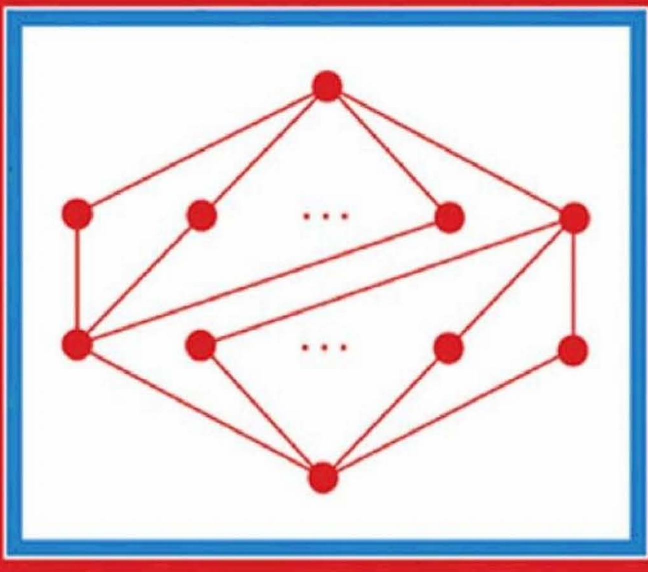
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 65

SPERNER THEORY

KONRAD ENGEL



CAMBRIDGE

more information – www.cambridge.org/9780521452069

The starting point of this book is Sperner's theorem, which answers the question: What is the maximum possible size of a family of pairwise unrelated (with respect to inclusion) subsets of a finite set? This theorem stimulated the development of a fast growing theory dealing with extremal problems on finite sets and, more generally, on finite partially ordered sets.

This book presents Sperner theory from a unified point of view, bringing combinatorial techniques together with methods from programming (e.g. flow theory and polyhedral combinatorics), from linear algebra (e.g. Jordan decompositions, Lie-algebra representations, and eigenvalue methods), from probability theory (e.g. limit theorems), and from enumerative combinatorics (e.g. Möbius inversion).

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

Volume 65

Sperner Theory

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

4 W. Miller, Jr. *Symmetry and separation of variables*
6 H. Minc *Permanents*
11 W. B. Jones and W. J. Thron *Continued fractions*
12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter, and S. D. Riemenschneider *Birkhoff interpolation*
21 W. T. Tutte *Graph theory*
22 J. R. Bastida *Field extensions and Galois theory*
23 J. R. Cannon *The one-dimensional heat equation*
25 A. Salomaa *Computation and automata*
26 N. White (ed.) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie, and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 N. White (ed.) *Combinatorial geometries*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations in several variables*
32 M. Kuczma, B. Chozewski, and R. Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen, and O. Staffans *Volterra integral and functional equations*
35 G. Gasper and M. Rahman *Basic hypergeometric series*
36 E. Torgersen *Comparison of statistical experiments*
37 A. Neumaier *Interval methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
40 N. White (ed.) *Matroid applications*
41 S. Sakai *Operator algebras in dynamical systems*
42 W. Hodges *Basic model theory*
43 H. Stahl and V. Totik *General orthogonal polynomials*
44 R. Schneider *Convex bodies*
45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
46 A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler *Oriented matroids*
47 G. A. Edgar and L. Sucheston *Stopping times and directed processes*
48 C. Sims *Computation with finitely presented groups*
49 T. Palmer *Banach algebras and the general theory of *-algebras*
50 F. Borceux *Handbook of categorical algebra I*
51 F. Borceux *Handbook of categorical algebra II*
52 F. Borceux *Handbook of categorical algebra III*
54 A. Katok and B. Hasselblatt *Introduction to the modern theory of dynamical systems*
55 V. N. Sachkov *Probabilistic methods of discrete mathematics*
56 V. N. Sachkov *Combinatorial methods of discrete mathematics*
57 P. M. Cohn *Skew fields*
58 Richard Gardner *Geometric tomography*
59 George A. Baker Jr. and Peter Graves-Morris *Padé approximants*
60 Jan Krajíček *Bounded arithmetic, propositional logic, and complexity theory*
61 H. Groemer *Geometric applications of Fourier series and spherical harmonics*
62 H. O. Fattorini *Infinite dimensional optimization and control theory*
63 A. C. Thompson *Minkowski geometry*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Sperner Theory

KONRAD ENGEL
Universität Rostock, Germany

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1997

Printed in the United States of America

Typeset in Times Roman

Library of Congress Cataloging-in-Publication Data

Engel, Konrad, 1956–

Sperner theory / Konrad Engel.

p. cm. – (Encyclopedia of mathematics and its applications; v. 65)

Includes bibliographical references and index.
ISBN 0 521 45206 6 (hardback)
1. Sperner theory. 2. Partially ordered sets. I. Title.
II. Series.
QA171.485.E537 1996 96-20967
511.3′2 – dc20

A catalog record of this book is available from the British Library

ISBN 0 521 45206 6 hardback

CONTENTS

PREFACE

Sperner theory is a very lively area of combinatorics and combinatorial optimization. Though nobody knows the exact age of it, we may assume that it was born in the end of the 1920s and that at least F.S. Macaulay and E. Sperner belong to the set of parents. Macaulay's contribution is discussed in the chapter on Macaulay posets. Sperner's theorem is the starting point of this book. It answers the following question: Given a family of pairwise unrelated (with respect to inclusion) subsets of a finite set, what is the maximum size of this family? In the 1930s the set of parents was enlarged by P. Erdős, C. Ko, and R. Rado, who studied intersection conditions, and in the beginning of the 1950s, R.P. Dilworth entered this set by proving his famous min–max theorem on partially ordered sets. We should not forget the grandfather. At the end of the last century R. Dedekind failed to find an explicit formula for the number of monotone Boolean functions, and this difficult problem was later attacked by several authors. It is impossible to mention all the offspring and descendants of Sperner theory. Many of them can be found in the references. The list of names there shows that many well-known mathematicians of this century have contributed to the development of the theory.

Several aspects make Sperner theory so interesting. The problems often may be clearly and simply formulated. For the solutions one indeed needs creativity, and the techniques are frequently surprising. Concerning applications and methods, there are several unexpected relations to other branches of mathematics, such as programming, algebra, probability theory, number theory, and geometry. Thus, studying Sperner theory means learning many important techniques in discrete mathematics and combinatorial optimization on a particular theme.

Ten years ago H.-D.O.F. Gronau and I wrote the monograph "Sperner theory in partially ordered sets" [161]. This work was followed by the books of B. Bollobás [73], I.A. Anderson [32], and C. Berge [50] that also discuss aspects of Sperner theory. Since I saw that a revision of [161] would be insufficient, I decided to write a new book. It has been my main goal to present a unifying theory that covers many seemingly distant and separate results, including as far as possible all important and famous theorems related to the subject. To meet this goal, I took a completely new approach toward selection, organization, and presentation of the material. Of course I also tried my best in order to give an updated presentation reflecting the modern development of the last ten years. More attention is paid to algorithmic aspects and much more space is devoted to the study of the Boolean lattice.

Up to a small epsilon everything is given with complete proofs (matched to the "theory"), so that the reader need not consult the original papers. The book is self-contained, and the reader needs only basic knowledge of mathematics. I hope that enthusiasm for the subject increases exponentially with the number of pages read.

Recognizing that it is impossible to refer to all of the large number of papers related to Sperner theory, I have put the emphasis on the main results and methods of the theory and not on a complete survey of all related results. Consequently, many topics are presented only in examples, and I apologize for omitting several interesting results.

I also intended to keep intersections with existing and forthcoming books small. For instance, I wrote nothing on dimension theory (W.T. Trotter [452]) and design theory (e.g., T. Beth, D. Jungnickel, H. Lenz [53]). The probabilistic methods are not discussed extensively because they are covered in the book by N. Alon and J.H. Spencer [29]. Some further algebraic methods will be contained in the forthcoming book by P. Frankl and L. Babai [35]; and L.H. Harper and J.D. Chavez [261] are building up a theory for discrete isoperimetric problems. Since there exist excellent books of problems as well as books containing problems with solutions like those of G.P. Gavrilov and A.A. Sapozhenko [220] and L. Lovász [354] and I.A. Anderson [32], I did not include exercises. However, several theorems, lemmata, propositions, corollaries, and claims in this book are suitable for exercises (which thus have complete solutions). Some *open problems* are mentioned in the text, and the corresponding page numbers can be found using the index.

The "easiest" examples for an author are those the author has studied. So my own list of references may seem inappropriately long. I included also some new results that will not be published elsewhere. Thus I hope the book might shed new light on some topics also for people who already do active research in the field.

I wish to thank Larry Harper, who initiated this project and who contributed with stimulating discussions. Many other mathematicians gave hints and indications to new results, and I gratefully acknowledge their contributions. Beyond that, I wish to express my gratitude to all my teachers, direct and indirect. In particular I would like to thank Christian Bey, Frank Ihlenburg, and Uwe Leck, who read the whole or at least parts of the manuscript very thoroughly. They found several unpleasant typographical and substantive errors, and I hope that their number is minimized now.

I am indebted to all involved in the production of this book. Cynthia Benn and Rena Wells were of great assistance in improving the style in many ways.

My deep thanks go to my parents Helga and Wolfgang Engel for (among many other things) introducing me early into mathematics as well as encouraging and supporting my occupation with it.

Above all I am grateful to my own family for opening widely happy new pages of my life, for their continuous encouragement and understanding. I leave it as an exercise to discover the names of my wife and my children in the book.

1

Introduction

1.1. Sperner's theorem

We start our investigations with the theorem that was the cornerstone for the whole theory. In the thirties, forties, and fifties few further results of a similar kind were published. But beginning with the sixties, the combinatorics of finite sets has undergone spectacular growth. Not only have subsets of a finite set been studied, but also more general objects like partially ordered sets. Many important results in this area can be found in this book.

Theorem 1.1.1 **(*Sperner*** **[436]).** *Let n be a positive integer and $\mathcal{F}$ be a family of subsets of $[n] := \{1, \ldots, n\}$ such that no member of $\mathcal{F}$ is included in another member of $\mathcal{F}$, that is, for all $X, Y \in \mathcal{F}$ we have $X \not\subseteq Y$. Then*
(a)

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

(b) Equality holds iff

$$\mathcal{F} = \begin{cases} \{X \subseteq [n] : |X| = \frac{n}{2}\} & \text{if } n \text{ even,} \\ \{X \subseteq [n] : |X| = \frac{n-1}{2}\} \text{ or } \{X \subseteq [n] : |X| = \frac{n+1}{2}\} & \text{if } n \text{ odd.} \end{cases}$$

Proof. The following presents Sperner's original approach. Clearly the families given in (b) satisfy the conditions of the theorem and have the corresponding size. Hence we must show that there do not exist "better" (resp. "other") families. Let

$\mathcal{F}$ be any family of maximum size satisfying the conditions of the theorem. Let

$$l(\mathcal{F}) := \min\{i : \text{ there is some } X \in \mathcal{F} \text{ with } |X| = i\},$$

$$u(\mathcal{F}) := \max\{i : \text{ there is some } X \in \mathcal{F} \text{ with } |X| = i\}.$$

For brevity we write l instead of $l(\mathcal{F})$ if $\mathcal{F}$ is clear from the context. Let

$$\mathcal{G} := \{X \in \mathcal{F} : |X| = l\},$$

$$\mathcal{H} := \{Y \subseteq [n] : |Y| = l + 1 \text{ and there is some } X \in \mathcal{G} \text{ with } X \subset Y\},$$

$$\mathcal{F}' := (\mathcal{F} - \mathcal{G}) \cup \mathcal{H}.$$

Claim 1. The family $\mathcal{F}'$ satisfies the conditions of the theorem.

Proof of Claim 1. The only obstacle could be the existence of some $Y \in \mathcal{H}$ and some $Z \in \mathcal{F} - \mathcal{G}$ such that $Y \subset Z$. But by definition of $\mathcal{H}$ we would find also some $X \in \mathcal{G} \subseteq \mathcal{F}$ with $X \subset Y$. Thus $X \subset Z$, contradicting the fact that $\mathcal{F}$ satisfies the conditions of the theorem. □

Claim 2. Let $l \le \frac{n-1}{2}$. Then $|\mathcal{F}'| \ge |\mathcal{F}|$, and $|\mathcal{F}'| = |\mathcal{F}|$ implies $l = \frac{n-1}{2}$.

Proof of Claim 2. Let us count the number N of pairs (X, Y) with $X \in \mathcal{G}$, $Y \in \mathcal{H}$, $X \subset Y$ in two different ways. For a fixed member X of $\mathcal{G}$, we can find exactly $n - l$ corresponding sets Y since Y can be obtained in a unique way from X by adding one element of $[n] - X$. Thus

$$N = |\mathcal{G}|(n - l). \tag{1.1}$$

For a fixed member Y of $\mathcal{H}$, we can find analogously $l + 1$ sets X with $X \subset Y$, $|X| = l$. But it is not necessary that all these sets X belong to $\mathcal{G}$. Thus

$$N \le |\mathcal{H}|(l + 1). \tag{1.2}$$

By (1.1) and (1.2) and because of $l \le \frac{n-1}{2}$,

$$|\mathcal{G}|(n - l) \le |\mathcal{H}|(l + 1), \tag{1.3}$$

$$\frac{|\mathcal{H}|}{|\mathcal{G}|} \ge \frac{n - l}{l + 1} \ge \frac{n - \frac{n-1}{2}}{\frac{n-1}{2} + 1} = 1,$$

where the last inequality is only an equality if $l = \frac{n-1}{2}$. Since $\mathcal{F}$ satisfies the conditions of the theorem, $\mathcal{F} \cap \mathcal{H} = \emptyset$. Thus

$$|\mathcal{F}'| = |\mathcal{F}| - |\mathcal{G}| + |\mathcal{H}| \ge |\mathcal{F}| \quad \left(\text{equality implies } l = \frac{n-1}{2}\right).$$

□

Recall that we have already chosen $\mathcal{F}$ as a family of maximum size that satisfies the conditions of the theorem. We obtain from Claims 1 and 2

$$l(\mathcal{F}) \geq \frac{n-1}{2} \text{ and (analogously) } u(\mathcal{F}) \leq \frac{n+1}{2}$$

because otherwise we could construct a family $\mathcal{F}'$ of larger size. If n is even, we are already done. So let n be odd. If $l(\mathcal{F}) = u(\mathcal{F})$, both values are either $\frac{n-1}{2}$ or $\frac{n+1}{2}$ and, consequently,

$$|\mathcal{F}| \leq \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}. \tag{1.4}$$

Thus assume that $l(\mathcal{F}) = \frac{n-1}{2}$, $u(\mathcal{F}) = \frac{n+1}{2}$. In this case we will obtain a contradiction. By Claim 2,

$$|\mathcal{F}| \leq |\mathcal{F}'| \leq \binom{n}{\frac{n+1}{2}}. \tag{1.5}$$

Since $\mathcal{F}$ is of maximum size, we must have equality in (1.5), thus also in (1.2), and this is possible only if for every $Y \in \mathcal{H}$ each l-element subset of Y belongs to $\mathcal{G}$. But consider under all pairs (Y, Z) with $Y \in \mathcal{H}$, $Z \in \mathcal{F} - \mathcal{G}$ such a pair for which $|Y \cap Z|$ is maximum. Since $|Y| = |Z| = l + 1$, $Y \neq Z$, there exist some $y \in Y - Z$ and some $z \in Z - Y$. In view of the preceding remarks, $Y - \{y\}$ must belong to $\mathcal{G}$; thus $Y' := (Y - \{y\}) \cup \{z\}$ belongs to $\mathcal{H}$. Now $|Y' \cap Z| = |Y \cap Z| + 1$ is a contradiction to the maximality of $|Y \cap Z|$. ■

This result (or at least part (a)) was obtained independently by several other mathematicians. As examples we mention here Gilbert [223] and the succeeding paper of Mikheev [369]. Using

$$\left\lfloor \frac{n}{2} \right\rfloor := \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

part (a) of Theorem 1.1.1 reads: $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's theorem can be always applied if one works with families of subsets that are pairwise incomparable with respect to inclusion. Here we consider only one example. It was found by Demetrovics [130] in a study of the relational model of data structures proposed by Codd [118] and Armstrong [33]. Suppose we are given m persons $P_1, \ldots, P_m$ and n attributes $A_1, \ldots, A_n$ like last name, first name, date of birth, place of birth, weight, and so forth. For each person, each attribute takes on a unique value. Using a right coding, we may suppose that each such value is a natural number. Thus all data on the persons can be represented by an $m \times n$-matrix $D = (d_{ij})$, where d_{ij} is the value of A_j for person P_i. We say that a set of attributes $\{A_j : j \in X\}$, and, briefly, the set $X \subseteq [n]$, is a *key* if for fixed values of A_j, $j \in X$, there exists at most one person P_i that has these values – that is, if

$$d_{ij} = d_{i'j} \text{ for all } j \in X \text{ implies } i = i'$$

(one already "knows" the person if one knows his values of the attributes of a key). A key $X \subseteq [n]$ is called a *minimal key* if there is no key X' with $X' \subset X$.

Corollary 1.1.1. *For n attributes the number of minimal keys is not greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and this bound is the best possible.*

Proof. It is trivial to see that the family $\mathcal{F}$ of *minimal* keys satisfies the conditions of Theorem 1.1.1, which proves the upper bound. To see that this bound can be attained, we must construct a corresponding matrix D. We do this here in a simple way with a large m, namely $m := \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Set all entries of the first row of D equal to 1. Then order all n-dimensional rows with $\lfloor \frac{n}{2} \rfloor - 1$ ones and $n + 1 - \lfloor \frac{n}{2} \rfloor$ zeros in any way, but count them from 2 up to $\binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Define the ith row of D to be the ith row from above, but with all zeros replaced by the number $i, i = 2, \ldots, \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Then every $\lfloor \frac{n}{2} \rfloor$-element subset of $[n]$ is a key since either we find in the corresponding places only ones – and this can be the case only in the first row, or we find some number $i \neq 1$ – and this can be the case only in the ith row. Moreover, it is easy to see that there is no key of size smaller than $\lfloor \frac{n}{2} \rfloor$; thus we have indeed $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ minimal keys. ∎

The preceding construction is due to Demetrovics and Katona [131]. For more information on similar combinatorial problems of data structures, see, for example, Demetrovics and Katona [131] and Demetrovics and Son [132].

1.2. Notation and terminology

The main objects considered in this book are *partially ordered sets* (abbreviated as *posets*), which are sets equipped with a reflexive, antisymmetric, and transitive relation (order relation). Throughout we suppose that the posets are *finite*. For the sake of brevity we will not distinguish between the poset and the underlying set. For two *comparable* (i.e., related) elements p, q of a poset P, we write in the usual way $p \leq q$ or, equivalently, $q \geq p$. Two posets P and Q are called *isomorphic* (denoted by $P \cong Q$) if there is a bijective mapping φ from P onto Q (called *isomorphism*) such that $p \leq q$ iff (i.e., if and only if) $\varphi(p) \leq \varphi(q)$. An *automorphism* of P is an isomorphism from P onto P.

Sometimes we study more general objects, namely graphs. An (*undirected, simple*) *graph* $G = (V, E)$ is a set V, called the *vertex set* or *point set*, together with a set E of two-element subsets of V, called the *edge set*. The *degree* $d(v)$ of a vertex $v \in V$ is defined as the number of edges containing v. The graph is called *regular of degree d* if $d(v) = d$ for all $v \in V$. We speak of *directed* *graphs* (*digraphs*) $G = (V, E)$ if E consists of (ordered) pairs (p, q) of different elements p, q of V and we call the elements of E *arcs*. For $e = \{p, q\}$ (p, q are the *endpoints* of e) (resp. $e = (p, q)$, p is the *starting point*, q is the *endpoint*), we write briefly pq, and in the directed case we use the notation $e^- := p, e^+ := q$. Moreover in the directed case we allow more than one arc between points p and q; thus E is a *multiset* of arcs.

The element q of a poset P is said to *cover* the element p (denoted by $p \lessdot q$ and $q \gtrdot p$) if $q > p$ and if $q \geq q' > p$ implies $q = q'$. Obviously, the order relation is the reflexive and transitive closure of the cover relation. A poset P can be illustrated by its *Hasse diagram*, which is a digraph $H(P) = (P, E(P))$ whose vertex set is P and whose arc set $E(P)$ consists of all pairs (p, q), where $p \lessdot q$. In figures we always have $q \gtrdot p$ if p and q are joined by a straight line and q lies higher than p. The *Hasse graph* is the underlying undirected graph of the Hasse diagram. An element p of P is called *minimal* (*maximal*) if $q \leq p$ ($q \geq p$) implies $q = p$. For two elements p, q of P, we define the *interval* $[p, q]$ to be the set of all elements of P lying between p and q; that is, $[p, q] = \{v \in P : p \leq v \leq q\}$.

A subset of pairwise comparable elements of a poset P is said to be a *chain*. We denote chains by $C = (c_0 < \cdots < c_h)$, which gives us not only the elements but also the relation between them. The number h is called the *length of* C. The *height function* assigns with each element of P the length of a longest chain with p at the top. A chain is called *saturated* if it has the form $C = (c_0 \lessdot \cdots \lessdot c_h)$, and it is called *maximal* if, in addition, c_0 and c_h are minimal and maximal elements of P, respectively.

An *antichain* is a subset of pairwise incomparable elements of P. Subsets of a poset will often be called *families* too (motivated by families of subsets of a set). Antichains are also called *Sperner families*. A *k-family* is a family in P containing no chain of $k+1$ elements in P, thus a 1-family is an antichain. Usually we denote families by roman letters F, G, and so on. If P is the Boolean lattice (to be defined in the next section) or if P is very similar to the Boolean lattice we also use script letters $\mathcal{F}$, $\mathcal{G}$, and so forth.

We speak of *maximal families* and *maximum families* satisfying various conditions. "Maximal" means not contained in any other; "maximum" means maximum-sized.

For graphs $G = (V, E)$, we define a subset C of V to be a *clique* if any two elements of C are joined by an edge (i.e., are *adjacent*), and a subset I of V is called *independent* if no two elements of I are adjacent. A *matching* in a graph $G = (V, E)$ is a subset M of E of pairwise *nonadjacent* edges; that is, no two edges of M have a common endpoint.

We often consider extremal problems not in a poset but in a *weighted poset* (P, w), which is a poset P together with a function (called a *weight function*) w from P into the set $\mathbb{R}_+$ of nonnegative real numbers. If $w(p) > 0$ for all $p \in P$, then (P, w) is called a *positively weighted poset*. The *weight* $w(F)$ of a family F of

(P, w) is defined by $w(F) := \sum_{p \in F} w(p)$. Every poset P can be considered as a weighted poset (P, w) where $w \equiv 1$; that is, $w(p) = 1$ for all $p \in P$. We identify P and $(P, 1)$. The maximum weights of an antichain and a k-family in (P, w) are denoted by $d(P, w)$ and $d_k(P, w)$, respectively. The parameter $d(P, w)$ is called the *width* of (P, w).

Given a weighted poset (P, w) and a subset F of P, the poset whose underlying set is F and whose elements are ordered and weighted as in (P, w) is called the *poset induced by* F. The *dual* (P^*, w) of (P, w) has the same underlying set and the same weight function as (P, w), but it is ordered by $p \leq_{P^*} q$ iff $p \geq_P q$.

The *(direct) product* $P \times Q$ *of the posets* P *and* Q is defined to be the set of all pairs (p, q), $p \in P, q \in Q$, with the order given by $(p, q) \leq_{P \times Q} (p', q')$ iff $p \leq_P p'$ and $q \leq_Q q'$. Moreover, the *product* $(P, v) \times (Q, w)$ *of the weighted posets* (P, v) *and* (Q, w) is the product of P and Q together with the weight function $v \times w$ defined by $(v \times w)(p, q) := v(p)w(q)$, $p \in P, q \in Q$. We denote a product of n copies of (P, w) by $(P, w)^n$, and for $(P_1, w_1) \times \cdots \times (P_n, w_n)$ we write briefly $\prod_{i=1}^n (P_i, w_i)$.

Given a group G of automorphisms of a poset P, a nonempty subset A of P is called an *orbit* if for all $p, q \in A$ there is some $\varphi \in G$ such that $\varphi(p) = q$ and if A is maximal with respect to this property. It is easy to see that the union of all orbits is a partition of P. Now the *quotient of* P *under* G (denoted by P/G) is the poset of all orbits ordered in the following way: $A \leq_{P/G} B$ iff there are some $a \in A, b \in B$ such that $a \leq_P b$ (it is easy to see that P/G is really a poset). The *weighted quotient* is the quotient together with the weight function w/G defined by $w/G(A) := |A|$, $A \in P/G$.

Given two posets P and Q, a mapping $\varphi : P \to Q$ is called *order preserving* if $p \leq q$ implies $\varphi(p) \leq \varphi(q)$. If Q is the set $\mathbb{R}$ with the natural ordering, we speak of *increasing functions*. *Decreasing functions* are defined in an analogous way. The *characteristic function* of a subset S of P is defined and denoted by

$$\varphi_S(p) := \begin{cases} 1 & \text{if } p \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The *support* of a function $f : P \to \mathbb{R}$ is the set $\operatorname{supp}(f) := \{p \in P : f(p) \neq 0\}$.

A subset F of a poset P is called a *filter* (*ideal*) if $p \in F$ and $q \geq p$ ($q \leq p$) imply $q \in F$. Sometimes filters (ideals) are also called *upper ideals* (*lower ideals*). A filter (ideal) F is said to be *generated by a subset* S *of* P if $F = \{p \in P : p \geq q \ (p \leq q)$ for some $q \in S\}$. If S contains only one element we speak of *principal* filters and ideals.

Given two elements p, q of P, the element v is called *supremum* (*infimum*) of p and q – denoted by $v = p \vee q$ ($v = p \wedge q$) – if $v \geq p$, $v \geq q$ and if $w \geq p$, $w \geq q$ imply $w \geq v$ (if $v \leq p$, $v \leq q$ and if $w \leq p$, $w \leq q$ imply $w \leq v$). In an analogous way we define the supremum (infimum) of any subset A of P, which we denote by $\sup A$ ($\inf A$). Most of the examples considered in this book are *lattices*, that is, posets P in which $p \vee q$ and $p \wedge q$ exist for all $p, q \in P$.

Further, almost all posets that we will study are *ranked posets*, that is, posets together with a rank function. Here a *rank function* of a poset P is a function r from P into the set $\mathbb{N}$ of all natural numbers such that $r(p) = 0$ for *some* minimal element p of P and $p \lessdot q$ implies $r(q) = r(p) + 1$. Note that we do not suppose – as traditionally – that $r(p) = 0$ for all minimal elements p of P. If in a ranked poset every minimal element has rank 0 and every maximal element has the same rank, we speak of a *graded poset* (note that in any poset there is at most one rank function with this property). Given a ranked poset P, r_P denotes throughout its rank function, but generally we omit the index P and merely write r. The number $r(P) := \max\{r(p) : p \in P\}$ is called the *rank of* P (note the difference from the weight $w(P)$ of a weighted poset (P, w), which we defined by $w(P) := \sum_{p \in P} w(p)$). Very often we set for the sake of brevity $n := r(P)$.

A subset F of a graded lattice is called *t-intersecting* (*t-cointersecting*) if $r(p \wedge q) \geq t$ ($r(p \vee q) \leq r(P) - t$) for all $p, q \in F$. *Intersecting* (*cointersecting*) is an abbreviation for 1-intersecting (1-cointersecting).

The *dual of a ranked poset* P is the dual P^* of P together with the rank function $r_{P^*} := r_P(P) - r_P(p)$ for all $p \in P$. Moreover, the *product of two ranked posets* P, Q is defined to be the poset $P \times Q$ together with the rank function $r_{P \times Q}$ given by $r_{P \times Q}(p, q) := r_P(p) + r_Q(q)$. For a ranked poset P, we define the *ith level* by $N_i(P) := \{p \in P : r(p) = i\}$; its size $W_i(P) := |N_i(P)|$ is called the *ith Whitney number*, $i = 0, \ldots, r(P)$ (when there is no danger of ambiguity, we write briefly N_i and W_i). It is useful to define $N_i := \emptyset$ and $W_i := 0$ if $i \notin \{0, \ldots, r(P)\}$. Obviously, each level of a ranked poset is an antichain, and the union of k levels is a k-family. The *rank-generating function* $F(P; x)$ of a ranked poset is defined by $F(P; x) := \sum_{p \in P} x^{r(p)} \big(= \sum_{i=0}^{r(P)} W_i x^i\big)$. It is easy to see that $F(P \times Q; x) = F(P; x) F(Q; x)$ if P and Q are ranked. For $S \subseteq \{0, \ldots, r(P)\}$, we define the *$S$-rank-selected subposet* (P_S, w_S) as the subposet induced by $P_S := \{p \in P : r(p) \in S\}$ together with the induced weights w_S.

For ranked posets P, Q of the same rank, we define the *rankwise (direct) product* $P \times_r Q$ to be the set $\bigcup_{i=0}^{r(P)} N_i(P) \times N_i(Q)$ together with the relation $(p, q) \leq_{P \times_r Q} (p', q')$ if $p \leq_P p'$ and $q \leq_Q q'$. If we have, in addition, weights v and w on P and Q, resp., then, as for usual products, $(v \times_r w)(p, q) := v(p) w(q)$.

Given a family F in a ranked poset P, the set of rank i elements of F is denoted by F_i, and the numbers $f_i := |F_i|$ are called *parameters of* F, $i = 0, \ldots, r(P)$. The vector $\boldsymbol{f} = (f_0, \ldots, f_{r(P)})^{\mathrm{T}}$ is called the *profile of* F. If $F = F_i$ for some i then we call F *i-uniform*.

For an element p of P, we define and denote the *upper* (resp. *lower*) *shadow of* p by $\nabla(p) := \{q \in P : q \gtrdot p\}$ (resp. $\Delta(p) := \{q \in P : q \lessdot p\}$). More generally, if P is ranked, let the *upper* (resp. *lower*) *k-shadow of* p be defined and denoted by $\nabla_{\to k}(p) := \{q \in N_k : q \geq p\}$ (resp. $\Delta_{\to k}(p) := \{q \in N_k : q \leq p\}$).

The $(k-)$ *shadows of a subset of* P are the unions of the $(k-)$ shadows of its elements.

More generally, given a weighted and ranked poset (P, w), the weight $w(N_i)$ of the ith level N_i of P is called the *weighted ith Whitney number*. We mostly use the following definitions in the $w \equiv 1$ case (where $W_i = w(N_i)$). The weighted and ranked poset (P, w) is said to have the *k-Sperner property* if the maximum weight of a k-family in (P, w) equals the largest sum of k weighted Whitney numbers in (P, w), that is, if

$$d_k(P, w) = \max\{w(N_{i_1}) + \cdots + w(N_{i_k}) : 0 \le i_1 < \cdots < i_k \le r(P)\}.$$

In the $k = 1$ case we also say briefly that (P, w) has the *Sperner property* or *(P, w) is Sperner*. Further, (P, w) has the *strong Sperner property* (*(P, w) is strongly Sperner*) if (P, w) has the k-Sperner property for all $k = 1, 2, \ldots$.

A sequence of nonnegative real numbers $\{a_n\}$ is called *unimodal* if there is a number h such that $a_i \le a_{i+1}$ for $i < h$ and $a_i \ge a_{i+1}$ if $i \ge h$. It is called *logarithmically concave* (or *log concave*) if $a_i^2 \ge a_{i-1}a_{i+1}$ for all i. For a finite sequence $(a_0, \ldots, a_n)$, we say that it is *symmetric* if $a_i = a_{n-i}$ for all i. If the (weighted) Whitney numbers of (P, w) are unimodal (resp. symmetric), then (P, w) is said to be *rank unimodal* (resp. *rank symmetric*). If $\{a_n\}$ and $\{b_n\}$ are two infinite sequences of real numbers, the following notations for $n \to \infty$ are well known: $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$; $a_n = O(b_n)$ if there exists some $c \in \mathbb{R}$ such that $|a_n| \le c|b_n|$ for all n; $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$ and $a_n \lesssim b_n$ if $a_n \le b_n(1 + o(1))$. All logarithms in this book are to the basis $e = 2.718\ldots$.

As usual, we denote the largest integer that is not greater than a given real number x by $\lfloor x \rfloor$. For the smallest integer that is not smaller than x, we write $\lceil x \rceil$. The set $\{1, \ldots, n\}$ we abbreviate by $[n]$. For the family of k-element subsets and for the power set of $[n]$, we use the notation $\binom{[n]}{k}$ (resp. $2^{[n]}$), which is motivated by the corresponding sizes. $A \subseteq B$ means that A is a subset of B, whereas strict inclusion is denoted by $A \subset B$. For the set difference of sets A and B, we write $A - B$. Moreover, we denote the complement of A in $[n]$ by $\overline{A}$; that is, $\overline{A} := [n] - A$. For $\mathcal{F} \subseteq 2^{[n]}$, let $\overline{\mathcal{F}} := \{\overline{A} : A \in \mathcal{F}\}$ be the *complementary family*.

Before considering some concrete examples of posets and lattices in the next section, let us look at some larger classes of ranked posets. For a general study of these lattices, see, for example, Aigner [21] and Stanley [441]. In a lattice P, the elements covering the minimal element are called *atoms*. The rank function of P is called *modular* (resp. *semimodular*) if

$$r(p \wedge q) + r(p \vee q) = (\text{ resp.} \le)\, r(p) + r(q) \text{ for all } p, q \in P.$$

A (finite) lattice is called *modular* if it has a modular rank function. Moreover, a lattice is said to be *geometric* if it has a semimodular rank function and every element is a supremum of atoms. Finally a lattice is called *distributive* if the following identities hold for all $p, q, v \in P$:

$$p \wedge (q \vee v) = (p \wedge q) \vee (p \wedge v),$$

$$p \vee (q \wedge v) = (p \vee q) \wedge (p \vee v).$$

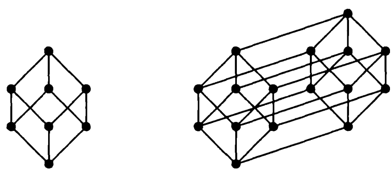
We note that each of these identities implies the other. All (finite) distributive lattices are ranked, for the proof see, for example, [21, p. 38]. Obviously, distributivity implies modularity.

1.3. The main examples

The following are several examples of posets we will consider in this book. Most of these posets can easily be shown to be lattices. Hence in all traditional examples the word *lattice* will be used instead of *poset*. Further, it is mentioned without proof that all following posets are ranked. The reader will learn in the book that all posets up to the last one have the Sperner property.

Example 1.3.1. *The Boolean lattice B_n.*

The poset of all subsets of an n-element set, ordered by inclusion, is the Boolean lattice. Obviously, B_n is isomorphic to $(0 < 1)^n$ as well as to the poset of all faces of an $(n-1)$-dimensional simplex (including the empty set as a face), ordered by inclusion. It is easy to see that $N_k(B_n)$ consists of all k-element subsets; thus $W_k(B_n) = \binom{n}{k}$. The Hasse diagrams of B_3 and B_4 are illustrated in Figure 1.1.



B_3 B_4

Figure 1.1

Example 1.3.2. *Chain products $S(k_1, \ldots, k_n)$.*

The poset $S(k_1, \ldots, k_n)$ consists of all n-tuples of integers $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $0 \le a_i \le k_i$, $i = 1, \ldots, n$, and we have $\boldsymbol{a} \le \boldsymbol{b}$ iff $a_i \le b_i$ for all i. We will adopt the convention $k_1 \ge \cdots \ge k_n$. Obviously, $S(k_1, \ldots, k_n) \cong \prod_{i=1}^{n}(0 < 1 < \cdots < k_i)$, and therefore $S(k_1, \ldots, k_n)$ is called a *chain product*. Given n distinct primes $p_1, \ldots, p_n$, $S(k_1, \ldots, k_n)$ is isomorphic to the *lattice* *of all divisors of* $p_1^{k_1} \dots p_n^{k_n}$, ordered by divisibility. The Boolean lattice B_n is isomorphic to $S(1, \dots, 1)$. From the product representation we obtain that the rank of an element $\boldsymbol{a}$ of $S(k_1, \dots, k_n)$ is given by $r(\boldsymbol{a}) = a_1 + \dots + a_n$, and the rank-generating function is $\prod_{i=1}^{n}(1 + x + \dots + x^{k_i})$.

Example 1.3.3. *The cubical poset Q_n.*

Q_n is the *poset of all faces of an n-dimensional cube* (not including the empty set as a face), ordered by inclusion. Here we consider only the discrete cube $\{\boldsymbol{a} = (a_1, \dots, a_n) : a_i \in \{0, 1\}\}$. Its faces are all subsets of the form $\{\boldsymbol{a} : a_i \in \{0, 1\}$ if $i \notin I, a_i = \alpha_i$ if $i \in I (i = 1, \dots, n)\}$, where I is a subset of $[n]$ and $\alpha_i (i \in I)$ are fixed elements of $\{0, 1\}$. Clearly the faces are exactly the intervals in the Boolean lattice B_n. If one notes that a face corresponds to an n-tuple $\boldsymbol{b} = (b_1, \dots, b_n)$ where $b_i = 2$ if $i \notin I$ and $b_i = \alpha_i$ if $i \in I$, then it is not difficult to see that Q_n is isomorphic to a product of n factors given in Figure 1.2. Thus we consider Q_n

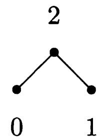


Figure 1.2

mostly as the set of all n-tuples $\boldsymbol{b} = (b_1, \dots, b_n)$ with $b_i \in \{0, 1, 2\}$ for all i and ordered by $\boldsymbol{b} \leq \boldsymbol{c}$ iff $c_i = 2$ or $b_i = c_i$ for all i. Obviously, the rank of $\boldsymbol{b}$ equals the number of "twos" in $\boldsymbol{b}$ and $W_k(Q_n) = \binom{n}{k} 2^{n-k}$.

We get the *cubical lattice* $\hat{Q}_n$ if we add to Q_n a minimal element (which is smaller than all elements of Q_n).

Example 1.3.4. *The function poset F_k^n.*

F_k^n consists of all partially defined functions of an n-element set into a k-element set. For a function f of F_k^n, let $D(f)$ be its domain. Two elements of F_k^n are ordered in the following way: $f \leq g$ iff $D(f) \subseteq D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$. A partially defined function of $\{x_1, \dots, x_n\}$ into $\{y_1, \dots, y_k\}$ can be represented as an n-tuple $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i = 0$ if $x_i \notin D(f)$ and $a_i = j$ if $x_i \in D(f)$ and $f(x_i) = y_j$. Thus F_k^n is isomorphic to a product of n factors given in Figure 1.3. It follows that F_2^n is isomorphic to the dual of Q_n. The rank of an element $\boldsymbol{a}$ of the poset F_k^n is given by the number of nonzero elements in $\boldsymbol{a}$, and we have $W_i(F_k^n) = \binom{n}{i} k^i$. If we add a maximal element to F_k^n, we get the *function lattice* $\hat{F}_k^n$.

Example 1.3.5. *Star products $T(k_1, \dots, k_n)$.*

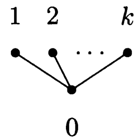


Figure 1.3

This poset consists of all n-tuples of integers $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $k_n - k_i \leq a_i \leq k_n$, $i = 1, \ldots, n$, and we have $\boldsymbol{a} \leq \boldsymbol{b}$ iff, for all i, $a_i = b_i$ or $b_i = k_n$. We suppose throughout that $k_1 \leq \cdots \leq k_n$. Obviously, $T(k_1, \ldots, k_n)$ is isomorphic to a product of stars presented in Figure 1.4. Thus it is a natural generalization of the cubical poset and the dual of the function poset. The rank of an element $\boldsymbol{a}$ of $T(k_1, \ldots, k_n)$ is given by the number of components equal to k_n, and the rank-generating function is $\prod_{i=1}^{n}(k_i + x)$.

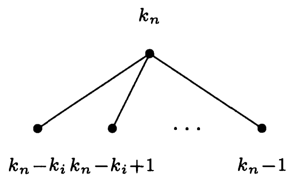


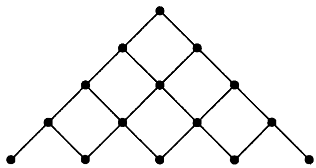
Figure 1.4

Example 1.3.6. *The poset $Int(S(k_1, \ldots, k_n))$.*

This is the poset of all nonempty intervals in $S(k_1, \ldots, k_n)$, ordered by inclusion; that is, its elements are sets of n-tuples of the form $\{\boldsymbol{a} : \alpha_i \leq a_i \leq \beta_i \ (i = 1, \ldots, n)\}$, where α_i and β_i are fixed integers, $0 \leq \alpha_i \leq \beta_i \leq k_i$ $(i = 1, \ldots, n)$. It is easy to see that $Int(S(k_i))$ is isomorphic to the "upper half" of $S(k_i, k_i)$ (see Figure 1.5). The corresponding isomorphism is the mapping that assigns the interval $\{\alpha, \alpha + 1, \ldots, \beta\}$, $0 \leq \alpha \leq \beta \leq k_i$, to the pair $(\beta, k_i - \alpha)$. If we interpret the elements of $S(k_1, \ldots, k_n)$ as points of a discrete (rectangular) parallelepipedon, then $Int(S(k_1, \ldots, k_n))$ consists of all *subparallelepipedons of a parallelepipedon*, ordered by inclusion. Note that $Int(S(1, \ldots, 1))$ is isomorphic to Q_n.

Example 1.3.7. *The poset $M(k_1, \ldots, k_n)$.*

The elements of this poset are subsets of $S(k_1, \ldots, k_n)$ of the form $\{\boldsymbol{a} \in S(k_1, \ldots, k_n) : a_i \in A_i,\ i = 1, \ldots, n\}$ where the A_i are nonempty subsets of $\{0, \ldots, k_i\}$, $i = 1, \ldots, n$, and the ordering is again by inclusion. It is easy to see that $M(k_1, \ldots, k_n)$ is isomorphic to $B'_{k_1+1} \times \cdots \times B'_{k_n+1}$ where B'_n denotes the Boolean lattice



$Int(S(4))$

Figure 1.5

B_n without the minimal element. In the case $n = 2$, this poset can be interpreted as the poset of submatrices of a matrix with $(k_1 + 1)$ rows and $(k_2 + 1)$ columns, ordered by containment. We speak also for $n \geq 2$ of the *poset of submatrices of a matrix.*

Example 1.3.8. *The posets $SQ_{k,n}$ and $SM_{k,n}$.*

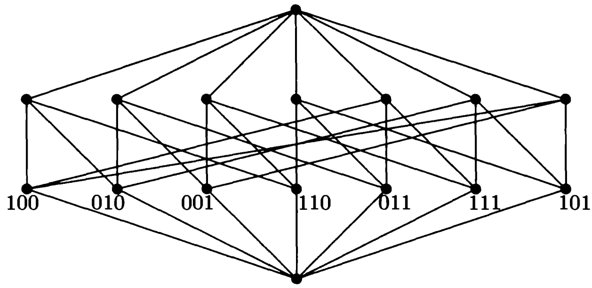
The elements are subsets of $S(k, \dots, k)$ of the form $\{\boldsymbol{a} \in S(k, \dots, k) : a_i \in A_i,\ i = 1, \dots, n\}$ where the A_i are for $SQ_{k,n}$ intervals in $[0, k]$ of same sizes and for $SM_{k,n}$ subsets of $[0, k]$ of same sizes, and the ordering is by inclusion. If we consider $S(k, \dots, k)$ as a discrete cube, then $SQ_{k,n}$ can be interpreted as the *poset of subcubes of a cube*. In the case $n = 2$, $SM_{k,n}$ can be interpreted as the *poset of square submatrices of a square matrix* (with $k + 1$ rows (resp. columns)), but we use this notation also for $n \geq 2$. It is easy to see that $SQ_{k,n} \cong Int(S(k)) \times_r \cdots \times_r Int(S(k))$ and $SM_{k,n} \cong B_{k+1} \times_r \cdots \times_r B_{k+1}$ (rankwise product).

Example 1.3.9. *The linear lattice $L_n(q)$.*

This is the poset of all subspaces of an n-dimensional vector space over a field of q elements (the Galois field $GF(q)$), ordered by inclusion. The rank of an element V of $L_n(q)$ equals its dimension dim V (as a subspace). In Figure 1.6 we illustrate $L_3(2)$ (the triples indicate the coordinates of vectors that generate the corresponding one-dimensional subspace). It is not difficult to see that each nonempty interval $[V, W]$ in $L_n(q)$ is isomorphic to $L_{\dim W - \dim V}(q)$ (consider the mapping that assigns to each $U \in [V, W]$ the factor space U/V). It is well known (cf. [21, p. 78]) that the ith Whitney number of $L_n(q)$ equals the Gaussian coefficient $\binom{n}{i}_q$, which is defined by

$$\binom{n}{i}_q := \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1)\cdots(q - 1)}, \qquad 0 \leq i \leq n.$$

The linear lattice $L_n(q)$ (as also B_n and $S(k_1, \dots, k_n)$) is isomorphic to its dual: If the vector space is realized as the set of all n-tuples with components from



$L_3(2)$

Figure 1.6

$GF(q)$, then an isomorphic image of some subspace V is the subspace $\{\boldsymbol{x} = (x_1, \dots, x_n)^{\mathrm{T}} : v_1x_1 + \cdots + v_nx_n = 0 \text{ for all } \boldsymbol{v} = (v_1, \dots, v_n)^{\mathrm{T}} \in V\}$.

Example 1.3.10. *Projective space lattices.*

A projective space lattice is the poset of all subspaces of a projective geometry, ordered by inclusion. For a definition of a projective geometry and its subspaces, see references [21, pp. 55ff] and [450, pp. 105ff]. Note here that if such a poset has rank ≥ 4, then it is isomorphic to some linear lattice $L_n(q)$ (see [450, p. 203] and note that we have the general supposition that all posets are finite). In the rank = 3 case, the projective geometry is a projective plane [450, Theorem 13.7]. Finally, in the rank = 2 case, projective space lattices have the form given in Figure 1.7.

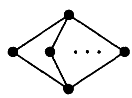


Figure 1.7

Example 1.3.11. *Modular geometric lattices.*

These are lattices that are both modular and geometric. We only need the fact that the modular geometric lattices are exactly the products of a Boolean lattice and projective space lattices, which was proved by Birkhoff (cf. [21, p. 64], where two-element chains are also considered as projective space lattices).

Example 1.3.12. *The affine poset $A_n(q)$.*

The poset of cosets of subspaces (affine subspaces) of an n-dimensional vector space over $GF(q)$, ordered by inclusion, is the affine poset $A_n(q)$. The rank of an element $a+V$ equals the dimension of V, and the ith Whitney number is given by $W_i(A_n(q)) = q^{n-i}\binom{n}{i}_q$. Every nonempty interval $[a+V,\ b+W]$ (i.e., we may suppose $a=b$) is obviously isomorphic to $L_{\dim W - \dim V}(q)$. If we add a unique minimal element, we obtain the affine lattice $\hat{A}_n(q)$.

Example 1.3.13. *The poset $L(m,n)$.*

The elements of the poset $L(m,n)$ are the n-tuples $\boldsymbol{a} = (a_1,\ldots,a_n)$ with $0 \le a_1 \le \cdots \le a_n \le m$. They are ordered componentwise; that is, $\boldsymbol{a} \le \boldsymbol{b}$ iff $a_i \le b_i$ for all i. $L(m,n)$ is isomorphic to the *poset of all ideals in* $S(m-1,n-1)$, ordered by inclusion (note that the mapping which assigns $\boldsymbol{a} \in L(m,n)$ to the ideal in $S(m-1,n-1)$ generated by the set $\{(a_1-1,n-1),(a_2-1,n-2),\ldots,(a_n-1,0)\}$ – an element $(-1,i)$ is considered nonexistent – is an isomorphism). The rank function is given by $r(\boldsymbol{a}) = a_1+\cdots+a_n$.

Example 1.3.14. *The poset $M(n)$.*

This is the poset of all n-tuples $\boldsymbol{a} = (a_1,\ldots,a_n)$ with $0 = a_1 = \cdots = a_h < a_{h+1} < \cdots < a_n \le n$, $h \in \{0,\ldots,n\}$ ($h=0$ means $a_1 > 0$), with the order given by $\boldsymbol{a} \le \boldsymbol{b}$ iff $a_i \le b_i$ for all i. For the rank of an element $\boldsymbol{a}$, we have $r(\boldsymbol{a}) = a_1+\cdots+a_n$; thus $r(M(n)) = \frac{n(n+1)}{2}$.

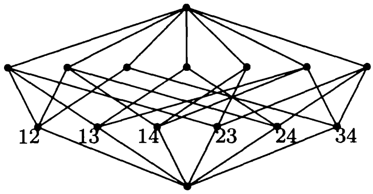
Example 1.3.15. *The graph poset G_n.*

The elements of the graph poset G_n are the simple graphs on n unlabeled vertices (i.e., the isomorphie classes of the graphs on the vertex set $[n]$ without loops and multiple edges). These graphs are ordered by containment (embedding). The rank of such a graph (as an element of the poset G_n) equals the number of edges in it.

Example 1.3.16. *The partition lattice Π_n.*

The elements of this lattice are the partitions of $[n]$, and they are ordered by refinement. We denote, for example, the partition $\{4\} \cup \{2,5,6\} \cup \{1,3\}$ of $[6]$ briefly by $4|256|13$ (or $526|13|4,\ldots$), and this partition covers $4|2|56|13$, $4|25|6|13$, $4|5|26|13$, $4|256|1|3$. Let $b(\pi)$ stand for the number of blocks in π. Then the rank function is given by $r(\pi) = n - b(\pi)$. Figure 1.8 illustrates Π_4 (the two-element blocks of the atoms are depicted). For the Whitney numbers, we have $W_i(\Pi_n) = S_{n,n-i}$, where $S_{n,k}$ is the corresponding *Stirling number of the second kind*, which is defined by the number of partitions of $[n]$ into exactly k blocks. If $\pi \le \sigma$ and $\sigma = B_1|\ldots|B_k$, where B_i contains n_i blocks of π, then obviously $[\pi,\sigma] \cong \Pi_{n_1} \times \cdots \times \Pi_{n_k}$. The following basic recurrence holds for the Stirling numbers:

$$S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}, \qquad n,k \ge 1, \tag{1.6}$$



Π_4

Figure 1.8

and the *Bell numbers* $B_n := |\Pi_n|$ (do not confuse them with the Boolean lattices B_n) can be determined by the formula of Dobinski (cf. [354, pp. 14f]):

$$B_n = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^n}{i!}. \tag{1.7}$$

It is well known [441, p. 127f] and can be easily proved directly that Π_n is a geometric lattice.

2

Extremal problems for finite sets

Many results in Sperner theory have their origin in theorems on families of subsets of a finite set, that is, in the special case of the Boolean lattice. So we start our investigations with the Boolean lattice. Because of the richness of the literature, we must be selective. In particular, this chapter presents several important methods (together with related results). The linear programming approach is discussed in a separate chapter.

2.1. Counting in two different ways

We have already seen this method in the proof of Claim 2 of Sperner's theorem. It can be illustrated with a *bipartite graph*, which is a graph $G = (V, E)$ whose vertex set V can be partitioned into two sets A and B such that $|e \cap A| = |e \cap B| = 1$ for all $e \in E$. The method of counting in two different ways is then given by the trivial equality

$$|E| = \sum_{v \in A} d(v) = \sum_{v \in B} d(v).$$

But the problem is the construction of the "right" vertex set $V = A \cup B$ and of the "right" edge set E. Edges in a bipartite graph can be written also as pairs of vertices. If we say in the following proofs that we count the number of pairs with "..." in two different ways, then the set A (domain of the first coordinate), the set B (domain of the second coordinate), and the edge set E (all pairs with the corresponding property) are implicitly given. For the sake of brevity the graph will not be presented (and drawn) explicitly.

2nd Proof of Theorem 1.1.1(a). Let S_n be, as usual, the set of all permutations of $[n]$. We count the number N of pairs (X, π) where $X \in \mathcal{F}, \pi \in S_n, X = \{\pi(1), \ldots, \pi(|X|)\}$, in two different ways. For fixed $X \in \mathcal{F}$, a corresponding permutation must map $\{1, \dots, |X|\}$ bijectively onto X and $\{|X|+1, \dots, n\}$ bijectively onto $[n] - X$. Thus we find $|X|!(n - |X|)!$ such permutations (this is also true for $X = \emptyset$), and it follows that

$$N = \sum_{X \in \mathcal{F}} |X|!(n - |X|)!. \tag{2.1}$$

However, for fixed π we find at most one corresponding $X \in \mathcal{F}$ because of the inclusion condition. Thus

$$N \leq \sum_{\pi \in S_n} 1 = n!. \tag{2.2}$$

From (2.1) and (2.2) we obtain (after division by $n!$)

$$\sum_{X \in \mathcal{F}} \frac{1}{\binom{n}{|X|}} \leq 1. \tag{2.3}$$

It is trivial to verify that the binomial coefficients are rank unimodal and rank symmetric; that is,

$$\binom{n}{0} \leq \binom{n}{1} \leq \cdots \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} \geq \cdots \geq \binom{n}{n-1} \geq \binom{n}{n}.$$

Thus (2.3) implies

$$\sum_{X \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1, \quad \text{i.e.,} \quad |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

■

This proof is due to Lubell [357]. The inequality (2.3) was also found (in a different way) by Yamamoto [471], and Meshalkin [364], and therefore inequality (2.3) is known as the *LYM-inequality*. Generalizations to other posets are discussed in Section 4.5.

Note that each permutation $\pi \in S_n$ corresponds in a bijective way to a maximal chain $C = (\emptyset \lessdot \{\pi(1)\} \lessdot \{\pi(1), \pi(2)\} \lessdot \cdots \lessdot \{\pi(1), \pi(2), \dots, \pi(n)\})$ in the Boolean lattice (realized on $2^{[n]}$). Thus, in another interpretation, we counted the number of pairs (X, C) with $X \in \mathcal{F}$, C a maximal chain, $X \in C$, in two different ways. In (2.2) we had only an inequality because there might exist maximal chains C that do not contain any member of $\mathcal{F}$. Now we change the situation in order to meet every chain exactly once. Let $\mathcal{D}$ be an ideal in $2^{[n]}$, and $\mathcal{U} := 2^{[n]} - \mathcal{D}$. Clearly, $\mathcal{U}$ is a filter. Moreover, let

$$E(\mathcal{D}, \mathcal{U}) := \{(Y, X) : Y \in \mathcal{D}, X \in \mathcal{U}, Y \lessdot X\}.$$

In the Hasse diagram of the Boolean lattice, this set is the set of arcs that leave $\mathcal{D}$ and enter $\mathcal{U}$.

Lemma 2.1.1. *If $\emptyset \in \mathcal{D}$ and $[n] \in \mathcal{U}$, then*

$$\sum_{(Y,X)\in E(\mathcal{D},\mathcal{U})} |Y|!(n-|X|)! = n!. \tag{2.4}$$

Proof. We count the number of pairs $((Y, X), C)$ with $(Y, X) \in E(\mathcal{D}, \mathcal{U})$, C a maximal chain, $Y, X \in C$, in two different ways. As before, we obtain that for fixed $(Y, X) \in E(\mathcal{D}, \mathcal{U})$ there exist exactly $|Y|!(n-|X|)!$ such chains. But since each maximal chain C starts in $\mathcal{D}$ and ends in $\mathcal{U}$ and since $\mathcal{D}$ is an ideal, C must leave $\mathcal{D}$ in a unique element Y and enter $\mathcal{U}$ in the next element X. Thus, for fixed C, there exists exactly one corresponding $(Y, X) \in E(\mathcal{D}, \mathcal{U})$. ■

The following identity was found by Ahlswede and Zhang [18], and can be considered a sharpening of the LYM-inequality.

Theorem 2.1.1 **(*AZ-identity*).** *Let $\mathcal{F}$ be a family of nonempty subsets of $[n]$. Let $w : 2^{[n]} \to \mathbb{N}$ be the weight function defined by*

$$w(X) := \Big|\bigcap_{A\in\mathcal{F}:A\subseteq X} A\Big|, \quad \textit{where } X \subseteq [n]$$

(here the intersection is defined to be empty if there is no $A \in \mathcal{F}$ with $A \subseteq X$). Then

$$\sum_{X\subseteq[n]} \frac{w(X)}{|X|\binom{n}{|X|}} = 1.$$

Proof. Let $\mathcal{U}$ be the filter generated by $\mathcal{F}$, and let $\mathcal{D} := 2^{[n]} - \mathcal{U}$. Then we may apply Lemma 2.1.1 to obtain (2.4). For fixed X, let $N(X)$ be the number of pairs $(Y, X) \in E(\mathcal{D}, \mathcal{U})$. It is easy to see that

$$N(X) = \begin{cases} 0 & \text{if } X \in \mathcal{D}, \\ |\{x \in X : X - \{x\} \in \mathcal{D}\}| & \text{if } X \in \mathcal{U}. \end{cases}$$

Clearly, $X \in \mathcal{D}$ iff there is no $A \in \mathcal{F}$ with $A \subseteq X$.

For $X \in \mathcal{U}$ and $x \in X$, we have $X - \{x\} \in \mathcal{D}$ iff $x \in A$ for all $A \in \mathcal{F}$ with $A \subseteq X$, that is, iff $x \in \bigcap_{A\in\mathcal{F}:A\subseteq X} A$. Consequently,

$$w(X) = N(X) \text{ for all } X \subseteq [n]. \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$\sum_{X\subseteq[n]} N(X)(|X|-1)!(n-|X|!) = n!,$$

$$\sum_{X\subseteq[n]} \frac{w(X)}{|X|\binom{n}{|X|}} = 1. \qquad ■$$

Note that the LYM-inequality follows indeed from Theorem 2.1.1 since for a Sperner family $\mathcal{F}$ and a member X of $\mathcal{F}$ clearly $w(X) = |X|$.

In the preceding investigations we worked with a set of arcs in the Hasse diagram that intersect (the arcs of) each maximal chain. Deletion of the set $E(\mathcal{D}, \mathcal{U})$ means that we "cut" every maximal chain. Instead of arcs we may consider also vertices. We say that a family $\mathcal{C}$ of subsets of $[n]$ is a *cutset* if each maximal chain meets $\mathcal{C}$. For instance, $\{\emptyset\}$ and $\{[n]\}$ are cutsets that are, in addition, minimum and, of course, minimal. Since we will study cutsets in more detail in Section 4.2, we consider here also one result for cutsets, though in the proof the method of counting in two different ways is not really used. Answering a question of Lih (see also [342]), Füredi, Griggs, and Kleitman [210] discovered that there exist "large" minimal cutsets. Let $c(n)$ be the maximum size of a minimal cutset in $2^{[n]}$.

Theorem 2.1.2. *We have*

$$c(n) \sim 2^n \textit{ as } n \to \infty.$$

Proof. First we will show that we may restrict ourselves to "special" values of n.

Claim 1. We have $\dfrac{c(n)}{2^n} \leq \dfrac{c(n+1)}{2^{n+1}}$ for all n.

Proof of Claim 1. Let $\mathcal{C}$ be a minimal cutset in $2^{[n]}$ of size $c(n)$. Then clearly $\emptyset, [n] \notin \mathcal{C}$. It is easy to verify that $\mathcal{C} \cup \{C \cup \{n+1\} : C \in \mathcal{C}\}$ is a minimal cutset in $2^{[n+1]}$. Thus $c(n+1) \geq 2c(n)$, which yields the assertion. □

By Claim 1, it is sufficient to find a sequence n_k with $n_k \to \infty$ as $k \to \infty$ such that

$$\frac{c(n_k)}{2^{n_k}} \geq 1 - o(1) \text{ as } k \to \infty. \tag{2.6}$$

Let us fix some $k \in \mathbb{N}$ and let $n := ak$ where a is some natural number that we will specify later. Let $\cup_{i=1}^{a} S_i$ be a partition of $[n]$ into classes of size k and let

$$\begin{aligned} \mathcal{C} := &\{C \subseteq [n] : 0 < |S_i \cap C| < k \text{ for all i}\} \\ &\cup \{C \subseteq [n] : |S_i \cap C| = 0, |S_j \cap C| = k \text{ for some } i \text{ and } j\}. \end{aligned}$$

Claim 2. The family $\mathcal{C}$ is a cutset.

Proof of Claim 2. Let $(\emptyset = X_0 \lessdot X_1 \lessdot \cdots \lessdot X_n = [n])$ be any maximal chain and define t as the largest integer such that X_t is still disjoint from some S_i. Then X_{t+1} intersects all S_i. If X_{t+1} does not contain any S_j, then it belongs to the first part of $\mathcal{C}$; otherwise X_t belongs to the second part of $\mathcal{C}$. □

Let

$$\begin{aligned} \mathcal{C}_0 := &\{C \subseteq [n] : 0 < |S_i \cap C| < k \text{ for all i}\} \\ &\cap \{C \subseteq [n] : |S_i \cap C| = 1, |S_j \cap C| = k-1 \text{ for some } i \text{ and } j\}. \end{aligned}$$

Claim 3. For any $C \in \mathcal{C}_0$, the family $\mathcal{C} - \{C\}$ is not a cutset.

Proof of Claim 3. Let $C \in \mathcal{C}_0$ where $|S_{i^*} \cap C| = 1, |S_{j^*} \cap C| = k-1$. Consider any chain of the form $(\cdots < C - S_{i^*} < C < C \cup S_{j^*} < \cdots)$. Assume that this chain contains a member X with $X \in \mathcal{C} - \{C\}$. For brevity we assume that $X < C$ (the case $X > C$ can be treated analogously). Obviously, $|X \cap S_{i^*}| = 0$; thus X could only belong to the second part of $\mathcal{C}$. But $|S_j \cap X| = k$ would imply $|S_j \cap C| = k$, a contradiction to $C \in \mathcal{C}_0$. □

By deleting certain members of $\mathcal{C}$ step by step, we obtain finally a minimal cutset $\mathcal{C}_1$. By Claim 3, $\mathcal{C}_1 \supseteq \mathcal{C}_0$; that is (using inclusion-exclusion),

$$\begin{aligned} c(n) &\geq |\mathcal{C}_0| = (2^k-2)^a - 2(2^k-k-2)^a + (2^k-2k-2)^a \\ &\geq (2^k-2)^a - 2(2^k-k)^a = 2^n\left(\left(1-\frac{2}{2^k}\right)^a - 2\left(1-\frac{k}{2^k}\right)^a\right). \end{aligned}$$

With $a := \left\lfloor \frac{2^k \log k}{k} \right\rfloor$ we obtain for the second factor

$$\left(1-\frac{2}{2^k}\right)^{\frac{2^k}{2}\frac{2\log k}{k}+O(1)} - 2\left(1-\frac{k}{2^k}\right)^{\frac{2^k}{k}\log k+O(1)} = 1-o(1) \text{ as } k \to \infty.$$

Thus, for $n_k := \left\lfloor \frac{2^k \log k}{k} \right\rfloor k$, the inequality (2.6) holds. ■

In fact, Füredi, Griggs, and Kleitman [210] strengthened the lower bound (using some random construction, for a similar result see Theorem 2.6.2) to

$$c(n) \geq 2^n\left(1-\gamma\frac{(\log n)^{3/2}}{\sqrt{n}}\right), \text{ where } \gamma \text{ is some constant.}$$

Let us return to the LYM-inequality and generalize it in another direction.

***Theorem 2.1.3* (*Bollobás* [71]).** *Let $\{(X_1, Y_1), \ldots, (X_m, Y_m)\}$ be a family of pairs of subsets of $[n]$ such that for all $1 \leq i, j \leq m$, $X_i \cap Y_j = \emptyset$ iff $i = j$. Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{|X_i|+|Y_i|}{|X_i|}} \leq 1.$$

Proof. We count in two different ways the number N of pairs (i, π), where $i \in [m]$, $\pi \in S_n$ with the following property:

$$\text{For all } k, l \in [n], \quad \pi(k) \in X_i, \pi(l) \in Y_i \text{ imply } k < l. \tag{2.7}$$

Let i be fixed. We may classify corresponding permutations by those sets Z_i that are mapped onto $X_i \cup Y_i$. Clearly, we have $\binom{n}{|X_i|+|Y_i|}$ possibilities to choose Z_i. If Z_i is fixed, then (because of property (2.7)) the first $|X_i|$ elements of Z_i must be mapped onto X_i (i.e., $|X_i|!$ possibilities), the rest of Z_i must be mapped onto Y_i

(i.e., $|Y_i|!$ possibilities), and $[n] - Z_i$ must be mapped onto $[n] - (X_i \cup Y_i)$ (i.e., $(n - (|X_i| + |Y_i|))!$ possibilities). Thus,

$$N = \sum_{i=1}^{m} \binom{n}{|X_i| + |Y_i|} |X_i|!|Y_i|!(n - |X_i| - |Y_i|)! = \sum_{i=1}^{m} \frac{n!}{\binom{|X_i|+|Y_i|}{|X_i|}}. \tag{2.8}$$

Now let π be fixed. Assume that there are two different numbers i, j corresponding to π. Since, by supposition, $X_i \cap Y_j \neq \emptyset$, we find some $h \in [n]$ such that $\pi(h) \in X_i$ and $\pi(h) \in Y_j$. But then in view of property (2.7), $\pi(k) \in X_j$ implies $k < h$, and $\pi(l) \in Y_i$ implies $h < l$. Consequently, $X_j \cap Y_i = \emptyset$, a contradiction. Thus

$$N \leq \sum_{\pi \in S_n} 1 = n!. \tag{2.9}$$

The assertion follows from (2.8) and (2.9). ∎

Let us emphasize that the LYM-inequality is a special case of Theorem 2.1.3. Indeed, let $\mathcal{F} = \{X_1, \ldots, X_m\}$ be a Sperner family of subsets of $[n]$. Then it is easy to verify that $\{(X_1, \overline{X}_1), \ldots, (X_m, \overline{X}_m)\}$ satisfies the conditions of Theorem 2.1.3. Consequently,

$$\sum_{X \in \mathcal{F}} \frac{1}{\binom{n}{|X|}} = \sum_{i=1}^{m} \frac{1}{\binom{|X_i|+|\overline{X}_i|}{|X_i|}} \leq 1.$$

Bollobás's theorem was published (as a lemma in a slightly modified version) one year before Lubell's proof of Sperner's theorem. As for the LYM-inequality, Ahlswede and Zhang [18] lifted the inequality in Theorem 2.1.3 to an identity, and Ahlswede and Cai [4] gave a further generalization. We omit the details. Instead, we consider another generalization. Earlier we worked with ordered partitions $(X, \overline{X})$ of $[n]$ into two classes. Now we look at ordered partitions $(X_1, \ldots, X_k)$ of $[n]$ into k classes where empty classes are allowed. We speak of *ordered k-partitions of* $[n]$. Two such partitions $(X_1, \ldots, X_k)$, $(Y_1, \ldots, Y_k)$ are called *inclusion-unrelated* (resp. *cross-unrelated*) if there is no $i \in [k]$ such that $X_i \subset Y_i$ or $Y_i \subset X_i$ (resp. if there are i, j, h, l such that $i < j,\ h > l,\ X_i \cap Y_j \neq \emptyset,\ X_h \cap Y_l \neq \emptyset$).

Theorem 2.1.4. *Let $\{(X_1^1, \ldots, X_k^1), \ldots, (X_1^m, \ldots, X_k^m)\}$ be a family of ordered k-partitions of $[n]$ that are pairwise inclusion-unrelated or cross-unrelated. Then*

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|X_1^i|, \ldots, |X_k^i|}} \leq 1.$$

Proof. The proof is analogous to the proof of Theorem 2.1.3. We only replace condition (2.7) by the following condition:

$$X_1^i = \{\pi(1), \ldots, \pi(|X_1^i|)\},\ X_2^i = \{\pi(|X_1^i| + 1), \ldots, \pi(|X_1^i| + |X_2^i|)\}, \ldots,$$
$$X_k^i = \{\pi(|X_1^i| + \cdots + |X_{k-1}^i| + 1), \ldots, \pi(n)\}.$$

Then it is easy to see that for fixed i there are exactly $|X_1^i|!\dots|X_k^i|!$ corresponding permutations, and for fixed π there is at most one i. This yields

$$\sum_{i=1}^{m}|X_1^i|!\cdots|X_k^i|!\leq n!,$$

and finally the assertion. ■

It is an easy exercise to verify that $\binom{n}{a_1,\dots,a_k}$ with $a_1+\cdots+a_k=n$ attains its maximum if all numbers a_j are "almost" equal. Since the ordered k-partitions $(X_1,\dots,X_k)$ of $[n]$ with $|X_j|=\lfloor\frac{n+j-1}{k}\rfloor$, $j=1,\dots,k$, are pairwise inclusion-unrelated as well as pairwise cross-unrelated, there follows a corollary.

Corollary 2.1.1. *The maximum size of a family of pairwise inclusion- or cross-unrelated ordered k-partitions of* $[n]$ *equals* $\binom{n}{a_1,\dots,a_k}$ *where* $a_j=\lfloor\frac{n+j-1}{k}\rfloor$, $j=1,\dots,k$.

In the case of pairwise inclusion-unrelated ordered k-partitions Corollary 2.1.1 (resp. Theorem 2.1.4) is due to Meshalkin [364] (resp. Bollobás [74]). For cross-unrelated ordered k-partitions, the results were obtained (in a slightly different notation) by Bandt, Burosch, and Drews [37].

Of course, one may assume also other conditions on the ordered (resp. unordered) k-partitions of $[n]$. For example, motivated by the study of independent random variables, Rényi [397, p. 16] called two unordered k-partitions $\{X_1,\dots, X_k\}$, $\{Y_1,\dots,Y_k\}$ of $[n]$ *qualitatively independent* if $X_i\cap Y_j\neq\emptyset$ for all $1\leq i, j\leq k$ (more generally, also k must not be constant). The determination of the maximum size of a family of k-partitions which are pairwise qualitatively independent is very difficult. Only for $k=2$ is an exact answer known (see the Profile-Polytope Theorem 3.3.1 and the third remark after it). For $k\geq 3$, only asymptotic bounds and estimates are known (see Poljak and Tuza [383], and Gargano, Kőrner, and Vaccaro [216, 217]). As an example we study the following condition: Two ordered 3-partitions (X_1,X_2,X_3), (Y_1,Y_2,Y_3) of $[n]$ are called *KS-independent* if $X_1\cap Y_3, X_2\cap Y_3, Y_1\cap X_3, Y_2\cap X_3\neq\emptyset$. Let $KS(n)$ be the maximum size of a family of pairwise KS-independent ordered 3-partitions of $[n]$. The following result is due to Kőrner and Simonyi [314]:

Theorem 2.1.5. *We have* $\lim_{n\to\infty}\sqrt[n]{KS(n)}=\frac{1+\sqrt{5}}{2}$.

Proof. First we show the inequality "$\leq$." Let $\mathcal{F}$ be a family of size $KS(n)$. We classify the members (X_1,X_2,X_3) of $\mathcal{F}$ with respect to the cardinalities of $|X_1|$ and $|X_2|$. Since these cardinalities belong to $\{0,\dots,n\}$ (more exactly to $\{1,\dots,n-1\}$), $(|X_1|,|X_2|)$ can take on only $(n+1)^2$ different pairs of values.

Thus there are numbers a, b, and a subfamily $\mathcal{F}_1$ of $\mathcal{F}$ such that $|X_1| = a$ and $|X_2| = b$ for all $(X_1, X_2, X_3) \in \mathcal{F}_1$ and

$$|\mathcal{F}_1| \geq \frac{KS(n)}{(n+1)^2}. \tag{2.10}$$

The families of pairs (X_1, X_3) and (X_2, X_3), where $(X_1, X_2, X_3) \in \mathcal{F}_1$, satisfy the conditions of Theorem 2.1.3; thus

$$|\mathcal{F}_1| \leq \binom{a + (n - (a+b))}{a} = \binom{n-b}{a}, \tag{2.11}$$

$$|\mathcal{F}_1| \leq \binom{b + (n - (a+b))}{b} = \binom{n-a}{b}. \tag{2.12}$$

Let $H(\lambda) := -\lambda \log \lambda - (1-\lambda)\log(1-\lambda),\ 0 < \lambda < 1$. In Section 2.6 (Corollary 2.6.2) we will prove that

$$\binom{n}{k} \leq e^{nH(k/n)}. \tag{2.13}$$

From (2.11)–(2.13) we obtain (with $\alpha := \frac{a}{n}, \beta := \frac{b}{n}$)

$$\begin{aligned} \frac{1}{n}\log|\mathcal{F}_1| &\leq \min\left\{(1-\alpha)H\left(\frac{\beta}{1-\alpha}\right), (1-\beta)H\left(\frac{\alpha}{1-\beta}\right)\right\} \\ &\leq \frac{1-\alpha}{2}H\left(\frac{\beta}{1-\alpha}\right) + \frac{1-\beta}{2}H\left(\frac{\alpha}{1-\beta}\right). \end{aligned} \tag{2.14}$$

Since $H(\lambda)$ is concave (see, e.g., the proof of Claim 1 in Theorem 2.6.6), we derive from Jensen's inequality (with $\gamma := \frac{\alpha+\beta}{2}$)

$$\frac{1}{n}\log|\mathcal{F}_1| \leq (1-\gamma)H\left(\frac{\gamma}{1-\gamma}\right).$$

Straightforward computations show that the RHS attains in $(0, 1)$ its maximum at $\gamma = \frac{1}{2} - \frac{\sqrt{5}}{10}$, and for this value we obtain (noting (2.10))

$$\begin{aligned} \frac{1}{n}\log|\mathcal{F}_1| &\leq \log\frac{1+\sqrt{5}}{2}, \\ \sqrt[n]{|\mathcal{F}_1|} &\leq \frac{1+\sqrt{5}}{2}, \\ \sqrt[n]{KS(n)} &\leq \sqrt[n]{(n+1)^2}\frac{1+\sqrt{5}}{2}. \end{aligned}$$

Now we prove the inequality "$\geq$." We construct a family whose size is large enough. Let $\mathcal{F}(n)$ be the family of all ordered 3-partitions (X_1, X_2, X_3) of $[n]$ (empty classes are allowed) with the property

$$1 \notin X_2, n \notin X_1 \text{ and } i \in X_1 \text{ iff } i+1 \in X_2, \qquad i = 1, \ldots, n-1.$$

It is easy to verify that

$$|\mathcal{F}(1)| = 1, |\mathcal{F}(2)| = 2, |\mathcal{F}(n+1)| = |\mathcal{F}(n-1)| + |\mathcal{F}(n)|.$$

Consequently $|\mathcal{F}(n)|$ equals the $(n+1)$th Fibonacci number, implying

$$\lim_{n\to\infty} \sqrt[n]{|\mathcal{F}(n)|} = \frac{1+\sqrt{5}}{2}.$$

As before, we may argue that there are numbers a, b and a subfamily $\mathcal{F}_1(n)$ of $\mathcal{F}(n)$ such that $|X_1| = a$ and $|X_2| = b$ for all $(X_1, X_2, X_3) \in \mathcal{F}_1(n)$ and $|\mathcal{F}_1(n)| \geq \frac{|\mathcal{F}(n)|}{(n+1)^2}$, which implies $\lim_{n\to\infty} \sqrt[n]{\mathcal{F}_1(n)} = \frac{1+\sqrt{5}}{2}$. Finally we classify the members $(X_1, X_2, X_3) \in \mathcal{F}_1(n)$ with respect to the sum of the elements of X_1. Clearly, the sum can take on at most $\frac{n(n+1)}{2}$ different values. Thus there exist a number c and a subfamily $\mathcal{F}_2(n)$ of $\mathcal{F}_1(n)$ such that $\sum_{i\in X_1} i = c$ for all members (X_1, X_2, X_3) of $\mathcal{F}_2(n)$ and $|\mathcal{F}_2(n)| \geq \frac{2|\mathcal{F}(n)|}{n(n+1)}$, which implies $\lim_{n\to\infty} \sqrt[n]{\mathcal{F}_2(n)} = \frac{1+\sqrt{5}}{2}$. Now it is sufficient to show that the members of $\mathcal{F}_2(n)$ are pairwise KS-independent. Let $(X_1, X_2, X_3), (Y_1, Y_2, Y_3)$ be different members of $\mathcal{F}_2(n)$. We have to show that $X_1 \cap Y_3 \neq \emptyset$ and $X_2 \cap Y_3 \neq \emptyset$. To show the first relation, let j be the smallest integer from $[n]$ such that $j \notin (X_1 \cap Y_1) \cup (X_2 \cap Y_2) \cup (X_3 \cap Y_3)$. Then $j \notin X_2 \cup Y_2$, since otherwise j could be replaced by $j-1$. If $j \in X_1$, then $j \in Y_3$, and $X_1 \cap Y_3 \neq \emptyset$ follows. Thus let $j \in X_3$ and $j \in Y_1$. Let $X_1 = \{u_1, \dots, u_a\}$, where $u_1 < \cdots < u_a$, and $Y_1 = \{v_1, \dots, v_a\}$, where $v_1 < \cdots < v_{k-1} < v_k = j < \cdots < v_a$ for some k. By the choice of j, we have $u_1 = v_1, \dots, u_{k-1} = v_{k-1}, j = v_k < u_k$, and by construction of $\mathcal{F}_2(n)$, $\sum_{i=1}^a u_i = \sum_{i=1}^a v_i$. Thus there must exist a smallest integer l such that $v_l > u_l$. Then it is easy to verify that $u_l \in Y_3$, hence also in this case $X_1 \cap Y_3 \neq \emptyset$. The relation $X_2 \cap Y_3 \neq \emptyset$ can be proved in the same way by looking at the corresponding *largest* integers from $[n]$ (note that also the sum of all elements in X_2 is constant for all $(X_1, X_2, X_3) \in \mathcal{F}_2(n)$ because of the construction of $\mathcal{F}(n)$). ■

Gargano, Kőrner, and Vaccaro [216]–[218] studied more general problems of the following kind: Let $G = (V, E)$ be a directed graph. What is the maximum size of a family of n-tuples of vertices of G such that for all $(v_1, \dots, v_n), (w_1, \dots, w_n) \in \mathcal{F}$ there are $i, j \in [n]$ such that $v_i w_i \in E$ and $w_j v_j \in E$ (resp. for all $(v_1, \dots, v_n), (w_1, \dots, w_n) \in \mathcal{F}$, and for all $e \in E$ there are $i, j \in [n]$ such that $e = v_i w_i = w_j v_j$).

Now we study a class of families that is as important as the Sperner families. In the Boolean lattice the family $\mathcal{F}$ of subsets of $[n]$ is *intersecting* if for all $X, Y \in \mathcal{F}$ it holds $X \cap Y \neq \emptyset$. In the case of cutsets we have seen that minimal families may have very different sizes. For maximal intersecting families, Erdős, Ko, and Rado [170] observed that we have a completely different situation:

Theorem 2.1.6. *Each maximal intersecting family in* $2^{[n]}$ *has size* 2^{n-1}.

Proof. Let $\mathcal{F}$ be a maximal intersecting family in $2^{[n]}$. We count in two different ways the number N of pairs $(X, (A, \overline{A}))$ with $X \in \mathcal{F}$, $A \subseteq [n]$, $X \in \{A, \overline{A}\}$. On the one hand, for fixed X, there exist exactly two "second components," namely $(X, \overline{X})$ and $(\overline{X}, X)$. Thus

$$N = 2|\mathcal{F}|. \tag{2.15}$$

On the other hand, for fixed $(A, \overline{A})$ there exists at most one $X \in \mathcal{F}$ with $X \in \{A, \overline{A}\}$ (note that $A, \overline{A}$ cannot both belong to $\mathcal{F}$ because of $A \cap \overline{A} = \emptyset$). Since we have 2^n pairs $(A, \overline{A})$ it follows that

$$N \leq 2^n. \tag{2.16}$$

From (2.15) and (2.16) we derived $|\mathcal{F}| \leq 2^{n-1}$. Assume that we have strict inequality. Then there must exist some A such that $A, \overline{A} \notin \mathcal{F}$. Since $\mathcal{F}$ is maximal, we cannot add A and $\overline{A}$ to $\mathcal{F}$. Thus there must exist some $X, Y \in \mathcal{F}$ such that $X \cap A = \emptyset, Y \cap \overline{A} = \emptyset$. But then $X \subseteq \overline{A}, Y \subseteq A$; that is, $X \cap Y = \emptyset$, a contradiction. ∎

Now we restrict ourselves to one level. The following result is a classical one. Erdős, Ko, and Rado [170] discovered it as early as in the thirties (see Erdős [169]). More than twenty years later it was published.

***Theorem 2.1.7* (*Erdős–Ko–Rado Theorem*).** *Let $\mathcal{F}$ be a k-uniform intersecting family in $2^{[n]}$. Then*

$$|\mathcal{F}| \leq \begin{cases} \binom{n-1}{k-1} & \text{if } k \leq \frac{n}{2}, \\ \binom{n}{k} & \text{otherwise,} \end{cases}$$

and the bound is the best possible.

Proof. In order to see that the bound cannot be improved take

$$\mathcal{F}^* := \begin{cases} \{X \in \binom{[n]}{k} : 1 \in X\} & \text{if } k \leq \frac{n}{2}, \\ \binom{[n]}{k} & \text{otherwise.} \end{cases}$$

Now we prove that the bound is true. The case $k > \frac{n}{2}$ is trivial. Thus let $k \leq \frac{n}{2}$. In contrast to the second proof of Sperner's theorem, we are working this time not with the set S_n of all permutations of $[n]$, but only with the set C_n of all cyclic permutations of $[n]$. Recall that a permutation ζ of $[n]$ is *cyclic* if n is the smallest integer with $\zeta^n(1) = 1$. We count the number N of pairs (X, ζ), where $X \in \mathcal{F}$, $\zeta \in C_n$, and there is some $i \in [n]$ with $X = \{\zeta^i(1), \zeta^{i+1}(1), \ldots, \zeta^{i+k-1}(1)\}$ in two different ways. For fixed X, we must determine the number of cyclic permutations in which the elements of X (i.e., also the elements of $[n] - X$) appear consecutively. We obtain $k!(n-k)!$ such permutations since we may permute the elements of X

and of $[n] - X$ independently, and each such ordering on X and $[n] - X$ yields a unique cyclic permutation. Consequently,

$$N = |\mathcal{F}|k!(n-k)!. \tag{2.17}$$

Next we have to estimate the number $d(\zeta)$ of members X of $\mathcal{F}$ for which there is some $i \in [n]$ such that $X = \{\zeta^i(1), \ldots, \zeta^{i+k-1}(1)\}$.

Claim. If $k \leq \frac{n}{2}$ then $d(\zeta) \leq k$.

Proof of Claim. Supposing that $d(\zeta) > 0$, we may – after a suitable renumbering – assume that ζ is the cycle $(12\ldots n)$, that is, $\zeta(i) = i + 1 (\operatorname{mod} n)$ and, moreover, that $X^* := \{1, \ldots, k\} \in \mathcal{F}$. Then $d(\zeta) = \{i \in [n] : \{i, i+1, \ldots, i + k - 1\} \in \mathcal{F}\}$ (the addition is always modulo n). Let

$$S := \{j \in [k] : \{j, j+1, \ldots, j+k-1\} \in \mathcal{F},$$
$$E := \{j \in [k] : \{j, j-1, \ldots, j-k+1\} \in \mathcal{F}.$$

If $X = \{i, i+1, \ldots, i+k-1\} \in \mathcal{F}$, then we must have $i \in S$ or $i+k-1 \in E$ since X must intersect X^*. Note that for X^* we have $1 \in S$ as well as $1+k-1 \in E$. Thus

$$d(\zeta) \leq |S| + |E| - 1. \tag{2.18}$$

Further, because of the intersection property of $\mathcal{F}$ and $k \leq \frac{n}{2}$, $j \in E$ implies $j + 1 \notin S$. Thus

$$|S| \leq k - (|E| - 1) \tag{2.19}$$

(we cannot exclude $k+1$). Together with (2.18) we obtain the assertion $d(\zeta) \leq k$. □

Since there are exactly $(n-1)!$ cyclic permutations, it follows from the claim that

$$N \leq k(n-1)!. \tag{2.20}$$

Finally, from (2.17) and (2.20) we derive

$$|\mathcal{F}|k!(n-k)! \leq k(n-1)!,$$

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

■

The method of proof is due to Katona [295] and known as *Katona's circle method.* It is fundamental for a whole theory that is presented in Chapter 3. The situation is more difficult if one replaces the condition $X \cap Y \neq \emptyset$ by $|X \cap Y| \geq t$, where t is some natural number. This case of t-intersecting families is treated in Sections 2.4, 2.5, 6.4, and 6.5.

Let $\mathcal{F}$ be a family in $2^{[n]}$. We say that a subfamily $\mathcal{F}_1$ of $\mathcal{F}$ is a *star* if there is some $i \in [n]$ such that $i \in X$ for all $X \in \mathcal{F}_1$. Let $\mathcal{I}_k := \binom{[n]}{0} \cup \cdots \cup \binom{[n]}{k}$. From Theorems 2.1.6 and 2.1.7, it easily follows that the maximum size of an intersecting family in $\mathcal{I}_k$ equals $\sum_{i=1}^{k} \binom{n-1}{i-1}$ if $k \leq \frac{n}{2}$ or $k = n$ (by the way, this result is also true for the remaining values of k, which follows from the Profile-Polytope Theorem 3.3.1). Obviously, the size of each maximal star in $\mathcal{I}_k$ also equals $\sum_{i=1}^{k} \binom{n-1}{i-1}$.

Thus we have seen that a special case of the following conjecture of Chvátal [101] is true: Let $\mathcal{I}$ be any ideal in $2^{[n]}$. Then the maximum size of an intersecting family in $\mathcal{I}$ equals the maximum size of a star in $\mathcal{I}$. For more information on this conjecture, we refer to Anderson [32], Berge [50], and Snevily [432]. Frankl [192] also mentions this and many other interesting problems.

Let μ be a positive integer. We say that a family $\mathcal{F}$ of subsets of $[n]$ is *μ-wise intersecting* (resp. *μ-wise cointersecting*) if, for all $X_1, \ldots, X_\mu \in \mathcal{F}$, there holds $X_1 \cap \cdots \cap X_\mu \neq \emptyset$ (resp. $X_1 \cup \cdots \cup X_\mu \neq [n]$). It is easy to see that the maximum size of a μ-wise intersecting (μ-wise cointersecting) family is 2^{n-1} if $\mu \geq 2$. The restriction to one level gives the following generalization of the Erdős–Ko–Rado Theorem, our 2.1.7, which was proved by Frankl [187] and in a different way by Gronau [247].

Theorem 2.1.8. *Let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$ and let $\mu \geq 2$.*

(a) If $\mathcal{F}$ is μ-wise intersecting, then

$$|\mathcal{F}| \leq \begin{cases} \binom{n-1}{k-1} & \text{if } k \leq \frac{\mu-1}{\mu} n, \\ \binom{n}{k} & \text{otherwise.} \end{cases}$$

(b) If $\mathcal{F}$ is μ-wise cointersecting, then

$$|\mathcal{F}| \leq \begin{cases} \binom{n-1}{k} & \text{if } k \geq \frac{n}{\mu}, \\ \binom{n}{k} & \text{otherwise.} \end{cases}$$

Both bounds are the best possible.

Proof. First observe that (a) and (b) are equivalent since $\mathcal{F}$ is μ-wise intersecting iff $\overline{\mathcal{F}}$ (the complementary family) is μ-wise cointersecting. Thus it remains to prove (b). The bound cannot be improved since

$$\mathcal{F}^* := \begin{cases} \{X \in \binom{[n]}{k} : 1 \notin X\} & \text{if } k \geq \frac{n}{\mu}, \\ \binom{[n]}{k} & \text{if } k < \frac{n}{\mu} \end{cases}$$

is μ-wise cointersecting. Further we follow the proof of Theorem 2.1.7. We have to replace $\frac{n}{2}$ by $\frac{n}{\mu}$ and (with the same notations) we then have to prove the following new claim:

Claim. If $k \geq \frac{n}{\mu}$ then $d(\zeta) \leq n - k$.

Proof of Claim. Suppose again that $\zeta = (12\dots n)$ and $X^* = \{1,\dots,k\}$. Let this time

$$S := \{i \in [n] : \{i, i+1, \dots, i+k-1\} \in \mathcal{F}\},$$

that is,

$$d(\zeta) = |S|. \tag{2.21}$$

Note that $1 \in S$. We partition $[n]$ into k sets $S_1, \dots, S_k$, namely, the residue classes modulo k. We show that

$$|S_j \cap S| \leq |S_j| - 1 \text{ for all } j \in [k]. \tag{2.22}$$

Assume the contrary, that is, $S_j = S \cap S_j$ for some $j \in [k]$. Then (with addition modulo n) the sets $X_1 := \{j, j+1, \dots, j+k-1\}$, $X_2 := \{j+k, \dots, j+2k-1\}, \dots, X_a := \{j+(a-1)k, \dots, j+ak-1\}$ would belong to $\mathcal{F}$, where $j+(a-1)k \leq n$, $j+ak-1 \geq n$. From the first inequality we derive $a-1 < \frac{n}{k}$, that is, $a \leq \mu$. If $ak \geq n$, then $\bigcup_{l=1}^{a} X_l = [n]$ in contradiction to the fact that $\mathcal{F}$ is μ-wise cointersecting. But if $ak < n$ then $a < \mu$, hence $a+1 \leq \mu$ and we have $\bigcup_{l=1}^{a} X_l \cup X^* = [n]$, which is again a contradiction. From (2.21) and (2.22) we obtain

$$d(\zeta) = \sum_{j=1}^{k} |S \cap S_j| \leq \sum_{j=1}^{k} (|S_j| - 1) = n - k.$$

□

The rest of the proof is analogous to the proof of Theorem 2.1.7. ■

If both conditions are combined we have the following result [162], which is a refined version of a result of Gronau [248] (based on a method of Frankl [188]).

Theorem 2.1.9. *Let $\mathcal{F}$ be a k-uniform μ-wise intersecting and ν-wise cointersecting family in $2^{[n]}$, and let $\mu, \nu \geq 4$. Then*

$$|\mathcal{F}| \leq \begin{cases} \binom{n-1}{k-1} & \text{if } k < \frac{n-1}{\nu} + 1, \\ \binom{n-2}{k-1} & \text{if } \frac{n-1}{\nu} + 1 \leq k \leq \frac{\mu-1}{\mu}(n-1), \\ \binom{n-1}{k} & \text{if } k > \frac{\mu-1}{\mu}(n-1). \end{cases}$$

The proof is omitted, but note that the first and third case follows from Theorem 2.1.8 and that in the second case an exchange operation is applied that is discussed in Section 2.3. For example, for $\mu = \nu = 3$ a complete solution is not known.

2.2. Partitions into symmetric chains

The following fact can be considered as a special case of the method of counting in two different ways. Suppose we are given a set S and look for the maximum size of a subset F of S satisfying a certain condition. To obtain an upper bound we partition S in an appropriate way into subsets $S_1, \ldots, S_k$ such that a "good" upper bound for $|F \cap S_j|$, $j \in [k]$, can be proved. If, for example, $|F \cap S_j| \leq a_j$ for all j, then $|F| \leq \sum_{j=1}^{k} a_j$ follows. Actually, we applied this method already in the proof of Theorem 2.1.8. Here we show that Sperner's theorem can also be proved in this way.

3rd Proof of Theorem 1.1.1(a). Take the set S to be $2^{[n]}$, and the classes S_j will be chains of the form $C = (X_0 \lessdot \cdots \lessdot X_h)$ such that $|X_0| + |X_h| = n$. These chains are called *symmetric*.

Claim. There exists a partition of $2^{[n]}$ into symmetric chains.

Proof of Claim. We proceed by induction on n and look only at the step $n \to n+1$. So let $\mathfrak{C}(n)$ be a symmetric chain partition of $2^{[n]}$. For each chain $C = (X_0 \lessdot \cdots \lessdot X_h) \in \mathfrak{C}(n)$, we construct new chains

$$C' := (X_0 \lessdot \cdots \lessdot X_h \lessdot X_h \cup \{n+1\}), \tag{2.23}$$

$$C'' := (X_0 \cup \{n+1\} \lessdot \cdots \lessdot X_{h-1} \cup \{n+1\}), \tag{2.24}$$

but C'' must be omitted (it does not exist) if $h = 0$. Now it is easy to verify that all these new chains form a partition $\mathfrak{C}(n+1)$ of $2^{[n+1]}$ into symmetric chains (distinguish the cases where members X of $2^{[n+1]}$ contain (resp. do not contain) $n+1$). □

Since each symmetric chain in $2^{[n]}$ contains exactly one member $X \in 2^{[n]}$ with $|X| = \lfloor \frac{n}{2} \rfloor$, each symmetric chain partition $\mathfrak{C}(n)$ of $2^{[n]}$ contains exactly $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains. If $\mathcal{F}$ is a Sperner family in $2^{[n]}$, then $|\mathcal{F} \cap C| \leq 1$ for each chain. Accordingly,

$$|\mathcal{F}| \leq \sum_{C \in \mathfrak{C}(n)} |\mathcal{F} \cap C| = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

■

This proof is due to de Bruijn, Tengbergen, and Kruyswijk [87] (who considered, more generally, chain products) and, independently, to Hansel [252]. Since there are many generalizations and related results, a separate chapter (Chapter 5) is devoted to this topic. Griggs [241] collected many interesting problems. Here only two applications will be presented. Let $\varphi(n)$ be the number of order-preserving maps from B_n into B_1, that is, the number of functions $f : 2^{[n]} \to \{0, 1\}$ such that $X \subseteq Y$ implies $f(X) \le f(Y)$ for all $X, Y \in 2^{[n]}$. If we interpret $f : 2^{[n]} \to \{0, 1\}$ as the characteristic function of a family $\mathcal{F}$, then obviously f is order preserving iff $\mathcal{F}$ is a filter. Since there is a bijection between filters and antichains (associate with a filter the set of its minimal elements), our number $\varphi(n)$ also equals the number of filters and the number of Sperner families in $2^{[n]}$. The problem of determining $\varphi(n)$ goes back to Dedekind [127] and is very difficult. The following bounds are due to Hansel [252].

Theorem 2.2.1. *For all* $n \ge 1$, $2^{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \varphi(n) \le 3^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$.

Proof. The lower bound is trivial since every subset of $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ is a Sperner family. To prove the upper bound, let $\mathfrak{C}(n)$ be the symmetric chain partition of $2^{[n]}$, which was constructed inductively in the third proof of Theorem 1.1.1(a).

Claim. For each chain C of $\mathfrak{C}(n)$ and three consecutive members $X_{a-1} \lessdot X_a \lessdot X_{a+1}$ of C, there is some $\hat{X}_a$ such that $X_{a-1} \lessdot \hat{X}_a \lessdot X_{a+1}$ and $\hat{X}_a$ is contained in a chain D of $\mathfrak{C}(n)$ with $|D| = |C| - 2$.

Proof of Claim. We proceed by induction on n and study the step $n \to n+1$.

Case 1. The three members X_{a-1}, X_a, X_{a+1} belong to the chain C' of type (2.23). If $n+1 \notin X_{a+1}$, then the three members belong to the chain C of $\mathfrak{C}(n)$, and we can find by the induction hypothesis a corresponding chain $D \in \mathfrak{C}(n)$ with the corresponding member $\hat{X}_a$ which also belongs to the chain $D' \in \mathfrak{C}(n+1)$. Since $|D| = |C| - 2$, also $|D'| = |C'| - 2$. If $n+1 \in X_{a+1}$, then $X_a = X_h$, and we can take $\hat{X}_a := X_{h-1} \cup \{n+1\}$. The corresponding chain is C''.

Case 2. The three members X_{a-1}, X_a, X_{a+1} belong to a chain C'' of type (2.24). Then $Y_j := X_j - \{n+1\}$, $j = a-1, a, a+1$, belong to $C \in \mathfrak{C}(n)$, and we find, by induction, a corresponding chain $D \in \mathfrak{C}(n)$ with the corresponding member $\hat{Y}_a$. Because Y_{a+1} cannot be the maximal element of C, the member $\hat{Y}_a$ is not the maximal element of D (otherwise we would have $|D| < |C| - 2$). Thus $\hat{X}_a := \hat{Y}_a \cup \{n+1\}$ belongs to D'', and $\hat{X}_a$ is the desired element since $|D''| = |D| - 1 = |C| - 3 = |C''| - 2$. □

We order the members of $\mathfrak{C}(n)$ by increasing size; that is, $\mathfrak{C}(n) = \{C_j : 1 \le j \le d\}$ and $|C_1| \le \cdots \le |C_d|$, where $d := \binom{n}{\lfloor \frac{n}{2} \rfloor}$. We construct (in a nondeterministic way) step by step all filters $\mathcal{F}$ in $2^{[n]}$ by deciding whether the members of C_j, $j = 1, \ldots, d$, should belong to $\mathcal{F}$ or not. For the chains of size at most two, we have at most three choices, namely, taking no, the maximal, or both members into $\mathcal{F}$. Suppose that we are done with all chains before $C = (X_0 \lessdot \cdots \lessdot X_h)$ and we have fixed some part of a filter $\mathcal{F}$. Let $\hat{X}_1, \ldots, \hat{X}_{h-1}$ be the corresponding members described in the claim. Of them we know already whether they belong to $\mathcal{F}$ or not. Let l (resp. u) be the largest (resp. smallest) index such that $\hat{X}_l \notin \mathcal{F}$ (resp. $\hat{X}_u \in \mathcal{F}$) (if there is no such index put $l := 0$ (resp. $u := h$)). Clearly, $u - l \leq 1$. In order to obtain a filter, we need that $X_1, \dots, X_{l-1} \notin \mathcal{F}$ and $X_{u+1}, \dots, X_h \in \mathcal{F}$ since $X_1 \lessdot \cdots \lessdot X_{l-1} \lessdot \hat{X}_l$ and $\hat{X}_u \lessdot X_{u+1} \lessdot \cdots \lessdot X_h$. Thus we have at most three choices to extend the fixed part of $\mathcal{F}$. This is clear if $l \geq u$. If $l + 1 = u$ at most X_l and X_u are free, but if $X_l \in \mathcal{F}$, then certainly $X_u \in \mathcal{F}$. Thus, all in all, we have at most $3^{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ possibilities for constructing all filters. ∎

From this theorem it follows that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \log 2 \leq \log \varphi(n) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \log 3.$$

Kleitman [301] found the asymptotic formula

$$\log \varphi(n) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor} \log 2 \text{ as } n \to \infty,$$

considering not only one fixed symmetric chain partition but all such partitions that arise by permuting the elements of $[n]$, and using complicated averaging arguments. An improvement of the remainder was given by Kleitman and Markowsky [308], and an asymptotic formula for $\varphi(n)$ itself was finally found by Korshunov [317] in a very complicated way. A simplification (in a more general form) was given by Sapozhenko [413]. More details on the history of this and related problems are contained in the survey of Korshunov [319].

***Theorem 2.2.2* (*Korshunov Theorem*).** *We have for $n \to \infty$ in the case of even resp. odd n*

$$\varphi(n) \sim \begin{cases} 2^{\binom{n}{\frac{n}{2}}} e^{\binom{n}{\frac{n}{2}-1}\left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n+4}}\right)}, \\ 2^{\binom{n}{\frac{n-1}{2}}+1} e^{\binom{n}{\frac{n-3}{2}}\left(\frac{1}{2^{(n+3)/2}} - \frac{n^2}{2^{n+6}} - \frac{n}{2^{n+3}}\right) + \binom{n}{\frac{n-1}{2}}\left(\frac{1}{2^{(n+1)/2}} + \frac{n^2}{2^{n+4}}\right)}. \end{cases}$$

Let us conclude this section by a game. Suppose there is given a filter $\mathcal{F}$ in $2^{[n]}$ (resp. an order-preserving map from B_n into B_1), but only a stage manager (our *oracle*), say Heidrun, knows it. Someone, say Sebastian, wants to get to know $\mathcal{F}$. He may pick, step by step, members of $2^{[n]}$, and Heidrun will tell him whether they belong to $\mathcal{F}$ or not. The clever Sebastian will certainly not test all elements of $2^{[n]}$. For example, if $\emptyset \in \mathcal{F}$, then automatically $\mathcal{F} = 2^{[n]}$. Let $\psi(\mathcal{F})$ be the smallest number of Heidrun answers that enable (using the right sequence of Sebastian questions) the determination of $\mathcal{F}$. Finally let

$$\psi(n) := \max\{\psi(\mathcal{F}) : \mathcal{F} \text{ is a filter in } 2^{[n]}\}.$$

The number $\psi(n)$ is sometimes called the *Shannon complexity of the problem of the recognition of monotone Boolean functions*. It was studied the first time by Korobkov and Reznik [316]. The following result is again due to Hansel [252], who improved an earlier estimation of Korobkov [315].

Theorem 2.2.3. *We have* $\psi(n) = \binom{n}{\lfloor \frac{n-1}{2} \rfloor} + \binom{n}{\lfloor \frac{n+1}{2} \rfloor}$.

Proof. "$\geq$." Let $\mathcal{F}^* := \{X \subseteq [n] : |X| \geq \lfloor \frac{n+1}{2} \rfloor\}$. Obviously, for all $X \subseteq [n]$ with $|X| = \lfloor \frac{n+1}{2} \rfloor$, also $\mathcal{F}^* - \{X\}$ is a filter, and, for all $X \subseteq [n]$ with $|X| = \lfloor \frac{n-1}{2} \rfloor$, $\mathcal{F}^* \cup \{X\}$ is a filter. Thus, knowing the Heidrun answers for all $Y \in 2^{[n]} - \{X\}$, where $X \in \binom{[n]}{\lfloor \frac{n-1}{2} \rfloor} \cup \binom{[n]}{\lfloor \frac{n+1}{2} \rfloor}$, Sebastian cannot determine the filter uniquely. Consequently, he must ask Heidrun about all these X.

"$\leq$." Sebastian asks Heidrun, step by step, about the members of the chains C_j of the symmetric chain partition considered in the proof of Theorem 2.2.1. From this proof it follows that at most two elements of C are "suspicious" if he already inspected all chains ahead of C. Thus, for chains of more than one element, two answers suffice. For one-element chains, Sebastian needs of course only one question. It is easy to see that there are $\binom{n}{\lfloor \frac{n+1}{2} \rfloor} - \binom{n}{\lfloor \frac{n-1}{2} \rfloor}$ one-element chains. Thus

$$\begin{aligned}\psi(n) &\leq \binom{n}{\lfloor \frac{n+1}{2} \rfloor} - \binom{n}{\lfloor \frac{n-1}{2} \rfloor} + 2\left(\binom{n}{\lfloor \frac{n}{2} \rfloor} - \binom{n}{\lfloor \frac{n+1}{2} \rfloor} + \binom{n}{\lfloor \frac{n-1}{2} \rfloor}\right) \\ &= \binom{n}{\lfloor \frac{n+1}{2} \rfloor} + \binom{n}{\lfloor \frac{n-1}{2} \rfloor}.\end{aligned}$$

∎

Sokolov [434] found another algorithm that achieves this bound. Alekseev [25] provided estimations for the recognition of order-preserving maps from $S(k_1, \dots, k_n)$ into $\{0, 1\}$ (in special cases, such as $k_1 = \cdots = k_n = 2$, he has exact values) and Gronau [244] slightly generalized the above proof to the recognition of order-preserving maps from B_n into $\{0, \dots, k\}$. Moreover, independently, Serzhantov [424] and I [151] each found exact values for the case of order-preserving maps from $S(k_1, \dots, k_n)$ into $S(k)$ if $k_1 = k_2, k_3 = k_4, \dots, k_{2j-1} = k_{2j}$ for some $j \leq \frac{n}{2}$ and $k_i \leq 2$ for all $2j < i \leq n$, where still some further generalization, for example, with results of Mahnke [359] can be formulated. Finally, Serzhantov [425] solved completely the case of order-preserving maps from $S(k, \dots, k)$ into $\{0, 1\}$. Applications and similar problems appear in the maximization of submodular functions on discrete structures; see, for example, Kovalëv and Moshchenski [323]; and in many other areas, see the survey of Sapir [411].

2.3. Exchange operations and compression

In the first proof of the Sperner Theorem 1.1.1 we selected a subfamily $\mathcal{G}$ of a Sperner family $\mathcal{F}$ and constructed a new family $\mathcal{H}$ (namely $\mathcal{H} := \nabla(\mathcal{G})$ or, though not explicitly mentioned, $\mathcal{H} := \Delta(\mathcal{G})$) such that $\mathcal{F}' := (\mathcal{F} - \mathcal{G}) \cup \mathcal{H}$ is still a Sperner family. Moreover, we had $\mathcal{H} \cap \mathcal{F} = \emptyset$, so that $|\mathcal{F}|$ and $|\mathcal{F}'|$ are related in the same way as $|\mathcal{G}|$ and $|\mathcal{H}|$. In other words, we exchanged the Sperner family by a new Sperner family, or we pushed (shifted) the old family upward (resp. downward). The general idea of exchanging something works in many cases. Here, we study that operation which was introduced by Erdős, Ko, and Rado [170]. Let for $i, j \in [n]$ the mapping $\lhd_{ij} : 2^{[n]} \to 2^{[n]}$ be defined by

$$\lhd_{ij}(X) := (X - \{j\}) \cup \{i\},$$

and, as usual, for a family $\mathcal{F} \subseteq 2^{[n]}$ let

$$\lhd_{ij}(\mathcal{F}) := \{\lhd_{ij}(X) : X \in \mathcal{F}\}.$$

We will apply this operator only to a part of $\mathcal{F}$, namely to

$$\mathcal{F}_{ij} := \{X \in \mathcal{F} : i \notin X, j \in X \text{ and } \lhd_{ij}(X) \notin \mathcal{F}\}.$$

Finally, the *ij-shifting* of the family $\mathcal{F}$ is defined by

$$s_{ij}(\mathcal{F}) := (\mathcal{F} - \mathcal{F}_{ij}) \cup \lhd_{ij}(\mathcal{F}_{ij}).$$

Hence $\mathcal{F}_{ij}$ and $\lhd_{ij}(\mathcal{F}_{ij})$ play the role of $\mathcal{G}$ and $\mathcal{H}$ from above. Note that by definition $|\mathcal{F}_{ij}| = |\lhd_{ij}(\mathcal{F}_{ij})|$ and $\lhd_{ij}(\mathcal{F}_{ij}) \cap \mathcal{F} = \emptyset$ implying

$$|s_{ij}(\mathcal{F})| = |\mathcal{F}|.$$

Lemma 2.3.1. *Let $i, j \in [n]$, $i \neq j$, and let $\mathcal{F}$ be intersecting (resp. cointersecting). Then $s_{ij}(\mathcal{F})$ is intersecting (resp. cointersecting), too.*

Proof. We prove the assertion only for intersecting families. Thus let $X, Y \in s_{ij}(\mathcal{F})$. If $X, Y \in \mathcal{F} - \mathcal{F}_{ij}$, then $X \cap Y \neq \emptyset$ by supposition, and if $X, Y \in \lhd_{ij}(\mathcal{F}_{ij})$, then $i \in X \cap Y$. Thus it remains to study the case $X \in \mathcal{F} - \mathcal{F}_{ij}$, $Y = \lhd_{ij}(Z)$ for some $Z \in \mathcal{F}_{ij}$.

Case 1. $i \in X$. Then $i \in X \cap Y$.

Case 2. $i \notin X$.

Case 2.1. $j \notin X$. Then $X \cap Y = X \cap Z \neq \emptyset$ by supposition.

Case 2.2. $j \in X$. Then $\lhd_{ij}(X) \in \mathcal{F}$ because otherwise $X \in \mathcal{F}_{ij}$. Since $\mathcal{F}$ is intersecting, there is some $l \in \lhd_{ij}(X) \cap Z$ where obviously $l \notin \{i, j\}$. But then also $l \in X \cap Y$. ∎

Lemma 2.3.2. *Let $i, j \in [n], i \neq j$, and let $\mathcal{F}$ be k-uniform. Then*

$$\Delta(s_{ij}(\mathcal{F})) \subseteq s_{ij}(\Delta(\mathcal{F})).$$

Proof. Let $X \in s_{ij}(\mathcal{F})$ and $Y \in \Delta(X)$. We must show that $Y \in s_{ij}(\Delta(\mathcal{F}))$. Let $X = Y \cup \{l\}, Y = X - \{l\}$.

Case 1. $X \in \mathcal{F} - \mathcal{F}_{ij}$. Then clearly $Y \in \Delta(\mathcal{F})$. We will show that

$$Y \notin (\Delta(\mathcal{F}))_{ij}. \tag{2.25}$$

which implies that $Y \in s_{ij}(\Delta(\mathcal{F}))$.

Case 1.1. $l = j$. Then $j \notin Y$, which shows (2.25).

Case 1.2. $l = i$.

Case 1.2.1. $j \in Y$. Then $\lhd_{ij}(Y) = X - \{j\} \in \Delta(X) \subseteq \Delta(\mathcal{F})$. Thus (2.25) holds.

Case 1.2.2. $j \notin Y$. Then (2.25) is trivial.

Case 1.3. $l \notin \{i, j\}$. We may suppose that $i \notin Y, j \in Y$ because otherwise (2.25) is clear. But then also $i \notin X, j \in X$. Since $X \notin \mathcal{F}_{ij}$, we have $\lhd_{ij}(X) \in \mathcal{F}$. But then also $\lhd_{ij}(Y) \in \Delta(\mathcal{F})$, which shows (2.25).

Case 2. $X = \lhd_{ij}(Z)$ for some $Z \in \mathcal{F}_{ij}$. Then $j \notin X$; that is, $l \neq j$.

Case 2.1. $l = i$. Then $Y = Z - \{j\}$; that is, $Y \in \Delta(Z) \subseteq \Delta(\mathcal{F})$. Since $j \notin Y$, it follows that $Y \notin (\Delta(\mathcal{F}))_{ij}$ and consequently $Y \in s_{ij}(\Delta(\mathcal{F}))$.

Case 2.2. $l \neq i$. Then $Y = \lhd_{ij}(Z - \{l\})$ and $Z - \{l\} \in \Delta(\mathcal{F})$.

Case 2.2.1. $Y \in \Delta(\mathcal{F})$. Then $Y \in s_{ij}(\Delta(\mathcal{F}))$ since clearly $\lhd_{ij}(Z - \{l\}) \notin (\Delta(\mathcal{F}))_{ij}$.

Case 2.2.2. $Y \notin \Delta(\mathcal{F})$. Then $(Z - \{l\}) \in (\Delta(\mathcal{F}))_{ij}$, which shows that $Y = \lhd_{ij}(Z - \{l\}) \in s_{ij}(\Delta(\mathcal{F}))$. ■

Now several results can be proved by applying induction and considering families that are invariant with respect to certain ij-shiftings. For $\mathcal{F} \subseteq 2^{[n]}$, let

$$\Sigma(\mathcal{F}) := \sum_{X \in \mathcal{F}} \sum_{i \in X} i.$$

2nd proof of Theorem 2.1.7. The only difficult part is the upper bound in the case $k \leq \frac{n}{2}$. For $n = 2k$, the proof is trivial since from each pair $(X, \overline{X})$ with $|X| = k$ at most one component may belong to a k-uniform intersecting family $\mathcal{F}$; that is, $|\mathcal{F}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. Now we proceed by induction on n and consider the step $n \to n + 1 > 2k$. Of all maximum k-uniform intersecting families, we choose such a family $\mathcal{F}$ for which $\Sigma(\mathcal{F})$ is minimum. Then

$$s_{i,n+1}(\mathcal{F}) = \mathcal{F} \text{ for all } i < n + 1 \tag{2.26}$$

since otherwise, in view of Lemma 2.3.1, $\mathcal{F}$ could be replaced by $s_{i,n+1}(\mathcal{F})$ for some i. Let

$$\mathcal{F}_1 := \{X \in \mathcal{F} : n+1 \notin X\},$$
$$\mathcal{F}_2 := \{X \in \mathcal{F} : n+1 \in X\}, \qquad \mathcal{F}_2' := \{X - \{n+1\} : X \in \mathcal{F}_2\}.$$

By the induction hypothesis (note that $\mathcal{F}_1 \subseteq 2^{[n]}$),

$$|\mathcal{F}_1| \leq \binom{n-1}{k-1}. \tag{2.27}$$

Claim. $\mathcal{F}_2'$ is intersecting.

Proof of Claim. Assume that there are $X_1, X_2 \in \mathcal{F}_2'$ such that $X_1 \cap X_2 = \emptyset$. Since $|X_1|+|X_2| = 2(k-1) \leq n-2$, there are different numbers $i_1, i_2 \in [n]$ such that $i_1 \notin X_1$, $i_2 \notin X_2$. Let $Y_1 := X_1 \cup \{n+1\}$, $Y_2 := X_2 \cup \{n+1\}$, $Z_1 := X_1 \cup \{i_1\}$, $Z_2 := X_2 \cup \{i_2\}$. Then $Y_1, Y_2 \in \mathcal{F}$ and $Z_1 = \lhd_{i_1,n+1}(Y_1)$, $Z_2 = \lhd_{i_2,n+1}(Y_2)$. We have $Z_1, Z_2 \in \mathcal{F}$ in view of (2.26). But obviously $Z_1 \cap Z_2 = \emptyset$, a contradiction. □

The induction hypothesis applied to $\mathcal{F}_2'$ yields

$$|\mathcal{F}_2| = |\mathcal{F}_2'| \leq \binom{n-1}{k-2}. \tag{2.28}$$

By (2.27) and (2.28),

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq \binom{n-1}{k-1} + \binom{n-1}{k-2} = \binom{n}{k-1}.$$

■

From (1.3) in the proof of Theorem 1.1.1 it follows that

$$|\nabla(\mathcal{G})| \geq |\mathcal{G}|\frac{n-l}{l+1},$$

if $\mathcal{G}$ is an l-uniform family. Analogously,

$$|\Delta(\mathcal{F})| \geq |\mathcal{F}|\frac{k}{n-k+1}$$

if $\mathcal{F}$ is a k-uniform family. Now we ask for a better lower bound for the size of the shadow if the size of $\mathcal{F}$ is fixed. First we prove a result of Lovász [354] which is a weaker version of the Kruskal–Katona Theorem, 2.3.6, but often easier to handle. Denote for real x and natural k,

$$\binom{x}{k} := \begin{cases} 1 & \text{if } k = 0, \\ \frac{x(x-1)\cdots(x-k+1)}{k!} & \text{if } k > 0, \\ 0 & \text{if } k < 0. \end{cases}$$

Theorem 2.3.1. *Let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$ and let $|\mathcal{F}| = \binom{x}{k}$, $k \le x \le n$. Then*

$$|\Delta(\mathcal{F})| \ge \binom{x}{k-1}.$$

Proof. We proceed by induction on the parameter $t := k + |\mathcal{F}|$. Note that we suppose $|\mathcal{F}| \ge 1$. If $t = 1$ we have $k = 0$, $|\mathcal{F}| = 1$, and all is trivial. Consider hence at the step $t \to t + 1 = k + |\mathcal{F}|$. Obviously, we may suppose $k > 0$. Under all k-uniform $|\mathcal{F}|$-element families with minimum shadow we choose such a family for which $\Sigma(\mathcal{F})$ is minimum. It is enough to prove the assertion for this family. We have

$$s_{1,i}(\mathcal{F}) = \mathcal{F} \text{ for all } i \in \{2, \dots, n\} \tag{2.29}$$

since otherwise, in view of Lemma 2.3.2, $\mathcal{F}$ could be replaced by $s_{1,i}(\mathcal{F})$ for some i. Let, similar to the previous proof,

$$\mathcal{F}_1 := \{X \in \mathcal{F} : 1 \notin X\},$$

$$\mathcal{F}_2 := \{X \in \mathcal{F} : 1 \in X\}, \mathcal{F}_2' := \{X - \{1\} : X \in \mathcal{F}_2\}.$$

Claim. We have $\Delta(\mathcal{F}_1) \subseteq \mathcal{F}_2'$.

Proof of Claim. Let $Y \in \Delta(\mathcal{F}_1)$; that is, $X := Y \cup \{l\} \in \mathcal{F}_1$ for some $l \in \{2, \dots, n\}$. Since, by (2.29), $\triangleleft_{1,l}(X) = Y \cup \{1\} \in \mathcal{F}$, it follows that $Y \in \mathcal{F}_2'$. □

Since $\mathcal{F} \ne \emptyset$ we have, in view of the claim, $\mathcal{F}_2' \ne \emptyset$.

Case 1. $|\mathcal{F}_2'| < \binom{x-1}{k-1}$. Then $|\mathcal{F}_2'| < \binom{x}{k} = |\mathcal{F}|$; that is, $\mathcal{F}_1 \ne \emptyset$.

Case 1.1. $k \le x < k + 1$. Then $|\mathcal{F}_2'| < k$. But $|\Delta(\mathcal{F}_1)| \ge k$, since one member of $\mathcal{F}_1$ already produces a k-element shadow. Thus the claim yields the contradiction $|\mathcal{F}_2'| \ge k$.

Case 1.2. $k+1 \le x$. We have $|\mathcal{F}_1| = |\mathcal{F}| - |\mathcal{F}_2| \ge \binom{x}{k} - \binom{x-1}{k-1} = \binom{x-1}{k}$. Since $1 \le |\mathcal{F}_1| < |\mathcal{F}|$ we may apply the induction hypothesis to obtain $|\Delta(\mathcal{F}_1)| \ge \binom{x-1}{k-1}$, and again the claim yields the contradiction $|\mathcal{F}_2'| \ge \binom{x-1}{k-1}$.

Case 2. $|\mathcal{F}_2'| \ge \binom{x-1}{k-1}$. By the induction hypothesis, $|\Delta(\mathcal{F}_2')| \ge \binom{x-1}{k-2}$. Hence

$$|\Delta(\mathcal{F})| = |\Delta(\mathcal{F}_2')| + |\mathcal{F}_2'| \ge \binom{x-1}{k-2} + \binom{x-1}{k-1} = \binom{x}{k-1}.$$

■

The preceding proof is due to Frankl [191]. Let us look at some consequences. Repeated application of Theorem 2.3.1 gives the following estimation:

Corollary 2.3.1. *Let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$ with $|\mathcal{F}| = \binom{x}{k}$, where $k \le x \le n$. Let $0 \le i \le k$. Then*

$$|\Delta_{\to i}(\mathcal{F})| \ge \binom{x}{i}.$$

Daykin [121] observed that the Erdős–Ko–Rado Theorem 2.1.7 can be easily derived from this corollary. Indeed, if $k \le \frac{n}{2}$ and $\mathcal{F}$ is a k-uniform intersecting family, then obviously $\mathcal{F} \cap \Delta_{\to k}(\overline{\mathcal{F}}) = \emptyset$. The assumption $|\mathcal{F}| > \binom{n-1}{k-1}$ (i.e., $|\overline{\mathcal{F}}| > \binom{n-1}{n-k}$) would yield in view of Corollary 2.3.1 (with $x \ge n-1$, $k := n-k$, $i := k$)

$$\binom{n}{k} \ge |\mathcal{F}| + |\Delta_{\to k}(\overline{\mathcal{F}})| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

But in our approach this is only an apparent simplification of the (original) second proof of Theorem 2.1.7, since we needed the exchange operation also to prove Theorem 2.3.1.

Corollary 2.3.2. *Let $\mathcal{F}$ be a nonempty k-uniform family in $2^{[n]}$, $n \ge 3$.*

(a) If $k \ge \frac{n}{2} + 1$ then $|\Delta(\mathcal{F})| - |\mathcal{F}| \ge \frac{n}{2}$.
(b) If $k \le \frac{n}{2} - 1$ then $|\nabla(\mathcal{F})| - |\mathcal{F}| \ge \frac{n}{2}$.

Proof. (a) Let $k \ge \frac{n}{2} + 1$ and $|\mathcal{F}| = \binom{x}{k}$, where $k \le x \le n$. By Theorem 2.3.1

$$|\Delta(\mathcal{F})| - |\mathcal{F}| \ge \binom{x}{k-1} - \binom{x}{k} = \frac{1}{k}\binom{x}{k-1}(2k-1-x) =: f(x).$$

The polynomial $f(x)$ of degree k has the k simple roots $0, 1, \dots, k-2, 2k-1$. Then $f'(x)$ obviously has in each interval between these roots a simple root. This shows that $f(x)$ is in $[k-2, 2k-1]$ increasing up to some value ξ and then decreasing. Consequently, for $k \le x \le n$ (note that $n \le 2k-2$),

$$f(x) \ge \min\{f(k), f(2k-2)\} = k-1 \ge \frac{n}{2}.$$

(b) Let $k \le \frac{n}{2} - 1$. Then by (a)

$$|\nabla(\mathcal{F})| - |\mathcal{F}| = |\Delta(\overline{\mathcal{F}})| - |\overline{\mathcal{F}}| \ge \frac{n}{2}.$$

■

Corollary 2.3.3. *Let $\mathcal{F}$ be a Sperner family in $2^{[n]}$, $l := \min\{i :$ there is some $X \in \mathcal{F}$ with $|X| = i\}$, $u := \max\{i :$ there is some $X \in \mathcal{F}$ with $|X| = i\}$.*

(*a*) *If* $l \leq \frac{n}{2} \leq u$ *then*

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{\lfloor \frac{n}{2} \rfloor} - \frac{(u-l)n}{2} & \textit{if } n \textit{ is even,} \\ \binom{n}{\lfloor \frac{n}{2} \rfloor} - \frac{(u-l-1)n}{2} & \textit{if } n \textit{ is odd.} \end{cases}$$

(*b*)

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{\lfloor \frac{n}{2} \rfloor} - \frac{(\lfloor \frac{n}{2} \rfloor - l)n}{2} & \textit{if } u \leq \frac{n}{2}, \\ \binom{n}{\lfloor \frac{n}{2} \rfloor} - \frac{(u - \lfloor \frac{n+1}{2} \rfloor)n}{2} & \textit{if } l \geq \frac{n}{2}. \end{cases}$$

Proof. The cases $n = 1, 2$ are trivial, hence let $n \geq 3$. The proof is analogous to the proof of Theorem 1.1.1. If we replace $\mathcal{F}$ by $\mathcal{F}' := (\mathcal{F} - \mathcal{G}) \cup \nabla(\mathcal{G})$, where $\mathcal{G} := \mathcal{F} \cap \binom{[n]}{l}$ (resp. $\mathcal{F}' := (\mathcal{F} - \mathcal{G}) \cup \Delta(\mathcal{G})$ where $\mathcal{G} := \mathcal{F} \cap \binom{[n]}{u}$), then we obtain a Sperner family for which by Corollary 2.3.2

$$|\mathcal{F}'| \geq |\mathcal{F}| + \frac{n}{2}.$$

An induction argument concludes the proof. ∎

Let us again play a game. This time suppose that there is given an ideal $\mathcal{I}$ in $2^{[n]}$. Only our amiable stage manager Heidrun knows it. Jana is more modest than Sebastian. She does not want to get to know $\mathcal{I}$ completely; she is only interested in an element of $\mathcal{I}$ of maximum rank. The rules are the same as in Section 2.2. Let $\chi(\mathcal{I})$ be the smallest number of Heidrun answers which enable the determination of an element of $\mathcal{I}$ of maximum rank. Then

$$\chi(n) := \max\{\chi(\mathcal{I}) : \mathcal{I} \text{ is a nonempty ideal in } 2^{[n]}\}$$

is the *Shannon complexity* of our problem. The following result is due to Katerinochkina [287]:

Theorem 2.3.2. *We have* $\chi(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor} + 1$.

Proof. Let $m := \lfloor \frac{n}{2} \rfloor$.

"$\geq$." Let $\mathcal{I}^* := \{X \subseteq [n] : |X| < m\}$. Obviously, for all $X \in \binom{[n]}{m}$, $\mathcal{I}^* \cup X$ is an ideal; moreover, $\mathcal{I}^* - \binom{[n]}{m-1}$ is an ideal. Thus Jana must ask Heidrun about all elements of $\binom{[n]}{m}$ and about at least one element of $\binom{[n]}{m-1}$ in order to be sure.

"$\leq$." Jana proceeds as follows. She chooses an arbitrary member X_m of $\binom{[n]}{m}$. If Heidrun says that $X_m \notin \mathcal{I}$, she takes an element X_{m-1} of $\Delta(X_m)$; if the answer for X_{m-1} is $X_{m-1} \notin \mathcal{I}$, she takes an element X_{m-2} of $\Delta(X_{m-1})$, and so on. Since we suppose that $\mathcal{I} \neq \emptyset$ (which is not really essential), Jana eventually chooses some

$X_{l-1} \in \Delta(X_l)$ for which Heidrun says: $X_{l-1} \in \mathcal{I}$ $(l \geq 1)$. Put $Y_{l-1} := X_{l-1}$. Now Jana inspects step by step the members of $\binom{[n]}{l}$ different from X_l. Say, for the members of the family $\mathcal{F}_l$, Heidrun's answers were $X \notin \mathcal{I}$, but Y_l is the first element of $\binom{[n]}{l}$ for which Jana hears $Y_l \in \mathcal{I}$. Then Jana inspects the members of $\binom{[n]}{l+1}$ that are not in the upper shadow of $\mathcal{F}_l \cup \{X_l\}$ (they cannot be in $\mathcal{I}$ – here the ideal property is used). Let $\mathcal{F}_{l+1}$ be the family of those X for which the answer is $X \notin \mathcal{I}$ and let Y_{l+1} be the first member with the answer $Y_{l+1} \in \mathcal{I}$. Further Jana inspects members of $\binom{[n]}{l+2}$ that are not in the upper shadow of $\mathcal{F}_l \cup \{X_l\} \cup \mathcal{F}_{l+1}$ (they cannot be in $\mathcal{I}$) and so on. This procedure leads to families $\mathcal{F}_l, \mathcal{F}_{l+1}, \ldots$, and sets $Y_{l-1}, Y_l, Y_{l+1}, \ldots$. Let us say that the procedure stops in the level $\binom{[n]}{u}$, that is, either Jana found some $Y_u \in \mathcal{I}$ but the $(u+1)$-shadow of $\mathcal{F}_l \cup \{X_l\} \cup \cdots \cup \mathcal{F}_u$ is the complete level $\binom{[n]}{u+1}$ or, though Jana inspected some family $\mathcal{F}_u$, she could not find an element $Y_u \in \mathcal{I}$. Obviously, Y_u (resp. Y_{u-1}) is the desired element of $\mathcal{I}$ of maximum rank. It is easy to see that

$$\mathcal{F} := \mathcal{F}_l \cup \{X_l\} \cup \mathcal{F}_{l+1} \cup \cdots \cup \mathcal{F}_u \cup \{Y_u\}$$

is a Sperner family (X_l and Y_u may not exist). The number of Heidrun answers is equal to $|\mathcal{F}| + (m-l) + (u-l) + 1$, and this number is not greater than $\binom{n}{m} + 1$ by Corollary 2.3.3 if $n \geq 4$ (in the case that already $X_m \in \mathcal{I}$, that is, $l - 1 = m$, one may delete the summand $m - l$, and in the case n odd, $l = m, u = m + 1$ one may apply Theorem 1.1.1(b)). The cases $n \leq 3$ can be easily settled separately.

■

Note that the problem of finding an element of an ideal with maximum rank occurs, for example, if one looks for solutions of the integer programming problem

$$\begin{aligned} A\boldsymbol{x} &\leq \boldsymbol{b}, \\ \boldsymbol{x} &\in B_n, \\ \mathbf{1}^{\mathrm{T}}\boldsymbol{x} &\to \max \end{aligned}$$

where A and $\boldsymbol{b}$ contain nonnegative numbers only, B_n is realized as the set of n-dimensional 0,1-vectors, and $\mathbf{1}^{\mathrm{T}} = (1, \ldots, 1)$. Katerinochkina [288] generalized her result to chain products $S(k, \ldots, k)$, and Kuzyurin [330] found an asymptotic formula for powers P^n of a finite poset P.

Now we look for the "best" lower bound for the shadow size. Instead of solving this problem directly, we study a more general problem. Let $G = (V, E)$ be an undirected graph and $A \subseteq V$. The *boundary* $B(A)$ of A is defined to be the set of vertices not in A that are adjacent to vertices in A; that is,

$$B(A) := \{v \in V - A : \text{ there is some } w \in A \text{ with } vw \in E\}.$$

Moreover, let $E_{in}(A)$ and $E_{out}(A)$ be the set of *inner edges* (resp. of *outer edges*);

that is,

$$E_{in}(A) := \{e \in E : e \subseteq A\},$$
$$E_{out}(A) := \{e \in E : |e \cap A| = 1\}.$$

Note that

$$\sum_{v \in A} d(v) = 2|E_{in}(A)| + |E_{out}(A)|. \tag{2.30}$$

Now the following problems are of interest (applications will be discussed later): Given $G = (V, E)$ and a natural number m, find

$\min\{|E_{out}(A)| : A \subseteq V, |A| = m\}$ *edge-isoperimetric problem (EIP)*,
$\min\{|B(A)| : A \subseteq V, |A| = m\}$ *vertex-isoperimetric problem (VIP)*,
$\max\{|E_{in}(A)| : A \subseteq V, |A| = m\}$ *maximum-edge problem (MEP)*.

If the graph G is regular, then in view of (2.30) the EIP and the MEP are equivalent. In general, these problems are difficult from an algorithmic point of view (NP-complete [215]). However, there is a whole theory for the solution of these problems for several classes of graphs and we recommend the surveys (resp. papers) of Aslanyan [34], Bezrukov [57, 58], and Ahlswede and Cai [5] as well as the forthcoming book of Harper and Chavez [261]. We will restrict ourselves to one special case, namely to the Hasse graph of the Boolean lattice B_n (which is regular of degree n). Let N be a finite set of natural numbers. On 2^N we define two new linear orders: For $X, Y \subseteq N$, $X \neq Y$, let

$X \prec_{rl} Y$ iff $\max(X - Y) < \max(Y - X)$,
$X \prec_{vip} Y$ iff $|X| < |Y|$ or ($|X| = |Y|$ and $Y \prec_{rl} X$)

(here we define $\max(Z) := \max\{i : i \in Z\}$ and set $\max(\emptyset) := 0$). The (reflexive) order $\preceq_{rl}$ is called the *reverse lexicographic order* (if we write the sets as 0,1-words, the most important letter is the last one). The two orders are illustrated in Figure 2.1 for B_4. Let $C_{rl}(m, 2^N)$ (resp. $C_{vip}(m, 2^N)$) be the families of the first m subsets of N with respect to $\preceq_{rl}$ (resp. $\preceq_{vip}$) (we omit in the following the indices rl (resp.

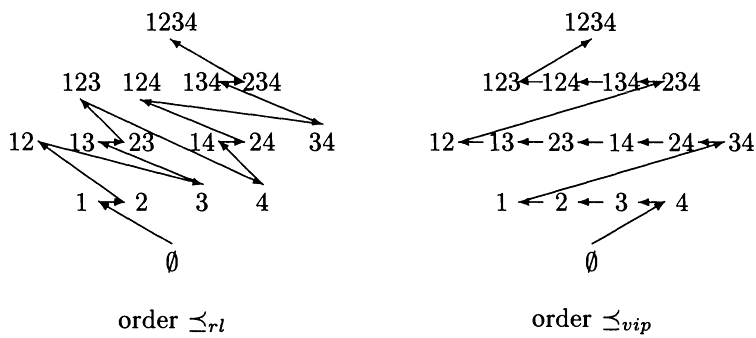


Figure 2.1

vip) if we mean both cases or if the case is clear from the context). The family $\mathcal{C}(m, 2^N)$ is called *compressed* and the operation of exchanging an m-element family $\mathcal{F}$ by $\mathcal{C}(m, 2^N)$ is called *compression*. Similarly, we define $\mathcal{C}(m, \binom{N}{k})$ and $\mathcal{L}(m, \binom{N}{k})$, where $1 \leq m \leq \binom{|N|}{k}$, to be the set of the first (resp. last) m elements of $\binom{N}{k}$, here only with respect to $\preceq_{rl}$. Families of the form $\mathcal{L}(m, \binom{N}{k})$ are called *$\mathcal{L}$-compressed*. If $\sum_{i=0}^{k-1} \binom{n}{i} < m \leq \sum_{i=0}^{k} \binom{n}{i}$, and $l := m - \sum_{i=0}^{k-1} \binom{n}{i}$, then obviously,

$$\mathcal{C}_{vip}(m, 2^N) = \mathcal{L}(l, \tbinom{N}{k}) \cup \{X \subseteq N : |X| < k\}, \tag{2.31}$$

$$B(\mathcal{C}_{vip}(m, 2^N)) = (\tbinom{N}{k} - \mathcal{L}(l, \tbinom{N}{k})) \cup \nabla(\mathcal{L}(l, \tbinom{N}{k})). \tag{2.32}$$

Lemma 2.3.3. *For any m, $1 \leq m \leq 2^{|N|}$, there exists some m' such that*

$$\mathcal{C}_{vip}(m, 2^N) \cup B(\mathcal{C}_{vip}(m, 2^N)) = \mathcal{C}_{vip}(m', 2^N).$$

Proof. In view of (2.31) and (2.32) it is enough to show that (with the notations from above) $\nabla(\mathcal{L}(l, \binom{N}{k}))$ is $\mathcal{L}$-compressed. We suppose $N = [n]$ and proceed by induction on n. Look at the step $n-1 \to n$. Let $\mathcal{F} := \mathcal{L}(l, \binom{N}{k})$ and let

$$\mathcal{F}_1 := \{X \in \mathcal{F} : n \notin X\},$$

$$\mathcal{F}_2 := \{X \in \mathcal{F} : n \in X\}, \qquad \mathcal{F}_2' := \{X - \{n\} : X \in \mathcal{F}_2\}.$$

Case 1. $\mathcal{F}_1 = \emptyset$. Then $\nabla(\mathcal{F}) = \{Y \cup \{n\} : Y \in \nabla'(\mathcal{F}_2')\}$, where ∇' means that we consider the upper shadow in $2^{[n-1]}$. Since $\mathcal{F}$ is $\mathcal{L}$-compressed, $\mathcal{F}_2'$ is also $\mathcal{L}$-compressed in $\binom{[n-1]}{k-1}$. By the induction hypothesis, $\nabla'(\mathcal{F}_2')$ is $\mathcal{L}$-compressed in $\binom{[n-1]}{k}$, and consequently also $\nabla(\mathcal{F})$ is $\mathcal{L}$-compressed in $\binom{[n]}{k+1}$.

Case 2. $\mathcal{F}_1 \neq \emptyset$. Then $\mathcal{F}$ contains all members of $\binom{[n]}{k}$ that contain n. It follows that $\nabla(\mathcal{F}) = \nabla'(\mathcal{F}_1) \cup \{X \in \binom{[n]}{k+1} : n \in X\}$. Since $\mathcal{F}$ is $\mathcal{L}$-compressed, $\mathcal{F}_1$ is also $\mathcal{L}$-compressed in $\binom{[n-1]}{k}$. The induction hypothesis implies that $\nabla'(\mathcal{F}_1)$ is $\mathcal{L}$-compressed in $\binom{[n-1]}{k+1}$ and consequently $\nabla(\mathcal{F})$ is $\mathcal{L}$-compressed in $\binom{[n]}{k+1}$. ∎

The solutions of the MEP (i.e., also of the EIP) and of the VIP are given by the following theorem, which is mainly due to Harper [254, 255] (the optimality of the ordering for the MEP was conjectured by Hales [254], and the MEP was also solved by Bernstein [51] and Hart [263]; the VIP was independently settled by Ahlswede, Gács, and Kőrner [11], Wang and Wang [459], and Gavrilov and Sapozhenko [220, Ch. I]).

Theorem 2.3.3. *Let N be a finite set of natural numbers, $V := 2^N$ and $E := \{XY : X, Y \subseteq N, |X - Y| + |Y - X| = 1\}$. Further let $1 \leq m \leq 2^{|N|}$. Then, for any m-element family $\mathcal{F}$ in 2^N,*

(*a*) $|E_{in}(\mathcal{F})| \leq |E_{in}(\mathcal{C}_{rl}(m, 2^N))|$,

(*b*) $|B(\mathcal{F})| \geq |B(\mathcal{C}_{vip}(m, 2^N))|$.

Geometrically, part (b) can be interpreted as follows: For given "discrete volume" (i.e., size), the minimum boundary is attained at an incomplete sphere.

Proof. We prove (a) and (b) simultaneously and proceed by induction on $n := |N|$. The cases $n = 1, 2$ are trivial, so we consider the step $n - 1 \to n \geq 3$. Obviously, we may suppose that $N = [n]$. For $X \subseteq [n]$, let $o(X)$ be the position of X in the linear order $\preceq$; that is, X is the $o(X)$th element in $\preceq$. Further let, for $\mathcal{F} \subseteq 2^{[n]}$,

$$o(\mathcal{F}) := \sum_{X \in \mathcal{F}} o(X).$$

Of all m-element families that have the maximum number of inner edges (resp. minimum) boundary, we choose a family $\mathcal{F}$ for which $o(\mathcal{F})$ is minimum. It is enough to show that $\mathcal{F} = \mathcal{C}(m, 2^{[n]})$. For $i \in [n]$, let

$$\mathcal{F}_1(i) := \{X \in \mathcal{F} : i \notin X\},$$

$$\mathcal{F}_2(i) := \{X \in \mathcal{F} : i \in X\}, \qquad \mathcal{F}_2'(i) := \{X - \{i\} : X \in \mathcal{F}_2(i)\}.$$

Moreover, let $m_1(i) := |\mathcal{F}_1(i)|, m_2(i) := |\mathcal{F}_2(i)|(= |\mathcal{F}_2'(i)|)$. Note that $\mathcal{F} = \mathcal{F}_1(i) \cup \mathcal{F}_2(i)$, $m = m_1(i) + m_2(i)$ for each i.

Claim. We have, for every i,

$$\mathcal{F}_1(i) = \mathcal{C}(m_1(i), 2^{[n]-\{i\}})$$

and

$$\mathcal{F}_2'(i) = \mathcal{C}(m_2(i), 2^{[n]-\{i\}}).$$

Proof of Claim. For brevity, we set

$$\mathcal{G}_1(i) := \mathcal{C}(m_1(i), 2^{[n]-\{i\}}),$$

$$\mathcal{G}_2'(i) := \mathcal{C}(m_2(i), 2^{[n]-\{i\}}), \qquad \mathcal{G}_2(i) := \{X \cup \{i\} : X \in \mathcal{G}_2'(i)\}.$$

Finally let $\mathcal{G} := \mathcal{G}_1(i) \cup \mathcal{G}_2(i)$. We will show that the objective functions for $\mathcal{G}$ are at least as good as for $\mathcal{F}$. Since obviously $o(\mathcal{G}) \leq o(\mathcal{F})$, the choice of $\mathcal{F}$ leads to the result $\mathcal{F} = \mathcal{G}$, which is our assertion. Here we study both parts of the theorem separately. The notations E_{in}' and B' mean that they are considered in $2^{[n]-\{i\}}$.

(a) We have

$$\begin{aligned}|E_{in}(\mathcal{F})| &= |E'_{in}(\mathcal{F}_1(i))| + |E'_{in}(\mathcal{F}'_2(i))| + |\mathcal{F}_1(i) \cap \mathcal{F}'_2(i)| \\ &\le |E'_{in}(\mathcal{G}_1(i))| + |E'_{in}(\mathcal{G}'_2(i))| + |\mathcal{G}_1(i) \cap \mathcal{G}'_2(i)| = |E_{in}(\mathcal{G})|\end{aligned}$$

(for the first two summands apply the induction hypothesis, and for the third summand note that $|\mathcal{F}_1(i) \cap \mathcal{F}'_2(i)| \le \min\{m_1(i), m_2(i)\} = |\mathcal{G}_1(i) \cap \mathcal{G}'_2(i)|$ since both parts of the last intersection are related by inclusion).

(b) Let, without loss of generality (w.l.o.g.), $m_1(i) \le m_2(i)$; that is, $\mathcal{G}_1(i) \subseteq \mathcal{G}'_2(i)$. Because of Lemma 2.3.3, $\mathcal{G}_1(i) \cup B'(\mathcal{G}_1(i))$ and $\mathcal{G}'_2(i)$ are related by inclusion.

Case 1. $\mathcal{G}_1(i) \cup B'(\mathcal{G}_1(i)) \subseteq \mathcal{G}'_2(i)$. Then

$$|B(\mathcal{G})| = |\mathcal{G}'_2(i)| - |\mathcal{G}_1(i)| + |B'(\mathcal{G}'_2(i))|$$

(the elements of $\mathcal{G}'_2(i)$ that are not in $\mathcal{G}_1(i)$ are in the boundary of $\mathcal{G}$, $|B'(\mathcal{G}'_2(i))|$ counts the members of the boundary of $\mathcal{G}$ that arise from $\mathcal{G}_2(i)$ and contain i). However,

$$|B(\mathcal{F})| \ge |\mathcal{F}'_2(i)| - |\mathcal{F}_1(i)| + |B'(\mathcal{F}'_2(i))|,$$

and the induction hypothesis gives $|B(\mathcal{F})| \ge |B(\mathcal{G})|$.

Case 2. $\mathcal{G}_1(i) \cup B'(\mathcal{G}_1(i)) \supseteq \mathcal{G}'_2(i)$. Then

$$|B(\mathcal{G})| = |B'(\mathcal{G}_1(i))| + |B'(\mathcal{G}'_2(i))|.$$

Clearly, however,

$$|B(\mathcal{F})| \ge |B'(\mathcal{F}_1(i))| + |B'(\mathcal{F}'_2(i))|,$$

and again the induction hypothesis yields $|B(\mathcal{F})| \ge |B(\mathcal{G})|$. □

From this claim we derive that

$$X \in \mathcal{F}, Y \preceq X, Y \ne \overline{X} \text{ imply } Y \in \mathcal{F} \tag{2.33}$$

since, if $Y \ne \overline{X}$, there is an $i \notin X \cup Y$ or a $j \in X \cap Y$ ($\mathcal{F}_1(i)$ (resp. $\mathcal{F}'_2(j)$) are compressed). Now we assume that $\mathcal{F} \ne C(m, 2^N)$. Then there must be a set $A \subseteq [n]$ with $A \notin \mathcal{F}$ for which the next set E (with respect to $\preceq$) belongs to $\mathcal{F}$. In view of (2.33), $E = \overline{A}$, and no further set X with $A \prec X$ can belong to $\mathcal{F}$. Moreover, every set X except A with $X \prec E$ belongs to $\mathcal{F}$. It is easy to see that a set A for which the next set is its complement is uniquely determined, namely

$$A = \begin{cases} \{1, \ldots, n-1\} & \text{for } \preceq_{rl}, \\ \begin{cases} \{1, \ldots, \frac{n-1}{2}\} & \text{if } n \text{ is odd,} \\ \{1, \ldots, \frac{n}{2} - 1, n\} & \text{if } n \text{ is even} \end{cases} & \text{for } \preceq_{vip}. \end{cases}$$

Without difficulties we may verify that for the family $\mathcal{G} := (\mathcal{F} - \{E\}) \cup \{A\}$ we have, respectively,

$$\begin{aligned} |E_{in}(\mathcal{G})| &= |E_{in}(\mathcal{F})| - 1 + n - 1 > |E_{in}(\mathcal{F})|, \\ |B(\mathcal{G})| &= |B(\mathcal{F})| - \left\lfloor \frac{n-1}{2} \right\rfloor < |B(\mathcal{F})|. \end{aligned}$$

This is a contradiction to the choice of $\mathcal{F}$. ∎

As we have seen, the sets

$$S_m := \mathcal{C}(m, 2^N)$$

are solutions of the EIP and MEP (resp. VIP). Obviously (we used this fact already),

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{2^n}.$$

We say that there is a *nested structure of solutions* (*NSS*) (which need not exist for other graphs). The existence of an NSS of the EIP for a graph $G = (V, E)$ enables the solution of an optimal numbering problem: Let $|V| = n$. We wish to minimize

$$a_G(l) := \sum_{vw \in E} |l(w) - l(v)|$$

over all bijective mappings $l : V \to [n]$.

Theorem 2.3.4. *Let $S_0 \subseteq \cdots \subseteq S_n$ be an NSS for the EIP of $G = (V, E)$, and let v_i be the (unique) element of $S_i - S_{i-1}$, $i = 1, \ldots, n$. Define the bijective mapping $l^* : V \to [n]$ by $l^*(v_i) := i$, $i = 1, \ldots, n$. Then*

$$a_G(l^*) \le a_G(l) \text{ for all bijective mappings } l : V \to [n].$$

Proof. Let $l : V \to [n]$ be bijective. We orient the edges of G such that $l(e^+) > l(e^-)$. This leads to a directed graph $\vec{G} = (V, \vec{E})$. Let $T_i := \{v \in V : l(v) \le i\}$. We have

$$\begin{aligned} a_G(l) &= \sum_{e \in \vec{E}} l(e^+) - l(e^-) \\ &= \sum_{e \in \vec{E}} \sum_{i : l(e^-) \le i < l(e^+)} 1 = \sum_{i=1}^{n-1} \sum_{e \in \vec{E} : l(e^-) \le i < l(e^+)} 1 \\ &= \sum_{i=1}^{n-1} |E_{out}(T_i)| \ge \sum_{i=1}^{n-1} |E_{out}(S_i)| = a_G(l^*). \end{aligned}$$

∎

Theorems 2.3.3 and 2.3.4 have the following application in coding theory: We want to encode the integers $0, \ldots, 2^n - 1$ by binary 0-1-vectors; that is, we look for a bijective mapping $2^{[n]} \longleftrightarrow \{0, \ldots, 2^n - 1\}$.

We suppose that during the transmission at most one digit may be received falsely, with the result that after the decoding a false integer is obtained. Our goal is that the difference to the right integer be small. More exactly, we want to minimize the expected value of the difference under an equidistribution assumption. It is easy to see that this expected value is $a_g(l)$ times a constant (we have here $l : 2^{[n]} \to \{0, \ldots, 2^n - 1\}$ instead of $l : 2^{[n]} \to [2^n]$, but this does not influence the solution). Theorems 2.3.3 and 2.3.4 show that, interestingly, the binary expansion of an integer is an optimal encoding (more exactly, Theorem 2.3.3 leads to the binary expansion, which must be read from the right to the left, but it does not matter to reverse all vectors). In [52] Bernstein, Hopcroft, and Steiglitz also admit other sets of integers to be encoded.

In addition to the NSS we have in the case of the VIP for the Hasse graph of the Boolean lattice in view of Lemma 2.3.3 an additional property:

$$\text{for all } i \text{ there is some } j \text{ such that } S_i \cup B(S_i) = S_j.$$

We speak of the *strong nested structure of solutions* (*SNSS*). Now we wish to minimize

$$b_G(l) := \max_{vw \in E} |l(w) - l(v)|$$

over all bijective mappings $l : V \to [n]$.

Theorem 2.3.5. *Let $S_0 \subseteq \cdots \subseteq S_n$ be an SNSS for the VIP of $G = (V, E)$ and let l^* be defined as in Theorem 2.3.4. Then*

$$b_G(l^*) \le b_G(l) \textit{ for all bijective mappings } l : V \to [n].$$

Proof. Let $l : V \to [n]$ be bijective and $T_i := \{v \in V : l(v) \le i\}, i = 1, \ldots, n$. Let $B(T_i) \ne \emptyset$. Since for all $w \in B(T_i)$, clearly $l(w) > i$, there must exist some $w \in B(T_i)$ with $l(w) \ge i + |B(T_i)|$. By definition of the boundary, w is adjacent to some $v \in T_i$. We have $l(w) - l(v) \ge i + |B(T_i)| - i = |B(T_i)|$ and thus (including also the cases $B(T_i) = \emptyset$)

$$b_G(l) \ge \max\{|B(T_i)|, i = 1, \ldots, n\} \ge \max\{|B(S_i)|, i = 1, \ldots, n\}. \qquad (2.34)$$

Furthermore, let vw be an edge with $|l^*(w) - l^*(v)| = b_G(l^*)$, where w.l.o.g., $l^*(v) < l^*(w)$. Let $i := l^*(v)$. Then $v \in S_i$, $w \in B(S_i)$. By the SNSS there is some j such that $S_i \cup B(S_i) = S_j$. Consequently, $l^*(w) \le j = i + |B(S_i)|$, which implies (noting (2.34))

$$b_G(l^*) = l^*(w) - l^*(v) \le |B(S_i)| \le \max\{|B(S_i)|, i = 1, \ldots, n\} \le b_G(l).$$

∎

With a graph and a numbering l of its vertices we may associate its adjacency matrix A_G, which is defined by

$$a_{ij} := \begin{cases} 1 & \text{if } vw \in E, \text{ where } l(v) = i, l(w) = j, \\ 0 & \text{otherwise.} \end{cases}$$

Now the (horizontal and vertical) distance of the entry a_{ij} to the main diagonal is $|i - j|$. The *bandwidth problem*, which is related to the efficiency of numerical algorithms, consists of finding such a numbering for which the maximum distance of a nonzero entry to the main diagonal is minimal. Thus Theorems 2.3.3 and 2.3.5 yield a solution of the bandwidth problem in the case of the Hasse graph of the Boolean lattice.

Theorems 2.3.4 and 2.3.5 are also due to Harper [254, 255], whose main goal was the solution of the optimal numbering problems that led to Theorem 2.3.3. As noticed by Björner [64] (see also [253, 336]), the following classical result was formulated (also in its numerical version, see Corollary 2.3.4) the first time by Schützenberger [422] (he used $(i, i+1)$-shiftings, but this need not result in a compressed family). Independently Kruskal [325], Harper (though not explicitly mentioned in [255]), Katona [292], and Clements and Lindström [117] (see Theorem 8.1.1) found the following theorem.

***Theorem 2.3.6* (*Kruskal–Katona Theorem*).** *Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $|\mathcal{F}| = m$. Then*

(a) $|\nabla(\mathcal{F})| \geq |\nabla(\mathcal{L}(m, \binom{[n]}{k}))|$,

(b) $|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{C}(m, \binom{[n]}{k}))|$.

Proof. (a) Let

$$\mathcal{F}_1 := \mathcal{F} \cup \{X : |X| < k\},$$

$$\mathcal{F}_2 := \mathcal{L}(m, \tbinom{[n]}{k}) \cup \{X : |X| < k\}.$$

Then, by (2.32) and Theorem 2.3.3(b),

$$|\nabla(\mathcal{F})| = |B(\mathcal{F}_1)| + m - \binom{n}{k} \geq |B(\mathcal{F}_2)| + m - \binom{n}{k} = |\nabla(\mathcal{L}(m, \tbinom{[n]}{k}))|.$$

(b) Note that $X \preceq_{rl} Y$ iff $\overline{Y} \preceq_{rl} \overline{X}$. Thus, by (a),

$$|\Delta(\mathcal{F})| = |\nabla(\overline{\mathcal{F}})| \geq |\nabla(\mathcal{L}(m, \tbinom{[n]}{n-k}))| = |\Delta(\mathcal{C}(m, \tbinom{[n]}{k}))|.$$

∎

Other proofs of this theorem were given by Hansel [253], Daykin [121], Eckhoff and Wegner [141], Hilton [267], and Frankl [191]. Mörs [374] and Füredi and

Griggs [209] determined the values of $m = |\mathcal{F}|$ such that $C(m, \binom{[n]}{k})$ is up to permutations of the elements the only family with minimum lower shadow.

We may calculate also the minimum size of the lower shadow of an m-element k-uniform family numerically. If $m = \binom{n}{k}$, then $\mathcal{F} = \binom{[n]}{k}$ and $|\Delta(\mathcal{F})| = \binom{n}{k-1}$. So let $1 \leq m < \binom{n}{k}$. Let $X^* = \{i_1, \ldots, i_k\}$, where $1 \leq i_1 < \cdots < i_k \leq n$, be the $(m+1)$th element in $\binom{[n]}{k}$ with respect to $\preceq_{rl}$. Further, let t be the smallest integer for which $i_t - 1 \geq t$. Obviously,

$$
\begin{aligned}
C\left(m, \binom{[n]}{k}\right) = &\left\{X \subseteq \binom{[n]}{k} : \max(X) < i_k\right\} \\
&\cup \left\{X \cup \{i_k\} : X \subseteq \binom{[n]}{k-1}, \quad \max(X) < i_{k-1}\right\} \\
&\cup \cdots \cup \left\{X \cup \{i_{t+1}, \ldots, i_k\} : X \subseteq \binom{[n]}{t}, \right. \\
&\left. \max(X) < i_t \right\},
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
\Delta\left(C\left(m, \binom{[n]}{k}\right)\right) = &\left\{X \subseteq \binom{[n]}{k-1} : \max(X) < i_k\right\} \\
&\cup \left\{X \cup \{i_k\} : X \subseteq \binom{[n]}{k-2}, \quad \max(X) < i_{k-1}\right\} \\
&\cup \cdots \cup \left\{X \cup \{i_{t+1}, \ldots, i_k\} : X \subseteq \binom{[n]}{t-1}, \right. \\
&\left. \max(X) < i_t \right\}.
\end{aligned}
\tag{2.36}
$$

Since on the RHS there are disjoint unions, we have

$$
m = \left|C\left(m, \binom{[n]}{k}\right)\right| = \binom{i_k - 1}{k} + \binom{i_{k-1} - 1}{k-1} + \cdots + \binom{i_t - 1}{t},
$$

$$
\left|\Delta\left(C\left(m, \binom{[n]}{k}\right)\right)\right| = \binom{i_k - 1}{k-1} + \binom{i_{k-1} - 1}{k-2} + \cdots + \binom{i_t - 1}{t-1}.
$$

Thus, we found a representation of m, called *k-representation of m*, in the form

$$
m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t} \quad \text{where } n \geq a_k > \cdots > a_t \geq t \geq 1 \tag{2.37}
$$

(in fact we have for $m < \binom{n}{k}$ the relation $n - 1 \geq a_k$, but we may include also $m = \binom{n}{k}$). The minimum size of the shadow of an m-element family in $\binom{[n]}{k}$ is then

$$
m' := \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_t}{t-1}. \tag{2.38}
$$

Lemma 2.3.4. *The k-representation (2.37) of an integer m with $1 \le m \le \binom{n}{k}$ is unique.*

Proof. We use the following identity, which can be easily proved by induction: For natural numbers u, v,

$$\binom{u}{0} + \binom{u+1}{1} + \cdots + \binom{u+v}{v} = \binom{u+v+1}{v}. \tag{2.39}$$

Assume that there exists a second representation

$$m = \binom{b_k}{k} + \cdots + \binom{b_s}{s} \quad \text{where } n \ge b_k > \cdots > b_s \ge s \ge 1.$$

It is easy to see that we may suppose, w.l.o.g., that there exists an integer l such that $a_k = b_k, \ldots, a_{l+1} = b_{l+1}, a_l > b_l$. Then

$$\begin{aligned} m - m &= \binom{a_l}{l} + \cdots + \binom{a_t}{t} - \left(\binom{b_l}{l} + \cdots + \binom{b_s}{s}\right) \\ &\ge \binom{a_l}{l} - \binom{b_l}{l} - \cdots - \binom{b_s}{s} \\ &> \binom{a_l}{l} - \binom{a_l - 1}{l} - \binom{a_l - 2}{l-1} - \cdots - \binom{a_l - l - 1}{0} = 0, \end{aligned}$$

a contradiction. ■

By the interpretation from which we derived the k-representation of m, it is obvious that a_k is the largest number for which $\binom{a_k}{k} \le m$, a_{k-1} is the largest number for which $\binom{a_{k-1}}{k-1} \le m - \binom{a_k}{k}$, and so forth.

If $t > 1$, then (2.38) is the $(k-1)$-representation of the minimum size of the shadow, and we may compute the minimum-size m'' of the shadow of the minimum-sized shadow:

$$m'' := \binom{a_k}{k-2} + \cdots + \binom{a_t}{t-2}.$$

However, if $t = 1$, we first must find the $(k-1)$-representation of m'. Let l be the smallest integer for which $a_{l+1} > a_l + 1$ (with $a_{k+1} := \infty$). Then, in view of the identity (2.39),

$$m' = \binom{a_k}{k-1} + \cdots + \binom{a_l + 1}{l},$$

which implies (again using (2.39))

$$\begin{aligned} m'' &= \binom{a_k}{k-2} + \cdots + \binom{a_l + 1}{l-1} \\ &= \binom{a_k}{k-2} + \cdots + \binom{a_l}{l-1} + \cdots + \binom{a_t}{t-2} \end{aligned}$$

(with $\binom{u}{v} := 0$ if $v < 0$). Iterated application of the preceding arguments leads to the following numerical version of the Kruskal–Katona Theorem:

Corollary 2.3.4. *Let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$, and let $|\mathcal{F}| = \binom{a_k}{k} + \cdots + \binom{a_t}{t}$ with $n \geq a_k > \cdots > a_t \geq t \geq 1$ be the k-representation of the size of $\mathcal{F}$. Then, for $0 \leq i \leq k$,*

$$|\Delta_{\to i}(\mathcal{F})| \geq \binom{a_k}{i} + \cdots + \binom{a_t}{t+i-k},$$

and the bound is the best possible.

Looking back at (2.35) we see that we may stop the classification of $C(m, \binom{[n]}{k})$ already at that stage where the containment of $i_{l+1}, \ldots, i_k$ is fixed ($l > t$). This leads to

$$m = \binom{i_k - 1}{k} + \cdots + \binom{i_{l+1} - 1}{l+1} + \binom{x}{l} \quad \text{where } i_l - 1 \leq x < i_l.$$

Now the same arguments as above combined with Theorem 2.3.1 (resp. Corollary 2.3.1) yield:

Corollary 2.3.5. *Let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$, and let $|\mathcal{F}| = \binom{a_k}{k} + \cdots + \binom{a_{l+1}}{l+1} + \binom{x}{l}$ where $n \geq a_k > \cdots > a_{l+1} > x \geq l$. Then, for $0 \leq i \leq k$,*

$$|\Delta_{\to i}(\mathcal{F})| \geq \binom{a_k}{i} + \cdots + \binom{a_{l+1}}{l+1+i-k} + \binom{x}{l+i-k}.$$

We encourage the reader to prove this corollary (for $i = k - 1$) directly by the proof method for Theorem 2.3.1. Katona [292] found the Kruskal–Katona Theorem when solving a problem of Erdős:

Theorem 2.3.7. *Let $\mathcal{F} = \{X_1, \ldots, X_m\}$ be a k-uniform family in $2^{[n]}$, $0 \leq i \leq k$, and let*

$$m^* := \binom{k+i}{k} + \binom{k+i-2}{k-1} + \binom{k+i-4}{k-2} + \cdots + \binom{k-i}{k-i}.$$

If $0 < m \leq m^*$ then there exist distinct sets $Y_1, \ldots, Y_m$ in $\binom{[n]}{k}$ such that $Y_j \subseteq X_j$ for all j. The bound is the best possible.

Proof. Because of Hall's theorem (Theorem 5.1.2) we only must show that

$$|\Delta_{\to i}(\mathcal{F}')| \geq |\mathcal{F}'| \text{ for each subfamily } \mathcal{F}' \text{ of } \mathcal{F}.$$

Since $|\mathcal{F}'| \leq |\mathcal{F}| \leq m^*$, there exist an integer $l \in \{k - i, \ldots, k\}$ and a real x with $l \leq x \leq 2l + i - k$ such that

$$|\mathcal{F}'| = \binom{k+i}{k} + \binom{k+i-2}{k-1} + \cdots + \binom{2l+2+i-k}{l+1} + \binom{x}{l}.$$

By Corollary 2.3.5,

$$|\Delta_{\to i}(\mathcal{F}')| \geq \binom{k+i}{i} + \binom{k+i-2}{i-1} + \cdots + \binom{2l+2+i-k}{l+1+i-k} + \binom{x}{l+i-k}.$$

Note that $\binom{x}{u} \geq \binom{x}{v}$ for $0 \leq u \leq v \leq x \leq u+v$, which can be checked in a straightforward manner. Hence

$$|\Delta_{\to i}(\mathcal{F}')| - |\mathcal{F}'| \geq \binom{x}{l+i-k} - \binom{x}{l} \geq 0.$$

The bound is the best possible since the i-shadow of $C(m^*+1, \binom{[n]}{k})$ is obviously smaller than m^*+1. ■

Leck [333] studied the problem of minimizing the size of the i-shadow of $\mathcal{F} = \{X_1, \dots, X_m\} \subseteq \binom{[n]}{k}$ for $m > m^*$ under the supposition that there are sets $Y_1, \dots, Y_m \in \binom{[n]}{i}$ such that $Y_j \subseteq X_j$ for all j.

2.4. Generating families

In the last decades several methods for proving theorems on families satisfying a certain intersection condition have been worked out. We have seen some of them in the previous sections. We will not discuss those methods that yield the maximum size of the families under consideration if n is sufficiently large. We refer here only to Frankl [189] and Schmerl [416] who built up a theory that is based on a theorem of Erdős–Rado [171]. The powerful method that is described in this section is very new. It was developed by Ahlswede and Khachatrian [15]. Though it is much more far-reaching, we will restrict ourselves to the highlight in extremal set theory: the complete determination of the maximum size of k-uniform t-intersecting families by Ahlswede and Khachatrian [15].

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. A family $\mathcal{G} \subseteq 2^{[n]}$ is called a *generating family for* $\mathcal{F}$ if $\nabla_{\to k}(\mathcal{G}) = \mathcal{F}$. Let $\mathfrak{G}(\mathcal{F})$ be the class of all generating families for $\mathcal{F}$. Note that $\mathcal{F} \in \mathfrak{G}(\mathcal{F})$. We will study generating families for special families. A family $\mathcal{F}$ is called *left* (resp. *right*) *shifted* if

$$s_{ij}(\mathcal{F}) = \mathcal{F} \text{ for all } 1 \leq i < j \leq n \text{ (resp. for all } 1 \leq j < i \leq n).$$

We may also define these families in an equivalent way as follows: On $2^{[n]}$ we consider the relation $\preceq_s$ which is the reflexive and transitive closure of the relation $\bigcup_{1 \leq i < j \leq n} \{(X, Y) : X = \lhd_{ij}(Y), i \notin Y, j \in Y\}$. Clearly, $(2^{[n]}, \preceq_s)$ is a poset, and the following proposition is immediate:

Proposition 2.4.1. *A family $\mathcal{F}$ is left (resp. right) shifted iff $\mathcal{F}$ is an ideal (resp. filter) in $(2^{[n]}, \preceq_s)$.*

In Section 2.3 we defined for $\mathcal{F} \subseteq 2^{[n]}$ the number $\Sigma(\mathcal{F}) := \sum_{X\in\mathcal{F}} \sum_{i\in X} i$. In addition, we introduce for $X \in 2^{[n]}$, $\mathcal{F} \subseteq 2^{[n]}$ the number

$$\max(\mathcal{F}) := \max_{X\in\mathcal{F}} \max(X) = \max_{X\in\mathcal{F}} \max_{i\in X} i.$$

Moreover, for $\mathcal{F} \subseteq \binom{[n]}{k}$, let

$$ak(\mathcal{F}) := \min_{\mathcal{G}\in\mathfrak{G}(\mathcal{F})} \max(\mathcal{G}).$$

Thus, $ak(\mathcal{F})$ denotes the smallest number i such that there is a family $\mathcal{G} \subseteq 2^{[i]}$ which is generating for $\mathcal{F}$.

Proposition 2.4.2. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be left shifted. Then there is a left-shifted family $\mathcal{G}$ that is generating for $\mathcal{F}$ such that* $\max(\mathcal{G}) = ak(\mathcal{F})$.

Proof. Under all families $\mathcal{G} \in \mathfrak{G}(\mathcal{F})$ with $\max(\mathcal{G}) = ak(\mathcal{F})$, take one of maximum size. This family $\mathcal{G}$ is left shifted. Assume the contrary. Then there are $X \notin \mathcal{G}$, $Y \in \mathcal{G}$ such that for some $1 \le i < j \le n$, $X = \lhd_{ij}(Y)$ where $i \notin Y$, $j \in Y$. If we can show that $\nabla_{\to k}(X) \subseteq \mathcal{F}$, we obtain the desired contradiction, since then $\mathcal{G} \cup \{X\}$ is also generating for $\mathcal{F}$ (recall that $\mathcal{G}$ is of maximum size). So let $X \subseteq Z \in \binom{[n]}{k}$. If $j \in Z$ then $Y \subseteq Z$, which implies $Z \in \mathcal{F}$ since $Y \in \mathcal{G}$; that is, $\nabla_{\to k}(Y) \subseteq \mathcal{F}$. If $j \notin Z$, let $W := \lhd_{ji}(Z)$. It is easy to see that $Y \subseteq W \in \mathcal{F}$. Because $\mathcal{F}$ is left shifted, it follows that $Z = \lhd_{ij}(W) \in \mathcal{F}$. ■

Since with each family $\mathcal{G}$, also the (Sperner) family $\mathcal{G}^*$ of minimal elements of $\mathcal{G}$ is generating for $\mathcal{F}$, we derive from Proposition 2.4.2:

Proposition 2.4.3. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be left shifted. Then there exists a Sperner family $\mathcal{G}^*$ that has the following properties:*

(*a*) *$\mathcal{G}^*$ is generating for $\mathcal{F}$,*

(*b*) $\max(\mathcal{G}^*) = ak(\mathcal{F})$,

(*c*) *if $Y \preceq_s X$ and $X \in \mathcal{G}^*$ then there exists some $Z \in \mathcal{G}^*$ such that $Z \subseteq Y$.*

We call a family $\mathcal{G}^*$ that has the properties described in Proposition 2.4.3 an *Ahlswede–Khachatrian family* (briefly *AK-family*) *for* $\mathcal{F}$. For $X \in \mathcal{G}^*$, let

$$\nabla'_{\to k}(X) := \{Y \in \binom{[n]}{k} : X \subseteq Y \quad \text{and} \quad Y - X \subseteq \{\max(X)+1, \ldots, n\}\},$$

and, as usual, we define $\nabla'_{\to k}(\mathcal{G}) := \bigcup_{X\in\mathcal{G}} \nabla'_{\to k}(X)$.

Proposition 2.4.4. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be left shifted and $\mathcal{G}^*$ an AK-family for $\mathcal{F}$. Then the sets $\nabla'_{\to k}(X)$, where $X \in \mathcal{G}^*$, partition the family $\mathcal{F}$.*

Proof. Since $\mathcal{G}^*$ is generating, $\nabla'_{\to k}(X) \subseteq \mathcal{F}$ for all $X \in \mathcal{G}^*$. First we show that all members of $\mathcal{F}$ are covered. Let $Z \in \mathcal{F}$. Since $\mathcal{G}^*$ is generating, there is some $X \in \mathcal{G}^*$ with $Z \in \nabla_{\to k}(X)$. Under all such sets X we take one for which $|X|$ is minimal and, in second instance, $|(Z - X) \cap \{1, \ldots, \max(X)\}|$ is minimal. It is enough to show that this intersection is empty. Assume there is some $i \in (Z - X) \cap \{1, \ldots, \max(X)\}$. Note that $i < \max(X)$. Let $Y := \lhd_{i,\max(X)}(X)$. Obviously, $Z \in \nabla_{\to k}(Y)$ and $|(Z - Y) \cap \{1, \ldots, \max(Y)\}|$ is smaller than the previous value for X. Moreover, $Y \in \mathcal{G}^*$ by Proposition 2.4.3(c) and the minimality of $|X|$. Thus we obtain a contradiction to the choice of X. Now we show that we have indeed a partition. Assume that there are different $X, Y \in \mathcal{G}^*$ and some $Z \in \binom{[n]}{k}$ such that $Z \in \nabla'_{\to k}(X) \cap \nabla'_{\to k}(Y)$. If $\max(X) = \max(Y)$, then clearly $Z - X = Z - Y$, which implies $X = Y$, a contradiction. If, for example, $\max(X) < \max(Y)$, then it is easy to see that $X \subseteq Y$, which is again a contradiction since $\mathcal{G}^*$ is a Sperner family. ■

Proposition 2.4.5. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be left shifted and $\mathcal{G}^*$ an AK-family for $\mathcal{F}$. Moreover, let $\mathcal{H}_i := \{X \in \mathcal{G}^* : |X| = i$ and $\max(X) = ak(\mathcal{F})(= \max(\mathcal{G}^*))\}$. Let $O_{\mathcal{H}_i}$ be the family of those members from $\mathcal{F}$ that are generated only by $\mathcal{H}_i$; that is,*

$$O_{\mathcal{H}_i} := \nabla_{\to k}(\mathcal{H}_i) - \nabla_{\to k}(\mathcal{G}^* - \mathcal{H}_i).$$

Then

$$O_{\mathcal{H}_i} = \nabla'_{\to k}(\mathcal{H}_i) \text{ and } |O_{\mathcal{H}_i}| = |\mathcal{H}_i|\binom{n - ak(\mathcal{F})}{k - i}.$$

Proof. The second equality is an immediate consequence of the first one, which we will prove now. Let $Z \in \nabla'_{\to k}(\mathcal{H}_i)$, that is, $Z \in \nabla'_{\to k}(X)$ for some $X \in \mathcal{H}_i$. Assume that $Z \in \nabla_{\to k}(Y)$ for some $Y \in \mathcal{G}^* - \mathcal{H}_i$. Then $Z - Y \supseteq Z - X$ since $\max(Y) \leq \max(X) = \max(\mathcal{G}^*)$. Consequently, $Y \subseteq X$, and since $\mathcal{G}^*$ is a Sperner family, $Y = X$, a contradiction. Thus $Z \in O_{\mathcal{H}_i}$. Let, conversely, $Z \in O_{\mathcal{H}_i}$. Assume that $Z \notin \nabla'_{\to k}(\mathcal{H}_i)$. Then by Proposition 2.4.4, there is some $Y \in \mathcal{G}^* - \mathcal{H}_i$ such that $Z \in \nabla'_{\to k}(Y)$, a contradiction. ■

Now we specialize the investigations to maximum k-uniform t-intersecting families. We may suppose throughout that $n > 2k - t$ since for $n \leq 2k - t$ every k-uniform family is automatically t-intersecting. As a candidate for a maximum family we have

$$\mathcal{S}_0 := \{X \in \binom{[n]}{k} : [t] \subseteq X\}.$$

Erdős, Ko, and Rado [170] have already shown that $\mathcal{S}_0$ is indeed a solution if n is sufficiently large. But there are other candidates, namely the families

$$\mathcal{S}_r := \{X \in \binom{[n]}{k} : |X \cap [t+2r]| \geq t+r\}, \quad r = 0, \ldots, k-t$$

which are easily seen to be t-intersecting. Frankl [188] conjectured that for any n the "best" of these families $\mathcal{S}_r$ is the solution. So let us first determine the best family $\mathcal{S}_r$.

Lemma 2.4.1. *We have*

(*a*) $|\mathcal{S}_r| = \sum_{i=0}^{r} \binom{t+2r}{t+r+i}\binom{n-t-2r}{k-t-r-i}$,

(*b*) $|\mathcal{S}_r| <$ *(resp.* $=$*)* $|\mathcal{S}_{r+1}|$ *iff* $n <$ *(resp.* $=$*)* $(k-t+1)\left(2+\frac{t-1}{r+1}\right)$.

Proof. (a) $\mathcal{S}_r$ is the disjoint union of the sets $\{X \in \binom{[n]}{k} : |X \cap [t+2r]| = t+r+i\}, i = 0, \ldots, r$.

(b) It is easy to see that

$$\begin{aligned} \mathcal{S}_{r+1} - \mathcal{S}_r &= \{X \in \binom{[n]}{k} : |X \cap [t+2r]| = t+r-1, \\ &\quad \{t+2r+1, t+2r+2\} \subseteq X\}, \\ \mathcal{S}_r - \mathcal{S}_{r+1} &= \{X \in \binom{[n]}{k} : |X \cap [t+2r]| = t+r, \\ &\quad \{t+2r+1, t+2r+2\} \cap X = \emptyset\}. \end{aligned}$$

Thus $|\mathcal{S}_r| <$ (resp. $=$) $|\mathcal{S}_{r+1}|$ is equivalent to the following relations:

$$\frac{|\mathcal{S}_r - \mathcal{S}_{r+1}|}{|\mathcal{S}_{r+1} - \mathcal{S}_r|} < \text{(resp. =)}\ 1,$$

$$\frac{\binom{t+2r}{t+r}\binom{n-t-2r-2}{k-t-r}}{\binom{t+2r}{t+r-1}\binom{n-t-2r-2}{k-t-r-1}} < \text{(resp. =)}\ 1,$$

$$n < \text{(resp. =)}\ (k-t+1)\left(2+\frac{t-1}{r+1}\right).$$

■

In particular it follows for $n \geq (k-t+1)(t+1)$ that $|\mathcal{S}_0| \geq |\mathcal{S}_1| > |\mathcal{S}_2| > \ldots$. Frankl [188] (for $t \geq 15$) and Wilson [470] (for any t) proved that $\mathcal{S}_0$ is indeed a maximum k-uniform t-intersecting family if $n \geq (k-t+1)(t+1)$. We will present Wilson's algebraic approach in Section 6.4. Some further special cases of Frankl's conjecture were settled by Frankl [188] and Frankl and Füredi [196]. But the complete solution is due to Ahlswede and Khachatrian [15]:

Theorem 2.4.1 (Complete Intersection Theorem). *Let* $1 \leq t \leq k \leq n$, $n > 2k - t$. *Let* $r \in \{0, \dots, k-t\}$ *be that number for which*

$$(k-t+1)\left(2 + \frac{t-1}{r+1}\right) \leq n < (k-t+1)\left(2 + \frac{t-1}{r}\right) \tag{2.40}$$

(with the definition $\infty := \frac{i}{0}$ *for all* $i \in \mathbb{N}$*). Then* $\mathcal{S}_r$ *is a maximum* k*-uniform* t*-intersecting family.*

Before we prove this theorem we need some further preparations.

Lemma 2.4.2.

(*a*) *Let* $i, j \in [n]$, $i \neq j$, *and let* $\mathcal{F}$ *be* k*-uniform* t*-intersecting. Then* $s_{ij}(\mathcal{F})$ *is* k*-uniform* t*-intersecting, too.*

(*b*) *There exists a maximum* k*-uniform* t*-intersecting family that is left shifted.*

Proof. (a) Generalize in an obvious way the proof of Lemma 2.3.1.

(b) Look at maximum families for which $\Sigma(\mathcal{F})$ is minimum. ■

Lemma 2.4.3. *Let* $\mathcal{F}$ *be a left-shifted* k*-uniform* t*-intersecting family and let* $\mathcal{G}^*$ *be an AK-family for* $\mathcal{F}$. *Let* $n > 2k - t$. *Then for all* $X_1, X_2 \in \mathcal{G}^*$,

(*a*) $|X_1 \cap X_2| \geq t$,

(*b*) $|X_1 \cap X_2| \geq t+1$ *if there are* $i, j \in [n]$ *with* $i < j$, $i \notin X_1 \cup X_2$, $j \in X_1 \cap X_2$.

Proof. (a) Assume that there are $X_1, X_2 \in \mathcal{G}^*$ with $|X_1 \cap X_2| \leq t - 1$. For the set $A := [n] - (X_1 \cup X_2)$, we have

$$\begin{aligned} |A| &= n - |X_1 \cup X_2| \geq 2k - (t-1) - |X_1 \cup X_2| \\ &= k - |X_1| + k - |X_2| + |X_1 \cap X_2| - (t-1). \end{aligned}$$

Thus we find in A two subsets Y_1, Y_2 with $|Y_l| = k - |X_l|$, $l = 1, 2$, and $|Y_1 \cap Y_2| \leq (t-1) - |X_1 \cap X_2|$. Since $\mathcal{G}^*$ is generating for $\mathcal{F}$, for $l = 1, 2$, $Z_l := X_l \cup Y_l$ belongs to $\mathcal{F}$. Obviously, $|Z_1 \cap Z_2| \leq t - 1$, a contradiction.

(b) Assume that there are $X_1, X_2 \in \mathcal{G}^*$, $i, j \in [n]$ with $|X_1 \cap X_2| = t$, $i < j$, $i \notin X_1 \cup X_2$ $j \in X_1 \cap X_2$. Let $Y_1 := \triangleleft_{ij}(X_1)$. Then $|Y_1 \cap X_2| = t - 1$, and by property (c) of Proposition 2.4.3, there is some $Z_1 \in \mathcal{G}^*$ with $Z_1 \subseteq Y_1$ implying $|Z_1 \cap X_2| \leq t - 1$. This contradicts (a). ■

Now we formulate the main lemma:

Lemma 2.4.4. *Let $\mathcal{F}$ be a left-shifted maximum k-uniform t-intersecting family and let $n > 2k - t$, $t \geq 1$. If for some $r \in \{0, \ldots, k - t\}$*

$$n > \frac{(k-t+1)(t+2r+1)}{r+1} \left(= (k-t+1)\left(2 + \frac{t-1}{r+1}\right)\right), \tag{2.41}$$

then

$$ak(\mathcal{F}) \leq t + 2r. \tag{2.42}$$

Proof. If $n = 2k - t + 1$, then by (2.41), $r \geq k - t + 1$, and hence (2.42) trivially holds. Thus let $n \geq 2k - t + 2$. Let $\mathcal{G}$ (we omit the asterisk) be an AK-family for $\mathcal{F}$ (see Proposition 2.4.3). Assume that

$$\max(\mathcal{G}) = t + 2r + \delta \text{ for some } \delta > 0. \tag{2.43}$$

Let

$$\mathcal{G}_1 := \{X \in \mathcal{G} : t + 2r + \delta \notin X\},$$

$$\mathcal{G}_2 := \{X \in \mathcal{G} : t + 2r + \delta \in X\}, \qquad \mathcal{G}_2' := \{X - \{t + 2r + \delta\} : X \in \mathcal{G}_2\}.$$

Since $\mathcal{G}$ is a Sperner family, $\mathcal{G}_1$ and $\mathcal{G}_2'$ are disjoint.

Claim 1. If $X_1, X_2 \in \mathcal{G}_1 \cup \mathcal{G}_2'$ and $|X_1 \cap X_2| < t$, then $X_1, X_2 \in \mathcal{G}_2'$, $|X_1 \cap X_2| = t - 1$, $|X_1| + |X_2| = 2t + 2r + \delta - 2$ and $|X_l| \geq k - (n - t - 2r - \delta)$, $l = 1, 2$.

Proof of Claim 1. By Lemma 2.4.3(a), $|X_1 \cap X_2| < t$ can hold only if $X_1, X_2 \in \mathcal{G}_2'$ and $(X_1 \cup \{t + 2r + \delta\}) \cap (X_2 \cup \{t + 2r + \delta\}) = t$, that is, $|X_1 \cap X_2| = t - 1$. But Lemma 2.4.3(b) implies that there is no $i < t + 2r + \delta$ with $i \notin X_1 \cup X_2$. Thus

$$\begin{aligned} |X_1| + |X_2| &= |X_1 \cap X_2| + |X_1 \cup X_2| \\ &= t - 1 + t + 2r + \delta - 1 = 2t + 2r + \delta - 2. \end{aligned}$$

If, for example, $|X_1| \leq k - (n - t - 2r - \delta) - 1$, then $|X_2| \geq n - k + t - 1 \geq k$ (recall $n > 2k - t$). This is a contradiction since obviously $|X_2| \leq k - 1$. □

Now we classify $\mathcal{G}_2$ and $\mathcal{G}_2'$ with respect to the size of the elements (and use a new letter):

$$\mathcal{H}_i := \{X \in \mathcal{G}_2 : |X| = i\}, \qquad \mathcal{H}_i' := \{X \in \mathcal{G}_2' : |X| = i - 1\}$$

(thus we make an exception from our general rules: The members of $\mathcal{H}_i'$ have size $i - 1$, but note that they can be obtained from the members of $\mathcal{H}_i$ by deleting the element $t + 2r + \delta$). Clearly we may restrict i to $t \leq i \leq t + 2r + \delta$.

If $\mathcal{H}_t \neq \emptyset$, then $|\mathcal{G}| = 1$ since $\mathcal{G}$ is a Sperner family that is by Lemma 2.4.3(a) t-intersecting. From Proposition 2.4.3(c) it follows that $\mathcal{G} = \{[t]\}$, in contradiction to (2.43).

If $\mathcal{H}_{t+2r+\delta} \neq \emptyset$, then $\mathcal{G}_2 = \{[t+2r+\delta]\} = \mathcal{G}$ because $\mathcal{G}$ is a Sperner family and $\max(\mathcal{G}) = t + 2r + \delta$. Since $\mathcal{F}$ has maximum size, necessarily $\delta = 0$, in contradiction to (2.43).

Thus we may suppose that

$$\mathcal{G}_2 = \bigcup_{t<i<t+2r+\delta} \mathcal{H}_i \quad \text{and} \quad \mathcal{G}_2' = \bigcup_{t<i<t+2r+\delta} \mathcal{H}_i'.$$

If $\mathcal{H}_i = \emptyset$ for all $i > k-(n-t-2r-\delta)$, then by Claim 1, $\mathcal{G}_1 \cup \mathcal{G}_2'$ is t-intersecting, and thus also $\mathcal{F}' := \nabla_{\to k}(\mathcal{G}_1 \cup \mathcal{G}_2')$ is t-intersecting. Obviously, $\mathcal{F} \subseteq \mathcal{F}'$. By the maximality of $\mathcal{F}$ we have $\mathcal{F} = \mathcal{F}'$. But $\mathcal{G}_1 \cup \mathcal{G}_2'$ is generating for $\mathcal{F} = \mathcal{F}'$, and $\max(\mathcal{G}_1 \cup \mathcal{G}_2') < \max(\mathcal{G}) = ak(\mathcal{F})$, a contradiction to the definition of $ak(\mathcal{F})$. If $\mathcal{H}_i = \emptyset$ for all i, then $\mathcal{G}_2 = \emptyset$; that is, $\max(\mathcal{G}) < t + 2r + \delta$, a contradiction to (2.43). Thus we may suppose that there is some i, such that

$$k - (n - t - 2r - \delta) < i < t + 2r + \delta \quad \text{and} \quad \mathcal{H}_i \neq \emptyset. \tag{2.44}$$

Case 1. $i \neq \frac{2t+2r+\delta}{2}$. Let

$$\mathcal{G}' := \mathcal{G}_1 \cup (\mathcal{G}_2 - (\mathcal{H}_i \cup \mathcal{H}_{2t+2r+\delta-i})) \cup \mathcal{H}_i',$$

$$\mathcal{G}'' := \mathcal{G}_1 \cup (\mathcal{G}_2 - (\mathcal{H}_i \cup \mathcal{H}_{2t+2r+\delta-i})) \cup \mathcal{H}_{2t+2r+\delta-i}'.$$

In view of Claim 1, $\mathcal{G}'$ and $\mathcal{G}''$ are both t-intersecting. Thus also

$$\mathcal{F}' := \nabla_{\to k}(\mathcal{G}') \quad \text{and} \quad \mathcal{F}'' := \nabla_{\to k}(\mathcal{G}'')$$

are t-intersecting. The next claim yields the desired contradiction to the maximality of $|\mathcal{F}|$.

Claim 2. We have $\max\{|\mathcal{F}'|, |\mathcal{F}''|\} > |\mathcal{F}|$.

Proof of Claim 2. The family $\mathcal{F} - \mathcal{F}'$ contains exactly those sets that are extensions only of the members of $\mathcal{H}_{2t+2r+\delta-i}$. By Proposition 2.4.5,

$$|\mathcal{F} - \mathcal{F}'| = |\mathcal{H}_{2t+2r+\delta-i}| \binom{n-t-2r-\delta}{k-2t-2r-\delta+i}. \tag{2.45}$$

Next we estimate $|\mathcal{F}' - \mathcal{F}|$. Let $X \in \mathcal{H}_i'$. We have $X \cup \{t+2r+\delta\} \in \mathcal{G}$, hence $X \notin \mathcal{G}$. Consequently, for all $Y \subseteq \{t+2r+\delta+1, \ldots, n\}$ with $|Y| = k-i+1$ we have $X \cup Y \in \mathcal{F}' - \mathcal{F}$ (if we had $X \cup Y \in \mathcal{F}$, then in view of $\max(\mathcal{G}) = t+2r+\delta$, it could be generated only by a set $Z \in \mathcal{G}$ with $Z \subseteq X$, but this is impossible since $\mathcal{G}$ is a Sperner family). Moreover, we have obviously

$$X \cup Y \neq X' \cup Y' \text{ for all } X, X' \in \mathcal{H}_i', Y, Y' \in \{t+2r+\delta+1, \ldots, n\}$$
$$\text{with } (X, Y) \neq (X', Y').$$

Therefore

$$|\mathcal{F}' - \mathcal{F}| \geq |\mathcal{H}_i| \binom{n-t-2r-\delta}{k-i+1}. \tag{2.46}$$

In the same way we may derive

$$|\mathcal{F} - \mathcal{F}''| = |\mathcal{H}_i|\binom{n-t-2r-\delta}{k-i}, \tag{2.47}$$

$$|\mathcal{F}'' - \mathcal{F}| \geq |\mathcal{H}_{2t+2r+\delta-i}|\binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1}. \tag{2.48}$$

Now assume that our claim is false; that is, $|\mathcal{F}' - \mathcal{F}| \leq |\mathcal{F} - \mathcal{F}'|$ and $|\mathcal{F}'' - \mathcal{F}| \leq |\mathcal{F} - \mathcal{F}''|$. In view of (2.45)–(2.48) it follows that

$$|\mathcal{H}_i|\binom{n-t-2r-\delta}{k-i+1} \leq |\mathcal{H}_{2t+2r+\delta-i}|\binom{n-t-2r-\delta}{k-2t-2r-\delta+i}, \tag{2.49}$$

$$|\mathcal{H}_{2t+2r+\delta-i}|\binom{n-t-2r-\delta}{k-2t-2r-\delta+i+1} \leq |\mathcal{H}_i|\binom{n-t-2r-\delta}{k-i}. \tag{2.50}$$

Because of (2.44) we have $\mathcal{H}_{2t+2r+\delta-i} \neq \emptyset$. Thus (2.49) and (2.50) imply

$$\begin{aligned}&(n+t-k-i)(n-t-2r-\delta-k+i)\\ &\leq (k-i+1)(k-2t-2r-\delta+i+1).\end{aligned}$$

But this is a contradiction since $n \geq 2k-t+2$ implying $n+t-k-i > k-i+1$ as well as $n-t-2r-\delta-k+i > k-2t-2r-\delta+i+1$. □

Case 2. $i = \frac{2t+2r+\delta}{2}$. Recall that for all $X \in \mathcal{H}'_{t+r+\frac{\delta}{2}}$, $|X| = t+r+\frac{\delta}{2}-1$ and $X \subseteq [t+2r+\delta-1]$. Let

$$\mathcal{K}_j := \{X \in \mathcal{H}'_{t+r+\frac{\delta}{2}} : j \notin X\}, \qquad j \in [t+2r+\delta-1].$$

Counting the number of pairs (X, j) with $X \in \mathcal{H}'_{t+r+\frac{\delta}{2}}$, $j \in [t+2r+\delta-1]$, $j \notin X$ in two different ways we obtain

$$\sum_{j=1}^{t+2r+\delta-1} |\mathcal{K}_j| = |\mathcal{H}'_{t+r+\frac{\delta}{2}}|\left(r+\frac{\delta}{2}\right).$$

Consequently, there is some j such that

$$|\mathcal{K}_j| \geq \frac{r+\frac{\delta}{2}}{t+2r+\delta-1}|\mathcal{H}'_{t+r+\frac{\delta}{2}}|. \tag{2.51}$$

By Lemma 2.4.3(b), $\mathcal{K}_j$ is t-intersecting. In view of Claim 1

$$\mathcal{G}' := (\mathcal{G} - \mathcal{H}_{t+r+\frac{\delta}{2}}) \cup \mathcal{K}_j$$

is t-intersecting; thus also

$$\mathcal{F}' := \nabla_{\to k}(\mathcal{G}')$$

is t-intersecting. The next claim also yields in this case the desired contradiction to the maximality of $|\mathcal{F}|$.

Claim 3. We have $|\mathcal{F}'| > |\mathcal{F}|$.

Proof of Claim 3. We partition $\mathcal{F}$ into the sets

$$\mathcal{L}_1 := \nabla_{\to k}(\mathcal{G} - \mathcal{H}_{t+r+\frac{\delta}{2}}),$$

$$\mathcal{L}_2 := \nabla_{\to k}(\mathcal{H}_{t+r+\frac{\delta}{2}}) - \nabla_{\to k}(\mathcal{G} - \mathcal{H}_{t+r+\frac{\delta}{2}}),$$

and $\mathcal{F}'$ into the sets $\mathcal{L}_1$ and

$$\mathcal{L}_3 := \nabla_{\to k}(\mathcal{K}_j) - \nabla_{\to k}(\mathcal{G} - \mathcal{H}_{t+r+\frac{\delta}{2}}).$$

Our claim is equivalent to

$$|\mathcal{L}_3| > |\mathcal{L}_2|. \tag{2.52}$$

By Proposition 2.4.5,

$$|\mathcal{L}_2| = |\mathcal{H}_{t+r+\frac{\delta}{2}}|\binom{n-t-2r-\delta}{k-t-r-\frac{\delta}{2}}. \tag{2.53}$$

Similarly to the estimation of $|\mathcal{F}' - \mathcal{F}|$ in Case 1 we lower estimate $\mathcal{L}_3$: For any $X \in \mathcal{K}_j$, $X \notin \mathcal{G}$. Consequently, for all $Y \subseteq \{t+2r+\delta, \dots, n\}$ with $|Y| = k-i+1$ we have $X \cup Y \in \mathcal{L}_3$, and different pairs (X, Y) yield different sets $X \cup Y$. Thus

$$|\mathcal{L}_3| \geq |\mathcal{K}_j|\binom{n-t-2r-\delta+1}{k-t-r-\frac{\delta}{2}+1}. \tag{2.54}$$

According to (2.51), (2.53), and (2.54) the following (equivalent) inequalities are sufficient for (2.52):

$$\frac{r+\frac{\delta}{2}}{t+2r+\delta-1}\binom{n-t-2r-\delta+1}{k-t-r-\frac{\delta}{2}+1} > \binom{n-t-2r-\delta}{k-t-r-\frac{\delta}{2}},$$

$$\left(r+\frac{\delta}{2}\right)(n-t-2r-\delta+1) > \left(k-t-r-\frac{\delta}{2}+1\right)(t+2r+\delta-1),$$

$$n > \frac{(k-t+1)(t+2r+\delta-1)}{r+\frac{\delta}{2}}. \tag{2.55}$$

Since δ is even (i is an integer) it follows that $\delta \geq 2$. Consequently (noting $t \geq 1$)

$$\frac{t+2r+1}{r+1} \geq \frac{t+2r+\delta-1}{r+\frac{\delta}{2}}. \tag{2.56}$$

From (2.41) and (2.56) we derive that (2.55) is true; hence also (2.52) and the claim are proved. □

Thus in both cases we obtained a contradiction. Hence our assumption $\delta > 0$ was false, and we have $\delta \leq 0$. ■

Lemma 2.4.5. *Let $\mathcal{F}$ be any k-uniform t-intersecting family and $\overline{\mathcal{F}}$ the complementary family. Let $\mathcal{G}$ and $\mathcal{H}$ be generating for $\mathcal{F}$ and $\overline{\mathcal{F}}$, respectively. Then, for all $X \in \mathcal{G}$, $Y \in \mathcal{H}$, we have $|X \cup Y| \geq n - k + t$.*

Proof. Assume the contrary, that is,

$$|X \cup Y| \leq n - k + t - 1 \quad \text{for some } X \in \mathcal{G}, Y \in \mathcal{H}.$$

Choose any $Z \supseteq X \cup Y$ with $|Z| = n-k+t-1$. The inequalities $n > 2k-t, t \geq 1$ imply that $n - k + t - 1 \geq \max\{k, n - k\}$. Thus we find some $X^* \in \nabla_{\to k}(X)$ and some $Y^* \in \nabla_{\to n-k}(Y)$ such that $X^* \subseteq Z$ and $Y^* \subseteq Z$. Since $\mathcal{G}$ and $\mathcal{H}$ are generating, we have $X^* \in \mathcal{F}$ and $Y^* \in \overline{\mathcal{F}}$, that is, $\overline{Y^*} \in \mathcal{F}$. Now the inequality

$$|X^* \cap \overline{Y^*}| = |X^* \cup Y^*| - |Y^*| \leq n - k + t - 1 - n - k = t - 1$$

contradicts the fact that $\mathcal{F}$ is t-intersecting. ■

After all we are able to prove the Complete Intersection Theorem:

Proof of Theorem 2.4.1. By Lemma 2.4.2(b) and Lemma 2.4.4 we find a maximum k-uniform t-intersecting family $\mathcal{F}$ that is left shifted and a generating family $\mathcal{G}$ for $\mathcal{F}$ such that

$$\max(\mathcal{G}) \leq \begin{cases} t + 2r & \text{if } n > (k - t + 1)\left(2 + \frac{t-1}{r+1}\right), \\ t + 2r + 2 & \text{if } n = (k - t + 1)\left(2 + \frac{t-1}{r+1}\right). \end{cases}$$

It is easy to see that the complementary family $\overline{\mathcal{F}}$ is a right-shifted maximum $(n - k)$-uniform $(n - 2k + t)$-intersecting family. If we replace in the members of $\overline{\mathcal{F}}$ every element i by $n + 1 - i$, then we obtain a maximum $(n - k)$-uniform $(n - 2k + t)$-intersecting family $\mathcal{F}^*$ that is left shifted. It is easy to verify that under our suppositions the inequality (2.40) is equivalent to

$$(k - t + 1)\left(2 + \frac{n - 2k + t - 1}{k - t - r + 1}\right)$$
$$< n \leq (k - t + 1)\left(2 + \frac{n - 2k + t - 1}{k - t - r}\right) \tag{2.57}$$

(also the case $r = k - t$ is covered by our definition $\infty := \frac{i}{0}$ for $i \in \mathbb{N}$). If we set $k' := n - k, t' := n - 2k + t, r' := k - t - r$, then (2.57) reads:

$$(k' - t' + 1)\left(2 + \frac{t' - 1}{r' + 1}\right) < n \leq (k' - t' + 1)\left(2 + \frac{t' - 1}{r'}\right). \tag{2.58}$$

By Lemma 2.4.4 we find a generating family $\mathcal{G}^*$ for $\mathcal{F}^*$ such that

$$\max(\mathcal{G}^*) \leq t' + 2r'.$$

Making again a replacement $i \to n+1-i$, we obtain a generating family $\mathcal{H}$ for $\overline{\mathcal{F}}$ such that (note $n+1-t'-2r' = t+2r+1$)

$$\mathcal{H} \subseteq 2^{\{t+2r+1,\ldots,n\}}.$$

Case 1. Both inequalities in (2.40) are strict. If $|X| \geq t+r$ for all $X \in \mathcal{G}$, then clearly $\mathcal{F} = \mathcal{S}_r$. Analogously, if $|Y| \geq t'+r' = n-k-r$ for all $Y \in \mathcal{G}^*$ then $\mathcal{F}^* = \mathcal{S}_{r'}$; that is, $|\mathcal{F}| = |\mathcal{F}^*| = |\mathcal{S}_{r'}| = |\overline{\mathcal{S}_r}| = |\mathcal{S}_r|$. Thus we may assume that there is some $X \in \mathcal{G}$ with $|X| \leq t+r-1$ and some $Y \in \mathcal{H}$ with $|Y| \leq n-k-r-1$. But then $|X \cup Y| \leq n-k+t-2$, which is a contradiction to Lemma 2.4.5.

Case 2. The first inequality in (2.40) is an equality. If $|X| \geq t+r+1$ for all $X \in \mathcal{G}$, then $\mathcal{F} = \mathcal{S}_{r+1}$, which by Lemma 2.4.1 has the same size as $\mathcal{S}_r$. The arguments of Case 1 explain here as well that we may assume that there is some $X \in \mathcal{G}$ with $X \leq t+r$ and some $Y \in \mathcal{H}$ with $|Y| \leq n-k-r-1$. But then $|X \cup Y| \leq n-k+t-1$, and we obtain again a contradiction to Lemma 2.4.5. ■

We mention that Ahlswede and Khachatrian [15] proved with some additional effort that the sets $\mathcal{S}_r$ (and in case of equal size also the sets $\mathcal{S}_{r+1}$) are – up to permutations of the elements – the only maximum k-uniform t-intersecting families if $n \geq 2k-t+2$ or $n = 2k-t+1$ and $t \geq 2$.

For the special case $t = 2, n = 4m, k = 2m$, we have obviously $r = m-1$. Thus $\left\{X \subseteq \binom{[4m]}{2m} : |X \cap [2m]| \geq m+1\right\}$ is a maximum $2m$-uniform $4m$-intersecting family. This was a famous conjecture of Erdős, Ko, and Rado [170]. Some previous results concerning this conjecture were obtained by Calderblank and Frankl [89].

A t-intersecting family $\mathcal{F}$ is called *nontrivial* if $|\bigcap_{X \in \mathcal{F}} X| < t$. By Theorem 2.4.1, in the case $n \leq (k-t+1)(t+1)$, there exist maximum k-uniform t-intersecting families that are nontrivial. For the remaining cases, the following family is also a candidate of a maximum nontrivial family:

$$\mathcal{S}_1' := \{X \in \binom{[n]}{k} : [t] \subseteq X, X \cap \{t+1,\ldots,k+1\} \neq \emptyset\}$$
$$\cup \{[k+1] - \{i\} : i \in [t]\}.$$

With the method of generating families Ahlswede and Khachatrian [16] found the following result, mentioned here without proof:

Theorem 2.4.2. *Let $n > (k-t+1)(t+1)$. Then in the case $k \leq 2t+1$ the family $\mathcal{S}_1$, and in the case $k > 2t+1$ one of the families $\mathcal{S}_1, \mathcal{S}_1'$ is a maximum nontrivial k-uniform t-intersecting family.*

This generalizes earlier results of Hilton and Milner [268] ($t = 1$) and Frankl [189] (n sufficiently large).

2.5. Linear independence

An important method for finding an upper bound for a family $\mathcal{F}$ of combinatorial objects can be described as follows: Try to find a vector space V of a certain dimension d and to associate with the elements of $\mathcal{F}$ injectively certain vectors of V. If the associated vectors are linearly independent, then clearly $|\mathcal{F}| \leq d$. In this section we will present some examples where this method can be applied successfully. Further algebraic methods, in particular the use of eigenvalues, are studied in Chapter 6. A systematic treatment of algebraic methods, including, for example, exterior algebra methods, can be found in the forthcoming book of Babai and Frankl [35].

The following result has its origin in the *design of experiments*. Fisher [181] found an inequality that was later reproved by Bose [80] in a very short way using the algebraic approach (in terms of matrices) sketched previously. First uniform families were considered, only. Nonuniform families were investigated starting with the paper of de Bruijn and Erdős [86]. Majumdar [360] and, independently, Isbell [276] adapted Bose's method to prove the succeeding theorem.

Theorem 2.5.1 **(*Nonuniform Fisher Inequality*).** *Let $\mathcal{F}$ be a family in $2^{[n]}$ with the property that $|X \cap Y| = t$ for all $X, Y \in \mathcal{F}$, $X \neq Y$. If $t > 0$ then $|\mathcal{F}| \leq n$.*

Proof. The "easiest" vector space is the vector space $V = \mathbb{R}^n$ of all column vectors with n real entries over the field $K = \mathbb{R}$, and the "easiest" way to associate a vector with a set $X \subseteq [n]$ is to take the characteristic vector. Let $\mathcal{F} = \{X_1, \ldots, X_m\}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ be the associated characteristic vectors. It is enough to show that these vectors are linearly independent for $t > 0$. Assume the contrary, that is (with $\mathbf{0} := (0, \ldots, 0)^{\mathrm{T}}$),

$$\mathbf{0} = \alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_m \boldsymbol{x}_m \tag{2.59}$$

where not all coefficients are zero. Obviously,

$$\boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j = \begin{cases} |X_i| & \text{if } i = j, \\ t & \text{otherwise.} \end{cases} \tag{2.60}$$

Consequently,

$$\begin{aligned} 0 = \mathbf{0}^{\mathrm{T}}\mathbf{0} &= \left(\sum_{i=1}^{m} \alpha_i \boldsymbol{x}_i^{\mathrm{T}}\right)\left(\sum_{j=1}^{m} \alpha_j \boldsymbol{x}_j\right) = \sum_{i=1}^{m} \alpha_i^2 |X_i| + \sum_{1 \leq i, j \leq m, i \neq j} \alpha_i \alpha_j t \\ &= \sum_{i=1}^{m} \alpha_i^2 (|X_i| - t) + t \left(\sum_{i=1}^{m} \alpha_i\right)^2 . \end{aligned} \tag{2.61}$$

Clearly, $|X_i| \geq t$ for all i and $|X_i| = t$ for at most one i, since otherwise the intersection condition would not be satisfied. But then the RHS is greater than 0, a contradiction. ■

It is interesting that we may generalize this theorem significantly if we restrict ourselves to uniform families. Let $L = \{l_1, \ldots, l_s\}$ be a set of integers, $0 \leq l_1 < \cdots < l_s \leq n$. A family $\mathcal{F}$ in $2^{[n]}$ is called *L-intersecting* if $|X \cap Y| \in L$ for all $X, Y \in \mathcal{F}, X \neq Y$. In the previous theorem we found a bound for $\{t\}$-intersecting families. Note that intersecting families are exactly the $\{1, 2, \ldots, n-1\}$-intersecting families. For $L \subseteq [n]$, let $\gcd(L)$ denote the greatest common divisor of the elements of L. The next result was found by Babai and Frankl [36] (for further information, refer also to Deza and Frankl [135]).

Theorem 2.5.2. *Let $\mathcal{F}$ be a k-uniform L-intersecting family in $2^{[n]}$ and let $k \not\equiv 0 \pmod{\gcd(L)}$. Then $|\mathcal{F}| \leq n$.*

Proof. We start as in the previous proof, but we work with $V = \mathbb{Q}^n$, $K = \mathbb{Q}$, that is, over the field of rational numbers. Then in (2.59) we may suppose that all coefficients α_i are integers (otherwise multiply by the least common multiple), and, in addition, we may suppose that

$$\gcd(\{\alpha_1, \ldots, \alpha_m\}) = 1 \tag{2.62}$$

(otherwise divide by the LHS). Instead of (2.61) we discuss the equalities ($i = 1, \ldots, m$)

$$0 = \boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{0} = \boldsymbol{x}_i^{\mathrm{T}}\left(\sum_{j=1}^{m} \alpha_j \boldsymbol{x}_j\right) = k\alpha_i + \sum_{j=1, j\neq i}^{m} \alpha_j \boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j. \tag{2.63}$$

With $l := \gcd(L)$ we may write (2.63) also in the form

$$k\alpha_i + a_i l = 0, \ i = 1, \ldots, m, \tag{2.64}$$

where a_i is some integer. Since $k \not\equiv 0 \pmod{l}$, there is some prime power p^b such that $l \equiv 0 \pmod{p^b}$, but $k \not\equiv 0 \pmod{p^b}$. Now (2.64) implies that $\alpha_i \equiv 0 \pmod{p}$ for all i, a contradiction to (2.62). ■

Up to now we worked with infinite fields. We want to emphasize that in particular in the case of finite fields linear independence is not the only way to obtain bounds. The following observation can be considered also as a "method": If the elements of a family $\mathcal{F}$ can be injectively associated with the elements of a d-dimensional vector space over the Galois field $GF(q)$, then $|\mathcal{F}| \leq q^d$. As an example we consider a variant of a theorem of Ahlswede, El Gamal and Pang [12] on pairs of binary codes (for generalizations and related problems, see Ahlswede [2] and

Ahlswede, Cai, and Zhang [7]). In the succeeding proof we follow the paper of Delsarte and Piret [129].

Theorem 2.5.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be families in $2^{[n]}$ with the property that $|X \cap Y|$ has the same parity for all $X \in \mathcal{F}$, $Y \in \mathcal{G}$. Then*

$$|\mathcal{F}||\mathcal{G}| \leq \begin{cases} 2^n & \text{for even parity,} \\ 2^{n-1} & \text{for odd parity.} \end{cases}$$

Proof. Let V be the n-dimensional vector space over $GF(2)$ whose elements are the n-dimensional 0,1-column vectors. With each $X \subseteq [n]$ we associate its characteristic vector $\boldsymbol{x}$ as an element of V. Let F and G be the sets of characteristic vectors of $\mathcal{F}$ and $\mathcal{G}$, respectively. Clearly, we may suppose that $F, G \neq \emptyset$. Let $\boldsymbol{y}_0$ be a fixed element of G and $G_0 := \{\boldsymbol{y} + \boldsymbol{y}_0 : \boldsymbol{y} \in G\}$. Moreover, let $\widetilde{F}$ and $\widetilde{G}_0$ be the subspaces generated by F and G_0, respectively. Then

$$|\mathcal{F}||\mathcal{G}| = |F||G| = |F||G_0| \leq |\widetilde{F}||\widetilde{G}_0| = 2^{\dim \widetilde{F}} 2^{\dim \widetilde{G}_0}. \tag{2.65}$$

The key point is that (in $GF(2)$)

$$\boldsymbol{x}^{\mathrm{T}} \boldsymbol{z} = 0 \text{ for all } \boldsymbol{x} \in F,\ \boldsymbol{z} \in G_0 \tag{2.66}$$

since (with $\boldsymbol{z} = \boldsymbol{y} + \boldsymbol{y}_0$, $\boldsymbol{y} \in G$) in view of the supposition

$$\boldsymbol{x}^{\mathrm{T}} \boldsymbol{z} = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{y} + \boldsymbol{x}^{\mathrm{T}} \boldsymbol{y}_0 \equiv |X \cap Y| + |X \cap Y_0| \equiv 0 \pmod 2.$$

Consequently,

$$\dim \widetilde{G}_0 \leq n - \dim \widetilde{F} \tag{2.67}$$

(the vectors $\boldsymbol{x}^{\mathrm{T}}$, $\boldsymbol{x} \in \mathcal{F}$, form a coefficient matrix of rank $\dim \widetilde{F}$, and all $\boldsymbol{z} \in G_0$ are solutions of the system (2.66) of linear equations). The inequalities (2.65) and (2.67) already yield $|\mathcal{F}||\mathcal{G}| \leq 2^n$. Thus let $|X \cap Y|$ be odd for all $X \in \mathcal{F}$, $Y \in \mathcal{G}$. Let $\boldsymbol{x}_0$ be a fixed element of F and $F_0 := \{\boldsymbol{x} + \boldsymbol{x}_0 : \boldsymbol{x} \in \mathcal{F}\}$. We have $F \cap F_0 = \emptyset$, since otherwise $\boldsymbol{x} + \boldsymbol{x}_0 = \boldsymbol{x}'$ for some $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{F}$, which implies for all $\boldsymbol{y} \in G$

$$0 \equiv |X \cap Y| + |X_0 \cap Y| \equiv (\boldsymbol{x} + \boldsymbol{x}_0)^{\mathrm{T}} \boldsymbol{y} = \boldsymbol{x}'^{\mathrm{T}} \boldsymbol{y} \equiv |X' \cap Y| \equiv 1 \pmod 2,$$

a contradiction. Let $F' := F \cup F_0$ and let $\widetilde{F}'$ be the subspace generated by F'. Then in (2.66) and (2.67) obviously F can be replaced by F'. As before, we obtain $|F'||G| \leq 2^n$. Now $|F| = \frac{1}{2}|F'|$, and the assertion follows. ■

Let us come back to linear independence. In the next examples the basic vector space is more difficult, so we need some preparation. We will work with subspaces of the vector space $R[x_1, \ldots, x_n]$ of all polynomials in the independent variables $x_1, \ldots, x_n$ over some field R (usually the real or rational numbers).

Lemma 2.5.1. *Let $p_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$, be polynomials. Suppose that there exist m n-tuples $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{in})$ such that*

$$p_i(\boldsymbol{a}_j) \begin{cases} = 0 & \text{if } 1 \le i < j \le m, \\ \neq 0 & \text{if } 1 \le i = j \le m. \end{cases}$$

Then the polynomials p_i, $i = 1, \ldots, m$, are linearly independent.

Proof. Assume the contrary. Then we find coefficients $\alpha_1, \ldots, \alpha_m$, not all equal to zero, such that $\sum_{i=1}^m \alpha_i p_i$ is the zero polynomial. Let i^* be the largest index with $\alpha_i \neq 0$. Then

$$\sum_{i=1}^m \alpha_i p_i(\boldsymbol{a}_{i^*}) = \alpha_{i^*} p_{i^*}(\boldsymbol{a}_{i^*}) \neq 0,$$

a contradiction. ■

Lemma 2.5.2. *Let $p_i(x_1, \ldots, x_n), i = 1, \ldots, m$, and $q_h(x_1, \ldots, x_n), h = 1, \ldots, t$, be polynomials. Suppose that there exist m n-tuples $\boldsymbol{a}_i$ and t n-tuples $\boldsymbol{b}_h$ such that for all possible i, j, h*

$$p_i(\boldsymbol{a}_j) \begin{cases} = 0 & \text{if } i \neq j, \\ \neq 0 & \text{if } i = j, \end{cases} \qquad q_h(\boldsymbol{a}_j) = 0, \qquad q_h(\boldsymbol{b}_j) \begin{cases} = 0 & \text{if } h < j, \\ \neq 0 & \text{if } h = j. \end{cases}$$

Then the polynomials p_i, $i = 1, \ldots, m$, and q_h, $h = 1, \ldots, t$, are linearly independent.

Proof. Assume the contrary. Then we find coefficients $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_t$, not all equal to zero, such that $\sum_{i=1}^m \alpha_i p_i + \sum_{h=1}^t \beta_h q_h$ is the zero polynomial. In particular, for $j = 1, \ldots, m$,

$$0 = \sum_{i=1}^m \alpha_i p_i(\boldsymbol{a}_j) + \sum_{h=1}^t \beta_h q_h(\boldsymbol{a}_j) = \alpha_j p_j(\boldsymbol{a}_j)$$

implying $\alpha_j = 0$ for all j. Thus $\sum_{h=1}^t \beta_h q_h$ is a nontrivial linear combination of the zero polynomial, a contradiction to Lemma 2.5.1. ■

A polynomial $p(x_1, \ldots, x_n)$ is called *homogenous of degree k* if

$$p(\lambda x_1, \ldots, \lambda x_n) = \lambda^k p(x_1, \ldots, x_n).$$

The homogenous polynomials of degree k form a subspace $R_k[x_1, \ldots, x_n]$ of $R[x_1, \ldots, x_n]$. All monomials of the form $x_1^{i_1} \cdots x_n^{i_n}$ with $i_1 + \cdots + i_n = k$ obviously yield a basis of this vector space. Consequently, for the dimension we have

$$\dim R_k[x_1, \ldots, x_n] = \binom{n+k-1}{k} \tag{2.68}$$

(k-combinations with repetitions; cf. [441, p. 15]).

The following theorem was stated by Frankl [190] and Kalai [283], but it was essentially prepared by a result of Lovász [353] who introduced exterior algebra methods. Fortunately the proof can be formulated in a more elementary way (we follow Babai and Frankl [35]).

Theorem 2.5.4. *Let $\{(A_1, B_1), \ldots, (A_m, B_m)\}$ be a family of pairs of subsets of $[n]$ such that $|A_i| = a$, $|B_i| = b$ for all i, and*

$$A_i \cap B_j \begin{cases} \neq \emptyset & \text{if } 1 \leq i < j \leq m, \\ = \emptyset & \text{if } 1 \leq i = j \leq m. \end{cases}$$

Then

$$m \leq \binom{a+b}{a}.$$

If $n \geq a + b$, the bound is the best possible.

Proof. We associate with each member (A_i, B_i) of the given family the homogenous polynomial in the variables $x_0, \ldots, x_b$ of degree a (over $R = \mathbb{R}$)

$$p_i(x_0, \ldots, x_b) := \prod_{l \in A_i} (l^0 x_0 + l^1 x_1 + \cdots + l^b x_b).$$

In view of (2.68) it is sufficient to prove that these polynomials are linearly independent. With every i we associate a nontrivial solution $\boldsymbol{y}_i = (y_{i0}, \ldots, y_{ib})$ of the system of b equations

$$l^0 y_{i0} + l^1 y_{i1} + \cdots + l^b y_{ib} = 0, \quad l \in B_i. \tag{2.69}$$

Then

$$p_i(\boldsymbol{y}_j) = 0 \text{ for } 1 \leq i < j \leq n \tag{2.70}$$

since we find by supposition some $l \in A_i \cap B_j$, and $l^0 y_{j0} + \cdots + l^b y_{jb} = 0$ by construction. However,

$$p_i(\boldsymbol{y}_i) \neq 0 \quad \text{for } 1 \leq i \leq n, \tag{2.71}$$

because otherwise there is some $l_{b+1} \in A_i$ such that $\boldsymbol{y}_i$ is a solution of (2.69) and, in addition, of

$$l_{b+1}^0 y_{i0} + l_{b+1}^1 y_{i1} + \cdots + l_{b+1}^b y_{ib} = 0. \tag{2.72}$$

But the determinant of the coefficient matrix of the system (2.69), (2.72) is Vandermonde's determinant, that is, nonzero; hence $\boldsymbol{y}_i = \boldsymbol{0}$, a contradiction. Now (2.70), (2.71), and Lemma 2.5.1 imply that $p_1, \ldots, p_m$ are linearly independent.

In order to see that the bound is best possible for $n \geq a + b$, take the family $\left\{(A, [a+b] - A) : A \subseteq \binom{[a+b]}{a}\right\}$. ■

Note that the result follows for the stronger condition $A_i \cap B_j = \emptyset$ iff $i = j$ from Theorem 2.1.3. Blokhuis [70] and in a similar way Yu [472] used resultants of polynomials (resp. Bezoutians) to prove these results and further a generalization of Alon [26]. For related questions, see Tuza [453].

We call a polynomial in the variables $x_1, \dots, x_n$ *multilinear* if it has the form

$$p(x_1, \dots, x_n) = \sum_{\boldsymbol{a} \in B_n} \alpha_{\boldsymbol{a}} x_1^{a_1} \cdots x_n^{a_n}$$

where B_n is realized as the set of all n-tuples $\boldsymbol{a} = (a_1, \dots, a_n)$ with entries 0,1, and the coefficients $\alpha_{\boldsymbol{a}}$ belong to the given field R. In other words, in a multilinear polynomial no variable appears in a power greater than 1. The degree of p is the largest value of $r(\boldsymbol{a}) = a_1 + \cdots + a_n$, extended over all $\boldsymbol{a} \in B_n$ with $\alpha_{\boldsymbol{a}} \neq 0$. Again, the set of all multilinear polynomials of degree at most k forms a subspace $R_{k,1}[x_1, \dots, x_n]$ of $R[x_1, \dots, x_n]$. Since the monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 + \cdots + a_n \leq k$, $\boldsymbol{a} \in B_n$, form a basis, we have

$$\dim R_{k,1}[x_1, \dots, x_n] = \sum_{i=0}^{k} \binom{n}{i}. \tag{2.73}$$

We will reprove the following result of Frankl and Wilson [202] and its predecessor, the theorem of Ray-Chaudhuri and Wilson [394], in a more general setting in Section 6.5.

Theorem 2.5.5. *Let $\mathcal{F}$ be an L-intersecting family in $2^{[n]}$. Then*

$$|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ and let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$ be the associated characteristic vectors. We may assume, w.l.o.g., that $|A_1| \geq \cdots \geq |A_m|$. With each A_i we associate first the polynomial

$$\overline{p}_i(x_1, \dots, x_n) := \prod_{l \in L: l < |A_i|} (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x} - l) \tag{2.74}$$

where, as usual, $\boldsymbol{x} = (x_1, \dots, x_n)^{\mathrm{T}}$. Then we expand $\overline{p}_i$ (theoretically) and replace every power x_i^b with $b \geq 1$ by x_i. This leads to a multilinear polynomial $p_i(x_1, \dots, x_n)$ of degree not greater than $|L|$. Note that for $\boldsymbol{a} \in B_n$, $i \in [m]$,

$$\overline{p}_i(\boldsymbol{a}) = p_i(\boldsymbol{a}) \tag{2.75}$$

since $x^2 = x$ for $x \in \{0, 1\}$. In view of (2.60), (2.75), and the ordering of the A_i, we have

$$p_i(\boldsymbol{a}_j) = \overline{p}_i(\boldsymbol{a}_j) \begin{cases} = 0 & \text{if } 1 \leq i < j \leq m, \\ \neq 0 & \text{if } 1 \leq i = j \leq m. \end{cases} \tag{2.76}$$

According to Lemma 2.5.1 these polynomials are linearly independent. Now the asserted inequality follows from (2.73). ■

The proofs of this and the next theorem are due to Alon, Babai, and Suzuki [28]. Snevily [433] conjectures that the upper bound can be improved to $\sum_{i=0}^{|L|}\binom{n-1}{i}$ if L contains only *positive* integers. This conjecture is a generalization of a conjecture of Frankl and Füredi [193], who considered the special case $L = \{1, \dots, k\}$. Up to now Snevily has proved his conjecture only for sufficiently large n as well as the Frankl–Füredi Conjecture for $k = 2$ (see [433]), and, after a result of Pyber [393], together with Lichtblau [340] for $n \le 2k + 3$.

***Theorem 2.5.6 (Ray-Chaudhuri and Wilson* [394]).** *Let $\mathcal{F}$ be a k-uniform L-intersecting family in $2^{[n]}$ where $k \ge |L|$. Then*

$$|\mathcal{F}| \le \binom{n}{|L|}.$$

Proof. Clearly we may suppose that $l < k$ for all $l \in L$. We start as in the previous proof and obtain polynomials $p_1, \dots, p_m$ and n-tuples $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m$. Note that, in view of the uniformity, (2.74) can be rewritten as

$$\overline{p}_i(x_1, \dots, x_n) = \prod_{l \in L}(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - l).$$

Following an idea of Blokhuis [69], we introduce more polynomials. Let $\{B_1, \dots, B_t\}$ be the family of all subsets of $[n]$ of size less than $|L|$ (do not confound these sets with the Boolean lattices), and let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_t$ be the associated characteristic vectors. Thus we have $t = \sum_{j=0}^{|L|-1}\binom{n}{j}$. Here we suppose that the members are numerated in such a way that $|B_1| \ge \cdots \ge |B_t|$. We define the polynomials $\overline{q}_h$, $h = 1, \dots, t$, by

$$\overline{q}_h(x_1, \dots, x_n) := \left(\sum_{j=1}^{n} x_j - k\right)\prod_{i \in B_h} x_i,$$

and reduce them, as before, using $x_j^2 = x_j$, to the multilinear polynomials $q_h(x_1, \dots, x_n)$. It is easy to verify that the conditions of Lemma 2.5.2 are satisfied. This lemma together with (2.73) implies

$$m + t \le \sum_{j=0}^{|L|}\binom{n}{j},$$
$$m \le \binom{n}{|L|}.$$

■

Note that for "bad" sets L the bound may be far from being best possible. If $L = \{0, 2, 4, \ldots, 2t\}(t \geq 1)$ and if k is odd, $k > 2t$, then according to Theorem 2.5.2 we get $|\mathcal{F}| \leq n$ whereas Theorem 2.5.6 gives $|\mathcal{F}| \leq \binom{n}{t+1}$ only. As for Theorem 2.5.1, there exist several modular variants of the preceding theorems, which are sometimes not as "easy" to prove as Theorem 2.5.2. We mention here only one "easy" variant:

Theorem 2.5.7. *Let p be a prime and L be a set of integers from $\{0, \ldots, p-1\}$. Let $\mathcal{F}$ be a family in $2^{[n]}$ with the property that for all different members X, Y of $\mathcal{F}$ there is some $l \in L$ with $|X \cap Y| \equiv l \pmod p$ and for all members X of $\mathcal{F}$ there is no $l \in L$ with $|X| \equiv l \pmod p$. Then*

$$|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}.$$

Proof. We modify slightly the proof of Theorem 2.5.5. First of all we are working this time over the Galois field $GF(p)$. We need not order the members of $\mathcal{F}$ by decreasing size. We may (again) define (instead of (2.74))

$$\overline{p}_i(x_1, \ldots, x_n) = \prod_{l \in L} (\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{x} - l),$$

and obtain as before the polynomials p_i. Then we have a still stronger relation than (2.76):

$$\overline{p}_i(\boldsymbol{a}_j) \begin{cases} = 0 & \text{if } i \neq j, \\ \neq 0 & \text{if } i = j \end{cases}$$

(note that $\boldsymbol{a}_i^{\mathrm{T}} \boldsymbol{a}_i \equiv |A_i| \not\equiv l \pmod p$ for all $l \in L$ by supposition). Lemma 2.5.1 and (2.73) yield the assertion. ■

Theorem 2.5.7 and related results have interesting geometric applications. Essential contributions in that direction were given by Frankl and Wilson [202]. The constructions may be often restricted to the case of primes because of the Prime Number Theorem (see, e.g., Krätzel [324, p. 116]):

Theorem 2.5.8 (Prime Number Theorem). *Let $\pi(x)$ be the number of primes less or equal to x. Then*

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to \infty.$$

Corollary 2.5.1. *Let p_n be the largest prime less or equal to n. Then*

$$p_n \sim n \text{ as } n \to \infty.$$

Proof. Choose any $\epsilon > 0$. We have to show that there exists some n_0 such that $p_n \geq (1-\epsilon)n$ for all $n \geq n_0$. By Theorem 2.5.8,

$$\frac{\pi(n)}{\pi((1-\epsilon)n)} \sim \frac{n}{(1-\epsilon)n} \frac{\log n + \log(1-\epsilon)}{\log n} > 1$$

if n is sufficiently large. Thus there exists a prime in the interval $((1-\epsilon)n, n]$. ■

The *diameter* of a finite set of points in the n-dimensional Euclidean space $\mathbb{R}^n$ is the maximum distance between any two points of the given set. Let $b_d(n)$ be the smallest number such that every finite set in $\mathbb{R}^n$ of diameter d can be partitioned into $b_d(n)$ sets of diameter smaller than d. Using contractions one can easily see that $b_d(n)$ does not depend on d, so we write briefly $b(n)$. Moreover, it is left to the reader to show that $b(n)$ is finite. Taking the vertices of a regular simplex in $\mathbb{R}^n$ one immediately obtains $b(n) \geq n+1$. In 1933 Borsuk [79] conjectured that equality holds (also for infinite, closed sets). Not until sixty years later did Kahn and Kalai [281] kill this conjecture and show that the lower bound can be significantly improved:

Theorem 2.5.9. *We have $b(n) \gtrsim (1.2)^{\sqrt{n}}$ as $n \to \infty$.*

Proof. First we prove the relation for a subsequence of the natural numbers. Note that $b(n)$ is clearly increasing. Let p be a prime (tending to infinity) and let $m := 4p$. We will construct a set T of points in the m^2-dimensional Euclidean space of diameter $d := \sqrt{2}m$. Let S be the set of points $\boldsymbol{x} = (x_1, \ldots, x_m)^{\mathrm{T}}$ for which $x_i \in \{-1, 1\}$ for all i, $x_1 = 1$, and the number of negative components equals $\frac{m}{2}$. Obviously, $|S| = \frac{1}{2}\binom{m}{\frac{m}{2}}$. With each $\boldsymbol{x} \in S$ we associate the set $X := \{i \in [m] : x_i = 1\} \in \binom{[m]}{\frac{m}{2}}$. For any $\boldsymbol{x}$, we put

$$\boldsymbol{x} * \boldsymbol{x} := (x_1x_1, x_1x_2, \ldots, x_1x_m, x_2x_1, \ldots, x_2x_m, \ldots, x_mx_m)^{\mathrm{T}}.$$

Now let $T := \{\boldsymbol{x} * \boldsymbol{x} : \boldsymbol{x} \in S\}$. Since $\boldsymbol{x} * \boldsymbol{x} \neq \boldsymbol{y} * \boldsymbol{y}$ if $\boldsymbol{x} \neq \boldsymbol{y}$,

$$|T| = \frac{1}{2}\binom{m}{\frac{m}{2}}. \tag{2.77}$$

In order to compute distances, we must be able to calculate the standard scalar product $\langle\ ,\ \rangle$. In a straightforward way one may verify the identity

$$\langle \boldsymbol{x} * \boldsymbol{x}, \boldsymbol{y} * \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2. \tag{2.78}$$

It is easy to see that for $\boldsymbol{x}, \boldsymbol{y} \in S$ and the associated sets X, Y (with $\overline{X} := [m] - X, \overline{Y} := [m] - Y$)

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y} \rangle &= |X \cap Y| + |\overline{X} \cap \overline{Y}| - (|X - Y| + |Y - X|) \\ &= m - 2|X| - 2|Y| + 4|X \cap Y|; \end{aligned}$$

that is,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4|X \cap Y| - m. \tag{2.79}$$

Thus we have for the distance ρ between two points $\boldsymbol{x} * \boldsymbol{x}$ and $\boldsymbol{y} * \boldsymbol{y}$ of T:

$$\begin{aligned}\rho^2(\boldsymbol{x} * \boldsymbol{x}, \boldsymbol{y} * \boldsymbol{y}) &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle^2 + \langle \boldsymbol{y}, \boldsymbol{y} \rangle^2 - 2\langle \boldsymbol{x}, \boldsymbol{y} \rangle^2 \\ &= m^2 + m^2 - 2(4|X \cap Y| - m)^2 \le 2m^2,\end{aligned}$$

and equality; that is, the diameter $d = \sqrt{2}m$ is attained iff $|X \cap Y| = \frac{m}{4} = p$.

Let U be a subset of T of diameter smaller than d and let $\mathcal{F}$ be the family of those sets X that are associated with the vectors $\boldsymbol{x}$ for which $\boldsymbol{x} * \boldsymbol{x} \in U$. From the preceding discussion we know that $\mathcal{F}$ is a $2p$-uniform family in $2^{[m]}$ with the property given in Theorem 2.5.7, where $L := \{1, \ldots, p-1\}$. Consequently,

$$\begin{aligned}|\mathcal{F}| &\le \sum_{i=0}^{p-1} \binom{m}{i} \le \binom{m}{p} \left(\frac{p}{m-p+1} + \frac{p}{m-p+1} \frac{p-1}{m-p+2} + \cdots \right) \\ &\le \binom{m}{p} \left(\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots \right) = \frac{1}{2} \binom{m}{p}.\end{aligned}$$

Thus, in order to partition T into sets of smaller diameter, we need at least $|T|/\frac{1}{2}\binom{m}{p}$ sets. Accordingly (noting 2.77))

$$b(m^2) \ge \frac{\binom{m}{\frac{m}{2}}}{\binom{m}{\frac{m}{4}}} \sim \frac{\sqrt{3}}{2} \left(\frac{3}{2\sqrt[4]{3}} \right)^m = \frac{\sqrt{3}}{2} (1.140)^m \quad (\text{as } p \to \infty)$$

(here we make an anticipation – the asymptotics follow from (7.7) or by Stirling's formula (Theorem 7.1.7)). Though we constructed T in the m^2-dimensional space, it is in fact contained in an affine subspace of dimension $\binom{m}{2}$ since we have for the components the equalities $x_i x_i = 1$ for all i and $x_i x_j = x_j x_i$ for all i, j. This gives, with $n' := \binom{m}{2}$,

$$b(n') \gtrsim \frac{\sqrt{3}}{2} \left(\frac{3}{2\sqrt[4]{3}} \right)^m \ge \frac{\sqrt{3}}{2} \left[\left(\frac{3}{2\sqrt[4]{3}} \right)^{\sqrt{2}} \right]^{\sqrt{n'}} \gtrsim (1.203)^{\sqrt{n'}} \quad \text{as } p \to \infty.$$

Finally, if n is arbitrary, let p be the largest prime such that $n' := \binom{4p}{2} \le n$. From Corollary 2.5.1 one obtains easily that $n' = n - o(n)$. Consequently,

$$b(n) \ge b(n') \gtrsim (1.203)^{\sqrt{n'}} \gtrsim (1.2)^{\sqrt{n}} \quad \text{as } n \to \infty.$$

■

The construction is a reformulation of that given by Kahn and Kalai [281]. For the computations, we used ideas of Nilli and his best friend Alon [378], who also noticed that one can enlarge the set S by taking those $\boldsymbol{x}$ for which the number of negative components is even (instead of $\frac{m}{2}$).

Let $c_d(n)$ be the smallest number of colors with which the points in $\mathbb{R}^n$ can be colored such that no two points of the same color have distance d. Again, $c_d(n)$ does not depend on d, so we write $c(n)$. It is an easy exercise to show that $c(n)$ is finite.

Corollary 2.5.2. *We have $c(n) \gtrsim (1.2)^{\sqrt{n}}$ as $n \to \infty$.*

Proof. Take a set of points of diameter d that cannot be partitioned into less than $b(n)$ sets of diameter smaller than d. Every "good" coloring (with respect to the distance d) of this set needs at least $b(n)$ colors since the color classes are sets of diameter smaller than d. Thus (by Theorem 2.5.9)

$$c(n) \geq b(n) \gtrsim (1.2)^{\sqrt{n}} \text{ as } n \to \infty.$$

■

This result is due to Frankl and Wilson [202], who proved a conjecture of Larman and Rogers [332] more than ten years before Kahn and Kalai [281] found Theorem 2.5.9. The original proof of Frankl and Wilson is similar to the proof of Theorem 2.5.9, only the construction is easier.

2.6. Probabilistic methods

Our general aim is the maximization of the size of families satisfying certain conditions. For example, for Sperner families and k-uniform intersecting families we could first "guess" a maximum family of the kind in question, and then prove that these families are optimal. But sometimes we do not have an idea how to produce such a large family. Here it is useful to apply the *probabilistic method*, sometimes also called the *Erdős method* in honor of P. Erdős, who not only is the founder of this area but also developed it with numerous deep results and still more intriguing problems. This method can be described as follows: In order to prove the existence of a family of "large" size satisfying the given condition, one has to construct an appropriate probability space and to show that a randomly chosen element (corresponding to a family) in this space has the desired properties with positive probability. Only a few instructive examples are presented here together with modifications of the method. The interested reader may learn much more in the monographs of Erdős and Spencer [172] and Alon and Spencer [29]. Certainly, also other facts from probability theory are useful in combinatorics. One example is the entropy inequality (Theorem 2.6.5). The application of limit theorems is studied in Chapter 7, where some standard definitions and results can also be found. Working in this area, one must be clever in calculus; in particular, it is useful to have many inequalities at hand. Here is an important one:

Proposition 2.6.1.

(*a*) *For all* $x \in \mathbb{R}$, $1 + x \le e^x$, *and equality holds only for* $x = 0$.

(*b*) *For all* $y > 0$, $\log y \le y - 1$, *and equality holds only for* $y = 1$.

Proof. (a) The function $f(x) := e^x - x - 1$ has a unique minimum at $x = 0$.
(b) By (a), $y \le e^{y-1}$; that is, for $y > 0$, $\log y \le y - 1$. ■

Our first example is due to Kleitman and Spencer [311] and also presented in the survey of Alon [27]. A family $\mathcal{F}$ in $2^{[n]}$ is called *k-independent* if, for every k distinct members $X_1, \ldots, X_k$ of $\mathcal{F}$, all the 2^k intersections $\cap_{i=1}^k Y_i$ are nonempty where each Y_i is either X_i or $\overline{X}_i$.

Theorem 2.6.1. *The maximum size of a k-independent family in* $2^{[n]}$ *is at least* $\lfloor \frac{1}{2} k!^{1/k} e^{n/(k2^k)} \rfloor$ *for all* $k \ge 2$.

Proof. Let, for $X \subseteq [n]$, $X^1 := X$ and $X^0 := \overline{X}$. Consider the random family $\mathcal{F} = \{X_1, \ldots, X_m\}$, where $(X_1, \ldots, X_m)$ is a sequence of m randomly, independently chosen subsets of $[n]$ and equidistribution is assumed (i.e., the probability of the choice of any particular subset equals 2^{-n}). More precisely, the elements of our probability space are the $n \times m$-arrays whose entries take on only the values 0,1, and the (classic) probability of such an array equals 2^{-nm}. The columns of the array are the characteristic vectors of the members of $\mathcal{F}$. Note that up to now the members are not necessarily different. We have

$$\begin{aligned} &P(\mathcal{F} \text{ is not } k\text{-independent}) \\ &\quad \le \sum_{1 \le i_1 < \cdots < i_k \le m} P(\{X_{i_1}, \ldots, X_{i_k}\} \text{ is not } k\text{-independent}) \\ &\quad \le \sum_{1 \le i_1 < \cdots < i_k \le m} \; \sum_{\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{0,1\}^k} P(X_{i_1}^{\epsilon_1} \cap \cdots \cap X_{i_k}^{\epsilon_k} = \emptyset). \end{aligned}$$

It is easy to see that $X_{i_1}^{\epsilon_1} \cap \cdots \cap X_{i_k}^{\epsilon_k} = \emptyset$ iff the corresponding subarray does not contain ϵ as a row. Hence the probability of this event equals $(1 - 2^{-k})^n$, and we conclude

$$\begin{aligned} P(\mathcal{F} \text{ is } k\text{-independent}) &= 1 - P(\mathcal{F} \text{ is not } k\text{-independent}) \\ &\ge 1 - \binom{m}{k} 2^k (1 - 2^{-k})^n. \end{aligned}$$

If $m = \lfloor \frac{1}{2} k!^{1/k} e^{n/(k2^k)} \rfloor$ then, in view of Proposition 2.6.1,

$$\binom{m}{k} 2^k (1 - 2^{-k})^n < \frac{m^k}{k!} 2^k e^{-n/2^k} \leq 1,$$

and finally,

$$P(\mathcal{F} \text{ is } k\text{-independent}) > 0.$$

The members of a k-independent family are automatically pairwise different; thus we find indeed a k-independent family of the size given in the assertion. ■

We already mentioned in Section 2 that for $k = 2$ (qualitative independence of unordered 2-partitions) the exact maximum size $\binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$ can be extracted from the Profile-Polytope Theorem 3.3.1.

Before the next example we will consider the probabilistic method in a more general form. Let S be an at most countable subset of $\mathbb{R}$ and let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a discrete random vector which takes on only values from S^n. Further let $g : \mathbb{R}^n \to \mathbb{R}$ be given. Then $g(\xi_1, \ldots, \xi_n)$ is a discrete random variable whose expected value equals

$$\mathrm{E}(g(\boldsymbol{\xi})) = \sum_{(s_1, \ldots, s_n) \in S^n} g(s_1, \ldots, s_n) P(\xi_1 = s_1, \ldots, \xi_n = s_n)$$

if the sum on the RHS is absolutely convergent. The "generalized" probabilistic method is the application of the trivial fact that there exist $s_1, \ldots, s_n \in S$ such that

$$g(s_1, \ldots, s_n) \geq \text{ (resp. } \leq) \ \mathrm{E}(g(\boldsymbol{\xi})).$$

In Theorem 2.3.3(b) there is given an exact solution of the vertex-isoperimetric problem for the Hasse graph of the Boolean lattice: Find

$$\min\{|(\Delta(\mathcal{F}) \cup \nabla(\mathcal{F})) - \mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } |\mathcal{F}| = m\}.$$

A related problem of Zuev [474] is the following: Find

$$\max\{|\Delta(\mathcal{F})| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is a Sperner family}\}.$$

If we take $\mathcal{F}$ to be a complete level in the middle of the Boolean lattice, then $\Delta(\mathcal{F})$ is again a complete level and using

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \sim \sqrt{\frac{2}{\pi n}} 2^n$$

(which follows from (7.6)) we obtain

$$|\Delta(\mathcal{F})| \geq c \frac{2^n}{\sqrt{n}} \text{ for some constant } c.$$

It was shown by Kospanov [320] and Füredi, Kahn, and Kleitman [211] that this bound can be improved significantly. The strongest lower bound is due to Chukhrov [100] (factor 0.2), but we follow the simpler proof of Füredi, Kahn, and Kleitman:

Theorem 2.6.2. *For sufficiently large n, there exists a Sperner family $\mathcal{F} \subseteq 2^{[n]}$ such that $|\Delta(\mathcal{F})| \geq 0.059 * 2^n$.*

Proof. Let $k := \left\lfloor \sqrt{\frac{n}{2}} \right\rfloor$, $l := \left\lceil \frac{n}{2} - \sqrt{\frac{n}{8}} \right\rceil$, and $u := \left\lceil \frac{n}{2} + \sqrt{\frac{n}{8}} \right\rceil$. We will restrict the shadow to the $[l, u]$-rank-selected subposet of B_n which we denote by Q. For its size, we have by (7.1)

$$
\begin{aligned}
|Q| &= \sum_{j=l}^{u} \binom{n}{j} \\
&= \sum_{-\frac{\sqrt{n}}{2}\sqrt{\frac{1}{2}}+\frac{n}{2} \leq j < \frac{\sqrt{n}}{2}\sqrt{\frac{1}{2}}+\frac{n}{2}} \binom{n}{j} \\
&\sim \left(\Phi\left(\sqrt{\frac{1}{2}}\right) - \Phi\left(-\sqrt{\frac{1}{2}}\right)\right) 2^n \gtrsim 0.5204 * 2^n. \qquad (2.80)
\end{aligned}
$$

We will see that for sufficiently large n there exists a Sperner family $\mathcal{F}$ in $2^{[n]}$ such that

$$|\Delta(\mathcal{F})| \geq 0.1146 * |Q|. \qquad (2.81)$$

Together with (2.80), this yields for sufficiently large n,

$$|\Delta(\mathcal{F})| \geq 0.059 * 2^n.$$

Let $t := u + 1 - l$, $c := e(\log(e+1) - 1) = 0.8515\ldots$, and p be the solution of

$$tp\binom{u}{k} = c.$$

Clearly, $\lim_{n\to\infty} \binom{u}{k} = \infty$; thus,

$$\lim_{n\to\infty} tp = 0. \qquad (2.82)$$

Let ξ be a random variable taking on values from $S := \{1, \ldots, t, t+1\}$ such that

$$P(\xi = i) = \begin{cases} p & \text{if } i = 1, \ldots, t, \\ 1 - tp & \text{if } i = t+1. \end{cases}$$

Let $\{\xi_K : K \in \binom{[n]}{k}\}$ be a set of (completely) independent copies of ξ (in the following K always belongs to $\binom{[n]}{k}$). For $j = l+1, \ldots, u+1$, we define the random families

$$\mathcal{F}_j := \{X \in \binom{[n]}{j} : \min\{\xi_K : K \subseteq X\} = j - l\},$$

$$\mathcal{F} := \bigcup_{j=l+1}^{u+1} \mathcal{F}_j.$$

Note that then the random family $\Delta(\mathcal{F})$ is contained in Q and that $|\Delta(\mathcal{F})|$ is a random variable which is a function of the ξ_K, $K \in \binom{[n]}{k}$.

Claim. $\mathcal{F}$ is a Sperner family in $2^{[n]}$.

Proof of Claim. The existence of some j, j' with $l + 1 \leq j < j' \leq u + 1$ and of some $X \in \mathcal{F}_j$, $X' \in \mathcal{F}_{j'}$ such that $X \subseteq X'$ could be the only obstacle. Since $X \in \mathcal{F}_j$, there is some $K \subseteq X$ such that $\xi_K = j - l$. But then also $K \subseteq X'$ and $\min\{\xi_K : K \subseteq X'\} \leq j - l < j' - l$ in contradiction to $X' \in \mathcal{F}_{j'}$. □

For each $A \in Q$, we define the random variable η_A by

$$\eta_A := \begin{cases} 1 & \text{if } A \in \Delta(\mathcal{F}), \\ 0 & \text{if } A \notin \Delta(\mathcal{F}). \end{cases}$$

Then

$$\mathrm{E}(|\Delta(\mathcal{F})|) = \mathrm{E}\left(\sum_{A \in Q} \eta_A\right) = \sum_{A \in Q} \mathrm{E}(\eta_A) = \sum_{A \in Q} P(A \in \Delta(\mathcal{F})).$$

If we can show that $P(A \in \Delta(\mathcal{F})) \geq 0.1146$ for all $A \in Q$, then there must be a realization of the ξ_K, $K \in \binom{[n]}{k}$, that is, also a realization of $\mathcal{F}$, such that (2.81) holds.

Claim. $P(A \in \Delta(\mathcal{F})) \geq 0.1146$ if $A \in Q$.

Proof of Claim. For sake of abbreviation, always let $i := j + 1 - l$. Let $A \subseteq [n]$, $|A| = j$, $l \leq j \leq u$. Then

$$\begin{aligned} P(A \in \Delta(\mathcal{F})) &= P(\exists b \in [n] - A : \min\{\xi_K : K \subseteq A \cup \{b\}\} = i) \\ &\geq P(\min\{\xi_K : K \subseteq A\} > i \text{ and} \\ &\qquad \exists b \in [n] - A : \min\{\xi_K : K \subseteq A \cup \{b\}\} = i) \\ &= P(\min\{\xi_K : K \subseteq A\} > i \text{ and} \\ &\qquad \exists b \in [n] - A : \min\{\xi_{K' \cup \{b\}} : K' \subseteq A, K' \in \tbinom{[n]}{k-1}\} = i). \end{aligned}$$

In the following, K' always denotes an element of $\binom{[n]}{k-1}$. The events in the last conjunction are independent since the index sets of the corresponding random variables, namely $\{K : K \subseteq A\}$ and $\{K' \cup \{b\} : K' \subseteq A, b \in [n] - A\}$ are disjoint and the ξ_K, $K \in \binom{[n]}{k}$ are (completely) independent. Thus we may continue:

$$P(A \in \Delta(\mathcal{F})) = P(\min\{\xi_K : K \subseteq A\} > i)P(\exists b \in [n] - A : \min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} = i). \tag{2.83}$$

Now we compute and estimate both factors. We have (note (2.82))

$$\begin{aligned} P(\min\{\xi_K : K \subseteq A\} > i) &= \prod_{K \subseteq A} P(\xi_K > i) = (1 - ip)^{\binom{j}{k}} \geq (1 - tp)^{\binom{j}{k}} \\ &\geq (1 - tp)^{\binom{u}{k}} = (1 - tp)^{c/tp} = e^{-c}(1 + o(1)). \end{aligned}$$

For the second factor, we need a little bit more effort:

$$\begin{aligned} &P(\exists b \in [n] - A : \min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} = i) \\ &= 1 - P(\forall b \in [n] - A : \min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} \neq i) \\ &= 1 - \prod_{b \in [n] - A} P(\min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} \neq i) \\ &= 1 - \prod_{b \in [n] - A} (1 - P(\min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} = i). \end{aligned} \tag{2.84}$$

Next we estimate for all $b \in [n] - A$ the probability in the RHS of formula (2.84). We have $\min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} = i$ iff for all $K' \subseteq A$, $\xi_{K' \cup \{b\}} \geq i$ and not for all $K' \subseteq A$, $\xi_{K' \cup \{b\}} > i$. Thus

$$\begin{aligned} P(\min\{\xi_{K' \cup \{b\}} : K' \subseteq A\} = i) &= (1 - (i - 1)p)^{\binom{j}{k-1}} - (1 - ip)^{\binom{j}{k-1}} \\ &\geq (1 - ip)^{\binom{j}{k-1}} \binom{j}{k-1} p \end{aligned}$$

(for the last estimation we used the Binomial Theorem and the relation $0 < 1-ip < 1$ for sufficiently large n in view of (2.82)). For the first factor of the RHS, we have by Bernoulli's formula,

$$\begin{aligned} 1 &\geq (1 - ip)^{\binom{j}{k-1}} \geq 1 - \binom{j}{k-1} ip \geq 1 - \binom{u}{k-1} tp \\ &= 1 - \binom{u}{k} tp \frac{k}{u - k + 1} = 1 + o(1). \end{aligned}$$

To estimate the second factor, we first observe

$$\begin{aligned} \binom{j}{k} &= \binom{u}{k} \frac{j}{u} \cdots \frac{j - k + 1}{u - k + 1} \geq \binom{u}{k} \left(\frac{j - k + 1}{u - k + 1}\right)^k \geq \binom{u}{k} \left(\frac{l - k + 1}{u - k + 1}\right)^k \\ &= \binom{u}{k} \left(1 - \frac{u - l}{u - k + 1}\right)^{\frac{u-k+1}{u-l} \frac{u-l}{u-k+1} k} = \binom{u}{k} e^{-1}(1 + o(1)). \end{aligned}$$

Thus

$$\binom{j}{k-1}p = \binom{j}{k}\frac{k}{j-k+1}p$$
$$\geq tp\binom{u}{k}\frac{kl}{t(j-k+1)}\frac{1}{l}e^{-1}(1+o(1)) = \frac{c}{el}(1+o(1)),$$

and we derive

$$P(\min\{\xi_{K'\cup\{b\}} : K' \subseteq A\} = i) \geq \frac{c}{el}(1+o(1)).$$

Insertion into (2.84) yields

$$P(\exists b \in [n] - A : \min\{\xi_{K'\cup\{b\}} : K' \subseteq A\} = i)$$
$$\geq 1 - (1 - \frac{c}{el}(1+o(1))^{n-j} \geq 1 - e^{-\frac{c}{el}(1+o(1))(n-j)}$$
$$= (1 - e^{-\frac{c}{e}})(1+o(1))$$

(here we used Proposition 2.6.1(a)). Thus the estimation of both factors in (2.83) yields

$$P(A \in \Delta(\mathcal{F})) \geq (1+o(1))e^{-c}(1 - e^{-\frac{c}{e}}).$$

For $c = e(\log(e+1) - 1)$, the function on the RHS attains its maximum $0.1147\ldots$. Thus, for sufficiently large n,

$$P(A \in \Delta(\mathcal{F})) \geq 0.1146.$$

■

Without proof we mention the following upper bounds of Kostochka [322]:

Theorem 2.6.3. *If n is sufficiently large then for any Sperner family $\mathcal{F}$ in $2^{[n]}$*

(a) $|\Delta(\mathcal{F})| \leq 0.725 * 2^n$,

(b) $|\Delta(\mathcal{F}) \cup \nabla(\mathcal{F})| \leq 0.9994 * 2^n$.

A related result for the boundary of a filter was previously obtained by Kostochka [321]. The next modification of the probabilistic method for the construction of large families without forbidden configurations has the following basic idea (see, e.g., Kleitman [305]): Construct a collection (i.e., repetitions are allowed) of m members entirely at random. Then compute the expected number of forbidden configurations in it (e.g., comparable elements, nonintersecting members, etc.). If this number is at most λm, then there must be some collection having no more than λm forbidden configurations, so that, by omitting at most λm members, we can obtain a collection (often automatically without repetitions, i.e., a family) of the desired form of size $(1-\lambda)m$. Now let us illustrate this idea (with a slight modification). We are looking for large (p, q)-Sperner families in B_n (see [157]). Here a family $\mathcal{F}$ of subsets of $[n]$ is called a (p, q)-*Sperner family* if the intersection of any p members of $\mathcal{F}$ is not contained in the union of any other q members of $\mathcal{F}$. Note that a $(1, 1)$-Sperner family is a usual Sperner family. Let $d_{n;p,q}$ be the maximum size of a (p, q)-Sperner family in B_n.

Theorem 2.6.4. *We have*

$$d_{n;p,q} \geq c\left(\frac{2^{p+q}}{2^{p+q}-1}\right)^{\frac{n}{p+q-1}},$$

where $c := (p+q-1)\left(\frac{p!q!}{(p+q)^{p+q}}\right)^{\frac{1}{p+q-1}}$.

Proof. Let $(X_1, \dots, X_p, X_{p+1}, \dots, X_{p+q})$ be a sequence of $p+q$ randomly, independently chosen subsets of $[n]$, where equidistribution is assumed (analogous to the proof of Theorem 2.6.1). Let $a := \frac{2^{p+q}}{2^{p+q}-1}$.

Claim. The probability of the event $X_1 \cap \dots \cap X_p \subseteq X_{p+1} \cup \dots \cup X_{p+q}$ equals $(\frac{1}{a})^n$.

Proof of Claim. As in the proof of Theorem 2.6.1, we consider the random sequence as a random 0, 1-array with n rows and $p+q$ columns where the jth column is the characteristic vector of X_j. We have $X_1 \cap \dots \cap X_p \subseteq X_{p+1} \cup \dots \cup X_{p+q}$ iff in the corresponding array there is no row of the form $(1, \dots, 1, 0, \dots, 0)$ (p ones and q zeros). This gives the probability $\left(\frac{2^{p+q}-1}{2^{p+q}}\right)^n$. □

Now we construct at random a sequence $(X_1, \dots, X_m)$ of m subsets of $[n]$ and after that we pick, step by step, $p+q$ of them in all possible ways, which gives $m(m-1)\cdots(m-p-q+1)$ sequences of the form $(X_{i_1}, \dots, X_{i_p}, X_{i_{p+1}}, \dots, X_{i_{p+q}})$. By Claim 1 we obtain for the expected number μ of events $X_{i_1} \cap \dots \cap X_{i_p} \subseteq X_{i_{p+1}} \cup \dots \cup X_{i_{p+q}}$ the value

$$\mu = m(m-1)\cdots(m-p-q+1)\left(\frac{1}{a}\right)^n.$$

For $\lambda > 0$, we choose the largest m such that $\mu \leq \lambda m$. We then have

$$m \geq (\lambda a^n)^{\frac{1}{p+q-1}}$$

since for $m \leq \lceil (\lambda a^n)^{\frac{1}{p+q-1}} \rceil$

$$\mu \leq m(m-1)^{p+q-1}\left(\frac{1}{a}\right)^n \leq m(\lambda a^n)\left(\frac{1}{a}\right)^n = \lambda m.$$

Clearly there must be a fixed sequence $(X_1, \dots, X_m)$ with at most λm "bad" events described above. We may destroy such an event by taking away, for example, X_{i_1}.

Then simultaneously $p!q!$ events are destroyed, namely those that are obtained by permuting $i_1, \ldots, i_p$ and $i_{p+1}, \ldots, i_{p+q}$, respectively. After all these events are destroyed, we retain a sequence whose members form a (p, q)-Sperner family of size at least $m - \frac{\lambda m}{p!q!}$ (note that repetitions always yield a "bad" event). Consequently,

$$d_{n;p,q} \geq f(\lambda) a^{\frac{n}{p+q-1}}$$

where $f(\lambda) = (1 - \frac{\lambda}{p!q!})\lambda^{\frac{1}{p+q-1}}$. The function $f(\lambda)$ attains its maximum in $\mathbb{R}_+$ at $\lambda = \frac{p!q!}{p+q}$. The corresponding value of f equals the constant c from the assertion. ■

The concluding example of this section shows that other parts of probability theory also can be applied successfully. First we need some preparations. Let S again be an at most countable subset of $\mathbb{R}$. Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ be a discrete random vector that takes on only values from S^n. Further let $(\eta_1, \ldots, \eta_r)$ be that random vector which takes on values from S^r with probabilities

$$P(\eta_1 = s_1, \ldots, \eta_r = s_r) = \sum_{(s_{r+1},\ldots,s_n)\in S^{n-r}} P(\xi_1 = s_1, \ldots, \xi_n = s_n).$$

Here we call this vector *marginal random vector*. Clearly, it can be considered as the restriction of the old vector to the first r components, so we denote also the new vector by $(\xi_1, \ldots, \xi_r)$ instead of $(\eta_1, \ldots, \eta_r)$. For $1 \leq i_1 < \cdots < i_r \leq n$, we may define in the same way the *marginal random vector* $(\xi_{i_1}, \ldots, \xi_{i_r})$. The components ξ_i can also be considered as one-dimensional marginal random vectors. It is easy to see that "being marginal" is transitive, in particular we have:

Proposition 2.6.2. *If $(\xi_1, \ldots, \xi_{n-1})$ is marginal to $(\xi_1, \ldots, \xi_n)$ and ξ_1 is marginal to $(\xi_1, \ldots, \xi_{n-1})$, then ξ_1 is marginal to $(\xi_1, \ldots, \xi_n)$.*

The *entropy* of any n-dimensional discrete random vector $\boldsymbol{\xi}$ taking on values from S^n is defined and denoted by

$$H(\boldsymbol{\xi}) := -\sum_{s\in S^n} P(\boldsymbol{\xi} = s) \log P(\boldsymbol{\xi} = s).$$

Here $0 \log 0$ is defined to be 0.

Theorem 2.6.5 (Entropy Inequality). *Let $\boldsymbol{\xi}$ be an n-dimensional discrete random vector and let $\xi_1, \ldots, \xi_n$ be its components, that is, the one-dimensional marginal vectors. Then*

$$H(\boldsymbol{\xi}) \leq \sum_{i=1}^{n} H(\xi_i).$$

Proof. We proceed by induction on n. The case $n = 1$ is clear. For the step $n-1 \to n$, let $\boldsymbol{\eta} := (\xi_1, \dots, \xi_{n-1})$ be the corresponding $(n-1)$-dimensional marginal random vector. We write briefly $\boldsymbol{t} := (s_1, \dots, s_{n-1})$, $p_{t,s_n} := P(\boldsymbol{\eta} = \boldsymbol{t}, \xi_n = s_n)$, $p_t := P(\boldsymbol{\eta} = \boldsymbol{t})$, $p_{s_n} := P(\xi_n = s_n)$. Then we have

$$H(\boldsymbol{\xi}) = -\sum_{(\boldsymbol{t},s_n)\in S^n} p_{t,s_n} \log p_{t,s_n},$$

$$H(\boldsymbol{\eta}) + H(\xi_n) = -\sum_{\boldsymbol{t}\in S^{n-1}} p_t \log p_t - \sum_{s_n\in S} p_{s_n} \log p_{s_n}$$

$$= -\sum_{(\boldsymbol{t},s_n)\in S^n} p_{t,s_n} \log p_t - \sum_{(\boldsymbol{t},s_n)\in S^n} p_{t,s_n} \log p_{s_n}.$$

We restrict the summation to those elements $(\boldsymbol{t}, s_n)$ for which $p_{t,s_n} > 0$. Then (noting Proposition 2.6.1(b))

$$H(\boldsymbol{\xi}) - (H(\boldsymbol{\eta}) + H(\xi_n)) = \sum p_{t,s_n} \log\left(\frac{p_t p_{s_n}}{p_{t,s_n}}\right) \le \sum p_{t,s_n}\left(\frac{p_t p_{s_n}}{p_{t,s_n}} - 1\right)$$

$$= \sum p_t p_{s_n} - \sum p_{t,s_n} = 1 - 1 = 0.$$

From the induction hypothesis and Proposition 2.6.2 we infer finally

$$H(\boldsymbol{\xi}) \le H(\boldsymbol{\eta}) + H(\xi_n) \le (H(\xi_1) + \cdots + H(\xi_{n-1})) + H(\xi_n).$$

∎

The notion of entropy stems from physics and is also used in information theory. The entropy inequality has many applications, for example, in search theory, cf. Aigner [22]. Here we will reflect briefly the set theoretic point of view. We use again the letter H for the definition of the function $H(\lambda) := -\lambda \log \lambda - (1 - \lambda)\log(1-\lambda),\ 0 < \lambda < 1$.

Corollary 2.6.1. *Let $\mathcal{F}$ be a family of subsets of $[n]$, $\mathcal{F}(i) := \{X \in \mathcal{F} : i \in X\}$, and $\alpha_i := \frac{|\mathcal{F}(i)|}{|\mathcal{F}|}$. Then*

$$\log |\mathcal{F}| \le \sum_{i=1}^{n} H(\alpha_i).$$

Proof. Let $m := |\mathcal{F}|$. We identify $\mathcal{F}$ with the set $C_{\mathcal{F}}$ of its characteristic vectors, that is, $C_{\mathcal{F}} \subseteq \{0,1\}^n$ and $m = |C_{\mathcal{F}}|$. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ be the discrete random vector that takes on values from $\{0,1\}^n$ with probabilities

$$P(\boldsymbol{\xi} = s) = \begin{cases} \frac{1}{m} & \text{if } s \in C_{\mathcal{F}}, \\ 0 & \text{otherwise.} \end{cases}$$

Then obviously $P(\xi_i = 1) = \alpha_i$, $P(\xi_i = 0) = 1 - \alpha_i$, $i = 1, \dots, n$. We have

$$H(\boldsymbol{\xi}) = -\sum_{s \in C_{\mathcal{F}}} \frac{1}{m} \log \frac{1}{m} = \log m$$

$$H(\xi_i) = -\alpha_i \log \alpha_i - (1 - \alpha_i) \log(1 - \alpha_i) = H(\alpha_i).$$

Now the asserted inequality is a direct consequence of Theorem 2.6.5. ∎

Frankl (see [27]) observed that the following frequently used inequality can be easily derived:

Corollary 2.6.2. *For every $0 \le p \le \frac{1}{2}$ and for every natural number n,*

$$\sum_{i \le np} \binom{n}{i} \le e^{nH(p)}.$$

Proof. We use the notations from Corollary 2.6.1. Let $\mathcal{F} := \{X \subseteq [n] : |X| \le np\}$. Counting the number of pairs (i, X) with $i \in [n]$, $X \in \mathcal{F}$, $i \in X$ in two different ways, we get

$$\sum_{i=1}^{n} |\mathcal{F}(i)| = \sum_{X \in \mathcal{F}} |X|. \tag{2.85}$$

According to $|\mathcal{F}(1)| = \cdots = |\mathcal{F}(n)|$ and $|X| \le pn$, we obtain for all i

$$n|\mathcal{F}(i)| \le pn|\mathcal{F}|,$$

$$\alpha_i \le p.$$

Since $H(\lambda)$ is increasing in $[0, \frac{1}{2}]$, it follows by Corollary 2.6.1 that

$$\log |\mathcal{F}| \le \sum_{i=1}^{n} H(\alpha_i) \le nH(p),$$

which is equivalent to the assertion. ∎

As a "real" combinatorial application of Corollary 2.6.1, we present a result of Kleitman, Shearer, and Sturtevant [310] answering a question of Erdős.

Theorem 2.6.6. *Let $k > 1$ and let $\mathcal{F}$ be a k-uniform family in $2^{[n]}$ such that all intersections of two different members of $\mathcal{F}$ are pairwise different. Then there is some $\epsilon > 0$ such that*

$$|\mathcal{F}| < (2 - \epsilon)^k.$$

Before we look at the proof of this theorem, note that the bound 2^k is trivial since for some fixed $X^* \in \mathcal{F}$ the sets $X \cap X^*$, $X \in \mathcal{F}$, are pairwise different subsets of X^*, and $|X^*| = k$.

Proof. Let us define $\mathcal{F}(i)$ and $\alpha_i, i = 1, \ldots, n$, as in Corollary 2.6.1, and let $m := |\mathcal{F}|$. Observe that we may suppose

$$m > \frac{1}{2}k^2, \tag{2.86}$$

since otherwise the proof is trivial. Moreover, we can suppose that $0 < \alpha_i < 1$ for all i; otherwise delete the corresponding element in the basic set $[n]$. Let $\mathcal{G}$ be the family of intersections of two different members of $\mathcal{F}$,

$$\mathcal{G}(i) := \{Y \in \mathcal{G} : i \in Y\}, \qquad \beta_i := \frac{|\mathcal{G}(i)|}{|\mathcal{G}|}, \quad i = 1, \ldots, n.$$

It is easy to see that

$$|\mathcal{G}| = \binom{m}{2}, \beta_i = \frac{\binom{|\mathcal{F}(i)|}{2}}{\binom{m}{2}} = \frac{\binom{\alpha_i m}{2}}{\binom{m}{2}} = \alpha_i^2 - \frac{\alpha_i(1-\alpha_i)}{m-1}, \qquad i = 1, \ldots, n.$$

We apply Corollary 2.6.1 to the family $\mathcal{G}$ with the corresponding numbers β_i and obtain

$$\log \binom{m}{2} \le \sum_{i=1}^{n} H(\beta_i). \tag{2.87}$$

In order to estimate the RHS we need some preparations: The equality (2.85) reads in the present case

$$\sum_{i=1}^{n} |\mathcal{F}(i)| = k|\mathcal{F}|, \quad \text{i.e.,} \quad \sum_{i=1}^{n} \alpha_i = k. \tag{2.88}$$

Claim 1. We have for all i, $H(\beta_i) \le H(\alpha_i^2) + \frac{1/e}{m-1}\alpha_i$.

Proof of Claim 1. We have for $0 < \lambda < 1$, $H'(\lambda) = -\log\lambda + \log(1-\lambda)$, $H''(\lambda) = -(\frac{1}{\lambda} + \frac{1}{1-\lambda}) < 0$; thus $H'(\lambda)$ is decreasing, and by the mean-value theorem,

$$\frac{H(\alpha_i^2) - H(\beta_i)}{\frac{\alpha_i(1-\alpha_i)}{m-1}} \ge H'(\alpha_i^2) \ge \log(1-\alpha_i).$$

Hence

$$H(\alpha_i^2) - H(\beta_i) \ge \frac{\alpha_i}{m-1}(1-\alpha_i)\log(1-\alpha_i) \ge -\frac{\alpha_i}{m-1}\frac{1}{e}.$$

□

Claim 2. We have $\sum_{i=1}^{n} H(\alpha_i^2) \le k\frac{H(\lambda^{*2})}{\lambda^*}$, where $\lambda^* := \sum_{i=1}^{n}(\frac{\alpha_i}{k})\alpha_i$.

Proof of Claim 2. Let $f(\lambda) := \frac{H(\lambda^2)}{\lambda}, 0 < \lambda < 1$. We have for $0 < \lambda < 1$, $f''(\lambda) = -\frac{2}{\lambda^3(1-\lambda^2)}\left(2\lambda^2 + (1-\lambda^2)\log(1-\lambda^2)\right) < 0$ since for the function

$g(l) := 2l + (1-l)\log(1-l)$ there holds $g(0) = 0$ and $g'(l) = 1 - \log(1-l) \geq 0$, $0 \leq l \leq 1$. Thus f is concave and by Jensen's inequality and (2.88) we have

$$\sum_{i=1}^{n} H(\alpha_i^2) = k\left(\sum_{i=1}^{n} \frac{\alpha_i}{k} f(\alpha_i)\right) \leq kf\left(\sum_{i=1}^{n} \frac{\alpha_i}{k}\alpha_i\right) = kf(\lambda^*) = k\frac{H(\lambda^{*2})}{\lambda^*}.$$

$\square$

From (2.87), (2.88), Claim 1, and Claim 2, we infer

$$\log\binom{m}{2} \leq \sum_{i=1}^{n} H(\alpha_i^2) + \sum_{i=1}^{n} \frac{1/e}{m-1}\alpha_i \leq k\frac{H(\lambda^{*2})}{\lambda^*} + \frac{k/e}{m-1}.$$

The function $f(\lambda)$ attains its maximum at $\lambda = 0.4914\ldots$, and its value is there $1.1249\ldots$. With (2.86) we obtain for $k \to \infty$

$$\binom{m}{2} \leq e^{1.125k+o(k)} \leq 3.08^{k+o(k)}.$$

Thus

$$m \leq \frac{1}{2} + \sqrt{\frac{1}{4} + 2 * 3.08^{k+o(k)}} \leq (2-\epsilon)^k \text{ for some } \epsilon > 0.$$

$\blacksquare$

3

Profile-polytopes for set families

In this chapter we will discuss from a linear programming point of view profiles of families of subsets of $[n]$ belonging to certain classes. Concerning the terminology we refer to Schrijver [421] and Nemhauser and Wolsey [377]. The investigations were initiated by Erdős, Frankl, and Katona [175].

With each family $\mathcal{F}$ we associate its *(reduced) profile* $\boldsymbol{f} = p(\mathcal{F})$, where $\boldsymbol{f} = (f_1, \dots, f_{n-1})^{\mathrm{T}}$ and, as always, $f_i := |\{X \in \mathcal{F} : |X| = i\}|$, $i = 0, \dots, n$. Since in some cases the empty set $\emptyset$ and the whole set $[n]$ play a special role, we will restrict ourselves to the *reduced* profiles – without f_0 and f_n – although for the sake of brevity we omit the word "reduced." In applications, one should not forget to add these special sets if possible and necessary. Given a class $\mathfrak{A}$ of families,

$$\mu(\mathfrak{A}) := \{p(\mathcal{F}) : \mathcal{F} \in \mathfrak{A}\}$$

is a finite set of points in $\mathbb{R}^{n-1}$. The convex hull of a set $\mu \subseteq \mathbb{R}^{n-1}$ is denoted as $\mathrm{conv}(\mu)$. (The *convex hull* of a set μ of points is the set of all *convex combinations* of finite subsets of μ, that is, the set $\{\lambda_1 \boldsymbol{f}_1 + \cdots + \lambda_r \boldsymbol{f}_r$ where $\{\boldsymbol{f}_1, \dots, \boldsymbol{f}_r\} \subseteq \mu$ and $0 \le \lambda_i$ for all i and $\sum_{i=1}^{r} \lambda_i = 1\}$.) The set $\mathrm{conv}(\mu(\mathfrak{A}))$ is called the *profile-polytope of* $\mathfrak{A}$. (A *polytope* is the convex hull of a finite set of points. The *dimension* of a polytope is the smallest dimension of an affine subspace of the Euclidean space containing the given polytope.) Let $\epsilon(\mu)$ be the set of extreme points of $\mathrm{conv}(\mu)$. (A point of a convex set is an *extreme point* if it cannot be written as a convex combination of two other points from the given set.) Clearly $\epsilon(\mu) \subseteq \mu$. It is easy to see that

$$\mathrm{conv}(\mu) = \mathrm{conv}(\epsilon(\mu)).$$

The profile-polytopes $\mathrm{conv}(\mu(\mathfrak{A}))$ for the class of Sperner families and the class of intersecting families, respectively, are illustrated for $n = 3$ in Figure 3.1 (the encircled points are the essential extreme points, which will be introduced later). Often there arise optimization problems of the following kind:

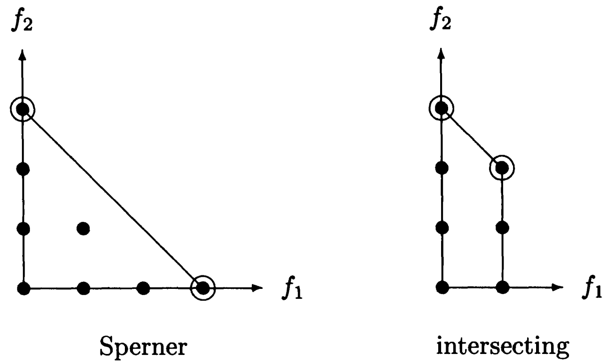


Figure 3.1

Let $w = (w_1, \ldots, w_{n-1})^{\mathrm{T}}$ be a given vector. Determine for a class $\mathfrak{A}$ of families

$$M(\mathfrak{A}, w) := \max\left\{\sum_{X \in \mathcal{F}} w_{|X|} : \mathcal{F} \in \mathfrak{A}\right\} = \max\{w^{\mathrm{T}} f : f \in \mu(\mathfrak{A})\}. \qquad (3.1)$$

Obviously,

$$\begin{aligned} M(\mathfrak{A}, w) &\leq \max\{w^{\mathrm{T}} f : f \in \operatorname{conv}(\mu(\mathfrak{A}))\} \\ &= \max\{w^{\mathrm{T}} f : f \in \operatorname{conv}(\epsilon(\mu(\mathfrak{A})))\}. \end{aligned}$$

Let $f^* \in \epsilon(\mu(\mathfrak{A}))$ be chosen such that

$$w^{\mathrm{T}} f^* \geq w^{\mathrm{T}} f' \text{ for all } f' \in \epsilon(\mu(\mathfrak{A})).$$

For $f = \sum_{f' \in \epsilon(\mu(\mathfrak{A}))} \lambda_{f'} f' \in \operatorname{conv}(\epsilon(\mu(\mathfrak{A})))$ (where $0 \leq \lambda_{f'}$ for all f' and $\sum \lambda_{f'} = 1$), we then have

$$w^{\mathrm{T}} f = \sum_{f' \in \epsilon(\mu(\mathfrak{A}))} \lambda_{f'} w^{\mathrm{T}} f' \leq \left(\sum_{f' \in \epsilon(\mu(\mathfrak{A}))} \lambda_{f'}\right) w^{\mathrm{T}} f^* = w^{\mathrm{T}} f^*,$$

hence

$$M(\mathfrak{A}, w) = \max\{w^{\mathrm{T}} f : f \in \epsilon(\mu(\mathfrak{A}))\}. \qquad (3.2)$$

In general we have far fewer extreme points than profiles and the maximization on the extreme points is only a numerical problem. So it is of great interest to have an overview of the extreme points for several classes. This is the main aim of this chapter.

Note that a polytope may be also defined as a bounded polyhedron where a *polyhedron* is the set of solutions of a system $Ax \leq b$; cf. [421, p. 89]. As before, any linear objective function attains its maximum at some extreme point.

A point $\boldsymbol{x} \in \mathbb{R}^n$ is an extreme point iff it satisfies all constraints and n *independent* constraints (i.e., the corresponding rows of A are linearly independent) are satisfied as equalities, cf. [421, p. 104].

Moreover, we would like to know "all" inequalities that are satisfied by all profiles of families belonging to some class. For instance, if $\mathfrak{A}$ is the class of Sperner families in $2^{[n]}$ we have derived the LYM-inequality $\sum_{i=1}^{n-1} f_i/\binom{n}{i} \leq 1$. The strongest inequalities are those which represent facets of $\mathrm{conv}(\mu(\mathfrak{A}))$. An inequality $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \leq b$ defines a *facet* F of a polytope $\mathcal{P}$ if this inequality is satisfied by all $\boldsymbol{x} \in \mathcal{P}$ and if $F := \{\boldsymbol{x} \in \mathcal{P} : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = b\}$ is one dimension lower than $\mathcal{P}$. All other *valid* inequalities (i.e., those that are satisfied by all elements of the polytope) can be obtained by taking a nonnegative linear combination of these facet-defining inequalities and by increasing the RHS; cf. [377, p. 88 ff] and [421, p. 93]. Note that, if the polytope $\mathcal{P}$ is given by a system $\mathcal{P} := \{\boldsymbol{x} : A\boldsymbol{x} \leq \boldsymbol{b}\}$, then all facets can be defined in the form $\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} = b_i$ for some i where $\boldsymbol{a}_i^{\mathrm{T}}$ (resp. b_i) is the ith row of A (resp. the ith coordinate of $\boldsymbol{b}$); cf. [377, p. 89]. The second objective of this chapter is to derive the facet-defining inequalities.

3.1. Full hereditary families and the antiblocking type

Let μ be a finite set of points in $\mathbb{N}^n$. This set μ is called *full* if all unit vectors $\boldsymbol{e}_i$ (with 1 at coordinate i) belong to μ, and it is called *hereditary* if $\boldsymbol{x} \in \mu, \boldsymbol{x}' \leq \boldsymbol{x}$, $\boldsymbol{x}' \in \mathbb{N}^n$ imply $\boldsymbol{x}' \in \mu$ (i.e., if it is an ideal in the direct product $\mathbb{N}^n$, where $\mathbb{N}$ is, as usual, linearly ordered). We will study full hereditary sets of points since we will investigate later on only full hereditary classes of families of subsets of $[n]$. Here a class $\mathfrak{A}$ is called *full* if for all $X \subseteq [n]$ the one-element family $\{X\}$ belongs to $\mathfrak{A}$, and it is called *hereditary* if $\mathcal{G} \subseteq \mathcal{F} \in \mathfrak{A}$ implies $\mathcal{G} \in \mathfrak{A}$. It is obvious that the set $\mu(\mathfrak{A})$ of points from $\mathbb{N}^{n-1}$ is full hereditary if $\mathfrak{A}$ is full hereditary (in our general investigations we work with $\mathbb{N}^n$ instead of $\mathbb{N}^{n-1}$). In fact, we may also replace everywhere $\mathbb{N}$ by $\mathbb{R}_+$, then in the definition of fullness we only need that $\alpha\boldsymbol{e}_i \in \mu$ for some $\alpha > 0$ $(i = 1, \ldots, n)$.

Lemma 3.1.1. *Let μ be full and hereditary.*

(*a*) *For all i, $x_i \geq 0$ is a facet-defining inequality for* $\mathrm{conv}(\mu)$.

(*b*) *All other facet-defining inequalities for* $\mathrm{conv}(\mu)$ *have the form $a_1x_1 + \cdots + a_nx_n \leq b$ with $a_i \geq 0$ for all i and $b > 0$.*

Proof. (a) Let us consider the inequality $x_1 \geq 0$, which is true for all $\boldsymbol{x} \in \mathrm{conv}(\mu)$. Let $F := \{\boldsymbol{x} \in \mathrm{conv}(\mu) : x_1 = 0\}$. The points $\boldsymbol{0}, \boldsymbol{e}_i, i = 2, \ldots, n$, are affine independent (i.e., $\boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent) and belong to F. Hence $\dim F = n - 1$ and F is a facet.

(b) Since μ is full, for all i, $x_i \leq 0$ cannot be a valid inequality, so all other facet-defining inequalities for $\text{conv}(\mu)$ must have the form $a_1x_1 + \cdots + a_nx_n \leq b$, where at least two coefficients, say a_1 and a_2, are nonzero. Assume, for example, that $a_1 < 0$. Let $F := \{\boldsymbol{x} \in \text{conv}(\mu) : a_1x_1 + \cdots + a_nx_n = b\}$ be a facet. There exists some extreme point $\boldsymbol{x}$ of F (which is also an extreme point of $\text{conv}(\mu)$, i.e., belongs to μ) with $x_1 > 0$ since otherwise F would satisfy two independent equalities $\sum_{i=1}^n a_ix_i = b$, $x_1 = 0$ and thus have dimension less than $n - 1$. Now let $\boldsymbol{x}'$ be the vector that can be obtained from $\boldsymbol{x}$ by changing the first coordinate into zero. Then $\boldsymbol{x}' \in \mu$, but $a_1x_1' + \cdots + a_nx_n' = b - a_1x_1 > b$, a contradiction. From $a_1 > 0$ we may derive $b > 0$ since $\boldsymbol{e}_1 \in \mu$. ■

Of course, we can divide the facet-defining inequalities of the form (b) in Lemma 3.1.1 by b, and the RHS becomes 1. Furthermore, for each i there must be some facet-defining inequality with $a_i > 0$, since otherwise there would be no restriction from above for x_i and $\text{conv}(\mu)$ would be unbounded. Thus we obtain:

Theorem 3.1.1. *Let μ be a full hereditary set of points in $\mathbb{N}^n$. Then there exists a nonnegative matrix A with no zero column such that*

$$\text{conv}(\mu) = \{\boldsymbol{x} \in \mathbb{R}_+^n : A\boldsymbol{x} \leq \mathbf{1}\}.$$

A *full-dimensional* polytope (i.e., a polytope of dimension n) in $\mathbb{R}^n$ of the form given in Theorem 3.1.1 is said to be of *antiblocking type*. The theory of blocking and antiblocking polyhedra was developed by Fulkerson in [206, 207, 208].

If, in general, $\mathcal{P}$ is any polytope of antiblocking type, $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}_+^n : A\boldsymbol{x} \leq \mathbf{1}\}$, where A is a nonnegative matrix with no zero columns, then the *antiblocker* $\mathcal{P}^c$ of $\mathcal{P}$ is the polytope

$$\mathcal{P}^c := \{\boldsymbol{y} \in \mathbb{R}_+^n : \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} \leq 1 \text{ for all } \boldsymbol{x} \in \mathcal{P}\}.$$

Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r$ be the extreme points of $\mathcal{P}$ and let C be the $(r \times n)$-matrix whose rows are $\boldsymbol{c}_1^{\mathrm{T}}, \ldots, \boldsymbol{c}_r^{\mathrm{T}}$.

Theorem 3.1.2. *We have*

(a) $\mathcal{P}^c = \{\boldsymbol{y} \in \mathbb{R}_+^n : C\boldsymbol{y} \leq \mathbf{1}\}$,

(b) $(\mathcal{P}^c)^c = \mathcal{P}$.

Proof. (a) Let $\boldsymbol{y} \in \mathcal{P}^c$. Then $\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{c}_i \leq 1$ for all i since the $\boldsymbol{c}_i$ are special elements of $\mathcal{P}$. Hence $C\boldsymbol{y} \leq \mathbf{1}$. Conversely, let $\boldsymbol{y} \in \mathbb{R}_+^n, C\boldsymbol{y} \leq \mathbf{1}$. Every point $\boldsymbol{x}$ of $\mathcal{P}$ can be written as a convex combination of the extreme points: $\boldsymbol{x} = \sum_{i=1}^r \alpha_i\boldsymbol{c}_i, \alpha_i \geq 0$ for all i, $\sum_{i=1}^r \alpha_i = 1$. Consequently, $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = \sum_{i=1}^r \alpha_i(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{c}_i) \leq \sum_{i=1}^r \alpha_i = 1$. Hence $\boldsymbol{y} \in \mathcal{P}^c$.

(b) Let $\boldsymbol{x} \in \mathcal{P}$. Then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} \le 1$ for all $\boldsymbol{y} \in \mathcal{P}^c$. Thus $\boldsymbol{x} \in (\mathcal{P}^c)^c$. The rows $\boldsymbol{a}_i^{\mathrm{T}}$ of A belong obviously to $\mathcal{P}^c$. Thus, if, conversely, $\boldsymbol{x} \in (\mathcal{P}^c)^c$, then $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a}_i = \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \le 1$ for all i; hence $A\boldsymbol{x} \le \mathbf{1}$, that is, $\boldsymbol{x} \in \mathcal{P}$. ■

In Figure 3.2, we illustrate the antiblockers $(\mathrm{conv}(\mu(\mathfrak{A})))^c$ where $\mathfrak{A}$ is – as for Figure 3.1 – the class of Sperner (resp. intersecting) families and $n = 3$.

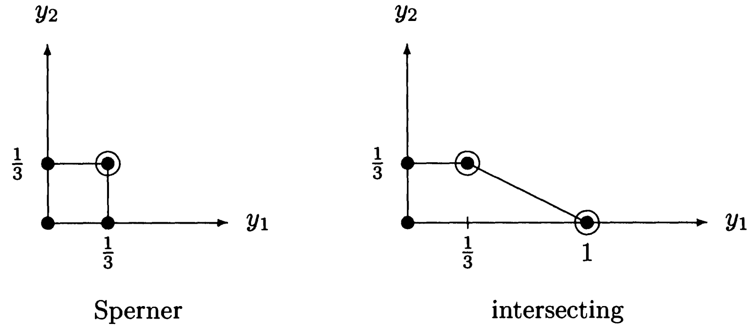


Figure 3.2

We call an extreme point $\boldsymbol{c}$ of a polytope $\mathcal{P}$ *essential* if there is no other point $\boldsymbol{d}$ of $\mathcal{P}$ with $\boldsymbol{c} \le \boldsymbol{d}$.

Theorem 3.1.3. *Let A be a nonnegative matrix with no zero column and define $\mathcal{P} := \{\boldsymbol{x} \in \mathbb{R}^n_+ : A\boldsymbol{x} \le \mathbf{1}\}$.*

(*a*) *If $\boldsymbol{c}$ is an extreme point of $\mathcal{P}$, then there exists some essential extreme point $\boldsymbol{d}$ of $\mathcal{P}$ such that, for all j, $c_j > 0$ implies $c_j = d_j$.*

(*b*) *Besides $\boldsymbol{y} \ge \mathbf{0}$ the facet-defining inequalities for $\mathcal{P}^c$ are given by*

$$\boldsymbol{d}_i^T \boldsymbol{y} \le 1, \quad i = 1, \dots, s,$$

where $\boldsymbol{d}_1, \dots, \boldsymbol{d}_s$ are the essential extreme points of $\mathcal{P}$ (these are called the essential facet-defining inequalities for $\mathcal{P}^c$).

Proof. (a) Let $\boldsymbol{c}$ be an extreme point of $\mathcal{P}$. Without loss of generality, we assume that $c_{t+1} = \cdots = c_n = 0$, but $c_1, \dots, c_t > 0$. Let $\boldsymbol{c}' := (c_1, \dots, c_t)^{\mathrm{T}}$ and $\boldsymbol{c}'' := (c_{t+1}, \dots, c_n)^{\mathrm{T}}$. In $A\boldsymbol{x} \le \mathbf{1}$ there must be t inequalities that are satisfied by $\boldsymbol{c}$ as equalities, and these equalities together with $x_{t+1} = \cdots = x_n = 0$ must be independent. Consequently we can divide A into the form $A = (A'|A'')$ and, more precisely,

$$A = \begin{pmatrix} A_1' & A_1'' \\ A_2' & A_2'' \end{pmatrix}$$

where $A_1'c' = \mathbf{1}$, rank $A_1' = t$, $A_2'c' < \mathbf{1}$. Since A_1' does not contain a zero column (otherwise we would have rank $A_1' < t$), we cannot increase in $\boldsymbol{c}$ one of the coordinates $j = 1, \ldots, t$ without changing the others. Now consider the coordinates $j = t+1, \ldots, n$.

Case 1. The jth column of A_1'' is not a zero column. Then we cannot increase coordinate j without changing the others.

Case 2. The jth column of A_1'' is the zero column, for example, for $j = t+1$. Then we can increase the $(t+1)$th coordinate until a strict inequality becomes a new equality (note that the $(t+1)$th column of A_2'' cannot be the zero column). The resulting point $\boldsymbol{d}$ is an extreme point since

$$\operatorname{rank}\begin{pmatrix} A_1' & \mathbf{0} \\ * \ldots * & \neq 0 \end{pmatrix} = t+1,$$

and we have, together with $x_{t+2} = \cdots = x_n = 0$, n linear independent equalities. So we have seen: Either there is no $\boldsymbol{d} \in \mathcal{P}$ with $\boldsymbol{d} \geq \boldsymbol{c}$, $\boldsymbol{d} \neq \boldsymbol{c}$ (and thus $\boldsymbol{c}$ is an essential extreme point) or there is some extreme point $\boldsymbol{d}$ with the property that for all j, $c_j > 0$ implies $c_j = d_j$ and $\boldsymbol{d}$ has one more nonzero coordinate than $\boldsymbol{c}$. Now we can argue for $\boldsymbol{d}$ in the same way and since there are at most n nonzero coordinates we find after a finite number of steps the desired essential extreme point.

(b) The inequalities $\boldsymbol{y} \geq \mathbf{0}$ are facet defining by a similar reason as in Lemma 3.1.1(a). Let $\boldsymbol{c}$ be an extreme point of $\mathcal{P}$ that is not essential and let $\boldsymbol{d}$ be an essential extreme point with the properties of (a). Suppose, w.l.o.g., that $c_1 = 0$ but $d_1 > 0$ and assume that $F := \{\boldsymbol{y} \in \mathcal{P}^c : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} = 1\}$ is a facet. Then there must be some $\boldsymbol{y}^* \in F$ with $y_1^* > 0$, since otherwise we would have two independent equalities $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} = 1$, $y_1 = 0$ for F, and thus F would be at most $n-2$ dimensional. But obviously $\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{d} = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{y}^* > \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}^* = 1$, a contradiction to $\boldsymbol{y}^* \in \mathcal{P}^c$. Now let $\boldsymbol{c}$ be an essential extreme point and assume that $F := \{\boldsymbol{y} \in \mathcal{P}^c : \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y} = 1\}$ is not a facet. We shall use the notations before Theorem 3.1.2 and suppose that $\boldsymbol{c} = \boldsymbol{c}_r$. Let $\mathcal{P}'$ be the convex hull of $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{r-1}$ and let C' be obtained from C by deleting the last row. Of course, $\mathcal{P}' \subsetneq \mathcal{P}$. But if F is not a facet, then $\mathcal{P}^c = \{\boldsymbol{y} \in \mathbb{R}_+^n : C'\boldsymbol{y} \leq \mathbf{1}\}$ and by Theorem 3.1.2(a), $\mathcal{P}'^c = \mathcal{P}^c$. In view of Theorem 3.1.2(b), $\mathcal{P} = (\mathcal{P}^c)^c = (\mathcal{P}'^c)^c = \mathcal{P}'$, a contradiction. ■

In form of Theorems 3.1.2 and 3.1.3 the facet-defining inequalities for $\mathcal{P}$ are given by the essential extreme points of $\mathcal{P}^c$. So our goal in this section is the following: Given a full hereditary set μ of points in $\mathbb{N}^n$, find the essential extreme points of $\operatorname{conv}(\mu)$ and $(\operatorname{conv}(\mu))^c$. Suppose we are given candidate lists $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s\}$ and $\{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_\sigma\}$ for these extreme points. How can we prove that these lists are correct and complete? Let μ' be the hereditary set in $\mathbb{N}^n$ (resp. $\mathbb{R}_+^n$) determined by $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s$; that is, let μ' be the ideal generated by $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s$, and let $\mathcal{P} := \operatorname{conv}(\mu')$. All our lists will have the property that μ' is full; thus by Theorem 3.1.1, $\mathcal{P}$ is of antiblocking type. Note that, by the definition of $\mathcal{P}$, its essential extreme points belong to $\{\boldsymbol{c}_1, \dots, \boldsymbol{c}_s\}$. Moreover, we define $\mathcal{Q}$ analogously, using $\gamma_1, \dots, \gamma_\sigma$. Our method of proof is as follows:

Step 1. Verify that $\boldsymbol{c}_i \in \mu$ for all i. This yields $\mathcal{P} \subseteq \operatorname{conv}(\mu)$.

Step 2. Verify that $\gamma_j^{\mathrm{T}}\boldsymbol{x} \leq 1$ for all j and for all $\boldsymbol{x} \in \mu$. This yields $\mu \subseteq \mathcal{Q}^c$; that is, $\operatorname{conv}(\mu) \subseteq \mathcal{Q}^c$.

Step 3. Show that $\mathcal{Q}^c \subseteq \mathcal{P}$ or $\mathcal{Q} \supseteq \mathcal{P}^c$. This yields (using Theorem 3.1.2(b)) that $\mathcal{P} \subseteq \operatorname{conv}(\mu) \subseteq \mathcal{Q}^c \subseteq \mathcal{P}$; that is, $\mathcal{P} = \operatorname{conv}(\mu)$ and $\mathcal{Q} = (\operatorname{conv}(\mu))^c$. Either we can show that for every element $\boldsymbol{x}$ of $\mathcal{Q}^c$ there exists a convex combination of $\boldsymbol{c}_1, \dots, \boldsymbol{c}_s$ which majorizes $\boldsymbol{x}$ (it suffices, of course, to test the maximal elements $\boldsymbol{x}$ of $\mathcal{Q}^c$) or we can determine algebraically the essential extreme points of $\mathcal{P}^c$ by considering the corresponding system of inequalities. We are done if these essential extreme points are contained in $\{\gamma_1, \dots, \gamma_\sigma\}$.

Step 4. Show that the candidate points are in fact essential extreme points. The test of being essential will be in our cases trivial. To see that the points from the list are extreme points we need only verify that they cannot be majorized by a nontrivial convex combination of other points from the same list, and this will also be easy. Of course, we could also verify that the points satisfy n (resp. for the reduced profiles of set families $n-1$) independent equalities in the system of linear inequalities defining $\mathcal{Q}^c$ (resp. $\mathcal{P}^c$).

Now let us come back to our set families. We denote the set of essential extreme points of $\operatorname{conv}(\mu(\mathfrak{A}))$ (resp. of $\operatorname{conv}(\mu(\mathfrak{A}))^c$) by $\epsilon^*(\mu(\mathfrak{A}))$ (resp. $\varphi^*(\mu(\mathfrak{A}))$). Note that our objective function (3.1) satisfies by (3.2) and Theorem 3.1.3(a),

$$M(\mathfrak{A}, \boldsymbol{c}) = \max\left\{\sum_{i=1}^{n-1} \max\{c_i, 0\} f_i \,:\, \boldsymbol{f} \in \epsilon^*(\mu(\mathfrak{A}))\right\}.$$

The determination of ϵ^* and φ^* is still too difficult. But we can reduce the problem.

3.2. Reduction to the circle

We start with a general fact for polytopes of antiblocking type. For a matrix D and a polytope $\mathcal{P}$, let $D\mathcal{P} := \{D\boldsymbol{x} : \boldsymbol{x} \in \mathcal{P}\}$.

Theorem 3.2.1. *Let $\mathcal{P} := \{\boldsymbol{x} \in \mathbb{R}_+^n : A\boldsymbol{x} \leq \mathbf{1}\}$ and $\mathcal{P}' := \{\boldsymbol{x} \in \mathbb{R}_+^n : A'\boldsymbol{x} \leq \mathbf{1}\}$, where A and A' are nonnegative matrices with no zero columns. Further let $D := \operatorname{diag}(d_1, \dots, d_n)$ be a diagonal matrix with positive diagonal elements. Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_s$ be the essential extreme points of $\mathcal{P}'$. If $\mathcal{P} \subseteq D\mathcal{P}'$ and $D\boldsymbol{c}_i \in \mathcal{P}$ for all i then*

(a) $\mathcal{P} = D\mathcal{P}'$ and $D\boldsymbol{c}_1, \dots, D\boldsymbol{c}_s$ are the essential extreme points of $\mathcal{P}$.

(b) $\mathcal{P}^c = D^{-1}\mathcal{P}'^c$.

Proof. (a) Let $\boldsymbol{x} = D\boldsymbol{x}'$ with $\boldsymbol{x}' \in \mathcal{P}'$. We have to show that $\boldsymbol{x} \in \mathcal{P}$. Since $\boldsymbol{x}'$ can be written as a convex combination of extreme points of $\mathcal{P}'$ we have by Theorem 3.1.3(a):

$$\boldsymbol{x}' \leq \sum_{i=1}^{s} \alpha_i \boldsymbol{c}_i \text{ with } \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{s} \alpha_i = 1.$$

Then $\boldsymbol{x} = D\boldsymbol{x}' \leq \sum_{i=1}^{s} \alpha_i D\boldsymbol{c}_i$. Since $\mathcal{P}$ is convex and in view of the supposition, the sum on the RHS belongs to $\mathcal{P}$, but then obviously $\boldsymbol{x}$ belongs to $\mathcal{P}$. The point $D\boldsymbol{c}_1$ is an extreme point of $\mathcal{P}$ since otherwise it could be written as a nontrivial convex combination $D\boldsymbol{c}_1 = \lambda\boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2$ with $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{P}$. With $\boldsymbol{x}_i = D\boldsymbol{x}'_i$, $\boldsymbol{x}'_i \in \mathcal{P}'$, $i = 1, 2$, we obtain $\boldsymbol{c}_1 = \lambda\boldsymbol{x}'_1 + (1-\lambda)\boldsymbol{x}'_2$, a contradiction. Moreover, $D\boldsymbol{c}_1$ is essential by an analogous argument.

(b) We have

$$\begin{aligned}
\mathcal{P}^c &= \{\boldsymbol{y} \in \mathbb{R}^n_+ : \boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} \leq 1 \text{ for all } \boldsymbol{x} \in \mathcal{P}\} \\
&= \{\boldsymbol{y} \in \mathbb{R}^n_+ : \boldsymbol{y}^{\mathrm{T}} D\boldsymbol{x}' \leq 1 \text{ for all } \boldsymbol{x}' \in \mathcal{P}'\} \\
&= \{D^{-1}\boldsymbol{z} \in \mathbb{R}^n_+ : \boldsymbol{z}^{\mathrm{T}}\boldsymbol{x}' \leq 1 \text{ for all } \boldsymbol{x}' \in \mathcal{P}'\} = D^{-1}\mathcal{P}'^c.
\end{aligned}$$

■

We use Katona's circle method [295] together with this theorem to solve a much easier problem for set families. Let ζ be a cyclic permutation of $[n]$. A subset X of $[n]$ is called *consecutive in* ζ if its elements are consecutive on the circle; see Figure 3.3. If $\mathfrak{A}$ is a class of families of subsets of $[n]$, let $\mathfrak{A}(\zeta)$ be the class of

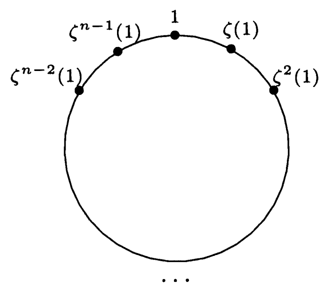


Figure 3.3

those families from $\mathfrak{A}$ whose elements are consecutive in ζ and let $\mu(\mathfrak{A}(\zeta)) := \{p(\mathcal{F}) : \mathcal{F} \in \mathfrak{A}(\zeta)\}$. For a permutation π of $[n]$, a subset X of $[n]$, a family $\mathcal{F}$ of subsets of $[n]$ and a class $\mathfrak{A}$ of such families we put $\pi(X) := \{\pi(i) : i \in X\}$, $\pi(\mathcal{F}) := \{\pi(X) : X \in \mathcal{F}\}$, and $\pi(\mathfrak{A}) := \{\pi(\mathcal{F}) : \mathcal{F} \in \mathfrak{A}\}$. The class $\mathfrak{A}$ is said to be *permutation invariant* if $\pi(\mathfrak{A}) = \mathfrak{A}$ for every permutation π of $[n]$.

Lemma 3.2.1. *Let $\mathfrak{A}$ be permutation invariant and let ζ_1, ζ_2 be two cyclic permutations. Then $\mu(\mathfrak{A}(\zeta_1)) = \mu(\mathfrak{A}(\zeta_2))$.*

Proof. Let π be the permutation that sends $\zeta_1^i(1)$ onto $\zeta_2^i(1)$, $i = 1, \dots, n$. Then $\mathcal{F} \in \mathfrak{A}(\zeta_1)$ iff $\pi(\mathcal{F}) \in \mathfrak{A}(\zeta_2)$. ∎

In the following we consider only permutation-invariant classes, so we restrict ourselves to the permutation

$$C := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & 1 \end{pmatrix}.$$

As for $\mathfrak{A}$, we can introduce the sets $\operatorname{conv}(\mu(\mathfrak{A}(C)))$, $\epsilon(\mu(\mathfrak{A}(C)))$, $\epsilon^*(\mu(\mathfrak{A}(C)))$. If $\mathfrak{A}$ is hereditary then also $\mathfrak{A}(C)$ is hereditary. If $\mathfrak{A}$ is full, then for every $X \subseteq [n]$ that is consecutive in C, the family $\{X\}$ belongs to $\mathfrak{A}(C)$. Thus under these two suppositions, $\operatorname{conv}(\mu(\mathfrak{A}(C)))$ also has the form of Theorem 3.1.1, and moreover there exists the antiblocker $(\operatorname{conv}(\mu(\mathfrak{A}(C))))^c$ and the set $\varphi^*(\mu(\mathfrak{A}(C)))$ of its essential extreme points.

Theorem 3.2.2. *Let $\mathfrak{A}$ be a full hereditary permutation-invariant class and let $D := \frac{1}{n}\operatorname{diag}(\binom{n}{1}, \dots, \binom{n}{n-1})$. Then $\operatorname{conv}(\mu(\mathfrak{A})) \subseteq D \operatorname{conv}(\mu(\mathfrak{A}(C)))$.*

Proof. Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_r$ be the extreme points of $\operatorname{conv}(\mu(\mathfrak{A}(C)))$. Let $\mathcal{F} \in \mathfrak{A}$ and let $\boldsymbol{f}$ be its profile. We have to show that $\boldsymbol{f}$ can be written as a convex combination of $D\boldsymbol{c}_1, \dots, D\boldsymbol{c}_r$. For any cyclic permutation ζ, let $\mathcal{F}(\zeta) := \{X \in \mathcal{F} : X \text{ is consecutive in } \zeta\}$ and let $\boldsymbol{f}(\zeta)$ be its profile. Since $\boldsymbol{f}(\zeta)$ belongs to $\mu(\mathfrak{A}(C))$, we can write $\boldsymbol{f}(\zeta)$ in the form

$$\boldsymbol{f}(\zeta) = \sum_{i=1}^{r} \alpha_i(\zeta)\boldsymbol{c}_i \quad \text{with } \alpha_i(\zeta) \geq 0 \text{ for all i, } \sum_{i=1}^{r} \alpha_i(\zeta) = 1.$$

With each $X \in \mathcal{F}$ we associate the profile $\boldsymbol{f}_X$ of the family $\{X\}$, which is a unit vector with 1 at coordinate $|X|$. Finally, let

$$E := \{(X, \zeta) : X \in \mathcal{F},\ \zeta \text{ is a cyclic permutation},\ X \text{ is consecutive in } \zeta\}.$$

Obviously,

$$\boldsymbol{f} = \sum_{X \in \mathcal{F}} \boldsymbol{f}_X.$$

For fixed ζ,

$$\boldsymbol{f}(\zeta) = \sum_{X:(X,\zeta)\in E} \boldsymbol{f}_X.$$

For fixed X, $X \neq \emptyset$, $X \neq [n]$, there exist $|X|!(n - |X|)! = (n - 1)!\frac{n}{\binom{n}{|X|}}$ cyclic permutations ζ with the property that X is consecutive in ζ. Hence

$$\frac{1}{(n-1)!} D \sum_{\zeta:(X,\zeta)\in E} f_X = f_X.$$

Summarizing we obtain

$$\begin{aligned}\frac{1}{(n-1)!}\sum_{\zeta} Df(\zeta) &= \frac{1}{(n-1)!}\sum_{\zeta}\sum_{X:(X,\zeta)\in E} Df_X \\ &= \sum_{X\in\mathcal{F}} \frac{1}{(n-1)!}\sum_{\zeta:(X,\zeta)\in E} Df_X = \sum_{X\in\mathcal{F}} f_X = f.\end{aligned}$$

So we have

$$f = \frac{1}{(n-1)!}\sum_{\zeta}\sum_{i=1}^{r}\alpha_i(\zeta)Dc_i = \sum_{i=1}^{r}\left(\frac{1}{(n-1)!}\sum_{\zeta}\alpha_i(\zeta)\right)Dc_i,$$

which is obviously a convex combination of the vectors Dc_i since there are in total $(n-1)!$ cyclic permutations. ∎

3.3. Classes of families arising from Boolean expressions

We say that a two-element family $\{X, Y\}$ of subsets of $[n]$ has the property

S	iff	$X \not\subseteq Y$ and $Y \not\subseteq X$	(Sperner),
I	iff	$X \cap Y \neq \emptyset$	(Intersecting),
C	iff	$X \cup Y \neq [n]$	(Cointersecting).

We speak of property $\overline{S}, \overline{I}, \overline{C}$ if $\{X, Y\}$ does not have property S, I, C, respectively. Let $B = B(S, I, C)$ be a Boolean expression of S, I, C; for example, $B = (S \wedge I) \vee \overline{C}$ (we write briefly $B = SI \vee \overline{C}$). A family $\mathcal{F}$ of subsets of $[n]$ is said to be a *B-family* if each two-element subfamily of $\mathcal{F}$ satisfies the Boolean expression B. For instance, $\mathcal{F}$ is an $(SI \vee \overline{C})$-family if for all different X, Y of $\mathcal{F}$, X, Y are not comparable with respect to inclusion and have a nonempty intersection or their union is the whole set. Note that the class $\mathfrak{A}_B$ of B-families is full, hereditary, and permutation invariant for every Boolean expression B. Determining the maximum size of B-families, Gronau [246] developed a reduction technique that allows us to consider only a small number of Boolean expressions instead of all 2^{2^3} Boolean expressions. This reduction method was generalized by Derbala and me [134] in connection with the study of profile-polytopes. I will not address these technical questions here. Instead we will study only those Boolean expressions that are most interesting from the point of view of profile-polytopes. Interesting, but unsolved, are the questions for $B = IC$ (cf. [199]) and $B = IC \vee \overline{I}\,\overline{C}$ (cf. [246]), so we will not discuss these expressions. The following theorem consists of a list containing the sets $\epsilon^*(\mu(\mathfrak{A}_B))$ and $\varphi^*(\mu(\mathfrak{A}_B))$ and the maximum size of B-families for eight Boolean expressions B. From the preceding section we know that this list gives deep informations for B-families. Certainly the complete proof will be tiring for the reader. Skipping the "long" cases at first reading is recommended. Later one can look, if necessary, at the other cases. We use the following notations for vectors and families of i-element subsets of $[n]$:

$$\boldsymbol{e}_i := (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathrm{T}}, \text{ where 1 is at coordinate } i,\ i = 1, \ldots, n-1,$$

$$\boldsymbol{u}_i := \binom{n}{i}\boldsymbol{e}_i \qquad \boldsymbol{v}_i := \binom{n-1}{i-1}\boldsymbol{e}_i \qquad \boldsymbol{w}_i := \binom{n-1}{i}\boldsymbol{e}_i$$

$$\boldsymbol{\mu}_i := \frac{1}{\binom{n}{i}}\boldsymbol{e}_i \qquad \boldsymbol{\nu}_i := \frac{1}{\binom{n-1}{i-1}}\boldsymbol{e}_i \qquad \boldsymbol{\omega}_i := \frac{1}{\binom{n-1}{i}}\boldsymbol{e}_i,$$

$$U_i := \binom{[n]}{i} \qquad V_i := \{X \in \binom{[n]}{i} : 1 \in X\} \qquad W_i := \{X \in \binom{[n]}{i} : 1 \notin X\}.$$

Clearly, $\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{w}_i$ are the profiles of U_i, V_i, W_i. Note that $\boldsymbol{v}_i + \boldsymbol{w}_i = \boldsymbol{u}_i, i = 1, \ldots, n-1$.

Theorem 3.3.1 (Profile-Polytope Theorem). *Let $\mathfrak{A}_B$ be the class of B-families. Then we have the following essential extreme points and essential facet-defining inequalities (given by the vector of coefficients) of the convex hull of the profiles of families from $\mathfrak{A}_B$; see Table 3.1.*

Here (and in the following proof) all vectors with index $\frac{n}{2}$ (resp. $\frac{n+1}{2}$) must be deleted if n is odd (resp. even). Sums in which the lower bound is greater than the upper bound are defined to be zero.

Before we start with the proof let us make several remarks.

(1) Throughout we may suppose that the B-families do not contain $\emptyset$ and $[n]$ as members since we are only considering reduced profiles (without f_0 and f_n).

(2) The difference between an I- and an $I \vee \overline{C}$-family is that pairs of complementary sets are allowed in the second one. The same is true for SI and $SI \vee \overline{C}$-families, which are both Sperner families.

(3) SIC-families are also called families of *qualitatively independent* sets. Here, two subsets X and Y of $[n]$ are said to be qualitatively independent if they divide $[n]$ into four nonempty parts $(X-Y,\ Y-X,\ X \cap Y,\ [n]-(X \cup Y))$. (It is easy to see that a family with the property that any two members are qualitatively independent is an SIC-family and vice versa.)

(4) It is easy to see that one can add to each member X of an $SIC \vee \overline{I}\,\overline{C}$-family its complement $\overline{X} := [n]-X$ such that the new family is still an $SIC \vee \overline{I}\,\overline{C}$-family. A *self-complementary Sperner family* is a Sperner family $\mathcal{F}$ with the property that $X \in \mathcal{F}$ implies $\overline{X} \in \mathcal{F}$. Obviously, $\mathcal{F}$ is a maximal self-complementary Sperner family iff it is a maximal $SIC \vee \overline{I}\,\overline{C}$-family.

(5) A *complement-free* Sperner family is a Sperner family $\mathcal{F}$ with the property that $X \in \mathcal{F}$ implies $\overline{X} \notin \mathcal{F}$. The $S(I \vee C)$-families are exactly the complement-free Sperner families.

Table 3.1

$\boxed{\mathrm{B}}$ $\max\lvert\mathcal{F}\rvert$	Essential extreme points $\epsilon^*(\mu(\mathfrak{A}_B))$	Essential facets $\varphi^*(\mu(\mathfrak{A}_B))$
$\boxed{S}$ $\binom{n}{\lfloor\frac{n}{2}\rfloor}$	$\{\mathbf{u}_i : 1 \le i \le n-1\}$	$\left\{\sum_{i=1}^{n-1} \boldsymbol{\mu}_i\right\}$
$\boxed{I}$ 2^{n-1}	$\left\{\sum_{j=i}^{n-i} \mathbf{v}_j + \sum_{j=n-i+1}^{n-1} \mathbf{u}_j : 1 \le i \le \frac{n+1}{2}\right\}$	$\left\{\frac{n}{i}\boldsymbol{\mu}_i : 1 \le i \le \frac{n}{2}\right\} \cup \left\{\frac{n-j}{n}(\boldsymbol{\nu}_i + \boldsymbol{\omega}_j) : j > \frac{n}{2}, i + j \le n\right\}$
$\boxed{I \vee \overline{C}}$ $2^{n-1} + \binom{n-1}{\lfloor\frac{n-2}{2}\rfloor}$	$\left\{\sum_{j=i}^{n-i-1} \mathbf{v}_j + \sum_{j=n-i}^{n-1} \mathbf{u}_j : 1 \le i \le \frac{n}{2}\right\}$	$\left\{\frac{n}{i}\boldsymbol{\mu}_i : 1 \le i < \frac{n}{2}\right\} \cup \left\{\frac{n-j}{n}(\boldsymbol{\nu}_i + \boldsymbol{\omega}_j) : j \ge \frac{n}{2}, i + j < n\right\} \cup \left\{\frac{1}{n}\boldsymbol{\omega}_{n-1}\right\}$
$\boxed{SIC}$ $\binom{n-1}{\lfloor\frac{n-2}{2}\rfloor}$	$\left\{\mathbf{v}_i : 1 \le i \le \frac{n}{2}\right\} \cup \left\{\mathbf{w}_j : \frac{n}{2} < j \le n-1\right\}$	$\left\{\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} \boldsymbol{\nu}_i + \sum_{j=\lfloor\frac{n}{2}\rfloor+1}^{n-1} \boldsymbol{\omega}_j\right\}$
$\boxed{SIC \vee \overline{I}\,\overline{C}}$ $2\binom{n-1}{\lfloor\frac{n-2}{2}\rfloor}$	$\left\{\mathbf{v}_i + \mathbf{w}_{n-i} : 1 \le i < \frac{n}{2}\right\} \cup \left\{\mathbf{u}_{\frac{n}{2}}\right\}$	$\left\{\sum_{i\in T} \boldsymbol{\nu}_i + \sum_{\frac{n}{2}<j\le n-1 : n-j\notin T} \boldsymbol{\omega}_j + \boldsymbol{\mu}_{\frac{n}{2}} : T \subseteq \left\{1, \dots, \lfloor\frac{n-1}{2}\rfloor\right\}\right\}$
$\boxed{S(I \vee C)}$ $\binom{n}{\lfloor\frac{n-1}{2}\rfloor}$	$\{\mathbf{u}_i : 1 \le i \le n-1\}$ if n is odd, $\left\{\mathbf{u}_i : 1 \le i \le n-1, i \ne \frac{n}{2}\right\} \cup \left\{\mathbf{v}_i + \mathbf{w}_j : 1 \le i < \frac{n}{2}, j = \frac{n}{2} \text{ or } i = \frac{n}{2}, \frac{n}{2} < j \le n-1\right\}$ if n is even	$\left\{\sum_{i=1}^{n-1} \boldsymbol{\mu}_i\right\}$ if n is odd, $\left\{\sum_{j=1}^{i}(\boldsymbol{\mu}_j + \boldsymbol{\mu}_{n-j}) + (1 - \frac{i}{n})\boldsymbol{\nu}_{\frac{n}{2}} + \frac{i}{n}\sum_{j=i+1}^{\frac{n}{2}-1}(\boldsymbol{\nu}_j + \boldsymbol{\omega}_{n-j}) : 1 \le i < \frac{n}{2}\right\} \cup \left\{\boldsymbol{\nu}_{\frac{n}{2}}\right\}$ if n is even

Table 3.1 (*cont.*)

$\boxed{SI}$ $\binom{n}{\lfloor\frac{n-1}{2}\rfloor}$	$\{u_i : \frac{n}{2} < i \le n-1\} \cup \{v_i + w_j : 1 \le i \le \frac{n}{2}, i+j > n\} \cup \{v_1\}$	$\left\{\sum_{i=1}^{i_1-1} \nu_i + \sum_{k=1}^{r}(1-\frac{i_k-1}{n}) * \sum_{i=i_k}^{i_{k+1}-1} \nu_i + \frac{i_k-1}{n} * \sum_{j=n-i_{k+1}+2}^{n-i_k+1} \omega_j + \mu_{\frac{n+1}{2}} : 1 < i_1 < \cdots < i_{r+1} = \lceil\frac{n+1}{2}\rceil\right\}$
$\boxed{SI \vee \overline{C}}$ $\binom{n}{\lfloor\frac{n}{2}\rfloor}$	$\{u_i : \frac{n}{2} \le i \le n-1\} \cup \{v_i + w_j : 1 \le i < \frac{n}{2}, i+j \ge n\}$	$\left\{\sum_{i=1}^{i_1-1} \nu_i + \sum_{k=1}^{r}(1-\frac{i_k}{n}) * \sum_{i=i_k}^{i_{k+1}-1} \nu_i + \frac{i_k}{n} * \sum_{j=n-i_{k+1}+1}^{n-i_k} \omega_j + \mu_{\frac{n}{2}} : 1 \le i_1 < \cdots < i_{r+1} = \lceil\frac{n}{2}\rceil\right\}$

(6) For $B = I$ and $B = I \vee \overline{C}$, we could have written in the third column ν_i instead of $\frac{n}{i}\mu_i$. However, we prefer our notation because of the nice analogy with $\epsilon^*(\mu(\mathfrak{A}_{SI\vee\overline{C}}))$ (resp. $\varphi^*(\mu(\mathfrak{A}_{SI})))$.

(7) If $\overline{B}$ is the Boolean expression that can be obtained from B by interchanging the letters C and I, then obviously $\mathcal{F}$ is a B-family iff $\overline{\mathcal{F}} := \{\overline{X} : X \in \mathcal{F}\}$ is a $\overline{B}$-family since $X \cap Y \neq \emptyset$ iff $\overline{X} \cup \overline{Y} \neq [n]$. Thus the extreme points for $\overline{B}$-families can be obtained by reversing the extreme points for B-families; that is, $(x_1, x_2, \ldots, x_{n-1})^{\mathrm{T}} \to (x_{n-1}, \ldots, x_2, x_1)^{\mathrm{T}}$ (note that $u_i \to u_{n-i}$, $v_i \to w_{n-i}$, $w_i \to v_{n-i}$, $\mu_i \to \mu_{n-i}$, $\nu_i \to \omega_{n-i}$, $\omega_i \to \nu_{n-i}$).

(8) Since many mathematicians contributed to this theorem, it cannot be attributed to any author in particular. Some historical remarks follow the proof.

Proof. The maximum sizes in the left column of the table follow from a calculation of $\max\{\mathbf{1}^{\mathrm{T}}\boldsymbol{c} : \boldsymbol{c} \in \epsilon^*(\mu(\mathfrak{A}_B))\}$, using the symmetry and unimodality of the binomial coefficients (by this maximization we find only maximum B-families that do not contain $\emptyset$ and $[n]$ since we consider only reduced profiles; only in the cases I and $I \vee \overline{C}$ we have to add $[n]$ to the families in order to obtain the real maximum). As an example we show this for $B = I \vee \overline{C}$. Let $\boldsymbol{c}_i$ be the ith vector in the set $\epsilon^*(\mu(\mathfrak{A}))$. Then we have $\boldsymbol{c}_{i+1} - \boldsymbol{c}_i = -\boldsymbol{v}_i + \boldsymbol{w}_{n-i-1}$; that is, $\mathbf{1}^{\mathrm{T}}(\boldsymbol{c}_{i+1} - \boldsymbol{c}_i) = -\binom{n-1}{i-1} + \binom{n-1}{i} \geq 0$ iff $1 \leq i \leq \frac{n}{2}$. Hence

$$\max\{\mathbf{1}^{\mathrm{T}}\boldsymbol{c} : \boldsymbol{c} \in \epsilon^*(\mu(\mathfrak{A}))\} = \mathbf{1}^{\mathrm{T}}\boldsymbol{c}_{\lfloor \frac{n}{2} \rfloor} = 2^{n-1} + \binom{n-1}{\lfloor \frac{n-2}{2} \rfloor} - 1$$

and the addition of 1 (corresponding to $[n]$) gives the maximum.

Let us note here, that the vectors given in the second column of the list are profiles of B-families. Indeed, all these vectors have the form

$$\boldsymbol{c} = \sum_{i \in J_1} \boldsymbol{u}_i + \sum_{i \in J_2} \boldsymbol{v}_i + \sum_{i \in J_3} \boldsymbol{w}_i,$$

and it is easy to see that, in each case, $\boldsymbol{c}$ is the profile of the B-family

$$\left(\bigcup_{i \in J_1} U_i\right) \cup \left(\bigcup_{i \in J_2} V_i\right) \cup \left(\bigcup_{i \in J_3} W_i\right).$$

To proceed with the proof, we introduce new vectors

$$\boldsymbol{u}'_i := n\boldsymbol{e}_i \qquad \boldsymbol{v}'_i := i\boldsymbol{e}_i \qquad \boldsymbol{w}'_i := (n-i)\boldsymbol{e}_i$$

$$\boldsymbol{\mu}'_i := \tfrac{1}{n}\boldsymbol{e}_i \qquad \boldsymbol{\nu}'_i := \tfrac{1}{i}\boldsymbol{e}_i \qquad \boldsymbol{\omega}'_i := \tfrac{1}{n-i}\boldsymbol{e}_i.$$

Obviously, $\boldsymbol{u}_i = D\boldsymbol{u}'_i, \boldsymbol{v}_i = D\boldsymbol{v}'_i, \boldsymbol{w}_i = D\boldsymbol{w}'_i, \boldsymbol{\mu}_i = D^{-1}\boldsymbol{\mu}'_i, \boldsymbol{\nu}_i = D^{-1}\boldsymbol{\nu}'_i, \boldsymbol{\omega}_i = D^{-1}\boldsymbol{\omega}'_i, i = 1, \ldots, n-1$, where $D := \frac{1}{n}\operatorname{diag}(\binom{n}{1}, \ldots, \binom{n}{n-1})$. Let us replace in the list of the theorem every vector by its "dashed" vector, for example, $\boldsymbol{u}_i \to \boldsymbol{u}'_i$. If we can show that the "*dashed*" *list* is the list for the classes $\mathfrak{A}_B(\mathcal{C})$, then we are done by Theorem 3.2.2, Theorem 3.2.1, and the fact that the undashed vectors belong to $\mu(\mathfrak{A}_B)$, that is, also to $\operatorname{conv}(\mu(\mathfrak{A}_B))$ – see the remarks earlier. So let us restrict ourselves to the classes $\mathfrak{A}_B(\mathcal{C})$ and read the list always as a dashed list. For $k, l \in [n]$, let

$$[k, l]_{\bmod n} := \begin{cases} \{k, k+1, \ldots, l\} & \text{if } k \leq l, \\ \{k, k+1, \ldots, n, 1, \ldots, l\} & \text{if } k > l. \end{cases}$$

Thus $[k, l]_{\bmod n}$ is consecutive in $\mathcal{C}$. For the sake of brevity, we will omit below the index $\bmod n$ since this will be clear from the context. We say that k is the *starting point* and l is the *endpoint* of $[k, l]_{\bmod n}$. Conversely, every consecutive set different from $[n]$ has a unique representation $[k, l]_{\bmod n}$.

For the proof, we have to verify that the dashed list is correct and complete. We are doing this by the method described after Theorem 3.1.3. We use also the notations $\mathcal{P}$ and $\mathcal{Q}$. Step 1 is easy to verify since we can argue as before for the undashed vectors. We have only to use the sets U'_i, V'_i, W'_i, which are defined the same as U_i, V_i, W_i but with the additional condition that the elements are consecutive in $\mathcal{C}$. Step 4 is easy, too, and therefore omitted here. We still have to verify Steps 2 and 3. As a warm-up, we start with the easiest case.

Case $B = S$.

Step 2. We have to show that $\sum_{i=1}^{n-1} f_i/n \leq 1$ for every $\mathcal{F} \in \mathfrak{A}_S(\mathcal{C})$, that is, for every Sperner family of being subsets of $[n]$ consecutive in $\mathcal{C}$. This inequality is true since no point can be a starting point more than once.

Step 3. We have $\boldsymbol{f} \in \mathcal{Q}^c$ iff $\boldsymbol{f} \geq \boldsymbol{0}$ and $\sum_{i=1}^{n-1} f_i/n \leq 1$. The convex combination $\sum_{i=1}^{n-2}(f_i/n)\boldsymbol{u}_i' + \left(1 - \sum_{i=1}^{n-2} f_i/n\right)\boldsymbol{u}_{n-1}'$ majorizes $\boldsymbol{f}$. This proves the case.

Before we consider the other cases, we will prove two lemmas. Let $N_i(\mathcal{C})$ be the family of all i-element subsets of $[n]$ that are in $\mathcal{C}$ consecutive. For $\mathcal{F} \subseteq N_i(\mathcal{C})$, let

$$\nabla_{\mathcal{C}}(\mathcal{F}) := \{Y \in N_{i+1}(\mathcal{C}) : Y \supseteq X \text{ for some } X \in \mathcal{F}\},$$

and let $\Delta_{\mathcal{C}}$ be defined analogously. We say that a set B of starting points of members of $\mathcal{F}$ forms a *block* if $B = [k, l]$ for some k and l but $k - 1 \pmod n$ and $l + 1 \pmod n$ are not starting points of any member of $\mathcal{F}$.

Lemma 3.3.1. *Let $\mathcal{F} \subseteq N_i(\mathcal{C})$, $i = 1, \ldots, n-1$, and suppose that the set of starting points of members of $\mathcal{F}$ consists of m blocks. Then*

$$|\nabla_{\mathcal{C}}(\mathcal{F})| = |\mathcal{F}| + m \text{ if } i \leq n-2,$$

$$|\Delta_{\mathcal{C}}(\mathcal{F})| = |\mathcal{F}| + m \text{ if } i \geq 2.$$

Proof. Under the conditions on i, a block $[k, l]$ of starting points for $\mathcal{F}$ yields the block $[k - 1 \pmod n, l]$ of starting points for $\nabla_{\mathcal{C}}(\mathcal{F})$ and the block $[k, l+1 \pmod n]$ of starting points for $\Delta_{\mathcal{C}}(\mathcal{F})$, and these new blocks are disjoint for different old blocks. ∎

Lemma 3.3.2. *Let $\mathcal{F} \subseteq N_i(\mathcal{C})$, $1 \leq i \leq \frac{n}{2}$, and let $\mathcal{F}$ be intersecting. Then*

(a) $|\mathcal{F}| \leq i$,

(b) $\frac{|\nabla_{\mathcal{C}}(\mathcal{F})|}{i+1} \geq \frac{|\mathcal{F}|}{i}$.

Proof. (a) Suppose, w.l.o.g., that $[1, i] \in \mathcal{F}$. By the suppositions of the lemma every member of $\mathcal{F}$ must have a starting point or an endpoint in $[1, i]$. Let S (resp. E) be the set of starting points (resp. endpoints) of elements of $\mathcal{F}$ lying in $[1, i]$. In particular, $1 \in S$, $i \in E$. Clearly, $[1, i]$ is the only element of $\mathcal{F}$ having both a starting point and an endpoint in $[1, i]$. Consequently, $|\mathcal{F}| = |S| + |E| - 1$. The elements of $E + 1 := \{k + 1 \pmod n : k \in E\}$ cannot be starting points by the suppositions. Thus $|S| \leq i - (|E| - 1)$ (we cannot exclude $i + 1$) and $|\mathcal{F}| \leq i - |E| + 1 + |E| - 1 = i$.

(b) Since $|\mathcal{F}| \le i$ and, by Lemma 3.3.1, $|\nabla_C(\mathcal{F})| \ge |\mathcal{F}| + 1$, there holds

$$\frac{|\nabla_C(\mathcal{F})|}{|\mathcal{F}|} \ge 1 + \frac{1}{|\mathcal{F}|} \ge \frac{i+1}{i}.$$

■

Cases $B = I$ and $B = I \vee \overline{C}$.

We treat these cases together. The proof is outlined for $B = I$; the particularities for $B = I \vee \overline{C}$ are attached in boxes.

Step 2. The inequalities $\frac{f_i}{i} \le 1$ are true for all $i \le \frac{n}{2}$ $\boxed{i < \frac{n}{2}}$, $\mathcal{F} \in \mathfrak{A}_B(C)$ by Lemma 3.3.2(a) $\boxed{\text{note that } \mathcal{F}_i \text{ is intersecting for } i < \frac{n}{2};\ f_{n-1} \le n \text{ is clear}}$. Moreover, we have to verify the inequalities

$$\frac{n-j}{ni} f_i + \frac{1}{n} f_j \le 1 \text{ for } j > \frac{n}{2}, \qquad i + j \le n \quad \boxed{j \ge \tfrac{n}{2}, i + j < n}$$

and, equivalently,

$$(n-j)\frac{f_i}{i} + f_j \le n.$$

Using Lemma 3.3.2(b) we obtain for our i and j

$$\frac{f_i}{i} = \frac{|\mathcal{F}_i|}{i} \le \frac{|\nabla_C^{n-j-i}(\mathcal{F}_i)|}{n-j}.$$

No element of $\nabla_C^{n-j-i}(\mathcal{F}_i)$ can be a complement of an element from $\mathcal{F}_j$ by our suppositions. Accordingly,

$$|\nabla_C^{n-j-i}(\mathcal{F}_i)| \le n - f_j,$$

and we conclude

$$\frac{n-j}{i} f_i \le n - f_j.$$

Step 3. Let f be a maximal vector satisfying the inequalities

$$\frac{f_i}{i} \le 1, \qquad 1 \le i \le \frac{n}{2}, \quad \boxed{1 \le i < \tfrac{n}{2}}$$

$$\frac{n-j}{ni} f_i + \frac{1}{n} f_j \le 1, \qquad j > \frac{n}{2}, i + j \le n$$

$$\boxed{j \ge \tfrac{n}{2}, i + j < n, \text{ and } f_{n-1} \le n}.$$

Since f is maximal we have

$$\frac{f_{i-1}}{i-1} \le \frac{f_i}{i}, \qquad 2 \le i \le \frac{n}{2} \quad \boxed{2 \le i < \tfrac{n}{2}}$$

(otherwise we could increase f_i up to $\frac{i}{i-1} f_{i-1}$ without disturbing the inequalities).

We denote the points from the list by

$$\boldsymbol{c}_i := \sum_{j=i}^{n-i} \boldsymbol{v}'_j + \sum_{j=n-i+1}^{n-1} \boldsymbol{u}'_j, \qquad 1 \le i \le \frac{n+1}{2}$$

$$\boxed{\boldsymbol{c}_i := \textstyle\sum_{j=i}^{n-i-1} \boldsymbol{v}'_j + \sum_{j=n-i}^{n-1} \boldsymbol{u}'_j,\ 1 \le i \le \frac{n}{2}}.$$

With

$$\alpha_1 := \frac{f_1}{1},$$

$$\alpha_i := \frac{f_i}{i} - \frac{f_{i-1}}{i-1}, \qquad 2 \le i < \left\lfloor \frac{n+1}{2} \right\rfloor \quad \boxed{2 \le i < \lfloor \tfrac{n}{2} \rfloor},$$

$$\alpha_{\lfloor \frac{n+1}{2} \rfloor} := 1 - \frac{f_{\lfloor \frac{n+1}{2} \rfloor - 1}}{\lfloor \frac{n+1}{2} \rfloor - 1} \quad \boxed{\alpha_{\lfloor \frac{n}{2} \rfloor} := 1 - \tfrac{f_{\lfloor \frac{n}{2} \rfloor - 1}}{\lfloor \frac{n}{2} \rfloor - 1}}$$

we obtain the desired convex combination, satisfying

$$\boldsymbol{f} \le \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \alpha_i \boldsymbol{c}_i \quad \boxed{\boldsymbol{f} \le \textstyle\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \alpha_i \boldsymbol{c}_i}.$$

To see this, consider the kth coordinate of the RHS. We have in the kth coordinate for $1 \le k \le \lfloor \frac{n+1}{2} \rfloor - 1$ $\boxed{1 \le k \le \lfloor \frac{n}{2} \rfloor - 1}$

$$k(\alpha_1 + \cdots + \alpha_k) = k\frac{f_k}{k} = f_k,$$

and for $\frac{n+1}{2} \le k \le n-1$ $\boxed{\frac{n}{2} \le k < n-1}$

$$k(\alpha_1 + \cdots + \alpha_{n-k}) + n(\alpha_{n-k+1} + \cdots + \alpha_{\lfloor \frac{n+1}{2} \rfloor})$$

$$\boxed{k(\alpha_1 + \cdots + \alpha_{n-k-1}) + n(\alpha_{n-k} + \cdots + \alpha_{\lfloor \frac{n}{2} \rfloor})},$$

which equals

$$k\frac{f_{n-k}}{n-k} + n\left(1 - \frac{f_{n-k}}{n-k}\right) \quad \boxed{k\tfrac{f_{n-k-1}}{n-k-1} + n\left(1 - \tfrac{f_{n-k-1}}{n-k-1}\right)},$$

and these terms are not smaller than f_k since $\boldsymbol{f}$ satisfies, in particular, the inequality

$$\frac{n-k}{n(n-k)} f_{n-k} + \frac{1}{n} f_k \le 1 \quad \boxed{\tfrac{n-k-1}{n(n-k-1)} f_{n-k-1} + \tfrac{1}{n} f_k \le 1}.$$

$\boxed{\text{If } k = n-1, \text{ we arrive at the inequality } f_{n-1} \le n, \text{ which is evident.}}$ If n is even $\boxed{\text{odd}}$ there remains the value $k = \frac{n}{2}$ $\boxed{k = \frac{n-1}{2}}$. Here we obtain for the kth coordinate k, which is not smaller than f_k by $\frac{f_k}{k} \le 1$.

Case $B = SIC$.

Step 2. We have to show that

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{f_i}{i} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \frac{f_j}{n-j} \leq 1 \text{ for all } \mathcal{F} \in \mu(\mathfrak{A}(\mathcal{C})).$$

Let $r := \min\{|X| : X \in \mathcal{F}\}$ and $s := \max\{|X| : X \in \mathcal{F}\}$. We may suppose that $r + s \leq n$, because otherwise we could consider the complementary family $\overline{\mathcal{F}}$ which is an SIC-family, too. Suppose further that $[1, r] \in \mathcal{F}$. By the Sperner property, every $i \in [n]$ can be at most once a starting point (resp. an endpoint) of a member of $\mathcal{F}$. Since $\mathcal{F}$ is an SIC-family, every member of $\mathcal{F}$ different from $[1, r]$ must have either the starting point or the endpoint in $[1, r]$. Let S (resp. E) be these sets of starting points (resp. endpoints) in $[1, r]$. The elements of $E + 1$ cannot be starting points since $\mathcal{F}$ is intersecting and cointersecting. By the arguments used in Lemma 3.3.2, $|\mathcal{F}| \leq r$. Consequently, using $r + s \leq n$,

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{f_i}{i} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n-1} \frac{f_j}{n-j} \leq \sum_{i=r}^{s} \frac{f_i}{r} = \frac{|\mathcal{F}|}{r} \leq 1.$$

Step 3. Since we have, besides $f \geq 0$, only one facet-defining inequality for $\mathcal{Q}^c$, the essential extreme points of $\mathcal{Q}^c$ must have exactly one nonzero coordinate and they must satisfy the essential facet-defining inequality. This gives the points from the list.

Case $B = SIC \vee \overline{I}\,\overline{C}$.

Step 2. As mentioned in the fourth remark after Theorem 3.3.1, we can add the complements of members of $\mathcal{F}$ to $\mathcal{F}$, and the resulting family is still a B-family. So we may suppose that $f_i = f_{n-i}$, $1 \leq i < \frac{n}{2}$. Consequently we only need to prove that

$$\sum_{1 \leq i < \frac{n}{2}} \frac{f_i}{i} + \frac{f_{\frac{n}{2}}}{n} \leq 1. \tag{3.3}$$

Let

$$\mathcal{F}_1 := \left\{X \in \mathcal{F} : |X| < \frac{n}{2}\right\}$$
$$\cup \left\{X \in \mathcal{F} : |X| = \frac{n}{2}, \quad 1 \in X\right\}, \qquad \mathcal{F}_2 := \mathcal{F} - \mathcal{F}_1.$$

Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are SIC-families. From Step 2 of the preceding case we obtain

$$\sum_{1 \leq i < \frac{n}{2}} \frac{f_i}{i} + \frac{f_{\frac{n}{2}}}{\frac{n}{2}} + \sum_{\frac{n}{2} < i \leq n-1} \frac{f_i}{n-i} \leq 2.$$

This verifies (3.3) because $f_i = f_{n-i}, \quad 1 \leq i < \frac{n}{2}$.

Step 3. We have $f \in \mathcal{Q}^c$ if $f \geq \mathbf{0}$ and

$$\sum_{i \in T} \frac{f_i}{i} + \sum_{\substack{\frac{n}{2} < i \leq n-1 \\ n-i \notin T}} \frac{f_i}{n-i} + \frac{f_{\frac{n}{2}}}{n} \leq 1 \text{ for all } T \subseteq \left\{1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \right\}.$$

Let

$$\alpha_i := \frac{1}{i} \max\{f_i, f_{n-i}\}, \; 1 \leq i < \frac{n}{2}, \qquad \alpha_{\frac{n}{2}} := \frac{f_{\frac{n}{2}}}{n}.$$

Then f is majorized by $\sum_{1 \leq i < \frac{n}{2}} \alpha_i(\boldsymbol{v}_i + \boldsymbol{w}_{n-i}) + \alpha_{\frac{n}{2}} \boldsymbol{u}_{\frac{n}{2}}$ and we have $\sum_{1 \leq i \leq \frac{n}{2}} \alpha_i \leq 1$ (take the set $T := \{i \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} : f_i > f_{n-i}\}$). Now we can increase one of the $\alpha's$ such that the sum becomes 1 and we get the desired convex combination majorizing f.

Case $B = S(I \vee C)$.

Let n be odd. Since the points $\boldsymbol{u}_i$, $1 \leq i \leq n-1$, belong to $\mu(\mathfrak{A}_{S(I \vee C)})$ we have $\operatorname{conv}(\mu(\mathfrak{A}_{S(I\vee C)})) \supseteq \operatorname{conv}(\mu(\mathfrak{A}_S))$. Every $S(I \vee C)$-family is an S-family; hence $\operatorname{conv}(\mu(\mathfrak{A}_{S(I\vee C)})) \subseteq \operatorname{conv}(\mu(\mathfrak{A}_S))$. Thus we can apply the results for $B = S$.

In the following let n be even, $n = 2m$.

Step 2. We have to verify that $f_m \leq m$ (this follows directly from Lemma 3.3.2) and that

$$\sum_{j=1}^{i} (f_j + f_{n-j}) + i \sum_{j=i+1}^{m-1} \left(\frac{f_j}{j} + \frac{f_{n-j}}{j} \right) + (n-i)\frac{f_m}{m} \leq n \text{ for all } \mathcal{F} \in \mathfrak{A}_B(C),$$
$$1 \leq i < m. \tag{3.4}$$

Take a family $\mathcal{F}$ for which the LHS is maximum. If $f_m = 0$ then the LHS is not greater than n, since we already derived for S-families the relation $\sum_{j=1}^{n-1} f_j \leq n$. So assume that $f_m > 0$.

Claim 1. If $[1, m] \in \mathcal{F}$, then either $[2, m+1]$ or $[m+2, 1]$ is in $\mathcal{F}$.

Proof of Claim 1. Suppose that $[m+2, 1] \notin \mathcal{F}$. We try to add $[2, m+1]$ to $\mathcal{F}$ (if it is not already there). The only obstacle could be a member X of $\mathcal{F}$ that is either contained in $[2, m+1]$ or contains $[2, m+1]$. In the first case, X has to contain $m+1$, because otherwise it would be a subset of $[1, m]$. Thus $X = [j, m+1]$ for some $2 < j \leq m+1$. Similarly, if X contains $[2, m+1]$ then $X = [2, l]$ holds for some $m+1 < l \leq n$. Since $\mathcal{F}$ is an S-family, at most one of these possible sets X can be in $\mathcal{F}$. Delete this X and add $[2, m+1]$ to $\mathcal{F}$. This change increases the LHS of (3.4), which is a contradiction to our choice of $\mathcal{F}$, and the claim is proved. $\square$

A pair of complementing, in C consecutive, m-element subsets is called an *equipartition*. We say that an equipartition is *represented in* $\mathcal{F}$ iff one of the parts is a member of $\mathcal{F}$. Claim 1 states that if an equipartition is represented in $\mathcal{F}$ then the neighboring equipartition is also represented. By induction, this results in

Claim 2. All equipartitions are represented in $\mathcal{F}$; that is, $f_m = m$. □

Claim 3. Let X and Y be members of $\mathcal{F}$ with sizes different from m. Then $X \cap Y \neq \emptyset$.

Proof of Claim 3. Let $|X| \leq |Y|$ and, w.l.o.g., $Y = [1, j]$. First suppose $j < m$. By Claim 2 and since $\mathcal{F}$ is an S-family we have $[j+1, j+m], [m+1, n] \in \mathcal{F}$. If X is disjoint to Y, then X must be contained in the union of $[j+1, j+m]$ and $[m+1, n]$, but neither of these sets can contain it alone. Hence $[m, j+m+1] \subseteq X$ follows. But then $|X| > j$, a contradiction. The case $j > m$ is easier. The set $[m+1, n]$ (belonging to $\mathcal{F}$, as above) covers the complement of Y; therefore X cannot be a subset of $\overline{Y}$ since $\mathcal{F}$ is an S-family. □

Claim 4. Let X and Y be members of $\mathcal{F}$ with sizes different from m. Then $X \cup Y \neq [n]$.

Proof of Claim 4. In Claims 1–3 we may replace everywhere the family $\mathcal{F}$ by the family $\overline{\mathcal{F}}$. However, Claim 3 with $\overline{\mathcal{F}}$ is equivalent to Claim 4 with $\mathcal{F}$. □

The last two claims result in:

Claim 5. $\mathcal{F} - \mathcal{F}_m$ is an SIC-family. □

Now we are in position to prove inequality (3.4). We use the corresponding inequality in Step 2 for $B = SIC$. We have

$$\begin{aligned}
&\sum_{j=1}^{i}(f_j + f_{n-j}) + i \sum_{j=i+1}^{m-1} \left(\frac{f_j}{j} + \frac{f_{n-j}}{j}\right) + (n-i)\frac{f_m}{m} \\
&\quad \leq i \sum_{j=1}^{m-1} \left(\frac{f_j}{j} + \frac{f_{n-j}}{j}\right) + (n-i)\frac{f_m}{m} \leq i + (n-i)\frac{f_m}{m} = n.
\end{aligned}$$

Step 3. We use the algebraic method to determine the essential extreme points of $\mathcal{P}^c$. The polytope is characterized by the system

$$\begin{aligned}
ny_i &\leq 1, 1 \leq i \leq n-1, && i \neq \frac{n}{2}, \\
iy_i + my_m &\leq 1, && 1 \leq i < \frac{n}{2}, \\
my_m + (n-j)y_j &\leq 1, && \frac{n}{2} < j \leq n-1, \\
y_i &\geq 0, && 1 \leq i \leq n-1.
\end{aligned}$$

Suppose that $\boldsymbol{y}$ is an essential extreme point. We have

$$y_k = \begin{cases} \min\{\frac{1}{n}, \quad \frac{1}{k}(1 - my_m)\}, & \text{if } 1 \leq k < \frac{n}{2}, \\ \min\{\frac{1}{n}, \quad \frac{1}{n-k}(1 - my_m)\}, & \text{if } \frac{n}{2} < k \leq n-1, \end{cases}$$

because otherwise we could increase y_k. If $y_m = 0$ then y is obviously not essential. If $y_m = \frac{1}{m}$ then $y_j = 0$, $1 \le j \le n-1$, $j \ne \frac{n}{2}$; that is, $y = \nu_m$. So let $0 < y_m < \frac{1}{m}$. Then there must exist some i with $1 \le i < \frac{n}{2}$ or some j with $\frac{n}{2} < j \le n-1$ such that

$$ny_i = 1 \text{ and } iy_i + my_m = 1 \text{ (resp. } ny_j = 1 \text{ and } my_m + (n-j)y_j = 1),$$

since otherwise we had at most $n-2$ linearly independent equalities for y. For the case $1 \le i < \frac{n}{2}$ (the case $\frac{n}{2} < j \le n-1$ can be treated analogously), we derive

$$y_j = \begin{cases} (1-\frac{i}{n})\frac{1}{m} & \text{if } j = m, \\ \frac{1}{n} & \text{if } 1 \le j \le i \text{ or } n-i \le j \le n-1, \\ \frac{i}{jn} & \text{if } i < j < m, \\ \frac{i}{(n-j)n} & \text{if } m < j < n-i, \end{cases}$$

that is, y has the form given in the table.

Cases $B = SI$ and $B = SI \vee \overline{C}$.

We treat these cases again together, the particularities for $B = SI \vee \overline{C}$ are attached in boxes. This time we change slightly our method of proof. In Step 2 we show that $f \in (\mathcal{P}^c)^c$ for all $\mathcal{F} \in \mathfrak{A}_B(\mathcal{C})$. This yields the desired inclusion $\operatorname{conv}(\mu(\mathfrak{A}_B(\mathcal{C}))) \subseteq \mathcal{P}$. In Step 3 we determine the extreme points of $\mathcal{P}^c$, which will be exactly the points from the table. For both steps, the following lemma is useful.

Lemma 3.3.3. *Let y be an essential extreme point of $\mathcal{P}^c$. Then*

(a) $y_1 = 1$ *and* $y_{\frac{n+1}{2}} = \frac{1}{n}$ $\boxed{y_{\frac{n}{2}} = \frac{1}{n}}$,

(b) $(i-1)y_{i-1} \ge iy_i$, $\quad 2 \le i \le \frac{n}{2}$ $\boxed{2 \le i < \frac{n}{2}}$,

(c) $(n-j)y_j \ge (n-j-1)y_{j+1}$, $\quad \frac{n}{2} < j \le n-2$,

(d) $iy_i + (i-1)y_{n-i+1} = 1$, $\quad 2 \le i \le \frac{n}{2}$ $\boxed{iy_i + iy_{n-i} = 1, \quad 1 \le i < \frac{n}{2}}$.

Proof. $\mathcal{P}^c$ is characterized by the following system:

$$y_1 \le 1, \tag{3.5}$$

$$ny_j \le 1, \quad \frac{n}{2} < j \le n-1 \quad \boxed{\tfrac{n}{2} \le j \le n-1}, \tag{3.6}$$

$$iy_i + (n-j)y_j \le 1, \quad 1 \le i \le \frac{n}{2}, \quad i+j > n$$

$$\boxed{1 \le i < \tfrac{n}{2}, \quad i+j \ge n}, \tag{3.7}$$

$$y_i \ge 0, \quad 1 \le i \le n-1. \tag{3.8}$$

Let y be an essential extreme point of $\mathcal{P}^c$.

(a) These equalities are satisfied, because we could otherwise increase the component up to 1 (resp. $\frac{1}{n}$) and no other inequality would be violated.

(b) If we had $(i-1)y_{i-1} < iy_i$, then we could increase y_{i-1} until we get equality in the last inequality without violating any other inequality, so y could not be essential.

(c) If we had $(n-j)y_j < (n-j-1)y_{j+1}$, then we could increase y_j and use analogous arguments.

(d) Suppose that $iy_i + (i-1)y_{n-i+1} < 1$ $\boxed{iy_i + iy_{n-i} < 1}$ for some i. Then we have by (c),

$$y_{n-1} \le 2y_{n-2} \le \cdots \le (i-1)y_{n-i+1} < 1 - iy_i$$

$$\boxed{y_{n-1} \le \cdots \le iy_{n-i} < 1 - iy_i}$$

so that we can increase y_i such that the supposed inequality becomes an equality without violating any other inequality of our system. ∎

Step 2. In order to prove $f \in (\mathcal{P}^c)^c$ it is sufficient to show that

$$y^{\mathrm{T}} f \le 1, \quad \text{i.e.,} \quad \sum_{X \in \mathcal{F}} y_{|X|} \le 1 \tag{3.9}$$

for all y satisfying the inequalities in Lemma 3.3.3 and the inequalities (3.5)–(3.8). Let $r := \min\{|X| : X \in \mathcal{F}\}$ and $s := \max\{|X| : X \in \mathcal{F}\}$. Recall that we may exclude the whole set $[n]$ to be a member of $\mathcal{F}$. Thus an $SI \vee \overline{C}$-family is always an S-family.

If $r > \frac{n}{2}$ $\boxed{r \ge \frac{n}{2}}$ then, by our conditions on y, we have $\sum_{X \in \mathcal{F}} y_{|X|} \le \sum_{X \in \mathcal{F}} \frac{1}{n}$, and the last sum is not greater than 1 since $\mathcal{F}$ is a Sperner family (look at the case $B = S$). The case $r = 1$ is trivial. So we can suppose that

$$2 \le r \le \frac{n}{2} \quad \boxed{2 \le r < \tfrac{n}{2}}.$$

Let, w.l.o.g., $[1, r] \in \mathcal{F}$.

Case 1. $r + s \le n + 1$ $\boxed{r + s \le n}$. (For $B = SI \vee \overline{C}$, we can suppose that $[r+1, n] \in \mathcal{F}$, since we could otherwise add this member to $\mathcal{F}$.) By the Sperner property no $i \in [n]$ can be more than once a starting point (resp. an endpoint) of a member of $\mathcal{F}$. By the intersecting property of $\mathcal{F}$ every member (up to $[r+1, n]$ in the case $B = SI \vee \overline{C}$) must have either a starting point or an endpoint in $[1, r]$. Let S (resp. E) be these sets of starting points (resp. endpoints) in $[1, r]$. Let

$$e_i := \begin{cases} y_{|X|} & \text{if } i \in E, \\ 0 & \text{if } i \notin E, \end{cases}$$

where in the first case X is the unique member of $\mathcal{F}$ having endpoint i,

$$s_i := \begin{cases} y_{|X|} & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

where in the first case X is the unique member of $\mathcal{F}$ with starting point i.

Claim 1. We have

$$e_i + s_{i+1} \le \frac{1}{r} + \frac{y_{n-r+1}}{r} \quad \boxed{e_i + s_{i+1} \le \tfrac{1}{r}}, \; 1 \le i \le r-1.$$

Proof of Claim 1. The case $e_i = s_{i+1} = 0$ is clear. If $e_i > 0$, $s_{i+1} = 0$, we have $e_i = y_{|X|}$ for some $X \in \mathcal{F}$ and consequently $e_i \le \frac{1}{n} \le \frac{1}{r}$ for $\frac{n}{2} < |X| \le n-1$ $\boxed{\frac{n}{2} \le |X| \le n-1}$ and $e_i \le \frac{1}{|X|} \le \frac{1}{r}$ for $1 \le |X| \le \frac{n}{2}$ $\boxed{1 \le |X| < \frac{n}{2}}$. If $e_i = 0$ and $s_{i+1} > 0$ we can use the same arguments.

So let $e_i = y_{|X|}$ and $s_{i+1} = y_{|Z|}$ for some $X, Z \in \mathcal{F}$. We must have $|X| + |Z| > n$ $\boxed{|X| + |Z| \ge n}$ in order to ensure the condition $X \cap Z \ne 0$ $\boxed{\text{or } X \cup Z = [n]}$. If X and Z have both cardinalities $> \frac{n}{2}$ $\boxed{\ge \frac{n}{2}}$ then $e_i + s_{i+1} \le \frac{2}{n} \le \frac{1}{r}$.

So let us, w.l.o.g., assume that $r \le |X| \le \frac{n}{2}$ $\boxed{r \le |X| < \frac{n}{2}}$ and $|Z| > \frac{n}{2}$ $\boxed{|Z| \ge \frac{n}{2}}$. Since we are in Case 1 we have

$$|Z| \le s \le n+1-r \quad \boxed{|Z| \le s \le n-r}.$$

If $|Z| \le n-r$ then by (3.7)

$$r(y_{|X|} + y_{|Z|}) \le |X| y_{|X|} + (n - |Z|) y_{|Z|} \le 1,$$

that is, $e_i + s_{i+1} \le \frac{1}{r}$. If $|Z| = n+1-r$, then by the inequalities in Lemma 3.3.3,

$$\begin{aligned} r(y_{|X|} + y_{|Z|}) &\le |X| y_{|X|} + r y_{n-r+1} \\ &\le r y_r + (r-1) y_{n-r+1} + y_{n-r+1} = 1 + y_{n-r+1}, \end{aligned}$$

that is, $e_i + s_{i+1} \le \frac{1}{r}(1 + y_{n-r+1})$. □

Now we can use this claim to derive for $B = SI$ (note Lemma 3.3.3(d))

$$\begin{aligned} \sum_{X \in \mathcal{F}} y_{|X|} &= y_r + \sum_{i=1}^{r-1} (e_i + s_{i+1}) \\ &\le \frac{1}{r}(r y_r + (r-1)(1 + y_{n-r+1})) \\ &= \frac{1}{r}(r - 1 + 1) = 1 \end{aligned}$$

and for $B = SI \vee \overline{C}$

$$\sum_{X \in \mathcal{F}} y_{|X|} = y_r + y_{n-r} + \sum_{i=1}^{r-1} (e_i + s_{i+1}) \leq \frac{1}{r} + (r-1)\frac{1}{r} = 1.$$

This shows (3.9).

Case 2. $r + s \geq n + 1$ $\boxed{r + s \geq n}$. We proceed by induction on $r + s$. We can start at $r + s = n + 1$ $\boxed{r + s = n}$, which we settled already in Case 1. So let $r + s \geq n + 2$ $\boxed{r + s \geq n + 1}$. Consider the subfamily $\mathcal{F}_s := \{X \in \mathcal{F} : |X| = s\}$ of $\mathcal{F}$. We can divide the set of starting points of members of $\mathcal{F}_s$ into blocks as before Lemma 3.3.1. Suppose we have m such blocks.

Case 2.1. Every block contains at most $n-s$ elements. It is easy to see that $\mathcal{F}' := \mathcal{F} - \mathcal{F}_s \cup \Delta_C(\mathcal{F}_s)$ is a B-family, too (the Sperner property is clear and property I $\boxed{\text{resp. } \overline{C}}$ follows by the inequality $r + s \geq n + 2$ $\boxed{r + s \geq n + 1}$).

Claim 2. We have

$$\frac{|\mathcal{F}_s|}{n - s} \leq \frac{\Delta_C(\mathcal{F}_s)}{n - s + 1}.$$

Proof of Claim 2. By Lemma 3.3.1, $|\Delta_C(\mathcal{F}_s)| = |\mathcal{F}_s| + m$. In view of the supposition of Case 2.1, $|\mathcal{F}_s| \leq m(n - s)$; hence

$$\frac{|\mathcal{F}_s|}{n - s} \leq m = |\Delta_C(\mathcal{F}_s)| - |\mathcal{F}_s|,$$

$$(n - s + 1)|\mathcal{F}_s| \leq (n - s)|\Delta_C(\mathcal{F}_s)|.$$

□

Now we have

$$\begin{aligned}\sum_{X \in \mathcal{F}} y_{|X|} &= \sum_{X \in \mathcal{F}'} y_{|X|} + |\mathcal{F}_s| y_s - |\Delta_C(\mathcal{F}_s)| y_{s-1} \\ &= \sum_{X \in \mathcal{F}'} y_{|X|} + \left(\frac{|\mathcal{F}_s|}{n - s}(n - s) y_s - \frac{|\Delta_C(\mathcal{F}_s)|}{n - s + 1}(n - s + 1) y_{s-1} \right) \leq 1\end{aligned}$$

by the induction hypothesis, Claim 2 and Lemma 3.3.3(c).

Case 2.2. There exists some block $\mathcal{B}$ of starting points of members of $\mathcal{F}_s$ with $|\mathcal{B}| > n - s$. In this case we do not need the induction hypothesis. Let us remember that $[1, r] \in \mathcal{F}$. The following starting points are possible for s-element members of $\mathcal{F}$ in order to ensure that the corresponding sets do not contain $[1, r]$:

$$2, 3, \ldots, n - s + 1 \text{ and } n - s + 2, n - s + 3, \ldots, r$$

$$\text{and } r + 1, r + 2, \ldots, r + n - s.$$

Since $\mathcal{B}$ contains more than $n - s$ elements, there must be some member of $\mathcal{F}_s$ with starting point $u \in \mathcal{B}$ satisfying $n - s + 2 \leq u \leq r$. Denote $R := [1, r]$, $Z := [u, u + s - 1 - n]_{\text{mod } n}$. Let us use the notion *boundary point* for either starting point or endpoint. Note that we have

$$R \cup Z = [n], \tag{3.10}$$

$$\text{any element of } [n] - Z \text{ is a boundary point of a member from } \mathcal{F}_s$$
$$\text{(consider the members having starting points in } \mathcal{B}), \tag{3.11}$$

$$R \cap Z \text{ is the union of two nonempty intervals}$$
$$I := [1, u + s - 1 - n] \text{ and } J := [u, r]. \tag{3.12}$$

Claim 3. There are at most $r + s + 1 - n$ members of $\mathcal{F}$ containing $[n] - Z$.

Proof of Claim 3. Let $W \neq R$ be a member of $\mathcal{F}$ satisfying $W \supset [n] - Z$. One boundary point of W must be in $I \cup J \cup \{u - 1, u + s - n\}$ because otherwise one of the conditions $W \supset [n] - Z$, $W \not\subset Z$, $W \not\supset R$ would be violated. Moreover, if both boundary points of W belong to $I \cup J \cup \{u - 1, u + s - n\}$, then they are both either in $I \cup \{u + s - n\}$ or in $J \cup \{u - 1\}$. Let $\psi(W)$ denote the boundary point of W being in $I \cup J \cup \{u - 1, u + s - n\}$ if there is only one. If there are two such boundary points, let $\psi(W)$ denote the one that is closer to $[n] - Z$; in other words, the boundary point that is larger in $I \cup \{u + s - n\}$ and that is smaller in $J \cup \{u - 1\}$. It is easy to check that $\psi(W)$ is an injection and that $\psi(W)$ cannot be 1 or r. Therefore $\psi(W)$ can have at most $|I \cup J| = u + s - 1 - n + r - u + 1 = r + s - n$ different values. Consequently, the number of sets $W \neq R$, $W \supset [n] - Z$, $W \in \mathcal{F}$ is at most $r + s - n$. Including R, we obtain the claim. □

Claim 4. For all $X \in \mathcal{F}$, there holds

$$y_{|X|} \leq \frac{1}{r}(1 - (r - 1)y_{n-r+1}) \quad \boxed{y_{|X|} \leq \tfrac{1}{r}(1 - r y_{n-r})}.$$

Proof of Claim 4. If $|X| \leq \frac{n}{2}$ $\boxed{|X| < \frac{n}{2}}$ then the inequality follows from (3.7). If $|X| > \frac{n}{2}$ $\boxed{|X| \geq \frac{n}{2}}$, then $y_{|X|} \leq \frac{1}{n}$ and $y_{n-r+1} \leq \frac{1}{n}$ $\boxed{y_{n-r} \leq \frac{1}{n}}$ yield the desired inequality since $\frac{1}{n} \leq \frac{1}{r}(1 - \frac{r-1}{n})$ $\boxed{\frac{1}{n} \leq \frac{1}{r}(1 - \frac{r}{n})}$. □

Claim 5. There are at most $2(n - s - 1)$ sets $X \in \mathcal{F}$ satisfying $X \cup Z \neq [n]$ and $X \neq Z$, and at least $n - s - 1$ of them are of size s.

Proof of Claim 5. Such a set X must have one of its boundary points in $[n] - Z$ (otherwise either $X \cup Z = [n]$ or $X \subset Z$). As always, by the Sperner property, every point of $[n] - Z$ is a starting point (resp. an endpoint) of at most one member of $\mathcal{F}$. The point $u + s - n$ cannot be a starting point and $u - 1$ cannot be an endpoint. Hence we have at most $2(n - s) - 2$ such sets. By (3.11) at least $n - s > n - s - 1$ of them are of size s. □

Now we have finally by Claims 3–5 and Lemma 3.3.3(c) (note $s > n - r + 1$ $\boxed{s > n - r}$) for $B = SI$

$$\begin{aligned}
\sum_{X \in \mathcal{F}} y_{|X|} &= y_{|Z|} + \sum_{X \in \mathcal{F}: X \cup Z = [n]} y_{|X|} + \sum_{X \in \mathcal{F}: X \cup Z \neq [n], X \neq Z} y_{|X|} \\
&\leq y_s + \frac{r+s+1-n}{r}(1-(r-1)y_{n-r+1}) + (n-s-1)y_s \\
&\quad + \frac{n-s-1}{r}(1-(r-1)y_{n-r+1}) \\
&= (n-s)y_s + 1 - (r-1)y_{n-r+1} \leq 1
\end{aligned}$$

and for $B = SI \vee \overline{C}$

$$\begin{aligned}
\sum_{X \in \mathcal{F}} y_{|X|} \leq\ & y_s + \frac{r+s+1-n}{r}(1 - r y_{n-r}) \\
&+ (n-s-1)y_s + \frac{n-s-1}{r}(1 - r y_{n-r}) \\
=\ & (n-s)y_s + 1 - r y_{n-r} \leq 1.
\end{aligned}$$

Step 3. We know already that every essential extreme point of $\mathcal{P}^c$ must satisfy the inequalities in Lemma 3.3.3 and (3.5)–(3.8). So there must exist integers $1 < i_1 < \cdots < i_r \leq \lfloor \frac{n}{2} \rfloor$ $\boxed{1 \leq i_1 < \cdots < i_r \leq \lfloor \frac{n-1}{2} \rfloor}$ such that with the definition $i_{r+1} := \lceil \frac{n+1}{2} \rceil$ $\boxed{i_{r+1} := \lceil \frac{n}{2} \rceil}$

$$\begin{aligned}
1 &= y_1 = 2y_2 = \cdots = (i_1 - 1)y_{i_1 - 1} > i_1 y_{i_1} = \cdots \\
&= (i_2 - 1)y_{i_2 - 1} > \cdots > i_k y_{i_k} = \cdots \\
&= (i_{k+1} - 1)y_{i_{k+1} - 1} > \ldots > i_r y_{i_r} = \cdots \\
&= (i_{r+1} - 1)y_{i_{r+1} - 1}
\end{aligned} \tag{3.13}$$

and, for $B = SI$,

$$\begin{aligned}
1 y_{n-1} &= \cdots = (i_1 - 2)y_{n-i_1+2} < (i_1 - 1)y_{n-i_1+1} = \cdots \\
&= (i_2 - 2)y_{n-i_2+2} < \cdots < (i_k - 1)y_{n-i_k+1} = \cdots \\
&= (i_{k+1} - 2)y_{n-i_{k+1}+2} < \cdots < (i_r - 1)y_{n-i_r+1} = \cdots \\
&= (i_{r+1} - 2)y_{n-i_{r+1}+2}
\end{aligned} \tag{3.14}$$

and, for $B = SI \vee \overline{C}$,

$$\begin{aligned}
1 y_{n-1} &= \cdots = (i_1 - 1)y_{n-i_1+1} < i_1 y_{n-i_1} = \cdots \\
&= (i_2 - 1)y_{n i_2 + 1} < \cdots < i_k y_{n-i_k} = \cdots \\
&= (i_{k+1} - 1)y_{n-i_{k+1}+1} < \cdots < i_r y_{n-i_r} = \cdots \\
&= (i_{r+1} - 1)y_{n-i_{r+1}+1}.
\end{aligned} \tag{3.14}$$

We must find $n-1$ linearly independent equalities satisfied by $\boldsymbol{y}$ in (3.5)–(3.8). We already have by Lemma 3.3.3(a) and (3.14)

$$y_{\frac{n+1}{2}} = \frac{1}{n} \quad \boxed{y_{\frac{n}{2}} = \frac{1}{n}}, \tag{3.15}$$

and for the variables $y_1, \ldots, y_{i_1-1}, y_{n-i_1+2}$ $\boxed{y_{n-i_1+1}}$, $\ldots, y_{n-1}$ the independent equalities (with the definition $y_n := 0$) for $B = SI$

$$iy_i + (i-1)y_{n-i+1} = 1, \qquad y_{n-i+1} = 0 \quad (1 \le i < i_1) \tag{3.16}$$

and, for $B = SI \vee \overline{C}$,

$$iy_i + iy_{n-i} = 1, \qquad y_{n-i} = 0 \quad (1 \le i < i_1). \tag{3.16}$$

Let us consider the $2(i_{k+1} - i_k)$ variables $y_{i_k}, \ldots, y_{i_{k+1}-1}, y_{n-i_k+1}$ $\boxed{y_{n-i_k}}$, $\ldots, y_{n-i_{k+1}+2}$ $\boxed{y_{n-i_{k+1}+1}}$, $k = 1, \ldots, r$. We have by Lemma 3.3.3(d) and (3.13),

$$\begin{aligned} iy_i + (n-j)y_j &= 1 \text{ for } i_k \le i \le i_{k+1} - 1 \\ &\qquad \text{and } n - i < j \le n - i_k + 1 \\ &\qquad \boxed{n - i \le j \le n - i_k}, \end{aligned} \tag{3.17}$$

and these are the only equalities in the inequalities (3.7) for the variables considered. If, conversely, (3.17) is satisfied then, for $B = SI$,

$$\begin{aligned} &i_k y_{i_k} = \cdots = (i_{k+1} - 1)y_{i_{k+1}-1}, \\ &(i_k - 1)y_{n-i_k+1} = \cdots = (i_{k+1} - 2)y_{n-i_{k+1}+2}, \; i_k y_{i_k} + (i_k - 1)y_{n-i_k+1} = 1, \end{aligned}$$

and, for $B = SI \vee \overline{C}$,

$$\begin{aligned} &i_k y_{i_k} = \cdots = (i_{k+1} - 1)y_{i_{k+1}-1}, \\ &i_k y_{n-i_k} = \cdots = (i_{k+1} - 1)y_{n-i_{k+1}+1}, \\ &i_k y_{i_k} + i_k y_{n-i_k} = 1. \end{aligned}$$

Accordingly, in the system (3.17) we can choose arbitrarily y_{n-i_k+1} $\boxed{y_{n-i_k}}$ and then the other variables are uniquely determined. Thus the rank of the system (3.17) is exactly $2(i_{k+1} - i_k) - 1$ and we need one more equality in (3.5)–(3.8) (independent from the others) for one of the considered variables. By (3.13) this can only be the equality

$$y_{n-i_k+1} = \frac{1}{n} \quad \boxed{y_{n-i_k} = \frac{1}{n}},$$

which implies

$$y_{i_k} = \frac{1}{i_k}\left(1 - \frac{i_k - 1}{n}\right) \quad \boxed{y_{i_k} = \frac{1}{i_k}(1 - \frac{i_k}{n})}. \tag{3.18}$$

We have seen: If y is an essential extreme point with associated indices $i_1, \dots, i_r$, then it must satisfy (3.15)–(3.18). Conversely, (3.15)–(3.18) yield a unique y, which satisfies (3.5)–(3.8), so it is an extreme point, which is easily seen to be essential. In detail, (3.15)–(3.18) yield

$$y_i = \frac{1}{i}, \qquad 1 \le i \le i_1 - 1,$$

$$y_j = 0, \qquad n-1 \ge j \ge n - i_1 + 2 \quad \boxed{n-1 \ge j \ge n - i_1 + 1}$$

and for $k = 1, \dots, r$,

$$y_i = \frac{1}{i}\left(1 - \frac{i_k - 1}{n}\right) \quad \boxed{y_i = \tfrac{1}{i}(1 - \tfrac{i_k}{n})}, \qquad i_k \le i \le i_{k+1} - 1,$$

and in the case $B = SI$,

$$y_j = \frac{1}{n-j}\frac{i_k - 1}{n}, \qquad n - i_k + 1 \ge j \ge n - i_{k+1} + 2$$

whereas in the case $B = SI \vee \overline{C}$,

$$y_j = \frac{1}{n-j}\frac{i_k}{n}, \qquad n - i_k \ge j \ge n - i_{k+1} + 1.$$

Consequently, we obtain the extreme points given in the "dashed" list. Thereby the Profile-Polytope Theorem is proved. ∎

Before examining some applications, let us consider the origins of this theorem. First the maximum size (together with some inequalities) of families satisfying certain condition was determined. The starting point was Sperner's theorem ($B = S$, easily this gives also the solution for $B = SI \vee \overline{C}$), and the trivial case $B = I$ was mentioned in [170]. The case $B = SI$ was settled by Milner [370] and the case $B = I \vee \overline{C}$ follows from a result of Erdős [167]. The problems for $B = SIC$ and $B = SIC \vee \overline{I}\,\overline{C}$ were solved by Brace and Daykin [83], Bollobás [72], Katona [297], Kleitman and Spencer [311], and Schönheim [418]. The solution for $B = S(I \vee C)$ was given by Purdy [392] (see also Greene and Hilton [231]). Concerning polytopes the case $B = S$ is trivial. As already mentioned, extreme points have been determined for the first time by Erdős, Frankl, and Katona [175] ($B = SI$), [176] ($B = I$). This was continued by Erdős and me [158] ($B = SIC$, $B = SIC \vee \overline{I}\,\overline{C}$, $B = S(I \vee C)$, by Derbala and me [133] ($B = SI \vee \overline{C}$), and by me [153] ($B = I \vee \overline{C}$). The first nontrivial complete description of essential facets was obtained by Katona and Schild [298] ($B = SI$). The rest was completed by me. We considered here only *some* conditions, though there exist several other interesting results. Concerning the maximum size of families satisfying various conditions we refer to West [464] and the references therein. Sometimes the polytopes may be studied also without Katona's circle method; see the papers of Frankl and Katona [199] and of Erdős and me [159].

Now let us come to some applications. As in the proof of the theorem we can find the maximum size of B-families $\mathcal{F}$ with the additional property that $a \le |X| \le b$ for all $X \in \mathcal{F}$ where a and b are some fixed numbers. To do so, one has to use in (3.2) the objective function given by the vector $\boldsymbol{w}$ with

$$w_i := \begin{cases} 1 & \text{if } a \le i \le b, \\ 0 & \text{otherwise.} \end{cases}$$

I will not list all the formulas here (for reference cf. [249]). Moreover, we can derive several LYM-type inequalities (originally not proved in this context) by specializing some facets, but this is not my aim. I provide instead four examples with a more interesting objective function $\boldsymbol{w}$.

Let $\xi_1, \dots, \xi_n$ be independent random variables, each taking value 1 with probability p and value 0 with probability $1 - p$. Liggett [341] proved the following lower bound for a convex combination of the ξ_i's, I [153] added the upper bound.

Theorem 3.3.2. *Let $\alpha_1, \dots, \alpha_n$ be positive real numbers with $\sum_{i=1}^{n} \alpha_i = 1$, and let $p \ge \frac{1}{2}$. Let*

$$R := \begin{cases} 0 & \text{if } n \text{ is even}, \\ \binom{n-1}{\frac{n-1}{2}-1} p^{\frac{n-1}{2}} (1-p)^{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$p \le P\big(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n \ge \tfrac{1}{2}\big) \le \sum_{i=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{n}{i} p^i (1-p)^{n-i} + R,$$

and the bounds are the best possible.

Proof. Let $\mathcal{F}_1 := \{X \subseteq [n] : \sum_{i \in X} \alpha_i < \frac{1}{2}\}$ and $\mathcal{F}_2 := \{X \subseteq [n] : \sum_{i \in X} \alpha_i \ge \frac{1}{2}\}$. Obviously, $\mathcal{F}_1$ is a C-family (containing $\emptyset$) and $\mathcal{F}_2$ is an $I \vee \overline{C}$-family (containing $[n]$). We have

$$P\big(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n < \tfrac{1}{2}\big) = \sum_{X \in \mathcal{F}_1} p^{|X|}(1-p)^{n-|X|},$$

$$P\big(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n \ge \tfrac{1}{2}\big) = \sum_{X \in \mathcal{F}_2} p^{|X|}(1-p)^{n-|X|}.$$

Let $\boldsymbol{f}_i$ be the full profile of $\mathcal{F}_i$ (i.e., with coordinates f_0 and f_n), and let $\boldsymbol{w} = (w_0, \dots, w_n)^{\mathrm{T}}$ be given by $w_i := p^i(1-p)^{n-i}$, $i = 0, \dots, n$. In order to obtain the lower bound (resp. the upper bound) we have to maximize $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f}$ for $\boldsymbol{f} \in \mu(\mathfrak{A}_C)$ (resp. for $\boldsymbol{f} \in \mu(\mathfrak{A}_{I \vee \overline{C}})$). For the lower bound, we take the "reversed" vectors from the table (extreme points) for $B = I$, add a 1 at a new

0-coordinate, a 0 at a new n-coordinate, yielding the vector

$$c_i := \left(1, \binom{n}{1}, \ldots, \binom{n}{i-1}, \binom{n-1}{i}, \ldots, \binom{n-1}{n-i}, 0, \ldots, 0\right)^{\mathrm{T}},$$
$$1 \le i \le \frac{n+1}{2}.$$

We then have to maximize $w^{\mathrm{T}} c_i$, $1 \le i \le \frac{n+1}{2}$. It is easy to calculate

$$\begin{aligned} w^{\mathrm{T}}(c_{i+1} - c_i) &= \binom{n-1}{i-1} p^i (1-p)^{n-i} - \binom{n-1}{n-i} p^{n-i}(1-p)^i \\ &= \binom{n-1}{i-1}\left(p^i(1-p)^{n-i}\left(1 - \left(\frac{p}{1-p}\right)^{n-2i}\right)\right) \le 0, \\ i &= 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor - 1. \end{aligned}$$

Hence the maximum is attained at the vector c_1, and we have

$$w^{\mathrm{T}} c_1 = \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-i} = 1 - p.$$

Consequently,

$$\begin{aligned} P\left(\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \ge \tfrac{1}{2}\right) &= 1 - P\left(\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n < \tfrac{1}{2}\right) \\ &\ge 1 - (1-p) = p. \end{aligned}$$

For the upper bound, we take the vectors from the table (extreme points) for $B = I \vee \overline{C}$, add a 0 at a new 0-coordinate, a 1 at a new n-coordinate. By the same method we find that the maximum is attained at the last vector of the list ($i = \lfloor \frac{n}{2} \rfloor$). This gives the value mentioned in the assertion. To see that the bounds are the best possible, take for the lower bound $\alpha_1 := 1, \alpha_2 := \cdots := \alpha_n := 0$ and for the upper bound $\alpha_1 := \cdots := \alpha_n := \frac{1}{n}$ if n is even and $\alpha_1 := \frac{2}{n+1}, \alpha_2 := \cdots := \alpha_n := \frac{1}{n+1}$ if n is odd. ■

Let $P = \{p_1, \ldots, p_n\}$ be a set of pairwise different primes. Let k be a positive integer and $m = p_1^k \cdots p_n^k$. We say that two divisors z and z' of m are *qualitatively independent* if there exist $p, q \in P$ with $p|z$, $p|z'$, $q \nmid z$, $q \nmid z'$ and if the existence of some $r \in P$ with $r|z$, $r \nmid z'$ implies the existence of some $s \in P$ with $s \nmid z$, $s|z'$.

Theorem 3.3.3. *Let Q be a set of pairwise qualitatively independent divisors of $m = p_1^k \cdots p_n^k$. Then*

$$|Q| \le \binom{n-1}{j^*} k^{j^*} \quad \text{where } j^* := \left\lceil (n-1)\frac{k}{k+1} \right\rceil,$$

and the bound is the best possible.

Proof. For $z \in Q$, let $D(z) := \{i \in [n] : p_i | z\}$. Obviously, $\mathcal{F}_Q := \{D(z) : z \in Q\}$ is an SIC-family. For $X \subseteq [n]$, there are exactly $k^{|X|}$ divisors z of m with $X = D(z)$. Thus $|Q| \leq \sum_{X \in \mathcal{F}_Q} k^{|X|}$. We take $\boldsymbol{w} = (w_1, \ldots, w_{n-1})^{\mathrm{T}}$ with $w_i := k^i$. To obtain an upper bound for $|Q|$ we have to maximize $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f}$ for $\boldsymbol{f} \in \mu(\mathfrak{A}_{SIC})$. Consider the extreme points of the list for $B = SIC$. It is not difficult to show that the upper bound is attained at $\boldsymbol{w}_{j^*}$ with $j^* = \lceil (n-1)\frac{k}{k+1} \rceil$. To see that the upper bound is the best possible, take $Q := \{z | m : |D(z)| = j^* \text{ and } p_1 \nmid z\}$. ■

A family F in $S = S(k_1, \ldots, k_n)$ is called *statically t-intersecting* if for all $\boldsymbol{x}, \boldsymbol{y} \in F$ there exist t coordinates $i_1, \ldots, i_t$ such that $x_{i_j}, y_{i_j} \geq 1$ for $j = 1, \ldots, t$. For $\boldsymbol{x} \in S$, we define the *support of* $\boldsymbol{x}$ by $\operatorname{supp}(\boldsymbol{x}) := \{i \in [n] : x_i \geq 1\}$ and the *support of* F by $\operatorname{supp}(F) := \{\operatorname{supp}(\boldsymbol{x}) : \boldsymbol{x} \in F\}$. The following fact is immediately clear:

Proposition 3.3.1. *$F \subseteq S$ is statically t-intersecting iff* $\operatorname{supp}(F)$ *is t-intersecting.*

For an application of the polytope method, we consider the special case $t = 1$ and $k = k_1 = \cdots = k_n$ and look for maximum statically 1-intersecting families in $N_i(S)$, $1 \leq i \leq kn$. Examples of such statically 1-intersecting families are the families $F_j := \{\boldsymbol{x} \in N_i(S) : j \leq |\operatorname{supp}(\boldsymbol{x})| \leq n - j \text{ and } x_1 \geq 1\} \cup \{\boldsymbol{x} \in N_i(S) : |\operatorname{supp}(\boldsymbol{x})| > n - j\}$.

Theorem 3.3.4. *One of the families F_j is a maximum statically 1-intersecting family in $N_i(S)$.*

Proof. Let $W_i(k, n) := |N_i(S(k, \ldots, k))|$. For any statically 1-intersecting family in $N_i(S)$ and for any $A \in \operatorname{supp}(F)$, we can add to F all $\boldsymbol{x} \in N_i(S)$ satisfying $\operatorname{supp}(\boldsymbol{x}) = A$. This gives $W_{i-|A|}(k-1, |A|)$ possibilities. Thus the maximum size of a statically 1-intersecting family is given by

$$\max \left\{ \sum_{A \in \mathcal{G}} W_{i-|A|}(k-1, |A|) : \mathcal{G} \subseteq 2^{[n]} \text{ is intersecting} \right\}.$$

Theorem 3.3.1, case I, shows that the maximum is attained at one of the families F_j. ■

It appears to be difficult to determine the right value of j exactly. In Section 7.3 we treat the problem (also for $t > 1$) from an asymptotic point of view. We will see that for $i = \lfloor \lambda n \rfloor$ in almost all cases the desired maximum is either asymptotically equal to F_1 or to $W_i(k, n)$.

The following fourth application was given by Erdős, Frankl, and Katona [175]. Consider *injective words* with letters from the alphabet $1, \ldots, n$, that is, sequences $\boldsymbol{x} = x_1 \ldots x_k$ with $x_i \in [n]$ for all i and $x_i \neq x_j$ whenever $i \neq j$. A *subword* of $\boldsymbol{x}$

is a sequence of the form $x_{i_1} \dots x_{i_l}$, $1 \le i_1 < \cdots < i_l \le k$. Let W_n be the poset of injective words over $[n]$ ordered by "being subwords."

Lemma 3.3.4. *The width of W_n is given by $d_1(W_n) = n!$.*

Proof. Writing the $n!$ permutations of $[n]$ as injective words gives an antichain of size $n!$; hence $d_1(W_n) \ge n!$. The $n!$ chains $x_1 < x_1x_2 < \cdots < x_1 \dots x_n$, where $x_1 \dots x_n$ is a permutation of $[n]$, cover the whole poset W_n; hence $d_1(W_n) \le n!$. ■

Theorem 3.3.5. *Let A be an antichain in W_n with the property that for any $\boldsymbol{x}, \boldsymbol{y} \in A$ there is some letter i being neither in $\boldsymbol{x}$ nor in $\boldsymbol{y}$. Then $|A| \le (n-1)!+1$, and the bound is the best possible.*

Proof. With each word $\boldsymbol{x} = x_1 \dots x_k$ we associate the set of its letters $L(\boldsymbol{x}) = \{x_1, \dots, x_k\}$. Let A be an antichain with the aforementioned properties, let $\mathcal{G} := \{L(\boldsymbol{x}) : \boldsymbol{x} \in A\}$, and let $\mathcal{F}$ be the family of maximal elements in $\mathcal{G}$. If $X \in \mathcal{F}$, then by Lemma 3.3.4, A can contain at most $|X|!$ members $\boldsymbol{x}$ with $L(\boldsymbol{x}) \subseteq X$. Consequently,

$$|A| \le \sum_{X \in \mathcal{F}} |X|!.$$

Obviously, $\mathcal{F}$ is an SC-family, and we can suppose $\emptyset, [n] \notin \mathcal{F}$. This time we take the weight-vector $\boldsymbol{w} = (w_1, \dots, w_{n-1})^{\mathrm{T}}$ with $w_i := i!$, and we have to maximize $\boldsymbol{w}^{\mathrm{T}}\boldsymbol{f}$ for $\boldsymbol{f} \in \mu(\mathfrak{A}_{SC})$. We take the reversed vectors from the list (extreme points) for $B = SI$. For the vectors $\boldsymbol{u}_i$, $1 \le i < \frac{n}{2}$, we have

$$i!\binom{n}{i} \le (n-1)!+1,$$

and for the vectors $\boldsymbol{v}_i + \boldsymbol{w}_j$, $i+j<n$, $\frac{n}{2} \le j \le n-1$ we have

$$\begin{aligned} i!\binom{n-1}{i-1} + j!\binom{n-1}{j} &= \frac{i(n-1)!}{(n-i)!} + \frac{(n-1)!}{(n-1-j)!} \\ &\le \frac{(n-j-1)(n-1)!}{(j+1)!} + \frac{(n-1)!}{(n-1-j)!} \\ &\le (n-1)!+1 \end{aligned}$$

(for $j = n-1$ and $j = n-2$ this can be calculated directly and for $\frac{n}{2} \le j < n-2$ one uses $\frac{n-j-1}{(j+1)!} \le \frac{1}{2}$ and $\frac{1}{(n-1-j)!} \le \frac{1}{2}$). Also the vector $\boldsymbol{w}_{n-1}$ gives only the value $(n-1)!\binom{n-1}{n-1} \le (n-1)!+1$. Consequently we derived the upper bound. To see that it is the best possible, take all words with $n-2$ letters from $[n-1]$ and the one-letter word n. ■

4

The flow-theoretic approach in Sperner theory

The *general Sperner problem* consists in determining the maximum weight of an antichain or a k-family in a weighted poset and in describing all or at least some optimal families (in the k-family case it is sometimes called the *Sperner–Erdős problem* – as early as in 1945 Erdős [165] proved that the Boolean lattices have the k-Sperner property). There are several other min (resp. max) problems related to the Sperner problem, and we will discuss them in this chapter, too.

If the poset is not specialized in any way, we can, of course, only derive some general formulas for the maximum weight. Min–max theorems are very important. In order to give the reader an impression of such theorems we discuss Dilworth's theorem, which is as fundamental as Sperner's theorem. It relates antichains of a poset to *partitions of P into chains* – that is, to sets $\mathfrak{D}$ of pairwise disjoint chains in P such that each element of P is contained in exactly one such chain.

***Theorem 4.0.1* (*Dilworth's theorem* [136]).** *The maximum size of an antichain in a finite poset P equals the minimum number of chains in a chain partition.*

Proof (Perles [381]). The inequality "$\leq$" follows from the fact that each chain can contain at most one element from some fixed antichain. Thus it is sufficient to find for each P an antichain A and a chain partition $\mathfrak{D}$ with $|A|$ chains. We prove this existence by induction on $|P|$. The case $|P| = 1$ is trivial. So consider the step $< |P| \to |P|$.

Case 1. There is a maximum antichain A that is different from the set of minimal elements and from the set of maximal elements of P. Let F (resp. I) be the filter (resp. ideal) generated by A. Clearly, $F \subset P$ and $I \subset P$ (strict inclusion). Moreover, $F \cap I = A$ ($p \in F \cap I$ implies the existence of $a_1, a_2 \in A$ with $a_1 \leq p \leq a_2$, i.e., $a_1 = p = a_2 \in A$; $A \subseteq F \cap I$ is clear) and $F \cup I = P$

(if there was some $p \in P - (F \cup I)$ then $A \cup \{p\}$ would be a larger antichain). Since A is also a maximum antichain in the posets induced by F and I there are chain partitions $\mathfrak{D}_F$ and $\mathfrak{D}_I$ of F and I into $|A|$ chains. Obviously, each chain of $\mathfrak{D}_F$ (resp. $\mathfrak{D}_I$) must start (resp. end) in A. Thus we may glue them together in the elements of A, which gives a chain partition of P into $|A|$ chains.

Case 2. If A is a maximum antichain, then A consists of all minimal or of all maximal elements of P. We choose a minimal element p and a maximal element q with $p \leq q$. Obviously, the maximum size of an antichain decreases from P to $P - \{p, q\}$ by 1. The induction hypothesis gives us a chain partition of $P - \{p, q\}$, which together with the chain $p \leq q$ is a desired chain partition of P. ∎

Generally, min–max theorems can be derived from the Duality Theorem in linear programming together with often nontrivial considerations on integer optimal solutions. Instead of this Duality Theorem our starting point will be the Max-Flow Min-Cut Theorem of Ford and Fulkerson [183] and Elias, Feinstein, and Shannon [145] and its generalization to min–cost flow problems.

Pioneering work in this direction was undertaken by Ford and Fulkerson [184] deriving Dilworth's theorem from the Max-Flow Min-Cut Theorem, and by Graham and Harper [228] relating the shifting technique to flow problems. Another pioneer, Frank [186], proved with the min-cost flow approach simultaneously deep min–max results for k-families (i.e., unions of k antichains) by Greene and Kleitman [233] and for unions of k chains by Greene [229]. The advantage of this method is that it not only gives the min–max results in question, but it also provides algorithms that efficiently determine the optima. Though this method was used mainly in the nonweighted case, I present it here for the weighted case because in several circumstances large posets can be reduced to small posets by flow morphisms, a notion that was introduced and first investigated by Harper [258]. The algorithms then run much more quickly if they are applied to the small weighted posets. For instance, if the small weighted posets are chains, the problems become more or less trivial and structural results can be derived for the original poset. Because of the importance of these flow techniques, we will not refer to other books but present main results in a separate section.

4.1. The Max-Flow Min-Cut Theorem and the Min-Cost Flow Algorithm

Let $G = (V, E)$ be a directed graph. Let s and t be two distinguished points of V, the *source* (resp. the *sink*), and let $c : E \to \mathbb{R}_+$ be a given nonnegative valued function, the *capacity*. Then the 5-tuple $N = (V, E, s, t, c)$ is called a *network*; in the following we are dealing with such networks. A *flow* on N is a function $f : E \to \mathbb{R}$ satisfying the following conditions:

$$\sum_{e^-=p} f(e) = \sum_{e^+=p} f(e) \text{ for all } p \in V - \{s, t\} \qquad \textit{(conservation of flow)} \tag{4.1}$$

and,

$$0 \le f(e) \le c(e) \text{ for all } e \in E \qquad \textit{(capacity constraints)}. \tag{4.2}$$

For any $A, B \subseteq V$, put

$$f(A, B) := \sum_{e^- \in A, e^+ \in B} f(e),$$

and let us define $c(A, B)$ analogously. An ordered partition (S, T) of V (i.e., $S \cap T = \emptyset$, $S \cup T = V$, and $S, T \neq \emptyset$) is called a *cut* if $s \in S$ and $t \in T$. For example, $(\{s\}, V - \{s\})$ and $(V - \{t\}, \{t\})$ are cuts. The *value* $v(f)$ *of the flow* f is defined by

$$v(f) := f(\{s\}, V - \{s\}) - f(V - \{s\}, \{s\}).$$

Moreover, $c(S, T)$ is called the *capacity of the cut* (S, T). Not surprisingly, the flow f is called *maximal* and the cut (S, T) is called *minimal* if $v(f) \ge v(f')$ for all flows f' and $c(S, T) \le c(S', T')$ for all cuts (S', T'). We speak of *integral flows* f (resp. *integral capacities* c) if $f, c : E \to \mathbb{N}$.

Lemma 4.1.1.

(*a*) *For any cut* (S, T) *and any flow* f *in* N, *we have* $f(S, T) - f(T, S) = v(f) \le c(S, T)$.

(*b*) *If for the flow* f *and the cut* (S, T) *the equalities*

$$f(e) = \begin{cases} c(e) & \textit{if } e^- \in S, e^+ \in T, \quad \textit{i.e., if } e \textit{ is saturated}, \\ 0 & \textit{if } e^+ \in S, e^- \in T, \quad \textit{i.e., if } e \textit{ is void} \end{cases}$$

are satisfied then f *is maximal and* (S, T) *is minimal. Furthermore,* $v(f) = c(S, T)$.

Proof. (a) In view of (4.1), (4.2), and $t \notin S$ we have

$$\begin{aligned} v(f) &= f(\{s\}, V - \{s\}) - f(V - \{s\}, \{s\}) + 0 \\ &= \sum_{e^-=s} f(e) - \sum_{e^+=s} f(e) + \left(\sum_{e^- \in S - \{s\}} f(e) - \sum_{e^+ \in S - \{s\}} f(e) \right) \\ &= \sum_{e^- \in S} f(e) - \sum_{e^+ \in S} f(e) \\ &= f(S, S) + f(S, T) - (f(S, S) + f(T, S)) \\ &= f(S, T) - f(T, S) \le c(S, T) - 0. \end{aligned}$$

(b) From the proof of (a) we observe that $v(f) = c(S, T)$ if $f(S, T) = c(S, T)$ and $f(T, S) = 0$ – that is, if the conditions of the lemma are satisfied. But then for any other flow f' and cut (S', T') because of (a)

$$v(f') \leq c(S, T) = v(f) \leq c(S', T').$$

■

Remark 4.1.1. *Note that we do not need the capacity constraints (4.2) to define $v(f)$. Also without these constraints we have $f(S, T) - f(T, S) = v(f)$, since in the proof of (a) we needed for this equality only the conservation of flow (4.1).*

Now we are able to prove the central result of Ford and Fulkerson [183] and Elias, Feinstein, and Shannon [145], which we formulate first only for integral capacities since this case is interesting for combinatorial applications.

Theorem 4.1.1 (Max-Flow Min-Cut Theorem). *Let $N = (V, E, s, t, c)$ be a network with integral capacity. The maximum value of an integral flow in N equals the minimum capacity of a cut in N.*

Proof. By Lemma 4.1.1 we only have to show that there is an integral flow f and a cut (S, T) such that the condition of Lemma 4.1.1(b) is satisfied. We determine f recursively. We start with $f_0 \equiv 0$ and assume that we have already constructed integral flows $f_0, \ldots, f_i$. Then we build recursively a vertex set S_i which we consider later as a set of *labeled vertices*. We start with $S_{i0} := \{s\}$ and assume that $S_{i0}, \ldots, S_{ij}$ are given. Let

$$F_{ij} := \{q \in V - S_{ij} : \text{ there is some } p \in S_{ij} \text{ such that } pq \in E \text{ and } f_i(pq) < c(pq) \text{ or } qp \in E \text{ and } f_i(qp) > 0\}.$$

Case 1. $F_{ij} \neq \emptyset$. Then we take some $q \in F_{ij}$ and set $S_{i,j+1} := S_{ij} \cup \{q\}$; that is, we label vertex q.

Case 2. $F_{ij} = \emptyset$; that is, for all $p \in S_{ij}, q \in V - S_{ij}$

$$f_i(pq) = \begin{cases} c(pq) & \text{if } pq \in E, \\ 0 & \text{if } qp \in E. \end{cases}$$

Since V is finite, Case 2 must hold for some j. At that moment or if earlier $t \in S_{ij}$ we put $S_i := S_{ij}$.

Now again two cases are possible:

Case 2.1. $t \in S_i$. Then there must be a sequence of pairwise distinct points $(p_0, p_1, \ldots, p_k)$ (called the *flow-augmenting path*) with an associated sequence $(e_1, \ldots, e_k)$ of arcs such that $p_0 = s$, $p_k = t$ and

either	$e_l = p_{l-1} p_l$	and	$f_i(e_l) < c(e_l)$	(e_l is a *forward arc*)
or	$e_l = p_l p_{l-1}$	and	$f_i(e_l) > 0$	(e_l is a *backward arc*).

Let

$$\begin{aligned}
\epsilon^+ &:= \min\{c(e_l) - f_i(e_l) &&: e_l \text{ is a forward arc}, && l = 1, \ldots, k\},\\
\epsilon^- &:= \min\{f_i(e_l) &&: e_l \text{ is a backward arc}, && l = 1, \ldots, k\},\\
\epsilon &:= \min\{\epsilon^+, \epsilon^-\}.
\end{aligned}$$

Then we obtain the new flow f_{i+1} by

$$f_{i+1}(e) := \begin{cases} f_i(e) + \epsilon & \text{if } e \text{ is a forward arc},\\ f_i(e) - \epsilon & \text{if } e \text{ is a backward arc},\\ f_i(e) & \text{if } e \text{ is not an arc in the flow-augmenting path.} \end{cases}$$

As illustrated in Figure 4.1, the flow conservation condition (4.1) is satisfied by f_{i+1}. The capacity constraints (4.2) are satisfied by definition of ϵ. Obviously, f_{i+1} is an integral flow since ϵ is an integer, and $v(f_{i+1}) = v(f_i) + \epsilon$. Note that ϵ is positive, hence $\epsilon \geq 1$.

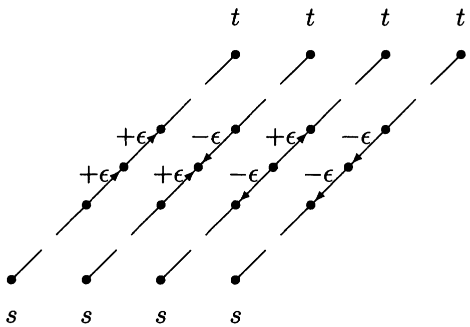


Figure 4.1

Case 2.2. $t \notin S_i$. Then $f = f_i$ and $(S, T) = (S_i, V - S_i)$ are a flow and a cut satisfying the conditions of Lemma 4.1.1(b).

Since the value of any flow is bounded, for example, by the finite capacity of the cut $c(\{s\}, V - \{s\})$ and since the flow value increases in each step by at least 1, after a finite number of steps Case 2.2 must appear. ∎

Remark 4.1.2. *(a) The Max-Flow Min-Cut Theorem remains true if there is not given for any arc e a capacity constraint $f(e) \leq c(e)$. For such arcs, we write $c(e) := \infty$. But we must suppose that there is at least one cut of finite capacity (this was the only condition we needed at the end of the proof).*

(b) Instead of integral flows and capacities we can work with rational flows and capacities. The result remains true since we can multiply by the common denominator of the capacities.

From the practical point of view it is enough to consider rational capacities. If we have real capacities we do not have a better lower bound for ϵ than 0. So we cannot show that the preceding procedure terminates. Moreover, there exist examples (cf. Lovász and Plummer [356, p. 47]) where the flow values $v(f_i)$ do not converge to the maximum flow value. But we still have some indeterminacy in our procedure. We have not specified which point q we label, that is, include into $S_{i,j+1}$ in Case 1 of our procedure. The possible points, the elements of F_{ij}, are called *fringe* vertices. We collect iteratively the fringe vertices in a queue and take them by the principle "first in, first out" (FIFO). Taking the point q from the queue and putting it into the set $S_{i,j+1}$, we obtain the following new fringe vertices, which we append at the end of the queue: $\{r \in V - S_{i,j+1} : qr \in E$ and $f_i(qr) < c(qr)$, or $rq \in E$ and $f_i(rq) > 0\}$. In order to find the flow-augmenting path in Case 2.1, we store in the queue together with the fringe vertices r the point q (with $+$ or $-$) which led to r (by a forward, resp. backward, arc). Then the path can be determined recursively beginning from t. Moreover, we store together with r the largest possible $\epsilon = \epsilon(r)$ with which the flow change works. Obviously we have $\epsilon(r) := \min\{\epsilon(q), c(qr) - f_i(qr)\}$, for a forward arc (resp. $\epsilon(r) := \min\{\epsilon(q), f_i(rq)\}$, for a backward arc). In Case 2.1 we have then $\epsilon := \epsilon(t)$. This is the so-called *labeling algorithm* using *breadth first search (BFS)*.

Without proof we mention the result of Edmonds and Karp [144]:

Theorem 4.1.2. *The labeling algorithm using BFS determines after $O(|V||E|^2)$ steps the maximum flow and minimum cut in the network $N = (V, E, q, s, c)$ with $c : E \to \mathbb{R}_+$.*

Since by this theorem the algorithm terminates (or because of a continuity argument) we have in particular:

Corollary 4.1.1. *Let $N = (V, E, s, t, c)$ be a network with $c : E \to \mathbb{R}_+$. Then* $\max\{v(f) : f$ *is a flow in* $N\} = \min\{c(S, T) : (S, T)$ *is a cut in* $N\}$.

The presented algorithm is easy to understand and to implement; see also Sedgewick [423] and Sysło, Deo, and Kowalik [449]. For more refined versions of flow algorithms, we refer to Tarjan [451], Jungnickel [279], and Mehlhorn [363].

Now let us assume that we have for N a further integral function $a : E \to \mathbb{N}$, the *cost function*. The *cost* $a(f)$ of the flow f is defined by

$$a(f) := \sum_{e \in E} f(e)a(e),$$

and we are interested in a flow having minimal cost and some value v_0 given in advance. The idea of Ford and Fulkerson [184] is to use a further function $\pi : V \to \mathbb{N}$, the so-called *potential function*, for a primal-dual approach. Let us first prove an important lemma.

Lemma 4.1.2. *Let f be a flow of value v_0 in the network N with integral cost a. Then f has minimum cost with respect to all other flows of value v_0 if there is a function $\pi : V \to \mathbb{N}$ with the following property:*

$$f(pq) = \begin{cases} 0 & \textit{if} \quad \pi(q) - \pi(p) < a(pq), \\ c(pq) & \textit{if} \quad \pi(q) - \pi(p) > a(pq) \end{cases} \qquad \textit{for all } pq \in E, \tag{4.3}$$

and

$$\pi(s) = 0. \tag{4.4}$$

Proof. For all $pq \in E$, define $\bar{a}(pq) := a(pq) + \pi(p) - \pi(q)$. Further let g be any flow of value v_0. Then (using Lemma 4.1.1(a) and the conservation of flow)

$$\begin{aligned} a(g) &= \sum_{e\in E} g(e)\bar{a}(e) - \sum_{p\in V} \pi(p) \sum_{e^-=p} g(e) + \sum_{q\in V} \pi(q) \sum_{e^+=q} g(e) \\ &= \sum_{e\in E} g(e)\bar{a}(e) + \sum_{p\in V} \pi(p)(g(V - \{p\}, \{p\}) - g(\{p\}, V - \{p\})) \\ &= \sum_{e\in E} g(e)\bar{a}(e) + \pi(t)v_0 - \pi(s)v_0 \\ &= \sum_{e\in E} g(e)\bar{a}(e) + v_0\pi(t). \end{aligned}$$

Consequently,

$$a(g) - a(f) = \sum_{e\in E} (g(e) - f(e))\bar{a}(e) \geq 0$$

since every item of the last sum is nonnegative:

If $\bar{a}(pq) < 0$, i.e., $\pi(q) - \pi(p) > a(pq)$ then $f(pq) = c(pq) \geq g(pq)$.
If $\bar{a}(pq) > 0$, i.e., $\pi(q) - \pi(p) < a(pq)$ then $f(pq) = 0 \leq g(pq)$.

∎

Remark 4.1.3. *If f and π satisfy the conditions of Lemma 4.1.2, then we derive from the proof*

$$a(f) = \sum_{e\in E:\bar{a}(e)<0} c(e)\bar{a}(e) + v_0\pi(t).$$

The following theorem shows that the converse of Lemma 4.1.2 is also true. Its algorithmic proof is of practical (and for us also of theoretical) interest.

Theorem 4.1.3. *Let N be a network with integral cost a. Let v^* be the maximum value of a flow in N and let $0 \leq v_0 \leq v^*$. Then there are a flow f in N of value v_0 and a function $\pi : V \to \mathbb{N}$ such that (4.3) and (4.4) hold.*

Proof. We will construct f and π iteratively, starting with $f_0 \equiv 0$ and $\pi_0 \equiv 0$. For fixed π, consider the network $N_\pi = (V, E_\pi, s, t, c)$, where $E_\pi := \{e \in E : \pi(e^+) - \pi(e^-) = a(e)\}$. Observe that changing a flow satisfying (4.3) and (4.4) on N_π does not violate these two conditions. Assume that we are given some flow f_k and some potential function π_k. Now we apply the preceding labeling algorithm to N_{π_k}, but starting with f_k. More exactly, in looking for a flow-augmenting path we admit only arcs from E_{π_k}; that is, a point q is a fringe vertex if there is some labeled point p such that $pq \in E_{\pi_k}$ and $f_k(pq) < c(pq)$ or $qp \in E_{\pi_k}$ and $f_k(qp) > 0$. As before, we come after finitely many steps (repeatedly augmenting the flow along paths) to a flow f_{k+1}, a set S_{k+1} of labeled vertices and a set T_{k+1} of unlabeled vertices with $t \in T_{k+1}$ such that

$$f_{k+1}(e) = \begin{cases} c(e) & \text{if } \pi_k(e^+) - \pi_k(e^-) = a(e) \text{ and } e^- \in S_{k+1}, e^+ \in T_{k+1}, \\ 0 & \text{if } \pi_k(e^+) - \pi_k(e^-) = a(e) \text{ and } e^- \in T_{k+1}, e^+ \in S_{k+1} \end{cases} \tag{4.5}$$

(the finiteness is clear if we have rational capacities, and for real capacities we can use the arguments of Edmonds and Karp in Theorem 4.1.2).

We change in this phase the flow value from $v(f_k)$ to $v(f_{k+1})$. Note that we can also obtain every value between these two values since in Case 2.1 we need not change the flow by the maximum possible value ϵ (we can take every ϵ' with $0 < \epsilon' < \epsilon$). Given f_{k+1}, S_{k+1}, and T_{k+1}, we change the potential function defining

$$\pi_{k+1}(p) := \begin{cases} \pi_k(p) & \text{if } p \in S_{k+1}, \\ \pi_k(p) + 1 & \text{if } p \in T_{k+1}. \end{cases}$$

It is important to note that (4.3) and (4.4) remain valid in this case: (4.3) could be violated (using the integrality of a and π) only if $\pi_k(q) - \pi_k(p) = a(pq)$ and $p \in S_{k+1}, q \in T_{k+1}$ (but then (4.5) gives $f_{k+1}(e) = c(pq)$) or if $\pi_k(q) - \pi_k(p) = a(pq)$ and $p \in T_{k+1}, q \in S_{k+1}$ (but then by (4.5), $f_{k+1}(e) = 0$). Now we may continue with the labeling algorithm on the new network $N_{\pi_{k+1}}$. We show that if f_{k+1} is not maximal in N, then we change the potential function only a finite number of times. Assume the contrary. Then we obviously have $S_{k+1} \subseteq S_{k+2} \subseteq \ldots$. So there must be an index l such that $S_l = S_{l+1} = \ldots$. But f_{k+1} is not maximal; hence by Lemma 4.1.1 there exists an edge $e \in E$ such that $f(e) < c(e)$ and $e^- \in S_l, e^+ \in T_l$ or $f(e) > 0$ and $e^+ \in S_l, e^- \in T_l$. In view of (4.3), (4.4), and (4.5) (with l instead of $k+1$), we must have $\pi_l(e^+) - \pi_l(e^-) < a(e)$ (resp. $\pi_l(e^+) - \pi_l(e^-) > a(e)$). So after a finite number of changes of the potential, the arc e is included into the network N_{π_m} for some $m \geq l$, but then $S_{m+1} \supseteq S_l \cup \{e^+\} \supset S_l$ (resp. $S_{m+1} \supseteq S_l \cup \{e^-\} \supset S_l$), a contradiction.

Summarizing, the algorithm works as follows: We change the flow a finite number of times, then the potential a finite number of times, then the flow a finite number of times, and so on such that (4.3) and (4.4) are satisfied. It is easy to see that there cannot be a flow-augmenting path in N_π from s to t if $\pi(t) > \sum_{e \in E} a(e)$. By the preceding remarks we come to the maximal flow after at most $\sum_{e \in E}$ a(e) changes of the potential function and then the algorithm terminates. ■

Remark 4.1.4. *If for fixed potential π the flow value in the network N_π is increased by ϵ, then the cost increases by $\pi(t)\epsilon$ since the first sum in Remark 4.1.3 remains constant.*

Let us prove finally a technical lemma concerning flows in networks on directed acyclic graphs (dags). Here a *dag* is a digraph $G = (V, E)$ in which there are no points $p_0, \dots, p_k$ such that $p_i p_{i+1} \in E, i = 0, \dots, k-1$, and $p_k p_0 \in E$. For example, the Hasse diagram of a poset is a dag. A *directed elementary flow (def)* in a network N is a flow f for which there exist points $s = p_0, p_1, \dots, p_l = t$ (forming a *directed chain*) such that $p_i p_{i+1} \in E, i = 0, \dots, l-1$, and

$$f(e) = \begin{cases} 1 & \text{if } e = p_i p_{i+1}, i = 0, \dots, l-1, \\ 0 & \text{otherwise.} \end{cases}$$

We define and denote the *support* of a flow in N by $\operatorname{supp}(f) := \{e \in E : f(e) > 0\}$.

Lemma 4.1.3. *Let N be a network on a dag with a directed chain from the source to the sink, and let f be a flow in N. Then f is a nonnegative combination of directed elementary flows; that is, there are defs $f_1, \dots, f_l$ and nonnegative real numbers $\lambda_1, \dots, \lambda_l$ such that $f = \lambda_1 f_1 + \dots + \lambda_l f_l$. If f is integral, the λ's can be taken to be integral, too.*

Proof. We construct this combination by decreasing recursively $|\operatorname{supp}(f)|$. We may assume that $|\operatorname{supp}(f)| > 0$. Then there is some arc $p_i p_{i+1}$ such that $f(p_i p_{i+1}) > 0$. Suppose first that $p_i \neq s$ and $p_{i+1} \neq t$. By the conservation of flow, (4.1), there must be arcs $p_{i-1} p_i$ and $p_{i+1} p_{i+2}$ with positive flow. In the same manner we find arcs $p_{i-2} p_{i-1}$ and $p_{i+2} p_{i+3}$ and so on. No point can appear twice, because we have a network on a dag. Thus the procedure must end after finitely many steps, say at $p_0 p_1$ and $p_{k-1} p_k$. Stop is possible only if $p_0 = s$ and $p_k = t$. Now define $\lambda_1 := \min\{f(p_i p_{i+1}), i = 0, \dots, k-1\}$. Obviously, $\lambda_1 > 0$. Further set

$$f_1(e) := \begin{cases} 1 & \text{if } e = p_i p_{i+1}, i = 0, \dots, k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $f' := f - \lambda_1 f_1(e)$ is a flow on N with smaller support. Now we do the same for f', finding λ_2 and f_2 and so on. We are done if $\operatorname{supp}(f) = \emptyset$. The claim concerning integrality is clear by construction. ■

4.2. The k-cutset problem

Although we are more interested in k-families, we start with k-cutsets (defined in this section) since the application of the flow method is a little bit easier for them, and of course we also need them later. For the poset P, let $\mathfrak{C}(P)$ be the set of all maximal chains in P. A subset F of P is called a *k-cutset* if $|F \cap C| \geq k$ for all $C \in \mathfrak{C}(P)$. Let k_0 be the smallest number of elements in a maximal chain of P. Obviously, k-cutsets exist only for $0 \leq k \leq k_0$. The *k-cutset problem* is the following: Given a weighted poset (P, w), determine

$$c_k(P, w) := \min\{w(F) : F \text{ is a } k\text{-cutset}\}.$$

Here we set $c_k(P, w) := \infty$ if $k > k_0$. The k-cutset problem can be considered as an integer linear programming problem. If we relax it we arrive at the definition of a *fractional k-cutset*. This is a function $x : P \to \mathbb{R}$ such that

$$\sum_{p \in C} x(p) \geq k \text{ for all } C \in \mathfrak{C}(P),$$

$$0 \leq x(p) \leq 1 \text{ for all } p \in P.$$

In the *fractional k-cutset problem* we have to determine

$$c_k^*(P, w) := \min\left\{\sum_{p \in P} w(p)x(p) : \quad x \text{ is a fractional } k\text{-cutset}\right\}.$$

Note that the fractional k-cutset problem reduces to the k-cutset problem if we require, in addition, that the solutions are integral.

Here and in the following we deal simultaneously with the given problem and its dual. Given a general *primal* linear programming problem of the form

$$\begin{aligned}
A\boldsymbol{x} + B\boldsymbol{y} + C\boldsymbol{z} &\leq \boldsymbol{a}\\
D\boldsymbol{x} + E\boldsymbol{y} + F\boldsymbol{z} &= \boldsymbol{b}\\
G\boldsymbol{x} + H\boldsymbol{y} + K\boldsymbol{z} &\geq \boldsymbol{c}\\
\boldsymbol{x} &\geq \boldsymbol{0}\\
\boldsymbol{z} &\leq \boldsymbol{0}\\
\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{y} + \boldsymbol{f}^{\mathrm{T}}\boldsymbol{z} &\to \max
\end{aligned}$$

where $A, \ldots, K$ are matrices and $\boldsymbol{a}, \ldots, \boldsymbol{f}$ are vectors of suitable dimension, its *dual* is defined to be the problem

$$\begin{aligned}
A^{\mathrm{T}}\boldsymbol{u} + D^{\mathrm{T}}\boldsymbol{v} + G^{\mathrm{T}}\boldsymbol{w} &\geq \boldsymbol{d}\\
B^{\mathrm{T}}\boldsymbol{u} + E^{\mathrm{T}}\boldsymbol{v} + H^{\mathrm{T}}\boldsymbol{w} &= \boldsymbol{e}
\end{aligned}$$

$$\begin{aligned} C^{\mathrm{T}}\boldsymbol{u} + F^{\mathrm{T}}\boldsymbol{v} + K^{\mathrm{T}}\boldsymbol{w} &\leq \boldsymbol{f} \\ \boldsymbol{u} &\geq \boldsymbol{0} \\ \boldsymbol{w} &\leq \boldsymbol{0} \\ \boldsymbol{a}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{v} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{w} &\rightarrow \min. \end{aligned}$$

It is easy to verify that the objective function at an admissible solution of the primal problem cannot be greater than the objective function at an admissible solution of the dual problem. The Duality Theorem in linear programming (cf. Schrijver [421]) states that both optima are equal to each other provided that both problems have admissible solutions or that one problem has an optimal solution.

The dual problem to the fractional k-cutset problem is the following: Find $y : \mathfrak{C}(P) \to \mathbb{R}_+$ and $z : P \to \mathbb{R}_+$ such that

$$\sum_{C \in \mathfrak{C}(P): p \in C} y(C) - z(p) \leq w(p) \text{ for all } p \in P, \tag{4.6}$$

and the function

$$\gamma_k(y, z) := k \sum_{C \in \mathfrak{C}(P)} y(C) - \sum_{p \in P} z(p)$$

is maximized.

By duality we have for any fractional k-cutset x (resp., in particular, for any k-cutset F) and any functions y, z satisfying (4.6)

$$\sum_{p \in P} w(p)x(p) \geq \gamma_k(y, z) \quad (\text{resp. } w(F) \geq \gamma_k(y, z)). \tag{4.7}$$

We can see this inequality also directly: We have

$$\begin{aligned} \gamma_k(y, z) &= k \sum_{C \in \mathfrak{C}(P)} y(C) - \sum_{p \in P} z(p) \\ &\leq \sum_{C \in \mathfrak{C}(P)} \left(\sum_{p \in C} x(p) y(C) \right) - \sum_{p \in P} z(p) \\ &= \sum_{p \in P} \left(\sum_{C \in \mathfrak{C}(P): p \in C} x(p) y(C) - z(p) \right) \leq \sum_{p \in P} x(p) w(p). \end{aligned}$$

Theorem 4.2.1. *Let k_0 be the smallest number of elements in a maximal chain of the weighted poset (P, w). Then for each $0 \leq k \leq k_0 - 1$ there exist a k-cutset F_k, a $(k+1)$-cutset F_{k+1} and functions $y : \mathfrak{C}(P) \to \mathbb{R}_+$, $z : P \to \mathbb{R}_+$ satisfying (4.6) such that*

$$w(F_k) = \gamma_k(y, z) \quad \textit{and} \quad w(F_{k+1}) = \gamma_{k+1}(y, z).$$

F_k and F_{k+1} are optimal k- (resp. $(k+1)$-) cutsets and y, z are optimal solutions of the dual to the fractional k- and $(k+1)$-cutset problem. In the case $k = 0$, we can choose $z \equiv 0$.

If w is integral then y and z can be chosen integral, too.

Proof. We will apply the algorithm of the proof of Theorem 4.1.3 in order to find F_k, F_{k+1}, y, and z. With our poset (P, w) we associate the network $N = (V, E, s, t, c)$ with cost function a as follows: $V := \{s, t\} \cup \{p : p \in P\} \cup \{p' : p \in P\}$ (where s and t are new vertices), $E := \{sp : p \text{ is minimal in } P\} \cup \{p't : p \text{ is maximal in } P\} \cup \{e_p^{(1)} = pp' : p \in P\} \cup \{e_p^{(2)} = pp' : p \in P\} \cup \{p_1' p_2 : p_1 \lessdot p_2\}$.

Here the union is understood in the multiset sense; we have two arcs from p to p', one denoted by $e_p^{(1)}$, the other by $e_p^{(2)}$. The capacity and cost are defined by

$$c(e) := \begin{cases} w(p) & \text{if } e = e_p^{(1)} \text{ for some } p \in P, \\ \infty & \text{otherwise}, \end{cases}$$

$$a(e) := \begin{cases} 1 & \text{if } e = e_p^{(2)} \text{ for some } p \in P, \\ 0 & \text{otherwise}. \end{cases}$$

The construction of the graph is presented in Figure 4.2 (the arcs are directed upward). Obviously, $G = (V, E)$ is a dag. Moreover, with a directed elementary

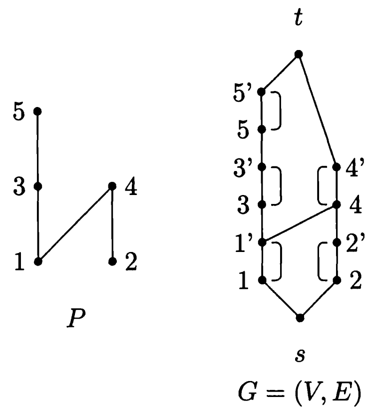


Figure 4.2

flow f along $s, p_1, p_1', p_2, p_2', \ldots, p_l, p_l', t$ of value 1 we can associate the (maximal) chain $C = (p_0 \lessdot p_1 \lessdot \cdots \lessdot p_l) \in \mathfrak{C}(P)$. If f is any flow in N then, by Lemma 4.1.3, f can be written in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_m f_m, \quad \lambda_i \geq 0, \quad f_i \text{ is a def for all } i.$$

Thus we may associate with f the function $y : \mathfrak{C}(P) \to \mathbb{R}_+$ defined by

$$y(C) := \begin{cases} \lambda_i & \text{if } C \text{ corresponds to } f_i \text{ as above,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that in this correspondence

$$f(e_p^{(1)}) + f(e_p^{(2)}) = \sum_{C \in \mathfrak{C}(P): p \in C} y(C), \tag{4.8}$$

$$a(f) = \sum_{p \in P} f(e_p^{(2)}), \tag{4.9}$$

$$v(f) = \sum_{C \in \mathfrak{C}(P)} y(C). \tag{4.10}$$

Moreover, if π is a fixed potential function and $\pi(t) < k_0$, then N_π has a cut of finite capacity, because otherwise there would have to be a flow with arbitrary large value. That is, there must be a def $s, p_1, p'_1, \ldots, p_l, p'_l, t$ going through $e_{p_1}^{(2)}, \ldots, e_{p_l}^{(2)}$ and this is by the construction of N_π only possible if $\pi(p_1) - \pi(s) = 0,\ \pi(p'_1) - \pi(p_1) = 1,\ \pi(p_2) - \pi(p'_1) = 0, \ldots,\ \pi(p'_l) - \pi(p_l) = 1,\ \pi(t) - \pi(p'_l) = 0$ which implies $\pi(t) - \pi(s) = \pi(t) = l \geq k_0 =$ minimal number of elements in a maximal chain, a contradiction to $\pi(t) < k_0$.

Accordingly, the finiteness of our algorithm is ensured until $\pi(t)$ is changed from $k_0 - 1$ to k_0. Now consider a situation where π is some actual potential with $\pi(t) = k, 0 \leq k \leq k_0$, and f is a corresponding actual flow. Define $F := \{p \in P : \pi(p') - \pi(p) = 1\}$.

Claim 1. F is a k-cutset.

Proof of Claim 1. During the algorithm, condition (4.3) is always satisfied; hence (since $c(e_p^{(2)}) = \infty$) $\pi(p') - \pi(p) \leq 1$ for all $p \in P$ and $\pi(e^+) - \pi(e^-) \leq 0$ for all arcs in E which are not of the form pp'. Consequently, if $C = (p_1 < p_2 < \cdots < p_l)$ is a maximal chain in P then $\pi(s) = 0,\ \pi(p_1) - \pi(s) \leq 0,\ \pi(p'_1) - \pi(p_1) \leq 1,\ \pi(p_2) - \pi(p'_1) \leq 0, \ldots,\ \pi(p'_l) - \pi(p_l) \leq 1,\ \pi(t) - \pi(p'_l) \leq 0$. Because the sum of the values on the LHS is $\pi(t) = k$, we must have at least k-times on the RHS a one; that is, we find at least k points p in $\{p_1, \ldots, p_l\}$ with $\pi(p') - \pi(p) = 1$. Therefore $|F \cap C| \geq k$. □

Claim 2. $w(F) = kv(f) - a(f)$.

Proof of Claim 2. Let $F_i := \{p \in F : \pi(p) = i \text{ (i.e., } \pi(p') = i + 1)\},\ i = 0, \ldots, k - 1$. Of course,

$$w(F) = \sum_{i=0}^{k-1} w(F_i). \tag{4.11}$$

The set S_i of all points of V having potential value $\leq i$ and the complement

$T_i := V - S_i$ define a cut (S_i, T_i) in the network N_π. By Lemma 4.1.1(a) we have

$$v(f) = f(S_i, T_i) - f(T_i, S_i), \quad i = 0, \dots, k-1. \tag{4.12}$$

Moreover, we have

$$f(T_i, S_i) = 0 \tag{4.13}$$

since for every arc $e \in E$ with $e^- \in T_i$, $e^+ \in S_i$ there holds $\pi(e^+) - \pi(e^-) \leq i - (i+1) < 0$, which implies by (4.3) that $f(e) = 0$. We have $f(S_i, T_i) = \sum_{p \in F_i} f(e_p^{(1)}) + f(e_p^{(2)})$ since (see the proof of Claim 1) the potential difference of the arcs can be at most 1, and it is equal to 1 only for arcs of the form pp'. Moreover, we know by (4.3) that the arcs $e_p^{(1)}$ with $p \in F_i$ must be saturated; consequently $f(e_p^{(1)}) = w(p)$ for all $p \in F_i$ and

$$f(S_i, T_i) = w(F_i) + \sum_{p \in F_i} f(e_p^{(2)}). \tag{4.14}$$

From (4.11)–(4.14) we derive that

$$kv(f) = w(F) + \sum_{p \in F} f(e_p^{(2)}).$$

Because of (4.3) we have

$$a(f) = \sum_{p \in P} f(e_p^{(2)}) = \sum_{p \in F} f(e_p^{(2)}),$$

which yields the assertion. □

The actual flow f corresponds to a function $y : \mathfrak{C}(P) \to \mathbb{R}_+$ as presented earlier. Define $z : P \to \mathbb{R}_+$ by $z(p) := f(e_p^{(2)})$. Then, in view of (4.8) and $f(e_p^{(1)}) \leq c(e_p^{(1)}) = w(p)$, y and z satisfy condition (4.6). Because of (4.9) and (4.10),

$$\gamma_k(y, z) = kv(f) - a(f).$$

Claim 2 then yields $w(F) = \gamma_k(y, z)$.

In the situation where the potential of t changes from k to $k+1$, $0 \leq k \leq k_0 - 1$, the actual flow f_{k+1} defines the functions y and z, and the two potentials π_k (before the change) and π_{k+1} (after the change) define as previously a k-cutset F_k and a $(k+1)$-cutset F_{k+1} with the desired properties. Since by (4.7), for any other k-cutset F'_k and for any other functions y', z' satisfying (4.6) it holds $w(F'_k) \geq \gamma_k(y, z) = w(F_k) \geq \gamma_k(y', z')$, F_k and y, z are optimal, and the same is true for F_{k+1}. Until $\pi(t)$ changes from 0 to 1, the arcs $e_p^{(2)}$, $p \in P$, are still void since they are not included in the graph G_π. Thus for $k = 0$ the defined function z satisfies $z(p) = 0$ for ail $p \in P$.

If w is integral, then the flow algorithm and the algorithm of Lemma 4.1.3 work only with integer values; thus also y and z are integral. ■

Many min–max theorems of this nature can be found in the survey of Schrijver [420]. The fact that there are y and z corresponding to a k-cutset as well to a $(k+1)$-cutset is called *t-phenomenon* by Hoffman and Schwartz [270]. The flow-theoretical approach gives a natural insight into this t-phenomenon.

Corollary 4.2.1.

(*a*) *We have* $c_k(P, w) = c_k^*(P, w)$ *for* $0 \leq k \leq k_0$.

(*b*) *There is a function* $y : \mathfrak{C}(P) \to \mathbb{R}_+$ *such that* $c_1(P, w) = \gamma_1(y, 0)$.

Proof. (a) Obviously, $c_k(P, w) \geq c_k^*(P, w)$. Take from Theorem 4.2.1 some k-cutset F_k and functions y, z. This gives, in view of (4.7),

$$c_k(P, w) \leq w(F_k) = \gamma_k(y, z) \leq c_k^*(P, w).$$

(b) We have in our construction $c_0(P, w) = \gamma_0(y, z)$ and $c_1(P, w) = \gamma_1(y, z)$ with $z = 0$. ■

Let us finally look at the important case when $w \equiv 1$ – that is, at unweighted posets P.

Corollary 4.2.2. *We have*

(*a*) $\min\{|F| :$ *F is a k-cutset in P* $\} = \max\{kt + |C_1 \cup \cdots \cup C_t| - \sum_{i=1}^{t} |C_i|\}$ *where the maximum ranges over all $t \geq 0$ and collections of maximal chains $C_1, \ldots, C_t$ in P. Moreover, the chains $C_1, \ldots, C_t$ can be chosen in such a way that for them the maximum in the above equation is attained not only for the parameter k but also for the parameter $k+1$.*

(*b*) *The minimum size of a 1-cutset equals the maximum number of pairwise disjoint maximal chains.*

Proof. (a) The LHS equals $w(F_k)$ (resp. $w(F_{k+1})$) in Theorem 4.2.1. So we have to show that the RHS equals $\gamma_k(y, z)$ (resp. $\gamma_{k+1}(y, z)$) in Theorem 4.2.1. Given our optimal integral y and z satisfying (4.6) we find a collection $C_1, \ldots, C_t$ of maximal chains by taking each chain $C \in \mathfrak{C}(P)$ exactly $y(C)$ times; that is, $t = \sum_{C \in \mathfrak{C}(P)} y(C)$. Let us put $\zeta(p) := |\{i : p \in C_i\}|$. Then because of (4.6), $0 \leq \zeta(p) \leq 1 + z(p)$. Moreover,

$$|\cup_{i=1}^{t} C_i| = |\{p \in P : \zeta(p) \geq 1\}|$$

and

$$\sum_{p \in P : \zeta(p) \geq 1} \zeta(p) = \sum_{p \in P} \zeta(p) = \sum_{i=1}^{t} |C_i|.$$

Thus

$$\begin{aligned}\gamma_k(y,z) &= kt - \sum_{p\in P:\zeta(p)=0} z(p) - \sum_{p\in P:\zeta(p)\geq 1} z(p) \leq kt - \sum_{p\in P:\zeta(p)\geq 1} (\zeta(p)-1)\\ &= kt - \sum_{i=1}^{t} |C_i| + |\cup_{i=1}^{t} C_i|.\end{aligned}$$

This means that the optimum $\gamma_k(y,z)$ is not greater than our RHS.

Conversely, if $C_1, \ldots, C_t$ is such an optimal collection of maximal chains, we define $y'(C)$ to be the number of times that C occurs in that collection and $z'(p) := \max\{0, \zeta(p)-1\}$, where $\zeta(p)$ is defined as above. Then y' and z' satisfy (4.6) and

$$\gamma_k(y', z') = kt - \sum_{p\in P:\zeta(p)\geq 1} (\zeta(p)-1) = kt - \sum_{i=1}^{t} |C_i| + |\cup_{i=1}^{t} C_i|.$$

This means that our RHS is not greater than $\gamma_k(y', z') \leq \gamma_k(y,z)$ where y and z are the optimal functions.

(b) Clearly, the maximum number of pairwise disjoint maximal chains is a lower bound for the minimum size of a 1-cutset. To prove that it is an upper bound we use (a). Consider an optimal collection $C_1, \ldots, C_t$ for the RHS in (a) with $k = 1$ where t is minimal. Then these chains are pairwise disjoint. Assume the contrary, for example, $C_{t-1} \cap C_t \neq \emptyset$. Then $|C_1 \cup \cdots \cup C_t| - |C_1 \cup \cdots \cup C_{t-1}| \leq |C_t| - 1$; hence, $(t-1) + |C_1 \cup \cdots \cup C_{t-1}| - \sum_{i=1}^{t-1} |C_i| \geq t + |C_1 \cup \cdots \cup C_t| - \sum_{i=1}^{t} |C_i|$, contradicting the optimality of $C_1, \ldots, C_t$ and the minimality of t. ■

By the way, it is not true that the minimum size of a family F in P meeting every maximal antichain at least once (called *fibre*) equals the maximum number of pairwise disjoint maximal antichains. This is shown by the following example of Lonc and Rival [351]; see Figure 4.3.

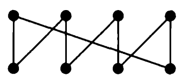


Figure 4.3

4.3. The k-family problem and related problems

In contrast to a k-cutset, a *k-family* is a subset F of P satisfying $|F \cap C| \leq k$ for all $C \in \mathfrak{C}^*(P)$, where $\mathfrak{C}^*(P)$ is the set of all chains in P. This definition is obviously equivalent to the one in Section 1.2. It is also easy to see that we could restrict ourselves to the set $\mathfrak{C}(P)$ of all maximal chains, but in the succeeding construction we find also some nonmaximal chains. Recall that the *k-family problem* is the following: Given a weighted poset (P, w) determine

$$d_k(P, w) := \max\{w(F) : F \text{ is a } k\text{-family}\}.$$

Before we proceed further with k-families, note that in a certain sense the k-family problem is nothing more than the 1-family problem – that is, *the Sperner problem*. Let $\mathfrak{A}(P)$ be the class of all antichains in P. It is easy to verify that $\mathfrak{A}(P)$ becomes a poset if we define for $A, A' \in \mathfrak{A}(P)$ that $A \le A'$ if for all $p \in A$ there is some $p' \in A'$ with $p \le p'$.

Lemma 4.3.1. *If F is a k-family in P, then there exist k pairwise disjoint antichains $A_1, \dots, A_k$ such that $A_k \le \cdots \le A_1$ and $F = \bigcup_{i=1}^k A_i$. Conversely, the union of k pairwise disjoint antichains is a k-family.*

Proof. The second claim is trivial and the first one follows by induction on k: Let A_1 be the set of all maximal elements of F (which is an antichain). Then $F - A_1$ is a $(k-1)$-family and by the induction hypothesis we find the corresponding antichains $A_2, \dots, A_k$. ■

Let C_k be the chain $(0 \lessdot 1 \lessdot \cdots \lessdot k-1)$. Given the poset (P, w) we define the poset (P_k, w_k) by $P_k := P \times C_k$, $w_k(p, i) := w(p)$ for all $p \in P, i \in C_k$. The following theorem goes back to an idea of Saks [405].

Theorem 4.3.1. *We have $d_k(P, w) = d_1(P_k, w_k)$ for $k = 1, 2, \dots$.*

Proof. "$\ge$:" Let A be an antichain in (P_k, w_k) such that $w_k(A) = d_1(P_k, w_k)$. Define $F := \{p \in P : (p, i) \in A \text{ for some } i\}$. Then F is a k-family in P with $w(F) = w_k(A)$.

"$\le$:" Let F be a k-family in P with $w(F) = d_k(P, w)$. We partition F according to Lemma 4.3.1 into k (possibly empty) antichains, where $A_k \le \cdots \le A_1$. Then we define $A := \bigcup_{i=1}^k (A_i, i)$ where $(A_i, i) := \{(p, i) : p \in A_i\}$. Then A is an antichain in P_k with $w_k(A) = w(F)$. ■

In the preceding construction we have constructed a "larger" poset. Berenguer, Diaz, and Harper [46] proposed a flow algorithm for the determination of $d_k(P, w)$ where the number of vertices is essentially $(k+1)|P|$. We may avoid the factor $k+1$ in the size of the network if we use the method described below, which also gives more theoretical insight into the optimization problems.

A *fractional k-family* is a function $x : P \to \mathbb{R}$ such that

$$\sum_{p \in C} x(p) \le k \text{ for all } C \in \mathfrak{C}^*(P),$$

$$0 \le x(p) \le 1 \text{ for all } p \in P,$$

and the *fractional k-family problem* consists in the determination of

$$d_k^*(P, w) := \max \left\{ \sum_{p \in P} w(p)x(p) : x \text{ is a fractional } k\text{-family} \right\}.$$

Dual to this linear programming problem is the following: Find $y : \mathfrak{C}^*(P) \to \mathbb{R}_+$ and $z : P \to \mathbb{R}_+$ such that

$$\sum_{C \in \mathfrak{C}^*(P): p \in C} y(C) + z(p) \geq w(p) \text{ for all } p \in P \tag{4.15}$$

and the function

$$\delta_k(y, z) := k \sum_{C \in \mathfrak{C}^*(P)} y(C) + \sum_{p \in P} z(p)$$

is minimized. It follows by duality (and can be seen, as for the k-cutsets, directly) that for any fractional k-family x (resp. in particular for any k-family F)

$$\sum_{p \in P} w(p)x(p) \leq \delta_k(y, z) \quad (\text{resp. } w(F) \leq \delta_k(y, z)). \tag{4.16}$$

Theorem 4.3.2. *For each $k = 0, 1, \ldots$, there exists a k-family F_k, a $(k+1)$-family F_{k+1} and functions $y : \mathfrak{C}^*(P) \to \mathbb{R}_+$, $z : P \to \mathbb{R}_+$ satisfying (4.15) such that $w(F_k) = \delta_k(y, z)$ and $w(F_{k+1}) = \delta_{k+1}(y, z)$. The families F_k and F_{k+1} are optimal k- (resp. $(k+1)$-) families and y, z are optimal solutions of the dual to the fractional k- and $(k+1)$-family problem. If w is integral, then y and z can be chosen integral, too.*

Proof. Let us apply again the algorithm of the proof of Theorem 4.1.3. This time we consider the following network $N = (V, E, s, t, c)$ with cost function a:

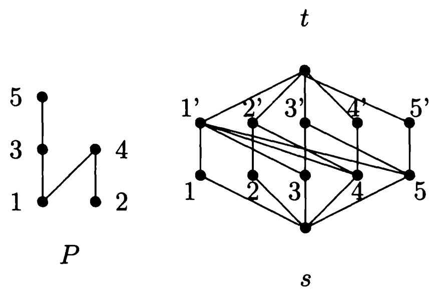
$$V := \{s, t\} \cup \{p : p \in P\} \cup \{p' : p \in P\} \qquad (s \text{ and } t \text{ are new vertices}),$$
$$E := \{sp : p \in P\} \cup \{p't : p \in P\} \cup \{pp' : p \in P\} \cup \{qp' : p < q\}.$$

The capacity and cost are defined by

$$c(e) := \begin{cases} w(p) & \text{if } e = sp \text{ or } e = p't \text{ for some } p \in P, \\ \infty & \text{otherwise,} \end{cases}$$

$$a(e) := \begin{cases} 1 & \text{if } e = pp' \text{ for some } p \in P, \\ 0 & \text{otherwise.} \end{cases}$$

For the construction of the network, see Figure 4.4. First of all we will discuss a correspondence between flows f in N and functions $y : \mathfrak{C}^*(P) \to \mathbb{R}_+$ and $z : P \to \mathbb{R}_+$.



$G = (V, E)$

Figure 4.4

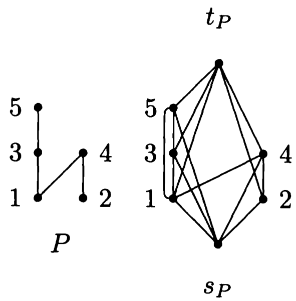
Let f be given with value $v(f)$ and cost $a(f)$. Let f_p be the directed elementary flow (def) through s, p, p', t. Obviously,

$$f' := f - \sum_{p \in P} f(pp') f_p$$

is a flow in N of value $v(f') = v(f) - a(f)$. Since $f'(pp') = 0$ for all $p \in P$ we can consider f' also as a flow in the network N' which arises from N by deleting all arcs pp'.

Now consider a third network $N_P = (V_P, E_P, s_P, t_P)$ (without a capacity), where $V_P := P \cup \{s_P, t_P\}$, $E_P := \{s_P p : p \in P\} \cup \{pt_P : p \in P\} \cup \{pq : p < q\}$.

For the construction of the new network, see Figure 4.5. For given f' in N'



N_P

Figure 4.5

(which was obtained from f in N), we define a flow $f_P \in N_P$ as follows: We set

$$f_P(e) := \begin{cases} f'(qp') & \text{if } e = pq, \\ w(p) - f(pp') - f'(sp) & \text{if } e = s_P p, \\ w(p) - f(pp') - f'(p't) & \text{if } e = pt_P. \end{cases}$$

Then the conservation of flow is satisfied for f_P, since for all $p \in P$

$$\begin{aligned}\sum_{e^-=p} f_P(e) &= w(p) - f(pp') - f'(p't) + \sum_{q>p} f'(qp') \\ &= w(p) - f(pp'),\end{aligned} \tag{4.17}$$

$$\begin{aligned}\sum_{e^+=p} f_P(e) &= w(p) - f(pp') - f'(sp) + \sum_{q<p} f'(pq') \\ &= w(p) - f(pp'),\end{aligned} \tag{4.18}$$

and the capacity constraints $f_P(p) \geq 0$ are satisfied by the definition of the capacity in N (resp. N'). We have

$$v(f_P) = \sum_{p \in P} f_P(s_P p) = w(P) - a(f) - v(f') = w(P) - v(f). \tag{4.19}$$

(V_P, E_P) is a dag; hence we can write again f_P as a nonnegative combination of defs:

$$f_P = \lambda_1 f_1 + \cdots + \lambda_m f_m, \lambda_i \geq 0.$$

Each f_i goes through certain vertices $s_P, p_1, \ldots, p_l, t_P$ where $C := (p_1 < \cdots < p_l) \in \mathfrak{C}^*(P)$.

Thus we may define

$$y(C) := \begin{cases} \lambda_i & \text{if } C \text{ corresponds to } f_i \text{ as above,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$v(f_P) = \lambda_1 + \cdots + \lambda_m = \sum_{C \in \mathfrak{C}^*(P)} y(C), \tag{4.20}$$

and in view of (4.17) and (4.18)

$$w(p) - f(pp') = \sum_{C \in \mathfrak{C}^*(P): p \in C} y(C). \tag{4.21}$$

Finally, if we define $z(p) := f(pp')$ for all $p \in P$, we have $y : \mathfrak{C}^*(P) \to \mathbb{R}_+$ and $z : P \to \mathbb{R}_+$ satisfying (4.15) *with equality* and

$$\delta_k(y, z) = kv(f_P) + a(f) = k(w(P) - v(f)) + a(f). \tag{4.22}$$

Of course, we can also go this way backward; that is, having functions y and z we can construct a flow f in our original network N such that (4.22) holds.

So our final task is the construction of a k-family F_k, a $(k+1)$-family F_{k+1} as well as a flow f in N such that

$$\begin{aligned} w(F_k) &= k(w(P) - v(f)) + a(f), \\ w(F_{k+1}) &= (k+1)(w(P) - v(f)) + a(f). \end{aligned}$$

Now let us run the algorithm of the proof of Theorem 4.1.3. Consider some actual potential π with $\pi(t) = k$ and some actual flow f. Define $F := \{p \in P : \pi(p') - \pi(p) = 1\}$.

Claim 1. F is a k-family.

Proof of Claim 1. Assume that there are $p_1, \ldots, p_{k+1} \in F$ such that $p_1 < \cdots < p_{k+1}$. Since the arcs $p_i p'_{i-1}, i = k+1, \ldots, 2$, have infinite capacity and because of (4.3) we have $\pi(p'_{i-1}) - \pi(p_i) \leq 0$ for all i. Consequently, $0 \leq \pi(p_1) = \pi(p'_1) - 1 \leq \pi(p_2) - 1 = \pi(p'_2) - 2 \leq \cdots \leq \pi(p_{k+1}) - k = \pi(p'_{k+1}) - (k+1) \leq \pi(t) - (k+1) < 0$, a contradiction. □

Claim 2. $w(F) = k(w(P) - v(f)) + a(f)$.

Proof of Claim 2. Let $F_i := \{p \in F : \pi(p) = i \text{ (i.e. } \pi(p') = i+1)\}$, $i = 0, \ldots, k-1$. We repeat the beginning of the proof of Claim 2 for k-cutsets (Theorem 4.2.1) and find that

$$v(f) = f(S_i, T_i), \tag{4.23}$$

where S_i is the set of all points of V having potential value $\leq i$, $T_i := V - S_i$, $i = 0, \ldots, k-1$.

Let $A_i := \{p \in P : \pi(p) > i\}$, $B_i := \{p \in P : \pi(p') \leq i\}$. Then

$$\begin{aligned} f(S_i, T_i) &= \sum_{p \in A_i} f(sp) + \sum_{p \in B_i} f(pt) + \sum_{p \in P - A_i, q \in P - B_i, q \leq p} f(pq') \\ &= w(A_i) + w(B_i) + \sum_{p \in F_i} f(pp') \end{aligned} \tag{4.24}$$

since the arcs sp, $p \in A_i$, and $p't$, $p \in B_i$, are saturated by (4.3) (note $\pi(s) = 0, \pi(t) = k$), and the only arcs pq' with $\pi(p) \leq i, \pi(q') > i$ are arcs pp' with $\pi(p) = i, \pi(p') = i+1$, again by (4.3) (these arcs all have infinite capacity). We have

$$F_i = P - (A_i \cup B_i), \tag{4.25}$$

since $p \in P - (A_i \cup B_i)$ iff $\pi(p) \leq i, \pi(p') > i$ – that is, iff $\pi(p) = i, \pi(p') = i+1$. Moreover,

$$w(A_i \cap B_i) = 0. \tag{4.26}$$

Assume the contrary. Then there is some $p \in A_i \cap B_i$ with $w(p) > 0$ and $\pi(p) > i, \pi(p') \leq i$. The arcs sp and $p't$ are saturated and the arc pp' has zero flow by (4.3). Thus there must be further points $q, r \in P$ such that $f(pq') > 0$ and $f(rp') > 0$ implying $\pi(q') = \pi(p) > i$ and $\pi(r) = \pi(p') \leq i$. But by the construction of the network, $q < p$ and $p < r$; that is, $q < r$ and $rq' \in E$. This is impossible since $\pi(q') - \pi(r) > 0$ and rq' has infinite capacity. From

(4.23)–(4.26) we derive

$$v(f) = w(P) - w(F_i) + \sum_{p \in F_i} f(pp'),$$

and summing up these equalities for $i = 0, \dots, k-1$ gives

$$kv(f) = kw(P) - w(F) + a(f),$$

$$w(F) = k(w(P) - v(f)) + a(f).$$

□

In the situation where the potential of t changes from k to $k+1$ we get our desired k-families F_k and F_{k+1}. The additional assertions in the theorem follow as for k-cutsets. ■

Corollary 4.3.1. *We have $d_k(P, w) = d_k^*(P, w)$ for all k.*

Let us look also at the case of unweighted posets – that is, $w \equiv 1$. Here Dilworth's Theorem 4.0.1 comes out very easily.

Corollary 4.3.2. *We have*

(*a*) (*Greene and Kleitman* [233]).

$$\max\{|F| : F \text{ is a } k\text{-family in } P\}$$

$$= \min\left\{\sum_{C \in \mathfrak{D}} \min\{|C|, k\} : \mathfrak{D} \text{ is a chain partition of } P\right\}.$$

Moreover, the chain partition $\mathfrak{D}$ can be chosen in such a way that the minimum in the above equation is attained not only for the parameter k but also for the parameter $k+1$.

(*b*) (*Dilworth* [136]). *The maximum size of an antichain equals the minimum number of chains in a chain partition of P.*

Proof. Of course, (b) follows from (a) with $k = 1$. So let us prove (a). If F is any k-family and $\mathfrak{D}$ is any chain partition, then

$$|F| = \sum_{C \in \mathfrak{D}} |F \cap C| \le \sum_{C \in \mathfrak{D}} \min\{|C|, k\}.$$

Furthermore, let y and z be our optimal functions from Theorem 4.3.2. In the proof we noted that y and z satisfy (4.15) with equality. Since $w \equiv 1$, $y : \mathfrak{C}^*(P) \to \{0, 1\}$ and $z : P \to \{0, 1\}$. If $y(C) = 1$, then $|C| \ge k$, since otherwise we could change $y(C)$ to zero and $z(p)$, $p \in C$, to 1 obtaining a smaller value in the objective function. The functions y, z provide the chain partition $\mathfrak{D}$ whose chains are all

$C \in \mathfrak{C}^*(P)$ with $y(C) = 1$ and the one-element chains $\{p\}$ with $z(p) = 1$. Finally,

$$\sum_{C \in \mathfrak{D}} \min\{|C|, k\} = \delta_k(y, z) = \max\{|F| : F \text{ is a } k\text{-family}\},$$

which proves the claim. ∎

A partition $\mathfrak{D}$ is called *k-saturated* if the minimum on the RHS of Corollary 4.3.2 is attained at $\mathfrak{D}$. This corollary says that for each k there exist chain partitions that are simultaneously k- and $(k+1)$-saturated. There are posets for which chain partitions being k-saturated for every k do not exist. An example was given by Greene and Kleitman [233]; see Figure 4.6. Moreover West [466] constructed

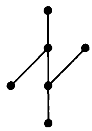


Figure 4.6

posets having no chain partition that is k-saturated for any two nonconsecutive values of k (not exceeding the largest number of elements in a chain).

Corollary 4.3.3. *The difference $d_{k+1}(P, w) - d_k(P, w)$ is decreasing.*

Proof. Again take the optimal families F_k, F_{k+1} from Theorem 4.3.2 and consider the flow f yielding these families. Then (note Claim 2 in the proof of Theorem 4.3.2)

$$\begin{aligned} d_{k+1}(P, w) - d_k(P, w) &= w(F_{k+1}) - w(F_k) \\ &= (k+1)(w(P) - v(f)) + a(f) \\ &\quad -(k(w(P) - v(f)) + a(f)) = w(P) - v(f). \end{aligned}$$

Since the flow value increases during the algorithm with increasing k, the difference $w(P) - v(f)$ decreases. ∎

We still continue the discussion of the proof of Theorem 4.3.2. An *antichain partition $\mathfrak{A}$ of P* is defined similarly to a chain partition. Remember that every k-family is a union of k antichains.

Corollary 4.3.4. *We have*

(*a*) (*Greene* [229])

$$\max\{|F| : F \textit{ is a union of } k \textit{ chains in } P\}$$
$$= \min\left\{\sum_{A \in \mathfrak{A}} \min\{|A|, k\} : \mathfrak{A} \textit{ is an antichain partition of } P\right\}.$$

Moreover, the antichain partition $\mathfrak{A}$ can be chosen in such a way that for it the minimum in the preceding equation is attained not only for the parameter k but also for the parameter $(k+1)$.

(*b*) (*Mirsky* [371]) *The maximum size of a chain equals the minimum number of antichains in an antichain partition of P.*

Proof. Again (b) can be derived from (a) with $k=1$, but I encourage the reader to prove (b) directly.

For any union of k chains and any antichain partition $\mathfrak{A}$, we have $|F| = \sum_{A\in\mathfrak{A}} |F \cap A| \leq \sum_{A\in\mathfrak{A}} \min\{|A|, k\}$ since each antichain A can contain at most $\min\{|A|, k\}$ elements of F. Consequently, max $\leq$ min. Let us look at the proof of Theorem 4.3.2 in order to prove max $\geq$ min . Since $w \equiv 1$, we are working with integral flows and integral functions y, z. For fixed potential π, let us increase the flow value always only by 1 (one could possibly do more along flow-augmenting paths, but in that case increase iteratively by 1). The maximum flow value in N is $w(P) = |P|$. We achieve a situation where $v(f) = |P| - k$. Let π be the actual potential. Then by (4.19)

$$v(f_P) = |P| - (|P| - k) = k.$$

In view of (4.20) there are at most k chains C with $y(C) > 0$. Thus the union F of these chains is a union of k (possibly empty) chains. Because of (4.15) F contains at least all elements p with $z(p) = 0$, hence

$$|F| \geq |P| - \sum_{p\in P} z(p) = |P| - a(f)$$

(remember that $z(p) = f(pp'))$.

We find as follows the partition into antichains: Suppose that $k_0 := \pi(t)$. In the proof of Claim 2 we worked with sets $F_i, i = 0, \ldots, k_0 - 1$. Each F_i is an antichain, since otherwise there would be $p, q \in F_i$ with $p < q$ and $\pi(p) = \pi(q) = i, \pi(p') = \pi(q') = i + 1$, which is impossible since qp' is an arc of infinite capacity but having potential difference greater than its cost. Look at the antichain partition $\mathfrak{A}$ whose antichains are $F_0, \ldots, F_{k_0-1}$ and the one-element classes $\{p\}$ with $p \in P - F$. Then, noting Claim 2,

$$\begin{aligned}\sum_{A\in\mathfrak{A}} \min\{|A|, k\} &\leq kk_0 + |P| - |F| \\ &= kk_0 + |P| - k_0(|P| - v(f)) - a(f) \\ &= kk_0 + |P| - k_0k - a(f) = |P| - a(f).\end{aligned}$$

Thus we found F and $\mathfrak{A}$ with

$$|F| \geq \sum_{A\in\mathfrak{A}} \min\{|A|, k\}$$

yielding the desired inequality max $\geq$ min.

The situation where for fixed potential π the flow value f changes from $|P| - (k+1)$ to $|P| - k$ yields the antichain partition $\mathfrak{A}$, which is optimal for the parameters k and $k+1$. ∎

There exist several proofs of the results in Corollaries 4.3.2–4.3.4. Let us mention here Fomin [182], Hoffman and Schwartz, [270], Saks [405], and in particular Frank [186], whose proof for the unweighted case is the basis of our proof of the weighted case. Generalizations and related results are by Edmonds and Giles [143], Linial [348], Berge [48], Hoffman [269], Saks [404], Gavril [219], Cameron [91], Cameron and Edmonds [90], Felsner [180], and Sarrafzadeh and Lou [414]. In particular for Dilworth's theorem there are a lot of different proofs, for example, by Dantzig and Hoffman [120], Fulkerson [205], Tverberg [454], and Harzheim [264]. For more informations, we refer to the above paper of Saks [404] and to the surveys of Hoffman [269], Schrijver [420], Berge [49], and West [465].

4.4. The variance problem

Given a poset P, a function $x : P \to \mathbb{R}$ is called a *representation of P* if

$$x(q) - x(p) \geq 1 \text{ for all } q > p.$$

This notion was introduced by Alekseev [23] in order to find an asymptotic formula for the width of products of posets; see Section 7.2. An example of a representation is the rank function r if P is ranked, and, in general, the height function h. Note that h can be calculated as a special case of the *critical path method* (cf. Sedgewick [423]): Label the minimal elements p of P with $h(p) := 0$. Let, after some steps, S be the set of labeled vertices. Then look in the next step for some minimal unlabeled point q (i.e., there is no unlabeled point below q) covering a labeled point p with largest label and put $h(q) := h(p) + 1$. Continue until all points are labeled. By induction it follows easily that the calculated value $h(p)$ is really the height of p.

The height function of a small poset is illustrated in Figure 4.7. Now consider again positively weighted posets (P, w). For the representation x of (P, w) (i.e., x is a representation of P), we define the *expected value* (also called *mean*) μ_x and the *variance* σ_x^2 by

$$\mu_x := \frac{1}{w(P)} \sum_{p \in P} w(p)x(p)$$

$$\sigma_x^2 := \frac{1}{w(P)} \sum_{p \in P} w(p)(x(p) - \mu_x)^2 = \frac{1}{w(P)} \sum_{p \in P} w(p)x^2(p) - \mu_x^2,$$

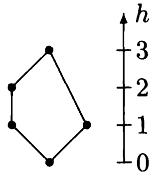


Figure 4.7

respectively. Note that μ_x and σ_x^2 can be defined for any function $x : P \to \mathbb{R}$. The *variance problem* is the following: Given a weighted poset (P, w), find a representation x such that σ_x^2 is minimal with respect to all representations. Such a representation is then called *optimal*. Though it follows easily from a compactness argument that optimal representations really exist – in other words, that the minimum is attained – we will derive this fact from the finiteness of the algorithm below. The variance of an optimal representation is called the *variance of* (P, w) and denoted by $\sigma^2(P, w)$. First we need some technical details. The following lemma is obvious.

Lemma 4.4.1. *Let x be a representation of (P, w), and let $y(p) = x(p) + c$ for all $p \in P$. Then also y is a representation of (P, w), and $\mu_y = \mu_x + c$, $\sigma_y^2 = \sigma_x^2$.*

For a nonempty subset F of P, we define its *expected value* by

$$\mu_x(F) := \frac{1}{w(F)} \sum_{p \in F} w(p)x(p),$$

and put $\mu_x(\emptyset) := \infty$. Note that $\mu_x = \mu_x(P)$.

Lemma 4.4.2. *Let $x : P \to \mathbb{R}$ be any function and let F and I be disjoint subsets of P with $\mu_x(F) < \mu_x$, and $\mu_x(I) > \mu_x$. Define $y_\epsilon : P \to \mathbb{R}$ by*

$$y_\epsilon(p) := \begin{cases} x(p) + \epsilon w(I) & \textit{if } p \in F, \\ x(p) - \epsilon w(F) & \textit{if } p \in I, \\ x(p) & \textit{otherwise.} \end{cases}$$

Then $\mu_{y_\epsilon} = \mu_x$ and $\sigma_{y_\epsilon}^2$ is strongly decreasing for $0 \le \epsilon \le \frac{1}{w(F)+w(I)}(\mu_x(I) - \mu_x(F)) =: \epsilon_0$.

Proof. The equality $\mu_{y_\epsilon} = \mu_x$ is easy. Moreover, if $0 < \epsilon < \frac{\mu_x(I)-\mu_x(F)}{w(F)+w(I)}$,

$$\begin{aligned}\frac{d}{d\epsilon}\sigma_{y_\epsilon}^2 &= \frac{d}{d\epsilon}\left(\sigma_x^2 + \frac{1}{w(P)}\left(\sum_{p\in F} w(p)((x(p)+\epsilon w(I))^2 - x^2(p))\right.\right.\\ &\quad \left.\left.+\sum_{p\in I} w(p)((x(p)-\epsilon w(F))^2 - x^2(p))\right)\right)\\ &= \frac{2w(I)w(F)}{w(P)}(w(F)+w(I))\left(\epsilon - \frac{\mu_x(I)-\mu_x(F)}{w(F)+w(I)}\right) < 0.\end{aligned}$$

■

Lemma 4.4.3. *If g is an increasing (resp. decreasing) function from P into $\mathbb{R}$, then g is the sum of a constant and a nonnegative combination of characteristic functions of filters (resp. ideals); that is, there are filters (resp. ideals) $F_1, \dots, F_l$ and real numbers $\lambda_0, \lambda_1, \dots, \lambda_l$ with $\lambda_1, \dots, \lambda_l \geq 0$ such that*

$$g = \lambda_0 + \sum_{i=1}^{l} \lambda_i \varphi_{F_i}.$$

Proof. We consider only increasing functions, the other case is analogous. It is enough to prove the statement for nonnegative functions g with $\lambda_0 = 0$ since each function can be written as a nonnegative function plus a constant, say λ_0. Now we prove this special case by induction on the support $|\operatorname{supp}(g)|$. The case $|\operatorname{supp}(g)| = 0$ is clear. If $|\operatorname{supp}(g)| > 0$, put $F_1 := \operatorname{supp}(g)$. Obviously, F_1 is a filter. Further put $\lambda_1 := \min\{g(p) : p \in F_1\}$. Then $g' = g - \lambda_1\varphi_{F_1}$ is nonnegative, increasing, and has smaller support size. The induction hypothesis applied to g' gives the result. ■

Let E be the arc set of the Hasse diagram $H(P)$ of P. For a given representation x, define

$$E_x := \{e \in E : x(e^+) - x(e^-) = 1\}.$$

The graph $G_x = (P, E_x)$ is said to be the *active graph*. The poset whose Hasse diagram is G_x is called the *active poset* and denoted by P_x. Finally, a function $f : P \to \mathbb{R}_+$ such that

$$\begin{aligned}\sum_{e^+=p} f(e) - \sum_{e^-=p} f(e) &= w(p)(x(p) - \mu_x) \quad \text{for all } p \in P,\\ f(e) &= 0 \quad \text{for all } e \in E - E_x\end{aligned}$$

is called a *representation flow on* (P, w) *relative to* x. For the sake of brevity, we set

$$p_f^{\pm} := \sum_{e^\pm = p} f(e).$$

The following theorem was proven for unweighted posets by Alekseev [23]. We present our own proof of the weighted case.

Theorem 4.4.1. *Let x be a representation of (P, w). Then the following conditions are equivalent:*

(i) x is an optimal representation.
(ii) $\sum_{p \in P} w(p)g(p)(x(p) - \mu_x) \geq 0$ for all increasing functions $g : P_x \to \mathbb{R}$.
(iii) $\mu_x(F) \geq \mu_x$ for all filters F in P_x.
(iv) $\mu_x(I) \leq \mu_x$ for all ideals I in P_x.
(v) There exists a representation flow on (P, w) relative to x.

Proof. In view of Lemma 4.4.1 we may restrict ourselves to representations x with $\mu_x = 0$.

(ii) $\to$ (i). Let y be another representation of P with $\mu_y = 0$. Then $g = y - x$ is an increasing function from P_x into $\mathbb{R}$ since for $e \in E_x$ we have $y(e^+) - y(e^-) \geq 1$ (y is a representation) but $x(e^+) - x(e^-) = 1$. We want to show that $\sigma_y^2 \geq \sigma_x^2$; that is,

$$\sum_{p \in P} w(p)y^2(p) \geq \sum_{p \in P} w(p)x^2(p). \tag{4.27}$$

From (ii) with $\mu_x = 0$ and $g = y - x$ we obtain

$$2\sum_{p \in P} w(p)y(p)x(p) \geq 2\sum_{p \in P} w(p)x^2(p). \tag{4.28}$$

Evidently,

$$\sum_{p \in P} w(p)(y(p) - x(p))^2 \geq 0. \tag{4.29}$$

Addition of (4.28) and (4.29) gives the desired inequality (4.27).

(i) $\to$ (iii). Let x be optimal and assume that there is some F in P_x with $\mu_x(F) < 0 = \mu_x$. Let $I := P - F$. Then obviously $\mu_x(I) > 0$. Let

$$\epsilon_1 := \frac{1}{w(I) + w(F)} \min\{x(e^+) - x(e^-) - 1 : \quad e \in E - E_x\}.$$

By definition of E_x, $\epsilon_1 > 0$. Now consider the function y_ϵ from Lemma 4.4.2 for $0 \leq \epsilon \leq \min\{\epsilon_0, \epsilon_1\}$. By the following reason, y_ϵ is a representation: Since x is a representation we must consider only arcs pq with $p \in F,\ q \in I$. The set F is a filter in P_x, thus $pq \in E - E_x$. Finally

$$\begin{aligned} y_\epsilon(q) - y_\epsilon(p) &= x(q) - x(p) - \epsilon(w(I) + w(F)) \\ &\geq x(q) - x(p) - \min\{x(e^+) - x(e^-) - 1 : e \in E - E_x\} \geq 1. \end{aligned}$$

From Lemma 4.4.2 we obtain $\sigma_{y_\epsilon}^2 < \sigma_x^2$, a contradiction.

(iii) → (ii). Write g in the form of Lemma 4.4.3. Then

$$\begin{aligned}\sum_{p\in P} w(p)g(p)x(p) &= \sum_{p\in P} w(p)\left(\lambda_0 + \sum_{i=1}^{l}\lambda_i\varphi_{F_i}\right)x(p)\\ &= \sum_{i=1}^{l}\lambda_i\left(\sum_{p\in P} w(p)x(p)\varphi_{F_i}(p)\right)\\ &= \sum_{i=1}^{l}\lambda_i w(F_i)\mu_x(F_i) \geq 0.\end{aligned}$$

(iii) ↔ (iv). This assertion follows from the facts that F is a filter in P_x iff $I = P - F$ is an ideal in P_x and $\mu_x(F) \geq \mu_x$ iff $\mu_x(P - F) \leq \mu_x$.

(iii) ↔ (v). For each representation x with $\mu_x = 0$, let us define the network $N = (V_N, E_N, s, t, c_x)$, where $V_N := P \cup \{s, t\}$(s and t are new vertices), $E_N := E \cup \{sp : p \in P\} \cup \{pt : p \in P\}$ (remember that $E = \{pq : p \lessdot q\}$),

$$c_x(e) := \begin{cases} \max\{0, -w(p)x(p)\} & \text{if } e = sp,\ p \in P,\\ \max\{0, w(p)x(p)\} & \text{if } e = pt,\ p \in P,\\ 0 & \text{if } e \in E - E_x,\\ \infty & \text{if } e \in E_x.\end{cases}$$

An example of the new network is illustrated in Figure 4.8.

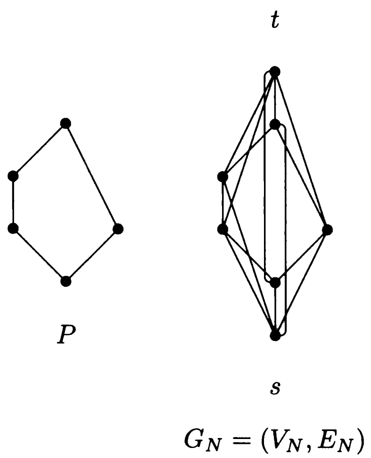


Figure 4.8

Claim. Condition (iii) holds iff $(\{s\}, V_N - \{s\})$ and $(V_N - \{t\}, \{t\})$ are minimal cuts in N.

Proof of Claim. First note that

$$\begin{aligned} c(V - \{t\}, \{t\}) - c(\{s\}, V - \{s\}) &= \sum_{p \in P} \max\{0, w(p)x(p)\} \\ &\quad - \max\{0, -w(p)x(p)\} \\ &= \sum_{p \in P} w(p)x(p) = 0; \end{aligned}$$

that is, both these cuts have equal capacity. Let (S, T) be any cut of finite capacity in N and let $F := S - \{s\}$, $I := T - \{t\}$ $(= P - F)$. Then F and I are a filter and an ideal in P_x, respectively, since otherwise there would be some $p \in F$, $q \in I$ with $p < q$ and $pq \in E_x$, but then $c(pq) = \infty$ in contradiction to the finiteness of the cut capacity. Conversely, if F is a filter in P_x and $I := P - F$, then $S := F \cup \{s\}$, $T := I \cup \{t\}$ define a cut of finite capacity. For these cuts, we have $c(\{s\}, V - \{s\}) \leq c(S, T)$ iff

$$\begin{aligned} \sum_{p \in P} c_x(sp) &\leq \sum_{p \in I} c_x(sp) + \sum_{p \in F} c_x(pt), \\ \sum_{p \in F} c_x(sp) &\leq \sum_{p \in F} c_x(pt), \\ 0 &\leq \sum_{p \in F} (\max\{0, w(p)x(p)\} - \max\{0, -w(p)x(p)\}), \\ \mu_x = 0 &\leq w(F)\mu_x(F) = \sum_{p \in P} w(p)x(p). \end{aligned}$$

Thus $(\{s\}, V - \{s\})$ has minimal capacity iff $\mu_x(F) \geq 0$ for every filter F in P_x. □

By the real version of the Max-Flow Min-Cut Theorem (Corollary 4.1.1) and in view of Lemma 4.1.1, the cuts $(\{s\}, V_N - \{s\})$ and $(V_N - \{t\}, \{t\})$ are minimal iff there exists a flow f_N in N (the maximal flow) such that the arcs sp and pt $(p \in P)$ are saturated. Given such a flow f_N, its restriction to E has the properties of a representation flow since for $e \in E$

$$\begin{aligned} c_x(sp) + p_f^+ &= p_f^- + c_x(pt), \\ p_f^+ - p_f^- &= \max\{0, w(p)x(p)\} - \max\{0, -w(p)x(p)\} = w(p)x(p) \end{aligned}$$

and, of course, $f(e) = 0$ (since $c(e) = 0$) for all $e \in E - E_x$. Conversely, given a function f satisfying the condition of a representation flow, we can extend f in a natural way to a flow f_N in N such that the arcs sp and pt $(p \in P)$ are saturated. ■

If we consider the variance problem as a quadratic optimization problem with linear constraints and convex objective function then the equivalence (i) $\leftrightarrow$ (v) follows from the theory of Kuhn and Tucker (cf. Martos [362, pp. 123 ff]): The *Lagrange function* is $L(x, f) := \sigma_x^2 + \sum_{e\in E} f(e)\left(1 - x(e^+) + x(e^-)\right)$, and the necessary and sufficient conditions for an optimal solution read:

$$x(e^+) - x(e^-) \geq 1 \text{ for all } e \in E,$$
$$p_f^+ - p_f^- = \frac{2}{w(P)} w(p)(x(p) - \mu_x) \text{ for all } p \in P,$$
$$\sum_{e\in E} f(e)\left(1 - x(e^+) + x(e^-)\right) = 0$$

with $f : E \to \mathbb{R}_+$. Of course, we may omit the factor $\frac{2}{w(P)}$ in the second condition, so the arc-values of the representation flow can be interpreted as the *Lagrange multipliers*.

We can use condition (iv) and the flow algorithm to construct an optimal representation together with the function f yielding then also $\sigma^2(P, w)$. In order to do this we need some further condition:

Let us say that (P, w) is *in equilibrium with respect to the representation* x if for every connected component C of the active graph G_x there holds $\mu_x(C) = 0$. Recall that a *(connected) component* in a directed graph is a maximal set C of vertices such that for any two elements v, w in C there exists a sequence $(v = v_0, v_1, \ldots, v_n = w)$ of vertices with the property that $v_i v_{i+1}$ is an edge in the underlying undirected graph for all $i \in \{0, \ldots, k-1\}$. The graph is called *connected* if it has only one connected component.

Lemma 4.4.4. *Let (P, w) be in equilibrium with respect to x. Then the values $x(p)$, $p \in P$, can be uniquely determined from the active graph.*

Proof. The value of x on any point of any component C determines the values of x on all other points of C. The value of the first point must be chosen in such a way that the expected value of C becomes zero. ■

Lemma 4.4.5. *Given a representation x, we can construct in a finite number of steps a representation y such that $\sigma_x^2 \geq \sigma_y^2$ and (P, w) is in equilibrium with respect to y.*

Proof. Noting Lemma 4.4.1, we may assume that $\mu_x = 0$. Let $C_1, \ldots, C_{k_x}$ be the components of G_x. Let

$$J_x^< := \{j \in \{1, \ldots, k_x\} : \mu_x(C_j) < 0\};\ F := \cup_{j\in J_x^<} C_j,$$

$$J_x^= := \{j \in \{1, \ldots, k_x\} : \mu_x(C_j) = 0\};\ R := \cup_{j\in J_x^=} C_j,$$

$$J_x^> := \{j \in \{1, \ldots, k_x\} : \mu_x(C_j) > 0\};\ I := \cup_{j\in J_x^>} C_j.$$

We may assume that $F \neq \emptyset$, that is, $I \neq \emptyset$ since otherwise (P, w) is already in equilibrium with respect to x. We put

$$\epsilon_1 := \frac{1}{w(I)+w(F)} \min\{x(e^+) - x(e^-) - 1 : \quad e^+ \in I, e^- \in F\},$$

$$\epsilon_2 := \frac{1}{w(I)} \min\{x(e^+) - x(e^-) - 1 : \quad e^+ \in R, e^- \in F\},$$

$$\epsilon_3 := \frac{1}{w(F)} \min\{x(e^+) - x(e^-) - 1 : \quad e^+ \in I, e^- \in R\},$$

$$\epsilon_4 := \frac{1}{w(I)} \min\{-\mu_x(C_j) : \quad j \in J_x^<\},$$

$$\epsilon_5 := \frac{1}{w(F)} \min\{\mu_x(C_j) : \quad j \in J_x^>\},$$

and carry out the shifting y_ϵ defined in Lemma 4.4.2 with $\epsilon := \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\}$, which is obviously greater than 0. By checking all types of arcs we see that y_ϵ is indeed a representation. Moreover,

$$\epsilon \leq \epsilon_5 \leq \frac{1}{w(F)}\mu_x(I) = \frac{1}{w(F)+w(I)}(\mu_x(I) - \mu_x(F)).$$

Lemma 4.4.2 implies that $\sigma^2_{y_\epsilon} < \sigma^2_x$.

Consider the parameter $\xi := 2k_x - |J_x^=|$ (k_x = number of components of G_x, $|J_x^=|$ = number of components of G_x in equilibrium). Of course, $1 \leq \xi_x \leq 2|P|$.

Claim. $\xi_{y_\epsilon} < \xi_x$.

Proof of Claim. Consider the new (undirected) graph $G_c = (V_c, E_c)$, where $V_c := \{1, \ldots, k_x\}$ and $jj' \in E_c$ iff there is some arc $e \in E$ such that $e^+ \in C_j$, $e^- \in C_{j'}$ and $y_\epsilon(e^+) - y_\epsilon(e^-) = 1$ (i.e., $\epsilon = \epsilon_1, \epsilon_2$ or ϵ_3). If $E_c = \emptyset$ then $\epsilon = \epsilon_4$ or ϵ_5, hence $|J_{y_\epsilon}| > |J_x^=|$ (the component for which the minimum is attained in the calculation of ϵ_4 (resp. ϵ_5) has after the shifting y_ϵ the expected value zero), but $k_{y_\epsilon} = k_x$ implying the statement in the claim.

If $E_c \neq \emptyset$ let α be the number of elements of $J_x^=$ that are an endpoint of some edge of G_c. If $\alpha = 0$, then $|J_{y_\epsilon}^=| = |J_x^=|$ but $k_{y_\epsilon} < k_x$ and we are done. If $\alpha > 0$, then $|J_{y_\epsilon}^=| \geq |J_x^=| - \alpha$ (if a component in equilibrium is joined with a component that is not in equilibrium, then the resulting set is not in equilibrium), but $k_{y_\epsilon} \leq k_x - \alpha$ (α components C_j with $j \in J_x^=$ are joined with other components). Consequently, $\xi_{y_\epsilon} \leq 2(k_x - \alpha) - (|J_x^=| - \alpha) = \xi_x - \alpha$. □

If we repeat this shifting several times, then, in view of the claim, after at most $2|P|$ steps we must arrive at a representation y such that (P, w) is in equilibrium with respect to y. We observe that in each shifting step the variance decreases; hence $\sigma_y^2 < \sigma_x^2$. ■

Theorem 4.4.2. *For any poset (P, w) and any real number μ, there is exactly one optimal representation with expected value μ, and this optimal representation can be constructed in a finite number of steps.*

Proof. We may suppose $\mu = 0$. Uniqueness: Assume that x and y are distinct optimal representations with expected value 0. Then $z = \frac{x+y}{2}$ is also a representation with expected value 0. But

$$\sigma_z^2 = \frac{1}{w(P)} \sum_{p \in P} w(p)z(p) = \frac{1}{w(P)} \sum_{p \in P} w(p) \left(\frac{x(p)+y(p)}{2}\right)^2$$
$$< \frac{1}{w(P)} \sum_{p \in P} \frac{x^2(p)+y^2(p)}{2} = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2\right) = \sigma_x^2,$$

which contradicts the assumption that x is optimal.

Existence and algorithm: Construct the height function h as in the beginning of this section and shift it down by the value μ_h to obtain a representation with expected value 0. Moreover, apply the procedure discussed in Lemma 4.4.5 to obtain a representation x such that (P, w) is in equilibrium with respect to x. Now iterate as follows: Apply the labeling algorithm to obtain a maximal flow in the network N_x defined in the proof of (iii) $\leftrightarrow$ (iv) of Theorem 4.4.1. If every arc sp, $p \in P$, is saturated, then stop because by this proof the actual representation is optimal. Otherwise we find in N_x a cut (S, T) of smaller capacity than the capacity of $(\{s\}, V - \{s\})$; that is, a filter $F := S - \{s\}$ with $\mu_x(F) < 0$. Carry out a shifting as in the proof of (i) $\rightarrow$ (iii) of Theorem 4.4.5 and change the resulting representation by the procedure from Lemma 4.4.5. This yields a new representation x' with the property that (P, w) is in equilibrium with respect to x', but $\sigma_{x'}^2 < \sigma_x^2$. Since the variance strongly decreases, in view of Lemma 4.4.4 active graphs cannot appear twice in this iteration. There is only a finite number of subgraphs of $H(P) = (P, E)$ (which are candidates for active graphs); hence this iteration ends after a finite number of steps. ■

In [156] we proved for the unweighted case that we have to call the max-flow algorithm at most $\frac{1}{21}|P|^6$ times to obtain the optimal representation. Test examples of Stendal [447] suggest that $O(|P|)$ calls already suffice. We do not know whether the algorithm of Theorem 4.4.2 is strongly polynomial, whether it has complexity $O(|P|^k)$ for some k.

4.5. Normal posets and flow morphisms

The following definition of a class of posets has its origin in the proof of Sperner [436] that the Boolean lattices possess the Sperner property (see the proof of Theorem 1.1.1). It was introduced by Graham and Harper [228].

Let (P, w) be a positively weighted poset with rank function r. For the sake of brevity we put throughout $n := r(P)$. We say that (P, w) has the *normalized matching property* (*NMP*) if

$$\frac{w(A)}{w(N_i)} \le \frac{w(\nabla(A))}{w(N_{i+1})} \quad \text{for all } A \subseteq N_i, \quad i = 0, \dots, n-1. \tag{4.30}$$

Briefly a poset with the NMP is called a *normal poset.*

Proposition 4.5.1. *Let (P, w) be a normal poset. Then all minimal and maximal elements of P have rank 0 and n, respectively; that is, (P, w) is graded.*

Proof. Assume $p \in P$ is minimal, $r(p) = i > 0$. Then

$$1 = \frac{w(N_{i-1})}{w(N_{i-1})} > \frac{w(\nabla(N_{i-1}))}{w(N_i)}, \quad \text{a contradiction.}$$

Assume $q \in P$ is maximal, $r(q) = j < n$. Then

$$\frac{w(\{q\})}{w(N_j)} > \frac{w(\nabla(\{q\}))}{w(N_{j+1})} = 0,$$

contradicting the NMP. ■

Proposition 4.5.2. *A ranked and positively weighted poset (P, w) is normal iff it has the dual normalized matching property, that is, iff*

$$\frac{w(A)}{w(N_i)} \le \frac{w(\Delta(A))}{w(N_{i-1})} \quad \textit{for all } A \subseteq N_i, \quad i = 1, \dots, n. \tag{4.31}$$

Proof. We only prove that (4.30) implies (4.31). Let $A \subseteq N_i$, $i = 1, \dots, n$, and observe that $\nabla(N_{i-1} - \Delta(A)) \subseteq N_i - A$. Thus by (4.30)

$$\begin{aligned} 1 - \frac{w(\Delta(A))}{w(N_{i-1})} &= \frac{w(N_{i-1} - \Delta(A))}{w(N_{i-1})} \le \frac{w(\nabla(N_{i-1} - \Delta(A)))}{w(N_i)} \le \frac{w(N_i - A)}{w(N_i)} \\ &= 1 - \frac{w(A)}{w(N_i)}, \end{aligned}$$

and (4.31) follows immediately. ■

Proposition 4.5.3. *Let (P, w) be normal. Then for any $S \subseteq [0, n]$ the S-rank-selected subposet is normal, too.*

Proof. We get the upper shadow in the subposet by taking several times (as often as necessary) the upper shadow in P. ■

Now we will discuss two other criteria of the NMP. Let (P, w) be ranked. For the sake of brevity let $N^p := N_{r(p)}$ (the level containing p). A *regular covering* of (P, w) is a function $f : \mathfrak{C}(P) \to \mathbb{R}_+$ such that

$$\sum_{C \in \mathfrak{C}(P): p \in C} f(C) = \frac{w(p)}{w(N^p)} \quad \text{for all } p \in P,$$

$$\sum_{C \in \mathfrak{C}(P)} f(C) = 1.$$

Moreover, we say that (P, w) satisfies the *k-LYM inequality* or briefly (P, w) *is k-LYM* if for all k-families F of P

$$\sum_{p \in F} \frac{w(p)}{w(N^p)} \le k. \tag{4.32}$$

In the case $k = 1$ we speak of LYM instead of 1-LYM. The following theorem is in the unweighted case due to Kleitman [304]. For arbitrary weights see also Harper [257] and [150].

Theorem 4.5.1. *Let (P, w) be graded and positively weighted. The following conditions are equivalent:*

(i) (P, w) is a normal poset,

(ii) (P, w) satisfies the LYM-inequality,

(iii) (P, w) satisfies the k-LYM inequality for every $k = 1, 2, \ldots,$

(iv) there exists a regular covering of (P, w).

Proof. (i) $\to$ (ii). Let F be an antichain (1-family). Let $l := \min\{i : F \cap N_i \neq \emptyset\}$. We prove (4.32) (for $k = 1$) by induction on $l = n, n-1, \ldots$. The case $l = n$ is trivial. Thus consider the step $l + 1 \to l < n$. Let $A := F \cap N_l$. We have $\nabla(A) \cap F = \emptyset$ because F is an antichain. Obviously, $F^* := (F - A) \cup \nabla(A)$ is an antichain, too. Using (4.30) and the induction hypothesis we obtain

$$\begin{aligned}\sum_{p \in F} \frac{w(p)}{w(N^p)} &= \frac{w(A)}{w(N_l)} + \sum_{p \in F - A} \frac{w(p)}{w(N^p)} \\ &\le \frac{w(\nabla(A))}{w(N_{l+1})} + \sum_{p \in F - A} \frac{w(p)}{w(N^p)} = \sum_{p \in F^*} \frac{w(p)}{w(N^p)} \le 1.\end{aligned}$$

(ii) $\to$(iii). This implication is clear since every k-family is a union of k antichains (see Lemma 4.3.1).

(iii) $\to$ (iv). We define the new weight v on P by $v(p) := \frac{w(p)}{w(N^p)}$ for all $p \in P$.

Claim. $c_1(P, v) = 1$.

Proof of Claim. Obviously N_0 is a 1-cutset of weight $v(N_0) = 1$. Let F be any other 1-cutset of P. Then $P - F$ is an n-family in P. By condition (iii) with

$k = n$, $v(P - F) = \sum_{p \in P-F} v(p) \leq n$, hence, $v(F) = v(P) - v(P - F) \geq (n+1) - n = 1$.

□

In view of Corollary 4.2.1(b), there exists $f : \mathfrak{C}(P) \to \mathbb{R}_+$ such that

$$\sum_{C \in \mathfrak{C}(P): p \in C} f(C) \leq v(p) \text{ for all } p \in P, \tag{4.33}$$

$$\sum_{C \in \mathfrak{C}(P)} f(C) = 1.$$

Assume that strict inequality holds for some $p^* \in N_i$ in (4.33). Since each maximal chain has exactly one member in the ith level of P, we conclude

$$1 = \sum_{C \in \mathfrak{C}(P)} f(C) = \sum_{p \in N_i} \sum_{C \in \mathfrak{C}(P): p \in C} f(C) < \sum_{p \in N_i} v(p) = \sum_{p \in N_i} \frac{w(p)}{w(N^p)} = 1,$$

a contradiction.

(iv) → (i). Let $A \subseteq N_i$, $B := N_{i+1} - \nabla(A)$, and let y be a regular covering of (P, w). Since each maximal chain in P contains at most one member of $A \cup B$,

$$1 = \sum_{C \in \mathfrak{C}(P)} f(C) \geq \sum_{p \in A \cup B} \sum_{C \in \mathfrak{C}(P): p \in C} f(C) = \sum_{p \in A} \frac{w(p)}{w(N_i)} + \sum_{p \in B} \frac{w(p)}{w(N_{i+1})}$$

$$= \frac{w(A)}{w(N_i)} + 1 - \frac{w(\nabla(A))}{w(N_{i+1})},$$

and (4.30) is proved. ■

Note that Wei [461] found a further equivalent condition in terms of solutions of a certain convex programming problem and its dual.

Let $G_i = (P_i, E_i)$ be the Hasse diagram of the $\{i, i+1\}$-rank-selected subposet of the graded poset (P, w); that is, $P_i = N_i \cup N_{i+1}$, $E_i = \{pq : p \in N_i, p \lessdot q\}$. Let w_i be the induced weight.

Corollary 4.5.1. *Let (P, w) be graded and positively weighted. Then (P, w) is normal iff for $i = 0, \ldots, n-1$ there exist functions $f_i : E_i \to \mathbb{R}_+$ such that*

$$p_{f_i}^- = \sum_{e \in E_i : e^- = p} f_i(e) = \frac{w(p)}{w(N_i)} \textit{ for all } p \in N_i,$$

$$p_{f_i}^+ = \sum_{e \in E_i : e^+ = p} f_i(e) = \frac{w(p)}{w(N_{i+1})} \textit{ for all } p \in N_{i+1}.$$

Proof. (P, w) is normal iff (P_i, w_i) is normal for all i, by definition. The arcs of E_i are exactly the maximal chains in (P_i, w_i). The result follows from the equivalence (i) ↔ (iv) in Theorem 4.5.1. ■

We encourage the reader to derive Corollary 4.5.1 directly from the real version of the Max-Flow Min-Cut Theorem (Corollary 4.1.1).

An unweighted poset P is called *regular* if P is graded and both the number of elements that cover any element p of P and the number of elements that are covered by p depend on the rank of p, only.

***Corollary* 4.5.2.** *Let P be a regular poset. Then P is normal.*

Proof. Define for $e \in E_i$, $i = 0, \dots, n-1$,

$$f_i(e) := \frac{1}{|E_i|}.$$

Then it is easy to see that the conditions of Corollary 4.5.1 are satisfied (with $w \equiv 1$). ∎

***Example* 4.5.1.** *The following posets are regular: The Boolean lattice B_n, the linear lattice $L_n(q)$, the affine poset $A_n(q)$, projective space lattices, the cubical poset Q_n and the function poset F_k^n.*

Using the existence of regular coverings we can estimate the weight of subsets (with certain properties) of normal posets by considering only the maximal chains.

***Theorem* 4.5.2 (*Kleitman* [304]).** *Let (P, w) be a normal poset and $G \subseteq P$. Then*

$$w(G) \le \max_{C \in \mathfrak{C}(P)} \sum_{p \in C \cap G} w(N^p).$$

Proof. Let f be a regular covering of (P, w) that exists by Theorem 4.5.1. Then

$$\begin{aligned} w(G) &= \sum_{p \in G} w(p) = \sum_{p \in G} \sum_{C \in \mathfrak{C}(P): p \in C} w(N^p) f(C) \\ &= \sum_{C \in \mathfrak{C}(P)} f(C) \sum_{p \in C \cap G} w(N^p) \le \max_{C \in \mathfrak{C}(P)} \sum_{p \in C \cap G} w(N^p). \end{aligned}$$

∎

In analogy to the k-Sperner property, we say that (P, w) (with $n = r(P)$) has the *k-cutset property* if

$$c_k(P, w) = \min_{0 \le i_1 < \cdots < i_k \le n} \left(w(N_{i_1}) + \cdots + w(N_{i_k}) \right).$$

It has the *strong cutset property* if it has the k-cutset property for every $k = 1, 2, \dots, n+1$.

Lemma 4.5.1.

(*a*) (P, w) *has the* k*-cutset property iff* (P, w) *has the* $(n+1-k)$*-Sperner property.*

(*b*) (P, w) *has the strong cutset property iff* (P, w) *has the strong Sperner property.*

Proof. Observe that F is an (optimal) k-cutset iff $P - F$ is an (optimal) $(n+1-k)$-family. ■

Corollary 4.5.3. Let (P, w) *be normal. Then it has the strong Sperner and the strong cutset property.*

Proof. Let F be a k-family, $1 \leq k \leq n+1$. Then every $C \in \mathfrak{C}(P)$ contains at most k elements. By Theorem 4.5.2

$$w(F) \leq \max_{0 \leq i_1 < \cdots < i_k \leq n} \left(w(N_{i_1}) + \cdots + w(N_{i_k}) \right).$$

The bound is attained by the union of the corresponding levels for which the maximum is attained. The strong cutset property follows from Lemma 4.5.1. ■

A subset F of the graded poset P is called a *mod-k-family* if $p_1, p_2 \in F$, $p_1 < p_2$ imply $r(p_2) - r(p_1) \geq k$.

Corollary 4.5.4 (Katona [293]). Let (P, w) *be normal and rank unimodal. Then the maximum weight of a mod-k-family equals the largest sum of the form* $\sum_i w(N_{h+ki})$.

Proof. Again by Theorem 4.5.2 the weight of any mod-k-family can be bounded by $\max_J \sum_{j \in J} w(N_j)$, where the maximum is extended over all $J \subseteq [0, n]$ with the property that $j_1, j_2 \in J$ imply $|j_1 - j_2| \geq k$. The unimodality permits the restriction to subsets J of the form $\{h, h+k, h+2k, \ldots\}$. The union of the corresponding levels for which the maximum is attained gives the optimal mod-k-family. ■

Up to now we used normality for upper estimates of certain families. Let us consider two results for lower estimates with the assumption of log concavity of the Whitney numbers. Examples of such posets can be found in the next section. First we need some preparations. Let I be an interval of integers.

Lemma 4.5.2. Let $\{a_i\}$, $i \in I$, *be a log concave sequence of positive numbers. Then the sequence* $\{\frac{1}{a_i}\}$ *is convex.*

Proof. We have for $i-1, i+1 \in I$

$$a_{i-1}a_{i+1} \le a_i\sqrt{a_{i-1}a_{i+1}} \le a_i\left(\frac{a_{i-1}+a_{i+1}}{2}\right),$$

hence

$$\frac{1}{a_i} \le \left(\frac{\frac{1}{a_{i+1}}+\frac{1}{a_{i-1}}}{2}\right).$$

■

Lemma 4.5.3. *Let $\{b_i\}$ be a convex sequence of positive numbers. Then for all $j,k,l \in I$ with $j \le k \le l$*

$$(l-k)b_j + (k-j)b_l \ge (l-j)b_k.$$

Proof. Though this inequality follows from a discrete version of Jensen's inequality, we give a direct proof. We may suppose $j < k < l$. We have by the convexity

$$b_{j+1}-b_j \le b_{j+2}-b_{j+1} \le \cdots \le b_k - b_{k-1} \le b_{k+1}-b_k.$$

Summation of the $k-j$ terms on the left yields

$$(b_k - b_j) \le (k-j)(b_{k+1}-b_k)$$

which is equivalent to

$$\frac{b_k-b_j}{k-j} \le \frac{b_{k+1}-b_j}{k+1-j}.$$

This gives finally an inequality which is equivalent to the asserted one:

$$\frac{b_k-b_j}{k-j} \le \frac{b_l-b_j}{l-j}.$$

■

Lemma 4.5.4. *Let $\{a_i\}, i \in I$, be a log concave sequence of positive numbers. Let the new sequence $\{b_i\}, i \in I$, be defined by*

$$b_i := \sum_{j\in I: j\le i} a_i.$$

Then for $j,k,l \in I$ with $j \le k \le l$

$$(a_l - a_k)b_j + (a_k - a_l)b_l \ge (a_l - a_j)b_k.$$

Proof. Let $j < k < l$. From the log concavity we derive easily

$$\frac{a_{j+1}-a_j}{a_{j+1}} \ge \frac{a_{j+2}-a_{j+1}}{a_{j+2}} \ge \cdots \ge \frac{a_k-a_{k-1}}{a_k} \ge \frac{a_{k+1}-a_k}{a_{k+1}}.$$

Similarly to Lemma 4.5.3 we obtain

$$\frac{a_k - a_j}{b_k - b_j} \geq \frac{a_{k+1} - a_k}{b_{k+1} - b_k}$$

hence

$$\frac{a_k - a_j}{b_k - b_j} \geq \frac{a_{k+1} - a_j}{b_{k+1} - b_j}.$$

This gives an inequality equivalent to the asserted one:

$$\frac{a_k - a_j}{b_k - b_j} \geq \frac{a_l - a_j}{b_l - b_j}.$$

■

Theorem 4.5.3. *Let (P, w) be normal with log concave weighted Whitney numbers. Let $\{w(N_i)\}, i = 0, \ldots, h$, be strictly increasing, let $0 \leq k \leq h$, and let $A \subseteq P$ be an antichain such that $w(A) \geq w(N_k)$. Finally let I be the ideal generated by A. Then*

(*a*) (*Kleitman and Milner* [309]) $\mu_r(A) \geq k$.

(*b*) (*Kleitman* [302]) $w(I) \geq \sum_{i=0}^{k} w(N_i)$.

Proof. Let $a_i := w(N_i)$ and $x_i := w(A \cap N_i)$, $i = 0, \ldots, n$. Then

$$0 \leq x_i \leq a_i, \tag{4.34}$$

in view of the supposition

$$\sum_{i=0}^{n} x_i \geq a_k, \tag{4.35}$$

and by the LYM-inequality, see Theorem 4.5.1,

$$\sum_{i=0}^{n} \frac{x_i}{a_i} \leq 1. \tag{4.36}$$

(a) We have to prove that

$$\sum_{i=0}^{n} i x_i \geq k \sum_{i=0}^{n} x_i. \tag{4.37}$$

Thus it is enough to show that the minimum of the objective function $\sum_{i=0}^{n}(i-k)x_i$ under the constraints (4.34), (4.35), and (4.36) is not less than zero. Because the minimum is attained at an extreme point of the polytope defined by (4.34)–(4.36) we have to prove (4.37) only for all these extreme points. Let $\boldsymbol{x} = (x_0, \ldots, x_n)$ be such an extreme point. It satisfies $n + 1$ independent inequalities from (4.34)–(4.36) as equalities, cf. p. 86.

Case 1. (4.35) or (4.36) is a strict inequality. Then in view of (4.34) and (4.36) at most one component of $\boldsymbol{x}$, say x_l, is nonzero. Moreover $a_l \geq x_l \geq a_k$ which implies $l \geq k$. Consequently,

$$\sum_{i=0}^{n}(i-k)x_i \geq (l-k)a_k \geq 0.$$

Case 2. (4.35) and (4.36) are satisfied as equalities. Then it is easy to see that at most two components of $\boldsymbol{x}$ are nonzero, say x_j and x_l. If $j, l \geq k$ then trivially $\sum_{i=0}^{n}(i-k)x_i \geq 0$. If $j, l < k$, then by the equations (4.35) and (4.36) and the strict monotony

$$1 = \frac{x_j}{a_j} + \frac{x_l}{a_l} < \frac{x_j + x_l}{a_k} = 1,$$

a contradiction. Thus we may assume that $j \leq k \leq l$. Assertion (4.37) reduces to

$$(j-k)x_j + (l-k)x_l \geq 0$$

and we know that by (4.35) and (4.36)

$$x_j + x_l = a_k, \ \frac{x_j}{a_j} + \frac{x_l}{a_l} = 1.$$

Solving this system of equations and inserting into the reduced assertion yields

$$(j-k)a_j(a_l - a_k) + (l-k)a_l(a_k - a_j) \geq 0,$$

which is equivalent to

$$(k-j)\frac{1}{a_l} + (l-k)\frac{1}{a_j} \geq (l-j)\frac{1}{a_k},$$

but this inequality is true because of Lemma 4.5.2 and Lemma 4.5.3.

(b) Let $y_i := w(I \cap N_i), i = 0, \ldots, n$. Obviously, $A_j := \{p \in A : r(p) \geq j\} \cup \{p \in N_j : p \notin I\}, j = 0, \ldots, n$, is an antichain. Again by the LYM-inequality

$$\sum_{i=j}^{n}\frac{x_i}{a_i} + 1 - \frac{y_j}{a_j} \leq 1;$$

that is,

$$\sum_{i=j}^{n}\frac{x_i}{a_i} \leq \frac{y_j}{a_j}.$$

We have, using $b_i := \sum_{j=0}^{i} a_j$,

$$w(I) = \sum_{j=0}^{n} y_j \geq \sum_{j=0}^{n} a_j \sum_{i=j}^{n}\frac{x_i}{a_i} = \sum_{i=0}^{n}\frac{x_i}{a_i}\sum_{j=0}^{i} a_j = \sum_{i=0}^{n}\frac{x_i}{a_i} b_i.$$

Consequently it is sufficient to prove that the minimum of the objective function $\sum_{i=0}^{n} \frac{x_i}{a_i} b_i$ under the constraints (4.34)–(4.36) is not less than b_k. Let us look again at an extreme point $\boldsymbol{x}$ of the polytope defined by (4.34)–(4.36). The Case 1 of (a) can be treated easily, so assume that (4.35) and (4.36) are satisfied as equalities. Moreover, as for (a), we may assume that

$$x_j = a_j \left(\frac{a_l - a_k}{a_l - a_j} \right), \qquad x_l = a_l \left(\frac{a_k - a_j}{a_l - a_j} \right), \qquad j \le k \le l,$$

and all other components of $\boldsymbol{x}$ are zero. Inserting these values into the objective function yields the asserted inequality

$$(a_l - a_k)b_j + (a_k - a_j)b_l \ge (a_l - a_j)b_k,$$

which is true by Lemma 4.5.4.

In the original proofs (in Greene and Kleitman [234] partially attributed to Odlyzko), duality arguments were used; that is, by an inspired guess one finds an admissible solution of the dual problem with a corresponding large value of the (dual) objective function. We used the primal approach to be a little bit more straightforward. In [174] Erdős, Faigle, and Kern generalized part (a) of Theorem 4.5.3 to the case of k-families.

In the proofs of Corollary 4.5.3 and 4.5.4, we considered instead of the poset P only chains C. This idea of reduction to smaller posets was generalized by Harper [258], who introduced the notion of flow morphisms: Let (P, v) and (Q, w) be weighted posets. A *flow morphism* is a mapping $\varphi : P \to Q$ such that

(i) φ is surjective,
(ii) $p_1 \lessdot_P p_2$ implies $\varphi(p_1) \lessdot_Q \varphi(p_2)$,
(iii) $w(q) = v(\varphi^{-1}(q))$ for all $q \in Q$,
(iv) the poset induced by $\varphi^{-1}(q_1) \cup \varphi^{-1}(q_2)$ is a normal poset of rank 1 for all $q_1, q_2 \in Q$ with $q_1 \lessdot_Q q_2$.

Here $\varphi^{-1}(q)$ is, as usual, the set of all $p \in P$ with $\varphi(p) = q$. We note that from condition (ii) it follows that the poset considered in (iv) cannot contain three distinct elements lying on a chain in P, but without (iv) it could be an antichain. Moreover, in view of Proposition 4.5.1, in that poset the elements of rank 0 and 1 are the elements of $\varphi^{-1}(q_1)$ and $\varphi^{-1}(q_2)$, respectively. Immediately from the definition we obtain:

Proposition 4.5.4. *The graded poset (P, v) is normal iff the rank function r is a flow morphism from (P, v) onto (C, w), where $C = (0 \lessdot 1 \lessdot \cdots \lessdot n)$ and $w(i) = v(N_i(P))$, $i = 0, \ldots, n$.*

Part (a) of the next theorem is due to Harper [258]. In the proof we use the approach of [150], [154].

Theorem 4.5.4. *If there is a flow morphism φ from (P, v) onto (Q, w), then for all k*

(a) $d_k(P, v) = d_k(Q, w)$,

(b) $c_k(P, v) = c_k(Q, w)$.

Proof. We prove only (a) since the proof for (b) is similar. We have

$$d_k(P, v) \geq d_k(Q, w)$$

since for any k-family F in Q, its preimage $\varphi^{-1}(F)$ is a k-family in P and $v(\varphi^{-1}(F)) = w(F)$. Thus we must show that

$$d_k(P, v) \leq d_k(Q, w).$$

Claim. If D is any chain in Q then the poset induced by $\varphi^{-1}(D)$ is normal.

Proof of Claim. Let D' be any maximal chain in Q containing D. Then, by our conditions, the poset induced by $\varphi^{-1}(D')$ is normal. Since $(\varphi^{-1}(D), v)$ is a rank-selected subposet of $(\varphi^{-1}(D'), v)$ (v is restricted to the corresponding sets), $(\varphi^{-1}(D), v)$ is normal, too (see Proposition 4.5.3). □

Let f_D be a regular covering of $(\varphi^{-1}(D), v)$ which exists by Theorem 4.5.1. This means that we have

$$\sum_{C \in \mathfrak{C}(\varphi^{-1}(D)): p \in C} f(C) = \frac{v(p)}{w(\varphi(p))} \text{ for all } p \in \varphi^{-1}(D),$$

$$\sum_{C \in \mathfrak{C}(\varphi^{-1}(D))} f(C) = 1.$$

Let further $y_Q : \mathfrak{C}^*(Q) \to \mathbb{R}_+$ and $z_Q : Q \to \mathbb{R}_+$ be functions such that

$$\sum_{D \in \mathfrak{C}^*(Q): q \in D} y_Q(D) + z_Q(q) \geq w(q) \text{ for all } q \in Q,$$

$$k \sum_{D \in \mathfrak{C}^*(Q)} y_Q(D) + \sum_{q \in Q} z_Q(q) = d_k(Q, w),$$

which exist by Theorem 4.3.2.

Define $y_P : \mathfrak{C}^*(P) \to \mathbb{R}_+$ and $z_P : P \to \mathbb{R}_+$ by

$$y_P(C) := f_{\varphi(C)}(C) y_Q(\varphi(C)), C \in \mathfrak{C}^*(P),$$

$$z_P(p) := \frac{v(p)}{w(\varphi(p))} z_Q(\varphi(p)), p \in P.$$

Then we have for all $p \in P$ (with the notation $q := \varphi(p)$)

$$\begin{aligned}
\sum_{C \in \mathfrak{C}^*(P): p \in C} y_P(C) + z_P(p) &= \sum_{D \in \mathfrak{C}^*(Q)} y_Q(D) \sum_{C \in \mathfrak{C}(\varphi^{-1}(D)): p \in C} f_D(C) \\
&\quad + \frac{v(p)}{w(q)} z_Q(q) \\
&= \sum_{D \in \mathfrak{C}^*(Q): q \in D} y_Q(D) \frac{v(p)}{w(q)} + \frac{v(p)}{w(q)} z_Q(q) \\
&\geq \frac{v(p)}{w(q)} w(q) = v(p);
\end{aligned}$$

that is, y_P, z_P satisfy the inequality (4.15).

Moreover,

$$\begin{aligned}
\delta_k(y_P, z_P) &= k \sum_{C \in \mathfrak{C}^*(P)} y_P(C) + \sum_{p \in P} z_P(p) \\
&= k \sum_{D \in \mathfrak{C}^*(Q)} \sum_{C \in \mathfrak{C}^*(P): \varphi(C) = D} f_D(C) y_Q(D) \\
&\quad + \sum_{q \in Q} \sum_{p \in P: \varphi(p) = q} \frac{v(p)}{w(q)} z_Q(q) \\
&= k \sum_{D \in \mathfrak{C}^*(Q)} y_Q(D) + \sum_{q \in Q} z_Q(q) = d_k(Q, w).
\end{aligned}$$

From Theorem 4.3.2 we finally derive

$$d_k(P, v) \leq \delta_k(y_P, z_P) = d_k(Q, w).$$

■

The reader should observe that we also may obtain Corollary 4.5.3 from Proposition 4.5.4 and Theorem 4.5.4 since the k-family problem on a chain is solved by taking those k elements of the chain whose weight sum is maximal (analogously for the k-cutset problem).

Let us weaken slightly condition (ii). We say that $\varphi : P \to Q$ is a *weak flow morphism* if it satisfies (i), (iii), (iv), and

(ii') $\quad p_1 <_P p_2$ implies $\varphi(p_1) <_Q \varphi(p_2)$.

An example of a weak flow morphism that is not a flow morphism is given in Figure 4.9. Note that in this example $c_1(P, v) = 6 \neq c_1(Q, w) = 3$. But:

Corollary 4.5.5. *If there is a weak flow morphism φ from (P, v) onto (Q, w), then $d_k(P, v) = d_k(Q, w)$ for all $k = 1, 2, \ldots$.*

Proof. Let (P^*, v) be the poset that has the same elements and weight as (P, v), but where $p_1 \lessdot_{P^*} p_2$ iff $p_1 \lessdot_P p_2$ and $\varphi(p_1) \lessdot_Q \varphi(p_2)$ (we omit in the Hasse

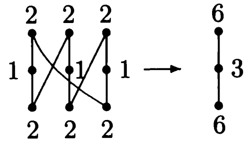


Figure 4.9

diagram of P those arcs that are not mapped onto an arc of the Hasse diagram of Q). By construction, φ is a flow morphism from (P^*, v) onto (Q, w). Accordingly, by Theorem 4.5.4 and the fact that each k-family in P is a k-family in P^*,

$$d_k(P, v) \leq d_k(P^*, v) = d_k(Q, w).$$

However, as in the proof of Theorem 4.5.4, the preimage of a k-family in Q is a k-family in P of the same weight; thus

$$d_k(P, v) \geq d_k(Q, w).$$

■

Theorem 4.5.5 (Weighted Quotient Theorem). *If G is a group of automorphisms of P, then $d_k(P) = d_k(P/G, w/G)$ for $k = 1, 2, \ldots$.*

Proof. [150] The mapping φ that assigns to each $p \in P$ the orbit containing p is a weak flow morphism from $(P, 1)$ onto $(P/G, w/G)$. Indeed, conditions (i)–(iii) are satisfied evidently and condition (iv) is satisfied because the poset induced by $\varphi^{-1}(A) \cup \varphi^{-1}(B)$, $A, B \in P/G$, $A \lessdot B$, is a regular poset of rank 1 and hence normal by Corollary 4.5.2. Theorem 4.5.4 yields the assertion. ■

Corollary 4.5.6 (Kleitman, Edelberg, and Lubell **[306]).** *For every poset P and for every positive integer k there exists a maximum k-family that is invariant under every automorphism of P.*

Proof. Let G be the group of all automorphisms of P and let F' be a maximum weighted k-family in $(P/G, w/G)$. Here F' is a set whose elements are orbits of P under G. Let F be the union of these orbits. Then F is a k-family in P that is invariant under every automorphism of P, and $|F| = w/G(F') = d_k(P/G, w/G) = d_k(P)$ by Theorem 4.5.5. Thus F is even a maximum k-family in P. ■

Freese [204] observed that this corollary can be derived in the $k = 1$ case from the result of Dilworth [137] that the maximum antichains of a poset form a distributive lattice.

In [152] we proved that not only the k-family problem but also the variance problem is preserved by weak flow morphisms:

Theorem 4.5.6. *Let x and y be optimal representations of (P, v) and (Q, w) with $\mu_x = \mu_y$. If there is a weak flow morphism φ from (P, v) onto (Q, w), then $x(p) = y(\varphi(p))$ for all $p \in P$. In particular, $\sigma^2(P, v) = \sigma^2(Q, w)$.*

Proof. If we define the function $z : P \to \mathbb{R}$ by $z(p) := y(\varphi(p))$ for all $p \in P$, then we must prove $z = x$. Since φ is a flow morphism we have $v(P) = w(Q)$,

$$\begin{aligned}\mu_z &= \frac{1}{v(P)} \sum_{p \in P} v(p) z(p) = \frac{1}{v(P)} \sum_{q \in Q} v(\varphi^{-1}(q)) y(q) \\ &= \frac{1}{w(Q)} \sum_{q \in Q} w(q) y(q) = \mu_y,\end{aligned}$$

$$\sigma_z^2 = \frac{1}{v(P)} \sum_{p \in P} v(p) z^2(p) - \mu_z^2 = \frac{1}{w(Q)} \sum_{q \in Q} w(q) y^2(q) - \mu_y^2 = \sigma_y^2.$$

By Theorem 4.4.2 it remains to show that z is an optimal representation. Since φ is a flow morphism, $p' > p$ implies $\varphi(p') > \varphi(p)$. Thus for all $p' > p$, $z(p') - z(p) = y(\varphi(p')) - y(\varphi(p)) \geq 1$; that is, z is a representation. In order to show that z is optimal we use the equivalence (i) $\leftrightarrow$ (v) of Theorem 4.4.1. It is easy to see that φ is a flow morphism from (P_z, v) onto (Q_y, w). For $q, q' \in Q$, $q \lessdot q'$ let $h_{qq'}$ be a regular covering of the poset induced by $\varphi^{-1}(q) \cup \varphi^{-1}(q')$ (note Theorem 4.5.1 and the definition of a flow morphism). So we have

$$\begin{aligned}\sum_{p' : p' \gtrdot p, \varphi(p') = q'} h_{qq'}(pp') &= \frac{v(p)}{w(\varphi(p))} \text{ for all } p \in \varphi^{-1}(q), \\ \sum_{p : p \lessdot p', \varphi(p) = q} h_{qq'}(pp') &= \frac{v(p')}{w(\varphi(p'))} \text{ for all } p' \in \varphi^{-1}(q').\end{aligned}$$

Let g be a representation flow on (Q, w) relative to y. We define $f : E(P) \to \mathbb{R}$ by

$$f(pp') := h_{\varphi(p)\varphi(p')}(pp') g(\varphi(p)\varphi(p')) \text{ for all } pp' \in E(P).$$

Then $z(p') - z(p) > 1$ implies $\varphi(p') - \varphi(p) > 1$; that is, $g(\varphi(p)\varphi(p')) = 0$ and $f(pp') = 0$. Moreover, by simple computation we obtain

$$p_f^+ - p_f^- = \frac{v(p)}{w(\varphi(p))} (\varphi(p)_g^+ - \varphi(p)_g^-) = v(p)(z(p) - \mu_z).$$

Consequently, f is a representation flow on (P, v) relative to z and hence by Theorem 4.4.1, z is an optimal representation. ∎

We define a graded poset (P, v) to be *rank compressed* if the rank function r is an optimal representation of (P, v) (of course r is a representation). It is easy to see that, for example, weighted chains are rank compressed. Note that the existence of a flow morphism (not of a *weak* flow morphism) implies that (P, v) is graded iff (Q, w) is graded. Directly from Theorem 4.5.6 and Proposition 4.5.4 we derive:

***Corollary* 4.5.7.** *If there exists a flow morphism from the graded poset (P, v) onto the graded poset (Q, w), then (P, v) is rank compressed iff (Q, w) is rank compressed. In particular, normal posets (P, v) are rank compressed.*

In connection with Section 4.3, consider the following conjecture of Griggs [235]: If P is a normal poset, then there is a partition into chains which is k-saturated for all $k = 1, 2, \ldots$.

Automorphisms are not only useful in the study of k-families (see Corollary 4.5.6), but they can be applied also to other classes of families. Let P be a ranked poset. We say that a group G of automorphisms of P is *rank transitive* if G acts transitively on the levels of P, that is, if P/G is a chain. For $p \in P$, let F_p be the principal filter generated by p, that is, $F_p := \{q \in P : q \geq p\}$, and let G_p be the subgroup of G that fixes the element p; that is, $G_p := \{\varphi \in G : \varphi(p) = p\}$. If there exists a rank-transitive automorphism group G of P, then obviously $F_p \cong F_q$ if $r(p) = r(q)$. In the following we use the notation $Q := F_{p_0}$ for some $p_0 \in N_0$ (up to isomorphism Q does not depend on the concrete element p_0 of N_0).

***Lemma* 4.5.5.** *Let G be a rank-transitive automorphism group of P. We have*

$$W_i(P) = \frac{|G|}{|G_p|} \text{ for all } p \in N_i(P), i = 0, \ldots, n.$$

Proof. We find a bijection between $N_i(P)$ and the left cosets of G relative to G_p by taking for $q \in N_i(P)$ the coset $\{\psi \in G : \psi(p) = q\}$. By Lagrange's Theorem the number of left cosets equals $\frac{|G|}{|G_p|}$, which implies the equality. ■

For a subset (family) F of P and an automorphism φ, let $\varphi(F) := \{\varphi(p) : p \in F\}$. Given an automorphism group G of P, we say that a class $\mathfrak{A}$ of families in P is G-*invariant* if $\varphi(F) \in \mathfrak{A}$ for all $F \in \mathfrak{A}$. Let us recall that $F_i := F \cap N_i(P)$ and that the parameters of F are defined by $f_i := |F_i|$. The following theorem is (in a different formulation) due to Erdős, Faigle, and Kern [173].

***Theorem* 4.5.7.** *Suppose that there exists a rank-transitive automorphism group of P. Let $C = (p_0 < p_1 < \cdots < p_n)$ be a fixed maximal chain in P, let Q be the filter generated by p_0, and let R be a system of representatives of the left cosets of G relative to G_{p_0} (i.e., $N_0(P) = \{\varrho(p_0) : \varrho \in R\}$, $\varrho_1(p_0) \neq \varrho_2(p_0)$ for $\varrho_1 \neq \varrho_2$).*

Let $w : P \to \mathbb{R}_+$ be defined by

$$w(p) := \frac{W_i(P)}{W_0(P)} \frac{1}{W_i(Q)} \text{ if } p \in N_i(P),$$

and let $\mathfrak{A}$ be a G-invariant class of families. If for all $F \in \mathfrak{A}$

$$\sum_{\varrho \in R} w(\varrho(C) \cap F) \leq 1, \tag{4.38}$$

then for all $F \in \mathfrak{A}$

$$\sum_{i=0}^{n} \frac{f_i}{W_i(Q)} \leq 1. \tag{4.39}$$

Proof. Since $\mathfrak{A}$ is G-invariant we have, by the supposition, that for all $\varphi \in G$

$$\sum_{\varrho \in R} w(\varrho(C) \cap \varphi^{-1}(F)) \ \leq 1;$$

that is,

$$\sum_{\varrho \in R} w(\varphi\varrho(C) \cap F) \ \leq 1,$$

$$\sum_{\varphi \in G} \sum_{\varrho \in R} \sum_{i=0}^{n} w(\varphi\varrho(p_i) \cap F_i) \leq |G|,$$

$$\sum_{i=0}^{n} \sum_{\varrho \in R} \frac{W_i(P)}{W_0(P)} \frac{1}{W_i(Q)} \sum_{\varphi \in G} |\varphi\varrho(p_i) \cap F_i| \ \leq |G|.$$

For a fixed element $q \in F_i$ there are $|G_{\varrho(p_i)}|$ many elements φ of G that map $\varrho(p_i)$ onto q (they form a left coset relative to $G_{\varrho(p_i)}$). By Lemma 4.5.5, $|G_{\varrho(p_i)}| = |G_{p_i}|$ for every $\varrho \in R$. Consequently,

$$\sum_{\varphi \in G} |\varphi\varrho(p_i) \cap F_i| = f_i |G_{p_i}|;$$

hence we may rewrite the last inequality (using Lemma 4.5.5 and $|R| = W_0(P)$):

$$\sum_{i=0}^{n} |R| \frac{W_i(P)}{W_0(P)} \frac{1}{W_i(Q)} f_i |G_{p_i}| \leq |G|,$$

$$\sum_{i=0}^{n} \frac{f_i}{W_i(Q)} \leq 1.$$

■

This theorem is relatively abstract. In applications we have to choose the chain C, the group G, and the system R of representatives in a convenient way.

A family F in the affine poset $A_q(n)$ is called *intersecting* if for all $X, Y \in A_q(n)$ the infimum $X \wedge Y$ exists – that is, if in a realization of the poset as affine subspaces of an n-dimensional vector space V_n, we have $X \cap Y \neq \emptyset$.

***Corollary 4.5.8 (Erdős, Faigle, and Kern* [173]).** *Let $F \subseteq A_q(n)$ be an intersecting Sperner family. Then*

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}_q} \leq 1.$$

Proof. Each bijective affine transformation of the underlying space V_n induces an automorphism of $P := A_q(n)$, and all such automorphisms form a rank-transitive automorphism group G. Note that the class of intersecting Sperner families is G-invariant. Let $C = (\{\mathbf{0}\} = X_0 \lessdot X_1 \lessdot \cdots \lessdot X_n = V_n)$ be a chain of linear (i.e., also affine) subspaces of V_n. Finally let R be the set of automorphisms ϱ of $A_q(n)$ that are induced by translations $\tau_{\boldsymbol{a}} : V_n \to V_n$, where $\tau_{\boldsymbol{a}}(\boldsymbol{x}) := \boldsymbol{x} + \boldsymbol{a}$ for all $\boldsymbol{x} \in V_n$ ($\boldsymbol{a} \in V_n$).

The filter F_{X_0} contains all affine subspaces with the zero vector; thus $Q = F_{X_0}$ is isomorphic to the linear lattice $L_n(q)$. Accordingly, $W_i(Q) = \binom{n}{i}_q$. Thus we must verify condition (4.38) of Theorem 4.5.7. For an affine subspace X of rank i we have

$$w(X) = \frac{W_i(P)}{W_0(P)} \frac{1}{W_i(Q)} = \frac{q^{n-i}\binom{n}{i}_q}{q^n} \frac{1}{\binom{n}{i}_q} = \frac{1}{q^i}.$$

The set $\left(\cup_{\varrho \in R} \varrho(C)\right) \cap F$ contains at most one element: Assume the contrary; then there exist $\boldsymbol{a}, \boldsymbol{b}, i, j$ such that different affine subspaces $Y = \boldsymbol{a} + X_i$ and $Z = \boldsymbol{b} + X_j$ belong to F. Since F is intersecting, there is some $\boldsymbol{c}$ such that $Y = \boldsymbol{c} + X_i$, $Z = \boldsymbol{c} + X_j$, and because F is a Sperner family we have $i = j$ and thus $Y = Z$, a contradiction. If the preceding set is empty, then (4.38) clearly holds, so let

$$\{Y\} = \left(\cup_{\varrho \in R} \varrho(C)\right) \cap F, \quad r(Y) = i.$$

We may obtain Y by a translation from X_i; that is, $Y = \boldsymbol{a} + X_i$, in exactly $|Y|$ ways (we need $\boldsymbol{a} \in Y$); hence in this case

$$\sum_{\varrho \in R} w(\varrho(C) \cap F) = |Y| \frac{1}{|Y|} = 1,$$

and (4.38) is verified. ■

For a second example let us look at the function poset F_k^n. A family F in F_k^n is called *intersecting* if for all $\boldsymbol{a}, \boldsymbol{b} \in F$, $\boldsymbol{a} \wedge \boldsymbol{b} \neq \mathbf{0}$ holds (equivalently, if for all $\boldsymbol{a}, \boldsymbol{b} \in F$ there is some index $i \in [n]$ such that $a_i = b_i \in \{1, \ldots, k\}$). The following corollary, which was proved by me in [147], was the predecessor of Theorem 4.5.7.

Corollary 4.5.9. *Let $F \subseteq F_k^n$ $(k \geq 2)$ be an intersecting Sperner family. Then*

$$\sum_{i=1}^{n} \frac{f_i}{\binom{n-1}{i-1} k^{i-1}} \leq 1.$$

Proof. We will apply Theorem 4.5.7 to the poset P which can be obtained from F_k^n by deleting the minimal element $\mathbf{0}$. In our usual terminology we then have $W_i(P) = W_{i+1}(F_k^n), i = 0, \ldots, n-1$. But in order to avoid this shift of the index we make the exception that the minimal elements of P have rank 1, and in Theorem 4.5.7 we work with $C = (p_1 \lessdot \cdots \lessdot p_n)$, G_{p_1} and so on. Then $W_i(P) = W_i(F_k^n) = \binom{n}{i} k^i$, $i = 1 \ldots, n$.

Let S_n be the symmetric group on $[n]$ and $S_{k,0}$ the set of permutations of $\{0, 1, \ldots, k\}$ that leave 0 invariant. For $\pi = (\pi_0, \pi_1, \ldots, \pi_n) \in S_n \times S_{k,0} \times \cdots \times S_{k,0}$ define $\varphi_\pi : F_k^n \to F_k^n$ by

$$\varphi_\pi(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \text{ iff } b_{\pi_0(i)} = \pi_i(a_i), i = 1, \ldots, n.$$

Thus $\varphi_\pi(\boldsymbol{a})$ can be obtained by applying simultaneously π_i to the ith component of $\boldsymbol{a}, i = 1, \ldots, n$, and then permuting the components according to π_0. It is easy to see that φ_π is an automorphism of F_k^n and that the set G of all such automorphisms is rank transitive. Again the class of all intersecting Sperner families is G-invariant. Let $C := (\boldsymbol{a}_1 \lessdot \cdots \lessdot \boldsymbol{a}_n)$, where $\boldsymbol{a}_i = (1, \ldots, 1, 0, \ldots, 0)$ (i ones), $i = 1, \ldots, n$.

The filter $Q = F_{\boldsymbol{a}_1}$ is obviously isomorphic to F_k^{n-1}, thus $W_i(Q) = \binom{n-1}{i-1} k^{i-1}$, $i = 1, \ldots, n$ (here again the minimal element is considered to have rank 1). Let ι be the identical permutation of $\{0, 1, \ldots, k\}$ and define

$$\zeta_0 := \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & 1 \end{pmatrix},$$

$$\zeta_n := \begin{pmatrix} 0 & 1 & 2 & \ldots & k-1 & k \\ 0 & 2 & 3 & \ldots & k & 1 \end{pmatrix}.$$

Finally let $\boldsymbol{\zeta} := (\zeta_0, \iota, \ldots, \iota, \zeta_n)$. The automorphism φ_ζ acts as follows on the elements of P: It first increases (modulo k) the nth component if it is not zero and it then shifts the components cyclically. We introduce for an element $(0, \ldots, 0, l, 0, \ldots, 0)$, where l is at the mth component, the short notation $\langle l, m \rangle$. So $\boldsymbol{a}_1 = \langle 1, 1 \rangle$. If we iteratively apply φ_ζ to $\boldsymbol{a}_1$ we obtain the "cycle" $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$, $\ldots$, $\langle 1, n \rangle$, $\langle 2, 1 \rangle$, $\ldots$, $\langle 2, n \rangle$, $\ldots$, $\langle k-1, n \rangle$, $\langle k, 1 \rangle$, $\ldots$, $\langle k, n \rangle$, $\langle 1, 1 \rangle$. Thus $R = \{\varphi_\zeta^1, \varphi_\zeta^2, \ldots, \varphi_\zeta^{kn}\}$ is a system of representatives of the left cosets of G relative to

G_{a_1}. It remains to verify (4.38). For $\boldsymbol{a} \in N_i(P)$ we have

$$w(\boldsymbol{a}) = \frac{\binom{n}{i}k^i}{\binom{n}{1}k^1\binom{n-1}{i-1}k^{i-1}} = \frac{1}{i}, \qquad i = 1, \ldots, n.$$

Let $\boldsymbol{a}$ be such an element of $J := \left(\cup_{\varrho \in R}\varrho(C)\right) \cap F$ that has smallest rank, say $r(\boldsymbol{a}) = j$.

Claim. $|J| \leq j$.

Proof of Claim. Obviously $\boldsymbol{a}_i = \langle 1, 1\rangle \vee \cdots \vee \langle 1, i\rangle$ and $\varphi_\zeta^l(\boldsymbol{a}_i) = \varphi_\zeta^l(\langle 1, 1\rangle) \vee \cdots \vee \varphi_\zeta^l(\langle 1, i\rangle)$; that is, $\varphi_\zeta^l(\boldsymbol{a}_i)$ corresponds to i consecutive elements from the preceding cycle, $i = 1, \ldots, n, l = 1, \ldots, kn$. Moreover, two such consecutive sets that belong to two different elements of J cannot have the same starting point or endpoint, because F (thus also J) is a Sperner family, and they must intersect since F (thus also J) is intersecting. Now we may use the arguments from the first proof of the Erdős–Ko–Rado Theorem (Theorem 2.1.7) to convince ourselves that $|J| \leq j$ (take for X the consecutive set corresponding to $\boldsymbol{a}$ and note that in the proof of the claim in the proof of Theorem 2.1.7 we needed only that the sets have size not greater than $\frac{n}{2}$, which is in our case $\frac{kn}{2}$). □

Now we quickly see that (4.38) is satisfied since

$$\sum_{\varrho \in R} w(\varrho(C) \cap F) \leq \frac{1}{j}| \cup_{\varrho \in R} \varrho(C) \cap F| \leq \frac{j}{j} = 1.$$

■

In the special case that $F \subseteq N_i(F_k^n)$ is intersecting, we clearly have $|F| \leq \binom{n-1}{i-1}k^{i-1}$. This result was independently obtained by Frankl [135] and up to some details by Meyer [368].

4.6. Product theorems

Let us first study the direct product in connection with flow morphisms.

***Theorem 4.6.1** (Harper [258]).*

(a) *If φ_i is a (weak) flow morphism from (P_i, v_i) onto (Q_i, w_i), $i = 1, 2$, then the mapping φ defined by $\varphi(p_1, p_2) := (\varphi_1(p_1), \varphi(p_2))$, $p_i \in P_i, i = 1, 2$, is a (weak) flow morphism from $(P_1, v_1) \times (P_2, v_2)$ onto $(Q_1, w_1) \times (Q_2, w_2)$.*

(b) *If φ and ψ are (weak) flow morphisms from (P, v) onto (Q, w) and from (Q, w) onto (R, u), respectively, then the mapping χ defined by $\chi(p) := \psi(\varphi(p))$, $p \in P$, is a (weak) flow morphism from (P, v) onto (R, u).*

Proof. In both cases it is easy to check that the conditions (i), (ii) resp. (ii'), (iii) of a (weak) flow morphism are satisfied. We prove (iv).

(a) Let $(q_1, q_2) \lessdot_{Q_1 \times Q_2} (q_1', q_2')$, that is, without loss of generality

$$q_1 \lessdot_{Q_1} q_1' \text{ and } q_2 = q_2'. \tag{4.40}$$

We have to show that

$$\frac{(v_1 \times v_2)(A)}{(w_1 \times w_2)(q_1, q_2)} \le \frac{(v_1 \times v_2)(\nabla(A))}{(w_1 \times w_2)(q_1', q_2')} \quad \text{for all } A \subseteq \varphi^{-1}(q_1, q_2)$$

where the upper shadow ∇ is taken in the poset induced by $\varphi^{-1}(q_1, q_2) \cup \varphi^{-1}(q_1', q_2')$. Let $\{p_2^1, p_2^2, \ldots, p_2^s\}$ be the list of second components appearing in A. In particular, $\varphi_2(p_2^j) = q_2$, $j = 1, \ldots, s$. Our set A can then be written as a disjoint union $A = (A_1, p_2^1) \cup \cdots \cup (A_s, p_2^s)$, where $A_j \subseteq \varphi^{-1}(q_1)$ and $(A_j, p_2^j) := \{(p_1, p_2^j) : p_1 \in A_j\}$, $j = 1, \ldots, s$. We have

$$\nabla(A) \supseteq (\nabla(A_1), p_2^1) \cup \cdots \cup (\nabla(A_s), p_2^s) \tag{4.41}$$

where this time the shadows $\nabla(A_j)$, $j = 1, \ldots, s$, are taken in the poset induced by $\varphi^{-1}(q_1) \cup \varphi^{-1}(q_1')$. Since φ_1 is a (weak) flow morphism we have

$$\frac{v_1(A_j)}{w_1(q_1)} \le \frac{v_1(\nabla(A_j))}{w_1(q_1')}, \qquad j = 1, \ldots, s. \tag{4.42}$$

Eventually, from (4.40)–(4.42) we obtain

$$\begin{aligned} \frac{(v_1 \times v_2)(A)}{(w_1 \times w_2)(q_1, q_2)} &= \sum_{j=1}^{s} \frac{v_1(A_j) v_2(p_2^j)}{w_1(q_1) w_2(q_2)} \\ &\le \sum_{j=1}^{s} \frac{v_1(\nabla(A_j)) v_2(p_2^j)}{w_1(q_1') w_2(q_2')} \le \frac{(v_1 \times v_2)(\nabla(A))}{(w_1 \times w_2)(q_1', q_2')}. \end{aligned}$$

(b) Let $r_1 \lessdot_R r_2$. By Theorem 4.5.1 we only have to show that there exists a regular covering of the poset induced by $\chi^{-1}(r_1) \cup \chi^{-1}(r_2)$ whose two levels are obviously $\chi^{-1}(r_1)$ and $\chi^{-1}(r_2)$. Let

$$E := \{p_1 p_2 : p_1 \lessdot_P p_2, \chi(p_i) = r_i, \quad i = 1, 2\}$$

and

$$F := \{q_1 q_2 : q_1 \lessdot_Q q_2, \psi(q_i) = r_i, \quad i = 1, 2\}.$$

Finally, let

$$E_{q_1 q_2} := \{p_1 p_2 \in E : \varphi(p_i) = q_i, \quad i = 1, 2\}.$$

Then $E = \bigcup_{q_1 q_2 \in F} E_{q_1 q_2}$ (disjoint union). Since ψ is a (weak) flow morphism, there exists a regular covering of the poset induced by $\psi^{-1}(r_1) \cup \psi^{-1}(r_2)$ – that is, a function $g : F \to \mathbb{R}_+$ such that

$$\sum_{q_2 : q_1 q_2 \in F} g(q_1 q_2) = \frac{w(q_1)}{u(r_1)} \text{ for all } q_1 \in \psi^{-1}(r_1),$$

$$\sum_{q_1 : q_1 q_2 \in F} g(q_1 q_2) = \frac{w(q_2)}{u(r_2)} \text{ for all } q_2 \in \psi^{-1}(r_2),$$

$$\sum_{q_1 q_2 \in F} g(q_1 q_2) = 1.$$

Analogously, for all $q_1 q_2 \in F$ there exists a regular covering $f_{q_1 q_2} : E_{q_1 q_2} \to \mathbb{R}_+$ of the poset induced by $\varphi^{-1}(q_1) \cup \varphi^{-1}(q_2)$. We define $f : E \to \mathbb{R}_+$ by

$$f(p_1 p_2) := f_{\varphi(p_1)\varphi(p_2)}(p_1 p_2) g(\varphi(p_1)\varphi(p_2)).$$

Then f is a regular covering of the poset induced by $\chi^{-1}(r_1) \cup \chi^{-1}(r_2)$ since (with $q_1 := \varphi(p_1)$)

$$\begin{aligned}\sum_{p_2 : p_1 p_2 \in E} f(p_1 p_2) &= \sum_{q_2 : q_1 q_2 \in F} \sum_{p_2 : p_1 p_2 \in E_{q_1 q_2}} f_{q_1 q_2}(p_1 p_2) g(q_1 q_2) \\ &= \sum_{q_2 : q_1 q_2 \in F} g(q_1 q_2) \frac{v(p_1)}{w(q_1)} = \frac{w(q_1)}{u(r_1)} \frac{v(p_1)}{w(q_1)} = \frac{v(p_1)}{u(r_1)}\end{aligned}$$

for all $p_1 \in \chi^{-1}(r_1)$, the corresponding inequality for all $p_2 \in \chi^{-1}(r_2)$ holds analogously, and

$$\sum_{p_1 p_2 \in E} f(p_1 p_2) = \sum_{q_1 q_2 \in F} \sum_{p_1 p_2 \in E_{q_1 q_2}} f_{q_1 q_2}(p_1 p_2) g(q_1 q_2) = \sum_{q_1 q_2 \in F} g(q_1 q_2) = 1.$$

■

Using these results for flow morphisms, we are able to prove in a relatively short way the following result of Harper [257], which was re-proved in a different way by Hsieh and Kleitman [273].

Theorem 4.6.2 (Product Theorem). *If (P, v) and (Q, w) are normal posets with log concave weighted Whitney numbers, then $(P, v) \times (Q, w)$ also is a normal poset with log concave weighted Whitney numbers.*

Proof. For the sake of brevity, we set $a_i := v(N_i(P))$, $b_i := w(N_i(Q))$, $c_i := (v \times w)(N_i(P \times Q))$, $i \in \mathbb{Z}$. At first we prove the statement (cf. Karlin [285, p. 394]) about the log concavity of the sequence $\{c_i\}$ (more information on log concavity (2-positivity) can be found, for instance, in the books of Karlin [285] and Brenti [84] as well as in the papers of Chase [99] and Bender and Canfield

[44]). Obviously, $c_i = \sum_{m=0}^{\infty} a_m b_{i-m}$ (there is only a finite number of nonzero items). We must prove that $c_i^2 \geq c_{i+1}c_{i-1}$ for all $i \in \mathbb{Z}$; that is,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_{i-m} a_n b_{i-n} \geq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_{i-m+1} a_n b_{i-n-1}. \tag{4.43}$$

Since the sequences $\{a_i\}$ and $\{b_i\}$ are log concave and we admit only positive weights,

$$\frac{a_1}{a_0} \geq \frac{a_2}{a_1} \geq \cdots \geq \frac{a_{r(P)}}{a_{r(P)-1}}, \tag{4.44}$$

$$\frac{b_1}{b_0} \geq \frac{b_2}{b_1} \geq \cdots \geq \frac{b_{r(Q)}}{b_{r(Q)-1}}. \tag{4.45}$$

Consequently,

$$(a_m a_n - a_{m-1} a_{n+1})(b_{i-m} b_{i-n} - b_{i-m+1} b_{i-n-1}) \geq 0 \tag{4.46}$$

for all integers m, n since the term in each parenthesis is nonnegative if $m \leq n$ and nonpositive, otherwise. The inequality (4.46) is equivalent to

$$\begin{aligned} &a_m b_{i-m} a_n b_{i-n} + a_{m-1} b_{i-m+1} a_{n+1} b_{i-n-1} \\ &\quad \geq a_{m-1} b_{i-m} a_{n+1} b_{i-m} + a_m b_{i-m+1} a_n b_{i-n-1}. \end{aligned}$$

Summing up the last inequality over all integers m, n and dividing the resulting inequality by 2 we obtain (4.43).

Now we prove the statement about normality. By Proposition 4.5.4 there exist flow morphisms φ_1 and φ_2 from (P, v) onto (C, v') and from (Q, w) onto (D, w') where $C = (0 \lessdot 1 \lessdot \cdots \lessdot r(P))$, $D = (0 \lessdot 1 \lessdot \cdots \lessdot r(Q))$, $v'(i) = a_i$, $w'(i) = b_i$, $i = 0, 1, \ldots$, and we have to show that there is a flow morphism χ from $(P, v) \times (Q, w)$ onto (E, u) where $E = (0 \lessdot 1 \lessdot \cdots \lessdot r(P) + r(Q)$, $u(i) = c_i, i = 0, 1, \ldots$. By Theorem 4.6.1(a) there exists a flow morphism φ from $(P, v) \times (Q, w)$ onto $(C, v') \times (D, w')$, and by Theorem 4.6.1(b) we must show only that there is a flow morphism ψ from $(C, v') \times (D, w')$ onto (E, u); that is, we must prove that $(C, v') \times (D, w')$ has the normalized matching property.

Let $A \subseteq N_k(C \times D) = \{(i, k-i) : \max\{0, k - r(Q)\} \leq i \leq \min\{r(P), k\}\}$. Suppose at first that

$$A = \{(m, k-m), (m+1, k-m-1), \ldots, (n, k-n)\}, \qquad m \leq n. \tag{4.47}$$

Then

$$\nabla(A) = \{(m, k-m+1), (m+1, k-m), \ldots, (n+1, k-n)\}, \tag{4.48}$$

and, if we set $S_k(h, j) := \sum_{i=h}^{j} a_i b_{k-i}$, we have to show the inequality in

$$\frac{(v' \times w')(A)}{(v' \times w')(N_k(C \times D))} = \frac{S_k(m,n)}{S_k(0,\infty)} \leq \frac{S_{k+1}(m,n+1)}{S_{k+1}(0,\infty)}$$
$$= \frac{(v' \times w')(\nabla(A))}{(v' \times w')(N_{k+1}(C \times D))}. \tag{4.49}$$

There holds

$$\frac{S_k(m,n)}{S_k(m,\infty)} = \frac{S_k(m,n)}{S_k(m,n+1)} \frac{S_k(m,n+1)}{S_k(m,n+2)} \cdots,$$

$$\frac{S_k(m,n)}{S_k(0,\infty)} = \frac{S_k(m,n)}{S_k(m,\infty)} \frac{S_k(m,\infty)}{S_k(m-1,\infty)} \frac{S_k(m-1,\infty)}{S_k(m-2,\infty)} \cdots \frac{S_k(1,\infty)}{S_k(0,\infty)}.$$

After writing $\frac{S_{k+1}(m,n+1)}{S_{k+1}(0,\infty)}$ as a telescoping product, too, we see that it suffices to prove that

$$\frac{S_k(m,j)}{S_k(m,j+1)} \leq \frac{S_{k+1}(m,j+1)}{S_{k+1}(m,j+2)} \quad \text{for all } j \geq n \tag{4.50}$$

$\left(\text{which implies } \frac{S_k(m,n)}{S_k(m,\infty)} \leq \frac{S_{k+1}(m,n+1)}{S_{k+1}(m,\infty)}\right)$ and

$$\frac{S_k(j,\infty)}{S_k(j-1,\infty)} \leq \frac{S_{k+1}(j,\infty)}{S_{k+1}(j-1,\infty)} \quad \text{for all } 1 \leq j \leq m. \tag{4.51}$$

The inequality (4.50) is equivalent to

$$\sum_{i=m}^{j} (a_{i+1}a_{j+1} - a_i a_{j+2}) b_{k-i} b_{k-j-1} + a_m a_{j+1} b_{k+1-m} b_{k-j-1} \geq 0,$$

and (4.51) is equivalent to

$$\sum_{i=j}^{\infty} (b_{k-i+1} b_{k-j+1} - b_{k-i} b_{k-j+2}) a_i a_{j-1} \geq 0.$$

These two inequalities are true because of (4.44) and $j \geq i$ $(i = m, m+1, \ldots, j)$ and because of (4.45) and $j \leq i$ $(i = j, j+1, \ldots)$, respectively. Hence (4.49) is true and the theorem is proved for the special case that A has the form (4.47). If A does not have that form we partition A into sets (blocks) $A_1, \ldots, A_t$ such that each A_j has the form

$$A_j = \{(m_j, k-m_j), (m_j+1, k-m_j-1), \ldots, (n_j, k-n_j)\},$$
$$m_1 \leq n_1 < m_2 \leq n_2 < \cdots < m_t \leq n_t,$$

and t is minimal with respect to this property (i.e., $m_{j+1} - n_j \geq 2,\ j = 1, \ldots, t-1$). Then $\nabla(A_1) \cup \cdots \cup \nabla(A_t)$ is a partition of $\nabla(A)$, and we have

$$\begin{aligned}\frac{(v' \times w')(A)}{c_k} &= \sum_{j=1}^{t} \frac{(v' \times w')(A_j)}{c_k} \\ &\leq \sum_{j=1}^{t} \frac{(v' \times w')(\nabla(A_j))}{c_{k+1}} = \frac{(v' \times w')(\nabla(A))}{c_{k+1}}\end{aligned}$$

since each A_j has the form (4.47). Thus $(C, v') \times (D, w')$ is normal, and the proof is complete. ■

Since the Product Theorem can be generalized by induction in a natural way to a product of n normal posets with log-concave (weighted) Whitney numbers, we can apply it to the following classes of (unweighted) posets:

Example 4.6.1. *The following posets are normal (i.e., also strongly Sperner) and have log-concave Whitney numbers: chain products $S(k_1, \ldots, k_n)$, the function poset F_k^n, star products $T(k_1, \ldots, k_n)$, the poset $Int(S(k_1, \ldots, k_n))$ (note that $Int(S(k_1, \ldots, k_n))$ is a product of "halved" posets $S(k_i, k_i)$), modular geometric lattices (see Examples 1.3.11 and 4.5.1; the log concavity can be easily verified).*

The case of chain products was already settled by Anderson [30]. See Section 6.3 for examples of a modular and of a geometric lattice that are not Sperner, hence, also not normal. In [259] Harper proved further that $Int(L_n(q))$, the poset of all nonempty intervals in the linear lattice $L_n(q)$ ordered by inclusion, is a normal poset with log-concave Whitney numbers.

Let us discuss further the unweighted posets. A ranked poset P is said to have the *strict k-Sperner property* if all maximum antichains are unions of complete levels. In [146, 149] we sharpened the definition of normality and log concavity to prove the strict k-Sperner property for several posets. In the following we denote the intervals $\{\alpha, \alpha+1, \ldots, \beta-1\}$ and $\{\alpha+1, \ldots, \beta-1\}$ by $[\alpha, \beta)$ and (α, β), respectively, and consider them to be empty if $\alpha \geq \beta$ or $\alpha \geq \beta - 1$, respectively. A graded poset P is said to be *I-normal* ($I \subseteq [0, n)$) if it is normal and if

$$\frac{|A|}{W_i} < \frac{|\nabla(A)|}{W_{i+1}} \text{ for all } \emptyset \neq A \subset N_i, \quad i \in [0, n) - I.$$

A *strictly normal poset* is an $\emptyset$-normal poset. There is an easy criterion for I-normality. Look at the Hasse diagram $G_i = (P_i, E_i)$ of the $\{i, i+1\}$-rank-selected subposet of P. We say that P is *I-level connected* if G_i is connected for all $i \in [0, n) - I$. In the case $I = \emptyset$ we speak of *strictly level connected posets*. Strengthening Corollary 4.5.1 we have:

Theorem 4.6.3. *Let P be a normal poset. The following conditions are equivalent:*

(*i*) *P is I-normal,*

(*ii*) *P is I-level connected and for all $i \in [0, n) - I$ there exist functions $f_i : E_i \to \mathbb{R}_+ - \{0\}$ such that*

$$\sum_{e \in E_i : e^- = p} f_i(e) = \frac{1}{W_i} \quad \textit{for all } p \in N_i,$$

$$\sum_{e \in E_i : e^+ = p} f_i(e) = \frac{1}{W_{i+1}} \quad \textit{for all } p \in N_{i+1}.$$

Proof. Throughout let $i \in [0, n) - I$.

(i) $\to$ (ii). Assume that G_i is not connected. Let F be a component of G_i. Then $\emptyset \neq A := F \cap N_i \neq N_i$. Moreover, $\nabla(A) \subseteq F$, but $\nabla(N_i - A) \cap F = \emptyset$. Consequently,

$$1 = \frac{|A| + |N_i - A|}{W_i} < \frac{|\nabla(A)| + |\nabla(N_i - A)|}{W_{i+1}} \leq 1,$$

a contradiction.

Consider the network $N = (V, E, s, t, c)$, where $V := N_i \cup N_{i+1} \cup \{s, t\}$, $E := E_i \cup \{sp : p \in N_i\} \cup \{pt : p \in N_{i+1}\}$,

$$c(e) := \begin{cases} \frac{1}{W_i} & \text{if } e = sp \text{ for some } p \in N_i, \\ \infty & \text{if } e \in E_i, \\ \frac{1}{W_{i+1}} & \text{if } e = qt \text{ for some } q \in N_{i+1}. \end{cases}$$

Let $\epsilon > 0$. Send, for every $pq \in E_i$, ϵ units of flow through s, p, q, t. This gives a flow f_ϵ with $f_\epsilon(e) = \epsilon > 0$ if $e \in E_i$ and $\lim_{\epsilon \to 0} f_\epsilon(e) = 0$ for every $e \in E$. Now change the capacity of N by defining the new capacity

$$c_\epsilon(e) := c(e) - f_\epsilon(e) \quad \text{for all } e \in E.$$

Claim. If ϵ is small enough, $(\{s\}, V - \{s\})$ (and $(V - \{t\}, \{t\})$) is a minimal cut.

Proof of Claim. We have $c_\epsilon(\{s\}, V - \{s\}) = 1 - \sum_{p \in N_i} f_\epsilon(sp) \leq 1$. In looking for cuts of smaller capacity we must consider only those cuts (S, T) for which $\emptyset \neq A := S \cap N_i \neq N_i$ and $\nabla(A) \subseteq S$. But we have

$$\begin{aligned} c_\epsilon(S, T) &\geq \frac{|N_i - A|}{W_i} - \sum_{p \in N_i - A} f_\epsilon(sp) + \frac{|\nabla(A)|}{W_{i+1}} - \sum_{q \in \nabla(A)} f_\epsilon(qt) \\ &= 1 + \left(\frac{|\nabla(A)|}{W_{i+1}} - \frac{|A|}{W_i} \right) - \sum_{p \in N_i - A} f_\epsilon(sp) - \sum_{q \in \nabla(A)} f_\epsilon(qt) \end{aligned}$$

tending to

$$1 + \frac{|\nabla(A)|}{W_{i+1}} - \frac{|A|}{W_i}$$

if $\epsilon \to 0$. Since there is only a finite number of subsets A of N_i and for every proper subset there holds $\frac{|\nabla(A)|}{W_{i+1}} - \frac{|A|}{W_i} > 0$, we find some small ϵ such that for every cut in question $c_\epsilon(S, T) > 1 \geq c_\epsilon(\{s\}, V - \{s\})$. □

By the Max-Flow Min-Cut Theorem (Theorem 4.1.1 and Corollary 4.1.1) and in view of Lemma 4.1.1(a), this claim implies that there is some flow g_ϵ in the new network such that the arcs sp, $p \in N_i$, and qt, $q \in N_{i+1}$, are saturated. The restriction of $f_\epsilon + g_\epsilon$ to the set E_i gives the desired function f_i.

(ii) $\to$ (i). Assume that there is some $A \subseteq N_i$ such that $\emptyset \neq A \neq N_i$ and $\frac{|A|}{W_i} = \frac{|\nabla(A)|}{W_{i+1}}$. Consider again the (first) network N. The cut (S, T) with $S := \{s\} \cup A \cup \nabla(A)$, $T := (N_i - A) \cup (N_{i+1} - \nabla(A)) \cup \{t\}$ has capacity $\frac{|N_i - A|}{W_i} + \frac{|\nabla(A)|}{W_{i+1}} = 1$. Extend in a natural way the function f_i to a flow f in N which has value $v(f) = 1$. By the suppositions there must be some arc from $N_i - A$ to $\nabla(A)$ which is not void, that is, having positive flow. From Lemma 4.1.1(a) we obtain $1 = f(S, T) - f(T, S)$ with $f(T, S) > 0$. Consequently, $c(S, T) \geq f(S, T) > 1$, a contradiction. ■

From the proof of Corollary 4.5.2 we easily derive:

Corollary 4.6.1. *A graded poset P is I-normal if it is regular and I-level connected.*

It is straightforward to verify that all posets given in Example 4.5.1 are strictly level connected and consequently strictly normal.

The Whitney numbers of a ranked poset P are called *I-log concave* if they are log concave and if $W_i^2 > W_{i-1}W_{i+1}$ for all $i \in (0, n) - I$. If $I = \emptyset$ we also say *strictly log concave*. In looking carefully for strict inequalities in the proof of the Product Theorem, we found the following result whose proof (with several technical details) is given in [149].

Theorem 4.6.4 (Strict Product Theorem). *Let P be an $[\alpha_1, \beta_1)$-normal poset with (α_2, β_2)-log-concave Whitney numbers, where $0 < \alpha_1 \leq r(P)$, $0 \leq \beta_1 < r(P)$, $0 \leq \alpha_2 \leq r(P)$, $0 \leq \beta_2 \leq r(P)$. Further, let Q be a strict normal poset with log-concave Whitney numbers. Then $P \times Q$ is $[\sigma, \tau)$-normal with (σ, τ)-log-concave Whitney numbers, where*

$$\sigma := \begin{cases} r(P) \\ r(Q) + \min\{\alpha_1, \alpha_2\} \end{cases}, \qquad \tau := \begin{cases} r(Q) & \text{if } r(P) \leq r(Q), \\ \max\{\beta_1, \beta_2\} & \text{if } r(P) > r(Q). \end{cases}$$

From this theorem one can derive by induction (see again [149]):

Corollary 4.6.2. *Let $Q_1, \dots, Q_n$, $n \geq 2$, be strict normal posets and let $P := \prod_{i=1}^{n} Q_i, r(Q_1) \leq \cdots \leq r(Q_n)$.*

(a) *If the Whitney numbers of the posets Q_i, $i = 1, \dots, n$, are log concave, then P is $[\alpha, \beta)$-normal and has (α, β)-log-concave Whitney numbers, where $\alpha := \sum_{i=1}^{n-1} r(Q_i)$, $\beta := r(Q_n)$.*

(b) *If the Whitney numbers of the posets Q_i, $i = 1, \dots, n$, are strictly log concave, then P is strictly normal and has strictly log-concave Whitney numbers.*

This corollary is helpful in proving the strict k-Sperner property:

Theorem 4.6.5. *If P is an $[\alpha, \beta)$-normal poset with (α, β)-log-concave Whitney numbers, then P has the strict k-Sperner property for all $k \geq \max\{1, \beta - \alpha + 1\}$. For these k, there are at most two maximum k-families.*

Proof. Let us look first at the case $k = 1$ and $\alpha \geq \beta$ – that is, P is strictly normal and has strictly log-concave Whitney numbers. It is easy to see that there is some index h such that $W_0 < W_1 < \cdots < W_h \geq W_{h+1} > \cdots > W_n$. From the LYM-inequality (see Theorem 4.5.1), we derive for every antichain F

$$\frac{|F|}{W_h} \leq \sum_{i=0}^{n} \frac{|F \cap N_i|}{W_i} \leq 1, \quad \text{and hence,} \quad |F| \leq W_h,$$

where the first inequality is an equality iff $F \subseteq N_h$ and $W_h > W_{h+1}$ or $F \subseteq N_h \cup N_{h+1}$ and $W_h = W_{h+1}$. In the first case we have only one maximum antichain, namely $F = N_h$. Assume that in the second case there is some maximum antichain different from N_h and N_{h+1}. Then $\emptyset \neq F \cap N_h \neq N_h$, consequently

$$\frac{|F \cap N_h|}{W_h} < \frac{|\nabla(F \cap N_h)|}{W_{h+1}},$$

but $\nabla(F \cap N_h)$ and $F \cap N_{h+1}$ are disjoint since F is an antichain. Thus $|F| = |F \cap N_h| + |F \cap N_{h+1}| < |\nabla(F \cap N_h)| + |F \cap N_{h+1}| \leq |N_{h+1}|$, a contradiction to the maximality of F.

Now let $1 < k \leq n$ and consider the chain $C_k = (0 \lessdot 1 \lessdot \cdots \lessdot k-1)$. Then $P_k := P \times C_k$ is strictly normal and has strictly log-concave Whitney numbers by Theorem 4.6.4. Let F be any maximum k-family in P. Then we construct the antichain A as in the proof of Theorem 4.3.1, which is by that theorem a maximum antichain in P_k. From the case $k = 1$ we know that we have at most two possibilities for A, and A must be a complete level in P_k. Thus F is a union of k complete (neighboring) levels in P, and there are at most two possibilities for F. ■

Example 4.6.2. *From Theorem 4.6.5, Corollary 4.6.2, and the remark after Corollary 4.6.1, we obtain that the following posets have the strict k-Sperner property and that there are at most two maximum k-families for all* $k = 1, 2, \ldots$*:* B_n, $L_n(q)$, $A_n(q)$, *projective space lattices,* Q_n, F_k^n, $Int(S(k_1, \ldots, k_n))$, *modular geometric lattices and* $S(k_1, \ldots, k_n)$ *if* $k_1 \geq \cdots \geq k_n$ *and* $k_2 + \cdots + k_n \geq k_1$ *(the last result is due to Clements* [110], *other proofs can be found in Katerinochkina* [289] *and Griggs* [240]).

In [239] Griggs proved the Sperner property for the following poset P_G: Suppose that $[n]$ is partitioned into sets $A_1, \ldots, A_m$, and let $I_i := [a_i, b_i]$, $i = 1, \ldots, m$, be nonempty intervals of numbers (we could also consider arithmetic progressions). Let P_G be the class of all subsets X of $[n]$ with the property $|X \cap A_i| \in I_i$, $i = 1, \ldots, m$, and order these elements (i.e., subsets) by inclusion.

Corollary 4.6.3. *The Griggs poset* P_G *is strictly normal and has strictly log-concave Whitney numbers.*

Proof. Let P_i be the poset of all subsets X_i of A_i with $|X_i| \in I_i$, ordered by inclusion. Obviously, P_i is isomorphic to the I_i-rank-selected subposet of the Boolean lattice $B_{|A_i|}$, which by the preceding remarks is strictly normal and has obviously strictly log-concave Whitney numbers. Moreover, P_G is isomorphic to $P_1 \times \cdots \times P_m$ where the isomorphism is given by $X \subseteq [n] \leftrightarrow (X_1 \cap A_1, \ldots, X \cap A_m)$. The statement follows from Corollary 4.6.2(b). ■

Instead of partitions, West, Harper, and Daykin [467] took chains $A_1 \subset A_2 \subset \cdots \subset A_m \subseteq [n]$ and defined similarly to the above poset P_{WHD} of all subsets X of $[n]$ with $|X \cap A_i| \in I_i$, $i = 1, \ldots, m$, ordered by inclusion.

Corollary 4.6.4. *The West–Harper–Daykin poset* P_{WHD} *is strictly normal and has strictly log-concave Whitney numbers.*

Proof. Observe at first that we can assume, w.l.o.g., that for our intervals $I_i = [a_i, b_i]$ the inequalities $a_1 \leq \cdots \leq a_m$ and $b_1 \leq \cdots \leq b_m$ hold and that $A_m = [n]$ (otherwise add $[n]$ as a new element of the chain together with the last interval I_m). Let P_k be the poset of all subsets X of A_k with $|X \cap A_i| \in I_i$, $i = 1, \ldots, k$.

Obviously, $P_{WHD} \cong P_m$. We prove by induction on k that P_k is strictly normal and has strictly log-concave Whitney numbers. The case $k = 1$ is easy. Now consider the step $1 \leq k - 1 \to k \leq m$. Let Q_k be the poset of all subsets of $A_k - A_{k-1}$, ordered by inclusion; that is, $Q_k \cong B_{|A_k| - |A_{k-1}|}$. By induction and Corollary 4.6.2(b) $P_{k-1} \times Q_k$ is strictly normal and has strictly log-concave Whitney numbers. But P_k is that rank-selected subposet of $P_{k-1} \times Q_k$ whose members have cardinality in I_k; thus the result follows. ■

The variance problem can be easier handled with respect to the direct product [152].

Theorem 4.6.6. *Let x and y be optimal representations of (P, v) and (Q, w), respectively. Then the function $z : P \times Q \to \mathbb{R}$ defined by $z(p, q) := x(p) + y(q)$ is an optimal representation of $(P, v) \times (Q, w)$. In particular, $\sigma^2((P, v) \times (Q, w)) = \sigma^2(P, v) + \sigma^2(Q, w)$.*

Proof. Obviously, z is a representation of $P \times Q$. Further it is easy to verify that $\mu_z = \mu_x + \mu_y$. Let f and g be representation flows on (P, v) relative to x and on (Q, w) relative to y, respectively, which exist by the supposition and by Theorem 4.4.1. Define the function $h : E(P \times Q) \to \mathbb{R}_+$ by

$$h((p, q)(p', q')) := \begin{cases} w(q) f(pp') & \text{if } pp' \in E(P_x) \text{ and } q = q', \\ v(p) g(qq') & \text{if } qq' \in E(Q_y) \text{ and } p = p', \\ 0 & \text{otherwise.} \end{cases}$$

Then $((p, q)(p', q')) \in E(P \times Q)$ and $z(p', q') - z(p, q) > 1$ imply $x(p') - x(p) > 1$ and $q = q'$ or $y(q') - y(q) > 1$ and $p = p'$; that is, $h((p, q)(p', q')) = 0$. Moreover,

$$\begin{aligned} (p, q)_h^+ - (p, q)_h^- &= w(q) p_f^+ + v(p) q_g^+ - (w(q) p_f^- + v(p) q_g^-) \\ &= w(q) v(p) (x(p) - \mu_x) + v(p) w(q) (y(q) - \mu_y) \\ &= (v \times w)(p, q)(z(p, q) - \mu_z); \end{aligned}$$

that is, h is a representation flow relative to z, and by Theorem 4.4.1 z is an optimal representation. Finally,

$$\begin{aligned} &\sigma^2((P, v) \times (Q, w)) \\ &= \frac{1}{(v \times w)(P \times Q)} \sum_{(p,q) \in P \times Q} (v \times w)(p, q) z^2(p, q) - \mu_z^2 \\ &= \frac{1}{v(P) w(Q)} \sum_{p \in P, q \in Q} v(p) w(q) (x(p) + y(q))^2 - (\mu_x + \mu_y)^2 \\ &= \sigma^2(P, v) + \sigma^2(Q, w), \end{aligned}$$

and the proof is complete. ■

We have seen that the direct product of normal posets is normal under some suppositions on the Whitney numbers. One cannot totally omit these suppositions. The product of the two unweighted normal posets in Figure 4.10 does not have

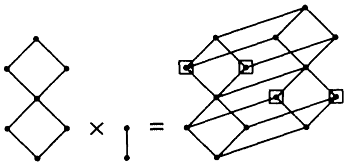


Figure 4.10

the Sperner property; that is, it is not normal. Because of Theorem 4.6.6 the direct product of rank-compressed posets is rank compressed. The situation is different if we consider rankwise direct products (this result is essentially due to Sali [409]).

Theorem 4.6.7. *If (P, v) and (Q, w) are normal posets of same rank, then $(P, v) \times_r (Q, w)$ is normal, too.*

Proof. Let $f_i : E_i(P) \to \mathbb{R}_+$ and $g_i : E_i(Q) \to \mathbb{R}_+$ be functions satisfying the conditions of Corollary 4.5.1, correspondingly. Define $h_i((p,q)(p'q')) : E_i(P \times_r Q) \to \mathbb{R}_+, i = 0, \dots, r(P) - 1$, by

$$h_i((p,q)(p',q')) := f_i(pp')g_i(qq'), \quad pp' \in E_i(P), \quad qq' \in E_i(Q).$$

Then

$$(p,q)^-_{h_i} = \sum_{\substack{p' > p, \\ q' > q}} f_i(pp')g_i(qq') = p^-_{f_i} q^-_{g_i} = \frac{v(p)}{v(N_i(P))} \frac{w(q)}{w(N_i(Q))}$$

$$= \frac{(v \times_r w)(p,q)}{(v \times_r w)(N_i(P \times_r Q))},$$

and the corresponding equality for $(p,q)^+_{h_i}$ holds, too. By Corollary 4.5.1, $(P, v) \times_r (Q, w)$ is normal. ■

More generally, using Theorem 4.6.3 and straightforward calculations, we obtain that the rankwise direct product preserves I-normality and I-log concavity.

Example 4.6.3. *The poset of subcubes of a cube $SQ(k, n)$ and the poset of square submatrices of a square matrix $SM(k, n)$ are strictly normal with strictly log concave Whitney numbers.*

However, the rank-compression property is not preserved by rankwise direct product that can be seen by the unweighted rank-compressed poset given in Figure 4.11 (see [59]): The indicated elements form a filter F. It is easy to see that $F \times_r F$ in

$P \times_r P$ does not satisfy the inequality (iii) in Theorem 4.4.1; that is, $P \times_r P$ is not rank compressed.

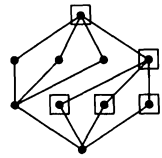


Figure 4.11

5

Matchings, symmetric chain orders, and the partition lattice

This chapter should be considered as a link between the preceding flow-theoretic (resp. linear programming) approach and the succeeding algebraic approach to Sperner-type problems. The main ideas are purely combinatorial. The powerful method of decomposing a poset into symmetric chains not only provides solutions of several problems, it is also very helpful for the insight into the algebraic machinery in Chapter 6. In particular, certain strings of basis vectors of a corresponding vector space will behave like (symmetric) chains.

5.1. Definitions, main properties, and examples

Throughout let P be a ranked poset of rank $n := r(P)$. We say that the level N_i can be matched into the level N_{i+1} (resp. N_{i-1}) if there is a matching of size W_i in the Hasse diagram $G_i = (P_i, E_i)$ (resp. $G_{i-1} = (P_{i-1}, E_{i-1})$) of the $\{i, i+1\}$- (resp. $\{i-1, i\}$-) rank-selected subposet (recall that a *matching* is a set of pairwise nonadjacent edges or arcs).

Theorem 5.1.1. *If there is an index h such that N_i can be matched into N_{i+1} for $0 \le i < h$ and N_i can be matched into N_{i-1} for $h < i \le n$, then P has the Sperner property.*

Proof. If we join the arcs of the corresponding matchings at all points that are both starting point and endpoint, we obtain a partition of P into saturated chains (isolated points are considered as 1-element chains). Each such chain has a common point with N_h. Consequently, we have W_h chains and Dilworth's Theorem 4.0.1 implies $d(P) \le W_h$, that is, $d(P) = W_h$. ∎

There exist several criteria for the existence of such matchings.

***Theorem 5.1.2* (*Hall's Theorem* [251]).** *Let P be a poset of rank $n = 1$. Then N_0 can be matched into N_1 iff for all $A \subseteq N_0$ there holds $|A| \le |\nabla(A)|$.*

Proof. The inequality $|A| \le |\nabla(A)|$ for every $A \subseteq N_0$ is clearly necessary, so let us look at the sufficiency. Let S be a maximum antichain in P, $A := N_0 \cap S$, $B := N_1 \cap S$. Then $\nabla(A) \cap B = \emptyset$; that is, $|S| = |A| + |B| \le |\nabla(A)| + |B| \le W_1$. Consequently, $d(P) = W_1$. By Dilworth's Theorem 4.0.1, P can be partitioned into W_1 chains. Obviously, every chain must contain exactly one element of N_1. The 2-element chains form the desired matching. ■

***Theorem 5.1.3* (*R. Canfield* [96]).** *Let P be a poset of rank $n = 1$ without isolated minimal points. If for all $p, q \in P$ with $p < q$ there holds $|\nabla(p)| \ge |\Delta(q)|$, then N_0 can be matched into N_1.*

Proof. We consider the network $N = (V, E, s, t, c)$ where $V := N_0 \cup N_1 \cup \{s, t\}$, $E := \{sp : p \in N_0\} \cup \{pq : p < q\} \cup \{qt : q \in N_1\}$ and $c(sp) := 1$ for $p \in N_0$, $c(pq) := \infty$ for $p < q$, $c(qt) := 1$ for $q \in N_1$. We define $f : E \to \mathbb{R}_+$ by $f(sp) := 1$ for $p \in N_0$, $f(pq) := \frac{1}{|\nabla(p)|}$ for $p < q$, $f(qt) := \sum_{p<q} \frac{1}{|\nabla(p)|}$ for $q \in N_1$. Then it is easy to see that f is a flow (note that by supposition

$$f(qt) \le \sum_{p<q} \frac{1}{|\Delta(q)|} = |\Delta(q)| \frac{1}{|\Delta(q)|} = 1).$$

We have $v(f) = W_0$. Consequently, the minimum capacity of a cut in N equals W_0, and by Theorem 4.1.1 there exists an integral, that is, 0,1-flow in N of value W_0. The arcs between N_0 and N_1 of flow value 1 form the desired matching. ■

We will apply this condition only later. First we will study cases where the construction of the matchings can be carried out in a certain global and symmetric way.

A chain C of P is called *symmetric* if it has the form $C = (p_0 \lessdot \cdots \lessdot p_h)$ with $r(p_0) + r(p_h) = n$. A set $\mathfrak{C}$ of chains of P is called a *symmetric chain partition* (or *symmetric chain decomposition*) if all chains $C \in \mathfrak{C}$ are symmetric and each element of P is contained in exactly one chain of $\mathfrak{C}$. Finally, our ranked poset P is said to be a *symmetric chain order* (briefly *sc-order*) if there exists a symmetric chain partition $\mathfrak{C}$ of P. We have already seen an example of an sc-order: From the third proof of Theorem 1.1.1 in the beginning of Section 2.2 it follows that the Boolean lattice is an sc-order. But we will find many more sc-orders. In the following we use the notation

$$\mathfrak{C}_i := \{C \in \mathfrak{C} : |C| = i\}, \quad i = 1, \ldots, n + 1.$$

Lemma 5.1.1. *If P is an sc-order then P is rank symmetric and rank unimodal.*

Proof. W_i equals the number of chains of a symmetric chain partition of P meeting N_i, $i = 0, \dots, n$. Now the rank symmetry follows from the fact that a symmetric chain meets N_i iff it meets N_{n-i}. The rank unimodality can be derived in the same way: If a symmetric chain has a rank i member, $i < \frac{n}{2}$, then it contains also a rank $i+1$ member. ■

Lemma 5.1.2. *Let $\mathfrak{C}$ be a symmetric chain partition of P. Then, for $i = 1, \dots, n+1$,*

$$|\mathfrak{C}_i| = \begin{cases} 0 & \text{if } n-i \text{ is even,} \\ W_{(n-i+1)/2} - W_{(n-i-1)/2} & \text{otherwise.} \end{cases}$$

Proof. Let $C = (p_0 \lessdot \cdots \lessdot p_{i-1})$ be any symmetric chain with $|C| = i$. Then $r(p_0) + r(p_{i-1}) = 2r(c_0) + i - 1 = n$; that is, $n - i$ is odd. Consequently, $\mathfrak{C}_i = \emptyset$ if $n - i$ is even. Moreover, C must start in $N_{(n-i+1)/2}$ and end in $N_{(n+i-1)/2}$. In order to count the number of such chains, observe that this number equals the number of chains meeting $N_{(n-i+1)/2}$ (i.e., $W_{(n-i+1)/2}$) reduced by the number of chains meeting $N_{(n-i-1)/2}$ (i.e., $W_{(n-i-1)/2}$). ■

Let us order the Whitney numbers such that they are decreasing. We can use the permutation π of $\{0, \dots, n+1\}$ defined by

$$\pi := \begin{cases} \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n-1 & n & n+1 \\ \frac{n}{2} & \frac{n-2}{2} & \frac{n+2}{2} & \frac{n-4}{2} & \frac{n+4}{2} & \cdots & 0 & n & n+1 \end{pmatrix} & \\ & \text{if } n \text{ is even,} \\ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & n-1 & n & n+1 \\ \frac{n-1}{2} & \frac{n+1}{2} & \frac{n-3}{2} & \frac{n+3}{2} & \frac{n-5}{2} & \cdots & 0 & n & n+1 \end{pmatrix} & \\ & \text{if } n \text{ is odd.} \end{cases}$$

The formula in Lemma 5.1.2 reads:

$$|\mathfrak{C}_i| = W_{\pi(i-1)} - W_{\pi(i)}, \, i = 1, \dots, n+1$$

(recall that $W_{n+1} = 0$ and note Lemma 5.1.1).

Now we present the key theorem.

Theorem 5.1.4. *If P is an sc-order, then P has the strong Sperner property.*

Proof. We must prove that the size of any k-family F in P is not greater than the largest sum of k Whitney numbers of P. In view of Lemma 5.1.1 we have to show that $|F| \leq \sum_{i=0}^{k-1} W_{\pi(i)}$. Let $\mathfrak{C}$ be a symmetric chain partition of P. Clearly, $|C \cap F| \leq \min\{|C|, k\}$ for all $C \in \mathfrak{C}$. Accordingly,

$$\begin{aligned}
|F| \leq \sum_{C \in \mathfrak{C}} \min\{|C|, k\} &= \sum_{i=1}^{k-1} i|\mathfrak{C}_i| + \sum_{i=k}^{n+1} k|\mathfrak{C}_i| \\
&= \sum_{i=1}^{k-1} i(W_{\pi(i-1)} - W_{\pi(i)}) + k\sum_{i=k}^{n+1}(W_{\pi(i-1)} - W_{\pi(i)}) \\
&= \sum_{i=0}^{k-1} W_{\pi(i)}.
\end{aligned}$$

■

Remark 5.1.1. *Note that we proved more: Every symmetric chain partition is k-saturated for every $k = 1, 2, \ldots$ (see the remarks after Corollary 4.3.2 and note that $\cup_{i=0}^{k-1} N_{\pi(i)}$ is a maximum k-family).*

Now, of course, it is interesting to have examples of sc-orders. A large class can be obtained in applying the following product theorem. The key idea goes back to de Bruijn, Tengbergen, and Kruyswijk [87], who considered products of chains. The product theorem was then proved (resp. re-proved) by Alekseev [24], Aigner [20], Griggs [236], Koester [313], and others.

Theorem 5.1.5. *If P and Q are sc-orders, then $P \times Q$ is an sc-order, too.*

Proof. Let $\mathfrak{C}$ and $\mathfrak{D}$ be symmetric chain partitions of P and Q, respectively. For each pair (C, D), where $C = (p_0 \lessdot \cdots \lessdot p_h) \in \mathfrak{C}$ and $D = (q_0 \lessdot \cdots \lessdot q_k) \in \mathfrak{D}$, we form the following chains of $P \times Q$ (see Figure 5.1).

$$\begin{aligned}
E_j(C, D) := ((p_0, q_j) &\lessdot \cdots \lessdot (p_{h-j}, q_j) \lessdot (p_{h-j}, q_{j+1}) \\
&\lessdot \cdots \lessdot (p_{h-j}, q_k)), \qquad j = 0, \ldots, \min\{h, k\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
r(p_0, q_j) + r(p_{h-j}, q_k) &= r(p_0) + (r(q_0) + j) + (r(p_h) - j) + r(q_k) \\
&= r(P) + r(Q) = r(P \times Q);
\end{aligned}$$

that is, the chains $E_j(C \times D)$ are symmetric. Moreover, $\mathfrak{E} := \cup_j E_j(C, D)$ is a partition of $C \times D$; that is, $\cup_{C \in \mathfrak{C}} \cup_{D \in \mathfrak{D}} \cup_j E_j(C, D)$ is a symmetric chain partition.

■

Example 5.1.1. *Clearly, Theorem 5.1.5 can be extended to the case of a product of n sc-orders. Hence the Boolean lattice B_n and, more generally, chain products $S(k_1, \ldots, k_n)$ are sc-orders.*

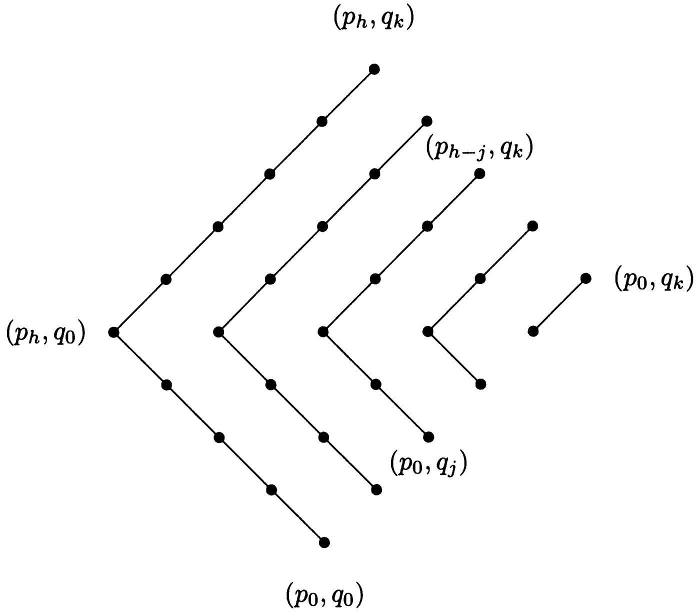


Figure 5.1

Using Theorem 5.1.5 we obtain a recursive procedure to determine a symmetric chain partition of $S(k_1, \dots, k_n)$. But there are also direct methods to do this. We present here the parenthesization method of Leeb (unpublished), Greene and Kleitman [232] and then calculate explicitly the upper and lower neighbors in such symmetric chains. This is closely related to the so-called lexicographic method of Aigner [19] (see also [21, pp. 432ff]).

With every $\boldsymbol{a} = (a_1, \dots, a_n) \in S(k_1, \dots, k_n)$ (here we do not need the general supposition $k_1 \geq \cdots \geq k_n$), we associate a sequence of $\sum_{i=1}^{n} k_i$ left and right parentheses as follows: first a_1 right parentheses, then $k_1 - a_1$ left, then a_2 right, $k_2 - a_2$ left, and so forth. For example, for $\boldsymbol{a} = (1, 2, 0, 1, 3) \in S(2, 3, 1, 2, 5)$ we obtain the following sequence:

$$) \; (\;) \;) \; (\; (\;) \; (\;) \;) \;) \; (\; ($$

Note that the number of right parentheses equals the rank of $\boldsymbol{a}$. Every sequence of left and right parentheses has a unique *parenthesization* obtained in the following way: Close all pairs of left and right parentheses that are either adjacent to or separated by other such pairs, repeating the process until no further pairing is possible (one may use stacks to produce this parenthesization algorithmically). The remaining unpaired parentheses form a sequence of right followed by left parentheses; in our example we have

$$) \;) \;) \; (\; ($$

Now we define the partition of $S(k_1, \ldots, k_n)$ by saying that two elements are in the same block iff they have the same parenthesization. Considering the remaining sequences of parentheses it is not difficult to see that each block is a symmetric chain (if we have $2x$ paired and y unpaired parentheses, then the first element of the chain has rank x, the last has rank $x + y$, and $x + x + y = 2x + y = \sum_{i=1}^{n} k_i$), and thus we have obtained a partition into symmetric chains. In the following table we see which elements lie in the same symmetric chain as $\boldsymbol{a}$ (in our example).

<table>
<tr><th>remaining sequence</th><th>sequence</th><th>element</th><th>rank</th></tr>
<tr><td>(((((</td><td>((|)((|(|)(|))(((</td><td>(0,1,0,1,2)</td><td>4</td></tr>
<tr><td>)((((</td><td>)(|)((|(|)(|))(((</td><td>(1,1,0,1,2)</td><td>5</td></tr>
<tr><td>))(((</td><td>)(|))(|(|)(|))(((</td><td>(1,2,0,1,2)</td><td>6</td></tr>
<tr><td>)))((</td><td>)(|))(|(|)(|)))((</td><td>(1,2,0,1,3)</td><td>7</td></tr>
<tr><td>))))(</td><td>)(|))(|(|)(|))))(</td><td>(1,2,0,1,4)</td><td>8</td></tr>
<tr><td>)))))</td><td>)(|))(|(|)(|)))))</td><td>(1,2,0,1,5)</td><td>9</td></tr>
</table>

How do we find for an element $\boldsymbol{a}$ the element $\boldsymbol{b}$ (if it exists) that covers $\boldsymbol{a}$ and lies in the same symmetric chain as $\boldsymbol{a}$? We already mentioned that one can use stacks, but it is also possible to calculate explicitly that component i for which $a_i = b_i - 1$ (for the other components we must have equality because of $\boldsymbol{a} < \boldsymbol{b}$). At least one of the $k_i - a_i$ left parentheses belonging to the ith component must remain unpaired if we build the parenthesization sequence from the left to the right. So we must have $k_i - a_i > a_{i+1}$. At the $(i + 1)$th component we close a_{i+1} right parentheses with the preceding left parentheses and create $k_{i+1} - a_{i+1}$ new left parentheses. If we recall that at least one left parenthesis from the ith component must remain unpaired, we see that we must have $k_i - a_i + k_{i+1} - a_{i+1} > a_{i+1} + a_{i+2}$. At the $(i + 2)$th component we close a_{i+2} right parentheses and create $k_{i+2} - a_{i+2}$ new left parentheses. If we continue in the same way, we see that we must have

$$\sum_{j=i}^{t-1}(k_j - a_j) > \sum_{j=i+1}^{t} a_j \text{ for every } i < t \leq n. \tag{5.1}$$

Since we come from $\boldsymbol{a}$ to $\boldsymbol{b}$ by changing the leftmost unpaired left parenthesis into a right parenthesis, the desired index i is the smallest integer in $[n]$ satisfying (5.1) (if there is not such an integer, we put $i := n + 1$). Let for $\boldsymbol{a} \in S(k_1, \ldots, k_n)$,

$$f_i(\boldsymbol{a}) := \sum_{j=1}^{i-1}(k_j - a_j) - \sum_{j=1}^{i} a_j, \; i = 1, \ldots, n+1,$$

where $a_{n+1} := 0$. It is easy to see that the searched after index i can be found by determining the *largest integer i for which $f_i(\boldsymbol{a})$ attains its minimum*. In a similar way we find for our element $\boldsymbol{a}$ the element $\boldsymbol{c}$ (if it exists) that is covered by $\boldsymbol{a}$ and lies in the same symmetric chain as $\boldsymbol{a}$ (we have to look into the sequences from the right to the left and must note that $\sum_{j=1}^{n}(k_j - a_j) - \sum_{j=1}^{n} a_j$ is constant). The index i for which $a_i = c_i + 1$ is the *smallest integer in $[n]$ for which $f_i(\boldsymbol{a})$ attains its minimum* (here $i = 0$ means that there is no such $\boldsymbol{c}$).

Note that the chains that are constructed by the parenthesization (resp. explicit) method are the same as those chains that are obtained if one applies Theorem 5.1.5. This follows easily by induction on n. From a more theoretical point of view, Gansner [214] studied the parenthesization and explicit methods in distributive lattices as well. Of course, we may specialize the preceding results to the Boolean lattice. Let us represent it as subsets of $[n]$ ($\boldsymbol{a} \in S(1, \ldots, 1)$ is the characteristic vector of the corresponding subset). We call the symmetric chain partition from above the *parenthesization partition of B_n*. Moreover, if S and $S' := S \cup \{i\}$ belong to the same symmetric chain, we speak of a *link* $S \to S'$ in a chain of the parenthesization partition. Later we will need the following proposition.

Proposition 5.1.1. *Let $\mathfrak{C}$ be the parenthesization partition of B_n.*

(*a*) *In each chain of $\mathfrak{C}$, elements are added in increasing order.*

(*b*) *n is an element of the last set in each chain.*

(*c*) *If $S \to S \cup \{i\}$ is a link in a chain of $\mathfrak{C}$, then $i + 1 \notin S$ and either $i = 1$ or $i - 1 \in S$.*

Proof. (a) We turn the unpaired parentheses from the left to the right.

(b) Either the last parenthesis is paired, and then it must be a right parenthesis, and thus n belongs to every set of the chain, or the last parenthesis is unpaired, and then we change it in the last step into a right parenthesis, and thus n belongs to the last set of the chain.

(c) Since $i \notin S$, the ith parenthesis is a left one. If we had $i + 1 \in S$ then the $(i + 1)$th parenthesis would be a right one; that is, the ith and the $(i + 1)$th parentheses were paired and thus the ith parenthesis could not be changed into a right one. If we had $i > 1$ and $i - 1 \notin S$, then the $(i - 1)$th parenthesis would be a left one. It cannot be paired since otherwise the ith parenthesis had also to be paired. Hence the next link would be $S \to S \cup \{i - 1\}$ instead of $S \to S \cup \{i\}$. ∎

Metropolis and Rota [365], [366], and Metropolis, Rota, Strehl, and White [367] worked out an analogous parenthesization algorithm to partition Q_n and F_k^n into chains. Vogt and Voigt [457] presented a similar explicit, but more difficult partition of $L_n(q)$. Not only with the help of the product theorem may we construct sc-orders. We also have a nice link to the normal posets.

***Theorem 5.1.6* (Griggs [235]).** *Every rank-symmetric and rank-unimodal normal poset P is an sc-order.*

Proof. We proceed by induction on $n := r(P)$. The case $n = 0$ is trivial. For $n = 1$ or $n = 2$, let $\mathfrak{D}$ be a 1- (resp. 2-) saturated partition of P into chains that exists by Corollary 4.3.2. We will show that $\mathfrak{D}$ is a symmetric chain partition. Let $\mathfrak{D}_i := \{D \in \mathfrak{D} : |D| = i\}, i = 1, 2, \ldots$. We recall that P has the strong Sperner property by Corollary 4.5.3. In the case $n = 1$ we have

$$W_0 = W_1 = d_1(P) = \sum_{D \in \mathfrak{D}} \min\{|D|, 1\} = |\mathfrak{D}_1| + |\mathfrak{D}_2|,$$

and since $\mathfrak{D}$ is a partition of P, the following holds:

$$W_0 + W_1 = |\mathfrak{D}_1| + 2|\mathfrak{D}_2|,$$

from where we derive $|\mathfrak{D}_1| = 0$; that is, all chains of $\mathfrak{D}$ are symmetric.

In the case $n = 2$ we have

$$W_0 + W_1 = W_1 + W_2 = d_2(P) = \sum_{D \in \mathfrak{D}} \min\{|D|, 2\} = |\mathfrak{D}_1| + 2|\mathfrak{D}_2| + 2|\mathfrak{D}_3|$$

and

$$W_0 + W_1 + W_2 = |\mathfrak{D}_1| + 2|\mathfrak{D}_2| + 3|\mathfrak{D}_3|.$$

We conclude that $|\mathfrak{D}_3| = W_0 = W_2$; that is, all elements of $N_0 \cup N_2$ are covered by 3-element chains. Consequently, there cannot be 2-element chains, and the 1-element chains consist of elements of rank 1. Thus all chains of $\mathfrak{D}$ are symmetric.

Now we consider the step $n-2 \to n$, where $n \geq 3$. By the induction hypothesis for the $\{1, \ldots, n-1\}$-rank-selected subposet (which is of course rank symmetric, rank unimodal, and normal) there exists a partition $\mathfrak{C}^*$ into symmetric chains. Again put $\mathfrak{C}^*_{n-1} := \{C \in \mathfrak{C}^* : |C| = n-1\}$. Let P' be the poset of rank 2 for which $N_0(P') := N_0(P)$, $N_1(P') := \mathfrak{C}^*_{n-1}$, $N_2(P') := N_n(P)$ holds and which is ordered in the following way: Let $p_0 \in N_0(P')$, $C = (p_1 \lessdot \cdots \lessdot p_{n-1}) \in N_1(P')$, $p_n \in N_2(P')$. Then

$$p_0 \lessdot_{P'} C \text{ iff } p_0 \lessdot_P p_1, \quad \text{and} \quad C \lessdot_{P'} p_n \text{ iff } p_{n-1} \lessdot_P p_n.$$

It is easy to see that P' is rank symmetric, rank unimodal, and normal. Consequently, by the already proved case $n = 2$, there is a symmetric chain partition $\mathfrak{D}$ of P'. If we consider the 1- and 3-element chains of $\mathfrak{C}^*$ in the obvious way as $(n-1)$- (resp. $(n+1)$-) element chains of P and add all chains of $\mathfrak{C}^* - \mathfrak{C}^*_{n-1}$ we obtain the searched after symmetric chain partition of P. ∎

Example 5.1.2. *The following posets are sc-orders: $L_n(q)$, projective space lattices, and modular geometric lattices (use Theorem 5.1.5).*

Sc-orders also behave nicely with respect to the rankwise direct product:

Theorem 5.1.7 **(*Sali* [409]).** *If P and Q are sc-orders (of the same rank) then $P \times_r Q$ is an sc-order, too.*

Proof. Let $n := r(P) = r(Q)$ and consider chains $C = (p_i \lessdot \cdots \lessdot p_{n-i})$, $r(p_i) = i$, and $D = (q_j \lessdot \cdots \lessdot q_{n-j}), r(p_j) = j$. We define $C \times_r D$ to be the chain $((p_k, q_k) \lessdot \cdots \lessdot (p_{n-k}, q_{n-k}))$ where $k := \max\{i, j\}$. Note that $C \times_r D$ is really a symmetric chain in $P \times_r Q$. Now it is easy to see that for given symmetric chain partitions $\mathfrak{C}$ and $\mathfrak{D}$ of P and Q, respectively, the set $\cup_{C\in\mathfrak{C}} \cup_{D\in\mathfrak{D}} C \times_r D$ is a symmetric chain partition of $P \times_r Q$. ■

It is still open whether the poset $L(m, n)$ and related posets are sc-orders as asked by Stanley [439]. Up to now the conjecture is confirmed for $\min\{m, n\} \leq 4$ by Riess [399], Lindström [346], and West [463].

5.2. More part Sperner theorems and the Littlewood–Offord problem

Symmetric chain partitions allow us to prove stronger forms of Sperner's theorem. These observations go back to Katona [291] and Kleitman [299]. We define a subset F of a product $P \times Q$ of two posets P and Q to be a *semiantichain* if there are no $(p_1, q_1), (p_2, q_2) \in F$ such that $p_1 = p_2$ and $q_1 < q_2$ or $p_1 < p_2$ and $q_1 = q_2$. We say that a product $P \times Q$ of two ranked posets has the *two-part Sperner property* if the largest size of a semiantichain equals the largest Whitney number of $P \times Q$.

Theorem 5.2.1 **(*Katona* [294]).** *If P and Q are sc-orders, then $P \times Q$ has the two-part Sperner property.*

Proof. Let $h = \left\lfloor \frac{r(P)+r(Q)}{2} \right\rfloor$. Clearly, $N_h(P \times Q)$ is a semiantichain since it is an antichain. Moreover, $W_h(P \times Q)$ is the largest Whitney number of $P \times Q$ (by Lemma 5.1.1 and Theorem 5.1.5). Let F be a semiantichain in $P \times Q$. We must prove $|F| \leq W_h(P \times Q)$. Let $\mathfrak{C}$ and $\mathfrak{D}$ be symmetric chain partitions of P and Q, respectively. In the proof of Theorem 5.1.5 we formed for every pair (C, D) of chains of the corresponding partitions $\min\{|C|, |D|\}$ chains $E_j(C, D)$, and taking these chains together we obtained a symmetric chain partition of $P \times Q$ that consists of exactly $W_h(P \times Q)$ chains since each chain meets $N_h(P \times Q)$. Consequently,

$$\sum_{C\in\mathfrak{C}} \sum_{D\in\mathfrak{D}} \min\{|C|, |D|\} = W_h(P \times Q).$$

Since F is a semiantichain, we have for every $C \in \mathfrak{C}$, $D \in \mathfrak{D}$

$$|F \cap (C \times D)| \leq \min\{|C|, |D|\}$$

(use Dirichlet's principle). Accordingly,

$$|F| = \sum_{C \in \mathfrak{C}} \sum_{D \in \mathfrak{D}} |F \cap (C \times D)| \leq \sum_{C \in \mathfrak{C}} \sum_{D \in \mathfrak{D}} \min\{|C|, |D|\} = W_h(P \times Q).$$

■

More generally, one may consider products $P := P_1 \times \cdots \times P_M$ of M ranked posets $P_1, \ldots, P_M$. A subset F of P is called an *M-part k-family* if there are no $k+1$ elements of F lying on a chain in P that are identical in $M-1$ components (thus a semiantichain is a two-part 1-family). In contrast to the k-family problem, Theorem 5.2.1 cannot be generalized to two-part k-families. All elements of a product P of two 3-element chains form a two-part 3-family, but its size is larger than the largest sum of three Whitney numbers of P (if one changes the definition of an M-part k-family such situations can be avoided, see Katona [294]). Also three-part 1-families do not behave as nicely. Look at a product P of three 2-element chains ($0 < 1$). The elements $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(0, 0, 0)$ form a three-part 1-family, but its size is greater than the largest Whitney number of P. Katona [296] and Griggs and Kleitman [243] added further conditions on a three-part 1-family in order to bound their size by the largest Whitney number of P.

Theorem 5.2.2 **(*Sali* [407]).** *Let $P_1, \ldots, P_M$ be sc-orders and let F be an M-part k-family in $P := P_1 \times \cdots \times P_M$. Then*

$$|F| \leq Mkd(P).$$

Proof. First we consider the special case that P_i is a chain of length $k_i, i = 1, \ldots, M$; that is, $P = S(k_1, \ldots, k_M)$. Take any chain of the symmetric chain partition that is constructed by the parenthesization method. If we are going from the bottom to the top we see (as in Proposition 5.1.1(a)) that the first component is changed (or not changed at all), then the second component is changed (or not changed at all), and so on. Hence the intersection of F and this chain cannot contain more than Mk elements. We have exactly $d(P)$ chains in the symmetric chain partition; hence,

$$|F| \leq Mkd(P) = MkW_{\lfloor \frac{n}{2} \rfloor},$$

where as always $n := r(P)$.

In the general case we work with symmetric chain partitions $\mathfrak{C}_i$ of $P_i, i = 1, \ldots, M$. If $C_i \in \mathfrak{C}_i, i = 1, \ldots, M$, we have by the preceding special case

$$|F \cap (C_1 \times \cdots \times C_M)| \leq Mk \left| \left\{ p \in C_1 \times \cdots \times C_M : r(p) = \left\lfloor \frac{n}{2} \right\rfloor \right\} \right|.$$

Hence by the pairwise disjointness of the chain products

$$\begin{aligned} |F| &= \sum_{C_1\in\mathfrak{C}_1,\dots,C_M\in\mathfrak{C}_M} |F\cap(C_1\times\cdots\times C_M)| \\ &\le Mk \sum_{C_1\in\mathfrak{C}_1,\dots,C_M\in\mathfrak{C}_M} \left|\left\{p\in C_1\times\cdots\times C_M : r(p)=\left\lfloor\frac{n}{2}\right\rfloor\right\}\right| \\ &= MkW_{\lfloor\frac{n}{2}\rfloor} = Mkd(P). \end{aligned}$$

■

In [408] Sali observed that one can replace the factor M by $c\sqrt{M}$, where the constant c is independent of P, M, and k. Moreover, infinitely many P's show that this result is best possible for every M and apart from the constant factor c. We postpone an analogous asymptotic result to Section 7.2 where we consider products of arbitrary posets of bounded size. P.L. Erdős, and Katona [177] and Sali [410] studied convex hulls of M-part k-families.

There is another class of posets which allows the determination of the maximum size of semiantichains. A ranked poset P is called a *skew chain order* if it can be partitioned into W_0 saturated chains (which then must contain an element of rank 0).

***Theorem 5.2.3* (*West and Kleitman* [468]).** *Let P and Q be skew chain orders. Then the* (*ground set of the*) *rankwise direct product $P \times_r Q$ is a semiantichain of maximum size.*

Because the proof is analogous to the proof of Theorem 5.1.5, it is left to the reader as an exercise. The result can be applied to $Int(S(k_1, k_2))$. Thus, in the poset of subrectangles of a rectangle (we speak of rectangles instead of parallelepipedons), the squares form a maximum semiantichain. Again one cannot generalize the result to higher dimensions, for example, to $Int(S(k_1, k_2, k_3))$. For example, in Q_3 ($\cong Int(S(1,1,1))$) N_1 is an antichain – that is, a three-part 1-family of size 12 – but $Q_1 \times_r Q_1 \times_r Q_1$ is of size 9. Peck [379] modified the conditions slightly to obtain a bound of this kind.

If one considers semiantichains as optimal integral solutions of a corresponding linear programming problem (packing problem) one has to ask also for the dual (covering) problem. The dual objects here are chains in which one of the coordinates remains fixed, called *unichains*. The following interesting problem of West and Saks (see [465]) is still open: Is it true that the largest size of a semiantichain in $P \times Q$ equals the minimum number of unichains needed to cover $P \times Q$? For several related results, we refer to West [465].

One of the earliest examples where Sperner-type theorems were applied is the *Littlewood–Offord problem* from 1943: Given complex numbers $z_1, \dots, z_n$ such that $|z_j| \ge 1$ for each j, how many sums of the form $\sum_{j=1}^{n} \epsilon_j z_j$, where $\epsilon_j \in \{-1, +1\}$ for each j, can lie inside any circle of radius r?

Littlewood and Offord [349] themselves found an upper bound of the form $Cr^{2^n} \log n/\sqrt{n}$, and in 1945 Erdős [165] showed – using Sperner's theorem – that the log term can be omitted. After that many further results were obtained. In the following we present some of the nicest. Instead of complex numbers we consider vectors $z_1, \ldots, z_n$ in the d-dimensional space $\mathbb{R}^d$; that is, $z_j = (z_{1,j}, \ldots, z_{d,j})$, $j = 1, \ldots, n$. Let $LO(n, d, r)$ be the smallest natural number such that for any choice of vectors $z_1, \ldots, z_n \in \mathbb{R}^d$ with $\|z_j\| \geq 1$ for each j and any choice of an open sphere of radius r there are no more than $LO(n, d, r)$ sums of the form $\sum_{j=1}^n \epsilon_j z_j$, where $\epsilon_j \in \{-1, +1\}$, in that sphere. We realize the Boolean lattice B_n as the chain product $(-1 \lessdot 1)^n$ and for $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in B_n$ we put briefly $s(\epsilon) := \sum_{j=1}^n \epsilon_j z_j$, where it is assumed that $z_1, \ldots, z_n$ are clear from the context.

First we study the case $r = 1$. Taking $z_j := (1, 0, \ldots, 0)$ for each j we see that

$$LO(n, d, 1) \geq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{5.2}$$

(note that $s(\epsilon)$ remains constant if ϵ runs through a fixed level of B_n).

If $d = 1$ the vectors z_j are real numbers (we write z_j), and we may assume that $z_j \geq 1$ for each j (if $z_j \leq -1$ for some j one has to take in the sums $-\epsilon_j$ instead of ϵ_j). For $\epsilon \lessdot \epsilon'$, we have $s(\epsilon') - s(\epsilon) \geq 2$ since $s(\epsilon') - s(\epsilon) = 2z_j$ for some j. Thus for each open (or half closed) interval I of length 2, the set $\{\epsilon \in B_n : s(\epsilon) \in I\}$ is an antichain. Sperner's theorem immediately yields (together with (5.2)):

Proposition 5.2.1. *We have* $LO(n, 1, 1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Let us mention that our function s divided by 2 is a representation of the Boolean lattice. In Section 7.2 we will define antichains in the same way using representations.

In the case $d = 2$ the two-part Sperner Theorem yields the same bound.

***Theorem 5.2.4 (Katona* [291], *Kleitman* [299]).** $LO(n, 2, 1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. As before, we may assume that $z_{1,j} \geq 0$ for each j. Moreover, we may assume $z_{2,1}, \ldots, z_{2,p} \geq 0$ and $z_{2,p+1}, \ldots, z_{2,n} < 0$. We consider the n-tuples ϵ as elements of $B_p \times B_{n-p}$. For different n-tuples ϵ, ϵ' with $\epsilon_j < \epsilon'_j$, $j \in J \subseteq \{p+1, \ldots, n\}$ and $\epsilon_j = \epsilon'_j$ otherwise, we have

$$s(\epsilon') - s(\epsilon) = 2\sum_{j \in J} (z_{1,j}, z_{2,j}).$$

Let $j^* \in J$. Clearly, $\sum_{j \in J} z_{1,j} \geq z_{1,j^*} \geq 0$ and $\sum_{j \in J} z_{2,j} \leq z_{2,j^*} < 0$. Consequently,

$$\|s(\epsilon') - s(\epsilon)\|^2 \geq 2^2 \|(z_{1,j^*}, z_{2,j^*})\|^2 \geq 4;$$

that is, s(ϵ) and s(ϵ') cannot both lie in an open sphere S of radius 1 (diameter 2). The same holds if $\epsilon_j < \epsilon'_j$ for $j \in J \subseteq \{1, \dots, p\}$ and $\epsilon_j = \epsilon'_j$ otherwise. Thus the set $\{\epsilon \in B_p \times B_{n-p} : s(\epsilon) \in S\}$ is a semiantichain that has by Theorem 5.2.1 (recall that B_n is an sc-order for any n) cardinality not larger than the largest Whitney number of $B_p \times B_{n-p}$, that is, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. ■

In the general case ($d \geq 1$) we define for a fixed choice of $z_1, \dots, z_n$ the graph $G = (V, E)$, where

$$V := \{\epsilon = (\epsilon_1, \dots, \epsilon_n) : \epsilon_j \in \{-1, 1\} \text{ for all } j\},$$

$$E := \{\epsilon\epsilon' : \|s(\epsilon') - s(\epsilon)\|^2 \geq 4\}.$$

For $d = 1$ and $d = 2$, any independent set of G (which is an antichain (resp. semiantichain) in B_n) has cardinality not greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. This is also true for $d > 2$:

Theorem 5.2.5 **(*Kleitman* [303]).** *We have* $LO(n, d, 1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. By (5.2) all we have to do is to show that our graph $G = G(n; z_1, \dots, z_n)$ can be covered by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ cliques. We construct our cliques in a similar way as the chains in a symmetric chain partition of B_n (see third proof of Theorem 1.1.1 in Section 2.2 and Theorem 5.1.5). Let us only look at the step $n \to n + 1$ and let $C = \{\epsilon_0, \dots, \epsilon_h\}$ be a clique that was produced for the parameter n. We assume, w.l.o.g., that $\langle s(\epsilon_h), z_{n+1} \rangle \geq \langle s(\epsilon_j), z_{n+1} \rangle$, $j = 0, \dots, h$ (here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$). We define the new sets

$$\bar{C} := \{(\epsilon_0, -1), \dots, (\epsilon_{h-1}, -1), (\epsilon_h, -1), (\epsilon_h, +1)\},$$

$$\bar{C}' := \{(\epsilon_0, +1), \dots, (\epsilon_{h-1}, +1)\}.$$

We claim that $\bar{C}$ and $\bar{C}'$ are cliques in $\bar{G} := G(n + 1; z_1, \dots, z_n, z_{n+1})$. This is clear for $\bar{C}'$ because for $0 \leq i < j \leq h - 1$ there holds

$$\begin{aligned}\|s(\epsilon_j, +1) - s(\epsilon_i, +1)\|^2 &= \|s(\epsilon_j) + z_{n+1} - (s(\epsilon_i) + z_{n+1})\|^2 \\ &= \|s(\epsilon_j) - s(\epsilon_i)\|^2 \geq 2^2.\end{aligned}$$

Up to the last element we can apply the same arguments to $\bar{C}$. Finally, for $0 \leq i \leq h$, we have

$$\begin{aligned}\|s(\epsilon_h, +1) - s(\epsilon_i, -1)\|^2 &= \|s(\epsilon_h) - s(\epsilon_i) + 2z_{n+1}\|^2 \\ &= \|s(\epsilon_h) - s(\epsilon_i)\|^2 + 4\|z_{n+1}\|^2 \\ &\quad + 4\langle s(\epsilon_h) - s(\epsilon_i), z_{n+1} \rangle \geq 0 + 4 + 0 = 4.\end{aligned}$$

Thus, as in the Boolean case, we obtained for a clique (chain) of size $h+1$ a new clique (chain) of size h (if $h \geq 1$) and a new clique (chain) of size $h+2$. It follows that the total number of cliques equals the total number of chains in a symmetric chain partition, and for B_n this number is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. ■

Now we study the case $r > 1$. As for $r = 1$ we obtain the lower bound

$$LO(n, d, r) \geq \text{ sum of the } \lceil r \rceil \text{ middle binomial coefficients.} \tag{5.3}$$

If $d = 1$ this bound is best possible:

Proposition 5.2.2 (Erdős [165]). *We have* $LO(n, 1, r) = \sum_{i=1}^{\lceil r \rceil} \binom{n}{\lfloor \frac{n - \lceil r \rceil}{2} \rfloor + i}$.

Proof. It is easy to see that for each open (or half closed) interval I of length $2r$, the set $\{\epsilon \in B_n : s(\epsilon) \in I\}$ is an $\lceil r \rceil$-family. Thus the result follows from the k-family property of B_n for every k (see Corollary 4.5.3 and Example 4.5.1). ■

For $d > 1$, we prove only an upper bound, which can be derived from the M-part Sperner Theorem and ideas of Griggs [237].

Theorem 5.2.6 (Sali [407]). *We have* $LO(n, d, r) \leq 2^{d-1} \lceil r\sqrt{d} \rceil \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Proof. As in the proof of Theorem 5.2.4 we assume that $z_{1,i} \geq 0$ for each i. We can partition the index set $[n]$ into $M := 2^{d-1}$ subsets (empty sets are allowed) such that for each such subset J the components $z_{2,j}$, $j \in J$, all have the same sign,. . ., the components $z_{d,j}$, $j \in J$, all have the same sign. Let S be any open sphere of radius r.

Claim. $F := \{\epsilon : s(\epsilon) \in S\}$ is an M-part $\lceil r\sqrt{d} \rceil$-family.

Proof of Claim. Let us look at one of the M subsets J. We suppose, w.l.o.g., that $z_{i,j} \geq 0$ for $i = 1, \ldots, d$, $j \in J$. Let $k := \lceil r\sqrt{d} \rceil$ and assume that there is a chain $\epsilon_0 < \cdots < \epsilon_k$ in F such that the components of the members are identical outside J. Then

$$s(\epsilon_k) - s(\epsilon_0) = 2 \sum_{j \in J'} z_j \text{ for some } J' \subseteq J, |J'| \geq k.$$

Geometrically we look at the projections onto the vector $\mathbf{1} = (1, \ldots, 1)$. We have

$$\langle z_j, \mathbf{1} \rangle = \sum_{i=1}^{d} z_{i,j} \geq \sqrt{z_{1,j}^2 + \cdots + z_{d,j}^2} \geq 1, j \in J'.$$

It follows that

$$\langle s(\epsilon_k) - s(\epsilon_0), \mathbf{1} \rangle = 2 \sum_{j \in J'} \langle z_j, \mathbf{1} \rangle \geq 2|J'|,$$

and finally by Cauchy's inequality

$$\|s(\epsilon_k) - s(\epsilon_0)\|^2 \geq \frac{1}{\|\mathbf{1}\|^2}\langle s(\epsilon_k) - s(\epsilon_0), \mathbf{1}\rangle^2 \geq \frac{4|J'|^2}{d} \geq \frac{4r^2 d}{d},$$

that is,

$$\|s(\epsilon_k) - s(\epsilon_0)\| \geq 2r,$$

a contradiction. □

The statement in the theorem now follows immediately from the claim and Theorem 5.2.2. ■

An almost final answer to the Littlewood–Offord problem was given by Frankl and Füredi [195].

Theorem 5.2.7. *Suppose that r is not an integer. Then*

(*a*) $LO(n,d,r) = (\lceil r \rceil + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ *as* $n \to \infty$.

(*b*) *If there is an integer s such that* $s - 1 < r < s - 1 + \frac{1}{10s^2}$, *then* $LO(n,d,r) = LO(n,1,r)$ *if n is sufficiently large.*

We omit the proof but mention that deeper geometrical insight is necessary to establish this result. Frankl and Füredi [195] posed the problem to determine $\lim_{n\to\infty} LO(n,d,r)/\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for integers r.

If one considers closed spheres instead of open spheres one has the same formula as in Theorem 5.2.7. One can use these formulas also for integers r if one replaces $\lceil r \rceil$ by $\lfloor r \rfloor + 1$.

Note also a "dual" variant of the Littlewood–Offord problem which goes back to Erdős, Sárközy, and Szemerédi [168]: Change the condition $\|z_j\| \geq 1$ into $\|z_j\| \leq 1$ for all j and ask for the number of sums $\sum_{j=1}^n \epsilon_j z_j$ (where $\epsilon_j \in \{-1,+1\}$ for all j) which lie in a sphere of radius r. The following result is presented without proof:

***Theorem 5.2.8 (Beck* [41]).** *Let $z_1, \ldots, z_n$ be vectors in $\mathbb{R}^d$ with $\|z_j\| \leq 1$ for all j. Then at least $c(d)2^n n^{-d/2}$ sums of the form $\sum_{j=1}^n \epsilon_j z_j$, where $\epsilon_j \in \{-1,+1\}$, lie in a closed sphere of radius $\sqrt{d}$ around the origin. The constant $c(d)$ depends only on the dimension.*

Sárközy and Szemerédi (unpublished) showed that for $d = 1$ this bound can be replaced by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ which has asymptotically the same order.

5.3. Coverings by intervals and sc-orders

Let us recall that Dilworth's Theorem 4.0.1 tells us how many chains we need to cover a given poset P. What is the situation if we replace chains by intervals? In [81] we showed together with Bouchemakh that the corresponding problems are in general NP-complete (cf. [215]). But if we restrict ourselves to special symmetric chain orders, we find some Sperner-type theorems.

Let P be a ranked poset and let $0 \leq l \leq u \leq n := r(P)$. Consider the $[l, u]$-rank-selected subposet $P_{[l,u]}$ of P (i.e. $P_{[l,u]} = \cup_{i=l}^{u} N_i$). Let $\varrho(P_{[l,u]})$ be the minimum number of intervals of the form $[p, q] := \{v \in P : p \leq v \leq q\}$, where $l \leq r(p) \leq r(q) \leq u$, which cover $P_{[l,u]}$. Since no two elements of N_l (resp. N_u) can be covered by one such interval, we have clearly

$$\varrho(P_{[l,u]}) \geq \max\{W_l, W_u\}. \tag{5.4}$$

We say that the ranked poset P has the *strong interval covering property* (briefly *strong IC-property*) if equality holds in (5.4) for all $0 \leq l \leq u \leq n$. To find examples, we introduce the following class: A ranked poset P is called a *special symmetric chain order* (briefly *ssc-order*) if there exists a symmetric chain partition $\mathfrak{C}$ of P with the additional property (*):

If $C = (p_1 \lessdot \cdots \lessdot p_h)$ is a chain of the partition that has less than maximum length then there exists a chain D of the partition such that $D = (q_0 \lessdot \cdots \lessdot q_{h+1})$ and $q_0 \lessdot p_1$ as well as $p_h \lessdot q_{h+1}$.

Theorem 5.3.1. *If P is an ssc-order, then P has the strong IC-property.*

Proof. First we consider the special case $l + u = n$ (i.e., $W_l = \max\{W_l, W_u\} = W_u$). Let $\mathfrak{C}$ be a symmetric chain partition of P satisfying property (*). With each $p \in N_i$ we associate the element $f(p)$ of N_{n-i} that lies on the same chain of $\mathfrak{C}$, $0 \leq i \leq \frac{n}{2}$. Then the W_l intervals $[p, f(p)]$, $p \in N_l$, cover $P_{[l,u]}$. This can be proved by induction on $n-2l$: The cases $n-2l = 0$ or 1 are easy. For the induction step, consider any element v of $P_{[l,n-l]}$. If $v \in N_l$ or N_{n-l} then v is clearly covered. Thus let $p \in P_{[l+1,n-(l+1)]}$. By the induction hypothesis v is covered by an interval $[q, f(q)]$, where $q \in N_{l+1}$. Let C be the chain of $\mathfrak{C}$ containing q and $f(q)$. If q has on C a predecessor p then v is covered by $[p, f(p)]$. If q is the starting point – that is, $f(q)$ is the endpoint of C – then there is, because of property (*), some chain D of $\mathfrak{C}$ of the form $D = (p \lessdot \cdots \lessdot f(p))$, where $p \in N_l$ and $p \lessdot q < f(q) \lessdot f(p)$. Hence v is covered by $[p, f(p)]$.

For the general case, let us assume, w.l.o.g., $l + u < n$, that is, $W_u = \max\{W_l, W_u\}$. Restricting the chains of $\mathfrak{C}$ to $P_{[l,n-u]}$ we find W_u chains of the form $(q \lessdot \cdots \lessdot p)$, where $p \in N_{n-u}$, which cover $P_{[l,n-u]}$. Now we join the chains $(q \lessdot \cdots \lessdot p)$ and $(p \lessdot \cdots \lessdot f(p))$ and replace the resulting chain by the interval $[q, f(p)]$. By the first part of the proof these W_u intervals cover $P_{[l,u]}$. ∎

Theorem 5.3.2. *Let P and Q be ssc-orders.*

(*a*) *Then $P \times Q$ is an ssc-order, too.*

(*b*) *If, in addition, $r(P) = r(Q)$, then $P \times_r Q$ is an ssc-order, too.*

Proof. (a) We take symmetric chain partitions $\mathfrak{C}$ and $\mathfrak{D}$ of P and Q, respectively, which have property (*). Then we construct the symmetric chain partition $\mathfrak{E}$ of $P \times Q$ as in the proof of Theorem 5.1.5. All we need is the verification of (*) for $\mathfrak{E}$. Property (*) is evident for the nonmaximal chains in the partition of each "rectangle" $C \times D$. Let us consider the maximal chain $E_0(C, D) = ((p_0, q_0) \lessdot \cdots \lessdot (p_h, q_k))$ of such a rectangle. If it does not have maximum length in $\mathfrak{E}$, then C or D does not have maximum length in $\mathfrak{C}$ or $\mathfrak{D}$, respectively. By symmetry, suppose this for C. Since $\mathfrak{C}$ has property (*), there must be a chain $C' = (p'_0 \lessdot \cdots \lessdot p'_{h+1})$ in $\mathfrak{C}$ such that $p'_0 \lessdot p_0$ and $p_h \lessdot p'_{h+1}$. But $E_0(C', D) = ((p'_0, q_0) \lessdot \cdots \lessdot (p'_{h+1}, q_k))$ belongs to $\mathfrak{E}$ and is thus the desired chain for $E_0(C, D)$.

(b) Here we use the construction in the proof of Theorem 5.1.7. The verification of (*) is an easy exercise. ■

Since single chains are clearly ssc-orders we derive immediately:

Example 5.3.1. *The following posets are ssc-orders, and thus have the strong IC-property:*

(*a*) *the Boolean lattice B_n and chain products $S(k_1, \ldots, k_n)$,*

(*b*) *the poset $SM_{k,n}$ of square submatrices of a square matrix.*

We found the last two theorems together with Bouchemakh [81]. The IC-property of B_n was first proved by Voigt and Wegener [458] using the parenthesization method. They used this result for the construction of minimal polynomials of symmetric Boolean functions. Let us describe this application in more detail:

Every Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ can be represented as a family $\mathcal{F}_f$ in B_n where B_n is represented as $\{0, 1\}^n$ and $\mathcal{F}_f := \{\boldsymbol{x} : f(\boldsymbol{x}) = 1\}$. A *monomial* is a conjunction of negated and nonnegated variables – for instance, $x_1\overline{x}_3x_4\overline{x}_6$. With each monomial we can associate in an obvious way a Boolean function f. The corresponding family $\mathcal{F}_f$ is obviously an interval in B_n. In Figure 5.2 we illustrate $\mathcal{F}_f$ for our example in B_6. A *polynomial* or *disjunctive normal form* is a disjunction of monomials. The corresponding family in B_n is a union of intervals. It is well known (and in this interpretation obvious) that each Boolean function can be represented as a polynomial. Let $d(f)$ be the smallest number k such that f can be represented by a polynomial with k monomials. Thus $\mathcal{F}_f$ is a union of $d(f)$ but not fewer intervals. Any polynomial representing f with the least possible number

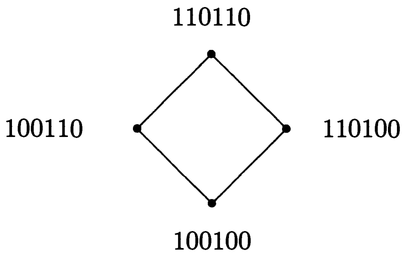


Figure 5.2

of monomials is called a *minimal polynomial.* Without proof we mention that by a result of Korshunov [318] there exist constants c_1, c_2 such that

$$c_1 \frac{2^n}{\log n \log \log n} < d(f) < c_2 \frac{2^n}{\log n \log \log n}$$

holds for almost all Boolean functions f. Sapozhenko [412] gave a simple algorithm that provides a polynomial with $\frac{c2^n}{\log n}$ monomials for some constant c and almost all Boolean functions. An exact result can be obtained for the following special case: A Boolean function f is called *symmetric* if the number of ones in the input completely determines its value (the value $f(x_1, \ldots, x_n)$ does not change if we permute the input variables). In other words, f is symmetric iff $\mathcal{F}_f$ is a union of complete levels of B_n. If, more precisely, $\mathcal{F}_f = B_{n[l_1,u_1]} \cup \cdots \cup B_{n[l_k,u_k]}$ where $0 \leq l_i \leq u_i \leq n, i = 1, \ldots, k$, and $u_i \leq l_{i+1} - 2, i = 1, \ldots, k-1$, then by the strong IC-property of B_n,

$$d(f) = \sum_{i=1}^{k} \max \left\{ \binom{n}{l_i}, \binom{n}{u_i} \right\}.$$

Let us mention in this context another result on Boolean functions. Suppose we are given some logical expression representing the function f. We say that a monomial is a *prime implicant* of this expression if trueness of the monomial implies trueness of the expression but deletion of variables in the monomial destroys this property. Obviously, a monomial is a prime implicant iff its corresponding interval belongs to $\mathcal{F}_f$ and if no extended interval has this property. Let $p(n)$ be the maximum number of prime implicants of a Boolean function with n variables.

***Theorem 5.3.3 (Vikulin* [456]).** *For any $\epsilon > 0$, there is some n_0 such that for $n > n_0$*

$$(c_1 - \epsilon)\frac{3^n}{n} < p(n) < (c_2 + \epsilon)\frac{3^n}{\sqrt{n}},$$

where $c_1 := \frac{3\sqrt{3}}{2\pi}$ and $c_2 := \frac{3}{2\sqrt{\pi}}$.

Proof. Let f be the symmetric Boolean function for which $\mathcal{F}_f = B_{n[l,n-l]}$ with $l := \lfloor \frac{n}{3} \rfloor$ holds. Counting the maximal intervals in $B_{n[l,n-l]}$ yields the lower bound $\binom{n}{l}\binom{n-l}{n-2l}$, which is by Stirling's formula (Theorem 7.1.7) asymptotically equal to the lower bound (one may derive this also from (7.7)).

For the upper bound, recall that the intervals in B_n are exactly the elements of the cubical poset Q_n. From the definition of a prime implicant we derive immediately that the corresponding elements in Q_n form an antichain. Thus

$$p(n) \leq d(Q_n).$$

It is easily seen that the maximum Whitney number in Q_n equals $\binom{n}{l}2^{n-l}$ with l as in the beginning of the proof. From Example 4.5.1 and Stirling's formula (resp. (7.7)) we obtain $d(Q_n) \sim c_2 \frac{3^n}{\sqrt{n}}$ (which follows also directly from Theorem 7.2.1). This proves the assertion. ■

See the end of this section for two further examples of ssc-orders.

***Example 5.3.2* (*Greene* [230]).** *The poset of shuffles W_{mn}.*

It is defined as follows: Let $\boldsymbol{x} = x_1 x_2 \ldots x_m$ and $\boldsymbol{y} = y_1 y_2 \ldots y_n$ be words formed from $m+n$ distinct letters. Now W_{mn} is the set of all words $\boldsymbol{w}$ with letters from $\boldsymbol{x}$ and $\boldsymbol{y}$ such that the restriction of $\boldsymbol{w}$ to the letters of $\boldsymbol{x}$ (resp. to $\boldsymbol{y}$) is a subword of $\boldsymbol{x}$ (resp. of $\boldsymbol{y}$), and we have $\boldsymbol{w} \leq \boldsymbol{w}'$ iff $\boldsymbol{w}'$ can be obtained from $\boldsymbol{w}$ by deleting letters of $\boldsymbol{x}$ and adding letters of $\boldsymbol{y}$. For example, if $\boldsymbol{x} = ABCDEFG$ and $\boldsymbol{y} = rstu$ then $BrDFuG \leq rDsuG$ in $W_{7,4}$. Note that $\boldsymbol{x}$ is the minimal and $\boldsymbol{y}$ is the maximal element in W_{mn} and that the structure of W_{mn} depends really only on m and n and not on the words $\boldsymbol{x}$ and $\boldsymbol{y}$. Given $\boldsymbol{w} \in W_{mn}$, the *interface* of $\boldsymbol{w}$, denoted by $I(\boldsymbol{w})$, is the set of all pairs of letters (y, x) where $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$, and x immediately follows y in $\boldsymbol{w}$. For any set I of pairs $(y, x), x \in \boldsymbol{x}, y \in \boldsymbol{y}$, we set $W_{mn}[I] := \{\boldsymbol{w} \in W_{mn} : I(\boldsymbol{w}) = I\}$. In our example we have $I = \{(r, D), (u, G)\}$. The minimal element of $W_{7,4}[I]$ is $ABCrDEFuG$ and the maximal element is $rDstuG$. In general, the minimal element of a nonempty $W_{mn}[I]$ has rank $|I|$ and the maximal element has rank $m+n-|I|$. Thus $W_{mn}[I]$ lies for every I in the "middle" of W_{mn}. For any word $\boldsymbol{w} \in W_{mn}$, let $R_{\boldsymbol{x}}(\boldsymbol{w})$ and $R_{\boldsymbol{y}}(\boldsymbol{w})$ be the set of letters of $\boldsymbol{x}$ (resp. of $\boldsymbol{y}$) which do not appear as components in the pairs of $I(\boldsymbol{w})$. For fixed I, we write also $R_{\boldsymbol{x}}(I)$ and $R_{\boldsymbol{y}}(I)$. It is not difficult to see that for $\boldsymbol{w}, \boldsymbol{w}' \in W_{mn}(I)$ we have $\boldsymbol{w} \leq \boldsymbol{w}'$ iff $R_{\boldsymbol{x}}(\boldsymbol{w}) \supseteq R_{\boldsymbol{x}}(\boldsymbol{w}')$ and $R_{\boldsymbol{y}}(\boldsymbol{w}) \subseteq R_{\boldsymbol{y}}(\boldsymbol{w}')$. Thus $W_{mn}[I]$ is isomorphic to the product of the dual of the subset lattice of $R_{\boldsymbol{x}}(I)$ with the subset lattice of $R_{\boldsymbol{y}}(I)$, that is, to $B_{m+n-2|I|}$. Since for different sets I and I' the sets $W_{mn}[I]$ and $W_{mn}[I']$ are disjoint and since each $W_{mn}[I]$ is an sc-order (the corresponding symmetric chains in $W_{mn}[I]$ are also symmetric in W_{mn}), W_{mn} is also an sc-order. Moreover, W_{mn} is an ssc-order: Since each $W_{mn}[I]$ is an ssc-order, for the proof we need only to examine the maximal chains C in the Boolean sublattices $W_{mn}[I]$, where $I \neq \emptyset$ (these chains are not maximal in W_{mn}). Delete in I exactly one pair yielding I'. Then the maximal chain D in $W_{mn}[I']$ (belonging to the symmetric chain partition) is a desired chain for property (*).

Note that Doran IV [139] introduced the "shuffling" of lattices. Here the poset W_{mn} is obtained by shuffling two Boolean lattices.

***Example 5.3.3* (*Simion and Ullmann* [431]).** *The lattice of noncrossing partitions $NC(n)$.*

The lattice $NC(n)$ is an induced subposet (moreover, a sublattice) of the partition lattice Π_n. Here a partition of $[n]$ is called *noncrossing* if, whenever $1 \leq a < b < c < d \leq n$ with a, c in the same block and b, d in the same block, then in fact all four elements are in the same block. Thus, for example, $138|2|4|57|6$ is a noncrossing partition of $[8]$ while $138|24|57|6$ is crossing. We decompose $NC(n)$ as $\bigcup_{i=1}^{n} R_i$, where $R_1 := \{\pi \in NC(n) : \{1\} \text{ is a block of } \pi\}$, and $R_i := \{\pi \in NC(n) : i = \min\{j : j \neq 1, j \text{ is in the same block of } \pi \text{ as } 1\}\}$ for $i \geq 2$. It is easy to see that $R_1 \cong R_2 \cong NC(n-1)$, and that $R_1 \cup R_2 \cong NC(n-1) \times (0 \lessdot 1)$ since each partition R_1 is covered by only one partition of R_2, namely the partition obtained by merging the block $\{1\}$ with the block containing the element 2. Moreover, for $i \geq 3$, $R_i \cong NC(i-2) \times NC(n-i+1)$ (realized as noncrossing partitions of $\{2, 3, \ldots, i-1\}$ and $\{i, i+1, \ldots, n\}$). The posets $R_1 \cup R_2$ and $R_i, i \geq 3$, lie in the "middle" of $NC(n)$ since in $R_1 \cup R_2$ the minimal (resp. maximal) element $1|2|\ldots|n$ (resp. $12\ldots n$) has rank 0 (resp. $r(NC(n)) = n-1$) and in R_i the minimal (resp. maximal) element $1i|2|\ldots|i-1|i+1|\ldots|n$ (resp. $1i(i+1)\ldots n|23\ldots(i-1)$) has rank 1 (resp. $n-2$). In a straightforward way we may apply induction and Theorem 5.3.2(a) to derive that $NC(n)$ is an ssc-order (note that in the symmetric chain partition the maximal chain in $R_1 \cup R_2$ is a desired chain concerning property (*) for the maximal chain in $R_i, i \geq 3$).

5.4. Semisymmetric chain orders and matchings

There exist some generalizations of sc-orders. For example, Griggs [238] introduced the *nested chain orders*, which are posets that can be partitioned into saturated chains such that for any two chains $C = (p_0 \lessdot \cdots \lessdot p_h)$ and $D = (q_0 \lessdot \cdots \lessdot q_k)$, the inequality $r(p_0) > r(q_0)$ implies $r(p_h) \leq r(q_k)$. It is left to the reader to prove that nested chain orders have the Sperner property. See also Griggs [242] who listed several properties of ranked posets together with their mutual relations. We consider in more detail another generalization, which implies the strong Sperner property at least in the lower half of the poset.

Let P be a ranked poset of rank $n := r(P)$. We call a chain C of P *semisymmetric* if it has the form $C = (p_0 \lessdot \cdots \lessdot p_h)$ with $r(p_0) + r(p_h) \geq n$, and we call P a *semisymmetric chain order* (*semi-sc-order*) if P can be partitioned into semisymmetric chains.

Theorem 5.4.1. *If P is a semi-sc-order then $W_0 \le W_1 \le \cdots \le W_{\lceil \frac{n}{2} \rceil}$, and the $[0, \lceil \frac{n}{2} \rceil]$-rank selected subposet Q (induced by $\cup_{i=0}^{\lceil \frac{n}{2} \rceil} N_i$) has the strong Sperner property.*

Proof. The inequality $W_i \le W_{i+1}, 0 \le i < \lceil \frac{n}{2} \rceil$, follows as in Lemma 5.1.1. Let us restrict the chains of the semi-sc-partition to Q. This yields a collection $\mathfrak{C}$ of chains. Obviously $\mathfrak{C}$ has $W_i - W_{i-1}$ chains of size $\lceil \frac{n}{2} \rceil - i + 1, i = 0, \ldots, \lceil \frac{n}{2} \rceil$. Thus for any k-family F in Q,

$$\begin{aligned}
|F| \le \sum_{C \in \mathfrak{C}} \min\{|C|, k\} &= \sum_{i=1}^{k-1} i\big(W_{\lceil \frac{n}{2} \rceil - i+1} - W_{\lceil \frac{n}{2} \rceil - i}\big) \\
&\quad + \sum_{i=k}^{\lceil \frac{n}{2} \rceil + 1} k\big(W_{\lceil \frac{n}{2} \rceil - i+1} - W_{\lceil \frac{n}{2} \rceil - i}\big) \\
&= \sum_{i=1}^{k} W_{\lceil \frac{n}{2} \rceil - i+1},
\end{aligned}$$

that is, Q has the strong Sperner property. ■

Theorem 5.4.2. *Let P and Q be semi-sc-orders.*

(a) Then $P \times Q$ is a semi-sc-order, too.

(b) If, in addition, $r(P) = r(Q)$ then $P \times_r Q$ is a semi-sc-order, too.

Proof. The constructions in the proofs of Theorem 5.1.5 and Theorem 5.1.7 yield the desired semi-sc-partitions. ■

Proposition 5.4.1. *Let P be a ranked poset of rank n. If, for $0 \le i < n$, N_i can be matched into N_{i+1} then P is a semi-sc-order.*

Proof. If we join the matchings as in the proof of Theorem 5.1.1 we obtain a partition of P into semisymmetric chains (each with top at level N_n). ■

It is not known whether each geometric lattice is a semi-sc-order, but an attractive special case is covered by the next theorem.

***Theorem 5.4.3 (Loeb, Damiani, and D'Antona* [350]).** *The partition lattice Π_n is a semi-sc-order.*

Proof. We will first partition Π_n (or better Π_{n+1}) into some smaller sets. In order to explain this we need some further definitions. For each $S \subseteq [n]$, we define recursively

$$c_i(S) := \begin{cases} 0 & \text{if } i \in S, \\ i - \sum_{j=1}^{i-1} c_j(S) & \text{if } i \notin S, \end{cases}$$

$i = 1, \ldots, n+1$. The vector $\boldsymbol{c}(S) = (c_1(S), \ldots, c_{n+1}(S))$ is called the *code of* S. For example, the code of $S = \{1, 2, 3, 7, 11, 12, 16, 18, 19\} \subseteq \{1, \ldots, 20\}$ is

$$\boldsymbol{c}(S) = (0, 0, 0, 4, 1, 1, 0, 2, 1, 1, 0, 0, 3, 1, 1, 0, 2, 0, 0, 3, 1).$$

We call a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{Z}^{n+1}$ a *coding vector* if $\alpha_{n+1} \neq 0$ and for all $i \in \{1, \ldots, n+1\}$ either $\alpha_i = 0$ or $\sum_{j=1}^{i} \alpha_j = i$ (in particular $\sum_{j=1}^{n+1} \alpha_j = n+1$). Induction on i easily yields that each component of a coding vector is nonnegative. Moreover, it is obvious from the definition that the code of any set S is a coding vector and that $c_i(S) = 0$ iff $i \in S$. Conversely, for each coding vector $\boldsymbol{\alpha}$ there exists exactly one set S such that $\boldsymbol{c}(S) = \boldsymbol{\alpha}$, namely $S := \{i : \alpha_i = 0\}$. Not only the zero elements of a coding vector $\boldsymbol{\alpha}$ determine the corresponding set S. It is enough to know the nonzero elements of $\boldsymbol{\alpha}$, since a nonzero element a must be preceded by exactly $a - 1$ zeros. Let $\boldsymbol{c}'(S)$ be the vector that can be obtained from $\boldsymbol{c}(S)$ by deleting all zeros. In our example we have

$$\boldsymbol{c}'(S) = (4, 1, 1, 2, 1, 1, 3, 1, 1, 2, 3, 1).$$

For a block B in a partition π, let $m(B) := \min\{i : i \in B\}$. We will enumerate the blocks $B_1, \ldots, B_k$ of π always in such a way that $m(B_1) < m(B_2) < \cdots < m(B_k)$. Under this supposition we may uniquely define the *antitype of* π by $at(\pi) := (|B_k|, |B_{k-1}|, \ldots, |B_1|)$. If, for example, $\pi \in \Pi_{21}$ equals $1|2, 5, 7|3, 16|4|6|8, 10, 12|9|11|13, 20|14|15|17, 18, 19, 21$, then $at(\pi) = (4, 1, 1, 2, 1, 1, 3, 1, 1, 2, 3, 1) = \boldsymbol{c}'(S)$ from our preceding example. Now we put for $S \subseteq [n]$

$$\Pi_S := \{\pi \in \Pi_{n+1} : at(\pi) = c'(S)\}.$$

By the preceding considerations, the sets Π_S, $S \subseteq \{1, \ldots, n\}$, partition the lattice Π_{n+1}. Each $\pi \in \Pi_S$ has as many blocks as $\boldsymbol{c}'(S)$ has components; that is, $n+1-|S|$. Consequently,

$$r(\pi) = n + 1 - ((n+1) - |S|) = |S|.$$

Let $\mathfrak{C}$ be a symmetric chain partition of $2^{[n]}$ obtained by the parenthesization method. For $C = (S_0 \lessdot \cdots \lessdot S_h)$, let

$$\Pi_C := \cup_{i=0}^{h} \Pi_{S_i}.$$

We will consider the subposets that are induced by $\Pi_C, C \in \mathfrak{C}$ (they partition also Π_{n+1}). If π (resp. σ) is a minimal (resp. maximal) element in Π_C then

$r(\pi) + r(\sigma) = n = r(\Pi_{n+1})$ since C is symmetric. Thus it is sufficient to show that each Π_C is a semi-sc-order. For that, we may apply Proposition 5.4.1. We have to find for each link $S \to S' := S \cup \{i\}$ in a chain of $\mathfrak{C}$ a matching from Π_S into $\Pi_{S'}$. Let $k := c_i(S)$. Note that $k \neq 0$. By Proposition 5.1.1(c) we have $i + 1 \notin S$, that is, $c_{i+1}(S) = 1$. Now, it is easy to verify that

$$c_j(S') = \begin{cases} 0 & \text{if } j = i, \\ k+1 & \text{if } j = i+1, \\ c_j(S) & \text{otherwise.} \end{cases}$$

Let $\pi \in \Pi_S$ and let A (resp. B) be the block of π that corresponds to the k (resp. the 1) in the substring $(\ldots k,\ 1\ldots)$ which is being replaced by $(\ldots 0,\ k+1\ldots)$. Let $f(\pi)$ be the partition that can be obtained from π by replacing the blocks A and B with their union $A \cup B$. It is easy to see that $\pi \lessdot f(\pi)$ and that $f(\pi) \in \Pi_{S'}$. Moreover, we can recover A and B from $A \cup B$ since B contains only one element, say b, and $b = \min\{j : j \in A \cup B\}$. Thus we can also recover π from $f(\pi)$ and consequently, $\{(\pi, f(\pi)) : \pi \in \Pi_s\}$ is a desired matching from Π_S into $\Pi_{S'}$. ∎

From the preceding theorem it follows in particular that in Π_n the level N_i can be matched into N_{i+1} for $i < \frac{n-1}{2}$. This was first proved by Kung [329] using an algebraic machinery. We even have the following stronger result:

Theorem 5.4.4 (*Canfield* [96]).

(*a*) *For every $\delta > 0$, there exists an $n_0(\delta)$ such that, for all $n \geq n_0(\delta)$, the following holds: In the partition lattice Π_n, the level N_i can be matched into N_{i+1} if $i < n - (1+\delta)\frac{n \log 4}{\log n}$.*

(*b*) *In Π_n the level N_i can be matched into N_{i-1} if $i \geq n - \frac{n \log 2}{\log n}$ and $n \geq 5$.*

Proof. (a) Let $\epsilon := (1+\delta)\frac{\log 4}{\log n}$, $b := (1 - \frac{\delta}{3})\frac{\log n}{\log 4}$, $i < n(1-\epsilon)$. Without loss of generality, we may assume $\delta \leq \frac{1}{2}$. We consider the subposet Q_i of rank 1 whose levels are N_i and N_{i+1}, and where $\pi < \sigma$ ($\pi \in N_i, \sigma \in N_{i+1}$) if π can be obtained from σ by splitting a block B of size not larger than $2b$ into two blocks B_1 and B_2 each of whose sizes is not larger than b. Obviously, $\pi \in N_i$ cannot contain more than $\frac{n}{b}$ blocks whose size exceeds b. Thus π contains at least $n - i - \frac{n}{b} \geq (n-i)\left(1 - \frac{1}{b\epsilon}\right)$ blocks of size not larger than b. By choosing any two of them we derive for Q_i that

$$|\nabla(\pi)| \geq \binom{(n-i)(1-\frac{1}{b\epsilon})}{2}.$$

We have $b\epsilon = (1+\delta)(1-\delta/3) \geq \frac{1}{1-\delta/3}$; that is, $1 - \frac{1}{b\epsilon} \geq \frac{\delta}{3}$. Let n be sufficiently large such that $\alpha := \delta(n-i) \geq 30$. An easy calculation shows that then $\binom{\alpha/3}{2} \geq \frac{\alpha^2}{20}$.

We conclude that

$$|\nabla(\pi)| \geq \frac{\alpha^2}{20}. \tag{5.5}$$

Now consider any $\sigma \in N_{i+1}$. It has $n-i-1 < n-i$ blocks. We find the predecessors of σ in Q_i by splitting a block of size $2b$ or less under additional conditions. Consequently,

$$|\Delta(\sigma)| \leq (n-i)2^{2b} = (n-i)n^{1-\delta/3}. \tag{5.6}$$

Let n be sufficiently large such that $\delta^2\epsilon \geq 20n^{-\delta/3}$. Then we obtain from (5.5) and (5.6) by straightforward computation

$$|\nabla(\pi)| \geq |\Delta(\sigma)|$$

and by Theorem 5.1.3 the level N_i can be matched into N_{i+1} in our poset Q_i – that is, also in Π_n.

(b) Since there are $2^{|B|-1}-1$ possibilities to partition a block B into two blocks, we have, for $\pi \in N_i$ and with $k := n-i =$ number of blocks in π,

$$|\Delta(\pi)| = \sum_{B \text{ block in } \pi} (2^{|B|-1}-1) \geq k(2^{n/k-1}-1)$$

where the last inequality follows by applying Jensen's inequality to the convex function $f(x) = 2^{x-1}-1$. We have $\frac{n}{k} \geq \frac{\log n}{\log 2}$, which implies $2^{n/k} \geq n$ and further

$$|\Delta(\pi)| \geq k\left(\frac{n}{2}-1\right).$$

Moreover, for $\sigma \in N_{i-1}$ obviously,

$$|\nabla(\sigma)| = \binom{k+1}{2}.$$

We have $\binom{k+1}{2} \leq k(\frac{n}{2}-1)$ iff $k \leq n-3$, and this is true for $n \geq 6$. Thus in this case $|\Delta(\pi)| \geq |\nabla(\sigma)|$, and Theorem 5.1.3 (in its dual form) gives the matchings. For $n = 5$, the assertion of the theorem is easily checked by inspection. ■

It is very interesting that the bounds in Theorem 5.4.4 are the best possible.

***Theorem 5.4.5 (Canfield* [96]).** *For every $\delta > 0$, there exists an $n_0(\delta)$ such that for all $n \geq n_0(\delta)$ there holds in the partition lattice Π_n:*

(*a*) *The level N_i cannot be matched into N_{i+1} if $i > n - (1-\delta)\frac{n \log 4}{\log n}$.*

(*b*) *The level N_i cannot be matched into N_{i-1} if $i < n - (1+\delta)\frac{n \log 2}{\log n}$.*

We will prove only part (a) by a construction that is partly due to Shearer [428]. The proof of part (b) relies on the application of a local limit theorem (together with an estimation of the remainder). Since we will not use the result of part (b) further in this book, the interested reader is referred to [96], but note that in Chapter 7 we will discuss applications of local limit theorems.

Proof of (a). Let, w.l.o.g., $\delta < 1$ and let $i > n-(1-\delta)\frac{n\log 4}{\log n}$. By Theorem 5.1.2 it is enough to find some $A \subseteq N_i$ such that $|\nabla(A)| < |A|$. Let

$$m := \left\lfloor \frac{\log n}{(1-\delta)\log 4} \right\rfloor.$$

Every element of N_i has $n-i < n\frac{(1-\delta)\log 4}{\log n} \leq \frac{n}{m}$ blocks. Let A be the set of those partitions from N_i at least $n^{1-\delta/2}$ of whose block sizes equal $2m$, at least $n^{1-\delta/2}$ of whose block sizes equal or exceed $3m$, and all of whose block sizes belong to the set $M := \{m, 2m, 3m, 3m+1, 3m+2, \ldots\}$. Since we have for every fixed $c > 0$ and sufficiently large n that

$$n^{1-\delta/2}cm \leq n\frac{c\log n}{n^{\delta/2}(1-\delta)\log 4} < n,$$

our set A is not empty if n is large enough. Let $\pi^* \in A$ and $\pi^* \lessdot \sigma$. Since the block in σ that was created by joining two blocks of π^* and the blocks in σ of size $2m$ can be partitioned in at least (resp. exactly) $\frac{1}{2}\binom{2m}{m}$ ways into two blocks of sizes belonging to M, we have

$$|\{\pi \in A : \pi \lessdot \sigma\}| \geq \frac{1}{2}\binom{2m}{m}n^{1-\delta/2} \text{ for all } \sigma \in \nabla(A).$$

(This bound is also easily seen to be true if σ was obtained from π^* by merging two blocks of size $2m$ because we have then an item $\frac{1}{2}\binom{4m}{2m}$.) Obviously,

$$|\{\sigma \in \nabla(A) : \sigma \gtrdot \pi\}| = \binom{n-i}{2} \leq \frac{1}{2}\left(\frac{n}{m}\right)^2 \text{ for all } \pi \in A.$$

Counting the pairs (π, σ) with $\pi \in A, \pi \lessdot \sigma$ in two different ways we obtain

$$|A|\frac{1}{2}\left(\frac{n}{m}\right)^2 \geq |\nabla(A)|\frac{1}{2}\binom{2m}{m}n^{1-\delta/2}. \tag{5.7}$$

Using the fact that $\binom{2m}{m}$ is the largest of $2m+1$ binomial coefficients whose sum is 2^{2m}, we obtain

$$\binom{2m}{m} > \frac{1}{2m+1}2^{2m} \geq \frac{1}{2m+1}2^{2\left(\frac{\log n}{(1-\delta)\log 4}-1\right)} = \frac{1}{4(2m+1)}n^{1/(1-\delta)}.$$

Consequently, by straightforward computation,

$$\frac{1}{2}\binom{2m}{m}n^{1-\delta/2} > \frac{1}{8(2m+1)}n^{1-\delta/2+1/(1-\delta)} > \frac{1}{2}\left(\frac{n}{m}\right)^2 \tag{5.8}$$

if n is sufficiently large. Now from (5.7) and (5.8) we derive $|A| > |\nabla(A)|$. ■

In addition to this result, Canfield [96] proved, using some earlier results of Mullin [376], that the transition from the existence to the nonexistence of matchings between neighboring levels is not chaotic but abrupt – that is, there are sequences L_n and R_n such that N_i can be matched (resp. cannot be matched) into N_{i+1} if $i < L_n$ (resp. $i \geq L_n$), and analogously for R_n and $N_i \to N_{i-1}$. Moreover, both L_n and R_n grow by at most 1 when n is increased by 1.

We may use this result to show that the partition lattice Π_n does not have the Sperner property if n is large enough. Before doing so, we need a localization of the largest Whitney number of Π_n, or, equivalently, of the maximum Stirling number of the second kind.

Lemma 5.4.1. *The sequence of Stirling numbers $\{S_{n,k} : k = 1, \dots, n\}$ is strictly log concave.*

Proof. We proceed by induction on n and use the basic recurrence $S_{n,k} = kS_{n-1,k} + S_{n-1,k-1}$ (applied to $S_{n,k}$, $S_{n,k-1}$, and $S_{n,k+1}$). The cases $n = 1, 2$ are trivial. Concerning the induction step we have (for $2 \leq k \leq n-1$)

$$\begin{aligned} S_{n,k}^2 - S_{n,k-1}S_{n,k+1} &= (S_{n-1,k-1}^2 - S_{n-1,k-2}S_{n-1,k}) \\ &\quad + (k^2 S_{n-1,k}^2 - (k^2-1)S_{n-1,k-1}S_{n-1,k+1}) \\ &\quad + (k+1)(S_{n-1,k-1}S_{n-1,k} - S_{n-1,k-2}S_{n-1,k+1}). \end{aligned}$$

The first two summands are positive by induction. Moreover, by induction,

$$S_{n-1,k-1}^2 S_{n-1,k}^2 \geq (S_{n-1,k-2}S_{n-1,k})(S_{n-1,k+1}S_{n-1,k-1}),$$

which implies that the third summand is also nonnegative. ∎

Note that this result shows that Π_n is rank unimodal. Let k_n be that number for which $S_{n,1} < S_{n,2} < \cdots < S_{n,k_n} \geq S_{n,k_n+1} > S_{n,k_n+2} > \cdots > S_{n,n}$; that is, W_{n-k_n} is the largest Whitney number of Π_n. Canfield [94] determined k_n precisely for sufficiently large n. Almost the same formula was obtained by Jichang and Kleitman [277]: Let x be the root of $(x + \frac{3}{2}) \log x = n - 2$. Then $k_n \in \{\lfloor x + \frac{1}{2} \rfloor, \lfloor x + \frac{1}{2} \rfloor + 1\}$. Moreover, $k_n = \lfloor x \rfloor + 1$ in "almost all cases." The proof is very complicated. From a recent result of Benoumhani [45] on the maximum coefficient of a polynomial whose roots are all negative real, it follows that (for all n) $|k_n - (n - \mu_r(\Pi_n))| \leq 1$ where $\mu_r(\Pi_n)$ is the expected value of the rank function of Π_n. With (6.50) this implies $|k_n - \frac{B_{n+1}}{B_n} + 1| \leq 1$. Here we need only a slightly weaker result:

Theorem 5.4.6 (Harper [256]). *We have $k_n \sim \frac{n}{\log n}$ as $n \to \infty$.*

Proof. We present a short proof that is essentially due to Rennie and Dobson [395].

It is enough to show that for every $\epsilon > 0$ we have

$$(1-\epsilon)\frac{n}{\log n} + O(1) \le k_n \le (1+\epsilon)\frac{n}{\log n} + O(1). \tag{5.9}$$

Claim 1. If $1 \le k \le n$, then $k^{n-k} \le S_{n,k} \le \binom{n-1}{k-1}k^{n-k}$.

Proof of Claim 1. Let us construct partitions with k blocks by putting n balls labeled with the numbers $1, \dots, n$ into k indistinguishable boxes such that each box obtains at least one ball. We put the balls with numbers $1, \dots, k$ into the boxes such that in each box there is exactly one ball. For the remaining $(n-k)$ balls, we have k^{n-k} possibilities for the distribution. This yields k^{n-k} pairwise different partitions. Thus the first inequality is proved. However, every partition can be constructed in the following way. Put ball number 1 into one box. Choose $k-1$ other balls (in $\binom{n-1}{k-1}$ different ways) and put them into the remaining boxes such that in each box there is exactly one ball. Distribute the remaining $n-k$ balls as before. This yields the upper bound $\binom{n-1}{k-1}k^{n-k}$. □

Claim 2. If $1 \le k < n$, then $\binom{n}{k} < \frac{n^n}{k^k(n-k)^{n-k}}$.

Proof of Claim 2. There are n^n possibilities to put n balls labeled with $1, \dots, n$ into n boxes labeled with $1, \dots, n$. Some of them can be obtained as follows: Choose k balls ($\binom{n}{k}$ possibilities). Put them into the boxes $1, \dots, k$ (k^k possibilities). Put the remaining $n-k$ balls into the boxes $k+1, \dots, n$ ($(n-k)^{n-k}$ possibilities). Clearly we obtain pairwise different distributions. Consequently, $\binom{n}{k}k^k(n-k)^{n-k} < n^n$, and Claim 2 is proved. □

Let

$$k_c := c\frac{n}{\log n} + O(1), \quad (c \text{ is a positive constant}),$$

$$k^* := \frac{n}{\log n} + O(1)$$

(we have the term $O(1)$ to ensure that k and k^* are integers).

Claim 3. We have $S_{n,k_c} < S_{n,k^*}$ if $c \ne 1$ and n is sufficiently large.

Proof of Claim 3. We write briefly $k := k_c$. In view of Claims 1 and 2 (together with the trivial inequality $\binom{n-1}{k-1} \le \binom{n}{k}$), it is enough to show that

$$\frac{n^n}{k^k(n-k)^{n-k}}k^{n-k} < k^{*n-k^*},$$

which is equivalent to

$$n\log n + (n-2k)\log k - (n-k)\log(n-k) < (n-k^*)\log k^*.$$

If we subtract the LHS from the RHS and use $\log(1+x) = x + o(x)$ we obtain the term

$$\begin{aligned} n & \left(1 - \frac{1}{\log n}\right)(\log n - \log\log n + o(1)) - n\log n \\ & - n\left(1 - \frac{2c}{\log n}\right)(\log c + \log n - \log\log n + o(1)) \\ & + n\left(1 - \frac{c}{\log n}\right)(\log n + o(1)) \\ & = n(c - \log c - 1 + o(1)). \end{aligned}$$

From Proposition 2.6.1(b) it follows that for sufficiently large n the last term is positive if $c \neq 1$, and this proves the statement in the claim. □

From Claim 3 we derive for $\epsilon > 0$ that $k_{1-\epsilon} < k^* < k_{1+\epsilon}$ which is our desired inequality (5.9). ■

As already mentioned we are now able to prove Canfield's result, which was a markstone in the development of the Sperner theory. It answers a question of Rota [402] negatively. Simplifications of the first complicated proof have been given by Shearer [428] and Jichang and Kleitman [277].

***Theorem 5.4.7* (*Canfield* [93]).** *For sufficiently large n, the partition lattice Π_n does not have the Sperner property.*

Proof. Take some small $\delta > 0$ and let n be large enough such that $(1-\delta) \times \log 4 \frac{n}{\log n} > k_n + 1$ (apply Theorem 5.4.6). Let $i := n - k_n - 1$ (i.e., W_{i+1} is the largest Whitney number of Π_n) and take the set $A \subseteq N_i$ from the proof of Theorem 5.4.5(a). We have seen that for sufficiently large n, $|A| > |\nabla(A)|$. Obviously, the set $S := A \cup (N_{i+1} - \nabla(A))$ is a Sperner family whose size is greater than W_{i+1}. ■

Theorem 5.4.7 and Corollary 4.5.3 imply that Π_n cannot be normal if n is sufficiently large. There is a sharper result:

***Theorem 5.4.8* (*Spencer* [435]).** *The partition lattice Π_n is not normal if $n \geq 20$.*

Proof. First let n be even. Let A be the set of those partitions from N_{n-2} both of whose block sizes equal $\frac{n}{2}$. Then $\Delta(A)$ consists of those partitions from N_{n-3} that have one block of size $\frac{n}{2}$. We have

$$W_{n-2} = S_{n,2} = \tfrac{1}{2}(2^n - 2),$$

$$W_{n-3} = S_{n,3} = \tfrac{1}{6}(3^n - 3\cdot 2^n + 3),$$

$$|A| = \frac{1}{2}\binom{n}{\frac{n}{2}},$$

$$|\Delta(A)| = \binom{n}{\frac{n}{2}} S_{\frac{n}{2},2}.$$

In a straightforward way one obtains

$$\frac{|A|}{W_{n-2}} \leq \frac{|\Delta(A)|}{W_{n-3}} \text{ iff } 3^{n-1} \leq 2^{3n/2} - 2^n - 2^{n/2+1} + 3,$$

which is false iff $n \geq 20$. For n odd, we may use the same arguments if we work with the set A of partitions from N_{n-3} that have $\{n\}$ as one block and whose other block sizes both equal $\frac{n-1}{2}$. ■

With the partition lattice one can also associate the *poset Pi_n of unordered partitions of an integer*, ordered by partitioning the items. More precisely, $Pi_n := \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : \sum_{i=1}^{n} i\sigma_i = n\}$, and we have $\boldsymbol{\sigma} \lessdot \boldsymbol{\tau}$ iff there are $i, j \in [n]$ such that $\sigma_{i+j} = \tau_{i+j} - 1$ and $\sigma_i = \tau_i + 1, \sigma_j = \tau_j + 1$ if $i \neq j$ (resp. $\sigma_i = \tau_i + 2$ if $i = j$). The symmetric group S_n on $[n]$ induces in an obvious way a group G of automorphisms of Π_n. It is easy to see that the quotient Π_n/G equals Pi_n. Moreover, if we define $w : Pi_n \to \mathbb{R}_+$ by

$$w(\boldsymbol{\sigma}) := \frac{n!}{\prod_{i=1}^{n} \sigma_i!(i!)^{\sigma_i}},$$

then (Pi_n, w) is the weighted quotient of $(\Pi_n, 1)$, and it does not have the Sperner property for large enough n by Theorem 4.5.6 and Theorem 5.4.7. However, the following problem is still open: Does the unweighted poset Pi_n of unordered partitions of an integer have the Sperner property?

6

Algebraic methods in Sperner theory

In this chapter we are concerned with unweighted, ranked posets only. Algebraic characterizations for the existence of certain matchings in graphs have been known for many years (cf. Lovász and Plummer [356], and note in particular the results of Kasteleyn [286], Perfect [380], and Edmonds [142]). We have seen that chains in posets can be constructed by joining matchings between consecutive levels. By Dilworth's theorem, the size of chain partitions is related to the width of a poset. For these (and other related) reasons, it is worthwhile to develop an algebraic machinery that answers many questions in Sperner theory. An essential step in this direction was undertaken by Stanley [439], who used deeper results from algebraic geometry (the Hard Lefschetz Theorem) in order to prove the Sperner property of several posets. Further work (in particular that of Proctor [388]) allows us to stay on a more or less elementary linear algebraic level. It is interesting that Erdős–Ko–Rado type theorems can also be proved using this approach. Important contributions were those of Lovász [355] and Schrijver [419] and some culmination was achieved by Wilson's [470] exact bound for the Erdős–Ko–Rado Theorem in the Boolean lattice. Besides this result, Wilson strongly influenced, in general, the development of the algebraic extremal set theory and, together with Kung (e.g., [327, 328]), the theory of geometric (and related) lattices. Kung's main tool is the finite Radon transform discovered by E. Bolker (see [328]), but we will formulate the results in the language of injective linear operators. Extremal set problems are discussed in Section 2.5. In this chapter (Section 6.5) we apply the fundamental observation that the image of a subspace of a vector space by a linear mapping has dimension not greater than the dimension of the preimage. For further information, the reader is referred to the book of Babai and Frankl [35].

Though the theory is easier to present under the supposition of rank symmetry we will consider first the general case and later specify the results to rank-symmetric posets. An important tool is the Jordan decomposition of nilpotent maps considered by Saks [406] and Gansner [213]. We introduce here the Jordan function in order to derive many results by algebraic manipulations only. We prefer to work throughout this chapter with linear operators instead of (incidence) matrices.

6.1. The full rank property and Jordan functions

Throughout, let P be a ranked poset with rank function r of rank $n := r(P)$. With P we associate the *poset space* $\widetilde{P}$, which is the vector space of all functions $\varphi : P \to \mathbb{R}$ with the usual vector space operations. With the element p of P we associate its *characteristic function* $\varphi_p \in \widetilde{P}$ defined by

$$\varphi_p(q) := \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

For the sake of brevity, we write $\widetilde{p}$ instead of φ_p, but $\widetilde{0}$ denotes the zero vector. Obviously, $\{\widetilde{p} : p \in P\}$ is a basis of $\widetilde{P}$. Thus every element φ of $\widetilde{P}$ has the form $\varphi = \sum_{p \in P} \mu_p \widetilde{p}$, where μ_p is a real number, and $\widetilde{P}$ can be considered as the vector space of all formal linear combinations of elements of P with real coefficients. With the standard scalar product

$$\left\langle \sum_{p \in P} \mu_p \widetilde{p}, \sum_{p \in P} \nu_p \widetilde{p} \right\rangle := \sum_{p \in P} \mu_p \nu_p$$

the space $\widetilde{P}$ becomes a Euclidean space (here we are working only with the field of real numbers). With a subset F of P we associate the subspace $\widetilde{F}$ of $\widetilde{P}$, which is generated by $\{\widetilde{p} : p \in F\}$. Obviously, the dimension of $\widetilde{F}$, denoted by $\dim \widetilde{F}$, equals $|F|$. Note that $\widetilde{P} = \widetilde{N}_0 \oplus \cdots \oplus \widetilde{N}_n$, where $\oplus$ denotes the direct sum. Any basis B of $\widetilde{P}$ with the property $B = \cup_{i=0}^{n} B \cap \widetilde{N}_i$, that is, all basis elements belong to some $\widetilde{N}_i$, is said to be a *ranked basis*. In particular, $\{\widetilde{p} : p \in \widetilde{P}\}$ is a ranked basis.

In the following we consider linear operators $\widetilde{P} \to \widetilde{P}$. We define $\Phi\Psi$ to be the operator for which $\Phi\Psi(\varphi) = \Phi(\Psi(\varphi))$ for all $\varphi \in \widetilde{P}$. Further let $\Phi^i := \Phi \cdots \Phi$ (i times) if $i \geq 1$. It is useful to define Φ^j to be the *identity operator* $\widetilde{I}$ if $j \leq 0$. For a subspace E of $\widetilde{P}$, let $\Phi(E) := \{\Phi(\varphi) : \varphi \in E\}$ and $\Phi|_E$ be the restriction of Φ to E. Let Φ^* be the *adjoint operator* of Φ, that is, $\Phi^* : \widetilde{P} \to \widetilde{P}$ with $\langle \Phi(\varphi), \psi \rangle = \langle \varphi, \Phi^*(\psi) \rangle$ for all $\varphi, \psi \in \widetilde{P}$.

If $j \notin \{0, \ldots, n\}$, let $\widetilde{N}_j = \{\widetilde{0}\}$. A linear operator $\widetilde{\nabla} : \widetilde{P} \to \widetilde{P}$ (resp. $\widetilde{\Delta} : \widetilde{P} \to \widetilde{P}$) is called a *raising* (resp. *lowering*) *operator* if $\widetilde{\nabla}(\widetilde{N}_i) \subseteq \widetilde{N}_{i+1}$ (resp. $\widetilde{\Delta}(\widetilde{N}_i) \subseteq \widetilde{N}_{i-1}$ for all i). A raising (resp. lowering) operator $\widetilde{\nabla}$ (resp. $\widetilde{\Delta}$) defined on the basis $\{\widetilde{p} : p \in P\}$ by

$$\widetilde{\nabla}(\widetilde{p}) := \sum_{q:q>p} c(p,q)\widetilde{q} \quad \left(\text{resp. } \widetilde{\Delta}(\widetilde{q}) := \sum_{p:p\lessdot q} d(p,q)\widetilde{p},\right) \tag{6.1}$$

with $c(p,q), d(p,q) \in \mathbb{R}$, is called an *order-raising* (resp. *order-lowering*) *operator* (here the empty sum is defined to be the zero vector). Thus order-raising (resp. -lowering) operators are defined by functions $E(P) \to \mathbb{R}$, where $E(P)$ is, as before, the arc set of the Hasse diagram of P. If $c(p,q) = 1$ (resp. $d(p,q) = 1$) for all $p \lessdot q$ we speak of the *Lefschetz raising* (resp. *lowering*) *operator*. In all that follows, $\widetilde{\nabla}$ (resp. $\widetilde{\Delta}$) denotes a raising (resp. lowering) operator, whereas $\widetilde{\nabla}_L = \widetilde{\nabla}_L(P)$ (resp. $\widetilde{\Delta}_L = \widetilde{\Delta}_L(P)$) denotes the Lefschetz raising (resp. lowering) operator in $\widetilde{P}$. Intuitively speaking, the Lefschetz raising operator $\widetilde{\nabla}_L$, when acting on a poset element, produces the sum of poset elements covering that element.

Lemma 6.1.1.

(*a*) *Let $\widetilde{\nabla}$ and $\widetilde{\Delta}$ be an (order-) raising (resp. lowering) operator. Then $\widetilde{\nabla}^*$ and $\widetilde{\Delta}^*$ is an (order-) lowering (resp. raising) operator.*

(*b*) *Let $\widetilde{\nabla}$ and $\widetilde{\Delta}$ be defined as in (6.1). If $c(p,q) = d(p,q)$ for all $p \lessdot q$, then $\widetilde{\Delta} = \widetilde{\nabla}^*$. In particular, $\widetilde{\Delta}_L = \widetilde{\nabla}_L^*$.*

Proof. (a) We consider only $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. Let $q \in N_{i+1}$ for some i. Then $\langle \widetilde{p}, \widetilde{\nabla}^*(\widetilde{q})\rangle = \langle \widetilde{\nabla}(\widetilde{p}), \widetilde{q}\rangle \neq 0$ only if $p \in N_i$ and $p \lessdot q$.

(b) We have for all $p, q \in P$

$$\langle \widetilde{\nabla}(\widetilde{p}), \widetilde{q}\rangle = \langle \widetilde{p}, \widetilde{\Delta}(\widetilde{q})\rangle = \begin{cases} c(p,q) = d(p,q) & \text{if } p \lessdot q, \\ 0 & \text{otherwise.} \end{cases}$$

■

Let $\widetilde{\nabla}_{ij} := \widetilde{\nabla}^{j-i}|_{\widetilde{N}_i}$ (resp. $\widetilde{\Delta}_{ji} := \widetilde{\Delta}^{j-i}|_{\widetilde{N}_j}$). Then $\widetilde{\nabla}_{ij} : \widetilde{N}_i \to \widetilde{N}_j$ (resp. $\widetilde{\Delta}_{ji} : \widetilde{N}_j \to \widetilde{N}_i$). If we fix the bases $\{\widetilde{p} : p \in N_i\}$ and $\{\widetilde{q} : q \in N_j\}$, we can, as usual, associate with the linear operator $\widetilde{\nabla}_{ij}$ (resp. $\widetilde{\Delta}_{ji}$) a matrix R_{ij} (resp. L_{ji}) whose columns (resp. rows) and rows (resp. columns) are indexed by rank i and rank j elements and whose entry in p's column (resp. row) and q's row (resp. column) equals $\langle \widetilde{\nabla}_{ij}(\widetilde{p}), \widetilde{q}\rangle$ (resp. $\langle \widetilde{p}, \widetilde{\Delta}_{ji}(\widetilde{q})\rangle$), $p \in N_i, q \in N_j$. If $\widetilde{\Delta} = \widetilde{\nabla}^*$, then

$$L_{ji} = R_{ij}^{\mathrm{T}} \text{ for all } i, j. \tag{6.2}$$

For a fixed raising (resp. lowering) operator $\widetilde{\nabla}$ (resp. $\widetilde{\Delta}$) and for $0 \le i \le j \le k \le n$, we have

$$\widetilde{\nabla}_{jk}\widetilde{\nabla}_{ij} = \widetilde{\nabla}_{ik} \quad (\text{resp. } \widetilde{\Delta}_{ji}\widetilde{\Delta}_{kj} = \widetilde{\Delta}_{ki}), \tag{6.3}$$

which implies

$$R_{jk}R_{ij} = R_{ik} \quad (\text{resp. } L_{ji}L_{kj} = L_{ki}). \tag{6.4}$$

If $\widetilde{\nabla}$ (resp. $\widetilde{\Delta}$) is the Lefschetz raising (resp. lowering) operator, we use the notation LR_{ij} (resp. LL_{ji}) instead of R_{ij} (resp. L_{ji}). In particular, $LR_{i,i+1}$ is the *incidence matrix* of rank $i+1$ versus rank i elements, and the entry in p's column (resp. row) and q's row (resp. column) of LR_{ij} (resp. LL_{ji}) equals the number of saturated chains from p to q, $p \in N_i$, $q \in N_j$, which can be easily derived from (6.4). Note that, as above, $LL_{ji} = LR_{ij}^{\mathrm{T}}$. We call the matrices LR_{ij} the *Lefschetz matrices of P*.

Let $\ker_{ij}(\widetilde{\nabla})$ be the *kernel* of $\widetilde{\nabla}_{ij}$, that is, $\ker_{ij}(\widetilde{\nabla}) := \{\varphi \in \widetilde{N}_i : \widetilde{\nabla}_{ij}(\varphi) = \widetilde{0}\}$, and let $\operatorname{rank}_{ij}(\widetilde{\nabla})$ be the *rank* of $\widetilde{\nabla}_{ij}$, that is, the dimension of $\widetilde{\nabla}_{ij}(\widetilde{N}_i)$. Often we use the shorthand notation $\ker_{ij}$ and rank_{ij}. From linear algebra we know that $\ker_{ij}$ is a subspace of $\widetilde{N}_i$ and that rank_{ij} equals the matrix theoretical rank of the matrix R_{ij} and, moreover,

$$\operatorname{rank}_{ij} + \dim \ker_{ij} = \dim \widetilde{N}_i = W_i, \tag{6.5}$$

$$\operatorname{rank}_{ij} = \operatorname{rank} R_{ij} \leq \min\{W_i, W_j\}. \tag{6.6}$$

If $\operatorname{rank}_{ij} = \min\{W_i, W_j\}$ we say that R_{ij} and $\widetilde{\nabla}_{ij}$ have *full rank*. Moreover, we say that $\widetilde{\nabla}$ has the *full rank property* if all $\widetilde{\nabla}_{ij}$, $0 \leq i \leq j \leq n$, have full rank. The definitions for lowering operators are analogous. For the sake of brevity, only raising operators are mentioned in the following. From (6.2), (6.5), (6.6), and Lemma 6.1.1 we derive immediately:

Lemma 6.1.2.

(a) $W_i \leq W_j$ and $\widetilde{\nabla}_{ij}$ has full rank iff $\widetilde{\nabla}_{ij}$ is injective,

(b) $W_i \geq W_j$ and $\widetilde{\nabla}_{ij}$ has full rank iff $\widetilde{\nabla}^*_{ji}$ is injective,

(c) $\widetilde{\nabla}$ is of full rank iff $\widetilde{\nabla}^*$ is of full rank.

Lemma 6.1.3. *If there exists in $\widetilde{P}$ a raising operator $\widetilde{\nabla}$ of full rank, then P is rank unimodal.*

Proof. Assume that there are $i \leq h \leq j$ such that $W_h < \min\{W_i, W_j\}$. Then by (6.4) $R_{ij} = R_{hj}R_{ih}$, implying $\min\{W_i, W_j\} = \operatorname{rank}_{ij} \leq \min\{\operatorname{rank} R_{hj}, \operatorname{rank} R_{ih}\} \leq W_h$, a contradiction. ∎

Later we will see that the full rank property is a criterion for the strong Sperner property. But let us look first at a weaker condition that already implies the Sperner property.

Theorem 6.1.1. *Let P be rank unimodal. If there exists an order-raising operator $\widetilde{\nabla}$ in $\widetilde{P}$ such that $\widetilde{\nabla}_{i,i+1}$ is of full rank for all $0 \leq i \leq n-1$, then P has the Sperner property.*

Proof. Let W_h be the largest Whitney number and let $S \subseteq N_i$. We show that for $i < h$, $|S| \leq |\nabla(S)|$, and for $i > h$, $|S| \leq |\Delta(S)|$. By Theorems 5.1.1 and 5.1.2 this is sufficient for the proof. Indeed, if $i < h$, $W_i \leq W_{i+1}$ implying that $\widetilde{\nabla}_{i,i+1}$ is injective. Since $\widetilde{\nabla}$ is order raising, $\widetilde{\nabla}_{i,i+1}(\widetilde{S})$ is a subspace of $\widetilde{\nabla(S)}$. Hence

$$|S| = \dim \widetilde{S} = \dim \widetilde{\nabla}_{i,i+1}(\widetilde{S}) \leq \dim \widetilde{\nabla(S)} = |\nabla(S)|.$$

If $i > h$, we may argue analogously using $\widetilde{\nabla}^*_{i+1,i}$. ■

For a product of two chains, there is a nice way to compute the rank of Lefschetz matrices:

Theorem 6.1.2. *Let $P = (0 \lessdot 1 \lessdot \cdots \lessdot n_1) \times (0 \lessdot 1 \lessdot \cdots \lessdot n_2)$. Then $\widetilde{\nabla}_L$ has the full rank property.*

Proof. Though this result can be obtained by calculating determinants with binomial coefficients as entries in an elementary, but nontrivial way, we will do a little bit more by giving a combinatorial interpretation of corresponding determinants. This approach is due to Gessel and Viennot [222]. Let $n := r(P) = n_1 + n_2$. First we show that the square matrices $LR_{k,n-k}$, $0 \leq k \leq \frac{n}{2}$, have nonzero determinant, that is, full rank. We index the elements in the kth resp. $(n-k)$th level from left to right according to Figure 6.1. Thus we have the elements $p_0, \ldots, p_k$ (resp. $q_0, \ldots, q_k$). Let $\mathfrak{C}_{k,n-k}$ be the set of all saturated chains C starting at some point $s(C)$ in the kth level and ending at some point $e(C)$ in the $(n-k)$th level. For each $C \in \mathfrak{C}_{k,n-k}$, let $A(C)$ be a $((k+1) \times (k+1))$-matrix whose entry $a_{ij}(C)$ equals 1 if $s(C) = p_j$ and $e(C) = q_i$, and 0 otherwise. Obviously,

$$LR_{k,n-k} = \sum_{C \in \mathfrak{C}_{k,n-k}} A(C).$$

By Leibniz's definition of a determinant

$$\begin{aligned} &\det LR_{k,n-k} \\ &= \sum_{\pi \in S_{k+1}} \chi(\pi) \left(\sum_{C \in \mathfrak{C}_{k,n-k}} a_{0,\pi(0)}(C) \right) \cdots \left(\sum_{C \in \mathfrak{C}_{k,n-k}} a_{k,\pi(k)}(C) \right) \\ &= \sum_{(C_0,\ldots,C_k) \in \mathfrak{C}^{k+1}_{k,n-k}} \sum_{\pi \in S_{k+1}} \chi(\pi) a_{0,\pi(0)}(C_0) \cdots a_{k,\pi(k)}(C_k), \end{aligned}$$

where S_{k+1} is the symmetric group on $\{0, \ldots, k\}$ and $\chi(\pi)$ is the character of π. It is easy to see that we may restrict the first sum to the set $\mathfrak{D}$ of $(k+1)$-tuples $(C_0, \ldots, C_k)$ for which $e(C_i) = q_i$ holds and for which the set of all starting points is the whole kth level. For each $(C_0, \ldots, C_k) \in \mathfrak{D}$, there exists exactly

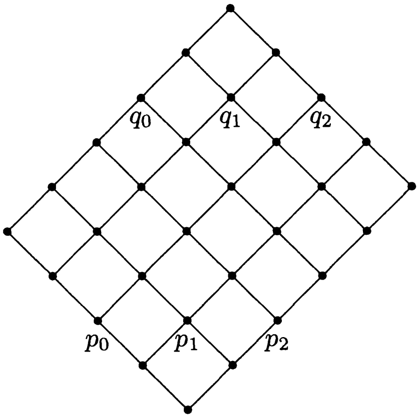


Figure 6.1

one permutation $\pi \in S_{k+1}$ such that $a_{0,\pi(0)}(C_0)\dots a_{k,\pi(k)}(C_k) = 1$ (we need $s(C_i) = p_{\pi(i)}, i = 0, \dots, k$), all other items in the second sum are zero. Let $\operatorname{sgn}(C_0, \dots, C_k)$ be the character of that permutation. So we have

$$\det LR_{k,n-k} = \sum_{(C_0,\dots,C_k)\in\mathfrak{D}} \operatorname{sgn}(C_0, \dots, C_k).$$

Now we define a mapping $\tau : \mathfrak{D} \to \mathfrak{D}$ in the following way: Let $(C_0, \dots, C_k) \in \mathfrak{D}$. If the chains $C_0, \dots, C_k$ are pairwise disjoint, let $\tau(C_0, \dots, C_k) := (C_0, \dots, C_k)$. Since we indexed the elements in the corresponding levels from the left to the right, this can be the case only if the corresponding permutation π is the identity – that is, only if $\operatorname{sgn}(C_0, \dots, C_k) = 1$. If the chains $C_0, \dots, C_k$ are not pairwise disjoint, look at all intersection points and choose the lexicographically smallest point p (first minimize the first coordinate, then the second coordinate). Let α be the least integer for which C_α and C_γ intersect in p for some $\gamma \neq \alpha$, and let β be the least integer for which C_α and C_β intersect in p. We construct C_α^* by first following C_α to p and then following C_β to the end. Similarly we construct C_β^* by following C_β to p and then C_α to the end. Now let

$$\tau(C_0, \dots, C_\alpha, \dots, C_\beta, \dots, C_k) := (C_0, \dots, C_\beta^*, \dots, C_\alpha^*, \dots, C_k).$$

Noting that the set of intersection points is invariant with respect to τ, it is easy to see that τ^2 is the identity map, thus τ is an *involution*. Moreover, in the case of intersection

$$\operatorname{sgn}(\tau(C_0, \dots, C_k)) = -\operatorname{sgn}(C_0, \dots, C_k)$$

since the corresponding new permutation is obtained by transposing two elements in the corresponding original permutation. Thus

$$\begin{aligned}\det LR_{k,n-k} &= \sum_{\substack{(C_0,\dots,C_k)\in\mathfrak{D}\\ \text{pairwise nonintersecting}}} 1 + \sum_{\substack{(C_0,\dots,C_k)\in\mathfrak{D}\\ \text{with intersections}}} \operatorname{sgn}(C_0,\dots,C_k)\\ &= |\{(C_0,\dots,C_k)\in\mathfrak{D} : \text{pairwise nonintersecting}\}|\\ &\quad+\frac{1}{2}\sum_{\substack{(C_0,\dots,C_k)\in\mathfrak{D}\\ \text{with intersections}}} (\operatorname{sgn}(C_0,\dots,C_k)+\operatorname{sgn}(\tau(C_0,\dots,C_k)))\\ &= |\{(C_0,\dots,C_k)\in\mathfrak{D} : \text{pairwise nonintersecting}\}|.\end{aligned}$$

Thus we obtained a nice combinatorial interpretation of our determinant. From our results on symmetric chain partitions of products (see the proof of Theorem 5.1.5), we know that there really exist $k+1$ pairwise nonintersecting saturated chains from the kth to the $(n-k)$th level; hence

$$\det LR_{k,n-k} \neq 0.$$

Let us now consider all matrices LR_{ij}, $0 \le i \le j \le n$. We assume that $i+j \le n$, since the other case can be treated analogously. Obviously, $W_i \le W_j$. From the symmetric case solved previously and the rank inequality for products of matrices, we obtain $W_i = \operatorname{rank} LR_{i,n-i} = \operatorname{rank} LR_{j,n-i} LR_{ij} \le \operatorname{rank} LR_{ij} \le W_i$; that is, $\operatorname{rank} LR_{ij} = W_i$. ■

Now we will use this result to calculate the ranks of our matrices (resp. operators) for products of arbitrary ranked posets. To do this, we need several preparations. Let $\widetilde{\nabla}$ be a raising operator in $\widetilde{P}$. A set $S = \{b^i, b^{i+1}, \dots, b^j\}$ of elements of $\widetilde{P}$ is called a *ranked Jordan string from $\widetilde{N}_i$ to $\widetilde{N}_j$ with respect to $\widetilde{\nabla}$* if $b^k \in \widetilde{N}_k$, $k = i, \dots, j$, and $\widetilde{\nabla}(b^k) = b^{k+1}$, $k = i, \dots, j-1$, $\widetilde{\nabla}(b^j) = \widetilde{0}$. Note that a ranked Jordan string generates an invariant subspace of $\widetilde{P}$ with respect to $\widetilde{\nabla}$, denoted in the following by V_{ijh} (h enumerates the ranked Jordan strings from $\widetilde{N}_i$ to $\widetilde{N}_j$). A basis of $\widetilde{P}$ that is a union of ranked Jordan strings is said to be a *ranked Jordan basis of $\widetilde{P}$ with respect to $\widetilde{\nabla}$*. Soon we will see that such a basis really exists. But let us first define

$$s_{ij}(\widetilde{\nabla}) := \begin{cases} \operatorname{rank}_{ij} + \operatorname{rank}_{i-1,j+1} - \operatorname{rank}_{i-1,j} - \operatorname{rank}_{i,j+1} & \text{if } 0 \le i \le j \le n,\\ 0 & \text{otherwise}\end{cases}$$

(if the operator $\widetilde{\nabla}$ is clear from the context we will write briefly s_{ij}).

Lemma 6.1.4. *In every ranked Jordan basis B of $\widetilde{P}$ with respect to $\widetilde{\nabla}$ there are exactly s_{ij} ranked Jordan strings from $\widetilde{N}_i$ to $\widetilde{N}_j$ with respect to $\widetilde{\nabla}$, where $0 \le i \le j \le n$.*

Proof. Let $B_i := B \cap \widetilde{N}_i$. Obviously, B_i is a basis of $\widetilde{N}_i$ and the elements of B_i belong to ranked Jordan strings starting in some $\widetilde{N}_u$ with $u \le i$. Obviously, $\widetilde{\nabla}_{ij}(\widetilde{N}_i)$ is generated by the elements $\widetilde{\nabla}_{ij}(b)$ with $b \in B_i$. We have $\widetilde{\nabla}_{ij}(b) = \widetilde{0}$ if b belongs to a ranked Jordan string ending in some $\widetilde{N}_v$ with $v < j$; otherwise $\widetilde{\nabla}_{ij}(b)$ belongs to B. Consequently, $\operatorname{rank}_{ij} = \dim \widetilde{\nabla}_{ij}(\widetilde{N}_i) =$ number of ranked Jordan strings from $\widetilde{N}_u$ to $\widetilde{N}_v$ with $u \le i, v \ge j$. Hence, $\operatorname{rank}_{ij} - \operatorname{rank}_{i,j+1}$ equals the number of ranked Jordan strings from $\widetilde{N}_u$ to $\widetilde{N}_j$ with $u \le i$ and finally $s_{ij} = (\operatorname{rank}_{ij} - \operatorname{rank}_{i,j+1}) - (\operatorname{rank}_{i-1,j} - \operatorname{rank}_{i-1,j+1})$ equals the number of ranked Jordan strings from $\widetilde{N}_i$ to $\widetilde{N}_j$, $0 \le i \le j \le n$. ■

Theorem 6.1.3. *Let $\widetilde{\nabla}$ be a raising operator in $\widetilde{P}$.*

(a) We have $s_{ij} \ge 0$ for all i, j.

(b) There exists a ranked Jordan basis of $\widetilde{P} = \widetilde{N}_0 \oplus \cdots \oplus \widetilde{N}_n$ with respect to $\widetilde{\nabla}$; that is, $\widetilde{P}$ can be written in the form $\widetilde{P} = \bigoplus_{0 \le i \le j \le n} \bigoplus_{h=1}^{s_{ij}} V_{ijh}$ where each V_{ijh} is a subspace generated by one of the s_{ij} ranked Jordan strings from $\widetilde{N}_i$ to $\widetilde{N}_j$.

Proof. Let $0 \le i \le j \le n$. Obviously, $\ker_{ij} \subseteq \ker_{i,j+1}$; hence we may look at the factor space $\ker_{i,j+1} / \ker_{ij}$. Its elements $\overline{v}$ are of the form $v + \ker_{ij}$ where $v \in \ker_{i,j+1}$ (v is a representative). Furthermore, $\widetilde{\nabla}(\ker_{i-1,j}) \subseteq \ker_{ij}$. So we define the mapping $\overline{\widetilde{\nabla}_{ij}} : \ker_{i-1,j+1} / \ker_{i-1,j} \to \ker_{i,j+1} / \ker_{ij}$ by $\overline{\widetilde{\nabla}_{ij}}(\overline{v}) := \overline{\widetilde{\nabla}(v)}$ where $v \in \ker_{i-1,j+1}$. This mapping is well defined since, for $v - w \in \ker_{i-1,j}$, we have $\widetilde{\nabla}(v - w) = \widetilde{\nabla}(v) - \widetilde{\nabla}(w) \in \ker_{ij}$. Furthermore $\overline{\widetilde{\nabla}_{ij}}$ is injective since $\overline{\widetilde{\nabla}_{ij}}(\overline{v}) = \overline{\widetilde{0}}$ iff $\widetilde{\nabla}(v) \in \ker_{ij}$ iff $v \in \ker_{i-1,j}$ iff $\overline{v} = \overline{\widetilde{0}}$. Consequently,

$$\begin{aligned}\dim \overline{\widetilde{\nabla}_{ij}}(\ker_{i-1,j+1} / \ker_{i-1,j}) &= \dim \ker_{i-1,j+1} / \ker_{i-1,j} \\ &= \dim \ker_{i-1,j+1} - \dim \ker_{i-1,j}\,.\end{aligned}$$

Let $\overline{\overline{B_{ij}}}$ be a basis of $\ker_{i,j+1} / \ker_{ij} / \overline{\widetilde{\nabla}_{ij}}(\ker_{i-1,j+1} / \ker_{i-1,j})$. Then

$$\begin{aligned}|\overline{\overline{B_{ij}}}| &= \dim \ker_{i,j+1} - \dim \ker_{ij} - \dim \ker_{i-1,j+1} + \dim \ker_{i-1,j} \\ &= \operatorname{rank}_{ij} + \operatorname{rank}_{i-1,j+1} - \operatorname{rank}_{i-1,j} - \operatorname{rank}_{i,j+1} = s_{ij}.\end{aligned}$$

In particular, $0 \le |\overline{\overline{B_{ij}}}| = s_{ij}$, so (a) is proved.

Now take for each element of $\overline{\overline{B_{ij}}}$ (which is a class of classes) one representative lying in $\ker_{i,j+1} / \ker_{ij}$ (which is a class), and then take for this class again a representative lying in $\ker_{i,j+1}$. This procedure gives us first a set $\overline{B_{ij}}$ and then a set B_{ij} of elements of $\widetilde{N}_i$ where $|B_{ij}| = s_{ij}$. Let

$$B := \bigcup_{0 \le i \le j \le n} \bigcup_{b \in B_{ij}} \{b, \widetilde{\nabla}(b), \ldots, \widetilde{\nabla}^{j-i}(b)\}.$$

We have finally to show that B is the desired ranked Jordan basis. First of all, the set $\{b, \dots, \widetilde{\nabla}^{j-i}(b)\}$, $b \in B_{ij}$, is a ranked Jordan string from $\widetilde{N}_i$ to $\widetilde{N}_j$ because (1) its elements belong to $\widetilde{N}_i, \dots, \widetilde{N}_j$, (2) they are not the zero vector because of $b \notin \ker_{ij}$, and (3) $\widetilde{\nabla}(\widetilde{\nabla}^{j-i}(b)) = \widetilde{\nabla}^{j+1-i}(b) = \widetilde{\nabla}_{i,j+1}(b) = \widetilde{0}$ because of $b \in \ker_{i,j+1}$.

In the following we often use the fact that, if U is a subspace of a vector space V and B_U is a basis of U and $B_{V/U}$ is a set of representatives of all elements from a basis of the factor space V/U, then $B_U \cup B_{V/U}$ is a basis of V. It is sufficient to show that $B \cap \widetilde{N}_i$ is a basis of $\widetilde{N}_i$ for $0 \le i \le n$. We proceed by induction on i. First note that for all i

$$\{\widetilde{0}\} = \ker_{ii} \subseteq \ker_{i,i+1} \subseteq \cdots \subseteq \ker_{i,n+1} = \widetilde{N}_i.$$

In order to realize the induction we prove a little bit more:

Claim. For all $0 \le i < j \le n+1$, the set

$$B \cap \ker_{ij} = \bigcup_{l=i}^{j-1} \left(B_{il} \cup \{\widetilde{\nabla}(b) : b \in B \cap (\ker_{i-1,l+1} - \ker_{i-1,l})\}\right)$$

is a basis of $\ker_{ij}$ and $B \cap (\ker_{ij} - \ker_{i,j-1})$ is a set of representatives of a basis of $\ker_{ij} / \ker_{i,j-1}$.

From this claim we obtain in particular that $B \cap \ker_{i,n+1} = B \cap \widetilde{N}_i$ is a basis of $\widetilde{N}_i$ and we are done.

Proof of Claim. Let $i = 0$. Note that $\ker_{-1,j} = \{\widetilde{0}\}$ for all j. So we have only to prove that $\cup_{l=0}^{j-1} B_{0,l}$ is a basis of $\ker_{0,j}$, but this follows easily by induction on j. In particular, $B_{0,j-1} = B \cap (\ker_{0,j} - \ker_{0,j-1})$, $j \ge 1$, is a set of representatives of a basis of $\ker_{0,j} / \ker_{0,j-1}$ by construction.

Now consider the step $i-1 \to i$. We prove our claim by induction on j. Usually we have to start with $j = i+1$. But we turn immediately to the induction step $j \to j+1$ since the arguments are analogous. The set $B \cap \ker_{i,j+1}$ can be obtained from $B \cap \ker_{ij}$ by adding the elements from $B_{ij} \cup \{\widetilde{\nabla}(b) : b \in B \cap (\ker_{i-1,j+1} - \ker_{i-1,j})\}$. Since by induction (on i) $B \cap (\ker_{i-1,j+1} - \ker_{i-1,j})$ is a set of representatives of a basis of $\ker_{i-1,j+1} / \ker_{i-1,j}$ and since $\overline{\widetilde{\nabla}}_{i,j}$ is injective, the set $\{\overline{\widetilde{\nabla}(b)} : b \in B \cap (\ker_{i-1,j+1} - \ker_{i-1,j})\}$ is a set of representatives of a basis of $\overline{\widetilde{\nabla}}_{i,j}(\ker_{i-1,j+1} / \ker_{i-1,j})$ that is a subset of $\ker_{i,j+1} / \ker_{i,j}$. This basis together with $\overline{B_{ij}}$ forms a basis of $\ker_{i,j+1} / \ker_{ij}$. Thus $B \cap (\ker_{i,j+1} - \ker_{ij}) = B_{ij} \cup \{\widetilde{\nabla}(b) : b \in B \cap (\ker_{i-1,j+1} - \ker_{i-1,j})\}$ is indeed a set of representatives of a basis of $\ker_{i,j+1} / \ker_{ij}$, and these elements together with a basis of $\ker_{ij}$ form a basis of $\ker_{i,j+1}$. ■

We call a subspace V of $\widetilde{P}$ a *ranked subspace* if V is the direct sum of subspaces of $\widetilde{N}_i$, $i = 0, \dots, n$ – that is, if $V = \oplus_{i=0}^{n} V \cap \widetilde{N}_i$. If, moreover, V is invariant with respect to some raising operator $\widetilde{\nabla}$, then by Theorem 6.1.3 there exists a ranked

Jordan basis of V (in the proof of Theorem 6.1.3 we must only always take $V \cap \widetilde{N}_i$ instead of $\widetilde{N}_i$).

With a raising operator $\widetilde{\nabla}$ in $\widetilde{P}$ we associate now its *Jordan function* $J(\widetilde{\nabla}, \widetilde{P}; x, y)$ defined by

$$J(\widetilde{\nabla}, \widetilde{P}; x, y) := \sum_{0 \le i \le j \le n} s_{ij}(x^i + x^{i+1} + \ldots + x^j)y^{i+j},$$

that is, a polynomial in the variables x and y with nonnegative (by Theorem 6.1.3(a)) integer coefficients. For an invariant ranked subspace V of $\widetilde{P}$, we define the Jordan function $J(\widetilde{\nabla}, V; x, y)$ in the same way; except that here s_{ij} counts the number of ranked Jordan strings from $\widetilde{N}_i$ to $\widetilde{N}_j$ belonging to a ranked Jordan basis of V. Let us write $J(\widetilde{\nabla}, \widetilde{P}; x, y)$ in the form

$$J(\widetilde{\nabla}, \widetilde{P}; x, y) = \sum_{k,l} a_{kl}(\widetilde{\nabla})x^k y^l,$$

where here and in the following, the summation is extended over all pairs of integers (resp. over all integers) (there is only a finite number of nonzero items). We call the numbers $a_{kl} = a_{kl}(\widetilde{\nabla})$ the *Jordan coefficients of* $\widetilde{\nabla}$, whereas the numbers $s_{ij} = s_{ij}(\widetilde{\nabla})$ are said to be the *string numbers of* $\widetilde{\nabla}$. Directly from the definitions and Theorem 6.1.3(a) we obtain:

Lemma 6.1.5. *The Jordan coefficients of $\widetilde{\nabla}$ have the following properties:*

(*a*) $a_{kl} = 0$ *if not* $0 \le k \le l \le k + n \le 2n$,

(*b*) $0 \le a_{kl} = a_{l-k,l} \le a_{k+1,l} = a_{l-k-1,l}$ *for all* $0 \le 2k < l \le 2n$,

(*c*) $a_{kl} = \sum_{i \le \min\{k,l-k\}} s_{i,l-i}$.

Knowing the Jordan coefficients, we may determine the string numbers, and from both we may derive the rank numbers:

Lemma 6.1.6. *Let a_{kl}, $k, l \in \mathbb{Z}$, be the Jordan coefficients of $\widetilde{\nabla}$. Then*

(*a*) $s_{ij} = a_{i,i+j} - a_{i-1,i+j}$ *if* $i \le j$,

(*b*) $rank_{ij} = \sum_{u \le i, v \ge j} s_{uv}$,

(*c*) $rank_{ij} = \sum_{l \le i+j} a_{jl} + \sum_{l \ge i+j+1} a_{il}$ *if* $i \le j$.

Proof. (a) If $i \le j$ then by Lemma 6.1.5(c)

$$a_{i,i+j} - a_{i-1,i+j} = \sum_{u \le i} s_{u,i+j-u} - \sum_{u \le i-1} s_{u,i+j-u} = s_{ij}.$$

(b) This fact follows from the proof of Lemma 6.1.4. It can also be derived by the observation that every other item is canceled in the summation if we write s_{uv} in the form given by its definition.

(c) From (a), (b), and Lemma 6.1.5(b) it follows that for $i \le j$:

$$\begin{aligned}
\mathrm{rank}_{ij} &= \sum_{u\le i, v\ge j} a_{u,u+v} - a_{u-1,u+v} \\
&= \sum_{u\le i, l-u\ge j} a_{ul} - a_{u-1,l} = \sum_{l} \sum_{u\le \min\{i,l-j\}} a_{ul} - a_{u-1,l} \\
&= \sum_{l-j\le i} a_{l-j,l} + \sum_{l-j>i} a_{il} \\
&= \sum_{l\le i+j} a_{jl} + \sum_{l\ge i+j+1} a_{il}.
\end{aligned}$$

■

The preceding results hold analogously for invariant ranked subspaces. Furthermore, if V is the direct sum of invariant ranked subspaces, $V = V_1 \oplus V_2$, then evidently, V is also an invariant ranked subspace and directly from the definition it follows that

$$J(\widetilde{\nabla}, V_1 \oplus V_2; x, y) = J(\widetilde{\nabla}, V_1; x, y) + J(\widetilde{\nabla}, V_2; x, y). \tag{6.7}$$

Let us consider chains of the form $C_{ij} := (i < i+1 < \cdots < j)$, where $r(l) = l,\ i \le l \le j$. Here make an exception from the usual condition that $r(p) = 0$ for some minimal element p. Obviously,

$$J(\widetilde{\nabla}_L, \widetilde{C}_{ij}; x, y) = (x^i + x^{i+1} + \cdots + x^j)y^{i+j}.$$

Moreover, by Theorem 6.1.2, $\widetilde{\nabla}_L$ has in $C_{i_1,j_1} \times C_{i_2,j_2}$ the full rank property. Hence, for $i_1 + i_2 \le k \le l \le j_1 + j_2$,

$$\mathrm{rank}_{kl} = \begin{cases} W_k & \text{if } k + l \le i_1 + i_2 + j_1 + j_2, \\ W_l & \text{otherwise} \end{cases}$$

(note that this chain product is rank symmetric and rank unimodal). Consequently,

$$s_{kl} = \begin{cases} W_k - W_{k-1} & \text{if } k + l = i_1 + i_2 + j_1 + j_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\begin{aligned}
(x^{i_1} + \cdots + x^{j_1})(x^{i_2} + \cdots + x^{j_2}) &= W_{i_1+i_2}x^{i_1+i_2} + W_{i_1+i_2+1}x^{i_1+i_2+1} \\
&\quad + \cdots + W_{j_1+j_2}x^{j_1+j_2},
\end{aligned}$$

we obtain

$$J(\widetilde{\nabla}_L, C_{i_1,j_1} \times C_{i_2,j_2}; x, y) = J(\widetilde{\nabla}_L, C_{i_1,j_1}; x, y)J(\widetilde{\nabla}_L, C_{i_2,j_2}; x, y). \tag{6.8}$$

This product property can be generalized to direct products of ranked posets. In order to avoid the notion of the tensor product, let $\iota : \widetilde{P_1} \times \widetilde{P_2} \to \widetilde{P_1 \times P_2}$ be the bilinear mapping defined by

$$\iota\left(\sum_{p_1 \in P_1} \alpha_{p_1}\widetilde{p_1}, \sum_{p_2 \in P_2} \beta_{p_2}\widetilde{p_2}\right) := \sum_{(p_1,p_2)\in P_1\times P_2} \alpha_{p_1}\beta_{p_2}\widetilde{(p_1, p_2)}.$$

Since $\dim \widetilde{P_1 \times P_2} = |P_1 \times P_2| = |P_1||P_2| = \dim \widetilde{P_1} \dim \widetilde{P_2}$, it follows easily by proving linear independence that for given bases B_m of $\widetilde{P}_m$, $m = 1, 2$, the set $B := \{\iota(b_1, b_2) : b_1 \in B_1, b_2 \in B_2\}$ is a basis of $\widetilde{P_1 \times P_2}$. For the sake of brevity, we omit in the following the letter ι and write (b_1, b_2) instead of $\iota(b_1, b_2)$ and so on. If V_m is a subspace of $\widetilde{P}_m$ and B'_m is a basis of V_m, $m = 1, 2$, then let $V_1 \times V_2$ be the subspace of $\widetilde{P_1 \times P_2}$ generated by $\{(b_1, b_2) : b_m \in B'_m, m = 1, 2\}$ (this definition is independent of the basis). In particular we can then write $\widetilde{P_1} \times \widetilde{P_2}$ instead of $\widetilde{P_1 \times P_2}$.

For linear operators $\Phi_m : \widetilde{P}_m \to \widetilde{P}_m$, $m = 1, 2$, we define their product $\Phi_1 \times \Phi_2 : \widetilde{P_1 \times P_2} \to \widetilde{P_1 \times P_2}$ on the basis $\{\widetilde{(p_1, p_2)} : (p_1, p_2) \in P_1 \times P_2\}$ by

$$(\Phi_1 \times \Phi_2)(\widetilde{(p_1, p_2)}) := (\Phi_1(p_1), \Phi_2(p_2)).$$

If $\widetilde{\nabla}_m$ is an (order-) raising operator and $\widetilde{I}_m$ the identity operator on $\widetilde{P}_m$, $m = 1, 2$, then obviously $\widetilde{\nabla}_1 \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_2$ is an (order-) raising operator on $\widetilde{P_1 \times P_2}$. Moreover, for the Lefschetz operators we have

$$\widetilde{\nabla}_L(P_1 \times P_2) = \widetilde{\nabla}_L(P_1) \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_L(P_2),$$

and analogous results hold for lowering operators.

Theorem 6.1.4. *Let $\widetilde{\nabla}_m$ be raising operators on $\widetilde{P}_m$, $m = 1, 2$. Then*

$$J(\widetilde{\nabla}_1 \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_2, \widetilde{P_1 \times P_2}; x, y) = J(\widetilde{\nabla}_1, \widetilde{P}_1; x, y)J(\widetilde{\nabla}_2, \widetilde{P}_2; x, y).$$

Proof. Roughly speaking, we are concerned with the generating function statement that a product of two direct sums of strings is the sum of the products. If we have, for example, $J(\widetilde{\nabla}_1, \widetilde{P}_1; x, y) = (x^0 + x^1 + x^2 + x^3)y^3 + (x^3 + x^4)y^7$ and $J(\widetilde{\nabla}_2, \widetilde{P}_2; x, y) = (x^0 + x^1 + x^2)y^2$, then multiplication with the result $(x^0 + x^1 + x^2 + x^3 + x^4 + x^5)y^5 + (x^1 + x^2 + x^3 + x^4)y^5 + (x^2 + x^3)y^5 + (x^3 + x^4 + x^5 + x^6)y^9 + (x^4 + x^5)y^9$ can be illustrated as in Figure 6.2. Briefly, let $P := P_1 \times P_2$ and $\widetilde{\nabla} := \widetilde{\nabla}_1 \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_2$. Let us write each $\widetilde{P}_m$, $m = 1, 2$, in the form of Theorem 6.1.3, where each $V_{i_m j_m h_m}$ is generated by some ranked Jordan string $\{b_m, \widetilde{\nabla}_m(b_m), \ldots, \widetilde{\nabla}_m^{j_m - i_m}(b_m)\}$ (b_m depends also on i_m, j_m, h_m)). It is easy to see that $V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2}$ is an invariant ranked subspace of $\widetilde{P}$ with

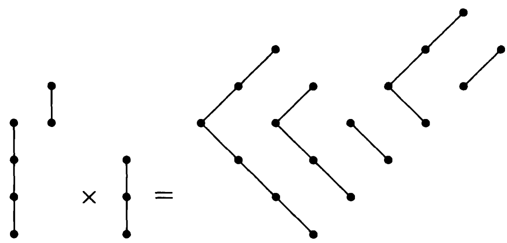


Figure 6.2

respect to $\widetilde{\nabla}$. We have

$$J(\widetilde{\nabla}_m, V_{i_m j_m h_m}; x, y) = (x^{i_m} + \cdots + x^{j_m}) y^{i_m + j_m}. \tag{6.9}$$

As before, let $C_{i_m j_m} := (i_m \lessdot \cdots \lessdot j_m)$ and $D := C_{i_1 j_1} \times C_{i_2 j_2}$. To calculate the Jordan function of $V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2}$ with respect to $\widetilde{\nabla}$, consider the linear mapping $f : V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2} \to \widetilde{D}$ given by

$$f(\widetilde{\nabla}_1^{u_1}(b_1), \widetilde{\nabla}_2^{u_2}(b_2)) := \widetilde{(u_1, u_2)}, \qquad 0 \le u_m \le j_m - i_m, \quad m = 1, 2.$$

Of course, f is bijective. But the most important fact is that f and the raising operators commute in the following sense:

$$f\widetilde{\nabla} = \widetilde{\nabla}_L(D) f, \quad \text{i.e.,} \quad f^{-1}(\widetilde{\nabla}_L(D)) f = \widetilde{\nabla}.$$

Consequently, applying f^{-1} to a ranked Jordan basis of $\widetilde{D}$ with respect to $\widetilde{\nabla}_L(D)$ gives a ranked Jordan basis of $V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2}$ with respect to $\widetilde{\nabla}$ with the same string numbers. In view of (6.8) and (6.9) we obtain

$$\begin{aligned} J(\widetilde{\nabla}, V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2}; x, y) &= J(\widetilde{\nabla}_L, \widetilde{D}; x, y) \\ &= J(\widetilde{\nabla}_1, V_{i_1 j_1 h_1}; x, y) J(\widetilde{\nabla}_2, V_{i_2 j_2 h_2}; x, y). \end{aligned} \tag{6.10}$$

For the sake of brevity, let for $m = 1, 2$,

$$Z_m := \{(i_m, j_m, h_m) : 0 \le i_m \le j_m \le r(P_m), \qquad 1 \le h_m \le s_{i_m j_m}(\widetilde{\nabla}_m)\}.$$

Now we have

$$\widetilde{P} = \bigoplus_{\substack{(i_1, j_1, h_1) \in Z_1 \\ (i_2, j_2, h_2) \in Z_2}} V_{i_1 j_1 h_1} \times V_{i_2 j_2 h_2}.$$

From (6.7) and (6.10) we finally obtain

$$
\begin{aligned}
&J(\widetilde{\nabla}, \widetilde{P}; x, y) \\
&= \sum_{\substack{(i_1,j_1,h_1)\in Z_1 \\ (i_2,j_2,h_2)\in Z_2}} J(\widetilde{\nabla}, V_{i_1j_1h_1} \times V_{i_2j_2h_2}; x, y) \\
&= \sum_{\substack{(i_1,j_1,h_1)\in Z_1 \\ (i_2,j_2,h_2)\in Z_2}} J(\widetilde{\nabla}_1, V_{i_1j_1h_1}; x, y) J(\widetilde{\nabla}_2, V_{i_2j_2h_2}; x, y) \\
&= \left(\sum_{(i_1,j_1,h_1)\in Z_1} J(\widetilde{\nabla}_1, V_{i_1j_1h_1}; x, y) \right) \left(\sum_{(i_2,j_2,h_2)\in Z_2} J(\widetilde{\nabla}_2, V_{i_2j_2h_2}; x, y) \right) \\
&= J(\widetilde{\nabla}_1, \widetilde{P}_1; x, y) J(\widetilde{\nabla}_2, \widetilde{P}_2; x, y).
\end{aligned}
$$

■

As in the case of normality, the situation is easier for rankwise direct products. Let P_m be ranked posets, $m = 1, 2$, and let $n := r(P_1) = r(P_2)$. As for direct products we use, for $i = 0, \ldots, n$, the short notation

$$
\left(\sum_{p_1 \in N_i(P_1)} \alpha_{p_1} \widetilde{p}_1, \sum_{p_2 \in N_i(P_2)} \beta_{p_2} \widetilde{p}_2 \right) := \sum_{(p_1,p_2)\in N_i(P_1 \times_r P_2)} \alpha_{p_1} \beta_{p_2} \widetilde{(p_1, p_2)}.
$$

If B_m is a ranked basis of $\widetilde{P}_m$, $m = 1, 2$, then $B := \cup_{i=0}^n \{(b_1, b_2) : b_1 \in \widetilde{N}_i(P_1) \cap B_1, b_2 \in \widetilde{N}_i(P_2) \cap B_2\}$ is a ranked basis of $\widetilde{P_1 \times_r P_2}$. Moreover, if $\Phi_m : \widetilde{P}_m \to \widetilde{P}_m$, $m = 1, 2$, are linear operators we may define $\Phi_1 \times_r \Phi_2 : \widetilde{P_1 \times_r P_2} \to \widetilde{P_1 \times_r P_2}$ on the basis $\cup_{i=0}^n \{\widetilde{(p_1, p_2)} : (p_1, p_2) \in N_i(P_1) \times N_i(P_2)\}$, $i = 0, \ldots, n$, by

$$
(\Phi_1 \times_r \Phi_2)(\widetilde{(p_1, p_2)}) := (\Phi_1(p_1), \Phi_2(p_2)), \qquad p_1 \in N_i(P_1), \quad p_2 \in N_i(P_2).
$$

If $\widetilde{\nabla}_m$ is an (order-) raising operator on $\widetilde{P}_m$, $m = 1, 2$, then $\widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2$ is an (order-) raising operator on $\widetilde{P_1 \times_r P_2}$; moreover,

$$
\widetilde{\nabla}_L(P_1 \times_r P_2) = \widetilde{\nabla}_L(P_1) \times_r \widetilde{\nabla}_L(P_2),
$$

and analogous relations hold for lowering operators.

Theorem 6.1.5. *Let $\widetilde{\nabla}_m$ be raising operators on $\widetilde{P}_m$, $m = 1, 2$, and let $r(P_1) = r(P_2)$.*

(*a*) *For $i \le j$, $\operatorname{rank}_{ij}(\widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2) = \operatorname{rank}_{ij}(\widetilde{\nabla}_1) \operatorname{rank}_{ij}(\widetilde{\nabla}_2)$.*

(*b*) *If $\widetilde{\nabla}_1$ and $\widetilde{\nabla}_2$ have the full rank property, then $\widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2$ has the full rank property, too.*

Proof. Clearly (b) follows from (a), so let us prove (a). Let $P := P_1 \times_r P_2$ and $\widetilde{\nabla} := \widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2$. Choose for each $\widetilde{\nabla}_m$, $m = 1, 2$, a ranked Jordan basis B_m with respect to $\widetilde{\nabla}_m$. Then we obtain a ranked Jordan basis B of $\widetilde{P}$ with respect to $\widetilde{\nabla}$ in the following way: Take all pairs of Jordan strings from B_1 (resp. B_2), for instance, the Jordan strings $\{b_m^{i_m}, b_m^{i_m+1}, \dots, b_m^{j_m}\}$, $m = 1, 2$. Let $i := \max\{i_1, i_2\}$, $j := \min\{j_1, j_2\}$, and form $\{(b_1^i, b_2^i), (b_1^{i+1}, b_2^{i+1}), \dots, (b_1^j, b_2^j)\}$. Then obviously $(b_1^k, b_2^k) \in \widetilde{N}_k(P_1 \times_r P_2)$, $\widetilde{\nabla}((b_1^k, b_2^k)) = (b_1^{k+1}, b_2^{k+1})$, $k = i, \dots, j-1$, and $\widetilde{\nabla}((b_1^j, b_2^j)) = \widetilde{0}$; that is, we obtain ranked Jordan strings. The union B of these strings has the form $B = \cup_{i=0}^n \{(b_1, b_2) : b_1 \in \widetilde{N}_i(P_1) \cap B_1,\ b_2 \in \widetilde{N}_i(P_2) \cap B_2\}$; that is, it is a ranked Jordan basis. The construction gives a ranked Jordan string meeting $\widetilde{N}_i(P_1 \times_r P_2)$ and $\widetilde{N}_j(P_1 \times_r P_2)$ iff the taken ranked Jordan strings also meet $\widetilde{N}_i(P_1)$ and $\widetilde{N}_j(P_1)$ (resp. $\widetilde{N}_i(P_2)$ and $\widetilde{N}_j(P_2)$). By Lemma 6.1.6(b) we have $\operatorname{rank}_{ij}(\widetilde{\nabla}_1)$ (resp. $\operatorname{rank}_{ij}(\widetilde{\nabla}_2)$) (independent) choices. Consequently, we obtain exactly $\operatorname{rank}_{ij}(\widetilde{\nabla}_1)\operatorname{rank}_{ij}(\widetilde{\nabla}_2)$ ranked Jordan strings meeting $\widetilde{N}_i(P_1 \times_r P_2)$ and $\widetilde{N}_j(P_1 \times_r P_2)$, and by Lemma 6.1.6(b) this number equals $\operatorname{rank}_{ij}(\widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2)$. ∎

The proof of Theorem 6.1.2 motivates the following definition. We say that the ranked poset P of rank $n := r(P)$ has *property* T if for all $0 \le i \le j \le n$ there exist $\min\{W_i, W_j\}$ pairwise disjoint saturated chains starting at some point in the ith level and ending at some point in the jth level. The following theorem is mainly due to Stanley [439] and Griggs [238] ((i) $\leftrightarrow$ (ii)); see also [391].

Theorem 6.1.6. *Let P be a ranked poset. The following conditions are equivalent:*
(i) P is rank unimodal and has the strong Sperner property,
(ii) P has property T*,*
(iii) there exists an order-raising operator $\widetilde{\nabla}$ on $\widetilde{P}$ with the full rank property,
(iv) there exists an order-lowering operator $\widetilde{\Delta}$ on $\widetilde{P}$ with the full rank property.

Before proving this theorem, we need a lemma that is interesting for itself.

Lemma 6.1.7. *If P has the strong Sperner property then for any $S \subseteq [0, n]$ the S-rank-selected subposet P_S has the strong Sperner property, too.*

Proof. We order the Whitney numbers using a permutation π of $[0, n] : W_{\pi(0)} \le \cdots \le W_{\pi(n)}$. Let $S = \{\pi(i_1), \dots, \pi(i_l)\}$ with $i_1 > \cdots > i_l$ and let $1 \le k \le l$. Assume that P_S has not the k-Sperner property. Then there is some k-family F_k in P_S such that $|F_k| > \sum_{j=1}^k W_{\pi(i_j)}$. But $F' := (\cup_{j=i_k}^n N_{\pi(j)} - \cup_{j=1}^k N_{\pi(i_j)}) \cup F_k$ is an $(n - i_k + 1)$-family of size greater than $\sum_{j=i_k}^n W_{\pi(j)}$, contradicting the $(n - i_k + 1)$-Sperner property of P. ∎

Proof of Theorem 6.1.6. (i) $\to$ (ii). Let $0 \le i \le j \le n$. By Lemma 6.1.7, in particular the $[i, j]$-rank selected subposet has the $(j - i)$-Sperner property and by Lemma 4.5.1(a) the 1-cutset property. Moreover $\min\{W_i, W_j\}$ is the smallest Whitney number of that subposet by rank unimodality. Now Corollary 4.2.2(b) implies the existence of $\min\{W_i, W_j\}$ maximal chains in that subposet.

(ii) $\to$ (iii). We proceed by induction on the rank n of P. If $n = 0$ we do not have to prove anything. Let $n = 1$. By the supposition, there exist $\min\{W_0, W_1\}$ pairwise disjoint chains $(p_j \lessdot q_j)$, $p_j \in N_0, q_j \in N_1$, $j = 1, \ldots, \min\{W_0, W_1\}$. If we define $\widetilde{\nabla}$ by

$$\widetilde{\nabla}(\widetilde{p}) := \begin{cases} \widetilde{q}_j & \text{if } p = p_j \text{ for some } j = 1, \ldots, \min\{W_0, W_1\}, \\ \widetilde{0} & \text{otherwise,} \end{cases}$$

then $\widetilde{\nabla}$ is obviously an order-raising operator with the full rank property. Now consider the step $n - 1 \to n$ $(n \geq 2)$. Suppose first that $W_0 \leq W_n$. Let P' be the $[1, n]$-rank selected subposet of P. Obviously, P' has property T, and by the induction hypothesis, there exists an order-raising operator $\widetilde{\nabla}'$ on $\widetilde{P}'$ with the full rank property. We extend $\widetilde{\nabla}'$ to an order-raising operator on $\widetilde{P}$ by setting $\widetilde{\nabla}'(\widetilde{p}) := \widetilde{0}$ if $p \in N_0$. By the supposition there exist W_0 pairwise disjoint chains $(p_{0,h} \lessdot \cdots \lessdot p_{n,h})$ such that $p_{k,h} \in N_k, k = 0, \ldots, n, h = 1, \ldots, W_0$. We define the order-raising operator $\widetilde{\nabla} = \widetilde{\nabla}(\alpha)$ on $\widetilde{P}$ depending on a real number α (to be defined later) in the following way: Let

$$\widetilde{\nabla}(\widetilde{p}) := \begin{cases} \widetilde{\nabla}'(\widetilde{p}) + \alpha \widetilde{p}_{k+1,h} & \text{if } p = p_{k,h} \ (k = 0, \ldots, n-1, h = 1, \ldots W_0), \\ \widetilde{\nabla}'(\widetilde{p}) & \text{otherwise.} \end{cases}$$

Obviously, $\widetilde{\nabla}$ is an order-raising operator for any $\alpha \in \mathbb{R}$. We have to prove that α can be chosen in such a way that $\widetilde{\nabla}$ has the full rank property. Let $1 \leq i \leq j \leq n$ and consider the associated matrix $R_{ij} = R_{ij}(\alpha)$. Since the entries of $R_{i,i+1}$ are linear functions of α and $R_{ij} = R_{j-1,j} \cdots R_{i+1,i+2} R_{i,i+1}$, the entries of R_{ij} are polynomials in α of degree not greater than $j - i$, in particular not greater than n. By the choice of $\widetilde{\nabla}'$ there exists in R_{ij} a minor of order $\min\{W_i, W_j\}$ which has nonzero determinant if $\alpha = 0$. Since this determinant is a polynomial in α of degree not greater than $n \min\{W_i, W_j\}$, it is nonzero for all but a finite number of values of α. Thus all matrices R_{ij}, that is, all operators $\widetilde{\nabla}_{ij}$, $1 \leq i \leq j \leq n$, have full rank for all but a finite number of values of α. Now consider $R_{0,j}$ and $\widetilde{\nabla}_{0,j}$, $1 \leq j \leq n$. If we write $\widetilde{\nabla}_{0,j}(\widetilde{p}_{0,k})$, $k = 1, \ldots, W_0$, as a linear combination of the basis vectors of $\widetilde{N}_j$, it is easy to see that the coefficient of $\widetilde{p}_{j,k}$ is a polynomial in α of degree j with leading coefficient 1, and all other coefficients are polynomials in α of degree less than j. Thus the minor of $R_{0,j}$ (of order W_0) which is formed by the columns indexed by $p_{0,k}$ and rows indexed by $p_{j,k}$, $k = 1, \ldots, W_0$, has the property that in each row and each column there exists exactly one entry that is a polynomial in α of degree j while all other entries are polynomials in α less than j. Consequently its determinant is a polynomial in α of degree jW_0 (and not the zero polynomial). It is nonzero; that is, $R_{0,j}$ and $\widetilde{\nabla}_{0,j}$ are of full rank for all but a finite number of values of α. Since we have only a finite number of exceptions, we find some $\alpha \in \mathbb{R}$ such that $\widetilde{\nabla}$ is of full rank. If $W_0 \geq W_n$ we may argue similarly using the $[0, n-1]$-rank-selected subposet.

(iii) $\leftrightarrow$ (iv). This equivalence follows from Lemma 6.1.2(c).

(iii) $\rightarrow$ (i). By Lemma 6.1.3, P is rank unimodal. Let $C_k := (0 \lessdot 1 \lessdot \cdots \lessdot k-1)$ and $P_k := P \times C_k$. By Theorem 4.3.1 we have $d_k(P) = d_1(P_k)$. Since each Whitney number of P_k equals the sum of at most k Whitney numbers of P, it is enough to show that P_k has the Sperner property for each k. Let $W_0(P) \leq \cdots \leq W_h(P) > W_{h+1}(P) \geq \cdots \geq W_n(P)$. Since $W_i(P_k) = W_{i-(k-1)}(P) + \cdots + W_i(P)$, we have $W_i(P_k) - W_{i-1}(P_k) = W_i(P) - W_{i-k}(P)$. Let h_k be the largest index such that this difference is nonnegative. Obviously, $h_k - k < h \leq h_k$. Let $\widetilde{\nabla}_k := \widetilde{\nabla} \times \widetilde{\nabla}_L(C_k)$.

Claim. We have

(a) $\operatorname{rank}_{i,i+1}(\widetilde{\nabla}_k) = W_i(P_k)$ if $i < h_k$.

(b) $\operatorname{rank}_{i,i+1}(\widetilde{\nabla}_k) = W_{i+1}(P_k)$ if $i \geq h_k$.

These two equalities imply that P_k is rank unimodal and that $\widetilde{\nabla}_{k_{i,i+1}}$ is of full rank for all i. Theorem 6.1.1 then gives the Sperner property and we are done.

Proof of Claim. We prove only (a) since (b) can be proved in the same way. Let $i < h_k$. In view of Theorem 6.1.4 we have

$$J(\widetilde{\nabla}_k, P_k; x, y) = \left(\sum_{m,l} a_{m,l}(\widetilde{\nabla}) x^m y^l\right)\left(1 + x + \cdots + x^{k-1}\right) y^{k-1}.$$

Hence the Jordan coefficients of $\widetilde{\nabla}_k$ are

$$a_{uv}(\widetilde{\nabla}_k) = a_{u,v-(k-1)}(\widetilde{\nabla}) + a_{u-1,v-(k-1)}(\widetilde{\nabla}) + \cdots + a_{u-(k-1),v-(k-1)}(\widetilde{\nabla}).$$

From this equality and Lemma 6.1.6(c) we may derive that our Claim(a), $\operatorname{rank}_{i,i+1}(\widetilde{\nabla}_k) = \operatorname{rank}_{i,i}(\widetilde{\nabla}_k)$ for $i < h_k$, is equivalent to

$$\sum_{l \leq 2i+1} a_{i+1,l}(\widetilde{\nabla}_k) + \sum_{l \geq 2i+2} a_{i,l}(\widetilde{\nabla}_k) = \sum_{l} a_{i,l}(\widetilde{\nabla}_k),$$

$$\sum_{l \leq 2i+1} (a_{i+1,l}(\widetilde{\nabla}_k) - a_{i,l}(\widetilde{\nabla}_k)) = 0,$$

$$\sum_{l \leq 2i+1} (a_{i+1,l-(k-1)}(\widetilde{\nabla}) - a_{i-(k-1),l-(k-1)}(\widetilde{\nabla})) = 0,$$

$$\sum_{l \leq 2i+1-(k-1)} (a_{i+1,l}(\widetilde{\nabla}) - a_{i-(k-1),l}(\widetilde{\nabla})) = 0. \tag{6.11}$$

By the choice of h_k we have $W_{i-(k-1)}(P) \leq W_{i+1}(P)$, and since $\widetilde{\nabla}$ is of full rank, it follows that

$$\operatorname{rank}_{i-(k-1),i+1}(\widetilde{\nabla}) = \operatorname{rank}_{i-(k-1),i-(k-1)}(\widetilde{\nabla})$$

which is in view of Lemma 6.1.6(c) equivalent to

$$\sum_{l\le 2i+1-(k-1)} a_{i+1,l}(\tilde{\nabla}) + \sum_{l\ge 2i+2-(k-1)} a_{i-(k-1),l}(\tilde{\nabla}) = \sum_l a_{i-(k-1),l}(\tilde{\nabla}),$$

$$\sum_{l\le 2i+1-(k-1)} (a_{i+1,l}(\tilde{\nabla}) - a_{i-(k-1),l}(\tilde{\nabla})) = 0.$$

Consequently, (6.11) and hence our claim are proved. ■

Theorem 6.1.7. *Let P be a ranked poset and $\tilde{\nabla}$ an order-raising operator on $\tilde{P}$. If $\tilde{\nabla}_{0,n}$ is injective, then there exist W_0 pairwise disjoint maximal chains in P.*

Proof. First note that W_0 is the smallest Whitney number in P since for all $0 \le i \le n$, $\tilde{\nabla}_{0,n} = \tilde{\nabla}_{i,n}\tilde{\nabla}_{0,i}$ and thus $\tilde{\nabla}_{0,i}$ is injective, implying $W_0 \le W_i$. Using the same arguments as in the proof (i) → (ii) of Theorem 6.1.6, we see that it is enough to prove that P has the n-Sperner property, and looking at the proof (iii) → (i) of that theorem, we see that we must prove the Sperner property of $P_n = P \times C_n$. But this follows also in the same way as in the aforementioned proof. We have here $k = n$, $h_k = n$, and for $i := n - 1$ we need our supposition $\operatorname{rank}_{0,n}(\tilde{\nabla}) = \operatorname{rank}_{0,0}(\tilde{\nabla})$. ■

The full rank property is preserved neither by direct product nor by rankwise direct product (consider the Lefschetz operators in the posets of Figure 6.3). If

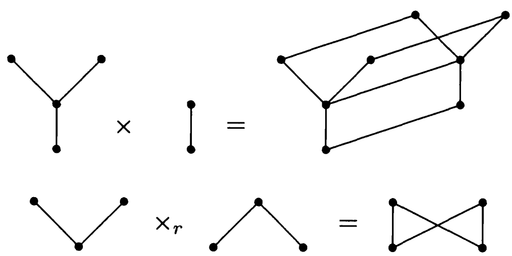


Figure 6.3

we have the additional condition that the posets are rank symmetric, then the full rank property is preserved by these operations but such posets we study in detail in the next section. In the case of rankwise direct product, a weaker condition is sufficient.

Theorem 6.1.8. *Let P_1, P_2 be ranked posets with $n := r(P_1) = r(P_2)$ and let $\tilde{\nabla}_1$, $\tilde{\nabla}_2$ be raising operators in $\tilde{P}_1$ and $\tilde{P}_2$, respectively. Moreover, let for all $i \le j$, $W_i(P_1) \le W_j(P_1)$ iff $W_i(P_2) \le W_j(P_2)$. Then the raising operator $\tilde{\nabla}_1 \times_r \tilde{\nabla}_2$*

on $\widetilde{P_1 \times_r P_2}$ has the full rank property if both $\widetilde{\nabla}_1$ and $\widetilde{\nabla}_2$ have the full rank property.

Proof. Let, w.l.o.g., $W_i(P_1 \times_r P_2) \le W_j(P_1 \times_r P_2), 0 \le i \le j \le n$. Since $W_k(P_1 \times_r P_2) = W_k(P_1)W_k(P_2), k = i, j$, by our condition $W_i(P_1) \le W_j(P_1)$ and $W_i(P_2) \le W_j(P_2)$. Using Theorem 6.1.5 and the full rank property of $\widetilde{\nabla}_1$ and $\widetilde{\nabla}_2$ we obtain

$$\begin{aligned} \operatorname{rank}_{ij}(\widetilde{\nabla}_1 \times_r \widetilde{\nabla}_2) &= \operatorname{rank}_{ij}(\widetilde{\nabla}_1)\operatorname{rank}_{ij}(\widetilde{\nabla}_2) \\ &= W_i(P_1)W_i(P_2) = W_i(P_1 \times_r P_2). \end{aligned}$$

■

Example 6.1.1. *The poset $SQ_{k,n}$ of subcubes of a cube has the full rank property, and consequently $SQ_{k,n}$ is strongly Sperner since it is a rankwise direct product of "halved" posets $S(k, k)$ and here the Lefschetz operator has the full rank property by Theorem 6.1.2.*

Let us finally study the full rank property with respect to quotients. We say that a group G of automorphisms of the ranked poset P is *rank preserving* if $r(p) = r(\varphi(p))$ for all $p \in P$ and $\varphi \in G$. Note that by our definition of a rank function not every automorphism group is rank preserving; see Figure 6.4. If G is rank

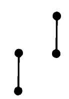


Figure 6.4

preserving we may define a rank function $r_{P/G}$ for the quotient P/G by

$$r_{P/G}([p]) := r(p),\ p \in P$$

where $[p]$ denotes the orbit containing p. The following theorem has its origin in a work of Pouzet [384] who proved the Sperner property of the graph poset G_n (see also Pouzet and Rosenberg [385]). Proofs were given by Harper [260] and Stanley [440].

Theorem 6.1.9. *Let G be a rank-preserving group of automorphisms of the ranked poset P and suppose that the Lefschetz raising operator $\widetilde{\nabla}_L$ on $\widetilde{P}$ has the full rank property. Then there exists an order-raising operator $\widetilde{\nabla}$ in $\widetilde{P/G}$ with the full rank property.*

Proof. The proof of the Weighted Quotient Theorem 4.5.5 mentioned that whenever $[p] \lessdot_{P/G} [q]$, the poset induced by $\{p^* \in P : p^* \in [p]\} \cup \{q^* \in P : q^* \in [q]\}$ is regular. We define for $[p] \lessdot_{P/G} [q]$,

$$c([p],[q]) := |\{q^* \in [q] : p \lessdot q^*\}|$$

(by the regularity $c([p],[q])$ is independent of the representative p). Moreover, we define the order-raising operator $\widetilde{\nabla}$ by

$$\widetilde{\nabla}(\widetilde{[p]}) := \sum_{[q]:[q]\gtrdot[p]} c([p],[q])\widetilde{[q]}$$

and we will show that $\widetilde{\nabla}$ has the full rank property. Let us define linear mappings $f : \widetilde{P/G} \to \widetilde{P}$ and $g : \widetilde{P/G} \to \widetilde{P}$ by

$$f(\widetilde{[p]}) := \frac{1}{|[p]|}\sum_{p'\in[p]} \widetilde{p}' \text{ and } g(\widetilde{[p]}) := \sum_{p'\in[p]} \widetilde{p}'.$$

Obviously, f and g are injective.

Claim. We have

(a) $f\widetilde{\nabla} = \widetilde{\nabla}_L f$,

(b) $g\widetilde{\nabla}^* = \widetilde{\Delta}_L g$.

Proof of Claim. (a) We have for all $[p] \in P/G$

$$f\widetilde{\nabla}(\widetilde{[p]}) = \sum_{[q]:[q]\gtrdot[p]} \frac{c([p],[q])}{|[q]|}\sum_{q'\in[q]} \widetilde{q}',$$

$$\widetilde{\nabla}_L f(\widetilde{[p]}) = \frac{1}{|[p]|}\sum_{p^*\in[p]}\sum_{q':q'\gtrdot p^*} \widetilde{q}'.$$

In both cases we have linear combinations of elements $\widetilde{q}'$ with the property $[q'] \gtrdot [p]$. The coefficient of $\widetilde{q}'$ equals

$$\frac{|\{q^* \in [q'] : p \lessdot q^*\}|}{|[q']|} \text{ for } f\widetilde{\nabla}(\widetilde{[p]}),$$

and

$$\frac{|\{p^* \in [p] : p^* \lessdot q'\}|}{|[p]|} \text{ for } \widetilde{\nabla}_L f(\widetilde{[p]}),$$

respectively. But these numbers are equal to each other by the regularity of the poset induced by $\{p^* \in P : p^* \in [p]\} \cup \{q^* \in P : q^* \in [q']\}$ (count the number of pairs (p^*, q^*) with $p^* \lessdot q^*$, $p^* \in [p]$, $q^* \in [q']$ in two different ways). So (a) is proved.

(b) As in (a) $g\widetilde{\nabla}^*(\widetilde{[p]})$ and $\widetilde{\Delta}_L g(\widetilde{[p]})$ are linear combinations of elements $\widetilde{q}'$ with $[q'] \lessdot [p]$. The coefficient of $\widetilde{q}'$ equals in both cases $|\{p^* \in [p] : p^* \gtrdot q'\}|$, so (b) is proved. □

From our claim it follows for $k \geq 0$ that

$$f\widetilde{\nabla}^k = \widetilde{\nabla}_L^k f \text{ and } g\widetilde{\nabla}^{*^k} = \widetilde{\Delta}_L^k g.$$

Let $0 \leq i \leq j \leq r(P)$. Then we have in particular

$$f\widetilde{\nabla}_{ij} = \widetilde{\nabla}_{L_{ij}} f \text{ and } g\widetilde{\nabla}^*_{ji} = \widetilde{\Delta}_{L_{ji}} g.$$

If $W_i(P) \leq W_j(P)$, then $\widetilde{\nabla}_{L_{ij}}$ is injective by supposition and Lemma 6.1.2(a). Since also f is injective, $\widetilde{\nabla}_{ij}$ must be injective, too – that is, of full rank. If $W_i \geq W_j$ then $\widetilde{\Delta}_{L_{ji}}$ is injective by supposition and Lemma 6.1.1(b) and Lemma 6.1.2(b). Since g is injective, $\widetilde{\nabla}^*_{ji}$ must be injective; that is, by Lemma 6.1.2(b), $\widetilde{\nabla}_{ij}$ is of full rank. ■

Applications of this theorem will be presented in the next section. We conclude this section with a proposition we will need in Section 6.4.

Proposition 6.1.1. *Let P be a ranked poset for which there exists a group G of automorphisms of P whose orbits are the levels of P, and let $0 \leq i \leq j \leq r(P)$. If $\operatorname{rank}_{ij}(\widetilde{\nabla}_L) < W_j$, then $\widetilde{p} \notin \widetilde{\nabla}_{L_{ij}}(\widetilde{N}_i)$ for all $p \in N_j$.*

Proof. Each element g of G defines a linear mapping $\gamma_g : \widetilde{P} \to \widetilde{P}$ by $\gamma_g(\widetilde{p}) := \widetilde{g(p)}$ for all $p \in P$. We have

$$\gamma_g \widetilde{\nabla}_L = \widetilde{\nabla}_L \gamma_g \text{ for all } g \in G$$

since for any $p \in P$

$$\gamma_g \widetilde{\nabla}_L(\widetilde{p}) = \sum_{q:q>p} \widetilde{g(q)} = \sum_{q:q>g(p)} \widetilde{q} = \widetilde{\nabla}_L \gamma_g(\widetilde{p}).$$

Consequently,

$$\gamma_g \widetilde{\nabla}_{L_{ij}}(\varphi) = \widetilde{\nabla}_{L_{ij}} \gamma_g(\varphi) \text{ for all } \varphi \in \widetilde{N}_i. \tag{6.12}$$

Assume that for some $p \in N_j$ there is a $\varphi \in \widetilde{N}_i$ such that $\widetilde{\nabla}_{L_{ij}}(\varphi) = \widetilde{p}$. Then for each $q \in N_j$, there exists by our supposition an element g of G such that $q = g(p)$. From (6.12) we obtain $\widetilde{q} = \gamma_g(\widetilde{p}) = \gamma_g \widetilde{\nabla}_{L_{ij}}(\varphi)$; that is, $\widetilde{q} \in \widetilde{\nabla}_{L_{ij}}(\widetilde{N}_i)$. Thus $\widetilde{N}_j = \widetilde{\nabla}_{L_{ij}}(\widetilde{N}_i)$, which contradicts $W_j > \operatorname{rank}_{ij}(\widetilde{\nabla}_L)$. ■

This proposition can be applied to B_n and $L_n(q)$ (take the automorphisms generated by permutations of the elements (resp. by bijective linear transformations)).

6.2. Peck posets and the commutation relation

In the preceding section we learned that in order to prove the strong Sperner property it is enough to find an order-raising (resp. order-lowering) operator with the full rank property. A first approach is given in Theorem 6.1.9. In this section we mainly restrict ourselves to rank-symmetric posets, for which it is easier to find such operators with the full rank property. In honor of the "dummy" mathematician G.W. Peck and his best friends Graham, West, Purdy, Erdős, Chung, and Kleitman, a ranked poset P is called a *Peck poset* if it is rank symmetric, rank unimodal, and if it has the strong Sperner property. One finds equivalent conditions in Theorem 6.1.6 where one has always to add rank symmetry. In particular, P is Peck iff it is rank symmetric and there exists an order-raising operator $\widetilde{\nabla}$ with the full rank property. If, in particular, the Lefschetz raising operator has the full rank property, then we speak (in the case of rank symmetry) of a *unitary Peck poset.*

Lemma 6.2.1. *Let P be a ranked poset of rank n. It is Peck iff there exists an order-raising operator $\widetilde{\nabla}$ such that for the Jordan function there holds $J(\widetilde{\nabla}, \widetilde{P}; x, y) = F(P; x)y^n$ where $F(P; x)$ is the rank-generating function of P. The poset P is unitary Peck iff $J(\widetilde{\nabla}_L, \widetilde{P}; x, y) = F(P; x)y^n$.*

Proof. Let P be Peck. Then we find a $\widetilde{\nabla}$ with the full rank property. We have $s_{ij} = 0$ if $i + j \neq n, i \leq j$. We verify this for $i + j < n$, the case $i + j > n$ is analogous. By Theorem 6.1.3 and Lemma 6.1.6, and in view of the supposition,

$$0 \geq W_i - W_{j+1} = \operatorname{rank}_{i,i} - \operatorname{rank}_{i,j+1} = \sum_{u \leq i, i \leq v \leq j} s_{uv} \geq s_{ij} \geq 0;$$

that is, $s_{ij} = 0$. Consequently (using Lemma 6.1.5(b)),

$$\operatorname{rank}_{jj} = \sum_{u \leq j, v \geq j} s_{uv} = \sum_{i \leq j, n-i \geq j} s_{i,n-i} = \sum_{i=0}^{\min\{j,n-j\}} s_{i,n-i},$$

and further

$$\begin{aligned} J(\widetilde{\nabla}, \widetilde{P}; x, y) &= \sum_{0 \leq i \leq \frac{n}{2}} s_{i,n-i}(x^i + \cdots + x^{n-i})y^n \\ &= \sum_{j=0}^{n} \left(\sum_{i=0}^{\min\{j,n-j\}} s_{i,n-i} \right) x^j y^n = \left(\sum_{j=0}^{n} \operatorname{rank}_{jj} x^j \right) y^n \\ &= \left(\sum_{j=0}^{n} W_j x^j \right) y^n = F(P; x)y^n. \end{aligned}$$

Let, conversely, $J(\widetilde{\nabla}, \widetilde{P}; x, y) = F(P; x)y^n$ for some order-raising operator $\widetilde{\nabla}$. Then

$$a_{kl}(\widetilde{\nabla}) = \begin{cases} 0 & \text{if } l \neq n, \\ W_k & \text{if } l = n. \end{cases}$$

From Lemma 6.1.5(b) we derive $0 \leq a_{0,n} = a_{n,n} \leq a_{1,n} = a_{n-1,n} \leq \cdots$; that is, $W_0 = W_n \leq W_1 = W_{n-1} \leq \cdots$, and thus P is rank symmetric and rank unimodal. By Lemma 6.1.6(c) we have for $0 \leq i \leq j \leq n$, in the case $i + j < n$, $\text{rank}_{ij} = a_{in} = W_i = \min\{W_i, W_j\}$, and in the case $i + j \geq n$, $\text{rank}_{ij} = a_{jn} = W_j = \min\{W_i, W_j\}$; that is, $\widetilde{\nabla}_{ij}$ has full rank. The unitary case follows easily from the general case. ■

Now we may derive quickly the following product theorem which was first proved by Canfield [95], and then in different ways by Proctor, Saks, and Sturtevant [391], by Saks [406] in a more general version, and by Proctor [388].

Theorem 6.2.1 (Peck Product Theorem). *If P_1 and P_2 are (unitary) Peck posets, then $P_1 \times P_2$ is a (unitary) Peck poset, too.*

Proof. Let $\widetilde{\nabla}_m$ be the corresponding order-raising operator of full rank on $\widetilde{P}_m$. Applying Lemma 6.2.1 and Theorem 6.1.4, we obtain for the order-raising operator $\widetilde{\nabla}_1 \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_2$ on $\widetilde{P_1 \times P_2}$:

$$\begin{aligned} J(\widetilde{\nabla}_1 \times \widetilde{I}_2 + \widetilde{I}_1 \times \widetilde{\nabla}_2; \widetilde{P_1 \times P_2}; x, y) &= J(\widetilde{\nabla}_1, \widetilde{P}_1; x, y) J(\widetilde{\nabla}_2, \widetilde{P}_2; x, y) \\ &= F(P_1; x) y^{r(P_1)} F(P_2; x) y^{r(P_2)} \\ &= F(P_1 \times P_2; x) y^{r(P_1 \times P_2)}; \end{aligned}$$

that is, by Lemma 6.2.1, $P_1 \times P_2$ is (unitary) Peck. ■

Example 6.2.1. *Since chains are clearly unitary Peck, we conclude from Theorem 6.2.1 that B_n and more generally $S(k_1, \ldots, k_n)$ are unitary Peck.*

The full rank of the Lefschetz (and, equivalently, incidence) matrices of B_n was discovered by Gottlieb [226] and, independently, by Kantor [284], who considered $L_n(q)$ as well.

A consequence of Theorem 6.1.9 is the following quotient theorem:

Theorem 6.2.2 (Peck Quotient Theorem). *Let P be a unitary Peck poset and G a rank-preserving group of automorphisms of P. Then the quotient P/G is a Peck poset.*

Proof. The proof of Theorem 6.1.9 already presented an order-raising operator $\widetilde{\nabla}$ in $\widetilde{P/G}$ with the full rank property. It remains to show that P/G is rank symmetric.

Since P is Peck, $W_i(P) = W_{n-i}(P)$, $0 \le i \le \frac{n}{2}$. In the end of the proof of Theorem 6.1.9 we found (with $j := n - i$) that $\widetilde{\nabla}_{i,n-i}$ and $\widetilde{\nabla}^*_{n-i,i}$ are injective; that is, $W_i(P/G) \le W_{n-i}(P/G)$ and $W_{n-i}(P/G) \le W_i(P/G)$, implying equality. ■

Let us mention that the quotient of a Peck poset is not necessarily a Peck poset and the quotient of a unitary Peck poset is not necessarily a unitary Peck poset. Examples are contributed by Stanley and can be found in [260].

Now we have the promised applications:

Example 6.2.2. *Let $P = S(m, \dots, m)$. To each permutation π of $\{1, \dots, n\}$ there corresponds an automorphism $\overline{\pi}$ of P defined by $\overline{\pi}(a_1, \dots, a_n) := (a_{\pi(1)}, \dots, a_{\pi(n)})$. Let G be the group of all such automorphisms of P. It is not difficult to see that $P/G \cong L(m, n)$. Since chain products are unitary Peck, $L(m, n)$ is a Peck poset.*

Example 6.2.3. *Let V be a set of n elements and P be the powerset of the set of all two-element subsets of V. Then an element of P can be considered as a (labeled) simple graph with vertex set V. We order P by inclusion. Then $P \cong B_{\binom{n}{2}}$. To every permutation π of V there corresponds a permutation $\overline{\pi}$ of $N_1(P)$ defined by $\overline{\pi}(\{x, y\}) := \{\pi(x), \pi(y)\}$, which generates an automorphism of P. Then each orbit can be considered as a simple graph on n unlabeled vertices; that is, $P/G \cong G_n$, the graph poset. Since $B_{\binom{n}{2}}$ is unitary Peck, G_n is Peck.*

In a straightforward way we obtain from Theorem 6.1.5 a further product theorem.

Theorem 6.2.3 (Peck Rankwise Product Theorem). *If P_1 and P_2 are (unitary) Peck posets, then $P_1 \times_r P_2$ is a (unitary) Peck poset, too.*

Example 6.2.4. *The poset of square submatrices of a square matrix $SM_{k,n}$ is unitary Peck.*

Though the poset of subsquares of a square $SQ_{k,n}$ is not Peck (it is not rank symmetric, because in the product representation the factors are not rank symmetric) we mention here that these posets are rank unimodal and strongly Sperner, which can be easily concluded from Theorem 6.1.2, Theorem 6.1.5, and Theorem 6.1.6.

From the definition of a Peck poset, Lemma 5.1.1 and Theorem 5.1.4, it follows that an sc-order is Peck. Now we introduce a notion that generalizes semi-sc-orders. We say that a ranked poset P of rank n is *semi-Peck* if there exists an order-raising operator $\widetilde{\nabla}$ such that $\text{rank}_{ij}(\widetilde{\nabla}) = W_i$ for all $0 \le i \le j \le n$ with $i + j \le n$ (again we add the word *unitary* if the preceding holds for $\widetilde{\nabla}_L$).

Proposition 6.2.1. *If P is a semi-sc-order, then it is semi-Peck.*

Proof. Let $p \in P$ and let C be the semisymmetric chain containing p. We set $\widetilde{\nabla}(\widetilde{p}) := \widetilde{0}$ if p is the maximal element of C and otherwise define $\widetilde{\nabla}(\widetilde{p}) := \widetilde{q}$, where q is the element of C that covers p. Then it is easy to see that $\operatorname{rank}_{ij} = W_i$ for $i + j \leq n$. ■

Theorem 5.4.1 remains valid also for this more general case.

Proposition 6.2.2. *If P is semi-Peck, then $W_0 \leq W_1 \leq \cdots \leq W_{\lceil \frac{n}{2} \rceil}$, and the $[0, \lceil \frac{n}{2} \rceil]$-rank selected subposet Q has the strong Sperner property.*

Proof. For $0 \leq i < \lceil \frac{n}{2} \rceil$, we have $W_i = \dim \widetilde{\nabla}_{i,i+1}(\widetilde{N_i}) \leq \dim \widetilde{N}_{i+1} = W_{i+1}$. The strong Sperner property now follows from Theorem 6.1.1. ■

Lemma 6.2.2. *Let P be a ranked poset of rank n. It is semi-Peck iff there is an order-raising operator $\widetilde{\nabla}$ such that y^n divides $J(\widetilde{\nabla}, \widetilde{P}; x, y)$.*

Proof. We have $s_{ij} = 0$ if $i + j < n$, since by Theorem 6.1.3(a) and Lemma 6.1.6(b), $0 = \operatorname{rank}_{ii} - \operatorname{rank}_{i,j+1} = \sum_{u \leq i, i \leq v \leq j} s_{uv} \geq s_{ij} \geq 0$. ■

In view of Theorem 6.1.4 and Lemma 6.2.2 there follows immediately:

Theorem 6.2.4. *If P_1 and P_2 are (unitary) semi-Peck posets, then $P_1 \times P_2$ is a (unitary) semi-Peck poset, too.*

From Theorem 6.1.5 we conclude that an analogous result holds for the rankwise direct product, too.

Example 6.2.5. *The posets F_k^n as well as the duals of the posets Q_n, $Int(S(k_1, \ldots, k_n))$, $M(k_1, \ldots, k_n)$, $SQ_{k,n}$ are semi-Peck.*

Now return to the symmetric case; that is, to Peck posets. What can we do if there is given a poset for which there does not exist a product or quotient representation as above? We may try to work with commutation relations as follows. Let P be a ranked poset of rank n and let $\widetilde{\nabla}$ and $\widetilde{\Delta}$ be a raising and lowering operator on $\widetilde{P}$, respectively. We say that the pair $(\widetilde{\nabla}, \widetilde{\Delta})$ has the *commutation property*, briefly *property* C (see [161]), if there are real numbers $\mu_0, \ldots, \mu_n$ such that

$$(\widetilde{\nabla}\widetilde{\Delta} - \widetilde{\Delta}\widetilde{\nabla})(\widetilde{p}) = \mu_{r(p)}\widetilde{p} \quad \text{for all } p \in P.$$

If in particular $(\widetilde{\nabla}_L, \widetilde{\Delta})$ has property C and if P has a unique minimal element, then P is called by Stanley [442] *μ-differential.* If in particular $\mu_i = 2i - n$, $i = 0, \ldots, n$, then we say that $(\widetilde{\nabla}, \widetilde{\Delta})$ has the *Lie-property.*

Example 6.2.6. *$(\widetilde{\nabla}_L, \widetilde{\Delta}_L)$ has the Lie-property if P is the Boolean lattice B_n, and it has property* C *with $\mu_i = \binom{i}{1}_q - \binom{n-i}{1}_q$ if P is the linear lattice $L_n(q)$. We verify this for the latter case: Take $U \in N_i(L_n(q))$ – that is,* $\dim U = i$ *– and write $\widetilde{\nabla}_L\widetilde{\Delta}_L(\widetilde{U})$ and $\widetilde{\Delta}_L\widetilde{\nabla}_L(\widetilde{U})$ as a linear combination of the basis vectors $\widetilde{V}$, $V \in N_i(L_n(q))$. Then the coefficient of $\widetilde{V}$ in $\widetilde{\nabla}_L\widetilde{\Delta}_L(\widetilde{U})$ (resp. $\widetilde{\Delta}_L\widetilde{\nabla}_L(\widetilde{U})$) is in both cases* 0 *if* $\dim V \cap U < i-1$, *it is* 1 *if* $\dim V \cap U = i-1$, *and $\binom{i}{1}_q$ (resp. $\binom{n-i}{1}_q$) if $V = U$. Hence $(\widetilde{\nabla}_L\widetilde{\Delta}_L - \widetilde{\Delta}_L\widetilde{\nabla}_L)(\widetilde{U}) = \left(\binom{i}{1}_q - \binom{n-i}{1}_q\right)\widetilde{U}$.*

Example 6.2.7. *Let $C = (p_0 \lessdot \cdots \lessdot p_n)$ be a chain and let $\widetilde{\Delta}(\widetilde{p}_i) := i(n-i+1)\widetilde{p}_{i-1}$, $i = 1, \ldots, n$, and $\widetilde{\Delta}(\widetilde{p}_0) := \widetilde{0}$. Then $(\widetilde{\nabla}_L, \widetilde{\Delta})$ has for $P := C$ the Lie-property.*

After the following preparations we will see that property C of $(\widetilde{\nabla}, \widetilde{\Delta})$ together with some additional conditions implies the full rank property of $\widetilde{\nabla}$ (and $\widetilde{\Delta}$).

Lemma 6.2.3. *Let f, g, h be natural numbers. If in $\widetilde{P}$ the pair $(\widetilde{\nabla}, \widetilde{\Delta})$ has property* C *with the numbers $\mu_0, \ldots, \mu_n$, then there exists a linear operator $\Omega_{f,g} : \widetilde{P} \to \widetilde{P}$ such that for all $\varphi \in \widetilde{N}_h$, $h = 0, \ldots, n$,*

$$\widetilde{\Delta}^f\widetilde{\nabla}^g(\varphi) = (\Omega_{f,g}\widetilde{\Delta} + \alpha_{f,g,h}\widetilde{\nabla}^{g-f})(\varphi) \tag{6.13}$$

where

$$\alpha_{f,g,h} = \begin{cases} 0 & \text{if } g > n-h \text{ or } f > g, \\ 1 & \text{if } g \le n-h \text{ and } f = 0, \\ \prod_{i=1}^f \left(\sum_{j=0}^{g-i} -\mu_{h+j}\right) & \text{if } 0 < f \le g \le n-h. \end{cases}$$

Proof. First, recall that we defined $\widetilde{\nabla}^{g-f}$ to be the identity operator if $g \le f$. The cases $f = 0$ or $g = 0$ are trivial, thus let $f \ne 0$ and $g \ne 0$. Since $\{\widetilde{p} : p \in P\}$ is a basis of $\widetilde{P}$, it is sufficient to consider the special elements $\varphi = \widetilde{p}$ where $p \in P$. We define

$$\Omega_{f,g}(\widetilde{q}) := \widetilde{0} \text{ if } r(q) \ge n-g.$$

Then (6.13) is satisfied for all $p \in P$ with $r(p) > n-g$ since then $\widetilde{\nabla}^g(\widetilde{p}) = \widetilde{0}$. Thus it remains to define $\Omega_{f,g}(\widetilde{q})$ for $r(q) < n-g$ and to verify (6.13) for all p with $r(p) \le n-g$.

We proceed by induction on f. Let $f = 1$. We proceed by induction on g. If $g = 1$, we define

$$\Omega_{1,1}(\widetilde{q}) := \widetilde{\nabla}(\widetilde{q}) \text{ if } r(q) < n-g = n-1.$$

Then indeed, for $r(p) \le n-g$, $\widetilde{\Delta}\widetilde{\nabla}(\widetilde{p}) = \widetilde{\nabla}\widetilde{\Delta}(\widetilde{p}) - \mu_h \widetilde{p}$ by property C. Now consider the step $g \to g+1$ $(g \ge 1)$. We set

$$\Omega_{1,g+1}(\widetilde{q}) := \widetilde{\nabla}\Omega_{1,g}(\widetilde{q}) \text{ if } r(q) < n-g-1.$$

Using property C and the induction hypothesis, we obtain for $h := r(p) \le n-g-1$

$$\begin{aligned} \widetilde{\Delta}\widetilde{\nabla}^{g+1}(\widetilde{p}) &= \widetilde{\Delta}\widetilde{\nabla}(\widetilde{\nabla}^g(\widetilde{p})) = \widetilde{\nabla}\widetilde{\Delta}(\widetilde{\nabla}^g(\widetilde{p})) - \mu_{h+g}\widetilde{\nabla}^g(\widetilde{p}) \\ &= \widetilde{\nabla}\Omega_{1,g}\widetilde{\Delta}(\widetilde{p}) + \alpha_{1,g,h}\widetilde{\nabla}^g(\widetilde{p}) - \mu_{h+g}\widetilde{\nabla}^g(\widetilde{p}) \\ &= \Omega_{1,g+1}\widetilde{\Delta}(\widetilde{p}) + \alpha_{1,g+1,h}\widetilde{\nabla}^g(\widetilde{p}). \end{aligned}$$

Finally consider the step $f \to f+1$ $(f \ge 1)$. We set

$$\Omega_{f+1,g}(\widetilde{q}) := \widetilde{\Delta}^f \Omega_{1,g}(\widetilde{q}) + \alpha_{1,g,h}\Omega_{f,g-1}(\widetilde{q}) \text{ if } h \le n-g.$$

We have (using the assertion for $f = 1$) for $h := r(p) \le n-g$

$$\widetilde{\Delta}^{f+1}\widetilde{\nabla}^g(\widetilde{p}) = \widetilde{\Delta}^f(\widetilde{\Delta}\widetilde{\nabla}^g(\widetilde{p})) = \widetilde{\Delta}^f\Omega_{1,g}\widetilde{\Delta}(\widetilde{p}) + \widetilde{\Delta}^f\alpha_{1,g,h}\widetilde{\nabla}^{g-1}(\widetilde{p}).$$

The induction hypothesis yields further

$$\widetilde{\Delta}^f\alpha_{1,g,h}\widetilde{\nabla}^{g-1}(\widetilde{p}) = \alpha_{1,g,h}(\Omega_{f,g-1}\widetilde{\Delta}(\widetilde{p}) + \alpha_{f,g-1,h}\widetilde{\nabla}^{g-1-f}(\widetilde{p})).$$

Consequently,

$$\begin{aligned} \widetilde{\Delta}^{f+1}\widetilde{\nabla}^g(\widetilde{p}) &= \Omega_{f+1,g}\widetilde{\Delta}(\widetilde{p}) + \alpha_{1,g,h}\alpha_{f,g-1,h}\widetilde{\nabla}^{g-1-f}(\widetilde{p}) \\ &= \Omega_{f+1,g}\widetilde{\Delta}(\widetilde{p}) + \alpha_{f+1,g,h}\widetilde{\nabla}^{g-(f+1)}(\widetilde{p}). \end{aligned}$$

■

We say that the sequence $(\mu_0, \ldots, \mu_n)$ is *regular* if for $0 \le l \le k \le n$

$$\mu_l + \mu_{l+1} + \cdots + \mu_k = 0 \text{ implies } l + k = n,$$

that is, if every consecutive subsequence that is not symmetric with respect to $\frac{n}{2}$ has nonzero sum.

Example 6.2.8. *The sequences* $\mu_i = 2i - n$ *(appearing in the Lie-property) and* $\mu_i = \binom{i}{1}_q - \binom{n-i}{1}_q$ *(appearing in Example 6.2.6) are regular.*

Lemma 6.2.4. *Let* $\alpha_{f,g,h}$ *be defined as in Lemma 6.2.3 and suppose that* $(\mu_0, \ldots, \mu_n)$ *is regular. Then, for* $0 < f \le g \le n - 2h$, $\alpha_{f,g,h} \ne 0$.

Proof. In the product representation for $\alpha_{f,g,h}$ in Lemma 6.2.3, every factor is nonzero by regularity. ■

Similarly to the previous notations we write $\ker_{t,t-1}(\widetilde{\Delta})$ for the kernel of $\widetilde{\Delta}_{t,t-1}$ (i.e., of $\widetilde{\Delta}|_{\widetilde{N}_t}$). Let

$$E_{i,t} := \widetilde{\nabla}_{t,i}(\ker_{t,t-1}(\widetilde{\Delta})), i \geq t. \tag{6.14}$$

Obviously, $E_{i,t} \subseteq \widetilde{N}_i$ for all $i \geq t$.

Lemma 6.2.5. *Suppose that $(\widetilde{\nabla}, \widetilde{\Delta})$ has in $\widetilde{P}$ property C with the sequence $(\mu_0, \dots, \mu_n)$. Let i, j, t be natural numbers with $i \geq t$ and let $\alpha_{f,g,h}$ be defined as in Lemma 6.2.3.*

(*a*) *All nonzero elements of $E_{i,t}$ are eigenvectors of $\widetilde{\Delta}^j \widetilde{\nabla}^j|_{\widetilde{N}_i}$ to the eigenvalue $\alpha_{j,j+i-t,t}$.*
(*b*) *All nonzero elements of $E_{i,t}$ are eigenvectors of $\widetilde{\nabla}^j \widetilde{\Delta}^j|_{\widetilde{N}_i}$ to the eigenvalue $\alpha_{j,i-t,t}$.*
(*c*) *If, in addition, $\widetilde{\nabla} = \widetilde{\Delta}^*$ then E_{i,t_1} is orthogonal to E_{i,t_2} for all $0 \leq t_1 < t_2 \leq i$.*

Proof. For (a) and (b), let $\varphi \in E_{i,t}$, that is, $\varphi = \widetilde{\nabla}^{i-t}(\varphi')$, where $\varphi' \in \widetilde{N}_t$, $\widetilde{\Delta}(\varphi') = \widetilde{0}$. By Lemma 6.2.3,
(a) $\widetilde{\Delta}^j \widetilde{\nabla}^j(\varphi) = \widetilde{\Delta}^j \widetilde{\nabla}^{j+i-t}(\varphi') = \alpha_{j,j+i-t,t} \widetilde{\nabla}^{i-t}(\varphi') = \alpha_{j,j+i-t,t}(\varphi)$,
(b) $\widetilde{\nabla}^j \widetilde{\Delta}^j(\varphi) = \widetilde{\nabla}^j \widetilde{\Delta}^j \widetilde{\nabla}^{i-t}(\varphi') = \widetilde{\nabla}^j \alpha_{j,i-t,t} \widetilde{\nabla}^{i-t-j}(\varphi') = \alpha_{j,i-t,t}\varphi$.
(c) Let $\varphi_l \in E_{i,t_l}$, that is, $\varphi \in \widetilde{\nabla}^{i-t_l}(\varphi_l')$ where $\varphi_l' \in \widetilde{N}_{t_l}$, $\widetilde{\Delta}(\varphi_l') = \widetilde{0}$, $l = 1, 2$. We have

$$\langle \varphi_1, \varphi_2 \rangle = \langle \widetilde{\nabla}^{i-t_1}(\varphi_1'), \widetilde{\nabla}^{i-t_2}(\varphi_2') \rangle = \langle \varphi_1', \widetilde{\Delta}^{i-t_1} \widetilde{\nabla}^{i-t_2}(\varphi_2') \rangle = \langle \varphi_1', \widetilde{0} \rangle = 0$$

since $i - t_1 > i - t_2$ and $\widetilde{\Delta}(\varphi_2') = \widetilde{0}$ (use again Lemma 6.2.3). ■

Theorem 6.2.5. *Under the supposition that $(\widetilde{\nabla}, \widetilde{\Delta})$ has in $\widetilde{P}$ property C with the regular sequence $(\mu_0, \dots, \mu_n)$, we have*

(*a*) *$\widetilde{N}_i = E_{i,0} \oplus \dots \oplus E_{i,i}$ (direct sum) for all $0 \leq i \leq \frac{n}{2}$,*
(*b*) *$\widetilde{\Delta}^j \widetilde{\nabla}^j|_{\widetilde{N}_i}$ is injective for all $0 \leq i \leq \frac{n}{2}$ and $0 \leq j \leq n - 2i$.*

Proof. In (b) we suppose $j > 0$ since the case $j = 0$ is trivial. We proceed by induction on $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$ and prove (a) and (b) simultaneously. Let $i = 0$. Then $E_{0,0} = \widetilde{N}_0$, and by Lemma 6.2.5(a), $E_{0,0}$ is the eigenspace of $\widetilde{\Delta}^j \widetilde{\nabla}^j|_{\widetilde{N}_0}$ to the eigenvalue $\alpha_{j,j,0}$, which is nonzero for all $0 \leq j \leq n$ by Lemma 6.2.4 (resp. (in the case $j = 0$) by definition). Thus $\widetilde{\Delta}^j \widetilde{\nabla}^j|_{\widetilde{N}_0}$ is injective for all $0 \leq j \leq n$. Now consider the step $< i \to i$ $(i \leq \frac{n}{2})$. From (6.5) (replace $\widetilde{\nabla}$ by $\widetilde{\Delta}$) we may derive

$$\dim \ker_{t,t-1}(\widetilde{\Delta}) = W_t - \operatorname{rank}_{t,t-1}(\widetilde{\Delta}) \geq W_t - W_{t-1}, \qquad t = 0, \dots, n.$$

Thus

$$\dim E_{i,i} = \dim \ker_{i,i-1}(\widetilde{\Delta}) \geq W_i - W_{i-1}. \tag{6.15}$$

By the induction hypothesis, $\widetilde{\Delta}^{n-2t}\widetilde{\nabla}^{n-2t}|_{\widetilde{N}_t}$ is injective for all $0 \leq t < i$. Since $t < i \leq \frac{n}{2}$ there holds $i - t \leq n - 2t$; thus also $\widetilde{\Delta}^{i-t}|_{\widetilde{N}_t}$ is injective. By the definition of $E_{i,t}$ (see (6.14)) we have

$$\dim E_{i,t} = \dim \ker_{t,t-1}(\widetilde{\Delta}) \geq W_t - W_{t-1}, \qquad 0 \leq t < i. \tag{6.16}$$

From (6.15) and (6.16) we derive

$$\sum_{t=0}^{i} \dim E_{i,t} \geq W_i = \dim \widetilde{N}_i.$$

Thus for (a) it is sufficient to prove that the sum $E_{i,0} + \cdots + E_{i,i}$ is direct (note that $E_{i,0} + \cdots + E_{i,i} \subseteq \widetilde{N}_i$). Let $\varphi_t \in E_{i,t}$, $t = 0, \ldots, i$. We have to show that $\varphi_0 + \cdots + \varphi_i = \widetilde{0}$ implies $\varphi_t = \widetilde{0}$ for every $t = 0, \ldots, i$. Let $\varphi_t = \widetilde{\nabla}^{i-t}(\varphi_t')$, where $\varphi_t' \in \widetilde{N}_t$, $\widetilde{\Delta}(\varphi_t') = \widetilde{0}$, $t = 0, \ldots, i$. Hence $\varphi_0 + \cdots + \varphi_i = \widetilde{0}$ is equivalent with

$$\widetilde{\nabla}^i(\varphi_0') + \cdots + \widetilde{\nabla}^1(\varphi_{i-1}') + \widetilde{\nabla}^0(\varphi_i') = \widetilde{0},$$

$$\widetilde{\nabla}\left(\widetilde{\nabla}^{i-1}(\varphi_0') + \cdots + \widetilde{\nabla}^0(\varphi_{i-1}')\right) + \varphi_i' = \widetilde{0}.$$

Applying $\widetilde{\Delta}$ to both sides we obtain

$$\widetilde{\Delta}\widetilde{\nabla}(\varphi_0'' + \cdots + \varphi_{i-1}'') = \widetilde{0},$$

where $\varphi_t'' = \widetilde{\nabla}^{i-1-t}(\varphi_t') \in E_{i-1,t}$, $0 \leq t \leq i-1$. By the induction hypothesis $\widetilde{\Delta}\widetilde{\nabla}|_{\widetilde{N}_{i-1}}$ is injective; consequently,

$$\varphi_0'' + \cdots + \varphi_{i-1}'' = \widetilde{0},$$

and since by the induction hypothesis $\widetilde{N}_{i-1} = E_{i-1,0} \oplus \cdots \oplus E_{i-1,i-1}$, it follows that $\varphi_t'' = \widetilde{0}$ implying (since $\varphi_t = \widetilde{\nabla}(\varphi_t'')$) that $\varphi_t = \widetilde{0}$, $t = 0, \ldots, i-1$. But then clearly also $\varphi_i = \widetilde{0}$.

For (b), it is sufficient to look at the eigenvalues $\alpha_{j,j+i-t,t}$ of $\widetilde{\Delta}^j\widetilde{\nabla}^j|_{\widetilde{N}_i}$ (see Lemma 6.2.5(a) and use the proved part (a)). In our case $0 \leq i \leq \frac{n}{2}$ and $0 \leq j \leq n - 2i$, hence by Lemma 6.2.4 (note that $j \leq j + i - t \leq n - 2t$) our eigenvalues are nonzero, implying the injectivity of $\widetilde{\Delta}^j\widetilde{\nabla}^j|_{\widetilde{N}_i}$. ■

The following theorem is mainly due to Proctor [387], who proved the equivalence (i) $\leftrightarrow$ (ii).

Theorem 6.2.6. *The following conditions are equivalent for a ranked poset P of rank n.*

(*i*) *P is a Peck poset,*

(*ii*) *there exist an order-raising operator $\widetilde{\nabla}$ and a lowering operator $\widetilde{\Delta}$ such that $(\widetilde{\nabla}, \widetilde{\Delta})$ has the Lie-property,* *(iii) there exist an order-raising operator $\widetilde{\nabla}$ and a lowering operator $\widetilde{\Delta}$ such that $(\widetilde{\nabla}, \widetilde{\Delta})$ has property* C *with a regular sequence* $(\mu_0, \ldots, \mu_n)$.

Proof. (i) $\rightarrow$ (ii). Let $\widetilde{\nabla}$ be the order-raising operator on $\widetilde{P}$ with the full rank property, which exists by the remarks in the beginning of this section. From Theorem 6.1.3 and Lemma 6.2.1 it follows that we can write $\widetilde{P}$ in the form

$$\widetilde{P} = \oplus_{0 \le i \le \frac{n}{2}} \oplus_{h=1}^{s_{i,n-i}} V_{i,h}$$

where each $V_{i,h}$ is generated by a ranked Jordan string from $\widetilde{N}_i$ to $\widetilde{N}_{n-i}$. We define the lowering operator on the elements of these strings, that is, on the corresponding ranked Jordan basis. To do so, consider any of the strings $S = \{b, \widetilde{\nabla}(b), \ldots, \widetilde{\nabla}^{n-2i}(b)\}$. We put (similarly to Example 6.2.7)

$$\widetilde{\Delta}(\widetilde{\nabla}^k(b)) := k(n-2i-k+1)\widetilde{\nabla}^{k-1}(b), \qquad k = 0, \ldots, n-2i.$$

Then it is easy to verify the Lie-property of $(\widetilde{\nabla}, \widetilde{\Delta})$.

(ii) $\rightarrow$ (iii). This is clear since the sequence $\mu_i = 2i - n$ is a special regular sequence (see Example 6.2.8).

(iii) $\rightarrow$ (i). By Theorem 6.2.5, in particular $\widetilde{\nabla}^j|_{\widetilde{N}_i}$ is injective for all $0 \le i \le \frac{n}{2}$, $0 \le j \le n-2i$. For the dual P^* of the poset P, the operators $\widetilde{\nabla}$ (resp. $\widetilde{\Delta}$) are lowering (resp. raising). It is easy to see that the pair $(\widetilde{\Delta}, \widetilde{\nabla})$ has in $\widetilde{P}^*$ property C with the regular sequence $(-\mu_n, \ldots, -\mu_0)$, hence again by Theorem 6.2.5, $\widetilde{\Delta}^j|_{\widetilde{N}_{n-i}}$ is injective for all $0 \le i \le \frac{n}{2}$, $0 \le j \le n-2i$. Taking $j := n-2i$ in both injections, we obtain $W_i \le W_{n-i}$ (resp. $W_{n-i} \le W_i$); that is, $W_i = W_{n-i}$, $0 \le i \le \frac{n}{2}$; thus P is rank symmetric. It remains to prove that $\widetilde{\nabla}$ has the full rank property. Consider $\widetilde{\nabla}_{ij} = \widetilde{\nabla}^{j-i}|_{\widetilde{N}_i}$, $0 \le i \le j \le n$. If $i + j \le n$, then $\widetilde{\nabla}_{ij}$ is injective by the preceding remarks; that is, it has full rank by Lemma 6.1.2(a). If $i + j > n$, we look again at $\widetilde{P}^*$. We know already that $(\widetilde{\nabla}^*, \widetilde{\Delta}^*)$ has in $\widetilde{P}^*$ property C with $(-\mu_n, \ldots, -\mu_0)$. Theorem 6.2.5 yields the injectivity of $(\widetilde{\nabla}^*)^{j-i}|_{\widetilde{N}_j}$ and by Lemma 6.1.2(b), $\widetilde{\nabla}_{ij}$ has full rank. ■

Remark 6.2.1. *P is unitary Peck iff conditions (ii) (resp. (iii)) in Theorem 6.2.6 hold, with $\widetilde{\nabla} = \widetilde{\nabla}_L$.*

Example 6.2.9. *In light of Example 6.2.6 and Example 6.2.8 we conclude immediately that the linear lattice $L_n(q)$ (and again that the Boolean lattice B_n) are unitary Peck. From the Peck Product Theorem 6.2.1 it follows, moreover, that modular geometric lattices are unitary Peck.*

One has still much freedom in constructing $(\widetilde{\nabla}, \widetilde{\Delta})$ satisfying conditions (ii) (resp. (iii)) of Theorem 6.2.6. But one has to solve a nonlinear (quadratic) system of equations with a very large number of variables (the unknown coefficients in the corresponding linear combinations). So let us restrict ourselves to the special case when $\widetilde{\nabla} = \widetilde{\nabla}_L$ and when $\widetilde{\Delta}$ is an *order*-lowering operator. Moreover, we will consider only modular lattices though the results can be given in a more general form (see Proctor [387]). Note that whenever p and q are two elements that cover (resp. which are covered by) a third element then necessarily this third element is $p \wedge q$ (resp. $p \vee q$), and $r(p) = r(q) = r(p \wedge q) + 1$ (resp. $r(p \vee q) - 1$). Since $r(p) + r(q) = r(p \wedge q) + r(p \vee q)$, p and q are covered by (resp. cover) a unique fourth element, namely $p \vee q$ (resp. $p \wedge q$).

We say that the modular lattice P of rank n is *edge-labelable* if there is a function $f : E(P) \to \mathbb{R}$ such that

$$f(p, p \vee q) = f(p \wedge q, q) \quad \text{for all } p, q \in P \text{ with } p, q \lessdot p \vee q \tag{6.17}$$

and

$$\sum_{e^+=p} f(e) - \sum_{e^-=p} f(e) = 2r(p) - n \quad \text{for all } p \in P. \tag{6.18}$$

Corollary 6.2.1. *Every edge-labelable modular lattice is unitary Peck.*

Proof. For $p \in P$, we define

$$\widetilde{\Delta}(\widetilde{p}) := \sum_{q : q \lessdot p} f(qp)\widetilde{q}.$$

It is easy to check that $(\widetilde{\nabla}_L, \widetilde{\Delta})$ has the Lie-property. ■

Before giving examples of edge-labelable modular lattices, let us study the connection to the variance problem.

Theorem 6.2.7. *Every Peck poset (with weight $w \equiv 1$) is rank compressed.*

Proof. A Peck poset P of rank n is rank symmetric and has property T (see Theorem 6.1.6). Hence there are W_i pairwise disjoint saturated chains from N_i to N_{n-i} ($i < \frac{n}{2}$) and consequently there is a bijection $\varphi : P \to P$ such that

$$r(p) + r(\varphi(p)) = n \quad \text{and} \quad p < \varphi(p) \quad \text{if } r(p) < \frac{n}{2}$$

(map a bottom element of such a chain onto its top element and conversely, the elements of $N_{\frac{n}{2}}$ one leaves invariant). We will verify condition (iii) of Theorem 4.4.1. Note that P equals the active poset P_r. Let F be any filter in P and define $F' := \{p \in F : r(p) < \frac{n}{2}\}$. Obviously, $\varphi(F') \subseteq F$ and $F' \cap \varphi(F') = \emptyset$.

Hence,

$$\mu_r(F) = \frac{1}{|F|}\sum_{p\in F} r(p) = \frac{1}{|F|}\left(\sum_{p\in F'}(r(p)+r(\varphi(p))) + \sum_{p\in F-(F'\cup\varphi(F'))} r(p)\right)$$

$$\geq \frac{1}{|F|}\left(|F'|n + (|F|-2|F'|)\frac{n}{2}\right) = \frac{n}{2} = \mu_r,$$

and the proof is complete. ■

A Peck poset with weight function $w \equiv 2$ is of course also rank compressed; hence by Theorem 4.4.1(i) $\leftrightarrow$ (v) there is a representation flow on $(P, 2)$ relative to r – that is, a function $f : E(P) \to \mathbb{R}_+$ such that

$$\sum_{e^+=p} f(e) - \sum_{e^-=p} f(e) = 2\left(r(p) - \frac{n}{2}\right) = 2r(p) - n \text{ for all } p \in P.$$

In the definition of an edge-labelable modular lattice we have exactly the same equation, but we did not require that $f(e) \geq 0$ for all $e \in E$. It is interesting that conditions (6.17) and (6.18) already imply this nonnegativity, which means that in edge-labelable modular lattices, we are working with representation flows:

Theorem 6.2.8. *Let P be a modular lattice and f a function from $E(P)$ into $\mathbb{R}$ satisfying (6.17) and (6.18). Then f is nonnegative.*

Proof. Let $e^* \in E(P)$ be any fixed arc with $\alpha := f(e^*) \neq 0$ and let $A := \{e \in E(P) : f(e) = \alpha\}$. Let I be the set of those elements of P that can be reached from a minimal element of P on a saturated chain not using arcs from A, and let $F := P - I$.

Claim 1. I is an ideal, that is, F is a filter.

Proof of Claim 1. Assume the contrary. Then there exist $p, q \in P$ with the property $p \lessdot q$, $p \notin I$, $q \in I$. We may assume that q has minimum rank with respect to this property. Let v be the predecessor of q on a chain from a minimal element of P to q, not using arcs from A. We have $v \lessdot q$ and $f(vq) \neq \alpha$. Clearly, $v \in I$, hence $v \neq p$. In view of $r(v) < r(q)$, $p \wedge v \lessdot v$, and the choice of q we have $p \wedge v \in I$. But because of (6.17), $f(p \wedge v, p) = f(v, p \vee v) = f(vq) \neq \alpha$, consequently also $p \in I$, a contradiction. □

Let $B := \{e \in E : e^- \in I, e^+ \in F\}$. Obviously, $B \subseteq A$, that is, for all $e \in B$, $f(e) = \alpha$.

Claim 2. $B \neq \emptyset$.

Proof of Claim 2. From all arcs of A take an arc pq that has an endpoint q of minimum rank. Then clearly $p \in I$. Moreover, $q \in F$; that is, $pq \in B$: Assume the contrary. Then there must be some $v \lessdot q$ with $f(vq) \neq \alpha$, that is, $v \neq p$. But then by (6.17), $f(pq) = f(p, p \vee v) = f(p \wedge v, v) = \alpha$ implying $(p \wedge v, v) \in A$, which is a contradiction to the choice of pq (note that $r(v) < r(q)$). □

To conclude the proof of the theorem, consider again the network $N = (V_N, E_N, s, t)$ from the proof (iii) $\leftrightarrow$ (v) in Theorem 4.4.1, but without capacities ($V_N := P \cup \{s, t\}$, $E_N := E \cup \{sp : p \in P\} \cup \{pt : p \in P\}$). We define $f_N : E_N \to \mathbb{R}$ by

$$f_N(e) := \begin{cases} f(e) & \text{if } e \in E, \\ \max\{0, n - 2r(p)\} & \text{if } e = sp,\ p \in P, \\ \max\{0, 2r(p) - n\} & \text{if } e = pt,\ p \in P. \end{cases}$$

Then f_N satisfies, because of (6.18), the conservation of flow (4.1). Consider the cut (S, T) in N where $S := I \cup \{s\}$, $T := F \cup \{t\}$. By Claim 1, $f_N(T, S) = 0$. By Remark 4.1.1 we have

$$\begin{aligned} v(f_N) &= \sum_{p \in P} \max\{0, 2r(p) - n\} = f_N(S, T) - f_N(T, S) \\ &= f_N(\{s\}, F) + f_N(I, F) + f_N(I, \{t\}) \\ &= \sum_{p \in F} \max\{0, n - 2r(p)\} + \alpha|B| + \sum_{p \in I} \max\{0, 2r(p) - n\}, \end{aligned}$$

which implies

$$\begin{aligned} \sum_{p \in F} \max\{0, 2r(p) - n\} &= \sum_{p \in F} \max\{0, n - 2r(p)\} + \alpha|B|, \\ \sum_{p \in F} 2r(p) - n &= \alpha|B|, \\ \mu_r(F) - \mu_r &= \alpha \frac{|B|}{2|F|}. \end{aligned}$$

Note that by Corollary 6.2.1 our poset P is Peck (we already replaced $\frac{n}{2}$ by μ_r); hence by Theorem 6.2.7 the LHS is nonnegative and from Claim 2 it follows that $\alpha \geq 0$. ■

In [389] Proctor proved this result for distributive lattices. We generalized it here to modular lattices.

Using the same construction of a representation flow in a product of two posets as introduced in the proof of Theorem 4.6.6, one obtains easily:

Theorem 6.2.9. *If P and Q are edge-labelable modular lattices, then $P \times Q$ is an edge-labelable modular lattice, too.*

Let us look now at some examples.

Theorem 6.2.10. *The posets $L(m,n)$ and $M(n)$ are edge-labelable modular lattices.*

Proof. Observe at first that, if $\boldsymbol{b} = (b_1, \ldots, b_n)$ covers $\boldsymbol{a} = (a_1, \ldots, a_n)$ in either $L(m,n)$ or $M(n)$, then there is some index i such that $a_i = b_i - 1$ and $a_j = b_j$ for all $j \neq i$. Put now for $L(m,n)$

$$f(\boldsymbol{a}, \boldsymbol{b}) := (m + n - a_i - i)(a_i + i)$$

and for $M(n)$

$$f(\boldsymbol{a}, \boldsymbol{b}) := \begin{cases} \frac{n(n+1)}{2} & \text{if } a_i = 0, \\ n(n+1) - a_i(a_i + 1) & \text{otherwise.} \end{cases}$$

We verify (6.17) and (6.18) only for $M(n)$ (similar steps work for $L(m,n)$, but we know already from Example 6.2.2 that $L(m,n)$ is Peck). Let $\boldsymbol{a}, \boldsymbol{b} \lessdot \boldsymbol{c} := \boldsymbol{a} \vee \boldsymbol{b}$. We pass from $\boldsymbol{c}$ to $\boldsymbol{a}$ by decreasing some component c_i by 1 (i.e., $a_i = c_i - 1$) and to $\boldsymbol{b}$ by decreasing some other component c_j $(i \neq j)$ by 1. Moreover, we pass from $\boldsymbol{b}$ to $\boldsymbol{d} := \boldsymbol{a} \wedge \boldsymbol{b}$ by decreasing the component $b_i = c_i$ by 1 (i.e., $d_i = c_i - 1 = a_i$). Consequently,

$$f(\boldsymbol{a}, \boldsymbol{c}) = f(\boldsymbol{d}, \boldsymbol{b}) = \begin{cases} \frac{n(n+1)}{2} & \text{if } a_i = 0, \\ n(n+1) - a_i(a_i + 1) & \text{otherwise;} \end{cases}$$

thus (6.17) is satisfied. To verify (6.18) consider any $\boldsymbol{a} = (a_1, \ldots, a_n) \in N_k$ where $0 = a_1 = \cdots = a_h < a_{h+1} < \cdots < a_n \leq n$, $h \in \{0, \ldots, n\}$ and define $a_0 := 0$, $a_{n+1} := n + 1$. We have

$$\sum_{e^+ = \boldsymbol{a}} f(e) = \sum_{\substack{h+1 \leq i \leq n: \\ a_{i-1} < a_i - 1}} n(n+1) - (a_i - 1)a_i \quad \left(+ \frac{n(n+1)}{2} \text{ if } a_{h+1} = 1 \right),$$

$$\sum_{e^- = \boldsymbol{a}} f(e) = \sum_{\substack{h+1 \leq i \leq n: \\ a_i + 1 < a_{i+1}}} n(n+1) - a_i(a_i + 1) \quad \left(+ \frac{n(n+1)}{2} \text{ if } a_{h+1} > 1 \right).$$

Now it is not difficult to see that in $\sum_{e^+ = \boldsymbol{a}} f(e) - \sum_{e^- = \boldsymbol{a}} f(e)$ we may drop the extra conditions on the summation (any new terms appearing will cancel each other), and we obtain

$$\sum_{e^+ = \boldsymbol{a}} f(e) - \sum_{e^- = \boldsymbol{a}} f(e) = \sum_{h+1 \leq i \leq n} -(a_i - 1)a_i + a_i(a_i + 1) - \frac{n(n+1)}{2}$$

$$= \sum_{i=h+1}^{n} 2a_i - \frac{n(n+1)}{2} = 2r(\boldsymbol{a}) - r(M(n)).$$

■

There is a nice application of the Peck property of $M(n)$ to a number-theoretic problem. Let $R = \{r_1, \dots, r_n\}$ be a set of n distinct positive real numbers and let α be any real number. Moreover, let $m(R, \alpha)$ denote the number of subsets of R adding up to α. Thus

$$m(R, \alpha) := \left| \left\{ X \subseteq [n] : \sum_{i \in X} r_i = \alpha \right\} \right|.$$

In 1963 Erdős and Moser [166] posed the problem of maximizing $m(R, \alpha)$ (in a slightly different form). First observe that there is a natural bijection $\varphi : 2^{[n]} \to M(n)$ given by $\varphi(X) := (0, \dots, 0, i_1, \dots, i_l)$ for any set $X = \{i_1, \dots, i_l\} \in 2^{[n]}$ with $i_1 < \dots < i_l$.

***Lemma 6.2.6 (Lindström* [344]).** *If $|R| = n$, then for all $\alpha \in \mathbb{R}$ the set $\{\varphi(X) : X \subseteq [n]$ and $\sum_{i \in X} r_i = \alpha\}$ is an antichain in $M(n)$.*

Proof. Let, w.l.o.g., $r_1 < \dots < r_n$. Assume, in the contrary, that there are $X = \{i_1, \dots, i_l\} \neq Y = \{j_1, \dots, j_m\} \in 2^{[n]}$, $i_1 < \dots < i_l$, $j_1 < \dots < j_m$, such that

$$\sum_{k=1}^{l} r_{i_k} = \sum_{k=1}^{m} r_{j_k} = \alpha,$$

but $\varphi(X) \leq_{M(n)} \varphi(Y)$. Then it is easy to see that $l \leq m$ and $i_l \leq j_m, i_{l-1} \leq j_{m-1}, \dots, i_1 \leq j_{m-l+1}$. If there was some strict inequality under these inequalities, then we would have $\alpha = \sum_{k=1}^{l} r_{i_k} \leq \sum_{k=1}^{l} r_{j_{m-l+k}} \leq \sum_{k=1}^{m} r_{j_k} = \alpha$ with at least one strict inequality, a contradiction. Consequently, $l = m$ and $i_l = j_l, \dots, i_1 = j_1$, that is, $\varphi(X) = \varphi(Y)$, implying $X = Y$, a contradiction. ■

Theorem 6.2.11. *If $|R| = n$, then for all $\alpha \in \mathbb{R}$,*

$$m(R, \alpha) \leq m\left([n], \left\lfloor \frac{n(n+1)}{4} \right\rfloor\right) = d(M(n)).$$

Proof. From Lemma 6.2.6 we derive directly

$$m(R, \alpha) \leq d(M(n)).$$

But $M(n)$ is Peck and of rank $\frac{n(n+1)}{2}$. Consequently,

$$\begin{aligned} d(M(n)) &= W_{\lfloor n(n+1)/4 \rfloor}(M(n)) = \left| \left\{ X \subseteq [n] : \sum_{i \in X} i = \left\lfloor \frac{n(n+1)}{4} \right\rfloor \right\} \right| \\ &= m\left([n], \left\lfloor \frac{n(n+1)}{4} \right\rfloor\right), \end{aligned}$$

and the proof is complete. ■

This and the next result are due to Stanley [439], who studied $M(n)$ from another point of view, and partly to Harper, who noticed that the Peck property of $M(n)$ together with Lemma 6.2.6 solves the modification of the Erdős–Moser problem. The original Erdős–Moser problem concerns integer sums including negative numbers. Let $S = \{s_1, \dots, s_n\}$ be a set of n distinct real numbers and α any real number. Let

$$em(S, \alpha) := \left| \left\{ X \subseteq [n] : \sum_{i \in X} s_i = \alpha \right\} \right|.$$

Theorem 6.2.12. *For all $\alpha \in \mathbb{R}$,*

$$em(S, \alpha) \leq em(\{-n, -n+1, \dots, 0, \dots, n-1, n\}, 0) \quad \text{if } |S| = 2n+1,$$
$$em(S, \alpha) \leq em(\{-n+1, \dots, 0, \dots, n-1, n\}, \lfloor \tfrac{n}{2} \rfloor) \quad \text{if } |S| = 2n.$$

Proof. Let us first suppose that $0 \notin S$. Moreover, let

$$s_1 < \cdots < s_v < 0 < s_{v+1} < \cdots < s_n.$$

A subset of S is uniquely characterized by a pair (X_1, X_2) with $X_1 \subseteq [v]$, $X_2 \subseteq \{v+1, \dots, n\}$ and, as previously, by a pair $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ with $\boldsymbol{a}_1 \in M(v)$, $\boldsymbol{a}_2 \in M(n-v)$. If we have another subset of S, characterized by (Y_1, Y_2) (resp. $(\boldsymbol{b}_1, \boldsymbol{b}_2)$) and having the same sum of the elements, then one can check similarly to the proof of Lemma 6.2.6 that we cannot have $\boldsymbol{a}_1 \geq_{M(v)} \boldsymbol{b}_1$ and $\boldsymbol{a}_2 \leq_{M(v)} \boldsymbol{b}_2$. Consequently, to a family of subsets of S that all have element sum α there corresponds an antichain in $M(v)^* \times M(n-v)$. Since with $M(v)$ its dual $M(v)^*$ (which is by the way isomorphic to $M(v)$) is also Peck, we have by Theorem 6.2.1,

$$em(S, \alpha) \leq d(M(v)^* \times M(n-v)) = h_n(v),$$

where

$$h_n(v) := W_{\lfloor ((v+1)v + (n-v+1)(n-v))/4 \rfloor}(M(v)^* \times M(n-v)).$$

Now consider the maximum of $h_n(v)$. Without loss of generality, we may assume that $v \leq n - v$, that is, $v \leq \lfloor \frac{n}{2} \rfloor$, and we will show that the maximum of $h_n(v)$ is attained at $v = \lfloor \frac{n}{2} \rfloor$. Otherwise we would have some $v < \lfloor \frac{n}{2} \rfloor$ with $h_n(v) > h_n(v+1)$. It is easy to see that the rank-generating function of $M(n)$ is given by

$$F(M(n); x) = (1+x)(1+x^2)\cdots(1+x^n),$$

and consequently,

$$F(M(v)^* \times M(n-v); x) = (1+x)\cdots(1+x^v)(1+x)\cdots(1+x^{n-v}).$$

Let

$$p(x) := F(M(v)^* \times M(n-v-1); x) = \beta_0 + \beta_1 x + \cdots + \beta_d x^d$$

with $d := \frac{(v+1)v+(n-v)(n-v-1)}{2}$. By rank symmetry and rank unimodality, $\beta_0 = \beta_d \le \beta_1 = \beta_{d-1} \le \cdots$, and $h_n(v)$ (resp. $h_n(v+1)$) is the largest (i.e., middle) coefficient in $p(x)(1+x^{n-v})$ (resp. $p(x)(1+x^{v+1})$). We put $\beta_i := 0$ if $i \notin \{0, \ldots, d\}$. We have

$$\begin{aligned} h_n(v) &= \beta_{\lfloor (d+n-v)/2 \rfloor} + \beta_{\lfloor (d+n-v)/2 \rfloor - (n-v)} \\ &\le \beta_{\lfloor (d+v+1)/2 \rfloor} + \beta_{\lfloor (d+v+1)/2 \rfloor - (v+1)} = h_n(v+1) \end{aligned}$$

since

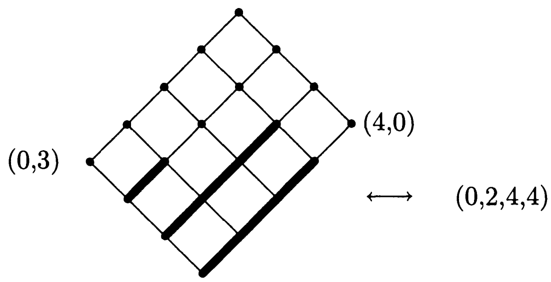
$$\beta_{\lfloor (d+n-v)/2 \rfloor} \le \beta_{\lfloor (d+v+1)/2 \rfloor}$$

in view of $\frac{d+n-v}{2} \ge \frac{d+v+1}{2} \ge \frac{d}{2}$ and $\beta_{\lfloor (d+n-v)/2 \rfloor - (n-v)} \le \beta_{\lfloor (d+v+1)/2 \rfloor - (v+1)}$ because of $\frac{d+n-v-2(n-v)}{2} = \frac{d+v-n}{2} \le \frac{d-v-1}{2} = \frac{d+v+1-2(v+1)}{2} \le \frac{d}{2}$, a contradiction. Consequently, indeed $h_n(\lfloor \frac{n}{2} \rfloor) = \max\{h_n(v) : 0 \le v \le \lfloor \frac{n}{2} \rfloor\}$.

Thus under the supposition $0 \notin S$, for $|S| = 2n$, $em(S, \alpha) \le h_{2n}(n) = em(\{-n, -n+1, \ldots, -1, 1, \ldots, n\}, 0)$ (note that the RHS counts the number of pairs $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in M(n) \times M(n)$ with $-r(\boldsymbol{a}_1) + r(\boldsymbol{a}_2) = 0$; that is, $(\frac{n(n+1)}{2} - r(\boldsymbol{a}_1)) + r(\boldsymbol{a}_2) = \frac{n(n+1)}{2})$, and for $|S| = 2n+1$, $em(S, \alpha) \le h_{2n+1}(n) = em(\{-n, \ldots, -1, 1, \ldots, n+1\}, \lfloor \frac{n+1}{2} \rfloor)$ (the RHS counts the number of pairs $(\boldsymbol{a}_1, \boldsymbol{a}_2) \in M(n) \times M(n+1)$ with $-r(\boldsymbol{a}_1) + r(\boldsymbol{a}_2) = \lfloor \frac{n+1}{2} \rfloor$; thus, $(\frac{n(n+1)}{2} - r(\boldsymbol{a}_1)) + r(\boldsymbol{a}_2) = \lfloor \frac{1}{2}(\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}) \rfloor$).

Finally we allow 0 to be an element of S. Clearly for $S = S' \cup \{0\}, 0 \notin S'$, $em(S, \alpha) = 2em(S', \alpha)$ (we can take the subsets with and without zero). For the sake of brevity, we consider only the case $|S| = 2n+1$. From the preceding we know that the best we can achieve when 0 is included is $2h_{2n}(n) = em(\{-n, \ldots, -1, 0, 1, \ldots, n\}, 0)$ and, without including 0, the optimum is $h_{2n+1}(n) = em(\{-n, \ldots, -1, 1, \ldots, n+1\}, \lfloor \frac{n+1}{2} \rfloor)$. But $2h_{2n}(n)$ is twice the largest (i.e., middle) coefficient in $(1+x)\cdots(1+x^n)(1+x)\cdots(1+x^n)$, which is not less than the largest coefficient $h_{2n+1}(n)$ in $(1+x)\cdots(1+x^n)(1+x)\cdots(1+x^n)(1+x^{n+1})$ (which is the sum of two coefficients from the preceding polynomial). Thus the result follows. ■

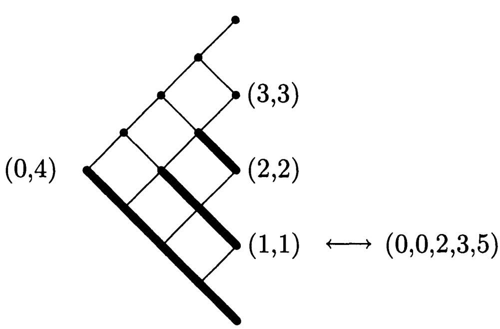
Let $\mathfrak{I}(P)$ be the poset of all ideals of P ordered by inclusion. It is well known (cf. Aigner [21, p. 33]) that $\mathfrak{I}(P)$ is a distributive lattice. Moreover, let $\mathfrak{I}^k(P) := \mathfrak{I}(\cdots(\mathfrak{I}(P))\cdots)$ (k-times). In Example 1.3.13 we already mentioned that $L(m, n)$ is isomorphic to the poset of all ideals of the product of two chains (with m (resp. n) elements), ordered by inclusion, that is, $L(m, n) \cong \mathfrak{I}(S(m-1, n-1))$ (recall the isomorphism: An ideal in $S(m-1, n-1)$ is uniquely characterized by the number of elements with second coordinate $i, 0 \le i \le n-1$; if we



$S(4,3)$

Figure 6.5

denote this number by a_{n-i}, we have $0 \leq a_1 \leq \cdots \leq a_n \leq m$). Figure 6.5 illustrates the isomorphism for $m = 5, n = 4$. Moreover, $M(n)$ is isomorphic to $\mathfrak{I}^2(S(n-2,1)) \cong \mathfrak{I}(L(n-1,2))$(note that an ideal in $L(n-1,2)$ is uniquely characterized by the number of elements with first coordinate i, $0 \leq i \leq n-1$. If we denote this number by a_{n-i} we have $0 \leq a_1 \leq \cdots \leq a_n \leq n$ with strict inequality in $a_i \leq a_{i+1}$ if $a_i \neq 0$; for an illustration see Figure 6.6). Without proof, let us note here that there are no other "interesting" edge-labelable distributive lattices:



$\mathfrak{I}(S(3,1)) \cong L(4,2)$

Figure 6.6

***Theorem 6.2.13* (*Proctor* [389]).** *The only edge-labelable distributive lattices are $\mathfrak{I}(S(m,n))$, $m,n \geq 0$, $\mathfrak{I}^2(S(n,1))$, $n \geq 0$, $\mathfrak{I}^k(S(1,1))$, $k \geq 1$, $\mathfrak{I}^3(S(1,2))$, $\mathfrak{I}^4(S(1,2))$, and products of these lattices.*

Let P be any ranked poset of rank n. Consider a further operator $H : \widetilde{P} \to \widetilde{P}$, defined on the basis elements

$$H(\widetilde{p}) := (2r(p) - n)\widetilde{p}.$$

If the pair of raising and lowering operators $(\widetilde{\nabla}, \widetilde{\Delta})$ has the Lie-property, then

$$\widetilde{\nabla}\widetilde{\Delta} - \widetilde{\Delta}\widetilde{\nabla} = H. \tag{6.19}$$

Moreover, it is easy to verify that

$$H\widetilde{\nabla} - \widetilde{\nabla}H = 2\widetilde{\nabla}, \tag{6.20}$$

$$H\widetilde{\Delta} - \widetilde{\Delta}H = -2\widetilde{\Delta}. \tag{6.21}$$

The short notion "Lie-property" is used here because of an important connection to Lie-algebras. The Lie-algebra $sl(2, \mathbb{C})$ consists of all 2×2 trace zero complex matrices with Lie-algebra multiplication given by $[A, B] := A \cdot B - B \cdot A$ where the dot product is the usual matrix product. The usual basis taken for $sl(2, \mathbb{C})$ is

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(cf. Humphreys [275]), and the relations $[R, L] = H, [H, R] = 2R, [H, L] = -2L$ completely describe the algebra structure of $sl(2, \mathbb{C})$. Now our three operators $\widetilde{\nabla}$, $\widetilde{\Delta}$, H satisfying (6.19), (6.20), (6.21) are said to be a *representation of* $sl(2, \mathbb{C})$ on the vector space $\widetilde{P}$. Since many such representations have been studied in algebra, one can go "backward": Look at any one of these representations on a vector space V and try to find a basis for V that is a poset P (i.e., $\widetilde{P} = V$), and in terms of which three operators on V behave like $\widetilde{\nabla}$, $\widetilde{\Delta}$, and H with regard to $\widetilde{P}$ (do not forget that for our theorems to hold, $\widetilde{\nabla}$ must be *order* raising). In this way we can find old and *new* posets which are, by our results, Peck. Using this approach, taking a certain subalgebra of the Lie-algebra $gl(n, \mathbb{C})$ (consisting of $n \times n$ complex matrices) and applying a construction of Gelfand and Zetlin [221], Proctor [390] could prove the following result (in a slightly generalized form), which is mentioned without proof:

Theorem 6.2.14. *The poset $\mathfrak{I}(S(m, n, r))$, $m, n, r \geq 0$, of all ideals in a product of three chains ordered by inclusion is Peck.*

It is still an interesting open problem of Stanley, and independently of Harper [260], whether the poset of all ideals in a product of more than three chains ordered by inclusion is Peck. The rank unimodality is open as well.

The posets from Theorem 6.2.13 are special cases of a more general class, namely *Bruhat orders*. These Bruhat orders (defined on Weyl groups) arise in the study of semisimple algebraic groups where they describe the inclusion relationships of certain subvarieties. A highlight in Sperner theory was the proof of Stanley [439] that all such Bruhat orders are Peck. In his proof Stanley used the Hard Lefschetz Theorem of algebraic geometry. Later Proctor developed the more elementary Lie-algebra approach.

Let us examine finally a special class of Bruhat orders – the orders of *type A*. The *poset of A-shuffles* $1^{k_1}2^{k_2}\dots m^{k_m}$ is the set of all $(k_1+\cdots+k_m)$-tuples with k_1 1's, ..., k_m m's where the ordering is the transitive closure of

$$(\dots, s_i, \dots, s_j, \dots) \le (\dots, s_j, \dots, s_i, \dots) \text{ when } s_j \le s_i.$$

If we interchange the entry s_i and the entry s_j, then we go "upward" (resp. "downward") if $s_j < s_i$ (resp. $s_j > s_i$). Obviously, $(m, \dots, m, \dots, 2, \dots, 2, 1, \dots, 1)$ is the unique minimal element and $(1, \dots, 1, 2, \dots, 2, \dots, m, \dots, m)$ is the unique maximal element. Let $\boldsymbol{s}$ be a fixed element. In $\boldsymbol{s}$, we have exactly k_j (ordered) entries j. Let a_{ij} be the number of those entries less than j that appear in $\boldsymbol{s}$ before the ith entry j, $1 \le i \le k_j$, $2 \le j \le m$. Obviously,

$$0 \le a_{1,j} \le \cdots \le a_{k_j,j} \le k_1 + \cdots + k_{j-1} \text{ for all } j. \tag{6.22}$$

Thus, for $\boldsymbol{s}$, we find a unique $(m-1)$-tuple of k_2-, ..., k_m-tuples of integers satisfying (6.22) and, conversely, given such an $(m-1)$-tuple, the element $\boldsymbol{s}$ is uniquely determined. For example, we have in $1^5 2^2 3^3 4^2$ the following correspondence:

$$(3, 1, 2, 1, 1, 4, 3, 3, 1, 1, 4, 2) \leftrightarrow ((1, 5), (0, 4, 4), (5, 9)).$$

Thus the elements $\boldsymbol{s}$ of $P := 1^{k_1}\dots m^{k_m}$ can be bijectively associated with the elements of $Q := L(k_1, k_2) \times L(k_1+k_2, k_3) \times \cdots \times L(k_1+\cdots+k_{m-1}, k_m)$, which we write in the form $\boldsymbol{a} = (\boldsymbol{a}_2, \dots, \boldsymbol{a}_m)$ where $\boldsymbol{a}_i$ is the corresponding k_i-tuple, $i = 2, \dots, m$.

Claim 1. Let $\boldsymbol{a}, \boldsymbol{b} \in Q$ and $\boldsymbol{s}, \boldsymbol{t}$ be the associated original elements in P. Then $\boldsymbol{a} \lessdot \boldsymbol{b}$ implies $\boldsymbol{s} \lessdot \boldsymbol{t}$.

Proof of Claim 1. Suppose that $a_{uv} = b_{uv} - 1$ and $a_{ij} = b_{ij}$ for all other pairs of indices. Then $\boldsymbol{s}$ can be obtained from $\boldsymbol{t}$ by interchanging the uth entry v with the last entry in $\boldsymbol{t}$ that appears in $\boldsymbol{t}$ before this entry v and is smaller than v (note that $a_{u-1,v} \le a_{uv}$ implying $b_{u-1,v} < b_{uv}$; consequently there must be an entry smaller than v between the $(u-1)th$ and the uth entry v if $u \ge 2$). □

Claim 2. P and Q have the same rank function; that is, if $\boldsymbol{s} \in P$ and $\boldsymbol{a}$ is the associated element in Q, then $r_P(\boldsymbol{s}) = r_Q(\boldsymbol{a})$.

Proof of Claim 2. Let $\boldsymbol{s}, \boldsymbol{t} \in P$ and $\boldsymbol{a}, \boldsymbol{b}$ be the associated elements of Q. Since $(m, \dots, m, \dots, 1, \dots, 1)$ and $((0, \dots, 0), \dots, (0, \dots, 0))$ are the corresponding unique minimal elements we must only show that $\boldsymbol{s} \lessdot \boldsymbol{t}$ implies $r(\boldsymbol{a}) + 1 = r(\boldsymbol{b})$. Thus let $\boldsymbol{s} \lessdot \boldsymbol{t}$. Then obviously $\boldsymbol{t}$ can be obtained from $\boldsymbol{s}$ by interchanging some entry $x := s_i$ and some entry $v := s_j$ (with $v < x$) and there is no k such that

$i < k < j$ and $v = s_j \leq s_k \leq s_i = x$. It is easy to see that $\boldsymbol{a}_l = \boldsymbol{b}_l$ for $l \in \{2, \ldots, m\} - \{x, v\}$. Suppose that at coordinate i there appears the wth entry x and at coordinate j there appears the uth entry v, and that there are α entries smaller than v between s_i and s_j (i.e., $\alpha := |\{k : i \leq k \leq j \text{ and } s_k < v\}|$). Then $b_{uv} = a_{uv} - \alpha$, $b_{wx} = a_{wx} + \alpha + 1$ and $b_{iv} = a_{iv}$, $b_{ix} = a_{ix}$ for all respective other indices i. Consequently,

$$r(\boldsymbol{b}) = \sum_{i,j} b_{ij} = \left(\sum_{i,j} a_{ij}\right) - \alpha + \alpha + 1 = r(\boldsymbol{a}) + 1.$$

□

By Claim 2, P has the same levels as Q, and by both claims, the Hasse diagram of Q can be obtained by deleting some arcs from the Hasse diagram of P. We say that Q is a *rank-preserving cover suborder of P* (in general, we really have to delete arcs; for example, in $1^1 2^1 3^1 4^1$ we have $(3, 1, 4, 2) \lessdot (2, 1, 4, 3)$, but for the associated elements we do not have $((1), (0), (2)) \lessdot ((0), (2), (2))$). Since Q is a Peck poset by Theorem 6.2.1 and Example 6.2.2, our poset of A-shuffles $P = 1^{k_1} \cdots m^{k_m}$ is Peck, too. (Take the "same" operators $(\widetilde{\nabla}, \widetilde{\Delta})$, and note that an order-raising operator $\widetilde{\nabla}$ on $\widetilde{Q}$ remains order raising for $\widetilde{P}$.)

In the special case $k_1 = \cdots = k_m = 1$, the poset Q is a product of chains. By Example 5.1.1 and Example 5.3.1, we obtain that our poset of A-shuffles is in this case an sc-order and has, moreover, the strong IC-property. Let us finally note that, if we restrict ourselves again to the case $k_1 = \cdots = k_m = 1$ but allow only interchanges between *neighboring* entries, then it is not known whether the thereby defined poset has the Sperner property. The dual of this poset (called *inversion poset* in [446] and *permutohedron lattice* in [335]) is the weak Bruhat order on a special Coxeter group, namely, the symmetric group. Answering a question of Björner [62], Leclerc [335] showed that the weak Bruhat order on another Coxeter group is not an sc-order. The more general unsolved problem is whether the weak Bruhat order on any Coxeter group is Peck.

6.3. Results for modular, geometric, and distributive lattices

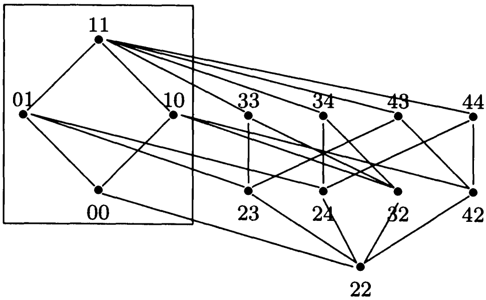
We know already that modular geometric lattices and modular edge-labelable lattices are Peck (see Example 6.2.9 and Corollary 6.2.1). When restricting ourselves to modular lattices (even to distributive lattices), we cannot derive the Sperner property. The example in Figure 4.10 shows a rank-symmetric, rank-unimodal distributive lattice without the Sperner property. Also for geometric lattices, the Sperner property does not hold in general. The following example is due to Dilworth and Greene [138]. Other examples have been given by Kahn in [280].

Example 6.3.1. *The Dilworth–Greene lattice.*

Construct the lattice DG_n as follows: Take a disjoint union of a Boolean lattice B_n (with elements of the form $\boldsymbol{a} = (a_1, \ldots, a_n)$, $a_i \in \{0, 1\}$ for all i) and of the dual Q_n^* of the cubical poset Q_n (with elements of the form $\boldsymbol{b} = (b_1, \ldots, b_n)$, $b_i \in \{2, 3, 4\}$ for all i, see Example 1.3.3 but replace 0, 1, 2 by 4, 3, 2, respectively). Further, add the following covering relations and take the transitive closure:

$$\boldsymbol{a} \gtrdot \boldsymbol{b} \text{ iff for all } i\ (a_i = 0 \Leftrightarrow b_i = 2), \qquad \boldsymbol{a} \in B_n, \quad \boldsymbol{b} \in Q_n^*.$$

In Figure 6.7, we illustrate DG_2.



DG_2

Figure 6.7

It is easy to verify that every element of DG_n is a supremum of atoms and that $r(p \wedge q) + r(p \vee q) \leq r(p) + r(q)$ for all $p, q \in DG_n$, that is, that DG_n is indeed a geometric lattice. Clearly,

$$W_k = \binom{n}{k-1} + \binom{n}{k} 2^k, \qquad k = 0, \ldots, n+1.$$

Taking in Q_n^* the kth level and in B_n the $(k-2)$th level (which is in DG_n in level $k-1$), we obtain an antichain A_k of size

$$|A_k| = \binom{n}{k-2} + \binom{n}{k} 2^k.$$

Simple computation shows that for $n = 10$, $\max\{W_k, k = 0, \ldots, 10\} = W_7 = 15{,}570$, but $|A_7| = 15{,}612 > W_7$; that is, DG_{10} does not have the Sperner property. Moreover, DG_n does not have the Sperner property for all $n \geq 12$ since $|A_k| > W_k$ if $k > \lfloor \frac{n+3}{2} \rfloor$, but $W_0 \leq W_1 \leq \cdots \leq W_{\lfloor \frac{n+3}{2} \rfloor + 1}$ if $n \geq 12$ (the computational details are omitted).

Let us note that Dilworth and Greene introduced DG_n in an "isomorphic way," namely, as the bond lattice of the graph given in Figure 6.8. Because of these counterexamples we will look at a further property for modular lattices. Let $d^+(p)$

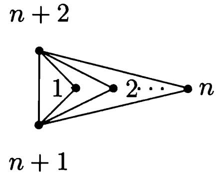


Figure 6.8

(resp. $d^-(p)$) be the *indegree* (resp. *outdegree*) of an element p of P in the Hasse diagram $H(P)$; that is,

$$d^{\pm}(p) := |\{e \in H(P) : e^{\pm} = p\}|.$$

A modular lattice is said to have the *degree-property* if there is an integer h such that

$$d^-(p) > d^+(p) \text{ for all } p \in P \text{ with } r(p) < h,$$

$$d^-(p) < d^+(p) \text{ for all } p \in P \text{ with } r(p) > h.$$

Theorem 6.3.1 **(*Stanley* [443]).** *Every modular lattice P with the degree-property has the Sperner property.*

Proof. In view of Lemma 6.1.2 and Theorem 6.1.1 it is enough to consider the Lefschetz operators and to show that $\widetilde{\nabla}_{L_{i,i+1}}$ is injective for $i < h$ and that $\widetilde{\Delta}_{L_{i+1,i}}$ is injective for $i \geq h$. In addition to these operators, we introduce the operator $H : \widetilde{P} \to \widetilde{P}$ defined on the basis $\{\widetilde{p} : p \in P\}$ by

$$H(\widetilde{p}) := (d^-(p) - d^+(p))\widetilde{p}.$$

Since P is modular we have

$$\widetilde{\Delta}_L \widetilde{\nabla}_L - \widetilde{\nabla}_L \widetilde{\Delta}_L = H, \quad \text{that is,} \quad \widetilde{\nabla}_L^* \widetilde{\nabla}_L - \widetilde{\nabla}_L \widetilde{\nabla}_L^* = H.$$

Let $i < h$ and assume that there is some $\varphi \in \widetilde{N}_i$ such that $\widetilde{\nabla}_L(\varphi) = \widetilde{0}$. Then

$$\begin{aligned} 0 &= \langle \widetilde{\nabla}_L(\varphi), \widetilde{\nabla}_L(\varphi) \rangle = \langle \widetilde{\nabla}_L^* \widetilde{\nabla}_L(\varphi), \varphi \rangle \\ &= \langle (\widetilde{\nabla}_L \widetilde{\nabla}_L^* + H)(\varphi), \varphi \rangle = \langle \widetilde{\nabla}_L^*(\varphi), \widetilde{\nabla}_L^*(\varphi) \rangle + \langle H(\varphi), \varphi \rangle. \end{aligned}$$

If $\varphi = \sum_{p \in N_i} \mu_p \widetilde{p}$ we obtain (noting that the scalar product is positive definite)

$$0 \geq \left\langle \sum_{p \in N_i} \mu_p (d^-(p) - d^+(p)) \widetilde{p}, \sum_{p \in N_i} \mu_p \widetilde{p} \right\rangle = \sum_{p \in N_i} (d^-(p) - d^+(p)) \mu_p^2$$

which implies by our supposition $d^-(p) > d^+(p)$ if $r(p) < h$ that $\mu_p = 0$ for all $p \in N_i$; that is, $\varphi = \widetilde{0}$. Consequently, $\widetilde{\nabla}_{L_{i,i+1}}$ is really injective if $i < h$. The case $i \geq h$ can be treated analogously. ■

We will apply this result to the lattice of subgroups of a certain group. First, note that the lattice of normal subgroups of any group (ordered by inclusion) is modular (cf. Kochendörffer [312, p. 293]). Consequently, the subgroup lattice of an abelian group is modular. If we restrict ourselves to finite *p-groups* (i.e., groups whose order is a power of a prime p) then a fundamental theorem about finite abelian p-groups [312, p. 107] states that any such group G is the direct product of cyclic p-groups:

$$G = (\mathbb{Z}/_{p^{\lambda_1}\mathbb{Z}}) \times \cdots \times (\mathbb{Z}/_{p^{\lambda_n}\mathbb{Z}}) =: G_{\boldsymbol{\lambda}}(p)$$

where, w.l.o.g., $\lambda_1 \leq \cdots \leq \lambda_n$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Thus any element of $G_{\boldsymbol{\lambda}}(p)$ can be written as an n-tuple $\boldsymbol{a} = (a_1, \ldots, a_n)$ where $a_i \in \{0, 1, \ldots, p^{\lambda_i} - 1\}$ for all i, and we have $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ iff $c_i = a_i + b_i \pmod{p^{\lambda_i}}$ for all i. Let $\boldsymbol{k} = (k, \ldots, k)$ (n times) where $k \in \mathbb{N}$.

Proposition 6.3.1. *The subgroup lattice of $G_{\mathbf{1}}(p)$ is isomorphic to $L_n(p)$.*

Proof. Because the elements of $G_{\mathbf{1}}(p)$ can be written as n-tuples, we may identify $G_{\mathbf{1}}(p)$ with an n-dimensional vector space V_n over $GF(p)$ (the addition is the same, and multiplication by a scalar is defined by $\alpha\boldsymbol{a} := \boldsymbol{a} + \cdots + \boldsymbol{a}$ (α times, $\alpha \in \{0, \ldots, p-1\}$)). Using the usual criteria it is easy to verify that a subset U of $G_{\mathbf{1}}$ (with addition) is a subgroup iff U (with addition and multiplication by a scalar) is a subspace of V_n. ■

By Example 6.2.9, $G_{\mathbf{1}}$ is unitary Peck and has, in particular, the Sperner property. However, the subgroup lattice of $G_{(1,2)}(p)$ does not have the Sperner property; see Figure 6.9. The nontrivial subgroups have the following form:

$$U_i := \langle\{(i, p)\}\rangle,\ i = 0, \ldots, p-1,\ U_p := \langle\{(1, 0)\}\rangle$$

$$V_i := \langle\{(i, 1)\}\rangle,\ i = 0, \ldots, p-1,\ V_p := \langle\{(1, 0), (0, p)\}\rangle$$

(where $U = \langle S\rangle$ means that U is generated by the elements of S).

***Corollary 6.3.1* (*Stanley* [443]).** *The subgroup lattice of $G_{\mathbf{2}}(p)$ has the Sperner property.*

Proof. By Theorem 6.3.1 it is sufficient to verify the degree property of the subgroup lattice P of $G_{\mathbf{2}}(p)$. Every element $\boldsymbol{a}$ of $G_{\mathbf{2}}(p)$ has a unique representation of the form $a_i = r_i + \alpha_i p,\ 0 \leq r_i,\ \alpha_i \leq p-1,\ 1 \leq i \leq n$; that is, $\boldsymbol{a} = \boldsymbol{r} + p\boldsymbol{\alpha}$ with

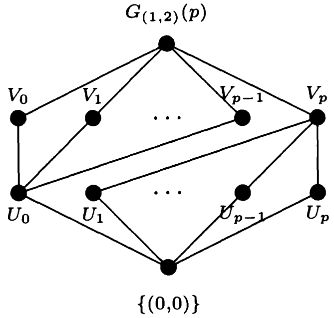


Figure 6.9

$\boldsymbol{r}, \boldsymbol{\alpha} \in V_n$ (the vector space of n-tuples with entries $0, \ldots, p-1$ over $GF(p)$). If U is a subgroup of $G_2(p)$ then obviously

$$R := \{\boldsymbol{r} \in V_n : \text{ there is some } \boldsymbol{\alpha} \in V_n \text{ with } \boldsymbol{r} + p\boldsymbol{\alpha} \in U\},$$

and

$$A := \{\boldsymbol{\alpha} \in V_n : p\boldsymbol{\alpha} \in U\}$$

are subspaces of V_n with $R \subseteq A$. Moreover, for fixed $\boldsymbol{r}$, the set $\{\boldsymbol{\alpha} \in V_n : \boldsymbol{r} + p\boldsymbol{\alpha} \in U\}$ is a residue class in V_n relative to A. Let φ denote the mapping that maps $\boldsymbol{r}$ onto this residue class. Obviously, φ is uniquely determined by the images of the basis elements of R.

It is easy to see that our subgroup U has the order

$$|U| = p^{\dim R} p^{\dim A} = p^{\dim R + \dim A}.$$

Conversely, if we are given subspaces R, A of V_n with $R \subseteq A$, a basis $B = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d\}$ of R and a mapping $\overline{\varphi} : B \to V_n/A$ (suppose that $\boldsymbol{\alpha}_i$ is a representative of $\overline{\varphi}(\boldsymbol{b}_i)$, $i = 1, \ldots d$), then

$$U := \{\lambda_1 \boldsymbol{b}_1 + \cdots + \lambda_d \boldsymbol{b}_d + p(\lambda_1 \boldsymbol{\alpha}_1 + \cdots + \lambda_d \boldsymbol{\alpha}_d) + p\boldsymbol{a} \pmod{p^2} :$$
$$\lambda_1, \ldots, \lambda_d \in \{0, \ldots, p-1\}, \boldsymbol{a} \in A\}$$

is a subgroup of $G_2(p)$ (which then defines $\varphi : R \to V_n/A$ completely; we have $\overline{\varphi} = \varphi|_B$). It is not difficult to see that we have in our subgroup lattice a covering relation $U_1 \lessdot U_2$ iff for the associated triples (R_i, A_i, φ_i), $i = 1, 2$, there holds

(1) $(R_1 \lessdot_{L_n(p)} R_2$ and $A_1 = A_2)$ or $(R_1 = R_2$ and $A_1 \lessdot_{L_n(p)} A_2)$

and

(2) $\varphi_1(\boldsymbol{r}) \subseteq \varphi_2(\boldsymbol{r})$ for each $\boldsymbol{r} \in R_1$.

Assume that U_1 is fixed. Let us count the number of successors of U_1; that is, $d^-(U_1)$. Let $d_i := \dim R_i$, $e_i := \dim A_i$, and B_i be a basis of R_i, $i = 1, 2$. For the first conjunction in Case 1, we have $\binom{e_1-d_1}{1}_p$ possibilities to choose R_2 (note that we need $R_2 \subseteq A_2 = A_1$). If then R_2 is fixed, B_2 is a basis of R_2 that extends B_1, then we may arbitrarily choose φ_2 on the additional basis element (we need $\varphi_1(\boldsymbol{b}) = \varphi_2(\boldsymbol{b})$ for all $\boldsymbol{b} \in B_1$), thus there are p^{n-e_1} choices for φ_2. For the second conjunction in Case 1, we have $\binom{n-e_1}{1}_p$ possibilities to choose A_2 and for each basis element $\boldsymbol{b}$ of B_1 we must choose the residue class relative to A_2 that contains $\varphi_1(\boldsymbol{b})$. Consequently,

$$d^-(U_1) = \frac{p^{e_1-d_1}-1}{p-1}p^{n-e_1} + \frac{p^{n-e_1}-1}{p-1} = \frac{p^{n-d_1}-1}{p-1}.$$

Now let U_2 be fixed and let us determine $d^+(U_2)$. The first conjunction of Case 1 gives $\binom{d_2}{1}_p$ possibilities and the second conjunction of Case 1 (together with Case 2) gives $\binom{e_2-d_2}{1}_p p^{d_2}$ possibilities (we need $R_1 = R_2 \subseteq A_1$ and may choose for every basis element $\boldsymbol{b}$ of B_1 a residue class relative to A_1 which is contained in $\varphi_2(\boldsymbol{b})$, the residue class relative to A_2). Consequently,

$$d^+(U_2) = \frac{p^{d_2}-1}{p-1} + \frac{p^{e_2-d_2}-1}{p-1}p^{d_2} = \frac{p^{e_2}-1}{p-1}.$$

If, finally, U is any subgroup of $G_2(p)$ with the associated vector spaces R and A of dimension d and e, respectively, then $r(U) = d + e$ and

$$d^-(U) > d^+(U) \text{ iff } p^{n-d} > p^e \text{ iff } r(U) < n,$$

$$d^-(U) < d^+(U) \text{ iff } p^{n-d} < p^e \text{ iff } r(U) > n;$$

thus, the degree-property holds. ■

It is conjectured (see [443]) that the subgroup lattice of $G_k(p)$ has the Sperner property for all positive integers k, but already the case $k = 3$ is open. On the other hand, Butler [88] proved the rank symmetry and rank unimodality of $G_{\boldsymbol{\lambda}}(p)$ for every $\boldsymbol{\lambda}$.

Let us now turn to geometric lattices. We already know that they do not, in general, have the Sperner property. But there is a nice conjecture of Kung [329]: Let P be a geometric lattice of rank n. Then the Lefschetz raising operator $\widetilde{\nabla}_L$ in $\widetilde{P}$ has the property that $\widetilde{\nabla}_{L_{i,i+1}}$ is injective (i.e., of full rank) for all $0 \le i < \frac{n}{2}$.

In addition, we conjecture that every geometric lattice is semi-Peck. Concerning the conjecture, there exist up to now only partial results, which we present now.

Lemma 6.3.1. *Let P be a geometric lattice of rank n and let $p, q \in P$.*

(a) If $r(p) = 1$ and $p \not\le q$ then $q \lessdot p \vee q$.

(b) if $r(p) = 1$ and $p \not\le q$ then $q \lessdot v$, $p \le v$ imply $v = p \vee q$.

(c) *If* $r(q) \le n-2$ *then* $d^-(q) \ge 2$.

Proof. (a) Clearly, $q < p \vee q$. Moreover, $r(p \vee q) \le r(p) + r(q) - r(p \wedge q) \le r(q) + 1$; that is, $p \lessdot p \vee q$.

(b) $q \lessdot v$, $p \le v$ and (a) imply $q \lessdot p \vee q \le v$; that is, $p \vee q = v$.

(c) Since $q \neq 1$ there must be some atom p with $p \not\le q$. By (a), $q \lessdot p \vee q$ and thus $p \vee q \neq 1$. Again there must be some atom p' with $p' \not\le p \vee q$. We have $q \lessdot p' \vee q$ and obviously $p' \vee q \neq p \vee q$. ■

***Theorem 6.3.2* (*Kung* [327]).** *Let P be a geometric lattice of rank $n > 1$. There exists an order-raising operator $\widetilde{\nabla}$ on $\widetilde{P}$ such that $\widetilde{\nabla}_{1,j}$ is injective for all $1 \le j \le n-1$.*

Proof. We define $\widetilde{\nabla}$ on the basis $\{\widetilde{p} : p \in P\}$ by

$$\widetilde{\nabla}(\widetilde{p}) := \frac{1}{(d^-(p)-1)} \sum_{p \lessdot q} \widetilde{q} \quad \text{if } r(p) \le n-2,$$

and on the elements $\widetilde{p}$ with $r(p) \ge n-1$ arbitrarily. By Lemma 6.3.1(c), $\widetilde{\nabla}$ is well defined. In order to prove the injectivity we introduce some other linear operators (see also Section 6.4). Let $\Psi_{1,j} : \widetilde{N}_1 \to \widetilde{N}_j$ be given by

$$\Psi_{1,j}(\widetilde{p}) := \sum_{q \in N_j : q \wedge p = 0} \widetilde{q}.$$

Moreover, let $\varphi_1 := \sum_{p \in N_1} \widetilde{p}$; that is, φ_1 is a fixed element of $\widetilde{N}_1$, and let $\Phi_1 : \widetilde{N}_1 \to \widetilde{N}_1$ be given by

$$\Phi_1(\widetilde{p}) := \varphi_1 - \widetilde{p}.$$

Finally, let $\widetilde{\Delta} := \widetilde{\nabla}^*$.

Claim. $\widetilde{\Delta}_{j,1}\Psi_{1,j} = \Phi_1$ for $1 \le j \le n-1$.

Proof of Claim. We proceed by induction on j. The case $j = 1$ is trivial. Since $\widetilde{\Delta}_{j+1,1}\Psi_{1,j+1} = \widetilde{\Delta}_{j,1}(\widetilde{\Delta}_{j+1,j}\Psi_{1,j+1})$, we need only verify the equality

$$\widetilde{\Delta}_{j+1,j}\Psi_{1,j+1} = \Psi_{1,j}, \qquad j = 1, \dots, n-2 \tag{6.23}$$

for the induction step. Let $p \in N_1,\ q \in N_j$ be arbitrary. Then

$$\begin{aligned}
\langle \widetilde{\Delta}\Psi_{1,j+1}(\widetilde{p}), \widetilde{q} \rangle &= \langle \Psi_{1,j+1}(\widetilde{p}), \widetilde{\nabla}(\widetilde{q}) \rangle \\
&= \sum_{v \in N_{j+1} : v \wedge p = 0} \sum_{w \gtrdot q} \left\langle \widetilde{v}, \frac{1}{d^-(q)-1}\widetilde{w} \right\rangle \\
&= \sum_{v \gtrdot q : v \wedge p = 0} \frac{1}{d^-(q)-1} = \begin{cases} 0 & \text{if } p \wedge q = p, \\ 1 & \text{if } p \wedge q = 0 \end{cases}
\end{aligned}$$

since for $p \wedge q = p$, the relation $v \gtrdot q$ implies $v \wedge p = p$ and for $p \wedge q = 0$ by Lemma 6.3.1(b) all but one of the successors of q are not related with p. The same RHS appears for $\langle \Psi_{1,j}(\widetilde{p}), \widetilde{q} \rangle$, thus (6.23) is true. □

It is straightforward to verify that Φ_1 is injective for $n > 1$ (note that then $W_1 > 1$). Consequently, also $\Phi_1^* = \Psi_{1,j}^* \widetilde{\nabla}_{1,j}$ and in particular $\widetilde{\nabla}_{1,j}$ are injective. ■

Corollary 6.3.2. *In every geometric lattice of rank n there exist W_1 pairwise disjoint saturated chains from level N_1 to level N_{n-1}.*

Proof. Consider the operator $\widetilde{\nabla}$ of the preceding theorem restricted to the $\{1, \dots, n-1\}$-rank selected subposet. It satisfies the condition of Theorem 6.1.7, which yields the assertion. ■

In order to generalize this result we need some facts about the Möbius function of a poset P introduced by Rota in [401]. The *Möbius function* $\mu : P \times P \to \mathbb{Z}$ can be defined inductively in the following way:

$$\mu(p,p) := 1 \text{ for all } p \in P,$$

$$\mu(p,q) := -\sum_{p \le v < q} \mu(p,v) \text{ for all } p < q,$$

$$\mu(p,q) := 0 \text{ for all } p, q \in P \text{ with } p \not\le q.$$

Lemma 6.3.2. *We have $\sum_{p \le v \le q} \mu(v,q) = 0$ for all $p < q$.*

Proof. We proceed by induction on $|[p,q]|$. The case $|[p,q]| = 2$; that is, $p \lessdot q$, is trivial, and the induction step works as follows:

$$\sum_{p \le v \le q} \mu(v,q) = 1 + \sum_{p \le v < q} \mu(v,q) = 1 - \sum_{p \le v < q} \sum_{v \le w < q} \mu(v,w)$$
$$= 1 - \sum_{p \le w < q} \sum_{p \le v \le w} \mu(v,w) = 1 - 1 = 0.$$

■

A key result for the Möbius function is the following formula:

Theorem 6.3.3 (Möbius inversion formula). *Let P be a finite poset and G an additive abelian group. Further let $f, g : P \to G$ be two functions. Then*

$$g(p) = \sum_{q : q \le p} f(q) \text{ for all } p \in P \tag{6.24}$$

iff

$$f(p) = \sum_{q:q \le p} g(q)\mu(q, p) \text{ for all } p \in P. \tag{6.25}$$

We may replace in both cases "$\le$" by "$\ge$" (and then, of course, $\mu(q, p)$ by $\mu(p, q)$).

Proof. If (6.24) holds, then by Lemma 6.3.2

$$\sum_{q \le p} g(q)\mu(q, p) = \sum_{q \le p} \sum_{v \le q} f(v)\mu(q, p) = \sum_{v \le p} f(v) \sum_{v \le q \le p} \mu(q, p) = f(p),$$

and if (6.25) holds then by the definition of μ,

$$\sum_{q \le p} f(q) = \sum_{q \le p} \sum_{v \le q} g(v)\mu(v, q) = \sum_{v \le p} g(v) \sum_{v \le q \le p} \mu(v, q) = g(p).$$

It is easy to see that we may everywhere replace "$\le$" by "$\ge$" together with the coordinate changes in μ. ■

There exist several techniques to calculate the Möbius function; cf. Stanley [441]. We need here only the following result.

***Theorem 6.3.4* (*Weisner* [462]).** *Let P be a finite lattice, $0 < p \le q$. Then*

$$\sum_{v \vee p = q} \mu(0, v) = 0.$$

Proof. We proceed by induction on $|[p, q]|$. If $|[p, q]| = 1$, that is, $p = q > 0$, then $\sum_{v \vee p = p} \mu(0, v) = \sum_{0 \le v \le p} \mu(0, v) = 0$ according to the definition of μ. If $|[p, q]| > 1$, that is, $0 < p < q$, then by definition of μ and by classifying with respect to $v \vee p$

$$0 = \sum_{0 \le v \le q} \mu(0, v) = \sum_{p \le w \le q} \left(\sum_{v \vee p = w} \mu(0, v) \right).$$

In view of the induction hypothesis, each term is 0 for $p \le w < q$; thus also the term with $w = q$ equals 0. ■

Theorem 6.3.5. *Let P be a geometric lattice of rank n. Then we have* $\operatorname{sgn}(\mu(0, 1)) = (-1)^n$ *and in particular,* $\mu(0, 1) \ne 0$.

Proof. We proceed by induction on n. The case $n = 1$ is trivial. For the induction step, consider a fixed atom p – that is, $r(p) = 1$. Then by Theorem 6.3.4

$$\sum_{v \vee p = 1} \mu(0, v) = 0. \tag{6.26}$$

We have $v \vee p = 1$ iff either $v = 1$ or $r(v) = n-1$, $p \not\leq v$ (the sufficiency is clear and for the necessity note that $r(v) \geq r(p \wedge v) + r(p \vee v) - r(p) \geq n-1$). Moreover, it is easy to see that the interval $[0, v]$ is a geometric lattice again. Consequently, by the induction hypothesis and (6.26)

$$\mu(0,1) = -\sum_{r(v)=n-1,\, p\not\leq v} \mu(0,v), \quad \operatorname{sgn}(\mu(0,1)) = -(-1)^{n-1} = (-1)^n.$$

■

Let us consider for a fixed geometric lattice P of rank n and a fixed number k, $0 < k < n$, the functions $\varphi, \psi, \chi : P \times P \to \mathbb{R}$ (depending on k) by

$$\varphi(p,q) := \begin{cases} 0 & \text{if } r(q) < n-k, \\ \mu(p,q) & \text{if } r(q) \geq n-k, \end{cases}$$

$$\psi(p,q) := \sum_{p \leq v \leq q} \varphi(p,v),$$

$$\chi(p,q) := \frac{\mu(q,1)}{\mu(p,1)} \psi(p,q). \tag{6.27}$$

Note that $\mu(p,1) \neq 0$ by Theorem 6.3.5 since the interval $[p,1]$ is also a geometric lattice.

Lemma 6.3.3. *We have*

(a) $\chi(p,q) = 0$ *if* $r(q) < n-k$,

(b) $\sum_{y:p\leq y} \chi(p,y) = 1$,

(c) $\sum_{y:y\geq p\vee w} \chi(p,y) = 0$ *if* $r(w) \leq k$, $w \not\leq p$.

Proof. (a) is clear, (b) and (c) we prove simultaneously by applying Möbius inversion to the interval $[x, 1]$, where briefly $x := p \vee w$. For $x \leq q \leq 1$, let

$$f_p(q) := \sum_{v\geq p: v\vee x = q} \varphi(p,v),$$

$$g_p(q) := \sum_{x\leq y\leq q} f_p(y).$$

Then

$$g_p(q) = \sum_{x\leq y\leq q} \sum_{v\geq p: v\vee x=y} \varphi(p,v) = \sum_{p\leq v\leq q} \varphi(p,v) = \psi(p,q).$$

Consequently, by Theorem 6.3.3

$$f_p(q) = \sum_{x\leq y\leq q} g_p(y)\mu(y,q) = \sum_{x\leq y\leq q} \psi(p,y)\mu(y,q).$$

In particular, for $q = 1$ it follows that

$$f_p(1) = \sum_{y:x\le y} \psi(p, y)\mu(y, 1) = \mu(p, 1) \sum_{y:x\le y} \chi(p, y). \tag{6.28}$$

However,

$$f_p(1) = \sum_{v\ge p:v\vee x=1} \varphi(p, v) = \sum_{v\ge p:r(v)\ge n-k, v\vee x=1} \mu(p, v).$$

If $v \ge p$, $v\vee x = 1$ and $r(w) \le k$, then $r(v) \ge r(v\vee w)+r(v\wedge w)-r(w) \ge n-k$. Hence we may continue, using Theorem 6.3.4 applied to the lattice $[p, 1]$, yielding

$$f_p(1) = \sum_{v\ge p:v\vee x=1} \mu(p, v) = \begin{cases} 0 & \text{if } x > p \quad (\text{i.e., } w \not\le p), \\ \mu(p, 1) & \text{if } x = p \quad (\text{i.e., } w \le p). \end{cases} \tag{6.29}$$

From (6.28) and (6.29) we may easily derive (c) and (with $w = 0$) also (b). ■

In addition to the Lefschetz operators we define the (linear) *rank i lowering (resp. raising) operator* $\widetilde{\Delta}_{\to i}$ (resp. $\widetilde{\nabla}_{\to i}$): $\widetilde{P} \to \widetilde{P}$ as follows:

$$\widetilde{\Delta}_{\to i}(\widetilde{p}) := \sum_{q\le p:r(q)=i} \widetilde{q} \qquad (\text{resp. } \widetilde{\nabla}_{\to i}(\widetilde{p}) := \textstyle\sum_{q\ge p:r(q)=i} \widetilde{q}).$$

More generally, if F, G are subsets of P then we define the operator $\widetilde{\Delta}_{F\to G}$: $\widetilde{F} \to \widetilde{G}$ and its adjoint $\widetilde{\nabla}_{G\to F}$ (for $p \in F$ (resp. $p \in G$)) by

$$\widetilde{\Delta}_{F\to G}(\widetilde{p}) := \sum_{q\in G:q\le p} \widetilde{q} \quad \left(\text{resp. } \widetilde{\nabla}_{G\to F}(\widetilde{p}) := \sum_{q\in F:q\ge p} \widetilde{q}\right). \tag{6.30}$$

So $\widetilde{\Delta}_{\to i}$ abbreviates $\widetilde{\Delta}_{P\to N_i}$. Moreover, let $\widetilde{\Delta}_{i\to j} := \widetilde{\Delta}_{N_i\to N_j}$ and $\widetilde{\nabla}_{i\to j} := \widetilde{\nabla}_{N_i\to N_j}$. Note that the matrix of $\widetilde{\nabla}_{i\to j}$ with respect to the bases $\{\widetilde{p} : p \in N_i\}$ and $\{\widetilde{p} : p \in N_j\}$ is exactly the incidence matrix of rank j versus rank i elements.

Lemma 6.3.4. *Let P be a geometric lattice of rank n, $0 < k < n$, and let $\chi : P \times P \to \mathbb{R}$ be defined as in (6.27). Then for $0 \le i \le k$,*

$$\widetilde{\Delta}_{\to i}(\widetilde{p}) = \widetilde{\Delta}_{\to i}\left(\sum_{y\ge p} \chi(p, y)\widetilde{y}\right).$$

Proof. Take any $w \in N_i$. Then by Lemma 6.3.3 (noting $i = r(w) \le k$)

$$\begin{aligned}\sum_{y\ge p} \chi(p, y)\langle\widetilde{\Delta}_{\to i}(\widetilde{y}), \widetilde{w}\rangle = \sum_{y\ge p\vee w} \chi(p, y) &= \begin{cases} 1 & \text{if } w \le p, \\ 0 & \text{if } w \not\le p \end{cases} \\ &= \langle\widetilde{\Delta}_{\to i}(\widetilde{p}), \widetilde{w}\rangle.\end{aligned}$$

■

***Theorem 6.3.6* (Dowling and Wilson [140]).** *Let P be a geometric lattice of rank n, $0 < k < n$.*

(a) The linear operator $\widetilde{\nabla}_{N_0\cup\cdots\cup N_k \to N_{n-k}\cup\cdots\cup N_n}$ is injective.

(b) The linear operator $\widetilde{\nabla}_{N_1\cup\cdots\cup N_k \to N_{n-k}\cup\cdots\cup N_{n-1}}$ is injective.

Proof. (a) Briefly let $T_0 := \widetilde{\nabla}_{N_0\cup\cdots\cup N_k \to N_{n-k}\cup\cdots\cup N_n}$. Let $\varphi = \sum_{r(p)\le k} \lambda_p \widetilde{p}$ be any element of $\widetilde{N}_0 \oplus \cdots \oplus \widetilde{N}_k$. We must show that $T_0(\varphi) = \widetilde{0}$ implies $\varphi = \widetilde{0}$. If we define $f, g : P \to \mathbb{R}$ by

$$f(p) := \begin{cases} \lambda_p & \text{if } r(p) \le k, \\ 0 & \text{otherwise,} \end{cases}$$

$$g(q) := \sum_{p\le q} f(p),$$

then $T_0(\varphi) = \sum_{r(q)\ge n-k} g(q)\widetilde{q}$. Let $T_0(\varphi) = \widetilde{0}$; that is, $g(q) = 0$ for all $q \in P$ with $r(q) \ge n-k$. We have to verify that $f \equiv 0$ (i.e., $\varphi = \widetilde{0}$). By Theorem 6.3.3 it is enough to show that $g \equiv 0$, that is, $g(q) = 0$ also for all $q \in P$ with $r(q) < n-k$. So let us consider such an element q. By Lemma 6.3.4 and Lemma 6.3.3(a) we have

$$\begin{aligned} \sum_{i=0}^{k} \widetilde{\Delta}_{\to i}(\widetilde{q}) &= \sum_{i=0}^{k} \sum_{y\ge q, r(y)\ge n-k} \chi(q, y)\widetilde{\Delta}_{\to i}(\widetilde{y}) \\ &= \sum_{y\ge q, r(y)\ge n-k} \chi(q, y) \sum_{i=0}^{k} \widetilde{\Delta}_{\to i}(\widetilde{y}). \end{aligned}$$

If we apply, finally, the linear operator $F : \widetilde{P} \to \mathbb{R}$ defined by

$$F(\widetilde{p}) := f(p) \quad \text{for all } p \in P$$

to the LHS and RHS (noting that $F\left(\sum_{i=0}^{k} \widetilde{\Delta}_{\to i}(\widetilde{z})\right) = \sum_{p\le z, r(p)\le k} f(p) = g(z)$), we obtain

$$g(q) = \sum_{y\ge q, r(y)\ge n-k} \chi(q, y)g(y) = 0. \tag{6.31}$$

(b) We may argue as in (a), but we may suppose that $f(0) = 0$ and must admit that $g(1) \ne 0$. Under these conditions, (6.31) reads

$$g(q) = \chi(q, 1)g(1) \quad \text{for all } q \in P.$$

For $q = 0$, we obtain in particular

$$0 = f(0) = g(0) = \chi(0, 1)g(1) = \frac{g(1)}{\mu(0, 1)} \sum_{r(q)\ge n-k} \mu(0, q).$$

If $g(1) = 0$ then, by the same reasons as in (a), $f \equiv 0$. So it remains to show that

$$\sum_{r(q) \geq n-k} \mu(0, q) \neq 0.$$

Let P^+ be the lattice that can be obtained from P by identifying all elements of rank at least $n-k$ (the *truncation to rank* $n-k$). Let μ^+ be the associated Möbius function. It is easy to verify that P^+ is a geometric lattice again; hence (using twice the definition of the Möbius function)

$$\sum_{r(q) \geq n-k} \mu(0, q) = -\sum_{r(q) < n-k} \mu(0, q) = -\sum_{r(q) < n-k} \mu^+(0, q) = \mu^+(0, 1),$$

and the RHS is nonzero by Theorem 6.3.5. ■

We say that the geometric lattice P of rank n has the *lowering property* if for all i, j, k with $0 \leq i \leq j \leq n-i \leq k \leq n-1$ there are linear operators $\Phi_{i,k,j} : \widetilde{N}_k \to \widetilde{N}_j$ such that

$$\widetilde{\Delta}_{j \to i} \Phi_{i,k,j} = \widetilde{\Delta}_{k \to i}. \tag{6.32}$$

For such lattices, we can prove a generalization of Kung's conjecture:

Theorem 6.3.7. *Let P be a geometric lattice of rank n. It has the lowering property iff $\widetilde{\nabla}_{i \to j}$ is injective for all i, j with $0 \leq i \leq j,\ i + j \leq n$.*

Proof. Suppose that P has the lowering property. The case $i = 0$ is trivial, so let $i > 0$. We have, by the lowering property,

$$\sum_{l=n-i}^{n-1} \widetilde{\Delta}_{j \to i} \Phi_{i,l,j} = \sum_{l=n-i}^{n-1} \widetilde{\Delta}_{l \to i}.$$

Taking the adjoint, we obtain

$$\left(\sum_{l=n-i}^{n-1} \Phi^*_{i,l,j} \right) \widetilde{\nabla}_{i \to j} = \sum_{l=n-i}^{n-1} \widetilde{\nabla}_{i \to l}.$$

It is easy to see that the RHS equals the operator T (with $k = i$) restricted to $\widetilde{N}_i$. Since this operator is injective by Theorem 6.3.6(b), $\widetilde{\nabla}_{i \to j}$ must be injective, too. Now suppose the injectivity of all the $\widetilde{\nabla}_{i \to j}$ and let $0 \leq i \leq j \leq n-i \leq k \leq n-1$. Let $p \in N_k$. Since $\widetilde{\nabla}_{i \to j}$ is injective, its adjoint $\widetilde{\Delta}_{j \to i}$ is surjective, consequently there exists some $\varphi_p \in \widetilde{N}_j$ such that $\widetilde{\Delta}_{j \to i}(\varphi_p) = \widetilde{\Delta}_{k \to i}(\widetilde{p})$. Thus we may define

$$\Phi_{i,k,j}(\widetilde{p}) := \varphi_p.$$

■

We conjecture that every geometric lattice has the lowering property.

Example 6.3.2. *The Boolean lattice B_n, the linear lattice $L_n(q)$, the affine poset $A_n(q)$, and the Dilworth–Greene lattice DG_n (see Example 6.3.1) have the lowering property.*

For B_n, take $\Phi_{i,k,j} := \frac{1}{\binom{k-i}{j-i}}\widetilde{\Delta}_{k\to j}$, for $L_n(q)$ and $A_n(q)$, take $\Phi_{i,k,j} := \frac{1}{\binom{k-i}{j-i}_q}\widetilde{\Delta}_{k\to j}$. For DG_n, we argue as follows: Let $1 \le i \le j \le n-i \le k \le n$ and let $p \in N_k$. Then p belongs either to the Q_n^*-part or to the B_n-part of DG_n, more precisely:

$$p \in N_k(Q_n^*) \quad \text{or} \quad p \in N_{k-1}(B_n).$$

We define

$$\Phi_{i,k,j} := \begin{cases} \frac{1}{\binom{k-i}{j-i}} \sum_{q \in N_j(Q_n^*): q \le p} \widetilde{q} \\ \qquad \text{if } p \in N_k(Q_n^*), \\ \frac{1}{\binom{k-i}{j-i}} \left(\sum_{q \in N_{j-1}(B_n): q \le p} \widetilde{q} + 2^{i-j} \sum_{q \in N_j(Q_n^*): q \le p} \widetilde{q} \right) \\ \qquad \text{if } p \in N_{k-1}(B_n). \end{cases}$$

Then (6.32) holds. This is easy to see for $p \in N_k(Q_n^*)$ since the ideal generated by p is a Boolean lattice. So let $p \in N_{k-1}(B_n)$ and take any $v \in N_i$. It is clear that

$$\langle \widetilde{\Delta}_{j\to i} \Phi_{i,k,j}(\widetilde{p}), \widetilde{v} \rangle = 0 \text{ if } v \not\le p.$$

Now let $v \le p$. If $v \in N_{i-1}(B_n)$, then

$$\begin{aligned} \langle \widetilde{\Delta}_{j\to i} \Phi_{i,k,j}(\widetilde{p}), \widetilde{v} \rangle &= \frac{1}{\binom{k-i}{j-i}} |\{q \in N_{j-1}(B_n) : v \le q \le p\}| \\ &= \frac{1}{\binom{k-i}{j-i}} \binom{k-1-(i-1)}{j-1-(i-1)} = 1, \end{aligned}$$

and if $v \in N_i(Q_n^*)$ then

$$\begin{aligned} \langle \widetilde{\Delta}_{j\to i} \Phi_{i,k,j}(\widetilde{p}), \widetilde{v} \rangle &= \frac{1}{\binom{k-i}{j-i}} \big(|\{q \in N_{j-1}(B_n) : v \le q \le p\}| \\ &\quad + 2^{i-j} |\{q \in N_j(Q_n^*) : v \le q \le p\}| \big) \\ &= \frac{1}{\binom{k-i}{j-i}} \left(\binom{k-1-i}{j-1-i} + 2^{i-j} \binom{k-1-i}{j-i} 2^{j-i} \right) = 1. \end{aligned}$$

The same result holds for $\langle \widetilde{\Delta}_{k\to i}(\widetilde{p}), \widetilde{v} \rangle$; thus (6.32) is verified.

It is not known whether the partition lattice Π_n has the lowering property, but Kung proved in [329] that in Π_n the operators $\widetilde{\nabla}_{i\to i+1} = \widetilde{\nabla}_{L_{i,i+1}}$ are injective for all $0 \le i < \frac{n}{2}$. In addition to Example 6.3.2, Bey [54] proved the full rank property of the Lefschetz operators on the affine lattice using some ideas we have examined here.

We conclude the study of geometric lattices with a theorem that was found by Vapnik and Chervonenkis [455] (in an implicit form), Sauer [415], Perles (see [429]), and Shelah [429] for the Boolean lattice. After Frankl and Pach [201] had given an algebraic proof for the theorem, it was Lefmann [338] who formulated and proved the result for geometric lattices. Much more information can be found in the survey by Füredi and Pach [212].

Theorem 6.3.8. *Let P be a geometric lattice of rank n and F a subset of P with $|F| > \sum_{i=0}^{s} W_i$. Then there is some $q \in N_{s+1}$ such that for every $v \in P$ with $v \le q$ there is some $p \in F$ such that $p \wedge q = v$.*

Proof. Briefly, let $G := \cup_{i=0}^{s} N_i$. Since $\dim \widetilde{F} = |F| > |G| = \dim \widetilde{G}$, there must be some nonzero element φ in $\widetilde{F}$ with

$$\widetilde{\Delta}_{F \to G}(\varphi) = \widetilde{0}. \tag{6.33}$$

Let $\varphi = \sum_{p \in F} \lambda_p \widetilde{p}$ and $F' := \{p \in F : \lambda_p \neq 0\}$. If we define for $q \in P$

$$\delta_q := \sum_{p \in F : p \ge q} \lambda_p,$$

then obviously

$$\widetilde{\Delta}_{F \to P}(\varphi) = \sum_{q \in P} \delta_q \widetilde{q}.$$

There exist elements $q \in P$ such that $\delta_q \neq 0$ because, for example, if q is a maximal element of F' then $\delta_q = \lambda_q \neq 0$. Under all elements q with $\delta_q \neq 0$ we choose one with minimum rank. For this element q^*, we have $r(q^*) \ge s + 1$ because otherwise $q^* \in G$ and thus $\langle \widetilde{\Delta}_{F \to P}(\varphi), \widetilde{q^*} \rangle = \langle \widetilde{\Delta}_{F \to G}(\varphi), \widetilde{q^*} \rangle = 0$ by (6.33). Though it may be that $r(q^*) > s + 1$, let us first prove the statement in the theorem for this element. Take any $v \in P$ with $v \le q^*$. We will apply Möbius inversion to the interval $[v, q^*]$. Let $f, g : [v, q^*] \to \mathbb{R}$ be defined by

$$f(x) := \sum_{p \in F : p \wedge q^* = x} \lambda_p, \quad g(x) := \sum_{x \le y \le q^*} f(y), \ x \in [v, q^*].$$

We have

$$g(x) = \sum_{p \in F : x \le p \wedge q^* \le q^*} \lambda_p = \sum_{p \in F : p \ge x} \lambda_p = \delta_x.$$

By Theorem 6.3.3 ,

$$f(x) = \sum_{x \le y \le q^*} \mu(x, y) g(y),$$

and in particular (noting the minimal choice of q^*)

$$f(v) = \sum_{v \leq y \leq q^*} \mu(v, y)\delta_y = \mu(v, q^*)\delta_{q^*}.$$

Since the interval $[v, q^*]$ is also a geometric lattice, we derive from Theorem 6.3.5 that $f(v) \neq 0$. But $f(v) = \sum_{p \in F: p \wedge q^* = v} \lambda_p$; thus the sum on the RHS cannot be empty. Consequently there really exists some $p \in F$ with $p \wedge q^* = v$. Finally take any $q \leq q^*$ with $r(q) = s + 1$. For $v \leq q \leq q^*$, there exists some $p \in F$ with $p \wedge q^* = v$, which implies $p \wedge q = v$. ■

Let us now study distributive lattices. Their width is "small" relative to their cardinality. This is a result of Kahn and Saks [282] mentioned here without proof:

Theorem 6.3.9. *For every $\epsilon > 0$, there exists an integer $n_0(\epsilon)$ with the property: If P is a distributive lattice of cardinality greater than n_0 then $\frac{d(P)}{|P|} < \epsilon$.*

Our main objective is the presentation of an inequality given by Ahlswede and Daykin [8] that has numerous applications. Though this inequality has a simple proof, much work preceded its publication. West [464] gave a good survey on the history of this inequality (see also Ahlswede and Daykin [9] and Graham [227]). Cornerstones of this theory are the starting result of Kleitman [300] on the intersection of ideals and filters in the Boolean lattice and the result of Fortuin, Kasteleyn, and Ginibre [185] who generalized Chebyshev's inequality for nondecreasing sequences in order to unify several results in statistical mechanics. However, the discovery of the Ahlswede–Daykin inequality was independent of these beginnings; it should be noted that there is a nonlattice theoretic background studied in detail in Ahlswede and Daykin [9].

Lemma 6.3.5. *Let a_i, b_i, c_i, d_i $(i = 1, 2)$ be nonnegative real numbers such that*

$$a_0 b_0 \leq c_0 d_0, \tag{6.34}$$

$$a_1 b_0 \leq c_1 d_0, \tag{6.35}$$

$$a_0 b_1 \leq c_1 d_0, \tag{6.36}$$

$$a_1 b_1 \leq c_1 d_1. \tag{6.37}$$

Then

$$(a_0 + a_1)(b_0 + b_1) \leq (c_0 + c_1)(d_0 + d_1). \tag{6.38}$$

Proof. If $a_1 = 0$ or $b_0 = 0$ the proof is trivial. Thus we may suppose $a_1 \neq 0$ and $b_0 \neq 0$. Then (6.35) yields $c_1 \neq 0$ and $d_0 \neq 0$. We derive from (6.34) and (6.37) that

$$\frac{a_0 a_1 b_0 b_1}{c_1 d_0} \leq c_0 d_1. \tag{6.39}$$

Further, (6.35), (6.36), and (6.39) imply

$$\begin{aligned} 0 \leq \frac{(a_1 b_0 - c_1 d_0)(a_0 b_1 - c_1 d_0)}{c_1 d_0} &= \frac{a_0 a_1 b_0 b_1}{c_1 d_0} - a_1 b_0 - a_0 b_1 + c_1 d_0 \\ &\leq c_0 d_1 - a_1 b_0 - a_0 b_1 + c_1 d_0, \end{aligned}$$

that is,

$$a_1 b_0 + a_0 b_1 \leq c_1 d_0 + c_0 d_1. \tag{6.40}$$

Summing up (6.34), (6.37), and (6.40) we obtain the assertion (6.38). ∎

Lemma 6.3.6. *Let $n \in \mathbb{N}$ and $\alpha, \beta, \gamma, \delta$ be functions from $2^{[n]}$ into $\mathbb{R}_+$ such that*

$$\alpha(X)\beta(Y) \leq \gamma(X \cup Y)\delta(X \cap Y) \textit{ for all } X, Y \in 2^{[n]}. \tag{6.41}$$

Then

$$\alpha(2^{[n]})\beta(2^{[n]}) \leq \gamma(2^{[n]})\delta(2^{[n]}), \tag{6.42}$$

where, as usual, $\alpha(2^{[n]}) = \sum_{X \in 2^{[n]}} \alpha(X)$ and so on.

Proof. We proceed by induction on n. If $n = 0$, that is, $2^{[n]} = \{\emptyset\}$, then (6.42) is the same as (6.41). Now consider the step $n-1 \to n$. We define $\alpha' : 2^{[n-1]} \to \mathbb{R}_+$ by

$$\alpha'(X') := \alpha(X') + \alpha(X' \cup \{n\}) \text{ for all } X' \in 2^{[n-1]},$$

and with β', γ', δ' we proceed analogously. Then

$$\alpha'(2^{[n-1]}) = \alpha(2^{[n]}) \text{ and so on.}$$

Thus, by the induction hypothesis, it is sufficient to prove that for all $X', Y' \in 2^{[n-1]}$

$$\alpha'(X')\beta'(Y') \leq \gamma'(X' \cup Y')\delta'(X' \cap Y');$$

that is,

$$\begin{aligned} &(\alpha(X') + \alpha(X' \cup \{n\}))(\beta(Y') + \beta(Y' \cup \{n\})) \\ &\quad \leq (\gamma(X' \cup Y') + \gamma(X' \cup Y' \cup \{n\})(\delta(X' \cap Y') + \delta(X' \cap Y' \cup \{n\})). \end{aligned} \tag{6.43}$$

Let $a_0 := \alpha(X')$, $a_1 := \alpha(X' \cup \{n\})$, $b_0 := \beta(Y')$, $b_1 := \beta(Y' \cup \{n\})$, $c_0 := \gamma(X' \cup Y')$, $c_1 := \gamma(X' \cup Y' \cup \{n\})$, $d_0 := \delta(X' \cap Y')$, $d_1 := \delta(X' \cap Y' \cup \{n\})$. Then (6.34)–(6.37) are satisfied by our supposition (6.41). Lemma 6.3.5 yields the asserted inequality (6.43). ∎

Theorem 6.3.10 (Four-Function Theorem of Ahlswede and Daykin). *Let P be a distributive lattice and $\alpha, \beta, \gamma, \delta$ functions from P into $\mathbb{R}_+$ such that*

$$\alpha(p)\beta(q) \le \gamma(p \vee q)\delta(p \wedge q) \quad \textit{for all } p, q \in P. \tag{6.44}$$

Then

$$\alpha(A)\beta(B) \le \gamma(A \cup B)\delta(A \wedge B) \textit{ for all } A, B \subseteq P \tag{6.45}$$

where $A \vee B := \{p \vee q : p \in A \text{ and } q \in B\}$, $A \wedge B := \{p \wedge q : p \in A \text{ and } q \in B\}$, and $\alpha(A) := \sum_{p \in A} \alpha(p)$ and so on.

Proof. By a theorem of Birkhoff (cf. [21, p. 33]), there is some set N, say $[n]$, and an injective mapping $\varphi : P \to 2^{[n]}$ such that

$$\varphi(p \vee q) = \varphi(p) \cup \varphi(q) \quad \text{and} \quad \varphi(p \wedge q) = \varphi(p) \cap \varphi(q). \tag{6.46}$$

Let $A, B \subseteq P$ be fixed and let $\varphi(A) := \{\varphi(p) : p \in A\}$ and so on. We define functions $\alpha', \beta', \gamma', \delta'$ from $2^{[n]}$ into $\mathbb{R}_+$ by

$$\alpha'(X) := \begin{cases} \alpha(\varphi^{-1}(X)) & \text{if } X \in \varphi(A), \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma'(X) := \begin{cases} \gamma(\varphi^{-1}(X)) & \text{if } X \in \varphi(A \cup B), \\ 0 & \text{otherwise} \end{cases}$$

(β', δ' are defined analogously). Then we have $\alpha(A) = \alpha'(2^{[n]})$, $\beta(B) = \beta'(2^{[n]})$, $\gamma(A \vee B) = \gamma'(2^{[n]})$, and $\delta(A \wedge B) = \delta'(2^{[n]})$. By Lemma 6.3.6, we must only prove that

$$\alpha'(X)\beta'(Y) \le \gamma'(X \cup Y)\delta'(X \cap Y) \text{ for all } X, Y \in 2^{[n]}.$$

This is trivially satisfied if $X \notin \varphi(A)$ or $Y \notin \varphi(B)$, and otherwise the last inequality follows from (6.46) and the supposition (6.44). ■

We will not discuss the questions of equality in this inequality. For a special case, Beck [40] derived necessary and sufficient conditions for the equality to hold.

Corollary 6.3.3. *Let P be the Boolean lattice B_n. Then*

(*a*) (*Kleitman* [300]) $|F \cap I||P| \le |F||I|$ *for all filters F and ideals I in P.*

(*b*) (*Marica and Schönheim* [361]) $|A| \le |A \ominus A|$ *for all $A \subseteq P = 2^{[n]}$ where $A \ominus A := \{X - Y : X, Y \in A\}$.*

Proof. Let in Theorem 6.3.10 $\alpha \equiv \beta \equiv \gamma \equiv \delta \equiv 1$. Then (6.44) is satisfied automatically.

(a) Put $A := F \cap I$ and note that $A \vee P \subseteq F$ and $A \wedge P \subseteq I$. Then by Theorem 6.3.10 $|A||P| \leq |A \vee P||A \wedge P| \leq |F||I|$.

(b) For a subset A of P, let $\overline{A}$ be the set of all complements of elements of A. Then $A \ominus A = A \wedge \overline{A} = \overline{A} \wedge A$. Now Theorem 6.3.10 yields $|A|^2 = |A||\overline{A}| \leq |A \vee \overline{A}||A \wedge \overline{A}| = |\overline{(A \vee \overline{A})}||A \ominus A| = |\overline{A} \wedge A||A \ominus A| = |A \ominus A|^2$. ∎

Part (b) of the preceding corollary was generalized by Daykin and Lovász [126] and by Ahlswede and Daykin [10] in several ways. Let us look at some nice implications of Part (a). Anderson [31] and Kleitman [234] observed that the following result of Seymour [426], Schönheim [417], Daykin, and Lovász [126], and Hilton [266] can be proved easily with this "theory."

Corollary 6.3.4. *Let $\mathcal{G} \subseteq 2^{[n]}$ be an intersecting, cointersecting family; that is, $X \cap Y \neq \emptyset$ and $X \cup Y \neq [n]$ for all $X, Y \in \mathcal{G}$. Then $|\mathcal{G}| \leq 2^{n-2}$, and this bound is the best possible.*

Proof. Let F (resp. I) be the filter (resp. ideal) generated by $\mathcal{G}$ in the Boolean lattice $P = B_n$. It is easy to see that F (resp. I) is an intersecting (resp. cointersecting) family and that $\mathcal{G} \subseteq F \cap I$. Consequently, by Corollary 6.3.3 and Theorem 2.1.6,

$$|\mathcal{G}| \leq |F \cap I| \leq \frac{1}{|P|}|F||I| \leq \frac{1}{2^n} 2^{n-1} 2^{n-1} = 2^{n-2}.$$

Because $\mathcal{G}^* := \{X \subseteq 2^{[n]} : 1 \in X, n \notin X\}$ is an intersecting, cointersecting family of size 2^{n-2}, the bound is the best possible. ∎

Part (a) of Corollary 6.3.3 (one of the origins of the Four-Function Theorem) originated when proving the following result:

Corollary 6.3.5 (Kleitman [300]). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting families in $2^{[n]}$. Then for all k, n with $1 \leq k \leq n$*

$$|\cup_{i=1}^{k} \mathcal{F}_i| \leq 2^n - 2^{n-k}.$$

Proof. We proceed by induction on k. The case $k = 1$ is given by Theorem 2.1.6. So let us consider the step $k-1 \to k$. We may assume that each $\mathcal{F}_i$ is a maximal intersecting family. Then, for $i = 1, \ldots, k$, $|\mathcal{F}_i| = 2^{n-1}$ (see Theorem 2.1.6) and $\mathcal{F}_i$ is a filter. Clearly, $F := \cup_{i=1}^{k-1} \mathcal{F}_i$ is a filter, too, and $I := 2^{[n]} - \mathcal{F}_k$ is an ideal. By the induction hypothesis, $|F| \leq 2^n - 2^{n-k+1}$ and by Corollary 6.3.3(a)

$$|F \cap I| \leq 2^{-n} 2^{n-1}(2^n - 2^{n-k+1}) = 2^{n-1} - 2^{n-k}.$$

Consequently,

$$|F \cap \mathcal{F}_k| = |F| - |F \cap I| \geq |F| - 2^{n-1} + 2^{n-k},$$

and finally

$$\begin{aligned}|\cup_{i=1}^{k}\mathcal{F}_i| &= |F\cup\mathcal{F}_k| = |F| + |\mathcal{F}_k| - |F\cap\mathcal{F}_k| \\ &\le |F| + 2^{n-1} - |F| + 2^{n-1} - 2^{n-k} = 2^n - 2^{n-k}.\end{aligned}$$

Taking $\mathcal{F}_i := \{X \subseteq 2^{[n]} : i \in X\}$, $i = 1, \ldots, k$, we see that the bound is the best possible. ∎

On the LHS of the Four-Function Theorem we sum over all pairs $(p, q) \in A \times B$ and on the RHS over all pairs of $(A \vee B) \times (A \wedge B)$. Reuter [398] showed that one can restrict the summation to pairs (p, q) satisfying additionally $p \wedge q = v$ and $p \vee q = w$ where v and w are fixed elements of P. The following $2m$-Function Theorem is an impressive generalization of the Four-Function Theorem. It was discovered by Rinott and Saks [400] and, independently, by Aharoni and Keich [1].

Theorem 6.3.11. *Let $m \geq 1$, let P be a distributive lattice, and let f_i, g_i, $i \in [m]$, be functions from P into $\mathbb{R}_+$ such that*

$$\prod_{i=1}^{m} f_i(p_i) \le \prod_{i=1}^{m} g_i\left(\bigvee_{S\subseteq\binom{[m]}{i}}\bigwedge_{j\in S} p_j\right) \quad \textit{for all } p_i \in P,\ i = 1,\ldots,m.$$

Then

$$\prod_{i=1}^{m} f_i(A_i) \le \prod_{i=1}^{m} g_i\left(\bigvee_{S\subseteq\binom{[m]}{i}}\bigwedge_{j\in S} A_j\right) \quad \textit{for all } A_i \subseteq P,\ i = 1,\ldots,m,$$

where $\bigvee_{S\subseteq\binom{[m]}{i}}\bigwedge_{j\in S} A_j := \{\bigvee_{S\subseteq\binom{[m]}{i}}\bigwedge_{j\in S} p_j : p_j \in A_j\}$, *and* $f_i(A_i) := \sum_{p_i\in A_i} f_i(p_i)$ *and so forth.*

I omit the proof of this result which is a culmination after preparatory work in particular by Daykin [124]. In fact, Rinott and Saks [400] proved more than Theorem 6.3.11 presenting an integral version of their result. Older versions including continuous versions of the inequalities are those of Harris [262], Holley [271], Preston [386], Ruzsa [403], and Batty and Bollmann [39]. A further important corollary of the Four-Function Theorem, due to Fortuin, Kasteleyn, and Ginibre [185], is, historically, a predecessor of it.

Theorem 6.3.12 (FKG-inequality). *Let f and g be both increasing (or both decreasing) functions on a distributive lattice P, and let $w : P \to \mathbb{R}_+$ satisfy the condition*

$$w(p)w(q) \le w(p\vee q)w(p\wedge q) \textit{ for all } p, q \in P. \tag{6.47}$$

Then

$$\left(\sum_{p\in P} f(p)w(p)\right)\left(\sum_{p\in P} g(p)w(p)\right) \le \left(\sum_{p\in P} f(p)g(p)w(p)\right)\left(\sum_{p\in P} w(p)\right). \tag{6.48}$$

Proof. We consider only the case of increasing functions. First we take characteristic functions $f = \varphi_F$ and $g = \varphi_G$, where F and G are filters of P (φ_F and φ_G are increasing). Taking $\alpha := \beta := \gamma := \delta := w$, $A := F$, $B := G$ in Theorem 6.3.10 and noting that $F \vee G$ is the set intersection of F and G, we obtain

$$\begin{aligned}&\left(\sum_{p\in P} \varphi_F(p)w(p)\right)\left(\sum_{p\in P} \varphi_G(p)w(p)\right)\\ &\qquad = w(F)w(G) \le w(F\vee G)w(F\wedge G)\\ &\qquad \le w(F\vee G)w(P)\\ &\qquad = \left(\sum_{p\in P} \varphi_F(p)\varphi_G(p)w(p)\right)\left(\sum_{p\in P} w(p)\right),\end{aligned}$$

and the FKG-inequality is proved in that special case. Now let f and g be arbitrary increasing functions. By Lemma 4.4.3 we can write f and g in the form

$$f = \lambda_0 + \sum_{i=1}^{h} \lambda_i \varphi_{F_i}, \qquad g = \nu_0 + \sum_{j=1}^{k} \nu_j \varphi_{G_j},$$

where F_i and G_j are suitable filters and $\lambda_i \ge 0$, $\nu_j \ge 0$, $i = 1, \ldots, h$, $j = 1, \ldots, k$. After a simple computation, we derive

$$\begin{aligned}&\left(\sum_{p\in P} f(p)w(p)\right)\left(\sum_{p\in P} g(p)w(p)\right) - \left(\sum_{p\in P} f(p)g(p)w(p)\right)\left(\sum_{p\in P} w(p)\right)\\ &\quad = \sum_{i=1}^{h}\sum_{j=1}^{k} \lambda_i\nu_j \left(\left(\sum_{p\in P} \varphi_{F_i}(p)w(p)\right)\left(\sum_{p\in P} \varphi_{G_j}(p)w(p)\right)\right.\\ &\qquad \left. - \left(\sum_{p\in P} \varphi_{F_i}(p)\varphi_{G_j}(p)w(p)\right)\left(\sum_{p\in P} w(p)\right)\right),\end{aligned}$$

which is nonpositive by the special case we proved earlier. ■

Corollary 6.3.6 (Chebyshev's inequality). *Let $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_n$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \le n\sum_{i=1}^{n} a_i b_i.$$

Proof. Let P be the chain $(1 \lessdot \cdots \lessdot n)$, define w, f, g on P by $w :\equiv 1$, $f(i) := a_i$, $g(i) := b_i$, and apply the FKG-inequality. ∎

As mentioned earlier, there are many applications of such inequalities – even in domains where one would not expect it. Concerning a result of Shepp [430] on linear extensions of posets, we refer to Anderson [32, p. 100]. Recently, Ahlswede and Khachatrian [14, 17] detected a remarkable connection to number theoretic inequalities. Here we present an application to Bernstein polynomials. Let $C[0, 1]$ denote the set of all continuous functions over $[0, 1]$. If $f \in C[0, 1]$, then the corresponding Bernstein polynomial $B_n f$ is defined by

$$(B_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

***Theorem 6.3.13 (Seymour and Welsh* [427]).** *Let $f, g \in C[0, 1]$ be increasing. Then for all $x \in [0, 1]$*

$$(B_n fg)(x) \geq ((B_n f)(x))((B_n g)(x)).$$

Proof. Let $x \in [0, 1]$ be arbitrary but fixed. We consider three functions $w : 2^{[n]} \to \mathbb{R}_+$ and $\hat{f}, \hat{g} : 2^{[n]} \to \mathbb{R}$, which are defined for all $A \subseteq [n]$ by

$$\begin{aligned} w(A) &:= x^{|A|}(1-x)^{n-|A|}, \\ \hat{f}(A) &:= f\left(\frac{|A|}{n}\right), \\ \hat{g}(A) &:= g\left(\frac{|A|}{n}\right). \end{aligned}$$

If $A \subseteq B \subseteq [n]$, then $\hat{f}(A) = f(\frac{|A|}{n}) \leq f(\frac{|B|}{n}) = \hat{f}(B)$ since f is increasing. Thus $\hat{f}$ (and analogously $\hat{g}$) is an increasing function on the Boolean lattice, that is, on $2^{[n]}$ ordered by inclusion. Moreover,

$$\begin{aligned} w(A)w(B) &= x^{|A|+|B|}(1-x)^{2n-|A|-|B|} \\ &= x^{|A\cup B|+|A\cap B|}(1-x)^{n-|A\cup B|+n-|A\cap B|} \\ &= w(A\cup B)w(A\cap B). \end{aligned}$$

The FKG-inequality yields

$$\begin{aligned} &\left(\sum_{A\subseteq[n]} \hat{f}(A)\hat{g}(A)w(A)\right)\left(\sum_{A\subseteq[n]} w(A)\right) \\ &\quad \geq \left(\sum_{A\subseteq[n]} \hat{f}(A)w(A)\right)\left(\sum_{A\subseteq[n]} \hat{g}(A)w(A)\right), \end{aligned}$$

that is,

$$\left(\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)g\left(\frac{k}{n}\right)x^k(1-x)^{n-k}\right)\left(\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right)$$
$$\geq\left(\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}\right)\left(\sum_{k=0}^{n}\binom{n}{k}g\left(\frac{k}{n}\right)x^k(1-x)^{n-k}\right),$$

and finally,

$$(B_n fg)(x) \geq ((B_n f)(x))((B_n g)(x)).$$

■

We say that a weighted poset (P, w) is an *FKG-poset* if for all increasing functions f and g the inequality (6.48) is true. The FKG-inequality says that (P, w) is an FKG-poset if P is a distributive lattice and w satisfies (6.47), for example, if $w \equiv 1$. It is interesting that we have again a product theorem:

***Theorem 6.3.14* (*Jones* [278]).** *If (P_i, w_i), $i = 1, 2$, are FKG-posets, then $(P_1, w_1) \times (P_2, w_2)$ is an FKG poset, too.*

Proof. Let $P := P_1 \times P_2$ and $w := w_1 \times w_2$. Let f and g be increasing on P. Then it is easy to see that the functions $f_1, g_1 : P_1 \to \mathbb{R}$ defined for all $p_1 \in P_1$ by

$$f_1(p_1) := \sum_{p_2 \in P_2} f(p_1, p_2)w_2(p_2),$$

$$g_1(p_1) := \sum_{p_2 \in P_2} g(p_1, p_2)w_2(p_2)$$

are increasing. Moreover, for fixed $p_1 \in P_1$ the functions $f(p_1, p_2)$ and $g(p_1, p_2)$ are increasing on P_2; hence (using the fact that (P_2, w_2) is an FKG-poset and that w_1, being a weight function, is nonnegative)

$$f_1(p_1)g_1(p_1) \leq \left(\sum_{p_2 \in P_2} f(p_1, p_2)g(p_1, p_2)w_2(p_2)\right)\left(\sum_{p_2 \in P_2} w_2(p_2)\right). \tag{6.49}$$

Using the fact that (P_1, w_1) is an FKG-poset, we obtain

$$\left(\sum_{(p_1,p_2)\in P} f(p_1, p_2)w(p_1, p_2)\right)\left(\sum_{(p_1,p_2)\in P} g(p_1, p_2)w(p_1, p_2)\right)$$
$$= \left(\sum_{p_1 \in P_1} w_1(p_1)f_1(p_1)\right)\left(\sum_{p_1 \in P_1} w_1(p_1)g_1(p_1)\right)$$

$$\le \left(\sum_{p_1 \in P_1} w_1(p_1) f_1(p_1) g_1(p_1)\right) w_1(P_1)$$

$$\le \left(\sum_{p_1 \in P_1} w_1(p_1) \sum_{p_2 \in P_2} f(p_1, p_2) g(p_1, p_2) w_2(p_2)\right) w_2(P_2) w_1(P_1)$$

$$= \left(\sum_{(p_1, p_2) \in P} f(p_1, p_2) g(p_1, p_2) w(p_1, p_2)\right) \left(\sum_{(p_1, p_2) \in P} w(p_1, p_2)\right).$$

■

Let us conclude this section with a study of the variance problem for our lattices (with weight $w \equiv 1$). Because modular geometric lattices are normal, they are also rank compressed by Corollary 4.5.7. Modular lattices are not necessarily rank compressed, however. To see this, consider the subgroup lattice P of $G_{(1,2)}(p)$ given in Figure 6.9 and take the filter $F := \{U_1, \ldots, U_p, V_p, G_{(1,2)}(p)\}$. We have

$$\mu_r(F) = \frac{p+5}{p+2} < \frac{3}{2} = \mu_r(P) \quad \text{if } p \ge 5.$$

Then by Theorem 4.4.1 the rank function is not an optimal representation if $p \ge 5$ (another counterexample was given by Stahl and Winkler (see [161, p. 84])). Moreover, geometric lattices are not necessarily rank compressed: Let us look at the Dilworth–Greene lattice DG_n and take $F := B_n$ (see Figure 6.7). Then

$$\mu_r(F) = \frac{n}{2} + 1 < \frac{(n+2)2^{n-1} + 2n3^{n-1}}{2^n + 3^n} = \mu_r(DG_n) \quad \text{if } n \ge 6;$$

that is, DG_n is not rank compressed if $n \ge 6$. However, we did not expect that distributive lattices are rank compressed. More generally, we have (see [148]):

Theorem 6.3.15. *If (P, w) is a positively weighted, ranked FKG-poset then (P, w) is rank compressed.*

Proof. Let r be the rank function of P. We verify condition (ii) of Theorem 4.4.1 in order to derive that r is an optimal representation. Let $g : P = P_r \to \mathbb{R}$ be increasing and take $f : P \to \mathbb{R}$ with $f(p) := r(p) - \mu_r$, $p \in P$. Clearly, f is increasing, too. Because (P, w) is an FKG-poset, we have

$$\sum_{p \in P} w(p) g(p)(r(p) - \mu_r)$$

$$\ge \frac{1}{w(P)} \left(\sum_{p \in P} w(p)(r(p) - \mu_r)\right) \left(\sum_{p \in P} w(p) g(p)\right) = 0$$

since $\sum_{p \in P} w(p)(r(p) - \mu_r) = 0$. ■

In Section 7.2 we will sketch a proof that the partition lattice Π_n is not rank compressed if n is sufficiently large. Following my paper [155], first we ask for filters F in Π_n that satisfy the necessary inequality $\mu_r(F) \geq \mu_r(\Pi_n)$. Because of the bijectivity between antichains A and filters F (generated by A) we study, more precisely, antichains.

Let λ be a positive integer. A λ-coloring of $[n]$ is as usual a function $c : [n] \to [\lambda]$. Given an antichain A in Π_n we define a *proper λ-coloring of A* to be a λ-coloring of $[n]$ such that for each element of A (which is a partition of $[n]$) there exists at least one block that is not monochromatic. Let $p(A, \lambda)$ be the number of proper λ-colorings of A. If, for example, A consists only of one partition π (with $b(\pi) := n - r(\pi)$ blocks), then obviously $p(\{\pi\}, \lambda) = \lambda^n - \lambda^{b(\pi)}$. Moreover, it is not difficult to see that for $A = \{\pi, \sigma\}$, the equation $p(\{\pi, \sigma\}, \lambda) = \lambda^n - \lambda^{b(\pi)} - \lambda^{b(\sigma)} + \lambda^{b(\pi \vee \sigma)}$ holds. To obtain a more general formula, we define for our antichain A the subposet L_A of Π_n by

$$L_A := \{\sup A' : A' \subseteq A\}$$

(here we put $\sup \emptyset := 0$, where $0 = 1|2|\cdots|n$ denotes the minimal element in Π_n). In the following we must distinguish between the expected value $\mu(.)$ and the Möbius function $\mu(.,.)$.

Theorem 6.3.16. *Given an antichain A in Π_n, we have*

$$p(A, \lambda) = \sum_{\alpha \in L_A} \mu(0, \alpha) \lambda^{b(\alpha)}.$$

Proof. For any coloring c of $[n]$, let $A(c)$ be the set of elements of A for which all blocks are monochromatic. Further, let $\varphi(c) := \sup A(c)$. It is easy to see that all blocks of $\alpha := \sup A'$ are monochromatic iff all blocks of each $\pi \in A'$ are monochromatic – that is, iff $A' \subseteq A(c)$. Thus we have, for any fixed coloring c,

$$\varphi(c) \geq_{L_A} \alpha \quad \text{iff the blocks of } \alpha \text{ are monochromatic.}$$

For $\alpha \in L_A$, let $f(\alpha)$ be the number of λ-colorings c of $[n]$ with $\varphi(c) = \alpha$. Obviously $f(0) = p(A, \lambda)$. Defining $g(\alpha) := \sum_{\beta \geq_{L_A} \alpha} f(\beta)$, we see by the preceding remarks that $g(\alpha)$ counts the number of λ-colorings for which the blocks of α are monochromatic. Hence $g(\alpha) = \lambda^{b(\alpha)}$. By Möbius inversion (Theorem 6.3.3),

$$p(A, \lambda) = f(0) = \sum_{\alpha \in L_A} \mu(0, \alpha) \lambda^{b(\alpha)}.$$

∎

Hence $p(A, \lambda)$ is a polynomial in λ that is called the *chromatic polynomial of* A. It is a natural generalization of the chromatic polynomial $p(G, \lambda)$ of a graph G (cf. [354]). If A is a set of rank-1 elements of Π_n, then A can be interpreted in the following way as a graph $G = (V, E)$ on $V = [n]$: Each element of A is a partition of $[n]$ with $n-2$ one-element blocks and 1 two-element block, and we put ij into E iff there is one element of A having ij as the two-element block. Then $p(A, \lambda) = p(G, \lambda)$.

Theorem 6.3.17. *Let A be an antichain in Π_n and F the filter generated by A. Then*

$$\mu_r(F) \geq \mu_r(\Pi_n) \textit{ iff } \sum_{i,j=1}^{\infty} \frac{i^n j^n}{i!j!}(i-j)\left(\frac{p(A,i)}{i^n} - \frac{p(A,j)}{j^n}\right) \geq 0.$$

Proof. We begin with the determination of $\mu_r(\Pi_n)$. We have $|\Pi_n| = B_n$ (the Bell number), and, using the recurrence of the Stirling numbers (1.6),

$$\sum_{\pi \in \Pi_n} r(\pi) = \sum_{i=0}^{n-1} i S_{n,n-i} = \sum_{k=1}^{n} (n-k) S_{n,k}$$
$$= \sum_{k=1}^{n} n S_{n,k} - S_{n+1,k} + S_{n,k-1} = n B_n - B_{n+1} + B_n.$$

Thus

$$\mu_r(\Pi_n) = n + 1 - \frac{B_{n+1}}{B_n}. \tag{6.50}$$

For $F \neq \Pi_n$ – that is, $A \neq \{0\}$ – we determine now $\sum_{\pi \in F} r(\pi)$. For $\pi \in \Pi_n$, let $A_\pi := \{\sigma \in A : \sigma \leq \pi\}$ and $\psi(\pi) := \sup A_\pi$. Obviously, $\psi(\pi) \in L_A$ and $\pi \geq \psi(\pi)$; that is, π lies in the filter generated by $\psi(\pi)$. For $\alpha \in L_A$, let

$$f_1(\alpha) := |\{\pi \in \Pi_n : \psi(\pi) = \alpha\}|, \qquad f_2(\alpha) := \sum_{\pi \in \Pi_n : \psi(\pi) = \alpha} r(\pi).$$

Obviously,

$$|F| = \sum_{\alpha \in L_A : \alpha \neq 0} f_1(\alpha), \qquad \sum_{\pi \in F} r(\pi) = \sum_{\alpha \in L_A : \alpha \neq 0} f_2(\alpha)$$

and with $g_i(\alpha) := \sum_{\beta \geq_{L_A} \alpha} f_i(\beta)$, $i = 1, 2$, we have

$$|F| = g_1(0) - f_1(0), \qquad \sum_{\pi \in F} r(\pi) = g_2(0) - f_2(0). \tag{6.51}$$

For fixed $\alpha \in L_A$, the set $F(\alpha) := \{\beta \in \Pi_n : \beta \geq \alpha\}$ forms an induced subposet of Π_n which is isomorphic to $\Pi_{b(\alpha)}$. We have $F(\alpha) = \{\beta \in \Pi_n : \psi(\beta) \geq \alpha\}$ because $\beta \geq \alpha$ implies $\psi(\beta) \geq \alpha$ and $\beta \geq \psi(\beta)$. We have

$$g_1(\alpha) = B_{b(\alpha)} \tag{6.52}$$

because

$$g_1(\alpha) = \sum_{\beta \geq_{L_A} \alpha} |\{\pi \in \Pi_n : \psi(\pi) = \beta\}|$$
$$= |\{\pi \in \Pi_n : \psi(\pi) \geq \alpha\}| = |\Pi_{b(\alpha)}| = B_{b(\alpha)},$$

and

$$g_2(\alpha) = nB_{b(\alpha)} - B_{b(\alpha)+1} + B_{b(\alpha)} \tag{6.53}$$

because

$$g_2(\alpha) = \sum_{\beta \geq_{L_A} \alpha} \sum_{\pi \in \Pi_n : \psi(\pi) = \beta} r(\pi) = \sum_{\pi \in \Pi_n : \psi(\pi) \geq \alpha} r(\pi) = \sum_{\pi \in \Pi_n : \pi \geq \alpha} r(\pi)$$
$$= \sum_{i=r(\alpha)}^{n-1} i S_{b(\alpha),n-i} = \sum_{k=1}^{b(\alpha)} (n-k) S_{b(\alpha),k} = nB_{b(\alpha)} - B_{b(\alpha)+1} + B_{b(\alpha)}$$

where the last equality follows as in the beginning of the proof. From (6.51), (6.52), and (6.53), we obtain via Möbius inversion

$$|F| = B_n - \sum_{\alpha \in L_A} \mu(0, \alpha) B_{b(\alpha)}, \tag{6.54}$$

$$\sum_{\pi \in F} r(\pi) = nB_n - B_{n+1} + B_n$$
$$- \sum_{\alpha \in L_A} \mu(0, \alpha)(nB_{b(\alpha)} - B_{b(\alpha)+1} + B_{b(\alpha)}). \tag{6.55}$$

From (6.50), (6.54), and (6.55) we obtain after a straightforward computation that

$$\mu_r(F) \geq \mu_r(\Pi_n)$$

iff

$$\sum_{\alpha \in L_A} \mu(0, \alpha)(B_{b(\alpha)+1} B_n - B_{b(\alpha)} B_{n+1}) \geq 0. \tag{6.56}$$

By Dobinski's formula (1.7) a product $B_a B_b$ can be written in the form

$$B_a B_b = \frac{1}{2}(B_a B_b + B_b B_a) = \frac{1}{2}\frac{1}{e^2} \sum_{i,j=0}^{\infty} \frac{i^a j^b}{i!j!} + \frac{i^b j^a}{i!j!}.$$

Thus (6.56) is equivalent to

$$\sum_{i,j=0}^{\infty} (i-j)\frac{1}{i!j!}\left(j^n \sum_{\alpha \in L_A} \mu(0, \alpha) i^{b(\alpha)} - i^n \sum_{\alpha \in L_A} \mu(0, \alpha) j^{b(\alpha)}\right) \geq 0,$$

and the last inequality is by Theorem 6.3.16 equivalent to

$$\sum_{i,j=1}^{\infty} \frac{i^n j^n}{i!j!}(i-j)\left(\frac{p(A,i)}{i^n} - \frac{p(A,j)}{j^n}\right) \geq 0.$$

■

We call an antichain A of Π_n *coloring-monotone* if

$$\frac{p(A,\lambda)}{\lambda^n} \leq \frac{p(A,\lambda+1)}{(\lambda+1)^n} \quad \text{for all } \lambda \in \mathbb{N}_+.$$

Hence A is coloring-monotone if the probability that there exists a nonmonochromatic block for a random coloring in each element of A increases with the number of colors. Directly from Theorem 6.3.17 we may derive:

Corollary 6.3.7. *Let A be an antichain in Π_n and F the filter generated by A. If A is coloring-monotone then*

$$\mu_r(F) \geq \mu_r(\Pi_n).$$

■

Proposition 6.3.2. *If $|A| \leq 3$ then A is coloring-monotone.*

Proof. The cases $|A| \in \{1,2\}$ are easy to verify, so let $A = \{\alpha, \beta, \gamma\}$. Generalizing the case $|A| \in \{1,2\}$ we may calculate $p(A,\lambda)$ also by the principle of inclusion and exclusion:

$$\begin{aligned} p(A,\lambda) &= \lambda^n - \lambda^{b(\alpha)}r(\alpha) - \lambda^{b(\beta)}r(\beta) - \lambda^{b(\gamma)}r(\gamma) \\ &\quad + \lambda^{b(\alpha\vee\beta)}r(\alpha\vee\beta) + \lambda^{b(\alpha\vee\gamma)}r(\alpha\vee\gamma) + \lambda^{b(\beta\vee\gamma)}r(\beta\vee\gamma) \\ &\quad - \lambda^{b(\alpha\vee\beta\vee\gamma)}r(\alpha\vee\beta\vee\gamma). \end{aligned}$$

We put $f(\lambda) := p(A,\lambda)/\lambda^n$. It is sufficient to prove that $f'(\lambda) \geq 0$ for $\lambda \geq 2$. We replace everywhere $b(\pi)$ by $n - r(\pi)$. In the calculation of $f'(\lambda)$ we omit the (positive) term with $\alpha \vee \beta \vee \gamma$ and decrease $\lambda^{-r(\alpha)}$ by $\lambda^{-r(\alpha\vee\beta)} + \lambda^{-r(\alpha\vee\gamma)}$ (note that $r(\alpha)+1 \leq r(\alpha\vee\beta)$, $r(\alpha)+1 \leq r(\alpha\vee\gamma)$, $\lambda \geq 2$) and do the same for $\lambda^{-r(\beta)}$ and $\lambda^{-r(\gamma)}$. We obtain

$$\begin{aligned} f'(\lambda) \geq \frac{1}{\lambda}\Big(&\lambda^{-r(\alpha\vee\beta)}(r(\alpha)+r(\beta)-r(\alpha\vee\beta)) \\ &+ \lambda^{-r(\alpha\vee\gamma)}(r(\alpha)+r(\gamma)-r(\alpha\vee\gamma)) \\ &+ \lambda^{-r(\beta\vee\gamma)}(r(\beta)+r(\gamma)-r(\beta\vee\gamma))\Big). \end{aligned}$$

The RHS of this inequality is nonnegative because the partition lattice is semimodular. ■

I leave it to the reader to verify for several classes of graphs (see [61] or [354]), like complete graphs, trees, circuits, wheels, interval graphs, and so forth, the coloring-monotonicity (recall that subsets of $N_1(\Pi_n)$ are interpreted as graphs). We studied the case $|A| = 1$ together with Bouroubi [82] in a direct way. Let me end with the remark that not all antichains are coloring-monotone if n is sufficiently large. Take for simplicity n even and $A \subseteq N_1$, whose two-element blocks have the form $\{i, j\}$ with $i \leq \frac{n}{2}$, $j > \frac{n}{2}$ (this can be interpreted as the complete bipartite graph on the vertex set $\{1, \dots, \frac{n}{2}\} \cup \{\frac{n}{2}+1, \dots, n\}$). Then it is easy to see that $p(A, 2) = 2$ and $p(A, 3) = 6 + 6(2^{\frac{n}{2}} - 2)$; that is,

$$\frac{p(A,2)}{2^n} \leq \frac{p(A,3)}{3^n} \text{ iff } 3^{n-1} \leq 2^{\frac{3n}{2}} - 2^n,$$

and this is false iff $n \geq 20$. Here is a nice analogy to the proof of Theorem 5.4.8.

6.4. The independence number of graphs and the Erdős–Ko–Rado Theorem

In this section we consider more general objects, namely simple graphs $G = (V, E)$ instead of posets. Let us recall that a subset $S \subseteq V$ is called independent if $vw \notin E$ for all $v,\ w \in S$. The *independence number of* G is defined by

$$\alpha(G) := \max\{|S| : S \text{ is independent}\}.$$

The determination of $\alpha(G)$ is, in general, difficult (NP-complete, cf. [215]), but here we are looking at special graphs that are, in particular, regular. We will consider an algebraic method for the the determination of upper bounds for $\alpha(G)$. It is based on the work of Hoffman [119, p. 115], Delsarte [128], Lovász [355], Schrijver [419], Wilson [470], and others. Though there are theories related to these questions – the theory of association schemes, cf. Bannai and Ito [38], and the theory of graph spectra, cf. Cvetkovič, Doob, and Sachs [119] (see also Haemers [250]) – we shall avoid building the whole machinery. Instead we'll examine some main ideas that are sufficient for our purposes. As throughout this chapter, linear operators are preferred to matrices.

We associate with a graph G the *graph space* $\widetilde{G}$ in the same way as it was done for posets (see Section 6.1). So $\widetilde{G}$ is freely generated by the vertices of G. The standard basis of $\widetilde{G}$ is given by $\{\widetilde{v} : v \in V\}$. Moreover, we consider $\widetilde{G}$ as a Euclidean space (with the standard scalar product). We will work with (linear) operators from $\widetilde{G}$ into $\widetilde{G}$. The identity operator on $\widetilde{G}$ is denoted by $\widetilde{I}$. Let

$$\varphi_G := \sum_{v \in V} \widetilde{v}.$$

We define the *all-one-operator* $\widetilde{J}$ by

$$\widetilde{J}(\widetilde{v}) := \varphi_G \text{ for all } v \in V.$$

Finally, an operator $A : \widetilde{G} \to \widetilde{G}$ is said to be an *adjacency operator* if for all $v \in V$

$$A(\widetilde{v}) = \sum_{w \in V : vw \in E} c(v, w)\widetilde{w},$$

where $c(v, w)$ are real numbers depending on v and w. Note that $\langle A(\widetilde{v}), \widetilde{w} \rangle = 0$ if $vw \notin E$. If for all $vw \in E$ there holds $c(v, w) = c(w, v)$, then $\langle A(\widetilde{v}), \widetilde{w} \rangle = \langle \widetilde{v}, A(\widetilde{w}) \rangle$; hence A is *self-adjoint*. For example, $\widetilde{J}$ is self-adjoint. If $c(v, w) = 1$ for all $vw \in E$, we speak (as for posets) of the *Lefschetz adjacency operator* A_L. It is well known that all eigenvalues of a self-adjoint operator A are real and that there exists a basis of $\widetilde{G}$ consisting of pairwise orthogonal eigenvectors of A. Recall that a self-adjoint operator M is *positive-semidefinite* if $\langle M(\varphi), \varphi \rangle \geq 0$ for all $\varphi \in \widetilde{G}$ and that this is equivalent to the fact that all eigenvalues of M are nonnegative.

Lemma 6.4.1. *Let A be a self-adjoint adjacency operator on a graph space $\widetilde{G}$ and let $M := \widetilde{I} + A - \frac{1}{\delta}\widetilde{J}$ be positive-semidefinite. Then $\alpha(G) \leq \delta$.*

Proof. Let S be any independent set in G and let $\varphi_S := \sum_{v \in S} \widetilde{v}$. Then

$$\langle A(\varphi_S), \varphi_S \rangle = \left\langle \sum_{v \in S} A(\widetilde{v}), \sum_{w \in S} \widetilde{w} \right\rangle = \sum_{v, w \in S} \langle A(\widetilde{v}), \widetilde{w} \rangle = 0$$

since $vw \notin E$ for all $v, w \in S$. Consequently,

$$\begin{aligned} 0 \leq \langle M(\varphi_S), \varphi_S \rangle &= \langle \varphi_S, \varphi_S \rangle + \langle A(\varphi_S), \varphi_S \rangle - \frac{1}{\delta}\langle \widetilde{J}(\varphi_S), \varphi_S \rangle \\ &= |S| - \frac{1}{\delta}\left\langle \sum_{v \in S} \widetilde{J}(\widetilde{v}), \varphi_S \right\rangle \\ &= |S| - \frac{1}{\delta}|S|\langle \varphi_G, \varphi_S \rangle = |S| - \frac{1}{\delta}|S|^2. \end{aligned}$$

This implies $|S| \leq \delta$. ■

Remark 6.4.1. *Note that in Lemma 6.4.1 $\alpha(G) = \delta$ iff $\langle M(\varphi_S), \varphi_S \rangle = 0$ for some independent set S in G.*

Up to now we have had much freedom in the choice of A, but we must choose A and δ such that the verification of the positive-semidefiniteness of M is possible. One way to do this is the construction of A as a linear combination of "elementary" adjacency operators for which the eigenvalues can be determined explicitly:

Lemma 6.4.2. *Let $A = \sum_{j=1}^{l} \beta_j B_j$, where each B_j is a self-adjoint adjacency operator on the graph space $\widetilde{G}$ of a graph G on n vertices. Suppose that the eigenspaces of the operators B_j are independent of j, and let $\lambda_{j,i}$, $i = 1, \ldots n$, be* *the eigenvalues of B_j, $j = 1, \ldots, l$. Finally suppose that φ_G is an eigenvector of B_j to the eigenvalue $\lambda_{j,1}$, $j = 1, \ldots, l$. Then the eigenvalues μ_i of $M := \widetilde{I} + A - \frac{1}{\delta}\widetilde{J}$ are given by*

$$\mu_i = \begin{cases} 1 + \sum_{j=1}^{l} \beta_j \lambda_{j,1} - \frac{n}{\delta} & \text{if } i = 1, \\ 1 + \sum_{j=1}^{l} \beta_j \lambda_{j,i} & \text{if } i = 2, \ldots, n. \end{cases}$$

Proof. By our suppositions, there exists a basis $\{\varphi_1 = \varphi_G, \varphi_2, \ldots, \varphi_n\}$ of pairwise orthogonal eigenvectors of B_j to the respective eigenvalues $\lambda_{j,1}, \ldots, \lambda_{j,n}$, $j = 1, \ldots, l$. We have $\widetilde{J}(\varphi_G) = \sum_{v \in V} \widetilde{J}(\widetilde{v}) = \sum_{v \in V} \sum_{w \in V} \widetilde{w} = n\varphi_G$ and $\widetilde{J}(\varphi_i) = \widetilde{0}$, $i = 2, \ldots, n$, since for each $v \in V$ it holds $\langle \widetilde{J}(\varphi_i), \widetilde{v} \rangle = \langle \varphi_i, \widetilde{J}(\widetilde{v}) \rangle = \langle \varphi_i, \varphi_G \rangle = 0$. Consequently,

$$M(\varphi_G) = \varphi_G + \sum_{j=1}^{l} \beta_j B_j(\varphi_G) - \frac{1}{\delta}\widetilde{J}(\varphi_G) = \left(1 + \sum_{j=1}^{l} \beta_j \lambda_{j,1} - \frac{n}{\delta}\right) \varphi_G,$$

$$M(\varphi_i) = \varphi_i + \sum_{j=1}^{l} \beta_j B_j(\varphi_i) - \frac{1}{\delta}\widetilde{J}(\varphi_i) = \left(1 + \sum_{j=1}^{l} \beta_j \lambda_{j,i}\right) \varphi_i, \ i = 2, \ldots, n.$$

■

With the assumptions of Lemma 6.4.2 we obtain the smallest possible upper bound on $\alpha(G)$ in determining

$$z := \max \left\{ \sum_{j=1}^{l} \beta_j \lambda_{j,1} \right\}$$

subject to

$$\sum_{j=1}^{l} \beta_j \lambda_{j,i} \geq -1, \qquad i = 2, \ldots, n.$$

Then we have $\mu_i \geq 0$ for $i = 2, \ldots, n$ and we may choose

$$\delta := \frac{n}{1+z}$$

(this is the smallest δ with the property $\mu_1 \geq 0$). We summarize the result:

Theorem 6.4.1. *Let $B_1, \ldots, B_l$ be self-adjoint adjacency operators on $\widetilde{G}$ whose eigenspaces are independent of the index. Let $\lambda_{j,i}$ ($i = 1, \ldots, n = |V(G)|$), be the eigenvalues of B_j, $j = 1, \ldots, l$, and let φ_G be an eigenvector of B_j to the eigenvalue of $\lambda_{j,1}$, $j = 1, \ldots, l$. Define $z := \max\{\sum_{j=1}^{l} \beta_j \lambda_{j,1} : \sum_{j=1}^{l} \beta_j \lambda_{j,i} \geq -1, \ i = 2, \ldots, n\}$. Then*

$$\alpha(G) \leq \frac{n}{1+z}.$$

Corollary 6.4.1 (Hoffman (unpublished)). *Let G be regular of degree d, $|V(G)| = n$, and let λ_n be the smallest eigenvalue of the Lefschetz adjacency operator A_L. Then*

$$\alpha(G) \le \frac{-n\lambda_n}{d - \lambda_n}.$$

Proof. In Theorem 6.4.1 we take $l := 1$ and $B_1 := A_L$. Our vector φ_G is an eigenvector of B_1 to the eigenvalue d since

$$A_L(\varphi_G) = \sum_{v \in V} A_L(\widetilde{v}) = \sum_{v \in V} \sum_{w \in V : vw \in E} \widetilde{w} = d \sum_{w \in V} \widetilde{w} = d\varphi_G.$$

We must calculate $\beta_1 d \to \max = z$ subject to $\beta_1 \lambda_{1,2} \ge -1, \dots, \beta_1 \lambda_{1,n} \ge -1$ where $\lambda_{1,2}, \dots, \lambda_{1,n} = \lambda_n$ are the eigenvalues of A_L (λ_n is the smallest one). To obtain a maximum we must clearly have $\beta_1 \ge 0$. Then the strongest restriction is the last one, namely $\beta_1 \lambda_n \ge -1$. Since the sum of the eigenvalues equals the trace of A_L, which is 0, λ_n is negative, and it follows that $\beta_1 \le -1/\lambda_n$ and $z = -(1/\lambda_n)d$. From Theorem 6.4.1 we derive

$$\alpha(G) \le \frac{n}{1 - \frac{d}{\lambda_n}} = \frac{-n\lambda_n}{d - \lambda_n}.$$

■

Now we will apply this method to prove Erdős–Ko–Rado type theorems. We will consider the Boolean lattice B_n and the linear lattice $L_n(q)$ simultaneously. More precisely, we will do the following calculations for $L_n(q)$. If we let $q \to 1$ we obtain the corresponding formulas for B_n. If the reader is only interested in B_n, he or she can prove some of the succeeding Lemmata more easily (again let everywhere $q \to 1$). For $L_n(q)$, we need the following theorem, which is due to Goldman and Rota [225]:

Theorem 6.4.2 (q-binomial Theorem). *Let $f_k(x, y) := (x - y)(x - qy) \cdots (x - q^{k-1}y)$, $k = 1, 2, \dots,$ and $f_0(x, y) := 1$. Then*

$$f_n(x, z) = \sum_{k=0}^{n} \binom{n}{k}_q f_k(x, y) f_{n-k}(y, z) \quad \text{for all } x, y, z \in \mathbb{R}.$$

Proof. Let V_n be a vector space of dimension n and let X, Y, Z be vector spaces of cardinalities x, y (resp. z) such that $Z \subseteq Y \subseteq X$ and $n \le \dim Z$. Let us count the number I of injective linear mappings $\Phi : V_n \to X$ such that the intersection of the image $\Phi(V_n)$ with Z equals the zero vector. Let $v_1, \dots, v_n$ be a basis of V_n. We count the ways of mapping this basis into a set of n independent vectors in X whose span $[\{\Phi(v_1), \dots, \Phi(v_n)\}]$ intersects Z in the zero vector. We need $\Phi(v_1) \in X - Z$; consequently there are $x - z$ choices for $\Phi(v_1)$. Further we need

$\Phi(v_2) \in X - [Z \cup \{\Phi(v_1)\}]$, so we may choose $\Phi(v_2)$ in $x - qz$ ways. Now we may continue with $\Phi(v_3) \in X - [Z \cup \{\Phi(v_1), \Phi(v_2)\}]$, that is, $x - q^2 z$ possibilities and so forth, and we get on the one hand

$$I = (x-z)(x-qz)\cdots(x-q^{n-1}z) = f_n(x,z).$$

On the other hand, we classify our mappings by $\Phi(V_n) \cap Y$, which is clearly a subspace of some dimension, say k, and hence it is the image of a k-dimensional subspace of V_n. We obtain all such mappings by first choosing a k-dimensional subspace W of V_n, then mapping W into Y in such a way that the intersection with Z is only the zero vector, and finally mapping $V_n - W$ into $X - Y$. We have $\binom{n}{k}_q$ choices for W. Let $w_1, \ldots, w_k$ be a basis of W and $w_1, \ldots, w_k, \ldots, w_n$ a basis of V_n. As before, we see that there are $(y-z)(y-qz)\cdots(y-q^{k-1}z) = f_k(y,z)$ possibilities to define $\Phi(w_1), \ldots, \Phi(w_k)$ and, independently, $(x-y)(x-qy)\cdots(x-q^{n-k-1}y) = f_{n-k}(x,y)$ choices for $\Phi(w_{k+1}), \ldots, \Phi(w_n)$. Consequently,

$$I = \sum_{k=0}^{n} \binom{n}{k}_q f_k(y,z) f_{n-k}(x,y) = \sum_{k=0}^{n} \binom{n}{k}_q f_{n-k}(y,z) f_k(x,y).$$

We proved our assertion for all numbers $x = q^a, y = q^b, z = q^c$ such that $n \le c \le b \le a$. For fixed a and b such that $b \ge 2n$, we have on both sides polynomials in z of degree at most n that are equal on $n+1$ points, namely $z = q^n, \ldots, q^{2n}$; hence they are identical. For fixed z and a such that $a \ge 2n$, we may conclude in the same way that the assertion holds for all y, z and then finally also for all x. ■

Corollary 6.4.2. *We have*

(*a*) $x^n = 1 + \sum_{k=1}^{n} \binom{n}{k}_q (x-1)(x-q)\cdots(x-q^{k-1})$,

(*b*) $(x-1)(x-q)\cdots(x-q^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} x^{n-k}$,

(*c*) $\binom{a+b}{n}_q = \sum_{k=0}^{n} \binom{a}{k}_q \binom{b}{n-k}_q q^{(a-k)(n-k)}$,

(*d*) $\binom{b-a}{n}_q = \sum_{k=0}^{n} (-1)^k \binom{a+k-1}{k}_q \binom{b}{n-k}_q q^{-ak-\binom{k}{2}-(a+k)(n-k)}$.

Proof. (a) Put in Theorem 6.4.2 $z := 0$ and $y := 1$.

(b) Put in Theorem 6.4.2 $z := 1$ and $y := 0$ and use $k := n - k$.

(c) Put in Theorem 6.4.2 $x := q^{a+b}$, $y := q^a$, $z := 1$ and use $k := n - k$.

(d) Replace in (c) the variable a by $-a$ and note that $\binom{-a}{k}_q = (-1)^k q^{-ak-\binom{k}{2}}$ $* \binom{a+k-1}{k}_q$. ■

As always we will use the letter P for both lattices $L_n(q)$ and B_n, and r denotes the rank function. Elements of P are denoted by X (subspaces of an n-dimensional vector space V over $GF(q)$ (resp. subsets of the n-element set $[n]$)). The standard basis of the poset space $\widetilde{P}$ is then $\{\widetilde{X} : X \in P\}$. Recall that $r(X) = \dim X$ (resp. $|X|$) and that a family $\mathcal{F} \subseteq P$ is called *k-uniform t-intersecting* if $r(X) = k$ for all $X \in \mathcal{F}$ and $r(X \wedge Y) \geq t$ for all $X, Y \in \mathcal{F}$. Let $G = G_{n,k,t}(P)$ be the graph on the vertex set N_k whose edge set E is given by $E := \{XY : X, Y \in N_k$ and $r(X \wedge Y) < t\}$. We call G briefly the *Johnson graph* because it arises in the Johnson scheme (see [38]). In the case $t = 1$ our graph G is called the *Kneser graph*. Clearly $\alpha(G)$ equals the maximum size of a k-uniform t-intersecting family in P. Our aim is the determination of $\alpha(G)$. (In the case of B_n we know already the solution by Theorem 2.4.1. We will derive this result again for $n \geq (k - t + 1)(t + 1)$ since the method gives in addition many interesting properties of the corresponding linear operators.) We may suppose throughout that $0 < t < k < (n + t)/2$ since otherwise the problem becomes trivial. We have

$$\alpha(G) \geq \binom{n-t}{k-t}_q \tag{6.57}$$

since we may take a fixed $X_0 \in P$ with $r(X_0) = t$ and obtain a k-uniform t-intersecting family $S := \{X \in N_k : X \supseteq X_0\}$ of size $\binom{n-t}{k-t}_q$. The question is for which parameters we have equality. We will see that this is the case if $n \geq 2k$ for $L_n(q)$ and if $n \geq (t + 1)(k - t + 1)$ for B_n.

In order to define the operators B_j of Theorem 6.4.1 (not to be confused with the Boolean lattice for which we write P here) we work with the operators $\widetilde{\nabla}_{i\to j}$ and $\widetilde{\Delta}_{i\to j}$ introduced after (6.30). Recall that, for example, $\widetilde{\Delta}_{i\to j}(\widetilde{X}) := \sum_{Y \subseteq X : r(Y)=j} \widetilde{Y}$, where $X \in N_i$. Moreover, in generalizing the operators from the proof of Theorem 6.3.2, we introduce $\Psi_{i\to j} : \widetilde{N}_i \to \widetilde{N}_j$ as follows:

$$\Psi_{i\to j}(\widetilde{X}) := \sum_{Y \in N_j : X \wedge Y = 0} \widetilde{Y} \quad \text{where } X \in N_i, \quad 0 \leq i,\ j \leq n.$$

Finally we define $B_j : \widetilde{N}_k \to \widetilde{N}_k$ by

$$B_j := \Psi_{j\to k}\widetilde{\Delta}_{k\to j}, \quad j = 0, \dots, k.$$

Lemma 6.4.3.

(*a*) *Let $X \in N_k, Y \in P, r(X \wedge Y) = m$, and $0 \leq j \leq k$. Then $|\{Z \in N_j : Z \leq X$ and $Z \wedge Y = 0\}| = q^{mj}\binom{k-m}{j}_q$.*

(*b*) $B_j(\widetilde{X}) = \sum_{i=0}^{k-j} \sum_{Y \in N_k : r(X\wedge Y)=i} q^{ij}\binom{k-i}{j}_q \widetilde{Y}$.

Proof. Obviously (b) follows from (a) with $m := i$. So let us prove (a). Let $W := X \wedge Y$. The number we are looking for is $\alpha := |\{Z \in N_j : Z \leq X, \quad Z \wedge W = 0\}|$. Since W consists of q^m vectors we may find $(q^k - q^m)\cdots(q^k - q^{m+j-1})$ ordered bases for a subspace $Z \in N_j$ with $Z \leq X, \quad Z \wedge W = 0$. On the other hand, each subspace Z has $(q^j-1)\cdots(q^j-q^{j-1})$ ordered bases. Consequently,

$$\alpha = \frac{(q^k-q^m)\cdots(q^k-q^{m+j-1})}{(q^j-1)\cdots(q^j-q^{j-1})} = q^{mj}\binom{k-m}{j}_q .$$

■

Lemma 6.4.3(b) tells us that B_j is self-adjoint for $j=0,\dots,k$ and that B_j is an adjacency operator for our Johnson graph space if $k-j<t$; that is, $j\geq k-t+1$. So we may use Theorem 6.4.1 with $B_{k-t+1},\dots,B_k$ if we are able to determine their eigenspaces and eigenvalues. This can be done using several identities.

Lemma 6.4.4. *We have*

(a) $\widetilde{\Delta}_{j\to l}\widetilde{\Delta}_{i\to j} = \binom{i-l}{j-l}_q\widetilde{\Delta}_{i\to l},\ 0\leq l\leq j\leq i\leq n,$

(b) $\widetilde{\nabla}_{j\to i}\widetilde{\nabla}_{l\to j} = \binom{i-l}{j-l}_q\widetilde{\nabla}_{i\to l},\ 0\leq l\leq j\leq i\leq n,$

(c) $\widetilde{\Delta}_{j\to l}\Psi_{i\to j} = q^{i(j-l)}\binom{n-l-i}{j-l}_q\Psi_{i\to l},\ 0\leq l\leq j\leq n,\ 0\leq i\leq n,$

(d) $\Psi_{j\to i}\widetilde{\nabla}_{l\to j} = q^{i(j-l)}\binom{n-l-i}{j-l}_q\Psi_{l\to i},\ 0\leq l\leq j\leq n,\ 0\leq i\leq n,$

(e) $\Psi_{i\to l} = \sum_{j=0}^{\min\{i,l\}}(-1)^j q^{\binom{j}{2}}\widetilde{\nabla}_{j\to l}\widetilde{\Delta}_{i\to j},\ 0\leq i,l\leq n,$

(f) $\widetilde{\Delta}_{i\to l} = \sum_{j=0}^{l}(-1)^j q^{\binom{j+1}{2}-lj}\widetilde{\nabla}_{j\to l}\Psi_{i\to j},\ 0\leq l\leq i\leq n.$

Proof. (a) and (b) follow from the fact that for $X\geq Y,\ r(X)=i,\ r(Y)=l$ it holds $|\{Z\in N_j : X\geq Z\geq Y\}| = \binom{i-l}{j-l}_q$.

(c) and (d): With a method analogous to the proof of Lemma 6.4.3(a) we obtain that, for fixed $Y\in N_l,\ X\in N_i$ with $Y\wedge X=0$, the equality $|\{Z\in N_j : Y\leq Z,\ Z\wedge X=0\}| = q^{i(j-l)}\binom{n-l-i}{j-l}_q$ holds. This yields the assertion.

(e) Let $X\in N_i,\ Y\in N_l,\ r(X\wedge Y)=m$. We have

$$\begin{aligned}\langle\widetilde{\nabla}_{j\to l}\widetilde{\Delta}_{i\to j}(\widetilde{X}),\widetilde{Y}\rangle &= |\{Z\in N_j : Z\leq X \text{ and } Z\leq Y\}|\\ &= |\{Z\in N_j : Z\leq X\wedge Y\}| = \binom{m}{j}_q .\end{aligned}$$

Consequently, putting in the polynomial identity from Corollary 6.4.2(a) $x:=0$ and $n:=m$ we obtain

$$\begin{aligned}\left\langle\left(\sum_{j=0}^{\min\{i,l\}}(-1)^j q^{\binom{j}{2}}\widetilde{\nabla}_{j\to l}\widetilde{\Delta}_{i\to j}\right)(\widetilde{X}),\widetilde{Y}\right\rangle &= \sum_{j=0}^{m}(-1)^j q^{\binom{j}{2}}\binom{m}{j}_q\\ &= \begin{cases}1 & \text{if } m=0,\\ 0 & \text{otherwise}\end{cases}\\ &= \langle\Psi_{i\to l}(\widetilde{X}),\widetilde{Y}\rangle.\end{aligned}$$

(f) Let $X \in N_i$, $Y \in N_l$, $r(X \wedge Y) = m$. We have

$$\begin{aligned}
\langle \widetilde{\nabla}_{j\to l}\Psi_{i\to j}(\widetilde{X}), \widetilde{Y}\rangle &= |\{Z \in N_j : Z \leq Y \text{ and } Z \wedge X = 0\}| \\
&= |\{Z \in N_j : Z \leq Y \text{ and } Z \wedge (X \wedge Y) = 0\}| \\
&= q^{mj}\binom{l-m}{j}_q
\end{aligned}$$

where the last identity follows from Lemma 6.4.3(a). Consequently,

$$\begin{aligned}
&\left\langle \left(\sum_{j=0}^{l}(-1)^j q^{\binom{j+1}{2}-lj}\widetilde{\nabla}_{j\to l}\Psi_{i\to j}\right)(\widetilde{X}), \widetilde{Y}\right\rangle \\
&\quad = \sum_{j=0}^{l-m}(-1)^j q^{\binom{j+1}{2}+(m-l)j}\binom{l-m}{j}_q.
\end{aligned}$$

We have to show that the RHS equals 1 if $m = l$, that is, $Y \leq X$, and equals 0 if $m < l$, that is, $Y \nleq X$. Putting $p := l - m$ and using the symmetry of the Gaussian coefficients we conclude

$$\begin{aligned}
\sum_{j=0}^{p}(-1)^j q^{\binom{j+1}{2}-pj}\binom{p}{j}_q &= \sum_{j=0}^{p}(-1)^{p-j} q^{\binom{p-j+1}{2}-p(p-j)}\binom{p}{j}_q \\
&= (-1)^p q^{\frac{p-p^2}{2}}\sum_{j=0}^{p}(-1)^j q^{\binom{j}{2}}\binom{p}{j}_q \\
&= \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

by the proof of (e). ■

Lemma 6.4.5. *We have for* $0 \leq j, l \leq k$,

$$\begin{aligned}
B_j B_l &= B_l B_j \\
&= q^{l(k-j)}\binom{n-j-l}{k-j}_q \sum_{i=0}^{\min\{j,l\}} (-1)^i q^{k(j-i)+\binom{i}{2}}\binom{n-k-i}{j-i}_q\binom{k-i}{l-i}_q B_i.
\end{aligned}$$

Proof. In view of Lemma 6.4.4 we have

$$\begin{aligned}
B_j B_l &= (\Psi_{j\to k}\widetilde{\Delta}_{k\to j})(\Psi_{l\to k}\widetilde{\Delta}_{k\to l}) = \Psi_{j\to k}(\widetilde{\Delta}_{k\to j}\Psi_{l\to k})\widetilde{\Delta}_{k\to l} \\
&= q^{l(k-j)}\binom{n-j-l}{k-j}_q \Psi_{j\to k}\Psi_{l\to j}\widetilde{\Delta}_{k\to l}
\end{aligned}$$

and further

$$\begin{aligned}
\Psi_{j\to k}\Psi_{l\to j}\widetilde{\Delta}_{k\to l} &= \Psi_{j\to k}\left(\sum_{i=0}^{\min\{j,l\}}(-1)^i q^{\binom{i}{2}}\widetilde{\nabla}_{i\to j}(\widetilde{\Delta}_{l\to i}\widetilde{\Delta}_{k\to l})\right)\\
&= \sum_{i=0}^{\min\{j,l\}}(-1)^i q^{\binom{i}{2}}\binom{k-i}{l-i}_q(\Psi_{j\to k}\widetilde{\nabla}_{i\to j})\widetilde{\Delta}_{k\to i}\\
&= \sum_{i=0}^{\min\{j,l\}}(-1)^i q^{\binom{i}{2}}\binom{k-i}{l-i}_q q^{k(j-i)}\binom{n-k-i}{j-i}_q \Psi_{i\to k}\widetilde{\Delta}_{k\to i}\\
&= \sum_{i=0}^{\min\{j,l\}}(-1)^i q^{k(j-i)+\binom{i}{2}}\binom{n-k-i}{j-i}_q\binom{k-i}{l-i}_q B_i.
\end{aligned}$$

The product $B_l B_j$ can be calculated analogously. ■

Lemma 6.4.6. *Let $0 \le j \le k \le n$ and $j + k \le n$. Then*

(a) $\Psi_{j\to k}$ *is injective,*

(b) $B_j(\widetilde{N}_k) = \widetilde{\nabla}_{j\to k}(\widetilde{N}_j)$.

Proof. (a) By Lemma 6.4.4(c) and (f) we have

$$\begin{aligned}
\widetilde{\Delta}_{j\to j} &= \sum_{i=0}^{j}(-1)^i q^{\binom{i+1}{2}-ji}\widetilde{\nabla}_{i\to j}\Psi_{j\to i}\\
&= \left(\sum_{i=0}^{j}(-1)^i q^{\binom{i+1}{2}-jk}\binom{n-i-j}{k-i}_q^{-1}\widetilde{\nabla}_{i\to j}\widetilde{\Delta}_{k\to i}\right)\Psi_{j\to k}.
\end{aligned}$$

Since $\widetilde{\Delta}_{j\to j}$ is the identity operator on $\widetilde{N}_j$ our operator $\Psi_{j\to k}$ must be injective.

(b) By definition and Lemma 6.4.4(a) and (e) we have

$$\begin{aligned}
B_j = \Psi_{j\to k}\widetilde{\Delta}_{k\to j} &= \sum_{i=0}^{j}(-1)^i q^{\binom{i}{2}}\binom{k-i}{j-i}_q\widetilde{\nabla}_{i\to k}\widetilde{\Delta}_{k\to i}\\
&= \widetilde{\nabla}_{j\to k}\left(\sum_{i=0}^{j}(-1)^i q^{\binom{i}{2}}\widetilde{\nabla}_{i\to j}\widetilde{\Delta}_{k\to i}\right),
\end{aligned}$$

hence

$$B_j(\widetilde{N}_k) \subseteq \widetilde{\nabla}_{j\to k}(\widetilde{N}_j). \tag{6.58}$$

Moreover, since $\widetilde{\Delta}_{k\to j}$ is a scalar multiple of the Lefschetz lowering operator restricted to $\widetilde{N}_k$, we derive from Theorem 6.2.5 that $\widetilde{\Delta}_{k\to j}$ is surjective and $\widetilde{\nabla}_{j\to k}$ is injective. Consequently (using (a)), $\dim B_j(\widetilde{N}_k) = \dim \Psi_{j\to k}(\widetilde{N}_j) = W_j = \dim \widetilde{\nabla}_{j\to k}(\widetilde{N}_j)$. This gives equality in (6.58). ■

Let us look at the spaces $E_{i,t}$ defined in (6.14). For our purposes, we use here other indices. Recall again that $\widetilde{\nabla}_{i\to k}$ and $\widetilde{\Delta}_{i\to i-1}$ are scalar multiples of $\widetilde{\nabla}_{L_{i,k}}$ (resp. $\widetilde{\Delta}_{L_{i,i-1}}$). Consequently, we have

$$E_{k,i} = \widetilde{\nabla}_{i\to k}(\ker(\widetilde{\Delta}_{i\to i-1})), \quad 0 \le i \le k, \tag{6.59}$$

and by Theorem 6.2.5

$$\widetilde{N}_k = E_{k,0} \oplus \cdots \oplus E_{k,k}, \quad 0 \le k \le \frac{n}{2}. \tag{6.60}$$

Theorem 6.4.3. *Let $0 \le i, j \le k \le \frac{n}{2}$. The spaces $E_{k,i}$ are the eigenspaces of B_j to the eigenvalue*

$$\lambda_{j,i} = (-1)^i q^{j(k-i)+\binom{i}{2}} \binom{n-i-j}{k-i}_q \binom{k-i}{j-i}_q$$

(here we set $\binom{k-i}{j-i}_q := 0$; that is, $\lambda_{j,i} = 0$ if $j < i$).

Proof. Let $\varphi \in E_{k,i}$; that is, $\varphi = \widetilde{\nabla}_{i\to k}(\varphi')$ for some $\varphi' \in \ker(\widetilde{\Delta}_{i\to i-1})$.

Case 1. $j < i$. We have as in the proof of Lemma 6.4.6(b)

$$\begin{aligned} B_j(\varphi) &= B_j \widetilde{\nabla}_{i\to k}(\varphi') \\ &= \sum_{l=0}^{j} (-1)^l q^{\binom{l}{2}} \binom{k-l}{j-l}_q \widetilde{\nabla}_{l\to k} \widetilde{\Delta}_{k\to l} \widetilde{\nabla}_{i\to k}(\varphi'). \end{aligned}$$

$\widetilde{\Delta}_{k\to l}\widetilde{\nabla}_{i\to k}(\varphi')$ is a scalar multiple of $\widetilde{\Delta}_L^{k-l}\widetilde{\nabla}_L^{k-i}(\varphi')$. Since $0 \le l \le j < i$ we have $k - l > k - i$. Lemma 6.2.3 implies that these terms equal the zero vector (note $\widetilde{\Delta}_L(\varphi') = \widetilde{0}$); that is, also $B_j(\varphi) = \widetilde{0}$ and $\lambda_{j,i} = 0$.

Case 2. $j \ge i$. Since $\varphi \in \widetilde{\nabla}_{i\to k}(\widetilde{N}_i)$, there is by Lemma 6.4.6(b) some $\varphi'' \in \widetilde{N}_k$ such that $B_i(\varphi'') = \varphi$. We first show that, for $0 \le l < i$, there holds $B_l(\varphi'') = \widetilde{0}$: In view of Lemma 6.4.5 and Case 1 we have

$$\widetilde{0} = B_l(\varphi) = B_l B_i(\varphi'') = B_i B_l(\varphi'').$$

Again by Lemma 6.4.6(b) there is some $\varphi''' \in \widetilde{N}_l$ such that $\widetilde{\nabla}_{l\to k}(\varphi''') = B_l(\varphi'')$. Using the definition of B_i, we conclude that

$$\widetilde{0} = \Psi_{i\to k} \widetilde{\Delta}_{k\to i} \widetilde{\nabla}_{l\to k}(\varphi''').$$

But $\widetilde{\Delta}_{k\to i}\widetilde{\nabla}_{l\to k}$ is injective according to Theorem 6.2.5 (note $l < i \le k$) and $\Psi_{i\to k}$ is injective by Lemma 6.4.6(a). Consequently, $\varphi''' = \widetilde{0}$ implying $B_l(\varphi'') = \widetilde{0}$. This result and Lemma 6.4.5 imply

$$\begin{aligned} B_j(\varphi) &= B_j B_i(\varphi'') = B_i B_j(\varphi'') \\ &= q^{j(k-i)} \binom{n-i-j}{k-i}_q \sum_{l=0}^{i} (-1)^l q^{k(i-l)+\binom{l}{2}} \binom{n-k-l}{i-l}_q \binom{k-l}{j-l}_q B_l(\varphi'') \\ &= q^{j(k-i)} \binom{n-i-j}{k-i}_q (-1)^i q^{\binom{i}{2}} \binom{k-i}{j-i}_q \varphi. \end{aligned}$$

■

For the application of Theorem 6.4.1, we need further that $\varphi_G = \sum_{X\in N_k} \widetilde{X}$ is an eigenvector. This is in the present situation the case because φ_G generates $E_{k,0}$. The corresponding eigenvalue is $\lambda_{j,0} := q^{jk}\binom{n-j}{k}_q \binom{k}{j}_q$. As mentioned previously, we will work with an operator of the form $A = \beta_{k-t+1}B_{k-t+1} + \cdots + \beta_k B_k$. Here we need numbers β_j, $j = k-t+1, \ldots, k$, such that

$$\sum_{j=k-t+1}^{k} \beta_j \lambda_{j,i} \geq -1, \quad i = 1, \ldots, k,$$

and $\sum_{j=k-t+1}^{k} \beta_j \lambda_{j,0}$ is as large as possible. To get an idea how to choose these β_j's, let us look back at the beginning of this section. We are looking for parameters n, k, t such that $\alpha(G) = \binom{n-t}{k-t}_q$. So we take in Lemma 6.4.1 $\delta := \binom{n-t}{k-t}_q$. Independent sets S of this size were represented after (6.57). The corresponding vector in $\widetilde{N}_k$ has the form $\varphi_S = \sum_{X\in N_k : X\supseteq Y} \widetilde{X}$, where $Y \in N_t$ is arbitrary. Obviously, $\varphi_S = \widetilde{\nabla}_{t\to k}(\widetilde{Y})$. By Remark 6.4.1, $\alpha(G) = \delta$ and S is maximum iff

$$\langle M(\varphi_S), \varphi_S\rangle = 0.$$

Moreover, M has to be positive-semidefinite. Hence the vectors φ_S are optimal solutions of the problem $\langle M(\varphi), \varphi\rangle \to \min$, where $\varphi \in \widetilde{N}_k$. This implies that $M(\varphi_S) = \widetilde{0}$ (either expand φ_S as a linear combination of eigenvectors or use the fact that the gradient of the corresponding quadratic objective function must be the zero vector).

Let $X \in N_k$ and $Y \in N_t$ be arbitrary. From the preceding remarks we obtain the condition

$$\left\langle M\widetilde{\nabla}_{t\to k}(\widetilde{Y}), \widetilde{X}\right\rangle = 0;$$

that is,

$$\left\langle \widetilde{Y}, \widetilde{\Delta}_{k\to t} M(\widetilde{X})\right\rangle = 0.$$

We have in view of Lemma 6.4.4(c)

$$\widetilde{\Delta}_{k\to t}B_j = \widetilde{\Delta}_{k\to t}\Psi_{j\to k}\widetilde{\Delta}_{k\to j} = q^{j(k-t)}\binom{n-j-t}{k-t}_q \Psi_{j\to t}\widetilde{\Delta}_{k\to j}.$$

Hence

$$\widetilde{\Delta}_{k\to t}M = \widetilde{\Delta}_{k\to t} + \sum_{j=k-t+1}^{k} \beta_j q^{j(k-t)}\binom{n-j-t}{k-t}_q \Psi_{j\to t}\widetilde{\Delta}_{k\to j} - \frac{1}{\delta}\widetilde{\Delta}_{k\to t}\widetilde{J}.$$

Let $m := r(X \wedge Y)$. Then

$$\left\langle \widetilde{Y}, \widetilde{\Delta}_{k\to t}(\widetilde{X})\right\rangle = \begin{cases} 1 & \text{if } Y \le X, \text{ i.e., } m = t,\\ 0 & \text{otherwise,}\end{cases}$$

$$\begin{aligned}\left\langle \widetilde{Y}, \Psi_{j\to t}\widetilde{\Delta}_{k\to j}(\widetilde{X})\right\rangle &= |\{Z \in N_j : Z \le X \text{ and } Z \wedge Y = 0\}|\\ &= q^{mj}\binom{k-m}{j}_q\end{aligned}$$

(this follows from Lemma 6.4.3(a)), and

$$\left\langle \widetilde{Y}, \widetilde{\Delta}_{k\to t}\widetilde{J}(\widetilde{X})\right\rangle = |\{Z \in N_k : Z \ge Y\}| = \binom{n-t}{k-t}_q = \delta.$$

Accordingly,

$$\begin{aligned}0 &= \left\langle \widetilde{Y}, \widetilde{\Delta}_{k\to t}M(\widetilde{X})\right\rangle\\ &= \begin{cases} 1 & \text{if } m = t,\\ 0 & \text{otherwise}\end{cases} + \sum_{j=k-t+1}^{k} \beta_j q^{j(k-t)}\binom{n-j-t}{k-t}_q q^{mj}\binom{k-m}{j}_q - \frac{1}{\delta}\delta.\end{aligned}$$

So we transformed our condition into the following system of equations:

$$\sum_{j=k-t+1}^{k} \beta_j q^{j(k-t+m)}\binom{n-j-t}{k-t}_q\binom{k-m}{j}_q = \begin{cases} 0 & \text{if } m = t,\\ 1 & \text{if } 0 \le m \le t-1.\end{cases}$$

For formal simplification, set $j := j-(k-t+1)$ and introduce $\gamma_j := \beta_{j+(k-t+1)}$, $j = 0, \dots, t-1$. We have to solve

$$\begin{aligned}&\sum_{j=0}^{t-1}\gamma_j q^{(j+(k-t+1))(k-t+m)}\binom{n-j-k-1}{k-t}_q\binom{k-m}{j+k-t+1}_q\\ &\qquad = \begin{cases} 0 & \text{if } m = t,\\ 1 & \text{if } 0 \le m \le t-1.\end{cases}\end{aligned}$$

Let us verify that a solution of this system is given by

$$\gamma_j := (-1)^j q^{-(k-t+1)j-\binom{j}{2}-(k-t+1+j)(k-j-1)} * \binom{j+k-t}{j}_q \binom{n-j-k-1}{k-t}_q^{-1}.$$

Indeed, insertion yields the LHS of our system:

$$\sum_{j=0}^{t-1}(-1)^j q^{-(k-t+1)j-\binom{j}{2}-(k-t+1+j)(t-1-m-j)}\binom{j+k-t}{j}_q \binom{k-m}{j+k-t+1}_q.$$

If $m = t$ this sum equals 0 since $\binom{k-m}{j+k-t+1}_q = 0$ for $k-m = k-t < j+k-t+1$. Let $0 \le m \le t-1$. By the same reason all summands with $j > t-1-m$ are zero, and thus we arrive at

$$\sum_{j=0}^{t-1-m}(-1)^j q^{-(k-t+1)j-\binom{j}{2}-(k-t+1+j)(t-1-m-j)} * \binom{j+k-t}{j}_q \binom{k-m}{t-1-m-j}_q.$$

If we put in Corollary 6.4.2(d) $a := k-t+1,\ b := k-m, n := t-1-m,\ k := j$, we see that the last sum equals $\binom{t-1-m}{t-1-m}_q = 1$, and the verification of our solution is complete.

Because we have chosen the γ's, that is, the β's, such that $\langle M(\varphi), \varphi\rangle = 0$ for all $\varphi \in \widetilde{\nabla}_{t\to k}(\widetilde{N}_t) = \widetilde{\nabla}_{t\to k}(E_{t,0} \oplus \cdots \oplus E_{t,t}) = E_{k,0} \oplus \cdots \oplus E_{k,t}$ (see (6.59) and (6.60)), we know already that

$$\sum_{j=k-t+1}^{k} \beta_j \lambda_{j,i} = \begin{cases} \dfrac{\binom{n}{k}_q}{\binom{n-t}{k-t}_q} - 1 & \text{if } i = 0, \\ -1 & \text{if } i = 1, \ldots, t. \end{cases}$$

In order to apply Theorem 6.4.1 we must finally investigate for which parameters k and t the inequalities

$$\sum_{j=k-t+1}^{k} \beta_j \lambda_{j,i} = \sum_{j=0}^{t-1} \gamma_j \lambda_{j+(k-t+1),i} \ge -1, \quad i = t+1, \ldots, k,$$

hold. Up to now all conclusions were true for $L_n(q)$ as well as for B_n. From now on we must distinguish between these cases.

Lemma 6.4.7. *Let $1 \le t < i \le k \le \frac{n}{2}$ and $q \ge 2$. Then*

$$\sum_{j=0}^{t-1} \gamma_j \lambda_{j+(k-t+1),i} > -1.$$

Proof. With the substitution $j := t-1-j$ and in view of Theorem 6.4.3 (recall that $\lambda_{k-j,i} = 0$ if $k-j < i$, i.e., $j > k-i$), we have

$$\sum_{j=0}^{t-1} \gamma_j \lambda_{j+(k-t+1),i} = \sum_{j=0}^{\min\{t-1,k-i\}} \gamma_{t-1-j}\lambda_{k-j,i}$$

$$= (-1)^i \sum_{j=0}^{\min\{t-1,k-i\}} (-1)^{t-1-j} q^{-(k-t+1)(t-1-j)-\binom{t-1-j}{2}+(k-j)(t-i-j)+\binom{i}{2}}$$

$$* \binom{k-1-j}{k-t}_q \binom{n-i-k+j}{k-i}_q \binom{k-i}{j}_q \binom{n-t+j-k}{k-t}_q^{-1}.$$

This is an alternating sum. It is sufficient to show that the terms decrease in absolute value and that the first term has absolute value smaller than 1. Note that

$$\frac{q^b-1}{q^a-1} < q^{b-a} \quad \text{for } 0 \le b < a,\ q > 1,$$

$$\frac{q^b-1}{q^a-1} < q^{b-a+1} \quad \text{for } a \ge 1,\ q \ge 2.$$

The absolute value of the ratio of consecutive terms (with index $j \ge 0$ and $j+1 \le \min\{t-1, k-i\}$) equals

$$q^{i+j-t} \frac{q^{t-j-1}-1}{q^{k-1-j}-1} \frac{q^{n-i-k+j+1}-1}{q^{n-2k+j+1}-1} \frac{q^{k-i-j}-1}{q^{j+1}-1} \frac{q^{n-2k+j+1}-1}{q^{n-t+j+1-k}-1}$$

$$< q^{i+j-t} q^{t-k} q^{k-i+1} q^{k-i-2j} q^{t-k} = q^{t-i-j+1} \le q^{-j} \le 1.$$

We are interested in the absolute value for $j = 0$. Considering the term as a function $g(i)$ where

$$g(i) := q^{-(k-t+1)(t-1)-\binom{t-1}{2}+k(t-i)+\binom{i}{2}} \binom{k-1}{k-t}_q \binom{n-i-k}{k-i}_q \binom{n-t-k}{k-t}_q^{-1}$$

we have to show that $g(t+1) < 1$ and $\frac{g(i+1)}{g(i)} \le 1$. Indeed,

$$g(t+1) = q^{-kt+t^2} \binom{k-1}{k-t}_q \frac{q^{k-t}-1}{q^{n-k-t}-1}$$

$$\le q^{-t(k-t)} \frac{(q^{k-1}-1)\cdots(q^t-1)}{(q^{k-t}-1)\cdots(q-1)} < q^{-t(k-t)} q^{t(k-t)} = 1$$

and

$$\frac{g(i+1)}{g(i)} = q^{i-k} \frac{q^{k-i}-1}{q^{n-i-k}-1} \le q^{i-k} \le 1.$$

This finishes the proof. ∎

Theorems 6.4.1 and 6.4.3 as well as Lemma 6.4.7 (with the choice of the β's from the preceding text) finally imply:

Theorem 6.4.4. *Let $0 \le 2k \le n$. Then the maximum size of a k-uniform t-intersecting family in $L_n(q)$ equals $\binom{n-t}{k-t}_q$.*

This theorem was first proved by Hsieh [272] for $n \ge 2k+1$, $q \ge 3$ and for $n \ge 2k+2$, $q = 2$ by difficult and lengthy combinatorial means. The case $t = 1$ can be treated more easily by first proving the theorem for $n = 2k$ (see Greene and Kleitman [234]) and then using an inductive argument (Deza and Frankl [135]). The proof presented before Theorem 6.4.4 is due to Frankl and Wilson [203]. They proved a little bit more:

Corollary 6.4.3. *Let $2k - t \le n \le 2k$. Then the maximum size of a k-uniform t-intersecting family $\mathcal{F}$ in $L_n(q)$ equals $\binom{2k-t}{k}_q$.*

Proof. For $X, Y \in \mathcal{F}$, we have by the modularity of the rank function $r(X \vee Y) = r(X) + r(Y) - r(X \wedge Y) \le 2k - t$. If we consider $\mathcal{F}$ as a family in the dual poset of $L_n(q)$ (which is isomorphic to $L_n(q)$), then $\mathcal{F}$ is an $(n-k)$-uniform $n - (2k - t)$-intersecting family. Since $n \le 2k$, we have $2(n - k) \le n$. By Theorem 6.4.4

$$|\mathcal{F}| \le \binom{n - (n - (2k - t))}{n - k - (n - (2k - t))}_q = \binom{2k - t}{k - t}_q = \binom{2k - t}{k}_q.$$

The bound is attained by the family $\mathcal{F} := \{X \le Y : r(X) = k\}$, where Y is fixed with $r(Y) = 2k - t$. ■

Now we will consider the Boolean lattice B_n. For the sake of brevity, we use (see the proof of Lemma 6.4.7)

$$\begin{aligned}\theta_i &:= \sum_{j=k-t+1}^{k} \beta_j \lambda_{j,i} \\ &= (-1)^i \sum_{j=0}^{t-1} (-1)^{t-1-j} \binom{k-1-j}{k-t} \binom{n-i-k+j}{k-i} \\ &\quad * \binom{k-i}{j} \binom{n-t+j-k}{k-t}^{-1}.\end{aligned}$$

Let us recall that we have to prove $\theta_i \ge -1$ for $1 \le t < i \le k \le \frac{n}{2}$.

Lemma 6.4.8. *Let $n = (t+1)(k-t+1)$ and $1 \le t \le k-2$. Then $\theta_{t+2} = -1$.*

Proof. Direct computation seems to be hopeless, so again we need some more theoretical insight. Let $Y \in N_{t+2}$ be fixed and let $S := \{X \in N_k : |X \cap Y| \geq t+1\}|$. It was mentioned already (before Lemma 2.4.1) that S is t-intersecting and from Lemma 2.4.1(b) it follows that $|S| = \binom{n-t}{k-t}$. Thus we work with

$$\varphi_S := \sum_{X \in S} \widetilde{X} = \sum_{X \in N_k : |X \cap Y| = t+1} \widetilde{X} + \sum_{X \in N_k : X \supseteq Y} \widetilde{X}.$$

It is easy to see that

$$\varphi_S = \widetilde{\nabla}_{t+1 \to k} \widetilde{\Delta}_{t+2 \to t+1}(\widetilde{Y}) - (t+1)\widetilde{\nabla}_{t+2 \to k}(\widetilde{Y}). \tag{6.61}$$

Hence, $\varphi_S \in \widetilde{\nabla}_{t+2 \to k}(\widetilde{N}_{t+2})$ (note Lemma 6.4.4(b)). Because $\widetilde{\nabla}_{t+2 \to k}(\widetilde{N}_{t+2}) = (E_{k,0} \oplus \cdots \oplus E_{k,t}) \oplus E_{k,t+1} \oplus E_{k,t+2}$ and $E_{k,0} \oplus \cdots \oplus E_{k,t} = \widetilde{\nabla}_{t \to k}(\widetilde{N}_t)$, we can write φ_S in the form

$$\varphi_S = \varphi + \varphi_{t+1} + \varphi_{t+2}, \text{ where } \varphi \in \widetilde{\nabla}_{t \to k}(\widetilde{N}_t), \varphi_{t+1} \in E_{k,t+1}, \varphi_{t+2} \in E_{k,t+2}.$$

Claim 1. $\varphi_{t+2} \neq \widetilde{0}$.

Proof of Claim 1. We have $\widetilde{\nabla}_{t+2 \to k}(\widetilde{Y}) \notin E_{k,0} \oplus \cdots \oplus E_{k,t+1} = \widetilde{\nabla}_{t+1 \to k}(\widetilde{N}_{t+1})$ because otherwise $\widetilde{\nabla}_{t+2 \to k}(\widetilde{Y}) = \widetilde{\nabla}_{t+2 \to k} \widetilde{\nabla}_{t+1 \to t+2}(\varphi)$ for some $\varphi \in \widetilde{N}_{t+1}$, which would imply by the injectivity of $\widetilde{\nabla}_{t+2 \to k}$ that $\widetilde{Y} \in \widetilde{\nabla}_{t+1 \to t+2}(\widetilde{N}_{t+1})$, and this contradicts Proposition 6.1.1 since $\operatorname{rank}(\widetilde{\nabla}_{L_{t+1,t+2}}) = W_{t+1} < W_{t+2}$. On the other hand, the first term on the RHS in (6.61) belongs to $E_{k,0} \oplus \cdots \oplus E_{k,t+1}$; thus $\varphi_{t+2} \neq \widetilde{0}$. □

Claim 2. $\varphi_{t+1} = \widetilde{0}$.

Proof of Claim 2. The eigenvalue of B_{t+1} to the eigenspace $E_{k,t+1}$ equals

$$\lambda_{t+1,t+1} = (-1)^{t+1} \binom{n - 2(t+1)}{k - (t+1)} \binom{k - (t+1)}{0} \neq 0$$

by Theorem 6.4.3. So it is sufficient to show that

$$B_{t+1}(\varphi_{t+1}) = \widetilde{0}.$$

Because

$$B_{t+1}(\varphi) \in E_{k,0} \oplus \cdots \oplus E_{k,t}, B_{t+1}(\varphi_{t+1}) \in E_{k,t+1}, B_{t+1}(\varphi_{t+2}) \in E_{k,t+2},$$

it is enough to show that

$$B_{t+1}(\varphi_S) \subseteq E_{k,0} \oplus \cdots \oplus E_{k,t} = \widetilde{\nabla}_{t \to k}(\widetilde{N}_t).$$

By Lemma 6.4.4(b), (d), and (e) and because B_{t+1} is self-adjoint, we have for $t+1 \le i \le k$

$$\begin{aligned} B_{t+1}\widetilde{\nabla}_{i\to k} &= \Psi_{t+1\to k}\widetilde{\Delta}_{k\to t+1}\widetilde{\nabla}_{i\to k} \\ &= \widetilde{\nabla}_{t+1\to k}(\Psi_{k\to t+1}\widetilde{\nabla}_{i\to k}) \\ &= \binom{n-i-(t+1)}{k-i}\widetilde{\nabla}_{t+1\to k}\Psi_{i\to t+1} \\ &= \binom{n-i-(t+1)}{k-i}\sum_{j=0}^{t+1}(-1)^j\binom{k-j}{t+1-j}\widetilde{\nabla}_{j\to k}\widetilde{\Delta}_{i\to j}. \end{aligned}$$

With $i = t+1$ and $i = t+2$ we obtain

$$\begin{aligned} B_{t+1}(\varphi_S) &= B_{t+1}\widetilde{\nabla}_{t+1\to k}\widetilde{\Delta}_{t+2\to t+1}(\widetilde{Y}) - (t+1)B_{t+1}\widetilde{\nabla}_{t+2\to k}(\widetilde{Y}) \\ &= \sum_{j=0}^{t+1}(-1)^j\binom{k-j}{t+1-j}\left[\binom{n-2t-2}{k-t-1}\binom{t+2-j}{t+1-j}\right. \\ &\qquad \left. -(t+1)\binom{n-2t-3}{k-t-2}\right]\widetilde{\nabla}_{j\to k}\widetilde{\Delta}_{t+2\to j}(\widetilde{Y}). \end{aligned}$$

We may restrict the summation to $j = 0, \dots, t$, since for $j = t+1$ the coefficient in brackets equals

$$\binom{n-2t-2}{k-t-1}1 - (t+1)\binom{n-2t-3}{k-t-2} = 0 \quad \text{for } n = (t+1)(k-t+1).$$

Consequently, $B_{t+1}(\varphi_S) \in \widetilde{\nabla}_{t\to k}(\widetilde{N}_t)$. □

Now we may conclude the proof of our Lemma. From the proof of Lemma 6.4.1 it follows that $\langle M(\varphi_S), \varphi_S\rangle = 0$ (we have here $\delta = |S| = \binom{n-t}{k-t}$). Furthermore, $\langle M(\varphi), \varphi\rangle = 0$ since we determined the numbers β_j from this equation. Because $M(\varphi) \in E_{k,0} \oplus \cdots \oplus E_{k,t}$ and $\varphi_{t+2} \in E_{k,t+2}$, we have $\langle M(\varphi), \varphi_{t+2}\rangle = \langle \varphi, M(\varphi_{t+2})\rangle = \langle \widetilde{J}(\varphi_{t+2}), \varphi_{t+2}\rangle = 0$ (recall that the eigenspaces $E_{k,0}, \dots, E_{k,k}$ are pairwise orthogonal; see Lemma 6.2.5(c)). Thus,

$$\begin{aligned} 0 &= \langle M(\varphi + \varphi_{t+2}), (\varphi + \varphi_{t+2})\rangle = \langle M(\varphi_{t+2}), \varphi_{t+2}\rangle \\ &= (1 + \theta_{t+2})\langle \varphi_{t+2}, \varphi_{t+2}\rangle. \end{aligned}$$

By Claim 1, $\theta_{t+2} = -1$. ■

Lemma 6.4.9. *Let $0 \le i \le k \le \frac{n}{2}$. Then*

$$\theta_i = (-1)^{t-1-i}\binom{i-1}{t-1}\sum_{j=0}^{t-1}\frac{i-t}{i-t+j}\binom{t-1}{j}\binom{k-t}{i-t+j}\binom{n-k-t+j}{i-t+j}^{-1}.$$

Direct computation yields

$$\binom{k-1-j}{k-t}\binom{n-i-k+j}{k-i}\binom{k-i}{j}\binom{n-t+j-k}{k-t}^{-1}$$
$$=\binom{i-1}{t-1}\binom{t-1}{j}\binom{k-1-j}{i-1}\binom{n-t+j-k}{i-t}^{-1}.$$

Consequently,

$$\theta_i = (-1)^{t-1-i}\binom{i-1}{t-1}\sum_{j=0}^{t-1}(-1)^j\binom{t-1}{j}\binom{k-1-j}{i-1}\binom{n-k-t+j}{i-t}^{-1}. \tag{6.62}$$

Consider the function

$$f = f(x,y;a,b) := \frac{1}{b}\binom{x}{a}\binom{y}{b}^{-1} \quad \text{with } x,y\in\mathbb{R},\ a,b\in\mathbb{N},\ y\geq b.$$

Let $If := f$ and

$$\begin{aligned} Xf(x,y;a,b) &:= f(x-1,y;a-1,b),\\ Yf(x,y;a,b) &:= f(x-1,y+1;a,b+1),\\ Zf(x,y;a,b) &:= f(x-1,y+1;a,b). \end{aligned}$$

In a straightforward way one may verify that

$$\begin{aligned} XYf &= YXf,\\ (I-Z)f &= (X+Y)f. \end{aligned}$$

Accordingly,

$$(I-Z)^{t-1}f = (X+Y)^{t-1}f;$$

that is,

$$\sum_{j=0}^{t-1}(-1)^j\binom{t-1}{j}Z^jf = \sum_{j=0}^{t-1}\binom{t-1}{j}X^{t-1-j}Y^jf,$$
$$\sum_{j=0}^{t-1}(-1)^j\binom{t-1}{j}f(x-j,y+j;a,b)$$
$$=\sum_{j=0}^{t-1}\binom{t-1}{j}f(x-t+1,y+j;\ a-t+1+j,b+j).$$

If we put here $x := k-1, y := n-k-t, a := i-1, b := i-t$, then the LHS becomes the same sum as in (6.62) divided by $i-t$, and the RHS becomes the sum in the assertion divided by $i-t$. ■

Lemma 6.4.10. *Let* $1 \le t < i \le k \le \frac{n}{2}$ *and* $n \ge (t+1)(k-t+1)$. *Then* $\theta_i \ge -1$.

Proof. From Lemma 6.4.9 we know that $\theta_{t+1}, \theta_{t+3}, \dots$ are nonnegative and that θ_{t+2} increases ($|\theta_{t+2}|$ decreases) if n increases. Lemma 6.4.8 says that $\theta_{t+2} \ge -1$ (if $t \le k-2$ and $n \ge (t+1)(k-t+1)$). So it is sufficient to show that

$$|\theta_i| > |\theta_{i+1}| \quad \text{for } i \ge t+2.$$

We work with the formula from Lemma 6.4.9. If $t = 1$, then

$$\frac{|\theta_{i+1}|}{|\theta_i|} = \frac{\binom{k-1}{i}\binom{n-k-1}{i-1}}{\binom{k-1}{i-1}\binom{n-k-1}{i}} = \frac{k-i}{n-k-i} \le 1$$

since $n \ge 2k$. Thus let $t \ge 2$. We show that the ratio of the summands with index j ($j \le \min\{k-(i+1), t-1\}$) in the formulas for θ_{i+1} and θ_i is smaller than 1. We have

$$\frac{\binom{i}{t-1}(i+1-t)\binom{t-1}{j}\binom{k-t}{i+1-t+j}(i-t+j)\binom{n-k-t+j}{i-t+j}}{\binom{i-1}{t-1}(i+1-t+j)\binom{n-k-t+j}{i+1-t+j}(i-t)\binom{t-1}{j}\binom{k-t}{i-t+j}}$$
$$= \frac{i-t+j}{i+1-t+j}\,\frac{i(k-i-j)}{(i-t)(n-k-i)} < \frac{i(k-i)}{(i-t)(n-k-i)}.$$

We have further

$$n-k-i \ge ((t+1)(k-t+1)-k-i)-(t-1)(i-(t+1)) = t(k-i),$$

and, because $i-t \ge 2,\ t \ge 2$,

$$(i-t)t \ge (i-t)+t = i.$$

Hence we may continue the estimation of our ratio:

$$\frac{i(k-i)}{(i-t)(n-k-i)} \le \frac{i(k-i)}{(i-t)t(k-i)} \le 1.$$

∎

All in all we proved (the special case of Theorem 2.4.1):

Theorem 6.4.5. *Let* $0 \le (t+1)(k-t+1) \le n$. *Then the maximum size of a* k*-uniform* t*-intersecting family in* B_n *equals* $\binom{n-t}{k-t}$.

This theorem is due to Wilson [470] (and to Frankl [188] for $t \ge 15$). Schrijver [419] had come close to the solution before (using Delsarte's and Lovász's ideas).

Of course, the presented method can also be applied to other structures. See, for example, the papers of Stanton [445], Moon [373], and Huang [274].

Finally, note that one may derive with this method that, for $n > 2k$ (resp. $n > (t+1)(k-t+1)$) in the case of $L_n(q)$ (resp. B_n), every maximum k-uniform t-intersecting family S has the structure $S = \{X \in N_k : X \geq X_0\}$ for some $X_0 \in N_t$. This can be shown using Remark 6.4.1. Strict inequalities in our estimates imply that $\sum_{X \in S} \widetilde{X} \in \widetilde{\nabla}_{t \to k}(\widetilde{N}_t)$ for every maximum family and then an induction argument yields this structure. We avoid detailed presentation and refer to Wilson [470].

6.5. Further algebraic methods to prove intersection theorems

As in the previous section, we will regard in the following the Boolean lattice B_n and the linear lattice $L_n(q)$ simultaneously and we will use the letter P for both B_n and $L_n(q)$. We have formally $\lim_{q \to 1} L_n(q) = B_n$. Recall that for a family $\mathcal{F}$ in P, $\widetilde{\mathcal{F}}$ denotes the subspace of $\widetilde{P}$ generated by $\{\widetilde{X} : X \in \mathcal{F}\}$. Assume that we have a second family $\mathcal{G}$ in P. In this section we will often work with the operator $\widetilde{\Delta}_{\mathcal{F} \to \mathcal{G}} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ and its adjoint $\widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}} : \widetilde{G} \to \widetilde{F}$, which were defined in (6.30) by

$$\widetilde{\Delta}_{\mathcal{F} \to \mathcal{G}}(\widetilde{X}) := \sum_{Y \in \mathcal{G} : Y \leq X} \widetilde{Y} \qquad \left(\text{resp. } \widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}}(\widetilde{X}) := \sum_{Y \in \mathcal{F} : X \leq Y} \widetilde{Y} \right)$$

(empty sums are defined to be the zero vector). The following idea allows us to compare the sizes of $\mathcal{F}$ and $\mathcal{G}$:

Proposition 6.5.1. *Suppose that there are operators $A : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ and $\Phi : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{F}}$ such that A is injective and $A(\widetilde{\mathcal{F}}) \subseteq \Phi(\widetilde{\mathcal{G}})$. Then $|\mathcal{F}| \leq |\mathcal{G}|$.*

Proof. We have

$$|\mathcal{F}| = \dim \widetilde{\mathcal{F}} = \dim A(\widetilde{\mathcal{F}}) \leq \dim \Phi(\widetilde{\mathcal{G}}) \leq \dim \widetilde{\mathcal{G}} = |\mathcal{G}|.$$

■

The operator A is determined by the matrix $M_A = \left(\langle A(\widetilde{X}), \widetilde{Y} \rangle\right)$, whose rows and columns are indexed by the elements X, Y of $\mathcal{F}$. The following fact is obvious:

Proposition 6.5.2. *The operator A is injective iff its matrix M_A has nonzero determinant, for instance, if M_A is a triangular matrix for which each diagonal element is nonzero.*

We will apply this idea to prove (or re-prove) some intersection theorems. Let $L = \{l_1, \dots, l_s\}$ be a set of integers, $0 \le l_1 < \dots < l_s \le n$. As in Section 2.5 we say that a family $\mathcal{F}$ in P is *L-intersecting* if $r(X \wedge Y) \in L$ for all $X, Y \in \mathcal{F}$, $X \ne Y$. Note that a t-intersecting family is the same as a $\{t, t+1, \dots, n-1\}$-intersecting family.

The following theorem is due to Frankl and Wilson [202] in the Boolean case (see Theorem 2.5.5). Lefmann [339] generalized it to $L_n(q)$.

Theorem 6.5.1. *Let $\mathcal{F}$ be an L-intersecting family in $L_n(q)$. Then*

$$|\mathcal{F}| \le \sum_{i=0}^{|L|} \binom{n}{i}_q .$$

The same result is true for B_n if we let $q \to 1$.

Proof. The set $\mathcal{G} := \cup_{i=0}^{|L|} N_i$ plays the role of $\mathcal{G}$ in Proposition 6.5.1. We define the operator A by

$$\langle A(\widetilde{X}), \widetilde{Y} \rangle := \prod_{l \in L : l < r(X)} \frac{q^{r(X \wedge Y)} - q^l}{q-1}, \quad X, Y \in \mathcal{F},$$

and for Φ we choose $\Phi := \widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}}$. If we order the elements of $\mathcal{F} = \{X_1, \dots, X_m\}$ (without repetitions) by nonincreasing rank, $r(X_1) \ge \dots \ge r(X_m)$, then

$$r(X_i \wedge X_j) < r(X_i) \quad \text{if } 1 \le i < j \le m.$$

Clearly $r(X_j \wedge X_j) = r(X_j)$; hence

$$\langle A(\widetilde{X_i}), \widetilde{X_j} \rangle \begin{cases} = 0 & \text{if } 1 \le i < j \le m, \\ \ne 0 & \text{if } 1 \le i = j \le m, \end{cases}$$

hence Proposition 6.5.2 can be applied for the verification of the injectivity of A. It remains to show that

$$A(\widetilde{\mathcal{F}}) \subseteq \widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}}(\widetilde{\mathcal{G}}). \tag{6.63}$$

To see this, we work with the operators $\widetilde{\Delta}_{\mathcal{F} \to i} := \widetilde{\Delta}_{\mathcal{F} \to N_i}$ and their adjoints $\widetilde{\nabla}_{i \to \mathcal{F}} := \widetilde{\nabla}_{N_i \to \mathcal{F}}$.

Claim. There exist real numbers $\alpha_{i,X}$, $i = 0, \dots, |L|$, $X \in \mathcal{F}$, such that

$$A(\widetilde{X}) = \sum_{i=0}^{|L|} \alpha_{i,X} \widetilde{\nabla}_{i \to \mathcal{F}} \widetilde{\Delta}_{\mathcal{F} \to i}(\widetilde{X}).$$

Proof of Claim. First observe that we have

$$\langle \widetilde{\nabla}_{i \to \mathcal{F}} \widetilde{\Delta}_{\mathcal{F} \to i}(\widetilde{X}), \widetilde{Y} \rangle = |\{Z \in N_i : Z \le X \wedge Y\}| = \binom{r(X \wedge Y)}{i}_q . \tag{6.64}$$

By the definition of A, for any $X \in \mathcal{F}$ there exists a polynomial f_X of degree at most $|L|$ such that

$$\langle A(\widetilde{X}), \widetilde{Y} \rangle = f_X(q^{r(X \wedge Y)}). \tag{6.65}$$

By Corollary 6.4.2(a) every monomial x^k can be written as a linear combination of the functions

$$\frac{(x-1)\cdots(x-q^{i-1})}{(q^i-1)\cdots(q^i-q^{i-1})}, \quad i = 0, \ldots, k$$

(for $i = 0$ this is the constant 1), that is, there exist numbers $\alpha_{i,X}$ such that

$$f_X = \sum_{i=0}^{|L|} \alpha_{i,X} \frac{(x-1)\cdots(x-q^{i-1})}{(q^i-1)\cdots(q^i-q^{i-1})}.$$

This implies by (6.64) and (6.65) that

$$\langle A(\widetilde{X}), \widetilde{Y} \rangle = \sum_{i=0}^{|L|} \alpha_{i,X} \binom{r(X \wedge Y)}{i}_q = \sum_{i=0}^{|L|} \alpha_{i,X} \langle \widetilde{\nabla}_{i \to \mathcal{F}} \widetilde{\Delta}_{\mathcal{F} \to i}(\widetilde{X}), \widetilde{Y} \rangle;$$

which yields the claim. □

Clearly,

$$\widetilde{\nabla}_{i \to \mathcal{F}} \widetilde{\Delta}_{\mathcal{F} \to i}(\widetilde{X}) \in \widetilde{\nabla}_{i \to \mathcal{F}}(\widetilde{N}_i) \subseteq \widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}}(\widetilde{\mathcal{G}}), \; i = 0, \ldots, |L|.$$

Thus also

$$A(\widetilde{X}) \subseteq \widetilde{\nabla}_{\mathcal{G} \to \mathcal{F}}(\widetilde{\mathcal{G}}),$$

and (6.63) is proved. ■

Corollary 6.5.1. *Let $\mathcal{F}$ be a k-uniform L-intersecting family in $L_n(q)$, $k \geq |L|$. Then*

$$|\mathcal{F}| \leq \binom{n}{|L|}_q.$$

The same result is true for B_n if we let $q \to 1$.

Proof. In Proposition 6.5.1 we take $\mathcal{G} := N_{|L|}$, A and Φ as in the proof of Theorem 6.5.1; that is, $\Phi := \widetilde{\nabla}_{|L| \to \mathcal{F}}$. As in Lemma 6.4.4(b), we have for $0 \leq i \leq |L|$

$$\widetilde{\nabla}_{i \to \mathcal{F}} = \frac{1}{\binom{k-i}{|L|-i}} \widetilde{\nabla}_{|L| \to \mathcal{F}} \widetilde{\nabla}_{i \to |L|}.$$

Thus $\widetilde{\nabla}_{i \to \mathcal{F}}(\widetilde{N}_i) \subseteq \widetilde{\nabla}_{|L| \to \mathcal{F}}(\widetilde{N}_{|L|})$ and the claim in the proof of Theorem 6.5.1 implies $A(\widetilde{\mathcal{F}}) \subseteq \widetilde{\nabla}_{|L| \to \mathcal{F}}(\widetilde{\mathcal{G}})$. ■

Originally this theorem was proved by Ray-Chaudhuri and Wilson [394] only for B_n (see Theorem 2.5.6); Frankl and Graham [198] generalized it to $L_n(q)$.

Now we admit more than one "special" size for the members of $\mathcal{F}$. The next result was again originally proved for B_n (but here in a more general version).

***Corollary 6.5.2** (**Alon, Babai, and Suzuki** [28]).* *Let $\mathcal{F}$ be an L-intersecting family in $L_n(q)$ such that $r(X) \in K$ for every $X \in \mathcal{F}$ where $K = \{k_1, \dots, k_r\}$, $n \geq k_1 > \cdots > k_r \geq |L| - r + 1$. Then*

$$|\mathcal{F}| \leq \sum_{j=1}^{r} \binom{n}{|L| + 1 - j}_q .$$

The same result is true for B_n if we let $q \to 1$.

Proof. We take $\mathcal{G} := \cup_{j=1}^{r} N_{|L|+1-j}$, A and Φ as in the proof of Theorem 6.5.1. If $i \geq |L| + 1 - r$ then $\widetilde{\nabla}_{i\to\mathcal{F}}(\widetilde{N}_i) \subseteq \widetilde{\nabla}_{\mathcal{G}\to\mathcal{F}}(\widetilde{\mathcal{G}})$. Thus let $i \leq |L| - r$; that is, $i < k_r$. Recall that $\mathcal{F}_{k_j} := \{X \in \mathcal{F} : r(X) = k_j\}$.

Claim. There are numbers $\beta_l, l = 1, \dots, r$, such that for $j = 1, \dots, r$, $\sum_{l=1}^{r} \beta_l \binom{k_j - i}{|L|+1-l-i}_q = 1$.

Before proving this claim, let us first finish the proof of the corollary. We have for $X \in N_i$

$$\begin{aligned}
\widetilde{\nabla}_{i\to\mathcal{F}}(\widetilde{X}) &= \sum_{j=1}^{r} \widetilde{\nabla}_{i\to\mathcal{F}_{k_j}}(\widetilde{X}) \\
&= \sum_{j=1}^{r} \left(\sum_{l=1}^{r} \beta_l \binom{k_j - i}{|L| + 1 - l - i}_q \right) \widetilde{\nabla}_{i\to\mathcal{F}_{k_j}}(\widetilde{X}) \\
&= \sum_{j=1}^{r} \sum_{l=1}^{r} \beta_l \widetilde{\nabla}_{|L|+1-l\to\mathcal{F}_{k_j}} \widetilde{\nabla}_{i\to|L|+1-l}(\widetilde{X}) \\
&= \sum_{j=1}^{r} \widetilde{\nabla}_{\mathcal{G}\to\mathcal{F}_{k_j}} \left(\sum_{l=1}^{r} \beta_l \widetilde{\nabla}_{i\to|L|+1-l}(\widetilde{X}) \right) \\
&= \widetilde{\nabla}_{\mathcal{G}\to\mathcal{F}} \left(\sum_{l=1}^{r} \beta_l \widetilde{\nabla}_{i\to|L|+1-l}(\widetilde{X}) \right) \subseteq \widetilde{\nabla}_{\mathcal{G}\to\mathcal{F}}(\widetilde{\mathcal{G}}),
\end{aligned}$$

which implies as previously $A(\widetilde{\mathcal{F}}) \subseteq \widetilde{\nabla}_{\mathcal{G}\to\mathcal{F}}(\widetilde{\mathcal{G}})$ (note that the summands with $l < |L| + 1 - k_j$ are zero (resp. the zero vector)).

Proof of Claim. We must prove only that the columns of the matrix of our system of equations are linearly independent. Assume the contrary. Then there exists a nontrivial solution of the system

$$\sum_{l=1}^{r} \gamma_l f_{a-l}(q^{h_j}) = 0, \quad j = 1, \dots, r, \tag{6.66}$$

where

$$f_m(x) := \frac{(x-1)\cdots(x-q^{m-1})}{(q^m-1)\cdots(q^m-q^{m-1})}, \quad a := |L|+1-i, \quad h_j := k_j - i$$

(in the Boolean case we have to work with the functions $\frac{1}{m!}x(x-1)\cdots(x-m+1)$ and to insert h_j instead of q^{h_j}). Because $k_j \geq |L|-r+1$, we have $q^{h_j} > q^{a-r-1}$ for all j. Dividing (6.66) by $(q^{h_j}-1)\cdots(q^{h_j}-q^{a-r-1})$, we obtain that then there also exists a nontrivial solution of the system

$$\delta_0 + \delta_1(q^{h_j}-q^{a-r}) + \cdots + \delta_{r-1}(q^{h_j}-q^{a-r})\cdots(q^{h_j}-q^{a-2}) = 0,$$
$$j = 1, \ldots, r.$$

But this is impossible since the nonzero polynomial $\delta_0 + \delta_1(x-q^{a-r}) + \cdots + \delta_{r-1}(x-q^{a-r})\cdots(x-q^{a-2})$ can have at most $r-1$ roots. ∎

Under some injectivity suppositions we may estimate shadows in B_n (Frankl and Füredi [194]) and $L_n(q)$ (Frankl and Graham [198]):

Theorem 6.5.2. *Let $0 \leq s \leq l \leq k \leq n$, $\mathcal{F}$ a k-uniform family in $L_n(q)$ and let $\widetilde{\Delta}_{\mathcal{F}\to s}$ be injective. Then*

$$\frac{|\Delta_{\to l}(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{k+s}{l}_q}{\binom{k+s}{s}_q}.$$

The same result is true for B_n if we let $q \to 1$.

Proof. Let $P = L_n(q)$ (resp. B_n). For $s = l$, we may use the standard observation that

$$|\mathcal{F}| = \dim \widetilde{\Delta}_{\mathcal{F}\to s}(\mathcal{F}) \leq \dim \widetilde{\Delta_{\to s}(\mathcal{F})} = |\Delta_{\to s}(\mathcal{F})|,$$

and for $k = 1$, the result is trivial. We apply induction on k and suppose that $s < l$. Let us fix some element Y in N_1. The filter generated by Y is again a linear lattice (resp. Boolean lattice) with the parameter $n-1$. We denote this filter by P_Y. Moreover, let $\mathcal{F}_Y := \mathcal{F} \cap P_Y$. The notations Δ^Y, $\widetilde{\Delta}^Y$, and so on mean that we consider the shadow (resp. shadow operator) only in P_Y (resp. $\widetilde{P}_Y$). Observe that $\mathcal{F}_Y$ is a $(k-1)$-uniform family in P_Y and that $N_{s+1} \cap P_Y$ is the sth level in P_Y which we denote by N_s^Y.

Claim. $\widetilde{\Delta}^Y_{\mathcal{F}_Y\to s}$ is injective.

Proof of Claim. We know that $\widetilde{\Delta}_{\mathcal{F}\to s}$ is injective, which implies that also $\widetilde{\Delta}_{\mathcal{F}_Y\to s}$ is injective; that is, $\widetilde{\nabla}_{s\to\mathcal{F}_Y}$ is surjective. It suffices to show that $\widetilde{\nabla}^Y_{s\to\mathcal{F}_Y}$ is surjective, and this is the case if

$$\widetilde{\nabla}_{s\to\mathcal{F}_Y}(\widetilde{X}) \in \widetilde{\nabla}^Y_{s\to\mathcal{F}_Y}(\widetilde{N}^Y_s) \quad \text{for all } X \in N_s. \tag{6.67}$$

If $X \notin P_Y$ then

$$\widetilde{\nabla}_{s\to\mathcal{F}_Y}(\widetilde{X}) = \widetilde{\nabla}^Y_{s\to\mathcal{F}_Y}(\widetilde{X\vee Y})$$

since for $Z \in \mathcal{F}_Y$ we have $Z \geq X$ iff $Z \geq X \vee Y$. If $X \in P_Y$ then as in Lemma 6.4.4(b)

$$\widetilde{\nabla}_{s\to\mathcal{F}_Y}(\widetilde{X}) = \frac{1}{\binom{k-1-(s-1)}{1}_q}\widetilde{\nabla}^Y_{s\to\mathcal{F}_Y}\widetilde{\nabla}^Y_{s-1\to s}(\widetilde{X}).$$

Thus in both cases (6.67) is proved. □

We may apply the induction hypothesis to the $(k-1)$-uniform family $\mathcal{F}_Y$ in P_Y to infer

$$|\Delta^Y_{\to l-1}(\mathcal{F}_Y)| \geq |\mathcal{F}_Y|\frac{\binom{k-1+s}{l-1}_q}{\binom{k-1+s}{k-1}_q}. \tag{6.68}$$

If we count the number of pairs (Y, X) with $N_1 \ni Y \leq X \in \mathcal{F}$ in two different ways we obtain that

$$\sum_{Y\in N_1}|\mathcal{F}_Y| = \binom{k}{1}_q|\mathcal{F}|, \tag{6.69}$$

and counting the number of pairs (Y, X) with $N_1 \ni Y \leq X \in \Delta_{\to l}(\mathcal{F})$ gives

$$\sum_{Y\in N_1}|\Delta^Y_{\to l-1}(\mathcal{F}_Y)| = \binom{l}{1}_q|\Delta_{\to l}(\mathcal{F})|. \tag{6.70}$$

Summing up (6.68) over $Y \in N_1$ yields, with (6.69) and (6.70),

$$\binom{l}{1}_q|\Delta_{\to l}(\mathcal{F})| \geq \binom{k}{1}_q|\mathcal{F}|\frac{\binom{k-1+s}{l-1}_q}{\binom{k-1+s}{k-1}_q}$$

which is equivalent to the statement in the theorem. ■

Note that the bound in Theorem 6.5.2 is the best possible. Take $n := k+s$, $\mathcal{F} := N_k$ and note that $\widetilde{\Delta}_{k\to s}$ is injective since $L_n(q)$ and B_n are unitary Peck (see Example 6.2.9). Part (b) of the next corollary is a result of Katona [290] in the Boolean case, which was proved by him without the algebraic approach. Frankl and Graham [198] also considered $L_n(q)$.

Corollary 6.5.3. *Let $\mathcal{F}$ be a k-uniform t-intersecting family in $L_n(q)$, $k-t \leq l \leq k$. Then*

(a) $\widetilde{\Delta}_{\mathcal{F}\to l}$ is injective,

(b) $\frac{|\Delta_{\to l}(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{l}_q}{\binom{2k-t}{k}_q}$.

The same result is true for B_n if we let $q \to 1$.

Proof. (a) Obviously $\mathcal{F}$ is L-intersecting where $L = \{t, t+1, \ldots, k-1\}$; that is, $|L| = k-t$. From the proof of Corollary 6.5.1 it follows that $\widetilde{\nabla}_{k-t\to\mathcal{F}}$ is surjective, because the corresponding operator $A : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$ is injective – that is, bijective. Consequently, $\widetilde{\Delta}_{\mathcal{F}\to k-t}$ is injective. Because $\binom{t}{l-k+t}_q \widetilde{\Delta}_{\mathcal{F}\to k-t} = \widetilde{\Delta}_{l\to k-t}\widetilde{\Delta}_{\mathcal{F}\to l}$, therefore $\widetilde{\Delta}_{\mathcal{F}\to l}$ is also injective.

(b) From (a) we know that $\widetilde{\Delta}_{\mathcal{F}\to k-t}$ is injective. The result follows from Theorem 6.5.2 with $s := k-t$. ■

The following classical theorem is due to Katona [290] in the Boolean case. I treated the linear lattice for $t = 1$ in [161], and Lefmann [337] generalized this to arbitrary t.

Theorem 6.5.3. *Let $\mathcal{F}$ be a t-intersecting family in $L_n(q)$, $t \geq 1$. Then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{k\geq\frac{n+t}{2}} \binom{n}{k}_q & \text{if } n+t \text{ is even,} \\ \binom{n-1}{\frac{n+t-1}{2}}_q + \sum_{k\geq\frac{n+t+1}{2}} \binom{n}{k}_q & \text{if } n+t \text{ is odd,} \end{cases}$$

and the bound is the best possible. The same result is true for B_n if we let $q \to 1$.

Proof. Under all maximum t-intersecting families in $L_n(q)$ (resp. B_n), we choose a family for which $k^* := \min\{k : \mathcal{F}_k \neq \emptyset\}$ is maximal. If $k^* > (n+t-1)/2$, we are done. If $k^* = (n+t-1)/2$, we only have to prove

Claim 1. $|\mathcal{F}_{(n+t-1)/2}| \leq \binom{n-1}{(n+t-1)/2}_q$.

If $k^* < (n+t-1)/2$ we will construct below a new t-intersecting family $\mathcal{F}'$ with $|\mathcal{F}'| \geq |\mathcal{F}|$ and $\min\{k : \mathcal{F}'_k \neq \emptyset\} > k^*$. This is then a contradiction to the choice of $\mathcal{F}$.

For $k^* \leq (n+t-1)/2$, let

$$\mathcal{G} := \{X \in N_{n+t-k^*-1} : r(X \wedge Y) = t-1 \quad \text{for some } Y \in \mathcal{F}_{k^*}\}.$$

Because $\mathcal{F}$ is t-intersecting,

$$\mathcal{F} \cap \mathcal{G} = \emptyset. \tag{6.71}$$

Claim 2. We have

$$|\mathcal{G}| \geq \frac{\binom{2k^*-t}{k^*-t+1}_q}{\binom{2k^*-t}{k^*}_q} |\mathcal{F}_{k^*}|.$$

Proof of Claim 2. Let $\mathcal{H} := \Delta_{\to k^*-t+1}(\mathcal{F}_{k^*})$. We have

$$\mathcal{G} \supseteq \{X \in N_{n+t-k^*-1} : \text{ there is some } Z \in \mathcal{H} \text{ with } X \wedge Z = 0\}$$

because for X, Z from the RHS the existence of some $Y \in \mathcal{F}_{k^*}$ with $Z \leq Y$ follows, and we have $r(X \vee Z) = n$; that is, also $r(X \vee Y) = n$, hence

$$\begin{aligned} r(X \wedge Y) &= r(X) + r(Y) - r(X \vee Y) \\ &= (n + t - k^* - 1) + k^* - n = t - 1. \end{aligned}$$

Recall the definition of $\Psi_{i \to j}$ before Lemma 6.4.3. Obviously

$$\Psi_{k^*-t+1 \to n-k^*+t-1}(\widetilde{\mathcal{H}}) \subseteq \widetilde{\mathcal{G}}.$$

Because $\Psi_{k^*-t+1 \to n-k^*+t-1}$ is injective by Lemma 6.4.6(a), it follows that

$$|\mathcal{G}| \geq |\mathcal{H}|.$$

Moreover, by Corollary 6.5.3(b),

$$|\mathcal{H}| \geq \frac{\binom{2k^*-t}{k^*-t+1}_q}{\binom{2k^*-t}{k^*}_q} |\mathcal{F}_{k^*}|.$$

□

Proof of Claim 1. For $L_n(q)$, the assertion follows directly from Corollary 6.4.3. For B_n, we work with $k^* := (n + t - 1)/2$ and $\mathcal{G}$ from above. Because of (6.71)

$$|\mathcal{F}_{k^*}| + |\mathcal{G}| \leq \binom{n}{k^*},$$

and in view of Claim 2 we infer

$$|\mathcal{F}_{k^*}| \left(1 + \frac{\binom{n-1}{\frac{n-t+1}{2}}}{\binom{n-1}{\frac{n+t-1}{2}}}\right) \leq \binom{n}{k^*}$$

which is equivalent to the assertion. □

For $k^* < (n + t - 1)/2$, let $\mathcal{F}' := (\mathcal{F} - \mathcal{F}_{k^*}) \cup \mathcal{G}$. Because of (6.71) and Claim 2 (note $k^* - t + 1 + k^* > 2k^* - t$) we have $|\mathcal{F}'| > |\mathcal{F}|$ (by the way, this can be proved also in an elementary way, see [337]). Thus it remains to prove the third claim.

Claim 3. $\mathcal{F}'$ is t-intersecting.

Proof of Claim 3. Let $X, Y \in \mathcal{F}'$. If $X, Y \in \mathcal{F}$, then $r(X \wedge Y) \geq t$ because $\mathcal{F}$ is t-intersecting. If $X \in \mathcal{F}$, $Y \in \mathcal{G}$, then $r(X \wedge Y) = r(X) + r(Y) - r(X \vee Y) \geq k^* + 1 + n + t - k^* - 1 - n = t$. If $X, Y \in \mathcal{G}$, we have also $r(X \wedge Y) = r(X) + r(Y) - r(X \vee Y) = 2(n + t - k^* - 1) - n > n + 2t - (n + t - 1) - 2 = t - 1$.

□

To see that the bound is the best possible, fix some $Y \in N_{n-1}$ and take

$$\mathcal{F}^* := \begin{cases} \left\{X : r(X) \geq \frac{n+t}{2}\right\} & \text{if } n+t \text{ is even,} \\ \left\{X : r(X) \geq \frac{n+t+1}{2}\right\} \cup \left\{X \in N_{\frac{n+t-1}{2}} : X \leq Y\right\} & \text{if } n+t \text{ is odd.} \end{cases}$$

■

7

Limit theorems and asymptotic estimates

Combinatorial formulas are often quite difficult and one has no idea of the growth of the corresponding functions. Therefore one is interested in asymptotic results where one or several parameters tend to infinity. In particular, this can be helpful in the production of counterexamples to some conjectures. With regard to our general theme we will restrict ourselves to the estimation of the (largest) Whitney numbers (or sums of them) and of antichains and related families in posets. The main idea is the application of corresponding limit theorems from probability theory as well as the use of variants of the saddle point method (cf. Berg [47] and de Bruijn [85]). This chapter lists some important theorems and then describes their application by means of some examples. Omitted are the proofs of those results contained in standard books on probability theory like Feller [178], [179], Rényi [396], and Petrov [382]. Bender's papers [42] and [43] are very helpful for asymptotic enumeration. In this chapter we will also see how the Sperner and the variance problems are related to each other.

7.1. Central and local limit theorems

In the following, we will be working mostly with *discrete random variables*, which are random variables ξ that take on at most countable many (in our cases only a finite number of) values $x_0, x_1, \ldots$ with probabilities $p_0, p_1, \ldots$ ($P(\xi = x_i) = p_i$, $i = 0, 1, \ldots$, and $\sum_i p_i = 1$).

Recall that the *distribution function* $F(x)$ of ξ is given by $F(x) := \sum_{i:x_i<x} p_i$. As limit distributions, there appear also *continuous random variables* – that is, random variables ξ for which there exists a *density function* f such that the distribution function $F(x) := P(\xi < x)$ can be calculated as an integral: $F(x) = \int_{-\infty}^{x} f(t)dt$.

If g is a continuous (possibly complex) function, then $g(\xi)$ is also a discrete (resp. continuous) random variable. The *expected value* of $g(\xi)$ is given by

$$\mathrm{E}(g(\xi)) := \int_{-\infty}^{\infty} g(x)dF(x),$$

which is the same as $\sum_i g(x_i)p_i$ in the discrete case and as $\int_{-\infty}^{\infty} g(x)f(x)dx$ in the continuous case (here it is supposed that the sum (resp. the integral) are absolutely convergent). $\mathrm{E}(\xi)$ and $\mathrm{V}(\xi) := \mathrm{E}((\xi - \mathrm{E}(\xi))^2)$ are called the *expected value* or the *mean of* ξ and the *variance of* ξ, respectively.

From the list of continuous distributions we need only the *Gaussian distribution* or *normal distribution* whose density function $f(x)$ and distribution function (here denoted by $\Phi(x)$) are given by

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt.$$

Given a sequence $\{\zeta_n\}$ of random variables with distribution functions $F_n(x)$ and sequences $\{\mu_n\}$ and $\{\sigma_n^2\}$ of real numbers, we say that $\{\zeta_n\}$ (or briefly ζ_n) is *asymptotically normal with mean* μ_n *and variance* σ_n^2 if the distribution function of $(\zeta_n - \mu_n)/\sigma_n$ tends to $\Phi(x)$ – that is, if

$$\lim_{n\to\infty} \int_{-\infty}^{\sigma_n x+\mu_n} dF_n(t) = \Phi(x) \quad \text{for every } x.$$

It is easy to see that this convergence is uniform for $x \in \mathbb{R}$ since $\Phi(x)$ is continuous.

The *characteristic function* $\varphi_\xi(t)$ of a random variable ξ is defined by

$$\varphi_\xi(t) := \mathrm{E}(e^{i\xi t}) = \int_{-\infty}^{\infty} e^{ixt}dF(x).$$

This is the Fourier–Stieltjes Transform of $F(x)$. It is well defined because $|e^{ixt}| = 1$. Conversely, the distribution function is uniquely determined by the characteristic function. We have

$$\varphi_{a\xi+b}(t) = e^{ibt}\varphi_\xi(at).$$

The characteristic function of the normal distribution is $e^{-t^2/2}$. The *Continuity Theorem* (see, e.g., Feller [179, p. 508]) implies:

Theorem 7.1.1. *The sequence* $\{\xi_n\}$ *is asymptotically normal with mean* μ_n *and variance* σ_n^2 *iff*

$$\lim_{n\to\infty} \varphi_{\frac{\xi_n-\mu_n}{\sigma_n}}(t) = e^{-t^2/2}.$$

The random variables $\xi_1, \dots, \xi_n$ are called (*completely*) *independent* if for all $x_1, \dots, x_n \in \mathbb{R}$

$$P(\xi_1 < x_1, \dots, \xi_n < x_n) = P(\xi_1 < x_1) \cdots P(\xi_n < x_n).$$

In the discrete case, this is equivalent to the equality

$$P(\xi_1 = x_1, \dots, \xi_n = x_n) = P(\xi_1 = x_1) \cdots P(\xi_n = x_n)$$

(of course it is enough to consider only those values $x_i \in \mathbb{R}$ for which $P(\xi_i = x_i) > 0$).

Theorem 7.1.2. *Let $\xi_1, \dots, \xi_n$ be independent random variables and let $\zeta_n := \xi_1 + \cdots + \xi_n$. Then*

$$\varphi_{\zeta_n}(t) = \varphi_{\xi_1}(t) \cdots \varphi_{\xi_n}(t).$$

Corollary 7.1.1. *If $\xi_1, \dots, \xi_n$ are independent random variables with the common distribution F, mean μ and variance σ^2 then $\zeta_n = \xi_1 + \cdots + \xi_n$ is asymptotically normal with mean $n\mu$ and variance $n\sigma^2$.*

As an example we consider the poset $P = S(k, \dots, k)$ which is a product of n $(k+1)$-element chains (the succeeding result may be generalized in a straightforward way to a product of n copies of any finite ranked poset). Each such $(k+1)$-element chain C has the rank-generating function $F(C; x) = \sum_{j=0}^{k} x^j$; hence $F(P; x) = \sum_{j=0}^{nk} W_j x^j = (F(C; x))^n$. We introduce the discrete independent random variables ξ_i, $i = 1, \dots, n$, which take on the values $0, \dots, k$ each with probability $\frac{1}{k+1}$. Let ξ be one of $\xi_1, \dots, \xi_n$. Note that

$$\varphi_\xi(t) = \sum_{j=0}^{k} \frac{1}{k+1} e^{ijt} = \frac{1}{k+1} F(C; e^{it}).$$

Moreover, $\zeta_n := \xi_1 + \cdots + \xi_n$ takes on the values $j = 0, \dots, nk$ each with probability $W_j(P)/(k+1)^n$. We have indeed

$$\varphi_{\zeta_n}(t) = \varphi_\xi^n(t) = \left(\frac{1}{k+1} F(C; e^{it})\right)^n = \frac{1}{(k+1)^n} F(P; e^{it}).$$

By Corollary 7.1.1, ζ_n is asymptotically normal with mean $\mu_n := n\frac{k}{2}$ and variance $\sigma_n^2 := n(k(k+2)/12)$.

In particular it follows that

$$\sum_{\sigma_n a + \mu_n \le j < \sigma_n b + \mu_n} W_j \sim (\Phi(b) - \Phi(a))(k+1)^n, \tag{7.1}$$

and for a sequence $x_n \to \infty$

$$\sum_{nk/2 - x_n\sqrt{n} \le j < nk/2 + x_n\sqrt{n}} W_j \sim (k+1)^n.$$

The last result states that almost nothing is far away from the mean. There exist better estimates for the portion that is far away; see Rényi [396, p. 324]:

Theorem 7.1.3. *Let $\xi_1, \ldots, \xi_n$ be independent random variables with the common distribution function $F(x)$, expected value μ, and variance σ^2. Suppose that there exists some M such that ξ_j takes on only values in the interval $[\mu - M, \mu + M]$. Then for $0 < \epsilon \leq \sigma^2/M$*

$$P(|\zeta_n - n\mu| \geq n\epsilon) \leq 2e^{-n\epsilon^2/2\sigma^2(1+\epsilon M/2\sigma^2)^2}.$$

If we apply to our preceding example this result with $\epsilon := n^{-1/2}x_n$, where $x_n \to \infty$ but $x_n = o(\sqrt{n})$, we obtain, with some constant $c > 0$,

$$\sum_{|j-nk/2|\geq x_n\sqrt{n}} W_j \leq 2(k+1)^n e^{-cx_n^2}.$$

Up to now we could only estimate sums of Whitney numbers. To obtain asymptotic formulas for single Whitney numbers we use local limit theorems. In the following, we work only with discrete random variables that take on only integer values. We call them briefly *integral* random variables. A sequence $\{\zeta_n\}$ of integral random variables is said to be *locally asymptotically normal with mean μ_n and variance σ_n^2* if

$$\lim_{n\to\infty} \sigma_n P(\zeta_n = \lfloor \sigma_n x + \mu_n \rfloor) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ uniformly for } x \in \mathbb{R}. \tag{7.2}$$

Under certain conditions, asymptotic normality implies local asymptotic normality. We say that a sequence $\{a_n(k)\}$ is *properly log concave in k* if $a_n^2(k) \geq a_n(k-1)a_n(k+1)$ for all k and the nonzero members $a_n(k)$, where n is fixed but arbitrary, appear consecutively.

***Theorem 7.1.4* (Bender [42]).** *Suppose that the integral random variables ζ_n are asymptotically normal with mean μ_n and variance σ_n^2 and that $\sigma_n^2 \to \infty$. If the sequence $a_n(k) := P(\zeta_n = k)$ is properly log concave in k, then ζ_n is locally asymptotically normal with mean μ_n and variance σ_n^2.*

Proof. First observe that for $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$,

$$P(\lfloor \mu_n + \alpha\sigma_n \rfloor < \zeta_n < \lfloor \mu_n + \beta\sigma_n \rfloor) = \Phi(\beta) - \Phi(\alpha) + o(1) \tag{7.3}$$

(here and in the following the convergence of the functions $o(1)$ is uniform over $\mathbb{R}$). In this formula also any relation "$<$" can be replaced by "$\leq$".

Let us fix some $\epsilon > 0$ and put $\gamma := \frac{\epsilon}{4}$, $\delta := \frac{\epsilon}{2}$. From (7.3) we obtain for sufficiently large n:

$$\sum_{\lfloor \mu_n \rfloor < k < \lfloor \mu_n + \gamma\sigma_n \rfloor} a_n(k) > \sum_{\lfloor \mu_n + \gamma\sigma_n \rfloor \leq k \leq \lfloor \mu_n + 2\gamma\sigma_n \rfloor} a_n(k).$$

On the LHS we have $\lfloor \mu_n + \gamma\sigma_n \rfloor - \lfloor \mu_n \rfloor - 1$ summands and on the RHS $\lfloor \mu_n + 2\gamma\sigma_n \rfloor - \lfloor \mu_n + \gamma\sigma_n \rfloor + 1$, that is, not less, summands. Consequently, one item of the first sum is greater than one item of the second sum, and the unimodality of the numbers $a_n(k)$ (which follows from the proper log concavity) implies that $a_n(k)$ is nonincreasing for $k \geq \lfloor \mu_n + \delta\sigma_n \rfloor$ if n is large enough. In the same way, we infer that $a_n(k)$ is nondecreasing for $k \leq \lfloor \mu_n - \delta\sigma_n \rfloor$ and large n.

Case 1. $|x| \geq \epsilon$. We study only $x \geq \epsilon$. Using (7.3) and the just proved monotonicity of the $a_n(k)$ for $k \geq \mu_n + (x - \delta)\sigma_n$ we derive

$$\begin{aligned}
&(\lfloor \mu_n + x\sigma_n \rfloor - \lfloor \mu_n + (x - \delta)\sigma_n \rfloor + 1) a_n(\lfloor \mu_n + x\sigma_n \rfloor) \\
&\quad \leq \sum_{\lfloor \mu_n + (x-\delta)\sigma_n \rfloor \leq k \leq \lfloor \mu_n + x\sigma_n \rfloor} a_n(k) = \Phi(x) - \Phi(x - \delta) + o(1) \qquad (7.4)
\end{aligned}$$

and

$$\begin{aligned}
&(\lfloor \mu_n + (x + \delta)\sigma_n \rfloor - \lfloor \mu_n + x\sigma_n \rfloor - 1) a_n(\lfloor \mu_n + x\sigma_n \rfloor) \\
&\quad \geq \sum_{\lfloor \mu_n + x\sigma_n \rfloor \leq k \leq \lfloor \mu_n + (x+\delta)\sigma_n \rfloor} a_n(k) = \Phi(x + \delta) - \Phi(x) + o(1). \qquad (7.5)
\end{aligned}$$

By Taylor's formula, for some $0 < \vartheta < 1$,

$$\Phi(x) - \Phi(x - \delta) = \frac{1}{\sqrt{2\pi}} \delta e^{-x^2/2} - \frac{\delta^2}{2} \frac{1}{\sqrt{2\pi}} (x + \vartheta\delta) e^{-(x+\vartheta\delta)^2/2}.$$

The function $te^{-t^2/2}$ has its maximum at $t = 1$, consequently

$$\Phi(x) - \Phi(x - \delta) = \frac{1}{\sqrt{2\pi}} \delta e^{-x^2/2} + O(\delta^2),$$

and analogously

$$\Phi(x + \delta) - \Phi(x) = \frac{1}{\sqrt{2\pi}} \delta e^{-x^2/2} + O(\delta^2),$$

where $O(\delta^2) = O(\epsilon^2)$ can be bounded independently of n. The first factors on the LHS of (7.4) and (7.5) are both $(1 + o(1))\delta\sigma_n$. We obtain

$$\sigma_n a_n(\lfloor \mu_n + x\sigma_n \rfloor) = \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} + O(\epsilon) \right) (1 + o(1)).$$

Case 2. $|x| \leq \epsilon$. We study only $0 \leq x \leq \epsilon$. Clearly $x + 2\epsilon \geq x + \epsilon \geq \epsilon$. Since $\lfloor \mu_n + x\sigma_n \rfloor + (\lfloor \mu_n + (x + 2\epsilon)\sigma_n \rfloor + 2) \geq 2\lfloor \mu_n + (x + \epsilon)\sigma_n \rfloor$, the proper log concavity, the monotonicity, and Case 1 yield

$$\begin{aligned}
&\sigma_n a_n(\lfloor \mu_n + x\sigma_n \rfloor) \\
&\quad \leq \sigma_n a_n(\lfloor \mu_n + (x + \epsilon)\sigma_n \rfloor) \frac{a_n(\lfloor \mu_n + (x + \epsilon)\sigma_n \rfloor)}{a_n(\lfloor \mu_n + (x + 2\epsilon)\sigma_n + 2 \rfloor)} \\
&\quad \leq \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} + O(\epsilon) \right) (1 + o(1)).
\end{aligned}$$

However, by analogous arguments,

$$\begin{aligned}
&(\sigma_n a_n(\lfloor \mu_n + x\sigma_n \rfloor))^2 \\
&\geq \sigma_n a_n(\lfloor \mu_n + (x+2\epsilon)\sigma_n \rfloor)\sigma_n a_n(\lfloor \mu_n + (x-2\epsilon)\sigma_n \rfloor - 1) \\
&\geq \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} + O(\epsilon)\right)^2 (1+o(1))^2.
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain the desired result. ■

Because the Whitney numbers of our example $P = S(k, \ldots, k)$ are properly log concave, we obtain by setting $x := 0$ in (7.2) (using Corollary 7.1.1, Theorem 7.1.4, and Example 4.6.1):

$$d(P) = W_{\lfloor nk/2 \rfloor} \sim \frac{1}{\sqrt{2\pi nk(k+2)/12}} (k+1)^n \quad \text{as } n \to \infty. \tag{7.6}$$

In order to have estimates of other Whitney numbers of P, for example, of $W_{\lfloor \lambda n \rfloor}$, we may shift the mean: We take the discrete independent random variables ξ_i where $P(\xi_i = j) = \alpha^j/(1+\alpha+\cdots+\alpha^k)$. We determine α as the solution of $\sum_{j=0}^k j\alpha^j / \sum_{j=0}^k \alpha^j = \lambda$; then each ξ_i has expected value λ and a certain variance σ^2 depending on λ and k. For $\zeta_n := \xi_1 + \cdots + \xi_n$, we have $P(\zeta_n = j) = (\alpha^j/(1+\alpha+\cdots+\alpha^k)^n W_j)$, and we obtain as before

$$W_{\lfloor \lambda n \rfloor} \sim \frac{1}{\sqrt{2\pi n}\sigma} \frac{(1+\alpha+\cdots+\alpha^k)^n}{\alpha^{\lfloor \lambda n \rfloor}} \quad \text{as } n \to \infty. \tag{7.7}$$

Now we discuss how the condition on proper log concavity can be replaced. A random variable ξ is said to have a *lattice distribution* if there exist constants a and $h > 0$ such that $(\xi - a)/h$ is an integral random variable. The quantity h is called the *span* of the distribution. If, additionally, there are no numbers a' and $h' > h$ such that $(\xi - a')/h'$ is an integral random variable, the span h is said to be *maximal*.

***Theorem 7.1.5* (Gnedenko [224, p. 249]).** *Let $\xi_1, \ldots, \xi_n$ be independent random variables with a common lattice distribution, constant a, span h, expected value μ, and variance $\sigma^2 > 0$, and let $\zeta_n := \xi_1 + \cdots + \xi_n$. Then the sequence $\{(\zeta_n - na)/h\}$ is locally asymptotically normal with mean $n(\frac{\mu - a}{h})$ and variance $n(\frac{\sigma^2}{h^2})$ iff the span h is maximal; this means we have uniformly in $N \in \mathbb{Z}$*

$$\lim_{n\to\infty} \left(\frac{\sigma\sqrt{n}}{h} P(\zeta_n = na + Nh) - \frac{1}{\sqrt{2\pi}} e^{-(na+Nh-n\mu)/2\sigma\sqrt{n}} \right) = 0$$

iff h is maximal.

Corollary 7.1.2. *Let ζ_n be defined as in Theorem 7.1.5 and let the span h of ξ_i be maximal, $i = 1, \ldots, n$. Let b_1, b_2 be fixed constants, $b_1 < b_2$, such that*

$b_2 - b_1 = jh$ for some natural number j. Then

$$P(b_1 + n\mu \le \zeta_n < b_2 + n\mu) \sim \frac{b_2 - b_1}{\sqrt{2\pi n}\sigma} \quad \text{as } n \to \infty.$$

Proof. Let

$$I := \left\{ N \in \mathbb{Z} : \frac{b_1 + n(\mu - a)}{h} \le N < \frac{b_2 + n(\mu - a)}{h} \right\}.$$

Since $\frac{b_2 - b_1}{h} = j$ we have $|I| = j$. For $N \in I$, the following holds:

$$\frac{b_1}{\sigma\sqrt{n}} \le \frac{na + Nh - n\mu}{\sigma\sqrt{n}} < \frac{b_2}{\sigma\sqrt{n}}.$$

Hence $\frac{na+Nh-n\mu}{\sigma\sqrt{n}} \to 0$ uniformly in $N \in I$. Theorem 7.1.5 implies

$$\begin{aligned} P(b_1 + n\mu \le \zeta_n < b_2 + n\mu) &= \sum_{N \in I} P(\zeta_n = na + Nh) \sim |I| \frac{h}{\sqrt{2\pi n}\sigma} \\ &= \frac{b_2 - b_1}{\sqrt{2\pi n}\sigma}. \end{aligned}$$

■

A counterpart of this corollary is the following theorem, which is a direct consequence of a result of Shepp (see [382, p. 214]).

Theorem 7.1.6. *Let $\xi_1, \ldots, \xi_n$ be independent random variables with a common nonlattice distribution, $\zeta_n := \xi_1 + \cdots + \xi_n$. Let b_1, b_2 be fixed constants, $b_1 < b_2$. Then*

$$P(b_1 + n\mu \le \zeta_n < b_2 + n\mu) \sim \frac{b_2 - b_1}{\sqrt{2\pi n}\sigma} \quad \text{as } n \to \infty.$$

Finally we study how the asymptotic normality of certain random variables can be shown using Theorem 7.1.1. We present a method of Hayman [265] on an example and investigate in detail the behavior of the Stirling numbers of the second kind. Let us recall Taylor's formula for holomorphic functions:

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \cdots + \frac{f^{(n)}(0)}{n!} z^n + \int_0^z \frac{f^{(n+1)}(t)}{n!} (z - t)^n dt, \qquad n \in \mathbb{N}. \tag{7.8}$$

Theorem 7.1.7 (Stirling's formula). *We have*

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty.$$

Proof (Model for the method of Hayman). Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C},$$

Cauchy's formula yields for every $r > 0$

$$\frac{1}{n!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{e^z}{z^{n+1}} dz.$$

Substituting $z = re^{i\varphi}$, we obtain

$$\frac{r^n}{n!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{re^{i\varphi} - ni\varphi} d\varphi.$$

We divide the integral on the RHS into two parts $I_1 + I_2$, where

$$I_1 = \int_{-\delta(n)}^{\delta(n)} e^{re^{i\varphi} - ni\varphi} d\varphi \quad \text{and} \quad I_2 = \int_{\delta(n) \le |\varphi| \le \pi} e^{re^{i\varphi} - ni\varphi} d\varphi.$$

Expanding the exponent by Taylor's formula, we have

$$\begin{aligned} re^{i\varphi} - ni\varphi &= r\left(1 + i\varphi - \frac{\varphi^2}{2} - ir \int_0^{\varphi} \frac{e^{it}}{2} (\varphi - t)^2 dt\right) - ni\varphi \\ &= r + i\varphi(r - n) - \frac{r\varphi^2}{2} + O(r\varphi^3). \end{aligned}$$

We determine r such that the (complex) linear term vanishes; that is, $r := n$. Finally we look for a $\delta(n)$ such that the omission of $O(r\varphi^3)$ and of I_2 does not influence the asymptotic behavior and I_1 can be easily approximated. Assume that we have such a $\delta(n)$. Then (with $\psi := \sqrt{r}\varphi$)

$$I_1 \sim \int_{-\delta(n)}^{\delta(n)} e^{r - r\varphi^2/2} d\varphi = \frac{e^r}{\sqrt{r}} \int_{-\sqrt{r}\delta(n)}^{\sqrt{r}\delta(n)} e^{-\psi^2/2} d\psi.$$

Because, for sufficiently large T,

$$\int_T^{\infty} e^{-\psi^2/2} d\psi \le \int_T^{\infty} \frac{1}{\psi^2} d\psi = \frac{1}{T},$$

it follows (with $r = n$) that

$$I_1 \sim \frac{e^r}{\sqrt{r}} \left(\int_{-\infty}^{\infty} e^{-\psi^2/2} d\psi + O\left(\frac{1}{\sqrt{r}\delta(n)}\right)\right) \sim \frac{e^n}{\sqrt{n}} \sqrt{2\pi}$$

if

Condition 1: $\sqrt{r}\delta(n) \to \infty$.

The omission of $O(r\varphi^3)$ in the integral I_1 has no influence if $e^{O(r\delta^3(n))} = 1 + o(1)$, that is, if

Condition 2: $r\delta^3(n) \to 0$.

Finally I_2 has to be small. We have

$$\begin{aligned}|I_2| &\le \int_{\delta(n)\le|\varphi|\le\pi} |e^{re^{i\varphi}-ni\varphi}|d\varphi \le 2\pi\, e^{\max\{\mathrm{Re}(re^{i\varphi}-ni\varphi):\delta(n)\le|\varphi|\le\pi\}}\\ &\le 2\pi\, e^r e^{\max\{r(\cos\varphi-1):\delta(n)\le|\varphi|\le\pi\}}\\ &= 2\pi\, e^n e^{r(\cos\delta(n)-1)} \le 2\pi\, e^n e^{-2r\delta^2(n)/\pi^2}\end{aligned}$$

(here we used that $\cos(\varphi) \le 1 - \frac{2}{\pi^2}\varphi^2$ if $|\varphi| \le \pi$). Thus $I_2 = o(I_1)$ if

$$e^{-2r\delta^2(n)/\pi^2} = o\left(\frac{1}{\sqrt{n}}\right),$$

and this is the case if

Condition 3: $r\delta^2(n) \ge n^\epsilon$ for some $\epsilon > 0$.

It is easy to see that the three conditions on $\delta(n)$ with $r = n$ are satisfied if, for example, $\delta(n) = n^{-5/12}$. Consequently,

$$\frac{n^n}{n!} \sim \frac{1}{2\pi} I_1 \sim \frac{1}{2\pi}\frac{e^n}{\sqrt{n}}\sqrt{2\pi},$$

which proves the assertion. ■

For us, it is sufficient to take the "simple" circle as the contour of integration. Of course, we could replace it by other contours around the origin. Moreover, in the *saddle point method* one looks for contours through the saddle point, which can be obtained by setting the derivative of the logarithm of the integrand to 0. In our case this is $\frac{d}{dz}\log(\frac{e^z}{z^{n+1}}) = 0$; that is, $z = n + 1$, thus we have integrated only "near" the saddle point.

This approach works not only for our function e^z but for a large class of holomorphic functions, called *admissible functions*. I will not present their definition (see [265]), but will introduce a sufficiently large subclass: Let $p(z)$ be a polynomial with real coefficients.

Basic functions: $e^{p(z)}$ is admissible if the coefficients a_n in the Taylor series $e^{p(z)} = \sum_{n=0}^{\infty} a_n z^n$ are positive for sufficiently large n.

Let $f(z)$ and $g(z)$ be admissible and let $h(z)$ be holomorphic and real for real z.

Construction 1: $e^{f(z)}$ and $f(z)g(z)$ are admissible.

Construction 2: If there is some $\delta > 0$ such that

$$\max_{|z|=r} |h(z)| = O(f^{1-\delta}(r)), \quad r \to \infty,$$

then $f(z) + h(z)$ is admissible. One can show that, in particular, $f(z) + p(z)$ is admissible.

Construction 3: If the leading coefficient of $p(z)$ is positive, then $p(z)f(z)$ and $p(f(z))$ are admissible.

Theorem 7.1.8 (Hayman [265]). *Let $f(z)$ be an admissible holomorphic function with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let $a(r) := r\frac{f'(r)}{f(r)} = r\frac{d}{dr}\log f(r)$ and $b(r) := ra'(r)$. Let r_n be the largest root of $a(r) = n$ (which can be shown to exist). Then*

$$a_n \sim \frac{1}{\sqrt{2\pi b(r_n)}} \frac{f(r_n)}{r_n^n} \text{ as } n \to \infty.$$

We will not develop here the theory of generating functions (cf. Aigner [21], Stanley [441], and Wilf [469]). As an example, we consider the Stirling numbers of the second kind.

Lemma 7.1.1. *We have*

$$\sum_{n,k=0}^{\infty} \frac{S_{n,k}}{n!} y^k z^n = e^{y(e^z-1)}.$$

Proof. Obviously,

$$\begin{aligned} e^{y(e^z-1)} &= \sum_{k=0}^{\infty} \frac{y^k}{k!}(e^z-1)^k \\ &= \sum_{k=0}^{\infty} \frac{y^k}{k!}\left(z+\frac{z^2}{2!}+\cdots\right)\cdots\left(z+\frac{z^2}{2!}+\cdots\right) \\ &= \sum_{k,n=0}^{\infty} \frac{1}{k!} \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geq 1}} \frac{1}{i_1!\cdots i_k!} y^k z^n. \end{aligned}$$

Now the assertion follows from the fact that

$$\begin{aligned} k!S_{n,k} &= |\{f : [n] \to [k] : f \text{ surjective}\}| \\ &= \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geq 1}} |\{f : [n] \to [k] : |f^{-1}(1)| = i_1, \ldots, |f^{-1}(k)| = i_k\}| \\ &= \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k\geq 1}} \frac{n!}{i_1!\cdots i_k!}. \end{aligned}$$

■

If we set $y := 1$, we obtain

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = e^{e^z-1}.$$

Obviously, this function is admissible. Thus we obtain from Theorem 7.1.8 for the size of the partition lattice Π_n

$$B_n \sim n! \frac{1}{\sqrt{2\pi(r^2+r)e^r}} \frac{e^{e^r-1}}{r^n} \quad \text{where } re^r = n.$$

Finally, using Stirling's formula (Theorem 7.1.7) and $e^r = \frac{n}{r}$, we derive a formula of Moser and Wyman [375]:

$$B_n \sim \frac{1}{\sqrt{r+1}} e^{n(r+1/r-1)-1}. \tag{7.9}$$

The following theorem is due (in a slightly different form) to Harper [256]. In the proof we follow Canfield [92] using Hayman's method.

Theorem 7.1.9. *Let the integral random variable ζ_n be defined by $P(\zeta_n = k) = \frac{S_{n,k}}{B_n}$. Let $\mu_n := e^r - 1$ and $\sigma_n^2 := \frac{e^r}{r+1} - 1 = \mu_n - \frac{n}{r+1}$, where r is the root of $re^r = n$. Then ζ_n is asymptotically normal with mean μ_n and variance σ_n^2.*

Proof. Let $p_n(y) := \sum_{k=0}^{\infty} S_{n,k} y^k$. Then the characteristic function of ζ_n is given by $\varphi_{\zeta_n} = p_n(e^{it})/p_n(1)$. Moreover,

$$\varphi_{(\zeta_n-\mu_n)/\sigma_n}(t) = e^{-it\mu_n/\sigma_n} p_n(e^{it/\sigma_n})/p_n(1). \tag{7.10}$$

By Theorem 7.1.1 it is sufficient to prove that the RHS tends to $e^{-t^2/2}$ if $n \to \infty$. Thus we will determine an asymptotic formula for $p_n(e^{it/\sigma_n})$. In view of Lemma 7.1.1

$$\sum_{n=0}^{\infty} p_n(y) \frac{z^n}{n!} = e^{y(e^z-1)}$$

which implies

$$\frac{p_n(y)}{n!} = \frac{1}{2\pi i} \oint_{|z|=r} \frac{e^{y(e^z-1)}}{z^{n+1}} dz = \frac{r^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{y(e^{re^{i\varphi}}-1)-ni\varphi} d\varphi.$$

As in the proof of Stirling's formula, we write the integral on the RHS as a sum $I_1 + I_2$, where I_1 is the integral over $|\varphi| \le \delta(n) =: \delta$ and I_2 is the integral over $\delta \le |\varphi| \le \pi$. First we investigate I_1. From (7.10) we know that we must later put $y := e^{it/\sigma_n}$ (resp. $y := 1$); that is, in both cases $y := e^{is}$ where $|s|$ is small ($s = t/\sigma_n$ or $s = 0$). Hence we expand also e^{is} (at least for members of higher order) in the Taylor expansion of the exponent of the integrand

$$g(s, \varphi) := e^{is}(e^{re^{i\varphi}} - 1) - ni\varphi.$$

We have

$$g(s,0) = e^{is}(e^r - 1),$$
$$\frac{\partial}{\partial\varphi}g(s,0) = e^{is}ire^r - in = i(re^r - n) - sre^r + re^r O(s^2).$$

We determine r such that the imaginary constant term vanishes; that is, $re^r = n$. Then

$$\frac{\partial}{\partial\varphi}g(s,0) = -sn + O(ns^2),$$
$$\frac{\partial^2}{\partial\varphi^2}g(s,0) = -e^{is}e^r(r^2 + r) = -n(r+1) + O(nrs),$$
$$\frac{\partial^3}{\partial\varphi^3}g(s,\varphi) = -ie^{is}e^{re^{i\varphi}}(r^3e^{3i\varphi} + 3r^2e^{2i\varphi} + re^{i\varphi}) = O(e^r r^3) = O(r^2 n).$$

We will take a δ such that $|s| \le \delta$ for large n. This is clear for $s = 0$. For $s = \frac{t}{\sigma_n}$, this inequality is satisfied if

Condition 0. $\delta^2(\frac{e^r}{r+1} - 1) \to \infty$, that is, $\delta^2\frac{n}{r^2} \to \infty$.

Moreover we have $|\varphi| \le \delta$. Thus Taylor's formula gives

$$\begin{aligned} g(s,\varphi) &= e^{is}(e^r - 1) - sn\varphi - n(r+1)\frac{\varphi^2}{2} + O(nr^2\delta^3) \\ &= e^{is}(e^r - 1) + \frac{s^2n^2}{2n(r+1)} - \frac{(\sqrt{n(r+1)}\varphi + sn/\sqrt{n(r+1)})^2}{2}. \end{aligned}$$

The substitution $\psi := \sqrt{n(r+1)}\varphi + sn/\sqrt{n(r+1)}$ yields the bounds of integration $\sqrt{n(r+1)}(\pm\delta + \frac{s}{r+1})$, which are in absolute value not less than $\sqrt{n(r+1)}\delta(1-\frac{1}{r})$. If we integrate from $-\infty$ to ∞ we obtain the value

$$\frac{1}{\sqrt{n(r+1)}}e^{e^{is}(e^r-1)+s^2n/2(r+1)}\sqrt{2\pi},$$

and this is the asymptotic value for I_1 uniformly for $|s| \le \delta$ if

Condition 1. $\sqrt{n(r+1)}\delta(1-\frac{1}{r}) \to \infty$; that is, $\sqrt{nr}\delta \to \infty$, and (omission of $O(nr^2\delta^3)$)

Condition 2. $nr^2\delta^3 \to 0$.

Again we want to make I_2 small. We have

$$I_2 = e^{g(s,0)}\int_{\delta\le|\varphi|\le\pi} e^{g(s,\varphi)-g(s,0)}d\varphi.$$

The absolute value of the second factor is not less than

$$2\pi e^{\max\{\mathrm{Re}\, e^{is}(e^{re^{i\varphi}} - e^r),\delta\le|\varphi|\le\pi\}}.$$

Using $1-\frac{\varphi^2}{2}\le\cos\varphi\le 1-\frac{2}{\pi^2}\varphi^2$, $\delta\le|\varphi|\le\pi$, and $|s|\le\delta$, we conclude

$$\begin{aligned}\operatorname{Re}(e^{is}(e^{re^{i\varphi}}-e^r)) &= e^{r\cos\varphi}\cos(r\sin\varphi+s)-e^r\cos s\\ &\le e^r(e^{r(\cos\varphi-1)}-\cos s)\\ &\le e^r\left(e^{-2r\delta^2/\pi^2}-\left(1-\frac{\delta^2}{2}\right)\right)\\ &\le e^r\left(\left(\frac{1}{2}-\frac{2}{\pi^2}r\right)\delta^2+O(r^2\delta^4)\right)\le -\delta^2e^r\end{aligned}$$

if n is large and

Condition 3.1. $r\delta^2\to 0$.

Under the preceding conditions on δ, we have, uniformly in $|s|\le\delta$,

$$\frac{|I_2|}{|I_1|}\lesssim\frac{1}{\sqrt{2\pi}}\sqrt{n(r+1)}e^{-s^2n/2(r+1)}2\pi e^{-\delta^2e^r}=o(1)$$

if $\sqrt{nr}e^{-\delta^2e^r}\to 0$, and this is the case if

Condition 3.2. $\frac{n}{r}\delta^2\ge n^\epsilon$ for some $\epsilon>0$.

It is easy to see that the conditions 0, 1, 2, 3.1, and 3.2 on $\delta=\delta(n)$ with $re^r=n$ are satisfied if, for example, $\delta=e^{-3r/8}$. Consequently, for $s=0$ and $s=\frac{t}{\sigma_n}$

$$p_n(e^{is})\sim\frac{n!}{\sqrt{2\pi}}\frac{r^{-n}}{\sqrt{n(r+1)}}e^{e^{is}(e^r-1)+s^2n/2(r+1)}.$$

Finally, by (7.10), using $e^{it/\sigma_n}=1+\frac{it}{\sigma_n}-\frac{t^2}{2\sigma_n^2}+O\left(\frac{t^3}{\sigma_n^3}\right)$ and $\mu_n=e^r-1$,

$$\begin{aligned}\varphi_{\frac{\zeta_n-\mu_n}{\sigma_n}} &\sim e^{-it\mu_n/\sigma_n}e^{(e^{it/\sigma_n}-1)(e^r-1)+(t^2/2)\left(\frac{n}{\sigma_n^2(r+1)}\right)}\\ &= e^{(t^2/2)\left(\frac{n-(r+1)\mu_n}{\sigma_n^2(r+1)}+o(1)\right)}\to e^{-t^2/2}.\end{aligned}$$

■

Corollary 7.1.3. *For the partition lattice Π_n, we have*

$$\max\{W_i(\Pi_n), i=0,\ldots,n-1\}\sim\frac{|\Pi_n|}{\sqrt{2\pi}\sigma_n}\quad \textit{as } n\to\infty$$

where $\sigma_n^2:=\frac{e^r}{r+1}-1$ and $re^r=n$.

Proof. The assertion follows from Theorem 7.1.4, Theorem 7.1.9, and Lemma 5.4.1. ■

Finally let us mention without proof some important generalizations of Theorem 7.1.8 and Theorem 7.1.9.

Theorem 7.1.10 **(*Canfield*** [92]**).** *Let $g(z)$ be a holomorphic function with $g(0) = 0$ and let*

$$e^{yg(z)} = \sum_{n,k=0}^{\infty} \frac{a_{n,k}}{n!} y^k z^n.$$

Further, suppose that $g(z)$ is either admissible or a polynomial (different from z) with real nonnegative coefficients such that the greatest common divisor of the indices of the nonzero coefficients equals 1. Let $A(r) := rg'(r)$, $B(r) := rA'(r)$, and r_n be the largest root of $A(r) = n$. Then the random variable ζ_n defined by $P(\zeta_n = k) := \frac{a_{n,k}}{\sum_i a_{n,i}}$ is asymptotically normal with mean $\mu_n := g(r_n)$ and variance $\sigma_n^2 := g(r_n) - \frac{A^2(r_n)}{B(r_n)}$.

Theorem 7.1.11 **(*Canfield and Harper*** [98]**).** *Let ν_0 be the ratio $\frac{x_2}{x_1}$ of the larger to the smaller of the two roots of the equation $x(1 - \log x) = \frac{5}{6}$. Let ϱ and ν be positive numbers with $\nu < \nu_0$ and let K be the (compact) set of sequences $\{A_j\}$ satisfying $\varrho \le A_j \le \nu\varrho$ for every j. Let*

$$e^{\sum_{j=1}^{\infty} y^{A_j} \frac{z^j}{j!}} = \sum_{n=0}^{\infty} \sum_{\alpha \in Z_n} \frac{a_{n,\alpha}}{n!} y^{\alpha} z^n$$

where $Z_n := \{A_{j_1} + \cdots + A_{j_k} : j_1 + \cdots + j_k = n, j_1, \ldots, j_k \ge 1, k = 1, 2, \ldots\}$, and let $re^r = n$. Then the random variable ζ_n defined by $P(\zeta_n = \alpha) = \frac{a_{n,\alpha}}{B_n}, \alpha \in Z_n$, is, uniformly over K, asymptotically normal with mean $\mu_n := \sum_{j=1}^{\infty} A_j \frac{r^j}{j!}$ and variance $\sigma_n^2 := \sum_{j=1}^{\infty} A_j^2 \frac{r^j}{j!} - \frac{1}{n(r+1)} \left(\sum_{j=1}^{\infty} jA_j \frac{r^j}{j!}\right)^2$.

Remark 7.1.1. *In* [97] *Canfield and Harper proved that we have for the variance of ζ_n from Theorem 7.1.11*

$$V(\zeta_n) = \sigma_n^2 + O(1) \text{ uniformly over } K.$$

The proof is based on a more precise formula for the Bell numbers (*cf.* (7.9)) *which is mainly due to Moser and Wyman* [375].

7.2. Optimal representations and limit Sperner theorems

Throughout this section let (P, v) be a positively weighted poset. For a representation x and a real number α, let

$$A_x(\alpha) := \left\{p \in P : \alpha - \tfrac{1}{2} \le x(p) < \alpha + \tfrac{1}{2}\right\}.$$

Clearly, $A_x(\alpha)$ is an antichain. Thus we have:

Proposition 7.2.1. *For every x and every $\alpha \in \mathbb{R}$,*

$$d(P, v) \geq v(A_x(\alpha)).$$

We will apply this lower bound to direct products of posets and to the partition lattice Π_n. First we need an auxiliary lemma that is similar to the shifting method. In contrast to the usual definition of a ranked poset Q, we work with *generalized rank functions*; that is, we admit also negative ranks. Hence Q can be partitioned into levels $N_i := \{q \in Q : r(q) = i\}$ with $i = \ldots, -2, -1, 0, 1, 2, \ldots$. We only require that $q \lessdot q'$ implies $r(q') = r(q) + 1$. In [152] we proved:

Lemma 7.2.1. *Let (Q, w) be a (generalized) ranked and weighted poset. Further let $g : E(Q) \to \mathbb{R}_+$ be such that for all $q \in Q$*

$$\sum_{e^+=q} g(e) \leq 1 \quad \text{and} \quad \sum_{e^-=q} g(e) \leq 1. \tag{7.11}$$

Define the new weights ∇w and Δw on Q by

$$\nabla w(q) := \max\left\{0, w(q) - \sum_{q':q' \gtrdot q} g(qq')w(q')\right\}, \tag{7.12}$$

$$\Delta w(q) := \max\left\{0, w(q) - \sum_{q':q' \lessdot q} g(q'q)w(q')\right\}. \tag{7.13}$$

Then, for all j,

$$d(Q, w) \leq w(N_j) + \sum_{i<j} \nabla w(N_i) + \sum_{i>j} \Delta w(N_i).$$

Proof. We proceed by induction on $|Q|$. If Q is an antichain, the proof is trivial; thus suppose the contrary. Let A be a maximum weighted antichain in (Q, w). It cannot contain all minimal and maximal elements of Q (otherwise Q would be an antichain). Suppose, w.l.o.g., that q^* is a minimal element of Q that is not a member of A. Let $q^* \in N_h$. Further let (Q', w') be the poset induced by $Q' := Q - \{q^*\}$; that is, $w'(q) = w(q)$ for all $q \in Q'$ (the levels of Q' are the sets $N_i - \{q^*\}$ if $i = h$ and N_i if $i \neq h$). Obviously,

$$w(A) = d(Q, w) = d(Q', w'). \tag{7.14}$$

In order to apply the induction hypothesis to (Q', w'), let g' be the restriction of g to $E(Q')$ ($\subseteq E(Q)$). Then the corresponding inequalities (7.11) are satisfied. We define, analogously to (7.12) and (7.13), the new weights $\nabla w'$ and $\Delta w'$. Obviously, $\nabla w(q) = \nabla w'(q)$ for all $q \in Q'$, $\Delta w'(q) \leq \Delta w(q) + g(q^*q)w(q^*)$ if $q^* \lessdot q$

and $\Delta w'(q) = \Delta w(q)$ otherwise. Let j be fixed. If $h < j$, then by the induction hypothesis and (7.14)

$$\begin{aligned} d(Q, w) &= d(Q', w') \\ &\le w'(N_j) + \sum_{i<j} \nabla w'(N_i - \{q^*\}) + \sum_{i>j} \Delta w'(N_i) \\ &\le w(N_j) + \sum_{i<j} \nabla w(N_i) + \sum_{i>j} \Delta w(N_i). \end{aligned}$$

Also, if $h \ge j$, in view of the induction hypothesis and (7.14) (note that $\Delta w(q^*) = w(q^*)$)

$$\begin{aligned} d(Q, w) &= d(Q', w') \\ &\le w'(N_j - \{q^*\}) + \sum_{i<j} \nabla w'(N_i) + \sum_{i>j} \Delta w'(N_i - \{q^*\}) \\ &\le w(N_j) + \sum_{i<j} \nabla w(N_i) + \sum_{i>j} \Delta w(N_i) - w(q^*) \\ &\quad + \sum_{q:q>q^*} g(q^*q) w(q^*) \\ &\le w(N_j) + \sum_{i<j} \nabla w(N_i) + \sum_{i>j} \Delta w(N_i) \end{aligned}$$

since $\sum_{q:q>q^*} g(q^*q) \le 1$ because of (7.11). ■

The following theorem is in the case $v \equiv 1$ due to Alekseev [23]. We proved the general case [152].

Theorem 7.2.1 (Asymptotic Product Theorem). *Let (P, v) be any positively weighted poset and assume that P is not an antichain. Then*

$$d((P, v)^n) \sim \frac{(v(P))^n}{\sqrt{2\pi n}\sigma(P, v)} \text{ as } n \to \infty,$$

where $v(P) := \sum_{p \in P} v(p)$ and $\sigma^2(P, v)$ is the variance of (P, v).

Proof. First observe that we may make the general supposition that $v(P) = 1$ (i.e., the weight function v is normalized) because $d((P, cv)^n) = c^n d((P, v)^n)$ for any positive constant c, $n = 1, 2, \ldots$. Let x be an optimal representation of (P, v) with $\mu_x = 0$ and define $y : P^n \to \mathbb{R}$ by $y(\boldsymbol{p}) := \sum_{i=1}^n x(p_i)$, where $\boldsymbol{p} = (p_1, \ldots, p_n) \in P^n$ (by Theorem 4.6.6, y is an optimal representation of $(P, v)^n$). To get a lower bound we take in $(P, v)^n$ the antichain $A := A_y(0)$ as defined at the beginning of this section. We consider the discrete random variable ξ defined by $P(\xi = \alpha) = v(\{p \in P : x(p) = \alpha\})$. The expected value μ of ξ

equals $\mu_x = 0$, and the variance σ^2 of ξ equals $\sigma_x^2 = \sigma^2(P, v)$. Let $\xi_1, \dots, \xi_n$ be independent copies of ξ and let $\zeta_n := \xi_1 + \cdots + \xi_n$. Then

$$(v \times \cdots \times v)(A) = P\left(-\tfrac{1}{2} \le \zeta_n < \tfrac{1}{2}\right).$$

If ξ has a nonlattice distribution, we can derive directly from Theorem 7.1.6 that

$$(v \times \cdots \times v)(A) \sim \frac{1}{\sqrt{2\pi n}\sigma(P, v)} \quad \text{as } n \to \infty. \tag{7.15}$$

If ξ has a lattice distribution, the same follows from Corollary 7.1.2 if the maximal span h of ξ has the form $h = \frac{1}{j}$ for some natural number j. Since P is not an antichain and x is an optimal representation of (P, v), there must be two elements $p, p' \in P$ such that $x(p') - x(p) = 1$. Because, by definition, $\frac{\xi - a}{h}$ is an integral random variable for some $a \in \mathbb{R}$, we have for some natural number N

$$\frac{x(p') - a}{h} - \frac{x(p) - a}{h} = N, \quad \text{i.e.,} \quad h = \frac{1}{N}.$$

Thus (7.15) is also proved in the case of a lattice distribution and we infer

$$d((P, v)^n) \gtrsim \frac{1}{\sqrt{2\pi n}\sigma(P, v)}.$$

It remains to show that

$$d((P, v)^n) \lesssim \frac{1}{\sqrt{2\pi n}\sigma(P, v)}. \tag{7.16}$$

Let $p_1, \dots, p_k$ be the elements of P and define $v_i := v(p_i), x_i := x(p_i), i = 1, \dots, k$. For an element $\boldsymbol{p} \in P^n$, let $a_i(\boldsymbol{p})$ be the number of occurrences of the element p_i in $\boldsymbol{p}, i = 1, \dots, k$. Let φ be the mapping that assigns to $\boldsymbol{p} \in P^n$ the k-tuple $(a_1(\boldsymbol{p}), \dots, a_k(\boldsymbol{p}))$. Let $Q = Q(n)$ be the set of all such k-tuples; that is,

$$Q := \{\boldsymbol{q} = (q_1, \dots, q_k) : q_1 + \cdots + q_k = n, q_i \in \{0, \dots, n\}\}.$$

We define a partial order on Q by setting $\boldsymbol{q}' <_Q \boldsymbol{q}$ iff there are indices $i \ne j$ such that

$$p_i <_P p_j \qquad \text{and } q_l = \begin{cases} q_l' & \text{if } l \ne i, j, \\ q_l' - 1 & \text{if } l = i, \\ q_l' + 1 & \text{if } l = j \end{cases}$$

(later it will be clear that there cannot be "cycles"; that is, the reflexive and transitive closure of "$<$" is really an order relation). Let ψ be the mapping from $E(Q)$ into $E(P)$ that assigns the arc $(p_i <_P p_j)$ to the arc $(\boldsymbol{q}' <_Q \boldsymbol{q})$. Finally we define a weight $w = w(n)$ on Q by setting

$$w(q_1, \dots, q_k) = \frac{n!}{q_1! \cdots q_k!} v_1^{q_1} \cdots v_k^{q_k}.$$

It is not difficult to see that the mapping φ defined after (7.16) is a flow morphism from $(P, v)^n$ onto (Q, w). (Note that for $\boldsymbol{q} = (q_1, \ldots, q_k)$ all elements of $\varphi^{-1}(\boldsymbol{q})$ have the weight $v_1^{q_1} \cdots v_k^{q_k}$ and that for $\boldsymbol{q}' \lessdot \boldsymbol{q}$ the poset induced by $\varphi^{-1}(\boldsymbol{q}') \cup \varphi^{-1}(\boldsymbol{q})$ is after a suitable normalization a regular, i.e., normal poset of rank 1.) By Theorem 4.5.4(a),

$$d((P, v)^n) = d(Q, w). \tag{7.17}$$

Define the function $z : Q \to \mathbb{R}$ by

$$z(q_1, \ldots, q_k) := \sum_{i=1}^{k} q_i x_i.$$

Since x is a representation of P, we have, for $\boldsymbol{q}' \lessdot \boldsymbol{q}$ the inequality $z(\boldsymbol{q}) - z(\boldsymbol{q}') \geq 1$; that is, there really cannot be cycles in the relation "$\lessdot_Q$" (using Theorem 4.5.6 one can prove easily that z is even an optimal representation of Q, but we do not need this fact here). We consider the active poset Q_z (see Section 4.4). Obviously,

$$d(Q, w) \leq d(Q_z, w). \tag{7.18}$$

The function $r : Q_z \to \mathbb{R}$ defined by $r(\boldsymbol{q}) := \lfloor z(\boldsymbol{q}) + \frac{1}{2} \rfloor$ is a (generalized) rank function because $\boldsymbol{q}' \lessdot_{Q_z} \boldsymbol{q}$ implies $z(\boldsymbol{q}) - z(\boldsymbol{q}') = 1$; that is, $r(\boldsymbol{q}) = r(\boldsymbol{q}') + 1$. Let $N_i := \{\boldsymbol{q} \in Q : r(\boldsymbol{q}) = i\}$ be the levels of Q_z. Further, let f be a representation flow on (P, v) relative to x (note Theorem 4.4.1) and set $F := \sum_{e \in E(P_x)} f(e)$. Obviously, if $e = \boldsymbol{q}'\boldsymbol{q} \in E(Q_z)$, then $\psi(e) = p_i p_j \in E(P_x)$. To apply Lemma 7.2.1, define $g : E(Q_z) \to \mathbb{R}_+$ by setting

$$g(e) := \frac{f(\psi(e))}{F}, \quad e \in E(Q_z).$$

Then, for all $\boldsymbol{q} = (q_1, \ldots, q_k) \in Q_z$,

$$\sum_{\boldsymbol{q}' : \boldsymbol{q}'\boldsymbol{q} \in E(Q_z)} g(\boldsymbol{q}'\boldsymbol{q}) = \sum_{\substack{i,j : p_i p_j \in E(P_x), \\ t_i < n, t_j > 0}} \frac{f(p_i p_j)}{F} \leq 1 \tag{7.19}$$

and, analogously,

$$\sum_{\boldsymbol{q}' : \boldsymbol{q}\boldsymbol{q}' \in E(Q_z)} g(\boldsymbol{q}\boldsymbol{q}') \leq 1, \tag{7.20}$$

where equality holds if $0 < q_i < n$ for all i. Accordingly, by Lemma 7.2.1

$$d(Q_z, w) \leq w(N_0) + \sum_{\substack{\boldsymbol{q} \in Q_z : \\ z(\boldsymbol{q}) < -\frac{1}{2}}} \nabla w(\boldsymbol{q}) + \sum_{\substack{\boldsymbol{q} \in Q_z : \\ z(\boldsymbol{q}) \geq \frac{1}{2}}} \Delta w(\boldsymbol{q}). \tag{7.21}$$

It is not difficult to see that $\varphi(A) = N_0$ for $A = A_y(0)$, where $A_y(0)$ is again the antichain defined in the beginning of the section. Since φ is a flow morphism,

$$(v \times \cdots \times v)(A) = w(N_0). \tag{7.22}$$

To prove (7.16) it is sufficient to show that

$$\sum_{q:z(q)<-\frac{1}{2}} \nabla w(\boldsymbol{q}) + \sum_{q:z(q)\geq\frac{1}{2}} \Delta w(\boldsymbol{q}) = o\left(\frac{1}{\sqrt{n}}\right) \tag{7.23}$$

(note (7.15), (7.17), (7.18), (7.21), and (7.22)). Let $\tau = \tau(n) := \sqrt{n}\log n$ and $G = G(n) := \{\boldsymbol{q} \in Q_z : |q_i - v_i n| \leq \tau \text{ for all } i = 1, \ldots, k\}$. First we prove that we may restrict both sums in (7.23) to elements $\boldsymbol{q}$ of G. To do so, let $\lambda^{(i)}$ be the random variable that takes on the values 1 and 0 with probabilities v_i and $1 - v_i$, respectively. Note that $\mathrm{E}(\lambda^{(i)}) = v_i$ and $V(\lambda^{(i)}) = v_i(1 - v_i)$. Let $\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}$ be independent copies of $\lambda^{(i)}$. Finally, let $\omega_n^{(i)} := \lambda_1^{(i)} + \cdots + \lambda_n^{(i)}$, $i = 1, \ldots, k$. Note that

$$\begin{aligned} P(\omega_n^{(i)} = T) &= \binom{n}{T} v_i^T (1 - v_i)^{n-T} \\ &= \binom{n}{T} v_i^T (v_1 + \cdots + v_{i-1} + v_{i+1} + \cdots + v_n)^{n-T} \\ &= v(\{\boldsymbol{p} \in P^n : a_i(\boldsymbol{p}) = T\}) \\ &= w(\{\boldsymbol{q} \in Q : q_i = T\}). \end{aligned}$$

This implies

$$\begin{aligned} w(Q - G) &= P\big(|\omega_n^{(1)} - v_1 n| > \tau \text{ or } \cdots \text{ or } |\omega_n^{(k)} - v_k n| > \tau\big) \\ &\leq \sum_{i=1}^{k} P\big(|\omega_n^{(i)} - v_i n| > \tau\big). \end{aligned}$$

By Theorem 7.1.3, for all i and sufficiently large n (we take $M := 1$, $\epsilon := \frac{\log n}{\sqrt{n}}$ and need $\epsilon \leq v_i(1 - v_i)$),

$$P\big(|\omega_n^{(i)} - v_i n| > \tau\big) \leq 2e^{-\log^2 n/(8v_i(1-v_i))} = o\left(\frac{1}{\sqrt{n}}\right).$$

Hence,

$$\sum_{\substack{q:q\notin G:\\ z(q)<-\frac{1}{2}}} \nabla w(\boldsymbol{q}) + \sum_{\substack{q:q\notin G:\\ z(q)\geq\frac{1}{2}}} \Delta w(\boldsymbol{q}) \leq \sum_{q:q\notin G} w(\boldsymbol{q}) = w(Q - G) = o\left(\frac{1}{\sqrt{n}}\right).$$

We conclude the proof of (7.23) by showing that

$$\Delta w(\boldsymbol{q}) \leq w(\boldsymbol{q}) o\left(\frac{1}{\sqrt{n}}\right) \tag{7.24}$$

uniformly in $\boldsymbol{q} \in G, z(\boldsymbol{q}) \geq \frac{1}{2}$ (the corresponding inequality for $\nabla w(\boldsymbol{q})$ can be proved analogously). Then the proof will be complete because (7.24) yields

$$\sum_{\substack{\boldsymbol{q}\in G:\\ z(\boldsymbol{q})<-\frac{1}{2}}} \nabla w(\boldsymbol{q}) + \sum_{\substack{\boldsymbol{q}\in G:\\ z(\boldsymbol{q})\geq\frac{1}{2}}} \Delta w(\boldsymbol{q}) \leq \sum_{\boldsymbol{q}:\boldsymbol{q}\in G} w(\boldsymbol{q}) o\left(\frac{1}{\sqrt{n}}\right) = o\left(\frac{1}{\sqrt{n}}\right)$$

(note that $w(Q) = (v \times \cdots \times v)(P^n) = (v(P))^n = 1$ by the general supposition). Let $\boldsymbol{q} \in G, z(\boldsymbol{q}) \geq \frac{1}{2}$. If $\Delta w(\boldsymbol{q}) = 0$, then (7.24) is trivially true; thus let $\Delta w(\boldsymbol{q}) > 0$. Then

$$\Delta w(\boldsymbol{q}) = w(\boldsymbol{q}) \left(1 - \sum g(\boldsymbol{q}'\boldsymbol{q}) \frac{w(\boldsymbol{q}')}{w(\boldsymbol{q})}\right). \tag{7.25}$$

Here and in the following, the sums are extended over all $\boldsymbol{q}'$ such that $\boldsymbol{q}' <_{Q_z} \boldsymbol{q}$, that is, over all pairs (i, j) such that $q_i < n, q_j > 0, p_i <_{P_x} p_j$. Note that $q_i < n$ and $q_j > 0$ are satisfied automatically for sufficiently large n since $\boldsymbol{q} \in G$. Hence we have equality in (7.19):

$$\sum g(\boldsymbol{q}'\boldsymbol{q}) = 1. \tag{7.26}$$

We have (for $\boldsymbol{q}' = (q_1, \ldots, q_i + 1, \ldots, q_j - 1, \ldots, q_k)$)

$$\begin{aligned}\frac{w(\boldsymbol{q}')}{w(\boldsymbol{q})} &= \frac{v_i q_j}{v_j (q_i + 1)}\\ &= 1 - \frac{1}{q_i + 1} + \frac{v_i}{(q_i + 1)n} \left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right) \left(n - \frac{q_i}{v_i} - \frac{1}{v_i}\right)\\ &\quad + \frac{1}{n} \left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right).\end{aligned}$$

It can be easily checked that, uniformly in $\boldsymbol{q} \in G$,

$$\frac{1}{q_i + 1} \lesssim \frac{1}{v_i n} = o\left(\frac{1}{\sqrt{n}}\right), \tag{7.27}$$

$$-\frac{v_i}{(q_i + 1)n} \left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right) \left(n - \frac{q_i}{v_i} - \frac{1}{v_i}\right) \lesssim \frac{\log^2 n}{n} \frac{v_i + v_j}{v_i^2 v_j} = o\left(\frac{1}{\sqrt{n}}\right). \tag{7.28}$$

With (7.25)–(7.28) we have already obtained

$$\Delta w(\boldsymbol{q}) \leq w(\boldsymbol{q}) \left(o\left(\frac{1}{\sqrt{n}}\right) - \frac{1}{n} \sum g(\boldsymbol{q}'\boldsymbol{q}) \left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right)\right),$$

uniformly in $\boldsymbol{q} \in G$, and it suffices to show

$$\sum g(\boldsymbol{q}'\boldsymbol{q}) \left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right) > 0 \text{ for all } \boldsymbol{q} \text{ with } z(\boldsymbol{q}) \geq \frac{1}{2}.$$

Since f is a representation flow on (P_x, v) we have by definition of g

$$\sum g(q'q)\left(\frac{q_j}{v_j} - \frac{q_i}{v_i}\right) = \sum \frac{1}{F} f(p_i p_j)\left(\frac{t_j}{v_j} - \frac{t_i}{v_i}\right)$$
$$= \frac{1}{F}\sum_{i=1}^{k} \frac{t_i}{v_i}(p_{i_f}^+ - p_{i_f}^-) = \frac{1}{F}\sum_{i=1}^{k} \frac{q_i}{v_i} v_i x_i = \frac{1}{F} z(q) > 0.$$

Thus not only the proof of (7.24), but also the proof of (7.16) and the proof of the theorem are complete. ■

In fact, in the case $v \equiv 1$, Alekseev used this result to derive an asymptotic formula for the number $\varphi_{P,Q}(n)$ of order-preserving maps from P^n into Q: Let S_Q be the (finite) set of all finite sequences $(H_0, H_1, \ldots, H_s)$ of subsets of Q such that $|H_0| = 1$, $|H_s| = 1$ and for all $i = 0, \ldots, s-1$, $p \in H_i$, $q \in H_{i+1}$ we have $p \le q$. Let

$$\tau(Q) := \max\{|H_0||H_1| \cdots |H_s| : (H_0, \ldots, H_s) \in S_Q\}.$$

Theorem 7.2.2 **(*Alekseev*** **[23]).** *If P and Q are not antichains, then*

$$\varphi_{P,Q} = \tau(Q)^{\frac{|P|^n}{\sqrt{2\pi n \sigma(P)}}(1+o(1))}.$$

In the proof, a method of Kleitman [301] is used. We will omit the proof but construct as many functions as given in the theorem: Let $\tau(Q)$ be attained by the sequence $(\{q_l\}, H_1, \ldots, H_{s-1}, \{q_u\})$ and let x be an optimal representation of $P = (P, 1)$. We take again the antichains

$$A_x(i) := \left\{p \in P^n : i - \tfrac{1}{2} \le x(p) < i + \tfrac{1}{2}\right\},$$

$i = 1, \ldots, s-1$, and put

$$A_x(< 0) := \left\{p \in P^n : x(p) < -\tfrac{1}{2}\right\},$$
$$A_x(> s-1) := \left\{p \in P^n : x(p) \ge s - \tfrac{1}{2}\right\}.$$

Obviously, any function that maps $A_x(< 0)$ onto $\{q_l\}$, $A_x(i)$ in any way into H_i, $i = 1, \ldots, s-1$, and $A_x(> 0)$ onto $\{q_u\}$, is order preserving. Consequently,

$$\varphi_{P,Q} \ge |H_1|^{|A_x(1)|} \cdots |H_{s-1}|^{|A_x(s-1)|}.$$

Exactly as in the proof of Theorem 7.2.1, we may derive from Corollary 7.1.2 that

$$|A_x(i)| = \frac{|P|^n}{\sqrt{2\pi n \sigma(P)}}(1 + o(1)) \quad \text{for every } i = 1, \ldots, s-1.$$

Consequently,

$$\varphi_{P,Q}(n) \ge (|H_1| \cdots |H_{s-1}|)^{\frac{|P|^n}{\sqrt{2\pi n\sigma(P)}}(1+o(1))} = \tau(Q)^{\frac{|P|^n}{\sqrt{2\pi n\sigma(P)}}(1+o(1))}.$$

Also without proof we mention another variant of Theorem 7.2.1 which we found together with Kuzyurin [163]:

Theorem 7.2.3. *Let $\{P_n\}$ be a sequence of posets ($v \equiv 1$) that are not antichains and whose sizes are bounded by some fixed constant. If $k = k(n) = o(\sqrt{n})$, then*

$$d_k(P_1 \times \cdots \times P_n) \sim \frac{k|P_1|\cdots|P_n|}{\sqrt{2\pi \sum_{i=1}^{n} \sigma^2(P_i)}} \quad \text{as } n \to \infty.$$

For a ranked and weighted poset (P, w), let

$$b(P, w) := \max_i w(N_i).$$

Proposition 7.2.2. *If $r(P) > 0$ then*

$$b((P, w)^n) \sim \frac{(w(P))^n}{\sqrt{2\pi n}\sigma_r} \quad \text{as } n \to \infty$$

where σ_r^2 is the variance of the rank function.

Proof. Apply Theorem 7.1.5 to the independent random variables ξ_j, $j = 1, \ldots, n$, where $P(\xi_j = i) = \frac{w(N_i)}{w(P)}$, $i = 0, \ldots, r(P)$. ∎

Note that $n\sigma_r^2$ is the variance of the rank function of $(P, w)^n$. Moreover, recall that we proved in Corollary 7.1.3 for the partition lattice Π_n with $w \equiv 1$ that also

$$b(\Pi_n) \sim \frac{|\Pi_n|}{\sqrt{2\pi}\sigma_n} \sim \frac{B_n}{\sqrt{2\pi}\sigma_{rf}}. \tag{7.29}$$

In contrast to the usual notation, we write σ_{rf}^2 instead of σ_r^2 in order to avoid confusion with the number r (the solution of $re^r = n$). Here we may replace σ_n by σ_{rf} because of Remark 7.1.1.

Let $\{(P_n, w_n)\}$ be a sequence of ranked and weighted posets. We say that (P_n, w_n) has the *asymptotic Sperner property* if

$$d(P_n, w_n) \sim b(P_n, w_n) \quad \text{as } n \to \infty.$$

Corollary 7.2.1. *Let (P, w) be a ranked and positively weighted poset. (P, w) is rank compressed iff $(P, w)^n$ has the asymptotic Sperner property.*

Proof. The case $r(P) = 0$ is trivial; for $r(P) > 0$ apply Theorem 7.2.1 and Proposition 7.2.2. ∎

Note that not all rank-compressed posets have the Sperner property and that the strong Sperner property does not imply rank compression; see Figure 7.1.

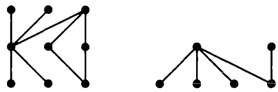


Figure 7.1

The question whether the partition lattice Π_n is asymptotically Sperner or rank compressed had been open for a long time. Finally Canfield and Harper [98] found that the answer in both cases is no.

***Theorem 7.2.4* (*Canfield and Harper*).** *We have for sufficiently large n*

$$\frac{d(\Pi_n)}{b(\Pi_n)} \geq n^{1/35} \quad \textit{and} \quad \frac{\sigma(\Pi_n)}{\sigma_{rf}(\Pi_n)} \leq \frac{1}{n^{1/35}}.$$

Proof. Let $\tau := \frac{e}{4}$, and r be the solution of $re^r = n$. We define the sequence $\{A_j^{(n)}\}$ by

$$A_j^{(n)} := \begin{cases} \frac{j}{r(1-\tau)} & \text{if } \lfloor \tau r \rfloor < j < \lfloor 2\tau r \rfloor, \\ \frac{1}{1-\tau} & \text{otherwise.} \end{cases}$$

Then we introduce the function $x : \Pi_n \to \mathbb{R}$ by

$$x(\pi) := A_{|B_1|}^{(n)} + \cdots + A_{|B_k|}^{(n)} \text{ if } \pi \in \Pi_n \text{ consists of the blocks } B_1, \ldots, B_k.$$

Note that if $A_j^{(n)}$ was proportional to all j, then $x(\pi)$ would be constant; that is, it would have variance 0. By checking several cases, one finds that x is a representation of Π_n^* (the dual of Π_n). The large interval in which $A_j^{(n)}$ is proportional to j yields a sufficiently small variance. If we define

$$Z_n := \{x(\pi) : \pi \in \Pi_n\}$$

and

$$a_{n,\alpha} := |\{\pi \in \Pi_n : x(\pi) = \alpha\}|,$$

then we may prove analogously to Lemma 7.1.1 that

$$e^{\sum_{j=1}^{\infty} y^{A_j}(z^j/j!)} = \sum_{k=0}^{\infty} \sum_{\alpha \in Z_n} \frac{a_{n,\alpha}}{n!} y^{\alpha} z^n.$$

By Theorem 7.1.11 (noting $2 < \nu_0$), the random variable ζ_n, where $P(\zeta_n = \alpha) = |\{\pi \in \Pi_n : x(\pi) = \alpha\}|/B_n$, is asymptotically normal with mean μ_n and variance σ_n^2 given in Theorem 7.1.11. Using Remark 7.1.1, we may calculate $V(\zeta_n)$ (which is the variance σ_x^2 of the representation x) with high precision: Let

$J := \lfloor \tau r \rfloor$, $K := \lfloor 2\tau r \rfloor$. Then

$$\sum_{j=1}^{\infty} (A_j^{(n)})^2 \frac{r^j}{j!}$$

$$= \frac{1}{(1-\tau)^2} \left(\sum_{j=1}^{\infty} \frac{j^2}{r^2} \frac{r^j}{j!} + \sum_{j=1}^{J} \left(1 - \frac{j^2}{r^2}\right) \frac{r^j}{j!} - \sum_{j=K}^{\infty} \left(\frac{j^2}{r^2} - 1\right) \frac{r^j}{j!} \right)$$

$$= \frac{1}{(1-\tau)^2} \left(\left(1 + \frac{1}{r}\right) e^r + O\left(\frac{r^J}{J!}\right) + O\left(\frac{r^K}{K!}\right) \right)$$

since the second and third sum are bounded by geometric series whose ratios are bounded away from 1. In a similar manner,

$$\sum_{j=1}^{\infty} j A_j^{(n)} \frac{r^j}{j!} = \frac{1}{1-\tau} \left((r+1)e^r + rO\left(\frac{r^J}{J!}\right) + rO\left(\frac{r^K}{K!}\right) \right).$$

Consequently,

$$V(\zeta_n) = O\left(\frac{r^J}{J!} + \frac{r^K}{K!}\right) + \frac{r}{n} O\left(\left(\frac{r^J}{J!} + \frac{r^K}{K!}\right)^2 \right) + O(1).$$

By Stirling's formula we find that both $\frac{r^J}{J!}$ and $\frac{r^K}{K!}$ are of order $O(\frac{4^{er/4}}{\sqrt{r}})$ (it was the aim to attain the same order during the determination of τ). Thus,

$$\frac{r^J}{J!} + \frac{r^K}{K!} = O\left(\frac{2^{er/2}}{\sqrt{r}}\right) = O\left(\frac{e^{(re\log 2)/2}}{\sqrt{r}}\right) = O\left(\frac{n^{(e\log 2)/2}}{r^\beta}\right)$$

where $\beta := \frac{1+e\log 2}{2}$ (we replaced e^r by $\frac{n}{r}$). Hence we are also able to express the variance as a power of n:

$$\sigma_x^2 = V(\zeta_n) = O\left(\frac{n^{(e\log 2)/2}}{r^\beta}\right).$$

By Theorem 7.1.9 and Remark 7.1.1, $\sigma_{rf}^2(\Pi_n) \sim \frac{n}{r^2}$, which yields

$$\frac{\sigma_x^2}{\sigma_{rf}^2} = O\left(n^{(e\log 2)/2-1} r^{2-\beta}\right) < \frac{1}{n^{2/35}} \quad \text{for } n \text{ large enough.} \tag{7.30}$$

Accordingly, the second inequality in the statement of the theorem is proved. To verify the first inequality, we apply Chebyshev's inequality: We have (with $\sigma_x^2 = V(\zeta_n)$)

$$P(|\zeta_n - \mathrm{E}(\zeta_n)| < 2\sigma_x) \geq \tfrac{3}{4}.$$

The interval $(\mathrm{E}(\zeta_n) - 2\sigma_x, \mathrm{E}(\zeta_n) + 2\sigma_x)$ can be covered by disjoint half open intervals of length ≤ 1, using at most $4\sigma_x + 1$ such intervals. Hence there exists at least one interval $I = \left[\alpha - \frac{1}{2},\ \alpha + \frac{1}{2}\right)$ such that

$$P(\zeta_n \in I) \geq \frac{\frac{3}{4}}{4\sigma_x + 1},$$

and Proposition 7.2.1 yields

$$d(\Pi_n) \geq B_n \frac{\frac{3}{4}}{4\sigma_x + 1}.$$

With (7.29) and (7.30) we derive for some constant $c > 0$

$$\frac{d(\Pi_n)}{b(\Pi_n)} \geq c\frac{\sigma_{rf}}{\sigma_x} > n^{1/35}.$$

■

For arbitrary posets, the ratio $\frac{d(P)}{b(P)}$ can be bounded from above in the following way (we omit the proof, see [164]).

Theorem 7.2.5. *Let P be a ranked poset of size k. Then*

$$(a) \quad \frac{d(P)}{b(P)} \leq \frac{1}{2}\left\lfloor \frac{k+2}{2} \right\rfloor,$$

$$(b) \quad \lim_{n\to\infty} \frac{d(P^n)}{b(P^n)} \leq \begin{cases} \sqrt{\frac{k+2}{4}} & \text{if } k \text{ is even,} \\ \sqrt{\frac{k+2+1/k}{4}} & \text{if } k \text{ is odd,} \end{cases}$$

and these bounds are the best possible.

7.3. An asymptotic Erdős–Ko–Rado Theorem

In this section we study maximum statically t-intersecting families in $N_i(S)$ where $S = S(k_1, \ldots, k_n)$ and $k_1 = \cdots = k_n = k$ (see the end of Section 3.3). The results we found together with Frankl [160]. Instead of $N_i(S)$ we write $N_i(n, k)$. Examples of statically t-intersecting families include the families $F_X := \{\boldsymbol{x} \in N_i(n, k) : \operatorname{supp}(\boldsymbol{x}) \supseteq X\}$, where $X \subseteq [n]$, $|X| = t$. In the following, we often take $X = [t] = \{1, \ldots, t\}$. Let $i = \lfloor \lambda n \rfloor$ where $0 < \lambda < k$. Moreover, let β_t be the unique positive solution of the equation

$$x + x^2 + \cdots + x^k = \frac{1}{t} \quad \left(\text{or, equivalently, } x^{k+1} = x\left(\frac{1}{t} + 1\right) - \frac{1}{t}\right)$$

and define

$$\lambda_t^* := \frac{\sum_{j=1}^k j\beta_t^j}{\sum_{j=0}^k \beta_t^j}.$$

Theorem 7.3.1. *Let k, t, and λ $(0 < \lambda < k)$ be fixed, and consider $n \to \infty$. Let $F \subseteq N_i(n,k)$ be a maximum statically t-intersecting family with $i = \lfloor \lambda n \rfloor$. Then we have*

$$
\begin{array}{lll}
(a) & |F| \sim |F_{[t]}| & \text{if } \lambda < \lambda_t^*, \\
(b) & |F| \gtrsim c|F_{[t]}| & \text{if } \lambda > \lambda_t^*, \text{ where } c > 1 \text{ is some constant,} \\
(c) & |F| \sim W_i(k,n) & \text{if } \lambda > \lambda_1^*.
\end{array}
$$

Proof. Let $\alpha = \alpha(\lambda)$ be the unique positive solution of

$$\frac{\sum_{j=1}^{k} j\alpha^j}{\sum_{j=0}^{k} \alpha^j} = \lambda$$

(observe that $g(x) := \sum_{j=1}^{k} jx^j / \sum_{j=0}^{k} x^j$ is continuous and increasing for $0 \leq x < \infty$ with $g(0) = 0$ and $\lim_{x\to\infty} g(x) = k$). Obviously we have

$$\alpha < \beta_t \text{ if } \lambda < \lambda_t^* \text{ and } \alpha > \beta_t \text{ if } \lambda > \lambda_t^*. \tag{7.31}$$

From (7.7) we know that

$$W_i(n,k) \sim \frac{1}{\sqrt{2\pi n \sigma}} \frac{(1+\alpha+\cdots+\alpha^k)^n}{\alpha^i} \tag{7.32}$$

where σ depends on k and λ, only. Let $p_0 := 1/\sum_{j=0}^{k} \alpha^j$. For $\boldsymbol{x} \in S$, let $z(\boldsymbol{x})$ be the number of zero coordinates of $\boldsymbol{x}$; that is, $z(\boldsymbol{x}) = n - |\operatorname{supp}(\boldsymbol{x})|$. Finally, for $\epsilon > 0$ let

$$
\begin{aligned}
N_{i,\epsilon}(n,k) &:= \{\boldsymbol{x} \in N_i(n,k) : z(\boldsymbol{x}) \notin [(p_0-\epsilon)n, (p_0+\epsilon)n]\}, \\
W_{i,\epsilon}(n,k) &:= |N_{i,\epsilon}(n,k)|.
\end{aligned}
$$

Claim 1. Let $0 < \epsilon < p_0(1-p_0)$. Then there exists some constant c such that $W_{i,\epsilon}(n,k)/W_i(n,k) \leq e^{-cn}$, that is, the number of elements $\boldsymbol{x}$ of $N_i(n,k)$ containing fewer than $(p_0-\epsilon)n$ or more than $(p_0+\epsilon)n$ zero coordinates is exponentially small in n with respect to $W_i(k,n)$.

Proof of Claim 1. Let η be a random variable with $P(\eta_0) = 1 - p_0$ and $P(\eta = 1) = p_0$. Let $\eta_1 \ldots, \eta_n$ be independent copies of η and let $\omega_n := \eta_1 + \cdots + \eta_n$. Then

$$
\begin{aligned}
P(\omega_n = h) &= \binom{n}{h} p_0^h (1-p_0)^{n-h} = \binom{n}{h} \frac{(\alpha+\alpha^2+\cdots+\alpha^k)^{n-h}}{(1+\alpha+\cdots+\alpha^k)^n} \\
&= \sum_{j=0}^{kn} \frac{\alpha^j}{(1+\alpha+\cdots+\alpha^k)^n} |\{\boldsymbol{x} \in N_j(n,k) : z(\boldsymbol{x}) = h\}|.
\end{aligned}
$$

Furthermore, in view of Theorem 7.1.3, for some constant $d > 0$,

$$
\begin{aligned}
2e^{-dn} &\geq P(|p_0 n - \omega_n| > \epsilon n) = \sum_{h \notin [(p_0-\epsilon)n,(p_0+\epsilon)n]} P(\omega_n = h) \\
&= \sum_{j=0}^{kn} \frac{\alpha^j}{(1+\alpha+\cdots+\alpha^k)^n} \sum_{h \notin [(p_0-\epsilon)n,(p_0+\epsilon)n]} |\{x \in N_j(n,k) : z(x) = h\}| \\
&\geq \frac{\alpha^i}{(1+\alpha+\cdots+\alpha^k)^n} W_{i,\epsilon}(n,k).
\end{aligned}
$$

With (7.32) we obtain, for some constant $0 < c < d$

$$
\frac{W_{i,\epsilon}(n,k)}{W_i(n,k)} \leq e^{-cn}.
$$

□

Claim 2. $|F_{[t]}| \geq W_i(n,k)/\binom{n}{t}$ if $i \geq kt$.

Proof of Claim 2. If $i \geq kt$ then $|\operatorname{supp}(x)| \geq t$ for all $x \in N_i(n,k)$. Thus $N_i(n,k) = \cup F_X$, implying $W_i(n,k) \leq \sum |F_X| = \binom{n}{t}|F_{[t]}|$ (the union and the sum are extended over all $X \in \binom{[n]}{t}$). □

(a) Let $\lambda < \lambda_t^*$. Then, by (7.31), $\alpha < \beta_t$ and, in view of the definition of β_t, $\alpha + \alpha^2 + \cdots + \alpha^k < \frac{1}{t}$. Consequently, $p_0 > \frac{t}{t+1}$. Let $0 < \epsilon < \min\{p_0(1-p_0), p_0 - \frac{t}{t+1}\}$. Finally let $F' := F - N_{i,\epsilon}(n,k)$ and $F'_{[t]} := F_{[t]} - N_{i,\epsilon}(n,k)$. Since by Claim 1 and 2

$$
\frac{W_{i,\epsilon}(n,k)}{|F|} \leq \frac{W_{i,\epsilon}(n,k)}{|F_{[t]}|} \leq \frac{\binom{n}{t}}{e^{cn}} \to 0,
$$

it follows that

$$
|F'| \sim |F| \text{ and } F'_{[t]} \sim F_{[t]}.
$$

Obviously, we have (compare with the proof of Theorem 3.3.4)

$$
|F'| = \sum_{j \in [(1-p_0-\epsilon)n,(1-p_0+\epsilon)n]} |\mathcal{G}_j| W_{i-j}(k-1,j) \tag{7.33}
$$

where $\mathcal{G}_j$ is some t-intersecting family of j-element subsets of $[n]$. However,

$$
|F'_{[t]}| = \sum_{j \in [(1-p_0-\epsilon)n,(1-p_0+\epsilon)n]} \binom{n-t}{j-t} W_{i-j}(k-1,j).
$$

Because there is some $\delta > 0$ such that $1 - p_0 + \epsilon < \frac{1}{t+1} - \delta$, the values j satisfy the inequality

$$
j \leq \left(\frac{1}{t+1} - \delta\right) n < \frac{n}{t+1} + t - 1.
$$

Theorem 6.4.5 implies $|\mathcal{G}_j| \le \binom{n-t}{j-t}$ and thus $|F'| \le |F'_{[t]}|$. Consequently, $|F| \lesssim |F_{[t]}|$, and the obvious relation $|F| \gtrsim |F_{[t]}|$ yields the assertion.

(b) Let $\lambda > \lambda_t^*$. We obtain (7.33) in the same way, but this time we work with some $\epsilon, \delta > 0$ such that

$$\frac{1}{t+1} + \delta < 1 - p_0 - \epsilon < 1 - p_0 + \epsilon < 1 - \delta \quad \text{and} \quad \epsilon < p_0(1 - p_0).$$

As on p. 53 we take the "Boolean" t-intersecting families

$$\mathcal{G}_{1,j} := \{X \in \binom{[n]}{j} : |X \cap [t+2]| \ge t+1\}$$

and define the statically t-intersecting family in $N_i(n,k)$ by

$$F^* := \{\boldsymbol{x} \in N_i(n,k) : \operatorname{supp}(\boldsymbol{x}) \in \cup_{j \in [(1-p_0-\epsilon)n, (1-p_0+\epsilon)n]} \mathcal{G}_{1,j}\}.$$

The size of F^* is given by (7.33), where we have to replace $|\mathcal{G}_j|$ by $|\mathcal{G}_{1,j}|$. A simple computation shows that, for $j := ln$,

$$\lim_{n\to\infty} \frac{|\mathcal{G}_{1,j}|}{\binom{n-t}{j-t}} = l(t+2-l(t+1)) > 1 \text{ uniformly in } \frac{1}{t+1} + \delta < l < 1 - \delta.$$

Thus there exists some constant $c > 1$ such that $|\mathcal{G}_{1,j}| > c\binom{n-t}{j-t}$ for $j \in [(1 - p_0 - \epsilon)n, (1 - p_0 + \epsilon)n]$ and n large enough. Consequently, for these n, $|F| \ge |F^*| > c|F'_{[t]}| \sim c|F_{[t]}|$.

(c) Let $\lambda > \lambda_1^*$. Analogously to (a) we obtain $p_0 < \frac{1}{2}$, and we find thus some $\delta, \epsilon > 0$ such that $\frac{1}{2} - \delta > p_0 + \epsilon$ and $\epsilon < p_0(1 - p_0)$. We put

$$F^* := \{\boldsymbol{x} \in N_i(n,k) : |\operatorname{supp}(\boldsymbol{x})| \ge (\tfrac{1}{2} + \delta)n\}.$$

Then, for sufficiently large n, F^* is a statically t-intersecting family because for $\boldsymbol{x}, \boldsymbol{y} \in F^*$ it follows that $|\operatorname{supp}(\boldsymbol{x}) \cap \operatorname{supp}(\boldsymbol{y})| \ge 2\delta n \ge t$. By Claim 1, $|F^*| \sim W_i(k,n)$. ∎

8

Macaulay posets

In this final chapter a theory is presented that is based on the Kruskal–Katona Theorem (Theorem 2.3.6) and its predecessor, the Macaulay Theorem (Corollary 8.1.1). The central objects are the Macaulay posets, and the main theorem says that chain and star products are Macaulay posets. These theorems provide solutions of the shadow-minimization problem, but several other existence and optimization problems can be solved in this theory as well. We restrict ourselves to chain and star products (and their duals) because these have many applications, and both posets are very natural generalizations of the Boolean lattice. Recall that we already know much about these special posets $S(k_1, \dots, k_n)$ and $T(k_1, \dots, k_n)$. We proved that they are normal and have log-concave Whitney numbers; see Example 4.6.1. In particular, they have the strong Sperner property. Furthermore, chain products are symmetric chain orders; see Example 5.1.1 (and, moreover, ssc-orders; see Example 5.3.1), and are unitary Peck; see Example 6.2.1.

A more detailed study of isoperimetric problems, also in other structures, such as toroidal grids, de Bruijn graphs, and so forth, will be presented in the forthcoming book of Harper and Chavez [261]. We refer also to Bezrukov [57, 58] and Ahlswede, Cai, Danh, Daykin, Khachatrian, and Thu [6]. In the case of chain products, we will further investigate two types of intersecting (resp. cointersecting) Sperner families and discuss some other properties. A complete table for several classes of families, as it was given in Chapter 3 for the Boolean lattice B_n, is not known and seems to be difficult to find since Katona's circle method does not work here. However, in some cases the succeeding theory may replace the circle method. Further results for families in chain products that satisfy certain conditions but are not Sperner families can be found, for example, in [161].

8.1. Macaulay posets and shadow minimization

Let P be a ranked poset and $\preceq$ a new linear order on P. Clearly, $\preceq$ induces also a linear order on each level N_i. For a subset F of P and a number m, let $\mathcal{C}(m, F)$ (resp. $\mathcal{L}(m, F)$) be the set of the first (resp. last) m elements of F with respect to $\preceq$. For a subset F of a level N_i, we abbreviate $\mathcal{C}(|F|, N_i)$ by $\mathcal{C}(F)$ (and, analogously, $\mathcal{L}(|F|, N_i)$ by $\mathcal{L}(F)$). For a still shorter notation, we will often write just $\mathcal{C}F$ (resp. $\mathcal{L}F$). Thus $\mathcal{C}F$ consists of the first $|F|$ elements in N_i. It is called the *compression of* F. Clearly, we have $\mathcal{C}\mathcal{C}F = \mathcal{C}F$; thus $\mathcal{C}$ can be interpreted as an *idempotent* operator on 2^{N_i}. The set $F \subseteq N_i$ is called *compressed* if $F = \mathcal{C}F$.

The ranked poset P is said to be a *Macaulay poset* if there exists a linear order $\preceq$ such that

$$\Delta(\mathcal{C}F) \subseteq \mathcal{C}(\Delta(F)) \text{ for all } F \subseteq N_i \quad \text{and all} \quad i \geq 1. \tag{8.1}$$

Though there may exist several linear orders such that (8.1) holds, we assume in the following that with a Macaulay poset always some fixed linear order $\preceq$ satisfying (8.1) is given; that is, we consider Macaulay posets as triples $(P, \leq, \preceq)$ and speak of Macaulay posets P *with associated linear orders* $\preceq$. The first proposition contains an equivalent definition of Macaulay posets:

Proposition 8.1.1. *Suppose that there is given a linear order $\preceq$ on the ranked poset P. Then (8.1) holds iff a compressed subset has minimum shadow with respect to all subsets of the same size; that is,*

$$|\Delta(\mathcal{C}F)| \leq |\Delta(F)| \quad \textit{for all } F \subseteq N_i, \quad i \geq 1, \tag{8.2}$$

and the shadow of every compressed subset is compressed; that is,

$$\Delta(\mathcal{C}F) = \mathcal{C}(\Delta(\mathcal{C}F)) \quad \textit{for all } F \subseteq N_i, \quad i \geq 1. \tag{8.3}$$

Proof. Assume that (8.1) holds. Then (8.2) is trivial. Moreover, since $\mathcal{C}$ is idempotent,

$$\Delta(\mathcal{C}F) = \Delta(\mathcal{C}\mathcal{C}F) \subseteq \mathcal{C}(\Delta(\mathcal{C}F)),$$

and (8.3) holds since $\Delta(\mathcal{C}F)$ and $\mathcal{C}(\Delta(\mathcal{C}F))$ have the same size by definition.

Now assume that (8.2) and (8.3) hold. Then

$$\Delta(\mathcal{C}F) = \mathcal{C}(\Delta(\mathcal{C}F)) \subseteq \mathcal{C}(\Delta(F))$$

since by (8.2) $\Delta(\mathcal{C}F)$, that is, $\mathcal{C}(\Delta(\mathcal{C}F))$ also has no more elements than $\mathcal{C}(\Delta(F))$.

■

The following fact was observed by Bezrukov in [55].

Proposition 8.1.2. *If P is a Macaulay poset of rank n, then so is its dual P^*.*

Proof. If $\preceq$ is the associated linear order on P, take the dual $\preceq^*$ as the associated linear order for P^*. By definition of the dual, it is enough to prove that

$$\nabla(\mathcal{L}F) \subseteq \mathcal{L}(\nabla(F)) \quad \text{for all } F \subseteq N_i, \quad i = 0, \dots, n-1. \tag{8.4}$$

Let $G := N_{i+1} - \nabla(F)$. Obviously, we have $\Delta(G) \subseteq N_i - F$. Accordingly, by (8.1),

$$\Delta(\mathcal{C}G) \subseteq \mathcal{C}(\Delta(G)) \subseteq \mathcal{C}(N_i - F) = N_i - \mathcal{L}F,$$

and consequently,

$$N_i - \Delta(\mathcal{C}G) \supseteq \mathcal{L}F. \tag{8.5}$$

Furthermore,

$$\nabla(N_i - \Delta(\mathcal{C}G)) \subseteq N_{i+1} - \mathcal{C}G = \mathcal{L}(\nabla(F)). \tag{8.6}$$

Now (8.5) and (8.6) yield (8.4). ■

The following proposition is an immediate consequence of the definition and Proposition 8.1.1.

Proposition 8.1.3. *Let P be a Macaulay poset with associated linear order $\preceq$. Then, for every $F \subseteq N_i$,*

(a) $\Delta_{\to k}(\mathcal{C}F) \subseteq \mathcal{C}(\Delta_{\to k}(F))$ *if* $k \le i$,

(b) $\nabla_{\to k}(\mathcal{L}F) \subseteq \mathcal{L}(\nabla_{\to k}(F))$ *if* $k \ge i$.

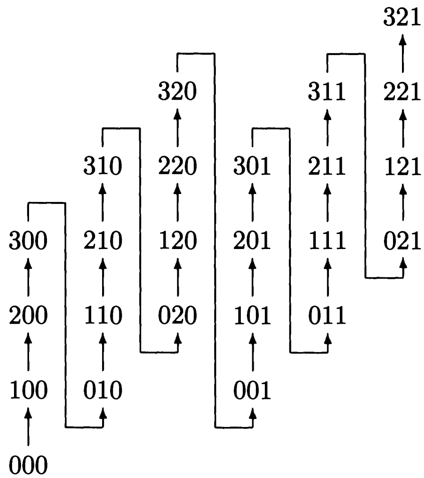
In the following we will show that chain products $S(k_1, \dots, k_n)$ and star products $T(k_1, \dots, k_n)$ are Macaulay posets. Recall that we always suppose $k_1 \ge \cdots \ge k_n$ for $S(k_1, \dots, k_n)$ and $k_1 \le \cdots \le k_n$ for $T(k_1, \dots, k_n)$. For the levels, we use sometimes the notation $N_i(k_1, \dots, k_n)$ to indicate the parameters (from the context it will be clear whether chain products or star products or both are considered). Note that the rank of $S(k_1, \dots, k_n)$ is not n, but $s := k_1 + \cdots + k_n$. First we need the corresponding linear orders $\preceq$. Let $\boldsymbol{a} \ne \boldsymbol{b}$.

In the case of $S(k_1, \dots, k_n)$ we put

$$\boldsymbol{a} \prec \boldsymbol{b} \text{ if } a_j < b_j \text{ where } j \text{ is the largest index with } a_j \ne b_j. \tag{8.7}$$

This is the *reverse lexicographic order on* $S(k_1, \dots, k_n)$. It is illustrated for $S(3, 2, 1)$ in Figure 8.1. For $\boldsymbol{a} \in T(k_1, \dots, k_n)$, we first define the sets $\boldsymbol{a}(l) := \{i \in [n] : a_i = l\}$. Recall that in Section 2.3 we already met the reverse lexicographic order on the subsets of $[n]$:

$$X \prec_{rl} Y \text{ if } \max(X - Y) < \max(Y - X).$$



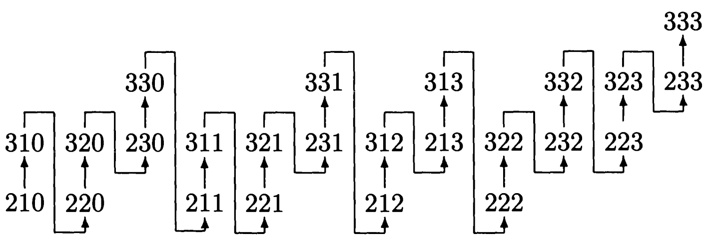
$S(3,2,1)$

Figure 8.1

Now, in the case of $T(k_1, \ldots, k_n)$, we put

$$\boldsymbol{a} \prec \boldsymbol{b} \text{ if } \boldsymbol{b}(l) \prec_{rl} \boldsymbol{a}(l) \text{ where } l \text{ is the smallest number with } \boldsymbol{a}(l) \neq \boldsymbol{b}(l). \qquad (8.8)$$

This is illustrated for $T(1,2,3)$ in Figure 8.2. For $k_1 = \cdots = k_n = 1$, these orders coincide. The following theorem is due to Clements and Lindström [117]



$T(1,2,3)$

Figure 8.2

for $S(k_1, \ldots, k_n)$, essentially to Lindström [345] for $T(2, \ldots, 2)$, to Leeb [336] and Bezrukov [56] for $T(k, \ldots, k)$, $k \geq 2$. Leeb [336] also mentioned without proof that the generalization to $T(k_1, \ldots, k_n)$ is possible. Leck [333] gave an explicit proof for this general case. In an early paper, Kruskal [326] stimulated the study of the cubical poset $Q_n \cong T(2, \ldots, 2)$.

Theorem 8.1.1. *Chain products $S(k_1, \ldots, k_n)$, and star products $T(k_1, \ldots, k_n)$ are Macaulay posets where the associated linear orders are given by (8.7) (resp. (8.8)).*

Proof. This proof is based on the original ideas of Clements and Lindström.

Claim 1. The shadow of every compressed family in N_i, $i \geq 1$, is compressed.

Proof of Claim 1. We use induction on n. The case $n = 1$ is trivial. Let $\boldsymbol{a}$ be the last element of the compressed family F.

First we consider $S(k_1, \ldots, k_n)$. We may suppose $k_n \geq 1$. Let

$$F_1 := \{\boldsymbol{b} \in N_i : b_n < a_n\},$$

$$F_2 := \{(b_1, \ldots, b_{n-1}) \in N_{i-a_n}(k_1, \ldots, k_{n-1}) : (b_1, \ldots, b_{n-1}, a_n) \in F\}.$$

Since F is compressed, we have $F_1 \subseteq F$. Moreover, F_2 is compressed, too. Obviously,

$$\Delta(F_1) = \{\boldsymbol{c} \in N_{i-1} : c_n < a_n\}$$

and

$$\Delta(F) = \Delta(F_1) \cup \{(c_1, \ldots, c_{n-1}, a_n) : (c_1, \ldots, c_{n-1}) \in \Delta(F_2)\}.$$

By the induction hypothesis, $\Delta(F_2)$ is compressed; thus $\Delta(F)$ is also compressed.

Now we consider $T(k_1, \ldots, k_n)$. For any vector $\boldsymbol{b}$ and any number l, let $\boldsymbol{b} \uparrow l$ be the vector that can be obtained from $\boldsymbol{b}$ by deleting all components equal to l. Let $l := \min\{a_j : j \in [n]\}$. Let

$$\begin{aligned} F_1 := \{\boldsymbol{b} \in N_i : \boldsymbol{b}(j) \neq \emptyset \text{ for some } j \in \{0, \ldots, l-1\}\} \\ \cup \{\boldsymbol{b} \in N_i : \boldsymbol{a}(l) \prec_{rl} \boldsymbol{b}(l)\}, \end{aligned}$$

$$\begin{aligned} F_2 := \{\boldsymbol{b} \uparrow l : \boldsymbol{b} \in F \text{ and } \boldsymbol{b}(j) = \emptyset \text{ for all } j \in \{0, \ldots, l-1\} \\ \text{and } \boldsymbol{b}(l) = \boldsymbol{a}(l)\}. \end{aligned}$$

Since F is compressed, $F_1 \subseteq F$. Let $\{j_1, \ldots, j_t\} = [n] - \boldsymbol{a}(l)$, where $j_1 < \cdots < j_t$. Note (for the induction) that $t < n$. The elements of F_2 have the form $(b_{j_1}, \ldots, b_{j_t})$ where $\max\{k_n - k_{j_h}, l + 1\} \leq b_{j_h} \leq k_n$, $h = 1, \ldots, t$. Let $\mu := \max\{k_n - k_{j_t}, l + 1\}$. Let F_2' be the family that can be obtained from F_2 by reducing each component of every element of F_2 by μ. It is easy to see that F_2' is a subset of $N_i(k_n - \max\{k_n - k_{j_1}, l + 1\}, \ldots, k_n - \max\{k_n - k_{j_t}, l + 1\})$, and F_2' is compressed since F is compressed. Obviously,

$$\begin{aligned} \Delta(F_1) := \{\boldsymbol{c} \in N_{i-1} : \boldsymbol{c}(j) \neq \emptyset \quad \text{for some } j \in \{0, \ldots, l-1\}\} \\ \cup \{\boldsymbol{c} \in N_{i-1} : \boldsymbol{a}(l) \prec_{rl} \boldsymbol{b}(l)\} \end{aligned}$$

and

$$\Delta(F) = \Delta(F_1) \cup \{\boldsymbol{c} \in N_{i-1} : \boldsymbol{c}(j) = \emptyset \text{ for all } j \in \{0, \dots, l-1\}$$
$$\text{and } \boldsymbol{c}(l) = \boldsymbol{a}(l) \text{ and } \boldsymbol{c} \uparrow l \in \Delta(F_2)\}. \tag{8.9}$$

The second set on the RHS results from $\Delta(F_2')$ by adding to each component of every element the number μ (which is at least $l+1$) and by introducing the new component l at the jth coordinate with $j \in \boldsymbol{a}(l)$. By the induction hypothesis, $\Delta(F_2')$ is compressed. Now it is not difficult to see that the RHS of (8.9), that is, $\Delta(F)$, is also compressed. □

By Proposition 8.1.1 it is now sufficient to prove (8.2).

Claim 2. The inequality (8.2) is true for $n = 1, 2$.

Proof of Claim 2. The case $n = 1$ is trivial; thus let $n = 2$.

$S(k_1, k_2)$: If $F \subseteq N_i(S(k_1, k_2))$ has the form $F = \{(m, i-m), (m+1, i-m-1), \dots, (p, i-p)\}$ (called *block*) then

$$|\Delta(F)| - |F| = \begin{cases} 1 & \text{if } m > 0, i-p > 0, \\ -1 & \text{if } m = 0, i-p = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8.10}$$

Observe that the difference can only be negative if F is a complete level – that is, if F is compressed. If F is not compressed and not of this special form, we partition F (similarly as A is partitioned in the proof of Theorem 4.6.2) into maximal blocks $B_1, \dots, B_t$. Then the sets $\Delta(B_i), i = 1, \dots, t$, partition $\Delta(F)$, and in view of (8.10)

$$|\Delta(F)| - |F| \ge |\Delta(B_1)| - |B_1| \ge |\Delta(\mathcal{C}F)| - |\mathcal{C}F|.$$

$T(k_1, k_2)$: The only interesting case is $i = 1$. Let $\alpha := |\{\boldsymbol{a} \in F : a_1 = k_2\}|$, $\alpha' := |\{\boldsymbol{a} \in \mathcal{C}F : a_1 = k_2\}|$, $\beta := |\{\boldsymbol{a} \in F : a_2 = k_2\}|$, $\beta' := |\{\boldsymbol{a} \in \mathcal{C}F : a_2 = k_2\}|$. It is easy to see that

$$\begin{aligned} |\Delta(F)| &= k_1\alpha + k_2\beta - \alpha\beta \\ &= k_1k_2 - (k_1 - \beta)(k_2 - \alpha), \\ |\Delta(\mathcal{C}F)| &= k_1k_2 - (k_1 - \beta')(k_2 - \alpha'). \end{aligned}$$

Hence

$$|\Delta(\mathcal{C}F)| \le |\Delta(F)| \text{ iff } (k_1 - \beta')(k_2 - \alpha') \ge (k_1 - \beta)(k_2 - \alpha). \tag{8.11}$$

Note that $\alpha + \beta = \alpha' + \beta'$; that is, $(k_1 - \beta) + (k_2 - \alpha) = (k_1 - \beta') + (k_2 - \alpha')$. We have $\mathcal{C}F = \{(k_2, 0), (k_2, 1), \dots, (k_2, k_2 - k_1), (k_2 - k_1, k_2), (k_2, k_2 - k_1 + 1), (k_2 - k_1 + 1, k_2), \dots\}$. If $\alpha' \le k_2 - k_1 + 1$ and $\beta' = 0$, then (8.11) follows immediately. If $\alpha' > k_2 - k_1 + 1$, then $(k_1 - \beta') - (k_2 - \alpha') \in \{0, 1\}$. A product with integral factors of constant sum attains its maximum if the factors are almost equal; thus (8.11) is proved also in this case. □

Now we prove (8.2) by induction on n. The cases $n = 1, 2$ are settled. So let us look at the step $n - 1 \to n \geq 3$. We still need some preparations. For $F \subseteq N_i$, we put

$$F_{j:d} := \{(a_1, \ldots, a_n) \in F : a_j = d\},$$

$$F_{j|d} := \{(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) \in F_{j:d}\}.$$

Let, for $S(k_1, \ldots, k_n)$ and $d \geq 1$,

$$\Delta_{j:d}(F_{j:d}) := \{(a_1, \ldots, d-1, \ldots, a_n) : (a_1, \ldots, d, \ldots, a_n) \in F_{j:d}\}$$

and, for $T(k_1, \ldots, k_n)$ and $d = k_n$,

$$\Delta_{j:d}(F_{j:d}) := \{(a_1, \ldots, e, \ldots, a_n) : (a_1, \ldots, d, \ldots, a_n) \in F_{j:d},\\ k_n - k_j \leq e \leq k_n - 1\}.$$

In all other cases let

$$\Delta_{j:d}(F_{j:d}) := \emptyset.$$

Furthermore, let

$$\Delta_{j|d}(F_{j:d}) := \{(b_1, \ldots, b_{j-1}, d, b_{j+1}, \ldots, b_n) :\\ (b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n) \in \Delta(F_{j|d})\}.$$

Note that

$$\Delta(F_{j:d}) = \Delta_{j:d}(F_{j:d}) \cup \Delta_{j|d}(F_{j:d}).$$

The set $\mathcal{C}(|F_{j:d}|, (N_i)_{j:d})$ of the first $|F_{j:d}|$ elements of $(N_i)_{j:d}$ is briefly denoted by $\mathcal{C}_{j:d}F_{j:d}$. We say that F is *j-compressed* if, for all possible d,

$$\mathcal{C}_{j:d}F_{j:d} = F_{j:d}$$

(all possible d means $d \in \{0, \ldots, k_j\}$ for $S(k_1, \ldots, k_n)$ and $d \in \{k_n - k_j, \ldots, k_n\}$ for $T(k_1, \ldots, k_n)$). For an element $\boldsymbol{a}$, let $o(\boldsymbol{a})$ be the position of $\boldsymbol{a}$ in the linear order $\preceq$; that is, $\boldsymbol{a}$ is the $o(\boldsymbol{a})$th element in $\preceq$. Further let

$$o(F) := \sum_{\boldsymbol{a} \in F} o(\boldsymbol{a}).$$

Assume that (8.2) is not true. Then let $F \subseteq N_i$ be a family for which

$$|\Delta(\mathcal{C}F)| > |\Delta(F)| \quad \text{and} \quad o(F) \text{ is minimum.} \tag{8.12}$$

Claim 3. F is j-compressed for every j.

Proof of Claim 3. Assume the contrary, that is,

$$F_{j:d^*} \neq \mathcal{C}_{j:d^*} F_{j:d^*}$$

for some j and d^*. Let $G := \Delta(F)$, and define

$$F' := \bigcup_d \mathcal{C}_{j:d} F_{j:d}, \qquad G' := \bigcup_d \mathcal{C}_{j:d} G_{j:d}.$$

Note that $|F'| = |F|$ and $|G'| = |G|$. Obviously, $o(F') < o(F)$. In the following we will show that

$$\Delta(F') \subseteq G'. \tag{8.13}$$

By the choice of F, this will be a contradiction to (8.12) because

$$|\Delta(\mathcal{C}F)| = |\Delta(\mathcal{C}F')| \leq |\Delta(F')| \leq |G'| = |G| = |\Delta(F)|.$$

So let us prove (8.13). We have

$$\Delta_{j|d}(F_{j:d}) \subseteq G_{j:d}.$$

The induction hypothesis (together with Claim 1 and Proposition 8.1.1) yields

$$\Delta_{j|d}(\mathcal{C}_{j:d} F_{j:d}) \subseteq \mathcal{C}_{j:d} G_{j:d}. \tag{8.14}$$

Furthermore we have (recall $G = \Delta(F)$)

$$\Delta_{j:d}(F_{j:d}) \subseteq \begin{cases} G_{j:(d-1)} & \text{for } S(k_1, \ldots, k_n), \quad d \geq 1, \\ \cup_{k_n - k_j \leq e \leq k_n - 1} G_{j:e} & \text{for } T(k_1, \ldots, k_n), \quad d = k_n. \end{cases}$$

Thus also

$$\Delta_{j:d}(\mathcal{C}_{j:d} F_{j:d}) \subseteq \begin{cases} \mathcal{C}_{j:(d-1)} G_{j:(d-1)} & \text{for } S(k_1, \ldots, k_n), \quad d \geq 1, \\ \cup_{k_n - k_j \leq e \leq k_n - 1} \mathcal{C}_{j:e} G_{j:e} & \text{for } T(k_1, \ldots, k_n), \quad d = k_n. \end{cases} \tag{8.15}$$

Now (8.14) and (8.15) imply for $S(k_1, \ldots, k_n)$

$$\Delta(\mathcal{C}_{j:d} F_{j:d}) \subseteq \begin{cases} \mathcal{C}_{j:d} G_{j:d} & \text{if } d = 0, \\ \mathcal{C}_{j:d} G_{j:d} \cup \mathcal{C}_{j:(d-1)} G_{j:(d-1)} & \text{if } d \geq 1, \end{cases}$$

and for $T(k_1, \ldots, k_n)$

$$\Delta(\mathcal{C}_{j:d} F_{j:d}) \subseteq \begin{cases} \mathcal{C}_{j:d} G_{j:d} & \text{if } d < k_n, \\ \mathcal{C}_{j:d} G_{j:d} \cup \left(\cup_{k_n - k_j \leq e \leq k_n - 1} \mathcal{C}_{j:e} G_{j:e}\right) & \text{if } d = k_n. \end{cases}$$

If we take the union over all possible d we obtain

$$\Delta(F') \subseteq G'.$$

□

Claim 4. F is compressed.

Then clearly $|\Delta(\mathcal{C}F)| = |\Delta(F)|$ in contradiction to (8.12), and the induction step is complete.

Proof of Claim 4. Assume that F is not compressed. Let, with respect to $\preceq$, $\boldsymbol{a}$ be the first element of N_i that is not contained in F and $\boldsymbol{e}$ be the last element of F. By our assumption, $\boldsymbol{a} \prec \boldsymbol{e}$. Claim 3 yields that $a_j \neq e_j$ for every j because otherwise $\boldsymbol{e} \in F$ would imply $\boldsymbol{a} \in F$. Let $F^* := (F - \{\boldsymbol{e}\}) \cup \{\boldsymbol{a}\}$. Clearly, $o(F^*) < o(F)$. We will show that

$$|\Delta(F^*)| \leq |\Delta(F)|. \tag{8.16}$$

As in the proof of Claim 3 this yields, by the choice of F, a contradiction to (8.12):

$$|\Delta(\mathcal{C}F)| = |\Delta(\mathcal{C}F^*)| \leq |\Delta(F^*)| \leq |\Delta(F)|.$$

Thus it remains to prove (8.16).

Here we separate the proof into two parts. At first we consider $S(k_1, \ldots, k_n)$.

Claim 5. We have $e_1 > 0$ and $a_1 < k_1$.

Proof of Claim 5. Assume $e_1 = 0$. Let

$$\boldsymbol{f} := \begin{cases} (0, k_2, \ldots, k_{n-1}, i - (k_2 + \cdots + k_{n-1})) & \text{if } k_2 + \cdots + k_{n-1} < i - a_n, \\ (0, f_2, \ldots, f_{n-1}, a_n) & \text{if } k_2 + \cdots + k_{n-1} \geq i - a_n, \end{cases}$$

where in the second case $(f_2, \ldots, f_{n-1})$ is, with respect to $\preceq$, the last element in $N_{i-a_n}(k_2, \ldots, k_{n-1})$. In the first case we have $f_n = i - (k_2 + \cdots + k_{n-1}) \leq e_n$ because $i = e_2 + \cdots + e_{n-1} + e_n \leq k_2 + \cdots + k_{n-1} + e_n$. Here equality holds only if $e_j = k_j$ for all $j \in \{2, \ldots, n-1\}$ – that is, if $\boldsymbol{e} = \boldsymbol{f}$. Thus $\boldsymbol{f} \preceq \boldsymbol{e}$. In the second case we also have $\boldsymbol{f} \preceq \boldsymbol{e}$, because $a_n < e_n$, which follows from $\boldsymbol{a} \prec \boldsymbol{e}$ and $a_n \neq e_n$. Thus, in both cases, $\boldsymbol{f} \in F$, because F is 1-compressed. In the second case we obviously have $\boldsymbol{a} \prec \boldsymbol{f}$, which implies $\boldsymbol{a} \in F$ because F is n-compressed, a contradiction. For the first case, we need a further intermediate element:

$$\boldsymbol{g} := (i - (k_2 + \cdots + k_{n-1} + a_n), k_2, \ldots, k_{n-1}, a_n).$$

Note that $\boldsymbol{g}$ belongs to $S(k_1, \ldots, k_n)$ because $i = a_1 + \cdots + a_{n-1} + a_n \leq k_1 + \cdots + k_{n-1} + a_n$; that is, $i - (k_2 + \cdots + k_{n-1} + a_n) \leq k_1$. It is easy to see that $\boldsymbol{a} \preceq \boldsymbol{g} \preceq \boldsymbol{f}$. It follows that $\boldsymbol{g}, \boldsymbol{a} \in F$ because F is 2- and n-compressed, a contradiction. Thus we know that $e_1 > 0$.

Assume $a_1 = k_1$. We may argue analogously. The intermediate element is this time

$$f := \begin{cases} (k_1, 0, \dots, 0, i - k_1) & \text{if } i - k_1 < e_n, \\ (k_1, f_2, \dots, f_{n-1}, e_n) & \text{if } i - k_1 \geq e_n, \end{cases}$$

where in the second case $(f_2, \dots, f_{n-1})$ is, with respect to $\preceq$, the first element in $N_{i-k_1-e_n}(k_2, \dots, k_{n-1})$, and for the first case we need the further intermediate element

$$\boldsymbol{g} := (i - e_n, 0, \dots, 0, e_n).$$

□

Now we continue the proof of Claim 4.

Case 1. $a_1 = 0$. We will show that $\Delta(F^*) \subseteq \Delta(F)$, which implies (8.16). It is enough to verify that $\Delta(\boldsymbol{a}) \subseteq \Delta(F)$. Thus let $\boldsymbol{a}' := (a_1, \dots, a_j - 1, \dots, a_n) \in \Delta(\boldsymbol{a})$, $j \in \{2, \dots, n\}$. Let $\boldsymbol{a}^* := (a_1 + 1, \dots, a_j - 1, \dots, a_n)$. Clearly, $\boldsymbol{a}^* \prec \boldsymbol{a}$. Since $\boldsymbol{a}$ is the first element not belonging to F, we have $\boldsymbol{a}^* \in F$. Hence $\boldsymbol{a}' \in \Delta(\boldsymbol{a}^*) \subseteq \Delta(F)$.

Case 2. $a_1 > 0$. Let

$$G := (\Delta(F) - \{(e_1 - 1, e_2, \dots, e_n)\}) \cup \{(a_1 - 1, a_2, \dots, a_n)\}.$$

We will show that $\Delta(F^*) \subseteq G$, which implies $|\Delta(F^*)| \leq |G| \leq |\Delta(F)|$. Obviously, $(e_1 - 1, e_2, \dots, e_n) \in \Delta(\boldsymbol{x})$ implies $\boldsymbol{e} \preceq \boldsymbol{x}$. Thus $\boldsymbol{e}$ is the only element of F whose shadow contains $(e_1 - 1, e_2, \dots, e_n)$, and therefore it is again enough to verify that $\Delta(\boldsymbol{a}) \subseteq \Delta(F) \cup \{(a_1 - 1, a_2, \dots, a_n)\}$. But this follows from the arguments in Case 1. □

Now we consider $T(k_1, \dots, k_n)$. Let l be the smallest number with $\boldsymbol{a}(l) \neq \boldsymbol{e}(l)$. Then $\boldsymbol{e}(l) \prec_{rl} \boldsymbol{a}(l)$, since $\boldsymbol{a} \prec \boldsymbol{e}$. Let j be the largest index such that $a_j = l \neq e_j$. Then, $a_{j+1}, \dots, a_n, e_{j+1}, \dots, e_n$ belong to $\{l + 1, \dots, k_n\}$ because of $a_{j+1} \neq e_{j+1}, \dots, a_n \neq e_n$.

Case 1. $e_s = k_n$ for some s, $s \neq j$.

We will show that $\Delta(F^*) \subseteq \Delta(F)$, which implies (8.16). It is enough to verify that $\Delta(\boldsymbol{a}) \subseteq \Delta(F)$. Let $\boldsymbol{a}' := (a_1, \dots, a_{r-1}, \alpha, a_{r+1}, \dots, a_n) \in \Delta(\boldsymbol{a})$ where $a_r = k_n$. Note that $s \neq r$ and that $a_s \neq k_n$. Let $\boldsymbol{a}^* = (a_1^*, \dots, a_n^*)$ where

$$a_h^* := \begin{cases} k_n & \text{if } h = s, \\ \alpha & \text{if } h = r, \\ a_h & \text{otherwise.} \end{cases}$$

We have $\boldsymbol{a}^* \prec \boldsymbol{e}$ since $\boldsymbol{e}(u) = \boldsymbol{a}^*(u)$ for $u \in \{0, \dots, l - 1\} - \{\alpha\}$ and in the case $\alpha < l$, $\boldsymbol{e}(\alpha) \prec_{rl} \boldsymbol{a}^*(\alpha)$, whereas in the case $\alpha \geq l$, $\boldsymbol{e}(l) \prec_{rl} \boldsymbol{a}^*(l)$. Since F is s-compressed and $a_s^* = e_s = k_n$, it follows that $\boldsymbol{a}^* \in F$. Thus $\boldsymbol{a}' \in \Delta(\boldsymbol{a}^*) \subseteq \Delta(F)$.

Case 2. $e_s < k_n$ for all $s \neq j$. Since we are working at least in the first level, that is, $i \geq 1$, we have $e_j = k_n$ and $i = 1$. Let $a_r = k_n, r \neq j$. Obviously,

$$|\Delta(F)| = |\Delta((F - \{e\}) \cup \{e\})| = |\Delta(F - \{e\})| + |\Delta(e) - \Delta(F - \{e\})|,$$

$$|\Delta(F^*)| = |\Delta((F - \{e\}) \cup \{a\})| = |\Delta(F - \{e\})| + |\Delta(a) - \Delta(F - \{e\})|.$$

Thus, in order to prove (8.16), we only must verify that

$$|\Delta(a) - \Delta(F - \{e\})| \leq |\Delta(e) - \Delta(F - \{e\})|. \tag{8.17}$$

Let $\mu(x) := \max\{x_h : h \in [n] - \{r, j\}\}$.

Claim 6. We have $|\Delta(a) - \Delta(F - \{e\})| \leq k_n - \mu(a)$.

Proof of Claim 6. It is enough to show that $a' := (a_1, \dots, a_{r-1}, \alpha, a_{r+1}, \dots, a_n) \in \Delta(F - \{e\})$ for $\alpha \in \{0, \dots, \mu(a) - 1\}$. Let $a_s = \mu(a)$ (note $s \neq r$), and define a^* by

$$a^*_h := \begin{cases} k_n & \text{if } h = s, \\ \alpha & \text{if } h = r, \\ a_h & \text{otherwise.} \end{cases}$$

Since a and a^* coincide in all but two components and $\alpha < \mu(a)$, it follows that $a^* \prec a$. By the choice of a, $a^* \in F - \{e\}$. Consequently, $a' \in \Delta(a^*) \subseteq \Delta(F - \{e\})$.

□

Claim 7. We have $|\Delta(e) - \Delta(F - \{e\})| \geq k_n - \mu(e) - 1$.

Proof of Claim 7. It is enough to show that $e' := (e_1, \dots, e_{j-1}, \epsilon, e_{j+1}, \dots, e_n) \notin \Delta(F - \{e\})$ for $\epsilon \in \{\mu(e) + 1, \dots, k_n - 1\}$. Thus assume that $e' \in \Delta(e^*)$ for some $e^* \in F - \{e\}$ where

$$e^*_h := \begin{cases} k_n & \text{if } h = s, \\ \epsilon & \text{if } h = j, \\ e_h & \text{otherwise.} \end{cases}$$

Since e and e^* coincide in all but two components and $\epsilon > \mu(e)$, it follows that $e \prec e^*$. By the choice of e, $e^* \notin F$, a contradiction. □

By Claim 6 and Claim 7, (8.17) is proved if we can show that

$$k_n - \mu(e) - 1 \geq k_n - \mu(a),$$

that is,

$$\mu(a) \geq \mu(e) + 1.$$

Assume the contrary. Then there must be some index $s \in [n] - \{r, j\}$ such that

$$e_s = \mu(\boldsymbol{e}) \geq \mu(\boldsymbol{a}) \geq a_s.$$

Let $\hat{\boldsymbol{e}}$ be that element which can be obtained from $\boldsymbol{e}$ by replacing the sth component e_s by a_s. Then clearly $\hat{\boldsymbol{e}} \prec \boldsymbol{e}$, and since F is j-compressed, $\hat{\boldsymbol{e}} \in F$. Obviously, $\hat{\boldsymbol{e}}(0) = \boldsymbol{a}(0), \ldots, \hat{\boldsymbol{e}}(l-1) = \boldsymbol{a}(l-1)$. Moreover, it is easy to verify that $\hat{\boldsymbol{e}}(l) \prec_{rl} \boldsymbol{a}(l)$. Thus $\boldsymbol{a} \prec \hat{\boldsymbol{e}}$. Since F is s-compressed, $\hat{\boldsymbol{e}} \in F$ implies $\boldsymbol{a} \in F$, a contradiction. □

Now the proof of Claim 4 for $T(k_1, \ldots, k_n)$, that is, the whole proof of the theorem, is complete. ■

In the case of chain products, Theorem 8.1.1 is known as the *Clements–Lindström Theorem*. In the following we discuss some immediate consequences of Theorem 8.1.1. Let $Mon(n)$ be the set of all monomials in the variables $x_1, \ldots, x_n$; thus, each element m of $Mon(n)$ has the form $m = x_1^{a_1} \cdots x_n^{a_n}$, $a_i \in \mathbb{N}$ for all i. $Mon(n)$ becomes a poset by defining $m \leq m'$ if $m | m'$, that is, we have

$$m = x_1^{a_1} \cdots x_n^{a_n} \leq m' = x_1^{a'_1} \cdots x_n^{a'_n} \text{ iff } a_i \leq a'_i \text{ for all } i.$$

Obviously, $Mon(n)$ is isomorphic to the infinite poset $S(\infty, \ldots, \infty)$, where ∞ means that we do not have upper bounds for the components of the n-tuples. The reverse lexicographic order given in (8.7) is also a linear order on $S(\infty, \ldots, \infty)$ (here the "priority" of the variables decreases from x_1 to x_n, but obviously one could also take any other priority order, or permutation, of the variables). Each level N_i of $S(\infty, \ldots, \infty)$ is clearly finite and equal to $N_i(S(k, \ldots, k))$ where k is large enough ($k \geq i$). Moreover, the compression of some $F \subseteq N_i$ considered in $S(\infty, \ldots, \infty)$ is the same as considered in $S(k, \ldots, k)$. Hence:

***Corollary 8.1.1* (Macaulay Theorem [358]).** *The poset $Mon(n)$ is a Macaulay poset.*

A shorter proof of this (probably oldest) result in our (Macaulay–)Sperner theory was obtained by Sperner [437] (to be precise; $Mon(n)^*$ was considered).

Note that the linear order for $T(2, \ldots, 2)$ on $N_0(T(2, \ldots, 2))$ coincides with the linear order for $S(1, \ldots, 1) \cong B_n$. Bezrukov [57] observed that Propositions 8.1.2, 8.1.3, and Theorem 8.1.1 yield (after exchanging 0 and 1):

Corollary 8.1.2. *Let F be a set of points in $\{0, 1\}^n$ and let $face_l(F)$ be the number of l-dimensional faces of the n-dimensional cube which contain at least one element of F. Then, for fixed size of F, $face_l(F)$ attains the minimum if F consists of the first $|F|$ elements with respect to the reverse lexicographic order.*

This result was proved by Bollobás and Radcliffe [78] without the use of Theorem 8.1.1 (for generalizations to $T(k, \dots, k)$ see Bollobás and Leader [75]).

For the next corollary, we first need another interpretation of the dual of $T(k_1, \dots, k_n)$. Suppose we are given a set A of $k_1 + \dots + k_n$ elements ($k_1 \le \dots \le k_n$) that is partitioned into n sets (color classes) A_j of size k_j, $j \in [n]$, respectively. Look at the family of subsets of A which have with each A_j at most one element in common. Without loss of generality, we may assume that $A_j = \{j, n+j, 2n+j, \dots, (k_j - 1)n + j\}$, $j = 1, \dots, n$. Thus our family can be defined by

$$\begin{aligned} Col(k_1, \dots, k_n) := \{X \subseteq [nk_n] :\ & \text{for all } x, x' \in X \text{ and,} \\ & \text{for all } r \in [n], x \equiv x' \equiv r (\operatorname{mod} n) \text{ implies } x = x' \le k_r n\}. \end{aligned}$$

With the inclusion relation, $Col(k_1, \dots, k_n)$ becomes a ranked poset that we call the *poset of colored subsets*. The rank is given by $r(X) = |X|$. The reverse lexicographic order $\prec_{rl}$ is a linear order on $Col(k_1, \dots, k_n)$.

Lemma 8.1.1. *The posets $Col(k_1, \dots, k_n)$ and $T(k_1, \dots, k_n)^*$ are isomorphic.*

Proof. We define $\varphi : Col(k_1, \dots, k_n) \to T(k_1, \dots, k_n)^*$ by setting $\varphi(X) = \boldsymbol{a}$ where

$$a_j := \begin{cases} k_n & \text{if there is no } x \in X \text{ with } x \equiv j (\operatorname{mod} n), \\ k_n - 1 - \frac{x-j}{n} & \text{if there is some } x \in X \text{ with } x \equiv j (\operatorname{mod} n). \end{cases}$$

It is an easy exercise to verify that $\boldsymbol{a} = \varphi(X)$ is well defined and that φ is an isomorphism. ■

Note that we have for $\boldsymbol{a} = \varphi(X)$ and $l \in \{0, \dots, k_n - 1\}$

$$j \in \boldsymbol{a}(l) \text{ iff } (k_n - 1 - l)n + j \in X.$$

Lemma 8.1.2. *Let φ be the isomorphism introduced in the proof of Lemma 8.1.1, and let $X, Y \in Col(k_1, \dots, k_n)$. Then*

$$X \prec_{rl} Y \textit{ iff } \varphi(Y) \prec \varphi(X)$$

where $\prec$ is defined by (8.8).

Proof. Let $X \prec_{rl} Y$, $\boldsymbol{a} := \varphi(X)$, $\boldsymbol{b} := \varphi(Y)$. It is enough to show that $\boldsymbol{b} \prec \boldsymbol{a}$. Let $y := \max(Y - X)$, $y = (k_n - 1 - q)n + j$, where $j \in [n]$, $q \in \{0, \dots, k_n - 1\}$. Then $j \in \boldsymbol{b}(q)$ but $j \notin \boldsymbol{a}(q)$. We show that for $l < q$, $\boldsymbol{a}(l) = \boldsymbol{b}(l)$. Assume the contrary.

Case 1. $\boldsymbol{a}(l) \prec_{rl} \boldsymbol{b}(l)$ for some $l < q$. Then there is some $t \in \boldsymbol{b}(l) - \boldsymbol{a}(l)$. This implies $(k_n - 1 - l)n + t \in Y - X$. But $(k_n - 1 - l)n + t \ge (k_n - 1 - q + 1)n + t = y + n + t - j > y$, in contrast to the definition of y.

Case 2. $\boldsymbol{b}(l) \prec_{rl} \boldsymbol{a}(l)$ for some $l < q$. In the same way we find an element x in $X - Y$ which is greater than y. This is a contradiction to $X \prec_{rl} Y$.

Finally we show that $\boldsymbol{a}(q) \prec_{rl} \boldsymbol{b}(q)$. We know already that $j \in \boldsymbol{b}(q) - \boldsymbol{a}(q)$. Assume that there is some $h \in \boldsymbol{a}(q) - \boldsymbol{b}(q), h > j$. Then $(k_n - 1 - q)n + h = y - j + h \in X - Y$, but since $y - j + h > y$, we have again a contradiction to $X \prec_{rl} Y$. ■

The following theorem was proved by Frankl, Füredi, and Kalai [197] in the case $k_n - k_1 \leq 1$, without the use of Theorem 8.1.1 (another proof was given by London [352]).

***Corollary 8.1.3* (Colored Kruskal–Katona Theorem).** *For all $F \subseteq N_i(Col(k_1, \ldots, k_n))$, $\Delta(\mathcal{C}F) \subseteq \mathcal{C}(\Delta(F))$, that is, $Col(k_1, \ldots, k_n)$ is a Macaulay poset with $\prec_{rl}$ as the associated linear order.*

Proof. By Lemmas 8.1.1 and 8.1.2 we have (with $\varphi(F) := \{\varphi(X) : X \in F\}$)

$$\Delta(\mathcal{C}F) \subseteq \mathcal{C}(\Delta(F)) \text{ iff } \nabla(\mathcal{L}\varphi(F)) \subseteq \mathcal{L}(\nabla(\varphi(F))),$$

and the last inclusion is true by Theorem 8.1.1 and Proposition 8.1.2. ■

Let P be a Macaulay poset with associated linear order $\preceq$. We say that a family $F \subseteq N_i$ is a *segment* if it consists of elements that are consecutive in N_i with respect to $\preceq$. If $F = \mathcal{C}F$ (resp. $F = \mathcal{L}F$) then F is an *initial* (resp. a *final*) segment. The *shadow function* sf_i assigns with each $F \subseteq N_i$ the number

$$sf_i(F) := |\Delta(\mathcal{C}F)|.$$

Obviously, sf_i depends only on the cardinality of F. We say that the shadow function sf_i is *little-submodular* if

$$sf_i(F_1) + sf_i(F_2) \geq sf_i(F_1 \cap F_2) + sf_i(F_1 \cup F_2) \text{ for all } F_1, F_2 \subseteq N_i$$
$$\text{with } F_1 \cap F_2 = \emptyset \text{ or } F_1 \cup F_2 = N_i. \tag{8.18}$$

Let $F \subseteq N_i$ be a segment and G (resp. H) be the set of elements in N_i that precede (resp. succeed) each element of F with respect to $\preceq$. The *new lower* (resp. *upper*) *shadow of F* is defined by

$$\Delta_{new}(F) := \Delta(F) - \Delta(G) \text{ (resp. } \nabla_{new}(F) := \nabla(F) - \nabla(H)).$$

Note that $\nabla_{new}(F)$ in P equals $\Delta_{new}(F)$ in the dual P^*. Thus we restrict ourselves to the lower shadows. Clearly,

$$|\Delta_{new}(F)| = |\Delta(F \cup G)| - |\Delta(G)|.$$

The shadow function sf_i is called *additive* (cf. Clements [114]) if

$$|\Delta_{new}(F_1)| \geq |\Delta_{new}(F_2)| \geq |\Delta_{new}(F_3)| \tag{8.19}$$

for all segments $F_1, F_2, F_3 \subseteq N_i$ with $|F_1| = |F_2| = |F_3|$ where F_1 is initial and F_3 is final.

Proposition 8.1.4. *The shadow function sf_i is little-submodular iff it is additive.*

Proof. Let sf_i be little-submodular. To prove (8.19), we associate with F_i the sets G_i, $i = 2, 3$, as shown previously. Then the assertion is equivalent to

$$|\Delta(F_1)| \geq |\Delta(F_2 \cup G_2)| - |\Delta(G_2)| \geq |\Delta(F_3 \cup G_3)| - |\Delta(G_3)|.$$

The first inequality follows from (8.18) applied to F_2 and G_2, and the second inequality follows from (8.18) applied to $F_2 \cup G_2$ and $N_i - F_1$ (note that $|N_i - F_1| = |G_3|$ and $|(F_2 \cup G_2) \cap (N_i - F_1)| = |(F_2 \cup G_2) - F_1| = |G_2|$).

Let, conversely, sf_i be additive.

Case 1. $F_1 \cap F_2 = \emptyset$. Put $H_1 := \mathcal{C}F_1$ and $H_2 := \mathcal{C}(F_1 \cup F_2) - \mathcal{C}F_2$. Then $|H_1| = |H_2|$, and by (8.19)

$$|\Delta_{new}(H_1)| \geq |\Delta_{new}(H_2)|,$$

that is,

$$|\Delta(\mathcal{C}F_1)| \geq |\Delta(\mathcal{C}(F_1 \cup F_2))| - |\Delta(\mathcal{C}F_2)|.$$

Case 2. $F_1 \cup F_2 = N_i$. Put $H_1 := \mathcal{C}F_1 - \mathcal{C}(F_1 \cap F_2)$, $H_2 := N_i - \mathcal{C}F_2$. Then $|H_1| = |F_1| - |F_1 \cap F_2| = |F_1 \cup F_2| - |F_2| = |H_2|$, and by (8.19)

$$|\Delta(\mathcal{C}F_1)| - |\Delta(\mathcal{C}(F_1 \cap F_2))| \geq |\Delta(\mathcal{C}(F_1 \cup F_2))| - |\Delta(\mathcal{C}F_2)|.$$

■

We call a Macaulay poset (with associated linear order $\preceq$) *little-submodular* if the shadow functions sf_i are little-submodular for all $i \geq 1$.

Proposition 8.1.5. *If P is a graded little-submodular Macaulay poset, then so is its dual P^*.*

Proof. As in the proof of Proposition 8.1.2 we take the dual $\preceq^*$ of $\preceq$ as the associated linear order for P^*. By Proposition 8.1.4 it is enough to show that

$$|\nabla_{new}(G_1)| \geq |\nabla_{new}(G_2)| \geq |\nabla_{new}(G_3)|$$

for all segments $G_1, G_2, G_3 \subseteq N_i$ with $|G_1| = |G_2| = |G_3|$ where G_1 is final and G_3 is initial, and for all $i \leq r(P) - 1$. Let $|G_1| > 0$ and $F_1 := \nabla_{new}(G_3)$,

$F_2 := \nabla_{new}(G_2)$, $F_3 := \nabla_{new}(G_1)$. It is easy to see that F_1, F_2, and F_3 are segments where F_1 is initial and F_3 is final. Moreover, $\Delta_{new}(F_3) \supseteq G_1$ and if $F_1 \neq \emptyset$ then $\Delta_{new}(F_1) = G_3$. Assume that the first inequality is false, that is $|F_3| < |F_2|$. Let F_2' be the set of the last $|F_3|$ elements of F_2 with respect to $\preceq$. Then $\Delta_{new}(F_2') \subsetneq G_2$ since the first element of F_2 (which does not belong to F_2') is related to some element of G_2. Consequently,

$$|\Delta_{new}(F_2')| < |G_2| = |G_1| \leq |\Delta_{new}(F_3)|,$$

a contradiction to the additivity of the shadow function sf_i. Now assume that the second inequality is false, that is, $|F_1| > |F_2|$. In view of the first inequality, G_2; hence also F_2, cannot be final. Let F_2'' be a segment of size $|F_1|$ containing F_2 and at least the next element p after the last element of F_2 with respect to $\preceq$. This element p must be related to some element q after G_2 because otherwise p is related to some element in G_2 and therefore belongs to $\nabla_{new}(G_2)$. It is easy to see that

$$\Delta_{new}(F_2'') \supseteq G_2 \cup \{q\}.$$

Consequently,

$$|\Delta_{new}(F_1)| = |G_3| < |G_2 \cup \{q\}| \leq |\Delta_{new}(F_2'')|,$$

a contradiction to the additivity of sf_i. ∎

The following result is for $S(k_1, \ldots, k_n)$ and $T(k, \ldots, k)$ mainly due to Clements [106, 115]. In the proof we use an idea of Kleitman (cf. [116, 112]), which significantly simplifies the original proofs of Clements [106, 115].

Theorem 8.1.2. *The Macaulay posets $S(k_1, \ldots, k_n)$, $Col(k_1, \ldots, k_n)$, and $T(k_1, \ldots, k_n)$ are little-submodular.*

Proof. For an n-tuple $\boldsymbol{a}$ and numbers i, j, let $(\boldsymbol{a}, i) := (a_1, \ldots, a_n, i)$ and $(\boldsymbol{a}, i, j) := (a_1, \ldots, a_n, i, j)$.

Case 1. $P = S(k_1, \ldots, k_n)$. We put $Q := S(k_1, \ldots, k_n, 1, 1)$. Let $F_1, F_2 \subseteq N_i(P)$ and define

$$G_1 := \{(\boldsymbol{a}, 1, 0) : \boldsymbol{a} \in \mathcal{C}F_1\},$$

$$G_2 := \{(\boldsymbol{a}, 0, 1) : \boldsymbol{a} \in \mathcal{C}F_2\},$$

$$G_3 := \{(\boldsymbol{a}, 0, 0) : \boldsymbol{a} \in N_{i+1}\}.$$

Then

$$|\Delta(G_1 \cup G_2 \cup G_3)| = |\Delta(G_3)| + |\Delta(\mathcal{C}F_1)| + |\Delta(\mathcal{C}F_2)|. \tag{8.20}$$

Further,

$$
\begin{aligned}
&\mathcal{C}(G_1 \cup G_2 \cup G_3)\\
&\quad = \begin{cases} G_3 \cup \{(\boldsymbol{a}, 1, 0) : \boldsymbol{a} \in \mathcal{C}(F_1 \cup F_2)\} \\ \qquad\qquad\qquad\qquad \text{if } F_1 \cap F_2 = \emptyset, \\ G_3 \cup \{(\boldsymbol{a}, 1, 0) : \boldsymbol{a} \in N_i\} \cup \{(\boldsymbol{a}, 0, 1) : \boldsymbol{a} \in \mathcal{C}(F_1 \cap F_2)\} \\ \qquad\qquad\qquad\qquad \text{if } F_1 \cup F_2 = N_i \end{cases}
\end{aligned}
$$

(note that in the second case $|F_1| + |F_2| = |N_i| + |F_1 \cap F_2|$). Moreover,

$$
\begin{aligned}
&|\Delta(\mathcal{C}(G_1 \cup G_2 \cup G_3))|\\
&\quad = \begin{cases} |\Delta(G_3)| + |\Delta(\mathcal{C}(F_1 \cup F_2))| & \text{if } F_1 \cap F_2 = \emptyset, \\ |\Delta(G_3)| + |\Delta(\mathcal{C}(F_1 \cup F_2))| + |\Delta(\mathcal{C}(F_1 \cap F_2))| & \text{if } F_1 \cup F_2 = N_i. \end{cases}
\end{aligned} \tag{8.21}
$$

Now the little-submodularity of sf_i follows from Theorem 8.1.1, and Equations (8.20) and (8.21).

Case 2. $P = T(k_1, \ldots, k_n)$. We may suppose $k_n > 1$, because otherwise $P = T(1, \ldots, 1) \cong S(1, \ldots, 1)$. We put $Q := T(k_1, \ldots, k_n, k_n)$. Let $F_1, F_2 \subseteq N_i(P)$ and define

$$
\begin{aligned}
G_1 &:= \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in \mathcal{C}F_1\},\\
G_2 &:= \{(\boldsymbol{a}, 1) : \boldsymbol{a} \in \mathcal{C}F_2\}.
\end{aligned}
$$

Then

$$
|\Delta(G_1 \cup G_2)| = |\Delta(\mathcal{C}F_1)| + |\Delta(\mathcal{C}F_2)|. \tag{8.22}
$$

Further,

$$
\begin{aligned}
&\mathcal{C}(G_1 \cup G_2)\\
&\quad = \begin{cases} \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in \mathcal{C}(F_1 \cup F_2)\} & \text{if } F_1 \cap F_2 = \emptyset, \\ \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in N_i\} \cup \{(\boldsymbol{a}, 1) : \boldsymbol{a} \in \mathcal{C}(F_1 \cap F_2)\} & \text{if } F_1 \cup F_2 = N_i. \end{cases}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&|\Delta(\mathcal{C}(G_1 \cup G_2))|\\
&\quad = \begin{cases} |\Delta(\mathcal{C}(F_1 \cup F_2))| & \text{if } F_1 \cap F_2 = \emptyset, \\ |\Delta(\mathcal{C}(F_1 \cup F_2))| + |\Delta(\mathcal{C}(F_1 \cap F_2))| & \text{if } F_1 \cup F_2 = N_i, \end{cases}
\end{aligned} \tag{8.23}
$$

and the little-submodularity of sf_i follows from Theorem 8.1.1, (8.22), and (8.23).

Case 3. $P = Col(k_1, \ldots, k_n)$. This case follows directly from Case 2, Lemmas 8.1.1, 8.1.2, and Proposition 8.1.5. ∎

The following example shows that submodularity (i.e., without the additional condition $F_1 \cap F_2 = \emptyset$ or $F_1 \cup F_2 = N_i$) does not hold. Let $F_j \subseteq B_n \cong S(1, \dots, 1) \cong T(1, \dots, 1) \cong Col(1, \dots, 1)$, $j = 1, 2$, be defined by

$$F_1 := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}, F_2 := \{\{1, 3\}, \{2, 3\}, \{1, 4\}\}.$$

Then

$$3 + 3 = sf_2(F_1) + sf_2(F_2) \not\geq sf_2(F_1 \cap F_2) + sf_2(F_1 \cup F_2) = 3 + 4.$$

One could not expect submodularity since the *Takagi function* as a limit function of a normalization of the function $sf_i(F) - |F|$ is not convex; for more details see Frankl, Matsumoto, Ruzsa, and Tokushige [200].

Finally, we call a Macaulay poset (with associated linear order $\preceq$) *shadow increasing* if

$$sf_i(F_2) \leq sf_{i+1}(F_1) \quad \text{for all } i \geq 1, F_1 \subseteq N_{i+1}, F_2 \subseteq N_i \text{ with } |F_1| = |F_2|. \tag{8.24}$$

Proposition 8.1.6. *Let P be a graded shadow increasing Macaulay poset. Then P is rank unimodal and*

$$sf_l(F_2) \leq sf_u(F_1) \textit{ for all } 1 \leq l < u \leq r(P),$$
$$F_1 \subseteq N_u, F_2 \subseteq N_l \textit{ with } |F_1| = |F_2|.$$

Proof. Assume that there is some h such that $W_{h-1} > W_h \leq W_{h+1}$. Let $F_2 := N_h$ and $F_1 \subseteq N_{h+1}$ such that $|F_1| = |F_2|$. Then, using that P is graded, $\Delta(\mathcal{C}F_2) = N_{h-1}$ and $\Delta(\mathcal{C}F_1) \subseteq N_h$. Consequently, $sf_h(F_2) = W_{h-1} > W_h \geq sf_{h+1}(F_1)$, a contradiction. Now the asserted inequality follows from an iterated application of (8.24). ∎

The following theorem is for S due to Clements [106] and for T and Col due to Leck [334].

Theorem 8.1.3. *The Macaulay posets $S(k_1, \dots, k_n)$, $Col(k_1, \dots, k_n)$, and $T(k_1, \dots, k_n)$ are shadow increasing.*

Proof. We again study the posets separately.

Case 1. $P = S(k_1, \dots, k_n)$. We put $Q := S(k_1, \dots, k_n, 1)$. For $F_1 \subseteq N_{i+1}(P)$ and $F_2 \subseteq N_i(P)$, we define

$$G_1 := \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in \mathcal{C}F_1\},$$

$$G_2 := \{(\boldsymbol{a}, 1) : \boldsymbol{a} \in \mathcal{C}F_2\}.$$

Clearly, $G_1, G_2 \subseteq N_{i+1}(Q)$, and both families are segments where G_1 is initial. Obviously,

$$\Delta_{new}(G_1) = \{(\boldsymbol{b}, 0) : \boldsymbol{b} \in \Delta(\mathcal{C}F_1)\},$$

$$\Delta_{new}(G_2) = \{(\boldsymbol{b}, 1) : \boldsymbol{b} \in \Delta(\mathcal{C}F_2)\}.$$

Theorem 8.1.2 and Proposition 8.1.4 imply for $|F_1| = |F_2|$, that is, $|G_1| = |G_2|$,

$$sf_i(F_2) = |\Delta(\mathcal{C}F_2)| = |\Delta_{new}(G_2)| \le |\Delta_{new}(G_1)| = |\Delta(\mathcal{C}F_1)| = sf_{i+1}(F_1).$$

Case 2. $P = T(k_1, \ldots, k_n)$. We put $Q := T(k_1, \ldots, k_n, k_n)$. For $F_1 \subseteq N_{i+1}(P)$ and $F_2 \subseteq N_i(P)$ with $|F_1| = |F_2|$, we define

$$G_0 := \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in \mathcal{C}F_1\},$$

$$G_{k_n} := \{(\boldsymbol{a}, k_n) : \boldsymbol{a} \in \mathcal{C}F_2\}.$$

Then $G_0, G_{k_n} \subseteq N_{i+1}(Q)$. Let $(\boldsymbol{e}, k_n)$ be the last element of G_{k_n} with respect to $\preceq$. Let, for $i = 1, \ldots, k_n - 1$,

$$G_i := \{(\boldsymbol{a}, i) : (\boldsymbol{a}, i) \preceq (\boldsymbol{e}, k_n)\},$$

$$G'_i := \{\boldsymbol{a} \in P : (\boldsymbol{a}, i) \in G_i\},$$

$$G := \cup_{i=1}^{k_n} G_i.$$

It is easy to see that each G'_i, $i = 1, \ldots, k_n - 1$, is compressed, that G is a segment in $N_{i+1}(Q)$, and that

$$\Delta_{new}(G) \supseteq \{(\boldsymbol{b}, k_n) : \boldsymbol{b} \in \Delta(\mathcal{C}F_2)\} \cup \left(\cup_{i=1}^{k_n-1} \{(\boldsymbol{b}, i) : \boldsymbol{b} \in \Delta(G'_i)\}\right). \quad (8.25)$$

Note that G has size $|F_1| + \sum_{i=1}^{k_n-1} |G_i|$. We partition $\mathcal{C}G$ into consecutive segments $H_0, \ldots, H_{k_n-1}$ such that $|H_0| = |F_1|$ and $|H_i| = |G_i|$, $i = 1, \ldots, k_n - 1$. Then $\mathcal{C}H_0 = G_0$ and, for $i = 1, \ldots, k_n - 1$, $\mathcal{C}H_i = \{(\boldsymbol{a}, 0) : \boldsymbol{a} \in G'_i\}$. Consequently,

$$|\Delta(\mathcal{C}H_0)| = |\Delta(\mathcal{C}F_1)|, \qquad |\Delta(\mathcal{C}H_i)| = |\Delta(G'_i)|, \quad i = 1, \ldots, k_n - 1. \quad (8.26)$$

From (8.25), (8.26), Theorem 8.1.2, and Proposition 8.1.4 we derive

$$\begin{aligned} |\Delta(\mathcal{C}F_1)| + \sum_{i=1}^{k_n-1} |\Delta(G'_i)| &= \sum_{i=0}^{k_n-1} |\Delta(\mathcal{C}H_i)| \\ &\ge |\Delta(\cup_{i=0}^{k_n-1} H_i)| = |\Delta(\mathcal{C}G)| \\ &\ge |\Delta_{new}(G)| \ge |\Delta(\mathcal{C}F_2)| + \sum_{i=1}^{k_n-1} |\Delta(G'_i)| \end{aligned}$$

and finally

$$sf_{i+1}(F_1) = |\Delta(\mathcal{C}F_1)| \geq |\Delta(\mathcal{C}F_2)| = sf_i(F_2).$$

Case 3. $P = Col(k_1, \ldots, k_n)$. We put $Q := Col(k_1, \ldots, k_n, k_n + 1)$, that is, $k_{n+1} := k_n + 1$. Let $\psi : [nk_n] \to [(n+1)(k_n+1)]$ be defined by

$$\psi(qn + r) := q(n+1) + r \quad (r \in [n]).$$

For $X \subseteq [nk_n]$, let $\psi(X) := \{\psi(x) : x \in X\}$. It is easy to verify that $X \in P$ implies $\psi(X) \in Q$. Thus ψ can be also considered as a function $\psi : P \to Q$. For a family F, let $\psi(F) := \{\psi(X) : X \in F\}$. Obviously, for $F \subseteq N_i(P)$,

$$\psi(\Delta(F)) = \Delta(\psi(F)).$$

For $F_1 \subseteq N_{i+1}(P)$, $F_2 \subseteq N_i(P)$, we define

$$G_1 := \{\psi(X) : X \in \mathcal{C}F_1\},$$

$$G_2 := \{\psi(X) \cup \{(n+1)k_{n+1}\} : X \in \mathcal{C}F_2\}.$$

Then $G_1, G_2 \subseteq N_{i+1}(Q)$ and G_2 is a segment. Obviously,

$$\Delta_{new}(G_2) = \{Y \cup \{(n+1)k_{n+1}\} : Y \in \Delta(\psi(\mathcal{C}F_2))\}.$$

Theorems 8.1.1, 8.1.2, and Proposition 8.1.4 imply for $|F_1| = |F_2|$, that is, $|G_1| = |G_2|$,

$$\begin{aligned} sf_i(F_2) &= |\Delta(\mathcal{C}F_2)| = |\Delta(\psi(\mathcal{C}F_2))| = |\Delta_{new}(G_2)| \leq |\Delta_{new}(\mathcal{C}G_1)| \\ &= |\Delta(\mathcal{C}G_1)| \leq |\Delta(G_1)| = |\Delta(\psi(\mathcal{C}F_1))| = |\Delta(\mathcal{C}F_1)| = sf_{i+1}(F_1). \end{aligned}$$

■

8.2. Existence theorems for Macaulay posets

In this section we will mainly characterize the profiles of ideals and antichains in Macaulay posets. Recall that

$$F_i = \{p \in F : r(p) = i\}, \quad f_i = |F_i|.$$

We say that a family $F \subseteq P$ is *compressed* if $\mathcal{C}F_i = F_i$ for all i. Obviously,

$$F \text{ is an ideal iff } \Delta(F_i) \subseteq F_{i-1} \text{ for all } i \geq 1. \tag{8.27}$$

For a natural number l, let $\Delta_i(l)$ be the value of the shadow function of an l-element family in N_i, that is,

$$\Delta_i(l) := |\Delta(\mathcal{C}(l, N_i))|.$$

Theorem 8.2.1. *Let P be a Macaulay poset. The following conditions are equivalent:*

(i) *f is the profile of an ideal in P,*
(ii) *f is the profile of a compressed ideal in P,*
(iii) *$\Delta_i(f_i) \leq f_{i-1}$ for all $i \geq 1$.*

Proof. (i) $\to$ (ii). Let F be an ideal with profile $\boldsymbol{f}$. Let $G := \cup_i \mathcal{C}F_i$. Then G is compressed and has also the profile $\boldsymbol{f}$. Moreover, by (8.1),

$$\Delta(G_i) = \Delta(\mathcal{C}F_i) \subseteq \mathcal{C}(\Delta(F_i)) \subseteq \mathcal{C}F_{i-1} = G_{i-1}, i \geq 1;$$

thus, in view of (8.27), G is an ideal.

(ii) $\leftrightarrow$ (iii). Let $F = \cup_i \mathcal{C}(f_i, N_i)$. Then F is compressed, and F is by (8.27) an ideal iff

$$\Delta(\mathcal{C}(f_i, N_i)) \subseteq \mathcal{C}(f_{i-1}, N_{i-1}) \text{ for all } i \geq 1.$$

Since by Proposition (8.1.1) the shadow of an initial segment is an initial segment, the last inclusion is equivalent to (iii).

(ii) $\to$ (i) is trivial. ■

Now we will derive a generalization of the equivalence (i) $\leftrightarrow$ (ii). Let $(P_1, \leq_1), \ldots, (P_s, \leq_s)$ be Macaulay posets with associated linear orders $\preceq_1, \ldots, \preceq_s$. Let $(Q, \leq) := (P_1, \leq_1) \times \cdots \times (P_s, \leq_s)$. We define on Q a new partial order by

$$(Q, \preceq) := (P_1, \preceq_1) \times \cdots \times (P_s, \preceq_s).$$

For a vector $\boldsymbol{i} = (i_1, \ldots, i_s)$, let

$$N_{\boldsymbol{i}}(Q) := N_{i_1}(P_1) \times \cdots \times N_{i_s}(P_s).$$

If we restrict $\preceq$ to $N_{\boldsymbol{i}}(Q)$, then $N_{\boldsymbol{i}}$ becomes a poset. For a family $F \subseteq Q$ and a vector $\boldsymbol{i}$, let

$$F_{\boldsymbol{i}} := \{\boldsymbol{p} \in F : \boldsymbol{p} \in N_{\boldsymbol{i}}(Q)\}, \; f_{\boldsymbol{i}} := |F_{\boldsymbol{i}}|.$$

The s-dimensional array whose entries are the numbers $f_{\boldsymbol{i}}$ is called the *generalized profile of F*. We say that the family F in Q is *compressed* iff $F_{\boldsymbol{i}}$ is an ideal in $N_{\boldsymbol{i}}(Q)$ for each possible $\boldsymbol{i}$. The following theorem was proved by Björner, Frankl, and Stanley [66] for the case that each P_j has the form $S(\infty, \ldots, \infty)$.

Theorem 8.2.2 (Colored Macaulay Theorem). *The array $(f_{\boldsymbol{i}})$ is the generalized profile of an ideal in Q iff it is the generalized profile of a compressed ideal in Q.*

Proof. It is enough to show that one can construct for each ideal F in Q a compressed ideal with the same generalized profile. Let, as in the proof of Theorem

8.1.1, for $p_j \in P_j$, $o_j(p_j)$ denote the position of p_j in the linear order $\preceq_j$, $j = 1, \dots, s$. Moreover, for $\boldsymbol{p} = (p_1, \dots, p_s) \in Q$ let $o(\boldsymbol{p}) := \sum_{i=1}^{s} o_j(p_j)$ and, for $F \subseteq Q$, $o(F) := \sum_{\boldsymbol{p} \in F} o(\boldsymbol{p})$.

Now, given an ideal F in Q with generalized profile (f_i), let G be an ideal in Q with generalized profile (f_i) for which $o(G)$ is minimum. Such a G exists since F is given. It remains to show that G is compressed. Assume the contrary. Then there are some $\boldsymbol{i}^*$, some $\boldsymbol{u} \in G_{\boldsymbol{i}^*}$ and some $\boldsymbol{v} \in N_{\boldsymbol{i}^*}(Q) - G_{\boldsymbol{i}^*}$ such that $\boldsymbol{v} \prec \boldsymbol{u}$. We may assume, w.l.o.g., that $v_1 \prec_1 u_1$, $v_2 = u_2, \dots, v_s = u_s$. For each $(p_2, \dots, p_n) \in P_2 \times \cdots \times P_n$, let

$$G_{i_1,(p_2,\dots,p_n)} := \{(a_1, p_2 \dots, p_n) \in G : r(a_1) = i_1\},$$

$$G'_{i_1,(p_2,\dots,p_n)} := \{a_1 \in N_{i_1}(P_1) : (a_1, p_2, \dots, p_n) \in G\}.$$

Taking all possible i_1 and $(p_2, \dots, p_n)$ yields a partition of G. We are going to compress each such class individually; that is, let

$$H'_{i_1,(p_2,\dots,p_n)} := \mathcal{C}G'_{i_1,(p_2,\dots,p_n)} \qquad \text{(compression in } N_{i_1}(P_1)),$$

$$H_{i_1,(p_2,\dots,p_n)} := \{(b_1, p_2, \dots, p_n) : b_1 \in H'_{i_1,(p_2,\dots,p_n)}\}.$$

Note that $o(H_{i_1,(p_2,\dots,p_n)}) \leq o(G_{i_1,(p_2,\dots,p_n)})$ and that at least for $i_1 = i_1^*$, $p_2 = u_2, \dots, p_s = u_s$ we have strict inequality. Let

$$H := \bigcup_{i_1} \bigcup_{(p_2,\dots,p_n)} H_{i_1,(p_2,\dots,p_n)}.$$

Then $o(H) < o(G)$. We finally show that H is an ideal, in contradiction to the choice of G (minimality of $o(G)$). So let

$$(p_1, \dots, p_n) \in H_{i_1,(p_2,\dots,p_n)}, \quad (q_1, \dots, q_s) < (p_1, \dots, p_s).$$

Case 1. $q_1 = p_1$. Let $t := o_1(p_1)$. Then $G'_{i_1,(p_2,\dots,p_n)}$ has at least t elements. These elements must belong to $G'_{i_1,(q_2,\dots,q_n)}$ since G is an ideal. Thus $G'_{i_1,(q_2,\dots,q_n)}$ has at least t elements, which implies that $q_1 \in H'_{i_1,(q_2,\dots,q_n)}$ – that is, $(q_1, q_2, \dots, q_n) \in H_{i_1,(q_2,\dots,q_n)} \subseteq H$.

Case 2. $q_1 < p_1$. Then $q_2 = p_2, \dots, q_s = p_s$. Since G is an ideal we have

$$\Delta\big(G'_{i_1,(p_2,\dots,p_n)}\big) \subseteq G'_{i_1-1,(p_2,\dots,p_n)}.$$

The relation (8.1) (P_1 is a Macaulay poset) yields

$$\begin{aligned}\Delta\big(H'_{i_1,(p_2,\dots,p_n)}\big) &= \Delta\big(\mathcal{C}G'_{i_1,(p_2,\dots,p_n)}\big) \subseteq \mathcal{C}(\Delta(G'_{i_1,(p_2,\dots,p_n)}))\\ &\subseteq \mathcal{C}G'_{i_1-1,(p_2,\dots,p_n)} = H_{i_1-1,(p_2,\dots,p_n)}.\end{aligned}$$

Thus $(q_1, p_2, \dots, p_n) \in H_{i_1-1,(p_2,\dots,p_n)} \subseteq H$. ∎

Now we look at antichains in Macaulay posets P of rank n. We say that a Sperner family F in P is *canonically compressed* if each F_i consists of, with respect to $\preceq$, the first f_i elements of N_i which are not in the (lower) shadow of $\cup_{j>i} F_j$. Thus (by induction and Proposition 8.1.1), F is canonically compressed iff $\Delta_{\to i}(F)$ is compressed for every i. The following result is in the case of $S(k_1, \ldots, k_n)$ due to Clements [104], and in the case of B_n to Daykin, Godfrey, and Hilton [125].

Theorem 8.2.3. *Let P be a Macaulay poset of rank n. The following conditions are equivalent:*

(*i*) *f is the profile of a Sperner family in P,*
(*ii*) *f is the profile of a canonically compressed Sperner family in P,*
(*iii*) *for all $i \in \{0, \ldots, n\}$ we have*

$$\Delta_{i+1}(\ldots \Delta_{n-2}(\Delta_{n-1}(\Delta_n(f_n) + f_{n-1}) + f_{n-2}) \cdots + f_{i+1}) + f_i \leq W_i.$$

Proof. (i) $\to$ (ii). We construct for a given Sperner family F with profile f the canonically compressed Sperner family G with profile f as follows: We define

$$G_n := \mathcal{C}(f_n, N_n),$$

and for $i = n-1, n-2, \ldots$

$$G_i := \mathcal{C}(f_i, N_i - \Delta_{\to i}(\cup_{j=i+1}^n G_j)) \tag{8.28}$$

(soon it will be clear that the construction works).

Claim. For all $i = n, n-1, \ldots$ for which the construction (8.28) is still possible (i.e., $f_i \leq |N_i| - |\Delta_{\to i}(\cup_{j=i+1}^n G_j)|$), there holds

(a) $H_i := \Delta_{\to i}(\cup_{j=i+1}^n G_j)$ is compressed,
(b) $H_i \subseteq \mathcal{C}\Delta_{\to i}(\cup_{j=i+1}^n F_j)$.

Proof of Claim. We proceed by induction on $i = n, n-1, \ldots$. The case $i = n$ is trivial (note $H_n = \emptyset$). Thus look at the step $i+1 \to i$.

(a) By the induction hypothesis and construction, $H_{i+1} \cup G_{i+1}$ is compressed. Since $\Delta_{\to i}(G_j) = \Delta(\Delta_{\to i+1}(G_j))$ for $j > i$, it follows that

$$H_i = \Delta(H_{i+1} \cup G_{i+1})$$

and because of Proposition 8.1.1, H_i is compressed.

(b) $\Delta_{\to i+1}(\cup_{j=i+2}^n F_j) \cap F_{i+1} = \emptyset$ since F is a Sperner family. Together with the induction hypothesis this provides

$$H_{i+1} \cup G_{i+1} \subseteq \mathcal{C}(\Delta_{\to i+1}(\cup_{j=i+1}^n F_j)).$$

Thus,

$$\begin{aligned} H_i &= \Delta(H_{i+1} \cup G_{i+1}) \subseteq \Delta(\mathcal{C}(\Delta_{\to i+1}(\cup_{j=i+1}^n F_j))) \\ &\subseteq \mathcal{C}(\Delta(\Delta_{\to i+1}(\cup_{j=i+1}^n F_j)) = \mathcal{C}(\Delta_{\to i}(\cup_{j=i+1}^n F_j)). \end{aligned}$$

Here the second inclusion follows from the fact that P is a Macaulay poset. □

As already mentioned, $F_i \cap \Delta_{\to i}(\cup_{j=i+1}^n F_j) = \emptyset$; hence

$$f_i \leq |N_i| - |\Delta_{\to i}(\cup_{j=i+1}^n F_j)| \leq |N_i| - |H_i|.$$

This shows that G_i can really be constructed for all $i \geq 0$. By construction, G is a Sperner family, and by (a) of the claim, $\Delta_{\to i}(G) = H_i \cup G_i$ is compressed for all $i \geq 0$.

(ii) $\to$ (iii). Induction on $i = n, n-1, \ldots$ easily yields that the LHS equals $|\Delta_{\to i}(F)|$, where F is the canonically compressed Sperner family with profile f.

(iii) $\to$ (ii). The condition says that the construction (8.28) is possible up to $i = 0$.

(ii) $\to$ (i) is trivial. ■

We use the notation $\mathcal{C}F$ for the canonically compressed Sperner family which has the same profile as F. From the proof of Theorem 8.2.3 (in particular Claim (b)) it follows easily that

$$\Delta_{\to i}(\mathcal{C}F) \subseteq \mathcal{C}(\Delta_{\to i}(F)) \text{ for all } i. \tag{8.29}$$

Without going into details we mention some applications in polyhedral combinatorics. Theorem 8.2.1 gives in the case $P = B_n$ a characterization of f-vectors (profiles) of simplicial complexes (Kruskal [325], Katona [292]) and of polyhedral complexes (Wegner [460]). In the case $P = S(\infty, \ldots, \infty)$, Theorem 8.2.1 together with Theorem 8.1.1 permits a characterization of the f-vectors of Cohen–Macaulay complexes (Stanley [438]), and in the case $P = Col(d, \ldots, d)$, these theorems allow a characterization of the f-vectors of $(d-1)$-dimensional completely balanced Cohen–Macaulay complexes (Frankl, Füredi, and Kalai [197]). A structural characterization of the f-vectors of balanced Cohen–Macaulay complexes is given by Theorem 8.2.2 applied to $P_i = S(\infty, \ldots, \infty)$, where the number of components may differ (Björner, Frankl, and Stanley [66]). Finally, in the case $P = B_n$, Theorem 8.2.3 provides a characterization of Betti sequences over some field of some simplicial complex and of some polyhedral complex on at most $n + 1$ vertices (Björner and Kalai [67]). We refer the reader to the surveys of Björner [63, 65], Björner and Kalai [68], and to the books of Stanley [444] and Ziegler [473].

8.3. Optimization problems for Macaulay posets

For $F \subseteq P, q \in P$, we write in the following $F \prec q$ (resp. $q \prec F$) if $p \prec q$ (resp. $q \prec p$) for all $p \in F$. We call a Macaulay poset P *rank greedy* if the associated linear order $\preceq$ is a *linear extension of the ordering* $\leq$ *of* P – that is, if

$$p \leq q \text{ implies } p \preceq q,$$

and if

$$\Delta(p) \prec q, r(p) > r(q) \text{ imply } p \prec q. \tag{8.30}$$

The motivation for this notion comes from the construction of linear extensions. Suppose that we have constructed already a set F of first several elements. The next element $p \in P - F$ must have the property $\Delta(p) \subseteq F$. From all elements with this property we take one with largest rank as the next element.

Proposition 8.3.1. *If P is a rank-greedy Macaulay poset, then so is its dual P^*.*

Proof. As in the proof of Proposition 8.1.2 we take the dual $\preceq^*$ of $\preceq$ as the associated linear order for P^*. Since $\preceq$ is a linear extension of $\leq$, trivially also $\preceq^*$ is a linear extension of $\leq^*$. We have to show that

$$q \prec \nabla(p), r(p) < r(q) \text{ imply } q \prec p.$$

We look for a counterexample (i.e., $q \prec \nabla(p), r(p) < r(q), p \preceq q$) where $r(q) - r(p)$ is minimal. We cannot have $\Delta(q) \prec p$ because otherwise $q \prec p$ by (8.30). Hence there is some q' with $q' \lessdot q$ and $p \preceq q'$. Note that $q' \preceq q$; that is, $q' \prec \nabla(p)$. If $r(q) - r(p) > 1$, we may replace the counterexample (p, q) by the counterexample (p, q'). This is a contradiction to the minimality of $r(q) - r(p)$. If $r(q) - r(p) = 1$, we obtain also a contradiction because the shadow of the initial segment with last element q is an initial segment which contains q' and hence also p (see Proposition 8.1.1). Thus $q \prec \nabla(p)$ is impossible. Consequently, there is no counterexample. ∎

Proposition 8.3.2. *The posets $S(k_1, \ldots, k_n)$, $T(k_1, \ldots, k_n)$, and $Col(k_1, \ldots, k_n)$ are rank greedy Macaulay posets.*

Proof. Obviously, the linear orders $\preceq$ (resp. $\preceq^*$) given in (8.7) and (8.8) are linear extensions of the corresponding posets. Now we prove (8.30) (in Cases 1 and 2) for $\boldsymbol{a} = p$ and $\boldsymbol{b} = q$. Let $\boldsymbol{c}$ be the last element with respect to $\preceq$ which belongs to $\Delta(\boldsymbol{a})$.

Case 1. $P = S(k_1, \ldots, k_n)$. Clearly, for some $m \in [n]$,

$$c_i = \begin{cases} a_i - 1 & \text{if } i = m, \\ a_i & \text{otherwise.} \end{cases}$$

It is easy to see that $a_i = 0$ for all $i < m$ (otherwise $\boldsymbol{c}$ would not be the last element in $\Delta(\boldsymbol{a})$). Since $\boldsymbol{c} \prec \boldsymbol{b}$, there must be some index j such that $c_j < b_j$ and $c_i = b_i$ for $i > j$. If $j > m$ then $\boldsymbol{a} \prec \boldsymbol{b}$ is obvious. If $j \leq m$ then $r(\boldsymbol{a}) - 1 = r(\boldsymbol{c}) = a_m - 1 + \sum_{i=m+1}^{n} a_i = a_m - 1 + \sum_{i=m+1}^{n} b_i < r(\boldsymbol{b})$, a contradiction to $r(\boldsymbol{a}) > r(\boldsymbol{b})$.

Case 2. $P = T(k_1, \ldots, k_n)$. Clearly, for some $m \in [n]$,

$$c_i = \begin{cases} a_i - 1 = k_n - 1 & \text{if } i = m, \\ a_i & \text{otherwise.} \end{cases}$$

It is easy to see that $a_i \neq k_n$ for all $i < m$. Since $\boldsymbol{c} \prec \boldsymbol{b}$, there must be some number l with $\boldsymbol{b}(l) \prec_{rl} \boldsymbol{c}(l), \boldsymbol{b}(i) = \boldsymbol{c}(i)$ for $i < l$. If $l < k_n - 1$ then $\boldsymbol{a} \prec \boldsymbol{b}$ is obvious. If $l = k_n - 1$ then there must be some index j such that $c_j = k_n - 1$, $b_j = k_n$, and $c_i = b_i$ for $i > j$. If $j > m$ then $\boldsymbol{a} \prec \boldsymbol{b}$ is obvious. If $j \leq m$ then $r(\boldsymbol{a}) - 1 = r(\boldsymbol{c}) < r(\boldsymbol{b})$, a contradiction.

Case 3. $P = Col(k_1, \ldots, k_n)$. This case follows directly from Case 2, Lemmas 8.1.1, 8.1.2, and Proposition 8.3.1. ■

For rank-compressed posets, we already derived the inequality

$$\mu_r(F) \leq \mu_r \text{ for every ideal } F \text{ in } P$$

(see Theorem 4.4.1). Now we ask for the maximum of $\mu_r(F)$ over all ideals of given size; that is, we are looking for

$$\max \left\{ \sum_{p \in F} r(p) : F \text{ is an ideal in } P \text{ with } |F| = m \right\}.$$

We will answer this question in a slightly generalized form:

Theorem 8.3.1. *Let P be a rank-greedy Macaulay poset, let $w : \{0, \ldots, r(P)\} \to \mathbb{R}$ be increasing and let $x : P \to \mathbb{R}$ be defined by $x(p) := w(r(p))$, that is, x is constant on the levels of P. Then, for all ideals F in P,*

$$x(F) \leq x(\mathcal{C}(|F|, P)).$$

Thus, taking the first $|F|$ elements in P with respect to the associated linear order $\preceq$ gives the largest weight of an ideal of given size.

Proof. Again, let $o(p)$ denote the position of p in the linear order $\preceq$, and, for $F \subseteq P$, $o(F) := \sum_{p \in F} o(p)$. Of all ideals F of given size for which $x(F)$ is maximum, take such an ideal F^* for which $o(F^*)$ is minimum. We will show that $F^* = \mathcal{C}(|F|, P)$. From Theorem 8.2.1 it follows that F^* is compressed (if we exchange an ideal by a compressed ideal with the same profile, we do not change the weight, but decrease the value $o(F)$). Let a be the first element of

P with respect to $\preceq$ that is not contained in F^*, and let e be the last element of F^*. Assume that $F^* \neq \mathcal{C}(|F|, P)$. Then $a \prec e$. Moreover, $\Delta(a) \subseteq F^* - \{e\}$ and e is a maximal element of F^* since $\preceq$ is a linear extension. Consequently, $F' := (F^* - \{e\}) \cup \{a\}$ is an ideal.

Claim. $r(e) \leq r(a)$.

Using this claim, we derive

$$x(F') = x(F^*) - x(r(e)) + x(r(a)) \geq x(F^*).$$

The relation

$$o(F') = o(F^*) - o(e) + o(a) < o(F^*)$$

yields the desired contradiction.

Proof of Claim. Assume that $r(e) > r(a)$. Let E be the ideal generated by e. Note that $E \subseteq F^*$. For all $f \in E$ with $r(f) = r(a)$, we have $f \prec a$ since otherwise $a \in F^*$ (recall that F^* is compressed). Let e^* be a minimal element of E with the property $a \prec e^*$. By the preceding remarks, $r(e^*) > r(a)$. The choice of e^* yields $\Delta(e^*) \prec a$. By (8.30), $e^* \prec a$, a contradiction. ∎

Several authors contributed to this theorem. Ahlswede and Katona [13] studied B_n and considered also other types of weight functions. Bezrukov and Voronin [60] then gave a generalization to $S(k_1, \ldots, k_n)$. After preparatory work of Kruskal [326] and Lindström [345], Bezrukov [56] settled $T(k, \ldots, k)$. He finally also proved Theorem 8.3.1 in an equivalent formulation [57]. Earlier special weight functions like $w(i) = i$ have been considered (e.g. for $S(k_1, \ldots, k_n)$) by Lindström and Zetterström [347] ($k_1 = \cdots = k_n$) and Clements and Lindström [117]).

This theorem provides a solution of the maximum-edge problem for certain graphs; see p. 40. First let G be the *Hamming graph* of $S(k_1, \ldots, k_n)$; that is, the vertex set V of G equals $S(k_1, \ldots, k_n)$, and we have $\boldsymbol{ab} \in E$ iff $|\{i : a_i \neq b_i\}| = 1$. Recall that for $F \subseteq V$, $E(F) := \{e \in E : e \subseteq F\}$.

Theorem 8.3.2. *For the Hamming graph of $S(k_1, \ldots, k_n)$ and for $0 \leq m \leq (k_1 + 1) \cdots (k_n + 1)$, we have*

$$\max\{|E(F)| : F \subseteq S(k_1, \ldots, k_n), |F| = m\} = |E(\mathcal{C}(m, S(k_1, \ldots, k_n)))|.$$

Thus, the maximum is attained by the set of the first m elements with respect to the reverse lexicographic order.

Proof. We proceed by induction on n. First we will show that we may assume that F is an ideal. Then we will apply Theorem 8.3.1 to prove the assertion. Let

$$F(i) := \{\boldsymbol{a} \in F : a_n = i\},$$

$$F'(i) := \{(a_1, \ldots, a_{n-1}) \in S(k_1, \ldots, k_{n-1}) : (a_1, \ldots, a_{n-1}, i) \in F\}.$$

Obviously, $|F(i)| = |F'(i)|$. Let π be a permutation of $\{0, \ldots, k_n\}$ for which

$$|F(\pi(0))| \geq \cdots \geq |F(\pi(k_n))|.$$

We have

$$\begin{aligned}
|E(F)| &= \sum_{i=0}^{k_n} |E(F(i))| + \sum_{0 \leq i < j \leq k_n} |\{\boldsymbol{ab} \in E : \boldsymbol{a} \in F(i), \boldsymbol{b} \in F(j)\}| \\
&= \sum_{i=0}^{k_n} |E(F'(i))| + \sum_{0 \leq i < j \leq k_n} |F'(i) \cap F'(j)| \\
&\leq \sum_{i=0}^{k_n} |E(F'(i))| + \sum_{0 \leq i < j \leq k_n} \min\{|F'(i)|, |F'(j)|\}
\end{aligned}$$

(note that $\boldsymbol{a} \in F(i), \boldsymbol{b} \in F(j), i \neq j, \boldsymbol{ab} \in E$ imply $(a_1, \ldots, a_{n-1}) = (b_1, \ldots, b_{n-1})$). Now we construct a new family G of the same size m. We use the same notations as for F, thus also $G(i)$ and $G'(i)$. The family G is uniquely determined if $G'(i)$ is given for $i = 0, \ldots, k_n$. We put

$$G'(i) := \mathcal{C}(|F'(\pi(i))|, \; S(k_1, \ldots, k_{n-1})).$$

Then, as above (and since $G'(i)$ and $G'(j)$ are related by inclusion),

$$E(G) = \sum_{i=0}^{k_n} |E(G'(i))| + \sum_{0 \leq i < j \leq k_n} \min\{|G'(i)|, |G'(j)|\},$$

and the induction hypothesis together with $|G'(i)| = |F'(\pi(i))|$ gives

$$|E(G)| \geq |E(F)|.$$

The sets $G'(i)$ are ideals by construction (recall that $\preceq$ is a linear extension; that is, the first elements of $S(k_1, \ldots, k_n)$ with respect to $\preceq$ always form an ideal). Moreover, $|G'(0)| \geq \cdots \geq |G(k_n)|$; that is $(a_1, \ldots, a_{n-1}, i+1) \in G$ implies also $(a_1, \ldots, a_{n-1}, i) \in G$. Consequently, G is an ideal. For any ideal G, $|E(G)|$ can be written also in the form

$$|E(G)| = \sum_{\boldsymbol{a} \in G} |\{\boldsymbol{b} \in G : \boldsymbol{b} < \boldsymbol{a}, \boldsymbol{ab} \in E(G)\}| = \sum_{\boldsymbol{a} \in G} r(\boldsymbol{a}).$$

Finally, by Theorem 8.3.1 and Proposition 8.3.2, we have $|E(G)| \leq |E(\mathcal{C}(m, S(k_1, \ldots, k_n)))|$. ■

Since the Hamming graph of $S(k_1, \ldots, k_n)$ is regular of degree $k_1 + \cdots + k_n$, Theorem 8.3.2 provides also a solution of the edge-isoperimetric problem; see p. 40. This is a result of Lindsey [343] (see also Clements [103] and Kleitman, Krieger, and Rothschild [307]).

Now let G be the Hasse graph of $T(k_1, \dots, k_n)$, which is the same as the Hasse graph of $Col(k_1, \dots, k_n)$. I prefer the formulation with $T(k_1, \dots, k_n)$ because of a succeeding application. Thus the vertex set of G equals $T(k_1, \dots, k_n)$, and we have $\boldsymbol{ab} \in E$ iff for all but one i, $a_i = b_i$, and for the exceptional i, $a_i = k_n$ and $b_i \neq k_n$ or $a_i \neq k_n$ and $b_i = k_n$.

Theorem 8.3.3. *We have in the case of the Hasse graph of $T(k_1, \dots, k_n)$ for $0 \leq m \leq (k_1 + 1) \cdots (k_n + 1)$,*

$$\max\{|E(F)| : F \subseteq T(k_1, \dots, k_n), |F| = m\} = |E(\mathcal{L}(m, T(k_1, \dots, k_n)))|.$$

Proof. The inductive proof is analogous to the proof of Theorem 8.3.2. With the same notations we get

$$|E(F)| \leq \sum_{i=0}^{k_n} |E(F'(i))| + \sum_{i=0}^{k_n - 1} \min\{|F'(i)|, |F'(k_n)|\}.$$

Let i^* be an index for which $|F(i^*)| \geq |F(i)|$ for all i. We define the new family G by

$$G'(i) := \begin{cases} \mathcal{L}(|F(i)|, T(k_1, \dots, k_{n-1})) & \text{if } i \notin \{i^*, k_n\}, \\ \mathcal{L}(|F(k_n)|, T(k_1, \dots, k_{n-1})) & \text{if } i = i^*, \\ \mathcal{L}(|F(i^*)|, T(k_1, \dots, k_{n-1})) & \text{if } i = k_n. \end{cases}$$

Then

$$\begin{aligned} |E(G)| &= \sum_{i=0}^{k_n} |E(G'(i))| + \sum_{i=0}^{k_n - 1} \min\{|G'(i)|, |G'(k_n)|\} \\ &= \sum_{i=0}^{k_n} |E(G'(i))| + \sum_{i=0}^{k_n - 1} |G'(i)| \geq |E(F)|. \end{aligned}$$

The sets $G'(i)$ are filters by construction. Moreover, $|G'(i)| \leq |G'(k_n)|$, that is, $(a_1, \dots, a_{n-1}, i) \in G$ implies $(a_1, \dots, a_{n-1}, k_n) \in G$. Consequently, G is a filter in $T(k_1, \dots, k_n)$, that is, an ideal in $T(k_1, \dots, k_n)^*$. Now we may conclude as in the proof of Theorem 8.3.2. ■

Finally, let G be the Hasse graph of $S(k_1, \dots, k_n)$. Thus we have for $\boldsymbol{a}, \boldsymbol{b} \in V$ that $\boldsymbol{ab} \in E$ iff $\sum_{i=1}^{n} |a_i - b_i| = 1$. In order to formulate the result we first look at a bijection φ between $S(k_1, \dots, k_n)$ and $T(k_n, \dots, k_1)$ given by

$$\varphi(a_1, \dots, a_n) := (k_1 - a_n, \dots, k_1 - a_2, k_1 - a_1).$$

The following assertions are easy to prove:

Lemma 8.3.1. *We have:*

(*a*) *If* $\varphi(\boldsymbol{a}) \geq \varphi(\boldsymbol{b})$, *then* $\boldsymbol{a} \leq \boldsymbol{b}$.

(*b*) *If* F *is an ideal in* $S(k_1, \dots, k_n)$, *then* $\varphi(F)$ *is a filter in* $T(k_n, \dots, k_1)$.

(*c*) *If* $\boldsymbol{a} \leq \boldsymbol{b}$ *then* $\varphi(\boldsymbol{b}) \preceq \varphi(\boldsymbol{a})$ (*where* $\prec$ *is given by* (8.8)).

(*d*) $\varphi^{-1}(\mathcal{L}(m, T(k_n, \dots, k_1)))$ *is an ideal in* $S(k_1, \dots, k_n)$ *for each* m.

Further, we will need the following lemma:

Lemma 8.3.2. *Let* $a_0, \dots, a_k$ *be real numbers and let* π *be a permutation of* $\{0, \dots, k\}$ *such that* $a_{\pi(0)} \geq \cdots \geq a_{\pi(k)}$. *Then*

$$\sum_{i=1}^{k} \min\{a_{i-1}, a_i\} \leq \sum_{i=1}^{k} \min\{a_{\pi(i-1)}, a_{\pi(i)}\}.$$

Proof. The RHS equals $\sum_{i=1}^{k} a_{\pi(i)}$. Let $s_i := \min\{a_{i-1}, a_i\}, i = 1, \dots, k$, and let σ be a permutation of $\{1, \dots, k\}$ such that $s_{\sigma(1)} \geq \cdots \geq s_{\sigma(k)}$. We only must show that $s_{\sigma(j)} \leq a_{\pi(j)}$ holds for $j = 1, \dots, k$. Assume the contrary, that is,

$$s_{\sigma(1)} \geq \cdots \geq s_{\sigma(j)} > a_{\pi(j)} \text{ for some } j.$$

In the following we consider the sets as multisets; that is, elements may appear repeatedly. Let $S_i := \{a_{i-1}, a_i\}, i = 1, \dots, k$. By our assumption, all elements of $\bigcup_{i=1}^{j} S_{\sigma(i)}$ are greater than $a_{\pi(j)}$. Since $a_{\pi(j)} \geq a_{\pi(j+1)} \geq \cdots$, it follows that $\bigcup_{i=1}^{j} S_{\sigma(i)} \subseteq \{a_{\pi(0)}, \dots, a_{\pi(j-1)}\}$. However, $\bigcup_{i=1}^{j} S_i$ clearly contains at least $j+1$ elements, a contradiction. ■

Theorem 8.3.4. *For the Hasse graph of* $S(k_1, \dots, k_n)$ *and for* $0 \leq m \leq (k_1 + 1) \cdots (k_n + 1)$, *we have*

$$\max\{|E(F)| : F \subseteq S(k_1, \dots, k_n), |F| = m\}$$
$$= |E(\varphi^{-1}(\mathcal{L}(m, T(k_n, \dots, k_1))))|.$$

Proof. We again use the approach and the notations from the proof of Theorem 8.3.2. Here we obtain

$$|E(F)| \leq \sum_{i=0}^{k_n} |E(F'(i))| + \sum_{i=1}^{k_n} \min\{|F'(i-1)|, |F'(i)|\}.$$

Let π be a permutation of $\{0, \dots, k_n\}$ for which

$$|F(\pi(0))| \geq \cdots \geq |F(\pi(k_n))|.$$

We define the new family G by

$$G'(i) := \varphi^{-1}(\mathcal{L}(|F'(\pi(i))|, T(k_{n-1}, \ldots, k_1))).$$

Then

$$|E(G)| = \sum_{i=0}^{k_n} |E(G'(i))| + \sum_{i=1}^{k_n} \min\{|G'(i-1)|, |G'(i)|\}.$$

By the induction hypothesis and Lemma 8.3.2, $|E(G)| \geq |E(F)|$. By Lemma 8.3.1(d) and construction, G is an ideal in $S(k_1, \ldots, k_n)$. We have for any ideal G in $S(k_1, \ldots, k_n)$ in view of Lemma 8.3.1(a) and (b),

$$\begin{aligned} |E(G)| &= \sum_{\boldsymbol{a} \in G} |\{\boldsymbol{b} \in G : \boldsymbol{b} < \boldsymbol{a} \text{ and } \sum_{i=1}^{n} |a_i - b_i| = 1\}|, \\ &= \sum_{\boldsymbol{a} \in G} |\{i \in [n] : a_i > 0\}|, \\ &= \sum_{\alpha \in \varphi(G)} |\{i \in [n] : \alpha_i < k_1\}| = |E(\varphi(G))|. \end{aligned}$$

From Theorem 8.3.3 we derive (noting Lemma 8.3.1(d))

$$\begin{aligned} |E(G)| &= |E(\varphi(G))| \leq |E(\mathcal{L}(m, T(k_n, \ldots, k_1)))| \\ &= |E(\varphi^{-1}(\mathcal{L}(m, T(k_n, \ldots, k_1))))|. \end{aligned}$$

■

In the case $k_1 = \cdots = k_n$ this theorem was proved in a different way by Bollobás and Leader [77]; the general case was settled by Ahlswede and Bezrukov [3]. The preceding proof is extracted from a more general approach, called *Variational Principle* by Bezrukov [55].

The graphs considered in the last two theorems are not regular (if not $k_1 = \cdots = k_n = 1$), hence we cannot derive a solution of the edge-isoperimetric problem. Moreover, this problem is here much more difficult because one has not always a nested structure of solutions; see Bollobás and Leader [77]. It is an interesting phenomenon, however, that omitting the bounds for the components yields an NSS; see [3].

In Theorem 8.3.1 we have already found ideals of given size with maximum weight. Now we study an analogous problem. Which antichain of given size generates an ideal of minimum weight? First we present a structural result that may be applied to S (which is selfdual by the succeeding Proposition 8.4.1) and to the dual pair T and Col. For S, the theorem was obtained by Clements [107], who significantly generalized preliminary results of Kleitman [302] (see Theorem 4.5.3(b)) and Daykin [123].

Theorem 8.3.5. *Let P and P^* be graded, little-submodular, and shadow-increasing Macaulay posets. Let $w : \{0, \ldots, r(P)\} \to \mathbb{R}_+$ be increasing, and let $x : P \to \mathbb{R}_+$ be defined by $x(p) := w(r(p))$; that is, x is constant on the levels of P. Let $0 \leq m \leq d(P)$, and let $\mathfrak{A}(m)$ be the class of all Sperner families in P of size m. Further, let $\mathfrak{A}_1(m)$ be the subclass of Sperner families from $\mathfrak{A}(m)$ that generate ideals of minimum weight. Then there are integers i and $1 \leq a \leq W_i$ such that*

$$\mathcal{C}(a, N_i) \cup (N_{i-1} - \Delta(\mathcal{C}(a, N_i))) \in \mathfrak{A}_1(m).$$

Proof. For any Sperner family F, let $I(F)$ be the ideal generated by F. Let $F \in \mathfrak{A}_1(m)$, and let $F' := \mathcal{C}F$. Recall that by (8.29) for all i

$$\Delta_{\to i}(F') \subseteq \mathcal{C}\Delta_{\to i}(F).$$

This implies

$$x(I(F')) = \sum_i w(i)|\Delta_{\to i}(F')| \leq \sum_i w(i)|\Delta_{\to i}(F)| = x(I(F)).$$

Consequently, F' also belongs to $\mathfrak{A}_1(m)$.

Let $\mathfrak{A}_2(m)$ be the class of all canonically compressed Sperner families from $\mathfrak{A}_1(m)$. We saw earlier that $\mathfrak{A}_2(m)$ is not empty. Let $\mathfrak{A}_3(m)$ be the class of families from $\mathfrak{A}_2(m)$ for which $r(I(F)) = \sum_{p \in I(F)} r(p)$ (r is the rank function) is minimum. For any F, let $l(F) := \min\{i : f_i \neq 0\}$ and $u(F) := \max\{i : f_i \neq 0\}$. We write briefly l and u if F is clear from the context.

Claim 1. For any $F \in \mathfrak{A}_3(m)$, $\Delta_{\to l}(F) = N_l$, or $F \subseteq N_l$ and $\Delta(F) = N_{l-1}$.

Proof of Claim 1. Assume the contrary and let i be the largest index for which $\Delta_{\to i}(F) = N_i$ (possibly $i = -1$). By our assumption, $i < l$. Since $\Delta_{\to i+1}(F)$ is compressed and not equal to N_{i+1}, the last element p of N_{i+1} (with respect to $\preceq$) is not related to any element of F. Let q be any element from F_u. Then $r(q) > r(p)$, since otherwise $F \subseteq N_l$, $\Delta(F) = N_{l-1}$. Now, $F' := (F - \{q\}) \cup \{p\}$ is a Sperner family. Let $F'' := \mathcal{C}F'$. Then

$$\begin{aligned} x(I(F'')) \leq x(I(F')) &= x(I(F)) - x(q) + x(p) \\ &= x(I(F)) - w(r(q)) + w(r(p)) \leq x(I(F)). \end{aligned}$$

Consequently, $F'' \in \mathfrak{A}_2(m)$. But in the same way we derive $r(F'') < r(F)$, and this is a contradiction to $F \in \mathfrak{A}_3(m)$. □

Claim 2. For any $F \in \mathfrak{A}_3(m)$, $u(F) - l(F) \leq 1$.

Proof of Claim 2. Assume the contrary, and let $F \in \mathfrak{A}_3(m)$ with $u - l \geq 2$.

Case 1. $|\Delta(F_u)| < |F_l|$. We will show that there exists a canonically compressed Sperner family F' with parameters

$$f'_i := \begin{cases} 0 & \text{if } i \geq u \text{ or } i < l, \\ |\Delta(F_u)| + f_{u-1} & \text{if } i = u-1, \\ f_i & \text{if } l+1 < i < u-1, \\ f_{l+1} + f_u & \text{if } i = l+1, \\ f_l - |\Delta(F_u)| & \text{if } i = l, \end{cases}$$

where in the case $u - l = 2$ the definition for $i = u - 1 = l + 1$ has to be replaced by $f'_i := f_i + |\Delta(F_u)| + f_u$. We only must convince ourselves that the construction described in (8.28) of the proof of Theorem 8.2.3 can be carried out. F'_{u-1} can be obviously constructed, and up to the level N_{l+2} the construction for F' is the same as for F (if $u - l > 2$). In particular we have (for $u - l > 2$) $\Delta_{\to l+2}(F) = \Delta_{\to l+2}(F')$. Let $X := N_{l+1} - \Delta_{\to l+1}(F)$. In order to see that the construction works until level $l + 1$, we must verify that $|X| \geq f_u$. Assume the contrary, $|X| < f_u$. By the construction of a canonically compressed Sperner family we have $F_l \subseteq \Delta_{new}(X)$ (moreover, we have equality because of Claim 1). Let X' be a compressed subset of F_u of size $|X|$. From Proposition 8.1.4 and Proposition 8.1.6 we conclude

$$|\Delta(F_u)| \geq |\Delta(X')| \geq |\Delta(CX)| = |\Delta_{new}(CX)| \geq |\Delta_{new}(X)| \geq |F_l|.$$

This is a contradiction since in our case $|\Delta(F_u)| < |F_l|$. Finally, we must verify that also F'_l can be constructed. We have by construction $F'_{l+1} = F_{l+1} \cup Y$ ($\cup \Delta(F_u)$ if $u - l = 2$), where Y consists of the next f_u elements after the last element of F_{l+1} (with respect to $\preceq$). The construction works on level N_l if

$$f_l - |\Delta(F_u)| \leq W_l - |\Delta_{\to l}(\cup_{j=l+1}^{u} F'_j)|. \tag{8.31}$$

The RHS equals

$$W_l - |\Delta_{\to l}(\cup_{j=l+1}^{u} F_j)| - |\Delta_{new}(Y)|.$$

Since we could construct F_l, we have

$$f_l \leq W_l - |\Delta_{\to l}(\cup_{j=l+1}^{u} F_j)|$$

(moreover, we have equality because of Claim 1). Thus the RHS of (8.31) is not less than $f_l - |\Delta_{new}(Y)|$. But this number is not less than the LHS of (8.31) since

$$|\Delta_{new}(Y)| \leq |\Delta_{new}(CY)| \leq |\Delta(F_u)|$$

by Proposition 8.1.4 and Proposition 8.1.6. We have (noting Claim 1)

$$x(I(F')) \leq x(I(F)) - f_u w(u) + f_u w(l+1) \leq x(I(F)),$$

and

$$r(I(F')) < r(I(F))$$

since $u > l+1$ and $f_u > 0$. This is a contradiction to $F \in \mathfrak{A}_3(m)$.

Case 2. $|\Delta(F_u)| \geq |F_l|$. Let X be the family of the last f_l elements of $\Delta(F_u)$ (with respect to $\preceq$), and let $Y := \nabla(X) \cap F_u$. We will show that there exists a Sperner family F' with parameters

$$f_i' := \begin{cases} 0 & \text{if } i > u \text{ or } i \leq l, \\ f_u - |Y| & \text{if } i = u, \\ f_{u-1} + f_l & \text{if } i = u-1, \\ f_i & \text{if } l+1 < i < u-1, \\ f_{l+1} + |Y| & \text{if } i = l+1, \end{cases}$$

where in the case $u - l = 2$ the definition for $i = u-1 = l+1$ has to be replaced by $f_i' := f_{u-1} + f_l + |Y|$. It is enough to find a family Z in $N_{l+1} - \Delta_{\to l+1}(F)$ with $|Z| = |Y|$ since then $F' := (F - F_l - Y) \cup X \cup Z$ is a desired family. We have

$$|N_{l+1} - \Delta_{\to l+1}(F)| \geq |\nabla(F_l)|. \tag{8.32}$$

Note that F_l is a final segment in N_l (with respect to $\preceq$) by Claim 1. In view of Proposition 8.1.4 and Proposition 8.1.6 applied to P^*,

$$|\nabla(F_l)| \geq |\nabla(\mathcal{L}X)| \geq |\nabla_{new}(X)|. \tag{8.33}$$

We have

$$\nabla_{new}(X) \supseteq \nabla(X) \cap F_u \tag{8.34}$$

since $\nabla(N_{u-1} - \Delta(F_u)) \cap F_u = \emptyset$, and $\nabla_{new}(X) = \nabla(X) - \nabla(N_{u-1} - \Delta(F_u))$ (note that $N_{u-1} - \Delta(F_u)$ is a final segment). The relations (8.32) to (8.34) imply

$$|N_{l+1} - \Delta_{\to l+1}(F)| \geq |\nabla(X) \cap F_u| = |Y|,$$

and thus the family Z can be found. Let $F'' := \mathcal{C}F'$. We have

$$x(I(F'')) \leq x(I(F')) \leq x(I(F)) - |Y|w(u) + |Y|w(l+1) \leq x(I(F)),$$

and

$$r(I(F'')) = r(I(F')) < r(I(F))$$

since $u > l+1$ and $|Y| > 0$. This is a contradiction to $F \in \mathfrak{A}_3(m)$. □

With Claim 1 and 2 the theorem is proved. ■

We note that, for Theorem 8.3.5, we do not need little-submodularity completely. It is sufficient to suppose the first inequality in (8.19) for P and P^*.

Corollary 8.3.1. *Let P and P^* be graded, little-submodular and shadow-increasing Macaulay posets. Then P and P^* have the Sperner property.*

Proof. Let $m := d(P)$. By Theorem 8.3.5, there exists a Sperner family F of size m such that for some i, $F \subseteq N_{i-1} \cup N_i$, $A := F \cap N_i$ is an initial segment and $B := F \cap N_{i-1}$ is a final segment. It is sufficient to show that $m \leq \max\{W_{i-1}, W_i\}$. Assume the contrary. Then $|\Delta(A)| < |A|$ since otherwise $|F| = |A| + |B| \leq |\Delta(A)| + |B| \leq W_{i-1}$. Analogously, $|\nabla(B)| < |B|$.

Case 1. $|A| \leq |N_i - A|$. Let A' be the set of the first $|A|$ elements of $N_i - A$ and let $F' := F \cup A' - \Delta_{new}(A')$. Then F' is obviously a Sperner family and by Proposition 8.1.4,

$$|F'| = |F| + |A'| - |\Delta_{new}(A')| \geq |F| + |A| - |\Delta_{new}(A)| > |F|,$$

a contradiction.

Case 2. $|A| > |N_i - A|$. Let A' be the set of the last $|N_i - A|$ elements of A and let $F' := (F - A') \cup \Delta_{new}(A')$. Again, F' is a Sperner family and by the second inequality in (8.19),

$$\begin{aligned} |F'| &= |F| + |\Delta_{new}(A')| - |A'| \geq |F| + |\Delta_{new}(N_i - A)| - |N_i - A| \\ &= |F| + |B| - |\nabla(B)| > |F|, \end{aligned}$$

a contradiction. ∎

Theorem 8.3.6. *The numbers i and a in Theorem 8.3.5 are uniquely determined if all weights are positive. We have $i = \min\{j : m \leq W_j\}$ and $a = \min\{b : b + W_{i-1} - |\Delta(\mathcal{C}(b, N_i))| = m\}$.*

Proof. First note that i is well defined by Corollary 8.3.1. Let W_h be the largest Whitney number. Then $i \leq h$. In order to see that a is also well defined, let

$$g(b) := b + W_{i-1} - |\Delta(\mathcal{C}(b, N_i))|.$$

We have $g(0) = W_{i-1} < m \leq g(W_i) = W_i$. Moreover, $g(b+1) - g(b) \leq 1$ since $|\Delta(\mathcal{C}(b, N_i))|$ cannot strictly decrease. Thus there is some b with $g(b) = m$ and hence a is well defined. The Sperner family $F := \mathcal{C}(a, N_i) \cup (N_{i-1} - \Delta(\mathcal{C}(a, N_i)))$ has size m, and we have

$$w(I(F)) = \sum_{j=0}^{i-1} w(j) W_j + w(i) a. \tag{8.35}$$

Let F' be a Sperner family of size m that generates an ideal of minimum weight and that has the form from Theorem 8.3.5:

$$F' = \mathcal{C}(a', N_{i'}) \cup (N_{i'-1} - \Delta(\mathcal{C}(a', N_{i'}))).$$

Claim 1. We have $i' \geq i$.

Proof of Claim 1. Assume the contrary. We consider the $[0, i']$-rank-selected subposet $P_{[0,i']}$. Clearly, F' is a Sperner family in $P_{[0,i']}$. Moreover, $P_{[0,i']}$ satisfies all conditions of Corollary 8.3.1; hence it has the Sperner property. Since $i \leq h$ and because of Proposition 8.1.6, $W_0 \leq \cdots \leq W_{i'}$. Consequently, $m = |F'| \leq W_{i'} < m$, a contradiction. □

Claim 2. We have $i' \leq i$.

Proof of Claim 2. Assume the contrary. Then

$$w(I(F')) = \sum_{j=0}^{i'-1} w(j)W_j + w(i')a' > \sum_{j=0}^{i} w(j)W_j \geq w(I(F)),$$

a contradiction. □

Now we know that $i' = i$, and it remains to show that $a' = a$. Obviously,

$$w(I(F')) - w(I(F)) = w(i)(a' - a).$$

Consequently, $a' \leq a$. But $a' < a$ and the definition of a imply $|F'| \neq m$, a contradiction. ■

We conclude this section with some remarks: Sturmfels, Weismantel, and Ziegler [448] used the preceding results (for S) to bound the cardinality of reduced Gröbner bases of n-dimensional lattices in $\mathbb{Z}^n$. For the Boolean lattice, Labahn [331] determined the maximum size of Sperner families having a lower shadow of exactly m elements in the ith level.

The solution of the vertex-isoperimetric problem for B_n (see Theorem 2.3.3(b)) can be generalized in a natural way to the Hasse graph of chain products. This was proved by Bollobás and Leader [76]. Chvátalová [102] ($n = 2$) and Moghadam [372] solved earlier the related bandwidth problem for this graph (cf. Theorem 2.3.5). Again, see the forthcoming book of Harper and Chavez [261].

8.4. Some further numerical and existence results for chain products

Until the end of this chapter, let $S := S(k_1, \ldots, k_n)$, $s := r(S) = k_1 + \cdots + k_n$. We may assume, w.l.o.g., that $k_n \geq 1$. Let $\preceq$ always be the reverse lexicographic order on S. We recall that the rank-generating function of S is given by

$F(S; x) = \prod_{i=1}^{n}(1 + x + \cdots + x^{k_i})$. For example, in $S(4, 3, 2)$ we have the following Whitney numbers:

$$\begin{aligned}\sum_{i=0}^{9} W_i x^i &= (1+x+x^2+x^3+x^4)(1+x+x^2+x^3)(1+x+x^2) \\ &= 1+3x+6x^2+9x^3+11x^4+11x^5+9x^6+6x^7+3x^8+x^9.\end{aligned}$$

We know from Theorem 8.1.1 that

$$\min\{|\Delta(F)| : F \subseteq N_i, |F| = m\} = |\Delta(\mathcal{C}(m, N_i))|.$$

So it is worthwhile to look for an algorithm that computes $|\Delta(\mathcal{C}(m, N_i))|$. For brevity, we use the notation

$$\binom{j}{i}_S := \begin{cases} W_i(k_1, \ldots, k_j) & \text{if } 1 \le j \le n, 0 \le i \le k_1 + \cdots + k_j, \\ 1 & \text{if } i = j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $S(\infty, \ldots, \infty)$ (resp. $k_1, \ldots, k_j$ sufficiently large), $\binom{j}{i}_S = \binom{j+i-1}{i}$ since $\binom{j}{i}$ counts the i-combinations of a j-element set with repetitions (i-multisets on $[j]$); cf. Stanley [441, p. 15]. Moreover, note that in the case $k_1 = \cdots = k_n = 1$ we obtain the usual binomial coefficients $\binom{j}{i}$. By definition, $\binom{j}{i}_S$ is the coefficient of x^i in the rank-generating function of $S(k_1, \ldots, k_j)$, that is, of $\prod_{l=1}^{j}(1 + x + \cdots + x^{k_l})$. It is easy to see that

$$\binom{j+1}{i}_S = \binom{j}{i}_S + \binom{j}{i-1}_S + \cdots + \binom{j}{i-k_{j+1}}_S, \tag{8.36}$$

which gives together with $\binom{0}{0}_S = 1$ a recursive method for the computation of $\binom{j}{i}_S$.

Algorithm. i-representation of m.
Input: $k_1, \ldots, k_n, i, m > 0$.
Put $l := 0$;
Repeat
    Put $l := l + 1$;
    Let j_l be the largest integer in $[n]$ such that $m \ge \binom{j_l}{i}_S$;
    Put $m := m - \binom{j_l}{i}_S$;
    Put $i := i - 1$
until $m = 0$.
Output: $j_1, j_2, \ldots, j_l$.

After the algorithm terminates, we have a representation of m in the form

$$m = \binom{j_1}{i}_S + \binom{j_2}{i-1}_S + \cdots + \binom{j_l}{i-l+1}_S. \tag{8.37}$$

The following theorem reflects results of Daykin [122, 123], Greene and Kleitman [234], and Clements [109].

Theorem 8.4.1. *Let $1 \le i \le s$ and $1 \le m \le \binom{n}{i}_S$. Then*

(*a*) *The algorithm "i-representation of m" terminates.*

(*b*) *It produces numbers $j_1, \ldots, j_l$ such that $n \ge j_1 \ge j_2 \ge \cdots \ge j_l$, $\binom{j_l}{i-l+1}_S > 0$ and $i \ge l$, and moreover, $j_t = j_{t+1} = \cdots = j_{t+u}$ implies $u < k_{j_t+1}$.*

(*c*) *The size of the shadow of the compressed m-element family in N_i equals $\binom{j_1}{i-1}_S + \binom{j_2}{i-2}_S + \cdots + \binom{j_l}{i-l}_S$.*

Proof. Let $F \subseteq N_i$, $|F| = m$, $F = CF$. If $m = \binom{n}{i}_S$ then $F = N_i$. The algorithm terminates after the first step and obviously $\Delta(F) = \binom{n}{i-1}_S$. So we suppose throughout the contrary case – that is, $m < \binom{n}{i}_S$. We assume that F contains all elements of N_i with last component $0, 1, \ldots, a_n - 1$, but not all elements with last component a_n ($a_n \le k_n$). The number of the first mentioned elements clearly equals $\binom{n-1}{i}_S, \binom{n-1}{i-1}_S, \ldots, \binom{n-1}{i-a_n+1}_S$ (here and in the following these numbers are considered as nonexistent if $a_n = 0$). In the first step (if $a_n > 0$) the algorithm determines $j_1 := n-1$ since $\binom{n-1}{i}_S \le m < \binom{n}{i}_S$. Suppose that we know already that

$$j_1 = \cdots = j_{t-1} = n-1, \qquad 2 \le t \le a_n.$$

Then, in the tth step, the algorithm determines $j_t := n-1$ since

$$m - \left(\binom{n-1}{i}_S + \cdots + \binom{n-1}{i-t+2}_S\right) \ge \binom{n-1}{i-t+1}_S$$

but

$$m - \left(\binom{n-1}{i}_S + \cdots + \binom{n-1}{i-t+2}_S\right) < \binom{n}{i-t+1}_S.$$

The last inequality is true because otherwise in view of (8.36)

$$\begin{aligned} m \ge & \binom{n-1}{i}_S + \cdots + \binom{n-1}{i-t+2}_S \\ & + \binom{n-1}{i-t+1}_S + \cdots + \binom{n-1}{i-t+1-k_n}_S \ge \binom{n}{i}_S. \end{aligned}$$

Thus $\binom{j_1}{i}_S + \cdots + \binom{j_{a_n}}{i-a_n+1}_S$ counts exactly the members of F having first coordinate $0, 1, \ldots,$ or $a_n - 1$. Now we look for the members of F having last coordinate

a_n and classify them with respect to the second last component. We put

$$m' := m - \left(\binom{n-1}{i}_S + \cdots + \binom{n-1}{i-a_n+1}_S\right).$$

If $m' = 0$, we have no members with first coordinate a_n, and the proof for (a) and (b) is complete. In the case $m' > 0$ assume that F contains all elements of N_i ending with $0, a_n,\quad 1, a_n, \ldots,\quad a_{n-1} - 1, a_n$ but not all elements ending with a_{n-1}, a_n ($a_{n-1} \leq k_{n-1}$); that is, $m' < \binom{n-1}{i-a_n}_S$. The number of the first mentioned elements equals $\binom{n-2}{i-a_n}_S, \binom{n-2}{i-a_n-1}_S, \ldots, \binom{n-2}{i-a_n-a_{n-1}+1}_S$. In the next step the algorithm determines $j_{a_n+1} := n-2$ since $\binom{n-2}{i-a_n}_S \leq m' < \binom{n-1}{i-a_n}_S$. Suppose we know already that $j_{a_n+1} = \cdots = j_{a_n+t-1} = n-2$, $2 \leq t \leq a_{n-1}$. Then in the (a_1+t)th step the algorithm determines $j_{a_n+t} := n-2$ since

$$m' - \left(\binom{n-2}{i-a_n}_S + \cdots + \binom{n-2}{i-a_n-t+2}_S\right) \geq \binom{n-2}{i-a_n-t+1}_S$$

but

$$m' - \left(\binom{n-2}{i-a_n}_S + \cdots + \binom{n-2}{i-a_n-t+2}_S\right) < \binom{n-1}{i-a_n-t+1}_S.$$

Again the last inequality is true because otherwise in view of (8.36)

$$\begin{aligned} m' &\geq \binom{n-2}{i-a_n}_S + \cdots + \binom{n-2}{i-a_n-t+2}_S \\ &\quad + \binom{n-2}{i-a_n-t+1}_S + \cdots + \binom{n-2}{i-a_n-t+1-k_{n-1}}_S \\ &\geq \binom{n-1}{i-a_n}_S. \end{aligned}$$

Thus $\binom{j_{a_n+1}}{i-a_n}_S + \cdots + \binom{j_{a_n+a_{n-1}}}{i-a_n-a_{n-1}+1}_S$ counts exactly the members of F having last coordinates $0, a_n,\quad 1, a_n,\ \ldots,\quad a_{n-1}-1, a_n$. Now we may continue in the same way, looking for members of F (which is compressed) having last coordinates a_{n-1}, a_n and classifying with respect to the third last coordinate and so on.

If $F \neq N_i$, $(a_1, a_2, \ldots, a_n)$ is the last element of F and if t is the largest index such that F contains all elements of N_i ending with $a_t, \ldots, a_n$ (note that $t \geq 2$), then induction (the first steps discussed earlier) easily yields the sequence of the numbers j determined by the algorithm:

$$\underbrace{n-1, \ldots, n-1}_{a_n}, \underbrace{n-2, \ldots, n-2}_{a_{n-1}}, \ldots, \underbrace{t-1, \ldots, t-1}_{a_t+1}.$$

Thus (a) and (b) are proved.

For (c), it is sufficient to observe that $\Delta(F)$ contains exactly all elements of N_{i-1} with last coordinate $0, 1, \ldots, a_n - 1$, with last two coordinates $0, a_n,\quad 1, a_n,\ \ldots,$

$a_{n-1} - 1, a_n$, and so on; thus

$$\binom{n-1}{i-1}_S + \cdots + \binom{n-1}{i-a_n}_S + \binom{n-2}{i-a_n-1}_S$$
$$+ \cdots + \binom{n-2}{i-a_n-a_{n-1}}_S + \cdots + \binom{t-1}{i-a_n-\cdots-a_t-1}_S$$

elements. ■

We emphasize that with the algorithm one may compute also the last vector of an m-element compressed family in N_i with respect to $\preceq$: The numbers $a_n, \ldots, a_{t+1}, a_t + 1$ can be obtained by counting the numbers j equal to $n-1$, $n-2, \ldots, t-1$. Clearly $(a_1, \ldots, a_{t-1})$ is the last vector in $N_{i-a_n-\cdots-a_t}(k_1, \ldots, k_{t-1})$, and thus we take a_{t-1} as large as possible, then a_{t-2} as large as possible and so on up to a_1.

The representation (8.37) of the number m with the properties of (b) in Theorem 8.4.1 is called (as the algorithm) the *i-representation of m*; see also p. 47. It is not difficult to see that this i-representation is unique (cf. [109]).

Example 8.4.1. *Computation of the size of the shadow.*

Consider $N_7(4,3,2)$ and $m := 5$. Note that $F = \{430, 421, 331, 412, 322\}$ is compressed and $\Delta(F) = \{420, 330, 411, 321, 231, 402, 312, 222\}$. The Whitney numbers $\binom{j}{i}_S$ are given in the following table.

i	$j=3$	$j=2$	$j=1$
0	1	1	1
1	3	2	1
2	6	3	**1**
3	9	4	$\boxed{\mathbf{1}}$
4	11	4	$\boxed{\mathbf{1}}$
5	11	**3**	$\boxed{0}$
6	9	$\boxed{\mathbf{2}}$	0
7	6	$\boxed{1}$	0

The algorithm determines $j_1 := 2$, $j_2 := 2$, $j_3 := 1$, $j_4 := 1$, $j_5 := 1$, that is,

$$5 = \binom{2}{7}_S + \binom{2}{6}_S + \binom{1}{5}_S + \binom{1}{4}_S + \binom{1}{3}_S.$$

For the size of the shadow, we obtain

$$8 = \binom{2}{6}_S + \binom{2}{5}_S + \binom{1}{4}_S + \binom{1}{3}_S + \binom{1}{2}_S.$$

An analogous study of the i-representation of a number m for the poset $T(k, \ldots, k)$ was done by Clements [113] and by Leck [333]. The technical details are more involved.

We have now an explicit formula for the minimum size of the shadow, but this formula is sometimes difficult to handle. Thus we will present easier estimates of the shadow. First recall that by the normality of S and in view of Proposition 4.5.2

$$\frac{|\nabla(F)|}{W_{i+1}(S)} \geq \frac{|F|}{W_i(S)} \text{ for all } F \subseteq N_i, \quad i = 0, \ldots, s-1, \tag{8.38}$$

$$\frac{|\Delta(F)|}{W_{i-1}(S)} \geq \frac{|F|}{W_i(S)} \text{ for all } F \subseteq N_i, \quad i = 1, \ldots, s. \tag{8.39}$$

From the rank symmetry and rank unimodality of S (cf. Proposition 5.1.1), we infer immediately

$$|\nabla(F)| \geq |F| \text{ for all } F \subseteq N_i, \quad i \leq \frac{s-1}{2}, \tag{8.40}$$

$$|\Delta(F)| \geq |F| \text{ for all } F \subseteq N_i, \quad i \geq \frac{s+1}{2}. \tag{8.41}$$

But Theorem 8.1.1 gives for "small" sizes of F a "best" normalized matching inequality:

Corollary 8.4.1. *Let $F \subseteq N_i(S(k_1, \ldots, k_n))$ and let $\boldsymbol{a} = (a_1, \ldots, a_n) \in S$ be the first vector in S such that $a_1 \geq \cdots \geq a_n$ and $|F| \leq W_i(S(a_1, \ldots, a_n))$. Then*

$$|\Delta(F)| \geq \frac{W_{i-1}(S(a_1, \ldots, a_n))}{W_i(S(a_1, \ldots, a_n))}|F|.$$

Proof. We must show that CF belongs to $S(a_1, \ldots, a_n)$ since then $\Delta(CF)$ belongs to $S(a_1, \ldots, a_n)$, too, and the assertion follows from Theorem 8.1.1 and (8.39). So assume the contrary. Let i be the largest index such that there is some $\boldsymbol{b} = (b_1, \ldots, b_n) \in CF$ with $b_i > a_i$. Then $CF \not\subseteq S(k_1, \ldots, k_{i-1}, a_i, \ldots, a_n)$; that is, $|F| = |CF| > W_i(S(k_1, \ldots, k_{i-1}, a_i, \ldots, a_n)) \geq W_i(S(a_1, \ldots, a_n))$, a contradiction. ■

Using a shifting technique similar to the proof of Lovász's theorem (Theorem 2.3.1) Björner, Frankl, and Stanley [66] found the following estimation for $S(\infty, \ldots, \infty)$ which we present without proof:

Theorem 8.4.2. *Let $F \subseteq N_i(S(\infty, \ldots, \infty))$, $|F| = \binom{x}{i}$, $x \in \mathbb{R}$, $x \geq i \geq 1$. Then $|\Delta(F)| \geq \binom{x-1}{i-1}$.*

By the way, Leck [333] used independently the shifting technique for another proof of the Clements–Lindström Theorem.

In Theorem 8.3.6 we provided formulas for the determination of i and a which are necessary for the "ideal minimization." For S, we will present an algorithm that computes a. In the case of B_n, Daykin [123] found an explicit formula for the minimum size of an ideal generated by an m-element Sperner family (and, more generally, m-element k-family). Clements [108, 111] also settled the case of S. First we need a representation of a number m' which is similar to the i-representation. Let

$$\begin{bmatrix} j \\ i \end{bmatrix}_S := \binom{j}{i}_S - \binom{j}{i-1}_S.$$

Algorithm. i-difference representation of m'.
Input: $k_1, \ldots, k_n, i, m' > 0$.
Put $l := 0$;
Repeat
  Put $l := l + 1$;
  Let j_l be the largest integer in $[n]$ such that the following conditions hold:
  (i) $m' \geq \begin{bmatrix} j_l \\ i \end{bmatrix}_S$, (ii) $j_l \leq j_{l-1}$ (if $l > 1$), (iii) $j_l < j_{l-k_{j_l+1}}$ (if $l > k_{j_l+1}$).
  Put $m' := m' - \begin{bmatrix} j_l \\ i \end{bmatrix}_S$;
  Put $i := i - 1$
until $m' = 0$.
Output: $j_1, j_2, \ldots, j_l$.

If the algorithm terminates, we have a representation of m' in the form

$$m' = \begin{bmatrix} j_1 \\ i \end{bmatrix}_S + \begin{bmatrix} j_2 \\ i-1 \end{bmatrix}_S + \cdots + \begin{bmatrix} j_l \\ i-l+1 \end{bmatrix}_S. \tag{8.42}$$

Theorem 8.4.3. *Let $0 < w(0) \leq \cdots \leq w(s)$, and let $x : S \to \mathbb{R}$ be defined by $x(\boldsymbol{a}) := w(r(\boldsymbol{a}))$. Let $\binom{n}{i-1}_S < m \leq \binom{n}{i}_S, i \leq \frac{s}{2}$. Then the algorithm i-difference representation of m' terminates for $m' := m - \binom{n}{i-1}_S$, $m' > 0$, and yields a representation (8.42). The minimum weight of an ideal generated by an m-element Sperner family equals*

$$\sum_{j=0}^{i-1} w(j)\binom{n}{j}_S + w(i)\left(\binom{j_1}{i}_S + \cdots + \binom{j_l}{i-l+1}_S\right).$$

Proof. The claim is essentially a reformulation of Theorems 8.3.5 and 8.3.6 (specialized to S). We have to show only that the algorithm terminates and that $a = \binom{j_1}{i}_S + \cdots + \binom{j_l}{i-l+1}_S$. The case $m = \binom{n}{i}_S$ is trivial; thus let $m < \binom{n}{i}_S$. Then, clearly, $j_1 \leq n-1$. Let $\boldsymbol{e}$ be the last element in $\mathcal{C}(a, N_i)$. Moreover, let t be the largest index such that $(e_1, \ldots, e_{t-1})$ is the last element in $N_{i-e_n-\cdots-e_t}(S(k_1, \ldots, k_{t-1}))$. From the proof of Theorem 8.4.1 we know that

$$\underbrace{n-1, \ldots, n-1}_{e_n}, \underbrace{n-2, \ldots, n-2}_{e_{n-1}}, \ldots, \underbrace{t-1, \ldots, t-1}_{e_t+1} \tag{8.43}$$

is the j-sequence for the i-representation of a. For the proof, it is enough to verify that the algorithm i-difference representation of m' determines exactly the same j-sequence. For brevity, we use the notation

$$h(b) := |\mathcal{C}(b, N_i)| - |\Delta(\mathcal{C}(b, N_i))| = b - |\Delta(\mathcal{C}(b, N_i))|.$$

Note that $h(a) = m'$ and a is the smallest natural number with this property. Let us assume that the algorithm i-difference representation of m' determined already the numbers

$$\underbrace{n-1, \ldots, n-1}_{e_n}, \ldots, \underbrace{u-1, \ldots, u-1}_{v}$$

in the right way, that is, as in (8.43) and with $u \geq t$. We have to show that the algorithm terminates in the next step with the number $u-1$ if $u = t$ and $v = e_u$ and that otherwise the next number is

$$\begin{cases} u-1 & \text{if } v < e_u, \\ u-2 & \text{if } v = e_u, u > t, \text{ and } e_{u-1} > 0, \\ u-z & \text{if } v = e_u, u > t, \text{ and } e_{u-1} = \cdots = e_{u-z+2} = 0, e_{u-z+1} > 0. \end{cases}$$

Case 1. $v < e_u$ or $u = t$ and $v = e_u$. Let $(f_1, \ldots, f_{u-1})$ be the last element in $N_{i-e_n-\cdots-e_{u+1}-v}(S(k_1, \ldots, k_{u-1}))$ and let $\boldsymbol{f} := (f_1, \ldots, f_{u-1}, v, e_{u+1}, \ldots, e_n)$. Let $\boldsymbol{f}$ be the bth element in N_i with respect to $\preceq$. Then $b \leq a$ and by definition of a, $h(b) \leq h(a)$. Consequently,

$$\begin{aligned} h(b) = \begin{bmatrix} n-1 \\ i \end{bmatrix}_S + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_S + \cdots + \begin{bmatrix} u-1 \\ i-e_n-\cdots-e_{u+1} \end{bmatrix}_S \\ + \cdots + \begin{bmatrix} u-1 \\ i-e_n-\cdots-e_{u+1}-v \end{bmatrix}_S \leq m'. \end{aligned}$$

Thus, if $v < e_u$ $(\leq k_u)$, the next element in the j-sequence in the algorithm i-difference representation of m' is $u-1$. If $u = t, v = e_u$, we have $\boldsymbol{f} = \boldsymbol{e}$. Furthermore, by definition of t, $e_u < k_u$. Thus the algorithm produces a last number $u-1$, the updated number m' becomes zero, and the algorithm terminates with the right sequence.

Case 2. $v = e_u$ and $u < t$. Then the algorithm still works since we had otherwise a contradiction to the definition of a and $\boldsymbol{e}$. We assume here, w.l.o.g., that $e_{u-1} > 0$ (otherwise we have to replace $u-1$ by $u-z$ and to add in the following vector f correspondingly more zeros). Let $i' := i - e_n - \cdots - e_u$ and let $f = (f_1, \dots, f_{u-2}, 0, e_u, \dots, e_t)$ where $(f_1, \dots, f_{u-2})$ is the last element in $N_{i'}(S(k_1, \dots, k_{u-2}))$. The arguments from Case 1 show also in this case that the next element in the j-sequence in the algorithm i-difference representation of m' is at least $u-2$. We only must prove that the next element cannot be $u-1$. This is clear if $e_u = k_u$ (see condition (iii) in the algorithm). Thus let $e_u < k_u$. Let $g = (g_1, \dots, g_{u-1}, e_u, \dots, e_t)$ where $(g_1, \dots, g_{u-1})$ is the last element in $N_{i'}(S(k_1, \dots, k_{u-1}))$. Let g be the bth element in N_i with respect to $\preceq$. Obviously, $a < b$. Let

$$A := \{(a_1, \dots, a_{u-1}) \in N_{i'}(S(k_1, \dots, k_{u-1})) : \\ (a_1, \dots, a_{u-1}, e_u, \dots, e_n) \in C(a, N_i)\}.$$

It is not difficult to see that

$$h(a) = \begin{bmatrix} n-1 \\ i \end{bmatrix}_S + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_S + \cdots + \begin{bmatrix} u-1 \\ i'+1 \end{bmatrix}_S + |A| - |\Delta(A)| = m',$$

$$h(b) = \begin{bmatrix} n-1 \\ i \end{bmatrix}_S + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_S + \cdots + \begin{bmatrix} u-1 \\ i'+1 \end{bmatrix}_S + \begin{bmatrix} u-1 \\ i' \end{bmatrix}_S.$$

So it remains to show that

$$|A| - |\Delta(A)| < \begin{bmatrix} u-1 \\ i' \end{bmatrix}_S. \tag{8.44}$$

Let $S' := S(k_1, \dots, k_{u-1})$. Note that $\begin{bmatrix} u-1 \\ j \end{bmatrix}_S = \begin{bmatrix} u-1 \\ j \end{bmatrix}_{S'} = W_j(S')$. Since S' is normal we have

$$\frac{|A|}{W_{i'}(S')} \le \frac{|\Delta(A)|}{W_{i'-1}(S')}.$$

This implies

$$|A| - |\Delta(A)| \le |A| \left(1 - \frac{W_{i'-1}(S')}{W_{i'}(S')}\right)$$
$$\begin{cases} \le 0 & \text{if } W_{i'-1}(S') \ge W_{i'}(S'), \\ < W_{i'}(S')\left(1 - \frac{W_{i'-1}(S')}{W_{i'}(S')}\right) = \begin{bmatrix} u-1 \\ i' \end{bmatrix}_S & \text{otherwise.} \end{cases}$$

In the first case we have a contradiction since we assumed that our algorithm worked until $i' + 1$; that is, we have

$$\begin{bmatrix} n-1 \\ i \end{bmatrix}_S + \cdots + \begin{bmatrix} u-1 \\ i'+1 \end{bmatrix}_S \leq m'$$

implying $|A| - |\Delta(A)| \geq 0$, and equality means that the algorithm has already terminated. The second case yields the desired inequality (8.44). ■

We conclude this section with a further existence theorem. For an element $\boldsymbol{a} = (a_1, \ldots, a_n) \in S$, let $\boldsymbol{a}^c := (k_1 - a_1, \ldots, k_n - a_n)$. This new element is called the *complement of* $\boldsymbol{a}$. For $F \subseteq S$, let $F^c := \{\boldsymbol{a}^c : \boldsymbol{a} \in F\}$ (the *complementary family*).

Proposition 8.4.1.

(*a*) *The mapping* $\boldsymbol{a} \mapsto \boldsymbol{a}^c$ *is an isomorphism between* $S(k_1, \ldots, k_n)$ *and its dual; thus chain products are self-dual.*

(*b*) *If* $\boldsymbol{a} \preceq \boldsymbol{b}$ *then* $\boldsymbol{b}^c \preceq \boldsymbol{a}^c$.

We omit the trivial proof. The following theorem of Daykin, Godfrey, and Hilton [125] (for B_n) (resp. of Clements [105] (for S)) says that with each Sperner family in S we may associate a "reflexed" Sperner family. The result had been conjectured by Kleitman and Milner [309].

Theorem 8.4.4. *If f is the profile of a Sperner family in S then so is f', where*

$$f_i' = \begin{cases} 0 & \text{if } i < \frac{s}{2}, \\ f_i & \text{if } i = \frac{s}{2}, \\ f_i + f_{s-i} & \text{if } i > \frac{s}{2}. \end{cases}$$

Proof. We assume s to be even. For s odd, the arguments are analogous. Let m be the largest integer for which $f_{\frac{s}{2}+m} + f_{\frac{s}{2}-m} > 0$. We proceed by induction on m. If $m = 0$, we may take $F' := F$. Now consider the step $m - 1 \to m$ where $1 \leq m \leq s/2$. Let $F^1 := \mathcal{C}F$ (recall that F^1 is the canonically compressed Sperner family with profile f which exists by Theorem 8.2.3). By the construction of F_1 we have

$$(F^1)_{\frac{s}{2}+m} = \mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m}).$$

Let

$$F^2 := (F^1 - \mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m})) \cup \Delta(\mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m})).$$

Since F^1 is a Sperner family, F^2 is a Sperner family, too. Next let us turn to the complements. Let $F^3 := (F^2)^c$. We again replace F^3 by the canonically compressed Sperner family $F^4 := \mathcal{C}F^3$. As for F^1 we have

$$(F^4)_{\frac{s}{2}+m} = \mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m}).$$

Let

$$F^5 := (F^4 - \mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m})) \cup \Delta(\mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m})).$$

For the parameters $f_0^5, f_1^5, \dots, f_s^5$ of the Sperner family F^5, we have

$$f_i^5 = \begin{cases} 0 & \text{if } i < \frac{s}{2} - m + 1 \text{ or } i > \frac{s}{2} + m - 1, \\ f_{s-i} + |\Delta(\mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m}))| & \text{if } i = \frac{s}{2} - m + 1, \\ f_{s-i} + |\Delta(\mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m}))| & \text{if } i = \frac{s}{2} + m - 1, \\ f_{s-i} & \text{if } \frac{s}{2} - m + 1 < i < \frac{s}{2} + m - 1. \end{cases}$$

Application of the induction hypothesis to F^5 gives a Sperner family F^6 with parameters $f_0^6, f_1^6, \dots, f_s^6$ satisfying

$$f_i^6 = \begin{cases} 0 & \text{if } i < \frac{s}{2}, \\ f_i^5 & \text{if } i = \frac{s}{2}, \\ f_i^5 + f_{s-i}^5 & \text{if } i > \frac{s}{2}. \end{cases}$$

These parameters already coincide with the asserted parameters f_i' for $i \notin \{\frac{s}{2}+m-1, \frac{s}{2}+m\}$. For the remaining indices, we have, with $a := |\Delta(\mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m}))| + |\Delta(\mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m}))|$,

$$\begin{aligned} f^6_{\frac{s}{2}+m-1} &= f_{\frac{s}{2}+m-1} + f_{\frac{s}{2}-m+1} + a, \\ f^6_{\frac{s}{2}+m} &= 0. \end{aligned}$$

Let $F^7 := \mathcal{C}F^6$. In F^7 we omit, with respect to $\preceq$, the first a elements of $(F^7)_{\frac{s}{2}+m-1}$ and add the first $f_{\frac{s}{2}-m} + f_{\frac{s}{2}+m}$ elements of $N_{\frac{s}{2}+m}$. This gives a family F^8, which already has the required parameters. We must show only that F^8 is a Sperner family. This is obviously the case if

$$\Delta(\mathcal{C}(f_{\frac{s}{2}-m} + f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m})) \subseteq \mathcal{C}(a, N_{\frac{s}{2}+m-1}). \tag{8.45}$$

But by Theorem 8.1.2,

$$\begin{aligned} |\Delta(\mathcal{C}(f_{\frac{s}{2}-m} + f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m}))| &\le |\Delta(\mathcal{C}(f_{\frac{s}{2}-m}, N_{\frac{s}{2}+m}))| \\ &\quad + |\Delta(\mathcal{C}(f_{\frac{s}{2}+m}, N_{\frac{s}{2}+m}))| = a. \end{aligned}$$

This gives (8.45). So we may indeed take $F' := F^8$. ∎

8.5. Sperner families satisfying additional conditions in chain products

Recall that by the normality of S and because of Theorem 4.5.1, for any Sperner family F in S,

$$\sum_{i=0}^{s} \frac{f_i}{W_i} \leq 1. \tag{8.46}$$

First we study *complement-free Sperner families*; that is, Sperner families for which $\boldsymbol{a} \in F$ implies $\boldsymbol{a}^c \notin F$. In the Boolean case $k_1 = k_n = 1$ we have by the Profile-Polytope Theorem 3.3.1 (note, in particular, Remark 5 following that theorem) a complete overview on LYM-type inequalities for such families. In the general case, we present only one inequality:

***Theorem 8.5.1* (*Clements and Gronau* [116]).** *If $F \subseteq S$ is a complement-free Sperner family with parameters $f_0, f_1, \ldots, f_s$, then*

$$\sum_{\substack{i=0 \\ i \neq \frac{s}{2}}}^{s} \frac{f_i}{W_i} + \frac{f_{\frac{s}{2}}}{W_{\lfloor \frac{s-1}{2} \rfloor}} \leq 1,$$

where $f_{\frac{s}{2}} := 0$ if s is odd.

Proof. If s is odd, the assertion is equivalent to (8.46). Thus let s be even. Let $F \subseteq S$ be a complement-free Sperner family. By Theorem 8.4.4, there exists a Sperner family F^1 whose parameters satisfy

$$f_i^1 = \begin{cases} 0 & \text{if } i < \frac{s}{2}, \\ f_i & \text{if } i = \frac{s}{2}, \\ f_i + f_{s-i} & \text{if } i > \frac{s}{2}. \end{cases}$$

Let $F^2 := (F^1)^c$. For the parameters of F^2, there holds $f_i^2 = f_{s-i}^1$, $i = 0, \ldots, s$. Let $F^3 := CF^2$ be the canonically compressed Sperner family with the same profile as F^2. Since F is complement free we have $f_{s/2}^3 = f_{s/2} \leq \frac{1}{2} W_{s/2}$. Because, moreover, $f_i^3 = f_i^2 = 0$ for $i > \frac{s}{2}$, the subfamily $(F^3)_{s/2}$ consists of the first $f_{s/2}^3$ elements of $N_{s/2}$ (with respect to $\preceq$). This implies that no vector of $(F^3)_{s/2}$ has k_n as last coordinate (more exactly, $k_n, k_n - 1, \ldots, \lceil \frac{k_n+1}{2} \rceil$ all can be excluded). Consequently,

$$(F^3)_{\frac{s}{2}} \cup \Delta((F^3)_{\frac{s}{2}}) \subseteq S(k_1, k_2, \ldots, k_n - 1) =: S'.$$

Since $r(S') = s - 1$, it follows that $\frac{s}{2} > \frac{1}{2} r(S')$, and by (8.41),

$$|\Delta((F^3)_{\frac{s}{2}})| \geq |(F^3)_{\frac{s}{2}}| = f_{\frac{s}{2}}. \tag{8.47}$$

Let $F^4 := (F^3 - (F^3)_{s/2}) \cup \Delta((F^3)_{s/2})$. Then F^4 is again a Sperner family, and its parameters are given by

$$f_i^4 = \begin{cases} f_i + f_{s-i} & \text{if } i \le \frac{s}{2} - 2, \\ f_{\frac{s}{2}-1} + f_{\frac{s}{2}+1} + |\Delta((F^3)_{\frac{s}{2}})| & \text{if } i = \frac{s}{2} - 1, \\ 0 & \text{if } i \ge \frac{s}{2}. \end{cases}$$

By (8.46), (8.47), and in view of the rank symmetry,

$$1 \ge \sum_{i=0}^{\frac{s}{2}-1} \frac{f_i^4}{W_i} = \sum_{\substack{i=0\\ i \ne \frac{s}{2}}}^{s} \frac{f_i}{W_i} + \frac{|\Delta((F^3)_{\frac{s}{2}})|}{W_{\frac{s}{2}-1}} \ge \sum_{\substack{i=0\\ i \ne \frac{s}{2}}}^{s} \frac{f_i}{W_i} + \frac{f_{\frac{s}{2}}}{W_{\lfloor \frac{s-1}{2} \rfloor}}.$$

■

Corollary 8.5.1. *The maximum size of a complement-free Sperner family in S equals $W_{\lfloor \frac{s-1}{2} \rfloor}$.*

Proof. The upper bound follows from Theorem 8.5.1 and the rank symmetry and rank unimodality of S. Clearly, $N_{\lfloor \frac{s-1}{2} \rfloor}$ is complement free; that is, the bound is really attained. ■

Note that Clements and Gronau [116] and Gronau [161] determined in several cases all maximum complement-free Sperner families.

A family $F \subseteq S$ is called *self-complementary* if $F = F^c$. In the Boolean case $k_1 = k_n = 1$ we know much about self-complementary Sperner families because of the Profile-Polytope Theorem (Theorem 3.3.1) and Remark 4 after it. In order to present some results for S we need some further notations. Let $t := 0$ if $k_1, \dots, k_n$ are even, and otherwise let $t = t(S)$ be the largest index such that k_t is odd. We define, for $\max\{t, 1\} \le i \le n$, subsets $S_i^* = S_i^*(k_1, \dots, k_n)$ of S as follows:

$$S_i^* := \left\{ \boldsymbol{a} \in S : a_i < \frac{k_i}{2} \text{ and } a_j = \frac{k_j}{2} \text{ for all } j = i+1, \dots, n \right\}$$

(in the case $i = n$ the second part of the conjunction has to be omitted). Moreover, we set

$$S_0^* := \begin{cases} \emptyset & \text{if } t > 0, \\ \{(\frac{k_1}{2}, \dots, \frac{k_n}{2})\} & \text{if } t = 0. \end{cases} \tag{8.48}$$

For a self-complementary Sperner family F in S, let

$$F^= := \{\boldsymbol{a} \in F : \boldsymbol{a} = \boldsymbol{a}^c\}.$$

Clearly,

$$F^{=} \subseteq S_0^*.$$

We partition $F - F^{=}$ into two sets

$$F^{-} := \left\{a \in F - F^{=} : r(a) < \frac{s}{2}\right\} \cup \left\{a \in F - F^{=} : r(a) = \frac{s}{2} \text{ and } a^c \prec a\right\},$$

$$F^{+} := \left\{a \in F - F^{=} : r(a) > \frac{s}{2}\right\} \cup \left\{a \in F - F^{=} : r(a) = \frac{s}{2} \text{ and } a \prec a^c\right\}.$$

Obviously, for each pair of complementary elements in $F - F^{=}$, exactly one member belongs to F^+ and the other belongs to F^-. The following result and its consequences are due to Gronau [245].

Theorem 8.5.2. *If $F \subseteq S$ is a self-complementary Sperner family, then*

$$\mathcal{C}(F^+) \subseteq \bigcup_{i=\max\{t,1\}}^{n} S_i^*.$$

Proof. By construction of the canonically compressed Sperner families and the definition of F^+, we have $\mathcal{C}(F^+) \subseteq \mathcal{C}F$. Let a be the last vector of $\mathcal{C}(F^+)$ (with respect to $\preceq$), and let $a \in N_m$ where $m \geq \frac{s}{2}$ by definition of F^+. It is easy to see that a is the last vector of $\Delta_{\to m}(\mathcal{C}(F^+))$. Let b be the first vector of $G := \mathcal{C}F - \mathcal{C}(F^+ \cup F^{=})$. Repeated application of (8.30) shows that $a \prec b$.

Claim. We have also $a \prec b^c$.

Proof of Claim. Since F is a self-complementary Sperner family, G^c is a Sperner family having the same parameters as F^+. Hence $\mathcal{C}(F^+) = \mathcal{C}(G^c)$. From (8.29) we infer

$$|\Delta_{\to m}(\mathcal{C}(F^+))| = |\Delta_{\to m}(\mathcal{C}(G^c))| \leq |\Delta_{\to m}(G^c)|.$$

If c is the last vector of $\Delta_{\to m}(G^c)$ then clearly $a \preceq c$. Moreover, $c \preceq b^c$ since b^c is the last vector of G^c. Hence $a \preceq b^c$. □

The relations $a \prec b, a \preceq b^c$ imply $a_j \leq k_j/2$ for all $j = 1, \ldots, n$ and $a_i < k_i/2$ for some $1 \leq i \leq n$. Obviously, the largest i with this property satisfies $i \geq t$. Thus

$$\mathcal{C}(F^+) \subseteq \bigcup_{\max\{t,1\}}^{n} S_i^*.$$

■

Obviously,

$$S_i^* \cong S\left(k_1, \ldots, k_{i-1}, \left\lfloor \frac{k_i - 1}{2} \right\rfloor\right), \tag{8.49}$$

but note that the minimal elements in S_i^* have rank $h_i := \frac{1}{2}\sum_{j=i+1}^{n} k_j$. Moreover, the sets S_i^* are pairwise disjoint. If F is a self-complementary Sperner family, then by Theorem 8.5.2, the sets $G_i := \mathcal{C}(F^+) \cap S_i^*$ are pairwise disjoint antichains whose union is $\mathcal{C}(F^+)$. Let $g_{i,0}, \ldots, g_{i,s}$ be the parameters of G_i. Note that

$$g_{i,j} = 0 \text{ if } j < \frac{s}{2} \quad \text{or} \quad j \geq s - h_{i-1}, \tag{8.50}$$

and that $\sum_{i=\max\{t,1\}}^{n} g_{i,j}$, $j = 0, \ldots, s$, are the parameters of F^+. The LYM-inequality (8.46) in each S_i^* yields

$$\sum_{\frac{s}{2} \leq j < s - h_{i-1}} \frac{g_{i,j}}{W_j(S_i^*)} \leq 1. \tag{8.51}$$

In the special case $t = n$; that is, k_n is odd, there is only one set S_n^*, and we have $F^= = \emptyset$. Moreover, F is the disjoint union of F^+ and F^- and $F^+ = (F^-)^c$. Hence we proved:

Corollary 8.5.2. *Let $F \subseteq S$ be a self-complementary Sperner family and let k_n be odd. Then*

$$\sum_{\frac{k_n}{2} < i < \frac{s}{2}} \frac{f_i}{W_{s-i}(S(k_1, \ldots, k_{n-1}, \frac{k_n - 1}{2}))}$$
$$+ \sum_{\frac{s}{2} \leq i < s - \frac{k_n}{2}} \frac{f_i}{W_i(S(k_1, \ldots, k_{n-1}, \frac{k_n - 1}{2}))} \leq 1 + 1 = 2.$$

In the Boolean case this inequality reads

$$\sum_{1 \leq i < \frac{n}{2}} \frac{f_i}{\binom{n-1}{i-1}} + \sum_{\frac{n}{2} \leq i \leq n-1} \frac{f_i}{\binom{n-1}{i}} \leq 2,$$

which is a result of Bollobás [72] and which can be also easily derived from the Profile-Polytope Theorem 3.3.1 (sum up for $B = SIC \vee \overline{I}\,\overline{C}$, two essential facet-defining inequalities with any T and its complement $\{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\} - T$).

The preceding approach gives also the maximum size of the families under consideration:

Theorem 8.5.3. *If $F \subseteq S$ is a self-complementary Sperner family then*

$$|F| \leq \begin{cases} W_{\frac{s}{2}} & \text{if } s \text{ is even}, \\ 2\sum_{i=t}^{n} W_{\frac{s+1}{2}}(S_i^*) & \text{if } s \text{ is odd}, \end{cases}$$

and this bound is the best possible.

Proof. We use the notations from above. If s is even then by the Sperner property, the rank symmetry and rank unimodality of S the inequality $|F| \le W_{s/2}$ is true for any Sperner family. Thus let s be odd. Then $t \ge 1$ and $F^{=} = \emptyset$. Since $|F| = |F^+| + |F^-|$ and $|F^+| = |F^-|$, it is enough to show that each G_i has size at most $W_{(s+1)/2}(S_i^*)$. Recall the isomorphism (8.49). The Whitney numbers $W_j(S_i^*)$ are decreasing for $\frac{s}{2} \le j < s - h_{i-1}$ since S_i^* is rank symmetric and rank unimodal, and the largest level of S_i^* is at rank $h_i + \lfloor \frac{1}{2}(k_1 + \cdots + k_{i-1} + \lfloor \frac{k_i - 1}{2} \rfloor) \rfloor \le \frac{s}{2}$. Because of (8.50) and (8.51) it follows that

$$|G_i| = \sum_{\frac{s+1}{2} \le j < s - h_{i+1}} g_{i,j} \le W_{\frac{s+1}{2}}(S_i^*).$$

That this bound is the best possible can be easily seen by taking $F := N_{\frac{s}{2}}$ if s is even and $F := G \cup G^c$ where $G := \cup_{i=t}^{n} N_{\frac{s+1}{2}}(S_i^*)$ if s is odd. ■

We say that a family F in S is *dynamically intersecting* (resp. *dynamically cointersecting*) if for all $\boldsymbol{x}, \boldsymbol{y} \in F$ there exists an index i such that $x_i + y_i > k_i$ (resp. $x_i + y_i < k_i$). In the Boolean case $k_1 = k_n = 1$ these notions coincide with the notions intersecting (resp. cointersecting). The following propositions are obvious.

Proposition 8.5.1. *A family F is dynamically intersecting iff its complement F^c is dynamically cointersecting.*

Proposition 8.5.2. *If the family F is dynamically intersecting then it is complement free.*

These propositions and Corollary 8.5.1 immediately imply:

Theorem 8.5.4. *The maximum size of a dynamically intersecting (resp. dynamically cointersecting) Sperner family in S equals $W_{\lfloor \frac{s-1}{2} \rfloor}$.*

Proposition 8.5.3. *Let F be a dynamically intersecting and cointersecting Sperner family. Then $F \cap F^c = \emptyset$, and $F \cup F^c \cup S_0^*$ is a self-complementary Sperner family, where S_0^* is given by (8.48).*

Proof. The disjointness of F and F^c follows from Proposition 8.5.2. Clearly, F and F^c are Sperner families and $F \cup F^c$ is self-complementary. Moreover, no element $\boldsymbol{x}$ of F (resp. F^c) is related to $(\frac{k_1}{2}, \dots, \frac{k_n}{2})$ because otherwise the pair $\boldsymbol{x}, \boldsymbol{y}$ with $\boldsymbol{y} := \boldsymbol{x}$ cannot satisfy the dynamic intersecting as well as the dynamic cointersecting condition. Hence it remains to show that there are no $\boldsymbol{x} \in F, \boldsymbol{y} \in F^c$ such that

$$\boldsymbol{x} \le \boldsymbol{y} \text{ or } \boldsymbol{x} \ge \boldsymbol{y}.$$

Assume the contrary. We have $y = z^c$ for some $z \in F$. Thus, for all i, $x_i \leq k_i - z_i$, or, for all i, $x_i \geq k_i - z_i$. In the first case F would not be dynamically intersecting, in the second case it would not be dynamically cointersecting, a contradiction.

■

Theorem 8.5.5. *If F is a dynamically intersecting and cointersecting Sperner family, then*

$$|F| \leq \begin{cases} \frac{1}{2} W_{\frac{s}{2}} & \text{if } s \text{ is even and } t > 0, \\ \frac{1}{2}(W_{\frac{s}{2}} - 1) & \text{if } s \text{ is even and } t = 0, \\ \sum_{i=t}^{n} W_{\frac{s+1}{2}}(S_i^*) & \text{if } s \text{ is odd,} \end{cases}$$

and the bound is the best possible.

Proof. The bound follows from Proposition 8.5.3 and Theorem 8.5.3. To see that it is the best possible take, for even s, from each pair of different complementary elements of $N_{s/2}$ exactly one member, and, for odd s, the set G from the end of the proof of Theorem 8.5.3. ■

In the light of the Erdős–Ko–Rado Theorem we study also the restriction to one level (here in the cointersecting case).

Theorem 8.5.6. *Let F be an l-uniform dynamically cointersecting family in S. Then*

$$|F| \leq \begin{cases} W_l & \text{if } l < \frac{s}{2}, \\ \sum_{i=\max\{t,1\}}^{n} W_l(S_i^*) & \text{if } l \geq \frac{s}{2}, \end{cases}$$

and the bound is the best possible.

Proof. The case $l < \frac{s}{2}$ is trivial, thus consider $l \geq \frac{s}{2}$. In this case F is automatically also dynamically intersecting, hence, by Proposition 8.5.3, $F \cup F^c$ is a self-complementary Sperner family (not containing $(\frac{k_1}{2}, \ldots, \frac{k_n}{2})$), and, by Theorem 8.5.2,

$$CF \subseteq \bigcup_{i=\max\{t,1\}}^{n} S_i^*.$$

Consequently,

$$|F| = |CF| \leq \sum_{i=\max\{t,1\}}^{n} W_l(S_i^*).$$

The family $\cup_{i=\max\{t,1\}}^{n} N_l(S_i^*)$ is obviously dynamically cointersecting; that is, the bound is the best possible. ■

In Section 7.3 we already studied statically t-intersecting families (see also Theorem 3.3.4). Here we investigate the case $t = 1$ repeating and extending the definition. A family F in S is called *statically intersecting* (resp. *statically cointersecting*) if for all $\boldsymbol{x}, \boldsymbol{y} \in F$ there exists some coordinate i such that $x_i, y_i \geq 1$ (resp. $x_i, y_i \leq k_i - 1$). Recall also the definition of the *support* as $\operatorname{supp}(\boldsymbol{x}) := \{i \in [n] : x_i \geq 1\}$, $\operatorname{supp}(F) := \{\operatorname{supp}(\boldsymbol{x}) : \boldsymbol{x} \in F\}$. The *cosupport* is defined similarly: $\operatorname{cosupp}(\boldsymbol{x}) := \{i \in [n] : x_i = k_i\}$, $\operatorname{cosupp}(F) := \{\operatorname{cosupp}(\boldsymbol{x}) : \boldsymbol{x} \in F\}$. As in Proposition 3.3.1 we have:

Proposition 8.5.4. *The family $F \subseteq S$ is statically intersecting (resp. statically cointersecting) iff* $\operatorname{supp}(F)$ *(resp.* $\operatorname{cosupp}(F)$*) is intersecting (resp. cointersecting).*

The following propositions are obvious and need no proof.

Proposition 8.5.5. *A family $F \subseteq S$ is statically intersecting iff its complement F^c is statically cointersecting.*

Proposition 8.5.6.

(*a*) *If $\boldsymbol{x}, \boldsymbol{y} \in S$ satisfy the statically intersecting (resp. cointersecting) condition, and if $z \geq \boldsymbol{x}$ (resp. $z \leq \boldsymbol{x}$), then $z, \boldsymbol{y}$ also satisfy the corresponding condition.*

(*b*) *If $r(\boldsymbol{x}) + r(\boldsymbol{y}) > s$ (resp. $r(\boldsymbol{x}) + r(\boldsymbol{y}) < s$), then $\boldsymbol{x}, \boldsymbol{y}$ satisfy the statically intersecting (resp. cointersecting) condition.*

If $k_1 = k_n = 1$; that is, if S is the Boolean lattice B_n, the statically intersecting (resp. cointersecting) condition coincides with the usual intersecting (resp. cointersecting) condition. In this case, the Profile-Polytope Theorem 3.3.1 gives us enough information about our families. In general, the statically intersecting and cointersecting conditions are more difficult to handle than the dynamic intersecting and cointersecting conditions. The reader may get an idea of the difficulties from the related Theorem 7.3.1. Only some partial results are presented here. In particular, we restrict ourselves to the case $k := k_1 = \cdots = k_n$. For even n, the set

$$\overline{S} := \{\boldsymbol{x} \in N_{\frac{s}{2}} : x_i \in \{0, k\},\ i = 1, \ldots, n\}$$

plays an exceptional role. Obviously, $|\overline{S}| = \binom{n}{n/2}$.

The last theorem in this section we obtained together with Gronau in [161].

Theorem 8.5.7. *Let $k_1 = k_n = k \geq 2$.*

(*a*) *If $F \subseteq S$ is a statically intersecting (resp. statically cointersecting) Sperner family then*

$$|F| \le \begin{cases} W_{\lceil \frac{s}{2} \rceil} & \text{if } n \text{ is odd,} \\ W_{\frac{s}{2}} - \frac{1}{2}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even and } k \ge 2n-1. \end{cases}$$

(*b*) *The same assertion is true if $F \subseteq S$ is a statically intersecting and cointersecting Sperner family.*

Proof. First we show that the bound can be attained. For odd n, let $F := N_{\lceil s/2 \rceil}$. Then, for all $\boldsymbol{x} \in F$, $|\operatorname{supp}(\boldsymbol{x})| > \frac{n}{2}$, which implies that F is statically intersecting. Moreover, we have for all $\boldsymbol{x} \in F$, $|\operatorname{cosupp}(\boldsymbol{x})| < \frac{n}{2}$ since otherwise (recall that n is odd), $r(\boldsymbol{x}) \ge \frac{n+1}{2}k > \lceil \frac{s}{2} \rceil$. Hence F is also statically cointersecting. If n is even, $\overline{S}$ consists of pairs of complementary elements. We partition $\overline{S}$ into two sets $\overline{S}_1$ and $\overline{S}_2$ by putting for each such pair one member into $\overline{S}_1$ and the other into $\overline{S}_2$. Then $|\overline{S}_1| = |\overline{S}_2| = \frac{1}{2}\binom{n}{n/2}$ and $\operatorname{supp}(\overline{S}_1)$ (resp. $\operatorname{cosupp}(\overline{S}_1)$) is intersecting (resp. cointersecting). Let $F := N_{s/2} - \overline{S}_2$. Then, for all $\boldsymbol{x} \in F$, $|\operatorname{supp}(\boldsymbol{x})| \ge \frac{n}{2}$, but $|\operatorname{supp}(\boldsymbol{x})| = \frac{n}{2}$ only if $\boldsymbol{x} \in \overline{S}_1$. Hence $|\operatorname{supp}(F)|$ is intersecting; that is, F is statically intersecting. In the same way it follows that F is statically cointersecting. Hence (a) and (b) are proved if we can show that the upper bound for (a) is correct. Because of Proposition 8.5.5 we may restrict ourselves to the cointersecting case.

For odd n, the upper bound is trivial since $W_{\lceil s/2 \rceil}$ is the maximum size of a Sperner family by the Sperner property of S. Thus let n be even. The case $n = 2$ can be checked easily; thus let $n \ge 4$.

For each family F, let $l = l(F)$ (resp. $u = u(F)$) be the smallest (resp. largest) index i such that $f_i > 0$. Let F be a maximum statically cointersecting family for which $u - l$ is minimum. It is enough to prove that $l = u = \frac{s}{2}$ since for $F \subseteq N_{s/2}$, there holds $|F \cap \overline{S}| \le \frac{1}{2}|\overline{S}|$ (from each pair of complementary elements of S at most one member may belong to F).

Claim 1. We have $u \le \frac{s}{2}$.

Proof of Claim 1. Assume $u \ge \frac{s}{2} + 1$. With the usual shifting we define the new family

$$F' := (F - F_u) \cup \Delta(F_u)$$

which is, in view of Proposition 8.5.6(a), still a statically cointersecting Sperner family whose size is at least as large as the size of F by (8.41). Since $u(F') - l(F') < u(F) - l(F)$, we have a contradiction to the choice of F. □

Using the same arguments and Proposition 8.5.6(b) we obtain immediately:
Claim 2. We have $l \ge \frac{s}{2} - 1$. □

So up to now it is clear that F consists only of $f_{s/2}$ elements of rank $\frac{s}{2}$ and $f_{s/2-1}$ elements of rank $\frac{s}{2} - 1$. We add one more condition on F: Of all maximum statically cointersecting families in $N_{s/2-1} \cup N_{s/2}$ we choose F such that $f_{s/2}$ is maximal (it remains to show that $f_{s/2-1} = 0$).

Claim 3. We have

$$f_{\frac{s}{2}} \geq W_{\frac{s}{2}}(S(\underbrace{k,\ldots,k}_{n-1},k-1)) \quad \text{and} \quad f_{\frac{s}{2}-1} \leq W_{\frac{s}{2}+1}(S(\underbrace{k,\ldots,k}_{n-1})).$$

Proof of Claim 3. Assume that $f_{\frac{s}{2}} < W_{\frac{s}{2}}(S(\underbrace{k,\ldots,k}_{n-1},k-1))$. We consider the canonically compressed Sperner family $\mathcal{C}F$ with the same profile as F. Because of the assumption, $(\mathcal{C}F)_{\frac{s}{2}}$ and $\Delta((\mathcal{C}F)_{\frac{s}{2}})$ are subsets of $S(\underbrace{k,\ldots,k}_{n-1},k-1)$. But $2\frac{s}{2} - 1 = s - 1 = k - 1 + (n-1)k$. Hence, by (8.41) with s replaced by $s - 1$

$$|\Delta((\mathcal{C}F)_{\frac{s}{2}})| \geq |(\mathcal{C}F)_{\frac{s}{2}}|.$$

Consequently (noting that $\mathcal{C}F$ is a Sperner family; that is, $\Delta((\mathcal{C}F)_{s/2}) \cap (\mathcal{C}F)_{s/2-1} = \emptyset$),

$$\begin{aligned} W_{\frac{s}{2}-1} &\geq |\Delta((\mathcal{C}F)_{\frac{s}{2}})| + |(\mathcal{C}F)_{\frac{s}{2}-1}| \\ &\geq |(\mathcal{C}F)_{\frac{s}{2}}| + |(\mathcal{C}F)_{\frac{s}{2}-1}| = |\mathcal{C}F| = |F|. \end{aligned} \tag{8.52}$$

However, consider the family $F' = F'_{\frac{s}{2}-1} \cup F'_{\frac{s}{2}}$ where

$$F'_{\frac{s}{2}-1} := \{x \in N_{\frac{s}{2}-1}(S) : x_1 = k\} \quad \text{and} \quad F'_{\frac{s}{2}} := N_{\frac{s}{2}}(S(\underbrace{k,\ldots,k}_{n-1},k-1)).$$

It is easy to see that F' is a statically cointersecting Sperner family. Since

$$N_{\frac{s}{2}-1}(S) = N_{\frac{s}{2}-1}(S(\underbrace{k,\ldots,k}_{n-1},k-1)) \cup F'_{\frac{s}{2}-1}$$

and

$$W_{\frac{s}{2}-1}(S(\underbrace{k,\ldots,k}_{n-1},k-1)) = W_{\frac{s}{2}}(S(\underbrace{k,\ldots,k}_{n-1},k-1))$$

it follows that

$$|F'| = W_{\frac{s}{2}-1}(S). \tag{8.53}$$

Hence, by (8.52) and (8.53), $|F'| \geq |F|$, and for the parameters we have

$$f'_{\frac{s}{2}} = W_{\frac{s}{2}}(S(\underbrace{k,\ldots,k}_{n-1},k-1)) > f_{\frac{s}{2}}$$

by our assumption. This contradicts the choice of F since $f_{s/2}$ has to be maximal.

Thus the first inequality is proved. In particular we know that $(\mathcal{C}F)_{s/2}$ contains all elements of $N_{s/2}(S)$ whose last coordinate is less than or equal to $k - 1$. By the construction of the canonically compressed Sperner family, every member of $(\mathcal{C}F)_{s/2-1}$ ends with k. Hence

$$f_{\frac{s}{2}-1} = |(\mathcal{C}F)_{\frac{s}{2}-1}| \le W_{\frac{s}{2}-1-k}(S(\underbrace{k,\dots,k}_{n-1})) = W_{\frac{s}{2}+1}(S(\underbrace{k,\dots,k}_{n-1})).$$

□

Claim 4. If $k \ge n$ and $0 \le i \le (n-1)k$, then

$$W_i(S(\underbrace{k,\dots,k}_{n-1})) \le W_i(S(\underbrace{k-1,\dots,k-1}_{n-1},n-1)).$$

Proof of Claim 4. Let, for $0 \le a \le n-1$,

$$S_a := S(\underbrace{k,\dots,k}_{n-a-1},\underbrace{k-1,\dots,k-1}_{a},a).$$

Obviously, it is enough to prove $W_i(S_a) \le W_i(S_{a+1})$ for $0 \le a \le n-2$, and this can be accomplished by constructing an injection φ from $N_i(S_a)$ into $N_i(S_{a+1})$. Here is such an injection: We put $\varphi(\boldsymbol{x}) := \boldsymbol{x}$ if $\boldsymbol{x} \in S_a \cap S_{a+1}$ and $\varphi(\boldsymbol{x}) := (x_1,\dots,x_{n-a-2},x_{n-a-1}-(a+1)+x_n,x_{n-a},\dots,x_{n-1},a+1)$ if $\boldsymbol{x} \in S_a - S_{a+1}$. Since we have for $\boldsymbol{x} \in S_a - S_{a+1}$ the relations $x_n \le a \le n-1,\ x_{n-a-1} = k \ge n$, it follows that $0 \le x_{n-a-1} - (a+1) + x_n \le k-1$. Thus, indeed $\varphi(\boldsymbol{x}) \in S_{a+1}$. The injectivity can be easily verified, and $r(\boldsymbol{x}) = r(\varphi(\boldsymbol{x}))$ is obvious. □

We conclude the proof of the theorem by showing that $f_{s/2-1} > 0$ yields a contradiction. With our assumption $f_{s/2-1} > 0$ there must exist some set $I \subseteq [n]$ of minimum size such that

$$G := \{\boldsymbol{x} \in F_{\frac{s}{2}-1} : \operatorname{cosupp}(\boldsymbol{x}) = I\} \ne \emptyset.$$

Let

$$H := \{\boldsymbol{x} \in \nabla(G) : \operatorname{cosupp}(\boldsymbol{x}) = I\}.$$

Claim 5. The family $F' := (F-G) \cup H$ is a statically cointersecting Sperner family.

Proof of Claim 5. By construction, $\operatorname{cosupp}(F) = \operatorname{cosupp}(F')$. Since F is cointersecting, Proposition 8.5.4 implies that also F' is cointersecting. The only obstacle to F' being a Sperner family could be the existence of some $\boldsymbol{x} \in F_{\frac{s}{2}-1} - G$ and some $\boldsymbol{y} \in H$ with $\boldsymbol{x} \le \boldsymbol{y}$. But by the choice of I,

$$|\operatorname{cosupp}(\boldsymbol{x})| \ge |\operatorname{cosupp}(\boldsymbol{y})| = |I| \quad \text{and} \quad \operatorname{cosupp}(\boldsymbol{x}) \ne I.$$

Hence, $\boldsymbol{x} \not\le \boldsymbol{y}$. □

Claim 6. We have $|F'| \ge |F|$.

Proof of Claim 6. Since $H \cap F = \emptyset$ (F is a Sperner family), the assertion is equivalent to $|G| \leq |H|$. Let $i := |I|$. With each $\boldsymbol{x} \in G \cup H$ we associate an element $\boldsymbol{x}'$ of $S(\underbrace{k-1, \ldots, k-1}_{n-i})$ by deleting all coordinates x_j of $\boldsymbol{x}$ for which $x_j = k$. Note that

$$r(\boldsymbol{x}') = r(\boldsymbol{x}) - ik. \tag{8.54}$$

We obtain families G' and H' with the obvious properties

$$|G'| = |G|, \qquad |H'| = |H|, \qquad \nabla(G') = H',$$

where the upper shadow is considered in $S(\underbrace{k-1, \ldots, k-1}_{n-i})$. Thus the assertion is equivalent to

$$|G'| \leq |\nabla(G')|. \tag{8.55}$$

By (8.40) and (8.54), the inequality (8.55) is satisfied if

$$2\left(\frac{s}{2} - 1 - ik\right) + 1 \leq (n-i)(k-1),$$

which is equivalent to

$$k \geq \frac{n-i}{i} - 1,$$

and this is true under our supposition $k \geq 2n-1$ (i.e., $k \geq n-2$) for $i \geq 1$. It remains to consider the case $i = 0$. We turn to the complements in $S(\underbrace{k-1, \ldots, k-1}_{n})$ and estimate the lower shadow. We have

$$(G')^c \subseteq N_{\frac{s}{2}+1-n}(S(\underbrace{k-1, \ldots, k-1}_{n})). \tag{8.56}$$

Clearly, $|(G')^c| = |G'| = |G| \leq f_{\frac{s}{2}-1}$, and using Claim 3 and Claim 4 we obtain (note $0 \leq \frac{s}{2} + 1 \leq (n-1)k$ because of $n \geq 4, k \geq 2$)

$$|(G')^c| \leq W_{\frac{s}{2}+1}(S_{n-1}). \tag{8.57}$$

Moreover,

$$W_{\frac{s}{2}+1}(S_{n-1}) \leq W_{\frac{s}{2}+1-n}(S_{n-1}) \tag{8.58}$$

since $(\frac{s}{2}+1) + (\frac{s}{2}+1-n) \geq n-1+(n-1)(k-1)$ by $k \geq 2n-1 \geq n-2$. From (8.56) to (8.58), we derive for the compression that

$$\mathcal{C}((G')^c) \subseteq N_{\frac{s}{2}+1-n}(S_{n-1}),$$

and using (8.41) and Theorem 8.1.1, we obtain

$$|\nabla(G')| = |\Delta((G')^c)| \geq |\Delta(\mathcal{C}((G')^c))| \geq |\mathcal{C}((G')^c)| = |(G')^c| = |G'|$$

since $2(\frac{s}{2}+1-n)-1 \geq n-1+(n-1)(k-1)$ is equivalent to $k \geq 2n-1$. Hence (8.55) is verified. □

Claim 5 and Claim 6 complete the proof of the theorem since $f_{\frac{s}{2}}{}' > f_{\frac{s}{2}}$, contradicting the choice of F. ■

We conjecture that Theorem 8.5.7 remains true for even n and $2 \leq k < 2n-1$.

NOTATION

We omit in the following list those symbols that are very well known, self-explanatory, or not used often enough to be worth listing.

Sets and Families

$\operatorname{conv}(\mu)$	convex hull of profiles	84
$\epsilon(\mu)$	extreme points of $\operatorname{conv}(\mu)$	84
$\mathcal{P}^c$	antiblocker	87
$\epsilon^*(\mu(\mathfrak{A}))$	essential extreme points	90
$\varphi^*(\mu(\mathfrak{A}))$	essential facets	90
supp	support	114
$\mathbb{Z}$	integers	168
cosupp	cosupport	384

Posets

$p \lessdot q$	p is covered by q	5
$H(P)$	Hasse diagram of P	5
$E(P)$	arc set of the Hasse diagram of P	5
$[p, q]$	interval between p and q	5
(P, w)	weighted poset	5
$w(F)$	weight of the family F	5
$d(P, w)$	width of (P, w)	6
$d_k(P, w)$	maximum weight of a k-family	6
P^*	dual poset	6
$P \times Q$	direct product	6
P/G	quotient of P under the group G	6
$p \vee q$	supremum of p and q	6
$p \wedge q$	infimum of p and q	6
$r(p)$	rank of the element p	7
$r(P)$	rank of the poset P	7
$N_i(P)$	ith level	7
$W_i(P)$	ith Whitney number	7
$F(P; x)$	rank-generating function	7
$P \times_r Q$	rankwise direct product	7
F_i	rank i elements of F	7
f_i	parameters of F	7
f	profile of F	7
∇, Δ	upper, lower shadow	7
$\nabla_{\to k}, \Delta_{\to k}$	upper, lower k-shadow	7
$\mathfrak{C}(P)$	maximal chains in P	125
$c_k(P, w)$	minimum weight of a k-cutset	125
$c_k^*(P, w)$	fractional k-cutset number	125
$\mathfrak{C}^*(P)$	chains in P	131

$\mathfrak{A}(P)$	antichains in P	132
$d_k^*(P, w)$	fractional k-family number	133
$h(p)$	height of p	140
μ_x	expected value of representation x	140
σ_x^2	variance of representation x	140
$\sigma^2(P, w)$	variance of (P, w)	141
$\mu_x(F)$	expected value of F w.r.t. x	141
$\mu(p, q)$	Möbius function	255
$\mathcal{C}(m, F)$	first m elements of F	333
$\mathcal{L}(m, F)$	last m elements of F	333
$\mathcal{C}F$	compression of F in a level	333
$\mathcal{L}F$	$\mathcal{L}$-compression of F in a level	333
sf_i	shadow function	345
∇_{new}, Δ_{new}	new upper, lower shadow	345
$\mathcal{C}F$	canonical compression of an antichain	355
$\boldsymbol{a}^c$	complement of $\boldsymbol{a}$ in S	376
F^c	complementary family in S	376

Examples

B_n	Boolean lattice	9
$S(k_1, \ldots, k_n)$	chain products	9
Q_n	cubical poset	10
F_k^n	function poset	10
$T(k_1, \ldots, k_n)$	star products	10
$Int(S(k_1, \ldots, k_n))$	poset of subparallelepipedons of a parallelepipedon	11
$M(k_1, \ldots, k_n)$	poset of submatrices of a matrix	11
$SQ_{k,n}$	poset of subcubes of a cube	12
$SM_{k,n}$	poset of square submatrices of a square matrix	12
$L_n(q)$	linear lattice	12
$A_n(q)$	affine poset	13
$L(m, n)$	the poset $L(m, n)$	14
$M(n)$	the poset $M(n)$	14
G_n	graph poset	14
Π_n	partition lattice	14
$Col(k_1, \ldots, k_n)$	poset of colored subsets	344

Graphs

$d(v)$	degree of vertex v	4
e^-, e^+	starting point, endpoint of an arc	5
$B(A)$	boundary	39
$E_{in}(A)$	inner edges	39
$E_{out}(A)$	outer edges	39
$v(f)$	value of flow	118
$c(S, T)$	capacity of cut	118
$a(f)$	cost of flow	121
$p_f^\pm$	inflow, outflow	142
$d^\pm(p)$	indegree, outdegree	250
$\alpha(G)$	independence number	276

Structures

$GF(q)$	Galois field of q elements	12
$R[x_1, \ldots, x_n]$	polynomials in the variables $x_1, \ldots, x_n$	63
$\langle \boldsymbol{x}, \boldsymbol{y} \rangle$	standard scalar product	69
$\widetilde{P}$	poset space	209
$\widetilde{p}$	poset space element corresponding to p	209
$\widetilde{F}$	subspace generated by the "elements" of F	209
$V \oplus W$	direct sum	209
$\widetilde{G}$	graph space	276

Sequences and Functions

supp	support	6
$a_n \sim b_n$	asymptotically equal	8
$a_n = O(b_n)$	Ooh of b_n	8
$a_n = o(b_n)$	little ooh of b_n	8
$a_n \lesssim b_n$	asymptotically not greater	8
log	logarithm to the basis $e = 2.718\ldots$	8
$\lfloor x \rfloor$	greatest integer $\leq x$	8
$\lceil x \rceil$	least integer $\geq x$	8
$\binom{n}{i}_q$	Gaussian coefficient	12
$S_{n,k}$	Stirling numbers of the second kind	14
B_n	Bell numbers	15
$\binom{x}{k}$	generalized binomial coefficient	35
$\Phi(x)$	Gaussian distribution function	305

$\varphi_\xi(t)$	characteristic function	305
$\binom{j}{i}_S$	Whitney number in S	368

Operators

$\Phi\vert_E$	restriction of Φ to a subspace E	209
Φ^*	adjoint operator	209
$\widetilde{\nabla}$, $\widetilde{\Delta}$	raising, lowering operator	209
$\widetilde{\nabla}_L$, $\widetilde{\Delta}_L$	Lefschetz raising, lowering operator	210
$\widetilde{\nabla}_{ij}$, $\widetilde{\Delta}_{ji}$	successive application of the raising, lowering operator	210
$\ker_{ij}$	kernel of $\widetilde{\nabla}_{ij}$	211
rank_{ij}	rank of $\widetilde{\nabla}_{ij}$	211
s_{ij}	string numbers	214
$J(\widetilde{\nabla}, \widetilde{P}; x, y)$	Jordan function	217
a_{kl}	Jordan coefficients	217
$\ker_{t,t-1}(\widetilde{\Delta})$	kernel of $\widetilde{\Delta}\vert_{\widetilde{N}_t}$	235
$\widetilde{\Delta}_{\to i}$, $\widetilde{\nabla}_{\to i}$	rank i lowering, raising operator	258
$\widetilde{\Delta}_{F\to G}$, $\widetilde{\nabla}_{G\to F}$	(F, G) lowering, (G, F) raising operator	258
$\widetilde{J}$	all-one-operator	276
A_L	Lefschetz adjacency operator	277
$\Psi_{i\to j}$	empty intersection operator	281
B_j	B_j-operator	281

Marks

□	end of proof of claim (as part of a proof of a theorem)	2
■	end of proof	3

BIBLIOGRAPHY

[1] R. Aharoni and U. Keich. A generalization of the Ahlwede–Daykin inequality. *Discrete Math.* **152**:1–12 (1996).

[2] R. Ahlswede. On code pairs with specified Hamming distances. In A. Hajnal and L. Lovász, editors, *Combinatorics (Proc. 7th Hungarian Colloq. on Combinatorics, Eger, 1987)*, vol. **52** of *Colloq. Math. Soc. János Bolyai*, pp. 9–47, North-Holland, Amsterdam (1988).

[3] R. Ahlswede and S. Bezrukov. Edge isoperimetric theorems for integer point arrays. *Appl. Math. Lett.*, **8**:75–80 (1995).

[4] R. Ahlswede and N. Cai. A generalization of the AZ identity. *Combinatorica*, **13**:241–7 (1993).

[5] R. Ahlswede and N. Cai. General edge-isoperimetric inequalities. Preprint 94-090, Univ. Bielefeld (1994).

[6] R. Ahlswede, N. Cai, N. Danh, D.E. Daykin, L.H. Khachatrian, and T.D. Thu. Report on work in progress in combinatorial extremal theory: Shadows, AZ-identities, matching. Ergänzungsreihe 95-004, Univ. Bielefeld (1995).

[7] R. Ahlswede, N. Cai, and Z. Zhang. A general 4-words inequality with consequences for 2-way communication complexity. *Adv. in Appl. Math.*, **10**:75–94 (1989).

[8] R. Ahlswede and D.E. Daykin. An inequality for the weights of two families of sets, their unions and intersections. *Z. Wahrsch. Verw. Gebiete*, **43**:183–5 (1978).

[9] R. Ahlswede and D.E. Daykin. Inequalities for a pair of maps $S \times S \to S$ with S a finite set. *Math. Z.*, **165**:267–89 (1979).

[10] R. Ahlswede and D.E. Daykin. The number of values of combinatorial functions. *Bull. London Math. Soc.*, **11**:49–51 (1979).

[11] R. Ahlswede, P. Gács, and J. Kőrner. Bounds on conditional probabilities with applications in multiuser communication. *Z. Wahrsch. Verw. Gebiete*, **34**:157–77 (1976).

[12] R. Ahlswede, A. El Gamal, and K.F. Pang. A two-family extremal problem in Hamming space. *Discrete Math.*, **49**:1–5 (1984).

[13] R. Ahlswede and G. Katona. Contributions to the theory of Hamming spaces. *Discrete Math.*, **17**:1–22 (1977).

[14] R. Ahlswede and L.H. Khachatrian. Number theoretic correlation inequalities for Dirichlet densities. Preprint 93-060, Univ. Bielefeld (1993).

[15] R. Ahlswede and L.H. Khachatrian. The complete intersection theorem for systems of finite sets. Preprint 95-066, Univ. Bielefeld (1995).

[16] R. Ahlswede and L.H. Khachatrian. The complete nontrivial intersection theorem for systems of finite sets. Preprint, Univ. Bielefeld (1995).

[17] R. Ahlswede and L.H. Khachatrian. Density inequalities for sets of multiples. *J. Number Theory*, **55**:170–80 (1995).

[18] R. Ahlswede and Z. Zhang. An identity in combinatorial extremal theory. *Adv. Math.*, **80**:137–51 (1990).
[19] M. Aigner. Lexicographic matching in Boolean algebras. *J. Combin. Theory Ser. B*, **14**:187–94 (1973).
[20] M. Aigner. Symmetrische Zerlegungen von Kettenprodukten. *Monatsh. Math.*, **79**:177–89 (1975).
[21] M. Aigner. *Combinatorial theory.* Springer-Verlag, Berlin, Heidelberg, New York (1979).
[22] M. Aigner. *Combinatorial Search.* B.G. Teubner and John Wiley & Sons, Stuttgart and Chichester–Singapore (1988).
[23] V.B. Alekseev. The number of monotone k-valued functions (in Russian). *Problemy Kibernet.*, **28**:5–24 (1974).
[24] V.B. Alekseev. Use of symmetry in finding the width of a partially ordered set (in Russian). *Diskret. Analiz*, **26** Grafy i Testy:20–35 (1974).
[25] V.B. Alekseev. The decipherment of certain classes of monotone multivalued functions (in Russian). *Zh. Vychisl. Mat. i Mat. Fiz.*, **16**:189–98 (1976).
[26] N. Alon. An extremal problem for sets with applications to graph theory. *J. Combin. Theory Ser. A*, **40**:82–9 (1985).
[27] N. Alon. Probabilistic methods in extremal finite set theory. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 39–57, Budapest (1994). János Bolyai Mathematical Society.
[28] N. Alon, L. Babai, and H. Suzuki. Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems. *J. Combin. Theory Ser. A*, **58**:165–80 (1991).
[29] N. Alon and J.H. Spencer. *The probabilistic method.* John Wiley & Sons, New York, Singapore (1992).
[30] I. Anderson. On the divisors of a number. *J. London Math. Soc.*, **43**:410–18 (1968).
[31] I. Anderson. Intersection theorems and a lemma of Kleitman. *Discrete Math.*, **16**:181–5 (1976).
[32] I. Anderson. *Combinatorics of finite sets.* Clarendon Press, Oxford (1987).
[33] W.W. Armstrong. Dependency structures of database relationships. In *Information Processing 74 (Proc. IFIP Congress, Stockholm, 1974)*, pp. 580–3, North-Holland, Amsterdam (1974).
[34] L.A. Aslanyan. An isoperimetric problem and related extremal problems for discrete spaces (in Russian). *Problemy Kibernet.*, **36**:85–127, 279 (1979).
[35] L. Babai and P. Frankl. Linear algebra methods in combinatorics (to appear).
[36] L. Babai and P. Frankl. On set intersections. *J. Combin. Theory Ser. A*, **28**:103–5 (1980).
[37] C. Bandt, G. Burosch, and K.-D. Drews. Conditionally monotone functions and three Sperner type conditions. *Elektron. Informationsverarb. Kybernet.*, **17**:523–37 (1981).
[38] E. Bannai and T. Ito. *Algebraic Combinatorics I. Association Schemes.* Benjamin/Cummings Publishing Company, London, Tokyo (1984).
[39] C.J.K. Batty and H.W. Bollmann. Generalized Holley-Preston inequalities on measure spaces and their products. *Z. Wahrsch. Verw. Gebiete*, **53**:157–73 (1980).
[40] I. Beck. An inequality for partially ordered sets. *J. Combin. Theory Ser. A*, **54**:123–8 (1990).
[41] J. Beck. On a geometric problem of Erdős, Sárkőzy, and Szemerédi concerning vector sums. *European J. Combin.*, **4**:1–10 (1983).
[42] E.A. Bender. Central and local limit theorems applied to asymptotic enumeration. *J. Combin. Theory Ser. A*, **15**:91–111 (1973).
[43] E.A. Bender. Asymptotic methods in enumeration. *SIAM Rev.*, **16**:485–515 (1974).
[44] E.A. Bender and E.R. Canfield. Log-concavity and related properties of the cycle index polynomials. *J. Combin. Theory Ser. A* (to appear).
[45] M. Benoumhani. Sur une propriété des polynômes à racines réelles négatives. Preprint, Université Claude Bernard, Lyon 1 (1995).
[46] X. Berenguer, J. Diaz, and L.H. Harper. A solution of the Sperner-Erdős problem. *Theoret. Comput. Sci.*, **21**:99–103 (1982).

[47] L. Berg. *Asymptotische Darstellungen und Entwicklungen.* Deutscher Verlag der Wissenschaften, Berlin (1968).

[48] C. Berge. k-optimal partitions of a directed graph. *European J. Combin.*, **3**:97–101 (1982).

[49] C. Berge. Path-partitions in directed graphs and posets. In I. Rival, editor, *Graphs and Order (Banff, Alta. 1984)*, vol. **147** of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pp. 447–64, Reidel, Dordrecht, Boston (1985).

[50] C. Berge. *Hypergraphs.* North-Holland, Amsterdam, Tokyo (1989).

[51] A.J. Bernstein. Maximally connected arrays on the n-cube. *SIAM J. Appl. Math.*, **15**:1485–9 (1967).

[52] A.J. Bernstein, K. Steiglitz, and J. Hopcroft. Encoding of analog signals for a binary symmetric channel. *IEEE Trans. Inform. Theory*, **12**:425–30 (1966).

[53] T. Beth, D. Jungnickel, and H. Lenz. *Design theory.* B.I.-Wissenschaftsverlag, Mannheim, Vienna, Zurich (1985).

[54] C. Bey. On incidence matrices of a finite affine geometry. *Ars Combin.* (to appear).

[55] S. Bezrukov. Variational principle in discrete extremal problems. Reihe Informatik Bericht tr-ri-94-152, Universität-GH-Paderborn (1994).

[56] S.L. Bezrukov. Minimization of the shadows of the partial mappings semilattice (in Russian). *Diskret. Analiz*, **46**:3–16 (1988).

[57] S.L. Bezrukov. Isoperimetric problems in discrete spaces. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 59–91, János Bolyai Mathematical Society, Budapest (1994).

[58] S.L. Bezrukov. Discrete extremal problems on graphs and posets. Habilitationsschrift, Universität-GH Paderborn (1995).

[59] S.L. Bezrukov and K. Engel. Properties of graded posets preserved by some operations. In R.L. Graham and J. Nešetřil, editors, *Mathematics of Paul Erdős*, Springer-Verlag, Berlin, New York (to appear).

[60] S.L. Bezrukov and V.P. Voronin. Extremal ideals of the lattice of multisets with respect to symmetric functionals (in Russian). *Diskret. Mat.*, **2**:50–8 (1990).

[61] N. Biggs. *Algebraic Graph Theory.* Cambridge Univ. Press, Cambridge (1974).

[62] A. Björner. Orderings of Coxeter groups. In C. Greene, editor, *Combinatorics and algebra*, vol. **34** of *Contemporary Mathematics*, pp. 175–95, Amer. Math. Soc., Providence, RI (1984).

[63] A. Björner. Face numbers of complexes and polytopes. In A.M. Gleason, editor, *Proceedings of the International Congress of Mathematicians, Berkeley, 1986*, vol. 2, pp. 1408–18, Amer. Math. Soc. (1987).

[64] A. Björner. The mathematical work of Bernt Lindström. *European J. Combin.*, **14**:143–9 (1993).

[65] A. Björner. Nonpure shellability, f-vectors, subspace arrangements and complexity. *DIMACS Series in Discrete Mathematics and Theoretical Computer Science* (to appear).

[66] A. Björner, P. Frankl, and R. Stanley. The number of faces of balanced Cohen–Macaulay complexes and a generalized Macaulay theorem. *Combinatorica*, **7**:23–34 (1987).

[67] A. Björner and G. Kalai. An extended Euler-Poincaré theorem. *Acta Math.*, **161**: 279–303 (1988).

[68] A. Björner and G. Kalai. On f-vectors and homology. In G. Bloom, R. Graham, and J. Malkevitch, editors, *Combinatorial Mathematics: Proc. 3rd Intern. Conf., New York, 1985*, vol. **555** of *Annals New York Academy of Sciences*, pp. 63–80 (1989).

[69] A. Blokhuis. *Few distance sets.* PhD thesis, Eindhoven Univ. Technology (1983).

[70] A. Blokhuis. Solution of an extremal problem for sets using resultants of polynomials. *Combinatorica*, **10**:393–6 (1990).

[71] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar.*, **16**:447–52 (1965).

[72] B. Bollobás. Sperner systems consisting of pairs of complementary subsets. *J. Combin. Theory Ser. A*, **15**:363–6 (1973).

[73] B. Bollobás. *Combinatorics.* Cambridge Univ. Press, Cambridge (1986).

[74] B. Bollobás. The number of unrelated partitions. *J. Combin. Theory Ser. A*, **49**:389–91 (1988).
[75] B. Bollobás and I. Leader. Exact face-isoperimetric inequalities. *European J. Combin.*, **11**:335–40 (1990).
[76] B. Bollobás and I. Leader. Compression and isoperimetric inequalities. *J. Combin. Theory Ser. A*, **56**:47–62 (1991).
[77] B. Bollobás and I. Leader. Edge-isoperimetric inequalities in the grid. *Combinatorica*, **11**:299–314 (1991).
[78] B. Bollobás and A.J. Radcliffe. Isoperimetric inequalities for faces of the cube and the grid. *European J. Combin.*, **11**:323–33 (1990).
[79] K. Borsuk. Drei Sätze über die n-dimensionale Sphäre. *Fund. Math.*, **20**:177–90 (1933).
[80] R.C. Bose. A note on Fisher's inequality for balanced incomplete block designs. *Ann. Math. Stat.*, **20**:619–20 (1949).
[81] I. Bouchemakh and K. Engel. Interval stability and interval covering property in finite posets. *Order*, **9**:163–75 (1992).
[82] S. Bouroubi. Etude du treillis des partitions. Master's thesis, U.S.T.H.B., Alger (1991).
[83] A. Brace and D.E. Daykin. Sperner-type theorems for finite sets. In D.J.A. Welsh and D.R. Woodall, editors, *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pp. 18–37, Inst. Math. Appl., Southend-on-Sea (1972).
[84] F. Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, **81**(413) (1989).
[85] N.G. de Bruijn. *Asymptotic Methods in Analysis*. North-Holland, Amsterdam (1958).
[86] N.G. de Bruijn and P. Erdős. On a combinatorial problem. *Indagationes Math.*, **10**:421–3 (1948).
[87] N.G. de Bruijn, C.A.v.E. Tengbergen, and D. Kruyswijk. On the set of divisors of a number. *Nieuw Arch. Wisk.*, **23**:191–3 (1951).
[88] L.M. Butler. A unimodality result in the enumeration of subgroups of a finite abelian group. *Proc. Amer. Math. Soc.*, **101**:771–5 (1987).
[89] A.R. Calderblank and P. Frankl. Improved upper bounds concerning the Erdős–Ko–Rado Theorem. *Combinatorics, Probability and Computing*, **1**:115–22 (1992).
[90] K. Cameron and J. Edmonds. Coflow polyhedra. *Discrete Math.*, **101**:1–21 (1992).
[91] K.B. Cameron. On k-optimal dipath partitions and partial k-colourings of acyclic digraphs. *European J. Combin.*, **7**:115–18 (1986).
[92] E.R. Canfield. Central and local limit theorems for the coefficients of polynomials of binomial type. *J. Combin. Theory Ser. A*, **23**:275–90 (1977).
[93] E.R. Canfield. On a problem of Rota. *Adv. Math.*, **29**:1–10 (1978).
[94] E.R. Canfield. On the location of the maximum Stirling number(s) of the second kind. *Stud. Appl. Math.*, **59**:83–93 (1978).
[95] E.R. Canfield. A Sperner property preserved by product. *Linear and Multilinear Algebra*, **9**:151–7 (1980).
[96] E.R. Canfield. Matchings in the partition lattice. *SIAM J. Discrete Math.*, **6**:100–9 (1993).
[97] E.R. Canfield and L.H. Harper. A simplified guide to large antichains in the partition lattice. *Congr. Numer.*, **100**:81–8 (1994).
[98] E.R. Canfield and L.H. Harper. Large antichains in the partition lattice. *Random Stuctures Algorithms*, **6**:89–104 (1995).
[99] P.J. Chase. Subsequence numbers and logarithmic concavity. *Discrete Math.*, **16**:123–40 (1976).
[100] I.P. Chukhrov. On the maximal size of the shadow of an antichain. *Diskret. Mat.*, **1** (4):78–85 (1989).
[101] V. Chvátal. Intersecting families of edges in hypergraphs having the hereditary property. In C. Berge and D. Ray-Chaudhuri, editors, *Hypergraph Seminar (Columbus, 1972)*, vol. **411** of *Lect. Notes Math.*, pp. 61–6, Springer-Verlag, Berlin (1974).
[102] J. Chvátalová. Optimal labelling of a product of two paths. *Discrete Math.*, **2**:249–53 (1975).

[103] G.F. Clements. Sets of lattice points which contain a maximal number of edges. *Proc. Amer. Math. Soc.*, **27**:13–15 (1971).

[104] G.F. Clements. A minimization problem concerning subsets of a finite set. *Discrete Math.*, **4**:123–8 (1973).

[105] G.F. Clements. An existence theorem for antichains. *J. Combin. Theory Ser. A*, **22**:368–71 (1977).

[106] G.F. Clements. More on the generalized Macaulay theorem II. *Discrete Math.*, **18**:253–64 (1977).

[107] G.F. Clements. The minimal number of basic elements in a multiset antichain. *J. Combin. Theory Ser. A*, **25**:153–62 (1978).

[108] G.F. Clements. Antichains in the set of subsets of a multiset. *Discrete Math.*, **48**:23–45 (1984).

[109] G.F. Clements. A generalization of the Kruskal–Katona theorem. *J. Combin. Theory Ser. A*, **37**:91–7 (1984).

[110] G.F. Clements. On uniqueness of maximal antichains of subsets of a multiset. *Period. Math. Hungar.*, **15**:307–13 (1984).

[111] G.F. Clements. Multiset antichains having minimal downsets. *J. Combin. Theory Ser. A*, **48**:255–8 (1988).

[112] G.F. Clements. On multiset k-families. *Discrete Math.*, **69**:153–64 (1988).

[113] G.F. Clements. Another generalization of the Kruskal–Katona theorem. *J. Combin. Theory Ser. A*, **68**:239–45 (1994).

[114] G.F. Clements. Additive Macaulay posets. *Order* (submitted).

[115] G.F. Clements. The cubical poset is additive. *Discrete Math.* (submitted).

[116] G.F. Clements and H.-D.O.F. Gronau. On maximal antichains containing no set and its complement. *Discrete Math.*, **33**:239–47 (1981).

[117] G.F. Clements and B. Lindström. A generalization of a combinatorial theorem of Macaulay. *J. Combin. Theory*, **7**:230–38 (1969).

[118] E.F. Codd. A relational model of data for large shared data banks. *Comm. ACM*, **13**:377–87 (1970).

[119] D.M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs. Theory and Application*. Deutscher Verlag der Wissenschaften, Berlin (1980).

[120] G.B. Dantzig and A.J. Hoffman. Dilworth's theorem on partially ordered sets. In H.W. Kuhn and A.W. Tucker, editors, *Linear inequalities and related systems*, vol. **38** of *Annals of Mathematics Study*, pp. 207–14, Princeton Univ. Press, Princeton (1956).

[121] D.E. Daykin. A simple proof of the Kruskal–Katona theorem. *J. Combin. Theory Ser. A*, **17**:252–3 (1974).

[122] D.E. Daykin. An algorithm for cascades giving Katona-type inequalities. *Nanta Math.*, **8**:78–83 (1975).

[123] D.E. Daykin. Antichains in the lattice of subsets of a finite set. *Nanta Math.*, **8**:84–94 (1975).

[124] D.E. Daykin. A hierarchy of inequalities. *Stud. Appl. Math.*, **63**:263–4 (1980).

[125] D.E. Daykin, J. Godfrey, and A.J.W. Hilton. Existence theorems for Sperner families. *J. Combin. Theory Ser. A*, **17**:245–51 (1974).

[126] D.E. Daykin and L. Lovász. The number of values of a Boolean function. *J. London Math. Soc. (2)*, **12**:225–30 (1976).

[127] R. Dedekind. Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler. In *Festschrift der Techn. Hochsch. Braunschweig bei Gelegenheit der 69. Versammlung deutscher Naturforscher und Ärzte*, pp. 1–40 (1897).

[128] P. Delsarte. An algebraic approach to the association schemes of coding theory. Philips Res. Rep. Suppl., 10 (1973).

[129] P. Delsarte and P. Piret. An extension of an inequality by Ahlswede, El Gamal and Pang for pairs of binary codes. *Discrete Math.*, **55**:313–15 (1985).

[130] J. Demetrovics. On the number of candidate keys. *Inform. Process. Lett.*, **7**:266–9 (1978).

[131] J. Demetrovics and G.O.H. Katona. Extremal combinatorial problems in relational data base. In *Fundamentals of computation theory (Szeged, 1981)*, vol. **117** of *Lecture Notes in Comput. Sci.*, pp. 110–19, Springer-Verlag, Berlin, New York (1981).

[132] J. Demetrovics and H.N. Son. Databases, closure operations and Sperner families. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 199–203, János Bolyai Mathematical Society, Budapest (1994).

[133] A. Derbala. Classes de familles satisfaisant une expression booléenne et leurs enveloppes convexes. Master's thesis, U.S.T.H.B., Alger (1991).

[134] A. Derbala and K. Engel. Algorithmic investigation of the weighted extremal set problem. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 205–15, János Bolyai Mathematical Society, Budapest (1994).

[135] M. Deza and P. Frankl. Erdős–Ko–Rado theorem – 22 years later. *SIAM J. Alg. Disc. Meth.*, **4**:419–31 (1983).

[136] R.P. Dilworth. A decomposition theorem for partially ordered sets. *Ann. of Math.*, **51**:161–6 (1950).

[137] R.P. Dilworth. Some combinatorial problems on partially ordered sets. In *Combinatorial Analysis*, vol. **10** of *Proc. Sympos. Appl. Math.*, pp. 85–90, Amer. Math. Soc., Providence, RI (1960).

[138] R.P. Dilworth and C. Greene. A counterexample to the generalization of Sperner's theorem. *J. Combin. Theory*, **10**:18–21 (1971).

[139] W.F.D. Doran IV. Shuffling lattices. *J. Combin. Theory Ser. A*, **66**:118–36 (1994).

[140] T. Dowling and R.M. Wilson. Whitney number inequalities for geometric lattices. *Proc. Amer. Math. Soc.*, **47**:504–12 (1975).

[141] J. Eckhoff and G. Wegner. Über einen Satz von Kruskal. *Period. Math. Hungar.*, **6**:137–42 (1975).

[142] J. Edmonds. Systems of distinct representatives and linear algebra. *J. Res. Nat. Bur. Standards Sect. B*, **71B**:241–7 (1967).

[143] J. Edmonds and R. Giles. A min–max relation for submodular functions on graphs. *Ann. Discrete Math.*, **1**:185–204 (1977).

[144] J. Edmonds and R.M. Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.*, **19**:248–64 (1972).

[145] P. Elias, A. Feinstein, and C.E. Shannon. A note on the maximum flow through a network. *IRE Trans. Inform. Theory*, **IT-2**:117–19 (1956).

[146] K. Engel. Strong properties in partially ordered sets, I. *Discrete Math.*, **47**:229–34 (1983).

[147] K. Engel. An Erdős–Ko–Rado theorem for the subcubes of a cube. *Combinatorica*, **4**:133–40 (1984).

[148] K. Engel. Optimal representations, LYM posets, Peck posets, and the Ahlswede-Daykin inequality. *Rostock. Math. Kolloq.*, **26**:63–8 (1984).

[149] K. Engel. Strong properties in partially ordered sets, II. *Discrete Math.*, **48**:187–96 (1984).

[150] K. Engel. A new proof of a theorem of Harper on the Sperner–Erdős problem. *J. Combin. Theory Ser. A*, **39**:19–24 (1985).

[151] K. Engel. Recognition of order-preserving maps. *Order*, **2**:41–7 (1985).

[152] K. Engel. Optimal representations of partially ordered sets and a limit Sperner theorem. *European J. Combin.*, **7**:287–302 (1986).

[153] K. Engel. Convex hulls for intersecting-or-noncointersecting families. *Rostock. Math. Kolloq.*, **46**:11–16 (1993).

[154] K. Engel. Flow morphisms, the k-cutset problem, and the mod k problem. Preprint, Univ. Rostock (1993).

[155] K. Engel. On the average rank of an element in a filter of the partition lattice. *J. Combin. Theory Ser. A*, **65**:67–78 (1994).

[156] K. Engel. An algorithm for the determination of the variance of a partially ordered set. *J. Algorithms*, 19:441–8 (1995).

[157] K. Engel. Interval packing and covering in the Boolean lattice. *Combinatorics, Probability and Computing* (to appear).

[158] K. Engel and P.L. Erdős. Sperner families satisfying additional conditions and their convex hulls. *Graphs Combin.*, **5**:47–56 (1989).

[159] K. Engel and P.L. Erdős. Polytopes determined by complementfree Sperner families. *Discrete Math.*, **81**:165–9 (1990).

[160] K. Engel and P. Frankl. An Erdős–Ko–Rado theorem for integer sequences of given rank. *European J. Combin.*, **7**:215–20 (1986).

[161] K. Engel and H.-D.O.F. Gronau. *Sperner Theory in Partially Ordered Sets*. BSB B.G. Teubner Verlagsgesellschaft, Leipzig (1985).

[162] K. Engel and H.-D.O.F. Gronau. An Erdős–Ko–Rado type theorem II. *Acta Cybernet.*, **7**:405–11 (1986).

[163] K. Engel and N.N. Kuzyurin. An asymptotic formula for the maximum size of an h-family in products of partially ordered sets. *J. Combin. Theory Ser. A*, **37**:337–47 (1984).

[164] K. Engel and N.N. Kuzyurin. About the ratio of the size of a maximum antichain to the size of a maximum level in finite partially ordered sets. *Combinatorica*, **5**:301–9 (1985).

[165] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, **51**:898–902 (1945).

[166] P. Erdős. Extremal properties in number theory. In A.L. Whiteman, editor, *Theory of numbers*, pp. 181–9, Amer. Math. Soc., Providence, RI (1965).

[167] P. Erdős. Aufgabe 545. *Elem. Math.*, **24**:136 (1969).

[168] P. Erdős. Some unsolved problems in graph theory and combinatorial analysis. In D.J.A. Welsh, editor, *Combinatorial Mathematics and Its Applications (Proc. Conf. Math. Inst., Oxford, 1969)*, pp. 97–109, Academic Press, London (1971).

[169] P. Erdős. Some of my favourite unsolved problems. In A. Baker, B. Bollobás, and A. Hajnal, editors, *A tribute to Paul Erdős*, pp. 467–78. Cambridge University Press, Cambridge (1990).

[170] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite set. *Quart. J. Math. Oxford Ser.*, **12**:313–20 (1961).

[171] P. Erdős and R. Rado. Intersection theorems for systems of sets. *J. London Math. Soc.*, **35**:85–90 (1960).

[172] P. Erdős and J. Spencer. *Probabilistic methods in combinatorics*. Akadémiai Kiadó, Budapest (1974).

[173] P.L. Erdős, U. Faigle, and W. Kern. A group-theoretical setting for some intersecting Sperner families. *Combinatorics, Probability and Computing*, **1**:323–34 (1992).

[174] P.L. Erdős, U. Faigle, and W. Kern. Note on the average rank of LYM-sets. *Discrete Math.*, **144**:11–22 (1995).

[175] P.L. Erdős, P. Frankl, and G.O.H. Katona. Intersecting Sperner families and their convex hulls. *Combinatorica*, **4**:21–34 (1984).

[176] P.L. Erdős, P. Frankl, and G.O.H. Katona. Extremal hypergraph problems and convex hulls. *Combinatorica*, **5**:11–26 (1985).

[177] P.L. Erdős and G.O.H. Katona. Convex hulls of more-part Sperner families. *Graphs Combin.*, **2**:123–34 (1986).

[178] W. Feller. *An Introduction to Probability Theory and Its Applications I*. John Wiley & Sons, New York (1950).

[179] W. Feller. *An Introduction to Probability Theory and Its Applications II*. John Wiley & Sons, New York (1966).

[180] S. Felsner. Orthogonal structures in directed graphs. *J. Combin. Theory Ser. B*, **57**: 309–21 (1993).

[181] R.A. Fisher. An examination of the different possible solutions of a problem in incomplete blocks. *Ann. Eugenics*, **10**:52–75 (1940).

[182] S.V. Fomin. Finite partially ordered sets and Young tableaux. *Soviet Math. Dokl.*, **19**: 1510–14 (1978).

[183] L.R. Ford Jr. and D.R. Fulkerson. Maximal flow through a network. *Canad. J. Math.*, **8**:399–404 (1956).

[184] L.R. Ford Jr. and D.R. Fulkerson. *Flows in Networks*. Princeton Univ. Press, Princeton, NJ (1962).
[185] C.M. Fortuin, P.W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, **22**:89–103 (1971).
[186] A. Frank. On chain and antichain families of a partially ordered set. *J. Combin. Theory Ser. B*, **29**:176–84 (1980).
[187] P. Frankl. On Sperner families satisfying an additional condition. *J. Combin. Theory Ser. A*, **20**:1–11 (1976).
[188] P. Frankl. The Erdős–Ko–Rado theorem is true for $n = ckt$. In A. Hajnal and V.T. Sós, editors, *Combinatorics (Proc. Fifth Hungarian Colloq. on Combinatorics, Keszthely, 1976)*, vol. **18** of *Colloq. Math. Soc. János Bolyai*, pp. 365–75, North-Holland, Amsterdam (1978).
[189] P. Frankl. On intersecting families of finite sets. *J. Combin. Theory Ser. A*, **24**:146–61 (1978).
[190] P. Frankl. An extremal problem for two families of sets. *European J. Combin.*, **3**:125–7 (1982).
[191] P. Frankl. A new short proof for the Kruskal–Katona theorem. *Discrete Math.*, **48**:327–9 (1984).
[192] P. Frankl. Old and new problems on finite sets. *Congr. Numer.*, **67**:243–56 (1988).
[193] P. Frankl and Z. Füredi. Families of finite sets with a missing intersection. In A. Hajnal, L. Lovász, and V.T. Sós, editors, *Finite and Infinite Sets (Proc. 6th Hung. Colloq. on Combinatorics, Eger, 1981)*, vol. **37**, pp. 305–18, North-Holland, Amsterdam (1984).
[194] P. Frankl and Z. Füredi. On hypergraphs without two edges intersecting in a given number of vertices. *J. Combin. Theory Ser. A*, **36**:230–6 (1984).
[195] P. Frankl and Z. Füredi. Solution of the Littlewood–Offord problem in high dimensions. *Ann. Math., II. Ser.*, **128**:259–70 (1988).
[196] P. Frankl and Z. Füredi. Beyond the Erdős–Ko–Rado theorem. *J. Combin. Theory Ser. A*, **56**:182–94 (1991).
[197] P. Frankl, Z. Füredi, and G. Kalai. Shadows of colored complexes. *Math. Scand.*, **63**: 169–78 (1988).
[198] P. Frankl and R.L. Graham. Intersection theorems for vector spaces. *European J. Combin.*, **6**:183–7 (1985).
[199] P. Frankl and G.O.H. Katona. Polytopes determined by hypergraph classes. *European J. Combin.*, **6**:233–43 (1985).
[200] P. Frankl, M. Matsumoto, I.Z. Ruzsa, and N. Tokushige. Minimum shadows in uniform hypergraphs and a generalization of the Takagi function. *J. Combin. Theory Ser. A*, **69**:125–48 (1995).
[201] P. Frankl and J. Pach. On the number of sets in a null t-design. *European J. Combin.*, **4**:21–3 (1983).
[202] P. Frankl and R.M. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, **1**:357–68 (1981).
[203] P. Frankl and R.M. Wilson. The Erdős–Ko–Rado theorem for vector spaces. *J. Combin. Theory Ser. A*, **43**:228–36 (1986).
[204] R. Freese. An application of Dilworth's lattice of maximal antichains. *Discrete Math.*, **7**:107–9 (1974).
[205] D.R. Fulkerson. Note on Dilworth's decomposition theorem. *Proc. Amer. Math. Soc.*, **7**:701–2 (1956).
[206] D.R. Fulkerson. Blocking polyhedra. In B. Harris, editor, *Graph Theory and Its Applications*, pp. 93–112, Academic Press, New York (1970).
[207] D.R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, **1**:168–94 (1971).
[208] D.R. Fulkerson. Anti-blocking polyhedra. *J. Combin. Theory Ser. B*, **12**:50–71 (1972).
[209] Z. Füredi and J.R. Griggs. Families of finite sets with minimum shadows. *Combinatorica*, **6**:335–54 (1986).

[210] Z. Füredi, J.R. Griggs, and D.J. Kleitman. A minimal cutset of the Boolean lattice with almost all members. *Graphs Combin.*, **5**:327–32 (1989).

[211] Z. Füredi, J. Kahn, and D.J. Kleitman. Sphere coverings of the hypercube with incomparable centers. *Discrete Math.*, **83**:129–34 (1990).

[212] Z. Füredi and J. Pach. Traces of finite sets: Extremal problems and geometric applications. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 251–82, János Bolyai Mathematical Society, Budapest (1994).

[213] E.R. Gansner. Acyclic digraphs, Young tableaux and nilpotent matrices. *SIAM J. Alg. Disc. Meth.*, **2**:429–40 (1981).

[214] E.R. Gansner. Parenthesizations of finite distributive lattices. *Algebra Universalis*, **16**:287–303 (1983).

[215] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Freeman, San Francisco (1979).

[216] L. Gargano, J. Kőrner, and U. Vaccaro. Qualitative independence and Sperner problems for directed graphs. *J. Combin. Theory Ser. A*, **61**:173–92 (1992).

[217] L. Gargano, J. Kőrner, and U. Vaccaro. Sperner capacities. *Graphs Combin.*, **9**:31–46 (1993).

[218] L. Gargano, J. Kőrner, and U. Vaccaro. Capacities: From information theory to extremal set theory. *J. Amer. Math. Soc.* (to appear).

[219] F. Gavril. Algorithms for maximum k-colorings and k-coverings of transitive graphs. *Networks*, **17**:465–70 (1987).

[220] G.P. Gavrilov and A.A. Sapozhenko. *Problem Book in Discrete Mathematics (in Russian)*. Nauka, Moscow (1977).

[221] I.M. Gelfand and M.L. Zetlin. Finite dimensional representations of the group of unimodular matrices (in Russian). *Dokl. Akad. Nauk SSSR*, **71**:825–8 (1950).

[222] I. Gessel and G. Viennot. Binomial determinants, paths, and hook length formulae. *Adv. Math.*, **58**:300–21 (1985).

[223] E.N. Gilbert. Lattice theoretic properties of frontal switching functions. *J. Math. Phys.*, **33**:57–67 (1954).

[224] B.W. Gnedenko. *Lehrbuch der Wahrscheinlichkeitsrechnung*. Akademie-Verlag, Berlin (1978).

[225] J. Goldman and G.-C. Rota. On the foundations of combinatorial theory IV. Finite vector spaces and Eulerian generating functions. *Stud. Appl. Math.*, **49**:239–58 (1970).

[226] D.H. Gottlieb. A certain class of incidence matrices. *Proc. Amer. Soc.*, **17**:1233–7 (1966).

[227] R.L. Graham. Linear extensions of partial orders and the FKG inequality. In I. Rival, editor, *Ordered Sets (Banff, Alta., 1981)*, vol. **83** of *NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.*, pp. 473–521, Reidel, Dordrecht, Boston (1982).

[228] R.L. Graham and L.H. Harper. Some results on matchings in bipartite graphs. *SIAM J. Appl. Math.*, **17**:1017–22 (1969).

[229] C. Greene. Some partitions associated with a partially ordered set. *J. Combin. Theory Ser. A*, **20**:69–79 (1976).

[230] C. Greene. Posets of shuffles. *J. Combin. Theory Ser. A*, **47**:191–206 (1988).

[231] C. Greene and A.J.W. Hilton. Some results on Sperner families. *J. Combin. Theory Ser. A*, **26**:202–9 (1979).

[232] C. Greene and D.J. Kleitman. Strong versions of Sperner's theorem. *J. Combin. Theory Ser. A*, **20**:80–8 (1976).

[233] C. Greene and D.J. Kleitman. The structure of Sperner k-families. *J. Combin. Theory Ser. A*, **20**:41–68 (1976).

[234] C. Greene and D.J. Kleitman. Proof techniques in the theory of finite sets. In G.-C. Rota, editor, *Studies in Combinatorics*, vol. **17** of *MAA Studies in Math.*, pp. 22–79, Math. Assoc. America, Washington, DC (1978).

[235] J.R. Griggs. Sufficient conditions for a symmetric chain order. *SIAM J. Appl. Math.*, **32**:807–9 (1977).

[236] J.R. Griggs. *Symmetric chain orders, Sperner theorems, and loop matchings*. PhD thesis, M.I.T. (1977).

[237] J.R. Griggs. The Littlewood-Offord problem: Tightest packing and an m-part Sperner theorem. *European J. Combin.*, **1**:225–34 (1980).

[238] J.R. Griggs. On chains and Sperner k-families in ranked posets. *J. Combin. Theory Ser. A*, **28**:156–68 (1980).

[239] J.R. Griggs. Collections of subsets with the Sperner property. *Trans. Amer. Math. Soc.*, **269**:575–91 (1982).

[240] J.R. Griggs. Maximum antichains in the product of chains. *Order*, **1**:21–8 (1984).

[241] J.R. Griggs. Problems on chain partitions. *Discrete Math.*, **72**:157–62 (1988).

[242] J.R. Griggs. Matchings, cutsets, and chain partitions in graded posets. *Discrete Math.*, 144:33–46 (1995).

[243] J.R. Griggs and D.J. Kleitman. A three part Sperner theorem. *Discrete Math.*, **17**:281–9 (1977).

[244] H.-D.O.F. Gronau. Recognition of monotone functions. *Acta Cybernet.*, **4**:279–81 (1979).

[245] H.-D.O.F. Gronau. On maximal antichains consisting of sets and their complements. *J. Combin. Theory Ser. A*, **29**:370–5 (1980).

[246] H.-D.O.F. Gronau. On maximal families of subsets of a finite set. *Discrete Math.*, **34**: 119–30 (1981).

[247] H.-D.O.F. Gronau. On Sperner families in which no k sets have an empty intersection, III. *Combinatorica*, **2**:25–36 (1981).

[248] H.-D.O.F. Gronau. An Erdős-Ko-Rado type theorem. In A. Hajnal, L. Lovász, and V.T. Sós, editors, *Finite and Infinite Sets (Proc. 6th Hung. Colloq. on Combinatorics, Eger, 1981)*, vol. **37** of *Colloq. Math. Soc. J. Bolyai*, pp. 333–42, North-Holland, Amsterdam (1984).

[249] H.-D.O.F. Gronau and C. Proske. Maximal families of restricted subsets of a finite set. *Acta Cybernet.*, **5**:441–51 (1982).

[250] W. Haemers. Eigenvalue methods. In A. Schrijver, editor, *Packing and Covering in Combinatorics*, vol. **106** of *Mathematical Centre Tracts*, pp. 15–38, Mathematical Centre, Amsterdam (1979).

[251] P. Hall. On representatives of subsets. *J. London Math. Soc.*, **10**:26–30 (1935).

[252] G. Hansel. Sur le nombre des fonctions Booléennes monotones de n variables. *C.R. Acad. Sci. Paris Sér. A-B*, **262**:1088–90 (1966).

[253] G. Hansel. Complexes et décompositions binomiales. *J. Combin. Theory Ser. A*, **12**:167–83 (1972).

[254] L.H. Harper. Optimal assignments of numbers to vertices. *SIAM J. Appl. Math.*, **12**:131–5 (1964).

[255] L.H. Harper. Optimal numberings and isoperimetric problems on graphs. *J. Combin. Theory*, **1**:385–93 (1966).

[256] L.H. Harper. Stirling behavior is asymptotically normal. *Ann. Math. Stat.*, **38**:410–14 (1967).

[257] L.H. Harper. The morphology of partially ordered sets. *J. Combin. Theory Ser. A*, **17**: 44–58 (1974).

[258] L.H. Harper. The global theory of flows in networks. *Adv. in Appl. Math.*, **1**:158–81 (1980).

[259] L.H. Harper. A Sperner theorem for the interval order of a projective geometry. *J. Combin. Theory Ser. A*, **28**:82–8 (1980).

[260] L.H. Harper. Morphisms for the strong Sperner property of Stanley and Griggs. *Linear and Multilinear Algebra*, **16**:323–37 (1984).

[261] L.H. Harper and J.D. Chavez. Discrete isoperimetric problems and pathmorphisms (to appear).

[262] T.E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, **56**:13–20 (1960).

[263] S. Hart. A note on the edges of the n-cube. *Discrete Math.*, **14**:157–63 (1976).
[264] E. Harzheim. Remarks on Dilworth's decomposition theorem. *Ars Combin.*, **16-B**:27–31 (1983).
[265] W.K. Hayman. A generalization of Stirling's formula. *J. Reine Angew. Math.*, **196**:67–95 (1956).
[266] A.J.W. Hilton. A theorem on finite sets. *Quart. J. Math. Oxford Ser. (2)*, **27**:33–6 (1976).
[267] A.J.W. Hilton. A simple proof of the Kruskal–Katona theorem and of some associated binomial inequalities. *Period. Math. Hungar.*, **10**:25–30 (1979).
[268] A.J.W. Hilton and E.C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, **18**:369–84 (1967).
[269] A.J. Hoffman. Extending Greene's theorem to directed graphs. *J. Combin. Theory Ser. A*, **34**:102–7 (1983).
[270] A.J. Hoffman and D.E. Schwartz. On partitions of a partially ordered set. *J. Combin. Theory Ser. B*, **23**:3–13 (1977).
[271] R. Holley. Remarks on the FKG inequalities. *Comm. Math. Phys.*, **36**:227–31 (1974).
[272] W.N. Hsieh. Intersection theorems for systems of finite vector spaces. *Discrete Math.*, **12**:1–16 (1975).
[273] W.N. Hsieh and D.J. Kleitman. Normalized matching in direct products of partial orders. *Stud. Appl. Math.*, **52**:285–9 (1973).
[274] T. Huang. An analogue of the Erdős-Ko-Rado theorem for the distance-regular graphs of bilinear forms. *Discrete Math.*, **64**:191–8 (1987).
[275] J. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, New York (1972).
[276] J.R. Isbell. An inequality for incidence matrices. *Proc. Amer. Math. Soc.*, **10**:216–18 (1959).
[277] S. Jichang and D.J. Kleitman. Superantichains in the lattice of partitions of a set. *Stud. Appl. Math.*, **71**:207–41 (1984).
[278] L.K. Jones. Sur une loi de probabilité conditionelle de Kleitman. *Discrete Math.*, **15**:107–8 (1976).
[279] D. Jungnickel. *Graphen, Netzwerke und Algorithmen*. Bibliographisches Institut & F.A. Brockhaus AG, Mannheim (1994).
[280] J. Kahn. Some non-Sperner paving matroids. *Bull. London Math. Soc.*, **12**:268 (1980).
[281] J. Kahn and G. Kalai. A counterexample to Borsuk's conjecture. *Bull. Amer. Math. Soc.*, **29**:60–2 (1993).
[282] J. Kahn and M. Saks. On the widths of finite distributive lattices. *Discrete Math.*, **63**:183–95 (1987).
[283] G. Kalai. Intersection patterns of convex sets. *Israel J. Math.*, **48**:161–74 (1984).
[284] W.M. Kantor. On incidence matrices of finite projective and affine spaces. *Math. Z.*, **124**:315–8 (1972).
[285] S. Karlin. *Total Positivity*. Stanford Univ. Press, Stanford (1968).
[286] P.W. Kasteleyn. The statistics of dimers on a lattice. I. The number of dimer arrangements on a quadratic lattice. *Physica*, **27**:1209–25 (1961).
[287] N.N. Katerinochkina. Search for a maximal upper zero of a monotone function of the algebra of logic (in Russian). *Dokl. Akad. Nauk SSSR*, **224**:557–60 (1975).
[288] N.N. Katerinochkina. Search for a maximal upper zero for a class of monotone functions in k-valued logic (in Russian). *Dokl. Akad. Nauk SSSR*, **234**:751–4 (1977).
[289] N.N. Katerinochkina. Sets that contain the greatest number of pairwise non-comparable n-dimensional k-ary collections (in Russian). *Mat. Zametki*, **24**:367–74 (1978).
[290] G.O.H. Katona. Intersection theorems for systems of finite sets. *Acta Math. Acad. Sci. Hung.*, **15**:329–37 (1964).
[291] G.O.H. Katona. On a conjecture of Erdős and a stronger form of Sperner's theorem. *Studia Sci. Math. Hungar.*, **1**:59–63 (1966).
[292] G.O.H. Katona. A theorem of finite sets. In P. Erdős and G. Katona, editors, *Theory of Graphs*, pp. 187–207, Akadémiai Kiadó and Academic Press, Budapest and New York (1968).

[293] G.O.H. Katona. Families of subsets having no subset containing another one with small difference. *Nieuw Arch. Wisk.*, **20**:54–67 (1972).

[294] G.O.H. Katona. A generalization of some generalizations of Sperner's theorem. *J. Combin. Theory Ser. B*, **12**:72–81 (1972).

[295] G.O.H. Katona. A simple proof of the Erdős–Ko–Rado theorem. *J. Combin. Theory Ser. B*, **13**:183–4 (1972).

[296] G.O.H. Katona. A three part Sperner theorem. *Studia Sci. Math. Hungar.*, **8**:379–90 (1973).

[297] G.O.H. Katona. Two applications (for search theory and truth functions) of Sperner type theorems. *Period. Math. Hungar.*, **3**:19–26 (1973).

[298] G.O.H. Katona and G. Schild. Linear inequalities describing the class of intersecting Sperner families of subsets, I. In R. Bodendieck and R. Henn, editors, *Topics in Combinatorics and Graph Theory. Essays in Honour of Gerhard Ringel*, pp. 413–20, Physica-Verlag, Heidelberg (1990).

[299] D.J. Kleitman. On a lemma of Littlewood and Offord on the distribution of certain sums. *Math. Z.*, **90**:251–9 (1965).

[300] D.J. Kleitman. Families of non-disjoint subsets. *J. Combin. Theory*, **1**:153–5 (1966).

[301] D.J. Kleitman. On Dedekind's problem: The number of monotone Boolean functions. *Proc. Amer. Math. Soc.*, **21**:677–82 (1969).

[302] D.J. Kleitman. On subsets contained in a family of non-commensurable subsets of a finite set. *J. Combin. Theory*, **7**:181–3 (1969).

[303] D.J. Kleitman. On a lemma of Littlewood and Offord on the distribution of linear combinations of vectors. *Adv. Math.*, **5**:155–7 (1970).

[304] D.J. Kleitman. On an extremal property of antichains in partial orders. The LYM property and some of its implications and applications. In M. Hall and J.H. van Lint, editors, *Combinatorics, Part 2: Graph theory, foundations, partitions and combinatorial geometry (Proc. NATO Advanced Study Inst., Breukelen, 1974)*, vol. **56** of *Math. Centre Tracts*, pp. 77–90, Math. Centrum, Amsterdam (1974).

[305] D.J. Kleitman. Methods for asymptotic counting. *Congr. Numer.*, **32**:63–85 (1981).

[306] D.J. Kleitman, M. Edelberg, and D. Lubell. Maximal sized antichains in partial orders. *Discrete Math.*, **1**:47–53 (1971).

[307] D.J. Kleitman, M.M. Krieger, and B.L. Rothschild. Configurations maximizing the number of pairs of Hamming-adjacent lattice points. *Stud. Appl. Math.*, **50**:115–19 (1971).

[308] D.J. Kleitman and G. Markowsky. On Dedekind's problem: The number of monotone Boolean functions, II. *Trans. Amer. Math. Soc.*, **213**:373–90 (1975).

[309] D.J. Kleitman and E.C. Milner. On the average size of sets in a Sperner family. *Discrete Math.*, **6**:141–7 (1973).

[310] D.J. Kleitman, J.B. Shearer, and D. Sturtevant. Intersection of k-element sets. *Combinatorica*, **1**:381–4 (1981).

[311] D.J. Kleitman and J. Spencer. Families of k-independent sets. *Discrete Math.*, **6**:255–62 (1973).

[312] R. Kochendörffer. *Lehrbuch der Gruppentheorie unter besonderer Berücksichtigung der endlichen Gruppen.* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig (1966).

[313] G. Koester. Über direkte Produkte Spernerscher Halbordnungen. In *Contributions to graph theory and its applications (Proc. Internat. Colloq., Oberhof, 1977)*, pp. 167–72, TH Ilmenau, Ilmenau (1977).

[314] J. Kőrner and G. Simonyi. A Sperner-type theorem and qualitative independence. *J. Combin. Theory Ser. A*, **59**:90–103 (1992).

[315] V.K. Korobkov. On monotone functions of the algebra of logic (in Russian). *Problemy Kibernet.*, **13**:5–28 (1965).

[316] V.K. Korobkov and T.L. Reznik. On some algorithms for the evaluation of monotone functions of the algebra of logic (in Russian). *Dokl. Akad. Nauk SSSR*, **147**:1022–5 (1962).

[317] A.D. Korshunov. The number of monotone Boolean functions (in Russian). *Problemy Kibernet.*, **38**:5–108 (1980).

[318] A.D. Korshunov. Complexity of shortest disjunctive normal forms of Boolean functions (in Russian). *Metody Diskret. Analiz.*, **37**:9–41, 85 (1981).

[319] A.D. Korshunov. Families of subsets of a finite set and closed classes of Boolean functions. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 375–96, János Bolyai Mathematical Society, Budapest (1994).

[320] Ë.Sh. Kospanov. On coverings by unit balls whose centers are incomparable. *Metody Diskret. Analiz.*, **44**:54–7 (1986).

[321] A.V. Kostochka. On the maximal cardinality of the boundary of a filter in the unit cube (in Russian). *Diskret. Analiz*, **40**:49–61 (1984).

[322] A.V. Kostochka. An upper bound for the size of the boundary of an antichain in the n-dimensional cube. *Diskret. Mat.*, **1** (3):53–61 (1989).

[323] M.M. Kovalëv and A.V. Moshchenski. Oracle complexity of maximization of submodular functions (in Russian). *Dokl. Akad. Nauk BSSR*, **36**:111–14 (1992).

[324] E. Krätzel. *Zahlentheorie*. Deutscher Verlag der Wissenschaften, Berlin (1981).

[325] J.B. Kruskal. The number of simplices in a complex. In R. Bellman, editor, *Mathematical Optimization Techniques*, pp. 251–78, Univ. of California Press, Berkeley, Los Angeles (1963).

[326] J.B. Kruskal. The number of s-dimensional spaces in a complex. An analogy between the complex and the cube. *J. Combin. Theory*, **6**:86–9 (1969).

[327] J.P.S. Kung. The Radon transforms of a combinatorial geometry, I. *J. Combin. Theory Ser. A*, **26**:97–102 (1979).

[328] J.P.S. Kung. Radon transforms in combinatorics and lattice theory. In I. Rival, editor, *Combinatorics and ordered sets (Proc. Summer research conference, Arcata, 1985)*, vol. **57** of *Contemp. Math.*, pp. 33–74, Amer. Math. Soc., Providence, RI (1986).

[329] J.P.S. Kung. The Radon transforms of the combinatorial geometry. II. Partition lattices. *Adv. Math.*, **101**:114–32 (1993).

[330] N.N. Kuzyurin. On the approximative search for a maximal upper zero of monotone functions in k-valued logic (in Russian). *Metody Diskret. Analiz.*, **33**:31–40 (1979).

[331] R. Labahn. Maximizing antichains in the cube with fixed size of a shadow. *Order*, **9**: 349–55 (1992).

[332] D.G. Larman and C.A. Rogers. The realization of distances within sets in Euclidean space. *Mathematica*, **19**:1–24 (1972).

[333] U. Leck. Extremalprobleme für den Schatten in Posets. PhD thesis, Freie Universität Berlin (1995).

[334] U. Leck. A property of colored complexes and their duals. Preprint 96/9, Univ. Rostock (1996).

[335] B. Leclerc. A finite Coxeter group the weak Bruhat order of which is not symmetric chain. *European J. Combin.*, **15**:181–5 (1994).

[336] K. Leeb. Salami-Taktik beim Quader-Packen. *Arbeitsber. Inst. Math. Masch. Datenverarb. (Inform.)*, **11** (5):1–15 (1978).

[337] H. Lefmann. On intersecting families of finite affine and linear spaces over $GF(q)$. In A. Hajnal and L. Lovász, editors, *Combinatorics (Proc. 7th Hungarian Colloq. on Combinatorics, Eger, 1987)*, vol. **52** of *Colloq. Math. Soc. János Bolyai*, pp. 365–74, North-Holland, Amsterdam (1988).

[338] H. Lefmann. An extremal problem for Graham–Rothschild parameter words. *Combinatorica*, **9**:153–60 (1989).

[339] H. Lefmann. On families in finite lattices. *European J. Combin.*, **11**:165–79 (1990).

[340] D. Lichtblau and H.S. Snevily. Intersecting families with restricted intersection values. *J. Combin. Theory Ser. A* (to appear).

[341] T.M. Liggett. Extensions of the Erdős-Ko-Rado theorem and a statistical application. *J. Combin. Theory Ser. A*, **23**:15–21 (1977).

[342] K.W. Lih. Problems in finite extremal set theory. *Surikaisekikenkyusho-Kokyuroku*, **607**:1–4 (1987).

[343] J.H. Lindsey II. Assignment of numbers to vertices. *Amer. Math. Monthly*, **71**:508–16 (1964).

[344] B. Lindström. Conjecture on a theorem similar to Sperner's. In R. Guy, H. Hanani, N. Sauer, and J. Schönheim, editors, *Combinatorial Structures and Their Applications*, p. 241, Gordon and Breach, New York (1970).

[345] B. Lindström. The optimal number of faces in cubical complexes. *Ark. Mat.*, **8**:245–57 (1971).

[346] B. Lindström. A partition of $L(3, n)$ into saturated symmetric chains. *European J. Combin.*, **1**:61–3 (1980).

[347] B. Lindström and H.-O. Zetterström. A combinatorial problem in the k-adic number system. *Proc. Amer. Math. Soc.*, **18**:166–70 (1967).

[348] N. Linial. Extending the Greene–Kleitman theorem to directed graphs. *J. Combin. Theory Ser. A*, **30**:331–4 (1981).

[349] J. Littlewood and C. Offord. On the number of real roots of a random algebraic equation III. *Mat. Sb.*, **12**:277–85 (1943).

[350] D. Loeb, E. Damiani, and O. d'Antona. Decompositions of B_n and Π_n using symmetric chains. *J. Combin. Theory Ser. A*, **65**:151–7 (1994).

[351] Z. Lonc and I. Rival. Chains, antichains, and fibres. *J. Combin. Theory Ser. A*, **4**:207–28 (1987).

[352] E. London. A new proof of the colored Kruskal–Katona theorem. *Discrete Math.*, **126**:217–23 (1994).

[353] L. Lovász. Flats in matroids and geometric graphs. In P.J. Cameron, editor, *Combinatorial Surveys (Proc. 6th British Comb. Conf., Egham 1977)*, pp. 45–86, Academic Press, London (1977).

[354] L. Lovász. *Combinatorial problems and exercises*. Akadémiai Kiadó, Budapest (1979).

[355] L. Lovász. On the Shannon capacity of a graph. *IEEE Trans. Inform. Theory*, **25**:1–7 (1979).

[356] L. Lovász and M.D. Plummer. *Matching theory*. Akadémiai Kiadó, Budapest (1986).

[357] D. Lubell. A short proof of Sperner's lemma. *J. Combin. Theory*, **1**:299 (1966).

[358] F.S. Macaulay. Some properties of enumeration in the theory of modular systems. *Proc. London Math. Soc.*, **26**:531–55 (1927).

[359] R. Mahnke. Einige Beispiele für rcc-symmetric chain Ordnungen. *Rostock. Math. Kolloq.*, **29**:29–32 (1986).

[360] K.N. Majumdar. On some theorems in combinatorics relating to incomplete block designs. *Ann. Math. Stat.*, **24**:377–89 (1953).

[361] J. Marica and J. Schönheim. Differences of sets and a problem of Graham. *Canad. Math. Bull.*, **12**:635–7 (1969).

[362] B. Martos. *Nonlinear programming. Theory and methods*. Akadémiai Kiadó, Budapest (1975).

[363] K. Mehlhorn. *Data Structures and Algorithms II: Graph Algorithms and NP-Completeness*. Springer-Verlag, Berlin, Tokyo (1984).

[364] L.D. Meshalkin. A generalization of Sperner's theorem on the number of subsets of a finite set (in Russian). *Teor. Veroyatnost. i Primenen.*, **8**:219–20 (1963).

[365] N. Metropolis and G.-C. Rota. Combinatorial structure of the faces of the n-cube. *SIAM J. Appl. Math.*, **35**:689–94 (1978).

[366] N. Metropolis and G.-C. Rota. On the lattice of faces of the n-cube. *Bull. Amer. Math. Soc.*, **84**:284–6 (1978).

[367] N. Metropolis, G.-C. Rota, V. Strehl, and N. White. Partitions into chains of a class of partially ordered sets. *Proc. Amer. Math. Soc.*, **71**:193–6 (1978).

[368] J.-C. Meyer. Quelques problèmes concernant les cliques des hypergraphes k-complets et q-parti h-complets. In C. Berge and D. Ray-Chaudhuri, editors, *Hypergraph Seminar (Columbus, 1972)*, vol. **411** of *Lecture Notes Math.*, pp. 127–39, Springer-Verlag, Berlin (1974).

[369] V.M. Mikheev. On sets containing the greatest number of pairwise noncomparable Boolean vectors. *Problemy Kibernetiki*, **2**:69–71 (1959).

[370] E.C. Milner. A combinatorial theorem on systems of sets. *J. London Math. Soc.*, **43**: 204–6 (1968).

[371] L. Mirsky. A dual of Dilworth's decomposition theorem. *Amer. Math. Monthly*, **78**: 876–7 (1971).

[372] H.S. Moghadam. Compression operators and a solution to the bandwidth problem of the product of n paths. PhD thesis, University of California, Riverside (1983).

[373] A. Moon. An analogue of the Erdős–Ko–Rado theorem for the Hamming schemes $H(n, q)$. *J. Combin. Theory Ser. A*, **32**:386–90 (1980).

[374] M. Mörs. A generalization of a theorem of Kruskal. *Graphs Combin.*, **1**:167–83 (1985).

[375] L. Moser and M. Wyman. An asymptotic formula for the Bell numbers. *Trans. Roy Soc. Canada*, **49**:49–54 (1955).

[376] R. Mullin. On Rota's problem concerning partitions. *Aequationes Math.*, **2**:98–104 (1969).

[377] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley & Sons, Chichester, Singapore (1988).

[378] A. Nilli. On Borsuk's problem. In H. Barcelo and G. Kalai, editors, *Proc. Jerusalem Combinatorics '93*, vol. **178** of *Contemporary Mathematics*, pp. 209–10. Amer. Math. Soc. (1994).

[379] G.W. Peck. Maximum antichains of rectangular arrays. *J. Combin. Theory Ser. A*, **27**:397–400 (1979).

[380] H. Perfect. Symmetrized form of P. Hall's theorem on distinct representatives. *Quart. J. Math. Oxford Ser. (2)*, **17**:303–6 (1966).

[381] M.A. Perles. A proof of Dilworth's decomposition theorem for partially ordered sets. *Israel J. Math.*, **1**:105–7 (1963).

[382] V.V. Petrov. *Sums of independent random variables*. Akademie-Verlag, Berlin (1975).

[383] S. Polyak and Z. Tuza. On the maximum number of qualitatively independent partitions. *J. Combin. Theory Ser. A*, **51**:11–116 (1989).

[384] M. Pouzet. Application d'une proprieté combinatoire des parties d'un ensemble aux groupes et aux relations. *Math. Z.*, **150**:117–34 (1976).

[385] M. Pouzet and I.G. Rosenberg. Sperner properties for groups and relations. *European J. Combin.*, **7**:349–70 (1986).

[386] C.J. Preston. A generalization of the FKG inequalities. *Comm. Math. Phys.*, **36**:233–41 (1974).

[387] R.A. Proctor. Representations of $sl(2, \mathbb{C})$ on posets and the Sperner property. *SIAM J. Alg. Disc. Meth.*, **3**:275–80 (1982).

[388] R.A. Proctor. Solution of two difficult problems with linear algebra. *Amer. Math. Monthly*, **89**:721–34 (1982).

[389] R.A. Proctor. A Dynkin diagram classification theorem arising from a combinatorial problem. *Adv. Math.*, **62**:103–17 (1986).

[390] R.A. Proctor. Solution of a Sperner conjecture of Stanley with a construction of Gelfand. *J. Combin. Theory Ser. A*, **54**:225–34 (1990).

[391] R.A. Proctor, M.E. Saks, and P.G. Sturtevant. Product partial orders with the Sperner property. *Discrete Math.*, **30**:173–80 (1980).

[392] G.B. Purdy. A result on Sperner collections. *Utilitas Math.*, **12**:95–9 (1977).

[393] L. Pyber. An extension of a Frankl–Füredi theorem. *Discrete Math.*, **52**:253–68 (1984).

[394] D.K. Ray-Chaudhuri and R.M. Wilson. On t-designs. *Osaka J. Math.*, **12**:737–44 (1975).

[395] B.C. Rennie and A.J. Dobson. On Stirling numbers of the second kind. *J. Combin. Theory*, **7**:116–21 (1969).

[396] A. Rényi. *Wahrscheinlichkeitsrechnung mit einem Anhang über Informationstheorie*. Deutscher Verlag der Wissenschaften, Berlin (1966).

[397] A. Rényi. *Foundations of Probability*. Holden-Day, San Francisco, Amsterdam (1970).

[398] K. Reuter. Note on the Ahlswede-Daykin inequality. *Discrete Math.*, **65**:209–12 (1987).

[399] W. Riess. Zwei Optimierungsprobleme auf Ordnungen. *Arbeitsber. Inst. Math. Masch. Datenverarb. (Inform.)*, **11** (5):1–59 (1978).
[400] Y. Rinott and M. Saks. Correlation inequalities and a conjecture for permanents. *Combinatorica*, **13**:269–77 (1993).
[401] G.-C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Z. Wahrsch. Verw. Gebiete*, **2**:340–68 (1964).
[402] G.-C. Rota. A generalization of Sperner's theorem. *J. Combin. Theory*, **2**:104 (1967).
[403] I.Z. Ruzsa. Probabilistic generalization of a number-theoretical inequality. *Amer. Math. Monthly*, **83**:723–5 (1976).
[404] M. Saks. Some sequences associated with combinatorial structures. *Discrete Math.*, **59**:135–66 (1986).
[405] M.E. Saks. A short proof of the existence of k-saturated partitions of partially ordered sets. *Adv. Math.*, **33**:207–11 (1979).
[406] M.E. Saks. Dilworth numbers, incidence maps and product partial orders. *SIAM J. Alg. Disc. Meth.*, **1**:211–15 (1980).
[407] A. Sali. Stronger form of an m-part Sperner theorem. *European J. Combin.*, **4**:179–83 (1983).
[408] A. Sali. A Sperner-type theorem. *Order*, **2**:123–7 (1985).
[409] A. Sali. Constructions of ranked posets. *Discrete Math.*, **70**:77–83 (1988).
[410] A. Sali. A note on convex hulls of more-part Sperner families. *J. Combin. Theory Ser. A*, **49**:188–90 (1988).
[411] M.V. Sapir. Recognition of monotone Boolean functions. Preprint, Medical Center "Bonum", Ekaterinburg (1995).
[412] A.A. Sapozhenko. The complexity of disjunctive normal forms that are obtainable by means of the gradient algorithm (in Russian). *Diskret. Analiz*, **21**:62–71 (1972).
[413] A.A. Sapozhenko. On the number of antichains in multilayered ranged sets (in Russian). *Diskret. Mat.*, **1**:110–28 (1989).
[414] M. Sarrafzadeh and R.D. Lou. Maximum k-covering of weighted transitive graphs with applications. *Algorithmica*, **9**:84–100 (1993).
[415] N. Sauer. On the density of families of sets. *J. Combin. Theory Ser. A*, **13**:145–7 (1972).
[416] J.H. Schmerl. A general framework for discovering and proving theorems of the Erdős-Ko-Rado type. *J. Combin. Theory Ser. A*, **61**:252–62 (1992).
[417] J. Schönheim. On a problem of Daykin concerning intersecting families of sets. In T.P. McDonough and V.C. Mavron, editors, *Combinatorics (Proc. Br. Combinatorics Conf. Aberystwyth 1973)*, vol. **13** of *London Math. Soc. Lecture Note Ser.*, pp. 139–40, Cambridge Univ. Press, London (1974).
[418] J. Schönheim. On a problem of Purdy related to Sperner systems. *Canad. Math. Bull.*, **17**:135–6 (1974).
[419] A. Schrijver. Association schemes and the Shannon capacity: Eberlein polynomials and the Erdős–Ko–Rado theorem. In L. Lovász and V.T. Sós, editors, *Algebraic methods in graph theory II (Proc. Conf. Szeged 1978)*, vol. **25** of *Colloq. Math. Soc. János Bolyai*, pp. 671–88, North-Holland, Amsterdam (1981).
[420] A. Schrijver. Min–max results in combinatorial optimization. In B. Korte, A. Bachem, and M. Grötschel, editors, *Mathematical Programming: The State of the Art*, pp. 439–500, Springer-Verlag, Berlin, New York (1983).
[421] A. Schrijver. *Theory of Linear and Integer Programming*. John Wiley & Sons, Chichester, Singapore (1986).
[422] M.P. Schützenberger. A characteristic property of certain polynomials of E. F. Moore and C. E. Shannon. In *RLE Quarterly Progress Report*, vol. **55**, pp. 117–18. Research Lab. of Electronics, MIT (1959).
[423] R. Sedgewick. *Algorithms*. Addison-Wesley, Bonn, Taipei (1991).
[424] A.V. Serzhantov. An optimal algorithm to decipher some classes of monotone functions (in Russian). *Zh. Vychisl. Mat. i Mat. Fiz.*, **23**:206–12 (1983).
[425] A.V. Serzhantov. On an optimal algorithm to decipher monotone functions (in Russian). *Dokl. Akad. Nauk SSSR*, **277**:304–7 (1984).

[426] P.D. Seymour. On incomparable collections of sets. *Mathematika*, **20**:208–9 (1973).
[427] P.D. Seymour and D.J.A. Welsh. Combinatorial applications of an inequality from statistical mechanics. *Math. Proc. Cambridge Philos. Soc.*, **77**:485–95 (1975).
[428] J.B. Shearer. A simple counterexample to a conjecture of Rota. *Discrete Math.*, **28**:327–30 (1979).
[429] S. Shelah. A combinatorial problem: Stability and order for models and theories in infinitary languages. *Pacific J. Math.*, **41**:247–61 (1972).
[430] L.A. Shepp. The XYZ conjecture and the FKG inequality. *Ann. Probab.*, **10**:824–7 (1980).
[431] R. Simion and D. Ullman. On the structure of the lattice of noncrossing partitions. *Discrete Math.*, **98**:193–206 (1991).
[432] H. Snevily. A new result on Chvátal's conjecture. *J. Combin. Theory Ser. A*, **61**:137–41 (1992).
[433] H.S. Snevily. On generalizations of the deBruijn-Erdős theorem. *J. Combin. Theory Ser. A*, **68**:232–8 (1994).
[434] N.A. Sokolov. Optimal recognition of monotone Boolean functions (in Russian). *Zh. Vychisl. Mat. i Mat. Fiz.*, **27**:1878–87 (1987).
[435] J.H. Spencer. A generalized Rota conjecture for partitions. *Stud. Appl. Math.*, **53**:239–41 (1974).
[436] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, **27**:544–8 (1928).
[437] E. Sperner. Über einen kombinatorischen Satz von Macaulay und seine Anwendung auf die Theorie der Polynomideale. *Abh. Math. Sem. Univ. Hamburg*, **7**:149–63 (1930).
[438] R. Stanley. Cohen-Macaulay complexes. In M. Aigner, editor, *Higher Combinatorics*, vol. **31** of *NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.*, pp. 51–62, Reidel, Dordrecht (1977).
[439] R.P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Alg. Disc. Meth.*, **1**:168–84 (1980).
[440] R.P. Stanley. Quotients of Peck posets. *Order*, **1**:29–34 (1984).
[441] R.P. Stanley. *Enumerative Combinatorics, Volume I.* Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA (1986).
[442] R.P. Stanley. Differential posets. *J. Amer. Math. Soc.*, **1**:919–61 (1988).
[443] R.P. Stanley. Some applications of algebra to combinatorics. *Discrete Appl. Math.*, **34**:241–77 (1991).
[444] R.P. Stanley. *Combinatorics and Commutative Algebra.* Birkhäuser, Boston (1996).
[445] D. Stanton. Some Erdős–Ko–Rado theorems for Chevalley groups. *SIAM J. Alg. Disc. Meth.*, **1**:160–3 (1980).
[446] D. Stanton and D. White. *Constructive Combinatorics.* Springer-Verlag, New York, Tokyo (1986).
[447] C. Stendal. Analyse eines Algorithmus zur Lösung eines speziellen quadratischen Optimierungsproblems. Master's thesis, Universität Rostock (1993).
[448] B. Sturmfels, R. Weismantel, and G.M. Ziegler. Gröbner bases of lattices, corner polyhedra, and integer programming. *Beiträge Algebra Geom.* (to appear).
[449] M.M. Sysło, N. Deo, and J.S. Kowalik. *Discrete Optimization Algorithms with Pascal Programs.* Prentice-Hall, Englewood Cliffs, NJ (1983).
[450] O. Tamaschke. *Projektive Geometrie I.* Bibliographisches Institut, Mannheim, Vienna, Zurich (1969).
[451] R.E. Tarjan. Algorithms for maximizing network flow. *Math. Programming*, **26**:1–11 (1986).
[452] W.T. Trotter. *Combinatorics of partially ordered sets: Dimension theory.* Johns Hopkins Univ. Press, Baltimore, London (1992).
[453] Z. Tuza. Application of the set-pair method in extremal hypergaph theory. In P. Frankl, Z. Füredi, G. Katona, and D. Miklós, editors, *Extremal problems for finite sets*, vol. **3** of *Bolyai Society Mathematical Studies*, pp. 479–514, János Bolyai Mathematical Society, Budapest (1994).

[454] H. Tverberg. On Dilworth's decomposition theorem for partially ordered sets. *J. Combin. Theory*, **3**:305–6 (1967).

[455] V.N. Vapnik and A. Ya. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory Probab. Appl.*, **16**:264–80 (1971).

[456] A.P. Vikulin. Bounds for the number of prime implicants (in Russian). *Problemy Kibernet.*, **29**:151–66 (1974).

[457] F. Vogt and B. Voigt. Symmetric chain decompositions of linear lattices. Preprint 1633, TH Darmstadt (1994).

[458] B. Voigt and I. Wegener. A remark on minimal polynomials of Boolean functions. In E. Borger, H. Kleine Buning, and M.M. Richter, editors, *CSL '88 (Proc. 2nd Workshop Computer, Science, Logic, Duisburg, 1988)*, vol. **385** of *Lecture Notes in Comput. Sci.*, pp. 372–83, Springer-Verlag, Berlin, New York (1989).

[459] D.-L. Wang and P. Wang. Discrete isoperimetric problems. *SIAM J. Appl. Math.*, **32**: 860–70 (1977).

[460] G. Wegner. Kruskal-Katona's theorem in generalized complexes. In A. Hajnal, L. Lovász, and V.T. Sós, editors, *Finite and Infinite Sets II (Proc. 6th Hung. Colloq. on Combinatorics, Eger, 1981)*, vol. **37** of *Colloq. Math. Soc. J. Bolyai*, pp. 821–7, North-Holland, Amsterdam (1984).

[461] V.K. Wei. A connection between a convex programming problem and the LYM property on perfect graphs. *J. Graph Theory*, **12**:571–87 (1988).

[462] L. Weisner. Abstract theory of inversion of finite series. *Trans. Amer. Math. Soc.*, **38**: 474–84 (1935).

[463] D.B. West. A symmetric chain decomposition of $L(4, n)$. *European J. Combin.*, **1**: 379–83 (1980).

[464] D.B. West. Extremal problems in partially ordered sets. In I. Rival, editor, *Ordered Sets (Banff, Alta., 1981)*, vol. **83** of *NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.*, pp. 473–521, Reidel, Dordrecht, Boston (1982).

[465] D.B. West. Parameters of partial orders and graphs: Packing, covering, and representation. In I. Rival, editor, *Graphs and Orders*, pp. 267–350, D. Reidel, Dordrecht, Boston (1985).

[466] D.B. West. "Poly-unsaturated" posets: The Greene–Kleitman theorem is best possible. *J. Combin. Theory Ser. A*, **41**:105–16 (1986).

[467] D.B. West, L.H. Harper, and D.E. Daykin. Some remarks on normalized matching. *J. Combin. Theory Ser. A*, **35**:301–8 (1983).

[468] D.B. West and D.J. Kleitman. Skew chain orders and sets of rectangles. *Discrete Math.*, **27**:99–102 (1979).

[469] H.S. Wilf. *Generatingfunctionology*. Academic Press, New York (1990).

[470] R.M. Wilson. The exact bound in the Erdős–Ko–Rado theorem. *Combinatorica*, **4**:247–57 (1984).

[471] K. Yamamoto. Logarithmic order of free distributive lattices. *J. Math. Soc. Japan*, **6**:343–53 (1954).

[472] J. Yu. An extremal problem for sets: A new approach via Bezoutians. *J. Combin. Theory Ser. A*, **62**:170–5 (1993).

[473] G.M. Ziegler. *Lectures on Polytopes*. Springer-Verlag, Berlin, Heidelberg, New York (1995).

[474] Yu.A. Zuev. Representation of Boolean functions by systems of linear inequalities. *Kibernetika*, **5**:7–9, 40 (1985).

INDEX