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Automorphic Pseudodifferential Analysis and Higher Level Weyl Calculi

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To the memory of
Laurent Schwartz

Contents

0 Foreword

The present foreword addresses itself to readers with a previous knowledge, or interest, in pseudodifferential analysis, modular form theory or quantization theory. The book itself, however, to start with the introduction which follows, has been written under no such assumption, and everything needed will be recalled in due time.

A. Pseudodifferential analysis: First and above all, this is the study of a certain class of pseudodifferential operators in one variable, namely those whose Weyl symbols are automorphic distributions on $\mathbb{R}^2$, *i.e.,* distributions invariant under the linear action of the group $\Gamma = SL(2,\mathbb{Z})$: these symbols are interesting, but singular objects. The spectral theory of the Euler operator in $L^2(\Gamma\backslash\mathbb{R}^2)$ – a Hilbert space the very definition of which may involve the Weyl calculus – shows that automorphic distributions are linear superpositions of the following elementary distributions: the Eisenstein distributions (of which a continuous family is needed) and the (exceptional) cusp-distributions. The main object of this study is the construction of a multiplication table – in other words a symbolic calculus – for the associated operators.

From the point of view of pseudodifferential analysis, one interest of this work lies in that it requires handling, and composing, extremely singular operators. When automorphic distributions are taken as symbols, a composition formula of the familiar type such as

$$h_1 \# h_2 \sim h_1 h_2 + (4i\pi)^{-1}\{h_1, h_2\} + \cdots \tag{0.1}$$

would be totally inappropriate, since none of the terms on the right-hand side could have in general any signification. This calls for a drastic change of point of view, putting the emphasis, on the phase space $\mathbb{R}^2$ too, on spectral-theoretic concepts rather than differential geometry. Also, it is useful to extend the Weyl calculus Op as a more general calculus Op^p depending on some integer $p \geq 0$: besides its role in smoothing up some of the difficulties inherent in the Weyl calculus proper, this extension may have some interest from the point of view of harmonic analysis – it yields a parameter-dependent generalization of the metaplectic representation – and appears in the analysis of certain relativistic wave equations; it is also the natural pseudodifferential analysis to use in problems dealing with functions on the real line, flat up to a specified order at zero. Other by-products include an improved version of Cotlar's lemma about sums of "almost-orthogonal" operators (in Section 10), and some understanding and applications, from the point of view of operator theory, of the complement of the symplectic Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ in the full linear algebra $\mathfrak{gl}(2n,\mathbb{R})$ (Section 12).

B. Non-holomorphic modular form theory: There is no *classical* way to turn spaces of automorphic functions on the half-plane into non-commutative (associative) algebras: the main point of the book is that, taking a *quantum* point of view, such a construction is possible, even quite natural. The trick is to associate operators to automorphic functions (relying on some symbolic calculus of operators) and consider the composition of operators. The composition formulas bring to light, in a novel way, much of the structure pertinent to the study of non-holomorphic modular form theory, such as the zeta function, Hecke's theory, L-functions and convolution L-functions. Even though we could have worked throughout on the half-plane, we have found it much better, for several reasons, to transfer automorphic functions to the plane $\mathbb{R}^2$, where they become objects Γ-invariant under the *linear* action of some arithmetic group. This forces one to work with distributions rather than functions, but apart from this harmless fact, it has only advantages: one of these is that the algebraic structure on the space of automorphic distributions can be defined solely in terms of the – immensely popular – Weyl calculus of operators (*cf. supra*). The main result (Sections 5 and 15) expresses the composition of any two Eisenstein distributions as the image of a quite canonical "Bezout distribution" (related to Poincaré-Selberg series) under a simple, but interesting, operator: more details can be found in the following introduction.

C. Quantization theory: Under this vocable, we mean the definition and study of rules of symbolic calculus associated with the consideration of nice "phase spaces": such a space could be $\mathbb{R}^2$ (on which there are more possibilities than what is usually believed, including the Op^p-calculus referred to in the first part of this foreword), or $\Gamma\backslash\mathbb{R}^2$ (the main object of study here), or a homogeneous space, or (tentatively, at least) some of the spaces of interest in algebraic geometry. Writing this monograph has confirmed again our feeling that deformation quantization (the "small parameter" point of view) may not be the more fruitful point of view. Since this position – also based on a fairly wide experimentation with the construction of alternative pseudodifferential analyses, over a span of years – lies outside the more popular trends, we have found it useful, in a last, largely self-contained, expository section, to indicate what could be some lines of a program in this direction. It is to be noted, in particular, that concepts relative to the composition of symbols are given a much wider realm than is usually the case, even (*cf.* Section 17) in the case of the one-dimensional Weyl calculus.

1 Introduction

The group $SL(2,\mathbb{R})$ acts on the Poincaré upper half-plane Π as a group of fractional-linear transformations $z \mapsto \frac{az+b}{cz+d}$. Given a discrete subgroup Γ of $SL(2,\mathbb{R})$ such as $\Gamma = SL(2,\mathbb{Z})$ (with one exception in Section 18, this will be the sole case considered in this book), the Γ-invariant functions f on Π are called *automorphic* functions: they can be identified to functions on any fundamental domain of the action of Γ, by which is meant any domain in Π containing essentially – *i.e.,* up to a negligible set – one point in each Γ-orbit. A *non-holomorphic modular form* is an automorphic function on Π which is at the same time a generalized eigenfunction of the Laplace-Beltrami operator Δ for some eigenvalue $\frac{1+\lambda^2}{4}$.

One of the themes of the present work is that moving from the half-plane Π to the plane $\mathbb{R}^2$, on which Γ acts in a linear way, is advantageous in several important aspects. Let us discard, to begin with, the first argument against this idea: there is no fundamental domain for the action of Γ in $\mathbb{R}^2$. True, and no continuous non-constant function on $\mathbb{R}^2$ qualifies as a Γ-invariant function. However, Γ-invariant *distributions* do exist, and are the central object of this study. Besides the Dirac mass at the origin, the simplest example available is the Dirac comb, supported in $\mathbb{Z}^2$. Since the Euler operator $\mathcal{E} = \frac{1}{2i\pi}\,(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} + 1)$ (the extra term 1 makes $\mathcal{E}$ a formally self-adjoint operator on $L^2(\mathbb{R}^2)$) commutes with the linear action of $SL(2,\mathbb{R})$, the terms $\mathfrak{E}^\sharp_{i\lambda}$ of the (continuous) decomposition of the Dirac comb into its homogeneous components of degrees $-1-i\lambda$ are themselves Γ-invariant: we call them the *Eisenstein distributions*.

It is very classical to denote as $L^2(\Gamma\backslash\Pi)$ the Hilbert space consisting of all automorphic functions whose restriction to some fundamental domain of Γ in Π is square-integrable with respect to the invariant measure on Π: indeed, this Hilbert space does not depend on which fundamental domain you choose. It is not as obvious, on the other hand, how to define a Hilbert space $L^2(\Gamma\backslash\mathbb{R}^2)$ of automorphic (*i.e.,* Γ-invariant) distributions on $\mathbb{R}^2$: but this can be done, which will be our first task.

A connection between automorphic distributions and non-holomorphic modular forms is best seen with the help of the (one-dimensional) Weyl calculus of pseudodifferential operators, a fixture of our story. This is a certain map which associates an operator $\mathrm{Op}(h)$ acting on functions of *one* variable to functions h of *two* variables. More precisely, $\mathrm{Op}(h)$ is well defined as a linear map from the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing C^∞ functions on the real line to the dual space $\mathcal{S}'(\mathbb{R})$ of tempered distributions whenever h lies in the space $\mathcal{S}'(\mathbb{R}^2)$: $\mathrm{Op}(h)$ is called the operator, or *pseudodifferential operator*, with symbol h. It is customary to set $\mathrm{Op}(h_1)\,\mathrm{Op}(h_2) = \mathrm{Op}(h_1 \# h_2)$ when the left-hand side makes sense: thus, the composition of operators gives rise to a partially defined bilinear composition $\#$ of distributions in two variables.

One of the niceties of the Weyl calculus is its *covariance* under the *metaplectic representation*, a concept which we now explain. It may be a little difficult to

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visualise a topological group which would be a two-fold cover of $SL(2,\mathbb{R})$, *i.e.*, a connected group $\widetilde{SL}(2,\mathbb{R})$ together with a homomorphism from $\widetilde{SL}(2,\mathbb{R})$ to $SL(2,\mathbb{R})$ the kernel of which would have exactly two elements. However, such a group exists, for reasons having to do with homotopy theory (the so-called fundamental group $\pi_1(SL(2,\mathbb{R}))$ is isomorphic to $\mathbb{Z}$, of which $\mathbb{Z}/2\mathbb{Z}$ is a quotient group): it is called the metaplectic group, and any of the two elements of $\widetilde{SL}(2,\mathbb{R})$ which go to some given element g of $SL(2,\mathbb{R})$ under the homomorphism referred to above is said to *lie above* g. Now, there is a canonical isomorphism between $\widetilde{SL}(2,\mathbb{R})$ and a group of unitary transformations of the Hilbert space $L^2(\mathbb{R})$, namely the group generated by the transformations $u \mapsto v$ with $v(at) = a^{-\frac{1}{2}} u(a^{-1}t)$ for some $a > 0$, or $v(t) = u(t) \exp i\pi ct^2$ for some $c \in \mathbb{R}$, or $v = e^{-\frac{i\pi}{4}} \mathcal{F}u$, where $\mathcal{F}$ is the Fourier transformation. Any unitary transformation U in the metaplectic group restricts as an isomorphism from the space $\mathcal{S}(\mathbb{R})$ onto itself, and extends as an isomorphism from the space $\mathcal{S}'(\mathbb{R})$ onto itself. The covariance property of the Weyl calculus refers to the formula $U\,\mathrm{Op}(h)\,U^{-1} = \mathrm{Op}(h \circ g^{-1})$: it is valid whenever h lies in $\mathcal{S}'(\mathbb{R}^2)$ and U lies above $g \in SL(2,\mathbb{R})$ in the metaplectic group.

It is to be noted that the two unitary transformations of $L^2(\mathbb{R})$ which lie, in the metaplectic group, above the same element g of $SL(2,\mathbb{R})$, are simply related since one is the product of the other by the transformation which consists in multiplying by -1: this will make the fact that the map $U \mapsto g \colon \widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is two-to-one rather than an isomorphism essentially harmless. Denote, for reasons to be apparent presently, as u_i the (even) function $t \mapsto e^{-\pi t^2}$ on $\mathbb{R}$, renormalized so as to have norm 1 in the space $L^2(\mathbb{R})$; in a similar way, denote as u_i^1 the renormalized version of the (odd) function $t \mapsto t\,e^{-\pi t^2}$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, and U is a unitary transformation in the metaplectic group, lying above the point g, it turns out that, up to the multiplication by some factor, depending only on g, of modulus 1, the functions Uu_i and Uu_i^1 agree with a pair of functions u_z and u_z^1, depending only on the point $z = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) . i = \frac{ai+b}{ci+d}$: note that the knowledge of $z = g.i$, on the other hand, only implies that of the class gK, with $K = SO(2)$, in $SL(2,\mathbb{R})$.

Then, any symbol $h \in \mathcal{S}'(\mathbb{R}^2)$ invariant under the linear action of the matrix $-I$ (in this case, $\mathrm{Op}(h)$ preserves the parity of functions) can be characterized by a pair of functions on Π, namely, with $p = 0$ or 1, the two functions $z \mapsto (u_z^p \,|\, \mathrm{Op}(h) u_z^p)$. The fundamental point (a consequence of the covariance of the Weyl calculus together with our construction of the functions u_z^p, $p = 0$ or 1) is that if h is an automorphic distribution in $\mathbb{R}^2$, these two functions on Π are automorphic in the usual sense: with the help of the Weyl calculus, we are thus in a position to establish a one-to-one correspondence between a space of automorphic distributions (on $\mathbb{R}^2$) and a space of pairs of automorphic functions on Π. Under this transfer, a certain Hilbert space of such pairs (for the cognoscenti only: the space of Cauchy data for the Lax-Phillips scattering theory) finally becomes the space $L^2(\Gamma \backslash \mathbb{R}^2)$ we have been looking for.

The structure of this space is, up to a point, well understood. Elements of this space are superpositions of the following building blocks, hereafter referred to as *elementary* automorphic distributions: the Eisenstein distributions, of which a continuous superposition is needed; and a countable family of much more mysterious *cusp-distributions*.

The next question is to better understand the cusp-distributions. Could one define a generating object for *all* elementary automorphic distributions, in a way comparable to the way the Dirac comb can be decomposed into Eisenstein distributions? The answer is yes, and is introduced early in this work under the name of *Bezout's distribution* and denoted as $\mathfrak{B}$. It can be thought of as being associated with a *comb of straight lines* in $\mathbb{R}^2$ in just the same way the Dirac comb is a comb of points.

That the above-defined correspondence between automorphic distributions (on $\mathbb{R}^2$) and pairs of automorphic functions on Π links the spectral theory of the Euler operator on $\mathbb{R}^2$ to that of the Laplacian on Π can be traced to the formula

$$\left(\Delta - \frac{1}{4}\right)\, (u_z^p \,|\, \mathrm{Op}(h) u_z^p) = (u_z^p \,|\, \mathrm{Op}(\pi^2\, \mathcal{E}^2\, h) u_z^p), \tag{1.1}$$

which expresses that if h happens to be homogeneous of degree $-1-i\lambda$, the scalar product $(u_z^p \,|\, \mathrm{Op}(h) u_z^p)$, as a function of z, is a (possibly generalized) eigenfunction of Δ for the eigenvalue $\frac{1+\lambda^2}{4}$. In particular, under this correspondence, the Eisenstein distribution $\mathfrak{E}_{i\lambda}^{\sharp}$ gives rise to a pair of modular forms both proportional to the usual Eisenstein series $E_{\frac{1-i\lambda}{2}}$: this is the modular form defined by complex continuation from the series (convergent when $\mathrm{Re}\, \nu < -1$)

$$E_{\frac{1-\nu}{2}}(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \left(\frac{|mz-n|^2}{\mathrm{Im}\, z} \right)^{\frac{\nu-1}{2}} . \tag{1.2}$$

It is now possible to point towards a few of the advantages offered by the whole construction. First, $\mathbb{R}^2$ has more symmetries than Π since, besides the linear action of $SL(2,\mathbb{R})$, it is also endowed with the action of $\mathbb{R}^2$ itself by translations. Next, the very concept of homogeneous automorphic distribution is slightly subtler than that of non-holomorphic modular form. For any given non-holomorphic modular form gives rise to *two* automorphic distributions (linked by the Fourier transformation on $\mathbb{R}^2$), each one corresponding to the choice of a square root of λ^2: in this way, the linearly independent automorphic distributions $\mathfrak{E}_{\pm i\lambda}^{\sharp}$ correspond to the linearly dependent Eisenstein series $E_{\frac{1\pm i\lambda}{2}}$, and the same goes for cusp-distributions versus non-holomorphic cusp-forms.

The fundamental novelty, which leads to most of the developments in the present paper, lies in the interpretation of automorphic distributions as symbols of operators. For, with any luck, such operators might be composed, and this would provide the space of automorphic distributions with the structure of an associative algebra. However, there are considerable difficulties, which we shall first overlook so as to present some of the final results: later in this introduction, we shall explain the origin of these difficulties and the way they are to be solved.

In linear analysis (and in quantum mechanics as well), it is always a good idea to substitute for the study of a given self-adjoint operator A that of any commutative algebra, as large as possible, of self-adjoint operators, containing A as an element. For the case of the operator Δ – which admits a self-adjoint realization in the space $L^2(\Gamma\backslash\Pi)$ of automorphic functions – such an algebra was introduced by Hecke: besides Δ, it consists of a sequence $(T_N)_{N\geq 1}$ of explicit (not differential) operators of an arithmetic nature, to be completed by one extra operator. All this transfers to the automorphic distribution level, to a sequence $(T_N^{\rm dist})_{N\geq 1}$ of operators, to be completed by the operator $T_{-1}^{\rm dist}$ such that $(T_{-1}^{\rm dist}\, h)(x,\xi) = h(-x,\xi)$: this last operator permits splitting automorphic distributions as sums of automorphic distributions of parities 0 and 1. Together with the Euler operator $\mathcal{E}$ and $T_{-1}^{\rm dist}$, the sequence $(T_N^{\rm dist})_{N\geq 1}$ generates a maximal commutative algebra of self-adjoint operators on the space $L^2(\Gamma\backslash\mathbb{R}^2)$, and the *elementary* automorphic distributions that have been alluded to above are exactly the joint eigenfunctions of all operators in this commutative algebra.

It is useful to introduce a generating series for all Hecke operators, setting $\mathcal{L}(s) = \sum_{N\geq 1} N^{-s} T_N^{\rm dist}$ for complex s with $\text{Re}\, s$ large. Multiplying $\mathcal{L}(s)$ by some Gamma-like function, in the spectral-theoretic sense, of the two operators $\mathcal{E}$ and $T_{-1}^{\rm dist}$, one finds an operator $\mathcal{L}'(s)$ the complex continuation of which, as a function of s, satisfies the simple functional equation $\mathcal{L}'(1-s) = \mathcal{L}'(s) \times (-1)^{\rm parity}$. The main formula (Theorem 15.1) of the book can be expressed as

$$\mathfrak{E}^\sharp_{i\lambda_1} \natural\, \mathfrak{E}^\sharp_{i\lambda_2} = \mathcal{L}'\left(\frac{1+i\,(\lambda+\lambda_2)}{2}\right) \mathcal{F}\,\mathcal{L}'\left(\frac{1+i\,(\lambda_1-\lambda_2)}{2}\right) \mathfrak{B} \\ + \text{ side term}\,, \tag{1.3}$$

where $\natural$ is, up to a slightly different normalization, just the sharp composition $\#$ of automorphic distributions corresponding under the Weyl calculus Op to the composition of operators, $\mathcal{F}$ is the Fourier transformation on $\mathbb{R}^2$, $\mathfrak{B}$ is the Bezout distribution the existence of which, as a canonical generator of all elementary automorphic distributions, has been asserted above, and the side term is a simple linear combination of four Eisenstein distributions.

It is a consequence of the covariance of the Weyl calculus under the metaplectic representation that $h_1 \# h_2$ (or $h_1 \natural\, h_2$) has to be an automorphic distribution whenever h_1 and h_2 are. The equation (1.3) is thus only the first entry – in some sense, but not all, the one most difficult to get at – in a multiplication table, the entries of which should give the decomposition into elementaries of the $\natural$-product of any two elementary automorphic distributions. This task has been pushed up to some point, but not fully completed, in Section 16. We have also, in the same section, hinted at a partly conjectural unified formula, stated in terms of Eulerian products.

Early in this work (in Section 5), we give a heuristic proof of the main formula (1.3), which makes a clear understanding of the role played by the two factors $\mathcal{L}(\frac{1+i(\lambda_1\pm\lambda_2)}{2})$ possible. However, the genuine proof of this formula entails considerable difficulties, and will be given later, in Sections 13 to 15. The most serious one arises from the fact that a $\natural$-product such as $\mathfrak{E}^\sharp_{i\lambda_1} \natural\, \mathfrak{E}^\sharp_{i\lambda_2}$ is not quite meaningful in the usual sense, since the associated operators cannot be composed. To lower our requirements, we may satisfy ourselves with the demand that all operators that have to be dealt with should act on the space linearly generated by the functions u_z^p, $p = 0$ or 1, $z \in \Pi$, and should be valued in the algebraic dual of this space: this is the bare minimum needed for applications to modular form theory. However, the operator whose Weyl symbol is an elementary automorphic distribution does not even send u_z into $L^2(\mathbb{R})$, which prevents us, if $p = 0$, from defining

$$(u_z^p \,|\, \mathrm{Op}(\mathfrak{E}^\sharp_{i\lambda_1})\, \mathrm{Op}(\mathfrak{E}^\sharp_{i\lambda_2})\, u_z^p) \colon = (\mathrm{Op}(\mathfrak{E}^\sharp_{-i\lambda_1})\, u_z^p \,|\, \mathrm{Op}(\mathfrak{E}^\sharp_{i\lambda_2})\, u_z^p) \tag{1.4}$$

as we would like to do: however, on the other hand, this definition is all right if $p = 1$.

There are two ways to circumvent this difficulty. The cheaper one, which we have used in the explicit computations, is based on the remark (Proposition 13.1) that one can sometimes define the image under the Euler operator $\mathcal{E}$ of the symbol h of some operator A without being able to define either h or A: the trick is to consider instead of A the operator $PAQ - QAP$, where Q and P are the two canonical generators of Heisenberg's representation in $L^2(\mathbb{R})$. This just works with our problem, and makes it possible to define the $\natural$-product of any two Eisenstein distributions modulo some distribution homogeneous of degree -1, which has to be a multiple of the Eisenstein distribution $\mathfrak{E}^\sharp_0$ if automorphic. Adopting this point of view, which we have done in all arithmetic computations, has the advantage that one may still work with the Weyl calculus, though with a rather indirect definition: another one is the explicit character of the composition formulas. A disadvantage is that the composition of operators used in this context exists only in such a weak sense that giving a meaning to the composition of *three* operators the Weyl symbols of which are elementary automorphic distributions – which would be necessary to support a claim of associativity – seems at best a remote possibility.

There is a deeper way to deal with the problem, based on the embedding of the Weyl calculus Op into a certain sequence $(\mathrm{Op}^p)_{p=0,1,\dots}$ of calculi. As soon as $p \geq 1$ (*i.e.*, with the exception of the Weyl calculus itself), the composition of two operators the symbols of which are elementary automorphic distributions has a genuine meaning in the Op^p-calculus, an even stronger one (Section 10) if $p \geq 2$: finally, any given number of operators with homogeneous automorphic Op^p-symbols can be composed (Theorem 10.7) if p is large enough. Just like the Weyl calculus, the Op^p-calculus benefits from some covariance property, linked to some (inequivalent) variant of the metaplectic representation: as a consequence, the composition, in the sense of the Op^p-symbolic calculus, of two automorphic distributions, is again (when well-defined) automorphic. Constructing the Op^p-calculi will take time: however, besides their role in smoothing up the Weyl calculus somewhat, when dealing with very singular symbols, these calculi may have an importance of their own. On the one hand, they seem to be the right pseudodifferential analyses in problems dealing with operators acting on functions flat up to a certain order at zero; next, their construction may be considered (Section 8) as paralleling Dirac's construction of the wave equation for the electron, except that instead of just one operator one has to consider a pair of non-commuting operators; finally (Section 7 about the horocyclic calculus) they provide a way, in quantization theory, of dealing with some of the not so nice features (the symbol map is not an isometry) inherent in the quantization of symmetric spaces.

We take this opportunity to give, in Section 17, a full proof of a theorem, somewhat carelessly treated in [62, Section 5], dealing with the one-dimensional Weyl calculus in general (not that associated with automorphic symbols): the point is that there exists a composition formula completely different from the well-known one, and that it is the one suitable when the metaplectic representation enters the picture in any serious way. Contrary to the usual (Moyal-type) formula $h_1 \# h_2 \sim h_1 h_2 + (4i\pi)^{-1}\{h_1, h_2\} + \cdots$, it has an extension to the Op^p-calculus, though we have not made the coefficients of this formula fully explicit (they are given instead by recurrence relations with respect to p). One may mention here too that, as has been proved by our student Bechata [5], this formula extends to the Weyl calculus on (complex-valued functions on) p-adic numbers, whereas the more familiar one would be meaningless too in this case.

We wish to strongly stress again that, in the arithmetic situation which is the environment of this paper, there *is* a symbolic calculus of operators, but the composition formula cannot bear any relation to the one we have grown accustomed to, or to any concept based on Taylor expansions and on classical objects such as the pointwise product of symbols: this composition formula is precisely the multiplication table the construction of which has begun here. This is not as exotic as one might think and, at the end of this work, we have inserted an informal section entitled "new perspectives in quantization theory", partly to show that the nature of composition formulas in alternative pseudodifferential analyses is much more varied, and linked to interesting spectral theory, than experience with the sole Weyl calculus might lead one to believe. The same section contains a small number of open problems, mostly of a harmonic analytic nature, some of which look quite feasible, even though extensive work may be required towards their solution.

Before we leave this introduction, we want to address ourselves, again, to our readers more interested in modular form theory, especially in facts about the Rankin-Cohen products. These are expressions

$$F_j^{k_1,k_2}(f_1,f_2) = \sum_{l=0}^{j} (-1)^l \binom{k_1+j-1}{l} \binom{k_2+j-1}{j-l} f_1^{(j-l)} f_2^{(l)} \qquad (1.5)$$

which permit building a sequence of *holomorphic* modular forms of weights $k_1 + k_2 + 2j$ from any pair of *holomorphic* modular forms f_1, f_2 of weights k_1 and k_2. These bilinear expressions were first introduced by H. Cohen [12] (Rankin had considered a special case in [40]) and enjoy quite a popularity at present [70, 13, 10, 11, 39]. We urge our readers interested in Rankin-Cohen products to have a look at the quantization Section 19, in particular the part of it relative to the composition formulas, to see why, in connection with some appropriate symbolic calculus, the composition formulas given in [63], the main ingredients of which were just Rankin-Cohen products, had a status quite comparable – but the symbolic calculus and the phase space were different – to, say, the composition formula from our present Section 17. Only, discrete Hilbert sums rather than direct integrals had to be considered there; in the arithmetic situation which is the most important object of the present work (Chapter 3), *both* discrete sums and integrals have to be considered simultaneously.

The methods in the present work, by definition, have to rely on ideas from two usually separated fields of activity. The present author is certainly more at ease with pseudodifferential analysis, which he has practised for decades, than with number theory. But he believes that his lack of competence in this latter domain is a guarantee that this book will be accessible to analysts in general: he can only hope that practitioners of modular form theory will view with a sympathetic eye these attempts, by an analyst, at familiarizing himself with some of the more elementary tools of their fascinating trade.

Chapter 1

Automorphic Distributions and the Weyl Calculus

2 The Weyl calculus, the upper half-plane, and automorphic distributions

The defining formula of the Weyl calculus [68] is

$$(\mathrm{Op}(h)u)(x) = \int h\left(\frac{x+y}{2}, \eta\right) u(y)\, e^{2i\pi(x-y)\eta}\, dy\, d\eta, \qquad u \in \mathcal{S}(\mathbb{R}). \tag{2.1}$$

The operator $\mathrm{Op}(h)$ is called the operator, or pseudodifferential operator, with symbol h. If $h \in \mathcal{S}'(\mathbb{R}^2)$, the space of tempered distributions on $\mathbb{R}^2$, then $\mathrm{Op}(h)$ is a well-defined linear operator from the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing C^∞ functions on $\mathbb{R}$ to $\mathcal{S}'(\mathbb{R})$, and the map Op so defined is an isomorphism. One way to see this is to introduce, for any pair u, v of functions in $\mathcal{S}(\mathbb{R})$, the *Wigner function* $W(v,u)$ on $\mathbb{R}^2$ defined as

$$W(v,u)(x,\xi) = \int_{-\infty}^{\infty} \bar{v}(x+t)\, u(x-t)\, e^{4i\pi t\xi}\, dt. \tag{2.2}$$

Then, as is easily seen, $W(v,u) \in \mathcal{S}(\mathbb{R}^2)$, and for every $h \in \mathcal{S}'(\mathbb{R}^2)$ the formula

$$\begin{aligned}(v|\mathrm{Op}(h)u) &= \langle h, W(v,u)\rangle \\ &= \int_{\mathbb{R}^2} h(x,\xi)\, W(v,u)(x,\xi)\, dx\, d\xi \end{aligned} \tag{2.3}$$

is a proper definition of $\mathrm{Op}(h)$: observe that we define scalar products $(\ |\ )$ as being antilinear with respect to the argument *on the left*. The symbol h lies in

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$L^2(\mathbb{R}^2)$ if and only if $\mathrm{Op}(h)$ extends as a Hilbert-Schmidt operator on $L^2(\mathbb{R})$. It is often useful to note that $W(v,u)$ is also the symbol of the rank-one operator $w \mapsto (v|w)u$.

The Weyl calculus of pseudodifferential operators was introduced by H.Weyl in 1926, as an answer to questions regarding the early theory of quantum mechanics; a somewhat similar motivation – with considerable incentive from harmonic analysis as well – will be present in Section 19. But the main current importance of pseudodifferential analysis lies in its role as the basic tool in the modern treatment of linear partial differential equations. One should mention that, though Weyl's formula was mentioned in the Kohn-Nirenberg foundational paper [28], it is mainly the *standard* calculus (which will be needed in (11.22), where it plays a minor role) that has been used, up to comparatively recently, by analysts: for this calculus, there is no covariance under the metaplectic representation. The literature on pseudodifferential analysis and its applications to partial differential equations is immense: our first choices would be the treatises [51, 24, 45] by Trèves, Hörmander, Shubin. A short introduction to the Weyl calculus, with more emphasis on harmonic analysis than on partial differential equations (thus closer from our present point of view) can be found in the Chapter 0 of [65]. No previous familiarity with the Weyl calculus, however, would be of much use in the present work, in which new methods had to be built from scratch.

Recall [67] that there exists a certain twofold covering $\widetilde{SL}(2,\mathbb{R})$ of the group $SL(2,\mathbb{R})$ and a unitary representation Met of $\widetilde{SL}(2,\mathbb{R})$ in $L^2(\mathbb{R})$, preserving the space $\mathcal{S}(\mathbb{R})$ and extending as a representation within the dual space $\mathcal{S}'(\mathbb{R})$ such that, for every $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ lying above some point $g \in SL(2,\mathbb{R})$, and every tempered distribution h on $\mathbb{R}^2$, the covariance rule

$$\mathrm{Met}(\tilde{g})\,\mathrm{Op}(h)\,\mathrm{Met}(\tilde{g})^{-1} = \mathrm{Op}(h \circ g^{-1}) \tag{2.4}$$

should hold: Met is the so-called *metaplectic* representation. The set of all unitary operators $\mathrm{Met}(\tilde{g})$, $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$, is generated as a group by the operators of the following three species:

(i) transformations $u \mapsto v$, $v(x) = a^{-\frac{1}{2}} u(a^{-1}x)$, $a > 0$;

(ii) multiplications by exponentials $\exp i\pi c x^2$, c real;

(iii) $e^{-\frac{i\pi}{4}}$ times the Fourier transformation.

These three transformations are associated with points $\tilde{g}$ that lie above the points $g = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$. As a consequence, Met is not an irreducible transformation, but acts within $L^2_{\mathrm{even}}(\mathbb{R})$ and $L^2_{\mathrm{odd}}(\mathbb{R})$ separately: the two terms can then be shown to be acted upon in an irreducible way. It is an easy task to check (2.4), starting from (2.1), in each of the three cases above.

A first consequence of the *covariance formula* (2.4) is that, given any two symbols h_1 and h_2, say in $L^2(\mathbb{R}^2)$ (the associated operators are then Hilbert-Schmidt operators on $L^2(\mathbb{R})$, thus can be composed in the usual sense), and given any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, the formula

$$(h_1 \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))\#(h_2 \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = (h_1 \# h_2) \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tag{2.5}$$

holds. In particular, if h_1 and h_2 are invariant under the action of a certain element of $SL(2,\mathbb{R})$, so is $h_1 \# h_2$, by definition the symbol of $\mathrm{Op}(h_1)\,\mathrm{Op}(h_2)$. From the second interpretation of the Wigner function given right after (2.3), and from the covariance formula (2.4), it also follows that

$$W(\mathrm{Met}(\tilde{g})\, v,\, \mathrm{Met}(\tilde{g})\, u) = W(v,u) \circ g^{-1} \tag{2.6}$$

if $u,\, v \in \mathcal{S}(\mathbb{R})$ and $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ lies above $g \in SL(2,\mathbb{R})$.

On $\mathbb{R}^2$, we shall always use the *symplectic* Fourier transformation $\mathcal{F}$, defined as

$$(\mathcal{F}h)(x,\xi) = \int_{\mathbb{R}^2} h(y,\eta)\, e^{2i\pi(x\eta - y\xi)}\, dy\, d\eta\,. \tag{2.7}$$

It is more intrinsic than the usual Fourier transformation, which depends on the choice of a scalar product on $\mathbb{R}^2$ rather than a two-form: in particular, it commutes with the linear action of the group $SL(2,\mathbb{R})$ on $\mathbb{R}^2$. In connection with the Weyl calculus, however, we shall often use instead $\mathcal{G} = 2^{2i\pi\mathcal{E}}\,\mathcal{F}$, (*cf. infra* for the signification of $2^{2i\pi\mathcal{E}}$ if in doubt): here $\mathcal{E}$ stands for Euler's operator, as defined by

$$\mathcal{E} = \frac{1}{2i\pi}\left(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} + 1\right) \tag{2.8}$$

(the extra constant makes it formally self-adjoint on $L^2(\mathbb{R}^2)$), so that

$$(\mathcal{G}h)(x,\xi) = 2\int_{\mathbb{R}^2} h(y,\eta)\, e^{4i\pi(x\eta - y\xi)}\, dy\, d\eta\,. \tag{2.9}$$

Concerning the use of $\mathcal{G}$, let us remark – the verification as a consequence of (2.1) is immediate – that, if $h \in \mathcal{S}'(\mathbb{R}^2)$, $\mathcal{G}h$ is the symbol of the operator $u \mapsto \mathrm{Op}(h)\check{u}$, where $\check{u}(x) = u(-x)$: in particular, $\mathcal{G}^2$ is the identity transformation. Then, operators with $\mathcal{G}$-invariant symbols vanish on $\mathcal{S}_{\mathrm{odd}}(\mathbb{R})$, and those whose symbols change to their negatives under $\mathcal{G}$ vanish on $\mathcal{S}_{\mathrm{even}}(\mathbb{R})$. On the other hand, even distributions on $\mathbb{R}^2$ are just the symbols of the operators that commute with the map $u \mapsto \check{u}$, in other words the operators which send $\mathcal{S}_{\mathrm{even}}(\mathbb{R})$ to $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$ and $\mathcal{S}_{\mathrm{odd}}(\mathbb{R})$ to $\mathcal{S}'_{\mathrm{odd}}(\mathbb{R})$. Even-even symbols h are those of operators which vanish on $\mathcal{S}_{\mathrm{odd}}(\mathbb{R})$ and send $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'_{\mathrm{even}}(\mathbb{R})$: they are characterized as being even distributions invariant under $\mathcal{G}$, and there is a similar notion of odd-odd symbol (even, invariant under $-\mathcal{G}$).

We assume some familiarity with the general spectral theory of self-adjoint operators, as can be found in treatises of functional analysis, for instance [41, 69], in particular with Stone's theorem about groups of unitary operators and their infinitesimal generators. The Euler operator $\mathcal{E}$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$, when $C_0^\infty(\mathbb{R}^2\backslash\{0\})$ is taken as its initial domain. One has

$$(t^{2i\pi\mathcal{E}}h)(x,\xi) = t\,h(tx,t\xi) \tag{2.10}$$

for every $h \in L^2(\mathbb{R}^2)$ and $t > 0$: besides the fact that the map $t \mapsto t^{2i\pi\mathcal{E}}$ is a group homomorphism, this means in particular that

$$\frac{d}{dt}\bigg|_{t=1}(t^{2i\pi\mathcal{E}}h) = 2i\pi\,\mathcal{E}\,h \quad \text{if} \quad h \in C_0^\infty(\mathbb{R}^2\backslash\{0\}).$$

The operator $t^{2i\pi\mathcal{E}}$ is a continuous endomorphism of the space $\mathcal{S}(\mathbb{R}^2)$, and extends as a continuous endomorphism of the dual space $\mathcal{S}'(\mathbb{R}^2)$, setting

$$\left\langle t^{2i\pi\mathcal{E}}\mathfrak{S},\, h\right\rangle = \left\langle \mathfrak{S},\, t^{-2i\pi\mathcal{E}}h\right\rangle \tag{2.11}$$

whenever $h \in \mathcal{S}(\mathbb{R}^2)$ and $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$, or

$$\left\langle t^{-1-2i\pi\mathcal{E}}\mathfrak{S},\, h\right\rangle = \left\langle \mathfrak{S},\, (x,\xi) \mapsto h(tx,t\xi)\right\rangle\,. \tag{2.12}$$

A distribution $\mathfrak{S}$ is homogeneous of degree $-1-\nu$ if and only if $t^{2i\pi\mathcal{E}}\mathfrak{S} = t^{-\nu}\,\mathfrak{S}$.

We now recall the spectral theory of the operator $\mathcal{E}$, *i.e.*, the decomposition of functions h in $L^2(\mathbb{R}^2)$ into homogeneous parts. Actually, we shall only have to deal with the subspace $L^2_{\text{even}}(\mathbb{R}^2)$ of $L^2(\mathbb{R}^2)$ consisting of even functions: this simplifies notation a little bit.

Any function $h \in \mathcal{S}(\mathbb{R}^2)$ can be decomposed as the integral

$$h = \int_{-\infty}^{\infty} h_\lambda\, d\lambda \tag{2.13}$$

into functions homogeneous of degrees $-1-i\lambda$, setting

$$h_\lambda(x,\xi) = \frac{1}{2\pi}\int_0^\infty t^{i\lambda}\,h(tx,t\xi)\,dt\ : \tag{2.14}$$

indeed (2.13) follows from (2.14) together with an application of the one-dimensional Fourier inversion formula to the function $f(\tau) = e^\tau\,h(e^\tau x, e^\tau \xi)$.

In the case when h is even, one can recover h_λ from the function $h_\lambda^\flat$ on the real line defined as

$$h_\lambda^\flat(s) = h_\lambda(s,1) \tag{2.15}$$

since one may write

$$h_\lambda(x,\xi) = |\xi|^{-1-i\lambda}\,h_\lambda^\flat\left(\frac{x}{\xi}\right)\,. \tag{2.16}$$

One then has

$$\|h\|^2_{L^2(\mathbb{R}^2)} = 4\pi\int_{-\infty}^{\infty}\|h_\lambda^\flat\|^2_{L^2(\mathbb{R})}\,d\lambda\,, \tag{2.17}$$

a formula which permits extending the definition of $h^\flat_\lambda$ as an element of $L^2(\mathbb{R})$, for almost every $\lambda \in \mathbb{R}$, whenever h lies in $L^2_{\text{even}}(\mathbb{R}^2)$: the formulas (2.13) and (2.17) together provide the spectral decomposition of Euler's operator in the space $L^2_{\text{even}}(\mathbb{R}^2)$. It will often be necessary to consider the complex continuation of the function $\lambda \mapsto h_\lambda$, getting the function $\nu \mapsto h_{-i\nu}$, with

$$h_{-i\nu}(x,\xi) = \frac{1}{2\pi} \int_0^\infty t^\nu \, h(tx, t\xi)\, dt\,, \tag{2.18}$$

as a result: if h lies in $\mathcal{S}(\mathbb{R}^2)$, the function $h_{-i\nu}$ is well defined for $\text{Re}\ \nu > -1$, a function homogeneous of degree $-1-\nu$ in $\mathbb{R}^2\backslash\{0\}$. Again, we set

$$h^\flat_{-i\nu}(s) = h_{-i\nu}(s,1)\,. \tag{2.19}$$

Recall that $\mathcal{G}$, rather than $\mathcal{F}$, plays an important role in the Weyl calculus. Coming back to the link between $\mathcal{G}$ and $\mathcal{F}$, as expressed just before (2.8), note that when acting on functions, the operator $2^{-\frac{1}{2}+i\pi\mathcal{E}}$ is given by the formula

$$\left(2^{-\frac{1}{2}+i\pi\mathcal{E}}\, h\right)(x,\xi) = h\left(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}\xi\right)\,. \tag{2.20}$$

There is no satisfactory way to get rid of the factor $\sqrt{2}$ entirely when dealing with the Weyl calculus in an arithmetic environment: ultimately, this is due to the fact that both the one-dimensional and the two-dimensional Fourier transformations play a role here, whereas the characters on the line that give rise to these two Fourier transforms are more naturally normalized in a different way (compare $\mathcal{F}$ and $\mathcal{G}$ in (2.9)). This is why, for short, we shall also use the modified version $\text{Op}_{\sqrt{2}}$ of the Weyl calculus, defined by

$$\text{Op}_{\sqrt{2}}(h) = \text{Op}\left(2^{-\frac{1}{2}+i\pi\mathcal{E}}\, h\right)\ : \tag{2.21}$$

the use of $\text{Op}_{\sqrt{2}}$ instead of Op will only serve to make a few formulas nicer, especially at the end of Section 16.

Remark. Some of our readers with a training in arithmetic will undoubtedly feel some reservations about the presence of the factor $\sqrt{2}$ in (2.20) and would find the change $(x,\xi) \mapsto (x, 2\xi)$ or $(2x,\xi)$ rather than $(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}\xi)$ preferable. They are of course right and, had we pursued the present work in the direction of congruence subgroups, this would have been our choice: however, when the emphasis is on $SL(2,\mathbb{Z})$, the present choice permits one not to have to bother with Hecke's subgroup $\Gamma_0(2)$.

We now introduce the Poincaré upper half-plane $\Pi = \{z \in \mathbb{C} \colon \text{Im}\ z > 0\}$, on which the group $SL(2,\mathbb{R})$ acts through fractional-linear transformations:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z = \frac{az+b}{cz+d}\,. \tag{2.22}$$

On Π, the Laplace-Beltrami operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \tag{2.23}$$

commutes with this action: in just the same way, the Euler operator $\mathcal{E}$ on $\mathbb{R}^2$ commutes with the linear action of $SL(2,\mathbb{R})$.

For every $z \in \Pi$, consider the two functions u_z and u_z^1 defined on the real line by

$$u_z(t) = 2^{\frac{1}{4}} \left(\mathrm{Im} \left(-\frac{1}{z} \right) \right)^{\frac{1}{4}} \exp \left(i\pi \frac{t^2}{\bar{z}} \right),$$

$$u_z^1(t) = 2^{\frac{5}{4}} \pi^{\frac{1}{2}} \left(\mathrm{Im} \left(-\frac{1}{z} \right) \right)^{\frac{3}{4}} t \exp \left(i\pi \frac{t^2}{\bar{z}} \right). \tag{2.24}$$

The family $(u_z)_{z\in\Pi}$ is total in $L^2_{\mathrm{even}}(\mathbb{R})$, and the family $(u_z^1)_{z\in\Pi}$ is total in $L^2_{\mathrm{odd}}(\mathbb{R})$. For our purposes, the best way to prove this is to use the metaplectic representation Met the existence of which has been recalled just around (2.4), and to show (this has been done in [62, p. 120–121]) that, if $\tilde{g} \in \widetilde{SL}(2,\mathbb{R})$ lies above $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and if $b > 0$, one has the formulas

$$\mathrm{Met}(\tilde{g})\, u_z = \pm \left(\frac{a + \frac{b}{\bar{z}}}{|a + \frac{b}{\bar{z}}|} \right)^{-\frac{1}{2}} u_{\frac{az+b}{cz+d}},$$

$$\mathrm{Met}(\tilde{g})\, u_z^1 = \pm \left(\frac{a + \frac{b}{\bar{z}}}{|a + \frac{b}{\bar{z}}|} \right)^{-\frac{3}{2}} u^1_{\frac{az+b}{cz+d}}, \tag{2.25}$$

in which the important fact is that the coefficients in front of the right-hand sides are constants (*i.e.*, depend only on z) of modulus one: this less precise result is quite easy to check directly, for one may satisfy oneself with doing this only when $\tilde{g}$ describes the set of generators of $\widetilde{SL}(2,\mathbb{R})$ given right after (2.4). Then, that each of the above-mentioned family is total in the appropriate space results from the well-known fact that, when restricted to even, or odd, functions in $L^2(\mathbb{R})$, the metaplectic representation is irreducible. In the odd case (only), one has a more precise result, to wit a *resolution of the identity*, obtained by polarizing the identity

$$\|v\|^2_{L^2(\mathbb{R})} = (8\pi)^{-1} \int_\Pi |(u_z^1|v)|^2\, d\mu(z). \tag{2.26}$$

Proposition 2.1. *An operator* $\mathrm{Op}(h)$, *with* $h \in \mathcal{S}'_{\mathrm{even}}(\mathbb{R}^2)$, *is characterized by the pair of functions* $z \mapsto (u_z|\mathrm{Op}(h)u_z)$ *and* $z \mapsto (u_z^1|\mathrm{Op}(h)u_z^1)$.

Proof. Using (2.24), one sees that the function $(z,w) \mapsto (u_w|\mathrm{Op}(h)u_z)$ (*resp.* $(u_w^1|\mathrm{Op}(h)u_z^1)$) is $(\mathrm{Im}\,(-\frac{1}{w})\,\mathrm{Im}\,(-\frac{1}{z}))^\delta$ with $\delta = \frac{1}{4}$ (*resp.* $\frac{3}{4}$) times a function holomorphic with respect to w and antiholomorphic with respect to z. Thus, if these two functions of (z,w) vanish on the diagonal of $\Pi \times \Pi$, they vanish on the whole of $\Pi \times \Pi$. Since h is even, one may assume that h is either an even-even or an odd-odd symbol: in either case, one may conclude, observing that the family $(u_z)_{z\in\Pi}$ (*resp.* $(u_z^1)_{z\in\Pi}$) is total in $\mathcal{S}_{\text{even}}(\mathbb{R})$ (*resp.* $\mathcal{S}_{\text{odd}}(\mathbb{R})$) as well. □

As a consequence of (2.2), it is easily found that

$$W(u_z, u_z)(x,\xi) = 2 \exp\left(-\frac{2\pi}{\text{Im } z}|x - z\,\xi|^2\right) \tag{2.27}$$

and

$$W(u_z^1, u_z^1)(x,\xi) = 2 \left[\frac{4\pi}{\text{Im } z}|x - z\,\xi|^2 - 1\right] \exp\left(-\frac{2\pi}{\text{Im } z}|x - z\,\xi|^2\right) : \tag{2.28}$$

actually, using (2.6), it is enough to check (2.27) and (2.28) when $z = i$, which simplifies the computation greatly.

A simple, if somewhat tedious, calculation (again, group-invariance arguments make it easier) shows that if $k(\frac{|x-z\xi|^2}{\text{Im } z})$ is a smooth function k of the indicated expression, one has

$$\left(\Delta - \frac{1}{4}\right) k\left(\frac{|x - z\xi|^2}{\text{Im } z}\right) = \pi^2\,\mathcal{E}^2\,.\,k\left(\frac{|x - z\xi|^2}{\text{Im } z}\right), \tag{2.29}$$

where the emphasis is put on the z-variable (*resp.* the (x,ξ)-variables) on the left-hand (*resp.* right-hand) side. This applies in particular to the pair of Wigner functions just displayed, yielding for every $h \in \mathcal{S}'(\mathbb{R}^2)$, and $p = 0$ or 1, the relation

$$(u_z^p|\text{Op}(\pi^2\,\mathcal{E}^2 h)u_z^p) = \left(\Delta - \frac{1}{4}\right)\,(z \mapsto (u_z^p|\text{Op}(h)u_z^p)) \tag{2.30}$$

as a consequence of (2.3) and (2.27), (2.28).

We conclude this section with the definition of automorphic distributions.

Definition 2.2. Set $\Gamma = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$. An automorphic distribution is any Γ-invariant tempered distribution on $\mathbb{R}^2$, *i.e.*, any distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$ with the property that

$$\begin{aligned} \langle \mathfrak{S} \circ g^{-1}, h\rangle : &= \langle \mathfrak{S}\,,\,h \circ g\rangle \\ &= \langle \mathfrak{S}, h\rangle \end{aligned} \tag{2.31}$$

for every $h \in \mathcal{S}(\mathbb{R}^2)$ and every $g \in \Gamma$.

Observe that since the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ lies in Γ, every automorphic distribution is an even distribution, which will make our life somewhat easier: it is only in Section 18 that we shall approach more general arithmetic groups and possibly odd distributions, but we shall not go far in this direction.

Theorem 2.3. *An automorphic distribution $\mathfrak{S}$ is characterized by the pair of functions $z \mapsto (u_z^p | \mathrm{Op}(\mathfrak{S}) u_z^p)$ $(p = 0, 1)$ on the upper half-plane: these two functions are automorphic. If $\mathfrak{S}$ is homogeneous of degree $-1 - i\lambda$, these functions are generalized eigenfunctions of Δ for the eigenvalue $\frac{1+\lambda^2}{4}$.*

Proof. Since an automorphic distribution is even, the first point follows from Proposition 2.1. Observe that neither of the extra factors, of modulus 1, that appear on the right-hand sides of (2.25), shows any longer in the scalar product $(u_z^p | \mathrm{Op}(\mathfrak{S}) u_z^p)$, since its two occurrences there cancel out: that such a scalar product is an automorphic function of z is thus a consequence of the covariance formula (2.4). It has to be a generalized eigenfunction of Δ for the eigenvalue $\frac{1+\lambda^2}{4}$ if $\mathfrak{S}$ is homogeneous of degree $-1 - i\lambda$, as a consequence of (2.30). □

3 Eisenstein distributions, Dirac's comb and Bezout's distribution

For $\mathrm{Re}\,\nu < -1$, $h \in \mathcal{S}(\mathbb{R}^2)$, one can define

$$\left\langle \mathfrak{E}_\nu^\sharp, h \right\rangle = \frac{1}{2} \sum_{|m|+|n| \neq 0} \int_{-\infty}^{\infty} |t|^{-\nu} h(tn, tm)\, dt\,, \tag{3.1}$$

a convergent expression. As shown in [62, Proposition 13.1], this defines an even tempered distribution $\mathfrak{E}_\nu^\sharp$, homogeneous of degree $-1-\nu$, and the function $\nu \mapsto \mathfrak{E}_\nu^\sharp$ extends as a holomorphic function of ν for $\nu \neq \pm 1$, with simple poles at $\nu = \pm 1$; the residues there are given as $\mathrm{Res}_{\nu=-1}\, \mathfrak{E}_\nu^\sharp = -1$ and $\mathrm{Res}_{\nu=1}\, \mathfrak{E}_\nu^\sharp = \delta$, the unit mass at the origin of $\mathbb{R}^2$. All that concerns the analytic continuation of the map $\nu \mapsto \mathfrak{E}_\nu^\sharp$ was derived in *loc.cit.* from (3.25) below, an identity obtained by the use of Poisson's formula. Also, $\mathcal{F}\mathfrak{E}_\nu^\sharp = \mathfrak{E}_{-\nu}^\sharp$. We shall also use, here, a different normalization, setting

$$\mathfrak{F}_\nu^\sharp = 2^{\frac{-1-\nu}{2}}\, \mathfrak{E}_\nu^\sharp\ : \tag{3.2}$$

since $\mathfrak{E}_\nu^\sharp$ is homogeneous of degree $-1-\nu$, this can also be written as

$$\mathfrak{F}_\nu^\sharp = 2^{-\frac{1}{2}+i\pi\mathcal{E}}\, \mathfrak{E}_\nu^\sharp\,, \tag{3.3}$$

in the sense of (2.12). Then,

$$\mathcal{G}\mathfrak{F}_\nu^\sharp = \mathfrak{F}_{-\nu}^\sharp\,, \tag{3.4}$$

with $\mathcal{G}$ as defined in (2.9). In view of (2.21), one has

$$\mathrm{Op}(\mathfrak{F}_\nu^\sharp) = \mathrm{Op}_{\sqrt{2}}(\mathfrak{E}_\nu^\sharp)\,. \tag{3.5}$$

Incidentally, the reader may wonder why we have felt it necessary to use both a change of typographical style and the superscript $\sharp$ to denote the lifted-up version $\mathfrak{E}_\nu^\sharp$ of the Eisenstein series $E_{\frac{1-\nu}{2}}$ (*cf.* Theorem 3.1) or the lifted-up version $\mathfrak{M}_j^\sharp$ (*cf. infra,* (4.4)) of a Maass cusp-form $\mathcal{M}_j$. This is so because we need to preserve the notation of [62], in which intermediary versions $\mathfrak{E}_\nu$ and $\mathfrak{M}_j$ (which we are dispensing with here, but which are likely to recur elsewhere) also played a role.

Remark on notation: to help the reader, let us make the following conventions. Fraktur-style capital letters with the superscript $\sharp$ denote *homogeneous* automorphic distributions. Non-homogeneous automorphic distributions, like the Dirac and Bezout distributions introduced later in this section, are denoted by fraktur-style capital letters without such a superscript; fraktur-style lower-case letters shall denote non-automorphic distributions, usually the pieces some more interesting (automorphic) distributions are made of. Calligraphic-style letters $\mathcal{F}$, $\mathcal{G}$ are reserved for versions of the Fourier transformation, $\mathcal{M}$ and $\mathcal{N}$ (in the next section) for Maass cusp-forms.

Introduce, whenever $\text{Re } \nu < 1$, the distribution $\mathfrak{q}_\nu$ such that

$$\langle \mathfrak{q}_\nu, h \rangle : = \frac{1}{2} \int_{-\infty}^{\infty} |t|^{-\nu}\, h(t, 0)\, dt\,, \tag{3.6}$$

which is just the term with $n = 1$, $m = 0$ from the right-hand side of (3.1). One may abbreviate $\mathfrak{q}_\nu$ as

$$\mathfrak{q}_\nu(x, \xi) = \frac{1}{2}\, |x|^{-\nu}\, \delta(\xi) \; : \tag{3.7}$$

here, of course, δ denotes the Dirac mass on $\mathbb{R}$. Given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$, one has $\left\langle \mathfrak{q}_\nu \circ g^{-1}, h \right\rangle = \langle \mathfrak{q}_\nu, h \circ g \rangle$ with

$$(h \circ g)(\xi) = h(a\xi_1 + b\xi_2, c\xi_1 + d\xi_2)\,, \tag{3.8}$$

so that

$$\left\langle \mathfrak{q}_\nu \circ g^{-1}, h \right\rangle = \frac{1}{2} \int_{-\infty}^{\infty} |t|^{-\nu}\, h(at, ct)\, dt\,. \tag{3.9}$$

As is usual, if $|m| + |n| \neq 0$, we shall denote as (m, n) the largest common divisor of m and n. When $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, $(m, n) = 1$, the pair $\left(\begin{smallmatrix} n & \cdot \\ m & \cdot \end{smallmatrix}\right)$ describes the set Γ / Γ_∞^o where $\Gamma = SL(2, \mathbb{Z})$ and $\Gamma_\infty^o = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{Z}\}$: our notation Γ_∞^o is meant to emphasize that this is only $\Gamma \cap N$, with $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) : b \in \mathbb{R}\}$ while the (twice as large) group $\Gamma_\infty = \{\pm g,\ g \in \Gamma_\infty^o\}$ is the subgroup of Γ whose associated group of fractional-linear transformations of Π (and of $P_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$) is the stabilizer of the point at infinity. Using also (3.2), one may thus write (setting in (3.1) $m = rm_1$, $n = rn_1$ with $r \geq 1$ and $(m_1, n_1) = 1$)

$$\mathfrak{F}_\nu^\sharp = 2^{\frac{-1-\nu}{2}}\, \zeta(1 - \nu) \sum_{g \in \Gamma / \Gamma_\infty^o} \mathfrak{q}_\nu \circ g^{-1}\,, \tag{3.10}$$

where ζ is the Riemann zeta function.

This series is weakly convergent in $\mathcal{S}'(\mathbb{R}^2)$ if $\text{Re }\nu < -1$. Introduce also

$$\mathfrak{r}_\nu = \mathcal{G}\,\mathfrak{q}_{-\nu} \ : \tag{3.11}$$

then, if $\text{Re }\nu > -1$ and $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathfrak{r}_\nu, h\rangle = \int_{-\infty}^{\infty} |t|^\nu\, dt \int_{\mathbb{R}^2} h(y,\eta)\, e^{-4i\pi t\eta}\, dy\, d\eta\,, \tag{3.12}$$

from which it follows that $\mathfrak{r}_\nu$ is a function:

$$\mathfrak{r}_\nu(x,\xi) = \pi^{-\nu-\frac{1}{2}}\, \frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(-\frac{\nu}{2})}\, |2\xi|^{-\nu-1}\,. \tag{3.13}$$

Using (3.10), (3.4) and (3.11), one thus gets when $\text{Re }\nu > 1$ the expression (a weakly convergent series in $\mathcal{S}'(\mathbb{R}^2)$)

$$\mathfrak{F}_\nu^\sharp = 2^{\frac{\nu-1}{2}}\, \zeta(1+\nu) \sum_{g\in\Gamma/\Gamma_\infty^o} \mathfrak{r}_\nu \circ g^{-1}\,. \tag{3.14}$$

No expansion comparable to (3.10) or (3.14), however, can be given when $-1 \leq \text{Re }\nu \leq 1$: in this range of values of ν, it is (3.25) below that has to be used.

In order to better understand the operator $\text{Op}(\mathfrak{F}_\nu^\sharp)$, we shall apply Proposition 2.1.

Theorem 3.1. *For $\nu \neq \pm 1$, one has*

$$(u_z \,|\, \text{Op}(\mathfrak{F}_\nu^\sharp) u_z) = E^*_{\frac{1-\nu}{2}}(z) \tag{3.15}$$

and

$$(u_z^1 \,|\, \text{Op}(\mathfrak{F}_\nu^\sharp) u_z^1) = -\nu\, E^*_{\frac{1-\nu}{2}}(z)\,, \tag{3.16}$$

where $E_{\frac{1-\nu}{2}}(z)$ is the classical Eisenstein series defined when $\text{Re }\nu < -1$ *as*

$$E_{\frac{1-\nu}{2}}(z) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\ (m,n)=1}} \left(\frac{|mz-n|^2}{\text{Im }z}\right)^{\frac{\nu-1}{2}} \tag{3.17}$$

and [48, p. 208]

$$\begin{aligned} E^*_{\frac{1-\nu}{2}}(z) :&= \zeta^*(1-\nu)\, E_{\frac{1-\nu}{2}}(z) \\ &= E^*_{\frac{1+\nu}{2}}(z) \end{aligned} \tag{3.18}$$

with

$$\begin{aligned} \zeta^*(s) :&= \pi^{-\frac{s}{2}}\, \Gamma(\frac{s}{2})\, \zeta(s) \\ &= \zeta^*(1-s)\,. \end{aligned} \tag{3.19}$$

Proof. Using analytic continuation, one may assume that $\mathrm{Re}\,\nu < -1$. From (2.3) together with (2.27) and (2.28), one has

$$(u_z \,|\, \mathrm{Op}(\mathfrak{F}^\sharp_\nu) u_z) = \left\langle \mathfrak{F}^\sharp_\nu \,,\, (x,\xi) \mapsto 2\,\exp(-\frac{2\pi}{\mathrm{Im}\,z}|x - z\xi|^2) \right\rangle \tag{3.20}$$

and

$$(u^1_z \,|\, \mathrm{Op}(\mathfrak{F}^\sharp_\nu) u^1_z) = \left\langle \mathfrak{F}^\sharp_\nu \,,\, (x,\xi) \mapsto 2 \left[\frac{4\pi}{\mathrm{Im}\,z}|x - z\xi|^2 - 1 \right] \exp(-\frac{2\pi}{\mathrm{Im}\,z}|x - z\xi|^2) \right\rangle . \tag{3.21}$$

Set, for $\varepsilon > 0$ and fixed $z \in \Pi$,

$$\begin{aligned} J(\varepsilon; x, \xi) &= \frac{1}{2} \int_{-\infty}^{\infty} |t|^{-\nu} . 2\,\exp\left(-\frac{2\pi\varepsilon\, t^2}{\mathrm{Im}\,z} |x - z\xi|^2 \right) dt \\ &= (2\pi)^{\frac{\nu-1}{2}}\, \Gamma\left(\frac{1-\nu}{2}\right) \varepsilon^{\frac{\nu-1}{2}} \left(\frac{|x - z\xi|^2}{\mathrm{Im}\,z} \right)^{\frac{\nu-1}{2}} . \end{aligned} \tag{3.22}$$

Then, from (3.10) and (3.9), then (3.20) and (3.21),

$$\begin{aligned} (u_z \,|\, \mathrm{Op}(\mathfrak{F}^\sharp_\nu) u_z) &= 2^{\frac{-1-\nu}{2}}\, \zeta(1-\nu) \sum_{(m,n)=1} J(1; n, m) \\ &= \pi^{\frac{\nu-1}{2}}\, \Gamma\left(\frac{\nu-1}{2}\right) \zeta(1-\nu) \times \frac{1}{2} \sum_{(m,n)=1} \left(\frac{|mz - n|^2}{\mathrm{Im}\,z} \right)^{\frac{\nu-1}{2}} \\ &= E^*_{\frac{1-\nu}{2}}(z) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} (u^1_z \,|\, \mathrm{Op}(\mathfrak{F}^\sharp_\nu) u^1_z) &= \left(-2\,\frac{d}{d\varepsilon} - 1 \right)\Big|_{\varepsilon=1} \left(2^{\frac{-1-\nu}{2}}\, \zeta(1-\nu) \sum_{(m,n)=1} J(\varepsilon; n, m) \right) \\ &= -\nu\, E^*_{\frac{1-\nu}{2}}(z) . \end{aligned} \tag{3.24}$$

□

We also gave in [62, Proposition 13.1] another expression of the Eisenstein distribution $\mathfrak{E}^\sharp_\nu$: namely, for $\mathrm{Re}\,\nu < 0$, $\nu \neq -1$, and $h \in \mathcal{S}(\mathbb{R}^2)$, one has

$$\begin{aligned} \langle \mathfrak{E}^\sharp_\nu, h \rangle &= \zeta(-\nu) \int_{-\infty}^{\infty} |t|^{-\nu-1} (\mathcal{F}_1^{-1} h)(0,t)\, dt + \zeta(1-\nu) \int_{-\infty}^{\infty} |t|^{-\nu}\, h(t,0)\, dt \\ &+ \sum_{n \neq 0} \sigma_\nu(|n|) \int_{-\infty}^{\infty} |t|^{-\nu-1} \left(\mathcal{F}_1^{-1} h\right) \left(\frac{n}{t}, t\right) dt , \end{aligned} \tag{3.25}$$

where the partial Fourier transform is defined as

$$(\mathcal{F}_1^{-1}h)(s,t) = \int_{-\infty}^{\infty} h(x,t)\, e^{2i\pi sx}\, dx\,, \tag{3.26}$$

and, for $n \geq 1$, $\sigma_\nu(n) = \sum_{\substack{d\geq 1\\ d|n}} d^\nu$.

Only the presence of the first two terms dictated the condition $\mathrm{Re}\ \nu < 0$ there, but analytic continuation makes it possible to drop this condition, assuming instead $\nu \neq -1, 0, 1, 2, \dots$. One may write (3.25) as

$$\mathfrak{E}_\nu^\sharp(x,\xi) = \zeta(-\nu)\,|\xi|^{-\nu-1} + \zeta(1-\nu)\,|x|^{-\nu}\,\delta(\xi) + \sum_{n\neq 0} \sigma_\nu(|n|)\,|\xi|^{-\nu-1}\, e^{2i\pi n\frac{x}{\xi}}\,. \tag{3.27}$$

Note that, at $\nu = 0$, the infinities from the first two terms cancel out since $\zeta(0) = -\frac{1}{2}$ and

$$-\frac{1}{2}\,|\xi|^{-\nu-1} - \nu^{-1}\,\delta(\xi) = \frac{1}{2\nu}\,\frac{d}{d\xi}\,[(|\xi|^{-\nu} - 1)\,\mathrm{sign}\,\xi] \ : \tag{3.28}$$

this explains why $\mathfrak{E}_0^\sharp$ is, indeed, meaningful.

Consider the following two distributions on $\mathbb{R}^2$:

$$\mathfrak{d} = \delta(x-1)\,\delta(\xi) \tag{3.29}$$

and

$$\mathfrak{b} = e^{2i\pi x}\,\delta(\xi - 1)\,, \tag{3.30}$$

i.e., the distributions defined, for $h \in \mathcal{S}(\mathbb{R}^2)$, by

$$\langle \mathfrak{d}, h\rangle = h(1,0) \tag{3.31}$$

and

$$\langle \mathfrak{b}, h\rangle = \int_{-\infty}^{\infty} h(x,1)\, e^{2i\pi x}\, dx\,. \tag{3.32}$$

One may observe that $\mathfrak{d}\circ g^{-1} = \mathfrak{d}$ whenever $g \in \Gamma_\infty^o$, the subgroup of Γ defined right after (3.9), and that the same invariance property holds if one substitutes $\mathfrak{b}$ for $\mathfrak{d}$. It thus makes sense, at least formally, to define the distributions

$$\mathfrak{D}^{\text{prime}} = 2\pi \sum_{g\in\Gamma/\Gamma_\infty^o} \mathfrak{d}\circ g^{-1} \tag{3.33}$$

and

$$\mathfrak{B} = \frac{1}{2} \sum_{g\in\Gamma/\Gamma_\infty^o} \mathfrak{b}\circ g^{-1}\,. \tag{3.34}$$

However, as we shall show in the next section, the first of these two distributions is just a slightly modified version of the Dirac comb alluded to above, while the second one incorporates *all* homogeneous automorphic distributions invariant under the symplectic Fourier transformation in its decomposition into homogeneous components. Otherwise stated, the first object, as we shall see, yields nothing but the Eisenstein series, while the second yields, in a sense, all non-holomorphic modular forms. The major task of the present work is to show how canonically $\mathfrak{B}$ generates not only automorphic distributions but also, in the sense of Weyl's calculus, sharp products of such.

Set

$$\mathfrak{D}(x,\xi) = 2\pi \sum_{|m|+|n|\neq 0} \delta(x-n)\,\delta(\xi-m) \tag{3.35}$$

(the Dirac comb on $\mathbb{R}^2$ with the mass at the origin deleted): then [62, Proposition 16.1]

$$\mathfrak{D} = 2\pi + \int_{-\infty}^{\infty} \mathfrak{E}^\sharp_{i\lambda}\, d\lambda \tag{3.36}$$

in a weak sense in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$; the proof is quite easy.

Since $\langle \mathfrak{d}\circ g^{-1}, h\rangle = \langle \mathfrak{d}, h\circ g\rangle$ and, as said just after (3.9), the set of pairs $\left(\begin{smallmatrix} n & \cdot \\ m & \cdot \end{smallmatrix}\right)$ is a set of representatives of Γ/Γ^o_∞, one has

$$\left\langle \mathfrak{D}^{\text{prime}}, h\right\rangle = 2\pi \sum_{(n,m)=1} h(n,m)\,. \tag{3.37}$$

The distribution just defined is related to the Dirac comb in an easy way. Actually, as shown in (*loc.cit.*, (16.57))

$$\mathfrak{D}^{\text{prime}} = \frac{12}{\pi} + \int_{-\infty}^{\infty} (\zeta(1-i\lambda))^{-1}\, \mathfrak{E}^\sharp_{i\lambda}\, d\lambda\,, \tag{3.38}$$

so that one could write in some sense

$$\mathfrak{D}^{\text{prime}} = (\zeta(1+2i\pi\mathcal{E}))^{-1}\,\mathfrak{D}\,. \tag{3.39}$$

As it turns out, the distribution $\mathfrak{D}^{\text{prime}}$ is somewhat more fundamental than $\mathfrak{D}$, and the definition of the Bezout distribution which follows bears more similarity to $\mathfrak{D}^{\text{prime}}$ than to $\mathfrak{D}$: we should really have denoted it as $\mathfrak{B}^{\text{prime}}$, but have not done so, for the sake of simpler notation, as it is by far the more important of the two related distributions, the second of which will be made explicit at the very end of this section.

We now turn to the proper definition of the Bezout distribution.

Definition 3.2. With $\mathfrak{b}$ defined as in (3.30), set, for every $\ell \geq 1$,

$$\mathfrak{b}^\ell = \pi^2\mathcal{E}^2\,(\pi^2\mathcal{E}^2+1)\,\cdots\,(\pi^2\mathcal{E}^2+(\ell-1)^2)\,\mathfrak{b}\,. \tag{3.40}$$

One then defines, for $\ell \geq 1$, the distribution

$$\mathfrak{B}^{\ell} = \frac{1}{2} \sum_{g \in \Gamma/\Gamma_{\infty}^{o}} \mathfrak{b}^{\ell} \circ g^{-1}\,, \tag{3.41}$$

a definition to be justified in the theorem that follows.

Actually, we are only interested in defining $\langle \mathfrak{B}^{\ell}, h \rangle$ when $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$; if $h \in \mathcal{S}_{\text{odd}}(\mathbb{R}^2)$, the series may not converge (though it does so if h lies in the image of $\mathcal{S}_{\text{odd}}(\mathbb{R}^2)$ under the operator $i\pi\mathcal{E}\,(\pi^2\mathcal{E}^2 + \frac{1}{4})$, a remark unimportant at this point), but it must be treated as zero since one may group the terms $\mathfrak{b}^{\ell} \circ g^{-1}$ and $\mathfrak{b}^{\ell} \circ (-g)^{-1}$ whose sum is an even distribution.

Theorem 3.3. *If $\ell \geq 1$, the series on the right-hand side of* (3.41) *is weakly convergent in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$, i.e., the series $\langle \mathfrak{B}^{\ell}, h \rangle = \frac{1}{2} \sum_{g \in \Gamma/\Gamma_{\infty}^{o}} \langle \mathfrak{b}^{\ell} \circ g^{-1}, h \rangle$ converges for every $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$.*

Proof. Our problem is to show that the series

$$\langle \mathfrak{B}, h \rangle = \frac{1}{2} \sum_{g \in \Gamma/\Gamma_{\infty}^{o}} \langle \mathfrak{b} \circ g^{-1}, h \rangle \tag{3.42}$$

converges whenever $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$ lies in fact in the image of the operator $\pi^2\,\mathcal{E}^2$. Completing the column $\left(\begin{smallmatrix} n \\ m \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix}\right)$, we thus have to examine

$$\langle \mathfrak{B}, h \rangle = \frac{1}{2} \sum_{(m,n)=1} I_{n,m}(h)\,, \tag{3.43}$$

with

$$I_{n,m}(h) \colon = \int_{-\infty}^{\infty} h(nx + n_1, mx + m_1)\, e^{2i\pi x}\, dx \ : \tag{3.44}$$

obviously, $I_{n,m}$ depends only on the pair n, m. This expression of $\mathfrak{B}$ (together with the condition $m_1 n - n_1 m = 1$) justifies the name of "Bezout distribution". If $|m| \geq |n|$, so that in particular $m \neq 0$, we write

$$I_{n,m}(h) = e^{-2i\pi \frac{\bar{n}}{m}} \int_{-\infty}^{\infty} h\left(nx - \frac{1}{m}, mx\right) e^{2i\pi x}\, dx\,, \tag{3.45}$$

where, as is usual, $\bar{n}$ is characterized by the condition $n\bar{n} \equiv 1 \bmod m$. In the case when $|n| > |m|$, we use instead of (3.45) the equation

$$I_{n,m}(h) = e^{2i\pi \frac{\bar{m}}{n}} \int_{-\infty}^{\infty} h\left(nx, mx + \frac{1}{n}\right) e^{2i\pi x}\, dx \tag{3.46}$$

with $m\bar{m} \equiv 1 \bmod n$ this time. Of course, $I_{n,m}(h)$ makes sense whether h is even or odd. We first need a lemma:

Lemma 3.4. *Let $h \in \mathcal{S}(\mathbb{R}^2)$. One has for some constant $C > 0$ the inequality*

$$|I_{n,m}(h)| \leq C\,(m^2+n^2)^{-\frac{1}{2}} \tag{3.47}$$

and, if h is an odd function,

$$|I_{n,m}(h)| \leq C\,(m^2+n^2)^{-1}\,. \tag{3.48}$$

Proof. We may assume that $|m| \geq |n|$: starting from (3.45), we get

$$\begin{aligned} |I_{n,m}(h)| &\leq C \int_{-\infty}^{\infty} (1+m^2x^2)^{-1}\,dx \\ &\leq C\,m^{-1}\,. \end{aligned} \tag{3.49}$$

Since, for t between 0 and $\frac{1}{m}$, one has

$$\int_{-\infty}^{\infty} [1+(nx-t)^2+m^2x^2]^{-1}\,dx \leq C\,m^{-1}\,,$$

it is clear that $I_{n,m}(h)$ agrees, up to a term less than $C\,m^{-2}$, with the integral

$$e^{-2i\pi\frac{\bar{n}}{m}} \int_{-\infty}^{\infty} h(nx,mx)\,e^{2i\pi x}\,dx\,,$$

which reduces if h is odd to

$$\begin{aligned} &i\,e^{-2i\pi\frac{\bar{n}}{m}} \int_{-\infty}^{\infty} (\sin 2\pi x)\,h(nx,mx)\,dx \\ &= \frac{i}{|m|}\,e^{-2i\pi\frac{\bar{n}}{m}} \int_{-\infty}^{\infty} \left(\sin\frac{2\pi x}{m}\right)\,h\left(\frac{n}{m}x,x\right)\,dx\,, \end{aligned} \tag{3.50}$$

clearly a $\mathrm{O}(m^{-2})$ for large m, as seen by splitting the integral into the parts where $|\frac{x}{m}| \leq 1$ and $|x| \geq |m|$. □

End of the proof of Theorem 3.3. Again, we consider the case when $|m| \geq |n|$. Let $f = f(s,\sigma)$ be any function in $\mathcal{S}(\mathbb{R}^2)$. Recalling that $h = 2i\pi\,\mathcal{E}\,f = (s\frac{\partial}{\partial s} + \sigma\frac{\partial}{\partial\sigma} + 1)\,f$, we have

$$(2i\pi\mathcal{E}\,f)\left(nx-\frac{1}{m},mx\right) = \left(x\frac{d}{dx}+1\right)\left(f\left(nx-\frac{1}{m},mx\right)\right) - \frac{1}{m}\frac{\partial f}{\partial s}\left(nx-\frac{1}{m},mx\right),$$

so that, after an integration by parts,

$$\begin{aligned} &I_{n,m}(2i\pi\,\mathcal{E}\,f) = \\ &e^{-2i\pi\frac{\bar{n}}{m}} \int_{-\infty}^{\infty} \left[-2i\pi\,x\,f\left(nx-\frac{1}{m},mx\right) - \frac{1}{m}\frac{\partial f}{\partial s}\left(nx-\frac{1}{m},mx\right)\right]\,e^{2i\pi x}\,dx\,, \end{aligned}$$

or

$$I_{n,m}\left(2i\pi\,\mathcal{E}\,f\right) = -\frac{2i\pi}{m}\,I_{n,m}\left(\left(\sigma + \frac{1}{2i\pi}\,\frac{\partial}{\partial s}\right)\,f\right), \tag{3.51}$$

finally leading to

$$I_{n,m}(-4\pi^2\mathcal{E}^2 f) = -\frac{4\pi^2}{m^2}\,I_{n,m}\left(\left(\sigma + \frac{1}{2i\pi}\frac{\partial}{\partial s}\right)^2 f\right) + \frac{2i\pi}{m}\,I_{n,m}\left(\left(\sigma - \frac{1}{2i\pi}\frac{\partial}{\partial s}\right) f\right), \tag{3.52}$$

where the last term arises from the commutation formulas $[2i\pi\,\mathcal{E},\sigma] = \sigma$ and $[2i\pi\,\mathcal{E}, \frac{\partial}{\partial s}] = -\frac{\partial}{\partial s}$.

It is then immediate from Lemma 3.4 that if $f \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$, one has

$$|I_{n,m}(-4\pi^2\mathcal{E}^2 f)| \leq C\,(m^2+n^2)^{-\frac{3}{2}},$$

from which Theorem 3.3 follows. □

Recall that the version $\mathrm{Op}_{\sqrt{2}}$ of the Weyl calculus was defined in (2.21). We now analyze the action of the operator with the Bezout distribution as its $\mathrm{Op}_{\sqrt{2}}$-symbol on the families of functions defined in (2.24).

Theorem 3.5. *For every $\ell \geq 1$, one has*

$$(u_z|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{B}^\ell)\,u_z) = (4\pi)^\ell\,\frac{\Gamma\left(\frac{1}{2}+\ell\right)}{\Gamma\left(\frac{1}{2}\right)}$$
$$\times\,\frac{1}{2}\sum_{\left(\begin{smallmatrix} n_1 & n_2 \\ m_1 & m_2\end{smallmatrix}\right)\in\Gamma/\Gamma_\infty^o}\left(\frac{y}{|-m_1 z+n_1|^2}\right)^{\ell+\frac{1}{2}}\exp 2i\pi\,\frac{m_2 z - n_2}{-m_1 z + n_1}, \tag{3.53}$$

and

$$(u_z^1|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{B}^\ell)\,u_z^1) = 0\,. \tag{3.54}$$

Proof. Of necessity, $(u_z^1|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_z^1) = 0$. For it is immediate that the distribution $\mathfrak{b}$ is invariant under the symplectic Fourier transformation $\mathcal{F}$ defined in (2.7), thus its image under $2^{-\frac{1}{2}+i\pi\mathcal{E}}$ is invariant under $\mathcal{G}$: as a consequence (*cf.* remark after (2.9)), $\mathrm{Op}(2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{b})$ vanishes on odd functions.

We now compute the interesting part. With $z = x+iy$ (we then use (s,σ) rather than (x,ξ) or (y,η) on $\mathbb{R}^2$), one has

$$\begin{aligned}(u_z|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_z) &= \left\langle\,\mathfrak{b},\,2^{-\frac{1}{2}-i\pi\mathcal{E}}\,W(u_z,u_z)\right\rangle\\ &= \left\langle\,\mathfrak{b},\,(s,\sigma)\mapsto\exp-\frac{\pi|s-z\sigma|^2}{y}\right\rangle :\end{aligned} \tag{3.55}$$

thus

$$\begin{aligned}(u_z|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_z) &= e^{-\pi y}\int_{-\infty}^{\infty} e^{2i\pi s}\, e^{-\pi\frac{(s-x)^2}{y}}\,ds\\ &= y^{\frac{1}{2}}\,e^{2i\pi z}\,. \end{aligned}\tag{3.56}$$

Then, as a consequence of the metaplectic covariance of the Weyl calculus together with (2.25)

$$\begin{aligned}(u_z|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b}\circ g^{-1})\,u_z) &= (u_{g^{-1}.z}|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_{g^{-1}.z})\\ &= \left(\frac{y}{|-m_1 z+n_1|^2}\right)^{\frac{1}{2}}\ \exp\,2i\pi\,\frac{m_2 z-n_2}{-m_1 z+n_1}\end{aligned}\tag{3.57}$$

if $g=\left(\begin{smallmatrix} n_1 & n_2\\ m_1 & m_2\end{smallmatrix}\right)\in\Gamma$.

Next, since $\mathcal{E}$ commutes with the action of Γ, one has

$$\pi^2\,\mathcal{E}^2\,(\pi^2\,\mathcal{E}^2+1)\,\cdots\,(\pi^2\,\mathcal{E}^2+(\ell-1)^2)\,2^{-\frac{1}{2}+i\pi\mathcal{E}}\,(\mathfrak{b}\circ g^{-1}) = 2^{-\frac{1}{2}+i\pi\mathcal{E}}\,(\mathfrak{b}^{\ell}\circ g^{-1})\,.$$

Also, as a consequence of (2.30), together with (2.4) and (2.25),

$$\begin{aligned}&(u_z|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b}^{\ell}\circ g^{-1})\,u_z)\\ &=\left(\Delta-\frac{1}{4}\right)\left(\Delta+\frac{3}{4}\right)\cdots\left(\Delta-\frac{1}{4}+(\ell-1)^2\right)\,(z\mapsto (u_{g^{-1}.z}|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_{g^{-1}.z}))\,.\end{aligned}\tag{3.58}$$

We note by induction that

$$\left(\Delta-\frac{1}{4}\right)\left(\Delta+\frac{3}{4}\right)\cdots\left(\Delta-\frac{1}{4}+(\ell-1)^2\right)\left(y^{\frac{1}{2}}\,e^{2i\pi z}\right)=(4\pi)^{\ell}\,\frac{\Gamma\left(\frac{1}{2}+\ell\right)}{\Gamma\left(\frac{1}{2}\right)}\,y^{\ell+\frac{1}{2}}\,e^{2i\pi z}.\tag{3.59}$$

The theorem follows, and one should recall at this point that the series on the right-hand side of (3.53) is a special case of a class of automorphic functions introduced by Selberg [43] (following the use of Poincaré's series in the holomorphic case), and used by many authors in the field ([15, 20] or [62, Section 20]).

A still special case is given by the family of functions (depending on $\nu\in\mathbb{C}$ with $\mathrm{Re}\,\nu<-1$ and on an integer N)

$$U_N\left(z,\frac{1-\nu}{2}\right)=\frac{1}{2}\sum_{\left(\begin{smallmatrix} n_1 & n_2\\ m_1 & m_2\end{smallmatrix}\right)\in\Gamma/\Gamma_\infty^o}\left(\frac{y}{|-m_1z+n_1|^2}\right)^{\frac{1-\nu}{2}}\ \exp 2i\pi N\,\frac{m_2 z-n_2}{-m_1 z+n_1}\,.\tag{3.60}$$

With this notation, the first statement of the theorem reduces to

$$(u_z|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{B}^\ell)\, u_z) = (4\pi)^\ell \, \frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\, U_1\left(z, \ell+\frac{1}{2}\right). \tag{3.61}$$

□

In the next section, we shall make the decomposition of the Bezout distribution into homogeneous components explicit, at the same time proving that not only $\mathfrak{B}^\ell$, with $\ell \geq 1$, but also $\mathfrak{B}$ itself makes sense as a tempered distribution.

Remark. One can also define $I_{n,m}$ under the sole assumption that $|m|+|n| \neq 0$ instead of $(m,n)=1$, setting

$$I_{n,m}(h) \colon = \int_{-\infty}^{\infty} h(nx + n_1\,(m,n), mx + m_1\,(m,n))\, e^{2i\pi x}\, dx\,, \tag{3.62}$$

where the matrix $\left(\begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix}\right)$ has determinant (m,n). The automorphic distribution $\zeta(1+2i\pi\,\mathcal{E})\,\mathfrak{B}$, which bears the same relation to $\mathfrak{B}$ as the relation of $\mathfrak{D}$ to $\mathfrak{D}^{\text{prime}}$ (*cf.* (3.39)), is just

$$\zeta(1+2i\pi\,\mathcal{E})\,\mathfrak{B} = \frac{1}{2} \sum_{|m|+|n|\neq 0} I_{n,m}\,. \tag{3.63}$$

4 The structure of automorphic distributions

We now need to refresh the reader's memory on Maass cusp-forms and the Roelcke-Selberg expansion – or, as the case may be, deliver him a crash course on the subject. On these classical matters, concerning the spectral theory of the modular Laplacian, several books are available, in particular [48, 25]. To begin with, recall that Π is endowed with an $SL(2,\mathbb{R})$-invariant Riemannian structure, for which the so-called "hyperbolic distance" is characterized by $\cosh d(i, x+iy) = \frac{1+x^2+y^2}{2y}$, together with its $SL(2,\mathbb{R})$-invariance property. The associated invariant measure is $d\mu(x+iy) = y^{-2} dx\, dy$.

There are infinitely many possible fundamental domains for the action of Γ on Π: the most usual one consists of all points $z \in \Pi$ with $-\frac{1}{2} < \mathrm{Re}\; z < \frac{1}{2}$ and $|z|^2 > 1$. The space $L^2(\Gamma\backslash\Pi)$ is the Hilbert space of automorphic (*i.e.*, Γ-invariant) functions on Π which are square-summable when restricted to any fundamental domain. In this space, the Laplacian Δ has a canonical self-adjoint realization, the spectral resolution of which has both a continuous and a discrete, countable, spectrum. Eisenstein series $E_{\frac{1-i\lambda}{2}}$, $\lambda \in \mathbb{R}$ (*cf.* (3.17)) serve as a (redundant, since $E_{\frac{1\pm i\lambda}{2}}$ are linearly dependent) family of generalized eigenfunctions, corresponding to the continuous part $[\frac{1}{4}, \infty[$ of the spectrum. Genuine eigenfunctions of Δ, *i.e.*, the ones in $L^2(\Gamma\backslash\Pi)$, belong to a Hilbert space which has an orthonormal basis *temporarily* denoted as $\{(\frac{3}{\pi})^{\frac{1}{2}}, \mathcal{M}_1, \mathcal{M}_2, \dots\}$, where the normalized constant corresponds to the eigenvalue 0, and $\mathcal{M}_j$ corresponds to the eigenvalue $\frac{1+\lambda_j^2}{4}$: the non-constant ones are called cusp-forms, and we shall always assume that $\lambda_j > 0$ and that the sequence (λ_j) is non-decreasing. It is known, by an application of Selberg's trace formula, that λ_j goes to infinity as $j \to \infty$ in such a way that $\frac{\lambda_j^2}{4} \sim 12\, j$ ([48, p. 290] or [25, p. 174]). The Roelcke-Selberg theorem is the assertion that any $f \in L^2(\Gamma\backslash\Pi)$ can be uniquely written as

$$f = \Phi^0 + \sum_{j\geq 1} \Phi^j \mathcal{M}_j + \frac{1}{8\pi} \int_{-\infty}^{\infty} \Phi(\lambda)\, E_{\frac{1-i\lambda}{2}}\, d\lambda\,, \tag{4.1}$$

where Φ^0, Φ^j are constants with $\sum |\Phi^j|^2 < \infty$, and $\Phi \in L^2(\mathbb{R})$ satisfies the symmetry property

$$\frac{\Phi(\lambda)}{\zeta^*(i\lambda)} = \frac{\Phi(-\lambda)}{\zeta^*(-i\lambda)}\,, \tag{4.2}$$

with ζ^* as defined in (3.19).

A cusp-form $\mathcal{M}_j$ associated with the eigenvalue $\frac{1+\lambda_j^2}{4}$ admits a Fourier series expansion (with respect to $\mathrm{Re}\ z$) given as

$$\mathcal{M}_j(x+iy) = y^{\frac{1}{2}} \sum_{n\neq 0} b_n\, K_{\frac{i\lambda_j}{2}}(2\pi\, |n| y)\, e^{2i\pi n x} \tag{4.3}$$

for some coefficients b_n, depending on $\mathcal{M}_j$. This is a simple consequence of the method of separation of variables: observe that if a function of y alone is an eigenfunction of Δ for the eigenvalue $\frac{1-\nu^2}{4}$, it has to be a linear combination of $y^{\frac{1-\nu}{2}}$ and $y^{\frac{1+\nu}{2}}$. Such a term is indeed present in the expansion (4.5) below of Eisenstein series but absent from the expansion (4.3) of cusp-forms, precisely so as to ensure that cusp-forms are square-integrable in the fundamental domain.

We now lift $\mathcal{M}_j$ to an automorphic distribution on $\mathbb{R}^2$. There is one subtle, but essential, point in this lifting, to wit: corresponding to *one* cusp-form, there are *two* distinct automorphic distributions. We shall assume that our sequence $(\lambda_j, \mathcal{M}_j)$ (note that in our present notation eigenvalues are to be repeated according to their multiplicity, though it is not known whether multiple eigenvalues do exist) has been parametrized by the set $\{j \colon j \geq 1\}$, and let us repeat that for each j the number λ_j has been chosen > 0 (only its square is known from the eigenvalue of Δ, or from the right-hand side of (4.3)). Then we define the distribution $\mathfrak{M}_j^\sharp$ on $\mathbb{R}^2$ through the equation

$$\left\langle \mathfrak{M}_j^\sharp, h \right\rangle = \frac{1}{2} \sum_{n\neq 0} |n|^{\frac{i\lambda_j}{2}}\, b_n \int_{-\infty}^{\infty} |t|^{-i\lambda_j - 1}\, (\mathcal{F}_1^{-1} h) \left(\frac{n}{t}, t\right) dt\,, \tag{4.4}$$

where the partial Fourier transform $\mathcal{F}_1^{-1} h$ has been defined in (3.26). It has been proved in [62, Proposition 13.1] that $\mathfrak{M}_j^\sharp$ is an even tempered distribution, homogeneous of degree $-1 - i\lambda_j$. We also define $\mathfrak{M}_{-j}^\sharp$ by the same formula, after

λ_j has been replaced by $-\lambda_j$: equation (4.4) is then valid for all $j \in \mathbb{Z}^\times$ if one defines $\lambda_{-j} : = -\lambda_j$. One may wish to call either of the two distributions $\mathfrak{M}^\sharp_{\pm j}$ a "cusp-distribution".

Eisenstein series, too, have classical Fourier series expansions, given by

$$E^*_{\frac{1-\nu}{2}}(x+iy) = \zeta^*(1-\nu)\, y^{\frac{1-\nu}{2}} \quad + \zeta^*(1+\nu)\, y^{\frac{1+\nu}{2}} \\ + 2\, y^{\frac{1}{2}} \sum_{n \neq 0} |n|^{-\frac{\nu}{2}}\, \sigma_\nu(|n|)\, K_{\frac{\nu}{2}}(2\pi |n| y)\, e^{2i\pi n x}\,. \tag{4.5}$$

Observe again that the presence of the first two terms distinguishes Eisenstein series from cusp-forms, as seen from the expansion (4.3). Comparing (4.3) and (4.4) on one hand, (4.5) and (3.25) on the other, one sees that the way $\mathcal{M}_j$ has been lifted to $\mathfrak{M}^\sharp_{\pm j}$ is quite similar to the way the Eisenstein series $E^*_{\frac{1-i\lambda}{2}}$ has been lifted to the pair of automorphic distributions $\mathfrak{E}^\sharp_{\pm i\lambda}$, even though the latter ones had an alternative definition (3.1) (not a direct one, though, as it relied on analytic continuation).

The following is a quotation from [62, Theorem 13.2 and Proposition 13.4], though we have changed the normalizations slightly so as to take the use of the quantization rule $\mathrm{Op}_{\sqrt{2}}$ into account. It is not going to be used in this book, but our readers with a previous interest in the Lax-Phillips scattering theory for the automorphic wave equation [30] may find the right-hand side of (4.8) quite natural.

Theorem 4.1. *The map $\mathfrak{S} \mapsto A = \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})$ establishes an isometry from the space of tempered distributions $\mathfrak{S}$ on $\mathbb{R}^2$ admitting decompositions*

$$\mathfrak{S} = c_0 + c_0'\,\delta + \frac{1}{8\pi}\int_{-\infty}^{\infty} \Psi(\lambda)\, \mathfrak{E}^\sharp_{i\lambda}\, d\lambda + \sum_{j\neq 0} c_j\, \mathfrak{M}^\sharp_j \tag{4.6}$$

with

$$|||\mathfrak{S}|||^2_\Gamma : = \frac{2\pi}{3}\left(|c_0|^2 + |c_0'|^2\right) + \frac{1}{8\,\pi}\int_{-\infty}^{\infty} |\Psi(\lambda)|^2\, |\zeta^*(i\lambda)|^2\, d\lambda \\ + 2\sum_{j\neq 0} |c_j|^2 < \infty \tag{4.7}$$

onto a linear space of operators A: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ commuting with all operators $\mathrm{Met}(g)$, $g \in \Gamma$, *and satisfying*

$$\|A\|^2_\Gamma : = \|\, z \mapsto (u_z | A u_z)\,\|^2_{L^2(\Gamma\backslash\Pi)} \\ + \frac{1}{4}\left\| \left|\Delta - \frac{1}{4}\right|^{-\frac{1}{2}} \left(z \mapsto (u_z^1 | A u_z^1)\right) \right\|^2_{L^2(\Gamma\backslash\Pi)} \\ < \infty\,. \tag{4.8}$$

Let us remark that the spaces of distributions or operators just introduced are not complete. This is so because we have decided, for simplicity, to consider only tempered distributions as symbols: a further extension would be necessary so as to get true Hilbert spaces. The absolute value around $\Delta - \frac{1}{4}$ in (4.8) is needed solely because of the eigenvalue 0.

Up to now, eigenvalues $\frac{1+\lambda_j^2}{4}$ have been repeated according to their multiplicity, and the cusp-forms $\mathcal{M}_j$ corresponding to the same eigenvalue have been chosen in an arbitrary way, with the sole condition that, together, they should make an orthonormal basis of the given eigenspace. More precision is now needed concerning the choice of the cusp-forms $\mathcal{M}_j$. First, we need to distinguish *even* cusp-forms $\mathcal{M}$, invariant under the map $\mathcal{M} \mapsto \check{\mathcal{M}}$ with $\check{\mathcal{M}}(z) = \mathcal{M}(-\bar{z})$, from *odd* cusp-forms, which change to their negatives under this transformation; even (*resp.* odd) eigenvalues are those for which there exists at least one even (*resp.* odd) eigenfunction. It is suitable to our needs to denote from now on as $(\lambda_k^+)_{k\geq 1}$ the increasing sequence of positive numbers such that $\frac{1+(\lambda_k^+)^2}{4}$ is an even eigenvalue of Δ: a similar convention holds for the sequence $(\lambda_k^-)_{k\geq 1}$ of odd eigenvalues; we still set $\lambda_{-k}^{\pm} = -\lambda_k^{\pm}$. Since possibly multiple eigenvalues (it is not known whether any such can exist, in the case of the full modular group Γ) are not repeated any more, we need to define for each k, in a specific way, an orthonormal basis $(\mathcal{M}_{k,\ell}^+)_{1\leq \ell \leq n_k^+}$ or $(\mathcal{M}_{k,\ell}^-)_{1\leq \ell \leq n_k^-}$ of even, or odd, cusp-forms corresponding to the given eigenvalue. This is made possible by Hecke's theory.

For every $N \geq 1$, the Hecke operator T_N is the linear operator on automorphic functions defined as

$$(T_N f)(z) = N^{-\frac{1}{2}} \sum_{\substack{ad=N, d>0 \\ b \bmod d}} f\left(\frac{az+b}{d}\right). \tag{4.9}$$

All relevant information concerning the T_N's can be found in [48, p. 241]; also recall for future use (*loc.cit.*, p. 238) that

$$M_N(\mathbb{Z}) = \bigcup_{\substack{ad=N, d>0 \\ b \bmod d}} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \tag{4.10}$$

where $M_N(\mathbb{Z})$ denotes the set of matrices with integer coefficients and determinant N. Then, for each eigenvalue $\frac{1+(\lambda_k^{\pm})^2}{4}$, one can select an orthonormal basis $(\mathcal{M}_{k,\ell}^{\pm})_{1\leq \ell \leq n_k^{\pm}}$ of the corresponding eigenspace, in such a way that each $\mathcal{M}_{k,\ell}^{\pm}$ should be a Maass-Hecke eigenform: this means that, besides being an eigenfunction of Δ, it should also be an eigenfunction of the Hecke operator T_N for each N.

Hecke's theory shows that if $\mathcal{M}$ is a Maass-Hecke form, then $b_{\pm 1} \neq 0$ and $T_N \mathcal{M} = \frac{b_N}{b_1} \mathcal{M} == \frac{b_{-N}}{b_{-1}} \mathcal{M}$. It is customary, when dealing with Hecke's theory, to consider *Maass-Hecke eigenforms* $\mathcal{N}$ *normalized by the condition* that $b_1 = 1$

(so that $T_N \mathcal{N} = b_N \mathcal{N}$ for all $N \geq 1$). Given an even eigenvalue $\frac{1+(\lambda_k^{\pm})^2}{4}$, with $k \geq 1$, we shall denote as $(\mathcal{N}_{k,\ell}^{\pm})_{1 \leq \ell \leq n_k^{\pm}}$ a basis of the corresponding (even or odd) eigenspace, consisting of Maass-Hecke forms normalized by the above-stated condition, to wit that the coefficient b_1 from the expansion (4.3) of $\mathcal{N}_{k,\ell}^{\pm}$ should be 1: then, for any given $\lambda_k^{\pm}$, the functions $\mathcal{N}_{k,\ell}^{\pm}$ are uniquely defined up to a permutation.

We now lift $\mathcal{N}_{k,\ell}^{\pm}$ to an automorphic distribution on $\mathbb{R}^2$. The recipe has been provided in (4.4) and remains the same: only the coefficients b_n have to be taken from the Fourier series expansion (4.3) of $\mathcal{N}_{k,\ell}^{\pm}$. In this way, we end up with two well-defined distributions $(\mathfrak{N}_{k,\ell}^{\pm})^{\sharp}$ and $(\mathfrak{N}_{-k,\ell}^{\pm})^{\sharp}$ (the one with the subscript $k \geq 1$ corresponds to the positive choice of a square root of $(\lambda_k^{\pm})^2$), homogeneous of degrees $-1 - i\lambda_k$ and $-1 - i\lambda_{-k} = -1 + i\lambda_k$ respectively.

The following result should be compared to Theorem 3.1.

Theorem 4.2. *For every $k \in \mathbb{Z}^{\times}$, and $\ell = 1, \dots, n_k^{\pm}$, one has*

$$\mathcal{F}\,(\mathfrak{N}_{k,\ell}^{\pm})^{\sharp} = (\mathfrak{N}_{-k,\ell}^{\pm})^{\sharp}\,, \tag{4.11}$$

where $\mathcal{F}$ is the symplectic Fourier transformation. Also, for every $z \in \Pi$,

$$(u_z \,|\, \mathrm{Op}_{\sqrt{2}}((\mathfrak{N}_{k,\ell}^{\pm})^{\sharp})\, u_z) = \mathcal{N}_{|k|,\ell}^{\pm}(z) \tag{4.12}$$

and

$$(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}((\mathfrak{N}_{k,\ell}^{\pm})^{\sharp})\, u_z^1) = -i\lambda_k^{\pm}\, \mathcal{N}_{|k|,\ell}^{\pm}(z)\,, \tag{4.13}$$

where $\mathrm{Op}_{\sqrt{2}}$ has been defined in (2.21).

Proof. It is just the same proof for even or odd cusp-forms: we shall thus drop the sign $\pm$. For $p = 0$ or 1, one has, using the Wigner function (2.27) or (2.28),

$$(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{N}_{k,\ell}^{\sharp})\, u_z^p) = \left\langle \mathfrak{N}_{k,\ell}^{\sharp}\,,\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z^p, u_z^p) \right\rangle\,, \tag{4.14}$$

with

$$2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z^p, u_z^p)(s,\sigma) = \left(-2\,\frac{d}{d\varepsilon} - 1\right)^p \Big|_{\varepsilon=1} \exp\left(-\frac{\pi\varepsilon}{y}|s - z\sigma|^2\right). \tag{4.15}$$

Now, with

$$\begin{aligned} h(s,\sigma) &= \exp\left(-\frac{\pi\varepsilon}{y}|s - z\sigma|^2\right) \\ &= \exp\left(-\frac{\pi\varepsilon}{y}\left[(s - x\sigma)^2 + y^2\sigma^2\right]\right), \end{aligned} \tag{4.16}$$

one has

$$(\mathcal{F}_1^{-1}h)\left(\frac{n}{t},t\right)=\left(\frac{y}{\varepsilon}\right)^{\frac{1}{2}}e^{2i\pi nx}\,e^{-\pi\varepsilon\,yt^2}\,e^{-\frac{\pi n^2 y}{\varepsilon t^2}} \tag{4.17}$$

and

$$\left(-2\frac{d}{d\varepsilon}-1\right)^p\Big|_{\varepsilon=1}(\mathcal{F}_1^{-1}h)\left(\frac{n}{t},t\right)=y^{\frac{1}{2}}\,e^{2i\pi nx}\left(2\pi y\left(t^2-\frac{n^2}{t^2}\right)\right)^p e^{-\pi y(t^2+\frac{n^2}{t^2})}. \tag{4.18}$$

Thus, with the coefficients b_n taken from (4.3),

$$(u_z^p\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{N}^\sharp_{k,\ell})\,u_z^p)=\frac{1}{2}\sum_{n\neq0}|n|^{\frac{i\lambda_k}{2}}\,b_n\,y^{\frac{1}{2}}\,e^{2i\pi nx}$$
$$\int_{-\infty}^{\infty}|t|^{-i\lambda_k-1}\left(2\pi\,y\left(t^2-\frac{n^2}{t^2}\right)\right)^p e^{-\pi y(t^2+\frac{n^2}{t^2})}\,dt\,. \tag{4.19}$$

Now, as a consequence of [31, p. 85],

$$\int_{-\infty}^{\infty}|t|^{-i\lambda_k-1}\,e^{-\pi y(t^2+\frac{n^2}{t^2})}\,dt=2\,|n|^{-\frac{i\lambda_k}{2}}\,K_{\frac{i\lambda_k}{2}}(2\pi|n|y)\,, \tag{4.20}$$

which yields the result if $p=0$. If $p=1$, we write

$$\int_{-\infty}^{\infty}\left(2\pi\,y\left(t^2-\frac{n^2}{t^2}\right)\right)|t|^{-i\lambda_k-1}\,e^{-\pi y(t^2+\frac{n^2}{t^2})}\,dt$$
$$=2\,|n|^{-\frac{i\lambda_k}{2}}\,(2\pi|n|y)\left[K_{\frac{i\lambda_k}{2}-1}(2\pi|n|y)-K_{\frac{i\lambda_k}{2}+1}(2\pi|n|y)\right]$$
$$=2\,(-i\lambda_k)\,|n|^{-\frac{i\lambda_k}{2}}\,K_{\frac{i\lambda_k}{2}}(2\pi|n|y)\,, \tag{4.21}$$

according to [31, p. 67]. This proves the two formulas (4.12) and (4.13).

To show that $\mathcal{F}\,\mathfrak{N}^\sharp_{k,\ell}=\mathfrak{N}^\sharp_{-k,\ell}$, or, what amounts to the same, that

$$\mathcal{G}\left(2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{N}^\sharp_{k,\ell}\right)=2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{N}^\sharp_{-k,\ell},$$

it suffices, according to Proposition 2.1, to show that

$$(u_z^p\,|\,\mathrm{Op}(\mathcal{G}\,(2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{N}^\sharp_{k,\ell}))\,u_z^p)=(u_z^p\,|\,\mathrm{Op}(2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{N}^\sharp_{-k,\ell})\,u_z^p) \tag{4.22}$$

for all $z\in\Pi$ and $p=0,1$. Now, from the remark which follows (2.9), the left-hand side is just $(-1)^p\,(u_z^p\,|\,\mathrm{Op}(2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{N}^\sharp_{k,\ell})\,u_z^p)$, so that what is needed in order to complete the proof of Theorem 4.2 is to remark that

$$(-1)^p\,(u_z^p\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{N}^\sharp_{k,\ell})\,u_z^p)=(u_z^p\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{N}^\sharp_{-k,\ell})\,u_z^p)\,, \tag{4.23}$$

a consequence of (4.12) and (4.13) since $\lambda_{-k}=-\lambda_k$. □

We now give the decomposition of the Bezout distribution into its homogeneous components. We should insist on the fact that, in $\mathfrak{N}^\sharp_{k,\ell}$, the subscript k can be any non-zero integer, while in $\mathcal{N}_{k,\ell}$, it has to be ≥ 1.

Theorem 4.3. *For $\ell \geq 1$, the Bezout distribution $\mathfrak{B}^\ell$ admits the weakly convergent decomposition, in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$,*

$$\begin{aligned}\mathfrak{B}^\ell = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\Gamma(\ell - \frac{i\lambda}{2})\,\Gamma(\ell + \frac{i\lambda}{2})}{\zeta^*(i\lambda)\,\zeta^*(-i\lambda)}\,\mathfrak{E}^\sharp_{i\lambda}\,d\lambda \\ + \frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} \frac{\Gamma(\ell - \frac{i\lambda_k^+}{2})\,\Gamma(\ell + \frac{i\lambda_k^+}{2})}{\|\mathcal{N}^+_{|k|,\ell}\|^2}\,(\mathfrak{N}^+_{k,\ell})^\sharp \\ + \frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} \frac{\Gamma(\ell - \frac{i\lambda_k^-}{2})\,\Gamma(\ell + \frac{i\lambda_k^-}{2})}{\|\mathcal{N}^-_{|k|,\ell}\|^2}\,(\mathfrak{N}^-_{k,\ell})^\sharp\,,\end{aligned} \tag{4.24}$$

where the norms are to be taken in the space $L^2(\Gamma\backslash\Pi)$.

Proof. We shall first show that the right-hand side of (4.24) is weakly convergent for every $\ell \geq 0$, then apply Proposition 2.1 to show that, if $\ell \geq 1$, its sum coincides with $\mathfrak{B}^\ell$. Incidentally, note that this will also give a meaning as an element of $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$ to $\mathfrak{B} = \mathfrak{B}^0$ itself: remark that

$$\frac{\Gamma(-\frac{i\lambda}{2})\,\Gamma(\frac{i\lambda}{2})}{\zeta^*(i\lambda)\,\zeta^*(-i\lambda)} = \frac{1}{\zeta(i\lambda)\,\zeta(-i\lambda)} \tag{4.25}$$

has no singularity on the real line and that, around (3.28), we emphasized that $\mathfrak{E}^\sharp_0$ is meaningful.

First, consider the integral term on the right-hand side of (4.24), and make $\mathfrak{E}^\sharp_{i\lambda}$ explicit with the help of (3.25). Using Pochhammer's symbols $(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)} = a(a+1)\cdots(a+\ell-1)$ and the definition (3.19) of the function ζ^*, we have to show, whenever $h \in \mathcal{S}(\mathbb{R}^2)$, the convergence of the following three integrals:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{(-\frac{i\lambda}{2})_\ell\,(\frac{i\lambda}{2})_\ell}{\zeta(i\lambda)}\,d\lambda \int_{-\infty}^{\infty} |t|^{-i\lambda-1}\,(\mathcal{F}_1^{-1}h)(0,t)\,dt\,, \\ I_2 &= \int_{-\infty}^{\infty} \frac{(-\frac{i\lambda}{2})_\ell\,(\frac{i\lambda}{2})_\ell}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,\zeta(1-i\lambda)\,d\lambda \int_{-\infty}^{\infty} |t|^{-i\lambda}\,h(t,0)\,dt\,, \\ I_3 &= \int_{-\infty}^{\infty} \frac{(-\frac{i\lambda}{2})_\ell\,(\frac{i\lambda}{2})_\ell}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,d\lambda \sum_{n\neq 0} \sigma_{i\lambda}(|n|) \int_{-\infty}^{\infty} |t|^{-i\lambda-1}\,(\mathcal{F}_1^{-1}h)\left(\frac{n}{t},t\right)\,dt\,. \end{aligned} \tag{4.26}$$

It is known [49, p. 161, 149] that $|\zeta(1 \pm i\lambda)|^{\pm 1} \leq C \log|\lambda|$ as $|\lambda| \to \infty$ so that, using the functional equation (3.19) of the zeta function and the estimate [31, p. 13] of the Gamma function on vertical lines, the same (even better, actually) inequality holds for $(\zeta(\pm i\lambda))^{-1}$. Pochhammer's symbols are taken care of, for large $|\lambda|$, by means of

$$|t|^{-i\lambda+r-1}\,(\operatorname{sign} t)^{\varepsilon} = \frac{1}{-i\lambda+r}\,\frac{d}{dt}\left(|t|^{-i\lambda+r}\,(\operatorname{sign} t)^{1-\varepsilon}\right), \tag{4.27}$$

with $\varepsilon = 0$ or 1, so that, saving an extra factor $\frac{1}{1+\lambda^2}$ at the end, and using an integration by parts, we are done for I_1 and I_2.

So far as I_3 goes, we start with the same integration by parts so as to compensate the Pochhammer's symbols, still saving an extra factor $\frac{1}{1+\lambda^2}$, next use the trivial inequality $|\sigma_{i\lambda}(|n|)| \leq 1 + \frac{\log n}{\log 2}$ and write $(r = 0, 1, \dots)$

$$\begin{aligned}
&\int_{-\infty}^{\infty} |t|^{r-1} \left| \left(\frac{d}{dt}\right)^j (\mathcal{F}_1^{-1} h)\left(\frac{n}{t}, t\right) \right| dt \\
&\leq C_N \sum_{m=0}^{j} \int_{-\infty}^{\infty} |t|^{r-1} \left|\frac{n}{t^2}\right|^m \left(1 + t^2 + \frac{n^2}{t^2}\right)^{-N} dt \\
&\leq C_N\, |n|^m\, (1+|n|)^{-\frac{N}{3}} \sum_{m=0}^{j} \int_{-\infty}^{\infty} |t|^{r-2m-1}\, (1+t^2)^{-\frac{N}{3}}\, (1+t^{-2})^{-\frac{N}{3}}\, dt
\end{aligned} \tag{4.28}$$

to conclude.

In view of (4.4), consider the term

$$\frac{1}{4} \sum_{k,\ell} \frac{\Gamma(\ell - \frac{i\lambda_k}{2})\,\Gamma(\ell + \frac{i\lambda_k}{2})}{\|\mathcal{N}_{|k|,\ell}\|^2} \sum_{n \neq 0} |n|^{\frac{i\lambda_k}{2}}\, b_n \int_{-\infty}^{\infty} |t|^{-i\lambda_k - 1}\, (\mathcal{F}_1^{-1} h)\left(\frac{n}{t}, t\right) dt, \tag{4.29}$$

coming from the application of the right-hand side of (4.24) to h: note that we have dropped the superscript $\pm$ since what follows applies just as well to the sum of automorphic distributions which corresponds to even, or odd, Maass-Hecke eigenforms. The study of this term goes along the same lines as that of I_3, except for the following: first, the $d\lambda$-integration is replaced by a summation with respect to k, ℓ; next, the coefficients b_n from the Fourier series expansion of $\mathcal{N}_{|k|,\ell}$ are unknown, and so is $\|\mathcal{N}_{|k|,\ell}\|^{-1}$. As indicated in [48, p. 220], $|b_n|$ is uniformly majorized by some fixed power of $|n|$ (the exponent $\frac{3}{10} + \varepsilon$ would do), and we have seen from the study of I_3 how to take care of powers of n. Next, very temporarily reverting to the notation $(\lambda_j, \mathcal{N}_j)$ instead of $(\lambda_k, \mathcal{N}_{k,\ell})$ so as to indicate that eigenvalues should be repeated according to their multiplicity, one has, as recalled just before (4.1), $\lambda_j \sim (48\,j)^{\frac{1}{2}}$ as $j \to \infty$, so that powers of λ_j (coming, say, from Pochhammer's symbols) in our series can be taken care of in exactly the same way powers of λ were compensated in the study of I_3. Finally, as a consequence of a result of R.A.Smith [46], quoted in [48, p. 247], one has for large $|k|$

$$\|\mathcal{N}_{|k|,\ell}\|^{-1} \leq C \left| \Gamma\left(\frac{i\lambda_k}{2}\right) \right|^{-1}, \tag{4.30}$$

which ends the proof that the series on the right-hand side of (4.24) is convergent.

To finish the proof of Theorem 4.3, we only have to show, with $p = 0$ or 1, that for all $z \in \Pi$, $(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{B}^\ell)\, u_z^p)$ agrees with the analogous scalar product, obtained after substituting for $\mathfrak{B}^\ell$ the right-hand side of (4.24). Recall from Theorem 3.1, and (3.5), that

$$(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{E}_{i\lambda}^\sharp)\, u_z^p) = (u_z^p \,|\, \mathrm{Op}(\mathfrak{F}_{i\lambda}^\sharp)\, u_z^p) \\ = (-i\lambda)^p\, E^*_{\frac{1-i\lambda}{2}}(z)\,. \tag{4.31}$$

From (4.12), (4.13),

$$(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}((\mathfrak{N}_{k,\ell}^\pm)^\sharp)\, u_z^p) = (-i\lambda_k)^p\, \mathcal{N}_{|k|,\ell}^\pm(z)\,. \tag{4.32}$$

Finally, using Theorem 3.5, what we have to show is that

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} (-i\lambda)^p\, \frac{\Gamma(\ell-\frac{i\lambda}{2})\,\Gamma(\ell+\frac{i\lambda}{2})}{\zeta^*(i\lambda)\,\zeta^*(-i\lambda)}\, E^*_{\frac{1-i\lambda}{2}}(z)\,d\lambda \\ +\frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (-i\lambda_k^+)^p\, \frac{\Gamma(\ell-\frac{i\lambda_k^+}{2})\,\Gamma(\ell+\frac{i\lambda_k^+}{2})}{\|\mathcal{N}_{|k|,\ell}^+\|^2}\, \mathcal{N}_{|k|,\ell}^+(z) \\ +\frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (-i\lambda_k^-)^p\, \frac{\Gamma(\ell-\frac{i\lambda_k^-}{2})\,\Gamma(\ell+\frac{i\lambda_k^-}{2})}{\|\mathcal{N}_{|k|,\ell}^-\|^2}\, \mathcal{N}_{|k|,\ell}^-(z) \tag{4.33}$$

is zero for $p = 1$, and agrees with $(4\pi)^\ell\, \frac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\frac{1}{2})}\, U_1(z, \ell+\frac{1}{2})$, as defined in (3.61), for $p = 0$.

The first point comes from the fact that, except for the factor $-i\lambda$, $-i\lambda_k^+$ or $-i\lambda_k^-$, all that remains under the integral or the series in each term on the right-hand side of (4.33) is an even function of λ, λ_k^+ or λ_k^-. The interesting part was proved by Selberg [43] and, rephrased as

$$\left(E_{\frac{1-i\lambda}{2}} \,|\, U_1\left(z, \ell+\frac{1}{2}\right)\right) = (4\pi)^{-\ell}\, \frac{\Gamma(\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})} \times 2\, \frac{\Gamma(\ell-\frac{i\lambda}{2})\,\Gamma(\ell+\frac{i\lambda}{2})}{\zeta^*(1+i\lambda)} \tag{4.34}$$

together with

$$\left(\mathcal{N}_{|k|,\ell}^\pm \,|\, U_1\left(z, \ell+\frac{1}{2}\right)\right) = (4\pi)^{-\ell}\, \frac{\Gamma(\frac{1}{2})}{\Gamma(\ell+\frac{1}{2})} \times \Gamma\left(\ell-\frac{i\lambda_k^\pm}{2}\right)\Gamma\left(\ell+\frac{i\lambda_k^\pm}{2}\right), \tag{4.35}$$

can also be found in [15, p. 246] or [20, p. 247]. □

It is natural to ask whether one could give the (integral) first term on the right-hand side of (4.24) a closed form, say when $\ell = 0$.

Proposition 4.4. *Set, for* $t \in \mathbb{R}$,

$$\phi(t) = \sum_{r \geq 1} \left(e^{2i\pi \frac{t}{r}} - 1 - 2i\pi \, \frac{t}{r} \right) \, \text{Möb}(r) \,, \tag{4.36}$$

where Möb(r) *is the arithmetic function (usually denoted as* $\mu(r)$*) which vanishes if* r *is not square free, and is* -1 *to the number of prime factors of* r *otherwise. Set*

$$\langle \mathfrak{S} \, , \, h \rangle = \frac{1}{2} \sum_{(m,n)=1} \int_{-\infty}^{\infty} \phi(t) \, h(tn, tm) \, dt, \qquad h \in \mathcal{S}(\mathbb{R}^2) \, , \tag{4.37}$$

a formula to be compared to the initial definition (3.1) *of Eisenstein distributions. Finally, set*

$$\mathfrak{B}_{\text{cont}} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta(i\lambda) \, \zeta(-i\lambda)} \, \mathfrak{E}_{i\lambda}^{\sharp} \, d\lambda \, , \tag{4.38}$$

which is (using (4.25)*) the continuous term in the spectral decomposition of* $\mathfrak{B}$*. Then*

$$\mathfrak{B}_{\text{cont}} = \mathfrak{S} + \frac{i}{2} \sum_{\zeta^*(\rho)=0} \text{Res}_{\mu = i\rho} [\, \zeta(i\mu) \, (\zeta(-i\mu))^{-1} \, \mathfrak{E}_{i\mu}^{\sharp} \,] \, . \tag{4.39}$$

Proof. Before we start it, note that the presence of the extra terms on the right-hand side of (4.39), associated to the non-trivial zeros of the zeta function, implies of course that $\mathfrak{S}$, contrary to $\mathfrak{B}_{\text{cont}}$, lies outside the Hilbert space of automorphic distributions introduced in Theorem 4.1.

Note that

$$\left| e^{2i\pi \frac{t}{r}} - 1 - 2i\pi \, \frac{t}{r} \right| \leq C \min \, (t^2, |t|), \qquad t \in \mathbb{R} \tag{4.40}$$

so that, for every a with $1 < a \leq 2$,

$$|\phi(t)| \leq C \min \, (t^2, |t|^a) \, . \tag{4.41}$$

Given any $h \in \mathcal{S}(\mathbb{R}^2)$, one has

$$\langle \mathfrak{S} \, , \, h \rangle = \int_{ia-\infty}^{ia+\infty} \langle \mathfrak{S} \, , \, h_{-\mu} \rangle \, d\mu \tag{4.42}$$

with

$$h_{-\mu}(x, \xi) = \frac{1}{2\pi} \int_0^{\infty} s^{-i\mu} \, h(sx, s\xi) \, ds \, , \tag{4.43}$$

thus

$$\langle \mathfrak{S} \, , \, h_{-\mu} \rangle = \frac{1}{4\pi} \sum_{(m,n)=1} \int_{-\infty}^{\infty} \phi(t) \, dt \int_0^{\infty} s^{-i\mu} \, h(stn, stm) \, ds \, . \tag{4.44}$$

If $1 < a = \mathrm{Im}\,\mu \leq 2$, one may write

$$\langle \mathfrak{S}\,,\, h_{-\mu}\rangle = \frac{1}{4\pi} \sum_{(m,n)=1} \int_0^\infty s^{-i\mu}\, \Phi(\mu)\, h(sn, sm)\, ds \tag{4.45}$$

with

$$\begin{aligned}
\Phi(\mu) &= \int_{-\infty}^{\infty} \phi(t)\, |t|^{i\mu-1}\, dt \\
&= \int_{-\infty}^{\infty} |t|^{i\mu-1}\, dt \sum_{r\geq 1} \left(e^{2i\pi \frac{t}{r}} - 1 - 2i\pi\, \frac{t}{r}\right) \mathrm{M\ddot{o}b}(r) \\
&= \int_{-\infty}^{\infty} |t|^{i\mu-1}\, dt \sum_{r\geq 1} (e^{2i\pi t} - 1 - 2i\pi\, t)\, r^{i\mu}\, \mathrm{M\ddot{o}b}(r) \\
&= \frac{1}{\zeta(-i\mu)} \int_{-\infty}^{\infty} |t|^{i\mu-1}\, (e^{2i\pi t} - 1 - 2i\pi\, t)\, dt \\
&= \frac{1}{\zeta(-i\mu)}\, \frac{\Gamma(i\mu)}{(2\pi)^{i\mu}}\, (e^{-\frac{\pi\mu}{2}} + e^{\frac{\pi\mu}{2}}) \\
&= \frac{1}{\zeta(-i\mu)}\, \pi^{\frac{1}{2}-i\mu}\, \frac{\Gamma(\frac{i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})} \\
&= \frac{\zeta(1-i\mu)}{\zeta(i\mu)\,\zeta(-i\mu)}
\end{aligned} \tag{4.46}$$

(we have used the duplication formula for the Gamma function [31, p. 3] together with the functional equation (3.19) of the zeta function and the well-known expansion as a Dirichlet series of $\frac{1}{\zeta}$).

Using the identity (a consequence of (3.1))

$$\left\langle \mathfrak{E}^\sharp_{i\mu}\,, h \right\rangle = \zeta(1-i\mu) \sum_{(m,n)=1} \int_0^\infty s^{-i\mu}\, h(sn, sm)\, ds\,, \tag{4.47}$$

we get, from (4.45),

$$\langle \mathfrak{S}\,,\, h_{-\mu}\rangle = \frac{1}{4\pi}\, \frac{1}{\zeta(i\mu)\,\zeta(-i\mu)} \left\langle \mathfrak{E}^\sharp_{i\mu}\,, h\right\rangle\ : \tag{4.48}$$

the theorem then follows from (4.41) and a contour deformation; note that the simple pole of $\mathfrak{E}^\sharp_{i\mu}$ at $\mu = i$ (*cf.* discussion between (3.1) and (3.2)) is killed by that of $\zeta(-i\mu)$. □

The distributions $\mathfrak{B}_{\mathrm{cont}}$ and $\mathfrak{S}$ are closely related to a "comb", *i.e.*, a Γ-invariant measure supported in $\mathbb{Z}^2\backslash\{0\}$.

Proposition 4.5. *Set*

$$\mathfrak{N}(x,\xi) = \frac{1}{2} \sum_{|n|+|m|\neq 0} (n,m)\,\text{Möb}(\,(n,m)\,)\,\delta(x-n)\,\delta(\xi-m)\,. \tag{4.49}$$

The decomposition of the distribution $\mathfrak{N}$ *into homogeneous components is*

$$\mathfrak{N} = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{1}{\zeta(1-i\lambda)\,\zeta(-i\lambda)}\,\mathfrak{E}^\sharp_{i\lambda}\,d\lambda \\ - \frac{i}{2}\sum_{\zeta^*(\rho)=0} \text{Res}_{\mu=i\rho}[\,(\zeta(1-i\mu)\,\zeta(-i\mu))^{-1}\,\mathfrak{E}^\sharp_{i\mu}\,]\,. \tag{4.50}$$

Proof. From the analogue of (4.44), one gets, if $\text{Im}\ \mu > 1$ and $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}\langle \mathfrak{N}\,, h_{-\mu}\rangle &= \frac{1}{4\pi}\sum_{|n|+|m|\neq 0} (n,m)\,\text{Möb}(\,(n,m)\,)\int_0^\infty s^{-i\mu}\,h(ns,ms)\,ds \\ &= \frac{1}{4\pi}\sum_{r\geq 1}\sum_{(n,m)=1} r\,\text{Möb}(r)\int_0^\infty s^{-i\mu}\,h(rns,rms)\,ds \\ &= \frac{1}{4\pi}(\zeta(-i\mu))^{-1}\sum_{(n,m)=1}\int_0^\infty s^{-i\mu}\,h(ns,ms)\,ds \\ &= \frac{1}{2}(\zeta(-i\mu))^{-1}\sum_{(n,m)=1} h_{-\mu}(n,m) \\ &= \frac{1}{4\pi}(\zeta(-i\mu))^{-1}\,\langle \mathcal{D}_{\text{prime}}\,,\,h_{-\mu}\rangle \\ &= \frac{1}{4\pi}(\zeta(-i\mu)\,\zeta(1-i\mu))^{-1}\,\langle \mathcal{D}\,,\,h_{-\mu}\rangle\,. \end{aligned} \tag{4.51}$$

Using (4.42), then the fact [62, (16.6)] that $\langle \mathcal{D}\,,\,h_{-\mu}\rangle = \left\langle \mathfrak{E}^\sharp_{i\mu}, h\right\rangle$ (proved in just the same way), one gets, if $a > 1$,

$$\langle \mathfrak{N}, h\rangle = \frac{1}{4\pi}\int_{ia-\infty}^{ia+\infty} (\zeta(-i\mu)\,\zeta(1-i\mu))^{-1}\left\langle \mathfrak{E}^\sharp_{i\mu}, h\right\rangle d\mu\,, \tag{4.52}$$

which leads to (4.50) after a contour deformation. □

Remark. In some sense, one may thus write

$$\mathfrak{S} = \frac{\zeta(1+2i\pi\,\mathcal{E})}{\zeta(-2i\pi\,\mathcal{E})}\,\mathfrak{N} = \pi^{\frac{1}{2}+2i\pi\,\mathcal{E}}\,\frac{\Gamma(-i\pi\,\mathcal{E})}{\Gamma(\frac{1}{2}+i\pi\,\mathcal{E})}\,\mathfrak{N}\,. \tag{4.53}$$

5 The main formula: a heuristic approach

With considerable precautions, one can give a meaning to the sharp product of two Eisenstein distributions $\mathfrak{E}^{\sharp}_{\nu_1}$ and $\mathfrak{E}^{\sharp}_{\nu_2}$. The "main formula" (5.38) or (5.62) referred to in the title above expresses the distribution $\mathfrak{E}^{\sharp}_{\nu_1} \# \mathfrak{E}^{\sharp}_{\nu_2}$ as the image of the Bezout distribution $\mathfrak{B}$ under a simple operator, quite interesting from an arithmetic point of view: the Bezout distribution has been built precisely towards this purpose.

We shall not prove the main formula in this section, only give a heuristic approach to it, with the following aims in mind: first, to introduce, at this early stage, the ingredients of the operator connecting $\mathfrak{E}^{\sharp}_{\nu_1} \# \mathfrak{E}^{\sharp}_{\nu_2}$ to $\mathfrak{B}$; next, to give a rough explanation of why the main formula should be expected, at the same time pointing towards the main tools as well as the main difficulties of the complete proof, which will be given in Sections 13 to 15. The formal arguments given here probably provide a better feeling about the role played by the "Dirichlet-Hecke" series $\mathcal{L}(s)$ (5.22) in (5.44) than the rigorous arguments to be developed later.

First, we transfer the usual Hecke operators T_N to our distribution setting.

Definition 5.1. Let $\mathcal{S}'^{\,\mathrm{per}}(\mathbb{R}^2)$ be the subspace of $\mathcal{S}'(\mathbb{R}^2)$ consisting of distributions invariant under the *linear* action of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, *i.e.*, under the transformation $(x,\xi) \mapsto (x+\xi, \xi)$: of course, it contains the space of Γ-invariant tempered distributions.

Given $\mathfrak{S} \in \mathcal{S}'^{\,\mathrm{per}}(\mathbb{R}^2)$ and an integer $N \geq 1$, we define the distribution $T_N^{\mathrm{dist}} \mathfrak{S}$ through

$$\left\langle T_N^{\mathrm{dist}} \mathfrak{S}, h \right\rangle : = N^{-\frac{1}{2}} \sum_{\substack{ad=N,\, d>0 \\ b \bmod d}} \left\langle \mathfrak{S},\, (x,\xi) \mapsto h\left(\frac{dx - b\xi}{\sqrt{N}}, \frac{a\xi}{\sqrt{N}} \right) \right\rangle . \tag{5.1}$$

In particular, if $\mathfrak{S}$ coincides with a function h_1, one has

$$(T_N^{\mathrm{dist}} h_1)(x,\xi) = N^{-\frac{1}{2}} \sum_{\substack{ad=N,\, d>0 \\ b \bmod d}} h_1 \left(\frac{ax + b\xi}{\sqrt{N}}, \frac{d\xi}{\sqrt{N}} \right) . \tag{5.2}$$

One also defines

$$\left\langle T_{-1}^{\mathrm{dist}} \mathfrak{S}, h \right\rangle : = \langle \mathfrak{S},\, (x,\xi) \mapsto h(-x,\xi) \rangle \ : \tag{5.3}$$

that $T_{-1}^{\mathrm{dist}}\, \mathfrak{S}$ is automorphic if $\mathfrak{S}$ is, is a consequence of the fact that the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ in $GL(2,\mathbb{R})$. One may also note that $T_{-1}^{\mathrm{dist}} T_N^{\mathrm{dist}} = T_N^{\mathrm{dist}} T_{-1}^{\mathrm{dist}}$, a consequence of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & -b \\ 0 & d \end{smallmatrix}\right) \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that it would be quite natural to set $T_{-N}^{\mathrm{dist}} : = T_{-1}^{\mathrm{dist}} T_N^{\mathrm{dist}}$ for all $N \geq 1$.

Remark. Recall that the involution $\mathfrak{S} \mapsto \mathcal{G}\mathfrak{S}$ admits a companion interpretation on the operator level since, as reported after (2.8), one has $\mathrm{Op}(\mathfrak{S})\check{u} = \mathrm{Op}(\mathcal{G}\mathfrak{S})u$ if $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$ and $u \in \mathcal{S}(\mathbb{R})$.

The involution T_{-1}^{dist} also has such an interpretation, to wit

$$(\mathrm{Op}(T_{-1}^{\text{dist}}\mathfrak{S})\,u)\check{}= \mathrm{Op}(\mathfrak{S})^*\,\tilde{u} \tag{5.4}$$

if $\tilde{u}(x) = \bar{u}(-x)$ and $\mathrm{Op}(\mathfrak{S})^*$ is the formal adjoint of $\mathrm{Op}(\mathfrak{S})$.

One can easily prove in a direct way that, for $N \geq 1$, $T_N^{\text{dist}}\,\mathfrak{S}$ is automorphic if $\mathfrak{S}$ is. However, one may dispense with the proof since it is a consequence of Proposition 2.1 and of the next proposition. Also, it is immediate that the Hecke operator T_N^{dist} commutes with $\mathcal{E}$. Finally, T_N^{dist} commutes with the symplectic Fourier transformation if $N \geq 1$ [62, Proposition 16.11], for general N when acting on even distributions only (in particular automorphic distributions).

Proposition 5.2. *For $p = 0$ or 1, and $z \in \Pi$, one has for every $\mathfrak{S} \in \mathcal{S}'^{\,\text{per}}(\mathbb{R}^2)$ and $N \geq 1$ the relation*

$$(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(T_N^{\text{dist}}\,\mathfrak{S})\,u_z^p) = T_N\,(z \mapsto\, (u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\,u_z^p)\,)\,, \tag{5.5}$$

where the operator T_N is defined by (4.9) on all functions f on Π invariant under the translation $z \mapsto z+1$: in particular, on automorphic functions, T_N is the usual Hecke operator. Also,

$$(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(T_{-1}^{\text{dist}}\,\mathfrak{S})\,u_z^p) = (u_{-\bar{z}}^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\,u_{-\bar{z}}^p)\,. \tag{5.6}$$

Proof. Starting in just the same way as in (4.14), (4.15), we get

$$\begin{aligned}
&(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(T_N^{\text{dist}}\,\mathfrak{S})\,u_z^p) \\
&= \left(-2\frac{d}{d\varepsilon} - 1\right)^p\Big|_{\varepsilon=1} \left\langle T_N^{\text{dist}}\,\mathfrak{S}\,,\,(s,\sigma) \mapsto \exp\left(-\frac{\pi\varepsilon}{y}\,|s - z\sigma|^2\right)\right\rangle \\
&= \left(-2\frac{d}{d\varepsilon} - 1\right)^p\Big|_{\varepsilon=1} N^{-\frac{1}{2}} \sum_{\substack{ad=N,\,d>0\\ b \bmod d}} \left\langle \mathfrak{S}\,,\,(s,\sigma) \mapsto \exp\left(-\frac{\pi\varepsilon}{Ny}\,|ds - b\sigma - az\sigma|^2\right)\right\rangle \\
&= \left(-2\frac{d}{d\varepsilon} - 1\right)^p\Big|_{\varepsilon=1} N^{-\frac{1}{2}} \sum_{\substack{ad=N,\,d>0\\ b \bmod d}} \left\langle \mathfrak{S}\,,\,(s,\sigma) \mapsto \exp\left(-\pi\varepsilon\,\frac{|s - \frac{az+b}{d}\sigma|^2}{\mathrm{Im}\,\frac{az+b}{d}}\right)\right\rangle \\
&= N^{-\frac{1}{2}} \sum_{\substack{ad=N,\,d>0\\ b \bmod d}} \left(u_{\frac{az+b}{d}}^p \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\,u_{\frac{az+b}{d}}^p\right)\,.
\end{aligned} \tag{5.7}$$

The second part is immediate. □

Note that

$$\left\langle N^{-i\pi\,\mathcal{E}}\,T_N^{\text{dist}}\,\mathfrak{S}\,,\,h\right\rangle = \sum_{\substack{ad=N,\,d>0\\ b \bmod d}} \langle \mathfrak{S}\,,\,(x,\xi) \mapsto h(dx - b\xi, a\xi)\rangle\,. \tag{5.8}$$

It has been observed in [62, Section 16] that Dirichlet series in the argument $2i\pi\,\mathcal{E}$, as well as the modified operator $N^{-i\pi\,\mathcal{E}}\,T_N^{\mathrm{dist}}$, act within the space of automorphic distributions supported in $\mathbb{Z}^2\backslash\{0\}$. One can prove something entirely analogous with the Bezout distribution substituted for the distribution $\mathfrak{D}^{\mathrm{prime}}$, *i.e.,* substituting combs of lines for combs of points. However, we shall satisfy ourselves with the following (needed) result.

Proposition 5.3. *Given n, m with $(n,m)=1$ and an integer $N \geq 1$, define the tempered distribution $I_{n,m}^N$ as*

$$\langle I_{n,m}^N, h\rangle = \int_{-\infty}^{\infty} h(nx+n_1, mx+m_1)\, e^{2i\pi Nx}\, dx\,, \tag{5.9}$$

where $\left(\begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix}\right) \in \Gamma$. Then

$$N^{i\pi\,\mathcal{E}}\,T_N^{\mathrm{dist}}\,\mathfrak{B} = \frac{1}{2}\sum_{(n,m)=1} I_{n,m}^N\,, \tag{5.10}$$

where both sides are considered as continuous linear forms on the image of $\mathcal{S}(\mathbb{R}^2)$ under the operator $\pi^2\,\mathcal{E}^2$.

Proof. Let ω denote a complete set of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with integer coefficients and $a \geq 1$, $ad = N$, $b \bmod d$, and let $\bar{\omega}$ denote a complete set of matrices $\left(\begin{smallmatrix} d & -b \\ 0 & a \end{smallmatrix}\right)$ under the same conditions: the map $g \mapsto \det(g).\,g^{-1}$ preserves $M_N(\mathbb{Z})$ (*cf.* (4.10)) and sends a set ω to a set $\bar{\omega}$. Thus

$$M_N(\mathbb{Z}) = \bigcup_{\gamma\in\omega} \Gamma\gamma = \bigcup_{\gamma\in\bar{\omega}} \gamma\Gamma\,. \tag{5.11}$$

Let A be a complete set of representatives of Γ/Γ_∞^0. Then

$$M_N(\mathbb{Z}) = \bigcup_{\substack{\gamma\in\omega\\ g\in A}} g\,\Gamma_\infty^0\,\gamma = \bigcup_{\substack{\gamma\in\bar{\omega}\\ g\in A}} \gamma\, g\,\Gamma_\infty^0\,. \tag{5.12}$$

Now $\cup_{\gamma\in\omega}\Gamma_\infty^0\,\gamma = \cup_{\gamma\in\bar{\omega}}\gamma\,\Gamma_\infty^0$ since both sets coincide with $\{\left(\begin{smallmatrix} a & \cdot \\ 0 & d \end{smallmatrix}\right) : a \geq 1,\ ad = N\}$. Thus

$$M_N(\mathbb{Z}) = \bigcup_{\substack{\gamma\in\bar{\omega}\\ g\in A}} g\,\gamma\,\Gamma_\infty^0 \tag{5.13}$$

too.

One then has, for every $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}\langle N^{-i\pi\,\mathcal{E}}\,T_N^{\mathrm{dist}}\,I_{1,0}^1\,, h\rangle &= \sum_{\substack{ad=N,\, d>0\\ b \bmod d}} \int_{-\infty}^{\infty} h(dx-b, a)\, e^{2i\pi x}\, dx \\ &= \sum_{\substack{ad=N,\, d>0\\ b \bmod d}} e^{2i\pi\frac{b}{d}} \int_{-\infty}^{\infty} h(dx, a)\, e^{2i\pi x}\, dx\,. \end{aligned}\tag{5.14}$$

Since $\sum_{b \bmod d} e^{2i\pi \frac{b}{d}} = 0$ unless $d = 1$, this reduces to

$$\begin{aligned}
&\int_{-\infty}^{\infty} h(x, N)\, e^{2i\pi x}\, dx \\
&= \int_{-\infty}^{\infty} (\, N^{-1+2i\pi\, \mathcal{E}}\, h) \left(\frac{x}{N}, 1\right)\, e^{2i\pi x}\, dx \\
&= \int_{-\infty}^{\infty} (\, N^{2i\pi\, \mathcal{E}}\, h)\, (x, 1)\, e^{2i\pi\, Nx}\, dx \ : \qquad (5.15)
\end{aligned}$$

thus

$$\begin{aligned}
\left\langle N^{-i\pi\, \mathcal{E}}\, T_N^{\mathrm{dist}}\, I_{1,0}^{1}\, , h \right\rangle &= \left\langle I_{1,0}^{N}, N^{2i\pi\, \mathcal{E}}\, h \right\rangle \\
&= \left\langle N^{-2i\pi\, \mathcal{E}}\, I_{1,0}^{N}, h \right\rangle \, . \qquad (5.16)
\end{aligned}$$

Letting Γ/Γ_∞^0 act and summing, we are done, in view of (3.41): the proof of Theorem 3.3 indeed justifies the convergence of both sides in the case when h lies in the image of $\mathcal{S}(\mathbb{R}^2)$ by the operator $\pi^2\, \mathcal{E}^2$.

Incidentally, recall that (4.24) from Theorem 4.3, in which the right-hand side is still meaningful when $\ell = 0$, allowed us to give a meaning to $\mathfrak{B}$ as a tempered distribution; the same could be done here with the right-hand side of (5.10), which would thus become an identity in $\mathcal{S}'_{\mathrm{even}}(\mathbb{R}^2)$. □

The action of the Hecke operators T_N^{dist} on Eisenstein distributions $\mathfrak{E}_\nu^\sharp$ or on cusp-distributions $(\mathfrak{N}_{k,\ell}^{\pm})^\sharp$ is easy to describe too. The first one has been given in [62, (16.88)]: for every $N \geq 1$,

$$T_N^{\mathrm{dist}}\, \mathfrak{E}_\nu^\sharp = N^{-\frac{\nu}{2}}\, \sigma_\nu(N)\, \mathfrak{E}_\nu^\sharp\, . \qquad (5.17)$$

Proposition 5.4. *One has, for every* $N \geq 1$,

$$T_N^{\mathrm{dist}}\, (\mathfrak{N}_{k,\ell}^{\pm})^\sharp = b_N\, (\mathfrak{N}_{k,\ell}^{\pm})^\sharp\, , \qquad (5.18)$$

where b_N *is the* N*-th coefficient from the Fourier expansion* (4.3) *of the cusp-form* $\mathcal{N}_{|k|,\ell}^{\pm}$.

Proof. According to Theorem 4.2 and Proposition 5.2, one has if $z \in \Pi$ and $p = 0$ or 1

$$\begin{aligned}
(u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}(T_N^{\mathrm{dist}}\, (\mathfrak{N}_{k,\ell}^{\pm})^\sharp)\, u_z^p) &= (-i\lambda_k)^p\, (T_N\, \mathcal{N}_{|k|,\ell}^{\pm})(z) \\
&= (-i\lambda_k)^p\, b_N\, \mathcal{N}_{|k|,\ell}^{\pm}(z) \\
&= b_N\, (u_z^p \,|\, \mathrm{Op}_{\sqrt{2}}((\mathfrak{N}_{k,\ell}^{\pm})^\sharp)\, u_z^p)\, , \qquad (5.19)
\end{aligned}$$

so that the proposition follows from Proposition 2.1. □

Proposition 5.5.

$$T_{-1}^{\text{dist}}\, \mathfrak{E}_\nu^\sharp = \mathfrak{E}_\nu^\sharp \tag{5.20}$$

and

$$T_{-1}^{\text{dist}}\, (\mathfrak{N}_{k,\ell}^+)^\sharp = (\mathfrak{N}_{k,\ell}^+)^\sharp\,, \qquad T_{-1}^{\text{dist}}\, (\mathfrak{N}_{k,\ell}^-)^\sharp = -(\mathfrak{N}_{k,\ell}^-)^\sharp\,. \tag{5.21}$$

Proof. This is a consequence of (5.6). □

As observed in [62, (16.83)], one can handle all Hecke operators simultaneously through the consideration of the one-parameter family of operators

$$\begin{aligned}\mathcal{L}(s)\colon &= \sum_{N\geq 1} N^{-s}\, T_N^{\text{dist}} \\ &= \prod_{p \text{ prime}} (1 - p^{-s} T_p^{\text{dist}} + p^{-2s})^{-1}\,.\end{aligned} \tag{5.22}$$

For $\text{Re}\,\nu < -1$ and $\text{Re}\,s > 1 - \frac{\text{Re}\,\nu}{2}$, one then has

$$\mathcal{L}(s)\, \mathfrak{E}_\nu^\sharp = \zeta\left(s - \frac{\nu}{2}\right)\, \zeta\left(s + \frac{\nu}{2}\right)\, \mathfrak{E}_\nu^\sharp\,, \tag{5.23}$$

and, for any cusp-distribution $(\mathfrak{N}_{k,\ell}^\pm)^\sharp$, one has for $\text{Re}\,s$ large enough

$$\mathcal{L}(s)\, (\mathfrak{N}_{k,\ell}^\pm)^\sharp = L(s, \mathcal{N}_{|k|,\ell}^\pm)\, (\mathfrak{N}_{k,\ell}^\pm)^\sharp\,, \tag{5.24}$$

where the L-function $L(s, \mathcal{M}_j^\pm)$ associated to a cusp-form admitting the Fourier series expansion on the right-hand side of (4.3) is defined, as is usual, by the Dirichlet series

$$L(s, \mathcal{M}_j^\pm) = \sum_{n\geq 1} b_n\, n^{-s}\ : \tag{5.25}$$

this is a consequence of (5.17) and (5.18).

Recall that if $\mathcal{M}_j^+$ is an *even* cusp-form, associated to the eigenvalue $\frac{1+(\lambda_j^+)^2}{4}$, $L^*(s, \mathcal{M}_j^+)$ extends as an entire function of s satisfying the functional equation [8, p. 107]

$$L^*(s, \mathcal{M}_j^+) = L^*(1-s, \mathcal{M}_j^+) \tag{5.26}$$

if one defines

$$L^*(s, \mathcal{M}_j^+) = \pi^{-s}\, \Gamma\left(\frac{s}{2} + \frac{i\lambda_j^+}{4}\right)\, \Gamma\left(\frac{s}{2} - \frac{i\lambda_j^+}{4}\right)\, L(s, \mathcal{M}_j^+)\,; \tag{5.27}$$

if $\mathcal{M}_j^-$ is an *odd* cusp-form, associated to the eigenvalue $\frac{1+(\lambda_j^-)^2}{4}$, one should set instead

$$L^*(s, \mathcal{M}_j^-) = \pi^{-s}\, \Gamma\left(\frac{s+1}{2} + \frac{i\lambda_j^-}{4}\right)\, \Gamma\left(\frac{s+1}{2} - \frac{i\lambda_j^-}{4}\right)\, L(s, \mathcal{M}_j^-)\,, \tag{5.28}$$

getting in this case the functional equation

$$L^*(s, \mathcal{M}_j^-) = -L^*(1-s, \mathcal{M}_j^-)\,. \tag{5.29}$$

Just in the same way as changing $L(s, \mathcal{M}_j^{\pm})$ to $L^*(s, \mathcal{M}_j^{\pm})$ made the usual functional equation possible, a modified version $\mathcal{L}'(s)$ of the Dirichlet-Hecke operator $\mathcal{L}(s)$ will be invariant under the symmetry $s \mapsto 1-s$.

Definition 5.6. An (even) distribution $\mathfrak{S} \in \mathcal{S}'_{\text{even}}(\mathbb{R}^2)$ shall be said to be of even (*resp.* odd) type if it is invariant (*resp.* changes to its negative) under the operator T_{-1}^{dist} introduced in (5.2).

From Proposition 5.5, it follows that Eisenstein distributions are of even type, and that a cusp-distribution $(\mathfrak{N}_{k,\ell}^+)^\sharp$ (*resp.* $(\mathfrak{N}_{k,\ell}^-)^\sharp$) is of even (*resp.* odd) type.

Definition 5.7. With $\mathcal{L}(s) = \sum_{N\geq 1} N^{-s}\, T_N^{\text{dist}}$, one defines

$$\mathcal{L}'(s) = \pi^{\frac{1}{2}-s}\, \frac{\Gamma\left(\frac{s+i\pi\,\mathcal{E}}{2}\right)}{\Gamma\left(\frac{1-s-i\pi\,\mathcal{E}}{2}\right)}\, \mathcal{L}_{\text{even}}(s) + \pi^{\frac{1}{2}-s}\, \frac{\Gamma\left(\frac{s+1+i\pi\,\mathcal{E}}{2}\right)}{\Gamma\left(\frac{2-s-i\pi\,\mathcal{E}}{2}\right)}\, \mathcal{L}_{\text{odd}}(s) \tag{5.30}$$

where $\mathcal{L}_{\text{even}}(s)$ (*resp.* $\mathcal{L}_{\text{odd}}(s)$) is the linear operator on automorphic distributions which coincides with $\mathcal{L}(s)$ on (even) distributions of the even (*resp.* odd) type and vanishes on distributions of the odd (*resp.* even) type.

In this definition, we are postponing all questions of convergence, or rather of analytic continuation. Assuming that this has been done, observe that

$$\mathcal{L}'(s) = \mathcal{L}'(1-s)\, T_{-1}^{\text{dist}}\,, \tag{5.31}$$

i.e., $\mathcal{L}'(s) = \pm\mathcal{L}'(1-s)$ according to type, a consequence of (5.23) and (5.24), (5.26) and (5.29).

Consider two Eisenstein distributions $\mathfrak{F}_{\nu_1}^\sharp$ and $\mathfrak{F}_{\nu_2}^\sharp$. From (3.14) and (3.13) it follows that, if $\text{Re}\,\nu > 1$,

$$\mathfrak{F}_\nu^\sharp = 2^{\frac{-\nu-3}{2}}\, \pi^{-\nu-\frac{1}{2}}\, \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma\left(-\frac{\nu}{2}\right)} \sum_{|n|+|m|\neq 0} |-mx+n\xi|^{-1-\nu}\ : \tag{5.32}$$

thus, if $\text{Re}\,\nu_1 > 1$ and $\text{Re}\,\nu_2 > 1$, one may expect the formula

$$\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp = 2^{\frac{-\nu_1-\nu_2-6}{2}}\, \pi^{-\nu_1-\nu_2-1}\, \frac{\Gamma\left(\frac{1+\nu_1}{2}\right)\Gamma\left(\frac{1+\nu_2}{2}\right)}{\Gamma\left(-\frac{\nu_1}{2}\right)\Gamma\left(-\frac{\nu_2}{2}\right)}$$
$$\times \sum_{\substack{|n_1|+|m_1|\neq 0\\ |n_2|+|m_2|\neq 0}} |-m_1x+n_1\xi|^{-1-\nu_1} \# |-m_2x+n_2\xi|^{-1-\nu_2}\,. \tag{5.33}$$

First, we split this sum as

$$\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2} = (\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{marg}} + (\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{main}}\,, \tag{5.34}$$

where the "marginal" terms are those with $n_1 m_2 - n_2 m_1 = 0$. Setting $n_j = r_j\, n$ and $m_j = r_j\, m$ with $r_1 \geq 1$, $r_2 \neq 0$ and $(n, m) = 1$, it is immediate that

$$(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{marg}} = 2^{\frac{-\nu_1-\nu_2-6}{2}}\, \pi^{-\nu_1-\nu_2-1}\, \frac{\Gamma\left(\frac{1+\nu_1}{2}\right)\Gamma\left(\frac{1+\nu_2}{2}\right)}{\Gamma\left(-\frac{\nu_1}{2}\right)\Gamma\left(-\frac{\nu_2}{2}\right)} \times 2\,\zeta(1+\nu_1)\,\zeta(1+\nu_2) \sum_{(n,m)=1} |-mx+n\xi|^{-\nu_1-\nu_2-2} \tag{5.35}$$

or, using (5.32) again,

$$(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{marg}} = \pi^{\frac{1}{2}}\, \frac{\Gamma\left(\frac{1+\nu_1}{2}\right)\Gamma\left(\frac{1+\nu_2}{2}\right)}{\Gamma\left(-\frac{\nu_1}{2}\right)\Gamma\left(-\frac{\nu_2}{2}\right)}\, \frac{\Gamma\left(-\frac{1+\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{2+\nu_1+\nu_2}{2}\right)} \times \frac{\zeta(1+\nu_1)\,\zeta(1+\nu_2)}{\zeta(2+\nu_1+\nu_2)}\, \mathfrak{F}^\sharp_{\nu_1+\nu_2+1}, \tag{5.36}$$

or finally, using the functional equation (3.19) of the zeta function,

$$(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{marg}} = \frac{\zeta(-\nu_1)\,\zeta(-\nu_2)}{\zeta(-1-\nu_1-\nu_2)}\, \mathfrak{F}^\sharp_{\nu_1+\nu_2+1}\,. \tag{5.37}$$

Our first aim in this section is to indicate why one may expect a formula such as

$$(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{main}} = \mathcal{L}'\left(\frac{1+\nu_1+\nu_2}{2}\right) \mathcal{G}\, \mathcal{L}'\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\, \mathfrak{B} + \text{side terms}\,, \tag{5.38}$$

where the side terms, similar to (5.37), cannot be made precise from the much too rough analysis that follows.

Let us just mention here, without proof, that the number of side terms, including (5.37), would be two or three in our case: in Sections 13 and 15, we shall find four side terms. The reason for this is that the number of side terms depends on the position of Re $(\nu_1 \pm \nu_2)$ with respect to ± 1: in Sections 13 to 15 (where proofs are complete), it is the case when $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$ that will be discussed.

The relation (5.38) would be even nicer if, in analogy with (3.3), we had introduced a special notation for the distribution $2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B}$: recall that $\mathfrak{F}^\sharp_\nu = 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{E}^\sharp_\nu$ is better adapted than $\mathfrak{E}^\sharp_\nu$ to the use of the Op-calculus: in the $\mathrm{Op}_{\sqrt{2}}$-calculus, it is the other way around (*cf.* (5.62) below).

We now study the (much harder) main part $(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})_{\mathrm{main}}$. It has been shown in [62, Lemma 5.1], and it will be reviewed with more care in Theorem

11.3, that if Re $\nu_1 > 1$, Re $\nu_2 > 1$ and ν_1, ν_2 are distinct from even integers, one has the formula

$$|x|^{-1-\nu_1} \# |\xi|^{-1-\nu_2} = \int_{-\infty}^{\infty} h_\lambda \, d\lambda \tag{5.39}$$

with

$$h_\lambda(x,\xi) = \sum_{j=0,1} C_j(\nu_1,\nu_2;i\lambda)\, |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\,, \tag{5.40}$$

where $|t|_j^\alpha := |t|^\alpha\,(\mathrm{sign}\, t)^j$ and

$$C_j(\nu_1,\nu_2;i\lambda) = 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}}\, \pi^{\frac{\nu_1+\nu_2-i\lambda}{2}}\, \frac{\Gamma\left(\frac{-\nu_1}{2}\right)\Gamma\left(\frac{-\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)}$$
$$\times\, i^j\, \frac{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda+2j)}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1-\nu_1-\nu_2+i\lambda+2j)}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)}\,. \tag{5.41}$$

Applying the covariance formula (2.5), it follows from (5.33) that

$$(\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp)_{\mathrm{main}} = 2^{\frac{-\nu_1-\nu_2-6}{2}}\, \pi^{-\nu_1-\nu_2-1}\, \frac{\Gamma\left(\frac{1+\nu_1}{2}\right)\Gamma\left(\frac{1+\nu_2}{2}\right)}{\Gamma\left(-\frac{\nu_1}{2}\right)\Gamma\left(-\frac{\nu_2}{2}\right)}$$
$$\int_{-\infty}^{\infty} \sum_{j=0,1} C_j(\nu_1,\nu_2;i\lambda) \sum_{n_1m_2-n_2m_1\neq 0} |n_1m_2-n_2m_1|_j^{\frac{-1-\nu_1-\nu_2+i\lambda}{2}}$$
$$|-m_1x+n_1\xi|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |-m_2x+n_2\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\, d\lambda\,. \tag{5.42}$$

We first refer to Definition 5.6 and observe that, when acting on even distributions, any of the operators $2i\pi\,\mathcal{E}$, $\mathcal{G}$ or $\mathcal{L}(s)$ preserves the type (even or odd) of distributions. Also, since changing m_1, m_2 to their negatives changes $n_1m_2-n_2m_1$ to its negative as well, it is clear that the last sum on the right-hand side of (5.42) has the type specified by j: it is of even type if $j=0$, of odd type if $j=1$. Next, going back to Definition 5.7 and remembering that $\mathcal{G}\,\mathcal{E} = -\mathcal{E}\,\mathcal{G}$, one sees that, when acting on distributions of the type specified by j,

$$\mathcal{L}'\left(\frac{1+\nu_1+\nu_2}{2}\right)\mathcal{G}\,\mathcal{L}'\left(\frac{1+\nu_1-\nu_2}{2}\right) = \pi^{-\nu_1}$$
$$\times\, \frac{\Gamma\left(\frac{1+\nu_1+\nu_2+2i\pi\mathcal{E}+2j}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-2i\pi\mathcal{E}+2j}{4}\right)}{\Gamma\left(\frac{1-\nu_1-\nu_2-2i\pi\mathcal{E}+2j}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+2i\pi\mathcal{E}+2j}{4}\right)}\,\mathcal{L}\left(\frac{1+\nu_1+\nu_2}{2}\right)\mathcal{G}\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right). \tag{5.43}$$

When acting on distributions homogeneous of degree $-1-i\lambda$, $2i\pi\,\mathcal{E}$ acts as the scalar $-i\lambda$, so that an admittedly formal argument would already reduce the formula to be proven to

$$\mathfrak{S} = \mathcal{L}\left(\frac{1+\nu_1+\nu_2}{2}\right) \mathcal{G}\,\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+2i\pi\,\mathcal{E}}\,\mathfrak{B} \\ + \text{side terms}\,, \tag{5.44}$$

with

$$\mathfrak{S} = \int_{-\infty}^{\infty} 2^{\frac{-9-i\lambda}{2}}\,\pi^{\frac{-2+\nu_1-\nu_2-i\lambda}{2}}\,d\lambda \sum_{j=0,1} \frac{\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)} \sum_{n_1m_2-n_2m_1>0} \\ (n_1m_2-n_2m_1)^{\frac{-1-\nu_1-\nu_2+i\lambda}{2}}\,|-m_1x+n_1\xi|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\,|-m_2x+n_2\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\,. \tag{5.45}$$

Next, with $g = \begin{pmatrix} n_2 & -n_1 \\ m_2 & -m_1 \end{pmatrix}$, so that $g^{-1} = (n_1m_2-n_2m_1)^{-1}\begin{pmatrix} -m_1 & n_1 \\ -m_2 & n_2 \end{pmatrix}$, we may write the sum on the last line, recalling that the set of matrices ω has been introduced in the beginning of the proof of Proposition 5.3, as

$$\sum_{N\geq 1} N^{\frac{-1-\nu_1-\nu_2+i\lambda}{2}} \sum_{g\in M_N(\mathbb{Z})} N^{-1-i\lambda}\left(|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\right)\circ g^{-1} \tag{5.46}$$

or, using (5.11), as

$$\sum_{N\geq 1} N^{\frac{-1-\nu_1-\nu_2+i\lambda}{2}} \sum_{\gamma\in\omega} f_j\circ\gamma \tag{5.47}$$

with

$$f_j(x,\xi) = \sum_{\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}\in\Gamma} |-m_1x+n_1\xi|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\,|-m_2x+n_2\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\,. \tag{5.48}$$

On the other hand, from (5.2),

$$T_N f_j = N^{\frac{i\lambda}{2}} \sum_{\gamma\in\omega} f_j\circ\omega\,. \tag{5.49}$$

Thus (5.44), to be proven, reduces to

$$\mathfrak{S}_1 = \mathcal{G}\,\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B} + \text{side terms}\,, \tag{5.50}$$

with

$$\mathfrak{S}_1 : = \int_{-\infty}^{\infty} d\lambda \sum_{j=0,1} 2^{\frac{-9-i\lambda}{2}}\, \pi^{\frac{-2+\nu_1-\nu_2-i\lambda}{2}}\, \frac{\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)}$$

$$\sum_{\left(\begin{smallmatrix} n_1 & n_2 \\ m_1 & m_2 \end{smallmatrix}\right)\in\Gamma} |-m_1 x + n_1 \xi|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |-m_2 x + n_2\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} . \tag{5.51}$$

As a next step, observe that

$$\mathcal{G}\left(|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \right) = (2\pi)^{i\lambda}$$
$$\times \frac{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)}\, |x|_j^{\frac{-1-\nu_1+\nu_2+i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2+i\lambda}{2}} \tag{5.52}$$

and recall that $\mathcal{G}$ commutes with the action of $SL(2,\mathbb{R})$: thus (changing λ to $-\lambda$ in the integral) $\mathfrak{S}_1 = \mathcal{G}\,\mathfrak{S}_1$, so that the identity to be proven is, finally,

$$\mathfrak{S}_1 = \mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\, \mathfrak{B} + \text{side terms}. \tag{5.53}$$

To try to justify this formula, we now start from the right-hand side, and from Proposition 5.3, thus getting if $\mathrm{Re}\,(\nu_1-\nu_2)>1$ the formula

$$N^{i\pi\,\mathcal{E}}\,\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right)\mathfrak{B} = \frac{1}{2}\sum_{N\geq 1} N^{\frac{-1-\nu_1+\nu_2}{2}} \sum_{(n,m)=1} I_{n,m}^N . \tag{5.54}$$

From (5.9), one has

$$I_{n,m}^N = I_{1,0}^N \circ g^{-1} \tag{5.55}$$

with $g = \left(\begin{smallmatrix} n & n_1 \\ m & m_1 \end{smallmatrix}\right) \in \Gamma$ (where the second column is arbitrary) and

$$I_{1,0}^N(x,\xi) = e^{2i\pi N x}\,\delta(\xi-1). \tag{5.56}$$

The decomposition of the distribution $(I_{1,0}^N)_{\text{even}}$, with

$$(I_{1,0}^N)_{\text{even}}(x,\xi) : = \frac{1}{2}[I_{1,0}^N(x,\xi) + I_{1,0}^N(-x,-\xi)],$$

into homogeneous components, is, using (2.13) and (2.14),

$$(I_{1,0}^N)_{\text{even}}(x,\xi) = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\xi|^{-1-i\lambda}\, e^{2i\pi N\frac{x}{\xi}}\, d\lambda , \tag{5.57}$$

an admittedly formal integral. If one could perform the sum of Γ/Γ_∞^0-transforms of this identity, one would thus get, but this is again essentially heuristic,

$$\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{B} = \frac{1}{8\pi}\sum_{N\geq 1}\int_{-\infty}^{\infty} N^{\frac{-1-\nu_1+\nu_2+i\lambda}{2}} \sum_{(n,m)=1} 2^{\frac{-1-i\lambda}{2}} |-mx+n\xi|^{-1-i\lambda}\, e^{2i\pi N\frac{m_1x-n_1\xi}{-mx+n\xi}}\, d\lambda\,, \quad (5.58)$$

still with $\begin{pmatrix} n & n_1 \\ m & m_1 \end{pmatrix} \in \Gamma$, and up to extra side terms. Obviously, a contour deformation would be needed in order to substitute for the series under the integral sign a convergent one: this (and the residue theorem) explains the disappearance of side terms from the result of our rough arguments. Let us mention at once that, though this would be quite feasible if non-trivial, we shall not try to repair the preceding arguments in the proof of the main formula to be started in Section 13, rather follow a quite different path, for reasons which will be given then.

We now split the distribution on the right-hand side of (5.58), noting that the set of matrices with integral entries and determinant -1 can be described either as the set of matrices $\begin{pmatrix} n & n_1 \\ -m & -m_1 \end{pmatrix}$, or the set of matrices $\begin{pmatrix} n & -n_1 \\ m & -m_1 \end{pmatrix}$, with $g = \begin{pmatrix} n & n_1 \\ m & m_1 \end{pmatrix} \in \Gamma$, and that changing g to the first of these matrices would be tantamount to changing (x,ξ) to $(-x,\xi)$, whereas changing g to the second one would be the same as changing N to $-N$. Thus, up to side terms,

$$\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{B} = \int_{-\infty}^{\infty} 2^{\frac{-9-i\lambda}{2}}\,\pi^{-1}\sum_{N\neq 0}\sum_{(n,m)=1} |N|_j^{\frac{-1-\nu_1+\nu_2+i\lambda}{2}} |-mx+n\xi|^{-1-i\lambda}\, e^{2i\pi N\frac{m_1x-n_1\xi}{-mx+n\xi}}\, d\lambda\,. \quad (5.59)$$

Now, it is an easy consequence of Poisson's formula (*cf.* for instance [62, (10.9), (10.10), (10.31), (10.32)]) that, for real t,

$$\sum_{N\neq 0} |N|_j^{s-1}\, e^{2i\pi Nt} = i^j\,\pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s+j}{2})}{\Gamma(\frac{1-s+j}{2})}\sum_{b\in\mathbb{Z}} |t+b|_j^{-s}\,, \quad (5.60)$$

with the following proviso: the domains of convergence of the two sides of this equation are disjoint, since the left-hand side (*resp.* the right-hand side) converges for Re $s<0$ (*resp.* Re $s>1$), and it is to be understood that, on both sides, analytic continuation has to be used. *Formally* plugging the result of this (genuine) equation into (5.59), we would get

$$\mathcal{L}\left(\frac{1+\nu_1-\nu_2}{2}\right) 2^{-\frac{1}{2}+i\pi\mathcal{E}}\,\mathfrak{B} = \int_{-\infty}^{\infty} 2^{\frac{-9-i\lambda}{2}} \sum_{j=0,1} \pi^{-\frac{-2+\nu_1-\nu_2-i\lambda}{2}}\,\frac{\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)} \sum_{(n,m)=1} |-mx+n\xi|^{-1-i\lambda} \sum_{b\in\mathbb{Z}} \left|\frac{m_1x-n_1\xi}{-mx+n\xi}+b\right|_j^{-\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\, d\lambda\,, \quad (5.61)$$

which is the same as (5.45), using again that the set of pairs n, m parametrizes Γ/Γ_∞^0.

Remark. As a conclusion to this heuristic section, let us briefly point towards the main difficulty, and the way it is going to be solved.

Proving the main formula (5.38) or, if one agrees to use the $\mathrm{Op}_{\sqrt{2}}$-calculus, the equivalent one, in which $\natural$ denotes the sharp product from the $\mathrm{Op}_{\sqrt{2}}$-calculus,

$$\mathfrak{E}_{\nu_1}^\sharp \natural\, \mathfrak{E}_{\nu_2}^\sharp = \mathcal{L}'\left(\frac{1+\nu_1+\nu_2}{2}\right) \mathcal{F}\mathcal{L}'\left(\frac{1+\nu_1-\nu_2}{2}\right) \mathfrak{B} + \frac{\zeta(-\nu_1)\,\zeta(-\nu_2)}{\zeta(-1-\nu_1-\nu_2)}\,\mathfrak{E}_{\nu_1+\nu_2+1}^\sharp + \text{other side terms} \tag{5.62}$$

(recall that the Fourier transformation has been defined in (2.7) and Bezout's distribution $\mathfrak{B}$ in the beginning of the proof of Theorem 4.3, and in (3.42) in an informal way) entails considerable difficulties.

Some of these are already apparent from the heuristic considerations that precede. They are connected to the lack of absolute convergence of series of integrals, and are to be taken care of by the (not quite) usual complex contour deformation methods. Another difficulty stems from the fact that we are dealing with rather singular distributions, typically something like $|x|_j^\alpha\,|\xi|_j^\beta$. To circumvent this difficulty, we shall appeal to Proposition 2.1 again, substituting for the analysis of $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$ the technically simpler analysis of functions $z \mapsto (u_z^p | \mathrm{Op}(\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp)\, u_z^p)$. We may then use results relative to the spectral theory of the automorphic Laplacian to proceed further.

But the main difficulty has not yet been singled out. It is the fact that $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$ is not really meaningful in the usual sense. Though $\mathrm{Op}(\mathfrak{F}_\nu^\sharp)$ is always well defined, for $\nu \neq \pm 1$, as a linear operator from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, it is not possible to find three "dense" spaces E_1, E_2, E_3 of functions or distributions, each of which would be invariant under the metaplectic representation, such that $\mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)$ should send E_1 to E_2, and $\mathrm{Op}(\mathfrak{F}_{\nu_1}^\sharp)$ should send E_2 to E_3. Not even choosing for E_1 the space algebraically generated by the u_z^p's, $p = 0$ or 1, and for E_3 the algebraic dual of this space, would do, for it will be proven in the beginning of Section 10 that $\mathrm{Op}(\mathfrak{F}_\nu^\sharp)$ does not even send u_z into $L^2(\mathbb{R})$: things behave better so far as odd states u_z^1 are concerned.

A spectral analysis of the problem of giving the sum on the right-hand side of (5.33) a meaning shows that there is a remaining difficulty, quite similar to the one that prevented us, in Theorem 3.3, from giving the series (3.41) defining $\mathfrak{B}^\ell$ a direct meaning in the case when $\ell = 0$. Namely, it is only after we have applied $i\pi\,\mathcal{E}$ or, more generally, $(i\pi\,\mathcal{E})_p$ for some $p \geq 1$ to each term, that a series such as $\sum |-m_1 x + n_1\xi|^{-1-\nu_1} \# |-m_2 x + n_2 \xi|^{-1-\nu_2}$ can be made to converge.

We must then give some answer to the following question: how can one, in certain cases, give a meaning to the image under $i\pi\,\mathcal{E}$ of the symbol h of some operator A, without being able to define either h or A? An answer is provided in Section 12 and the very beginning of Section 13: though this is not strictly unavoidable, a much better understanding of this issue can be obtained from the introduction of the "higher-level Weyl calculi", to be developed in the next seven sections. For, with the Op^p-calculus, there is a naturally associated pair of sets of functions $(u_z^p)_{z\in\Pi}$, $(u_z^{p+1})_{z\in\Pi}$, such that, extending the concept of Wigner function to the Op^p-theory, the Wigner function associated with a pair (u_z^p, u_z^p) (*resp.* with a pair (u_z^{p+1}, u_z^{p+1})) should be essentially the image of the usual Wigner function $W(u_z, u_z)$ under the operator $(-i\pi\,\mathcal{E})_p$ (*resp.* $(-i\pi\,\mathcal{E})_{p+1}$).

The Op^p-calculus has other applications, and may not be devoid of interest from the point of view of elementary harmonic analysis, relativistic quantum mechanics or quantization theory. However, readers only interested in the automorphic *Weyl* calculus may be advised to jump to Section 13, after having familiarized themselves only with Definition (6.13) and with Proposition 12.1.

Chapter 2

A Higher-level Weyl Calculus of Operators

N.B. The level alluded to here is an energy level (Theorem 8.1).

6 A tamer version of the Weyl calculus: the horocyclic calculus

The Op^p-calculus, to be introduced in Section 9 with a first approach in the present section, smoothes up the most serious difficulties inherent in the automorphic pseudodifferential analysis in two ways. First, as soon as $p \geq 1$, it "forgets" all distributions homogeneous of degree -1: and a detailed spectral analysis of our problem shows that this part of the decomposition is indeed the major obstacle to defining the sharp product of, say, two Eisenstein distributions. Next, its use makes it possible to substitute for the collection of functions (ψ_z) a set of functions $\psi_z^{\tau+1}$ which are just as nice as the ψ_z's outside zero but which, moreover, vanish up to a certain order at zero (*cf.* (6.2)). In the next seven sections, we shall develop this calculus to a further extent than what is strictly needed for its application to the automorphic pseudodifferential calculus.

The horocyclic calculus was first introduced in [56, Theorem 6.1], but [62, Section 17] gives a more self-contained introduction. It depends on some real parameter $\tau > -1$, and on the consideration of the Hilbert space $H_{\tau+1}$ of (classes of) measurable functions on the half-line $(0, \infty)$ such that

$$\|v\|_{\tau+1}^2 := \int_0^\infty |v(s)|^2 s^{-\tau}\, ds < \infty. \tag{6.1}$$

There exists a unitary projective representation $\mathcal{D}_{\tau+1}$ of $SL(2,\mathbb{R})$ in this space, equivalent to a representation taken from the projective discrete series of $SL(2,\mathbb{R})$.

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The horocyclic calculus is a symbolic calculus of operators on $H_{\tau+1}$, in which symbols live on $\mathbb{R}^2$: it satisfies a covariance property fully analogous to (2.4), the representation $\mathcal{D}_{\tau+1}$ taking the place of the metaplectic representation. As we wish to rapidly familiarize the reader with the main features of the horocyclic calculus, we postpone the definition of $\mathcal{D}_{\tau+1}$ to the next section.

It may not be necessary at present to recall the original definition of the horocyclic calculus, which depends on the use of the Radon transformation, from functions on Π to even functions on $\mathbb{R}^2$, and on a connection between the horocyclic calculus and any of the symbolic calculi available with Π as a phase space. Instead, we shall give a set of characteristic properties, which the reader may take as an axiomatic definition of the calculus. The existence and uniqueness will be shown by other, more interesting, means in Section 9, at least in the special case when 2τ is an odd integer: it will be sufficient to deal with this case in the present work, even though, as shown in Section 8, part of the structure with p replaced by an arbitrary real number may be useful in the study of some relativistic wave equations.

We first define, for any $\tau > -1$ and $z \in \Pi$, the function

$$\psi_z^{\tau+1}(s) = (\Gamma(\tau+1))^{-\frac{1}{2}}\,(4\pi)^{\frac{\tau+1}{2}}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\frac{\tau+1}{2}} s^{\tau}\, e^{2i\pi s\bar{z}^{-1}} \;: \tag{6.2}$$

it has norm 1, and the representation $\mathcal{D}_{\tau+1}$ essentially permutes the elements of this family: for, given any $z \in \Pi$ and $g \in SL(2,\mathbb{R})$, there exists $\omega \in \mathbb{C}$, $|\omega| = 1$ depending on (τ, g, z), such that

$$\mathcal{D}_{\tau+1}(g)\psi_z^{\tau+1} = \omega\, \psi_{g.z}^{\tau+1}\,. \tag{6.3}$$

For any pair w, z of points of Π, we consider the rank-one operator

$$P_{w,z} \colon v \mapsto (\psi_w^{\tau+1}|v)\psi_z^{\tau+1}, \tag{6.4}$$

where the scalar product (antilinear on the left) is of course that associated with the Hilbert space $H_{\tau+1}$. The following, reproducing [62, Theorem 17.5 and Proposition 17.6], is a characterization of the horocyclic calculus.

Theorem 6.1. *For every $\tau > -1$, there exists a unique isometry* $\mathrm{symb}_{\tau+1}$ *from the Hilbert space of all Hilbert-Schmidt operators on $H_{\tau+1}$ into $L^2_{\mathrm{even}}(\mathbb{R}^2)$ with the following properties: first, the image of* $\mathrm{symb}_{\tau+1}$ *consists of all functions in $L^2_{\mathrm{even}}(\mathbb{R}^2)$ invariant under the (unitary) symmetry*

$$\frac{\Gamma(i\pi\mathcal{E})}{\Gamma(-i\pi\mathcal{E})}\,\frac{\Gamma\left(\tau+\frac{1}{2}-i\pi\mathcal{E}\right)}{\Gamma\left(\tau+\frac{1}{2}+i\pi\mathcal{E}\right)}\,\mathcal{G}\,. \tag{6.5}$$

Next, the function $p_{w,z} = \mathrm{symb}_{\tau+1}(P_{w,z})$, where $P_{w,z}$ is the rank-one operator defined in (6.4), *is characterized, with the notation introduced in* (2.14) and

(2.15), *by the equation*

$$(p_{w,z})^\flat_\lambda(s) = \gamma_\lambda^{\tau+1}\,(\psi_w^{\tau+1},\psi_z^{\tau+1})\left(\frac{i}{2}\right)^{\frac{1}{2}+\frac{i\lambda}{2}} \frac{(w-\bar z)^{\frac{1}{2}+\frac{i\lambda}{2}}}{(w-s)^{\frac{1}{2}+\frac{i\lambda}{2}}(s-\bar z)^{\frac{1}{2}+\frac{i\lambda}{2}}}\,, \tag{6.6}$$

where the scalar product, taken in the space $H_{\tau+1}$, can be made explicit as

$$(\psi_w^{\tau+1},\psi_z^{\tau+1}) = \frac{(\bar w^{-1}-w^{-1})^{\frac{\tau+1}{2}}(\bar z^{-1}-z^{-1})^{\frac{\tau+1}{2}}}{(\bar z^{-1}-w^{-1})^{\tau+1}}\,, \tag{6.7}$$

and the constant $\gamma_\lambda^{\tau+1}$ is given by

$$\gamma_\lambda^{\tau+1} = 2^{-\frac{1}{2}}\,(2\pi)^{-1-\frac{i\lambda}{2}}\,\frac{\Gamma\left(\frac{1}{2}+\frac{i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)}\,\frac{\Gamma\left(\tau+\frac{1}{2}+\frac{i\lambda}{2}\right)}{\Gamma(\tau+1)}\,. \tag{6.8}$$

The function $\mathrm{symb}_{\tau+1}(B)$ *is called the horocyclic symbol of the operator B.*

Given $\tau > -1$, the two maps $\mathit{Sq}_{\text{even}}^{\tau+1}$ and $\mathit{Sq}_{\text{odd}}^{\tau+1}$ defined by

$$(\mathit{Sq}_{\text{even}}^{\tau+1}\,v)(t) = 2^{\frac{\tau-1}{2}}\,|t|^{\frac{1}{2}-\tau}\,v\left(\frac{t^2}{2}\right) \tag{6.9}$$

and

$$(\mathit{Sq}_{\text{odd}}^{\tau+1}\,v)(t) = 2^{\frac{\tau-1}{2}}\,|t|^{\frac{1}{2}-\tau}\,v\left(\frac{t^2}{2}\right)\,\mathrm{sign}(t) \tag{6.10}$$

define two isometries, the first one from $H_{\tau+1}$ onto $L^2_{\text{even}}(\mathbb{R})$, the second one from $H_{\tau+1}$ onto $L^2_{\text{odd}}(\mathbb{R})$. In the even case, we shall choose $\tau = -\frac{1}{2}+2\ell$ with $\ell = 0, 1, \ldots$ and set

$$\begin{aligned} u_z^{2\ell}(t) &= (\mathit{Sq}_{\text{even}}^{\tau+1}\,\psi_z^{\tau+1})(t) \\ &= \left(\frac{(2\pi)^{2\ell+\frac{1}{2}}}{\Gamma(2\ell+\frac{1}{2})}\right)^{\frac{1}{2}}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\ell+\frac{1}{4}} t^{2\ell}\,e^{i\pi\frac{t^2}{z}}\,; \end{aligned} \tag{6.11}$$

in the odd case, we shall choose $\tau = \frac{1}{2}+2\ell$ and set

$$\begin{aligned} u_z^{2\ell+1}(t) &= (\mathit{Sq}_{\text{odd}}^{\tau+1}\,\psi_z^{\tau+1})(t) \\ &= \left(\frac{(2\pi)^{2\ell+\frac{3}{2}}}{\Gamma(2\ell+\frac{3}{2})}\right)^{\frac{1}{2}}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\ell+\frac{3}{4}} t^{2\ell+1}\,e^{i\pi\frac{t^2}{z}}\,. \end{aligned} \tag{6.12}$$

The two formulas (6.11) and (6.12) of course reduce to

$$u_z^p(t) = \left(\frac{(2\pi)^{p+\frac{1}{2}}}{\Gamma(p+\frac{1}{2})}\right)^{\frac{1}{2}}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\frac{p}{2}+\frac{1}{4}} t^p\,e^{i\pi\frac{t^2}{z}} \tag{6.13}$$

whatever the parity of $p = 0, 1, \ldots$. We also remark that our present notation extends the meaning of u_z^p, as it has been defined in (2.24) in the case when $p = 0$ or 1. The reader should not mistake the sequence $(u_i^p)_{p \geq 0}$ for an orthogonal sequence of Hermite functions, despite the coincidence for $p = 0$ or 1.

We shall now substitute for the Weyl calculus a generalized version: $\mathrm{op}^{2\ell}$ will do when dealing with operators acting within spaces of even functions only, and $\mathrm{op}^{2\ell+1}$ will work in the odd case. Mark the lower-case, meant to emphasize that, so to speak, in any case, only "one fourth" of a complete calculus of operators on $L^2(\mathbb{R})$ is realized in this way.

Definition 6.2. We denote as $\mathrm{op}^{2\ell}$ the isometry from the subspace of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions invariant under the symmetry (6.5), where $\tau = -\frac{1}{2} + 2\ell$, onto the space of even-even Hilbert-Schmidt operators on $L^2(\mathbb{R})$ (*cf.* discussion following (2.9)), characterized by

$$A = \mathrm{op}^{2\ell}(h) \Longleftrightarrow h = \mathrm{symb}_{2\ell+\frac{1}{2}}\left(\left(\mathcal{S}q_{\text{even}}^{2\ell+\frac{1}{2}}\right)^{-1} A\, \mathcal{S}q_{\text{even}}^{2\ell+\frac{1}{2}}\right), \tag{6.14}$$

and we denote as $\mathrm{op}^{2\ell+1}$ the isometry from the subspace of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions invariant under the symmetry (6.5), where $\tau = \frac{1}{2} + 2\ell$, onto the space of odd-odd Hilbert-Schmidt operators on $L^2(\mathbb{R})$ characterized by

$$A = \mathrm{op}^{2\ell+1}(h) \Longleftrightarrow h = \mathrm{symb}_{2\ell+\frac{3}{2}}\left(\left(\mathcal{S}q_{\text{odd}}^{2\ell+\frac{3}{2}}\right)^{-1} A\, \mathcal{S}q_{\text{odd}}^{2\ell+\frac{3}{2}}\right). \tag{6.15}$$

Note that only the intertwining operator occurring in (6.14) or (6.15) distinguishes the op^p-calculus from the horocyclic calculus. From [62, Theorem 17.7], it follows that the op^0-(*resp.* the op^1)-calculus is just the same as the even-even (*resp.* the odd-odd) part of the Weyl calculus. But, first, one should check that the symmetry (6.5) reduces to $\mathcal{G}$ when $\tau = -\frac{1}{2}$, to $-\mathcal{G}$ when $\tau = \frac{1}{2}$: again, one should go back to the discussion following (2.9) to note that, indeed, this is all right.

To develop the op^p-calculus further, we first compute Wigner functions. The concept is the same as in (2.3), only substituting op^p for Op. Consider the involution (6.5)

$$\Sigma_p \colon = \frac{\Gamma(i\pi\,\mathcal{E})\,\Gamma(p - i\pi\,\mathcal{E})}{\Gamma(-i\pi\,\mathcal{E})\,\Gamma(p + i\pi\,\mathcal{E})}\,\mathcal{G} = \frac{(-i\pi\,\mathcal{E})_p}{(i\pi\,\mathcal{E})_p}\,\mathcal{G}, \tag{6.16}$$

using on the right-hand side the notation provided by the use of Pochhammer's symbols.

The Wigner function $\mathrm{wig}^p(v, u)$, where v, u lie in $\mathcal{S}(\mathbb{R})$ and have the parity associated with p, is characterized by the property that one should have

$$(v|\mathrm{op}^p(h)\,u) = \int_{\mathbb{R}^2} h(x,\xi)\,\mathrm{wig}^p(v,u)(x,\xi)\,dx\,d\xi \tag{6.17}$$

for every $h \in L^2_{\text{even}}(\mathbb{R}^2)$ invariant under Σ_p, and that it should also be invariant under the same symmetry.

One of the most pleasant properties of the horocyclic calculus (a property not shared by any of the calculi available with Π as a phase space, for instance [6, 7] or [55, 58]) is that the symbol map is an isometry from a space of Hilbert-Schmidt operators to some closed subspace of $L^2_{\text{even}}(\mathbb{R}^2)$. This also holds for the op^p-calculus in view of the intertwining formula (6.14) or (6.15). A consequence, by a trivial Hilbert space argument, is that the Wigner function $\text{wig}^p(v,u)$ is also the op^p-symbol of the rank-one operator $w \mapsto (v|w)u$. From Theorem 6.1, we get in particular (with the notation (2.14), (2.15))

$$[\text{wig}^p(u_z^p, u_z^p)]^\flat_\lambda(s) = \gamma_\lambda^{p+\frac{1}{2}} \left(\frac{|z-s|^2}{\text{Im }z}\right)^{\frac{-1-i\lambda}{2}} . \tag{6.18}$$

Thus, from (2.16) and (6.8),

$$[\text{wig}^p(u_z^p, u_z^p)]_\lambda(x,\xi) = \frac{\pi^{-\frac{1}{2}}}{2} \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(p+\frac{i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(p+\frac{1}{2}\right)} \left(\frac{2\pi\,|x-z\xi|^2}{\text{Im }z}\right)^{\frac{-1-i\lambda}{2}} . \tag{6.19}$$

Recall (2.13) that

$$\text{wig}^p(u_z^p, u_z^p) = \int_{-\infty}^{\infty} [\text{wig}^p(u_z^p, u_z^p)]_\lambda \, d\lambda \tag{6.20}$$

and that $2i\pi\,\mathcal{E}\,h = -i\lambda\,h$ if h is a distribution on $\mathbb{R}^2$ homogeneous of degree $-1-i\lambda$. It thus follows that

$$\begin{aligned} \text{wig}^p(u_z^p, u_z^p) &= \frac{\Gamma(p-i\pi\,\mathcal{E})}{\Gamma(-i\pi\,\mathcal{E})} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(p+\frac{1}{2}\right)} W(u_z,u_z) \\ &= \frac{(-i\pi\,\mathcal{E})_p}{\left(\frac{1}{2}\right)_p} W(u_z,u_z)\,, \end{aligned} \tag{6.21}$$

with (recalling (2.27))

$$W(u_z,u_z)(x,\xi) = 2\,\exp\left(-\frac{2\pi\,|x-z\xi|^2}{\text{Im }z}\right) . \tag{6.22}$$

The equation (6.21), together with the $\mathcal{G}$-invariance of $W(u_z,u_z)$ and the relation $\mathcal{G}(i\pi\,\mathcal{E})\,\mathcal{G}^{-1} = -i\pi\,\mathcal{E}$, makes it possible to check that, indeed, $\text{wig}^p(u_z,u_z)$ is invariant under Σ_p. As Σ_p is its own transpose, when acting on $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$, one sees that defining $\text{op}^p(h)$ by (6.16) is also meaningful whenever $h \in L^2_{\text{even}}(\mathbb{R}^2)$: the result is just the same as $\text{op}^p(\frac{1}{2}(h+\Sigma_p h))$.

We already observed in (2.29) that applying the operator $\pi^2\,\mathcal{E}^2$ to a function of $q = \frac{2\pi\,|x-z\xi|^2}{\text{Im }z}$, viewed as a function of x,ξ, was equivalent to applying it, viewed as a function of z, the operator $\Delta - \frac{1}{4}$. Here, taking the first point of view, one gets

$$i\pi\,\mathcal{E}\,.\,k(q) = \left(q\frac{d}{dq} + \frac{1}{2}\right)\,.\,k(q)\,. \tag{6.23}$$

We have thus proved

Proposition 6.3. *For any* $p = 0, 1, \dots,$ *and* $z \in \Pi$,

$$\mathrm{wig}^p(u_z^p, u_z^p)(x,\xi) = (-1)^p \frac{\left(q\frac{d}{dq}+\frac{1}{2}\right)\left(q\frac{d}{dq}-\frac{1}{2}\right)\cdots\left(q\frac{d}{dq}-p+\frac{3}{2}\right)}{\frac{1}{2}\cdot\frac{3}{2}\cdots\left(-\frac{1}{2}+p\right)}\,.\,2\,e^{-q}\,, \tag{6.24}$$

with $q = \frac{2\pi\,|x-z\xi|^2}{\mathrm{Im}\ z}$.

Note that the right-hand side is the product of e^{-q} by a polynomial in q. Using a "sesquiholomorphic" argument, one immediately finds $\mathrm{wig}^p(u_w^p, u_z^p)$ for any pair w, z as well: indeed, since u_z^p is $(\mathrm{Im}\ (-\frac{1}{z}))^{\frac{p}{2}+\frac{1}{4}}$ times an antiholomorphic function of z, (6.17) shows that $\mathrm{wig}^p(u_w^p, u_z^p)$ is $(\mathrm{Im}\ (-\frac{1}{w})\,\mathrm{Im}\ (-\frac{1}{z}))^{\frac{p}{2}+\frac{1}{4}}$ times a function which is holomorphic in w and antiholomorphic in z. Thus $\mathrm{wig}^p(u_w^p, u_z^p)$ is obtained from (6.24) by substituting

$$2\pi\,\frac{(x-w\xi)\,(x-\bar{z}\xi)}{\frac{w-\bar{z}}{2i}}$$

for q, and multiplying the whole new function obtained by (compare (6.7))

$$\frac{(\bar{w}^{-1}-w^{-1})^{\frac{p}{2}+\frac{1}{4}}(\bar{z}^{-1}-z^{-1})^{\frac{p}{2}+\frac{1}{4}}}{(\bar{z}^{-1}-w^{-1})^{p+\frac{1}{2}}}\,.$$

This makes it possible, for any tempered (this is far from necessary) distribution $h \in \mathcal{S}'(\mathbb{R}^2)$, invariant under the symmetry (6.16), to define $\mathrm{op}^p(h)$ in the following *minimal* sense: as a linear operator from the space algebraically generated by the u_z^p's, $z \in \Pi$, to the weak dual of this space.

We now generalize Theorem 3.1 and Theorem 4.2. First, just as in the Weyl calculus, we set (*cf.* (2.21))

$$\mathrm{op}^p_{\sqrt{2}}(h) = \mathrm{op}^p(2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,h)\,. \tag{6.27}$$

Theorem 6.4. *For every* $p = 0, 1, \dots$ *and* $z \in \Pi$,

$$(u_z^p|\mathrm{op}^p_{\sqrt{2}}(\mathfrak{E}_\nu^\sharp)\,u_z^p) = \frac{\left(-\frac{\nu}{2}\right)_p}{\left(\frac{1}{2}\right)_p}\,E^*_{\frac{1-\nu}{2}}(z) \tag{6.28}$$

if $\nu \neq \pm 1$, *and*

$$(u_z^p|\mathrm{op}^p_{\sqrt{2}}((\mathfrak{N}^\pm_{k,\ell})^\sharp)\,u_z^p) = \frac{\left(-\frac{i\lambda_k}{2}\right)_p}{\left(\frac{1}{2}\right)_p}\,\mathcal{N}^\pm_{|k|,\ell}(z) \tag{6.29}$$

for any cusp-distribution $(\mathfrak{N}^\pm_{k,\ell})^\sharp$.

Proof.

$$\begin{aligned}(u_z^p|\mathrm{op}^p_{\sqrt{2}}(\mathfrak{E}^\sharp_\nu)\, u_z^p) &= \left\langle \mathfrak{E}^\sharp_\nu\,,\, 2^{-\frac{1}{2}-i\pi\,\mathcal{E}}\,\mathrm{wig}^p(u_z^p, u_z^p)\right\rangle \\ &= \left\langle 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\frac{(i\pi\,\mathcal{E})_p}{\left(\frac{1}{2}\right)_p}\,\mathfrak{E}^\sharp_\nu\,,\, W(u_z, u_z)\right\rangle \\ &= \frac{\left(-\frac{\nu}{2}\right)_p}{\left(\frac{1}{2}\right)_p}\,\langle \mathfrak{F}^\sharp_\nu\,,\, W(u_z, u_z)\rangle \end{aligned} \tag{6.30}$$

since $\mathfrak{E}^\sharp_\nu$ is homogeneous of degree $-1-\nu$, and it suffices to apply Theorem 3.1 for the first part. The second part is proven in the same way, using Theorem 4.2. □

7 The higher-level metaplectic representations

The "square root construction" to be introduced now is very similar to the one which yields first-order systems such as Dirac's equation from second-order equations such as Klein-Gordon's. Only, the starting point here will be not one operator, but the set of infinitesimal operators from a pair of representations taken from the discrete series of $SL(2,\mathbb{R})$. In the next section, as a development unrelated to our current automorphic endeavours, we shall briefly describe how this construction indeed produces the radial parts of the energy operators from Dirac's equation (a wave equation for the electron-positron) or from Weyl's equation for the neutrino.

Our present motivations are different: the main point is to smooth up a little bit the Weyl calculus of operators while preserving its fundamental covariance property. To this effect, we first need to generalize the metaplectic representation Met to a sequence $(\mathrm{Met}_p)_{p\geq 0}$: this will be obtained by piecing together two irreducible representations $\mathcal{D}_{p+\frac{1}{2}}$ and $\mathcal{D}_{p+\frac{3}{2}}$ from the holomorphic discrete series of $SL(2,\mathbb{R})$. We reproduce the (classical) definition of this latter concept, from [62, Section 17]. Recall that a projective representation is almost like a representation, except for the fact that the homomorphism property is weakened by allowing extra "phase" factors, complex numbers of modulus one (*cf.* 1. in Proposition 7.1). Also, the projective representation $\mathcal{D}_{\tau+1}$ defined as follows is square-integrable only in the case when $\tau > 0$, but we shall, for simplicity, call the whole family (where $\tau > -1$) the projective holomorphic discrete series, or simply the discrete series.

Proposition 7.1. *Let τ be a real number > -1, and let $H_{\tau+1}$ be the Hilbert space of all (classes of) measurable functions on the half-line $(0,\infty)$ such that*

$$\|v\|^2_{\tau+1} := \int_0^\infty |v(s)|^2 s^{-\tau}\, ds < \infty. \tag{7.1}$$

There exists a unitary projective representation $\pi = \mathcal{D}_{\tau+1}$ of $SL(2,\mathbb{R})$ in $H_{\tau+1}$ with the following properties, in the statement of which $g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$:

1. *for every pair* (g, g_1) *of elements of* $SL(2,\mathbb{R})$, *the complex number* $\pi(gg_1)^{-1}\pi(g)\pi(g_1)$ *belongs to the group* $\exp(2i\pi\tau\mathbb{Z})$;
2. *for every* g *with* $b < 0$, $\pi(g) = e^{i\pi(\tau+1)}\pi(-g)$;
3. *if* $b = 0$, $a > 0$, $(\pi(g)v)(s) = a^{\tau-1}v(a^{-2}s)e^{2i\pi\frac{c}{a}s}$;
4. *if* $b > 0$, *and* $v \in C_0^\infty(]0,\infty[)$,

$$(\pi(g)v)(s) = e^{-i\pi\frac{\tau+1}{2}}\frac{2\pi}{b}\int_0^\infty v(t)\,\left(\frac{s}{t}\right)^{\frac{\tau}{2}}\,\exp\left(2i\pi\frac{ds+at}{b}\right)\,J_\tau\left(\frac{4\pi}{b}\sqrt{st}\right)dt. \tag{7.2}$$

Proof. That one can give a real-type realization of the holomorphic discrete series in which the integral kernels of the unitary operators $\pi(g)$ are realized with the help of Bessel-type functions is well known in a much more general context [21, 22]. An elementary reference is ([55], Proposition 1.5) (note that in the latter reference, the representation was denoted as $g \mapsto M_{g^{-1}}$ and that our present $\mathcal{D}_{\tau+1}(g)$ is just M_{g_1} with $g_1 = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)g^{-1}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, and τ substituted for λ).

Of course, if τ is an integer, $\mathcal{D}_{\tau+1}$ can be chosen as a genuine representation in a unique way; in any case, $\mathcal{D}_{\tau+1}(g)$ is uniquely defined up to the multiplication by some number in $\exp(2i\pi\tau\mathbb{Z})$. □

Proposition 7.2. *Under the assumption that* $\tau > 0$, *consider the Hilbert space* $\tilde{H}_{\tau+1}$ *of all holomorphic functions* f *in* Π *with*

$$\|f\|^2 := \int_\Pi |f(z)|^2(\text{Im } z)^{\tau+1}\,d\mu(z) < \infty \tag{7.3}$$

together with the map $v \mapsto f$,

$$f(z) = (4\pi)^{\frac{\tau}{2}}\,(\Gamma(\tau))^{-\frac{1}{2}}\,z^{-\tau-1}\int_0^\infty v(s)\,e^{-2i\pi sz^{-1}}\,ds. \tag{7.4}$$

The map just defined is an isometry from $H_{\tau+1}$ *onto* $\tilde{H}_{\tau+1}$: *it intertwines the representation* $\mathcal{D}_{\tau+1}$ *of* $SL(2,\mathbb{R})$ *in* $H_{\tau+1}$ *and a representation* $\tilde{\mathcal{D}}_{\tau+1}$ *of* $SL(2,\mathbb{R})$ *in* $\tilde{H}_{\tau+1}$, *taken from the holomorphic (projective) discrete series, characterized up to scalar factors in the group* $\exp(2i\pi\tau\mathbb{Z})$ *by the fact that*

$$(\tilde{\mathcal{D}}_{\tau+1}(g)f)(z) = (-cz+a)^{-\tau-1}f\left(\frac{dz-b}{-cz+a}\right) \tag{7.5}$$

if $c < 0$. *In all this the fractional powers which occur are those associated with the principal determination of the logarithm in* Π.

Let us emphasize that, as one should be, we have been *very* careful with the phase factors, especially in view of the fact that the values of τ of main interest to us are $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, ... Not everyone normalizes the holomorphic discrete series in the same way, as the conjugation under some inner automorphism of $SL(2,\mathbb{R})$ may occur: here, no choice could possibly be arbitrary, in view of the necessity of coherence with the Weyl symbolic calculus.

It is important, first, to understand the infinitesimal operators of the representation $\mathcal{D}_{\tau+1}$, *i.e.*, the infinitesimal operators, in the sense of Stone's theorem, of the one-parameter groups of unitaries $t \mapsto \pi(\exp tX)$, $X \in \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. For instance, if $X = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, and $g = \exp \beta X = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}$, the operator $\pi(g)$ given in (7.2) is exactly, when $\beta < 0$, the operator $\exp(i\beta A)$, where A is the appropriate self-adjoint realization of the formal differential operator

$$A = -\frac{1}{2\pi}\left(s\frac{d^2}{ds^2} + (1-\tau)\frac{d}{ds}\right) \tag{7.6}$$

on the half-line. One can also establish the following: if A is a certain self-adjoint realization of the formal differential operator

$$A = -\frac{1}{2\pi}\left(s\frac{d^2}{ds^2} + (1-\tau)\frac{d}{ds} - 4\pi^2\, s\right), \tag{7.7}$$

then

$$e^{i\pi\frac{\tau+1}{2}}\left(\exp\left(-\frac{i\pi}{2}\,A\right)\, v\right)(s) = 2\pi\int_0^\infty \left(\frac{s}{t}\right)^{\frac{\tau}{2}} J_\tau(4\pi\sqrt{st})\, v(t)\, dt\,, \tag{7.8}$$

so that $\exp(-\frac{i\pi}{2}\,A)$ is the unitary $\pi(g)$, in the sense of Proposition 7.1, corresponding to $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is much easier to check that if $X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\exp(\alpha X) = \begin{pmatrix} e^{-\alpha} & 0 \\ 0 & e^{\alpha} \end{pmatrix}$, then $\pi(g)$, given in Proposition 7.1 as

$$(\pi(g)v)(s) = e^{\alpha(1-\tau)}\, v(e^{2\alpha s})\,, \tag{7.9}$$

can also be expressed as $\pi(g) = \exp(i\alpha A)$, with

$$A = -i\left(2s\frac{d}{ds} + 1 - \tau\right); \tag{7.10}$$

also, that if $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so that $\exp(\gamma X) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, one has $\pi(\exp \gamma X) = \exp(i\gamma\, A)$ with $(Av)(s) = 2\pi\, s\, v(s)$.

In the three cases when X is the element $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of $\mathfrak{g}$, we have thus made a certain operator A, such that $\pi(e^{tX}) = \exp(itA)$, explicit: the correspondence $X \mapsto iA$ from $\mathfrak{g}$ to a certain space of second-order differential operators, linearly extended, is usually denoted as $X \mapsto d\pi(X)$: it is the infinitesimal representation associated with π. It is then Stone's theorem on one-parameter groups of unitaries which permits one to define, for each formal differential operator A in the linear space under consideration, the self-adjoint realization one has in mind: it will not be necessary to characterize the domain of iA in each case.

We can now begin our construction of the representation Met_p, $p = 0, 1, \ldots$

Definition 7.3. Given $p = 0, 1, \ldots,$ we consider the space $\mathcal{S}_p(\mathbb{R}) = \{x \mapsto x^p\, v(x),\ v \in \mathcal{S}(\mathbb{R})\}$, and the operators of "position and momentum" Q and P, defined on $\mathcal{S}_p(\mathbb{R})$ by the equations

$$\begin{aligned} Qu &= xu\,, \\ Pu &= \frac{1}{2i\pi}\left[\frac{du}{dx} + (-1)^{p+1}\,\frac{p}{x}\,\check{u}\right]. \end{aligned} \tag{7.11}$$

Note that, if $u = x^p\, v$,

$$\begin{aligned} 2i\pi\,(Pu)(x) &= \frac{d}{dx}\,(x^p\, v(x)) + (-1)^{p+1}\,\frac{p}{x}\,(-x)^p\, v(-x) \\ &= x^p\,\frac{dv}{dx} + px^{p-1}\,[v(x) - v(-x)]\,. \end{aligned} \tag{7.12}$$

This implies that $Pu \in \mathcal{S}_p(\mathbb{R})$.

The space $\mathcal{S}_p(\mathbb{R})$ is provided with the topology which makes the multiplication by x^p an isomorphism from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}_p(\mathbb{R})$.

Of course, P and Q reduce to the usual position and momentum operators in the case when $p = 0$: then, the operators iP, iQ and iI linearly generate, over $\mathbb{R}$, the set of infinitesimal operators of the so-called Heisenberg representation of the three-dimensional Heisenberg group in $L^2(\mathbb{R})$. When $p \neq 0$, P and Q do not generate a finite-dimensional Lie algebra, but we have no need for it. It is handy to write all operators on $\mathcal{S}_p(\mathbb{R})$ in block-matrix form, corresponding to the decomposition $\mathcal{S}_p(\mathbb{R}) = (\mathcal{S}_p(\mathbb{R}))_{\text{even}} \oplus (\mathcal{S}_p(\mathbb{R}))_{\text{odd}}$. To avoid carrying the coefficient $(-1)^{p+1}$ throughout, or splitting the discussion at every step, or using very cumbersome terminology, **we shall usually assume that p is even**: however, when p is odd, the situation is fully similar except for the fact that one must switch the two terms of the direct sum above instead. **All** statements given in the even case will then remain valid in the odd case.

The following formal computations are straightforward, if somewhat tedious.

Proposition 7.4. *With respect to the decomposition $\mathcal{S}_p(\mathbb{R}) = (\mathcal{S}_p(\mathbb{R}))_{\text{even}} \oplus (\mathcal{S}_p(\mathbb{R}))_{\text{odd}}$ in the case when p is even, $\mathcal{S}_p(\mathbb{R}) = (\mathcal{S}_p(\mathbb{R}))_{\text{odd}} \oplus (\mathcal{S}_p(\mathbb{R}))_{\text{even}}$ when p is odd, one has*

$$Q = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \qquad P = \frac{1}{2i\pi}\begin{pmatrix} 0 & \frac{d}{dx} + \frac{p}{x} \\ \frac{d}{dx} - \frac{p}{x} & 0 \end{pmatrix},$$

$$[P, Q] = \frac{1}{2i\pi}\begin{pmatrix} 1 + 2p & 0 \\ 0 & 1 - 2p \end{pmatrix},$$

$$Q^2 = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}, \qquad PQ + QP = \frac{1}{2i\pi}\begin{pmatrix} 2x\,\frac{d}{dx} + 1 & 0 \\ 0 & 2x\,\frac{d}{dx} + 1 \end{pmatrix},$$

$$P^2 = -\frac{1}{4\pi^2}\begin{pmatrix} \frac{d^2}{dx^2} + \frac{p(1-p)}{x^2} & 0 \\ 0 & \frac{d^2}{dx^2} - \frac{p(1+p)}{x^2} \end{pmatrix}. \tag{7.13}$$

One must note that all "second-order" operators are diagonal in block-matrix form, contrary to P and Q; they generate a Lie algebra, isomorphic to $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, given by the relations:

Proposition 7.5.

$$\begin{aligned} [P^2, Q^2] &= \frac{1}{i\pi}\,(PQ+QP)\,,\\ [P^2, PQ+QP] &= \frac{2}{i\pi}\,P^2\,,\\ [Q^2, PQ+QP] &= -\frac{2}{i\pi}\,Q^2\,. \end{aligned} \tag{7.14}$$

Finally,

Proposition 7.6.

$$\begin{aligned} -[Q^2, P] = [PQ+QP, Q] &= \frac{1}{i\pi}\,Q\,,\\ [P^2, Q] = -[PQ+QP, P] &= \frac{1}{i\pi}\,P\,. \end{aligned} \tag{7.15}$$

We now connect the operators in the linear space generated by P^2, $PQ + QP$, Q^2 to the infinitesimal operators of the representations $\mathcal{D}_{\tau+1}$, $\tau = p \pm \frac{1}{2}$. Define, in block-matrix form, the unitary operator

$$\Phi = \begin{pmatrix} \mathcal{S}q_{\text{even}}^{p+\frac{1}{2}} & 0\\ 0 & \mathcal{S}q_{\text{odd}}^{p+\frac{3}{2}} \end{pmatrix} \tag{7.16}$$

from $H_{p+\frac{1}{2}} \oplus H_{p+\frac{3}{2}}$ onto $L^2_{\text{even}}(\mathbb{R}) \oplus L^2_{\text{odd}}(\mathbb{R})$. Under this isomorphism, one may transfer the operators P^2, $PQ+QP$, Q^2. The result of a simple formal computation is the following:

Proposition 7.7.

$$\begin{aligned} \Phi^{-1}\,(\pi\,P^2)\,\Phi &= \begin{pmatrix} -\frac{1}{2\pi}\,[s\frac{d^2}{ds^2} + (\frac{3}{2}-p)\,\frac{d}{ds}] & 0\\ 0 & -\frac{1}{2\pi}\,[s\frac{d^2}{ds^2} + (\frac{1}{2}-p)\,\frac{d}{ds}] \end{pmatrix},\\ \Phi^{-1}\,(\pi\,(PQ+QP))\,\Phi &= \begin{pmatrix} -i\,(2s\,\frac{d}{ds} + \frac{3}{2} - p) & 0\\ 0 & -i\,(2s\,\frac{d}{ds} + \frac{1}{2} - p) \end{pmatrix},\\ \Phi^{-1}\,(\pi\,Q^2)\,\Phi &= \begin{pmatrix} 2\pi\,s & 0\\ 0 & 2\pi\,s \end{pmatrix}. \end{aligned} \tag{7.17}$$

In the upper-left corner (*resp.* the lower-right corner), one recognizes in each case the operator A, listed above, corresponding to the case when $X = \left(\begin{smallmatrix} 0 & -1\\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 0\\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0\\ 1 & 0 \end{smallmatrix}\right)$, and to the value $\tau = p - \frac{1}{2}$ (*resp.* $p + \frac{1}{2}$). Since the operators $-i\,d\pi(X)$, with $\pi = \mathcal{D}_{p+\frac{1}{2}}$ or $\mathcal{D}_{p+\frac{3}{2}}$, have already been provided with an unambiguous definition as self-adjoint (rather than only symmetric) operators, we can, under the transfer by Φ, give all operators in the linear space generated by P^2, $PQ+QP$ and Q^2 an unambiguous definition as a self-adjoint operator on $L^2(\mathbb{R})$ just as well.

Definition 7.8. The p-metaplectic representation Met_p is the unitary projective representation on $L^2(\mathbb{R})$ defined by

$$\mathrm{Met}_p(g) = \Phi \begin{pmatrix} \mathcal{D}_{p+\frac{1}{2}}(g) & 0 \\ 0 & \mathcal{D}_{p+\frac{3}{2}}(g) \end{pmatrix} \Phi^{-1} \tag{7.18}$$

for all $g \in SL(2,\mathbb{R})$, where $\mathcal{D}_\tau$ has been introduced in Definition 7.1.

Remark that, since $\tau = p \pm \frac{1}{2}$ and $p \in \mathbb{Z}$, Met_p lifts up, for all values of p, as a genuine representation of a twofold covering of $SL(2,\mathbb{R})$, but not of $SL(2,\mathbb{R})$ itself.

Theorem 7.9. *For $p = 0, 1, \dots$, the unitary operator $\mathrm{Met}_p(g)$ preserves the subspace $\mathcal{S}_p(\mathbb{R})$ of $L^2(\mathbb{R})$ for every $g \in SL(2,\mathbb{R})$.*

Proof. With $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the case when $b = 0$ is trivial, so we can assume that $b > 0$ and write $\mathrm{Met}_p(g) = \begin{pmatrix} U_{p-\frac{1}{2}} & 0 \\ 0 & U_{p+\frac{1}{2}} \end{pmatrix}$ with

$$(U_\tau u)(x) = e^{-i\pi\frac{\tau+1}{2}}\,\frac{\pi}{b}\int_{-\infty}^{\infty} |xy|^{\frac{1}{2}}\,(\operatorname{sign} xy)^j \exp\left(i\pi\frac{dx^2+ay^2}{b}\right) J_\tau\left(\frac{2\pi\,|xy|}{b}\right) u(y)\,dy\,, \tag{7.19}$$

with $j = 0$ or $1, j \equiv \tau + \frac{1}{2} \bmod 2$. Setting $q = p$ when dealing with functions of the parity related to p (*i.e.*, even if p is even, odd if p is odd), $q = p+1$ when dealing with functions of the opposite parity, performing the change of variable $y \mapsto \frac{by}{2\pi}$ and changing the constants a and d, we thus have to show that, for every $q = 0, 1, \dots$, the operator V_q defined by

$$(V_q u)(x) = \int_{-\infty}^{\infty} e^{i\pi\,(dx^2+ay^2)}\,(xy)^q\,\frac{J_{q-\frac{1}{2}}(|xy|)}{|xy|^{q-\frac{1}{2}}}\,u(y)\,dy \tag{7.20}$$

preserves the space $\mathcal{S}_q(\mathbb{R})$: let us not forget that a function in $\mathcal{S}_p(\mathbb{R})$ of any given parity is actually in $\mathcal{S}_q(\mathbb{R})$.

Since [31, p. 72] one has for some constant C and every $r > 0$

$$\frac{J_{q-\frac{1}{2}}(r)}{r^{q-\frac{1}{2}}} = C\left(r^{-1}\frac{d}{dr}\right)^q\left(\frac{\sin r}{r}\right), \tag{7.21}$$

and the function $(r^{-1}\frac{d}{dr})^q\,(\frac{\sin r}{r})$ extends as an even C^∞ function of r on the real line, it is clear that, as soon as u lies in $\mathcal{S}(\mathbb{R})$, the function $x \mapsto \frac{(V_q u)(x)}{x^q}$ is well defined and C^∞ on $\mathbb{R}$: it remains to be shown that it is rapidly decreasing at infinity.

First assume that u is zero in a neighbourhood of 0. Then, for $|x| \geq 1$, an integration by parts, applied to the equation

$$x^{-q}\,(V_q u)(x) = C \int_{-\infty}^{\infty} e^{i\pi\,(dx^2+ay^2)}\, y^q \left(x^{-2}y^{-1}\frac{d}{dy}\right)^q \left(\frac{\sin xy}{xy}\right) u(y)\,dy\,, \quad (7.22)$$

makes it possible to isolate of the factor $\sin xy$; next, a new integration by parts, based on the relation $\sin xy = (x^{-1}\frac{d}{dy})^{4N}\,\sin xy$, shows that the function $x^{-q}V_q u$ is also rapidly decreasing at infinity (together with all its derivatives).

We now drop the assumption that u vanishes in a neighbourhood of 0, still assuming, of course, that $u \in \mathcal{S}_q(\mathbb{R})$. Let w be an arbitrary function in $\mathcal{S}_q(\mathbb{R})$, and denote as yw the function $y \mapsto y\,w(y)$. Assuming $q \geq 1$, and using an integration by parts (based on the use of the operator $\frac{d}{dy}(y.)$) and the equation [31, p. 67]

$$|xy|\,J'_{q-\frac{1}{2}}(|xy|) = |xy|\,J_{q-\frac{3}{2}}(|xy|) - \left(q-\frac{1}{2}\right) J_{q-\frac{1}{2}}(|xy|)\,, \quad (7.23)$$

one may write, starting from (7.19) and (7.20),

$$\begin{aligned}(V_q((yw)'))(x) = -\int_{-\infty}^{\infty} & e^{i\pi\,(dx^2+ay^2)}\,|xy|^{\frac{1}{2}}\,(\operatorname{sign} xy)^q \\ & [\,2i\pi\,ay^2\,J_{q-\frac{1}{2}}(|xy|) + (1-q)\,J_{q-\frac{1}{2}}(|xy|) + |xy|\,J_{q-\frac{3}{2}}(|xy|)\,]\,w(y)\,dy\,, \quad (7.24)\end{aligned}$$

in other words

$$(V_q\,(yw' + (2-q)w + 2i\pi\,ay^2 w)\,)(x) = -x\,V_{q-1}(yw)\,. \quad (7.25)$$

Now, given any $u \in \mathcal{S}_q(\mathbb{R})$, solve the equation

$$yw' + (2-q)w + 2i\pi\,ay^2 w = u \quad (7.26)$$

with $w(0) = 0$, so that

$$w(y) = y^{q-2}\,e^{-i\pi ay^2} \int_0^y t^2\,e^{i\pi at^2}\,dt\,. \quad (7.27)$$

Then, with $\phi \in C_0^\infty(\mathbb{R})$ with $\phi(y) = 1$ in a neighbourhood of 0, u can be written as $u = u_1 + u_2$, where u_1 (*resp.* u_2) is the image under the operator $y\frac{d}{dy} + 2 - q + 2i\pi\,ay^2$ of the function ϕw (*resp.* $(1-\phi)\,w$). The function $V_q u_2$ lies in $V_q(\mathbb{R})$ because u_2 does and, moreover, vanishes in a neighbourhood of 0. From (7.25) one has $V_q u_1 = -x\,V_{q-1}\,(y\phi w)$, so that the proof of Theorem 7.9 follows by induction: the case when $q = 0$ reduces to the well-known fact that unitaries in the image of the (usual) metaplectic representation preserve the space $\mathcal{S}(\mathbb{R})$. □

We now come to the discussion of the adjoint action of unitaries in the p-metaplectic group on the operators P and Q: at the same time, this will provide us with an easy mnemonic for remembering the group homomorphism θ (a twofold covering) extending the map $\mathrm{Met}_p(g) \mapsto g$.

Theorem 7.10. *Given an operator*

$$\tilde{X} = \pi\,[\alpha\,(PQ+QP) + \beta\,P^2 + \gamma\,Q^2] \tag{7.28}$$

in the Lie algebra (over $\mathbb{R}$*) generated by the operators* P^2, Q^2, $PQ+QP$, *set*

$$X = \begin{pmatrix} -\alpha & -\beta \\ \gamma & \alpha \end{pmatrix} \in \mathfrak{g}\ : \tag{7.29}$$

this is the matrix of the operator $i\,ad\,\tilde{X}$, *acting on the space generated by* Q *and* P, *in the basis* $(-P, Q)$. *Given any operator* $-yP+\eta Q$ *in this space, one has, for every* $t \in \mathbb{R}$,

$$e^{it\tilde{X}}\,(-yP+\eta Q)\,e^{-it\tilde{X}} = -y'P+\eta' Q \tag{7.30}$$

with $\begin{pmatrix} y' \\ \eta' \end{pmatrix} = (\exp tX)\begin{pmatrix} y \\ \eta \end{pmatrix}$.

Proof. From Proposition 7.6, one has

$$\begin{aligned} [\tilde{X}, P] &= i\,(\alpha P + \gamma Q)\,, \\ [\tilde{X}, Q] &= -i\,(\alpha Q + \beta P)\,, \end{aligned} \tag{7.31}$$

which proves the first point. The second one is a consequence of the relation

$$e^{it\tilde{X}}\,(-yP+\eta Q)\,e^{-it\tilde{X}} = e^{it\,ad\,\tilde{X}}\,(-yP+\eta Q)\,, \tag{7.32}$$

where $i\,ad\,\tilde{X}$ is represented by the matrix X in the given basis. □

In particular, defining σ_τ by the equation (7.8), *i.e.*,

$$(\sigma_\tau u)(s) = 2\pi \int_0^\infty \left(\frac{s}{t}\right)^{\frac{\tau}{2}} J_\tau(4\pi\sqrt{st})\,u(t)\,dt\,, \tag{7.33}$$

one defines the p-Fourier transformation $\mathcal{F}_p$ as

$$\mathcal{F}_p = \Phi \begin{pmatrix} \sigma_{p-\frac{1}{2}} & 0 \\ 0 & -i\,\sigma_{p+\frac{1}{2}} \end{pmatrix} \Phi^{-1}\ : \tag{7.34}$$

then

$$\mathcal{F}_p = e^{\frac{i\pi}{2}(p+\frac{1}{2})}\,\exp\left(-\frac{i\pi}{2}\,(\pi\,(Q^2+P^2))\right), \tag{7.35}$$

and

$$\mathcal{F}_p^{-1}\, Q\, \mathcal{F}_p = P\,. \tag{7.36}$$

When $p = 0$, $\mathcal{F}_p$ is the usual Fourier transformation $\mathcal{F}$ on $L^2(\mathbb{R})$, defined as

$$(\mathcal{F}\, u)(y) = \int e^{-2i\pi\, xy}\, u(x)\, dx\,. \tag{7.37}$$

In all cases, $\mathcal{F}_p^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, the block-matrix representation of the map $u \mapsto \check{u}$.

Making $\mathcal{F}_p$ explicit will help understand in which sense it generalizes the Fourier transformation.

Proposition 7.11. *Let $u \in C_0^\infty(\mathbb{R}\backslash\{0\})$. Assume that p is even. If u is even, one has*

$$(\mathcal{F}_p u)(x) = 2\pi \int_0^\infty (|x|y)^{\frac{1}{2}}\, J_{p-\frac{1}{2}}(2\pi|x|y)\, u(y)\, dy \tag{7.38}$$

and if u is odd,

$$(\mathcal{F}_p u)(x) = -2i\pi\, (\mathrm{sign}\, x) \int_0^\infty (|x|y)^{\frac{1}{2}}\, J_{p+\frac{1}{2}}(2\pi|x|y)\, u(y)\, dy\,. \tag{7.39}$$

Proof. From (7.34) and the Definition (7.16) of Φ, one gets

$$\mathcal{F}_p = Sq_{\text{even}}^{p+\frac{1}{2}}\ \sigma_{p-\frac{1}{2}}\ (Sq_{\text{even}}^{p+\frac{1}{2}})^{-1} \qquad \text{on even functions}$$

and

$$\mathcal{F}_p = -i\, Sq_{\text{odd}}^{p+\frac{3}{2}}\ \sigma_{p+\frac{1}{2}}\ (Sq_{\text{odd}}^{p+\frac{3}{2}})^{-1} \qquad \text{on odd functions}\,. \tag{7.40}$$

Proposition 7.11 is then a consequence of (7.33) together with the Definition (6.9), (6.10) of $Sq_{\text{even}}^{p+\frac{1}{2}}$ and $Sq_{\text{odd}}^{p+\frac{3}{2}}$: the computation is straightforward, if somewhat tedious. □

From (7.32), it is clear that the map $\theta : \pm\mathrm{Met}_p(g) \mapsto g \in SL(2,\mathbb{R})$ is characterized as follows: with $U = \pm\mathrm{Met}_p(g)$, $\theta(U)$ is the matrix of the transformation $Ad\, U$:

$$(Ad\, U)(-yP + \eta Q) = U\,(-yP + \eta Q)\, U^{-1} \tag{7.41}$$

in the basis $(-P, Q)$.

Now, consider the effect of p-metaplectic unitaries (*i.e.*, operators $\omega\,\mathrm{Met}_p(g)$ for some $g \in SL(2,\mathbb{R})$ and $\omega \in \mathbb{C}$, $|\omega| = 1$) on functions u_z^p or u_z^{p+1} as introduced in (6.13).

Theorem 7.12. *Given a p-metaplectic unitary U, with $\theta(U)$ (as defined above (7.41)) $= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, one has*

$$U\, u_z^p = \omega\, u^p_{\frac{az+b}{cz+d}} \tag{7.42}$$

and

$$U\, u_z^{p+1} = \omega'\, u^{p+1}_{\frac{az+b}{cz+d}} \tag{7.43}$$

for some "phase factors" ω or $\omega' \in \mathbb{C}$, with $|\omega| = |\omega'| = 1$.

Proof. This is a consequence of (6.13) and of the Definition 7.8 of the p-metaplectic representation: however, we prefer to give a direct proof for a set of generators of the p-metaplectic group (this is sufficient), computing the phase factors in the cases considered. If $U = \exp(i\gamma\,.\pi\, Q^2)$, so that $\theta(U) = \left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right)$, one immediately gets, according to Proposition 7.4,

$$\begin{aligned}(U\, u_z^p)(x) &= e^{i\pi\,\gamma\, x^2}\, u_z^p(x) \\ &= u^p_{z'}(x)\end{aligned} \tag{7.44}$$

with $\frac{1}{\bar{z}'} = \frac{1}{\bar{z}} + \gamma$, *i.e.*, $z' = \frac{z}{\gamma z+1} = \left(\begin{smallmatrix} 1 & 0 \\ \gamma & 1 \end{smallmatrix}\right).z$, and the same goes for the action on u_z^{p+1}. If $U = \exp(i\alpha\,.\pi\,(PQ + QP))$, so that $\theta(U) = \left(\begin{smallmatrix} e^{-\alpha} & 0 \\ 0 & e^{\alpha} \end{smallmatrix}\right)$, one gets, starting from (6.13) and using Proposition 7.4 again,

$$(U\, u_z^p)(x) = u^p_{e^{-2\alpha} z}(x)\,, \tag{7.45}$$

and the same goes with the odd states u_z^{p+1}.

Things are more complicated in the case when $U = \exp(i\beta\,.\pi\, P^2)$, so that $\theta(U) = \left(\begin{smallmatrix} 1 & -\beta \\ 0 & 1 \end{smallmatrix}\right)$. Assuming, say, $\beta < 0$, and using, with $\tau = p \pm \frac{1}{2}$, the formula provided by (7.19), we get, with $C = \left(\frac{(2\pi)^{\tau+1}}{\Gamma(\tau+1)}\right)^{\frac{1}{2}}$, the equation

$$\begin{aligned}(U_\tau\, u_z^{\tau+\frac{1}{2}})(x) = C \left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{\frac{\tau+1}{2}} & e^{-i\pi\frac{\tau+1}{2}}\, \frac{\pi}{|\beta|} \int_{-\infty}^{\infty} |xy|^{\frac{1}{2}}\, (\mathrm{sign}\, xy)^j \\ &\exp\left(-i\pi\, \frac{x^2+y^2}{\beta}\right)\, J_\tau\left(\frac{2\pi\,|xy|}{|\beta|}\right)\, y^{\tau+\frac{1}{2}}\, e^{i\pi\frac{y^2}{\bar{z}}}\, dy\,,\end{aligned} \tag{7.46}$$

with $j = \tau - p + \frac{1}{2}$ ($= 0$ or 1). This integral is essentially found in [31, p. 93], which yields the result

$$\begin{aligned}(U_\tau\, u_z^{\tau+\frac{1}{2}})(x) = C \left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{\frac{\tau+1}{2}} & \left(-i\left(1-\frac{\beta}{\bar{z}}\right)\right)^{-\tau-1}_{\text{right}} \\ e^{-i\pi\frac{\tau+1}{2}}\, |x|^{\tau+\frac{1}{2}}\, (\mathrm{sign}\, x)^j\, &\exp\left(i\pi\, x^2\left(\frac{1}{\beta(1-\frac{\beta}{\bar{z}})} - \frac{1}{\beta}\right)\right)\,,\end{aligned} \tag{7.47}$$

where the subscript in $(\ )_{\text{right}}^{-\tau-1}$ means that the principal determination of the power function on the right-half plane is to be chosen. Now, with $\frac{1}{\bar{z}'} = \frac{1}{\beta(1-\frac{\beta}{\bar{z}})} - \frac{1}{\beta}$, so that $z' = z - \beta$, one has $\mathrm{Im}\,(-\frac{1}{z'}) = |1 - \frac{\beta}{\bar{z}}|^{-2}\,\mathrm{Im}\,(-\frac{1}{z})$ so that, with $\zeta = \frac{i\,(\frac{\beta}{\bar{z}}-1)}{|\frac{\beta}{\bar{z}}-1|}$ and $\omega = e^{-i\pi\frac{\tau+1}{2}}\,\zeta_{\text{right}}^{-\tau-1}$, we get $|\omega| = 1$ and the relation $U_\tau\, u_z^{\tau+\frac{1}{2}} = \omega\, u_{z-\beta}^{\tau+\frac{1}{2}}$. If $\beta > 0$, the formula is almost the same: only $e^{-i\pi\frac{\tau+1}{2}}$ has to be changed to its inverse.

We shall also need the formulas

$$\mathcal{F}_p^{-1} u_z^p = \left(\frac{-iz}{|z|}\right)_{\text{right}}^{-\frac{1}{2}-p} u_{-\frac{1}{z}}^p$$

and

$$\mathcal{F}_p^{-1} u_z^{p+1} = i\left(\frac{-iz}{|z|}\right)_{\text{right}}^{-\frac{3}{2}-p} u_{-\frac{1}{z}}^{p+1}, \tag{7.48}$$

proved in the same way, starting from (7.38), (7.39). □

In connection with the representation Met_p, the families of functions $(u_z^p)_{z\in\Pi}$ and $(u_z^{p+1})_{z\in\Pi}$ will be used in exactly the same way the families (u_z) and (u_z^1) have been used in connection with the usual metaplectic representation.

Proposition 7.13. *Assume that p is even. Then the set $(u_z^p)_{z\in\Pi}$ is total in $(\mathcal{S}^p(\mathbb{R}))_{\text{even}}$ and the set $(u_z^{p+1})_{z\in\Pi}$ is total in $(\mathcal{S}^p(\mathbb{R}))_{\text{odd}}$.*

Proof. Dividing functions in the appropriate space by x^p or x^{p+1} reduces the question to the case when $p = 0$, in which it is well known to be true. □

In all that precedes, and in all to be coming with the exception of Section 12 (which depends on induction on p starting from $p = 0$), the assumption that p is an integer is not necessary: we now substitute for p any real number s (one should assume $s > -\frac{1}{2}$ if an appeal to the projective representation $\mathcal{D}_{s+\frac{1}{2}}$ is needed) and extend Proposition 7.4 as a definition, setting in particular

$$P_s = \frac{1}{2i\pi}\begin{pmatrix} 0 & \frac{d}{dr} + \frac{s}{r} \\ \frac{d}{dr} - \frac{s}{r} & 0 \end{pmatrix}: \tag{7.49}$$

we shall regard P_s as acting on functions u only defined on $(0,\infty)$, which is not a genuine difference from our first interpretation, in which each component of u was assumed to have a definite parity; observe that we changed the variable x to r.

The spectral structure of the operator P_s (and, as a consequence of the analogue of (7.30) concerning the s-metaplectic representation, that of any operator in the real vector space generated by P_s and $Q_s = Q$) is of course an easy matter, in view of the formula

$$P_s^2 = -\frac{1}{4\pi^2}\begin{pmatrix} \frac{d^2}{dr^2} - \frac{s(s-1)}{r^2} & 0 \\ 0 & \frac{d^2}{dr^2} - \frac{s(s+1)}{r^2} \end{pmatrix} \tag{7.50}$$

taken from (7.13).

Another interesting operator is the operator $\Lambda = \pi\,(P_s^2 + Q^2)$, which occurs in (7.35), hereafter referred to as the Op^s-version of the harmonic oscillator. From (7.13),

$$\Lambda = \begin{pmatrix} L_s & 0 \\ 0 & L_{s+1} \end{pmatrix} \tag{7.51}$$

with

$$L_s = -\frac{1}{4\pi}\left[\frac{d^2}{dr^2} - \frac{s(s-1)}{r^2} - 4\pi^2 r^2\right]. \tag{7.52}$$

The standard WKB method shows that, among the solutions of $(L_s - \lambda)\,\phi = 0$ on $]0, \infty[$, there is one which goes to zero, as $r \to \infty$, like $r^{\lambda - \frac{1}{2}} e^{-\pi r^2}$ and another one which goes to infinity like $r^{\lambda-\frac{1}{2}} e^{\pi r^2}$. On the other hand, at $r = 0$, the roots of the indicial equation are s and $1-s$. *We assume that $s > \frac{1}{2}$ from now on.* If $s > \frac{3}{2}$, it is clear from what precedes that L_s, initially defined on $C_0^\infty(]0, \infty[)$, is essentially self-adjoint on $L^2((0, \infty); dr)$; if $\frac{1}{2} < s \leq \frac{3}{2}$, we fix a self-adjoint extension of L_s by the boundary condition $r\phi' - s\phi = 0$ at 0, which selects the solutions of $(L_s - \lambda)\,\phi = 0$ behaving like r^s at zero. One can show that L_s is unitarily equivalent to the operator which is the generator, in the sense of Stone's theorem, of the unitary group $\theta \mapsto \mathcal{D}_{s+\frac{1}{2}}(\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right))$ associated with the representation $\mathcal{D}_{s+\frac{1}{2}}$ from the projective discrete series. However, it suffices for our purposes to note that the spectrum of L_s is the sequence $(s + \frac{1}{2} + 2n)_{n=0,1,\dots}$ and that an eigenfunction corresponding to the eigenvalue $s + \frac{1}{2} + 2n$ is the function

$$\phi_n(r) = e^{-\pi r^2}\, r^s\, L_n^{(s-\frac{1}{2})}(2\pi r^2)\,, \tag{7.53}$$

which can be found in [31, p. 243]: here $L_n^{(s-\frac{1}{2})}$ is a generalized Laguerre polynomial.

The matrix structure, on the other hand, of the operator P_s, makes the study of the "generalized eigenvalue equation" $P_s u = Mu$ in which M may be a general hermitian matrix, interesting as well. The following proposition will be useful in the next section.

Proposition 7.14. *Assume $s > \frac{1}{2}$, and let $M = \left(\begin{smallmatrix} a & -ib \\ ib & d \end{smallmatrix}\right)$ be an arbitrary hermitian matrix with $d \neq 0$. The equation*

$$P_s u = \frac{1}{2}\, Mu \tag{7.54}$$

has a non-trivial solution in the space $(L^2((0,\infty);dr))^2$ *if and only if* $b>0$ *and the condition*

$$\det M + \left(\frac{bs}{s+n}\right)^2 = 0 \tag{7.55}$$

is satisfied for some $n=0,1,\ldots$. *If* (7.55) *holds, the corresponding space of solutions is one-dimensional, generated by*

$$u_n(r) = e^{-\pi\mu r}\begin{pmatrix} r^s\, L_n^{(2s-1)}(2\pi\mu r) \\ \frac{2\pi\mu^2}{isd}\, r^{s+1}\, L_{n-1}^{(2s+1)}(2\pi\mu r)\end{pmatrix} \tag{7.56}$$

where $\mu = (-\det M)^{\frac{1}{2}} = \frac{bs}{s+n}$: *the second row should be interpreted as zero in the case when* $n=0$.

Proof. Assume that $\det M < 0$ to start with, and set $\det M = -\mu^2$, $\mu>0$. Consider the isometry $\theta\colon (L^2((0,\infty);dr))^2 \to (L^2((0,\infty);\frac{2x^2}{\mu}\,dx))^2$ defined by

$$(\theta u)(x)\colon = x^{-\frac{1}{2}}\, u\left(\frac{x^2}{\mu}\right)$$

or

$$(\theta^{-1}v)(r) = (\mu r)^{\frac{1}{4}}\, v((\mu r)^{\frac{1}{2}})\,. \tag{7.57}$$

A straightforward computation shows that

$$\theta\, P_s\, \theta^{-1} = \frac{1}{4i\pi}\begin{pmatrix} 0 & \frac{\mu}{x}\frac{d}{dx} + \frac{\mu(\frac{1}{2}+2s)}{x^2} \\ \frac{\mu}{x}\frac{d}{dx} + \frac{\mu(\frac{1}{2}-2s)}{x^2} & 0\end{pmatrix}. \tag{7.58}$$

Thus, with $v=\theta u$, the equation $P_s u = \frac{1}{2}\, Mu$ is equivalent to

$$\mathcal{B}_s v = \mu^{-1} M\, x\, v \tag{7.59}$$

with

$$\begin{aligned}\mathcal{B}_s\colon &= \frac{1}{2i\pi}\begin{pmatrix} 0 & \frac{d}{dx} + \frac{\frac{1}{2}+2s}{x} \\ \frac{d}{dx} + \frac{\frac{1}{2}-2s}{x} & 0\end{pmatrix} \\ &= (x^{-\frac{1}{2}})\, P_{2s}\, (x^{\frac{1}{2}})\,,\end{aligned} \tag{7.60}$$

where $(x^{\frac{1}{2}})$ denotes the operator of multiplication by $x^{\frac{1}{2}}$. We also set

$$\mathcal{B}_s^* = \frac{1}{2i\pi}\begin{pmatrix} 0 & \frac{d}{dx} + \frac{2s-\frac{1}{2}}{x} \\ \frac{d}{dx} - \frac{2s+\frac{1}{2}}{x} & 0\end{pmatrix}: \tag{7.61}$$

this is the adjoint of $\mathcal{B}_s$ in $(L^2((0,\infty);dx))^2$, not in the Hilbert space referred to in the beginning of the present proof. It is immediate that

$$\mathcal{B}_s^* \mathcal{B}_s = -\frac{1}{4\pi^2}\begin{pmatrix} \frac{d^2}{dx^2} - \frac{(2s-\frac{1}{2})(2s-\frac{3}{2})}{x^2} & 0 \\ 0 & \frac{d^2}{dx^2} - \frac{(2s+\frac{1}{2})(2s+\frac{3}{2})}{x^2}\end{pmatrix}. \tag{7.62}$$

Also, setting

$$M^{\mathrm{cof}} = \begin{pmatrix} d & ib \\ -ib & a \end{pmatrix}, \tag{7.63}$$

so that in particular $M^{\mathrm{cof}}\, M = -\mu^2 I$, a straightforward computation shows that

$$M^{\mathrm{cof}}(x)\,\mathcal{B}_s - \mathcal{B}_s^*\, M(x) = -\frac{2bs}{\pi}\, I\ : \tag{7.64}$$

thus

$$\pi\,(\mathcal{B}_s^* + \mu^{-1} M^{\mathrm{cof}}(x))\,(\mathcal{B}_s - \mu^{-1} M(x)) = \pi\,\mathcal{B}_s^*\,\mathcal{B}_s + \pi\,(x^2) - \frac{2bs}{\mu}\, I\,, \tag{7.65}$$

and (7.59) implies

$$\pi\,(\mathcal{B}_s^*\,\mathcal{B}_s + (x^2))\, v = \frac{2bs}{\mu}\, v\,. \tag{7.66}$$

At this point, we may briefly remark that if $\det M$ had been non-negative, we would have been led with the help of analogous transformations to a second-order differential equation without any L^2 solution near infinity (the same left-hand side with a term $-(x^2)$, or no such term at all, instead of (x^2)). It was thus no loss of generality to assume that $\det M < 0$, and we now come back to the discussion of this case.

Comparing (7.62) and (7.52), one sees that

$$\pi\,(\mathcal{B}_s^*\,\mathcal{B}_s + (x^2)) = \begin{pmatrix} L_{2s-\frac{1}{2}} & 0 \\ 0 & L_{2s+\frac{3}{2}} \end{pmatrix}. \tag{7.67}$$

Actually, with $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$, the problem $\mathcal{B}_s v = \mu^{-1} M x v$ fully reduces – formally to start with – to the eigenfunction equation

$$L_{2s-\frac{1}{2}}\, v_1 = \frac{2bs}{\mu}\, v_1\,, \tag{7.68}$$

since v_2 can then be obtained as

$$v_2 = \left[\frac{\mu}{2i\pi d}\left(x^{-1}\frac{d}{dx} + \frac{\frac{1}{2} - 2s}{x^2}\right) - \frac{ib}{d}\right] v_1\,. \tag{7.69}$$

Since $2s - \frac{1}{2} > \frac{1}{2}$ as well, we discussed the equation (7.68) in $L^2((0,\infty);dx)$ just before the statement of Proposition 7.14. It should be observed that since

$s > \frac{1}{2}$, only a solution v_1 equivalent near 0 to a constant times $x^{2s-\frac{1}{2}}$, not $x^{\frac{3}{2}-2s}$, can yield a v_2 in the required space $L^2((0,1); x^2\,dx)$ near zero: it is thus, even when $s \leq \frac{3}{2}$, the self-adjoint extension of $L_{2s-\frac{1}{2}}$ discussed before the statement of Proposition 7.14 that we are indeed interested in.

We thus get, from (7.53),

$$bs = \mu\,(s+n) \qquad \text{for some } n = 0, 1, \dots \tag{7.70}$$

and, up to a multiplicative constant,

$$v_1(x) = e^{-\pi x^2}\, x^{2s-\frac{1}{2}}\, L_n^{(2s-1)}(2\pi x^2)\,, \tag{7.71}$$

which implies (using (7.69) and (7.70))

$$\frac{id}{\mu}\, v_2(x) = e^{-\pi x^2}\, x^{2s-\frac{1}{2}} \left[2\,(L_n^{(2s-1)})'(2\pi x^2) + \frac{n}{s}\, L_n^{(2s-1)}(2\pi x^2)\right]. \tag{7.72}$$

Since (a consequence of some formulas in [31, p. 241])

$$2s\,\frac{d}{dt}\, L_n^{(2s-1)}(t) + n\, L_n^{(2s-1)}(t) + t\, L_{n-1}^{(2s+1)}(t) = 0\,, \tag{7.73}$$

one has

$$v_2(x) = -\frac{2i\pi\mu}{sd}\, e^{-\pi x^2}\, x^{2s+\frac{3}{2}}\, L_{n-1}^{(2s+1)}(2\pi x^2) \tag{7.74}$$

if $n \geq 1$, and $v_2 = 0$ if $n = 0$. Computing $\mu^{-s}\,\theta^{-1} v$, we are done. □

Remark. The Op^s-version of the harmonic oscillator Λ, introduced in (7.51), has a neater structure than any of its two non-zero entries, actually just as neat as that of the ordinary harmonic oscillator. For its eigenfunctions can be constructed by the usual procedure applicable to Hermite functions: the proof of the proposition that follows is immediate by induction. It should be emphasized that it works notwithstanding the fact that Q and P_s do not generate a finite-dimensional Lie algebra in the case when $s \neq 0$.

Proposition 7.15. *Set $A = \pi^{\frac{1}{2}}(Q + iP_s)$ (the annihilation operator) and $A^* = \pi^{\frac{1}{2}}(Q - iP_s)$ (the creation operator, adjoint to the preceding one in $(L^2((0,\infty); dr))^2$). One has*

$$\begin{aligned} \Lambda &= A^*A + \begin{pmatrix} \frac{1}{2}+s & 0 \\ 0 & \frac{1}{2}-s \end{pmatrix} \\ &= AA^* - \begin{pmatrix} \frac{1}{2}+s & 0 \\ 0 & \frac{1}{2}-s \end{pmatrix}. \end{aligned} \tag{7.75}$$

Set $\Psi_0(r) = \begin{pmatrix} r^s e^{-\pi r^2} \\ 0 \end{pmatrix}$ and, for $k \geq 0$, $\Psi_k = {A^*}^k \Psi_0$. Then $A\Psi_0 = 0$; also

$$A\Psi_k = \begin{cases} k\,\Psi_{k-1} &, \quad k \text{ even} \geq 2 \\ (k+2s)\,\Psi_{k-1} &, \quad k \text{ odd} \end{cases}$$

and

$$\Lambda \Psi_k = (k + \frac{1}{2} + s)\, \Psi_k, \qquad \text{all } k \geq 0\,. \tag{7.76}$$

Finally

$$\begin{aligned} \|\Psi_{2j+1}\|^2 &= (2j+1+s)\, \|\Psi_{2j}\|^2\,, \\ \|\Psi_{2j+2}\|^2 &= (2j+2)\, \|\Psi_{2j+1}\|^2\,, \qquad j = 0, 1, \dots \end{aligned} \tag{7.77}$$

and the sequence of functions $\left(\frac{\Psi_k}{\|\Psi_k\|}\right)_{k\geq 0}$ *constitutes an orthonormal basis of the space* $(L^2((0,\infty); dr))^2$.

8 The radial parts of relativistic wave operators

The present section is an excursion into quantum mechanics, leading to another interpretation of the constructions from the last section: it is unrelated to modular form, or automorphic distribution, theory.

It is a very classical fact that, when the space $L^2(\mathbb{R}^d)$ is decomposed according to the action of the rotation group, the Fourier transformation (that associated with the Euclidean structure of $\mathbb{R}^d$) decomposes into summands, each of which is related under the elementary change of coordinate $s \mapsto s^2$ to the Hankel transform $\pi(g)$ as defined in (7.2), with $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$: more specifically, if $r = |x|$, $x \in \mathbb{R}^d$ and f is the product of a function $U(r)$ by some harmonic homogeneous polynomial of degree k, then the Fourier transform of f is $V(r)$ times the same polynomial, with

$$V(r) = 2\pi\, i^{-k}\, r^{1-\frac{d}{2}-k} \int_0^\infty U(t)\, t^{\frac{d}{2}+k}\, J_{\frac{d}{2}-1+k}(2\pi\, rt)\, dt\ : \tag{8.1}$$

note that the subscript $\tau = \frac{d}{2} - 1 + k$ is an integer or half an integer according to whether the dimension d is even or odd. The whole representation $\pi = \mathcal{D}_{\tau+1}$ can also be interpreted as a summand in the decomposition of the part of the metaplectic representation on $L^2(\mathbb{R}^d)$ commuting with rotations.

In the preceding section, we have been led to analyzing a certain realization of the direct sum of $\mathcal{D}_{p+\frac{1}{2}}$ and $\mathcal{D}_{p+\frac{3}{2}}$: in the case when $p = 0$, it was especially natural to identify the spaces $H_{\frac{1}{2}}$ and $H_{\frac{3}{2}}$ with the even and odd parts of $L^2(\mathbb{R})$. We now give an alternative interpretation of the direct sum of $\mathcal{D}_{p+\frac{1}{2}}$ and $\mathcal{D}_{p+\frac{3}{2}}$ in the case when $p \geq 1$, showing that the operator P as introduced in (7.13), or a certain linear combination of P and $[P, Q]$, can be thought of as a radial part of a first-order system occurring in a wave equation from relativistic quantum mechanics.

The analogy between the original construction of the Dirac wave equation and our definition of the operators P, Q is related to the fact that, in both cases, a "square root construction" is needed. In our case, Q^2, $PQ + QP$ and P^2 were known before P and Q were (and, indeed, are directly related to the infinitesimal operators of some pair of representations from the discrete series of $SL(2,\mathbb{R})$). We very briefly recall now how the Dirac equation and, in a similar way, the Weyl equation (possibly for the neutrino: it describes a massless particle and has no invariance under the parity transform, which makes it suitable for the description of a particle playing a role in weak interactions) answer a square root problem: we shall use results and methods from Thaller's book [50], in particular pages 125 to 129.

With $\psi = \psi(t,\boldsymbol{x})$, $t \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, both wave equations can be written (*loc.cit.,* p. 3 and 4) as

$$i\hbar\,\frac{\partial\psi}{\partial t} = H_0\,\psi\,, \tag{8.2}$$

with an operator H_0 acting only in the space coordinates $\boldsymbol{x}$ to be defined now. Recall the definition of Pauli matrices

$$\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \qquad \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \qquad \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\ : \tag{8.3}$$

one also introduces the vectors $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ as well as their "scalar product" $\boldsymbol{\sigma}.\nabla = \sum \sigma_j \frac{\partial}{\partial x_j}$. Then, Weyl's equation is just (8.2) with the choice

$$H_0 = -i\hbar c\,\boldsymbol{\sigma}.\nabla\,, \tag{8.4}$$

in other words, with $\partial_j = \frac{\partial}{\partial x_j}$,

$$H_0 = -i\hbar c \begin{pmatrix} \partial_3 & \partial_1 - i\,\partial_2 \\ \partial_1 + i\,\partial_2 & -\partial_3 \end{pmatrix}\ : \tag{8.5}$$

of course, $\hbar$ and c denote Planck's constant and the velocity of light respectively. It is immediate that

$$H_0^2 = -\hbar^2 c^2 \begin{pmatrix} \boldsymbol{\Delta} & 0 \\ 0 & \boldsymbol{\Delta} \end{pmatrix}\,, \tag{8.6}$$

where $\boldsymbol{\Delta}$ is the usual (negative !) Laplacian on $\mathbb{R}^3$. Thus the construction of H_0 just answers the problem of extracting a square root of (some two-dimensional matrix extension of) $\boldsymbol{\Delta}$. We also denote as $\mathfrak{Q}$ the operator

$$\mathfrak{Q} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}\,, \tag{8.7}$$

a square root of the corresponding two-dimensional matrix extension of the operator of multiplication by $|x|^2$.

Remarks.

1. One may observe that the operators H_0^2, $\mathfrak{Q}^2$ and

$$\begin{aligned}[H_0^2\,,\mathfrak{Q}^2] &= -4\,\hbar^2 c^2 \left(\sum x_j\partial_j + \frac{3}{2}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ &= -2i\hbar c\,(H_0\,\mathfrak{Q} + \mathfrak{Q}\,H_0)\end{aligned} \tag{8.8}$$

constitute a linear basis of the set of infinitesimal operators of some (reducible) unitary representation of the twofold covering of $SL(2,\mathbb{R})$.

2. Contrary to the triple $\{H_0^2\,,\,\mathfrak{Q}^2\,,\,H_0\,\mathfrak{Q}+\mathfrak{Q}\,H_0\}$, H_0 and $\mathfrak{Q}$ do not fit within any finite-dimensional Lie algebra. However, under the adjoint action, the Lie algebra above acts on the linear space generated by H_0 and $\mathfrak{Q}$, in view of the formulas

$$\begin{aligned}-[\mathfrak{Q}^2\,,\,H_0] &= [H_0\,\mathfrak{Q}+\mathfrak{Q}\,H_0\,,\,\mathfrak{Q}] = -2i\hbar c\,\mathfrak{Q}\,,\\ [H_0^2\,,\,\mathfrak{Q}] &= -[H_0\,\mathfrak{Q}+\mathfrak{Q}\,H_0\,,\,H_0] = -2i\hbar c\,H_0\ : \end{aligned}\tag{8.9}$$

these formulas are completely analogous to (7.15), which will be explained by Theorem 8.1.

In the case of Dirac's equation, four-component spinors are needed: then, $\mathbf{1}$ denoting the 2×2-identity matrix and $\boldsymbol{\sigma}.\nabla$ denoting also the diagonal 2×2-matrix the non-zero entries of which coincide with $\boldsymbol{\sigma}.\nabla$ as previously defined, one sets

$$H_0 = \begin{pmatrix} mc^2\,\mathbf{1} & -i\hbar c\,\boldsymbol{\sigma}.\nabla \\ -i\hbar c\,\boldsymbol{\sigma}.\nabla & -mc^2\,\mathbf{1}\end{pmatrix}\ : \tag{8.10}$$

m is a positive number, to be interpreted as the mass of the electron. It is then immediate that

$$H_0^2 = \begin{pmatrix} (-\hbar^2c^2\,\boldsymbol{\Delta}+m^2c^4)\,\mathbf{1} & 0 \\ 0 & (-\hbar^2c^2\,\boldsymbol{\Delta}+m^2c^4)\,\mathbf{1}\end{pmatrix}\,, \tag{8.11}$$

showing, as is of course well known, that in this case too the construction of H_0 answers a square root problem.

We now show that, in the Weyl case, the operators which occur in the decomposition of the operator H_0 under the rotation group (more precisely under its twofold covering $SU(2)$) are nothing but cases of the operator P introduced in the present section: with a slight modification, the same will do in connection with the Dirac equation.

Following Thaller's notation and methods, we introduce polar coordinates with $\boldsymbol{x}(r,\theta,\phi)$ defined by

$$\begin{aligned} x_1(r,\theta,\phi) &= r\,\sin\theta\,\cos\phi\,,\\ x_2(r,\theta,\phi) &= r\,\sin\theta\,\sin\phi\,,\\ x_3(r,\theta,\phi) &= r\,\cos\theta\,, \end{aligned}\tag{8.12}$$

and the vector $\boldsymbol{e}_r = \frac{\boldsymbol{x}}{r}$, the unit vector in the radial direction. Then, with $\boldsymbol{p} = -i\hbar\,\nabla$ and $\boldsymbol{L} = \boldsymbol{x}\wedge\boldsymbol{p}$ (the *orbital* angular momentum), one has [50, (4.101)]

$$-i\hbar\,\nabla = -i\hbar\,\boldsymbol{e}_r\,\frac{\partial}{\partial r} - \frac{1}{r}\,(\boldsymbol{e}_r\wedge\boldsymbol{L})\ : \tag{8.13}$$

this equation holds for scalar functions on $\mathbb{R}^3$ or, just as well, componentwise for vector-valued functions. Given any two vectors $\boldsymbol{A}$, $\boldsymbol{B}$ in $\mathbb{C}^3$, one has the immediate relation

$$(\boldsymbol{\sigma}.\boldsymbol{A})\,(\boldsymbol{\sigma}.\boldsymbol{B}) = (\boldsymbol{A}.\boldsymbol{B})\,\mathbf{1} + i\,\boldsymbol{\sigma}.(\boldsymbol{A}\wedge\boldsymbol{B})\,, \tag{8.14}$$

so that (8.13) implies

$$-i\hbar\,\boldsymbol{\sigma}.\nabla = -i\hbar\,(\boldsymbol{\sigma}.\boldsymbol{e}_r)\,\frac{\partial}{\partial r} + \frac{i}{r}\,(\boldsymbol{\sigma}.\boldsymbol{e}_r)\,(\boldsymbol{\sigma}.\boldsymbol{L})\ : \tag{8.15}$$

thus, in Weyl's case,

$$H_0 = -ic\,(\boldsymbol{\sigma}.\boldsymbol{e}_r)\left(\hbar\,\frac{\partial}{\partial r} - \frac{1}{r}\,\boldsymbol{\sigma}.\boldsymbol{L}\right)\,, \tag{8.16}$$

the analogue of *loc.cit.*, (4.104). It is immediate, from (8.5) and (8.7), to compute the commutator

$$[H_0, \mathfrak{Q}] = \frac{c}{i}\,(3\hbar + 2\boldsymbol{\sigma}.\boldsymbol{L})\,. \tag{8.17}$$

Next, a separation of variables is considered, embedding $L^2((0,\infty), dr)\otimes L^2(S^2)$ (where S^2 is the unit sphere) into $L^2(\mathbb{R}^3)$ via the map such that $f\otimes\Psi\mapsto\psi$, with

$$\psi(\boldsymbol{x}(r,\theta,\phi)) = r^{-1}f(r)\,\Psi(\theta,\phi)\,. \tag{8.18}$$

On S^2, the usual (scalar) spherical harmonics Y_l^m, with $l = 0, 1, 2, \dots$ and $m = -l, -l+1, \dots, +l$, are considered: recall (*loc.cit.*, p. 126) that, for $m \geq 0$,

$$Y_l^m(\theta,\phi) = \left[\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} e^{im\phi}\,P_l^m(\cos\theta) \tag{8.19}$$

in terms of the Legendre polynomials

$$P_l^m(x) = \frac{(-1)^m}{2^l\,l!}\,(1-x^2)^{\frac{m}{2}}\,\frac{d^{m+l}}{dx^{m+l}}\,(x^2-1)^l\,, \tag{8.20}$$

and that

$$Y_l^{-m} = (-1)^m\,\overline{Y_l^m}\,. \tag{8.21}$$

Next, one introduces, for $j = \frac{1}{2}, \frac{3}{2}, \dots$ and $m_j = -j, -j+1, \dots, +j$, the two-component functions $\Psi_{j\pm\frac{1}{2}}^{m_j}$ on S^2, with

$$\Psi_{j-\frac{1}{2}}^{m_j} = \frac{1}{\sqrt{2j}}\begin{pmatrix}\sqrt{j+m_j}\,Y_{j-\frac{1}{2}}^{m_j-\frac{1}{2}}\\ \sqrt{j-m_j}\,Y_{j-\frac{1}{2}}^{m_j+\frac{1}{2}}\end{pmatrix} \tag{8.22}$$

and

$$\Psi_{j+\frac{1}{2}}^{m_j} = \frac{1}{\sqrt{2j+2}}\begin{pmatrix}\sqrt{j+1-m_j}\,Y_{j+\frac{1}{2}}^{m_j-\frac{1}{2}}\\ -\sqrt{j+1+m_j}\,Y_{j+\frac{1}{2}}^{m_j+\frac{1}{2}}\end{pmatrix} \tag{8.23}$$

(again, with Thaller's notation). It has been shown in *loc.cit.*, p. 127, that the functions $\Psi^{m_j}_{j\pm\frac{1}{2}}$ are simultaneous eigenfunctions of the operators (well defined on $(L^2(S^2))^2$) $\hbar^{-1}J^2 := \hbar^{-1}L^2 + \boldsymbol{\sigma}.\boldsymbol{L} + \frac{3}{4}\hbar$ (where L^2 is the sum of squares of the components of $\boldsymbol{L}$), $J_3 := L_3 + \hbar\sigma_3/2$ and $\boldsymbol{\sigma}.\boldsymbol{L} + \hbar$: the eigenvalues are $\hbar$ times respectively $j(j+1)$, m_j and $-\kappa$, where $\kappa = -(j+\frac{1}{2})$ in the case of the function $\Psi^{m_j}_{j-\frac{1}{2}}$ and $\kappa = j + \frac{1}{2}$ in the case of the function $\Psi^{m_j}_{j+\frac{1}{2}}$. Also (*loc.cit.*, (4.121)),

$$\boldsymbol{\sigma}.\boldsymbol{e}_r\, \Psi^{m_j}_{j\pm\frac{1}{2}} = \Psi^{m_j}_{j\mp\frac{1}{2}}\,. \tag{8.24}$$

The Hilbert space $(L^2(S^2))^2$ thus appears as the Hilbert direct sum of the spaces $\mathfrak{H}_{j,m_j}$ with $j = \frac{1}{2}, \frac{3}{2}, \dots$ and $m_j = -j, \dots, +j$, where $\mathfrak{H}_{j,m_j}$ is the two-dimensional space generated by the functions $\Psi^{m_j}_{j\pm\frac{1}{2}}$. Then the space $(L^2(\mathbb{R}^3))^2$ appears as the Hilbert direct sum of the subspaces $L^2((0,\infty), dr)\otimes\mathfrak{H}_{j,m_j}$ if each of these spaces is identified with its image under the embedding (8.17): by means of the basis $\{\Psi^{m_j}_{j-\frac{1}{2}}, \Psi^{m_j}_{j+\frac{1}{2}}\}$ of $\mathfrak{H}_{j,m_j}$, the space $L^2((0,\infty), dr)\otimes\mathfrak{H}_{j,m_j}$ can of course be identified with $(L^2((0,\infty), dr))^2$, which provides a realization of endomorphisms of this space by 2×2-matrices of operators on $L^2((0,\infty), dr)$.

The following theorem shows that the pair (P, Q) introduced in Definition 7.3 is nothing but a restriction of the pair of operators

$$(2\pi\hbar c)^{-1}\, H_0 = \frac{1}{2i\pi}\begin{pmatrix} \partial_3 & \partial_1 - i\,\partial_2 \\ \partial_1 + i\,\partial_2 & -\partial_3 \end{pmatrix} \quad , \quad \mathfrak{Q} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \tag{8.25}$$

to some space in an orthogonal sequence of Hilbert spaces, all isomorphic to $(L^2((0,\infty), dr))^2$, associated with the decomposition under the rotation group of a Hilbert space of solutions of the Weyl wave equation for the neutrino.

Theorem 8.1. *The restriction to the space* $L^2((0,\infty),dr)\otimes\mathfrak{H}_{j,m_j} \sim (L^2((0,\infty),dr))^2$ *of the operator* H_0 *which occurs on the right-hand side of Weyl's equation* (8.2) *agrees with* $2\pi\hbar c\, P$, *where* P *is the operator introduced in* (7.13) *corresponding to the value* $p = j + \frac{1}{2}$ *of the integral parameter there. The operator* $\mathfrak{Q}$ *defined in* (8.7) *preserves the same space and, under the identification of this space with* $(L^2((0,\infty), dr))^2$, *becomes the operator* $Q = \left(\begin{smallmatrix} 0 & r \\ r & 0 \end{smallmatrix}\right)$ *introduced in* (7.13) *as well.*

Proof. As an immediate consequence of (8.15), (8.24) together with the actual value of κ as recalled above, and not forgetting that, because of the embedding (8.18), it is the operator $\frac{d}{dr} - \frac{1}{r}$, acting on functions of r alone, that transfers to the operator $\frac{\partial}{\partial r}$ acting on functions on $\mathbb{R}^3$, one finds the equation

$$H_0 = -i\hbar c\begin{pmatrix} 0 & \frac{d}{dr} - \frac{1}{r} + \frac{j+\frac{3}{2}}{r} \\ \frac{d}{dr} - \frac{1}{r} - \frac{j-\frac{1}{2}}{r} & 0 \end{pmatrix} : \tag{8.26}$$

of course, here, one has $r > 0$ whereas x could be positive or negative in (7.13): however, this is not a genuine difference since, in (7.13), we only considered vector-valued functions on the real line each component of which had a specified parity.

Concerning $\mathfrak{Q}$, written in polar coordinates, in the original basis for spinors, as

$$\mathfrak{Q} = r \begin{pmatrix} \cos\theta & e^{-i\phi}\,\sin\theta \\ e^{-i\phi}\,\sin\theta & -\cos\theta \end{pmatrix}, \tag{8.27}$$

one has to check the relations

$$\mathfrak{Q}\,(f(r)\,\Psi^{m_j}_{j-\frac{1}{2}}(\theta,\phi)) = r\,f(r)\,\Psi^{m_j}_{j+\frac{1}{2}}(\theta,\phi)$$

and

$$\mathfrak{Q}\,(f(r)\,\Psi^{m_j}_{j+\frac{1}{2}}(\theta,\phi)) = r\,f(r)\,\Psi^{m_j}_{j-\frac{1}{2}}(\theta,\phi)\,. \tag{8.28}$$

Taking (8.22), (8.23) into account, as well as (8.19), one sees that the first of these two relations reduces to the pair of equations

$$\begin{aligned}&(j-m+1)\,P^{m-\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) - (j+m)\,\cos\theta\,P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) = \sin\theta\,P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\,,\\ &\cos\theta\,P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) - P^{m+\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) = (j+m)\,\sin\theta\,P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\end{aligned} \tag{8.29}$$

and the second one to the pair of equations

$$\begin{aligned}&(j-m+1)\,\cos\theta\,P^{m-\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) - (j+m)\,P^{m-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) = \sin\theta\,P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\,,\\ &P^{m+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta) - \cos\theta\,P^{m+\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta) = (j-m+1)\,\sin\theta\,P^{m-\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta)\ :\end{aligned} \tag{8.30}$$

all these equations can be found in [31, p. 171].

For safety, one may check that

$$[P,Q] = \frac{1}{2i\pi}\begin{pmatrix} 1+2p & 0 \\ 0 & 1-2p \end{pmatrix}$$

or

$$[H_0,\mathfrak{Q}] = \frac{\hbar c}{i}\begin{pmatrix} 1+2p & 0 \\ 0 & 1-2p \end{pmatrix} \tag{8.31}$$

in the matrix realization of operators in the space $L^2((0,\infty),dr)\otimes\mathfrak{H}_{j,m_j}$. From (8.17), this reduces to remarking that

$$\boldsymbol{\sigma}.\boldsymbol{L} + \hbar = \hbar\begin{pmatrix} j+\frac{1}{2} & 0 \\ 0 & -(j+\frac{1}{2}) \end{pmatrix}, \tag{8.32}$$

which was indeed mentioned between (8.23) and (8.24). □

When dealing with the operator H_0 taken from Dirac's equation, the computations are slightly more complicated but are done in full in [50]. One first introduces the four-component functions

$$\Phi^{+}_{m_j,\mp(j+\frac{1}{2})} = \begin{pmatrix} \Psi^{m_j}_{j\mp\frac{1}{2}} \\ 0 \end{pmatrix}, \qquad \Phi^{-}_{m_j,\mp(j+\frac{1}{2})} = \begin{pmatrix} 0 \\ \Psi^{m_j}_{j\pm\frac{1}{2}} \end{pmatrix} : \tag{8.33}$$

note (comparing this to *loc.cit.*, (4.111)) the disappearance of the coefficient i from the non-zero component of the first of these two functions. This time, as explained in *loc.cit.*, p. 127, one must decompose $(L^2(S^2))^4$ by means of the simultaneous consideration of the operators J^2, J_3 and K, the so-called spin-orbit operator. This introduces a family of two-dimensional subspaces $\mathfrak{K}_{m_j,\kappa_j}$ (parametrized by j and m_j as above, together with $\kappa_j = \mp(j+\frac{1}{2})$) of $(L^2(S^2))^4$, in which we take $\{\Phi^{+}_{m_j,\kappa_j}, \Phi^{-}_{m_j,\kappa_j}\}$ as a basis. With the same meaning as in (8.26), one finds that, in the tensor product of $L^2((0,\infty),dr)$ by this space, the operator H_0 is represented by the matrix

$$h_{\kappa_j} = \begin{pmatrix} mc^2 & -i\hbar c\left(\frac{d}{dr} - \frac{\kappa_j}{r}\right) \\ -i\hbar c\left(\frac{d}{dr} + \frac{\kappa_j}{r}\right) & -mc^2 \end{pmatrix}. \tag{8.34}$$

This can be found from *loc.cit.*, (4.134), with the following two differences: we have not set $\hbar = 1$, and there is an extra coefficient i in the vector $\Phi^{+}_{m_j,\mp(j+\frac{1}{2})}$ as taken there, so that (4.122) from *loc.cit.* must be replaced, with $\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}$, by the equation

$$\boldsymbol{\alpha}.\boldsymbol{e}_r\, \Phi^{\pm}_{m_j,\kappa_j} = \Phi^{\mp}_{m_j,\kappa_j}\,. \tag{8.35}$$

With $p = -\kappa_j$, a non-zero integer, one finds that this operator can also be written as

$$h_{\kappa_j} = 2\pi\hbar c\, P - \frac{mc^2}{2\kappa_j}\,(2i\pi\,[P,Q] - \mathbf{1}) \tag{8.36}$$

with the operators P and Q as taken from (7.13): in this case, p could also be a negative integer, but it would then suffice to switch the two basis vectors to change this if so wished.

We also introduce, this time, the operator $\left(\begin{smallmatrix} 0 & \mathfrak{Q} \\ \mathfrak{Q} & 0 \end{smallmatrix}\right)$ and note, as an easy consequence of Theorem 8.1, that, again, it can be represented by the operator-matrix $\left(\begin{smallmatrix} 0 & r \\ r & 0 \end{smallmatrix}\right)$ within the space $\mathfrak{K}_{m_j,\kappa_j}$ provided with the isomorphism with $(L^2((0,\infty),dr))^2$ associated to the basis $\{\Phi^{+}_{m_j,\kappa_j}, \Phi^{-}_{m_j,\kappa_j}\}$. Since, in the original four-component spinor representation,

$$\left[H_0\,, \begin{pmatrix} 0 & \mathfrak{Q} \\ \mathfrak{Q} & 0 \end{pmatrix}\right] = \begin{pmatrix} \frac{c}{i}\,(3\hbar + 2\boldsymbol{\sigma}.\boldsymbol{L}) & 2\,mc^2\,\mathfrak{Q} \\ -2\,mc^2\,\mathfrak{Q} & \frac{c}{i}\,(3\hbar + 2\boldsymbol{\sigma}.\boldsymbol{L}) \end{pmatrix} \tag{8.37}$$

as follows from (8.17), one may check that the matrix representation, within $\mathfrak{K}_{m_j,\kappa_j}$, of this commutator is the matrix

$$\frac{\hbar c}{i}\begin{pmatrix} 1+2p & 0 \\ 0 & 1-2p \end{pmatrix} + mc^2\begin{pmatrix} 0 & 2r \\ -2r & 0 \end{pmatrix} :$$

one may thus verify that, indeed, this is the same as the bracket

$$\left[2\pi\hbar c\, P + \frac{mc^2}{2p}\,(2i\pi\,[P,Q]-1)\,,\, Q\right]$$

computed within the algebra generated by the operators P and Q in Proposition 7.4.

We finally consider Dirac's equation for the electron subjected to an attractive Coulomb potential: the operator H_0 in (8.10) must be replaced by

$$H = H_0 + \frac{\gamma}{r}\,, \tag{8.38}$$

with $\gamma < 0$. We set

$$\alpha\colon = -\frac{\gamma}{\hbar c}\ : \tag{8.39}$$

in the case of the Coulomb problem associated with the hydrogen atom, one has $\gamma = -\frac{e^2}{4\pi\varepsilon_0}$, where e is the charge of the electron and ε_0 is the fundamental unit so denoted, so that α is in this case the fine structure constant, a dimensionless constant $\sim \frac{1}{137.036}$.

The same additional term $\frac{\gamma}{r}$ then arises in the radial components of the operator under study, so that, from (8.34), one ends up with the problem of computing the eigenvalues and eigenfunctions in $(L^2((0,\infty),dr))^2$ of the new operator

$$h_\kappa = \begin{pmatrix} mc^2 + \frac{\gamma}{r} & -i\hbar c\left(\frac{d}{dr} - \frac{\kappa}{r}\right) \\ -i\hbar c\left(\frac{d}{dr} + \frac{\kappa}{r}\right) & -mc^2 + \frac{\gamma}{r} \end{pmatrix}. \tag{8.40}$$

Taking when necessary the conjugate under the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, it is no loss of generality to assume that $\kappa < 0$, which we do from now on.

Following (with slight modifications) the computations in [50, p. 209], one is led to introducing, with

$$s = \sqrt{\kappa^2 - \alpha^2}\,, \tag{8.41}$$

the matrix

$$A\colon = \begin{pmatrix} s-\kappa & -i\alpha \\ i\alpha & s-\kappa \end{pmatrix}, \tag{8.42}$$

which diagonalizes the matrix $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\begin{pmatrix} \gamma & i\hbar c\kappa \\ -i\hbar c\kappa & \gamma \end{pmatrix} = \hbar c\left(\begin{smallmatrix} \kappa & -i\alpha \\ -i\alpha & -\kappa \end{smallmatrix}\right)$, where the second matrix on the left-hand side arose as the coefficient of $\frac{1}{r}$ in h_κ. Of course, this requires that one should have $\alpha < |\kappa|$, and we assume the stronger inequality $\alpha < \sqrt{\kappa^2 - \frac{1}{4}}$: it is certainly satisfied in the case of the equation modelling the hydrogen atom, since κ is a non-zero integer, and it implies that $s > \frac{1}{2}$. Following the trick in *loc.cit.*, one is led to computing

$$\left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right) A^{-1} \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) h_\kappa\, A = \begin{pmatrix} mc^2 & -i\hbar c\left(\frac{d}{dr} + \frac{s}{r}\right) \\ -i\hbar c\left(\frac{d}{dr} - \frac{s}{r}\right) & -mc^2 \end{pmatrix}\ : \tag{8.43}$$

with $w = Au$, the eigenvalue equation $(h_\kappa - E)\,w = 0$ takes the more pleasant form

$$\begin{pmatrix} mc^2 & -i\hbar c\left(\frac{d}{dr}+\frac{s}{r}\right) \\ -i\hbar c\left(\frac{d}{dr}-\frac{s}{r}\right) & -mc^2 \end{pmatrix} u = \frac{E}{s}\begin{pmatrix} -\kappa & \frac{i\gamma}{\hbar c} \\ -\frac{i\gamma}{\hbar c} & -\kappa \end{pmatrix} u\,, \tag{8.44}$$

or

$$2\pi\hbar c\, P_s u = N u\,, \tag{8.45}$$

with

$$N = \begin{pmatrix} -mc^2 & 0 \\ 0 & mc^2 \end{pmatrix} + \frac{E}{s}\begin{pmatrix} \sqrt{\alpha^2+s^2} & -i\alpha \\ i\alpha & \sqrt{\alpha^2+s^2} \end{pmatrix} : \tag{8.46}$$

note that

$$\det N = E^2 - m^2c^4\,. \tag{8.47}$$

Setting $M = (\pi\hbar c)^{-1}N$, we have to solve the equation

$$P_s u = \frac{1}{2}\, M u$$

in $(L^2((0,\infty);dr))^2$: we may thus apply Proposition 7.14, with

$$\mu^2 = \frac{m^2c^4 - E^2}{(\pi\hbar c)^2} \qquad \text{and} \qquad b = \frac{E\alpha s^{-1}}{\pi\hbar c}\,, \tag{8.48}$$

finding in particular that the set of eigenvalues of h_κ is the sequence $(E_n)_{n=0,1,\dots}$ with $E_n > 0$ and $m^2c^4 - E_n^2 = \frac{\alpha^2E_n^2}{(s+n)^2}$ and getting the well-known fact [50, p. 214] that

$$E_n = \frac{mc^2(s+n)}{\sqrt{\alpha^2+(s+n)^2}} \tag{8.49}$$

as a result.

The spectral decomposition of the energy operator occurring in the equation of the relativistic hydrogen atom is of course well known: we only deemed it worthwhile to emphasize its link with the study of the operator P_s, showing also that the analysis of the radial parts of the problem reduces to that of the Op^s-version of the harmonic oscillator.

9 The higher-level Weyl calculi

One of several ways to characterize the Weyl calculus is by the property that it sends the symbol $(x,\xi) \mapsto e^{2i\pi\,(ax+b\xi)}$ to the operator $\exp 2i\pi\,(ax + \frac{b}{2i\pi}\,\frac{d}{dx})$. We extend this to a definition of the Op^p-calculus.

Definition 9.1. Given a function h on $\mathbb{R}^2$ the (symplectic) Fourier transform $\mathcal{F}h$ of which lies in $L^1(\mathbb{R}^2)$, one defines

$$\mathrm{Op}^p(h) = \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta)\, \exp\left(2i\pi\,(\eta Q - yP)\right) dy\, d\eta\,, \tag{9.1}$$

with P and Q as introduced in (7.11).

We must first give the operator $\exp(2i\pi\,(\eta Q - yP))$ a meaning for every $(y,\eta) \in \mathbb{R}^2$, in other words define a self-adjoint realization of the operator $\eta Q - yP$: since, obviously, the operator Q (of multiplication by x) is essentially self-adjoint in $L^2(\mathbb{R})$ when initially defined on $\mathcal{S}_p(\mathbb{R})$, the same goes for all operators $\eta Q - yP$. For, on one hand, Theorem 7.9 shows that all p-metaplectic unitaries preserve the space $\mathcal{S}_p(\mathbb{R})$; on the other hand, each operator $\eta Q - yP$ is the image of Q under the adjoint action of such a unitary. Then all operators $\exp(2i\pi\,(\eta Q - yP))$ are well defined unitary operators in $L^2(\mathbb{R})$, so that $\mathrm{Op}^p(h)$ is well defined as a bounded linear operator in this space.

Theorem 9.2. *The* Op^p*-calculus is covariant under the action of the p-metaplectic representation,* i.e., *given h with* $\mathcal{F}h \in L^1(\mathbb{R}^2)$ *and a metaplectic unitary* U, *one has*

$$U\,\mathrm{Op}^p(h)\,U^{-1} = \mathrm{Op}^p(h \circ \theta(U)^{-1})\,, \tag{9.2}$$

where $\theta(U) \in SL(2,\mathbb{R})$ *has been defined in* (7.41).

Proof. Since $U\,(-yP + \eta Q)\,U^{-1} = -y'P + \eta' Q$ with $\begin{pmatrix} y' \\ \eta' \end{pmatrix} = \theta(U)\, \left(\begin{smallmatrix} y \\ \eta \end{smallmatrix}\right)$, one has, with (y',η') and (y,η) linked by this relation,

$$\begin{aligned} U\,\mathrm{Op}^p(h)\,U^{-1} &= \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta)\, \exp\left(2i\pi\,(\eta' Q - y'P)\right) dy\, d\eta \\ &= \int_{\mathbb{R}^2} ((\mathcal{F}h) \circ \theta(U^{-1}))(y,\eta)\, \exp\left(2i\pi\,(\eta Q - yP)\right) dy\, d\eta\ : \end{aligned} \tag{9.3}$$

since $\mathcal{F}$ commutes with the linear action of $SL(2,\mathbb{R})$ in $\mathbb{R}^2$, we are done. □

In particular, even symbols give rise to operators which send even (*resp.* odd) functions to functions with the same parity.

Theorem 9.3. *Given h satisfying the hypotheses of Definition* 9.1, *and* $z \in \Pi$, *one has*

$$\begin{aligned} (u_z^p \,|\, \mathrm{Op}^p(h)\, u_z^p) = \int_{\mathbb{R}^2} &(\mathcal{F}h)(y,\eta) \\ &\frac{(i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p} \left((y,\eta) \mapsto \exp\left(-\frac{\pi\,|y - z\eta|^2}{2\,\mathrm{Im}\ z} \right) \right) dy\, d\eta \end{aligned} \tag{9.4}$$

and

$$(u_z^{p+1}\,|\,\mathrm{Op}^p(h)\,u_z^{p+1}) = \int_{\mathbb{R}^2} (\mathcal{F}\,h)(y,\eta)\\ \frac{(i\pi\,\mathcal{E})_{p+1}}{\left(\frac{1}{2}\right)_{p+1}}\left((y,\eta)\mapsto \exp\left(-\frac{\pi\,|y-z\eta|^2}{2\,\mathrm{Im}\;z}\right)\right)\,dy\,d\eta\,, \quad (9.5)$$

where we have used Pochhammer's symbols again.

Proof. Let us first compute $(u_z^p|e^{2i\pi\,Q}\,u_z^p)$. With $C(p)=\left(\frac{(2\pi)^{p+\frac{1}{2}}}{\Gamma(p+\frac{1}{2})}\right)^{\frac{1}{2}}$, one has, as a consequence of (6.13),

$$\begin{aligned}&(u_z^p|e^{2i\pi\,Q}\,u_z^p)\\ &= (C(p))^2\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{p+\frac{1}{2}}\int_{-\infty}^{\infty} t^{2p}\,e^{-2\pi\,t^2\,\mathrm{Im}\,(-\frac{1}{z})}\,e^{2i\pi\,t}\,dt\\ &= (C(p))^2\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{p+\frac{1}{2}}\left(\frac{1}{2i\pi}\frac{d}{d\varepsilon}\right)^{2p}\Big|_{\varepsilon=1}\int_{-\infty}^{\infty} e^{-2\pi\,t^2\,\mathrm{Im}\,(-\frac{1}{z})}\,e^{2i\pi\varepsilon\,t}\,dt\\ &= (C(p))^2\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{p+\frac{1}{2}}\left(\frac{1}{2i\pi}\frac{d}{d\varepsilon}\right)^{2p}\Big|_{\varepsilon=1}\\ &\qquad\left[\left(2\,\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}\exp\left(-\frac{\pi\,\varepsilon^2}{2\,\mathrm{Im}\,(-\frac{1}{z})}\right)\right]. \end{aligned} \quad (9.6)$$

Now, given any point $(y,\eta)\in\mathbb{R}^2\backslash\{0\}$, choose a p-metaplectic unitary U with $\eta Q - yP = U\,Q\,U^{-1}$, and set $\theta(U)=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{R})$. Then (Theorem 7.12), for some $\omega\in\mathbb{C}$, $|\omega|=1$, depending only on z and p, one has $U^{-1}\,u_z^p=\omega\,u^p_{\frac{dz-b}{-cz+a}}$ and one can write

$$\begin{aligned}&(u_z^p|e^{2i\pi\,(\eta Q-yP)}\,u_z^p) = \left(u^p_{\frac{dz-b}{-cz+a}}\,|\,e^{2i\pi\,Q}\,u^p_{\frac{dz-b}{-cz+a}}\right)\\ &= (C(p))^2\left(\mathrm{Im}\,\left(\frac{cz-a}{dz-b}\right)\right)^p\left(-\frac{1}{4\pi^2}\frac{d^2}{d\varepsilon^2}\right)^p\Big|_{\varepsilon=1}\left(2^{-\frac{1}{2}}\exp\left(-\frac{\pi\,\varepsilon^2}{2\,\mathrm{Im}\,(\frac{cz-a}{dz-b})}\right)\right).\end{aligned} \quad (9.7)$$

Since, as a consequence of Theorem 7.10, one has $\left(\begin{smallmatrix}y\\ \eta\end{smallmatrix}\right)=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)$, *i.e.*, $b=y$ and $d=\eta$, this leads to

$$\begin{aligned}(u_z^p|\mathrm{Op}^p(h)\,u_z^p) &= \frac{(2\pi)^p\,\pi^{\frac{1}{2}}}{\Gamma(p+\frac{1}{2})}\left(\frac{\mathrm{Im}\;z}{|\eta z-y|^2}\right)^p\\ &\times\int_{\mathbb{R}^2}(\mathcal{F}\,h)(y,\eta)\left(-\frac{1}{4\pi^2}\frac{d^2}{d\varepsilon^2}\right)^p\Big|_{\varepsilon=1}\left(\exp\left(-\pi\varepsilon^2\,\frac{|\eta z-y|^2}{2\,\mathrm{Im}\;z}\right)\right)\,dy\,d\eta\,.\end{aligned} \quad (9.8)$$

To finish the proof, we need a lemma.

Lemma 9.4.

$$\left(-\frac{1}{4\pi^2}\frac{d^2}{d\varepsilon^2}\right)^p\Big|_{\varepsilon=1} e^{-\frac{\pi\varepsilon^2}{2}(y^2+\eta^2)} \times (y^2+\eta^2)^{-p} = (2\pi)^{-p}\,(i\pi\,\mathcal{E})_p\,(e^{-\frac{\pi}{2}(y^2+\eta^2)}). \tag{9.9}$$

Proof. Applying Rodrigues' formula relative to Hermite polynomials [31, p. 252], one can write the left-hand side as

$$(-1)^p\,2^{-3p}\,\pi^{-p}\,e^{-\frac{\pi}{2}(y^2+\eta^2)}\,H_{2p}\left(\left(\frac{\pi}{2}(y^2+\eta^2)\right)^{\frac{1}{2}}\right). \tag{9.10}$$

Connecting this to generalized Laguerre functions (*loc.cit.*, p. 250), this can also be written as

$$(2\pi)^{-p}\,p!\,e^{-\frac{\pi}{2}(y^2+\eta^2)}\,L_p^{(-\frac{1}{2})}\left(\frac{\pi}{2}(y^2+\eta^2)\right). \tag{9.11}$$

Noting that $i\pi\,\mathcal{E} = \frac{1}{2}(y\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial\eta}+1)$ acts on functions of $t=\frac{\pi}{2}(y^2+\eta^2)$ as $t\frac{d}{dt}+\frac{1}{2}$, we see that the lemma reduces to the assertion that

$$\left(t\frac{d}{dt}+\frac{1}{2}\right)\left(t\frac{d}{dt}+\frac{3}{2}\right)\dots\left(t\frac{d}{dt}+p-\frac{1}{2}\right)e^{-t} = p!\,e^{-t}\,L_p^{(-\frac{1}{2})}(t). \tag{9.12}$$

By induction, we thus have to show that

$$\left(t\frac{d}{dt}+p+\frac{1}{2}\right)\left(e^{-t}\,L_p^{(-\frac{1}{2})}(t)\right) = (p+1)\,e^{-t}\,L_{p+1}^{(-\frac{1}{2})}(t). \tag{9.13}$$

This is a consequence of the two relations

$$t\frac{d}{dt}L_p^{(-\frac{1}{2})} = p\,L_p^{(-\frac{1}{2})} - \left(p-\frac{1}{2}\right)L_{p-1}^{(-\frac{1}{2})} \tag{9.14}$$

and

$$(p+1)\,L_{p+1}^{(-\frac{1}{2})} = \left(2p+\frac{1}{2}-t\right)L_p^{(-\frac{1}{2})} - \left(p-\frac{1}{2}\right)L_{p-1}^{(-\frac{1}{2})} \tag{9.15}$$

to be found in (*loc.cit.*, p. 241). □

End of the proof of Theorem 9.3. Applying the lemma to (9.8), we get

$$(u_i^p|\mathrm{Op}^p(h)\,u_i^p) = \frac{(2\pi)^p}{(\frac{1}{2})_p}\int_{\mathbb{R}^2}(\mathcal{F}\,h)(y,\eta)\,\frac{(i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p}\left((y,\eta)\mapsto e^{-\frac{\pi}{2}(y^2+\eta^2)}\right)dy\,d\eta\,, \tag{9.16}$$

from which (9.4) follows, using the covariance Theorem 9.2. The formula (9.5) (in which $p+1$ has been substituted for p) follows just the same lines, since Theorem 7.12 applies to both cases. □

Corollary 9.5. *Assume that p is even. If $h \in L^2(\mathbb{R}^2)$, satisfying the hypotheses of Definition* 9.1, *is an even function on $\mathbb{R}^2$, then $\mathrm{Op}^p(h)$ coincides with $\mathrm{op}^p(h)$ (as introduced in Definition* 6.2*) on $(\mathcal{S}^p(\mathbb{R}))_{\text{even}}$, and $\mathrm{Op}^p(h)$ coincides with $\mathrm{op}^{p+1}(h)$ on $(\mathcal{S}^p(\mathbb{R}))_{\text{odd}}$ $(= (\mathcal{S}^{p+1}(\mathbb{R}))_{\text{odd}})$.*

Proof. Since

$$\left(\mathcal{F}\left((y,\eta) \mapsto \exp\left(-\pi\,\frac{|y-\eta z|^2}{2\,\mathrm{Im}\ z}\right)\right)\right)(x,\xi) = 2\ \exp\left(-2\pi\,\frac{|x-z\xi|^2}{\mathrm{Im}\ z}\right) \tag{9.17}$$

and $\mathcal{F}\,(i\pi\,\mathcal{E})_p = (-i\pi\,\mathcal{E})_p$, one can write (9.4) as

$$\begin{aligned}(u_z^p|\mathrm{Op}^p(h)\,u_z^p) &= \int_{\mathbb{R}^2} h(x,\xi)\,\frac{(-i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p}\,.\,2\ \exp\left(-2\pi\,\frac{|x-z\xi|^2}{\mathrm{Im}\ z}\right)\,dx\,d\xi \\ &= \int_{\mathbb{R}^2} h(x,\xi)\,\frac{(-i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p}\,W(u_z,u_z)(x,\xi)\,dx\,d\xi \\ &= \int_{\mathbb{R}^2} h(x,\xi)\,\mathrm{wig}^p(u_z^p,u_z^p)(x,\xi)\,dx\,d\xi\,, \end{aligned} \tag{9.18}$$

where we have used (2.27) and (6.21). Since both $\mathrm{Op}^p(h)$ and $\mathrm{op}^p(h)$ act continuously from $(\mathcal{S}^p(\mathbb{R}))_{\text{even}}$ to $L^2_{\text{even}}(\mathbb{R})$, and agree on each function u_z^p, they coincide on $(\mathcal{S}^p(\mathbb{R}))_{\text{even}}$ according to Proposition 7.13; the same goes with the odd-odd part. □

Remarks. Though possibly somewhat disconcerting, it would be fully correct to simultaneously set

$$W^p(u_z^p,u_z^p) = \mathrm{wig}^p(u_z^p,u_z^p) \quad \text{and} \quad W^p(u_z^{p+1},u_z^{p+1}) = \mathrm{wig}^{p+1}(u_z^{p+1},u_z^{p+1}).$$

The Op^p-calculus, restricted to even symbols, contains both the op^p-calculus and the op^{p+1}-calculus.

Starting from (6.16), one sees after some formal manipulations that an even function h on $\mathbb{R}^2$ can be uniquely rebuilt from the pair of (respectively Σ_p-invariant and Σ_{p+1}-invariant) functions

$$h^p :\, = \frac{1+\Sigma_p}{2}\,h \qquad \text{and} \qquad h^{p+1} :\, = \frac{1+\Sigma_{p+1}}{2}\,h \tag{9.19}$$

as

$$h = (-i\pi\,\mathcal{E})^{-1}\left[(p-i\pi\,\mathcal{E})\,h^p - (p+i\pi\,\mathcal{E})\,h^{p+1}\right]. \tag{9.20}$$

This can be meaningful, for instance in the case when h^p and h^{p+1}, both lying in $L^2_{\text{even}}(\mathbb{R}^2)$, are such that $\lambda = 0$ does not lie in the support of the spectral decomposition of the sum on the right-hand side of (9.20). In this case, Theorem 6.1 (the proof of which has not been reproduced here, and goes back to [56] together with [62, Section 17]) shows that the square of the norm of $\mathrm{Op}^p(h)$ as a Hilbert-Schmidt operator in $L^2(\mathbb{R})$ is $\|h^p\|^2_{L^2(\mathbb{R}^2)} + \|h^{p+1}\|^2_{L^2(\mathbb{R}^2)}$. It is only when $p = 0$ that this reduces to $\|h\|^2_{L^2(\mathbb{R}^2)}$ (since $\Sigma_0 = \mathcal{G}$ and $\Sigma_1 = -\mathcal{G}$).

To develop the Op^p-calculus further, we need to make $\exp 2i\pi\,(\eta Q - yP)$ explicit for almost all (y,η), taking advantage of Theorem 7.10. To provide for our needs in the present work, however, it suffices to consider only even symbols h, which calls for making only $\cos 2\pi(\eta Q - yP)$ explicit rather than the exponential: this simplifies matters greatly, though the sine part can also be analyzed (using Legendre functions Q_n of the second kind, involving logarithms unless $p = 0$).

In our case, only Legendre polynomials P_n $(n = 0, 1, \dots)$ will have to be used. Recall for instance [31, p. 232] Rodrigues' formula

$$P_n(x) = \frac{1}{2^n\, n!}\,\frac{d^n}{dx^n}\,[(x^2-1)^n]\,. \tag{9.21}$$

The domain on which Legendre polynomials naturally live as a sequence of orthogonal polynomials is the interval $[-1, 1]$.

Theorem 9.6. *Assume that p is even ≥ 2. Let h be an even function on $\mathbb{R}^2$ satisfying the assumption of Definition 9.1 and assume that the function $y^{-1}\,(y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \eta} + 1)\,\mathcal{F}h = y^{-1}\,(2i\pi\,\mathcal{E})\,\mathcal{F}h$ lies in $L^1(\mathbb{R}^2)$ too. Then, for every $u \in (\mathcal{S}_p(\mathbb{R}))_{\mathrm{even}}$, $\mathrm{Op}^p(h)$ is given by the equation*

$$(\mathrm{Op}^p(h)\,u)(x) = \int_{\mathbb{R}^2} (\mathcal{F}\,(i\pi\,\mathcal{E}h))(y,\eta)\,\frac{dy\,d\eta}{y} \int_{|x-y|}^{|x+y|} e^{\frac{i\pi\,\eta(x^2-t^2)}{y}}\,P_{p-1}\left(\frac{x^2+t^2-y^2}{2xt}\right)u(t)\,dt\,, \tag{9.22}$$

and if $u \in (\mathcal{S}_p(\mathbb{R}))_{\mathrm{odd}}$, the same formula works after one has changed the subscript $p-1$ of the Legendre polynomial to p.

Proof. If $\eta \neq 0$, set $\beta = -\frac{y}{\eta}$, and start from the equation

$$\exp 2i\pi\,(\eta Q - yP) = e^{i\pi\beta P^2}\,e^{2i\pi\eta Q}\,e^{-i\pi\beta P^2}\,, \tag{9.23}$$

a consequence of (7.30); recall that $\theta(e^{-i\pi\beta P^2}) = \left(\begin{smallmatrix}1 & \beta\\ 0 & 1\end{smallmatrix}\right)$. If $\beta > 0$, the operator $e^{-i\pi\beta P^2}$ is given, on the even or odd subspace, by the formula (7.19) for U_τ with $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & \beta\\ 0 & 1\end{smallmatrix}\right)$: recall that the value of τ corresponding to the even (*resp.* odd) subspace is $\tau = p - \frac{1}{2}$ (*resp.* $p + \frac{1}{2}$). The operator $\cos 2\pi\eta Q$ preserves each of the two subspaces: its effect is to multiply a function of s by $\cos 2\pi\eta s$. We may thus set

$$\cos 2\pi\,(\eta Q - yP) = e^{i\pi\beta P^2}\,\cos(2\pi\eta Q)\,e^{-i\pi\beta P^2} = \begin{pmatrix} A^{p-\frac{1}{2}} & 0\\ 0 & A^{p+\frac{1}{2}}\end{pmatrix} \tag{9.24}$$

(according to the decomposition $\mathcal{S}_p(\mathbb{R}) = (\mathcal{S}_p(\mathbb{R}))_{\text{even}} \oplus (\mathcal{S}_p(\mathbb{R}))_{\text{odd}}$), with the formula, valid for $x > 0$,

$$(A^\tau u)(x) = \frac{4\pi^2}{\beta^2} \int_0^\infty (xs)^{\frac{1}{2}}\, e^{-i\pi \frac{x^2+s^2}{\beta}}\, J_\tau\left(\frac{2\pi xs}{|\beta|}\right) ds$$

$$\int_0^\infty (st)^{\frac{1}{2}}\, \cos(2\pi\eta s)\, e^{i\pi\frac{s^2+t^2}{\beta}}\, J_\tau\left(\frac{2\pi st}{|\beta|}\right) u(t)\, dt \;: \qquad (9.25)$$

this formula is also valid if $\beta < 0$ since the two factors $e^{\pm\frac{i\pi}{2}(\tau+1)}$ cancel out.

Now $2\pi s\, \cos(2\pi\eta s) = \frac{\partial}{\partial\eta} \sin(2\pi\eta s)$, and the operator $\frac{\partial}{\partial\eta}$ relative to the pair of coordinates $(\beta = -\frac{y}{\eta}, \eta)$ is the same as the operator $\eta^{-1}\, (y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\eta})$ relative to the coordinates (y, η).

On the other hand, the integral

$$\int_0^\infty \sin(2\pi\eta s)\, J_\tau\left(\frac{2\pi xs}{|\beta|}\right) J_\tau\left(\frac{2\pi st}{|\beta|}\right) ds \qquad (9.26)$$

(semi-convergent unless $t = |x \pm y|$) can be found in [31, p. 426]: its value is, if $\tau > -\frac{1}{2}$,

$$\frac{1}{4\pi}\, \frac{|\beta|\, \mathrm{sign}\, \eta}{(xt)^{\frac{1}{2}}}\, P_{\tau-\frac{1}{2}}\left(\frac{x^2+t^2-\beta^2\eta^2}{2xt}\right) \times \mathrm{char}\left(|\frac{x^2+t^2-\beta^2\eta^2}{2xt}| \leq 1\right)$$

$$= \frac{1}{4\pi}\, \frac{|y|}{(xt)^{\frac{1}{2}}}\, P_{\tau-\frac{1}{2}}\left(\frac{x^2+t^2-y^2}{2xt}\right) \times \mathrm{char}\,(|x-|y|| \leq t \leq x+|y|)\,. \qquad (9.27)$$

Thus, for u of a given parity (related to τ in the above-mentioned way), and $x > 0$, one has

$$(\mathrm{Op}^p(h)u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta)\, \frac{dy\, d\eta}{y^2}$$

$$\left(y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\eta}\right) \left(y \int_{|x-y|}^{|x+y|} e^{\frac{i\pi\eta(x^2-t^2)}{y}}\, P_{\tau-\frac{1}{2}}\left(\frac{x^2+t^2-y^2}{2xt}\right) u(t)\, dt\right). \qquad (9.28)$$

This leads to Theorem 9.6 after it has been observed that $y^{-2}\, (y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\eta})\, (y.) = y^{-1}\, (2i\pi\, \mathcal{E})$ has the operator $-(2i\pi\, \mathcal{E})\, (y^{-1}.)$ as a transpose, and that $-i\pi\mathcal{E}\mathcal{F} = i\pi\mathcal{F}\mathcal{E}$.

The proof is over but, for one's peace of mind, one may check that in the case of the odd-odd part of the Weyl calculus ($p = 0$, $P_{\tau-\frac{1}{2}} = 1$), one gets the correct formula: this can be done, letting the operator $y^{-1}\, (y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial\eta})$ remain on the same side as in (9.28). The same works, provided one does not appeal to a non-existent polynomial P_{-1}, with the even-even part of the Weyl calculus, after one has computed the elementary case $\tau = -\frac{1}{2}$ of the integral (9.26), and found that it is

$$\frac{1}{8\pi} \frac{|y|}{(xt)^{\frac{1}{2}} \eta} \sum_{\varepsilon_1^2 = \varepsilon_2^2 = 1} \text{sign}\,(|y| + \varepsilon_1 x + \varepsilon_2 t)\,. \tag{9.29}$$

□

Contrary to the Weyl calculus, the Op^p-calculus does not benefit, when $p \neq 0$, from the Heisenberg covariance, which entails a few differences. In particular, the space $\mathcal{S}_{\bullet}(\mathbb{R}^2)$, to be defined now, is better adapted to the new situation than $\mathcal{S}(\mathbb{R}^2)$. It is defined as the space of all functions $h = h(y,\eta) \in C^\infty(\mathbb{R}^2 \backslash \{0\})$, all derivatives of which are rapidly decreasing as $|y| + |\eta| \to \infty$, and which have the following behaviour at zero: for every pair (α, β) of non-negative integers, and every polynomial $P(y,\eta)$ homogeneous of degree $\alpha+\beta$, the function $P(y,\eta)\,\partial_y^\alpha \partial_\eta^\beta h$ extends as a continuous function on $\mathbb{R}^2$. The space $\mathcal{S}_{\bullet}(\mathbb{R}^2)$ has a natural Fréchet topology, and we denote as $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$ its topological dual, a subspace of $\mathcal{S}'(\mathbb{R}^2)$.

Proposition 9.7. *The definition* 9.1 *of* $\text{Op}^p(h)$ *extends to the case when* h *is an even tempered distribution with* $\mathcal{F}h \in (\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$, *defining a weakly continuous linear operator from* $\mathcal{S}_p(\mathbb{R})$ *to the dual space of* $\mathcal{S}_p(\mathbb{R})$.

Proof. Since, under the assumptions of Definition 9.1, one has, for u and $v \in \mathcal{S}_p(\mathbb{R})$,

$$(v|\,\text{Op}^p(h)u) = \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta)\,(v|\,\exp 2i\pi\,(\eta Q - yP)\,u)\,dy\,d\eta\,, \tag{9.30}$$

one has to show that the function $V(y,\eta)$ (the Fourier transform of the Op^p-concept of Wigner function) defined as

$$(V(v,u))(y,\eta) = (v|\,\exp 2i\pi\,(\eta Q - yP)\,u) \tag{9.31}$$

lies in $\mathcal{S}_{\bullet}(\mathbb{R}^2)$. Obviously, the function $V(v,u)$ is continous and bounded. One has

$$\frac{1}{2i\pi}\left(y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial \eta}\right) . \, e^{2i\pi\,(\eta Q - yP)} = (\eta Q - yP)\, e^{2i\pi\,(\eta Q - yP)}\,, \tag{9.32}$$

and from the equation (*cf.* Theorem 7.10)

$$\exp 2i\pi\,(\eta Q - yP) = e^{-i\pi \frac{y}{\eta} P^2}\, e^{2i\pi\eta Q}\, e^{i\pi \frac{y}{\eta} P^2}\,, \tag{9.33}$$

valid if $\eta \neq 0$, we get

$$\frac{1}{i\pi}\,\eta\frac{\partial}{\partial y} . \, e^{2i\pi\,(\eta Q - yP)} = -P^2\, e^{2i\pi\,(\eta Q - yP)} + e^{2i\pi\,(\eta Q - yP)}\, P^2\,. \tag{9.34}$$

In view of (7.36), there is a similar formula which permits us to express the $y\,\frac{\partial}{\partial \eta}$-derivative of $\exp(2i\pi\,(\eta Q - yP))$. Finally, $[\eta\,\frac{\partial}{\partial y}, y\,\frac{\partial}{\partial \eta}] = \eta\,\frac{\partial}{\partial \eta} - y\,\frac{\partial}{\partial y}$, so that for every operator D in the linear space generated by $y\,\frac{\partial}{\partial y}$, $y\,\frac{\partial}{\partial \eta}$, $\eta\,\frac{\partial}{\partial y}$, $\eta\,\frac{\partial}{\partial \eta}$, one can express $(DV)(y,\eta)$ as a linear combination (the coefficients of which are polynomials in y,η) of expressions $(Av \mid \exp(2i\pi(\eta Q - yP))\,Bu)$, where A and B are operators in the algebra generated by P and Q (thus acting as endomorphisms of $\mathcal{S}_p(\mathbb{R})$). Then the same holds when substituting for D any element in the *algebra* generated by the four first-order differential operators above.

What remains to be shown is that, given v and $u \in \mathcal{S}_p(\mathbb{R})$, the function $V(v,u)(y,\eta)$ is rapidly decreasing as $|y|+|\eta| \to \infty$. To do this, we use the explicit formula for $V(v,u)$ provided by Theorem 9.6 (compare (9.30) and (9.22)). Since the operator $y\,\frac{\partial}{\partial y} + \eta\,\frac{\partial}{\partial \eta}$ has been taken care of by (9.32), this amounts to showing that the integral

$$y^{-1} \int_0^\infty \int_0^\infty \frac{\bar{v}(x)\,u(t)}{xt}\, e^{\frac{i\pi\eta(x^2-t^2)}{y}}\, P_{\tau-\frac{1}{2}}\left(\frac{x^2-y^2+t^2}{2xt}\right) \times \mathrm{char}\,(|x-|y|| < t < x+|y|)\,dx\,dt \qquad (9.35)$$

is rapidly decaying to zero as $|y|+|\eta| \to \infty$. This is clear as $y \to \infty$ since on the support of the integrand, either x or t is no smaller than $\frac{1}{2}|y|$; to arrive at the same conclusion when $|\eta| \to \infty$, the easiest way is to use (7.36) again. □

For instance, since the first Definition (3.1) of the Eisenstein distribution $\mathfrak{E}_\nu^\sharp$ is actually that of a *measure*, it is clear that $\mathfrak{E}_\nu^\sharp$ lies in the space $(\mathcal{S}'_{\boldsymbol{\cdot}}(\mathbb{R}^2))_{\text{even}}$ when $\mathrm{Re}\ \nu < -1$. This still holds when $\mathrm{Re}\ \nu < 0$, $\nu \neq -1$ according to (3.25), but not in general.

For our purposes in the present paper, however, it is much more convenient to be satisfied with the following "minimal" definition (*cf.* remark following Proposition 6.3) of an operator $\mathrm{Op}^p(h)$: as a weakly continuous operator from the linear space algebraically generated by the sets of functions u_z^p and u_z^{p+1}, $z \in \Pi$, to the algebraic dual of this space. For, then, Theorem 9.3, together with a "sesquiholomorphic" argument (*cf.* what follows Proposition 6.3) gives $\mathrm{Op}^p(h)$ a meaning whenever $h \in \mathcal{S}'(\mathbb{R}^2)$. Observe from (9.18) that if h is homogeneous of degree $-1-2j$ for some $j = 0,1,\dots,p-1$, *i.e.*, $(i\pi\mathcal{E})_p h = 0$, then $\mathrm{Op}^p(h) = 0$: in particular $\mathcal{E}h = 0$ implies $\mathrm{Op}^p(h) = 0$ for every $p \geq 1$.

In particular, if $h = h(x)$ with $h \in \mathcal{S}'_{\text{even}}(\mathbb{R})$, the operator $\mathrm{Op}^p(h)$ is simply the operator of multiplications by h. This is well known in the case of the Weyl calculus; on the other hand, a formal argument, starting from (9.1), is immediate. However, a more careful proof will at the same time provide us with a formula which will be useful later. Using the decomposition (2.13) of a Wigner function into its homogeneous parts, one sees that it is sufficient to examine the case when h itself is homogeneous, or only that when $h(x,\xi) = |x|^{-1-\nu}$ with $\mathrm{Re}\ \nu < 0$, in which case $\mathrm{Op}(h)$ is just the operator that multiplies a function of x by $|x|^{-1-\nu}$: for, as a tempered distribution, $|x|^{-1-\nu}$ depends on ν in a holomorphic way in the set defined by $\nu \neq 0, 2, \dots$.

To begin with, one has for every $z \in \Pi$ and $p \geq 0$,

$$(u_z^p|\,|x|^{-1-\nu}u_z^p) = \frac{(2\pi)^{p+\frac{1}{2}}}{\Gamma(p+\frac{1}{2})}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{p+\frac{1}{2}}\int_{-\infty}^{\infty}|x|^{2p-1-\nu}\,e^{-2\pi x^2\,\mathrm{Im}\,(-\frac{1}{z})}\,dx$$

$$= (2\pi)^{\frac{\nu+1}{2}}\,\frac{\Gamma(p-\frac{\nu}{2})}{\Gamma(p+\frac{1}{2})}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\frac{\nu+1}{2}}. \tag{9.36}$$

When $p = 0$, *i.e.*, in the case of the Weyl calculus, this of course coincides with the scalar product $(u_z|\,\mathrm{Op}(|x|^{-1-\nu})u_z)$.

Then, (6.21) and Theorem 9.3 (or Corollary 9.5) yield

$$(u_z^p|\,\mathrm{Op}^p(|x|^{-1-\nu})u_z^p) = \left\langle |x|^{-1-\nu},\,\frac{(-i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p}\,W(u_z,u_z)\right\rangle$$

$$= \left\langle \frac{(i\pi\,\mathcal{E})_p}{(\frac{1}{2})_p}\,|x|^{-1-\nu},\,W(u_z,u_z)\right\rangle$$

$$= \frac{(-\frac{\nu}{2})_p}{(\frac{1}{2})_p}\,(2\,\pi)^{\frac{\nu+1}{2}}\,\frac{\Gamma(-\frac{\nu}{2})}{\Gamma(\frac{1}{2})}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{\frac{\nu+1}{2}}, \tag{9.37}$$

and it suffices to compare (9.36) to (9.37): the same can be done with the functions u_z^{p+1} instead of the functions u_z^p.

We finally give a rather explicit, if in inductive form, connection of the Op^p-calculus to the Weyl calculus. It is essential, at this point, to consider simultaneously even and odd values of p: we thus call *functions of the parity related to* p the functions on $\mathbb{R}$ which are even (*resp.* odd) according to whether p is. Note that, when changing the parity of p, nothing is changed to the fact that, at least for the time being, only even symbols on $\mathbb{R}^2$ are considered.

One may rewrite Theorem 9.6 as follows: if $p \geq 1$, $u \in \mathcal{S}_p(\mathbb{R})$ has the parity related to p and $x > 0$, one has

$$(\mathrm{Op}^p(h)u)(x) = 2\int_0^{\infty}(\mathcal{F}_2^{-1}(i\pi\,\mathcal{E}h))\left(\frac{x^2-t^2}{2y},y\right)\frac{dy}{y}$$

$$\int_{|x-y|}^{|x+y|}P_{p-1}\left(\frac{x^2+t^2-y^2}{2xt}\right)u(t)\,dt\,, \tag{9.38}$$

where $\mathcal{F}_2^{-1}$ denotes the inverse Fourier transform with respect to the second variable; if $u \in \mathcal{S}_p(\mathbb{R})$ has the other parity, the formula remains true after one has replaced p by $p+1$ on the right-hand side.

Lemma 9.8. *If $h \in (\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$ is homogeneous of degree $-1-\nu$, $|\text{Re}\,\nu| < 1$, one has for $x > 0$,*

$$(\text{Op}^p(h)u)(x) = \left(-\frac{\nu}{2}\right)_p \int_0^\infty (4\pi xt)^{\frac{1}{2}}\, |x^2 - t^2|^{-\frac{1}{2}} \left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}} (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right) \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right) u(t)\, dt \quad (9.39)$$

if $u \in \mathcal{S}_p(\mathbb{R})$ has the parity related to p; if u has the other parity, the formula remains true provided one changes p to $p+1$ on the right-hand side.

Proof. Actually, the integral should not be considered as giving a *pointwise* value of $\text{Op}^p(h)u$, rather (after it has been extended to $x < 0$ by parity) a weak definition in $\mathcal{S}'_p(\mathbb{R})$: the same remains valid in the computations which follow.

One has $i\pi\,\mathcal{E}h = -\frac{\nu}{2}\,h$, and

$$(\mathcal{F}_2^{-1}(i\pi\,\mathcal{E}h))\left(\frac{x^2-t^2}{2y}, y\right) = -\frac{\nu}{2}\left|\frac{x-t}{y}\right|^{-\nu} (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right), \quad (9.40)$$

so that, for u with the parity related to p and $x > 0$,

$$(\text{Op}^p(h)u)(x) = -\nu \int_0^\infty y^{\nu-1}\, (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right) |x-t|^{-\nu}\, dy \int_{|x-y|}^{|x+y|} P_{p-1}\left(\frac{x^2+t^2-y^2}{2xt}\right) u(t)\, dt \quad (9.41)$$

or

$$(\text{Op}^p(h)u)(x) = \int_0^\infty (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right) k(x,t)\, u(t)\, dt \quad (9.42)$$

with

$$\begin{aligned} k(x,t) &= -\nu\, |x-t|^{-\nu} \int_{|x-t|}^{x+t} y^{\nu-1}\, P_{p-1}\left(\frac{x^2+t^2-y^2}{2xt}\right) dy \\ &= -\nu\, xt\, |x-t|^{-\nu}\, (x^2+t^2)^{\frac{\nu-2}{2}} \int_{-1}^1 \left(1 - \frac{2xt}{x^2+t^2}\, s\right)^{\frac{\nu-2}{2}} P_{p-1}(s)\, ds\,. \end{aligned} \quad (9.43)$$

According to [31, p. 231],

$$\int_{-1}^1 \exp\left(r \cdot \frac{2xt}{x^2+t^2}\, s\right) P_{p-1}(s)\, ds = \left(\frac{\pi\,(x^2+t^2)}{rxt}\right)^{\frac{1}{2}} I_{p-\frac{1}{2}}\left(\frac{2rxt}{x^2+t^2}\right). \quad (9.44)$$

As $\text{Re}\,\nu < 2$, we can then use the Gamma integral to find

$$\begin{aligned} k(s,t) = &-\frac{\nu}{\Gamma(\frac{2-\nu}{2})}\, \pi^{\frac{1}{2}}\, (xt)^{\frac{1}{2}}\, |x-t|^{-\nu}\, (x^2+t^2)^{\frac{\nu-1}{2}} \\ &\times \int_0^\infty r^{\frac{-\nu-1}{2}}\, e^{-r}\, I_{p-\frac{1}{2}}\left(\frac{2rxt}{x^2+t^2}\right) dt\,, \end{aligned} \quad (9.45)$$

where, thanks to [31, p. 92], the value of the integral is

$$\Gamma\left(p-\frac{\nu}{2}\right)\left(\frac{|x^2-t^2|}{x^2+t^2}\right)^{\frac{\nu-1}{2}}\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right): \tag{9.46}$$

this proves the lemma. □

Remark. Though the case when $p=0$ (the even-even part of the Weyl calculus) has been excluded from our computations (since we have used Theorem 9.6, which does not apply in this case), the result of Lemma 9.8 is valid nevertheless, since [31, p. 172]

$$\mathfrak{P}^{\frac{1}{2}}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right)=(4\pi xt)^{-\frac{1}{2}}\,|x^2-t^2|^{\frac{1}{2}}\left[\left|\frac{x-t}{x+t}\right|^{\frac{\nu}{2}}+\left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}}\right]; \tag{9.47}$$

also, since h is homogeneous of degree $-1-\nu$, one has

$$\left|\frac{x+t}{x-t}\right|^{\nu}(\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2},x-t\right)=(\mathcal{F}_2^{-1}h)\left(\frac{x-t}{2},x+t\right). \tag{9.48}$$

Now, on the whole real line, $(\mathcal{F}_2^{-1}h)(\frac{x+t}{2},x-t)$ is the well-known integral kernel of $\mathrm{Op}(h)$.

The same verification (though, this time, it is not really necessary) would work with the odd-odd part of the Weyl calculus, using (*loc. cit.*)

$$\nu\,\mathfrak{P}^{-\frac{1}{2}}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right)=(\pi xt)^{-\frac{1}{2}}\,|x^2-t^2|^{\frac{1}{2}}\left[\left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}}-\left|\frac{x+t}{x-t}\right|^{-\frac{\nu}{2}}\right]. \tag{9.49}$$

Theorem 9.9. *Assume that $p\geq 1$, and let h satisfy the same assumptions as in Lemma 9.8. On functions $u\in\mathcal{S}_p(\mathbb{R})$ with the parity related to p, $\mathrm{Op}^p(h)u$ is the same as $\mathrm{Op}^{p-1}(h)u$. On functions with the other parity, one has*

$$\begin{aligned}\mathrm{Op}^p(h)u&=\frac{1-p+\frac{\nu}{2}}{p+\frac{\nu}{2}}\,\mathrm{Op}^{p-1}(h)u\\&\quad+\frac{p-\frac{1}{2}}{p+\frac{\nu}{2}}\left[x\,\mathrm{Op}^{p-1}(h)(x^{-1}u)+x^{-1}\,\mathrm{Op}^{p-1}(h)(xu)\right].\end{aligned} \tag{9.50}$$

Proof. That $\mathrm{Op}^p(h)$ and $\mathrm{Op}^{p-1}(h)$ agree as operators acting on functions with the parity related to p is a consequence of Corollary 9.5, which works just as well when $p-1$ is odd (with the usual switch of the two terms in the decomposition of $\mathcal{S}_{p-1}(\mathbb{R})$). Assume now that $p\geq 2$, and that u has the parity contrary to that related to p. We start from Lemma 9.8, using [31, p. 165]

$$\begin{aligned}&\left(p-\frac{\nu}{2}\right)\left(-p-\frac{\nu}{2}\right)\mathfrak{P}^{-\frac{1}{2}-p}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right)\\&\quad=\mathfrak{P}^{\frac{3}{2}-p}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right)+(1-2p)\,\frac{x^2+t^2}{2xt}\,\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-\nu-1}{2}}\left(\frac{x^2+t^2}{|x^2-t^2|}\right),\end{aligned} \tag{9.51}$$

so that, for $x > 0$, recalling from the proof of Lemma 9.8 that the computations which follow must be considered as valid in the weak sense in $\mathcal{S}'_p(\mathbb{R})$,

$$\begin{aligned}(\mathrm{Op}^p(h)u)(x) = -\frac{-\frac{\nu}{2}+p-1}{p+\frac{\nu}{2}} \left(-\frac{\nu}{2}\right)_{p-1} \int_0^\infty (4\pi xt)^{\frac{1}{2}} \, |x^2-t^2|^{-\frac{1}{2}} \\ \left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}} \mathfrak{P}^{\frac{3}{2}-p}_{\frac{-\nu-1}{2}} \left(\frac{x^2+t^2}{|x^2-t^2|}\right) (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right) u(t)\, dt \\ + \frac{2p-1}{p+\frac{\nu}{2}} \left(-\frac{\nu}{2}\right)_p \int_0^\infty (4\pi xt)^{\frac{1}{2}} \, |x^2-t^2|^{-\frac{1}{2}} \left(\frac{x}{2t}+\frac{t}{2x}\right) \\ \left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}} \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-\nu-1}{2}} \left(\frac{x^2+t^2}{|x^2-t^2|}\right) (\mathcal{F}_2^{-1}h)\left(\frac{x+t}{2}, x-t\right) u(t)\, dt\,.\end{aligned} \tag{9.52}$$

Paying much attention to the parity of functions involved, we end up with the formula (9.50).

A slightly different proof is required for the part of the Op^1-calculus dealing with even functions. This time, we use the identity (*loc.cit.*)

$$\begin{aligned}\frac{2\nu xt}{|x^2-t^2|} \mathfrak{P}^{-\frac{3}{2}}_{\frac{-\nu-1}{2}} \left(\frac{x^2+t^2}{|x^2-t^2|}\right) = (\pi xt)^{-\frac{1}{2}} \, |x^2-t^2|^{\frac{1}{2}} \\ \times \left[(\nu+2)^{-1} \left(\left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}+1} - \left|\frac{x+t}{x-t}\right|^{-\frac{\nu}{2}-1} \right) \right. \\ \left. -(\nu-2)^{-1} \left(\left|\frac{x+t}{x-t}\right|^{\frac{\nu}{2}-1} - \left|\frac{x+t}{x-t}\right|^{-\frac{\nu}{2}+1} \right) \right] :\end{aligned} \tag{9.53}$$

the proof is quite similar, but we must also use (9.48) again. □

Corollary 9.10. *Let h be an even tempered distribution with $\mathcal{F}h \in \mathcal{S}'_{\bullet}(\mathbb{R}^2)$. Then, for every integer $p \geq 1$, the operator $\mathrm{Op}^p(h)$ defined according to Proposition 9.7 agrees on functions the parity of which is unrelated to p with the operator*

$$\begin{aligned}\frac{1}{2} \left[(x)\, \mathrm{Op}^{p-1}\left(\frac{2p-1}{p-i\pi\,\mathcal{E}}\, h\right) (x^{-1}) + (x^{-1})\, \mathrm{Op}^{p-1}\left(\frac{2p-1}{p-i\pi\,\mathcal{E}}\, h\right) (x)\right] \\ + \mathrm{Op}^{p-1}\left(\frac{1-p-i\pi\,\mathcal{E}}{p-i\pi\,\mathcal{E}}\, h\right).\end{aligned} \tag{9.54}$$

It agrees with $\mathrm{Op}^{p-1}(h)$ on functions the parity of which is related to p.

Proof. We first observe that the operator $(p-i\pi\,\mathcal{E})^{-1}$, defined on $L^2(\mathbb{R}^2)$ by

$$((p-i\pi\,\mathcal{E})^{-1}\, h)(x,\xi) = 2\int_1^\infty t^{-2p}\, h(tx, t\xi)\, dt\,, \tag{9.55}$$

sends the space $\mathcal{S}_{\bullet}(\mathbb{R}^2)$ defined just before Proposition 9.7 to itself; since the transpose $(p+i\pi\,\mathcal{E})^{-1}$ of $(p-i\pi\,\mathcal{E})^{-1}$ is the same as the conjugate of $(p-i\pi\,\mathcal{E})^{-1}$ under $\mathcal{F}$, the operator $(p-i\pi\,\mathcal{E})^{-1}$ is also well defined as an endomorphism of the space of even tempered distributions h such that $\mathcal{F}h \in \mathcal{S}'_{\bullet}(\mathbb{R}^2)$. Formula (9.54) is essentially, then, a rephrasing of (9.50), after a decomposition of functions in the space $\mathcal{S}_{\bullet}(\mathbb{R}^2)$ into their homogeneous components has taken place. □

Remark. It must be emphasized that Corollary 9.10 does not provide a link between the Op^p-symbol and symbols of lower level of *the same* operator: in other words, there is no simple formula for the Weyl symbol of an operator such as $(x)\,\mathrm{Op}(h)\,(x^{-1})$. This explains why, in Section 12, it will not be an easy task to extend our results on the Weyl sharp product of two power functions in Section 11 to the analogous question in the Op^p-calculus.

10 Can one compose two automorphic operators?

In this section, we indicate why it is not possible to compose, in the usual sense, two operators the Weyl symbols of which are Eisenstein distributions or cusp-distributions; but that one can do so if one substitutes for the Weyl calculus the Op^p-calculus, for some number $p \geq 2$. If one is satisfied with a *minimal* definition of such a product (this concept is introduced just after Proposition 6.3 and recalled between (9.35) and (9.36)), the condition $p \geq 1$ suffices. Actually, in Definition 13.2, we shall show how a rather indirect definition makes it possible to finally work with the Weyl ($p=0$) case. The Op^p-calculus, however, seems to be unavoidable (Theorem 10.7) if one wishes to compose a number $N \geq 3$ of operators with homogeneous automorphic symbols: as mentioned in the introduction, this has some bearing on the question of associativity for the *partially defined* sharp operation on automorphic symbols.

Consider the distribution $\mathfrak{b}_n^\nu$ $(n \neq 0)$ defined as

$$\langle \mathfrak{b}_n^\nu, h\rangle = |n|^{\frac{\nu}{2}} \int_{-\infty}^{\infty} |t|^{-\nu-1}\,(\mathcal{F}_1^{-1}h)\left(\frac{n}{t}, t\right)\,dt\,, \tag{10.1}$$

in other words

$$\mathfrak{b}_n^\nu(x,\xi) = |n|^{\frac{\nu}{2}}\,|\xi|^{-\nu-1}\,e^{2i\pi n\frac{x}{\xi}}\,. \tag{10.2}$$

It occurs in the Fourier series decomposition (3.25) of the Eisenstein distribution $\mathfrak{E}_\nu^\sharp$, also in that (4.4) of any cusp-distribution. We are interested in it as a constituent of a Γ-invariant distribution, so it is just as well to consider instead the distribution

$$\mathfrak{a}_n^\nu = \mathfrak{b}_n^\nu \circ \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\,, \tag{10.3}$$

i.e.,

$$\mathfrak{a}_n^\nu(x,\xi) = |n|^{\frac{\nu}{2}}\,|x|^{-\nu-1}\,e^{-2i\pi n\frac{\xi}{x}}\,. \tag{10.4}$$

We want to compute $\mathrm{Op}^p(\mathfrak{a}_n^\nu)$ by an application of Theorem 9.6, thus starting with the elementary observation that $\mathcal{F}\mathfrak{a}_n^\nu = \mathfrak{a}_n^{-\nu}$. Let us point out that, in the present section, we have to deal with symbols in $\mathcal{S}'(\mathbb{R}^2)$ not in $\mathcal{S}'_{\cdot}(\mathbb{R}^2)$. Nevertheless, they give rise, in the Op^p-calculus, to operators at least in the minimal sense: we can still use Lemma 9.8 for the computations, as an easy continuity argument shows.

We first compute the Weyl operator $\mathrm{Op}(\mathfrak{a}_n^\nu) = \mathrm{Op}^0(\mathfrak{a}_n^\nu)$, using (9.1): since (a remembrance from Heisenberg's representation)

$$\left(\exp 2i\pi\left(\eta x - \frac{y}{2i\pi}\frac{\partial}{\partial x}\right) u\right)(x) = u(x-y)\, e^{2i\pi(x-\frac{y}{2})\eta}\,, \tag{10.5}$$

we find, if $\mathrm{Re}\,\nu > 0$,

$$(\mathrm{Op}(\mathfrak{a}_n^\nu)u)(x) = |n|^{-\frac{\nu}{2}} \int_{\mathbb{R}^2} |y|^{\nu-1}\, u(x-y)\, e^{2i\pi\eta(x-\frac{y}{2}-\frac{n}{y})}\, dy\, d\eta \tag{10.6}$$

(this is of course meant in the sense that the result has to be tested against a function of x in the space $\mathcal{S}(\mathbb{R})$). Now the equation $\frac{y}{2}+\frac{n}{y} = x$ is solvable only if $x^2 \geq 2n$ and, if $x^2 > 2n$, it has two solutions $y = x \pm \sqrt{x^2-2n}$; also, $y^2-2n = \pm 2y\sqrt{x^2-2n}$ and $|\frac{dy}{dx}| = \frac{y^2}{|y^2-2n|} = \frac{|y|}{\sqrt{x^2-2n}}$. This yields the result, after complex continuation has been used:

Proposition 10.1. *For every $n \neq 0$, $\nu \in \mathbb{C}$, and $u \in \mathcal{S}(\mathbb{R})$, $\mathrm{Op}(\mathfrak{a}_n^\nu)u$ is a function, given as*

$$\begin{aligned}(\mathrm{Op}(\mathfrak{a}_n^\nu)u)(x) &= |n|^{-\frac{\nu}{2}}\, \mathrm{char}\,(x^2 > 2n)\,(x^2-2n)^{-\frac{1}{2}} \\ &\times \left[\left|x+\sqrt{x^2-2n}\right|^\nu u\left(-\sqrt{x^2-2n}\right) + \left|x-\sqrt{x^2-2n}\right|^\nu u\left(\sqrt{x^2-2n}\right)\right].\end{aligned} \tag{10.7}$$

This explains why we cannot use the Weyl calculus in any direct way in our present investigations. For, if $n > 0$ and $u \in \mathcal{S}(\mathbb{R})$ does not vanish at zero, the function $\mathrm{Op}(\mathfrak{a}_n^\nu)u$ will never lie in $L^2(\mathbb{R})$, because of the factor $(x^2-2n)^{-\frac{1}{2}}$: it just fails. This absolutely prevents us from giving a meaning to a product like $\mathrm{Op}(\mathfrak{a}_{n_1}^{\nu_1})\,\mathrm{Op}(\mathfrak{a}_{n_2}^{\nu_2})$ with $n_2 > 0$ and $n_1 < 0$, not even in the minimal sense, which would call (starting from the observation that the adjoint of $\mathrm{Op}(\mathfrak{a}_n^\nu)$ is $\mathrm{Op}(\mathfrak{a}_{-n}^{\bar{\nu}})$) for giving at least

$$(u_z|\,\mathrm{Op}(\mathfrak{a}_{n_1}^{\nu_1})\,\mathrm{Op}(\mathfrak{a}_{n_2}^{\nu_2})\,u_z) = (\mathrm{Op}(\mathfrak{a}_{-n_1}^{\bar{\nu}_1})\,u_z\,|\,\mathrm{Op}(\mathfrak{a}_{n_2}^{\nu_2})\,u_z) \tag{10.8}$$

a meaning for all z: note that we have not even come, yet, to the problem of summing with respect to n, which in this case of course makes things even worse.

The most salient features of the operators $\mathrm{Op}(\mathfrak{a}_n^\nu)$ in (10.7) or $\mathrm{Op}^p(\mathfrak{a}_n^\nu)$ to be analyzed presently, are easily explained in relation to the following lemma.

Lemma 10.2. *Given $\rho \in]0,1[$, define the following two Hilbert spaces of measures:*

$$L^{2,\rho}_{\text{even}} = \left\{ u = \sum_{m=0}^{\infty} a_m \left[\delta\left(x - \sqrt{2(\rho+m)}\right) + \delta\left(x + \sqrt{2(\rho+m)}\right) \right] : \right.$$
$$\left. \|u\|^2_{\rho,\text{even}} : = 2^{\frac{3}{2}} \sum_0^{\infty} (\rho+m)^{\frac{1}{2}} |a_m|^2 < \infty \right\} \tag{10.9}$$

and

$$L^{2,\rho}_{\text{odd}} = \left\{ u = \sum_{m=0}^{\infty} a_m \left[\delta\left(x - \sqrt{2(\rho+m)}\right) - \delta\left(x + \sqrt{2(\rho+m)}\right) \right] : \right.$$
$$\left. \|u\|^2_{\rho,\text{odd}} : = 2^{\frac{3}{2}} \sum_0^{\infty} (\rho+m)^{\frac{1}{2}} |a_m|^2 < \infty \right\} . \tag{10.10}$$

Then $L^2(\mathbb{R})$ can be written as the Hilbert direct integral

$$L^2(\mathbb{R}) = \oplus \int_0^1 (L^{2,\rho}_{\text{even}} \oplus L^{2,\rho}_{\text{odd}})\, d\rho\,. \tag{10.11}$$

Proof. Split $u \in L^2(\mathbb{R})$ into its even and odd parts u_{even} and u_{odd}. Defining $u_{\rho,\text{even}}$ and $u_{\rho,\text{odd}}$ as the sums within the right-hand sides of (10.9) and (10.10), setting $a_m = \frac{u_{\text{even}}(\sqrt{2(\rho+m)})}{\sqrt{2(\rho+m)}}$ in the first case, and using in the second one the same formula with u_{odd} substituted for u_{even}, it suffices to verify the formulas

$$u = \int_0^1 u_{\rho,\text{even}}\, d\rho + \int_0^1 u_{\rho,\text{odd}}\, d\rho\,,$$
$$\|u\|^2 = \int_0^1 [\, \|u_{\rho,\text{even}}\|^2_{\rho,\text{even}} + \|u_{\rho,\text{odd}}\|^2_{\rho,\text{odd}}\,]\, d\rho\,. \tag{10.12}$$

□

Now, consider the commutative subgroup $C = \{\varepsilon \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right); c \in \mathbb{Z},\ \varepsilon = \pm 1\}$ of Γ: recall that the p-metaplectic unitaries above $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ are the two transformations $u \mapsto \pm e^{i\pi c x^2} u$, and that the transformations above $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are $u \mapsto \pm i\check{u}$, since the transformations above $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ are $\pm\, e^{-i\pi(\frac{p}{2}+\frac{1}{4})}\, \mathcal{F}_p$ (a consequence of (7.8), (7.33), (7.34)). However, it is simpler (so as not to have to use any two-fold cover) to use on C the modified representation which agrees with the metaplectic representation on elements $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ and simplifies on the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to the unitary transform $u \mapsto \check{u}$. The Hilbert space decomposition referred to in Lemma 10.2 is none other than the decomposition associated with the set of characters of C, *i.e.*, with pairs $(\pm 1, \rho)$, where ρ is a real number mod 1: indeed, on $L^{2,\rho}_{\text{even}} \oplus L^{2,\rho}_{\text{odd}}$, the transformation $u \mapsto e^{i\pi c x^2} u$ acts as the multiplication by $e^{2i\pi c\rho}$, and the transform $u \mapsto \check{u}$ of course acts on each of the two spaces in the last sum as $(-1)^{\text{parity}}$.

As shown by (10.4), $\mathfrak{a}_n^\nu \circ \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)^{-1} = \mathfrak{a}_n^\nu$ when $c \in \mathbb{Z}$: the covariance of the Op^p-calculus under the p-metaplectic representation thus shows that the operator $\mathrm{Op}^p(\mathfrak{a}_n^\nu)$ must commute with the decomposition referred to in Lemma 10.2.

Theorem 10.3. *Let* $\mathfrak{a}_n^\nu$ *be given by* (10.4), *with* $|\mathrm{Re}\,\nu| < 1$ *and* $n \in \mathbb{Z}^\times$. *For every even* $p \geq 0$, *and* $u \in (\mathcal{S}_p(\mathbb{R}))_{\text{even}}$, *one has*

$$(\mathrm{Op}^p(\mathfrak{a}_n^\nu)\, u)(x) = 2^{\frac{\nu+1}{2}} \left(\frac{\pi}{|n|}\right)^{\frac{1}{2}} \left(-\frac{\nu}{2}\right)_p \mathrm{char}\,(x^2 > 2n)$$
$$\left(\frac{|x|}{\sqrt{x^2 - 2n}}\right)^{\frac{1}{2}} \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}} \left(\left|1 - \frac{x^2}{n}\right|\right) u\left(\sqrt{x^2 - 2n}\right) \quad (10.13)$$

and, if $u \in (\mathcal{S}_p(\mathbb{R}))_{\text{odd}}$, *the same formula holds after one has changed* p *to* $p+1$ *on the right-hand side and inserted the factor* $\mathrm{sign}\, x$.

Proof. We apply Lemma 9.8, with

$$(\mathcal{F}_2^{-1}\mathfrak{a}_n^\nu)\left(\frac{x+t}{2}, x-t\right) = |n|^{\frac{\nu}{2}} \left|\frac{x+t}{2}\right|^{-\nu-1} \delta\left(x - t - \frac{2n}{x+t}\right). \quad (10.14)$$

Now the equation $t + \frac{2n}{x+t} = x$ is solvable if and only if $x^2 \geq 2n$, and its non-negative solution is $t = \sqrt{x^2 - 2n}$, Also,

$$\left|\frac{dt}{d(x - t - \frac{2n}{x+t})}\right| = \frac{x + \sqrt{x^2 - 2n}}{2\sqrt{x^2 - 2n}}. \quad (10.15)$$

This immediately leads to the sought-after result. □

We prepare for the analysis of $\mathrm{Op}^p(\mathfrak{E}_\nu^\sharp)$ by the remark that, since $\mathfrak{E}_\nu^\sharp$ is invariant under the linear action of the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$, (3.25) may be rewritten, if $\nu \neq -1, 0, 1, 2, \ldots$, as

$$\langle \mathfrak{E}_\nu^\sharp, h\rangle = \zeta(-\nu) \int_{-\infty}^{\infty} |t|^{-\nu-1} (\mathcal{F}_2^{-1} h)(t, 0)\, dt + \zeta(1-\nu) \int_{-\infty}^{\infty} |t|^{-\nu}\, h(0,t)\, dt$$
$$+ \sum_{n \neq 0} |n|^{-\frac{\nu}{2}}\, \sigma_\nu(|n|)\ \langle \mathfrak{a}_n^\nu, h\rangle\ , \quad (10.16)$$

in other words

$$\mathfrak{E}_\nu^\sharp(x,\xi) = \zeta(-\nu)\, |x|^{-\nu-1} + \zeta(1-\nu)\, \delta(x)\, |\xi|^{-\nu}$$
$$+ \sum_{n \neq 0} |n|^{-\frac{\nu}{2}}\, \sigma_\nu(|n|)\ \mathfrak{a}_n^\nu(x,\xi). \quad (10.17)$$

In Theorem 10.3, we computed the operator $A_n^\nu \colon = \mathrm{Op}^p(\mathfrak{a}_n^\nu)$. We now compute the operator corresponding to the second exceptional term.

Proposition 10.4. *Assume that* $|\text{Re}\,\nu| < 1$, $\nu \neq 0$ *and that* p *is even. The operator* $A_0^\nu{:} = \text{Op}^p(\delta(x)\,|\xi|^{-\nu})$ *is given as follows: on even functions lying in* $\mathcal{S}_p(\mathbb{R})$, *it coincides with the operator*

$$u \mapsto c_p(\nu)\,|x|^{\nu-1}\,u \tag{10.18}$$

with

$$c_p(\nu) = 2^\nu\,\pi^{\nu-\frac{1}{2}}\,\frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{\nu}{2})}\,\frac{(-\frac{\nu}{2})_p}{(\frac{\nu}{2})_p}\,; \tag{10.19}$$

on odd functions, it coincides with the operator

$$u \mapsto c_{p+1}(\nu)\,|x|^{\nu-1}\,u \tag{10.20}$$

where $c_{p+1}(\nu)$ *is obtained by substituting* $p+1$ *for* p *in* $c_p(\nu)$.

Proof. It is easily found that

$$\mathcal{G}\,(\delta(x)\,|\xi|^{-\nu}) = 2^\nu\,\pi^{\nu-\frac{1}{2}}\,\frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{\nu}{2})}\,|x|^{\nu-1}\,. \tag{10.21}$$

Recall from what immediately follows (6.22) that, on even functions, an operator $\text{Op}^p(h)$ coincides with $\text{Op}^p(\Sigma_p h)$ with $\Sigma_p = \frac{(-i\pi\,\mathcal{E})_p}{(i\pi\,\mathcal{E})_p}\,\mathcal{G}$. The same goes for odd functions after Σ_{p+1} has been substituted for Σ_p. Proposition 10.4 thus follows from the fact, proved just in (9.36)–(9.37), that if h depends only on x, the operator $\text{Op}^p(h)$ is the operator of multiplication by the function h. □

Because of Proposition 10.4, the first two terms on the right-hand side of (10.17) are quite obvious to deal with: the first one yields, under Op^p, a bounded operator from $\mathcal{S}_p(\mathbb{R})$ to $L^2(\mathbb{R})$ provided $\text{Re}\,\nu < p - \frac{1}{2}$, and the second one does the same if $\text{Re}\,\nu > \frac{1}{2} - p$. Under the condition $|\text{Re}\,\nu| < p - \frac{1}{2}$, one sees also that each of the first two terms on the right-hand side of (10.17) yields, under Op^p, a bounded operator from $L^2(\mathbb{R})$ to $\mathcal{S}'_p(\mathbb{R})$ since, as a consequence of (10.1) together with the fact that the operators $\exp(2i\pi\,(\eta Q - yP))$ are unitary, the adjoint of an operator $\text{Op}^p(h)$ is $\text{Op}^p(\bar{h})$, just as in the Weyl calculus.

The image, under Op^p, of the series on the right-hand side of (10.17), though of course more difficult to analyze, actually behaves better in a way.

To prepare for the proof of a precise result, we need a certain estimate ((10.27) below) which goes more than half-way towards the proof of a certain version of Cotlar's lemma, all the practitioners of pseudodifferential analysis will find essentially familiar: this lemma, proven by Cotlar [14] and shown in the same paper to be relevant for the study of such operators as the Hilbert transform, seems to have been first applied towards the study of *general* pseudodifferential operators in the paper [9] of Calderon-Vaillancourt. Note, however, that our present version (which we used to teach ten years ago, but never properly published) is *both* stronger *and* easier to prove than the more familiar one (*cf.* remark below): this is why we cannot resist adding the few extra lines from (10.27) to (10.28) so as to complete the proof.

Lemma 10.5. *Let* H *be a Hilbert space, and* $(A_n)_{n\geq 1}$ *a sequence of bounded linear operators on* H. *Set*

$$k_1(n,m) = \| \, |A_n|^{\frac{1}{2}} \, |A_m|^{\frac{1}{2}} \, \|$$

and

$$k_2(n,m) = \| \, |A_n^*|^{\frac{1}{2}} \, |A_m^*|^{\frac{1}{2}} \, \|, \tag{10.22}$$

where $|B| := (B^*B)^{\frac{1}{2}}$ *for every bounded operator* B. *Assume that there exist two constants* $C_1 > 0$ *and* $C_2 > 0$ *such that, with* $j = 1, 2$,

$$\sum_{n,m} k_j(n,m) \, |z_n| \, |w_m| \leq C_j \left(\sum_n |z_n|^2 \right)^{\frac{1}{2}} \left(\sum_m |w_m|^2 \right)^{\frac{1}{2}} \tag{10.23}$$

for all sequences of numbers $(z_n)_{n\geq 1}$ *and* $(w_m)_{m\geq 1}$. *Then the operator* A *defined, in the weak sense on* H, *as* $A = \sum_n A_n$, *is bounded on* H, *and satisfies the estimate* $\|A\| \leq (C_1 C_2)^{\frac{1}{2}}$.

Proof. We use the so-called *polar decomposition* $B = U|B|$ of any bounded linear operator B [41], where U is a partial isometry: it is characterized by the given relation on the closure of the image of $|B|$ and the fact that it vanishes on $\operatorname{Ker}|B|$. One then easily sees (taking the square) that $|B^*| = U \, |B| \, U^*$. As a consequence, one finds, for any pair u, v of vectors in H, that

$$|(v \mid Bu)| \leq \| \, |B|^{\frac{1}{2}} u \, \| \, \| \, |B^*|^{\frac{1}{2}} v \, \| \ : \tag{10.24}$$

indeed,

$$|(v \mid Bu)| = |(v, U|B|u)| \leq \| \, |B|^{\frac{1}{2}} u \, \| \, \| \, |B|^{\frac{1}{2}} U^* v \, \| \, ; \tag{10.25}$$

also,

$$\| \, |B|^{\frac{1}{2}} U^* v \, \|^2 = (v \mid U|B|U^* v) = (v \mid |B^*| v) = \| \, |B^*|^{\frac{1}{2}} v \, \|^2 . \tag{10.26}$$

Thus, for every n,

$$\sum_n |(v \mid A_n u)| \leq \left(\sum_n (u \mid |A_n| u) \right)^{\frac{1}{2}} \left(\sum_n (v \mid |A_n^*| v) \right)^{\frac{1}{2}} \ : \tag{10.27}$$

this reduces the problem to the case when $A_n = A_n^*$ for all n, a positive semi-definite operator, in which one may then conclude by

$$\begin{aligned}\| Au \|^2 &= \sum_{n,m} (A_m u \,|\, A_n u) \\ &= \sum_{n,m} \left(A_n^{\frac{1}{2}} A_m^{\frac{1}{2}} A_m^{\frac{1}{2}} u \,|\, A_n^{\frac{1}{2}} u \right) \\ &\leq \sum_{n,m} k_1(n,m) \, \| A_m^{\frac{1}{2}} u \| \, \| A_n^{\frac{1}{2}} u \| \\ &\leq C_1 \sum \| A_n^{\frac{1}{2}} u \|^2 \\ &= C_1 \sum_n (u \,|\, A_n u) \\ &= C_1 \, (u, Au) \\ &\leq C_1 \, \| Au \| \, \| u \| \,. \end{aligned} \tag{10.28}$$

□

Remark. In the usual Cotlar's lemma, instead of the kernels $k_1(n,m)$ and $k_2(n,m)$, one has to deal with $k_1'(n,m) = \| A_n A_m^* \|^{\frac{1}{2}}$ and $k_2'(n,m) = \| A_n^* A_m \|^{\frac{1}{2}}$. The preceding lemma is stronger since on one hand

$$\| \, |A_n|^{\frac{1}{2}} \, |A_m|^{\frac{1}{2}} \, \|^2 = \| \, |A_m|^{\frac{1}{2}} \, |A_n| \, |A_m|^{\frac{1}{2}} \, \| \,, \tag{10.29}$$

and an application of Hadamard's three line theorem (which can be found in many places, for instance as Theorem XII.1.3 in [73]) to the function

$$F(z) = (v \,|\, |A_m|^{\frac{1}{2}+z} \, |A_n| \, |A_m|^{\frac{1}{2}-z} \, u) \tag{10.30}$$

permits us to prove that

$$\| \, |A_m|^{\frac{1}{2}} \, |A_n| \, |A_m|^{\frac{1}{2}} \, \| \; \leq \| \, |A_n| \, |A_m| \, \| \; : \tag{10.31}$$

with partial isometries U_n and U_m such that $|A_n| = U_n A_n$ and $|A_m| = |A_m|^* = A_m^* U_m^*$, one also gets

$$\begin{aligned}\| \, |A_n| \, |A_m| \, \| &= \| \, U_n A_n A_m^* U_m^* \, \| \\ &\leq \| \, A_n A_m^* \, \| \,, \end{aligned} \tag{10.32}$$

which justifies this remark.

Lemma 10.5 will not really be needed here: only the fact (a consequence of (10.27)) that the two conditions $\sum \| \, |A_n| \, \| < \infty$ and $\sum \| \, |A_n^*| \, \| < \infty$ are sufficient to ensure the weak convergence of the series $\sum A_n$.

Theorem 10.6. *Consider, for* $\varepsilon > 0$, *the Hilbert space* $H = L^2(\mathbb{R}, (1+x^2)^\varepsilon\, dx) = \{u \text{ on } \mathbb{R}\colon \int_{-\infty}^{\infty} (1+x^2)^\varepsilon\, |u(x)|^2\, dx < \infty\}$. *Consider the operator*

$$A = \sum_{n \neq 0} |n|^{-\frac{\nu}{2}}\, \sigma_\nu(|n|)\, \mathrm{Op}^p(\mathfrak{a}_n^\nu)\,, \tag{10.33}$$

the Op^p*-symbol of which is the sum of the series on the right-hand side of* (10.17) *(from which the first two terms have been excluded). If* $p \geq 2$, *and* $|\mathrm{Re}\ \nu| < \min(1, p-\frac{3}{2})$, *the operator is well defined, in the weak sense, as a bounded operator from* H *to* $L^2(\mathbb{R})$, *provided that* $\varepsilon > 1 + |\mathrm{Re}\ \nu|$. *In particular, in the case of the Eisenstein distribution* $\mathfrak{E}_{i\lambda}^\sharp$, $\lambda \in \mathbb{R}$, *this works as soon as* $p \geq 2$, *and* ε *only has to be* > 1. *If* $\mathfrak{M}_j^\sharp$ *is an arbitrary cusp-distribution, as defined by the series* (4.4), *one can define in the same way the operator* $\mathrm{Op}^p(\mathfrak{M}_j^\sharp)$ *as a bounded operator from* H *to* $L^2(\mathbb{R})$ *as soon as* $p \geq 3$ *and* $\varepsilon > \frac{13}{10}$; *this result could be improved to* $p \geq 2$ *and* $\varepsilon > 1$ *if the non-holomorphic Ramanujan-Petersson conjecture had been proved.*

Proof. Forgetting the constant (*i.e.*, independent of x and n) coefficients on the right-hand side of (10.13) (one may then assume that $\nu \neq 0$ if $p \geq 1$), set

$$(B_n^\nu\, u)(x) = |n|^{-\frac{1}{2}}\, \mathrm{char}\,(x^2 > 2n) \left(\frac{x}{\sqrt{x^2-2n}}\right)^{\frac{1}{2}} \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}} \left(\frac{x^2-n}{|n|}\right)\, u\left(\sqrt{x^2-2n}\right) \tag{10.34}$$

where u has the parity related to p, and $x > 0$; the case when the parity of u is opposed to that of p works even better (with the same proof), and we shall not worry about it. After a change of variable $x \mapsto \sqrt{x^2+2n}$, it is immediate that the adjoint $(B_n^\nu)^*$ of B_n^ν in $L^2(\mathbb{R})$ is given as

$$(B_n^\nu)^* = B_{-n}^{\bar{\nu}}\,. \tag{10.35}$$

The two transformations $x \mapsto \sqrt{x^2+2n}$ and $x \mapsto \sqrt{x^2-2n}$ destroy each other in the product $B_{-n}^{\bar{\nu}}\, B_n^\nu$, and $|B_n^\nu| = (B_{-n}^{\bar{\nu}}\, B_n^\nu)^{\frac{1}{2}}$ reduces to the operator of multiplication by the function

$$x \mapsto |n|^{-\frac{1}{2}}\, \mathrm{char}\,(x^2 > -2n) \left| \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}} \left(\frac{x^2+n}{|n|}\right)\right|. \tag{10.36}$$

Recall from [31, p. 153] that

$$\Gamma\left(\frac{1}{2}+p\right)\, \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}(s) = \left(\frac{s-1}{s+1}\right)^{\frac{p}{2}-\frac{1}{4}}\, {}_2F_1\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{1}{2}+p; \frac{1-s}{2}\right). \tag{10.37}$$

When $s = \frac{x^2+n}{|n|}$ and $x^2 > -2n$, one has $s \geq 1$ and $\frac{1-s}{2} \leq 0$: recall that the hypergeometric function is well defined and C^∞ on $]-\infty, 1[$. When $1 \leq s \leq 2$, we simply write

$$|\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}(s)| \leq C\,(s-1)^{\frac{p}{2}-\frac{1}{4}}\,. \tag{10.38}$$

For $s \geq 2$, we use the "linear transformation" of the hypergeometric function [31, p. 48] (recalling that $|\mathrm{Re}\ \nu| < 1,\ \nu \neq 0$)

$$\begin{aligned}&{}_2F_1\left(\frac{1+\nu}{2}, \frac{1-\nu}{2}; \frac{1}{2}+p; \frac{1-s}{2}\right) = \\ &\sum_{\pm}\left(\frac{2}{1+s}\right)^{\frac{1\mp\nu}{2}} \frac{\Gamma(\frac{1}{2}+p)\,\Gamma(\pm\nu)}{\Gamma(\frac{1\pm\nu}{2})\,\Gamma(p\pm\frac{\nu}{2})}\ {}_2F_1\left(\frac{1\mp\nu}{2}, p\mp\frac{\nu}{2}; 1\mp\nu; \frac{2}{1+s}\right)\end{aligned} \tag{10.39}$$

to get the estimate

$$|\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}(s)| \leq C\,(1+s)^{\frac{|\mathrm{Re}\ \nu|-1}{2}}\,. \tag{10.40}$$

We are actually asserting that the operator A will be bounded not as an operator in $L^2(\mathbb{R})$, but as an operator from $L^2(\mathbb{R}, (1+x^2)^\varepsilon\,dx)$ to $L^2(\mathbb{R})$ for suitable $\varepsilon > 0$. This amounts to a claim concerning the operator $A\,(1+x^2)^{-\frac{\varepsilon}{2}}$ as an operator in $L^2(\mathbb{R})$. We thus set

$$C_n^\nu = B_n^\nu\,(1+x^2)^{-\frac{\varepsilon}{2}}\,, \tag{10.41}$$

so that $|C_n^\nu|$ is the operator of multiplication by the function

$$x \mapsto f_{n,\nu}(x) :\ = |n|^{-\frac{1}{2}}\,\mathrm{char}\,(x^2 > -2n)\,(1+x^2)^{-\frac{\varepsilon}{2}}\left|\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}\left(\frac{x^2+n}{|n|}\right)\right| \tag{10.42}$$

and $|(C_n^\nu)^*|$ is the operator of multiplication by the function

$$x \mapsto |n|^{-\frac{1}{2}}\,\mathrm{char}\,(x^2 > 2n)\,(1+x^2-2n)^{-\frac{\varepsilon}{2}}\left|\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}\left(\frac{x^2-n}{|n|}\right)\right|. \tag{10.43}$$

Since both operators $|C_n^\nu|$ and $|(C_n^\nu)^*|$ are multiplication operators, finding their operator-norms is a trivial task: as is easily seen, all we have to do is to consider the supremum of the function on the right-hand side of (10.42) in the case when $n \geq 1$. It is no loss of generality either, diminishing ε if needed, to assume that

$$|\mathrm{Re}\ \nu| + 1 < \varepsilon < p - \frac{1}{2}\,. \tag{10.44}$$

From our preceding review of the hypergeometric function,

$$f_{n,\nu}(x) \leq C\,n^{-\frac{1}{2}}\,(1+x^2)^{-\frac{\varepsilon}{2}}\left(\frac{x^2}{n}\right)^{\frac{p}{2}-\frac{1}{4}} \tag{10.45}$$

if $\frac{x^2}{n} \leq 1$, and

$$f_{n,\nu}(x) \leq C\, n^{-\frac{1}{2}}\,(1+x^2)^{-\frac{\varepsilon}{2}}\left(1+\frac{x^2}{n}\right)^{\frac{|\mathrm{Re}\ \nu|-1}{2}} \tag{10.46}$$

if $\frac{x^2}{n} \geq 1$. In view of our assumption that $|\mathrm{Re}\ \nu| < p - \frac{3}{2}$, the estimate (10.45), together with

$$|n|^{-\frac{\nu}{2}}\,\sigma_\nu(|n|) = \mathrm{O}(|n|^{\frac{|\mathrm{Re}\ \nu|}{2}}\log|n|), \qquad |n| \to \infty, \tag{10.47}$$

is sufficient in the case when $|x| \leq 1$; if $|x| \geq 1$ but $\frac{x^2}{n} \leq 1$, we multiply the right-hand side of (10.45) by $(\frac{n}{x^2})^{\frac{1}{2}(p-\frac{1}{2}-\varepsilon)}$, getting the estimate

$$f_{n,\nu}(x) \leq C\, n^{\frac{-1-\varepsilon}{2}}, \tag{10.48}$$

sufficient for our purposes thanks to (10.47), as a result. Finally, if $\frac{x^2}{n} \geq 1$, we multiply the right-hand side of (10.46) by $(\frac{n}{x^2})^{\frac{-\varepsilon+|\mathrm{Re}\ \nu|-1}{2}}$ and find (10.48) again. This concludes the proof of the part of Theorem 10.6 dealing with the operators associated to the Eisenstein distributions.

In the case of a cusp-distribution $\mathfrak{M}_j^\sharp$ in the place of the Eisenstein distribution, there are only two changes: first, the two exceptional terms from the analogue of (10.17) are absent (*cf.* (4.4)); next, the coefficient $|n|^{-\frac{\nu}{2}}\,\sigma_\nu(|n|)$ has to be replaced by the coefficient b_n from the Fourier series decomposition of $\mathcal{M}_j$. Now, it is no loss of generality to assume that $\mathcal{M}_j$ is a Maass-Hecke eigenform, in which case, while waiting for a proof of the non-holomorphic Ramanujan conjecture, one can be satisfied with the inequality $|b_n| \leq C\,|n|^{\frac{3}{10}+\varepsilon}$, proved by Selberg [43], as quoted in [48, p. 220]. □

As soon as an operator $\mathrm{Op}^p(\mathfrak{S})$, where $\mathfrak{S} \in \mathcal{S}'_{\text{even}}(\mathbb{R}^2)$, acts from $\mathcal{S}_p(\mathbb{R})$ to $L^2(\mathbb{R})$, the operator $\mathrm{Op}^p(\overline{\mathfrak{S}})$ acts from $L^2(\mathbb{R})$ to $\mathcal{S}'_p(\mathbb{R})$. Theorem 10.6 thus gives a meaning, as an operator from $\mathcal{S}_p(\mathbb{R})$ to $\mathcal{S}'_p(\mathbb{R})$, to a product $\mathrm{Op}(h_1)\,\mathrm{Op}(h_2)$ where $p \geq 2$ and each h_j is an Eisenstein distribution $\mathfrak{E}_\nu^\sharp$ with $|\mathrm{Re}\ \nu| < 1$, or when $p \geq 3$ and each h_j is a cusp-distribution or an Eisenstein distribution.

The Op^p-calculus not only makes it possible to compose two operators with homogeneous automorphic symbols: it even permits taking the composition of any given number of such operators, *provided that p is large enough.* To see this, introduce for $\varepsilon \geq 0$, $\delta \geq 0$ the space

$$H_{\varepsilon,\delta} = L^2(\mathbb{R}, |x|^{-2\delta}(1+x^2)^{\varepsilon}\, dx)\,. \tag{10.49}$$

Theorem 10.7. *Let* (ε, δ) *and* (ε', δ') *satisfy for some* $\alpha < 1$ *the following set of inequalities:*

$$\begin{aligned} \delta - \delta' &\geq \alpha + 1, \qquad & (\varepsilon - \delta) - (\varepsilon' - \delta') &> \alpha + 1, \\ p &\geq \frac{1}{2} - \delta, \qquad & p &\geq \frac{1}{2} + \delta', \\ p &> \varepsilon' - \delta' + \alpha + \frac{3}{2}, \qquad & p &> \delta - \varepsilon + \alpha + \frac{3}{2}. \end{aligned} \tag{10.50}$$

Let $\mathfrak{M}^\sharp$ *be any homogeneous automorphic distribution, the Fourier coefficients of which satisfy the estimates* $|b_n| \leq C\,|n|^{\frac{\alpha}{2}}$ *for* $n \neq 0$*: any* $\alpha > |\mathrm{Re}\ \nu|$ *will do in the case when* $\mathfrak{M}^\sharp = \mathfrak{E}_\nu^\sharp$*, and any* $\alpha > \frac{3}{5}$ *will do in the case of a cusp-distribution. Then the operator* $\mathrm{Op}^p(\mathfrak{M}^\sharp)$ *is bounded as a linear operator from* $H_{\varepsilon,\delta}$ *to* $H_{\varepsilon',\delta'}$*.*

Proof. Before we give it, observe that given any pair (ε', δ'), one can find (ε, δ) and p such that all the inequalities (10.50) are satisfied: this proves our assertion concerning the possibility to compose any number of operators with homogeneous automorphic symbols in the appropriate Op^p-calculus.

Instead of the operator C_n^ν in (10.41), we now have to consider the operator

$$\begin{aligned} D_n^\nu &= |x|^{-\delta'}(1+x^2)^{\frac{\varepsilon'}{2}}\, B_n^\nu\, |x|^\delta (1+x^2)^{-\frac{\varepsilon}{2}} \\ &= B_n^\nu\, |x|^\delta (x^2+2n)^{-\frac{\delta'}{2}}(1+x^2)^{-\frac{\varepsilon}{2}}(1+x^2+2n)^{\frac{\varepsilon'}{2}}\, \mathrm{char}(x^2 > -2n)\,. \end{aligned} \tag{10.51}$$

Then $|D_n^\nu|$ is the operator of multiplication by the function

$$\begin{aligned} g_{n,\nu}(x) &= |n|^{-\frac{1}{2}}\, \mathrm{char}(x^2 > -2n) \\ &\quad \times |x|^\delta (x^2+2n)^{-\frac{\delta'}{2}}(1+x^2)^{-\frac{\varepsilon}{2}}(1+x^2+2n)^{\frac{\varepsilon'}{2}} \left| \mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}\left(\frac{x^2+n}{|n|}\right)\right|, \end{aligned} \tag{10.52}$$

and $|(D_n^\nu)^*|$ is obtained in the same way, substituting $-n$ for n and the pairs $(-\varepsilon', -\delta')$ and $(-\varepsilon, -\delta)$ for (ε, δ) and (ε', δ') respectively: this substitution leaves the set of inequalities (10.50) invariant. Also, the case when $n \leq -1$ reduces by a change of variables to that when $n \geq 1$: using (10.38) and (10.40) as

$$|\mathfrak{P}^{\frac{1}{2}-p}_{\frac{-1-\nu}{2}}(s)| \leq C\,(s-1)^{\frac{p}{2}-\frac{1}{4}}\,(s+1)^{-\frac{p}{2}+\frac{|\mathrm{Re}\ \nu|}{2}-\frac{1}{4}}\,, \tag{10.53}$$

we find

$$\begin{aligned} g_{n,\nu}(x) &\leq C\,n^{-\frac{1}{2}}\,|x|^\delta (x^2+2n)^{-\frac{\delta'}{2}}(1+x^2)^{-\frac{\varepsilon}{2}}(1+x^2+2n)^{\frac{\varepsilon'}{2}} \\ &\qquad \left(\frac{x^2}{n}\right)^{\frac{p}{2}-\frac{1}{4}} \left(\frac{x^2+2n}{n}\right)^{-\frac{p}{2}+\frac{|\mathrm{Re}\ \nu|}{2}-\frac{1}{4}}\,. \end{aligned} \tag{10.54}$$

When $|x| \leq 1$, since $\delta + p - \frac{1}{2} \geq 0$, this is less than $C\, n^{-\frac{1}{4}+\frac{\varepsilon'-\delta'}{2}-\frac{p}{2}}$, so that

$$\sum_{n\geq 1} n^{\frac{\alpha}{2}} \sup_{|x|\leq 1} g_{n,\nu}(x) < \infty \tag{10.55}$$

under the assumptions made. When $|x| \geq 1$,

$$g_{n,\nu}(x) \leq C\, n^{-\frac{|\mathrm{Re}\ \nu|}{2}} (x^2)^{\frac{\delta-\varepsilon}{2}+\frac{p}{2}-\frac{1}{4}} (x^2+2n)^{\frac{\varepsilon'-\delta'}{2}+\frac{|\mathrm{Re}\ \nu|-p}{2}-\frac{1}{4}}\,. \tag{10.56}$$

If the exponent of x^2 is ≥ 0, this is

$$\begin{aligned} &\leq C\, n^{-\frac{|\mathrm{Re}\ \nu|}{2}} (x^2+2n)^{\frac{(\varepsilon'-\delta')-(\varepsilon-\delta)}{2}+\frac{|\mathrm{Re}\ \nu|-1}{2}} \\ &\leq C\, n^{\frac{(\varepsilon'-\delta')-(\varepsilon-\delta)-1}{2}} \end{aligned} \tag{10.57}$$

and the same holds in the case of a cusp-distribution $\mathfrak{M}_j^\sharp$, in which we must substitute $i\lambda_j$ for ν. If, on the contrary, $p < \frac{1}{2} + \varepsilon - \delta$, one has

$$g_{n,\nu}(x) \leq C\, n^{\frac{\varepsilon'-\delta'-p}{2}-\frac{1}{4}}\,. \tag{10.58}$$

In both cases,

$$\sum_{n\geq 1} n^{\frac{\alpha}{2}} \sup_{|x|\geq 1} g_{n,\nu}(x) < \infty\,. \tag{10.59}$$

The assumption $\delta - \delta' \geq |\mathrm{Re}\ \nu| + 1$ is needed only in the case of an Eisenstein distribution where, in conjunction with the inequality $(\varepsilon-\delta)-(\varepsilon'-\delta') \geq |\mathrm{Re}\ \nu|-1$, a stronger version of which has already been assumed, it lets the two exceptional terms, which are (Proposition 10.4) the multiplication operators by $|x|^{\pm\nu-1}$, act from $H_{\varepsilon,\delta}$ to $H_{\varepsilon',\delta'}$. □

11 The sharp product of two power-functions: the Weyl case

In this section and the next one, as a preparation for the composition formula of Section 17, we examine a sharp product $|x|^{-1-\nu_1} \overset{p}{\#} |\xi|^{-1-\nu_2}$, where $\overset{p}{\#}$ of course denote the sharp product in the Op^p-analysis, characterized by the formula

$$\mathrm{Op}^p(h_1 \overset{p}{\#} h_2) = \mathrm{Op}^p(h_1)\, \mathrm{Op}^p(h_2).$$

It is sometimes convenient to analyze separately the commutator and anticommutator part and, in a corresponding way, to make use, on Π, of the Poisson bracket of two functions together with their pointwise product.

Definition 11.1. Given two even symbols h_1 and h_2, such that the product, in any order, of $\mathrm{Op}^p(h_1)$ and $\mathrm{Op}^p(h_2)$ makes sense, we set

$$h_1 \underset{j}{\#^p} h_2 = \begin{cases} \frac{1}{2}\left(h_1 \overset{p}{\#} h_2 + h_2 \overset{p}{\#} h_1\right) & \text{if } j = 0\,, \\ \frac{1}{2}\left(h_1 \overset{p}{\#} h_2 - h_2 \overset{p}{\#} h_1\right) & \text{if } j = 1\ : \end{cases} \tag{11.1}$$

when dealing with the Weyl calculus (the $p = 0$ case), we shall usually drop the superscript p. Given any two functions $f_1,\, f_2 \in C^\infty(\Pi)$, we set

$$f_1 \underset{j}{\times} f_2 = \begin{cases} f_1 f_2 & \text{if } j = 0\,, \\ \frac{1}{2}\{f_1, f_2\} & \text{if } j = 1\,, \end{cases} \tag{11.2}$$

where the "Poisson bracket" is defined, with $z = x + iy$, as

$$\{f_1, f_2\} : \ = y^2 \left(-\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} \right). \tag{11.3}$$

We need to briefly refresh the reader's memory on the spectral decomposition of the (non-automorphic) Laplace-Beltrami operator Δ on Π.

As a consequence of spherical representation theory, if $f \in C_0^\infty(\Pi)$, one may write (Mehler's formula)

$$f(z) = \int_0^\infty f_\lambda(z) \left(\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2} \right) d\lambda \tag{11.4}$$

with

$$f_\lambda(z) = \frac{1}{4\pi^2} \int_\Pi f(w)\, \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w))\, d\mu(w), \tag{11.5}$$

where $\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}$ is the usual Legendre function, d stands for the geodesic distance on Π, and the numerical factor on the right-hand side of (11.4) is $|c(\frac{\lambda}{2})|^{-2}$ in terms of Harish-Chandra's c-function. One may note the formula

$$\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2} = \pi\, \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)}\, \frac{\Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(\frac{-i\lambda}{2}\right)}\,. \tag{11.6}$$

Let $\mathcal{H}_{i\lambda}$ be the completion of the space of all f_λ ($f \in C_0^\infty(\Pi)$) under the norm defined by

$$\begin{aligned} \|f_\lambda\|^2_{\mathcal{H}_{i\lambda}} &= (4\pi^2)^{-2} \int_{\Pi\times\Pi} f(z)\, \bar{f}(w)\, \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w))\, d\mu(z)\, d\mu(w) \\ &= (4\pi^2)^{-1}\, (f, f_\lambda)_{L^2(\Pi)}\,. \end{aligned} \tag{11.7}$$

Then

$$\|f\|^2_{L^2(\Pi)} = 4\pi^2 \int_0^\infty \|f_\lambda\|^2_{\mathcal{H}_{i\lambda}} \left(\frac{\pi\lambda}{2}\tanh\frac{\pi\lambda}{2}\right) d\lambda, \tag{11.8}$$

and since

$$\Delta f_\lambda = \frac{1}{4}(1+\lambda^2) f_\lambda, \tag{11.9}$$

(11.4) and (11.8) express the decomposition of $L^2(\Pi)$ as a Hilbert direct integral of eigenspaces of Δ.

One way to prove (11.4)–(11.5) without harmonic analysis – but with the help of a few special function formulas – is based on the Stieltjes-Stone-Kodaira-Titschmarsh theorem [41] or [48, p. 111], which gives the spectral density (when it exists)

$$\frac{dE_\rho}{d\rho} = \frac{1}{2i\pi}\left[(T-(\rho+i0)\,I)^{-1} - (T-(\rho-i0)\,I)^{-1}\right] \tag{11.10}$$

relative to the spectral decomposition

$$I = \int dE_\rho, \qquad T = \int \rho\, dE_\rho \tag{11.11}$$

of some self-adjoint operator T: in our case, $T=\Delta$, acting on the Hilbert space $L^2(\Pi)$, so that the $d\rho$-integral only takes place on $[\frac{1}{4},\infty[$ and we may set $\rho = \frac{1+\lambda^2}{4}$; we are more interested, actually, in $\frac{d}{d\lambda}E_{\frac{1+\lambda^2}{4}} = \frac{\lambda}{2}\left(\frac{dE_\rho}{d\rho}\right)(\rho = \frac{1+\lambda^2}{4})$. Now the resolvant of Δ, *i.e.*, the operator $(\Delta - \frac{1-\nu^2}{4})^{-1}$, is given if $\mathrm{Re}\,\nu < 0$ as

$$\left(\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1} f\right)(z) = \frac{1}{2\pi}\int_\Pi f(w)\,\mathfrak{Q}_{-\frac{1}{2}-\frac{\nu}{2}}(\cosh d(z,w))\,d\mu(w) \tag{11.12}$$

in terms of some Legendre function: a proof of this can be found, for instance, in [29], or [48, p. 270] or [62, p. 206]. If $\lambda > 0$ and $\nu = -0 \pm i\lambda$, one has $\frac{1-\nu^2}{4} = \frac{1+\lambda^2}{4} \pm i0$, so that the integral kernel of $\frac{dE_\rho}{d\rho}$ at $\rho = \frac{1+\lambda^2}{4}$ is given as

$$\frac{1}{2i\pi}\cdot\frac{1}{2\pi}\left[\mathfrak{Q}_{-\frac{1}{2}-\frac{i\lambda}{2}}(\delta) - \mathfrak{Q}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta)\right] \tag{11.13}$$

with $\delta(z,w) = \cosh d(z,w)$. One can conclude with the help of the equation

$$\pi\,\frac{\Gamma(\frac{1+i\lambda}{2})\,\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\,\Gamma(-\frac{i\lambda}{2})} \times \frac{1}{4\pi^2}\,\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta) = \frac{1}{4\pi^2 i}\left[\mathfrak{Q}_{-\frac{1}{2}-\frac{i\lambda}{2}}(\delta) - \mathfrak{Q}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta)\right] \times \frac{\lambda}{2} \tag{11.14}$$

to be found in [31, p. 164].

One should be careful to notice that, in the spectral decomposition of a function f on Π, the subscript λ in f_λ always satisfies $\lambda \geq 0$, and points towards the (generalized) eigenvalue $\frac{1+\lambda^2}{4}$ of Δ. On the other hand (*cf.* (2.13), (2.14)), on (even) functions h on $\mathbb{R}^2$, the subscript λ in h_λ can be any real number, and points towards the (generalized) eigenvalue $-\frac{\lambda}{2\pi}$ of $\mathcal{E}$, corresponding to functions homogeneous of degree $-1-i\lambda$.

Recall from (9.37) the formula

$$(u_z|\mathrm{Op}(|x|^{-1-\nu})u_z) = 2^{\frac{\nu+1}{2}}\,\pi^{\frac{\nu}{2}}\,\Gamma\left(-\frac{\nu}{2}\right)\left(\frac{|z|^2}{\mathrm{Im}\ z}\right)^{\frac{-\nu-1}{2}}, \tag{11.15}$$

$\nu \neq 0, 2, \ldots$. We need to extend (11.5) as a definition for some more general functions f, built from analogues of (11.15), not even square-integrable in general. Even though $\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}$ is an even function of λ, we shall always, for clarity (so that it should be clear that we are dealing with a spectral decomposition on Π, not $\mathbb{R}^2$), assume that $\lambda \geq 0$ in f_λ.

Lemma 11.2. *Let ε_1 and ε_2 satisfy $\varepsilon_1+\varepsilon_2 > -1$, $|\varepsilon_1-\varepsilon_2| < 1$, and let a continuous function f on Π satisfy*

$$|f(z)| \leq C\left(\frac{|z|^2}{\mathrm{Im}\ z}\right)^{-\frac{\varepsilon_1}{2}-\frac{1}{2}}(\mathrm{Im}\ z)^{\frac{\varepsilon_2+1}{2}} \tag{11.16}$$

or

$$|f(z)| \leq C\,|\mathrm{Re}\ z|\left(\frac{|z|^2}{\mathrm{Im}\ z}\right)^{-\frac{\varepsilon_1}{2}-\frac{3}{2}}(\mathrm{Im}\ z)^{\frac{\varepsilon_2+1}{2}} \tag{11.17}$$

for some constant $C>0$. Then, for every $\lambda \geq 0$, one may define $f_\lambda(z)$ by (11.5), *where the integral is convergent.*

Proof. According to the decomposition of

$$\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(t) = {}_2F_1\left(\frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; 1; \frac{1-t}{2}\right) \tag{11.18}$$

provided by the first two lines of [31, p. 48], one sees, remembering that the hypergeometric series takes the value 1 at the origin, that

$$|\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(t)| \leq C\,t^{-\frac{1}{2}}, \qquad t \to \infty\,. \tag{11.19}$$

Then

$$\begin{aligned} |\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w))| &\leq C\,(\cosh d(z,w))^{-\frac{1}{2}} \\ &\leq C\,e^{\frac{d(i,z)}{2}}\,(\cosh d(i,w))^{-\frac{1}{2}} \\ &= C\,e^{\frac{d(i,z)}{2}}\left(\frac{1+(\mathrm{Im}\ w)^2+(\mathrm{Re}\ w)^2}{2\,\mathrm{Im}\ w}\right)^{-\frac{1}{2}}. \end{aligned} \tag{11.20}$$

If (11.16) is satisfied, we thus have to bound the integral

$$\int_{\Pi} \left(\frac{x^2+y^2}{y} \right)^{-\frac{\varepsilon_1}{2}-\frac{1}{2}} y^{\frac{\varepsilon_2+1}{2}} \left(\frac{1+x^2+y^2}{y} \right)^{-\frac{1}{2}} \frac{dx\,dy}{y^2}$$
$$= \int_0^{\pi} \sin^{\frac{\varepsilon_1+\varepsilon_2-1}{2}} \theta \, d\theta \int_0^{\infty} r^{\frac{\varepsilon_2-\varepsilon_1-1}{2}} (1+r^2)^{-\frac{1}{2}} \, dr \,. \tag{11.21}$$

When (11.17) is satisfied, there is an extra factor $\frac{xy}{x^2+y^2}$, which is less than 1. □

We shall begin our analysis of a product $\mathrm{Op}^p(|x|^{-1-\nu_1})\,\mathrm{Op}^p(|\xi|^{-1-\nu_2})$ by the Weyl ($p=0$) case. This case is much easier in view of the *second* (the first, historically!) covariance property of this calculus, to wit that related to the Heisenberg representation. Indeed, recall that the Weyl calculus is related in an easy way to the *standard*, or convolution-first, calculus, the definition of which is given by

$$(\mathrm{Op}_{\mathrm{std}}(h)u)(x) = \int h(x,\xi)\, u(y)\, e^{2i\pi(x-y)\xi}\, dy\, d\xi, \qquad u \in \mathcal{S}(\mathbb{R}). \tag{11.22}$$

It is immediate that both operators $\mathrm{Op}(h)$ and $\mathrm{Op}_{\mathrm{std}}(h)$ can be defined, whenever $h \in \mathcal{S}'(\mathbb{R}^2)$, as linear operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. Also, as is well known (*cf. e.g.* [52, p.15]), the standard symbol f and the Weyl symbol g of *the same* operator are linked by the formula

$$g = \exp\left(-\frac{1}{4i\pi} \frac{\partial^2}{\partial x \partial \xi} \right) f \,, \tag{11.23}$$

or

$$(\mathcal{F}_1 g)(\eta,\xi) = (\mathcal{F}_1 f)\left(\eta, \xi - \frac{\eta}{2}\right) , \tag{11.24}$$

where $\mathcal{F}_1$ denotes the Fourier transformation with respect to the first variable, defined as $(\mathcal{F}_1 g)(\eta,\xi) = \int g(x,\xi)\, e^{-2i\pi x\eta}\, dx$. The easiest way to check this formula is to check, from (11.22) and (2.1), that (with $\hat{u} = \mathcal{F}u$)

$$\mathcal{F}(\mathrm{Op}_{\mathrm{std}}(h)u)(\eta) = \int (\mathcal{F}_1 f)(\eta-\xi,\xi)\, \hat{u}(\xi)\, d\xi$$

and

$$\mathcal{F}(\mathrm{Op}(h)u)(\eta) = \int (\mathcal{F}_1 f)\left(\eta-\xi, \frac{\eta+\xi}{2} \right) \hat{u}(\xi)\, d\xi \,. \tag{11.25}$$

Defining the product of two operators $\mathrm{Op}(h_1)\,\mathrm{Op}(h_2)$ is always possible when the two symbols lie in $\mathcal{S}'(\mathbb{R}^2)$ and are such that the first one (as a function of x,ξ) depends only on x, and the second only on ξ: for, in this case, one may take Schwartz's space $\mathcal{O}_M$ [42, p. 99] as an intermediary space (one would take $\mathcal{O}'_C$ if one were interested in the product in the reverse order). Moreover, in this case, the standard symbol of the product of operators above coincides with the pointwise product of the two symbols, and (11.24) makes it possible to compute the Weyl symbol of the product as well: this is the method we shall use presently. More generally, we shall say that a symbol h is *polarized* if it depends only on some (non-zero) linear combination $ax + b\xi$ of x and ξ. Two polarized symbols are *transversally polarized* if the linear forms they depend on are linearly independent: in this case, the two associated operators can be composed in a meaningful way, since the conjugation by some metaplectic transformation (recall that all metaplectic transformations preserve the space $\mathcal{S}(\mathbb{R})$ as well as its dual) reduces the problem to the particular situation of a pair of symbols the first of which depends only on x, the second only on ξ (the intermediary space, however, is not preserved). One may recall (2.5) at this point.

The next theorem is a special case of Lemma 5.1 in [62]. We take this opportunity to give a few more details about the proof, the previous version of which made use of semi-convergent only integrals. We first set

$$|t|_j^\alpha : = |t|^\alpha\,(\mathrm{sign}(t))^j, \qquad \alpha \in \mathbb{C}, \quad j = 0, 1\,. \tag{11.26}$$

As a matter of fact, in view of our investigations with automorphic distributions, it would be sufficient to consider only sharp products of non-signed powers, *i.e.*, of the kind $|x|^{-1-\nu_1} \# |\xi|^{-1-\nu_2}$. However, the consideration of the more general case will be unavoidable in the induction procedure to be carried in the next section in relation to the Op^p-calculus. Before stating the theorem, let us recall that $x \mapsto |x|_k^{-1-\nu}$ is well defined as a tempered distribution, depending holomorphically on ν, for $\nu \neq k, k+2, \dots$.

Theorem 11.3. *Let $k = 0$ or 1, and let ν_1 and ν_2 be complex numbers with $|\mathrm{Re}\,(\nu_1 - \nu_2)| < 1$, $\mathrm{Re}\,(\nu_1 + \nu_2) > -1$, and $\nu_1 \neq k, k+2, \dots$, $\nu_2 \neq k, k+2, \dots$. For $j = 0$ or 1, and $\lambda \in \mathbb{R}$, set*

$$C_j(\nu_1, \nu_2; k; i\lambda) = 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}}\, \pi^{\frac{\nu_1+\nu_2-i\lambda}{2}}\, \frac{\Gamma(\frac{-\nu_1+k}{2})\Gamma(\frac{-\nu_2+k}{2})}{\Gamma(\frac{\nu_1+k+1}{2})\Gamma(\frac{\nu_2+k+1}{2})}$$
$$\times\, i^{j+k-2jk}\, \frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2(j+k-2jk)}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4})}{\Gamma(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4})\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2(j+k-2jk)}{4})\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4})}\,. \tag{11.27}$$

Let $h_1(x,\xi) = |x|_k^{-1-\nu_1}$ and $h_2(x,\xi) = |\xi|_k^{-1-\nu_2}$. Then one has, in the weak sense in $\mathcal{S}'(\mathbb{R}^2)$,

$$h_1 \# h_2 = \int_{-\infty}^{\infty} h_\lambda\, d\lambda, \tag{11.28}$$

with

$$h_\lambda(x,\xi) = \sum_{j=0,1} C_j(\nu_1, \nu_2; k; i\lambda)\, |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\,. \tag{11.29}$$

Proof. The claim is that, for every function $W \in \mathcal{S}(\mathbb{R}^2)$, one has

$$\langle h_1 \# h_2, W \rangle = \int_{-\infty}^{\infty} \langle h_\lambda, W \rangle \, d\lambda \,. \tag{11.30}$$

Using the estimate [31, p. 13]

$$|\Gamma(x+iy)| \sim (2\pi)^{\frac{1}{2}} \, e^{-\frac{\pi}{2}|y|} \, |y|^{x-\frac{1}{2}}, \qquad |y| \to \infty \,, \tag{11.31}$$

we first remark that the coefficient $|C_j(\nu_1, \nu_2; k; i\lambda)|$ is majorized by some constant times $|\lambda|^{\frac{1}{2}\mathrm{Re}\,(\nu_1+\nu_2)}$. Since

$$\left(x\frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} \right) \left(|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \right) \\ = (-1-i\lambda) \, |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \,, \tag{11.32}$$

the right-hand side of (11.30) can also be written for any positive integer N as

$$\int_{-\infty}^{\infty} \sum_{j=0,1} C_j(\nu_1, \nu_2; k; i\lambda) \, (1+i\lambda)^{-N} \, d\lambda \\ \left\langle |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \,, \left(x\frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} + 2 \right)^N W \right\rangle , \tag{11.33}$$

a convergent integral under the assumptions made on W and ν_1, ν_2. On the other hand, as a consequence of (11.23),

$$\langle h_1 \# h_2, W \rangle = \left\langle |x|^{-1-\nu_1} \, |\xi|^{-1-\nu_2} \,, \exp\left(-\frac{1}{4i\pi} \frac{\partial^2}{\partial x \partial \xi} \right) W \right\rangle , \tag{11.34}$$

a holomorphic function of ν_1, ν_2 in the domain indicated. This makes it possible to assume, without loss of generality, that $-1 - \mathrm{Re}\,\nu_2 < \mathrm{Re}\,\nu_1 < \mathrm{Re}\,\nu_2 < 0$. Under this condition, one may write, using (11.24),

$$(h_1 \# h_2)(x, \xi) = (-i)^k \, \pi^{\nu_1 + \frac{1}{2}} \, \frac{\Gamma(\frac{-\nu_1+k}{2})}{\Gamma(\frac{\nu_1+1+k}{2})} \int_{-\infty}^{\infty} e^{2i\pi x \eta} \, |\eta|_k^{\nu_1} \, \left| \xi - \frac{\eta}{2} \right|_k^{-1-\nu_2} d\eta \,, \tag{11.35}$$

as was done in [62], (5.28), as well as

$$(h_1 \# h_2)^\flat_\lambda(s) = \frac{1}{2\pi} \int_0^\infty r^{i\lambda} \, (h_1 \# h_2)(rs, r) \, dr \\ = \frac{1}{2} \, (-i)^k \, \pi^{\nu_1 - \frac{1}{2}} \, \frac{\Gamma(\frac{-\nu_1+k}{2})}{\Gamma(\frac{\nu_1+1+k}{2})} \int_{-\infty}^{\infty} |\eta|_k^{\nu_1} \, d\eta \int_0^\infty r^{i\lambda} \, e^{2i\pi r s \eta} \, \left| r - \frac{\eta}{2} \right|_k^{-1-\nu_2} dr \,, \tag{11.36}$$

as written in *loc.cit.*, (5.30). To improve the convergence still, one may write

$$(h_1 \# h_2)(x,\xi) = \lim_{\varepsilon\to 0}(h_1\# h_2)^\varepsilon(x,\xi)\,, \tag{11.37}$$

with

$$(h_1\# h_2)^\varepsilon(x,\xi) = e^{-2\pi\varepsilon\xi^2}\,(h_1\# h_2)(x,\xi)\,. \tag{11.38}$$

Substituting $(h_1\#h_2)^\varepsilon$ for $h_1\#h_2$, one is led to inserting the extra factor $e^{-2\pi\varepsilon r^2}$ under the last integral on the right-hand side of (11.36): then one is dealing with a genuinely convergent double integral, and the rest of the proof of Lemma 5.1 in [62] goes without change. □

Reading the rather technical rest of this section is not necessary for the sequel. The main interest of Theorem 11.4 is to show why, when coupled with spectral decomposition, the bilinear operations $\underset{j}{\#}$ are so closely related to the pointwise product and Poisson bracket on Π. A similar phenomenon will show up in Section 17. Theorem 11.4 and its corollary are also necessary if one tries to repair the heuristic proof, given in Section 5, of our main formula: we have chosen a different path, in Sections 13 to 15, but the one not expounded would be needed if one wanted to treat the case of two Eisenstein factors $\mathfrak{E}_{\nu_1}$ and $\mathfrak{E}_{\nu_1}$ with $\text{Re}\,\nu_1 > 1$, $\text{Re}\,\nu_2 > 1$.

Theorem 11.4. *Set*

$$f_j^{\nu_1,\nu_2}(z) = (u_z\,|\,\text{Op}(|x|^{-1-\nu_1})u_z) \underset{j}{\times} (u_z\,|\,\text{Op}(|\xi|^{-1-\nu_2})u_z)\,. \tag{11.39}$$

Assume that $|\text{Re}\,(\nu_1\pm\nu_2)| < 1$, $\nu_1\neq 0$, $\nu_2\neq 0$. *Then, for all* $z\in\Pi$, *one has*

$$(u_z|\text{Op}(|x|^{-1-\nu_1}\#|\xi|^{-1-\nu_2})u_z) = \pi^2\sum_{j=0}^{1}(-i)^j\int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(\frac{1-\nu_1-\nu_2+i\lambda+2j}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1+\nu_2+i\lambda+2j}{4}\right)} \left(f_j^{\nu_1,\nu_2}\right)_{|\lambda|}\,d\lambda\,, \tag{11.40}$$

where the operation indicated by the last subscript $|\lambda|$ *is defined according to Lemma* 11.2, i.e., *by equation* (11.5).

On the other hand,

$$(u_z|\mathrm{Op}(|x|_1^{-1-\nu_1}\#|\xi|_1^{-1-\nu_2})u_z)$$
$$= \pi^2 \sum_{j=0}^{1} i^{j+1} \frac{\Gamma(\frac{1-\nu_1}{2})\,\Gamma(\frac{1-\nu_2}{2})\,\Gamma(\frac{1+\nu_1}{2})\,\Gamma(\frac{1+\nu_2}{2})}{\Gamma(-\frac{\nu_1}{2})\,\Gamma(-\frac{\nu_2}{2})\,\Gamma(\frac{2+\nu_1}{2})\,\Gamma(\frac{2+\nu_2}{2})} \int_{-\infty}^{\infty} \frac{\Gamma(\frac{3+\nu_1+\nu_2-i\lambda-2j}{4})}{\Gamma(\frac{3-\nu_1-\nu_2+i\lambda-2j}{4})}$$
$$\times \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4})\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2+i\lambda+2j}{4})}$$
$$(f_j^{\nu_1,\nu_2})_{|\lambda|}\, d\lambda\,. \tag{11.41}$$

Substituting u_z^1 for u_z on the left-hand side, one has in both cases a similar identity after one has plugged the extra factor $-i\lambda$ on the right-hand side.

Proof. With $C_j(\nu_1,\nu_2;k;i\lambda)$ as defined in (11.27), and $h_1(x,\xi) = |x|_k^{-1-\nu_1}$, $h_2(x,\xi) = |\xi|_k^{-1-\nu_2}$, one has, according to Theorem 11.3,

$$(u_z|\mathrm{Op}((h_1\#h_2)_\lambda)u_z) = \sum_{j=0,1} C_j(\nu_1,\nu_2;k;i\lambda)\, I^j_{\nu_1,\nu_2;i\lambda}(z)\,, \tag{11.42}$$

where

$$I^j_{\nu_1,\nu_2;i\lambda}(z) := \left(u_z \,|\, \mathrm{Op}\left(|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}\right) u_z\right). \tag{11.43}$$

From (2.3) and (2.27),

$$I^j_{\nu_1,\nu_2;i\lambda}(z) = 2\int_{\mathbb{R}^2} |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}} \exp\left(-\frac{2\pi}{\mathrm{Im}\ z}|x-z\xi|^2\right) dx\, d\xi\,. \tag{11.44}$$

Setting $s = \frac{x}{\xi} \in \mathbb{R}$, $r = x^2+\xi^2 > 0$, so that $x = \frac{r^{\frac12} s}{(1+s^2)^{\frac12}}$, $\xi = \frac{r^{\frac12}}{(1+s^2)^{\frac12}}$, $dx\,d\xi = \frac{dr\,ds}{2(1+s^2)}$ and $|x-z\xi|^2 = \frac{r|z-s|^2}{1+s^2}$, we get

$$I^j_{\nu_1,\nu_2;i\lambda}(z)$$
$$= 2\int_{-\infty}^{\infty} |s|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} (1+s^2)^{\frac{-1+i\lambda}{2}}\, ds \int_0^\infty r^{\frac{-1-i\lambda}{2}} \exp\left(-\frac{2\pi\, r\,|z-s|^2}{(\mathrm{Im}\ z)\,(1+s^2)}\right) dr$$
$$= 2\,(2\pi)^{\frac{-1+i\lambda}{2}}\, \Gamma\left(\frac{1-i\lambda}{2}\right) \int_{-\infty}^{\infty} |s|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \left(\frac{|z-s|^2}{\mathrm{Im}\ z}\right)^{\frac{-1+i\lambda}{2}} ds\,. \tag{11.45}$$

On the other hand, the case $p=0$ from (9.37) gives $(u_z|\mathrm{Op}(h_1)u_z)$: in view of (7.48), $\mathcal{F}u_z$ agrees with $u_{-\frac{1}{z}}$ up to some constant of modulus one, so that, from the covariance formula (2.4),

$$(u_z \,|\, \mathrm{Op}(|\xi|^{-1-\nu_2}) u_z) = (\mathcal{F} u_z \,|\, \mathrm{Op}(|x|^{-1-\nu_2}) \mathcal{F} u_z) \\ = (u_{-\frac{1}{z}} \,|\, \mathrm{Op}(|x|^{-1-\nu_2}) u_{-\frac{1}{z}}) . \qquad (11.46)$$

One thus has

$$f_0^{\nu_1,\nu_2}(z) = 2\,(2\pi)^{\frac{\nu_1+\nu_2}{2}}\, \Gamma\left(-\frac{\nu_1}{2}\right) \Gamma\left(-\frac{\nu_2}{2}\right) |z|^{-\nu_1-1} (\mathrm{Im}\ z)^{\frac{\nu_1+\nu_2+2}{2}} \qquad (11.47)$$

and, as a small computation involving the use of the Poisson bracket (11.3) shows,

$$f_1^{\nu_1,\nu_2}(z) \\ = -\frac{1}{2} (2\pi)^{\frac{\nu_1+\nu_2}{2}} (1+\nu_1)(1+\nu_2) \Gamma\left(-\frac{\nu_1}{2}\right) \Gamma\left(-\frac{\nu_2}{2}\right) (\mathrm{Re}\ z) |z|^{-\nu_1-3} (\mathrm{Im}\ z)^{\frac{\nu_1+\nu_2+4}{2}} . \qquad (11.48)$$

We may then compute $(f_0^{\nu_1,\nu_2})_{|\lambda|}$ and $(f_1^{\nu_1,\nu_2})_{|\lambda|}$ by an application of (11.5), in which we have substituted for the Legendre function $\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w))$ its integral expression

$$\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{|z-s|^2}{\mathrm{Im}\ z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} \left(\frac{|w-s|^2}{\mathrm{Im}\ w}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} ds \qquad (11.49)$$

proved in [62, (4.38)]. We thus get

$$(f_0^{\nu_1,\nu_2})_{|\lambda|}(z) = 2^{\frac{\nu_1+\nu_2-2}{2}}\, \pi^{\frac{\nu_1+\nu_2}{2}-3} \Gamma\left(-\frac{\nu_1}{2}\right) \Gamma\left(-\frac{\nu_2}{2}\right) \int_\Pi |w|^{-\nu_1-1} (\mathrm{Im}\ w)^{\frac{\nu_1+\nu_2+2}{2}} (w) \\ d\mu \int_{-\infty}^{\infty} \left(\frac{|z-s|^2}{\mathrm{Im}\ z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} \left(\frac{|w-s|^2}{\mathrm{Im}\ w}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} ds . \qquad (11.50)$$

We integrate first with respect to w, picking from [62], (8.22) the formula

$$\int_\Pi \left(\frac{|w|^2}{\mathrm{Im}\ w}\right)^{\frac{-\nu_1-1}{2}} \left(\frac{|w-s|^2}{\mathrm{Im}\ w}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} (\mathrm{Im}\ w)^{\frac{\nu_2+1}{2}}\, d\mu(w) = \frac{\pi^{\frac{1}{2}}}{2} |s|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \\ \times \frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2+i\lambda}{4})}{\Gamma(\frac{1+\nu_1}{2})\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1+\nu_2}{2})} , \qquad (11.51)$$

so that

$$(f_0^{\nu_1,\nu_2})_{|\lambda|}(z) = 2^{\frac{\nu_1+\nu_2-4}{2}}\, \pi^{\frac{\nu_1+\nu_2-5}{2}}\, \Gamma\left(-\frac{\nu_1}{2}\right) \Gamma\left(-\frac{\nu_2}{2}\right) \\ \times \frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda}{4})\Gamma(\frac{1+\nu_1+\nu_2+i\lambda}{4})}{\Gamma(\frac{1+\nu_1}{2})\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1+\nu_2}{2})} \\ \int_{-\infty}^{\infty} |s|^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \left(\frac{|z-s|^2}{\mathrm{Im}\ z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} ds . \qquad (11.52)$$

The computation of $(f_1^{\nu_1,\nu_2})_{|\lambda|}(z)$ calls for that of the integral

$$\int_\Pi (\text{Re } w) \left(\frac{|w|^2}{\text{Im } w}\right)^{\frac{-\nu_1-3}{2}} \left(\frac{|w-s|^2}{\text{Im } w}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} (\text{Im } w)^{\frac{\nu_2+1}{2}} d\mu(w)$$

instead of that on the right-hand side of (11.50). Writing

$$(\text{Re } w) \left(\frac{|w|^2}{\text{Im } w}\right)^{\frac{-\nu_1-3}{2}} = -\frac{1}{\nu_1+1} (\text{Im } w) \frac{\partial}{\partial \text{Re } w} \left(\frac{|w|^2}{\text{Im } w}\right)^{\frac{-\nu_1-1}{2}}$$

and using an integration by parts, so as to finally substitute $-\frac{\partial}{\partial s}$ for $\frac{\partial}{\partial \text{Re } w}$, one can see that this integral is

$$\frac{\pi^{\frac{1}{2}}}{2} \frac{\Gamma(\frac{3+\nu_1-\nu_2+i\lambda}{4})\Gamma(\frac{3+\nu_1+\nu_2-i\lambda}{4})\Gamma(\frac{3-\nu_1+\nu_2+i\lambda}{4})\Gamma(\frac{3+\nu_1+\nu_2+i\lambda}{4})}{\Gamma(\frac{3+\nu_1}{2})\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{3+\nu_2}{2})} |s|_1^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}.$$

Thus $(f_1^{\nu_1,\nu_2})_{|\lambda|}(z)$ has an expression fully similar to the right-hand side of (11.52), except that each of the four Gamma factors $\Gamma(\frac{1\pm\nu_1\pm\nu_2\pm i\lambda}{4})$ has to be replaced by $\Gamma(\frac{3\pm\nu_1\pm\nu_2\pm i\lambda}{4})$, and there is an extra factor $-\text{sign}(s)$ in the integrand.

Setting, for $\lambda \in \mathbb{R}$,

$$(f_j^{\nu_1,\nu_2})_{|\lambda|}(z) = B_j(\nu_1,\nu_2;i\lambda)\, I^j_{\nu_1,\nu_2;i\lambda}(z)\,, \tag{11.53}$$

a definition of the first factor on the right-hand side, we thus have, comparing (11.45) and (11.52),

$$B_j(\nu_1,\nu_2;i\lambda) = (-1)^j\, 2^{\frac{\nu_1+\nu_2-i\lambda-5}{2}}\, \pi^{\frac{\nu_1+\nu_2-i\lambda-4}{2}}\, \frac{\Gamma(-\frac{\nu_1}{2})\,\Gamma(-\frac{\nu_2}{2})}{\Gamma(\frac{1+\nu_1}{2})\,\Gamma(\frac{1+\nu_2}{2})}$$
$$\frac{\Gamma(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2j}{4})\Gamma(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2+i\lambda+2j}{4})}{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}. \tag{11.54}$$

In view of (11.42), we thus have

$$(u_z|\text{Op}((h_1\#h_2)_\lambda)u_z) = \sum_{j=0,1} \frac{C_j(\nu_1,\nu_2;k;i\lambda)}{B_j(\nu_1,\nu_2;i\lambda)}\, (f_j^{\nu_1,\nu_2})_{|\lambda|}(z) \tag{11.55}$$

and, using (11.27) and (11.54), we immediately get

$$\frac{C_j(\nu_1,\nu_2;k;i\lambda)}{B_j(\nu_1,\nu_2;i\lambda)} = \pi^2\,(-i)^j$$
$$\times \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2j}{4})\Gamma(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4})\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4})\Gamma(\frac{1+\nu_1+\nu_2+i\lambda+2j}{4})}, \tag{11.56}$$

which proves (11.40).

When changing k from 0 to 1, the change in the left-hand side (from (11.40) to (11.41)) is due to the change in the coefficient $C_j(\nu_1,\nu_2;k;i\lambda)$ from (11.27). There is an extra factor of

$$\frac{\Gamma(\frac{1-\nu_1}{2})\,\Gamma(\frac{1-\nu_2}{2})\,\Gamma(\frac{1+\nu_1}{2})\,\Gamma(\frac{1+\nu_2}{2})}{\Gamma(\frac{2+\nu_1}{2})\,\Gamma(\frac{2+\nu_2}{2})\,\Gamma(-\frac{\nu_1}{2})\,\Gamma(-\frac{\nu_1}{2})}\; i^{1-2j} \times \frac{\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2-2j}{4})\,\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2j}{4})}{\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2-2j}{4})\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2j}{4})}\,. \tag{11.57}$$

Observing that one of the four Gamma factors below is not the same in the right-hand sides of (11.40) and (11.41), we see that, going from the right-hand side of (11.40) to that of (11.41), there is an extra factor of

$$i^{2j+1}\,\frac{\Gamma(\frac{1-\nu_1}{2})\,\Gamma(\frac{1-\nu_2}{2})\,\Gamma(\frac{1+\nu_1}{2})\,\Gamma(\frac{1+\nu_2}{2})}{\Gamma(\frac{2+\nu_1}{2})\,\Gamma(\frac{2+\nu_2}{2})\,\Gamma(-\frac{\nu_1}{2})\,\Gamma(-\frac{\nu_1}{2})}\,\frac{\Gamma(\frac{3+\nu_1+\nu_2-i\lambda-2j}{4})}{\Gamma(\frac{3-\nu_1-\nu_2+i\lambda-2j}{4})}\times\frac{\Gamma(\frac{1-\nu_1-\nu_2+i\lambda+2j}{4})}{\Gamma(\frac{1+\nu_1+\nu_2-i\lambda+2j}{4})}\,. \tag{11.58}$$

The expressions (11.57) and (11.58) agree, which concludes the proof of the part of Theorem 11.4 dealing with the functions u_z.

When substituting the functions u_z^1 for the functions u_z, one can see, using (2.27) and (2.28), that all one has to do is to replace the factor $\exp(-\frac{2\pi}{\mathrm{Im}\,z}|x-z\xi|^2)$ from the integrand of (11.44) by its image under the operator $-2i\pi\,\mathcal{E}$: one can also, instead, let the operator $2i\pi\,\mathcal{E}$ act on $|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}$, ending up with the extra factor $-i\lambda$. □

Corollary 11.5. *The formulas* (11.40) *and* (11.41) *still hold if we substitute for the assumptions* $|\mathrm{Re}\,(\nu_1\pm\nu_2)|<1$, $\nu_1\neq 0$, $\nu_2\neq 0$, *the assumptions*

$$|\mathrm{Re}\,(\nu_1-\nu_2)|<1, \quad \mathrm{Re}\,(\nu_1+\nu_2)>-1 \tag{11.59}$$

together with $\nu_1\neq k,k+2,\dots,\nu_2\neq k,k+2,\dots$ *where* k *has the value* 0 *or* 1 *according to whether we are dealing with the formula* (11.40) *or* (11.41).

Proof. As a distribution-valued function, the map $\nu_1\mapsto |x|_k^{-1-\nu_1}$ is holomorphic for $\nu_1\neq k,k+2,\dots$. Thus, the left-hand side of (11.40) or (11.41) is, for fixed $z\in\Pi$, a holomorphic function of ν_1, ν_2 in the indicated domain. So as to simplify notation, we consider only the case when $k=0$ from now on, as there is no longer any difference between the two cases. In view of (9.37), nothing is changed in the computation of $f_0^{\nu_1,\nu_2}(z)$, $f_1^{\nu_1,\nu_2}(z)$ which led to (11.47) and (11.48). Getting rid of the factors which depend only on ν_1, ν_2, which are all holomorphic, we see that all that has to be done is to show that the integral

$$\int_{-\infty}^{\infty}\frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{1+i\lambda-\nu_1-\nu_2+2j}{4})\Gamma(\frac{1-i\lambda-\nu_1+\nu_2+2j}{4})\Gamma(\frac{1-i\lambda+\nu_1-\nu_2+2j}{4})\Gamma(\frac{1+i\lambda+\nu_1+\nu_2+2j}{4})}\,d\lambda\int_\Pi \mathfrak{P}_{-\frac12+\frac{i\lambda}{2}}(\cosh d(z,w))\,f(w)\,d\mu(w)\,, \tag{11.60}$$

meant as a superposition of integrals (integrating with respect to $d\mu(w)$ first), converges and depends on ν_1, ν_2 in a holomorphic way, in the case when $f = g^{\nu_1,\nu_2}$ or $f = g_1^{\nu_1,\nu_2}$, with

$$g^{\nu_1,\nu_2}(z) = (z\bar{z})^{\frac{-\nu_1-1}{2}} (z-\bar{z})^{\frac{\nu_1+\nu_2+2}{2}}$$

and

$$g_1^{\nu_1,\nu_2}(z) = (z+\bar{z})\,(z\bar{z})^{\frac{-\nu_1-3}{2}} (z-\bar{z})^{\frac{\nu_1+\nu_2+4}{2}} . \tag{11.61}$$

Now, the convergence of the integral over Π on the right-hand side of (11.60) is just a consequence of Lemma 11.2, and it remains to take care of the $d\lambda$-integration as well. In view of (11.31), whenever ν_1, ν_2 lies within a compact subset of the domain under consideration, one has for some constant C the estimate

$$\left| \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{1+i\lambda-\nu_1-\nu_2+2j}{4})\Gamma(\frac{1-i\lambda-\nu_1+\nu_2+2j}{4})\Gamma(\frac{1-i\lambda+\nu_1-\nu_2+2j}{4})\Gamma(\frac{1+i\lambda+\nu_1+\nu_2+2j}{4})} \right| \leq C\,(1+|\lambda|)^{1-2j} . \tag{11.62}$$

On the other hand, an elementary, if tedious, computation, shows (writing $\Delta = (z-\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}$) that $\Delta g^{\nu_1,\nu_2}$ is a linear combination of g^{ν_1,ν_2} and g^{ν_1+2,ν_2+2}, and that $\Delta g_1^{\nu_1,\nu_2}$ is a linear combination of $g_1^{\nu_1,\nu_2}$, $g_1^{\nu_1+2,\nu_2+2}$ and g^{ν_1+2,ν_2+2}. Since

$$f_{|\lambda|} = \left(\frac{1+\lambda^2}{4}\right)^{-2} (\Delta^2 f)_{|\lambda|} , \tag{11.63}$$

the $d\lambda$-integration is taken care of as well. □

12 Beyond the symplectic group

The main aim of this section is to extend Theorem 11.3, the composition of the (Weyl) sharp product of two power functions, to the Op^p-calculus. A crucial tool for this lies in the understanding of a link between the operators $\mathrm{Op}^p(h)$ and $\mathrm{Op}^p(2i\pi\,\mathcal{E}\,h)$, h being any distribution in $\mathcal{F}(\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$.

It is our feeling that these latter considerations may be of some independent interest in the Weyl calculus itself, in which it is not more difficult to tackle with the n-dimensional case.

Recall that the definition of the n-dimensional Weyl calculus is just the same as (2.1), except for the fact that the exponent $2i\pi(x-y)\eta$ must be replaced by $2i\pi\,\langle x-y,\eta\rangle$, the integration taking place on $\mathbb{R}^{2n}$. Again, the linear map Op is an isomorphism from $\mathcal{S}'(\mathbb{R}^{2n})$ onto the space of weakly continuous linear operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. There is one pair (Q_j, P_j) of position and momentum operators for each coordinate: Q_j is the operator which multiplies a function of x by x_j, and $P_j = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}$; the Weyl symbols of these two operators are the two functions $(x,\xi) \mapsto x_j$ and $(x,\xi) \mapsto \xi_j$. Multiplying an operator $\mathrm{Op}(h)$, on the left or on the right, by P_j or Q_j, can be traced on the level of symbols by the set of formulas

$$\begin{aligned} \xi_j \# h &= \xi_j\, h + \frac{1}{4i\pi}\frac{\partial h}{\partial x_j} \quad , \quad & x_j \# h &= x_j\, h - \frac{1}{4i\pi}\frac{\partial h}{\partial \xi_j}\,, \\ h \# \xi_j &= \xi_j\, h - \frac{1}{4i\pi}\frac{\partial h}{\partial x_j} \quad , \quad & h \# x_j &= x_j\, h + \frac{1}{4i\pi}\frac{\partial h}{\partial \xi_j}\,. \end{aligned} \tag{12.1}$$

Let $\mathfrak{h}$ be the ($(2n+1)$-dimensional) Lie algebra of the Heisenberg group: since $\{\xi_j, x_k\} = \delta_{jk}$, $\mathfrak{h}$ can be identified with the space of real-valued affine functions $(x,\xi) \mapsto \langle \alpha, x\rangle + \langle \beta, \xi\rangle + c$ on $\mathbb{R}^{2n}$, provided with the structure associated to the Poisson bracket. One has $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the subspace of linear functions on $\mathbb{R}^{2n}$.

Elements in the symmetric second power of $\mathfrak{p}$ (over $\mathbb{R}$) can be identified to homogeneous polynomials of degree 2 with real coefficients, *i.e.,* the functions ℓ,

$$\ell(x,\xi) = \langle ax, x\rangle + 2\,\langle bx, \xi\rangle + \langle c\xi, \xi\rangle\ , \tag{12.2}$$

where a, b, c are real $n \times n$ matrices with real entries and, denoting as a' the transpose of a, one has $a' = a$ and $c' = c$. Then, given any symbol $h \in \mathcal{S}'(\mathbb{R}^{2n})$, one has (a consequence of (12.1))

$$i\pi\,(\ell \# h - h \# \ell) = \left\langle bx + c\xi, \frac{\partial h}{\partial x}\right\rangle - \left\langle ax + b'\xi, \frac{\partial h}{\partial \xi}\right\rangle\ , \tag{12.3}$$

a function which shall also be denoted as $\mathrm{ad}\, i\pi\ell\,(h)$. This is the image of h under the differential operator $\mathrm{ad}\, i\pi\ell$ with real linear coefficients on $\mathbb{R}^{2n}$ associated with the matrix

$$\begin{pmatrix} b & c \\ -a & -b' \end{pmatrix} \in \mathfrak{sp}(n,\mathbb{R})\ : \tag{12.4}$$

actually, this matrix is the generic element of the Lie algebra of the symplectic group, the dimension of which is $2n^2 + n$.

Setting $A = \mathrm{Op}(h)$ and $L = \mathrm{Op}(i\pi\ell)$, one may write (12.3) as

$$\mathrm{ad}\, L\,(A) = \mathrm{Op}\,(\mathrm{ad}\, i\pi\ell\,(h))\,. \tag{12.5}$$

This is the infinitesimal version of the (n-dimensional version of) the covariance relation (2.4) of the Weyl calculus under the metaplectic representation: indeed, the operators $(i\pi)^{-1}L$ are just the infinitesimal operators, in the sense of Stone's theorem, of the unitaries arising from the metaplectic representation.

As is well known, the analysis of the effect, on an operator, of the operations of commutation with a prescribed set of operators plays an important role in pseudodifferential analysis, where it may serve to characterize classes of operators defined by conditions relative to their symbols, as R. Beals showed [4].

We now want to address the following question: is there any interpretation of the second *antisymmetric* power of $\mathfrak{p}$? Alternatively, can one find an operator-theoretic interpretation of the action on symbols of elements in the Lie algebra $\mathfrak{gl}\,(2n,\mathbb{R})$?

Proposition 12.1. *With any element*

$$\Lambda = i\pi \sum_k R_k \wedge S_k\,, \tag{12.6}$$

with $\sum_k R_k \wedge S_k$ *lying in the second antisymmetric power, over* $\mathbb{R}$, *of the linear space generated by all the* Q_j*'s,* P_j*'s, associate the linear operator* $\mathrm{mad}\,(\Lambda)$ *("*mad*" stands for "mixed adjoint") acting on the space of linear operators from* $\mathcal{S}(\mathbb{R}^n)$ *to* $\mathcal{S}'(\mathbb{R}^n)$, *defined as*

$$\mathrm{mad}\,(\Lambda)(A) = i\pi \sum_k (R_k\, A\, S_k - S_k\, A\, R_k)\,. \tag{12.7}$$

Setting $R_k = \mathrm{Op}(r_k)$, $S_k = \mathrm{Op}(s_k)$, *one has*

$$\mathrm{mad}\,(\Lambda)(\mathrm{Op}(h)) = \mathrm{Op}(\mathrm{mad}\,(i\pi\lambda)(h)) \tag{12.8}$$

with

$$\mathrm{mad}\,(i\pi\lambda)(h) \colon = i\pi \sum_k (r_k\#h\#s_k - s_k\#h\#r_k)\,, \tag{12.9}$$

and the space of all operators $\mathrm{mad}\,(i\pi\lambda)$ *describes a supplementary subspace* $\mathfrak{op}\,(n,\mathbb{R})$ *of* $\mathfrak{sp}\,(n,\mathbb{R})$ *in* $\mathfrak{gl}\,(2n,\mathbb{R})$, *it being understood that the differential operator associated with* $M = (M_{jk})_{1\le j,k\le 2n} \in \mathfrak{gl}\,(2n,\mathbb{R})$ *is* $\sum M_{jk} X_k \frac{\partial}{\partial X_j} + \frac{1}{2}\sum M_{jj}$, *with* $X_j = x_j$ *if* $j \le n$, $X_j = \xi_{j-n}$ *if* $n+1 \le j \le 2n$.

Proof. Only the last assertion requires a proof. It is a consequence of the relations (themselves a consequence of (12.1))

$$\begin{aligned}
x_j\#h\#x_k - x_k\#h\#x_j &= \frac{1}{2i\pi}\left(x_j\,\frac{\partial h}{\partial \xi_k} - x_k\,\frac{\partial h}{\partial \xi_j}\right)\,,\\
\xi_j\#h\#\xi_k - \xi_k\#h\#\xi_j &= -\frac{1}{2i\pi}\left(\xi_j\,\frac{\partial h}{\partial x_k} - \xi_k\,\frac{\partial h}{\partial x_j}\right)\,,\\
x_j\#h\#\xi_k - \xi_k\#h\#x_j &= -\frac{1}{2i\pi}\left(x_j\,\frac{\partial h}{\partial x_k} + \xi_k\,\frac{\partial h}{\partial \xi_j} + \delta_{jk}\,h\right)\,.
\end{aligned} \tag{12.10}$$

Comparing (12.10) to (12.4), one sees that the first two lines of (12.10) provide for *antisymmetric* substitutes for the blocks c and $-a$ taken from the block-matrix in (12.4), whereas the third line permits one to obtain matrices of the species $\left(\begin{smallmatrix} b & 0 \\ 0 & b' \end{smallmatrix}\right)$:

thus, we get a full supplementary space, in $\mathfrak{gl}\,(2n,\mathbb{R})$, of the Lie algebra $\mathfrak{sp}\,(n,\mathbb{R})$. It is to be noted that, since the linear first-order operator on the right-hand side of the last line of (12.10) has to be formally self-adjoint on $L^2(\mathbb{R}^{2n})$, it is natural that the constant δ_{jk} should occur there. □

Remarks.

1. In particular, we now have an operator interpretation for the action on symbols of the operator $2i\pi\,\mathcal{E} = \sum x_j \frac{\partial}{\partial x_j} + \sum \xi_j \frac{\partial}{\partial \xi_j} + n$. Only, one should be careful that the operators with symbols h and $(x,\xi) \mapsto t^n h(tx, t\xi)$ with $t > 0$ are not unitarily equivalent in general: this transformation is tantamount to a change in "Planck's constant", coupled if needed (*i.e,* if one wants the rescaling to alter only the ξ-variables) with a (symplectic) map of the kind $h \mapsto\ \ ((x,\xi) \mapsto h(t^{-1}x, t\xi))$.
2. You may think of $\mathfrak{dp}\,(n,\mathbb{R})$ as the "diaplectic" subspace of $\mathfrak{gl}\,(2n,\mathbb{R})$, where the prefix is justified by (12.7). The subalgebra $\mathfrak{sp}\,(n,\mathbb{R})$ normalizes $\mathfrak{dp}\,(n,\mathbb{R})$ in the Lie algebra $\mathfrak{gl}\,(2n,\mathbb{R})$. This may be checked from (12.3) and (12.10) by a case-by-case study, or by the remark that if R, S, T lie in the linear space generated by the Q_j's and the P_j's, so does $RTS - TRS = R\,[S,T] + [R,T]\,S$, accompanied by the formula $[\mathrm{ad}\,(RS), \mathrm{mad}\,(T \wedge V)] = \mathrm{mad}\,((RST - TRS) \wedge V) + \mathrm{mad}\,((VRS - RSV) \wedge T)$.

In the one-dimensional case, the Euler operator $2i\pi\,\mathcal{E}$ alone enables one to bridge the gap from $\mathfrak{sl}\,(2,\mathbb{R})$ to $\mathfrak{gl}\,(2,\mathbb{R})$. We proceed towards an analysis of an operator-theoretic interpretation of this operator in the Op^p-calculus: this is not as easy if $p > 0$ as in the Weyl case, mostly in view of the more complicated commutation relation (7.13) between P and Q.

Lemma 12.2. *In the* Op^p*-calculus, the operator*

$$P\, e^{2i\pi\,(\eta Q - yP)}\, Q - Q\, e^{2i\pi\,(\eta Q - yP)}\, P - (\eta Q - yP)\, e^{2i\pi\,(\eta Q - yP)} \tag{12.11}$$

coincides, on functions with a parity related to p*, with the operator*

$$\frac{1+2p}{2i\pi}\, e^{2i\pi\,(\eta Q - yP)} + \frac{p}{2i\pi}\left[e^{-2i\pi\,(\eta Q - yP)} - e^{2i\pi\,(\eta Q - yP)}\right]\,; \tag{12.12}$$

the same formula is valid for the action of the operator on functions with a parity contrary to p*, after one has substituted* $-p$ *for* p *in* (12.12).

Proof. Set, for $t \in \mathbb{R}$,

$$R(t) = P\, e^{2i\pi t\,(\eta Q - yP)}\, Q - Q\, e^{2i\pi t\,(\eta Q - yP)}\, P\,. \tag{12.13}$$

Then

$$\begin{aligned} R'(t) &= 2i\pi\, P\,(\eta Q - yP)\, e^{2i\pi t\,(\eta Q - yP)}\, Q - 2i\pi\, Q\,(\eta Q - yP)\, e^{2i\pi t\,(\eta Q - yP)}\, P \\ &= 2i\pi\,(\eta Q - yP)\, R(t) + 2i\pi\,[P,Q]\, e^{2i\pi t\,(\eta Q - yP)}\,(\eta Q - yP)\,. \end{aligned} \tag{12.14}$$

Also, $R(0) = [P,Q]$ so that, regarding (12.14) as a differential equation, we easily get

$$R(t) = e^{2i\pi t\,(\eta Q - yP)}\,[P,Q] + \\ 2i\pi\, e^{2i\pi t\,(\eta Q - yP)} \int_0^t e^{-2i\pi s\,(\eta Q - yP)}\,[P,Q]\,e^{2i\pi s\,(\eta Q - yP)}\,ds\;(\eta Q - yP)\,. \qquad (12.15)$$

Next, we compute $2i\pi\, e^{-2i\pi\eta\, Q}\,[P,Q]\,e^{2i\pi\eta\, Q}$: since Q is the operator of multiplication by x, the block-form of this matrix is (using also (7.13))

$$\begin{pmatrix} \cos 2\pi\eta x & -i\,\sin 2\pi\eta x \\ -i\,\sin 2\pi\eta x & \cos 2\pi\eta x \end{pmatrix} \begin{pmatrix} 1+2p & 0 \\ 0 & 1-2p \end{pmatrix} \begin{pmatrix} \cos 2\pi\eta x & i\,\sin 2\pi\eta x \\ i\,\sin 2\pi\eta x & \cos 2\pi\eta x \end{pmatrix}$$
$$= I + 2p \begin{pmatrix} \cos 4\pi\eta x & i\,\sin 4\pi\eta x \\ -i\,\sin 4\pi\eta x & -\cos 4\pi\eta x \end{pmatrix}$$
$$= I + 2p\, e^{-4i\pi\eta\, Q} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (12.16)$$

In view of (9.23), here recalled (valid when $\eta \neq 0$)

$$e^{2i\pi\,(\eta Q - yP)} = e^{-i\pi \frac{y}{\eta}\,P^2}\,e^{2i\pi\eta\, Q}\,e^{i\pi\frac{y}{\eta}\,P^2} \qquad (12.17)$$

and of the fact that P^2 commutes with $[P,Q]$, one has

$$2i\pi\, e^{-2i\pi\,(\eta Q - yP)}\,[P,Q]\,e^{2i\pi\,(\eta Q - yP)} = I + 2p\, e^{-4i\pi\,(\eta Q - yP)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (12.18)$$

an identity also valid if $\eta = 0$ (using (7.36)). Using also the fact that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, which is the block-matrix form of the endomorphism $u \mapsto (-1)^p\,\check{u}$ of $\mathcal{S}_p(\mathbb{R})$, anticommutes with $\eta Q - yP$, we can make (12.15) explicit as

$$R(t) = e^{2i\pi t\,(\eta Q - yP)}\,[P,Q] + t\,e^{2i\pi t\,(\eta Q - yP)}\,(\eta Q - yP) \\ + \frac{p}{2i\pi}\left(e^{-2i\pi t\,(\eta Q - yP)} - e^{2i\pi t\,(\eta Q - yP)}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (12.19)$$

Specializing to $t = 1$, we get (12.12). □

Lemma 12.3. *Let h be an even distribution in $\mathcal{F}(\mathcal{S}'_{\boldsymbol{.}}(\mathbb{R}^2))_{\text{even}}$ and let $A = \text{Op}^p(h)$. The operators $\text{Op}^p((2p + 2i\pi\,\mathcal{E})\,h)$ and $2i\pi\,(PAQ - QAP)$ agree on the subspace of $\mathcal{S}_p(\mathbb{R})$ consisting of functions with the parity related to p, and the operators $\text{Op}^p((-2p + 2i\pi\,\mathcal{E})\,h)$ and $2i\pi\,(PAQ - QAP)$ agree on the subspace of $\mathcal{S}_p(\mathbb{R})$ consisting of functions with the parity contrary to p. Also, $(2p + 2i\pi\,\mathcal{E})\,h$ is Σ_p-invariant if and only if h is Σ_{p+1}-invariant, and $(-2p+2i\pi\,\mathcal{E})\,h$ is Σ_{p+1}-invariant if and only if h is Σ_p-invariant.*

Proof. First note that $\mathrm{Op}^p(h)$ and $\mathrm{Op}^p(2i\pi\mathcal{E}\,h)$ are well defined by Proposition 9.7. If u and v lie in $\mathcal{S}_p(\mathbb{R})$ and have the parity related to p, one has

$$(v|\,\mathrm{Op}^p(h)u) = \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta)\,(v|\,\exp 2i\pi\,(\eta Q - yP)\,u)\,dy\,d\eta \tag{12.20}$$

so that, from Lemma 12.2, using the fact that h is even,

$$2i\pi\,(v|(PAQ - QAP)u) = 2i\pi \int_{\mathbb{R}^2} (\mathcal{F}h)(y,\eta) \\ (v|\,[(\eta Q - yP)\,e^{2i\pi(\eta Q - yP)} + (1+2p)\,e^{2i\pi(\eta Q - yP)}]\,u)\,dy\,d\eta\,. \tag{12.21}$$

From (9.32),

$$2i\pi\,(\eta Q - yP)\,e^{2i\pi(\eta Q - yP)} = (2i\pi\,\mathcal{E} - 1)\,e^{2i\pi(\eta Q - yP)}\,, \tag{12.22}$$

where the operator $2i\pi\,\mathcal{E} - 1$ transfers to $-2i\pi\,\mathcal{E} - 1$ under transposition, then to $2i\pi\,\mathcal{E} - 1$ again under commutation with $\mathcal{F}$. Thus

$$2i\pi\,(v|(PAQ - QAP)u) \\ = \int_{\mathbb{R}^2} \mathcal{F}((2p + 2i\pi\,\mathcal{E})\,h)(y,\eta)\,(v|\,\exp 2i\pi\,(\eta Q - yP)\,u)\,dy\,d\eta\,. \tag{12.23}$$

Concerning the second statement, we note that, as a consequence of (6.16), one has

$$\Sigma_p\,(p + i\pi\,\mathcal{E}) = \frac{(p + i\pi\,\mathcal{E})}{(p - i\pi\,\mathcal{E})}\,\Sigma_{p+1}\,(p + i\pi\,\mathcal{E}) \\ = (p + i\pi\,\mathcal{E})\,\Sigma_{p+1}$$

and

$$\Sigma_{p+1}\,(p - i\pi\,\mathcal{E}) = (p - i\pi\,\mathcal{E})\,\Sigma_p\,. \tag{12.24}$$

□

It is convenient, at this point, to introduce some terminology. First, recall from the proof of Corollary 9.10 that, if $p \geq 1$, the operator $(p + i\pi\,\mathcal{E})^{-1}$ is well defined as an endomorphism of $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$: as a consequence, if $h \in \mathcal{F}\mathcal{S}'_{\bullet}(\mathbb{R}^2)$, *i.e.*, $\mathcal{G}h \in \mathcal{S}'_{\bullet}(\mathbb{R}^2)$, one can always define $\Sigma_p h = \frac{(-i\pi\,\mathcal{E})_p}{(i\pi\,\mathcal{E})_p}\,\mathcal{G}h \in \mathcal{S}'_{\bullet}(\mathbb{R}^2)$. If $\Sigma_p h$, too, lies in the space $\mathcal{F}\mathcal{S}'_{\bullet}(\mathbb{R}^2)$ (in particular, this is the case if h is Σ_p-invariant), then $h = \frac{(-i\pi\,\mathcal{E})_p}{(i\pi\,\mathcal{E})_p}\,\mathcal{G}(\Sigma_p h)$ is in $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$. All this may look just a little bit confusing, but this is only due to the fact that the useful space is $\mathcal{F}\mathcal{S}'_{\bullet}(\mathbb{R}^2)$ rather than $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$. Anyway, the developments in this section will yield at each step only symbols actually lying in the intersection $\mathcal{F}\mathcal{S}'_{\bullet}(\mathbb{R}^2) \cap \mathcal{S}'_{\bullet}(\mathbb{R}^2)$. Given a continuous linear operator A from $\mathcal{S}_p(\mathbb{R})$ to $\mathcal{S}'_p(\mathbb{R})$, commuting with the map $u \mapsto \check{u}$, we shall say that A admits a *lower-type* symbol if there exists an even symbol $h \in \mathcal{F}\mathcal{S}'_{\bullet}(\mathbb{R}^2)$ satisfying $\Sigma_p h = h$ such that the restrictions of A and $\mathrm{Op}^p(h)$ to the subspace of $\mathcal{S}_p(\mathbb{R})$ consisting of all functions with a parity related to p agree: of necessity, the symbol h is unique, up to the addition of a linear combination of distributions homogeneous of degrees $-1-2j, 0 \leq j \leq p-1$ (*cf.* (9.18)), connected to a horocyclic symbol, as follows from Definition 6.2 and Corollary 9.5. Similarly, we define the concept of *higher-type* symbol of A, in relation to the restriction of A to the subspace of $\mathcal{S}_p(\mathbb{R})$ consisting of all functions with a parity contrary to p: of course, it is then the involution Σ_{p+1} that has to be considered instead of Σ_p. Formula (9.20) permits us, in principle, to fully rebuild any even symbol from its lower and higher parts: but, in order that an operator admit an Op^p-symbol, it may not be sufficient that it should admit both a lower-type symbol and a higher-type symbol, since there is the problem of applying $(i\pi\,\mathcal{E})^{-1}$ to the sum on the right-hand side of (9.20).

One other word of caution is necessary: it is only in the Weyl ($p=0$) case that any symbol reduces to the sum of its lower and higher parts, for then $\Sigma_1 = -\Sigma_0$. Also, it is not true unless $p=0$ that an operator with a Σ_p-invariant symbol should vanish on functions with the parity contrary to p (or vice-versa).

Remark. The following is an immediate consequence of Lemma 12.3: let $p \geq 1$, and let A be a bounded linear operator from $\mathcal{S}_p(\mathbb{R})$ to $\mathcal{S}'_p(\mathbb{R})$, commuting with the map $u \mapsto \check{u}$. If the operator $2i\pi\,(QAP - PAQ)$ admits a lower-type symbol h_{low}, then A admits a higher-type symbol, the image under $(2p + 2i\pi\,\mathcal{E})^{-1}$ of h_{low}.

Within some fixed Op^p-calculus, we set, for $k=0$ or 1 and $\nu \in \mathbb{C}$, $\nu \neq k, k+2, \ldots$,

$$|Q|_k^{-1-\nu} = \mathrm{Op}^p(|x|_k^{-1-\nu})\,, \qquad |P|_k^{-1-\nu} = \mathrm{Op}^p(|\xi|_k^{-1-\nu})\,, \tag{12.25}$$

after having noted that the symbols involved are well defined as tempered distributions depending analytically on ν in the given range of values of (k,ν): actually, we take $|Q|_k^{-1-\nu}$ to mean the operator of multiplication by $|x|_k^{-1-\nu}$ even when $\mathrm{Re}\;\nu > 0$ and define (*cf.* (7.36)) $|P|_k^{-1-\nu} := \mathcal{F}_p^{-1}\,|Q|_k^{-1-\nu}\,\mathcal{F}_p$; this is the only place in this section where we consider symbols possibly not in $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$. Note that in the case when $k=1$ the symbols in (12.25) are *odd* functions on $\mathbb{R}^2$, so that the associated operators change the parity of functions. Their definition can nevertheless be regarded as an extension of (9.1): alternatively, one may use the formulas

$$|Q|_1^{-1-\nu} = Q\,|Q|^{-2-\nu} = |Q|^{-2-\nu}\,Q\,, \qquad |P|_1^{-1-\nu} = P\,|P|^{-2-\nu} = |P|^{-2-\nu}\,P\,. \tag{12.26}$$

We also set

$$A^p(\nu_1,\nu_2;k) = |Q|_k^{-1-\nu_1}\,|P|_k^{-1-\nu_2}\,, \tag{12.27}$$

assuming throughout this section that, besides $\nu_1 \neq k, k+2, \ldots$, $\nu_2 \neq k, k+2, \ldots$, one also has $|\text{Re } (\nu_1 - \nu_2)| < 1$, $\text{Re } (\nu_1 + \nu_2) > -1$. Even though we are really interested only in the case when $k = 0$, the consideration of the case $k = 1$ is unavoidable. Our main problem is to show that, for all p, the operator $A^p(\nu_1, \nu_2; k)$ has an Op^p-symbol $h^p(\nu_1, \nu_2; k)$; also, to relate this symbol to the Weyl symbol $h^0(\nu_1, \nu_2; k) = |x|_k^{-1-\nu_1} \# |\xi|_k^{-1-\nu_2}$: that this latter symbol is well defined as a tempered distribution is a consequence of the remark following (11.20).

It is, however, necessary for our purposes to remark a little more, namely that both $h^0(\nu_1, \nu_2; k)$ and the Fourier tranform (or, what amounts to the same, the $\mathcal{G}$-transform) of this symbol lie in the space $(\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$ used in Proposition 9.7. To that effect, one may use Theorem 11.3. Indeed, on one hand any symbol $|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}$ lies in this space since it is integrable against any continuous function rapidly decreasing at infinity; on the other hand the summation with respect to λ can be taken care of by means of (11.27). This shows that $h^0(\nu_1, \nu_2; k)$ is in $(\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$. On the other hand, from Theorem 11.3 again, we get the weak decomposition in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$:

$$\mathcal{G}h^0(\nu_1, \nu_2; k) = \int_{-\infty}^{\infty} (\mathcal{G}h^0(\nu_1, \nu_2; k))_\lambda \, d\lambda \tag{12.28}$$

with

$$(\mathcal{G}h^0(\nu_1, \nu_2; k))_\lambda(x, \xi) = \sum_{j=0,1} D_j(\nu_1, \nu_2; k; i\lambda) \, |x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}} \, |\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}, \tag{12.29}$$

and

$$D_j(\nu_1, \nu_2; k; i\lambda) = (2\pi)^{-i\lambda} \frac{\Gamma(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}) \, \Gamma(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4})}{\Gamma(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}) \, \Gamma(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4})} \times C_j(\nu_1, \nu_2; k; -i\lambda) \, : \tag{12.30}$$

the same reasoning as the one applied to $h^0(\nu_1, \nu_2; k)$ above can now be applied to its $\mathcal{G}$-transform.

We now show by induction on p that, under the hypotheses indicated right after (12.27), any operator $A^p(\nu_1, \nu_2; k)$ has a lower-type symbol as well as a higher-type symbol.

Lemma 12.4. *Under the assumptions that* $\nu_1 \neq k, k+2, \ldots$ *and* $\nu_2 \neq k, k+2, \ldots$, *together with* $|\text{Re } (\nu_1 - \nu_2)| < 1$, $\text{Re } (\nu_1 + \nu_2) > -1$, *one has*

$$A^p(\nu_1, \nu_2; k) \, Q = Q \, A^p(\nu_1, \nu_2; k) - \frac{1+\nu_2}{2i\pi} \, Q \, A^p(\nu_1 + 1, \nu_2 + 1; 1 - k) \, . \tag{12.31}$$

Proof. When acting on functions of t of any given parity, $P = (2i\pi)^{-1}(\frac{d}{dt} \pm \frac{p}{t})$, so that

$$P\left(|t|^{-1-\nu}u\right) = |t|^{-1-\nu}\, Pu - \frac{1+\nu}{2i\pi}\, |t|_1^{-2-\nu}\, u\,, \tag{12.32}$$

in other words

$$|Q|^{-1-\nu}\, P = P\, |Q|^{-1-\nu} + \frac{1+\nu}{2i\pi}\, |Q|_1^{-2-\nu}\,. \tag{12.33}$$

Using

$$P = \mathcal{F}_p^{-1}\, Q\, \mathcal{F}_p\,, \qquad -Q = \mathcal{F}_p^{-1}\, P\, \mathcal{F}_p\,, \tag{12.34}$$

we get

$$|P|^{-1-\nu}\, Q = Q\, |P|^{-1-\nu} - \frac{1+\nu}{2i\pi}\, |P|_1^{-2-\nu}\,. \tag{12.35}$$

Thus

$$\begin{aligned} A^p(\nu_1,\nu_2;0)\, Q &= |Q|^{-1-\nu_1}\, |P|^{-1-\nu_2}\, Q \\ &= |Q|^{-1-\nu_1}\, Q\, |P|^{-1-\nu_2} - \frac{1+\nu_2}{2i\pi}\, |Q|^{-1-\nu_1}\, |P|_1^{-2-\nu_2} \\ &= Q\left(|Q|^{-1-\nu_1}\, |P|^{-1-\nu_2} - \frac{1+\nu_2}{2i\pi}\, |Q|_1^{-2-\nu_1}\, |P|_1^{-2-\nu_2}\right)\,, \end{aligned} \tag{12.36}$$

which gives the case $k = 0$ of the lemma. The proof of the case $k = 1$ is identical. □

Lemma 12.5. *Let $p \geq 1$ be given. Assume that whenever $k = 0$ or 1, and ν_1, ν_2 are complex numbers such that $\nu_1 \neq k, k+2, \ldots$, $\nu_2 \neq k, k+2, \ldots$ and $|\mathrm{Re}\,(\nu_1 - \nu_2)| < 1$, $\mathrm{Re}\,(\nu_1 + \nu_2) > -1$, the operator $A^p(\nu_1,\nu_2;k)$ has a lower-type symbol $h^p_{\text{low}}(\nu_1,\nu_2;k)$. Then, such an operator also has a higher-type symbol, given by the formula*

$$\begin{aligned} &(2p + 2i\pi\,\mathcal{E})\, h^p_{\text{high}}(\nu_1,\nu_2;k) \\ &= -(1+\nu_1+\nu_2)\, h^p_{\text{low}}(\nu_1,\nu_2;k) + \frac{(1+\nu_1)(1+\nu_2)}{2i\pi}\, h^p_{\text{low}}(\nu_1+1,\nu_2+1;1-k)\,. \end{aligned} \tag{12.37}$$

Proof. First recall from the remarks which followed (12.24) that $(2p + 2i\pi\,\mathcal{E})^{-1}$ can indeed be applied to the right-hand side of (12.37) if $p \neq 0$, thus making this equation a valid definition of $h^p_{\text{high}}(\nu_1,\nu_2;k)$: (12.24) then shows that, indeed, this symbol is Σ_{p+1}-invariant. It is immediate that

$$Q\, A^p(\nu_1,\nu_2;k)\, P = A^p(\nu_1-1,\nu_2-1;1-k)\,. \tag{12.38}$$

On the other hand, using (12.33), then Lemma 12.4 (twice),

$$
\begin{aligned}
P\, A^p(\nu_1,\nu_2;k)\, Q &= P\,|Q|_k^{-1-\nu_1}\,|P|_k^{-1-\nu_2}\, Q \\
&= |Q|_k^{-1-\nu_1}\,|P|_{1-k}^{-\nu_2}\, Q - \frac{1+\nu_1}{2i\pi}\,|Q|_{1-k}^{-2-\nu_1}\,|P|_k^{-1-\nu_2}\, Q \\
&= Q^{-1}\, A^p(\nu_1-1,\nu_2-1;1-k)\, Q - \frac{1+\nu_1}{2i\pi}\, Q^{-1}\, A^p(\nu_1,\nu_2;k)\, Q \\
&= A^p(\nu_1-1,\nu_2-1;1-k) - \frac{\nu_2}{2i\pi}\, A^p(\nu_1,\nu_2;k) \\
&\quad - \frac{1+\nu_1}{2i\pi}\, A^p(\nu_1,\nu_2;k) - \frac{(1+\nu_1)(1+\nu_2)}{4\pi^2}\, A^p(\nu_1+1,\nu_2+1;1-k)\,. \qquad (12.39)
\end{aligned}
$$

Using the remark just before (12.25), we are done. □

The preceding lemma gives a pair of equations ($k=0$ or 1) entirely concerned with the Op^p-theory only: we now connect the Op^{p+1}-theory to the Op^p-theory.

When acting on functions with the parity related to $p+1$, the operators $A^p(\nu_1,\nu_2;0)$ and $A^{p+1}(\nu_1,\nu_2;0)$ (products of two factors with even symbols) agree. This is so fundamental that we give another short proof of this fact, independent of Corollary 9.5. It suffices to show that, on the given space of functions, the two operators P^2, taken from the Op^p-theory or the Op^{p+1}-theory, agree. Now, from (7.13), $P^2 = -\frac{1}{4\pi^2}\begin{pmatrix} A_p & 0 \\ 0 & D_p \end{pmatrix}$ with $A_p = (\frac{d}{dx}+\frac{p}{x})(\frac{d}{dx}-\frac{p}{x}) = \frac{d^2}{dx^2} + \frac{p(1-p)}{x^2}$, and $D_p = (\frac{d}{dx}-\frac{p}{x})(\frac{d}{dx}+\frac{p}{x}) = \frac{d^2}{dx^2} - \frac{p(p+1)}{x^2}$. Thus $D_p = A_{p+1}$.

Things are different so far as the operators $A^p(\nu_1,\nu_2;1)$ and $A^{p+1}(\nu_1,\nu_2;1)$ are concerned. Indeed,

$$
\begin{aligned}
A^p(\nu_1,\nu_2;1) &= |Q|_1^{-1-\nu_1}\,|P|_1^{-1-\nu_2} \\
&= |Q|^{-2-\nu_1}\, QP\, |P|^{-2-\nu_2}\,. \qquad (12.40)
\end{aligned}
$$

Now, in the Op^p-theory, one has

$$
QP = \frac{1}{2i\pi}\begin{pmatrix} x\frac{d}{dx} - p & 0 \\ 0 & x\frac{d}{dx} + p \end{pmatrix},
$$

whereas

$$
QP = \frac{1}{2i\pi}\begin{pmatrix} x\frac{d}{dx} - p - 1 & 0 \\ 0 & x\frac{d}{dx} + p + 1 \end{pmatrix} \qquad (12.41)
$$

in the Op^{p+1}-theory. This leads to the following:

Lemma 12.6. *Let $p \geq 0$ be given. Assume that whenever $k=0$ or 1, and ν_1, ν_2 are complex numbers such that $\nu_1 \neq k, k+2, \ldots$, $\nu_2 \neq k, k+2, \ldots$ and* $|\mathrm{Re}\,(\nu_1 - \nu_2)| < 1$, $\mathrm{Re}\,(\nu_1+\nu_2) > -1$, *the operator $A^p(\nu_1,\nu_2;k)$ has a higher-type symbol*

$h^p_{\text{high}}(\nu_1, \nu_2; k)$. *Then, under the same assumptions regarding* ν_1, ν_2, *any operator* $A^{p+1}(\nu_1, \nu_2; k)$ *also has a lower-type symbol, given as*

$$h^{p+1}_{\text{low}}(\nu_1, \nu_2; 0) = h^p_{\text{high}}(\nu_1, \nu_2; 0) \tag{12.42}$$

or

$$h^{p+1}_{\text{low}}(\nu_1, \nu_2; 1) = h^p_{\text{high}}(\nu_1, \nu_2; 1) - \frac{2p+1}{2i\pi}\, h^p_{\text{high}}(\nu_1 + 1, \nu_2 + 1; 0)\,. \tag{12.43}$$

Proposition 12.7. *Let* $k = 0$ *or* 1. *For every* $p = 0, 1, \dots,$ *and every pair* (ν_1, ν_2) *of complex numbers with* $\nu_1 \neq k, k+2, \dots,$ $\nu_2 \neq k, k+2, \dots$ *and* $|\text{Re}\ (\nu_1 - \nu_2)| < 1$, $\text{Re}\ (\nu_1 + \nu_2) > -1$, *the operator* $A^p(\nu_1, \nu_2; k)$ *admits a unique* Op^p*-symbol* $h^p(\nu_1, \nu_2; k)$ *in the space* $(\mathcal{S}'_{\bullet}\mathbb{R}^2))_{\text{even}}$, *with* $\mathcal{G}h^p(\nu_1, \nu_2; k)$ *in the same space. For* $p \geq 0$, *the symbol* $(1 + i\pi\,\mathcal{E})_p\, h^p_{\text{high}}(\nu_1, \nu_2; k)$ *is a linear combination, the coefficients of which are polynomials in* ν_1, ν_2 *of degree* $\leq p$ *depending on* p, k, r, *of the symbols* $h^0_{\text{high}}(\nu_1 + r, \nu_2 + r; k')$ *with* $r = 0, 1, \dots, 2p$, $k' = 0$ *or* 1, $r + k' \equiv k$ *mod* 2*; for* $p \geq 1$, *the symbol* $(1 + i\pi\,\mathcal{E})_{p-1}\, h^p_{\text{low}}(\nu_1, \nu_2; k)$ *is a linear combination, the coefficients of which are polynomials in* ν_1, ν_2 *of degree* $\leq p$ *depending on* p, k, r, *of the symbols* $h^0_{\text{high}}(\nu_1 + r, \nu_2 + r; k')$ *with* $r = 0, 1, \dots, 2p-1$, $k' = 0$ *or* 1, $r + k' \equiv k$ mod 2.

Proof. All linear combinations to be discussed in the present proof are supposed to have coefficients which are polynomials in ν_1, ν_2. We know from the considerations which preceded Lemma 12.4 that, with $h^0(\nu_1, \nu_2; k) = |x|_k^{-1-\nu_1} \# |\xi|_k^{-1-\nu_2}$, both $h^0(\nu_1, \nu_2; k)$ and its $\mathcal{G}$-transform lie in $(\mathcal{S}'_{\bullet}(\mathbb{R}^2))_{\text{even}}$, and the same holds with $h^0_{\text{low}}(\nu_1, \nu_2; k)$ and $h^0_{\text{high}}(\nu_1, \nu_2; k)$, the images of $h^0(\nu_1, \nu_2; k)$ under the operators $\frac{1}{2}\,(I \pm \mathcal{G})$. Using (12.37), (12.42) or (12.43), and remembering that, for $p \geq 1$, $(p + i\pi\,\mathcal{E})^{-1}$ can be applied to any symbol h in $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$, we may inductively construct two sequences $(h^p_{\text{low}}(\nu_1, \nu_2; k))_{p \geq 0}$ and $(h^p_{\text{high}}(\nu_1, \nu_2; k))_{p \geq 0}$ of lower-type and higher-type symbols of the sequence of operators $(A^p(\nu_1, \nu_2; k))_{p \geq 0}$. Indeed, (12.37) makes it possible to construct the higher-type symbol of $A^p(\nu_1, \nu_2; k)$ from the knowledge, for all admissible pairs ν_1, ν_2, of the corresponding lower-type symbol, while (12.42) and (12.43) make it possible to compute a lower-type symbol in terms of some higher-type symbols, while raising the level p.

The assertion concerning

$$(1 + i\pi\,\mathcal{E})_p\, h^p_{\text{high}}(\nu_1, \nu_2; k),\ \ p \geq 0, \quad \text{or} \quad (1 + i\pi\,\mathcal{E})_{p-1}\, h^p_{\text{low}}(\nu_1, \nu_2; k),\ \ p \geq 1,$$

is immediate by induction, a consequence again of (12.37), (12.42) and (12.43), and from what has been said in the paragraph following (12.24), all symbols $h^p_{\text{low}}(\nu_1, \nu_2; k)$, $h^p_{\text{high}}(\nu_1, \nu_2; k)$, $\mathcal{G}h^p_{\text{low}}(\nu_1, \nu_2; k)$ and $\mathcal{G}h^p_{\text{high}}(\nu_1, \nu_2; k)$ lie in $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$. The symbol $h^p(\nu_1, \nu_2; k)$ can then be obtained from (9.20), as

$$\begin{aligned} h^p(\nu_1, \nu_2; k) = (-2i\pi\,\mathcal{E})^{-1}\, &[(2p - 2i\pi\,\mathcal{E})\, h^p_{\text{low}}(\nu_1, \nu_2; k) \\ &- (2p + 2i\pi\,\mathcal{E})\, h^p_{\text{high}}(\nu_1, \nu_2; k)]\ : \end{aligned} \tag{12.44}$$

admittedly, it is only for $p > 0$ that the operator $(p + i\pi\,\mathcal{E})^{-1}$ is well defined as an endomorphism of $\mathcal{S}'_{\bullet}(\mathbb{R}^2)$. However, we only have to apply $(2i\pi\,\mathcal{E})^{-1}$, here, to symbols of the species $h^0_{\mathrm{high}}(\nu_1 + r, \nu_2 + r; k')$, and it follows from (12.5) and (12.6), together with

$$h^0_{\mathrm{high}}(\nu_1 + r, \nu_2 + r; k') = \frac{1}{2}\,\left[h^0(\nu_1 + r, \nu_2 + r; k') - \mathcal{G}h^0_{\mathrm{high}}(\nu_1 + r, \nu_2 + r; k')\right]\,, \tag{12.45}$$

that the function

$$\frac{1}{2}\,[\,C_j(\nu_1 + r, \nu_2 + r; k'; i\lambda) - D_j(\nu_1 + r, \nu_2 + r; k'; i\lambda)\,]$$

is the spectral density of $h^0_{\mathrm{high}}(\nu_1 + r, \nu_2 + r; k')$ against the family of homogeneous distributions $|x|_j^{\frac{-1-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_j^{\frac{-1+\nu_1-\nu_2-i\lambda}{2}}$ and that it remains a C^∞ function of λ after it has been divided by $-i\lambda$. □

Lemma 12.8. *For every integer $r \geq 0$, and $k = 0$ or 1, one can uniquely define four polynomials $A_{r,k}$, $B_{r,k}$, $C_{r,k}$ and $D_{r,k}$ in three indeterminates, with* degree$(A_{r,k}) \leq r$, degree$(C_{r,k}) \leq r$, degree$(B_{r,k}) \leq r - 1$, degree$(D_{r,k}) \leq r - 1$, *with the following properties: set $k' = 0$ or 1, $r + k' \equiv k$ mod 2, and let ν_1, $\nu_2 \in \mathbb{C}$ satisfy the assumptions of Proposition* 12.6. *Then*

$$\begin{aligned} &h^0_{\mathrm{high}}(\nu_1 + r, \nu_2 + r; k') = [(\nu_1 + 1)_r\,(\nu_2 + 1)_r]^{-1} \times \\ &\quad[\,A_{r,k}(\nu_1, \nu_2; 2i\pi\,\mathcal{E})\,h^0_{\mathrm{high}}(\nu_1, \nu_2; k) + 2i\pi\,\mathcal{E}\,B_{r,k}(\nu_1, \nu_2; 2i\pi\,\mathcal{E})\,h^0_{\mathrm{low}}(\nu_1, \nu_2; k)\,] \end{aligned} \tag{12.46}$$

and

$$\begin{aligned} &h^0_{\mathrm{low}}(\nu_1 + r, \nu_2 + r; k') = [(\nu_1 + 1)_r\,(\nu_2 + 1)_r]^{-1} \times \\ &\quad[\,C_{r,k}(\nu_1, \nu_2; 2i\pi\,\mathcal{E})\,h^0_{\mathrm{low}}(\nu_1, \nu_2; k) + 2i\pi\,\mathcal{E}\,D_{r,k}(\nu_1, \nu_2; 2i\pi\,\mathcal{E})\,h^0_{\mathrm{high}}(\nu_1, \nu_2; k)\,]\,. \end{aligned} \tag{12.47}$$

Proof. This is obtained by induction, starting from the two equations

$$\begin{aligned} &(2i\pi)^{-1}\,h^0_{\mathrm{high}}(\nu_1 + 1, \nu_2 + 1; k) = \\ &\frac{1 + \nu_1 + \nu_2}{(1 + \nu_1)(1 + \nu_2)}\,h^0_{\mathrm{high}}(\nu_1, \nu_2; 1 - k) + \frac{1}{(1 + \nu_1)(1 + \nu_2)}\,2i\pi\,\mathcal{E}h^0_{\mathrm{low}}(\nu_1, \nu_2; 1 - k) \end{aligned} \tag{12.48}$$

and

$$\begin{aligned} &(2i\pi)^{-1}\,h^0_{\mathrm{low}}(\nu_1 + 1, \nu_2 + 1; k) = \\ &\frac{1 + \nu_1 + \nu_2}{(1 + \nu_1)(1 + \nu_2)}\,h^0_{\mathrm{low}}(\nu_1, \nu_2; 1 - k) + \frac{1}{(1 + \nu_1)(1 + \nu_2)}\,2i\pi\,\mathcal{E}h^0_{\mathrm{high}}(\nu_1, \nu_2; 1 - k)\,, \end{aligned} \tag{12.49}$$

the second of which is the case $p = 0$ of (12.37): in a similar way, the first one follows from Lemma 12.3. Alternatively, one can derive (12.48) and (12.49) from the identity

$$h^0(\nu_1+1,\nu_2+1;k) = 2i\pi\,\frac{1+\nu_1+\nu_2+2i\pi\,\mathcal{E}}{(1+\nu_1)(1+\nu_2)}\,h^0(\nu_1,\nu_2;1-k) \tag{12.50}$$

and its consequence

$$\mathcal{G}h^0(\nu_1+1,\nu_2+1;k) = 2i\pi\,\frac{1+\nu_1+\nu_2-2i\pi\,\mathcal{E}}{(1+\nu_1)(1+\nu_2)}\,\mathcal{G}h^0(\nu_1,\nu_2;1-k)\,. \tag{12.51}$$

The easiest way to prove (12.50) is, using Theorem 11.3, to note that the coefficients $C_j(\nu_1,\nu_2;k;i\lambda)$ in (11.22) satisfy

$$\begin{aligned}\frac{C_j(\nu_1+1,\nu_2+1;1;i\lambda)}{C_j(\nu_1,\nu_2;0;i\lambda)} &= \frac{C_j(\nu_1+1,\nu_2+1;0;i\lambda)}{C_j(\nu_1,\nu_2;1;i\lambda)}\\ &= 2i\pi\,\frac{1+\nu_1+\nu_2-i\lambda}{(1+\nu_1)(1+\nu_2)}\,.\end{aligned} \tag{12.52}$$

□

Theorem 12.9. *For $k = 0$ or 1, and $p = 0,1,\dots$, one can find two polynomials $M_{p;k}(\nu_1,\nu_2;X)$ and $N_{p;k}(\nu_1,\nu_2;X)$ in ν_1, ν_2 and some indeterminate X, the degrees of which are at most $2p$, such that, whenever k, ν_1, ν_2 satisfy the assumptions of Proposition 12.7, the identity*

$$\begin{aligned}&h^p(\nu_1,\nu_2;k)\\ &= \left[(\nu_1+1)_{2p}\,(\nu_2+1)_{2p}\,(1+i\pi\,\mathcal{E})_p\right]^{-1} M_{p;k}(\nu_1,\nu_2;2i\pi\,\mathcal{E})\,h^0_{\text{low}}(\nu_1,\nu_2;k)\\ &+ \left[(\nu_1+1)_{2p}\,(\nu_2+1)_{2p}\,(i\pi\,\mathcal{E})_p\right]^{-1} N_{p;k}(\nu_1,\nu_2;2i\pi\,\mathcal{E})\,h^0_{\text{high}}(\nu_1,\nu_2;k)\end{aligned} \tag{12.53}$$

holds. Recall that

$$h^0_{\text{low}}(\nu_1,\nu_2;k) = \frac{1}{2}\,\left[\left(|x|_k^{-1-\nu_1}\#|\xi|_k^{-1-\nu_2}\right) + \mathcal{G}\left(|x|_k^{-1-\nu_1}\#|\xi|_k^{-1-\nu_2}\right)\right] \tag{12.54}$$

and that

$$h^0_{\text{high}}(\nu_1,\nu_2;k) = \frac{1}{2}\,\left[\left(|x|_k^{-1-\nu_1}\#|\xi|_k^{-1-\nu_2}\right) - \mathcal{G}\left(|x|_k^{-1-\nu_1}\#|\xi|_k^{-1-\nu_2}\right)\right]\,. \tag{12.55}$$

Proof. It suffices to combine Proposition 12.7 and Lemma 12.8. □

Chapter 3

The Sharp Composition of Automorphic Distributions

13 The Roelcke-Selberg expansion of functions associated with $\mathfrak{E}^\sharp_{\nu_1} \# \mathfrak{E}^\sharp_{\nu_2}$: the continuous part

In this section, we begin our proof of the main formula (5.38) (or (5.62). First, there are two reasons why we do not follow the lines of the heuristic approach in Section 5, neither of which has to do with the difficulty of the approach. The first one is that, from the very start, the original definition as a series (3.1) or (5.32) of $\mathfrak{F}^\sharp_\nu$ is only valid if $\mathrm{Re}\ \nu > 1$ while, from the point of view of the spectral analysis of automorphic distributions, the case when ν is pure imaginary is more important: the method below will take us directly to this case. Still, let us observe that it is precisely because our heuristic section was based on the Definition (5.32) of $\mathfrak{F}^\sharp_\nu$ that it allowed us to get some true understanding of the role played by the Dirichlet-Hecke operators $\mathcal{L}(s)$ in the formula. The second, and related, reason is that a proof based on (5.32) would depend on a definition of Eisenstein series which could not generalize to more general automorphic distributions. On the contrary, our present proof is based on the Fourier series expansion (3.25), which generalizes to the case of cusp-distributions (4.4).

As said at the end of Section 5, one possible way, avoiding the use of the Op^p-calculus (*cf.* Theorems 10.6 and 10.7) to try and define the sharp product of two Eisenstein distributions could be based on an answer to the question: how can one (sometimes) give a definition of the image under $2i\pi\,\mathcal{E}$ of the symbol h of some operator A, without being able to define either h or A? Section 12 gives the answer:

A. Unterberger, *Automorphic Pseudodifferential Analysis and Higher Level Weyl Calculi*

Proposition 13.1. *Let* $\mathfrak{S} \in \mathcal{S}'_{\text{even}}(\mathbb{R}^2)$. *For every* $z \in \Pi$, *one has*

$$(u_z|\text{Op}(2i\pi\,\mathcal{E}\,\mathfrak{S})\,u_z) = (u_z^1|\text{Op}(\mathfrak{S})\,u_z^1) \tag{13.1}$$

and

$$(u_z^1|\text{Op}(2i\pi\,\mathcal{E}\,\mathfrak{S})\,u_z^1) = -3^{\frac{1}{2}}\left[(u_z|\text{Op}(\mathfrak{S})\,u_z^2) + (u_z^2|\text{Op}(\mathfrak{S})\,u_z)\right] + 3\,(u_z^2|\text{Op}(\mathfrak{S})\,u_z^2)\,. \tag{13.2}$$

Proof. Recalling that in the Weyl calculus, when acting on functions of x, the operators Q and P are respectively the operators of multiplication by x and $\frac{1}{2i\pi}\frac{d}{dx}$, one gets from (6.13) the relations

$$\begin{aligned} Pu_z &= \frac{1}{2\pi^{\frac{1}{2}}}\left(\text{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}\bar{z}^{-1}\,u_z^1\,,\\ Qu_z &= \frac{1}{2\pi^{\frac{1}{2}}}\left(\text{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}u_z^1\,. \end{aligned} \tag{13.3}$$

Since $z^{-1} - \bar{z}^{-1} = -2i\,\text{Im}\,(-\frac{1}{z})$, this entails the relation

$$(Pu_z|A\,Qu_z) - (Qu_z|A\,Pu_z) = \frac{1}{2i\pi}\,(u_z^1|Au_z^1)\,, \tag{13.4}$$

so that (13.1) is a consequence of Lemma 12.3: observe that in the case of the Weyl calculus this lemma reduces to the formula

$$\text{Op}(2i\pi\,\mathcal{E}\,h) = 2i\pi\,(P\,\text{Op}(h)\,Q - Q\,\text{Op}(h)\,P)\,. \tag{13.5}$$

Recall from Proposition 12.1 that the right-hand side of (13.5) was denoted as $\text{mad}\,(2i\pi\,(P\wedge Q))\,\text{Op}(h)$ there.

The formula (13.2) is proved in the same way, using

$$\begin{aligned} Qu_z^1 &= \frac{3^{\frac{1}{2}}}{2\,\pi^{\frac{1}{2}}}\left(\text{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}u_z^2\,,\\ Pu_z^1 &= \frac{1}{i\pi^{\frac{1}{2}}}\left(\text{Im}\left(-\frac{1}{z}\right)\right)^{\frac{1}{2}}u_z + \frac{3^{\frac{1}{2}}}{2\pi^{\frac{1}{2}}}\left(\text{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}\bar{z}^{-1}\,u_z^2\,. \end{aligned} \tag{13.6}$$

□

It is essential to note, now, that the right-hand sides of (13.1) and (13.2) can very well be meaningful, thus giving the operator $\text{Op}(2i\pi\,\mathcal{E}\,\mathfrak{S})$ a *minimal* sense (*i.e.,* a sense as a linear space from the linear space generated by the functions u_z, u_z^1 to the algebraic dual of this latter space), without it being necessary to assign the would-be operator $\text{Op}(\mathfrak{S})$ *any* meaning. Also observe that a sesqui-holomorphic argument permits to find the function $(w,z) \mapsto (u_w^1|\text{Op}(\mathfrak{S})\,u_z^1)$ from the knowledge of the function $z \mapsto (u_z^1|\mathrm{Op}(\mathfrak{S})\, u_z^1)$, and that something similar holds for the right-hand side of (13.2). This scheme will work in our present investigations: however, it will be necessary to consider the right-hand side of (13.2) as given as just one integral, not the sum of three (divergent) ones.

Definition 13.2. Let A_1 and A_2 be two linear operators: $\mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, commuting with the map $u \mapsto \check{u}$, and let A_1^* be the formal adjoint of A_1. We shall say that the operator $\mathrm{mad}\,(2i\pi\,(P \wedge Q))\,(A_1 A_2)$ is well defined in the minimal sense if, for every point $z \in \Pi$, and $p = 0$, 1 or 2, the distributions $A_1^* u_z^p$ and $A_2 u_z^p$ are actually (locally summable) functions on $\mathbb{R}$, and the integrals

$$\int_{-\infty}^{\infty} \overline{(A_1^* u_z^1)(t)}\,(A_2 u_z^1)(t)\,dt \tag{13.7}$$

and

$$\int_{-\infty}^{\infty} \Big[3\,\overline{(A_1^* u_z^2)(t)}\,(A_2 u_z^2)(t) - 3^{\frac12}\,\overline{(A_1^* u_z)(t)}\,(A_2 u_z^2)(t) - 3^{\frac12}\,\overline{(A_1^* u_z^2)(t)}\,(A_2 u_z)(t) \Big]\, dt \tag{13.8}$$

are convergent as improper integrals. We shall say that the operator $\mathrm{mad}\,(2i\pi\,(P \wedge Q))\,(A_1 A_2)$ admits a symbol $\mathfrak{T}$ in the minimal sense if there exists a (necessarily unique) symbol $\mathfrak{T} \in \mathcal{S}'_{\mathrm{even}}(\mathbb{R}^2)$ such that these two integrals coincide with $(u_z\,|\,\mathrm{Op}(\mathfrak{T})u_z)$ and $(u_z^1\,|\,\mathrm{Op}(\mathfrak{T})u_z^1)$ respectively.

Remark. Again, the operator $\mathrm{mad}\,(2i\pi\,(P \wedge Q))\,(A_1 A_2)$ can be well defined in the minimal sense without $A_1 A_2$ being necessarily so. However, if the first-mentioned operator admits a symbol $\mathfrak{T}$ in the minimal sense and if the operator $(2i\pi\,\mathcal{E})^{-1}$ can be applied to $\mathfrak{T}$ from the consideration of the decomposition of $\mathfrak{T}$ into homogeneous components, then one may define $(2i\pi\,\mathcal{E})^{-1}\,\mathfrak{T}$ as a symbol of $A_1 A_2$. This symbol will at best be defined only up to the addition of an arbitrary (tempered) distribution of degree -1: but this is quite coherent with the fact, observed after (9.35), that, for $p \geq 1$, the Op^p-calculus "forgets" all such distributions. For Proposition 13.1 is just a devious trick to avoid the use of the Op^1-calculus and stay within the Weyl calculus proper.

Such a definition of a symbol of $A_1 A_2$ is somewhat indirect, but it is in full analogy with the definition of Bezout's distribution in Section 4: from the proof of Theorem 3.3, we found that only $(2i\pi\,\mathcal{E})^2\,\mathfrak{B}$, rather than $\mathfrak{B}$, could be given a direct definition by a convergent series as in (3.41): but after we found, for $\ell \geq 1$, the spectral decomposition of $\mathfrak{B}^\ell$ in Theorem 4.3, it appeared that it also made sense for $\ell = 0$.

Proposition 13.3. *If* ν_1, ν_2 *are complex numbers with* $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$, *the operator* $\mathrm{mad}\,(2i\pi\,(P \wedge Q))\,(\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_1})\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2}))$ *is well defined in the minimal sense.*

Proof. Recall from (3.2), (3.27) and (10.4) that, if $\nu \neq -1, 0, 1, \dots,$

$$\mathfrak{F}_\nu^\sharp = 2^{\frac{-1-\nu}{2}} \left[\zeta(-\nu)\, |x|^{-\nu-1} + \zeta(1-\nu)\, \delta(x)\, |\xi|^{-\nu} + \sum_{n\neq 0} \frac{\sigma_\nu(|n|)}{|n|^{\frac{\nu}{2}}}\, \mathfrak{a}_n^\nu \right] \tag{13.9}$$

and from (10.7) that

$$(\mathrm{Op}(\mathfrak{a}_n^\nu)u)(t) = |n|^{-\frac{\nu}{2}}\, \mathrm{char}(t^2 > 2n)\, (t^2 - 2n)^{-\frac{1}{2}} \times \sum_{\varepsilon=\pm 1} \left| t - \varepsilon\sqrt{t^2-2n} \right|^\nu u\left(\varepsilon\sqrt{t^2-2n}\right). \tag{13.10}$$

To better understand the difficulty involved in defining the product $\mathrm{Op}(\mathfrak{F}_{\nu_1}^\sharp)$ $\mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)$, let us first examine (from the consideration of (13.9)) the operator $\mathrm{Op}(\sum_{n\leq -1} \frac{\sigma_\nu(|n|)}{|n|^{\frac{\nu}{2}}}\, \mathfrak{a}_n^\nu)$: clearly, it sends $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ since, if $|u(t)| \leq C(1+|t|)^{-2N}$, one has, for $n < 0$,

$$|(\mathrm{Op}(\mathfrak{a}_n^\nu)u)(t)| \leq C\, |n|^{\frac{|\mathrm{Re}\ \nu|}{2}}\, (t^2 + 2|n|)^{-N-1} \left[|t| + \sqrt{t^2+2|n|} \right]^{|\mathrm{Re}\ \nu|}, \tag{13.11}$$

and a term-by-term differentiation can be carried out without any trouble. Since (from (13.10) or, in a simpler way, (10.4))

$$\mathrm{Op}\left(\sum_{n\geq 1} \frac{\sigma_\nu(n)}{n^{\frac{\nu}{2}}}\, \mathfrak{a}_n^\nu \right) = \left(\mathrm{Op}(\sum_{n\geq 1} \frac{\sigma_{\bar\nu}(n)}{n^{\frac{\bar\nu}{2}}}\, \mathfrak{a}_{-n}^{\bar\nu}) \right)^*, \tag{13.12}$$

the operator just mentioned sends $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$. As one can thus expect, it will indeed be the sum $\sum_{\substack{n_1\leq -1\\ n_2\geq 1}} \frac{\sigma_{\nu_1}(|n_1|)\,\sigma_{\nu_2}(n_2)}{|n_1|^{\frac{\nu_1}{2}}\, n_2^{\frac{\nu_2}{2}}}\, \mathfrak{a}_{n_1}^{\nu_1} \# \mathfrak{a}_{n_2}^{\nu_2}$ that will create difficulties; in other words, when examining $(\mathrm{Op}(\mathfrak{F}_{\bar\nu_1}^\sharp)\, u_z^p \,|\, \mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)\, u_z^p)$ and expanding the two symbols under consideration by means of (13.9), it is the terms with $n_1 \geq 1$ and $n_2 \geq 1$ which we must be careful of.

Concerning the second term on the right-hand side of (13.9), it is immediate from (2.1), or even more so from (10.19) and (10.20), that if $\nu \notin \mathbb{Z}$,

$$\mathrm{Op}(\delta(x)\, |\xi|^{-\nu})\, u = 2^\nu\, \pi^{\nu-\frac{1}{2}}\, \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{\nu}{2})}\, |t|^{\nu-1}\, \check{u}\,. \tag{13.13}$$

So as to simplify notation somewhat, we shall change z to $-\frac{1}{z}$ before studying the expressions (13.7) and (13.8), in which $\mathrm{Op}(\mathfrak{F}_{\nu_j}^\sharp)$ has been substituted for A_j: we thus set

$$f_{\nu_1,\nu_2}^1(z) = \int_{-\infty}^{\infty} \overline{(\mathrm{Op}(\mathfrak{F}_{\bar\nu_1}^\sharp)\, u_{-\frac{1}{z}}^1)(t)}\, (\mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)\, u_{-\frac{1}{z}}^1)(t)\, dt \tag{13.14}$$

and

$$f^2_{\nu_1,\nu_2}(z) = \int_{-\infty}^{\infty} \left[3\, \overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})\, u^2_{-\frac{1}{z}})(t)}\, (\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\, u^2_{-\frac{1}{z}})(t) \right.$$
$$\left. - 3^{\frac12} \sum_{p=0,2} \overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})\, u^p_{-\frac{1}{z}})(t)}\, (\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\, u^{2-p}_{-\frac{1}{z}})(t) \right] dt\,. \qquad (13.15)$$

We first examine the simpler integral (13.14), a genuine one as it will turn out. Since, with $z = x + iy$,

$$u^1_{-\frac{1}{z}}(t) = C(1)\, y^{\frac34}\, t\, e^{-i\pi \bar z t^2} \qquad (13.16)$$

with $C(1) = 2^{\frac54}\, \pi^{\frac12}$, where we have set, more generally, (*cf.* (6.13))

$$C(p) = \left(\frac{(2\pi)^{p+\frac12}}{\Gamma(p+\frac12)} \right)^{\frac12}, \qquad (13.17)$$

(13.16), together with (13.9), (13.10), (13.13) and the functional equation (3.19) of the zeta function, yields, if $|\mathrm{Re}\, \nu| < 1$, recalling (11.26) for the notation concerning signed powers,

$$\left(\mathrm{Op}(\mathfrak{F}^\sharp_\nu)\, u^1_{-\frac{1}{z}}\right)(t) = C(1)\, y^{\frac34}\, [2^{\frac{-\nu-1}{2}}\, \zeta(-\nu)\, |t|_1^{-\nu} - 2^{\frac{\nu-1}{2}}\, \zeta(\nu)\, |t|_1^{\nu}]\; e^{-i\pi\bar z t^2}$$
$$+ 2^{\frac{-\nu-1}{2}}\, C(1)\, y^{\frac34} \sum_{n\neq 0} \frac{\sigma_\nu(|n|)}{|n|^\nu}\, \mathrm{char}(t^2 > 2n)$$
$$\times \left[\left|t - \sqrt{t^2-2n}\right|^\nu - \left|t + \sqrt{t^2-2n}\right|^\nu \right] e^{-i\pi \bar z (t^2-2n)}\,. \qquad (13.18)$$

The first two terms add up to a function which is an $\mathrm{O}(|t|^{-|\mathrm{Re}\,\nu|})$ near zero, and rapidly decreasing at infinity; it has also been observed that the sum of all terms with $n < 0$ in the sum above is a smooth function, rapidly decreasing at infinity. The terms with $n > 0$, however, must be examined more carefully: first, their sum has a mild singularity, to wit a lack of differentiability at the points $\pm\sqrt{2n}$, $n \geq 1$. Since $|e^{-i\pi\bar z(t^2-2n)}| = e^{-\pi y(t^2-2n)}$ and the number of terms to be considered is $\leq \frac{t^2}{2}$, only the n's with $t^2 - 2n = \mathrm{O}(\log t)$ may contribute to a total not less, say, than $\mathrm{O}(t^{-2})$. But when $t^2 - 2n = \mathrm{O}(\log t)$, $t \to \infty$, one has

$$\left|t - \sqrt{t^2-2n}\right|^\nu - \left|t + \sqrt{t^2-2n}\right|^\nu = \mathrm{O}\left(t^{\mathrm{Re}\,\nu - 1}\, \sqrt{t^2-2n} \right)\,;$$

also, given $C > 0$, the sum $\sum \sqrt{t^2-2n}$ extended to all integers n such that $0 \leq t^2 - 2n \leq C\, \log t$ is an $\mathrm{O}((\log t)^{\frac32})$. Finally, $\frac{\sigma_\nu(n)}{n^\nu} = \sigma_{-\nu}(n)$ is, for large n, an $\mathrm{O}(\log n)$ if $\mathrm{Re}\,\nu \geq 0$, an $\mathrm{O}(n^{-\mathrm{Re}\,\nu}\, \log n)$ if $\mathrm{Re}\,\nu < 0$, so that the terms with $n > 0$ from (13.18) contribute to $(\mathrm{Op}(\mathfrak{F}_\nu^\sharp)\, u^1_{-\frac{1}{z}})(t)$ a term which is an $\mathrm{O}(|t|^{|\mathrm{Re}\ \nu|-1}\,(\log t)^{\frac{5}{2}})$ for large t.

This estimate shows that, when $|\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2| < 1$, the right-hand side of (13.14) is a bona fide integral. The improper integral (13.15) is slightly more difficult to deal with. Our claim is that the dt-integral, taken from $-a$ to a ($-a_1$ to a_2, with a_1 and a_2 independent, would be just as well) has a limit as $a \to \infty$, and that this limit is also the sum of the double series of integrals obtained from the decomposition (13.19) below: the simple trick is to perform an integration by parts (in the integral $\int_{-a}^{a} \dots$) in *some* of the terms involved, namely, with the notation below, those with $n_1 \geq 1$ and $n_2 \geq 1$.

For $p = 0$ or 2, and $|\mathrm{Re}\ \nu| < 1$,

$$\left(\mathrm{Op}(\mathfrak{F}_\nu^\sharp)\, u^p_{-\frac{1}{z}}\right)(t) = C(p)\, y^{\frac{2p+1}{4}} \left[2^{\frac{-\nu-1}{2}}\, \zeta(-\nu)\, |t|^{p-\nu-1} + 2^{\frac{\nu-1}{2}}\, \zeta(\nu)\, |t|^{p+\nu-1}\right]$$
$$\times\, e^{-i\pi\bar{z}t^2} + 2^{\frac{-\nu-1}{2}}\, C(p)\, y^{\frac{2p+1}{4}} \sum_{n\neq 0} \frac{\sigma_\nu(|n|)}{|n|^\nu}\, \mathrm{char}(t^2 > 2n)$$
$$\times \left[\left|t - \sqrt{t^2-2n}\right|^\nu + \left|t + \sqrt{t^2-2n}\right|^\nu\right] (t^2-2n)^{\frac{p-1}{2}}\, e^{-i\pi\bar{z}(t^2-2n)}\,. \quad (13.19)$$

We develop $(\mathrm{Op}(\mathfrak{F}_{\nu_1}^\sharp)\, u^{p_1}_{-\frac{1}{z}})(t)$ (*resp.* $(\mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)\, u^{p_2}_{-\frac{1}{z}})(t)$), with p_1 (*resp.* p_2) $= 0$ or 2 but $p_1 + p_2 \geq 2$ according to the preceding expansion, with n_1 (*resp.* n_2) substituted for n: as explained right after (13.12), only the terms with $n_1 \geq 1$, $n_2 \geq 1$ are worrisome in the expansion of the product $\overline{(\mathrm{Op}(\mathfrak{F}_{\nu_1}^\sharp)\, u^{p_1}_{-\frac{1}{z}})(t)} \times (\mathrm{Op}(\mathfrak{F}_{\nu_2}^\sharp)\, u^{p_2}_{-\frac{1}{z}})(t)$: since $\frac{C(2)}{C(0)} = \frac{4\pi}{3^{\frac{1}{2}}}$, they contribute to the integrand on the right-hand side of (13.15) the function

$$g(t) = 2^{\frac{-\nu_1-\nu_2-2}{2}}\, 3^{\frac{1}{2}}\, C(0)\, C(2)\, y^{\frac{3}{2}} \sum_{n_1,\, n_2 \geq 1} \frac{\sigma_{\nu_1}(n_1)\, \sigma_{\nu_2}(n_2)}{n_1^{\nu_1}\, n_2^{\nu_2}}$$
$$\mathrm{char}(t^2 > 2\max(n_1, n_2)) \sum_{\substack{\varepsilon_1 = \pm 1\\ \varepsilon_2 = \pm 1}} |t - \varepsilon_1\sqrt{t^2-2n_1}|^{\nu_1}\, |t - \varepsilon_2\sqrt{t^2-2n_2}|^{\nu_2}$$
$$\times \left[4\pi y\, (t^2-2n_1)^{\frac{1}{2}}\, (t^2-2n_2)^{\frac{1}{2}} - \left(\frac{t^2-2n_2}{t^2-2n_1}\right)^{\frac{1}{2}} - \left(\frac{t^2-2n_1}{t^2-2n_2}\right)^{\frac{1}{2}}\right]$$
$$\times\, e^{i\pi z(t^2-2n_1)}\, e^{-i\pi\bar{z}(t^2-2n_2)}\,. \quad (13.20)$$

This series cannot be integrated on the real line (though each term can) only because of the behaviour of g at infinity. However, note that

$$e^{i\pi z(t^2-2n_1)}\, e^{-i\pi\bar{z}(t^2-2n_2)} = e^{-2\pi y(t^2-n_1-n_2)}\, e^{2i\pi(n_2-n_1)x} \quad (13.21)$$

and that the last line in the preceding expression of $g(t)$ can be written as

$$2^{2i\pi(n_2-n_1)x} \times \left(-t^{-1}\frac{d}{dt}\right) \left[(t^2-2n_1)^{\frac{1}{2}}\,(t^2-2n_2)^{\frac{1}{2}}\,e^{-2\pi y(t^2-n_1-n_2)}\right]. \tag{13.22}$$

Since

$$\begin{aligned}&\frac{d}{dt}\left(\left|t-\varepsilon_1\sqrt{t^2-2n_1}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n_2}\right|^{\nu_2}\right)\\ &\quad=-\left[\frac{\varepsilon_1\nu_1}{\sqrt{t^2-2n_1}}+\frac{\varepsilon_2\nu_2}{\sqrt{t^2-2n_2}}\right]\left|t-\varepsilon_1\sqrt{t^2-2n_1}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n_2}\right|^{\nu_2},\end{aligned} \tag{13.23}$$

one may write $\int_{-\infty}^{\infty} g(t)\,dt$ as $\int_{-\infty}^{\infty} h(t)\,dt$ with

$$\begin{aligned}h(t) = -2^{\frac{-\nu_1-\nu_2-2}{2}}\,3^{\frac{1}{2}}\,C(0)\,C(2)\,y^{\frac{3}{2}}&\sum_{n_1,\,n_2\geq 1}\frac{\sigma_{\nu_1}(n_1)\,\sigma_{\nu_2}(n_2)}{n_1^{\nu_1}\,n_2^{\nu_2}}\,e^{2i\pi(n_2-n_1)x}\\ \operatorname{char}(t^2>2\max(n_1,n_2))&\sum_{\substack{\varepsilon_1=\pm1\\ \varepsilon_2=\pm1}}\left|t-\varepsilon_1\sqrt{t^2-2n_1}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n_2}\right|^{\nu_2}\\ \times\left[\frac{\varepsilon_1\nu_1\,t^{-1}}{\sqrt{t^2-2n_1}}+\frac{\varepsilon_2\nu_2\,t^{-1}}{\sqrt{t^2-2n_2}}+t^{-2}\right]&(t^2-2n_1)^{\frac{1}{2}}\,(t^2-2n_2)^{\frac{1}{2}}\,e^{-2\pi y(t^2-n_1-n_2)}.\end{aligned} \tag{13.24}$$

Splitting the last exponential as $e^{-\pi y(t^2-2n_1)}\,e^{-\pi y(t^2-2n_2)}$ and using the Cauchy-Schwarz inequality for series together with the estimate

$$\begin{aligned}\sum_{n\geq 1}|\sigma_{-\nu}(n)|^2\operatorname{char}(t^2>2n)\left[|t|\pm\sqrt{t^2-2n}\right]^{2\,\mathrm{Re}\,\nu}&t^{-2}\left[1+(t^2-2n)\right]e^{-2\pi y(t^2-2n)}\\ &=\mathrm{O}\left(|t|^{2\,|\mathrm{Re}\,\nu|-2}\,(\log t)^{\frac{7}{2}}\right),\qquad |t|\to\infty,\end{aligned} \tag{13.25}$$

obtained, as in the study of the integral (13.14), by remarking that only the terms with $t^2-2n=\mathrm{O}(\log t)$ are important in the sum, we see that, when $|\mathrm{Re}\,\nu_1|+|\mathrm{Re}\,\nu_2|<1$, the function h is integrable. □

Lemma 13.4. *The functions* $f^1_{\nu_1,\nu_2}$ *and* $f^2_{\nu_1,\nu_2}$ *introduced in* (13.14) *and* (13.15) *are* Γ*-invariant.*

Proof. This is obvious for the first one, in view of the covariance of the Weyl calculus under the metaplectic representation and the absolute convergence of the integral (13.14). Note that the invariance of $f^2_{\nu_1,\nu_2}$ under translations by integers is obvious too.

After the preparation provided by the proof of Lemma 13.3, one may write the function $2^{\frac{\nu_1+\nu_2+2}{2}} f^2_{\nu_1,\nu_2}(z)$ as a series, the main part of which being

$$\sum_{n_1,\,n_2\neq 0} \frac{\sigma_{\nu_1}(|n_1|)\,\sigma_{\nu_2}(|n_2|)}{|n_1|^{\frac{\nu_1}{2}}\,|n_2|^{\frac{\nu_2}{2}}} \Big[3\,(\mathrm{Op}(\mathfrak{a}^{\bar{\nu_1}}_{-n_1})\,u^2_{-\frac{1}{z}}\,|\,\mathrm{Op}(\mathfrak{a}^{\nu_2}_{n_2})\,u^2_{-\frac{1}{z}}) \\ - 3^{\frac{1}{2}} \sum_{p=0,2} (\mathrm{Op}(\mathfrak{a}^{\bar{\nu_1}}_{-n_1})\,u^p_{-\frac{1}{z}}\,|\,\mathrm{Op}(\mathfrak{a}^{\nu_2}_{n_2})\,u^{2-p}_{-\frac{1}{z}})\Big]\,, \tag{13.26}$$

where each scalar product within the bracket is meaningful: this is the same as

$$\sum_{n_1,\,n_2\neq 0} \frac{\sigma_{\nu_1}(|n_1|)\,\sigma_{\nu_2}(|n_2|)}{|n_1|^{\frac{\nu_1}{2}}\,|n_2|^{\frac{\nu_2}{2}}} \left(u^1_{-\frac{1}{z}}\,|\,\mathrm{mad}(2i\pi(P\wedge Q))\,(\mathrm{Op}(\mathfrak{a}^{\nu_1}_{n_1})\,\mathrm{Op}(\mathfrak{a}^{\nu_2}_{n_2}))\,u^1_{-\frac{1}{z}}\right). \tag{13.27}$$

Now, by the metaplectic covariance property,

$$\left(u^1_{-\frac{1}{z}}\,|\,\mathrm{mad}(2i\pi(P\wedge Q))\,(\mathrm{Op}(\mathfrak{a}^{\nu_1}_{n_1})\,\mathrm{Op}(\mathfrak{a}^{\nu_2}_{n_2}))\,u^1_{-\frac{1}{z}}\right) \\ = \left(u^1_z\,|\,\mathrm{mad}(2i\pi(P\wedge Q))\,(\mathrm{Op}(\mathfrak{b}^{\nu_1}_{n_1})\,\mathrm{Op}(\mathfrak{b}^{\nu_2}_{n_2}))\,u^1_z\right) \tag{13.28}$$

with $\mathfrak{b}^\nu_n = \mathfrak{a}^\nu_n \circ \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, which concludes the proof since one also has

$$\mathrm{mad}(2i\pi(P\wedge Q))\,(\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_1})\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})) = 2^{\frac{-2-\nu_1-\nu_2}{2}} \sum_{n_1,\,n_2\neq 0} \frac{\sigma_{\nu_1}(|n_1|)\,\sigma_{\nu_2}(|n_2|)}{|n_1|^{\frac{\nu_1}{2}}\,|n_2|^{\frac{\nu_2}{2}}} \\ \mathrm{mad}(2i\pi(P\wedge Q))\,(\mathrm{Op}(\mathfrak{b}^{\nu_1}_{n_1})\,\mathrm{Op}(\mathfrak{b}^{\nu_2}_{n_2})) + \text{extra terms} \tag{13.29}$$

(where the extra terms arise from the consideration, for each factor, of the first two terms on the right-hand side of (13.9): they are exactly the transforms, under the map $z \mapsto -\frac{1}{z}$, of the terms we have neglected writing in (13.26)): this series expansion has to be understood in the minimal sense, *i.e.*, when testing against a pair (u_z, u_z) or (u^1_z, u^1_z). □

Under the assumption that $|\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2| < 1$, we shall now consider the function $f^1_{\nu_1,\nu_2}$ and expand it as a Fourier series

$$f^1_{\nu_1,\nu_2}(z) = \sum_{m\in\mathbb{Z}} a^1_m(y)\,e^{2i\pi mx}\,. \tag{13.30}$$

Using (13.21) again, we find, starting from (13.18),

$$a_0^1(y) = (C(1))^2\, y^{\frac{3}{2}} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2 \int_{-\infty}^{\infty} e^{-2\pi y t^2}\, dt$$

$$\times\Bigg[2^{\frac{-\varepsilon_1\nu_1-\varepsilon_2\nu_2-2}{2}}\, \zeta(-\varepsilon_1\,\nu_1)\,\zeta(-\varepsilon_2\,\nu_2)\, |t|^{-\varepsilon_1\,\nu_1-\varepsilon_2\,\nu_2} + 2^{\frac{-\nu_1-\nu_2-2}{2}} \sum_{n\neq 0} e^{4\pi\, ny}$$

$$\frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}}\, \mathrm{char}(t^2>2n)\, \left|t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1} \left|t-\varepsilon_2\sqrt{t^2-2n}\right|^{\nu_2}\Bigg] \tag{13.31}$$

and, for $m\neq 0$,

$$a_m^1(y) = a^1(m;\nu_1,\nu_2;y) + a^1(-m;\nu_2,\nu_1;y) + \sum_{\substack{n\neq 0\\ n\neq -m}} a_n^1(m;\nu_1,\nu_2;y) \tag{13.32}$$

with

$$a^1(m;\nu_1,\nu_2;y) = (C(1))^2\, y^{\frac{3}{2}} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2\; 2^{\frac{-\varepsilon_1\nu_1-\nu_2-2}{2}}\, \zeta(-\varepsilon_1\,\nu_1)\, \frac{\sigma_{\nu_2}(|m|)}{|m|^{\nu_2}}$$

$$\int_{-\infty}^{\infty} \mathrm{char}(t^2>2m)\, |t|^{-\varepsilon_1\nu_1} \left|t-\varepsilon_2\sqrt{t^2-2m}\right|^{\nu_2} e^{-2\pi y(t^2-m)}\, dt \tag{13.33}$$

and

$$a_n^1(m;\nu_1,\nu_2;y) = (C(1))^2\, y^{\frac{3}{2}} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2\; 2^{\frac{-\nu_1-\nu_2-2}{2}}\, \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)}{|n|^{\nu_1}\,|n+m|^{\nu_2}}$$

$$\int \left|t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1} \left|t-\varepsilon_2\sqrt{t^2-2n-2m}\right|^{\nu_2} e^{-2\pi y(t^2-2n-m)}\, dt\,, \tag{13.34}$$

where the last integral is taken on the domain characterized by $t^2>\max(2n, 2n+2m)$.

Lemma 13.5. *If* $|\mathrm{Re}\,(\nu_1\pm\nu_2)|<1$, *the function*

$$g_{\nu_1,\nu_2}^1(z){:}\; = f_{\nu_1,\nu_2}^1(z) - 2\sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}$$

$$\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right) \zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)\, E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(z) \tag{13.35}$$

lies in $L^2(\Gamma\backslash\Pi)$. *It remains in this space after it has been applied the Laplacian* Δ *any number of times.*

Proof. Recall that a fundamental domain of Γ in Π is $\{z \in \Pi\colon |z| > 1,\ -\frac{1}{2} < \mathrm{Re}\, z < \frac{1}{2}\}$. In view of the classical Fourier expansion (4.5) of $E^*_{\frac{1-\nu}{2}}(z) = \zeta^*(1-\nu)\, E_{\frac{1-\nu}{2}}(z)$, it is then immediate that the function

$$2\,\varepsilon_1\,\varepsilon_2\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)\,E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(z) \tag{13.36}$$

agrees, up to some error term in $L^2(\Gamma\backslash\Pi)$, with the function

$$2\,\varepsilon_1\,\varepsilon_2\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)\,y^{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,, \tag{13.37}$$

which is the same as the term

$$\varepsilon_1\,\varepsilon_2\,(C(1))^2\,y^{\frac{3}{2}}\,2^{\frac{-\varepsilon_1\nu_1-\varepsilon_2\nu_2-2}{2}}\,\zeta(-\varepsilon_1\,\nu_1)\,\zeta(-\varepsilon_2\,\nu_2)\int_{-\infty}^{\infty}|t|^{-\varepsilon_1\,\nu_1-\varepsilon_2\,\nu_2}\,e^{-2\pi y t^2}\,dt \tag{13.38}$$

from the right-hand side of the expansion (13.31) of $a_0^1(y)$.

It thus remains to be shown that the major term $\sum_{n\neq 0}\cdots$ of $a_0^1(y)$, together with the series $\sum_{m\neq 0} a_m^1(y)\,e^{2i\pi m x}$, add up to some function in $L^2(\Gamma\backslash\Pi)$. Let us first estimate, when z lies in the usual fundamental domain of Γ in Π, so that $y > \frac{\sqrt{3}}{2}$, and $n \neq 0$, the term

$$I_n(y) = \int_{t^2>2n}\sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}}\varepsilon_1\,\varepsilon_2\left|t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n}\right|^{\nu_2}e^{-2\pi y(t^2-2n)}\,dt\,, \tag{13.39}$$

or

$$\begin{aligned} I_n(y) = 2\int_{\sqrt{2n}}^{\infty}\Bigg[&\left|t-\sqrt{t^2-2n}\right|^{\nu_1+\nu_2} - |2n|^{\nu_1}\left|t-\sqrt{t^2-2n}\right|^{-\nu_1+\nu_2}\\ &-|2n|^{\nu_2}\left|t-\sqrt{t^2-2n}\right|^{\nu_1-\nu_2} + |2n|^{\nu_1+\nu_2}\left|t-\sqrt{t^2-2n}\right|^{-\nu_1-\nu_2}\Bigg]\,e^{-2\pi y(t^2-2n)}\,dt\,. \end{aligned} \tag{13.40}$$

When $n<0$, setting $t=\sqrt{2|n|}\,\sinh s$, we find

$$\int_0^{\infty}\left(\sqrt{t^2-2n}-t\right)^{\nu}e^{-2\pi y(t^2-2n)}\,dt = (2|n|)^{\frac{\nu+1}{2}}\int_0^{\infty}e^{-\nu s}\,e^{-4\pi|n|y\,\cosh^2 s}\,\cosh s\,ds\,, \tag{13.41}$$

so that each of the four terms on the right-hand side of (13.40) contributes to $a_0^1(y)$ a term which is an $\mathrm{O}(e^{-2\pi n y})$ in the fundamental domain.

When $n>0$, setting $t=\sqrt{2n}\,\cosh s$, one finds

$$\int_{\sqrt{2n}}^{\infty}\left(t-\sqrt{t^2-2n}\right)^{\nu}e^{-2\pi y(t^2-2n)}\,dt = (2n)^{\frac{\nu+1}{2}}\int_0^{\infty}e^{-\nu s}\,e^{-4\pi n y\,\sinh^2 s}\,\sinh s\,ds\,, \tag{13.42}$$

so that

$$I_n(y) = 8\,(2n)^{\frac{\nu_1+\nu_2+1}{2}} \int_0^\infty (\sinh \nu_1 s)\,(\sinh \nu_2 s)\, e^{-4\pi |n| y \sinh^2 s}\, \sinh s\, ds \ : \tag{13.43}$$

it is clear that the integral is an $\mathrm{O}((ny)^{-2})$.

Since $\frac{\sigma_{\nu_1}(|n|)}{|n|^{\nu_1}} = \sigma_{-\nu_1}(|n|) = \mathrm{O}(|n|^{\max(0, -\mathrm{Re}\ \nu_1)} \log |n|)$ and $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$, the series

$$y^{\frac{3}{2}} \sum_{n \neq 0} \left| \frac{\sigma_{\nu_1}(|n|)\, \sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}} \right| |n|^{\frac{\mathrm{Re}\ (\nu_1+\nu_2+1)}{2}}\, (ny)^{-2} \tag{13.44}$$

converges and contributes an $\mathrm{O}(y^{-\frac{1}{2}})$, a function in $L^2(\Gamma\backslash\Pi)$, to $a_0^1(y)$.

The series $\sum_{m \neq 0} a_m^1(y)\, e^{2i\pi m x}$ remains to be analyzed. Set

$$I_{n,m}(y) = \int \sum_{\substack{\varepsilon_1 = \pm 1 \\ \varepsilon_2 = \pm 1}} \varepsilon_1\, \varepsilon_2 \\ \left| t - \varepsilon_1 \sqrt{t^2 - 2n} \right|^{\nu_1} \left| t - \varepsilon_2 \sqrt{t^2 - 2n - 2m} \right|^{\nu_2} e^{-2\pi y (t^2 - 2n - m)}\, dt\,, \tag{13.45}$$

where the integration takes place on the domain defined by $t^2 > \max(2n, 2n{+}2m)$. Since, then, $t^2 - 2n - m > |m|$, one gets, using the Cauchy-Schwarz inequality,

$$|I_{n,m}(y)| \leq e^{-\pi |m| y} \left[\int_{t^2 > 2n} \left| \sum_{\varepsilon_1 = \pm 1} \varepsilon_1 \left| t - \varepsilon_1 \sqrt{t^2 - 2n} \right|^{\nu_1} \right|^2 e^{-\pi y (t^2 - 2n)}\, dt \right]^{\frac{1}{2}} \\ \times \left[\int_{t^2 > 2n+2m} \left| \sum_{\varepsilon_2 = \pm 1} \varepsilon_2 \left| t - \varepsilon_2 \sqrt{t^2 - 2n - 2m} \right|^{\nu_2} \right|^2 e^{-\pi y (t^2 - 2n - 2m)}\, dt \right]^{\frac{1}{2}} . \tag{13.46}$$

The summability with respect to n is obtained in the same way as the one relative to $I_n(y)$ above, and the extra factor $e^{-\pi |m| y}$ takes care of the m-summability, at the same time providing for the behaviour of the term $\sum_{n \neq 0, -m} a_n^1(m; \nu_1, \nu_2; y)$ from (13.32) at infinity in the fundamental domain. The contribution of the term $a^1(m; \nu_1, \nu_2; y)$ is taken care of by the estimate $t^2 - m \geq |m|$, valid when $t^2 > 2m$.

That the application of the Laplacian does not destroy the estimates in all that precedes is immediate. □

From the Definition (13.35) of $g^1_{\nu_1,\nu_2}(z)$, (13.30) and (4.5), one has the Fourier expansion

$$g^1_{\nu_1,\nu_2}(z) = \sum_{m \in \mathbb{Z}} b_m^1(y)\, e^{2i\pi m x}\,, \tag{13.47}$$

with

$$b_0^1(y) = (C(1))^2\, y^{\frac{3}{2}} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2 \int_{-\infty}^{\infty} \Big[2^{\frac{-\nu_1-\nu_2-2}{2}} \sum_{n\neq 0} \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}}$$

$$\mathrm{char}(t^2>2n)\, \left|t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1} \left|t-\varepsilon_2\sqrt{t^2-2n}\right|^{\nu_2} \Big]\, e^{-2\pi y(t^2-2n)}\, dt$$

$$-\,2 \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2\, \pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\, \Gamma\!\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right) \zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)$$

$$\times\, \frac{\zeta^*(-\varepsilon_1\nu_1-\varepsilon_2\nu_2)}{\zeta^*(2+\varepsilon_1\nu_1+\varepsilon_2\nu_2)}\, y^{-\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}} \qquad (13.48)$$

and, for $m \neq 0$,

$$b_m^1(y) = a_m^1(y) - \frac{4}{\zeta^*(2+\varepsilon_1\nu_1+\varepsilon_2\nu_2)} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2$$

$$\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\, \Gamma\!\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right) \zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)$$

$$y^{\frac{1}{2}}\, |m|^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\, \sigma_{-1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}(|m|)\, K_{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(2\pi|m|y)\,. \qquad (13.49)$$

We shall apply to the function $g^1_{\nu_1,\nu_2}$ the Roelcke-Selberg theorem, already quoted in (4.1), in the following version: one has the spectral decomposition

$$g^1_{\nu_1,\nu_2} = \Phi^0 + \sum_{k\geq 1} (g^1_{\nu_1,\nu_2})_k + \frac{1}{8\pi} \int_{-\infty}^{\infty} \Phi(\lambda)\, E_{\frac{1-i\lambda}{2}}\, d\lambda\,, \qquad (13.50)$$

where Φ^0 is a constant, $\Phi \in L^2(\mathbb{R})$ satisfies the symmetry property (4.2) and, for every $k = 1, 2, \dots$ the function $(g^1_{\nu_1,\nu_2})_k$ denotes the projection of $g^1_{\nu_1,\nu_2}$ on the finite-dimensional space of $L^2(\Gamma\backslash\Pi)$ corresponding to the discrete eigenvalue $\frac{1+\lambda_k^2}{4}$: here, of course, $(\frac{1+\lambda_k^2}{4})_{k\geq 1}$ denotes the sequence of eigenvalues of the modular Laplacian enumerated *without repetition.*

Here is a recipe for computing the coefficients of the Roelcke-Selberg decomposition of any function $g \in L^2(\Gamma\backslash\Pi)$ with $\Delta g \in L^2(\Gamma\backslash\Pi))$, with the Fourier series expansion

$$g(z) = \sum_{m\in\mathbb{Z}} b_m(y)\, e^{2i\pi mx}\,, \qquad (13.51)$$

taken from [62, Theorem 7.3, Theorem 7.4]: it may be considered as an extension of the Rankin-Selberg unfolding method, and is in full generality best phrased in hyperfunction-theoretic terms, though a simpler concept will do here. First, the coefficients Φ^0 and $\Phi(\lambda)$: we shall worry about the discrete part in the next section. With the help of the coefficient $b_0(y)$ from (13.51), set

$$C_0^+(\mu) = \frac{1}{8\pi} \int_0^1 b_0(y)\, y^{-\frac{3}{2}} \frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\, dy$$

and

$$C_0^-(\mu) = -\frac{1}{8\pi} \int_1^\infty b_0(y)\, y^{-\frac{3}{2}} \frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\, dy\,. \tag{13.52}$$

Then the function C_0^+ is holomorphic in the half-plane $\text{Im}\ \mu > 1$ and extends as a meromorphic function to the half-plane $\text{Im}\ \mu > 0$, with an only possible (simple) pole at $\mu = i$: the residue there is $(-4i\pi)^{-1}\,\Phi^0$. The function C_0^- is holomorphic in the half-plane $\text{Im}\ \mu < 0$. If each of the two functions $C_0^\pm$ happens to extend up to the real line as a continuous function, the jump

$$C_0(\lambda) = C_0^+(\lambda + i0) - C_0^-(\lambda - i0) \tag{13.53}$$

is related to the spectral density Φ by the relation

$$C_0(\lambda) = \frac{1}{8\pi} \frac{\pi^{-\frac{i\lambda}{2}}}{\Gamma(-\frac{i\lambda}{2})}\, \Phi(-\lambda) = \frac{1}{8\pi} \frac{\zeta(-i\lambda)}{\zeta^*(1-i\lambda)}\, \Phi(\lambda)\,. \tag{13.54}$$

Remark. In particular, assume that the coefficient $b_0(y)$ from the Fourier series expansion (13.51) of some function $g \in L^2(\Gamma\backslash\Pi)$, $\Delta g \in L^2(\Gamma\backslash\Pi)$, can be written as a finite sum

$$b_0(y) = c_0(y) + \sum_\ell c_\ell\, y^{\alpha_\ell}\,, \tag{13.55}$$

where $|\text{Re}\ \alpha_\ell| < \frac{1}{2}$ for all ℓ and the integral $\int_1^\infty c_0(y)\, y^{-\frac{3}{2}-\frac{i\mu}{2}}\, dy$ converges when $\text{Im}\ \mu < 2$. Then the integral

$$\phi(\mu) = \int_0^\infty c_0(y)\, y^{-\frac{3}{2}-\frac{i\mu}{2}}\, dy \tag{13.56}$$

is convergent for μ in the non-void strip $\max(1, 1 - 2\min_\ell(\text{Re}\ \alpha_\ell)) < \text{Im}\ \mu < 2$. It extends as a meromorphic function, still denoted as ϕ, in the upper half-plane, without poles except i and the numbers $i(1-2\alpha_\ell)$, and one has

$$\Phi(-\lambda) = \phi(\lambda + i0) \tag{13.57}$$

for every $\lambda \in \mathbb{R}$. Indeed, on the one hand, if $\text{Im}\ \mu > \max(1, 1 - 2\min_\ell(\text{Re}\ \alpha_\ell))$ (a number < 2), one has

$$\frac{1}{8\pi} \int_0^1 c_0(y)\, y^{-\frac{3}{2}} \frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\, dy = C_0^+(\mu) - \frac{1}{8\pi} \frac{\pi^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})} \sum_\ell \frac{c_\ell}{-\frac{1}{2} + \alpha_\ell - \frac{i\mu}{2}} \tag{13.58}$$

where the left-hand side must be a convergent integral. On the other hand, in the lower half-plane,

$$-\frac{1}{8\pi}\int_1^\infty c_0(y)\,y^{-\frac{3}{2}}\,\frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\,dy = C_0^-(\mu) - \frac{1}{8\pi}\,\frac{\pi^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\sum_\ell \frac{c_\ell}{-\frac{1}{2}+\alpha_\ell-\frac{i\mu}{2}}\,, \tag{13.59}$$

a holomorphic function. Thus $C_0^-(\mu)$ extends as a meromorphic function in the half-plane $\mathrm{Im}\ \mu < 2$, with the points $i(1-2\,\alpha_\ell)$ as only possible poles, and the function $C_0^+(\mu) - C_0^-(\mu)$, meromorphic in the strip $0 < \mathrm{Im}\ \mu < 2$ with no possible poles except i and the points $i(1-2\,\alpha_\ell)$, coincides in the strip $\max(1, 1 - 2\min_\ell(\mathrm{Re}\ \alpha_\ell)) < \mathrm{Im}\ \mu < 2$ with the integral $\frac{1}{8\pi}\int_0^\infty c_0(y)\,y^{-\frac{3}{2}}\,\frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\,dy$. Considering $C_0^+(\lambda+i0) - C_0^-(\lambda+i0)$ and using (13.54), we find the expression for $\Phi(-\lambda)$.

One should also note that, under the conditions above, the constant Φ^0 in the Roelcke-Selberg decomposition of g is given as $-4i\pi$ times the residue of the function $\frac{1}{8\pi}\,\frac{\pi^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\,\phi(\mu)$ at $\mu = i$.

Theorem 13.6. *If* $|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1$, *the continuous part of the Roelcke-Selberg decomposition* (13.50) *of the function* $g^1_{\nu_1,\nu_2}$ *introduced in* (13.35) *is given by the spectral density*

$$\Phi(\lambda) = i\lambda\,\pi^{-\frac{i\lambda}{2}}\,\frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)} \times \sum_{\varepsilon=\pm1} \varepsilon\,\zeta\left(\frac{1-i\varepsilon\lambda+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda+\nu_1-\nu_2}{2}\right)$$
$$\times\zeta\left(\frac{1+i\varepsilon\lambda-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\lambda-\nu_1-\nu_2}{2}\right). \tag{13.60}$$

The constant term Φ^0 *is zero.*

Proof. In view of the expression (13.48) of $b_0^1(y)$, the remark above applies: indeed, on the one hand $|\mathrm{Re}\ (\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2})| < \frac{1}{2}$; on the other hand, setting

$$c_0^1(y) = (C(1))^2\,y^{\frac{3}{2}}\int_{-\infty}^\infty e^{-2\pi y(t^2-2n)}\,dt \sum_{\substack{\varepsilon_1=\pm1\\ \varepsilon_2=\pm1}} \varepsilon_1\,\varepsilon_2\,2^{\frac{-\nu_1-\nu_2-2}{2}} \sum_{n\neq0}$$
$$\frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}}\,\mathrm{char}(t^2>2n)\left|t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n}\right|^{\nu_2}, \tag{13.61}$$

we proved in (13.44) that $c_0^1(y) = \mathrm{O}(y^{-\frac{1}{2}})$ for $y \to \infty$, which proves the convergence of the integral $\int_1^\infty c_0^1(y)\,y^{-\frac{3}{2}-\frac{i\mu}{2}}\,dy$ when $\mathrm{Im}\ \mu < 2$. We may thus set

$$\phi^1(\mu) = \int_0^\infty c_0^1(y)\,y^{-\frac{3}{2}-\frac{i\mu}{2}}\,dy\,, \tag{13.62}$$

a convergent integral when $1 + \frac{|\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_1|}{2} < \mathrm{Im}\ \mu < 2$, which can also be written (integrating first with respect to dy) as

$$\phi^1(\mu) = 2^{\frac{1-\nu_1-\nu_2+i\mu}{2}}\, \pi^{\frac{i\mu}{2}}\, \Gamma\left(\frac{2-i\mu}{2}\right) \sum_{n \neq 0} \frac{\sigma_{\nu_1}(|n|)\, \sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}}$$
$$\int_{t^2>2n} (t^2-2n)^{\frac{i\mu-2}{2}} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\, \varepsilon_2 \left|t - \varepsilon_1 \sqrt{t^2-2n}\right|^{\nu_1} \left|t - \varepsilon_2 \sqrt{t^2-2n}\right|^{\nu_2} dt\,. \tag{13.63}$$

Denoting as $I_n^1(\nu_1, \nu_2)$ the integral which appears on the second line of this last formula, one has, if $n \geq 1$,

$$I_n^1(\nu_1,\nu_2) = (2n)^{\frac{-1+\nu_1+\nu_2+i\mu}{2}} \int_{-\infty}^{\infty} |\sinh s|^{i\mu-1} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\, \varepsilon_2\, e^{-(\varepsilon_1 \nu_1 + \varepsilon_2 \nu_2) s}\, ds \tag{13.64}$$

and

$$I_{-n}^1(\nu_1,\nu_2) = (2n)^{\frac{-1+\nu_1+\nu_2+i\mu}{2}} \int_{-\infty}^{\infty} (\cosh s)^{i\mu-1} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\, \varepsilon_2\, e^{-(\varepsilon_1 \nu_1 + \varepsilon_2 \nu_2) s}\, ds\,. \tag{13.65}$$

Both integrals converge if $-1 + |\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2| < \mathrm{Im}\ \mu < 2$ (for the first one, the sum must be expressed as the product of two hyperbolic sines, as in (13.43)). Also, it is immediate that

$$\begin{aligned}\int_{-\infty}^{\infty} |\sinh s|^{i\mu-1}\, e^{-\nu s}\, ds &= 2^{-i\mu} \int_0^{\infty} t^{\frac{-i\mu-\nu-1}{2}}\, |1-t|^{i\mu-1}\, dt \\ &= 2^{-i\mu}\, \Gamma(i\mu) \left[\frac{\Gamma(\frac{1-i\mu-\nu}{2})}{\Gamma(\frac{1+i\mu-\nu}{2})} + \frac{\Gamma(\frac{1-i\mu+\nu}{2})}{\Gamma(\frac{1+i\mu+\nu}{2})}\right]\end{aligned} \tag{13.66}$$

if $-1 + |\mathrm{Re}\ \nu| < \mathrm{Im}\ \mu < 0$, and that

$$\int_{-\infty}^{\infty} (\cosh s)^{i\mu-1}\, e^{-\nu s}\, ds = 2^{-i\mu} \frac{\Gamma(\frac{1-i\mu-\nu}{2})\, \Gamma(\frac{1-i\mu+\nu}{2})}{\Gamma(1-i\mu)} \tag{13.67}$$

if $-1 + |\mathrm{Re}\ \nu| < \mathrm{Im}\ \mu$.

Observe that these two integrals add up to

$$\begin{aligned} 2^{-i\mu}\, \Gamma(i\mu)\, \Gamma(\frac{1-i\mu-\nu}{2})\, \Gamma(\frac{1-i\mu+\nu}{2}) \Bigg[&\frac{1}{\Gamma(\frac{1-i\mu+\nu}{2})\, \Gamma(\frac{1+i\mu-\nu}{2})} \\ &+ \frac{1}{\Gamma(\frac{1-i\mu-\nu}{2})\, \Gamma(\frac{1+i\mu+\nu}{2})} + \frac{1}{\Gamma(i\mu)\, \Gamma(1-i\mu)} \Bigg], \end{aligned} \tag{13.68}$$

which is the same as

$$\frac{2^{-i\mu}}{\pi}\Gamma(i\mu)\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)\left[\cos\frac{\pi(i\mu-\nu)}{2}+\cos\frac{\pi(i\mu+\nu)}{2}+\sin\pi i\mu\right]$$
$$=\frac{2^{1-i\mu}}{\pi}\Gamma(i\mu)\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)\cos\frac{\pi(i\mu)}{2}\left[\cos\frac{\pi\nu}{2}+\sin\frac{\pi i\mu}{2}\right]$$
$$=\frac{2^{2-i\mu}}{\pi}\Gamma(i\mu)\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)$$
$$\times\cos\frac{\pi(i\mu)}{2}\cos\frac{\pi(1-i\mu+\nu)}{4}\cos\frac{\pi(1-i\mu-\nu)}{4}$$
$$=2^{2-i\mu}\pi^2\frac{\Gamma(i\mu)\Gamma(\frac{1-i\mu-\nu}{2})\Gamma(\frac{1-i\mu+\nu}{2})}{\Gamma(\frac{1+i\mu}{2})\Gamma(\frac{1-i\mu}{2})\Gamma(\frac{3-i\mu+\nu}{4})\Gamma(\frac{1+i\mu-\nu}{4})\Gamma(\frac{3-i\mu-\nu}{4})\Gamma(\frac{1+i\mu+\nu}{4})}. \tag{13.69}$$

Using the duplication formula for the Gamma function [31, p. 3]

$$\Gamma(z)=2^{z-1}\pi^{-\frac{1}{2}}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right), \tag{13.70}$$

this is the same as

$$2^{-i\mu}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{i\mu}{2})\Gamma(\frac{1-i\mu-\nu}{4})\Gamma(\frac{1-i\mu+\nu}{4})}{\Gamma(\frac{1-i\mu}{2})\Gamma(\frac{1+i\mu-\nu}{4})\Gamma(\frac{1+i\mu+\nu}{4})}. \tag{13.71}$$

Thus, from this result and (13.63), when $1+|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2|<\mathrm{Im}\ \mu<2$, one has

$$\phi^1(\mu)=2\pi^{\frac{1+i\mu}{2}}\frac{\Gamma(\frac{i\mu}{2})\Gamma(\frac{2-i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})}\sum_{\varepsilon=\pm1}\varepsilon\frac{\Gamma(\frac{1-i\mu-\nu_1-\varepsilon\nu_2}{4})\Gamma(\frac{1-i\mu+\nu_1+\varepsilon\nu_2}{4})}{\Gamma(\frac{1+i\mu-\nu_1-\varepsilon\nu_2}{4})\Gamma(\frac{1+i\mu+\nu_1+\varepsilon\nu_2}{4})}$$
$$\times\sum_{n\geq1}n^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}\sigma_{\nu_1}(n)\sigma_{\nu_2}(n). \tag{13.72}$$

On the other hand ([48, p. 163] or [25, p. 232] or [62, p. 144]) a formula due to Ramanujan yields

$$\sum_{n\geq1}n^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}\sigma_{\nu_1}(n)\sigma_{\nu_2}(n)=(\zeta(1-i\mu))^{-1}\times$$
$$\zeta\left(\frac{1-i\mu+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\mu+\nu_1-\nu_2}{2}\right)\zeta\left(\frac{1-i\mu-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\mu-\nu_1-\nu_2}{2}\right) \tag{13.73}$$

if $\mathrm{Im}\ \mu>1+|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2|$.

From the functional equation (3.19) of the zeta function,

$$\frac{\Gamma(\frac{1-i\mu-\nu_1-\varepsilon\nu_2}{4})\,\Gamma(\frac{1-i\mu+\nu_1+\varepsilon\nu_2}{4})}{\Gamma(\frac{1+i\mu+\nu_1+\varepsilon\nu_2}{4})\,\Gamma(\frac{1+i\mu-\nu_1-\varepsilon\nu_2}{4})}\,\zeta\left(\frac{1-i\mu-\nu_1-\varepsilon\nu_2}{2}\right)\zeta\left(\frac{1-i\mu+\nu_1+\varepsilon\nu_2}{2}\right)$$
$$= \pi^{-i\mu}\,\zeta\left(\frac{1+i\mu+\nu_1+\varepsilon\nu_2}{2}\right)\zeta\left(\frac{1+i\mu-\nu_1-\varepsilon\nu_2}{2}\right) \quad (13.74)$$

and

$$\frac{\Gamma(\frac{i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})}\,(\zeta(1-i\mu))^{-1} = \frac{\pi^{i\mu-\frac{1}{2}}}{\zeta(i\mu)}\,. \quad (13.75)$$

Thus

$$\phi^1(\mu) = (-i\mu)\,\pi^{\frac{i\mu}{2}}\,\frac{\Gamma(\frac{-i\mu}{2})}{\zeta(i\mu)} \times \sum_{\varepsilon=\pm1}\varepsilon$$
$$\zeta\left(\frac{1+i\varepsilon\mu+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\mu+\nu_1-\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\mu-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\mu-\nu_1-\nu_2}{2}\right), \quad (13.76)$$

which concludes the proof of Theorem 13.6, with the help of (13.57). □

Remark. As this is immediate, one may observe (though this has to be true by the very construction of Φ) that the function

$$\zeta^*(-i\lambda)\,\Phi(\lambda) = i\lambda\,\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{-i\lambda}{2}\right) \times \sum_{\varepsilon=\pm1}\varepsilon$$
$$\zeta\left(\frac{1-i\varepsilon\lambda+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda+\nu_1-\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\lambda-\nu_1-\nu_2}{2}\right) \quad (13.77)$$

is even.

Theorem 13.7. *Assume that* $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$ *and set*

$$g^2_{\nu_1,\nu_2}(z) := f^2_{\nu_1,\nu_2}(z) + 2\sum_{\substack{\varepsilon_1=\pm1\\ \varepsilon_2=\pm1}} (1+\varepsilon_1\nu_1+\varepsilon_2\nu_2)\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}$$
$$\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\zeta(-\varepsilon_1\nu_1)\,\zeta(-\varepsilon_2\nu_2)\,E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(z) \quad (13.78)$$

where $f^2_{\nu_1,\nu_2}$ *has been introduced in* (13.15). *Then, for every* $N \geq 0$, *the function* $\Delta^N g^2_{\nu_1,\nu_2}$ *lies in* $L^2(\Gamma\backslash\Pi)$. *The continuous part of the Roelcke-Selberg decomposi-*

tion, analogous to (13.50), *of* $g^2_{\nu_1,\nu_2}$, *is given by the spectral density*

$$\Phi(\lambda) = -\lambda^2 \pi^{-\frac{i\lambda}{2}} \frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)} \times \sum_{\varepsilon=\pm 1}$$
$$\zeta\left(\frac{1-i\varepsilon\lambda+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda+\nu_1-\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\lambda-\nu_1-\nu_2}{2}\right). \tag{13.79}$$

The constant term Φ^0 *is zero.*

Proof. Just as in (13.30), we now set

$$f^2_{\nu_1,\nu_2}(z) = \sum_{m\in\mathbb{Z}} a^2_m(y)\, e^{2i\pi mx}\,. \tag{13.80}$$

Rather than redo the whole proof, let us indicate what is to be changed. From the expansion (13.27) of the main contribution to $f^2_{\nu_1,\nu_2}(z)$, and the remark that a similar treatment applies to the terms we have neglected writing in (13.16), it is clear that, changing $a^1_0(y)$, $a^1_m(y)$, $a^1(m;\nu_1,\nu_2;y)$ and $a^1_n(m;\nu_1,\nu_2;y)$ to $a^2_0(y)$, $a^2_m(y)$, $a^2(m;\nu_1,\nu_2;y)$ and $a^2_n(m;\nu_1,\nu_2;y)$, the calculations between (13.30) and (13.34) will apply to $f^2_{\nu_1,\nu_2}$ after we have made the modifications that correspond to applying $\mathrm{mad}\,(2i\pi(P\wedge Q))$ to each operator involved. For any operator A,

$$\left(u^1_{-\frac{1}{z}}\,|\,2i\pi(PAQ-QAP)u^1_{-\frac{1}{z}}\right) = \left(-2i\pi Pu^1_{-\frac{1}{z}}\,|\,AQu^1_{-\frac{1}{z}}\right) - \left(Qu^1_{-\frac{1}{z}}\,|\,A(2i\pi P)u^1_{-\frac{1}{z}}\right) \tag{13.81}$$

and applying Q (*resp.* $2i\pi\,P$) to $u^1_{-\frac{1}{z}}(t)$ amounts to multiplying it by t (*resp.* $t^{-1} - 2i\pi\bar{z}t$). This provides us with the extra factors that must be inserted below the various dt-integrals between (13.31) and (13.34) so as to change the notions relative to $f^1_{\nu_1,\nu_2}$ to the corresponding ones relative to $f^2_{\nu_1,\nu_2}$: within the first term on the right-hand side of (13.31), which gives $a^1_0(y)$, insert the extra factor

$$\varepsilon_1\,\varepsilon_2\,[-(t^{-1}+2i\pi zt)t - t(t^{-1}-2i\pi\bar{z}t)] = \varepsilon_1\,\varepsilon_2\,[-2+4\pi yt^2]\,. \tag{13.82}$$

The presence of $\varepsilon_1\,\varepsilon_2$ in this extra factor ought to be explained: it originates from the application of (13.13) (the same will occur presently with (13.10) instead) and the fact that $Pu^1_{-\frac{1}{z}}$ and $Qu^1_{-\frac{1}{z}}$, contrary to $u^1_{-\frac{1}{z}}$, are even functions. Within the second term from the same integral, insert

$$-\left[\frac{\varepsilon_1}{\sqrt{t^2-2n}} + 2i\pi z\varepsilon_1\sqrt{t^2-2n}\right]\varepsilon_2\sqrt{t^2-2n}$$
$$-\,\varepsilon_1\sqrt{t^2-2n}\left[\frac{\varepsilon_2}{\sqrt{t^2-2n}} - 2i\pi\bar{z}\varepsilon_2\sqrt{t^2-2n}\right]$$
$$= \varepsilon_1\varepsilon_2\,\left[-2+4\pi y(t^2-2n)\right]\,. \tag{13.83}$$

Under the integral on the right-hand side of (13.33), which gives $a^1(m;\nu_1,\nu_2;y)$, insert the extra factor

$$\varepsilon_1\varepsilon_2\left[-\frac{(t^2-2m)^{\frac{1}{2}}}{t}-\frac{t}{(t^2-2m)^{\frac{1}{2}}}+4\pi y\,t\,(t^2-2m)^{\frac{1}{2}}\right]\ : \tag{13.84}$$

finally, within the integral (13.34), insert the extra factor

$$\varepsilon_1\varepsilon_2\left[-\left(\frac{t^2-2n-2m}{t^2-2n}\right)^{\frac{1}{2}}-\left(\frac{t^2-2n}{t^2-2n-2m}\right)^{\frac{1}{2}}+4\pi y(t^2-2n)^{\frac{1}{2}}(t^2-2n-2m)^{\frac{1}{2}}\right]. \tag{13.85}$$

In the proof of Lemma 13.5, the main term (13.37), differing by some error term in $L^2(\Gamma\backslash\Pi)$ from the Eisenstein series (13.36), originated from the integral (13.38): in view of the correcting factor (13.82), this integral has to be replaced, now, by

$$\begin{aligned}
&(C(1))^2 y^{\frac{3}{2}}\, 2^{\frac{-\varepsilon_1\nu_1-\varepsilon_2\nu_2-2}{2}}\,\zeta(-\varepsilon_1\nu_1)\zeta(-\varepsilon_2\nu_2)\int_{-\infty}^{\infty}(-2+4\pi y t^2)\,|t|^{-\varepsilon_1\nu_1-\varepsilon_2\nu_2}\,e^{-2\pi y t^2}\,dt\\
&=4\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,\zeta(-\varepsilon_1\,\nu_1)\,\zeta(-\varepsilon_2\,\nu_2)\\
&\qquad\times\left[-\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)+\Gamma\left(\frac{3-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\right]\,y^{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\\
&=-2\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,(1+\varepsilon_1\nu_1+\varepsilon_2\nu_2)\\
&\qquad\times\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\,\zeta(-\varepsilon_1\,\nu_1)\,\zeta(-\varepsilon_2\,\nu_2)\,y^{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\ :
\end{aligned} \tag{13.86}$$

up to an error term in $L^2(\Gamma\backslash\Pi)$, all this coincides with

$$\begin{aligned}
&-2\,\pi^{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}\,(1+\varepsilon_1\nu_1+\varepsilon_2\nu_2)\\
&\qquad\times\Gamma\left(\frac{1-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{2}\right)\,\zeta(-\varepsilon_1\,\nu_1)\,\zeta(-\varepsilon_2\,\nu_2)\,E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(z)\,,
\end{aligned} \tag{13.87}$$

which provides the linear combination of Eisenstein series to be considered on the right-hand side of (13.78).

The rest of the proof of Lemma 13.5 called for the examination of the second term $\sum_{n\neq 0}\cdots$ on the right-hand side of (13.32) together with that of the series $\sum_{m\neq 0} a_m^1(y)\,e^{2i\pi mx}$. As observed in the proof of Proposition 13.3, factors like t^2-2n_1 or t^2-2n_2 are rather harmless in all estimates, since all can be assumed to be $\mathrm{O}(\log t)$ in the domain of the t-variable where the exponential is not an

$O(t^{-N})$ for N as large as one pleases: however, the presence of $\varepsilon_1\varepsilon_2$ in the extra factors which precede requires that, as in the proof of Proposition 13.3, we should appeal to the identity

$$\left[4\pi y\,(t^2-2n_1)^{\frac{1}{2}}\,(t^2-2n_2)^{\frac{1}{2}}-\left(\frac{t^2-2n_2}{t^2-2n_1}\right)^{\frac{1}{2}}-\left(\frac{t^2-2n_1}{t^2-2n_2}\right)^{\frac{1}{2}}\right]e^{-2\pi y(t^2-n_1-n_2)}$$
$$=\left(-t^{-1}\frac{d}{dt}\right)\left[(t^2-2n_1)^{\frac{1}{2}}\,(t^2-2n_2)^{\frac{1}{2}}\,e^{-2\pi y(t^2-n_1-n_2)}\right] \tag{13.88}$$

already used between (13.20) and (13.22), and perform, in the terms with $n_1\geq 1$, $n_2\geq 1$, the corresponding integration by parts. Let us also recall, from (13.23), that

$$\left(t^{-1}\frac{d}{dt}-t^{-2}\right)\left(\left|t-\varepsilon_1\sqrt{t^2-2n_1}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n_2}\right|^{\nu_2}\right)$$
$$=-\left[\frac{\varepsilon_1\nu_1\,t^{-1}}{\sqrt{t^2-2n_1}}+\frac{\varepsilon_2\nu_2\,t^{-1}}{\sqrt{t^2-2n_2}}+t^{-2}\right]\left|t-\varepsilon_1\sqrt{t^2-2n_1}\right|^{\nu_1}\left|t-\varepsilon_2\sqrt{t^2-2n_2}\right|^{\nu_2}, \tag{13.89}$$

a function which, after having been multiplied by $(t^2-2n_1)^{\frac{1}{2}}\,(t^2-2n_2)^{\frac{1}{2}}$ (a factor originating from (13.88)) is an $O(t^{|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_1|-1})$ in a way uniform with respect to $n_1\geq 1$, $n_2\geq 1$.

Then the end of the proof of Lemma 13.5 extends to the new situation, which proves the first part of Theorem 13.7.

The second part is of course modelled on the proof of Theorem 13.6. The function $c_0^1(y)$ introduced in (13.61) must be replaced by $c_0^2(y)$, obtained by plugging the extra factor $\varepsilon_1\,\varepsilon_2\,[-2+4\pi y(t^2-2n)]$, from (13.82), under the integral, in all terms with $n<0$. When $n>0$, on the other hand, we must use the integration by parts provided by (13.88) and (13.89), substituting for the preceding expression the new extra factor (which can also be read directly from a comparison between (13.20) and (13.24)), in the case when $n_1=n_2=n$,

$$-\varepsilon_1\,\varepsilon_2\,[(\varepsilon_1\nu_1+\varepsilon_2\nu_2)t^{-1}(t^2-2n)^{\frac{1}{2}}+t^{-2}(t^2-2n)]\,. \tag{13.90}$$

After the dt-integration has been carried, one sees, observing also that

$$\int_0^\infty e^{-2\pi y(t^2-2n)}\,[-2+4\pi y(t^2-2n)]\,y^{-\frac{i\mu}{2}}\,dy=-i\mu\int_0^\infty e^{-2\pi y(t^2-2n)}\,y^{-\frac{i\mu}{2}}\,dy\,, \tag{13.91}$$

that $\phi^1(\mu)$ must be replaced by $\phi^2(\mu)$, a function which can be decomposed in an analogous way, separating the terms with $n<0$ from those with $n>0$: the integrals $I_n^1(\nu_1,\nu_2)$ and $I_{-n}^1(\nu_1,\nu_2)$, with $n \geq 1$, have to be replaced by $I_n^2(\nu_1,\nu_2)$ and $I_{-n}^2(\nu_1,\nu_2)$, where $I_{-n}^2(\nu_1,\nu_2)$ is obtained from $I_{-n}^1(\nu_1,\nu_2)$, in (13.65), by deleting the factor $\varepsilon_1\,\varepsilon_2$ and multiplying the result by $-i\mu$; also, since the factor (13.90) becomes $-\varepsilon_1\,\varepsilon_2\,[(\varepsilon_1\nu_1+\varepsilon_2\nu_2)\,\tanh s+(\tanh s)^2]$ when $t=(2n)^{\frac{1}{2}}\,\cosh s$,

$$I_n^2(\nu_1,\nu_2) = -(2n)^{\frac{-1+\nu_1+\nu_2+i\mu}{2}} \int_{-\infty}^{\infty} |\sinh s|^{i\mu}\,(\cosh s)^{-1} \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} [\varepsilon_1\nu_1+\varepsilon_2\nu_2\ +\tanh s]\,e^{-(\varepsilon_1\nu_1+\varepsilon_2\nu_2)\,s}\,ds\ : \tag{13.92}$$

it is essential to note that the integral still converges if $-1+|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2| < \mathrm{Im}\ \mu < 2$ since

$$-\sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} (\varepsilon_1\nu_1+\varepsilon_2\nu_2)\,e^{-(\varepsilon_1\nu_1+\varepsilon_2\nu_2)\,s} = 2(\nu_1+\nu_2)\,\sinh(\nu_1+\nu_2)s + 2(\nu_1-\nu_2)\,\sinh(\nu_1-\nu_2)s\,. \tag{13.93}$$

However, to *compute* $I_n^2(\nu_1,\nu_2)$, it is much easier, relying on an argument of analytic continuation at the end, to assume to start with that $-1+|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2| < \mathrm{Im}\ \mu < 0$, in which case the calculations from the proof of Theorem 13.6 can still be used: when going from $I_n^1(\nu_1,\nu_2)$ to $I_n^2(\nu_1,\nu_2)$, all that has to be done, again, is to forget the factor $\varepsilon_1\,\varepsilon_2$ and multiply the result by $-i\mu$. Thus, to modify the expression (13.76) so as to get $\phi^2(\mu)$ in place of $\phi^1(\mu)$, it suffices to delete the factor ε and multiply the result by $-i\mu$: this leads to (13.78). □

14 The Roelcke-Selberg expansion of functions associated with $\mathfrak{E}_{\nu_1}^\sharp \# \mathfrak{E}_{\nu_2}^\sharp$: the discrete part

We now come to the computation of the orthogonal projection $(g_{\nu_1,\nu_2}^1)_k$ of g_{ν_1,ν_2}^1 onto the finite-dimensional subspace of $L^2(\Gamma\backslash\Pi)$ corresponding to the discrete eigenvalue $\frac{1+\lambda_k^2}{4}$: we repeat that all the λ_k's are distinct. Again, we start from the Fourier series expansion (13.47), which defines the coefficients $b_m^1(y)$.

The recipe [62, Theorem 7.4] calls for the examination of the analytic continuation as meromorphic functions in the entire plane of the functions ($m \neq 0$)

$$C_m(\mu) = \frac{1}{8\pi}\int_0^\infty b_m^1(y)\,y^{-\frac{3}{2}}\,\frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\,dy\,, \tag{14.1}$$

initially defined when $\mathrm{Im}\ \mu > 0$: then one has

$$(g_{\nu_1,\nu_2}^1)_k(z) = y^{\frac{1}{2}} \sum_{m\neq 0} d_m\,K_{\frac{i\lambda_k}{2}}(2\pi|m|y)\,e^{2i\pi mx} \tag{14.2}$$

with

$$d_m = -8i\pi\,|m|^{-\frac{i\lambda_k}{2}} \times \text{residue of } C_m(\mu) \quad \text{at } \mu = \lambda_k\,. \tag{14.3}$$

Actually, the theorem quoted is slightly different: there, it is assumed that one has *first* substracted from the function $g = g^1_{\nu_1,\nu_2} \in L^2(\Gamma\backslash\Pi)$ to be analyzed the continuous part g_{cont} of its Roelcke-Selberg expansion, and that the coefficients b^1_m are relative to the difference $g - g_{\text{cont}}$. Thus, the contribution to the integral $C_m(\mu)$ of what comes from the function $\frac{1}{8\pi}\int_{-\infty}^{\infty}\Phi(\lambda)\,E_{\frac{1-i\lambda}{2}}\,d\lambda$ must be analyzed too: this is possible since, from Theorem 13.6, we now know $\Phi(\lambda)$ explicitly. In a quite comparable situation, the relevant contour deformation argument has been given in [62, Theorem 11.1], and there is no need to repeat the rather lengthy proof here: in view of the behaviour [31, p. 13] at infinity of the Gamma function on vertical lines, and of the classical estimates [49, p. 149, 161] of $\zeta(\sigma+it)$, $t \to \infty$, within the critical strip, or of $(\zeta(it))^{-1}$, (13.60) gives an inequality $|\Phi(\lambda) \leq C\,e^{-\frac{\pi|\lambda|}{4}}\,|\lambda|^b$, $|\lambda| \to \infty$, for some b, and the proof of the quoted theorem used exactly the same [62, (11.8)]. Let us just emphasize, as a consequence of the analysis, that g_{cont} contributes to $C_m(\mu)$ a term which has no poles in some open half-plane containing the closed upper half-plane except points $-i\omega$, ω a non-trivial zero of the zeta function. Since none of these poles is real, this is of no consequence for our present purposes.

Despite the fact that $f^1_{\nu_1,\nu_2}$, contrary to $g^1_{\nu_1,\nu_2}$, does not belong to $L^2(\Gamma\backslash\Pi)$, we may also substitute for C_m the function defined in the same way (but only when $\text{Im}\,\mu$ is large) after $a^1_m(y)$ has been substituted for $b^1_m(y)$: this would amount to neglecting the extra term on the right-hand side of (13.49), the complex continuation of which indeed does not contribute to $C_m(\mu)$ any pole on the real line since [31, p. 91], for $\text{Im}\,\mu > 1 + |\text{Re}\,\nu_1| + |\text{Re}\,\nu_2|$,

$$\begin{aligned}&\int_0^\infty y^{-1-\frac{i\mu}{2}}\,K_{\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(2\pi|m|y)\,dy\\ &= 2^{-\frac{i\mu}{2}-2}\,(2\pi|m|)^{\frac{i\mu}{2}}\,\Gamma\left(\frac{1-i\mu+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{4}\right)\Gamma\left(\frac{-1-i\mu-\varepsilon_1\nu_1-\varepsilon_2\nu_2}{4}\right).\end{aligned} \tag{14.4}$$

The integrals involved in this section, contrary to the ones that led to Theorems 13.6 and 13.7, cannot be computed explicitly: the good news is that, now, only the *singularities* of $C_m(\mu)$ as μ crosses the real line are of interest, so that we may neglect all terms which extend holomorphically a little below $\mathbb{R}$. In particular, we may substitute for $C_m(\mu)$ the function

$$\tilde{C}_m(\mu) = \frac{1}{8\pi}\int_0^\infty \sum_{\substack{n\neq 0\\ n\neq -m}} a^1_n(m;\nu_1,\nu_2;y)\,y^{-\frac{3}{2}}\,\frac{(\pi y)^{-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})}\,dy\,, \tag{14.5}$$

where $a^1_n(m;\nu_1,\nu_2;y)$ was defined in (13.34).

We first remark that, when $\mathrm{Im}\ \mu$ is large enough, one can exchange the signs of summation and integration, in other words that the series

$$\sum_{\substack{n\neq 0\\ n\neq -m}} \left| \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)}{|n|^{\nu_1}\,|n+m|^{\nu_2}} \right| \times \int_0^\infty y^{\frac{\mathrm{Im}\ \mu}{2}} \left| \int \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2 \right.$$
$$\left. \left| t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1} \left| t-\varepsilon_2\sqrt{t^2-2n-2m}\right|^{\nu_2} e^{-2\pi y(t^2-2n-m)}\, dt \right| dy, \tag{14.6}$$

in which the last integral is carried over the set defined by $t^2 > \max(2n, 2n+2m)$, is convergent: this has actually been done in (13.46).

Starting from (13.34), with $C(1) = 2^{\frac{5}{4}}\,\pi^{\frac{1}{2}}$, using (14.5) and performing the dy-integration first, we get, for $\mathrm{Im}\ \mu$ large enough,

$$\tilde{C}_m(\mu) = 2^{\frac{-\nu_1-\nu_2+i\mu-5}{2}}\,\pi^{-1} \left(-\frac{i\mu}{2}\right) \sum_{\substack{n\neq 0\\ n\neq -m}} \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)}{|n|^{\nu_1}\,|n+m|^{\nu_2}}\, I_{\nu_1,\nu_2}(n;m)\,, \tag{14.7}$$

with

$$I_{\nu_1,\nu_2}(n;m) :=$$
$$\int \sum_{\substack{\varepsilon_1=\pm 1\\ \varepsilon_2=\pm 1}} \varepsilon_1\,\varepsilon_2 \left| t-\varepsilon_1\sqrt{t^2-2n}\right|^{\nu_1} \left| t-\varepsilon_2\sqrt{t^2-2n-2m}\right|^{\nu_2} (t^2-2n-m)^{\frac{i\mu}{2}-1}\, dt\,, \tag{14.8}$$

where the integral is carried over the same set as above: recall that $t^2-2n-m > |m|$ on this set.

Our problem (still under the assumption $|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1$) is to understand the function $\tilde{C}_m$ up to an error term which holomorphically extends slightly below the real line: it should be emphasized that m is kept fixed throughout. Since

$$\frac{\sigma_\nu(|n|)}{|n|^\nu} = \sigma_{-\nu}(|n|) = \mathrm{O}(|n|^{\max(0,-\mathrm{Re}\ \nu)}\,\log|n|)$$

as $|n| \to \infty$, we should begin with an expression of $I_{\nu_1,\nu_2}(n;m)$ up to an error term in $\mathrm{O}(|n|^\alpha)$, with $\alpha < -1 + \min(0, \mathrm{Re}\ \nu_1) + \min(0, \mathrm{Re}\ \nu_2)$. As

$$I_{\nu_1,\nu_2}(n;-m) = I_{\nu_2,\nu_1}(n-m;m)\,, \tag{14.9}$$

it suffices to consider the case when $m > 0$.

The integral $I_{\nu_1,\nu_2}(n;m)$ converges as soon as

$$\mathrm{Im}\ \mu > |\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2| - 1.$$

When $n \to \infty$, we set $t = \pm(2n+2m)^{\frac{1}{2}} \cosh s$, so that

$$I_{\nu_1,\nu_2}(n;m) = -4\,(2n+2m)^{\frac{-1+\nu_1+\nu_2+i\mu}{2}} \int_0^\infty \sum_{\varepsilon_1=\pm 1} \varepsilon_1 \sinh(\nu_2 s)$$
$$\left|\cosh s - \varepsilon_1 \left(\sinh^2 s + \frac{m}{n+m}\right)^{\frac{1}{2}}\right|^{\nu_1} \left(\sinh^2 s + \frac{m}{2n+2m}\right)^{\frac{i\mu}{2}-1} \sinh s\, ds\,; \tag{14.10}$$

when $n \to -\infty$, setting $t = |2n+2m|^{\frac{1}{2}} \sinh s$,

$$I_{\nu_1,\nu_2}(n;m) = -4\,|2n+2m|^{\frac{-1+\nu_1+\nu_2+i\mu}{2}} \int_0^\infty \sum_{\varepsilon_1=\pm 1} \varepsilon_1 \sinh(\nu_2 s)$$
$$\left|\left(\cosh^2 s + \frac{m}{|n+m|}\right)^{\frac{1}{2}} - \varepsilon_1 \sinh s\right|^{\nu_1} \left(\cosh^2 s + \frac{m}{|2n+2m|}\right)^{\frac{i\mu}{2}-1} \cosh s\, ds\,. \tag{14.11}$$

When $n \to \infty$, $\sum \varepsilon_1\, |\cosh s - \varepsilon_1\, (\sinh^2 s + \frac{m}{n+m})^{\frac{1}{2}}|^{\nu_1}$ goes to $-2\sinh(\nu_1 s)$ and the same holds, when $t \to -\infty$, for $\sum \varepsilon_1\, |(\cosh^2 s + \frac{m}{|n+m|})^{\frac{1}{2}} - \varepsilon_1 \sinh s|^{\nu_1}$: thus, if $\mathrm{Im}\ \mu < 2$, as well as $\mathrm{Im}\ \mu > |\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2| - 1$,

$$I_{\nu_1,\nu_2}(n;m) \sim 2^{\frac{\nu_1+\nu_2+i\mu+5}{2}}\, |n|^{\frac{-1+\nu_1+\nu_2+i\mu}{2}}\, C^{\pm}(\nu_1,\nu_2;\mu), \qquad n \to \pm\infty\,, \tag{14.12}$$

with

$$C^{+}(\nu_1,\nu_2;\mu) = \frac{1}{2}\int_{-\infty}^{\infty} \sinh(\nu_1 s)\, \sinh(\nu_2 s)\, |\sinh s|^{i\mu-1}\, ds = 2^{-i\mu-2}\Gamma(i\mu)$$
$$\times \left[\frac{\Gamma(\frac{1-i\mu-\nu_1-\nu_2}{2})}{\Gamma(\frac{1+i\mu-\nu_1-\nu_2}{2})} + \frac{\Gamma(\frac{1-i\mu+\nu_1+\nu_2}{2})}{\Gamma(\frac{1+i\mu+\nu_1+\nu_2}{2})} - \frac{\Gamma(\frac{1-i\mu+\nu_1-\nu_2}{2})}{\Gamma(\frac{1+i\mu+\nu_1-\nu_2}{2})} - \frac{\Gamma(\frac{1-i\mu-\nu_1+\nu_2}{2})}{\Gamma(\frac{1+i\mu-\nu_1+\nu_2}{2})}\right] \tag{14.13}$$

(a function without singularity at $\mu = 0$ or i), where we have used (13.66), and

$$C^{-}(\nu_1,\nu_2;\mu) = \frac{1}{2}\int_{-\infty}^{\infty} \sinh(\nu_1 s)\, \sinh(\nu_2 s)\, (\cosh s)^{i\mu-1}\, ds = \frac{2^{-i\mu-2}}{\Gamma(1-i\mu)}$$
$$\times \left[\Gamma\!\left(\frac{1-i\mu-\nu_1-\nu_2}{2}\right)\Gamma\!\left(\frac{1-i\mu+\nu_1+\nu_2}{2}\right) - \Gamma\!\left(\frac{1-i\mu-\nu_1+\nu_2}{2}\right)\Gamma\!\left(\frac{1-i\mu+\nu_1-\nu_2}{2}\right)\right]. \tag{14.14}$$

In the case when $\mathrm{Im}\,\mu < 1$, the error term that remains when replacing $I_{\nu_1,\nu_2}(n;m)$ by the right-hand side of (14.12) is an $\mathrm{O}(|n|^{\frac{-3+\mathrm{Re}\,(\nu_1+\nu_2)-\mathrm{Im}\,\mu}{2}})$, and the exponent $\frac{-3+\mathrm{Re}\,(\nu_1+\nu_2)-\mathrm{Im}\,\mu}{2}$ is indeed less than $-1+\min(0,\mathrm{Re}\,\nu_1)+\min(0,\mathrm{Re}\,\nu_2)$ (*cf.* what immediately precedes (14.9)) as soon as $\mathrm{Im}\,\mu > |\mathrm{Re}\,\nu_1|+|\mathrm{Re}\,\nu_2|-1$. As we assume $|\mathrm{Re}\,(\nu_1\pm\nu_2)|<1$, it follows from (14.7) and (14.12) that, when applying (14.2) and (14.3), we may substitute for $C_m(\mu)$ the function $D_m(\mu)$ defined as

$$D_m(\mu) = 2^{i\mu}\,\pi^{-1}\left(-\frac{i\mu}{2}\right)\Bigg[\sum_{\substack{n\geq 1\\ n\neq -m}} \sigma_{\nu_1}(n)\,\sigma_{\nu_2}(|n+m|)\,n^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}$$
$$\times\, C^+(\nu_1,\nu_2;\mu) + \sum_{\substack{n\leq -1\\ n\neq -m}} \sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)\,|n|^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}\,C^-(\nu_1,\nu_2;\mu)\Bigg]. \tag{14.15}$$

A neat simplification occurs when we consider, instead of $C^+(\nu_1,\nu_2;\mu)$ and $C^-(\nu_1,\nu_2;\mu)$, their sum and difference. In view of the expression (13.71) for the sum of the right-hand sides of (13.66) and (13.67), we indeed get

$$C^+(\nu_1,\nu_2;\mu)+C^-(\nu_1,\nu_2;\mu) = 2^{-i\mu-2}\,\pi^{\frac12}\,\frac{\Gamma(\frac{i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})}$$
$$\times\left[\frac{\Gamma(\frac{1-i\mu-\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\mu+\nu_1+\nu_2}{4})}{\Gamma(\frac{1+i\mu-\nu_1-\nu_2}{4})\,\Gamma(\frac{1+i\mu+\nu_1+\nu_2}{4})} - \frac{\Gamma(\frac{1-i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\mu-\nu_1+\nu_2}{4})}{\Gamma(\frac{1+i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{1+i\mu-\nu_1+\nu_2}{4})}\right]. \tag{14.16}$$

Similarly, the *difference* of the right-hand sides of (13.66) and (13.67), *i.e.*,

$$2^{-i\mu}\,\Gamma(i\mu)\,\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)$$
$$\times\left[\frac{1}{\Gamma(\frac{1-i\mu+\nu}{2})\,\Gamma(\frac{1+i\mu-\nu}{2})}+\frac{1}{\Gamma(\frac{1-i\mu-\nu}{2})\,\Gamma(\frac{1+i\mu+\nu}{2})}-\frac{1}{\Gamma(i\mu)\,\Gamma(1-i\mu)}\right], \tag{14.17}$$

can be written, following the same computations as those between (13.68) and (13.71), as

$$\frac{2^{-i\mu}}{\pi}\Gamma(i\mu)\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)\left[\cos\frac{\pi(i\mu-\nu)}{2}+\cos\frac{\pi(i\mu+\nu)}{2}-\sin\pi i\mu\right]$$
$$=\frac{2^{2-i\mu}}{\pi}\Gamma(i\mu)\Gamma\left(\frac{1-i\mu-\nu}{2}\right)\Gamma\left(\frac{1-i\mu+\nu}{2}\right)\cos\frac{\pi(i\mu)}{2}\sin\frac{\pi(1-i\mu+\nu)}{4}\sin\frac{\pi(1-i\mu-\nu)}{4}$$
$$=2^{2-i\mu}\,\pi^2\,\frac{\Gamma(i\mu)\,\Gamma(\frac{1-i\mu-\nu}{2})\,\Gamma(\frac{1-i\mu+\nu}{2})}{\Gamma(\frac{1+i\mu}{2})\,\Gamma(\frac{1-i\mu}{2})\,\Gamma(\frac{1-i\mu+\nu}{4})\,\Gamma(\frac{3+i\mu-\nu}{4})\,\Gamma(\frac{1-i\mu-\nu}{4})\,\Gamma(\frac{3+i\mu+\nu}{4})}, \tag{14.18}$$

which reduces to

$$2^{-i\mu}\,\pi^{\frac{1}{2}}\,\frac{\Gamma(\frac{i\mu}{2})\,\Gamma(\frac{3-i\mu-\nu}{4})\,\Gamma(\frac{3-i\mu+\nu}{4})}{\Gamma(\frac{1-i\mu}{2})\,\Gamma(\frac{3+i\mu-\nu}{4})\,\Gamma(\frac{3+i\mu+\nu}{4})}\,. \tag{14.19}$$

It follows that

$$C^+(\nu_1,\nu_2;\mu)-C^-(\nu_1,\nu_2;\mu)=2^{-i\mu-2}\,\pi^{\frac{1}{2}}\,\frac{\Gamma(\frac{i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})}$$
$$\times\left[\frac{\Gamma(\frac{3-i\mu-\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\mu+\nu_1+\nu_2}{4})}{\Gamma(\frac{3+i\mu-\nu_1-\nu_2}{4})\,\Gamma(\frac{3+i\mu+\nu_1+\nu_2}{4})}-\frac{\Gamma(\frac{3-i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1+\nu_2}{4})}{\Gamma(\frac{3+i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{3+i\mu-\nu_1+\nu_2}{4})}\right]. \tag{14.20}$$

Finally, we see that (14.2) is valid with

$$d_m=-8i\pi\,|m|^{-\frac{i\lambda_k}{2}}\times\text{residue of } D_m(\mu)\quad\text{at } \mu=\lambda_k \tag{14.21}$$

if

$$D_m(\mu)=2^{i\mu-1}\,\pi^{-1}\left(-\frac{i\mu}{2}\right)\sum_{j=0,1}\left[C^+(\nu_1,\nu_2;\mu)+(-1)^j C^-(\nu_1,\nu_2;\mu)\right]$$
$$\sum_{\substack{n\neq 0\\ n\neq -m}}\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)\,|n|_j^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}\,, \tag{14.22}$$

recalling that signed powers have been defined in (11.26).

To get at the poles and residues of the two Dirichlet series on the right-hand side of (14.22), we shall appeal to the pointwise product and Poisson bracket (11.3) of the two Eisenstein series $E^*_{\frac{1-\nu_1}{2}}$ and $E^*_{\frac{1-\nu_2}{2}}$. On one hand, trying to find, by an application of the same method, the discrete summands in the spectral decomposition of these two functions will depend on an examination of the same series: only the coefficients will differ. On the other hand, the Roelcke-Selberg decomposition of these two functions is known [62, Section 14]. Recall from (*loc.cit.*, Section 12), that the Poisson bracket under consideration lies in $L^2(\Gamma\backslash\Pi)$ and, from (*loc.cit.*, Proposition 14.2) that the ordinary product $E^*_{\frac{1-\nu_1}{2}}\,E^*_{\frac{1-\nu_2}{2}}$ lies in this space after one has subtracted from it a sum of four Eisenstein series. For short, we shall still call Roelcke-Selberg decomposition of the product the sum of the four Eisenstein series (which lies outside $L^2(\Gamma\backslash\Pi)$) and of the usual Roelcke-Selberg decomposition of the remainder.

For every k, let us denote as $(g^{\mathrm{sym}}_{\nu_1,\nu_2})_k$ (*resp.* $(g^{\mathrm{antisym}}_{\nu_1,\nu_2})_k$) the projection onto the subspace of $L^2(\Gamma\backslash\Pi)$ corresponding to the discrete eigenvalue $\frac{1+\lambda_k^2}{4}$ of the product $E^*_{\frac{1-\nu_1}{2}}\,E^*_{\frac{1-\nu_2}{2}}$ (*resp.* the halved Poisson bracket $\frac{1}{2}\,\{E^*_{\frac{1-\nu_1}{2}}\,,\,E^*_{\frac{1-\nu_2}{2}}\}$). The method expounded in (14.1), (14.2), (14.3) is still applicable. As just said, however, it is necessary to first subtract from the product (this is not necessary when dealing with the Poisson bracket) a certain linear combination of four Eisenstein series $\sum c(\varepsilon_1\nu_1, \varepsilon_2\nu_2)\, E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}$ (the coefficients of which can be found in [62, (9.10)] but are not needed), so as to get a function in $L^2(\Gamma\backslash\Pi)$ as a result: this is in full analogy with the way (13.35) we built the function $g^1_{\nu_1,\nu_2}$ from $f^1_{\nu_1,\nu_2}$. We denote as b_m^{sym}, $C_m^{\mathrm{sym}}(\mu)$, d_m^{sym} the Fourier coefficients of the product, the function built by (14.1) from these coefficients, finally the coefficient defined by (14.3) from the residues of the function C_m^{sym}; something entirely analogous goes for the Poisson bracket, using the superscript "antisym" everywhere.

Lemma 14.1. *Assume* $|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1$. *One has*

$$C_m^{\mathrm{sym}}(\mu) \sim \frac{2^{-4}\pi^{-\frac{3}{2}}}{\Gamma(-\frac{i\mu}{2})\,\Gamma(\frac{1-i\mu}{2})} \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)\,|n|^{\frac{-1-\nu_1-\nu_2+i\mu}{2}} \times$$
$$\Gamma\left(\frac{1-i\mu+\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu+\nu_1-\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu-\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu-\nu_1-\nu_2}{4}\right) \tag{14.23}$$

up to an error term which extends holomorphically to the half-plane $\mathrm{Im}\ \mu > -1 + |\mathrm{Re}\ \nu_1| + |\mathrm{Re}\ \nu_2|$. *In a similar way,*

$$C_m^{\mathrm{antisym}}(\mu) \sim \frac{2^{-4}\, i\, \pi^{-\frac{3}{2}}}{\Gamma(-\frac{i\mu}{2})\,\Gamma(\frac{1-i\mu}{2})} \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)\,|n|_1^{\frac{-1-\nu_1-\nu_2+i\mu}{2}} \times$$
$$\Gamma\left(\frac{3-i\mu+\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{3-i\mu+\nu_1-\nu_2}{4}\right)\Gamma\left(\frac{3-i\mu-\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{3-i\mu-\nu_1-\nu_2}{4}\right) \tag{14.24}$$

up to an error term which extends holomorphically to the same half-plane.

Proof. From (4.5), one sees that $b_m^{\mathrm{sym}}(y)$, the coefficient of $e^{2i\pi m x}$ in the expansion

$$E^*_{\frac{1-\nu_1}{2}}(z)\, E^*_{\frac{1-\nu_2}{2}}(z) = \sum_{m\in\mathbb{Z}} b_m^{\mathrm{sym}}(y)\, e^{2i\pi m x}\,, \tag{14.25}$$

is given for $m \neq 0$ as the sum of three terms: those which come from the "constant" (*i.e.*, independent of x) terms in the expansion (4.5) of one, or the other, of the two factors, which do not contribute to the real poles of C_m^{sym} (by (14.4) again), and the interesting part

$$b_m^{\mathrm{sym}}(y) \sim 4\,y \sum_{\substack{n\neq 0\\ n\neq -m}} \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)}{|n|^{\frac{\nu_1}{2}}\,|n+m|^{\frac{\nu_2}{2}}}\, K_{\frac{\nu_1}{2}}(2\pi|n|y)\, K_{\frac{\nu_2}{2}}(2\pi|n+m|y)\,. \tag{14.26}$$

Then

$$C_m^{\rm sym}(\mu) \sim \frac{1}{2}\,\frac{\pi^{-1-\frac{i\mu}{2}}}{\Gamma(-\frac{i\mu}{2})} \sum_{\substack{n\neq 0\\ n\neq -m}} \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)}{|n|^{\frac{\nu_1}{2}}\,|n+m|^{\frac{\nu_2}{2}}} \times \int_0^\infty y^{\frac{-1-i\mu}{2}}\, K_{\frac{\nu_1}{2}}(2\pi|n|y)\, K_{\frac{\nu_2}{2}}(2\pi|n+m|y)\, dy\,. \quad (14.27)$$

The so-called Weber-Schafheitlin integral on the last line can be expressed [31, p. 101] as

$$\frac{1}{8}\,\pi^{\frac{-1+i\mu}{2}} \left(\Gamma\left(\frac{1-i\mu}{2}\right)\right)^{-1} |n|^{\frac{-1+i\mu-\nu_2}{2}}\, |n+m|^{\frac{\nu_2}{2}} \times\Gamma\left(\frac{1-i\mu+\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu+\nu_1-\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu-\nu_1+\nu_2}{4}\right)\Gamma\left(\frac{1-i\mu-\nu_1-\nu_2}{4}\right) \times {}_2F_1\left(\frac{1-i\mu+\nu_1+\nu_2}{4},\frac{1-i\mu-\nu_1+\nu_2}{4};\frac{1-i\mu}{2};1-\left(\frac{n+m}{n}\right)^2\right). \quad (14.28)$$

Since, for fixed m and $n\to\pm\infty$, the argument $1-(\frac{n+m}{n})^2$ of the hypergeometric function is quite close to 0, the value of this function is close to 1, and the error committed in the integral by substituting the constant 1 for the hypergeometric function would be $\mathrm{O}(|n|^{\frac{-1-\mathrm{Im}\,\mu}{2}-1})$; since $\frac{\sigma(|n|)}{|n|^{\frac{\nu}{2}}} = \mathrm{O}(|n|^{\frac{|\mathrm{Re}\,\nu|}{2}})$, the error committed in $C_m^{\rm sym}(\mu)$ by this substitution extends as a holomorphic function of μ for $\mathrm{Im}\,\mu > -1+|\mathrm{Re}\,\nu_1|+|\mathrm{Re}\,\nu_2|$, a neighborhood of the closed upper half-plane. This leads to (14.23).

We emphasize that, so as to be in position to apply [62, Theorem 7.4], we should really consider instead of $d_m^{\rm sym}$ the coefficient $\tilde{d}_m^{\rm sym}$ obtained as the residue, as in (14.3), of the function $\tilde{C}_m$ associated to the *difference* between $E^*_{\frac{1-\nu_1}{2}}\, E^*_{\frac{1-\nu_2}{2}}$ and the linear combination of four Eisenstein series above: however, from an equation similar to (14.4), one sees that the function C_m associated through (14.1) to an Eisenstein distribution $E_{1+\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}$ has no real poles.

Using the formula that gives the derivative of a Bessel function [31, p. 67], we get

$$\frac{d}{dy}\,(y^{\frac{1}{2}}\, K_{\frac{\nu}{2}}(2\pi|n|y) = -\pi\,|n|\,y^{\frac{1}{2}}\,(K_{\frac{\nu-2}{2}}(2\pi|n|y)+K_{\frac{\nu+2}{2}}(2\pi|n|y)) + \frac{1}{2}\,y^{-\frac{1}{2}}\,K_{\frac{\nu}{2}}(2\pi|n|y)\ : \quad (14.29)$$

this entails (using (4.5))

$$\{E^*_{\frac{1-\nu_1}{2}}\,,\,E^*_{\frac{1-\nu_2}{2}}\} \sim \\ 8i\pi \sum_{n_1 n_2 \neq 0} |n_1|^{-\frac{\nu_1}{2}}\,|n_2|^{-\frac{\nu_2}{2}}\,\sigma_{\nu_1}(|n_1|)\,\sigma_{\nu_2}(|n_2|)\,e^{2i\pi(n_1+n_2)x}\,c^{n_1,n_2}_{\nu_1,\nu_2}(y)\,, \quad (14.30)$$

with

$$c^{n_1,n_2}_{\nu_1,\nu_2}(y) = \frac{n_1-n_2}{2}\,y^2\,K_{\frac{\nu_1}{2}}(2\pi|n_1|y)\,K_{\frac{\nu_2}{2}}(2\pi|n_2|y) \\ +\pi\,|n_1|\,n_2\,y^3\,(K_{\frac{\nu_1-2}{2}}(2\pi|n_1|y)+K_{\frac{\nu_1+2}{2}}(2\pi|n_1|y)\,K_{\frac{\nu_2}{2}}(2\pi|n_2|y) \\ -\pi\,n_1\,|n_2|\,y^3\,K_{\frac{\nu_1}{2}}(2\pi|n_1|y)\,(K_{\frac{\nu_2-2}{2}}(2\pi|n_2|y)+K_{\frac{\nu_2+2}{2}}(2\pi|n_2|y)\,, \quad (14.31)$$

and we have to evaluate the integral

$$I^{n_1,n_2}_{\nu_1,\nu_2} = \int_0^\infty y^{\frac{-3-i\mu}{2}}\,c^{n_1,n_2}_{\nu_1,\nu_2}(y)\,dy\,. \quad (14.32)$$

It is out of the question to write its lengthy expression, provided by the Weber-Schafheitlin integral (14.28) as soon as Im $\mu > -3 + |\text{Re}\ \nu_1| + |\text{Re}\ \nu_2|$ (yes, the situation has improved): because of the presence of factors like n_1, n_2, or $|n_1|\,n_2$, or $n_1\,|n_2|$, ordinary powers $|\ |^\alpha$ as well as signed powers $|\ |_1^\alpha$ do appear in the result of the computation; on the other hand, some of the Gamma factors are to be evaluated at $\frac{3-i\mu\pm\nu_1\pm\nu_2}{4}$, and some at $\frac{7-i\mu\pm\nu_1\pm\nu_2}{4}$. Keeping only the "main term" just as we have done after (14.28), *i.e.*, replacing all the hypergeometric functions by 1, we find

$$I^{n_1,n_2}_{\nu_1,\nu_2} \sim \frac{1}{2^4}\,\pi^{\frac{-3+i\mu}{2}} \\ \times \frac{\Gamma(\frac{3-i\mu+\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1-\nu_2}{4})}{\Gamma(\frac{3-i\mu}{2})} \\ \times \Bigg[|n_1|_1^{\frac{-1+i\mu-\nu_2}{2}}\,|n_2|^{\frac{\nu_2}{2}} - |n_1|^{\frac{-3+i\mu-\nu_2}{2}}\,|n_2|_1^{\frac{\nu_2+2}{2}} + \frac{4}{3-i\mu} \\ \times \Bigg[\Bigg(\frac{3-i\mu-\nu_1+\nu_2}{4}\,\frac{3-i\mu-\nu_1-\nu_2}{4} \\ + \frac{3-i\mu+\nu_1+\nu_2}{4}\,\frac{3-i\mu+\nu_1-\nu_2}{4} \Bigg)\,|n_1|^{\frac{-3+i\mu-\nu_2}{2}}\,|n_2|_1^{\frac{\nu_2+2}{2}} \\ - \frac{3-i\mu+\nu_1-\nu_2}{4}\,\frac{3-i\mu-\nu_1-\nu_2}{4}\,|n_1|_1^{\frac{-1+i\mu-\nu_2}{2}}\,|n_2|^{\frac{\nu_2}{2}} \\ - \frac{3-i\mu+\nu_1+\nu_2}{4}\,\frac{3-i\mu-\nu_1+\nu_2}{4}\,|n_1|_1^{\frac{-5+i\mu-\nu_2}{2}}\,|n_2|^{\frac{\nu_2+4}{2}} \Bigg] \Bigg]\,. \quad (14.33)$$

We need to substitute for the pair (n_1, n_2) a pair $(-n, n+m)$ with $n \neq 0, -m$, and find a main term as $n \to \pm\infty$: thus, since m is to be kept fixed,

$|n_1|^\alpha = |n|^\alpha$, $|n_2|^\alpha = |n+m|^\alpha$, $|n_2|_1^\alpha = |n+m|_1^\alpha$ but $|n_1|_1^\alpha = -|n|_1^\alpha$. After this substitution, the main term of $I_{\nu_1,\nu_2}^{n_1,n_2}$ becomes

$$\frac{1-i\mu}{2^4}\,\pi^{\frac{-3+i\mu}{2}}\,|n|_1^{\frac{-1+i\mu}{2}} \times \frac{\Gamma(\frac{3-i\mu+\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1-\nu_2}{4})}{\Gamma(\frac{3-i\mu}{2})}, \tag{14.34}$$

up to an error term in $\mathrm{O}(|n|^{\frac{-3-\mathrm{Im}\ \mu}{2}})$: since

$$\sum_{n\neq 0,\,-m} |n|^{\frac{1}{2}(|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2|)-\frac{1}{2}(3+\mathrm{Im}\ \mu)} < \infty \tag{14.35}$$

as soon as $\mathrm{Im}\ \mu > -1+|\mathrm{Re}\ \nu_1|+|\mathrm{Re}\ \nu_2|$, (14.30) shows that, up to an error term which extends as a holomorphic function of μ a little below the real line, one has

$$C_m^{\mathrm{antisym}}(\mu) \sim \frac{i(1-i\mu)}{2^5}\,\frac{\pi^{-\frac{3}{2}}}{\Gamma(-\frac{i\mu}{2})} \times \frac{\Gamma(\frac{3-i\mu+\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1+\nu_2}{4})\,\Gamma(\frac{3-i\mu-\nu_1-\nu_2}{4})}{\Gamma(\frac{3-i\mu}{2})} \times \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n+m|)\,|n|_1^{\frac{-1-\nu_1-\nu_2+i\mu}{2}} \ : \tag{14.36}$$

this ends the proof of Lemma 14.1. □

So as to state the two theorems in this section, some reminders are necessary, concerning the spectral theory of the modular Laplacian Δ on $L^2(\Gamma\backslash\Pi)$. Recall from the discussion between (4.8) and (4.11) that $(\frac{1+(\lambda_k^\pm)^2}{4})$ is the sequence of (not repeated) even (*resp.* odd) eigenvalues of Δ and that the image of the orthogonal projection operator $P_{\lambda_k^+}$ (*resp.* $P_{\lambda_k^-}$) onto the corresponding eigenspace, consisting solely of even (*resp.* odd) cusp-forms, has a basis of Maass-Hecke forms denoted as $(\mathcal{N}_{k,\ell}^+)_{1\leq\ell\leq n_k^+}$ (*resp.* $(\mathcal{N}_{k,\ell}^-)_{1\leq\ell\leq n_k^-}$): it is assumed that $\lambda_k^\pm > 0$ for all k. So far as we know, it has never been proved that the sets of even and odd eigenvalues are disjoint, so that the preceding operators should really have been denoted as $P_{\lambda_k^+}^+$ and $P_{\lambda_k^-}^-$: but no confusion can arise. Also, the definition of the L-function $L^*(.,\mathcal{N}^\pm)$ associated with an even or odd cusp-form has been recalled in (5.27) and (5.28): there is a slight difference in the Archimedean factor between the two cases.

Theorem 14.2. *Assuming* $|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1$, *let* $\frac{1+(\lambda_k^+)^2}{4}$ *be an even eigenvalue of the modular Laplacian, and let*

$$(g_{\nu_1,\nu_2}^1)_{k,+} = P_{\lambda_k^+}\, g_{\nu_1,\nu_2}^1\,, \tag{14.37}$$

where the function $g^1_{\nu_1,\nu_2}$ *has been introduced in* (13.35)*. One has*

$$(g^1_{\nu_1,\nu_2})_{k,+} = \sum_{\varepsilon=\pm 1} \frac{\pi\,\Gamma(\frac{i\lambda_k^+}{2})\,\Gamma(\frac{-i\lambda_k^+}{2})\,(-\frac{i\varepsilon\lambda_k^+}{2})}{\Gamma(\frac{1+i\varepsilon\lambda_k^+-\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\varepsilon\lambda_k^++\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\varepsilon\lambda_k^+-\nu_1+\nu_2}{4})\,\Gamma(\frac{1+i\varepsilon\lambda_k^++\nu_1+\nu_2}{4})} \times \sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right) L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right) \|\mathcal{N}^+_{k,\ell}\|^{-2}\,\mathcal{N}^+_{k,\ell}\,. \tag{14.38}$$

In a similar way, if $\frac{1+(\lambda_k^-)^2}{4}$ *is an odd eigenvalue, set*

$$(g^1_{\nu_1,\nu_2})_{k,-} = P_{\lambda_k^-}\, g^1_{\nu_1,\nu_2}\,. \tag{14.39}$$

Then

$$(g^1_{\nu_1,\nu_2})_{k,-} = \sum_{\varepsilon=\pm 1} \frac{\pi\,\Gamma(\frac{i\lambda_k^-}{2})\,\Gamma(\frac{-i\lambda_k^-}{2})\,(\frac{i\varepsilon\lambda_k^-}{2})}{\Gamma(\frac{3+i\varepsilon\lambda_k^--\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\varepsilon\lambda_k^-+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\varepsilon\lambda_k^--\nu_1+\nu_2}{4})\,\Gamma(\frac{3+i\varepsilon\lambda_k^-+\nu_1+\nu_2}{4})} \times \sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right) L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right) \|\mathcal{N}^-_{k,\ell}\|^{-2}\,\mathcal{N}^-_{k,\ell}\,. \tag{14.40}$$

Proof. One should first realize that, since an Eisenstein series is an even modular form (under the symmetry $z \mapsto -\bar{z}$), so is the product of two Eisenstein series, while their Poisson bracket is odd. A function such as $g^1_{\nu_1,\nu_2}$ (*resp.* $g^2_{\nu_1,\nu_2}$), or $f^1_{\nu_1,\nu_2}$ (*resp.* $f^2_{\nu_1,\nu_2}$) (*cf.* (13.14) and (13.15)), on the other hand, has cusp-forms of both types in its decomposition, since it has to do with the composition of two operators.

The Roelcke-Selberg decomposition of the pointwise product $E^*_{\frac{1-\nu_1}{2}}\,E^*_{\frac{1-\nu_2}{2}}$ has been given in [62, Section 14]: with our present notation, introduced just before Lemma 14.1, one would write (*loc.cit.*, (14.9) and Theorem 14.5) as

$$(g^{\rm sym}_{\nu_1,\nu_2})_{k,+} = \frac{1}{2}\sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right) L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right) \|\mathcal{N}^+_{k,\ell}\|^{-2}\mathcal{N}^+_{k,\ell}. \tag{14.41}$$

On the other hand, the Fourier coefficients of $(g^1_{\nu_1,\nu_2})_{k,+}$ and $(g^{\rm sym}_{\nu_1,\nu_2})_{k,+}$ are given in terms of the residues at $\mu = \lambda_k^+$ of two functions (taking the place of C_k in (14.1)) which are respectively expressed as the even (*i.e.*, $j = 0$) term in (14.22), and (14.23): now the Dirichlet series involved in both functions are the same, only the coefficients differ. Taking the ratio of these two coefficients (recall that $C^+(\nu_1,\nu_2;\mu)+C^-(\nu_1,\nu_2;\mu)$ has been made explicit in (14.16)), we find (14.38).

In a similar way, [62, Theorem 14.8] together with (*loc.cit.*, (12.1) and (14.61)) gives the Roelcke-Selberg decomposition of the Poisson bracket $\{E^*_{\frac{1-\nu_1}{2}}, E^*_{\frac{1-\nu_2}{2}}\}$: with our present notations, given an odd eigenvalue $\frac{1+(\lambda_k^-)^2}{4}$,

$$(g^{\text{antisym}}_{\nu_1,\nu_2})_{k,-}=\frac{1}{2i}\sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right)L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right)\|\mathcal{N}^-_{k,\ell}\|^{-2}\mathcal{N}^-_{k,\ell}. \tag{14.42}$$

Now, it is the coefficient of the odd (*i.e.*, $j=1$) term from (14.23) that should be compared to (14.24), using also (14.20): this leads to (14.40). □

Theorem 14.3. *Under the same assumptions as in Theorem* 14.2, *set*

$$(g^2_{\nu_1,\nu_2})_{k,+}=P_{\lambda_k^+}\,g^2_{\nu_1,\nu_2}\,,\qquad (g^2_{\nu_1,\nu_2})_{k,-}=P_{\lambda_k^-}\,g^2_{\nu_1,\nu_2}\,, \tag{14.43}$$

where $g^2_{\nu_1,\nu_2}$ *was introduced in* (13.78). *Then*

$$\begin{aligned}(g^2_{\nu_1,\nu_2})_{k,+}\\ =\sum_{\varepsilon=\pm1}&\frac{\pi\,\Gamma(\frac{i\lambda_k^+}{2})\,\Gamma(\frac{-i\lambda_k^+}{2})\,(-\frac{(\lambda_k^+)^2}{2})}{\Gamma(\frac{1+i\varepsilon\lambda_k^+-\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\varepsilon\lambda_k^++\nu_1-\nu_2}{4})\,\Gamma(\frac{1-i\varepsilon\lambda_k^+-\nu_1+\nu_2}{4})\,\Gamma(\frac{1+i\varepsilon\lambda_k^++\nu_1+\nu_2}{4})}\\ &\times\sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right)L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^+_{k,\ell}\right)\|\mathcal{N}^+_{k,\ell}\|^{-2}\mathcal{N}^+_{k,\ell}\end{aligned} \tag{14.44}$$

and

$$\begin{aligned}(g^2_{\nu_1,\nu_2})_{k,-}\\ =\sum_{\varepsilon=\pm1}&\frac{\pi\,\Gamma(\frac{i\lambda_k^-}{2})\,\Gamma(\frac{-i\lambda_k^-}{2})\,(\frac{(\lambda_k^-)^2}{2})}{\Gamma(\frac{3+i\varepsilon\lambda_k^--\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\varepsilon\lambda_k^-+\nu_1-\nu_2}{4})\,\Gamma(\frac{3-i\varepsilon\lambda_k^--\nu_1+\nu_2}{4})\,\Gamma(\frac{3+i\varepsilon\lambda_k^-+\nu_1+\nu_2}{4})}\\ &\times\sum_\ell L^*\left(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right)L^*\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^-_{k,\ell}\right)\|\mathcal{N}^-_{k,\ell}\|^{-2}\mathcal{N}^-_{k,\ell}\,.\end{aligned} \tag{14.45}$$

Proof. The proof of Theorem 14.2 can be modified into a proof of Theorem 14.3 along the lines of the end of the proof of Theorem 13.7. To start with, there is an extra factor $-i\mu$ with the same origin as in (13.91). Next, if we consider the analogue of (14.8) occurring in our present computation, all we have to do, besides plugging the extra factor $-i\mu$, is forgetting the factor $\varepsilon_1\varepsilon_2$, thus substituting $\cosh(\nu_1 s)$ and $\cosh(\nu_2 s)$ for the corresponding hyperbolic sines in (14.13) and (14.14): this has already been explained after (13.91). The net result is that, in (14.13), (14.14), (14.16) and (14.20), the minus sign in front of the product of Gamma factors depending on $\nu_1 - \nu_2$ must disappear: this leads to (14.44) and (14.45), from a comparison with (14.38) and (14.40). □

15 A proof of the main formula

In this section, we state and prove the composition formula announced in (5.38) or (5.62): one may also look at (15.33) for a fully expanded version of this result. We refer to Definition 13.2 for a proper understanding of the following statement.

Theorem 15.1. *Let ν_1 and ν_2 be complex numbers with $|\mathrm{Re}\,(\nu_1 \pm \nu_2)| < 1$. There exists an automorphic (tempered) distribution $\mathfrak{S}$ satisfying for every $z \in \Pi$ the identities*

$$(u_z|\mathrm{Op}(2i\pi\,\mathcal{E}\,\mathfrak{S})\,u_z) = \int_{-\infty}^{\infty} \overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})u_z^1)(t)}\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\,u_z^1)(t)\,dt \tag{15.1}$$

and

$$(u_z^1|\mathrm{Op}(2i\pi\,\mathcal{E}\,\mathfrak{S})\,u_z^1) = \int_{-\infty}^{\infty} \Big[3\,\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})u_z^2)(t)}\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\,u_z^2)(t)$$
$$-3^{\frac12}\,\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})u_z)(t)}\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\,u_z^2)(t)\; - 3^{\frac12}\,\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_{\bar\nu_1})u_z^2)(t)}\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2})\,u_z)(t)\Big]\,dt\ : \tag{15.2}$$

the distribution $\mathfrak{S}$ is unique up to the addition of a multiple of the Eisenstein distribution $\mathfrak{E}_0^\sharp$. One choice of $\mathfrak{S}$, denoted as $\mathfrak{F}^\sharp_{\nu_1}\#\mathfrak{F}^\sharp_{\nu_2}$, is given as

$$\mathfrak{F}^\sharp_{\nu_1}\#\mathfrak{F}^\sharp_{\nu_2} = \sum_{\varepsilon=\pm1}\left[\frac{\zeta(\varepsilon\nu_1)\,\zeta(\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1+\nu_2)-1)}\,\mathfrak{F}^\sharp_{1-\varepsilon(\nu_1+\nu_2)} + \frac{\zeta(\varepsilon\nu_1)\,\zeta(-\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1-\nu_2)-1)}\,\mathfrak{F}^\sharp_{\varepsilon(\nu_1-\nu_2)-1}\right]$$
$$+\mathcal{L}'\left(\frac{1+\nu_1+\nu_2}{2}\right)\,\mathcal{G}\,\mathcal{L}'\left(\frac{1+\nu_1-\nu_2}{2}\right)\,2^{-\frac12+i\pi\,\mathcal{E}}\,\mathfrak{B}\,, \tag{15.3}$$

where $\mathfrak{B} = \mathfrak{B}^0$ is the Bezout distribution properly introduced in the proof of Theorem 4.3, *and $\mathcal{L}'(s)$ was introduced in Definition* 5.7 .

Remarks. By a sesquiholomorphic argument, the pair (z, z) may be replaced by (w, z). The conditions above exactly mean, in the sense of Definition 13.2, that the operator $\mathrm{mad}\,(2i\pi\,(P\wedge Q))\,(\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_1})\,\mathrm{Op}(\mathfrak{F}^\sharp_{\nu_2}))$ admits the symbol $2i\pi\,\mathcal{E}\,(\mathfrak{F}^\sharp_{\nu_1}\#\mathfrak{F}^\sharp_{\nu_2})$ in the minimal sense.

The *almost*-uniqueness only is due to the equation $i\pi\,\mathcal{E}\,\mathfrak{E}_0^\sharp = 0$.

All the difficulties of the proof of Theorem 15.1 are behind us: what remains is only a matter of piecing various notions and results together. It is useful to give another normalization of cusp-distributions, defining

$$(\mathfrak{F}^+_{k,\ell})^\sharp = 2^{\frac{-1-i\lambda_k^+}{2}}\,\zeta^*(i\lambda_k^+)\,\zeta^*(-i\lambda_k^+)\,\|\mathcal{N}^+_{|k|,\ell}\|^{-2}\,(\mathfrak{N}^+_{k,\ell})^\sharp\;: \tag{15.4}$$

when $k \geq 1$, $\lambda_k^+ > 0$ and all concepts are introduced between (4.10) and (4.11); next, $\lambda_{-k} = -\lambda_k$ and $(\mathfrak{N}^+_{-k,\ell})^\sharp = \mathcal{F}\,(\mathfrak{N}^+_{k,\ell})^\sharp$. We make exactly the analogous definition of $(\mathfrak{F}^-_{k,\ell})^\sharp$, substituting the superscript $-$ for $+$ everywhere in (15.4).

The case $\ell = 0$ of Theorem 4.3 may then be recalled as follows, recalling also (*cf.* (3.2)) that $\mathfrak{E}^\sharp_\nu = 2^{\frac{1+\nu}{2}}\,\mathfrak{F}^\sharp_\nu$:

$$\begin{aligned} 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B} &= \frac{1}{4\pi}\int_{-\infty}^{\infty} (\zeta(i\lambda)\,\zeta(-i\lambda))^{-1}\,\mathfrak{F}^\sharp_{i\lambda}\,d\lambda \\ &\quad + \frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (\zeta(i\lambda_k^+)\,\zeta(-i\lambda_k^+))^{-1}\,(\mathfrak{F}^+_{k,\ell})^\sharp \\ &\quad + \frac{1}{2}\sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (\zeta(i\lambda_k^-)\,\zeta(-i\lambda_k^-))^{-1}\,(\mathfrak{F}^-_{k,\ell})^\sharp\,. \end{aligned} \tag{15.5}$$

Lemma 15.2. *One has*

$$\mathcal{G}\,\mathfrak{F}^\sharp_\nu = \mathfrak{F}^\sharp_{-\nu}, \qquad \mathcal{G}\,(\mathfrak{F}^\pm_{k,\ell})^\sharp = (\mathfrak{F}^\pm_{-k,\ell})^\sharp\,. \tag{15.6}$$

Proof. The first equation was already given in (3.4). Since (*cf.* (2.7)) $\mathcal{G} = 2^{2i\pi\,\mathcal{E}}\,\mathcal{F}$ and (*cf.* (4.4)) $(\mathfrak{N}^\pm_{k,\ell})^\sharp$ is homogeneous of degree $-1-i\lambda_k^\pm$, one has (not forgetting that $\mathcal{E}\mathcal{F} = -\mathcal{F}\mathcal{E}$), $\mathcal{G}\,(\mathfrak{N}^\pm_{k,\ell})^\sharp = 2^{i\lambda_k^\pm}\,(\mathfrak{N}^\pm_{-k,\ell})^\sharp$, from which the second equation follows. □

Lemma 15.3. *One has*

$$\begin{aligned} \mathcal{L}'(s)\,\mathfrak{F}^\sharp_\nu &= \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s}{2}-\frac{\nu}{4})}{\Gamma(\frac{1-s}{2}+\frac{\nu}{4})}\,\zeta\left(s-\frac{\nu}{2}\right)\zeta\left(s+\frac{\nu}{2}\right)\mathfrak{F}^\sharp_\nu\,, \\ \mathcal{L}'(s)\,(\mathfrak{F}^+_{k,\ell})^\sharp &= \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s}{2}-\frac{i\lambda_k^+}{4})}{\Gamma(\frac{1-s}{2}+\frac{i\lambda_k^+}{4})}\,L(s,\mathcal{N}^+_{|k|,\ell})\,(\mathfrak{F}^+_{k,\ell})^\sharp\,, \\ \mathcal{L}'(s)\,(\mathfrak{F}^-_{k,\ell})^\sharp &= \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s+1}{2}-\frac{i\lambda_k^-}{4})}{\Gamma(\frac{2-s}{2}+\frac{i\lambda_k^-}{4})}\,L(s,\mathcal{N}^+_{|k|,\ell})\,(\mathfrak{F}^-_{k,\ell})^\sharp\,. \end{aligned} \tag{15.7}$$

Proof. Since $\mathfrak{F}^\sharp_\nu$ and $(\mathfrak{F}^\pm_{k,\ell})^\sharp$ are proportional respectively to $\mathfrak{E}^\sharp_\nu$ and $(\mathfrak{N}^\pm_{k,\ell})^\sharp$, the first identity is a consequence of (5.23) and (5.30); the other two are a consequence of (5.24) and (5.30). □

The definition of the function $s \mapsto L^*(s,\mathcal{M})$ associated with a cusp-form $\mathcal{M}$, recalled in (5.27) and (5.28), is very classical. There is a more precise variant adapted to cusp-*distributions* $\mathfrak{M}^\sharp$ only: recall from (4.4) that any cusp-form $\mathcal{M}_j$ associated to the eigenvalue $\frac{1+\lambda_j^2}{4}$, with $j \geq 1$ and $\lambda_j > 0$, yields *two* cusp-distributions $(\mathfrak{M}_{\pm j})^\sharp$, homogeneous of degrees $-1-i\lambda_j$ and $-1+i\lambda_j$ respectively. The map $(\mathfrak{M}_j)^\sharp \mapsto \mathcal{M}_j$, on the contrary, is well defined.

Definition 15.4. For any even cusp-distribution $(\mathfrak{M}^+)^\sharp$, homogeneous of degree $-1-i\lambda^+$, $\lambda^+ \in \mathbb{R}$, we set

$$\Lambda(s,(\mathfrak{M}^+)^\sharp) = \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s}{2}-\frac{i\lambda^+}{4})}{\Gamma(\frac{1-s}{2}+\frac{i\lambda^+}{4})}\,L(s,\mathcal{M}^+)\,, \tag{15.8}$$

where $\mathcal{M}^+$ is the cusp-form associated with $(\mathfrak{M}^+)^\sharp$; similarly, if $(\mathfrak{M}^-)^\sharp$ is an odd cusp-distribution homogeneous of degree $-1-i\lambda^-$, associated with the cusp-form $\mathcal{M}^-$, we set

$$\Lambda(s,(\mathfrak{M}^-)^\sharp) = \pi^{\frac{1}{2}-s}\,\frac{\Gamma(\frac{s+1}{2}-\frac{i\lambda^-}{4})}{\Gamma(\frac{2-s}{2}+\frac{i\lambda^-}{4})}\,L(s,\mathcal{M}^-)\,. \tag{15.9}$$

Observe that, since

$$\pi^{\frac{1}{2}}\,L^*(s,\mathcal{M}^+) = \Gamma\left(\frac{1-s}{2}+\frac{i\lambda^+}{4}\right)\Gamma\left(\frac{s}{2}+\frac{i\lambda^+}{4}\right)\Lambda(s,(\mathfrak{M}^+)^\sharp)$$

and

$$\pi^{\frac{1}{2}}\,L^*(s,\mathcal{M}^-) = \Gamma\left(\frac{2-s}{2}+\frac{i\lambda^-}{4}\right)\Gamma\left(\frac{s+1}{2}+\frac{i\lambda^-}{4}\right)\Lambda(s,(\mathfrak{M}^-)^\sharp)\,, \tag{15.10}$$

the function $\Lambda(s,(\mathfrak{M}^+)^\sharp)$, just like $L^*(s,\mathcal{M}^+)$, is invariant under the symmetry $s \mapsto 1-s$, and that $\Lambda(s,(\mathfrak{M}^-)^\sharp)$ changes to its negative under the same symmetry.

To *define* the action of the operator $\mathcal{L}'(s)$ on an automorphic tempered distribution $\mathfrak{S}$, rather than trying to give a global definition, we assume that $\mathfrak{S}$ has, in the weak sense in $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$, a decomposition into homogeneous components, just like $\mathfrak{B}^\ell$ in (4.24), and we define $\mathcal{L}'(s)$, componentwise, by means of the preceding lemma. In this way:

Lemma 15.5. *Assume that* $|\text{Re}\,(\nu_1 \pm \nu_2)| < 1$. *Then*

$$2i\pi\,\mathcal{E}\,\mathcal{L}'\left(\frac{1+\nu_1+\nu_2}{2}\right)\mathcal{G}\,\mathcal{L}'\left(\frac{1+\nu_1-\nu_2}{2}\right)2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B} = \mathfrak{T}_{\text{cont}} + \mathfrak{T}^+_{\text{disc}} + \mathfrak{T}^-_{\text{disc}} \tag{15.11}$$

with

$$\mathfrak{T}_{\rm cont} = \frac{1}{4\pi} \int_{-\infty}^{\infty} (-i\lambda)\, \mathfrak{F}^{\sharp}_{i\lambda} \frac{\zeta(\frac{1+\nu_1+\nu_2+i\lambda}{2})\, \zeta(\frac{1+\nu_1-\nu_2-i\lambda}{2})\, \zeta(\frac{1-\nu_1+\nu_2-i\lambda}{2})\, \zeta(\frac{1-\nu_1-\nu_2+i\lambda}{2})}{\zeta(i\lambda)\, \zeta(-i\lambda)}\, d\lambda\,, \tag{15.12}$$

and

$$\mathfrak{T}^{+}_{\rm disc} = \frac{1}{2} \sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (-i\lambda_k^+)\, (\mathfrak{F}^{+}_{k,\ell})^{\sharp} \times \frac{\Lambda(\frac{1+\nu_1+\nu_2}{2},\, (\mathfrak{N}^{+}_{k,\ell})^{\sharp})\, \Lambda(\frac{1+\nu_1-\nu_2}{2},\, (\mathfrak{N}^{+}_{-k,\ell})^{\sharp})}{\zeta(i\lambda_k^+)\, \zeta(-i\lambda_k^+)}\,, \tag{15.13}$$

finally

$$\mathfrak{T}^{-}_{\rm disc} = \frac{1}{2} \sum_{\substack{k,\ell \\ k\in\mathbb{Z}^\times}} (-i\lambda_k^-)\, (\mathfrak{F}^{-}_{k,\ell})^{\sharp} \times \frac{\Lambda(\frac{1+\nu_1+\nu_2}{2},\, (\mathfrak{N}^{-}_{k,\ell})^{\sharp})\, \Lambda(\frac{1+\nu_1-\nu_2}{2},\, (\mathfrak{N}^{-}_{-k,\ell})^{\sharp})}{\zeta(i\lambda_k^-)\, \zeta(-i\lambda_k^-)}\,. \tag{15.14}$$

Proof. That the series and integrals involved are weakly convergent in $\mathcal{S}'_{\rm even}(\mathbb{R}^2)$ follows the lines of the beginning of the proof of Theorem 4.3. The three terms of the decomposition of course refer to the continuous part and the two discrete parts of the decomposition of the distribution under study into homogeneous parts. From Lemmas 15.2 and 15.3, we find that the image of $\mathfrak{F}^{\sharp}_{i\lambda}$ under the operator

$$S_{\nu_1,\nu_2} \colon = 2i\pi\, \mathcal{E}\, \mathcal{L}' \left(\frac{1+\nu_1+\nu_2}{2} \right) \mathcal{G}\, \mathcal{L}' \left(\frac{1+\nu_1-\nu_2}{2} \right) \tag{15.15}$$

is

$$\pi^{-\nu_1} \frac{\Gamma(\frac{1+\nu_1-\nu_2-i\lambda}{4})\, \Gamma(\frac{1+\nu_1+\nu_2+i\lambda}{4})}{\Gamma(\frac{1-\nu_1+\nu_2+i\lambda}{4})\, \Gamma(\frac{1-\nu_1-\nu_2-i\lambda}{4})} (i\lambda)\, \mathfrak{F}^{\sharp}_{-i\lambda} \times \zeta\left(\frac{1+\nu_1-\nu_2-i\lambda}{2} \right) \zeta\left(\frac{1+\nu_1-\nu_2+i\lambda}{2} \right) \zeta\left(\frac{1+\nu_1+\nu_2+i\lambda}{2} \right) \zeta\left(\frac{1+\nu_1+\nu_2-i\lambda}{2} \right) \tag{15.16}$$

(do not forget that $2i\pi\, \mathcal{E}\, \mathfrak{F}^{\sharp}_{i\lambda} = -i\lambda\, \mathfrak{F}^{\sharp}_{i\lambda}$). Using the functional equation of the zeta function in the form

$$\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\, \zeta(s) = \pi^{s-\frac{1}{2}}\, \zeta(1-s) \tag{15.17}$$

this simplifies to

$$\zeta\left(\frac{1-\nu_1+\nu_2+i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1-\nu_2+i\lambda}{2}\right)\zeta\left(\frac{1-\nu_1-\nu_2-i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1+\nu_2-i\lambda}{2}\right)(i\lambda)\mathfrak{F}^\sharp_{-i\lambda}; \tag{15.18}$$

finally, we change λ to $-\lambda$ in the integral, finding (15.12) as a result.

There is no need to redo the computation in order to find

$$\begin{aligned} S_{\nu_1,\nu_2}\,(\mathfrak{F}^+_{k,\ell})^\sharp = \pi^{-\nu_1}\,&\frac{\Gamma(\frac{1+\nu_1-\nu_2-i\lambda_k^+}{4})\,\Gamma(\frac{1+\nu_1+\nu_2+i\lambda_k^+}{4})}{\Gamma(\frac{1-\nu_1+\nu_2+i\lambda_k^+}{4})\,\Gamma(\frac{1-\nu_1-\nu_2-i\lambda_k^+}{4})}\,(i\lambda_k^+)\,(\mathfrak{F}^+_{-k,\ell})^\sharp \\ &\times L\left(\frac{1+\nu_1+\nu_2}{2},\mathcal{N}^+_{|k|,\ell}\right)\,L\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^+_{|k|,\ell}\right) \end{aligned} \tag{15.19}$$

and

$$\begin{aligned} S_{\nu_1,\nu_2}\,(\mathfrak{F}^-_{k,\ell})^\sharp = \pi^{-\nu_1}\,&\frac{\Gamma(\frac{1+\nu_1-\nu_2-i\lambda_k^-}{4})\,\Gamma(\frac{1+\nu_1+\nu_2+i\lambda_k^-}{4})}{\Gamma(\frac{1-\nu_1+\nu_2+i\lambda_k^-}{4})\,\Gamma(\frac{1-\nu_1-\nu_2-i\lambda_k^-}{4})}\,(i\lambda_k^+)\,(\mathfrak{F}^-_{-k,\ell})^\sharp \\ &\times L\left(\frac{1+\nu_1+\nu_2}{2},\mathcal{N}^-_{|k|,\ell}\right)\,L\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^-_{|k|,\ell}\right)\,. \end{aligned} \tag{15.20}$$

Recalling Definition 15.4, we may write (since $\lambda^\pm_{-k}=-\lambda^\pm_k$)

$$S_{\nu_1,\nu_2}(\mathfrak{F}^+_{k,\ell})^\sharp=\Lambda\left(\frac{1+\nu_1+\nu_2}{2},(\mathfrak{N}^+_{-k,\ell})^\sharp\right)\Lambda\left(\frac{1+\nu_1-\nu_2}{2},(\mathfrak{N}^+_{k,\ell})^\sharp\right)(i\lambda_k^+)(\mathfrak{F}^+_{-k,\ell})^\sharp \tag{15.21}$$

and, in a similar way,

$$S_{\nu_1,\nu_2}(\mathfrak{F}^-_{k,\ell})^\sharp=\Lambda\left(\frac{1+\nu_1+\nu_2}{2},(\mathfrak{N}^-_{-k,\ell})^\sharp\right)\Lambda\left(\frac{1+\nu_1-\nu_2}{2},(\mathfrak{N}^-_{k,\ell})^\sharp\right)(i\lambda_k^-)(\mathfrak{F}^-_{-k,\ell})^\sharp. \tag{15.22}$$

This leads to (15.13) and (15.14) after we have changed k to $-k$ in the series. □

Lemma 15.6. *One has*

$$\begin{aligned} (u_z|\mathrm{Op}(\mathfrak{F}^\sharp_\nu)u_z) &= \zeta^*(\nu)\,E_{\frac{1-\nu}{2}}(z)\,, \\ (u_z^1|\mathrm{Op}(\mathfrak{F}^\sharp_\nu)u_z^1) &= -\nu\,\zeta^*(\nu)\,E_{\frac{1-\nu}{2}}(z)\,, \\ (u_z|\mathrm{Op}((\mathfrak{F}^\pm_{k,\ell})^\sharp)\,u_z) &= \zeta^*(i\lambda_k^\pm)\,\zeta^*(-i\lambda_k^\pm)\,\|\mathcal{N}^\pm_{|k|,\ell}\|^{-2}\,\mathcal{N}^\pm_{|k|,\ell}(z) \end{aligned}$$

and

$$(u_z^1|\mathrm{Op}((\mathfrak{F}^\pm_{k,\ell})^\sharp)\,u_z^1) = -i\lambda_k^\pm\,\zeta^*(i\lambda_k^\pm)\,\zeta^*(-i\lambda_k^\pm)\,\|\mathcal{N}^\pm_{|k|,\ell}\|^{-2}\,\mathcal{N}^\pm_{|k|,\ell}(z)\,. \tag{15.23}$$

Proof. The first two equations were given in (3.15) and (3.16), to be completed by (3.18) and (3.19). Next, recall from (2.21) that $\mathrm{Op}_{\sqrt{2}}(h) = \mathrm{Op}(2^{-\frac{1}{2}+i\pi\,\mathcal{E}}h)$ so that (4.12) reads

$$(u_z \,|\, \mathrm{Op}(2^{\frac{-1-i\lambda_k^{\pm}}{2}}\,(\mathfrak{N}_{k,\ell}^{\pm})^{\sharp})\,u_z) = \mathcal{N}_{|k|,\ell}^{\pm}(z)\,. \tag{15.24}$$

The third equation (15.23) is thus a consequence of the Definition (15.4) of $(\mathfrak{F}_{k,\ell}^{\pm})^{\sharp}$. According to (4.13), substituting u_z^1 for u_z on the left-hand side of this last equation just calls for the extra factor $-i\lambda_k^{\pm}$ on the right-hand side. □

Proof of Theorem 15.1. Denote as $\mathfrak{T}$ the image of the right-hand side under $2i\pi\,\mathcal{E}$, *i.e.*,

$$\begin{aligned}\mathfrak{T} = \sum_{\varepsilon=\pm 1} \Bigg[& (\varepsilon(\nu_1+\nu_2)-1)\,\frac{\zeta(\varepsilon\nu_1)\,\zeta(\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1+\nu_2)-1)}\,\mathfrak{F}^{\sharp}_{1-\varepsilon(\nu_1+\nu_2)}\\ & +(1-\varepsilon(\nu_1-\nu_2))\,\frac{\zeta(\varepsilon\nu_1)\,\zeta(-\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1-\nu_2)-1)}\,\mathfrak{F}^{\sharp}_{\varepsilon(\nu_1-\nu_2)-1}\Bigg]\\ & + \mathfrak{T}_{\mathrm{cont}} + \mathfrak{T}^{+}_{\mathrm{disc}} + \mathfrak{T}^{-}_{\mathrm{disc}}\,,\end{aligned} \tag{15.25}$$

where the last three terms have been defined in Lemma 15.5: we may denote as $\mathfrak{T}_{\mathrm{side}}$ the first term.

What we have to show is that $(u_z|\mathrm{Op}(\mathfrak{T})u_z)$ agrees with the function $f^1_{\nu_1,\nu_2}(z)$ introduced in (13.14), and that $(u_z^1|\mathrm{Op}(\mathfrak{T})u_z^1)$ agrees with the function $f^2_{\nu_1,\nu_2}(z)$ introduced in (13.15). First, the side terms: we use Lemma 15.6. Since (3.18)

$$\zeta^*(1-\varepsilon(\nu_1+\nu_2))\,E_{\frac{\varepsilon(\nu_1+\nu_2)}{2}} = \zeta^*(\varepsilon(\nu_1+\nu_2)-1)\,E_{1-\frac{\varepsilon(\nu_1+\nu_2)}{2}}\,, \tag{15.26}$$

one has

$$\begin{aligned}&(u_z|\mathrm{Op}(\mathfrak{T}_{\mathrm{side}})u_z)\\ &\quad= \sum_{\varepsilon=\pm 1}\Bigg[(\varepsilon(\nu_1+\nu_2)-1)\,\frac{\zeta(\varepsilon\nu_1)\,\zeta(\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1+\nu_2)-1)}\,\zeta^*(\varepsilon(\nu_1+\nu_2)-1)\,E_{1-\frac{\varepsilon(\nu_1+\nu_2)}{2}}(z)\\ &+(1-\varepsilon(\nu_1-\nu_2))\,\frac{\zeta(\varepsilon\nu_1)\,\zeta(-\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1-\nu_2)-1)}\,\zeta^*(\varepsilon(\nu_1-\nu_2)-1)\,E_{1-\frac{\varepsilon(\nu_1-\nu_2)}{2}}(z)\Bigg]\end{aligned} \tag{15.27}$$

or

$$\begin{aligned}&(u_z|\mathrm{Op}(\mathfrak{T}_{\mathrm{side}})u_z)\\ &\quad= 2\sum_{\varepsilon=\pm 1}\Bigg[\pi^{\frac{1-\varepsilon(\nu_1+\nu_2)}{2}}\,\Gamma\left(\frac{1+\varepsilon(\nu_1+\nu_2)}{2}\right)\,\zeta(\varepsilon\nu_1)\,\zeta(\varepsilon\nu_2)\,E_{1-\frac{\varepsilon(\nu_1+\nu_2)}{2}}(z)\\ &\quad-\pi^{\frac{1-\varepsilon(\nu_1-\nu_2)}{2}}\,\Gamma\left(\frac{1+\varepsilon(\nu_1-\nu_2)}{2}\right)\,\zeta(\varepsilon\nu_1)\,\zeta(-\varepsilon\nu_2)\,E_{1-\frac{\varepsilon(\nu_1-\nu_2)}{2}}(z)\Bigg]\,.\end{aligned} \tag{15.28}$$

This clearly identifies to the function $f^1_{\nu_1,\nu_2}(z) - g^1_{\nu_1,\nu_2}(z)$ as defined in (13.35): indeed, when $\varepsilon_1 = \varepsilon_2 = -\varepsilon$ in (13.35), one gets the first term on the right-hand side of (15.28); when $\varepsilon_1 = -\varepsilon_2 = -\varepsilon$, one gets the second one. Comparing the exceptional terms on the right-hand sides of (13.35) and (13.78), one sees that, in order to find the second ones in terms of the first, one has to insert the factor $-\frac{1+\varepsilon_1\nu_1+\varepsilon_2\nu_2}{\varepsilon_1\varepsilon_2}$, which is $\varepsilon(\nu_1+\nu_2)-1$ when $\varepsilon_1 = \varepsilon_2 = -\varepsilon$, $1-\varepsilon(\nu_1-\nu_2)$ when $\varepsilon_1 = -\varepsilon_2 = -\varepsilon$: these are precisely the extra factors to be inserted in the two terms on the right-hand side of (15.28) when changing u_z to u_z^1 on the left-hand side, as it follows from a comparison between the first two lines of (15.23). Thus $(u_z^1|\mathrm{Op}(\mathfrak{T}_{\mathrm{side}})u_z^1)$ agrees with $f^2_{\nu_1,\nu_2}(z) - g^2_{\nu_1,\nu_2}(z)$.

Next, the continuous part. Using (15.12) and the first formula (15.23) on one hand, (13.50) and (13.60) on the other hand, we must identify the integrals

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}(-i\lambda)\,\frac{\zeta^*(i\lambda)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,E_{\frac{1-i\lambda}{2}}(z)$$
$$\frac{\zeta\left(\frac{1+\nu_1+\nu_2+i\lambda}{2}\right)\,\zeta\left(\frac{1+\nu_1-\nu_2-i\lambda}{2}\right)\,\zeta\left(\frac{1-\nu_1+\nu_2-i\lambda}{2}\right)\,\zeta\left(\frac{1-\nu_1-\nu_2+i\lambda}{2}\right)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,d\lambda \tag{15.29}$$

and

$$\frac{1}{8\pi}\int_{-\infty}^{\infty} i\lambda\,\pi^{-\frac{i\lambda}{2}}\,\frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)}\times\sum_{\varepsilon=\pm1}\varepsilon\,E_{\frac{1-i\lambda}{2}}(z)$$
$$\zeta\left(\frac{1-i\varepsilon\lambda+\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda+\nu_1-\nu_2}{2}\right)\zeta\left(\frac{1+i\varepsilon\lambda-\nu_1+\nu_2}{2}\right)\zeta\left(\frac{1-i\varepsilon\lambda-\nu_1-\nu_2}{2}\right)d\lambda, \tag{15.30}$$

which is immediate since $\zeta^*(i\lambda)\,E_{\frac{1-i\lambda}{2}}(z) = \pi^{-\frac{i\lambda}{2}}\,\frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)}\,E_{\frac{1-i\lambda}{2}}(z)$ is an even function of λ. Substituting the study of $(u_z^1|\mathrm{Op}(\mathfrak{T}_{\mathrm{cont}})u_z^1)$ for that of $(u_z|\mathrm{Op}(\mathfrak{T}_{\mathrm{cont}})u_z)$, we must insert under the first of these two integrals the extra factor $-i\lambda$, thus changing $i\varepsilon\lambda$ to $-\lambda^2$ in the second: comparing (13.79) to (13.60), we are done.

Finally, the discrete parts: we must draw the reader's attention to the fact that, in Lemma 15.5, the index k can be positive or negative: but, in Theorems 14.2 and 14.3, one has $k \geq 1$ since we are dealing with the Roelcke-Selberg decomposition on Π, not with decompositions into homogeneous parts on $\mathbb{R}^2$. From (15.13) and Lemma 15.6, immediately reducing the domain of k in (15.13) to $\{1,2,\dots\}$, one gets

$$(u_z|\mathrm{Op}(\mathfrak{T}^+_{\mathrm{disc}})u_z) = \frac{1}{2}\sum_{\substack{k,\ell\\ \ell\geq1}}\sum_{\varepsilon=\pm1}(-i\varepsilon\lambda_k^+)\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\Gamma\left(-\frac{i\lambda_k^+}{2}\right)$$
$$\times\Lambda\left(\frac{1+\nu_1+\nu_2}{2},(\mathfrak{N}^+_{\varepsilon k,\ell})^\sharp\right)\Lambda\left(\frac{1+\nu_1-\nu_2}{2},(\mathfrak{N}^+_{-\varepsilon k,\ell})^\sharp\right)\|\mathcal{N}^+_{|k|,\ell}\|^{-2}\mathcal{N}^+_{|k|,\ell}(z)\,. \tag{15.31}$$

Since, from (15.10), the product of the two Λ-functions on the last line is also

$$\frac{\pi\, L^*\left(\frac{1+\nu_1+\nu_2}{2}, \mathcal{N}^+_{|k|,\ell}\right)\, L^*\left(\frac{1+\nu_1-\nu_2}{2}, \mathcal{N}^+_{|k|,\ell}\right)}{\Gamma\left(\frac{1-\nu_1-\nu_2+i\varepsilon\lambda_k^+}{4}\right)\Gamma\left(\frac{1+\nu_1+\nu_2+i\varepsilon\lambda_k^+}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2-i\varepsilon\lambda_k^+}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-i\varepsilon\lambda_k^+}{4}\right)}, \tag{15.32}$$

the right-hand side of (15.31) is just the same as that of (14.38). Exactly the same computation works for the $\mathfrak{T}^-_{\text{disc}}$ part: only

$$L^*\left(\frac{1+\nu_1+\nu_2}{2}, \mathcal{N}^-_{|k|,\ell}\right) = -L^*\left(\frac{1-\nu_1-\nu_2}{2}, \mathcal{N}^-_{|k|,\ell}\right):$$

finally, from Lemma 15.6, substituting u_z^1 for u_z again calls for inserting the extra factor $-i\lambda_k^\pm$ ($k \in \mathbb{Z}^\times$), *i.e.*, $-i\varepsilon\lambda_k^\pm$ after we have reduced k to the set $\{1, 2, \dots\}$, and a comparison between Theorems 14.2 and 14.3 finishes the proof of Theorem 15.1. □

Remarks.

1. It is instructive to check the coherent way under which the data and results of the formula

$$\begin{aligned}\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2} &= \sum_{\varepsilon=\pm 1}\left[\frac{\zeta(\varepsilon\nu_1)\,\zeta(\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1+\nu_2)-1)}\,\mathfrak{F}^\sharp_{1-\varepsilon(\nu_1+\nu_2)} + \frac{\zeta(\varepsilon\nu_1)\,\zeta(-\varepsilon\nu_2)}{\zeta(\varepsilon(\nu_1-\nu_2)-1)}\,\mathfrak{F}^\sharp_{\varepsilon(\nu_1-\nu_2)-1}\right]\\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathfrak{F}^\sharp_{i\lambda}\,\frac{\zeta(\frac{1+\nu_1+\nu_2+i\lambda}{2})\,\zeta(\frac{1+\nu_1-\nu_2-i\lambda}{2})\,\zeta(\frac{1-\nu_1+\nu_2-i\lambda}{2})\,\zeta(\frac{1-\nu_1-\nu_2+i\lambda}{2})}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,d\lambda\\ &+ \frac{1}{2}\sum_{k,\ell,\pm}(\mathfrak{F}^\pm_{k,\ell})^\sharp \times \frac{\Lambda(\frac{1+\nu_1+\nu_2}{2}, (\mathfrak{N}^\pm_{k,\ell})^\sharp)\,\Lambda(\frac{1+\nu_1-\nu_2}{2}, (\mathfrak{N}^\pm_{-k,\ell})^\sharp)}{\zeta(i\lambda_k^\pm)\,\zeta(-i\lambda_k^\pm)}\end{aligned} \tag{15.33}$$

transform under the symmetries $\mathcal{G}$ and T^{dist}_{-1} discussed in the remark following (5.3), as well as under the complex conjugation. From the functional equation $\text{Op}(\mathcal{G}\mathfrak{S}) = \text{Op}(\mathfrak{S})\,C = C\,\text{Op}(\mathfrak{S})$, with $Cu = \check{u}$, valid for every automorphic distribution $\mathfrak{S}$, and from Lemma 15.2, one expects the formula $\mathfrak{F}^\sharp_{-\nu_1} \# \mathfrak{F}^\sharp_{-\nu_2} = \mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2}$. Using the equation $\Lambda(1-s, (\mathfrak{N}^\pm_{k,\ell})^\sharp) = \pm\Lambda(s, (\mathfrak{N}^\pm_{k,\ell})^\sharp)$, one may check that it is indeed satisfied. The equation $\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{-\nu_2} = \mathcal{G}(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})$ may be verified in the same way, though it is slightly more fun to check. From the equation $\text{Op}(T^{\text{dist}}_{-1}\mathfrak{S})\,u = \overline{\text{Op}(\mathfrak{S})^*\bar{u}}$ – note that this is also $\text{Op}(\mathfrak{S})'u$ where $\text{Op}(\mathfrak{S})'$ is the transpose of $\text{Op}(\mathfrak{S})$ – together with $T^{\text{dist}}_{-1}\mathfrak{F}^\sharp_\nu = \mathfrak{F}^\sharp_\nu$, we see that we must expect that $\mathfrak{F}^\sharp_{\nu_2} \# \mathfrak{F}^\sharp_{\nu_1} = T^{\text{dist}}_{-1}(\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2})$, which can indeed be asserted from (15.33) and the equation $T^{\text{dist}}_{-1}\,(\mathfrak{F}^\pm_{k,\ell})^\sharp = \pm(\mathfrak{F}^\pm_{k,\ell})^\sharp$. Finally, since taking the complex conjugate of some symbol amounts to taking the adjoint of the associated operator, one should expect the relation $\overline{\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2}} = \mathfrak{F}^\sharp_{\bar\nu_2} \# \mathfrak{F}^\sharp_{\bar\nu_1}$. To verify this, one may successively check that $\overline{\mathcal{N}^\pm_{|k|,\ell}} = \pm \mathcal{N}^\pm_{|k|,\ell}$ (from (4.3)), $\overline{(\mathfrak{N}^\pm_{k,\ell})^\sharp} = \pm(\mathfrak{N}^\pm_{-k,\ell})^\sharp$ (from (4.12)–(4.13)), $\overline{(\mathfrak{F}^\pm_{k,\ell})^\sharp} = \pm(\mathfrak{F}^\pm_{-k,\ell})^\sharp$ (from (15.4)), finally use the already quoted relation $\Lambda(1-s, (\mathfrak{N}^\pm_{k,\ell})^\sharp) = \pm \Lambda(s, (\mathfrak{N}^\pm_{k,\ell})^\sharp)$.

2. The first and second lines of the right-hand side of (15.33) are related. Indeed, if one introduces the function

$$F(\mu) = \frac{1}{4\pi}\, \mathfrak{F}^\sharp_{i\mu}\, \frac{\zeta\left(\frac{1+\nu_1+\nu_2+i\mu}{2}\right) \zeta\left(\frac{1+\nu_1-\nu_2-i\mu}{2}\right) \zeta\left(\frac{1-\nu_1+\nu_2-i\mu}{2}\right) \zeta\left(\frac{1-\nu_1-\nu_2+i\mu}{2}\right)}{\zeta(i\mu)\,\zeta(-i\mu)} \tag{15.34}$$

which appears, when $\mu = \lambda \in \mathbb{R}$, as the integrand in the second line, the first one is exactly $2i\pi$ times the sum of the residues of F at the poles of the product of four zeta functions upstairs; the other poles of F, which originate from the zeros of $\zeta(i\mu)\,\zeta(-i\mu)$, do not contribute to (15.33).

The reason for all this is the following: Theorem 15.1 extends to more general values of (ν_1, ν_2), provided only that Re $(\nu_1 \pm \nu_2) \neq \pm 1$. But the number of exceptional terms (the ones on the first line of the right-hand side of (15.33)) is in general four minus the smallest number of times one must cross the union of the four lines Re $(\nu_1 \pm \nu_2) = \pm 1$ to link the actual value of (Re ν_1, Re ν_2) to $(0,0)$. Each exceptional term on the right-hand side of (15.33) is connected not to a change of contour in the $d\lambda$-integral, but to a discontinuity of the $d\lambda$-integral, always taken on $\mathbb{R}$, whenever (Re ν_1, Re ν_2) crosses any of the four lines Re $(\nu_1 \pm \nu_2) = \pm 1$; something similar occurs in [62, Proposition 14.2].

16 Towards the completion of the multiplication table

In this section, written for the sake of (almost) completeness, we compute, with the same meaning as in Theorem 15.1, a sharp product such as $\mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^\pm_{r,\ell})^\sharp$ instead of $\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2}$. An identity

$$\mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^+_{r,\ell})^\sharp = \mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right) . 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\, \mathfrak{B}\,, \tag{16.1}$$

similar to (15.3), will be written: constructing the operator $\mathcal{R}'(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp)$ taking the place formerly taken by the operator $\mathcal{L}'(\frac{1+\nu_1+\nu_2}{2})\, \mathcal{G}\, \mathcal{L}'(\frac{1+\nu_1-\nu_2}{2})$ will depend on "convolution L-functions", a classical concept recalled in (16.19).

At the end of the section, using Eulerian products, we shall show that – with one possible, but unlikely, exception – one can give a unified expression for all coefficients involved in the decomposition into homogeneous components of sharp products of "elementary" automorphic distributions.

Before embarking on the rather lengthy, if straightforward, computations, there is one point which we wish to stress. One rather interesting feature of the main formula (15.3) for $\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2}$ was that it has been possible to factorize, as $\mathcal{L}'(\frac{1+\nu_1+\nu_2}{2})\, \mathcal{G}\, \mathcal{L}'(\frac{1+\nu_1-\nu_2}{2})$, the operator transforming $2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B}$ into the main part of $\mathfrak{F}^\sharp_{\nu_1} \# \mathfrak{F}^\sharp_{\nu_2}$: this decomposition was explained, in a heuristic way, in Section 5.

No such decomposition is possible in our present investigations regarding $\mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^\pm_{r,\ell})^\sharp$. The reason is that, for Eisenstein distributions $\mathfrak{E}^\sharp_\nu$ only, the Hecke polynomial (corresponding to $\frac{1}{2}\, E^*_{\frac{1-\nu}{2}}$, a normalization chosen so that the first Fourier coefficient should be 1)

$$1 - \frac{\sigma_\nu(p)}{p^{\frac{\nu}{2}}}\, X + X^2 = (1 - p^{-\frac{\nu}{2}}\, X)\,(1 - p^{\frac{\nu}{2}}\, X) \tag{16.2}$$

(where p is prime) factors as a product of two meaningful polynomials: for giving $\mathfrak{E}^\sharp_\nu$, rather than $E^*_{\frac{1-\nu}{2}}$ only, singles out the degree of homogeneity $-1-\nu$ from the pair $(-1-\nu,\, -1+\nu)$. We do not know how to split into the product of two distinguishable factors the Hecke polynomials relative to cusp-distributions, a rather deep question which will have to be raised again at the very end of this section.

Coming back to more down-to-earth developments, recall from (4.3), (4.4) that if $\mathcal{M}_{|r|}$ is a cusp-form with a Fourier series expansion

$$\mathcal{M}_{|r|}(x+iy) = y^{\frac{1}{2}} \sum_{n\neq 0} b_n\, K_{\frac{i\lambda_r}{2}}(2\pi\,|n|y)\, e^{2i\pi n x}\,, \tag{16.3}$$

the associated cusp-distribution $\mathfrak{M}^\sharp_r$ homogeneous of degree $-1-i\lambda_r$ (where λ_r can be positive or negative) is given by

$$\langle \mathfrak{M}^\sharp_r, h\rangle = \frac{1}{2} \sum_{n\neq 0} |n|^{\frac{i\lambda_r}{2}}\, b_n \int_{-\infty}^{\infty} |t|^{-i\lambda_r-1}\, (\mathcal{F}_1^{-1} h)\left(\frac{n}{t}, t\right) dt \tag{16.4}$$

or, in view of (10.1) and (10.3), together with a Γ-invariance argument,

$$\mathfrak{M}^\sharp_r = \frac{1}{2} \sum_{n\neq 0} b_n\, \mathfrak{a}_n^{i\lambda_r}\,, \tag{16.5}$$

where $\mathfrak{a}_n^{i\lambda_r}$ has been defined in (10.4).

Recall from (13.9) that the "non-constant" part of $\mathfrak{F}^\sharp_\nu$, *i.e.*, the sum of all terms from the right-hand side of (13.9) with the exception of the first two terms (the analogues of which are absent from the expansion of cusp-distributions) is given by the same expansion, with ν substituted for $i\lambda_r$ and

$$b_n = 2^{\frac{1-\nu}{2}}\, \frac{\sigma_\nu(|n|)}{|n|^{\frac{\nu}{2}}}\,. \tag{16.6}$$

In order to complete our multiplication table (with respect to the sharp product of automorphic distributions), thus arriving at a genuine symbolic calculus of (pseudodifferential) operators with automorphic symbols, what would remain to be done is a set of explicit formulas for the sharp product of an Eisenstein distribution and a cusp-distribution (in any order), or that of two cusp-distributions. We shall satisfy ourselves with the first of these two problems: also, let us remark that it was precisely with this purpose in mind that we chose the method of proof of our main theorem which was developed in the Sections 13–15, rather than that suggested by the heuristic considerations of Section 5.

Indeed, there is very little to modify in our preceding computations, besides changing the coefficients b_n. As a matter of fact, the new situation is easier, since there are no "constant" terms to worry about, and we do not need to substract any linear combination of Eisenstein series from the analogues of $f^1_{\nu_1,\nu_2}$ and $f^2_{\nu_1,\nu_2}$ so as to get a result in the space $L^2(\Gamma\backslash\Pi)$; moreover, in order to find the integral term of the decomposition, one may appeal to the usual Rankin-Selberg method rather than the more elaborate version between (13.51) and (13.59).

Consider the case of a sharp product $\mathfrak{S} = \mathfrak{F}^\sharp_\nu \# \mathfrak{M}^\sharp_r$, with $\mathfrak{F}^\sharp_\nu$ as is usual and $\mathfrak{M}^\sharp_r$ given by (16.4): we assume that $\mathfrak{M}^\sharp_r$ is associated with an even or odd cusp-form, *i.e.*, that the coefficients b_n are even or odd functions of n. It should be understood right away that the meaning to be ascribed to this is only indirect, just as in Proposition 13.2: that is, we define only $2i\pi\,\mathcal{E}\,\mathfrak{S}$ in the minimal sense, denoting as $f^1_{\nu,\mathfrak{M}^\sharp_r}$ and $f^2_{\nu,\mathfrak{M}^\sharp_r}$ the two functions on the right-hand sides of (13.1) and (13.2).

Following the same proof as that of Theorem 13.6, and using (16.6), we must substitute for the series $2^{\frac{2-\nu_1-\nu_2}{2}} \sum_{n\neq 0} \frac{\sigma_{\nu_1}(|n|)\,\sigma_{\nu_2}(|n|)}{|n|^{\nu_1+\nu_2}}$ which occurs on the right-hand side of (13.63) the series

$$2^{\frac{1-\nu}{2}} \sum_{n\neq 0} \frac{\sigma_\nu(|n|)\,b_n}{|n|^{\nu+\frac{i\lambda_r}{2}}}\,, \tag{16.7}$$

besides substituting ν (*resp.* $i\lambda_r$) for ν_1 (*resp.* ν_2) everywhere. In the case when $\mathcal{M}_{|r|}$ is an even cusp-form, say $\mathcal{M}_{|r|} = \mathcal{N}^+_{|r|,\ell}$, we must thus substitute for the series $\sum_{n\geq 1} n^{\frac{-1-\nu_1-\nu_2+i\mu}{2}}\,\sigma_{\nu_1}(n)\,\sigma_{\nu_2}(n)$ on the right-hand side of (13.72) the expression

$$2^{\frac{-1+i\lambda_r}{2}} \sum_{n\geq 1} n^{\frac{-1-\nu+i\mu}{2}}\,\sigma_\nu(n)\,b_n\,, \tag{16.8}$$

which coincides when $\mathrm{Im}\,\mu$ is large [62, (14.37)] with

$$2^{\frac{-1+i\lambda_r}{2}}\,(\zeta(1-i\mu))^{-1}\,L\left(\frac{1-i\mu+\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right)\,L\left(\frac{1-i\mu-\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right)\,. \tag{16.9}$$

In the case of the odd cusp-form $\mathcal{M} = \mathcal{N}^-_{r,\ell}$, the calculation is just the same, except for the fact that the difference, as computed between (14.17) and (14.19), of the two integrals in (13.64) and (13.65), rather than their sum as computed between (13.68) and (13.71), should occur: comparing the results of the two calculations, we see that the odd case differs from the even case only in that each of four Gamma factors on the right-hand side of (13.72) has to be computed at the point $\frac{3\pm i\mu\pm(\nu_1+\varepsilon\nu_2)}{4}$ rather than $\frac{1\pm i\mu\pm(\nu_1+\varepsilon\nu_2)}{4}$.

Proposition 16.1. *Let ν be a complex number with $|\mathrm{Re}\,\nu|<1$, and consider the cusp-distribution $(\mathfrak{F}^+_{r,\ell})^\sharp$ as defined in* (15.4)*: r can be either positive or negative, in other words so can be the sign of λ_r. In analogy with* (13.14) *and* (13.15)*, set*

$$f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z)=\int_{-\infty}^{\infty}\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_\nu)\,u^1_{-\frac{1}{z}})(t)}\,(\mathrm{Op}((\mathfrak{F}^+_{r,\ell})^\sharp)\,u^1_{-\frac{1}{z}})(t)\,dt \tag{16.10}$$

and

$$f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z)=\int_{-\infty}^{\infty}\Big[3\,\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_\nu)\,u^2_{-\frac{1}{z}})(t)}\,(\mathrm{Op}((\mathfrak{F}^+_{r,\ell})^\sharp)\,u^2_{-\frac{1}{z}})(t)$$
$$-3^{\frac{1}{2}}\sum_{p=0,2}\overline{(\mathrm{Op}(\mathfrak{F}^\sharp_\nu)\,u^p_{-\frac{1}{z}})(t)}\,(\mathrm{Op}((\mathfrak{F}^+_{r,\ell})^\sharp)\,u^{2-p}_{-\frac{1}{z}})(t)\Big]\,dt\ : \tag{16.11}$$

these two functions lie in $L^2(\Gamma\backslash\Pi)$.

The spectral density Φ_1 in the Roelcke-Selberg decomposition (4.1) *of the first of these two functions is given by the equation*

$$\Phi_1(\lambda)=i\lambda\,\pi^{-\frac{i\lambda}{2}}\,\frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)}\,\frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda^+_r)\,\zeta^*(-i\lambda^+_r)}\times\sum_{\varepsilon=\pm1}\varepsilon$$
$$\Lambda\left(\frac{1+i\lambda+\nu}{2},\,(\mathfrak{F}^+_{-\varepsilon r,\ell})^\sharp\right)\Lambda\left(\frac{1+i\lambda-\nu}{2},\,(\mathfrak{F}^+_{\varepsilon r,\ell})^\sharp\right) \tag{16.12}$$

(recall that the functions $\Lambda(s,(\mathfrak{M}^+_r)^\sharp)$ have been defined in (15.8)*), and that associated with the function $f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$ is given by the formula*

$$\Phi_2(\lambda)=-\lambda^2\,\pi^{-\frac{i\lambda}{2}}\,\frac{\Gamma(\frac{i\lambda}{2})}{\zeta(-i\lambda)}\,\frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda^+_r)\,\zeta^*(-i\lambda^+_r)}\times\sum_{\varepsilon=\pm1}$$
$$\Lambda\left(\frac{1+i\lambda+\nu}{2},\,(\mathfrak{F}^+_{-\varepsilon r,\ell})^\sharp\right)\Lambda\left(\frac{1+i\lambda-\nu}{2},\,(\mathfrak{F}^+_{\varepsilon r,\ell})^\sharp\right). \tag{16.13}$$

In the case when attention is paid to an odd cusp-distribution $(\mathfrak{F}^-_{r,\ell})^\sharp$, the spectral density associated with the function $f^1_{\nu,(\mathfrak{F}^-_{r,\ell})^\sharp}$ or $f^2_{\nu,(\mathfrak{F}^-_{r,\ell})^\sharp}$ is given by the same pair of formulas, only substituting λ^-_r for λ^+_r.

Proof. Start with the function $f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$. Following the proof of Theorem 13.6 and using also (16.9) together with (15.4) (which provides the extra factor from $(\mathfrak{N}^+_{r,\ell})^\sharp$ to $(\mathfrak{F}^+_{r,\ell})^\sharp$), one finds that $\Phi_1(\lambda) = \phi_1(-\lambda)$, where $\phi_1(\mu)$ is defined as

$$\phi_1(\mu) = \pi^{\frac{1+i\mu}{2}} \frac{\Gamma(\frac{i\mu}{2})\,\Gamma(\frac{2-i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})} \sum_{\varepsilon=\pm1} \varepsilon\, \frac{\Gamma(\frac{1-i\mu-\nu-\varepsilon i\lambda_r^+}{4})\,\Gamma(\frac{1-i\mu+\nu+\varepsilon i\lambda_r^+}{4})}{\Gamma(\frac{1+i\mu-\nu-\varepsilon i\lambda_r^+}{4})\,\Gamma(\frac{1+i\mu+\nu+\varepsilon i\lambda_r^+}{4})}$$
$$\frac{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)}{\zeta(1-i\mu)}\, \|\mathcal{N}^+_{|r|,\ell}\|^{-2}\, L\left(\frac{1-i\mu+\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right) L\left(\frac{1-i\mu-\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right). \tag{16.14}$$

The definition (15.8) of the functions $\Lambda(s, (\mathfrak{M}_r^+)^\sharp)$ yields the identity

$$\pi^{i\mu} \frac{\Gamma(\frac{1-i\mu-\nu-\varepsilon i\lambda_r^+}{4})\Gamma(\frac{1-i\mu+\nu+\varepsilon i\lambda_r^+}{4})}{\Gamma(\frac{1+i\mu-\nu-\varepsilon i\lambda_r^+}{4})\Gamma(\frac{1+i\mu+\nu+\varepsilon i\lambda_r^+}{4})} \times L\left(\frac{1-i\mu+\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right) L\left(\frac{1-i\mu-\nu}{2}, \mathcal{N}^+_{|r|,\ell}\right)$$
$$= \Lambda\left(\frac{1-i\mu+\nu}{2}, (\mathfrak{N}^+_{-\varepsilon r,\ell})^\sharp\right) \Lambda\left(\frac{1-i\mu-\nu}{2}, (\mathfrak{N}^+_{\varepsilon r,\ell})^\sharp\right), \tag{16.15}$$

from which one easily gets the first of the four equations stated. When changing $f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$ to $f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$, all that has to be done (following the proof of Theorem 13.7)) is to suppress the sign ε right after the related summation sign, and multiply the net result by $i\lambda$.

When substituting an odd cusp-form $(\mathfrak{F}^-_{r,\ell})^\sharp$ for $(\mathfrak{F}^+_{r,\ell})^\sharp$, one sees (from what has been said immediately after (16.9)) that each of the four Gamma factors $\Gamma(\frac{1\pm i\mu\pm(\nu+i\varepsilon\lambda_r^+)}{4})$ has to be replaced by $\Gamma(\frac{3\pm i\mu\pm(\nu+i\varepsilon\lambda_r^-)}{4})$: but this difference disappears from the final formula, since it is taken care of by the extra factors (in (15.8) and (15.9)) from $L(s, \mathcal{M}^\pm)$ to $\Lambda(s, (\mathfrak{M}_r^\pm)^\sharp)$. □

Remarks.

1. For one's peace of mind, one should check that each of the two functions $\lambda \mapsto \zeta^*(-i\lambda)\,\Phi_j(\lambda)$ is even. In the first case for instance, using the functional equation of the function $s \mapsto \Lambda(s, (\mathfrak{F}^+_{\varepsilon r,\ell})^\sharp)$, one may write

$$\zeta^*(-i\lambda)\,\Phi_1(\lambda) = i\lambda\, \frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)} \times \sum_{\varepsilon=\pm1} \varepsilon$$
$$\Lambda\left(\frac{1+i\lambda+\nu}{2}, (\mathfrak{F}^+_{-\varepsilon r,\ell})^\sharp\right) \Lambda\left(\frac{1-i\lambda+\nu}{2}, (\mathfrak{F}^+_{\varepsilon r,\ell})^\sharp\right), \tag{16.16}$$

from which this is immediate.

2. The factor $\frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)}$ can also be written as $\|(\mathfrak{F}^+_{\pm r,\ell})^\sharp\|_\Gamma^{-2}$, where $\|\ \|_\Gamma$ denotes the norm in the space $L^2(\Gamma\backslash\mathbb{R}^2)$ introduced in Theorem 4.1: this is a consequence of (4.7) and (4.4).

We prepare for the analysis of the discrete parts from the Roelcke-Selberg decompositions of $f^1_{\nu,(\mathfrak{F}^{\pm}_{r,\ell})^\sharp}$ and $f^2_{\nu,(\mathfrak{F}^{\pm}_{r,\ell})^\sharp}$ by that relative to the pointwise product and Poisson bracket of the two functions $E^*_{\frac{1-\nu}{2}}$ and $\mathcal{N}^{\pm}_{|r|,\ell}$. First, note that

$$\int_{\Gamma\backslash\Pi} E_{\frac{1-\nu}{2}} \{\mathcal{M}_1, \overline{\mathcal{M}_2}\}\, d\mu = \int_{\Gamma\backslash\Pi} \{E_{\frac{1-\nu}{2}}, \mathcal{M}_1\}\, \overline{\mathcal{M}_2}\, d\mu \tag{16.17}$$

for any two cusp-forms $\mathcal{M}_1$ and $\mathcal{M}_2$. Indeed, we may assume that each of the two forms has a given parity (relative to its transformation under the map $z \mapsto -\bar{z}$). Next, the Poisson bracket of two functions with the same (*resp.* the opposite) parity is odd (*resp.* even): it follows that, if $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same parity, the integrands on the two sides of (16.17) are always odd functions, so that both integrals are zero. We thus may assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ have opposite parities, and write the difference of the two sides of (16.17) as $I = \int_{\Gamma\backslash\Pi}\{\mathcal{M}_1,\, E_{\frac{1-\nu}{2}}\, \overline{\mathcal{M}_2}\}\, d\mu$. One of the two functions $\mathcal{M}_1$ and $E_{\frac{1-\nu}{2}}\, \overline{\mathcal{M}_2}$ must vanish on the arc $|z| = 1$, $-\frac{1}{2} < x < \frac{1}{2}$ since it is odd as well as invariant under the map $z \mapsto -\frac{1}{z}$: this, and the periodicity, permits us to evaluate the integral I by parts (going back to the definition (11.3) of Poisson brackets) without any boundary term, thus proving (16.17).

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two cusp-forms with the Fourier expansions

$$\mathcal{M}_1(z) = y^{\frac{1}{2}} \sum_{n\neq 0} b_n\, K_{\frac{i\lambda_1}{2}}(2\pi|n|y)\, e^{2i\pi nx}$$

and

$$\mathcal{M}_2(z) = y^{\frac{1}{2}} \sum_{n\neq 0} c_n\, K_{\frac{i\lambda_2}{2}}(2\pi|n|y)\, e^{2i\pi nx}\,, \tag{16.18}$$

it is customary to define the "convolution L-function" associated with these two functions [8, p. 73] or [26, p. 231] (the references just quoted emphasize, rather, the case of two holomorphic cusp-forms) by the formula, valid for $\mathrm{Re}\, s$ large enough, but the function obtained extends as a meromorphic function in the entire plane,

$$L(s, \mathcal{M}_1 \times \mathcal{M}_2) = \zeta(2s) \sum_{n\geq 1} b_n c_n\, n^{-s}\,. \tag{16.19}$$

From the references just given, in particular Lemma 1.6.1 in the first one, one sees that, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are Hecke forms, so that their L-functions have Eulerian products:

$$\begin{aligned} L(s, \mathcal{M}_1) &= \prod_p (1 - b_p p^{-s} + p^{-2s})^{-1}\,, \\ L(s, \mathcal{M}_2) &= \prod_p (1 - c_p p^{-s} + p^{-2s})^{-1}\,, \end{aligned} \tag{16.20}$$

then the convolution L-function also has such a product, given as

$$L(s, \mathcal{M}_1 \times \mathcal{M}_2) = \prod_p \prod_{\substack{\varepsilon_1=\pm 1 \\ \varepsilon_2=\pm 1}} (1 - \beta_p^{\varepsilon_1} \gamma_p^{\varepsilon_2} p^{-s})^{-1}, \tag{16.21}$$

where the coefficients β_p and γ_p are obtained from the Hecke polynomials

$$\begin{aligned} 1 - b_p X + X^2 &= (1 - \beta_p X)(1 - \beta_p^{-1} X), \\ 1 - c_p X + X^2 &= (1 - \gamma_p X)(1 - \gamma_p^{-1} X). \end{aligned} \tag{16.22}$$

The standard Rankin-Selberg method [71, p. 268] makes it possible to prove the following lemma, the first half of which could also be derived from Moreno's computation [36] of the spectral decomposition of the pointwise product of an Eisenstein series by a (Maass) cusp-form.

Lemma 16.2. *Assume* $|\mathrm{Re}\ \nu| < 1$. *Then, for any two even Maass-Hecke cusp-forms* $\mathcal{N}^{\pm}_{k_1,\ell_1}$ *and* $\mathcal{N}^{\pm}_{k_2,\ell_2}$, *of the same parity, one has (with* $\mathcal{N}_{k_j,\ell_j} = \mathcal{N}^{\pm}_{k_j,\ell_j}$ *and* $\lambda_{k_j} = \lambda^{\pm}_{k_j}$*)*

$$\begin{aligned} &\int_{\Gamma\backslash\Pi} E_{\frac{1-\nu}{2}}(z)\, \mathcal{N}_{k_1,\ell_1}(z)\, \overline{\mathcal{N}}_{k_2,\ell_2}(z)\, d\mu(z) = \frac{1}{4} \frac{\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{1-\nu}{2})} \\ &\Gamma\left(\tfrac{1-\nu+i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu+i(\lambda_{k_1}-\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu+i(-\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu-i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \\ &\qquad \times (\zeta(1-\nu))^{-1}\, L\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right). \end{aligned} \tag{16.23}$$

In the case of two Maass-Hecke cusp-forms of different parity, one has the formula

$$\begin{aligned} &\frac{1}{2} \int_{\Gamma\backslash\Pi} E_{\frac{1-\nu}{2}}(z)\, \{\mathcal{N}_{k_1,\ell_1}, \overline{\mathcal{N}}_{k_2,\ell_2}\}(z)\, d\mu(z) = \frac{1}{4i} \frac{\pi^{\frac{\nu-1}{2}}}{\Gamma(\frac{1-\nu}{2})} \\ &\Gamma\left(\tfrac{3-\nu+i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu+i(\lambda_{k_1}-\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu+i(-\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu-i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \\ &\qquad \times (\zeta(1-\nu))^{-1}\, L\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right). \end{aligned} \tag{16.24}$$

Proof. With $\mathcal{M}_1 = \mathcal{N}^{\pm}_{k_1,\ell_1}$ and $\mathcal{M}_2 = \mathcal{N}^{\pm}_{k_2,\ell_2}$, as in (16.18), the "constant" (*i.e.*, independent of x) term in the Fourier expansion of the product $\mathcal{M}_1 \overline{\mathcal{M}_2}$ is (since the Fourier coefficients of the Hecke form $\mathcal{M}_2$ are real numbers)

$$a_0(y) = y \sum_{n \neq 0} b_n\, c_n\, K_{\frac{i\lambda_1}{2}}(2\pi|n|y)\, K_{\frac{i\lambda_2}{2}}(2\pi|n|y). \tag{16.25}$$

Thus, the Rankin-Selberg method (*loc.cit.*) or the extension described between (13.51) and (13.59) permits to find the spectral density Φ of the Roelcke-Selberg decomposition (13.50) of $\mathcal{M}_1 \overline{\mathcal{M}_2}$ from the formula

$$\begin{aligned}\Phi(-\lambda) &= (E_{\frac{1+i\lambda}{2}} \,|\, \mathcal{M}_1 \overline{\mathcal{M}_2}) \\ &= \int_{\Gamma\backslash\Pi} E_{\frac{1-i\lambda}{2}}(z)\, \mathcal{M}_1(z)\, \overline{\mathcal{M}_2}(z)\, d\mu(z) \\ &= \int_0^\infty a_0(y)\, y^{\frac{-3-i\lambda}{2}}\, dy\,, \end{aligned} \tag{16.26}$$

where the last integral is meant as the value at $\mu = \lambda$ obtained by analytic continuation from its value for $\mathrm{Im}\,\mu$ large. From the Weber-Schafheitlin integral [31, p. 101], already used in (14.27), we get, if $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same parity,

$$\Phi(-\lambda) = \frac{1}{4} \frac{\pi^{\frac{i\lambda-1}{2}}}{\Gamma(\frac{1-i\lambda}{2})} \sum_{n\geq 1} b_n\, c_n\, n^{\frac{i\lambda-1}{2}} \times$$
$$\Gamma\left(\tfrac{1-i\lambda+i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-i\lambda+i(\lambda_{k_1}-\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-i\lambda+i(-\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-i\lambda-i(\lambda_{k_1}+\lambda_{k_2})}{4}\right), \tag{16.27}$$

where, again, the sum of the Dirichlet series really means the value at $\mu = \lambda$ "from above": (16.23) follows, using (16.19) and analytic continuation again.

When integrating the Poisson bracket of two Hecke cusp-forms against an Eisenstein series, only the case when the two cusp-forms have a distinct parity can yield a non-zero result. With $\mathcal{M}_1 = \mathcal{N}^{\pm}_{k_1,\ell_1}$ and $\mathcal{M}_2 = \mathcal{N}^{\mp}_{k_2,\ell_2}$, the constant coefficient of interest this time is

$$a_0(y) = \sum_{n\neq 0} b_n\, c_n \left\{ y^{\frac{1}{2}}\, K_{\frac{i\lambda_1}{2}}(2\pi|n|y)\, e^{2i\pi nx}\,,\, y^{\frac{1}{2}}\, K_{\frac{i\lambda_2}{2}}(2\pi|n|y)\, e^{-2i\pi nx} \right\}. \tag{16.28}$$

This can be written as

$$a_0(y) = 2i\pi \sum_{n\neq 0} b_n\, c_n\, c^{n,-n}_{i\lambda_1,i\lambda_2}(y)\,, \tag{16.29}$$

where the last factor has the same signification as the particular case of (14.31): thus

$$\int_0^\infty a_0(y)\, y^{\frac{-3-i\lambda}{2}}\, dy = 2i\pi \sum_{n\neq 0} b_n\, c_n\, I^{n,-n}_{i\lambda_1,i\lambda_2}\,, \tag{16.30}$$

where the last factor has the same signification as in (14.32).

Fortunately, there is no need to redo the quite heavy calculations between (14.28) and (14.34). We must of course change the coefficient $2\, \frac{\sigma_{\nu_1}(|n|)}{|n|^{\frac{\nu_1}{2}}}$ from the Fourier expansion (4.5) of the Eisenstein series $E^*_{\frac{1-\nu_1}{2}}(z)$ to b_n, and do something similar with the second factor which occurred in our previous computations. Next, looking at (14.28), it is immediate that the integer m which occurred there has to be replaced by 0 (we are now dealing with the "constant" part from a Fourier series expansion), so that the hypergeometric function reduces to 1 exactly, and not only approximately; finally, the equation (14.34) becomes exact for the same reason (in Section 13, we were quite satisfied with a main term and an estimate, since we were only interested in the *singularities* of the relevant function of μ when crossing the real line, not in its exact value). This leads to (16.24), not forgetting a last sign change, due to the fact that the case $m = 0$ from Section 13 would correspond to the pair $(-n, n)$, not $(n, -n)$. □

Remark. From (16.23) and (16.24), together with the functional equation (3.18) of the function $E_{\frac{1-\nu}{2}}$, one sees that the function $L^*(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2})$ defined as

$$L^*\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right) = \pi^{\nu-1} L\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right) \times$$
$$\Gamma\left(\tfrac{1-\nu+i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu+i(\lambda_{k_1}-\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu+i(-\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{1-\nu-i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \tag{16.31}$$

if $\mathcal{N}_{k_1,\ell_1}$ and $\mathcal{N}_{k_2,\ell_2}$ have the same parity, and

$$L^*\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right) = \pi^{\nu-1} L\left(\frac{1-\nu}{2}, \mathcal{N}_{k_1,\ell_1} \times \mathcal{N}_{k_2,\ell_2}\right) \times$$
$$\Gamma\left(\tfrac{3-\nu+i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu+i(\lambda_{k_1}-\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu+i(-\lambda_{k_1}+\lambda_{k_2})}{4}\right) \Gamma\left(\tfrac{3-\nu-i(\lambda_{k_1}+\lambda_{k_2})}{4}\right) \tag{16.32}$$

if the parities of $\mathcal{N}_{k_1,\ell_1}$ and $\mathcal{N}_{k_2,\ell_2}$ are opposite, is invariant under the symmetry $\nu \mapsto -\nu$. This, or at least half of it (Poisson brackets are probably less used than pointwise products in number theory, though just as important in pseudodifferential analysis), is of course certainly well known.

Recall that the projection operators $P_{\lambda_k^+}$ and $P_{\lambda_k^-}$ onto the eigenspaces of the modular Laplacian consisting solely of even, or odd, cusp-forms, have been defined just before Theorem 14.2. From the preceding lemma, together with (16.17), we shall get the images, under each of these two operators, of the functions (where $r \geq 1$)

$$g^{\text{sym}}_{\nu,\mathcal{N}_{r,\ell}} := E^*_{\frac{1-\nu}{2}}\, \mathcal{N}_{r,\ell} \qquad \text{and} \qquad g^{\text{antisym}}_{\nu,\mathcal{N}_{r,\ell}} := \frac{1}{2}\left\{E^*_{\frac{1-\nu}{2}}, \mathcal{N}_{r,\ell}\right\} : \tag{16.33}$$

here, $\mathcal{N}_{r,\ell}$ could be an $\mathcal{N}^+_{r,\ell}$ or an $\mathcal{N}^-_{r,\ell}$: in the first case, the function $g^{\text{sym}}_{\nu,\mathcal{N}_{r,\ell}}$ will only have possible non-zero images under operators $P_{\lambda_k^+}$, and the function $g^{\text{antisym}}_{\nu,\mathcal{N}_{r,\ell}}$ will only have possible non-zero images under operators $P_{\lambda_k^-}$; if $\mathcal{N}_{r,\ell}$ is a $\mathcal{N}^-_{r,\ell}$, it is the other way around.

We denote as $(g^{\text{sym}}_{\nu,\mathcal{N}_{r,\ell}})_k$ the image of the function $g^{\text{sym}}_{\nu,\mathcal{N}_{r,\ell}}$ under the projection $P_{\lambda_k^+}$ (*resp.* $P_{\lambda_k^-}$) according to whether $\mathcal{N}_{r,\ell}$ is an even or odd Maass-Hecke cusp-form; we denote as $(g^{\text{antisym}}_{\nu,\mathcal{N}_{r,\ell}})_k$ the image of the function $g^{\text{antisym}}_{\nu,\mathcal{N}_{r,\ell}}$ under the projection $P_{\lambda_k^-}$ (*resp.* $P_{\lambda_k^+}$) according to whether $\mathcal{N}_{r,\ell}$ is even or odd. In each case, in what follows, λ_r or λ_k should be provided with the only superscript $+$ or $-$ which is meaningful.

Lemma 16.3. *Assuming* $|\text{Re}\,\nu| < 1$, *one has with the notation just explained*

$$(g^{\text{sym}}_{\nu,\mathcal{N}_{r,\ell}})_k = \frac{1}{4}\sum_{\ell'} L^*\left(\frac{1-\nu}{2},\, \mathcal{N}_{r,\ell}\times\mathcal{N}_{k,\ell'}\right)\, \|\mathcal{N}_{k,\ell'}\|^{-2}\,\mathcal{N}_{k,\ell'} \tag{16.34}$$

and

$$(g^{\text{antisym}}_{\nu,\mathcal{N}_{r,\ell}})_k = \frac{1}{4i}\sum_{\ell'} L^*\left(\frac{1-\nu}{2},\, \mathcal{N}_{r,\ell}\times\mathcal{N}_{k,\ell'}\right)\, \|\mathcal{N}_{k,\ell'}\|^{-2}\,\mathcal{N}_{k,\ell'}\,. \tag{16.35}$$

Proof. This is an immediate consequence of Lemma 16.2 and (16.17). □

Proposition 16.4. *Assume that* $|\text{Re}\,\nu| < 1$ *and, with* $r \in \mathbb{Z}^\times$, *consider a cusp-distribution* $(\mathfrak{F}^+_{r,\ell})^\sharp$ *as has been defined in* (15.4). *Recall that the functions* $f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$ *and* $f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}$ *have been defined in* (16.10) *and* (16.11). *The projections of each of these two functions onto the (discrete) eigenspaces of the modular Laplacian are given by the formulas*

$$\begin{aligned}(f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,+} &= \zeta^*(-i\lambda_r^+)\,\zeta^*(i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\times\\ \sum_{\varepsilon=\pm1} &\frac{\frac{\pi}{2}\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\Gamma\left(\frac{-i\lambda_k^+}{2}\right)\left(-\frac{i\varepsilon\lambda_k^+}{2}\right)}{\Gamma\left(\frac{1+i\varepsilon\lambda_k^+-\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{1-i\varepsilon\lambda_k^+ +\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{1-i\varepsilon\lambda_k^+-\nu+i\lambda_r^+}{4}\right)\Gamma\left(\frac{1+i\varepsilon\lambda_k^+ +\nu+i\lambda_r^+}{4}\right)}\\ &\times\sum_{\ell'} L^*\left(\frac{1-\nu}{2},\,\mathcal{N}^+_{|r|,\ell}\times\mathcal{N}^+_{k,\ell'}\right)\|\mathcal{N}^+_{k,\ell'}\|^{-2}\,\mathcal{N}^+_{k,\ell'}\end{aligned} \tag{16.36}$$

together with

$$\begin{aligned}(f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,-} &= \zeta^*(-i\lambda_r^+)\,\zeta^*(i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\times\\ \sum_{\varepsilon=\pm1} &\frac{\frac{\pi}{2}\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\Gamma\left(\frac{-i\lambda_k^-}{2}\right)\left(\frac{i\varepsilon\lambda_k^-}{2}\right)}{\Gamma\left(\frac{3+i\varepsilon\lambda_k^--\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{3-i\varepsilon\lambda_k^- +\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{3-i\varepsilon\lambda_k^--\nu+i\lambda_r^+}{4}\right)\Gamma\left(\frac{3+i\varepsilon\lambda_k^- +\nu+i\lambda_r^+}{4}\right)}\\ &\times\sum_{\ell'} L^*\left(\frac{1-\nu}{2},\,\mathcal{N}^+_{|r|,\ell}\times\mathcal{N}^-_{k,\ell'}\right)\|\mathcal{N}^-_{k,\ell'}\|^{-2}\,\mathcal{N}^-_{k,\ell'}\end{aligned} \tag{16.37}$$

for the first one, and

$$(f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,+} = \zeta^*(-i\lambda_r^+)\,\zeta^*(i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\times$$

$$\sum_{\varepsilon=\pm1}\frac{\frac{\pi}{2}\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\Gamma\left(\frac{-i\lambda_k^+}{2}\right)\left(-\frac{(\lambda_k^+)^2}{2}\right)}{\Gamma\left(\frac{1+i\varepsilon\lambda_k^+-\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{1-i\varepsilon\lambda_k^++\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{1-i\varepsilon\lambda_k^+-\nu+i\lambda_r^+}{4}\right)\Gamma\left(\frac{1+i\varepsilon\lambda_k^++\nu+i\lambda_r^+}{4}\right)}$$

$$\times\sum_{\ell'}L^*\left(\frac{1-\nu}{2},\,\mathcal{N}^+_{|r|,\ell}\times\mathcal{N}^+_{k,\ell'}\right)\|\mathcal{N}^+_{k,\ell'}\|^{-2}\,\mathcal{N}^+_{k,\ell'} \tag{16.38}$$

together with

$$(f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,-} = \zeta^*(-i\lambda_r^+)\,\zeta^*(i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\times$$

$$\sum_{\varepsilon=\pm1}\frac{\frac{\pi}{2}\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\Gamma\left(\frac{-i\lambda_k^-}{2}\right)\left(\frac{(\lambda_k^-)^2}{2}\right)}{\Gamma\left(\frac{3+i\varepsilon\lambda_k^--\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{3-i\varepsilon\lambda_k^-+\nu-i\lambda_r^+}{4}\right)\Gamma\left(\frac{3-i\varepsilon\lambda_k^--\nu+i\lambda_r^+}{4}\right)\Gamma\left(\frac{3+i\varepsilon\lambda_k^-+\nu+i\lambda_r^+}{4}\right)}$$

$$\times\sum_{\ell'}L^*\left(\frac{1-\nu}{2},\,\mathcal{N}^+_{|r|,\ell}\times\mathcal{N}^-_{k,\ell'}\right)\|\mathcal{N}^-_{k,\ell'}\|^{-2}\,\mathcal{N}^-_{k,\ell'} \tag{16.39}$$

for the second one.

Proof. Let (b_n) be the sequence of Fourier coefficients relative to the Maass-Hecke cusp-form $\mathcal{N}^+_{|r|,\ell}$: the coefficients b'_n relative to the cusp-distribution $(\mathfrak{F}^+_{r,\ell})^\sharp$, homogeneous of degree $-i-i\lambda_r^+$, are given (a consequence of (15.4)) as

$$b'_n = 2^{\frac{-1-i\lambda_r^+}{2}}\,\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\,b_n\,. \tag{16.40}$$

Following the computations in Section 13, only substituting ν (*resp.* $i\lambda_r^+$) for ν_1 (*resp.* ν_2), and substituting $2^{\frac{i\lambda_r^+-1}{2}}\,b'_{n+m}$ for $\frac{\sigma_{\nu_2}(|n+m|)}{|n+m|^{\frac{\nu_2}{2}}}$, we find that

$$(f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,+}(z) = y^{\frac{1}{2}}\sum_{m\neq0}d_m\,K_{\frac{i\lambda_k}{2}}(2\pi|m|y)\,e^{2i\pi mx} \tag{16.41}$$

with

$$d_m = -8i\pi\,|m|^{-\frac{i\lambda_k}{2}}\times\text{residue of }D_m(\mu)\quad\text{at }\mu=\lambda_k\,, \tag{16.42}$$

where (compare (14.22))

$$D_m(\mu) = 2^{i\mu-2}\,\pi^{-1}\,(-\frac{i\mu}{2})\,\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}$$
$$\times \sum_{j=0,1} [C^+(\nu, i\lambda_r^+;\mu) + (-1)^j C^-(\nu, i\lambda_r^+;\mu)] \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_\nu(|n|)\, b_{n+m} |n|_j^{\frac{-1-\nu+i\mu}{2}}\,. \tag{16.43}$$

On the other hand, we need to generalize Lemma 14.1, denoting this time as $C_m^{\mathrm{sym}}(\mu)$ and $C_m^{\mathrm{antisym}}(\mu)$ the functions associated by (14.1) to the functions $g^{\mathrm{sym}}_{\nu,\mathcal{N}_{|r|,\ell}}$ or $g^{\mathrm{antisym}}_{\nu,\mathcal{N}_{|r|,\ell}}$ as defined in (16.33). Here, we must replace $\frac{\sigma_{\nu_2}(|n+m|)}{|n+m|^{\frac{\nu_2}{2}}}$ by $\frac{1}{2}\, b_{n+m}$ in order to transform (14.23) and (14.24) into equations valid in our present case, thus getting

$$C_m^{\mathrm{sym}}(\mu) \sim \frac{2^{-5}\pi^{-\frac{3}{2}}}{\Gamma(-\frac{i\mu}{2})\,\Gamma(\frac{1-i\mu}{2})} \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_\nu(|n|)\, b_{n+m}\, |n|^{\frac{-1-\nu+i\mu}{2}}$$
$$\times\,\Gamma\left(\tfrac{1-i\mu+\nu+i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{1-i\mu+\nu-i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{1-i\mu-\nu+i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{1-i\mu-\nu-i\lambda_r^+}{4}\right) \tag{16.44}$$

and

$$C_m^{\mathrm{antisym}}(\mu) \sim \frac{2^{-5}\, i\,\pi^{-\frac{3}{2}}}{\Gamma(-\frac{i\mu}{2})\,\Gamma(\frac{1-i\mu}{2})} \sum_{\substack{n\neq 0\\ n\neq -m}} \sigma_\nu(|n|)\, b_{n+m}\, |n|_1^{\frac{-1-\nu+i\mu}{2}}$$
$$\times\,\Gamma\left(\tfrac{3-i\mu+\nu+i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{3-i\mu+\nu-i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{3-i\mu-\nu+i\lambda_r^+}{4}\right)\Gamma\left(\tfrac{3-i\mu-\nu-i\lambda_r^+}{4}\right) \tag{16.45}$$

up to an error term which extends holomorphically to the half-plane $\mathrm{Im}\ \mu > -1 + |\mathrm{Re}\ \nu|$. □

In order to properly state the theorem concerning the decomposition of $\mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^+_{r,\ell})^\sharp$ into homogeneous components, it is useful to introduce a definition, which should be compared to Definition 5.7 of $\mathcal{L}(s)$ and $\mathcal{L}'(s)$: actually, it is the product $\mathcal{L}'(\frac{1+\nu_1+\nu_2}{2})\,\mathcal{G}\,\mathcal{L}'(\frac{1+\nu_1-\nu_2}{2})$ from the right-hand side of (15.3) which can be so generalized, not its two individual factors on both sides.

Definition 16.5. Let $\mathfrak{M}^\sharp_r$ be a cusp-distribution, homogeneous of degree $-1-i\lambda_r^+$, arising by (4.4) from some even Maass-Hecke cusp-form and the choice of a square root of $(\lambda_r^+)^2$. As an operator acting on automorphic (tempered) distributions, we set, when $\mathrm{Re}\ s$ is large,

$$\mathcal{R}(s, \mathfrak{M}_r^\sharp) = \zeta(2s) \sum_{N \geq 1} \frac{b_N}{N^s}\, T_N^{\mathrm{dist}}, \tag{16.46}$$

where the set $(b_N)_{N \in \mathbb{Z}^\times}$ is the sequence of Fourier coefficients of the Maass-Hecke form under consideration. We also set

$$\mathcal{R}'(s, \mathfrak{M}_r^\sharp) = 2^{\frac{-1+i\lambda_r^+}{2}}\, \pi^{1-2s} \times \left[\frac{\Gamma\left(\frac{s+i\pi\,\mathcal{E}}{2} - \frac{i\lambda_r^+}{4}\right) \Gamma\left(\frac{s-i\pi\,\mathcal{E}}{2} + \frac{i\lambda_r^+}{4}\right)}{\Gamma\left(\frac{1-s+i\pi\,\mathcal{E}}{2} - \frac{i\lambda_r^+}{4}\right) \Gamma\left(\frac{1-s-i\pi\,\mathcal{E}}{2} + \frac{i\lambda_r^+}{4}\right)} \right.$$
$$\left. \times\, \mathcal{R}_{\mathrm{even}}(s, \mathfrak{M}_r^\sharp) - \frac{\Gamma\left(\frac{s+1+i\pi\,\mathcal{E}}{2} - \frac{i\lambda_r^+}{4}\right) \Gamma\left(\frac{s+1-i\pi\,\mathcal{E}}{2} + \frac{i\lambda_r^+}{4}\right)}{\Gamma\left(\frac{2-s+i\pi\,\mathcal{E}}{2} - \frac{i\lambda_r^+}{4}\right) \Gamma\left(\frac{2-s-i\pi\,\mathcal{E}}{2} + \frac{i\lambda_r^+}{4}\right)}\, \mathcal{R}_{\mathrm{odd}}(s, \mathfrak{M}_r^\sharp) \right], \tag{16.47}$$

where $\mathcal{R}_{\mathrm{even}}(s, \mathfrak{M}_r^\sharp)$ (*resp.* $\mathcal{R}_{\mathrm{odd}}(s, \mathfrak{M}_r^\sharp)$) is the linear operator on automorphic distributions which coincides with $\mathcal{R}(s, \mathfrak{M}_r^\sharp)$ on distributions of the even (*resp.* odd) type and vanishes on distributions of the odd (*resp.* even) type (*cf.* Definition 5.6).

One can show that $\mathcal{R}'(s, \mathfrak{M}_r^\sharp)$ extends as an operator-valued meromorphic function in the entire s-plane and that it satisfies the equation (which plays the role formerly played by (5.31))

$$\mathcal{R}'(s, \mathfrak{M}_r^\sharp) = \mathcal{R}'(1-s, \mathcal{G}\, \mathfrak{M}_r^\sharp). \tag{16.48}$$

The easiest way to do this is to calculate the effect of $\mathcal{R}'(s, \mathfrak{M}_r^\sharp)$ on the distributions $\mathfrak{F}_{i\lambda}^\sharp$, $(\mathfrak{F}_{k,\ell'}^+)^\sharp$ and $(\mathfrak{F}_{k,\ell'}^-)^\sharp$, assuming, say, that $\mathfrak{M}_r^\sharp = (\mathfrak{N}_{r,\ell}^+)^\sharp$.

Lemma 16.6. *When* $\mathrm{Re}\ s$ *is large, one has*

$$\mathcal{R}(s, (\mathfrak{N}_{r,\ell}^+)^\sharp)\, \mathfrak{F}_{i\lambda}^\sharp = L\left(s + \frac{i\lambda}{2}, \mathcal{N}_{|r|,\ell}^+\right) L\left(s - \frac{i\lambda}{2}, \mathcal{N}_{|r|,\ell}^+\right) \mathfrak{F}_{i\lambda}^\sharp \tag{16.49}$$

and

$$\mathcal{R}(s, (\mathfrak{N}_{r,\ell}^+)^\sharp)\, (\mathfrak{F}_{k,\ell'}^\pm)^\sharp = L(s, \mathcal{N}_{|r|,\ell}^+ \times \mathcal{N}_{|k|,\ell'}^\pm)\, (\mathfrak{F}_{k,\ell'}^\pm)^\sharp. \tag{16.50}$$

Proof. Let $(b_N)_{N \in \mathbb{Z}^\times}$ and $(c_N)_{N \in \mathbb{Z}^\times}$ be the sequences of Fourier coefficients of the Maass-Hecke cusp-forms $\mathcal{N}_{|r|,\ell}^+$ and $\mathcal{N}_{|k|,\ell'}^+$. Then, for $N \geq 1$, $T_N^{\mathrm{dist}}\, (\mathfrak{F}_{k,\ell'}^\pm)^\sharp = c_N\, (\mathfrak{F}_{k,\ell'}^\pm)^\sharp$ as a consequence of Theorem 4.2 and Proposition 5.2 (together with the fundamental equation $T_N\, \mathcal{N}_{|k|,\ell'}^\pm = c_N\, \mathcal{N}_{|k|,\ell'}^\pm$): (16.50) thus follows from (16.46) and (16.19). On the other hand, $T_N^{\mathrm{dist}}\, \mathfrak{E}_\nu^\sharp = N^{-\frac{\nu}{2}}\, \sigma_\nu(N)\, \mathfrak{E}_\nu^\sharp$ for comparable reasons (or *cf.* [62, (16.88)]), and (16.49) follows from the equation (*loc.cit.,* (4.37))

$$\sum_{N \geq 1} N^{-\frac{\nu}{2}-s}\, \sigma_\nu(N)\, b_N = (\zeta(2s))^{-1}\, L\left(s + \frac{\nu}{2}, \mathcal{N}_{|r|,\ell}^+\right) L\left(s - \frac{\nu}{2}, \mathcal{N}_{|r|,\ell}^+\right). \tag{16.51}$$

□

With the help of (16.47), (15.4) and (5.27), one derives from (16.49) the equation

$$\mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right) \mathfrak{F}^\sharp_{i\lambda} = \frac{\pi}{2}\,\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\,\mathfrak{F}^\sharp_{i\lambda}$$
$$\times \frac{L^*\left(\frac{1-\nu-i\lambda}{2}, \mathcal{N}^+_{|r|,\ell}\right) L^*\left(\frac{1-\nu+i\lambda}{2}, \mathcal{N}^+_{|r|,\ell}\right)}{\Gamma\left(\frac{1+\nu-i(\lambda+\lambda_r^+)}{4}\right)\Gamma\left(\frac{1-\nu+i(\lambda-\lambda_r^+)}{4}\right)\Gamma\left(\frac{1-\nu-i(\lambda-\lambda_r^+)}{4}\right)\Gamma\left(\frac{1+\nu+i(\lambda+\lambda_r^+)}{4}\right)}. \tag{16.52}$$

From (16.50) together with (16.31) and (16.32), one finds

$$\mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right) (\mathfrak{F}^+_{k,\ell'})^\sharp = \frac{\pi}{2}\,\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\,(\mathfrak{F}^+_{k,\ell'})^\sharp$$
$$\times \frac{L^*\left(\frac{1-\nu}{2}, \mathcal{N}^+_{|r|,\ell} \times \mathcal{N}^+_{|k|,\ell'}\right)}{\Gamma\left(\frac{1+\nu-i(\lambda_k^++\lambda_r^+)}{4}\right)\Gamma\left(\frac{1-\nu+i(\lambda_k^+-\lambda_r^+)}{4}\right)\Gamma\left(\frac{1-\nu-i(\lambda_k^+-\lambda_r^+)}{4}\right)\Gamma\left(\frac{1+\nu+i(\lambda_k^++\lambda_r^+)}{4}\right)} \tag{16.53}$$

and

$$\mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right) (\mathfrak{F}^-_{k,\ell'})^\sharp = -\frac{\pi}{2}\,\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)\,\|\mathcal{N}^+_{|r|,\ell}\|^{-2}\,(\mathfrak{F}^-_{k,\ell'})^\sharp$$
$$\times \frac{L^*\left(\frac{1-\nu}{2}, \mathcal{N}^+_{|r|,\ell} \times \mathcal{N}^-_{|k|,\ell'}\right)}{\Gamma\left(\frac{3+\nu-i(\lambda_k^-+\lambda_r^+)}{4}\right)\Gamma\left(\frac{3-\nu+i(\lambda_k^--\lambda_r^+)}{4}\right)\Gamma\left(\frac{3-\nu-i(\lambda_k^--\lambda_r^+)}{4}\right)\Gamma\left(\frac{3+\nu+i(\lambda_k^-+\lambda_r^+)}{4}\right)}. \tag{16.54}$$

Together with Theorem 4.1 (the decomposition of automorphic distributions into homogeneous components), the equations (16.52), (16.53) and (16.54) provide the analytic continuation of $\mathcal{R}'(\frac{s}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp)$ as an (operator-valued) function of s. The functional equation (16.48) is a consequence of the relation $\mathcal{G}\,(\mathfrak{F}^+_{r,\ell})^\sharp = (\mathfrak{F}^+_{-r,\ell})^\sharp$ from Lemma 15.2, the relation $\lambda^+_{-r} = -\lambda^+_r$ as defined right after (4.4), and of the functional equations relative to the functions $s \mapsto L^*(s, \mathcal{N}^+_{|r|,\ell})$ and $s \mapsto L^*(s, \mathcal{N}^+_{|r|,\ell} \times \mathcal{N}^\pm_{|k|,\ell'})$.

Theorem 16.7. *Let* $\nu \in \mathbb{C}$ *satisfy* $|\mathrm{Re}\,\nu| < 1$. *With the same meaning as in Theorem* 15.1, *only substituting* $\mathfrak{F}^\sharp_\nu$ *and* $(\mathfrak{F}^+_{r,\ell})^\sharp$ *for* $\mathfrak{F}^\sharp_{\nu_1}$ *and* $\mathfrak{F}^\sharp_{\nu_2}$, *one can uniquely – up to the addition of a multiple of* $\mathfrak{E}^\sharp_0$ *– define the automorphic distribution* $\mathfrak{S} = \mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^+_{r,\ell})^\sharp$, *satisfying the identities analogous to* (15.1) *and* (15.2). *One has*

$$\mathfrak{F}^\sharp_\nu \# (\mathfrak{F}^+_{r,\ell})^\sharp = \mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right) . 2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B}. \tag{16.55}$$

Proof. Just as in Lemma 15.5, we write (starting from (15.51))

$$\begin{aligned}\mathfrak{T}\colon &= 2i\pi\,\mathcal{E}\,\mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right).\,2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B}\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{-i\lambda\,\Psi(\lambda)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,\mathfrak{F}^\sharp_{i\lambda}\,d\lambda + \frac{1}{2}\sum_{\substack{k,\ell'\\k\neq 0}}\frac{-i\lambda_k^+\,a_{k,\ell'}}{\zeta(i\lambda_k^+)\,\zeta(-i\lambda_k^+)}\,(\mathfrak{F}^+_{k,\ell'})^\sharp\\ &\qquad + \frac{1}{2}\sum_{\substack{k,\ell'\\k\neq 0}}\frac{-i\lambda_k^-\,b_{k,\ell'}}{\zeta(i\lambda_k^-)\,\zeta(-i\lambda_k^-)}\,(\mathfrak{F}^-_{k,\ell'})^\sharp,\end{aligned} \tag{16.56}$$

with

$$\Psi(\lambda)\,\mathfrak{F}^\sharp_{i\lambda} = \mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right)\,\mathfrak{F}^\sharp_{i\lambda}, \tag{16.57}$$

$$a_{k,\ell'}\,(\mathfrak{F}^+_{k,\ell'})^\sharp = \mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right)\,(\mathfrak{F}^+_{k,\ell'})^\sharp, \tag{16.58}$$

$$b_{k,\ell'}\,(\mathfrak{F}^-_{k,\ell'})^\sharp = \mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}^+_{r,\ell})^\sharp\right)\,(\mathfrak{F}^-_{k,\ell'})^\sharp, \tag{16.59}$$

as computed in (16.52), (16.53) and (16.54).

What we need to prove is the pair of equations

$$\begin{aligned}(u_z\,|\,\mathrm{Op}(\mathfrak{T})u_z) &= f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z),\\ (u^1_z\,|\,\mathrm{Op}(\mathfrak{T})u^1_z) &= f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z),\end{aligned} \tag{16.60}$$

where the Roelcke-Selberg expansion of the right-hand sides have been computed in Proposition 16.1 and Proposition 16.4. Using Lemma 15.6, one sees that what remains to be done is to verify the pair of equations

$$\begin{aligned}f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z) &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{-i\lambda\,\Psi(\lambda)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,E^*_{\frac{1-i\lambda}{2}}(z)\,d\lambda\\ &\quad + \frac{1}{2}\sum_{k,\ell'}(-i\lambda_k^+)\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\Gamma\left(-\frac{i\lambda_k^+}{2}\right)\,a_{k,\ell'}\,\|\mathcal{N}^+_{|k|,\ell'}\|^{-2}\,\mathcal{N}^+_{|k|,\ell'}(z)\\ &\quad + \frac{1}{2}\sum_{k,\ell'}(-i\lambda_k^-)\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\Gamma\left(-\frac{i\lambda_k^-}{2}\right)\,b_{k,\ell'}\,\|\mathcal{N}^-_{|k|,\ell'}\|^{-2}\,\mathcal{N}^-_{|k|,\ell'}(z)\end{aligned} \tag{16.61}$$

and

$$f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp}(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{-\lambda^2\,\Psi(\lambda)}{\zeta(-i\lambda)}\,\pi^{-\frac{i\lambda}{2}}\,\Gamma\left(\frac{i\lambda}{2}\right)\,E^*_{\frac{1-i\lambda}{2}}(z)\,d\lambda$$
$$-\frac{1}{2}\sum_{k,\ell'}(\lambda_k^+)^2\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\,\Gamma\left(-\frac{i\lambda_k^+}{2}\right)\,a_{k,\ell'}\,\|\mathcal{N}^+_{|k|,\ell'}\|^{-2}\,\mathcal{N}^+_{|k|,\ell'}(z)$$
$$-\frac{1}{2}\sum_{k,\ell'}(\lambda_k^-)^2\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\,\Gamma\left(-\frac{i\lambda_k^-}{2}\right)\,b_{k,\ell'}\,\|\mathcal{N}^-_{|k|,\ell'}\|^{-2}\,\mathcal{N}^-_{|k|,\ell'}(z)\,. \tag{16.62}$$

So far as the continuous parts from the Roelcke-Selberg decompositions are concerned, we must thus check (reducing the integral to that of an even function) that

$$\frac{\Phi_1(\lambda)}{\zeta^*(i\lambda)} = \frac{i\lambda\,(\Psi(-\lambda)-\Psi(\lambda))}{\zeta(i\lambda)\,\zeta(-i\lambda)}$$

and

$$\frac{\Phi_2(\lambda)}{\zeta^*(i\lambda)} = \frac{-\lambda^2\,(\Psi(-\lambda)+\Psi(\lambda))}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,, \tag{16.63}$$

where the functions Φ_1 and Φ_2 have been introduced and computed in Proposition 16.1, or, what amounts to the same in view of the functional equations satisfied by these two functions (*cf.* remark preceding (16.16)) that

$$\Psi(\lambda) = -\frac{1}{2}\,\frac{\pi^{\frac{i\lambda}{2}}\,\zeta(-i\lambda)}{\Gamma(\frac{i\lambda}{2})}\,\left[\frac{\Phi_1(\lambda)}{i\lambda}+\frac{\Phi_2(\lambda)}{\lambda^2}\right]. \tag{16.64}$$

Using (16.57), (16.52) and the equation (a consequence of (15.4) and (15.10)

$$\Lambda\left(\frac{1-\nu-i\lambda}{2},(\mathfrak{F}^+_{r,\ell})^\sharp\right)\,\Lambda\left(\frac{1-\nu+i\lambda}{2},(\mathfrak{F}^+_{-r,\ell})^\sharp\right)$$
$$= \frac{\frac{\pi}{2}\,(\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+))^2\,\|\mathcal{N}^+_{|r|,\ell}\|^{-4}\,L^*\left(\frac{1-\nu-i\lambda}{2},\mathcal{N}^+_{|r|,\ell}\right)\,L^*\left(\frac{1-\nu+i\lambda}{2},\mathcal{N}^+_{|r|,\ell}\right)}{\Gamma\left(\frac{1-\nu-i\lambda+i\lambda_r^+}{4}\right)\,\Gamma\left(\frac{1+\nu+i\lambda+i\lambda_r^+}{4}\right)\,\Gamma\left(\frac{1-\nu+i\lambda-i\lambda_r^+}{4}\right)\,\Gamma\left(\frac{1+\nu-i\lambda-i\lambda_r^+}{4}\right)}\,, \tag{16.65}$$

one immediately derives (16.64) from (16.12) and (16.13) (the terms with $\varepsilon = 1$ from the right-hand sides of these two equations cancel out).

Next, the discrete parts: with $a_{k,\ell'}$ and $b_{k,\ell'}$ as defined in (16.58), (16.59) and computed in (16.53), (16.54), we must check that, for $k \geq 1$, one has

$$\frac{i\lambda_k^+}{2}\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\,\Gamma\left(-\frac{i\lambda_k^+}{2}\right)\,\sum_{\ell'}(a_{-k,\ell'}-a_{k,\ell'})\,\|\mathcal{N}^+_{k,\ell'}\|^{-2}\,\mathcal{N}^+_{k,\ell'}(z)$$
$$= (f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,+}(z) \tag{16.66}$$

and

$$\frac{i\lambda_k^-}{2}\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\Gamma\left(-\frac{i\lambda_k^-}{2}\right)\sum_{\ell'}(b_{-k,\ell'}-b_{k,\ell'})\,\|\mathcal{N}_{k,\ell'}^-\|^{-2}\,\mathcal{N}_{k,\ell'}^-(z)$$
$$=(f^1_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,-}(z)\,, \tag{16.67}$$

finally that

$$-\frac{(\lambda_k^+)^2}{2}\,\Gamma\left(\frac{i\lambda_k^+}{2}\right)\Gamma\left(-\frac{i\lambda_k^+}{2}\right)\sum_{\ell'}(a_{-k,\ell'}+a_{k,\ell'})\,\|\mathcal{N}_{k,\ell'}^+\|^{-2}\,\mathcal{N}_{k,\ell'}^+(z)$$
$$=(f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,+}(z) \tag{16.68}$$

and

$$-\frac{(\lambda_k^-)^2}{2}\,\Gamma\left(\frac{i\lambda_k^-}{2}\right)\Gamma\left(-\frac{i\lambda_k^-}{2}\right)\sum_{\ell'}(b_{-k,\ell'}+b_{k,\ell'})\,\|\mathcal{N}_{k,\ell'}^-\|^{-2}\,\mathcal{N}_{k,\ell'}^-(z)$$
$$=(f^2_{\nu,(\mathfrak{F}^+_{r,\ell})^\sharp})_{k,-}(z)\,. \tag{16.69}$$

With the help of (16.53) and (16.54) on one side, (16.36)–(16.39) on the other side, the verification is straightforward. □

Proposition 16.8. *The continuous part of the decomposition of* $\mathfrak{F}_\nu^\sharp\#(\mathfrak{F}^+_{r,\ell})^\sharp$ *into homogeneous components can also be written as*

$$(\mathfrak{F}_\nu^\sharp\#(\mathfrak{F}^+_{r,\ell})^\sharp)_{\rm cont}=\frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)}$$
$$\times\,\Lambda\left(\frac{1+\nu+2i\pi\,\mathcal{E}}{2},(\mathfrak{F}^+_{r,\ell})^\sharp\right)\,\mathcal{G}\,\Lambda\left(\frac{1+\nu+2i\pi\,\mathcal{E}}{2},(\mathfrak{F}^+_{-r,\ell})^\sharp\right)\,.\,\left(2^{-\frac12+i\pi\,\mathcal{E}}\mathfrak{B}\right)_{\rm cont}\,. \tag{16.70}$$

Proof. Using the spectral decomposition (15.5) of $2^{-\frac12+i\pi\,\mathcal{E}}\mathfrak{B}$ and the invariance under $\mathcal{G}$ of this distribution, one may write the right-hand side of (16.70) as

$$\frac{\|\mathcal{N}^+_{|r|,\ell}\|^2}{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)}\times\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\Lambda\left(\frac{1+\nu+i\lambda}{2},(\mathfrak{F}^+_{r,\ell})^\sharp\right)\Lambda\left(\frac{1+\nu-i\lambda}{2},(\mathfrak{F}^+_{-r,\ell})^\sharp\right)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\,\mathfrak{F}^\sharp_{i\lambda}\,d\lambda\,. \tag{16.71}$$

Now (15.4) and (15.10) yield

$$\Lambda\left(\frac{1+\nu-i\lambda}{2},(\mathfrak{F}^+_{-r,\ell})^\sharp\right)=2^{\frac{-1+i\lambda_r^+}{2}}\,\frac{\zeta^*(i\lambda_r^+)\,\zeta^*(-i\lambda_r^+)}{\|\mathcal{N}^+_{|r|,\ell}\|^2}$$
$$\times\,\frac{\pi^{\frac12}\,L^*\left(\frac{1+\nu-i\lambda}{2},\mathcal{N}^+_{|r|,\ell}\right)}{\Gamma\left(\frac{1+\nu-i\lambda-i\lambda_r^+}{4}\right)\Gamma\left(\frac{1-\nu+i\lambda-i\lambda_r^+}{4}\right)} \tag{16.72}$$

and, to get the first Λ-factor on the right-hand side of (16.71), it suffices to change (λ, λ_r^+) into $(-\lambda, -\lambda_r^+)$. This yields (16.70) after one has used (16.52) to compute the left-hand side

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\mathcal{R}'\left(\frac{1-\nu}{2}, (\mathfrak{F}_{r,\ell}^+)^\sharp\right)}{\zeta(i\lambda)\,\zeta(-i\lambda)}\, \mathfrak{F}_{i\lambda}^\sharp \, d\lambda\,,$$

as given by (16.56), of this equation. □

The structure of the composition formulas may show in a clearer way from a reformulation of the preceding results *in terms of Eulerian products*: this will occupy us for the remainder of this section.

It will be convenient, here, to use the normalisation $\mathrm{Op}_{\sqrt{2}}$ of the Weyl calculus as defined in (2.21), and the associated $\natural$-product, linked to the usual sharp product by the equation

$$f_1 \natural f_2 = 2^{\frac{1}{2}-i\pi\,\mathcal{E}}\left((2^{-\frac{1}{2}+i\pi\,\mathcal{E}} f_1)\#(2^{-\frac{1}{2}+i\pi\,\mathcal{E}} f_2)\right) \; : \tag{16.73}$$

this will permit us to get rid of the extra operator $2^{-\frac{1}{2}+i\pi\,\mathcal{E}}$ in all its occurrences.

Recall that our constant use, in these last two sections, of the distributions $\mathfrak{F}_\nu^\sharp$ and $(\mathfrak{F}_{k,\ell}^\pm)^\sharp$ was due to the role they play in the expansion (15.5) of $2^{-\frac{1}{2}+i\pi\,\mathcal{E}}\,\mathfrak{B}$.

We now choose another normalization, paying interest, rather, to the distributions $\frac{1}{2}\,\mathfrak{E}_\nu^\sharp$ and $(\mathfrak{N}_{k,\ell}^\pm)^\sharp$. The reason for this is that, when $p = 0$ or 1 and $z \in \Pi$, one has

$$\left(u_z^p|\mathrm{Op}_{\sqrt{2}}\left(\frac{1}{2}\,\mathfrak{E}_\nu^\sharp\right)\, u_z^p\right) = (-\nu)^p\,\frac{1}{2}\,E^*_{\frac{1-\nu}{2}}(z)$$

and

$$\left(u_z^p|\mathrm{Op}_{\sqrt{2}}\left((\mathfrak{N}_{k,\ell}^\pm)^\sharp\right)\, u_z^p\right) = (-i\lambda_k^\pm)^p\,\mathcal{N}_{|k|,\ell}^\pm(z) \tag{16.74}$$

as reported in (3.15), (3.16) and (4.12), (4.13): now, the functions $\frac{1}{2}\,E^*_{\frac{1-\nu}{2}}$ and $\mathcal{N}_{|k|,\ell}^\pm$ are normalized by the fact that the coefficient of $y^{\frac{1}{2}}\,K_{\frac{\nu}{2}}(2\pi y)\,e^{2i\pi x}$ or $y^{\frac{1}{2}}\,K_{\frac{i\lambda_k^\pm}{2}}(2\pi y)\,e^{2i\pi x}$ in their Fourier series expansions (4.5) or (4.3) is 1. As a consequence, the corresponding L-functions have (for Re s large) the Eulerian product expansions

$$\begin{aligned} L(s, \frac{1}{2}\,E^*_{\frac{1-\nu}{2}}) &= \sum_{n\geq 1} n^{-s}\,n^{-\frac{\nu}{2}}\,\sigma_\nu(n) \\ &= \zeta\left(s-\frac{\nu}{2}\right)\,\zeta\left(s+\frac{\nu}{2}\right) \\ &= \prod_{p\ \mathrm{prime}} (1-p^{\frac{\nu}{2}-s})^{-1}\,(1-p^{-\frac{\nu}{2}-s})^{-1} \end{aligned} \tag{16.75}$$

and, if (4.3) is the Fourier series expansion of $\mathcal{N}^{\pm}_{|k|,\ell}$,

$$\begin{aligned} L(s, \mathcal{N}^{\pm}_{|k|,\ell}) &= \sum_{n\geq 1} b_n \, n^{-s} \\ &= \prod_{p \text{ prime}} (1 - p^{-s}\, b_p + p^{-2s})^{-1} \\ &= \prod_{p \text{ prime}} (1 - p^{-s}\, \beta_p)^{-1} \, (1 - p^{-s}\, \beta_p^{-1})^{-1} \end{aligned} \tag{16.76}$$

if the Hecke polynomial $1 - b_p\, X + X^2$ factors as $(1 - \beta_p\, X)\,(1 + \beta_p^{-1}\, X)$ (*cf.* [8, p. 119]). Let us, for clarity, refer to the unordered pair (β_p, β_p^{-1}) as the p^{th}-*Hecke pair relative to* $\mathcal{N}^{\pm}_{|k|,\ell}$: in view of (16.75), the pair $(p^{\frac{\nu}{2}}, p^{-\frac{\nu}{2}})$ should be thought of as the p^{th}-*Hecke pair relative to* $\frac{1}{2}\, E^*_{\frac{1-\nu}{2}}$.

With the help of Eulerian products, we would like to get a better grasp of the coefficients $c(\mathfrak{N}; \mathfrak{N}_1, \mathfrak{N}_2)$ which should make the formula

$$\begin{aligned} \mathfrak{N}_1 \natural \, \mathfrak{N}_2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} c\left(\frac{1}{2}\, \mathfrak{E}^{\sharp}_{i\lambda}; \mathfrak{N}_1, \mathfrak{N}_2\right) \, \mathfrak{E}^{\sharp}_{i\lambda} \, \frac{d\lambda}{\zeta(i\lambda)\, \zeta(-i\lambda)} \\ + \frac{1}{2} \sum_{k,\ell,\pm} c((\mathfrak{N}^{\pm}_{k,\ell})^{\sharp}; \mathfrak{N}_1, \mathfrak{N}_2) \, \frac{\Gamma\left(\frac{i\lambda_k^{\pm}}{2}\right) \Gamma\left(-\frac{i\lambda_k^{\pm}}{2}\right)}{\|\mathcal{N}^{\pm}_{|k|,\ell}\|^2} \, (\mathfrak{N}^{\pm}_{k,\ell})^{\sharp} \\ + \text{ exceptional terms} \end{aligned} \tag{16.77}$$

valid in general: here, $\mathfrak{N}_1$ or $\mathfrak{N}_2$ could be an $\frac{1}{2}\, \mathfrak{E}^{\sharp}_{\nu}$ or an $(\mathfrak{N}^{\pm}_{r,\ell})^{\sharp}$, and the exceptional terms are present only when both $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are Eisenstein distributions. One should have another look at (4.33) to understand the reason for the presence of the extra factors (independent of $\mathfrak{N}_1$, $\mathfrak{N}_2$) in the integral or series.

Before doing this, however, let us remark that building our multiplicative table – a task which we shall leave uncompleted – is a lengthy job, involving difficulties of several natures: after reduction by some symmetry considerations of the kind used at the end of Section 15, we are left with an overall task of completing about 19 entries $c(\mathfrak{N}; \mathfrak{N}_1, \mathfrak{N}_2)$, of which we have made only 7 fully explicit. To start with, let us observe that changing an $(\mathfrak{N}^{+}_{k,\ell})^{\sharp}$ to an $(\mathfrak{N}^{-}_{k,\ell})^{\sharp}$ is not quite as trivial as it seems since, besides having to be very careful with signs, one must also, here and there, trade a pointwise product of automorphic functions for half a Poisson bracket of such, or the other way around. Next, the case when $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are all Eisenstein distributions was, from the point of view of analysis, the most difficult one, because of severe divergences: it was at this point that the usual Rankin-Selberg method proved insufficient. On the other hand, difficulties *of a number-theoretic nature* increase when the number κ of cusp-distributions among $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ does. For $\kappa = 0, 1$ or 2 respectively, $c(\mathfrak{N}; \mathfrak{N}_1, \mathfrak{N}_2)$ can be expressed, up to Gamma factors, as a product of four zeta functions, a product of two L-functions or a convolution L-function. As a conjecture, would it be too much to expect that, when $\kappa = 3$, the value at $s = \frac{1}{2}$ of a *triple L-function* as defined in [8, p. 386] should be required?

Relying on the present state of our multiplication table, here is a list of coefficients $c(\mathfrak{N}; \mathfrak{N}_1, \mathfrak{N}_2)$. From (15.33), and the fact that $2^{\frac{1}{2}-i\pi\mathcal{E}}\,\mathfrak{F}^\sharp_\nu = \mathfrak{E}^\sharp_\nu$, one gets

$$\begin{aligned}
&c\left(\frac{1}{2}\,\mathfrak{E}^\sharp_{i\lambda};\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_1},\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_2}\right)\\
&=\frac{1}{4}\zeta\left(\frac{1+\nu_1+\nu_2+i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1-\nu_2-i\lambda}{2}\right)\zeta\left(\frac{1-\nu_1+\nu_2-i\lambda}{2}\right)\zeta\left(\frac{1-\nu_1-\nu_2+i\lambda}{2}\right)\\
&=\frac{1}{4}\,\pi^{-\nu_1}\,\frac{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda}{4}\right)\,\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda}{4}\right)}{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda}{4}\right)\,\Gamma\left(\frac{1-\nu_1-\nu_2+i\lambda}{4}\right)}\\
&\quad\times\zeta\left(\frac{1+\nu_1-\nu_2+i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1-\nu_2-i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1+\nu_2-i\lambda}{2}\right)\zeta\left(\frac{1+\nu_1+\nu_2+i\lambda}{2}\right)
\end{aligned}\tag{16.78}$$

or, finally,

$$\begin{aligned}
c\left(\frac{1}{2}\,\mathfrak{E}^\sharp_{i\lambda};\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_1},\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_2}\right) &= \frac{1}{4}\,\pi^{-\nu_1}\,\frac{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda}{4}\right)\,\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda}{4}\right)}{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda}{4}\right)\,\Gamma\left(\frac{1-\nu_1-\nu_2+i\lambda}{4}\right)}\\
&\quad\times L\left(\frac{1+\nu_1+\nu_2}{2},\frac{1}{2}\,E^*_{\frac{1-i\lambda}{2}}\right)\,L\left(\frac{1+\nu_1-\nu_2}{2},\frac{1}{2}\,E^*_{\frac{1-i\lambda}{2}}\right).
\end{aligned}\tag{16.79}$$

Still from (15.33), and the fact that

$$2^{\frac{1}{2}-i\pi\mathcal{E}}\,(\mathfrak{F}^\pm_{k,\ell})^\sharp = \frac{\zeta^*(i\lambda_k^\pm)\,\zeta^*(-i\lambda_k^\pm)}{\|\mathcal{N}^+_{|k|,\ell}\|^2}\,(\mathfrak{N}^\pm_{k,\ell})^\sharp\,,$$

one gets

$$\begin{aligned}
c((\mathfrak{N}^\pm_{k,\ell})^\sharp;\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_1},\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_2}) &= \frac{1}{4}\,\Lambda\left(\frac{1+\nu_1+\nu_2}{2},(\mathfrak{N}^\pm_{k,\ell})^\sharp\right)\,\Lambda\left(\frac{1+\nu_1-\nu_2}{2},(\mathfrak{N}^\pm_{-k,\ell})^\sharp\right)\\
&= \frac{1}{4}\,\pi^{-\nu_1}\,\frac{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda_k^\pm+2\epsilon}{4}\right)\,\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda_k^\pm+2\epsilon}{4}\right)}{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda_k^\pm+2\epsilon}{4}\right)\,\Gamma\left(\frac{1-\nu_1-\nu_2+i\lambda_k^\pm+2\epsilon}{4}\right)}\\
&\quad\times L\left(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}^\pm_{|k|,\ell}\right)\,L\left(\frac{1+\nu_1+\nu_2}{2},\mathcal{N}^\pm_{|k|,\ell}\right),
\end{aligned}\tag{16.80}$$

with $\epsilon = 0$ if dealing with $(\mathfrak{N}^+_{k,\ell})^\sharp$, $\epsilon = 1$ if dealing with $(\mathfrak{N}^-_{k,\ell})^\sharp$. We may already observe that, when going from $c(\frac{1}{2}\,\mathfrak{E}^\sharp_{i\lambda};\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_1},\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_2})$ to $c((\mathfrak{N}^+_{k,\ell})^\sharp;\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_1},\frac{1}{2}\,\mathfrak{E}^\sharp_{\nu_2})$, the Gamma factors are identical, except for the substitution of λ_k^+ for λ; also, if (β_p, β_p^{-1}) is the p^{th}-th Hecke pair relative to $\frac{1}{2}\, E^*_{\frac{1-i\lambda}{2}}$ in the first case, and to $\mathcal{N}^{\pm}_{|k|,\ell}$ in the second one, the products of two L-functions apparent in (16.79) and (16.80) can both be written as

$$\left[\prod_{\text{prime}} \left(1 - p^{\frac{-1-\nu_1-\nu_2}{2}}\, \beta_p\right) \left(1 - p^{\frac{-1-\nu_1-\nu_2}{2}}\, \beta_p^{-1}\right) \times \left(1 - p^{\frac{-1-\nu_1+\nu_2}{2}}\, \beta_p\right) \left(1 - p^{\frac{-1-\nu_1+\nu_2}{2}}\, \beta_p^{-1}\right)\right]^{-1}. \tag{16.81}$$

Thus, when examined with the help of Eulerian products, *the formulas for* $c\left(\frac{1}{2}\, \mathfrak{E}^\sharp_{i\lambda};\, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu_1}, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu_2}\right)$ *and* $c\left((\mathfrak{N}^+_{k,\ell})^\sharp;\, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu_1}, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu_2}\right)$ *are identical.* However, there is no way to get rid of the Gamma factors in the second one, whereas this would be possible in the first.

Starting from (16.70), one finds that

$$\begin{aligned} c\left(\frac{1}{2}\, \mathfrak{E}^\sharp_{i\lambda};\, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu}, (\mathfrak{N}^+_{r,\ell})^\sharp\right) &= \frac{1}{4}\, \Lambda\left(\frac{1+\nu+i\lambda}{2}, (\mathfrak{N}^+_{r,\ell})^\sharp\right) \Lambda\left(\frac{1+\nu-i\lambda}{2}, (\mathfrak{N}^+_{-r,\ell})^\sharp\right) \\ &= \frac{1}{4}\, \pi^{-\nu}\, \frac{\Gamma\left(\frac{1+\nu-i\lambda_r^+ + i\lambda}{4}\right) \Gamma\left(\frac{1+\nu+i\lambda_r^+ - i\lambda}{4}\right)}{\Gamma\left(\frac{1-\nu_+ i\lambda_r^+ - i\lambda}{4}\right) \Gamma\left(\frac{1-\nu_- i\lambda_r^+ + i\lambda}{4}\right)} \\ &\quad \times L\left(\frac{1+\nu-i\lambda}{2}, \mathcal{N}^+_{|r|,\ell}\right) L\left(\frac{1+\nu+i\lambda}{2}, \mathcal{N}^+_{|r|,\ell}\right). \end{aligned} \tag{16.82}$$

If (β_p, β_p^{-1}) is the p^{th}-th Hecke pair relative to $\mathcal{N}^+_{|r|,\ell}$, one may write the product of two L-functions as

$$\left[\prod_{\text{prime}} \left(1 - p^{\frac{-1-\nu+i\lambda}{2}}\, \beta_p\right) \left(1 - p^{\frac{-1-\nu+i\lambda}{2}}\, \beta_p^{-1}\right) \times \left(1 - p^{\frac{-1-\nu-i\lambda}{2}}\, \beta_p\right) \left(1 - p^{\frac{-1-\nu-i\lambda}{2}}\, \beta_p^{-1}\right)\right]^{-1} : \tag{16.83}$$

looking also at the product of Gamma factors on the right-hand side of equation (16.82), one sees that, again, the Eulerian product version of the coefficient $c\left(\frac{1}{2}\, \mathfrak{E}^\sharp_{i\lambda};\, \frac{1}{2}\, \mathfrak{E}^\sharp_{\nu}, (\mathfrak{N}^+_{r,\ell})^\sharp\right)$ has exactly the same structure as the first two coefficients discussed.

Finally, let us compute and discuss the coefficient $c\left((\mathfrak{N}_{k,\ell'}^{+})^{\sharp}; \frac{1}{2}\,\mathfrak{E}_{\nu}^{\sharp}, (\mathfrak{N}_{r,\ell}^{+})^{\sharp}\right)$. From (16.56), (16.58) and (16.53), we see that the coefficient of $\frac{1}{2}\,\frac{(\mathfrak{F}_{k,\ell'}^{+})^{\sharp}}{\zeta(i\lambda_k^{+})\,\zeta(-i\lambda_k^{+})}$ in the expansion of $\mathfrak{F}_{\nu}^{\sharp}\#(\mathfrak{F}_{r,\ell}^{+})^{\sharp}$ is

$$\frac{\pi}{2}\,\zeta^*(i\lambda_r^{+})\,\zeta^*(-i\lambda_r^{+})\,\|\mathcal{N}_{|r|,\ell}^{+}\|^{-2} \times \frac{L^*\left(\frac{1+\nu}{2}, \mathcal{N}_{|r|,\ell}^{+}\times\mathcal{N}_{|k|,\ell'}^{+}\right)}{\Gamma\left(\frac{1+\nu-i(\lambda_k^{+}+\lambda_r^{+})}{4}\right)\Gamma\left(\frac{1-\nu+i(\lambda_k^{+}-\lambda_r^{+})}{4}\right)\Gamma\left(\frac{1-\nu-i(\lambda_k^{+}-\lambda_r^{+})}{4}\right)\Gamma\left(\frac{1+\nu+i(\lambda_k^{+}+\lambda_r^{+})}{4}\right)}, \tag{16.84}$$

where we have used the invariance under $s\mapsto 1-s$ of a function $L^*(s,\mathcal{N}^{+}\times\mathcal{N}^{+})$. From (16.31), this is

$$\frac{1}{2}\,\pi^{-\nu}\,\zeta^*(i\lambda_r^{+})\,\zeta^*(-i\lambda_r^{+})\,\|\mathcal{N}_{|r|,\ell}^{+}\|^{-2}\,L\left(\frac{1+\nu}{2}, \mathcal{N}_{|r|,\ell}^{+}\times\mathcal{N}_{|k|,\ell'}^{+}\right) \times \frac{\Gamma\left(\frac{1+\nu-i\lambda_r^{+}+i\lambda_k^{+}}{4}\right)\Gamma\left(\frac{1+\nu+i\lambda_r^{+}-i\lambda_k^{+}}{4}\right)}{\Gamma\left(\frac{1-\nu+i\lambda_r^{+}-i\lambda_k^{+}}{4}\right)\Gamma\left(\frac{1-\nu-i\lambda_r^{+}+i\lambda_k^{+}}{4}\right)}. \tag{16.85}$$

Thus

$$c\left((\mathfrak{N}_{k,\ell'}^{+})^{\sharp}; \frac{1}{2}\,\mathfrak{E}_{\nu}^{\sharp}, (\mathfrak{N}_{r,\ell}^{+})^{\sharp}\right) = \frac{1}{4}\,\pi^{-\nu}\,\frac{\Gamma\left(\frac{1+\nu-i\lambda_r^{+}+i\lambda_k^{+}}{4}\right)\Gamma\left(\frac{1+\nu+i\lambda_r^{+}-i\lambda_k^{+}}{4}\right)}{\Gamma\left(\frac{1-\nu+i\lambda_r^{+}-i\lambda_k^{+}}{4}\right)\Gamma\left(\frac{1-\nu-i\lambda_r^{+}+i\lambda_k^{+}}{4}\right)} \times L\left(\frac{1+\nu}{2}, \mathcal{N}_{|r|,\ell}^{+}\times\mathcal{N}_{|k|,\ell'}^{+}\right). \tag{16.86}$$

Again, the Archimedean factor is exactly the same as before, *mutatis mutandis*. Introducing the p^{th}-th Hecke pair (β_p, β_p^{-1}) relative to $\mathcal{N}_{|r|,\ell}^{+}$ as before, together with the pair $(\alpha_p, \alpha_p^{-1})$ relative to $\mathcal{N}_{|k|,\ell'}^{+}$, one gets (using this time (16.21))

$$L\left(\frac{1+\nu}{2}, \mathcal{N}_{|r|,\ell}^{+}\times\mathcal{N}_{|k|,\ell'}^{+}\right) = \left[\prod_{\text{prime}}\left(1-p^{\frac{-1-\nu}{2}}\,\alpha_p\beta_p\right)\left(1-p^{\frac{-1-\nu}{2}}\,\alpha_p\beta_p^{-1}\right) \times\left(1-p^{\frac{-1-\nu}{2}}\,\alpha_p^{-1}\beta_p\right)\left(1-p^{\frac{-1-\nu}{2}}\,\alpha_p^{-1}\beta_p^{-1}\right)\right]^{-1}, \tag{16.87}$$

the same as (16.83) except for the substitution of $(\alpha_p, \alpha_p^{-1})$ for $(p^{\frac{i\lambda}{2}}, p^{-\frac{i\lambda}{2}})$.

Thus a clear conjecture concerning the general structure of coefficients $c(\mathfrak{N};\mathfrak{N}_1,\mathfrak{N}_2)$ emerges. Of course, one has to be careful with the Gamma factors (*cf.* (16.57) and (16.39) on one hand, (16.54) on the other, as well as (16.80)) when cusp-distributions of the odd type are concerned, and we have to admit that the case when $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are all cusp-distributions might present some more difficulties (the other cases are only a question of how patient you are) if one wishes to connect the result to the value at $s=\frac{1}{2}$ of some triple L-function, not only to the integral on $\Gamma\backslash\Pi$ of the product of three Hecke forms or to the integral of a Hecke form times the Poisson bracket of two Hecke forms or to the iterated Poisson bracket of three Hecke forms.

Remark. It is clear that all coefficients $c(\mathfrak{N};\mathfrak{N}_1,\mathfrak{N}_2)$ which occur in the decomposition into homogeneous components of a product $\mathfrak{N}_1 \natural \mathfrak{N}_2$ of "elementary" automorphic distributions *i.e.*, $\frac{1}{2}\,\mathfrak{E}_\nu^\sharp$'s or $(\mathfrak{N}_{k,\ell}^\pm)^\sharp$'s, have nice Eulerian products, with a fully identical structure: there is one possible exception in the case when all three entries are cusp-distributions.

Our proofs, however, could not be based on (arithmetic) localisation techniques: only an adelic setting, not a classical distribution setting, might make this possible, but we would then not be dealing with the same problem.

An interesting question regards the splitting of the Hecke polynomial $1-b_p\,X+X^2$ into two *distinguishable* factors. An adelic point of view is required here, but much more is involved: let us only remark that, so far as the Archimedean place is concerned, this is exactly what we have done throughout the present work (*cf.* (16.2)), substituting decompositions into homogeneous components (on $\mathbb{R}^2$) for spectral decompositions with respect to Δ (on Π).

We hope to succeed in carrying out a similar program in general in the not too distant future.

Chapter 4

Further Perspectives

17 Another way to compose Weyl symbols

This section serves several purposes. The first one is to give a detailed proof of the first theorem to follow, a rather careless version of which was indicated in [62], Theorem 5.3. Next, we shall observe (with fewer details) that, contrary to the usual formula for the composition of symbols, this theorem extends to the Op^p-calculus. The reason why this is so is that, for $p \geq 1$, the Op^p-calculus, while still covariant under the p-metaplectic representation, does not admit any covariance under the Heisenberg representation: indeed, the point $0 \in \mathbb{R}$ plays a very special role in this calculus, which has therefore no translation invariance. In Section 19, we shall indicate why, in the general context of quantization, this type of formula is prevalent. Last, let us indicate that, as has been proved by Bechata [5], the present formula extends to the p-adic Weyl calculus (dealing with complex-valued functions on p-adic numbers), while, again, the Moyal-type formula would be meaningless.

In the second part of this section, we shall see that there is a close relation between the composition of Weyl symbols on one hand, the pair constituted by the pointwise product and Poisson bracket of functions on Π on the other hand. This phenomenon has already been encountered in the last part of Section 11. The same holds in the quite different Γ-invariant environment, as a comparison, say, between our present Theorem 13.6 and Theorem 9.6 of [62], or between Theorem 14.2 and Theorem 14.5 from *loc.cit.* would show. In this direction, let us emphasize that the arithmetic case cannot be reduced to that which deals with Δ on the "open space" Π, since one is concerned there with two quite different realizations of the Laplacian. On the other hand, we fully used in the present work (especially in Section 14) the resources provided by the results from *loc.cit.*

The reason for all these resemblances is that, on a rank-one symmetric space such as Π, there are very few possible covariant bilinear machines: more precisely,

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there are essentially two possibilities (one symmetric, one antisymmetric) as soon as a maximal spectral decomposition has been carried, with respect to both the input and output; of course, there is a considerable variety of possibilities in *global* formulas, *i.e.*. those which do not use the spectral decomposition.

Theorem 17.1. *Let* h_1 *and* h_2 *be two symbols in* $\mathcal{S}_{\rm even}(\mathbb{R}^2)$. *Then one has, in the weak sense in* $\mathcal{S}'(\mathbb{R}^2)$,

$$h_1 \# h_2 = \int_{-\infty}^{\infty} h_\lambda \, d\lambda, \tag{17.1}$$

where h_λ *is associated through* (2.16) *to the function* $h^\flat_\lambda$,

$$h^\flat_\lambda(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1 \, d\lambda_2 \int_{\mathbb{R}^2} K_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s) \; (h_1)^\flat_{\lambda_1}(s_1)(h_2)^\flat_{\lambda_2}(s_2) ds_1 \, ds_2, \tag{17.2}$$

and the integral kernel $K_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s)$ *is given as*

$$\begin{aligned} K_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s) &= 2^{-\frac{3}{2}} \, (2\pi)^{\frac{i(-\lambda+\lambda_1+\lambda_2)-2}{2}} \\ &\times \sum_{j=0}^{1} (-i)^j \, \frac{\Gamma\left(\frac{1+i(\lambda+\lambda_1-\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda-\lambda_1+\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(-\lambda-\lambda_1-\lambda_2)+2j}{4}\right)}{\Gamma\left(\frac{1+i(-\lambda-\lambda_1+\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(-\lambda+\lambda_1-\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda+\lambda_1+\lambda_2)+2j}{4}\right)} \\ &\times \chi^j_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s), \end{aligned} \tag{17.3}$$

with

$$\begin{aligned} \chi^j_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s) :&= |s_1 - s_2|^{\frac{1}{2}(-1+i(\lambda_1+\lambda_2+\lambda))} |s_1 - s|^{\frac{1}{2}(-1+i(\lambda_1-\lambda_2-\lambda))} \\ &\times |s_2 - s|^{\frac{1}{2}(-1+i(-\lambda_1+\lambda_2-\lambda))} \left[\mathrm{sign}\left(\frac{s_1 - s_2}{(s - s_1)(s_2 - s)} \right) \right]^j . \end{aligned} \tag{17.4}$$

Actually, in [62, Theorem 5.3], we gave a more general formula, since symbols not necessarily even were considered as well: this will not be needed here, and it would complicate notations a little bit. However, though all computations have been properly carried in *loc.cit.*, we have not been very careful there in giving all justifications, especially in view of the fact that semi-convergent integrals had to be used. This was not really important in *loc.cit.*, where the main reason for our statement of Theorem 5.3 was to give some extra incentive towards the interest in the integral kernels in (17.4), which played an important role in other parts of this work.

We take this opportunity to complete the justification of the theorem under discussion. Our first lemma to that effect is Theorem 11.3 or, rather, the case when $k = 0$ of this theorem (then the function $C_j(\nu_1, \nu_2; k; i\lambda)$ from (11.27) simplifies a little bit, to the function $C_j(\nu_1, \nu_2; i\lambda)$ in (5.41)). In any case, using signed powers, as defined in (11.26) is necessary even when $k = 0$.

The next lemma is just (a particular case since we here consider only even symbols) Lemma 5.2 in [62], the proof of which involved only (absolutely) convergent integrals, and will not be rewritten here. We first generalize the Definitions (17.4) and (17.3) of $\chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)$ and $K_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)$, letting complex numbers ν_1 and ν_2 with possibly non-zero real parts be substituted for $i\lambda_1$ and $i\lambda_2$ everywhere. Of course, we denote the new functions obtained as $\chi^j_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)$ and $K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)$.

Lemma 17.2. *Assume that*

$$|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1, \quad \mathrm{Re}\ \nu_1 < 0, \quad \mathrm{Re}\ \nu_2 < 0\,. \tag{17.5}$$

Then, with $h_1(x,\xi) = |x|^{-1-\nu_1}$, $h_2(x,\xi) = |\xi|^{-1-\nu_2}$, *one has in the weak sense in* $\mathcal{S}'(\mathbb{R}^2)$

$$h_1 \# h_2 = \int_{-\infty}^{\infty} h_\lambda\, d\lambda\,, \tag{17.6}$$

with

$$h^\flat_\lambda(s) = \int_{\mathbb{R}^2} K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)\, |s_1|^{-1-\nu_1}\, ds_1\, ds_2\,. \tag{17.7}$$

Corollary 17.3. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be given, with* $ad - bc \neq 0$. *With the same assumptions as in Theorem* 11.3, *one has*

$$|ax + b\xi|^{-1-\nu_1} \# |cx + d\xi|^{-1-\nu_2} = \int_{-\infty}^{\infty} g_\lambda(x,\xi)\, d\lambda\,, \tag{17.8}$$

with

$$g^\flat_\lambda(s) = \int_{\mathbb{R}^2} K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)\, |as_1 + b|^{-1-\nu_1}\, |cs_2 + d|^{-1-\nu_2}\, ds_1\, ds_2\,. \tag{17.9}$$

Proof. Setting $\left(\begin{smallmatrix} c \\ d \end{smallmatrix}\right) = (ad - bc)\left(\begin{smallmatrix} c' \\ d' \end{smallmatrix}\right)$, it is no loss of generality to assume that $ad - bc = 1$. As a consequence of (2.5),

$$|ax + b\xi|^{-1-\nu_1} \# |cx + d\xi|^{-1-\nu_2} = \int_{-\infty}^{\infty} h_\lambda(ax + b\xi, cx + d\xi)\, d\lambda \tag{17.10}$$

and from (2.16), then (17.7),

$$\begin{aligned} h_\lambda(ax + b\xi, cx + d\xi) &= |cx + d\xi|^{-1-i\lambda}\, h^\flat_\lambda\left(\frac{ax + b\xi}{cx + d\xi}\right) \\ &= |cx + d\xi|^{-1-i\lambda} \int_{\mathbb{R}^2} K_{\nu_1,\nu_2;i\lambda}\left(s_1, s_2; \frac{ax + b\xi}{cx + d\xi}\right) |s_1|^{-1-\nu_1}\, ds_1\, ds_2\,. \end{aligned} \tag{17.11}$$

Now the kernel $K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)$ has the fundamental covariance property (immediate, *cf.* [62], (5.16)) that

$$K_{\nu_1,\nu_2;i\lambda}\left(\frac{as_1+b}{cs_1+d},\frac{as_2+b}{cs_2+d};\frac{a\frac{x}{\xi}+b}{c\frac{x}{\xi}+d}\right)$$
$$= |cs_1+d|^{1-\nu_1}\,|cs_2+d|^{1-\nu_2}\left|c\frac{x}{\xi}+d\right|^{1+i\lambda} K_{\nu_1,\nu_2;i\lambda}\left(s_1,s_2;\frac{x}{\xi}\right). \quad (17.12)$$

Performing a change of variables in (17.11), we thus get

$$h_\lambda(ax+b\xi,cx+d\xi) = |cx+d\xi|^{-1-i\lambda}$$
$$\times\int_{\mathbb{R}^2}|as_1+b|^{-1-\nu_1}\,|cs_2+d|^{-1-\nu_2}\left|c\frac{x}{\xi}+d\right|^{-1+i\lambda} K_{\nu_1,\nu_2;i\lambda}\left(s_1,s_2;\frac{x}{\xi}\right)ds_1\,ds_2\,, \quad (17.13)$$

i.e.,

$$|ax+b\xi|^{-1-\nu_1}\#|cx+d\xi|^{-1-\nu_2} = \int_{-\infty}^{\infty} g_\lambda(x,\xi)\,d\lambda \quad (17.14)$$

with

$$g_\lambda(x,\xi) = |\xi|^{-1-i\lambda}\int_{\mathbb{R}^2}|as_1+b|^{-1-\nu_1}\,|cs_2+d|^{-1-\nu_2}K_{\nu_1,\nu_2;i\lambda}\left(s_1,s_2;\frac{x}{\xi}\right)ds_1\,ds_2\,. \quad (17.15)$$

Applying (2.15) again, we are done. □

We need another lemma, so as to bound the last integral in (17.2).

Lemma 17.4. *Set*

$$I:=\int_{\mathbb{R}^3}|s_1-s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}$$
$$|s_2-s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\,|s-s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}\,|u_1(s_1)\,u_2(s_2)\,u(s)|\,ds_1\,ds_2\,ds\,. \quad (17.16)$$

Set

$$|||u_1|||_1 = \sup((1+|s_1|)^{1+\varepsilon_1}\,|u_1(s_1)|)\,,$$
$$|||u_2|||_2 = \sup((1+|s_2|)^{1+\varepsilon_2}\,|u_2(s_2)|)\,. \quad (17.17)$$

Assume that $|\varepsilon_1\pm\varepsilon_2|<1$, *so that in particular* $\|u_j\|_{L^\infty}\leq |||\,u_j|||_j$, $j=1,2$. *Then*

$$I\leq C\,|||u_1|||_1\;|||u_2|||_2\;\|u\|_{L^2}\,, \quad (17.18)$$

with a constant C *depending only on* ε_1, ε_2.

Proof. Set

$$v_1(s_1) = |s_1|^{-1-\varepsilon_1}\, u_1(s_1^{-1}),$$
$$v_2(s_2) = |s_2|^{-1-\varepsilon_2}\, u_2(s_2^{-1}),$$
$$v(s) = |s|^{-1}\, u(s^{-1})\,, \qquad (17.19)$$

so that $|||v_1|||_1 = |||u_1|||_1$, $|||v_2|||_2 = |||u_2|||_2$, $\|v\|_{L^2} = \|u\,\|_{L^2}$. If A is a measurable subset of $\mathbb{R}^3$, denote as I_A the same integral as I after the domain of integration has been changed to A. Set $A' = \{(s_1, s_2, s)\colon (s_1^{-1}, s_2^{-1}, s^{-1}) \in A\}$. Then an obvious change of variables shows that $I_{A'}$ is just the same as I_A after u_1, u_2 and u have been replaced by v_1, v_2 and v. To prove Lemma 17.4, it thus suffices to bound by the required product of norms the integral I_A in the case when $A = \{(s_1, s_2, s)\colon |s_1| \le 2,\ |s_2| \le 2\}$ or when $A = \{(s_1, s_2, s)\colon |s_1| \le 1,\ |s_2| \ge 2\}$. Consider the first case to start with: set

$$J(s) = \int_{\substack{|s_1|\le 2\\ |s_2|\le 2}} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, |s_2 - s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s - s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}\, ds_1\, ds_2\,. \qquad (17.20)$$

If $|s| \le 3$, then

$$J(s) \le \int_{\substack{|s_1|\le 5\\ |s_2|\le 5}} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, |s_2|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}\, ds_1\, ds_2$$
$$\le \int_{|s_1|\le 5} |s_1|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, ds_1 \int_{|t|\le \frac{5}{|s_1|}} |t|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |1-t|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, dt\,. \qquad (17.21)$$

Now, in view of the assumptions made about ε_1, ε_2, the last integral is less than a constant if $\varepsilon_2 < 0$, less than $C_1 + C_2\, |\log |s_1||$ if $\varepsilon_2 = 0$, and less than a constant times $|s_1|^{-\varepsilon_2}$ if $\varepsilon_2 > 0$. This shows that $J(s)$ is bounded for $|s| \le 3$; if $|s| \ge 3$,

$$J(s) \le C\, |s|^{-1} \int_{\substack{|s_1|\le 2\\ |s_2|\le 2}} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, ds_1\, ds_2$$
$$\le C\, |s|^{-1} \qquad (17.22)$$

since $|\varepsilon_1 + \varepsilon_2| < 1$, so that $J(s) \le C\,(1 + |s|)^{-1}$ for all s and

$$I_A \le \sup\nolimits_{|s_1|\le 2} |u_1(s_1)| \times \sup\nolimits_{|s_2|\le 2} |u_2(s_2)| \times \int_{-\infty}^{\infty} (1 + |s|)^{-1}\, |u(s)|\, ds$$
$$\le C\, \|u_1\|_{L^\infty}\ \|u_2\|_{L^\infty}\ \|u\|_{L^2}\,. \qquad (17.23)$$

We now assume that $A = \{(s_1, s_2, s)\colon |s_1| \le 1,\ |s_2| \ge 2\}$. Then

$$I_A \le C\, \|u_1\|_{L^\infty} \int_{\substack{|s_1|\le 1\\ |s_2|\ge 2}} |s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, |s_2 - s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s - s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}$$
$$|u_2(s_2)\, u(s)|\, ds_1\, ds_2\, ds\,. \qquad (17.24)$$

Now

$$\begin{aligned}
&\int_{|s_2|\geq 2} (1+|s_2|)^{-1-\varepsilon_2}\, |s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}}\, |s_2-s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, ds_2 \\
&\leq \int_{|s_2|\geq 2} |s_2|^{\frac{-3+\varepsilon_1-\varepsilon_2}{2}}\, |s_2-s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, ds_2 \\
&\leq C \int_{2\leq |s_2|\leq \frac{|s|}{2}} |s_2|^{\frac{-3+\varepsilon_1-\varepsilon_2}{2}}\, (1+|s|)^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, ds_2 \\
&\quad + C \int_{|s_2|\geq \max(2,\frac{|s|}{2})} (1+|s|)^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s_2|^{-1+\varepsilon_1-\varepsilon_2}\, |s_2-s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, ds_2 \\
&\leq C\,(1+|s|)^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}} \qquad (17.25)
\end{aligned}$$

since $|\varepsilon_1-\varepsilon_2|<1$. Thus

$$\begin{aligned}
I_A &\leq C\, \|u_1\|_{L^\infty} \int_{|s_1|\leq 1} (1+|s|)^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s-s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}\, |u(s)|\, ds_1\, ds \\
&\quad \times \sup\left((1+|s_2|)^{1+\varepsilon_2}\, |u_2(s_2)|\right) \\
&\leq C\, \|u_1\|_{L^\infty}\, |||u_2|||_2 \times \int_{-\infty}^{\infty} K(s)\, |u(s)|\, ds \qquad (17.26)
\end{aligned}$$

with

$$\begin{aligned}
K(s) &= \int_{|s_1|\leq 1} (1+|s|)^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}\, |s-s_1|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}}\, ds_1 \\
&\leq C\,(1+|s|)^{-1}\,. \qquad (17.27)
\end{aligned}$$

□

We shall apply Lemma 17.4 presently with $\varepsilon_1 = \text{Re}\ \nu_1$, $\varepsilon_2 = \text{Re}\ \nu_2$ and $u_1 = (h_1)^\flat_{-i\nu_1}$, $u_2 = (h_2)^\flat_{-i\nu_2}$, finally $u = (h_3)^\flat_\lambda$, with h_1, h_2, h_3 three functions in $\mathcal{S}_{\text{even}}(\mathbb{R}^2)$. Recall from (2.18), (2.19) that

$$u_1(s_1) = \frac{1}{2\pi} \int_0^\infty t^{\nu_1}\, h_1(ts_1, t)\, dt\ : \qquad (17.28)$$

then w_1, defined as $w_1(s_1) = |s_1|^{-1-\nu_1}\, u_1(s_1^{-1})$, can be written as

$$w_1(s_1) = \frac{1}{2\pi} \int_0^\infty t^{\nu_1}\, h_1(t, ts_1)\, dt \qquad (17.29)$$

(one may compare w_1 to v_1 in (17.19)), and the norm $|||u_1|||_1$ as defined in (17.17) is equivalent to $\|u_1\|_{L^\infty} + \|w_1\|_{L^\infty}$. Given $\varepsilon_1 \in]-1,1[$, one clearly has, since $\text{Re}\ \nu_1 = \varepsilon_1$,

$$\begin{aligned}
|||u_1|||_1 &\leq \sup\left(s_1 \mapsto \int_0^\infty t^{\varepsilon_1}\, [\,|h_1(ts_1,t)| + |h_1(t,ts_1)|\,]\, dt\right) \\
&\leq C(\varepsilon_1)\, \sup((1+x^2+\xi^2)\, |h_1(x,\xi)|)\,. \qquad (17.30)
\end{aligned}$$

Of course, something entirely similar holds with u_2 or u, after one has substituted h_2 or h_3 for h_1, and ν_2 or $i\lambda$ for ν_1.

The basic idea towards the proof of Theorem 17.1 is to reduce it to the case when the two factors are polarized in the sense described between (11.25) and (11.26), *i.e.,* when each of them only depends on some linear combination of x and ξ. The formula (2.9) for the $\mathcal{G}$-transform provides such a decomposition for each factor: indeed, it suffices to write

$$h(x,\xi) = 2\int_{\mathbb{R}^2} (\mathcal{G}h)(y,\eta)\, e^{4i\pi(x\eta - y\xi)}\, dy\, d\eta\,, \tag{17.31}$$

and to use polar coordinates, to get the decomposition

$$h(x,\xi) = \int_0^\pi h^\theta(x\sin\theta - \xi\cos\theta)\, d\theta\,, \tag{17.32}$$

with

$$h^\theta(x) = 2\int_{-\infty}^{\infty} (\mathcal{G}h)(t\cos\theta, t\sin\theta)\, e^{4i\pi tx}\, |t|\, dt\,. \tag{17.33}$$

One may observe that (still under the assumption that h lies in $\mathcal{S}(\mathbb{R}^2)$), the function $h^\theta(x)$ is a C^∞ function of (θ, x), and that it satisfies the estimates

$$x \mapsto (1+|x|)^{\alpha+j} \left(\frac{\partial}{\partial\theta}\right)^j h^\theta(x) \in L^2(\mathbb{R}), \qquad \alpha < \frac{3}{2}\,. \tag{17.34}$$

Conversely, assume that a function h on $\mathbb{R}^2$ can be written as

$$h(x,\xi) = \int_0^\pi g^\theta(x\sin\theta - \xi\cos\theta)\, d\theta \tag{17.35}$$

for some function $g^\theta \in L^1(\mathbb{R})$, depending as such in a continuous way on θ. We then compute $\mathcal{G}h$: if $f(x,\xi) = a(x)$, it is immediate that $(\mathcal{G}f)(x,\xi) = \delta(x)\, \hat{a}(2\xi)$, from which, after a rotation of coordinates, one gets

$$\langle \mathcal{G}h, \phi\rangle = \int_0^\pi d\theta \int_{-\infty}^{\infty} \widehat{g^\theta}(2\xi)\, \phi(\xi\cos\theta, \xi\sin\theta)\, d\xi \tag{17.36}$$

for any function $\phi \in \mathcal{S}(\mathbb{R}^2)$. Thus

$$|\xi|\, (\mathcal{G}h)(\xi\cos\theta, \xi\sin\theta) = \widehat{g^\theta}(2\xi) \ : \tag{17.37}$$

it follows that g^θ has to coincide with h^θ as defined by (17.33).

We now need to connect the decomposition of a function $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$ into homogeneous parts to the decomposition of a symbol into polarized ones, as provided by (17.32), (17.33). Using (17.33) as a definition of $(h_{-i\nu})^\theta$, and remembering that $\mathcal{G}\mathcal{E} = -\mathcal{E}\mathcal{G}$, one thus gets, for every complex ν with $-1 < \mathrm{Re}\ \nu < 1$ and $x \neq 0$,

$$\begin{aligned}(h_{-i\nu})^\theta(x) &= 2\int_{-\infty}^{\infty} (\mathcal{G}h)_{i\nu}(t\cos\theta, t\sin\theta)\, e^{4i\pi tx}\, |t|\, dt \\ &= \frac{1}{\pi}\int_0^\infty r^{-\nu}\, dr \int_{-\infty}^{\infty} e^{4i\pi tx}\, |t|\, (\mathcal{G}h)(rt\cos\theta, rt\sin\theta)\, dt \\ &= \frac{1}{2\pi}\int_0^\infty r^{-\nu-2}\, h^\theta\left(\frac{x}{r}\right) dr \end{aligned} \tag{17.38}$$

or, finally, noting from (17.33) that, in our case, $(h_{-i\nu})^\theta$ is an even function,

$$(h_{-i\nu})^\theta(x) = |x|^{-1-\nu} \times \frac{1}{2\pi}\int_0^\infty r^\nu\, h^\theta(r)\, dr\,, \tag{17.39}$$

where the integral converges thanks to (17.34). Since

$$(h_{-i\nu})(x, \xi) = \int_0^\pi (h_{-i\nu})^\theta(x\sin\theta - \xi\cos\theta)\, d\theta\,, \tag{17.40}$$

one has

$$\begin{aligned} h^\flat_{-i\nu}(s) &= \int_0^\pi (h_{-i\nu})^\theta(s\sin\theta - \cos\theta)\, d\theta \\ &= \int_0^\pi \left[|s\sin\theta - \cos\theta|^{-1-\nu} \times \frac{1}{2\pi}\int_0^\infty r^\nu\, h^\theta(r)\, dr \right] d\theta\,. \end{aligned} \tag{17.41}$$

Proof of Theorem 17.1. Set

$$\begin{aligned} A_j(\nu_1, \nu_2; i\lambda) &= 2^{-\frac{3}{2}}\, (2\pi)^{\frac{\nu_1+\nu_2-i\lambda-2}{2}} \\ &\times (-i)^j\, \frac{\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda+2j}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda+2j}{4}\right)\Gamma\left(\frac{1-\nu_1-\nu_2-i\lambda+2j}{4}\right)}{\Gamma\left(\frac{1-\nu_1+\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2-i\lambda+2j}{4}\right)\Gamma\left(\frac{1+\nu_1+\nu_2+i\lambda+2j}{4}\right)}\,, \end{aligned} \tag{17.42}$$

so that (17.3), extended by complex continuation as explained just before Lemma 17.2, reads

$$K_{\nu_1,\nu_2;i\lambda}(s_1, s_2; s) = \sum_{j=0}^{1} A_j(\nu_1, \nu_2; i\lambda)\, \chi^j_{\nu_1,\nu_2;i\lambda}(s_1, s_2; s)\,. \tag{17.43}$$

Set $\nu_1 = \varepsilon_1 + i\lambda_1$ and $\nu_2 = \varepsilon_2 + i\lambda_2$, and assume in all that follows that $|\varepsilon_1 \pm \varepsilon_2| < 1$. We first note the estimate

$$\begin{aligned} |A_j(\nu_1, \nu_2; i\lambda)| &\le C\left|1 + \frac{\nu_1 - \nu_2 + i\lambda}{2}\right|^{\frac{\varepsilon_1-\varepsilon_2}{2}} \\ &\times \left|1 + \frac{-\nu_1 + \nu_2 + i\lambda}{2}\right|^{\frac{-\varepsilon_1+\varepsilon_2}{2}} \left|1 + \frac{\nu_1 + \nu_2 + i\lambda}{2}\right|^{\frac{-\varepsilon_1-\varepsilon_2}{2}}\,, \end{aligned} \tag{17.44}$$

a consequence of (11.31); a rougher estimate, sufficient for our purposes, is

$$|A_j(\nu_1,\nu_2;i\lambda)| \leq C\ \left[(1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda^2)\right]^{\frac{3}{4}}\,. \tag{17.45}$$

The claim in Theorem 17.1 is that, given any three functions h_1, h_2, $h_3 \in \mathcal{S}(\mathbb{R}^2)$, the first two even (it then does not harm the generality to assume that so is the third one), one has

$$\langle h_1 \# h_2, h_3\rangle = \int_{-\infty}^{\infty} \langle h_\lambda, h_3\rangle\ d\lambda\,, \tag{17.46}$$

where $h_\lambda^\flat$ is the integral defined by (17.2).

We first show that, indeed, $h_\lambda^\flat$ is well defined, for all λ, as an element of $L^2(\mathbb{R})$. Since $(\mathcal{E}h_j)_{\lambda_j}^\flat = -\frac{\lambda_j}{2\pi}(h_j)_{\lambda_j}^\flat$, (17.2) can also be written as

$$h_\lambda^\flat(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{j=0,1} \left(1-\frac{i\lambda_1}{2}\right)^{-3} \left(1-\frac{i\lambda_2}{2}\right)^{-3} A_j(i\lambda_1,i\lambda_2;i\lambda)\,d\lambda_1\,d\lambda_2$$

$$\int_{\mathbb{R}^2} \chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)\ ((1+i\pi\mathcal{E})^3(h_1))_{\lambda_1}^\flat(s_1)\,((1+i\pi\mathcal{E})^3(h_2))_{\lambda_2}^\flat(s_2)\,ds_1\,ds_2\,. \tag{17.47}$$

Now, in view of Lemma 17.4 and (17.30), the last integral on the right defines a function of s in $L^2(\mathbb{R})$, with a norm bounded by a constant independent of λ_1, λ_2: using then (17.45), one can carry the $d\lambda_1\,d\lambda_2$-integration, which shows that $h_\lambda^\flat$ is well defined as an element of $L^2(\mathbb{R})$, with a norm bounded by a constant times $(1+\lambda^2)^{\frac{3}{4}}$. Now, using (2.16), next (2.14),

$$\begin{aligned}\langle h_\lambda, h_3\rangle &= \int_{\mathbb{R}^2} h_3(x,\xi)\,|\xi|^{-1-i\lambda}\,h_\lambda^\flat\left(\frac{x}{\xi}\right)\,dx\,d\xi\\ &= 2\int_{-\infty}^{\infty} h_\lambda^\flat(s)\,ds\int_0^\infty h_3(st,t)\,t^{-i\lambda}\,dt\\ &= 4\pi\int_{-\infty}^{\infty} h_\lambda^\flat(s)\,(h_3)_{-\lambda}^\flat(s)\,ds\,.\end{aligned} \tag{17.48}$$

Using again $(h_3)_{-\lambda}^\flat = (1+\frac{i\lambda}{2})^{-3}\,((1+i\pi\mathcal{E})^3\,h_3)_{-\lambda}^\flat$, one sees that the integral on the right-hand side of (17.46) is convergent, and that it can be written as

$$\int_{-\infty}^{\infty}\langle h_\lambda, h_3\rangle\,d\lambda = 4\pi\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}h_\lambda^\flat(s)\,(h_3)_{-\lambda}^\flat(s)\,ds$$

$$= 4\pi\int_{-\infty}^{\infty}d\lambda\int_{-\infty}^{\infty}(h_3)_{-\lambda}^\flat(s)\,ds\int_{\mathbb{R}^2}\sum_{j=0}^{1}A_j(i\lambda_1,i\lambda_2;i\lambda)\,d\lambda_1\,d\lambda_2$$

$$\int_{\mathbb{R}^2}\chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)(h_1)_{\lambda_1}^\flat(s_1)\,(h_2)_{\lambda_2}^\flat(s_2)\,ds_1\,ds_2\,, \tag{17.49}$$

a convergent 6-tuple integral.

The estimates which led to (17.49), primarily based on Lemma 17.4 and (17.45), show that, fixing ε_1 and ε_2 with $|\varepsilon_1 \pm \varepsilon_2| < 1$ and substituting everywhere $\nu_1 = \varepsilon_1 + i\lambda_1$ and $\nu_2 = \varepsilon_2 + i\lambda_2$ for $i\lambda_1$ and $i\lambda_2$ would let the integral on the right-hand side of (17.49) remain convergent. Then a contour deformation shows (using (17.43) again) that

$$\int_{-\infty}^{\infty} \langle h_\lambda, h_3\rangle \, d\lambda = 4\pi \int_{\mathrm{Re}\ \nu_1=\varepsilon_1} \int_{\mathrm{Re}\ \nu_2=\varepsilon_2} \frac{d\nu_1}{i}\,\frac{d\nu_2}{i}$$
$$\int_{-\infty}^{\infty} d\lambda \int_{\mathbb{R}^3} K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)\,(h_1)^\flat_{-i\nu_1}(s_1)\,(h_2)^\flat_{-i\nu_2}(s_2)\,(h_3)^\flat_{-\lambda}(s)\,ds_1\,ds_2\,ds \tag{17.50}$$

for any such pair $(\varepsilon_1,\,\varepsilon_2)$. We choose $\varepsilon_1 < 0$, $\varepsilon_2 < 0$ (still with $\varepsilon_1 + \varepsilon_2 > -1$) to be in a position to apply Lemma 17.4 presently.

Next, decompose $(h_1)_{-i\nu_1}$ and $(h_2)_{-i\nu_2}$ into polarized symbols according to (17.40):

$$(h_1)_{-i\nu_1}(x,\xi) = \int_0^\pi ((h_1)_{-i\nu_1})^{\theta_1}(x\sin\theta_1 - \xi\cos\theta_1)\,d\theta_1\ : \tag{17.51}$$

in the corresponding decomposition of $(h_2)_{-i\nu_2}$, we use of course the angle θ_2 instead of θ_1. Then, according to (17.39),

$$(h_1)^\flat_{-i\nu_1}(s_1) = \int_0^\pi b(\nu_1,\theta_1)\ |s_1\sin\theta_1 - \cos\theta_1|^{-1-\nu_1}\,d\theta_1\,, \tag{17.52}$$

where we have set

$$b(\nu_1,\theta_1) = \frac{1}{2\pi}\int_0^\infty r_1^{\nu_1}\,(h_1)^{\theta_1}(r_1)\,dr_1\,. \tag{17.53}$$

From (17.34) it follows that, given $\varepsilon_1 \in]-1,1[$, $b(\nu_1,\theta_1)$ is bounded for $\theta_1 \in [0,\pi]$ and $\mathrm{Re}\,\nu_1 = \varepsilon_1$: powers of $\mathrm{Im}\ \nu_1$ can also be saved, so as to ensure the convergence of the following integral with respect to $\frac{d\nu_1}{i}$, by the same trick as the one in (17.47). Thus

$$\int_{-\infty}^{\infty} \langle h_\lambda, h_3\rangle\, d\lambda = 4\pi \int_{\substack{\mathrm{Re}\ \nu_1=\varepsilon_1\\ \mathrm{Re}\ \nu_2=\varepsilon_2}} \frac{d\nu_1}{i}\,\frac{d\nu_2}{i}\int_0^\pi\int_0^\pi b(\nu_1,\theta_1)\,b(\nu_2,\theta_2)\,d\theta_1\,d\theta_2$$
$$\int_{-\infty}^{\infty} d\lambda\int_{\mathbb{R}^3} K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)\,|s_1\sin\theta_1-\cos\theta_1|^{-1-\nu_1}\,|s_2\sin\theta_2-\cos\theta_2|^{-1-\nu_2}$$
$$(h_3)^\flat_{-\lambda}(s)\,ds_1\,ds_2\,ds\,, \tag{17.54}$$

provided we show the convergence of the new integral: this requires some more care, in view of the $d\theta_1\,d\theta_2$-integration, and of the singularity which is to be expected when $\theta_2 - \theta_1 \in \pi\mathbb{Z}$, *i.e.*, when the linear forms $x \sin\theta_1 - \xi\cos\theta_1$ and $x\sin\theta_2 - \xi\cos\theta_2$ are not transversal (all our preceding lemmas have been based on the reduction to the case when these two forms were simply x and ξ). Recalling (17.43), and noting that

$$|\chi^j_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)| = \chi^0_{\varepsilon_1,\varepsilon_2;0}(s_1,s_2;s) \tag{17.55}$$

when $\mathrm{Re}\,\nu_1 = \varepsilon_1$, $\mathrm{Re}\,\nu_2 = \varepsilon_2$, one sees that one has to find an appropriate bound for the integral

$$I(\theta_1,\theta_2) = \int_{\mathbb{R}^3} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}} |s_1 - s|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}} |s_2 - s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}$$
$$|s_1\sin\theta_1 - \cos\theta_1|^{-1-\varepsilon_1} |s_2\sin\theta_2 - \cos\theta_2|^{-1-\varepsilon_2} |(h_3)^\flat_{-\lambda}(s)|\, ds_1\, ds_2\, ds\,. \tag{17.56}$$

Performing when $\theta_2 - \theta_1 \notin \mathbb{Z}$ the change of variables $s_1 \mapsto \frac{as_1+b}{cs_1+d}$, $s_2 \mapsto \frac{as_2+b}{cs_2+d}$, $s \mapsto \frac{as+b}{cs+d}$, where $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G$ is defined as $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} -\delta^{-1}\cos\theta_2 & \cos\theta_1 \\ -\delta^{-1}\sin\theta_2 & \sin\theta_1\end{smallmatrix}\right)$ with $\delta = \sin(\theta_2 - \theta_1)$, one sees that $s_1\sin\theta_1 - \cos\theta_1$ transforms to $\frac{s_1}{cs_1+d}$, that $s_2\sin\theta_2 - \cos\theta_2$ transforms to $\frac{\delta}{cs_2+d}$, thus

$$I(\theta_1,\theta_2) = |\delta|^{-1-\varepsilon_2} \int_{\mathbb{R}^3} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}} |s_1 - s|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}} |s_2 - s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}$$
$$|s_1|^{-1-\varepsilon_1} |cs+d|^{-1} \left|(h_3)^\flat_{-\lambda}\left(\frac{as+b}{cs+d}\right)\right| ds_1\, ds_2\, ds\,. \tag{17.57}$$

Now, in the proof of [62], Lemma 5.2 (our present Lemma 17.2), we have computed, under the assumptions $\varepsilon_1 < 0$, $\varepsilon_2 < 0$, $|\varepsilon_1 - \varepsilon_2| < 1$, the integral

$$\int_{\mathbb{R}^3} |s_1 - s_2|^{\frac{-1+\varepsilon_1+\varepsilon_2}{2}} |s_1 - s|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}} |s_2 - s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}} |s_1|^{-1-\varepsilon_1}\, ds_1\, ds_2$$
$$= C(\varepsilon_1,\, \varepsilon_2)\, |s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}} \tag{17.58}$$

for some constant $C(\varepsilon_1,\, \varepsilon_2)$. Since, as a consequence of (2.14) and of the fact that $h_3 \in \mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$, one has $|(h_3)^\flat_{-\lambda}(s)| \leq C\,(1+s^2)^{-\frac{1}{2}}$ for some $C > 0$, it follows that

$$I(\theta_1,\theta_2) \leq C(\varepsilon_1,\, \varepsilon_2)\, |\delta|^{-1-\varepsilon_2} \times \int_{-\infty}^{\infty} |s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}} |cs+d|^{-1} \left(1 + (\frac{as+b}{cs+d})^2\right)^{-\frac{1}{2}} ds$$
$$= C(\varepsilon_1,\, \varepsilon_2)\, |\delta|^{-1-\varepsilon_2}\, J(\theta_1,\theta_2)\,, \tag{17.59}$$

with

$$J(\theta_1,\theta_2) = \int_{-1}^{1} |s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}} \left[(as+b)^2 + (cs+d)^2\right]^{-\frac{1}{2}} ds$$
$$+ \int_{-1}^{1} |s|^{\frac{-1+\varepsilon_1-\varepsilon_2}{2}} \left[(bs+a)^2 + (ds+c)^2\right]^{-\frac{1}{2}} ds\,. \tag{17.60}$$

On the one hand, the L^2-norm of the function $s \mapsto [(as+b)^2+(cs+d)^2]^{-\frac{1}{2}}$ or $[(bs+a)^2+(ds+c)^2]^{-\frac{1}{2}}$ does not depend on the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. Its L^∞-norm, however, does, as follows: the minimum of the function $s \mapsto (as+b)^2+(cs+d)^2$ as s describes the real line turns out to be $(a^2+c^2)^{-1}$, in our case δ^2; similarly, the minimum of $(bs+a)^2+(ds+c)^2$ is 1. Thus

$$\max_{s\in\mathbb{R}} \left[(as+b)^2+(cs+d)^2\right]^{-\frac{1}{2}} \leq |\delta|^{-1} \tag{17.61}$$

and, if $\frac{1}{p}+\frac{1}{q}=1$, $p \in]2,\infty[$, one has $\|s \mapsto [(as+b)^2+(cs+d)^2]^{-\frac{1}{2}}\|_{L^p} \leq \pi^{\frac{1}{p}}\,|\delta|^{\frac{2-p}{p}}$. Provided that the function $s \mapsto |s|^{\frac{-1-\varepsilon_1+\varepsilon_2}{2}}$ lies in $L^q([-1,1])$, *i.e.*, $\frac{1-\varepsilon_1+\varepsilon_2}{2} > \frac{1}{p}$, one thus has $J(\theta_1,\theta_2) \leq C\,|\delta|^{\frac{2-p}{p}}$ and $I(\theta_1,\theta_2) \leq C\,|\delta|^{-2-\varepsilon_2+\frac{2}{p}}$. Since $\delta = \sin(\theta_2-\theta_1)$, the integral on the right-hand side of (17.54) will be convergent if one can choose p with $\frac{p}{2} > \frac{1}{1-\varepsilon_1+\varepsilon_2}$ and $\varepsilon_2 < \frac{2}{p}-1$, *i.e.*, $\frac{p}{2} < \frac{1}{1+\varepsilon_2}$ (still with $|\varepsilon_1 \pm \varepsilon_2| < 1$, $\varepsilon_1 < 0$ and $\varepsilon_2 < 0$), which is possible.

In accordance with (17.8), we then set, if $\theta_1-\theta_2 \notin \pi\mathbb{Z}$,

$$|x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,\#|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2} = \int_{-\infty}^{\infty} (g_{\theta_1,\theta_2}^{\nu_1,\nu_2})_\lambda(x,\xi)\,d\lambda\,, \tag{17.62}$$

a definition of the function $(g_{\theta_1,\theta_2}^{\nu_1,\nu_2})_\lambda$: (17.48) and Corollary 17.3 show that

$$\begin{aligned}\left\langle \left(g_{\theta_1,\theta_2}^{\nu_1,\nu_2}\right)_\lambda, h_3 \right\rangle &= 4\pi \int_{-\infty}^{\infty} (g_{\theta_1,\theta_2}^{\nu_1,\nu_2})^\flat_\lambda(s)\,(h_3)^\flat_{-\lambda}(s)\,ds \\ &= 4\pi \int_{-\infty}^{\infty} (h_3)^\flat_{-\lambda}(s)\,ds \int_{\mathbb{R}^2} K_{\nu_1,\nu_2;i\lambda}(s_1,s_2;s)\,|s_1\sin\theta_1-\cos\theta_1|^{-1-\nu_1} \\ &\qquad |s_2\sin\theta_2-\cos\theta_2|^{-1-\nu_2}\,ds_1\,ds_2\,. \end{aligned}\tag{17.63}$$

Thus (17.54) reduces to

$$\begin{aligned}\int_{-\infty}^{\infty} \langle h_\lambda, h_3\rangle\,d\lambda = \int_{\mathrm{Re}\ \nu_1=\varepsilon_1} \int_{\mathrm{Re}\ \nu_2=\varepsilon_2} \frac{d\nu_1}{i}\,\frac{d\nu_2}{i} \int_0^\pi \int_0^\pi b(\nu_1,\theta_1)\,b(\nu_2,\theta_2) \\ \left\langle\, |x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,\#|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2}\,,\, h_3 \,\right\rangle\,d\theta_1\,d\theta_2\,.\end{aligned}\tag{17.64}$$

On the other hand, using (2.13) and a contour integration, next (17.40), finally (17.41) and (2.16), we get, whenever $\varepsilon_1 > -1$,

$$\begin{aligned} h_1(x,\xi) &= \int_{\mathrm{Re}\ \nu_1=\varepsilon_1} (h_1)_{-i\nu_1}(x,\xi)\,\frac{d\nu_1}{i} \\ &= \int_{\mathrm{Re}\ \nu_1=\varepsilon_1} \frac{d\nu_1}{i} \int_0^\pi (h_1)^{\theta_1}_{-i\nu_1}(x\sin\theta_1-\xi\cos\theta_1)\,d\theta_1 \\ &= \int_{\mathrm{Re}\ \nu_1=\varepsilon_1} \frac{d\nu_1}{i} \int_0^\pi b(\nu_1,\theta_1)\,|x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,d\theta_1\,. \end{aligned}\tag{17.65}$$

Thus

$$\langle h_1 \# h_2, h_3\rangle = \int_{\mathrm{Re}\ \nu_1=\varepsilon_1}\int_{\mathrm{Re}\ \nu_2=\varepsilon_2} \frac{d\nu_1}{i}\,\frac{d\nu_2}{i} \\ \left\langle \left(\int_0^\pi b(\nu_1,\theta_1)\,|x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,d\theta_1\right) \# \right. \\ \left. \left(\int_0^\pi b(\nu_2,\theta_2)|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2}\,d\theta_2\right)\ ,\ h_3\right\rangle\ , \tag{17.66}$$

and the only remaining problem is showing that, given ν_1 and ν_2 with $-1 < \mathrm{Re}\ \nu_1 < 0$, $-1 < \mathrm{Re}\ \nu_2 < 0$, $|\mathrm{Re}\ (\nu_1 \pm \nu_2)| < 1$, and $h_3 \in \mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$, one has

$$\left\langle \left(\int_0^\pi b(\nu_1,\theta_1)\,|x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,d\theta_1\right) \# \right. \\ \left. \left(\int_0^\pi b(\nu_2,\theta_2)\,|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2}\,d\theta_2\right)\ ,h_3\right\rangle = \int_0^\pi\int_0^\pi b(\nu_1,\theta_1)\,b(\nu_2,\theta_2) \\ \left\langle\, |x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,\#|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2}\ ,\ h_3\,\right\rangle\,d\theta_1\,d\theta_2\,. \tag{17.67}$$

We first show that the two sides of this equation depend analytically on ν_1, ν_2 in the larger domain $-1 < \mathrm{Re}\ \nu_1 < 0$, $-1 < \mathrm{Re}\ \nu_2 < 0$, $|\mathrm{Re}\ (\nu_1 - \nu_2)| < 1$, $|\mathrm{Re}\ (\nu_1 - \nu_2)| - \mathrm{Re}\ (\nu_1 + \nu_2) < 2$. Concerning the left-hand side, recall from (17.52) that the integral that appears as the first factor there is none other than $(h_1)_{-i\nu_1}(x,\xi)$: in view of the direct Definition (2.18) of such a symbol, it can be decomposed, since it is smooth outside 0 and homogeneous of degree $-1-\nu_1$, as the sum of an integrable symbol and a smooth symbol with bounded derivatives of all orders: each of the two terms thus produces a bounded operator on $L^2(\mathbb{R})$.

To study the right-hand side, we need a lemma.

Lemma 17.5. *Assume that* $-1 < \mathrm{Re}\ \nu_1 < 0$, $-1 < \mathrm{Re}\ \nu_2 < 0$, $|\mathrm{Re}\ (\nu_1 - \nu_2)| < 1$, $|\mathrm{Re}\ (\nu_1 - \nu_2)| - \mathrm{Re}\ (\nu_1 + \nu_2) < 2$, *and let* $h_3 \in \mathcal{S}(\mathbb{R}^2)$. *One has, for all* θ_1 *and* θ_2 *in* $]0,\pi[$,

$$\left|\left\langle\, |x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1}\,\#|x\sin\theta_2-\xi\cos\theta_2|^{-1-\nu_2}\ ,\ h_3\,\right\rangle\right| \\ \leq C\,|\sin(\theta_2-\theta_1)|^{-1-\min(\mathrm{Re}\ \nu_1,\mathrm{Re}\ \nu_2)} \tag{17.68}$$

with some $C > 0$ *independent of* θ_1, θ_2.

Proof. We first give a slightly modified version of Theorem 11.3, using a contour deformation $i\lambda \mapsto -\varepsilon + i\lambda$, $0 < \varepsilon < 1$, ending up with

$$|x|^{-1-\nu_1}\#|\xi|^{-1-\nu_2} = \int_{-\infty}^{\infty} h_{i\varepsilon+\lambda}\,d\lambda, \tag{17.69}$$

with

$$h_{i\varepsilon+\lambda}(x,\xi) = \sum_{j=0,1} C_j(\nu_1,\nu_2;-\varepsilon+i\lambda)\,|x|_j^{\frac{-1+\varepsilon-\nu_1+\nu_2-i\lambda}{2}}\,|\xi|_j^{\frac{-1+\varepsilon+\nu_1-\nu_2-i\lambda}{2}}\,. \quad (17.70)$$

Looking at (11.27) and (11.29), we find that this integral decomposition is valid under the new conditions

$$-1 < \text{Re}\,\nu_1 < 0,\ -1 < \text{Re}\,\nu_2 < 0,\ |\text{Re}\,(\nu_1-\nu_2)| < 1-\varepsilon,\ \ |\text{Re}\,(\nu_1+\nu_2)| > -1-\varepsilon\,. \quad (17.71)$$

This leads to the following modified version of (17.62):

$$|x\sin\theta_1 - \xi\cos\theta_1|^{-1-\nu_1}\,\#|x\sin\theta_2 - \xi\cos\theta_2|^{-1-\nu_2} = \int_{-\infty}^{\infty} (g_{\theta_1,\theta_2}^{\nu_1,\nu_2})_{i\varepsilon+\lambda}(x,\xi)\,d\lambda\,, \quad (17.72)$$

with

$$(g_{\theta_1,\theta_2}^{\nu_1,\nu_2})_{i\varepsilon+\lambda}(x,\xi) = |\sin(\theta_2-\theta_1)|_j^{\frac{-1-\varepsilon-\nu_1-\nu_2+i\lambda}{2}} \sum_{j=0,1} C_j(\nu_1,\nu_2;-\varepsilon+i\lambda)$$
$$|x\sin\theta_1 - \xi\cos\theta_1|_j^{\frac{-1+\varepsilon-\nu_1+\nu_2-i\lambda}{2}}\,|x\sin\theta_2 - \xi\cos\theta_2|_j^{\frac{-1+\varepsilon+\nu_1-\nu_2-i\lambda}{2}}\,, \quad (17.73)$$

where $C_j(\nu_1,\nu_2;-\varepsilon+i\lambda)$ is defined from (11.27) by analytic continuation: indeed, when the matrix $\begin{pmatrix}\sin\theta_1 & -\cos\theta_1\\ \sin\theta_2 & -\cos\theta_2\end{pmatrix}$ lies in G, *i.e.*, when $\sin(\theta_2-\theta_1)=1$, this follows from (17.70) together with (2.5). If this is not the case, a simple homogeneity argument will do, after one has divided the first row of the matrix that precedes by $\sin(\theta_2-\theta_1)$.

Consider, for large N, the integral

$$I = \int_{\mathbb{R}^2} |x\sin\theta_1 - \xi\cos\theta_1|^{\frac{-1+\varepsilon-\text{Re}\,(\nu_1-\nu_2)}{2}}\,|x\sin\theta_2 - \xi\cos\theta_2|^{\frac{-1+\varepsilon+\text{Re}\,(\nu_1-\nu_2)}{2}}$$
$$(1+x^2+\xi^2)^{-N}\,dx\,d\xi\,. \quad (17.74)$$

Assuming for instance Re $\nu_2 \le$ Re ν_1, one may write the integral I, setting $\theta=\theta_2-\theta_1$, as

$$\int_{\mathbb{R}^2} |\xi|^{\frac{-1+\varepsilon-\text{Re}\,(\nu_1-\nu_2)}{2}}\,|x\sin\theta - \xi\cos\theta|^{\frac{-1+\varepsilon+\text{Re}\,(\nu_1-\nu_2)}{2}}\,(1+x^2+\xi^2)^{-N}\,dx\,d\xi$$
$$\le C\,|\sin\theta|^{\frac{-1+\varepsilon+\text{Re}\,(\nu_1-\nu_2)}{2}}\,\sup_\xi \int_{-\infty}^{\infty}(1+x^2)^{-\frac{N}{2}}\,|x-\xi\,\text{cotan}\,\theta|^{\frac{-1+\varepsilon+\text{Re}\,(\nu_1-\nu_2)}{2}}\,dx$$
$$\le C\,|\sin\theta|^{\frac{-1+\varepsilon+\text{Re}\,(\nu_1-\nu_2)}{2}}\,. \quad (17.75)$$

Thus, recalling from (11.31) that the coefficient $C_j(\nu_1,\nu_2;-\varepsilon+i\lambda)$ is bounded by a constant times $(1+|\lambda|)^{\frac{1}{2}(\mathrm{Re}\ (\nu_1+\nu_2)-\varepsilon)}$, we get

$$\begin{aligned}&\left|\left\langle (g_{\theta_1,\theta_2}^{\nu_1,\nu_2})_{i\varepsilon+\lambda}, h_3\right\rangle\right|\\&\qquad\le C\,|\sin\theta|^{\frac{-1-\varepsilon-\mathrm{Re}\ (\nu_1+\nu_2)}{2}}\,(1+|\lambda|)^{\frac{1}{2}\mathrm{Re}\ (\nu_1+\nu_2)}\,|\sin\theta|^{\frac{-1+\varepsilon+\mathrm{Re}\ (\nu_1-\nu_2)}{2}}\\&\qquad= C\,|\sin\theta|^{-1-\mathrm{Re}\ \nu_2}\,(1+|\lambda|)^{\frac{1}{2}\mathrm{Re}\ (\nu_1+\nu_2)}\,. \qquad (17.76)\end{aligned}$$

Finally, the $d\lambda$-summability is achieved by the use of the integration by parts associated with (11.32). □

End of the proof of Theorem 17.1. Since the exponent of $|\sin(\theta_2-\theta_1)|$ on the right-hand side of (17.68) is >-1, and the two functions $b(\nu_j,\theta_j)$ are bounded, the integral on the right-hand side of (17.67) is convergent and (just like the left-hand side, as observed above), depends analytically on ν_1, ν_2 in the domain characterized by (17.71). One may thus be satisfied with proving (17.67) under the additional assumption that $\mathrm{Re}\,\nu_1<-\frac{1}{2}$ and $\mathrm{Re}\,\nu_2<-\frac{1}{2}$. To that effect, introduce the harmonic oscillator $L=\mathrm{Op}(\pi(x^2+\xi^2))$ and its domain $D(L)$ as a self-adjoint operator in $L^2(\mathbb{R})$. Then $D(L)\subset L^\infty(\mathbb{R})$ and $D(L)$, with its proper Hilbert space structure, is acted upon in an isometric way by all operators $\mathrm{Met}(\tilde{g})$, $\tilde{g}$ lying above the subgroup $K=SO(2)$ of G. It then follows (by a reduction to the case when $\theta_1=\frac{\pi}{2}$) that, since $-1<\mathrm{Re}\ \nu_1<-\frac{1}{2}$, the operator $A_{\nu_1,\theta_1}:=\mathrm{Op}(|x\sin\theta_1-\xi\cos\theta_1|^{-1-\nu_1})$ sends $D(L)$ to $L^2(\mathbb{R})$, as well as $L^2(\mathbb{R})$ to $D(L^{-1})$, with a norm independent of θ_1. Recalling that the two functions $b(\nu_j,\theta_j)$ are bounded, we may thus write

$$\begin{aligned}&\left(\int_0^\pi b(\nu_1,\theta_1)\,A_{\nu_1,\theta_1}\,d\theta_1\right)\left(\int_0^\pi b(\nu_2,\theta_2)\,A_{\nu_2,\theta_2}\,d\theta_2\right)\\&\qquad=\int_0^\pi\int_0^\pi b(\nu_1,\theta_1)\,b(\nu_2,\theta_2)\,A_{\nu_1,\theta_1}\,A_{\nu_2,\theta_2}\,d\theta_1\,d\theta_2\,,\end{aligned} \qquad (17.77)$$

an identity between two bounded operators from $D(L)$ to $D(L^{-1})$: this concludes the proof of Theorem 17.1. □

Remark. Recall from (11.1) that

$$h_1\underset{j}{\#}h_2=\begin{cases}\frac{1}{2}(h_1\#h_2+h_2\#h_1) & \text{if } j=0\,,\\ \frac{1}{2}(h_1\#h_2-h_2\#h_1) & \text{if } j=1\,.\end{cases} \qquad (17.78)$$

It is immediate that

$$\chi^j_{i\lambda_2,i\lambda_1;i\lambda}(s_2,s_1;s)=(-1)^j\,\chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)\,. \qquad (17.79)$$

From this equation, and Theorem 17.1, it is clear that one gets for $h_1 \underset{0}{\#} h_2$ (*resp.* $h_1 \underset{1}{\#} h_2$) the same expression as that provided by (17.1), (17.2), after one has kept, on the right-hand side of (17.3), only the terms with $j = 0$ (*resp.* $j = 1$).

We must now recall a few facts concerning the Radon transformation, and the result of some computations performed in [62, Section 4]. We parametrize the generic elements of the subroups N, A, K occurring in the Iwasawa decomposition of $G = SL(2, \mathbb{R}) = NAK$ as

$$n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{-\frac{r}{2}} \end{pmatrix}, \quad k = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad . \tag{17.80}$$

Following the normalizations in ([23], ch.II, §3), we set $dn = \pi^{-1} db$, $da = \pi\, dr$, $dk = (4\pi)^{-1} d\theta$, which corresponds to the choice of the Haar measure

$$dg = e^{-2\rho(\log a)}\, dn\, da\, dk \tag{17.81}$$

on $G = NAK$. Recall that ρ, the positive half-root, is the element of $\mathfrak{a}^*$ characterized by $\rho\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = 1$.

The homogeneous space G/K is identified with the Poincaré upper half-plane Π in the standard way: its base point is i. Since the class of $g \in G$ in G/N is characterized by the left column of the matrix g, the space $\Xi = G/MN$, with $M = \{\pm I\}$, can be identified with the quotient of $\mathbb{R}^2 \backslash \{0\}$ by the equivalence relation which identifies two points, the negative of each other: in other words, functions on Ξ are to be identified with even functions on $\mathbb{R}^2 \backslash \{0\}$), and the correct measure to be used on Ξ is [62, (4.3)] that which corresponds to the Lebesgue integral of even functions on the whole of $\mathbb{R}^2$. Also, the natural base-point of Ξ is $\pm(1, 0)$.

One then defines the Radon transform V from functions f on Π to functions on Ξ by the formula

$$(Vf)(g.(1,0)) = \int_N f((gn).i)\, dn \tag{17.82}$$

and its dual transform V^* (from functions h on Ξ to functions on Π) as

$$(V^*h)(g.i) = \int_K h((gk).(1,0)\, dk. \tag{17.83}$$

The two operators V and V^* are formally adjoint of each other under the given normalizations of the measures on Π and Ξ. However, V is not an isometry, and does not have a dense range in any meaning: the second of these two facts is fundamental, but the first one is a defect, which can be repaired with the help of the operator

$$T = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2} - i\pi\mathcal{E}\right)}{\Gamma(-i\pi\mathcal{E})} = \pi^{-\frac{1}{2}}(-i\pi\mathcal{E}) \int_0^\infty t^{-\frac{1}{2}}(1+t)^{-1+i\pi\mathcal{E}}\, dt. \tag{17.84}$$

Recall that $\mathcal{E}$ is essentially self-adjoint on $L^2_{\text{even}}(\mathbb{R}^2)$ when given the initial domain $C^\infty_{\text{even}}(\mathbb{R}^2\backslash\{0\})$, and that $t^{2i\pi\mathcal{E}}$ was made explicit in (2.10). The operator TV is an isometry from $L^2(\Pi)$ to some subspace of $L^2_{\text{even}}(\mathbb{R}^2)$, which we shall describe presently. Set, as an operator on $L^2_{\text{even}}(\mathbb{R}^2)$,

$$\kappa = \frac{\Gamma\left(\frac{1}{2}+i\pi\mathcal{E}\right)}{\Gamma\left(\frac{1}{2}-i\pi\mathcal{E}\right)}\,(2\pi)^{-2i\pi\mathcal{E}}\mathcal{G}, \tag{17.85}$$

an involutive unitary transformation. To understand κ, one may note ([56], Proposition 4.1) that if $h(x,\xi) = h_0(x^2+\xi^2)$, then $(\kappa h)(x,\xi) = (x^2+\xi^2)^{-1}h_0((x^2+\xi^2)^{-1})$.

The following is taken from [62, Theorem 4.1].

Theorem 17.6. *The unitary transformation TV, initially defined on the space of continuous functions on Π with a compact support, extends as an isometry from $L^2(\Pi)$ onto the subspace* $\text{Ran}(TV)$ *of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions invariant under the symmetry $T\kappa T^{-1}$. The operator V^*T^* extends on* $\text{Ran}(TV)$ *as the inverse of TV, and is zero on the subspace* $(\text{Ran}(TV))^\perp$ *of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions h with $T\kappa T^{-1}h = -h$. Moreover, the isometry TV intertwines the two quasi-regular actions of $G = SL(2,\mathbb{R})$ on $L^2(\Pi)$ and $L^2(\Xi)$ respectively, and transforms the operator $\Delta - \frac{1}{4}$ on $L^2(\Pi)$ into the operator $\pi^2\mathcal{E}^2$ on $L^2(\Xi)$.*

We need to recall from [62, (4.34) and (4.36)] the formulas

$$(TVf)^\flat_\lambda(s) = \frac{1}{2}(2\pi)^{-\frac{3}{2}}\,\frac{\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)}\int_\Pi \left(\frac{|z-s|^2}{\text{Im}\, z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} f(z)\,d\mu(z) \tag{17.86}$$

and

$$(V^*T^*h_\lambda)(z) = (2\pi)^{-\frac{1}{2}}\,\frac{\Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(-\frac{i\lambda}{2}\right)}\int_{-\infty}^\infty h^\flat_\lambda(s)\left(\frac{|z-s|^2}{\text{Im}\, z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} ds. \tag{17.87}$$

Also, from (11.4), (11.6), together with Theorem 17.6, it follows that

$$V^*T^*(TVf)_\lambda = \frac{\pi}{2}\,\frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{-i\lambda}{2}\right)}\,f_{|\lambda|}\,. \tag{17.88}$$

We are now in a position to relate the Weyl sharp product on even functions on $\mathbb{R}^2$ to the pointwise products and Poisson brackets of functions on Π.

Lemma 17.7. *Given $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$, one has for all $\lambda\in\mathbb{R}$,*

$$(u_z|\text{Op}(h_\lambda)u_z) = 2\,(2\pi)^{\frac{i\lambda}{2}}\,\Gamma\left(-\frac{i\lambda}{2}\right)\,(V^*T^*h_\lambda)(z) \tag{17.89}$$

and

$$(u^1_z|\text{Op}(h_\lambda)u^1_z) = -2i\lambda\,(2\pi)^{\frac{i\lambda}{2}}\,\Gamma\left(-\frac{i\lambda}{2}\right)\,(V^*T^*h_\lambda)(z)\,. \tag{17.90}$$

Proof. Recall that the role of the Wigner function $W(u,v)$ associated with a pair of functions $u,\ v\in\mathcal{S}(\mathbb{R})$ was explained in (2.3),and that $W(u_z,u_z)$ and $W(u_z^1,u_z^1)$ were made explicit in (2.27) and (2.28). We need the decomposition of these two functions into homogeneous components: with the notation (2.13)–(2.16), it has been found in ([62], (13.14), (13.15)) that

$$(W(u_z,u_z))_\lambda^\flat(s)=(2\pi)^{\frac{-i\lambda-3}{2}}\,\Gamma\left(\frac{1+i\lambda}{2}\right)\left(\frac{|s-z|^2}{\mathrm{Im}\ z}\right)^{\frac{-1-i\lambda}{2}} \tag{17.91}$$

and

$$(W(u_z^1,u_z^1))_\lambda^\flat=i\lambda\,(W(u_z,u_z))_\lambda^\flat\,. \tag{17.92}$$

Then, using also (17.48), one has for every $h\in\mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$,

$$\begin{aligned}(u_z|\mathrm{Op}(h_\lambda)u_z)&=\langle h_\lambda,W(u_z,u_z)\rangle\\&=4\pi\int_{-\infty}^{\infty}h_\lambda^\flat(s)\,W(u_z,u_z)_{-\lambda}^\flat(s)\,ds\,,\end{aligned} \tag{17.93}$$

so that (17.89) follows from (17.95) and (17.87); (17.90) then follows from (17.96). □

The following theorem should be compared to Theorem 11.4:

Theorem 17.8. *Given h_1 and $h_2\in\mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$, one has for all $\lambda\in\mathbb{R}$, and all $z\in\Pi$, the equations*

$$\begin{aligned}(u_z|\mathrm{Op}((h_1\#h_2)_\lambda)u_z)=\pi^2\sum_{j=0}^{1}(-i)^j\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\\\frac{\Gamma\left(\frac{1+i\lambda}{2}\right)\Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\lambda-\lambda_1+\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\lambda+\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1+\lambda_2)+2j}{4}\right)}\\\left[(u_z|\mathrm{Op}((h_1)_{\lambda_1})u_z)\underset{j}{\times}(u_z|\mathrm{Op}((h_2)_{\lambda_2})u_z)\right]_{|\lambda|}\,d\lambda_1\,d\lambda_2\,,\end{aligned} \tag{17.94}$$

and $(u_z^1|\mathrm{Op}((h_1\#h_2)_\lambda)u_z^1)$ is obtained by the same formula, substituting u_z^1 for u_z everywhere and inserting the extra factor $\frac{i\lambda}{\lambda_1\lambda_2}$ under the integral.

Proof. To start with, we need to recall again two formulas from [62]. Propositions 8.1 and 8.2 there deal with two functions u_1 and u_2 on the real line, and with their lifted-up versions $u_1^\sharp$ and $u_2^\sharp$ as homogeneous even functions on $\mathbb{R}^2$ of degrees $-1-i\lambda_1$ and $-1-i\lambda_2$: the operation $u\mapsto u^\sharp$ is the reciprocal of the operation $h_\lambda\mapsto h_\lambda^\flat$ defined in (2.15), and we shall assume here, to start with, that $u_1=(h_1)_{\lambda_1}^\flat$ and $u_2=(h_2)_{\lambda_2}^\flat$ for some functions h_1 and $h_2\in\mathcal{S}_{\mathrm{even}}(\mathbb{R}^2)$, so that $u_j^\sharp=(h_j)_{\lambda_j}$. The regularity assumption in the two quoted propositions is that u_1 and u_2 should belong to the spaces $C_{i\lambda_1}^\infty$ and $C_{i\lambda_2}^\infty$ of C^∞-vectors of two representations $\pi_{i\lambda_1}$

and $\pi_{i\lambda_2}$ taken from the class one principal series of G: there is no need to define these spaces here, as it suffices to recall from [62, (13.17)] that, indeed, $h_\lambda^\flat$ does belong to the appropriate space $C_{i\lambda}^\infty$ whenever $h \in \mathcal{S}_{\text{even}}(\mathbb{R}^2)$. We may then quote the two propositions under consideration in one stroke, using the notation (11.2): if h_1 and h_2 lie in $\mathcal{S}_{\text{even}}(\mathbb{R}^2)$, one has for almost all s, and $j = 0, 1$,

$$\int_{\mathbb{R}^2} \chi^j_{i\lambda_1, i\lambda_2; i\lambda}(s_1, s_2; s)\, (h_1)^\flat_{\lambda_1}(s_1)\, (h_2)^\flat_{\lambda_2}(s_2)\, ds_1\, ds_2 = 2^{\frac{9}{2}}\, \pi^2$$
$$\times \frac{\Gamma\left(\frac{-i\lambda_1}{2}\right) \Gamma\left(\frac{-i\lambda_2}{2}\right) \Gamma\left(\frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1-i(\lambda+\lambda_1+\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda-\lambda_1+\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda+\lambda_1-\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)+2j}{4}\right)}$$
$$\left(TV((V^*T^*(h_1)_{\lambda_1}) \underset{j}{\times} (V^*T^*(h_2)_{\lambda_2}))\right)^\flat_\lambda (s)\,. \tag{17.95}$$

Thus, from (17.99) and Theorem 17.1,

$$(h_1 \# h_2)_\lambda = 4\pi \sum_{j=0}^{1} (-i)^j \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{\frac{i(-\lambda+\lambda_1+\lambda_2)}{2}}$$
$$\frac{\Gamma\left(\frac{-i\lambda_1}{2}\right) \Gamma\left(\frac{-i\lambda_2}{2}\right) \Gamma\left(\frac{i\lambda}{2}\right)}{\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(-\lambda-\lambda_1+\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(-\lambda+\lambda_1-\lambda_2)+2j}{4}\right) \Gamma\left(\frac{1+i(\lambda+\lambda_1+\lambda_2)+2j}{4}\right)}$$
$$\left(TV((V^*T^*(h_1)_{\lambda_1}) \underset{j}{\times} (V^*T^*(h_2)_{\lambda_2}))\right)_\lambda \, d\lambda_1\, d\lambda_2\,. \tag{17.96}$$

Next, using Lemma 17.7,

$$(u_z | \mathrm{Op}((h_1 \# h_2)_\lambda) u_z) = 2\, (2\pi)^{\frac{i\lambda}{2}}\, \Gamma\left(-\frac{i\lambda}{2}\right)\, (V^*T^*(h_1 \# h_2)_\lambda)(z)\,. \tag{17.97}$$

Using also

$$V^*T^* \left(TV((V^*T^*(h_1)_{\lambda_1}) \underset{j}{\times} (V^*T^*(h_2)_{\lambda_2}))\right)_\lambda$$
$$= \frac{\pi}{2} \frac{\Gamma(\frac{1+i\lambda}{2})\, \Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\, \Gamma(-\frac{i\lambda}{2})} \left[(V^*T^*(h_1)_{\lambda_1}) \underset{j}{\times} (V^*T^*(h_2)_{\lambda_2}) \right]_\lambda, \tag{17.98}$$

a consequence of (17.88), finally, in the same way as in (17.101),

$$(V^*T^*(h_j)_{\lambda_j})(z) = \frac{1}{2} \frac{(2\pi)^{-\frac{i\lambda_j}{2}}}{\Gamma\left(-\frac{i\lambda_j}{2}\right)} (u_z | \mathrm{Op}((h_j)_{\lambda_j}) u_z)\,, \tag{17.99}$$

one proves the first part of Theorem 17.8. The second one is the result of a comparison of (17.89) and (17.90). □

The game can be played in reverse, yielding

$$(u_z|\mathrm{Op}((h_1)_{\lambda_1})u_z) \underset{j}{\times} (u_z|\mathrm{Op}((h_2)_{\lambda_2})u_z) = \frac{1}{2\pi}\, i^j \int_{-\infty}^{\infty}$$

$$\frac{\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\lambda-\lambda_1+\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\lambda+\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1+\lambda_2)+2j}{4}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{-i\lambda}{2}\right)}$$

$$\left(u_z\Big|\mathrm{Op}\left(\left((h_1)_{\lambda_1}\underset{j}{\#}(h_2)_{\lambda_2}\right)_\lambda\right)u_z\right)d\lambda\,. \tag{17.100}$$

Consider for instance two homogeneous symbols in the commutant of the harmonic oscillator, introduced just after (17.76) and needed to complete the proof of Theorem 17.1:

Theorem 17.9. *Let $\pi\ell(x,\xi)=\pi(x^2+\xi^2)$ be the symbol of the harmonic oscillator L. Assume that $-1<\mathrm{Re}\,\nu_1<1$, $-1<\mathrm{Re}\,\nu_2<1$. Let $h=\ell^{\frac{-1-\nu_1}{2}}\#\ell^{\frac{-1-\nu_2}{2}}$ be the composition of the two Weyl symbols under consideration. Then*

$$h_\lambda = \frac{1}{4}(2\pi)^{\frac{\nu_1+\nu_2-i\lambda-1}{2}}\,\ell^{\frac{-1-i\lambda}{2}} \times \frac{\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda}{4}\right)\Gamma\left(\frac{1-\nu_1-\nu_2-i\lambda}{4}\right)}{\Gamma\left(\frac{1+\nu_1}{2}\right)\Gamma\left(\frac{1+\nu_2}{2}\right)\Gamma\left(\frac{1-i\lambda}{2}\right)}\,. \tag{17.101}$$

Proof. From ([53], (1.6) or (1.8)), one has

$$\mathrm{Op}(e^{-2\pi s\ell}) = (1-s^2)^{-\frac{1}{2}}\left(\frac{1-s}{1+s}\right)^L, \qquad 0<s<1. \tag{17.102}$$

Setting, for $0<s_1,\ s_2<1$,

$$\frac{(1-s_1)(1-s_2)}{(1+s_1)(1+s_2)} = \frac{1-s}{1+s}, \tag{17.103}$$

i.e., $s=\frac{s_1+s_2}{1+s_1s_2}$, hence $1-s^2=\frac{(1-s_1^2)(1-s_2^2)}{(1+s_1s_2)^2}$, one sees that

$$e^{-2\pi s_1\ell}\#e^{-2\pi s_2\ell} = (1+s_1s_2)^{-1}\,e^{-2\pi\frac{s_1+s_2}{1+s_1s_2}\ell}, \tag{17.104}$$

a formula still valid, by analytic continuation, for $s_1>0,\ s_2>0$. Since

$$\ell^{\frac{-1-\nu}{2}} = \frac{(2\pi)^{\frac{\nu+1}{2}}}{\Gamma\left(\frac{\nu+1}{2}\right)}\int_0^\infty e^{-2\pi s\ell}\,s^{\frac{\nu-1}{2}}\,ds, \qquad \mathrm{Re}\,\nu>-1, \tag{17.105}$$

one has

$$\begin{aligned} h:&=\ell^{\frac{-1-\nu_1}{2}}\#\ell^{\frac{-1-\nu_2}{2}} \\ &= \frac{(2\pi)^{\frac{\nu_1+\nu_2+2}{2}}}{\Gamma\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)}\int_0^\infty\int_0^\infty s_1^{\frac{\nu_1-1}{2}}\,s_2^{\frac{\nu_2-1}{2}}\,e^{-2\pi\frac{s_1+s_2}{1+s_1s_2}\ell}\,\frac{ds_1\,ds_2}{1+s_1s_2}\,. \end{aligned} \tag{17.106}$$

Thus

$$
\begin{aligned}
h_\lambda &= \frac{(2\pi)^{\frac{\nu_1+\nu_2}{2}}}{\Gamma\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)} \int_0^\infty \int_0^\infty \int_0^\infty s_1^{\frac{\nu_1-1}{2}} s_2^{\frac{\nu_2-1}{2}} t^{i\lambda} e^{-2\pi t^2 \frac{s_1+s_2}{1+s_1 s_2}\ell} \frac{ds_1\, ds_2\, dt}{1+s_1 s_2} \\
&= \frac{1}{2}(2\pi)^{\frac{\nu_1+\nu_2-i\lambda-1}{2}} \frac{\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)} \ell^{\frac{-1-i\lambda}{2}} I(\nu_1, \nu_2; \lambda),
\end{aligned}
\tag{17.107}
$$

with

$$
\begin{aligned}
I(\nu_1, \nu_2; \lambda) &= \int_0^\infty \int_0^\infty s_1^{\frac{\nu_1-1}{2}} s_2^{\frac{\nu_2-1}{2}} (s_1+s_2)^{\frac{-i\lambda-1}{2}} (1+s_1 s_2)^{\frac{1\lambda-1}{2}} ds_1\, ds_2 \\
&= \frac{1}{2} \frac{\Gamma\left(\frac{1+\nu_1+\nu_2-i\lambda}{4}\right)\Gamma\left(\frac{1+\nu_1-\nu_2+i\lambda}{4}\right)\Gamma\left(\frac{1-\nu_1+\nu_2+i\lambda}{4}\right)\Gamma\left(\frac{1-\nu_1-\nu_2-i\lambda}{4}\right)}{\Gamma\left(\frac{1-i\lambda}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)},
\end{aligned}
\tag{17.108}
$$

where the last equality is obtained by the change of variables $s_1 = t^{\frac{1}{2}} x^{-\frac{1}{2}}$, $s_2 = t^{\frac{1}{2}} x^{\frac{1}{2}}$. □

Remark. One cannot help noticing a certain analogy between the coefficient in (17.101) and the one in the integral term of (15.33). This is quite justified, since in both cases we are interested in the expansion into homogeneous components of the sharp product of two homogeneous objects: only, these are Γ-invariant in (15.33) and K-invariant ($K = SO(2)$) in (17.101).

Theorem 17.9 gives a quite different proof of the following formula, first given by Mizony, whose proof [33, 34] depends on an ingenious identity concerning the function ${}_3F_2$:

Corollary 17.10. *For every $\delta \geq 1$, one has*

$$
\begin{aligned}
&\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda_1}{2}}(\delta)\, \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda_2}{2}}(\delta) = \frac{1}{8\pi^2} \int_0^\infty \\
&\frac{\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(-\lambda-\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(-\lambda+\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1+\lambda_2)}{4}\right)}{\Gamma\left(\frac{i\lambda}{2}\right)\Gamma\left(\frac{-i\lambda}{2}\right)} \\
&\times \frac{\Gamma\left(\frac{1+i(-\lambda+\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda-\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1-i(\lambda+\lambda_1+\lambda_2)}{4}\right)}{\Gamma\left(\frac{1-i\lambda_1}{2}\right)\Gamma\left(\frac{1-i\lambda_2}{2}\right)\Gamma\left(\frac{1+i\lambda_1}{2}\right)\Gamma\left(\frac{1+i\lambda_2}{2}\right)} \\
&\times \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta)\, d\lambda.
\end{aligned}
\tag{17.109}
$$

Proof. From (17.89), then (17.87), it follows that

$$\begin{aligned}(u_z|\mathrm{Op}(\ell^{\frac{-1-i\lambda}{2}})u_z) &= 2\,(2\pi)^{i\lambda}\,\Gamma\left(-\frac{i\lambda}{2}\right)\,(V^*T^*\ell^{\frac{-1-i\lambda}{2}})(z)\\ &= 2\,(2\pi)^{i\lambda-\frac{1}{2}}\,\Gamma\left(\frac{1-i\lambda}{2}\right)\int_{-\infty}^{\infty}(1+s^2)^{\frac{-1-i\lambda}{2}}\left(\frac{|z-s|^2}{\mathrm{Im}\,z}\right)^{\frac{i\lambda-1}{2}}ds\\ &= (2\pi)^{i\lambda+\frac{1}{2}}\,\Gamma\left(\frac{1-i\lambda}{2}\right)\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(i,z))\,, \qquad (17.110)\end{aligned}$$

where d is the hyperbolic distance on Π: the last equation is a special case of the formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{|z-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}}\left(\frac{|w-s|^2}{\mathrm{Im}\,w}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}}ds = \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)) \qquad (17.111)$$

proved in [62, (4.38)]. The corollary is then a consequence of Theorem 17.9, (17.100) and (17.110), setting $\delta = \cosh d(i,z)$, together with the observation that the integrand on the right-hand side of (17.109) is invariant under the change of λ to $-\lambda$. Of course, a symbol such as $\ell^{\frac{-1-i\lambda}{2}}$ does not lie in $\mathcal{S}(\mathbb{R}^2)$, but this is easy to repair, using instead the approximation

$$(\varepsilon+\ell)^{\frac{-1-i\lambda}{2}}\,e^{-2\pi\varepsilon\ell} = \frac{(2\pi)^{\frac{1+i\lambda}{2}}}{\Gamma\left(\frac{1+i\lambda}{2}\right)}\int_0^{\infty} e^{-2\pi(s+\varepsilon)\,\ell}e^{-2\pi\varepsilon s}\,s^{\frac{i\lambda-1}{2}}\,ds\,.$$

□

Remarks. A few remarks concerning either the Op^p-calculus or a comparison between the results of Sections 15 and 17, are in order.

Theorem 17.1 extends to the Op^p-calculus, with the sole difference that the Gamma factors on the right-hand side of (17.3) (the coefficients denoted as $A_j(\nu_1,\nu_2;i\lambda)$ in (17.43)) should be replaced by some coefficients $A_j^p(\nu_1,\nu_2;i\lambda)$: for the proof of Theorem 17.1 relied entirely on Theorem 11.3 (the spectral decomposition of the sharp product of two power functions), and it has been proved in Theorem 12.9 that Theorem 11.3 extends to the Op^p-calculus, at the sole price of changing the coefficients $C_j(\nu_1,\nu_2;i\lambda)$ to p-dependent coefficients. That we do not write the Op^p-version of Theorem 17.1 explicitly is only due to the fact that the results of Section 12 do not make the coefficients $C_j^p(\nu_1,\nu_2;i\lambda)$ fully explicit: they are given instead by recurrence relations. It is only when $p=0$ (the Weyl case) that the formulas which are the results of Theorems 17.1 and 17.8 are actually simpler for the full calculus than for its even-even and odd-odd parts considered separately. This may be the occasion to state that Theorem 17.1 extends to the case when symbols which are not necessarily invariant under the map $(x,\xi)\mapsto(-x,-\xi)$ are considered. Indeed, this more general case is considered in [62, Theorem 5.3]: even though the proof there is far from being as complete as that of Theorem 17.1 in the present work, only the algebra differs.

Concerning the sharp product of *automorphic* symbols, our first guess of Theorem 15.1 was based on a simultaneous consideration of Theorem 17.1 (in its earlier version from *loc. cit.*) and of the results of [62] regarding the Roelcke-Selberg decomposition of the product or Poisson bracket of two Eisenstein series. Indeed, denote as Θ the map introduced in Proposition 2.1, which associates with an automorphic distribution h a pair of automorphic functions on Π, namely the two functions $z \mapsto (u_z^p | \mathrm{Op}(h) u_z^p)$, $p = 0$ or 1. Then Theorem 17.8, a consequence of Theorem 17.1, would seem to relate the Roelcke-Selberg decomposition of the image under Θ of the symbol $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$ to that of the two functions $E^*_{\frac{1-\nu_1}{2}} \cdot E^*_{\frac{1-\nu_2}{2}}$ and $\{E^*_{\frac{1-\nu_1}{2}}, E^*_{\frac{1-\nu_2}{2}}\}$. Only Theorems 17.1 and 17.8 are very far from being applicable to such singular distributions as $\mathfrak{F}_{\nu_1}^\sharp$ and $\mathfrak{F}_{\nu_2}^\sharp$. Still, disregarding difficulties and concentrating on the computations, we certainly got in this way the right result: this was a third verification of the formula for the sharp product of two Eisenstein distributions since, besides the genuine proof in Sections 12 to 14, we already gave a completely different heuristic "proof" of it in Section 5.

It is clear, from such formulas as (6.21), that the two main theorems regarding the existence of explicit formulas for $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$ or $\mathfrak{F}_\nu^\sharp \# (\mathfrak{F}_{r,\ell}^\pm)^\sharp$ should admit extensions to the case when $\#^p$ $(p = 1, 2, \dots)$ is substituted for $\#$, in other words in the Op^p-calculus. That we have not made these formulas explicit is due to the following two reasons: first, more readers are likely to be interested in new facts concerning the ubiquitous Weyl calculus than in the structure of new calculi; next, when $p \geq 1$, the formulas should again be more complicated and involve coefficients only determined by recurrence relations. Still, it is important to note that, contrary to the case of the Weyl calculus in which the definition of $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$ had to be somewhat indirect, based as it was on Proposition 13.1 (a definition of $\mathcal{E}(\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp)$ rather than of $\mathfrak{F}_{\nu_1}^\sharp \# \mathfrak{F}_{\nu_2}^\sharp$), the definition of $\mathfrak{F}_{\nu_1}^\sharp \#^p \mathfrak{F}_{\nu_2}^\sharp$ would be perfectly natural for $p \geq 2$, in view of Theorem 10.6.

18 Odd automorphic distributions and modular forms of non-zero weight

Our purpose in this section is to stress that the whole Weyl calculus, not only the part associated with even symbols, has interesting connections with modular form theory. Our emphasis is *not* on the interesting new arithmetic facts that emerge when substituting for Γ a congruence subgroup Γ', or when twisting the definition by characters of Γ': only, doing this is unavoidable in the present context.

As a consequence of Proposition 2.1, an automorphic distribution $\mathfrak{S}$ can be characterized by a pair of non-holomorphic modular forms f^0 and f^1, where $f^p(z) = (u_z^p | \mathrm{Op}_{\sqrt{2}}(\mathfrak{S}) u_z^p)$. That one gets a characterization is due to the fact that, by definition, an automorphic distribution is even (since the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ lies in Γ): consequently, the associated operator commutes with the map $u \mapsto \check{u}$.

We now generalize the concept of automorphic distribution, allowing a subgroup Γ' of Γ and a character χ of Γ' to enter the picture: a Γ'-automorphic distribution $\mathfrak{S}$ with character χ shall be any tempered distribution satisfying the equation

$$\mathfrak{S} \circ g = \chi(g)\, \mathfrak{S} \tag{18.1}$$

for all $g \in \Gamma'$; the left-hand side of (18.1) is still defined by the equation $\langle \mathfrak{S} \circ g, h \rangle = \langle \mathfrak{S}, h \circ g^{-1} \rangle$ for every $h \in \mathcal{S}(\mathbb{R}^2)$. Introducing a non-trivial character is unavoidable if one wants, so as to give $\mathbb{R}^2$ (as opposed to Π) its full role, to consider odd symbols as well. Indeed, the equation (18.1) cannot be valid, if $\mathfrak{S} \neq 0$ is odd, unless the condition $\chi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = -1$ is satisfied (assuming of course that $-I \in \Gamma'$).

In the odd case, instead of the above-defined functions f^0 and f^1, we must now consider the functions $f^{0,1}$ and $f^{1,0}$, where

$$f^{p,1-p}(z) :\, = (u_z^p | \mathrm{Op}_{\sqrt{2}}(\mathfrak{S}) u_z^{1-p}) \ : \tag{18.2}$$

again they characterize the (odd) distribution $\mathfrak{S}$. As will be seen, the two functions so defined will be modular forms of weights ∓ 1 and character χ. We take from [8, p. 135] the definition of a Γ'-modular form of weight k and character χ, with $\chi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = (-1)^k$, as a C^∞ function f on Π, satisfying the automorphy condition

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{a+bz^{-1}}{|a+bz^{-1}|}\right)^k \chi(g)\, f(z), \qquad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma', \tag{18.3}$$

at the same time of moderate growth at infinity (*i.e.*, bounded by some power of $\mathrm{Im}\, z$ as $\mathrm{Im}\, z \to \infty$) and an eigenfunction of the Maass differential operator [8, p. 129]

$$\Delta_k = \Delta + i\, k\, \mathrm{Im} \left(-\frac{1}{z}\right) \left(z^2 \frac{\partial}{\partial z} + \bar{z}^2 \frac{\partial}{\partial \bar{z}}\right). \tag{18.4}$$

One may remember that the usual definition (*loc.cit.*) involves the multiplier $(\frac{cz+d}{|cz+d|})^k$ instead of the one above: the two notions are related by the fact that, with

$$(S_k f)(z) = \left(\frac{z}{|z|}\right)^k f(z), \tag{18.5}$$

the function $S_k^{-1} f$ is automorphic in the usual sense if and only if the function f is automorphic in the sense of (18.3). Our present slash operator thus has to be defined as

$$\left(f \Big|_k g\right)(z) = \left(\frac{a+bz^{-1}}{|a+bz^{-1}|}\right)^{-k} f\left(\frac{az+b}{cz+d}\right) \tag{18.6}$$

and the operator Δ_k is the conjugate, under the S_k-transform, of the usual operator $\tilde{\Delta}_k = \Delta + ik\, (\mathrm{Im}\ z)(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})$: it is also the conjugate of $\tilde{\Delta}_k$ under the map $f \mapsto \check{f}$, $\check{f}(z) = f(-\frac{1}{z})$.

One may also recall (*loc.cit.*, p. 144) that the study of Maass forms of general integral weights can be essentially reduced to that of forms of degrees 0 and 1. As a matter of fact, it is quite possible to obtain Maass forms of arbitrary integral weights $j-k$ in a direct way, substituting for $(u_z^p|\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})u_z^p)$ or $(u_z^p|\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})u_z^{1-p})$, with $p=0$ or 1, more general expressions $(v_z^j|\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})v_z^k)$, where the function v_z^j (not to be mistaken for u_z^j except in the cases when $j=0$ or 1) is the j-th eigenfunction of the z-dependent harmonic oscillator $L_z=\mathrm{Op}(\pi\,\frac{|x-z\xi|^2}{\mathrm{Im}\ z})$. However, this may not be necessary since in any case an even or odd automorphic distribution is fully characterized by two of the four functions $(u_z^p|\mathrm{Op}_{\sqrt{2}}(\mathfrak{S})u_z^q)$ with $p=0$ or 1, $q=0$ or 1.

Concerning the unavoidable occurrence of the intertwining operator S_k, we must emphasize again that the constraint of compatibility with the Weyl calculus left us no choice whatsoever (it is essential that one should have $\mathrm{Met}(g)\,u_z=\theta(g,z)\,u_{g.z}$ for some complex number $\theta(g,z)$ of absolute value 1): to see it in a different light, let us mention that some formulas would have been simpler if we could have worked throughout – as we did in [55] – with the right half-plane rather than the upper half-plane. On the other hand, one may observe that no automorphy condition could be simpler than (18.1), which concerns automorphic *distributions* rather than modular forms.

It may be interesting to start with an effective construction of all characters of Γ, which can be neatly done with the help of the metaplectic representation Met: recall that this is a representation, in $\mathcal{S}'(\mathbb{R})$, of the two-fold covering $\widetilde{SL}(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, and we are interested in its restriction to the part $\tilde{\Gamma}$ of $\widetilde{SL}(2,\mathbb{R})$ that lies above Γ. Though it is rather delicate (the Maslov index theory is needed in general, and quadratic residue symbols are necessary [44] for the arithmetic case) to make the representation Met explicit without any ambiguity, it is quite easy to describe it up to the ambiguity factors ± 1: indeed, right after (2.4), we have recalled that the unitary transformations associated with points $\tilde{g}$ that lie above the points $g=\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ of Γ (these two matrices of course generate Γ) are respectively the operator of multiplication by the exponential $\exp i\pi x^2$ and $e^{-\frac{i\pi}{4}}$ times the Fourier transformation. Now, if E is any *even*-dimensional subspace of $\mathcal{S}'(\mathbb{R})$ invariant under the metaplectic transformation, and if Met_E is the metaplectic representation considered as acting only on E, it is clear that, defining for any $\tilde{g}\in\tilde{\Gamma}$ the number $\chi(\tilde{g})$ as the determinant of $\mathrm{Met}_E(\tilde{g})$, one gets a character of $\tilde{\Gamma}$ which actually can be identified to a character of Γ (since, up to the possible multiplication by -1, $\mathrm{Met}_E(\tilde{g})$ only depends on the image of $\tilde{g}$ in Γ).

Proposition 18.1. *Given any even number $N\geq 2$, the linear subspace E^N of $\mathcal{S}'(\mathbb{R})$ a basis of which is provided by the distributions $v_{N,q}$ (q mod N) with*

$$v_{N,q}(x)=\sum_{j\in\mathbb{Z}}\delta\left(x-\left(j+\frac{q}{N}\right)\sqrt{N}\right) \tag{18.7}$$

is invariant under the metaplectic representation restricted to $\tilde{\Gamma}$.

More precisely, for every q*, one has*

$$e^{i\pi x^2}\, v_{N,q} = e^{i\pi\,\frac{q^2}{N}}\, v_{N,q}$$

and

$$e^{-\frac{i\pi}{4}}\,\mathcal{F}\, v_{N,q} = e^{-\frac{i\pi}{4}}\, N^{-\frac{1}{2}} \sum_{r \bmod N} e^{-2i\pi\,\frac{qr}{N}}\, v_{N,r}\,. \tag{18.8}$$

The associated character $\chi_N = \det \operatorname{Met}_{E^N}$*, as explained just before the statement of Proposition* 18.1*, is given on generators of* Γ *by*

$$\chi_N\left(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\right) = e^{i\pi\,\frac{(N-1)(2N-1)}{6}} \qquad \text{and} \qquad \chi_N\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right) = i^{N+3}\,. \tag{18.9}$$

If $6|N$*,* χ_N *generates the group of all characters of* Γ*.*

Proof. The first of the two formulas (18.8) is trivial. For the second one, we use Poisson's formula

$$\sum_k e^{2i\pi akx} = |a|^{-1} \sum_j \delta\left(x - \frac{j}{a}\right)\,, \tag{18.10}$$

valid for any real number $a \neq 0$, to find

$$\begin{aligned}(\mathcal{F}v_{N,q})(x) &= \sum_k e^{-2i\pi\,(k+\frac{q}{N})\,\sqrt{N}\,x}\\ &= e^{-2i\pi\frac{qx}{\sqrt{N}}} \sum_k e^{-2i\pi\,k\,\sqrt{N}\,x}\\ &= N^{-\frac{1}{2}}\, e^{-2i\pi\frac{qx}{\sqrt{N}}} \sum_k \delta\left(x - \frac{k}{\sqrt{N}}\right)\\ &= N^{-\frac{1}{2}} \sum_k e^{-2i\pi\frac{qk}{N}}\, \delta\left(x - \frac{k}{\sqrt{N}}\right)\,.\end{aligned} \tag{18.11}$$

Setting $k = Nj + r$, $r \bmod N$, we get the second of the two formulas (18.8). The determinant of the matrix associated with the multiplication by $e^{i\pi x^2}$ is $\exp\frac{i\pi}{N}(1^2 + \cdots + (N-1)^2) = \exp i\pi\frac{(N-1)(2N-1)}{6}$. The determinant of the matrix associated with the transformation $e^{-\frac{i\pi}{4}}\,\mathcal{F}$ is $e^{-\frac{i\pi N}{4}}\, N^{-\frac{N}{2}}$ times the determinant $\Delta = \det\left((\theta^{qk})_{0\leq q,k\leq N-1}\right)$, where θ is a primitive N^{th}-root of unity. In view of the relation $(e^{-\frac{i\pi}{4}}\,\mathcal{F})^2 = -iC$ with $Cv = \check{v}$, together with $Cv_{N,q} = v_{N,-q}$, it is easier to compute Δ^2, which is $\det\left((b_{qk})_{0\leq q,k\leq N-1}\right)$ with $b_{qk} = 0$ unless $q + k = 0$ mod N, in which case $b_{qk} = N$; thus $\Delta^2 = (-1)^{\frac{(N-1)(N-2)}{2}}\, N^N$. This yields $\chi_N(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)) = \varepsilon i$ for some $\varepsilon = \pm 1$, since $i^{-N} \times (-1)^{\frac{(N-1)(N-2)}{2}} = -1$ as N is even.

Set $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, so that $s^2 = (st)^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus, for any character χ of Γ, $\chi(s)$ must be a fourth root of unity, and $\chi(st)$ must be a sixth root of unity; also, Γ can have at most 12 characters. Now, as shown by what has already been proved of (18.9), $\chi_N(s)$ is a primitive fourth root of unity for every N, and $\chi_N(t)$ is a primitive third root of unity whenever N is divisible by 3, which proves the last statement in Proposition 18.1. Finally, since $\chi(-st) = \chi(s^3 t) = -\varepsilon i\, e^{i\pi \frac{(N-1)(2N-1)}{6}}$ is a third root of unity, it follows that $\varepsilon = (-1)^{\frac{N}{2}+1}$. □

Remark. In particular, there are six characters χ of Γ such that $\chi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = -1$.

We first justify the claim above concerning the link between Γ'-automorphic distributions with character χ and the corresponding modular forms.

Theorem 18.2. *If $\mathfrak{S}$ is a Γ'-automorphic distribution with character χ, homogeneous of degree $-1-i\lambda$, the function*

$$f^{0,1}(z) = \left(u_z | \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\, u_z^1\right) = \left\langle \mathfrak{S},\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z, u_z^1)\right\rangle \tag{18.12}$$

is a Γ'-modular form of weight -1 with character χ, on which the operator Δ_{-1} takes the eigenvalue $\frac{1+\lambda^2}{4}$.

The same holds with the function $f^{1,0}(z) = (u_z^1 | \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\, u_z)$, only changing the weight -1 to 1.

Proof. First, we establish the equation

$$W(u_z, u_z^1)(x,\xi) = 2\pi^{\frac{1}{2}} \left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \frac{i}{z}\,(x - z\xi)\, \exp -2\pi \frac{|x-z\xi|^2}{\mathrm{Im}\; z} \quad : \tag{18.13}$$

to do this, we note that the case when $z = i$, *i.e.*,

$$W(u_i, u_i^1)(x,\xi) = 2\pi^{\frac{1}{2}}\,(x - i\xi)\, e^{-2\pi(x^2+\xi^2)} \tag{18.14}$$

is an easy consequence of (2.2). Next, observe that when restricted to the subgroup $N'A$ of $SL(2,\mathbb{R}) = NAK$, where N' is the image of N under the conjugation by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the metaplectic representation can be defined by the formulas which follow (2.4), without any need for lifting them to some cover of the (simply connected!) group $N'A$. In particular, setting

$$\left(\mathrm{Met}\left(\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right)\right) u\right)(x) = e^{i\pi \frac{c}{a} x^2}\, a^{-\frac{1}{2}}\, u(a^{-1}x)\,, \tag{18.15}$$

one sees from (2.24) that $\mathrm{Met}(\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right))\, u_i^p = u_z^p$, $p = 0$ or 1, provided that $z = \frac{a^2}{ac-i}$. Then, from the covariance formula (2.6) and the fact that $W(\phi,\psi)$ is also the Weyl symbol of the rank-one operator $v \mapsto (\phi|v)\psi$, one finds

$$W(u_z, u_z^1)(x,\xi) = 2\pi^{\frac{1}{2}}\,(a^{-1}x + i(cx - a\xi)) \times \exp -2\pi[a^{-2}x^2 + (cx - a\xi)^2]\,, \tag{18.16}$$

the same as (18.13) in view of the equation defining z in terms of a, c.

From this identity, one routinely checks the transformation formula (using (18.6) with $k = -1$)

$$\begin{aligned} W(u_z, u_z^1)(x,\xi)\Big|_{-1} g &= W(u_z, u_z^1)(dx - b\xi, -cx + a\xi) \\ &= W(u_z, u_z^1)(g^{-1}.(x,\xi)) \end{aligned} \tag{18.17}$$

and the same applies after one has substituted $2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z, u_z^1)$ for $W(u_z, u_z^1)$. Then,

$$\begin{aligned} \left(f^{0,1}\Big|_{-1} g\right)(z) &= \frac{a + bz^{-1}}{|a + bz^{-1}|}\, f^{0,1}\left(\frac{az+b}{cz+d}\right) \\ &= \frac{a + bz^{-1}}{|a + bz^{-1}|} \left\langle \mathfrak{S}\,,\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_{g.z}, u_{g.z}^1)\right\rangle \\ &= \left\langle \mathfrak{S}\,,\, (x,\xi) \mapsto 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z, u_z^1)(x,\xi)\Big|_{-1} g \right\rangle \\ &= \left\langle \mathfrak{S}\,,\, (2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z, u_z^1)) \circ g^{-1} \right\rangle\,, \end{aligned} \tag{18.18}$$

where we have used (18.17) at the end. If $\mathfrak{S}$ is automorphic, the last expression coincides with $\chi(g)\, f^{0,1}(z)$ when $g \in \Gamma'$, so we are done for the first part, for what concerns the transformation rule. The differential equation will be a consequence of the more general lemma that follows.

Next, $\overline{\mathfrak{S}}$ is automorphic with character $\bar{\chi}$ if $\mathfrak{S}$ is automorphic with character χ, and $f^{1,0}(z)$ is the complex conjugate of $(u_z|\mathrm{Op}_{\sqrt{2}}(\overline{\mathfrak{S}})\, u_z^1)$, which reduces all that has to be proven about $f^{1,0}$ to the similar facts regarding $f^{0,1}$. □

Looking at the formula (18.13) for $W(u_z, u_z^1)(x,\xi)$, we need to establish that all functions in a related class satisfy a certain differential equation, comparable to (2.29) but more complicated.

Lemma 18.3. *Given any C^∞ function ϕ on $[0,\infty[$, the images under the operators $\Delta_{-1} - \frac{1}{4}$ and $\pi^2\,\mathcal{E}^2$ of the function*

$$\frac{x - z\xi}{z}\left(\mathrm{Im}\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \phi\left(\frac{|x - z\xi|^2}{\mathrm{Im}\ z}\right) \tag{18.19}$$

*are the same. It should be understood, of course, that the first (*resp. *the second) of the two operators acts on the given function when considered as a function of* z *(*resp. (x,ξ)*).*

Proof. Start from the function

$$\psi(z) = \left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \phi\left(\frac{1}{\mathrm{Im}\,(-\frac{1}{z})}\right) : \tag{18.20}$$

it is the transform, under the involution $f \mapsto \check{f}$ (*cf.* what follows (18.6)), of the function $(\mathrm{Im}\,z)^{-\frac{1}{2}}\,\phi(\frac{1}{\mathrm{Im}\,z})$, whose image under the operator $\tilde{\Delta}_k$ mentioned right after (18.6) is easily computed to be

$$-\frac{3}{4}\,(\mathrm{Im}\,z)^{-\frac{1}{2}}\,\phi\left(\frac{1}{\mathrm{Im}\,z}\right) - 3\,(\mathrm{Im}\,z)^{-\frac{3}{2}}\,\phi'\left(\frac{1}{\mathrm{Im}\,z}\right) - (\mathrm{Im}\,z)^{-\frac{5}{2}}\,\phi''\left(\frac{1}{\mathrm{Im}\,z}\right). \tag{18.21}$$

It follows that

$$\begin{aligned}(\Delta_k\,\psi)(z) = &-\frac{3}{4}\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \phi\left(\frac{1}{\mathrm{Im}\,(-\frac{1}{z})}\right)\\ &-3\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{3}{2}} \phi'\left(\frac{1}{\mathrm{Im}\,(-\frac{1}{z})}\right) - \left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{5}{2}} \phi''\left(\frac{1}{\mathrm{Im}\,(-\frac{1}{z})}\right).\end{aligned} \tag{18.22}$$

Now, choosing some matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\xi & x \\ c & d \end{pmatrix} \tag{18.23}$$

such that $\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right) = g^{-1}.\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$, we get

$$\left(\psi\Big|_{-1} g\right)(z) = \frac{x - z\xi}{z}\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \phi\left(\frac{|x - z\xi|^2}{\mathrm{Im}\,z}\right). \tag{18.24}$$

We can then apply the fact that the slash operation of weight -1 commutes with the operator Δ_{-1}, finding

$$\begin{aligned}\Delta_{-1}&\left[\frac{x - z\xi}{z}\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}} \phi\left(\frac{|x - z\xi|^2}{\mathrm{Im}\,z}\right)\right] = \frac{x - z\xi}{z}\left(\mathrm{Im}\,\left(-\frac{1}{z}\right)\right)^{-\frac{1}{2}}\\ &\times\left[-\frac{3}{4}\phi\left(\frac{|x-z\xi|^2}{\mathrm{Im}\,z}\right) - 3\frac{|x-z\xi|^2}{\mathrm{Im}\,z}\phi'\left(\frac{|x-z\xi|^2}{\mathrm{Im}\,z}\right) - \left(\frac{|x-z\xi|^2}{\mathrm{Im}\,z}\right)^2 \phi''\left(\frac{|x-z\xi|^2}{\mathrm{Im}\,z}\right)\right].\end{aligned} \tag{18.25}$$

On the other hand, since x, ξ only enter the function under consideration through two homogeneous combinations, it is not difficult to find

$$\pi^2\, \mathcal{E}^2 \left[\frac{x-z\xi}{z} \left(\mathrm{Im} \left(-\frac{1}{z} \right) \right)^{-\frac{1}{2}} \phi \left(\frac{|x-z\xi|^2}{\mathrm{Im}\ z} \right) \right] = \frac{x-z\xi}{z} \left(\mathrm{Im} \left(-\frac{1}{z} \right) \right)^{-\frac{1}{2}}$$
$$\times \left[-\phi \left(\frac{|x-z\xi|^2}{\mathrm{Im}\ z} \right) - 3 \frac{|x-z\xi|^2}{\mathrm{Im}\ z} \phi' \left(\frac{|x-z\xi|^2}{\mathrm{Im}\ z} \right) - \left(\frac{|x-z\xi|^2}{\mathrm{Im}\ z} \right)^2 \phi'' \left(\frac{|x-z\xi|^2}{\mathrm{Im}\ z} \right) \right]. \tag{18.26}$$

□

Needless to say, since $W(u_z^1, u_z) = \overline{W}(u_z, u_z^1)$, the consideration of $f^{1,0}$ instead of $f^{0,1}$ leads to the consideration of the class of functions analogous to (18.9), with $z^{-1}\,(x - z\xi)$ replaced by its complex conjugate: then, the same result applies, only substituting Δ_1 for Δ_{-1}.

Remark. Let L be the standard harmonic oscillator considered in Theorem 17.9 or in the end of the proof of Theorem 17.1, also as the case $s = 0$ of the operator Λ of Proposition 7.15: $L = \mathrm{Op}(x^2 + \xi^2)$. A complete orthonormal basis of $L^2(\mathbb{R})$ is provided by the sequence $(v^k)_{k \geq 0}$, with $v^k = (k!)^{-\frac{1}{2}} A^{*k} v$, where $A^* = \pi^{\frac{1}{2}} (x - \frac{1}{2\pi} \frac{d}{dx})$ and $v(x) = 2^{\frac{1}{4}} e^{-\pi x^2}$. In terms of the ordinary Hermite polynomials H_k, one has

$$v^k(x) = 2^{\frac{1}{4} - \frac{k}{2}}\, (k!)^{-\frac{1}{2}}\, e^{-\pi x^2}\, H_k \left(\sqrt{2\pi}\, x \right). \tag{18.27}$$

More generally, given any point $z \in \Pi$, consider the oscillator

$$L_z = \mathrm{Op} \left(\pi \frac{|x - z\xi|^2}{\mathrm{Im}\ z} \right). \tag{18.28}$$

Considering the image under the metaplectic representation of some $\tilde{g}$ lying above the matrix $\left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix} \right)$, one sees that, if $z = \frac{a^2}{ac - i}$, the functions v_z^k defined as

$$v_z^k(x) = e^{i\pi \frac{c}{a} x^2}\, a^{-\frac{1}{2}}\, v^k(a^{-1} x) \tag{18.29}$$

constitute a complete orthonormal basis of eigenfunctions of the oscillator L_z. Then, substituting for the function $f^{0,1}$ in (18.12) the function

$$f^{j,k}(z) = (v_z^j | \mathrm{Op}_{\sqrt{2}}(\mathfrak{S})\, v_z^k), \tag{18.30}$$

Theorem 18.2 fully extends, only substituting the weight $j - k$ and the operator Δ_{j-k} for -1 and Δ_{-1}.

One can now generalize the Dirac and Bezout distributions: given a subgroup Γ' of Γ and a character χ of Γ' trivial on the subgroup $\Gamma'^{o}_{\infty} = \Gamma' \cap \Gamma^{o}_{\infty}$, we set, with $\mathfrak{d}$ and $\mathfrak{b}$ defined as in (3.29) and (3.30),

$$\mathfrak{D}_\chi^{\text{prime}} = 2\pi \sum_{g\in\Gamma'/\Gamma'^o_\infty} \chi(g)\,\mathfrak{d}\circ g^{-1}$$

and, formally at least,

$$\mathfrak{B}_\chi = \frac{1}{2} \sum_{g\in\Gamma'/\Gamma'^o_\infty} \chi(g)\,\mathfrak{b}\circ g^{-1}. \tag{18.31}$$

These are Γ'-automorphic distributions with character χ as defined in (18.1).

There is of course no convergence problem (in the space of tempered distributions) regarding the first of these two series, but we must again consider the second one more carefully. Assuming that $-I = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right) \in \Gamma'$, the case when $\chi(-I) = 1$ is taken care of by Theorem 3.3, requiring that one should consider, for $\ell \geq 1$, the distribution

$$\mathfrak{B}_\chi^\ell = \frac{1}{2} \sum_{g\in\Gamma'/\Gamma'^o_\infty} \chi(g)\,\mathfrak{b}^\ell\circ g^{-1} \tag{18.32}$$

instead of $\mathfrak{B}_\chi$.

In the case when $\chi(-I) = -1$, we need to reconsider our definition slightly (*cf.* the observation that immediately precedes the statement of Theorem 3.3), and we introduce instead of $\mathfrak{b}^\ell$ and $\mathfrak{B}_\chi^\ell$ the distributions

$$\mathfrak{c}^\ell = \left(\pi^2\,\mathcal{E}^2 + \frac{1}{4}\right)\left(\pi^2\,\mathcal{E}^2 + \frac{9}{4}\right)\cdots\left(\pi^2\,\mathcal{E}^2 + \left(\ell - \frac{1}{2}\right)^2\right)\mathfrak{b} \tag{18.33}$$

(compare (3.40)) and

$$\mathfrak{C}_\chi^\ell = \frac{1}{2} \sum_{g\in\Gamma'/\Gamma'^o_\infty} \chi(g)\,\mathfrak{c}^\ell\circ g^{-1}\ : \tag{18.34}$$

we can now give an analogue of Theorem 3.3.

Theorem 18.4. *Assume that $\ell \geq 1$. The series defining $\langle \mathfrak{C}_\chi^\ell\,, h\rangle$ converges whenever the function $h \in \mathcal{S}(\mathbb{R}^2)$ lies in the image of $\mathcal{S}(\mathbb{R}^2)$ by the operator $2i\pi\,\mathcal{E}$.*

Proof. Starting from (3.52), and noting that the operator $[2i\pi\,\mathcal{E}\,, \sigma - \frac{1}{2i\pi}\frac{\partial}{\partial s}] = \sigma + \frac{1}{2i\pi}\frac{\partial}{\partial s}$ is also the operator which occurs on the right-hand side of (3.51), one finds the equation (applying (3.52) to $h = 2i\pi\mathcal{E}\,f$ rather than f)

$$\begin{aligned} I_{n,m}(2i\pi\,\mathcal{E}\,[-4\pi^2\mathcal{E}^2 - 1]\,f) = &-\frac{4\pi^2}{m^2}\, I_{n,m}\left(\left(\sigma + \frac{1}{2i\pi}\frac{\partial}{\partial s}\right)^2 .\,2i\pi\,\mathcal{E}\,f\right)\\ &+\frac{4\pi^2}{m^2}\, I_{n,m}\left(\left(\sigma^2 + \frac{1}{4\pi^2}\frac{\partial^2}{\partial s^2}\right) f\right), \end{aligned} \tag{18.35}$$

from which the theorem follows, with the help of Lemma 3.4. □

One should note, however, that the function $f^{0,1}$ associated with $\mathfrak{C}_\chi^\ell$ – contrary to $f^{1,0}$ – will be zero for an arbitrary choice of χ. For the distribution $\mathfrak{b}$ introduced in (3.30) is invariant under the Fourier transformation $\mathcal{F}$ in (2.7), hence $2^{-\frac{1}{2}-i\pi\mathcal{E}}\,\mathfrak{b}$ is invariant under $\mathcal{G}$ and, as explained right after (2.9), the operator $\mathrm{Op}_{\sqrt{2}}(\mathfrak{b}) = \mathrm{Op}(2^{-\frac{1}{2}-i\pi\mathcal{E}}\,\mathfrak{b})$ vanishes on odd functions; so does, then, $\mathrm{Op}_{\sqrt{2}}(\mathfrak{B}_\chi)$ considered on any space of odd functions on which it is meaningful. However, assuming $\chi(-I) = -1$, a non-trivial result may be obtained from the consideration of

$$(u_z^1\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\,u_z) = \left\langle \mathfrak{b}\,,\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\,W(u_z^1, u_z)\right\rangle\,, \tag{18.36}$$

where, setting $z = x + iy$ again,

$$(2^{-\frac{1}{2}-i\pi\mathcal{E}}\,W(u_z^1, u_z))(s,\sigma) = \left(\frac{\pi}{2y}\right)^{\frac{1}{2}} |z|\,\frac{s-\bar{z}\sigma}{i\bar{z}}\,\exp -\pi\frac{|s-z\sigma|^2}{y}\,. \tag{18.37}$$

Theorem 18.5. *For any integer $N \geq 1$, and any complex number ν such that* $\mathrm{Re}\,\nu < -1$*, set*

$$V_{N,\chi}\left(z, \frac{1-\nu}{2}\right) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{z}{|z|} \sum_{g=\left(\begin{smallmatrix} n_1 & n_2 \\ m_1 & m_2\end{smallmatrix}\right)\in\Gamma'/\Gamma'^{o}_{\infty}} \chi(g)\left(\frac{y}{|-m_1z+n_1|^2}\right)^{\frac{1-\nu}{2}} \times \frac{|-m_1z+n_1|}{-m_1z+n_1}\,\exp 2i\pi N\,\frac{m_2z-n_2}{-m_1z+n_1}\,. \tag{18.38}$$

Then, for every integer $\ell \geq 1$, one has, with $\mathfrak{C}_\chi^\ell$ defined in (18.34)*, the identity*

$$(u_z^1\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{C}_\chi^\ell)\,u_z) = (4\pi)^\ell\,\frac{\Gamma\left(\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,V_{1,\chi}(z,\ell+1)\,. \tag{18.39}$$

Proof. Before giving it, we make a few observations. First, the extra factor $\frac{z}{|z|}$ in (18.38) accounts for the operator S_1 in (18.5): dropping this factor, one gets a Γ'-modular form of weight 1 with character χ in the usual sense, actually, again, a Poincaré-Selberg series. The number N in the exponent is not needed in the present theorem, but would occur, as shown in Proposition 5.3, when the action of Hecke's operators is considered. Finally, the series that defines $(u_z^1\,|\,\mathrm{Op}_{\sqrt{2}}(\mathfrak{C}_\chi^\ell)\,u_z)$ is, indeed, convergent, though a direct claim to Theorem 18.4 is not possible. For the function $W(u_i^1, u_i)$ conjugate to the one in (18.14) does not lie in the image under the operator $2i\pi\,\mathcal{E}$ of any function in $\mathcal{S}(\mathbb{R}^2)$: it can, however, with $\varepsilon = 0$ or 1, be written as the image of the function

$$(s,\sigma) \mapsto 2^{-\frac{3}{2}}\pi^{-\frac{1}{2}}\,(s-i\sigma)^{-1}\left(\varepsilon - e^{-2\pi\,(s^2+\sigma^2)}\right)\,,$$

either of which is either slightly singular at zero or not rapidly decreasing at infinity.

We start with the equation (a consequence of (18.36), (18.37))

$$\begin{aligned}(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\, u_z) &= \frac{|z|}{i\bar{z}} \left(\frac{\pi}{2y}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} (s-\bar{z})\, e^{2i\pi s}\, e^{-\frac{\pi|s-z|^2}{y}}\, ds \\ &= (2\pi)^{\frac{1}{2}} \frac{yz}{|z|}\, e^{2i\pi z}\,, \end{aligned} \tag{18.40}$$

a result to be compared to (3.56). Then

$$\begin{aligned}&\left(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{c}_\ell)\, u_z\right) \\ &= \left(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}} \left(\left(\pi^2\mathcal{E}^2 + \frac{1}{4}\right) \cdots \left(\pi^2\mathcal{E}^2 + \left(\ell - \frac{1}{2}\right)^2\right) \mathfrak{b}\right) u_z\right) \\ &= \left\langle \mathfrak{b}\,,\, \left(\pi^2\mathcal{E}^2 + \frac{1}{4}\right) \cdots \left(\pi^2\mathcal{E}^2 + \left(\ell - \frac{1}{2}\right)^2\right) . 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z^1, u_z) \right\rangle . \end{aligned} \tag{18.41}$$

Applying (18.40) and Lemma 18.3, we get

$$\left(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{c}_\ell)\, u_z\right) = \Delta_1\, (\Delta_1 + 2) \cdots (\Delta_1 + \ell(\ell-1))\, \left(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{b})\, u_z\right) . \tag{18.42}$$

We thus compute, with $f(z) = \frac{z}{|z|}\, e^{2i\pi z}$, recalling from (18.4) that $\Delta_1 = \Delta + iy\,(\frac{z}{\bar{z}} \frac{\partial}{\partial z} + \frac{\bar{z}}{z} \frac{\partial}{\partial \bar{z}})$,

$$\begin{aligned}\Delta_1\, (y^\ell\, f(z)) &= y^\ell \left[\Delta f - 2\ell y \frac{\partial f}{\partial y} - \ell(\ell-1) f + iy \left(\frac{z}{\bar{z}} \frac{\partial f}{\partial z} + \frac{\bar{z}}{z} \frac{\partial f}{\partial \bar{z}}\right) + \frac{\ell}{2} \left(\frac{z}{\bar{z}} - \frac{\bar{z}}{z}\right) f\right], \\ &= \left[2\pi\, (2\ell-1)\, y^{\ell+1} - \ell(\ell-1)\, y^\ell\right] \frac{z}{|z|}\, e^{2i\pi z}\,, \end{aligned} \tag{18.43}$$

so that

$$\Delta_1 (\Delta_1 + 2) \cdots (\Delta_1 + \ell(\ell-1)) \left(y \frac{z}{|z|} e^{2i\pi z}\right) = (4\pi)^\ell\, \frac{1}{2} . \frac{3}{2} \cdots \left(\ell - \frac{1}{2}\right) y^{\ell+1} \frac{z}{|z|} e^{2i\pi z} \tag{18.44}$$

and

$$\left(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{c}_\ell)\, u_z\right) = (2\pi)^{\frac{1}{2}}\, (4\pi)^\ell\, \frac{\Gamma\left(\ell + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\, y^{\ell+1}\, \frac{z}{|z|}\, e^{2i\pi z}\,. \tag{18.45}$$

On the other hand, using (18.13), we get

$$\begin{aligned}(u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{c}_\ell \circ g^{-1})\, u_z) &= \left\langle \mathfrak{c}_\ell \circ g^{-1}\,,\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z^1, u_z)\right\rangle \\ &= \left\langle \mathfrak{c}_\ell\,,\, 2^{-\frac{1}{2}-i\pi\mathcal{E}}\, W(u_z^1, u_z) \circ g\right\rangle \\ &= (u_z^1 \,|\, \mathrm{Op}_{\sqrt{2}}(\mathfrak{c}_\ell)\, u_z)\Big|_1 g^{-1}\,. \end{aligned} \tag{18.46}$$

The last equation, together with (18.45), leads to Theorem 18.5. □

19 New perspectives and problems in quantization theory

The word "quantization" covers a variety of activities. We take it to mean, approximately (and half-jokingly), pseudodifferential analysis plus harmonic analysis minus applications to partial differential equations. The rule of the game is to emulate, as best one can, Weyl's 1926 definition of his symbolic calculus. Experience shows that there are many possible directions in which this can be achieved. We are concerned, here, only with situations in which a good group invariance is present, and we shall be especially interested in the structure of the composition formulas. We shall also raise a number of questions, some of which seem quite tractable.

First, let us make it clear from the very start that we do not place much emphasis on Planck's constant (or on so-called deformation quantization either): as it is necessary to explain our reasons, we shall, very temporarily, introduce such a constant and denote it as ϵ, since h has been consistently used in this text to denote functions on $\mathbb{R}^2$. The ϵ-dependent one-dimensional Weyl's quantization rule is (compare (2.1))

$$(\mathrm{Op}(h)\,u)(x) = \epsilon^{-1} \int_{\mathbb{R}^2} h\left(\frac{x+y}{2}, \xi\right) e^{\frac{2i\pi}{\epsilon}(x-y)\xi}\, u(y)\, dy\, d\xi\,. \tag{19.1}$$

Let us start with a discussion of the variety of composition formulas available in the Weyl calculus, *i.e.,* the variety of ways to analyze the sharp operation, defined by

$$\mathrm{Op}(h_1 \# h_2) = \mathrm{Op}(h_1)\,\mathrm{Op}(h_2)\,. \tag{19.2}$$

Introducing the symplectic form $[\ ,\ ]$ defined by

$$[(y,\eta),(z,\zeta)] = -y\zeta + z\eta\,, \tag{19.3}$$

the first formula (hereafter referred to as the global integral composition formula) appears in two fully equivalent versions, to wit

$$(h_1 \# h_2)(X) = \frac{4}{\epsilon^2} \int_{\mathbb{R}^4} h_1(Y)\, h_2(Z)\, e^{-\frac{4i\pi}{\epsilon}[Y-X,Z-X]}\, dY\, dZ \tag{19.4}$$

(where it has been found convenient to denote the pair (x,ξ) as $X \in \mathbb{R}^2$), and

$$(h_1 \# h_2)(X) = \{e^{i\pi\epsilon L}(h_1(X+Y)\, h_2(X+Z))\}\,(Y=Z=0)\,, \tag{19.5}$$

where L stands for the operator, on functions of $(Y;Z) = ((y,\eta);(z,\zeta))$, defined by

$$i\pi L = (4i\pi)^{-1}\left(-\frac{\partial^2}{\partial y\,\partial\zeta} + \frac{\partial^2}{\partial z\,\partial\eta}\right)\,. \tag{19.6}$$

The Moyal formula is obtained after one has expanded the exponential $\exp(i\pi\epsilon L)$ as a power series in ϵ: it reads

$$(h_1\#h_2)(x,\xi) = \sum_{n\geq 0}\left(\frac{\epsilon}{4i\pi}\right)^n \sum_{j+k=n} \frac{(-1)^j}{j!\,k!}\left(\frac{\partial}{\partial x}\right)^j\left(\frac{\partial}{\partial \xi}\right)^k h_1(x,\xi)\left(\frac{\partial}{\partial x}\right)^k\left(\frac{\partial}{\partial \xi}\right)^j h_2(x,\xi)\,. \tag{19.7}$$

Of course, the right-hand side makes sense, as a finite sum, in the case when the symbols h_1 and h_2 are polynomials with respect to, say, the second variable, with smooth functions of the first variable as coefficients: these are just the symbols of linear differential operators. That the formula (with $\epsilon = 1$) is still correct, in some suitable asymptotic sense, when h_1 and h_2 lie in appropriate classes of symbols, is part of the subject of pseudodifferential analysis, and may require some care.

The role of the parameter ϵ is fully justified in the Weyl calculus itself, as it is the parameter which distinguishes one particular representation in the main "series" of irreducible representations of Heisenberg's group. That a small parameter should be considered in the so-called semi-classical analysis of general and difficult problems of partial differential equations (the literature on this is immense) is also, of course, beyond doubt. We are not forgetting, either, that the pioneers (Bohr, Dirac ...) of quantum mechanics did put emphasis, in their "correspondence principle", on ϵ as making (through a limiting process) the connection from quantum mechanics to classical mechanics possible. However, it would lead to a considerable loss of information (and interest) to systematize the role of such "small" parameters in situations where a large group is present, in which one can do much better, relying instead on the combined resources of harmonic analysis and spectral theory.

In this text, we made almost no use of the Heisenberg representation (of Heisenberg's group) into $L^2(\mathbb{R})$, which one may regard as a projective representation (*i.e.*, a representation up to the correction by complex factors of modulus one) of the additive group of $\mathbb{R}^2$ into $L^2(\mathbb{R})$, namely that defined (we now fix $\epsilon = 1$) by

$$(\tau(y,\eta)u)(t) = u(t-y)\,e^{2i\pi(t-\frac{y}{2})\eta}\,. \tag{19.8}$$

Starting from the function u_i (*cf.* (2.23)), hereafter denoted as ψ, such that $\psi(t) = 2^{\frac{1}{4}}\,e^{-\pi t^2}$, one gets a total set (ψ_z) in $L^2(\mathbb{R})$ if one sets $\psi_{x+iy} = \tau(x,y)\,\psi$. More precisely, $\|u\|^2 = \int_{\mathbb{C}} |(\psi_z|u)|^2\,dx\,dy$ for every $u \in L^2(\mathbb{R})$ or, equivalently,

$$u = \int_{\mathbb{C}} (\psi_z|u)\psi_z\,dx\,dy\,. \tag{19.9}$$

Of course, it should be clear to the reader that the subscript $z \in \mathbb{C}$ in (19.9) bears no relation to the subscript $z \in \Pi$ in u_z, in (2.23). In the first case, it is the Heisenberg representation that is involved; in the second case, the metaplectic one.

Another symbolic rule (though one can hardly consider it as a symbolic calculus of operators) is that which assigns to an operator A on $L^2(\mathbb{R})$ (under extremely weak assumptions regarding A) its Wick symbol f, the function on $\mathbb{C}$ defined by $f(z) = (\psi_z | A\psi_z)_{L^2(\mathbb{R})}$. In the reverse direction, the operator with anti-Wick symbol g (g living on $\mathbb{C}$ too) is defined as

$$\mathrm{Op}_{\mathrm{anti-Wick}}(g)\, u = \int_{\mathbb{C}} g(x+iy)\, (\psi_{x+iy}|u)\, \psi_{x+iy}\, dx\, dy\,. \tag{19.10}$$

Recall that, with $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ on $\mathbb{C} = \mathbb{R}^2$, any operator with an anti-Wick symbol g has a Weyl symbol $h = \exp\left(-\frac{\Delta}{8\pi}\right) g$, and every operator with a Weyl symbol h has a Wick symbol $f = \exp\left(-\frac{\Delta}{8\pi}\right) h$. Going backwards, however, is impossible.

The rank-one projection operator $u \mapsto (\psi_z|u)\,\psi_z$ has the Dirac mass at the point z as its anti-Wick symbol. In the Weyl calculus, the Dirac mass at $z = x+iy$ is associated to the involutive operator σ_z with

$$(\sigma_z u)(t) = u(2x-t)\, e^{4i\pi(x-t)y} \quad : \tag{19.11}$$

this operator is closely linked to the symmetry $S_z \colon w \mapsto 2z - w$ on the phase space $\mathbb{C}$ since

$$\sigma_z\, \mathrm{Op}(h)\, \sigma_z = \mathrm{Op}(h \circ S_z) \tag{19.12}$$

for every symbol h.

In all that precedes, it is not more difficult to start with $\mathbb{R}^n$ rather than $\mathbb{R}$. This is of course not the case in the non-commutative generalizations about which we shall briefly report now, and the rank-one situation, in particular that concerned with the upper half-plane Π, has been considerably more developed than the higher-rank analogues.

Given any hermitian symmetric space Π (admitting a bounded realization), the reproducing kernel property of the Bergman space of Π or, more generally, of Hilbert spaces H of holomorphic L^2 sections of appropriate line bundles on Π, makes it possible to assign to each point z of Π a vector ψ_z in a canonical way, so that the Wick-anti-Wick calculus of operators can be generalized right away: this was done by Berezin [6, 7], the pair of symbols taking the name of covariant and contravariant symbols. The connected component G of the identity in the group $\mathrm{Aut}(\Pi)$ of complex automorphisms of Π is a Lie group, acting transitively on Π (then $\Pi = G/K$ with K compact), and there is enough group structure in the whole construction to enable $\mathrm{Aut}(\Pi)$ to act on H through a unitary representation π, or at least a projective unitary representation. Instead of generalizing the Wick calculus, one can generalize the Weyl calculus in such a way that, just as in (19.11)–(19.12), the Dirac mass at $z \in \Pi$ should give rise to the operator σ_z, a self-adjoint operator associated through the representation π to the geodesic symmetry S_z on Π around z.

It is not at all our point, in this section, to discuss the merits of the two quantization rules, generalizing Wick's and Weyl's: let us just mention that, in parallel with the two species (covariant and contravariant) of symbols in the generalization of Wick's, there is a pair of (active and passive) symbols in the generalization of Weyl's. When viewed *globally* (we shall explain what this means in a moment), the generalization of the Weyl calculus certainly qualifies as a pseudodifferential analysis, at least in the examples which have been treated, while the Berezin calculus does not, as it suffers from the same defect as the Wick calculus: there is no going back from the covariant symbol to a contravariant symbol.

The only hermitian symmetric space Π which has been considered in this work is the upper half-plane: the family of Hilbert spaces to be considered is then the family $(H_{\tau+1},\ \tau > -1)$, introduced in Proposition 7.1 (the projective discrete series) and the main features of the calculi generalizing Wick's, or Weyl's, have in this case been recalled in [62, Section 17].

What we wish to emphasize in this last section is that the quantization method suggests a variety of problems in harmonic analysis, having to do with the concrete spectral decomposition of several interesting operators: a few of them have been treated by several authors.

A typical example of such problems is the spectral characterization of the operators *on the phase space* (this is where symbols live, as opposed to the *configuration space* where, in some realization as functions, elements of H do) linking two species of symbols of the same operator: for instance the contravariant and covariant ones, or the active and passive ones. The first of these two problems has attracted a lot of attention in recent years. Actually, a formula was given by Berezin himself [7], who did not have the time to write a proof. A complete proof, not forgetting the exceptional domains either, was subsequently given in [64], and more general situations (for instance that of matrix-valued Berezin kernels) were studied, very recently, by van Dijk, Hille, Molchanov, Pevzner [16, 17, 18], Zhang [72], Neretin [37, 38]. Arazy and Upmeier [2] went further in that they first considered *real* symmetric domains (where, of course, one is not, in general, properly dealing with symbols of *operators*, but some of the theory subsists): they also solved [1, 3], for all rank-one domains, the question of spectrally analyzing the link between the active and passive symbols of one operator. This latter, more difficult, problem, had only been solved, previously, in the case of the upper half-plane [55].

Many questions, concerned with the explicit spectral theory of invariant (not differential) operators in homogeneous spaces, are of great significance in this direction. For instance, can one have a theory that would be valid for phase spaces of the symmetric non-Riemannian type? A first approach in this direction was given in [35, 60], where the case of the phase space G/MA with $G = SL(2, \mathbb{R})$ (the one-sheeted hyperboloid, when viewed as an orbit in the coadjoint representation of G), was considered. As it turns out, this is the good choice of a phase space if you wish to have a symbolic calculus of operators acting on the Hilbert space of a representation taken from the *principal* series of representations of G (recall that the representation $\mathcal{D}_{\tau+1}$, defined in Proposition 7.1, for which the Wick-type or Weyl type calculi are available, is taken from the projective *discrete* series).

Another question concerns the use of the horocyclic space $G/MN = \mathbb{R}^2/(x,\xi) \sim (-x,-\xi)$ and its higher-rank generalizations. It is important for the following reasons. The operator, referred to above, linking the active and passive symbols of the same operator, contrary to that linking the contravariant and covariant symbols, is a very nice invertible operator [55], which would (as well as its inverse) be called a pseudodifferential operator in any symbolic calculus of operators one would like to make available, with Π taking the role of the *configuration*, rather than phase, space. However, it would be even nicer if this operator, linking the two species of symbols, were simply the identity operator. First, one should realize that the map: active symbol $\mapsto$ operator and the map: operator $\mapsto$ passive symbol are the adjoint of each other (the same goes with the pair contravariant-covariant) if one considers on one side the space of L^2-functions on the phase space, on the other side the space of Hilbert-Schmidt operators on the Hilbert space H of functions on which operators are supposed to act. It then becomes clear that only one species of symbol will do if it realizes an isometry between the Hilbert spaces of symbols and operators under consideration: this is satisfied, for instance, in the case of the Weyl calculus, and it is a very desirable property. Now, using the method of [56], one should be able to implement such a program if, instead of using a hermitian symmetric space G/K as a phase space, one would use the appropriate space of horocycles. In the case when $G = SL(2,\mathbb{R})$, this led to the definition of the horocyclic calculus (*cf.* Section 6); in the higher-rank case, it should work too: the proper harmonic analysis on the phase space itself can be found in [23, Chapter 2], and its two most important properties are that invariant operators on the horocyclic space are much easier to describe than on the space G/K, next that an appropriate Radon transformation connects the analysis on the two homogeneous domains.

We now come to the question of composition formulas: as will be seen, it raises a host of interesting questions, again of a spectral-theoretic nature, in a harmonic analysis setting. From the beginning of this section, one may expect that several answers to this question may be possible. First, we discard the Moyal-type formula (19.7) as well as the attempts at generalizing it with hermitian symmetric spaces as phase spaces: no such formula can be relevant. Berezin did suggest one in [7], but that was after he had identified the ratio of two Gamma functions with the asymptotic expansion one can get from an application of Stirling's formula: his point of view in this early work was that the parameter which we have denoted as τ in the case when $G = SL(2,\mathbb{R})$ should be understood as the inverse of a "Planck's constant" and, true enough, if you neglect remainders formally lying in $\mathcal{O}(\tau^{-\infty})$, you will be able to give some kind of justification to Berezin's composition formula. However, we now explain why this is not a correct point of view. First, there is no such thing as a calculus of operators using solely, say, the covariant species of symbol: there is a trivial formula enabling one to find the covariant symbol of the product of two operators with given contravariant symbols, but there is no going backwards from a covariant symbol to a contravariant symbol. So, let us take instead a calculus in which such a composition of symbols can exist, for instance the one defined by the passive symbol. In the case when $G = SL(2, \mathbb{R})$, a fully explicit "global" such formula was given, a generalization of (19.5). Let us observe that the formula (19.5) has three ingredients: a chart $\Phi_X : \mathbb{R}^2 \mapsto \mathbb{R}^2$ (that defined by $\Phi_X(Y) = X + Y$) for every point X of the phase space, the ϵ-dependent exponential function, and the operator $i\pi L$ on $\mathbb{R}^2$. In the case when the phase space is the upper half-plane, it was found in [58] (one can also find the formula in [62, (17.36)]) that a formula with the same kind of structure, and exactly the same operator $i\pi L$, works, with the following modifications: first, the chart Φ_z becomes a z-dependent chart (easily defined in terms of the normal geodesic coordinates around z on Π) from $\mathbb{R}^2$ onto Π. But next, and most important, the exponential (ϵ-dependent) function must be replaced by a certain explicit function $E_\tau(i\pi L)$ with an essential singularity at $\tau = \infty$ when its dependence with respect to this parameter is concerned: thus, Taylor expansions with respect to τ^{-1} are essentially meaningless.

We shall not raise the question whether analogues of the formula we have just been discussing to the higher-rank case can be found: this is probably untractable, and we shall suggest in a moment a different, deeper formulation of the composition problem. Meanwhile, let us note that some questions concerning the active (or passive, there is no distinction at this point) calculus have been answered in the affirmative in some higher-rank cases. The following question is quite natural in view of experience with pseudodifferential analysis, and its solution relies as much on methods developed in this context [53], in particular the systematic use of families of coherent states and the appropriate concept of Wigner function, than on harmonic analysis: is it true that symbols on Π which are C^∞, and satisfy the property that they remain bounded functions after having been applied any operator in the differential algebra $D(G/K)$, give rise to bounded operators (on the relevant Hilbert space)? For two different series of hermitian symmetric domains, the positive answer was given in [57], then [66]. A more detailed analysis of the characterization of certain classes of operators by properties of their symbols has been developed in the case of the upper-half plane[58]. For contravariant symbols, of course, no such question is of any interest since very bad symbols yield very good operators, in general: of course, poor operators may have excellent covariant symbols.

There is no doubt that Moyal's type formulas like (19.7), which give some approximation of the symbol of the product of two operators in terms of *local* operations (such as Taylor expansions), are the quintessence of the way pseudodifferential analysis is applied to problems in partial differential equations. It is true, too, even though this may have only historical value by now, that the founding fathers of quantum mechanics viewed the quantization problem as a link between operator theory and *geometry* on the phase space, based on the use of such tools as Poisson brackets and related concepts (canonical transformations ...). However, our point is that one should more properly view the quantization method as a link between the spectral theory of operators on the configuration space and the phase space respectively.

Consider the case when the phase space $\mathcal{X}$ is acted upon, in a transitive way, by some group G of measure-preserving transformations: assume, also, that the symbolic calculus you are considering is covariant (*cf.* (2.4)) under this action of G and a certain representation π of G in H. This means that $\pi(g)\,\mathrm{Op}(h)\,\pi(g)^{-1} = \mathrm{Op}(h \circ g^{-1})$ for every admissible symbol h and every $g \in G$. In the nicer cases, the regular representation of G into $L^2(\mathcal{X})$ will decompose as a superposition (both a continuous integral part and a series will be needed in general) of *irreducible* unitary representations of G. It thus makes sense to ask for a formula that would give the composition of symbols in relation to such a decomposition: in other words, we want to put our hands on the terms of the decomposition of $h_1 \# h_2$ in terms of those of h_1 and h_2.

We want to briefly report about cases when this program has been implemented, and tell why we view this way of looking at composition formulas as a fundamental one. First, consider the case of the Weyl calculus (the n-dimensional case would work just as well). If you are interested in its covariance under the Heisenberg representation, the group G to be considered is $\mathbb{R}^2$, acting upon itself by translations. Now the differential operators on the phase space $\mathbb{R}^2$ which commute with this action are the differential operators with constant coefficients, and the joint (generalized) eigenfunctions of this algebra of operators are just the exponentials $X \mapsto e^{2i\pi\langle A,X\rangle}$, $A \in \mathbb{R}^2$: this provides the decomposition of $L^2(\mathbb{R}^2)$ into irreducibles. The formula we are asking for reduces in this case to

$$e^{2i\pi\langle A,X\rangle} \# e^{2i\pi\langle B,X\rangle} = e^{i\pi[A,B]}\, e^{2i\pi\langle A+B,X\rangle} \ : \tag{19.13}$$

now the integral composition formula (19.4) is nothing else than this formula, coupled with the Fourier inversion formula on $\mathbb{R}^2$. A totally different way to look at the (one-dimensional, in this case) Weyl calculus is to forget about the Heisenberg representation and interest oneself in the metaplectic representation instead: then the geometric action of the group $G = SL(2,\mathbb{R})$ on $\mathbb{R}^2$ is the linear action, the Euler operator generates the algebra of invariant differential operators on the phase space, and the decomposition of $L^2(\mathbb{R}^2)$ into irreducibles is the decomposition into homogeneous components (there are some extra signs to be considered unless, as we have done in this work, one is only interested in even symbols: *cf* [62, Section 5]). Then the composition formula is nothing but the one the proof of which has been detailed as Theorem 17.1. Recall from the remark at the end of Section 17 that the last formula has an extension to the Op^p-calculi.

Consider now the quantization rules associated with various homogeneous spaces of $G = SL(2,\mathbb{R})$. Let us start with $\Pi = G/K$, and the symbolic calculus defined by means of the passive symbol in the $\mathcal{D}_{\tau+1}$ (*cf.* Proposition 7.1)-calculus: Section 17 in [62] contains a complete set of formulas, of a spectral-theoretic nature, relating the various calculi available in relation to the representation $\mathcal{D}_{\tau+1}$. The Laplacian Δ generates the algebra of G-invariant differential operators on Π, and the question is to give the spectral decomposition, as defined by Mehler's formula (11.4)–(11.5), of the sharp product (from the $\mathcal{D}_{\tau+1}$-calculus) of two symbols f_1 and f_2, (generalized) eigenfunctions of the Laplacian for the eigenvalues $\frac{1+\lambda_1^2}{4}$ and $\frac{1+\lambda_2^2}{4}$ respectively. Here it is, in the case when $\tau = \pm\frac{1}{2}$:

$$\left(f_1 \underset{\tau}{\#} f_2 \right)_\lambda = \frac{2^{-\tau-1}}{\pi} \sum_{\varepsilon=\pm 1} \sum_{j=0}^{1} (-i)^j \left(\frac{i\varepsilon\lambda_1\lambda_2}{\lambda} \right)^{\tau+\frac{1}{2}}$$
$$\times \frac{\Gamma\left(\frac{\tau}{2}+\frac{1}{4}+\frac{i\lambda}{4}\right)\Gamma\left(\frac{\tau}{2}+\frac{1}{4}-\frac{i\lambda}{4}\right)\Gamma\left(\frac{-\tau}{2}+\frac{1}{4}+\frac{i\lambda_1}{4}\right)\Gamma\left(\frac{-\tau}{2}+\frac{1}{4}-\frac{i\lambda_1}{4}\right)\Gamma\left(\frac{-\tau}{2}+\frac{1}{4}+\frac{i\lambda_2}{4}\right)\Gamma\left(\frac{-\tau}{2}+\frac{1}{4}-\frac{i\lambda_2}{4}\right)}{\Gamma\left(\frac{1+i(\varepsilon\lambda-\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\varepsilon\lambda-\lambda_1+\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(-\varepsilon\lambda+\lambda_1-\lambda_2)+2j}{4}\right)\Gamma\left(\frac{1+i(\varepsilon\lambda+\lambda_1+\lambda_2)+2j}{4}\right)}$$
$$\times \left(f_1 \underset{j}{\times} f_2 \right)_\lambda \quad : \qquad (19.14)$$

recall from (11.2) that $f_1 \underset{0}{\times} f_2$ and $f_1 \underset{1}{\times} f_2$ denote the pointwise product and half the Poisson bracket of the two functions under consideration. We shall not prove this formula here, only explain how this can be done. First, one should break the formulas in Theorem 17.1 into two parts, so as to consider the even-even and odd-odd parts of the Weyl calculus separately. Next, use the maps $\mathcal{S}q^{\frac{1}{2}}_{\text{even}}$ and $\mathcal{S}q^{\frac{3}{2}}_{\text{odd}}$, as defined in (6.9) and (6.10), to transfer the two parts of the even Weyl calculus to two calculi acting on the spaces associated with the two representations $\mathcal{D}_{\frac{1}{2}}$ and $\mathcal{D}_{\frac{3}{2}}$ respectively: what one gets is the two horocyclic calculi (Theorem 6.1) relative to these representations. Now, [62, (17.20) and (17.15)] relates the horocyclic symbol of an operator to its passive symbol, in terms making use of the Radon transformation. Finally, (17.95) and (17.98) permit one to relate the bilinear operations on Π in (19.15) to the operations on $\mathbb{R}^2$ associated with the integral kernels $\chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1, s_2; s)$ which occur in the formulation of Theorem 17.1. In the case when $\tau = 0$ (the Hardy space), a formula analogous to (19.14) has been given by our student Marzi: the coefficients to be substituted for the products of Gamma factors, however, involve [32] two-dimensional integrals.

From the point of view of harmonic analysis – though possibly not from that of pseudodifferential analysis – this way of looking at the composition formula has some features which are more essential than the global formula referred to above: for it does not really care about which (covariant) symbolic calculus you are using. For instance, if you wish to use the active symbol rather than the passive symbol, some complicated change is needed in the global formula: but in (19.14), all you have to do is to insert three extra factors, depending on λ_1, λ_2 and on λ respectively, making up for the connecting link (expressed in spectral-theoretic terms [62, (17.15)]) between the two calculi under consideration.

Still in the case when $G = SL(2,\mathbb{R})$, one can do the same with the phase space $\mathcal{X} = G/MA$ (a one-sheeted hyperboloid) instead of G/K. Now [47], the decomposition of $L^2(G/MA)$ involves both a continuous part and a discrete part: the latter one is the Hilbert sum of the representations $\mathcal{D}_{2k}$, $k = 1, 2, \ldots$. As said before, the space G/MA is a good phase space for the symbolic calculi of operators acting on the Hilbert spaces of representations taken from the principal series $(\pi_{i\lambda})$ (the complementary series would do just as well). As has been proved in [63], the discrete part of $L^2(G/MA)$ is closed under the sharp product of symbols, and there are some fully explicit series of *differential* bilinear operators permitting one to compute the components (under the action of G) of the sharp product of any two such symbols: in this case, and for this part of the calculus, there is a Moyal-type series (but, again, certainly not a power series in λ^{-1}: the Gamma-like coefficients, as functions of λ, have an essential singularity at infinity), and it even converges in the L^2 sense.

One of our incentives towards obtaining such a formula had been the following fact, established as a lemma by H.Cohen in [12], and also reported about in [71]: if f_1 and f_2 are holomorphic modular forms (with respect to any arithmetic group) of weights k_1 and k_2 and j is a non-negative integer, the function

$$F_j^{k_1,k_2}(f_1,f_2) = \sum_{l=0}^{j}(-1)^l \binom{k_1+j-1}{l} \binom{k_2+j-1}{j-l} f_1^{(j-l)} f_2^{(l)} \qquad (19.15)$$

is a holomorphic modular form of weight k_1+k_2+2j. One cannot fail to recognize this fact as a fact of covariance and, indeed, the various terms in the composition formula, in the case under consideration, are just provided by the Cohen bilinear machine, up to the (λ-dependent) coefficients [63]. A related idea (without any symbolic calculus of *operators*, though) was developed, independently, in [13].

This may be a good place to start introducing considerations relative to automorphic function theory. But before doing this, we may raise the question of generalizing this way to look at the composition formula to higher-rank cases. As said before, we consider it as both more tractable and deeper, at least from the point of view of harmonic analyis, than the search for global formulas. It might start with the systematic search, on homogeneous spaces $\mathcal{X}$, for invariant (in some obvious sense) bilinear operations, defined on pairs of (generalized) joint eigenfunctions of the algebra $D(\mathcal{X})$ of invariant differential operators: in the case when $\mathcal{X}$ is the upper half-plane, there are essentially two such operations, namely the operations $(f_1,f_2) \mapsto (f_1 \underset{j}{\times} f_2)_\lambda$ which occur in (19.14). More operations are to be expected in the higher-rank cases.

Lifting automorphic functions from $\Gamma\backslash\Pi$ to functions on $\Gamma\backslash G$, as is usually done [8, p. 242], one may look at the problem of developing automorphic pseudodifferential analysis from a fairly general point of view, very similar to what precedes: namely, modular forms appear as the building blocks in the decomposition of the right regular representation of G in the appropriate L^2-space, and the question is to compute and decompose the sharp product (with respect to some calculus) of any two such forms. There are many obvious questions which may be raised in this context. In the present work, we have considered only the case when $\Gamma = SL(2,\mathbb{Z})$, certainly an interesting one since both a continuous and a discrete part are involved. Could one do something similar, at least, say, when $SL(2, \mathbb{Z})$ is replaced by some congruence subgroup? In the cocompact case, could one find interesting formulas in connection with, say, quaternion algebras? Since any Riemann surface with genus ≥ 2 can be realized as a quotient of Π by such an arithmetic group, this would be a good way to answer the problem of quantizing all Riemann surfaces. Of course, higher-rank analogues can be expected to be much more complicated, since Siegel's domains are to replace the upper half-plane: but, looking back at the introduction of [67], one may remember that A.Weil had such modular forms in mind when he introduced the metaplectic representation. Needless to say, the introduction of a "small parameter" would be even more counterproductive in the arithmetic context.

Finally, let us just mention that, when the parameter τ in $\mathcal{D}_{\tau+1}$ goes to ∞, the representation has a non-degenerate limit or, more properly said, contraction: in this case it is a representation of the (3-dimensional) Poincaré group of lowest dimension. There is, then, a perfectly canonical symbolic calculus "in the limit" which, under increasing degrees of generality, was developed in [54, 59, 65] under the name of "Fuchs calculus". Contrary to all the other calculi mentioned in the present section (with the exception of the Weyl calculus), the Fuchs calculus admits composition formulas of the Moyal type, valid in the asymptotic sense for appropriate classes of symbols, which should make it useful as a pseudodifferential analysis, though applications to partial differential equations have not yet been carried far enough.

As a byproduct of [59], symbolic calculi meant as calculi of observables fully compatible with the principles of special relativity have been developed (the Klein-Gordon and the Dirac calculi): a very short exposition of the fundamentals of these two calculi can be found in [61]. It should be observed that these depend on two constants (Planck's constant and the velocity of light), and that the Weyl calculus is of course the limit of the Klein-Gordon calculus as $c \to \infty$: the point of this remark is to recall that, in physics, Planck's constant is far from being the only "small constant" of interest. It is only after we had developed the Op^p-calculi in view of their applications to the automorphic pseudodifferential analysis that we realized that, again, these calculi can be viewed as the calculi of observables (satisfying a certain superselection rule) associated with some elementary particle, in this case the neutrino, as was shown in Section 8.

Index of Notation

Subject Index

Bibliography

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