frozen), Γ^a (adiabatic and impermeable).

In the first example the boundary conditions are

$$\lambda_i \frac{\partial \theta_i}{\partial n} + \alpha(\lambda_i \theta_i - \bar{u}) = 0$$

$\bar{u}$ is a given function ($\bar{u} > 0$ on Γ^+ and $\bar{u} < 0$ on Γ^-)

$$m \frac{\partial h}{\partial n} + \beta(m(h+d) - \bar{H}) = 0 \text{ on } \Gamma^+, \quad m(h+d) = \bar{H} = 0 \text{ on } \Gamma^-,$$

$\bar{H}$ is a given function ($\bar{H} > 0$ on Γ^+)

$$\frac{\partial \theta_i}{\partial n} = \frac{\partial h}{\partial n} = 0 \text{ on } \Gamma_a,$$

$$e(\theta(0), \mu(0)) = e_0, \tag{20}$$

where e_0 is the given initial energy. To specify the energy is equivalent to specify the temperature and the water content.

In the second example the boundary and initial conditions are

$$\frac{\partial \theta_i}{\partial n} = \frac{\partial h}{\partial n} = 0 \text{ on } \Gamma_a, \ \lambda_i \theta_i = \bar{u}, \ m(h+\ell) = \bar{H} \text{ on } \bar{\Gamma} \cup \Gamma^+, \ e(\theta(0)\mu(0)) = e_0. \tag{21}$$

The unknowns of the problem are e,h or θ,μ and h and Ω_1, Ω_2 and Ω_3. The data are Ω, $\Gamma^+,\Gamma^-,\Gamma^a,\varepsilon,m,\ell^e,\lambda_i,c_i,\rho^s,\rho^w,\alpha,\beta,\bar{u},\bar{H},e_0,r$.

To simply the equations (15) to (21) let us define

$$u = \lambda_1\theta^+ - \lambda_2\theta^-, \ H = m(h+d), \ b_i = \frac{\rho c_i}{\lambda_i}, \ \bar{\ell} = \rho\ell, \hat{\ell} = \rho^w\ell^w,$$

$$E(u,\mu) = b_1u^+ - b_2u^- + \bar{\ell}(\mu-1) \text{ with } \mu = 0 \text{ if } u^- > 0, \ \mu = 1 \text{ if } u^+ > 0,$$

$0 \le \mu \le 1$ if $u = 0$,

$$(u^+ = \mathrm{Sup}\{u,0\},\ u^- = \mathrm{Sup}\{-u,0\}). \tag{22}$$

The new equation equivalent to equations (15) to (21) are the equations (22) and

$$E > 0,\ \frac{\partial E}{\partial t} - \Delta u = r,\ -\Delta H = 0 \text{ in } Q_1, \tag{23}$$

$$E < \bar{\ell},\ \frac{\partial E}{\partial t} - \Delta u = r,\ H = 0 \text{ in } Q_2, \tag{24}$$

$$-\bar{\ell} \le E \le 0,\ \frac{\partial E}{\partial t} - \hat{\ell}\,\Delta H = r \text{ in } Q_3, \tag{25}$$

at the frost line, we have $u_1 = u_2 = u_3 = 0$ and the transmission conditions:

$$[\mathrm{grad}u + \hat{\ell}\ \mathrm{grad}H]\cdot\vec{N} = [E]\vec{W}\cdot\vec{N}. \tag{26}$$

The boundary conditions are

$$\frac{\partial u}{\partial n} + \alpha(u-\bar{u}) = 0 \text{ on } \Gamma^{\pm},\ \frac{\partial u}{\partial n} = \frac{\partial H}{\partial n} \text{ on } \Gamma_a,$$

$$\frac{\partial H}{\partial n} + \beta(H-\bar{H}) = 0 \text{ on } \Gamma^+,\ H = \bar{H} = 0 \text{ on } \Gamma^-. \tag{27}$$

or

$$u = \bar{u},\ H = \bar{H} \text{ on } \Gamma^{\pm},\ \frac{\partial u}{\partial n} = \frac{\partial H}{\partial n} \text{ on } \Gamma_a, \tag{28}$$

The initial condition is

$$E(u(0),\mu(0)) = E_0. \tag{29}$$

The unknowns are E,H or u,μ,H. The data are $Q,\Gamma^+,\Gamma^-,\Gamma^a,\bar{\ell},\hat{\ell},b_1,b_2,\alpha,\beta,\bar{u},\bar{H}$, E_0,r.

4. MATHEMATICAL RESULTS

4.1. THERMAL MODEL

The thermal model is the Stefan problem. It is a classical problem for which many results are known. The reader is referred to papers and their

bibliography in this book (Caffarelli, Di Benedetto, Evans, Niezgodgka, Pawlow,...) and the works of Bossavit, Brezis, Caffarelli, Cannon, Damlamian, Duvaut, Fasano, Friedman, Kamenomostskaya, Lions, Primicerio, Rubinstein and many others.

4.2. THERMAL AND HYDRAULIC MODEL

As far as we know, the mathematical aspect of this model has not been widely studied. Some particular cases have been reported [2]. The mathematical results we present here are obtained by a variational formulation.

4.2.1. Some notations

Let there be φ and ψ smooth functions. We define

$$(\varphi,\psi) = \int_\Omega \varphi\,\psi\,d\Omega,\quad |\varphi| = \left(\int_\Omega \varphi^2\,d\Omega\right)^{\frac{1}{2}},$$

$$a_1(\varphi,\psi) = \int_\Omega \mathrm{grad}\varphi\ \mathrm{grad}\psi\ d\Omega + \alpha\int_{\Gamma^\pm}\varphi\,\psi\,d\Gamma,$$

$$a_2(\varphi,\psi) = \int_\Omega \mathrm{grad}\varphi\ \mathrm{grad}\psi\ d\Omega + \beta\int_{\Gamma^+}\varphi\,\psi\,d\Gamma,$$

$$a(\varphi,\psi) = \int_\Omega \mathrm{grad}\varphi\ \mathrm{grad}\psi\ d\Omega,$$

$$R(\varphi) = (r,\varphi),\ L_1(\varphi) = \alpha\int_{\Gamma^\pm}\bar{u}\varphi\,d\Gamma,\ L_2(\varphi) = \beta\int_{\Gamma^\pm}\bar{H}\varphi\,d\Gamma,$$

$$\mathcal{L} = R + L_1 + \hat{\ell}L_2,$$

and

$$\Sigma^+ = \Gamma^+\times]0,T[,\ \Sigma^- = \Gamma^-\times]0,T[,\ \Sigma^\pm = \Sigma^+ \cup \Sigma^-\ .$$

4.2.2. The variational formulation

We give the variational formulation for the boundary conditions (27); it is easy to write it for the boundary conditions (28).

The physical description of the problem introduces the set Q_1, Q_2 and Q_3. This description suitable for a simple situation, becomes too complicated for a general description. For instance the frost lines are not very well defined and they are not very easy to handle.

The variational formulation introduces more convenient new sets close to Q_i. They are related to the water content μ and not to the temperature θ.

More precisely we wish to find two open sets Q^+ and Q^- of $\bar{Q}$ and three functions u, μ and H which verify the following equations

$$\text{suppu}^+ \subset \complement Q^- \subset \text{suppu}^+ \cup Z \cup N^+, \quad u \leq 0 \text{ on } Q^-,$$

$$\text{supp}^- \subset \complement Q^+ \subset \text{suppu}^- \cup Z \cup N^-, \quad u \geq 0 \text{ on } Q^+,$$

where N^+ and N^- are two sets of measure zero and Z the set of the points where $u = 0$: $Z = \{(x,t) \in \bar{Q};\ u(x,t) = 0\}$ (suppφ is the support of the function φ).

The set Q^+ (or Q^-) is up to a measure zero set the set of the points where u is negative (or positive) plus a set where u is zero (figure 2).

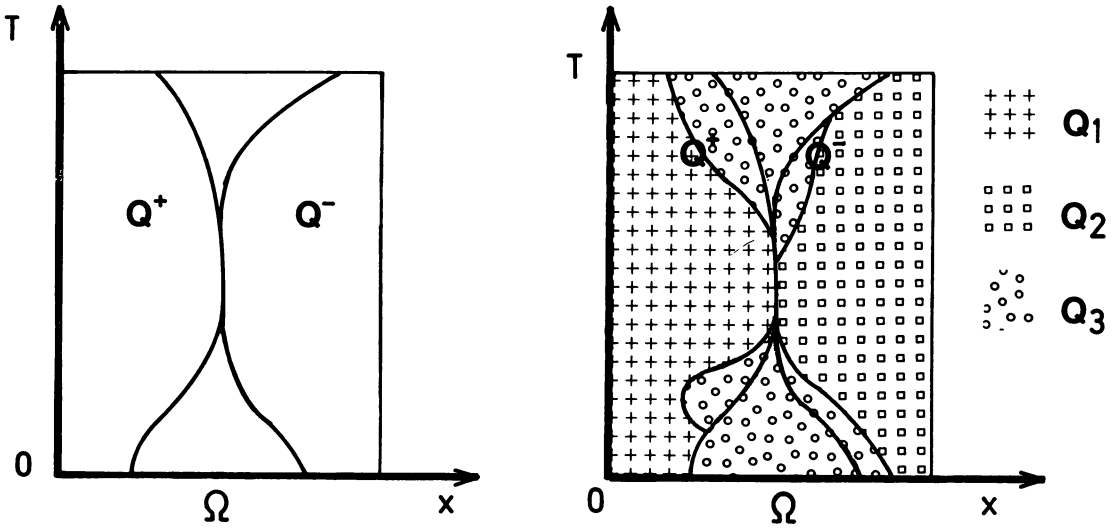

Figure 2. The zones Q^+, Q^- and $Q_1(\theta > 0)$, $Q_2(\theta < 0)$, $Q_3(\theta = 0)$.

$\mu = 1$ on Q^+, $\mu = 0$ on $\overset{\circ}{\overbrace{\text{suppu}^-}}$ (on the interior of the support of u^-).

The part Q^+ is not frozen; there can exist zones at the temperature 0 in Q^+ but the water is only liquid. The water content μ is zero where the structure is frozen. The zone where μ is between 0 and 1 is in $\complement Q^+$ where $u = 0$. Let us also remark that $Q_1 \subset Q^+$ and $Q_2 \subset Q^+$ and that the cloud Q^3 is in Q^+, Q^- and $\complement Q^+ \cup Q^-$.

Let us multiply the last equation of (23) by φ. Choose φ a smooth function with support in Q^+, we have by integration by parts

$$\int_0^T a_2(H,\varphi)\,dt = \int_0^T L_2(\varphi)\,dt.$$

Let us multiply the last equation of (23) by $\hat{\ell}$ and add to the preceding one; we obtain

$$\frac{\partial E}{\partial t} - \Delta u - \hat{\ell}\Delta H = r \text{ in } Q_1, \tag{30}$$

a relation which is true in Q_2 and Q_3. The transmission conditions (26) show that it is true in Q. Choose ψ a smooth function with $\psi(T) = 0$, let us multiply the equation (30) by ψ and integrate by parts. By using the transmission conditions (26), the boundary (27) and initial (29) conditions we obtain

$$\int_0^T \{-(E, \frac{d\psi}{dt}) + a_2(u,\psi) + \hat{\ell}a_1(H,\psi)\}dt = \int_0^T \mathcal{L}(\varphi)\,dt + (E_0,\psi(0)).$$

We replace the condition $H = 0$ in $\overset{\circ}{\text{supp}}u^- = Q_2$ by

$$\text{grad}H \mid \overset{\circ}{\text{supp}}u^- = 0,\ H|\Sigma^- = 0.$$

The two conditions are in practice equivalent. This formulation has a more practical aspect because assuming that there exist water in the frozen zone [21] it expresses that this water does not move. This was in fact the condition we wanted to express by letting $H = 0$ in Q_2.

Note The set Q^+ in this formulation replaces Q_1. In Q^+ the equation of diffusion of the water is verified, the temperature is positive or equal to zero and there is no phase change. The set $\overset{\circ}{Q}^+$ is the set where there can exist a phase change but only where the temperature is zero.

4.2.3. Some functional spaces

In the following we will need the spaces defined below. Let O be an open set of $\bar{Q}$, we define

$$\mathcal{D}(\mathcal{O}) = \{\varphi \mid \varphi \in C^{\infty}(\bar{Q}) \ ; \ \varphi|\Sigma^{-} = 0 \ ; \ \mathrm{supp}(\varphi) \subset \mathcal{O}\}$$

$$\mathcal{D}_{0}(\mathcal{O}) = \{\varphi \mid \varphi \in C^{\infty}(\bar{Q}) ; \ \mathrm{supp}(\varphi) \subset \mathcal{O}; \ \varphi|\Sigma^{\pm} = 0\},$$

$$v(\mathcal{O}) = \overline{\mathcal{D}(\mathcal{O})} \text{ for the topology of } L^{2}(0,T;H^{1}(\Omega)),$$

$$V_{0}(\mathcal{O}) = \overline{\mathcal{D}_{0}(\mathcal{O})} \text{ for the topology of } L^{2}(0,T;H^{1}(\Omega)),$$

$$W = \{\psi \mid \psi \in L^{2}(0,T;H^{1}(\Omega)) \ ; \ \frac{d\psi}{dt} \in L^{2}(Q) \ ; \ \psi(T) = 0\},$$

$$W_{0} = \{\psi \mid \psi \in W \ ; \ \psi|\Gamma^{\pm} = 0\}.$$

The proof of the theorems below uses the following lemma [18], [19].

Lemma Let there be an open set $\mathcal{O}$ and $\mathcal{O}_{n}$ on a sequence of open sets of $\bar{Q}$ which tends towards $\mathcal{O}$ for the Haussdorff topology [8], we have then

$$\forall \varphi \in V(\mathcal{O}) \ (\text{or } V_{0}(\mathcal{O})), \ \exists \varphi_{n} \in V(\mathcal{O}_{n}) \ (\text{or } \varphi_{n} \in V_{0}(\mathcal{O}_{n}))$$

such that $\varphi_{n} \to \varphi$ for the strong topology of $L^{2}(0,T;H^{1}(\Omega))$.

4.2.4. Mathematical results [13]

We give two theorems; the first one for the boundary conditions (27) and the second one for the boundary conditions (28).

Theorem 1 We assume that the open set Ω is smooth, $E_{0} \in L^{2}(\Omega)$, $r \in L^{2}(Q)$; $\bar{u}$ and $\bar{H} \in L^{2}(\Sigma^{\pm})$; $\bar{H}|\Sigma^{-} = 0$.

There exist two open sets of $\bar{Q}$, Q^{+} and Q^{-} ($F^{+} = \complement Q^{-}$, $F^{-} = \complement Q^{+}$) and three functions u, H and μ such that

$$u, H \in L^{2}(0,T;H^{1}(\Omega)), \ u \in L^{\infty}(0,T;L^{2}(\Omega)), \ \mu \in L^{\infty}(Q),$$

$$0 \leq \mu \leq 1, \ \mu|Q^{+} = 1, \ \mu|\overset{\circ}{\overbrace{\mathrm{suppu}}}^{-} = 0, \qquad \text{I}$$

$$\mathrm{suppu}^{-} \subset F^{-} \subset \mathrm{suppu}^{-} \cup Z \cup N^{-}, \ u|Q^{-} \leq 0,$$

$$\mathrm{suppu}^{+} \subset F^{+} \subset \mathrm{suppu}^{+} \cup Z \cup N^{+}, \ u|Q^{+} \geq 0,$$

where $Z = \{(x,t) \in \bar{Q} | u(x,t) = 0\}$ and N^{+}, N^{-} two zero measure sets,

$$\text{gradH} \mid \overset{\circ}{\widehat{\text{supp}u}}^{-} = 0, \quad H|\Sigma^{-} = 0.$$

If $\bar{H} \geq 0$, $H \geq 0$,

$$\forall \varphi \in V(Q^{+}), \quad \int_0^T a_2(H,\varphi)\,dt = \int_0^T L^2(\varphi)\,dt.$$

II

$$\forall \psi \in W, \quad \int_0^T \{-(E(u,\mu),\frac{d\psi}{dt}) + a_1(u,\psi) + \hat{\ell}a_2(H,\psi)\}\,dt = \int_0^T \mathcal{L}(\varphi)\,dt + (E_0,\psi(0)).$$

Theorem 2 If the assumptions of theorem 1 are verified and if there exist two functions $\mathfrak{E}$ and q and a number $\gamma > 0$ such that

$$\mathfrak{E} \in L^2(0,T;H^1(\Omega)), \quad \frac{d\mathfrak{E}}{dt} \in L^2(Q), \quad q \in L^\infty(Q), \quad \text{gradq} \in L^\infty(Q),$$

$$\bar{u}|\Sigma^{+} = \frac{\mathfrak{E}}{b_1}|\Sigma^{+} \geq \gamma, \quad \bar{u}|\Sigma^{-} = \frac{\mathfrak{E}|\Sigma^{-} - \bar{\ell}}{b_2} \leq -\gamma,$$

and

$$q|\Sigma^{\pm} = \frac{\bar{H}}{\bar{u}}, \quad \bar{H} \geq 0, \text{ q and gradq are not too large,}$$

the conclusions I of theorem 1 are verified and

$$H|\Sigma^{\pm} = \bar{H}, \quad u|\Sigma^{\pm} = \bar{u},$$

$$\forall \varphi \in V_0(Q^{+}), \quad \int_0^T a(H,\varphi)\,dt = 0,$$

$$\forall \psi \in W_0, \quad \int_0^T \{-(E(u,\mu),\frac{d\psi}{dt}) + a(u,\psi) + \hat{\ell}a(H,\psi)\}dt = \int_0^T R(\psi)\,dt + (E_0,\psi(0)).$$

Note It is important for the head of water to be greater or equal to $-d$ $(H \geq 0)$. This means that the frost line is a sink for the water and this is a physical necessity for the solution to be a realistic description of soil freezing.

4.2.5. Proof outlines of theorems 1 and 2

We use a regularization of the energy and a penalization of the condition

gradH = O. We introduce two regularization $M_n(x)$ and $P_n(x)$ of the opposite of the Heaviside function such that $|M_n'(x)| \leq nP_n(x)$.

We define the approximated energy by

$$E_n(u) = b_1 u^+ - b_2 u^- - \bar{\ell} M_n(u).$$

We solve the two approximated problems:

Find u_n and H_n such that

$$\forall \varphi \in W, \int_O^T \{-(E_n(u_n), \frac{d\varphi}{dt}) + a_2(u_n,\varphi) + \hat{\ell} a_2(H_n,\varphi)\}dt = (E_O, \varphi(O)) + \int_O^T \mathcal{L}(\varphi)\,dt,$$

$$\text{with } E_n(u_n(O)) = E_O, \tag{31}$$

$$\forall \psi \in L^2(O,T;H^1(\Omega)),$$

$$\int_O^T \{a_2(H_n,\varphi) + n^2(P_n^2(u_n)\,\text{grad}H_n, \text{grad}\psi)\}dt + n^2 \int_{\Sigma^-} H_n^2 \psi\, d\Sigma = \int_O^T L_2(\psi)\,dt, \tag{32}$$

We obtain a priori estimates by letting $\varphi = E_n$ in (31) and $\psi = H_n$ in (32) and by using the relation between M_n and P_n. The proof of theorem 1 is obtained by using the lemma, the a priori estimates and by letting $n \to +\infty$. The theorem 2 is proved by letting α and $\beta \to +\infty$, (the function q is used to obtain a priori estimates for H_n).

Notes The above thermal and hydraulic model describes well the freezing of a soil. If we want to describe thawing, it is necessary to assume that the depression d is a function of the time t. This depression which is almost constant during freezing decreases greatly during thaw (it has even discontinuities [6]). It is easy to make the necessary assumption for theorems similar to theorems 1 and 2.

5. THE HORIZONTAL DISTRIBUTION OF THE PERMAFROST

This problem of practical importance is described by Xu Xiaozu, using the Bergren formula and a statistical method to calculate the lower limit altitude of continuous and discontinuous permafrost [9], [22].

6. NUMERICAL RESULTS (Aronson, to appear)

Aronson described a variational inequality which solves the one phase quasi-static thermal model using the freezing index [10], [12]

$$u(x,t) = \int_0^t \theta(x,\tau)\,d\tau.$$

He gave results which show the regularity and dependence on the data of the solution of the variational inequality. He described a numerical method [10] to approximate the solution and mentioned analog physical problems such as Hele Shaw flow which can be used to simulate any of them.

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APPENDIX

Determination of the horizontal distribution of permafrost and of its maximum depth during seasonal thaw by Xu Xiaozu (Lanzhou Institute of Glaciology and Cryopedology - P.R.C.)

Both the range of the horisontal distribution of permafrost and its maximum depth of seasonal thaw are F.B.P. of frost-thaw interface both in horizontal and vertical directions in permafrost districts.

The changes of the two free boundaries have their duality: regularity and randomness. Namely, they exhibit a certain law of variation from one district to another, and at the same time there are some random factors piled on the background of this regular change. The problems, therefore, are much complicated.

Here we discuss how to deal with the two F.B.P. practically, by means of a mathematical analysis coupled with the use of a statistic method.

1. The existence of permafrost must follow the annual heat balance

$$Q^- - Q_g = Q^+ \tag{1}$$

where Q^-, Q^+ and Q_g denote the heat going out of the ground surface, that going into it and that coming from underground, during a year, respectively. Now let

$$\begin{aligned} Q^- &= h_f(Q+C^- t^-/2), \qquad h_f = (2\lambda^- t^- \tau^-/Q + C^- t^-/2)^{\frac{1}{2}}, \\ Q^+ &= h_t(Q+C^+ t^+/2), \qquad h_t = (2\lambda^+ t^+ \tau^+/Q + C^+ t^+/2)^{\frac{1}{2}}. \end{aligned} \tag{2}$$

Then

$$(\lambda^- t^- \tau^- (2Q+C^- t^-)/\lambda^+ t^+ \tau^+ (2Q+C^+ t^+))^{\frac{1}{2}} - Q_g^2/(\lambda^+ t^+ \tau^+ (2Q+C^+ t^+))^{\frac{1}{2}} = 1. \tag{3}$$

If we introduce the following expression for the equivalent annual mean ground surface temperature

$$\bar{t}_s = (\lambda^- t^- \tau^- + \lambda^+ t^+ \tau^+)/\lambda^- \tau_o, \tag{4}$$

we have

$$\bar{t}_s = -((\lambda^- t^- \tau^- (2Q-C^- t^-) Q_g^2)^{\frac{1}{2}}/\lambda^- Q\tau_o + Q_g^2/2Q\tau_o\lambda^- \tag{5}$$

where h_f and h_t are the frost and thaw depth of ground respectively.

C, λ, t, τ are the volume heat capacity, the heat conductivity, the temperature and the duration and signs + and - refer to liquid and solid phases, respectively. Q is the latent heat, and τ_o is the period equal to one year.

It is well known that the value of the annual mean ground temperature ($\bar{t}_o$) being equal to zero can be taken as a divisional index for dividing the perenially and seasonally frozen ground. When $\bar{t}_o = 0$, Q_g is invariably equal to 0, and from equation (5), $\bar{t}_s = 0$; thus, the value $\bar{t}_s = 0$ can be used for the same purpose.

On the premises mentioned above, and according to the good relationships between $\bar{t}_s$ and the yearly mean air temperature and between the latter and latitude and altitude, we obtained the expressions by means of statistic method for calculating the elevation of continuous and discontinuous permafrost in China.

Region	Expression for the elevation	
	continuous permafrost	discontinuous permafrost
Heilonjiang	H = (44.04 - 0.825L)/0.410	H = (42.04 - 0.825L)/0.410
Xinjiang	H = (66.50 - 1.173L)/0.516	H = (64.90 - 1.173L)/0.516
Qinghai	H = (60.12 - 1.020L)/0.562	H = (58.02 - 1.020L)/0.562
Xizhang	H = (45.24 - 0.491L)/0.572	H = (42.24 - 0.491L)/0.572

where H is the elevation in hundreds of meters and L the latitude in degrees.

2. The equation of heat balance at the frost-thaw interface

$$-\lambda^- \frac{\partial t^-}{\partial h} + {}^+ \frac{\partial t^+}{\partial h} = (Q+C^+t^+/2) \frac{dh}{d\tau} \quad . \tag{6}$$

To simplify, we neglect the gradient in the frozen part and $C^+t^+/2$ and we get

$$h = (2\lambda^+t^+\tau^+/Q)^{\frac{1}{2}}. \tag{7}$$

This is the well known Stefan formula. To rise the accuracy of (7) and to make the parameters more easily determined we put two revised coefficients in (7)

$$h = K_1 \ (2\lambda^{+} K_2 \Omega / Q)^{\frac{1}{2}} \tag{8}$$

where K_1 is the revised coefficient of depth, K_2 is the revised coefficient of conversion from the thawing index of ground surface to that of temperature and $\Omega(t^{+}, \tau^{+})$ is the thawing index of air temperature. These coefficients and parameters have been determined for the different regions.

From the above discussion, we find that mathematical analysis in combination with statistical methods is useful in order to obtain the behaviour of the frost-thaw interface in the various districts of the permafrost area. The results we obtain are useful for engineering practice.

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D DING

Physical nature of frost processes

INTRODUCTION

The prerequisite for the development of further studying the frost processes quantitatively and experimentally is to present its physical nature. The frost process is composed of some comprehensive physical processes interrelated to each other closely, so that it must be described by a group of combination equations. Modelling test must follow the similar conditions of physical processes, so it is necessary for us to set up similar criterions completely.

At present the approximate analysis solution still has its life in the study of frost processes, especially for the open system. But the different deal ways must be searched for different frost subjects.

It is worthy to give more attention to the probability applicating thermodynamics to the study of frost processes because of its advantages in the calculation of the subject with phase change and frost heave.

Symbols

X : coordinate of a given soil layer, which original point is at the ground surface, (m);

τ : time, (h_r);

r_o : dry density of soils, (kg/m^3);

r : bulk density of soils, (kg/m3);

t : temperature in soils, ($^\circ C$);

T : absolute temperature, ($^\circ C$);

a : thermal diffusivity of soils, (M^2/h_r);

: thermal conductivity of soils, ($k_{cal}./m.h_r.^\circ C$);

C : voluminal specific heat, ($k_{cal}./m^3.^\circ C$);

L : latent heat of phase change of water, ($k_{cal}./kg$);

w : water content, (%);

w : amount of migration water in unit volume of soil, (%);

k : moisture conductivity, ($m^2/h_r.$);

g : isobaric potential;

u : chemical potency;

Q : quantity of heat;

H : total heat;

u : intrinsical energy;

S : entropy;

p : pressure;

p_i : steam pressure;

v : volume;

N : composition of mass;

R : general fitting constant of gas;

C : geometric proportion coefficient;

: quantity of finalizing geometric type;

-,+: subscripts referring to the frozen and thawed state of soils;

1,2: subscripts referring to the phase of unfrozen water and ice.

PHYSICAL NATURE IN FROST PROCESSES

The exchange of energy and mass is actively undertaken between the surface layer of the earth's crust and the atmosphere, and between that and the underlying lithosphere, governing the thermodynamic condition of the surface layer itself. As the exchange reaches up to a critical state, the surface layer goes into a frozen state, and consequently forms a frozen surface layer and a series of phenomena caused by freezing. Therefore the frozen surface layer is a geological and geographical system.

Frozen soils are mainly composed of mineral particles, unfrozen water, ice (with plasticity) and gas. All the components are interrelated and interacted on each other closely, resulting in a nonhomogeneous capillary-porous texture. Especially, as there are fine grains, free ions, polarized water molecules, and ice crystals in frozen soils, so we can consider the frozen soils as a physicochemistry system.

Frost action is a comprehensively physical and chemical process. Considering the chemical transformation in soils during freezing being insignificant, the author considers the frost process as only a comprehensively physical process, which includes the transformation of heat, the phase change of water, the water migration, as well as various mechanical processes under the certain conditions of loading, temperature and moisture. Among which, the change of heat and the phase transition are primary. Each

process is independent, and the law of its change depends upon its intrinsic properties. The important thing, however, is that these processes are interrelated to each other, resulting in a complicated feedback system, and deciding the characteristics of the whole soil layer. Generally speaking, the interactions within a process tend to keep its balance, on the contrary, the actions outside the process tend to destroy the balance. And as a result of that, the frost processes irreversibly evolve to an advanced stage.

The study on frost processes includes three main aspects, i.e. the power or mechanism, the state parameters and determination of their values, and the main laws or natures of the process. Above all, the frost action is a nature-history process, and that is to say that the frost processes just develop under a certain nature-history condition, resulting in forming the geological and geomorphological phenomena in a frozen ground region. For a seasonally frozen layer, the frost processes include the changes of temperature and moisture field, the formation of ice layers, and the frost heave process.

One of the most important advances of the current studies on cryopedology is to study the frost processes quantitatively and experimentally. The study methods, to be specific, state as follows. Firstly, the state parameters of the process should be determined. For example, according to the thermodynamics view, determining the parameters of $\bar{P}$,V,N and T is sufficient and complete under the case of without external electrical (or magnetic) field action. Secondly, the mathematical model has to be established, i.e. the relationship between the time-and-space-dependent parameters and their influenced factors must be established by means of the physical-mechanical laws so as to found a set of combining equations. And finally, one has to work out the solutions by experiments and analytical methods.

For the frost processes of a seasonally frozen layer, we give the following equations. Among which, the equations referring to the heat and moisture changes, respectively, are

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} \tag{1}$$

$$\frac{\partial w}{\partial \tau} = k \frac{\partial^2 w}{\partial x^2} \tag{2}$$

the equation to the phase transition of water, and to ice or to the change of unfrozen water content is

$$dg_1 = dg_2 \ , \tag{3}$$

and the equation to the frost heave process or the amount of frost heave is

$$d\Omega = du + pdv \tag{4}$$

where g, Ω and u are all as functions of t and w, and the relations between t and w can be expressed by the heat-water balance equations at the frost-thaw interface.

$$\Sigma\lambda \frac{\partial t}{\partial x} = [Lr_o(w_h+k^+ \frac{\partial w}{\partial x}^+) + Q_c] \frac{dh}{d\tau} \tag{5}$$

$$w(h+\Delta h, \ \tau+\Delta\tau) = w_h + \Sigma k \frac{\partial w}{\partial x} \tag{6}$$

The comprehensively discussed processes cannot be dealt with by an analytic method, but the model test research is available. In the light of the nature of frost process, the views and ways of system theory are perhaps the most prospective. At present, among the analytic ways, with which the various processes can be dealt with independently, both the approximate solution and numerical solution are of great importance. The application of approximate solution will be discussed in the next section. Here we will discuss the similar criterion of model test on frost process.

From Equations (1) and (2), we obtain $a\tau/l^2$ = idem; $k\tau/l^2$ = idem. and from Equations (5) and (4), we have $\lambda t\tau/Lrwl^2$ = idem. P/Lrw = idem.

According to the "κ" theorem on dimensional analysis, there are nine physical quantities and five dimensions appeared in the above equations altogether, and, therefore, the four criterions above are complete and independent. From the criterions, we can obtain the following equations of similar multiple

$$C_l^2 = C_a C_\tau, \quad C_l^2 = C_k C_\tau, \quad C_l^2 = C_\lambda C_t C_\tau / C_L C_r C_w, \quad C_p = C_L C_r C_w \ .$$

Because of the above four restraining conditions, the similar multiples have only five variables, and the other three are determined by the above

mentioned conditions.

If the model experiments are performed on the undisturbed soil samples, then we have

$$C_\lambda = C_a = C_k = C_r = C_L = 1,$$

and from the above, we obtain

$$C_t = 1, \quad C_w = 1, \quad C_\tau = C_l^2.$$

The other similar multiples can be determined by the dimensional combination method. Here, we only give the similar multiple for stress $C_p = 1$, and that for water flow $C_i = 1/C_l$.

THE APPROXIMATE CALCULATION OF FROST PROCESSES

Frost penetration depth and amount of migrating water

If we take a section of soil layers from the ground surface (X=O) to the top of ground water capillary zone (X=H) as a system, then, according to the nature of transferring heat and moisture in the system, it can be divided into the three interrelated subsystems, i.e., the frozen soil section ($X=X_k$), the freezing soil section (X_k-h) and the thawed soil section (h-H).

To make the subject simple, the author proposes the premises as follows

1. The whole system is homogeneous, the heat transmission takes conduction as the main way, the thermal effect during water migration is reflected only in the latent heat of phase transition of water, and, at the same time, the heat change within the frozen soils is taken into consideration.

2. The temperature at the upper and lower boundary of the system, t_o and t_H, are taken their average values during the freezing process, the temperature at the interface of phase change of water, t_h, is $0^\circ C$, and the change of temperature in both the frozen and thawed regions (t^- and t^+) follow the linear laws

$$t^- = - t_o/h \cdot (x-h) \; ; \quad t^+ = t_H/(H-h) \cdot (x-h) \cdot \tag{2.1}$$

3. The moisture migration takes place in both the freezing and thawed soil regions. The upper boundary of the freezing region, X_k, is determined

by the level of the temperature t_k, which is corresponding to the critical value (w_k) of unfrozen water in the temperature field

$$X_k = (t_o - t_k)/t_o \cdot h \ .$$

The water content (w_H) at the lower boundary is a constant, and the water content at the interface of phase change of water is $w_h = w_o - w$, where w_o being the initial water content, and w the amount of water flowing out of the system during thawing. The bonding water content (w_{kp}) of the soils is deducted from the total water content. The water contents in the two regions, w^- and w^+, change lineally as follows

$$\begin{aligned} w^- &= w_k + (w_h - w_k)/(h - X_k) \cdot (X - X_k); \\ w^+ &= w_H - (w_H - w_h)/(H - h) \cdot (H - x). \end{aligned} \tag{2.2}$$

Under the above premises, the balance of heat and moisture at the interface of phase change of water can be described by the equations

$$\lambda^- \frac{\partial w^+}{\partial x} - \lambda^+ \frac{\partial t^+}{\partial x} = [Lr_o(w_h + \Delta w) + \tfrac{1}{2} C^- |t_o|] \frac{dh}{d\tau} \ . \tag{2.3}$$

$$k^+ \frac{\partial w^+}{\partial x} - k^- \frac{\partial w^-}{\partial x} = \Delta w \frac{dh}{d\tau} \ . \tag{2.4}$$

Considering the equations (1) and (2), and letting

$$a = \lambda^- |t_o| \ ; \quad b = \lambda^+ t_H + Lr_o F \ ;$$

$$C = Lr_o w_h + C^-/2.|t_o| \ ; \quad E = k^- t_o/t. \ (w_h - w_k) \ ;$$

$$F = k^+ (w_H - w_h).$$

we obtain the equation describing the rate of frost penetration from equations (2.3) and (2.4)

$$dh/d = [aH - (a + b) h] /h (H - h) c. \tag{2.5}$$

Integrating the equation (2.5), we obtain the equation for determining

the depth of frost penetration

$$\tau = -C\left[\frac{bH}{(a+b)^2} \cdot h - \frac{h^2}{2(a+b)} + \frac{abH^2}{(a+b)^3} \cdot \ell n\ (1 - \frac{a+b}{aH} h)\right], \tag{2.6}$$

and that for determining the amount of migrating water

$$Q = wr_o = r_o C \frac{(E+F)h-Eh}{aH-(a+b)h} \quad . \tag{2.7}$$

From equations (2.3) and (2.4), it can be seen that the applicable range of the equations (2.6) and (2.7) is

$$E/(E+F) \cdot H \leq h \leq \lambda^- |t_o| / (\lambda^- |t_o| + \lambda^+ t_H) \cdot H \tag{2.8}$$

If H is very great, and the moisture migration in the freezing region is not taken into consideration, we have the revised Stephen formula:

$$h = [2\lambda^- |t_o| \tau / (Lr_o w_o + \tfrac{1}{2} C|t_o|)]^{\frac{1}{2}} \tag{2.9}$$

The state forming the ice layer can be analysed by equations (2.6) and (2.7), and from the full condition forming the ice layer, $r_o w_h + Q \geq r_o w_B$, we can obtain the location where the ice layer is formed.

$$h^1 \geq [a(w_B - w_h) + CE] / [C(E+F) + (a+b)(w_B - w_h)] \cdot H, \tag{2.10}$$

and, furthermore, obtain the thickness of the formed ice layer equals

$$L = r_o/\bar{r}_w [1.09C\ r_o/r_w \int_{h'}^{h} \frac{(E+F)h-EH}{aH-(a+b)h}\ dh + \int_{h'}^{h} (0.09w_h - w_B)\ dh]$$

$$= r_o/\bar{r}_w [(w_B - 0.09w_h - 1.09C \frac{E+F}{a+b})(h'-h) + 1.09C\ (\frac{EH}{a+b} - \frac{(E+F)aH}{(a+b)^2}) \cdot \tag{2.11}$$

$$\ell n\ (\frac{aH-(a+b)h}{aH-(a+b)h'})]$$

Unfrozen water content and amount of frost heave

As for the problem of phase change of water in frost process, from the point of view of cryopedology, it is considered as the problem of the formation and distribution of ice, and from that of physics, it is considered as the

problem of the balance and transformation of bonding water, unfrozen water and ice in the soil system. This problem is discussed under following premises: Soils system is homogeneous, ice is transformed from pure water (including the migrating water), and the change of gasses in this process is negligible. So the ice in pore of this system possesses the same properties as it being in free state. The nature of unfrozen water is governed by the features of particle surface, and is characterized by magnetic parameters (thermodynamic parameters). And the discussed problem is turned into the problem of phase transition of water-to-ice in the case of without external electrical or magnetic field.

According to equations (1-3), we have the following relation

$$(\partial g_1/\partial T)_{pN}dT + (\partial g_1/\partial p)_{TN}dp + (\partial g_1/\partial N)_{Tp}dN = (\partial g_2/\partial T)_{pN}dT + (\partial g_2/\partial p)_{TN}dp + (\partial g_2/\partial N)_{Tp}dN. \tag{2.12}$$

According to the well-known thermodynamic relation, equation (2.12) can be written as

$$(S_2 - S_1)dT + (V_1 + V_2)dp + (U_1 - U_2)dN = 0 \tag{2.13}$$

For the temperature and applied pressure being constant, we have

$$(\partial N/\partial p)_T = (V_1 - V_2)/(U_2 - U_1) = -\Delta V/\Delta U. \tag{2.14}$$

$$(\partial N/\partial T)p = (S_2 - S_1)/(U_2 - U_1) = \Delta S/\Delta U. \tag{2.15}$$

Since $\Delta S = \Delta H/T$, and $\Delta U = RT\ \ell n(P_2/P_1)$, we obtain

$$(\partial N/\partial T)_p = \Delta H/RT^2\ \ell n(P_2/P_1)\ ; \tag{2.16}$$

$$(\partial N/\partial P)_T = -\Delta V/RT\ \ell n(P_2/P_1)\ . \tag{2.17}$$

In equations (2.16) and (2.17), because of $\Delta H < 0$, $P_2 < P_1$, and $\Delta V > 0$, so $(\partial N/\partial T)_p > 0$, $(\partial N/\partial P)_T > 0$. Thus, unfrozen water content increases with the temperature or pressure increasing, i.e. in this case ice is transformed into unfrozen water.

Integrating equations (2.16) and (2.17), we can obtain the relations for determining the unfrozen water content (or ice content) in soils at a certain temperature and pressure. At present, however, there are some difficulties in the experimental determinations of a few of necessary parameters.

If a soil layer suffered to freezing, because of the moisture redistribution (migration), ice formation and pressure produced during the frost process, the volume and density of the soil layer must have changed, and it is called the frost heave of soils, and the corresponding frost heave increment in the vertical direction is so-called the amount of frost heave.

The increase of ice volume is the necessary condition for producing the frost heave deformation, and the full condition is that the increase of ice volume must be greater than the sum of the decrease of pore volume in the soil and the compressive amount of the underlying thawed soils during freezing.

For a semi-infinite ground body suffered an external load of p, if we take the ice within the soil layer of the thickness of dh which is taking place the phase change as a system, and the upper and lower soil layers as surroundings, the first thermodynamic law must take the form of equation (1.4) where

$$du = \Delta u r_o (w_h + \Delta w)\, dh \tag{2.18}$$

in which, Δu is the intrinsical energy increment of the unit weight of water during phase change; and

$$dQ = \Delta H r_o (w_h + \Delta w)\, dh \tag{2.19}$$

in which, ΔH is the entropy increment of unit weight of water during phase change.

ΣPdV is the sum of the volume expansion work, and it includes the following two parts:

$P_1 dV_1$ is the expansion work of the system done by overcoming the soil weight and the applied load, and in which includes the expressive deformation work of the underlying soil layer, and that is

$$p_1 dV_1 = [P + r_o (1 + w_h + \Delta w)] d\ell \tag{2.20}$$

in which, ℓ is the deformation of the soil layer, i.e. the amount of frost heave of the soil layer.

And $P_2 dV_2$ is the expansion work of the system done by expanding to the pore of soil, and we have

$$p_2 dV_2 = [P + r_o (Hw_h + \Delta w)] \Delta n dh \tag{2.21}$$

in which, Δn is the change of pore volume of the soil during freezing, and we have

$$\Delta n = -[\ell/h - (1.09\Delta w + 0.09 w_h)]$$

Substituting the equations (2.18) - (2.21) in equation (1.4), we obtain the relation for determining the frost-heaving rate.

$$d\ell/dh = \frac{(\Delta u + \Delta H) r (w + \Delta w)}{p + r_o (1 + w_h + \Delta w) h} + \ell/h - (1.09\Delta w + 0.09 w_h) \tag{2.22}$$

Integrating the above relation, we obtain the equation for determining the amount of frost heave

$$\ell = \int_o^h d\ell/dh \, dh \tag{2.23}$$

If it is known that how the w varies with depth, the above equation can be solved.

For saturated soils (h=0), the amount of migrating water is a constant, and then the amount of frost heave can be calculated by

$$\ell = \frac{\Delta u - \Delta H}{L + w_h + \Delta w} \ln \left(\frac{p + (1 + w_h + \Delta w) r_o h}{p} \right) . \tag{2.24}$$

CONCLUSIONS

In this paper a mathematical model describing the frost processes in principle has been presented after the analysis of physical nature in frost processes; a set of similar criterions, used for modelling test of frost processes, has been founded on the basis of a group of equations, and the relationship of similar multiple, used for undisturbed solids in modelling test, has been also given out.

For the open system with the constant under-ground water level, the obtained approximative solution, used for calculating the freezing penetration depth and the quantity of moisture migration, is clear in physical conception. The formula of unfrozen water content, obtained by thermodynamical method, can reflect the effects of temperature, pressure and soil properties, and the formula of calculating the amount of frost heave by the same method is founded under the consideration of factors of porosity change, solidification of underlying soils, etc.

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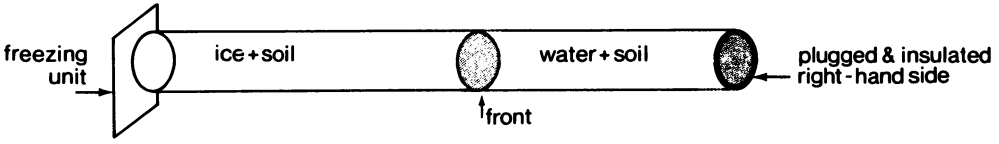
R B GUENTHER

Freezing in a bounded medium

1. INTRODUCTION

In constructing mathematical models describing freezing processes, one is usually led to one or two phase Stefan problems. If, however, the density of the frozen material is strictly less than that of the thawed material and if the process is taking place in a restricted medium, then the standard assumptions leading to a Stefan problem are no longer valid and the model must be modified. Furthermore, in the case of water or a water-saturated soil, the freezing point is pressure dependent.

In this article we consider the case of an insulated tube, half of which is initially filled with frozen soil and the other half with soil and water. The process will be regarded as one dimensional and the freezing accomplished by keeping the left hand side at a small temperature. The right hand side of the tube is insulated and no mass is allowed to escape from the tube.



It will be assumed that the soil-water phase is compressible but that the ice-soil phase is incompressible and that the density of the solid phase is constant and less than that of the unfrozen phase.

As the freezing proceeds, then, the unfrozen mass is compressed into a smaller and smaller volume. Because of the density difference, the density of the unfrozen material will increase and pressures will build up. The freezing point will decrease. At some point in time, the pressures will either become so large that the ice-plug will be pushed back or the freezing process will come to a stop. It is this process which, under certain additional assumptions, will be modelled here.

2. THE MATHEMATICAL PROBLEM

Suppose that the length of the pipe is two units. Set $x = s(t)$ denote the location of the ice-water at time t and suppose $s(o) = 1$. In the sequel, the subscript i will refer to the frozen phase and w to the unfrozen. In the ice-soil phase we assume the physical parameters c_i, the specific heat, ρ_i, the density, and k_i, the thermal conductivity to be constant and arrive at the equation

$$\frac{\partial u_i}{\partial t} = a_i \frac{\partial^2 u_i}{\partial x_i^2}, \qquad a_i = \frac{k_i}{c_i \rho_i}, \tag{2.1}$$

governing the temperature u_i in the solid phase. Initial and boundary conditions at $t = 0$, $0 < x < 1$ and $x = 0$, $t > 0$ must also be prescribed. The rest of the equations are all closely related. Beginning with the simplest, we assume that the temperature is continuous at the front so that

$$u_i(s(t),t) = u_w(s(t),t) = T(p), \tag{2.2}$$

where $u_w(x,t)$ represents the temperature of the water-soil phase and $T(p)$ represents the temperature of the freezing point at the pressure p. It will be assumed that $T(p)$ is a twice continuously differentiable function of p and that it is decreasing. In the applications T can be represented quite well by an exponential function of the form

$T(p) = -\alpha\exp(\sigma p) + \alpha$, α, σ positive constants, over a wide pressure range.

Next, the usual energy balance across the front is assumed and one has

$$\dot{s}(t) = C\left[k_i \frac{\partial}{\partial x} u_i(s(t),t) - k_w \frac{\partial u_w(s(t),t}{\partial x}\right]. \tag{2.3}$$

Here $C = \frac{1}{\lambda \rho_i}$, λ being the latent heat, is assumed constant and $C > 0$. Differentiation with respect to t will be denoted by a dot. To obtain additional equations, an expression for the density of the water-soil medium is needed. Let $\rho_w = \rho$ denote the density of the water-soil medium. It will be assumed that the tube is sufficiently small that ρ depends only on t and not on the space coordinate. The initial mass is given by

$$m = \rho_i + \rho(0)$$

and since the mass remains constant, at time t it will be

$$m = \rho_i s(t) + \rho(t)(2 - s(t)),$$

so that

$$\rho(t) = \frac{m - \rho_i s(t)}{2 - s(t)} . \tag{2.4}$$

Since $\rho_i < \rho(0)$, it is obvious that $\rho(t)$ tends to infinity as $s(t)$ approaches 2. The compressibility of the unfrozen medium is now taken into account by assuming a linear relationship between the pressure and density

$$\rho(t) = \rho_o + \gamma p, \quad \rho_o = \rho(0), \tag{2.5}$$

where γ is the coefficient of compressibility and $p = 0$ is atmospheric pressure.

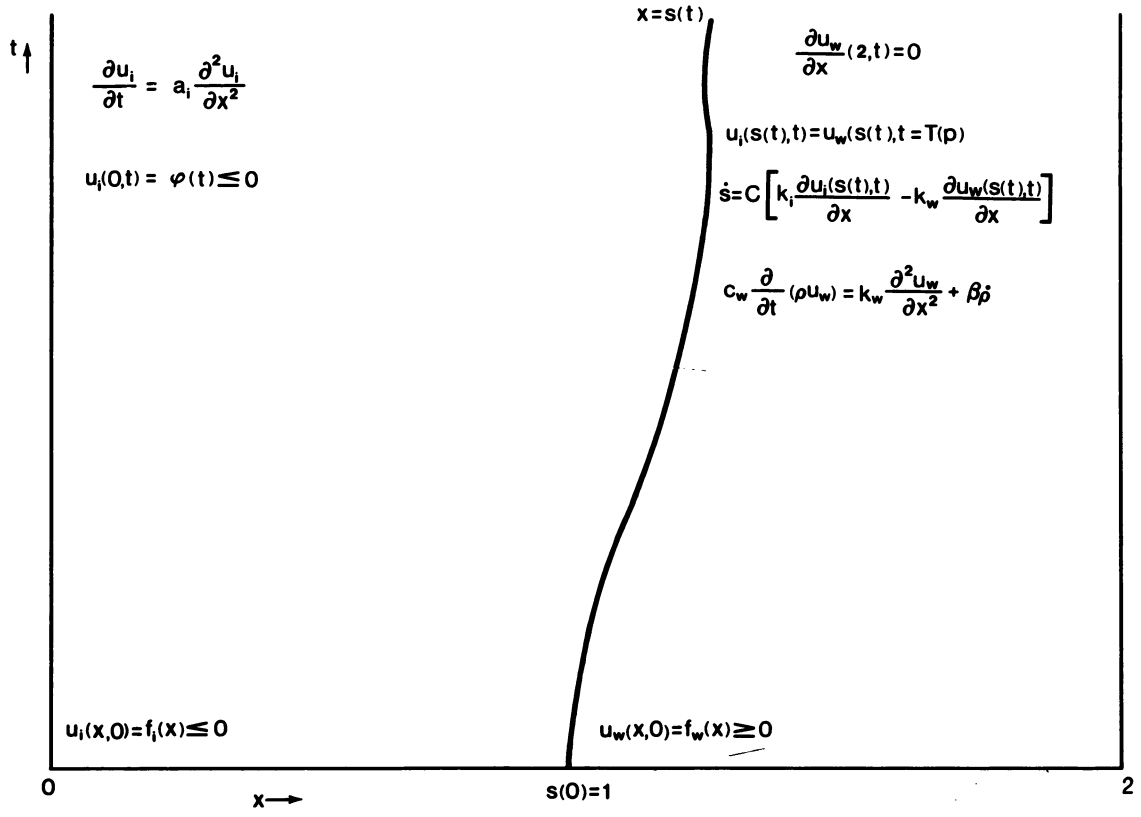
In order to derive an equation for the temperature of the water-soil phase, one carries out the usual energy balance. However, due to the fact that the medium is being compressed, a certain amount of heat will be released. The rate at which heat is released due to compression per unit length comes in as a source term and we assume that it is proportional to $\dot{\rho}(t)$. The convection term due to the motion of the medium will be ignored. The equation governing $u_w(x,t)$ is thus

$$c_w \frac{\partial}{\partial t} (\rho(t) u_w) = k_w \frac{\partial^2 u_w}{\partial x^2} + \beta \dot{\rho}, \tag{2.6}$$

where c and k_w are constants. Initially u_w is assumed to be known and at $x = 2$, $\frac{\partial u_w}{\partial x}$ vanishes. Schematically, we seek functions $u_i(x,t)$, $u_w(x,t)$, $s(t)$, $\rho(t)$, $p(t)$ such that

$$\rho(t) = \rho_o + \gamma p(t)$$

$$m = \rho_i + \rho_o, \ \rho_i < \rho_o ,$$



$$\rho(t) = \frac{m - \rho_i\, s(t)}{2 - s(t)}$$

$$T = T(p) \text{ with } T(0) = 0 \text{ given.}$$

The $f_i(x)$ and $f_w(x)$ refer, respectively to the initial conditions for u_i and u_w, of course, and $\varphi(t)$ to the boundary condition for u_i at $x = 0$.

3. EXISTENCE AND UNIQUENESS THEORY

The method of solution is based upon integral equations and as a consequence only the salient steps will be given here. The first step is to derive a set of integral equations which are equivalent to the differential equations. Assume, therefore, that a solution exists. Let for $t > t$ fixed

$$D_i = \{(x,t) : 0 < x < s(t), \quad 0 < t < t\}$$

$$D_w = \{(x,t) : s(t) < x < 2, \quad 0 < t < t\}$$

Further, set $\mu(t) = \dfrac{\partial u_i}{\partial x}(s(t),t)$ and $\nu(t) = \dfrac{\partial u_w}{\partial x}(s(t),t)$.
Next, let

$$k(x,t) = (4\P t)^{-\frac{1}{2}} \exp(-x^2/4t)$$

denote the fundamental solution for the heat equation and introduce the functions

$$G^i(x,t;\, y,\T) = k(x-y,\, a_i(t-\T)) - k(x+y,\, a_i(t-\T))$$

$$N^i(x,t;\, y,\T) = k(x-y,\, a_i(t-\T)) + k(x+y,\, a_i(t-\T)).$$

Integrating Green's identity with u_i and G^i over D_i yields the representation for u_i

$$u_i(x,t) = \int_0^l f_i(y) G^i(x,t;y,0)\,dy + a_i\int_0^t G^i_y(x,t;0,\T)\varphi(\T)\,d\T$$

$$+ \int_0^t T(p)\; G^i(x,t;s(\T),\T)\dot{s}(\T)\,d\T$$

$$+ a_i\int_0^t [\mu(\T)\; G^i(x,t;s(\T),\T) - T(p)\; G^i_y(x,t;s(\T),\T)]\,d\T.$$

Differentiate with respect to x and let x tend to s(t) from the left. Assuming that f_i and φ are differentiable and that $f_i(0) = \varphi(0)$ yields

$$\frac{1}{2}\mu(t) = \int_0^l f_i(y) N^i(s(t),t;y,0)\,dy - \int_0^t N^i(s(t),t;0,\T)\varphi'(\T)\,d\T$$

$$+ \int_0^t T'(p)\dot{p}N(s(t),t;s(\T),\T)\,d\T + a\int_{io}^t \mu(\T) G^i_x(s(t),t;s(\T),\T)\,d\T. \tag{3.1}$$

To obtain a similar formula for $u_w(x,t)$ and hence for $\nu(t)$, it is simplest to make a change of dependent variable and set

$$w(x,t) = c_w\rho(t)u_w(x,t) - \beta\rho(t).$$

Then w satisfies the differential equation

$$\frac{\partial}{\partial t} w = a(t) \frac{\partial^2 w}{\partial x^2}, \quad a(t) = \frac{k_w}{c_w\rho(t)} .$$

Now set

$$\omega(t) = w(s(t),t)$$

and

$$G^w(x,t;y,T) = k(x-y, A) - k(x+y-4, A),$$

$$N^w(x,t;y,T) = k(x-y, A) + k(x+y-4, A)$$

with

$$A = \int_T^t a(\theta)\,d\theta.$$

Apply Green's identity with w and N^w and use the fact that $\frac{\partial w}{\partial x}$ vanishes at $x = 2$ to obtain the representation

$$w(x,t) = \int_1^2 w(y,0)N^w(x,t;y,0)\,dy - \int_0^t \omega(t)N^w(x,t;s(T),T)\dot{s}(T)\,dT$$

$$- \int_0^t a(T)w_y(s(T),T)N^w(x,t;s(T),T)\,dT$$

$$+ \int_0^t a(T)\omega(T)\, N_y^w(x,t;s(T),dT.$$

Differentiate with respect to x and let x approach s(t) from the right to obtain after an integration by parts and assuming that $f_w'(2) = 0$ to obtain

$$\frac{k_w c_w \rho(t)}{2} \nu(t) = k_w c_w \rho_o \int_1^2 f'(y)\, G^w(s(t),t;y,0)\,dy$$

$$- k_w^2 \int_o^t \nu(T)\, N_x^w(s(t),t;s(T),T)\,dT \qquad (3.2)$$

$$- k_w \int_o^t \dot{\omega}(T)\, G^w(s(t),t;s(T),T)\,dT.$$

Integrating the differential equation for s(t) yields an integral equation for s(t). (3.1) and (3.2) yield after some rearrangements

$$\mu(t) = 2\int_o^1 N^i(s(t),t;y,0)\, f_1'(y)\,dy - 2\int_o^t N^i(s(t),t;0,T)\omega'(T)\,dT$$

$$+ 2\int_o^t T'(p)\dot{p}(T) N^i(s(t),t;s(T),dT$$

$$+ 2a_i \int_o^t G_x^i(s(t),t;s(T),T)\mu(T)\,dT$$

and

$$\nu(t) = \frac{2c_w\rho_o a(t)}{k_w}\int_1^2 G^w(s(t),t;y,0) f_w'(y)\,dy$$

$$- 2a(t)\int_o^t N_x^w(s(t),t;s(T),T)\nu(T)\,dT$$

$$- \frac{2a(t)}{k_w}\int_o^t \dot{\omega}(T)\, G^w(s(t),t;s(T),T)\,dT.$$

This set of integral equations was obtained assuming that a solution to the system of differential equations exists. The converse is also true; namely, a solution to the system of integral equations will also yield a solution to the system of differential equations, in other words the two problems are equivalent. Assuming now that $f_i(x) \le 0$ and $f_w(x) \ge 0$ are twice continuously

differentiable and $\varphi(t) \leq 0$ once with $f_i(1) = f_w(1) = 0$, $f_i(0) = \varphi(0)$, $f_w'(2) = 0$ and that T is twice continuously differentiable, $T(0) = 0$ and $T'(p) \leq 0$, one can prove the following theorem by iteration.

Theorem There exists a value $\tilde{t} > 0$ depending on the data such that for $0 \leq t \leq \tilde{t}$ there exists a unique solution to the system of integral equations and hence to the original free boundary value problem.

4. CONCLUDING REMARKS

Several of the assumptions leading to the above model are not entirely realistic; in particular, the assumption that p and ρ are functions of t alone and independent of x should be dispensed with. Nevertheless, numerical experiments with the data chosen so that $\rho_i < \rho(0)$ and $s(0) > 0$ indicate that the pressures do in fact build up as they should as the front moves to the right.

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J R OCKENDON

Generalised phase-change problems

This discussion group was inaugurated by the presentation to the organisers by Profs. Kamin and Rubinstein of a small bell, the "Lucca Bell". This was intended both as a token of appreciation for the way Profs. Fasano and Primicerio had arranged the meeting and as a means of controlling the noisier participants at the discussion groups.

The presentations at this consequently orderly session all concern models which reduce to the classical Stefan problem in special cases. The different generalisations are motivated either by shortcomings in the formulation of this classical problem or by the need to model more complicated and realistic physical situations.

In its simplest form, the classical Stefan problem is for a function $u(\underset{\sim}{x},t)$ which satisfies

$$\rho c \frac{\partial u}{\partial t} = \text{div } (k \text{ grad } u) \tag{1}$$

except at a smooth unknown interface $f(\underset{\sim}{x},t) = 0$. At this moving boundary

$$u \text{ is continuous and equal to zero} \tag{2}$$

but u has a jump in its normal derivative

$$[k \text{ grad } u. \text{ grad } f]^{+}_{-} = -L \frac{\partial f}{\partial t} \tag{3}$$

where $[\ ]^{+}_{-}$ refers to a jump crossing $f = 0$ from the side in which $u < 0$ to the side in which $u > 0$; ρc, k and L are prescribed positive constants. Also u satisfies some initial condition compatible with $f(\underset{\sim}{x},0) = 0$, and boundary conditions at prescribed boundaries.

This moving boundary problem is so important because, with various choices of the parameters, (1)-(3) may be the simplest model for any of the following situations: u could be the temperature in a heat conducting material undergoing a phase change, a concentration in a liquor in which a crystal with boundary $f = 0$ is dissolving, the pressure in a Hele-Shaw cell containing a

contracting cavity f = O, the potential in an electrochemical machining process, the pressure in the saturated part of a porous medium, the remainder of which is dry, or the probability density associated with an optimal stopping time problem.

In spite of this wide applicability, the existence of u is still an open question in two or more space dimensions $\underset{\sim}{x}$, although the weak solution which is naturally suggested by the heat conduction problem formulated in terms of the enthalpy exists and is unique. Also, blow up, and non-existence may occur even in one dimension when L < O, which is the case for superheated or supercooled phase change problems, expanding cavities in Hele-Shaw flow, etc. Another limitation on the use of (1)-(3) is that common situations say in metallurgy and nuclear engineering involve both coupled heat and mass transfer phenomena and surface effects such as surface tension and energy sources at f = O, none of which are catered for in (2), (3). The following papers are addressed to ways of overcoming some of these deficiences.

A glimpse of the possible behaviour of two-dimensional solutions of (1)-(3) when L < O can be obtained by considering perturbation solutions when c = O, f = O is nearly a plane and the problem is a "one phase" problem in which u ≡ O in f > O. We write

$$f \sim x - Vt - \varepsilon e^{\sigma t} \sin y + \ldots, \quad \varepsilon << 1, \quad n > O, \quad V < O, \tag{4}$$

and

$$u \sim -\frac{LV}{k}(x-Vt) + \varepsilon A e^{\sigma t} \sin y \, e^{n(x-Vt)} \ldots, \tag{5}$$

thinking of u as the temperature of a supercooled liquid out of which a heat flux LV is removed at large distances, $x-Vt \to -\infty$. Expanding formally and equating coefficients of ε yields

$$\sigma = -nV \tag{6}$$

so that the growth of "fingers" is possible as in Figure 1. Physically, the heat flow from the points marked A in the transverse y direction increases as the fingers grow longer. This result, which predicts that fingers of arbitrarily small wavelength will grow arbitrarily rapidly is not altered quantitatively by the reintroduction of a heat capacity c but it is affected by a surface tension Γ. If this is modelled by replacing (2) by

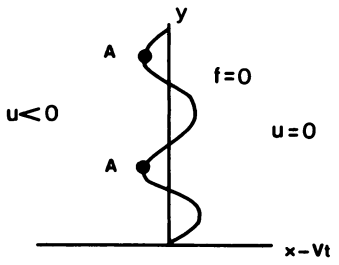

Figure 1

$$u = \kappa\,\Gamma, \kappa \text{ = curvature of } f = 0, \tag{7}$$

then in the stability analysis of the one phase problem, (6) becomes

$$\sigma = -nV - k\Gamma n^3. \tag{8}$$

There is now a "most dangerous" wavelength $2\pi\sqrt{-k\Gamma/V}$ and very short wavelength fingers are damped.

This naive localised linear analysis has been put on a more satisfactory basis by including weakly nonlinear effects [5] and also by considering the global stability of a nearly one-dimensional initial value problem [4]. Another important global effect, that of the "unperturbed" boundary curvature say for a growing cylindrical or spherical crystal, was considered at the meeting by Prof. Chadam, who presented several interesting solutions of the linear stability equations. Some special solutions valid over appreciable time intervals were discussed but arbitrarily growing nearly spherical crystals could only be analysed locally in time. In both cases a critical crystal radius was found to exist below which the one phase Stefan problem (1), (3), (7) was stable. Such a result is consistent with (8) because the unstable low wavelengths predicted by (8) will not exist on a sphere of sufficiently small radius.

In connection with modelling surface effects in phase change problems, numerous modifications have been suggested both to (2) and (3), [3]. Most of these models are not obviously amenable to any mathematical theory. However, the possibility of modelling surface tension as an energy conservation statement together with an entropy maximisation condition, which would have

considerable mathematical advantages over (7) is discussed in the subsequent paper by Prof. Rogers.

Stability considerations also play an important role in the modelling of alloy solidification which, in the simplest case of a dilute binary alloy, involves the solution of a vector system of parabolic equations (1). Here the components of the unknown vector, namely the concentration and temperature, have to be coupled by new experimental evidence. This is conventionally taken to be the assumption of thermodynamic equilibrium between the phases at the phase boundary and is explained very clearly in the paper by Drs. Wilson and Solomon. However, a new mathematical problem immediately arises. The above-mentioned existence and uniqueness features of the weak solution of the scalar Stefan problem cannot be carried over to the vector system in any obvious way. This has not prevented the definition of ad hoc weak solutions in special cases and numerical experimentation with these definitions has proved encouraging, as shown by our two alloy solidification papers. Indeed, the paper by Profs. Bermudez and Saguez contains some very useful results about the discretisation of one "weak formulation", even though the solution of this formulation may not exist! These two papers can be construed as a plea for more mathematicians to work on the weak solutions of vector diffusion equations.

Another intriguing possibility, which is touched upon in the two alloy papers, is that of the stabilisation of what would be unstable solutions of the classical alloy model by making allowance for the existence of "mushy regions". Such regions can be envisaged most easily in the scalar problem (1) when u is interpreted as a temperature and the phase change is brought about by volumetric heating. This results in a nonhomogeneous field equation and any solution with a sharp phase boundary can be seen necessarily to have a superheated solid region, in which the temperature exceeds zero. Such a sharp boundary would be unstable by an argument similar to (6), but it has been suggested in [1] that the weak solution does not have a sharp boundary, at least initially. Instead u vanishes on a region of the (x,t) plane in the one-dimensional case, and Prof. Rogers has made allowance for this possibility when surface tension effects are included.

When the linear stability calculation (6) is carried out for the classical alloy solidification model, the "Mullins-Sekerka" instability is found if material regions exist which, instead of being simply supercooled in the

monocomponent situation of Figure 1, are in the unstable or "constitutionally supercooled" regions of the equilibrium diagram (Figure (2) of the paper by Dr. Wilson). Now alloy solidification can be observed with at least three phase boundary morphologies. For very slow and carefully controlled solidification, the boundary can be smooth and almost planar but with faster freezing with many nucleation sites the regular or irregular dendritic patterns of Figures 6 and 7 of Prof. Szekely's paper may be observed. These last two indicate instability of a planar phase boundary. It would be a triumph for mathematics if "mushy regions" could be obtained using the ad hoc weak formulations mentioned above and used to describe, even crudely, these two situations.

We have mentioned that the introduction of volumetric heating into the classical model (1) does not effect the mathematical theory of the weak solution and the associated numerical schemes in any serious way. However, the effect of a surface energy source, as in an ablation problem, makes the weak solution approach impossible in more than one space dimension. Problems involving such effects do have to be solved in practice, in the field of reactor safety amongst others, and Dr. Turland's paper describes two possible numerical schemes, inevitably of the front-tracking type. At present, very few convergence or stability results are available for such schemes, and the only available yardstick for comparative assessment is their performance on test problems.

The use of these schemes is also discussed by Dr. Peckover in a paper which also mentions a fascinating but almost untouched generalisation of the Stefan problem. This the problem of regelation, where the phase change temperature is not constant because sufficiently large pressures are applied, say by pulling a thin wire through ice. A decription of a model for this situation is given in [2]; because the molten layer is very thin, the Navier-Stokes equations for the velocity can be solved explicitly under the "lubrication approximation". This velocity is a functional of the phase boundary position so that back substitution into the energy equation can give a Stefan problem in which the phase boundary appears explicitly in the field equation. Such a model may be applicable not only to the ball and wax problem described by Dr. Peckover, but also to geological problems involving magma migration.

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A BERMUDEZ & C SAGUEZ

Mathematical formulation and numerical solution of an alloy solidification problem

INTRODUCTION

The problem of solidification of a binary alloy differs from the classical two-phase Stefan problem by the following points:

- the temperature of solidification is unknown and depends on the concentration,
- we have a coupled system (temperature - concentration),
- the concentration is discontinuous along the interface liquid-solid.

By using the chemical activity, as in A. B. Crowley - J. R. Ockendon [3], G. J. Fix [4], a mathematical formulation close to quasi-variational inequalities is obtained. An interpretation of this formulation, when a mushy region exists, is given.

A problem discretized in time is introduced. By using the properties of monotonicity of the system, we prove, in a constructive way, the existence of maximal and minimal solutions. A numerical algorithm is directly deduced and numerical results are presented.

I. PHYSICAL PROBLEM AND MATHEMATICAL FORMULATION

I.1. Physical equations

Let Ω be an open bounded subset of $\mathbb{R}^n$, $\Omega_1(t)$ (resp. $\Omega_2(t)$) the liquid (resp. solid) region and $S(t)$ the free-boundary (interface liquid-solid).

Let θ_i (resp. c_i) denote the temperature (resp. the concentration) in $\Omega_i(t)$ $(i=1,2)$. We have the following equations

$$\rho k \frac{\partial \theta_i}{\partial t} - \operatorname{div}(\alpha_i \operatorname{grad}\theta_i) = 0 \text{ in } \bigcup_t (\Omega_i(t) \times \{t\}) \tag{1.1}$$

$$\frac{\partial c_i}{\partial t} - \operatorname{div}(\beta_i \operatorname{grad} c_i) = 0 \text{ in } \bigcup_t (\Omega_i(t) \times \{t\}) \tag{1.2}$$

Along the free-boundary, we have the conditions:

$$(\alpha_1 \overrightarrow{\operatorname{grad}\theta}_1 - \alpha_2 \overrightarrow{\operatorname{grad}\theta}_2).\vec{n} = -\rho L \vec{V}.\vec{n} \tag{1.3}$$

$$(\beta_1 \overrightarrow{\text{gradc}_1} - \beta_2 \overrightarrow{\text{gradc}_2}) . \vec{n} = -(c_1 - c_2)\,|_{S(t)}\ \vec{V}.\vec{n} \tag{1.4}$$

$$\theta_1 = \theta_2|_{S(t)} \tag{1.5}$$

$$c_1 = \sigma_1 \theta_1|_{S(t)} \tag{1.6}$$

$$c_2 = \sigma_2 \theta_2|_{S(t)} \tag{1.7}$$

where $\frac{1}{\sigma_1}$, $\frac{1}{\sigma_2}$ are the slopes of the phase-change diagram, as shown in Figure 1.1.

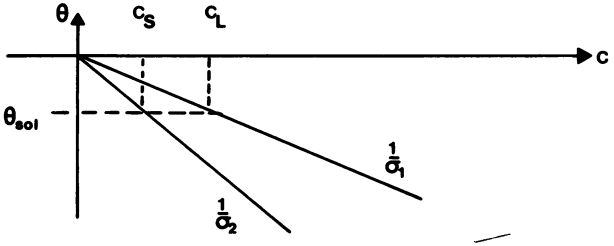


Figure 1.1: Phase-change diagram

We consider the following boundary conditions:

$$\alpha \frac{\partial \theta}{\partial n} = q \quad \text{on} \quad \Sigma = \partial\Omega \times]0,T[\tag{1.8}$$

$$\frac{\partial c}{\partial n} = 0 \quad \text{on} \quad \Sigma \tag{1.9}$$

and the initial conditions

$$\theta(x,0) = \theta_0(x) \ ; \ c(x,0) = c_0(x) \tag{1.10}$$

Remark 1.1. Here we have supposed that ρ and k are constant. This work can be immediately extended when ρ and k are functions of θ.

Remark 1.2. The results given in this paper, can be also extended for a more

general phase change diagram:

- the assumption of linearity of the diagrams can be omitted.
  (A. Bermudez - C. Saguez [2])
- the solidification temperature when c = 0 can be different from zero.

I.2. Mathematical formulation

As in A. B. Crowley - J. R. Ockendon [2], G. J. Fix [4], we introduce the function w defined in each phase by $w_i = \frac{c_i}{\sigma_i}$ (i=1,2). Then w is a continuous function. By some calculations, we obtain the following equations in the sense of distributions

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - \operatorname{div}(\alpha \operatorname{grad}\theta) = 0 \quad \text{in } Q = \Omega \times]0,T[&& (1.11)\\ &\frac{\partial v}{\partial t} - \operatorname{div}(\gamma \operatorname{grad} w) = 0 \quad \text{in } Q && (1.12)\\ &u \in H_w(\theta) \ ; \ v \in G_\theta(w) && (1.13) \end{aligned}\right\}$$

where

$$\alpha(\theta,w) = \begin{cases} \alpha_1 & \text{if } \theta > w \\ \alpha_2 & \text{if } \theta < w \end{cases}$$

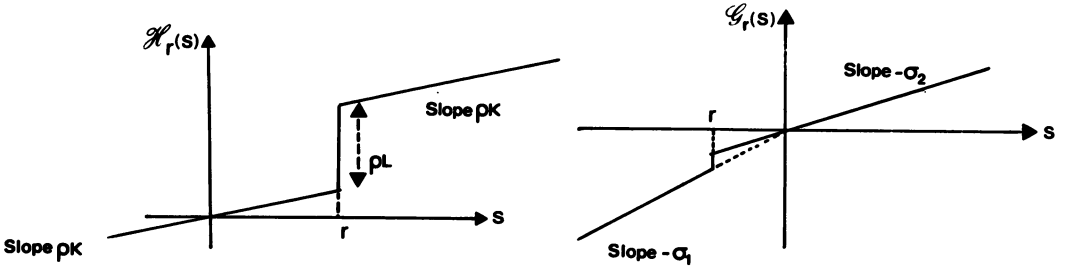
$$\gamma(w,\theta) = \begin{cases} -\sigma_1\beta_1 & \text{if } \theta > w \\ -\sigma_2\beta_2 & \text{if } \theta < w \end{cases}$$

and

$$H_w(\theta) = \{u \in L^2(\Omega) \,|\, u(x) \in \mathcal{H}_{w(x)}(\theta(x))\}$$

$$G_\theta(w) = \{v \in L^2(\Omega) \,|\, v(x) \in \mathcal{G}_{\theta(x)}(w(x))\}$$

with $\mathcal{H}_r(s)$, $\mathcal{G}_r(s)$ the following maximal monotone graphs



Remark 1.3. u corresponds to the enthalpy of the system, and -v to the concentration.

Remark 1.4. We have for the liquid domain and the solid domain

$$\Omega_1(t) = \{x \mid \theta(x,t) > w(x,t)\}$$

$$\Omega_2(t) = \{x \mid \theta(x,t) < w(x,t)\}$$

I.3. Case of a mushy region

We indicate how the system (1.11)-(1.13) can be generalized when a mushy region exists, i.e. when

$$m_6(\{(x,t) \mid \theta(x,t) = w(x,t)\}) > 0 \tag{1.14}$$

We consider a small volume element containing both solid and liquid phase. We denote by $f(x,t)$ the liquid fraction and by $\bar{c}(x,t)$ the average concentration (i.e. $\bar{c} = f\, c_1 + (1-f)c_2$). Then we have

$$\frac{\partial \bar{c}}{\partial t} - \operatorname{div}(f\, \beta_1 \operatorname{grad} c_1 + (1-f)\, \beta_2 \operatorname{grad} c_2) = 0 \tag{1.15}$$

and because we suppose that liquid and solid are in equilibrium $\left(w = \theta = \frac{c_1}{\sigma_1} = \frac{c_2}{\sigma_2}\right)$ we obtain

$$\frac{\partial}{\partial t}((-f\sigma_1 - (1-f)\sigma_2)w) - \operatorname{div}(\gamma \operatorname{grad} w) = 0 \tag{1.16}$$

with

$$\gamma = -f\,\beta_1\sigma_1 - (1-f)\,\beta_2\sigma_2 \ . \tag{1.17}$$

We have

$$-f\sigma_1 - (1-f)\sigma_2 w \in G_\theta(w) \tag{1.18}$$

because $w = \theta$ and $0 \le f \le 1$.

So the system (1.11)-(1.13) holds in the case of a mushy region with γ given by (1.17), $\alpha = f\alpha_1 + (1-f)\alpha_2$, and $f = \frac{u-\rho k\theta}{\rho L}$.

II. STUDY OF A DISCRETIZED PROBLEM

We define the associated discretized problem

$$\left|\begin{array}{l} \dfrac{u^{n+1}-u^n}{\Delta t} + A^n\,\theta^{n+1} = h^{n+1} \ ; \quad u^{n+1} \in H_{w^{n+1}}(\theta^{n+1}) \qquad (2.1) \\[2ex] \dfrac{v^{n+1}-v^n}{\Delta t} + B^n w^{n+1} = 0 \ ; \quad v^{n+1} \in G_{\theta n+1}(w^{n+1}) \qquad n=0,\ldots,N_T-1 \qquad (2.2) \\[2ex] u^0 = u_0(x) \in H_{w^0}(\theta^0) \ ; \quad v^0 = v_0(x) \in G_{\theta^0}(w^0) \qquad (2.3) \end{array}\right.$$

with

$$(A^n\theta, z)_{V',V} = \int_\Omega \alpha^n \mathrm{grad}\theta\,\mathrm{grad}z\,d\Omega \qquad (V = H^1(\Omega),\ V' \text{ dual of } V) \tag{2.4}$$

$$(B^n w, z)_{V',V} = \int_\Omega \gamma^n \mathrm{grad}w\,\mathrm{grad}z\,d\Omega \tag{2.5}$$

and h^{n+1} defined by

$$(h^{n+1}, z)_{V',V} = \frac{1}{\Delta t}\int_{\partial\Omega} \left(\int_{n\Delta t}^{(n+1)\Delta t} g(\tau)\,d\tau\right) z\,d\Gamma \tag{2.6}$$

where

$$\left|\begin{array}{l} \alpha^n = f^n\alpha_1 + (1-f^n)\alpha_2 \\ \\ \gamma^n = -f^n\,\beta_1\sigma_1 - (1-f^n)\beta_2\sigma_2 \\ \\ f^n = X\left(\dfrac{u^n-\rho k\theta^n}{\rho L}\right)\ ;\ X(r) = \left|\begin{array}{lll} 0 & \text{if} & r \le 0 \\ r & \text{if} & 0 \le r \le 1 \\ 1 & \text{if} & r \ge 1 \end{array}\right. \end{array}\right.$$

II.1. A result of existence

We have the following proposition

Proposition 2.1 The problem (2.1)-(2.3) has a maximal solution and a minimal solution.

We give the principal steps of the proof. Firstly, we have the following lemmas

Lemma 2.1 For w^{n+1} fixed, the variational inequality (2.1) has a unique solution θ^{n+1} and the application $T_1 : w^{n+1} \to \theta^{n+1}$ verifies

$$w_1^{n+1} \le w_2^{n+1} \Rightarrow T_1 w_1^{n+1} \le T_1 w_2^{n+1}$$

Lemma 2.2 For θ^{n+1} fixed, the variational inequality (2.2) has a unique solution w^{n+1} and the application $T_2 : \theta^{n+1} \to w^{n+1}$ verifies

$$\theta_1^{n+1} \le \theta_2^{n+1} \Rightarrow T_2\theta_1^{n+1} \le T_2\theta_2^{n+1}$$

Lemma 2.3 There exist two functions $\underline{w}$ and $\bar{w}$ such that

$$\forall \theta \in L^2(\Omega) \quad \underline{w} \le T_2\theta \le \bar{w} \quad \text{a.e.}$$

So we can define the sequences $\{\theta_j\}$ and $\{w_j\}$ by

$$w_0 = \underline{w}\ ;\ \theta_0 = T_1\underline{w}$$

$$w_{j+1} = T_2\theta_j\ ;\ \theta_{j+1} = T_1 w_{j+1}$$

Then we prove successively the following points

i) by lemma 2.1, 2.2

$$\underline{w} \leq w_1 \leq \ldots \leq w_j \leq w_{j+1} \leq \ldots \leq \bar{w}$$

$$\underline{\theta} = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_j \leq \theta_{j+1} \leq \ldots \leq \bar{\theta} = T_1\bar{w}$$

ii) $\{w_j\}$ and $\{\theta_j\}$ are bounded in $H^1(\Omega)$ (the proof uses classical techniques to obtain a priori estimates).

iii) By i) and ii) the sequences $\{w_j\}$ and $\{\theta_j\}$ converge in the following sense

$$w_j \longrightarrow \underset{\sim}{w} \quad \text{in } H^1(\Omega) \quad \text{weakly}$$

$$\theta_j \longrightarrow \underset{\sim}{\theta} \quad \text{in } H^1(\Omega) \quad \text{weakly}$$

iv) We pass to the limit in the two variational inequalities and we obtain that $(\underset{\sim}{w},\underset{\sim}{\theta})$ is solution of the system (2.1)-(2.3).

v) $(\underset{\sim}{w},\underset{\sim}{\theta})$ is minimal solution (if (w,θ) is another solution, we have

$w_j \leq w \ \forall\, j$ and $\theta_j \leq \theta \quad \forall j$)

vi) the existence of a maximal solution is obtained in the same way by using a dual problem.

For the details of the demonstration, we refer to A. Bermudez - C. Saguez [2].

II.2. Numerical algorithm

From the proof of the proposition 2.1, we deduce the following algorithm (we give the algorithm to compute the minimal solution). At each step of time we use the iterations

1) Compute $\underline{w}^{n+1}$, solution

$$\frac{\sigma_2 \underline{w}^{n+1}}{\Delta t} + B^n \underline{w}^{n+1} = \frac{v^n}{\Delta t}$$

2) $j = 0$; $w_0^{n+1} = \underline{w}^{n+1}$

3) Compute θ_j^{n+1}, solution of the variational inequality

$$\frac{u_j^{n+1}}{\Delta t} + A^n \theta_j^{n+1} = h^{n+1} + \frac{u^n}{\Delta t} \ ; \ u_j^{n+1} \in H_{w_j^{n+1}}(\theta_j^{n+1})$$

4) Compute w_{j+1}^{n+1}, solution of the variational inequality

$$\frac{v_{j+1}^{n+1}}{\Delta t} + B^n w_{j+1}^{n+1} = \frac{v^n}{\Delta t} \; ; \; v_{j+1}^{n+1} \in G_{\theta_j^{n+1}}(w_{j+1}^{n+1})$$

5) Test of convergence

if verified, n = n+1

if not, j = j+1, go to 3).

From the proposition 2.1, we have

Proposition 2.2 The algorithm given above converges to the minimal solution of the problem (2.1)-(2.3).

At the points 3) and 4) we compute the solution of a variational inequality. Suppose that we have solved the variational inequality

$$u + A\theta = f \; ; \; u \in H(\theta)$$

We use the following algorithm

1) Compute θ^ℓ solution of the P.D.E.

$$A\theta^\ell + \omega\theta^\ell = f - y^\ell$$

2) $g^{\ell+1} = \delta H^{\omega,\lambda}(\theta^\lambda + \lambda y^\ell) + (1-\delta) y^\ell$

where $H^{\omega,\lambda}$ is the Yosida approximation of $H - \omega I$.

We have the result of convergence (A. Bermudez - C Sanguez [1], C. Sanguez [6]).

Proposition 2.3 For $2\omega > \frac{1}{\lambda} \geq (\omega - m)^+$ (m coercivity coefficient of A) and $\delta \in (0,1]$ we have

$$\theta^\ell \to \theta \quad \text{in } H^1(\Omega) \text{ strongly}$$

$$u^\ell = y^\ell + \omega\theta^\ell \longrightarrow u \text{ in } L^2(\Omega) \text{ weakly.}$$

III.3. Numerical results

We consider the 1-dimensional case.

We take the following data

$$\begin{cases} \Omega =]0,1[\\ \alpha_1 = \alpha_2 = 1 \ ; \ \beta_1 = \beta_2 = 1 \\ \rho = 1 \ ; \ \ k = 1 \\ L = 1 \ ; \ \sigma_1 = -2 \ ; \ \sigma_2 = -1 \end{cases}$$

$$q(t) = -1$$

$$\theta_0(x) = 0.5 \ ; \ c_0(x) = 1$$

The numerical parameters are

$$\Delta t = 1/10 \ ; \ \Delta x = 1/20$$

$$\lambda = 0.16 \ ; \ \delta = 1 \ ; \ \omega = \frac{1}{2\lambda}$$

The results are presented figures 2.1, 2.2.

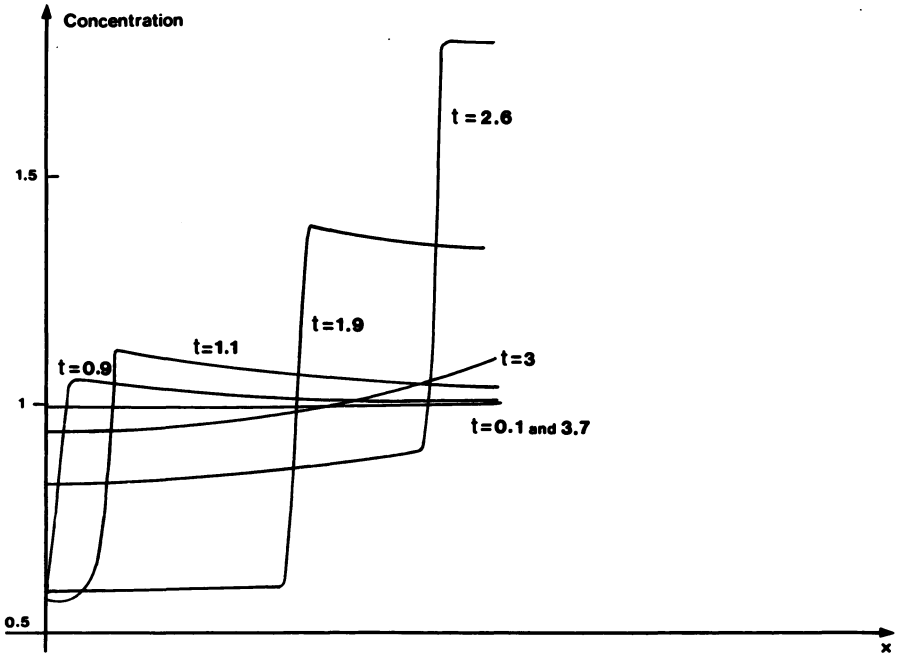


Figure 2.1. Concentration for t = 0.1; 0.9; 1.1; 1.9; 2.6; 3; 3.7.

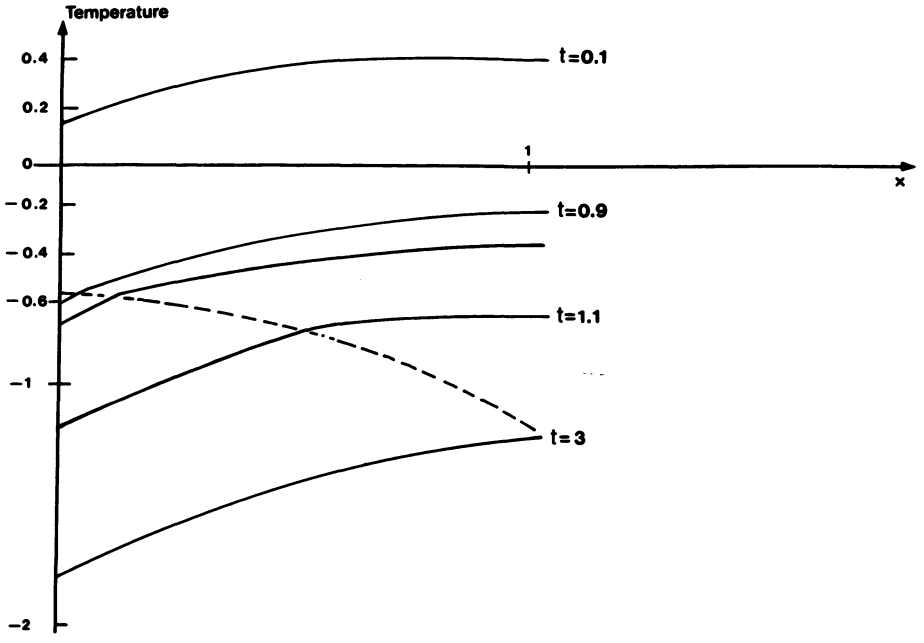


Figure 2.2. Temperature for t = 0.1; 0.9; 1.1; 1.9; 3.

Other examples, in particular in the 2-dimensional case, are given in in A. Bermudez - C. Saguez [2].

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R S PECKOVER

The modelling of some melting problems

1. INTRODUCTION

A wide range of free boundary problems arise in reactor safety studies - in the movements of solid, liquid and gas phases through pools of coolant, in the quenching of hot solids and in the melting attack on solid structures. Of these, two are discussed in this paper viz. (i) the movement of hot particles in low melting point solids and (ii) the ablation of a solid slab when one face is strongly heated. Particular regimes are considered for idealized models which have interesting features.

2. THE DESCENT OF AN INTERNALLY HEATED PARTICLE THROUGH A LOW MELTING POINT SOLID

The essential characteristic of the descent through a low melting point solid of a hot particle is that it does so by melting a transient liquid cavity in which it falls under gravity. After an initial transient period of acceleration, a particle containing a constant heat source falls at a steady velocity with an invariant cavity shape. For a given weight of particle and strength of heat source, the fall velocity in a given bed material is determined. The weight of the particle is balanced by the drag exerted on it by the flow of liquid in the cavity. The drag depends on the shape of the particle and of the enveloping cavity, which is determined by the heat transfer from the particle; this is itself influenced by the rate of fall. This regelation problem is related to the classical experiment of a wire pulled through ice [1]; and to similar experiments with spheres [2]. However for a hot particle the melted cavity is produced by the sphere's internal heating and not by pressure melting. For ice skating, friction provides the heat source, which can produce the lubricating water film even when pressure melting is negligible [3].

2.1. Formulation

Here the particle is taken to be spherical, of radius a, and it is assumed that the thermophysical properties of the solid bed material (including the

thermal diffusivity κ) do not change on melting. It is convenient to work with dimensionless variables, measuring length and time in multiples of a and a^2/κ respectively. The dimensionless temperature θ has been scaled so that $\theta = 1$ is the melting point of the solid and $\theta = 0$ is the ambient temperature far from the particle. The configuration is axisymmetric; spherical polar co-ordinates centred on the particle are taken with $\underline{r} \equiv (r,\phi)$ where r and ϕ are the radius and co-latitude respectively. When the particle is falling steadily, we have in the frame of reference of the particle

$$(\underline{u}.\nabla)\theta = \nabla^2\theta \qquad (r \geq 1) \tag{2.1}$$

and in the liquid cavity, $\underline{u}$ satisfies the Navier-Stokes equation

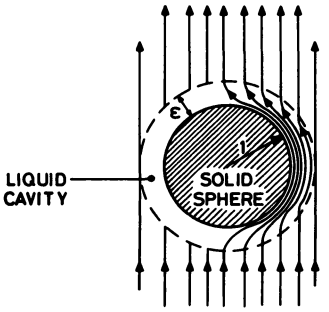

Figure 1. The motion of the melting substrate material in the frame of reference of the sphere.

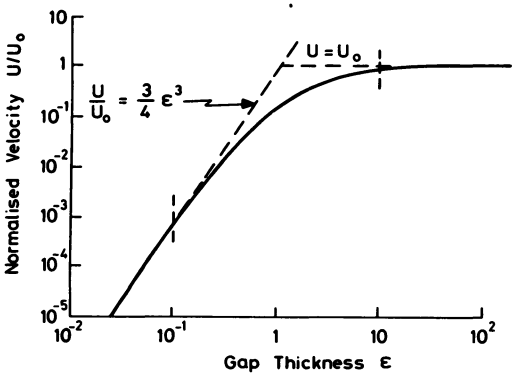

Figure 2. The fall velocity U as a function of the normalized gap thickness ε; U_o is the Stokes velocity.

$$p^{-1}(\underline{u}.\underline{\nabla})\underline{u} = \nabla^2\underline{u} - \nabla\Pi, \qquad \text{div } \underline{u} = 0 \tag{2.2}$$

where Π is the normalized deviation of the pressure from its ambient value and p is the Prandtl number. The boundary conditions are (i) $\theta \to 0$ as $r \to \infty$. (ii) $\underline{u} = \underline{U}$, $\theta = 1$ and $-[\partial\theta/\partial n] = S\,\underline{U}.\hat{\underline{n}}$ on the cavity boundary $\underline{r} \equiv \underline{s}$. Here S is the Stefan number, $[.]$ means jump across discontinuity, and $\underline{U}$ is the steady vertical velocity with which the bed approaches the particle (in the latter's frame of reference). (iii) The total (normalized) rate of heat production rate in the particle is Q. Below we use $\partial\theta/\partial r = -Q/4\pi$ on $r = 1$, but in general the particle's temperature distribution depends on α, the ratio of particle thermal conductivity to that in the bed material. (iv) $\underline{u} = 0$ on $r = 1$. In steady fall, the balance between the reduced weight of the particle and the drag exerted on it can be written in dimensionless form as $12pU_o = c_D U^2$ where U_o is the Stokes terminal velocity of the particle in an infinite medium, and is a measure of the particle's weight. c_D is the drag coefficient.

If $\{\psi\}$ represents the set of parameters $\{Q,S,p,U_o,\alpha\}$ then it is clear that $\underline{u} = \underline{u}(\underline{r};\{\psi\})$, $\theta = \theta(\underline{r};\{\psi\})$, $\underline{s} = \underline{s}(\phi,\{\psi\})$ and $U = U(\{\psi\})$. Depending on the parameters in $\{\psi\}$ a number of different regimes may be discerned.

2.2. Solutions

A. No melting One solution regime involves conduction through the bed material without melting. If the heat source distribution is spherically symmetric, then outside the particle, $\theta = Q/r$; thus $\theta < 1$ on $r = 1$, provided $Q < 1$. For this regime the remainder of the parameters in $\{\psi\}$ are not relevant.

B. Slow Descent A second solution regime is that for which U is sufficiently small that (i) equation (2.1) reduces to Laplace's equation; (ii) the Reynolds number (U/p) is small so that Stokes theory is valid in the cavity; and (iii) the discontinuity in the temperature gradient at the cavity interface can be neglected so that the cavity remains spherical. If $Q = 1 + \varepsilon$, then $\theta = (1 + \varepsilon)/r$ is the solution of (2.1); thus the cavity radius is $1 + \varepsilon$, and the cavity gap thickness is ε. To relate the heat source strength to the fall velocity it is necessary to calculate the velocity field in the cavity and thus the drag on the particle. The

configuration is shown in Figure 1.

An axisymmetric Stokes stream function ψ may be introduced such that

$$r\sin\phi(u_r, u_\phi) = (r^{-1}\partial/\partial\phi, \partial/\partial r)\psi \tag{2.3}$$

where ϕ is the co-latitude. The velocity $\underline{u}$ satisfies (2.2) with the l.h.s. neglected and the boundary conditions $\underline{u} = U$ on $r = 1$ and $\underline{u} = \underline{U}$ on $r = Q$. The solution is $\psi = \psi_1 + \psi_2 + \psi_3$ where

$$\left.\begin{aligned} \psi_1 &= \tfrac{1}{2}\lambda_1 U\sigma^2[1 + \tfrac{1}{2}r^{-3} - 3r/2] \\ \psi_2 &= \tfrac{1}{2}\lambda_2 U\sigma^2[1 - r^{-3}] \\ \psi_3 &= \tfrac{3}{4}\lambda_2 U\sigma^2[1 - r^2] \end{aligned}\right\} \tag{2.4}$$

in which $\sigma = r\sin\phi$, $\lambda_1 = 4Q(Q^5 - 1)/D$, $\lambda_2 = 2Q(Q^2 - 1)/D$ and $D = (Q-1)^4(4Q^2+7Q+4)$. The first term ψ_1 corresponds to the viscous flow around a rigid stationary sphere of radius unity with uniform free stream velocity $\lambda_1 U$[4,p231]. The second term ψ_2 represents an irrotational flow around the unit sphere with free stream velocity $\lambda_2 U$[4,p452]. The third term ψ_3 corresponds to a distribution of vorticity $\omega = 15\lambda_2 U\sigma/2$ (cf Hill's spherical vortex [4,p526].) The values of λ_1 and λ_2 are determined by the outer boundary condition only, since λ_1 and $(\lambda_2 + \lambda_3)$ satisfy the boundary conditions on $r = 1$ independently. The vorticity has only an azimuthal component of magnitude $\tfrac{1}{2}U\sin\phi(15\lambda_2 r - 3\lambda_1 r^{-2})$ and the corresponding pressure distribution is $-\tfrac{1}{2}U\cos\phi(30\lambda_2 r + 3\lambda_1 r^{-2})$.

On the particle surface, the normalized force F per unit area exerted on the sphere is $-\Pi\hat{\underline{n}} - (\hat{\underline{n}}\wedge\omega)$ [4,p178]; the total drag exerted by the fluid can be obtained by integrating this expression over the surface of the sphere. It is however more illuminating to consider separately the three contributions ψ_1, ψ_2 and ψ_3 to the stream function. From d'Alembert's Paradox, the irrotational flow ψ_2 cannot contribute to the net drag. For ψ_3, the vertical component of the force acting at a point on the particle surface is $15\lambda_2 U.P_2(\cos\phi)$ where P_2 is a Legendre polynomial; when this is integrated over the surface, the resultant vanishes. The sole contribution to the drag consequently arises from ψ_1, which is the stream function for the regular Stokes problem. Thus $U_o = \lambda_1 U$, or

$$U/U_o = (Q-1)^4(4Q^2+7Q+4)/4Q(Q^5-1) \qquad (2.5)$$

which is the required relationship between U and Q. When $\varepsilon \to \infty$, U tends to the Stokes velocity U_o. When $\varepsilon << 1$, $(U/U_o) = \frac{3}{4}\varepsilon^3$. A cubic dependence on ε is also obtained in [5] for the thin spherical cavity, and in [6] for lubrication flow beneath a plane circular disc. Figure 2 shows (U/U_o) as a function of ε; the limiting forms for $\varepsilon >> 1$ and $\varepsilon << 1$ are good approximations except when $10^{-1} < \varepsilon < 10$.

In the solid, the Peclet number is simply U. In the spherical shell of liquid, the flow is most rapid past the equator of the particle, and the average flow velocity U_m there satisfies $U_m(Q^2-1) = \pi UQ^2$. If $\varepsilon < 1$, $U_m\varepsilon$ is the appropriate liquid Peclet number. When ε is small, $U_m\varepsilon = \frac{1}{2}U$; when $\varepsilon = 1$, $U_m\varepsilon = 4U/3$. Thus the condition $U << 1$ is sufficient to ensure that the l.h.s. of (2.1) is negligible in both solid and melt. Similarly the condition $U << p$ ensures that lubrication theory is valid in the liquid shell. From boundary condition (ii), the jump in the temperature gradient at the melt-front is negligible provided $SU << 1$. Finally equation (2.5) implies $U < U_o$. Hence the condition for the slow descent solution to be valid is

$$U << U^* \equiv \min [1, p, S^{-1}, U_o] \qquad (2.6)$$

holds and (2.5) is true.

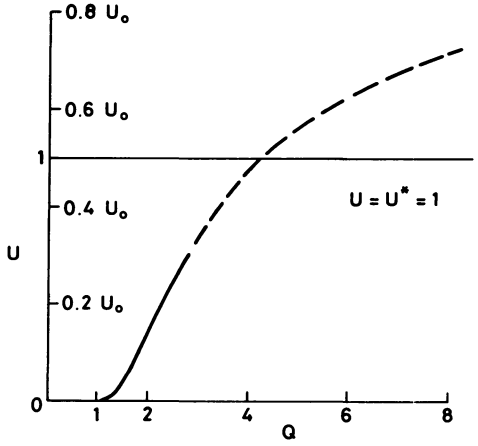

Figure 3. The fall velocity U as a function of heating rate Q when the Stokes velocity $U_o = 2$. For $p > 1$ and $S < 1$, $U^* = 1$. The solution is valid for $U << U^*$.

2.3. Comment

Note that for a sufficiently light particle in a viscous fluid with a low Stefan number (i.e. $U_o << 1$, $S < 1$, $p > 1$), the two regimes - (A) no melting and (B) slow descent - constitute the complete solution for all values of Q. For other values of the parameter set $\{\psi\}$, the non-linearities in this free boundary problem cannot be circumvented; these will be considered elsewhere. Some experimental results for the descent of internally heated spheres are presented in [7].

Note also that whereas the Stokes analysis for an isolated sphere in an infinite medium is only valid for $U_o << p$, the analysis in §2.2B is valid for $\varepsilon^3 U_o << p$, i.e. it is the gap size which is restricted, not the particle size.

3. THE ABLATION OF A PLANE SLAB

The one-dimensional melting of a slab of finite thickness, when the melt is instantaneously removed, is a classical problem for which accurate numerical solutions are available and for which uniqueness has been established [8]. A number of different regimes may be discerned, depending on the governing parameters, and approximate methods may be used in those which enable the functional dependence of the solution on the characteristic dimensionless parameters to be clarified. One such method *the pseudo-separation of variables* is discussed and used in this context below in §3.3.

3.1. Formulation and Melting Regimes

If a normalization is chosen so that the slab is initially of unit thickness and zero temperature (equal to that of the coolant), that the melting point of the slab material is unity, and that time is measured in units of the thermal diffusion time for the slab, then the equations for this one phase Stefan problem are

Fourier equation: $\theta_\tau = \theta_{xx}$ $X(\tau) < x < 1$ (3.1)

Front face: $\Psi = Su - \theta_x$ (3.2a)

$\theta = \theta_o(\tau)$ (3.2b)

at $x = X(\tau)$, $\tau \geq 0$

Back face: Newton's law of cooling : $\theta_x + H\theta = 0$ (3.3)

Initial Conditions: $\theta = 0$ $0 < x < 1$, $\tau = 0$ (3.4)

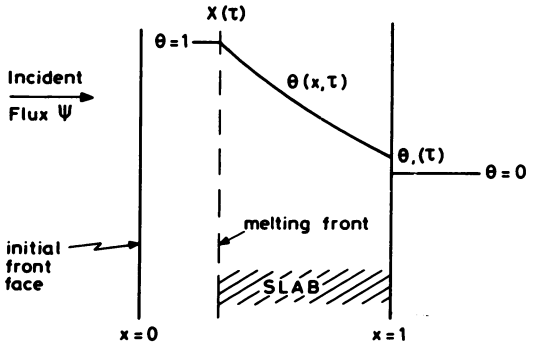

Figure 4. The configuration for the melting of a plane slab

$X(\tau)$ is the location of the front face of the solid, and $u \equiv dX/d\tau$. During the initial heat-up phase ($0 \leq \tau \leq \tau_p$), θ_o is below the melting point, $X = 0$, and $u = 0$. For $\tau > \tau_p$, $\theta_o \equiv 1$ and melting occurs so that $X > 0$ and $u > 0$. The configuration is shown in figure 4.

From these governing equations it is apparent that θ and X have the functional forms $\theta \equiv \theta(x,\tau;\Psi,H,S)$, $X \equiv X(\tau;\Psi,H,S)$ where Ψ is the dimensionless heat flux at the front face, H is the Biot number, i.e. the dimensionless heat transfer coefficient at the rear face, and S is the Stefan number, the dimensionless latent heat of melting. The time for melt-through τ_m is given functionally by $\tau_m \equiv \tau_m(\Psi,H,S)$.

When the slab has completely melted, the sensible and latent heat absorbed by the slab is 1 + S. The rate at which heat is conducted into the coolant is $H\theta_1$ where $\theta_1(\tau)$ is the temperature at the rear face. The energy balance at melt-through thus gives

$$\Psi.\tau_m = 1 + S + H \int_o^{\tau_m} \theta_1(\tau)\,d\tau \qquad (3.5)$$

If $Q \equiv \Psi/(1+S)$, then $\tau_m = Q^{-1}$ for $H = 0$. However there are some ranges of Ψ and H for which τ_m does not exist. It is straightforward to show [9] (i) melting can only occur if and only if $\Psi > H/(H+1)$ (ii) complete melting can only occur if and only if $\Psi > H$ (iii) when equilibrium is reached with only partial melting, the fraction melted $\gamma = 1 + H^{-1} - \Psi^{-1}$.

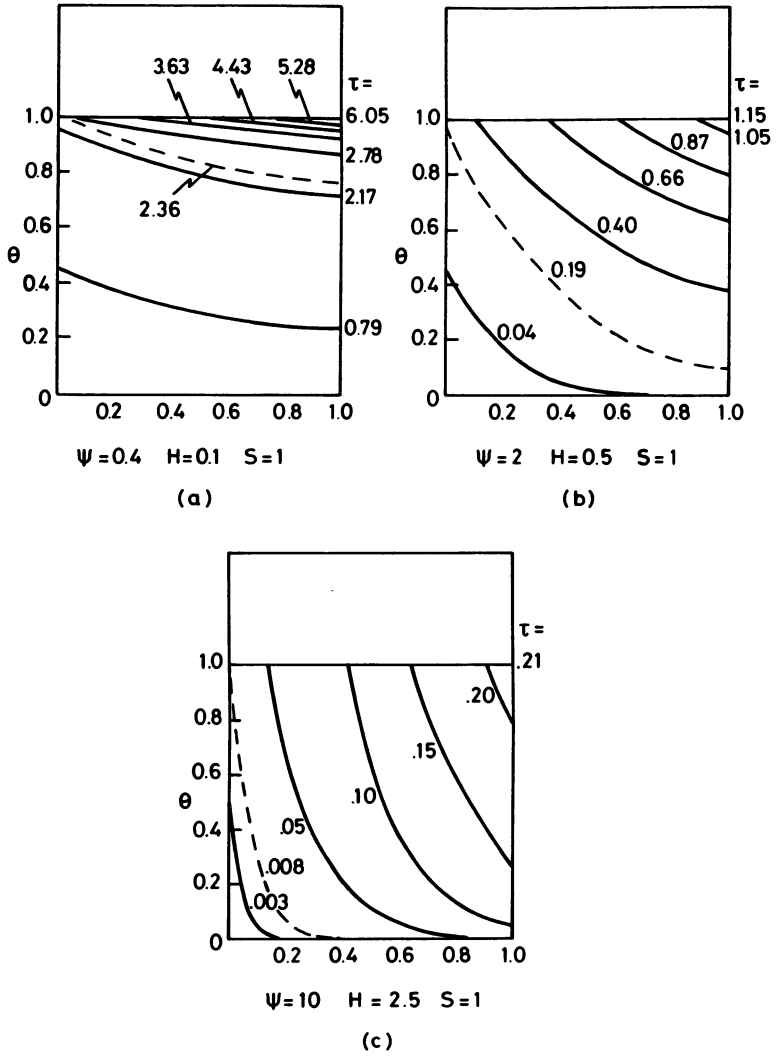

Figure 5. Evolution of temperature profiles for 3 cases when $H/\Psi = 0.25$.

Numerical integration of (3.1)-(3.4) has been carried out using the MELTIN code [10] which uses Crank-Nicolson with front tracking. Sufficiently small space and time steps have been used to assure the accuracy of the method. Figure 5 shows temperature profiles when $S = 1$ and $H/\Psi = 0.25$. These illustrate clearly the marked difference in the evolution of the temperature profiles when $\Psi << 1$ and $\Psi << 1$. The dashed profile shows the

temperature distribution when melting first begins. A parametric study was carried out [9] using MELTIN for S = 0.37, a value appropriate for steel. The lines of constant melt-through time ('isochrones') obtained therein are reproduced in Figure 6.

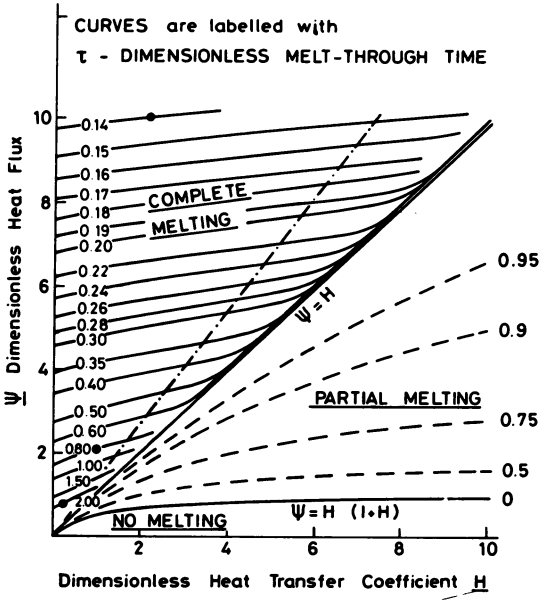

Figure 6. Dependence of melt-through time τ_m on H and Ψ. The dashed curves (---) indicate the fractions melted when melting is incomplete. The line (- - -) is $\Psi = H(1+S)$.

3.2. Premelting

Prior to melting, the temperature profile can be calculated as a Fourier sum [11,p125]. The time τ_p before melting is implicitly given by

$$\gamma = \sum_{n=0}^{\infty} \beta_n \exp(-\alpha_n^2 \tau_p) \qquad (3.6)$$

where $\gamma = 1 + H^{-1} - \Psi^{-1}$ as above, $\beta_n = 2(\alpha_n^2 + H^2)/\{\alpha_n^2(H + H^2 + \alpha_n^2)\}$ and the α_n are the roots of $\alpha \tan \alpha = H$ [11,p491].

When $H \to 0$ and $\Psi \lesssim \pi/2$, then τ_p is sufficiently large for all except the first term in the Fourier sum to be negligible and the profile at the onset of melting is essentially parabolic (see also [12]).

$$\theta = \Psi\{\frac{1}{3} - x) + (\frac{x^2}{2} + \tau_p)\} = 1 + \Psi[\frac{x^2}{2} - x] \tag{3.7}$$

where

$$\tau_p = \frac{1}{\Psi} - \frac{1}{3} : \qquad \Psi \lesssim \pi/2,\ H = 0 \tag{3.8}$$

When $\Psi >> 1$, the rapid temperature rise prior to melting occurs only in a thin boundary layer and is thus independent of H. The temperature profile for an infinite half space [11,p75] is appropriate in this case. It follows that

$$\tau_p = \pi/(4\Psi^2) \qquad \Psi \lesssim \pi/2,\ H = 0 \tag{3.9}$$

The heat-balance integral method [12] underestimates (3.9) by $\sim$ 15%, giving $2/(3\Psi^2)$. Figure 7 shows that (3.8) and (3.9) together cover the whole range of Ψ and match well with a difference of $\sim$ 5% at $\Psi = \pi/2$.

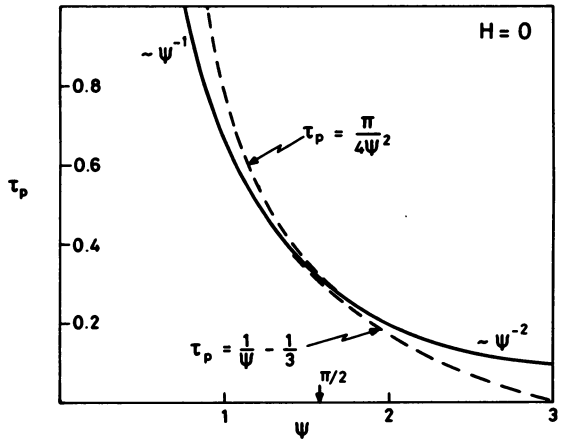

Figure 7. Time to first melting when H = 0.

3.3. High Heat Flux: $Q >> 1$

It is convenient to renormalize into variables $z \equiv Q(1-x)$, $\sigma = Q^2\tau$, $U \equiv dZ/d\sigma$ where Z is the position of the melting front. Then (3.1) to (3.4) become

$$\left.\begin{aligned} &\theta_\sigma = \theta_{zz} \quad 0 < z < Z\ (\sigma);\ \theta_z = \mu\theta \text{ on } z = 0; \\ &\theta = 1 \text{ and } 1 + S = -SU + \theta_z \text{ on } z = Z; \end{aligned}\right\} \tag{3.10}$$

where $\mu \equiv H/Q$, the slab thickness is now Q, and the 'applied flux' is $1 + S$. In this normalization the time σ_m to melt completely a slab with insulated rear face (μ=0) is Q.

When Q is large, the melting of the slab can be divided into four stages. (i) pre-melting, lasting $\{\pi/4(1+S)^2\} < 1$ - see above; (ii) an acceleration stage during which the melting front speed increases to its steady value. This has been analysed when $S >> 1$, by Andrews and Atthey [13] and their solution shows that $|U|$ approaches 1 when $\sigma \sim 1$. The solution by Landau [14,15] confirms this for the full range of S. (iii) a period $\sim Q$ during which the front advances steadily towards the rear face. (iv) a final stage in which the rear boundary condition exerts a significant influence. This stage can be rather short if $H \to 0$, or infinite as $H \to \Psi$. These phases are clearly discernible in Figure 8 for $Q = 10$. We are concerned here with the analysis of the final stage.

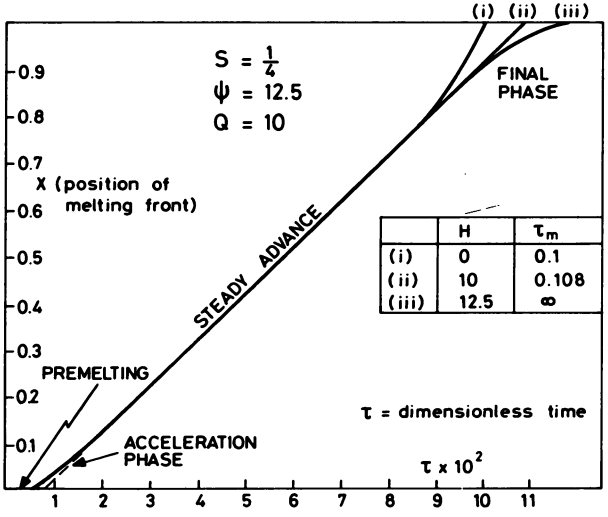

Figure 8. Position of melting front as a function of time.

An important special case is $\mu = 1$ (i.e. $H = \Psi/(1+S)$). After a time period $\sigma \sim O(1)$, the front will have accelerated and the temperature profile relaxed to

$$\theta = \exp(z - Z) \tag{3.11}$$

giving $U = -1$. This profile satisfies the rear boundary condition at all

times when $\mu = 1$, and so the speed of the front is unchanged as it approaches the rear face. The normalized enthalpy in the thermal front is 1 (unity), and this is equal to that conducted through the rear face prior to complete melting. Thus

$$(1 + S)\sigma_m = Q(1 + S) + 1 \qquad (\mu = 1) \tag{3.12}$$

For general μ, the method of *pseudo-separation of variables* may be used. In this we write $\theta = \theta(\sigma_1 z) = G(\sigma)\ F(nz)$ where $n = n(\sigma)$ is a slowly varying function of time, to be determined. Substitution into the Fourier equation (3.10) gives $G_\sigma/G + F_\sigma/F = F_{zz}/F$ or

$$G_\sigma/(n^2 G) + (n_\sigma z/n^2) F_y/F = F_{yy}/F \tag{3.13}$$

where $y \equiv nz$. Provided the second term on the left hand side is negligible then a kind of separable solution can be obtained. Introduction of a separation constant C^2, gives $F \sim \exp(\pm Cnz)$. Since n is to be determined, C can be absorbed in n and chosen to be unity. The second term is then negligible provided $n_o z/n^2 \ll 1$.

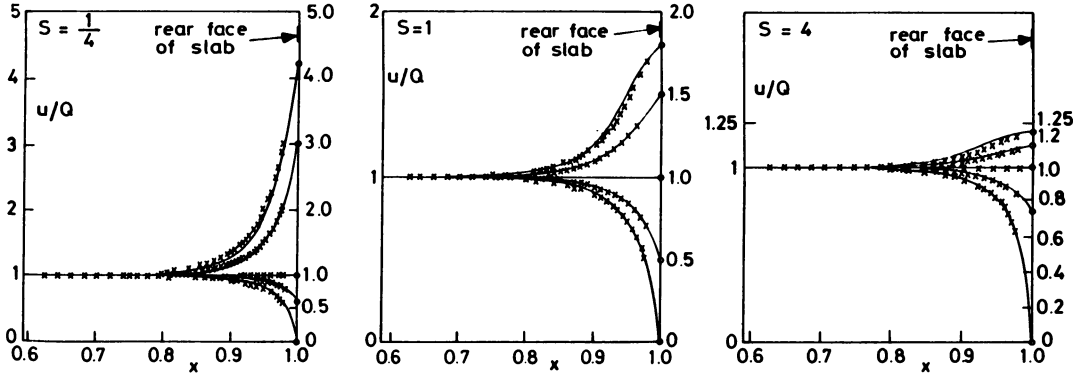

Figure 9. Plots of the melt front velocity against position. The initial position of the front face is x = 0. Smooth curves are from (3.20); for comparison the accurate results from MELTIN are denoted by xxx.

In order to satisfy the rear boundary condition at all times, the following Ansatz is made for θ:

$$\theta = \theta_1(\sigma) \left\{\cosh(nz) + (\mu/n) \sinh(nz)\right\} \tag{3.14}$$

where θ_1 can be identified as the temperature at the rear face. At the melt-front

$$1 = \theta_1 \left\{\cosh(nZ) + (\mu/n)\sinh(nZ)\right\} \tag{3.15}$$

Let

$$\Gamma = \frac{\sinh(nZ) + (\mu/n)\cosh(nZ)}{\cosh(nZ) + (\mu/n)\sinh(nZ)} \tag{3.16}$$

then

$$\theta_z = n\Gamma \text{ at } z = Z \tag{3.17}$$

and the Stefan condition becomes

$$1 + S + SU = n\Gamma \tag{3.18}$$

Since $\theta = 1$ at the front at all times then in the frame of reference of the front, $U\theta_\zeta + \theta_{\zeta\zeta} \to 0$ as $\zeta \to 0$ where $\zeta = z - Z$. Insertion of the ansatz (3.14) gives

$$Un\Gamma + n^2 = 0 \tag{3.19}$$

The 3 equations (3.16), (3.18), (3.19) can be manipulated into the form

$$\left.\begin{aligned} U &= -(1 + S)/(\Gamma^2 + S) \\ n &= -U\Gamma \\ Z &= (1/2n) \log\left(\left\{\frac{\Gamma+1}{\Gamma-1} \quad \frac{\mu-n}{\mu+n}\right\}\right) \end{aligned}\right\} \tag{3.20}$$

which for fixed S and μ give U and Z parametrically in terms of Γ. Figure 9 shows U as a function of Z for a range of values of μ. The smooth curves have been calculated using (3.20), the 'crosses' are the exact solution calculated using MELTIN [10]. The agreement in general is good. It is simple to confirm that those solutions which differ from the true ones to some significant extent are precisely those where the coupling terms in (3.13) is not negligible. The steadily advancing solution corresponds to $\mu = \Gamma = n = -U = 1$ giving Z indeterminate. When $\mu \neq 1$, the solution far from the rear face is also a steady advance : $\Gamma = n = -U = 1$.

When U is small, expansions in the small parameter Γ^{-1} are appropriate. For n and U, only the first term is needed; for Z it is necessary to keep the first two terms. Elimination of n and Γ gives the approximate formula

$$Z + \sigma_a U = Q(1 - \gamma) \equiv Z_o \tag{3.21}$$

where $\sigma_a \equiv \{S\Delta + (1 - \Delta^3)/3\}/(1 + S)^2$ and $\Delta \equiv \Psi/H$. The solution of (3.21) is

$$Z = Z_o + B \exp(-\sigma/\sigma_a) \tag{3.22}$$

where B is a positive constant determined by matching to the appropriate solution at earlier times. When $Z_o < 0$, the rear face is reached in finite time with velocity $U = Z_o/\sigma_a$. For $Z_o > 0$, the slab reaches equilibrium when $Z = Z_o$ after a logarithmically infinite time with only the fraction γ melted. When $Z_o = 0$, both U and Z decay exponentially, and the slab takes an infinite time to melt completely. When Z_o is small and negative i.e. Ψ is close to H, the dominant contribution to the time for melt-through is

$$\sigma_m \sim - \left\{S/(1 + S)^2\right\} \log(\Psi - H) \tag{3.23}$$

Thus the Ansatz (3.14) leading to equations (3.20) gives a good representation of the solution as the melting front approaches the rear face, except when μ is small.

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J C W ROGERS

The Stefan problem with surface tension

1. INTRODUCTION

Consider the classical two-phase Stefan problem, in which a fixed bounded region Ω is divided into two time-varying parts $\Omega_1(t)$ and $\Omega_2(t)$, occupied respectively by the "solid" and "liquid" phases of a substance. The classical formulation of the problem involves solving the equations

$$T_t(x,t) = D_i \Delta T(x,t),\ x \in \mathrm{Int}(\Omega_1(t)),\ i = 1, 2,\ \Omega_1(t) \cup \Omega_2(t) = \Omega, \quad (1.1a)$$

subject to the continuity condition at the interface between the regions

$$T(x,t) = T_M,\ x \in \partial\Omega_1(t) \cap \partial\Omega_2(t). \quad (1.1b)$$

The interface evolves through the motion of its points with an outward normal velocity $v \cdot \nu_1$ given by

$$Lv \cdot \nu_1 = D_1 c_1 (\nabla T \cdot \nu_1)_1 - D_2 c_2 (\nabla T \cdot \nu_1)_2, \quad (1.1c)$$

where the subscripts denote the sides of the interface on which the derivatives are evaluated. T is the temperature, L is the latent heat of melting, T_M is the temperature of melting, D_1, D_2, c_1, c_2 are the respective thermal diffusion coefficients and heat capacities in the solid and liquid phases. For any region, ν will denote the unit outward normal; in particular, ν_1 is the unit outward normal of Ω_1. (1.1) is to be supplemented by suitable initial conditions for $T(x,0)$ and boundary conditions for $T(x,t)$, $x \in \partial\Omega$.

Define the energy u by

$$u = c_1(T - T_M)(1 - \chi) + [L + c_2(T - T_M)]\chi, \quad (1.2)$$

where χ is the characteristic function of Ω_2. Then, it follows from (1.1a) and (1.1c), for V any smooth domain contained in Ω, that

$$\frac{d}{dt}\int_V u(x,t)\,dx = \int_{\partial V} (c_1 D_1 (1 - x) + c_2 D_2)\,\nabla T \cdot \nu\, dS$$
$$- \int_{V \cap \partial\Omega_1 \cap \partial\Omega_2} (c_2 - c_1)(T - T_M) v \cdot \nu_1\, dS. \tag{1.3}$$

If the condition (1.1b) holds at the interface, the second term on the right-hand side of (1.3) vanishes and (1.3) has the form of a "conservation law" for $\int_V u(x,t)\,dx$.

In some recent papers [3, 1], the Stefan problem has been treated directly as a conservation law

$$u_t = \Delta f(u),\ x \in \Omega, \tag{1.4a}$$

where u is given by (1.2) and f(u) is given by

$$f(u) = \begin{cases} D_1 u & u < 0 \\ 0 & 0 < u < L \\ D_2(u - L) & u > L \end{cases} \tag{1.4b}$$

$\chi(u)$ is determined in terms of u through

$$\chi(u) = \min\left(\left(\frac{u}{L}\right)_+, 1\right). \tag{1.4c}$$

In the papers referred to, algorithms satisfying the principle of energy conservation, for the approximate solution of the problem in the form (1.4), have been presented. Convergence of these algorithms has been proved [1] and in special cases error estimates have been derived [3].

The statement (1.4) has several advantages over (1.1), and these have been described elsewhere. We will content ourselves here to list the following facts:

(i) The formulation (1.4) is independent of the topology of the free boundary

(ii) Cases where the free boundary is diffuse and χ takes on values in (0,1), which can arise, for example, when there are heat sources, are amenable to theoretical and numerical treatment by slight modification of (1.4) and the associated algorithms [9].

(iii) The approximations to the solution of (1.4) exhibit some remarkable algebraic properties which, because of the fact of convergence, carry over to the exact solutions of (1.4) [2]. Thus, in several respects the essential mathematical structure of the Stefan problem is expressed more naturally in the form (1.4) than in the classical statement (1.1).

The treatment of systems of conservation laws is much more difficult than that of a single conservation law like (1.4). We have been interested, from a mathematical point of view, in investigating a hierarchy of progressively more complicated problems, starting with the Stefan problem. In this spirit, we have found, for example, that the problem of ablation of a solid under heating at its surface has the structure of a conservation law with an irreversibility constraint added [10].

A natural problem to study in any such sequence of problems is the Stefan problem with surface tension. There has been a recent formulation of this problem by Langer [7]. Because we are interested in the problem from a different direction, especially with regard to the possibility of a geometry-independent treatment of the problem, our first task will be to explore the suitability of Langer's statement of the problem for our purposes.

Energy is still conserved when there is surface tension. To the expression $\int_V u(x,t)\,dx$ formerly given for the energy in a domain V, we should add the energy of surface tension to get

$$U(V) = \int_V u(x,t)\,dx + \int_{V\cap\partial\Omega_1\cap\partial\Omega_2} \gamma\,dS, \tag{1.5}$$

where γ is the coefficient of surface tension [6]. Thus, instead of (1.3), we should have

$$\frac{d}{dt}U(V) = \int_{\partial V} (c_1D_1(1-\chi) + c_2D_2\chi)\nabla T\cdot\nu\,dS. \tag{1.6}$$

We note that

$$\frac{d}{dt}\int_{V\cap\partial\Omega_1\cap\partial\Omega_2}\gamma\,dS = \int_{V\cap\partial\Omega_1\cap\partial\Omega_2}\gamma\kappa v\cdot\nu_1\,dS, \tag{1.7}$$

where κ is the sum of the principal curvatures of $\partial\Omega_1\cap\partial\Omega_2$, considered positive when the center of curvature lies toward Ω_1. If we compare (1.5) and (1.7) with (1.3), we see that (1.6) will hold in general only if (1.1c)

is replaced by

$$[L + (c_2 - c_1)(T - T_M) - \gamma\kappa]v \cdot \nu_1 = D_1 c_1 (\nabla T \cdot \nu_1)_1 - D_2 c_2 (\nabla T \cdot \nu_1)_2. \quad (1.8)$$

However, according to equation (2.2) of Langer, in the case of surface tension the motion of the interface is still given by (1.1c), and the temperature of the interface satisfies ((2.3) of Langer)

$$T - T_M = -T_M \frac{\gamma\kappa}{L} . \quad (1.9)$$

Thus, the equations given by Langer are not generally consistent with energy conservation. That will be a serious defect from our point of view, and we will have to make the necessary modifications, if we are to have a formulation of the problem consistent with conservation of energy.

For our purposes, Langer's treatment is also unsatisfactory because of examples of the following sort: Consider the case of radial symmetry in which a ball of ice of initial radius R_o and initial temperature $T_M\left(1 - \left(\frac{2\gamma}{R_o L}\right)\right)$ is melted by contact with a liquid whose temperature at $|x| = 2R_o$ is held at a constant above T_M for all time. It is not hard to see, by simple monotonicity arguments, that, under the governance of Langer's equations, the ball of ice will completely vanish at some critical time t_c. In the process, according to (1.9), the temperature cannot be bounded from below, completely violating known physical constraints and forcing us to deal with mathematical pathologies which have no physical foundation.

In the sequel, we will explore alternatives to Langer's formulation of the Stefan problem with surface tension, with the aim of identifying the paramount physical principles to which our treatment should be faithful.

2. ENTROPY MAXIMIZATION

In order to provide an adequate alternative, we need to know from what assumptions (1.9) was derived. (1.9) follows as a first-order correction to the case when $\gamma = 0$ from general thermodynamic laws, as well as from specific thermodynamic properties of the solid-liquid mixture [5]. Basically, there are three thermodynamic conditions to be satisfied at the interface: equality of the temperatures on the two sides, a jump condition for the pressures p in terms of the interface geometry, and equality of the chemical potentials

μ on the two sides. For a homogeneous substance, the chemical potential reduces to the Gibbs free energy per molecule, and the pressure may be thought of as a function of T and μ [11]. Accordingly, at the interface one obtains

$$p_1 = p_1(T,\mu), \quad p_2 = p_2(T,\mu), \tag{2.1}$$

from which μ may be eliminated and T determined, if p_1 and p_2 are known. By such a method (1.9) has been derived.

All these interface conditions can be derived from a single entropy maximization principle [6]: The entropy is maximized with respect to local variations in the energy, the volume, and the number density of the phases. Nonlocal energy interchanges take place through the thermal diffusion and do not enter into the set of variations under which the entropy attains a maximum, for in fact we are not dealing with true thermodynamic (actually, "thermostatic") equilibrium, but rather with an evolution of the system slowly modulated by thermal diffusion. Thus, the entropy is really changing slowly with the time. We have, for $V \subset \Omega$,

$$\int_V T\frac{dS(dV)}{dt} = \int_{\partial V} (c_1 D_1 (1 - \chi) + c_2 D_2 \chi) \nabla T \cdot \nu dS, \tag{2.2}$$

where S(dV) is the entropy associated with the domain dV of volume dx. The attainment of a maximum of the entropy with respect to a class of variations is equivalent to the balancing of the rapid and competing molecular processes which cause these variations.

To make our discussion more definite, we consider the case for which the internal energies of the solid and liquid are proportional to their temperatures and quantites

$$U_i(dV) = n_i(dV) k_i T_i, \quad i = 1, 2, \tag{2.3}$$

where dV is a volume, U_i is the energy of the appropriately numbered phase in dV, $n_i(dV)$ is the number of molecules of phase i in dV, and T_i is the corresponding temperature. It follows from thermodynamic considerations [11] that the entropy functional has the form

$$S(\Omega) = \int_{\Omega} \left\{ k_1 n_1(dV) \left(\ell n \frac{U_1(dV)}{n_1(dV)} - \ell n \phi_1 \left(\frac{V_1(dV)}{n_1(dV)} \right) \right) + k_2 n_2(dV) \left(\ell n \frac{U_2(dV)}{n_2(dV)} - \ell n \phi_2 \left(\frac{V_2(dV)}{n_2(dV)} \right) \right) \right\} \tag{2.4}$$

where $V_1(dV)$ is the volume occupied by molecules of solid in dV and $V_2(dV)$ is the volume occupied by molecules of liquid in dV. If we use χ to denote the average over dV of the characteristic function of Ω_2, as we did in (1.2), we see that

$$V_2(dV) = \chi dx, \; V_1(dV) = (1 - \chi) dx. \tag{2.5}$$

The functions ϕ_1 and ϕ_2 which occur in (2.4) are to be determined by the equations of state of the solid and liquid, respectively. The pressures p_1 and p_2 are given by [11]

$$\frac{\phi_i' \left(\frac{V_i(dV)}{n_i(dV)} \right)}{\phi_i \left(\frac{V_i(dV)}{n_i(dV)} \right)} = - \frac{p_i}{k_i T_i} . \tag{2.6}$$

If, for example, solid and liquid are assumed to be incompressible and to have a common volume α per molecule, we may consider ϕ_i, i = 1,2, to be given by the limit as $\varepsilon \to 0$ of

$$- \frac{\phi_i'(\xi)}{\phi_i(\xi)} = \begin{cases} 0 & \xi > \alpha \\ \frac{1}{\varepsilon} \left(\frac{1}{\xi} - \frac{1}{\alpha} \right) & \xi < \alpha \end{cases} \tag{2.7}$$

In the limit as $\varepsilon \to 0$ we will get $n_i(dV) = \frac{1}{\alpha} V_i(dV)$ and, with $k_i = \alpha c_i$, i = 1, 2, $U_1(dV) + U_2(dV)$ will be identical with udx where u is given in (1.2), except for an unimportant constant multiplying dx. The incompressible limit will be the regime of primary importance to us in the sequel.

The equality of T_1 and T_2 at a point follows from the requirement that $S(\Omega)$ be a maximum for variations $\delta U_1(dV) = -\delta U_2(dV)$ with $V_i(dV)$ and $n_i(dV)$

fixed. The equality of μ_1 and μ_2 at a point follows for variations $\delta n_1(dV) = -\delta n_2(dV)$ with $V_i(dV)$ and $U_i(dV)$ held fixed. The relation between p_1 and p_2 at a point follows from maximization of $S(\Omega)$ with respect to variations in which $\delta V_1(dV) = -\delta V_2(dV)$, $\delta U_2(dV) = 0$, $\delta n_i(dV) = 0$, $i = 1, 2$, and with total energy contained in dV held fixed. This total energy has, in addition to $U_1(dV)$ and $U_2(dV)$, a contribution from the surface-tension energy in dV which may vary in some way as χ varies, and this variation will have to be studied in order to find a relation between p_1 and p_2. This is done in the next section.

The variations have to be taken under the constraints that $U_i(dV) \geq 0$, $n_i(dV) \geq 0$, and $V_i(dV) \geq 0$, $i = 1, 2$. It may happen that the maximum of $S(\Omega)$ is achieved at an extreme point on one of these constraining sets, as for example in $\mathrm{Int}(\Omega_1)$ where $\chi = 0$ and in $\mathrm{Int}(\Omega_2)$ where $\chi = 1$.

Details regarding the role of surface-tension energy will have to be examined further. However, it seems clear that, whereas the structure of the classical Stefan problem is that of a single conservation law, the Stefan problem with surface tension has the structure of a conservation law in tandem with a principle of entropy maximization. Energy is propagated from point to point in conservative fashion by means of the thermal diffusion processes, that is, (1.6) holds for all sufficiently smooth domains V in Ω. At the same time, values of χ and T in a volume dV are determined by maximizing the entropy with respect to a class of variations for which U(dV), dx, and $n(dV) = n_1(dV) + n_2(dV)$ are kept fixed. The appearance of an entropy principle in the Stefan problem with surface tension is reminiscent of the use of entropy principles in the theory of hyperbolic conservation laws [8].

3. ENERGY OF SURFACE TENSION

As we have pointed out, for each volume dV the total energy U(dV) has three components,

$$U(dV) = U_1(dV) + U_2(dV) + U_s(dV), \tag{3.1}$$

where $U_s(dV)$ is the surface-tension energy contained in dV. The relation between p_1 and p_2 at a point follows from requiring that $S(\Omega)$ be a maximum with respect to variations with $\delta U(dV) = \delta U_2(dV) = 0$ and $\delta(V_1(dV) + V_2(dV)) = 0$. From (2.5), it follows that in this case $\delta(dx) = 0$, and

$$\delta V_1(dV) = -(\delta\chi)\,dx, \quad \delta V_2(dV) = (\delta\chi)\,dx, \tag{3.2}$$

that is, the volume variation may be thought of as a variation of χ. If $S(\Omega)$ is given by an expression like (2.4), one will need to know how $U_1(dV)$ varies as χ varies, and, since

$$\delta U_1(dV) = -\delta U_s(dV), \tag{3.3}$$

one will need to know the dependence of $U_s(dV)$ on χ.

When the free boundary is sharp and γ is independent of T, there is no component of entropy for the interface [6], and

$$\frac{d}{dt}\int_V (U_1(dV) + U_2(dV)) = \int_V T\frac{dS(dV)}{dt} - \int_V \left(p_1 \frac{d(V_1(dV))}{dt} + p_2 \frac{d(V_2(dV))}{dt} \right) \tag{3.4}$$

reduces, because of (2.2) and (1.6) to

$$-\int_V \left(p_1 \frac{dV_1(dV)}{dt} + p_2 \frac{dV_2(dV)}{dt} \right) + \frac{d}{dt}\int_{V\cap\partial\Omega_1\cap\partial\Omega_2} \gamma dS = 0. \tag{3.5}$$

Here we have used (3.1), and for $U_s(V)$ we have substituted the second term on the right-hand side of (1.5). From (3.5) and (1.7) it follows, in the incompressible limit, that

$$p_1 - p_2 = \gamma\kappa. \tag{3.6}$$

Although $U_s(V)$ will be given by the second term on the right-hand side of (1.5) when the interface is sharp, it will be necessary for us to replace this by an expression which is meaningful for values of χ in (0,1), if the approach we develop is to be geometry-independent. The expression which comes most readily to mind is

$$U_s(V) = \int_V \gamma|\nabla\chi|\,dx. \tag{3.7}$$

Indeed, with this expression the maximization of $S(\Omega)$ as given by (2.4) becomes a problem in the calculus of variations, and the Euler equation that results leads to (3.6) in the appropriate limiting case of a sharp interface.

However, there are two problems which we have in working with (3.7). First, we have a singularity of energy at a sharp interface, and there is some question about the physical origin of such a singularity, as well as with regard to a mechanism for its propagation. Requiring that the interface have a finite thickness does not seem to be a satisfactory way of resolving this difficulty. A second problem is that, with the surface-tension energy (3.7), the interpretation of a value of $\chi \in (0,1)$ as representing an actual combination of microscopic volumes of solid ($\chi = 0$) and liquid ($\chi = 1$) is meaningless, as (3.7) would ascribe to such a mixture an infinite surface-tension energy.

The origin of the phenomenon of surface tension appears to be the following [4]: When liquid molecules become solid, they release a certain amount of energy and enter bound molecular states. This binding is a nonlocal effect, characterized by a small distance λ. The solid molecules at the surface are not bound to as many other solid molecules as are those in the interior. Thus, although they have less energy than the liquid molecules, relative to the solid molecules in the interior they have an energy excess, which is called the "surface energy". When one is dealing with length scales large compared to λ, this "surface energy" appears to be concentrated on a sharp surface.

The effect of surface tension is thus a molecular effect. We are interested in phenomena on a length scale large compared to λ, so that we are justified in using a continuum approach. But we do not want the formulation which we use to describe macroscopic effects to behave pathologically when the functions we deal with fail to be smooth on a small scale. The expression we use for the surface-tension energy will be chosen so as to look very much like the energy corresponding to the picture given above when one is dealing with length scales large compared to λ, but it will not represent the actual physical situation on molecular length scales, although it will give a mathematically well-behaved treatment of phenomena on those length scales.

Since the surface-tension energy density will be proportional to the number of solid molecules at a point, with density $(1 - \chi)/\alpha$ (in the incompressible case), and will exist because of the proximity of molecules of liquid, with density χ/α, to which the solid molecules are not bound, and since the energy of binding per molecule of the solid molecules to $\frac{4}{3}\pi\lambda^3/\alpha$

other solid molecules within a distance λ would be αL, we write for the surface-tension energy for the mixture occupying a domain V,

$$U_s(V) = \int_V \frac{3}{4\pi\lambda^3} \int_{|y-x|<\lambda} \beta L(1 - \chi(x))\chi(y)dydx. \tag{3.8}$$

Here β is a geometric factor, depending on the crystalline structure of the solid [4].

If we use (3.8) to evaluate $\delta U_s(dV)$ resulting from the variation of χ at different points and use (3.3), we will find that the requirement that $S(\Omega)$ be a maximum with respect to such variations leads to the following equation when changes in T over a distance λ can be ignored:

$$p_1(x) = p_2(x) - \frac{3L\beta}{4\pi\lambda^3} \int_{|y|<\lambda} (1 - 2\chi(x + y))dy. \tag{3.9}$$

In particular, in the solid region ($\chi = 0$), away from the interface, (3.9) gives $p_1(x) = p_2(x) - \beta L$, and in the liquid region ($\chi = 1$), away from the interface, we get $p_1(x) = p_2(x) + \beta L$. In the case when $\chi = 0$ for $|x| < R_o$ and $\chi = 1$ for $|x| > R_o$, with $R_o > \lambda$, we find for $|x| = R_o$ that

$$p_1(x) = p_2(x) + \frac{3}{8}\beta L\frac{\lambda}{R_o}. \tag{3.10}$$

This is a result like (3.6) if we let

$$\gamma = \frac{3}{16}\beta L\lambda. \tag{3.11}$$

In a similar way we examine the consequence of the fact that $S(\Omega)$ is a maximum with respect to transfers of volume between points x_1 and x_2, without variation of χ anywhere. Neglecting the difference in temperatures between x_1 and x_2, we get

$$p(x_1) = p(x_2) + \frac{3\beta L}{4\pi\lambda^3}\chi(x_1) \int_{|y|<\lambda} (1 - \chi(x_1 + y))dy$$

$$+ \frac{3\beta L}{4\pi\lambda^3}(1 - \chi(x_1)) \int_{|y|<\lambda} \chi(x_1 + y)dy$$

$$- \frac{3\beta L}{4\pi\lambda^3} \chi(x_2) \int_{|y|<\lambda} (1 - \chi(x_2 + y))dy$$

$$- \frac{3\beta L}{4\pi\lambda^3} (1 - \chi(x_2)) \int_{|y|<\lambda} \chi(x_2 + y)dy, \tag{3.12}$$

where

$$p(x) = (1 - \chi(x))p_1(x) + \chi(x)p_2(x). \tag{3.13}$$

4. NUMERICAL TREATMENT

If we try to describe solidification processes numerically with a spatial grid of size Δx, we cannot expect to be able to treat phenomena occurring on a smaller scale. Each cell will have a single value of χ, and thus the numerical scheme will inevitably mix quantities over a distance Δx. Accordingly, details of scale smaller than Δx will appear as "slush". Nuclei of smaller scale cannot be treated, and the maximum supercooling of liquid which could be effectively handled would be $\frac{\gamma}{L(\Delta x)} T_M$.

Since the interface will have to be spread out over at least a distance Δx, which will generally be much larger than λ, for numerical purposes one would replace (3.8) by

$$U_s(V) \sim \int_V \frac{3}{4\pi a^3} \int_{|y-x|<a} \frac{\beta\lambda L}{a}(1 - \chi(x))\chi(y)dydx, \tag{4.1}$$

where a is a length which should be chosen (in the lowest-order integration scheme) to be large compared to Δx, but small compared to the scale of details of interest. β has been multiplied by the factor λ/a so as to keep γ, given by (3.11), invariant.

In a numerical alogrithm based on the expression of the problem as a conservation law plus an entropy maximization, energy would be transferred from one cell to another by means of the thermal diffusion. The entropy functional would then be maximized with respect to internal transfers of energy, volume, and particles between the phases in each cell, and with respect to transfers of volume from one cell to another.

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L RUBINSTEIN

On mathematical models for solid–liquid zones in a two-phase monocomponent system and in binary alloys

INTRODUCTION

In what follows phenomenological models are proposed, describing creation and development of solid-liquid zones (mushy regions) in the course of solidification/melting of a monocomponent medium or binary alloys. These models are macroscopic ones and include phenomenologically defined kinetic equations, describing the interphase heat and mass exchange within the mushy regions, as well as effective parameters which cannot be determined otherwise than experimentally.

The main feature of the model proposed consists in distinguishing the temperature and concentrations in the bulk of the liquid/solid phases within the mushy region and those at the phase interfaces. It is in contrast with models described in a relevant literature (Tien and Geiger, 1967; Licht and Kuzminskaja, 1961; Atthey, 1974; Avdonin, 1980) where the temperature and concentrations in the bulk of both the phases are supposed to be satisfying the diagram of the isobaric thermodynamic equilibrium.

First we formulate equations of the problem for a binary alloy capable of forming a continuous set of solid solutions. Setting in these equations the concentration of a more refractory component (the solvent) in the solid phase equal to one, we obtain the model for a mushy region in a binary alloy capable of forming a simple eutectic mixture. If this concentration is identically equal to the one in liquid phase too, the equations become those for a mushy region in a monocomponent medium. All equations are derived assuming that all hydrodynamic effects may be disregarded (first of all those caused by the jump of densities at phase interfaces and the variation of densities caused by the change of concentration) and that there are distributed volume sources/sinks of heat. If these sources/sinks of heat are different from zero then one may pass from the equations of "two-temperature" models to those of a standard "one-temperature" formalism, where no distinguishing the temperature in a bulk of the solid and the liquid phase is performed. These equations may be obtained in all cases by equalizing temperature of both the phases within the mushy region and

introducing the relevant effective parameters. If there are no distributed volume sources/sinks of heat equations of the one-temperature formalism remain applicable in all cases except for the case of creation and development of a mushy region in a monocomponent medium. In this case there are neither sources of a latent heat to be absorbed in a course of melting, nor receiver of a latent heat released in the course of solidification.

1. MODELLING THE MUSHY REGION IN A BINARY ALLOY CAPABLE OF FORMING A CONTINUOUS SET OF SOLID SOLUTIONS

We consider a binary alloy A:B occupying the region G:

$$G = G_s \cup G_f \cup G_{sf} \tag{1}$$

where G_s and G_f are subregions occupied by a pure solid and pure liquid phase, and G_{sf} - is a mushy region. Superscripts s and f refer to values in G_s and G_f respectively. Subscripts s and f refer to the solid and liquid phases in G_{sf}. Subscript m refers to values at the phase interfaces within the mushy region and on its boundaries with G_s and G_f. Let, further, u, v and σ be the temperature, concentration of the component A and the volume concentration of a solid phase within the solid-liquid zone.

The solid-liquid zone (mushy region) is understood below as a region where particles of both the phases, being of a macroscopic size, are distributed chaotically. Since every particle is considered as a macroscopic one, the usual partial equations of heat and mass transfer are assumed to be applicable to it. However, particles boundaries, which are phase interfaces, are surfaces of strong or at least weak discontinuities of all thermodynamic parameters, determining the state of the system in whole. Therefore the mathematical description of the process evolution in such heterogeneous medium is practically impossible without using the homogenization procedure. This procedure consists in replacing the real heterogeneous medium by a fictitious one. Equations of mass and heat transfer within this fictitious medium are obtained by means of the averaging procedure, applied to the integral form of the respective laws of conservation. Below we follow the averaging algorithm as it is described in L. Rubinstein (1972) where the procedure of taking into account the change of the form of local conservation laws due to variation of the phase concentration is strongly formalized. Equations of the heat and mass transfer within every phase of the fictitious

homogeneous medium contain source-like terms, imitating the interphase exchange. These terms have to be defined by phenomenological kinetic equations containing effective parameters, which may only be prescribed by the use of the experimental data.

The simplest, but the most useful, form of these kinetic equations shows that the rate of the heat/mass transfer from the bulk of the phase under consideration to the phase interfaces is proportional to the difference between the temperature/concentration in the bulk of the phase and the respective local equilibrium values, which are values keeping at the phase interfaces. This form of kinetic equations coincides, in its essence, with that in the theory of a dynamics of sorption of a gas from a stream of poisoned air, injected into the layer of a porous adsorbent (Zuchovitski et al, 1945; Tichonov et al, 1946). This theory has perfectly withstood the comparison with a lot of experimental data. The analogy existing between this sorption-diffusion process and the process of solidification/melting within the mushy regions allows to expect that such a form of phenomenological kinetic equations is a satisfactory one.

After these remarks we may write all system of equations of the problem under consideration. Comments to conditions on boundaries of G_{sf}, as well as on the character of initial conditions will be done later. We have

A. Equations of heat transfer within the solid-liquid phase of the mushy region

$$\begin{aligned} &\mathrm{div}(k_f(1-\sigma)\,\mathrm{grad}u_f) - \mathrm{div}(k_f(u_f-u_m)\,\mathrm{grad}\sigma) - SJ_f + (1-\sigma)F_f \\ &\quad = c_f\rho_f[(1-\sigma)u_{ft} + (u_f-u_m)\sigma_t]; \\ &\mathrm{div}(k_s\,\sigma\,\mathrm{grad}u_s) + \mathrm{div}(k_s(u_s-u_m)\,\mathrm{grad}\,\sigma) + SJ_s + \sigma F_s \\ &\quad = c_s\rho_s[(\sigma\,u_{st} + (u_s-u_m)]\sigma_t\ . \end{aligned} \tag{2}$$

Here F_f and F_s are density of distributed sources of heat per unit of volume. J_f and J_s denote fluxes of heat to/from the phase-interfaces from/into the bulk of the respective phase. According to what is said above

$$J_s = \alpha_s(u_m - u_s);\quad J_f = \alpha_f(u_f-u_m) \tag{3}$$

S-denotes the specific surface of interfaces within G_{sf}.

B. Equations of mass transfer within the solid/liquid phases of G_{sf}.

In what follows diffusion in a solid phase is taken into account since D_s may be comparable with D_f. For instance the coefficient of diffusion of Au in Pb at 285°C is equal to 4.6 10^{-6} cm^2/sec which is only one order smaller than characteristic values of coefficients of diffusion in liquids. Thus we have

$$\begin{aligned} &\mathrm{div}(D_s \sigma\, \mathrm{grad} v_s) + \mathrm{div}(D_s(v_s - \Psi_s(u_m))\, \mathrm{grad}\sigma) + SI_s \\ &\quad = v_{st} + (v_s - \Psi_s(u_m))\, \sigma_t; \\ &\mathrm{div}(D_f(1-\sigma)\mathrm{grad} v_f) - \mathrm{div}(D(v_f - \Psi_f(u_m))\mathrm{grad}\sigma) - SI_f \\ &\quad = v_{ft} + (\Psi_f(u_m) - v_f)\, \sigma_t \end{aligned} \tag{4}$$

where the mass exchange terms are defined as

$$I_s = \beta_s(\Psi_s(u_m) - v_s);\ I_f = \beta_f(v_f - \Psi_f(u_m)) \tag{5}$$

C. Equations determining the rate of the change of concentration of the solid phase within G_{sf}

$$\gamma\rho_s\sigma_t = S(J_s - J_f);\ (v_s - v_f)\sigma_t = S(I_f - I_s) \tag{6}$$

These equations replace thermal and diffusion Stefan conditions, which are valid if the process of solidification/melting evolves without creation of the solid-liquid zone.

D. Diagram of the phase equilibrium (equations of solidus and liquidus)

$$v_{sm} = \Psi_s(u_m);\ v_{fm} = \Psi_f(u_m) \tag{7}$$

where Ψ_s and Ψ_f are monotonically decreasing functions and $\Psi_f < \Psi_s$ for $0 < v_f < 1$.

E. Equations of the heat and mass transfer in G_s and G_f

$$\mathrm{div}(k^i \mathrm{grad} u^i) + F^i = c^i\rho^i u^i_t;\ \mathrm{div}(D^i \mathrm{grad} v^i) = v^i_t;\ i = s,f \tag{8}$$

F. Initial conditions in G_{sf}

There exists a serious difficulty in prescribing realistic initial location

of a mushy region and distribution of the temperature and concentrations within it, when one considers the appearance of mushy region in a course of the process evolution. Indeed, each process of the change of the phase state of the first kind, i.e. accompanied by release or absorption of a latent heat cannot evolve without some undercooling or overheating. The time Δt of the exit from the undercooled (overheated) state is much shorter than the duration of the process of solidification (melting). Therefore this exit may be considered as a spontaneous (instant) one. The depth of undercooling (overheating) must be described phenomenologically, or, better still, with a direct use of experimental data. In any case it cannot be evaluated in the course of solution of any Stefan-like problem.

As a first hypothesis, serving for determining the initial location of a mushy region, we propose to accept the following

1. The mushy region at the moment of its creation may be described as a region where the spontaneous exit takes place from the undercooled (overheated) state.

2. At the moment of spontaneous exit from this state the critical depth of undercooling (overheating) is reached at least at one point of that region.

This means the following. Let $u = g_f(v)$ be the equation of liquidus, solved with respect to the temperature u. Assume that the mixture of solid and liquid particles appears at the moment $t = 0$ in the region G°_{sf}. Then everywhere in G°_{sf}

$$\sigma = 0;\ g_f(v^f) - \varepsilon \le u^f < g_f(v^f) \text{ at } t = 0 \tag{9}$$

and

$$u^f = g_f(v^f) - \varepsilon \text{ at least in one point of } G_{sf} \text{ at } t = 0 \tag{9*}$$

Here $\varepsilon > 0$ is a maximal depth of undercooling.

Analogous conditions may be proposed for the case of exit from the overheated state in the course of melting.

G. Conditions on the boundaries of G_{sf}

Let

$$S_f = \bar{G}_f \cap \bar{G}_{sf}; \qquad S_s = \bar{G}_s \cap \bar{G}_{sf} \tag{10}$$

be the boundaries separating the solid-liquid zone from the pure liquid and

respectively pure solid zones. In a real heterogeneous medium the continuity of the temperature, concentration and their fluxes take place at all points of $S_s(S_f)$ where $S_s(S_f)$ separates $G_s(G_f)$ from solid (liquid) particles of G_{sf}. In contrast to this at all points of $S_s(S_f)$ where $S_s(S_f)$ separates $G_s(G_f)$ from liquid (solid) particles of G_{sf} temperature changes continuously, but concentrations satisfy equations of solidus and liquidus respectively, whereas fluxes of heat and mass satisfy the thermal and diffusion Stefan conditions. Taking this into account and applying the averaging procedure we find that in a fictitious homogeneous medium must be valid the following conditions of conjugation

$$u^s = u_s \overset{\text{def}}{=} u_m;\ k^s u^s_n - k_s u_{sn} = O;\ k^s u^s_n - k_f u_{fn} = \gamma\rho_s n_t$$

$$v^s = v_s = \Psi_s(u_m);\ v_f = \Psi_f(u_m);\ (v_s - v_f)n_t = D_f v_{fn} - D^s v^s_n \qquad \text{on } S_s \qquad (11)$$

$$u^f = u_f \overset{\text{def}}{=} u_m;\ k_s u_{sn} - k^f u^f_n = \gamma\rho_s n_t;\ k_f u_{fn} - k^f u^f_n = O$$

$$v^f = v_f = \Psi_f(u_m);\ v_s = \Psi_s(u_m);\ D^f v^f_n - D_f v_{fn} = O;\ (v_s - v_f)n_t \qquad \text{on } S_f \qquad (12)$$

$$= D^f v^f_n - D_s v_{sn}$$

Let us emphasize that $k^s,\ldots,D^f$ are physical parameters whereas $k_s,\ldots D_f$ are no more than effective ones, dependent on the connectedness of the subset of points of real mushy region, belonging to each of the phases. The specific surface S if the interfaces in G_{sf} cannot also be prescribed theoretically, which makes the model above the semi-empirical only. Whether this model is sufficiently realistic or not may be only elucidated by juxtaposition of results of numerical evaluations and the respective experimental data. We hope that such a juxtaposition will be performed in the future.

2. MODELLING THE MUSHY REGION IN BINARY ALLOYS CAPABLE OF FORMING A SIMPLE EUTECTIC MIXTURE AND IN MONOCOMPONENT MEDIUM. PASSAGE TO THE ONE-TEMPERATURE FORMALISM.

As it is said in the introduction models for mushy regions in binary alloys capable of forming a simple eutectic mixture, and in a mono-component medium may be obtained by setting

$$v^s \equiv v_s = 1 \text{ for a binary alloy}$$

$$v^s \equiv v_s \equiv v^f \equiv v_f = 1 \text{ for a monocomponent medium} \tag{13}$$

In these cases respective equations serving for definition of concentrations in solid/solid and liquid phases must be omitted.

In order to pass to a standard one-temperature models define

$$g = \sigma g_s + (1 - \sigma) g_f \text{ for } g = k,\ \rho,\ c\rho \text{ or } F, \tag{14}$$

and equalize the temperature of both the phases

$$u_s = u_f = u_m \overset{\text{def}}{=} u \tag{15}$$

Then summing equations (2) and taking into account (6) we obtain

$$\text{div}(k\,\text{grad}u) + \gamma\rho_s\sigma_t + F = c_\rho u_t \text{ in } G_{sf} \tag{16}$$

In a monocomponent medium

$$u_m \equiv \text{const} \tag{17}$$

Hence (16) turns into

$$\gamma\rho_s\sigma_t + F = 0 \tag{18}$$

If follows from here that in the absence of distributed sources of heat, i.e. when $F \equiv 0$,

$$\sigma \equiv \text{const} \tag{19}$$

which shows that the standard one-temperature model is unapplicable for describing the creation and development of mushy region in monocomponent media in the absence of distributed volume sources/sinks of heat.

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J SZEKELY

Some mathematical physical and engineering aspects of melting and solidification problems

INTRODUCTION - PERSPECTIVES

The purpose of this paper is to discuss some problems that arise in the tackling of the engineering aspects of melting and solidification problems, where compromises must be made regarding both the faithful representation of physical reality and mathematical rigor. This dilemma is sketched in Figure 1.

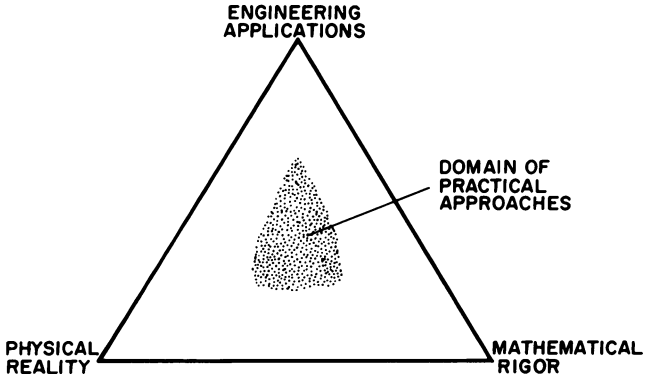

Figure 1. The conflict between physical reality, mathematical rigor and engineering applications.

The rather more pragmatic approach which is necessarily required, in order to obtain engineering answers to practical problems may be in quite sharp contrast to the elegant mathematical contributions to this volume, nonetheless it may provide a perspective and perhaps also indicate a useful source of problems which may benefit from the attention of those with mathematical skills.

Figure 2 shows a selection of sketches of practical problems, involving melting or solidification, which have received or are receiving attention at this time. It is seen that these systems may differ quite appreciably regarding the characteristic dimension of the phase in which the phase change occurs, with a corresponding major difference in the time scale. The solidification of an ingot will take several hours, while the time span of

interest in rapid solidification is in the microsecond range.

A very important point has to be stressed here, is that in many cases in these systems the principal practical problem of interest is not solidification per se, but rather some other property of the system, such as the solidified structure, the distribution of impurities or the mechanical characteristics of the solidified product. However, in other cases, principally in new process development the rate of solidification (or melting) still remains a point of interest.

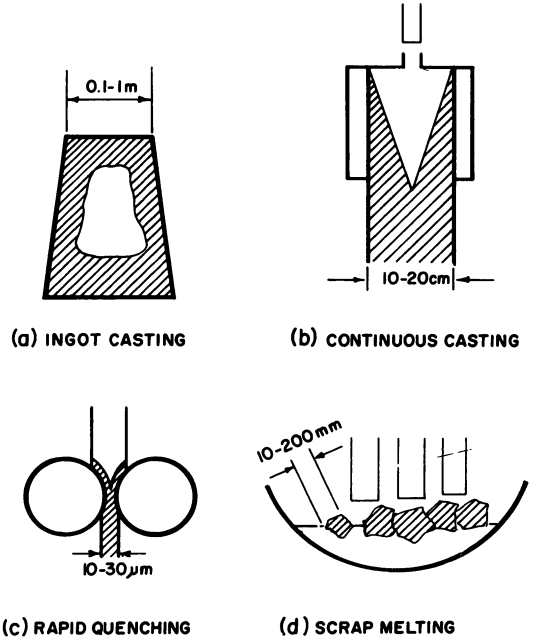

Figure 2. Some practical problems involving melting and solidification.

In the following we shall consider a range of melting and solidification problems, using the classical Stefan problem as the benchmark and will explore the possible complicating factors that may arise in practice and will briefly discuss the means that have been devised for handling these.

THE STATEMENT OF MELTING OR SOLIDIFICATION PROBLEMS

Figure 3 shows a sketch of the classical Stefan Problem, a semi infinite medium, extending from $y = 0$ to $y = \infty$ is initially at some uniform temperature $T_{m,i}$. At time = 0 the temperature of the surface corresponding to

$y = 0$ is suddenly reduced to $T_{s,o}$ which is below the melting point, T_{imp}. As a result a solidified layer is formed (indicated by the cross hatched lines) the thickness of which increases with time.

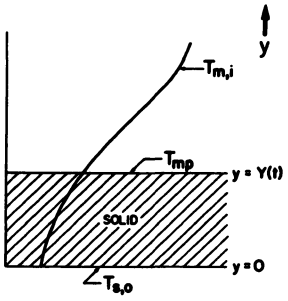

Figure 3. Sketch of the classical Stefan problem

As it has been fully documented in the literature (2.3), such problems are readily formulated by writing the Fourier's equation for each phase,

$$\alpha_i \frac{\partial^2 T_i}{\partial y^2} = \frac{\partial T_i}{\partial t} \tag{1}$$

$$i = s,m$$

with the boundary conditions being given as

$$T_s = T_{s,o} \qquad \text{at} \quad y = 0 \tag{2}$$

$$T_m = T_{m,i} \qquad \text{at} \quad t = 0 \tag{3}$$

and

$$T_m \to T_{m,i} \qquad \text{as } y \to \infty \tag{4}$$

The boundary conditions at the solid-melt boundary, i.e. at $y = Y(t)$ take the following form

$$T_s = T_m = T_{mp} \text{ at } y = Y(t) \tag{5}$$

$$k_m \frac{\partial T_m}{\partial y} - k_s \frac{\partial T_s}{\partial y} = \rho_s \Delta H \frac{dY}{dt} \qquad \text{at } y = Y(t) \tag{6}$$

Here Equations (5) and (6) express the fact that at the melt - solid boundary both the solid and the liquid phases are at the melting point and that thermal energy is being conserved at the melt line respectively.

As discussed in Carslaw and Jaeger (2) analytical solutions are available for the system of Equations (1-6) and these result in the well known parabolic growth curves for the solidification front. It is of interest to note that these linear equations represent the very few cases for which closed form analytical solutions are available.

Let us now turn our attention to the complexities that have to be considered in the representation of practical problems.

Convection in the Molten Phase

In many practical systems involving melting or solidification the molten phase will not be stationary but will undergo convective motion. Under these conditions Equation (1) and the boundary condition contained in Equation (6) will be inappropriate because thermal energy will be transferred to the phase boundary by convection, that is the bulk motion in the fluid, rather than by convection.

Thus under these conditions Equation (6) would have to be replaced by

$$h(T_b - T_{mp}) - k_s \frac{\partial T_s}{\partial y} = \rho \Delta H \frac{dy}{dt} \qquad (7)$$

where h is a heat transfer coefficient.

The temperature profiles corresponding to this situation are sketched in Figure 4, where it is seen that the temperature gradients are confined to the solid phase and to the immediate vicinity of the melt - solid interface.

h, the heat transfer coefficient may be regarded either as an empirical parameter, which has to be determined experimentally; alternatively we may define h in the following manner

$$h = \left(\frac{q_y}{\frac{\partial T_m}{\partial y}} \right)_{y=Y} \equiv \frac{q_n}{\frac{\partial T_m}{\partial n}} \qquad (8)$$

here q_y or q_n represent the convective heat flux from the melt to the melt - solid interface. q_n may be evaluated from the solution of the (turbulent) thermal energy balance equation, viz.

$$\frac{DT_m}{DT} = \nabla\alpha_m \nabla T_m \tag{9}$$

which in turn requires the knowledge of the velocity field in the melt and hence the solution of the turbulent Navier-Stokes equations

$$\frac{D\underset{\sim}{U}}{Dt} = \underset{\sim}{\nabla} \cdot \underset{\sim}{\tau}^t + \rho\underset{\sim}{g} \tag{10}$$

Work of this type has been reported for certain specific systems. (4,5,6)

It should be noted that when there is convection in the molten phase, which tends to be the case for most practical systems, one either has to resort to empiricism, or has to face the much greater complexity inherent in solving both fluid flow and convective heat flow equations in the molten state, in order to represent the conditions at the melt - solid interface.

The Behavior of Multi-Component Systems

Much of the literature of melting and solidification problems has been devoted to the study of one component, (i.e. pure) systems, which are characterized by uniquely defined, single melting temperature. However, virtually all practical systems encountered in processing applications involve multi-component systems, the behavior of which differs appreciably from pure materials.

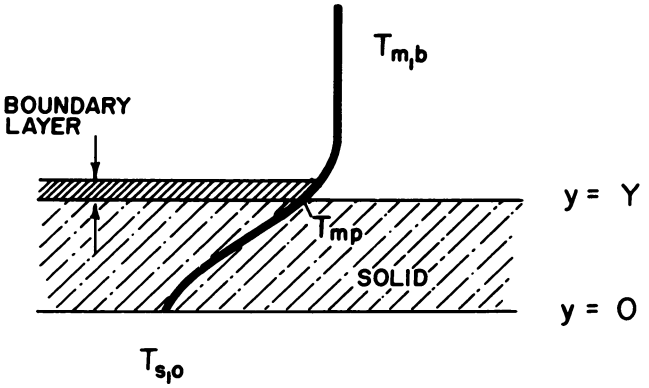

Figure 4. The temperature profile in case of convection in the molten phase.

Figure 5 shows the phase diagram of the iron-carbon system and it is seen that even relatively small changes in the composition of the melt have a marked effect on the melting temperature. Thus an iron melt containing as little as 0.05 wt % carbon has to be regarded as a binary.

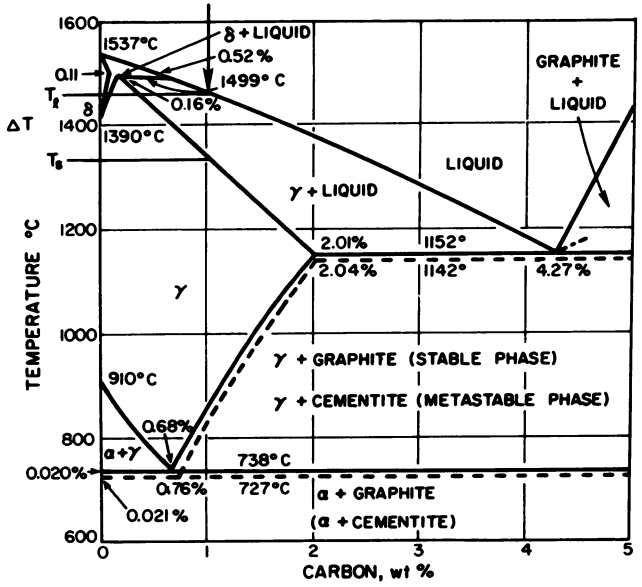

Figure 5. The iron - carbon phase diagram

The key aspects of the solidification of multi-component, or even binary systems may be summarized as follows

(a) In multi-component or in binary systems melting or solidification does not occur at a fixed temperature, but rather, phase change occurs over a temperature range which is composition dependent.

(b) In the majority of cases, rather than having a well defined boundary between the molten and the solid phases, one has to contend with three regions, which as sketched in Figure 6 involve

a solid phase

a two phase region, where molten and solid phases coexist and

a molten phase

The actual structure of the two phase region, that is the size and the spacing of the solid "fingers" termed dendrites may play a very important rule in determining the properties of the solidified material. This point needs to be emphasized further, by stating that the distribution of impurities in the interdendritic space may be crucial regarding the quality

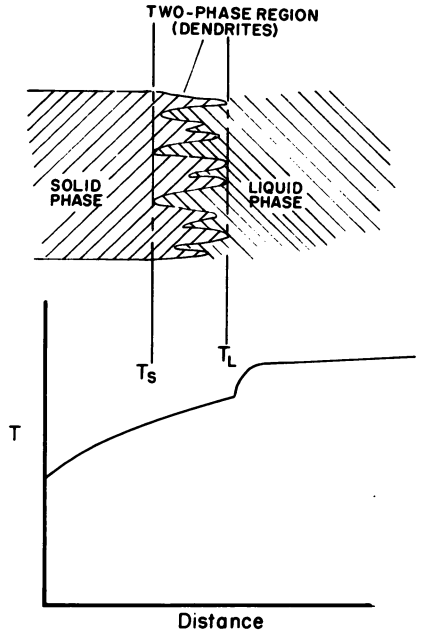

Figure 6. The dendritic solidification of a two-component mixture

of the material produced.

A number of very important problems still require resolution regarding the solidification of multi-component systems. Some of these are listed in the following

(i) The quantitative criteria have to be established to relate the formation of dendrites and the structure of the dendritic zones to the conditions of solidification. (The problems of morphological stability). (1,7)

(ii) The distribution and redistribution of rejected solutes in the interdendritic regions still need further attention, notwithstanding the progress that has been made in this area.

(iii) When the fluid velocities in the melt are high, the tips of the dendrites may be sheared off, thus a two phase region may exist, both in the vicinity of the solid and also in the bulk of the melt, as sketched in Figure 7. (8,9).

The definition of the precise conditions, which cause the shearing off of the dendrites would also deserve further study.

Figure 7. Suspended solid crystals in an agitated melt.

Non-Equilibrium Systems

The mathematical representation of melting and solidification processes described up to the present assumed the existence of thermodynamic equilibrium between the molten and the solid phases. There are, however, a number of applications when these conditions are not met. Two such systems will be noted briefly here.

(i) At extremely rapid cooling rates, say 10^5 - 10^6 °C/s rather than forming solid crystal, a glass is produced. An important consequence of this is that under these conditions no latent heat or solidification needs to be absorbed. Certain glassy materials, in particular metallic glasses, have very attractive mechanical properties.

(ii) A second group of non-equilibrium processes, worthy of mention, involve the super-cooling of melts. Under certain conditions it is possible to cool a metallic melt several hundred degrees °C below the freezing temperature, before freezing is actually initiated. Frequently the onset of freezing is a nucleation process, which then involves the liberation of the latent heat of freezing, such as sketched in Figure 8. The mathematical modelling of both these systems would be a very fruitful area of further research.

CONCLUDING REMARKS

In this brief review, the author has attempted to highlight some of the melting and solidification problems which are of current engineering interest. Perhaps the most important of these are solidification in the presence of

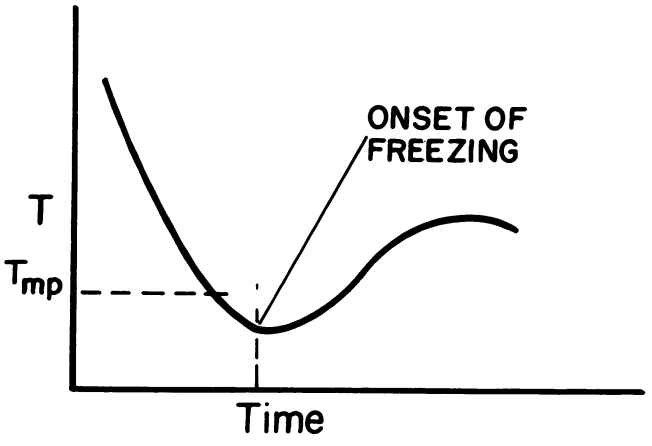

Figure 8. The time dependent temperature profile experienced when a melt is sub-cooled.

convection the behavior of multi-component systems (viz dendrite formation!) and the nonequilibrium systems such as glass formation and supercooling.

If this brief summary has stimulated interest among the mathematicians in tackling these fascinating but very complex problems, the author's time will have been very well spent in preparing this compilation.

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B D TURLAND

Finite difference front-tracking algorithms for Stefan problems arising in reactor safety studies

1. INTRODUCTION

Assessments of the residual risk from nuclear power plants require the modelling of highly unlikely accidents in which part of the reactor core melts. The molten material, in which heat is generated by the decay of unstable fission products, contacts structures within the reactor vessel, and if the vessel is penetrated, could interact with the basemat. Large computer codes exist that predict the response of reactor containments to such accidents and these require algorithms for the various melting interactions. It is necessary that the algorithms contain sufficient detail to avoid gross inaccuracy, but should not lead to large increases in running time for the codes in which they are incorporated. (In some cases simple heat balances are sufficient.) Various approximate analytic [1] and numerical procedures have been investigated. For efficient numerical algorithms we require (i) good conservation properties (ii) good discretization of the thermal flux, particularly close to the melt front, and (iii) engineering accuracy (i.e. $\sim$3% in front propagation speed) with a limited number of nodes and reasonably large timesteps.

The problems to be solved can be divided into two groups. In the first, heat penetrates the whole medium before significant melting occurs and convective cooling of the far boundary of the medium is significant (e.g. thermal attack on a thin steel slab). An algorithm (the 'gamma' method) based on modifications to a standard finite difference approximation of the heat equation on a fixed domain has been developed for this group of problems (see section 2). In the second group a thermal front, associated with the moving melt-front, propagates into an ambient medium (e.g. growth of a melt-pool in the basemat), and a conservative form of the isotherm migration method (IMM) suited to this group is presented in section 3. Solutions for test problems demonstrate the efficiency of these algorithms.

2. THE 'GAMMA' METHOD: FRONT TRACKING ON A FIXED MESH IN TWO DIMENSIONS

Consider the melting (ablation) problem shown in Figure 1. ϕ_z is a heat flux

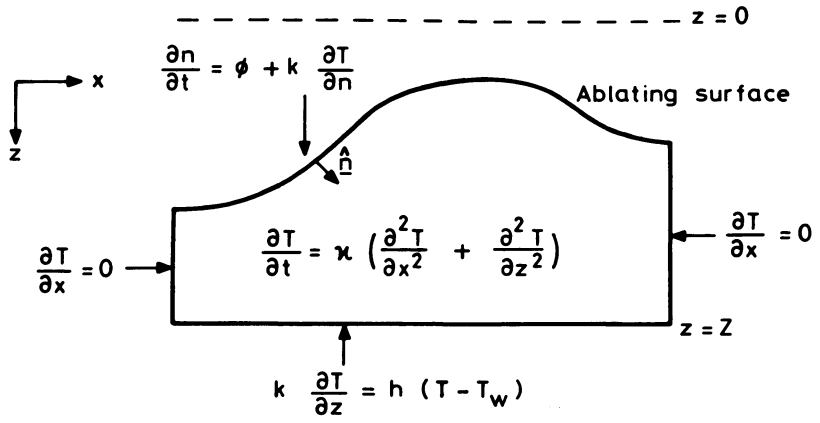

Figure 1. Typical ablation problem

density applied to the upper surface which, once melting has started, is at temperature T_m. The ADI method [2] is the basis of this algorithm for solving the heat equation and modifications required to account for the varying domain are outlined below. The computational mesh is shown in Figure 2. The melt-front position $z = \zeta(x,t)$ must be a single valued

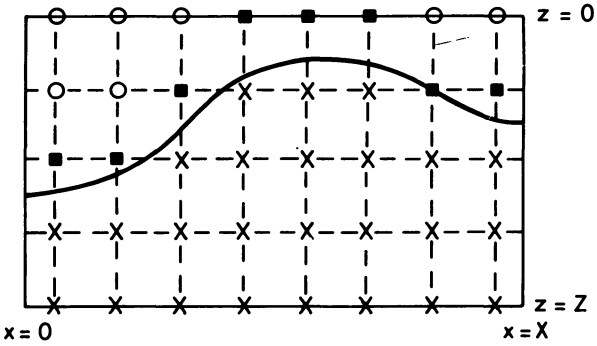

Figure 2. The mesh used in the 'gamma' method. x are normal mesh points in the slab; ■ are guard points, which incorporate the moving boundary condition, and o are not used in the calculation.

function of x at all times, and at time t it is discretized as ζ_i, from which two sets of parameters Γ_i, γ_i are derived

$$\zeta_i = (\Gamma_i - \gamma_i)\delta z \qquad (2.1)$$

where Γ_i is an integer and $0.2 \leq \gamma_i < 1.2$ to restrict the size of the coefficients appearing in equation (2.2) below.

The key to the method is to note that the discretized sets of equations for the domain interior (i.e. for all i, $\Gamma_i < j \leq J$) may be closed by using the usual guard values on the fixed boundaries and by using guard values at nodes (i,j) where $j = \Gamma_i$ obtained by a quadratic extrapolation of the temperature profile through points (i,j+2), (i,j+1) and noting that $T = T_m$ when $z = \zeta_i$. (This is the Lagrange interpolation of Crank [3].) Using (2.1) the following relationship for the guard point is obtained

$$T_{i,j} = a[\gamma_i]T_m + b[\gamma_i]T_{i,j+1} + c[\gamma_i]T_{i,j+2}; \quad j = \Gamma_i \tag{2.2}$$

where

$$a[\gamma] = 2/(\gamma+\gamma^2); \; b[\gamma] = -2(1-\gamma)/\gamma; \; c[\gamma] = (1-\gamma)/(1+\gamma) \tag{2.3}$$

The Stefan condition on the ablation front may be re-written in the form

$$\frac{\partial \zeta}{\partial t} = \phi_z + k\left(1 + \left(\frac{\partial \zeta}{\partial x}\right)^2\right)\left.\frac{\partial T}{\partial z}\right|_{z=\zeta} \quad \text{where } \phi_z = ((1 + \frac{\partial \zeta}{\partial x})^2)^{\frac{1}{2}}\phi \tag{2.4}$$

The discretized form of this equation is always advanced explicitly at the beginning of each half timestep to obtain values of ζ_i, γ_i and Γ_i at the end of the half step. A second-order discretization of $\partial T/\partial z$ is used

$$\frac{\partial T}{\partial z} \approx \frac{T_{i,j+2} - T_{i,j}}{\delta z} - \frac{\gamma}{\delta z}\left(T_{i,j+2} - 2T_{i,j+1} + T_{i,j}\right); \quad j = \Gamma_i \tag{2.5}$$

The half of the ADI with z-discretization implicit uses previously calculated temperature values for x-discretization including old guard point values, but uses the updated values of γ_i and Γ_i to close the tridiagonal system of equations. $\partial \zeta/\partial t$ may then be re-evaluated for the end of the half time step using previously calculated values of γ and Γ. At this stage an iteration on γ and Γ is possible, however, the chosen procedure is to make corrections for changes in $\partial \zeta/\partial t$ during the timestep, and use timestep control to ensure that these correction terms remain small.

The second half of the ADI (x-discretization implicit) is performed in a similar manner. First, equation 2.4 is discretized to obtain an estimate of the domain at the end of the time-step. The tridiagonal systems are then

solved sequentially, starting with j = J reducing j by one each time. This allows updated guard point values (calculated on the estimated domain) to be used when they are required. This effectively masks out the points shown o in Figure 2. At the end of the timestep $\partial\zeta/\partial t$ is re-evaluated, and γ and Γ are recalculated as above.

Further details of the method will be published elsewhere. A one-dimensional version [4] which uses a Crank-Nicolson discretization has proved to be successful and efficient for practical problems [5]. The two-dimensional code has been tested using ablation problems and encouraging results obtained, however results are only given in this paper for the Bonnerot-Jamet problem [6].

3. CONSERVATION FORM OF THE ISOTHERM MIGRATION METHOD (IMM)

The IMM in one dimension [7,8] may be interpreted as a co-ordinate transformation, whereby temperature is replaced as the dependent variable by a spatial co-ordinate (the position of the isotherm). Turland has shown that if the resulting non-linear equation is discretized in a conservative manner the method works well for Stefan problems involving the propagation of a thermal front into an ambient medium [9]. The IMM can be extended to two and three dimensions in a number of ways [10,11,9]. One way is to choose an appropriate orthogonal curvilinear co-ordinate system, denoted (u,v,w), and enact a change of variable making one spatial co-ordinate (u, say) the dependent variable by treating temperature as an independent variable [9].

It can be shown that the heat equation

$$\rho c \frac{\partial T}{\partial t} = \underset{\sim}{\nabla} \cdot (k \underset{\sim}{\nabla} T) \tag{3.1}$$

where ρ is constant but c and k may be functions of temperature, transforms to

$$\frac{\partial \psi}{\partial t} = -\left(\frac{\partial \Phi^T}{\partial T} + \frac{\partial \Phi^v}{\partial v} + \frac{\partial \Phi^w}{\partial w} \right) \tag{3.2}$$

where

$$\psi(v,w,T,t) = \rho c(T) \int_o^u h_1 h_2 h_3 \, du' \tag{3.3}$$

for $h_1 = h_1(u',v,w)$ etc

$$\Phi^T = \frac{h_2 h_3}{h_1} \frac{k}{u_T} \left[1 + \frac{h_1^2}{h_2^2} u_v^2 + \frac{h_1^2}{h_3^2} u_w^2 \right] \tag{3.4}$$

$$\Phi^v = - \frac{h_1 h_3}{h_2} k \, u_v ; \; \Phi^w = - \frac{h_1 h_2}{h_3} k \, u_w \tag{3.5}$$

and

$$h_1 = \left(\frac{\partial s}{\partial u} \right)_{v,w} ; \; h_2 = \left(\frac{\partial s}{\partial v} \right)_{u,w} ; \; h_3 = \left(\frac{\partial s}{\partial w} \right)_{u,v} \tag{3.6}$$

where s is a measure of distance. Non-numeric subscripts denote partial differentiation. The Stefan condition

$$\rho L(\partial n/\partial t) = \underset{\sim}{\phi} \cdot \hat{\underset{\sim}{n}} + k(\partial T/\partial n) \tag{3.7}$$

transforms to

$$\frac{L}{c} \frac{\partial \psi}{\partial t} = F\phi + \Phi^T ; \; F = h_2 h_3 \left(1 + \frac{h_1^2}{h_2^2} u_v^2 + \frac{h_1^2}{h_3^2} u_w^2 \right) \tag{3.8}$$

where quantities are evaluated at the melting temperature.

If w is an ignorable co-ordinate the appropriate forms of ψ, Φ^T,Φ^v and F are given in Table 1. From equation 3.2 a conservative scheme for the quantity ψ(closely related to the heat content) can be constructed using a control volume method in (v,T) space.

	CARTESIAN (u≡y, v≡x)	CYLINDRICAL (u≡r, v≡θ)	SPHERICAL (u≡r, v≡θ)
ψ	ρcy	$\rho cr^2/2$	$\rho cr^3 \sin\theta/3$
Φ^T	$k(1+y_x^2)/y_T$	$k(r^2+r_\theta^2)/rr_T$	$k(r^2+r_\theta^2)\sin\theta/r_T$
Φ^v	$-ky_x$	$-kr_\theta/r$	$-kr_\theta \sin\theta$
F	$(1+y_x^2)^{\frac{1}{2}}$	$(r^2+r_\theta^2)^{\frac{1}{2}}$	$r\sin\theta(r^2+r_\theta^2)^{\frac{1}{2}}$

TABLE 1. Form of ψ, Φ^T, Φ^v and F used in CONIMM.

This has been implemented for problems in two spatial variables in the CONIMM code which uses the three co-ordinate systems given in Table 1. Rather than use an explicit scheme CONIMM uses a time-centred discretization (alternating direction implicit method) with appropriate linearization in the evaluation of some of the derivatives of u. (Linearization is not needed for the temperature dependence of the thermal properties.) This removes the timestep constraint imposed by an explicit method, but also destroys exact conservation: however, in practice this scheme gives answers close to those from an explicit scheme.

4. RESULTS FOR TEST PROBLEMS

It is instructive to determine the performance of these methods and codes on problems already in the literature or with analytic solutions. Three such problems are considered here. The first does not involve melting, but demonstrates the performance of the CONIMM code on a thermal front propagation problem. (In the IMM the movement of the melt-front is simple: the test demonstrates that the method does not degrade the solution of the heat equation.) It is solved using CONIMM's spherical polar option. The second problem is the freezing of a square prism, which is solved with CONIMM's cylindrical option. The third problem is that of Bonnerot and Jamet [6], which can be envisaged as a freezing problem with the liquid at the freezing point. The main difficulty lies in the chosen initial conditions and the steep gradient the melt-front makes with the cartesian co-ordinate system at early times. This problem has been solved using both CONIMM's cartesian option and by the gamma method.

Problem 1 Diffusion of heat from a hot sphere into an ambient medium with temperature dependent thermal properties.
It is required to solve numerically

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 k \frac{\partial T}{\partial R}\right) \tag{4.1}$$

in $1 < R < \infty$ with initial conditions $T = 1$ at $t = 0$ for $R > 1$, and boundary conditions $T = 2$ at $R = 1$ for $t > 0$, and $T \to 1$ as $R \to \infty$ for $t > 0$, where $\rho = 1$ and $c(T)=k(T)=T$. The analytic solution is

$$T(R,t) = \left[1 + \frac{3}{R}\,\mathrm{erfc}\left\{\frac{R-1}{2t^{\frac{1}{2}}}\right\}\right]^{\frac{1}{2}} \qquad (4.2)$$

This problem may be solved using one spatial variable, but if the co-ordinate system is offset by R_s it provides a test of two dimensional spherical polar algorithms. The mesh used in the IMM is obtained by the intersection of a set of radial lines with the isotherms (see Figure 3). The calculated

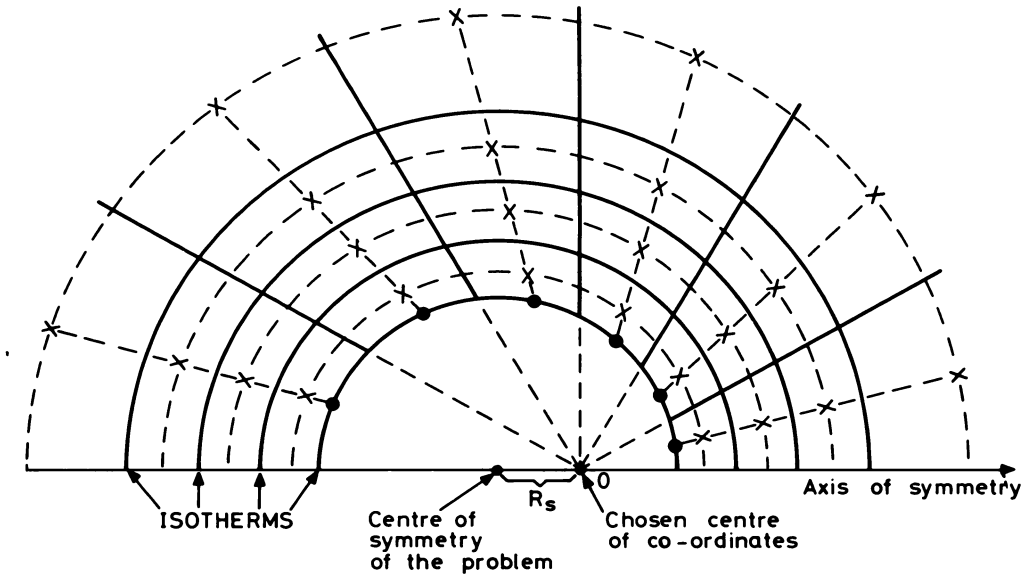

Figure 3. IMM mesh for problem 1. The solid isotherm and radial lines define cells for the method. The mesh points (x) used in the calculation are located at the centre of these blocks in temperature-angle space. Boundary conditions are imposed at nodes ●.

isotherm positions are for the centres of mesh cells and correspond to block averages. Equation 4.1 was inverted numerically to give R(T,t) for the chosen isotherms at t = 0.005. CONIMM was run with 16 isotherms until t = 1.0 with (a) R_s = 0 and (b) R_s = 0.5 using 10 radial lines. The numerical results are compared in Table 2 with (i) the analytic solution obtained from (4.1) and (ii) block averages calculated from the analytic solution for the mesh that was used (denoted $\bar{R}_{16}$). (In CONIMM the position of the outermost isotherm is determined by conservation and by setting

Temperature T	Starting Profile R(t=0.005)	Analytic Solution R(t=-1.0)	Numerical Solutions R(t=1.0)	Analytic Solution $\bar{R}_{16}$(t=1.0)
1.969	1.0046	1.0271	1.0273±0.0002	1.0276
1.844	1.0230	1.147	1.148±0.001	1.147
1.656	1.051	1.368	1.371±0.001	1.369
1.344	1.092	1.920	1.932±0.002	1.922
1.094	1.177	2.864	2.918±0.005	2.881
1.031	1.223	3.522	3.636±0.020	3.700

TABLE 2: Results for problem 1. Numerical solution is given for R_s=0, and the maximum difference between the R_s=0 and R_s=0.5 solutions is shown. $\bar{R}_{16}$ is a block average.

Φ^T = 0 at the outer (ambient) boundary. There is no need for the penetration distance introduced in [7].) The main source of error arises because of the difference between the spot value and block average in the outermost block. Run (b) shows good agreement with run (a) indicating that CONIMM handles the varying curvature of the isotherms effectively. Run (a) required 85 time-steps and took 12 seconds of CPU on the PRIME 500, run (b) required 89 time-steps and used 40 seconds of CPU. This example demonstrates the efficiency of CONIMM for problems involving temperature dependent thermal properties and propagation into an ambient medium. A relatively minor modification (an option in the code) allows the condition at the inner surface (T = 2, R = 1) to be replaced by the Stefan condition.

Problem 2 The freezing of a square prism.

This problem (see Figure 4) has been solved using a variety of numerical techniques [10,11,12]. At t = 0 the whole prism is liquid at its freezing point. The boundary temperature is then depressed and held constant while the liquid solidifies. The IMM solution employs a cylindrical polar co-ordinate system centred on x = 0, y = 0. From symmetry the temperature field need only be solved over one-eighth of the prism. The IMM requires a starting solution, and the most suitable one-dimensional solution to use is that of Neumann [13] for freezing from a plane wall. At short times this is accurate everywhere except in the corners. The Neumann solution for L = 1.561 (other properties being set to one) was discretized to find isotherm positions at t = 0.001. The IMM gives a good discretization of the

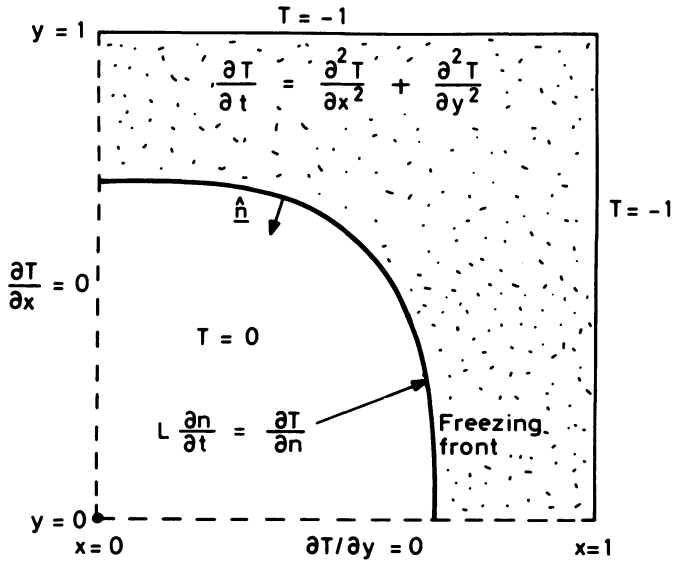

Figure 4. The freezing of a square prism

	5 × 4 mesh			10 × 10 mesh		
TIME	x_A^F	x_D^F	N	x_A^F	x_D^F	N
0.001	0.967	0.912	0	0.967	0.939	0
0.1	0.673	0.634	121	0.673	0.630	241
0.3	0.426	0.358	205	0.426	0.352	420
0.5	0.209	0.157	265	0.208	0.154	518
0.6	0.066	0.047	299	0.067	0.048	575
0.61	0.043	0.032	304	0.046	0.033	584

TABLE 3. Freezing of a square prism. x_A^F and x_D^F are the values of the x-co-ordinate of the freezing front on axis and diagonal respectively. N is the number of timesteps (this depends on a user specified accuracy parameter). The discrepancy in the initial values of x_D^F arises from the interpretation of the starting solution.

temperature profile at all times, but may lead to penalties arising from the limited timestep if neighbouring points are too close.

Results for a coarse mesh (4 isotherms and 5 radial lines) and for a finer mesh (10 isotherms and 10 radial lines) are given in Table 3. These agree well with each other, when differences in the interpretation of the starting solution have decayed, and with Crowley's enthalpy method results [12]. They

show greater consistency on mesh refinement than the results obtained with the IMM of Crank and Crowley [11], whose results are consistent with those presented here. No stability problems were encountered in the present calculations. The predicted time for complete freezing is 0.6296±0.0005 (coarse mesh) and 0.6327±0.0002 (for finer mesh), compared with Crowley's value of ∿0.625. The run for the finer mesh took 160 seconds CPU time on a PRIME 500.

Problem 3 The Bonnerot-Jamet problem [6].
This is illustrated in Figure 5. The problem involves a rapid redistribution of the initial temperature profile which leads to little movement of the boundary near x = 0 and rapid movement near x = 1 at early times. Both methods described above have been applied to this problem. For the 'gamma' method a fixed mesh on the domain $0 < x < 1$, $0 < y < 4$ was used with a fixed timestep $\delta t = 0.01$. No stability difficulties were encountered. Results are given in Table 4. The cartesian version of CONIMM was also used to solve this problem. The selection of small timesteps by the code for the early

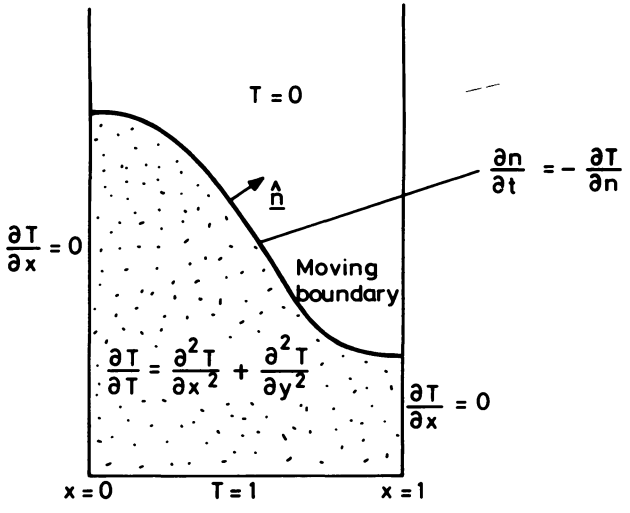

Figure 5. Problem 3: The initial position of the boundary is $y = 2 + \cos\pi x$ and the initial temperature distribution is linear in y.

part of the evolution was indicative of difficulties at this stage. Results are given in Table 4. Furzeland [14] has compiled a set of results for this problem, obtained using transform and enthalpy methods, which may be compared with Table 4. Combining all these studies, the best approximation to the

	'GAMMA' METHOD			FINITE-ELEMENTS [6]		
x	δx=1/10	δx=1/15	δx=1/20	δx=1/8	δx=1/16	δx=1/64
0	3.030	3.037	3.042	3.181	3.122	3.068
0.5	2.770	2.774	2.779	2.978	2.902	2.810
1.0	2.583	2.596	2.601	2.712	2.635	2.610

x	δx=1/5	δx=1/8	δx=1/10	δx=1/12	
0	3.161	3.132	3.120	3.111	CONIMM
0.5	2.888	2.856	2.836	2.822	(Cartesian option)
1.0	2.487	2.513	2.529	2.540	

TABLE 4. Moving boundary position at t=2 for problem 3. The CONNIM calculations were done with equal numbers of isotherms and radial lines (δx is the spacing in x).

position of the front at t = 2 appears to be y = 3.055 (x = 0), y = 2.795 (x = 0.5), and y = 2.600 (x = 1), where the error in assessment is likely to be less than 0.015. Using this solution as a basis for comparison, the transform methods [14] and the 'gamma' method perform well, as does the enthalpy method (although the interpolation used to obtain the frontal position in the enthalpy solution is not discussed). The IMM gives reasonable solutions, even on a coarse mesh but appears to be less accurate than the solutions discussed above (the use of block averages may degrade the solution and conservation is unimportant). All methods compare favourably with the original finite element method of Bonnerot and Jamet [6].

4. DISCUSSION AND CONCLUSIONS

Two methods of solving ablation and Stefan-like problems in 2 space dimensions using finite difference methods have been presented. The IMM is best suited to problems with constant temperature boundary conditions (including the Stefan condition) and requires monotonicity of the temperature field in one of the chosen spatial variables. It is efficient for problems with propagating thermal fronts, and the conservative formulation outlined above removes any arbitrariness in the condition on the outermost isotherm. The formulation implemented in CONIMM treats temperature dependent properties and requires no interpolation. The examples discussed above illustrate the range of problems that can be treated by CONIMM and show that in all cases

reasonable accuracy can be obtained with a limited amount of computation. The performance on problem 1 is particularly impressive, compared with conventional methods (although the accuracy is no better than 1%). Although this example did not include melting, the addition of an ablation condition does not lead to additional complications or sources of inaccuracy. Problem 2 illustrates the high accuracy that can be achieved when the temperature profiles are relatively simple: the main source of inaccuracy here is the initial singularity caused by the corner. Problem 3 is less suited to the IMM, but again reasonable results were obtained.

The 'gamma' method is related to a number of other finite difference methods for moving boundary problems. In practice the equations solved are identical to those arising in the enthalpy method except close to the moving boundary where knowledge of the intercept of the front should, in principle, lead to better discretizations of the thermal fluxes in this region. Ablation examples (not discussed here) and problem 3 illustrate the potential of this method, which per timestep requires only a small amount of additional computation over enthalpy schemes, and less than any transform method. The main drawback, as with most front tracking schemes, is that the moving boundary must be a single valued function of one of the co-ordinates; but this can often be arranged in practical problems. The method is suitable for problems that cannot be treated by the enthalpy method such as ablation problems, and can be generalized in many ways. It may be applied in other co-ordinate systems and with other conditions on the free boundary (e.g. for porous flow problems) and could be used to find solutions of free boundary (time-independent) problems by pseudo-timestepping. The computer coding requirements are relatively small, being modifications to the boundary conditions in an otherwise conventional scheme. Generalization to multi-phase problems and three dimensions are also possible.

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Progress with simple binary alloy solidification problems

I. INTRODUCTION

In this article we give a preliminary report of our studies on binary alloy solidification which is essentially that which was reported orally in Montecatini.

We are not the first to consider the alloy solidification problem. In 1967 Rubinstein [4] gave a similarity solution for a problem representing the crystallization of a binary alloy in a semi-infinite domain. For this solution the melt temperature and the concentration of the solidified alloy are constants. Fix [3], in the proceedings of the 1977 meeting in Gatlinburg, gave an analysis of numerical methods for alloy solidification. His comment, "... one must be prepared to expect nonuniqueness through branching and in fact unstable branches in alloy solidification," although applied in his paper to multidimensional problems, gives us a warning about even our simple one dimensional one.

More recently Crowley and Ockendon [2], and also Bermudez and Saguez [1], have considered binary alloy solidification problems which are quite similar to ours. Their approaches are similar in spirit to ours, and their numerical results are qualitatively similar to ours. However, both of these analyses are limited by the assumption that the heat capacities in the solid and liquid phases are the same. Because of the different way we state our weak formulation we are able to avoid this unrealistic assumption.

In one sense this article presents one more logical step in analyzing moving boundary problems which are descended from the classical Stefan problem. But in another sense the step we have taken is not logical at all. For we have proceeded to construct a computer model of a physical process which couples heat and mass transfer at vastly different rates by analogy with the models we have used to analyze the uncoupled processes.

Our model is intuitively appealing, arguably plausible and in a certain sense self-consistent. But we do not attempt to defend it on purely logical grounds.

In what follows we state the mathematical problem, which we assume

represents the solidification of a simple binary alloy; we give a very brief introduction to two component phase diagrams, whose relations we assume hold at the solidification front; we describe our "phlogiston formulation" of the problem, which is derived by analogy with the enthalpy method for simpler problems; we describe the finite difference scheme, with which we compute the various constituents of our phlogiston formulation; and finally we relate our computational experience with the model thus far.

A number of physical assumptions contribute to the formulation of the mathematical model, which may itself be questionable. We do not address this issue. We have not exercised great creativity in devising a difference scheme with which to compute.

The items of interest are the mathematical problem itself and the weak formulations analogous to the well known enthalpy approach. We would like to promote a discussion on this problem. In particular we would be interested to know if the existence and uniqueness of a solution to the mathematical problem are limited to a region of the set of parameter values for which the rates of diffusion of energy and material are not too different. (It is evident that the problem has no solution in the extreme case of zero material diffusivity in the liquid.) We are also very interested in defining and justifying an appropriate weak formulation.

These interests are obviously related. It may be that limitations on the ranges of parameter values necessary to insure existence and uniqueness for a strong solution are superfluous for a properly posed weak formulation. There also arises the question of the physical interpretation to be attached to a weak solution.

These interests are motivated by our numerical experience which indicates that when the diffusion rates are not too different, plausible distributions of temperature and concentration can be computed using our scheme; but when the rates for these processes differ by four orders of magnitude, as they do for real materials, then some anomalies appear that are open to various interpretations.

John Ockendon would like to interpret these numerical results as indicative of the existence of a "mushy zone". We are inclined to agree, but the evidence is far from conclusive.

II. STATEMENT OF THE MATHEMATICAL PROBLEM

The problem which we consider consists of two diffusion problems, one for heat and the other for concentration, posed in a one-dimensional finite box which is divided into two regions by a free boundary. In this simplest mathematical model of alloy solidification the equations governing heat and mass transfer are assumed to couple only at the free boundary. Conservation relations for energy and mass determine the movement of the free boundary. The solidification temperature and the limiting values of solute concentration at the free boundary from the solid and liquid phases are related by the phase diagram relations, to be explained in more detail in section III. Neither material nor heat can escape at the right hand boundary. At the left hand boundary no material enters and the temperature is a specified constant. Initially the material is all in the liquid phase at uniform temperature and concentration which are related by the equation for the liquidus line. That is, initially the liquid is just ready to precipitate solid material.

The precise statement of the problem which we consider is as follows. The diffusion equations in the solid and liquid regions are

$$\rho\, c_L\, T_t(t,x) = (K_L\, T_x(t,x))_x \quad \text{for } t > 0, \text{ and } x \in (0(S(t)), \tag{2.1}$$

$$C_t(t,x) = (D_L\, C_x(t,x))_x \quad \text{for } t > 0, \text{ and } x \in (0,S(t)), \tag{2.2}$$

$$\rho\, c_S\, T_t(t,x) = (K_S\, T_x(t,x))_x \quad \text{for } t > 0, \text{ and } x \in (S(t),L), \tag{2.3}$$

$$C_t(t,x) = (D_S\, C_x(t,x))_x \quad \text{for } t > 0, \text{ and } x \in (S(t),L). \tag{2.4}$$

The boundary conditions at the two fixed boundaries are

$$T(t,0) = T_0,\ C_x(t,0) = 0, \tag{2.5}$$

$$T_x(t,L) = 0,\ C_x(t,L) = 0. \tag{2.6}$$

The initial conditions are

$$S(0) = 0,\ T(0,x) = T_I,\ C(0,x) = C_I, \text{ and } T_I = \gamma_L\, C_I + T_A. \tag{2.7}$$

The relations at the free boundary are

$$\rho\, H\, \dot{S}(t) = K_S\, T_x(t,S^-) - K_L\, T_x(t,S^+)\ , \tag{2.8}$$

$$[C(t,S^-) - C(t,S^+)]\dot{S}(t) = D_L C_x(t,S^+) - D_S C_x(t,S^-), \tag{2.9}$$

and

$$T(t,S^-) = T(t,S^+) = \gamma_S C(t,S^-) + T_A = \gamma_L C(t,S^+) + T_A. \tag{2.10}$$

Equations 8 and 9 are statements of conservation of energy and material respectively at the phase front. In equation 10 the statements are that the temperature is continuous across the phase front and that the temperature and concentration at the phase front are related by the phase diagram relations. Here we have explicitly assumed the linearity of the solidus and liquidus lines.

This completes the statement of the mathematical problem. However, the constants T_O, T_A, T_I, γ_L, γ_S, and C_I cannot be completely arbitrary. The temperatures must satisfy the order relations $T_O < T_A < T_I$, and the slopes of the liquidus and solidus lines must be such that the former is above the latter. In addition it is necessary that T_I and C_I be such that initially the material is in the liquid state. We have imposed this condition explicitly by making part of the initial conditions, equations 7, that (C_I,T_I) lie on the liquidus line. Here again we have assumed the linearity of the liquidus line.

III. A BRIEF INTRODUCTION TO SIMPLE TWO COMPONENT PHASE DIAGRAMS

A simple two component phase diagram of the kind which we shall consider shows three things, namely: the freezing temperature of a liquid of specified concentration, the liquidus line; the melting temperature of a solid of specified concentration, the solidus line; and the equilibrium concentrations of a solid and liquid which may coexist at a specified temperature.

Figure 1 shows a two component equilibrium phase diagram for the case in which the two constituents form solid solutions in all proportions. The upper curve is the liquidus line and the lower curve is the solidus line. It is customary to assume that these curves are smooth and monotone increasing. The coordinate axes are labelled "Concentration" and "Temperature". The former is to be measured in some dimensionless fraction of component B (the higher melting). Both mole fraction and weight fraction seem to be in use for this purpose.

The interpretation of this diagram is as follows. If one has a volume of this A/B mixture in thermodynamic equilibrium with uniform temperature

equal to T* and overall concentration of component B equal to C*, then exactly one of the following three alternatives must hold

1. If the point on the diagram corresponding to (C*,T*) lies on or above the liquidus line, then the entire volume of material will be in the liquid phase.
2. If the point of the diagram corresponding to (C*,T*) lies on or below the solidus line, then the entire volume of material will be in the solid phase.
3. If the point on the diagram corresponding to (C*,T*) lies between the solidus and liquidus lines, then both solid and liquid phases will be present in the volume. In this case the concentration of the liquid will be C_L, where the point (C_L,T*) lies on the liquidus line, and the concentration of the solid will be C_S, where the point (C_S,T*) lies on the solidus line. If one assumes, as we shall, that the two phases have the same density, then one could compute the volume of each phase present from simple conservation of mass relations.

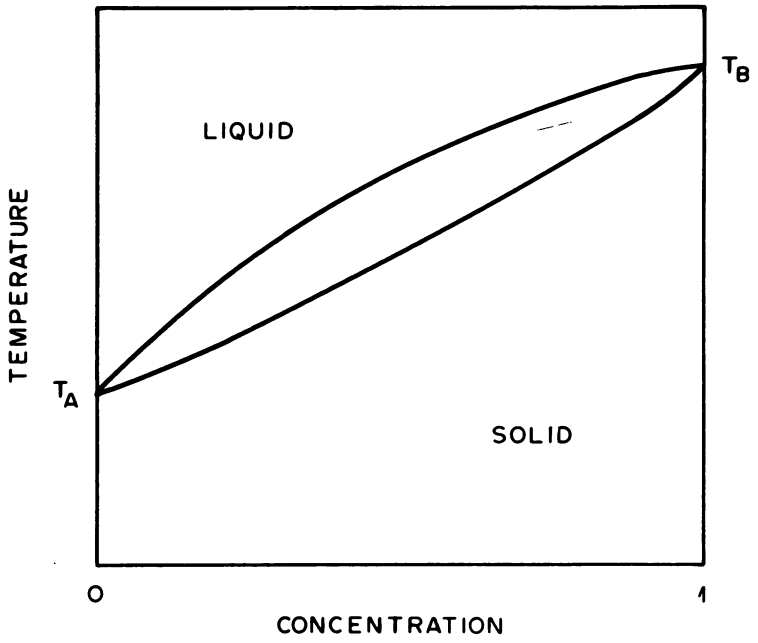

Figure 1. A simple two component phase diagram for the case in which the two constituents form solid solutions in all proportions.

The phrase "at thermodynamic equilibrium" is a crucial part of this explanation. It must be emphasized that these phase diagrams are statements about equilibrium conditions. If one has a volume of the A/B mixture at a particular C* and T* lying between the liquidus and solidus curves and alternative 3 above does not hold, then the system is not in equilibrium. In this case one must have supercooling or supersaturation or some other nonequilibrium condition.

Observe that at thermodynamic equilibrium there must be a discontinuity in concentration across the phase front. This follows from the liquid and solid in equilibrium at the same temperature having different concentrations. Furthermore, this discontinuity is far from constant. It is just the horizontal gap between the liquidus and solidus lines which varies as the process continues and the freezing temperature changes.

We shall limit our attention to a neighborhood of the point on the diagram labelled T_A, the melting point of component A (the lower melting). Our motivation for doing this is multifaceted. Partly we wish to justify the assumption we make that the solidus and liquidus curves are linear. But also, since from physical considerations we expect the liquid to be depleted in component B (the higher melting) as solidification proceeds, we wish to avoid the unhappy prospect of this approximation getting progressively worse as the process continues.

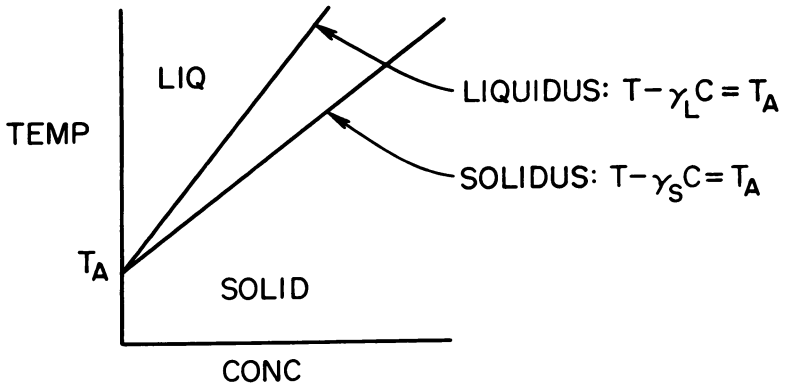

Figure 2. A linear phase diagram for a dilute binary alloy

In a neighbourhood of the point $(0,T_A)$ both phases are dilute solutions of component B in component A. We assume that the solidus and liquidus curves are linear with slopes γ_S and γ_L respectively as indicated in figure 2. In this case the jump in concentration across the phase front varies linearly with the freezing temperature.

This concludes our commentary on phase diagrams.

IV. OUR "PHLOGISTON FORMULATION"

The enthalpy formulation for Stefan-like problems is well known. Although it comes in several variations, the basic ideas are: (1) the enthalpy, E, is defined in terms of temperature, T, so that in each phase the heat capacity of the material is the partial derivative of the enthalpy with respect to the temperature; (2) a jump in the value of the enthalpy equal to the latent heat

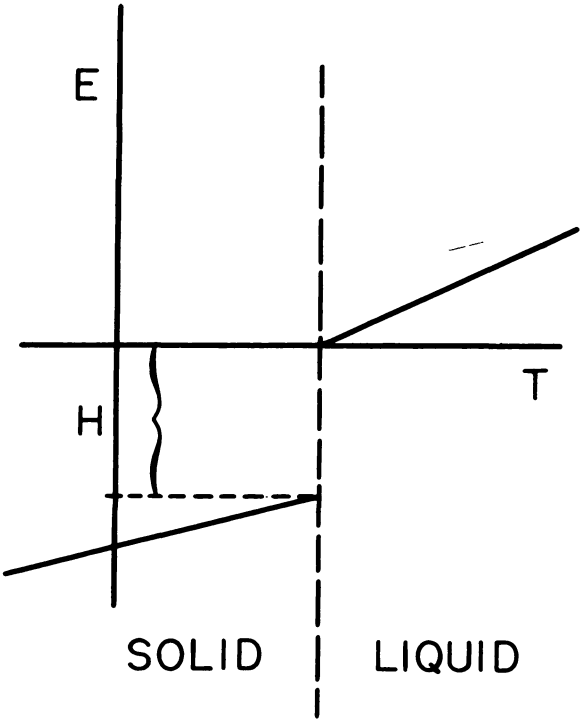

Figure 3. A typical enthalpy as a function of temperature relation for a pure material. H is the latent heat of the phase transition. The slope of the E vs T line in each phase is the heat capacity of the material in that phase.

of the phase transition occurs at the critical temperature; (3) instead of modeling the heat equation in each phase and tracking the phase front, the equation $E_t = \text{div}(k_p \text{grad } T)$ is modeled everywhere. The subscript p, on the thermal conductivity here, indicates the phase. A diagram of a typical enthalpy versus temperature relation is shown in figure 3.

We proceed to mimic this prescription to define a weak formulation of our binary alloy solidification problem. Firstly, we define the "chemical potential", W, which is related to the concentration, C, as temperature is related to enthalpy by the equation

$$W = \begin{cases} \gamma_L C & \text{in the liquid} \quad (T > T_A + \gamma_L C), \\ \gamma_S C & \text{in the solid} \quad (T < T_A + \gamma_S C). \end{cases} \tag{4.1}$$

Here again we have explicitly made use of the assumed linearity of the liquidus and solidus lines.

Observe, that because we assume that the phase diagram relations hold at the phase change front, W is continuous just as T is. On the other hand C has a jump across the phase change front just as E does. The jump in C is not constant but is determined by the phase diagram.

In addition, if we drew a new phase diagram with C replaced by W, the solid and liquid regions would be divided by the single line $T - W = T_A$.

For reasons which we hope will become clear as the narrative continues, we prefer to give the concentration in terms of the chemical potential. This is what we will use in the difference scheme.

$$C = \begin{cases} \gamma_L^{-1} W, & \text{when} \quad T > T_A + W, \\ \gamma_S^{-1} W, & \text{when} \quad T < T_A + W. \end{cases} \tag{4.2}$$

Figure 4 shows C as a function of T and W. It is obvious that this relation is intimately connected with the phase diagram. If one considers C as a function of W for constant T, then the resulting relation is of the same form as the enthalpy/temperature relation, shown in Figure 2, except that the jump is not constant. In computing with the difference scheme we compute C at an advanced time step from T and W values at the preceding time step. However, we also use the relation the other way in computing T and W

at an advanced time step from C and the phlogiston at the same time.

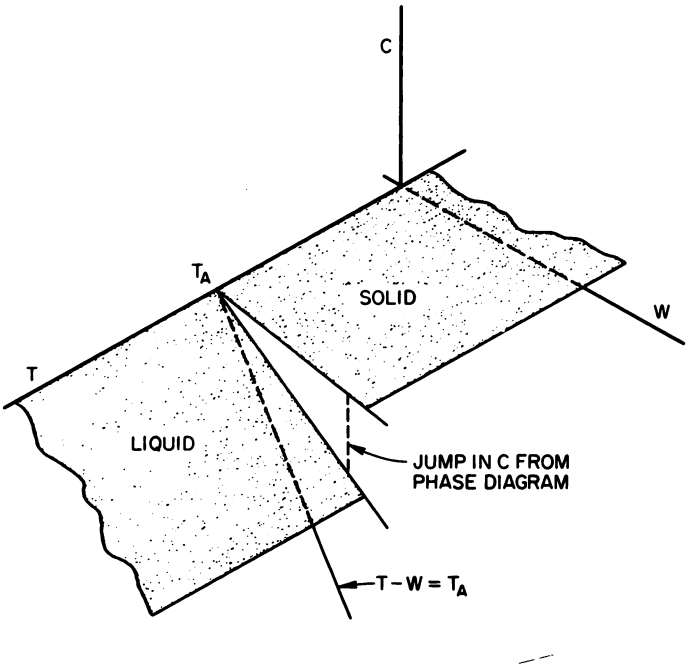

Figure 4. Concentration shown as a function of temperature and chemical potential. The line $T - W = T_A$ is in the T/W plane. The quarter planes marked "SOLID" and "LIQUID" are cut directly over this line.

We next incorporate the linear phase diagram into the definition of our phlogiston, Φ, preserving the characteristics of the enthalpy method stated above. The plan is to lay the phase diagram down in a horizontal plane and to define the phlogiston over it so that in each phase the partial derivative of the phlogiston with respect to the temperature is the heat capacity in that phase.

Figure 5 depicts the desired relationship. The phlogiston is represented as a pair of half planes over the (C,T) plane separated by a gap corresponding to the discontinuity in the concentration across the phase front. In each separate phase the inclination of the appropriate half plane to the (C,T) plane, along a line of constant concentration, is just the heat capacity in that phase.

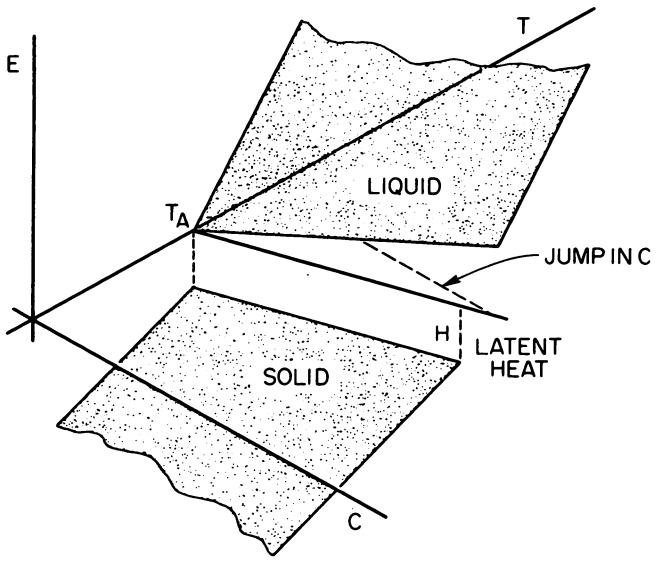

Figure 5. Phlogiston, here denoted by "E", as a function of temperature and concentration.

The equation defining Φ in terms of temperature and concentration is as follows

$$\Phi(t,x) = \begin{cases} \rho\, c_L\, (T(t,x) - T_A - \gamma_L\, C(t,x)), & \text{in the liquid,} \\ \\ \rho\, c_S\, (T(t,x) - T_A - \gamma_S\, C(t,x)) - \rho\, H, & \text{in the solid.} \end{cases} \tag{4.3}$$

This defines algebraically the split half planes shown in figure 5. Here, and in equation 2 which follows, c_L and c_S are the heat capacities of the liquid and solid respectively.

In terms of the two continuous variables, temperature and chemical potential, the relation is

$$\Phi = \begin{cases} \rho\, c_L\, (T - T_A - W) \ , & \text{when } T - W > T_A, \\ \\ \rho\, c_S\, (T - T_A - W) - \rho\, H \ , & \text{when } T - W < T_A. \end{cases} \tag{4.4}$$

Equations 2 and 4 are what we discretize for the finite difference scheme.

We point out that c_L need not equal c_S, and that liquid which is about to precipitate solid material, i.e. has T and W satisfying $T - W = T_A$, always has zero phlogiston. On the other hand newly formed solid always has philogiston equal to $-\rho H$. Furthermore, H need not be constant but, because of the other linear assumptions we have made, H may varv at most linearly with chemical potential (and hence with concentration). These are the characteristics which distinguish our weak formulation from that of Crowley and Ockendon [2] and from that of Bermudez and Saguez [1].

The program is to act as if our phlogiston and concentration are analogous to the enthalpy and to proceed accordingly. Namely, we discretize equations 2 and 4 and compute without tracking the boundary. After concentration and phlogiston are evaluated, then chemical potential and temperature are obtained. In this particular instance we have used an explicit difference scheme but obviously this is not required.

V. THE FINITE DIFFERENCE SCHEME

Our finite difference scheme is undoubtedly not in its final form. We have consciously decided to compute with a simple scheme initially in order to gain some appreciation for the idiosyncracies of the problem. Accordingly we have implemented what is essentially a straightforward explicit discretization of our weak formulation.

Briefly, we have discretized the equations for Φ and C in terms of T and W. The parameters entering these explicit time difference equations are evaluated at each node. Values appropriate for the solid and liquid regions are used when a node is definitely in either a solid or liquid region. The liquid region is defined by the condition $\Phi \geq 0$, the solid region is defined by the condition $\Phi \leq -\rho H$; and the condition $0 > \Phi > -\rho H$ defines our "mushy zone". In the mushy zone the parameters are a linear combination of those for solid and liquid. The coefficients in this linear combination are just $-\Phi/(\rho H)$ and $1 + \Phi/(\rho H)$, based on the assumption that in the mushy zone

$-\Phi/(\rho H)$ represents the frozen fraction.

After Φ and C are computed at an (n+1)th time step, using T and W values at the nth time step in the difference equations, T and W are computed at the advanced time from them. This is done for nodes not in the mushy zone by using the relations between T, Φ, and C and between W and C given in the preceding section and depicted in figures 4 and 5. For nodes in the mushy zone we have again assumed that $-\Phi/(\rho H)$ represents the frozen fraction, f; we have extended the relation between W and C and used $W = C/[(f/\gamma_S) + (1-f)/\gamma_L]$; and we have taken $T = T_A + W$.

At the fixed boundaries we have taken centered difference approximations to the derivatives T_x and C_x in applying the boundary conditions $T_x = 0$ and $C_x = 0$. In the initial array of C values we set the concentration at the first node equal to the solid concentration which can exist in equilibrium with the initial liquid concentration according to the phase diagram. Thus in effect we assume that the first node has already solidified.

The equations for the difference scheme are as follows.

$$\Phi_i^{n+1} = \Phi_i^n + \lambda[k(n,i)\ (T_{i+1}^n - T_i^n) - k(n,i-1)\ (T_i^n - T_{i-1}^n) - d_1(n,i)\ (W_{i+1}^n - W_i^n) + d_1(n,i-1)\ (W_i^n - W_{i-1}^n)], \tag{5.1}$$

for $i = 2,\ldots, I-1$, $n = 1, 2, \ldots$; and

$$C_i^{n+1} = C_i^n + \lambda\ [d_2(n,i)\ (W_{i+1}^n - W_i^n) - d_2(n,i-1)(W_i^n - W_{i-1}^n)], \tag{5.2}$$

for $i = 1,\ldots, I$, $n = 1, 2, \ldots$.

Here λ stands for $\Delta t/\Delta x^2$, and the coefficients $k(n,i)$, $d_1(n,i)$ and $d_2(n,i)$ are given in terms of the frozen fraction, f_i^n, and the parameters of the original problem by the relations

$$f_i^n = \begin{cases} 0 & \text{if } \Phi_i^n \geq 0, \\ -\Phi_i^n/(\rho H) & \text{if } -\rho H < \Phi_i^n < 0, \\ 1 & \text{if } \Phi_i^n \leq -\rho H ; \end{cases}$$

$$k(n,i) = [k_S\ (f_i^n + f_{i+1}^n) + k_L\ (2 - f_i^n - f_{i+1}^n)]/2\ ;$$

$$d_j(n,i) = [d_{jS}\ (f_i^n + f_{i+1}^n) + d_{jL}\ (2 - f_i^n - f_{i+1}^n)]/2,$$

for $j = 1,2$, with d_{1S}, d_{1L}, d_{2S} and d_{2L} defined by

$$d_{1S} = \rho\ c_S\ D_S.\quad d_{1L} = \rho\ c_L\ D_L\ ,$$

$$d_{2S} = D_S\ \gamma_S^{-1}\ ,\quad d_{2L} = D_L\ \gamma_L^{-1}\ .$$

The boundary conditions are

$$T_1^n = T_O,\quad \Phi_1^{n+1} = \rho\ c_S(T_O - T_A - \gamma_S\ C_1^{n+1}) - \rho\ H\ ,$$

$$C^{n+1} = C_1^n + 2\ \lambda\ [d_2(n,i)\ (W_2^n - W_1^n)]\ ,$$

$$\Phi_I^{n+1} = \Phi_I^n - 2\ \lambda\ [k(n,I-1)\ (T_I^n - T_{I-1}^n) - d_1(n,I-1)\ (W_I^n - W_{I-1}^n)]\ ,$$

$$C_I^{n+1} = C_I^n - 2\ \lambda\ [d_2(n,I-1)\ (W_I^n - W_{I-1}^n)]\ ,$$

for $n = 1, 2, \ldots$.

The initial conditions are

$$C_1^1 = \gamma_S^{-1}\ (T_{Initial} - T_A)\ ,$$

$$C_i^1 = \gamma_L^{-1}(T_{Initial} - T_A),\ \text{and}\ T_i = T_{Initial},$$

for $i = 2, \ldots, I$.

After Φ and C have been updated for a particular time step, W and T are computed at the advanced time step as follows.

In the solid, defined by $\Phi_i^{n+1} \leq -\rho\ H$:

$$W_i^{n+1} = \gamma_S\ C_i^{n+1}\ ,$$

$$T_i^{n+1} = T_A + \gamma_S\ C_i^{n+1} + (\rho\ H + \Phi_i^{n+1})/(\rho\ c_S)\ .$$

In the liquid, defined by $\Phi_i^{n+1} \geq 0$

$$W_i^{n+1} = \gamma_L \, C_i^{n+1} ,$$

$$T_i^{n+1} = T_A + \gamma_L \, C_i^{n+1} + \Phi_i^{n+1}/(\rho \, c_L) .$$

In the mushy zone, defined by $-\rho H < \Phi_i^{n+1} < 0$

$$W_i^{n+1} = C_i^{n+1} \; / \; [f_i^{n+1} \; / \; \gamma_S + (1 - f_i^{n+1}) / \; \gamma_L] ,$$

$$T_i^{n+1} = T_A + W_i^{n+1} .$$

VI. NUMERICAL EXPERIENCE WITH THE MODEL

Succinctly stated, our numerical experience has been that when the diffusivities for the thermal and concentration problems are taken equal, or not significantly different, then our model produces appealing results. In particular, the region for which phlogiston values are between 0 and $-\rho H$, which could be identified as a mushy zone, is limited to one or two mesh widths. Furthermore, the temperature values at these mesh points remain essentially constant as the value of the phlogiston drops; and the values of the concentration variable show a jump across the span of these mesh widths which is as expected from the linearized phase diagram. However, when the diffusivity for the thermal problem is made 10,000 times the diffusivity for the concentration problem then, although the program produces results which one could still argue are acceptable, there are some anomalies. In particular the region where phlogiston values are between 0 and $-\rho H$ grows rather extensive. We have taken Δx to be 1/20 and 1/40 and the maximum Courant number equal to .4 to insure stability. Under these circumstances the putative mushy region has typically grown to approximately half of the grid. If this represents reality, then the mushy region grows to be about half of the region before the material is half solidified.

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