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Contents

Introduction

The present work is intended to develop three main results:

(1) An extension of the standard basis theorem, going back to Doubilet, Rota, and Stein, and eventually to Capelli and Young, to algebras containing positively signed and negatively signed variables, or superalgebras as we call them. Such an extension has required a rethinking of some of the basic concepts of linear algebra, such as "matrix" and "coordinate system," along lines that we believe to be new, and which we hope will lead to an extension to "signed" modules of the entire apparatus of linear algebra. The standard basis theorem, which we prove, is characteristic-free and includes, besides the classical case, straightening algorithms, which apply to permanents as well as to determinants, as well as a mixed generalization of the notion of both determinant and permanent, called the biproduct, which differs from the Berezin determinant.

(2) A rigorous presentation of the symbolic method of invariant theory for symmetric tensors, in characteristic zero. The results here offer no great novelty over the nineteenth century, except rigor.

(3) A new symbolic method (foreshadowed by Weitzenböck) for the representation of invariants of skew-symmetric tensors. Here, the results turn out to be more satisfactory. Symbolic expressions for the invariants of skew-symmetric tensors are more manageable and easier to compute than those for symmetric tensors. In fact, in contrast to symmetric tensors, the "meaning" of the vanishing of an invariant can be more easily gleaned from the symbolic representation, as we show by several examples.

In both instances, the actual invariant is obtained from the symbolic representation by applying an operator which we call the umbral operator. Invariants of symmetric tensors are obtained by applying the umbral operator to certain polynomials in a commutative algebra, whereas invariants of skew-symmetric tensors are obtained by applying the umbral operator to polynomials in an anticommutative algebra. Thus, the umbral operator can be viewed as mapping an anticommutative algebra into a commutative algebra, and vice versa. It is an instance of a Schur functor.

The umbral operator thus establishes a cryptomorphism between commutative and anticommutative algebras, and can be used to systematically develop anticommutative analogs of concepts of algebraic geometry. This, we believe, may ultimately turn out to be the main byproduct of the present investigation.

In the exposition, we have preferred to use algebras generated by an alphabet over algebras generated by a free module. The results can, however, be recast in a basis-free language: the choice between these two equivalent languages is largely a matter of taste and of the objectives at hand.

Synopsis

We prove two distinct but closely related results. The first is the extension of the standard basis theorem to superalgebras (defined below). The second is the application of the standard basis theorem to the computation of invariants (and, more generally, of covariants) of symmetric and skew-symmetric tensors. In this synopsis we give an informal description of the main ideas and results which can be read independently of the body of the work and which can be used as a guideline to the text.

We begin by recalling the three fundamental algebraic systems of invariant theory: the symmetric algebra, the divided powers algebra, and the exterior algebra.

(1) Given an alphabet A^0 (that is, a set A^0 whose elements are to be viewed as "variables"), the symmetric algebra $\mathrm{Symm}(A^0)$ generated by A^0 is the familiar commutative algebra of polynomials in the variables A^0. The coefficients of these polynomials will be integers, although (here and everywhere below) an arbitrary commutative ring with identity could be taken as the ring of coefficients.

(2) Given an alphabet A^-, the exterior algebra $\mathrm{Ext}(A^-)$ is the algebra generated by the variables A^-, subject to the identities $ab = -ba$ and $a^2 = 0$ for $a, b \in A^-$. Thus, $\mathrm{Ext}(A^-)$ is the algebra of "polynomials in anticommutative variables A^-." A nonzero monomial in $\mathrm{Ext}(A^-)$ is a product of a finite sequence of variables

$$a_1 a_2 \cdots a_n, \qquad a_i \in A^-,$$

where no two a_i coincide, and two monomials are related by the familiar "sign law"

$$a_1 a_2 \cdots a_n = (\operatorname{sgn}\sigma) a_{\sigma 1} a_{\sigma 2} \cdots a_{\sigma n}$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$.

Given an alphabet A^+, the divided powers algebra $\mathrm{Div}(A^+)$ is the commutative algebra generated by the variables $a^{(i)}$, as a ranges over A and as $i = 0, 1, 2, \ldots$. We set $a^{(0)} = 1$ and $a^{(1)} = a$. The idea is that $a^{(i)}$ is to satisfy the same identities

(3) as $a^i/i!$. Thus we impose the identities

$$(*) \qquad a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)}$$

(other identities usually impose in the definition of the divided power algebra will not be needed, and need not be recalled here).

We wish to develop a suitable notation for the tensor product of these three algebras, for three disjoint alphabets A^0, A^-, and A^+. The usual notation of tensor products proves unwieldy, and we choose to describe the tensor product by a more direct route, namely, as the monoid algebra of a certain monoid to be presently defined. Thus, after taking the disjoint sum $A^0 \oplus A^- \oplus A^+ = A$ we consider a monoid Mon$[A]$ which is "almost" the free monoid generated by A. The words in Mon$[A]$ shall be products of variables in A^0 and in A^-, and of the divided powers $a^{(i)}$ for $a \in A^+$. Thus a word appears as a product, e.g.,

$$w = abc^{(i)}de^{(j)}.$$

The identities among these monomials are those that follow from the following commutation relations $ab = ba$ in all cases except when both a and b belong to A^-, in which case we set $ab = -ba$. Thus, if $a, b \in A^-$, if $c, d \in A^0$, and if $e, f \in A^+$ then we have, for example,

$$(**) \qquad dac^2e^{(3)}bf^{(5)} = -c^2bf^{(5)}ae^{(3)}d.$$

The product of two words is juxtaposition, except that products of divided powers are to be simplified by $(*)$. Thus, for example, $(ae)(be) = 2abe^{(2)}$. The length of a monomial is computed taking into account the fact that the divided power $a^{(i)}$ is to be considered of Length i. Thus, the monomial $(**)$ is of length (= degree) 13. With these conventions, the monoid algebra of Mon$[A]$ is well defined by taking formal linear combinations and products. We call it the *superalgebra* Super$[A]$ generated by the *signed alphabet* A. The variables of the alphabet A will be designated as neutral, negatively signed, and positively signed, respectively.

The superalgebra Super$[A]$, although a worthwhile subject of investigation, is not the immediate object of the present study. We need to define a more complex structure, which will be called the *fourfold algebra*. To this end, we consider two signed alphabets $L = L^+ \oplus L^-$ and $P = P^+ \oplus P^-$, called *proper*, because neither has neutral elements. Their elements will be called *letters* and *places*, respectively. From these two alphabets we define a third signed alphabet $[L \mid P]$, the *letterplace alphabet*. The definition of $[L \mid P]$ is fundamental. The elements of $[L \mid P]$ will be pairs $(x \mid \alpha)$, where $x \in L$ and $\alpha \in P$; these pairs will

sometimes be called letterplaces (the geometric motivation for this term will be given shortly). Their signatures are determined by the following rules:

(i) $(x \mid \alpha) \in [L \mid P]^+$ if $x \in L^+$ and $\alpha \in P^+$,
(ii) $(x \mid \alpha) \in [L \mid P]^0$ if $x \in L^-$ and $\alpha \in P^-$,
(iii) $(x \mid \alpha) \in [L \mid P]^-$ otherwise.

The Superalgebra Super$[L \mid P]$ is called the *fourfold algebra*. Our objective will be to show that the fourfold algebra is the suitable machinery to develop the invariant theory of symmetric and skew-symmetric tensors, and, as a byproduct, to formulate a "signed" generalization of linear algebra.

All theorems of linear and multilinear algebras can be viewed as consequences of the Laplace expansions for determinants. This sweeping assertion is in part made precise by the second fundamental theorem of invariant theory, and it will serve as motivation for the generalization of some such theorem to superalgebras that we shall develop below.

In ordinary linear algebra, one deals with vectors (here denoted by letters) and coordinates (also known as linear functionals, here denoted by places). The αth coordinate of a vector x is a *scalar* (here denoted by a neutral variable). Thus, vectors must be taken as negatively signed letters, and coordinates must be taken as negatively signed places. The value of a vector x at the coordinate α is denoted by $(x \mid \alpha)$ and is a scalar (that is, a neutral element). A *matrix* is a set $\{(x \mid \alpha); x \in X,\ \alpha \in A\}$ where X and A are respectively linearly ordered sets of letters and places of the same size, i.e., $|X| = |A| = n$. Relabeling the letters $x_1, \ldots, x_n$ and the places $\alpha_1, \ldots, \alpha_n$ by their linear order, the determinant of a matrix is then defined as usual (except for sign) as

$$(-1)^{n(n-1)/2} \sum_\sigma (\operatorname{sgn}\sigma)(x_1 \mid \alpha_{\sigma 1})(x_2 \mid \alpha_{\sigma 2}) \cdots (x_n \mid \alpha_{\sigma n}),$$

where σ ranges over all permutations of the set $\{1, 2, \ldots, n\}$ of indices. One verifies that the above expression equals (expansion by columns instead of rows)

$$(-1)^{n(n-1)/2} \sum_\sigma (\operatorname{sgn}\sigma)(x_{\sigma 1} \mid \alpha_1)(x_{\sigma 2} \mid \alpha_2) \cdots (x_{\sigma n} \mid \alpha_n).$$

We wish to recast this definition in a form that will be suitable for generalization to the fourfold algebra. To this end, we write the above determinant by the notation $(x_1 x_2 \cdots x_n \mid \alpha_1 \alpha_2 \cdots \alpha_n)$ and we note that this expression extends to a bilinear map from the exterior algebra Ext(X) in the letters, and from the exterior algebra Ext(A) in the places, with values in the symmetric algebra in the letterplaces $(x \mid \alpha)$. We are thus led to generalize such a bilinear map to superalgebras by suitably generalizing the notion of determinant and of the Laplace expansion that characterizes it. We first have to recast the classical Laplace expansions of a determinant in a language that is suitable for generalization. It turns out that the language of Hopf algebras is eminently suited to this purpose. We first review this language in the (classical) case of two exterior algebras Ext(X) and Ext(A) or, what is equivalent, to the case of the superalgebra Super[$L \mid P$] where $L^0 = L^+ = P^0 = P^+ = \emptyset$. In this special case, the determinant can be viewed as a bilinear form from Super[L] $\times$ Super[P] to Super[$L \mid P$]. Let w be a word in Mon[L] and let w' be a word in Mon[P]. We shall define $(w \mid w')$ as an element of Super[$L \mid P$]. The bilinear function $(w \mid w')$ shall satisfy the following conditions:

(i) $(w \mid w') = 0$ unless the words w and w' have the same length.

(ii) $(w \mid w') = (x \mid \alpha)$ if $w = x$ and $w' = \alpha$, that is, if the words are of length one, that is, if the words reduce to a single letter and place.

(iii) $(1 \mid 1) = 1$, where 1, the identity of the algebra, can be identified with a word of length zero.

Finally, we shall impose on $(w \mid w')$ the analog of Laplace expansions. To state these properly we recall that if w is a word in an exterior algebra, the *coproduct*

$$\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$$

is the sum over all pairs of such words $w_{(1)}$ and $w_{(2)}$ such that $w_{(1)}w_{(2)} = w$, with suitable signs (incorporated in the notation) which need only be specified in the text.

In the notation of Hopf algebras, the Laplace expansion of a determinant $(w \mid w'w'')$ takes the pleasing form

$$\sum_w \pm (w_{(1)} \mid w')(w_{(2)} \mid w''),$$

where again we do not yet worry about signs. There is a similar expansion "by columns" that is similar to the present expansion by rows.

We now generalize this notation to an arbitrary fourfold algebra Super[$L \mid P$]. The fourfold algebra is the tensor product of four algebras, each one generated by those letterplaces $(x \mid \alpha)$ where x is either positive or negative, and α either positive or negative (recall that there are no neutral letters or places). Each factor in such a tensor product is an exterior algebra, a divided powers algebra, or a symmetric algebra.

Similarly, each of the superalgebras Super[L] and Super[P] is the tensor product of an exterior algebra and a divided powers algebra.

We now extend the definition of a "determinant" (which we call a *biproduct*) to the fourfold algebra as follows. If $w \in$ Super[L] and $w', w'' \in$ Super[P], we again define a bilinear form $(w \mid w')$ taking values in Super[$L \mid P$] (note that we now allow L and P to be arbitrary proper alphabets, that is, alphabets without neutral elements). The biproduct $(w \mid w')$ will be subject to (i), (ii), and (iii) exactly as above, and to the technical condition

(iv) $(x^{(n)} \mid \alpha^{(n)}) = (x \mid \alpha)^{(n)}$ for divided powers of positively signed letters and places.

Finally, one assumes analogs of the Laplace expansions in the form

$$(*) \qquad (w \mid w'w'') = \sum_w \pm (w_{(1)} \mid w')(w_{(2)} \mid w''),$$

with suitable signs, where

$$\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$$

is the coproduct in the Hopf algebra Super[L] (see below for further explanations). One also assumes a similar Laplace expansion where the roles of letters and places are interchanged.

The sum on the right of the Laplace expansion $(*)$, as well as the sign of each term, can be understood without knowledge of Hopf algebras as follows. The sum on the right of $(*)$ ranges over all distinct ordered pairs $w_{(1)}$, $w_{(2)}$ of subwords of w such that $w_{(1)}w_{(2)} = w$. By (i), the only nonzero terms are given by pairs $w_{(1)}, w_{(2)}$ such that Length$(w_{(2)})$ = Length(w') and Length$(w_{(2)})$ = Length(w''). The sign of each term on the right of $(*)$ is determined by the following rule. In the monoid Mon[$L \oplus P$] generated by the disjoint sum of the alphabets L and P, consider the words $ww'w''$ and $w_{(1)}w'w_{(2)}w''$. Evidently the second word can be obtained from the first by a succession of transpositions of adjacent elements of $L \oplus P$ (with proper care given to divided powers). The sign of the corresponding term on the right of $(*)$ is then the parity of the number of such transpositions, for which the two adjacent elements of $L \oplus P$, which are being transposed, are both negatively signed. We stress the fact that of the two letters being transposed, one may belong to L and another to P.

EXAMPLE. Suppose all letters are positively signed, and places are of arbitrary signature. Then

$$(a^{(2)}b \mid \alpha\beta\gamma) = (a^{(2)} \mid \alpha\beta)(b \mid \gamma) + (ab \mid \alpha\beta)(a \mid \gamma).$$

(v) Finally, one assumes a dual Laplace expansion

$$(w'w'' \mid w) = \sum \pm (w' \mid w_{(1)})(w'' \mid w_{(2)})$$

with a similar rule for the signs.

We show in the text that these expansion rules are consistent. They define a bilinear form, which we call the *biproduct*, which can be viewed as a signed generalization of the determinant. When w and w' are products of distinct positively signed letters and places, then the biproduct $(w \mid w')$ reduces to the permanent. This is not the case, however, for $(w \mid w')$ when w and w' are arbitrary products of divided powers. In this case, the pair $(w \mid w')$ can in no way be seen as the generalization of the determinant or permanent of a matrix (unless one is willing to consider the possibility that the set of rows and the set of columns of such a "matrix" shall be allowed to become multisets).

The biproduct preserves all formal properties (that is, the Laplace expansions) of a determinant, except for signs. As already remarked, all of linear and multilinear algebra can be recast in terms of Laplace identities. It stands to reason, therefore, to expect that the biproduct will yield a generalization of linear algebra to "signed" vector spaces and signed modules generally. In this paper we take two steps toward the realization of this program, which we now proceed to informally describe.

The first step is the extension to the fourfold algebra of the standard basis theorem of Doubilet, Rota, and Stein (a special case of our result, which in the present notation corresponds to setting $L = L^-$ and $P = P^-$). Such a generalization proceeds as follows. Define a Young diagram over L (and similarly over P) as a sequence $D = (w_1, w_2, \ldots, w_n)$ of words w_i in Mon$[L]$, such that $\lambda_i = \text{Length}(w_i)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The vector $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the *shape* of the Young diagram. If

$$w_i = x_{i1}x_{i2}\cdots x_{i\lambda i}, \qquad x_{ij} \in L,$$

where we have written out each divided power by repeating the same letter (strictly speaking, this is an abuse of notation), then the words

$$\tilde{w}_j = x_{ij}x_{2j}\cdots x_{\lambda j \cdot j}$$

define the *dual diagram* $\tilde{D}$ of shape $\tilde{\lambda}$.

A Young diagram D over L (or over P) will be said to be *standard* when, for every pair of words

$$w_i = x_1x_2\cdots x_r, \qquad x_j \in L,$$
$$\tilde{w}_i = y_1y_2\cdots y_s, \qquad y_j \in L,$$

where w_i is in D and $\tilde{w}_i$ is in $\tilde{D}$ the following conditions are satisfied. Let us refer to w_i as a *row* of the diagram and to $\tilde{w}_i$ as a *column* of the diagram. Choose a linear order on the alphabet L, which will remain fixed from now on. Then

(i) two successive letters x_i and x_{i+1} in the same row are in nondecreasing order ($x_i \leq x_{i+1}$) if they are both positively signed and in strictly increasing order ($x_i < x_{i+1}$) if they are both negatively signed.

(ii) two successive letters y_i and y_{i+1} in the same column of the tableau D (i.e., in the same row of the tableau $\tilde{D}$) are in strictly increasing order ($y_i < y_{i+1}$) if they are both positively signed and in nondecreasing order ($y_i \leq y_{i+1}$) if they are both negatively signed.

(iii) two successive letters in the same row (or in the same column) which are of different signatures are in strictly increasing order.

A similar definition is given for Young diagrams made out of P.

Now let D and $E = [w'_1, \ldots, w'_n]$ be Young diagrams of the same shape made out of the alphabets L and P, respectively. We define the *Young tableau* of the diagram pair D, E to be

$$\text{Tab}(D \mid E) = \pm(w_1 \mid w'_1)(w_2 \mid w'_2)\cdots(w_n \mid w'_n).$$

(This is a slight oversimplification, because of the possible occurrence of divided powers—the text gives the precise definition.)

A *standard Young tableau* Tab$(D \mid E)$ is a Young tableau where both diagrams D and E are standard. Our main result states that standard Young tableaux are an integral basis of the superalgebra Super$[L \mid P]$. Actually, the final result is stronger. It states that a (not necessarily standard) tableau Tab$(D \mid E)$ is uniquely expressible as a linear combination with integer coefficients of standard Young tableaux. If Tab$(D \mid E)$ is of shape λ, then the only standard Young tableaux occurring with nonzero coefficients are of shapes which are greater than λ in the dominance order of shapes. Thus the submodules, spanned by all tableaux whose shapes form an order ideal in the partially ordered set of shapes under the dominance order, span an invariant submodule of Super$[L \mid P]$, under permutations of both letters and places.

An immediate corollary of this theorem is that the number of standard Young diagrams is independent of the linear ordering chosen for the alphabet. No direct combinatorial proof of this surprising fact is known at present.

The proof of the standard basis theorem proceeds in two steps. The fact that the standard tableaux span the superalgebra is a consequence of the following *straightening formula*

$$\sum_w \pm(vw_{(1)} \mid u')(w_{(2)}w \mid u) = \sum_{u,v} \pm(wv_{(1)} \mid u'u_{(1)})(v_{(2)}w' \mid u_{(2)}).$$

(For the actual computation of the signs, consult the text.)

The straightening formula leads to a recursive algorithm (ultimately going back to Alfred Young) to express any monomial in the superalgebra as an integral linear combination of standard tableaux.

The fact that the standard tableaux are linearly independent is subtler; it is based on a refined duality. One defines a dual alphabet $L^* = L^{*+} \cup L^{*-}$ with L^{*+} in bijective correspondence with L^- and L^{*-} with L^+. Thus, if x is a letter, then x^* is a letter of the opposite sign; similarly, for P^*. One then defines a scalar bilinear form

$$\langle p, q\rangle, \qquad p \in \text{Super}[L^* \mid P^*],\ q \in \text{Super}[L \mid P],$$

by the following rules:

(i) $\langle (x^* \mid \alpha^*), (y \mid \beta)\rangle = 0$ if $x \neq y$ or $\alpha \neq \beta$, and

(ii) $\langle (x^* \mid \alpha^*), (x \mid \alpha)\rangle = \pm 1$, where the sign on the right is negative if and only if both x and α^* are negatively signed. To see the motivation for this rule, we (quite unrigorously) rewrite the left side as

$$(x^* \mid \alpha^*)(x \mid \alpha) = \pm(x^* \mid x)(\alpha^* \mid \alpha),$$

and we see that the minus sign is due to the "fact" that the elements x and α^* have been commuted.

(iii) Analogs of Laplace expansions. For example, if $w'w'' \in \text{Super}[L^* \mid P^*]$ and $w \in \text{Super}[L \mid P]$, we set

$$\langle w'w'', w\rangle = \sum_w \pm\langle w', w_{(1)}\rangle\langle w'', w_{(2)}\rangle.$$

Here, the coproduct

$$\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$$

is taken in the Hopf algebra Super$[L \mid P]$, which is the tensor product of two Hopf algebras; that is, the pairs $w_{(1)}$ and $w_{(2)}$ range over all subwords of w, for $w \in$ Super$[L \mid P]$, such that $w = w_{(1)}w_{(2)}$. Again, the signs on the right are determined according to the number of transpositions of adjacent negatively signed elements (letters or places).

The byproduct of this seemingly unusual setup for duality of two algebras is the generalization to superalgebras of the celebrated result of A. Young (in Young's language, stating that $P(12)N(12) = 0$). In this present context, we prove that $\langle (w^* \mid u^*), (w' \mid u') \rangle = 0$ whenever the length of the biproducts is two or more. This leads to a considerable simplification of the computation of

$$\langle \mathrm{Tab}(D^* \mid E^*),\ \mathrm{Tab}(D' \mid E') \rangle.$$

One finds upon evaluating that on the right side only sums of certain simple monomials appear (which we have called Gale-Ryser interpolants after a famous theorem of Gale and Ryser on the existence of 0-1-matrices having given row and column sums). One goes on to establish a triangular biorthogonality property for standard tableaux, from which linear independence is inferred.

In order to describe the invariant-theoretic results, we shall assume from now on that $P = P^- = \{1, 2, \ldots, n\}$. We define a *bracket* (of length n) to be the element $[w] = (w \mid 1\,2 \cdots n)$ of Super$[L \mid P]$. Clearly the bracket will vanish unless the word w in Super$[L]$ is of length n. From now on we shall separately consider either of the two special cases $L = L^-$ or $L = L^+$. In the case $L = L^-$, the bracket coincides with the classical bracket first defined by Cayley; that is, if $w = x_1 x_2 \cdots x_n$, the bracket is the determinant of the matrix of n row vectors x_i which have the entries

$$x_i = ((x_i \mid 1), (x_i \mid 2), \ldots, (x_i \mid n)).$$

Note that because $L = L^-$ and $P = P^-$, each entry is a neutral element, and thus can legitimately be called a scalar. The straightening formula now specializes to give identities on brackets of the form

$$(**) \qquad \sum_w \pm [v w_{(1)}][w_{(2)} w'] = \sum_v \pm [w v_{(1)}][v_{(2)} w'],$$

which, as is known, can be used for an abstract characterization of the bracket. All of linear algebra is "coded" in these identities.

In the second case, $L = L^+$, one obtains a remarkable generalization of the determinant. The bracket $[w]$, for $w \in$ Super$[L] =$ Super$[L^+]$, will now no longer be scalar valued but will take its value in an exterior algebra. We shall call it the *skew bracket*. It can no longer be viewed as the determinant or the permanent of a "matrix", unless w is a product of n distinct letters. One computes, for example, $[a^{(n)}] = (a \mid 1)(a \mid 2) \cdots (a \mid n)$, where each $(a \mid i)$ is an element of degree one of an exterior algebra. More generally, one has

$$[w][w'] = (-1)^n [w'][w],$$

and identities of the form $(**)$ hold for skew brackets (with fewer signs) and in fact can be used to characterize skew brackets. Thus, the possibility arises that skew brackets are the syntax for a semantic construct which will be a "skew" analog of linear algebra. We shall leave this enticing possibility untouched for the moment, and proceed directly to the invariant-theoretic applications of the bracket (both classical and skew).

For ease of exposition, we deal separately with symmetric and skew-symmetric tensors, keeping the narrative as informal as possible, at the cost of some inaccuracies which are corrected in the text.

Given a vector space V of dimension n over a field of characteristic zero, a symmetric tensor t over V (that is, a homogeneous element of the symmetric algebra $S(V)$) has, relative to a basis $e_1, e_2, \ldots, e_n$, the coordinates $a_{j_1 j_2 \cdots j_n}$, so that

$$t = \sum \binom{k}{j_1, j_2, \ldots, j_n} a_{j_1 j_2 \cdots j_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n},$$

where the sum ranges over all n-tuples $(j_1 \cdots j_n)$; note that almost all the terms equal zero (because the multinomial coefficient vanishes). The integer k is the degree or *step* of the tensor. An *invariant* I of t is a polynomial in the coordinates $a_{j_1 j_2 \cdots j_n}$ which is independent of the choice of the basis, except for a constant factor. Thus, if I is an invariant, then the *condition* $I = 0$ is *geometric*; that is, it is independent of the coordinate system. The idea of invariant theory is to express by the vanishing of invariants certain logically or combinatorially defined properties of tensors such as decomposability, rank, divisibility by another tensor, etc. To this end, a slight extension of the notion of invariant is needed, namely, the notion of a covariant.

Observe that a *joint invariant* of a set of tensors may be defined in the obvious way. When the coordinates of one or more of the tensors of such a set are allowed to be independent transcendentals (adjoined to the base field), a joint invariant will turn out to be a polynomial in these independent transcendentals. Such a polynomial is called a *covariant*, with the understanding that a covariant vanishes when it vanishes identically as a polynomial in its independent transcendentals.

We next describe the symbolic method for the representation of all invariants (and hence all covariants) of symmetric tensors. To every (symmetric) tensor t we associate an indefinite supply of letters (or *symbols*) belonging to a *negatively signed* alphabet L^-, and we say that a letter "belongs" to the tensor t.

We next define an operator U, the *umbral operator*, from the superalgebra Super$[L \mid P]$ to the "scalar" coordinates of a tensor of step k. We simply set

$$\langle U, (a \mid 1)^{j_1} (a \mid 2)^{j_2} \cdots (a \mid n)^{j_n} \rangle = a_{j_1 j_2 \cdots j_n},$$

where a is any of the letters of L belonging to a tensor t of step k, and where $j_1+j_2+\cdots+j_n=k$. If $j_1+j_2+\cdots+j_n\neq k$, the right side is set equal to zero. Note that all $(a \mid i)$ are neutral elements.

If w is any monomial in Super$[L \mid P]$ (a symmetric algebra) we can write $w=w(a)w(b)\cdots$ when $w(a)$ is the product of all letterplaces $(a \mid i)$ containing the letter a, etc. We set

$$\langle U,w\rangle=\langle U,w(a)w(b)\cdots\rangle=\langle U,w(a)\rangle\langle U,w(b)\rangle\cdots.$$

Finally, we extend the definition of U to all of Super$[L \mid P]$ by linearity. The main result states that a polynomial I is an invariant of a set of symmetric tensors if and only if $I=\langle U,p\rangle$, where p is a polynomial in brackets (in the present case of symmetric tensors, the brackets are nothing but ordinary determinants).

The classification of invariants can be further simplified by applying the standard basis theorem. According to this theorem, any polynomial in brackets can be integrally expressed as an integral linear combination of bracket products which correspond to standard tableaux, in the given linear ordering of the alphabet L. Thus, it suffices (by and large) to consider the case where p is a standard tableau in Super$[L \mid P]$, where each row is of length n (thus, the n places $1,2,\ldots,n$ appear in each row on the right side of the tableau). If x_i is a transcendental, we can safely set

$$\langle U,(x \mid i)^j\rangle=x_i^j,$$

thereby identifying x with a symbol. If we further agree that the linear order of L shall be chosen in such a way as to place last all symbols corresponding to transcendentals (that is, after all symbols corresponding to genuine tensors), then a standard tableau for a covariant is partitioned into an upper part, containing all symbols corresponding to the tensor(s), and a lower part, containing only (symbols for) transcendentals. For all practical purposes, only the upper part will matter in the description of the covariant. We thus succeed in associating to every covariant a standard Young diagram in the familiar form such as

$$\begin{matrix} abcd \\ abe \\ bfg. \end{matrix}$$

The invariant theory of symmetric tensors can be carried much farther by the symbolic method, but the present exposition now turns to the much less explored subject of skew-symmetric tensors, where previous work (done largely in the nineteenth century) was considerably more limited in scope. Our objectives will be, first, to develop a symbolic method that will be strikingly analogous to the symbolic method for symmetric tensors, and second, to show by examples that the method is (for skew-symmetric tensors better than for symmetric tensors) effective both for the construction and for the interpretation of invariants and covariants.

The method is "dual" to the one described above for symmetric tensors, in the sense that the words "commutative" and "anticommutative" are judiciously interchanged. It is not clear a priori, however, how such an interchange should be carried out, and in fact the method we are about to describe reveals new clues to how this interchange may be effective in disparate situations.

A skew-symmetric tensor t of step k is a homogeneous element of the exterior algebra $\bigwedge(V)$. Relative to a basis $e_1,e_2,\ldots,e_n$ of V, the tensor can be written in the form

$$t=\sum_{i_1<i_2<\cdots<i_k}(a_{i_1i_2\cdots i_k})e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}$$

where the sum ranges over all subsets $\{i_1,i_2,\ldots,i_k\}$ of $\{1,2,\ldots,n\}$, with the customary conventions relative to signs, namely.

$$a_{\sigma_1\sigma_2\cdots\sigma_k}=(\operatorname{sgn}\sigma)a_{12\cdots k}.$$

An invariant I of t is a polynomial in the coordinates $a_{i_1i_2\cdots i_k}$ which is independent of the coordinates, except for a constant factor, so that (again) the condition $I=0$ is geometric. Joint invariants and covariants are defined as in the symmetric case.

We now describe the delicate symbolic method by which all invariants (and covariants) of skew-symmetric tensors can be represented. Choose a linearly ordered alphabet L which is now *positively signed* ($L=L^+$). Recall that the place alphabet P is the *negatively signed* set $\{1,2,\ldots,n\}$. To every (skew-symmetric) tensor t we assign an infinite supply of letters "belonging" to it, and we assume that the supply is ample enough for all purposes. In addition, we assume that L contains an indefinitely large supply of letters belonging to an indefinitely large supply of independent transcendentals that may be adjoined to the ground field if and when necessary. We next define an umbral operator U that (again) maps the superalgebra Super$[L \mid P]$ to the field of scalars, eventually with transcendentals adjoined.

If a is a letter belonging to the skew-symmetric tensor t of step k, we set

$$\langle U,(a^{(k)} \mid i_1i_2\cdots i_k)\rangle=a_{i_1i_2\cdots i_k}.$$

On the right side is the scalar component of the tensor t relative to the basis $e_1,e_2,\ldots,e_n$, which will remain fixed from now on. On the left side, $a^{(k)}$ is the kth divided power of the letter a, so that (by the properties of the byproduct)

$$(a^{(k)} \mid i_1i_2\cdots i_k)=(a \mid i_1)(a \mid i_2)\cdots(a \mid i_k).$$

Note that for $i\neq j$ we have $(a \mid i)(a \mid j)=-(a \mid j)(a \mid i)$, since $(a \mid i)$ is negatively signed to all $a\in L$ and $i\in P$.

Next, set $\langle U,(a^{(j)} \mid i_1i_2\cdots i_j)\rangle=0$, if $j\neq k$.

Now let n be an arbitrary monomial in Super$[L \mid P]$. By altering the sign of m, if necessary, we can write

$$m=\pm m(a)m(b)\cdots,$$

where $m(a)$ is the product of all letter places $(a \mid i)$ containing the letter a, etc., and, what is essential, where $a < b < \cdots$ in the linear order of L. Thus,

$$m(a) = (a^{(j)} \mid i_1 i_2 \cdots i_j)$$

for some j, similarly for $m(b)$, etc.

Now set

$$\langle U, m(a)m(b)\cdots\rangle = \langle U, m(a)\rangle\langle U, m(b)\rangle\cdots,$$

and extend to Super$[L \mid P]$ by linearity. We prove that the operator U is well defined. For example, let a and b be both letters belonging to the tensor t, and suppose that $a < b$. Then

$$\langle U, (a^{(k)} \mid 12\cdots k)(b^{(k)} \mid 12\cdots k)\rangle = (a_{12\cdots k})^2,$$

but

$$\langle U, (b^{(k)} \mid 12\cdots k)(a^{(k)} \mid 12\cdots k)\rangle = -(a_{12}\cdots k)^2.$$

if k is odd. As another example, let u and v be vectors in V, that is, skew-symmetric tensors of step one. Then we have

$$\langle U, (a \mid i)\rangle = u_i, \qquad \langle U, (b \mid i)\rangle = v_i,$$

if a belongs to u and b belongs to v. Thus, if $a < b$, then

$$\langle U, (ab \mid ij)\rangle = \langle U, (a \mid i)(b \mid j) + (b \mid i)(a \mid j)\rangle = u_i v_j - u_j v_i,$$

and we find (spectacularly enough) that

$$\langle U, (ab \mid ij)\rangle = \langle U, (ba \mid ij)\rangle,$$

as we might well expect, since $(ab \mid ij) = (ba \mid ij)$ in Super$[L \mid P]$, and since U is well defined.

More generally, let $u^1, u^2, \ldots, u^n$ be vectors, and let $a_1 < a_2 < \cdots < a_n$ be symbols (letters) belonging to each. Then we find

$$\langle U, [a_1 a_2 \cdots a_n]\rangle = [u^1 u^2 \cdots u^n]$$

where the left bracket is the bracket in Super$[L \mid P]$, and the right bracket is the ordinary determinant of the vectors $u^1, \ldots, u^n$ relative to the basis $e_1, \ldots, e_n$. In particular, we find that

$$\langle U, [a_{\sigma 1} a_{\sigma 2} \cdots a_{\sigma n}]\rangle = [u^1 u^2 \cdots u^n]$$

for any permutation σ. This point is worth stressing, because it is the point at which Weitzenböck's attempt to develop a symbolic method for skew-symmetric tensors failed. Weitzenböck (like all other classical invariant theorists) failed to distinguish between a vector u and a symbol a representing u, and used the same notation for both. As a consequence, his brackets $[u^1 u^2 \cdots u^n]$ were to be taken sometimes as symmetric and sometimes as skew-symmetric relative to permutations of $\{1, 2, \ldots, n\}$ depending on the context! Small wonder that few invariants (other than those previously known) should have been computed by such a technique.

Our main result is formally similar to the main theorem for symmetric tensors. It states that every invariant (or joint invariant) of a skew-symmetric tensor t can be written in the form $\langle U, p\rangle$, where p is a polynomial in skew brackets containing only symbols belonging to t.

Again, the listing of invariants is simplified by applying the standard basis theorem to the subalgebra of Super$[L \mid P]$ spanned by polynomials in skew brackets. A skew bracket monomial, that is, a product of skew brackets

$$m = [w_1][w_2]\cdots[w_j],$$

where w_i is a word in Super$[L]$ which is of Length n, stands for a Young diagram $D = (w_1, w_2, \ldots, w_j)$. Let us write $n = \text{Tab}(D)$. (The tableaux in the places need not be written out, since they all consist of successions of the single word $w' = 12\cdots n$.) By the standard basis theorem, those monomials $m = \text{Tab}(D)$, where D is a standard Young diagram, span the subspace of Super$[L \mid P]$ of bracket polynomials. Since $L = L^+$, a diagram D will be standard if along each row the letters appear in nondecreasing order, and along each column the letters appear in strictly increasing order. In other words, two successive letters a and b in a word w_i must satisfy the condition $a \le b$, whereas the jth letters c and d of two successive words w_i and w_{i+1} must satisfy the condition $c < d$.

For covariants, it is again convenient to let the letters belonging to vectors (and tensors) with independent transcendental entries be last in the order of the alphabet L, so that D partitions into an upper and a lower part as before. The upper part of D suffices to define a covariant. In other words, a covariant is to be specified by a Young diagram $D' = (w'_1, \ldots, w'_j)$ where Length$(w'_i) \le n$. The covariant will be $\text{Tab}(w'_1 w''_1, \ldots, w'_j w''_j)$, where $w'_i w''_i$ is of length n, and where the words w''_i are made up of symbols (letters) belonging to independent transcendentals.

Contrary to what happens for symmetric tensors, the translation of a geometric property into the vanishing of covariants turn out to be successful for skew-symmetric tensors, and we devote Chapter 5 to the discussion of a number of such examples old and new.

For a tensor t of step 2, or bivector, the only invariant is easily proved to be the Pfaffian, which for $n = 2k$ is symbolically represented by

$$a_1^{(2)} a_2^{(2)} \cdots a_k^{(2)},$$

where a_i are distinct letters belonging to t. The covariants of t are represented by incomplete Pfaffian

$$a_1^{(2)} a_2^{(2)} \cdots a_j^{(2)}, \qquad j \le k,$$

and (as is well known) they give the rank of the tensor.

For a tensor of step k, the covariants

$$C_p:\begin{array}{l} a_1a_2\cdots a_p\\ a_1^{(k-1)}\\ \vdots\\ a_p^{(k-1)}\end{array}$$

determine the rank of the tensor: if C_{r+1} vanishes but C_r does not vanish, then the tensor is of rank r.

The classical Grassmann conditions for a tensor to be decomposable translate into the vanishing of either of the two covariants

$$\begin{array}{l} a^{(k)}b\\ b^{(k-1)}\end{array} \quad \text{or} \quad \begin{array}{l} a^{(k)}b^{(2)}\\ b^{(k-2)}\end{array}$$

where a and b are distinct letters belonging to the tensor.

The condition that a tensor of step 3 be divisible by a vector is the vanishing of the covariant

$$\begin{array}{l} a^{(2)}b^{(3)}\\ ac^{(3)}\end{array}$$

where a, b, c are symbols belonging to the same tensor. These conditions are generalized in the text to give covariants specifying whether a given tensor of arbitrary step is divisible by one, two, etc., linearly independent vectors.

In conclusion, we list those covariants which specify each of the canonical forms for a tensor of step 3 in dimension $n = 6$ and 7.

For $n = 6$ the covariants that specify each of the four canonical forms are

$$\begin{array}{l} a^{(3)}b^{(2)}\\ b;\end{array}$$

$$\begin{array}{l} a^{(3)}b^{(2)}c\\ bc^{(2)};\end{array}$$

and

$$\begin{array}{l} a^{(3)}b^{(2)}c\\ bc^{(2)}d^{(3)}.\end{array}$$

We find it remarkable that each of these turns out to be a standard Young diagram. A similar phenomenon happens for $n = 7$.

Notation Guide

Dedicated
to the memory
of
Alfredo Capelli,
H. W. Turnbull,
Alfred Young,
Roland Weitzenböck.

1. The Superalgebra Super$[A]$

0. Introduction. We define in this chapter a generalization of the ordinary algebra of polynomials in a set of A variables with integer coefficients. Our variables shall be of three kinds: positively signed, neutral, and negatively signed: $A = A^+ \cup A^0 \cup A^-$.

Positively signed variables are the least familiar: they are the *divided powers.* To every positively signed variable a we assign a sequence $a^{(1)}, a^{(2)}, a^{(3)}, \ldots$ of divided powers, which behave algebraically "as if" $a^{(i)}$ were to equal $a^i/i!$ Then for example we have the "rules"

$$a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)}$$
$$(a^{(i)})^{(j)} = \frac{(ij)!}{j!(i!)^j} a^{(ij)},$$

and

$$(a+b)^{(i)} = \sum_{j+k=i} a^{(j)} b^{(k)}.$$

Positively signed variables and their divided powers commute. This seemingly artificial device is essential in making invariant theory characteristic-free.

Neutral variables behave like ordinary polynomial variables; in particular they also commute.

Negatively signed variables a, b anticommute: $ab = -ba$ and $a^2 = b^2 = 0$. However, we have $ac = ca$ when a is negatively signed and c is a variable (or a divided power) of either of the two other kinds.

The superalgebra Super$[A]$ is the algebra spanned by monomials obtained by multiplying variables of each of the three kinds—for positively signed variables, one multiplies divided powers. The variables appearing in a monomial can be permuted at will, subject only to the condition that a minus sign be prefixed every time two negatively signed variables are permuted.

Much of the work in this chapter goes in showing that the structure of a Hopf algebra can be given to the superalgebra Super[A] by setting

$$\Delta(a)^{(i)} = \sum_{j+k=i} a^{(j)} \otimes a^{(k)}, \qquad a \in A^+,$$

$$\Delta a = 1 \otimes a + a \otimes 1, \qquad a \in A^0 \cup A^-,$$

and extending Δ so that it is an algebra homomorphism from Super[A] to Super[$A \oplus A$]. The only delicate point arises in the commutation rule $(1 \otimes a)(b \otimes 1) = -b \otimes a$ when both a and b are negatively signed. Because of this feature, the superalgebra Super[$A \oplus A$] cannot be identified with the ordinary tensor product Super[A] $\otimes$ Super[A] of commutative algebra, but rather it is a "signed" tensor product.

In geometric and combinatorial interpretations, the neutral elements will not directly appear; they appear indirectly as "scalars." Remarkably, positively signed variables give the symbolic representation of skew-symmetric tensors, and negatively signed variables give the symbolic representation of symmetric tensors.

Combinatorially, positively signed variables relate to the algebra of multisets (or bosons), and negatively signed variables relate to the algebra of sets (or fermions).

The material in this chapter offers no great novelty. Readers acquainted with the techniques of Hopf algebras may simply skim over the definitions and proceed to the next chapter.

1. Definitions. Let A be a set. We denote by Mon(A) the *free monoid generated by A*. We recall that the elements of Mon(A) are finite sequences of elements in A. If $w \in$ Mon(A) with $w(1) = x_1$, $w(2) = x_2, \ldots, w(n) = x_n$, then we shall call w a *word* and denote it by $w = x_1x_2\cdots x_n$. Two such words $w = x_1x_2\cdots x_n$ and $w' = y_1y_2\cdots y_m$ are *equal* when $n = m$ and $x_1 = y_1$, $x_2 = y_2, \ldots, x_n = y_n$.

If $w = x_1x_2\cdots x_n$ is a word in Mon(A), we shall call $x_1x_2\cdots x_n$ its *display* and n its *length*. The words having length 1 will be identified with the elements of A. We shall allow w to be the empty sequence, in this case taking n to be 0, and denoting this word by 1. (To avoid confusion, we shall always assume that $1 \notin A$.)

The *product* of two words $w = x_1x_2\cdots x_n$ and $w' = y_1y_2\cdots y_m$ is the word $ww' = x_1x_2\cdots x_ny_1y_2\cdots y_m$. Endowed with this operation, Mon(A) is an assocîatiave (but not commutative) semigroup with identity.

A *multiset* on a set A is a function $m\colon A \to \mathbf{Z}^+$ where $\mathbf{Z}^+$ is the set of nonnegative integers. For $a \in A$, the integer $m(a)$ is called the *multiplicity* of the element a in the multiset m. There is a useful way to visualize multisets, which we shall often tacitly appeal to. Suppose, for example, that $m(a) = 2$, $m(b) = 1, m(c) = 3$ and that $m(x) = 0$ for all other $x \in A$. Then we describe the multiset by the display $\{a, b, c, a, c, c\}$.

Let $w \in$ Mon(A). We associate to w the multiset, cont(w), on A defined by

cont($w; a$) = cont(w)(a) = the number of occurrences of the element a among the elements x_i in the display of w.

Sometimes, in this situation, we say the *content* of the word w is the multiset cont(w).

A *signed set* A is a set together with three disjoint subsets, denoted by A^+, A^0, and A^-, whose union is A. We shall call the elements in A^+ "positive," those in A^0 "neutral," and those in A^- "negative." A function $f\colon A \to B$ from a signed set A to a signed set B will always be assumed to satisfy the conditions $f(A^+) \subset B^+$, $f(A^0) \subset B^0$, and $f(A^-) \subset B^-$. The *direct sum* of two disjoint signed sets A and B is the signed set $A \cup B$, denoted by $A \oplus B$, where

$$(A \oplus B)^+ = A^+ \cup B^+, \qquad (A \oplus B)^0 = A^0 \cup B^0, \qquad (A \oplus B)^- = A^- \cup B^-.$$

The signed set A is said to be *proper* if A^0 is empty. In this case, we define the *adjoint* of the signed set A in the following way. Let A^* be a set isomorphic to A via a mapping $\phi\colon A \to A^*$. We make A^* into a signed set by defining $(A^*)^-$ to be $\phi(A^+)$ and $(A^*)^+$ to be $\phi(A^-)$. The mapping ϕ induces a map (also denoted by ϕ) from Mon(A) to Mon(A^*), obtained by setting $\phi(x_1x_2\cdots x_n) = \phi(x_1)\phi(x_2)\cdots\phi(x_n) = x_1^*x_2^*\cdots x_n^*$.

The basic concept we shall use is that of an *extended signed set.* Consider a signed set A. The extended signed set of A, denoted by A_d, is defined as follows:

$$(A_d)^- = A^-, \quad (A_d)^0 = A^0, \quad (A_d)^+ = A^+ \times \mathbf{N}$$

where $\mathbf{N}$ = natural numbers = $\{1, 2, \ldots\}$. If $a \in A^+$, we shall write a instead of $(a, 1)$ and, otherwise, $a^{(n)}$ instead of (a, n). We shall call $a^{(n)}$ the nth *divided power of* the element a.

If A is a signed set, we denote by Div(A) the free monoid generated by the extended signed set A_d. A homomorphism Disp: Div(A) $\to$ Mon(A) is defined as follows. If $a \in A^-$ or $a \in A^0$, set Disp(a) = a. If $a \in A^+$, set Disp($a^{(n)}$) = $aa\cdots a$ (n-times). Using the homomorphism Disp, we define, for $w \in$ Div(A), Length(w) = length(Disp(w)) and Cont(w) = cont(Disp(w)).

To illustrate these ideas, let $w = bc^{(3)}a^{(2)}a^{(3)}cd$ be in Div(A). Then, Disp(w) = $bcccaaaaacd$ and Length(w) = 11. Also, w has a content m where $m(a) = 5, m(b) = 1, m(c) = 4$, and $m(d) = 1$.

Different words in Div(A) may have the same "display," i.e., Disp(w) may equal Disp(w') even though $w \neq w'$. For example, take w as above and w' to be $bc^{(3)}a^{(5)}cd$. Nevertheless, one can define an inverse mapping from the set Mon(A) to the set Div(A), denoted by $w \to$ stand(w), which is however not a homomorphism of monoids. For $w \in$ Mon(A), define stand(w) to be the unique word in Div(A) such that

(a) Disp(stand(w)) = w;

(b) if stand(w) = $y_1y_2\cdots y_k$ where $y_i \in A_d$ and if $y_j = a^{(i)}$ for some $a \in A^+$, $i \in \mathbf{N}$, then $y_{j-1} \neq a^{(r)}$ and $y_{j+1} \neq a^{(s)}$ for all integers r and s.

So, with w and w' as in the above example, we have stand(Disp(w)) = w'.

The heart of many of the computations of invariant theory is the calculation of the sign of certain expressions. If $w \in \mathrm{Div}(A)$, we define the *parity* of w, denoted by $|w|$, to be 0 or 1 depending on whether $\sum_{a\in A^-} \mathrm{Cont}(w;a)$ is even or odd. In the first case, we say that w is *positive* and in the second, *negative*. Arithmetical operations on the parity symbol will always be carried out in $\mathbf{Z}_2$; for example, we have $|ww'| = |w| + |w'|$.

If k is an integer, we set $\mathrm{sign}(k) = (-1)^k$.

2. Construction of the superalgebra Super[A]**.** Let A be a signed set. We construct a $\mathbf{Z}$-algebra, Tens[A], as follows: As a $\mathbf{Z}$-module, Tens[A] is free with basis consisting of the words in Div(A). Thus, if $p \in \mathrm{Tens}[A]$, then p can be written uniquely as a finite sum, $p = \sum c_i w_i$ where $c_i \in \mathbf{Z}$ and $w_i \in \mathrm{Div}(A)$. Multiplication in Tens[A] is defined by extending the multiplication in Div(A). So, if p is as above and $q = \sum d_j w_j$ is another element in Tens[A], then $pq = \sum c_i d_j w_i w_j$ where the product $w_i w_j$ is taken as in Div(A). In this way, Tens[A] becomes an associative algebra with identity.

In Tens[A], we define an ideal I_A (or simply I) to be the ideal generated by all expressions of the following forms:

(I1) $uv - \mathrm{sign}(|u|\,|v|)vu$ for $u, v \in \mathrm{Div}(A)$,

(I2) aa, where $a \in A^-$,

(I3) $a^{(i)}a^{(j)} - \binom{i+j}{i}a^{(i+j)}$ where $a \in A^+$.

The quotient algebra Tens[A]$/I$ will be denoted by Super[A].

When the signed set A is proper, then in the algebra Super[A] we define a sequence of many operations $p \to p^{(i)}$, where $p \in \mathrm{Super}[A]$, as follows. Such an element p can be written as a finite linear combination

$$p = \sum_j c_j \mathrm{mon}(w_j)$$

where $c_j \in \mathbf{Z}$ and $\mathrm{mon}(w_j)$ is the canonical image of $w_j \in \mathrm{Tens}[A]$ in Super[A]. Whenever no confusion is possible, we shall write w_j in place of $\mathrm{mon}(w_j)$, thus

$$p = \sum_j c_j w_j.$$

We set $p^{(0)} = 1$, the identity of the superalgebra Super[A]; $p^{(1)} = p$. If i is a positive integer other than 1, the definition of $p^{(i)}$ is accomplished in the following steps:

Step 1. If $p = a$, with $a \in A^-$, a set $p^{(i)} = 0$ if $i > 1$.

Step 2. If $p = a^{(k)}$ with $a \in A^+$, set

$$p^{(i)} = (a^{(k)})^{(i)} = \frac{(ik)!}{i!(k!)^i}a^{(ki)}.$$

Step 3. If $p = y_1 y_2 \cdots y_n$, with $y_i \in A_d$, set $p^{(i)} = y_1^{(i)} y_2^{(i)} \cdots y_n^{(i)}$.

Step 4. If $p = c\,\mathrm{mon}(w)$, where $c \in \mathbf{Z}$ and $w \in \mathrm{Div}(A)$, set $p^{(i)} = c^i w^{(i)}$.

Step 5. If $p = \sum_{j=1}^n c_j w_j = \sum_{j=1}^n p_j$, where $p_j = c_j w_j$, then set

$$p^{(i)} = \sum p_1^{(i_1)} p_2^{(i_2)} \cdots p_n^{(i_n)},$$

where the sum ranges over all n-tuples of nonnegative integers $i_1, i_2, \ldots, i_n$ such that $i_1 + i_2 + \cdots + i_n = i$.

When A is a signed set, the algebra Super[A] will be called the *superalgebra* generated by the signed set A. When A is a *proper* signed set, the superalgebra Super[A] will be tacitly assumed to be endowed with the operation $p \to p^{(i)}$, where it will be called the *divided power operation.*

Observe the following properties of Super[A].

(1) We can set $\mathrm{Cont}(\mathrm{mon}(w)) = \mathrm{Cont}(w)$ and $\mathrm{Length}(\mathrm{mon}(w)) = \mathrm{Length}(w)$ for $w \in \mathrm{Div}[A]$, as well as $|\mathrm{mon}(w)| = |w|$. Thus, fixing a multiset m on A, the set of all $\mathrm{mon}(w)$ for $w \in \mathrm{Div}[A]$ with $\mathrm{Cont}(w) = m$ spans a submodule of Super[A], which we call the submodule of all monomials of content m.

Indeed, let m be a multiset on A. We consider all words $w \in \mathrm{Div}(A)$ so that $\mathrm{Cont}(w) = m$. These words, together with 0, span a submodule $\mathrm{Tens}_m[A]$ of Tens[A]. Furthermore, $\mathrm{Tens}[A] = \sum_m \mathrm{Tens}_m[A]$ where the sum is a direct sum. Now, let $i \in I$. Then $i = \sum i_m$ where $i_m \in \mathrm{Tens}_m[A]$. Each i_m is in I since this property holds for the generators of I. It follows that if $w, w' \in \mathrm{Div}(A)$ with $w \equiv w' \pmod I$ but $w \notin I$, then $\mathrm{Cont}(w) = \mathrm{Cont}(w')$, $\mathrm{Length} w = \mathrm{Length}\, w'$, and $|w| = |w'|$. We sometimes speak of the "parity of a word in Super[A]."

(2) If $a, b \in A_d$, then $ab = ba$ in Super[A] unless a and b are negative in which case $ab = -ba$. (This follows from (I1).)

Examples. (1) If $A = A^0$, then $\mathrm{Super}[A] = \mathrm{Symm}(A)$, the $\mathbf{Z}$-algebra consisting of all polynomials with "variables" the elements $a \in A$, often called the symmetric algebra generated by the set A.

(2) If $A = A^-$, then $\mathrm{Super}[A] = \mathrm{Wedge}(A)$, the exterior algebra generated by the set A, that is, the $\mathbf{Z}$-algebra consisting of all skew-symmetric monomials in the "variables" A. Thus, a monomial in Super[A], say $a_1 a_2 \ldots a_k$, satisfies the relation $a_1 a_2 \cdots a_k = \mathrm{sgn}(\sigma) a_{\sigma 1} a_{\sigma 2} \cdots a_{\sigma k}$ where σ is any permutation of $\{1, 2, \ldots, k\}$.

(3) If $A = A^+$, then Super[A] is the divided powers algebra Divp[A] generated by the set A. We recall that, in characteristic zero (that is, over the field $\mathbf{Q}$ of rational numbers), the divided power $a^{(i)}$ can be identified with $a^i/i!$.

We omit the proof of the following two elementary facts.

PROPOSITION 1. *An injection $f\colon B \to A$ induces an isomorphism*

$$\hat{f}\colon \mathrm{Super}[B] \to \mathrm{Super}[A].$$

PROPOSITION 2. *For any signed set A, we have a natural isomorphism of* Super[A] *with the tensor product*

$$\mathrm{Divp}[A^+] \otimes \mathrm{Wedge}[A^-] \otimes \mathrm{Symm}[A^0].$$

It follows from Proposition 2 that if $a_1, a_2, \ldots, a_r$ are distinct elements in A^+, $b_1, b_2, \ldots, b_s$ are distinct elements in A^-, and $c_1, c_2, \ldots, c_u$ are distinct elements in A^0, then $a_1^{(e_1)} a_2^{(e_2)} \cdots a_r^{(e_r)} b_1 b_2 \cdots b_s c_1^{m_1} c_2^{m_2} \cdots c_u^{m_u}$ is a nonzero element in Super[A].

In addition, all elements of the above form constitute a basis for Super[A].

3. Tensor products.

PROPOSITION 3. *Let A and B be disjoint signed sets. Then the* $\mathbf{Z}$*-modules* Super[A] $\otimes$ Super[B] *and* Super[$A \oplus B$] *are isomorphic.*

PROOF. By Proposition 1, we may consider Super[A] and Super[B] as subsets of Super[$A \oplus B$]. Then, the multiplication map

$$\text{Super}[A] \times \text{Super}[B] \to \text{Super}[A \oplus B],$$

$(u, v) \to uv$, gives a mapping ϕ from Super[A] $\otimes$ Super[B] onto Super[$A \oplus B$].

Clearly, a monomial in Super[A] can be written in many ways as a product of letters in A_d^+, A^-, and A^0. We call such a monomial "canonical" if it has the form

$$a_1^{(e_1)} a_2^{(e_2)} \cdots a_r^{(e_r)} b_1 b_2 \cdots b_s c_1^{m_1} c_2^{m_2} \cdots c_u^{m_u},$$

where the a_i are distinct elements in A^+, the b_i are distinct elements in A^-, and the c_i are distinct elements in A^0. Such monomials span the module Super[A]. Similarly, canonical monomials span the module Super[B].

To show that the mapping ϕ is one-to-one, suppose that $\phi(\sum_{i=1}^n c_i r_i \otimes s_i) = 0$, where $c_i \in \mathbf{Z}$, r_i is a canonical monomial in Super[A], and s_i is a canonical monomial in Super[B]. As remarked in §2, we may assume that all $r_i s_i$ have the same content. But then $r_1 = \pm r_2 = \cdots = \pm r_n$, $s_1 = \pm s_2 = \cdots = \pm s_n$ so that $n - 1$. But $\phi(r \otimes s) = rs \neq 0$ as remarked above. This completes the proof.

Since Super[$A \oplus B$] is an algebra, we may use the isomorphism of the preceding proposition to define a multiplication on the tensor product Super[A]$\otimes$Super[B]. Namely, if

$$r = \sum_i c_i w_i \otimes w_i' \quad \text{and} \quad s = \sum_j d_j w_j \otimes w_j',$$

then set

$$rs = \sum_{i,j} c_i d_j \text{sign}(|w_i'|\,|w_j|) w_i w_j \otimes w_i' w_j'.$$

Note that the module structure on Super[A] $\otimes$ Super[B] agrees with the usual tensor product module structure. But, the multiplication on Super[A]$\otimes$Super[B] is not the usual multiplication on the tensor product, because of notable differences in sign.

Some consequences of the preceding proposition will be frequently used below. First, let r be a monomial in Super[A] and s be a monomial in Super[B]. We may then define $|r \otimes s|$ to be $|r| + |s|$ and $\text{Cont}(r \otimes s) = \text{Cont}(r) * \text{Cont}(s)$.

Second, let A, B, and C be mutually disjoint signed sets. The $\mathbf{Z}$-module Super[A] $\otimes$ Super[B] $\otimes$ Super[C] can be made into an algebra through its identification with either Super[$A \oplus B$] $\otimes$ Super[C] or Super[A] $\otimes$ Super[$B \oplus C$]. In either case we have

$$(w_1 \otimes w_2 \otimes w_3)(w_1' \otimes w_2' \otimes w_3') = \varepsilon w_1 w_1' \otimes w_2 w_2' \otimes w_3 w_3',$$

where $\varepsilon = \text{sign}(|w_2|\,|w_1'| + |w_3|\,|w_1'| + |w_3|\,|w_2'|)$. Also, $|w_1 \otimes w_2 \otimes w_3| = |w_1| + |w_2| + |w_3|$.

Third, we may define a multiplication on the $\mathbf{Z}$-module Super[A] $\otimes$ Super[A] by (the device of) viewing B as a set isomorphic to but disjoint from A and proceeding as above to get

$$(w_1 \otimes w_2)(w_1' \otimes w_2') = \text{sign}(|w_2|\,|w_1'|) w_1 w_1' \otimes w_2 w_2'.$$

Again, $|w_1 \otimes w_2| = |w_1| + |w_2|$. Similarly, we may define a multiplication on Super[A] $\otimes$ Super[A] $\otimes$ Super[A] (and longer tensor products) getting formulas similar to the above.

4. The coproduct Δ.

Let A be a signed set. We define an algebra homomorphism ϕ: Tens[A] $\to$ Super[A] $\otimes$ Super[A] as follows:

(a) $\phi(1) = 1 \otimes 1$,
(b) if $a \in A^- \cup A^0$, set $\phi(a) = 1 \otimes a + a \otimes 1$,
(c) if $a \in A^+$, set $\phi(a^{(n)}) = 1 \otimes a^{(n)} + a^{(1)} \otimes a^{(n-1)} + a^{(2)} \otimes a^{(n-2)} + \cdots + a^{(n-1)} \otimes a^{(1)} + a^{(n)} \otimes 1$.

If u is a word in Tens[A], observe that $\phi(u)$ is a sum of terms each of which has parity $|u|$. We shall sometimes write $|\phi(u)| = |u|$.

We next show that ϕ sends I_A to 0. Let u, v be words in Tens[A], say $u = x_1 x_2 \cdots x_n$ and $v = y_1 y_2 \cdots y_m$. We show that ϕ sends $uv - \text{sign}(|u|\,|v|)vu$ to 0. Now, $\phi(u) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$ is a sum of terms $r \otimes s$ each of parity

$$|u| = |x_1| + |x_2| + \cdots + |x_n| = |r \otimes s| = |r| + |s|.$$

Similary, $\phi(v)$ is a sum of terms $r' \otimes s'$ each of parity

$$|v| = |y_1| + |y_2| + \cdots + |y_m| = |r' \otimes s'| = |r'| + |s'|.$$

Therefore, $\phi(uv - \text{sign}(|u|\,|v|)vu)$ is a sum of terms of the following forms:

$$\begin{aligned}
&(r \otimes s)(r' \otimes s') - \text{sign}(|u|\,|v|)(r' \otimes s')(r \otimes s) \\
&\quad = \text{sign}(|r'|\,|s|) rr' \otimes ss' - \text{sign}(|u|\,|v| + |s'|\,|r|) r'r \otimes s's \\
&\quad = (\text{sign}(|r'|\,|s|) - \text{sign}(|u|\,|v| + |s'|\,|r| + |r'|\,|r| + |s'|\,|s|) rr' \otimes ss' \\
&\quad = (\text{sign}(|r'|\,|s|) - \text{sign}(|r'|\,|s|)) rr' \otimes ss' = 0.
\end{aligned}$$

Second, observe that if $a \in A^-$, then

$$\begin{aligned}
\phi(a^2) &= \phi(a)\phi(a) = (1 \otimes a + a \otimes 1)(1 \otimes a + a \otimes 1) \\
&= 1 \otimes a^2 - a \otimes a + a \otimes a + a^2 \otimes 1 = 0.
\end{aligned}$$

Third, let us show that $\phi(a^{(i)}a^{(j)}) = \binom{i+j}{i}\phi(a^{(i+j)})$. Let $n = i + j$. Then $\phi(a^{(i)}a^{(j)})$ equals

$$[a^{(i)} \otimes 1 + a^{(i-1)} \otimes a + \cdots + 1 \otimes a^{(i)}][a^{(j)} \otimes 1 + a^{(j-1)} \otimes a + \cdots + 1 \otimes a^{(j)}],$$

which is a sum of terms of the form

$$\binom{n-k-l}{i-k}\binom{k+l}{k}a^{(n-k-l)} \otimes a^{(k+l)},$$

where $k = 0, \ldots, i$ and $l = 0, \ldots, j$. The coefficient of a term $a^{(n-u)} \otimes a^{(u)}$ is (easily) seen to be $\binom{n}{i}$.

Since ϕ sends the generators of I_A to 0, it sends I_A to 0 and defines an algebra homomorphism Δ: Super[A] $\to$ Super[A] $\otimes$ Super[A] such that:

(a) $\Delta(1) = 1 \otimes 1$,

(b) $\Delta(a) = 1 \otimes a + a \otimes 1$ if $a \in A^-$ or $a \in A^0$,

(c) $\Delta(a^{(n)}) = a^{(n)} \otimes 1 + a^{(n-1)} \otimes a + \cdots + 1 \otimes a^{(n)}$ if $a \in A^+$.

If u is a word in Super[A], then Δu is a sum of terms of the form $r \otimes s$, say, $\Delta u = r_1 \otimes s_1 + r_2 \otimes s_2 + \cdots + r_p \otimes s_p$.

We shall denote this series by the *Sweedler notation*,

$$\Delta u = \sum_u u_{(1)} \otimes u_{(2)}.$$

We turn to the question of uniqueness in this expansion.

PROPOSITION 4. *Let π: Tens[A] $\to$ Super[A] be the natural mapping. Then the mapping $\pi \otimes \pi$: Tens[A] $\otimes$ Tens[A] $\to$ Super[A] $\otimes$ Super[A] has kernel $J_A = I_A \otimes$ Tens[A] + Tens[A] $\otimes I_A$.*

PROOF. Clearly $J_A \subset \ker(\pi \otimes \pi)$. By Proposition 2, we may choose elements $e_1, e_2, \ldots$ in Tens[A] so that $\pi(e_1), \pi(e_2), \ldots$ is a basis for Super[A]. Hence, each element in Tens[A] has the form $a + c_1e_1 + \cdots + c_ne_n$ where $a \in I_A$ and $c_j \in \mathbf{Z}$. If $v \in$ Tens[A] $\otimes$ Tens[A] but $v \notin J_A$, then $v = a + \sum c_{ij}e_i \otimes e_j$ where $a \in J_A, c_{ij} \in \mathbf{Z}$, and some $c_{ij} \neq 0$. But then $\pi(v) = \sum c_{ij}\pi(e_i) \otimes \pi(e_j) \neq 0$.

The same reasoning shows that the kernel of $\pi \otimes \cdots \otimes \pi$ (n-times): Tens[A] $\otimes \cdots \otimes$ Tens[A] $\to$ Super[A] $\otimes \cdots \otimes$ Super[A] is $I_A \otimes$ Tens[A] $\otimes \cdots \otimes$ Tens[A] + Tens[A] $\otimes I_A \otimes$ Tens[A] $\otimes \cdots \otimes$ Tens[A] $+ \cdots +$ Tens[A] $\otimes \cdots \otimes$ Tens[A] $\otimes I_A$.

Let m be a multiset on A; let Tens$_m$[A] be the submodule of Tens[A] spanned by 0 and all words $w \in$ Div(A) such that Cont(w) $= m$. Then, Tens[A] is the direct sum of all the Tens$_m$[A]. Furthermore, I_A is the direct sum of all the $I_m = I_A \cap$ Tens$_m$[A] (as remarked in §2). Next, let m_1 and m_2 be multisets on A and set $T(m_1, m_2) =$ Tens$_{m_1}$[A] $\otimes$ Tens$_{m_2}$[A]. Then, Tens[A] $\otimes$ Tens[A] is the direct sum of the modules $T(m_1, m_2)$ as m_1 and m_2 range over all multisets in A. Furthermore, $J_A = I_A \otimes$ Tens[A] + Tens[A] $\otimes I_A$ is the direct sum of all the modules $J(m_1, m_2) = J_A \cap T(m_1, m_2)$.

Now, let u be a word in Super[A]. By the definition of Δ, we see that in $\Delta u = \sum u_{(1)} \otimes u_{(2)}$, the monomials $u_{(1)}, u_{(2)}$ may be chosen so that Cont($u_{(1)}$)+

Cont($u_{(2)}$) = Cont(u) for each term $u_{(1)} \otimes u_{(2)}$. Let $\Delta u = \sum v_{(1)} \otimes v_{(2)}$ be another representation of Δu where the $v_{(1)}, v_{(2)}$ are words. Then, in Tens[A] $\otimes$ Tens[A],

$$\sum u_{(1)} \otimes u_{(2)} = \sum v_{(1)} \otimes v_{(2)} + j,$$

where j is an element in the ideal $J_A = I_A \otimes$ Tens[A] + Tens[A] $\otimes I_A$. Comparing contents on both sides of this equation, we see that those terms $v_{(1)} \otimes v_{(2)}$ with Cont($v_{(1)} \otimes v_{(2)}$) $\neq$ Cont(u) must be in J_A and have image 0 under $\pi \otimes \pi$. So, these terms can be omitted from any expression for Δu. With this convention, we see that any representation of Δu as a sum of terms $u_{(1)} \otimes u_{(2)}$ satisfies Cont($u_{(1)} \otimes u_{(2)}$) = Cont($u_{(1)}$) + Cont($u_{(2)}$) = Cont(u). Similarly, we may suppose that Length($u_{(1)}$) + Length($u_{(2)}$) = Length(u) and $|u_{(1)}| + |u_{(2)}| = |u|$.

5. The Coassociative Law. Let A be a signed set. Then the homomorphisms $1 \otimes \Delta$ and $\Delta \otimes 1$ both send Super[A] $\otimes$ Super[A] to Super[A] $\otimes$ Super[A] $\otimes$ Super[A].

PROPOSITION 5 (COASSOCIATIVE LAW). *Let A be a signed set. The homomorphisms $(1 \otimes \Delta) \cdot \Delta$ and $(\Delta \otimes 1) \cdot \Delta$ from Super[A] to Super[A] $\otimes$ Super[A] $\otimes$ Super[A] coincide.*

PROOF. Let $w \in$ Div(A), with $w = x_1x_2 \cdots x_n$. We show by induction on Length w that $(\Delta \otimes 1)(\Delta w) = (1 \otimes \Delta)\Delta w$. If Length $w = 1$, this equality follows from the definition of Δ by a simple calculation.

Now, suppose that Length(w) $= n > 1$ and that equality holds for all words of Length $< n$. If $w = a^{(n)}$, then one may check that

$$(\Delta \otimes 1)(\Delta w) = (1 \otimes \Delta)(\Delta w) = \sum a^{(i)} \otimes a^{(j)} \otimes a^{(k)},$$

where the sum ranges over all nonnegative integers i, j, k such that $i + j + k = n$. Otherwise, $w = uv$ where both u and v have Length $< n$. Write

$$\Delta u = \sum_u u_{(1)} \otimes u_{(2)} \quad \text{and} \quad \Delta v = \sum_v v_{(1)} \otimes v_{(2)}.$$

By the induction hypothesis

(1) $\sum_u \Delta u_{(1)} \otimes u_{(2)} = \sum_u u_{(1)} \otimes \Delta u_{(2)}$

and

(2) $\sum_v \Delta v_{(1)} \otimes v_{(2)} = \sum_v v_{(1)} \otimes \Delta v_{(2)}$.

Now,

$$\Delta w = \Delta u \Delta v = \sum_{u,v} \mathrm{sign}(|u_{(2)}|\,|v_{(1)}|)u_{(1)}v_{(1)} \otimes u_{(2)}v_{(2)}.$$

Hence,

$$(\Delta \otimes 1)\Delta w = \sum_{u,v} \mathrm{sign}(|u_{(2)}|\,|v_{(1)}|)\Delta(u_{(1)}v_{(1)}) \otimes u_{(2)}v_{(2)},$$

and

$$(1\otimes\Delta)\Delta w=\sum_{u,v}\operatorname{sign}(|u_{(2)}|\,|v_{(1)}|)u_{(1)}v_{(1)}\otimes\Delta(u_{(2)}v_{(2)}).$$

These two previous expressions are equal as follows from multiplying (1) and (2) and recalling that $|\Delta v_{(1)}|=|v_{(1)}|$ and $|\Delta u_{(2)}|=|u_{(2)}|$. This completes the proof.

Next, define algebra homomorphisms $\varepsilon\colon \mathrm{Super}[A]\to\mathbf{Z}$ and $S\colon \mathrm{Super}[A]\to \mathrm{Super}[A]$ as follows: $\varepsilon(w)=1$ if Length$(w)=0$ and $\varepsilon(w)=0$ if Length$(w)>0$; $S(w)=\operatorname{sign}(\mathrm{Length}(w))w$. These maps along with Δ give Super[A] the structure of a Hopf algebra. In other words, the following diagrams are commutative:

$$\begin{tikzcd}
 & \mathrm{Super}[A] \arrow[dl, "\mathrm{id}"'] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
\mathbf{Z}\otimes\mathrm{Super}[A] & \mathrm{Super}[A]\otimes\mathrm{Super}[A] \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & \mathrm{Super}[A]\otimes\mathbf{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Super}[A] \arrow[dl, "\varepsilon"'] \arrow[r, "\Delta"] \arrow[d, "S"] & \mathrm{Super}[A]\otimes\mathrm{Super}[A] \arrow[d, "S\otimes S"] \\
\mathbf{Z} & \mathrm{Super}[A] \arrow[l, "\varepsilon"] \arrow[r, "\Delta"] & \mathrm{Super}[A]\otimes\mathrm{Super}[A]
\end{tikzcd}$$

Lastly, we define a sequence of algebra homomorphisms $\Delta^{(n)}\colon \mathrm{Super}[A]\to \mathrm{Super}[A]\otimes\cdots\otimes\mathrm{Super}[A]$ (n-times) by repeatedly using the Coassociative Law. Set $\Delta^{(1)}=\mathrm{id},\Delta^{(2)}=\Delta,\Delta^{(3)}=(1\otimes\Delta)\cdot\Delta=(\Delta\otimes 1)\cdot\Delta,\dots,\Delta^{(n)}=(1\otimes\Delta)\cdot\Delta^{(n-1)}$. We conclude with a property of the antipode $S(w)$ which will be used repeatedly in Chapter 3.

PROPOSITION 6. *Let w be a word in* Div(A). *Let $\Delta w=\sum_w w_{(1)}\otimes w_{(2)}$. Then $\sum_w w_{(1)}S(w_{(2)})=\varepsilon(w)$.*

PROOF. We use induction on Length(w); the case Length$(w)\le 1$ follows at once from the definitions. In general, if $w=a^{(n)}$, then $\Delta w=\sum_{i=0}^n a^{(i)}\otimes a^{(n-i)}$. Therefore,

$$\begin{aligned}\sum_w w_{(1)}S(w_{(2)})&=\sum_{i=0}^n\operatorname{sign}(n-i)a^{(i)}a^{(n-i)}\\&=a^{(n)}\sum_{i=0}^n\operatorname{sign}(n-i)\binom{n}{i}=0.\end{aligned}$$

Otherwise, $w=w'w''$ where Length$(w')<n$ and Length$(w'')<n$. Let us omit

the parentheses and write

$$\Delta(w')=\sum_{w'}w_1'\otimes w_2'\quad\text{and}\quad\Delta(w'')=\sum_{w''}w_1''\otimes w_2''.$$

Then,

$$\Delta(w)=\Delta(w')\Delta(w'')=\sum\operatorname{sign}(|w_2'|\,|w_1''|)w_1'w_1''\otimes w_2'w_2''.$$

So,

$$\begin{aligned}\sum_w w_{(1)}S(w_{(2)})&=\sum\operatorname{sign}(|w_2'|\,|w_1''|+\mathrm{Length}(w_2'w_2''))w_1'w_1''w_2'w_2''\\&=\sum w_1'\operatorname{sign}(\mathrm{Length}(w_2'))w_2'w_1''\operatorname{sign}(\mathrm{Length}(w_2''))w_2''=0.\end{aligned}$$

2. Laplace Pairings

0. Introduction. In this chapter we introduce a far-reaching generalization of the classical notion of the determinant of a matrix. By way of motivation, we review the ordinary determinant in the present notation.

Suppose we are given a matrix whose rows are labeled $x_1, x_2, \ldots, x_n$, where $x_i \in L^-$ and where L is a proper signed set, and whose columns are labeled $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $\alpha_i \in P^-$ and where P is another proper signed set. The entries of the matrix are denoted by $(x_i \mid \alpha_j)$; we consider them as neutral elements of another (improper!) signed set $[L \mid P]$. Since neutral elements behave like "scalars," that is $(x_i \mid \alpha_i) \in [L \mid P]^0$, we operate with the "variables" $(x_i \mid \alpha_i)$ as we do with ordinary numbers.

We denote the determinant of the matrix $(x_i \mid \alpha_i)$ by the expression

$$(x_1 x_2 \cdots x_n \mid \alpha_1 \alpha_2 \cdots \alpha_n).$$

Again, $x_1 \cdots x_n \in \mathrm{Div}[L^-]$ and $\alpha_1 \cdots \alpha_n \in \mathrm{Div}[P^-]$. In the case at hand, we can define the determinant by the classical formula

$$(*) \qquad \begin{aligned}(x_1 \cdots x_n \mid \alpha_1 \cdots \alpha_n) &= \sum_\sigma \pm (x_{\sigma 1} \mid \alpha_1)(x_{\sigma 2} \mid \alpha_2) \cdots (x_{\sigma n} \mid \alpha_n) \\ &= \sum_\sigma \pm (x_1 \mid \alpha_{\sigma 1})(x_2 \mid \alpha_{\sigma 2}) \cdots (x_n \mid \alpha_{\sigma n});\end{aligned}$$

however, we shall follow another more circuitous route for what seems like an obvious definition. Let us recall the Laplace expansion of a determinant (by rows first) and express it in the present notation. Using the notation of Hopf algebras developed in the preceding chapter, set

$$x_1 \cdots x_i = u, \qquad x_{i+1} \cdots x_n = u', \qquad \alpha_1 \cdots \alpha_n = w.$$

One can then verify that a Laplace expansion by rows can be elegantly rewritten (forgetting about the signs for the moment) as

$$(**) \qquad (uu' \mid w) = \sum_w \pm (u \mid w_{(1)})(u' \mid w_{(2)}).$$

A similar identity holds for Laplace expansions by columns.

Multiple Laplace expansions can be similarly expressed. On the right side of $(**)$, expand $(u \mid w_{(i)})$ by a further Laplace expansion, say writing $u = u''u'''$, thus:

$$(u \mid w_{(1)}) = (u''u''' \mid w_{(1)}) = \sum_{w_{(1)}} \pm(u'' \mid w_{(11)})(u''' \mid w_{(12)}).$$

Substituting in $(**)$, we obtain

$$(uu' \mid w) = \sum_{w} \sum_{w_{(i)}} \pm(u'' \mid w_{(11)})(u''' \mid w_{(12)})(u' \mid w_{(2)}).$$

Using the Coassociative Law of the coproduct, the right side can be rewritten in the form

$$\sum_{w} \pm(u'' \mid w_{(1)})(u''' \mid w_{(2)})(u' \mid w_{(3)}),$$

and it can be verified that the double Laplace expansion is consistent; that is, the same expression would have been obtained by expanding $(u' \mid w_{(2)})$ first. Thus, by a succession of iterated Laplace expansions, we eventually arrive at the ordinary expression $(*)$ for the determinant, and it does not matter in what order we take our Laplace expansions. In other words, the fact that the determinant is well defined is tantamount to verifying that the order in which successive Laplace expansions of minors is carried out is immaterial, and all lead eventually to $(*)$.

In the present content, we wish to generalize the notion of determinant by allowing $x_i \in L^+ \cup L^-$ and $\alpha_j \in P^+ \cup P^-$. Specifically, we wish to define a *biproduct* $(w \mid u)$ where w is any word in $\mathrm{Div}[L]$ and u any word in $\mathrm{Div}[P]$, positive or negative. One cannot do this by generalizing $(*)$, because the algebraic properties of the divided powers $x^{(i)}$ for $x \in L^+$ all but force the definition

$$(x^{(i)} \mid \alpha_1\alpha_2\cdots\alpha_i) = (x \mid \alpha_1)(x \mid \alpha_2)\cdots(x \mid \alpha_i),$$

where $\alpha_i \in P^-$, and $(x^{(i)} \mid \alpha^{(i)}) = (x \mid \alpha)^{(i)}$ where $\alpha \in P^+$. We must resort therefore to another device: we define symbols $(w \mid u)$ and subject them to suitable generalizations of the Laplace expansions which take the varying signs into account. The problem is then that of proving that successive Laplace expansions can be carried out in any order, giving a consistent result. This is, in brief, the content of the present chapter. The assignment of signs is carried out by a simple idea. A biproduct $(w \mid u)$ is given the sign $|w| + |u|$. Two biproducts commute if not both of them have negative signs and anticommute if both have negative signs. One carries out a Laplace expansion by noting the number of commutations of negative letters in the biproducts that are necessary to carry out the expansion. For example, if $x, y \in L^+$ and $\alpha, \beta \in P^-$ and we want to expand the biproduct $(xy \mid \alpha\beta)$, we may choose to expand it by "rows" and obtain $(x \mid \alpha)(y \mid \beta) + (y \mid \alpha)(x \mid \beta)$; here all the signs are positive because x and y have been permuted in the expansion, and they are both positive. In other words, all the signs are $+$ because no pairs of negative entries have been permuted. On the other hand, we may choose to expand by "columns" and then obtain

$(x \mid \alpha)(y \mid \beta) - (x \mid \beta)(y \mid \alpha)$. Here, the minus sign arises from the fact that two negatively signed elements, namely α and β, have been permuted. Because $(x \mid \beta)$ and $(y \mid \alpha)$ are negatively signed, we have $(x \mid \beta)(y \mid \alpha) = -(y \mid \alpha)(x \mid \beta)$, and the two expressions coincide, giving a consistent definition.

The basic rule for Laplace expansions is the following: if $u = u'u''$ then

$$(w \mid u'u'') = \sum_{w} \mathrm{sign}(|w_{(2)}|\,|u'|)(w_{(1)} \mid u')(w_{(2)} \mid u'').$$

The "reason" for the sign can be viewed as follows. In passing from $(w_{(1)} \otimes w_{(2)} \mid u' \otimes u'')$ to $(w_{(1)} \mid u')(w_{(2)} \mid u'')$ one "commutes" the words $w_{(2)}$ and u'.

There is a dual rule when the roles of w and u are exchanged.

The resulting definition of a biproduct $(w \mid u)$ includes not only the ordinary determinant, when all letters in w and u are negatively signed, and the permanent, which occurs when all letters in both w and u are positively signed *and distinct*, but also a characteristic-free generalization of the permanent, which occurs when all letters in both w and u are positively signed but not necessarily distinct.

The most interesting new special case, however, occurs when all letters in w are positively signed and all letters in u are negatively signed. We obtain then a generalization on the notion of determinant (foreshadowed, but only in characteristic zero, by Doubilet and Rota) which, as we shall see, plays a role at least as great as that of Cayley's "bracket" in invariant theory. The results in this chapter are believed to be new and are published here for the first time. The notion of Laplace pairing can be given in a more general context of Hopf algebras, but we have deliberately restricted the exposition to the cases needed in invariant theory.

1. The fourfold algebra. Let L and P be disjoint proper signed sets (so that $L^0 = P^0 = \varnothing$). We shall call the elements of L *letters* and the elements of P *places*. We define a new signed set, which is not proper, by $[L \mid P] = \{(x \mid \alpha) \colon x \in L, \alpha \in P\}$. Here, the elements of $[L \mid P]$ are denoted by symbols $(x \mid \alpha)$. The positive, neutral, and negative elements in $[L \mid P]$ are defined as follows:

(i) if $x \in L^+$, $\alpha \in P^+$, then $(x \mid \alpha) \in [L \mid P]^+$;
(ii) if $x \in L^-$, $\alpha \in P^-$, then $(x \mid \alpha) \in [L \mid P]^0$;
(iii) otherwise, $(x \mid \alpha) \in [L \mid P]^-$.

The $\mathbf{Z}$-algebra $\mathrm{Super}[L \mid P]$ will be called the *fourfold algebra*. By Propositions 2 and 3, we have an isomorphism of the fourfold algebra with the fourfold tensor product.

$$\begin{aligned}\mathrm{Super}[L \mid P] = {} & \mathrm{Super}[L^- \mid P^-] \otimes \mathrm{Super}[L^- \mid P^+] \\ & \otimes \mathrm{Super}[L^+ \mid P^-] \otimes \mathrm{Super}[L^+ \mid P^+].\end{aligned}$$

In the fourfold algebra $\mathrm{Super}[L \mid P]$ we have $(x \mid \alpha)(y \mid \beta) = \pm(y \mid \beta)(x \mid \alpha)$ with a minus sign only when exactly one of x and α is negative *and* exactly one of y and β is negative. We shall use this fact often in what follows.

2. The Laplace pairing. Let A be any signed set. If $k \in \mathbf{Z}^+$, we denote by $\mathrm{Tens}_k[A]$ the submodule of $\mathrm{Tens}[A]$ spanned by 0 and all $w \in \mathrm{Div}(A)$ such that $\mathrm{Length}(w) = k$, and similarly we define $\mathrm{Super}_k[A]$.

Let L and P be proper signed sets. For $k, l \in \mathbf{Z}^+$ we define a bilinear mapping Ω from $\mathrm{Super}_k[L] \times \mathrm{Super}_l[P]$ to $\mathrm{Super}[L \mid P]$. This will be done by first defining a mapping, which we also denote by Ω, from $\mathrm{Tens}_k[L] \times \mathrm{Tens}_l[P]$ to $\mathrm{Super}[L \mid P]$, and then by verifying that the required conditions are satisfied to induce a bilinear mapping on the superalgebras. The bilinear mapping Ω is defined by the following rules:

Rule 1. If $k \neq l$, then $\Omega \equiv 0$.

For $k = l = n$, say, we define Ω inductively. It suffices to define $\Omega(w, u)$ for words w and u. We shall denote $\Omega(w, u)$ by $(w \mid u)$, and we call $(w \mid u)$ the *biproduct* of w and u.

In the inductive definition of Ω, we shall check that the following two conditions hold at each step n.

(i) If $w \in I_L$ and $u \in \mathrm{Tens}_n[P]$, then $\Omega(w, u) = 0$. If $u \in I_P$ and $w \in \mathrm{Tens}_n[L]$, then $\Omega(w, u) = 0$.

(ii) (Commutation rule for biproducts) Let w, w', u, u' be words of Length $\leq n$. Then

$$(w \mid u)(w' \mid u') = \varepsilon (w' \mid u')(w \mid u),$$

where $\varepsilon = \mathrm{sign}(|w'|\,|u| + |w'|\,|w| + |u'|\,|u| + |u'|\,|w|)$.

Rule 2. $(1 \mid 1) = 1$.

Rule 3. For $x \in L$, $\alpha \in P$ set $\Omega(x, \alpha) = (x \mid \alpha)$.

Condition (i) holds for $n = 0, 1$ since nonzero elements in I_L or I_P are sums of words having Length ≥ 2. Condition (ii) holds for $n = 0, 1$ as remarked at the end of §1.

Suppose that Ω has been defined on $\mathrm{Tens}_k[L] \times \mathrm{Tens}_k[P]$ for $k < n$ in such a way that conditions (i) and (ii) hold. We then define Ω on $\mathrm{Tens}_n[L] \times \mathrm{Tens}_n[P]$ by the following three rules:

Rule 4. $(x^{(n)} \mid \alpha^{(n)}) = (x \mid \alpha)^{(n)}$.

Rule 5. $(w \mid u'u'') = \sum_w \mathrm{sign}(|w_{(2)}|\,|u'|)(w_{(1)} \mid u')(w_{(2)} \mid u'')$,
where $\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$.

Rule 6. $(w'w'' \mid u) = \sum_u \mathrm{sign}(|w''|\,|u_{(1)}|)(w' \mid u_{(1)})(w'' \mid u_{(2)})$,
where $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$.

We now check that these rules give a consistent definition of Ω and then that conditions (i) and (ii) hold. To show that Ω is well defined, we need to establish the following facts.

(1) Rule 5 does not depend on the representation of Δw as a sum, $\sum_w w_{(1)} \otimes w_{(2)}$. Rule 6 does not depend on the representation of Δu.

(2) Rule 5 does not depend on the particular factorization of $u = u'u''$.
Rule 6 does not depend on the particular factorization of $w = w'w''$.

(3) In computing $(ww' \mid uu')$ the same answer is arrived at whether Rule 5 or Rule 6 is applied.

PROOF OF (1). We prove (1) for Rule 5, the proof for Rule 6 being similar. We begin by noting that since condition (i) holds, we may view w and u as elements of $\mathrm{Super}[L]$ and $\mathrm{Super}[P]$, respectively, when writing $(w \mid u)$. Thus, the expressions $(w_{(1)} \mid u')$ and $(w_{(2)} \mid u'')$ are well defined.

Let $\mathrm{Length}(u') = k_1 < n$ and $\mathrm{Length}(u'') = k_2 < n$. Then u' defines a linear mapping $\mathrm{Tens}[L] \to \mathrm{Super}[L \mid P], w \to (w \mid u')$. Similarly, u'' defines a linear mapping $\mathrm{Tens}[L] \to \mathrm{Super}[L \mid P]$ by $w \to (w \mid u'')$. These two linear mappings in turn define a linear map from $\mathrm{Tens}[L] \otimes \mathrm{Tens}[L]$ to $\mathrm{Super}[L \mid P]$ by $w_1 \otimes w_2 \to (w_1 \mid u')(w_2 \mid u'')$.

Now let $\Delta w = \sum_w w_{(1)} \otimes w_{(2)} = \sum_w v_{(1)} \otimes v_{(2)}$ be two representations of Δw. We apply the remarks following Proposition 4. Considering these representations as elements in $\mathrm{Tens}[L] \otimes \mathrm{Tens}[L]$, we may equate terms in $T(m_1, m_2)$ where m_1, m_2 are any two given multisets on L. Then, there is an element $j \in J_L \cap T(m_1, m_2)$ so that

$$\sum_w{}' (w_{(1)} \otimes w_{(2)} = \sum_w{}' v_{(1)} \otimes v_{(2)} + j,$$

where the sums are restricted to terms in $T(m_1, m_2)$. Applying the linear mapping defined by u' and u'' to the equation above and recalling condition (i), we see that

$$\sum{}' (w_{(1)} \mid u')(w_{(2)} \mid u'') = \sum{}' (v_{(1)} \mid u')(v_{(2)} \mid u'').$$

Since m_2 is the content of each $w_{(2)}$ and $v_{(2)}$, we may multiply this equation by $\mathrm{sign}\,(|u'|\,w_{(2)}|)$. This completes the proof of (1).

PROOF OF (2). To prove (2) for Rule 5, we shall show that $(w \mid (ab)c) = (w \mid a(bc))$. First, we recall that if $\Delta w = \sum w_{(1)} \otimes w_{(2)}$, then

$$\begin{aligned}(\Delta \otimes 1)(\Delta w) &= \sum \Delta w_{(1)} \otimes w_{(2)} = \sum w_{(11)} \otimes w_{(12)} \otimes w_{(2)} \\ &= (1 \otimes \Delta)(\Delta w) = \sum w_{(1)} \otimes \Delta w_{(2)} \\ &= \sum w_{(1)} \otimes w_{(21)} \otimes w_{(22)}.\end{aligned}$$

Therefore,

$$\begin{aligned}(w \mid (ab)c) &= \sum_w \mathrm{sign}(|ab|\,|w_{(2)}|)(w_{(1)} \mid ab)(w_{(2)} \mid c) \\ &= \sum \mathrm{sign}(|ab|\,|w_{(2)}| + |a|\,|w_{(12)}|)(w_{(11)} \mid a)(w_{(12)} \mid b)(w_{(2)} \mid c).\end{aligned}$$

Similarly,

$$\begin{aligned}(w \mid a(bc)) &= \sum_w \mathrm{sign}(|a|\,|w_{(2)}|)(w_{(1)} \mid a)(w_{(2)} \mid bc) \\ &= \sum \mathrm{sign}(|a|\,|w_{(2)}| + |b|\,|w_{(22)}|)(w_{(1)} \mid a)(w_{(21)} \mid b)(w_{(22)} \mid c).\end{aligned}$$

To see that these expressions are equal, let us change notation and set $\Delta^{(3)} w = \sum_w w_{(1)} \otimes w_{(2)} \otimes w_{(3)}$. Each of the expressions above has the form $(*)$

$$\sum_w \mathrm{sign}(|a|\,|w_{(2)}| + |a|\,|w_{(3)}| + |b|\,|w_{(3)}|)(w_{(1)} \mid a)(w_{(2)} \mid b)(w_{(3)} \mid c)$$

since $|w_{(2)}| = |w_{(21)}| + |w_{(22)}|$. But an argument just as in the proof of (1) shows that $(*)$ does not depend on the representation of $\Delta^{(3)}w$.

PROOF OF (3). In computing $(ww' \mid uu')$ we may apply Rule 5 or Rule 6. Let us begin by using Rule 5. Then

$$(ww' \mid uu') = \sum \operatorname{sign}(|u|\,|(ww')_{(2)}|)((ww')_{(1)} \mid u)((ww')_{(2)} \mid u').$$

If $\Delta w = \sum w_{(1)} \otimes w_{(2)}$ and $\Delta w' = \sum w'_{(1)} \otimes w'_{(2)}$, then

$$\Delta(ww') = \Delta w \Delta w' = \sum \operatorname{sign}(|w_{(2)}|\,|w'_{(1)}|) w_{(1)}w'_{(1)} \otimes w_{(2)}w'_{(2)}.$$

Hence, applying Rule 6 we obtain

$$\begin{aligned}(ww' \mid uu') &= \sum \operatorname{sign}(|u|\,|w_{(2)}| + |u|\,|w'_{(2)}| + |w_{(2)}|\,|w'_{(1)}|) \\ &\qquad\qquad \times (w_{(1)}w'_{(1)} \mid u)(w_{(2)}w'_{(2)} \mid u') \\ &= \sum \varepsilon (w_{(1)} \mid u_{(1)})(w'_{(1)} \mid u_{(2)})(w_{(2)} \mid u'_{(1)})(w'_{(2)} \mid u'_{(2)})\end{aligned}$$

where

$$\varepsilon = \operatorname{sign}(|u|\,|w_{(2)}| + |u|\,|w'_{(2)}| + |w_{(2)}|\,|w'_{(1)}| + |u_{(1)}|\,|w'_{(1)}| + |u'_{(1)}|\,|w'_{(2)}|).$$

Next, we begin with Rule 6. Then

$$(ww' \mid uu') = \sum \operatorname{sign}(|(uu')_{(1)}|\,|w'|)(w \mid uu')_{(1)})(w \mid (uu')_{(2)}).$$

We calculate $\Delta(uu') = \Delta u \Delta u'$ and then apply Rule 5 and condition (ii) to $(w_{(2)} \mid u'_{(1)})(w'_{(1)} \mid u_{(2)})$ to conclude the proof.

There are two useful formulas which follow by induction from these arguments.

(a)

$$(x_1x_2\cdots x_m \mid u) = \sum_u \operatorname{sign}\left(\sum_{i>j} |x_i|\,|u_{(j)}|\right)(x_1 \mid u_{(1)})(x_2 \mid u_{(2)})\cdots(x_m \mid u_{(m)}),$$

where $\Delta^{(m)}(u) = \sum_u u_{(1)} \otimes u_{(2)} \otimes \cdots \otimes u_{(m)}$.

(b)

$$(x \mid u_1u_2\cdots u_m) = \sum_x \operatorname{sign}\left(\sum_{i>j} |x_{(i)}|\,|u_j|\right)(x_{(1)} \mid u_1)(x_{(2)} \mid u_2)\cdots(x_{(m)} \mid u_m),$$

where $\Delta^{(m)}(x) = \sum_x x_{(1)} \otimes x_{(2)} \otimes \cdots \otimes x_{(m)}$.

PROOF OF CONDITION (i). We shall prove that if $u \in I_P$ and if w is any word in $\mathrm{Tens}_n[L]$, then $\Omega(w,u) = 0$. The other statement in (i) is proved in the same way. We may assume that $u \in I_P \cap \mathrm{Tens}_n[P]$ by Rule 1. If w factors, say $w = x_1x_2\cdots x_m$, then we see that $\Omega(w,u) = 0$ by applying formula (a) above since $\Delta^{(m)}(u) = 0$ for $m > 0$. Otherwise, $w = x^{(n)}$. Let r, u, v, s be words in $\mathrm{Tens}[P]$ of Lengths h, i, j, k so that $h + i + j + k = n$. Then, applying formula (b) and Rule 1, we see that

$$\Omega(x^{(n)}, ruvs) = (x^{(h)} \mid r)(x^{(i)} \mid u)(x^{(j)} \mid v)(x^{(k)} \mid s).$$

Also,

$$\Omega(x^{(n)}, rvus) = (x^{(h)} \mid r)(x^{(j)} \mid v)(x^{(i)} \mid u)(x^{(k)} \mid s)$$

and, applying the commutation rule for biproducts, we see that this is

$$\operatorname{sign}(|u|\,|v|)(x^{(h)} \mid r)(x^{(i)} \mid u)(x^{(j)} \mid v)(x^{(k)} \mid s).$$

Therefore,

$$\Omega(x^{(n)}, ruvs - \operatorname{sign}(|u|\,|v|)rvus) = 0.$$

Furthermore,

$$\begin{aligned}(x^{(n)} \mid r\alpha^{(i)}\alpha^{(j)}s) &= (x^{(h)} \mid r)(x^{(i)} \mid \alpha^{(i)})(x^{(j)} \mid \alpha^{(j)})(x^{(k)} \mid s) \\ &= (x^{(h)} \mid r)(x \mid \alpha)^{(i)}(x \mid \alpha)^{(j)}(x^{(k)} \mid s) \\ &= \tbinom{i+j}{i}(x^{(h)} \mid r)(x \mid \alpha)^{(i+j)}(x^{(k)} \mid s) \\ &= (x^{(n)} \mid r\tbinom{i+j}{i}\alpha^{(i+j)}s).\end{aligned}$$

Similarly, if $u \in P^-$ then $(x^{(n)} \mid ru^2s) = (x^{(h)} \mid r)(x \mid u)(x \mid u)(x^{(k)} \mid s) = 0$. This completes the proof of condition (i).

PROOF OF CONDITION (ii). If either $(w \mid u)$ or $(w' \mid u')$ is $(x^{(n)} \mid \alpha^{(n)}) = (x \mid \alpha)^{(n)}$, the condition holds since $(x \mid \alpha)^{(n)}$ is in the center of $\mathrm{Super}[L \mid P]$. Otherwise, we may apply Rules 5 and 6 and use the induction hypothesis. This completes the proof of condition (ii).

We have completed the argument showing that the bilinear mapping Ω from $\mathrm{Tens}[L] \times \mathrm{Tens}[P]$ to $\mathrm{Super}[L \mid P]$ is well defined. In fact, since condition (i) holds, we may consider Ω as a bilinear mapping from $\mathrm{Super}[L] \times \mathrm{Super}[P]$ to $\mathrm{Super}[L \mid P]$.

Notation. We shall denote sign $(|w_{(2)}|\,|u'|)(w_{(1)}|u')(w_{(2)} \mid u'')$ by

$$\left(\begin{array}{c|c} w_{(1)} & u' \\ w_{(2)} & u'' \end{array}\right)$$

and write Rule 5 as

$$(w \mid u'u'') = \sum_w \left(\begin{array}{c|c} w_{(1)} & u' \\ w_{(2)} & u'' \end{array}\right).$$

Similarly, Rule 6 may be written as

$$(w'w'' \mid u) = \sum_u \left(\begin{array}{c|c} w' & u_{(1)} \\ w'' & u_{(2)} \end{array}\right).$$

Extending this idea, we define the tableau

$$\left(\begin{array}{c|c} w_1 & u_1 \\ w_2 & u_2 \\ \vdots & \\ w_n & u_n \end{array}\right)$$

to be sign $(\sum_{i>j}|w_i|\,|u_j|)(w_1 \mid u_1)(w_2 \mid u_2)\cdots(w_n \mid u_n)$. Formula (a) can then be written as

$$(x_1x_2\cdots x_m \mid u) = \sum_u \begin{pmatrix} x_1 & u_{(1)} \\ x_2 & u_{(2)} \\ \vdots & \vdots \\ x_m & u_{(m)} \end{pmatrix}$$

and formula (b) becomes

$$(x \mid u_1u_2\cdots u_m) = \sum_x \begin{pmatrix} x_{(1)} & u_1 \\ x_{(2)} & u_2 \\ \vdots & \vdots \\ x_{(m)} & u_m \end{pmatrix}.$$

3. Examples.

EXAMPLE 1. Let us calculate $(xy \mid \alpha\beta)$ where $x, y \in L^+$ and $\alpha, \beta \in P^-$. First, we shall apply Rule 5. To use this, we begin by observing that

$$\begin{aligned}\Delta w = \Delta x \Delta y &= (1\otimes x + x\otimes 1)(1\otimes y + y \otimes 1)\\ &= 1\otimes xy + x\otimes y + y\otimes x + xy\otimes 1.\end{aligned}$$

Then, Rule 5 along with Rule 1 gives

$$(xy \mid \alpha\beta) = (x \mid \alpha)(y \mid \beta) + (y \mid \alpha)(x \mid \beta).$$

Second, we shall do the same calculation but now using Rule 6. Here,

$$\begin{aligned}\Delta u = \Delta\alpha\Delta\beta &= (1\otimes\alpha + \alpha\otimes 1)(1\otimes\beta + \beta\otimes 1)\\ &= 1\otimes\alpha\beta + \alpha\otimes\beta - \beta\otimes\alpha + \alpha\beta\otimes 1.\end{aligned}$$

So, $(xy \mid \alpha\beta) = (x \mid \alpha)(y \mid \beta) - (x \mid \beta)(y \mid \alpha)$ which, according to the commutation rule for biproducts, is $(x \mid \alpha)(y \mid \beta) + (y \mid \alpha)(x \mid \beta)$.

EXAMPLE 2. Let $x_1, \ldots, x_n$ and $\alpha_1, \ldots, \alpha_n$ be distinct elements in L^+ and P^+, respectively. Then

$$(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n) = \sum_\sigma (x_1 \mid \alpha_{\sigma1})(x_2 \mid \alpha_{\sigma2})\cdots(x_n \mid \alpha_{\sigma n}),$$

where σ ranges over all permutations of $\{1, 2, \ldots, n\}$.

PROOF. The proof is by induction on n and we sketch the details. (It is similar to the complete argument given for Example 5.) In Rule 6, we take $w' = x_1$ and $w'' = x_2x_3\cdots x_n$. Now

$$\begin{aligned}\Delta(\alpha_1\alpha_2\cdots\alpha_n) &= \alpha_1\otimes\alpha_2\alpha_3\cdots\alpha_n + \alpha_2\otimes\alpha_1\hat{\alpha}_2\alpha_3\cdots\alpha_n\\ &\quad + \cdots + \alpha_n\otimes\alpha_1\alpha_2\cdots\alpha_{n-1} + (\text{terms of wrong length}).\end{aligned}$$

Here, the symbol $\hat{\alpha}_i$ means α_i is not included in the product. Therefore,

$$\begin{aligned}(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n) &= (x_1 \mid \alpha_1)(x_2\cdots x_n \mid \alpha_2\cdots\alpha_n)\\ &\quad + (x_1 \mid \alpha_2)(x_2\cdots x_n \mid \alpha_1\hat{\alpha}_2\alpha_3\cdots\alpha_n)\\ &\quad + \cdots + (x_1 \mid \alpha_n)(x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_{n-1})\\ &= \sum_\sigma (x_1 \mid \alpha_{\sigma1})(x_2 \mid \alpha_{\sigma2})\cdots(x_n \mid \alpha_{\sigma n}),\end{aligned}$$

using the induction hypothesis. We call $(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n)$ the *permanent* of x_i, α_j and denote it by $\operatorname{perm}(x_i \mid \alpha_j)$.

EXAMPLE 3. Let $x_1, \ldots, x_n$ and $\alpha_1, \ldots, \alpha_n$ be distinct elements in L^- and P^-, respectively. Then

$$(x_1x_2\cdots x_n \mid \alpha_n\alpha_{n-1}\cdots\alpha_1) = \sum_\sigma \operatorname{sgn}\sigma(x_1 \mid \alpha_{\sigma1})(x_2 \mid \alpha_{\sigma2})\cdots(x_n \mid \alpha_{\sigma n}),$$

where σ ranges over all permutations of $\{1, 2, \ldots, n\}$.

PROOF. The proof is by induction on n and we sketch the details. We shall apply Rule 6 with $w' = x_1$ and $w'' = x_2\cdots x_n$, noting that $|w''| = 0$ or 1 depending on whether $n-1$ is even or odd. Now

$$\Delta(\alpha_n\alpha_{n-1}\cdots\alpha_1) = \sum_{i=1}^n (-1)^{n-i}\alpha_i\otimes\alpha_n\alpha_{n-1}\cdots\hat{\alpha}_i\cdots\alpha_1 + \cdots.$$

Therefore,

$$\begin{aligned}(x_1x_2\cdots x_n \mid \alpha_n\alpha_{n-1}\cdots\alpha_1) &= \sum_{i=1}^n (-1)^{i-1}(x_1 \mid \alpha_i)(x_2\cdots x_n \mid \alpha_n\cdots\hat{\alpha}_i\cdots\alpha_1)\\ &= \sum_\sigma \operatorname{sgn}\sigma(x_1 \mid \alpha_{\sigma1})\cdots(x_n \mid \alpha_{\sigma n}).\end{aligned}$$

EXAMPLE 4. Let $x \in L^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in P^-$. Then

$$(x^{(n)} \mid \alpha_1\alpha_2\cdots\alpha_n) = (x \mid \alpha_1)(x \mid \alpha_2)\cdots(x \mid \alpha_n).$$

PROOF. The proof is by induction on n, the case $n = 1$ being trivial. In general, we apply Rule 5 with $u' = \alpha_1$ and $u'' = \alpha_2\cdots\alpha_n$. We recall that $\Delta x^{(n)} = \sum_{i=0}^n x^{(i)}\otimes x^{(n-i)}$. Therefore,

$$\begin{aligned}(x^{(n)} \mid \alpha_1\alpha_2\cdots\alpha_n) &= (x \mid \alpha_1)(x^{(n-1)} \mid \alpha_2\cdots\alpha_n)\\ &= (x \mid \alpha_1)(x \mid \alpha_2)\cdots(x \mid \alpha_n).\end{aligned}$$

EXAMPLE 5 (DOUBILET-ROTA, 1976). Let $x_1, x_2, \ldots, x_n \in L^+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in P^-$. Then

$$\begin{aligned}(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n) &= \sum_\sigma \operatorname{sgn}\sigma(x_1 \mid \alpha_{\sigma1})(x_2 \mid \alpha_{\sigma2})\cdots(x_n \mid \alpha_{\sigma n})\\ &= \sum_\sigma (x_{\sigma1} \mid \alpha_1)(x_{\sigma2} \mid \alpha_2)\cdots(x_{\sigma n} \mid \alpha_n),\end{aligned}$$

where σ ranges over all permutations of $\{1, 2, \ldots, n\}$.

PROOF. We shall prove the first equality by induction on n, applying Rule 6. The second equality is proved in a similar manner but by applying Rule 5. The case $n = 1$ is trivial so we may suppose that the first equality holds for all positive integers $< n$. We now apply Rule 6 with $w' = x_1$ and $w'' = x_2\cdots x_n$. Now,

$$\Delta(\alpha_1\alpha_2\cdots\alpha_n) = \sum_{i=1}^n (-1)^{i-1}\alpha_i\otimes\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n + (\text{terms of wrong length}).$$

So,

$$\begin{aligned}(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n) &= \sum_{i=1}^{n}(-1)^{i-1}(x_1 \mid \alpha_i)(x_2\cdots x_n \mid \alpha_1\cdots\hat{\alpha}_i\cdots\alpha_n)\\ &= \sum_{i=1}^{n}(-1)^{i-1}(x_1 \mid \alpha_i)\sum_{\tau}(\operatorname{sgn}\tau)(x_2 \mid \alpha_{\tau 1})\cdots(x_n \mid \alpha_{\tau n}),\end{aligned}$$

where the second sum on the right-hand side is over all permutations τ of $\{1,\dots,\hat{i},\dots,n\}$.

Let us put $\sigma = \tau(1i)$. We note that $\sigma(1) = i$. Furthermore, as i runs over $1,2,\dots,n$ and τ runs through all permutations of $\{1,\dots,\hat{i},\dots,n\}$, the formula $\sigma = \tau(1i)$ gives back exactly once each permutation σ of $\{1,2,\dots,n\}$. In this notation, we have

$$\begin{aligned}(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n) &= \sum_{i=1}^{n}(-1)^{i-1}(x_1 \mid \alpha_i)\sum_{\tau}(\operatorname{sgn}\tau)\\ &\quad\times(x_2 \mid \alpha_{\tau 1})\cdots(x_i \mid \alpha_{\tau(i-1)})(x_{i+1} \mid \alpha_{\tau(i+1)})\cdots(x_n \mid \alpha_{\tau n})\\ &= \sum_{\sigma}(x_1 \mid \alpha_{\sigma 1})(-1)^i(\operatorname{sgn}\sigma)\\ &\quad\times(x_2 \mid \alpha_{\sigma i})\cdots(x_i \mid \alpha_{\sigma(i-1)})(x_{i+1} \mid \alpha_{\sigma(i+1)})\cdots(x_n \mid \alpha_{\sigma n}).\end{aligned}$$

Now, let $\mu = (i, i-1\cdots 2)$. We note that $\operatorname{sgn}\mu = (-1)^i$. The expression above becomes

$$\sum_{\sigma}\operatorname{sgn}(\sigma\mu)(x_1 \mid \alpha_{\sigma\mu 1})(x_2 \mid \alpha_{\sigma\mu 2})\cdots(x_n \mid \alpha_{\sigma\mu n})$$

as desired.

EXAMPLE 6. We shall calculate $(x^{(2)}y \mid \alpha^{(2)}\beta)$ where y and β are negative. Now,

$$\begin{aligned}\Delta(\alpha^{(2)}\beta) = \alpha^{(2)}\otimes\beta + \alpha\beta\otimes\alpha + \alpha^{(2)}\beta\otimes 1 + \alpha\otimes\alpha\beta\\ + 1\otimes\alpha^{(2)}\beta + \beta\otimes\alpha^{(2)}.\end{aligned}$$

Hence, by Rule 6,

$$\begin{aligned}(x^{(2)}y \mid \alpha^{(2)}\beta) &= (x^{(2)} \mid \alpha^{(2)})(y \mid \beta) - (x^{(2)} \mid \alpha\beta)(y \mid \alpha)\\ &= (x \mid \alpha)^{(2)}(y \mid \beta) - (x \mid \alpha)(x \mid \beta)(y \mid \alpha).\end{aligned}$$

4. The tableau of two words. In this section, we shall continue to let L and P be disjoint proper signed sets. Let $w = x_1x_2\cdots x_n$ and $u = \alpha_1\alpha_2\cdots\alpha_n$ be words in $\mathrm{Mon}(L)$ and $\mathrm{Mon}(P)$, respectively. Then $\mathrm{stand}(w)$ and $\mathrm{stand}(u)$ are words in $\mathrm{Div}(L)$ and $\mathrm{Div}(P)$, respectively. We define the tableau of w and u, written $\mathrm{tab}(w \mid u)$, by

$$\mathrm{tab}(w \mid u) = (\mathrm{stand}(w) \mid \mathrm{stand}(u)),$$

where the right-hand side is the Laplace pairing Ω from $\mathrm{Super}[L]\times\mathrm{Super}[P]$ to $\mathrm{Super}[L \mid P]$.

Note. If each $x_i \in L^+$ and each $\alpha_j \in P^+$, we define the *characteristic-free permanent* to be $\mathrm{tab}(x_1x_2\cdots x_n \mid \alpha_1\alpha_2\cdots\alpha_n)$. A calculation with such an expression was made in Example 2, above.

Now, for a moment, let A be any signed set. We may take a basis for $\mathrm{Super}[A]$ over $\mathbf{Z}$ as remarked at the end of §2. Chapter 1, namely, elements of the form

$$a_1^{(e_1)}\cdots a_r^{(e_r)}b_1\cdots b_s c_1^{m_1}\cdots c_u^{m_u},$$

where the a's (resp. c's) are distinct elements in A^+ (resp. A^0) and the b's are distinct elements in A^-. We then may extend this basis to a $\mathbf{Z}$-basis of $\mathrm{Super}[A]\otimes\cdots\otimes\mathrm{Super}[A]$. We shall call this basis the *natural basis* of $\mathrm{Super}[A]\otimes\cdots\otimes\mathrm{Super}[A]$.

The purpose of this section is to examine $\Delta^{(m)}\mathrm{tab}(w \mid u)$ in anticipation of some of the arguments in Chapter 3. We begin by noting two properties of Δ. (Their easy proofs are omitted.)

(1) If $a \in A^-$ or A^0, then

$$\Delta^{(m)}(a) = a\otimes 1\otimes\cdots\otimes 1 + 1\otimes a\otimes 1\otimes\cdots\otimes 1 + \cdots + 1\otimes\cdots\otimes 1\otimes a.$$

If $a \in A^+$, then

$$\Delta^{(m)}(a^{(e)}) = \sum a^{(i_1)}\otimes\cdots\otimes a^{(i_m)},$$

where the sum is over all m-tuples of nonnegative integers $(i_1,\dots,i_m)$ such that $i_1+\cdots+i_m = e$.

(2) Let w be a word in $\mathrm{Div}(A)$ with

$$w = x_1^{(e_1)}\cdots x_r^{(e_r)}y_1\cdots y_s$$

and where the x's (resp. y's) are distinct elements in A^+ (resp. A^-). Then

$$\Delta(w) = \Delta(x_1^{(e_1)})\cdots\delta(y_s) = \sum \pm w_1\otimes w_2,$$

where w_1 and w_2 range over all distinct words such that $\mathrm{Cont}(w_1)+\mathrm{Cont}(w_2) = \mathrm{Cont}(w)$.

PROPOSITION 7. *Let L and P be disjoint proper signed sets. Let $w \in \mathrm{Mon}(L)$, $w = x_1x_2\cdots x_n$, where w has no repeated negative letter. Let $u \in \mathrm{Mon}(P)$, $u = \alpha_1\alpha_2\cdots\alpha_n$, where u has no repeated negative letter. Let $m \geq n$ and let $\Delta^{(m)}(\mathrm{stand}(w) \mid \mathrm{stand}(u))$ be written as a linear combination of basis elements in the natural basis for $\mathrm{Super}[L \mid P]\otimes\cdots\otimes\mathrm{Super}[L \mid P]$, say,*

$$\Delta^{(m)}(\mathrm{stand}(w) \mid \mathrm{stand}(u)) = \sum c_v v,\ c_v \in \mathbf{Z}.$$

(a) *If $v = v_1\otimes\cdots\otimes v_m$ where $\mathrm{Length}(v_i) \leq 1$ for each $i = 1,\dots,m$ and if $c_v \neq 0$, then v has the following form.*

(*) *Let σ be a permutation of $\{1,2,\dots,n\}$ and let $\{i_1,i_2,\dots,i_n\}$ be a subset of $\{1,2,\dots,m\}$. Let $j \in \{1,2,\dots,m\}$. If $j = i_k$, put $v_j = (x_k \mid \alpha_{\sigma k})$; otherwise, put $v_j = 1$. Let $v = v_1\otimes\cdots\otimes v_m$.*

(b) *Conversely, if v has the form (*), then $c_v = \pm 1$.*

PROOF. We proceed by induction on n, the cases $(x \mid \alpha)$ and $(x^{(n)} \mid \alpha^{(n)})$ following from property (1), above. Hence, we may assume, e.g., that $\text{stand}(w) = w'w''$ where w', w'' have no common letters and have Lengths smaller than $\text{Length}(w)$. Apply Rule 6, §1, to see that

$$\Delta^{(m)}(w'w'' \mid \text{stand}(u)) = \sum \pm \Delta^{(m)}(w' \mid u_{(1)}) \Delta^{(m)}(w'' \mid u_{(2)}).$$

The proof now follows by applying the induction hypothesis and property (2), above.

An immediate consequence of part (b) of the Theorem is that $\text{tab}(w \mid u) \neq 0$ if neither w nor u has repeated negative letters.

3. The Standard Basis Theorem

0. Introduction. In this chapter we derive a generalization of the classical straightening algorithm whose idea goes back to Clebsch, Capelli, and Young. We consider double Young tableaux, namely, products

$$(w_1 \mid u_1)(w_2 \mid u_2) \cdots (w_n \mid u_n)$$

of biproducts of w_i in $\text{Div}[L]$ and u_i in $\text{Div}[P]$, where L and P are proper alphabets. If $\text{Length}(w_i) = \lambda_i$ and if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then such a product of biproducts can be viewed as an operation which assigns a member of $\text{Super}[L \mid P]$ to a pair of triangular arrays D-E (or Young diagrams) of the same shape $\lambda = (\lambda_1, \lambda_2, \ldots)$.

The rows of the diagrams D are the displays of the words $w_1, \ldots, w_n$, lined up one beneath the other; in this way the columns of D are defined. The diagram E is similarly made up out of the words $u_1, u_2, \ldots, u_n$. The two diagrams D and E are of the same display shape λ, where λ_i is the number of entries of the ith row. The novelty of the present approach consists in dealing with Young diagrams which are filled with both positively and negatively signed letters. This leads to a definition of a standard Young diagram (sometimes also called a Young tableau) which differs from the ones previously used, and which, as will be proved below, is the only one compatible with entires of two different signatures. The definition of a standard Young diagram requires both alphabets L and P to be linearly ordered sets. A diagram D filled with letters from the alphabet L, say, is defined to be standard where:

(1) if two letters x, y are next to each other on the same row of the diagram, and y follows x, then we must have $x \leq y$ unless both x and y are negatively signed, in which case we require $x < y$.

(2) if two letters x, y are next to each other on the same column, and if y is on a lower row than x, then we have $x \leq y$ unless both x and y are positively signed, in which case we require that $x < y$.

Our main result (the general Standard Basis Theorem) states that double standard diagrams (standard in both letters and places) are an integral basis for the underlying module of the superalgebra $\text{Super}[L \mid P]$. The elements of a standard basis thus depend upon the linear orders assigned to L and P. Such orderings of both L and P can be chosen arbitrarily independently of the position of positively and negatively signed elements; thus, depending on this choice, the set of standard Young diagrams can strikingly differ. For example all positively signed letters may be placed before all negatively signed letters in the linear order, or the linear order may be so chosen that every letter is immediately followed by a letter of the opposite parity. By the invariance of a dimension of a free module, it follows from our Standard Basis Theorem that the number of standard Young diagrams is the same irrespective of the linear ordering chosen, a fact that is far from obvious combinatorially. Among several special cases of the Standard Basis Theorem, at least two are worthy of mention and lead to new results in the invariant theory of tensors. When all letters are positive and all places negative, we obtain a generalization of the determinant that will be useful to derive the invariant theory of skew-symmetric tensors in arbitrary characteristics in much the same way as the classical case ($L = L^-$ and $P = P^-$, as in Doubilet-Rota-Stein) relates to the invariant theory of symmetric tensors (but in this latter case only in characteristic zero).

Secondly, when all letters and places are positive, we obtain a straightening algorithm for permanents, applicable in arbitrary characteristic, a result which a priori may seem unexpected.

1. Definitions, statement of theorem. In this chapter, we shall develop the main combinatoric tools to be applied to invariant theory. We begin with some important terminology.

Let A (for a moment) be any set. A *Young diagram* on A, $D = (w_1, w_2, \ldots, w_n)$ is a sequence of words $w_i \in \mathrm{Mon}(A)$ such that $\mathrm{length}(w_1) \geq \mathrm{length}(w_2) \geq \cdots \geq \mathrm{length}(w_n)$. The integer n is called the *number of rows* of the Young diagram D and the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called the *shape* of D; here $\lambda_i = \mathrm{length}(w_i)$. We define "content" in this setting by $\mathrm{cont}(D) = \mathrm{cont}(w_1) + \mathrm{cont}(w_2) + \cdots + \mathrm{cont}(w_n)$.

If $D = (w_1, w_2, \ldots, w_n)$ where $w_1 = x_1 x_2 \cdots x_{\lambda_1}$, $w_2 = y_1 y_2 \cdots y_{\lambda_2}$, then we envision D as follows

$$\begin{array}{rcccc} & x_1 & x_2 & \cdots & x_{\lambda_1} \\ D\colon & y_1 & y_2 & \cdots & y_{\lambda_2} \\ & & \vdots & & \end{array}$$

The shapes of Young diagrams may be partially ordered. Indeed, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_m)$, then we write $\lambda \leq \lambda'$ when $\sum_i \lambda_i = \sum_i \lambda'_i$ and $\lambda_1 \leq \lambda'_1$, $\lambda_1 + \lambda_2 \leq \lambda'_1 + \lambda'_2$, and so on.

Let $D = (w_1, w_2, \ldots, w_n)$ be a Young diagram with $w_1 = x_{11} x_{12} \cdots x_{1\lambda_1}$, $w_2 = x_{21} x_{22} \cdots x_{2\lambda_2}, \ldots$. We define a Young diagram $\tilde{D} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_k)$, called the *dual* of D, by

$$\tilde{w}_1 = x_{11} x_{21} \cdots x_{n1}, \qquad \tilde{w}_2 = x_{12} x_{22} \cdots$$

and so on. The shape of $\tilde{D}$ is $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k)$ where $\tilde{\lambda}_i$ is the number of λ_j which are $\geq i$. We note here that $\tilde{\tilde{D}} = D$ and $\mathrm{cont}(\tilde{D}) = \mathrm{cont}(D)$. We call $k = \lambda_1$ the *number of columns* of D. Furthermore, we say that two entries of D are in the same row (resp. column) if they are in the same w_i (resp. $\tilde{w}_i$). Also, in an obvious way, we may speak of "adjacent entries" in D.

A linearly ordered proper signed set A is called an *alphabet.*

Let A be an alphabet. Let $D = (w_1, w_2, \ldots, w_n)$ be a Young diagram on A and let $\tilde{D} = (\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_k)$ be its dual. The Young Diagram D is said to be *standard* when, for each of the words

$$w_i = x_1 x_2 \cdots x_r, \qquad x_j \in A,$$
$$\tilde{w}_i = y_1 y_2 \cdots y_s, \qquad y_j \in A,$$

the following conditions are satisfied:

(1) if $x_j \in A^+$, then $x_j \leq x_{j+1}$,
(2) if $x_j \in A^-$, then $x_j < x_{j+1}$,
(3) if $y_j \in A^+$, then $y_j < y_{j+1}$,
(4) if $y_j \in A^-$, then $y_j \leq y_{j+1}$.

Now, let L and P be disjoint alphabets. In Chapter 2, we defined the signed set $[L \mid P]$ and constructed the Laplace pairing

$$\Omega\colon \mathrm{Super}[L] \times \mathrm{Super}[P] \to \mathrm{Super}[L \mid P].$$

Let $D = (w_1, w_2, \ldots, w_n)$ and $E = (w'_1, w'_2, \ldots, w'_n)$ be Young diagrams, of the same shape, on L and P, respectively. The *Young tableau* of the diagram pair D-E is defined by $\mathrm{Tab}(D \mid E) = (\mathrm{stand}(w_1) \mid \mathrm{stand}(w'_1)) \cdots (\mathrm{stand}(w_n) \mid \mathrm{stand}(w'_n))$. With all this terminology in place, we may now state the main theorem in this chapter.

THEOREM 8. *Let L and P be disjoint alphabets. The* $\mathrm{Tab}(D_i \mid E_i)$*, where D_i and E_i are standard and have the same shape, form a basis of* $\mathrm{Super}[L \mid P]$ *over* $\mathbf{Z}$. *In particular, let D and E be Young diagrams, having the same shape, on L and P, respectively. Then*

$$\mathrm{Tab}(D \mid E) = \sum_i c_i \,\mathrm{Tab}(D_i \mid E_i)$$

in $\mathrm{Super}[L \mid P]$ *where*

(i) *D_i and E_i are standard on L and P, respectively;*
(ii) $\mathrm{cont}(D_i) = \mathrm{cont}(D)$ *and* $\mathrm{cont}(E_i) = \mathrm{cont}(E)$;
(iii) $\mathrm{shape}(D) \leq \mathrm{shape}(D_i)$;
(iv) *the integer coefficients c_i are unique.*

The existence of such a representation will be proved in §2. The uniqueness of the coefficients c_i will be proved in §§3, 4, and 5.

NOTES. (1) If $L = L^-$ and $P = P^-$, we get the classical straightening formula proved by Doubilet, Rota, and Stein in 1974.

(2) The case where $L = L^+$ and $P = P^-$ was proved in characteristic zero only by Doubilet and Rota in 1976. The characteristic-free result given above is new.

(3) Permanents may be straightened by applying the theorem when $L = L^+$ and $P = P^+$. Our result is however more general, since it applies to characteristic-free permanents as well.

(4) The dimension of the space spanned by $\mathrm{Tab}(D_i \mid E_i)$, where D_i and E_i are standard of some given shape, is independent of the orderings on L and P. (It does depend, of course, on the choice of subsets L^+, L^-, P^+, and P^-.) One interesting order is when $L^+ < L^-$.

We conclude this section with some technical comments on a content function for $L \cup P$. These will be useful to us in the proofs to follow.

Let $w \in \mathrm{Div}([L \mid P])$. For $x \in L$, we define

$$\mathrm{Cont}(w)(x) = \sum_{\alpha \in P} \mathrm{Cont}(w)(x \mid \alpha).$$

Similarly, for $\alpha \in P$ we may define $\mathrm{Cont}(w)(\alpha)$. In this way, we obtain a multiset, $\mathrm{Cont}(w)$, on $L \cup P$. The following properties of this multiset will be useful to us. (Their elementary proofs are omitted.)

(1) If $z \in L \cup P$, then $\mathrm{Cont}(ww')(z) = \mathrm{Cont}(w)(z) + \mathrm{Cont}(w')(z)$.

(2) Let m be a given multiset on $L \cup P$. We consider all words $w \in \mathrm{Div}([L \mid P])$ such that $\mathrm{Cont}(w) = m$. These words, together with 0, span a submodule of $\mathrm{Tens}[L \mid P]$ which we shall denote by $\mathrm{Tens}_m[L \mid P]$. Furthermore, $\mathrm{Tens}[L \mid P]$ is the direct sum of all these submodules.

Let I be the ideal in $\mathrm{Tens}[L \mid P]$ defining $\mathrm{Super}[L \mid P]$. If $i \in I$ and $i = \sum i_m$, where $i_m \in \mathrm{Tens}_m[L \mid P]$, then each i_m is in I since this is true for the generators of I. It follows that if $w, w' \in \mathrm{Div}([L \mid P])$ and $w \equiv w' \pmod I$ but $w \notin I$, then $\mathrm{Cont}(w)(z) = \mathrm{Cont}(w')(z)$ for all $z \in L \cup P$.

(3) Let w, u be words of the same Length in $\mathrm{Super}[L]$ and $\mathrm{Super}[P]$, respectively. Applying induction on Length w and Rules 4, 5, and 6 in Chapter 2, §1, we may show that for each term W in the expansion of $(w \mid u)$, $\mathrm{Cont}(W)(x) = \mathrm{Cont}(w)(x)$ for all $x \in L$ and $\mathrm{Cont}(W)(\alpha) = \mathrm{Cont}(u)(\alpha)$ for all $\alpha \in P$. Sometimes, we denote this by $\mathrm{Cont}(w \mid u)(x) = \mathrm{Cont}(w)(x)$ and $\mathrm{Cont}(w \mid u)(\alpha) = \mathrm{Cont}(u)(\alpha)$.

(4) Let $w \in \mathrm{Super}[L \mid P]$ and let

$$\Delta^{(m)}(w) = \sum w_{(1)} \otimes \cdots \otimes w_{(m)}.$$

In Chapter 1, we saw that (we may assume that)

$$\sum_i \mathrm{Cont}\, w_{(i)}(x \mid \alpha) = \mathrm{Cont}(w)(x \mid \alpha).$$

From this it follows that we may assume that

$$\sum \mathrm{Cont}(w_{(i)})(z) = \mathrm{Cont}(w)(z)$$

for all $z \in L \cup P$.

Let D and E be Young diagrams, having the same shape, on L and P, respectively. Let $x \in L$. It follows from the preceding properties that

$$\mathrm{Cont}(\mathrm{Tab}(D \mid E))(x) = \mathrm{cont}(D)(x).$$

Similarly, if $\alpha \in P$, then

$$\mathrm{Cont}(\mathrm{Tab}(D \mid E))(\alpha) = \mathrm{cont}(E)(\alpha).$$

This has the following important consequence. The proof of Theorem 8 will be carried out in some fixed submodule of $\mathrm{Super}[L \mid P]$ of given Content on $L \cup P$.

2. Spanning. In this section, we shall show that $\mathrm{Tab}(D \mid E)$ can be written as a linear combination of the $\mathrm{Tab}(D_i \mid E_i)$, where D_i and E_i are standard Young diagrams. For a moment, let A be any signed set. We recall that in Chapter 1, we defined a mapping $S\colon \mathrm{Super}[A] \to \mathrm{Super}[A]$ by $S(w) = \mathrm{sign}(\mathrm{Length}(w))w$. In this section, we shall denote $S(w)$ by $\langle w \rangle$. We also defined $\varepsilon(w) = 1$ if $\mathrm{Length}(w) = 0$ and $\varepsilon(w) = 0$, otherwise.

In Chapter 2, we defined the tableau

$$\left(\begin{array}{c|c} w_1 & u_1 \\ w_2 & u_2 \\ \vdots & \vdots \\ w_n & u_n \end{array}\right)$$

to be sign $(\sum_{i>j} |w_i|\,|u_j|)(w_1 \mid u_1) \cdots (w_n \mid u_n)$. We shall be working with such expressions throughout this section.

Again, let A be any signed set. If $z \in \mathrm{Super}[A]$, then we have written $\Delta z = \sum_z z_{(1)} \otimes z_{(2)}$. In this section only, we change this notation and write

$$\Delta z = \sum_z z_1 \otimes z_2.$$

Similarly, since $\Delta^{(3)}(z) = (1 \otimes \Delta)\Delta(z)$, we shall write

$$\Delta^{(3)}(z) = \sum_{z, z_2} z_1 \otimes z_{21} \otimes z_{22}$$

and so on. (We shall use repeatedly fact (1) following Rule 6 in Chapter 2.)

REMARK. An induction argument on Length z shows that

$$\Delta z = \sum_z z_1 \otimes z_2 = \sum \mathrm{sign}(|z_1|\,|z_2|) z_2 \otimes z_1.$$

PROPOSITION 9. *Let d, c be words in* $\mathrm{Super}[L]$ *with* $\Delta^{(3)}(d) = \sum_d d_1 \otimes d_2 \otimes d_3$. *Let y, z be words in* $\mathrm{Super}[P]$ *with* $\Delta^{(3)}(z) = \sum z_1 \otimes z_2 \otimes z_3$. *Then*

$$\sum_{d,z} \left(\begin{array}{c|c} d_1 & y \\ d_2 & z_1 \\ \langle d_3 \rangle & z_2 \\ c & z_3 \end{array}\right) = \left(\begin{array}{c|c} d & y \\ c & z \end{array}\right).$$

PROOF. The left-hand side of this equation may be rewritten as

$$\sum_{d,z}\sum_{z_1}\left(\begin{array}{c|c} d_1 & y \\ d_2 & z_{11} \\ \langle d_3\rangle & z_{12} \\ c & z_2 \end{array}\right).$$

Applying Rule 6 in Chapter 2, we see this is equal to

$$\sum_{d,z}\left(\begin{array}{c|c} d_1 & y \\ d_2\langle d_3\rangle & z_1 \\ c & z_2 \end{array}\right).$$

Relabeling the d-variables, the expression above is

$$\sum_{d,z}\sum_{d_2}\left(\begin{array}{c|c} d_1 & y \\ d_{21}\langle d_{22}\rangle & z_1 \\ c & z_2 \end{array}\right).$$

According to Proposition 6, we have $\sum_{d_2} d_{21}\langle d_{22}\rangle = \varepsilon(d_2)$. So, the expression above becomes

$$\sum_{d,z}\left(\begin{array}{c|c} d_1 & y \\ \varepsilon(d_2) & z_1 \\ c & z_2 \end{array}\right).$$

The only nonzero term of this sum occurs when $d_1 = d$, $d_2 = 1$, $z_1 = 1$, and $z_2 = z$. This proves the Proposition.

PROPOSITION 10. *Let a, b, c be words in* Super$[L]$ *with $\Delta a = \sum_a a_1 \otimes a_2$ and $\Delta b = \sum_b b_1 \otimes b_2$. Let x and y be words in* Super$[P]$ *with $\Delta y = \sum_y y_1 \otimes y_2$. Then*

$$\sum_b \left(\begin{array}{c|c} ab_1 & x \\ b_2c & y \end{array}\right) = \operatorname{sign}(|a|\,|b|)\sum_{a,y}\left(\begin{array}{c|c} ba_1 & xy_1 \\ \langle a_2\rangle c & y_2 \end{array}\right).$$

PROOF. We being by applying Rule 6, Chapter 2, §2, to the terms $(ab_1 \mid x)$ and $(b_2c \mid y)$. The left-hand side of the equation stated in the theorem then becomes

$$\sum_{b,x,y}\left(\begin{array}{c|c} a & x_1 \\ b_1 & x_2 \\ b_2 & y_1 \\ c & y_2 \end{array}\right).$$

Permuting the first row with the second and third rows (via condition (ii) in §2 of Chapter 2), this becomes

$$\sum_{b,x,y} \operatorname{sign}(|a|\,|b|)\left(\begin{array}{c|c} b_1 & x_2 \\ b_2 & y_1 \\ a & x_1 \\ c & y_2 \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_1|).$$

Next, we apply Proposition 9 to the last two rows with $d = a$, $y = x_1$, $c = c$, and $z = y_2$; we obtain

$$\sum_{b,x,y}\sum_{y_2,a} \operatorname{sign}(|a|\,|b|)\left(\begin{array}{c|c} b_1 & x_2 \\ b_2 & y_1 \\ a_1 & x_1 \\ a_2 & y_{21} \\ \langle a_3\rangle & y_{22} \\ c & y_{23} \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_1|).$$

But,

$$\sum_{y,y_2} y_1 \otimes y_{21} \otimes y_{22} \otimes y_{23} = \sum_{y,y_3} y_1 \otimes y_2 \otimes y_{31} \otimes y_{32}.$$

We apply Rule 6 to the last two rows of the expression above and relabel the y-variables (as allowed by fact (1) following Rule 6 in Chapter 2) to obtain

$$\operatorname{sign}(|a|\,|b|)\sum_{a,b,x,y}\left(\begin{array}{c|c} b_1 & x_2 \\ b_2 & y_1 \\ a_1 & x_1 \\ a_2 & y_2 \\ \langle a_3\rangle c & y_3 \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_1|).$$

We relabel the a-variables to see this equals

$$\operatorname{sign}(|a|\,|b|)\sum_{a,b,x,y}\left(\begin{array}{c|c} b_1 & x_2 \\ b_2 & y_1 \\ a_{11} & x_1 \\ a_{12} & y_2 \\ \langle a_2\rangle c & y_3 \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_1|).$$

Two further applications of Rule 5 simplify this to

$$\operatorname{sign}(|a|\,|b|)\sum_{a,x,y}\left(\begin{array}{c|c} b & x_2y_1 \\ a_1 & x_1y_2 \\ \langle a_2\rangle c & y_3 \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_1|).$$

Relabeling the y-variables, we see that this equals

$$\operatorname{sign}(|a|\,|b|)\sum_{a,x,y}\left(\begin{array}{c|c} b & x_2y_{11} \\ a_1 & x_1y_{12} \\ \langle a_2\rangle c & y_2 \end{array}\right) \operatorname{sign}(|x_1|\,|x_2| + |x_1|\,|y_{11}|).$$

Now,

$$\Delta(xy_1) = \left(\sum_x x_1 \otimes x_2\right)\left(\sum_{y_1} y_{11} \otimes y_{12}\right) = \sum_{x,y_1} \operatorname{sign}(|x_2|\,|y_{11}|)x_1y_{11} \otimes x_2y_{12}.$$

We apply the remark made just before Proposition 9 to see that $\Delta(xy_1)$ is also

$$\left(\sum_x \operatorname{sign}(|x_2|\,|x_1|)x_2 \otimes x_1\right)\left(\sum_{y_1} y_{11} \otimes y_{12}\right)$$
$$= \sum_{x,y_1} \operatorname{sign}(|x_2|\,|x_1| + |x_1|\,|y_{11}|)x_2y_{11} \otimes x_1y_{12}.$$

Therefore, the expression above is

$$\operatorname{sign}(|a|\,|b|)\sum_{a,x,y}\left(\begin{array}{c|c} b & x_1y_{11} \\ a_1 & x_2y_{12} \\ \langle a_2\rangle c & y_2 \end{array}\right)\operatorname{sign}(|x_2|\,|y_{11}|).$$

One final application of Rule 6 shows that this equals

$$\operatorname{sign}(|a|\,|b|)\sum_{a,y}\left(\begin{array}{c|c} ba_1 & xy_1 \\ \langle a_2\rangle c & y_2 \end{array}\right)$$

as desired.

NOTE. Applying the same reasoning with the roles of the left and right tableaux interchanged, we see that

$$\sum_{y}\left(\begin{array}{c|c} a & xy_1 \\ b & y_2z \end{array}\right) = \operatorname{sign}(|x|\,|y|)\sum_{x,b}\left(\begin{array}{c|c} ab_1 & yx_1 \\ b_2 & \langle x_2\rangle z \end{array}\right).$$

PROOF OF THEOREM 8, (i), (ii), AND (iii). Let $D = (w_1, w_2, \ldots, w_n)$ and $E = (w'_1, w'_2, \ldots, w'_n)$ be Young diagrams, having the same shape, on the alphabets L and P, respectively, Suppose that the shape of $D = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

If $n = 1$, then we are finished since in the expression $(\operatorname{stand}(w_1) \mid \operatorname{stand}(w'_1))$ terms can be rearranged in Super$[L]$ or Super$[P]$ and $a^{(i)}a^{(j)}$ can be replaced by $\binom{i+j}{i}a^{(i+j)}$.

Next, suppose $n > 1$. As noted at the end of §1, we shall carry out this proof in a submodule of Super$[L \mid P]$ of given Content on $L \cup P$. We may assume the theorem holds for all such Young diagrams of shape bigger than $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We now suppose that the theorem is false for some diagram pairs of shape λ. We order such diagram pairs via the lexicographic ordering obtained by lining up the words. That is, suppose that $D = (w_1, w_2, \ldots, w_n)$ and $E = (w'_1, w'_2, \ldots, w'_n)$ where $w_i = x_{i1}x_{i2}\cdots x_{i\lambda_i}$ and $w'_i = x'_{i1}x'_{i2}\cdots x'_{i\lambda_i}$. Then, we apply the lexicographic order to the sequence

$$x_{11}, x_{12}, \ldots, x_{1\lambda_1}, x_{21}, \ldots, x'_{n\lambda_n}.$$

Among the diagram pairs D-E of shape λ for which the theorem is false, we choose the smallest pair D-E in this lexicographic order. It follows that the rows of D and E are nondecreasing, i.e.,

$$x_{i1} \le x_{i2} \le \cdots \quad \text{and} \quad x'_{i1} \le x'_{i2} \le \cdots.$$

If either D or E has a row with a repeated negative letter, then $\operatorname{Tab}(D \mid E) = 0$ and the theorem holds. Otherwise, since either D or E is not standard, conditions (3) or (4) of the definition in §1 must be violated. We shall assume that the first such violation occurs in D. Changing notation, let us assume this first violation occurs in rows

$$x_1x_2\cdots x_mx_{m+1}x_{m+2}\cdots x_n$$

$$y_1y_2\cdots y_my_{m+1}y_{m+2}\cdots y_r$$

when comparing x_{m+1} to y_{m+1}.

Case 1. $x_{m+1} > y_{m+1}$. We begin by noting that

$$x_1 \le \cdots \le x_m < x_{m+1} \le \cdots \le x_n,$$

$$y_1 \le \cdots \le y_m \le y_{m+1} = \cdots = y_{m+k} < y_{m+k+1} \le \cdots \le y_r$$

(where $k \ge 1$) and

$$y_1 \le \cdots \le y_m \le y_{m+1} < x_{m+1} \le \cdots \le x_n.$$

In the statement of Proposition 10, we take

$$a = \operatorname{stand}(x_1\cdots x_m), \qquad c = \operatorname{stand}(y_{m+k+1}\cdots y_r),$$

and $b = de$, where

$$d = \operatorname{stand}(x_{m+1}\cdots x_n), \qquad e = \operatorname{stand}(y_1\cdots y_{m+k}).$$

Let $\Delta(de) = \sum b_1 \otimes b_2$. Then (we may assume that) there is one and only one term with $b_1 = d$ and $b_2 = e$.

We now examine both sides of the equation in Proposition 10. On the left-hand side, the term with $b_1 = d$, $b_2 = e$ appears once. In all the other terms on the left-hand side, b_1 must contain letters from e; the corresponding Young diagram is smaller than D in the lexicographic order. On the right-hand side, the length of the first row is

$$\operatorname{Length}(de) + \operatorname{Length}(a_1) \ge \operatorname{Length}(d) + \operatorname{Length}(e) > \operatorname{Length}(ad).$$

All the corresponding Young diagrams have longer shapes than that of D. Combining these two statements, we see that the theorem holds for D-E (which is a contradiction).

Case 2. $x_{m+1} = y_{m+1} = z =$ a positive element in L. Let us suppose that the rows in question are

$$x_1\cdots x_m\overbrace{z\cdots z}^{i}x_{m+i+1}\cdots x_n$$

and

$$y_1\cdots y_k\underbrace{z\cdots z\cdots z}_{j}y_{k+j+1}\cdots y_r,$$

where

$$x_1 \le \cdots \le x_m < z < x_{m+i+1} \le \cdots \le x_n$$

and

$$y_1 \le \cdots \le y_k < z < y_{k+j+1} \le \cdots \le y_r.$$

Again, we use Proposition 10, but now with

$$a = \operatorname{stand}(x_1\cdots x_m),$$

$$b = z^{(i+j)}\operatorname{stand}(x_{m+i+1}\cdots x_n)\operatorname{stand}(y_1\cdots y_k),$$

and

$$c = \operatorname{stand}(y_{k+j+1}\cdots y_r).$$

On the left-hand side, the term

$$\left(\begin{array}{l|l} az^{(i)}\,\mathrm{stand}(x_{m+i+1}\cdots x_n) & \\ & \cdots \\ \quad\mathrm{stand}(y_1\cdots y_k)z^{(j)}c & \end{array}\right)$$

appears once; if there are any other terms on the left-hand side, they correspond to Young diagrams shorter than D in the Lexicographic order. All the terms on the right-hand side correspond to Young diagrams having shape larger than that of D. This completes the proof of (i), (ii), and (iii).

3. A bilinear Z-valued form. Let A be any proper signed set. We define the *adjoint* of A in the following way. Let A^* be a set which is isomorphic to A via a mapping $\phi\colon A \to A^*$. We make A^* into a signed set by defining $(A^*)^-$ to be $\phi(A^+)$ and $(A^*)^+$ to be $\phi(A^-)$. The mapping ϕ induces a map (also denoted by ϕ) from $\mathrm{Mon}(A)$ to $\mathrm{Mon}(A^*)$ by

$$\phi(x_1x_2\cdots x_n) = \phi(x_1)\phi(x_2)\cdots\phi(x_n) = x_1^*x_2^*\cdots x_n^*.$$

If $w \in \mathrm{Mon}(A)$, we shall denote $\phi(w)$ by w^*. For a Young diagram $D = (w_1, w_2, \ldots, w_n)$ on A, we set $D^* = \phi(D) = (w_1^*, w_2^*, \ldots, w_n^*)$. If D and E are Young diagrams, of the same shape, on disjoint alphabets L and P, then we define

$$\mathrm{Tab}^+(D \mid E) = \mathrm{Tab}(\tilde{D}^* \mid \tilde{E}^*).$$

As usual, let L and P be disjoint alphabets. If $w \in \mathrm{Div}(L)$ and $w' \in \mathrm{Div}(P)$ with $\mathrm{Length}(w) = \mathrm{Length}(w')$, then we define $\mathrm{Length}(w \mid w') = \mathrm{Length}(w) = \mathrm{Length}(w')$. Indeed, according to (3), §1, we may assume that each nonzero term in the expansion of $(w \mid w')$ has Length equal to $\mathrm{Length}(w)$.

PROPOSITION 11. *For $p \in \mathrm{Super}[L^* \mid P^*]$ and $q \in \mathrm{Super}[L \mid P]$, a bilinear* **Z**-*valued form is consistently defined by the following algorithm:*

(1) $\langle p, q\rangle = 0$ *if* $\mathrm{Length}(p) \neq \mathrm{Length}(q)$;

(2) $\langle 1, 1\rangle = 1$;

(3) $\langle (x^* \mid \alpha^*), (x \mid \alpha)\rangle = \mathrm{sign}(|x|\,|\alpha^*|)$,
$\langle (x^* \mid \alpha^*), (y \mid \beta)\rangle = 0$ *if* $y \neq x$ *or* $\beta \neq \alpha$;

(4) $\langle (x^* \mid \alpha^*)^{(n)}, (y \mid \beta)^{(n)}\rangle = 0$ *if* $n \geq 2$;

(5) $\langle w, u', u''\rangle = \sum_w \mathrm{sign}(|w_{(2)}|\,|u'|)\langle w_{(1)}, u'\rangle\langle w_{(2)}, u''\rangle$

where $\Delta w = \sum_w w_{(1)} \otimes w_{(2)}$;

(6) $\langle w'w'', u\rangle = \sum_u \mathrm{sign}(|w''|\,|u_{(1)}|)\langle w', u_{(1)}\rangle\langle w'', u_{(2)}\rangle$

where $\Delta u = \sum_u u_{(1)} \otimes u_{(2)}$.

The proof of Proposition 11 is just like that given for the mapping Ω in Chapter 2, §2, and we shall omit it except for four comments. First, both conditions (i) and (ii) must be checked at each step n. (Since **Z** is commutative, ε can be -1 in condition (ii) only when both sides of the equation are 0.) Second, the analogues of Formulas (a) and (b) hold and will be important to us later. Third, if $(x^* \mid \alpha^*)$ is positive, then $(x \mid \alpha)$ is neutral. Fourth, $|(x \mid \alpha)| = |x| + |\alpha|$.

PROPOSITION 12. *Let $w, w' \in \mathrm{Div}(L)$, let $u, u' \in \mathrm{Div}(P)$, and let the Lengths of all four words be at least 2. Then $\langle (w^* \mid u^*), (w' \mid u')\rangle = 0$.*

PROOF. Suppose for a moment that we have proved the theorem when all the words have Length $= 2$. Then, applying (4), (5), and (6) in Proposition 11, we obtain the proof for any Length ≥ 3. So, we shall assume that all the words have Length $= 2$. The argument in this case is a long computation whose main points we shall indicate here.

(I) Let A be any signed set and let $x, y \in A$. Then

$$\begin{aligned}\Delta(xy) &= (x\otimes 1 + 1\otimes x)(y \otimes 1 + 1 \otimes y)\\ &= xy \otimes 1 + \mathrm{sign}(|x|\,|y|)y \otimes x\\ &\quad + x \otimes y + 1 \otimes xy.\end{aligned}$$

(II) To prove the theorem, it is enough to show that

$$\langle (x'y' \mid \alpha'\beta'), (xy \mid \alpha\beta)\rangle = 0,$$

where the letters and places above are not necessarily distinct. Now, applying Rule 5, Chapter 2, we see that

$$\begin{aligned}(xy \mid \alpha\beta) &= \mathrm{sign}(|x|\,|y| + |x|\,|\alpha|)(y \mid \alpha)(x \mid \beta)\\ &\quad + \mathrm{sign}(|y|\,|\alpha|)(x \mid \alpha)(y \mid \beta),\end{aligned}$$

so

$$\begin{aligned}&\langle (x'y' \mid \alpha'\beta'), (xy \mid \alpha\beta)\rangle\\ &\quad = \langle (x'y' \mid \alpha'\beta'), \mathrm{sign}(|x|\,|y| + |x|\,|\alpha|)(y \mid \alpha)(x \mid \beta)\\ &\qquad\qquad + \mathrm{sign}(|y|\,|\alpha|)(x \mid \alpha)(y \mid \beta)\rangle.\end{aligned}$$

(III) We now apply (5), Proposition 11, calculating $\Delta((x'y' \mid \alpha'\beta'))$ to be

$$\begin{aligned}&\mathrm{sign}(|x'|\,|y'| + |x'|\,|\alpha'|)[\mathrm{sign}(|(y' \mid \alpha')|\,|(x' \mid \beta')|)\\ &\qquad \times (x' \mid \beta') \otimes (y' \mid \alpha') + (y' \mid \alpha') \otimes (x' \mid \beta')]\\ &\quad + \mathrm{sign}(|y'|\,|\alpha'|)[\mathrm{sign}(|(x' \mid \alpha')|\,|(y' \mid \beta')|)\\ &\qquad \times (y' \mid \beta') \otimes (x' \mid \alpha') + (x' \mid \alpha') \otimes (y' \mid \beta')]\\ &\quad + (\text{terms of wrong shape}).\end{aligned}$$

When (5) is applied, we obtain eight terms.

$$\begin{aligned}&\mathrm{sign}(|x|\,|y| + |x|\,|\alpha| + |x'|\,|y'| + |x'|\,|\alpha'|\\ (1)\qquad &\qquad + |(y' \mid \alpha')|\,|(x' \mid \beta')| + |(y \mid \alpha)|\,|(y' \mid \alpha')|)\\ &\quad \times \langle (x' \mid \beta'), (y \mid \alpha)\rangle\langle (y' \mid \alpha'), (x \mid \beta)\rangle;\end{aligned}$$

$$\text{(2)}\quad \begin{aligned}&\operatorname{sign}(|x|\,|y| + |x|\,|\alpha| + |x'|\,|y'| + |x'|\,|\alpha'| + |(y \mid \alpha)|\,|(x' \mid \beta')|)\\ &\quad\times \langle (y' \mid \alpha'), (y \mid \alpha)\rangle \langle (x' \mid \beta'), (x \mid \beta)\rangle;\end{aligned}$$

$$\text{(3)}\quad \begin{aligned}&\operatorname{sign}(|x|\,|y| + |x|\,|\alpha| + |y'|\,|\alpha'| + |(x' \mid \alpha')|\,|(y' \mid \beta')| + |(y \mid \alpha)|\,|(x' \mid \alpha')|)\\ &\quad\times \langle (y' \mid \beta'), (y \mid \alpha)\rangle \langle (x' \mid \alpha'), (x \mid \beta)\rangle;\end{aligned}$$

$$\text{(4)}\quad \begin{aligned}&\operatorname{sign}(|x|\,|y| + |x|\,|\alpha| + |y'|\,|\alpha'| + |(y \mid \alpha)|\,|(y' \mid \beta')|)\\ &\quad\times \langle (x' \mid \alpha'), (y \mid \alpha)\rangle \langle (y' \mid \beta'), (x \mid \beta)\rangle;\end{aligned}$$

$$\text{(5)}\quad \begin{aligned}&\operatorname{sign}(|y|\,|\alpha| + |x'|\,|y'| + |x'|\,|\alpha'| + |(y' \mid \alpha')|\,|(x' \mid \beta')| + |(x \mid \alpha)|\,|(y' \mid \alpha')|)\\ &\quad\times \langle (x' \mid \beta'), (x \mid \alpha)\rangle \langle (y' \mid \alpha'), (y \mid \beta)\rangle;\end{aligned}$$

$$\text{(6)}\quad \begin{aligned}&\operatorname{sign}(|y|\,|\alpha| + |x'|\,|y'| + |x'|\,|\alpha'| + |(x \mid \alpha)|\,|(x' \mid \beta')|)\\ &\quad\times \langle (y' \mid \alpha'), (x \mid \alpha)\rangle \langle (x' \mid \beta'), (y \mid \beta)\rangle;\end{aligned}$$

$$\text{(7)}\quad \begin{aligned}&\operatorname{sign}(|y|\,|\alpha| + |y'|\,|\alpha'| + |(x' \mid \alpha')|\,|(y' \mid \beta')| + |(x \mid \alpha)|\,|(x' \mid \alpha')|)\\ &\quad\times \langle (y' \mid \beta'), (x \mid \alpha)\rangle \langle (x' \mid \alpha'), (y \mid \beta)\rangle;\end{aligned}$$

$$\text{(8)}\quad \begin{aligned}&\operatorname{sign}(|y|\,|\alpha| + |y'|\,|\alpha'| + |(x \mid \alpha)|\,|(y' \mid \beta')|)\\ &\quad\times \langle (x' \mid \alpha'), (x \mid \alpha)\rangle \langle (y' \mid \beta'), (y \mid \beta)\rangle.\end{aligned}$$

(IV) Next, we pair off the eight terms above as follows: (1) and (7), (2) and (8), (3) and (5), (4) and (6). To see how the proof is concluded, let us focus in on terms (1) and (7). The inner products in question give 0 unless $x' = y^*$, $y' = x^*$, $\beta' = \alpha^*$, and $\alpha' = \beta^*$. This is true for *both* terms (1) and (7). If the terms are not zero, we calculate each one using (3), Proposition 11, and the equalities

$$|x^*| = |x| + 1, \qquad |(x \mid \alpha)| = |x| = |\alpha|.$$

The net result is that these two terms always cancel each other. This completes the proof.

4. Gale-Ryser interpolating matrices. For a moment, let A be any signed set. A matrix $w = \{w_{ij}\colon i, j = 1, 2, 3, \ldots\}$ is said to be a *Gale-Ryser matrix* when

(1) $w_{ij} \in \operatorname{Mon}(A)$ and length $(w_{ij}) \le 1$;

(2) for fixed i, almost all w_{ij} equal 1 and for fixed j, almost all w_{ij} equal 1.

Under these conditions, the row products

$$r_i = w_{i1}w_{i2}\cdots, \qquad i = 1, 2, \ldots$$

and the column products

$$c_j = w_{1j}w_{2j}\cdots, \qquad j = 1, 2, \ldots$$

are well defined. We display this as follows:

$$\begin{array}{c|cccc} \vdots & \vdots & \vdots & \vdots & \\ r_3 & w_{31} & w_{32} & w_{33} & \cdots \\ r_2 & w_{21} & w_{22} & w_{23} & \cdots \\ r_1 & w_{11} & w_{12} & w_{13} & \cdots \\ \hline & c_1 & c_2 & c_3 & \cdots \end{array}$$

A Gale-Ryser matrix W *interpolates* the pair of Young diagrams $D = (w_1, w_2, \ldots)$ and $D' = (u_1, u_2, \ldots)$ on A when $\operatorname{cont}(w_j) = \operatorname{cont}(c_j)$ and $\operatorname{cont}(u_i) = \operatorname{cont}(r_i)$ for all i and j. The pair (D, D') is a *Gale-Ryser pair* when there exists a Gale-Ryser matrix which interpolates (D, D').

PROPOSITION 13. *Let D and D' be Young diagrams over A having shapes λ and λ', respectively. The pair $(D, \tilde{D}')$ is a Gale-Ryser pair only if $\lambda \le \lambda'$.*

PROOF. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda' = (\mu_1, \mu_2, \ldots)$. Let $W = (w_{ij})$ be the Gale-Ryser matrix which interpolates the pair $(D, \tilde{D}')$. The number of rows in $\tilde{D}'$ is μ_1; therefore in the display of W, μ_1 is the smallest integer n so that each row above r_n consists only of 1's. On the other hand, λ_1 is the number of entries, different from 1, in column 1, From this, we see that $\lambda_1 \le \mu_1$.

Next, suppose that $\lambda_1 \le \mu_1, \ldots, \lambda_1 + \lambda_2 + \cdots + \lambda_{j-1} \le \mu_1 + \mu_2 + \cdots + \mu_{j-1}$. We shall prove

$$(*) \qquad \lambda_j \le \mu_j + (\mu_1 + \cdots + \mu_{j-1}) - (\lambda_1 + \cdots + \lambda_{j-1}).$$

To do this, we need to describe μ_j and λ_j as the cardinalities of certain sets of row indices. First,

$$\begin{aligned}\mu_j &= \text{number of words in } \tilde{D}' \text{ of length} \ge j\\ &= \operatorname{Card}\{i\colon \text{number of entires} \ne 1 \text{ in row } i \text{ is} \ge j\}.\end{aligned}$$

Second,

$$\lambda_j = \operatorname{Card}\{i\colon w_{ij} \ne 1\}.$$

Now, let i be a row index appearing for λ_j (so that $w_{ij} \ne 1$). We distinguish two cases.

Case 1. $\operatorname{Card}\{k\colon w_{ik} \ne 1\} \ge j$. In this case, i is an index appearing in the description of μ_j.

Case 2. $\operatorname{Card}\{k\colon w_{ik} \ne 1\} = l + 1 < j$. In this case, i appears in the descriptions of $\mu_1, \ldots, \mu_{l+1}$ but in only l of the $\lambda_1, \ldots, \lambda_{j-1}$.

This completes the proof of the theorem.

Let L be an alphabet. Let $D = (w_1, w_2, \ldots, w_n)$ and $D' = (w'_1, w'_2, \ldots, w'_n)$ be Young diagrams on L having the same shapes. Suppose that

$$w_i = x_{i1}x_{i2}\cdots x_{i\lambda_i} \quad \text{and} \quad w'_i = x'_{i1}x'_{i2}\cdots x'_{i\lambda_i}.$$

We order the row sequences $(x'_{11}, x'_{12}, \ldots, x'_{1\lambda_1}, x'_{21}, \ldots)$ and $(x_{11}, x_{12}, \ldots, x_{1\lambda_1}, x_{21}, \ldots)$ lexicographically. We say that for $D' < D$ if in this order, the sequence for D' is less than the sequence for D.

PROPOSITION 14. *Let $D = (w_1, w_2, \dots, w_n)$ and $D' = (w'_1, w'_2, \dots, w'_n)$ be standard Young diagrams, having the same shape $\lambda = (\lambda_1, \dots, \lambda_n)$, on an alphabet L. Suppose that there exists a Gale-Ryser matrix $W = (w_{ij})$ which interpolates D and $\tilde{D}'$. Then either $D' < D$ or $D' = D$ and $w_j = w_{1j}w_{2j}\cdots w_{\lambda_j j}$ for each $j = 1, \dots, n$.*

PROOF. We proceed by induction on n, the case $n = 1$ being immediate. Otherwise, let

$$w_i = x_{i1}x_{i2}\cdots x_{i\lambda_i}, \qquad w'_i = x'_{i1}x'_{i2}\cdots x'_{i\lambda_i}$$

and $\tilde{D}' = (u'_1, \dots, u'_{\lambda_1})$ where $u'_r = x'_{1r}x'_{2r}\cdots$. Since W exists, we note that D and D' have the same content.

Let $x_1, x_2, \dots, x_m$ be the distinct elements in L appearing in D and suppose that $x_1 < x_2 < \cdots < x_m$. Since D is standard, negative elements cannot be repeated in the rows of D and positive elements cannot be repeated in the columns of D. (The same holds for D'.)

If x_1 is negative, then x_1 can only appear once in w_1; also, among the u'_r, x_1 can only appear in u'_1. Therefore, $w_{11} = x_1 = x_{11} = x'_{11}$.

If x_1 is positive, then let $\operatorname{cont}(w_1; x_1) = k \geq 1$. if x_1 appears in u'_r, $r > k$, then $x_1 = x'_{11} = \cdots = x'_{1r}$ and $D' < D$. Otherwise, x_1 appears only in $u'_1, \dots, u'_k$ and

$$x_1 = w_{11} = \cdots = w_{k1} = x_{11} = \cdots = x_{1k} = x'_{11} = \cdots = x'_{1k}.$$

Next, let x_i be the smallest element after x_1 appearing in w_1. We shall assume that

$$(*) \qquad x_1 = w_{11} = \cdots = w_{k1} = x_{11} = \cdots = x_{1k} = x'_{11} = \cdots = x'_{1k},$$

where $k = 1$ if x_1 is negative.

First, suppose that x_i is negative. If x_i appears in u'_r, $r > k+1$, then $x'_{1k+1} < x_i$ and $D' < D$. Otherwise, x_i appears only in $u'_1, \dots, u'_{k+1}$. Since $(*)$ holds, we must have $w_{k+1,1} = x_i$. Now, $x'_{1k+1} \leq x_i$. If $x'_{1k+1} < x_i$ then $D' < D$. Otherwise, we have $x'_{1k+1} = x_{1k+1} = w_{k+11}$.

Second, let us assume that x_i is positive and that $\operatorname{cont}(w_1; x_i) = l \geq 1$. If x_i appears in u'_r, $r > k + l$, then $x'_{1r} \leq x_i$ and $D' < D$. Otherwise, x_i only appears among u'_j, $j = 1, \dots, k + l$. Since $(*)$ holds, we must have

$$w_{k+11} = \cdots = w_{k+l1} = x_i = x_{1k+1} = \cdots = x_{1k+l}.$$

Also, $x'_{1j} \leq x_i$ for $j = k+1, \dots, k+l$. If some $x'_{1j} < x_i$, we have $D' < D$.

We may continue on in this way to show that either $D' < D$ or $D' = D$ and $w_1 = w'_1 = w_{11}\cdots w_{\lambda_1 1}$. In the latter case, the matrix (w_{ij}), $j \geq 2$, interpolates $D_1 = (w_2, \dots, w_n)$ and $\tilde{D}'_1$ where $D'_1 = (w'_2, \dots, w'_n)$. We may now apply the induction hypothesis to complete the proof.

5. Proof of Theorem 8, (iv). Throughout this section, we shall let L and P be disjoint alphabets. Before giving the proof of Theorem 8, (iv), we need one more result.

PROPOSITION 15. *Let D, D' be standard Young diagrams on L and let E, E' be standard Young diagrams on P.*

(a) *Then $\langle \mathrm{Tab}^+(D' \mid E'), \mathrm{Tab}(D \mid E)\rangle = 0$ unless both $(D, \tilde{D}')$ and $(E, \tilde{E}')$ are Gale-Ryser pairs.*

(b) *Also, $\langle \mathrm{Tab}^+(D \mid E), \mathrm{Tab}(D \mid E)\rangle = \pm 1$.*

PROOF. Let $P^1, P^2, \dots, P^r$ be the biproducts such that $\mathrm{Tab}(D \mid E) = P^1P^2\cdots P^r$ and, similarly, let $\mathrm{Tab}^+(D' \mid E') = N^1N^2\cdots N^s$. We apply formulas (a) and (b) of §3 to see that

$$\begin{aligned}
&\langle \mathrm{Tab}^+(D' \mid E'), \mathrm{Tab}(D \mid E)\rangle \\
&= \langle N^1N^2\cdots N^{ws}, P^1P^2\cdots P^r\rangle \\
&= \sum_{P^1,\dots,P^r} \pm\langle N^1, P^1_{(1)}\cdots P^r_{(1)}\rangle\cdots\langle N^s, P^1_{(s)}\cdots P^r_{(s)}\rangle,
\end{aligned}$$

where we used the fact that

$$\Delta^{(s)}(P^1\cdots P^r) = \Delta^{(s)}(P^1)\cdots\Delta^{(s)}(P^r).$$

Continuing on, we see that this is the sum over all $P^1, \dots, P^r$, $N^1, \dots, N^s$ of terms

$$(\neq) \qquad \begin{aligned}
&\pm\langle N^1_{(1)}, P^1_{(1)}\rangle\langle N^1_{(2)}, P^2_{(1)}\rangle\cdots\langle N^1_{(r)}, P^r_{(1)}\rangle \\
&\times\langle N^2_{(1)}, P^1_{(2)}\rangle\langle N^2_{(2)}, P^2_{(2)}\rangle\cdots\langle N^s_{(r)}, P^r_{(s)}\rangle.
\end{aligned}$$

According to Proposition 12, all the expressions $\langle N^i_{(j)}, P^j_{(i)}\rangle$ are 0 unless $\mathrm{Length}(N^i_{(j)}) = \mathrm{Length}(P^j_{(i)}) \leq 1$. Furthermore, if this expression is not 0, then

$$N^i_{(j)} = (x^* \mid y^*) \text{ and } P^j_{(i)} = (x \mid y).$$

Now, let us suppose that $\langle \mathrm{Tab}^+(D' \mid E'), \mathrm{Tab}(D \mid E)\rangle$ is not zero. Then, there is a nonzero term of the form $(\neq)$ in the expansion above. For such a term, let us put $w_{ij} = x$ if $P^j_{(i)} = (x \mid y)$. We shall show that the matrix $W = (w_{ij})$ interpolates the pair $(D, \tilde{D}')$.

To see this, let $D = (w_1, w_2, \dots)$, $D' = (w'_1, w'_2, \dots)$, and $\tilde{D}' = (u'_1, u'_2, \dots)$. Then, for each $x \in L$, we have

$$\begin{aligned}
\operatorname{cont}(w_{1j}w_{2j}\cdots)(x) &= \sum_i \operatorname{cont} w_{ij}(x) = \sum_i \operatorname{Cont} P^j_{(i)}(x) \\
&= \operatorname{Cont}(P^j_{(1)}\cdots)(x) = \operatorname{Cont} P^j(x) = \operatorname{cont} w_j(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\operatorname{cont}(w_{i1}w_{i2}\cdots)(x) &= \sum_j \operatorname{cont} w_{ij}(x) = \sum_j \operatorname{Cont} P^j_{(i)}(x) \\
&= \sum_j \operatorname{Cont} N^i_{(j)}(x^*) = \operatorname{Cont} N^i(x^*) = \operatorname{cont} u'_i(x).
\end{aligned}$$

Hence, w_{ij} interpolates $(D, \tilde{D}')$. In a similar manner, we may find a matrix which interpolates $(E, \tilde{E}')$. This completes the proof of (a).

Next, we prove (b). First, let us introduce some notation and write $D = (w_1, \ldots, w_n)$ and $E = (u_1, \ldots, u_n)$, where $w_i = x_{i1} \cdots x_{i\lambda_i}$ and $u_i = \alpha_{i1} \cdots \alpha_{i\lambda_i}$. We shall use the notation introduced in the first part of this proof, e.g.,

$$P^j = (\text{stand}(w_j) \mid \text{stand}(u_j)).$$

Now, we apply Proposition 7 (Chapter 2) to the expansions of $\Delta^{(\lambda_1)}(P^j)$ and $\Delta^{(n)}(N^i)$. According to Proposition 7, in these expansions there is one and only one nonzero term such that

$$P^j_{(i)} = (x_{ji} \mid \alpha_{ji}) \quad \text{and} \quad N^i_{(j)} = (x^*_{ji} \mid \alpha^*_{ji}).$$

The corresponding term $(\neq)$ in the expansion of $\langle \text{Tab}^+(D \mid E), \text{Tab}(D \mid E) \rangle$ equals ± 1. But this is the only possible nonzero term in the expansion according to Proposition 14. (Any nonzero term gives an interpolating matrix which is unique by Proposition 14.)

This completes the proof of Proposition 15.

PROOF OF THEOREM 8, (iv). We begin by ordering pairs (D, E) of Young diagrams, where D and E have the same shape, according to the following rules.

(i) If D has shape λ and D' has shape λ' and if $\lambda < \lambda'$, then $(D, E) < (D', E')$.

(ii) If D and D' have the same shape, then $(D, E) < (D', E')$ if in the lexicographic ordering on the word sequences for (D, E) and $(D'E')$, (D, E) is larger than (D', E').

Now suppose (in the terminology of Theorem 8) that

$$\text{Tab}(D \mid E) = \sum_i c_i \, \text{Tab}(D_i \mid E_i),$$

where the diagram pairs $(D_i \mid E_i)$ are ordered from smallest to largest. Then

$$\langle \text{Tab}^+(D_j \mid E_j), \text{Tab}(D \mid E) \rangle = \sum_i c_i \langle \text{Tab}^+(D_j \mid E_j), \text{Tab}(D_i \mid E_i) \rangle.$$

According to Propositions 13, 14, and 15 these equations for the c_i give a triangular system of linear equations with ± 1 entries on the diagonal. Thus, the system has a unique solution. This completes the proof.

4. Invariant Theory

0. Introduction. In this chapter we prove the fundamental result of invariant theory for both symmetric and skew-symmetric tensors, namely, the fact that all invariants can be symbolically represented by polynomials in brackets. Even though the field is assumed to be of characteristic zero, the result given below is new for skew-symmetric tensors (a previous attempt by Weitzenböck to develop a symbolic calculus contains an inconsistency, as we show in detail in the next chapter). We give here an informal description of the symbolic method. A symmetric tensor in a vector space V over a fixed K can be written, relative to a basis $e_1, e_2, \ldots, e_n$ of V, in the form

$$\begin{aligned} t &= \sum_{j_1 \cdots j_n} \binom{k}{j_1 \cdots j_n} t_{j_1 \cdots j_n} e_1^{j_1} \cdots e_n^{j_n} \\ &= \sum_{j_1 \cdots j_n} \binom{k}{j_1 \cdots j_n} (t \mid j_1 \cdots j_n) e_1^{j_1} \cdots e_n^{j_n}, \end{aligned}$$

where $t_{j_1 \cdots j_n} = (t \mid j_1 \cdots j_n)$ are the *coordinates* of the tensor t. The tensor is a homogeneous element of degree k (or step k, as we shall say) of the symmetric algebra over the vector space V.

Similarly, a skew-symmetric tensor is an element of step k of the exterior algebra over the vector space V. Relative to the given basis, we can write

$$t = \sum_{j_1 < j_2 < \cdots < j_k} (t \mid j_1 j_2 \cdots j_k) e_{j_1} e_{j_2} \cdots e_{j_k},$$

where the exterior product is indicated by juxtaposition.

The idea is to view the coordinates $(t \mid j_1 \cdots j_k)$ as the images under the action of a linear functional, which we call the umbral linear functional, of the letters of an alphabet $\text{Div}[L \mid P]$. The set $P = P^-$ will consist of the letters $\{1, 2, \ldots, n\}$, the letters of L^+ will name the skew-symmetric tensors, and the letters of L^- will name the symmetric tensors. The crucial feature of the symbolic method is the use of several distinct letters for the same tensor. Thus, for $a \in L^+$, we set

$$U((a^{(k)} \mid j_1 \cdots j_k)) = (t \mid j_1 \cdots j_k),$$

and for $a \in L^-$ we set

$$U((a \mid 1)^{j_1} \cdots (a \mid n)^{j_n}) = (t \mid j_1 \cdots j_n).$$

We say that the letter a is a *symbol* for the tensor t. One defines the bracket

$$[w] = (w \mid 12 \cdots n), \qquad w \in L,$$

and one proves that every invariant, or every joint invariant of a set of tensors, can be written as the image of a bracket polynomial under the action of the umbral linear functional. Detailed applications of the method are worked out in the next chapter.

We note the fact that in the umbral representation skew-symmetric tensors are represented by positively signed letters and symmetric tensors by negatively signed letters. Because of this fact, the computation of invariants for skew-symmetric tensors turns out to be simpler than for symmetric tensors.

1. Definitions. Let K be an infinite field; let W be a finite-dimensional vector space over K; and let $\{w_1, \cdots, w_m\}$ be any basis for W. Let $K[X_1, \ldots, X_m]$ be the polynomial algebra in m variables $X_1, \ldots, X_m$ over K. A function $f: W \to K$ is called a *polynomial function on* W if there is a polynomial $p \in K[X_1, \ldots, X_m]$ such that for every $w \in W$, $w = \sum_{i=1}^m a_i w_i$, we have $f(w) = p(a_1, \cdots, a_m)$. It is easy to see that this definition does not depend on the choice of a basis for W. We shall denote the algebra of all polynomial functions on W by $K[W]$.

Next let G be a group which acts on W via a representation $\rho: G \to \mathrm{GL}(W)$. The action of G on W gives rise to an action of G on $K[W]$ by defining $(\gamma \cdot f)(w) = f(\rho(\gamma)^{-1} w)$ for all $\gamma \in G, f \in K[W], w \in W$.

Note. The algebra $K[W]$ is (naturally) isomorphic to $K[X_1, \ldots, X_m]$. Also, if $\rho(\gamma^{-1}) w_i = \sum_{j=1}^m c_{ji} w_j$, then $\gamma \cdot X_k = \sum_{i=1}^m c_{ki} x_i$. The action of G on $K[W]$ has the following properites.

(i) If e is the identity in G, then $e \cdot f = f$ for all $f \in K[W]$.

(ii) If $\gamma_1, \gamma_2 \in G$, then $(\gamma_1 \gamma_2) \cdot f = \gamma_1 \cdot (\gamma_2 \cdot f)$ for all $f \in K[W]$.

(iii) For each $\gamma \in G$, the mapping $f \to \gamma \cdot f$ defines an automorphism of $K[W]$.

Let K^* denote the multiplicative group of nonzero elements in K and let $\chi: G \to K^*$ be a homomorphism of G. An element $f \in K[W]$ is called a (*relative*) *invariant*, corresponding to χ, if $\gamma \cdot f = \chi(\gamma) f$ for all $\gamma \in G$. If $\chi = 1$, we sometimes call f an *absolute invariant.*

We shall be interested, here, in the group $G = \mathrm{GL}_n(K)$ and certain (explicit) actions of G which we now describe. Let V be a vector space of dimension n over the field K and let $\{e_1, \ldots, e_n\}$ be a given basis for V. In the usual way, we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}_n(K)$ using this basis.

A *skew-symmetric tensor t of step k* is an element of the kth homogeneous component $\bigwedge^k(V)$ of the exterior algebra $\bigwedge(V)$. Relative to the given basis, we have

$$t = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

where $a_{i_1 \cdots i_k} \in K$. We shall denote the corresponding coordinate functions by $t_{i_1 \cdots i_k}$ so, in the notation just given,

$$t_{i_1 \cdots i_k}(t) = a_{i_1 \cdots i_k}.$$

An element $N \in \mathrm{GL}_n(K)$ induces a linear transformation $\bigwedge^k(N): \bigwedge^k(V) \to \bigwedge^k(V)$. We may rephrase our basic definition in this context as follows. A polynomial $I(t)$ in the variables $t_{i_1 \cdots i_k}$ is said to be an *invariant* when there is a positive integer g such that

$$I(\bigwedge^k(N)t) = (\det N)^g I(t)$$

for all $N \in \mathrm{GL}_n(K), t \in \bigwedge^k(V)$. Sometimes we call such a polynomial an invariant of skew-symmetric tensors t of step k or, more simply, an invariant of t. The major goal of these lectures will be to describe explicitly all such invariants.

We shall also be interested in actions of $\mathrm{GL}_n(K)$ on the symmetric algebra $S(V)$. A *symmetric tensor t of step k* is an element of the kth homogeneous component $S^k(V)$ of the symmetric algebra $S(V)$. Relative to the given basis $\{e_1, \ldots, e_n\}$ of V, we may write

$$t = \sum_{\substack{j_1, \ldots, j_n \\ j_1 + \cdots + j_n = k}} \binom{k}{j_1 \cdots j_n} a_{j_1 \cdots j_n} e_1^{j_1} \cdots e_n^{j_n}$$

where $a_{j_1 \cdots j_n} \in K$. We shall assume that each multinomial coefficient appearing above is nonzero in K. We shall denote the corresponding coordinate functions by $t_{j_1 \cdots j_n}$ so, in the notation just given,

$$t_{j_1 \cdots j_n}(t) = a_{j_1 \cdots j_n}.$$

An element $N \in \mathrm{GL}_n(K)$ induces a linear transformation $S^k(N): S^k(V) \to S^k(V)$. Then, a polynomial $I(t)$ in the variables $t_{j_1 \cdots j_n}$ is said to be an *invariant* when there is a positive integer g such that

$$I(S^k(N)t) = (\det N)^g I(t)$$

for all $N \in \mathrm{GL}_n(K), t \in S^k(V)$. Sometimes we shall call such a polynomial an invariant of symmetric tensors of step k or, more simply, an invariant of t.

The actions above extend to actions of $\mathrm{GL}_n(K)$ on products

$$(+) \qquad W = \bigwedge^{k_1}(V) \times \ldots \times \bigwedge^{k_r}(V) \times S^{h_1}(V) \times \cdots \times S^{h_p}(V);$$

namely,

$$N \cdot (w_1, \ldots, w_r, w_1', \ldots, w_p') = (Nw_1, \ldots, Nw_r, Nw_1', \ldots, Nw_p').$$

So, we may speak of invariants in this setting; namely, a polynomial I in $K[W]$ is called invariant (or, in the classical terminology, *joint invariant*) if there is a positive integer g so that

$$I(Nw) = (\det N)^g I(w)$$

for all $N \in \mathrm{GL}_n(K), w \in W$. In actions on such W, we shall always assume that each multinomial coefficient $\binom{h}{j_1 j_2 \cdots j_n}$ is nonzero in K if some $S^h(V), h > 1$, appears among the components of W.

2. The action of $\mathrm{GL}_n(K)$ **on** Super$[L \mid P]$. Let L be a proper signed set (where L^- may be empty). Let $P = P^- = \{1, 2, \ldots, n\}$ be a proper signed set, disjoint from L, with $1 < 2 < \cdots < n$. We construct the algebra Super$[L \mid P]$ as in Chapter 2. It is important to notice that there are no positive elements in $[L \mid P]$. We shall now define an action of $\mathrm{GL}_n(K)$ on Super$[L \mid P]$.

Let $N \in \mathrm{GL}_n(K)$ with, say, $N^{-1}e_j = \sum_{i=1}^n a_{ij}e_i$. We define a K-algebra automorphism of Tens$[L \mid P] \otimes K$ by $N(x \mid i) = \sum_{j=1}^n a_{ij}(x \mid j)$. We note in passing that $N_1(N_2(x \mid i)) = (N_1N_2)(x \mid i)$ for all $N_1, N_2 \in \mathrm{GL}_n(K)$.

Next, let I be the ideal in Tens $[L \mid P]$ which defines Super$[L \mid P]$. Then N sends $I \otimes K$ to itself (as is easily checked) and, so, defines a mapping, also denoted by N, of Super$[L \mid P] \otimes K$ to itself.

Note. According to Theorem 8, a basis for the Z-algebra Super$[L \mid P]$ consists of all Tab$(D \mid E)$ where D, E are standard. Then (according to a standard theorem on tensor products), the Tab$(D \mid E)$ forms a basis for the K-vector space Super$[L \mid P] \otimes K$.

THEOREM 16. *Let w be a word in* Super$[L]$ *having* Length $= n$. *Let* $N \in \mathrm{GL}_n(K)$. *Then*

$$N(w \mid 12 \cdots n) = (\det N)^{-1}(w \mid 12 \cdots n).$$

PROOF. According to formula (b), Chapter 2, §2, we have

$$(w \mid 12 \cdots n) = \sum_w \pm (w_{(1)} \mid 1) \cdots (w_{(n)} \mid n).$$

Hence, $N(w \mid 12 \cdots n)$ is

$$\sum \pm a_{1j_1} \cdots a_{nj_n}(w_{(1)} \mid j_1) \cdots (w_{(n)} \mid j_n),$$

where the sum is over w and all $j_1, \ldots, j_n$. This expression equals

$$\sum_{j_1, \ldots, j_n} a_{1j_1} \cdots a_{nj_n}(w \mid j_1 \cdots j_n).$$

Since all the j's are negative, the term $(w \mid j_1 \cdots j_n)$ is 0 unless the j's are distinct, i.e., $j_1 = \sigma(1), \ldots, j_n = \sigma(n)$, where σ is some permutation of $\{1, 2, \ldots, n\}$. Therefore, the preceding sum is

$$\sum_\sigma a_{1\sigma(1)} \cdots a_{n\sigma(n)}\mathrm{sgn}(\sigma)(w \mid 12 \cdots n) = (\det N)^{-1}(w \mid 12 \cdots n).$$

This completes the proof of Theorem 16.

Let $E = (u_1, u_2, \ldots, u_g)$ be the Young diagram on P where each u_i is $12 \cdots n$. Let $D_1, \ldots, D_r$ be standard Young diagrams on L having the same shape as E. Let $F' \in$ Super$[L \mid P] \otimes K$,

$$(*,) \qquad F' = \sum_{s=1}^r c_s \mathrm{Tab}(D_s \mid E)$$

where each $c_s \in K$. Applying Theorem 16, we see that

$$N \cdot F' = (\det N)^{-g} F'$$

for each $N \in \mathrm{GL}_n(K)$.

THEOREM 17. *Let* $F' \in$ Super$[L \mid P] \otimes K$ *satisfy* $N \cdot F' = (\det N)^{-g}F'$ *for each* $N \in \mathrm{GL}_n(K)$. *Then* F' *has the form* $(*)$ *above.*

PROOF. Using the straightening formula (Theorem 8), we may write

$$(**) \qquad F' = \sum_s c_s \mathrm{Tab}(D_s \mid E_s)$$

where $c_s \in K^*$ and D_s, E_s are standard.

For fixed j, $1 \le j \le n$, let $j(s)$ be the number of times j appears in E_s, that is (in the terminology of §1, Chapter 3),

$$j(s) = \mathrm{Cont}(\mathrm{Tab}(D_s \mid E_s))(j) = \mathrm{cont}(E_s)(j).$$

Now, let b be any element in K^* and define $N \in \mathrm{GL}_n(K)$ as

$$N^{-1}e_i = e_i \quad \text{if } i \neq j \quad \text{and} \quad N^{-1}e_j = be_j.$$

Then

$$N(x \mid i) = (x \mid i) \quad \text{if } i \neq j \quad \text{and} \quad N(x \mid j) = b(x \mid j).$$

We apply N to $(**)$, always bearing in mind property (3), §1, Chapter 3. Then

$$b^g F' = \sum_s c_s b^{j(s)} \mathrm{Tab}(D_s \mid E_s).$$

Comparing coefficients of Tab$(D_s \mid E_s)$ and remembering that K is infinite, we see that each $j(s)$ is equal to g.

Since each j, $1 \le j \le n$, appears g times in each E_s, the minimum number of rows of E_s is g. If each E_s has exactly g rows, then $E_s = E$ and we are finished.

Otherwise, let us look at an E_s having more than g rows. Since E_s is standard, all the 1's occur in the first column. Let l be the first number in the first column following the run of g 1's. Then, all of the numbers 1 to $l-1$ occur in the first g rows of E_s. Moreover, if q is the number of l's in the first column, the remaining

$g-q$ l's must all occur in the first $g-q$ rows. This situation is summarized by

$$\begin{array}{ll}
g\left\{\begin{array}{l}
\begin{array}{ccccc} 1 & \ldots & l-1 & l & \ldots \\ & & \ldots & & \\ 1 & \ldots & l-1 & l & \ldots \end{array} \\
q\left\{\begin{array}{ccccc} 1 & \ldots & l-1 & m & \ldots \\ & & \ldots & & \\ 1 & & l-1 & & \end{array}\right.
\end{array}\right. \\
\qquad q\left\{\begin{array}{lll} l & & \ldots \\ \vdots & & \\ l & \ldots & \end{array}\right. \\
\qquad\quad \vdots
\end{array}$$

where $m \geq l+1$. Such an E_s will be called *an l-critical diagram of parameter q*.

Let j be the smallest index such that there exists a j-critical tableau in the expansion of F'. We break up the expansion of F' into

$$F' = \sum_s c_s \mathrm{Tab}(D_s \mid E_s) + \sum_t c_t \mathrm{Tab}(D_t \mid E_t) + G,$$

where the first summation is over all the indices s such that E_s is j-critical, the second summation is over all t such that E_t is l-critical for some $l > j$, and G is the linear combination of all the $\mathrm{Tab}(D_s \mid E_s)$ where $E_s = E$.

Let $b \in K^*$. We define $N \in \mathrm{GL}_n(K)$ by $N^{-1}e_j = be_{j-1} + e_j, N^{-1}e_k = e_k$ for $k \neq j$. Then,

$$N(x \mid j-1) = (x \mid j-1) + b(x \mid j), \qquad N(x \mid k) = (x \mid k) \quad \text{for } k \neq j-1.$$

If $w \in \mathrm{Super}[L]$, then N fixes an expression like $(w \mid \cdots j-1\ j \cdots)$. Indeed, applying formula (b), §2, Chapter 2, we see that N sends $(w \mid \cdots j-1\ j \cdots)$ to

$$(w \mid \cdots j-1\ j \cdots) + b(w \mid \cdots j\ j \cdots).$$

The second term is 0 since $j^2 = 0$.

With this in mind, we apply N to F' and see that N fixes each term in the second summation and each term in G. For the j-critical E_s, we have

$$N\,\mathrm{Tab}(D_s \mid E_s) = \mathrm{Tab}(D_s \mid E_s) + \sum_r b^{e(r)} \mathrm{Tab}(D_s \mid E_s^r),$$

where E_s^r has the form

$$\begin{array}{ll}
g\left\{\begin{array}{l}
\begin{array}{ccccc} 1 & \ldots & j-1 & j & \ldots \\ & \ldots & & & \\ 1 & \ldots & j-1 & j & \ldots \end{array} \\
q\left\{\begin{array}{ccccc} 1 & \ldots & * & m & \\ & \ldots & & & \\ 1 & \ldots & * & & \ldots \end{array}\right.
\end{array}\right. \\
\qquad q\left\{\begin{array}{ll} j & \ldots \\ \vdots & \\ j & \ldots \end{array}\right. \\
\qquad\quad \vdots
\end{array}$$

where $e(r)$ of the $*$ are j, $e(r) \geq 1$, and the rest of the $*$ are $j-1$. The largest $e(r)$ is q and this occurs when all the $*$ are chosen to be j; let us denote this (standard) Young diagram by E_s''.

Now we consider the equation $N \cdot F' = F'$. Since K is infinite, we may think of this as a polynomial in b and set the various coefficients equal to 0. Let q' be the largest possible parameter appearing in the j-critical diagrams. Then the highest power of b is $b^{q'}$ and its coefficient is $\sum_s c_s \mathrm{Tab}(D_s \mid E_s'')$ where the sum is over all pairs (D_s, E_s) where E_s is j-critical of parameter q'. We conclude that $\sum_s c_s \mathrm{Tab}(D_s \mid E_s'') = 0$; since all the D_s and E_s'' are standard, we have $c_s = 0$, contradicting our original assumption (i.e., $c_s \in K^*$).

3. The umbral linear functional. Throughout this section, we fix an action of $\mathrm{GL}_n(K)$ on a vector space W having the form

$$(+) \qquad W = \bigwedge^{k_1}(V) \times \cdots \times \bigwedge^{k_r}(V) \times S^{h_1}(V) \times \cdots \times S^{h_p}(V).$$

As always, we shall assume that each $\binom{h}{j_1 \cdots j_n}$ is nonzero in K, if some $S^h(V), h > 1$, appears among the components of W. Also, if V, itself, appears among the components of W, then we shall assume it is written as $\bigwedge^1(V)$.

Given such an action, we shall assume that we have available an infinite alphabet L which satisfies the following conditions.

(L1) There is a subset L_∞ of L so that both L_∞^+ and L_∞^- are infinite.

(L2) There is a subset L'_∞ of L_∞^+ such that $L_\infty^+ - L'_\infty$ is infinite. Furthermore, each $a \in (L - L_\infty) \cup L'_\infty$ is associated to one and only one of the components of W, i.e., an $S^h(V)$ or a $\bigwedge^k(V)$. We say that *a belongs* to this component.

(L3) Each component of W has infinitely many elements belonging to it.

(L4) If $a \in L$ belongs to an $S^h(V), h > 1$, then $a \in L^-, a \notin L_\infty$.

(L5) If $a \in L$ belongs to a $\bigwedge^k(V), k > 1$, then $a \in L^+, a \notin L_\infty$.

(L6) If $a \in L$ belongs to a copy of V (i.e., some $\bigwedge^1(V)$), then $a \in L'_\infty$.

Let $P = P' = \{1, 2, \ldots, n\}$ be a finite signed alphabet with $1 < 2 < \cdots < n$. The *umbral linear functional* U: $\text{Super}[L \mid P] \otimes K \to K[W]$, denoted by $f \to \langle U, f \rangle$, is defined as follows.

(1) Let a belong to some $\bigwedge^k(V)$. Then $\langle U, (a^{(k)} \mid i_1 \cdots i_k) \rangle$ is the coordinate function $t_{i_1 \cdots i_k}$ on that component $\bigwedge^k(V)$. If $j \neq k$, then $\langle U, (a^{(j)} \mid i_1 \cdots i_j) \rangle$ is 0.

(2) Let a belong to some $S^h(V)$. If $j_1 + j_2 + \cdots + j_n = h$, then

$$\langle U, (a \mid 1)^{j_1} (a \mid 2)^{j_2} \cdots (a \mid n)^{j_n} \rangle$$

is the coordinate function $t_{j_1 j_2 \cdots j_n}$ on that component $S^h(V)$. If $j_1 + j_2 + \cdots + j_n \neq h$, then $\langle U, (a \mid 1)^{j_1} (a \mid 2)^{j_2} \cdots (a \mid n)^{j_n} \rangle$ is 0.

(3) Let $a \in L_\infty$ and suppose that a does not belong to any component of W. Then $\langle U, (a \mid i_1) \cdots (a \mid i_k) \rangle$ is 0.

(4) Let us order $[L \mid P]$ by saying $(x \mid \alpha) < (x' \mid \alpha')$ if either $x < x'$ or $x = x'$ and $\alpha < \alpha'$. As usual, we choose a basis of $\text{Super}[L \mid P]$ consisting of words $(z_1 \mid \gamma_1) \cdots (z_n \mid \gamma_n)$ but now we shall also assume that $(z_1 \mid \gamma_1) \leq (z_2 \mid \gamma_2) \leq \cdots \leq (z_n \mid \gamma_n)$.

The umbral linear functional U is defined on such an element by:

$$\begin{aligned}\langle U, (x \mid \gamma_{i_1}) \cdots (x \mid \gamma_{i_p})(y \mid \gamma_{j_1}) \cdots (y \mid \gamma_{j_q}) \cdots \rangle \\ = \langle U, (x \mid \gamma_{i_1}) \cdots (x \mid \gamma_{i_p}) \rangle \cdot \langle U, (y \mid \gamma_{j_1}) \cdots (y \mid \gamma_{j_q}) \rangle \cdots\end{aligned}$$

where $x < y < \cdots$, $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots \leq \gamma_{i_p}$, $\gamma_{j_1} \leq \cdots \leq \gamma_{j_q}, \ldots$, and the individual terms are calculated using rules (1), (2), and (3).

It follows that U is well defined by these rules. The following properties of U will also be important later.

(U1) The mapping U is onto $K[W]$.

(U2) For all $N \in \text{GL}_n(K)$, $f \in \text{Super}[L \mid P] \otimes K$, we have $\langle U, Nf \rangle = N\langle U, f \rangle$.

PROOF. To prove (U1), we need only notice that the coordinate functions on $\bigwedge^k(V)$ are the images under U of the various $(a^{(k)} \mid i_1 \cdots i_k)$ and that the coordinate functions on $S^h(V)$ are the images under U of the various $(a \mid 1)^{j_1} \cdots (a \mid n)^{j_n}$.

To prove (2), we may assume that f is a monomial consisting of products of terms having either the form $(a^{(k)} \mid i_1 \cdots i_k)$ or $(a \mid 1)^{j_1} \cdots (a \mid n)^{j_n}$. That (2) holds for such terms individually follows by a direct (and somewhat tedious) computation which we omit. That (2) holds for f then follows from the definition of U, in particular (4).

4. The first fundamental theorem. We shall continue to follow the notation introduced in §3. In addition, throughout this section, we shall assume that the characteristic of the field K is 0. If $w \in \text{Super}[L]$, let us define $[w] \in \text{Super}[L \mid P]$ to be $(w \mid 12 \cdots n)$. An element $p \in \text{Super}[L \mid P] \otimes K$ is called a *bracket monomial* if $p = [w_1] \cdots [w_k]$ and a *bracket polynomial* if it is a linear combination of bracket monomials.

THEOREM 18. *If* $\text{char}\, K = 0$, *then every (joint, relative) invariant of* $\text{GL}_n(K)$ *acting on* W *can be written as* $\langle U, p \rangle$ *where* p *is a bracket polynomial in* $\text{Super}[L \mid P] \otimes K$.

PROOF. Let $F' \in \text{Super}[L \mid P] \otimes K$ satisfy $N \cdot F' = (\det N)^{-g} F'$ for each $N \in \text{GL}_n(K)$. According to Theorem 17, F' is a bracket polynomial. Furthermore,

$$N\langle U, F' \rangle = \langle U, N \cdot F' \rangle = (\det N)^{-g} \langle U, F' \rangle.$$

To finish the proof, we need only show that each invariant polynomial in $K[W]$ is the image under U of an invariant polynomial in $\text{Super}[L \mid P] \otimes K$. This fact follows immediately from the complete reduciblity of $\text{GL}_n(K)$. However, we can give a complete proof, using the ideas developed in this monograph, and avoiding the appeal to representation theory. We turn to this argument now.

1) There is a K-algebra homomorphism φ from $\text{Super}[L \mid P] \otimes K$ to $\text{Super}[L^* \mid P^*] \otimes K$ which extends the mapping $(x \mid \alpha) \to (x^* \mid \alpha^*)$. Furthermore, $\Delta\varphi = (\varphi \otimes \varphi)\Delta$.

PROOF. The mapping $(x \mid \alpha) \to (x^* \mid \alpha^*)$ gives a homomorphism of Tens $[L \mid P] \otimes K$ to $\text{Super}[L^* \mid P^*] \otimes K$ which factors through $\text{Super}[L \mid P] \otimes K$. The equality $\Delta\varphi = (\varphi \otimes \varphi)\Delta$ holds on the generators $(x \mid \alpha)$ of $\text{Super}[L \mid P]$ and, hence, is true in general since all the maps are algebra homomorphisms.

2) For basis words $w, w' \in \text{Super}[L \mid P]$, we define $(w, w') = \langle \varphi(w), w' \rangle$ where the expression on the right-hand side is the scalar product of §3, Chapter 3. We extend this scalar product to $\text{Super}[L \mid P] \otimes K$. Let

$$w = (x_1 \mid \alpha_1)^{e_1} \cdots (x_r \mid \alpha_r)^{e_r} (y_1 \mid \beta_1) \cdots (y_k \mid \beta_k),$$

where $x_i \in L^-$, $y_j \in L^+$, and the $(x_1 \mid \alpha_1), \ldots, (y_k \mid \beta_k)$ are distinct. Then $(w, w') = 0$ if $w \neq w'$ and

$$(w, w) = e_1! \cdots e_r! \operatorname{sign}(1/2(k-1)k).$$

PROOF. If w and w' are of different Lengths, then $(w, w') = 0$. So, we may assume that Length w = Length $w' = n$ with, say, $w' = w'_1 w'_2 \cdots w'_n$ and Length $w'_i = 1$ for all $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}(w, w') &= \langle \varphi(w), w' \rangle \\ &= \sum_{\varphi(w)} \operatorname{sign}\left(\sum_{i>j} |\varphi(w)_{(i)}| \, |w'_j|\right) \langle \varphi(w)_{(1)}, w'_1 \rangle \cdots \langle \varphi(w)_{(n)}, w'_n \rangle.\end{aligned}$$

For a term in this summation to be nonzero, we need $\varphi(w)_{(i)} = \varphi(w'_i)$ for each $i = 1, \ldots, n$. But then

$$\text{Cont}\, \varphi(w) = \sum_i \text{Cont}\, \varphi(w)_{(i)} = \sum_i \text{Cont}\, \varphi(w'_i) = \text{Cont}\, \varphi(w')$$

and, so, $w = w'$. Now we calculate (w, w). If $x \in L^-$, we have

$$\varphi(x \mid \alpha)^e = (x^* \mid \alpha^*) \cdots (x^* \mid \alpha^*) = e!(x^* \mid \alpha^*)^{(e)}.$$

Therefore, in the expansion of $\Delta^{(n)}(\varphi(w))$, the term

$$(x_1^* \mid \alpha_1^*) \otimes \cdots \otimes (x_r^* \mid \alpha_r^*) \otimes (y_1^* \mid \beta_1^*) \otimes \cdots \otimes (y_k^* \mid \beta_k^*)$$

appears once and only once and has coefficient $e_1! \cdots e_r!$

3) Let V be any finite-dimensional subspace of Super$[L \mid P] \otimes K$ spanned by basis words. The restriction of $(\, , \,)$ to V is a nondegenerate, symmetric, bilinear form on V. (This follows from 2).) Furthermore, for any $N \in \mathrm{GL}_n(K), w, w' \in$ Super$[L \mid P]$, we have $({}^tNw, w') = (w, Nw')$.

PROOF. The group $\mathrm{GL}_n(K)$ acts on Super$[L \mid P] \otimes K$ and its tensor products. Furthermore, $(N \otimes N)\Delta = \Delta N$ since this equality holds for the generators $(x \mid i)$ of Super$[L \mid P]$. We shall use this and the fact that $(\varphi \otimes \varphi)\Delta = \Delta\varphi$ in the computation that follows. In the Sweedler notation, we have

$$\sum_w Nw_{(1)} \otimes Nw_{(2)} = \sum_{Nw} (Nw)_{(1)} \otimes (Nw)_{(2)}$$

and

$$\sum_w \varphi w_{(1)} \otimes \varphi w_{(2)} = \sum_{\varphi(w)} (\varphi(w))_{(1)} \otimes (\varphi(w))_{(2)}.$$

We prove by induction on Length w that $({}^tNw, w') = (w, Nw')$. The case Length $w = 1$ is an easy calculation. Now in general,

$$\begin{aligned}({}^tNw, w'w'') &= \langle \varphi({}^tNw), w'w'' \rangle \\ &= \sum_w \mathrm{sign}(|w'|\,|w_{(2)}|)({}^tNw_{(1)}, w')({}^tNw_{(2)}, w'') \\ &= \sum_w \mathrm{sign}(|w'|\,|w_{(2)}|)(w_{(1)}, Nw')(w_{(2)}, Nw'') \\ &= (w, N(w'w'')).\end{aligned}$$

4) Let F be an invariant polynomial in $K[W]$. We may choose a finite-dimensional subspace V in Super$[L \mid P]$ such that (i) V is spanned by basis words, (ii) V is stable under $\mathrm{GL}_n(K)$, (iii) the image of V under U contains F. Let us denote this image by $U(V)$ and the kernel of U on V by $\ker(U)$.

The bilinear form $(\, , \,)$ is nondegenerate on $\ker U$. Indeed, $\ker U$ has a basis consisting of expressions having two possible forms, namely:

(i) words $w \in V$ such that $U(w) = 0$,

(ii) differences $w_1 - w_2$ where w_1, w_2 are words in V such that $U(w_1) = U(w_2) \neq 0$.

Since $(\, , \,)$ is nondegenerate on V and $\ker U$, we see that $V = (\ker U) \otimes (\ker U)^\perp$. It follows from 3) that $\mathrm{GL}_n(K)$ sends $(\ker U)^\perp$ to itself. Hence, the $\mathrm{GL}_n(K)$-modules $U(V)$ and $(\ker U)^\perp$ are isomorphic and F is the image under U of an invariant polynomial in Super$[L \mid P]$.

What happens to the proof above if $\mathrm{char}\, K = p > 0$? First, in order that U send Super$[L \mid P] \otimes K$ onto $K[W]$ we need to require that the multinomial coefficients $\binom{h}{j_1 \cdots j_n}$ be nonzero in K if $S^h(V)$ is a component of W. With this assumption, the entire argument goes through except the statement that $(\, , \,)$ is nondegenerate on $\ker U$. In fact, simple examples show that $(\, , \,)$ may be degenerate on $\ker U$. (See Example 2, §4, Chapter 5.)

Next, let R denote the subalgebra of $K[W]$ generated by all invariant polynomials. Is R finitely generated over K? That is, are there finitely many invariant polynomials $f_1, \ldots, f_m$ in R so that $R = K[f_1, \ldots, f_m]$? In 1939, H. Weyl showed that the answer is affirmative when K is the field of complex numbers. Weyl's result has been extended in recent times to arbitrary algebraically closed fields. This major accomplishment of Nagata, Mumford, and Haboush is part of modern geometric invariant theory. In this setting, Theorem 18 says that there are finitely many *bracket* monomials $p_1, \ldots, p_m$ so that $R = K[f_1, \ldots, f_m]$ where $f_i = \langle U, p_i \rangle$. Of course, it would be desirable to have a constructive proof of the existence of $p_1, \ldots, p_m$.

5. Covariants. Throughout this section, we shall fix an action of $\mathrm{GL}_n(K)$ on a vector space W having the form

$$W = \bigwedge^{k_1}(V) \times \cdots \times \bigwedge^{k_r}(V) \times S^{h_1}(V) \times \cdots \times S^{h_p}(V).$$

We shall assume that V, itself, is not one of the components of W. Let L be the alphabet constructed for W as in §3 and let U: Super$[L \mid P] \otimes K \to K[W]$ be the corresponding umbral linear functional.

Let $w \in \mathrm{Div}(L)$ with Length$(w) = k < n$. Let $a_1, a_2, \ldots, a_{n-k}$ be distinct symbols in L_∞^+. We extend U so that each a_i belongs to a distinct copy of V. Then the expression $\langle U, [wa_1 \cdots a_{n-k}] \rangle$ gives an invariant polynomial on $W \times V \times \cdots \times V$, where there are $(n-k)$ copies of V. Since this invariant polynomial is alternating and multilinear on $V \times \cdots \times V$, we may view it as an invariant polynomial on $W \times \bigwedge^{n-k}(V)$ which is linear in the coordinates on $\bigwedge^{n-k}(V)$. In this setting, the invariant polynomial will be denoted by $\mathrm{cov}(w)$.

We may extend this construction as follows. Let $D = (w_1, w_2, \ldots, w_k)$ be a Young diagram on L with length$(w_i) = \lambda_i \leq n$. Let us choose distinct symbols a_{ij} in L_∞^+ where $1 \leq i \leq k$, $1 \leq j \leq n - \lambda_i$, and construct a new Young diagram $D' = (w_1', \ldots, w_k')$ where

$$\begin{aligned} w_1' &= w_1 a_{11} \cdots a_{1,n-\lambda_1} \\ w_2' &= w_2 a_{21} \cdots a_{2,n-\lambda_2} \\ &\vdots \\ w_k' &= w_k a_{k1} \cdots a_{k,n-\lambda_k}. \end{aligned}$$

We shall denote the expression $[\mathrm{stand}(w_1')] \cdots [\mathrm{stand}(w_k')]$ in Super$[L \mid P]$ by $\mathrm{cov}(D)$ and call it a *covariant* of W. Applying the umbral operator to $\mathrm{cov}(D)$, we obtain an invariant polynomial on $W \times \bigwedge^{n-\lambda_1}(V) \times \cdots \times \bigwedge^{n-\lambda_k}(V)$ which is linear in the coordinate functions on the various $\bigwedge^{n-\lambda_i}(V)$. This invariant polynomial may also be denoted (on occasion) by $\mathrm{cov}(D)$ if no confusion seems possible. Let $\alpha \in W$. We shall write $\operatorname{cov}(D)(\alpha) = 0$ if for all $\gamma_i \in \bigwedge^{n-\lambda_i}(V)$, we have $\operatorname{cov}(D)(\alpha, \gamma_1, \dots, \gamma_k) = 0$.

Note. In Chapter 5, we shall give many examples of covariants as just defined. For this note only, however, we shall adopt a slightly more general definition.

Let V be a finite-dimensional vector space with $\dim V = n$. Let W be a vector space having the form $W = \bigwedge^{k_1}(V) \times \cdots \times \bigwedge^{k_r}(V) \times S^{h_1}(V) \times \cdots \times S^{h_p}(V)$. A *covariant* of $\mathrm{GL}(V)$ acting on W is an invariant of $\mathrm{GL}(V)$ on $W \times V \times \cdots \times V$. Let C be such a covariant with, say, m copies of V. Let $\alpha \in W$. We write $C \sim 0$ at α if for all $\gamma_1, \dots, \gamma_m \in V$ we have $C(\alpha, \gamma_1, \dots, \gamma_m) = 0$.

Let X be a $\mathrm{GL}(V)$-stable subset of W. Suppose that there are polynomial functions $f_1, \dots, f_r, f_{r+1}, \dots, f_{r+s}$ on W so that

$$X = \{\alpha \in W: f_i(\alpha) = 0 \textit{ for each } i = 1, \dots, r \textit{ and there exists } j,\ 1 \le j \le s, \textit{ so that } f_{r+j}(\alpha) \neq 0\}.$$

Then there exist covariants $C_1, \dots, C_r, C_{r+1}, \dots, C_{r+s}$ on $W \times V \times \cdots \times V$, where there are n copies of V, so that

$$X = \{\alpha \in W: C_i \sim 0 \textit{ at } \alpha \textit{ for each } i = 1, \dots, r \textit{ and there exists } j,\ 1 \le j \le s, \textit{ so that } C_{r+j} \not\sim 0 \textit{ at } \alpha\}.$$

PROOF. Let f be any polynomial function on W. We define a function f^* on $W \times \mathrm{GL}(V)$ by $f^*(\alpha, N) = f(N^{-1}\alpha)$ for all $\alpha \in W, N \in \mathrm{GL}(V)$. The function f^* is $\mathrm{GL}(V)$-invariant, i.e., for any $N' \in \mathrm{GL}(V)$ we have $f^*(N'\alpha, N'N) = f^*(\alpha, N)$. The set $W \times \mathrm{GL}(V)$ is open in $W \times V \times \cdots \times V$, where there are n copies of V. In fact,

$$W \times \mathrm{GL}(V) = \{(\alpha, \gamma_1, \dots, \gamma_n) \in W \times V \times \cdots \times V: \det(\gamma_1, \dots, \gamma_n) \neq 0\}.$$

According to a standard algebraic fact, there is an integer m so that the function $\det^m f^*$ is a polynomial function on $W \times V \times \cdots \times V$. Let us call the invariant polynomial function $\det^m f^*$ the covariant associated to f.

Let X be defined as above. Let $C_i = \det^{m_i} f_i^*$ be the covariant associated to f_i for $i = 1, \dots, r, r+1, \dots, s$. We shall show that these covariants define X.

First, let $\alpha \in x$ and $N \in \mathrm{GL}(V)$. For each i, $1 \le i \le r$ we have

$$C_i(\alpha, N) = (\det N)^{m_i} f_i(N^{-1}\alpha) = 0$$

since $N^{-1}\alpha \in X$. Since C_i vanishes on $\{\alpha\} \times \mathrm{GL}(V)$, we have $C_i \sim 0$ at α. Next, suppose that $f_{r+j}(\alpha) \neq 0$ for some j, $1 \le j \le s$. Let e denote the identity element in $\mathrm{GL}(V)$. Then

$$C_{r+j}(\alpha, e) = f_{r+j}(\alpha) \neq 0.$$

Secondly, let $\alpha \in W$; suppose that $C_i \sim 0$ at α for each $i = 1, \dots, r$ and that there exists j, $1 \le j \le s$, so that $C_{r+j} \not\sim 0$ at α. We shall show that $\alpha \in X$. Indeed, let N be any element in $\mathrm{GL}(V)$. Then for each i, $1 \le i \le r$, we have

$$0 = C_i(\alpha, N) = (\det N)^{m_i} f_i(N^{-1}\alpha)$$

so each $f_i(N^{-1}\alpha) = 0$. Also, since $C_{r+j} \not\sim 0$ at α, there is an N in $\mathrm{GL}(V)$ so that $C_{r+j}(\alpha, N) \neq 0$. Then

$$0 \neq C_{r+j}(\alpha, N) = (\det N)^{m_{r+j}} f_{r+j}(N^{-1}\alpha)$$

so that $f_{r+j}(N^{-1}\alpha) \neq 0$. By definition, $N^{-1}\alpha \in X$. But X is $\mathrm{GL}(V)$-stable so $\alpha \in X$.

6. The classical groups. Throughout this section, we fix an action of $\mathrm{GL}_n(K)$ on a vector space W having the form

$$(+) \qquad W = \bigwedge^{k_1}(V) \times \cdots \times \bigwedge^{k_r}(V) \times S^{h_1}(V) \times \cdots \times S^{h_p}(V).$$

We shall assume that K is of characteristic 0. Also, if V, itself, appears among the components of W, then we shall assume it is written as $\bigwedge^1(V)$.

Let G be any subgroup of $\mathrm{GL}_n(K)$. Then G acts on W in the natural way. Later on in this section, it will be important to distinguish between the absolute invariants of G and the relative invariants of G. An element $f \in K[W]$ is called an *absolute invariant* of G if $f(Nw) = f(w)$ for all N in G. An element $f \in K[W]$ is called a *relative invariant* of G if there is a positive integer g so that $f(Nw) = (\det N)^g f(w)$ for all N in G. We shall be interested here in the case where G is one of the classical groups.

The symplectic group. We assume here that $n = 2m$. An element $\sum a_{ij} e_i \wedge e_j$ in $\bigwedge^2(V)$ can be identified with the skew-symmetric matrix (a_{ij}). The group $\mathrm{GL}_n(K)$ acts on the vector space of $n \times n$ skew-symmetric matrices via $N \cdot J = NJ({}^tN)$. The identification above establishes a $\mathrm{GL}_n(K)$-isomorphism between $\bigwedge^2(V)$ and the vector space of $n \times n$ skew-symmetric matrices.

Let $v_0 = \sum a_{ij} e_i \wedge e_j$ be defined as follows: if $1 \le i \le m$ and $j = 2m - i + 1$, then $a_{ij} = 1$; otherwise $a_{ij} = 0$. The *symplectic group*, $\mathrm{Sp}_{2m}(K)$, consists of all $N \in \mathrm{GL}_n(K)$ such that $Nv_0 = v_0$. If J_0 is the skew-symmetric matrix corresponding to v_0, then $\mathrm{Sp}_{2m}(K)$ consists of all $N \in \mathrm{GL}_n(K)$ so that $NJ_0({}^tN) = J_0$. It is known that $\det N = 1$ for all $N \in \mathrm{Sp}_{2m}(K)$.

THEOREM 19. *Let $\mathrm{Sp}_{2m}(K)$ act on the vector space W. Let* $\mathrm{Super}[L \mid P]$ *be constructed for the action of $\mathrm{GL}_{2m}(K)$ on $W \times \bigwedge^2(V)$. Let U' be the corresponding umbral linear functional except if a is a symbol corresponding to the (added) component $\bigwedge^2(V)$, then*

$$\langle U', (a^{(2)} \mid ij)\rangle = 1 \quad \textit{if } 1 \le i \le m \textit{ and } j = 2m - i + 1,$$
$$\langle U', (a^{(2)} \mid ij)\rangle = 0 \quad \textit{otherwise}.$$

Then every (absolute) invariant of $\mathrm{Sp}_{2m}(K)$ acting on W can be written as $\langle U', p\rangle$ where p is a bracket polynomial in $\mathrm{Super}[L \mid P] \times K$.

PROOF. The umbral linear functional U constructed in §3 sends $\mathrm{Super}[L \mid P] \otimes K$ onto $K[W \times \bigwedge^2(V)]$. The operator U' sends $\mathrm{Super}[L \mid P] \otimes K$ onto $K[W]$ since for each $w \in W, p \in \mathrm{Super}[L \mid P]$, we have $\langle U', p\rangle(w) = \langle U, p\rangle(w, v_0)$.

Now let p be a bracket polynomial in Super$[L \mid P] \otimes K$. Let $F = \langle U, p\rangle$ and define $f \in K[W]$ by $f(w) = F(w, v_0)$. Let $N \in \mathrm{Sp}_{2m}(K)$. Then

$$\begin{aligned} f(Nw) &= F(Nw, v_0) = F(Nw, Nv_0) \\ &= F(w, v_0) \quad (\text{since } \det N = 1) = f(w). \end{aligned}$$

Therefore, f is an absolute invariant of $\mathrm{Sp}_{2m}(K)$ and we have $f(w) = F(w, v_0) = \langle U, p\rangle(w, v_0) = \langle U', p\rangle(w)$. That any absolute invariant $f \in K[W]$ can be constructed in this manner follows immediately from [**Gr**, (1.2)] via standard arguments in algebraic geometry.

The orthogonal group. An element $\sum a_{ij}e_ie_j$ in $S^2(V)$ can be identified with the symmetric matrix (a_{ij}). The group $\mathrm{GL}_n(K)$ acts on the vector space of $n \times n$ symmetric matrices via $N \cdot J = NJ({}^tN)$. The identification just given establishes a $\mathrm{GL}_n(K)$-isomorphism between $S^2(V)$ and the vector space of $n \times n$ symmetric matrices.

Let $v_0 = \sum e_ie_i$. The *orthogonal group*, $\mathrm{O}_n(K)$, consists of all $N \in \mathrm{GL}_n(K)$ such that $Nv_0 = v_0$. If we denote the $n \times n$ identity matrix by I_n, then $\mathrm{O}_n(k)$ consists of all $N \in \mathrm{GL}_n(K)$ such that $NI_n({}^tN) = I_n$. It is easy to see that $\det N = \pm 1$ for all $N \in \mathrm{O}_n(K)$.

THEOREM 20. *Let $\mathrm{O}_n(K)$ act on the vector space W. Let* Super$[L \mid P]$ *be constructed for the action of* $\mathrm{GL}_n(K)$ *on* $W \times S^2(V)$. *Let U' be the corresponding umbral linear functional except if a is a symbol corresponding to the (added) component $S^2(V)$, then*

$$\begin{aligned} &\langle U', (a \mid 1)^{j_1}(a \mid 2)^{j_2} \cdots (a \mid n)^{j_n}\rangle \\ &= 1 \quad \text{if some } j_k = 2 \text{ and } j_1 + j_2 + \cdots + j_n = 2 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then every absolute invariant of $\mathrm{O}_n(K)$ acting on W can be written as $\langle U', p\rangle$ where p is a bracket polynomial in Super$[L \mid P] \otimes K$ *such that each monomial in p has an even number of brackets.*

This theorem is proved in a manner similar to the proof given for Theorem 19. We may also be interested in the relative invariants of $\mathrm{O}_n(K)$ acting on W. These, together with the absolute invariants, give the absolute invariants of $\mathrm{SO}_n(K)$ acting on W; here, $\mathrm{SO}_n(K) = \{N \in \mathrm{O}_n(K) \colon \det N = 1\}$.

THEOREM 21. *Let $\mathrm{O}_n(K)$ act on the vector space W. Let* Super$[L \mid P]$ *be constructed for the action of* $\mathrm{GL}_n(K)$ *on* $W \times S^2(V)$. *Let U' be as in Theorem 20. Then every relative invariant of $\mathrm{O}_n(K)$ acting on W can be written as $\langle U', p\rangle$ where p is a bracket polynomial in* Super$[L \mid P] \otimes K$ *such that each monomial in p has an odd number of brackets.*

Once again, we may prove this result by rewriting the proof for Theorem 19. We should also observe that any relative invariant f of $\mathrm{O}_n(K)$ acting on W gives an absolute invariant $t_{i_1 \cdots i_n} f$ of $\mathrm{O}_n(K)$ acting on $W \times \bigwedge^n(V)$.

5. Examples

1. Skew-symmetric tensors. Let K be an infinite field such that char $K \neq 2$. Let V be a finite-dimensional vector space over K with, say, $\dim V = n$. We shall fix a basis $\{e_1, \ldots, e_n\}$ of V.

Let $\bigwedge(V)$ be the exterior algebra over V. In general (with the exception of the e_i's), we shall denote the elements of $\bigwedge(V)$ by Greek letters $\alpha, \beta, \gamma, \ldots$. The product of two elements in $\bigwedge(V)$ will denoted by juxtaposition instead of a wedge, i.e., the product of elements α, β in $\bigwedge(V)$ will be denoted by $\alpha\beta$ instead of $\alpha \wedge \beta$. (The wedge symbol will be used to denote another operation.)

We saw in Example (2), §2, Chapter 1, that $\bigwedge(V)$ can be written as Super$[A] \otimes K$ where $A = A^- = \{e_1, \ldots, e_n\}$. So, $\bigwedge(V)$ becomes a Hopf algebra with an algebra homomorphism $\Delta \colon \bigwedge(V) \to \bigwedge(V) \otimes \bigwedge(V)$ such that for each μ in V, we have $\Delta(\mu) = \mu \otimes 1 + 1 \otimes \mu$.

Recall that a skew-symmetric tensor α of *step* k is an element of $\bigwedge^k(V)$. For such an α we sometimes write step$(\alpha) = k =$ length(α). Relative to the given basis $\{e_1, \cdots, e_n\}$ of V, we have the expansion

$$\alpha = \sum_{i_1 < \cdots < i_k} (\alpha \mid i_1 \cdots i_k) e_{i_1} \cdots e_{i_k},$$

where the coefficients, denoted by $(\alpha \mid i_1 \cdots i_k)$, are in K. If α is of step n, then $\alpha = (\alpha \mid 1 \cdots n) e_1 \cdots e_n$.

If step$(\alpha) = k$ and step$(\beta) = n-k$, we shall often write the scalar $(\alpha\beta \mid 1 \cdots n)$ simply as $\alpha\beta$.

An element $\alpha \in \bigwedge^k(V)$ is said to be *divisible* by a vector $\beta \in V$ if there is an element $\gamma \in \bigwedge^{k-1}(V)$ so that $\alpha = \beta\gamma$. The tensor α is called *decomposable* if there are vectors $\beta_1, \ldots, \beta_k$ in V so that $\alpha = \beta_1 \cdots \beta_k$. We shall repeatedly make use of the following facts concerning division of tensors:

(D1) A vector $\mu \in V$ divides an element $\alpha \in \bigwedge^k(V)$ if and only if $\mu\alpha = 0$.

(D2) Let $\alpha \in \bigwedge^k(V)$. Suppose there exist linearly independent vectors $\mu_1, \ldots, \mu_r$ which divide α. Then there is an element $\gamma \in \bigwedge^{k-r}(V)$ so that $\alpha = \mu_1 \cdots \mu_r \gamma$.

The proof of (D1) follows from taking any basis of V containing μ and expressing α in terms of the corresponding basis for $\bigwedge^k(V)$. The proof of (D2) is similar; choose any basis of V containing $\mu_1, \ldots, \mu_r$.

PROPOSITION 22. *Let* $\alpha \in \bigwedge^k(V)$. *Let* $V_\alpha = \{\mu \in V : \alpha\mu = 0\}$.
(i) $\dim V_\alpha \le k$.
(ii) $\dim V_\alpha = k$ *if and only if* α *is decomposable.*
(iii) *If* $\dim V_\alpha < k$, *then* $\dim V_\alpha \le k-2$.
(iv) *Let* $W_\alpha = \{\alpha\mu : \mu \in V\}$. *Then* $\dim W_\alpha \ge n-k$.
Furthermore, $\dim W_\alpha = n-k$ *if and only if the tensor* α *is decomposable. If* $\dim W_\alpha > n-k$, *then* $\dim W_\alpha \ge n-k+2$.

PROOF. According to (D1), the vector space V_α consists of all μ which divide α. Statements (i) and (ii) follow from (D2). To prove (iii), let us assume that $\dim V_\alpha = k-1$ and that $\{\beta_1, \ldots, \beta_{k-1}\}$ is basis for V_α. Extending this to a basis for V and writing α in terms of this basis, we see that $\alpha = \beta_1 \cdots \beta_{k-1}\beta$ for some β in V. But then $\dim V_\alpha = k$. Statement (iv) is a consequence of statements (i), (ii), (iii) applied to the mapping $\psi: V \to \bigwedge^{k+1}(V), \mu \to \alpha\mu$. For $n = \dim(\text{image}\,\psi) + \dim V_\alpha$.

2. The meet. Let $\alpha \in \bigwedge^i(V)$ and $\beta \in \bigwedge^j(V)$. The *meet* of α and β, denoted by $\alpha \wedge \beta$, is that tensor defined as follows:
(a) if $i+j<n$, then $\alpha \wedge \beta = 0$;
(b) if $i+j \ge n$, then $\alpha \wedge \beta = \sum_\alpha (\alpha_{(1)}\beta \mid 1 \cdots n)\alpha_{(2)}$.

Since this operation will play a major role in what follows, perhaps a word of explanation as to why it is well defined is in order. For given β, formula (b) is a composite mapping: $\text{Super}[A] \to \text{Super}[A] \otimes \text{Super}[A] \to K \otimes \text{Super}[A] \to \text{Super}[A]$. Reading from left to right, the first map is $\alpha \to \Delta\alpha$, the second map is $\gamma \otimes \delta \to (\gamma\beta \mid 1 \cdots n) \otimes \delta$, and the third map is multiplication.

The following properties of the meet operation are important; their proofs follow quickly from the definition and are omitted.

(M1) If $\text{step}\,\alpha + \text{step}\,\beta \ge n$, then $\text{step}\,(\alpha \wedge \beta) = \text{step}\,\alpha + \text{step}\,\beta - n$.
(M2) The mapping $\bigwedge(V) \times \bigwedge(V) \to \bigwedge(V)$, $(\alpha, \beta) \to \alpha \wedge \beta$, is bilinear.
(M3) Let $A = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$ with $i_1 < \cdots < i_k$; let $e_A = e_{i_1} \cdots e_{i_k}$. (If $A = \varnothing$, put $e_A = 1$.)
(i) If $A \cup B \ne \{1, \ldots, n\}$, then $e_A \wedge e_B = 0$.
(ii) If $A \cup B = \{1, \ldots, n\}$, then $e_A \wedge e_B = \text{sign}(k) e_{A \cap B}$ where k is the number of inversions in the sequence obtained by first writing in order $A - B$ and then writing in order $B - A$.

(M4) Let $\text{step}\,\alpha = i$ and $\text{step}\,\beta = j$. Let $p = i+j-n$. Then $\alpha \wedge \beta = \text{sign}(k)\beta \wedge \alpha$ where $k = (i-p)(j-p)$.
(M5) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.
(M6) Let $N \in GL_n(K)$. Then $N\alpha \wedge N\beta = (\det N) \cdot N(\alpha \wedge \beta)$.
(M7) $\alpha \wedge \beta = \sum_\beta (\alpha\beta_{(2)} \mid 1 \cdots n)\beta_{(1)}$.

Let $\alpha \in \bigwedge^k(V)$. We define the *span* of α, denoted by $\text{span}(\alpha)$, to be the subspace of V consisting of all vectors having the form $\alpha \wedge \beta$ for $\beta \in \bigwedge^{n-k+1}(V)$. We define $\text{rank}\,\alpha = \dim(\text{span}\,\alpha)$. If α is decomposable, say $\alpha = \beta_1 \cdots \beta_k$, then (it is easy to show that) $\text{span}\,\alpha$ has a basis $\{\beta_1, \ldots, \beta_k\}$.

The relationship between the meet and the ordinary product of two skew-symmetric tensors may be viewed using a mapping ϕ defined as follows. Let $A = \{i_1, \ldots, i_p\}$ be a subset of $\{1, \ldots, n\}$ with $i_1 < \cdots < i_p$. Let A' be the complement of A in $\{1, \ldots, n\}$, say $A' = \{j_1, \ldots, j_r\}$ with $j_1 < \cdots < j_r$. We define a linear transformation $\phi: \bigwedge(V) \to \bigwedge(V)$ by $\phi(e_A) = \text{sign}(k)e_{A'}$, where k is the number of inversions in the sequence $j_1, \ldots, j_r, i_1, \ldots, i_p$. We note that ϕ is a vector space isomorphism which sends $\bigwedge^p(V)$ to $\bigwedge^{n-p}(V)$. If $\alpha \in \bigwedge^k(V)$, then $\phi(\phi(\alpha)) = \text{sign}(k(n-k))\alpha$. If $\beta \in \bigwedge^j(V)$, then $\phi(\alpha \wedge \beta) = \phi(\alpha)\phi(\beta)$ and $\phi(\alpha\beta) = \phi(\alpha) \wedge \phi(\beta)$.

PROPOSITION 23. *Let* $\alpha \in \bigwedge^k(V)$. *Then*
(i) $\text{rank}\,\alpha \ge k$;
(ii) $\text{rank}\,\alpha = k$ *if and only if* α *is decomposable;*
(iii) *if* $\text{rank}\,\alpha > k$, *then* $\text{rank}\,\alpha \ge k+2$.

PROOF. We examine the kernel of the mapping $\psi: \bigwedge^{n-k+1}(V) \to \text{span}\,\alpha$ given by $\psi(\beta) = \alpha \wedge \beta$. An element β is in the kernel of ψ if and only if $\phi(\alpha)\phi(\beta) = 0$, i.e., if and only if for all $\mu \in V$ we have $0 = \phi(\alpha)\mu\phi(\beta)$. Hence,

$$\dim(\ker\psi) = \binom{n}{n-k+1} - \dim\{\phi(\alpha)\mu : \mu \in V\}$$

and

$$\begin{aligned}\text{rank}\,\alpha &= \binom{n}{n-k+1} - \dim(\ker\psi)\\ &= \dim\{\phi(\alpha)\mu : \mu \in V\}.\end{aligned}$$

We apply Proposition 22, (iv), to see that $\text{rank}\,\alpha = k$ or $\text{rank}\,\alpha \ge k+2$.

If α is decomposable, we noted above that $\text{rank}\,\alpha = k$. If, on the other hand, $\text{rank}\,\alpha = k$, then the equation above shows that $\phi(\alpha)$ is decomposable. Therefore,

$$\dim\{\phi(\alpha) \wedge \delta : \delta \in \bigwedge^{n-1}(V)\} = n-k.$$

Indeed, if $\phi(\alpha) = \gamma_1 \cdots \gamma_{n-k}$, then a basis for this vector space is

$$\gamma_2 \cdots \gamma_{n-k},\ \gamma_1\gamma_3 \cdots \gamma_{n-k},\ \gamma_1 \cdots \gamma_{n-k-1}.$$

But this vector space is isomorphic via ϕ to $\{\alpha\mu : \mu \in V\}$. Thus, α is decomposable by Proposition 22, (iv).

Note. In the course of this proof, we showed that α is decomposable if and only if $\phi(\alpha)$ is decomposable.

(M8) If $\alpha \in \bigwedge^{n-1}(V)$, then α is decomposable.

PROOF. We note that $\phi(\alpha)$, being a vector, is decomposable.

(M9) Let $\alpha \in \bigwedge^k(V), \beta \in \bigwedge^{n-1}(V)$. Then $\alpha \wedge \beta \wedge \beta = 0$.

PROOF. We apply ϕ to see that $\phi(\alpha)\phi(\beta)\phi(\beta) = \phi(\alpha) \cdot 0 = 0$.

(M10) Let $\alpha \in \bigwedge^k(V)$ and $\gamma \in \bigwedge^{n-1}(V)$. Let $\beta = \alpha \wedge \gamma \in \bigwedge^{k-1}(V)$. If α is decomposable, then so is β.

PROOF. First, $\operatorname{span} \beta \subset \operatorname{span} \alpha$ according to (M5). If α is decomposable, then $\operatorname{rank} \alpha = k$. Now, $\operatorname{rank} \beta = k-1$ or $\operatorname{rank} \beta \geq k+1$ according to Proposition 23. It follows that $\operatorname{rank} \beta = k-1$ since $\operatorname{rank} \beta \leq \operatorname{rank} \alpha$.

3. Covariants. Let W be a vector space having the form

$$\overset{k_1}{\bigwedge}(V) \times \cdots \times \overset{k_r}{\bigwedge}(V).$$

We shall assume that if V, itself, appears as a component of W, then it is written as $\bigwedge^1(V)$. let L be the alphabet constructed for W as in Chapter 4, §3. Let $D = (w_1, \ldots, w_k)$ be a Young diagram on L with $\operatorname{length}(w_i) = \lambda_i \leq n$. In Chapter 4, §5, we used D to define an invariant polynomial on

$$W \times \overset{n-\lambda_1}{\bigwedge}(V) \times \cdots \times \overset{n-\lambda_k}{\bigwedge}(V)$$

called $\operatorname{cov}(D)$.

For $\alpha \in W$, we have defined $\operatorname{cov}(D)(\alpha) = 0$ to mean that for all $\gamma_i \in \bigwedge^{n-\lambda_i}(V)$, $i = 1, \ldots, k$, we have $\operatorname{cov}(D)(\alpha, \gamma_1, \ldots, \gamma_k) = 0$. We shall write $D \sim 0$ at α instead of $\operatorname{cov}(D)(\alpha) = 0$.

This may also be interpreted as follows. In the expansion of $\operatorname{cov}(D)$, the coefficients of the various monomials in the γ_i's are given by the expressions

$$(*) \qquad \langle U, (\operatorname{stand}(w_1) \mid i_1 \cdots i_{\lambda_1})\rangle \cdots \langle U, (\operatorname{stand}(w_k) \mid j_1 \cdots j_{\lambda_k})\rangle(\alpha),$$

where $\{i_1, \ldots, i_{\lambda_1}\}, \ldots, \{j_1, \ldots, j_{\lambda_k}\}$ are any k subsets of $\{1, 2, \ldots, n\}$. (This follows from Rule 6, §2, Chapter 2.) Hence $D \sim 0$ at α means that each term $(*)$ is 0.

The relationship between the meet and certain covariants will be very important later. In stating the main result below, we (tacitly) take W to be $\bigwedge^k(V) \times \bigwedge^j(V) \times \bigwedge^p(V)$ where $k + j > n$ and $p = 2n - k - j$.

THEOREM 24. *Let a, b, c be in L with $a < b < c$. Suppose that a belongs to $\bigwedge^k(V)$, b belongs to $\bigwedge^j(V)$, and c belongs to $\bigwedge^p(V)$ where $p = 2n - k - j$. Let $\alpha \in \bigwedge^k(V), \beta \in \bigwedge^j(V), \gamma \in \bigwedge^p(V)$. Let $\varepsilon = \operatorname{sign}[(k + j - n)j]$.*

(i) *Let $r = k + j - n$ and let $\{i_1, \ldots, i_r\}$ be any subset of $\{1, \ldots, n\}$ with $i_1 < \cdots < i_r$. Then*

$$\langle U, [a^{(n-j)}b^{(j)}](a^{(r)} \mid i_1 \cdots i_r)\rangle(\alpha, \beta) = \varepsilon(\alpha \wedge \beta \mid i_1 \cdots i_r);$$

(ii) $\langle U, [a^{(n-j)}b^{(j)}][a^{(r)}c^{(p)}] > (\alpha, \beta, \gamma) = \varepsilon(\alpha \wedge \beta)\gamma$;

(iii) $(\alpha \wedge \beta)\gamma = \varepsilon'(\alpha \wedge \gamma)\beta$ *where* $\varepsilon' = \operatorname{sign}[(n-j)(n-p)]$.

PROOF. Let us introduce some notation. First, let a be any symbol and let $A = \{i_1, \ldots, i_k\}$ be any subset of $\{1, \ldots, n\}$ with $i_1 < \cdots < i_k$. We shall denote

$(a^{(k)} \mid i_1 \cdots i_k)$ by $(a^{(k)} \mid A)$. Now, it is enough to prove (i) when $\alpha = e_A$ and $\beta = e_B$. Then

$$\begin{aligned}&\langle U, [a^{(n-j)}b^{(j)}](a^{(r)} \mid i_1 \cdots i_r)\rangle(\alpha, \beta)\\ &\quad = \sum \pm \langle U, (a^{(n-j)} \mid h'_1 \cdots h'_{n-j})(b^{(j)} \mid h_1 \cdots h_j)(a^{(r)} \mid i_1 \cdots i_r)\rangle(\alpha, \beta).\end{aligned}$$

The only possible nonzero term in this sum would have $B = \{h_1, \ldots, h_j\}$, $A - B = \{h'_1, \ldots, h'_{n-j}\}$, and $A \cap B = \{i_1, \ldots, i_r\}$. This term gives $\varepsilon(e_A \wedge e_B \mid i_1 \cdots i_r)$. Statement (ii) is proved similarly and (iii) follows by interchanging the two brackets in (ii).

The theorem above extends to more than two brackets. (Both the statement and proof are easily generalized.) For example, let a, b, c, d be symbols in L with $a < b < c < d$. Let a belong to $\bigwedge^k(V)$, b belong to $\bigwedge^p(V)$, c belong to $\bigwedge^r(V)$, and d belong to $\bigwedge^s(V)$ where $s = 3n - kp - r$. Let $\alpha \in \bigwedge^k(V), \beta \in \bigwedge^p(V), \gamma \in \bigwedge^r(V), \delta \in \bigwedge^s(V)$. Then

$$\langle U, [a^{(n-p)}b^{(p)}][a^{(n-r)}c^{(r)}][a^{(n-s)}d^{(s)}]\rangle$$

at $(\alpha, \beta, \gamma, \delta)$ is $\pm(\alpha \wedge \beta \wedge \gamma)\delta$ where the sign depends only on k, p, r, and s.

4. Examples of invariants.

EXAMPLE 1. Suppose $\operatorname{char} K = 0$. Let F be an invariant polynomial on $\bigwedge^k(V)$, homogeneous of degree d, such that $F(N\alpha) = (\det N)^r F(\alpha)$ for all $\alpha \in \bigwedge^k(V), N \in \mathrm{GL}_n(K)$. Then $rn = dk$. (Recall that $n = \dim V$.)

PROOF. Let p be a bracket monomial in $\operatorname{Super}[L \mid P] \otimes K$, say $p = [w_1] \cdots [w_r]$. There are rn symbols appearing in the words $w_1, \ldots, w_r$ and each symbol appears k times if $\langle U, p\rangle \neq 0$. The degree of $\langle U, p\rangle$ is rn/k and the example is proved by applying Theorem 18.

EXAMPLE 2. Let $\dim V = n = 2r$, $\operatorname{char} K = 0$. If F is any homogeneous invariant polynomial in $\bigwedge^2(V)$, then there is a positive integer g so that $F = c\mathrm{Pf}^g$ where $c \in K$ and Pf is the *Pfaffian.*

PROOF. We begin by recalling the definition of the Pfaffian,

$$r!\mathrm{Pf} = \sum \operatorname{sgn} \sigma (\alpha \mid \sigma 1 \sigma 2)(\alpha \mid \sigma 3 \sigma 4) \cdots (\alpha \mid \sigma 2r - 1 \sigma 2r),$$

where the sum is over all permutations σ of $\{1, \ldots, 2r\}$ such that $\sigma 1 < \sigma 2, \sigma 3 < \sigma 4, \ldots, \sigma(2r-1) < \sigma(2r)$. We observe that each monomial in the sum appears $r!$ times, each with the same sign, so that all the coefficients of Pf are ± 1.

Let $a_1, \ldots, a_r$ be symbols belonging to $\bigwedge^2(V)$ such that $a_1 < \cdots < a_r$. Applying formula (b), Chapter 2, §2, we see that $\langle U, [a_1^{(2)} \cdots a_r^{(2)}]\rangle = r!\mathrm{Pf}$ and (by Theorem 16) that $N\mathrm{Pf} = (\det N)^{-1}\mathrm{Pf}$.

It is well known that $\mathrm{GL}_n(k)$ has an open orbit in $\bigwedge^2(V)$, say $\mathrm{GL}_n(K)\alpha_0$, where we may assume that $\mathrm{Pf}(\alpha_0) = 1$. Let F be any invariant polynomial on $\bigwedge^2(V)$ with, say, $F(N\alpha) = (\det N)^g F(\alpha)$. Then $F = F(\alpha_0)\mathrm{Pf}^g$. Indeed,

$$\begin{aligned}F(N\alpha_0) &= (\det N)^g F(\alpha_0) = (\det N)^g \mathrm{Pf}(\alpha_0)^g F(\alpha_0)\\ &= F(\alpha_0)(\mathrm{Pf}(N\alpha_0))^g.\end{aligned}$$

Thus, $F(\alpha) = F(\alpha_0)\mathrm{Pf}(\alpha)^g$ for all $\alpha \in \mathrm{GL}_n(K)\cdot\alpha_0$; since the orbit of α_0 is open, $F(\alpha) = F(\alpha_0)\mathrm{Pf}(\alpha)^g$ everywhere.

Note 1. Suppose $\operatorname{char} K = p < r$. It still can be shown that Pf is invariant. Since its degree is r, it would only be the image of $[a_1^{(2)}\cdots a_r^{(2)}]$ in Super$[L \mid P]\otimes K$. But $\langle U, [a_1^{(2)}\cdots a_r^{(2)}]\rangle = r!\mathrm{Pf} = 0$. We conclude that Theorem 18 cannot be true for $\operatorname{char} K = p > 0$.

Note 2. Suppose $\operatorname{char} K \neq 2$ and $\dim V = n = 2r+1$. It can be shown that $\mathrm{SL}_n(k)$ has an open orbit in $\bigwedge^2(V)$, say $\mathrm{SL}(K)\cdot\alpha_0$. Therefore, any invariant polynomial F is constant since $F(N\alpha_0) = (\det N)^g F(\alpha_0) = 1^g F(\alpha_0) = F(\alpha_0)$.

It is possible to give a combinatoric proof for Example 2 which avoids the use of orbits. We turn to this proof now, beginning with a lemma interesting in its own right.

EXCHANGE LEMMA. *Let u, v, w be words in* Super$[L]$. *Then*

$$\sum_u [u_{(1)}v][u_{(2)}w] = \operatorname{sign}(k)\sum_v [v_{(1)}u][v_{(2)}w],$$

where $k = n - \mathrm{Length}(w)$.

PROOF. This lemma follows at once from Proposition 10 by taking $x = y = 1\cdots n$, $a = v$, $b = u$, and $c = w$.

PROOF OF EXAMPLE 2. Let F be a homogeneous, invariant polynomial on $\bigwedge^2(V)$. Then, by Theorem 18, F can be written as $\langle U, p\rangle$ where p is a bracket polynomial in Super$[L \mid P]\otimes K$. A typical bracket appearing in p has the form $[a_1^{(2)}\cdots a_j^{(2)}a_{j+1}\cdots a_{j+k}]$ where the a_i's are distinct symbols belonging to $\bigwedge^2(V)$ and $2j + k = n$. If $2j = n$ in every such bracket, then we are finished. Otherwise, there are brackets with $2j < n$; we shall say that such a bracket "splits" the symbols $a_{j+1},\dots,a_{j+k}$.

Let $a_1,\dots,a_m$, $a_1 < \cdots < a_m$, be all the symbols appearing in p. If a_r is split by some bracket in p but $a_1,\dots,a_{r-1}$ are not, then we shall say that p has property $P(r-1)$. Let r be the largest integer so that F is the image of a bracket polynomial, say p, in $a_1,\dots,a_m$ satisfying $P(r)$. Suppose that $r < m+1$.

Let $[w_1]\cdots[w_k]$ be a bracket monomial in p such that w_1 splits a_r. We may assume that w_2 also splits a_r and that

$$[w_1] = [b_1^{(2)}\cdots b_s^{(2)}a_r v], \qquad [w_2] = [d_1^{(2)}\cdots d_t^{(2)}a_r v']$$

where $b_1,\dots,b_s$ (resp. $d_1,\dots,d_t$) are all the symbols in w_1 (resp. w_2) $< a_r$. We apply the Exchange Lemma to $[w_1][w_2]$ taking $u = b_1^{(2)}\cdots b_s^{(2)}a_r^{(2)}$, $v = v$, and $w = d_1^{(2)}\cdots d_t^{(2)}v'$. Then

$$\sum_u [u_{(1)}v][u_{(2)}w][w_3]\cdots[w_k] = \pm\sum_v [v_{(1)}u][v_{(2)}w][w_3]\cdots[w_k].$$

Each term on the left-hand side of this equation has image $\langle U, [w_1]\cdots[w_k]\rangle$. Each term on the right-hand side satisfies property $P(r+1)$. Hence, $r = m+1$ and the example is proved.

5. Examples of covariants. We shall assume that K is an infinite field and $\operatorname{char} K \neq 2$.

EXAMPLE 1. Let $p \le n$ and let $a_1,\dots,a_p$ be p distinct symbols belonging to $\bigwedge^k(V)$. Let C_p be the covariant defined by the diagram

$$C_p: \quad \begin{matrix} a_1a_2\cdots a_p \\ a_1^{(k-1)} \\ \vdots \\ a_p^{(k-1)} \end{matrix}$$

Let $\alpha \in \bigwedge^k(V)$. Then α has rank r if and only if $C_r \not\sim 0$ at α but $C_{r+1} \sim 0$ at α.

PROOF. Let us choose p distinct symbols $b_1,\dots,b_p$ belonging to p distinct copies of $\bigwedge^{n-k+1}(V)$. For simplicity, let us suppose that $a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$. Let $\beta_1,\cdots,\beta_p$ be p elements in the copies of $\bigwedge^{n-k+1}(V)$. We evaluate

$$\langle U, [a_1^{(k-1)}b_1^{(n-k+1)}]\cdots[a_p^{(k-1)}b_p^{(n-k+1)}](a_1\cdots a_p \mid i_1\cdots i_p)\rangle$$

at $(\alpha,\beta_1,\dots,\beta_p)$. By Example 5, Chapter 2, §3,

$$(a_1\cdots a_p \mid i_1\cdots i_p) = \sum_\sigma \operatorname{sgn}\sigma (a_1 \mid i_{\sigma 1})\cdots(a_p \mid i_{\sigma p})$$

where σ ranges over all permutations of $\{1,\dots,p\}$. So, the expression above is

$$\pm\sum_\sigma \operatorname{sgn}\sigma \langle U, [a_1^{(k-1)}b_1^{(n-k+1)}](a_1 \mid i_{\sigma 1})\rangle\cdots\langle U, [a_p^{(k-1)}b_p^{(n-k+1)}](a_p \mid i_{\sigma p})\rangle$$

at $(\alpha,\beta_1,\dots,\beta_p)$. By Theorem 24(i), this is

$$\pm\sum_\sigma \operatorname{sgn}\sigma (\alpha\wedge\beta_1)_{i_{\sigma 1}}\cdots(\alpha\wedge\beta_p)_{i_{\sigma p}}.$$

EXAMPLE 2. Let a belong to $\bigwedge^k(V)$, b belong to $\bigwedge^r(V)$, and c belong to $\bigwedge^{k+r}(V)$. Suppose that $a < b$. Let $\alpha \in \bigwedge^k(V), \beta \in \bigwedge^r(V)$. Then

$$\langle U, (c^{(k+r)} \mid i_1\cdots i_{k+r})\rangle(\alpha\beta) = \langle U, (a^{(k)}b^{(r)} \mid i_1\cdots i_{k+r})\rangle(\alpha,\beta).$$

PROOF. The top expression is (by definition) $(\alpha\beta \mid i_1\cdots i_{k+r})$. We now apply Rule 6, Chapter 2, §2 to see that

$$(a^{(k)}b^{(r)} \mid i_1\cdots i_{k+r}) = \sum \pm(a^{(k)} \mid j_1\cdots j_k)(b^{(r)} \mid h_1\cdots h_r),$$

where $\{j_1,\dots,j_k\}\cup\{h_1,\dots,h_r\} = \{i_1,\dots,i_{k+r}\}$. If we assume that $j_1 < \cdots < j_k$ and $h_1 < \cdots < h_r$, then the sign is determined by the number of transpositions in the sequence $j_1,\dots,j_k,h_1,\dots,h_r$. If we apply U and evaluate at (α,β), the expression becomes

$$\sum \pm(\alpha \mid j_1\cdots j_k)(\beta \mid h_1\cdots h_r) = (\alpha\beta \mid i_1\cdots i_{k+r}).$$

EXAMPLE 3. Let a belong to $\bigwedge^k(V)$, b belong to $\bigwedge^j(V)$, c belong to $\bigwedge^i(V)$, and d belong to $\bigwedge^p(V)$ where $i+j+k+p=2n$. Let $\alpha \in \bigwedge^k(V), \beta \in \bigwedge^j(V), \gamma \in \bigwedge^i(V)$, and $\delta \in \bigwedge^p(V)$. Then

$$\langle U, [a^{(n-i-j)}b^{(j)}c^{(i)}][a^{(k+i+j-n)}d^{(p)}]\rangle$$

evaluated at $(\alpha,\beta,\gamma,\delta)$ is $\pm(\alpha \wedge \beta\gamma)\delta = \pm(\alpha\wedge\delta)\beta\gamma$.

PROOF. Let e belong to $\bigwedge^{i+j}(V)$. We apply Theorem 24 and Example 2 to see that

$$\langle U, [a^{(n-i-j)}b^{(j)}c^{(i)}][a^{(k+i+j-n)}d^{(p)}]\rangle$$

evaluated at $(\alpha,\beta,\gamma,\delta)$ is

$$\pm\langle U, [a^{(n-i-j)}e^{(i+j)}][a^{(k+i+j-n)}d^{(p)}]\rangle(\alpha,\beta\gamma,\delta) = \pm(\alpha\wedge\beta\gamma)\delta.$$

EXAMPLE 4. Let a belong to $\bigwedge^k(V)$, b belong to V. Let C be the covariant determined by (the word) $a^{(k)}b$. Let $\alpha \in \bigwedge^k(V), \mu \in V$. Then μ divides α if and only if $C \sim 0$ at (α,μ).

PROOF. This follows immediately from (D1), §1, and Example 2.

EXAMPLE 5 (THE GRASSMANN CONDITION). Let a, b belong to $\bigwedge^k(V)$. Let C be the covariant determined by the diagram.

$$C:\quad \begin{matrix} a^{(k)}b \\ b^{(k-1)}. \end{matrix}$$

Let $\alpha \in \bigwedge^k(V)$. Then α is decomposable if and only if $C \sim 0$ at α.

PROOF. We apply Example 3 to see that $C \sim 0$ at α if and only if for any $\gamma \in \bigwedge^{n-k-1}(V), \delta \in \bigwedge^{n-k+1}(V)$, we have

$$0 = (\alpha\wedge\alpha\gamma)\delta = (\alpha\wedge\delta)\alpha\gamma.$$

Now, suppose that α is decomposable, say, $\alpha = \alpha_1\cdots\alpha_k$. Then $\alpha\wedge\delta \in \operatorname{span}\alpha$ and, so, is a linear combination of $\alpha_1,\cdots,\alpha_k$. Hence, $(\alpha\wedge\delta)\alpha = 0$.

Conversely, if $(\alpha\wedge\delta)\alpha\gamma = 0$ for all γ,δ, then every vector $\alpha\wedge\delta$ divides α. Since rank $\alpha \geq k$, we see (using (D2), §1) that rank $\alpha = k$ and α is decomposable.

EXAMPLE 6. Let a, b belong to distinct copies of $\bigwedge^k(V)$. Let C be the covariant defined by the diagram

$$C:\quad \begin{matrix} a^{(k)}b \\ b^{(k-1)}. \end{matrix}$$

Let $\alpha,\beta \in \bigwedge^k(V)$ with $\alpha,\beta \neq 0$. If $C \sim 0$ at (α,β), then α is decomposable and $\beta = c\alpha$ for some $c \in K$.

PROOF. According to Example 3, $C \sim 0$ at (α,β) if and only if for all $\gamma \in \bigwedge^{n-k-1}(V), \delta \in \bigwedge^{n-k+1}(V)$, we have

$$0 = (\beta\wedge\alpha\gamma)\delta = (\beta\wedge\delta)\alpha\gamma.$$

If $\beta \neq 0$, then α is divisible by each vector $\beta\wedge\delta$ in span β. But rank $\beta \geq k$ and, applying (D2) in §1, we see that rank $\beta = k$. This shows (by Proposition 23) that β and, then, α are decomposable and $\beta = c\alpha$.

EXAMPLE 7. (i) Let $\alpha \in \bigwedge^k(V)$ where $k \leq n-3$. Then α is decomposable if and only if $\alpha\mu$ is decomposable for all $\mu \in V$.

(ii) Let $\alpha \in \bigwedge^k(V)$ where $k \geq 3$. Then α is decomposable if and only if $\alpha\wedge\delta$ is decomposable for all $\delta \in \bigwedge^{n-1}(V)$.

PROOF. Statement (ii) follows from (i) by applying ϕ. To prove (i), it is (obviously) enough to show that α is decomposable if $\alpha\mu$ is decomposable for all $\mu \in V$.

LEMMA. *Let $\alpha,\beta \in \bigwedge^k(V)$ be decomposable tensors with $\alpha = \alpha_1\cdots\alpha_k$ and $\beta = \beta_1\cdots\beta_k$ where $\alpha_i,\beta_j \in V$. Then $\alpha+\beta$ is decomposable if and only if* $\dim(\operatorname{span}\alpha \cap \operatorname{span}\beta) \geq k-1$.

PROOF. Let $W = \operatorname{span}\alpha\cap\operatorname{span}\beta$. If $\dim W = k-1$, we choose a basis of W, say $\{\gamma_1,\dots,\gamma_{k-1}\}$ and extend this to bases $\{\gamma_1,\dots,\gamma_{k-1},\gamma'\}, \{\gamma_1,\dots,\gamma_{k-1},\gamma''\}$ of span α and span β, respectively, so that

$$\alpha = \gamma_1\cdots\gamma_{k-1}\gamma' \quad\text{and}\quad \beta = \gamma_1\cdots\gamma_{k-1}\gamma''.$$

Then, $\alpha+\beta = \gamma_1\cdots\gamma_{k-1}(\gamma'+\gamma'')$.

On the other hand, suppose that $\dim W = r < k-1$ but $\alpha+\beta$ is decomposable. We may assume that

$$\alpha = \alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_k, \qquad \beta = \alpha_1\cdots\alpha_r\beta_{r+1}\cdots\beta_k,$$

where $\{\alpha_1,\dots,\alpha_r\}$ is a basis for W. Let $\alpha+\beta = \alpha_1\cdots\alpha_r\gamma_{r+1}\cdots\gamma_k$ for some $\gamma_i \in V$; we put $\gamma = \gamma_{r+1}$. Then

$$0 = (\alpha+\beta)\gamma = \alpha\gamma+\beta\gamma = \alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_k\gamma + \alpha_1\cdots\alpha_r\beta_{r+1}\cdots\beta_k\gamma.$$

If $\{\alpha_1,\dots,\alpha_r,\alpha_{r+1},\dots,\alpha_k,\beta_{r+1},\dots,\beta_k,\gamma\}$ is linearly independent, we obtain a contradiction at once from this equation. So, we may assume that

$$\gamma = \sum_{i=1}^{r} c_i\alpha_i + \sum_{i=r+1}^{k} c_i'\alpha_i + \sum_{i=r+1}^{k} c_i''\beta_i$$

for some c_i, c_i', c_i'' in K. Then each of the terms appearing in $\alpha_1\cdots\alpha_r\alpha_{r+1}\cdots\alpha_k\gamma$ contains one β_j while each of the terms appearing in $\alpha_1\cdots\alpha_r\beta_{r+1}\cdots\beta_k\gamma$ contains at least two β_j's (since $r < k-1$). Hence, each $c_i'' = 0$. It then follows that each $c_i' = 0$ and, so, that $\gamma \in W$, which is a contradiction.

We now prove Example 7. In this argument, for convenience, we denote elements in V by $a, b, \dots, u, v, \dots$. First, let $u \in V$ be chosen so that $\alpha u \neq 0$. Let $v \in V$ satisfy $\alpha uv = 0$. We shall show that there is a scalar $c \in K$ so that $\alpha v = c\alpha u$. If $\alpha v = 0$ we may take $c = 0$. If $v = cu$, then $\alpha v = c\alpha u$. Otherwise, by assumption, $\alpha u + \alpha v = \alpha(u+v)$ is decomposable. Applying the previous lemma, we see that there are elements $b', b'', v_2, \dots, v_{k-1}$ in V so that

$$\alpha u = b'v_2\cdots v_{k-1}uv \quad\text{and}\quad \alpha v = b''v_2\cdots v_{k-1}uv.$$

If $\{b', b'', v_2, \ldots, v_{k-1}, u, v\}$ is linearly dependent, we have $\alpha v = c\alpha u$. Otherwise, we may extend this set to a basis of V, say $\{b', b'', v_2, \ldots, v_{k-1}, u, v, w_1, \ldots, w_r\}$ with $r - n - k - 2 > 0$. Writing α in terms of this basis, we have

$$\alpha = b'' v_2 \cdots v_{k-1} u + uv\beta - b' v_2 \cdots v_{k-1} v$$

for some $\beta \in \bigwedge^{k-2}(V)$. Then

$$\alpha w_1 = b'' v_2 \cdots v_{k-1} u w_1 + uv\beta w_1 - b' v_2 \cdots v_{k-1} v w_1.$$

Let $\gamma = u w_2 \cdots w_r$. We calculate $\alpha w_1 \wedge \alpha w_1 \gamma$. First,

$$b'' v_2 \cdots v_{k-1} u w_1 \wedge \alpha w_1 \gamma = c u v_2 \cdots v_{k-1} w_1$$

with $c \in K, c \neq 0$. Second, each term appearing in $uv\beta w_1 \wedge \alpha w_1 \gamma$ is divisible by uvw_1. Third, $b' v_2 \cdots v_{k-1} v w_1 \wedge \alpha w_1 \gamma = 0$. Hence, $\alpha w_1 \wedge \alpha w_1 \gamma$ contains the term $u v_2 \cdots v_{k-1} w_1$ and $\alpha w_1 \wedge \alpha w_1 \gamma \neq 0$. However, applying Example 5, we have $\alpha w_1 \wedge \alpha w_1 \gamma = 0$. This contradiction shows that $\alpha v = c\alpha u$ with $c \in K$.

Now, we show that α is decomposable. Let $V_\alpha = \{w \in V \colon \alpha w = 0\}$. We shall show that $\dim V_\alpha = k$. Let u be any vector in V so that $\alpha u \neq 0$. Let $T \colon V \to \bigwedge^{k+2}(V)$ be defined by $T(v) = (\alpha u) v$. Since αu is decomposable, $\dim(\operatorname{kernel} T) = k + 1$. On the other hand, let $v \in \operatorname{kernel} T$. If $\alpha v \neq 0$, then (by what was just proved), $\alpha v = c\alpha u$ for some $c \in K$ and $\alpha(v - cu) = 0$. It follows that $\dim(\operatorname{kernel} T / V_\alpha) = 1$ and $\dim V_\alpha = k$.

EXAMPLE 8. Let a, b be symbols belonging to $\bigwedge^k(V)$ where $k = 2, \ldots, n-2$. Let C_k be the covariant defined by the diagram

$$C_k\colon \quad \begin{matrix} a^{(k)} b^{(2)} \\ b^{(k-2)} \end{matrix}$$

Let $\alpha \in \bigwedge^k(V)$. Then α is decomposable if and only if $C_k \sim 0$ at α.

PROOF. We begin with a lemma which will also be used later.

LEMMA. Let $\alpha \in \bigwedge^k(V), \mu \in V, \delta \in \bigwedge^{n-k+1}(V)$, and $\gamma \in \bigwedge^{n-k-3}(V)$. Then $(\alpha\mu \wedge \delta)(\alpha\mu\gamma) = (\alpha \wedge \mu\delta)(\alpha\mu\gamma)$.

PROOF. Since Δ is an algebra homomorphism, we have

$$\begin{aligned} \Delta(\alpha\mu) = \Delta\alpha\Delta\mu &= \left(\sum_\alpha \alpha_{(1)} \otimes \alpha_{(2)}\right)(\mu \otimes 1 + 1 \otimes \mu) \\ &= \sum_\alpha (\pm \alpha_{(1)}\mu \otimes \alpha_{(2)} + \alpha_{(1)} \otimes \alpha_{(2)}\mu). \end{aligned}$$

It follows that

$$\begin{aligned} \alpha\mu \wedge \delta &= \sum_{\alpha\mu} ((\alpha\mu)_1 \delta \mid 1 \cdots n)(\alpha\mu)_{(2)} \\ &= \sum_\alpha (\alpha_{(1)}\mu\delta \mid 1 \cdots n)\alpha_{(2)} + \sum_\alpha (\alpha_{(1)}\delta \mid 1 \cdots n)\alpha_{(2)}\mu \\ &= (\alpha \wedge \mu\delta) + (\alpha \wedge \delta)\mu. \end{aligned}$$

The lemma follows immediately since $\mu^2 = 0$.

To prove Example 8, we begin by noting that $C_k \sim 0$ at α means that $(\alpha \wedge \alpha\gamma)\delta = 0$ for all $\gamma \in \bigwedge^{n-k-2}(V), \delta \in \bigwedge^{n-k+2}(V)$. If α is decomposable, say $\alpha = \alpha_1 \cdots \alpha_k$ with $\alpha_i \in V$, then

$$\alpha \wedge \alpha\gamma = \sum_\alpha (\alpha_{(1)}\alpha\gamma \mid 1 \cdots n)\alpha_{(2)},$$

which is 0 since $\alpha_{(1)} = \alpha_i\alpha_j$ and, so, $\alpha_{(1)}\alpha = 0$.

We prove the converse by induction on k with $k = n-2, n-3, \ldots, 2$. For $k = n-2$, we need to show that α is decomposable if $\alpha \wedge \alpha = 0$. Applying ϕ, we see that it is enough to show that $\alpha \in \bigwedge^2(V)$ is decomposable if $\alpha^2 = 0$. According to Example 5, we need to show that

$$0 = \langle U, (a^{(2)} b \mid ijk)(b \mid h)\rangle(\alpha) = (*)$$

where $i < j < k$. But, a (short) calculation shows that the coefficient of $e_i e_j e_k e_h$ in α^2 is $2(*)$. Since $\operatorname{char} K \neq 2$, $(*) = 0$ and α is decomposable.

In general, let $\alpha \in \bigwedge^k(\nabla)$ and suppose that $C_k \sim 0$ at α. Let $\mu \in V$. Applying the lemma, we see that $C_{k+1} \sim 0$ at $\alpha\mu$ since for $\gamma \in \bigwedge^{n-k-3}(V), \delta \in \bigwedge^{n-k+1}(V)$, we have $(\alpha\mu \wedge \alpha\mu\gamma)\delta = (\alpha\mu \wedge \delta)(\alpha\mu\gamma) = (\alpha \wedge \mu\delta)(\alpha\mu\gamma) = 0$. By induction, $\alpha\mu$ is decomposable. We now apply Example 7.

EXAMPLE 9. Let a, b, c belong to $\bigwedge^3(V)$ and let $\alpha \in \bigwedge^3(V)$. Let C be the covariant defined by the diagram

$$C\colon \quad \begin{matrix} a^{(2)} b^{(3)} \\ a c^{(3)} \end{matrix}$$

Then α is divisible by a vector if and only if $C \sim 0$ at α.

PROOF. According to Example 3, $C \sim 0$ at α if and only if $(\alpha \wedge \alpha\gamma)(\alpha\delta) = 0$ for all $\gamma \in \bigwedge^{n-5}(V), \delta \in \bigwedge^{n-4}(V)$. Now, suppose that $\alpha = \mu\beta$ where $\mu \in V$, $\beta \in \bigwedge^2(V)$. Then

$$\Delta\alpha = \Delta\mu\Delta\beta = (\mu \otimes 1 + 1 \otimes \mu)\left(\beta \otimes 1 + 1 \otimes \beta + \sum \beta_{(1)} \otimes \beta_{(2)}\right)$$

where $\beta_{(1)}, \beta_{(2)}$ are all nonzero vectors. By definition,

$$\alpha \wedge \alpha\gamma = \sum_\alpha (\alpha_{(1)}\alpha\gamma \mid 1 \cdots n)\alpha_{(2)}.$$

But each $\alpha_{(1)}$ of step 2 is either β or of the form $\mu\beta_{(1)}$. In either case, $\alpha_{(1)}\alpha = 0$.

If $\alpha \wedge \alpha\gamma = 0$ for all $\gamma \in \bigwedge^{n-5}(V)$, then α is decomposable by Example 8. Otherwise, suppose that there is a γ^* with $\alpha \wedge \alpha\gamma^* \neq 0$. By assumption, $(\alpha \wedge \alpha\gamma^*)\alpha = 0$ so the vector $\alpha \wedge \alpha\gamma^*$ divides α.

EXAMPLE 10. Let $\alpha \in \bigwedge^3(V)$. Suppose that α is not decomposable but is divisible by a vector $\mu \in V$. Let a, b belong to $\bigwedge^3(V)$, and u belong to $\bigwedge^{n-5}(V)$.

There is a $\gamma^* \in \bigwedge^{n-5}(V)$ so that the vector with ith coordinate

$$\langle U, [a^{(3)}b^{(2)}u^{(n-5)}](b \mid i)\rangle(\alpha, \gamma^*)$$

is a nonzero scalar multiple of μ.

PROOF. This follows at once from the proof of Example 9. Indeed, $C \sim 0$ at α since μ divides α. Since α is not decomposable, it is divisible by a vector of the form $\alpha \wedge \alpha\gamma^*$.

EXAMPLE 11. Suppose $\dim V = 5$. If $\alpha \in \bigwedge^3(V)$, then α is divisible by a vector.

PROOF. A direct computation shows that $(\alpha \wedge \alpha)\alpha = 0$. We now apply Example 9.

EXAMPLE 12. Let a, b, c, d belong to $\bigwedge^{(4)}(V)$. Let C_1, C_2, C_3 be the covariants defined by the following diagrams:

$$C_1: \begin{array}{l} a^{(4)}b^{(2)} \\ \\ b^{(2)} \end{array} \qquad C_2: \begin{array}{l} a^{(4)}b^{(2)} \\ bc^{(4)} \\ b \end{array} \qquad C_3: \begin{array}{l} a^{(4)}b^{(2)} \\ bc^{(4)} \\ b\,d^{(4)} \end{array}$$

Let $\alpha \in \bigwedge^4(V)$.

(i) If $C_1 \sim 0$ at α, then α is decomposable.

(ii) If $C_1 \nsim 0$ at α but $C_2 \sim 0$ at α, then α is divisible by exactly two linearly independent vectors.

(iii) If $C_1 \nsim 0$ at α and $C_2 \nsim 0$ at α, but $C_3 \sim 0$ at α then α is divisible by exactly one nonzero vector.

PROOF. According to Example 8, $C_1 \sim 0$ at α if and only if α is decomposable. If $C_1 \nsim 0$ at α, then there is a $\gamma^* \in \bigwedge^{n-6}(V)$ so that $\alpha \wedge \alpha\gamma^* \neq 0$. Now, $\alpha \wedge \alpha\gamma^* \in \bigwedge^2(V)$ and has rank ≥ 2. If $C_2 \sim 0$ at α, then $((\alpha \wedge \alpha\gamma^*) \wedge \delta')\alpha\delta = 0$ for all $\delta \in \bigwedge^{n-5}(V), \delta' \in \bigwedge^{n-1}(V)$. Thus, α is divisible by each vector in the span of $\alpha \wedge \alpha\gamma^*$. Since α is not decomposable, it is not divisible by three linearly independent vectors. Hence, rank $(\alpha \wedge \alpha\gamma^*) = 2$ and (ii) is proved.

We note that $C_2 \sim 0$ at α means that $\alpha \wedge \alpha\gamma_1 \wedge \alpha\gamma_2 = 0$ for all $\gamma_1 \in \bigwedge^{n-6}(V)$, $\gamma_2 \in \bigwedge^{n-5}(V)$. Now, suppose that α is divisible by two linearly independent vectors, say $\alpha = \mu_1\mu_2\beta, \mu_i \in V, \beta \in \bigwedge^2(V)$. Then for $\gamma_1 \in \bigwedge^{n-6}(V)$ we have

$$\alpha \wedge \alpha\gamma_1 = \sum(\alpha_{(1)}\alpha\gamma_1 \mid 1 \cdots n)\alpha_{(2)} = 0$$

since either μ_1 or μ_2 or β appears in $\alpha_{(1)}$. So, $C_2 \sim 0$ at α.

Finally, suppose that $C_1 \nsim 0$ at $\alpha, C_2 \nsim 0$ at α, but $C_3 \sim 0$ at α. Then, there are elements $\gamma_1^* \in \bigwedge^{n-6}(V), \gamma_2^* \in \bigwedge^{n-5}(V)$ so that the vector $\mu = \alpha \wedge \alpha\gamma_1^* \wedge \alpha\gamma_2^* \neq 0$. Since $C_3 \sim 0$ at α, we have $0 = (\alpha \wedge \alpha\gamma_1^* \wedge \alpha\gamma_2^*)\alpha\delta$ for all $\delta \in \bigwedge^{n-5}(\nabla)$, so μ divides α.

EXAMPLE 13. Let $a, b, c, d, e, \ldots$ belong to $\bigwedge^k(V)$. let $C_1, C_2, \ldots, C_{k-1}$ be the covariants defined by the diagrams

$$C_1: \begin{array}{l} a^{(k)}b^{(2)} \\ b^{(k-2)} \end{array}$$

$$C_2: \begin{array}{l} a^{(k)}b^{(2)} \\ bc^{(k)} \\ b^{(k-3)} \end{array}$$

$$C_3: \begin{array}{l} a^{(k)}b^{(2)} \\ bc^{(k)} \\ bd^{(k)} \\ b^{(k-4)} \end{array}$$

$$\vdots$$

$$C_{k-2}: \begin{array}{l} a^{(k)}b^{(2)} \\ bc^{(k)} \\ bd^{(k)} \\ \vdots \\ b \end{array}$$

$$\vdots$$

$$C_{k-1}: \begin{array}{l} a^{(k)}b^{(2)} \\ bc^{(k)} \\ bd^{(k)} \\ \vdots \\ be^{(k)} \end{array}$$

Let $\alpha \in \bigwedge^k(V)$.

(i) If $C_1 \sim 0$ at α, then α is decomposable.

(ii) If $C_1 \nsim 0$ at α but $C_2 \sim 0$ at α, then α is divisible by exactly $k-2$ linearly independent vectors.

(iii) If $C_1 \nsim 0$ at α, $C_2 \nsim 0$ at α, but $C_3 \sim 0$ at α, then α is divisible by exactly $k-3$ linearly independent vectors.

PROOF. This is similar to that given when $k = 4$ so we omit it.

EXAMPLE 14. Let $\alpha \in \bigwedge^2(V)$. Then rank α is even, say, rank $\alpha = 2r$. Furthermore, there is a basis $\{\beta_1, \ldots, \beta_{2r}\}$ of span α so that $\alpha = \beta_1\beta_2 + \cdots + \beta_{2r-1}\beta_{2r}$.

We begin by proving the following result, which will also be useful later.

LEMMA. *Let $\alpha \in \bigwedge^k(V), k \geq 2$. There exist $\mu \in V$ and $\delta \in \bigwedge^{n-1}(V)$ so that the following conditions hold:*

(i) $\alpha\mu \wedge \delta = (-1)^k\alpha + (\alpha \wedge \delta)\mu$;

(ii) $W = \operatorname{span}(\alpha\mu \wedge \delta) + \operatorname{span}(\alpha \wedge \delta) \subset \operatorname{span}\alpha$;

(iii) $\mu \notin W$;

(iv) $\dim W = \operatorname{rank}\alpha - 1$;

(v) *if α is not decomposable and if $k \geq 3$, then δ may be chosen so that $\alpha \wedge \delta$ is not decomposable.*

PROOF. Let $\delta \in \bigwedge^{n-1}(V)$ be chosen so that $\alpha \wedge \delta \neq 0$. If α is not decomposable, we may assume (by Example 7) that $\alpha \wedge \delta$ is not decomposable. Let $\gamma^* \in \bigwedge^{n-k+1}(V)$ be chosen so that

$$(-1)^{k-1}(\alpha \wedge \delta)\gamma^* = (\alpha \wedge \gamma^*)\delta = 1.$$

Let $\mu = \alpha \wedge \gamma^*$. Statement (i) follows as in the proof of the Lemma, Example 8.

We have $\operatorname{span}(\alpha \wedge \delta) \subset \operatorname{span}\alpha$ by (M5). To show that $\operatorname{span}(\alpha\mu \wedge \delta) \subset \operatorname{span}\alpha$, let $\gamma \in \bigwedge^{n-k+1}(V)$. Then, using the Lemma, Example 8, we have

$$\begin{aligned}(\alpha\mu \wedge \delta) \wedge \gamma &= (-1)^k\alpha \wedge \gamma + ((\alpha \wedge \delta)\mu) \wedge \gamma \\ &= (-1)^k\alpha \wedge \gamma - (\alpha \wedge \delta \wedge \mu\gamma) + (\alpha \wedge \delta \wedge \gamma)\mu.\end{aligned}$$

Since $\alpha \wedge \delta \wedge \gamma$ is a scalar and $\mu = \alpha \wedge \gamma^*$, we may conclude that $\operatorname{span}(\alpha\mu \wedge \delta) \subset \operatorname{span}\alpha$. This proves (ii). Statement (iv) will follow from the equation above for $(\alpha\mu \wedge \delta) \wedge \gamma$ once (iii) is proven.

By definition, $\mu = \alpha \wedge \gamma^* \in \operatorname{span}\alpha$ and $\mu\delta = 1$. However, for any $\gamma \in \bigwedge^{n-k+1}(V)$, we have

$$(\alpha\mu \wedge \delta \wedge \gamma)\delta = \pm((\alpha\mu \wedge \delta) \wedge \delta)\gamma = \pm(\alpha\mu \wedge \delta \wedge \delta)\gamma = 0$$

according to (M9). Similarly, for any $\gamma \in \bigwedge^{n-k+2}(V)$, we have

$$(\alpha \wedge \delta \wedge \gamma)\delta = \pm(\alpha \wedge \delta \wedge \delta)\gamma = 0.$$

PROOF OF EXAMPLE 14. We proceed by induction on rank α. If rank $\alpha = 2$, then α is decomposable. Otherwise, let us choose $\delta \in \bigwedge^{n-1}(V)$ and $\gamma^* \in \bigwedge^{n-1}(V)$ as in the proof of the lemma above. As before, let $\mu = \alpha \wedge \gamma^*$. Then $\beta = \alpha\mu \wedge \delta$ is in $\bigwedge^2(V)$ but rank $\beta <$ rank α by (iv). By induction, $\operatorname{rank}\beta = 2r - 2$.

If μ_1 and μ_2 are any elements in V and if δ' is any element in $\bigwedge^{n-1}(V)$, then (by the definition of the meet) we have

$$\mu_1\mu_2 \wedge \delta' = (\mu_1\delta')\mu_2 - (\mu_2\delta')\mu_1.$$

Hence,

$$\begin{aligned}\beta \wedge \gamma^* &= \alpha \wedge \gamma^* + ((\alpha \wedge \delta)\mu) \wedge \gamma^* \\ &= \alpha \wedge \gamma^* + ((\alpha \wedge \delta)\gamma^*)\mu - (\mu\gamma^*)(\alpha \wedge \delta) \\ &= \mu - \mu - (\mu\gamma^*)(\alpha \wedge \delta).\end{aligned}$$

Since γ^* is in $\bigwedge^{n-1}(V)$, it is decomposable and $\mu\gamma^* = (\alpha \wedge \gamma^*)\gamma^* = 0$. Hence, $\beta \wedge \gamma^* = 0$. Then, for any $\delta' \in \bigwedge^{n-1}(V)$, we have $(\beta \wedge \delta')\gamma^* = -(\beta \wedge \gamma^*)\delta' = 0$. Since $(\alpha \wedge \delta)\gamma^* = -1$, it follows that $\alpha \wedge \delta$ is not in $\operatorname{span}\beta$. Applying statements (ii), (iii), and (iv) in the lemma, we see that rank $\alpha = 2r$. We now apply the induction hypothesis to the equation $\alpha = \beta + (\alpha \wedge \gamma^*)(\alpha \wedge \delta)$.

Note. The proof above gives something more which will be used later. Namely, let $\alpha \in \bigwedge^2(V)$ with rank $\alpha = 2r$. Let $\beta_1 = \alpha \wedge \delta_1$ and $\beta_2 = \alpha \wedge \delta_2$ be in span α. If $(\alpha \wedge \delta_1)\delta_2 \neq 0$, then there is a scalar $c \in K$ and vectors $\beta_3, \ldots, \beta_{2r}$ in span α so that $\alpha = c\beta_1\beta_2 + \cdots + \beta_{2r-1}\beta_{2r}$. (To see this, just take $\delta_1 = \delta$ and $\gamma^* = -c\delta_2$.)

EXAMPLE 15. Let $\alpha \in \bigwedge^k(V)$ and $W = \operatorname{span}\alpha$. Then $\alpha \in \bigwedge^k(W)$.

PROOF. We proceed by induction on step α and, for fixed step, by induction on rank α. If $k = 1$, the statement is trivial and if $k = 2$, the result follows from Example 14. So, let $k \geq 3$. If rank $\alpha = k$, then α is decomposable with, say, $\alpha = \beta_1 \cdots \beta_k$. Since span α has basis $\{\beta_1, \ldots, \beta_k\}$, our example holds.

We may now assume $\alpha \in \bigwedge^k(V)$ and rank $\alpha > k$. We apply the Lemma, Example 14, to see that we may write $\alpha\mu \wedge \delta = (-1)^k\alpha + (\alpha \wedge \delta)\mu$. By induction, the example holds for $\alpha\mu \wedge \delta$ (since $\operatorname{rank}(\alpha\mu \wedge \delta) < \operatorname{rank}\alpha$) and $\alpha \wedge \delta$ (since $\alpha \wedge \delta \in \bigwedge^{k-1}(V)$). Now, $\mu \in \operatorname{span}\alpha$ and $\operatorname{span}(\alpha\mu \wedge \delta) + \operatorname{span}(\alpha \wedge \delta) \subset \operatorname{span}\alpha$. Hence, $\alpha \in \bigwedge^k(\operatorname{span}\alpha)$.

EXAMPLE 16. We shall determine canonical forms for elements of $\bigwedge^3(V)$ when $\dim V = 3, 4, 5$, and 6. If $\alpha \in \bigwedge^3(V)$ and $\dim V = 3$ or 4, then $\operatorname{rank}\alpha = 3$ and α is decomposable. If $\dim V = 5$, then α is divisible by a vector (Example 11). Thus, we may assume that $\dim V = 6$.

It will simplify our notation if we abandon the Greek alphabet and, instead, denote elements in $\bigwedge^k(V)$ by symbols $a^{(k)}$. If $k = 1$, we write simply $b, c, d, \ldots$. We shall prove that the following four canonical forms suffice.

I. $a^{(3)} = bcd$.

II. $a^{(3)} = bcd + bef$.

III. $a^{(3)} = bcd + bfg + cef$.

IV. $a^{(3)} = bcd + efg$.

If rank $a^{(3)} = 3$, then $a^{(3)}$ has form I. Hence, we may assume that rank $a^{(3)} = 5$ or 6. If rank $\alpha = 5$, then α has form I or II (by Example 15). So, we may assume rank $\alpha = 6$. According to the previous lemma, we may write

$$a^{(3)} = bv^{(2)} + c^{(3)},$$

where $W = \operatorname{span} c^{(3)} + \operatorname{span} v^{(2)} \subset \operatorname{span} a^{(3)}$, $b \notin W$, $\dim W = \operatorname{rank} a^{(3)} - 1$, and $v^{(2)}$ is not decomposable. Since $v^{(2)}$ is not decomposable, rank $v^{(2)} = 4$ by Example 14.

LEMMA 1. *Let $v^{(2)} \in \bigwedge^2(V)$ have* rank 4. *Let v and w be any linearly independent vectors in* span $v^{(2)}$. *Then there are vectors b, c, so that either*

(i) *$v^{(2)} = c_1vw + bc$ for some $c_1 \in K$ or*

(ii) *$v^{(2)} = vb + cw$.*

PROOF. We may assume that $\dim V = 4$. Let $v = v^{(2)} \wedge d_1^{(3)}$ and $w = v^{(2)} \wedge d_2^{(3)}$ where $d_1^{(3)}$ and $d_2^{(3)}$ are linearly independent elements in $\bigwedge^3(V)$. If $vd_2^{(3)} \neq 0$, then we obtain (i) from the note following Example 14. So, let us assume that $vd_2^{(3)} = 0$. Let $d_3^{(3)} \in \bigwedge^3(V)$ be chosen so that $vd_3^{(3)} = 0$ and $wd_3^{(3)} = 1$. Then, again by the note following Example 14, the tensor $w^{(2)} = v^{(2)} - w(v^{(2)} \wedge d_3^{(3)})$ has rank 2. To obtain case (ii), we need only prove that v is in span $w^{(2)}$. But

$$\begin{aligned} w^{(2)} \wedge d_1^{(3)} &= v^{(2)} \wedge d_1^{(3)} - (w(v^{(2)} \wedge d_3^{(3)})) \wedge d_1^{(3)} \\ &= v^{(2)} \wedge d_1^{(3)} - (wd_1^{(3)})(v^{(2)} \wedge d_3^{(3)}) + ((v^{(2)} \wedge d_3^{(3)})\, d_1^{(3)})w \end{aligned}$$

which is $v^{(2)} \wedge d_1^{(3)} = v$ since

$$wd_1^{(3)} = (v^{(2)} \wedge d_2^{(3)})\, d_1^{(3)} = -(v^{(2)} \wedge d_1^{(3)})\, d_2^{(3)} = 0$$

and

$$(v^{(2)} \wedge d_3^{(3)})\, d_1^{(3)} = -(v^{(2)} \wedge d_1^{(3)})\, d_3^{(3)} = -vd_3^{(3)} = 0.$$

Case 1. $c^{(3)}$ is decomposable.

Since rank $v^{(2)} = 4$ and rank $c^{(3)} = 3$, we see that $\dim(\operatorname{span} v^{(2)} \cap \operatorname{span} c^{(3)}) = 2$. Let v_1, v_2 be a basis of this intersection. There is a vector $d \in V$ so that $c^{(3)} = dv_1v_2$. Furthermore (by Lemma 1), we may find $v_3, v_4 \in \operatorname{span} v^{(2)}$ so that either $v^{(2)} = v_1v_2 + v_3v_4$ or $v^{(2)} = v_1v_3 + v_2v_4$. In the first case, $a^{(3)}$ has the form IV and in the second case, form III.

Case 2. $c^{(3)}$ is not decomposable.

From the dimension identity

$$\begin{aligned} &\dim \operatorname{span} c^{(3)} + \dim \operatorname{span} v^{(2)} \\ &= \dim((\operatorname{span} c^{(3)}) \wedge (\operatorname{span} v^{(2)})) + \dim((\operatorname{span} c^{(3)}) \vee (\operatorname{span} v^{(2)})) \end{aligned}$$

(where meets and joins here are meant in the lattice sense, as meets and joins of subspaces of a vector space) we infer

$$5 + 4 = \dim((\operatorname{span} c^{(3)}) \wedge (\operatorname{span} v^{(2)})) + 5;$$

hence $\dim((\operatorname{span} c^{(3)}) \wedge (\operatorname{span} v^{(2)})) = 4$. This implies that $\operatorname{span} v^{(2)} \subseteq \operatorname{span} c^{(3)}$.

Since $c^{(3)}$ is of rank 5, it is divisible by a vector, say, $c^{(3)} = dw^{(2)}$, where $w^{(2)}$ is a bivector of rank 4, which can be written as the sum of two decomposable bivectors $w^{(2)} = w_1^{(2)} + w_2^{(2)}$. Again from the dimension identity

$$4 + 3 = \dim((\operatorname{span} v^{(2)}) \vee (\operatorname{span}(dw_i^{(2)})) + \dim((\operatorname{span} v^{(2)}) \wedge (\operatorname{span}(dw_i^{(2)})),$$

we infer that $\dim(\operatorname{span}(v^{(2)}) \vee \operatorname{span}(dw_i^{(2)})) = 5$ either for $i = 1$ or for $i = 2$; for otherwise, if $\dim(\operatorname{span}(v^{(2)}) \vee (\operatorname{span} dw_i^{(2)})) = 4$ for $i = 1$ and $i = 2$, it would follow that $\dim(\operatorname{span}(v^{(2)}) \wedge \operatorname{span}(dw_i^{(2)}) = 3$ for $i = 1$ and $i = 2$. This would in turn entail that span $(dw_i^{(2)}) \subseteq \operatorname{span}(v^{(2)})$ for $i = 1$ and $i = 2$, whence $\operatorname{span}(dw^{(2)}) \subseteq \operatorname{span}(v^{(2)})$, and $\operatorname{span}(c^{(3)}) \subseteq \operatorname{span}(dw^{(2)}) \vee \operatorname{span}(v^{(2)}) =$

$\operatorname{span}(v^{(2)})$. Thus, rank $c^{(3)} \leq$ rank $v^{(2)} = 4$, contradicting the assumption that rank $c^{(3)} = 5$. We therefore may and will assume henceforth that

$$\dim(\operatorname{span}(v^{(2)}) \vee \operatorname{span}(dw_1^{(2)})) = 5,$$

and, as a consequence, that

$$\dim((\operatorname{span}(v^{(2)}) \wedge \operatorname{span}(dw_1^{(2)})) = 2.$$

There are two subcases.

Case 2.1. $\dim((\operatorname{span} v^{(2)}) \wedge (\operatorname{span} dw_2^{(2)})) = 3$.

We then have that $\operatorname{span}(dw_2^{(2)}) \subseteq \operatorname{span}(v^{(2)})$, and hence $d \in \operatorname{span}(v^{(2)})$. Choose a basis d, f_1, f_2, f_3, f_4 of the subspace $\operatorname{span}(v^{(2)}) \vee \operatorname{span}(dw^{(2)})$ such that:

(a) d, f_1 span $\operatorname{span}(v^{(2)}) \wedge \operatorname{span}(dw_1^{(2)})$,
(b) d, f_2, f_3 span $dw_2^{(2)}$,
(c) d, f_1, f_4 span $dw_1^{(2)}$
(d) d, f_1, f_2, f_3 span $\operatorname{span} v^{(2)}$.

There are now two possibilities: either

Case 2.1.1. $v^{(2)} = k_1\, df_1 + k_2 f_2 f_3$, for some scalars k_1 and k_2, or else

Case 2.1.2. $v^{(2)} = k_1\, df_2 + k_2 f_1 f_3$.

In case 2.1.1 we find

$$a^{(3)} = (k_1 b + f_4)\, df_1 + (k_2 b + d) f_2 f_3,$$

which gives canonical form IV, and in case 2.1.2 we find

$$a^{(3)} = (k_1 b + f_3)\, df_2 + k_2 b f_1 f_3 + df_1 f_4,$$

whence, setting $f_5 = k_1 b + k_2 f_3$,

$$a^{(3)} = f_5\, df_2 + b f_1 f_5 + d_1 f_1 f_4,$$

which gives canonical form III.

Case 2.2. $\dim(\operatorname{span}(v^{(2)}) \wedge \operatorname{span}(dw_2^{(2)}) = 2$.

Again we have two sub-subcases.

Case 2.2.1. $d \notin \operatorname{span}(v^{(2)})$.

We can then choose f_1, f_2 in $\operatorname{span}(v^{(2)})$ so that $dw_1^{(2)} = df_1 f_2$ and f_3, f_4 in $\operatorname{span}(v^{(2)})$ so that $dw_2^{(2)} = df_3 f_4$. The vectors f_1, f_2, f_3, f_4 are linearly independent, hence they span $\operatorname{span}(v^{(2)})$. It follows that $\operatorname{span}(v^{(2)}) = \operatorname{span}(w^{(2)})$. We can therefore find a basis g_1, g_2, g_3, g_4 of $\operatorname{span}(v^{(2)})$ such that, for suitable scalars k_i:

$$\begin{aligned} v^{(2)} &= k_1 g_1 g_2 + k_2 g_3 g_4, \\ w^{(2)} &= k_3 g_1 g_2 + k_4 g_3 g_4, \end{aligned}$$

so that

$$a^{(3)} = (k_1 b + k_3 d) g_1 g_2 + (k_2 b + k_4 d) g_3 g_4,$$

which is canonical form IV.

Case 2.2.2. $d \in \operatorname{span}(V^{(2)})$.

Then, for suitable g_i, $v^{(2)} = dg_1 + g_2g_3$, and $a^{(3)} = bg_2g_3 + d(w^{(2)} + bg_1)$, which is, after a change of lettering, case 1. The proof is therefore complete.

We can now proceed to express the preceding results in terms of the vanishing of covariants. Consider the covariants

$$C_1: \begin{array}{l} a^{(3)}b^{(2)} \\ b \end{array}$$

$$C_2: \begin{array}{l} a^{(3)}b^{(2)}c \\ bc^{(2)} \end{array}$$

$$C_3: \begin{array}{l} a^{(3)}b^{(2)}c \\ bc^{(2)}\,d^{(3)} \end{array}$$

when all symbols are equivalent. We again assume that $n = 6$. Clearly the vanishing of C_i implies the vanishing of C_{i+1}. If $C_3 \nsim 0$ then the tensor $a^{(3)}$ has the canonical form IV. One verifies that if $a^{(3)}$ has the canonical form III then $C_3 \sim 0$. Conversely, suppose that $C_3 \sim 0$. From our previous classification, we infer that the tensor must have one of the canonical forms I, II, or III. Similarly, $C_2 \sim 0$ if and only if the tensor has one of the canonical forces I or II, and $C_1 \sim 0$ if and only if the tensor has the canonical form I; that is, if and only if it is decomposable.

Then we see that each of the canonical forms for a tensor of step 3 in dimension 6 can be characterized by the vanishing or nonvanishing of a covariant.

Consider now the following covariants:

$$C_4: \begin{array}{l} a^{(3)}b^{(2)}c^{(2)} \\ b \\ c \end{array}$$

$$C_5: \begin{array}{l} a^{(3)}b^{(2)}c^{(2)} \\ b\,d^{(3)} \\ c \end{array}$$

$$C_6: \begin{array}{l} a^{(3)}b^{(2)}c^{(2)} \\ b\,d^{(3)}e^{(2)} \\ ce \end{array}$$

$$C_7: \begin{array}{l} a^{(3)}b^{(2)}c^{(2)} \\ b\,d^{(3)}e^{(2)}f \\ cef^{(2)} \end{array}$$

$$C_8: \begin{array}{l} a^{(3)}b^{(2)}c^{(2)} \\ b\,d^{(3)}e^{(2)}f \\ cef^{(2)}g^{(3)} \end{array}$$

We state without proof the classification of tensors of step 3 in dimension 7. To the previously found canonical forms we now add the following canonical forms:

V. $aqp + brp + csp$.

VI. $qrs + aqp + brp + csp$.

VII. $abc + qrs + aqp$.

VIII. $abc + qrs + aqp + brp$.

IX. $abc + qrs + aqp + brp + csp$.

Here all symbols are of step 1 and all symbols are equivalent.

Again, one verifies directly that the most general tensor of step 3 in dimension 7 is of the form IX. If $C_8 \sim 0$ but $C_7 \nsim 0$ then the canonical form is VIII; if $C_7 \sim 0$ but $C_6 \nsim 0$ then the canonical form is VII, etc. The proof that these canonical forms are the only ones will not be given here.

For a tensor of type IX, the value of the covariant C_8 is the single term

$$\begin{array}{c} (aqp)(bc)(rs) \\ a(qrs)(bc)p \\ qa(br)(csp). \end{array}$$

For a vector of step 3 in dimension 8 we have the following canonical forms:

X. $aqp + brp + csp + crt$.

XI. $qrs + aqp + brp + csp + crt$.

XII. $abc + qrs + aqp + crt$.

XIII. $abc + qrs + aqp + brp + crt$.

XIV. $abc + qrs + aqp + brp + csp + crt$.

XV. $aqp + bst + crt$.

XVI. $aqp + brp + bst + crt$.

XVII. $qrs + aqp + brp + bst + crt$.

XVIII. $aqp + brp + csp + bst + crt$.

XIX. $qrs + aqp + brp + csp + bst + crt$.

XX. $abc + qrs + aqp + bst + crt$.

XXI. $abc + qrs + aqp + brp + bst + crt$.

XXII. $abc + qrs + aqp + brp + csp + bst + crt$.

The covariants corresponding to each of these canonical forms will be described elsewhere.

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