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NON-ASSOCIATIVE NORMED ALGEBRAS

Volume 1: The Vidav–Palmer and Gelfand–Naimark Theorems

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NON-ASSOCIATIVE NORMED ALGEBRAS

Volume 1: The Vidav–Palmer and Gelfand–Naimark Theorems

This first systematic account of the basic theory of normed algebras, without assuming associativity, includes many new and unpublished results and is sure to become a central resource for researchers and graduate students in the field.

This first volume focuses on the non-associative generalizations of (associative) C*-algebras provided by the so-called non-associative Gelfand–Naimark and Vidav–Palmer theorems, which give rise to alternative C*-algebras and non-commutative JB*-algebras, respectively. The relationship between non-commutative JB*-algebras and JB*-triples is also fully dicussed. The second volume covers Zel'manov's celebrated work in Jordan theory to derive classification theorems for non-commutative JB*-algebras and JB*-triples, as well as other topics.

The book interweaves pure algebra, geometry of normed spaces, and complex analysis, and includes a wealth of historical comments, background material, examples, and exercises. The authors also provide an extensive bibliography.

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To Ana María and Inés

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Preface

Reviewing the non-associative part of a monograph by Irving Kaplansky

In 1970, Irving Kaplansky published his small monograph [762] on *Algebraic and analytic aspects of operator algebras*, and devoted its last section to providing the reader with his impressions concerning non-associative normed algebras. Actually, he began the section by saying:

> I predict that when the time is ripe there is going to be quite a flurry of activity concerning nonassociative Banach algebras in general, and nonassociative C^*-algebras in particular. Let me take the space to speculate a little on what we may see some day.

Many years have passed since the publication of [762] and, as a matter of fact, most of Kaplansky's predictions have come true, some even exceeding the original expectations. The diverse results corroborating the accuracy of Kaplansky's predictions will be used to illustrate the content of the book we are introducing. Therefore, let us continue reproducing Kaplansky's words in short excerpts, and insert some clarifying comments.

> The speculation can start encouragingly, with a fact. The (complex) Gelfand–Mazur Theorem works fine: a normed division algebra must be the complex numbers. Of course, we must agree on what a division algebra A is to be. We take it to mean that for any nonzero x, both R_x and L_x are one-to-one and onto, where R_x (L_x) denotes right (left) multiplication by x. (Actually, for the proof all we need is R_x.)

With the words 'for the proof all we need is R_x', Kaplansky is suggesting the notion of a right-division algebra.

> Suppose then that A is a normed division algebra. We claim that A is one-dimensional, and by way of contradiction we assume that x and y are linearly independent. Then for every complex scalar λ, $R_{x-\lambda y} = R_x - \lambda R_y$ is a bounded operator on A which is one-to-one and onto. The same is true of $R_x R_y^{-1} - \lambda I$. But $R_x R_y^{-1}$ must have something in its spectrum.

It seems obvious to us that Kaplansky implicitly assumes that the (possibly non-associative) normed algebra A above is complete, to be sure that 'the same is true of $R_x R_y^{-1} - \lambda I$' concerning boundedness. Under this assumption, Kaplansky's claim (that complete normed one-sided division complex algebras are isomorphic to $\mathbb{C}$) is stated in Corollary 2.7.3, whereas the non-complete case is raised as Problem 2.7.4 (see also Theorem 4.1.63 for a partial affirmative answer). Actually, a result better than Kaplansky's claim holds. Indeed, if A is a complete normed complex algebra, and if it is a quasi-division algebra (which means that, for every nonzero $x \in A$, at least one of the operators L_x, R_x is one-to-one and onto), then $\dim(A) \leqslant 2$ (Theorem 2.7.7) and, in general, no more can be said (Example 2.5.36). Moreover, if in addition A is unital or nearly associative, then A is isomorphic to $\mathbb{C}$ (Corollaries 2.7.9 and 2.7.10). It is also worth mentioning that, for associative (and even alternative) algebras, the notions of division, one-sided division, and quasi-division coincide, and also coincide with the classical notion of a division algebra in this setting – namely that the algebra is unital and each nonzero element has an inverse (Proposition 2.5.38).

Let us continue with Kaplansky's words:

> On the other hand, the real Gelfand–Mazur theorem does not seem to have received any attention. The conjecture is that any real normed division algebra is finite-dimensional, after which the topologists would teach us that the dimension is 1, 2, 4, or 8.

The conjecture that normed division real algebras are finite-dimensional was first formulated by Wright [640], after proving it in the particular case of absolute-valued algebras (Corollary 2.6.24). The general case of the conjecture, even with the additional requirement of completeness, remains open (Problem 2.7.45). Nevertheless, normed one-sided division real algebras need not be finite-dimensional, even if they are absolute-valued and complete (Theorem 2.7.38). The enormous theorem of 'the topologists' (that finite-dimensional division real algebras are of dimension 1, 2, 4, or 8) is stated without proof in Theorem 2.6.51, referring the reader to the whole of Chapter 11 of [727] for a complete proof. The particularization to absolute-valued algebras is much more elementary, and is stated in Fact 2.6.50.

Kaplansky continues as follows:

> There is a related circle of ideas which has received a good deal of attention. In [337], Inglestam proved the following pretty theorem: if an associative real Banach algebra A is built on a Hilbert space and has a unit of norm 1, then in the first place A is finite-dimensional; moreover A must be the reals, complexes, or quaternions in their ordinary norm. An alternative account was given by Smiley [590]. The crucial point here is the behaviour of the unit element relative to convexity, and nonassociative generalizations have been given by Strzelecki [605, 606] and Inglestam [338]. (One should note the difference between this problem and that of Urbanik and Wright [620], who do not assume a Hilbert space but deduce it from the equality $\|xy\| = \|x\|\,\|y\|$.)

Ingelstam's theorem (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras whose norm comes from an inner product) is stated in Corollary 2.6.22. Here $\mathbb{H}$ and $\mathbb{O}$ stand for the algebra of Hamilton's quaternions and the algebra of Cayley numbers, respectively. Strzelecki's generalization (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras whose closed unit ball has a unique tangent hyperplane at the unit) is stated as the equivalence (ii)$\Leftrightarrow$(iii) in Theorem 2.6.21.

In our opinion, the non-commutative Urbanik–Wright theorem (that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique unital absolute-valued real algebras) becomes one of the jewels of the theory of non-associative normed algebras. Therefore, we devote special attention to it. We state it as the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.6.21, and provide a second proof in §2.7.67. The non-commutative Urbanik–Wright theorem had also been predicted by Kaplansky in [377], who, by means of a general theorem on the so-called unital composition algebras, reduced the proof to the case that the norm comes from an inner product. The two proofs we have given of the non-commutative Urbanik–Wright theorem follow Kaplansky's indication. We also state the commutative Urbanik–Wright theorem, proved in [620] as well, that $\mathbb{R}$, $\mathbb{C}$, and another natural two-dimensional algebra, are the unique absolute-valued commutative real algebras (Theorem 2.6.41).

> Now let us go beyond Gelfand–Mazur considerations to general theory. I divide the remarks under three headings, corresponding to the three classes of nonassociative algebras that have withstood the test of time.
>
> (1) *Lie algebras*. Let me be honest. I have nothing to say, even by way of the wildest speculation, about a possible theory of Banach Lie algebras.

Banach Lie algebras will be discussed in our book only in an incidental way. Nevertheless, let us say that a complete structure theory for Lie H^*-algebras has been developed (cf. Remark 2.6.54 for references), and that different aspects of general or particular Banach Lie algebras have been considered in [47, 96, 97, 98, 99, 130, 175, 188, 253, 254, 452, 453, 569, 604, 615, 627]. For a comprehensive account, the reader is referred to the books [687, 688, 740].

> (2) *Alternative*. Alternative rings are only a slight generalization of associative rings. It is therefore a reasonable presumption that most standard associative results will survive, perhaps in a suitably altered form, and perhaps with a lot of extra proof.

The 'reasonable presumption' above is indeed correct. As a relevant sample, Kaplansky's celebrated theorem [375], that normed associative real algebras with no nonzero topological divisor of zero are isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, has its 'suitably altered form' for alternative algebras (Theorem 2.5.50).

> Alternative C^*-algebras look like a plausible topic. Over the complex numbers the essentially new algebra – the Cayley matrix algebra – has every right to be called a C^*-algebra. But the subject should be developed in real style, so as to allow the Cayley division algebra to survive.

We feel that Kaplansky assumes the current Gelfand–Naimark characterization of closed $*$-invariant subalgebras of operators on complex Hilbert spaces and that consequently, by an 'alternative C^*-algebra', he means a complete normed alternative complex algebra A endowed with a conjugate-linear algebra involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$. If this is so, then the complex Cayley matrix algebra $C(\mathbb{C})$ is indeed an alternative C^*-algebra (Proposition 2.6.8). Moreover, $C(\mathbb{C})$ becomes the unique 'essentially new' alternative C^*-algebra. Indeed, $C(\mathbb{C})$ is the unique prime alternative C^*-algebra which is not associative and, by a standard C^*-argument, the theory of general alternative C^*-algebras reduces to the cases of prime associative C^*-algebras and that of $C(\mathbb{C})$ [125, 481]. It is also worth mentioning that, in the case of unital algebras, there are relevant redundancies in the definition of an alternative C^*-algebra suggested above. The redundances are so severe that, as we prove in Theorem 3.2.5, the alternative identities $x^2y = x(xy)$ and $yx^2 = (yx)x$ follow from the remaining requirements (see also Theorem 3.5.53 for a non-unital variant).

As far as we know, a systematic treatment of real alternative C^*-algebras has not been done to date. Anyway, real alternative C^*-algebras can be introduced as closed $*$-invariant real subalgebras of complex ones (see Definition 4.2.45). With this convention, clearly, the algebra $\mathbb{O}$ of Cayley numbers 'survives'.

> Alternative AW^*-algebras probably have a decisive structure theory. (I feel sure that subject will be developed in AW^* rather than in W^* style – unless somebody cares to work up the theory of Cayley Hilbert space.) There ought to be a unique direct sum decomposition into four homogeneous pieces: (a) real Cayley matrix, (b) Cayley division, (c) complex Cayley matrix, (d) associative.

As a matter of fact, concerning 'AW^* style', no progress has been made on the above prediction, even in the complex case. However, the 'W^* style' has become extremely successful. Indeed, as shown by G. Horn [331], every (complex) alternative W^*-algebra has a unique direct sum decomposition into two pieces: (a) the algebra of all continuous functions from a suitable compact Hausdorff hyper-Stonean space to $C(\mathbb{C})$; (b) an associative von Neumann algebra. It can be derived from Horn's theorem that every real alternative W^*-algebra has 'a unique direct sum decomposition into four pieces': (a) the algebra of all continuous functions from a suitable compact Hausdorff hyper-Stonean space to the real Cayley matrix algebra $C(\mathbb{R})$; (b) the same with $\mathbb{O}$ instead of $C(\mathbb{R})$; (c) the same with $C(\mathbb{C})$ instead of $C(\mathbb{R})$; (d) a real associative von Neumann algebra. Thus, 'in W^* style', Kaplansky's prediction is right.

> (3) *Jordan*. In saying that Jordan Banach algebras ought to be studied we have the blessing of the Master himself [461]. In recent years Topping [813, 614], Effros and Størmer [228] and Størmer [601, 602, 603] have made significant progress. In one way, however, they made an undesirable retreat from [461]. By assuming from the start that they were dealing with operators on a Hilbert space, they ruled out the exceptional Jordan algebra, and left to the future the possibility of a theorem asserting that suitable infinite-dimensional Jordan Banach algebras are special. Admittedly I am prejudiced, but perhaps the AW^* point of view is a good one here.

Eight years after the above paragraph was written, Alfsen, Shultz, and Størmer [15] removed the 'undesirable retreat from [461]' by introducing the so-called *JB*-algebras. *JB*-algebras are complete normed Jordan real algebras which include both Jordan algebras of self-adjoint 'operators on a Hilbert space' (Corollary 3.1.2) and 'the exceptional Jordan algebra' (Example 3.1.56). *JB*-algebras enjoy a deep and complete structure theory which has been comprehensively developed in Hanche-Olsen and Størmer [738], and has been revisited recently in Alfsen and Shultz [673]. Different approaches to *JB*-algebras can be found in Ayupov, Rakhimov, and Usmanov [684], and Iochum [748]. Since we are unable to organize the basic theory of *JB*-algebras in a better way than that of [738], we have limited ourselves to state without proof those results which are needed for our actual purposes, and to complement the theory in some aspects (originated by the papers of Wright [641] and Wright and Youngson [643]) which are not covered by the Hanche-Olsen and Størmer book. This is done from Section 3.1. Among the results taken from [738], we emphasize the one asserting that *JB*-algebras generated by two elements are (isometrically isomorphic to) Jordan algebras of self-adjoint operators on a

Hilbert space (Proposition 3.1.3). Thus, 'suitable infinite-dimensional Jordan Banach algebras are special'.

Kaplansky concludes with the following comment:

> Kadison's beautiful results [358, 359] on isometries of C^*-algebras deserve to be generalized to Jordan C^*-algebras if only to encompass the exceptional Jordan algebra.

What Kaplansky means by a 'Jordan C^*-algebra' is not in doubt because he himself later introduced this notion in detail in his final lecture to the 1976 St. Andrews Colloquium of the Edinburgh Mathematical Society, and pointed out its potential importance. Jordan C^*-algebras (called JB^*-algebras since Youngson's paper [652]) were first studied by Wright [641], who proved that the passing from each JB^*-algebra to its self-adjoint part establishes a bijective categorical correspondence between JB^*-algebras and JB-algebras (see Corollary 3.4.3 and Theorem 3.4.8).

Kadison's celebrated Theorem A of [358] is fully discussed in Section 2.2. Actually, we prove the unit-free Paterson–Sinclair generalization [480] of Kadison's result (Theorem 2.2.19), starting from a non-associative germ (Theorem 2.2.9). An appropriate version for JB-algebras of the Kadison–Paterson–Sinclair theorem, following the arguments in [643, 223, 342], is stated in Theorem 3.1.21. Therefore, as first proved in [643], for isometries preserving units of unital JB^*-algebras, Kadison's theorem survives (see Proposition 3.4.25) 'encompassing the exceptional Jordan algebra'. Nevertheless, Kadison's theorem does not survive for general isometries, nor even in the case of closed $*$-invariant unital Jordan subalgebras of operators on Hilbert spaces. The reason is that, as shown by Braun, Kaup, and Upmeier [126], two such algebras can be linearly isometric without being $*$-isomorphic (see Antitheorem 3.4.34). Anyway, the Kadison–Paterson–Sinclair theorem survives verbatim in the case of alternative C^*-algebras and also, in a suitably altered form, in the case of JB^*-algebras which are dual Banach spaces [366].

About the core of the book

Now that we have concluded our review of the non-associative part of Kaplansky's monograph [762], let us comment about the leitmotiv of the present book. (To this end, we found the Introduction of [366] useful.) Our aim is to deal with non-associative generalizations of C^*-algebras. To this end, we realize that most generalizations appearing in the literature, like JB^*-algebras, JB-algebras (both had already been discussed when we reviewed [762]), and JB^*-triples, contain C^*-algebras, but only after suitable manipulations. Thus C^*-algebras become JB^*-algebras after replacing the associative product xy with the Jordan product

$$x \bullet y := \frac{1}{2}(xy + yx).$$

They are JB-algebras after the same replacement and then passing to the self-adjoint part, and they are also JB^*-triples after replacing the product with the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

As a matter of fact, many years ago we tried to approach the non-associative generalizations of C^*-algebras in a somewhat more ingenuous way. Indeed, we removed associativity in the abstract characterizations of unital (associative) C^*-algebras given either by the Gelfand–Naimark theorem or by the Vidav–Palmer theorem, and studied (possibly non-unital) closed $*$-invariant subalgebras of the Gelfand–Naimark or Vidav–Palmer algebras born after removing associativity.

To be more precise, for a norm-unital complete normed (possibly non-associative) complex algebra A, we considered the following conditions:

(GN) (Gelfand–Naimark axiom). *There is a conjugate-linear vector space involution $*$ on A satisfying $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a\|^2$ for every a in A.*

(VP) (Vidav–Palmer axiom). $A = H(A, \mathbf{1}) + iH(A, \mathbf{1})$.

In both conditions, $\mathbf{1}$ denotes the unit of A, whereas, in (VP), $H(A, \mathbf{1})$ stands for the closed real subspace of A consisting of those elements h in A such that $f(h)$ belongs to $\mathbb{R}$ for every bounded linear functional f on A satisfying $\|f\| = f(\mathbf{1}) = 1$.

As we said before, if the norm-unital complete normed complex algebra A above is associative, then (GN) and (VP) are equivalent conditions, both providing nice characterizations of unital C^*-algebras (see Lemma 2.2.5 and Theorems 1.2.3 and 2.3.32). In the general non-associative case we were considering, things began to be more amusing. Indeed, it is easily seen that (GN) implies (VP) (refer again to Lemma 2.2.5), but the converse implication is not true (see Example 2.3.65).

The amusing aspect of the non-associative consideration of the Vidav–Palmer and the Gelfand–Naimark axioms greatly increased thanks to the fact (explained in what follows) that Condition (VP) (respectively, (GN)) on a norm-unital complete normed complex algebra A implies that A is 'nearly' (respectively, 'very nearly') associative. To specify our last assertion, let us recall some elementary concepts of non-associative algebra. *Alternative algebras* are defined as those algebras A satisfying $a^2b = a(ab)$ and $ba^2 = (ba)a$ for all a, b in A. By Artin's theorem (stated in Theorem 2.3.61), an algebra A is alternative (if and) only if, for all a, b in A, the subalgebra of A generated by $\{a, b\}$ is associative. According to Definition 2.4.9 and Proposition 3.2.1, *non-commutative Jordan algebras* can be introduced as those algebras A satisfying the *Jordan identity* $(ab)a^2 = a(ba^2)$ and the *flexibility* condition $(ab)a = a(ba)$. As shown in Proposition 2.4.19, non-commutative Jordan algebras are *power-associative* (i.e. all subalgebras generated by a single element are associative) and, as a consequence of Artin's theorem, alternative algebras are non-commutative Jordan algebras. For an element a in an algebra A, we denote by U_a the mapping $b \to a(ab + ba) - a^2b$ from A to A. By means of Definitions I and II below, we provide the algebraic notions just introduced with analytic robes. We note that the notion of an alternative C^*-algebra, emphasized in Definition I, had already appeared when we reviewed the monograph [762].

Definition I By an *alternative C^*-algebra* we mean a complete normed alternative complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for every a in A.

Definition II By a *non-commutative JB^*-algebra* we mean a complete normed non-commutative Jordan complex algebra (say A) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_a(a^*)\| = \|a\|^3$ for every a in A.

Since the equality $U_a(b) = aba$ holds for all elements a, b in an alternative algebra, it is not difficult to realize that alternative C^*-algebras become particular examples of non-commutative JB^*-algebras. Actually, alternative C^*-algebras are precisely those non-commutative JB^*-algebras which are alternative (see Fact 3.3.2). Now, by means of Theorems GN and VP which follow, we can specify how the behaviour of the Gelfand–Naimark and Vidav–Palmer axioms in the non-associative setting were clarified.

Theorem GN *Norm-unital complete normed complex algebras fulfilling the Gelfand–Naimark axiom are nothing other than unital alternative C^*-algebras.*

Theorem VP *Norm-unital complete normed complex algebras fulfilling the Vidav–Palmer axiom are nothing other than unital non-commutative JB^*-algebras.*

Now, keeping in mind Theorems GN and VP above, together with the obvious fact that closed $*$-invariant subalgebras of an alternative C^*-algebra (respectively, of a non-commutative JB^*-algebra) are alternative C^*-algebras (respectively, non-commutative JB^*-algebras), there is no doubt that, even in the non-unital case, both alternative C^*-algebras and non-commutative JB^*-algebras become reasonable non-associative generalizations (the latter containing the former) of classical C^*-algebras. Therefore the main goal of our book will be to prove Theorems GN and VP (see Theorems 3.2.5 and 3.3.11), together with their non-unital variants (see Theorem 3.5.53 and [365]), and to describe alternative C^*-algebras and non-commutative JB^*-algebras by means of the so-called representation theory.

It is worth mentioning that although our approach to the non-associative generalizations of C^*-algebras is different from those of JB^*-algebras, JB-algebras, and JB^*-triples, in the end all approaches give rise essentially to the same mathematical creature. Indeed, Kaplansky's JB^*-algebras are nothing other than those non-commutative JB^*-algebras which are commutative. On the other hand, every non-commutative JB^*-algebra becomes a JB^*-algebra after symmetrizing its product (see Fact 3.3.4); JB^*-algebras and JB-algebras coincide after a categorical correspondence (a fact already noted when we reviewed [762]); non-commutative JB^*-algebras become JB^*-triples in a natural way (see Theorem 4.1.45); and every JB^*-triple can be seen as a closed subtriple of a suitable JB^*-algebra (a fact collected without proof in Theorem 4.1.113). Therefore most basic results in the classical theory of JB^*-algebras, JB-algebras, and JB^*-triples will be involved in our development.

Since JB^*-triples had not appeared when we reviewed [762], let us comment about them briefly. Roughly speaking, JB^*-triples become a functional-analytic solution to the problem of the classification of all 'bounded symmetric domains' in complex Banach spaces. Partial solutions to this problem in the same line are due to Loos [772], who settled the finite-dimensional case, and to Harris [313], who proved that the open unit ball of each norm-closed subspace of any C^*-algebra, which is also closed under the triple product $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$, is a bounded symmetric domain. The definitive solution, due to Kaup [380, 381], asserts that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable complex Banach space, and that if the open unit ball of a complex Banach space is a bounded symmetric domain, then the Banach space itself is *almost* a C^*-algebra, and there is an intrinsically defined triple product $\{\cdots\}$ on it which behaves algebraically and geometrically like the one obtained from the binary product of a C^*-algebra by taking $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$. The precise formulation of this last fact gives rise to the definition of a JB^*-triple (see §2.2.27 and Fact 4.1.41).

JB^*-triples have been intensively studied in recent years and their basic theory can be found in the monographs of Chu [710], Dineen [721], Friedman and Scarr [732], Iordanescu [750], Isidro and Stachó [751], and Upmeier [814, 815], as well as in the survey papers of Chu and Mellon [173], Kaup [384], Rodríguez [525], and Russo [547]. The initial binary approach of our book complements these works.

About the organization of the book

The work we are introducing is covered in two volumes. Roughly speaking, the dividing line between the two can be drawn between what can be done before and after involving the holomorphic theory of JB^*-triples and the structure theory of non-commutative JB^*-algebras. Volume 1 is now concluded, whereas Volume 2 exists today only in the authors' minds. Therefore we are going to describe in detail the content of the first volume (Chapters 1–4), and announce in a less precise form what we intend to do in the second (Chapters 5–8).

Volume 1

In Chapter 1, we develop the basic theory of normed algebras, putting special emphasis on the cases of complete normed unital associative complex algebras and of (associative) C^*-algebras. Non-associative normed algebras are considered here only when they do not offer special difficulties, or difficulties can be overcome in an elementary way. Thus, the first three sections of the chapter are mainly devoted to attracting the attention of the non-expert reader. The chapter is complemented with a fourth section where some selected topics in the theory of compact and weakly compact operators (including recent developments [441, 596]) are discussed.

Chapter 2 is essentially devoted to settling the two first steps in the proof of the 'non-associative Vidav–Palmer theorem' (Theorem VP), namely that the natural involution of any Vidav–Palmer algebra is an algebra involution (see Theorem 2.3.8) and that Vidav–Palmer algebras are non-commutative Jordan algebras (see Theorem 2.4.11). Among the applications of these results, we emphasize the Kadison–Paterson–Sinclair theorem on isometries of C^*-algebras [358, 480] proved in Theorem 2.2.19, the Blecher–Ruan–Sinclair non-associative characterization of (associative) C^*-algebras [106] proved in Theorem 2.4.27, and the non-commutative Urbanik–Wright theorem (already discussed when we reviewed [762]) proved in Theorem 2.6.21. (We missed the formulation and a proof of this last theorem in the delightful book *Numbers* [727].) Applications of the study of contractive projections on C^*-algebras are also made (see Theorems 2.3.68 and 2.4.24). As absolute-valued algebras arise naturally in the Urbanik–Wright theorem, we devote special attention to them (see Sections 2.6, 2.7, and 2.8). Since the Vidav–Palmer axiom depends only on the normed space of the algebra and on the unit, we follow [22, 425] to develop, where possible, the theory of numerical ranges of elements of norm-unital normed algebras in the more general setting of a normed space, in which a norm-one element has been distinguished (see Sections 2.1 and 2.9). This approach (which involves relevant results of pure geometry of normed spaces [56, 268, 287, 291, 293, 299]) complements those of Bonsall–Duncan [694, 695, 696], Doran–Belfi [725], and Palmer [786, 787] in their books.

In Chapter 3 our development depends heavily on the basic theory of JB-algebras, which is taken without proof from Hanche-Olsen and Størmer [738], and is complemented in some aspects not covered by that book (see Section 3.1 and Subsection 3.4.1). In particular, surjective linear isometries between JB-algebras are described in detail (see Theorem 3.1.21), and Wright's categorical correspondence between JB-algebras and JB^*-algebras [641] is established (see Fact 3.4.9). Chapter 3 is essentially devoted to proving Theorem GN (see Theorem 3.2.5), concluding the proof of Theorem VP (see Theorem 3.3.11), developing the theory of alternative C^*-algebras and of non-commutative JB^*-algebras in those aspects which do not involve the so-called 'Jordan spectral theory' (see Subsections 3.4.2, 3.4.4, 3.5.1, 3.5.3, and 3.6.2), and proving the unit-free variant of Theorem GN (see Theorem 3.5.53). The behaviour of the original Gelfand–Naimark axiom $\|a^*a\| = \|a^*\|\|a\|$ in the non-associative setting is fully discussed, generalizing the associative forerunners due to Glimm and Kadison [290] and Vowden [629] (see Subsections 3.5.2, 3.5.4, and 3.5.6). Some auxiliary results, taken from the non-geometric theory of non-associative normed algebras, are also included. Thus Dixmier's fundamental theorem [723] on continuous automorphisms, which are exponentials of continuous derivations, is proved (see Theorem 3.4.49).

In Chapter 4, the Jacobson–McCrimmon notion of the Jordan inverse [754, 433, 436] is involved to derive a spectral theory for normed non-commutative Jordan algebras, which generalizes the one developed in Sections 1.1 and 1.3 for normed associative algebras. This is done in Subsections 4.1.1, 4.1.2, 4.1.4, and 4.1.5. Jordan spectral theory is applied to continue the development of the basic theory of alternative C^*-algebras and of non-commutative JB^*-algebras. In particular, the relationship between non-commutative JB^*-algebras and JB^*-triples is settled (see Theorems 4.1.45 and 4.1.55). The functional-analytic treatment of JB^*-triples is continued in Section 4.2, where Kaup's commutative Gelfand–Naimark type theorems for JB^*-triples [380, 381] are proved (see Theorems 4.2.7 and 4.2.9) and then, following [126, 269, 385, 655], the convexity properties of the closed unit balls of JB^*-triples and of non-commutative JB^*-algebras are established (see Theorems 4.2.24, 4.2.28, 4.2.34, and 4.2.36). Following [77, 78], we describe C^*-algebras and JB^*-algebras generated by a non-self-adjoint idempotent (see Theorems 4.3.11, 4.3.16, 4.3.29, and 4.3.32), and derive Spitkovsky's theorem [595], that C^*-algebras generated by a non-self-adjoint idempotent are generated by two self-adjoint idempotents (see Corollary 4.3.17); we also discuss the appropriate variant of Spitkovsky's theorem for JB^*-algebras (see Corollary 4.3.34). We prove that non-commutative JB^*-algebras have minimum norm topology (see Theorem 4.4.29) and minimality of norm (see Proposition 4.4.34). These results generalize associative forerunners by Cleveland [176] and Bonsall [111], respectively.

We state Behncke's theory of complete normed hermitian Jordan complex $*$-algebras [82], which is proved following the Aupetit–Youngson arguments [48], and is presented in a somewhat new way involving the so-called JB^*-representations (see Theorem 4.5.29 and Corollary 4.5.30). Then a theory of complete normed hermitian alternative complex $*$-algebras is derived (see Theorem 4.5.37 and Corollary 4.5.39) in such a way that it contains the classical associative forerunners [493, 565] as stated in Bonsall–Duncan [696, Section 41]. Generalizing Sakai's theorem [807], we prove that domains of closed densely defined derivations of any unital non-commutative JB^*-algebra are closed under the functional calculus of class C^2 at self-adjoint elements (see Theorem 4.6.63). The chapter also includes some auxiliary results taken from the general theory of non-associative normed algebras. Thus the non-associative generalization [516] of Johnson's uniqueness-of-norm theorem [353] (see also [696, 715, 786]), as well as Aupetit's celebrated forerunner for non-commutative Jordan algebras [40], are settled (see Section 4.4). Along the same lines, a non-associative version of [522], as well as non-associative applications of Bollobás' extremal algebra [110, 182], are discussed (see Section 4.6).

Volume 2

Chapter 5 will be devoted to proving what can be seen as a unit-free version of the non-associative Vidav–Palmer theorem, namely that non-commutative JB^*-algebras are precisely those complete normed complex algebras having an approximate unit bounded by one, and whose open unit ball is a bounded symmetric domain [365]. Some ingredients in the long proof of this result have been already established in Volume 1. This is the case of the Bohnenblust–Karlin Corollary 2.1.13, the non-associative Vidav–Palmer theorem (Theorem 3.3.11) as well as its dual version (Corollary 3.3.26), Proposition 3.5.23, Theorem 4.1.45, and the equivalence (ii)⇔(vii) in the Braun–Kaup–Upmeier Theorem 4.2.24. The new relevant ingredients to be proved in the chapter are: (i) the Chu–Iochum–Loupias result that bounded linear operators from a JB^*-triple to its dual are weakly compact [172] (equivalently, via [717, Corollary on p. 12], that all continuous products on the Banach space of a JB^*-triple are Arens regular); (ii) Kaup's theorem that JB^*-triples are precisely those complex Banach spaces whose open unit ball is a bounded symmetric domain [381]; (iii) the contractive projection theorem for JB^*-triples collected without proof in Theorem 2.3.74; and (iv) Dineen's celebrated result that the bidual of a JB^*-triple is a JB^*-triple [213].

Chapter 6 will contain the representation theory for alternative C^*-algebras, already sketched when we reviewed [762], and the representation theory for non-commutative JB^*-algebras, following [19, 124, 222, 481, 482, 641]. In these papers a precise classification of certain prime non-commutative JB^*-algebras (the so-called 'non-commutative JBW^*-factors') is obtained, and the fact that every non-commutative JB^*-algebra has a faithful family of the so-called 'Type I' factor representations is proven. When these results specialize for classical C^*-algebras, Type I non-commutative JBW^*-factors are nothing other than the (associative) W^*-factors consisting of all bounded linear operators on some complex Hilbert space [738, Proposition 7.5.2], and, consequently, Type I factor representations are precisely irreducible representations on Hilbert spaces. The chapter will contain also a classification of all prime non-commutative JB^*-algebras, obtained in [255, 363] by applying Zel'manov's purely algebraic techniques in [437, 662, 663]. Many applications of the representation theory will be discussed. Among them, we emphasize the generalization to non-commutative JB^*-algebras [340] of Kaplansky's characterization of commutativity of C^*-algebras by means of the absence of isotropic elements [761, Theorem B in Appendix III].

In Chapters 7 and 8, we will discuss selected topics in the theory of non-associative normed algebras, which need not be directly related to the non-associative generalizations of C^*-algebras. Chapter 7 will deal with the analytic treatment of Zel'manov's prime theorems for Jordan structures, already sketched in Chapter 6. Thus several direct contributions of Zel'manov to the theory of normed Jordan algebras [147, 538] (one of which was refined later in [447]) and the binary results in [146, 151, 152, 539] will be included with proofs. The ternary results in [448, 449] will be simply surveyed. The survey papers [145, 450, 532] could provide the reader with a more detailed overview of the intended content of the whole of Chapter 7.

The concluding Chapter 8 will deal with miscellany in the theory of non-associative normed algebras. We will complement our knowledge on non-associative generalizations of Rickart's dense-range-homomorphism theorem (see Theorem 4.1.19 and Proposition 4.1.108) with those obtained in [165, 529]. Complementing Corollary 4.4.55, some automatic continuity theorems for homomorphisms 'into', taken from [165, 462, 529], will also be included. Automatic continuity of Lie homomorphisms, culminating in [130] through the papers of Berenguer–Villena [97, 98] and Aupetit–Mathieu [47] already discussed in Subsection 4.4.5, will also receive special attention. As an auxiliary tool for the proof of the main result in [130], we will incorporate the discussion in [91] about normed Jordan algebras 'with finite spectrum'. Actually, the theory of normed Jordan structures subjected to 'finiteness conditions' would merit being systematically organized. Nevertheless, we will not be doing this, and will limit ourselves to surveying this matter by reviewing results from Aupetit [43, 44], Aupetit–Baribeau [45], Aupetit–Maouche [46], Benslimane–Boudi [87, 88], Benslimane–Fernández–Kaidi [89], Benslimane–Jaa–Kaidi [90], Benslimane–Kaidi [91], Benslimane–Rodríguez [95], Boudi [119], Boudi–Marhnine–Zarhouti–Fernández–García [120], Bouhya–Fernández [121], Fernández [250, 251, 252], Fernández–García–Sánchez [256], Fernández–Rodríguez [258], Hessenberger [322, 323, 324, 325], Hessenberger–Maouche [326], Loos [402, 404, 405], Maouche [413, 412], Pérez–Rico–Rodríguez–Villena [489], and Wilkins [636]. Another favourite topic to be included in this chapter is that of the general theory of non-associative H^*-algebras, which incidentally appears in Remark 2.6.54, Lemma 2.7.50, Subsection 2.8.2, and Corollary 4.1.104 of this volume. This will be done by taking the appropriate material from the papers [142, 144, 148, 149, 198, 199, 259, 526, 624] (see also [687, Chapters 7 and 8] and [525, Section E]). The chapter will conclude with the non-associative discussion of the Rota–Strang paper [544] covered in [452, 453].

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1

Foundations

1.1 Rudiments on normed algebras

Introduction In this section we develop the basic theory of normed algebras, putting special emphasis on the case of complete normed unital associative complex algebras. Non-associative normed algebras are considered only when they do not offer special difficulties, or when the difficulties can be overcome in an elementary way. Thus, this section is mainly devoted to attracting the attention of the non-expert reader, although the expert reader should browse through it in order to become familiar with the definitions and symbols introduced here, which need to be kept in mind throughout the whole book.

Subsection 1.1.1 deals with the basic spectral theory, and culminates with the proof in Theorem 1.1.46 of the celebrated Gelfand–Beurling formula. In Subsection 1.1.2, we prove Rickart's dense-range-homomorphism theorem. Subsection 1.1.3 deals with the Gelfand theory for complete normed unital associative and commutative complex algebras, as stated in Theorem 1.1.73, and some applications are discussed. In Subsection 1.1.4, we introduce topological divisors of zero in a normed algebra, and involve this notion to prove in Corollary 1.1.95 that (bounded linear) operators on a Banach space are neither bounded below nor surjective whenever they lie in the boundary of the set of all bijective operators. Subsections 1.1.5 and 1.1.6 discuss the complexification, the unital extension, and the completion of a normed algebra. These tools allow us to show how many results, proved originally for complete normed unital associative complex algebras, remain true (sometimes in a suitably altered form) for general normed associative algebras. This section, and throughout, concludes with a subsection of historical notes and comments.

1.1.1 Basic spectral theory

Throughout this work, $\mathbb{K}$ will stand for the field of real or complex numbers. Given vector spaces X,Y over $\mathbb{K}$, we denote by $L(X,Y)$ the vector space over $\mathbb{K}$ of all linear mappings from X to Y, and we set $L(X) := L(X,X)$.

By an *algebra* over $\mathbb{K}$ we mean a vector space A over $\mathbb{K}$ endowed with a bilinear mapping $(a,b) \to ab$ from $A \times A$ to A, which is called the *product* or the *multiplication* of A. An algebra is said to be *associative* (respectively, *commutative*) if its product is associative (respectively, commutative). An element e of an algebra A is said to be a *unit* for A if $ea = ae = a$ for every $a \in A$. Clearly, an algebra has at most a unit. The algebra A is said to be *unital* if it has a nonzero unit (equivalently, if A has a unit and $A \neq 0$). The unit of a given unital algebra will be denoted by $\mathbf{1}$ unless otherwise stated. Given subsets B, C of an algebra A, we set

$$BC := \{xy : (x,y) \in B \times C\}.$$

Exceptionally, but never in Chapter 1, we will consider algebras over an arbitrary field $\mathbb{F}$. These are defined as above with $\mathbb{F}$ instead of $\mathbb{K}$.

Example 1.1.1 (a) Let E be a non-empty set. Then the set $F^{\mathbb{K}}(E)$ (of all functions from E to $\mathbb{K}$), with operations defined pointwise, becomes a unital associative and commutative algebra over $\mathbb{K}$.

(b) Let X be a nonzero vector space over $\mathbb{K}$. Then the vector space $L(X)$, with the product defined as the composition of mappings, becomes a unital associative algebra over $\mathbb{K}$. It is easily realized that $L(X)$ is commutative if and only if $\dim(X) = 1$. We denote by I_X the unit of $L(X)$, namely the identity mapping on X.

(c) By a *subalgebra* of an algebra A over $\mathbb{K}$ we mean a (vector) subspace (say B) of A such that $BB \subseteq B$. In this way, subalgebras of algebras become new examples of algebras.

Let A and B be algebras over $\mathbb{K}$. By an *algebra homomorphism* from A to B we mean a linear mapping $F : A \to B$ satisfying $F(xy) = F(x)F(y)$ for all $x, y \in A$. We say that A and B are *(algebra) isomorphic* if there exists a bijective algebra homomorphism from A to B.

Exercise 1.1.2 Prove that every one-dimensional algebra over $\mathbb{K}$ with nonzero product is isomorphic to $\mathbb{K}$.

A norm $\|\cdot\|$ on (the vector space of) an algebra A over $\mathbb{K}$ is said to be an *algebra norm* if the inequality $\|ab\| \leqslant \|a\|\|b\|$ holds for all $a, b \in A$. By a *normed algebra* we mean an algebra A over $\mathbb{K}$ endowed with an algebra norm. A normed algebra A is said to be *complete* if it becomes a complete metric space under the distance $d(a,b) := \|a - b\|$, i.e. if the normed space underlying A is a Banach space.

§**1.1.3** It is clear that the product of any normed algebra is continuous. Actually, the axiom $\|ab\| \leqslant \|a\|\|b\|$ of normed algebras does not give much more. Indeed, if $|||\cdot|||$ is a norm on an algebra A over $\mathbb{K}$ making the product of A continuous (say $|||ab||| \leqslant M|||a|||\,|||b|||$ for all $a, b \in A$ and some positive number M), then by setting $\|\cdot\| := M|||\cdot|||$, we are provided with an equivalent norm on A converting A into a normed algebra.

Given a normed space X over $\mathbb{K}$, we denote by

$$\mathbb{B}_X := \{x \in X : \|x\| \leqslant 1\}$$

the closed unit ball of X, by

$$\mathbb{S}_X := \{x \in X : \|x\| = 1\}$$

the unit sphere of X, and by X' the (topological) dual of X. When necessary, every normed space X will be seen as a subspace of its bidual X''. Given normed spaces

X, Y over $\mathbb{K}$, we denote by $BL(X,Y)$ the normed space over $\mathbb{K}$ of all bounded linear mappings from X to Y, and we set $BL(X) := BL(X,X)$.

Example 1.1.4 (a) Let E be a locally compact Hausdorff topological space. Then the subalgebra $C_0^{\mathbb{K}}(E)$ of $F^{\mathbb{K}}(E)$ (consisting of all $\mathbb{K}$-valued continuous functions on E vanishing at infinity), endowed with the norm

$$\|x\| := \max\{|x(t)| : t \in E\},$$

becomes a complete normed associative and commutative algebra over $\mathbb{K}$. This algebra is unital if and only if E is compact. When this is the case, we also write $C^{\mathbb{K}}(E)$ instead of $C_0^{\mathbb{K}}(E)$. We note that, by taking E equal to $\mathbb{N}$ endowed with the discrete topology, we obtain that the real or complex Banach space c_0 (of all null sequences in $\mathbb{K}$) naturally becomes a complete normed associative and commutative algebra over $\mathbb{K}$.

(b) Let X be a nonzero normed space over $\mathbb{K}$. Then $BL(X)$ is a subalgebra of $L(X)$, and, endowed with the operator norm

$$\|F\| := \sup\{\|F(x)\| : x \in \mathbb{B}_X\},$$

becomes a normed unital associative algebra over $\mathbb{K}$. Moreover, the normed algebra $BL(X)$ is complete if and only if X is a Banach space. Involving the Hahn–Banach theorem, it is easily realized that $BL(X)$ is commutative if and only if $\dim(X) = 1$.

(c) By restricting the norm, any subalgebra of a normed algebra will be seen without notice as a new normed algebra.

(d) Let E be a topological space, and let A be a normed algebra over $\mathbb{K}$. Then, by the continuity of the product of A, the vector space $C(E,A)$ of all continuous functions from E to A becomes an algebra over $\mathbb{K}$ under the product defined pointwise. The subalgebra $C_b(E,A)$ of $C(E,A)$, consisting of all bounded continuous functions from E to A, becomes a normed algebra over $\mathbb{K}$ under the sup norm.

§1.1.5 An element e of an algebra is said to be an *idempotent* if $e^2 = e$. If e is a nonzero idempotent in a normed algebra, then we clearly have $\|e\| \geqslant 1$. In particular, the unit $\mathbf{1}$ of a normed unital algebra satisfies $\|\mathbf{1}\| \geqslant 1$. Moreover, no more can be said. Indeed, if M is any real number with $M \geqslant 1$, and if for $\lambda \in \mathbb{K}$ we set $\|\lambda\| := M|\lambda|$, where $|\cdot|$ stands for the usual module on $\mathbb{K}$, then $\|\cdot\|$ becomes an algebra norm on $\mathbb{K}$ satisfying $\|1\| = M$.

Fact 1.1.6 *Let E be a connected topological space, let A be a normed algebra over $\mathbb{K}$, let t_0 be in E, and let B stand for the subalgebra of $C(E,A)$ consisting of those continuous functions from E to A vanishing at t_0. Then B has no nonzero idempotent.*

Proof Assume to the contrary that there is a nonzero idempotent $e \in B$. Then $e(t)$ is an idempotent in A for every $t \in E$, and $e(t)$ is nonzero for some $t \in E$. Therefore, since $e(t_0) = 0$, the continuous mapping $t \to \|e(t)\|$ from E to $\mathbb{R}$ would have a disconnected range (cf. §1.1.5 above), contradicting the connectedness of E. □

As an application of §1.1.3, we have the following.

Proposition 1.1.7 *Let A be a finite-dimensional algebra over $\mathbb{K}$. Then A can be provided with an algebra norm.*

Proof Let $\{u_1,\ldots,u_n\}$ be a basis of A. Then, for $j,k=1,\ldots,n$, we have $u_ju_k = \sum_{i=1}^n \rho_i^{jk}u_i$, for suitable $\rho_1^{jk},\ldots,\rho_n^{jk}\in\mathbb{K}$. Now, for $a=\sum_{i=1}^n \lambda_i u_i$, set $|||a||| := \sum_{i=1}^n |\lambda_i|$. Then, for a as above and $b=\sum_{i=1}^n \mu_i u_i$, we have

$$ab = \textstyle\sum_{i=1}^n \tau_i u_i, \quad \text{with} \quad \tau_i := \sum_{j,k=1}^n \lambda_j\mu_k\rho_i^{jk},$$

and hence

$$|||ab||| = \sum_{i=1}^n |\tau_i| \leqslant M \sum_{j,k=1}^n |\lambda_j||\mu_k| = M|||a||||||b|||,$$

where $M := n\max\{|\rho_i^{jk}| : i,j,k=1,\ldots,n\}$ does not depend on the couple (a,b). Finally, apply §1.1.3. □

Lemma 1.1.8 *Let X be a Banach space over $\mathbb{K}$, let Y,Z be normed spaces over $\mathbb{K}$, and let $f : X\times Y\to Z$ be a separately continuous bilinear mapping. Then f is (jointly) continuous.*

Proof For $u\in X$ (respectively, $v\in Y$), let us denote by f_u (respectively, f_v) the bounded linear mapping from Y to Z (respectively, from X to Z) defined by $f_u(y) := f(u,y)$ for every $y\in Y$ (respectively, $f_v(x) := f(x,v)$ for every $x\in X$). Then $\mathscr{F} := \{f_v : v\in\mathbb{B}_Y\}$ is a pointwise bounded family of bounded linear mappings from the Banach space X to the normed space Z. Indeed, for each $x\in X$ and every $v\in\mathbb{B}_Y$ we have

$$\|f_v(x)\| = \|f(x,v)\| = \|f_x(v)\| \leqslant \|f_x\|.$$

It follows from the uniform boundedness principle that $\mathscr{F}$ is uniformly bounded on $\mathbb{B}_X$. This implies the existence of a positive number M satisfying

$$\|f(x,y)\| \leqslant M\|x\|\|y\| \text{ for every } (x,y)\in X\times Y. \qquad \square$$

By combining §1.1.3 and Lemma 1.1.8, we obtain the following.

Proposition 1.1.9 *Let A be an algebra over $\mathbb{K}$ endowed with a complete norm $\|\cdot\|$ making the product of A separately continuous. Then, up to the multiplication of $\|\cdot\|$ by a suitable positive number, A becomes a complete normed algebra.*

Definition 1.1.10 Let A be an algebra over $\mathbb{K}$. The *annihilator*, $\mathrm{Ann}(A)$, of A is defined by

$$\mathrm{Ann}(A) := \{a\in A : aA = Aa = 0\}.$$

By a *centralizer* on A we mean a linear mapping (say f) from A to A satisfying $f(ab) = f(a)b = af(b)$ for all $a,b\in A$. The set Γ_A of all centralizers on A is a subalgebra of $L(A)$ containing I_A. This subalgebra is called the *centroid* of A. The algebra A is said to be *central* over $\mathbb{K}$ whenever $\Gamma_A = \mathbb{K}I_A$.

The next proposition contains an easy ‘automatic continuity theorem’.

Proposition 1.1.11 *Let A be an algebra over $\mathbb{K}$ with $\mathrm{Ann}(A)=0$. We have:*

(i) *Γ_A is a commutative algebra.*
(ii) *If A is complete normed, then $\Gamma_A\subseteq BL(A)$.*

Proof Given $f,g \in \Gamma_A$ and $a,b \in A$, we see that

$$\begin{aligned}(f \circ g)(a)b &= f(g(a))b = g(a)f(b) = g(af(b)) \\ &= g(f(a)b) = g(f(a))b = (g \circ f)(a)b\end{aligned}$$

and

$$\begin{aligned}b(f \circ g)(a) &= bf(g(a)) = f(b)g(a) = g(f(b)a) \\ &= g(bf(a)) = bg(f(a)) = b(g \circ f)(a).\end{aligned}$$

It follows from the arbitrariness of b in A that $(f \circ g - g \circ f)(a)$ belongs to $\mathrm{Ann}(A)$. Therefore $(f \circ g - g \circ f)(a) = 0$. Now, since a is arbitrary in A, we conclude that $f \circ g = g \circ f$. Thus Γ_A is a commutative algebra.

Assume that A is complete normed. Let f be in Γ_A, and let a_n be a sequence in A with $a_n \to 0$ and $f(a_n) \to b \in A$. Then, for every $a \in A$ we have

$$0 \leftarrow f(a)a_n = af(a_n) \to ab \quad \text{and} \quad 0 \leftarrow a_n f(a) = f(a_n)a \to ba,$$

hence $ab = ba = 0$. Since a is arbitrary in A, and A has zero annihilator, we get that $b = 0$. Thus the continuity of f follows from the closed graph theorem. □

Let A be a unital associative algebra. An element $x \in A$ is said to be *invertible* in A if there exists $y \in A$ such that $xy = yx = \mathbf{1}$. If x is invertible, then the element y above is unique, is called the *inverse* of x, and is denoted by x^{-1}. We denote by $\mathrm{Inv}(A)$ the set of all invertible elements of A.

Example 1.1.12 (a) Let E be a non-empty set, let $F^{\mathbb{K}}(E)$ be as in Example 1.1.1(a), and let x be in $F^{\mathbb{K}}(E)$. Then $x \in \mathrm{Inv}(F^{\mathbb{K}}(E))$ if and only if $x(t) \neq 0$ for every $t \in E$.

(b) Let X be a nonzero vector space over $\mathbb{K}$, and let F be in $L(X)$. Then $F \in \mathrm{Inv}(L(X))$ if and only if F is bijective.

(c) Let E be a compact Hausdorff topological space, let $C^{\mathbb{K}}(E)$ be as in Example 1.1.4(a), and let x be in $C^{\mathbb{K}}(E)$. Then $x \in \mathrm{Inv}(C^{\mathbb{K}}(E))$ if and only if $x(t) \neq 0$ for every $t \in E$. Therefore $x \in \mathrm{Inv}(C^{\mathbb{K}}(E))$ if and only if $x \in \mathrm{Inv}(F^{\mathbb{K}}(E))$.

(d) Let X be a nonzero normed space over $\mathbb{K}$, and let F be in $BL(X)$. Then $F \in \mathrm{Inv}(BL(X))$ if and only if F is bijective and F^{-1} is continuous. Therefore, in the case that X is in fact a Banach space, the Banach isomorphism theorem gives that $F \in \mathrm{Inv}(BL(X))$ if and only if F is bijective, and hence $F \in \mathrm{Inv}(BL(X))$ if and only if $F \in \mathrm{Inv}(L(X))$.

Lemma 1.1.13 *Let A be a normed unital associative algebra over $\mathbb{K}$, and let a and b be in $\mathrm{Inv}(A)$. Then we have:*

(i) $\|a^{-1} - b^{-1}\| \leqslant \|a^{-1}\|\,\|b^{-1}\|\,\|a - b\|$.
(ii) $\left| \frac{1}{\|a^{-1}\|} - \frac{1}{\|b^{-1}\|} \right| \leqslant \|a - b\|$.
(iii) *If $\|a - b\| < \frac{1}{\|a^{-1}\|}$, then $\|b^{-1}\| \leqslant \frac{\|a^{-1}\|}{1 - \|a^{-1}\|\|a-b\|}$.*

Proof We have

$$\|a^{-1} - b^{-1}\| = \|a^{-1}(b-a)b^{-1}\| \leqslant \|a^{-1}\|\,\|b^{-1}\|\,\|a - b\|,$$

which proves assertion (i). Now, keeping in mind that

$$\|b^{-1}\| - \|a^{-1}\| \leqslant \|a^{-1} - b^{-1}\|,$$

it follows from assertion (i) that

$$\frac{1}{\|a^{-1}\|} - \frac{1}{\|b^{-1}\|} \leqslant \|a-b\|. \tag{1.1.1}$$

The proof of assertion (ii) is concluded by combining the inequality (1.1.1) with the one obtained by interchanging the roles of a and b. On the other hand, it follows from the inequality (1.1.1) that

$$\frac{1-\|a^{-1}\|\|a-b\|}{\|a^{-1}\|} = \frac{1}{\|a^{-1}\|} - \|a-b\| \leqslant \frac{1}{\|b^{-1}\|}. \tag{1.1.2}$$

Since the condition $\|a-b\| < \frac{1}{\|a^{-1}\|}$ leads to $1-\|a^{-1}\|\|a-b\| > 0$, assertion (iii) follows from (1.1.2). □

Corollary 1.1.14 *Let A be a normed unital associative algebra over $\mathbb{K}$, let a be in A, and let z be in $\mathbb{K}$ such that $a - z\mathbf{1} \in \mathrm{Inv}(A)$ and $|z| > \|\mathbf{1}\|\,\|a\|$. Then*

$$\|(a-z\mathbf{1})^{-1}\| \leqslant \frac{\|\mathbf{1}\|}{|z| - \|\mathbf{1}\|\,\|a\|}.$$

Proof We have $\|-z\mathbf{1}-(a-z\mathbf{1})\| = \|a\| < \dfrac{1}{\|(z\mathbf{1})^{-1}\|}$, and hence, by Lemma 1.1.13(iii),

$$\|(a-z\mathbf{1})^{-1}\| \leqslant \frac{\|(z\mathbf{1})^{-1}\|}{1-\|(z\mathbf{1})^{-1}\|\|a\|} = \frac{\|\mathbf{1}\|}{|z|-\|\mathbf{1}\|\,\|a\|}.$$ □

Let A be a unital associative algebra over $\mathbb{K}$. It is straightforward that ab and a^{-1} belong to $\mathrm{Inv}(A)$ whenever a, b are in $\mathrm{Inv}(A)$. As a consequence, the set $\mathrm{Inv}(A)$ is a group with respect to the product of A. We recall that a *topological group* is a group G endowed with a topology making the mappings $(x,y) \to xy$ from $G \times G$ to G, and $x \to x^{-1}$ from G to G, continuous.

Proposition 1.1.15 *Let A be a normed unital associative algebra over $\mathbb{K}$. Then* $\mathrm{Inv}(A)$ *is a topological group in the induced topology from A.*

Proof Keeping in mind the continuity of the product of A, it only remains to verify the continuity of the mapping $x \to x^{-1}$ from $\mathrm{Inv}(A)$ to A. By Lemma 1.1.13(ii), the mapping $x \to \frac{1}{\|x^{-1}\|}$ from $\mathrm{Inv}(A)$ to $\mathbb{R}$ is continuous, and hence so is the mapping $x \to \|x^{-1}\|$. Now, the proof concludes by invoking Lemma 1.1.13(i). □

§**1.1.16** Let A be a normed associative algebra over $\mathbb{K}$, and let a be in A. We define powers of a by $a^1 := a$ and $a^{n+1} := aa^n$. It is easily realized that $a^{n+m} = a^n a^m$ for all $n, m \in \mathbb{N}$. Now assume that A is normed. Then we define the *spectral radius* $\mathfrak{r}(a)$ of a by

$$\mathfrak{r}(a) := \inf\left\{\|a^n\|^{\frac{1}{n}} : n \in \mathbb{N}\right\}.$$

Obviously, $\mathfrak{r}(a) \leqslant \|a\|$ and $\mathfrak{r}(\lambda a) = |\lambda|\mathfrak{r}(a)$ for $\lambda \in \mathbb{K}$. It is also clear that, as the infimum of a family of continuous functions, $\mathfrak{r}(\cdot)$ becomes an upper semicontinuous function on A.

Lemma 1.1.17 *Let α_n be a sequence of non-negative real numbers satisfying*

$$\alpha_{n+m} \leqslant \alpha_n \alpha_m \ \textit{for all} \ \ n,m \in \mathbb{N}.$$

Then the limit $\lim \alpha_n^{\frac{1}{n}}$ *exists and is equal to* $\inf\{\alpha_n^{\frac{1}{n}} : n \in \mathbb{N}\}$.

Proof Write $\alpha = \inf\{\alpha_n^{\frac{1}{n}} : n \in \mathbb{N}\}$ and let $\varepsilon > 0$. Fix k such that $\alpha_k^{\frac{1}{k}} < \alpha + \varepsilon$. Any natural number $n \geqslant k$ can be written uniquely in the form $n = q(n)k + r(n)$, where $q(n) \in \mathbb{N}$ and $0 \leqslant r(n) \leqslant k-1$, and hence, setting α_0 equal to 1, we obtain

$$\alpha_n \leqslant \alpha_{r(n)} \alpha_k^{q(n)} \leqslant \max\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} (\alpha + \varepsilon)^{q(n)k}.$$

Since $\frac{r(n)}{n} \to 0$, we have $\frac{q(n)k}{n} \to 1$ as $n \to \infty$, and hence

$$\alpha_n^{\frac{1}{n}} \leqslant \max\{1, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}\}^{\frac{1}{n}} (\alpha + \varepsilon)^{\frac{q(n)k}{n}} \to \alpha + \varepsilon$$

as $n \to \infty$. Thus $\limsup \alpha_n^{\frac{1}{n}} \leqslant \alpha + \varepsilon$ and, since ε was arbitrary and $\alpha \leqslant \alpha_n^{\frac{1}{n}}$ for every n, we conclude that $\lim \alpha_n^{\frac{1}{n}} = \alpha$. □

Corollary 1.1.18 *Let A be a normed associative algebra over $\mathbb{K}$, and let a be in A. We have:*

(i) $\mathfrak{r}(a) = \lim \|a^n\|^{\frac{1}{n}}$.
(ii) *If* $\mathfrak{r}(a) < 1$, *then the sequence a^n converges to zero.*

Proof Assertion (i) follows from Lemma 1.1.17 above and the fact that

$$\|a^{n+m}\| \leqslant \|a^n\| \, \|a^m\| \quad \text{for all} \quad n,m \in \mathbb{N}.$$

Assume that $\mathfrak{r}(a) < 1$. Choose $\mathfrak{r}(a) < \eta < 1$. By assertion (i), we have $\|a^n\|^{\frac{1}{n}} < \eta$ for $n \in \mathbb{N}$ large enough, and hence $\|a^n\| < \eta^n \to 0$. Thus assertion (ii) has been proved. □

Corollary 1.1.19 *Let A and B be normed associative algebras over $\mathbb{K}$, let $F : A \to B$ be a continuous algebra homomorphism, and let a be in A. Then $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$. As a consequence, every equivalent algebra norm on A gives rise to the same spectral radius on A.*

Proof For $n \in \mathbb{N}$, we have

$$\|F(a)^n\| = \|F(a^n)\| \leqslant \|F\| \|a^n\|.$$

Therefore, by taking nth roots, and letting $n \to \infty$, Corollary 1.1.18(i) gives $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$. □

Lemma 1.1.20 (von Neumann) *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let a be in A with $\mathfrak{r}(a) < 1$. Then $\mathbf{1} - a \in \mathrm{Inv}(A)$ and*

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n,$$

where $a^0 := \mathbf{1}$.

Proof Choose η with $\mathfrak{r}(a) < \eta < 1$. By Corollary 1.1.18(i), we have $\|a^n\| \leqslant \eta^n$ for n large enough, and therefore the series $\sum \|a^n\|$ converges. It follows from the completeness of A that the series $\sum a^n$ is convergent in A. Since for each n we have

$$(\mathbf{1}-a)(\mathbf{1}+a+\cdots+a^n) = (\mathbf{1}+a+\cdots+a^n)(\mathbf{1}-a) = \mathbf{1}-a^{n+1},$$

it follows that

$$(\mathbf{1}-a)\left(\sum_{n=0}^{\infty} a^n\right) = \left(\sum_{n=0}^{\infty} a^n\right)(\mathbf{1}-a) = \mathbf{1}.$$

Thus $\mathbf{1}-a$ is an invertible element of A, and its inverse is $\sum_{n=0}^{\infty} a^n$. □

Corollary 1.1.21 *Let A be a complete normed unital associative algebra over $\mathbb{K}$. We have:*

(i) *If $a \in A$ satisfies $\|\mathbf{1}-a\| < 1$, then $a \in \mathrm{Inv}(A)$.*
(ii) *If $a \in \mathrm{Inv}(A)$, and if $b \in A$ satisfies $\|a-b\| < \frac{1}{\|a^{-1}\|}$, then $b \in \mathrm{Inv}(A)$.*

Proof Assertion (i) follows by writing $a = \mathbf{1}-(\mathbf{1}-a)$ and by applying Lemma 1.1.20. Given $a \in \mathrm{Inv}(A)$ and $b \in A$ such that $\|a-b\| < \frac{1}{\|a^{-1}\|}$, we see that

$$\|\mathbf{1}-a^{-1}b\| = \|a^{-1}(a-b)\| \leqslant \|a^{-1}\|\|a-b\| < 1.$$

Therefore, by assertion (i), $a^{-1}b \in \mathrm{Inv}(A)$, and so $b = a(a^{-1}b) \in \mathrm{Inv}(A)$. □

Lemma 1.1.22 *Let A be a unital associative algebra over $\mathbb{K}$, and let x and y be in* $\mathrm{Inv}(A)$. *Then*

$$x^{-1}-y^{-1}-y^{-1}(y-x)y^{-1} = y^{-1}(y-x)x^{-1}(y-x)y^{-1}.$$

Proof We have

$$\begin{aligned} x^{-1}-y^{-1}-y^{-1}(y-x)y^{-1} &= x^{-1}(y-x)y^{-1}-y^{-1}(y-x)y^{-1} \\ &= (x^{-1}-y^{-1})(y-x)y^{-1} \\ &= y^{-1}(y-x)x^{-1}(y-x)y^{-1}. \end{aligned}$$ □

Let X,Y be normed spaces over $\mathbb{K}$, let Ω be a non-empty open subset of X, let x_0 be in Ω, and let $f:\Omega \to Y$ be a function. We recall that f is said to be *(Fréchet) differentiable at* x_0 if there exists $T \in BL(X,Y)$ such that

$$\lim_{\substack{x\to x_0 \\ x\in\Omega\setminus\{x_0\}}} \frac{\|f(x)-f(x_0)-T(x-x_0)\|}{\|x-x_0\|} = 0.$$

In this case, the operator T is unique, and is called the *(Fréchet) derivative of f at* x_0. When $X = \mathbb{K}$, the natural identification $BL(\mathbb{K},Y) \equiv Y$ allows us to see the derivative of f at x_0 as the element $f'(x_0) \in Y$ given by

$$f'(x_0) := \lim_{\substack{x\to x_0 \\ x\in\Omega\setminus\{x_0\}}} \frac{f(x)-f(x_0)}{x-x_0}.$$

Theorem 1.1.23 *Let A be a complete normed unital associative algebra over $\mathbb{K}$. Then $\mathrm{Inv}(A)$ is open in A. Moreover, the mapping $x \to x^{-1}$ from $\mathrm{Inv}(A)$ to A is differentiable at any point $a \in \mathrm{Inv}(A)$, with derivative equal to the mapping $x \to -a^{-1}xa^{-1}$ from A to A.*

Proof The first conclusion follows from Corollary 1.1.21(ii). Let us fix $a \in \mathrm{Inv}(A)$. Then, by Lemma 1.1.22, for each $x \in \mathrm{Inv}(A)$ we have

$$x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}] = a^{-1}(x-a)x^{-1}(x-a)a^{-1},$$

and hence

$$\|x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}]\| \leqslant \|a^{-1}\|^2 \, \|x^{-1}\| \, \|x-a\|^2.$$

Since the mapping $x \to \|x^{-1}\|$ is continuous (by Lemma 1.1.13(ii)), we derive

$$\lim_{\substack{x \to a \\ x \in \mathrm{Inv}(A)\setminus\{a\}}} \frac{\|x^{-1} - a^{-1} - [-a^{-1}(x-a)a^{-1}]\|}{\|x-a\|} = 0.$$

Therefore, the mapping $x \to x^{-1}$ is differentiable at a with derivative the mapping $T \in BL(A)$ given by $T(x) = -a^{-1}xa^{-1}$. □

§**1.1.24** Let A be an algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Since the intersection of any family of subalgebras of A is again a subalgebra of A, it follows that the intersection of all subalgebras of A containing S is the smallest subalgebra of A containing S. This subalgebra is called *the subalgebra of A generated by S*, and is denoted by $A(S)$.

Exercise 1.1.25 Let A be a unital algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Prove that $A(S \cup \{\mathbf{1}\}) = \mathbb{K}\mathbf{1} + A(S)$.

Now, let A be a normed algebra, and let S be a non-empty subset of A. Since the intersection of any family of closed subalgebras of A is again a closed subalgebra of A, it follows that the intersection of all closed subalgebras of A containing S is the smallest closed subalgebra of A containing S. This subalgebra is called *the closed subalgebra of A generated by S*, and is denoted by $\bar{A}(S)$.

Exercise 1.1.26 Let A be a normed algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Prove that:

(i) If S is a subalgebra of A, then so is $\bar{S}$.
(ii) $\bar{A}(S) = \overline{A(S)}$.
(iii) If A is unital, then $\bar{A}(S \cup \{\mathbf{1}\}) = \mathbb{K}\mathbf{1} + \bar{A}(S)$.

§**1.1.27** As usual, we denote by $\mathbb{K}[\mathbf{x}]$ the algebra of all polynomials in the indeterminate $\mathbf{x}$ with coefficients in $\mathbb{K}$. Let A be a unital associative algebra over $\mathbb{K}$, and let $a \in A$. Given a polynomial $p(\mathbf{x}) = \sum_{k=0}^{n} \alpha_k \mathbf{x}^k$ with coefficients $\alpha_k \in \mathbb{K}$, we denote by $p(a)$ the element of A given by $p(a) = \sum_{k=0}^{n} \alpha_k a^k$. It is clear that the mapping $p \to p(a)$ is a unit-preserving algebra homomorphism from $\mathbb{K}[\mathbf{x}]$ onto the subalgebra of A generated by $\mathbf{1}$ and a.

If A is a unital associative algebra and if B is a subalgebra of A containing the unit of A, then it is clear that $\mathrm{Inv}(B) \subseteq B \cap \mathrm{Inv}(A)$. The next example shows that the reverse inclusion may not, in general, be true, even in a complete normed context.

Example 1.1.28 Consider the complete normed unital associative and commutative algebra $C^{\mathbb{C}}(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let u be the element of $C^{\mathbb{C}}(\mathbb{T})$ given by $u(z) = z$ for every $z \in \mathbb{T}$. It is clear that u is invertible in $C^{\mathbb{C}}(\mathbb{T})$ and that the inverse of u is the function defined by $u^{-1}(z) = \frac{1}{z}$ for every $z \in \mathbb{T}$. Let B (respectively, C) denote the subalgebra (respectively, closed subalgebra) of $C^{\mathbb{C}}(\mathbb{T})$ generated by $\{\mathbf{1}, u\}$. Note that B is nothing other than the subalgebra of $C^{\mathbb{C}}(\mathbb{T})$ consisting of all complex polynomial functions, and that $C = \bar{B}$ because of Exercise 1.1.26(ii). If u were invertible in C, then we would have $u^{-1} \in C$, and therefore there would be a polynomial function p satisfying $\|u^{-1} - p\| < 1$. Thus, for $z \in \mathbb{T}$ we would have $|\frac{1}{z} - p(z)| < 1$, and hence $|1 - zp(z)| < 1$. Then, by the maximum modulus principle, the inequality $|1 - zp(z)| < 1$ would be true for every $z \in \mathbb{B}_{\mathbb{C}}$, and in particular $1 = |1 - 0p(0)| < 1$. This contradiction shows that u is not invertible in C.

§1.1.29 Given an element a in a complete normed unital associative algebra A, $\exp(a)$ is defined as the element of A given by

$$\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

where $a^0 := \mathbf{1}$.

Exercise 1.1.30 Let a and b be commuting elements of a complete normed unital associative algebra A. Prove that

$$\exp(a+b) = \exp(a)\exp(b), \quad \exp(a) \in \mathrm{Inv}(A), \quad \text{and} \quad \exp(a)^{-1} = \exp(-a).$$

Let A be a unital associative algebra over $\mathbb{K}$. By a *one-parameter semigroup* in A we mean a mapping $S : \mathbb{R}_0^+ \to A$ satisfying

(i) $S(0) = \mathbf{1}$.
(ii) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \in \mathbb{R}_0^+$.

If A is complete normed, and if a is any element of A, then it is clear that the mapping $S : \mathbb{R}_0^+ \to A$ defined by $S(t) := \exp(ta)$ becomes a continuous one-parameter semigroup in A. Conversely, we have the following.

Theorem 1.1.31 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let $S : \mathbb{R}_0^+ \to A$ be a continuous one-parameter semigroup in A. Then there exists an element a in A such that $S(t) = \exp(ta)$ for every $t \in \mathbb{R}_0^+$. Moreover, this element is given by the formula*

$$a = \lim_{t \to 0} \frac{S(t) - \mathbf{1}}{t}.$$

Proof Since S is continuous, the integral $\int_\alpha^\beta S(t)\,dt$ exists for all $\alpha, \beta \in \mathbb{R}_0^+$ and is an element of A. Further, by the fundamental theorem of calculus, we have

$$\lim_{\beta \to \alpha} \frac{1}{\beta - \alpha} \int_\alpha^\beta S(t)\,dt = S(\alpha)$$

for each $\alpha \in \mathbb{R}_0^+$. In particular, since $S(0) = \mathbf{1}$, it follows that there exists a $\delta > 0$ such that

$$\left\| \frac{1}{\beta} \int_0^\beta S(t)\,dt - \mathbf{1} \right\| < 1$$

for every β with $0 < \beta < \delta$. By Corollary 1.1.21(i), we realize that

$$\frac{1}{\beta} \int_0^\beta S(t)\,dt \in \mathrm{Inv}(A)$$

for every β with $0 < \beta < \delta$. On the other hand, it is clear that

$$\begin{aligned} \int_0^\beta S(t+r)\,dr - \int_0^\beta S(r)\,dr &= \int_0^\beta S(t)S(r)\,dr - \int_0^\beta S(r)\,dr \\ &= (S(t) - \mathbf{1}) \int_0^\beta S(r)\,dr, \end{aligned}$$

and

$$\begin{aligned} \int_0^\beta S(t+r)\,dr - \int_0^\beta S(r)\,dr &= \int_t^{\beta+t} S(r)\,dr - \int_0^\beta S(r)\,dr \\ &= \int_\beta^{\beta+t} S(r)\,dr - \int_0^t S(r)\,dr. \end{aligned}$$

Therefore, we have

$$\frac{1}{t}(S(t) - \mathbf{1}) \int_0^\beta S(r)\,dr = \frac{1}{t} \int_\beta^{\beta+t} S(r)\,dr - \frac{1}{t} \int_0^t S(r)\,dr. \tag{1.1.3}$$

For $\beta \in]0, \delta[$ we obtain

$$\frac{1}{t}(S(t) - \mathbf{1}) = \left[\frac{1}{t} \int_\beta^{\beta+t} S(r)\,dr - \frac{1}{t} \int_0^t S(r)\,dr \right] \left[\int_0^\beta S(r)\,dr \right]^{-1}.$$

When $t \to 0$ the right-hand side tends to a limit since the first factor tends to $S(\beta) - \mathbf{1}$. It follows that

$$a := \lim_{t \to 0} \frac{S(t) - \mathbf{1}}{t} \tag{1.1.4}$$

exists and is an element of A. Keeping in mind (1.1.4), and again taking the limit as $t \to 0$ in (1.1.3), we obtain

$$S(\beta) = \mathbf{1} + a \int_0^\beta S(r)\,dr \tag{1.1.5}$$

for every $\beta \in \mathbb{R}_0^+$. Note that, for each $t \in \mathbb{R}_0^+$ we have

$$S'(t) = \lim_{r \to 0} \frac{S(t+r) - S(t)}{r} = \lim_{r \to 0} \frac{S(t)S(r) - S(t)}{r} = aS(t),$$

and hence, for every $k \in \mathbb{N} \cup \{0\}$ we have

$$\frac{d}{dr}\left[-\frac{(t-r)^{k+1}}{k+1} S(r) \right] = (t-r)^k S(r) - \frac{(t-r)^{k+1}}{k+1} aS(r),$$

and consequently

$$\int_0^t (t-r)^k S(r)\,dr = \frac{t^{k+1}}{k+1}\mathbf{1} + \frac{a}{k+1} \int_0^t (t-r)^{k+1} S(r)\,dr. \tag{1.1.6}$$

Iterated substitution of (1.1.6) (with $k = 0,\dots,n-1$) in (1.1.5) leads to

$$S(t) = \mathbf{1} + \frac{t}{1!}a + \frac{t^2}{2!}a^2 + \cdots + \frac{t^n}{n!}a^n + \frac{a^{n+1}}{n!}\int_0^t (t-r)^n S(r)\,dr$$

for every $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\left\| S(t) - \left(\mathbf{1} + \frac{t}{1!}a + \frac{t^2}{2!}a^2 + \cdots + \frac{t^n}{n!}a^n\right)\right\| &= \left\|\frac{a^{n+1}}{n!}\int_0^t (t-r)^n S(r)\,dr\right\| \\ &\leqslant \frac{\|a\|^{n+1}}{n!}\left\|\int_0^t (t-r)^n S(r)\,dr\right\| \\ &\leqslant \frac{\|a\|^{n+1}}{n!}t^{n+1}\max\{\|S(r)\| : 0 \leqslant r \leqslant t\}.\end{aligned}$$

Letting $n \to \infty$ we obtain that

$$S(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}a^n = \exp(ta),$$

which concludes the proof. □

Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A. We define the *spectrum of a relative to A* (denoted by $\mathrm{sp}(A,a)$, or simply $\mathrm{sp}(a)$ when the algebra A is without doubt) as the subset of $\mathbb{K}$ given by

$$\mathrm{sp}(A,a) := \{\mu \in \mathbb{K} : a - \mu\mathbf{1} \notin \mathrm{Inv}(A)\}.$$

Example 1.1.32 (a) Let E be a non-empty set, and let x be in $F^{\mathbb{K}}(E)$. As a consequence of Example 1.1.12(a), we have $\mathrm{sp}(F^{\mathbb{K}}(E),x) = \{x(t) : t \in E\}$.

(b) Let X be a nonzero vector space over $\mathbb{K}$, and let F be in $L(X)$. As a consequence of Example 1.1.12(b), we have

$$\mathrm{sp}(L(X),F) = \{\mu \in \mathbb{K} : F - \mu I_X \text{ is not bijective}\}.$$

(c) Let E be a compact Hausdorff topological space, and let x be in $C^{\mathbb{K}}(E)$. As a consequence of Example 1.1.12(c), we have

$$\mathrm{sp}(C^{\mathbb{K}}(E),x) = \{x(t) : t \in E\},$$

and hence $\mathrm{sp}(C^{\mathbb{K}}(E),x) = \mathrm{sp}(F^{\mathbb{K}}(E),x)$.

(d) Let X be a nonzero normed space over $\mathbb{K}$, and let F be in $BL(X)$. As a consequence of Example 1.1.12(d), we have

$$\mathrm{sp}(BL(X),F) = \{\mu \in \mathbb{K} : F - \mu I_X \text{ has not a continuous inverse}\}.$$

In the case that X is in fact a Banach space, we actually have

$$\mathrm{sp}(BL(X),F) = \{\mu \in \mathbb{K} : F - \mu I_X \text{ is not bijective}\},$$

and hence $\mathrm{sp}(BL(X),F) = \mathrm{sp}(L(X),F)$.

Fact 1.1.33 *Let A be a unital associative algebra over $\mathbb{K}$, and let a and b be in A. Then we have:*

(i) $\mathbf{1}-ab \in \mathrm{Inv}(A)$ *if and only if* $\mathbf{1}-ba \in \mathrm{Inv}(A)$.
(ii) $\mathrm{sp}(A,ab)\backslash\{0\} = \mathrm{sp}(A,ba)\backslash\{0\}$.

Proof Assertion (i) follows from the observation that if $\mathbf{1}-ab$ has inverse c, then $\mathbf{1}-ba$ has inverse $\mathbf{1}+bca$. Assertion (ii) follows from assertion (i). □

The following lemma is straightforward.

Lemma 1.1.34 *Let A and B be unital associative algebras over $\mathbb{K}$, and let $\phi : A \to B$ be a unit-preserving algebra homomorphism. Then*

(i) $\phi(\mathrm{Inv}(A)) \subseteq \mathrm{Inv}(B)$. *More precisely, if $a \in \mathrm{Inv}(A)$, then $\phi(a) \in \mathrm{Inv}(B)$ with* $\phi(a)^{-1} = \phi(a^{-1})$.
(ii) *For each $a \in A$,* $\mathrm{sp}(B,\phi(a)) \subseteq \mathrm{sp}(A,a)$.

Lemma 1.1.35 *Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A. Then a is invertible in A if (and only if) there exists a unique $b \in A$ such that $ab = \mathbf{1}$.*

Proof Assume that there exists a unique $b \in A$ such that $ab = \mathbf{1}$. Then $aba = a$, so $a(b+ba-\mathbf{1}) = \mathbf{1}$. It follows from the uniqueness of b that $b = b+ba-\mathbf{1}$, and hence $ba = \mathbf{1}$. □

§1.1.36 Let A be an algebra over $\mathbb{K}$. We define the *opposite algebra* of A as the algebra over $\mathbb{K}$ consisting of the vector space of A and the product $(a,b) \to ba$, and denote it by $A^{(0)}$. Given another algebra B over $\mathbb{K}$, algebra homomorphisms from A to $B^{(0)}$ are called *algebra antihomomorphisms* from A to B. Given an element a in an algebra A, we denote by L_a (respectively, R_a) the operator of left (respectively, right) multiplication by a on A. The mappings $L : a \to L_a$ and $R : a \to R_a$ from A to $L(A)$ are linear, with $L_{\mathbf{1}} = R_{\mathbf{1}} = I_A$ whenever A is unital. Moreover A is associative if and only if L is an algebra homomorphism, if and only if R is an algebra antihomomorphism. If A is normed, then L_a and R_a lie in $BL(A)$ for every $a \in A$ and, as mappings from A to $BL(A)$, L and R are contractive.

Lemma 1.1.37 *Let A be a unital associative algebra over $\mathbb{K}$, and let $a \in A$. We have:*

(i) $a \in \mathrm{Inv}(A)$ *if and only if* $L_a \in \mathrm{Inv}(L(A))$. *Moreover, if* $a \in \mathrm{Inv}(A)$, *then* $L_a^{-1} = L_{a^{-1}}$.
(ii) $\mathrm{sp}(A,a) = \mathrm{sp}(L(A),L_a)$.

Proof Since the mapping $x \to L_x$ is a unit-preserving algebra homomorphism from A to $L(A)$, it follows from Lemma 1.1.34(i) that $a \in \mathrm{Inv}(A)$ leads to $L_a \in \mathrm{Inv}(L(A))$ and $L_a^{-1} = L_{a^{-1}}$. Conversely, if $L_a \in \mathrm{Inv}(L(A))$, then, by Lemma 1.1.35, $a \in \mathrm{Inv}(A)$. □

Invoking Example 1.1.32(d), we get the following.

Corollary 1.1.38 *Let A be a normed unital associative algebra over $\mathbb{K}$, and let $a \in A$. The following conditions are equivalent:*

(i) $a \in \mathrm{Inv}(A)$.
(ii) $L_a \in \mathrm{Inv}(BL(A))$.
(iii) $L_a \in \mathrm{Inv}(L(A))$.

As a consequence, $\mathrm{sp}(A,a) = \mathrm{sp}(BL(A), L_a) = \mathrm{sp}(L(A), L_a)$.

Lemma 1.1.39 *Let A be a normed unital associative algebra over $\mathbb{K}$, let a be in A, and let W be an open subset of $\mathbb{K}$ contained in $\mathbb{K}\backslash\mathrm{sp}(A,a)$. Then the mapping $f : W \to A$ given by $f(\lambda) = (a-\lambda\mathbf{1})^{-1}$ is differentiable at any point $\lambda \in W$ with derivative $f'(\lambda) = (a-\lambda\mathbf{1})^{-2}$.*

Proof Let us fix $\lambda \in W$. Then, for $\mu \in W$ we have

$$\begin{aligned} f(\mu) - f(\lambda) &= (a-\mu\mathbf{1})^{-1} - (a-\lambda\mathbf{1})^{-1} \\ &= (a-\mu\mathbf{1})^{-1}\,[(a-\lambda\mathbf{1}) - (a-\mu\mathbf{1})]\,(a-\lambda\mathbf{1})^{-1} \\ &= (\mu-\lambda)(a-\mu\mathbf{1})^{-1}(a-\lambda\mathbf{1})^{-1}, \end{aligned}$$

and, keeping Proposition 1.1.15 in mind, we derive that

$$f'(\lambda) = \lim_{\mu\to\lambda} (a-\mu\mathbf{1})^{-1}(a-\lambda\mathbf{1})^{-1} = (a-\lambda\mathbf{1})^{-2}. \qquad \square$$

Proposition 1.1.40 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let a be in A. Then $\mathrm{sp}(A,a)$ is a compact subset of $\mathbb{K}$ contained in $\mathfrak{r}(a)\mathbb{B}_{\mathbb{K}}$. Moreover, the mapping $\lambda \to (a-\lambda\mathbf{1})^{-1}$ from $\mathbb{K}\backslash\mathrm{sp}(A,a)$ to A is differentiable.*

Proof Let $\lambda \in \mathbb{K}$ with $|\lambda| > \mathfrak{r}(a)$. Then $\mathfrak{r}(\lambda^{-1}a) < 1$, and so, by Lemma 1.1.20, $a - \lambda\mathbf{1} = -\lambda(\mathbf{1} - \lambda^{-1}a) \in \mathrm{Inv}(A)$, i.e. $\lambda \notin \mathrm{sp}(A,a)$. This proves that $\mathrm{sp}(A,a)$ is a subset of $\mathfrak{r}(a)\mathbb{B}_{\mathbb{K}}$, and consequently that $\mathrm{sp}(A,a)$ is bounded in $\mathbb{K}$. Now, in view of Lemma 1.1.39, to conclude the proof it is enough to show that $\mathbb{K}\backslash\mathrm{sp}(A,a)$ is an open subset of $\mathbb{K}$. But this follows from the fact that $\mathrm{Inv}(A)$ is an open subset of A (by Theorem 1.1.23) and the equality

$$\mathbb{K}\backslash\mathrm{sp}(A,a) = h^{-1}(\mathrm{Inv}(A)),$$

where h denotes the continuous function $\lambda \to a - \lambda\mathbf{1}$ from $\mathbb{K}$ to A. $\square$

Now we prove one of the fundamental results in the theory of normed algebras.

Theorem 1.1.41 *Let A be a normed unital associative complex algebra, and let a be in A. Then $\mathrm{sp}(A,a) \neq \emptyset$.*

Proof To derive a contradiction, suppose that $\mathrm{sp}(A,a) = \emptyset$. Then, by Lemma 1.1.39, the mapping $f : \mathbb{C} \to A$ given by $f(z) = (a - z\mathbf{1})^{-1}$ is an entire function. On the other hand, by Corollary 1.1.14, for each $z \in \mathbb{C}$ with $|z| > \|\mathbf{1}\|\,\|a\|$, we have $\|f(z)\| \leqslant \frac{\|\mathbf{1}\|}{|z| - \|\mathbf{1}\|\,\|a\|}$, and as a consequence

$$\lim_{z\to\infty} f(z) = 0.$$

It follows from Liouville's theorem that $f(z) = 0$ for each $z \in \mathbb{C}$. In particular,

$$a^{-1} = f(0) = 0,$$

a contradiction. $\square$

Definition 1.1.42 Let A be an associative algebra over $\mathbb{K}$. We say that A is a *division* algebra if A is unital and $\mathrm{Inv}(A) = A \setminus \{0\}$.

The next corollary is known as the *complex Gelfand–Mazur theorem*.

Corollary 1.1.43 *Let A be a normed division associative complex algebra. Then A is isomorphic to $\mathbb{C}$.*

Proof For every $a \in A$ there exists $\lambda \in \mathrm{sp}(A,a)$, hence

$$a - \lambda \mathbf{1} \notin \mathrm{Inv}(A) = A \setminus \{0\},$$

and so $a = \lambda \mathbf{1}$. Now, it is clear that the mapping $\lambda \to \lambda \mathbf{1}$ is a bijective algebra homomorphism from $\mathbb{C}$ to A. □

Combining Proposition 1.1.40 and Theorem 1.1.41 we have the following.

Corollary 1.1.44 *Let A be a complete normed unital associative complex algebra, and let $a \in A$. Then $\mathrm{sp}(A,a)$ is a non-empty compact subset of $\mathbb{C}$ contained in $\mathfrak{r}(a)\mathbb{B}_{\mathbb{C}}$.*

Let K be a non-empty compact subset of $\mathbb{K}$. Then there exists a complete normed unital associative and commutative algebra A over $\mathbb{K}$, and an element $a \in A$, such that $\mathrm{sp}(A,a) = K$. Indeed, according to Example 1.1.32(c), it is enough to take $A := C^{\mathbb{K}}(K)$, and a equal to the element of $C^{\mathbb{K}}(K)$ defined by $a(t) := t$ for every $t \in K$. If we dispense the commutativity, then the algebra A can be chosen independent of K. This is realized by means of the following.

Exercise 1.1.45 Let K be a non-empty compact subset of $\mathbb{K}$, and let H stand for the infinite-dimensional separable Hilbert space over $\mathbb{K}$. Prove that there exists $T \in BL(H)$ such that $\mathrm{sp}(BL(H),T) = K$.

Solution We may take $H = \ell_2$. Note that each bounded sequence α_n in $\mathbb{K}$ gives rise to a bounded linear operator on H, just the one $\{\mu_n\} \to \{\alpha_n\mu_n\}$. Now, let $n \to \lambda_n$ be a mapping from $\mathbb{N}$ onto a dense subset of K, and consider the operator $T \in BL(H)$ defined by $T(\{\mu_n\}) := \{\lambda_n\mu_n\}$. Then, since $T - \lambda_n I_H$ is not injective for each n, we have $\{\lambda_n : n \in \mathbb{N}\} \subseteq \mathrm{sp}(BL(H),T)$, and hence, by Proposition 1.1.40, also $K \subseteq \mathrm{sp}(BL(H),T)$. To prove the converse inclusion, note that, if $\lambda \in \mathbb{K} \setminus K$, then the sequence $\frac{1}{\lambda_n - \lambda}$ is bounded, and hence we can consider the operator $T_\lambda \in BL(H)$ defined by $T_\lambda(\{\mu_n\}) := \{\frac{\mu_n}{\lambda_n - \lambda}\}$, which satisfies $T_\lambda(T - \lambda I_H) = (T - \lambda I_H)T_\lambda = I_H$. □

Theorem 1.1.46 (Gelfand–Beurling) *Let A be a complete normed unital associative complex algebra, and let a be in A. Then we have*

$$\mathfrak{r}(a) = \max\{|\lambda| : \lambda \in \mathrm{sp}(A,a)\}.$$

Proof It follows from Corollary 1.1.44 that

$$\rho(a) := \max\{|\lambda| : \lambda \in \mathrm{sp}(A,a)\} \leqslant \mathfrak{r}(a).$$

In order to prove the converse inequality, consider the set

$$W := \{z \in \mathbb{C} : \mathbf{1} - za \in \mathrm{Inv}(A)\}.$$

Note that, for $z \in \mathbb{C}\backslash\{0\}$, we have that $z \in W$ if and only if $z^{-1} \in \mathbb{C}\backslash \mathrm{sp}(A,a)$. Therefore, it follows from Corollary 1.1.44 that W is an open subset of $\mathbb{C}$ containing $\{z \in \mathbb{C} : |z| < \rho(a)^{-1}\}$. Now, consider the mapping $h : W \to A$ given by $h(z) := (\mathbf{1} - za)^{-1}$. Note that $h(z) = -z^{-1}(a - z^{-1}\mathbf{1})^{-1}$ for every $z \in W\backslash\{0\}$, and hence, by Proposition 1.1.40, h is differentiable in $W\backslash\{0\}$. Since, by Proposition 1.1.15, h is continuous in W, it follows that h is differentiable in W. [Alternatively, by Theorem 1.1.23, h is differentiable in W.] Keeping in mind Lemma 1.1.20, for each $z \in \mathbb{C}$ with $|z| < \mathfrak{r}(a)^{-1}$ we can write

$$h(z) = \sum_{n=0}^{\infty} a^n z^n.$$

Therefore, for each $f \in A'$, the mapping $f \circ h$ is analytic in W and

$$(f \circ h)(z) = \sum_{n=0}^{\infty} f(a^n) z^n$$

is the power series expansion of $f \circ h$ at $z = 0$. By Taylor's theorem, this power series expansion converges for $z \in \mathbb{C}$ with $|z| < \rho(a)^{-1}$. Therefore, for each $z \in \mathbb{C}$ with $|z| < \rho(a)^{-1}$, the sequence $f(a^n)z^n$ converges to 0, and therefore is bounded. Since this is true for each $f \in A'$, it follows from the principle of uniform boundedness that $a^n z^n$ is a bounded sequence. Hence, there is a positive number M (depending of z, of course) such that $\|a^n z^n\| \leqslant M$ for all n, and therefore $\|a^n\|^{\frac{1}{n}} \leqslant M^{\frac{1}{n}} \frac{1}{|z|}$. Consequently, $\mathfrak{r}(a) \leqslant \frac{1}{|z|}$. Finally, by letting $|z| \to \rho(a)^{-1}$, we conclude that $\mathfrak{r}(a) \leqslant \rho(a)$. □

Theorem 1.1.46, together with Example 1.1.32(d), provides us with the following purely algebraic characterization of the spectral radius of a bounded linear operator on a complex Banach space.

Corollary 1.1.47 *Let X be a complex Banach space, and let F be in $BL(X)$. Then*

$$\mathfrak{r}(F) = \max\{|\mu| : \mu \in \mathrm{sp}(L(X), F)\}.$$

1.1.2 Rickart's dense-range-homomorphism theorem

Let A be an algebra over $\mathbb{K}$. A subset I of A is called a *left* (respectively, *right*) *ideal* of A if I is a subspace of A and $AI \subseteq I$ (respectively, $IA \subseteq I$). The subset I is said to be a (*two-sided*) *ideal* of A if it is both a left and right ideal of A.

Exercise 1.1.48 Let A be a normed algebra over $\mathbb{K}$, and let I be a $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A. Prove that the closure of I in A is a $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A.

A left, right or two-sided ideal I of an algebra A is called *proper* if $I \neq A$. In the case that A has a unit $\mathbf{1}$, it is clear that I is proper if and only if $\mathbf{1} \notin I$. §1.1.36 should be kept in mind for the proof of the next proposition.

Proposition 1.1.49 *Let A be a complete normed unital algebra over $\mathbb{K}$. Then the closure of a proper* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A is a proper* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A.*

Proof Assume for example that I is a left ideal of A, and suppose that $\bar{I} = A$. Then, choose $b \in I$ such that $\|\mathbf{1} - b\| < 1$, and note that

$$\|I_A - R_b\| = \|R_{\mathbf{1}} - R_b\| = \|R_{\mathbf{1}-b}\| \leqslant \|\mathbf{1} - b\| < 1.$$

By Corollary 1.1.21(i), we find that the element R_b is invertible in $BL(A)$, hence R_b is bijective, and so $A = R_b(A) = Ab \subseteq I$. □

The assumption in the above proposition that A is unital cannot be removed, even if A is associative and commutative. A first counterexample is given by any infinite-dimensional Banach space, endowed with the zero product, by taking a dense proper subspace. A less trivial counterexample is obtained by taking A equal to c_0 (cf. Example 1.1.4(a)), and thinking about the ideal c_{00} of all quasi-null sequences. Another counterexample is the following.

Example 1.1.50 Let A stand for the closed subalgebra of $C^{\mathbb{K}}([0,1])$ consisting of those $x \in C^{\mathbb{K}}([0,1])$ such that $x(0) = 0$, and let I denote the proper ideal of A consisting of those $y \in C^{\mathbb{K}}([0,1])$ for which there is $0 < \delta_y \leqslant 1$ such that $y([0, \delta_y]) = 0$. We are going to show that I is dense in A. To this end, for $n \in \mathbb{N}$, let y_n be the element of I defined by

$$y_n(t) = \begin{cases} 0 & \text{if} \quad t \in \left[0, \frac{1}{n+1}\right[\\ n(n+1)\left(t - \frac{1}{n+1}\right) & \text{if} \quad t \in \left[\frac{1}{n+1}, \frac{1}{n}\right[\\ 1 & \text{if} \quad t \in \left[\frac{1}{n}, 1\right]. \end{cases}$$

Then, for all $x \in A$ and $n \in \mathbb{N}$ we have

$$\|xy_n - x\| \leqslant \max\left\{|x(t)| : t \in \left[0, \frac{1}{n}\right]\right\}.$$

Since x is a continuous function with $x(0) = 0$, it follows that $xy_n \to x$. Finally, since $xy_n \in I$ for every n, and x is arbitrary in A, we deduce that I is dense in A.

Definition 1.1.51 Let A be an algebra over $\mathbb{K}$, and let I be a $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A. We say that I is a *maximal* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal* of A if I is proper and if the only proper $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A containing I is I itself.

As an easy application of Zorn's lemma, one derives the following.

Fact 1.1.52 *Let A be a unital algebra over $\mathbb{K}$. Then there are maximal* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideals of A. More precisely, every proper* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A is contained in a maximal* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A.*

As a straightforward consequence of Proposition 1.1.49, we get the following.

Corollary 1.1.53 *Let A be a complete normed unital algebra over $\mathbb{K}$. Then every maximal* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A is closed.*

Let A be an algebra over $\mathbb{K}$, and let I be an ideal of A. Then the quotient vector space A/I becomes an algebra over $\mathbb{K}$, called the *quotient algebra*, for the (well-defined) product given by

$$(a+I)(b+I) := ab+I \quad \text{for all} \quad a+I, b+I \in A/I.$$

The mapping $\pi : A \to A/I$ defined by $\pi(a) := a+I$ becomes an algebra homomorphism, called the *quotient mapping* or *canonical homomorphism*. If A is unital, and if I is proper, then the quotient algebra A/I is unital, and the canonical homomorphism preserves units.

An algebra A is said to be *simple* if A has nonzero product and 0 is the only proper ideal of A.

Lemma 1.1.54 *Let A be a unital algebra over $\mathbb{K}$, and let M be a maximal ideal of A. Then the quotient algebra A/M is simple.*

Proof Clearly A/M has nonzero product. Let I be a proper ideal of A/M. Then $\pi^{-1}(I)$ is a proper ideal of A containing M, so we have $\pi^{-1}(I) = M$ by maximality of M, and so $I = 0$. □

§**1.1.55** Let A be a normed algebra, and let I be a closed ideal of A. Then the quotient algebra A/I becomes a normed algebra for the quotient norm

$$\|a+I\| := \inf\{\|a+x\| : x \in I\} \quad \text{for every} \quad a+I \in A/I.$$

Indeed, for $a, b \in A$ and $x, y \in I$, we have

$$\|ab+I\| \leqslant \|ab+xb+ay+xy\| = \|(a+x)(b+y)\| \leqslant \|a+x\|\|b+y\|,$$

and it is enough to take infimum in x, y to get $\|(a+I)(b+I)\| \leqslant \|a+I\|\|b+I\|$. The quotient mapping $\pi : A \to A/I$ is a continuous open mapping with $\|\pi\| \leqslant 1$.

A relevant notion in the automatic continuity theory is that of the *separating space* $\mathfrak{S}(\Phi)$ of a linear mapping Φ from a normed space X into a normed space Y, which is defined by

$$\mathfrak{S}(\Phi) := \{y \in Y : \text{ there exists } x_n \to 0 \text{ in } X \text{ with } \Phi(x_n) \to y\}.$$

With this notion in mind, the closed graph theorem reads as follows.

Fact 1.1.56 *Let X and Y be Banach spaces over $\mathbb{K}$, and let $\Phi : X \to Y$ be a linear mapping. Then Φ is continuous if (and only if) $\mathfrak{S}(\Phi) = 0$.*

Lemma 1.1.57 *Let X and Y be normed spaces over $\mathbb{K}$, and let $\Phi : X \to Y$ be a linear mapping. Then $\mathfrak{S}(\Phi)$ is a closed subspace of Y.*

Proof Clearly $\mathfrak{S}(\Phi)$ is a subspace of Y. Assume that $y \in Y$ is such that there exists a sequence y_n in $\mathfrak{S}(\Phi)$ converging to y. Then, for each $n \in \mathbb{N}$, we can choose x_n in X such that $\|x_n\| < \frac{1}{n}$ and $\|\Phi(x_n) - y_n\| < \frac{1}{n}$. Therefore $x_n \to 0$ and $\Phi(x_n) - y_n \to 0$.

Since $y_n \to y$, we obtain that $x_n \to 0$ and $\Phi(x_n) \to y$, and we conclude that $y \in \mathfrak{S}(\Phi)$. Thus $\mathfrak{S}(\Phi)$ is a closed subspace of Y. $\square$

Lemma 1.1.58 *Let A and B be normed algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be an algebra homomorphism with dense range. Then $\mathfrak{S}(\Phi)$ is an ideal of B.*

Proof Let b be in $\mathfrak{S}(\Phi)$, so that there is a sequence x_n in A such that $x_n \to 0$ and $\Phi(x_n) \to b$. Then, for each $a \in A$ we see that $ax_n \to 0$, $x_n a \to 0$,

$$\Phi(ax_n) = \Phi(a)\Phi(x_n) \to \Phi(a)b, \text{ and } \Phi(x_n a) = \Phi(x_n)\Phi(a) \to b\Phi(a).$$

Therefore $\Phi(a)b$ and $b\Phi(a)$ lie in $\mathfrak{S}(\Phi)$. Now, given $c \in B$, choose a sequence a_n in A such that $\Phi(a_n) \to c$, and note that by the above $\Phi(a_n)b$ and $b\Phi(a_n)$ lie in $\mathfrak{S}(\Phi)$. Since $\Phi(a_n)b \to cb$ and $b\Phi(a_n) \to bc$, and $\mathfrak{S}(\Phi)$ is closed in B (by Lemma 1.1.57), we realize that cb and bc lie in $\mathfrak{S}(\Phi)$. It follows from the arbitrariness of c in B that $\mathfrak{S}(\Phi)$ is an ideal of B. $\square$

Lemma 1.1.59 and Proposition 1.1.60 immediately below could have been proved much earlier.

Lemma 1.1.59 *Let A be a unital associative algebra over $\mathbb{K}$, and let a in A. If there exist $b, c \in A$ such that $ab = ca = \mathbf{1}$, then a is invertible in A.*

Proof Assume the existence of $b, c \in A$ satisfying $ab = ca = \mathbf{1}$. Then,

$$b = \mathbf{1}b = (ca)b = c(ab) = c\mathbf{1} = c,$$

and hence $a \in \mathrm{Inv}(A)$. $\square$

Proposition 1.1.60 *Let A be a complete normed complex algebra, let B be a complete normed unital associative complex algebra, let $\Phi : A \to B$ be an algebra homomorphism, and let a be in A. Then*

$$\mathfrak{r}(\Phi(a)) \leqslant \|a\|. \tag{1.1.7}$$

As a consequence,

$$1 \leqslant \|a\| + \|\mathbf{1} - \Phi(a)\|. \tag{1.1.8}$$

Proof We claim that $\mathbf{1} - \Phi(x)$ is invertible in B whenever x is in A and satisfies $\|x\| < 1$. Let x be in A with $\|x\| < 1$. Then we have $\|L_x\| < 1$ and $\|R_x\| < 1$, so that by Lemma 1.1.20 and Example 1.1.12(d), $I_A - L_x$ and $I_A - R_x$ are bijective operators on A. Therefore there exist $y, z \in A$ satisfying $xy = y - x$ and $zx = z - x$. Now we have

$$(\mathbf{1} - \Phi(x))(\mathbf{1} + \Phi(y)) = (\mathbf{1} + \Phi(z))(\mathbf{1} - \Phi(x)) = \mathbf{1},$$

and hence, by Lemma 1.1.59, $\mathbf{1} - \Phi(x)$ is invertible in B, as desired. Now, taking $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$, and applying the claim just proved to $x := \frac{a}{\lambda}$, we derive that $\mathrm{sp}(B, \Phi(a)) \subseteq \|a\|\mathbb{B}_{\mathbb{C}}$, so that the inequality (1.1.7) follows from Theorem 1.1.46.

By Theorem 1.1.41, we can choose $\lambda \in \mathrm{sp}(B, \Phi(a))$, and then clearly $1 - \lambda \in \mathrm{sp}(B, \mathbf{1} - \Phi(a))$. It follows from Corollary 1.1.44 and the inequality (1.1.7) already proved that

$$1 \leqslant |\lambda| + |1 - \lambda| \leqslant \mathfrak{r}(\Phi(a)) + \mathfrak{r}(\mathbf{1} - \Phi(a)) \leqslant \|a\| + \|\mathbf{1} - \Phi(a)\|,$$

and the proof of the proposition is complete. $\square$

Definition 1.1.61 Let A be a unital algebra over $\mathbb{K}$. The *strong radical*, or the *Brown–McCoy radical*, of A, denoted by s-Rad(A), is defined as the intersection of all maximal ideals of A. The algebra A is called *strongly semisimple* if s-Rad$(A) = 0$.

Theorem 1.1.62 (Rickart's dense-range-homomorphism theorem) *Let A be a complete normed complex algebra, let B be a complete normed unital associative complex algebra, and let $\Phi : A \to B$ be an algebra homomorphism with dense range. Assume that B is strongly semisimple. Then Φ is continuous.*

Proof Assume at first that B is actually simple. By Lemma 1.1.58, $\mathfrak{S}(\Phi)$ is an ideal of B, and hence $\mathfrak{S}(\Phi) = 0$ or B because of the simplicity of B. Suppose that $\mathfrak{S}(\Phi) = B$. Then we can choose a sequence a_n in A such that $a_n \to 0$ and $\Phi(a_n) \to \mathbf{1}$, so that, by the inequality (1.1.8) in Proposition 1.1.60, we have $1 \leqslant \|a_n\| + \|\mathbf{1} - \Phi(a_n)\| \to 0$, a contradiction. Therefore $\mathfrak{S}(\Phi) = 0$, and Φ is continuous in view of Fact 1.1.56.

Now assume that B is strongly semisimple but not necessarily simple. Let a_n be a sequence in A with $a_n \to 0$ and $\Phi(a_n) \to b \in B$. In view of the closed graph theorem, to prove that Φ is continuous it is enough to show that $b = 0$. Let M be a maximal ideal of B. By Lemma 1.1.54, the quotient algebra B/M is a simple unital associative complex algebra. Moreover, by Corollary 1.1.53, M is closed in B, and hence, by §1.1.55, B/M is a complete normed algebra. Since the quotient mapping $\pi : A \to A/M$ is a continuous unit-preserving surjective algebra homomorphism, it follows that $\pi \circ \Phi$ is a unit-preserving algebra homomorphism from A to B/M with dense range. Therefore, by the first paragraph in the proof, $\pi \circ \Phi$ is continuous. As a consequence, we have $(\pi \circ \Phi)(a_n) \to 0$. But, on the other hand, we deduce from the continuity of π that $(\pi \circ \Phi)(a_n) = \pi(\Phi(a_n)) \to \pi(b)$. Therefore $\pi(b) = 0$, and so $b \in M$. Now, keeping in mind the arbitrariness of the maximal ideal M, and the strong semisimplicity of B, we conclude that $b = 0$, as desired. □

Corollary 1.1.63 *Let A be a strongly semisimple unital associative complex algebra. Then A has at most one complete algebra norm topology.*

Proof Let $\|\cdot\|$ and $|||\cdot|||$ be two complete algebra norms on A. Then, by Theorem 1.1.62 applied to the identity mapping $I_A : (A, \|\cdot\|) \to (A, |||\cdot|||)$ and to its inverse mapping, we obtain that both norms are equivalent. □

1.1.3 Gelfand's theory

By a *character* on an algebra A over $\mathbb{K}$ we mean a nonzero algebra homomorphism from A to $\mathbb{K}$.

Corollary 1.1.64 *Let A be a complete normed unital algebra over $\mathbb{K}$, and let φ be a character on A. Then φ is continuous with $\|\varphi\| \leqslant 1$.*

Proof Since $\ker(\varphi)$ is both an ideal and a maximal subspace, it is a maximal ideal of A. Therefore, by Corollary 1.1.53, $\ker(\varphi)$ is closed in A, and hence φ is continuous. Given $a \in A$ and $n \in \mathbb{N}$, we have

$$|\varphi(a)|^n = |\varphi(a)^n| = |\varphi(L_a^{n-1}(a))| \leqslant \|\varphi\|\,\|a\|^n,$$

and hence $|\varphi(a)| \leqslant \|\varphi\|^{\frac{1}{n}}\|a\|$. By letting $n \to \infty$, we obtain that $|\varphi(a)| \leqslant \|a\|$. Thus $\|\varphi\| \leqslant 1$. □

Proposition 1.1.65 *Let A be a unital associative and commutative algebra over $\mathbb{K}$. We have:*

(i) *If A is simple, then A is a field extension of $\mathbb{K}$.*
(ii) *If M is a maximal ideal of A, then A/M is a field extension of $\mathbb{K}$.*

Proof Assume that A is simple. Let a be a nonzero element of A. Then aA is a nonzero ideal of A, and hence $A = aA$ by simplicity. By taking $b \in A$ such that $\mathbf{1} = ab$, we realize that a is invertible in A with inverse b. This shows that A is a field, and the proof of assertion (i) is complete. Assertion (ii) follows from assertion (i) by invoking Lemma 1.1.54. □

Combining Corollaries 1.1.43 and 1.1.53 and Proposition 1.1.65, we derive the following.

Corollary 1.1.66 *Let A be a complete normed unital associative and commutative complex algebra, and let M be a maximal ideal of A. Then A/M is isomorphic to $\mathbb{C}$.*

On the other hand, as a consequence of Lemma 1.1.34(ii), we have the following.

Corollary 1.1.67 *Let A be unital associative algebra over $\mathbb{K}$, and let φ be a character on A. Then $\varphi(a) \in \mathrm{sp}(A,a)$, for every $a \in A$.*

In what follows, for any unital associative and commutative complex algebra A, we denote by $\Delta = \Delta_A$ the set of all characters on A.

Proposition 1.1.68 *Let A be a complete normed unital associative and commutative complex algebra. We have:*

(i) *The mapping $\varphi \to \ker(\varphi)$ defines a bijection from Δ onto the set of all maximal ideals of A.*
(ii) *For every $a \in A$, $\mathrm{sp}(A,a) = \{\varphi(a) : \varphi \in \Delta\}$.*
(iii) *For every $a \in A$, $\mathfrak{r}(a) = \max\{|\varphi(a)| : \varphi \in \Delta\}$.*

Proof We already know that $\ker(\varphi)$ is a maximal ideal of A whenever φ is in Δ. Let M be a maximal ideal of A. Then, by Corollary 1.1.66, there exists an algebra isomorphism $\phi : A/M \to \mathbb{C}$. If $\pi : A \to A/M$ denotes the quotient mapping, then it is clear that $\varphi = \phi \circ \pi$ is a character on A with $\ker(\varphi) = M$. On the other hand, if φ_1, φ_2 are characters on A such that $\ker(\varphi_1) = \ker(\varphi_2)$, then there exists $\lambda \in \mathbb{C}$ such that $\varphi_1 = \lambda\varphi_2$. Since $\varphi_1(\mathbf{1}) = \varphi_2(\mathbf{1}) = 1$, it follows that $\lambda = 1$, and hence $\varphi_1 = \varphi_2$. Thus, assertion (i) has been proved.

In order to prove assertion (ii), let us fix an element a in A. If $\lambda \in \mathrm{sp}(A,a)$, then $A(a - \lambda\mathbf{1})$ is a proper ideal of A, and hence, by Fact 1.1.52, there exists a maximal ideal of A containing $a - \lambda\mathbf{1}$. Therefore, by assertion (i), there exists a character φ on A such that $\varphi(a - \lambda\mathbf{1}) = 0$, and hence $\varphi(a) = \lambda$. This shows that the inclusion $\mathrm{sp}(A,a) \subseteq \{\varphi(a) : \varphi \in \Delta\}$ holds. The reverse inclusion follows from Corollary 1.1.67.

Finally, keeping in mind Theorem 1.1.46, assertion (iii) is a consequence of assertion (ii). □

Corollaries 1.1.69 and 1.1.70 (immediately below) follow from Fact 1.1.52 and assertion (i) in Proposition 1.1.68.

Corollary 1.1.69 *If A is a complete normed unital associative and commutative complex algebra, then Δ is not empty.*

Corollary 1.1.70 *If A is a complete normed unital associative and commutative complex algebra, then s-$\mathrm{Rad}(A) = \bigcap_{\varphi\in\Delta} \ker(\varphi)$.*

If A is a complete normed unital associative and commutative complex algebra, it follows from Corollary 1.1.64 that Δ is contained in A'. Thus, we can endow Δ with the relative w^*-topology. This topology on Δ is called the *Gelfand topology*, and Δ, equipped with the Gelfand topology, is called the *carrier space* or *Gelfand space* of A.

Proposition 1.1.71 *If A is a complete normed unital associative and commutative complex algebra, then the carrier space Δ of A is a non-empty compact Hausdorff topological space.*

Proof Δ is not empty by Corollary 1.1.69, and is Hausdorff by definition. It follows from Corollary 1.1.64 that Δ is in fact contained in the closed unit ball $\mathbb{B}_{A'}$ of A'. Since, by the Banach–Alaoglu theorem, $\mathbb{B}_{A'}$ is compact in the w^*-topology, it suffices to show that Δ is w^*-closed in A'. Note that, for each $a,b\in A$, the mapping $h_{a,b} : f \to f(ab) - f(a)f(b)$ from A' to $\mathbb{C}$ is w^*-continuous, and hence the set $H_{a,b} := \{f \in A' : h_{a,b}(f) = 0\}$ is w^*-closed. Since $\Delta\cup\{0\} = \bigcap_{(a,b)\in A\times A} H_{a,b}$, it follows that $\Delta\cup\{0\}$ is w^*-closed. Moreover, the mapping $h : f \to f(\mathbf{1})$ from A' to $\mathbb{C}$ is w^*-continuous, and hence the set

$$H := \left\{f\in A' : h(f) = 1\right\}$$

is w^*-closed. Therefore, $\Delta = H\cap(\Delta\cup\{0\})$ is w^*-closed in A'. □

Definition 1.1.72 Let A be a unital associative algebra over $\mathbb{K}$, and let B be a subalgebra of A. It is clear that, if B contains the unit of A, then the inclusion $\mathrm{Inv}(B)\subseteq B\cap\mathrm{Inv}(A)$ holds. We say that B is a *full subalgebra* of A if B contains the unit of A and the equality $\mathrm{Inv}(B) = B\cap\mathrm{Inv}(A)$ actually holds. It is clear that B is a full subalgebra of A if and only if it contains the unit of A and we have $\mathrm{sp}(A,x) = \mathrm{sp}(B,x)$ for every $x\in B$.

Let A be a complete normed unital associative and commutative complex algebra. It is clear from the definition of the Gelfand topology that, for each $a\in A$, the mapping $G(a) : \Delta\to\mathbb{C}$ defined by

$$G(a)(\varphi) := \varphi(a) \text{ for every } \varphi\in\Delta,$$

is continuous. The element $G(a)$ of the complete normed algebra $C^{\mathbb{C}}(\Delta)$ is called the *Gelfand transform* of a. The mapping $G : a \to G(a)$ of A into $C^{\mathbb{C}}(\Delta)$ is called the *Gelfand representation* of A. Now, we summarize the so-called 'Gelfand theory' in Theorem 1.1.73 below. We recall that a family $\mathscr{F}$ of mappings from a set E to a set F is said to *separate the points* of E whenever for all $x_1, x_2\in E$ with $x_1\neq x_2$ there exists $f\in\mathscr{F}$ such that $f(x_1)\neq f(x_2)$.

Theorem 1.1.73 *Let A be a complete normed unital associative and commutative complex algebra, with carrier space Δ and Gelfand representation $G : A \to C^{\mathbb{C}}(\Delta)$. Then*

(i) *Δ is a non-empty compact Hausdorff topological space.*
(ii) *The Gelfand representation G is a contractive unit-preserving algebra homomorphism whose range is a full subalgebra of $C^{\mathbb{C}}(\Delta)$ separating the points of Δ.*
(iii) *$G(a)(\Delta) = \mathrm{sp}(A, a)$ for every $a \in A$.*
(iv) *$\|G(a)\| = \mathfrak{r}(a)$ for every $a \in A$.*
(v) $\ker(G) = s\text{-Rad}(A) = \{a \in A : \mathfrak{r}(a) = 0\}$.

Proof Assertion (i) was proved in Proposition 1.1.71, whereas assertions (iii) and (iv) were proved in Proposition 1.1.68(ii)–(iii). On the other hand, it is clear that assertion (iv) implies that G is contractive. The remaining conclusions in assertion (ii) can be easily checked. Finally, assertion (v) follows from Corollary 1.1.70 and assertion (iv). □

Corollary 1.1.74 *Let A be a complete normed strongly semisimple unital associative and commutative complex algebra, and let B be any subalgebra of A containing the unit of A. Then B is strongly semisimple.*

Proof By Theorem 1.1.73, B is isomorphic to a subalgebra of $C^{\mathbb{C}}(E)$ (for some compact Hausdorff topological space E) containing the unit. But such a subalgebra (say C) is always strongly semisimple. Indeed, the valuations of elements of C at points of E are characters, and kernels of characters are maximal ideals. □

Theorem 1.1.75 (Gelfand homomorphism theorem) *Let A be a complete normed complex algebra, let B be a complete normed strongly semisimple unital associative and commutative complex algebra, and let $\Phi : A \to B$ be an algebra homomorphism. Then Φ is continuous.*

Proof Regarding Φ as a mapping from A to $\overline{\Phi(A)}$, and keeping in mind Corollary 1.1.74, we may additionally assume that Φ has dense range. Then the result follows from Theorem 1.1.62. □

Given a compact Hausdorff topological space E and a point $t \in E$, we denote by $\varphi_t : C^{\mathbb{C}}(E) \to \mathbb{C}$ the valuation mapping at t, i.e. the mapping $f \to f(t)$ from $C^{\mathbb{C}}(E)$ to $\mathbb{C}$.

Proposition 1.1.76 *Let E be a compact Hausdorff topological space. Then the mapping $t \to \varphi_t$ is a homeomorphism from E onto $\Delta_{C^{\mathbb{C}}(E)}$.*

Proof The mapping is injective, because if t_1, t_2 are distinct points of E, then by Urysohn's lemma there is a function $f \in C^{\mathbb{C}}(E)$ such that $f(t_1) = 0$ and $f(t_2) = 1$, and therefore $\varphi_{t_1} \neq \varphi_{t_2}$.

Now we show the surjectivity of the mapping. For each $f \in C^{\mathbb{C}}(E)$, we will consider the function $f^* \in C^{\mathbb{C}}(E)$ defined by $f^*(t) := \overline{f(t)}$, and the closed subset $Z(f)$ of E given by $Z(f) := \{t \in E : f(t) = 0\}$. Let $\varphi \in \Delta_{C^{\mathbb{C}}(E)}$, and let $\{f_1, \dots, f_n\}$

be a finite subset of $\ker(\varphi)$. If $\bigcap_{k=1}^{n} Z(f_k) = \emptyset$, then $g := \sum_{k=1}^{n} f_k^* f_k$ is strictly positive on E, hence $g \in \mathrm{Inv}(C^{\mathbb{C}}(E))$, and so

$$0 \neq \varphi(g) = \sum_{k=1}^{n} \varphi(f_k^*)\varphi(f_k) = 0.$$

This contradiction shows that every finite subset of $\ker(\varphi)$ has a common zero. Since E is compact, all the functions in $\ker(\varphi)$ have a common zero, say t_0, and thus $\ker(\varphi) \subseteq \ker(\varphi_{t_0})$. Since $\varphi(\mathbf{1}) = \varphi_{t_0}(\mathbf{1}) = 1$, it follows that $\varphi = \varphi_{t_0}$.

Therefore, the mapping $t \to \varphi_t$ is a bijection from E onto $\Delta_{C^{\mathbb{C}}(E)}$. Since it is clearly continuous, and both E and $\Delta_{C^{\mathbb{C}}(E)}$ are compact Hausdorff topological spaces, we conclude that it is a homeomorphism. □

Let E be a compact Hausdorff topological space, and set $A := C^{\mathbb{C}}(E)$. It is straightforward that, up to the identification $C^{\mathbb{C}}(E) \equiv C^{\mathbb{C}}(\Delta_A)$ (as normed algebras) induced by the identification $E \equiv \Delta_A$ (as topological spaces) given by Proposition 1.1.76 (see §1.2.27 below for details), the Gelfand representation of A becomes the identity. Another relevant consequence of Proposition 1.1.76 is the following.

Corollary 1.1.77 *Let E and F be locally compact Hausdorff topological spaces, and let $\Phi : C_0^{\mathbb{C}}(E) \to C_0^{\mathbb{C}}(F)$ be a bijective algebra homomorphism. Then there exists a homeomorphism $\eta : F \to E$ such that*

$$\Phi(f)(t) = f(\eta(t)) \quad \textit{for all} \quad f \in C_0^{\mathbb{C}}(E) \quad \textit{and} \quad t \in F. \tag{1.1.9}$$

Proof First note that since the algebras $C_0^{\mathbb{C}}(E)$ and $C_0^{\mathbb{C}}(F)$ are isomorphic, they are either unital or not at the same time, so E and F are either compact or not at the same time.

Assume that E and F are compact. Then, clearly, $\varphi \circ \Phi$ lies in $\Delta_{C^{\mathbb{C}}(E)}$ whenever φ is in $\Delta_{C^{\mathbb{C}}(F)}$, and the mapping $\tau : \varphi \to \varphi \circ \Phi$ from $\Delta_{C^{\mathbb{C}}(F)}$ to $\Delta_{C^{\mathbb{C}}(E)}$ is a homeomorphism. Now, let $\sigma_E : E \to \Delta_{C^{\mathbb{C}}(E)}$ and $\sigma_F : F \to \Delta_{C^{\mathbb{C}}(F)}$ be the homeomorphisms given by Proposition 1.1.76, and set $\eta := \sigma_E^{-1} \circ \tau \circ \sigma_F$. Then η is a homeomorphism from F onto E satisfying (1.1.9).

Now assume that E and F are not compact. Let $\hat{E}$ and $\hat{F}$ stand for their respective one-point compactifications, and regard $C_0^{\mathbb{C}}(E)$ and $C_0^{\mathbb{C}}(F)$ as ideals of $C^{\mathbb{C}}(\hat{E})$ and $C^{\mathbb{C}}(\hat{F})$, respectively, in the natural way. Then Φ extends uniquely to a bijective algebra homomorphism $\hat{\Phi} : C^{\mathbb{C}}(\hat{E}) \to C^{\mathbb{C}}(\hat{F})$. By the above paragraph, there exists a homeomorphism $\hat{\eta} : \hat{F} \to \hat{E}$ such that $\hat{\Phi}(f)(t) = f(\hat{\eta}(t))$ for all $f \in C^{\mathbb{C}}(\hat{E})$ and $t \in \hat{F}$. Moreover, since $\hat{\Phi}(C_0^{\mathbb{C}}(E)) = C_0^{\mathbb{C}}(F)$, we have $\hat{\eta}(F) = E$, so that, by restricting $\hat{\eta}$ to F, we are provided with a homeomorphism η from F onto E satisfying (1.1.9). □

Let A be an algebra over $\mathbb{K}$, and let S be a non-empty subset of A. The *commutant* of S is defined as the set

$$S^c := \{a \in A : ax = xa \ \text{ for every } \ x \in S\}.$$

We write S^{cc} instead of $(S^c)^c$ for the *second commutant* or *bicommutant* of S. The subset S is called *commutative* if it consists of pairwise commuting elements (equivalently, if $S \subseteq S^c$). The following result, whose proof is left to the reader, contains elementary properties of commutants in associative algebras.

Proposition 1.1.78 *Let A be an associative algebra over $\mathbb{K}$, and let S, S_1, S_2 non-empty subsets of A. We have:*

(i) *S^c is a subalgebra of A, containing $\mathbf{1}$ if A is unital. Moreover, if in addition A is a normed algebra, then S^c is a closed subalgebra of A.*
(ii) *$S_2^c \subseteq S_1^c$ whenever $S_1 \subseteq S_2$.*
(iii) *$S \subseteq S^{cc}$ and $S^c = S^{ccc}$.*
(iv) *S is commutative if and only if S^{cc} is commutative.*

Corollary 1.1.79 *Let A be a (normed) associative algebra over $\mathbb{K}$, and let S be a non-empty commutative subset of A. Then the (closed) subalgebra of A generated by S is commutative.*

Proof By Proposition 1.1.78, S^{cc} is a (closed) commutative subalgebra of A containing S. □

We recall that commutants of subsets in unital associative algebras are subalgebras containing the unit.

Lemma 1.1.80 *Let A be a unital associative algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Then S^c is a full subalgebra of A.*

Proof Let x be in $S^c \cap \mathrm{Inv}(A)$. For each $b \in S$, by multiplying the equality $xb = bx$ on both sides by x^{-1}, we get $bx^{-1} = x^{-1}b$. Thus $x^{-1} \in S^c$, and $x \in \mathrm{Inv}(S^c)$. Therefore $S^c \cap \mathrm{Inv}(A) \subseteq \mathrm{Inv}(S^c)$. □

Corollary 1.1.81 *Let A be a complete normed unital associative complex algebra, and let a, b be commuting elements of A. Then we have*

(i) $\mathrm{sp}(a+b) \subseteq \mathrm{sp}(a) + \mathrm{sp}(b)$ *and* $\mathrm{sp}(ab) \subseteq \mathrm{sp}(a)\,\mathrm{sp}(b)$.
(ii) $\mathfrak{r}(a+b) \leqslant \mathfrak{r}(a) + \mathfrak{r}(b)$ *and* $\mathfrak{r}(ab) \leqslant \mathfrak{r}(a)\,\mathfrak{r}(b)$.

Proof Keeping in mind Theorem 1.1.46, assertion (ii) is a consequence of assertion (i). In order to prove assertion (i), consider the commutative set $S := \{a, b\}$, and set $B := S^{cc}$. By Proposition 1.1.78, B is a closed unital commutative subalgebra of A containing S. Let Δ stand for the carrier space of B, and let $G : B \to C^{\mathbb{C}}(\Delta)$ denote the Gelfand representation. By Theorem 1.1.73(ii)–(iii), we see that

$$\begin{aligned}\mathrm{sp}(B, a+b) = G(a+b)(\Delta) &= (G(a) + G(b))(\Delta) \\ &\subseteq G(a)(\Delta) + G(b)(\Delta) = \mathrm{sp}(B, a) + \mathrm{sp}(B, b).\end{aligned}$$

In the same way, $\mathrm{sp}(B, ab) \subseteq \mathrm{sp}(B, a)\,\mathrm{sp}(B, b)$. Now, the result follows by invoking Lemma 1.1.80. □

To conclude the present subsection, let us specialize Gelfand's theory to the case of algebras 'generated' by a single element. Given a (normed) algebra A over $\mathbb{K}$ and a subset S of A, we say that A is generated by S as a (normed) algebra if the (closed) subalgebra of A generated by S is the whole algebra A. The following lemma will often be applied.

Lemma 1.1.82 *Let A and B be (normed) algebras over $\mathbb{K}$, let S be a subset of A, and assume that A is generated by S as a (normed) algebra. We have:*

(i) *If $\phi, \psi : A \to B$ are (continuous) algebra homomorphisms such that $\phi(s) = \psi(s)$ for every $s \in S$, then $\phi = \psi$.*
(ii) *If $\phi : A \to B$ is a (continuous) algebra homomorphism, then $\phi(A)$ is contained in the (closed) subalgebra of B generated by $\phi(S)$.*

Proof Let ϕ and ψ be as in assertion (i). Then $\ker(\phi - \psi)$ is a (closed) subalgebra of A containing S, and hence $\ker(\phi - \psi) = A$, which proves assertion (i).

Now let $\phi : A \to B$ be a (continuous) algebra homomorphism. Let C stand for the (closed) subalgebra of B generated by $\phi(S)$. Then $\phi^{-1}(C)$ is a (closed) subalgebra of A containing S, and hence $\phi^{-1}(C) = A$. Thus $\phi(A) \subseteq C$, which proves assertion (ii). □

Theorem 1.1.83 *Let A be a complete normed unital associative complex algebra, let a be in A such that A is generated by $\{\mathbf{1}, a\}$ as a normed algebra, and denote by u the inclusion mapping $\mathrm{sp}(A,a) \hookrightarrow \mathbb{C}$. Then there exists a unique continuous unit-preserving algebra homomorphism $\Phi : A \to C^{\mathbb{C}}(\mathrm{sp}(A,a))$ such that $\Phi(a) = u$. Moreover we have:*

(i) *Φ is contractive.*
(ii) $\ker(\Phi) = s\text{-}\mathrm{Rad}(A)$.
(iii) *The range of Φ is a full subalgebra of $C^{\mathbb{C}}(\mathrm{sp}(A,a))$.*
(iv) $\mathrm{sp}(A,x) = \Phi(x)(\mathrm{sp}(A,a))$ *for every $x \in A$.*

Proof Since A is commutative, we may consider the carrier space Δ of A, and the Gelfand representation $G : A \to C^{\mathbb{C}}(\Delta)$. By Theorem 1.1.73, the Gelfand transform $G(a)$ of a is a continuous mapping from Δ onto $\mathrm{sp}(A,a)$. Moreover, if $\varphi_1, \varphi_2 \in \Delta$ satisfy $\varphi_1(a) = \varphi_2(a)$, then, since $\varphi_1(\mathbf{1}) = \varphi_2(\mathbf{1})$, we have $\varphi_1 = \varphi_2$ (by Lemma 1.1.82(i)). Therefore, $G(a)$ is injective, and consequently a homeomorphism. Now, the homeomorphism $G(a) : \Delta \to \mathrm{sp}(A,a)$ induces an isometric surjective algebra homomorphism $G(a)^t : C^{\mathbb{C}}(\mathrm{sp}(A,a)) \to C^{\mathbb{C}}(\Delta)$ (see §1.2.27 below for details). By setting

$$\Phi := (G(a)^t)^{-1} \circ G, \tag{1.1.10}$$

we realize that Φ is a continuous unit-preserving algebra homomorphism from A to $C^{\mathbb{C}}(\mathrm{sp}(A,a))$ satisfying $\Phi(a) = u$. On the other hand, the uniqueness of such a homomorphism follows by again invoking Lemma 1.1.82(i). Thus the first conclusion in the theorem has been proved. Finally, since the algebras $C^{\mathbb{C}}(\mathrm{sp}(A,a))$ and $C^{\mathbb{C}}(\Delta)$ are isomorphic via $G(a)^t$, and $G(a)^t$ is an isometry, the second conclusion follows from (1.1.10) and Theorem 1.1.73. □

Remark 1.1.84 Let A and a be as in Theorem 1.1.83. Then the linear hull of the set $E := \{\exp(za) : z \in \mathbb{C}\}$ is dense in A. Indeed, the linear hull of E is a subalgebra of A containing $\mathbf{1}$, and its closure contains a because

$$a = \lim_{z \to 0} \frac{\exp(za) - \mathbf{1}}{z}.$$

1.1.4 Topological divisors of zero

Definition 1.1.85 Let A be an algebra over $\mathbb{K}$. An element a of A is said to be a *left* (respectively, *right*) *divisor of zero* in A if there exists $b \in A \setminus \{0\}$ such that $ab = 0$ (respectively, $ba = 0$). Elements of A which are left or right (respectively, left and right) divisors of zero are called *one-sided divisors of zero* (respectively, *two-sided divisors of zero*) in A. We say that an element $a \in A$ is a *joint divisor of zero* in A if there is $b \in A \setminus \{0\}$ such that $ab = 0$ and $ba = 0$. We note that A has no nonzero left divisor of zero if and only if A has no nonzero right divisor of zero. When this is the case, we simply say that A *has no nonzero divisor of zero*.

Exercise 1.1.86 Let A be a nonzero finite-dimensional complex algebra with no nonzero divisor of zero. Prove that A is isomorphic to $\mathbb{C}$.

Solution Fix $y \in A \setminus \{0\}$, and let x be in A. Then the characteristic polynomial of the operator $L_y^{-1}L_x$ must have a complex root (say λ), and hence there exists $z \in A \setminus \{0\}$ such that $(L_y^{-1}L_x - \lambda I_A)(z) = 0$. This implies that $(x - \lambda y)z = 0$, and hence that $x = \lambda y$. Since x is arbitrary in A, we have $A = \mathbb{C}y$. Now, A is a one-dimensional complex algebra with nonzero product, so the proof is concluded by applying Exercise 1.1.2. □

Let us recall that a linear mapping F between normed spaces X and Y is called *bounded below* if there is $m > 0$ such that $\|F(x)\| \geqslant m\|x\|$ for every $x \in X$.

Definition 1.1.87 Let A be a normed algebra over $\mathbb{K}$. An element a of A is said to be a *left* (respectively, *right*) *topological divisor of zero* in A if there exists a sequence a_n of norm-one elements of A satisfying $\lim aa_n = 0$ (respectively, $\lim a_n a = 0$). Hence left (respectively, right) topological divisors of zero in A are nothing other than those elements a of A such that the operator L_a (respectively R_a) is not bounded below. Elements of A which are left or right (respectively, left and right) topological divisors of zero are called *one-sided topological divisors of zero* (respectively, *two-sided topological divisors of zero*) in A. We say that an element $a \in A$ is a *joint topological divisor of zero* in A if there is a sequence a_n of norm-one elements of A satisfying $aa_n \to 0$ and $a_n a \to 0$. In the case that A is commutative, all notions introduced above coincide, and therefore we will simply speak about *topological divisors of zero*.

Exercise 1.1.88 Let a be an element in a normed algebra A. We have:

(i) If a is a left (respectively, joint) divisor of zero in A, then a is a left (respectively, joint) topological divisor of zero in A. Moreover the converse is true when A is finite-dimensional.
(ii) The element a is not a joint topological divisor of zero in A if and only if there exists a positive number m satisfying $m\|b\| \leqslant \|ab\| + \|ba\|$ for every $b \in A$.
(iii) If A is unital and associative, and if a is a one-sided topological divisor of zero in A, then $a \notin \mathrm{Inv}(A)$.

Hint for assertion (ii) Assume that there is no $m > 0$ satisfying

$$m\|b\| \leqslant \|ab\| + \|ba\| \quad \text{for every} \quad b \in A.$$

Then, for each $n \in \mathbb{N}$, there exists $b_n \in A$ such that $\frac{1}{n}\|b_n\| > \|ab_n\| + \|b_n a\|$. By setting $a_n := \frac{b_n}{\|b_n\|}$, we have $\|a_n\| = 1$ for every $n \in \mathbb{N}$, and the sequences aa_n and $a_n a$ converge to zero in A. Therefore a is a joint topological divisor of zero in A. □

Exercise 1.1.89 Let A be a normed associative algebra over $\mathbb{K}$, and let x be in A with $\mathfrak{r}(x) = 0$, then x is a joint topological divisor of zero in A.

Solution If $x^n = 0$ for some $n \in \mathbb{N}$, then x is a joint divisor of zero. Otherwise, for $n \in \mathbb{N}$ we set $x_n := \frac{x^n}{\|x^n\|}$, so that we have $\|x_n\| = 1$ and

$$0 \leqslant \liminf \|xx_n\| = \liminf \|x_n x\| = \liminf \frac{\|x^{n+1}\|}{\|x^n\|} \leqslant \lim \|x^n\|^{\frac{1}{n}} = \mathfrak{r}(x) = 0,$$

and therefore x is a joint topological divisor of zero in A. □

Proposition 1.1.90 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let a be in the boundary of* $\operatorname{Inv}(A)$ *relative to A. Then a is a joint topological divisor of zero in A.*

Proof Since a lies in the boundary of $\operatorname{Inv}(A)$ relative to A, it follows from Theorem 1.1.23 that $a \notin \operatorname{Inv}(A)$ and there exists a sequence a_n in $\operatorname{Inv}(A)$ with $a_n \to a$. By Lemma 1.1.13(ii), for $n, m \in \mathbb{N}$ we have

$$\left| \frac{1}{\|a_n^{-1}\|} - \frac{1}{\|a_m^{-1}\|} \right| \leqslant \|a_n - a_m\|.$$

Therefore $\frac{1}{\|a_n^{-1}\|}$ is a Cauchy sequence in $\mathbb{R}$. Put $\lambda = \lim \frac{1}{\|a_n^{-1}\|}$. Assume that $\lambda \neq 0$. Then $\|a_n^{-1}\|$ converges to λ^{-1}, and hence is bounded. Let $M \in \mathbb{R}$ such that $\|a_n^{-1}\| \leqslant M$ for every $n \in \mathbb{N}$. By Lemma 1.1.13(i), for all $n, m \in \mathbb{N}$ we have $\|a_n^{-1} - a_m^{-1}\| \leqslant M^2 \|a_n - a_m\|$, and hence a_n^{-1} is a Cauchy sequence in A. If $a_n^{-1} \to b \in A$, then $a_n a_n^{-1} \to ab$ and $a_n^{-1} a_n \to ba$, so $ab = ba = \mathbf{1}$, and so $a \in \operatorname{Inv}(A)$, a contradiction. Therefore $\lambda = 0$, i.e. $\frac{1}{\|a_n^{-1}\|} \to 0$. Now, setting $b_n := \frac{a_n^{-1}}{\|a_n^{-1}\|}$, and writing

$$ab_n = (a - a_n)b_n + \frac{1}{\|a_n^{-1}\|}\mathbf{1} \quad \text{and} \quad b_n a = b_n(a - a_n) + \frac{1}{\|a_n^{-1}\|}\mathbf{1},$$

we realize that $ab_n \to 0$ and $b_n a \to 0$. Thus a is a joint topological divisor of zero in A. □

Corollary 1.1.91 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, let a be an element of A, and let λ lie in the boundary of* $\operatorname{sp}(A, a)$ *relative to $\mathbb{K}$. Then $a - \lambda\mathbf{1}$ is a joint topological divisor of zero in A.*

For a subset S of a vector space X over $\mathbb{K}$, $\operatorname{co}(S)$ will denote the convex hull of S.

Lemma 1.1.92 *Let X be a nonzero normed space over $\mathbb{K}$, let K be a compact subset of X, and let ∂K denote the boundary of K relative to X. We have:*

(i) $K \subseteq \operatorname{co}(\partial K)$.

(ii) *If L is a compact subset of X satisfying $K \subseteq L$ and $\partial L \subseteq \partial K$, then* $\operatorname{co}(K) = \operatorname{co}(L)$.

Proof We may assume that $K \neq \emptyset$. Fix a nonzero element u in X. For each $x \in K$, the set $S := \{t \in \mathbb{R} : x + tu \in K\}$ is a compact subset of $\mathbb{R}$ with $0 \in S$. If $\alpha = \min S$ and $\beta = \max S$, then $x + \alpha u, x + \beta u \in \partial K$ and $x \in [x + \alpha u, x + \beta u]$. Therefore $x \in \operatorname{co}(\partial K)$. Thus assertion (i) is proved.

In order to prove assertion (ii), assume that L is a compact set in X such that $K \subseteq L$ and $\partial L \subseteq \partial K$. Keeping in mind assertion (i), we have

$$\begin{array}{ccc} \operatorname{co}(K) & \subseteq & \operatorname{co}(L) \\ || & & || \\ \operatorname{co}(\partial K) & \supseteq & \operatorname{co}(\partial L), \end{array}$$

and the proof concludes. □

Let K be a non-empty compact set in $\mathbb{K}$. Recall that the bounded connected components of $\mathbb{K} \setminus K$ are called the *holes* of K (in $\mathbb{K}$).

Proposition 1.1.93 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let B be a closed subalgebra of A containing the unit of A. We have:*

(i) $\operatorname{Inv}(B)$ *is a clopen subset of* $\operatorname{Inv}(A) \cap B$*, and the boundary of* $\operatorname{Inv}(B)$ *relative to* B *is contained in the boundary of* $\operatorname{Inv}(A)$ *relative to* A.

(ii) *For each* $b \in B$*, we have* $\operatorname{sp}(A, b) \subseteq \operatorname{sp}(B, b)$*, and the boundary of* $\operatorname{sp}(B, b)$ *relative to* $\mathbb{K}$ *is contained in the boundary of* $\operatorname{sp}(A, b)$ *relative to* $\mathbb{K}$*. As a consequence,* $\operatorname{co}(\operatorname{sp}(A, b)) = \operatorname{co}(\operatorname{sp}(B, b))$.

(iii) *If* b *is in* B *and if* $\operatorname{sp}(A, b)$ *has no hole in* $\mathbb{K}$*, then* $\operatorname{sp}(A, b) = \operatorname{sp}(B, b)$.

Proof Keeping in mind Theorem 1.1.23, it is clear that $\operatorname{Inv}(B)$ is an open set in $\operatorname{Inv}(A) \cap B$. To see that it is also closed, let b_n be a sequence in $\operatorname{Inv}(B)$ converging to a point $b \in \operatorname{Inv}(A) \cap B$. Then b_n^{-1} converges to b^{-1} in A, so $b^{-1} \in B$, which implies that $b \in \operatorname{Inv}(B)$. Hence, $\operatorname{Inv}(B)$ is a clopen subset of $\operatorname{Inv}(A) \cap B$. Moreover, if b is in the boundary of $\operatorname{Inv}(B)$ relative to B, then, by Proposition 1.1.90, b is a joint topological divisor of zero in B, so b is a joint topological divisor of zero in A, and so $b \notin \operatorname{Inv}(A)$ (by Exercise 1.1.88(iii)). Since there exists a sequence b_n in $\operatorname{Inv}(B) \subseteq \operatorname{Inv}(A)$ such that b_n converges to b, it follows that b lies in the boundary of $\operatorname{Inv}(A)$ relative to A.

The inclusion $\operatorname{Inv}(B) \subseteq \operatorname{Inv}(A)$, and its consequence that $\operatorname{sp}(A, b) \subseteq \operatorname{sp}(B, b)$ for every $b \in B$, are well-known facts to us. Moreover, if b is in B, and if λ is in the boundary of $\operatorname{sp}(B, b)$ relative to $\mathbb{K}$, then $b - \lambda \mathbf{1}$ is in the boundary of $\operatorname{Inv}(B)$ relative to B. Therefore, by assertion (i), $b - \lambda \mathbf{1}$ is in the boundary of $\operatorname{Inv}(A)$ relative to A, hence $\lambda \in \operatorname{sp}(A, b)$, and so λ lies in the boundary of $\operatorname{sp}(A, b)$ relative to $\mathbb{K}$. Now, by Lemma 1.1.92, we get $\operatorname{co}(\operatorname{sp}(A, b)) = \operatorname{co}(\operatorname{sp}(B, b))$.

Finally, let b be in B such that $\operatorname{sp}(A, b)$ has no hole in $\mathbb{K}$. If $\mathbb{K} = \mathbb{R}$, then $\operatorname{sp}(A, b)$ is connected, and hence, by assertion (ii), we have $\operatorname{sp}(A, b) = \operatorname{sp}(B, b)$. If $\mathbb{K} = \mathbb{C}$, then $\mathbb{C} \setminus \operatorname{sp}(A, b)$ is connected. Since $\mathbb{C} \setminus \operatorname{sp}(B, b)$ is a clopen subset of $\mathbb{C} \setminus \operatorname{sp}(A, b)$ by assertions (i) and (ii), it follows that $\mathbb{C} \setminus \operatorname{sp}(A, b) = \mathbb{C} \setminus \operatorname{sp}(B, b)$, and hence $\operatorname{sp}(A, b) = \operatorname{sp}(B, b)$. □

Let X, Y be normed spaces over $\mathbb{K}$. Given $F \in BL(X, Y)$, we denote by $F' : Y' \to X'$ the transpose operator of F. We recall that the mapping $F \to F'$ from $BL(X, Y)$ to $BL(Y', X')$ becomes a linear isometry, and that, in the case that $Y = X$, this isometry is an algebra antihomomorphism from $BL(X)$ to $BL(X')$.

Proposition 1.1.94 *Let X be a normed space over $\mathbb{K}$, and let F be in $BL(X)$. We have:*

(i) *F is a left topological divisor of zero in $BL(X)$ if and only if F is not bounded below.*
(ii) *If F is a right topological divisor of zero in $BL(X)$, then F is not an open mapping.*

Proof Assume that F is a left topological divisor of zero in $BL(X)$. Then there exists a sequence F_n of norm-one elements of $BL(X)$ such that $FF_n \to 0$. For $n \in \mathbb{N}$, choose a norm-one element $y_n \in X$ such that $\|F_n(y_n)\| \geqslant \frac{n}{n+1}$, and set $x_n := \frac{F_n(y_n)}{\|F_n(y_n)\|}$. Then x_n becomes a sequence of norm-one elements of X satisfying $F(x_n) \to 0$, and hence F is not bounded below.

Now assume that F is not bounded below. Then there is a sequence x_n of norm-one elements of X such that $F(x_n) \to 0$. Take a norm-one element $f \in X'$ and, for $n \in \mathbb{N}$, consider the bounded linear operator F_n on X defined by $F_n(x) := f(x)x_n$ for every $x \in X$. Then F_n becomes a sequence of norm-one elements of $BL(X)$ satisfying $FF_n \to 0$, and hence F is a left topological divisor of zero in $BL(X)$.

Finally assume that F is a right topological divisor of zero in $BL(X)$. Then F' is a left topological divisor of zero in $BL(X')$. Therefore, by the first paragraph in the proof, there exists a sequence f_n of norm-one elements of X' satisfying $F'(f_n) \to 0$. If F were an open mapping (say $F(\mathbb{B}_X) \supseteq \delta\mathbb{B}_X$ for a suitable $\delta > 0$), then, for every $n \in \mathbb{N}$, we would have

$$\|F'(f_n)\| = \sup\{|f_n(F(x))| : x \in \mathbb{B}_X\} \geqslant \delta \sup\{|f_n(x)| : x \in \mathbb{B}_X\} = \delta\|f_n\| = \delta,$$

a contradiction. □

The following corollary follows straightforwardly from the open mapping theorem and Propositions 1.1.90 and 1.1.94.

Corollary 1.1.95 *Let X be a Banach space over $\mathbb{K}$, and let F be in the boundary of* $\mathrm{Inv}(BL(X))$ *relative to $BL(X)$. Then F is neither bounded below nor surjective.*

Corollary 1.1.96 *Let X be a complex normed space, and let T be in $BL(X)$. Then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = \mathfrak{r}(T)$ and such that $T - \lambda I_X$ is not bounded below.*

Proof Let $\hat{X}$ stand for the completion of X, and let $\hat{T}$ be the unique bounded linear operator on $\hat{X}$ which extends T. According to Theorem 1.1.46, there is $\lambda \in \mathrm{sp}(BL(\hat{X}), \hat{T})$ with $|\lambda| = \mathfrak{r}(\hat{T})$, and such a λ lies in the boundary of $\mathrm{sp}(BL(\hat{X}), \hat{T})$ relative to $\mathbb{C}$. Then, since $\mathfrak{r}(\hat{T}) = \mathfrak{r}(T)$, we have $|\lambda| = \mathfrak{r}(T)$. On the other hand, by Corollary 1.1.95, $\hat{T} - \lambda I_{\hat{X}}$ is not bounded below, i.e. there is a sequence y_n in $\mathbb{S}_{\hat{X}}$ such that $\hat{T}(y_n) - \lambda y_n \to 0$. For $n \in \mathbb{N}$, take $x_n \in \mathbb{S}_X$ with $\|x_n - y_n\| \leqslant \frac{1}{n}$. Then $T(x_n) - \lambda x_n \to 0$. It follows that $T - \lambda I_X$ is not bounded below. □

1.1.5 The complexification of a normed real algebra

Let X, Y, Z, U be vector spaces over $\mathbb{K}$, and let $F : X \to Z$ and $G : Y \to U$ be linear mappings. We denote by $F \otimes G$ the linear mapping from $X \otimes Y$ to $Z \otimes U$

determined by

$$(F \otimes G)(x \otimes y) = F(x) \otimes G(y) \text{ for every } (x,y) \in X \times Y.$$

Now, let X, Y be normed spaces over $\mathbb{K}$. We recall that the *projective tensor norm* $\|\cdot\|_\pi$ on $X \otimes Y$ is defined by

$$\|\alpha\|_\pi := \inf \left\{ \sum_{i=1}^{n} \|x_i\|\|y_i\| : \alpha = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The normed space $(X \otimes Y, \|\cdot\|_\pi)$ will be called the *projective tensor product* of X and Y, and will be denoted by $X \otimes_\pi Y$. It is well known and easy to see that the equality $\|x \otimes y\|_\pi = \|x\|\|y\|$ holds for every $(x,y) \in X \times Y$. Moreover, if Z and U are normed spaces over $\mathbb{K}$, and if $F : X \to Z$ and $G : Y \to U$ are bounded linear mappings, then the linear mapping $F \otimes G$ from $X \otimes_\pi Y$ to $Z \otimes_\pi U$ is bounded with $\|F \otimes G\| = \|F\|\|G\|$.

Let X be a real vector space. Then the *complexification* of X is defined as the real vector space $\mathbb{C} \otimes X$, endowed with its natural structure of complex vector space deriving from the multiplication of vectors by complex numbers determined by $\lambda(\mu \otimes x) = (\lambda\mu) \otimes x$. The space X will be regarded as a real subspace of its complexification via the imbedding $x \to 1 \otimes x$, and then the clear equality $\mathbb{C} \otimes X = X \oplus iX$ gives rise to a unique conjugate-linear involutive mapping $\sharp$ on $\mathbb{C} \otimes X$ such that X becomes the set of fixed points for $\sharp$. This involutive mapping will be called the *canonical involution of the complexification* of X.

Lemma 1.1.97 *Let X be a real normed space. Then $\mathbb{C} \otimes_\pi X$ becomes a complex normed space. Moreover, both the natural imbedding $X \hookrightarrow \mathbb{C} \otimes_\pi X$ and the canonical involution $\sharp$ of $\mathbb{C} \otimes_\pi X$ are isometries. Therefore, the direct sum $\mathbb{C} \otimes_\pi X = X \oplus iX$ is topological, and hence $\mathbb{C} \otimes_\pi X$ is complete if and only if so is X.*

Proof To prove the first conclusion it is enough to show that, for $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the mapping $\phi : \alpha \to \lambda\alpha$ from $\mathbb{C} \otimes_\pi X$ to $\mathbb{C} \otimes_\pi X$ is an isometry. But, for such a λ, we have $\phi = L_\lambda \otimes I_X$ and $\phi^{-1} = L_{\bar{\lambda}} \otimes I_X$, so $\|\phi\| = 1$ and $\|\phi^{-1}\| = 1$, and so ϕ is an isometry, as required. That the natural imbedding $X \hookrightarrow \mathbb{C} \otimes_\pi X$ is an isometry becomes obvious. On the other hand, since $\sharp = \tau \otimes I_X$ (where τ stands for the conjugation of $\mathbb{C}$), we derive that $\sharp$ is isometric. Finally, for $x, y \in X$ we have $x = \frac{1}{2}[(x+iy) + (x+iy)^\sharp]$, and hence $\|x\| \leqslant \|x+iy\|$, which shows that the direct sum $\mathbb{C} \otimes_\pi X = X \oplus iX$ is topological. □

In view of the above lemma, the complexification of a real normed space X, endowed with the projective tensor norm, will be called the (*projective*) *normed complexification* of X, and will be denoted by $X_\mathbb{C}$.

Let A, B be algebras over $\mathbb{K}$. As usual, the vector space $A \otimes B$ will be considered without notice as a new algebra over $\mathbb{K}$ with product determined by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$. Now, assume that A and B are normed. Then it is straightforward that the projective tensor norm on $A \otimes B$ is an algebra norm, so that $A \otimes_\pi B$ will be seen without notice as a normed algebra.

Now, let A be a real algebra. Then $\mathbb{C} \otimes A$ is both a complex vector space and a real algebra, and it is straightforward that both structures behave smoothly, so the complexification of A becomes naturally a complex algebra containing A as a real subalgebra via the natural imbedding $A \hookrightarrow \mathbb{C}\otimes A$, and the canonical involution of $\mathbb{C}\otimes A$ becomes a conjugate-linear algebra automorphism. Moreover, $\mathbb{C}\otimes A$ is associative, commutative, or unital, if and only if so is A.

Keeping in mind the above ideas and Lemma 1.1.97, we derive the following.

Proposition 1.1.98 *Let A be a normed real algebra. Then its normed complexification $A_{\mathbb{C}} := \mathbb{C}\otimes_{\pi} A$ becomes naturally a normed complex algebra containing isometrically A as a real subalgebra. Moreover, the direct sum $A_{\mathbb{C}} = A \oplus iA$ is topological, and hence $A_{\mathbb{C}}$ is complete if and only if so is A. In addition, $A_{\mathbb{C}}$ is associative, commutative, or unital, if and only if so is A.*

Lemma 1.1.99 *Let A be a unital associative algebra over $\mathbb{K}$, and let a and b be in A. We have:*

(i) *a and b are invertible in A if (and only if) so are ab and ba.*
(ii) *When a and b commute, a and b are invertible in A if (and only if) so is ab.*

Proof Assume that ab and ba lie in $\mathrm{Inv}(A)$. Then

$$a[b(ab)^{-1}] = (ab)(ab)^{-1} = \mathbf{1} \text{ and } [(ba)^{-1}b]a = (ba)^{-1}(ba) = \mathbf{1}.$$

Therefore, by Lemma 1.1.59, $a \in \mathrm{Inv}(A)$. Analogously, $b \in \mathrm{Inv}(A)$. □

Proposition 1.1.100 *Let A be a unital associative real algebra, and let a be in A. Then*

$$\mathrm{sp}(\mathbb{C}\otimes A, a) = \{\alpha + i\beta : \alpha, \beta \in \mathbb{R} \textit{ such that } (a-\alpha\mathbf{1})^2 + \beta^2\mathbf{1} \notin \mathrm{Inv}(A)\}.$$

Proof Using that the canonical involution $\sharp$ of $\mathbb{C}\otimes A$ is a conjugate-linear algebra automorphism, we realize that an element $x \in \mathbb{C}\otimes A$ is invertible in $\mathbb{C}\otimes A$ if and only if so is $x^\sharp$, and that, if this is the case, then we have $(x^\sharp)^{-1} = (x^{-1})^\sharp$. Then we deduce that A becomes a full real subalgebra of $\mathbb{C}\otimes A$, and that for $x \in \mathbb{C}\otimes A$ we have $\mathrm{sp}(\mathbb{C}\otimes A, x^\sharp) = \{\bar{\lambda} : \lambda \in \mathrm{sp}(\mathbb{C}\otimes A, x)\}$ (which implies that $\mathrm{sp}(\mathbb{C}\otimes A, a)$ is invariant under the conjugation of $\mathbb{C}$). Now, keeping in mind that for $\alpha, \beta \in \mathbb{R}$ we have

$$[a-(\alpha+i\beta)\mathbf{1}][a-(\alpha-i\beta)\mathbf{1}] = (a-\alpha\mathbf{1})^2 + \beta^2\mathbf{1} \in A,$$

it follows from Lemma 1.1.99(ii) that

$$\alpha + i\beta \in \mathrm{sp}(\mathbb{C}\otimes A, a) \Leftrightarrow \alpha - i\beta \in \mathrm{sp}(\mathbb{C}\otimes A, a) \Leftrightarrow (a-\alpha\mathbf{1})^2 + \beta^2\mathbf{1} \notin \mathrm{Inv}(A).$$

□

Now we can prove the real variant of Theorem 1.1.46.

Corollary 1.1.101 *Let A be a complete normed unital associative real algebra, and let a be in A. Then*

$$\mathfrak{r}(a) = \max\{|\alpha + i\beta| : \alpha, \beta \in \mathbb{R} \textit{ with } (a-\alpha\mathbf{1})^2 + \beta^2\mathbf{1} \notin \mathrm{Inv}(A)\}. \tag{1.1.11}$$

Proof By Proposition 1.1.98 and Theorem 1.1.46, we have

$$\mathfrak{r}(a) = \max\{|\mu| : \mu \in \mathrm{sp}(A_{\mathbb{C}}, a)\},$$

and (1.1.11) follows from Proposition 1.1.100. □

Now, invoking Corollary 1.1.101 and Example 1.1.12(d), we get the following real variant of Corollary 1.1.47.

Corollary 1.1.102 *Let X be a real Banach space, and let F be in $BL(X)$. Then*

$$\mathfrak{r}(F) = \max\{|\alpha + i\beta| : \alpha, \beta \in \mathbb{R} \text{ with } (F - \alpha I_X)^2 + \beta^2 I_X \notin \mathrm{Inv}(L(X))\}.$$

We conclude the present subsection with the following.

Exercise 1.1.103 Apply Proposition 1.1.98 to show that Proposition 1.1.60 remains true when we replace 'complex algebras' with 'real algebras'. Then look at the proof of Theorem 1.1.62 to realize that this theorem also remains true with the same replacement.

1.1.6 The unital extension and the completion of a normed algebra

§1.1.104 Let A be an algebra over $\mathbb{K}$. The *unital extension* of A is defined as the algebra over $\mathbb{K}$ whose vector space is $\mathbb{K} \times A$ and whose product is given by

$$(\lambda, x)(\mu, y) := (\lambda\mu, \lambda y + \mu x + xy).$$

We will denote by $A_{\mathbb{1}}$ the unital extension of A. $A_{\mathbb{1}}$ is always a unital algebra, whose unit is $\mathbb{1} := (1, 0)$. Moreover, the mapping $a \to (0, a)$ from A to $A_{\mathbb{1}}$ identifies A with an ideal of $A_{\mathbb{1}}$. With this identification in mind, we have

$$A_{\mathbb{1}} = \mathbb{K}\mathbb{1} \oplus A.$$

Up to an algebra isomorphism, the unital extension of A is the unique unital algebra over $\mathbb{K}$ containing A as a subalgebra and enjoying the following universal property: *every algebra homomorphism from A to any unital algebra B over $\mathbb{K}$ extends in a unique way to a unit-preserving algebra homomorphism from $A_{\mathbb{1}}$ to B*. It is straightforward that $A_{\mathbb{1}}$ is associative or commutative, if and only if so is A.

Let B, C be algebras over $\mathbb{K}$. The *(algebra) direct product* of B and C is defined as the algebra over $\mathbb{K}$ whose vector space is $B \times C$, and whose product is defined coordinate-wise. Set $A := B \times C$. Regarding B and C as subsets of A by means of the natural imbeddings $B \hookrightarrow A$ and $C \hookrightarrow A$, they become ideals of A such that $A = B \oplus C$ (which implies $BC = CB = 0$).

§1.1.105 Conversely, let A be an algebra over $\mathbb{K}$, and let B, C be ideals of A such that $A = B \oplus C$. Then the mapping $(b, c) \to b + c$ from the algebra direct product $B \times C$ to A becomes a bijective algebra homomorphism taking B onto B and C onto C.

Now, let B, C be associative algebras over $\mathbb{K}$. Then $B \times C$ is an associative algebra. Assume additionally that B and C are unital. Then $B \times C$ is a unital associative algebra, and an element $(b, c) \in B \times C$ is invertible in $B \times C$ if and only if b is invertible in B and c is invertible in C. As a consequence, for $(b, c) \in B \times C$ we have

$$\mathrm{sp}(B \times C, (b, c)) = \mathrm{sp}(B, b) \cup \mathrm{sp}(C, c). \tag{1.1.12}$$

Proposition 1.1.106 *Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A. Then* $\mathrm{sp}(A, a) \cup \{0\} = \mathrm{sp}(A_{\mathbb{1}}, a)$.

Proof Noticing that A and $\mathbb{K}(\mathbb{1}-\mathbf{1})$ are ideals of $A_{\mathbb{1}}$ such that

$$A_{\mathbb{1}} = A \oplus \mathbb{K}(\mathbb{1}-\mathbf{1}),$$

and that A and $\mathbb{K}(\mathbb{1}-\mathbf{1})$ are unital algebras, the result follows from §1.1.105 and the equality (1.1.12). $\square$

By a *norm-unital normed algebra* we mean a normed unital algebra A such that $\|\mathbf{1}\| = 1$. Now, the following proposition is straightforward.

Proposition 1.1.107 *Let A be a (complete) normed algebra over $\mathbb{K}$. Then its unital extension $A_{\mathbb{1}} = \mathbb{K}\mathbb{1} \oplus A$, endowed with the norm*

$$\|\lambda\mathbb{1} + a\| := |\lambda| + \|a\|, \tag{1.1.13}$$

becomes a norm-unital (complete) normed algebra over $\mathbb{K}$ containing isometrically A as a closed proper ideal.

Keeping in mind the universal property of the unital extension, the next unit-free version of Corollary 1.1.64 follows straightforwardly from Corollary 1.1.64 itself and Proposition 1.1.107.

Corollary 1.1.108 *Let A be a complete normed algebra over $\mathbb{K}$, and let φ be a character on A. Then φ is continuous with $\|\varphi\| \leqslant 1$.*

Another relevant consequence of Proposition 1.1.107 is the following.

Corollary 1.1.109 *Two complete algebra norms on an associative algebra A over $\mathbb{K}$ give rise to the same spectral radius on A.*

Proof In view of Proposition 1.1.98, we may assume that $\mathbb{K} = \mathbb{C}$. Let $\|\cdot\|$ and $|||\cdot|||$ be complete algebra norms on A, and denote by the same symbols the corresponding complete algebra norms on $A_{\mathbb{1}}$ given by Proposition 1.1.107. By Theorem 1.1.46, for $a \in A$ we have

$$\mathfrak{r}_{\|\cdot\|}(a) = \max\{|\lambda| : \lambda \in \operatorname{sp}(A_{\mathbb{1}}, a)\} = \mathfrak{r}_{|||\cdot|||}(a). \qquad \square$$

§1.1.110 Let A be a norm-unital normed algebra over $\mathbb{K}$, and let M be a closed proper ideal of A. Then the normed algebra A/M is norm-unital. Indeed, A/M is unital with unit $\mathbf{1}+M$, and, by §1.1.5, we have

$$1 \leqslant \|\mathbf{1}+M\| \leqslant \|\mathbf{1}\| = 1.$$

Proposition 1.1.111 *Let A be a normed unital algebra over $\mathbb{K}$. Then there exists an equivalent norm $|||\cdot|||$ on A converting A into a norm-unital normed algebra over $\mathbb{K}$. A choice of such a norm $|||\cdot|||$ is given by*

$$|||a||| = \inf\{|\lambda| + \|a - \lambda\mathbf{1}\| : \lambda \in \mathbb{K}\} \tag{1.1.14}$$

for every $a \in A$.

Proof Set $M := \mathbb{K}(\mathbb{1}-\mathbf{1}) \subseteq A_{\mathbb{1}}$. Then A and M are ideals of $A_{\mathbb{1}}$ such that $A_{\mathbb{1}} = A \oplus M$. Let $\pi : A_{\mathbb{1}} \to A_{\mathbb{1}}/M$ be the quotient homomorphism. It follows that $\pi_{|A}$ is a bijective algebra homomorphism from A to $A_{\mathbb{1}}/M$. Everything asserted until now made no use of the fact that the algebra A is normed. Now, consider

$A_{\mathbb{1}}$ as a normed algebra under the norm given by Proposition 1.1.107, so that, by §1.1.110, $A_{\mathbb{1}}/M$ becomes a norm-unital normed algebra. Since A is closed in $A_{\mathbb{1}}$, and M is finite-dimensional, the direct sum $A_{\mathbb{1}} = A \oplus M$ is topological, and hence the bijective algebra isomorphism $\pi_{|A}$ is a homeomorphism. It follows that $|||a||| := \|\pi_{|A}(a)\|$ defines an equivalent norm on A converting A into a norm-unital normed algebra over $\mathbb{K}$. Finally, the equality (1.1.14) follows from the ones $|||a||| = \inf\{\|a + \lambda(\mathbb{1} - \mathbf{1})\| : \lambda \in \mathbb{K}\}$ and (1.1.13). □

Lemma 1.1.112 *Let X, Y, Z be normed spaces over $\mathbb{K}$, with completions $\hat{X}, \hat{Y}, \hat{Z}$, respectively, and let $f : X \times Y \to Z$ be a continuous bilinear mapping. Then there exists a unique continuous mapping $\hat{f} : \hat{X} \times \hat{Y} \to \hat{Z}$ extending f. Moreover $\hat{f}$ is bilinear with $\|\hat{f}\| = \|f\|$.*

Proof For each $x \in X$, the mapping $f_x : Y \to \hat{Z}$ defined by $f_x(y) = f(x,y)$ is a continuous linear mapping. Therefore, there exists a unique continuous linear mapping $\hat{f}_x : \hat{Y} \to \hat{Z}$ such that $\hat{f}_{x|Y} = f_x$ and $\|\hat{f}_x\| = \|f_x\|$. Consider the mapping $g : X \times \hat{Y} \to \hat{Z}$ defined by $g(x,y) = \hat{f}_x(y)$. It is routine to verify that g is a continuous bilinear mapping such that $g_{|X \times Y} = f$ and $\|g\| = \|f\|$. Now, for each $y \in \hat{Y}$, consider de mapping $g_y : X \to \hat{Z}$ defined by $g_y(x) = g(x,y)$ for every $x \in X$. Since g_y is a continuous linear mapping, there exists a unique continuous linear mapping $\hat{g}_y : \hat{X} \to \hat{Z}$ such that $\hat{g}_{y|X} = g_y$ and $\|\hat{g}_y\| = \|g_y\|$. Again, the mapping $\hat{f} : \hat{X} \times \hat{Y} \to \hat{Z}$, defined by $\hat{f}(x,y) = \hat{g}_y(x)$ for all $x \in \hat{X}, y \in \hat{Y}$, is a continuous bilinear mapping such that $\hat{f}_{|X \times \hat{Y}} = g$ and $\|\hat{f}\| = \|g\|$. Therefore, $\hat{f}$ extends f and satisfies $\|\hat{f}\| = \|f\|$. Finally, the fact that $\hat{f}$ is the only continuous mapping from $\hat{X} \times \hat{Y}$ to $\hat{Z}$ extending f follows from the fact that $X \times Y$ is dense in $\hat{X} \times \hat{Y}$. □

Definition 1.1.113 Let A be a normed algebra over $\mathbb{K}$. The *(algebra) completion* of A is defined as the complete normed algebra whose Banach space is the completion of the normed space of A, and whose product is the unique continuous extension of that of A given by Lemma 1.1.112. Clearly, if A is associative, commutative, or unital, then so is the completion of A.

Corollary 1.1.114 *Let A be a complete normed associative algebra over $\mathbb{K}$, let B be a normed associative algebra over $\mathbb{K}$, and let $F : A \to B$ be an algebra homomorphism. Then*

$$\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a) \text{ for every } a \in A.$$

Proof Regarding F as a mapping from A to the completion of B, F remains an algebra homomorphism, and hence we may assume that B is complete. Moreover, we may additionally assume that $\mathbb{K} = \mathbb{C}$. Indeed, if $\mathbb{K} = \mathbb{R}$, then it is enough to replace A and B with their normed complexifications (see Proposition 1.1.98), and F with $I_{\mathbb{C}} \otimes F$. Now, consider the unital extensions $A_{\mathbb{1}}$ and $B_{\mathbb{1}}$ of A and B, respectively (which, by Proposition 1.1.107, are complete normed associative complex algebras), and the unique unit-preserving algebra homomorphism $G : A_{\mathbb{1}} \to B_{\mathbb{1}}$ extending F. Then, invoking Lemma 1.1.34(ii) and Theorem 1.1.46, for $a \in A$ we have

$$\begin{aligned} \mathfrak{r}(F(a)) = \mathfrak{r}(G(a)) &= \max\{|\mu| : \mu \in \operatorname{sp}(B_{\mathbb{1}}, G(a))\} \\ &\leqslant \max\{|\lambda| : \lambda \in \operatorname{sp}(A_{\mathbb{1}}, a)\} = \mathfrak{r}(a). \end{aligned}$$ □

We note that Corollary 1.1.109 follows straightforwardly from Corollary 1.1.114 above.

Arguments such as those in the proof of Corollary 1.1.114 also allow us to derive the following from Corollary 1.1.81(ii).

Corollary 1.1.115 *Let A be an associative normed algebra over $\mathbb{K}$, and let a,b be commuting elements of A. Then*

$$\mathfrak{r}(a+b) \leqslant \mathfrak{r}(a)+\mathfrak{r}(b) \ \text{ and } \ \mathfrak{r}(ab) \leqslant \mathfrak{r}(a)\,\mathfrak{r}(b).$$

1.1.7 Historical notes and comments

The material in this section has been elaborated mainly from the books of Berberian [689], Bonsall and Duncan [696], Bourbaki [697], Conway [711], Helemskii [742], Hille and Phillips [746], Müller [780], Murphy [781], Rickart [795], Rudin [804], and Sinclair [810]. Other sources are quoted in what follows.

For functional analysts, Proposition 1.1.7 is folklore because of the elementary observation in §1.1.3 and the fact that bilinear mappings starting from a product of two finite-dimensional normed spaces and valued into a normed space are automatically continuous. However, for people working only in pure algebra, Proposition 1.1.7 'needs a proof'. The elementary one given here is close to that of Albert's paper [9].

For the roots of Theorem 1.1.31, the reader is referred to the beginning of Notes and Remarks VIII.10 in Dunford–Schwartz [726]. The proof given here is taken from Theorem 9.4.2 in Hille–Phillips [746]. According to the authors of [746], the main points in this proof are due to Nagumo [457], who, according to [804, p. 373], initiated the abstract study of normed algebras. Another proof of Theorem 1.1.31 will be discussed in Subsection 1.3.3. Besides Theorem 1.1.31, the exponential function on complete normed unital associative algebras has relevant applications. A sample, also due to Nagumo [457] (see also [820, Theorem 7.3]), is formulated in the following.

Theorem 1.1.116 *Let A be a complete normed unital associative algebra. Then the connected component of the unit in the topological group* $\mathrm{Inv}(A)$ *coincides with the subgroup of* $\mathrm{Inv}(A)$ *generated by the set* $\{\exp(a) : a \in A\}$.

According to the preface of Zelazko's book [820], the foundations of the theory of normed associative algebras are due to Gelfand [284]. In particular, Theorem 1.1.41 and the proof given here are due to him. A proof, avoiding complex function theory, was later given by Rickart [503], and is included in [795, Theorem 1.6.3] and [696, Theorem 5.7]. In fact, Rickart's proof gives additional information, formulated as follows.

Theorem 1.1.117 *Let A be a normed unital associative complex algebra, and let a be in A. Then there exists $\lambda \in \mathrm{sp}(A,a)$ with $|\lambda| \geqslant \mathfrak{r}(a)$.*

According to [795, p. 38], the 'spectral radius formula' of Theorem 1.1.46, which was proved by Gelfand [284] in the general case of a complete normed unital associative complex algebra, was proved for the algebra of absolutely convergent trigonometric series by Beurling [101]. With Theorem 1.1.117 in mind, the Gelfand–Beurling formula follows straightforwardly from Proposition 1.1.40. After the

Gelfand–Beurling formula is proved in one or other manner, Theorem 1.1.117 can be refined by replacing in its formulation $|\lambda| \geqslant \mathfrak{r}(a)$ with $|\lambda| = \mathfrak{r}(a)$. Indeed, this follows by applying Theorem 1.1.46, with the completion of A (say $\hat{A}$) instead of A, and by noticing that $\operatorname{sp}(\hat{A}, a) \subseteq \operatorname{sp}(A, a)$.

A direct proof of Corollary 1.1.43, avoiding both Theorem 1.1.41 and complex function theory, can be found in Mazet's paper [430].

Example 1.1.50 contains the solution of the undergraduate student A. Molino to an exercise raised in the class.

§1.1.118 Theorem 1.1.62 (in the particular case that the algebra A is associative) and Corollary 1.1.63 are due to Rickart [501]. The actual version of Theorem 1.1.62 is taken from [540]. We do not know whether Theorem 1.1.62 remains true when associativity of the algebra B is removed altogether. By looking at the proof, this would be the case whenever the inequality (1.1.8) in Proposition 1.1.60 could be proved without involving the associativity of B. But this last problem remains open to date.

§1.1.119 According to [804, p. 373], 'what really got the subject of associative normed algebras was Gelfand's discovery of the important role played by the maximal ideals of a [complete normed associative and] commutative algebra [284] and his construction of what is now known as the Gelfand transform'. In particular, Theorem 1.1.73 and the associative forerunner of Theorem 1.1.75 are due to Gelfand [284].

According to [790, p. 111], Corollary 1.1.77 is due to Gelfand and Kolmogorov [286].

The notion of a topological divisor of zero was introduced by Shilov [564]. Propositions 1.1.90 and 1.1.93 are due to Rickart [500].

In relation to Proposition 1.1.93, it is worth mentioning the following.

Proposition 1.1.120 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, let B be a closed subalgebra of A containing the unit of A, and let b be in B. We have:*

(i) $\operatorname{sp}(B, b)$ *is the union of* $\operatorname{sp}(A, b)$ *and a (possibly empty) collection of holes of* $\operatorname{sp}(A, b)$.

(ii) *If $\mathbb{K} = \mathbb{C}$, and if B is generated by $\{\mathbf{1}, b\}$ as a normed algebra, then* $\operatorname{sp}(B, b)$ *has no hole in $\mathbb{C}$.*

Assertion (i) in the above proposition follows easily from the first assertion in Proposition 1.1.93(ii) and the purely topological fact that, *if V and W are open sets in some topological space, if $V \subseteq W$, and if W contains no boundary point of V, then V is a union of connected components of W* [804, Lemma 10.16]. For a proof of assertion (ii) in Proposition 1.1.120, the reader is referred to [696, Theorem 19.5]. As a straightforward consequence of Proposition 1.1.120(i), we deduce the following.

Corollary 1.1.121 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, and let a be in* $\operatorname{Inv}(A)$. *If* 0 *does not belong to any hole of* $\operatorname{sp}(A, a)$, *then a^{-1} lies in the closed subalgebra of A generated by $\{\mathbf{1}, a\}$.*

Corollaries 1.1.95 and 1.1.96 become the rudiments of the so-called operator theory. Additional basic information on this topic can be found in Section 1.4 below, as well as in [689, 726, 795]. For deeper developments, the reader is referred to [767, 780].

Due to the power of complex methods, the possibility (assured by Proposition 1.1.98) of regarding (isometrically) any real normed algebra A as a real subalgebra of a normed complex algebra becomes a relevant fact. Our method of converting the algebraic complexification of A into a complex normed algebra by means of the projective tensor norm, is no other than that of Bonsall and Duncan in [696, Proposition 13.3]. This method has the advantage that it works without problems in the non-associative setting. It is worth mentioning that other earlier methods, like the one in Theorem 1.3.2 of Rickart's book [795], only work in the associative setting. All facts about the projective tensor product stated without proof in our development are straightforward. Anyway, the reader can consult Section 3 of the Defant–Floret book [717] for details and complements.

Proposition 1.1.111 was proved first by Ocaña in his PhD thesis [784] with a proof different from ours. Ocaña's proof was included in [452].

§**1.1.122** In the associative case, the existence of a norm-unital equivalent renorming of a normed unital algebra can be proved as follows:

> Let A be a normed unital algebra, and, for $a \in A$, set $|||a||| := \|L_a\|$. Then we have $|||\mathbf{1}||| = 1$ and
>
> $$|||a||| = \|L_a\| \leqslant \|a\| = \|L_a(\mathbf{1})\| \leqslant \|L_a\|\|\mathbf{1}\| = \|\mathbf{1}\|\,|||a|||$$
>
> for every $a \in A$, so that $|||\cdot|||$ is an equivalent norm on A. Finally, if A is associative, then, as pointed out in §1.1.36, the mapping $a \to L_a$ is an algebra homomorphism, which implies that $|||\cdot|||$ is an algebra norm. □

1.2 Introducing C^*-algebras

Introduction We introduce C^*-algebras, and develop their basic theory, focussing on the unital commutative case. Some applications to the non-commutative case, like the continuous functional calculus in a normal element, are considered. The passing from a C^*-algebra to its C^*-algebra unital extension is also discussed in order to profit from the non-unital case. As in the case of Section 1.1, the present section is mainly devoted to attracting the attention of the non-expert reader. Deeper developments will be discussed later.

1.2.1 The results

We begin this section by introducing the adjoint of a bounded linear operator between Hilbert spaces.

Proposition 1.2.1 *Let H,K,L be Hilbert spaces over $\mathbb{K}$.*

(i) *If $F \in BL(H,K)$, then there is a unique element $F^* \in BL(K,H)$ such that*

$$(F(x)|y) = (x|F^*(y)) \text{ for all } x \in H \text{ and } y \in K.$$

(ii) *The mapping $F \to F^*$ from $BL(H,K)$ to $BL(K,H)$ is conjugate-linear. Moreover, for $F \in BL(H,K)$ we have $F^{**} = F$ and*

$$\|F\| = \|F^*\| = \|F^*F\|^{\frac{1}{2}}.$$

(iii) *If $F \in BL(H,K)$, and if $G \in BL(K,L)$, then $(GF)^* = F^*G^*$.*

Proof If $F \in BL(H,K)$ and $y \in K$, then the function $x \to (F(x)|y)$ is a continuous linear functional on H, so, by Riesz' representation theorem, there is a unique element $F^*(y) \in H$ such that $(F(x)|y) = (x|F^*(y))$ for every $x \in H$. Moreover,

$$\|F^*(y)\| = \sup\{|(F(x)|y)| : x \in \mathbb{B}_H\} \leqslant \|F\|\|y\|.$$

It is routine that the map $F^* : K \to H$ is linear, and from the above we realize that F^* is bounded with $\|F^*\| \leqslant \|F\|$. Thus, F^* satisfies the condition in assertion (i), which yields the uniqueness of F^*.

Let F be in $BL(H,K)$. Then, for $x \in \mathbb{B}_H$ we have

$$(F(x)|F(x)) = (x|F^*F(x)) \leqslant \|F^*F\|,$$

so

$$\|F\|^2 = \sup\left\{\|F(x)\|^2 : x \in \mathbb{B}_H\right\} \leqslant \|F^*F\| \leqslant \|F\|^2.$$

Hence, $\|F\| = \|F^*F\|^{\frac{1}{2}}$. The other properties in assertion (ii) have routine verifications.

Assertion (iii) follows from the fact that, for $F \in BL(H,K)$ and $G \in BL(K,L)$ we have

$$(GF(x)|z) = (F(x)|G^*(z)) = (x|F^*G^*(z))$$

for all $x \in H$ and $z \in L$. □

Let H and K be Hilbert spaces over $\mathbb{K}$. For each $F \in BL(H,K)$, the operator $F^* \in BL(K,H)$ is called the *adjoint* of F.

By an involution on a set E, we mean a mapping $x \to x^*$ from E to E satisfying $x^{**} = x$ for every $x \in E$. Given a linear or conjugate-linear involution $*$ on a vector space X over $\mathbb{K}$, we denote by $H(X,*)$ the real subspace of X consisting of all $*$-invariant elements of X. In the case $K = \mathbb{C}$ and $*$ is conjugate-linear, we have $X = H(X,*) \oplus iH(X,*)$. Given two sets E and F, each of which is endowed with an involution $*$ and a mapping $f : E \to F$, we say that f is a $*$-mapping if the equality $f(x^*) = f(x)^*$ holds for every $x \in E$.

A linear or conjugate-linear involution $*$ on an algebra A over $\mathbb{K}$ is said to be an *algebra involution* if the equality $(ab)^* = b^*a^*$ holds for all $a, b \in A$.

§1.2.2 By a $*$-*algebra* over $\mathbb{K}$ we mean an algebra over $\mathbb{K}$ endowed with a conjugate-linear algebra involution $*$. The $*$-invariant subalgebras of a given $*$-algebra will be called $*$-*subalgebras*. As usual, by a C^*-*algebra* we mean a complete normed associative complex $*$-algebra A satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$. The straightforward fact that the involution of any C^*-algebra is an isometry will be applied without notice. On the other hand, it is clear that closed $*$-subalgebras of C^*-algebras are C^*-algebras. Moreover, it follows from

Proposition 1.2.1 that, given a complex Hilbert space H, the passing from each element $F \in BL(H)$ to its adjoint F^* becomes a conjugate-linear algebra involution on $BL(H)$, and that, endowed with this involution and the operator norm, $BL(H)$ turns out to be a C^*-algebra.

The celebrated *non-commutative Gelfand–Naimark theorem* establishes the reciprocal.

Theorem 1.2.3 *Every C^*-algebra is isometric and algebra $*$-isomorphic to a closed $*$-subalgebra of $BL(H)$ for some complex Hilbert space H.*

For the moment, we will not discuss the proof of Theorem 1.2.3.

Let E be a non-empty set, and let $F^{\mathbb{C}}(E)$ stand for the algebra of all complex-valued functions on E (see Example 1.1.1(a)). For each $x \in F^{\mathbb{C}}(E)$, we denote by x^* the element in $F^{\mathbb{C}}(E)$ defined by $x^*(t) = \overline{x(t)}$ for every $t \in E$. It is clear that the mapping $x \to x^*$ is a conjugate-linear algebra involution on $F^{\mathbb{C}}(E)$. In the case that E is endowed with a locally compact Hausdorff topology, the subalgebra $C_0^{\mathbb{C}}(E)$ of $F^{\mathbb{C}}(E)$ (see Example 1.1.4(a)) is $*$-invariant, and, endowed with the involution $*$ and its canonical norm $\|x\| := \max\{|x(t)| : t \in E\}$, becomes a commutative C^*-algebra.

According to the celebrated *commutative Gelfand–Naimark theorem*, there are no more examples of commutative C^*-algebras. Indeed, we have the following.

Theorem 1.2.4 *Every commutative C^*-algebra is isometric and algebra $*$-isomorphic to $C_0^{\mathbb{C}}(E)$ for some locally compact Hausdorff topological space E.*

Theorem 1.2.4 above follows easily from Theorem 1.2.23 and Proposition 1.2.44 below. In the commutative case, the proof of the non-commutative Gelfand–Naimark theorem (Theorem 1.2.3) is very easy. Indeed, we have the following.

Exercise 1.2.5 Let E be a locally compact Hausdorff topological space. Then there exists a complex Hilbert space H such that the C^*-algebra $C_0^{\mathbb{C}}(E)$ is isometrically algebra $*$-isomorphic to a closed $*$-subalgebra of $BL(H)$.

Solution Take H equal to the complex Hilbert space $\ell_2(E)$, and for $x \in C_0^{\mathbb{C}}(E)$ let $\Phi(x) \in BL(H)$ be defined by $\Phi(x)(\{\lambda_t\}_{t\in E}) := \{x(t)\lambda_t\}_{t\in E}$ for every $\{\lambda_t\}_{t\in E} \in H$. Then the mapping $x \to \Phi(x)$ becomes an isometric algebra $*$-homomorphism from $C_0^{\mathbb{C}}(E)$ to $BL(H)$. $\square$

One of the ingredients in the proof of Theorem 1.2.4 is the so-called *Stone–Weierstrass theorem* (Theorem 1.2.10), which is proved in what follows.

§1.2.6 Let E be a compact Hausdorff topological space, and let f and g be in $C^{\mathbb{R}}(E)$. As usual, we say that $f \leqslant g$ whenever $f(t) \leqslant g(t)$ for every $t \in E$. Then $\leqslant$ is a partial order on $C^{\mathbb{R}}(E)$, and in fact $(C^{\mathbb{R}}(E), \leqslant)$ is a lattice for the operations $\vee$ and $\wedge$ given by

$$(f \vee g)(t) = \max\{f(t), g(t)\} \text{ and } (f \wedge g)(t) = \min\{f(t), g(t)\}.$$

Lemma 1.2.7 *Let E be a compact Hausdorff topological space and let X be a subset of $C^{\mathbb{R}}(E)$. If $f \vee g$ and $f \wedge g$ belong to X for all f and g in X, then every continuous function on E that can be approximated from X in every pair of points in E can in fact be approximated uniformly from X.*

Proof Let f be a function that can be approximated as described above. For every $\varepsilon > 0$ and u, v in E there is thus an $f_{u,v}$ in X such that both u and v are in the sets

$$V^{u,v} = \{t \in E : f(t) < f_{u,v}(t) + \varepsilon\} \text{ and } V_{u,v} = \{t \in E : f_{u,v}(t) < f(t) + \varepsilon\}.$$

For fixed u and variable v the open sets $V^{u,v}$ cover E. Since E is compact, we can therefore find $v_1, \dots, v_n \in E$ such that $E = \bigcup V^{u,v_k}$. By the assumption on X, we have $f_u := \vee f_{u,v_k} \in X$, and we see that $f(t) < f_u(t) + \varepsilon$ for every t in E. At the same time $f_u(t) < f(t) + \varepsilon$ for every t in $W_u = \bigcap V_{u,v_k}$ which is an open neighbourhood of u. Varying now u, we find $u_1, \dots, u_m$ such that $E = \bigcup W_{u_k}$, and we have

$$f_\varepsilon := \wedge f_{u_k} \in X \text{ with } f_\varepsilon(t) - \varepsilon < f(t) < f_\varepsilon(t) + \varepsilon \text{ for every } t \text{ in } E. \qquad \square$$

Lemma 1.2.8 *Let E be a compact Hausdorff topological space. If A is a closed subalgebra of $C^{\mathbb{R}}(E)$, then A is stable under the lattice operations $f \vee g$ and $f \wedge g$ in $C^{\mathbb{R}}(E)$.*

Proof For $\varepsilon > 0$ consider the function $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = \varepsilon^2 + x^2$. Note that for every $x \in \mathbb{R}$ we have

$$h(x) = 1 + \varepsilon^2 + (x^2 - 1) = (1 + \varepsilon^2)(1 + u(x)),$$

where $u : \mathbb{R} \to \mathbb{R}$ is the function given by $u(x) = \frac{x^2-1}{1+\varepsilon^2}$. Since, for each $x \in [-1,1]$ we have $|u(x)| \leqslant \frac{1}{1+\varepsilon^2} < 1$, it follows that the power series expansion of $h(x)^{\frac{1}{2}}$ at 0 converges uniformly on $[-1,1]$. We can thus find a polynomial p such that $|(\varepsilon^2 + x^2)^{\frac{1}{2}} - p(x)| < \varepsilon$ for every x in $[-1,1]$, in particular, $p(0) < 2\varepsilon$. Set $q(x) = p(x) - p(0)$. Note that for every $x \in \mathbb{R}$ we have $0 \leqslant (\varepsilon^2 + x^2)^{\frac{1}{2}} - |x| \leqslant \varepsilon$, and consequently $|\,q(x) - |x|\,| \leqslant \varepsilon + 2\varepsilon + \varepsilon = 4\varepsilon$. Since $q(0) = 0$, we know that $q(f) \in A$ for every f in A. Now take f in A with $\|f\| \leqslant 1$. Then

$$\|\,q(f) - |f|\,\| \leqslant 4\varepsilon.$$

Since ε is arbitrary and A is closed in $C^{\mathbb{R}}(E)$, it follows that $|f| \in A$ for every f in A. As

$$f \vee g = \tfrac{1}{2}(f + g + |f - g|) \text{ and } f \wedge g = \tfrac{1}{2}(f + g - |f - g|),$$

we immediately have the desired conclusions. $\square$

§1.2.9 Let $A = C^{\mathbb{C}}(E)$, where E is a compact Hausdorff topological space. Then $H(A, *)$ is a closed real subalgebra of A isomorphic to $C^{\mathbb{R}}(E)$. Therefore, we can write $C^{\mathbb{C}}(E) = C^{\mathbb{R}}(E) \oplus iC^{\mathbb{R}}(E)$.

Theorem 1.2.10 (Stone–Weierstrass) *Let E be a compact Hausdorff topological space and let A be a $*$-subalgebra of $C^{\mathbb{C}}(E)$ containing the constant functions and separating the points of E. Then A is dense in $C^{\mathbb{C}}(E)$.*

Proof The closure $\bar{A}$ of A in $C^{\mathbb{C}}(E)$ is still a $*$-subalgebra of $C^{\mathbb{C}}(E)$, so the set $H(\bar{A},*)$ of real-valued functions from $\bar{A}$ is a closed subalgebra of $C^{\mathbb{R}}(E)$. By Lemma 1.2.8 it is therefore stable under the lattice operations $f \vee g$ and $f \wedge g$ in $C^{\mathbb{R}}(E)$. Let t_1 and t_2 be in E with $t_1 \neq t_2$. By assumption, there is $g \in A$ such that $g(t_1) \neq g(t_2)$. Since the real and imaginary parts of g belong to $H(A,*)$ we see that $H(\bar{A},*)$ separates points in E and contains the constant real-valued functions. Given α, β real numbers, we note that there exists $h \in H(\bar{A},*)$ such that $h(t_1) = \alpha$ and $h(t_2) = \beta$. Indeed, there exists $g \in H(\bar{A},*)$ such that $g(t_1) \neq g(t_2)$, and then the function

$$h = \frac{\alpha - \beta}{g(t_1) - g(t_2)} g + \left(\alpha - \frac{\alpha - \beta}{g(t_1) - g(t_2)} g(t_1) \right) \mathbf{1}$$

lies in $H(\bar{A},*)$ and satisfies $h(t_1) = \alpha$ and $h(t_2) = \beta$. Given $f \in C^{\mathbb{R}}(E)$ we can therefore choose h in $H(\bar{A},*)$ such that $h(t_1) = f(t_1)$ and $h(t_2) = f(t_2)$. Thus $H(\bar{A},*)$ fulfills the assumptions in Lemma 1.2.7, whence $f \in H(\bar{A},*)$. Consequently,

$$C^{\mathbb{C}}(E) = H(\bar{A},*) + iH(\bar{A},*) = \bar{A}. \qquad \square$$

Definition 1.2.11 Let A be a C^*-algebra. An element $a \in A$ is said to be *self-adjoint* (respectively, *normal*) if $a^* = a$ (respectively, $a^*a = aa^*$). We note that self-adjoint elements are normal, and that, for $a \in A$, a^*a is self-adjoint.

Lemma 1.2.12 *Let A be a C^*-algebra, and let a be a normal element of A. Then* $\mathfrak{r}(a) = \|a\|$.

Proof We have

$$\begin{aligned} \|a^2\|^2 &= \|(a^2)^* a^2\| = \|a^* a^* a a\| = \|a^* a a^* a\| \\ &= \|(a^* a)^* (a^* a)\| = \|a^* a\|^2 = \|a\|^4, \end{aligned}$$

and hence $\|a^2\| = \|a\|^2$. Since, for $n \in \mathbb{N}$, a^n is also a normal element of A, the last equality becomes the starting point for an induction argument showing that

$$\|a^{2^n}\| = \|a\|^{2^n} \text{ for every } n \in \mathbb{N}.$$

Therefore, by Corollary 1.1.18(i), we have

$$\mathfrak{r}(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|. \qquad \square$$

Proposition 1.2.13 *Let A be a C^*-algebra, and let a be in A. Then*

$$\|a\| = \sqrt{\mathfrak{r}(a^* a)}.$$

Proof By Lemma 1.2.12 applied to the self-adjoint element a^*a, we have

$$\mathfrak{r}(a^* a) = \|a^* a\| = \|a\|^2. \qquad \square$$

Corollary 1.2.14 *Let A and B be C^*-algebras, and let $\Phi : A \to B$ be an algebra $*$-homomorphism. Then Φ is contractive. As a consequence, if Φ is bijective, then Φ is an isometry.*

Proof Let a be in A. By Proposition 1.2.13 and Corollary 1.1.114, we have

$$\|\Phi(a)\|^2 = \mathfrak{r}(\Phi(a)^* \Phi(a)) = \mathfrak{r}(\Phi(a^* a)) \leqslant \mathfrak{r}(a^* a) = \|a\|^2. \qquad \square$$

Exercise 1.2.15 Let a be an element in the C^*-algebra $M_2(\mathbb{C})$ of all 2×2 complex matrices. Prove that

$$2\|a\| = (\tau+2\delta)^{\frac{1}{2}} + (\tau-2\delta)^{\frac{1}{2}}, \tag{1.2.1}$$

where $\tau = \text{trace}(a^*a)$ and $\delta = |\det(a)|$.

Solution Given a self-adjoint matrix $h = \begin{pmatrix} r & z \\ \bar{z} & s \end{pmatrix}$ where r,s are real and z is complex, it is enough to compute the roots of the characteristic polynomial of h, and apply Theorem 1.1.46 and Proposition 1.2.13, to get

$$2\|h\| = |r+s| + [(r-s)^2 + 4|z|^2]^{\frac{1}{2}}.$$

Now (1.2.1) follows by taking $h = a^*a$ after elementary arrangements. □

Lemma 1.2.16 *Let A be a unital associative algebra over $\mathbb{K}$, and let a be in* $\text{Inv}(A)$. *Then*

$$\text{sp}(A,a^{-1}) = \left\{\lambda^{-1} : \lambda \in \text{sp}(A,a)\right\}.$$

Proof It is enough to show that, for $\lambda \in \mathbb{K}\backslash\{0\}$, the condition $a-\lambda\mathbf{1} \in \text{Inv}(A)$ implies that $a^{-1}-\lambda^{-1}\mathbf{1} \in \text{Inv}(A)$. But this follows from the fact that

$$a^{-1}-\lambda^{-1}\mathbf{1} = -\lambda^{-1}a^{-1}(a-\lambda\mathbf{1}).$$ □

Exercise 1.2.17 Let A be a complete normed unital associative $*$-algebra over $\mathbb{K}$ whose involution is continuous. Prove that $\exp(a)^* = \exp(a^*)$ for every $a \in A$.

§1.2.18 Let A be a unital C^*-algebra. Then A is norm-unital. Indeed, we have $\mathbf{1}^* = \mathbf{1}$, so $\|\mathbf{1}\| = \|\mathbf{1}^*\mathbf{1}\| = \|\mathbf{1}\|^2$. An element u in A is said to be *unitary* if $u^*u = uu^* = \mathbf{1}$, i.e. if $u \in \text{Inv}(A)$ and $u^{-1} = u^*$. If u is a unitary element of A, then $\|u\| = 1$, since $\|u\|^2 = \|u^*u\| = \|\mathbf{1}\| = 1$. As a consequence of Exercises 1.1.30 and 1.2.17, $\exp(ih)$ is unitary for every self-adjoint element h in A.

Exercise 1.2.19 Let H be a complex Hilbert space. Use the polarization law to realize that linear isometries from H to itself are precisely those operators $u \in BL(H)$ such that $u^*u = I_H$. Conclude then that surjective linear isometries from H to itself are nothing other than unitary elements of $BL(H)$.

Proposition 1.2.20 *Let A be a unital C^*-algebra. Then*

(i) $\text{sp}(A,u) \subseteq \mathbb{S}_{\mathbb{C}}$, *for every unitary element u of A.*
(ii) $\text{sp}(A,h) \subseteq \mathbb{R}$, *for every self-adjoint element h of A.*

Proof Let u be a unitary element of A, and let λ be in $\text{sp}(A,u)$. Since $\|u\| = 1$, it follows from Proposition 1.1.40 that $|\lambda| \leqslant 1$. By Lemma 1.2.16, $\lambda^{-1} \in \text{sp}(A,u^*)$. Since u^* is also unitary, it follows that $|\lambda^{-1}| \leqslant 1$, and so we conclude that $|\lambda| = 1$.

Let h be a self-adjoint element of A, and let $\alpha + i\beta$ be in $\text{sp}(A,h)$, where $\alpha,\beta \in \mathbb{R}$. Thus, for any $\gamma \in \mathbb{R}$ we have $\alpha + i(\beta+\gamma) \in \text{sp}(A, h+i\gamma\mathbf{1})$. By Proposition 1.1.40 we have $|\alpha + i(\beta+\gamma)| \leqslant \|h+i\gamma\mathbf{1}\|$, and hence

$$\begin{aligned}\alpha^2 + (\beta+\gamma)^2 = |\alpha+i(\beta+\gamma)|^2 \leqslant \|h+i\gamma\mathbf{1}\|^2 &= \|(h+i\gamma\mathbf{1})^*(h+i\gamma\mathbf{1})\| \\ &= \|(h-i\gamma\mathbf{1})(h+i\gamma\mathbf{1})\| = \|h^2+\gamma^2\mathbf{1}\| \leqslant \|h\|^2+\gamma^2.\end{aligned}$$

Subtracting γ^2 on both sides we are left with $\alpha^2+\beta^2+2\beta\gamma \leqslant \|h\|^2$. Since this is satisfied for all $\gamma \in \mathbb{R}$ we conclude $\beta = 0$. □

Exercise 1.2.21 Let X and Y be complex vector spaces endowed with conjugate-linear involutions $*$, and let $\Phi : X \to Y$ be a linear mapping. Prove that Φ is a $*$-mapping if and only if $\Phi(H(X,*)) \subseteq H(Y,*)$.

Combining Proposition 1.2.20(ii) with Corollary 1.1.67, we deduce the following consequence of Exercise 1.2.21.

Corollary 1.2.22 *Let A be a unital C^*-algebra, and let φ be a character on A. Then φ is a $*$-mapping.*

Now we prove the unital version of Theorem 1.2.4.

Theorem 1.2.23 *Let A be a unital commutative C^*-algebra. Then the Gelfand representation is an isometric algebra $*$-isomorphism from A onto $C^{\mathbb{C}}(\Delta)$, where Δ stands for the carrier space of A.*

Proof By Theorem 1.1.73, the Gelfand representation $G : A \to C^{\mathbb{C}}(\Delta)$ is a contractive unit-preserving algebra homomorphism whose range separates the points of Δ, and the equality $\|G(a)\| = \mathfrak{r}(a)$ holds for every $a \in A$. This equality, together with Lemma 1.2.12, gives us that G is an isometry. On the other hand, by Corollary 1.2.22, we have that

$$G(a^*)(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{G(a)(\varphi)} = G(a)^*(\varphi)$$

for all $a \in A$ and $\varphi \in \Delta$. Thus G is a $*$-mapping. It follows that $G(A)$ is a closed $*$-subalgebra of $C^{\mathbb{C}}(\Delta)$ containing the constant functions and separating the points of Δ. The Stone–Weierstrass theorem implies, therefore, that $G(A) = C^{\mathbb{C}}(\Delta)$. □

Now we are going to take advantage, in the non-commutative case, of the commutative Gelfand–Naimark theorem just proved. In relation to the next result, we note that, in view of Example 1.1.28, unital closed subalgebras of unital C^*-algebras need not be full subalgebras.

Proposition 1.2.24 *Let A be a unital C^*-algebra. Then we have:*

(i) *Each non-invertible element of A is a one-sided topological divisor of zero in A.*
(ii) *Each closed $*$-subalgebra of A containing $\mathbf{1}$ is a full subalgebra of A.*

Proof Let x be in $A \setminus \mathrm{Inv}(A)$. Then, by Lemma 1.1.99(i), x^*x or xx^* is not invertible in A. Assume that x^*x is not invertible. Then, by Proposition 1.2.20(ii), x^*x lies in the boundary of $\mathrm{Inv}(A)$ relative to A (indeed, $x^*x = \lim(x^*x - \frac{i}{n}\mathbf{1})$). Therefore, by Proposition 1.1.90, there exists a sequence x_n of norm-one elements of A such that $x^*xx_n \to 0$. Since

$$\|xx_n\|^2 = \|x_n^*x^*xx_n\| \leqslant \|x^*xx_n\|,$$

we conclude that $xx_n \to 0$, which shows that x is a left topological divisor of zero in A. Now assume that xx^* is not invertible. Then a similar argument gives that x is a right topological divisor of zero in A. Thus, in any case, x is a one-sided topological divisor of zero in A, which concludes the proof of assertion (i).

Now, let B be any closed $*$-subalgebra of A containing $\mathbf{1}$, and let b be an element of $B\setminus\mathrm{Inv}(B)$. By assertion (i) just proved (with B instead of A), b is a one-sided topological divisor of zero in B, so it is a one-sided topological divisor of zero in A, and so, by Exercise 1.1.88(iii), b is not invertible in A. Thus B is a full subalgebra of A, and the proof of assertion (ii) is concluded. □

Proposition 1.2.25 *Let A be a $*$-algebra over $\mathbb{K}$, and let S be a non-empty $*$-invariant subset of A. Then the subalgebra of A generated by S is $*$-invariant. If in addition A is a normed algebra, and if $*$ is continuous, then the closed subalgebra of A generated by S is $*$-invariant.*

Proof Let B denote the subalgebra of A generated by S. Since $S\subseteq B$, we have $S=S^*\subseteq B^*$, and hence B^* is a subalgebra of A containing S. Therefore $B\subseteq B^*$, and as a result $B=B^*$. Now assume that in addition A is a normed algebra, and $*$ is continuous. Then $\bar{B}^*\subseteq\overline{B^*}=\bar{B}$, and hence $\bar{B}^*=\bar{B}$. The proof concludes by invoking Exercise 1.1.26(ii). □

Combining Corollary 1.1.79 and Proposition 1.2.25 above, we find the following.

Corollary 1.2.26 *Let A be a unital C^*-algebra, and let a be a normal element of A. Then the closed subalgebra of A generated by $\{\mathbf{1},a,a^*\}$ is a commutative $*$-subalgebra of A.*

§1.2.27 Let E,F be compact Hausdorff topological spaces, and let $\tau:E\to F$ be a continuous mapping. Then the *transpose mapping* $\tau^t:C^{\mathbb{C}}(F)\to C^{\mathbb{C}}(E)$, defined by $\tau^t(f)=f\circ\tau$ for every $f\in C^{\mathbb{C}}(F)$, becomes a unit-preserving contractive algebra $*$-homomorphism. Moreover, if τ is surjective, τ^t is isometric, and if τ is bijective, then τ^t is bijective.

Theorem 1.2.28 *Let A be a unital C^*-algebra, let a be a normal element of A, and denote by u the inclusion mapping $\mathrm{sp}(A,a)\hookrightarrow\mathbb{C}$. Then there exists a unique unit-preserving algebra $*$-homomorphism $\Phi:C^{\mathbb{C}}(\mathrm{sp}(A,a))\to A$ such that $\Phi(u)=a$. Moreover, Φ is isometric, and its range coincides with the closed subalgebra of A generated by $\{\mathbf{1},a,a^*\}$.*

Proof Denote by B the closed subalgebra of A generated by $\{\mathbf{1},a,a^*\}$. By Corollary 1.2.26, B is a commutative $*$-subalgebra of A. Consider the carrier space Δ of B, and the Gelfand representation $G:B\to C^{\mathbb{C}}(\Delta)$. By Theorem 1.2.23, G is an isometric surjective algebra $*$-homomorphism. On the other hand, by Proposition 1.2.24(ii), we have $\mathrm{sp}(A,a)=\mathrm{sp}(B,a)$. By Theorem 1.1.73, the Gelfand transform $G(a)$ of a is a continuous mapping from Δ onto $\mathrm{sp}(a)$. Moreover, if φ_1,φ_2 are in Δ and satisfy $\varphi_1(a)=\varphi_2(a)$, then we have $\varphi_1(a^*)=\varphi_2(a^*)$ (by Corollary 1.2.22) and $\varphi_1(\mathbf{1})=\varphi_2(\mathbf{1})$, and hence $\varphi_1=\varphi_2$ (by Lemma 1.1.82(i)). Therefore, $G(a)$ is injective, and consequently a homeomorphism. Now, as noticed in §1.2.27, the homeomorphism $G(a):\Delta\to\mathrm{sp}(a)$ induces an isometric surjective algebra $*$-homomorphism $G(a)^t:C^{\mathbb{C}}(\mathrm{sp}(a))\to C^{\mathbb{C}}(\Delta)$. Now, set $\Phi:=J\circ G^{-1}\circ G(a)^t$, where J stands for the inclusion mapping $B\hookrightarrow A$. Then, clearly, Φ is a unit-preserving isometric algebra $*$-homomorphism from $C^{\mathbb{C}}(\mathrm{sp}(a))$ to A with $\Phi(C^{\mathbb{C}}(\mathrm{sp}(a)))=B$. Moreover,

$\Phi(u) = G^{-1}(G(a)^t(u)) = G^{-1}(G(a)) = a$. This concludes the proof concerning the existence, and also proves the last conclusion in the theorem.

Let $\Psi : C^{\mathbb{C}}(\mathrm{sp}(A,a)) \to A$ be any unit-preserving algebra $*$-homomorphism with $\Psi(u) = a$. Then Ψ and Φ coincide on $\{\mathbf{1}, u, u^*\}$. Therefore, since Ψ is automatically continuous (by Corollary 1.2.14), and $C^{\mathbb{C}}(\mathrm{sp}(a))$ is generated by $\{\mathbf{1}, u, u^*\}$ as a normed algebra (by the Stone–Weierstrass theorem), it follows from Lemma 1.1.82(i) that $\Psi = \Phi$. □

§**1.2.29** As in Theorem 1.2.28, let a be a normal element of a unital C^*-algebra A, and let u be the inclusion mapping $\mathrm{sp}(A,a) \hookrightarrow \mathbb{C}$. The unique unit-preserving algebra $*$-homomorphism Φ from $C^{\mathbb{C}}(\mathrm{sp}(A,a))$ into A such that $\Phi(u) = a$ will be called the *continuous functional calculus* at a. If p is a complex polynomial (regarded as a complex-valued continuous function on $\mathrm{sp}(A,a)$), then we have $\Phi(p) = p(a)$ in the sense of §1.1.27. This is one of the reasons why, for $f \in C^{\mathbb{C}}(\mathrm{sp}(A,a))$, the symbol $\Phi(f)$ is usually replaced with $f(a)$. Note that, since $f(a)$ lies in a commutative $*$-subalgebra, $f(a)$ is a normal element of A.

Corollary 1.2.30 *Let E be a compact Hausdorff topological space, and let a be in $C^{\mathbb{C}}(E)$. Then*

(i) $f(a) = f \circ a$ *and* $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$ *for every* $f \in C^{\mathbb{C}}(\mathrm{sp}(a))$.
(ii) $(g \circ f)(a) = g(f(a))$ *for all* $f \in C^{\mathbb{C}}(\mathrm{sp}(a))$ *and* $g \in C^{\mathbb{C}}(\mathrm{sp}(f(a)))$.

Proof We begin by noticing that, by Example 1.1.32(c),

$$\mathrm{sp}(C^{\mathbb{C}}(E), a) = a(E),$$

and hence we can consider the mapping $f \to f \circ a$ from $C^{\mathbb{C}}(\mathrm{sp}(a))$ to $C^{\mathbb{C}}(E)$, which becomes an isometric unit-preserving algebra $*$-homomorphism taking u to a. Now, by Theorem 1.2.28, we find that $f(a) = f \circ a$ for every $f \in C^{\mathbb{C}}(\mathrm{sp}(a))$. As a consequence, for each $f \in C^{\mathbb{C}}(\mathrm{sp}(a))$ we see that

$$\mathrm{sp}(f(a)) = (f \circ a)(E) = f(a(E)) = f(\mathrm{sp}(a)).$$

Thus assertion (i) is proved.

Let f be in $C^{\mathbb{C}}(\mathrm{sp}(a))$, and let g be in $C^{\mathbb{C}}(\mathrm{sp}(f(a)))$. It follows from assertion (i) that $(g \circ f)(a) = (g \circ f) \circ a = g \circ (f \circ a) = g(f(a))$, and the proof of assertion (ii) is complete. □

Corollary 1.2.31 *Let A be a unital C^*-algebra, and let a be in A. Then we have:*

(i) *a is unitary if and only if a is normal and* $\mathrm{sp}(A,a) \subseteq \mathbb{S}_{\mathbb{C}}$.
(ii) *a is self-adjoint if and only if a is normal and* $\mathrm{sp}(A,a) \subseteq \mathbb{R}$.

Proof The ‘only if’ parts were proved in Proposition 1.2.20. In order to prove the ‘if’ parts, suppose that a is normal, and consider the continuous functional calculus at a. If $\mathrm{sp}(A,a) \subseteq \mathbb{S}_{\mathbb{C}}$, then u is a unitary element in $C^{\mathbb{C}}(\mathrm{sp}(A,a))$, and hence $a = \Phi(u)$ is a unitary element in A. If $\mathrm{sp}(A,a) \subseteq \mathbb{R}$, then u is a self-adjoint element in $C^{\mathbb{C}}(\mathrm{sp}(A,a))$, and hence $a = \Phi(u)$ is a self-adjoint element in A. □

Proposition 1.2.32 *Let A and B be unital C^*-algebras, let a be a normal element of A, and let $\phi : A \to B$ be a unit-preserving algebra $*$-homomorphism. Then $\phi(a)$ is a normal element of B, $\mathrm{sp}(B, \phi(a)) \subseteq \mathrm{sp}(A, a)$, and we have $\phi(f(a)) = f(\phi(a))$ for every $f \in C^{\mathbb{C}}(\mathrm{sp}(A, a))$.*

Proof The fact that $\phi(a)$ is a normal element of B is clear, whereas the inclusion $\mathrm{sp}(B, \phi(a)) \subseteq \mathrm{sp}(A, a)$ follows from Lemma 1.1.34(ii). Now the mapping $f \to f_{|\mathrm{sp}(B,\phi(a))}$ from $C^{\mathbb{C}}(\mathrm{sp}(A, a))$ to $C^{\mathbb{C}}(\mathrm{sp}(B, \phi(a)))$ becomes a contractive unit-preserving algebra $*$-homomorphism. Since ϕ is also contractive (by Corollary 1.2.14), it follows from Theorem 1.2.28 that the mappings $f \to \phi(f(a))$ and $f \to f(\phi(a))$ become contractive unit-preserving algebra $*$-homomorphisms from $C^{\mathbb{C}}(\mathrm{sp}(A, a))$ to B taking u to $\phi(a)$. Since $C^{\mathbb{C}}(\mathrm{sp}(A, a))$ is generated by $\{\mathbf{1}, u, u^*\}$ as a normed algebra (by the Stone–Weierstrass theorem), it follows from Lemma 1.1.82(i) that $\phi(f(a)) = f(\phi(a))$ for every $f \in C^{\mathbb{C}}(\mathrm{sp}(A, a))$. □

Corollary 1.2.33 *Let A be unital C^*-algebra, let a be a normal element of A, and let B be any closed $*$-subalgebra of A containing $\{\mathbf{1}, a\}$. Then we have $\mathrm{sp}(A, a) = \mathrm{sp}(B, a)$, and the continuous functional calculuses at a, relative to A and B, coincide.*

Proof The equality $\mathrm{sp}(A, a) = \mathrm{sp}(B, a)$ follows from Proposition 1.2.24(ii). Now, keeping in mind that the inclusion $B \hookrightarrow A$ becomes a unit-preserving algebra $*$-homomorphism, the remaining part of the conclusion follows by invoking Proposition 1.2.32. □

Proposition 1.2.34 *Let a be a normal element of a unital C^*-algebra A, and let $f \in C^{\mathbb{C}}(\mathrm{sp}(a))$. Then $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$* ('spectral mapping theorem'). *Moreover, if $g \in C^{\mathbb{C}}(\mathrm{sp}(f(a)))$, then $(g \circ f)(a) = g(f(a))$.*

Proof Let B stand for the closed subalgebra of A generated by $\{\mathbf{1}, a, a^*\}$. By Corollary 1.2.26, B is a commutative $*$-subalgebra of A, so that, by the commutative Gelfand–Naimark theorem (Theorem 1.2.23), the Gelfand representation is an isometric algebra $*$-isomorphism from B onto $C^{\mathbb{C}}(\Delta)$, where Δ stands for the carrier space of B. Now, the result follows from Corollaries 1.2.30 and 1.2.33. □

Let A be a unital C^*-algebra. An element a of A is said to be *positive* if a is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}_0^+$. We write $a \geqslant 0$ to mean that a is positive, and denote by A^+ the set of positive elements of A. We note that, as a consequence of Proposition 1.2.24(ii), if B is a closed $*$-subalgebra of A containing the unit of A, then $B^+ = A^+ \cap B$. In the case that $A = C^{\mathbb{C}}(E)$, where E is a compact Hausdorff topological space, A^+ consists precisely of those functions $f \in A$ such that $f(E) \subseteq \mathbb{R}_0^+$. Recall that each real-valued continuous function h on E can be written in a unique way as a difference of two positive continuous functions $h = h^+ - h^-$ such that $h^+h^- = 0$.

As a first application of the continuous functional calculus we have the following result.

Corollary 1.2.35 *If a is a self-adjoint element in a unital C^*-algebra A, then there are unique positive elements u, v in A such that $a = u - v$ and $uv = vu = 0$.*

Proof Recall that $\mathrm{sp}(a) \subseteq \mathbb{R}$ by Proposition 1.2.20(ii). If we consider the real-valued continuous functions f and g defined on $\mathbb{R}$ by $f(t) := \frac{1}{2}(|t| + t)$ and

$g(t) := \frac{1}{2}(|t| - t)$, then $fg = 0$ and $f(t) - g(t) = t$ for every $t \in \mathbb{R}$. It follows that $u = f(a)$ and $v = g(a)$ have the asserted properties.

To show the uniqueness, let u_1, v_1 be positive elements in A such that

$$a = u_1 - v_1 \text{ and } u_1v_1 = v_1u_1 = 0.$$

Let C be the closed subalgebra of A generated by the set $\{\mathbf{1}, u_1, v_1\}$. It follows from Corollary 1.1.79 and Proposition 1.2.25 that C is a commutative $*$-subalgebra of A. Since $a \in C$, it follows that the closed subalgebra of A (say B) generated by $\{\mathbf{1}, a\}$ is contained in C, and hence $u, v \in C$ by Theorem 1.2.28. On the other hand, by Theorem 1.2.23, the Gelfand representation $G : C \to C^{\mathbb{C}}(\Delta_C)$ is an algebra $*$-isomorphism. The uniqueness now follows from the uniqueness statement for $C^{\mathbb{C}}(\Delta_C)$. □

The decomposition $a = u - v$ of a self-adjoint element a given in Corollary 1.2.35 is sometimes called the *orthogonal decomposition* of a. The elements u and v are called the *positive* and *negative parts* of a and, according to §1.2.29, are denoted by a^+ and a^-, respectively. Note that $|a| = a^+ + a^-$.

Let $A = C^{\mathbb{C}}(E)$, where E is a compact Hausdorff topological space. If $f \in A$ is positive, and if n is a natural number, then f has a unique positive nth root in A, namely the function $t \to \sqrt[n]{f(t)}$.

Corollary 1.2.36 *If a is a positive element in a unital C^*-algebra A, and if $n \in \mathbb{N}$, then there is a unique positive element b in A such that $a = b^n$.*

Proof If we consider the real-valued continuous function f defined on $\mathbb{R}_0^+$ by $f(t) := \sqrt[n]{t}$, then $f(t)^n = t$ for every $t \in \mathbb{R}_0^+$. It follows from Proposition 1.2.34 that $b = f(a)$ is positive and $b^n = a$. To show the uniqueness, assume that c is a positive element in A such that $a = c^n$, and consider the closed subalgebra (say C) of A generated by the set $\{\mathbf{1}, c\}$. It follows from Corollary 1.2.26 that C is a $*$-invariant commutative subalgebra of A. Since $a = c^n \in C$, it follows that the closed subalgebra of A (say B) generated by $\{\mathbf{1}, a\}$ is contained in C, and hence $b \in C$ by Theorem 1.2.28. Finally, keeping in mind that the Gelfand representation $G : C \to C^{\mathbb{C}}(\Delta_C)$ is an algebra $*$-isomorphism (by Theorem 1.2.23), the uniqueness now follows from the uniqueness statement for $C^{\mathbb{C}}(\Delta_C)$. □

If a is a positive element in a unital C^*-algebra A, then the unique element b obtained in Corollary 1.2.36 is called the *positive nth root* of a and, according to §1.2.29, is denoted by $a^{\frac{1}{n}}$.

§1.2.37 If h is a self-adjoint element of the closed unit ball of a unital C^*-algebra A, then $\mathbf{1} - h^2$ is a positive element of A, and the element $u = h + i\sqrt{\mathbf{1} - h^2}$ is unitary and satisfies $h = \frac{1}{2}(u + u^*)$. Therefore, the unitaries linearly span A, a result that is frequently useful.

Let $A = C^{\mathbb{C}}(E)$, where E is a compact Hausdorff topological space. The positive condition for a real-valued function $f \in A$ can also be expressed in terms of the norm. Indeed, given $t \in \mathbb{R}$ with $\|f\| \leqslant t$, we have that

$$f \text{ is positive if and only if } \|f - t\mathbf{1}\| \leqslant t.$$

Proposition 1.2.38 *Let A be a unital C^*-algebra, let a be a self-adjoint element in A, and let t be a real number such that $\|a\| \leqslant t$. Then $a \geqslant 0$ if and only if $\|a - t\mathbf{1}\| \leqslant t$.*

Proof By considering the closed subalgebra of A generated by the set $\{\mathbf{1}, a\}$, and keeping in mind Corollary 1.2.26, we may suppose that $A = C^{\mathbb{C}}(E)$, for a suitable compact Hausdorff topological space. The result now follows from the comment immediately before the statement of the present proposition. □

§1.2.39 Let X be a real vector space, and let C be a non-empty subset of X. We say that C is a *cone* in X if $\lambda x \in C$ whenever $x \in C$ and $\lambda \geqslant 0$. If moreover $x, y \in C$ implies $x + y \in C$, then we say that C is a *convex cone*. If C is a cone, and if it contains no entire straight line (i.e. $C \cap (-C) = 0$), then C is said to be a *proper cone*. When a proper convex cone C is given in X, we make X a partially ordered set by defining $x \leqslant y$ to mean $y - x \in C$. The relation $\leqslant$ is translation-invariant; that is, $x \leqslant y$ implies $x + z \leqslant y + z$ for every $z \in X$. Also, $x \leqslant y$ implies $tx \leqslant ty$ for every $t \in \mathbb{R}_0^+$, and $x \leqslant y$ if and only if $-y \leqslant -x$. Moreover, if:

- X is in fact a normed space;
- the proper convex cone C is closed in X;
- x_n and y_n are sequences in X converging to x and y, respectively; and
- $x_n \leqslant y_n$ for every $n \in \mathbb{N}$;

then $x \leqslant y$.

Proposition 1.2.40 *If A is a unital C^*-algebra, then A^+ is a closed proper convex cone in $H(A, *)$.*

Proof That A^+ is a closed subset of A follows immediately from the continuity of the involution and Proposition 1.2.38. It is obvious that if $a \in A^+$ and $\lambda \geqslant 0$, then $\lambda a \in A^+$. Let $a, b \in A^+$. By Proposition 1.2.38,

$$\|a - \|a\|\mathbf{1}\| \leqslant \|a\| \text{ and } \|b - \|b\|\mathbf{1}\| \leqslant \|b\|,$$

so

$$\|a + b - (\|a\| + \|b\|)\mathbf{1}\| \leqslant \|a - \|a\|\mathbf{1}\| + \|b - \|b\|\mathbf{1}\| \leqslant \|a\| + \|b\|.$$

By Proposition 1.2.38 again, $a + b \geqslant 0$. Finally if $a \in A^+ \cap (-A^+)$, then $\operatorname{sp}(a) \subseteq \mathbb{R}_0^+$ and $-\operatorname{sp}(a) \subseteq \mathbb{R}_0^+$, hence $\operatorname{sp}(a) = 0$. Since a is self-adjoint, Lemma 1.2.12 applies, so that $\|a\| = \mathfrak{r}(a) = 0$, and hence $a = 0$. □

§1.2.41 According to the above proposition and §1.2.39, if A is a unital C^*-algebra, then $H(A, *)$ will be seen without notice as a partially ordered set.

As pointed out in §1.1.36, if A is an associative algebra over $\mathbb{K}$, then the mapping $a \to L_a$ from A to $L(A)$ becomes an algebra homomorphism. In the case of C^*-algebras, we have the following additional information.

Lemma 1.2.42 *Let A be a C^*-algebra. Then the mapping $a \to L_a$ from A to $BL(A)$ becomes an isometry. As a consequence, the set*

$$\mathscr{B} := \{\lambda I_A + L_a : \lambda \in \mathbb{C}, a \in A\}$$

is a closed subalgebra of $BL(A)$.

Proof For $a \in A$, we have

$$\|a\|^2 = \|aa^*\| = \|L_a(a^*)\| \leqslant \|L_a\|\|a^*\| = \|L_a\|\|a\|,$$

and hence $\|a\| = \|L_a\|$. Now $\mathscr{B}$ is the sum of the closed subspace $\{L_a : a \in A\}$ of $BL(A)$ and the finite-dimensional subspace $\mathbb{C}I_A$, and therefore it is closed in $BL(A)$. □

§1.2.43 Let B, C be C^*-algebras. Then the algebra direct product $B \times C$ becomes a C^*-algebra under the involution defined coordinate-wise, and the norm $\|(b,c)\| := \max\{\|b\|, \|c\|\}$.

Proposition 1.2.44 *Let A be a C^*-algebra. Then there are a unique involution and a unique norm on the unital extension $A_{\mathbb{1}}$ of A extending the involution and the norm of A and converting $A_{\mathbb{1}}$ into a C^*-algebra.*

Proof It is clear that the unique conjugate-linear algebra involution $*$ on $A_{\mathbb{1}} = \mathbb{C}\mathbb{1} \oplus A$ extending that of A is given by

$$(\lambda \mathbb{1} + a)^* := \bar{\lambda}\mathbb{1} + a^*.$$

Then the uniqueness of the desired norm on $A_{\mathbb{1}}$ follows from Corollary 1.2.14. To prove the existence of the desired norm on $A_{\mathbb{1}}$, we distinguish two cases depending on whether or not A has a unit.

First assume that A has a unit $\mathbf{1}$. Then A and $\mathbb{C}(\mathbb{1} - \mathbf{1})$ are ideals of $A_{\mathbb{1}}$ such that $A_{\mathbb{1}} = A \oplus \mathbb{C}(\mathbb{1} - \mathbf{1})$, and therefore, since

$$(a + \lambda(\mathbb{1} - \mathbf{1}))^* = a^* + \bar{\lambda}(\mathbb{1} - \mathbf{1}),$$

it is enough to apply §§1.1.105 and 1.2.43 to realize that the desired extended norm on $A_{\mathbb{1}}$ is given by

$$\|a + \lambda(\mathbb{1} - \mathbf{1})\| := \max\{\|a\|, |\lambda|\}.$$

As a result, the unique C^*-algebra structure on $A_{\mathbb{1}}$ extending that of A is given by the involution in the first paragraph of the proof and the norm

$$\|\lambda \mathbb{1} + a\| := \max\{\|a + \lambda \mathbf{1}\|, |\lambda|\}.$$

Now assume that A does not have a unit. Then the mapping $\lambda \mathbb{1} + a \to \lambda I_A + L_a$ from $A_{\mathbb{1}}$ to $BL(A)$ becomes an injective algebra homomorphism, and hence, applying Lemma 1.2.42, we realize that

$$\|\lambda \mathbb{1} + a\| := \|\lambda I_A + L_a\| \tag{1.2.2}$$

defines a complete algebra norm on $A_{\mathbb{1}}$ extending the norm on A. Moreover, given $\lambda \mathbb{1} + a \in A_{\mathbb{1}}$, and setting $T := \lambda I_A + L_a$ and $T^* := \bar{\lambda} I_A + L_{a^*}$, for every $x \in A$ we have

$$\|T(x)\|^2 = \|(\lambda x + ax)^*(\lambda x + ax)\| = \|x^*(T^*T)(x)\| \leqslant \|T^*T\|\|x\|^2,$$

so $\|T\|^2 \leqslant \|T^*T\|$, and so, invoking (1.2.2),

$$\|\lambda \mathbb{1} + a\|^2 \leqslant \|(\lambda \mathbb{1} + a)^*(\lambda \mathbb{1} + a)\|.$$ □

From now on, given a C^*-algebra A, the unital extension $A_{\mathbb{1}}$ of A will be seen without notice as a C^*-algebra containing A as a closed $*$-subalgebra, as shown by Proposition 1.2.44.

Now, we prove that complex associative and commutative algebras have at most one C^*-algebra structure.

Proposition 1.2.45 *Let A and B be commutative C^*-algebras, and let $\Phi : A \to B$ be a bijective algebra homomorphism. Then Φ preserves involutions and norms.*

Proof By passing to C^*-algebra unital extensions if necessary, we may assume that A and B are unital. Moreover, by Theorem 1.2.23, we may also assume that $A = C^{\mathbb{C}}(E)$ and $B = C^{\mathbb{C}}(F)$ for suitable compact Hausdorff topological spaces E and F. Then the result follows straightforwardly from Corollary 1.1.77. □

Given an associative algebra A and $a \in A$, we know that $0 \in \mathrm{sp}(A_{\mathbb{1}}, a)$.

Lemma 1.2.46 *Let A be a C^*-algebra, let a be a normal element of A, let f be in $C^{\mathbb{C}}(\mathrm{sp}(A_{\mathbb{1}}, a))$, and let $f(a)$ be the element of $A_{\mathbb{1}}$ given by the continuous functional calculus* (see §1.2.29). *Then $f(a)$ lies in A if and only if $f(0) = 0$.*

Proof Thinking about the unit-preserving algebra $*$-homomorphism ϕ from $A_{\mathbb{1}} = \mathbb{C}\mathbb{1} \oplus A$ to $\mathbb{C}$ defined by $\phi(\lambda \mathbb{1} + x) := \lambda$, the result follows from Proposition 1.2.32. □

§1.2.47 Let A be a non-unital C^*-algebra. Then $H(A, *)$ will be seen without notice as a partially ordered set relative to the order induced by that of the C^*-algebra unital extension of A (cf. §1.2.41). Now, with Lemma 1.2.46 in mind, the following proposition follows from Corollaries 1.2.35 and 1.2.36.

Proposition 1.2.48 *Let A be a C^*-algebra. We have:*

(i) *If a is a self-adjoint element of A, then there are unique positive elements u, v in A such that $a = u - v$ and $uv = vu = 0$.*
(ii) *If a is a positive element of A, and if $n \in \mathbb{N}$, then there is a unique positive element b in A such that $a = b^n$.*

Proposition 1.2.49 *Let A be a C^*-algebra, and let e be a non-self-adjoint idempotent in A. Then $\mathrm{sp}(A_{\mathbb{1}}, i(e - e^*))$ is a symmetric subset of $\mathbb{R}$, and the mapping $\lambda \to 1 + \lambda^2$ becomes a surjection from $\mathrm{sp}(A_{\mathbb{1}}, i(e - e^*)) \setminus \{0\}$ onto $\mathrm{sp}(A_{\mathbb{1}}, e^*e) \setminus \{0, 1\}$. Consequently, we have:*

(i) $\|e\|^2 = 1 + \|e - e^*\|^2$.
(ii) $\{0, \|e\|^2\} \subseteq \mathrm{sp}(A_{\mathbb{1}}, e^*e) \subseteq \{0\} \cup [1, \|e\|^2]$.

Proof A straightforward computation shows that, for $\lambda \in \mathbb{C}$, we have

$$\lambda(1 + \lambda^2)[i(e - e^*) - \lambda \mathbb{1}] = (e^* - i\lambda \mathbb{1})[e^*e - (1 + \lambda^2)\mathbb{1}](e + i\lambda \mathbb{1}). \tag{1.2.3}$$

Now, let λ be in $\mathbb{C} \setminus \{0, i, -i\}$. Then we have $\lambda(1 + \lambda^2) \neq 0$. Moreover, since 0 and 1 are the unique possible numbers in the spectrum of an idempotent, both $e^* - i\lambda \mathbb{1}$ and $e + i\lambda \mathbb{1}$ are invertible in $A_{\mathbb{1}}$. It follows from (1.2.3) that $i(e - e^*) - \lambda \mathbb{1}$ is invertible in $A_{\mathbb{1}}$ if and only if so is $e^*e - (1 + \lambda^2)\mathbb{1}$. Therefore, since

$$\mathrm{sp}(A_{\mathbb{1}}, i(e - e^*)) \subseteq \mathbb{R}$$

(by Proposition 1.2.20(ii)), $e^*e - (1 - \rho)\mathbb{1}$ is invertible in $A_{\mathbb{1}}$ for every $\rho \in \mathbb{R}^+ \setminus \{1\}$. Now, keeping in mind that $\mathrm{sp}(A_{\mathbb{1}}, e^*e) \subseteq \mathbb{R}$, we easily derive that $\mathrm{sp}(A_{\mathbb{1}}, i(e - e^*))$ is symmetric (relative to zero), and that the mapping $\lambda \to 1 + \lambda^2$ is a surjection from $\mathrm{sp}(A_{\mathbb{1}}, i(e - e^*)) \setminus \{0\}$ onto $\mathrm{sp}(A_{\mathbb{1}}, e^*e) \setminus \{0, 1\}$. The consequences, listed in the statement of the present proposition, follow from Lemma 1.2.12 and Theorem 1.1.46. □

As a straightforward consequence, we get the following.

Corollary 1.2.50 *A nonzero idempotent e in a C^*-algebra is self-adjoint if (and only if) $\|e\| = 1$.*

Proposition 1.2.51 *Let A be a commutative C^*-algebra. Then for every algebra norm $|||\cdot|||$ on A we have $\|\cdot\| \leqslant |||\cdot|||$.*

Proof By Propositions 1.1.107 and 1.2.44, we may assume that A is unital. Let $|||\cdot|||$ be an algebra norm on A. Let Δ stand for the carrier space of A (cf. Proposition 1.1.71), let Δ_1 denote the subset of Δ consisting of all $|||\cdot|||$-continuous characters on A, let B stand for the completion of $(A, |||\cdot|||)$, and let Δ_B denote the carrier space of B. We note that, by Corollary 1.1.64, the mapping $\psi \to \psi_{|A}$ becomes a bijection from Δ_B to Δ_1, and that, since $A = C^{\mathbb{C}}(\Delta)$ in a natural way (cf. Theorem 1.2.23), complex-valued continuous functions on Δ can be seen as elements of B.

We claim that Δ_1 is dense in Δ. Assume, to derive a contradiction, that $\overline{\Delta_1} \subsetneqq \Delta$. Then there is an open subset Ω of Δ such that $\bar{\Omega} \subseteq \Delta \setminus \overline{\Delta_1}$, together with real-valued continuous functions x, y on Δ such that $x(\phi) = 1$ if $\phi \in \overline{\Delta_1}$, $x(\phi) = 0$ if $\phi \in \bar{\Omega}$, $y \neq 0$, and $y(\phi) = 0$ if $\phi \in \Delta \setminus \Omega$. Since $xy = 0$, and $y \neq 0$, x cannot be invertible in B. Therefore, by Proposition 1.1.68(ii), there exists $\eta \in \Delta_B$ such that $\eta(x) = 0$, and hence $\phi_0 := \eta_{|A}$ lies in Δ_1 and satisfies $x(\phi_0) = 0$, which is a contradiction.

Now that the claim has been proved, it is enough to apply Proposition 1.1.68(iii) to realize that for every $a \in A$ we have

$$|||a||| \geqslant \mathfrak{r}_{|||\cdot|||}(a) = \sup_{\psi \in \Delta_B} |\psi(a)| = \sup_{\phi \in \Delta_1} |a(\phi)| = \sup_{\phi \in \Delta} |a(\phi)| = \|a\|. \qquad \square$$

Now we can refine Corollary 1.2.14 as follows.

Corollary 1.2.52 *Let A and B be C^*-algebras, and let $\Phi : A \to B$ be an algebra $*$-homomorphism. Then Φ is contractive. Moreover, if Φ is injective, then Φ is an isometry.*

Proof The first conclusion follows from Corollary 1.2.14. Assume that Φ is injective, and let a be in A. Then $\|\Phi(\cdot)\|$ is an algebra norm on the closed subalgebra of A generated by a^*a, which is a commutative C^*-algebra. Therefore, by Proposition 1.2.51, we have

$$\|a\|^2 = \|a^*a\| \leqslant \|\Phi(a^*a)\| = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a)\|^2.$$

Hence $\|\Phi(a)\| = \|a\|$ because of the first conclusion. $\square$

1.2.2 Historical notes and comments

The material in this section has been elaborated mainly from the books of Berberian [689], Bonsall and Duncan [696], Bourbaki [697], Choquet [709], Dixmier [724], Helemskii [742], Murphy [781], and Pedersen [789]. Other sources are quoted as appropriate.

The original abstract characterization of closed self-adjoint subalgebras of bounded linear operators on complex Hilbert spaces, given by Gelfand and Naimark [285], has been refined with time, giving rise to the current notion of a C^*-algebra, as defined in §1.2.2. The reader is referred to Chapter 1 of Doran–Belfi [725] for a detailed account of how the Gelfand–Naimark axioms were successively weakened, and of the several authors who contributed to this process.

Theorem 1.2.10 becomes the unital version of the Stone–Weierstrass theorem. A unit-free version follows almost straightforwardly. Indeed, we have the following.

Corollary 1.2.53 *Let E be a locally compact Hausdorff topological space, and let A be a $*$-subalgebra of $C_0^{\mathbb{C}}(E)$ satisfying*

(i) *for each $t \in E$ there is $f \in A$ such that $f(t) \neq 0$;*
(ii) *A separates the points of E.*

Then A is dense in $C_0^{\mathbb{C}}(E)$.

Proof Let E_∞ be the one point compactification of E and identify $C_0^{\mathbb{C}}(E)$ with

$$\left\{ f \in C^{\mathbb{C}}(E_\infty) : f(\infty) = 0 \right\}.$$

Then $A + \mathbb{C}\mathbf{1}$ is a $*$-subalgebra of $C^{\mathbb{C}}(E_\infty)$ and separates points not only in E but in E_∞. Indeed, if $t \in E$, there is by assumption a g in A with $g(t) \neq 0$, whereas $g(\infty) = 0$ since $g \in C^{\mathbb{C}}(E_\infty)$. By Theorem 1.2.10, for each $f \in C_0^{\mathbb{C}}(E)$ and $\varepsilon > 0$, there are $g \in A$ and $\lambda \in \mathbb{C}$ such that $\|f - (g + \lambda\mathbf{1})\| < \frac{\varepsilon}{2}$. As $f(\infty) = g(\infty) = 0$, we see that $|\lambda| < \frac{\varepsilon}{2}$, whence $\|f - g\| < \varepsilon$. □

Exercise 1.2.15 is taken from the Duncan–Taylor paper [218].

Proposition 1.2.49 is taken from the Becerra–Rodríguez paper [77], where some previous ideas on p. 28 of [798] are exploited. Corollary 1.2.50 is folklore. Indeed, it follows easily from the non-commutative Gelfand–Naimark theorem, stated in Theorem 1.2.3 (see for example [711, Proposition II.3.3]).

Proposition 1.2.51 is due to Kaplansky [375] and is included in several books (see for example [806, Theorem 1.2.4] and [810, Theorem 10.1]). A full generalization of Kaplansky's result was later proved by Rickart [502], who incorporated it in his book [795, Corollary 3.7.7].

1.3 The holomorphic functional calculus

Introduction The holomorphic functional calculus in a single element of a complete normed unital associative complex algebra is one of the most useful tools in spectral theory. After developing the algebraic forerunners in Subsection 1.3.1, Subsection 1.3.2 provides a detailed exposition of the holomorphic functional calculus, and indicates some applications. Deeper applications will be discussed later (see, for example, Subsection 3.4.3).

1.3.1 The polynomial and rational functional calculuses

As stated in §1.1.27, given an element a of a unital associative algebra A over $\mathbb{K}$, the mapping

$$p(\mathbf{x}) = \sum_{k=0}^{n} \alpha_k \mathbf{x}^k \to p(a) = \sum_{k=0}^{n} \alpha_k a^k$$

is a unit-preserving algebra homomorphism from $\mathbb{K}[\mathbf{x}]$ onto the subalgebra of A generated by $\mathbf{1}$ and a. This homomorphism is called the *polynomial functional calculus* at a. As a first outstanding application of this functional calculus, we are going to prove Theorem 1.3.2 below.

Two idempotents u, v in an algebra are said to be *orthogonal* if $uv = vu = 0$.

Lemma 1.3.1 *Let A be a unital associative algebra over $\mathbb{K}$, let $\alpha_1, \dots, \alpha_n$ be in $\mathbb{K}$, let $e_1, \dots, e_n$ be pairwise orthogonal idempotents in A such that $\mathbf{1} = \sum_{k=1}^n e_k$, and set $a =: \sum_{k=1}^n \alpha_k e_k$. Then for every $q \in \mathbb{K}[\mathbf{x}]$ we have $q(a) = \sum_{k=1}^n q(\alpha_k) e_k$.*

Proof The assumptions become the cases $m = 0, 1$ of the equality

$$a^m = \sum_{k=1}^n \alpha_k^m e_k \quad (m \in \mathbb{N} \cup \{0\}),$$

which is then proved by an easy induction. Now the result follows straightforwardly. $\square$

Theorem 1.3.2 *Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A. Then the following conditions are equivalent:*

(i) *a is a linear combination of pairwise orthogonal idempotents.*
(ii) *There exist pairwise different numbers $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ such that*

$$\prod_{k=1}^n (a - \alpha_k \mathbf{1}) = 0.$$

Moreover, if $\alpha_1, \dots, \alpha_n$ are in $\mathbb{K}$ in such a way that $\prod_{k=1}^n (a - \alpha_k \mathbf{1}) = 0$, then:

(iii) *There are unique pairwise orthogonal idempotents $e_1, \dots, e_n \in A$ such that*

$$\mathbf{1} = \sum_{k=1}^n e_k \text{ and } a = \sum_{k=1}^n \alpha_k e_k.$$

(iv) *The idempotents e_k can be explicitly computed by means of the equality*

$$e_k = \frac{\prod_{j \neq k}(a - \alpha_j \mathbf{1})}{\prod_{j \neq k}(\alpha_k - \alpha_j)}.$$

Proof (i)⇒(ii) Assume that a is a linear combination of pairwise orthogonal idempotents $e_1, \dots, e_m$, say $a = \sum_{k=1}^m \alpha_k e_k$ for suitable $\alpha_1, \dots, \alpha_m \in \mathbb{K}$. By setting

$$\alpha_{m+1} := 0 \text{ and } e_{m+1} := \mathbf{1} - \sum_{k=1}^m e_k,$$

$e_1, \dots, e_m, e_{m+1}$ become pairwise orthogonal idempotents such that

$$\mathbf{1} = \sum_{k=1}^{m+1} e_k \text{ and } a = \sum_{k=1}^{m+1} \alpha_k e_k.$$

Let n be the minimum natural number such that there are $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and pairwise orthogonal idempotents $e_1, \dots, e_n$ in A such that

$$\mathbf{1} = \sum_{k=1}^n e_k \text{ and } a = \sum_{k=1}^n \alpha_k e_k.$$

Then $\alpha_1, \dots, \alpha_n$ are pairwise different. Indeed, if for example the equality $\alpha_1 = \alpha_2$ were true, then a would be a linear combination of the $n-1$ pairwise orthogonal idempotents $e_1 + e_2, e_3, \dots, e_n$, the sum of which is $\mathbf{1}$, contrarily to the choice of n. Now let p be in $\mathbb{K}[\mathrm{x}]$ defined by $p(\mathbf{x}) := \prod_{k=1}^n (\mathbf{x} - \alpha_k)$. Then, by Lemma 1.3.1, we have $p(a) = 0$.

(ii)⇒(i) The proof of this implication is contained in the next paragraph.

Suppose that $\alpha_1,\dots,\alpha_n$ are in $\mathbb{K}$ in such a way that $\prod_{k=1}^n(a-\alpha_k\mathbf{1})=0$. To prove (iii) and (iv), we may assume that $n\geqslant 2$. For $k=1,\dots,n$, set $p_k(\mathbf{x}):=\frac{\prod_{j\neq k}(\mathbf{x}-\alpha_j)}{\prod_{j\neq k}(\alpha_k-\alpha_j)}$. Then

$$\textstyle\sum_{k=1}^n p_k(\mathbf{x})-1 \text{ and } \sum_{k=1}^n \alpha_k p_k(\mathbf{x})-\mathbf{x}$$

are polynomials of degree $\leqslant n-1$ which vanish at n different points, namely $\alpha_1,\dots,\alpha_n$, so $\sum_{k=1}^n p_k(\mathbf{x})=1$ and $\sum_{k=1}^n \alpha_k p_k(\mathbf{x})=\mathbf{x}$, hence $\sum_{k=1}^n p_k(a)=\mathbf{1}$ and $\sum_{k=1}^n \alpha_k p_k(a)=a$. On the other hand, we clearly have $(a-\alpha_k\mathbf{1})p_k(a)=0$, so $ap_k(a)=\alpha_k p_k(a)$, and so $a^m p_k(a)=\alpha_k^m p_k(a)$ for every non-negative integer m, hence $q(a)p_k(a)=q(\alpha_k)p_k(a)$ for every $q\in\mathbb{K}[\mathbf{x}]$. Therefore, by taking $q=p_k$, we obtain that $p_k(a)^2=p_k(a)$, and hence $p_k(a)$ is an idempotent in A. Moreover, for $j\neq k$ we clearly have $p_j(a)p_k(a)=0$. Now, by setting $e_k:=p_k(a)$, assertion (iii) has been proved in what concerns the existence, and assertion (iv) becomes clear by the definition of the polynomials p_k. Now it only remains to prove assertion (iii) in what concerns the uniqueness. Let $u_1,\dots,u_n$ be pairwise orthogonal idempotents in A such that $\mathbf{1}=\sum_{k=1}^n u_k$ and $a=\sum_{k=1}^n \alpha_k u_k$. It follows from Lemma 1.3.1 that $q(a)=\sum_{k=1}^n q(\alpha_k)u_k$ for every $q\in\mathbb{K}[\mathbf{x}]$. By taking $q=p_k$, we get that $e_k=u_k$, as desired. □

The next proposition becomes the version for operators of Theorem 1.3.2 above.

Proposition 1.3.3 *Let X be a vector space over $\mathbb{K}$, and let T be in $L(X)$ such that $\prod_{k=1}^n(T-\alpha_k I_X)=0$ for pairwise different numbers $\alpha_1,\dots,\alpha_n\in\mathbb{K}$. Then*

$$X=\textstyle\bigoplus_{k=1}^n X_k, \text{ where } X_k:=\{x\in X: T(x)=\alpha_k x\} \text{ for } k=1,\dots,n. \tag{1.3.1}$$

Moreover, if for each k we denote by P_k the projection from X onto X_k corresponding to the decomposition (1.3.1), then P_k can be explicitly computed by means of the equality

$$P_k=\frac{\prod_{j\neq k}(T-\alpha_j I_X)}{\prod_{j\neq k}(\alpha_k-\alpha_j)}.$$

Proof For $k=1,\dots,n$, set $P_k:=\frac{\prod_{j\neq k}(T-\alpha_j I_X)}{\prod_{j\neq k}(\alpha_k-\alpha_j)}$. Then, according to Theorem 1.3.2, $\{P_k: k=1,\dots,n\}$ is a family of pairwise orthogonal linear projections on X satisfying

$$I_X=\sum_{k=1}^n P_k, \tag{1.3.2}$$

and consequently we have $X=\bigoplus_{k=1}^n P_k(X)$. Therefore, to conclude the proof it is enough to show that $P_k(X)=X_k$ for every $k=1,\dots,n$. Let k be in $\{1,\dots,n\}$. Since $(T-\alpha_k I_X)P_k=0$, we have $TP_k=\alpha_k P_k$, which implies $P_k(X)\subseteq X_k$. Conversely, if x is in X_k, then $(T-\alpha_k I_X)(x)=0$, so $P_j(x)=0$ whenever $j\neq k$, and so $x=P_k(x)$ (by (1.3.2)), hence $x\in P_k(X)$. □

Now we deal with the spectral behaviour of the polynomial functional calculus.

Proposition 1.3.4 *Let A be a unital associative algebra over $\mathbb{K}$, let a be in A, and let p be in $\mathbb{K}[\mathbf{x}]$. We have:*

(i) $p(\mathrm{sp}(A,a)) \subseteq \mathrm{sp}(A,p(a))$.
(ii) *If* $\mathbb{K} = \mathbb{C}$ *and if* $\mathrm{sp}(A,a) \neq \emptyset$*, then*

$$\mathrm{sp}(A,p(a)) = p(\mathrm{sp}(A,a)). \tag{1.3.3}$$

Proof First of all, note that the result holds in the case in which p is constant. Assume that p is non-constant. Then, for each $\lambda \in \mathbb{K}$, there exists $q \in \mathbb{K}[\mathbf{x}]$ such that $p(\mathbf{x}) - p(\lambda) = (\mathbf{x} - \lambda)q(\mathbf{x})$, and hence

$$p(a) - p(\lambda)\mathbf{1} = (a - \lambda\mathbf{1})q(a) = q(a)(a - \lambda\mathbf{1}).$$

Therefore, if $p(\lambda) \notin \mathrm{sp}(A,p(a))$, then, by Lemma 1.1.99(ii), we have that $a - \lambda\mathbf{1} \in \mathrm{Inv}(A)$, and so $\lambda \notin \mathrm{sp}(A,a)$. Thus assertion (i) follows. Now, assume that $\mathbb{K} = \mathbb{C}$ and that $\mathrm{sp}(A,a) \neq \emptyset$. Let λ be in $\mathbb{C}$. Then we can write

$$p(\mathbf{x}) - \lambda = \beta(\mathbf{x} - \alpha_1)\cdots(\mathbf{x} - \alpha_n)$$

for some $\beta, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $\beta \neq 0$, $n \in \mathbb{N}$. Hence

$$p(a) - \lambda\mathbf{1} = \beta(a - \alpha_1\mathbf{1})\cdots(a - \alpha_n\mathbf{1}).$$

By Lemma 1.1.99(ii), we find that $p(a) - \lambda\mathbf{1} \notin \mathrm{Inv}(A)$ if and only if at least one of the factors $a - \alpha_i\mathbf{1}$ is non-invertible. Thus $\lambda \in \mathrm{sp}(A,p(a))$ if and only if $p(z) - \lambda = 0$ for some $z \in \mathrm{sp}(A,a)$. Hence $p(\mathrm{sp}(A,a)) = \mathrm{sp}(A,p(a))$. □

Corollary 1.3.5 *Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A such that there exists a non-constant polynomial $p \in \mathbb{K}[\mathbf{x}]$ with $p(a) = 0$. Then*

$$\mathrm{sp}(A,a) \subseteq \{\lambda \in \mathbb{K} : p(\lambda) = 0\}.$$

If in addition p is of the minimum possible degree, then

$$\mathrm{sp}(A,a) = \{\lambda \in \mathbb{K} : p(\lambda) = 0\}.$$

Proof Since $p(a) = 0$, it follows from Proposition 1.3.4(i) that

$$p(\mathrm{sp}(A,a)) \subseteq \mathrm{sp}(A,p(a)) = \mathrm{sp}(A,0) = \{0\},$$

and hence $\mathrm{sp}(A,a) \subseteq \{\lambda \in \mathbb{K} : p(\lambda) = 0\}$. Let $\lambda_1, \ldots, \lambda_n$ be the roots of p in $\mathbb{K}$ of multiplicities $m_1, \ldots, m_n$, and write

$$p(\mathbf{x}) = q(\mathbf{x})(\mathbf{x} - \lambda_1)^{m_1}\cdots(\mathbf{x} - \lambda_n)^{m_n}$$

for some nonzero $q \in \mathbb{K}[\mathbf{x}]$ with no root in $\mathbb{K}$. Then we have

$$0 = p(a) = q(a)(a - \lambda_1\mathbf{1})^{m_1}\cdots(a - \lambda_n\mathbf{1})^{m_n}.$$

Therefore, if p is of the minimum possible degree, then none of the factors $a - \lambda_i\mathbf{1}$ can be invertible, and hence $\lambda_i \in \mathrm{sp}(A,a)$ for every $i \in \{1, \cdots, n\}$. □

Corollary 1.3.6 *Every element in a finite-dimensional unital associative algebra over $\mathbb{K}$ has finite spectrum.*

As usual, we denote by $\mathbb{K}(\mathbf{x})$ the field of fractions of the integral domain $\mathbb{K}[\mathbf{x}]$. Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A. Consider the subset $\mathscr{Q}_a$ of $\mathbb{K}(\mathbf{x})$ consisting of those fractions which can be written as

$$\frac{p(\mathbf{x})}{q(\mathbf{x})} \text{ with } p, q \in \mathbb{K}[\mathbf{x}] \text{ and } q(a) \in \mathrm{Inv}(A).$$

Then, clearly, $\mathscr{Q}_a$ is a subalgebra of $\mathbb{K}(\mathbf{x})$ containing $\mathbb{K}[\mathbf{x}]$. Moreover, using the commutativity of the set $\{\mathbf{1}, a\}$, together with Proposition 1.1.78 and Lemma 1.1.80, it is routine to verify that the correspondence

$$f = \frac{p}{q} \to f(a) := p(a)q(a)^{-1}$$

becomes a well-defined algebra homomorphism from $\mathscr{Q}_a$ to A, extending the polynomial functional calculus at a. This extended algebra homomorphism is called the *rational functional calculus* at a. We note that the range of the rational functional calculus at a becomes a commutative subalgebra of A. We also note that, if $f = \frac{p}{q}$ is in $\mathscr{Q}_a$, the fact that $q(a) \in \mathrm{Inv}(A)$ implies that $q(\alpha) \neq 0$ for every $\alpha \in \mathrm{sp}(A, a)$ (by Proposition 1.3.4(i)), and hence that, for such an α, the symbol $f(\alpha) := \frac{p(\alpha)}{q(\alpha)}$ has a sense as an element of $\mathbb{K}$.

Exercise 1.3.7 Let A be a unital associative algebra over $\mathbb{K}$, and let a be in A such that $\prod_{k=1}^{n}(a - \alpha_k \mathbf{1}) = 0$ for pairwise different numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Apply Lemma 1.3.1 and Theorem 1.3.2 to show that there are unique pairwise orthogonal idempotents $e_1, \ldots, e_n \in A$ such that $f(a) = \sum_{k=1}^{n} f(\alpha_k) e_k$ for every $f \in \mathbb{K}(\mathbf{x})$ with poles outside $\{\alpha_1, \ldots, \alpha_n\}$.

Now we prove the so-called *spectral mapping theorem* for the rational functional calculus.

Proposition 1.3.8 *Let A be a unital associative complex algebra, let $a \in A$ with* $\mathrm{sp}(A, a) \neq \emptyset$, *and let f be in $\mathscr{Q}_a$. Then*

$$\mathrm{sp}(A, f(a)) = f(\mathrm{sp}(A, a)).$$

Proof Write $f = \frac{p}{q}$ with $p, q \in \mathbb{C}[\mathbf{x}]$ and $q(a) \in \mathrm{Inv}(A)$. Note that, for each $z \in \mathbb{C}$ we have

$$f(a) - z\mathbf{1} = p(a)q(a)^{-1} - z\mathbf{1} = (p(a) - zq(a))q(a)^{-1} = (p - zq)(a)q(a)^{-1},$$

and hence, by Lemma 1.1.99(ii), $f(a) - z\mathbf{1}$ is invertible in A if and only if so is $(p - zq)(a)$. Therefore, $\lambda \in \mathrm{sp}(A, f(a))$ if and only if $0 \in \mathrm{sp}(A, (p - \lambda q)(a))$. Now note that by Proposition 1.3.4 we have

$$\mathrm{sp}(A, (p - \lambda q)(a)) = (p - \lambda q)(\mathrm{sp}(A, a)) = \{p(\alpha) - \lambda q(\alpha) : \alpha \in \mathrm{sp}(A, a)\},$$

and that, as we already know, $q(\alpha) \neq 0$ for every $\alpha \in \mathrm{sp}(A, a)$. As a result, λ belongs to $\mathrm{sp}(A, f(a))$ if and only if there exists $\alpha \in \mathrm{sp}(A, a)$ such that $0 = p(\alpha) - \lambda q(\alpha)$, that is to say $\lambda = \frac{p(\alpha)}{q(\alpha)} = f(\alpha)$. □

Now, invoking Theorem 1.1.41, we get the following.

Corollary 1.3.9 *Let A be a normed unital associative complex algebra, and let $a \in A$. If f is in $\mathscr{Q}_a$, then*

$$\operatorname{sp}(A, f(a)) = f(\operatorname{sp}(A, a)).$$

In particular, if p is in $\mathbb{C}[\mathbf{x}]$, then

$$\operatorname{sp}(A, p(a)) = p(\operatorname{sp}(A, a)).$$

1.3.2 The main results

In a purely algebraic setting, besides the rational functional calculus, little or nothing can be done. However, if A is a complete normed unital associative complex algebra, the possibility of introducing convergence processes opens new trails that we will now explore. For example, if $a \in A$, and if f is an entire function represented by the power series $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \quad (z \in \mathbb{C})$, then the series $\sum_{n \geqslant 0} \alpha_n a^n$ is absolutely convergent, a fact that enables the definition

$$f(a) := \sum_{n=0}^{\infty} \alpha_n a^n.$$

In this way we obtain an algebra homomorphism from the algebra of all entire functions to A, called the *entire functional calculus* at a. A more general approach can be made keeping in mind that, if f is a function analytic on a disc $\{z \in \mathbb{C} : |z - z_0| < R\} \supseteq \operatorname{sp}(A, a)$, and if $f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$ is the Taylor expansion of f, then, by Theorem 1.1.46 and Corollary 1.1.18(i), the series $\sum_{n=0}^{\infty} \alpha_n (a - z_0\mathbf{1})^n$ converges in A, and its sum is naturally denoted by $f(a)$. The functional calculus obtained in this way extends the polynomial functional calculus, but not the rational functional calculus. Indeed, if $A = C^{\mathbb{C}}(\mathbb{T})$, and if $a(z) := z$ for every $z \in \mathbb{T}$, then the fraction $f(\mathbf{x}) := \frac{1}{\mathbf{x}}$ lies in $\mathscr{Q}_a$. Nevertheless, as shown in Example 1.1.28, $f(a) = a^{-1}$ (in the sense of the rational functional calculus) does not lie in the closed subalgebra of A generated by $\{\mathbf{1}, a\}$, and hence cannot be of the form $\sum_{n=0}^{\infty} \alpha_n (a - z_0\mathbf{1})^n$.

Recall that a *curve* in $\mathbb{C}$ is a continuous mapping $\gamma : [0,1] \to \mathbb{C}$. Sometimes, when there is no risk of confusion, we will identify a given curve γ with its range $\gamma([0,1])$. The curve γ is *closed* if $\gamma(0) = \gamma(1)$. A *contour* Γ in $\mathbb{C}$ consists of the union of a finite family $\{\Gamma_1, \ldots, \Gamma_n\}$ of pairwise disjoint, piecewise continuously differentiable, closed curves in $\mathbb{C}$. For a contour Γ in $\mathbb{C}$ and $z_0 \in \mathbb{C} \setminus \Gamma$, we define the *index* of z_0 with respect to Γ to be

$$\operatorname{Ind}_\Gamma(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - z_0}.$$

Let K be a compact set in $\mathbb{C}$, and let Ω be an open set in $\mathbb{C}$ containing K. A contour Γ *surrounds* K in Ω whenever $\Gamma \subseteq \Omega \setminus K$ and

$$\operatorname{Ind}_\Gamma(z) = \begin{cases} 1 & \text{if} \quad z \in K \\ 0 & \text{if} \quad z \in \mathbb{C} \setminus \Omega. \end{cases}$$

The existence of contours surrounding K in Ω is well known (see, for example [680, Lemma 3.4.5]).

For a non-empty open subset Ω of $\mathbb{C}$, we denote by $\mathscr{H}(\Omega)$ the algebra (under pointwise operations) of all complex-valued holomorphic functions on Ω, endowed with the topology of the uniform convergence on compact sets.

Lemma 1.3.10 *Let A be a complete normed unital associative complex algebra, let a be in A, and let Ω be an open set in $\mathbb{C}$ containing $\mathrm{sp}(A,a)$. We have:*

(i) *For each $f \in \mathscr{H}(\Omega)$, and for each contour Γ surrounding $\mathrm{sp}(A,a)$ in Ω, the integral*
$$\frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz$$
is well defined and does not depend on the choice of Γ.

(ii) *If Ω_1, Ω_2 are open sets in $\mathbb{C}$ such that $\mathrm{sp}(A,a) \subseteq \Omega \subseteq \Omega_1 \cap \Omega_2$, if $f_1 \in \mathscr{H}(\Omega_1)$ and $f_2 \in \mathscr{H}(\Omega_2)$ satisfy $f_{1|\Omega} = f_{2|\Omega}$, and if Γ_1, Γ_2 are contours surrounding $\mathrm{sp}(A,a)$ in Ω_1, Ω_2, respectively, then*
$$\frac{1}{2\pi i}\int_{\Gamma_1} f_1(z)(z\mathbf{1}-a)^{-1}dz = \frac{1}{2\pi i}\int_{\Gamma_2} f_2(z)(z\mathbf{1}-a)^{-1}dz.$$

Proof Let f be in $\mathscr{H}(\Omega)$, and let Γ be a contour surrounding $\mathrm{sp}(A,a)$ in Ω. By Proposition 1.1.15, the function $f(z)(z\mathbf{1}-a)^{-1}$ is continuous on Γ, so that the integral
$$\frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz$$
exists and defines an element of A. Moreover, the integrand is actually a holomorphic A-valued function in $\Omega \setminus \mathrm{sp}(A,a)$ (by Proposition 1.1.40). The Cauchy theorem implies therefore that the integral is independent of the choice of Γ, provided only that Γ surrounds $\mathrm{sp}(A,a)$ in Ω.

Now, assume that Ω_1, Ω_2 are open sets in $\mathbb{C}$ such that
$$\mathrm{sp}(A,a) \subseteq \Omega \subseteq \Omega_1 \cap \Omega_2,$$
that $f_1 \in \mathscr{H}(\Omega_1)$ and $f_2 \in \mathscr{H}(\Omega_2)$ satisfy $f_{1|\Omega} = f_{2|\Omega}$, and that Γ_1, Γ_2 are contours surrounding $\mathrm{sp}(A,a)$ in Ω_1, Ω_2, respectively. Fix a contour Γ surrounding $\mathrm{sp}(A,a)$ in Ω. Then, by assertion (i), we have
$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_1} f_1(z)(z\mathbf{1}-a)^{-1}dz &= \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz\\ &= \frac{1}{2\pi i}\int_{\Gamma_2} f_2(z)(z\mathbf{1}-a)^{-1}dz.\end{aligned}$$
□

Let A be a complete normed unital associative complex algebra, and let a be in A. According to the above lemma, if f is a holomorphic function on an open neighbourhood Ω of $\mathrm{sp}(A,a)$, then we can define $f(a)$ by means of the following Cauchy integral
$$f(a) := \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz, \tag{1.3.4}$$
where Γ is any contour surrounding $\mathrm{sp}(A,a)$ in Ω.

The definition (1.3.4) coincides with the previous definition for polynomials. Indeed, we have the following.

Proposition 1.3.11 *Let a be an element of a complete normed unital associative complex algebra A, and let Γ be a contour surrounding* $\operatorname{sp}(A,a)$ *in $\mathbb{C}$. If $p(\mathbf{x}) = \sum_{k=0}^{n} \alpha_k \mathbf{x}^k$ is in $\mathbb{C}[\mathbf{x}]$, then*

$$\frac{1}{2\pi i}\int_\Gamma p(z)(z\mathbf{1}-a)^{-1}dz = \sum_{k=0}^{n} \alpha_k a^k.$$

In particular

$$\frac{1}{2\pi i}\int_\Gamma (z\mathbf{1}-a)^{-1}dz = \mathbf{1}.$$

Proof It is sufficient to show that

$$\frac{1}{2\pi i}\int_\Gamma z^k(z\mathbf{1}-a)^{-1}dz = a^k$$

for every $k \geqslant 0$. Fix $R > \mathfrak{r}(a)$. By Lemma 1.3.10(i), we have

$$\frac{1}{2\pi i}\int_\Gamma z^k(z\mathbf{1}-a)^{-1}dz = \frac{1}{2\pi i}\int_{|z|=R} z^k(z\mathbf{1}-a)^{-1}dz.$$

On the other hand, by Lemma 1.1.20, for each z with $|z| > \mathfrak{r}(a)$ we have

$$(z\mathbf{1}-a)^{-1} = \frac{1}{z}\left(\mathbf{1}-\frac{a}{z}\right)^{-1} = \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}.$$

Since this series converges uniformly on $\{z \in \mathbb{C} : |z| = R\}$, it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=R} z^k(z\mathbf{1}-a)^{-1}dz &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|z|=R}\frac{a^n}{z^{n-k+1}}\,dz\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z^{n-k+1}}\right)a^n.\end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{dz}{z^{n-k+1}} = \begin{cases} 1 & \text{if} \quad n=k \\ 0 & \text{if} \quad n\neq k, \end{cases}$$

the result follows. □

The definition (1.3.4) coincides with the previous definition for fractions. Indeed, we have the following.

Proposition 1.3.12 *Let A be a complete normed unital associative complex algebra, let a be in A, let f be in $\mathscr{Q}_a$ (say $f = \frac{p}{q}$ with $p, q \in \mathbb{C}[\mathbf{x}]$ and $q(a) \in$* $\operatorname{Inv}(A)$*), and let Γ be a contour surrounding* $\operatorname{sp}(A,a)$ *in the open set $\mathbb{C}\setminus\{z \in \mathbb{C} : q(z) = 0\}$. Then*

$$\frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz = p(a)q(a)^{-1}.$$

Proof Let $\lambda_1,\ldots,\lambda_n$ be the roots of q of multiplicities $m_1,\ldots,m_n$. Then $\frac{p(z)}{q(z)}$ can be expressed as

$$\frac{p(z)}{q(z)} = p_1(z) + \sum_{j=1}^{n}\sum_{k=1}^{m_j}\frac{c_{j,k}}{(z-\lambda_j)^k}$$

for some polynomial p_1 and complex numbers $c_{j,k}$, and as a consequence

$$p(a)q(a)^{-1} = p_1(a) + \sum_{j=1}^{n}\sum_{k=1}^{m_j} c_{j,k}(a-\lambda_j\mathbf{1})^{-k}.$$

Since $p_1(a) = \frac{1}{2\pi i}\int_\Gamma p_1(z)(z\mathbf{1}-a)^{-1}dz$ (by Proposition 1.3.11), to conclude the proof it is enough to show that, for $k \in \mathbb{N}$ and $\lambda \notin \mathrm{sp}(A,a)$, we have

$$(a-\lambda\mathbf{1})^{-k} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{(z-\lambda)^k}(z\mathbf{1}-a)^{-1}dz,$$

where Γ is any contour surrounding $\mathrm{sp}(A,a)$ in $\mathbb{C}\setminus\{\lambda\}$.

Let λ and Γ be as above. We claim that the function

$$z \to \left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1}$$

is (the restriction to $\mathbb{C}\setminus(\mathrm{sp}(A,a)\cup\{\lambda\})$ of) a holomorphic function on $\mathbb{C}\setminus\{\lambda\}$. Indeed, by writing

$$\begin{aligned}&\left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1}\\ &\qquad = \frac{1}{(z-\lambda)^k}(a-\lambda\mathbf{1})^{-k}\left[(a-\lambda\mathbf{1})^k - (z-\lambda)^k\mathbf{1}\right](z\mathbf{1}-a)^{-1},\end{aligned}$$

and by keeping in mind that

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1})$$

for all commuting $x, y \in A$, we realize that

$$\begin{aligned}&\left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1}\\ &= \frac{1}{(z-\lambda)^k}(a-\lambda\mathbf{1})^{-k}\left[(a-\lambda\mathbf{1})^k - (z-\lambda)^k\mathbf{1}\right](z\mathbf{1}-a)^{-1}\\ &= \frac{1}{(z-\lambda)^k}(a-\lambda\mathbf{1})^{-k}(a-z\mathbf{1})\Big[(a-\lambda\mathbf{1})^{k-1} + (a-\lambda\mathbf{1})^{k-2}(z-\lambda)\\ &\quad + \cdots + (a-\lambda\mathbf{1})(z-\lambda)^{k-2} + (z-\lambda)^{k-1}\mathbf{1}\Big](z\mathbf{1}-a)^{-1}\\ &= -\frac{1}{(z-\lambda)^k}(a-\lambda\mathbf{1})^{-k}\Big[(a-\lambda\mathbf{1})^{k-1} + (a-\lambda\mathbf{1})^{k-2}(z-\lambda)\\ &\quad + \cdots + (a-\lambda\mathbf{1})(z-\lambda)^{k-2} + (z-\lambda)^{k-1}\mathbf{1}\Big].\end{aligned}$$

Therefore, the function $z \to \left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1}$ is the product of the holomorphic function in $\mathbb{C}\setminus\{\lambda\}$ given by

$$z \to -\frac{1}{(z-\lambda)^k}(a-\lambda\mathbf{1})^{-k}$$

by the polynomial function

$$z \to (a-\lambda\mathbf{1})^{k-1} + (a-\lambda\mathbf{1})^{k-2}(z-\lambda) + \cdots + (a-\lambda\mathbf{1})(z-\lambda)^{k-2} + (z-\lambda)^{k-1}\mathbf{1},$$

and hence the function

$$z \to \left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1} \text{ is holomorphic in } \mathbb{C}\setminus\{\lambda\}.$$

Now that the claim has been proved, the Cauchy theorem implies that

$$\int_\Gamma \left[\frac{1}{(z-\lambda)^k}\mathbf{1} - (a-\lambda\mathbf{1})^{-k}\right](z\mathbf{1}-a)^{-1}dz = 0,$$

and hence, by Proposition 1.3.11,

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{1}{(z-\lambda)^k}(z\mathbf{1}-a)^{-1}dz &= \frac{1}{2\pi i}\int_\Gamma (a-\lambda\mathbf{1})^{-k}(z\mathbf{1}-a)^{-1}dz \\ &= (a-\lambda\mathbf{1})^{-k}\frac{1}{2\pi i}\int_\Gamma (z\mathbf{1}-a)^{-1}dz = (a-\lambda\mathbf{1})^{-k},\end{aligned}$$

and the proof is complete. □

Now, we formulate and prove the main result in this section.

Theorem 1.3.13 *Let A be a complete normed unital associative complex algebra, let a be in A, let Ω be an open subset of $\mathbb{C}$ containing $\mathrm{sp}(A,a)$, and let u stand for the inclusion mapping $\Omega \hookrightarrow \mathbb{C}$. Then there is a unique continuous unit-preserving algebra homomorphism $f \to f(a)$ from $\mathscr{H}(\Omega)$ into A taking u to a. Furthermore, we have:*

(i) $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz$ *for every $f \in \mathscr{H}(\Omega)$, where Γ is any contour surrounding $\mathrm{sp}(A,a)$ in Ω.*

(ii) $f(a) = \displaystyle\sum_{n=0}^{\infty} c_n(a - z_0\mathbf{1})^n$ *when Ω is the open disc of centre z_0 and radius $R \leqslant \infty$,*

and $f \in \mathscr{H}(\Omega)$ is represented by the power series $\displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$.

(iii) $f(a) \in \{a\}^{cc}$ *for each $f \in \mathscr{H}(\Omega)$.*

(iv) $\mathrm{sp}(A, f(a)) = f(\mathrm{sp}(A,a))$ *for each $f \in \mathscr{H}(\Omega)$.*

(v) *If $f \in \mathscr{H}(\Omega)$ and if Ω_1 is an open set in $\mathbb{C}$ such that $f(\Omega) \subseteq \Omega_1$, then*

$$(g \circ f)(a) = g(f(a)) \textit{ for every } g \in \mathscr{H}(\Omega_1).$$

Proof According to (1.3.4) and to validate assertion (i), we define

$$f(a) := \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz,$$

where Γ is a contour surrounding $\mathrm{sp}(A,a)$ in Ω. The linearity of the mapping $f \to f(a)$ is clear. Moreover, $\mathbf{1} \to \mathbf{1}$ and $u \to a$, by Proposition 1.3.11. Assume that f_n is a sequence of holomorphic functions on Ω converging uniformly to f on compact subsets of Ω. Then f is a holomorphic function on Ω, and f_n converges uniformly on Γ. Therefore we have

$$2\pi\|f_n(a) - f(a)\| = \|\int_\Gamma [f_n(z) - f(z)](z\mathbf{1}-a)^{-1}dz\| \leqslant M\ell(\Gamma)\|f_n - f\|_\Gamma \to 0,$$

as $n \to \infty$, where $M = \max\{\|(z\mathbf{1}-a)^{-1}\| : z \in \Gamma\}$, $\ell(\Gamma)$ denotes the length of Γ and $\|f_n - f\|_\Gamma = \sup\{|f_n(z) - f(z)| : z \in \Gamma\}$. Since the topology of uniform convergence on compact sets of Ω is metrizable, we conclude that the mapping $f \to f(a)$ is continuous. It remains to be proved that $f \to f(a)$ is multiplicative. Explicitly, if $f, g \in \mathscr{H}(\Omega)$, and $h(z) = f(z)g(z)$ for every $z \in \Omega$, it has to be shown that

$$h(a) = f(a)g(a). \tag{1.3.5}$$

If f and g are rational functions with poles outside Ω, then (1.3.5) holds because of Proposition 1.3.12. In the general case, by Runge's theorem, there are sequences f_n and g_n of rational functions, with poles outside Ω, converging uniformly on compact subsets of Ω to f and g, respectively. Then $f_n g_n$ converges to h in the same manner, and (1.3.5) follows from the continuity of the mapping $f \to f(a)$.

Assume that Φ is a continuous unit-preserving algebra homomorphism from $\mathscr{H}(\Omega)$ into A such that $\Phi(u) = a$. Then, it is clear that $\Phi(p) = p(a)$ for every polynomial p, and that $\Phi(\frac{1}{q}) = q(a)^{-1}$ if q is a polynomial without zeros in Ω. Hence $\Phi(f) = f(a)$ for every rational function f with poles outside Ω. Now, from the continuity and the Runge theorem, we obtain $\Phi(f) = f(a)$ for every $f \in \mathscr{H}(\Omega)$.

Assume that Ω is a disc of centre $z_0 \in \mathbb{C}$ and radius $R \leqslant \infty$ and that $f \in \mathscr{H}(\Omega)$ is represented by the power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$. For each $n \in \mathbb{N}$, consider the polynomial $p_n(z) = \sum_{k=0}^{n} c_k (z - z_0)^k$. Since p_n converges uniformly to f on compact subsets of Ω, it follows from the first paragraph in the proof and Proposition 1.3.11 that

$$f(a) = \lim p_n(a) = \lim \sum_{k=0}^{n} c_k (a - z_0 \mathbf{1})^k = \sum_{n=0}^{\infty} c_n (a - z_0 \mathbf{1})^n,$$

and assertion (ii) is proved.

Assertion (iii) follows directly from the definition since $\{a\}^{cc}$ is a closed subalgebra of A containing $(z\mathbf{1} - a)^{-1}$ for every $z \in \Gamma$ (by Proposition 1.1.78(i) and Lemma 1.1.80).

Now, we will prove assertion (iv). If $\lambda \notin f(\mathrm{sp}(A,a))$, then

$$g(z) = (f(z) - \lambda)^{-1}$$

is a holomorphic function on an open neighbourhood of $\mathrm{sp}(A,a)$ contained in Ω. Keeping in mind Lemma 1.3.10(ii), we see that

$$(f(a) - \lambda \mathbf{1})g(a) = g(a)(f(a) - \lambda \mathbf{1}) = \mathbf{1},$$

and hence $\lambda \notin \mathrm{sp}(A, f(a))$. Conversely, if $\lambda \in f(\mathrm{sp}(A,a))$, then there exists $z_0 \in \mathrm{sp}(A,a)$ with $f(z_0) = \lambda$, and consequently $f(z) - \lambda = (z - z_0)g(z)$ for some function $g \in \mathscr{H}(\Omega)$. Then $f(a) - \lambda \mathbf{1} = (a - z_0 \mathbf{1})g(a) = g(a)(a - z_0 \mathbf{1})$ and, since $a - z_0 \mathbf{1} \notin \mathrm{Inv}(A)$, Lemma 1.1.99(ii) applies to conclude that $f(a) - \lambda \mathbf{1} \notin \mathrm{Inv}(A)$. Hence $\lambda \in \mathrm{sp}(A, f(a))$.

Finally, in order to prove assertion (v), assume that $f \in \mathscr{H}(\Omega)$, and that Ω_1 is an open set in $\mathbb{C}$ such that $f(\Omega) \subseteq \Omega_1$. Consider the mapping

$$\Phi : \mathscr{H}(\Omega_1) \to A$$

defined by $\Phi(g) = (g \circ f)(a)$. It can immediately be verified that Φ is a continuous unit-preserving algebra homomorphism such that $\Phi(u_1) = f(a)$, where u_1 stands for the inclusion mapping of Ω_1 in $\mathbb{C}$. Note that, by assertion (iv), we have $\mathrm{sp}(A, f(a)) = f(\mathrm{sp}(A,a)) \subseteq \Omega_1$. Therefore, it follows from the uniqueness of the functional calculus that $\Phi(g) = g(f(a))$ for every $g \in \mathscr{H}(\Omega_1)$, that is to say $(g \circ f)(a) = g(f(a))$ for every $g \in \mathscr{H}(\Omega_1)$. $\square$

The algebra homomorphism $f \to f(a)$ in Theorem 1.3.13 is called the *holomorphic functional calculus* at a. Assertion (iv) in Theorem 1.3.13 is called the *spectral mapping theorem* (for the holomorphic functional calculus).

Exercise 1.3.14 Let A be a complete normed unital associative complex algebra, and let a be in A such that $\prod_{k=1}^{n}(a - \alpha_k \mathbf{1}) = 0$ for pairwise different numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Note that, by Corollary 1.3.5, $\mathrm{sp}(A,a) \subseteq \{\alpha_1, \ldots, \alpha_n\}$, and then apply Exercise 1.3.7, Theorem 1.3.13, and Runge's theorem to show that there are unique pairwise orthogonal idempotents $e_1, \ldots, e_n \in A$ such that $f(a) = \sum_{k=1}^{n} f(\alpha_k) e_k$ for every complex-valued holomorphic function f on some open subset of $\mathbb{C}$ containing $\{\alpha_1, \ldots, \alpha_n\}$.

The Cauchy integral formulae for derivatives also hold for the holomorphic functional calculus. Indeed, we have the following.

Proposition 1.3.15 *Let A be a complete normed unital associative complex algebra, let $a \in A$, let Ω be an open subset of $\mathbb{C}$ containing $\mathrm{sp}(A,a)$, and let f be in $\mathscr{H}(\Omega)$. Then, for each natural number k we have*

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_\Gamma f(z)(z\mathbf{1} - a)^{-k-1}\, dz,$$

where Γ is any contour surrounding $\mathrm{sp}(A,a)$ in Ω.

Proof Let Γ be a contour surrounding $\mathrm{sp}(A,a)$ in Ω, and let $k \in \mathbb{N}$. Then, since

$$f^{(k)}(a) = \frac{1}{2\pi i} \int_\Gamma f^{(k)}(w)(w\mathbf{1} - a)^{-1}\, dw$$

and

$$(z\mathbf{1} - a)^{-k-1} = \frac{1}{2\pi i} \int_\Gamma (z - w)^{-k-1}(w\mathbf{1} - a)^{-1}\, dw,$$

Fubini's rule gives

$$\begin{aligned} f^{(k)}(a) &= \frac{1}{2\pi i} \int_\Gamma \left[\frac{k!}{2\pi i} \int_\Gamma f(z)(z-w)^{-k-1}\, dz \right] (w\mathbf{1} - a)^{-1}\, dw \\ &= \frac{k!}{2\pi i} \int_\Gamma \left[\frac{1}{2\pi i} \int_\Gamma (z-w)^{-k-1}(w\mathbf{1} - a)^{-1}\, dw \right] f(z)\, dz \\ &= \frac{k!}{2\pi i} \int_\Gamma f(z)(z\mathbf{1} - a)^{-k-1}\, dz. \end{aligned}$$ $\square$

Other consequences of Theorem 1.3.13 are compiled in what follows.

Corollary 1.3.16 *Let A, B be complete normed unital associative complex algebras, let $\phi : A \to B$ be a continuous unit-preserving algebra homomorphism, let $a \in A$, let Ω be an open set in $\mathbb{C}$, and suppose that $\mathrm{sp}(A,a) \subseteq \Omega$. Then $\mathrm{sp}(B, \phi(a)) \subseteq \Omega$, and $\phi(f(a)) = f(\phi(a))$ for every $f \in \mathscr{H}(\Omega)$.*

Proof By Lemma 1.1.34(ii), $\mathrm{sp}(B,\phi(a)) \subseteq \mathrm{sp}(A,a)$, hence $\mathrm{sp}(B,\phi(a)) \subseteq \Omega$ and any contour surrounding $\mathrm{sp}(A,a)$ in Ω is a contour surrounding $\mathrm{sp}(B,\phi(a))$ in Ω. Let f be in $\mathscr{H}(\Omega)$, and let Γ be a contour surrounding $\mathrm{sp}(A,a)$ in Ω. By Theorem 1.3.13(i), we have

$$\begin{aligned}\phi(f(a)) &= \frac{1}{2\pi i}\int_\Gamma \phi(f(z)(z\mathbf{1}-a)^{-1})\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-\phi(a))^{-1}dz = f(\phi(a)).\end{aligned}$$ □

The following result is easily deduced from Theorems 1.3.13 and 1.2.28.

Fact 1.3.17 *Let a be a normal element of a unital C^*-algebra A, let Ω be an open subset of $\mathbb{C}$ such that $\mathrm{sp}(A,a) \subseteq \Omega$, and let f be in $\mathscr{H}(\Omega)$. Then $f(a)$ has the same meaning in both continuous functional calculus and holomorphic functional calculus at a.*

Proof Since the mappings $f \to f_{|\mathrm{sp}(A,a)}$, from $\mathscr{H}(\Omega)$ to $C^{\mathbb{C}}(\mathrm{sp}(A,a))$, and $g \to g(a)$, from $C^{\mathbb{C}}(\mathrm{sp}(A,a))$ to A, are continuous unit-preserving algebra homomorphisms, we realize that the mapping $f \to f_{|\mathrm{sp}(A,a)}(a)$ from $\mathscr{H}(\Omega)$ to A is a continuous unit-preserving algebra homomorphism taking u to a. □

As a consequence of Corollary 1.2.30 and Fact 1.3.17 we have the following.

Corollary 1.3.18 *Let E be a compact Hausdorff topological space, let $a \in C^{\mathbb{C}}(E)$, let Ω be an open set in $\mathbb{C}$, and suppose that $a(E) \subseteq \Omega$. Then $f(a) = f \circ a$ for every $f \in \mathscr{H}(\Omega)$.*

Applying Corollaries 1.3.16 and 1.3.18, we obtain the next result.

Corollary 1.3.19 *Let A be a complete normed unital associative and commutative complex algebra, with carrier space Δ and Gelfand representation $G : A \to C^{\mathbb{C}}(\Delta)$, let $a \in A$, let Ω be an open set in $\mathbb{C}$, and suppose that $\mathrm{sp}(A,a) \subseteq \Omega$. Then $G(f(a)) = f \circ G(a)$ for every $f \in \mathscr{H}(\Omega)$.*

Proposition 1.3.20 *Let A be a complete normed unital associative complex algebra, let a be in A, and let Ω be an open subset of $\mathbb{C}$ containing $\mathrm{sp}(A,a)$. Then there exists $\varepsilon > 0$ such that $\mathrm{sp}(A,b) \subseteq \Omega$ for every $b \in A$ with $\|b-a\| < \varepsilon$.*

Proof Suppose on the contrary that for every n there exist $x_n \in A$ and $\lambda_n \in \mathrm{sp}(A,x_n) \setminus \Omega$ such that $\|x_n - a\| < \frac{1}{n}$. Then $|\lambda_n| \leqslant \|x_n\| \leqslant \|a\| + 1$, and so there exists a subsequence of λ_n converging to some $\lambda \in \mathbb{C} \setminus \Omega$. Since $x_n - \lambda_n\mathbf{1} \notin \mathrm{Inv}(A)$, we have $a - \lambda\mathbf{1} \notin \mathrm{Inv}(A)$ by Theorem 1.1.23. Thus $\lambda \in \mathrm{sp}(A,a)$ and $\lambda \notin \Omega$, which is a contradiction with the assumption that $\mathrm{sp}(A,a)$ is contained in Ω. □

§1.1.36 should be kept in mind for the formulation and proof of the next theorem.

Theorem 1.3.21 *Let A be a complete normed unital associative complex algebra, let Ω be a non-empty open set in $\mathbb{C}$, and let f be in $\mathscr{H}(\Omega)$. Then*

$$A_\Omega := \{x \in A : \mathrm{sp}(A,x) \subseteq \Omega\}$$

is a non-empty open subset of A, and the mapping $\tilde{f}: x \to f(x)$ from A_Ω to A is holomorphic. Moreover, for each $a \in A_\Omega$, the Fréchet derivative of $\tilde{f}$ at a is represented by the following $BL(A)$-valued integral

$$D\tilde{f}(a) = \frac{1}{2\pi i}\int_\Gamma f(z) L_{(z\mathbf{1}-a)^{-1}} R_{(z\mathbf{1}-a)^{-1}}\, dz,$$

where Γ is any contour that surrounds $\operatorname{sp}(A,a)$ *in Ω.*

Proof Clearly $\Omega\mathbf{1} \subseteq A_\Omega$, and hence A_Ω is not empty. Moreover, by Proposition 1.3.20, A_Ω is an open subset of A. Let a be in A_Ω, and let Γ be a contour surrounding $\operatorname{sp}(A,a)$ in Ω. Since the mapping $x \to x^{-1}$ is continuous on $\operatorname{Inv}(A)$, the integrand $f(z)L_{(z\mathbf{1}-a)^{-1}}R_{(z\mathbf{1}-a)^{-1}}$ is continuous on Γ, so that the integral

$$\frac{1}{2\pi i}\int_\Gamma f(z) L_{(z\mathbf{1}-a)^{-1}} R_{(z\mathbf{1}-a)^{-1}}\, dz$$

exists and defines an element (say T) of $BL(A)$. Moreover, the integrand is actually a holomorphic $BL(A)$-valued function on $\Omega \setminus \operatorname{sp}(A,a)$ (by Proposition 1.1.40). The Cauchy theorem implies therefore that the integral is independent of the choice of Γ, provided that Γ surrounds $\operatorname{sp}(A,a)$ in Ω. For each $h \in A$, the valuation of elements of $BL(A)$ at h becomes a continuous linear mapping from $BL(A)$ to A, and hence

$$\begin{aligned}
T(h) &= \left(\frac{1}{2\pi i}\int_\Gamma f(z) L_{(z\mathbf{1}-a)^{-1}} R_{(z\mathbf{1}-a)^{-1}} dz\right)(h)\\
&= \frac{1}{2\pi i}\int_\Gamma f(z) L_{(z\mathbf{1}-a)^{-1}} R_{(z\mathbf{1}-a)^{-1}}(h)\, dz\\
&= \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-1} h (z\mathbf{1}-a)^{-1} dz.
\end{aligned}$$

Since Γ is a compact subset of $\mathbb{C} \setminus \operatorname{sp}(A,a)$, there exists a positive number M such that $\|(z\mathbf{1}-a)^{-1}\| \leqslant M$ for every $z \in \Gamma$. Let z be in Γ and let h be in A with $\|h\| < \frac{1}{2M}$. Then we have

$$\|(z\mathbf{1}-a)-(z\mathbf{1}-a-h)\| = \|h\| < \frac{1}{2M} < \frac{1}{M} \leqslant \frac{1}{\|(z\mathbf{1}-a)^{-1}\|}.$$

Therefore, by Corollary 1.1.21(ii), $z\mathbf{1}-a-h$ is invertible in A, and then, by Lemma 1.1.13(iii),

$$\|(z\mathbf{1}-a-h)^{-1}\| \leqslant \frac{\|(z\mathbf{1}-a)^{-1}\|}{1-\|(z\mathbf{1}-a)^{-1}\|\|h\|} \leqslant \frac{M}{1-\frac{1}{2}} = 2M.$$

Since

$$\begin{aligned}
&(z\mathbf{1}-a-h)^{-1} - (z\mathbf{1}-a)^{-1} - (z\mathbf{1}-a)^{-1}h(z\mathbf{1}-a)^{-1}\\
&\quad = (z\mathbf{1}-a)^{-1}h(z\mathbf{1}-a-h)^{-1}h(z\mathbf{1}-a)^{-1}
\end{aligned}$$

(by Lemma 1.1.22), we derive that

$$\begin{aligned}
&\|(z\mathbf{1}-a-h)^{-1} - (z\mathbf{1}-a)^{-1} - (z\mathbf{1}-a)^{-1}h(z\mathbf{1}-a)^{-1}\|\\
&\quad \leqslant \|(z\mathbf{1}-a)^{-1}\|^2\|h\|^2\|(z\mathbf{1}-a-h)^{-1}\| \leqslant 2M^3\|h\|^2.
\end{aligned}$$

Since Γ surrounds $\mathrm{sp}(A,a)$ in Ω, the set $W := \{z \in \mathbb{C} : \mathrm{Ind}_\Gamma(z) = 1\}$ is open in $\mathbb{C}$ and satisfies $\mathrm{sp}(A,a) \subseteq W \subseteq \Omega$. By Proposition 1.3.20, there exists $\varepsilon > 0$ such that $\mathrm{sp}(A,b) \subseteq W$ for every $b \in A$ with $\|b-a\| < \varepsilon$. Therefore, if in addition we assume that $\|h\| < \varepsilon$, then $\mathrm{sp}(A,a+h) \subseteq W$, hence Γ is a contour surrounding $\mathrm{sp}(A,a+h)$ in Ω, and so

$$\begin{aligned} &f(a+h) - f(a) - T(h) \\ &\quad = \frac{1}{2\pi i}\int_\Gamma f(z)\left[(z\mathbf{1}-a-h)^{-1} - (z\mathbf{1}-a)^{-1} - (z\mathbf{1}-a)^{-1}h(z\mathbf{1}-a)^{-1}\right]dz. \end{aligned}$$

It follows that

$$\|f(a+h) - f(a) - T(h)\| \leqslant \frac{1}{2\pi}\ell(\Gamma)\max\{|f(z)| : z \in \Gamma\}2M^3\|h\|^2,$$

where $\ell(\Gamma)$ denotes the length of Γ. Hence

$$\lim_{h\to 0}\frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0. \qquad \square$$

Corollary 1.3.22 *Let A be a complete normed unital associative complex algebra, let Ω be a non-empty open set in $\mathbb{C}$, and let f be a biholomorphic function from Ω onto $f(\Omega)$. Then the mapping $\tilde{f} : a \to f(a)$ is biholomorphic from A_Ω onto $A_{f(\Omega)}$.*

Proof Let $g : f(\Omega) \to \Omega$ be the inverse of f. Since $g \circ f$ and $f \circ g$ are the identity mappings on Ω and $f(\Omega)$, respectively, Theorem 1.3.13(v) shows that $\tilde{g} \circ \tilde{f}$ and $\tilde{f} \circ \tilde{g}$ are the identity mappings on A_Ω and $A_{f(\Omega)}$, respectively. Now $\tilde{f}$ and its inverse $\tilde{g}$ are holomorphic (by Theorem 1.3.21), and the proof is complete. $\square$

This section closes with two typical applications of the holomorphic functional calculus. If A is a unital associative algebra over $\mathbb{K}$, and if e is an idempotent in A different from 0 and $\mathbf{1}$, then, by Corollary 1.3.5, we have $\mathrm{sp}(A,e) = \{0,1\}$, and hence A has some element whose spectrum is disconnected, provided such an idempotent e does exist. Conversely, we have the following.

Proposition 1.3.23 *Let A be a complete normed unital associative complex algebra. If the spectrum of some element of A is not connected, then A contains an idempotent different from* 0 *and* $\mathbf{1}$. *More precisely, if $a \in A$ is such that $\mathrm{sp}(A,a) = F \cup G$ for some disjoint non-empty closed subsets F,G of $\mathrm{sp}(A,a)$, then there is an idempotent $e \neq 0,\mathbf{1}$ in $\{a\}^{cc}$ satisfying:*

(i) *If $a_1 := ae$ and $a_2 := a(\mathbf{1}-e)$, then $a = a_1 + a_2$ and $a_1a_2 = a_2a_1 = 0$.*
(ii) $\mathrm{sp}(A,a_1) = F \cup \{0\}$ *and* $\mathrm{sp}(A,a_2) = G \cup \{0\}$.

Proof Let a be in A such that $\mathrm{sp}(A,a) = F \cup G$, where F and G are disjoint non-empty closed subsets of $\mathrm{sp}(A,a)$. Let Ω_1, Ω_2 be disjoint open subsets of $\mathbb{C}$ such that $F \subseteq \Omega_1$ and $G \subseteq \Omega_2$. Set $\Omega = \Omega_1 \cup \Omega_2$, and consider the function $f : \Omega \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 1 & \text{if} \quad z \in \Omega_1 \\ 0 & \text{if} \quad z \in \Omega_2. \end{cases}$$

Then $f \in \mathscr{H}(\Omega)$ and satisfies that $f^2 = f$. Therefore, by Theorem 1.3.13, $e := f(a)$ is an idempotent in $\{a\}^{cc}$ with $\mathrm{sp}(A,e) = \{0,1\}$ (which implies $e \neq 0,\mathbf{1}$). Now, noticing that $e(1-e)=0=(1-e)e$, assertion (i) follows. Consider the functions $f_1(z) = zf(z)$ and $f_2(z)=z(1-f(z))$. It follows from Theorem 1.3.13 that $a_1 = f_1(a)$ and $a_2 = f_2(a)$. Hence

$$\mathrm{sp}(A,a_1) = f_1(\mathrm{sp}(A,a)) = F \cup \{0\} \text{ and } \mathrm{sp}(A,a_2) = f_2(\mathrm{sp}(A,a)) = G \cup \{0\},$$

which concludes the proof. □

Definition 1.3.24 Let A be an associative algebra, and let a,b be in A. We say that b is an nth *root* of a in A if $a=b^n$. If in addition A is complete normed and unital, and if $a=\exp(b)$, then we say that b is a *logarithm* of a in A. We note that the two meanings of $\exp(b)$, given by §1.1.29 and Theorem 1.3.13, do agree (in view of Theorem 1.3.13(ii)).

Proposition 1.3.25 *Let A be a complete normed unital associative complex algebra, and let a be an element of A. If zero lies in the unbounded connected component of $\mathbb{C} \setminus \mathrm{sp}(A,a)$, then*

(i) *a has roots of all orders in A.*
(ii) *a has a logarithm in A.*

Moreover, if a satisfies the condition $\mathrm{sp}(A,a) \subseteq \mathbb{R}^+$, then the roots in assertion (i) can be chosen so as to satisfy the same condition.

Proof Assume that zero lies in the unbounded connected component of $\mathbb{C} \setminus \mathrm{sp}(A,a)$. Then there is a function f, holomorphic in a simply connected open set Ω containing $\mathrm{sp}(A,a)$, which satisfies $\exp(f(z)) = z$. It follows from Theorem 1.3.13(v) that $b = f(a)$ is a logarithm of a. If $\mathrm{sp}(A,a) \subseteq \mathbb{R}^+$, then f can be chosen so as to be real on $\mathrm{sp}(A,a)$, so that $\mathrm{sp}(A,b)$ lies in $\mathbb{R}$, by the spectral mapping theorem. If $c = \exp(\frac{1}{n}b)$, then $c^n = a$, and another application of the spectral mapping theorem shows that $\mathrm{sp}(A,c) \subseteq \mathbb{R}^+$ if $\mathrm{sp}(A,b) \subseteq \mathbb{R}$. □

Remark 1.3.26 Let A be a complete normed non-unital associative complex algebra, let a be in A, let Ω be an open set in $\mathbb{C}$, containing $\mathrm{sp}(A_{\mathbb{1}},a)$ (so that, $0 \in \Omega$), let f be in $\mathscr{H}(\Omega)$, and, keeping in mind Proposition 1.1.107, let $f(a)$ be the element of $A_{\mathbb{1}}$ given by the holomorphic functional calculus. It is easily realized that $f(a) \in A$ if and only if $f(0)=0$. Indeed, consider the continuous unit-preserving algebra homomorphism $\phi : A_{\mathbb{1}} = \mathbb{C}\mathbb{1} \oplus A \to \mathbb{C}$ defined by $\phi(\lambda\mathbb{1} + x) := \lambda$, and apply Corollary 1.3.16.

1.3.3 Historical notes and comments

The material in this section has been elaborated mainly from the books of Conway [711], Dales [715], Doran and Belfi [725], Müller [780], Palmer [786], and Rudin [804]. Other sources are quoted in what follows.

Theorem 1.3.2 and Proposition 1.3.3 are folklore. Proposition 1.3.3 can be found in [792, Corollary 5.1], where it is derived from a more general result [792, Theorem 5.1].

According to [780, C.1.18], ‘the [holomorphic] functional calculus for one [bounded linear] operator [on a complex Banach space] was first used by Riesz

[504] for construction of the spectral projection of a compact operator [cf. Proposition 1.3.23]. The general spectral mapping theorem [Theorem 1.3.13(iv)] was proved by Dunford [219]'. For a more detailed history, the reader is referred to [786, pp. 343–4].

In relation to Proposition 1.3.23, it is worth mentioning the following theorem, which is due to Koliha [391].

Theorem 1.3.27 *Let A be a complete normed unital associative complex algebra, let a be in A, and let μ be in $\mathbb{C}$. Then μ is an isolated point of* $\mathrm{sp}(A,a)$ *if (and only if) there exists a nonzero idempotent $e \in A$ satisfying $ae = ea$, $\mathfrak{r}((a-\mu\mathbf{1})e) = 0$, and $e + a - \mu\mathbf{1} \in \mathrm{Inv}(A)$.*

Now let us conclude this subsection by taking the following short proof of Theorem 1.1.31 from [726], which relies on the holomorphic functional calculus.

Proof Let A and S be as in Theorem 1.1.31. In view of Proposition 1.1.98, to prove the conclusion in that theorem, we may assume that the complete normed unital associative algebra A is complex. Set $\Omega := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \frac{\pi}{2}\}$. Since S is continuous and $S(0) = \mathbf{1}$, there exists $\varepsilon > 0$ such that $\|S(t) - \mathbf{1}\| < 1$ for $0 \leqslant t \leqslant \varepsilon$. Therefore

$$\mathrm{sp}(A, S(t)) \subseteq \{z \in \mathbb{C} : |z-1| < 1\} \subseteq \Omega,$$

and $L(t) := \log S(t)$ is defined and continuous for $0 \leqslant t \leqslant \varepsilon$. Here, arg and log stand for the principal values of the argument and logarithmic functions, respectively, and, for $x \in A$ with $\mathrm{sp}(A,x) \subseteq \Omega$, $\log x$ has to be understood in the sense of the holomorphic functional calculus.

Let t be in $\mathbb{R}_0^+$ and let n be in $\mathbb{N}$ such that $nt \leqslant \varepsilon$. Note that for every $k \in \{1, \ldots, n\}$ we have

$$\mathrm{sp}(A, S(t))^k = \mathrm{sp}(A, S(t)^k) = \mathrm{sp}(A, S(kt)) \subseteq \Omega.$$

Noticing that $|\arg(\lambda)| < \frac{\pi}{2(k+1)}$ whenever $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ satisfy $|\arg(\lambda)| < \frac{\pi}{2k}$ and $|\arg(\lambda^{k+1})| < \frac{\pi}{2}$, a finite induction gives that

$$\mathrm{sp}(A, S(t)) \subseteq \Omega_n := \left\{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \frac{\pi}{2n} \right\}.$$

Since $\log z^n = n \log z$ on Ω_n, it follows that $\log(S(t)^n) = n \log S(t)$, and hence

$$L(nt) = \log S(nt) = \log(S(t)^n) = n \log S(t) = nL(t).$$

As a consequence of the above paragraph, we have $L(t) = nL(\frac{1}{n}t)$ for all $0 \leqslant t \leqslant \varepsilon$ and $n \in \mathbb{N}$. Now, for each rational number $\frac{m}{n}$ with $0 \leqslant \frac{m}{n} \leqslant 1$, and each t in the interval $[0, \varepsilon]$, we have $\frac{m}{n}L(t) = mL(\frac{1}{n}t) = L(\frac{m}{n}t)$, and in particular $\frac{m}{n}L(\varepsilon) = L(\frac{m}{n}\varepsilon)$. From the continuity of L it follows that $sL(\varepsilon) = L(s\varepsilon)$ for $0 \leqslant s \leqslant 1$, and in particular $\frac{t}{\varepsilon}L(\varepsilon) = L(t)$ for $0 \leqslant t \leqslant \varepsilon$. Set $a := \frac{1}{\varepsilon}L(\varepsilon)$. Then $L(t) = ta$ for $0 \leqslant t \leqslant \varepsilon$, and hence

$$S(t) = \exp(ta) \tag{1.3.6}$$

for $0 \leqslant t \leqslant \varepsilon$. If $t \in \mathbb{R}^+$ is arbitrary then $\frac{1}{n}t < \varepsilon$ for some sufficiently large $n \in \mathbb{N}$, and so we have

$$S(t) = S\left(\frac{1}{n}t\right)^n = \exp\left(\frac{1}{n}ta\right)^n = \exp(ta).$$

Thus (1.3.6) holds for every $t \in \mathbb{R}_0^+$. □

1.4 Compact and weakly compact operators

Introduction We introduce compact and weakly compact operators between (possibly non-complete) normed spaces, and discuss both old and recent selected topics in their theory. As far as possible, results on compact operators are obtained from those on weakly compact operators.

1.4.1 Operators from a normed space to another

Let X and Y be normed spaces over $\mathbb{K}$. By a *compact* (respectively, *weakly compact*) *operator* from X to Y we mean a linear mapping $T : X \to Y$ such that the closure of $T(\mathbb{B}_X)$ in Y is a compact (respectively, weakly compact) subset of Y. We denote by $\mathfrak{K}(X,Y)$ (respectively, $\mathfrak{W}(X,Y)$) the set of all compact (respectively, weakly compact) operators from X to Y.

The following fact is straightforward.

Fact 1.4.1 *Let X and Y be normed spaces over $\mathbb{K}$. We have:*

(i) *$\mathfrak{K}(X,Y)$ and $\mathfrak{W}(X,Y)$ are subspaces of $L(X,Y)$ satisfying*

$$\mathfrak{K}(X,Y) \subseteq \mathfrak{W}(X,Y) \subseteq BL(X,Y).$$

(ii) *If T is in $\mathfrak{K}(X,Y)$ (respectively, $\mathfrak{W}(X,Y)$), if U and V are normed spaces over $\mathbb{K}$, if F is in $BL(Y,V)$, and if G is in $BL(U,X)$, then $F \circ T \circ G$ belongs to $\mathfrak{K}(U,V)$ (respectively, $\mathfrak{W}(U,V)$).*

(iii) *If T is a compact (respectively, weakly compact) operator from X to Y, then the restriction of T to any subspace of X is a compact (respectively, weakly compact) operator.*

(iv) *If T is a compact (respectively, weakly compact) operator from X to Y, and if Z is any subspace of Y containing the closure of $T(\mathbb{B}_X)$ in Y (for example, if Z is a closed subspace of Y containing $T(X)$), then T, regarded as a mapping from X to Z, is a compact (respectively, weakly compact) operator.*

An easy, but not so straightforward result, is the following.

Proposition 1.4.2 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be a compact (respectively, weakly compact) operator from X to Y. If Y is infinite-dimensional (respectively, non-reflexive), then $T(X)$ is of the first category in Y.*

Proof Assume that $T(X)$ is of the second category in Y. Then it follows from the equality

$$T(X) = \bigcup_{n\in\mathbb{N}} nT(\mathbb{B}_X)$$

that the closure of $T(\mathbb{B}_X)$ in Y (say C) contains a closed ball in Y. Since C is compact (respectively, weakly compact), such a ball also becomes compact (respectively, weakly compact), so that, by the Riesz sphere theorem (respectively, by Goldstine's theorem), Y is finite-dimensional (respectively, reflexive). $\square$

It follows from Proposition 1.4.2 that, *if Y is a normed space of the second category in itself, and if there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y, then Y is finite-dimensional (respectively, reflexive)*. As a consequence, we have the following.

Corollary 1.4.3 *Let Y be a Banach space such that there exists a surjective compact (respectively, weakly compact) operator from some normed space to Y. Then Y is finite-dimensional (respectively, reflexive).*

Both compact and weakly compact versions of Corollary 1.4.3 do not remain true if the assumption that Y is a Banach space is relaxed to the assumption that Y is an arbitrary normed space. Actually, as we are going to show in Proposition 1.4.4 immediately below, there are abundant examples of infinite-dimensional normed spaces X and Y such that we can find a bijective compact (hence weakly compact) operator from X to Y. Of course, the space Y in these examples cannot be complete (much less reflexive).

Proposition 1.4.4 *Let X be a separable Banach space, and let Y be an infinite-dimensional Banach space. Then there exists a bijective compact operator from X' to some subspace of Y.*

Proof Take a normalized basic sequence $(y_n)_{n\in\mathbb{N}}$ in Y, as well as a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{S}_X$ whose linear hull is dense in X. Then the mapping

$$T : x' \to \sum_{n=1}^{\infty} \frac{1}{n^2} x'(x_n) y_n$$

becomes an injective linear operator from X' to Y. Moreover, since T is the uniform limit on $\mathbb{B}_{X'}$ of a sequence of weak*-to-norm continuous functions, the restriction of T to $\mathbb{B}_{X'}$ is weak*-to-norm continuous, and hence $T(\mathbb{B}_{X'})$ is (norm-)compact. It follows from Fact 1.4.1(iv) that T, regarded as an operator from X' to $T(X')$, becomes a bijective compact operator. □

The bijective compact operators given by the above proposition start from Banach spaces. Now, the existence of bijective compact operators starting from non-complete normed spaces follows from the following.

Proposition 1.4.5 *Let $(X, \|\cdot\|)$ be a normed space, and let f be a $\|\cdot\|$-discontinuous linear funcional on X. Then the norm $|||\cdot|||$ on X defined by*

$$|||x||| := \|x\| + |f(x)|$$

is not complete. Moreover, compact (respectively, weakly compact) operators starting from $(X, \|\cdot\|)$ remain compact (respectively, weakly compact) when they are regarded as operators starting from $(X, |||\cdot|||)$.

Proof Since $\mathbb{B}_{(X,|||\cdot|||)} \subseteq \mathbb{B}_{(X,\|\cdot\|)}$, the last conclusion in the statement becomes clear. Assume that the norm $|||\cdot|||$ is complete. Then, since f is $|||\cdot|||$-continuous, $\ker(f)$ is $|||\cdot|||$-complete. Keeping in mind that $\|\cdot\|$ and $|||\cdot|||$ coincide on $\ker(f)$, we deduce that $\ker(f)$ is closed in $(X, \|\cdot\|)$, an hence that f is $\|\cdot\|$-continuous, contrary to the assumption. □

Now we deal with useful characterizations of compact and weakly compact operators.

Proposition 1.4.6 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. Then T is a weakly compact operator if and only if $T''(X'') \subseteq Y$.*

Proof Let C stand for the closure of $T(\mathbb{B}_X)$ in Y.

Assume that T is weakly compact, so that C is a weakly compact subset of Y. Since the weak topology of Y is the restriction to Y of the weak* topology of Y'', it follows that C is weak*-closed in Y''. Therefore, since T'' is weak*-to-weak* continuous and $\mathbb{B}_X$ is weak* dense in $\mathbb{B}_{X''}$, we derive that $T''(\mathbb{B}_{X''}) \subseteq C \subseteq Y$, and hence that $T''(X'') \subseteq Y$.

Now assume that $T''(X'') \subseteq Y$. Then, since $\mathbb{B}_{X''}$ is a weak*-compact subset of X'', we see that $T''(\mathbb{B}_{X''})$ is a weak*-compact subset of Y'' contained in Y, and hence it is a weakly compact subset of Y containing $T(\mathbb{B}_X)$. It follows that C is a weakly compact subset of Y. □

Remark 1.4.7 Let X, Y, and T be as in Proposition 1.4.6. Although not emphasized in the statement, the above proof shows that T is weakly compact if and only if $T''(\mathbb{B}_{X''})$ is contained in the closure of $T(\mathbb{B}_X)$ in Y. As a consequence, if T is weakly compact, then so is T''. As a partial converse, if Y is in fact a Banach space, and if T'' is weakly compact, then, since Y is closed in Y'', and T'' coincides with T on X, it follows from Fact 1.4.1(iii)–(iv) that T is weakly compact.

Again, let X, Y, and T be as in Proposition 1.4.6. One can wonder what has to be added to the condition $T''(X'') \subseteq Y$ in order to characterize the compactness of T. An answer is provided by the following.

Theorem 1.4.8 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. Then the following conditions are equivalent:*

(i) *T is a compact operator. $T''(X'') \subseteq Y$, and the restriction of T'' to $\mathbb{B}_{X''}$ is weak*-to-norm continuous.*
(ii) *For every sequence x_n in $\mathbb{B}_X$, the sequence $T(x_n)$ has a convergent subsequence in Y.*

Proof Again, let C stand for the closure of $T(\mathbb{B}_X)$ in Y.

(i)⇒(ii) Assume that T is compact. Since C is norm-compact, and the restriction to C of the weak* topology of Y'' is Hausdorff and weaker than the norm topology, we derive that the norm and weak* topologies of Y'' coincide on C. On the other hand, according to Remark 1.4.7 immediately above, we have $T''(\mathbb{B}_{X''}) \subseteq C$. Since T'' is weak*-to-weak* continuous, it follows that the restriction of T'' to $\mathbb{B}_{X''}$ is weak*-to-norm continuous. Finally, the inclusion $T''(X'') \subseteq Y$ follows from the one $T''(\mathbb{B}_{X''}) \subseteq C$ just pointed out (or directly from Proposition 1.4.6).

(ii)⇒(iii) Let x_n be a sequence in $\mathbb{B}_X$, and take a weak*-cluster point x'' to the sequence x_n in $\mathbb{B}_{X''}$. Then by assumption (ii), $T''(x'')$ lies in Y and becomes a norm-cluster point to the sequence $T(x_n)$ in Y. Since the norm topology of Y is metrizable, it follows that $T''(x'')$ is the limit in Y of a suitable subsequence of the sequence $T(x_n)$.

(iii)⇒(i) Let y_n be any sequence in C. For each n, choose $x_n \in \mathbb{B}_X$ such that $\|y_n - T(x_n)\| < \frac{1}{n}$. By assumption (iii), some subsequence of $T(x_n)$ (say $T(x_{n_k})$) has a limit in Y, which clearly lies in C. It follows that y_{n_k} has a limit in C. Thus every sequence in C has a convergent subsequence in C, hence C is compact. □

§1.4.9 Let X, Y, and T be as in the above theorem. It follows from the implication (i)⇒(ii) that, if T is compact, then T'' is compact and T is *completely continuous* (meaning that the restriction of T to $\mathbb{B}_X$ is weak-to-norm continuous). By the way, it is clear that, if X is reflexive, and if T is completely continuous, then T is compact. The verification of this last result shows that actually, if X is reflexive, and if T is completely continuous, then $T(\mathbb{B}_X)$ is compact. Hence, in light of Fact 1.4.1(iv), we realize that T, regarded as a mapping from X to $T(X)$, is a (surjective) compact operator. To convert it into a bijective compact operator it is enough to think about the induced mapping $X/\ker(T) \to T(X)$.

Corollary 1.4.10 *Let X be a normed space over $\mathbb{K}$, let Y be a Banach space over $\mathbb{K}$, and let T be in $BL(X,Y)$. Then the following conditions are equivalent:*

(i) *T is a compact operator.*
(ii) *The restriction of T'' to $\mathbb{B}_{X''}$ is weak*-to-norm continuous.*
(iii) *T'' is a compact operator.*

Proof (i)⇒(ii) By the implication (i)⇒(ii) in Theorem 1.4.8.

(ii)⇒(iii) As suggested in §1.4.9 immediately above, this implication follows from the weak*-compactness of $\mathbb{B}_{X''}$.

(iii)⇒(i) Since Y is closed in Y'', and T'' coincides with T on X, this implication follows from Fact 1.4.1(iii)–(iv). □

Keeping in mind that every Banach space is closed in its bidual (respectively, that the uniform limit of any sequence of continuous functions is a continuous function), Proposition 1.4.6 (respectively, the equivalence (i)⇔(ii) in Corollary 1.4.10) yields the following.

Corollary 1.4.11 *Let X be a normed space over $\mathbb{K}$, and let Y be a Banach space over $\mathbb{K}$. Then both $\mathfrak{K}(X,Y)$ and $\mathfrak{W}(X,Y)$ are closed in $BL(X,Y)$.*

§1.4.12 Let X and Y be normed spaces over $\mathbb{K}$. By a *finite-rank operator* from X to Y we mean a bounded linear mapping $T : X \to Y$ such that $T(X)$ is a finite-dimensional subspace of Y. We denote by $\mathfrak{F}(X,Y)$ the set of all finite-rank operators from X to Y, and note that $\mathfrak{F}(X,Y)$ is a subspace of $BL(X,Y)$. It is also worth remarking that, for $y \in Y$ and $f \in X'$, the mapping $y \otimes f : x \to f(x)y$ from X to Y is a finite-rank operator. Actually, as we show in Fact 1.4.13 immediately below, there are not many more finite-rank operators.

Fact 1.4.13 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $\mathfrak{F}(X,Y)$. Then there are $y_1, \ldots, y_n \in Y$ and $f_1, \ldots, f_n \in X'$ such that $T = \sum_{i=1}^n y_i \otimes f_i$.*

Proof We may assume that $T \neq 0$. Let $\{y_1, \ldots, y_n\}$ a basis of $T(X)$, let $\{g_1, \ldots, g_n\}$ stand for the corresponding dual basis of the algebraic dual of $T(X)$ (equal to $T(X)'$ by Tihonov's theorem), and, for $1 \leqslant i \leqslant n$, set $f_i := g_i \circ T$. Then f_i belongs to X' for every i, and we straightforwardly realize that $T = \sum_{i=1}^n y_i \otimes f_i$. □

Let X be a normed space over $\mathbb{K}$, let M be a nonzero finite-dimensional subspace of X, and let T be a continuous linear projection on X with $T(X) = M$. The above proof shows that $T = \sum_{i=1}^{n} y_i \otimes f_i$, where $\{y_1, \ldots, y_n\}$ is any basis of M, $\{g_1, \ldots, g_n\}$ stands for the corresponding dual basis, and, for $1 \leqslant i \leqslant n$, f_i is an element of X' which extends g_i. Thanks to the Hahn–Banach theorem, this argument can be reverted to get the well-known fact which follows.

Fact 1.4.14 *Let X be a normed space over $\mathbb{K}$, and let M be a finite-dimensional subspace of X. Then M is complemented in X* (i.e. there exists a continuous linear projection on X whose range equals M).

Given normed spaces X and Y over $\mathbb{K}$, we denote by $\overline{\mathfrak{F}(X,Y)}$ the closure of $\mathfrak{F}(X,Y)$ in $BL(X,Y)$. As a straightforward consequence of Fact 1.4.13 (or directly by Tihonov's theorem), we get that *finite-rank operators are compact operators*. Therefore, invoking Corollary 1.4.11, we derive the following.

Corollary 1.4.15 *Let X be a normed space over $\mathbb{K}$, and let Y be a Banach space over $\mathbb{K}$. Then*

$$\overline{\mathfrak{F}(X,Y)} \subseteq \mathfrak{K}(X,Y).$$

The next proposition becomes a refinement of the so-called *Banach–Steinhaus closure theorem* [817, Theorem 3.3.13].

Proposition 1.4.16 *Let X be a Banach space over $\mathbb{K}$, let Y be a normed space over $\mathbb{K}$, and let F_n be a sequence in $BL(X,Y)$ converging pointwise to a mapping $F : X \to Y$. Then F lies in $BL(X,Y)$, and F_n converges to F uniformly in each compact subset of X.*

Proof By the uniform boundedness principle, there exists $M > 0$ such that $\|F_n(x)\| \leqslant M\|x\|$ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, since F is clearly lineal, F lies in $BL(X,Y)$ and $\|F\| \leqslant M$. Now, let C be a compact subset of X, and let $\varepsilon > 0$. Then there exists a finite subset D of X such that $C \subseteq D + \frac{\varepsilon}{3M}\mathbb{B}_X$. Moreover, since F_n converges pointwise to F, there is $m \in \mathbb{N}$ such that $\|F(d) - F_n(d)\| \leqslant \frac{\varepsilon}{3}$ whenever d is in D and $n \geqslant m$. Therefore, for $c \in C$ and $n \geqslant m$, it is enough to choose $d \in D$ and $y \in \frac{\varepsilon}{3M}\mathbb{B}_X$ with $c = d + y$, to get

$$\begin{aligned}
\|F(c) - F_n(c)\| &\leqslant \|F(c-d)\| + \|F(d) - F_n(d)\| + \|F_n(d-c)\| \\
&\leqslant \|F\|\|y\| + \|F(d) - F_n(d)\| + \|F_n\|\|y\| \\
&\leqslant 2M\|y\| + \|F(d) - F_n(d)\| \\
&\leqslant \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

□

Exercise 1.4.17 Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $\mathfrak{K}(X,Y)$. Prove that the closure of $T(X)$ in Y is separable.

Hint Recall that every compact metric space is separable, and keep in mind that the closure of $T(\mathbb{B}_X)$ in Y is a compact metric space. □

The inclusion $\overline{\mathfrak{F}(X,Y)} \subseteq \mathfrak{K}(X,Y)$ in Corollary 1.4.15 is often an equality. Indeed, we have the following.

Proposition 1.4.18 *Let X be a normed space over $\mathbb{K}$, and let Y be a Banach space over $\mathbb{K}$. Then the equality $\overline{\mathfrak{F}(X,Y)} = \mathfrak{K}(X,Y)$ holds if Y has a Schauder basis, or if Y is a Hilbert space.*

Proof Assume that Y has a Schauder basis $(y_n)_{n\in\mathbb{N}}$, let $(y'_n)_{n\in\mathbb{N}}$ stand for the sequence of biorthogonal functionals on Y associated to $(y_n)_{n\in\mathbb{N}}$, and for $n \in \mathbb{N}$ set

$$P_n := \sum_{i=1}^{n} y_i \otimes y'_i \in \mathfrak{F}(Y,Y).$$

Then the sequence P_n converges pointwise to I_Y. Let T be in $\mathfrak{K}(X,Y)$. It follows from Proposition 1.4.16 that P_n converges uniformly to I_Y on $T(\mathbb{B}_X)$. But this means that $P_n \circ T$ converges to T in the norm topology of $BL(X,Y)$. Since $P_n \circ T \in \mathfrak{F}(X,Y)$, it follows that $T \in \overline{\mathfrak{F}(X,Y)}$.

Now assume that Y is a Hilbert space, let T be in $\mathfrak{K}(X,Y) \setminus \mathfrak{F}(X,Y)$, and let Z stand for the closure of $T(X)$ in Y. Then, by Exercise 1.4.17, Z is an infinite-dimensional separable Hilbert space, and hence has a Schauder basis. It follows from Fact 1.4.1(iv) and the above paragraph that T, regarded as an operator from X to Z, is the limit in $BL(X,Z)$ of a suitable sequence F_n in $\mathfrak{F}(X,Z)$. Therefore, denoting by ι the inclusion $Z \hookrightarrow Y$, $\iota \circ F_n$ becomes a sequence in $\mathfrak{F}(X,Y)$ converging to T in the norm topology of $BL(X,Y)$. Hence $T \in \overline{\mathfrak{F}(X,Y)}$. □

1.4.2 Operators from a normed space to itself

Given a normed space X, we set $\mathfrak{F}(X) := \mathfrak{F}(X,X)$, $\mathfrak{K}(X) := \mathfrak{K}(X,X)$, and $\mathfrak{W}(X) := \mathfrak{W}(X,X)$, and note that, in view of Fact 1.4.1(i)–(ii), $\mathfrak{F}(X)$, $\mathfrak{K}(X)$, and $\mathfrak{W}(X)$ are ideals of the algebra $BL(X)$.

Proposition 1.4.19 *Let X be a normed space over $\mathbb{K}$, let T be in $\mathfrak{W}(X)$, and let λ be in $\mathbb{K} \setminus \{0\}$. We have:*

(i) $\ker(T - \lambda I_X)$ *is a reflexive Banach space, and the equality*

$$\ker(T - \lambda I_X) = \ker(T'' - \lambda I_{X''})$$

holds.

(ii) *The following conditions are equivalent:*

 (a) $T - \lambda I_X$ *is surjective.*
 (b) $T'' - \lambda I_{X''}$ *is surjective.*
 (c) $T - \lambda I_X$ *is open.*

(iii) *If $T - \lambda I_X$ is bijective, then $(T - \lambda I_X)^{-1}$ is continuous.*

Proof We can assume that $\lambda = 1$.

Set $M := \ker(T - I_X)$. Then T becomes the identity on M, which implies (since M is closed in X and T is weakly compact) that $\mathbb{B}_M$ is weakly compact or, equivalently, that the normed space M is reflexive. On the other hand, the equality $\ker(T - I_X) = \ker(T'' - I_{X''})$ follows from the inclusion $T''(X'') \subseteq X$ (cf. Proposition 1.4.6). Thus assertion (i) has been proved.

Now we proceed to prove assertion (ii).

(a)⇒(b) Let x'' be in X''. Set $y'' := (T'' - I_{X''})(x'')$. Then we have

$$x'' = T''(x'') - y'' \in X + (T'' - I_{X''})(X'').$$

On the other hand, by assumption (a), we have

$$X = (T - I_X)(X) \subseteq (T'' - I_{X''})(X'').$$

It follows that x'' lies in $(T'' - I_{X''})(X'')$.

(b)⇒(c) By assumption (b) and the open mapping theorem, there exists a positive number k such that $k\mathbb{B}_{X''} \subseteq (T'' - I_{X''})(\mathbb{B}_{X''})$. Thus, for x in $k\mathbb{B}_X$, there is some $x'' \in \mathbb{B}_{X''}$ such that $T''(x'') - x'' = x$, which implies that x'' lies in X, and hence that x belongs to $(T - I_X)(\mathbb{B}_X)$. Therefore we have $k\mathbb{B}_X \subseteq (T - I_X)(\mathbb{B}_X)$, and $T - I_X$ indeed becomes open.

(c)⇒(a) This is clear.

Finally, assertion (iii) follows from the implication (a)⇒(c) in (ii). □

Somehow, the next theorem complements the above proposition, since the case $\lambda = 0$ has been not discussed there.

Theorem 1.4.20 *Let X be a normed space, and let T be a surjective weakly compact operator from X to X. Then $X/\ker(T)$ is a reflexive Banach space.*

Proof For any normed space Y, let Δ_Y stand for the open unit ball of Y. Let K denote the closure in X of $T(\Delta_X)$. Then K is weakly compact and we have $X = \bigcup_{n\in\mathbb{N}} nK$. Since

$$K \subseteq X = T(X) = \bigcup_{n\in\mathbb{N}} T(nK),$$

and for every $n \in \mathbb{N}$ the set $T(nK)$ is weakly compact, it follows from the Baire category theorem for compact spaces that there exists $m \in \mathbb{N}$ such that $K \cap T(mK)$ has non-empty interior in K, when K is endowed with the weak topology. This means that there exists a weakly open (so norm-open) subset U of X satisfying

$$\emptyset \neq K \cap U \subseteq T(mK). \tag{1.4.1}$$

Since K is the closure of $T(\Delta_X)$ in X, we can find $y \in T(\Delta_X)$ and $r \in \mathbb{R}^+$ such that

$$y + r\Delta_X \subseteq U. \tag{1.4.2}$$

Now, take $x \in \Delta_X$ with $T(x) = y$, and use the continuity of T to find $s \in \mathbb{R}^+$ such that

$$x + s\Delta_X \subseteq \Delta_X \text{ and } T(s\Delta_X) \subseteq r\Delta_X. \tag{1.4.3}$$

It follows from (1.4.1), (1.4.2), and (1.4.3) that

$$T(x + s\Delta_X) \subseteq T(\Delta_X) \cap (y + r\Delta_X) \subseteq T(mK),$$

which reads as

$$x + s\Delta_X \subseteq mK + \ker(T). \tag{1.4.4}$$

Now, set $\hat{X} := X/\ker(T)$, and let $\pi : X \to \hat{X}$ stand for the natural quotient mapping. It follows from (1.4.4) that

$$\pi(x) + s\Delta_{\hat{X}} \subseteq \pi(mK).$$

Keeping in mind that $\pi(mK)$ is weakly compact, it follows from the above inclusion that $\mathbb{B}_{\hat{X}}$ is weakly compact or, equivalently, that $\hat{X}$ is reflexive. □

We already know that there are examples of non-complete (hence non-reflexive) normed spaces X and Y such that we can find a bijective compact (so weakly compact) operator from X to Y (cf. Corollary 1.4.3 and Propositions 1.4.4 and 1.4.5). As an outstanding consequence of the above theorem, we are going to realize that the equality $X = Y$ cannot happen in these examples. Indeed, we have the following.

Corollary 1.4.21 *Let X be a normed space over $\mathbb{K}$. We have:*

(i) *If there exists a bijective weakly compact operator from X to X, then X is reflexive.*
(ii) *If there exists a surjective compact operator from X to X, then X is finite-dimensional.*

Proof Assertion (i) follows straightforwardly from Theorem 1.4.20.

Assume that there exists a surjective compact operator $T : X \to X$. Then, by Theorem 1.4.20, $X/\ker(T)$ is a Banach space. On the other hand, denoting by $\pi : X \to X/\ker(T)$ the natural quotient mapping, $\pi \circ T$ becomes a surjective compact operator (cf. Fact 1.4.1(ii)). It follows from Corollary 1.4.3 that $X/\ker(T)$ is finite-dimensional. Since T induces a linear bijection from $X/\ker(T)$ to X, we conclude that X is finite-dimensional. □

The next proposition shows that assertion (i) in Corollary 1.4.21 does not remain true when surjectivity replaces bijectivity.

Proposition 1.4.22 *Let X be a reflexive Banach space over $\mathbb{K}$ containing closed subspaces Y and Z such that Y has a Schauder basis, Z is isomorphic to X, and $X = Y \oplus Z$. Then there exists a couple (M,T), where M is a dense proper subspace of X, and T is a surjective weakly compact operator from M to M.*

Proof Let $(y_n)_{n\in\mathbb{N}}$ be the Schauder basis of Y whose existence is assumed, and let $(y'_n)_{n\in\mathbb{N}}$ be the sequence of biorthogonal functionals on Y associated to $(y_n)_{n\in\mathbb{N}}$. Then, since the sequence $y_n \otimes y'_n$ converges pointwise (to zero), it follows from the uniform boundedness principle that there exists a positive number k such that $\|y_n\|\|y'_n\| = \|y_n \otimes y'_n\| \leqslant k$ for every $n \in \mathbb{N}$, so that we can consider the linear mapping

$$F : y \to \sum_{n=1}^{\infty} \frac{1}{n^2} y'_n(y) y_n$$

from Y to Y, whose range is a dense subspace of Y. Moreover, since F is compact (by Corollary 1.4.15), it follows from Corollary 1.4.3 that $F(Y) \neq Y$. Therefore, $G := F \oplus I_Z$ is a bounded linear operator on X such that $M := G(X)$ is a dense proper subspace of X. Now, let Φ be the isomorphism from Z onto X whose existence is assumed, and let $T : M \to M$ be the linear operator defined by $T := G \circ \Phi \circ \pi$, where π stands for the restriction to M of the projection from X onto Z corresponding to the decomposition $X = Y \oplus Z$. Then, since $M = F(Y) \oplus Z$, we have $\pi(M) = Z$, so $(\Phi \circ \pi)(M) = \Phi(Z) = X$, and so $T(M) = G(X) = M$. This shows that T is surjective. Moreover T is weakly compact because it factors through a reflexive Banach space. □

We note that all requirements on the space X in the above proposition are fulfilled in the case $X = \ell_p(I)$, where I is any infinite set, and $1 < p < \infty$. Therefore we are provided with surjective weakly compact operators on non-complete normed spaces of arbitrary density character.

§**1.4.23** Let X be a nonzero vector space over $\mathbb{K}$, and let T be in $L(X)$. We denote by $\mathrm{sp}(T)$ the spectrum of T relative to $L(X)$, i.e. the set of those $\lambda \in \mathbb{K}$ such that $T - \lambda I_X$ is not bijective (cf. Example 1.1.32(b)). We will apply without notice the fact that, if X is a Banach space, then $\mathrm{sp}(T)$ coincides with the spectrum of T relative to $BL(X)$ (cf. Example 1.1.32(d)).

Fact 1.4.24 *Let X be a nonzero Banach space over $\mathbb{K}$, and let T be in $BL(X)$. Then*

$$\mathrm{sp}(T) = \mathrm{sp}(T').$$

Proof It suffices to show that T is bijective if and only if so is T'.

Since the mapping $F \to F'$ from $BL(X)$ to $BL(X')$ is a unit-preserving algebra antihomomorphism, it follows that T' is bijective whenever T is.

Now, assume that T' is bijective. By the above paragraph, T'' is bijective, and hence, by the Banach isomorphism theorem, T'' (and hence T) is bounded below. This implies that T is injective and that $T(X)$ is closed in X. On the other hand, $T(X)$ is dense in X because, if f is in X' with $f(T(X)) = 0$, then $T'(f)(X) = 0$, so $T'(f) = 0$, hence $f = 0$ (by the injectivity of T'), and the Hahn–Banach theorem applies. It follows that T is bijective. □

The next corollary shows that, concerning spectrum, weakly compact operators on a (possibly non-complete) normed space become like bounded linear operators on a Banach space.

Corollary 1.4.25 *Let X be a nonzero normed space over $\mathbb{K}$, and let T be in $\mathfrak{W}(X)$. Then*

$$\mathrm{sp}(BL(X), T) = \mathrm{sp}(T) = \mathrm{sp}(T').$$

As a consequence, $\mathrm{sp}(T)$ *is a compact subset of* $\mathbb{K}$*, and is non-empty whenever* $\mathbb{K} = \mathbb{C}$.

Proof In view of Fact 1.4.24, to prove the equality $\mathrm{sp}(T) = \mathrm{sp}(T')$ it is enough to show that $\mathrm{sp}(T) = \mathrm{sp}(T'')$. The equality $\mathrm{sp}(T) \setminus \{0\} = \mathrm{sp}(T'') \setminus \{0\}$ follows from Proposition 1.4.19(i)–(ii). On the other hand, in view of the inclusion $T''(X'') \subseteq X$, we have that $0 \notin \mathrm{sp}(T'')$ if and only if X is reflexive and $0 \notin \mathrm{sp}(T)$. But Corollary 1.4.21(i) asserts that X is reflexive whenever $0 \notin \mathrm{sp}(T)$.

Now we proceed to prove that $\mathrm{sp}(BL(X), T) = \mathrm{sp}(T)$. The equality

$$\mathrm{sp}(BL(X), T) \setminus \{0\} = \mathrm{sp}(T) \setminus \{0\}$$

follows from Proposition 1.4.19(iii). On the other hand, $0 \notin \mathrm{sp}(BL(X), T)$ if and only if $0 \notin \mathrm{sp}(T)$ and T^{-1} is continuous. But, if $0 \notin \mathrm{sp}(T)$, then, as a by-product of Corollary 1.4.21(i), X is a Banach space, and hence T^{-1} is continuous.

The consequence follows from the equality $\mathrm{sp}(T) = \mathrm{sp}(T')$ already proved, by keeping in mind Proposition 1.1.40 and Theorem 1.1.41. □

For the notion of a full subalgebra of a unital associative algebra, the reader is referred to Definition 1.1.72.

Corollary 1.4.26 *Let X be a nonzero normed space over $\mathbb{K}$. Then both $\mathfrak{K}(X) + \mathbb{K}I_X$ and $\mathfrak{W}(X) + \mathbb{K}I_X$ are full subalgebras of the algebra of all (possibly discontinuous) linear operators on X.*

Proof Let B stand indistinctly for $\mathfrak{K}(X) + \mathbb{K}I_X$ or $\mathfrak{W}(X) + \mathbb{K}I_X$, and let $F \in B$ be a bijective operator. We must show that F^{-1} lies in B. Write $F = T + \lambda I_X$ with $T \in \mathfrak{W}(X)$ (occasionally, $T \in \mathfrak{K}(X)$) and $\lambda \in \mathbb{K}$. First assume that $\lambda \neq 0$. Then, by Proposition 1.4.19(iii), F^{-1} is continuous, and hence $T \circ F^{-1}$ is weakly compact (occasionally, compact). Therefore we have

$$F^{-1} = -\lambda^{-1} T \circ F^{-1} + \lambda^{-1} I_X \in B,$$

as desired. Now assume that $\lambda = 0$. Then, by Corollary 1.4.21, X is reflexive (occasionally, finite-dimensional), so that, clearly, F^{-1} is weakly compact (occasionally, compact), and hence belongs to B. □

The information about weakly compact operators stated in Proposition 1.4.19 and Corollary 1.4.25 can be complemented and refined in the particular case of compact operators. This is done in Theorem 1.4.27 and Corollary 1.4.29 below.

Theorem 1.4.27 *Let X be a nonzero normed space over $\mathbb{K}$, let T be in $\mathfrak{K}(X)$, and let λ be in $\mathbb{K} \setminus \{0\}$. We have:*

(i) $\ker(T - \lambda I_X)$ *is finite-dimensional.*
(ii) *If Y is any closed subspace of X, then $(T - \lambda I_X)(Y)$ is closed in X; in particular, $T - \lambda I_X$ has closed range.*
(iii) *$T - \lambda I_X$ is injective if and only if $T - \lambda I_X$ is surjective.*

Proof We may assume that $\lambda = 1$. Denote by N and R the kernel and the range, respectively, of the operator $T - I_X$.

Note that T becomes the identity on N, which implies (since N is closed in X and T is compact) that $\mathbb{B}_N$ is compact or, equivalently, that N is finite-dimensional. This proves assertion (i).

Now we proceed to prove assertion (ii). Let Y be a closed subspace of X. Suppose first that the restriction to Y of $T - I_X$ is injective. Let z be in $\overline{(T - I_X)(Y)}$, so that there is a sequence y_n in Y such that the sequence $(T - I_X)(y_n)$ converges to z. Suppose, to derive a contradiction, that $\|y_n\| \to \infty$. Then we may assume that $y_n \neq 0$ for every $n \in \mathbb{N}$, and we have $\frac{1}{\|y_n\|} \to 0$, so that $(T - I_X)(\frac{y_n}{\|y_n\|}) \to 0$. Since $\frac{y_n}{\|y_n\|}$ lies in $\mathbb{S}_X$ and $T \in \mathfrak{K}(X)$, by passing to a subsequence if necessary, we may assume that $T(\frac{y_n}{\|y_n\|})$

converges to some $x \in X$. Then

$$\frac{y_n}{\|y_n\|} = T\left(\frac{y_n}{\|y_n\|}\right) - (T - I_X)\left(\frac{y_n}{\|y_n\|}\right) \to x,$$

and hence $x \in \mathbb{S}_Y$. It follows from the continuity of T that $T(\frac{y_n}{\|y_n\|}) \to T(x)$, hence $T(x) = x$, and so $x \in \ker(T - I_X)_{|Y}$, which contradicts that $(T - I_X)_{|Y}$ is injective. Thus $\|y_n\| \nrightarrow \infty$, and hence, passing again to a suitable subsequence, we may assume that the sequence $\|y_n\|$ is bounded. By the compactness of T, we may assume in addition that $T(y_n)$ converges to some $u \in X$. Then, we see that

$$y_n = T(y_n) - (T - I_X)(y_n) \to u - z,$$

hence $u - z \in Y$, since Y is closed, and $(T - I_X)(y_n) \to (T - I_X)(u - z)$, since $T - I_X$ is continuous. Therefore $z = (T - I_X)(u - z) \in (T - I_X)(Y)$. It follows from the arbitrariness of z in $\overline{(T - I_X)(Y)}$ that $(T - I_X)(Y)$ is closed in X. Now, remove the additional assumption that $(T - I_X)_{|Y}$ is injective, and set $Z := N \cap Y$. It follows from assertion (i) and Fact 1.4.14 that $Y = Z \oplus M$ for some closed subspace M of Y. Note that M is a closed subspace of X and $(T - I_X)_{|M}$ is a injective. Therefore, by the previously studied case, $(T - I_X)(M)$ is a closed subspace of X. By noticing that $(T - I_X)(Y) = (T - I_X)(M)$, we conclude that $(T - I_X)(Y)$ is a closed subspace of X.

Finally, we proceed to prove assertion (iii). Suppose that $N = 0$ and $R_1 := R \neq X$. Then, according to assertion (ii), $R_2 := (T - I_X)(R_1)$ is a closed subspace of R_1 and $R_2 \neq R_1$, since $T - I_X$ is injective. In this way, by denoting $R_n := (T - I_X)^n(X)$, we obtain a strictly decreasing sequence of closed subspaces. As a consequence of the Riesz lemma, we choose $u_n \in R_n$ such that $d(u_n, R_{n+1}) \geqslant \frac{1}{2}$ and $\|u_n\| = 1$. Then, if $p > q$,

$$T(u_p) - T(u_q) = (T - I_X)(u_p) - (T - I_X)(u_q) + u_p - u_q = z - u_q$$

with $z \in R_{p+1} + R_{q+1} + R_p \subseteq R_{q+1}$, so that $\|T(u_p) - T(u_q)\| \geqslant \frac{1}{2}$, which is impossible, since T is compact. This shows that $R = X$ if $N = 0$. Conversely, suppose that $R = X$ and $N_1 := N \neq 0$. Then $N_2 := \ker(T - I_X)^2$ is a closed subspace of X and $N_2 \neq N_1$, since $T - I_X$ is surjective. In this way, by denoting $N_n := \ker(T - I_X)^n$, we obtain a strictly increasing sequence of closed subspaces. As a consequence of the Riesz lemma, we choose $v_n \in N_{n+1}$ such that $d(v_n, N_n) \geqslant \frac{1}{2}$ and $\|v_n\| = 1$. Then, if $p < q$,

$$T(v_p) - T(v_q) = (T - I_X)(v_p) - (T - I_X)(v_q) + v_p - v_q = z - v_q$$

with $z \in N_p + N_q + N_{p+1} \subseteq N_q$, so that $\|T(v_p) - T(v_q)\| \geqslant \frac{1}{2}$, which is impossible, since T is compact. This shows that $N = 0$ if $R = X$. $\square$

Let X be a vector space over $\mathbb{K}$, and let $T : X \to X$ be a linear mapping. A number $\lambda \in \mathbb{K}$ is said to be an *eigenvalue* of T if $T - \lambda I_X$ is not injective. If λ is an eigenvalue of T, then the elements of $\ker(T - \lambda I_X)$ are called the *eigenvectors* of T corresponding to λ.

Lemma 1.4.28 *Let X be a vector space over $\mathbb{K}$, and let $T : X \to X$ be a linear mapping. If $\lambda_1, \ldots, \lambda_n$ are pairwise different eigenvalues of T, and if, for $1 \leqslant i \leqslant n$,*

x_i is a nonzero eigenvector of T corresponding to λ_i, then the vectors $x_1,\dots,x_n$ are linearly independent.

Proof We proceed by induction. The result is clearly true if $n=1$; assume the result holds for $n-1$, let $\lambda_1,\dots,\lambda_n$ be pairwise different eigenvalues of T, and, for $1\leqslant i\leqslant n$, let x_i be a nonzero eigenvector of T corresponding to λ_i. If $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ satisfy that $\sum_{i=1}^n \alpha_i x_i=0$, then we have

$$0=(T-\lambda_n I_X)\left(\sum_{i=1}^n \alpha_i x_i\right)=\sum_{i=1}^{n-1}\alpha_i(\lambda_i-\lambda_n)x_i,$$

therefore $\alpha_i=0$ $(1\leqslant i\leqslant n-1)$, hence $\alpha_n x_n=0$, and so we also have $\alpha_n=0$. Thus $x_1,\dots,x_n$ are linearly independent vectors. □

Corollary 1.4.29 *Let X be a nonzero normed space over $\mathbb{K}$, and let T be in $\mathfrak{K}(X)$. Then we have:*

(i) $\mathrm{sp}(T)\setminus\{0\}$ *coincides with the set of all nonzero eigenvalues of T.*
(ii) $\mathrm{sp}(T)$ *is either finite (possibly empty) or countably infinite.*
(iii) *if* $\mathrm{sp}(T)$ *is empty, then $\mathbb{K}=\mathbb{R}$ and X is finite-dimensional.*
(iv) *If* $\mathrm{sp}(T)$ *is countably infinite, then $0\in\mathrm{sp}(T)$ and, for each bijection $n\to\lambda_n$ from $\mathbb{N}$ to* $\mathrm{sp}(T)\setminus\{0\}$, *the sequence λ_n converges to zero.*

Proof Assertion (i) follows from Theorem 1.4.27(iii), whereas assertion (iii) follows from Corollaries 1.4.21(ii) and 1.4.25.

In order to prove assertions (ii) and (iv) we claim that, for each $\delta\in\mathbb{R}^+$, the set

$$\Lambda_\delta:=\{\lambda\in\mathbb{K}:\lambda \text{ is an eigenvalue of } T \text{ with } |\lambda|>\delta\}$$

is finite. To derive a contradiction, suppose the existence of $\delta_0\in\mathbb{R}^+$ such that the set Λ_{δ_0} is infinite. There is then a sequence λ_n of pairwise different elements in Λ_{δ_0}. For each natural number n, choose a nonzero eigenvector x_n of T corresponding to λ_n, and consider the subspace M_n of X generated by $x_1,\dots,x_n$. By Lemma 1.4.28, M_n becomes a strictly increasing sequence of finite-dimensional subspaces which are invariant under T. As a consequence of the Riesz lemma, we choose $y_n\in M_{n+1}$ such that $d(y_n,M_n)\geqslant\frac{1}{2}$ and $\|y_n\|=1$. Let p,q be in $\mathbb{N}$. Then

$$T(y_p)-T(y_q)=\lambda_{p+1}y_p+(T-\lambda_{p+1}I_X)(y_p)-T(y_q)=\lambda_{p+1}(y_p+z),$$

where

$$z:=\lambda_{p+1}^{-1}[(T-\lambda_{p+1}I_X)(y_p)-T(y_q)].$$

Note that $y_p=\sum_{i=1}^{p+1}\alpha_i x_i$ for suitable $\alpha_1,\dots,\alpha_{p+1}\in\mathbb{K}$, and consequently

$$(T-\lambda_{p+1}I_X)(y_p)=\sum_{i=1}^{p}\alpha_i(\lambda_i-\lambda_{p+1})x_i\in M_p.$$

Since $T(y_q)\in M_{q+1}$, it follows that, for $q<p$, we have $z\in M_p$, and hence

$$\|T(y_p)-T(y_q)\|=|\lambda_{p+1}|\|y_p+z\|>\frac{1}{2}\delta_0,$$

which is impossible, since T is compact. Now that the claim is proved, keeping in mind that $\mathrm{sp}(T)\setminus\{0\}=\cup_{n\in\mathbb{N}}\Lambda_{\frac{1}{n}}$ (by assertion (i)), assertion (ii) follows. Assume from now on that $\mathrm{sp}(T)$ is countably infinite. Let $n\to\lambda_n$ be any bijection from $\mathbb{N}$ to $\mathrm{sp}(T)\setminus\{0\}$. Then, according to assertion (i) and the claim, the set $\{n\in\mathbb{N}:|\lambda_n|>\varepsilon\}$ is finite for every $\varepsilon>0$, and so the sequence λ_n converges to zero. Moreover, since $\mathrm{sp}(T)$ is closed in $\mathbb{K}$ (by Corollary 1.4.25), we derive that $0\in\mathrm{sp}(T)$, and the proof is complete. □

The next exercise complements assertion (iv) in Corollary 1.4.29 above.

Exercise 1.4.30 Let λ_n be any sequence in $\mathbb{K}\setminus\{0\}$ converging to zero, and let H stand for the infinite-dimensional separable Hilbert space over $\mathbb{K}$. Prove that there exists an injective compact operator $T:H\to H$ such that

$$\mathrm{sp}(T)=\{0\}\cup\{\lambda_n:n\in\mathbb{N}\}.$$

Solution We may take $H=\ell_2$. Then, according to the solution of Exercise 1.1.45, the mapping $T:H\to H$ defined by $T(\{\mu_n\}):=\{\lambda_n\mu_n\}$ becomes a bounded linear operator on H such that $\mathrm{sp}(T)=\{0\}\cup\{\lambda_n:n\in\mathbb{N}\}$. Moreover, clearly, T is injective. For $n,m\in\mathbb{N}$, define $\rho_{nm}\in\mathbb{K}$ by $\rho_{nm}=\lambda_n$ if $n\leqslant m$, and $\rho_{nm}=0$ otherwise, and let T_m stand for the finite-rank operator on H given by $T_m(\{\mu_n\}):=\{\rho_{nm}\mu_n\}$. Then we easily realize that, for $m\in\mathbb{N}$, we have $\|T-T_m\|=\max\{|\lambda_k|:k\geqslant m+1\}$. Since the right-hand side of this equality tends to zero as $m\to\infty$, it follows from Corollary 1.4.15 that T is a compact operator. □

Definition 1.4.31 A normed algebra A is said to be *topologically simple* if A has nonzero product and has no nonzero closed proper ideal.

Proposition 1.4.32 *Let X be a nonzero normed space over $\mathbb{K}$, and let A be a subalgebra of $BL(X)$ containing $\mathfrak{F}(X)$. Then $\mathfrak{F}(X)$ is contained in any nonzero ideal of A. As a consequence, $\mathfrak{F}(X)$ is a simple algebra, and the closure in $BL(X)$ of $\mathfrak{F}(X)$ is a topologically simple normed algebra.*

Proof Let I be a nonzero ideal of A. Take $F\in I$ and $x_0\in X$ such that $F(x_0)\neq 0$, and invoke the Hahn–Banach theorem to find $f_0\in X'$ with $f_0(F(x_0))=1$. Then, for every $(x,f)\in X\times X'$ we have

$$x\otimes f=(x\otimes f_0)\circ F\circ(x_0\otimes f)\in I,$$

so that it is enough to apply Fact 1.4.13 to conclude that $\mathfrak{F}(X)\subseteq I$. By taking A equal to $\mathfrak{F}(X)$ or $\overline{\mathfrak{F}(X)}$, the consequence follows straightforwardly. □

By combining Corollary 1.4.11 and Propositions 1.4.18 and 1.4.32, we get the following.

Corollary 1.4.33 *Let $X\neq 0$ be a Hilbert space, or a Banach space having a Schauder basis. Then $\mathfrak{K}(X)$ is a topologically simple complete normed algebra, as well as the smallest nonzero closed proper ideal of $BL(X)$.*

1.4.3 Discussing the inclusion $\overline{\mathfrak{F}(X,Y)} \subseteq \mathfrak{K}(X,Y)$ in the non-complete setting

In this subsection, we are going to show that the completeness of the space Y in Corollaries 1.4.11 and 1.4.15 is very far from being removable. To this end, we begin by realizing that the limit of a sequence of finite-rank operators need not be a compact operator, even if the operator starts from a Banach space and its range lives in a pre-Hilbert space. Indeed, we have the following.

Example 1.4.34 Set $X := c_0$, and let $T : X \to \ell_2$ stand for the bounded linear operator defined by $T(\{\mu_n\}) := \{\frac{1}{n}\mu_n\}$ for every $\{\mu_n\} \in X$. For natural numbers n,m, define $\rho_{nm} \in \mathbb{K}$ by $\rho_{nm} = \frac{1}{n}$ if $n \leqslant m$, and $\rho_{nm} = 0$ otherwise, and let $T_m : X \to \ell_2$ stand for the finite-rank operator given by $T_m(\{\mu_n\}) := \{\rho_{nm}\mu_n\}$ for every $\{\mu_n\} \in X$. Then we easily realize that, for $m \in \mathbb{N}$, we have

$$\|T - T_m\| \leqslant \sqrt{\sum_{k=m+1}^{\infty} \frac{1}{k^2}},$$

so that T is the limit in $BL(X,\ell_2)$ of the sequence T_m. Now set $Y := T(X)$, and note that $T_m(X) \subseteq Y$ for every $m \in \mathbb{N}$. It follows that T, regarded as a mapping from X to Y, lies in $\overline{\mathfrak{F}(X,Y)}$. However T, regarded as a mapping from X to Y, is not a compact operator. Indeed, for $m \in \mathbb{N}$, let x_m be the element of $\mathbb{B}_X$ having 1 in its first m entries and 0 otherwise. If T, regarded as a mapping from X to Y, were compact, then the sequence $T(x_m)$ would have a subsequence converging to some $y \in Y$. But, since the sequence $T(x_m)$ converges to $\{\frac{1}{n}\}$ in ℓ_2, we would have $\{\frac{1}{n}\} = y \in Y = T(X)$, and this would imply the existence of some $\{\mu_n\} \in X = c_0$ with $\frac{1}{n}\mu_n = \frac{1}{n}$ for every $n \in \mathbb{N}$, which is impossible.

The above example shows that Corollary 1.4.15, and consequently the compact version of Corollary 1.4.11, do not remain true if the completeness of the space Y is removed. Indeed, if X,Y,T are as in Example 1.4.34, then T lies in $\overline{\mathfrak{F}(X,Y)}$, and hence in the closure of $\mathfrak{K}(X,Y)$ in $BL(X,Y)$, but $T \notin \mathfrak{K}(X,Y)$.

Given normed spaces X and Y over $\mathbb{K}$, we denote by $\overline{\mathfrak{K}(X,Y)}$ (respectively, $\overline{\mathfrak{W}(X,Y)}$) the closure in $BL(X,Y)$ of $\mathfrak{K}(X,Y)$ (respectively, $\mathfrak{W}(X,Y)$). The inclusion $\overline{\mathfrak{K}(X,Y)} \subseteq \overline{\mathfrak{W}(X,Y)}$ being clear, we have also the following.

Fact 1.4.35 *Let X and Y be normed spaces over $\mathbb{K}$. We have:*

(i) *If T is in $\overline{\mathfrak{K}(X,Y)}$, then the restriction of T'' to $\mathbb{B}_{X''}$ is weak*-to-norm continuous, and hence $T'' \in \mathfrak{K}(X'',Y'')$.*
(ii) $\overline{\mathfrak{K}(X,Y)} \setminus \mathfrak{K}(X,Y) \subseteq \overline{\mathfrak{W}(X,Y)} \setminus \mathfrak{W}(X,Y)$, *and hence $\mathfrak{K}(X,Y)$ is closed in $BL(X,Y)$ whenever so is $\mathfrak{W}(X,Y)$.*

Proof Set $S := \{T \in BL(X,Y) : (T'')_{|\mathbb{B}_{X''}}$ is weak*-to-norm continuous$\}$ and note that S is closed in $BL(X,Y)$ and that, by Proposition 1.4.6 and the equivalence (i)$\Leftrightarrow$(ii) in Theorem 1.4.8, we have $\mathfrak{K}(X,Y) = S \cap \mathfrak{W}(X,Y)$. □

By combining Example 1.4.34 and Fact 1.4.35, we realize that both Corollary 1.4.10 and the weakly compact version of Corollary 1.4.15 do not remain true if the completeness of the space Y is removed. Indeed, if X,Y,T are as in Example 1.4.34, then

$$T \notin \mathfrak{K}(X,Y), \;\; T'' \in \mathfrak{K}(X'',Y''), \;\text{ and }\; T \in \overline{\mathfrak{W}(X,Y)} \setminus \mathfrak{W}(X,Y).$$

Our discussion centres now in showing that Example 1.4.34 is not anecdotic. Indeed, as a consequence of Theorem 1.4.39 and Remark 1.4.40 below, given an arbitrary non-reflexive Banach space X, we can find a pre-Hilbert space Y and a non-compact operator $T \in \overline{\mathfrak{F}(X,Y)}$. Of course, the construction is more involved, and begins with the following refinement of the part of Proposition 1.4.18 involving Hilbert spaces.

Proposition 1.4.36 *Let X be a normed space over $\mathbb{K}$, let H be a Hilbert space over $\mathbb{K}$, and let T be in $\mathfrak{K}(X,H)$. Then T, regarded as an operator from X to $T(X)$, lies in $\overline{\mathfrak{F}(X,T(X))}$.*

Proof We may assume that $T \notin \mathfrak{F}(X,H)$. Set $Z := T(X)$. Then, by Exercise 1.4.17, Z is an infinite-dimensional separable pre-Hilbert space, and hence has a countably infinite orthonormal basis (say $\{z_n : n \in \mathbb{N}\}$). For each $n \in \mathbb{N}$, let π_n stand for the orthogonal projection from H onto the linear hull of $\{z_1,\ldots,z_n\}$, and set $S_n := \pi_n \circ T$. Then for $n \in \mathbb{N}$ we have $S_n(X) \subseteq Z$, and hence S_n will be seen as an element of $\mathfrak{F}(X,Z)$. Moreover, since $\{z_n : n \in \mathbb{N}\}$ remains an orthonormal basis for the closure of Z in H (say $\bar{Z}$), for each $y \in \bar{Z}$ we have

$$(I_H - \pi_n)(y) = \sum_{i=1}^{\infty}(y|z_i)\,z_i - \sum_{i=1}^{n}(y|z_i)\,z_i = \sum_{i=n+1}^{\infty}(y|z_i)\,z_i \to 0$$

as $n \to \infty$. Therefore, by Proposition 1.4.16, the sequence $I_H - \pi_n$ converges to zero uniformly on compact subsets of $\bar{Z}$, so in particular on the closure of $T(\mathbb{B}_X)$ in H. Noticing that for $x \in \mathbb{B}_X$ and $n \in \mathbb{N}$ we have

$$\|T(x) - S_n(x)\| = \|T(x) - \pi_n(T(x))\| = \|(I_H - \pi_n)(T(x))\|,$$

it follows that S_n converges to T (regarded as an operator from X to Z) in $BL(X,Z)$. □

Lemma 1.4.37 *Let X and Y be normed spaces over $\mathbb{K}$, let T be in $BL(X,Y)$, and let Z be a subspace of Y containing $T(X)$. Then the following conditions are equivalent:*

(i) *T, regarded as a mapping from X to Z, is weakly compact.*
(ii) *$T''(X'') \subseteq Z$.*

Proof Let F stand for T regarded as a mapping from X to Z, and let i denote the inclusion $Z \hookrightarrow Y$. Then we have $T = i \circ F$ and $i'' = j \circ \Phi$, where $Z^{\circ\circ}$ denotes the bipolar of Z in Y'', $\Phi : Z'' \to Z^{\circ\circ}$ is the natural identification, and j stands for the inclusion $Z^{\circ\circ} \hookrightarrow Y''$. Therefore $T'' = j \circ \Phi \circ F''$. On the other hand, by Proposition 1.4.6, condition (i) is equivalent to $F''(X'') \subseteq Z$. Since both Z'' and $Z^{\circ\circ}$ contain Z in a natural way, and Φ becomes the identity on Z, it follows that condition (i) is equivalent to condition (ii). □

Proposition 1.4.38 *Let X be a non-reflexive Banach space over $\mathbb{K}$, and let Y be an infinite-dimensional Banach space over $\mathbb{K}$. Then there exists $T \in \overline{\mathfrak{F}(X,Y)}$ such that T, regarded as an operator from X to $T(X)$, is not compact.*

Proof Since X is non-reflexive, neither is X', and hence $\mathbb{B}_{X'}$ is not weakly compact. By the Eberlein–Šmulyan Theorem (see for example [729, Theorem 3.109]), there exists a sequence z'_n in $\mathbb{B}_{X'}$ with no cluster point in the weak topology of X'. But certainly z'_n has a cluster point (say z') in the weak* topology of X'. Therefore $x'_n := z'_n - z$ is a bounded sequence in X' which has zero as a cluster point in the weak* topology but not in the weak topology. Since zero is not a cluster point to x'_n in the weak topology, there exist $x''_1, \ldots, x''_m \in X''$ and $p \in \mathbb{N}$ such that

$$\max\left\{|x''_k(x'_n)| : 1 \leqslant k \leqslant m\right\} \geqslant 1 \ \text{ for every } \ n \geqslant p. \tag{1.4.5}$$

Now, take a normalized basic sequence $(y_n)_{n\in\mathbb{N}}$ in Y, and consider the operator $T \in \overline{\mathfrak{F}(X,Y)}$ defined by

$$T(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} x'_n(x) y_n.$$

To obtain a contradiction, suppose that T, regarded as an operator from X onto $T(X)$, is compact. Then, by Lemma 1.4.37, $T''(X'') \subseteq T(X)$, hence for each $k \in \{1, \ldots, m\}$ there exists $x_k \in X$ such that $T''(x''_k) = T(x_k)$, and so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x''_k(x'_n) y_n = \sum_{n=1}^{\infty} \frac{1}{n^2} x'_n(x_k) y_n.$$

Therefore $x''_k(x'_n) = x'_n(x_k)$ for all $k \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$, and consequently, by (1.4.5), $\max\{|x'_n(x_k)| : 1 \leqslant k \leqslant m\} \geqslant 1$ for all $n \geqslant p$, which is impossible because zero is a cluster point to x'_n in the weak* topology of X'. □

Theorem 1.4.39 *Let X be a Banach space over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *X is reflexive.*
(ii) *$\mathfrak{W}(X,Y) = BL(X,Y)$ for every normed space Y over $\mathbb{K}$.*
(iii) *$\mathfrak{W}(X,Y)$ is closed in $BL(X,Y)$ for every normed space Y over $\mathbb{K}$.*
(iv) *$\mathfrak{K}(X,Y)$ is closed in $BL(X,Y)$ for every normed space Y over $\mathbb{K}$.*
(v) *$\overline{\mathfrak{F}(X,Y)} \subseteq \mathfrak{K}(X,Y)$ for every normed space Y over $\mathbb{K}$.*

Proof The implications (i)⇒(ii)⇒(iii), and (iv)⇒(v) are clear, whereas the implication (iii)⇒(iv) follows from Fact 1.4.35(ii).

(v)⇒(i) Assume that X is not reflexive. Then, by Proposition 1.4.38, there exists $T \in \mathfrak{K}(X, \ell_2)$ such that T, regarded as an operator from X to $Y := T(X)$, does not belong to $\mathfrak{K}(X,Y)$. But, by Proposition 1.4.36, T, regarded as an operator from X to Y, lies in $\overline{\mathfrak{F}(X,Y)}$. □

Remark 1.4.40 Looking at the above proof, we realize that Theorem 1.4.39 remains true if in conditions (ii) to (v) we replace 'for every normed space Y over $\mathbb{K}$' with 'for every separable pre-Hilbert space Y over $\mathbb{K}$'.

Corollary 1.4.21(ii) could be a sample of how results concerning compact operators between normed spaces X and Y, which are true when Y is complete but need not be true in general (cf. Corollary 1.4.3 and Proposition 1.4.4), are indeed true when $X = Y$. Therefore, thinking about Corollary 1.4.15 and Example 1.4.34, one can wonder whether the inclusion $\overline{\mathfrak{F}(X)} \subseteq \mathfrak{K}(X)$ holds for every normed space X. As a matter of fact, the answer to this question is negative, even if X is a pre-Hilbert space. Indeed, we have the following.

Example 1.4.41 Set $X := c_{00}$ regarded as a subspace of ℓ_2, and consider the bounded linear operator $T : X \to X$ defined by $T(\{\mu_n\}) := \{\frac{1}{n}\mu_n\}$ for every $\{\mu_n\} \in X$. For $n, m \in \mathbb{N}$, define $\rho_{nm} \in \mathbb{K}$ by $\rho_{nm} = \frac{1}{n}$ if $n \leqslant m$, and $\rho_{nm} = 0$ otherwise, and let $T_m : X \to X$ stand for the finite-rank operator given by $T_m(\{\mu_n\}) := \{\rho_{nm}\mu_n\}$ for every $\{\mu_n\} \in X$. Then for $m \in \mathbb{N}$ we have $\|T - T_m\| = \frac{1}{m+1}$, so that T is the limit in $BL(X)$ of the sequence T_m, and hence $T \in \overline{\mathfrak{F}(X)}$. However $T \notin \mathfrak{K}(X)$ because T is bijective and Corollary 1.4.21(ii) applies. (An elementary verification of the non-compactness of T, avoiding Corollary 1.4.21(ii), is left to the reader.)

Let X and $T \in \overline{\mathfrak{F}(X)} \setminus \mathfrak{K}(X)$ be as in the above example. Then, clearly, we have $T \in \overline{\mathfrak{K}(X)} \setminus \mathfrak{K}(X)$. Moreover, invoking Fact 1.4.35, we realize that $T'' \in \mathfrak{K}(X'')$ and that $T \in \overline{\mathfrak{W}(X)} \setminus \mathfrak{W}(X)$. Therefore, in the absence of completeness of the space Y, neither Corollary 1.4.10 nor Corollary 1.4.11 remain true even if $X = Y$.

1.4.4 Historical notes and comments

According to Taylor and Lay [811, p. 293],

> The essential notion of a compact mapping was introduced by David Hilbert in 1906; eleven years later F. Riesz [505] published a thorough study of compact operators. In 1930, J. Schauder [555] added some refinements to the theory by proving that the conjugate of a compact operator is itself compact.

For a much more detailed history of the so-called *Riesz–Schauder theory* for compact operators, the reader is referred to Pietsch's book [790, Section 2.6]. We completely agree with Pietsch's words in this section that 'The theory of compact operators is a convincing example that deep and important mathematics can be – or should I say *must be* – elegant'.

The theory of compact operators is often developed in the setting of possibly non-complete normed spaces (the books [805, 811] are samples of such an approach). This is so because most important results in that theory can be proved without requiring completeness, and such results become widely useful in the study of relevant integral equations of mathematical physics and theoretical mechanics, whose solutions are modelled through non-complete normed spaces (see [811, p. 306] for details). Thus for example, Schauder's theorem, quoted in the above Taylor–Lay review, does not require completeness. We will not discuss the proof of Schauder's theorem here, but include it as the first assertion in Theorem 1.4.42 below.

Theorem 1.4.42 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X)$. We have:*

(i) *If T is compact, then so is T'.*
(ii) *If Y is complete and if T' is compact, then T is compact.*

Proof For the proof of assertion (i), the reader is referred to [811, Theorem V.7.3].

Assertion (ii) follows from the implication (iii)⇒(i) in Corollary 1.4.10 by applying assertion (i) with T' instead of T. □

To realize that the completeness of Y in assertion (ii) of the above theorem cannot be removed altogether, Fact 1.4.43 immediately below will be useful. Given normed spaces X,Y,Z over $\mathbb{K}$, and $T \in BL(X,Y)$, we say that T *factors through* Z if there exist $F \in BL(X,Z)$ and $G \in BL(Z,Y)$ such that $T = G \circ F$.

Fact 1.4.43 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. We have:*

(i) *T lies in $\mathfrak{F}(X,Y)$ if and only if it factors through a finite-dimensional space over $\mathbb{K}$.*
(ii) *If T belongs to $\mathfrak{F}(X,Y)$, then T' lies in $\mathfrak{F}(Y',X')$.*
(iii) *If T belongs to $\overline{\mathfrak{F}(X,Y)}$, then T' lies in $\overline{\mathfrak{F}(Y',X')}$.*
(iv) *If T belongs to $\overline{\mathfrak{F}(X,Y)}$, then T' lies in $\mathfrak{K}(Y',X')$.*

Proof Assertion (i) is straightforward.

Assume that $T \in \mathfrak{F}(X,Y)$. Then, by assertion (i), there is a finite-dimensional space Z over $\mathbb{K}$, together with $F \in BL(X,Z)$ and $G \in BL(Z,Y)$, such that $T = G \circ F$. Therefore $T' = F' \circ G'$ factors through the finite-dimensional space Z', and lies indeed in $\mathfrak{F}(Y',X')$.

Keeping in mind that the mapping $F \to F'$ from $BL(X,Y)$ to $BL(Y',X')$ is continuous, assertion (iii) follows from assertion (ii).

Assertion (iv) follows from assertion (iii) and Corollary 1.4.15. □

Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. According to Examples 1.4.34 and 1.4.41 and Fact 1.4.43(iv), there are choices of X,Y,T such that T' is compact but T is not, and either X is complete or $X = Y$. Moreover, invoking Theorem 1.4.42(i), Fact 1.4.43(iv), and the implication (v)⇒(i) in Theorem 1.4.39, we get the following.

Corollary 1.4.44 *A Banach space X over $\mathbb{K}$ is reflexive if and only if, for every normed space Y over $\mathbb{K}$ and every operator $T \in BL(X,Y)$ such that $T' \in \mathfrak{K}(Y',X')$, we have $T \in \mathfrak{K}(X,Y)$.*

According to [790, Subsection 4.8.5], weakly compact (referred to as *weakly completely continuous*) operators on Banach spaces were introduced by Kakutani and Yosida for the purpose of the so-called ergodic theory (see [790, §5.3.5.4] and references therein).

Our starting approach to compact operators through weakly compact ones between (possibly non-complete) normed spaces follows the lines of the Mena and Rodríguez paper [441]. This paper was motivated by Spurný's recent theorem [596] (stated in Corollary 1.4.21(ii)) that *compact operators from an infinite-dimensional normed space to itself cannot be surjective.* We note that Spurný's theorem implies that zero belongs to the 'algebraic' spectrum (cf. §1.4.23) of any compact operator on an infinite-dimensional normed space, a fact that concludes the classical spectral theory of compact operators in the non-complete setting. Concerning classical results, we have taken guidance from the books of Cerdà [706], Conway [711], Dunford and Schwartz [726], Hille and Phillips [746], Rynne and Youngson [805], and Taylor and Lay [811].

Results from Proposition 1.4.2 to Proposition 1.4.5 are taken almost verbatim from [441]. Some of them are classical. Thus, for example, the compact version of Corollary 1.4.3 can be found in [811, Theorem V.7.4]. The existence of bijective compact operators between infinite-dimensional normed spaces is implicitly known in the classical literature. (Indeed, according to §1.4.9, every compact non-finite-rank operator starting from a reflexive space gives rise naturally to a bijective compact operator between infinite-dimensional spaces.) Nevertheless, we did not seen this fact emphasized until Spurný's paper [596]. Although strangely introduced, the starting space in Spurný's example is (isometrically isomorphic to) ℓ_2. Proposition 1.4.4 provides us with many more similar examples, where again the starting space is complete. Note that, if every surjective compact operator had to start from a Banach space, then Spurný's theorem reviewed in the preceding paragraph would follow straightforwardly from the compact version of Corollary 1.4.3. As a matter of fact, adding Proposition 1.4.5 to Proposition 1.4.4, we are provided with bijective compact operators starting from a non-complete space.

According to [726, p. 539], in the complete normed setting, Proposition 1.4.6 is due to Nakamura [458]. Nevertheless, its proof, which can be seen for example in [726, Theorem VI.4.2], requires no changes when the non-complete normed case is considered.

A Schauder type theorem also holds for weakly compact operators. Indeed, we have the following.

Theorem 1.4.45 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. We have*

(i) *If T is weakly compact, then T' is weak*-to-weak continuous.*
(ii) *If T' is weak*-to-weak continuous, then T' is weakly compact.*
(iii) *If Y is complete, and if T' is weakly compact, then T is weakly compact.*

Proof Assume that T is weakly compact, let y'_λ be a net in Y' convergent to zero in the weak*-topology of Y', and let x'' be in X''. In order to prove assertion (i), we must show that $x''(T'(y'_\lambda)) \to 0$. But, in view of Proposition 1.4.6, we have $y := T''(x'') \in Y$, hence

$$x''(T'(y'_\lambda)) = T''(x'')(y'_\lambda) = y'_\lambda(y) \to 0,$$

as desired.

Since $\mathbb{B}_{Y'}$ is weak*-compact, assertion (ii) becomes clear.

Finally assume that T' is weakly compact. Then, by assertions (i) and (ii) (with T' instead of T), T'' is weakly compact, so that, if in addition Y is complete, the weak compactness of T follows from Remark 1.4.7. □

Let X,Y, and T be as in the above theorem. According to [726, p. 539], the consequence that, *when Y is complete, T is weakly compact if and only if so is T'* (respectively, that, *when Y is complete, T is weakly compact if and only if T' is weak*-to-weak continuous*) is due to Gantmacher [279] (respectively, to Nakamura [458]).

A later characterization of weakly compact operators, due to Davis, Figiel, Johnson, and Pelczyński [209] (see also [711, Theorem VI.5.4] or [729, Theorem 13.33]), reads as follows.

Theorem 1.4.46 *Let X and Y be Banach spaces over $\mathbb{K}$, and let $T : X \to Y$ be a bounded linear operator. Then T is a weakly compact operator (if and) only if it factors through a reflexive Banach space over $\mathbb{K}$.*

We will not discuss the proof of the above theorem here, but note that it remains true in the non-complete setting. Indeed, if X and Y are normed spaces over $\mathbb{K}$, and if $T : X \to Y$ is a weakly compact operator, then it follows from Proposition 1.4.6 and Remark 1.4.7 that $T''(X'') \subseteq Y$ and that T'' is weakly compact, so that, by considering the inclusion mapping $X \hookrightarrow X''$, and applying Theorem 1.4.46 (with X'', Y'', T'' instead of X, Y, T), we realize that T factors through a reflexive Banach space.

Results from Theorem 1.4.8 to Corollary 1.4.15 are classical, although the equivalence (i)$\Leftrightarrow$(ii) in Theorem 1.4.8 could not be explicitly recorded anywhere. Actually, Schauder's theorem, together with arguments like those in the proof of the implication (i)$\Rightarrow$(ii) in Theorem 1.4.8, allow us to prove the following refinement of Corollary 1.4.10.

Theorem 1.4.47 *Let X and Y be normed spaces over $\mathbb{K}$, and let T be in $BL(X,Y)$. We have:*

(i) *If T is compact, then the restriction of T' to $\mathbb{B}_{Y'}$ is weak*-to-norm continuous.*
(ii) *If Y is complete, and if the restriction of T' to $\mathbb{B}_{Y'}$ is weak*-to-norm continuous, then T is compact.*

Proof Assume that T is compact. Then, by Schauder's theorem (Theorem 1.4.42(i)), T' is compact. Let C stand for the norm-closure of $T'(\mathbb{B}_{Y'})$ in X'. Since C is norm-compact, and the restriction to C of the weak* topology of X' is Hausdorff and weaker than the norm topology, we derive that the norm and weak* topologies of X' coincide on C. Since T' is weak*-to-weak* continuous and $T'(\mathbb{B}_{Y'}) \subseteq C$, it follows that the restriction of T' to $\mathbb{B}_{Y'}$ is weak*-to-norm continuous.

Now assume that the restriction of T' to $\mathbb{B}_{Y'}$ is weak*-to-norm continuous. Then, since $\mathbb{B}_{Y'}$ is weak*-compact, $T'(\mathbb{B}_{Y'})$ is a norm-compact subset of X', and hence T' is a compact operator. Therefore, if in addition Y is complete, then, by the converse of Schauder's theorem (Theorem 1.4.42(ii)), T is compact. □

We have derived the above theorem from Schauder's theorem, but, conversely, Schauder's theorem follows straightforwardly from assertion (i) in the above theorem. Although straightforward, we emphasize the following.

Corollary 1.4.48 *Let X be a normed space over $\mathbb{K}$, let Y be a Banach space over $\mathbb{K}$, and let T be in $BL(X,Y)$. Then T is compact if and only if the restriction of T' to $\mathbb{B}_{Y'}$ is weak*-to-norm continuous.*

If the space Y above is separable, then $\mathbb{B}_{Y'}$ is metrizable, and hence we have the following.

Corollary 1.4.49 *Let X be a normed space over $\mathbb{K}$, let Y be a separable Banach space over $\mathbb{K}$, and let T be in $BL(X,Y)$. Then T is compact if and only if, whenever y'_n is a sequence in Y' converging to some $y' \in Y'$ in the weak*-topology of Y', the sequence $T'(y'_n)$ converges to $T'(y')$ in the norm topology of X'.*

Corollary 1.4.48 seems to be due to Dunford and Schwartz [726, Theorem VI.5.6]. According to them [726, p. 539], Corollary 1.4.49 had been proved earlier by Gelfand [283].

Results from Propositions 1.4.16–1.4.18 are classical. Thus, for example, Proposition 1.4.16 can be found in [729, Corollary 3.87].

The question of whether the equality $\overline{\mathfrak{F}(X,Y)} = \mathfrak{K}(X,Y)$ holds for all normed spaces X and all Banach spaces Y remained an open problem for many years. This question, known as *the approximation problem*, goes back to the Polish School in Lwów, and appeared in Banach's book in 1932 (see [685, pp. 146 and 179]). According to the comments by Pelczyński and Bessaga in [685, pp. 179–88], Banach, Mazur, and Schauder had already observed that the approximation problem was related to the problem of the existence of a basis, as well as to some questions regarding the approximation of continuous functions. The relation between the approximation problem and the problem of the existence of a basis becomes clear. Indeed, if the approximation problem had a negative answer with Y separable, then, by Proposition 1.4.18, there would exist a separable Banach space without a Schauder basis. Concerning the relation between the approximation problem and questions on the approximation of continuous functions, we follow Pietsch [790, p. 285] according to whom Mazur knew that a negative answer to the approximation problem would imply a negative response to the following question:

§**1.4.50** *Given a real-valued continuous function $f = f(s,t)$ defined on the square $[0,1] \times [0,1]$ and a number $\varepsilon > 0$, do there exist numbers $a_1,\dots,a_n$, $b_1,\dots,b_n$ in $[0,1]$ and $c_1,\dots,c_n$ in $\mathbb{R}$ with the property that*

$$\left| f(s,t) - \sum_{k=1}^{n} c_k f(a_k,t) f(s,b_k) \right| < \varepsilon$$

for all $s,t \in [0,1]$?

On November 6, 1936, Mazur, offering an exceptional prize (a live goose), had entered his question in the famous 'Scottish book' of open problems kept at the Scottish Coffee House in Lwów by Banach, Mazur, Ulam, and other mathematicians in their circle (see [776, Problem 153]).

A first systematic study of the approximation problem was initiated by Grothendieck [736] in 1955. Let Y be a Banach space over $\mathbb{K}$, and let us say that Y has the *approximation property* if for every compact subset K of Y and every $\varepsilon > 0$ there is an operator $T \in \mathfrak{F}(Y)$ such that $\|T(y) - y\| \leqslant \varepsilon$ for every $y \in K$. The outstanding result of Grothendieck (which can also be seen in [729, Theorem 16.35] or [717, Proposition 5.3(1)]) asserts that *Y has the approximation property if (and only if) the equality $\overline{\mathfrak{F}(X,Y)} = \mathfrak{K}(X,Y)$ holds for all normed spaces X.*

The approximation problem (equivalently, whether every Banach space has the approximation property) had been considered as one of the central open problems of Functional Analysis, and it was not until 1972 that Enflo [245], using a very complicated and delicate construction, produced the first example of a Banach space which fails the approximation property. Since Enflo's space is separable, he also provided the mathematical community with the first example of a separable Banach space without a Schauder basis. About a year after solving the problem, Enflo travelled to

Warsaw to give a lecture on his solution, after which, since Enflo's example answered Mazur's question 1.4.50 negatively, Mazur handed him a white goose in a basket. Soon after some other 'artificial' examples were constructed. However, Szankowski [608] showed that nature provides its own examples when he demonstrated that the space $BL(H)$ of all bounded linear operators on any infinite-dimensional Hilbert space H fails the approximation property. He also pointed out that his argument allowed him to construct the first example of a separable C^*-algebra failing the approximation property, which consequently had no Schauder basis.

For the state-of-the-art on the approximation problem, we refer the reader to the fine surveys written by Casazza [161] and Oja [465], and to the books written by Lindenstrauss and Tzafriri [769, 770], Diestel and Uhl [720], Defant and Floret [717], Megginson [778], Pietsch [790], and Diestel, Fourie, and Swart [719]. These sources have been widely used in our text.

Most results from Proposition 1.4.19 to Corollary 1.4.26 are taken from [441]. Nevertheless, we must say that, as acknowledged in [441], the argument in the proof of Theorem 1.4.20, as well as the one to derive Corollary 1.4.21(ii) from that theorem, involve only slight changes in Spurný's original proof of Corollary 1.4.21(ii) [596]. Fact 1.4.24 is well known (see for example [711, Proposition VI.1.9]). Although easily derivable from the results of [441], the equality $\mathrm{sp}(BL(X),T) = \mathrm{sp}(T)$ in Corollary 1.4.25 went unnoticed there. We note that, before [596], this equality was unknown even if the weakly compact operator T was actually compact. Results from Theorem 1.4.27 to Exercise 1.4.30 (with $\mathrm{sp}(BL(X),T)$ instead of $\mathrm{sp}(T)$ when $\mathrm{sp}(T)$ appears) are classical.

The scarcity of the spectrum of a compact operator T from a normed space X to itself, assured by Corollary 1.4.29(ii), is complemented by the fact that, for $0 \neq \lambda \in \mathrm{sp}(T)$, the non-bijectivity of $T - \lambda I_X$ centres in a finite-dimensional subspace. Indeed, we have the following theorem, which culminates the Riesz–Schauder theory.

Theorem 1.4.51 *Let X be a normed space over $\mathbb{K}$, let T be in $\mathfrak{K}(X)$, and let λ be in $\mathbb{K} \setminus \{0\}$. Then X becomes the topological direct sum of two T-invariant subspaces Y and Z such that Y is finite-dimensional, $T - \lambda I_X$ is nilpotent on Y, and $T - \lambda I_X$ is bijective on Z.*

Sketch of proof A fundamental step in the argument is of a purely algebraic nature. So let us forget for the moment that X is endowed with a norm, and let $F : X \to X$ be any linear mapping. The *descent* $d(F)$ of F is defined by the equality

$$d(F) := \min \left\{ n \in \mathbb{N} \cup \{0\} : R(F^n) = R(F^{n+1}) \right\},$$

with the convention that $\min \emptyset = \infty$. Analogously, the *ascent* $a(F)$ of F is defined by the equality

$$a(F) := \min \left\{ n \in \mathbb{N} \cup \{0\} : \ker(F^n) = \ker(F^{n+1}) \right\},$$

with the same convention. The outstanding fact is that, if both $d(F)$ and $a(F)$ are finite, then these two numbers coincide, and that, denoting by m their common value, we have $X = \ker(F^m) \oplus F^m(X)$ (see for example [811, Theorems V.6.2 and V.7.9]). If this is the case, then clearly both $\ker(F^m)$ and $F^m(X)$ are F-invariant subspaces, and moreover F is nilpotent on $\ker(F^m)$ (since $m = a(F)$) and bijective on $F^m(X)$ (since $m = d(F)$).

Now, recall that X is a normed space, let T and λ be as in the statement, and set $F := T - \lambda I_X$. Then minor changes to the proof of assertion (iii) in Theorem 1.4.27 show that both $d(F)$ and $a(F)$ are finite. It follows that, setting $Y := \ker(F^m)$ and $Z := F^m(X)$ with m as above, Y becomes a finite-dimensional subspace (by Theorem 1.4.27(i)) and we have $X = Y \oplus Z$, the sum being topological because Z is closed (by Theorem 1.4.27(ii)) and finite-codimensional in X. □

It is easily realized that the injective compact operators on the complex Hilbert space ℓ_2 constructed in the solution of Exercise 1.4.30 are normal (cf. §1.2.2 and Definition 1.2.11). Actually, there are not much more normal compact operators on complex Hilbert spaces. Indeed, *if K is any complex Hilbert space, and if $F : K \to K$ is a normal compact operator which is not a finite-rank operator, then there is a copy of ℓ_2 (say H) in K such that both H and its orthogonal $H^\perp$ are F-invariant subspaces, $F = 0$ on $H^\perp$, and $F = T$ on H, where T is an injective compact operator like those in the solution of Exercise 1.4.30* (see for example [711, Theorem II.7.6]).

Exercise 1.4.30 also raises the question of characterizing those Banach spaces X such that there exists an injective compact operator from X to itself. Separability of X is a sufficient condition, but not a necessary one. Indeed the mapping $\{\mu_n\} \to \{\frac{1}{n}\mu_n\}$ from ℓ_∞ to itself is an injective compact operator. More generally, if X is the dual of any separable Banach space, then, by Proposition 1.4.4, there exists an injective compact operator from X to itself. A natural class of Banach spaces containing both separable ones and their duals is that of Banach spaces with w^*-separable duals. We note that, thanks to the bipolar theorem, the w^*-separability of the dual X' of a Banach space X can be characterized by the existence of a sequence $(f_n)_{n\in\mathbb{N}}$ in X' such that $\bigcap_{n\in\mathbb{N}} \ker(f_n) = 0$. Now we follow [304] to provide us with a reasonable answer to the question we are dealing with. Indeed, we have the following.

Proposition 1.4.52 *For a Banach space X, the following conditions are equivalent:*

(i) *X' is w^*-separable.*

(ii) *For every infinite-dimensional Banach space Y, there exists an injective compact operator from X to Y.*

(iii) *There exists an injective compact operator from X to X.*

(iv) *There exists an injective compact operator from X to some Banach space.*

(v) *There exists an injective bounded linear operator from X to some separable Banach space.*

(vi) *There exists an injective bounded linear operator from X to ℓ_∞.*

(vii) *There exists an injective bounded linear operator from X to some Banach space whose dual is w^*-separable.*

Proof (i)⇒(ii) Let Y be any infinite-dimensional Banach space. Take a normalized basic sequence $(y_n)_{n\in\mathbb{N}}$ in Y, and invoke assumption (i) to find a sequence $(f_n)_{n\in\mathbb{N}}$ in

$\mathbb{B}_{X'}$ such that $\bigcap_{n\in\mathbb{N}} \ker(f_n) = 0$. Then the mapping

$$T : x \to \sum_{n=1}^{\infty} \frac{1}{n^2} f_n(x) y_n$$

becomes an injective compact operator from X to Y.

(ii)⇒(iii) Keeping in mind that condition (iii) is automatically fulfilled if X is finite-dimensional, this implication becomes clear.

(iii)⇒(iv) This is clear.

(iv)⇒(v) Since the closure of the range of a compact operator is separable (cf. Exercise 1.4.17).

(v)⇒(vi) Since separable Banach spaces imbed (isometrically) into ℓ_∞ (see for example [671, Theorem 2.5.7]).

(vi)⇒(vii) Since the dual of ℓ_∞ is w^*-separable.

(vii)⇒(i) By assumption (vii), there exists an injective bounded linear operator T from X to a Banach space Y such that Y' is w^*-separable. By taking a sequence $(g_n)_{n\in\mathbb{N}}$ in Y' such that $\bigcap_{n\in\mathbb{N}} \ker(g_n) = 0$, and setting $f_n := g_n T$, the sequence $(f_n)_{n\in\mathbb{N}} \subset X'$ satisfies $\bigcap_{n\in\mathbb{N}} \ker(f_n) = 0$. □

By keeping in mind the compact version of Corollary 1.4.3, the implication (i)⇒(iii) in the above proposition yields the following.

Corollary 1.4.53 *A Banach space X is finite-dimensional if (and only if) X' is w^*-separable and every injective bounded linear operator from X to X is surjective.*

§**1.4.54** It is conjectured in [304] that the w^*-separability of the dual in the above corollary cannot be altogether removed. This conjecture has been proved recently by Avilés and Koszmider [49]. Indeed, they have built an infinite compact Hausdorff topological space E such that every injective bounded linear operator from $C^{\mathbb{K}}(E)$ to itself is surjective. For additional information about Banach spaces with w^*-separable duals, the reader is referred to [729, Subsection 3.11.5].

§**1.4.55** Set $X := c_0$ or ℓ_p $(1 \leqslant p < \infty)$. According to Corollary 1.4.33, $\mathfrak{K}(X)$ is the smallest nonzero closed ideal of $BL(X)$. It is worth mentioning that, as proved by Gohberg, Markus, and Fel'dman [294] (see also [715, Theorem 2.5.9] or [703, Section 5.4]), $\mathfrak{K}(X)$ is also the largest proper ideal of $BL(X)$. In the case that X is equal to the complex space ℓ_2, this result had been proved earlier by Calkin [157], and, since then, the complete normed simple complex algebra $BL(\ell_2)/\mathfrak{K}(\ell_2)$ has been known as the *Calkin algebra*. We will see later (cf. Proposition 2.3.43) that the Calkin algebra is a C^*-algebra in a natural way.

Most results in Subsection 1.4.3 are new. We thank J. F. Mena for his collaboration in the writing of this subsection. In particular, Example 1.4.41 is due to him.

2

Beginning the proof of the non-associative Vidav–Palmer theorem

2.1 Basic results on numerical ranges

Introduction In this section, we begin to enter the fascinating world of the geometry of norm-unital normed algebras around their units, which originated in the Bohnenblust–Karlin paper [108]. To this end, we develop the basic theory of numerical ranges of elements in a pointed normed space, of elements in a norm-unital normed algebra, and, in particular, of bounded linear operators on a normed space. One of the star results in this section is Theorem 2.1.27, which states a celebrated Banach space characterization, also proved in [108], of unitary elements in unital C^*-algebras.

2.1.1 Algebra numerical ranges

By a *numerical-range space* over $\mathbb{K}$ we mean a couple (X,u), where X is a normed space over $\mathbb{K}$, and u is a fixed norm-one element in X (called the *distinguished element* of X). Let (X,u) be a numerical-range space. By a *state of* X *relative to* u we mean an element $f \in \mathbb{B}_{X'}$ satisfying $f(u) = 1$. We denote by $D(X,u)$ the set of all states of X relative to u, and remark that, by the Hahn–Banach and Banach–Alaoglu theorems, $D(X,u)$ becomes a non-empty w^*-compact convex subset of X'. Given $x \in X$, we define the *numerical range* of x (denoted by $V(X,u,x)$, or simply $V(x)$ when the couple (X,u) is without doubt) as the non-empty compact convex subset of $\mathbb{K}$ given by

$$V(X,u,x) := \{f(x) : f \in D(X,u)\}.$$

Given $\lambda \in \mathbb{K}$ and $r \geqslant 0$, we denote by $B_{\mathbb{K}}(\lambda,r)$ the closed ball in $\mathbb{K}$ with centre λ and radius r.

Proposition 2.1.1 *Let (X,u) be a numerical-range space over $\mathbb{K}$, and let x be in X. Then we have*

$$V(X,u,x) = \bigcap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, \|x-\lambda u\|).$$

Proof For all $f \in D(X,u)$ and $\lambda \in \mathbb{K}$, we see that

$$|f(x)-\lambda| = |f(x)-\lambda f(u)| = |f(x-\lambda u)| \leqslant \|f\|\|x-\lambda u\| = \|x-\lambda u\|,$$

and hence $f(x) \in B_{\mathbb{K}}(\lambda, \|x-\lambda u\|)$. Thus $V(X,u,x) \subseteq \bigcap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, \|x-\lambda u\|)$. In order to prove the converse inclusion, assume first that $x \in \mathbb{K}u$. If $x = \alpha u$, then $\|x-\lambda u\| = |\alpha - \lambda|$ for every $\lambda \in \mathbb{K}$, and hence

$$\bigcap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, \|x-\lambda u\|) = \bigcap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, |\alpha-\lambda|) \subseteq B_{\mathbb{K}}(\alpha, 0) = \{\alpha\} = V(X,u,x).$$

Now, assume that $x \notin \mathbb{K}u$. Let z be in $\bigcap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, \|x-\lambda u\|)$. Consider the linear mapping $g : \mathbb{K}u + \mathbb{K}x \to \mathbb{K}$ given by $g(\alpha u + \beta x) := \alpha + \beta z$. Since, for all $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K} \backslash \{0\}$ we have

$$|g(\alpha u)| = |\alpha| = \|\alpha u\|$$

and

$$|g(\alpha u + \beta x)| = |\alpha + \beta z| = |\beta| |\tfrac{\alpha}{\beta} + z| \leqslant |\beta| \|\tfrac{\alpha}{\beta} u + x\| = \|\alpha u + \beta x\|,$$

we get $\|g\| \leqslant 1$. Now, if f is a Hahn–Banach extension of g, then we realize that $f \in D(X,u)$ and, consequently, that $z = g(x) = f(x) \in V(X,u,x)$. □

A straightforward consequence of Proposition 2.1.1 is the following.

Corollary 2.1.2 *Let (X,u) and (Y,v) be numerical-range spaces over $\mathbb{K}$, and let T be a linear operator from X to Y such that $T(u) = v$. We have:*

(i) *If T is contractive, then $V(Y,v,T(x)) \subseteq V(X,u,x)$ for every $x \in X$.*
(ii) *If T is an isometry, then $V(Y,v,T(x)) = V(X,u,x)$ for every $x \in X$.*

As a consequence, if Z is a subspace of X with $u \in Z$, then

$$V(Z,u,z) = V(X,u,z) \ \text{ for every } \ z \in Z.$$

Corollary 2.1.3 *Let X be a nonzero normed space, and let T be in $BL(X)$. Then we have*

$$V(BL(X'), I_{X'}, T') = V(BL(X), I_X, T).$$

Proof Since the mapping $F \to F'$ form $BL(X)$ to $BL(X')$ is a linear isometry sending I_X to $I_{X'}$, the result follows from Corollary 2.1.2(ii). □

For a complex (normed) vector space X, we will denote by $X_{\mathbb{R}}$ the real (normed) vector space obtained when multiplication of vectors by scalars is restricted to $\mathbb{R} \times X$. Given a complex number z, we denote by $\Re(z)$ the real part of z.

Proposition 2.1.4 *Let (X,u) be a complex numerical-range space, and let x be in X. Then we have*

$$V(X_{\mathbb{R}}, u, x) = \Re(V(X,u,x)). \tag{2.1.1}$$

Proof Since the mapping $f \to \Re \circ f$ is a surjective linear isometry from $(X')_{\mathbb{R}}$ to $(X_{\mathbb{R}})'$, we have $D(X_{\mathbb{R}}, u) = \{\Re \circ f : f \in D(X,u)\}$, and the equality (2.1.1) follows. □

Proposition 2.1.5 *Let (X,u) be a numerical-range space over $\mathbb{K}$, and let x be in X. Then we have*

$$\max \Re(V(X,u,x)) = \inf\left\{ \frac{\|u+rx\|-1}{r} : r > 0 \right\} = \lim_{r \to 0^+} \frac{\|u+rx\|-1}{r}.$$

Proof In view of Proposition 2.1.4, we may assume that $\mathbb{K}=\mathbb{R}$. Since, by Proposition 2.1.1, we have

$$V(X,u,x)=\bigcap_{t\in\mathbb{R}}[t-\|x-tu\|,t+\|x-tu\|],$$

it follows that

$$\max V(X,u,x)=\inf\{t+\|x-tu\| : t\in\mathbb{R}\}. \tag{2.1.2}$$

Note that, for $t_1,t_2\in\mathbb{R}$ such that $t_1\leqslant t_2$, we have

$$\|x-t_1u\|=\|x-t_2u+(t_2-t_1)u\|\leqslant\|x-t_2u\|+\|(t_2-t_1)u\|=\|x-t_2u\|+t_2-t_1,$$

hence $t_1+\|x-t_1u\|\leqslant t_2+\|x-t_2u\|$, and so the mapping $t\to t+\|x-tu\|$ from $\mathbb{R}$ to $\mathbb{R}$ is increasing. Therefore, by (2.1.2), we derive

$$\max V(X,u,x)=\inf\{t+\|x-tu\| : t\in\mathbb{R}^-\}=\inf\left\{\frac{\|u+rx\|-1}{r} : r\in\mathbb{R}^+\right\},$$

and

$$\max V(X,u,x)=\lim_{t\to-\infty}(t+\|x-tu\|)=\lim_{r\to 0^+}\frac{\|u+rx\|-1}{r}.$$ □

Corollary 2.1.6 *Let (X,u) be a numerical-range space, let I be an interval of $\mathbb{R}$ such that $0\in I$ and $I\cap\mathbb{R}^+\neq\emptyset$, and let $f:I\to X$ be a mapping having a right derivative at 0 and satisfying $f(0)=u$. Then we have*

$$\max\Re(V(X,u,f'_+(0)))=\lim_{r\to 0^+}\frac{\|f(r)\|-1}{r}.$$

Proof Considering the mapping $f_0:I\cap\mathbb{R}^+\to X$ defined by

$$f_0(r):=\frac{f(r)-u}{r},$$

and noticing that, for $r\in I\cap\mathbb{R}^+$ we have

$$f(r)=u+rf'_+(0)+r(f_0(r)-f'_+(0)),$$

we get

$$\|u+rf'_+(0)\|-r\|f_0(r)-f'_+(0)\|\leqslant\|f(r)\|\leqslant\|u+rf'_+(0)\|+r\|f_0(r)-f'_+(0)\|.$$

Therefore we have

$$\begin{aligned}\frac{\|u+rf'_+(0)\|-1}{r}-\|f_0(r)-f'_+(0)\|&\leqslant\frac{\|f(r)\|-1}{r}\\&\leqslant\frac{\|u+rf'_+(0)\|-1}{r}+\|f_0(r)-f'_+(0)\|\end{aligned}$$

for every $r\in I\cap\mathbb{R}^+$. By keeping in mind Proposition 2.1.5 and the fact that

$$\lim_{r\to 0^+}\|f_0(r)-f'_+(0)\|=0,$$

the conclusion follows by letting $r\to 0^+$. □

Every norm-unital normed algebra will be considered without notice as a numerical-range space with $\mathbf{1}$ as its distinguished element. The next proposition becomes a basic result for the development of the theory of numerical ranges in norm-unital normed algebras.

Proposition 2.1.7 *Let A be a norm-unital complete normed associative algebra, and let a be in A. Then we have*

$$\max \Re(V(A,\mathbf{1},a)) = \sup\left\{\frac{1}{r}\log\|\exp(ra)\| : r>0\right\} = \lim_{r\to 0^+}\frac{1}{r}\log\|\exp(ra)\|.$$

Proof By Corollary 2.1.6, we have

$$\lim_{r\to 0^+}\frac{\|\exp(ra)\|-1}{r} = \max \Re(V(A,\mathbf{1},a)).$$

Therefore, by the chain rule for right derivatives, we also have

$$\lim_{r\to 0^+}\frac{1}{r}\log\|\exp(ra)\| = \max \Re(V(A,\mathbf{1},a)). \tag{2.1.3}$$

Let r be in $\mathbb{R}^+$. Since

$$e^{\frac{1}{r}}\exp(a) = \exp\left(\frac{1}{r}\mathbf{1}\right)\exp(a) = \exp\left(\frac{1}{r}\mathbf{1}+a\right),$$

we have

$$e^{\frac{1}{r}}\|\exp(a)\| = \left\|\exp\left(\frac{1}{r}\mathbf{1}+a\right)\right\| \leqslant e^{\|\frac{1}{r}\mathbf{1}+a\|}.$$

Therefore

$$\|\exp(a)\| \leqslant e^{\|\frac{1}{r}\mathbf{1}+a\|-\frac{1}{r}} = e^{\frac{\|\mathbf{1}+ra\|-1}{r}},$$

and hence

$$\log\|\exp(a)\| \leqslant \frac{\|\mathbf{1}+ra\|-1}{r}.$$

Keeping in mind Proposition 2.1.5, and letting $r\to 0$, we get

$$\log\|\exp(a)\| \leqslant \max \Re(V(A,\mathbf{1},a)).$$

Now, given $r\in\mathbb{R}^+$, the above inequality, with ra instead of a, reads as

$$\log\|\exp(ra)\| \leqslant \max \Re(V(A,\mathbf{1},ra)) = r\max \Re(V(A,\mathbf{1},a)),$$

and hence

$$\frac{1}{r}\log\|\exp(ra)\| \leqslant \max \Re(V(A,\mathbf{1},a)). \tag{2.1.4}$$

By combining (2.1.3) and (2.1.4), the result follows. □

Definition 2.1.8 Let (X,u) be a numerical-range space over $\mathbb{K}$. An element x of X is said to be *dissipative* if $\max \Re(V(X,u,x)) \leqslant 0$. Assume in addition that $\mathbb{K}=\mathbb{C}$. An element $x\in X$ is said to be *hermitian* if $V(X,u,x)\subseteq\mathbb{R}$.

For a complex (normed) algebra A, we will denote by $A_{\mathbb{R}}$ the real (normed) algebra obtained when the multiplication of elements of A by scalars is restricted to $\mathbb{R}\times A$.

Corollary 2.1.9 *Let A be a norm-unital complete normed associative algebra, and let a be in A. We have:*

(i) *a is dissipative if and only if $\|\exp(ra)\| \leqslant 1$ for every $r \in \mathbb{R}^+$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then $V(A, \mathbf{1}, a) = 0$ if and only if $\|\exp(ra)\| = 1$ for every $r \in \mathbb{R}$.*
(iii) *If $\mathbb{K} = \mathbb{C}$, then a is hermitian if and only if $\|\exp(ira)\| = 1$ for every $r \in \mathbb{R}$.*

Proof Assertion (i) is a straightforward consequence of Proposition 2.1.7. Assertion (ii) follows from (i) by keeping in mind that, for $r \in \mathbb{R}$, we have $[\exp(ra)]^{-1} = \exp(-ra)$. Assertion (iii) follows from (ii) because, by Proposition 2.1.4, a is hermitian if and only if $V(A_{\mathbb{R}}, \mathbf{1}, ia) = 0$. □

The following lemma specifies §1.1.36 in the case of norm-unital normed algebras, and becomes the key tool for transferring results on algebra numerical ranges from the associative case to the non-associative one.

Lemma 2.1.10 *Let A be a norm-unital normed algebra. Then the mappings*

$$a \to L_a \text{ and } a \to R_a$$

from A to $BL(A)$ become lineal isometries preserving distinguished elements. As a consequence, for $a \in A$ we have

$$V(A, \mathbf{1}, a) = V(BL(A), I_A, L_a) = V(BL(A), I_A, R_a).$$

Proof The first conclusion is straightforward, whereas the consequence follows from Corollary 2.1.2(ii). □

Let (X, u) be a numerical-range space. For $x \in X$ we define the *numerical radius*, $v(x) = v(X, u, x)$, of x by

$$v(x) := \max\{|z| : z \in V(x)\}.$$

The *numerical index, $n(X, u)$, of X at u*, is defined as the maximum non-negative real number k satisfying $k\|x\| \leqslant v(x)$ for every $x \in X$. We note that $v(\cdot)$ becomes a seminorm on X such that $v(\cdot) \leqslant \|\cdot\|$, and that, if Z is a subspace of X with $u \in Z$, then, by Corollary 2.1.2, we have

$$n(Z, u) \geqslant n(X, u). \tag{2.1.5}$$

Proposition 2.1.11 *Let A be a norm-unital normed complex algebra. Then*

$$n(A, \mathbf{1}) \geqslant \frac{1}{e}.$$

Proof In view of the above comments, we may assume that A is complete.

First assume that A is associative. Let a be in A. We must show that

$$\|a\| \leqslant ev(a). \tag{2.1.6}$$

By Proposition 2.1.7, we have $\log\|\exp(a)\| \leqslant \max \Re(V(A, \mathbf{1}, a)) \leqslant v(a)$. Replacing a with za, for $z \in \mathbb{C}$, we get

$$\|\exp(za)\| \leqslant e^{|z|v(a)}.$$

Consider the entire function $f : \mathbb{C} \to A$ given by $f(z) = \exp(za)$. Then, for $r \in \mathbb{R}^+$, the first Cauchy estimate gives

$$\|a\| = \|f'(0)\| \leqslant \frac{1}{r}\max\{\|\exp(za)\| : |z| = r\} \leqslant \frac{1}{r}e^{rv(a)}.$$

If $v(a) = 0$, then letting $r \to +\infty$ we get $a = 0$, and hence (2.1.6) holds. Otherwise, taking $r := \frac{1}{v(a)}$, (2.1.6) is obtained.

Now assume that A is not associative. Then, by Lemma 2.1.10, we can see A as a subspace of $BL(A)$ containing I_A. Therefore, by the above paragraph and (2.1.5), we obtain $n(A, \mathbf{1}) \geqslant n(BL(A), I_A) \geqslant \frac{1}{e}$. □

Definition 2.1.12 Let X be a normed space over $\mathbb{K}$, and let u be in X. We say that u is a *vertex* of $\mathbb{B}_X$ if $\|u\| = 1$ and $D(X,u)$ separates the points of X. Assume that $\|u\| = 1$. We note that the condition $n(X,u) > 0$ implies that u is a vertex of $\mathbb{B}_X$. Assume in addition that $\mathbb{K} = \mathbb{C}$. We denote by $H(X,u)$ the closed real subspace of X consisting of all hermitian elements of X. It is easily realized that u is a vertex of $\mathbb{B}_X$ if and only if $H(X,u) \cap iH(X,u) = 0$. Therefore we have the following.

Corollary 2.1.13 *Let A be a norm-unital normed complex algebra. Then $\mathbf{1}$ is a vertex of $\mathbb{B}_A$ (equivalently, $H(A,\mathbf{1}) \cap iH(A,\mathbf{1}) = 0$).*

§**2.1.14** Let X be a normed space, and let Y be a subspace of X'. We say that Y is *almost norming* if there exists $m > 0$ such that

$$m\|x\| \leqslant \sup\{|f(x)| : f \in \mathbb{B}_Y\} \tag{2.1.7}$$

for every $x \in X$. When the number m above is equal to 1, we say that Y is *norming*. Now, let u be a norm-one element in X, and set

$$D^Y(X,u) := D(X,u) \cap Y.$$

If $D^Y(X,u) = \emptyset$, then we set $n^Y(X,u) := 0$. Otherwise, we define $n^Y(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leqslant v^Y(x) := \sup\left\{|f(x)| : f \in D^Y(X,u)\right\}$$

for every $x \in X$. We note that

$$n^Y(X,u) \leqslant n(X,u), \tag{2.1.8}$$

and that, if $n^Y(X,u) > 0$ (respectively, $n^Y(X,u) = 1$), then Y is almost norming (respectively, norming).

Given a subset S of a vector space X over $\mathbb{K}$, $|\mathrm{co}|(S)$ will stand for the absolutely convex hull of S in X. If X is normed, then $\overline{\mathrm{co}}(S)$ and $\overline{|\mathrm{co}|}(S)$ will denote the closed convex and closed absolutely convex hull of S in X, respectively.

Lemma 2.1.15 *Let X be a normed space, let u be a norm-one element in X, and let Y be an almost norming closed subspace of X' such that the linear hull of $D^Y(X,u)$ is equal to Y. Then $n^Y(X,u) > 0$.*

Proof Since the linear hull of $D^Y(X,u)$ is equal to Y, we realize that $\overline{|\mathrm{co}|}(D^Y(X,u))$ is a barrel in Y. Therefore, since barrels in a Banach space are neighbourhoods of zero, there exists $k > 0$ such that $k\mathbb{B}_Y \subseteq \overline{|\mathrm{co}|}(D^Y(X,u))$. On the other hand, since Y is almost norming, the inequality (2.1.7) holds for some $m > 0$ and all $x \in X$. It follows that $mk\|x\| \leqslant v^Y(x)$ for every $x \in X$, and hence that $n^Y(X,u) > 0$. □

Definition 2.1.16 Let X be a normed space. An element $u \in X$ is said to be a *geometrically unitary element* of X if $\|u\| = 1$ and the linear hull of $D(X,u)$ equals the whole space X'. Clearly, *geometrically unitary elements of X are vertices of $\mathbb{B}_X$.*

Theorem 2.1.17 *Let u be a norm-one element in a normed space X over $\mathbb{K}$. We have:*

(i) *u is a geometrically unitary element of X if and only if $n(X,u) > 0$.*
(ii) *If $n(X,u) > 0$, then we have*

$$\mathrm{int}(\mathbb{B}_{X'}) \subseteq \frac{1}{n(X,u)}|\mathrm{co}|(D(X,u)), \tag{2.1.9}$$

and hence

$$\mathbb{B}_{X'} \subseteq \frac{1}{n(X,u)}\overline{|\mathrm{co}|}(D(X,u)).$$

(iii) *If $\mathbb{K} = \mathbb{R}$, and if $n(X,u) > 0$, then we have in fact*

$$\mathbb{B}_{X'} \subseteq \frac{1}{n(X,u)}|\mathrm{co}|(D(X,u)), \tag{2.1.10}$$

and hence, for each $f \in X'$, there are $\alpha_1, \alpha_2 \geqslant 0$ and $f_1, f_2 \in D(X,u)$ such that

$$f = \alpha_1 f_1 - \alpha_2 f_2 \quad \textit{and} \quad \alpha_1 + \alpha_2 \leqslant \tfrac{\|f\|}{n(X,u)}.$$

(iv) *If $\mathbb{K} = \mathbb{C}$, and if $n(X,u) > 0$, then, for each $f \in X'$, there are $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geqslant 0$ and $f_1, f_2, f_3, f_4 \in D(X,u)$ such that*

$$f = \alpha_1 f_1 - \alpha_2 f_2 + i(\alpha_3 f_3 - \alpha_4 f_4) \quad \textit{and} \quad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leqslant \tfrac{\sqrt{2}\|f\|}{n(X,u)}.$$

(v) *$n(X'',u) = n(X,u)$.*
(vi) *u is a geometrically unitary element of X'' if and only if it is a geometrically unitary element of X.*

Proof Assume that u is a geometrically unitary element of X. Then, by Lemma 2.1.15, we have $n(X,u) > 0$.

To prove the converse, note that, in the duality (X',X), the set

$$B := \{x \in X : v(x) \leqslant 1\}$$

is the absolute polar of $D(X,u)$ in X, and that the inclusion $n(X,u)B \subseteq \mathbb{B}_X$ holds. It follows from the bipolar theorem that

$$n(X,u)\mathbb{B}_{X'} \text{ is contained in the } w^*\text{-closure of } |\mathrm{co}|(D(X,u)). \tag{2.1.11}$$

Assume that $\mathbb{K} = \mathbb{R}$ and that $n(X,u) > 0$. Then, $|\mathrm{co}|(D(X,u))$ is w^*-compact in X' because

$$|\mathrm{co}|(D(X,u)) = \mathrm{co}(D(X,u) \cup -D(X,u)) \tag{2.1.12}$$

and $D(X,u)$ is w^*-compact, and hence (2.1.10) follows from (2.1.11). Now the proof of assertion (iii) is concluded by noticing that for $f \in X' \setminus \{0\}$ we have $\frac{1}{\|f\|}f \in \mathbb{B}_{X'}$, and applying (2.1.10) and (2.1.12). On the other hand, noticing that (2.1.10) implies

(2.1.9), and that (2.1.9) implies that u is a geometrically unitary element of X, assertion (ii) and the 'if' part of assertion (i) are proved in the case $\mathbb{K}=\mathbb{R}$.

Now, assume that $\mathbb{K}=\mathbb{C}$ and that $n(X,u)>0$. Let $\varepsilon>0$, and take $n\in\mathbb{N}$ such that $\mathbb{B}_{\mathbb{C}}\subseteq(1+\varepsilon)\mathrm{co}(\{z_1,\ldots,z_n\})$, where $z_1,\ldots,z_n$ are the nth roots of 1 in $\mathbb{C}$. Then we have

$$|\mathrm{co}|(D(X,u))=\mathrm{co}(\mathbb{B}_{\mathbb{C}}D(X,u))\subseteq(1+\varepsilon)\mathrm{co}(\cup_{i=1}^n z_iD(X,u)).$$

Since $\mathrm{co}(\cup_{i=1}^n z_iD(X,u))$ is w^*-compact, it follows from (2.1.11) that

$$n(X,u)\mathbb{B}_{X'}\subseteq(1+\varepsilon)\mathrm{co}(\cup_{i=1}^n z_iD(X,u)).$$

Since $\mathrm{co}(\cup_{i=1}^n z_iD(X,u))\subseteq|\mathrm{co}|(D(X,u))$, we derive that

$$n(X,u)\mathbb{B}_{X'}\subseteq(1+\varepsilon)|\mathrm{co}|(D(X,u)).$$

Therefore, since $\mathrm{int}(\mathbb{B}_{X'})\subseteq\cup_{\varepsilon>0}\frac{1}{1+\varepsilon}\mathbb{B}_{X'}$, we have

$$n(X,u)\mathrm{int}(\mathbb{B}_{X'})\subseteq|\mathrm{co}|(D(X,u)).$$

This proves assertion (ii) and the 'if' part of assertion (i) in the case $\mathbb{K}=\mathbb{C}$. On the other hand, since

$$|\mathrm{co}|(D(X,u))\subseteq\sqrt{2}\mathrm{co}\,(D(X,u)\cup -D(X,u)\cup iD(X,u)\cup -iD(X,u))$$

because

$$|\mathrm{co}|(D(X,u))=\mathrm{co}(\mathbb{B}_{\mathbb{C}}D(X,u))\ \text{ and }\ \mathbb{B}_{\mathbb{C}}\subseteq\sqrt{2}\mathrm{co}(\{1,-1,i,-i\}),$$

and

$$\mathrm{co}(D(X,u)\cup -D(X,u)\cup iD(X,u)\cup -iD(X,u))$$

is w^*-compact, it follows from (2.1.11) that

$$\mathbb{B}_{X'}\subseteq\frac{\sqrt{2}}{n(X,u)}\mathrm{co}(D(X,u)\cup -D(X,u)\cup iD(X,u)\cup -iD(X,u)),$$

which, after straightforward computations, proves assertion (iv).

By (2.1.5), we have $n(X'',u)\leqslant n(X,u)$. To prove the converse inequality, we may assume that $n(X,u)>0$. Let x'' be in X''. Then we have

$$\begin{aligned}v(X'',u,x'')&=\max\{|f(x'')|:f\in D(X'',u)\}\geqslant\sup\{|f(x'')|:f\in D(X,u)\}\\&=\sup\{|f(x'')|:f\in\overline{|\mathrm{co}|}(D(X,u))\}\geqslant n(X,u)\sup\{|f(x'')|:f\in\mathbb{B}_{X'}\}\\&=n(X,u)\|x''\|,\end{aligned}$$

the last inequality being true because of assertion (ii). Since x'' is arbitrary in X'', we derive $n(X'',u)\geqslant n(X,u)$, as desired.

Finally, assertion (vi) follows from assertions (i) and (v). □

The next example shows that, when $\mathbb{K}=\mathbb{C}$, the inclusion

$$\mathbb{B}_{X'}\subseteq\frac{1}{n(X,u)}|\mathrm{co}|(D(X,u))$$

cannot be expected in assertion (ii) of Theorem 2.1.17, even if $n(X,u)=1$. In the example, X is in fact a unital commutative C^*-algebra and u is the unit of X.

Example 2.1.18 Let X stand for the complex Banach space of all convergent sequences $(a_n)_{n\in\mathbb{N}}$ of complex numbers, and let $u \in \mathbb{S}_X$ be the sequence constantly equal to 1. Then, since the coordinate projections on X are elements of $D(X,u)$, we have $n(X,u) = 1$. Moreover, thanks to the natural identification of X' with the complex Banach space $\ell_1(\mathbb{N}\cup\{0\})$, $D(X,u)$ becomes the set of those sequences $(b_n)_{n\in\mathbb{N}\cup\{0\}} \in \mathbb{S}_{\ell_1(\mathbb{N}\cup\{0\})}$ such that $b_n \geqslant 0$ for every $n \in \mathbb{N}\cup\{0\}$. Now set

$$f := \left(\frac{1}{2^{n+1}} e^{i\frac{1}{n+1}}\right)_{n\in\mathbb{N}\cup\{0\}} \in \mathbb{S}_{\ell_1(\mathbb{N}\cup\{0\})}.$$

We are going to realize that f does not belong to $|\mathrm{co}|(D(X,u))$. Assume to the contrary that $f \in |\mathrm{co}|(D(X,u))$, so that $f = \sum_{k=1}^m \lambda_k f_k$ for suitable $m \in \mathbb{N}$, $f_k \in D(X,u)$, and $\lambda_k \in \mathbb{C}\setminus\{0\}$ with $\sum_{k=1}^m |\lambda_k| \leqslant 1$. For $k = 1,\dots,m$, write $f_k = (b_{nk})_{n\in\mathbb{N}\cup\{0\}} \in \mathbb{S}_{\ell_1(\mathbb{N}\cup\{0\})}$ with $b_{nk} \geqslant 0$ for every $n \in \mathbb{N}\cup\{0\}$. Then, since

$$1 = \|f\| = \sum_{n=0}^{\infty}\left|\sum_{k=1}^{m} \lambda_k b_{nk}\right| \leqslant \sum_{n=0}^{\infty}\sum_{k=1}^{m} |\lambda_k| b_{nk} = \sum_{k=1}^{m}\sum_{n=0}^{\infty} |\lambda_k| b_{nk} = \sum_{k=1}^{m} |\lambda_k| \leqslant 1,$$

we deduce that $|\sum_{k=1}^m \lambda_k b_{nk}| = \sum_{k=1}^m |\lambda_k| b_{nk}$ for every $n \in \mathbb{N}\cup\{0\}$. Therefore for each $n \in \mathbb{N}\cup\{0\}$ there exists $\rho_n \in \mathbb{S}_{\mathbb{C}}$ such that

$$\lambda_k b_{nk} = \rho_n |\lambda_k| b_{nk} \text{ for every } k = 1,\dots,m. \tag{2.1.13}$$

Let n be in $\mathbb{N}\cup\{0\}$. Since

$$\rho_n \sum_{k=1}^{m} |\lambda_k| b_{nk} = \sum_{k=1}^{m} \lambda_k b_{nk} = \frac{1}{2^{n+1}} e^{i\frac{1}{n+1}},$$

we deduce that $\rho_n = e^{i\frac{1}{n+1}}$ and that there is $k(n) \in \{1,\dots,m\}$ such that $b_{nk(n)} > 0$. It follows from (2.1.13) that $e^{i\frac{1}{n+1}} = \frac{\lambda_{k(n)}}{|\lambda_{k(n)}|}$. But this is a contradiction because the set $\left\{e^{i\frac{1}{n+1}} : n \in \mathbb{N}\cup\{0\}\right\}$ is infinite and the set $\left\{\frac{\lambda_{k(n)}}{|\lambda_{k(n)}|} : n \in \mathbb{N}\cup\{0\}\right\}$ is finite.

Thinking about positive results for norm-unital normed algebras, it is enough to combine Proposition 2.1.11 and Theorem 2.1.17 to achieve the following.

Corollary 2.1.19 *Let A be a norm-unital normed complex algebra. Then* $\mathbf{1}$ *is a geometrically unitary element of A. More precisely, we have*

$$\mathrm{int}(\mathbb{B}_{A'}) \subseteq e|\mathrm{co}|(D(A,\mathbf{1})).$$

Moreover, for each $f \in A'$, there are $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \geqslant 0$ and $f_1,f_2,f_3,f_4 \in D(A,\mathbf{1})$ such that

$$f = \alpha_1 f_1 - \alpha_2 f_2 + i(\alpha_3 f_3 - \alpha_4 f_4) \ \text{ and } \ \alpha_1+\alpha_2+\alpha_3+\alpha_4 \leqslant e\sqrt{2}\|f\|.$$

§2.1.20 Let A be a norm-unital normed associative algebra. An element $u \in A$ is said to be *algebraically unitary* if u is invertible in A with $\|u\| = \|u^{-1}\| = 1$. If u is an algebraically unitary element of A, then L_u is a surjective linear isometry taking $\mathbf{1}$ to u. Therefore we have the following.

Corollary 2.1.21 *Let A be a norm-unital normed associative complex algebra, and let u be an algebraically unitary element of A. Then all conclusions in Corollary 2.1.19 remain true when we replace $\mathbf{1}$ with u.*

By combining Lemma 2.1.15, the inequality (2.1.8), and Theorem 2.1.17(i), we find the following.

Corollary 2.1.22 *Let X, u, and Y be as in Lemma 2.1.15. Then u is a geometrically unitary element of X.*

Lemma 2.1.23 *Let (X,u) and (Y,v) be numerical-range spaces over $\mathbb{K}$. Assume that v is a geometrically unitary element of Y, and that there exists a bounded below linear contraction T from X to Y such that $T(u) = v$. Then u is a geometrically unitary element of X.*

Proof By Corollary 2.1.2(i), we have $v(T(x)) \leqslant v(x)$ for every $x \in X$. Let $k > 0$ be such that $\|T(x)\| \geqslant k\|x\|$ for every $x \in X$. Then, for each $x \in X$ we have

$$kn(Y,v)\|x\| \leqslant n(Y,v)\|T(x)\| \leqslant v(T(x)) \leqslant v(x).$$

Thus $kn(Y,v) \leqslant n(X,u)$. Now, the result follows from Theorem 2.1.17(i). □

Corollary 2.1.24 *Let X be a normed space, and let F be in $BL(X)$. If F' is a geometrically unitary element of $BL(X')$, then F is a geometrically unitary element of $BL(X)$.*

Proof The mapping $T : G \to G'$ from $BL(X)$ to $BL(X')$ is a linear isometry with $T(F) = F'$. Now, apply Lemma 2.1.23. □

Lemma 2.1.25 *Let X be a normed space over $\mathbb{K}$, and let u be a vertex of $\mathbb{B}_X$. Then u is an extreme point of $\mathbb{B}_X$.*

Proof Let x, y be in $\mathbb{B}_X$ such that $u = \frac{1}{2}(x+y)$, and let f be in $D(X,u)$. Then we have $1 = f(u) = \frac{1}{2}(f(x) + f(y))$ with $f(x), f(y) \in \mathbb{B}_{\mathbb{K}}$. Since 1 is an extreme point of $\mathbb{B}_{\mathbb{K}}$, we get $f(u-x) = 0$. Finally, since f is arbitrary in $D(X,u)$, and u is a vertex of $\mathbb{B}_X$, we derive $x = u$. □

Lemma 2.1.26 [806, Proposition 1.6.1 and Theorem 1.6.4] *The closed unit ball of a nonzero C^*-algebra A has extreme points if and only if A is unital. In this case, the extreme points of $\mathbb{B}_A$ are precisely the elements $u \in A$ such that*

$$(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0.$$

A proof of the above lemma will be given much later (see §4.2.37).

Theorem 2.1.27 *Let A be a unital C^*-algebra, and let u be in A. Then the following conditions are equivalent:*

(i) *u is unitary (i.e. $u^*u = uu^* = \mathbf{1}$).*
(ii) *u is algebraically unitary.*
(iii) *u is geometrically unitary.*
(iv) *u is a vertex of the closed unit ball of A.*

Proof The implications (i)⇒(ii) and (iii)⇒(iv) are clear, whereas the implication (ii)⇒(iii) follows from Corollary 2.1.21.

(iv)⇒(i) Assume that u is a vertex of $\mathbb{B}_A$. By Lemmas 2.1.25 and 2.1.26, we have $(\mathbf{1}-uu^*)A(\mathbf{1}-u^*u)=0$. Therefore $u^*(\mathbf{1}-uu^*)u(\mathbf{1}-u^*u)=0$. Since

$$[u(\mathbf{1}-u^*u)]^* = (\mathbf{1}-uu^*)u^* = u^*(\mathbf{1}-uu^*),$$

using the C^*-identity we get $u(\mathbf{1}-u^*u)=0$. Note that, for any $\lambda\in\mathbb{C}$ and $f\in D(A,u)$, we have

$$\begin{aligned}
|1+\lambda f(\mathbf{1}-uu^*)|^2 &= |f(u+\lambda(\mathbf{1}-uu^*))|^2 \leqslant \|u+\lambda(\mathbf{1}-uu^*)\|^2 \\
&= \|[u+\lambda(\mathbf{1}-uu^*)]^*[u+\lambda(\mathbf{1}-uu^*)]\| \\
&= \|[u^*+\bar\lambda(\mathbf{1}-uu^*)][u+\lambda(\mathbf{1}-uu^*)]\| \\
&= \|u^*u+\lambda u^*(\mathbf{1}-uu^*)+\bar\lambda(\mathbf{1}-uu^*)u+|\lambda|^2(\mathbf{1}-uu^*)^2\| \\
&= \|u^*u+|\lambda|^2(\mathbf{1}-uu^*)\| \leqslant 1+|\lambda|^2.
\end{aligned}$$

For $\lambda=\overline{f(\mathbf{1}-uu^*)}$, this implies $|f(\mathbf{1}-uu^*)|=0$. Since u is a vertex of $\mathbb{B}_A$ and $f\in D(A,u)$ was arbitrary, this shows that $\mathbf{1}-uu^*=0$. Similarly we get $\mathbf{1}-u^*u=0$, so u is unitary. □

2.1.2 Operator numerical ranges

Now we are going to construct a norm-unital complete normed associative (necessarily real) algebra such that its unit is a vertex of the closed unit ball but is not a geometrically unitary element. Since such an algebra will be of the form $BL(X)$ for some real Banach space X, we are pleasantly obliged to do a quick incursion into the world of numerical ranges of operators.

Lemma 2.1.28 *Let A be a norm-unital normed real algebra, and let a be in A with $V(A,\mathbf{1},a)=0$ and such that a^2 lies in the linear hull of $\{a,\mathbf{1}\}$. Then we have $a^2=-\|a\|^2\mathbf{1}$ and $\|a+s\mathbf{1}\|^2=\|a\|^2+s^2$ for every $s\in\mathbb{R}$.*

Proof The assumption that a^2 lies in the linear hull of $\{a,\mathbf{1}\}$ implies that this linear hull is an associative subalgebra of A, so that we may assume that A is associative and complete. Moreover, by the same assumption, there are real numbers α,μ such that

$$(a-\alpha\mathbf{1})^2=\mu\mathbf{1}. \tag{2.1.14}$$

On the other hand, since $V(A,\mathbf{1},a)=0$, Corollary 2.1.9(ii) applies, so that we have

$$e^{\alpha r}\|\exp[r(a-\alpha\mathbf{1})]\| = \|\exp(ra)\| = 1 \quad\text{for every}\quad r\in\mathbb{R}. \tag{2.1.15}$$

If $a\in\mathbb{R}\mathbf{1}$, then $a=0$ because $V(A,\mathbf{1},a)=0$, and the result follows. Therefore, in what follows we will assume that $a\notin\mathbb{R}\mathbf{1}$, and hence that there exists $M>0$ such that

$$M|s| \leqslant \|s\mathbf{1}+t(a-\alpha\mathbf{1})\| \quad\text{for all}\quad s,t\in\mathbb{R}. \tag{2.1.16}$$

Suppose that $\mu=0$. Then, since by (2.1.14) we have

$$\exp[r(a-\alpha\mathbf{1})]=\mathbf{1}+r(a-\alpha\mathbf{1}),$$

it follows from (2.1.15) that

$$e^{\alpha r}\|\mathbf{1}+r(a-\alpha\mathbf{1})\|=1 \ \text{ for every } \ r\in\mathbb{R},$$

which, in view of (2.1.16), implies $\alpha=0$. Therefore we have $\|\mathbf{1}+ra\|=1$ for every $r\in\mathbb{R}$, so $a=0$, contradicting our previous assumption that $a\notin\mathbb{R}\mathbf{1}$.

Now, suppose that $\mu>0$ (say $\mu=\lambda^2$ with $\lambda>0$). Then, since by (2.1.14) we have

$$\exp[r(a-\alpha\mathbf{1})]=(\cosh r\lambda)\mathbf{1}+\frac{1}{\lambda}(\sinh r\lambda)(a-\alpha\mathbf{1}),$$

it follows from (2.1.15) that

$$e^{\alpha r}\left\|(\cosh r\lambda)\mathbf{1}+\frac{1}{\lambda}(\sinh r\lambda)(a-\alpha\mathbf{1})\right\|=1 \ \text{ for every } \ r\in\mathbb{R},$$

which, in view of (2.1.16), implies $Me^{\alpha r}\cosh r\lambda\leqslant 1$ for every $r\in\mathbb{R}$, another contradiction.

It follows from the two preceding paragraphs that $\mu<0$ (say $\mu=-\lambda^2$ with $\lambda>0$). Then, since by (2.1.14) we have

$$\exp[r(a-\alpha\mathbf{1})]=(\cos r\lambda)\mathbf{1}+\frac{1}{\lambda}(\sin r\lambda)(a-\alpha\mathbf{1}),$$

it follows from (2.1.15) that

$$e^{\alpha r}\left\|(\cos r\lambda)\mathbf{1}+\frac{1}{\lambda}(\sin r\lambda)(a-\alpha\mathbf{1})\right\|=1 \ \text{ for every } \ r\in\mathbb{R},$$

which, in view of (2.1.16), implies $\alpha=0$. Therefore we have

$$\left\|(\cos r\lambda)\mathbf{1}+\frac{1}{\lambda}(\sin r\lambda)a\right\|=1 \ \text{ for every } \ r\in\mathbb{R}, \tag{2.1.17}$$

and taking $r:=\frac{\pi}{2\lambda}$, we derive that $\|a\|=\lambda$, which together with (2.1.17) implies easily that $\|a+s\mathbf{1}\|^2=\|a\|^2+s^2$ for every $s\in\mathbb{R}$, and also, by (2.1.14), that $a^2=-\|a\|^2\mathbf{1}$, as required. □

Let X be a normed space. We define the *spatial numerical index*, $N(X)$, of X by $N(X):=n(BL(X),I_X)$. Since $BL(X)$ can be seen as a subspace of $BL(X')$ containing $I_{X'}$ (via the mapping $T\to T'$ form $BL(X)$ to $BL(X')$), the inequality (2.1.5) applies, so that we have

$$N(X)\geqslant N(X'). \tag{2.1.18}$$

Corollary 2.1.29 *Let X be a normed two-dimensional real space. Then X is a Hilbert space if and only if $N(X)=0$.*

Proof Assume that X is a Hilbert space. Then $X=\mathbb{C}_{\mathbb{R}}$, and hence the operator T of multiplication by the imaginary unit becomes an element of $BL(X)$ with $V(BL(X),I_X,T)=0$. Indeed, we have

$$\lim_{r\to 0}\frac{\|I_X+rT\|-1}{r}=\lim_{r\to 0}\frac{\sqrt{1+r^2}-1}{r}=0,$$

and Proposition 2.1.5 applies. Now, we have $v(T)=0$ and $T\neq 0$, which implies $N(X)=0$.

Assume that $N(X) = 0$. Then, since $BL(X)$ is finite-dimensional, there exists a norm-one element $T \in BL(X)$ such that $v(T) = 0$. Moreover, since X is two-dimensional, T^2 belongs to the linear hull of $\{I_X, T\}$. It follows from Lemma 2.1.28 that $T^2 = -I_X$ and that

$$\|(\cos\varphi)I_X + (\sin\varphi)T\| = 1 \text{ for every } \varphi \in \mathbb{R}. \tag{2.1.19}$$

As a consequence, for $\varphi \in \mathbb{R}$, the operator $T_\varphi := (\cos\varphi)I_X + (\sin\varphi)T$ is bijective with $T_\varphi^{-1} = T_{-\varphi}$ and $\|T_\varphi\| = \|T_\varphi^{-1}\| = 1$. Now, since T_φ is a surjective linear isometry on X, it is enough to take a norm-one element $u \in X$, to have $\|(\cos\varphi)u + (\sin\varphi)T(u)\| = \|T_\varphi(u)\| = 1$ for every $\varphi \in \mathbb{R}$, which, again by the two-dimensionality of X, shows ostensibly that X is a copy of the Euclidean plane. □

§**2.1.30** Given a normed space X, we denote by $\Pi(X)$ the subset of $X \times X'$ defined by

$$\Pi(X) := \{(x, f) : x \in \mathbb{S}_X, f \in D(X, x)\}.$$

For any subset Γ of $\Pi(X)$, we set $\pi_1(\Gamma) := \{x : (x, f) \in \Gamma \text{ for some } f\}$.

Proposition 2.1.31 *Let X be a normed space, let Γ be a subset of $\Pi(X)$ such that its natural projection $\pi_1(\Gamma)$ is dense in $\mathbb{S}_X$, and let T be in $BL(X)$. Then we have*

$$V(BL(X), I_X, T) = \overline{\text{co}}\{f(T(x)) : (x, f) \in \Gamma\}.$$

Proof Let $\mu = \sup \Re(\{f(T(x)) : (x, f) \in \Gamma\})$. Since, for $(x, f) \in \Gamma$, the mapping $F \to f(F(x))$ belongs to $D(BL(X), I_X)$, Proposition 2.1.5 applies, so that we have

$$\mu \leqslant \inf\left\{\frac{\|I_X + rT\| - 1}{r} : r > 0\right\}. \tag{2.1.20}$$

The case $T = 0$ is obvious; so assume that $T \neq 0$. Let $0 < r < \|T\|^{-1}$, $\varepsilon > 0$, $x \in \mathbb{S}_X$. Since $\pi_1(\Gamma)$ is dense in $\mathbb{S}_X$, there exists $(y, g) \in \Gamma$ such that $\|x - y\| < \varepsilon$. We have $\Re(g(T(y))) \leqslant \mu \leqslant \|T\|$, and so

$$\|(I_X - rT)(y)\| \geqslant \Re(g((I_X - rT)(y)) = 1 - r\Re(g(T(y))) \geqslant 1 - r\mu > 0.$$

Therefore

$$\|(I_X - rT)(x)\| \geqslant 1 - r\mu - \|I_X - rT\|\varepsilon.$$

Since ε is arbitrary, this gives $\|(I_X - rT)(x)\| \geqslant 1 - r\mu$, and therefore

$$\|(I_X - rT)(x)\| \geqslant (1 - r\mu)\|x\| \text{ for every } x \in X.$$

If we replace x by $(I_X + rT)(x)$, this gives

$$\|(I_X + rT)(x)\| \leqslant (1 - r\mu)^{-1}\|(I_X - r^2T^2)(x)\| \text{ for every } x \in X,$$

and so

$$\|I_X + rT\| \leqslant \frac{1 + r^2\|T^2\|}{1 - r\mu}.$$

Therefore

$$\frac{\|I_X + rT\| - 1}{r} \leqslant \frac{\mu + r\|T^2\|}{1 - r\mu},$$

and this, together with (2.1.20), proves that

$$\mu = \inf\left\{ \frac{\|I_X + rT\| - 1}{r} : r > 0 \right\}.$$

Now, again by Proposition 2.1.5, we have

$$\max \Re(V(BL(X), I_X, T)) = \sup \Re(\{f(T(x)) : (x, f) \in \Gamma\}).$$

The proof is easily completed by replacing T with appropriate scalar multiples of T and using the fact that $V(BL(X), I_X, T)$ is a closed convex set. □

§**2.1.32** Given a normed space X, and a bounded linear operator T on X, we define the *spatial numerical range*, $W(T)$, of T by

$$W(T) := \{f(T(x)) : (x, f) \in \Pi(X)\}.$$

Corollary 2.1.33 *Let X be a normed space, and let T be in $BL(X)$. Then we have*

$$V(BL(X), I_X, T) = \overline{\text{co}} W(T).$$

Proof By the Hahn–Banach theorem, we have $\pi_1(\Pi(X)) = \mathbb{S}_X$. Therefore the result follows by applying Proposition 2.1.31 with $\Gamma = \Pi(X)$. □

The next corollary has its own interest.

Corollary 2.1.34 *Let X be a Banach space, and let F be in $BL(X')$. Then we have*

$$V(BL(X'), I_{X'}, F) = \overline{\text{co}}\{F(f)(x) : (x, f) \in \Pi(X)\}.$$

Proof The set $\Gamma' := \{(f, x) : (x, f) \in \Pi(X)\}$ is a subset of $\Pi(X')$ whose projection into the first coordinate is dense in $\mathbb{S}_{X'}$ (thanks to the Bishop–Phelps theorem). Therefore the result follows by applying Proposition 2.1.31 with (X', Γ', F) instead of (X, Γ, T). □

For a convex subset S of a vector space X, we denote by $\text{ext}(S)$ the set of all extreme points of S.

Corollary 2.1.35 *Let X be a Banach space, and let T be in $BL(X)$. Then we have*

$$v(BL(X), I_X, T) = \sup\{|x''(T'(x'))| : x' \in \text{ext}(\mathbb{B}_{X'}), x'' \in \text{ext}(\mathbb{B}_{X''}), x''(x') = 1\}.$$

Proof As a consequence of Corollary 2.1.33, we have

$$v(T) := v(BL(X), I_X, T) = \sup\{\phi(x) : x \in \mathbb{S}_X\}, \tag{2.1.21}$$

where, for $x \in \mathbb{S}_X$, $\phi(x)$ is defined by

$$\phi(x) := \sup\{|x'(T(x))| : x' \in D(X, x)\}.$$

Now recall that $D(X, x)$ is a w^*-compact convex subset of X', and note that, for $x \in \mathbb{S}_X$, the mapping $x' \to |x'(T(x))|$ from X' to $\mathbb{R}$ is convex and w^*-continuous, and that consequently the set $\{x' \in D(X, u) : |x'(T(x))| = \phi(x)\}$ is a w^*-closed face of $D(X, u)$. It follows from the Krein–Milman theorem that, for each $x \in \mathbb{S}_X$, there exists $x' \in \text{ext}(D(X, u))$ such that $|x'(T(x))| = \phi(x)$. Since $D(X, u)$ is a face of $\mathbb{B}_{X'}$, we derive from (2.1.21) that

$$v(T) = \sup\{|x'(T(x))| : x' \in \text{ext}(\mathbb{B}_{X'}), x \in \mathbb{S}_X, x'(x) = 1\}.$$

As a consequence, we have

$$v(T) \leqslant \sup\{|x''(T'(x'))| : x' \in \mathrm{ext}(\mathbb{B}_{X'}), x'' \in D(X', x')\} \leqslant v(T') = v(T),$$

the equality at the end being true because of Corollary 2.1.3. Therefore we can write

$$v(T) = \sup\{\psi(x') : x' \in \mathrm{ext}(\mathbb{B}_{X'})\},$$

where now, for $x' \in \mathrm{ext}(\mathbb{B}_{X'})$, $\psi(x')$ is defined by

$$\psi(x') := \sup\{|x''(T'(x'))| : x'' \in D(X', x')\}.$$

The proof is concluded by applying to ψ the same argument as that used for ϕ. □

Fact 2.1.36 *Let n be a positive integer greater than or equal to 2, and let m be in $\mathbb{Z}$. Then we have*

$$\left|\cos\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right).$$

If in addition n is even, then we also have

$$\left|\sin\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right).$$

Proof We observe that the inequality

$$\left|\cos\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right)$$

is equivalent to the fact that

$$\frac{m\pi}{n} + \frac{\pi}{2n} \notin \left]-\frac{\pi}{2n}, \frac{\pi}{2n}\right[+ \pi\mathbb{Z},$$

which is equivalent to $2m+1 \notin\ ]-1,1[+ 2n\mathbb{Z}$, and this last statement is obviously true. Analogously, we note that the inequality

$$\left|\sin\left(\frac{m\pi}{n} + \frac{\pi}{2n}\right)\right| \leqslant \cos\left(\frac{\pi}{2n}\right)$$

is equivalent to

$$\frac{m\pi}{n} + \frac{\pi}{2n} \notin \left]\frac{\pi}{2} - \frac{\pi}{2n}, \frac{\pi}{2} + \frac{\pi}{2n}\right[+ \pi\mathbb{Z},$$

which is equivalent to $2m+1 \notin\ ]n-1, n+1[+ 2n\mathbb{Z}$, a statement which is true when n is even. □

Let n be a positive integer greater than or equal to 2. For $k = 1, 2, \ldots, 2n$, we write

$$x_k := \left(\cos\left(\frac{k\pi}{n}\right), \sin\left(\frac{k\pi}{n}\right)\right),$$
$$x'_k := \frac{1}{\cos\left(\frac{\pi}{2n}\right)}\left(\cos\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right), \sin\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right)\right),$$

and we define X_n to be the two-dimensional real Banach space such that

$$\mathrm{ext}(\mathbb{B}_{X_n}) = \{x_k : k = 1, 2, \ldots, 2n\}. \tag{2.1.22}$$

Lemma 2.1.37 *Let n be an even positive integer, and let X_n be defined as above. Then $N(X_n) \leqslant \tan\left(\frac{\pi}{2n}\right)$.*

Proof For $j,k \in \{1,2,\ldots,2n\}$ we have

$$|x_k'(x_j)| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left| \cos\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right) \cos\left(\frac{j\pi}{n}\right) + \sin\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right) \sin\left(\frac{j\pi}{n}\right) \right|$$
$$= \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left| \cos\left(\frac{(k-j)\pi}{n} + \frac{\pi}{2n}\right) \right|$$

so, the preceding fact tells us that $|x_k'(x_j)| \leqslant 1$. Moreover, for $k \in \{1,2,\ldots,2n\}$ we have $x_k'(x_k) = 1$ and $x_k'(x_{k+1}) = 1$ (with the identification $x_{2n+1} = x_1$). Therefore,

$$\mathrm{ext}(\mathbb{B}_{X_n'}) = \{x_k' \,:\, k = 1,2,\ldots,2n\}. \tag{2.1.23}$$

It follows from (2.1.22) and (2.1.23) that, for every $T \in BL(X)$ we have

$$\|T\|_n = \max\{|x_k'(T(x_j))| \,:\, j,k = 1,2,\ldots,2n\}, \tag{2.1.24}$$

and, by applying Corollary 2.1.35, that

$$v_n(T) = \max\{|x_k'(T(x_j))| \,:\, k = 1,2,\ldots,2n;\ j = k,k+1\}. \tag{2.1.25}$$

Now, consider the operator $U \in BL(X_n)$ represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In order to compute the norm $\|\cdot\|_n$ and the numerical radius $v_n(\cdot)$ of U we need an explicit formula for $x_k'(U(x_j))$. For $j,k \in \{1,2,\ldots,2n\}$ we have

$$x_k'(U(x_j))$$
$$= \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left(-\cos\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right) \sin\left(\frac{j\pi}{n}\right) + \sin\left(\frac{k\pi}{n} + \frac{\pi}{2n}\right) \cos\left(\frac{j\pi}{n}\right) \right),$$

and hence, we deduce

$$x_k'(U(x_j)) = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \sin\left(\frac{(k-j)\pi}{n} + \frac{\pi}{2n}\right). \tag{2.1.26}$$

As a consequence, for every $k \in \{1,\ldots,2n\}$ we have

$$|x_k'(U(x_k))| = |x_k'(U(x_{k+1}))| = \tan\left(\frac{\pi}{2n}\right),$$

and hence, by (2.1.25), we get $v_n(U) = \tan\left(\frac{\pi}{2n}\right)$. On the other hand, using the equality (2.1.26) and Fact 2.1.36, we have

$$|x_k'(U(x_j))| = \frac{1}{\cos\left(\frac{\pi}{2n}\right)} \left| \sin\left(\frac{(k-j)\pi}{n} + \frac{\pi}{2n}\right) \right| \leqslant 1 \qquad j,k \in \{1,\ldots,2n\},$$

and the equality holds for $k = \frac{n}{2}$ and $j = 2n$, so that, by (2.1.24), we have $\|U\|_n = 1$. Now that we know $v_n(U)$ and $\|U\|_n$, the inequality $N(X_n) \leqslant \tan\left(\frac{\pi}{2n}\right)$ becomes obvious. □

Given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of normed spaces over $\mathbb{K}$, we denote by $\oplus_{\lambda \in \Lambda}^{\ell_1} X_\lambda$, the ℓ_1-sum of the family. In the case that the family has just two elements (say X, Y), we use the simpler notation $X \oplus_1 Y$.

Lemma 2.1.38 *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of normed spaces over $\mathbb{K}$, and let Z stand for $\oplus_{\lambda\in\Lambda}^{\ell_1} X_\lambda$. Then we have*

$$N(Z) = \inf_\lambda N(X_\lambda).$$

Moreover, if $N(X_\lambda) > 0$ for every $\lambda \in \Lambda$, then I_Z is a vertex of the closed unit ball of $BL(Z)$.

Proof For any normed spaces X and Y, let E stand for $X \oplus_1 Y$. We first check that $N(E) \leqslant N(X)$ by showing that $N(E) \leqslant v(S)$ for any $S \in BL(X)$ with $\|S\| = 1$. For such an operator S, let $T \in BL(E)$ be given by $T(x,y) = (S(x),0)$. Then $\|T\| = 1$ and hence $N(E) \leqslant v(T)$, so that, by Corollary 2.1.33, for every $\varepsilon > 0$ we may find $z = (x,y) \in \mathbb{S}_E$ and $z' = (x',y') \in \mathbb{S}_{E'}$ such that

$$x'(x) + y'(y) = \|x'\|\|x\| + \|y'\|\|y\| = 1 \tag{2.1.27}$$

and

$$N(E) - \varepsilon \leqslant \left|z'(T(z))\right| = |x'(S(x))| \leqslant \left|\frac{x'}{\|x'\|}\left(S\left(\frac{x}{\|x\|}\right)\right)\right|$$

(we may assume that $N(E) > 0$) implying that $N(E) - \varepsilon \leqslant v(S)$, and therefore $N(E) \leqslant v(S)$, because $x'(x) = \|x'\|\|x\|$ by (2.1.27).

For fixed $\lambda_0 \in \Lambda$ we clearly have

$$Z = \left[\oplus_{\lambda\neq\lambda_0}^{\ell_1} X_\lambda\right] \oplus_1 X_{\lambda_0},$$

so, by the above paragraph, we get $N(Z) \leqslant N(X_{\lambda_0})$ and it follows that

$$N(Z) \leqslant \inf_\lambda N(X_\lambda).$$

Now, let us prove the reverse inequality. Let T be in $BL(Z)$. Then T can be seen as a family $(T_\lambda)_{\lambda\in\Lambda}$ where $T_\lambda \in BL(X_\lambda, Z)$ for every $\lambda \in \Lambda$, and $\|T\| = \sup_\lambda \|T_\lambda\|$. Let $\varepsilon > 0$. Then we can find $\lambda_0 \in \Lambda$ such that $\|T_{\lambda_0}\| > \|T\| - \varepsilon$, and we write $Z = X_{\lambda_0} \oplus_1 Y$, where $Y := \oplus_{\lambda\neq\lambda_0}^{\ell_1} X_\lambda$, and $T_{\lambda_0} = (A,B)$ where $A \in BL(X_{\lambda_0})$ and $B \in BL(X_{\lambda_0}, Y)$. Now we choose $x_0 \in \mathbb{S}_{X_{\lambda_0}}$ such that

$$\|T_{\lambda_0}(x_0)\| = \|A(x_0)\| + \|B(x_0)\| > \|T\| - \varepsilon,$$

find $a_0 \in \mathbb{S}_{X_{\lambda_0}}$, $y' \in \mathbb{S}_{Y'}$ satisfying

$$\|A(x_0)\|a_0 = A(x_0) \ \text{ and } \ y'(B(x_0)) = \|B(x_0)\|,$$

and define an operator $S \in BL(X_{\lambda_0})$ by $S(x) = A(x) + y'(B(x))a_0$ for every $x \in X_{\lambda_0}$. Then

$$\|S\| \geqslant \|S(x_0)\| = \Big\|A(x_0) + \|B(x_0)\|a_0\Big\| = \|A(x_0)\| + \|B(x_0)\| > \|T\| - \varepsilon,$$

so we may find $x \in X_{\lambda_0}$, $x' \in X'_{\lambda_0}$ such that

$$\|x\| = \|x'\| = x'(x) = 1 \ \text{ and } \ |x'(S(x))| \geqslant N(X_{\lambda_0})[\|T\| - \varepsilon].$$

For $z=(x,0)\in\mathbb{S}_Z$ and $z'=(x',x'(a_0)y')\in\mathbb{S}_{Z'}$ we clearly have $z'(z)=1$ and

$$\begin{aligned}|z'(T(z))| &= |x'(A(x))+x'(a_0)y'(B(x))|\\ &= |x'(S(x))|\\ &\geqslant N(X_{\lambda_0})[\|T\|-\varepsilon]. \qquad (2.1.28)\end{aligned}$$

The desired inequality $N(Z)\geqslant \inf_\lambda N(X_\lambda)$ follows. Moreover, if $T\neq 0$, and if $N(X_\lambda)>0$ for every $\lambda\in\Lambda$, then the inequality (2.1.28) shows that $v(T)>0$. Since T is arbitrary in $BL(Z)$, it follows that I_Z is a vertex of the closed unit ball of $BL(Z)$. □

Proposition 2.1.39 *There exists a real Banach space Z such that I_Z is a vertex of the closed unit ball of $BL(Z)$, but is not a geometrically unitary element of $BL(Z)$.*

Proof By Corollary 2.1.29 and Lemma 2.1.37, for each $n\in 2\mathbb{N}$, we can find a two-dimensional real normed space X_n with $N(X_n)>0$, in such a way that the sequence $N(X_n)$ converges to zero. Now, set $Z:=\oplus_{n\in 2\mathbb{N}}^{\ell_1}X_n$. It follows from Lemma 2.1.38 that I_Z is a vertex of the closed unit ball of $BL(Z)$ and that $N(Z)=0$. But, by Theorem 2.1.17(i), the equality $N(Z)=0$ is equivalent to the fact that I_Z is not a geometrically unitary element of $BL(Z)$. □

Let X be a normed space, and let u be in X. The element $u\in X$ is said to be a *strongly extreme point* of $\mathbb{B}_X$ if $\|u\|=1$ and, whenever x_n and y_n are sequences in $\mathbb{B}_X$ with $\lim_{n\to\infty}\frac{1}{2}(x_n+y_n)=u$, we have $\lim_{n\to\infty}(x_n-y_n)=0$. Clearly, strongly extreme points of $\mathbb{B}_X$ are extreme points of $\mathbb{B}_X$. Assume that $\|u\|=1$. We define the *modulus of midpoint local convexity* of X at u as the function $\delta_X(u,\cdot):\mathbb{R}^+\to\mathbb{R}$ given by

$$\delta_X(u,\varepsilon):=\inf\left\{\max_{\pm}\|u\pm x\|-1 : x\in\varepsilon\mathbb{S}_X\right\}.$$

It is clear that, if Y is a subspace of X containing u, then we have

$$\delta_X(u,\cdot)\leqslant\delta_Y(u,\cdot). \qquad (2.1.29)$$

Lemma 2.1.40 *Let X be a normed space, and let u be a norm-one element of X. Then u is a strongly extreme point of $\mathbb{B}_X$ if and only if $\delta_X(u,\varepsilon)>0$ for every $\varepsilon\in\mathbb{R}^+$.*

Proof If for some $\varepsilon\in\mathbb{R}^+$ we have $\delta_X(u,\varepsilon)=0$, then for each $n\in\mathbb{N}$ there exists $z_n\in\varepsilon\mathbb{S}_X$ such that $\|u\pm z_n\|-1\leqslant\frac{1}{n}$. For each $n\in\mathbb{N}$, define

$$x_n:=\left(1+\frac{1}{n}\right)^{-1}(u+z_n)\ \text{ and }\ y_n:=\left(1+\frac{1}{n}\right)^{-1}(u-z_n).$$

It is clear that $x_n,y_n\in\mathbb{B}_X$ and that $\frac{1}{2}(x_n+y_n)=(1+\frac{1}{n})^{-1}u$ converges to u, but $x_n-y_n=2(1+\frac{1}{n})^{-1}z_n$ does not converge to 0. Thus u is not a strongly extreme point of $\mathbb{B}_X$.

Conversely, assume that u is not a strongly extreme point of $\mathbb{B}_X$, so that there are sequences x_n and y_n in $\mathbb{B}_X$ satisfying

$$\lim_{n\to\infty}\frac{1}{2}(x_n+y_n)=u\ \text{ and }\ \lim_{n\to\infty}(x_n-y_n)\neq 0.$$

By passing to subsequences if necessary, we may assume that $\|x_n - y_n\| > 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n - y_n\| = \varepsilon > 0$. Note that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\| u \pm \frac{\varepsilon}{2} \frac{x_n - y_n}{\|x_n - y_n\|} \right\| \\
&= \left\| u - \frac{1}{2}(x_n + y_n) + \frac{1}{2}(x_n + y_n) \pm \frac{\varepsilon}{2} \frac{x_n - y_n}{\|x_n - y_n\|} \right\| \\
&\leqslant \left\| u - \frac{1}{2}(x_n + y_n) \right\| + \left\| \frac{1}{2}\left(1 \pm \frac{\varepsilon}{\|x_n - y_n\|}\right) x_n + \frac{1}{2}\left(1 \mp \frac{\varepsilon}{\|x_n - y_n\|}\right) y_n \right\| \\
&\leqslant \left\| u - \frac{1}{2}(x_n + y_n) \right\| + \frac{1}{2}\left| 1 \pm \frac{\varepsilon}{\|x_n - y_n\|} \right| + \frac{1}{2}\left| 1 \mp \frac{\varepsilon}{\|x_n - y_n\|} \right| \\
&= \left\| u - \frac{1}{2}(x_n + y_n) \right\| + \frac{1}{2}\left| 1 + \frac{\varepsilon}{\|x_n - y_n\|} \right| + \frac{1}{2}\left| 1 - \frac{\varepsilon}{\|x_n - y_n\|} \right|,
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \max_{\pm} \left\| u \pm \frac{\varepsilon}{2} \frac{x_n - y_n}{\|x_n - y_n\|} \right\| - 1 = 0.$$

As a result, $\delta_X(u, \frac{\varepsilon}{2}) = 0$. □

Proposition 2.1.41 *Let X be a normed space over $\mathbb{K}$, and let u be a norm-one element of X. Then we have*

$$\delta_X(u, \varepsilon) \geqslant \sqrt{1 + n(X, u)^2 \varepsilon^2} - 1. \tag{2.1.30}$$

As a consequence, if u is a geometrically unitary element of X, then u is a strongly extreme point of $\mathbb{B}_X$.

Proof Let $\varepsilon > 0$, let x be in $\varepsilon \mathbb{S}_X$, and let f be in $D(X, u)$. Then we have

$$\max_{\pm} \|u \pm x\|^2 \geqslant \max_{\pm} |f(u \pm x)|^2 \geqslant \frac{1}{2}(|f(u + x)|^2 + |f(u - x)|^2) = 1 + |f(x)|^2.$$

Taking supremum over $f \in D(X, u)$, we get

$$\max_{\pm} \|u \pm x\|^2 \geqslant 1 + |v(X, u, x)|^2 \geqslant 1 + n(X, u)^2 \varepsilon^2,$$

so

$$\max_{\pm} \|u \pm x\| - 1 \geqslant \sqrt{1 + n(X, u)^2 \varepsilon^2} - 1,$$

and (2.1.30) follows by taking infimum in $x \in \varepsilon \mathbb{S}_X$. The consequence in the statement of the proposition follows from (2.1.30) by keeping in mind Theorem 2.1.17(i) and Lemma 2.1.40. □

Corollary 2.1.42 *Let A be a norm-unital normed algebra over $\mathbb{K}$. Then the unit of A is a strongly extreme point of $\mathbb{B}_A$. More precisely, we have*

$$\delta_A(\mathbf{1}, \varepsilon) \geqslant \sqrt{1 + e^{-2} \varepsilon^2} - 1. \tag{2.1.31}$$

Proof If $\mathbb{K} = \mathbb{C}$, then the result follows from Corollary 2.1.19 and Propositions 2.1.11 and 2.1.41. In the case $\mathbb{K} = \mathbb{R}$, the result follows from the complex case, keeping in mind that: (i) both the notion of strongly extreme point and the definition of modulus of midpoint local convexity involve only real scalars; (ii) $A_{\mathbb{C}}$ is a norm-unital normed algebra (by Proposition 1.1.98); and (iii) $\delta_A(\mathbf{1},\varepsilon) \geqslant \delta_{A_{\mathbb{C}}}(\mathbf{1},\varepsilon)$ (by (2.1.29)). $\square$

The next example shows that, in general, a vertex of the closed unit ball of a Banach space X need not be a strongly extreme point of $\mathbb{B}_X$.

Example 2.1.43 Let X be the space of all sequences in $\mathbb{K}$ converging to zero, with norm defined by

$$\|(a_n)_{n\in\mathbb{N}}\| := \sup\left\{\frac{1}{n}(|a_1|+|a_n|)+\left(1-\frac{1}{n}\right)\max\{|a_1|,|a_n|\} : n=2,3,\ldots\right\}.$$

Notice that

$$\frac{1}{2}\sup\{|a_n| : n=2,3,\ldots\} \leqslant \|(a_n)_{n\in\mathbb{N}}\| \leqslant |a_1|+2\sup\{|a_n| : n=2,3,\ldots\},$$

so X is isomorphic to c_0 and X' is isomorphic to ℓ^1. Let $\{e_n\}$ be the standard Schauder basis of X and $\{e_n'\}$ the standard Schauder basis of X'; that is, $e_m'((a_n)_{n=1}^{\infty}) = a_m$. It is easy to check that the following functionals are in $D(X,e_1)$:

$$e_1', e_1'+\frac{1}{2}e_2', \ldots, e_1'+\frac{1}{n}e_n', \ldots$$

and the linear hull of these functionals is w^*-dense in X', so e_1 is a vertex. The point e_1 is however not strongly extreme. Indeed, since

$$\|e_1 \pm e_n\| = \frac{2}{n}+\left(1-\frac{1}{n}\right) \longrightarrow 1 \ \text{ while } \ \|e_n\| = 1,$$

it is enough to take $x_n := \frac{e_1+e_n}{\|e_1+e_n\|}$ and $y_n := \frac{e_1-e_n}{\|e_1-e_n\|}$ to realize that

$$\lim_{n\to\infty}\frac{1}{2}(x_n+y_n) = e_1 \ \text{ but } \ \lim_{n\to\infty}\|x_n-y_n\| = 1.$$

Now, we are going to summarize and clarify several results in the current section by discussing the diagram in §2.1.44 immediately below.

§2.1.44 Let X be a nonzero normed space over $\mathbb{K}$, and let u be in X. Then, keeping in mind Lemma 2.1.25 and Proposition 2.1.41, the implications in the following diagram hold.

$$\begin{array}{ccc}
\boxed{u \text{ geometrically unitary}} & \Longrightarrow & \boxed{u \text{ vertex of } \mathbb{B}_X} \\
\Downarrow & & \Downarrow \\
\boxed{u \text{ strongly extreme point of } \mathbb{B}_X} & \Longrightarrow & \boxed{u \text{ extreme point of } \mathbb{B}_X}
\end{array}$$

If X is a norm-unital normed complex algebra, and if u is the unit of X, then, by Corollary 2.1.19, the strongest condition in the diagram is automatically true. On the other hand, by Proposition 2.1.39 and Corollary 2.1.42, there exists a norm-unital normed (necessarily real) algebra X whose unit u is both a vertex and a strongly extreme point of $\mathbb{B}_X$, but not a geometrically unitary element of X. Thus the horizontal implication in the top row and the vertical implication in the left column of the above diagram are simultaneously non-reversible, even in the setting of norm-unital normed real algebras.

According to Corollary 2.1.42 and Example 2.1.43, there is a choice of (X,u) (necessarily outside the setting of norm-unital normed algebras) such that u is a vertex of $\mathbb{B}_X$ but not a strongly extreme point of $\mathbb{B}_X$. As a consequence, the horizontal implication in the bottom row is not reversible.

To conclude the discussion of the above diagram, note that, by taking $X = \mathbb{C}_{\mathbb{R}}$, and $u = 1$, we are provided with an example where u is a strongly extreme point of $\mathbb{B}_X$ but not a vertex of $\mathbb{B}_X$. As a consequence, the vertical implication in the right column is not reversible, even in the setting of (necessarily real) norm-unital normed algebras.

2.1.3 Historical notes and comments

This section, as well as a great part of the present work, is tributary to the codification of the theory of numerical ranges done in the Bonsall and Duncan books [694, 695, 696]. Most results from Proposition 2.1.1 to Corollary 2.1.13, including their proofs, are taken almost verbatim from those books. The material just quoted is originally due to Bohnenblust and Karlin [108], who seem to have been the first authors to consider algebra numerical ranges. Proposition 2.1.5 is folklore. It follows, for example, from [726, Theorem V.9.5].

Let X be a complex Banach space. As a consequence of Corollary 2.1.13, I_X is a vertex of the closed unit ball of $BL(X)$. Actually, I_X remains a vertex of the closed unit ball of a much larger Banach space. More precisely, by passing to restrictions to $\mathbb{B}_X$, the space $BL(X)$ can be identified with a closed subspace of the sup-normed Banach space $A_0(X)$ of all bounded continuous functions from $\mathbb{B}_X$ to X which are holomorphic in the open unit ball of X and vanish at zero, and it follows straightforwardly from Proposition 2 in Harris' paper [312] that I_X becomes a vertex of $\mathbb{B}_{A_0(X)}$. In fact, minor changes to the proof of [312, Proposition 2] allow us to realize that I_X is a vertex of $\mathbb{B}_{A(X)}$, where $A(X)$ stands for the Banach space of all bounded continuous functions from $\mathbb{B}_X$ to X which are holomorphic in the open unit ball of X. A sketch of the proof could be the following:

Let F be in $A(X)$ such that $V(A(X),I_X,F) = 0$. We must show that $F = 0$. To this end, we take a norm-one element $x \in X$ and $\ell \in D(X,x)$, and consider the function $f : \mathbb{S}_{\mathbb{C}} \to \mathbb{C}$ defined by $f(\lambda) := \lambda^{-1}\ell(F(\lambda x))$. Since, for $\lambda \in \mathbb{S}_{\mathbb{C}}$, the mapping $G \to \lambda^{-1}\ell(G(\lambda x))$ from $A(X)$ to $\mathbb{C}$ is an element of $D(A(X),I_X)$, we have $f(\lambda) = 0$ (and hence $\ell(F(\lambda x)) = 0$) for every $\lambda \in \mathbb{S}_{\mathbb{C}}$. By the Cauchy formula for the circumference, we have in fact $\ell(F(\lambda x)) = 0$ for every $\lambda \in \mathbb{B}_{\mathbb{C}}$. With the notation in [312, pp. 1006–7], the above implies that

$$W(F_s) := \{\ell(F_s(x)) : \|x\| = 1, \ell \in D(X,x)\} = 0 \text{ for every } s \in]0,1[.$$

Finally, by [312, Theorem 1], we have $F = 0$, as required. $\square$

Lemma 2.1.15 is taken from the Bandyopadhyay–Jarosz–Rao paper [56]. Theorem 2.1.17 is due to Martínez, Mena, Payá, and Rodríguez [425], and was partially rediscovered in [56]. Theorem 2.1.17 is derived in [425] from the following result, also proved there.

Theorem 2.1.45 *Let (X,u) be a numerical-range space over $\mathbb{K}$, let C be a convex closed subset of $\mathbb{K}$ with non-empty interior, and let x'' be in X'' with $\{x''(f) : f \in D(X,u)\} \subseteq C$ (which happens for example if x'' is in X'' with $V(X'',u,x'') \subseteq C$). Then there exists a net x_λ in X such that $V(X,u,x_\lambda) \subseteq C$ for every λ, and $x'' = w^*-\lim_\lambda x_\lambda$.*

This generalizes a similar result previously proved by Smith [592] for norm-unital complete normed associative complex algebras. Theorem 2.1.17 is nothing other than an abstract version of Corollary 2.1.19, proved previously and independently by Moore [446] and Sinclair [579], and shows that the Moore–Sinclair theorem and Proposition 2.1.11 are 'equivalent'.

Example 2.1.18 has been pointed out to us by M. Martín (recent private communication). Corollaries 2.1.21–2.1.24 are taken from [56]. Lemma 2.1.23 is taken from [332].

Lemma 2.1.26 is due to Kadison [358]. Theorem 2.1.27 is originally due to Bohnenblust and Karlin [108] (1955). At that time, neither Theorem 2.1.17 nor even its Corollary 2.1.21 (a straightforward consequence of the earlier Moore–Sinclair Corollary 2.1.19) were known. However, the particularization of Corollary 2.1.21 to unital C^*-algebras (applied in our proof) was always C^*-folklore. Although included in Palmer's book [787] (2001), Theorem 2.1.27 remained forgotten by many people until its rediscovery by Akemann and Weaver [5] (2002). However, the Akemann–Weaver paper actually contains new relevant results along the lines of characterizing in Banach space terms the elements in certain distinguished subsets of C^*-algebras. As a sample, we have the following.

Theorem 2.1.46 *A norm-one element u of a C^*-algebra A is a partial isometry (i.e. $uu^*u = u$) if and only if the sets*

$$\{x \in A : \text{ there exists } \alpha > 0 \text{ with } \|u+\alpha x\| = \|u-\alpha x\| = 1\}$$

and

$$\{x \in A : \|u+\beta x\| = \max\{1, \|\beta x\|\} \text{ for all } \beta \in \mathbb{C}\}$$

coincide.

The Akemann–Weaver paper [5] has had the additional merit of encouraging the study of geometrically unitary elements in normed spaces, started earlier in [425], giving rise to a large series of papers, such as [56], already quoted, and [293] and [399], which will be discussed later in some detail (see Sections 2.9 and 3.1, respectively).

§2.1.47 It was an open problem for many years whether the inequality (2.1.18) (that $N(X) \geqslant N(X')$ for every nonzero normed space X) is in fact an equality. An affirmative answer was even claimed without proof in [216] (1987). As a matter of fact, Boyko, Kadets, Martín, and Werner [122] (2007) found an example of a nonzero Banach space X with $N(X) > N(X')$. Namely, *the Banach space*

$$X = \{(x,y,z) \in c \oplus_\infty c \oplus_\infty c : \lim x + \lim y + \lim z = 0\}$$

satisfies that $N(X) = 1$ *and* $N(X') < 1$ *in both the real and the complex case.* Actually, essentially with the same techniques, it is shown in [122] that *there are examples of a real Banach space* Y*, and of a complex Banach space* Z*, both isomorphic to* c_0*, such that*

$$N(Y) = 1, N(Y') = 0, \text{ and } N(Z) = 1, N(Z') = \frac{1}{e}.$$

Keeping in mind Theorem 2.1.17(i), the real Banach space Y above satisfies that I_Y is a geometrically unitary element of $BL(Y)$, but $I_{Y'}$ is not a geometrically unitary element of $BL(Y')$. (As a consequence, the converse of Corollary 2.1.24 does not hold.) In fact, *there is a real Banach space* X *such that* I_X *is a geometrically unitary element of* $BL(X)$ *(with* $N(X) = 1$*), whereas* $I_{X'}$ *is not even a vertex of* $\mathbb{B}_{BL(X')}$. This follows from Theorem 3.3 in Martín's paper [417] by taking E equal to the two-dimensional Euclidean real space, and X equal to the space $X(E)$ in that theorem.

Corollary 2.1.29 is nothing other than the particularization to the case $n = 2$ of the following theorem of Rosenthal [542]. For any real normed space X, set $Z(X) := \{T \in BL(X) : V(T) = 0\}$.

Theorem 2.1.48 *A real normed space* X *of dimension* $n \in \mathbb{N}$ *is a Hilbert space provided* $\dim(Z(X)) > \frac{(n-1)(n-2)}{2}$*, in which case* $\dim(Z(X)) = \frac{n(n-1)}{2}$.

A rediscovery of Rosenthal's theorem, and related results, can be found in [420]. Corollary 2.1.29 can be also derived from [731, Proposition 9.4.5] (a result also due to Rosenthal [543]). The proof given here, based on the rather technical Lemma 2.1.28, could be new.

The spacial numerical range of a bounded linear operator on a normed space was introduced by Bauer [61], extending Toeplitz' numerical range of matrices [613], and, concerning most applications, it is equivalent to Lumer's numerical range [407]. This is so because of Proposition 2.1.31, essentially proved in [407]. The actual formulation of Proposition 2.1.31, as well as Corollaries 2.1.33 and 2.1.34 and their proofs, are taken from [694, Section 9].

§2.1.49 Let X be a normed space over $\mathbb{K}$, and let Y be a (possibly non-closed) subspace of X. We write $\Pi(Y,X)$ to denote the subset of $\mathbb{S}_Y \times \mathbb{S}_{X'}$ given by

$$\Pi(Y,X) := \{(y,x') \in \mathbb{S}_Y \times \mathbb{S}_{X'} : x' \in D(X,y)\},$$

so that, in the case $Y = X$, we have $\Pi(X,X) = \Pi(X)$ in the sense of §2.1.30. Given a function $f : \mathbb{S}_Y \to X$, we define the *spatial numerical range*, $W(f)$, of f by

$$W(f) := \{x'(f(y)) : (y,x') \in \Pi(Y,X)\}.$$

This notion applies in particular to (possibly unbounded) linear operators T from Y to X by simply considering the restriction of T to $\mathbb{S}_Y$, and consequently setting $W(T) := W(T_{|\mathbb{S}_Y})$. Applying this convention in the case that $Y = X$ and the linear operator T is bounded, we re-encounter the particular notion introduced in §2.1.32. From now on, assume that X is a Banach space. In the case that Y is dense in X, spatial numerical ranges of (possibly unbounded) linear operators from Y to X have been successfully considered by Lumer and Phillips [409] and Arendt, Chernoff, and Kato [27]. The case that Y is a closed proper subspace of X and that the linear operator $T : Y \to X$ is bounded has been considered by Harris [315], and more recently by Martín, Merí, and Payá [419], in order to discuss whether Corollary 2.1.33 remains true in this more general context. Denote by I_Y^X the inclusion mapping $Y \hookrightarrow X$. It is proved in [315] that *the equality*

$$\overline{\mathrm{co}}(W(T)) = V(BL(Y,X), I_Y^X, T) \tag{2.1.32}$$

holds for every $T \in BL(X,Y)$ if either Y is finite-dimensional or X is uniformly smooth. (For the notion of a uniformly smooth normed space, see Definition 2.9.45 below.) In turn, the authors of [419] prove that, *fixing Y arbitrarily in the class of all infinite-dimensional Banach spaces, there is a choice of X in such a way that there exists $T \in BL(Y,X)$ such that the equality (2.1.32) fails*, and that, *fixing X arbitrarily in the class of all non-reflexive Banach spaces, up to equivalent renorming, there is a choice of Y in such a way that there exists $T \in BL(Y,X)$ such that the equality (2.1.32) fails*. Moreover, they introduce a sufficient condition on the couple (Y,X) (weaker than the uniform smoothness of X or the finite dimensionality of Y) assuring that the equality (2.1.32) holds for every $T \in BL(Y,X)$.

Now, let X be a Banach space. Denote by $B(\mathbb{S}_X, X)$ the Banach space of all bounded functions from $\mathbb{S}_X$ to X, endowed with the natural sup norm, and by I_X, the inclusion mapping $\mathbb{S}_X \hookrightarrow X$. Then, for every $f \in B(\mathbb{S}_X, X)$, we clearly have

$$\overline{\mathrm{co}}(W(f)) \subseteq V(B(\mathbb{S}_X, X), I_X, f). \tag{2.1.33}$$

In general, the above inclusion cannot be expected to be an equality. Indeed, as we will prove in Theorem 2.9.56, *the inclusion (2.1.33) becomes an equality for every $f \in B(\mathbb{S}_X, X)$ if and only if X is uniformly smooth*. The inclusion (2.1.33) is known to be an equality when X is complex and f is (the restriction to $\mathbb{S}_X$ of) a uniformly continuous function from $\mathbb{B}_X$ to X which is holomorphic on the interior of $\mathbb{B}_X$ [312]. More generally, the inclusion (2.1.33) becomes an equality whenever $f : \mathbb{S}_X \to X$ is any uniformly continuous bounded function [315]. In fact, as shown later in [534], we are provided with the following generalization of Proposition 2.1.31.

Theorem 2.1.50 *Let X be a Banach space, and let f be any uniformly continuous bounded function $f : \mathbb{S}_X \to X$. Then we have*

$$V(B(\mathbb{S}_X, X), I_X, f) = \overline{\mathrm{co}}\{x'(f(x)) : (x,x') \in \Gamma\},$$

where Γ is any subset of $\Pi(X)$ such that its natural projection $\pi_1(\Gamma)$ is dense in $\mathbb{S}_X$.

Corollary 2.1.35 can be derived from a result of Lima (see [400, Theorem 8]). The proof given here is taken from Martín's PhD thesis [774].

Fact 2.1.36 and Lemma 2.1.37 are due to Martín and Merí [418], who actually prove the following.

Theorem 2.1.51 *Let n be a positive integer greater than or equal to 2. For $k = 1, 2, \ldots, 2n$, set*

$$x_k := \left(\cos\left(\frac{k\pi}{n}\right), \sin\left(\frac{k\pi}{n}\right)\right),$$

and let X_n stand for the two-dimensional real Banach space such that

$$\operatorname{ext}(\mathbb{B}_{X_n}) = \{x_k \, : \, k = 1, 2, \ldots, 2n\}.$$

Then $N(X_n) = \tan\left(\frac{\pi}{2n}\right)$ if n is even, and $N(X_n) = \sin\left(\frac{\pi}{2n}\right)$ if n is odd.

Lemma 2.1.38 and Proposition 2.1.39 are due to Martín and Payá [421], who, concerning Lemma 2.1.38, take some ideas from the proof of Wojtaszczyk's Theorem 1 in [637]. The original proof of Proposition 2.1.39 does not invoke Corollary 2.1.29 and Lemma 2.1.37 (as we have done in our proof), but the real part of the following theorem of Duncan, McGregor, Pryce, and White [217].

Theorem 2.1.52 *For each $t \in [0,1]$ in the real case (respectively, $t \in [1/e, 1]$ in the complex case) there is a two-dimensional Banach space X with $N(X) = t$.*

The notion of a strongly extreme point seems to have been introduced by Lumer [407], who proved the non-quantitative complex version of Corollary 2.1.42. Lumer's result is included in [694, Theorem 4.5]. The non-quantitative real version of Corollary 2.1.42, although straightforwardly deducible from the complex one by passing to normed complexification, seems to appear first in the Harmand–Rao paper [311]. The modulus of midpoint local convexity was introduced by Milman [444]. Lemma 2.1.40 seems to be folklore. The quantitative version of Corollary 2.1.42 is due to Kadets, Katkova, Martín, and Vishnyakova [357] in both the real and complex cases. The non-quantitative version of Proposition 2.1.41, as well as Example 2.1.43, is due to Bandyopadhyay, Jarosz, and Rao [56]. As far as we know, the quantitative version of Proposition 2.1.41 does not appear formulated anywhere, although it is nothing other than the core of the argument in [357] to prove Corollary 2.1.42. For additional information about strongly extreme points, the reader is referred to the papers of Kunen and Rosenthal [395] and McGuigan [438].

Let X be a normed space, and let u be in $\mathbb{S}_X$. The element u is said to be a *strongly exposed point* of $\mathbb{B}_X$ if there exists $g \in \mathbb{S}_{X'}$ with the property that, whenever x_n is a sequence in $\mathbb{B}_X$ such that $g(x_n) \to 1$, we have $x_n \to u$. It is well known that, if u is a strongly exposed point, then u is a *denting point* of $\mathbb{B}_X$, which means that there are slices of $\mathbb{B}_X$ of arbitrarily small diameter which contain u. On the other hand, if u is a denting point of $\mathbb{B}_X$, then u is a strongly extreme point of $\mathbb{B}_X$ [438, Theorem 2.4]. Now, recall that, by Corollary 2.1.42, the unit of any norm-unital normed algebra A is a strongly extreme point of $\mathbb{B}_A$. Under renorming, a better result holds. Indeed, as shown in the Moreno–Rodríguez paper [452], we have the following.

Theorem 2.1.53 *Let A be a norm-unital normed algebra. Then there exists an equivalent algebra norm $|||\cdot|||$ on A arbitrarily close to $\|\cdot\|$, satisfying $|||\mathbf{1}||| = 1$, and such that $\mathbf{1}$ becomes a strongly exposed point of $\mathbb{B}_{(A,|||\cdot|||)}$.*

We note that, even if the norm-unital normed algebra A is associative, $\mathbf{1}$ need not be a denting point (much less a strongly exposed point) of $\mathbb{B}_A$. Indeed, the Banach algebra of all bounded linear operators on any infinite-dimensional complex Hilbert space has no denting point in its closed unit ball [301]. More generally, the closed unit ball of a C^*-algebra A is *dentable* (i.e. there are slices of $\mathbb{B}_A$ of arbitrarily small diameter) if and only if A is finite-dimensional [67].

§2.1.54 By a *unitary normed algebra* we mean a norm-unital normed associative algebra A such that $\mathbb{B}_A$ equals the closed convex hull of the set of all algebraically unitary elements of A (in the sense of §2.1.20). Unital C^*-algebras (see Lemma 2.3.28 below), as well as real or complex discrete group algebras $\ell_1(G)$ for every group G, are relevant examples of unitary Banach algebras. According to the Becerra–Cowell–Rodríguez–Wood paper [65], we have the following.

Theorem 2.1.55 *Let A be a unitary Banach algebra over $\mathbb{K}$ such that $\mathbb{B}_A$ is dentable, and let u be a norm-one element of A. Then the following conditions are equivalent:*

(i) *u is an algebraically unitary element of A.*
(ii) *u is a denting point of $\mathbb{B}_A$.*
(iii) *$\mathbb{B}_A$ equals the closed convex hull of the orbit of u under the group of all surjective linear isometries on A.*

We do not know if Theorem 2.1.27 remains true whenever 'unital C^*-algebra' is replaced with 'unitary complex Banach algebra' and condition (i) is erased, nor even if a Banach space characterization of algebraic unitaries in unitary complex Banach algebras can be found.

Unitary Banach algebras were first considered in Cowie's PhD thesis [713], but most of her results were not published elsewhere. Indeed, in her paper [181] we only find some incidental references to unitary Banach algebras. More recently, unitary Banach algebras have been reconsidered by Hansen and Kadison [309] without noticing Cowie's forerunner. One of the main goals in both [713] and [309] is to obtain characterizations of unital C^*-algebras among unitary Banach algebras by some extra conditions (a question raised by Gorin [296]). An example along these lines is the following proposition, proved in the works of both Cowie and Hansen–Kadison.

Proposition 2.1.56 *Let A be a unitary closed subalgebra of a C^*-algebra B. Then A is $*$-invariant, and hence it is a C^*-algebra.*

Proof The unit $\mathbf{1}$ of A is a norm-one idempotent in B, and hence, by Corollary 1.2.50, it is self-adjoint. Therefore, replacing B with the C^*-algebra $\mathbf{1}B\mathbf{1}$ if necessary, we may assume that B is unital with the same unit as that of A. Let u be an algebraically unitary element of A. Then u remains algebraically unitary in B with the same inverse, and hence, by the implication (ii)⇒(i) in Theorem 2.1.27, we have $u^*u = uu^* = \mathbf{1}$, which implies that $u^* = u^{-1} \in A$. Since $\mathbb{B}_A$ is the closed convex hull of the set of all algebraically unitary elements of A, it follows that A is $*$-invariant. □

For a survey of the more relevant results in [181, 309, 713] the reader is referred to [639]. Incidentally, unitary normed algebras will be considered again in Corollary 2.4.31, Remark 2.4.33, and Theorem 3.5.70. For further developments of the theory, see [62, 63, 64, 65, 80].

2.2 An application to Kadison's isometry theorem

Introduction We prove in Theorem 2.2.9(i) a non-associative germ of the celebrated Kadison theorem [358] on surjective linear isometries between unital C^*-algebras, and state in Theorem 2.2.19 the non-unital version of Kadison's theorem, due to Paterson and Sinclair [480]. Hermitian operators on C^*-algebras, as well as on non-associative algebras close to them, are also described (see Theorem 2.2.9(ii) and Proposition 2.2.26).

2.2.1 Non-associative results

We begin with the following easy but useful result.

Proposition 2.2.1 *Let (X,u) be a complex numerical-range space, and let T be a dissipative element in the numerical-range space $(BL(X),I_X)$ such that $T(u)=0$. Then*

$$T(H(X,u)) \subseteq H(X,u).$$

Proof First assume that X is a Banach space. Let $r>0$. Since T is dissipative in $(BL(X),I_X)$, Corollary 2.1.9(i) applies, so that we have $\|\exp(rT)\| \leqslant 1$. On the other hand, since $T(u)=0$, we have $\exp(rT)(u)=u$. It follows from Corollary 2.1.2 that $V(X,u,\exp(rT)(x)) \subseteq V(X,u,x)$ for every $x\in X$. As a consequence, we derive that $\exp(rT)(H(X,u)) \subseteq H(X,u)$. Keeping in mind that $H(X,u)$ is a closed real subspace of X, and that

$$T(x) = \lim_{r\to 0^+} \frac{1}{r}[(\exp(rT))(x)-x]$$

for every $x\in X$, we obtain $T(H(X,u)) \subseteq H(X,u)$, as required.

Now remove the assumption of completeness of X. Let $\hat{X}$ stand for the completion of X. Then, by Corollary 2.1.2, we have $V(X,u,x) = V(\hat{X},u,x)$ for every $x\in X$, and hence $H(X,u) = X\cap H(\hat{X},u)$. On the other hand, if for $F\in BL(X)$ we consider the unique operator $\hat{F}\in BL(\hat{X})$ extending F, then the mapping $F\to\hat{F}$ becomes a linear isometry from $BL(X)$ to $BL(\hat{X})$ preserving distinguished elements. Therefore, again by Corollary 2.1.2, we have $V(BL(X),I_X,F) = V(BL(\hat{X}),I_{\hat{X}},F)$ for every $F\in BL(X)$. As a consequence, since T is dissipative in $BL(X)$, so is $\hat{T}$ in $BL(\hat{X})$, and moreover, since $T(u)=0$, $\hat{T}(u)=0$ also. The proof is now concluded by applying the above paragraph with $(\hat{X},\hat{T})$ instead of (X,T). □

Corollary 2.2.2 *Let (X,u) be a complex numerical-range space, and let T be in $H(BL(X),I_X)$ such that $T(u)=0$. Then $iT(H(X,u)) \subseteq H(X,u)$.*

Proof The assumption that T is hermitian implies that iT is dissipative. Now, apply Proposition 2.2.1. □

Lemma 2.2.3 *Let A be a norm-unital normed complex algebra, and let h,k be in $H(A,\mathbf{1})$. We have:*

(i) *If B is any norm-unital normed complex algebra, and if $F:A\to B$ is a surjective linear isometry such that $F(\mathbf{1})=\mathbf{1}$, then*

$$i(F(hk)-F(h)F(k)) \in H(B,\mathbf{1}).$$

(ii) *If T belongs to $H(BL(A),I_A)$ with $T(\mathbf{1})=0$, then*

$$-T(hk)+T(h)k+hT(k)\in H(A,\mathbf{1}).$$

Proof Let B and F be as in assertion (i). Keeping in mind Corollary 2.1.2 and Lemma 2.1.10, we have

$$F(h)\in H(B,\mathbf{1}),\ \ L_h\in H(BL(A),I_A),\ \text{ and }\ L_{F(h)}\in H(BL(B),I_B).$$

Moreover, since the mapping $G\to F^{-1}GF$ from $BL(B)$ to $BL(A)$ is a linear isometry preserving distinguished elements, we also have

$$F^{-1}L_{F(h)}F\in H(BL(A),I_A).$$

Therefore $S:=L_h-F^{-1}L_{F(h)}F\in H(BL(A),I_A)$. Since $S(\mathbf{1})=0$, Corollary 2.2.2 applies, so that we have

$$i(hk-F^{-1}(F(h)F(k))=iS(k)\in H(A,\mathbf{1}).$$

Finally, again by Corollary 2.1.2, we get $i(F(hk)-F(h)F(k))\in H(B,\mathbf{1})$, as required.

To prove assertion (ii), as in the proof of Proposition 2.2.1, we may assume that A is complete. Let T be as in assertion (ii). Let r be in $\mathbb{R}\setminus\{0\}$. Then, by Corollary 2.1.9(iii), $\exp(irT)$ is a unit-preserving surjective linear isometry on A. Therefore, by assertion (i), we have

$$i\,[\exp(irT)(hk)-\exp(irT)(h)\exp(irT)(k)]\in H(A,\mathbf{1}),$$

and hence

$$-T(hk)+T(h)k+hT(k)=i\lim_{r\to 0}\frac{\exp(irT)(hk)-\exp(irT)(h)\exp(irT)(k)}{r}$$

lies in $H(A,\mathbf{1})$. □

§2.2.4 Given an element a in an algebra A, we denote by U_a the mapping

$$b\to a(ab+ba)-a^2b$$

from A to A. Thus $U_a=L_a(L_a+R_a)-L_{a^2}$.

Lemma 2.2.5 *Let A be a normed unital complex algebra, and let $*$ be a conjugate-linear vector space involution on A such that $\mathbf{1}^*=\mathbf{1}$. Assume that at least one of the following conditions holds:*

(i) $\|a^*a\|=\|a\|^2$ *for every $a\in A$,*
(ii) $\|U_a(a^*)\|=\|a\|^3$ *for every $a\in A$.*

Then we have $\|\mathbf{1}\|=1$ and $H(A,*)=H(A,\mathbf{1})$. Therefore

$$A=H(A,\mathbf{1})\oplus iH(A,\mathbf{1}).$$

Proof That $\|\mathbf{1}\|=1$ is straightforward. Assume that (i) holds. Let h and r be in $H(A,*)$ and $\mathbb{R}\setminus\{0\}$, respectively. Then we have

$$\|\mathbf{1}+irh\|^2=\|(\mathbf{1}-irh)(\mathbf{1}+irh)\|=\|\mathbf{1}+r^2h^2\|,$$

and hence

$$\frac{\|\mathbf{1}+irh\|^2-1}{r} = r\frac{\|\mathbf{1}+r^2h^2\|-1}{r^2}.$$

Therefore, by Proposition 2.1.5, we get $\max\Re(V(A,\mathbf{1},ih)) = 0$. Applying this last equality with $-h$ instead of h, we conclude that $\Re(V(A,\mathbf{1},ih)) = 0$, that is $h \in H(A,\mathbf{1})$. By the arbitrariness of $h \in H(A,*)$, we derive that $H(A,*) \subseteq H(A,\mathbf{1})$. In order to prove the converse inclusion, assume that $h \in H(A,\mathbf{1})$, and write $h = h_1 + ih_2$ with $h_1, h_2 \in H(A,*)$. Then, $h - h_1 = ih_2 \in H(A,\mathbf{1}) \cap iH(A,\mathbf{1})$, and therefore, by Corollary 2.1.13, we have $h = h_1 \in H(A,*)$.

Assume that (ii) holds. Let h and r be in $H(A,*)$ and $\mathbb{R}\setminus\{0\}$, respectively. Then we have

$$\|\mathbf{1}+irh\|^3 = \|U_{\mathbf{1}+irh}(\mathbf{1}-irh)\| = \|\mathbf{1}+irh+r^2h^2+ir^3(2hh^2-h^2h)\|,$$

and hence

$$\frac{\|\mathbf{1}+irh\|^3-1}{r} = \frac{\|\mathbf{1}+irh+r^2h^2+ir^3(2hh^2-h^2h)\|-1}{r}.$$

Therefore, by Corollary 2.1.6, we get

$$3\max\Re(V(A,\mathbf{1},ih)) = \max\Re(V(A,\mathbf{1},ih)),$$

whence $\max\Re(V(A,\mathbf{1},ih)) = 0$. The rest of the proof runs as above. □

Remark 2.2.6 Let A be a normed unital complex algebra, and let $*$ be a conjugate-linear vector space involution on A such that $\mathbf{1}^* = \mathbf{1}$. We will see much later (cf. Corollary 3.2.7) that condition (i) in Lemma 2.2.5 implies condition (ii) in the same lemma. Indeed, as we will prove in Theorem 3.2.5, condition (i) implies that A is 'very nearly' associative, and that $*$ is an algebra involution on A. By the way, according to Corollary 2.4.12 below, condition (ii) implies that A is 'nearly' associative, and that $*$ is an algebra involution on A.

Let A be an algebra over $\mathbb{K}$. We denote by A^{sym} the algebra consisting of the vector space of A and the product $a \bullet b := \frac{1}{2}(ab+ba)$, and remark that, if A is normed, then so is A^{sym} under the given norm on A. By a *derivation* of A we mean a linear operator D on A satisfying $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. By a *Jordan derivation* of A we mean a derivation of A^{sym}. We note that Jordan derivations of A are precisely those linear mappings $D : A \to A$ such that $D(a^2) = 2a \bullet D(a)$ for every $a \in A$. Now, let A and B be algebras over $\mathbb{K}$. By a *Jordan homomorphism* from A to B we mean an algebra homomorphism from A^{sym} to B^{sym}. We note that Jordan homomorphisms from A to B are precisely those linear mappings $F : A \to B$ such that $F(a^2) = F(a)^2$ for every $a \in A$.

Definition 2.2.7 By a *nice algebra* we mean a normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying

$$(a^2)^* = (a^*)^2 \text{ for every } a \in A \tag{2.2.1}$$

and some of the following conditions:

(i) $\|a^*a\| = \|a\|^2$ for every $a \in A$,
(ii) $\|U_a(a^*)\| = \|a\|^3$ for every $a \in A$.

According to Remark 2.2.6, the requirement (2.2.1) in the definition of a nice algebra is superfluous, but this is a deep fact which does not matter for the moment.

Lemma 2.2.8 *Let A be a nice algebra. Then we have:*

(i) $(x \bullet y)^* = x^* \bullet y^*$ *for all* $x, y \in A$.
(ii) $\mathbf{1}^* = \mathbf{1}$.
(iii) *A is norm-unital and* $H(A,*) = H(A,\mathbf{1})$.
(iv) *If h, k are in $H(A,\mathbf{1})$, then $h \bullet k$ lies in $H(A,\mathbf{1})$.*

Proof Assertion (i) is a straightforward consequence of the requirement $(a^2)^* = (a^*)^2$ for every $a \in A$. Since the unit $\mathbf{1}$ of A becomes a unit for A^{sym}, it follows from assertion (i) that $\mathbf{1}^*{=}\mathbf{1}$, which proves assertion (ii). Therefore, by the remaining requirements on nice algebras, and Lemma 2.2.5, we have $\|\mathbf{1}\| = 1$ and $H(A,*) = H(A,\mathbf{1})$, which proves (iii). Finally, (iv) follows from (i) and (iii). □

The above lemma must be kept in mind throughout the proof of Theorem 2.2.9 immediately below, and will not be explicitly invoked in that proof.

Theorem 2.2.9 *Let A be a nice algebra. We have:*

(i) *If B is a nice algebra, and if $F : A \to B$ is a unit-preserving surjective linear isometry, then F is a Jordan-$*$-homomorphism.*
(ii) *If T is in $H(BL(A), I_A)$, then there are $h \in H(A,*)$ and a continuous Jordan derivation D of A such that $T = L_h + D$ and $D(a^*) = -D(a)^*$ for every $a \in A$.*

Proof Let B be a nice algebra, and let $F : A \to B$ be a unit-preserving surjective linear isometry. By Corollary 2.1.2, we have $F(H(A,\mathbf{1})) \subseteq H(B,\mathbf{1})$, and hence F is $*$-mapping. Let h, k be in $H(A,*)$. Again by Corollary 2.1.2, we have

$$F(h \bullet k) - F(h) \bullet F(k) \in H(B,\mathbf{1}).$$

On the other hand, keeping in mind that

$$H(A,\mathbf{1}) = H(A^{\mathrm{sym}},\mathbf{1}) \ \text{ and } \ H(B,\mathbf{1}) = H(B^{\mathrm{sym}},\mathbf{1}),$$

Lemma 2.2.3(i) applies, so that we have

$$i(F(h \bullet k) - F(h) \bullet F(k)) \in H(B,\mathbf{1}).$$

It follows from Corollary 2.1.13 that

$$F(h \bullet k) - F(h) \bullet F(k) = 0.$$

Since h, k are arbitrary elements of $H(A,*)$, and $A = H(A,*) + iH(A,*)$, the proof of assertion (i) is concluded.

Now, let T be in $H(BL(A), I_A)$. Since the mapping $G \to G(\mathbf{1})$ from $BL(A)$ to A is a linear contraction preserving distinguished elements, Corollary 2.1.2 applies, so that

$h := T(\mathbf{1})$ lies in $H(A, \mathbf{1})$. Set $D := T - L_h$. Then $D(\mathbf{1}) = 0$ and, by Lemma 2.1.10, D belongs to $H(BL(A), I_A)$. By Proposition 2.2.1, we have

$$iD(k) \in H(A, *) \text{ for every } k \in H(A, *),$$

which implies that iD is a $*$-mapping (equivalently, $D(a^*) = -D(a)^*$ for every $a \in A$). As a consequence, for $k_1, k_2 \in H(A, *)$, we see that

$$i(-D(k_1 \bullet k_2) + D(k_1) \bullet k_2 + k_1 \bullet D(k_2)) \in H(A, \mathbf{1}),$$

and on the other hand, by Lemma 2.2.3(ii), we have

$$-D(k_1 \bullet k_2) + D(k_1) \bullet k_2 + k_1 \bullet D(k_2) \in H(A, \mathbf{1}).$$

It follows from Corollary 2.1.13 that

$$-D(k_1 \bullet k_2) + D(k_1) \bullet k_2 + k_1 \bullet D(k_2) = 0.$$

Since k_1, k_2 are arbitrary elements of $H(A, *)$, and $A = H(A, *) + iH(A, *)$, the proof of assertion (ii) is concluded. □

2.2.2 The Kadison–Paterson–Sinclair theorem

Now, we are going to apply Theorem 2.2.9 to derive the Paterson–Sinclair results concerning isometries and hermitian operators on C^*-algebras.

Lemma 2.2.10 *Let A be a C^*-algebra, let a be in A, and let T_a stand for the bounded linear operator on A defined by $T_a(b) := \frac{1}{2}(aa^*b + ba^*a)$. Then we have $\mathfrak{r}(T_a) = \|a\|^2$.*

Proof For $n \in \mathbb{N}$, we have $T_a^n(a) = a(a^*a)^n$, so

$$\begin{aligned}\|T_a^n\|^2\|a\|^2 \geqslant \|T_a^n(a)\|^2 = \|a(a^*a)^n\|^2 = \|(a^*a)^n a^* a(a^*a)^n\| \\ = \|(a^*a)^{2n+1}\| = \|a\|^{2(2n+1)},\end{aligned}$$

and so $\|T_a^n\| \geqslant \|a\|^{2n}$. Therefore we have $\mathfrak{r}(T_a) \geqslant \|a\|^2$. On the other hand, we clearly have $\mathfrak{r}(T_a) \leqslant \|T_a\| \leqslant \|a\|^2$. □

§2.2.11 Let X, Y, Z be normed spaces over $\mathbb{K}$, and let $m : X \times Y \to Z$ be a bounded bilinear mapping. The *adjoint operation* $m' : Z' \times X \to Y'$ is defined, for $(z', x) \in Z' \times X$, by

$$m'(z', x)(y) := z'(m(x, y)) \text{ for every } y \in Y. \tag{2.2.2}$$

It is straightforward that m' just defined becomes a bounded bilinear mapping with $\|m'\| = \|m\|$. The construction of m' from m can be iterated to get bounded bilinear mappings

$$m'' : Y'' \times Z' \to X' \text{ and } m^t := m''' : X'' \times Y'' \to Z''$$

satisfying $\|m''\| = \|m^t\| = \|m\|$. It is easily realized that m^t extends m.

Indeed, given $(x,y,z') \in X \times Y \times Z'$, and regarding $x, y, m(x,y)$ as elements of X'', Y'', Z'', respectively, we have

$$m'''(x,y)(z') = x(m''(y,z')) = m''(y,z')(x) = y(m'(z',x))$$
$$= m'(z',x)(y) = z'(m(x,y)) = m(x,y)(z'),$$

and hence $m'''(x,y) = m(x,y)$.

The extension m^t of m is called the *(first) Arens extension* of m. In the case that m is the product of a given normed algebra A, the product m^t on A'' will be called the *(first) Arens product* of A'', A'' will be considered without notice as a normed algebra (relative to its Arens product) containing A as a subalgebra, and the product of A'' will be denoted by juxtaposition.

Lemma 2.2.12 *Let X,Y,Z be normed spaces over $\mathbb{K}$, and let $m : X \times Y \to Z$ be a bounded bilinear mapping. We have:*

(i) *The adjoint $m' : Z' \times X \to Y'$ is w^*-continuous in its first variable.*
(ii) *The second adjoint $m'' : Y'' \times Z' \to X'$ is w^*-continuous in its second variable whenever the first one is fixed in Y.*
(iii) *The Arens extension m^t is the unique mapping from $X'' \times Y''$ to Z'' satisfying the following properties:*
   (a) *m^t extends m.*
   (b) *m^t is w^*-continuous in its first variable.*
   (c) *m^t is w^*-continuous in its second variable whenever the first one is fixed in X.*

Proof Let x be in X. By the definition (2.2.2) of m', the composition of the mapping $m'(\cdot,x) : Z' \to Y'$ with any w^*-continuous linear functional on Y' is a w^*-continuous linear functional on Z'. This proves assertion (i).

According again to the definitions, for $(x,y,z') \in X \times Y \times Z'$ we have

$$m''(y,z')(x) = y(m'(z',x)) = m'(z',x)(y) = z'(m(x,y)),$$

and hence the composition of the mapping $m''(y,\cdot) : Z' \to X'$ with any w^*-continuous linear functional on X' is a w^*-continuous linear functional on Z'. This proves assertion (ii).

Property (iii)(a) is already known. Property (iii)(b) follows by applying assertion (i), with m'' instead of m, whereas property (iii)(c) follows by applying assertion (ii), with m' instead of m. Now that we know that properties (iii)(a–c) are fulfilled, let g be any mapping from $X'' \times Y''$ to Z'' satisfying these properties with g instead of m^t. Then, for $x \in X$, the set

$$\{y'' \in Y'' : g(x,y'') = m^t(x,y'')\}$$

is w^*-closed in Y'' and contains Y. Since Y is w^*-dense in Y'', we deduce

$$g(x,y'') = m^t(x,y'') \text{ for every } (x,y'') \in X \times Y''.$$

Now, for $y'' \in Y''$, the set

$$\{x'' \in X'' : g(x'',y'') = m^t(x'',y'')\}$$

is w^*-closed in X'' and contains X. Since X is w^*-dense in X'', we deduce

$$g(x'',y'') = m^t(x'',y'') \text{ for every } (x'',y'') \in X'' \times Y''. \qquad \square$$

Corollary 2.2.13 *Let A be a normed unital algebra over $\mathbb{K}$. Then the unit of A remains a unit for A''.*

Proof By Lemma 2.2.12, the sets

$$\{a'' \in A'' : a''\mathbf{1} = a''\} \text{ and } \{a'' \in A'' : \mathbf{1}a'' = a''\}$$

are w^*-closed in A''. Since they contain A, and A is w^*-dense in A'', they are the whole algebra A''. $\square$

Definition 2.2.14 Let X,Y,Z be vector spaces over $\mathbb{K}$, and let $m : X \times Y \to Z$ be a bilinear mapping. We shall denote by m^r the mapping $(y,x) \to m(x,y)$ from $Y \times X$ into Z. Now, assume in addition that X,Y,Z are normed, and that m is bounded, and, according to §2.2.11, let $m^t : X'' \times Y'' \to Z''$ stand for the Arens extension of m. We will say that m is *Arens regular* if $m^{rt} = m^{tr}$. Note that m^{rtr} extends m because m^{rt} extends m^r. The extension m^{rtr} of m is called the *second Arens extension* of m. Thus, the Arens regularity of m is equivalent to the coincidence of its first and second Arens extensions. In the case that m is the product of a given normed algebra A, the product m^{rtr} on A'' will be called the *second Arens product* of A'', and A is said to be *Arens regular* if m is Arens regular (equivalently, if the first and second Arens products of A'' coincide).

The following theorem will be proved later (see §2.3.53). For the moment, the reader is referred to [806, Theorem 1.17.2].

Theorem 2.2.15 *Let A be a nonzero C^*-algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital C^*-algebra. As a consequence, A is Arens regular.*

§2.2.16 Let A be a nonzero C^*-algebra. It follows from Theorem 2.2.15 above that the set

$$\{a'' \in A'' : a''A \subseteq A \text{ and } Aa'' \subseteq A\}$$

becomes a closed $*$-subalgebra of A'' containing the unit of A'', and containing A as an ideal. This subalgebra of A'' is called the *C^*-algebra of multipliers* of A, and will be denoted by $M(A)$. We note that, by Corollary 2.2.13, if A has a unit, then $M(A) = A$. More information about $M(A)$ will be given in Propositions 2.3.56 and 2.3.57.

Given elements a,b in an algebra, we define the *commutator* $[a,b]$ of a,b by $[a,b] := ab - ba$.

Lemma 2.2.17 *Let A be a C^*-algebra. We have:*

(i) $M(A) = \{a'' \in A'' : a'' \bullet A \subseteq A\}$.
(ii) *If u is a unitary element of A'' such that $au^*a \in A$ and $ua^*u \in A$ for every $a \in A$, then u lies in $M(A)$.*

Proof Let a'' be in A'' such that $a'' \bullet A \subseteq A$. Then, for $b \in A$ we have

$$[a'',b^2] = 2[a'' \bullet b, b] \in [A,A] \subseteq A.$$

Since A is the linear hull of the set $\{b^2 : b \in A\}$, we derive $[a'', A] \subseteq A$. This inclusion, together with the assumed one $a'' \bullet A \subseteq A$, leads to $a''A \subseteq A$ and $Aa'' \subseteq A$. This proves the inclusion $\{a'' \in A'' : a'' \bullet A \subseteq A\} \subseteq M(A)$. Since the converse inclusion is clear, the proof of assertion (i) is concluded.

Let u be a unitary element of A'' such that $au^*a \in A$ and $ua^*u \in A$ for every $a \in A$. Then, for every $*$-invariant element $b \in A$ we have that $c := ubu$ lies in A, so $\frac{1}{2}(cu^*b + bu^*c)$ also lies in A, and so

$$u \bullet b^2 = \frac{1}{2}(ub^2 + b^2u) = \frac{1}{2}((ubu)u^*b + bu^*(ubu)) = \frac{1}{2}(cu^*b + bu^*c) \in A.$$

Since A is the linear hull of the set $\{b^2 : b^* = b \in A\}$ (by Proposition 1.2.48), we derive that $u \bullet A \subseteq A$. Now, applying assertion (ii), we get $u \in M(A)$, concluding the proof of assertion (ii). □

Remark 2.2.18 As a matter of fact, the first requirement in assertion (ii) of Lemma 2.2.17 (that $au^*a \in A$ for every $a \in A$) implies the second (that $ua^*u \in A$ for every $a \in A$). Indeed, this is a consequence of [3, Proposition 4.4] or [366, 7.7], which will be proved in Volume 2 of our work.

For elements a, b, c in a C^*-algebra, we set $\{abc\} := \frac{1}{2}(ab^*c + cb^*a)$.

Theorem 2.2.19 (Kadison–Paterson–Sinclair) *Let A and B be C^*-algebras, and let $F : A \to B$ be a bijective linear mapping. Then the following conditions are equivalent:*

(i) *F is an isometry.*
(ii) *There exists a unitary element $u \in M(B)$, and a bijective Jordan-$*$-homomorphism $G : A \to B$, satisfying $F(a) = uG(a)$ for every $a \in A$.*
(iii) *$F(\{xyz\}) = \{F(x)F(y)F(z)\}$ for all $x, y, z \in A$.*

Proof (i)⇒(ii) Assume that condition (i) holds. Then the bitranspose mapping $F'' : A'' \to B''$ is a surjective linear isometry. Therefore, by Theorem 2.1.27, $u := F''(\mathbf{1})$ is a unitary element of B''. Now, consider the unit-preserving surjective linear isometry $H : a'' \to u^*F''(a'')$ from A'' to B''. Then, by Theorem 2.2.9(i), H is a bijective Jordan-$*$-homomorphism, and we have clearly $F''(a'') = uH(a'')$ for every $a'' \in A''$. For an arbitrary element $b \in B$, we can write $b = F(a)$ for a suitable $a \in A$, so that

$$bu^*b = F(a)u^*F(a) = uH(a)u^*uH(a) = uH(a)^2 = uH(a^2) = F(a^2) \in B.$$

On the other hand, for $a'' \in A''$ we have

$$uF''(a'')^*u = uH(a'')^* = uH(a''^*) = F''(a''^*),$$

which implies $uF(a)^*u = F(a^*)$ for every $a \in A$, and hence $uBu \subseteq B$. It follows from Lemma 2.2.17(ii) that u belongs to $M(B)$. Therefore H maps A onto B, and the mapping $G : a \to H(a)$ from A onto B is a bijective Jordan-$*$-homomorphism.

(ii)⇒(iii) Assume that (ii) holds. Let x, y, z be in A. Then since

$$\{xyz\} := x \bullet (y^* \bullet z) + z \bullet (y^* \bullet x) - (x \bullet z) \bullet y^*,$$

and G is a Jordan-$*$-homomorphism, we have $G(\{xyz\}) = \{G(x)G(y)G(z)\}$. On the other hand, since u is a unitary element of $M(B)$, for $b_1, b_2, b_3 \in B$, the equality $u\{b_1b_2b_3\} = \{(ub_1)(ub_2)(ub_3)\}$ is straightforwardly verified. It follows that $F(\{xyz\}) = \{F(x)F(y)F(z)\}$, as required.

(iii)⇒(i) Assume that (iii) holds. Let a be in A, and let T_a be as in Lemma 2.2.10. Then, since for $x \in A$ we have $T_a(x) = \{aax\}$, we get $T_{F(a)} = FT_aF^{-1}$. According to Example 1.1.1(b), for every vector space X over $\mathbb{K}$, let $L(X)$ stand for the associative algebra over $\mathbb{K}$ consisting of all linear operators on X. Since the mapping $T \to FTF^{-1}$ is a bijective algebra homomorphism from $L(A)$ onto $L(B)$, we have $\mathrm{sp}(L(A),T_a) = \mathrm{sp}(L(B),T_{F(a)})$, and hence, by Corollary 1.1.47, $\mathfrak{r}(T_a) = \mathfrak{r}(T_{F(a)})$. Therefore, by Lemma 2.2.10, we have $\|a\| = \|F(a)\|$. □

Corollary 2.2.20 *Let A and B be C^*-algebras, and let $F : A \to B$ be a surjective linear isometry. Then $F''(M(A)) = M(B)$.*

Proof By the proof of the implication (i)⇒(ii) in Theorem 2.2.19, there exists $u \in M(B)$, together with a bijective Jordan-$*$-homomorphism $H : A'' \to B''$, such that $F''(a'') = uH(a'')$ for every $a'' \in A''$. Since, by Lemma 2.2.17(i), we have $H(M(A)) = M(B)$, we derive $F''(M(A)) = M(B)$, as required. □

Lemma 2.2.21 *Let A be a complete normed algebra, and let D be a continuous derivation of A. Then* $\exp(D)$ *is a bijective algebra endomorphism on A.*

Proof Let a,b be in A. By the Leibniz rule, for $n \in \mathbb{N}$ we have

$$D^n(ab) = \sum_{i=0}^{n} \binom{n}{i} D^i(a)D^{n-i}(b).$$

Therefore

$$\begin{aligned}\exp(D)(ab) &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{1}{i!(n-i)!} D^i(a)D^{n-i}(b) = \sum_{i,j=0}^{\infty} \frac{1}{i!j!} D^i(a)D^j(b) \\ &= \left(\sum_{i=0}^{\infty} \frac{1}{i!} D^i(a)\right)\left(\sum_{j=0}^{\infty} \frac{1}{j!} D^j(b)\right) = \exp(D)(a)\exp(D)(b).\end{aligned}$$ □

§2.2.22 We recall that an algebra A is said to be *semiprime* if, for every nonzero ideal I of A, there are $x,y \in I$ such that $xy \neq 0$. Note that C^*-algebras are semiprime.

Lemma 2.2.23 [678, Theorem 6.3.11] *Jordan derivations of semiprime associative algebras are derivations.*

Lemma 2.2.24 *Let X be a normed space, let S be in $BL(X)$, and let M be an S-invariant subspace of X, then we have*

$$V(BL(M), I_M, S_M) \subseteq V(BL(X), I_X, S),$$

where S_M stands for the restriction of S to M, regarded as an operator on M.

Proof The set A of those operators $T \in BL(X)$ such that $T(M) \subseteq M$ is a subspace of $BL(X)$ containing I_X, and the mapping $T \to T_M$ from A to $BL(M)$ is a linear contraction taking I_X to I_M. Therefore, by Corollary 2.1.2, we have

$$V(BL(M), I_M, S_M) \subseteq V(A, I_X, S) = V(BL(X), I_X, S).$$ □

Lemma 2.2.25 immediately below will be proved later, as a consequence of a more general result (see §3.1.52). For the moment, the reader is referred to the original proof which can be seen in [806, Lemma 4.1.3].

Lemma 2.2.25 *Derivations of C^*-algebras are continuous.*

Proposition 2.2.26 *Let A be a C^*-algebra, and let $T : A \to A$ be a mapping. Then the following conditions are equivalent:*

(i) *T belongs to $H(BL(A), I_A)$.*
(ii) *There are $h \in H(M(A), *)$ and a derivation D of A such that $D(a^*) = -D(a)^*$ and $T(a) = ha + D(a)$ for every $a \in A$.*

Proof (i)⇒(ii) Assume that (i) holds. Then, by Corollary 2.1.9(iii), for $r \in \mathbb{R}$, $F := \exp(irT)$ is a surjective linear isometry on A, and hence, by Corollary 2.2.20,

$$\exp(irT'')(M(A)) = F''(M(A)) \subseteq M(A).$$

Therefore, since for $a'' \in A''$ we have

$$iT''(a'') = \lim_{r \to 0^+} \frac{1}{r}\left[(\exp(irT''))(a'') - a''\right],$$

we obtain $T''(M(A)) \subseteq M(A)$. On the other hand, since the mapping $G \to G''$ from $BL(A)$ to $BL(A'')$ is a linear isometry preserving distinguished elements, Corollary 2.1.2 applies, so that $T'' \in H(BL(A''), I_{A''})$. It follows from Lemma 2.2.24 that $(T'')_{M(A)} \in H(BL(M(A)), I_{M(A)})$. Applying Theorem 2.2.9, we find $h \in H(M(A), *)$ and a Jordan derivation D of $M(A)$ such that

$$T''(a'') = ha'' + D(a'') \ \text{ and } \ D(a''^*) = -D(a'')^* \ \text{ for every } \ a'' \in M(A).$$

As a consequence,

$$T(a) = T''(a) = ha + D(a) \ \text{ and } \ D(a^*) = -D(a)^* \ \text{ for every } \ a \in A.$$

Finally, by Lemma 2.2.23, D is a derivation of $M(A)$, and, since

$$D(a) = T(a) - ha \in A \ \text{ for every } \ a \in A,$$

we can see D as a derivation of A.

(ii)⇒(i) Assume that condition (ii) holds. By Lemma 2.2.8, h belongs to $H(M(A), \mathbf{1})$, where $\mathbf{1}$ stands for the unit of $M(A)$. Therefore, by Lemma 2.1.10, the mapping $L_h : x \to hx$ from $M(A)$ to $M(A)$ belongs to $H(BL(M(A)), I_{M(A)})$, and hence $(L_h)_A : a \to ha$ belongs to $H(BL(A), I_A)$. Now, to conclude the proof, it is enough to show that D lies in $H(BL(A), I_A)$. By Lemmas 2.2.21 and 2.2.25, for $r \in \mathbb{R}$, $\exp(irD)$ is a bijective $*$-homomorphism on A. Therefore, by the implication (ii)⇒(i) in Theorem 2.2.19, we have

$$\|\exp(irD)\| = 1 \ \text{ for every } \ r \in \mathbb{R},$$

and hence, by Corollary 2.1.9(iii), $D \in H(BL(A), I_A)$, as required. □

2.2.3 Historical notes and comments

Proposition 2.2.1 seems to us to be new. As far as we know, it has been never published before the present work, and helps in a relevant way to simplify the proofs of several published results.

Theorem 2.2.9 is due to Kaidi, Martínez, and Rodríguez [362]. The proof we have given, taken from [362], is close to Paterson's proof [479] of Kadison's isometry theorem [358]. A forerunner of Theorem 2.2.9(i) (respectively, Theorem 2.2.9(ii)) is due to Wright and Youngson [643] (respectively, Youngson [654]), and will be discussed in detail as a part of Proposition 3.4.25 (respectively, Proposition 3.4.28). For a '$*$-free' generalization of the Wright–Youngson result just quoted, the reader is referred to the Arazy–Soler paper [26].

The notion and elementary results concerning the adjoint of a bounded bilinear mapping between normed spaces are taken from Arens' memorable papers [28, 29].

Lemma 2.2.25 is due to Sakai [549], and is included in several books (see for example [678, 723, 806]). As we said, a proof following the ideas of [678] will be given in §3.1.52. A more general result, asserting the automatic continuity of derivations of complete normed semisimple associative algebras, is due to Johnson and Sinclair [355], and is included in several books (see for example [715, 786]). (For the meaning of a semisimple algebra, the reader is referred to Definition 3.6.12 below.)

Theorem 2.2.19 and Proposition 2.2.26 are due to Paterson and Sinclair [480], although the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.2.19 is not explicitly formulated there. The most relevant generalization of the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.2.19 is due to Kaup [381].

§**2.2.27** To provide the reader with a precise formulation of Kaup's result, let us recall that *JB^*-triples* are defined as those complex Banach spaces X endowed with a continuous *triple product* $\{\cdots\}: X\times X\times X\to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(i) For all x in X, the mapping $y\to\{xxy\}$ from X to X is a hermitian operator on X and has non-negative spectrum.

(ii) The equality
$$\{ab\{xyz\}\}=\{\{abx\}yz\}-\{x\{bay\}z\}+\{xy\{abz\}\}$$
holds for all a,b,x,y,z in X.

(iii) $\|\{xxx\}\|=\|x\|^3$ for every x in X.

JB^*-triples generalize C^*-algebras because, as we will prove in Fact 4.1.41, each C^*-algebra becomes a JB^*-triple under the triple product
$$\{xyz\}:=\frac{1}{2}(xy^*z+zy^*x).$$
Now, Kaup's generalization of the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.2.19 reads as follows.

Theorem 2.2.28 *Surjective linear isometries between JB^*-triples are precisely those bijective linear mappings which preserve triple products.*

A forerunner of Kaup's theorem (generalizing as well the equivalence (i)$\Leftrightarrow$(iii) in Theorem 2.2.19) is due to Harris [314]. The particularization of Theorem 2.2.19 to the unital case is the celebrated Kadison isometry theorem [358]. We formulate it here for the sake of completeness.

Theorem 2.2.29 *Surjective linear isometries between unital C^*-algebras A and B are precisely the mappings of the form $a \to uF(a)$, where $F : A \to B$ is a bijective Jordan-$*$-homomorphism, and u is a unitary element of B.*

Kadison's theorem is included, with two different proofs, in Fleming and Jamison (see [730, Theorem 6.2.5 and p. 156]), who also include the Paterson–Sinclair Theorem 2.2.19, without any proof in this case (see [730, Theorem 6.2.6]). The proofs we have given of Theorem 2.2.19 and Proposition 2.2.26 are essentially the original ones in [480]. For an alternative proof of Theorem 2.2.19, see Sherman's paper [563]. For a '$*$-free' generalization of Theorem 2.2.29, the reader is referred to Arazy's paper [25]. For another generalization, see Proposition 2.3.20 below. For more generalizations of Theorem 2.2.29, and precise formulations of some of those quoted here, the reader is referred to [730, pp. 171–9]. The unital forerunner of Proposition 2.2.26 is due to Sinclair [578].

Lemma 2.2.23 is originally due to Cusack [202], and, as we pointed out in its formulation, is included with proof in Ara and Mathieu [678]. Cusack's result generalizes Herstein's forerunner [321] for prime algebras (see §2.5.41 below for the definition of a prime algebra). Actually, for our purposes, the later and much weaker result of Sinclair [578] (that Jordan derivations of complete normed semisimple associative algebras are derivations) would have been enough.

2.3 The associative Vidav–Palmer theorem, starting from a non-associative germ

Introduction In this section, we continue developing the theory of algebra numerical ranges, to be applied to the study of the so-called Vidav algebras (also known as V-algebras). These algebras are defined as norm-unital normed complex algebras satisfying a certain condition, which depends only on their normed spaces and their units, and are automatically endowed with a natural conjugate-linear vector space involution. As the main result, we prove in Theorem 2.3.8 that the natural involution of a V-algebra is an algebra involution. Then, we complete the proof of the so-called associative Vidav–Palmer theorem, which identifies complete associative V-algebras with unital C^*-algebras. The associative Vidav–Palmer theorem and the technology in its proof are applied to complement the general theory of C^*-algebras begun in Section 1.2 and continued in Theorem 2.1.27 and Section 2.2. Among the applications of the associative Vidav–Palmer theorem, we emphasize the one asserting that closed ideals of C^*-algebras are $*$-invariant (see Proposition 2.3.43), as well as the one involved in the proof of Theorem 2.2.15 (that biduals of nonzero C^*-algebras are unital C^*-algebras in a natural way) given in §2.3.53. Then we deduce from Theorem 2.2.19 the Banach–Stone theorem on isometries of $C_0^{\mathbb{C}}(E)$-spaces. Finally, we introduce alternative algebras, derive the Vidav–Palmer type theorem for unital alternative C^*-algebras, and prove a result on ranges of unit-preserving contractive linear projections on alternative C^*-algebras (see Theorem 2.3.68), which, even in the particular associative case, generalizes the classical one by Hamana [305].

2.3.1 Natural involutions of V-algebras are algebra involutions

Lemma 2.3.1 *Let A be a norm-unital normed complex algebra, and let h,k be in $H(A,\mathbf{1})$. Then $i[h,k]$ lies in $H(A,\mathbf{1})$.*

Proof By Lemma 2.1.10, the operator $L_h - R_h$ is hermitian, and clearly vanishes at $\mathbf{1}$. Therefore, by Corollary 2.2.2 we have

$$i[h,k] = i(L_h - R_h)(k) \in H(A,\mathbf{1}). \qquad \square$$

Given a complex number z, we denote by $\Im(z)$ the imaginary part of z. The next result is proved in [695, Lemma 26.3] for the case $X = \mathbb{C}$. The general case follows from the particular one just quoted, and the Hahn–Banach theorem.

Lemma 2.3.2 *Let X be a complex Banach space, let $f : \mathbb{C} \to X$ be an entire function, and let φ be in $\mathbb{R}\backslash\pi\mathbb{Z}$. Assume that $\|f(z)\| \leqslant e^{|\Im(z)|}$ for every $z \in \mathbb{C}$. Then*

$$\cos(\varphi)f(0) + \sin(\varphi)f'(0) = \sum_{n\in\mathbb{Z}} \gamma_n f(n\pi+\varphi), \ \textit{where} \ \gamma_n := \frac{(-1)^n \sin^2(\varphi)}{(n\pi+\varphi)^2}.$$

Lemma 2.3.3 *Let A be a norm-unital complete normed associative complex algebra, and let h be in $H(A,\mathbf{1})$. Then $\|\exp(zh)\| \leqslant e^{|\Re(z)|v(h)}$.*

Proof By Proposition 2.1.7, for $z \in \mathbb{C}$ we have

$$\|\exp(zh)\| \leqslant e^{\max \Re(V(A,\mathbf{1},zh))}.$$

Since

$$\begin{aligned} \max \Re(V(A,\mathbf{1},zh)) &= \max\{\Re(z\xi) : \xi \in V(A,\mathbf{1},h)\} \\ &\leqslant \max\{\Re(z\xi) : \xi \in [-v(h),v(h)]\} = |\Re(z)|v(h), \end{aligned}$$

it follows that $\|\exp(zh)\| \leqslant e^{|\Re(z)|v(h)}$. $\square$

Proposition 2.3.4 *Let A be a norm-unital normed complex algebra, let h be in $H(A,\mathbf{1})$, and let λ,μ be complex numbers. Then*

$$v(\lambda\mathbf{1}+\mu h) = \|\lambda\mathbf{1}+\mu h\|.$$

Proof In view of Corollary 2.1.2, we may assume that A is complete.

First assume that A is associative. We can also assume that μ is nonzero and that $\mathbf{1}$ and h are linearly independent, since otherwise the conclusion is clear. Moreover, after multiplying by a complex number of modulus 1, we can assume that $\lambda = r + is$ and $\mu = it$ for suitable $r,s,t \in \mathbb{R}$, so that $t \neq 0$. By Proposition 2.1.11, we have $v(s\mathbf{1}+th) > 0$, and so we can consider

$$k := \frac{s\mathbf{1}+th}{v(s\mathbf{1}+th)}.$$

Clearly $k \in H(A,\mathbf{1})$ and $v(k) \leqslant 1$. Consider the entire function $f : \mathbb{C} \to A$ given by $f(z) = \exp(izk)$. By Lemma 2.3.3 we have $\|f(z)\| \leqslant e^{|\Im(z)|}$ for every $z \in \mathbb{C}$. Therefore, by Lemma 2.3.2, for $\varphi \in \mathbb{R}\backslash\pi\mathbb{Z}$, we have

$$\cos(\varphi)\mathbf{1} + i\sin(\varphi)k = \sum_{n\in\mathbb{Z}} \gamma_n f(n\pi+\varphi),$$

where

$$\gamma_n = \frac{(-1)^n \sin^2(\varphi)}{(n\pi+\varphi)^2}.$$

Since, by Corollary 2.1.9(iii), we have $\|f(n\pi+\varphi)\| = 1$ for every $n \in \mathbb{Z}$, we deduce that

$$\|\cos(\varphi)\mathbf{1} + i\sin(\varphi)k\| \leqslant \sum_{n\in\mathbb{Z}} |\gamma_n|.$$

But,

$$\sum_{n\in\mathbb{Z}} |\gamma_n| = \sin^2(\varphi) \sum_{n\in\mathbb{Z}} \frac{1}{(n\pi+\varphi)^2} = 1,$$

and consequently $\|\cos(\varphi)\mathbf{1} + i\sin(\varphi)k\| \leqslant 1$. Now, a continuity argument gives that

$$\|\cos(\varphi)\mathbf{1} + i\sin(\varphi)k\| \leqslant 1 \ \text{ for every } \ \varphi \in \mathbb{R}. \tag{2.3.1}$$

Note that

$$\lambda\mathbf{1} + \mu h = r\mathbf{1} + i(s\mathbf{1} + th) = r\mathbf{1} + iv(s\mathbf{1} + th)k = \rho(\cos(\varphi)\mathbf{1} + i\sin(\varphi)k),$$

where $\rho = |r + iv(s\mathbf{1} + th)|$ and φ is an argument of $r + iv(s\mathbf{1} + th)$. Therefore

$$\begin{aligned}
\|\lambda\mathbf{1} + \mu h\| &= \|\rho(\cos(\varphi)\mathbf{1} + i\sin(\varphi)k)\| \leqslant \rho = [r^2 + v(s\mathbf{1} + th)^2]^{\frac{1}{2}} \\
&= \max\{r^2 + (s + t\alpha)^2 : \alpha \in V(A, \mathbf{1}, h)\}^{\frac{1}{2}} \\
&= \max\{|r + i(s + t\alpha)|^2 : \alpha \in V(A, \mathbf{1}, h)\}^{\frac{1}{2}} \\
&= \max\{|\lambda + \mu\alpha| : \alpha \in V(A, \mathbf{1}, h)\} = v(\lambda\mathbf{1} + \mu h),
\end{aligned}$$

and hence $\|\lambda\mathbf{1} + \mu h\| \leqslant v(\lambda\mathbf{1} + \mu h)$. But the converse inequality is clear.

Now assume that A is not associative. Then, by Lemma 2.1.10 and the above paragraph, we have

$$v(\lambda\mathbf{1} + \mu h) = v(\lambda I_A + \mu L_h) = \|\lambda I_A + \mu L_h\| = \|\lambda\mathbf{1} + \mu h\|. \qquad \square$$

Corollary 2.3.5 *Let A be a norm-unital normed complex algebra, let h, k be hermitian elements in A, and let λ be a real number. Then*

(i) $\|k\| \leqslant \|h + ik\|$.
(ii) $\|h + i\lambda\mathbf{1}\|^2 = \|h\|^2 + \lambda^2$.
(iii) $\|h + ik\| \leqslant 2v(h + ik)$.
(iv) $\|h - ik\| \leqslant 2\|h + ik\|$.

Proof Assertions (i) and (ii) are straightforward consequences of Proposition 2.3.4. But Proposition 2.3.4 also gives

$$\|h + ik\| \leqslant \|h\| + \|k\| = v(h) + v(k) \leqslant 2v(h + ik) = 2v(h - ik) \leqslant 2\|h - ik\|,$$

which proves assertions (iii) and (iv). $\square$

Definition 2.3.6 By a V-*algebra* we mean a norm-unital normed complex algebra A satisfying $A = H(A,\mathbf{1}) + iH(A,\mathbf{1})$. Let A be a V-algebra. Then, by Corollary 2.1.13, we have in fact $A = H(A,\mathbf{1}) \oplus iH(A,\mathbf{1})$, so there is a unique conjugate-linear vector space involution $*$ on A such that $H(A,*) = H(A,\mathbf{1})$. This involution will be called the *natural involution* of A.

Without enjoying their name, V-algebras have already appeared in our development (see the last conclusion in Lemma 2.2.5). Assertions (iii) and (iv) in Corollary 2.3.5 lead straightforwardly to assertions (i) and (ii), respectively, in the following.

Lemma 2.3.7 *Let A be a V-algebra. Then we have:*

(i) $n(A,\mathbf{1}) \geqslant \frac{1}{2}$.
(ii) *The natural involution $*$ of A satisfies $\|a^*\| \leqslant 2\|a\|$ for every $a \in A$, and hence is continuous.*

The main result in this section is the following.

Theorem 2.3.8 *The natural involution of any V-algebra is an algebra involution.*

The strategy for the proof of this theorem will consist of the consideration of the bilinear mappings h, k from $H(A,\mathbf{1}) \times H(A,\mathbf{1})$ into $H(A,\mathbf{1})$ (A is the V-algebra under consideration) defined by

$$xy = h(x,y) + ik(x,y) \ \text{ for all } \ x,y \in H(A,\mathbf{1}). \tag{2.3.2}$$

It is not difficult to show (see the end of the proof) that the assertion of Theorem 2.3.8 is equivalent to the following relations:

$$h(x,y) = h(y,x) \ \text{ for all } \ x,y \ \text{ in } \ H(A,\mathbf{1}); \tag{2.3.3}$$

$$k(x,y) = -k(y,x) \ \text{ for all } \ x,y \ \text{ in } \ H(A,\mathbf{1}). \tag{2.3.4}$$

The relation (2.3.3) is easy. Indeed, we have the following.

Lemma 2.3.9 *Let A be a V-algebra, and let h be defined by* (2.3.2). *Then we have $h(x,y) = h(y,x)$ for all x,y in $H(A,\mathbf{1})$.*

Proof Let x,y be in $H(A,\mathbf{1})$. From the equality

$$i[x,y] = i(h(x,y) - h(y,x)) - (k(x,y) - k(y,x))$$

and Lemma 2.3.1, we obtain

$$h(x,y) - h(y,x) \in H(A,\mathbf{1}) \cap iH(A,\mathbf{1}) = 0. \qquad \square$$

In view of this lemma and the above comments we concentrate our attention on the function k. We first obtain, from the theory of numerical ranges, enough information about this function to reduce the proof of (2.3.4) to normed space results.

Lemma 2.3.10 *Let A be a V-algebra. Then*

$$\|k(x,y) + \lambda x + \mu y\|^2 \leqslant (\|x\|^2 + \mu^2)(\|y\|^2 + \lambda^2)$$

for all x,y in $H(A,\mathbf{1})$ and λ, μ in $\mathbb{R}$.

Proof Let x, y be in $H(A, \mathbf{1})$. The equality

$$(x+i\mu\mathbf{1})(y+i\lambda\mathbf{1}) = (h(x,y) - \lambda\mu\mathbf{1}) + i(k(x,y) + \lambda x + \mu y),$$

together with Corollary 2.3.5(i), gives

$$\|k(x,y) + \lambda x + \mu y\| \leqslant \|(x+i\mu\mathbf{1})(y+i\lambda\mathbf{1})\| \leqslant \|x+i\mu\mathbf{1}\|\|y+i\lambda\mathbf{1}\|,$$

and Corollary 2.3.5(ii) concludes the proof. □

As mentioned above, the information about the function k given by Lemma 2.3.10 will allow us to reduce the proof of the statement (2.3.4) to the context of normed space theory. So we begin the second (and final) step in the proof of Theorem 2.3.8. We shall show that any bilinear mapping on a real normed space satisfying the property asserted for k in Lemma 2.3.10 must be antisymmetric.

Notation 2.3.11 Let X be a real normed space, and let $f : X \times X \to X$ be a bilinear mapping. We shall say that the pair (X, f) satisfies the *α-property* whenever the inequality

$$\|f(x,y) + \lambda x + \mu y\|^2 \leqslant (\|x\|^2 + \mu^2)(\|y\|^2 + \lambda^2) \qquad (\alpha)$$

holds for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$. It is clear that, if f satisfies the α-property, then f is continuous (take $\lambda = \mu = 0$). With the terminology just introduced, Lemma 2.3.10 reads as follows: *if A is a V-algebra, then the pair $(H(A, \mathbf{1}), k)$ satisfies the α-property.*

From now until the conclusion of the proof of Corollary 2.3.18, X will denote a real Banach space, f will be a bilinear mapping from $X \times X$ into X, and we shall assume that the pair (X, f) satisfies the α-property. For x in X, f_x will denote the bounded linear operator on X defined by $f_x(y) = f(x,y)$ for every $y \in X$.

Lemma 2.3.12 *Let x be an extreme point in the closed unit ball of X. Then $f(x,x) = 0$.*

Proof Take $y = x$ and $\lambda = \mu = \pm 1$ in the inequality (α). □

Lemma 2.3.13 $V(BL(X), I_X, f_x) = 0$ *for every $x \in X$.*

Proof Let r be a positive real number, let x be in X, and let us take $\lambda = 0$ and $\mu = r^{-1}$ in (α) to obtain

$$\|(I_X + rf_x)(y)\| \leqslant (1 + r^2\|x\|^2)^{\frac{1}{2}}\|y\| \text{ for every } y \in X,$$

and so $\|I_X + rf_x\| \leqslant (1 + r^2\|x\|^2)^{\frac{1}{2}}$. By Proposition 2.1.5, we have

$$\max V(f_x) = \lim_{r \to 0^+} \frac{\|I_X + rf_x\| - 1}{r} \leqslant \lim_{r \to 0^+} \frac{(1 + r^2\|x\|^2)^{\frac{1}{2}} - 1}{r} = 0.$$

The proof is concluded by using $-x$ instead of x. □

In agreement with the notation in Definition 2.2.14, f^r will stand for the mapping $(x,y) \to f(y,x)$ from $X \times X$ to X.

Lemma 2.3.14 *Assume that X is complete, and that $f = f^r$, and let x be an extreme point of $\mathbb{B}_X$. Then $f_x = 0$.*

Proof Let r and y be in $\mathbb{R}$ and X, respectively. By Lemma 2.3.13, we have $V(f_y) = 0$, so, applying Corollary 2.1.9(ii), we obtain $\|\exp(rf_y)\| \leqslant 1$, and so $\exp(rf_y)$ is a surjective linear isometry on X. Therefore, since x is an extreme point of $\mathbb{B}_X$, so is $\exp(rf_y)(x)$, and hence, by Lemma 2.3.12 we have

$$f(\exp(rf_y)(x), \exp(rf_y)(x)) = 0.$$

Taking derivatives at $r = 0$, and keeping in mind that $f = f^r$, we obtain $f(x, f_y(x)) = 0$, that is $f(x, f_x(y)) = 0$, whence $f_x^2 = 0$. On the other hand, by Lemma 2.3.13, we have $V(f_x) = 0$. It follows from Lemma 2.1.28 that $f_x = 0$, as required. □

In agreement with the notation in §2.2.11, $f^t : X'' \times X'' \to X''$ will stand for the Arens extension of f.

Lemma 2.3.15 *The pair (X'', f^t) satisfies the α-property.*

Proof Let x and y'' be arbitrarily fixed elements in X and X'', respectively, and let λ, μ be fixed real numbers. Since the norm of X'' is w^*-lower semicontinuous, and the mapping $z'' \to f^t(x, z'')$ from X'' to X'' is w^*-continuous (by Lemma 2.2.12(iii)(c)), the function

$$z'' \to \|f^t(x, z'') + \lambda x + \mu z''\|$$

from X'' to $\mathbb{R}$ is w^*-lower semicontinuous. Therefore, the set

$$L = \{z'' \in X'' : \|f^t(x, z'') + \lambda x + \mu z''\|^2 \leqslant (\|x\|^2 + \mu^2)(\|y''\|^2 + \lambda^2)\}$$

is w^*-closed. Since (X, f) satisfies the α-property, we have $\|y''\|\mathbb{B}_X \subseteq L$, and hence $\|y''\|\mathbb{B}_{X''} \subseteq L$ (because $\mathbb{B}_{X''}$ is the w^*-closure of $\mathbb{B}_X$ in X''). In particular, $y'' \in L$, and we have proved that

$$\|f^t(x, y'') + \lambda x + \mu y''\|^2 \leqslant (\|x\|^2 + \mu^2)(\|y''\|^2 + \lambda^2) \qquad (\alpha')$$

holds for all λ, μ in $\mathbb{R}$, x in X, and y'' in X''.

Now we fix x'', y'' in X'' and λ, μ in $\mathbb{R}$. Since the mapping $z'' \to f^t(z'', y'')$ from X'' to X'' is w^*-continuous (by Lemma 2.2.12(iii)(b)), the function

$$z'' \to \|f^t(z'', y'') + \lambda z'' + \mu y''\|$$

from X'' to $\mathbb{R}$ is w^*-lower semicontinuous. Therefore, the set

$$M = \{z'' \in X'' : \|f^t(z'', y'') + \lambda z'' + \mu y''\|^2 \leqslant (\|x''\|^2 + \mu^2)(\|y''\|^2 + \lambda^2)\}$$

is w^*-closed. Since $\|x''\|\mathbb{B}_X \subseteq M$ (by (α')), it follows that $\|x''\|\mathbb{B}_{X''} \subseteq M$. Therefore $x'' \in M$, which completes the proof. □

Lemma 2.3.16 *The pair $(X, \frac{1}{2}(f + f^r))$ satisfies the α-property.*

Proof Interchange x with y and λ with μ in the inequality (α), and sum up the resulting inequality with the previous one. □

Proposition 2.3.17 *For all x and y in X we have*

$$f(x, y) = -f(y, x).$$

Proof Write $g = \frac{1}{2}(f^t + f^{tr})$. By Lemmas 2.3.15 and 2.3.16 the pair (X'', g) satisfies the α-property. On the other hand, the equality $g = g^r$ is clear. Therefore, by Lemma 2.3.14, we have $g_{x''} = 0$ for every extreme point x'' of $\mathbb{B}_{X''}$. In particular, we have $g(x'', y) = 0$, that is $f^t(x'', y) = -f^t(y, x'')$ for every $y \in X$. For any fixed y the last equality holds by linearity for all x'' in the linear hull of the set of all extreme points of $\mathbb{B}_{X''}$, and by w^*-continuity (thanks to Lemma 2.2.12) it also holds in the w^*-closure of this linear hull, that is for all x'' in X''. In particular, $f(x,y) = -f(y,x)$ for all x, y in X. □

Corollary 2.3.18 *The mapping f is Arens regular.*

Proof By Lemma 2.3.15 and Proposition 2.3.17 we have $f^r = -f$ and $f^{tr} = -f^t$, so $f^{rt} = f^{tr}$, which is the Arens regularity of f. □

With Proposition 2.3.17 above, the proof of Theorem 2.3.8 is essentially completed. However, a brief summary might be useful.

End of the proof of Theorem 2.3.8 Let A be the V-algebra under consideration so that, by Lemma 2.3.10, the pair $(H(A, \mathbf{1}), k)$ satisfies the α-property. Let x, y be in $H(A, \mathbf{1})$. By Lemma 2.3.9 (respectively, Proposition 2.3.17), we have $h(x,y) = h(y,x)$ (respectively, $k(x,y) = -k(y,x)$). Therefore, we get

$$(xy)^* = h(x,y) - ik(x,y) = h(y,x) + ik(y,x) = yx,$$

where $*$ stands for the natural involution of A. Now, let a, b be arbitrary elements in A, and write $a = x + iy$ and $b = z + it$ with $x, y, z, t \in H(A, \mathbf{1})$. Then we have

$$(ab)^* = (xz - yt + i(xt + yz))^* = zx - ty - i(tx + zy) = b^* a^*. \qquad \square$$

From now on, every V-algebra will be seen endowed with its natural involution. Now note that, if A is a V-algebra, and if $*$ stands for its natural involution, then, by Theorem 2.3.8, the conclusion in Lemma 2.2.8 holds. Therefore, repeating the arguments in the proof of Theorem 2.2.9, we obtain the following.

Corollary 2.3.19 *Let A be a V-algebra. We have:*

(i) *If B is a V-algebra, and if $F : A \to B$ is a unit-preserving surjective linear isometry, then F is a Jordan-$*$-homomorphism.*
(ii) *If T is in $H(BL(A), I_A)$, then there are $h \in H(A, *)$ and a continuous Jordan derivation D of A such that $T = L_h + D$ and $D(a^*) = -D(a)^*$ for every $a \in A$.*

Proposition 2.3.20 *Let A be a norm-unital normed complex algebra, let B be a C^*-algebra, and let $F : A \to B$ be a surjective linear isometry. Then B has a unit, and there exists a unitary element $u \in B$, and a surjective isometric Jordan homomorphism $G : A \to B$, satisfying $F(a) = uG(a)$ for every $a \in A$.*

Proof By Corollary 2.1.13, the unit $\mathbf{1}$ of A is a vertex of $\mathbb{B}_A$, and hence $u := F(\mathbf{1})$ is a vertex of $\mathbb{B}_B$. As a first consequence, by Lemma 2.1.25, $\mathbb{B}_B$ has extreme points, and therefore, by Lemma 2.1.26, B has a unit. Now, by Theorem 2.1.27, u is a unitary element of B, and hence the mapping $G : a \to u^* F(a)$ from A to B becomes a unit-preserving surjective linear isometry. Since B is a V-algebra (by Lemma 2.2.5), and A is unit-preserving linearly isometric to B, we see that B is also a V-algebra. Therefore, by Corollary 2.3.19, G is a surjective isometric Jordan-$*$-homomorphism, and we clearly have $F(a) = uG(a)$ for every $a \in A$. $\square$

2.3.2 The associative Vidav–Palmer theorem

In what follows, we are going to complete the proof of the associative Vidav–Palmer theorem.

Lemma 2.3.21 *Let A be a norm-unital complete normed associative algebra over $\mathbb{K}$, and let a be in A. Then* $\mathrm{sp}(a) \subseteq V(a)$.

Proof Let μ be in $\mathrm{sp}(a)$. Then, for every $\lambda \in \mathbb{K}$, $\mu - \lambda$ lies in $\mathrm{sp}(a - \lambda\mathbf{1})$, and hence, by Proposition 1.1.40, we have $|\mu - \lambda| \leqslant \|a - \lambda\mathbf{1}\|$. Therefore we obtain $\mu \in \cap_{\lambda \in \mathbb{K}} B_{\mathbb{K}}(\lambda, \|a - \lambda\mathbf{1}\|)$, and Proposition 2.1.1 applies. $\square$

From now on, we assume that the reader is familiarized with the holomorphic functional calculus in a single element of a unital complete normed associative complex algebra A, as stated in Section 1.3. We recall that, roughly speaking, the holomorphic functional calculus assigns to each element $a \in A$, to each open subset Ω of $\mathbb{C}$ containing $\mathrm{sp}(a)$, and to each holomorphic mapping $f : \Omega \to \mathbb{C}$, an element $f(a) \in A$, in such a way that, when the triple (a, Ω, f) moves, things behave reasonably. The holomorphic functional calculus will be involved without notice in the proofs of Proposition 2.3.22 and Lemma 2.3.28 below.

Proposition 2.3.22 *Let A be a norm-unital normed associative complex algebra, and let h be in $H(A, \mathbf{1})$. Then* $\mathfrak{r}(h) = \|h\|$.

Proof We may assume that A is complete, and that $\mathfrak{r}(h) < \frac{\pi}{2}$. Then, by Lemma 2.3.21, we have $\mathrm{sp}(h) \subseteq]-\frac{\pi}{2}, \frac{\pi}{2}[$, and hence

$$h = \arcsin(\sin h) = \sum_{n=0}^{\infty} \alpha_n (\sin h)^{2n+1},$$

where $\alpha_n := \frac{(2n)!}{2^{2n}(n!)^2(2n+1)}$. On the other hand, by Corollary 2.1.9(iii), we have $\|\sin h\| \leqslant 1$. It follows that $\|h\| \leqslant \sum_{n=0}^{\infty} \alpha_n = \frac{\pi}{2}$. $\square$

Proposition 2.3.23 *Let A be a norm-unital complete normed associative complex algebra, and let h be in $H(A, \mathbf{1})$. Then* $V(h) = \mathrm{co}(\mathrm{sp}(h))$.

Proof Clearly, $V(h) = [s, t]$ for suitable $s, t \in \mathbb{R}$. Therefore, by Lemma 2.3.21 and Proposition 2.3.22, we derive

$$\mathfrak{r}(h - s\mathbf{1}) = v(h - s\mathbf{1}) = t - s = v(h - t\mathbf{1}) = \mathfrak{r}(h - t\mathbf{1}).$$

Since $\mathrm{sp}(h - s\mathbf{1}) \subseteq [0, t - s]$ and $\mathrm{sp}(h - t\mathbf{1}) \subseteq [s - t, 0]$, it follows that

$$t - s \in \mathrm{sp}(h - s\mathbf{1}) \text{ and } s - t \in \mathrm{sp}(h - t\mathbf{1}),$$

that is, both t and s lie in $\mathrm{sp}(h)$. Thus we have shown that $V(h) \subseteq \mathrm{co}(\mathrm{sp}(h))$. The converse inclusion follows by a new application of Lemma 2.3.21. $\square$

Lemma 2.3.24 *Let A be a complete associative and commutative V-algebra, let $*$ denote the natural involution of A, and let Δ stand for the carrier space of A. Then the Gelfand transform $a \to G(a)$ from A to $C^{\mathbb{C}}(\Delta)$ becomes a continuous bijective algebra $*$-homomorphism. Moreover, for $h \in H(A, \mathbf{1})$ we have $V(h) = \mathrm{co}(G(h)(\Delta))$.*

Proof The last conclusion follows from Proposition 2.3.23 and Theorem 1.1.73(iii). As a consequence, we have $G(H(A,*)) \subseteq H(C^{\mathbb{C}}(\Delta),*)$, which implies that the Gelfand transform is an algebra $*$-homomorphism. Now, by Theorem 1.1.73(ii), $G(A)$ becomes a $*$-subalgebra of $C^{\mathbb{C}}(\Delta)$ containing the constant functions and separating the points of Δ, so that, by the Stone–Weierstrass theorem (cf. Theorem 1.2.10), $G(A)$ is dense in $C^{\mathbb{C}}(\Delta)$. On the other hand, for $a = h + ik \in A$, with $h, k \in H(A, \mathbf{1})$, we have $G(a) = G(h) + iG(k)$ with $G(h), G(k) \in H(C^{\mathbb{C}}(\Delta),*)$, and therefore, invoking Proposition 2.3.22 and Theorem 1.1.73(iv), we get

$$\|a\| \leqslant \|h\| + \|k\| = \mathfrak{r}(h) + \mathfrak{r}(k) = \|G(h)\| + \|G(k)\| \leqslant 2\|G(a)\| = 2\mathfrak{r}(a) \leqslant 2\|a\|,$$

so the Gelfand transform is a linear homeomorphism from A onto $G(A)$, and so $G(A)$ is closed in $C^{\mathbb{C}}(\Delta)$. It follows that $G(A) = C^{\mathbb{C}}(\Delta)$. □

Remark 2.3.25 We note that the above proof does not involve Theorem 2.3.8. Therefore, it becomes an autonomous proof of that theorem in the particular associative and commutative case.

Lemma 2.3.26 *Let A be a complete associative V-algebra, let $*$ stand for the natural involution of A, and let a be in A. Then $V(a^*a)$ consists only of non-negative real numbers.*

Proof First of all, note that, since $*$ is an algebra involution (by Theorem 2.3.8), x^*x is a hermitian element of A whenever x is in A. Let us denote by $K(A)$ the set of those elements $p \in A$ such that $V(p) \subseteq \mathbb{R}_0^+$. Let b be in A. Write $b = h + ik$ with $h, k \in H(A, \mathbf{1})$. Then we have

$$b^*b = 2h^2 + 2k^2 - bb^*. \tag{2.3.5}$$

Now let B stand for the closed subalgebra of A generated by $\{h, \mathbf{1}\}$. Then, B is associative and commutative. Moreover, since $*$ is a continuous algebra involution (by Lemma 2.3.7), B is $*$-invariant. Therefore, keeping in mind Corollary 2.1.2, it is easily realized that B is a complete V-algebra whose natural involution is the restriction of $*$ to B. Since both h and h^2 are hermitian elements of B, Lemma 2.3.24 applies (with B instead of A), so that we have

$$V(A, \mathbf{1}, h^2) = V(B, \mathbf{1}, h^2) \subseteq \mathbb{R}_0^+,$$

that is $h^2 \in K(A)$. Analogously, we get $k^2 \in K(A)$. Assume that $-b^*b \in K(A)$. Then, since $\mathrm{sp}(b^*b) \setminus \{0\} = \mathrm{sp}(bb^*) \setminus \{0\}$ (by Fact 1.1.33), it follows from Proposition 2.3.23 that $-bb^* \in K(A)$, and hence, by (2.3.5), that $b^*b \in K(A)$. But $-b^*b \in K(A)$ by assumption, so $V(b^*b) = 0$. Analogously, $V(bb^*) = 0$, so $V(h^2 + k^2) = 0$. Therefore $V(h^2) = V(k^2) = 0$, and, by Corollary 2.1.13, $h^2 = k^2 = 0$. By Proposition 2.3.22, we finally have $h = k = 0$, and hence $b = 0$.

Now, let a be the arbitrary element of A in the statement. Then, keeping in mind that $a^*a \in H(A, \mathbf{1})$, and thinking about the closed subalgebra of A generated by

$\{a^*a, \mathbf{1}\}$, we can apply Lemma 2.3.24 (with this subalgebra instead of A) to get the existence of $p, q \in K(A)$ such that $a^*a = p - q$ and $pq = qp = 0$. Set $b := aq$. Then $-b^*b = -qa^*aq = q^3 \in K(A)$, and hence $b = 0$ by the above paragraph. Consequently, we have $a^*aq = a^*b = 0$, so $q^2 = 0$, and so $q = 0$. Thus we get $a^*a = p \in K(A)$, as required. □

Lemma 2.3.27 *Let A be a C^*-algebra, and let $|\cdot|$ be an algebra norm on A such that $|\cdot| \leqslant \|\cdot\|$. Then $|\cdot| = \|\cdot\|$.*

Proof Let a be in A. Then the closed subalgebra of A generated by a^*a is a commutative C^*-algebra, and hence, by Proposition 1.2.51, we have

$$\|a\|^2 = \|a^*a\| \leqslant |a^*a| \leqslant |a^*||a| \leqslant \|a^*\||a| = \|a\||a|,$$

and hence $\|a\| \leqslant |a|$. □

The above lemma will be generalized later in Proposition 4.4.34.

Lemma 2.3.28 (Russo–Dye theorem) *Let A be a unital C^*-algebra. Then $\mathbb{B}_A = \overline{\text{co}}(U)$, where U stands for the set of all unitary elements of A.*

Proof Let x be in A such that $0 < \|x\| < 1$. Then $\mathfrak{r}(x^*) = \mathfrak{r}(x) < 1$, and $\mathfrak{r}(xx^*) = \mathfrak{r}(x^*x) = \|x\|^2 < 1$. Let D stand for the open disc in $\mathbb{C}$ with centre 0 and radius $\|x\|^{-1}$. Then we may define $F : D \to A$ by

$$F(\lambda) = (\mathbf{1} - xx^*)^{-\frac{1}{2}}(\lambda\mathbf{1} + x)(\mathbf{1} + \lambda x^*)^{-1}(\mathbf{1} - x^*x)^{\frac{1}{2}} \quad \text{for every } \lambda \in D.$$

It is routine that F is holomorphic on D. Let λ be in $\mathbb{S}_{\mathbb{C}}$. The following equalities are elementary:

$$(\mathbf{1} + \lambda x^*)^{-1}(\lambda\mathbf{1} + x) = x + \lambda(\mathbf{1} + \lambda x^*)^{-1}(\mathbf{1} - x^*x), \tag{2.3.6}$$

$$(\lambda\mathbf{1} + x)(\mathbf{1} + \lambda x^*)^{-1} = x + \lambda(\mathbf{1} - xx^*)(\mathbf{1} + \lambda x^*)^{-1}. \tag{2.3.7}$$

Since $\mathfrak{r}(x) < 1$, $F(\lambda)$ is invertible, and by (2.3.6) and (2.3.7),

$$\begin{aligned}
(F(\lambda)^{-1})^* &= (\mathbf{1} - xx^*)^{\frac{1}{2}}(\bar{\lambda}\mathbf{1} + x^*)^{-1}(\mathbf{1} + \bar{\lambda}x)(\mathbf{1} - x^*x)^{-\frac{1}{2}} \\
&= (\mathbf{1} - xx^*)^{\frac{1}{2}}(\mathbf{1} + \lambda x^*)^{-1}(\lambda\mathbf{1} + x)(\mathbf{1} - x^*x)^{-\frac{1}{2}} \\
&= (\mathbf{1} - xx^*)^{-\frac{1}{2}}a(\mathbf{1} - x^*x)^{\frac{1}{2}},
\end{aligned}$$

where

$$\begin{aligned}
a &:= (\mathbf{1} - xx^*)(\mathbf{1} + \lambda x^*)^{-1}(\lambda\mathbf{1} + x)(\mathbf{1} - x^*x)^{-1} \\
&= (\mathbf{1} - xx^*)[x + \lambda(\mathbf{1} + \lambda x^*)^{-1}(\mathbf{1} - x^*x)](\mathbf{1} - x^*x)^{-1} \\
&= x + \lambda(\mathbf{1} - xx^*)(\mathbf{1} + \lambda x^*)^{-1} \\
&= (\lambda\mathbf{1} + x)(\mathbf{1} + \lambda x^*)^{-1}.
\end{aligned}$$

This shows that $F(\lambda)$ is unitary for each $\lambda \in \mathbb{S}_{\mathbb{C}}$. Since F is holomorphic, we have

$$F(0) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})\, d\theta.$$

Consideration of the series expansion of $(\mathbf{1}-y)^{\frac{1}{2}}$ for $\|y\|<1$ gives

$$(\mathbf{1}-xx^*)^{\frac{1}{2}}x = x(\mathbf{1}-x^*x)^{\frac{1}{2}}.$$

Therefore $x = F(0) \in \overline{\mathrm{co}}(U)$. $\square$

Proposition 2.3.29 (Russo–Dye–Palmer theorem) *Let A be a unital C^*-algebra. Then $\mathbb{B}_A = \overline{\mathrm{co}}(E)$, where*

$$E := \{\exp(ih) : h \in H(A,*)\}.$$

Proof We retain the notation in the proof of Lemma 2.3.28. Let u be in U. By that lemma, it is enough to show that $u \in \overline{\mathrm{co}}(E)$. Since u lies in a commutative C^*-subalgebra of A containing $\mathbf{1}$, we may assume that A is commutative. Let Δ stand for the carrier space of A. By Lemmas 2.2.5 and 2.3.24, the Gelfand transform $a \to G(a)$ becomes a continuous bijective algebra $*$-homomorphism from A to $C^{\mathbb{C}}(\Delta)$. Given $0<t<1$, set $x := tu$. Let λ be in $\mathbb{S}_{\mathbb{C}}$. Then we have

$$\begin{aligned}(\lambda\mathbf{1}+F(\lambda))(\mathbf{1}+\lambda x^*) &= \left[\lambda\mathbf{1}+(\lambda\mathbf{1}+x)(\mathbf{1}+\lambda x^*)^{-1}\right](\mathbf{1}+\lambda x^*)\\ &= \lambda\mathbf{1}+\lambda^2x^*+\lambda\mathbf{1}+x = 2\lambda\left[\mathbf{1}+\frac{1}{2}(\lambda x^*+\bar{\lambda}x)\right].\end{aligned}$$

Since $\|x\|<1$, it follows that $\lambda\mathbf{1}+F(\lambda)$ is invertible, and hence, since $F(\lambda)$ is a unitary element of A, we derive that $\mathrm{sp}(F(\lambda)) \subseteq \mathbb{S}_{\mathbb{C}} \setminus \{-\lambda\}$. Now, take a continuous logarithm $\log : \mathbb{S}_{\mathbb{C}} \setminus \{-\lambda\} \to \mathbb{C}$. Then $g := -i\log \circ G(F(\lambda))$ is a real-valued continuous function on Δ satisfying $\exp(ig) = G(F(\lambda))$. Now, if h denotes the unique element in A such that $G(h) = g$, then we have $h \in H(A,*)$ and $\exp(ih) = F(\lambda)$. Therefore $F(\lambda)$ lies in E and, as in Lemma 2.3.28, $x \in \overline{\mathrm{co}}(E)$. Finally, let $t \to 1$. $\square$

Definition 2.3.30 Let A be a complex $*$-algebra, and let f be a linear functional on A. We say that f is *positive* if, for every $a \in A$, we have $f(a^*) = \overline{f(a)}$ and $f(a^*a) \geqslant 0$. By a C^*-*seminorm* (respectively, a C^*-*norm*) on A we mean a submultiplicative seminorm (respectively, norm) $|||\cdot|||$ on A satisfying

$$|||a^*a||| = |||a|||^2 \text{ for every } a \in A.$$

Lemma 2.3.31 *Let A be a normed unital associative complex $*$-algebra whose involution is continuous, let P stand for the set of those continuous positive linear functionals f on A satisfying $f(\mathbf{1}) = 1$, and for $a \in A$ set*

$$|||a||| := \sup\{\sqrt{f(a^*a)} : f \in P\}.$$

Then $|||\cdot|||$ is a continuous C^*-seminorm on A.

Proof Let f be a positive linear functional on A. Then $(a,b) \to f(b^*a)$ becomes a non-negative hermitian sesquilinear form on A, and hence the mapping $a \to \sqrt{f(a^*a)}$ is a seminorm on A, and moreover, by the Cauchy–Schwartz inequality, for $a \in A$ we have

$$|f(a)|^2 = |f(\mathbf{1}^*a)|^2 \leqslant f(\mathbf{1})f(a^*a). \tag{2.3.8}$$

Therefore, if in addition f is continuous, then we have

$$|f(a)|^2 \leqslant f(\mathbf{1})\|f\|\|a^*\|\|a\| \leqslant f(\mathbf{1})\|f\|\|*\|\|a\|^2$$

for every $a \in A$, which implies

$$\|f\| \leqslant f(\mathbf{1})\|*\|, \tag{2.3.9}$$

so

$$f(a^*a) \leqslant \|f\|\|a^*a\| \leqslant f(\mathbf{1})\|*\|^2\|a\|^2,$$

and so $|||\cdot||| \leqslant \|*\|\|\cdot\|$ on A, which shows that $|||\cdot|||$ is indeed a continuous seminorm.

Let a,b be in A, and let f be in P. We claim that

$$f((ab)^*ab) \leqslant |||a|||^2 f(b^*b). \tag{2.3.10}$$

To prove the claim, consider the mapping $g : x \to f(b^*xb)$ from A to $\mathbb{C}$, and note that g is a continuous positive linear functional on A. If $g = 0$, then the assertion in the claim holds in an obvious way. Otherwise, by (2.3.9), we have $f(b^*b) = g(\mathbf{1}) \neq 0$, and consequently $\frac{1}{f(b^*b)}g$ is an element of P. Therefore

$$\frac{f((ab)^*ab)}{f(b^*b)} = \frac{g(a^*a)}{f(b^*b)} \leqslant |||a|||^2,$$

which concludes the proof of the claim. Now that the claim is proved, it is enough to take suprema in (2.3.10), when f runs over P, to get that $|||\cdot|||$ is a submultiplicative seminorm on A.

Now, let a be in A, and let f be in P. Writing (2.3.8) with a^*a instead of a, we get $f(a^*a)^2 \leqslant f(a^*aa^*a)$, which implies $|||a|||^2 \leqslant |||a^*a|||$. Since $|||\cdot|||$ is a submultiplicative seminorm, the above implies in its turn that $|||a|||^2 = |||a^*a|||$. □

Theorem 2.3.32 (Vidav–Palmer) *Let A be a complete associative V-algebra. Then A, endowed with its natural involution and its own norm, becomes a C^*-algebra.*

Proof Let $*$ stand for the natural involution of A. Then, by Lemma 2.3.7(ii) and Theorem 2.3.8, $*$ is a continuous algebra involution, a fact that must be kept in mind throughout the remaining part of the proof. Let P and $|||\cdot|||$ be as in Lemma 2.3.31. Then, by Lemma 2.3.26, we have $D(A,\mathbf{1}) \subseteq P$, and hence, by Lemma 2.3.7(i) and (2.3.8), for $a \in A$ we get

$$\begin{aligned}\frac{1}{2}\|a\| &\leqslant v(a) = \sup\{|f(a)| : f \in D(A,\mathbf{1})\}\\ &\leqslant \sup\{\sqrt{f(a^*a)} : f \in D(A,\mathbf{1})\} \leqslant |||a|||.\end{aligned}$$

It follows from Lemma 2.3.31 that $|||\cdot|||$ is an equivalent C^*-norm on A. Therefore $(A,*,|||\cdot|||)$ is a C^*-algebra, and hence, by Proposition 2.3.29 and Corollary 2.1.9(iii), we have

$$\mathbb{B}_{(A,|||\cdot|||)} = \overline{\text{co}}\{\exp(ih) : h \in H(A,*)\} = \overline{\text{co}}\{\exp(ih) : h \in H(A,\mathbf{1})\} \subseteq \mathbb{B}_A.$$

As a result, we have $\|\cdot\| \leqslant |||\cdot|||$, and the proof is concluded by applying Lemma 2.3.27. □

Remark 2.3.33 The application of Lemma 2.3.27, just done at the end of the above proof, can be avoided. Indeed, by Lemma 2.2.5, we have

$$H((A,|||\cdot|||),\mathbf{1})) = H(A,*) = H(A,\mathbf{1}),$$

and hence, by Proposition 2.3.22, we get $|||h||| = \|h\|$ for every $h \in H(A,*)$. Therefore, if for some $a \in A$ we had $\|a\| < |||a|||$, then we would have

$$\|a^*a\| \leqslant \|a^*\|\|a\| < |||a^*||| |||a||| = |||a^*a||| = \|a^*a\|,$$

a contradiction.

2.3.3 Complements on C^*-algebras

§2.3.34 Let (X,u) be a real numerical-range space such that u is a vertex of $\mathbb{B}_X$. Then, clearly, the set $C := \{x \in X : V(X,u,x) \subseteq \mathbb{R}_0^+\}$ is a closed proper convex cone in X. Therefore, according to §1.2.39, X naturally becomes an ordered set in such a way that $C = \{x \in X : x \geqslant 0\}$. The order of X just reviewed will be called *the numerical-range order*.

The following facts follow straightforwardly.

Fact 2.3.35 *Let (X,u) and (Y,v) be real numerical-range spaces such that u and v are vertices of $\mathbb{B}_X$ and $\mathbb{B}_Y$, respectively, and let $T : X \to Y$ be a linear mapping with $T(u) = v$. We have:*

(i) *If T is contractive, then T preserves numerical-range orders.*
(ii) *If in fact $n(Y,v) = 1$, and if T preserves numerical-range orders, then T is contractive.*

Fact 2.3.36 *Let (X,u) be a real numerical-range space with $n(X,u) = 1$. Then $0 \leqslant x \leqslant y$ in the numerical-range order of X implies $\|x\| \leqslant \|y\|$.*

§2.3.37 Now, let A be a norm-unital normed complex algebra. Then $(H(A,\mathbf{1}),\mathbf{1})$ is a real numerical-range space, and, by Corollary 2.1.2 and Propositions 2.1.4 and 2.3.4, we have $n(H(A,\mathbf{1})) = 1$. Therefore, §2.3.34 and Fact 2.3.35 apply appropriately. We also note that if A is in fact a unital C^*-algebra, then, by Lemma 2.2.5, we have $H(A,*) = H(A,\mathbf{1})$.

Keeping in mind Lemma 2.2.5 and Proposition 2.3.23, we get the following.

Proposition 2.3.38 *Let A be a unital C^*-algebra. Then the usual order of $H(A,*)$* (cf. §1.2.41) *coincides with the numerical-range order.*

Now, we summarize some facts about the order in the self-adjoint part of a (possibly non-unital) C^*-algebra (cf. §1.2.47).

Proposition 2.3.39 *Let A be a C^*-algebra. We have:*

(i) *For $h \in H(A,*)$, the following conditions are equivalent:*
   (a) *$h \geqslant 0$.*
   (b) *$h = k^2$ for some $k \in H(A,*)$.*
   (c) *$h = a^*a$ for some $a \in A$.*
(ii) *If $a,b \in H(A,*)$ with $a \leqslant b$, and if c is in A, then $c^*ac \leqslant c^*bc$.*
(iii) *If $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$.*

(iv) *If A is unital, and if a,b are positive invertible elements with $a \leqslant b$, then* $0 \leqslant b^{-1} \leqslant a^{-1}$.
(v) *If* $0 \leqslant a \leqslant b$, *then* $a^{\frac{1}{2}} \leqslant b^{\frac{1}{2}}$.

Proof By Proposition 1.2.48(ii), every positive element of A is of the form k^2 for some $k \in H(A,*)$, and hence of the form a^*a for some $a \in A$. Conversely, by considering the C^*-algebra unital extension of A if necessary, and applying Lemma 2.3.26 and Proposition 2.3.38, we obtain $a^*a \geqslant 0$ for every $a \in A$. This proves assertion (i).

Let a,b in $H(A,*)$ such that $a \leqslant b$, and let c be in A. Keeping in mind assertion (i), we realize that there exists $x \in A$ such that $b-a = x^*x$, hence

$$c^*bc - c^*ac = c^*(b-a)c = c^*x^*xc = (xc)^*xc \geqslant 0.$$

This proves assertion (ii).

Passing to the C^*-algebra unital extension of A if necessary, assertion (iii) follows from Proposition 2.3.38 and Fact 2.3.36.

Assume that A is unital. To prove assertion (iv) we first observe that if $c \geqslant \mathbf{1}$, then c is invertible and $c^{-1} \leqslant \mathbf{1}$. This follows from assertion (ii) since

$$c^{-1} = c^{-\frac{1}{2}}\mathbf{1}c^{-\frac{1}{2}} \leqslant c^{-\frac{1}{2}}cc^{-\frac{1}{2}} = \mathbf{1}.$$

Now, suppose that a,b are invertible and $0 \leqslant a \leqslant b$. Then, we have

$$\mathbf{1} = a^{-\frac{1}{2}}aa^{-\frac{1}{2}} \leqslant a^{-\frac{1}{2}}ba^{-\frac{1}{2}}.$$

It follows from the above particular case that $(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-1} \leqslant \mathbf{1}$, that is

$$a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}} \leqslant \mathbf{1}.$$

Hence $b^{-1} \leqslant (a^{\frac{1}{2}})^{-1}(a^{\frac{1}{2}})^{-1} = a^{-1}$.

Finally, in order to prove assertion (v), note that it suffices to show that for positive elements $a,b \in A$, the condition $a^2 \leqslant b^2$ implies $a \leqslant b$. Moreover, by considering the C^*-algebra unital extension of A if necessary, we may assume that A is unital. Let $t > 0$, and write $(t\mathbf{1}+b+a)(t\mathbf{1}+b-a) = h+ik$ with $h,k \in H(A,*)$. Then

$$\begin{aligned} h &= \frac{1}{2}[t\mathbf{1}+b+a)(t\mathbf{1}+b-a)+(t\mathbf{1}+b-a)(t\mathbf{1}+b+a)] \\ &= t^2\mathbf{1}+2tb+b^2-a^2 \geqslant t^2\mathbf{1}. \end{aligned}$$

Consequently, h is both invertible and positive. Since

$$\mathbf{1}+ih^{-\frac{1}{2}}kh^{-\frac{1}{2}} = h^{-\frac{1}{2}}(h+ik)h^{-\frac{1}{2}},$$

and $\mathbf{1}+ih^{-\frac{1}{2}}kh^{-\frac{1}{2}}$ is invertible (by Proposition 1.2.20(ii)), we get that $h+ik$ is invertible. It follows that $c(t\mathbf{1}+b-a) = \mathbf{1} = (t\mathbf{1}+b-a)c^*$ for some $c \in A$, and therefore $(t\mathbf{1}+b-a)$ is invertible because of Lemma 1.1.59. Consequently, $-t \notin \mathrm{sp}(b-a)$. Hence $\mathrm{sp}(b-a) \subseteq \mathbb{R}_0^+$, so $b-a$ is positive, that is, $a \leqslant b$. □

Exercise 2.3.40 Prove that, for elements a,b in an arbitrary C^*-algebra, the implication

$$0 \leqslant a \leqslant b \implies a^2 \leqslant b^2$$

is not true.

Solution For example, consider the C^*-algebra $A = M_2(\mathbb{C})$, and take the self-adjoint matrices $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. □

Now we are going to study closed ideals of C^*-algebras.

Lemma 2.3.41 *Let A be a V-algebra, let B be a norm-unital normed complex algebra, and let $T : A \to B$ be a unit-preserving surjective linear contraction. Then B is a V-algebra, and T preserves natural involutions.*

Proof Since $A = H(A,\mathbf{1}) + iH(A,\mathbf{1})$, and $T(H(A,\mathbf{1})) \subseteq H(B,\mathbf{1})$ (by Corollary 2.1.2(i)), we have $B = T(A) \subseteq H(B,\mathbf{1}) + iH(B,\mathbf{1}) \subseteq B$, and hence B is a V-algebra. Since $H(A,*) = H(A,\mathbf{1})$ and $H(B,*) = H(B,\mathbf{1})$, it follows from Exercise 1.2.21 that T is a $*$-mapping. □

If X is a vector space over $\mathbb{K}$, if $*$ is a conjugate-linear involution on X, and if M is a $*$-invariant subspace of X, then the mapping $x + M \to x^* + M$ becomes a conjugate-linear involution on X/M, which will be called *the quotient involution*.

Corollary 2.3.42 *Let A be a V-algebra, and let M be a closed proper ideal of A. Then M is invariant under the natural involution of A, and A/M is a V-algebra whose natural involution is the quotient involution of that of A.*

Proof Set $B := A/M$. By §1.1.110, B is a norm-unital normed complex algebra. Now, the result follows from Lemma 2.3.41 by taking $T : A \to B$ equal to the quotient algebra homomorphism, and noticing that $M = \ker(T)$. □

Proposition 2.3.43 *Let A be a C^*-algebra, and let M be a closed ideal of A. Then M is $*$-invariant, and A/M is a C^*-algebra for the quotient norm and the quotient involution.*

Proof We may assume that M is proper. If A is unital, then the result follows from Lemma 2.2.5, Corollary 2.3.42, and Theorem 2.3.32.

Assume that A is not unital. Then the C^*-algebra unital extension $A_{\mathbb{1}}$ of A is a unital C^*-algebra containing A (isometrically) as a $*$-subalgebra (by Proposition 1.2.44), and M becomes a closed proper ideal of $A_{\mathbb{1}}$. By the above paragraph, M is $*$-invariant, and $A_{\mathbb{1}}/M$ is a C^*-algebra in the natural way. Moreover, since A/M can be seen isometrically as a $*$-subalgebra of $A_{\mathbb{1}}/M$, A/M is a C^*-algebra in the natural way. □

Now, we are going to prove Theorem 2.2.15 (that biduals of C^*-algebras are C^*-algebras in a natural way). To this end, the reader should recall the definition of the adjoint of a bounded bilinear mapping (see §2.2.11) and, in particular, should regard the bidual of any normed algebra as a new normed algebra (relative to the Arens product). Moreover, the consequence of Corollary 2.2.13, that the bidual of a norm-unital normed algebra A is a norm-unital normed algebra with the same unit as that of A, should be kept in mind.

Lemma 2.3.44 *Let A be a norm-unital normed algebra over $\mathbb{K}$, and let a'' be in A''. Then $V(A'',\mathbf{1},a'')$ equals the closure in $\mathbb{K}$ of the set*

$$V^{w^*}(A'',\mathbf{1},a'') := \{a''(f) : f \in D(A,\mathbf{1})\}.$$

Proof Let m denote the product of A. For $(b,b') \in A \times A'$, we have

$$m'(b',\mathbf{1})(b) = b'(m(\mathbf{1},b)) = b'(b),$$

and hence $m'(b',\mathbf{1}) = b'$. Therefore, for $(b',b'') \in \Pi(A')$ we have

$$m''(b'',b')(\mathbf{1}) = b''(m'(b',\mathbf{1})) = b''(b') = 1,$$

so $m''(b'',b') \in D(A,\mathbf{1})$.

On the other hand, invoking Lemma 2.1.10, and applying Corollary 2.1.34 with $X := A'$ and $F := L_{a''} : A'' \to A''$, we have

$$\begin{aligned} V(A'',\mathbf{1},a'') &= V(BL(A''),I_{A''},L_{a''}) \\ &= \overline{\mathrm{co}}\{m'''(a'',b'')(b') : (b',b'') \in \Pi(A')\} \\ &= \overline{\mathrm{co}}\{a''(m''(b'',b')) : (b',b'') \in \Pi(A')\}. \end{aligned}$$

Since the set $V^{w^*}(A'',\mathbf{1},a'')$ is convex, it follows from the first paragraph in the proof that $V(A'',\mathbf{1},a'')$ is contained in the closure in $\mathbb{K}$ of $V^{w^*}(A'',\mathbf{1},a'')$. The converse inclusion is also true because $V(A'',\mathbf{1},a'')$ is closed in $\mathbb{K}$ and, for $f \in D(A,\mathbf{1})$, the mapping $b'' \to b''(f)$ from A'' to $\mathbb{K}$ is an element of $D(A'',\mathbf{1})$. □

§**2.3.45** Let X be a normed space over $\mathbb{K}$ endowed with a continuous conjugate-linear involution $*$. For $f \in X'$ we define $f^* \in X'$ by $f^*(x) = \overline{f(x^*)}$ for every $x \in X$. Then $f \to f^*$ is a continuous conjugate-linear involution on X' which we shall call the *transpose* of the one on X. In what follows, X' and X'' will be seen without notice endowed with the transpose and bitranspose involutions, respectively, and the fact that the bitranspose involution on X'' extends the one given on X should be kept in mind.

Let X,Y,Z be normed spaces over $\mathbb{K}$, each of which is endowed with a continuous conjugate-linear involution $*$, and let $m : X \times Y \to Z$ be a continuous bilinear mapping. Then m^* will stand for the continuous bilinear mapping $(x,y) \to (m(x^*,y^*))^*$ from $X \times Y$ into Z.

Lemma 2.3.46 *Let X,Y,Z be normed spaces over $\mathbb{K}$, each of which is endowed with a continuous conjugate-linear involution $*$, and let $m : X \times Y \to Z$ be a continuous bilinear mapping. Then $m'^* = m^{*\prime}$. As a consequence, $m^{t*} = m^{*t}$.*

Proof The fist conclusion follows from the fact that, for all $x \in X$, $y \in Y$, and $z' \in Z'$, we have

$$\begin{aligned} m'^*(z',x)(y) &= (m'(z'^*,x^*))^*(y) = \overline{m'(z'^*,x^*)(y^*)} = \overline{z'^*(m(x^*,y^*))} \\ &= z'((m(x^*,y^*))^*) = z'(m^*(x,y)) = m^{*\prime}(z',x)(y). \end{aligned}$$

As a consequence, we have

$$m^{t*} = m'''^* = m''^{*\prime} = m'^{*\prime\prime} = m^{*\prime\prime\prime} = m^{*t}. \qquad \square$$

Corollary 2.3.47 *Let A be a normed $*$-algebra over $\mathbb{K}$ whose involution is continuous. Then the bitranspose involution on A'' is an algebra involution if and only if A is Arens regular.*

Proof Let m denote the product of A. Since $*$ is an algebra involution on A, we have $m^* = m^r$. Then, by Lemma 2.3.46, the equality $m^{t*} = m^{rt}$ holds, and so $m^{rt} = m^{tr}$ (the Arens regularity of A) is equivalent to $m^{t*} = m^{tr}$ (i.e. the bitranspose involution is an algebra involution on A''). □

Proposition 2.3.48 *Let A be a V-algebra. Then the bidual of A is a V-algebra whose natural involution coincides with the bitranspose of the natural involution of A. As a consequence, A is Arens regular.*

Proof Let $*$ denote indistinctly the natural involution of A and the first and second transpose ones. Let (a'', f) be in $H(A'', *) \times D(A, \mathbf{1})$. Then clearly $f \in H(A', *)$, so $a''(f) \in \mathbb{R}$, and so $a'' \in H(A'', \mathbf{1})$ (by Lemma 2.3.44). Therefore we have $H(A'', *) \subseteq H(A'', \mathbf{1})$, which proves the first conclusion. Now, the Arens regularity of A follows from the fact that $*$ is an algebra involution on A'' (by Theorem 2.3.8) and Lemma 2.3.47. □

Lemma 2.3.49 *Let A be a normed associative algebra over $\mathbb{K}$. Then the normed algebra A'' is associative.*

Proof Let m denote the product of A. Since A is associative, for all $a, b, c \in A$ we have

$$m(m(a,b),c) = m(a,m(b,c)).$$

Therefore, for every $a' \in A'$ we see that

$$\begin{aligned} m'(a',m(a,b))(c) &= a'(m(m(a,b),c)) = a'(m(a,m(b,c))) \\ &= m'(a',a)(m(b,c)) = m'(m'(a',a),b)(c), \end{aligned}$$

and hence

$$m'(a',m(a,b)) = m'(m'(a',a),b). \tag{2.3.11}$$

Applying (2.3.11), for every $c'' \in A''$ we get

$$\begin{aligned} m'(m''(c'',a'),a)(b) &= m''(c'',a')(m(a,b)) = c''(m'(a',m(a,b))) \\ &= c''(m'(m'(a',a),b)) = m''(c'',m'(a',a))(b), \end{aligned}$$

and hence

$$m'(m''(c'',a'),a) = m''(c'',m'(a',a)). \tag{2.3.12}$$

Using (2.3.12), for every $b'' \in A''$ we obtain

$$\begin{aligned} m''(b'',m''(c'',a'))(a) &= b''(m'(m''(c'',a'),a)) = b''(m''(c'',m'(a',a))) \\ &= m^t(b'',c'')(m'(a',a)) = m''(m^t(b'',c''),a')(a), \end{aligned}$$

and hence

$$m''(b'',m''(c'',a')) = m''(m^t(b'',c''),a'). \tag{2.3.13}$$

Finally, applying (2.3.13), for every $a'' \in A''$ we conclude that

$$m^t(m^t(a'',b''),c'')(a') = m^t(a'',b'')(m''(c'',a')) = a''(m''(b'',m''(c'',a')))$$
$$= a''(m''(m^t(b'',c''),a')) = m^t(a'',m^t(b'',c''))(a'),$$

and hence

$$m^t(m^t(a'',b''),c'') = m^t(a'',m^t(b'',c'')),$$

i.e. A'' is associative. □

Lemma 2.3.50 *Let X be a normed space over $\mathbb{K}$ endowed with a continuous conjugate-linear involution $*$, and let Y be a $*$-invariant subspace of X. Then the bipolar $Y^{\circ\circ}$ of Y in X'' is invariant under the bitranspose involution, and the natural Banach space identification of $Y^{\circ\circ}$ with Y'' preserves bitranspose involutions.*

Proof For the sake of convenience, let us take different symbols for the w^*-topology of X'' (say $\sigma(X'',X')$) and the w^*-topology of Y'' (say analogously $\sigma(Y'',Y')$). Let $*$ also stand for the bitranspose involution on X''. Since $*$ is $\sigma(X'',X')$-continuous on X'', and Y is $*$-invariant, the set $\{x \in Y^{\circ\circ} : x^* \in Y^{\circ\circ}\}$ is $\sigma(X'',X')$-closed in X'' and contains Y. Therefore, since Y is $\sigma(X'',X')$-dense in $Y^{\circ\circ}$, we deduce that $Y^{\circ\circ}$ is $*$-invariant. Now, denote also by $*$ the bitranspose involution on Y'' of the involution of Y (as a $*$-invariant subspace of X), and recall that it is $\sigma(Y'',Y')$-continuous. Since the natural identification $\Phi : Y^{\circ\circ} \to Y''$ is $\sigma(X'',X')_{|Y^{\circ\circ}}$-to-$\sigma(Y'',Y')$ continuous and becomes the identity on Y, the set $\{x \in Y^{\circ\circ} : \Phi(x^*) = \Phi(x)^*\}$ is $\sigma(X'',X')$-closed in X'' and contains Y. The $\sigma(X'',X')$-density of Y in $Y^{\circ\circ}$ gives $\Phi(x^*) = \Phi(x)^*$ for every $x \in Y^{\circ\circ}$. □

Lemma 2.3.51 *Let X,Y,Z be normed spaces over $\mathbb{K}$, and let $m : X \times Y \to Z$ be a bounded bilinear mapping. The following conditions are equivalent:*

(i) *The Arens extension $m^t : X'' \times Y'' \to Z''$ is w^*-continuous in its second variable.*
(ii) *The Arens extension m^t is separately w^*-continuous.*
(iii) *There exists a separately w^*-continuous mapping from $X'' \times Y''$ to Z'' which extends m.*
(iv) *m is Arens regular.*

Proof (i)⇒(ii) By Lemma 2.2.12(iii)(b).

(ii)⇒(iii) This is clear.

(iii)⇒(iv) Assume that there exists a separately w^*-continuous mapping $g : X'' \times Y'' \to Z''$ which extends m. Then, by Lemma 2.2.12(iii), we have $g = m^t$ and $g^r = m^{rt}$. Since the first equality implies $g^r = m^{tr}$, we deduce $m^{tr} = m^{rt}$, i.e. m is Arens regular.

(iv)⇒(i) By Lemma 2.2.12(iii)(b), m^{rt} is w^*-continuous in its first variable. Therefore, if m is Arens regular (i.e. $m^{tr} = m^{rt}$), then m^t is w^*-continuous in its second variable. □

Lemma 2.3.52 *Let B be a normed algebra over $\mathbb{K}$, and let A be a subalgebra of B. Then the bipolar $A^{\circ\circ}$ of A in B'' is a subalgebra of B'', and the natural Banach space identification of $A^{\circ\circ}$ with A'' becomes a bijective algebra homomorphism. As a consequence A is Arens regular whenever so is B.*

Proof The set $\{y \in A^{\circ\circ} : Ay \subseteq A^{\circ\circ}\}$ contains A and, by Lemma 2.2.12(iii)(c), is $\sigma(B'',B')$-closed in B''. Therefore, since A is $\sigma(B'',B')$-dense in $A^{\circ\circ}$, we deduce that $AA^{\circ\circ} \subseteq A^{\circ\circ}$. Now, the set $\{z \in A^{\circ\circ} : zA^{\circ\circ} \subseteq A^{\circ\circ}\}$ contains A and, by Lemma 2.2.12(iii)(b), is $\sigma(B'',B')$-closed in B''. The $\sigma(B'',B')$-density of A in $A^{\circ\circ}$ gives that $A^{\circ\circ}A^{\circ\circ} \subseteq A^{\circ\circ}$, i.e. $A^{\circ\circ}$ is a subalgebra of B''. Assume that B is Arens regular. Then, by Lemma 2.3.51, the product of B'' is separately $\sigma(B'',B')$-continuous. Now, since the natural identification $\Phi : A^{\circ\circ} \to A''$ is $\sigma(B'',B')_{|A^{\circ\circ}}$-to-$\sigma(A'',A')$ bicontinuous and becomes the identity on A, the mapping $(u,v) \to \Phi(\Phi^{-1}(u)\Phi^{-1}(v))$ from $A'' \times A''$ to A'' is separately $\sigma(A'',A')$-continuous and extends the product of A. It follows from Lemma 2.3.51 that A is Arens regular. □

§2.3.53 *Proof of Theorem 2.2.15* Let A be a nonzero C^*-algebra.

Assume at first that A is unital. Then, by Lemma 2.2.5, A is a V-algebra. Therefore, by Proposition 2.3.48, Lemma 2.3.49, and Theorem 2.3.32, A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital C^*-algebra, and moreover A is Arens regular.

To conclude the proof, we must show that the same conclusion holds if A is not unital. In this case, we consider the C^*-algebra unital extension $A_{\mathbb{1}}$ of A (see Proposition 1.2.44), so that, by the above paragraph, the desired conclusion holds with $A_{\mathbb{1}}$ instead of A. Therefore, since A is a $*$-subalgebra of $A_{\mathbb{1}}$, Lemmas 2.3.50 and 2.3.52 apply to get that the bipolar $A^{\circ\circ}$ of A in $(A_{\mathbb{1}})''$ is a $*$-subalgebra of the C^*-algebra $(A_{\mathbb{1}})''$, that the natural Banach space identification $\Phi : A^{\circ\circ} \to A''$ becomes a bijective algebra $*$-homomorphism, and that A is Arens regular. Now, since Φ is an isometry which becomes the identity on A, it turns out clear that A'' is a C^*-algebra in the natural way. Finally, since $\mathbb{B}_{A''}$ has extreme points (by the Banach–Alaoglu and Krein–Milman theorems), it follows from Lemma 2.1.26 that A'' is unital. □

Now that Theorem 2.2.15 has been proved, we are going to provide some complements on the C^*-algebra of multipliers of a given C^*-algebra in order to derive the *Banach–Stone theorem*, on isometries of $C_0^{\mathbb{C}}(E)$-spaces, from the Kadison–Paterson–Sinclair theorem (cf. Theorem 2.2.19).

Definition 2.3.54 *Flexible algebras* are defined as those algebras satisfying the 'flexibility' condition $(ab)a = a(ba)$.

Fact 2.3.55 *Let A be a flexible algebra over $\mathbb{K}$, let a be in A, and let I be an ideal of A having a unit element e. Then $ae = ea$.*

Proof Since ae and ea lie in I, and e is the unit of I, we have

$$ae = e(ae) = (ea)e = ea.$$ □

An ideal I of an algebra A over $\mathbb{K}$ is said to be *essential* if $I \cap J \neq 0$ for every nonzero ideal J of A. Now, let A be a C^*-algebra. We recall that closed ideals of A are $*$-invariant (cf. Proposition 2.3.43) (and hence are C^*-algebras), and that the C^*-algebra $M(A)$ of multipliers of A was defined as the closed $*$-subalgebra of A'' given by

$$\{a'' \in A'' : a''A \subseteq A \text{ and } Aa'' \subseteq A\}$$

(cf. §2.2.16).

Proposition 2.3.56 *Let A be a C^*-algebra. We have:*

(i) *Every closed $*$-subalgebra of $M(A)$ containing A becomes a C^*-algebra containing A as an essential ideal.*
(ii) *Conversely, any C^*-algebra containing A as an essential ideal can be seen in a natural way as a closed $*$-subalgebra of $M(A)$ containing A. More precisely, if B is a C^*-algebra, if $\phi : A \to B$ is an isometric algebra $*$-homomorphism such that $\phi(A)$ is an essential ideal of B, then there is a unique algebra $*$-homomorphism $\eta : B \to M(A)$ satisfying $\eta \circ \phi = \imath$, where $\imath$ denotes the inclusion $A \hookrightarrow M(A)$, and moreover η is an isometry.*

Proof Let C be any closed $*$-subalgebra of $M(A)$ containing A. Clearly A is an ideal of C. Now let I be an ideal of C with $I \cap A = 0$. Then we have $AI = 0$, so $A''I = 0$ because the product of A'' is w^*-continuous in its first variable and A is w^*-dense in A''. Therefore $I^*I = 0$, hence $I = 0$. This proves assertion (i).

Let B be a C^*-algebra, and let $\phi : A \to B$ be an isometric algebra $*$-homomorphism such that $\phi(A)$ is an essential ideal of B. Since A'' and B'' are C^*-algebras with separately w^*-continuous products and w^*-continuous involutions (cf. Theorem 2.2.15 and the implication (iv)$\Rightarrow$(ii) in Lemma 2.3.51), ϕ'' becomes an isometric algebra $*$-homomorphism from A'' to B'' whose range is a w^*-closed ideal of B''. Keeping in mind the Banach–Alaoglu and Krein–Milman theorems, it follows from Lemma 2.1.26 that $\phi''(A'')$ has a unit (say e), which becomes clearly an idempotent of B'' satisfying $\phi''(A'') = B''e$. From now on, Fact 2.3.55 should be omnipresent in the remaining part of the proof. Thus $(B''(\mathbf{1} - e)) \cap B$ is an ideal of B with zero intersection with $\phi(A)$ so, since $\phi(A)$ is an essential ideal of B, we have $(B''(\mathbf{1} - e)) \cap B = 0$ and therefore, by Corollary 1.2.52, the mapping $\psi : x \to xe$ from B into $\phi''(A'')$ is an isometric algebra $*$-homomorphism. Then the mapping $\eta = (\phi'')^{-1} \circ \psi$ is an isometric algebra $*$-homomorphism from B to A'' whose range is contained in $M(A)$, and routinely we see that $\eta \circ \phi = \imath$, where $\imath$ denotes the inclusion $A \hookrightarrow M(A)$. Assume that $\zeta : B \to M(A)$ is an algebra $*$-homomorphism satisfying $\zeta \circ \phi = \imath$. Since $\phi(A)$ is an ideal of B, given $a \in A$ and $b \in B$ there is $c \in A$ such that $\phi(a)b = \phi(c)$, and we have

$$\begin{aligned} a\eta(b) = \imath(a)\eta(b) = \eta(\phi(a))\eta(b) = \eta(\phi(a)b) = \eta(\phi(c)) = \imath(c) \\ = \zeta(\phi(c)) = \zeta(\phi(a)b) = \zeta(\phi(a))\zeta(b) = \imath(a)\zeta(b) = a\zeta(b). \end{aligned}$$

It follows from the arbitrariness of $a \in A$ that $A(\eta(b) - \zeta(b)) = 0$. Now, keeping in mind that the product of A'' is w^*-continuous in its first variable and that A is w^*-dense in A'', we see that $A''(\eta(b) - \zeta(b)) = 0$, and hence $\eta(b) - \zeta(b) = 0$. It follows from the arbitrariness of $b \in B$ that $\zeta = \eta$, and the proof is complete. □

Given a topological space E, we denote by $C_b^{\mathbb{C}}(E)$ the C^*-algebra of all complex-valued bounded continuous functions on E.

Proposition 2.3.57 *Let E be a locally compact Hausdorff topological space. Then the C^*-algebra $C_b^{\mathbb{C}}(E)$ identifies naturally with the C^*-algebra of multipliers of $C_0^{\mathbb{C}}(E)$.*

Proof Set $A := C_0^{\mathbb{C}}(E)$ and $B := C_b^{\mathbb{C}}(E)$. Then it easily realized that A is an essential ideal of B. Therefore, by Proposition 2.3.56, there is a unique isometric $*$-homomorphism $\eta : B \to M(A)$ satisfying $\eta \circ \phi = \iota$, where ϕ and ι denote the inclusions $A \hookrightarrow B$ and $A \hookrightarrow M(A)$, respectively. Thus, to conclude the proof it is enough to show that η is surjective. By Goldstine's theorem, there is a net u_λ in $\mathbb{B}_A$ converging to $\mathbf{1}$ in the weak* topology of A''. Then $u_\lambda a''$ converges to a'' in the weak* topology of A'' for every $a'' \in A''$, so $u_\lambda a$ converges to a in the weak topology of A for every $a \in A$; hence

$$u_\lambda a \quad \text{converges to } a \text{ pointwise for every } a \in A. \tag{2.3.14}$$

Now let x be in $M(A)$. Then, for each λ, $u_\lambda x$ is a function from E to $\|x\|\mathbb{B}_{\mathbb{C}}$, i.e. $u_\lambda x \in (\|x\|\mathbb{B}_{\mathbb{C}})^E$. Therefore, since $(\|x\|\mathbb{B}_{\mathbb{C}})^E$ is compact for the topology of the pointwise convergence, the net $u_\lambda x$ has a cluster point h in $(\|x\|\mathbb{B}_{\mathbb{C}})^E$ relative to that topology. Let a be in A. It follows that ha is a cluster point to the net $(u_\lambda x)a$ in the topology of the pointwise convergence of $\mathbb{C}^E$. But, by (2.3.14), $u_\lambda(xa)$ converges to xa pointwise. Since $(u_\lambda x)a = u_\lambda(xa)$ in $M(A)$, it follows that $ha = xa$ in $\mathbb{C}^E$. Now we claim that h is continuous. Let t_μ be a net in E converging to some $t \in E$. We must show that $h(t_\mu)$ converges to $h(t)$ in $\mathbb{C}$. Take a compact neighbourhood K of t in E, so that we may assume that t_μ lies in K for every μ, and use Urysohn's lemma to find $a \in A$ such that $a = 1$ on K. Then we have

$$\lim_\mu h(t_\mu) = \lim_\mu h(t_\mu)a(t_\mu) = \lim_\mu (xa)(t_\mu) = (xa)(t) = h(t)a(t) = h(t),$$

as desired. Now that the claim has been proved, we have that h lies in B. Finally, the equality $\eta(h) = x$ is easily realized. Indeed, keeping in mind that η becomes the identity on A, for an arbitrary function $a \in A$ we have

$$\eta(h)a = \eta(h)\eta(a) = \eta(ha) = ha = xa,$$

so $(\eta(h) - x)A = 0$, hence $\eta(h) = x$, as desired. □

Combining Corollary 1.1.77, Theorem 2.2.19, and Proposition 2.3.57, we get the following.

Theorem 2.3.58 (Banach–Stone) *Let E and F be locally compact Hausdorff topological spaces, and let $\Psi : C_0^{\mathbb{C}}(E) \to C_0^{\mathbb{C}}(F)$ be a surjective linear isometry. Then there exist a continuous function $g : F \to \mathbb{S}_{\mathbb{C}}$ and a homeomorphism $\eta : F \to E$ such that*

$$\Psi(f)(t) = g(t)f(\eta(t)) \ \textit{for all} \ f \in C_0^{\mathbb{C}}(E) \ \textit{and} \ t \in F.$$

Corollary 2.3.59 *Two locally compact Hausdorff topological spaces E and F are homeomorphic if (and only if) the Banach spaces $C_0^{\mathbb{C}}(E)$ and $C_0^{\mathbb{C}}(F)$ are linearly isometric.*

2.3.4 Introducing alternative C^*-algebras

Let A be an algebra over $\mathbb{K}$. Given elements $a, b, c \in A$, we define the *associator* $[a, b, c]$ of a, b, c by

$$[a, b, c] := (ab)c - a(bc).$$

We say that A is *alternative* if the equalities $a^2b = a(ab)$ and $ba^2 = (ba)a$ hold for all $a, b \in A$. These identities can be written in the form

$$[a,a,b] = 0 \text{ and } [b,a,a] = 0.$$

The first of these identities is called the *left alternative identity*, and the second is the *right alternative identity*.

Linearizing the left and right alternative identities, we obtain the identities

$$[a,c,b] + [c,a,b] = 0 \text{ and } [b,c,a] + [b,a,c] = 0,$$

from which it follows that in an alternative algebra, the associator is an alternating function of its arguments. In particular, every alternative algebra is flexible (cf. Definition 2.3.54). In view of flexibility, in an alternative algebra we can (and shall) subsequently write the product aba without indicating an arrangement of parentheses.

Lemma 2.3.60 *In every alternative algebra A over $\mathbb{K}$ the following identities are valid:*

$$\begin{aligned} x(yzy) &= [(xy)z]y, && \textit{the right Moufang identity,} \\ (yzy)x &= y[z(yx)], && \textit{the left Moufang identity,} \\ (xy)(zx) &= x(yz)x, && \textit{the middle Moufang identity.} \end{aligned}$$

Proof We shall first prove that the algebra A satisfies the identity

$$[x^2, y, x] = 0. \tag{2.3.15}$$

By the flexible and left alternative identities we have

$$(x^2y)x = [x(xy)]x = x(xyx) = x^2(yx),$$

that is, (2.3.15) is proved. Now by (2.3.15)

$$xy^3 = (xy^2)y - [x, y^2, y] = (xy^2)y = [(xy)y]y.$$

Linearizing the identity $xy^3 - [(xy)y]y = 0$ we find that

$$\begin{aligned} 0 &= x(zy^2) + x(yzy) + x(y^2z) - (xz)y^2 - [(xy)z]y - (xy^2)z \\ &= x(yzy) - [(xy)z]y - [x, y^2, z] - [x, z, y^2] = x(yzy) - [(xy)z]y, \end{aligned}$$

and so the right Moufang identity is proved. Left Moufang identity is the reciprocal relationship (obtained by passing to the opposite algebra, which is alternative since the defining identities are reciprocal). Finally,

$$\begin{aligned} (xy)(zx) - x(yz)x &= -[xy, z, x] + [x, y, z]x = [z, xy, x] + [z, x, y]x \\ &= [z(xy)]x - z(xyx) + [(zx)y]x - [z(xy)]x = 0. \end{aligned}$$ □

The right Moufand identity is equivalent to

$$[x, yz, y] = -[x, y, z]y, \tag{2.3.16}$$

since $[x, yz, y] = [x(yz)]y - x(yzy) = [x(yz)]y - [(xy)z]y = -[x, y, z]y$. The linearized form of (2.3.16) is

$$[x, yz, u] + [x, uz, y] = -[x, y, z]u - [x, u, z]y. \tag{2.3.17}$$

Now we prove the so-called ‘Artin’s theorem’.

Theorem 2.3.61 *An algebra A over $\mathbb{K}$ is alternative (if and) only if the subalgebra of A generated by any two elements of A is associative.*

Proof The ‘if’ part is clear. If a, b are two elements of A, we denote by $p = p(a,b)$ any non-associative product $c_1 c_2 \cdots c_t$ (with some distribution of parentheses) of t factors c_k, each of which is equal to a or b. Also we denote the length t of such a product by $\deg(p)$. In order to prove the ‘only if’ part it clearly suffices to show that $[p,q,r] = 0$ for all non-associative products $p = p(a,b)$, $q = q(a,b)$, and $r = r(a,b)$. We shall prove this assertion by induction on the number

$$n = \deg(p) + \deg(q) + \deg(r).$$

If $n = 3$, then everything follows from the left and right alternative and flexible identities. Now let $n > 3$, and assume inductively that $[u,v,w] = 0$ for all non-associative products $u = u(a,b)$, $v = v(a,b)$, and $w = w(a,b)$ such that $\deg(u) + \deg(v) + \deg(w) < n$. Since $\max\{\deg(p), \deg(q), \deg(r)\} < n$, the induction hypothesis implies, by the usual argument which yields the generalized associative law from the associative law, that parentheses are not necessary in the writing of the products $p = p(a,b)$, $q = q(a,b)$, and $r = r(a,b)$.

Now two of the products p, q, r must begin with the same letter, say a. Moreover, since associators alternate in A, we may assume that q and r begin with a. If $\deg(q) > 1$ and $\deg(r) > 1$, then $q = aq_0$ and $r = ar_0$, where $\deg(q_0) = \deg(q) - 1$ and $\deg(r_0) = \deg(r) - 1$. Setting $x = ar_0$, $z = q_0$, and $u = p$ in (2.3.17), we have

$$\begin{aligned}
[p,q,r] &= [p, aq_0, ar_0] = -[ar_0, aq_0, p] \\
&= [ar_0, pq_0, a] + [ar_0, a, q_0]p + [ar_0, p, q_0]a \\
&= -[pq_0, ar_o, a] = [pq_0, a, r_0]a = 0
\end{aligned}$$

by (2.3.16) and the assumption of the induction. If only one of the products q, r has degree > 1 (say $q = aq_0$), then (2.3.16) implies that

$$[p,q,r] = [p, aq_0, a] = -[p, a, q_0]a = 0$$

by the assumption of the induction. The easiest case $\deg(q) = \deg(r) = 1$ is given by the right alternative law: $[p,q,r] = [p,a,a] = 0$. □

§2.3.62 By an *alternative C^*-algebra* we mean a complete normed alternative complex $*$-algebra A satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$.

Now, the generalization of the associative Vidav–Palmer theorem to the alternative case is easy. Indeed, we have the following.

Corollary 2.3.63 *Let A be a complete alternative V-algebra. Then A, equipped with its natural involution $*$, becomes an alternative C^*-algebra.*

Proof That $*$ is an algebra involution follows from Theorem 2.3.8. Let a be an element of A, and let B stand for the closed subalgebra of A generated by $\{a, a^*, \mathbf{1}\}$. Then, by Theorem 2.3.61, B is associative. Moreover, since $*$ is a continuous algebra involution (by Lemma 2.3.7), B is $*$-invariant. Now, keeping in mind Corollary 2.1.2, it is easily realized that B is a complete V-algebra whose natural involution is the restriction of $*$ to B. Therefore, since B is associative, Theorem 2.3.32 applies, so that we have $\|x^*x\| = \|x\|^2$ for every $x \in B$. Consequently, $\|a^*a\| = \|a\|^2$. □

Let A be an algebra over $\mathbb{K}$, and let $\pi : A \to A$ be a linear projection. Then $\pi(A)$ becomes naturally an algebra over $\mathbb{K}$ under the product $\odot^\pi$ defined by

$$x \odot^\pi y := \pi(xy)$$

for all $x, y \in \pi(A)$. If A has a unit $\mathbf{1}$ and if $\pi(\mathbf{1}) = \mathbf{1}$, then $\mathbf{1}$ remains a unit for $(\pi(A), \odot^\pi)$. If A is normed, and if π is contractive, then $(\pi(A), \odot^\pi)$ becomes a normed algebra (under the restriction of the norm of A). Therefore, if in addition A is norm-unital and π is unit-preserving, then $(\pi(A), \odot^\pi)$ is a norm-unital normed algebra. Moreover, invoking Corollary 2.1.2(i), we straightforwardly derive the following.

Proposition 2.3.64 *Let A be a V-algebra, and let $\pi : A \to A$ be a unit-preserving contractive linear projection. Then $(\pi(A), \odot^\pi)$ is a V-algebra whose natural involution coincides with the restriction to $\pi(A)$ of the natural involution of A.*

Since unital alternative C^*-algebras are V-algebras (by Lemma 2.2.5), and complete alternative V-algebras are alternative C^*-algebras (by Corollary 2.3.63), Proposition 2.3.64 above could lead to expect that ranges of unit-preserving contractive linear projections on unital alternative C^*-algebras are alternative C^*-algebras. As the following example shows, this is no longer true, even in the case of (associative) C^*-algebras.

Example 2.3.65 Let A stand for the unital C^*-algebra $M_2(\mathbb{C})$, and let π be the unit-preserving linear projection on A defined by

$$\pi \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} := \begin{pmatrix} \frac{\alpha_{11}+\alpha_{22}}{2} & \alpha_{12} \\ \alpha_{21} & \frac{\alpha_{11}+\alpha_{22}}{2} \end{pmatrix}.$$

With the help of Exercise 1.2.15, it is easily realized that π is contractive. Let x, y be the elements of $\pi(A)$ given by $x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we have $x \odot^\pi x = 0$ and $y \odot^\pi x = \frac{1}{2}\mathbf{1}$, and hence $y \odot^\pi (x \odot^\pi x) = 0$ and $(y \odot^\pi x) \odot^\pi x = \frac{1}{2}x$. Therefore $(\pi(A), \odot^\pi)$ is not associative, nor even alternative. Moreover, noticing that $y = x^*$, we have $\|x^* \odot^\pi x\| = \frac{1}{2}$, whereas $\|x\|^2 = 1$. Thus, in view of Proposition 2.3.64, $(\pi(A), \odot^\pi)$ becomes an example of a V-algebra whose norm is not a C^*-norm for the natural involution. The last reason of this pathology will be found later in Theorem 3.2.5.

§**2.3.66** We note that, as in the associative case, if A is a unital alternative C^*-algebra, then, by Lemma 2.2.5, we have $H(A, *) = H(A, \mathbf{1})$, and hence, by §§2.3.37 and 2.3.34, $H(A, *)$ is canonically endowed with the numerical-range order.

Corollary 2.3.67 *Let A be a unital alternative C^*-algebra and let a be in A. Then we have $a^*a \geqslant 0$ in the numerical-range order of $H(A, *)$.*

Proof By Theorem 2.3.61, the closed subalgebra of A (say B) generated by $\{\mathbf{1}, a, a^*\}$ is a C^*-algebra. Therefore, by Corollary 2.1.2 and Propositions 2.3.38 and 2.3.39(i), we have $V(A, \mathbf{1}, a^*a) = V(B, \mathbf{1}, a^*a) \subseteq \mathbb{R}_0^+$. It follows that $a^*a \geqslant 0$ in $H(A, *)$. □

Theorem 2.3.68 *Let A be a unital alternative C^*-algebra, and let $\pi : A \to A$ be a unit-preserving contractive linear projection such that*

$$\pi(\pi(a)^*\pi(a)) \leqslant \pi(a^*a) \tag{2.3.18}$$

for every $a \in A$. Then $\pi(A)$ is a $*$-invariant subspace of A, and $\pi(A)$ becomes a unital alternative C^*-algebra under the norm and the involution of A, and the product $x \odot^\pi y := \pi(xy)$. Moreover, if A is associative, then so is the algebra $(\pi(A), \odot^\pi)$.

Proof We note at first that, by Lemma 2.2.5, Corollary 2.1.2(i), and Exercise 1.2.21, π becomes a $*$-mapping. Let f be in $D(A, \mathbf{1})$. Then $g := f\pi$ lies in $D(A, \mathbf{1})$, so, by Corollary 2.3.67, the mapping $(a, b) \to g(b^*a)$ becomes a non-negative hermitian sesquilinear form on (the vector space of) A, and so, by passing to quotient by the subspace $M := \{a \in A : g(a^*a) = 0\}$, we get a complex pre-Hilbert space X with well-defined inner product $(\hat{a}|\hat{b}) := g(b^*a)$, where, for $c \in A$, $\hat{c}$ stands for the canonical image of c in X. Since $g\pi = g$, it follows from (2.3.18) that, for $a \in A$, we have

$$\|\widehat{\pi(a)}\|^2 = g(\pi(a)^*\pi(a)) = g[\pi(\pi(a)^*\pi(a))] \leqslant g(\pi(a^*a)) = g(a^*a) = \|\hat{a}\|^2.$$

Therefore $\hat{a} \to \widehat{\pi(a)}$ becomes a well-defined contractive linear projection on X, whose continuous extension to the Hilbert space completion of X (say H) will be denoted by P. By Corollary 1.2.50, P is a self-adjoint operator on H, and hence, for $a, b \in A$ we have

$$\begin{aligned} g(\pi(a)b) = g(\pi(a^*)^*b) &= (\hat{b}|\widehat{\pi(a^*)}) = (\hat{b}|P(\widehat{a^*})) \\ &= (P(\hat{b})|\widehat{a^*}) = (\widehat{\pi(b)}|\widehat{a^*}) = g(a\pi(b)). \end{aligned}$$

Since f is an arbitrary element of $D(A, \mathbf{1})$, and $g = f\pi$, it follows from Corollary 2.1.13 that

$$\pi(\pi(a)b) = \pi(a\pi(b)) \tag{2.3.19}$$

for all $a, b \in A$. Now, invoking that A is an alternative algebra and (2.3.19), for $x, y \in \pi(A)$, we get that

$$\begin{aligned} x \odot^\pi (y \odot^\pi y) &= \pi(x\pi(y^2)) = \pi(\pi(x)y^2) = \pi(xy^2) \\ &= \pi((xy)y) = \pi((xy)\pi(y)) = \pi(\pi(xy)y) = (x \odot^\pi y) \odot^\pi y, \end{aligned}$$

and similarly $(y \odot^\pi y) \odot^\pi x = y \odot^\pi (y \odot^\pi x)$. This shows that $(\pi(A), \odot^\pi)$ is an alternative algebra. Since $(\pi(A), \odot^\pi)$ is a complete V-algebra (by Lemma 2.2.5 and Proposition 2.3.64), it follows from Corollary 2.3.63 that $(\pi(A), \odot^\pi)$ is an alternative C^*-algebra.

Assume that A is actually associative. Then, invoking again (2.3.19), for $x,y,z \in \pi(A)$ we obtain that

$$x \odot^{\pi} (y \odot^{\pi} z) = \pi(x\pi(yz)) = \pi(\pi(x)yz) = \pi(xyz)$$
$$= \pi(xy\pi(z)) = \pi(\pi(xy)z) = (x \odot^{\pi} y) \odot^{\pi} z,$$

and hence the alternative C^*-algebra $(\pi(A), \odot^{\pi})$ is associative. □

Keeping in mind Fact 2.3.35(i), Theorem 2.3.68 implies the following.

Corollary 2.3.69 *Let A be a unital C^*-algebra, and let $\pi : A \to A$ be a unit-preserving contractive linear projection such that*

$$\pi(a)^*\pi(a) \leqslant \pi(a^*a) \tag{2.3.20}$$

for every $a \in A$. Then $\pi(A)$ is a $*$-invariant subspace of A, and $\pi(A)$ becomes a unital C^*-algebra under the norm and the involution of A, and the product $x \odot^{\pi} y := \pi(xy)$.

2.3.5 Historical notes and comments

Proposition 2.3.4 is a light non-associative version of the following theorem due to Sinclair [580].

Theorem 2.3.70 *Let A be a norm-unital normed associative complex algebra, let k be a hermitian element of A, and let λ, μ be complex numbers. Then $\mathfrak{r}(\lambda\mathbf{1} + \mu k) = \|\lambda\mathbf{1} + \mu k\|$.*

The argument given here is taken from [695, Section 26], where techniques in Browder's paper [133] are incorporated. The conclusion of the proof of Theorem 2.3.70, following the lines just quoted, goes as follows.

Proof We may assume that A is complete. Moreover, a normalization process, like the one in the proof of Proposition 2.3.4, allows us to assume in addition that $\|k\| = 1$ and that $(\lambda, \mu) = (\cos(\varphi), i\sin(\varphi))$ for some φ in $\mathbb{R}$. Then, by Proposition 2.3.22, we have $\mathfrak{r}(k) = 1$, and hence, since $\mathrm{sp}(k) \subseteq \mathbb{R}$ (by Lemma 2.3.21), we also have $\mathfrak{r}(\cos(\varphi)\mathbf{1} + i\sin(\varphi)k) \geqslant 1$. Therefore, invoking the inequality (2.3.1) in the proof of Proposition 2.3.4, we deduce

$$\mathfrak{r}(\cos(\varphi)\mathbf{1} + i\sin(\varphi)k) = \|\cos(\varphi)\mathbf{1} + i\sin(\varphi)k\|.$$ □

Complete associative V-algebras were first considered by Vidav [623]. Let A be a complete associative V-algebra such that

(V) for every hermitian element $h \in A$ we have $h^2 = x + iy$ with x, y commuting hermitian elements of A.

Vidav proved that the natural involution $*$ of A is an algebra involution, and that there is an equivalent algebra norm $|||\cdot|||$ on A satisfying $|||a^*a||| = |||a|||^2$ for every $a \in A$, and $|||h||| = \|h\|$ for every $h \in H(A, \mathbf{1})$. According to [694, p. 57], 'E. Berkson [100] and B. W. Glickfeld [289] showed independently (and by different methods) that actually we have

(BG) $|||a||| = \|a\|$ for every $a \in A$.

Finally, T. W. Palmer [474] showed that condition (V) is unnecessary and he also gave the simplest proof of assertion (BG).' The whole result just reviewed (called the associative Vidav–Palmer theorem, and stated in Theorem 2.3.32), together with Lemma 2.2.5, can be read as that *unital C^*-algebras and complete associative V-algebras are the same*.

As we have just said, the associative case of Theorem 2.3.8 (that natural involutions of V-algebras are algebra involutions) is due to Vidav [623] and Palmer [474]. The general non-associative version stated here is due to Rodríguez [515], whose proof has been reproduced almost verbatim. Particular non-associative forerunners had been obtained earlier in [113, 362, 422, 514].

Proposition 2.3.20 is taken from [528]. A non-unital version, also established in [528], will be proved in Volume 2 of our work.

The consequence of Theorem 2.3.70, given by Proposition 2.3.22, was proved independently by Sinclair and Browder in the already cited papers [580] and [133], respectively. The proof given here is that of Bonsall and Crabb [115], as modified in [695, Section 26]. Proposition 2.3.23 is due to Vidav [623]. Lemma 2.3.26 is an adaptation of Kaplansky's argument to show that a certain requirement in the Gelfand–Naimark abstract characterization of C^*-algebras [285] is redundant. Indeed, the fact given by Proposition 2.3.39(i) that $a^*a \geqslant 0$ for every element a in a C^*-algebra, was taken by Gelfand and Naimark as an axiom. Kaplansky's result, based on previous ideas of Fukamiya [273] and Kelley and Vaught [387], was recorded in Schatz' review [554] of Fukamiya's paper. Lemma 2.3.27 is due to Bonsall [111]. Lemma 2.3.28 is due to Russo and Dye [548], whereas the refinement given by Proposition 2.3.29 is due to Palmer [475]. The commutative forerunner of the Russo–Dye theorem is due to Phelps [491]. The elementary proof included here (as well as in the books [695, 696, 725, 787]) of the Russo–Dye–Palmer result is due to Harris [313], who works in a more general setting. The whole proof of the associative Vidav–Palmer theorem given here (based on the results already quoted in the current paragraph) is essentially Palmer's original proof, as included in the books [694, 695, 696, 725, 786, 787]. Nevertheless, we have introduced some simplifications, like the one given by Lemma 2.3.24 (an adaptation of the commutative Gelfand–Naimark Theorem 1.2.23) and the construction of the equivalent C^*-norm in the conclusion of proof of Theorem 2.3.32, which, through Lemma 2.3.31, is taken from [510]. For a wide generalization of Lemma 2.3.26 the reader is referred to the Shirali–Ford theorem [565], and the alternative proofs of this theorem given by Harris [313] and Pták [494] (see also [696, Theorem 41.5]). This generalization will be discussed later in detail (see Corollary 4.5.39(ii)).

The Russo–Dye theorem, stated in Lemma 2.3.28, has been refined by Robertson [506], Popa [492], Gardner [281], and Kadison and Pedersen [360], among others. Some of these refinements will be discussed later in a non-associative setting (see Subsection 4.2.3). The Russo–Dye theorem does not remain true for real C^*-algebras (i.e. closed real $*$-subalgebras of C^*-algebras). Indeed, $C^{\mathbb{R}}([0,1])$ is a unital real C^*-algebra with no unitary elements other than $\pm\mathbf{1}$. Nevertheless, the Russo–Dye theorem survives for finite-dimensional real C^*-algebras [62], and, more generally, for von Neumann real algebras [459].

To conclude our historical review of the associative Vidav–Palmer theorem, let us comment on the Vidav–Palmer original proof that the natural involution of a complete associative V-algebra is an algebra involution. The break point in the Vidav–Palmer argument is the following generalization of Vidav's Proposition 2.3.23, also proved in [623].

Lemma 2.3.71 *Let A be a norm-unital complete normed associative complex algebra, and let a be in A such that $a = h + ik$ for commuting elements $h,k \in H(A,\mathbf{1})$. Then $V(a) = \mathrm{co}(\mathrm{sp}(a))$.*

Proof Since both $\mathrm{sp}(h)$ and $\mathrm{sp}(k)$ consist only of real numbers (by Lemma 2.3.21), and the inclusions $\mathrm{sp}(a) \subseteq \mathrm{sp}(h) + i\mathrm{sp}(k)$ and $\mathrm{sp}(h) \subseteq \mathrm{sp}(a) - i\mathrm{sp}(k)$ hold (by Corollary 1.1.81(i)), we get

$$\max \mathrm{sp}(h) = \max \Re(\mathrm{sp}(a)).$$

On the other hand, by Corollary 2.1.9(iii), for $r \in \mathbb{R}$ we have

$$\|\exp(ra)\| = \|\exp(rh)\exp(irk)\| = \|\exp(rh)\|.$$

It follows from Propositions 2.1.7 and 2.3.23 that

$$\begin{aligned}\max \Re(V(a)) &= \sup\left\{\frac{1}{r}\log\|\exp(ra)\| : r > 0\right\}\\ &= \sup\left\{\frac{1}{r}\log\|\exp(rh)\| : r > 0\right\}\\ &= \max V(h) = \max \mathrm{sp}(h) = \max \Re(\mathrm{sp}(a)).\end{aligned}$$

By replacing a with za for $z \in \mathbb{B}_{\mathbb{C}}$, the result follows. □

Now, the Vidav–Palmer argument concludes as follows.

§2.3.72 (First of all, we note that the associative versions of Corollary 2.1.13 and Lemma 2.3.1, which will be applied here, were known by Vidav in [623].) Let A be a complete associative V-algebra. To prove that the natural involution of A is an algebra involution it is enough to realize that

(i) $i[h,k]$ lies in $H(A,\mathbf{1})$ whenever h,k are in $H(A,\mathbf{1})$.
(ii) a^2 lies in $H(A,\mathbf{1})$ whenever a is in $H(A,\mathbf{1})$.

Since we know that assertion (i) holds (cf. Lemma 2.3.1), it only remains to show that assertion (ii) also holds. Let a be in $H(A,\mathbf{1})$, and write $a^2 = h + ik$ with $h,k \in H(A,\mathbf{1})$. Then, according to Palmer's notice in [474], we have $0 = [a,a^2] = [a,h] + i[a,k]$, which, together with assertion (i) and Corollary 2.1.13, yields $[a,k] = 0$, and hence $[h,k] = [a^2 - ik, k] = 0$. Therefore Lemma 2.3.71 applies, so that we have $V(a^2) = \mathrm{co}(\mathrm{sp}(a^2))$. On the other hand, we know that $\mathrm{sp}(a) \subseteq \mathbb{R}$ (by Lemma 2.3.21), which implies that $\mathrm{sp}(a^2) \subseteq \mathbb{R}_0^+$. It follows that $V(a^2) \subseteq \mathbb{R}_0^+$, and hence that a^2 lies in $H(A,\mathbf{1})$, as desired.

Now results from Lemma 2.3.21 to Theorem 2.3.32, Lemma 2.3.71, and §2.3.72 provide the reader with a complete autonomous proof of the associative Vidav–Palmer theorem, avoiding the non-associative germ stated in Theorem 2.3.8.

For later reference, we include here the following straightforward consequence of Lemmas 2.2.5 and 2.3.71.

Corollary 2.3.73 *Let A be a unital C^*-algebra, and let a be a normal element of A. Then $V(a) = \mathrm{co}(\mathrm{sp}(a))$.*

Today, Proposition 2.3.39 is folklore in the theory of C^*-algebras. Our approach follows that of Conway [711] and Murphy [781]. In relation to Proposition 2.3.39, we reproduce with minor changes a paragraph from pp. 250–1 of the Kadison–Ringrose book [758]:

> Suppose that f is a continuous real-valued function defined on a subset S of the real line. We say that f is *operator-monotonic increasing* on S if $f(a) \leqslant f(b)$ whenever a and b are self-adjoint elements of a C^*-algebra, $a \leqslant b$ and $\mathrm{sp}(A,a) \cup \mathrm{sp}(A,b) \subseteq S$. With $g(t) = t^{\frac{1}{2}}$ $(t \geqslant 0)$ and $h(t) = -\frac{1}{t}$ $(t > 0)$, g and h are operator-monotonic increasing on their respective domains of definition, by assertions (v) and (iv) in Proposition 2.3.39. It is clear that an operator-monotonic increasing function on S is monotonic increasing (in the elementary sense) on S. However, as shown by Exercise 2.3.40, the converse is false. For further information on this subject, see [310, 616].

Proposition 2.3.43 is also folklore. The proof given here (through Lemma 2.3.41, Corollary 2.3.42, and the associative Vidav–Palmer theorem) is due to Palmer [476] and is included in [694, Theorem 7.7]. For the classical proof, the reader is referred to [724].

The proof of Theorem 2.2.15 given here (which consists of results from Lemma 2.3.44 to §2.3.53, plus the associative Vidav–Palmer theorem) is built from [476, Proposition 3.9] (see also [694, Section 12]), [514, Section 3], [786, Corollary 1.4.12] (see also [715, Corollary 2.6.18]), [787, Proposition 9.140], [365, Lemma 2.3], and folklore results about the Arens adjoint of a bounded bilinear mapping, already contained in Arens' pioneering paper [29]. As a whole, it could be new. Although straightforward, Lemma 2.3.46 (that biduals of associative normed algebras are associative) is relevant. Indeed, as proved by Arens [29], biduals of associative and commutative normed algebras need not be commutative. A simpler counterexample has been provided by Zalduendo [658] (see also [786, 1.4.9]). We note that the bidual of a commutative normed algebra A is commutative if and only if A is Arens regular (see Lemma 3.5.24(i) below for details).

Proposition 2.3.56 is originally due to Busby [138]. Our proof becomes an adaptation of an argument of Edwards [223]. Proposition 2.3.57 is due to Johnson [352]. With minor changes, our proof is that of Murphy [781, Example 3.1.3]. Theorem 2.3.58 goes back to Banach [685, Theorem IX.4.3] and Stone [600].

Artin's theorem, stated in Theorem 2.3.61, tells us that alternative algebras are very close to associative algebras. Our proof has been taken from [808, pp. 29–30]. A similar (maybe better formalized) proof, involving the notions in §§2.8.17 and 2.8.26 below, can be found in [822, pp. 36–37]. The first known non-associative alternative algebra is the so-called algebra of Cayley numbers. This algebra will be introduced in Section 2.5, and its history can be seen in Subsection 2.5.4 below.

Corollary 2.3.63 is due to Rodríguez [514] (see also Braun [125]). Despite the application of this corollary to prove Theorem 2.3.68 of the current section, another nontrivial application will be done in the proof of Proposition 2.6.8 below, asserting that the complexification of the algebra of Cayley numbers can be converted into an alternative C^*-algebra.

Looking at Proposition 2.3.64 and Example 2.3.65, we realize that, concerning ranges of contractive projections, the class of complete V-algebras is more stable than the subclass consisting of unital C^*-algebras. The most relevant result in this direction is the following one, proved independently by Kaup [382] and Stachó [597].

Theorem 2.3.74 *Let X be a JB^*-triple, and let $\pi : X \to X$ be a contractive linear projection. Then $\pi(X)$ becomes a JB^*-triple under the triple product*

$$\{xyz\}^{\pi} := \pi(\{xyz\}).$$

We remark that, as we reviewed in Subsection 2.2.3, C^*-algebras are JB^*-triples in a natural way, and that, as we will prove much later (see Theorems 3.3.11 and 4.1.45), every complete V-algebra becomes a JB^*-triple under a suitable triple product.

Corollary 2.3.69 is due to Hamana [305]. The refinement, given by Theorem 2.3.68, is new (even in the associative setting), although the core of the argument is taken from Hamana's original proof of Corollary 2.3.69. Since the assumption (2.3.20) in Corollary 2.3.69 is stronger than the one (2.3.18) in Theorem 2.3.68 (by Fact 2.3.35(i)), the application of the associative Vidav–Palmer theorem (via Corollary 2.3.63) done in the proof of Theorem 2.3.68 can be avoided if one is interested only in Corollary 2.3.69. Indeed, following Hamana's original argument again, we have the following.

Proof of Corollary 2.3.69 Let A and π be as in Corollary 2.3.69. Then, for $x \in \pi(A)$ we have

$$x^*x = \pi(x)^*\pi(x) \leqslant \pi(x^*x) = x^* \odot^{\pi} x,$$

so, invoking Fact 2.3.36, we derive

$$\|x\|^2 = \|x^*x\| \leqslant \|x^* \odot^{\pi} x\| = \|\pi(x^*x)\| \leqslant \|x^*x\| = \|x\|^2,$$

and so $\|x^* \odot^{\pi} x\| = \|x\|^2$. Since the product $\odot^{\pi}$ is associative (by the last paragraph in the proof of Theorem 2.3.68), we see that $(\pi(A), \odot^{\pi})$ is a C^*-algebra. $\square$

The argument above works verbatim if 'alternative C^*-algebra' replaces 'C^*-algebra' in Corollary 2.3.69. In this case, the appropriate conclusion (that $(\pi(A), \odot^{\pi})$ is an alternative algebra) can be obtained without any resource to the proof of Theorem 2.3.68. Indeed, it follows directly from Theorem 3.2.5 below.

2.4 V-algebras are non-commutative Jordan algebras

Introduction We introduce non-commutative Jordan algebras, and prove as the main result that V-algebras are non-commutative Jordan algebras (see Theorem 2.4.11). Then we apply this result to prove a theorem (originally due to Choi and Effros [169]) asserting that ranges of unit-preserving 2-contractive linear projections on unital C^*-algebras naturally become C^*-algebras (see Theorem 2.4.24), as well as a refined version of a non-associative characterization of unital C^*-algebras, originally due to Blecher, Ruan, and Sinclair [106] (see Theorem 2.4.27).

2.4.1 The main result

We begin this subsection with the following.

Lemma 2.4.1 *Let A be a norm-unital normed complex algebra, and let a,b,c be in $H(A,\mathbf{1})$. Then $[a,b,c]$ lies in $H(A,\mathbf{1})$.*

Proof By Lemma 2.1.10, L_a, R_c belong to $H(BL(A),I_A)$, and hence, by Lemma 2.3.1, we have $-i[R_c,L_a] \in H(BL(A),I_A)$. Therefore, since $-i[R_c,L_a](\mathbf{1}) = 0$, Corollary 2.2.2 applies, so that

$$[a,b,c] = -i^2[R_c,L_a](b) \in H(A,\mathbf{1}). \qquad \square$$

Proposition 2.4.2 *Let X be a complex normed space, and let T be in $BL(X)$. We have:*

(i) *If T is a dissipative element in $(BL(X),I_X)$, then*

$$\|T(x)\|^2 \leqslant 8\|x\|\|T^2(x)\| \text{ for every } x \in X.$$

(ii) *If T is a hermitian element in $(BL(X),I_X)$, then*

$$\|T(x)\|^2 \leqslant 4\|x\|\|T^2(x)\| \text{ for every } x \in X.$$

Proof Assume that T is dissipative. Let x be in X, and let $\delta \geqslant 0$. We claim that

$$\|T^2(x)\| \geqslant \delta(\|\delta x + T(x)\| - \delta\|x\|).$$

To this end, we set $u := \delta x + T(x)$. If $u = 0$, the required inequality is obvious. Assume that $u \neq 0$, take $v := \frac{1}{\|u\|}u$, and choose $f \in D(X,v)$. Then

$$\|u\| = \|u\|f(v) = f(u) = \delta f(x) + f(T(x)),$$

and so

$$\|u\|f(T(v)) = f(T(u)) = \delta f(T(x)) + f(T^2(x)) = \delta\|u\| - \delta^2 f(x) + f(T^2(x)).$$

Therefore, since $\Re f(T(v)) \leqslant 0$ (because the mapping $F \to f(F(v))$ belongs to $D(BL(X),I_X)$), we have

$$\begin{aligned}\|T^2(x)\| &\geqslant |f(T^2(x))| \geqslant -\Re f(T^2(x)) \geqslant \delta\|u\| - \delta^2\Re f(x)\\ &\geqslant \delta\|u\| - \delta^2\|x\| = \delta(\|\delta x + T(x)\| - \delta\|x\|),\end{aligned}$$

which proves the claim. Now, if $x \neq 0$, assertion (i) follows from the claim by taking $\delta := \frac{\|T(x)\|}{4\|x\|}$.

Now, assume that T is hermitian. Then the claim is applicable to both iT and $-iT$, so that we have

$$\|T^2(x)\| \geqslant \delta(\|iT(x) + \delta x\| - \delta\|x\|),$$

and

$$\|T^2(x)\| \geqslant \delta(\|iT(x) - \delta x\| - \delta\|x\|).$$

Since $\|iT(x) + \delta x\| + \|iT(x) - \delta x\| \geqslant 2\|T(x)\|$, this gives

$$\|T^2(x)\| \geqslant \delta(\|T(x)\| - \delta\|x\|).$$

Therefore, if $x \neq 0$, assertion (ii) follows by taking $\delta := \frac{\|T(x)\|}{2\|x\|}$. $\square$

Corollary 2.4.3 *Let A be a norm-unital normed complex algebra, let a be in A, and let h be in $H(A,\mathbf{1})$. Then $\|[h,a]\|^2 \leqslant 4\|a\|\|[h,[h,a]]\|$.*

Proof Set $T := L_h - R_h$. Then, by Lemma 2.1.10, T lies in $H(BL(A), I_A)$. Therefore, by Proposition 2.4.2(ii), we have

$$\|[h,a]\|^2 = \|T(a)\|^2 \leqslant 4\|a\|\|T^2(a)\| = 4\|a\|\|[h,[h,a]]\|. \qquad \square$$

Lemma 2.4.4 *Let A be a commutative algebra, let D be a derivation of A, and let a be an element of A. If L_aD is a derivation of A, then, for every $b \in A$, we have $[a,D(b),b] = 0$.*

Proof For $b \in A$, we have

$$\begin{aligned}2(aD(b))b = 2(L_aD)(b)b = (L_aD)(b^2) &= L_a(D(b^2)) \\ = L_a(2D(b)b) &= 2a(D(b)b).\end{aligned} \qquad \square$$

Lemma 2.4.5 *Let A be a commutative algebra such that, for all $a,b \in A$, $[L_a,L_b]$ is a derivation of A. Then, for all $a,b \in A$, we have $[a,[a^2,b,a],b] = 0$.*

Proof Let a,c be in A. Since $[L_c,L_a]$ is a derivation of A, we have

$$[L_c,L_a](a^2) = 2a[L_c,L_a](a),$$

that is

$$ca^3 - (ca^2)a = 2(ca^2)a - 2((ca)a)a,$$

and hence

$$L_{a^3} + 2L_a^3 = 3L_aL_{a^2}.$$

Since $[L_a^3, L_a] = 0$, we derive

$$[L_{a^3},L_a] = 3[L_aL_{a^2},L_a] = 3L_a[L_{a^2},L_a].$$

Therefore $[L_{a^2},L_a]$ is a derivation such that $L_a[L_{a^2},L_a]$ is a new derivation, and Lemma 2.4.4 applies. $\square$

Lemma 2.4.6 *Let A be a commutative V-algebra and let a,b be in A. Then $[L_a,L_b]$ is a derivation of A.*

Proof We may assume that a and b are hermitian. Then, by Lemmas 2.1.10 and 2.3.1, $i[L_a,L_b]$ is a hermitian operator. On the other hand, since A is commutative, we have $i[L_a,L_b](\mathbf{1}) = 0$. It follows from Corollary 2.3.19(ii) that $[L_a,L_b]$ is a derivation of A. $\square$

Although straightforward, the following fact is very useful.

Fact 2.4.7 *Let A be an algebra, and let $D: A \to A$ be a linear mapping. Then D is a derivation of A if and only if the equality $[D,L_a] = L_{D(a)}$ holds for every $a \in A$.*

Jordan algebras over $\mathbb{K}$ are defined as those commutative algebras over $\mathbb{K}$ satisfying the so-called 'Jordan identity', namely $(ab)a^2 = a(ba^2)$.

We note that, if A is a V-algebra, then A^{sym} is a commutative V-algebra. In this way, the following result becomes a cornerstone in our present development.

Lemma 2.4.8 *Let A be a commutative V-algebra. Then A is a Jordan algebra.*

Proof Let a,b be in A. By Lemmas 2.4.5 and 2.4.6, we have

$$[a,[a^2,b,a],b]=0,$$

that is $[L_b,L_a][L_a,L_{a^2}](b)=0$. It follows from Lemma 2.4.6 and Fact 2.4.7 that

$$[[L_b,L_a],[[L_a,L_{a^2}],L_b]]=[[L_b,L_a],L_{[L_a,L_{a^2}](b)}]=L_{[L_b,L_a][L_a,L_{a^2}](b)}=0.$$

Now, suppose that $a\in H(A,\mathbf{1})$, take $b:=a^2$, and set $T:=i[L_{a^2},L_a]$. Then we have $[T,[T,L_{a^2}]]=0$. Note that, by Theorem 2.3.8, $a^2\in H(A,\mathbf{1})$, and hence, by Lemmas 2.1.10 and 2.3.1, L_a, L_{a^2}, and T belong to $H(BL(A),I_A)$. Therefore, by Corollary 2.4.3, we get $[T,L_{a^2}]=0$, that is $[L_{a^2},[L_{a^2},L_a]]=0$. Applying Corollary 2.4.3 again, we conclude that $[L_{a^2},L_a]=0$, and hence $[a,b,a^2]=0$ for every $b\in A$. Now, a standard linearization argument, such as that in [808, p. 91], will show that A is a Jordan algebra. Indeed, replacing a in this equality by $a+tc$ ($t\in\mathbb{R}$), the coefficient of t is 0, and we have

$$2[a,b,ca]+[c,b,a^2]=0 \quad \text{for all } a,c \text{ in } H(A,\mathbf{1}) \text{ and } b \text{ in } A.$$

Replacing again a by $a+td$ ($t\in\mathbb{R}$), we have similarly (after dividing by 2) the multilinear identity

$$[a,b,cd]+[d,b,ca]+[c,b,ad]=0 \quad \text{for all } a,c,d \text{ in } H(A,\mathbf{1}) \text{ and } b \text{ in } A.$$

Since $A=H(A,\mathbf{1})+iH(A,\mathbf{1})$, and the product and the associator are $\mathbb{C}$-multilinear, we see that

$$[a,b,cd]+[d,b,ca]+[c,b,ad]=0 \quad \text{for all } a,b,c,d \text{ in } A.$$

Finally, by taking $a=c=d$, we conclude that A satisfies the Jordan identity, as required. □

Definition 2.4.9 By a *Jordan-admissible algebra* we mean an algebra A such that A^{sym} is a Jordan algebra. We define *non-commutative Jordan algebras* as those algebras which are both flexible (cf. Definition 2.3.54) and Jordan-admissible.

As a consequence of Theorem 2.3.61, we have the following.

Corollary 2.4.10 *Alternative algebras are non-commutative Jordan algebras.*

Theorem 2.4.11 *Every V-algebra is a non-commutative Jordan algebra.*

Proof Let A be a V-algebra. Then, by Lemma 2.4.8, A^{sym} is a Jordan algebra, i.e. A is Jordan-admissible. Therefore, to conclude the proof it is enough to show that A is flexible. Let $*$ stand for the natural involution of A, and let a,b be in A. Since $*$ is an algebra involution (by Theorem 2.3.8), we have $[a^*,b^*,a^*]^*=-[a,b,a]$. On the other hand, by Corollary 2.4.1 we also have $[a^*,b^*,a^*]^*=[a,b,a]$. It follows that $[a,b,a]=0$. □

By combining Lemma 2.2.5 and Theorems 2.3.8 and 2.4.11, we derive the following.

Corollary 2.4.12 *Let A be a normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying*

$$\mathbf{1}^* = \mathbf{1} \ \text{ and } \ \|U_a(a^*)\| = \|a\|^3 \ \text{ for every } \ a \in A.$$

Then A is a non-commutative Jordan algebra, and $*$ is an algebra involution on A.

The same conclusion holds (with the same argument) if we replace the requirement that $\|U_a(a^*)\| = \|a\|^3$ for every $a \in A$ with the one that $\|a^*a\| = \|a\|^2$ for every $a \in A$. We have not emphasized this fact here because, as we will see in Theorem 3.2.5 below, a better result holds.

Power-associative algebras are defined as those algebras A such that the subalgebra of A generated by each element of A is associative.

Proposition 2.4.13 *Let A be a Jordan algebra. Then:*

(i) *A is power-associative.*
(ii) *For $a \in A$ and $n,m \in \mathbb{N}$ we have $[L_{a^n}, L_{a^m}] = 0$.*
(iii) *For $a \in A$ and $n \in \mathbb{N}$, L_{a^n} lies in the subalgebra of $L(A)$ generated by $\{L_a, L_{a^2}\}$.*

Proof Since $[a,b,a^2] = 0$ for all $a,b \in A$, we can argue as in the proof of Lemma 2.4.8 to obtain

$$2[a,b,ca] + [c,b,a^2] = 0 \ \text{ for all } \ a,b,c \in A \tag{2.4.1}$$

and

$$[a,b,cd] + [d,b,ca] + [c,b,ad] = 0 \ \text{ for all } \ a,b,c,d \in A. \tag{2.4.2}$$

But the identity (2.4.2) is equivalent to

$$L_{ab}L_c - L_aL_bL_c + L_{ca}L_b - L_{b(ca)} + L_{cb}L_a - L_cL_bL_a = 0 \tag{2.4.3}$$

for all $a,b,c \in A$. Now let a be in A, define powers of a by $a^1 := a$ and $a^{n+1} := a^n a$, let B stand for the subalgebra of $L(A)$ generated by $\{L_a, L_{a^2}\}$, and note that, thanks to the Jordan identity, B is commutative. For $n \geqslant 2$, set $b = a$ and $c = a^{n-1}$ in (2.4.3) to obtain

$$L_{a^{n+1}} = L_{a^2}L_{a^{n-1}} - L_a^2 L_{a^{n-1}} - L_{a^{n-1}}L_a^2 + 2L_{a^n}L_a. \tag{2.4.4}$$

By induction on n we see that L_{a^n} lies in B for every $n \in \mathbb{N}$, and hence that $[L_{a^n}, L_{a^m}] = 0$. Thus assertions (ii) and (iii) have been proved.

To prove assertion (i) it is enough to show that $a^{n+m} = a^n a^m$ for all $n,m \in \mathbb{N}$ (for then the linear hull of the powers of a becomes an associative subalgebra of A containing a). For $m = 1$ this holds by definition of a^n. Now assume that $a^{n+m} = a^n a^m$. Then

$$a^{n+m+1} = a^{n+m}a = (a^n a^m)a = L_a L_{a^n}(a^m) = L_{a^n}L_a(a^m) = a^n(aa^m) = a^n a^{m+1}.$$

Hence $a^{n+m} = a^n a^m$ for all $n,m \in \mathbb{N}$, as desired. □

Lemma 2.4.14 *An algebra A over $\mathbb{K}$ is flexible if and only if, for all $a,b \in A$, the equality $[a \bullet b, b] = b \bullet [a,b]$ holds.*

Proof Decodifying the equality $[a \bullet b, b] = b \bullet [a,b]$, according to the meaning of the commutator $[\cdot,\cdot]$ and that of the symmetrized product $\bullet$, we realize that it is nothing other than a rewriting of the flexibility condition $(ab)a = a(ba)$. □

Lemma 2.4.15 *Let A be an algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *A is flexible.*
(ii) *For each $a \in A$, the mapping $b \to [a,b]$ is a Jordan derivation of A.*

Proof (i)⇒(ii) Assume that A is flexible. Then, linearizing the flexibility condition $(ab)a = a(ba)$ in the variable a, we get

$$(ab)c + (cb)a = a(bc) + c(ba),$$

and, taking $c = b$, we obtain

$$[a,b^2] = (ab)b - b(ba) = (ab+ba)b - b(ba+ab) = 2[a \bullet b, b].$$

It follows from Lemma 2.4.14 that $[a,b^2] = 2b \bullet [a,b]$, and hence that condition (ii) holds.

(ii)⇒(i) Assume that condition (ii) holds. Then, for $a,b \in A$, we have

$$[a \bullet b, b] = a \bullet [b,b] + [a,b] \bullet b = b \bullet [a,b],$$

so that, by Lemma 2.4.14, A is flexible. □

An algebra A is said to be *power-commutative* if the subalgebra of A generated by each element of A is commutative.

Corollary 2.4.16 *Let A be a flexible algebra. Then maximal commutative subsets of A are subalgebras of A. As a consequence, the subalgebra of A generated by any commutative subset of A is commutative. In particular, A is power-commutative.*

Proof Let S be a maximal commutative subset of A. Then, as happens for any maximal commutative subset of any algebra, S is a subspace of A. Let x,y be in S. Then, by Lemma 2.4.15, for every $z \in S$ we have

$$[z,xy] = [z, x \bullet y] = x \bullet [z,y] + [z,x] \bullet y = 0,$$

so that $S \cup \{xy\}$ is a commutative subset of A. It follows from the maximality of S that xy lies in S. Thus S is a subalgebra of A.

Let T be a commutative subset of A. Then, by Zorn's lemma, there exists a maximal commutative subset S of A containing T. Since S is a subalgebra of A (by the above paragraph), it follows that the subalgebra of A generated by T is commutative. □

Lemma 2.4.17 *Let A be a power-commutative algebra over $\mathbb{K}$, and let a be in A. Then the subalgebra of A generated by a coincides, as an algebra, with the subalgebra of A^{sym} generated by a.*

Proof Let $A(a)$ (respectively, $A^{\mathrm{sym}}(a)$) stand for the subalgebra of A (respectively, of A^{sym}) generated by a. Since all subalgebras of A are subalgebras of A^{sym}, we have

$A^{\mathrm{sym}}(a) \subseteq A(a)$ as subsets. But, since A is power-commutative, $A(a)$ is a commutative subset of A, so, by the above inclusion, $A^{\mathrm{sym}}(a)$ is also a commutative subset of A, and so it is in fact a subalgebra of A, and therefore $A(a) \subseteq A^{\mathrm{sym}}(a)$. It follows that $A(a) = A^{\mathrm{sym}}(a)$ as algebras. □

Corollary 2.4.18 *Let A be an algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *A is power-associative.*
(ii) *A is power-commutative, and A^{sym} is power-associative.*

Proof Assume that A is power-associative. Then, since associative algebras generated by a single element are commutative, A is power-commutative, and then, by Lemma 2.4.17, A^{sym} is power-associative.

Now, assume that A is power-commutative and that A^{sym} is power-associative. Then, by Lemma 2.4.17, A is power-associative. □

Proposition 2.4.19 *Non-commutative Jordan algebras are power-associative.*

Proof Let A be a non-commutative Jordan algebra. Since A^{sym} is a Jordan algebra, A^{sym} is a power-associative algebra (by Proposition 2.4.13(i)). On the other hand, since A is flexible, Corollary 2.4.16 applies, so that A is power-commutative. It follows from Corollary 2.4.18 that A is power-associative. □

Corollary 2.4.20 *Let A be a complete V-algebra, let $*$ stand for the natural involution of A, and let h be a hermitian element in A. Then the closed subalgebra of A generated by h and $\mathbf{1}$ is $*$-invariant, and, endowed with the restriction of $*$, becomes a commutative C^*-algebra.*

Proof Let B stand for the closed subalgebra of A generated by $\{h, \mathbf{1}\}$. Then, by Theorem 2.4.11 and Proposition 2.4.19, B is associative and commutative. Moreover, since $*$ is a continuous algebra involution (by Lemma 2.3.7 and Theorem 2.3.8), B is $*$-invariant. Now, keeping in mind Corollary 2.1.2, it is easily realized that B is a complete V-algebra whose natural involution is the restriction of $*$ to B. Therefore, since B is associative, Theorem 2.3.32 applies, so that B is a C^*-algebra. □

2.4.2 Applications to C^*-algebras

Let X be a vector space over $\mathbb{K}$, and let n be a natural number. Then the vector space $M_n(X)$ of all $n \times n$ matrices with entries in X becomes an $M_n(\mathbb{K})$-bimodule by denoting αx and $x\alpha$ (for $\alpha \in M_n(\mathbb{K})$ and $x \in M_n(X)$) the elements of $M_n(X)$ formed by left and right multiplication of x by α in the obvious sense of matrix multiplication. Given $p, q \in \mathbb{N}$ with $p + q = n$, $y \in M_p(X)$, and $z \in M_q(X)$, we denote by $y \oplus z$ the element of $M_n(X)$ whose principal diagonal blocks (from top to bottom) are y and z, and whose off-diagonal blocks have 0 (in X) at each entry. Now let Y be another vector space over $\mathbb{K}$, and let $F : X \to Y$ be a linear mapping. We denote by F_n the linear mapping from $M_n(X)$ to $M_n(Y)$ which consists of applying F to each entry of the matrix in $M_n(X)$. In what follows, $M_n(\mathbb{C})$ will be seen endowed with the involution and the corresponding C^*-norm $|\cdot|_n$ deriving from its natural identification with the algebra of all linear operators on the complex Hilbert space $\mathbb{C}^n$.

Lemma 2.4.21 *Let X be a complex normed space, let n be in $\mathbb{N}$, and let $\|\cdot\|_n$ be a norm on $M_n(X)$ satisfying the following conditions:*

(i) $\|\alpha x\beta\|_n \leqslant |\alpha|_n\|x\|_n|\beta|_n$ *for every $x \in M_n(X)$ and all $\alpha,\beta \in M_n(\mathbb{C})$.*
(ii) $\|\mathrm{diag}\{y,0,\ldots,0\}\|_n = \|y\|$ *for every $y \in X$.*

Then, the natural vector space identification $M_n(X) \equiv X^{n^2}$ becomes bicontinuous when $M_n(X)$ is endowed with the topology of the norm $\|\cdot\|_n$ and X^{n^2} is endowed with the product topology.

Proof For $i,j = 1,\ldots,n$, and $y \in X$, let $y[ij]$ stand for the element of $M_n(X)$ having y in the ij entry and 0 otherwise, and let $u_{ij} := 1[ij]$ be the usual ij matrix unit in $M_n(\mathbb{C})$. Note that $u_{i1}(y[11])u_{1j} = y[ij]$ and that, by condition (ii), $\|y[11]\|_n = \|y\|$. Now, let $x = (x_{ij})$ be in $M_n(X)$. Then we have

$$\|x\|_n \leqslant \sum_{i,j}\|x_{ij}[ij]\|_n = \sum_{i,j}\|u_{i1}(x_{ij}[11])u_{1j}\|_n \leqslant \sum_{i,j}\|x_{ij}[11]\|_n = \sum_{i,j}\|x_{ij}\|,$$

the last inequality being true thanks to condition (i). On the other hand, since $u_{1i}xu_{j1} = x_{ij}[11]$, we have

$$\|x_{ij}\| = \|x_{ij}[11]\|_n = \|u_{1i}xu_{j1}\|_n \leqslant \|x\|_n,$$

and hence

$$\sum_{i,j}\|x_{ij}\| \leqslant n^2\|x\|_n. \qquad \square$$

Let A be an algebra over $\mathbb{K}$, and let n be in $\mathbb{N}$. We already know that $M_n(A) = M_n(\mathbb{K}) \otimes A$ naturally becomes an algebra over $\mathbb{K}$. When A is in fact a unital algebra, then the algebra structure of $M_n(A)$ overlaps its $M_n(\mathbb{K})$-bimodule structure. Indeed, in this case $M_n(\mathbb{K})$ can be seen as a subalgebra of $M_n(A)$, by means of the embedding $\alpha \to \alpha\mathbf{1}_n = \mathbf{1}_n\alpha$ (where $\mathbf{1}_n$ stands for the unit of $M_n(A)$), and then the left and right module operations on $M_n(A)$ coincide with the operators of left and right multiplication by elements of $M_n(\mathbb{K})$ on the algebra $M_n(A)$.

Now, let A be a $*$-algebra over $\mathbb{K}$, and let n be in $\mathbb{N}$. Then $M_n(A)$ has a *canonical* conjugate-linear algebra *involution* (also denoted by $*$), namely the one consisting of transposing the matrix and applying the involution of A to each entry. Thus, $M_n(A)$ will be considered without notice as a new $*$-algebra.

We recall that an associative complex $*$-algebra has at most a complete C^*-norm.

Proposition 2.4.22 *Let A be a C^*-algebra. Then, for each $n \in \mathbb{N}$, $M_n(A)$ becomes a C^*-algebra for a suitable (unique) norm $\|\cdot\|_n$. Moreover, for $n,m \in \mathbb{N}$, we have:*

(i) $\|\alpha a\beta\|_n \leqslant |\alpha|_n\|a\|_n|\beta|_n$ *for every $a \in M_n(A)$ and all $\alpha,\beta \in M_n(\mathbb{C})$.*
(ii) $\|b \oplus c\|_{n+m} = \max\{\|b\|_n,\|c\|_m\}$ *for all $b \in M_n(A)$ and $c \in M_m(A)$.*

Proof By Proposition 1.2.44, we may assume that A is unital.

Let n be in $\mathbb{N}$. For $a = (a_{ij})_{1\leqslant i,j\leqslant n} \in M_n(A)$, set $\|a\| := \sum_{1\leqslant i,j\leqslant n}\|a_{ij}\|$. Then $\|\cdot\|$ becomes a complete algebra norm on $M_n(A)$ making the involution $*$ of $M_n(A)$

isometric. Let P_n stand for the set of all $\|\cdot\|$-continuous positive linear functionals g on $M_n(A)$ satisfying $g(\mathbf{1}_n) = 1$, and, for $a \in M_n(A)$, set

$$\|a\|_n := \sup\{\sqrt{g(a^*a)} : g \in P_n\}.$$

By Lemma 2.3.31, $\|\cdot\|_n$ becomes a $\|\cdot\|$-continuous C^*-seminorm on $M_n(A)$. Now, for $a = (a_{ij})_{1\leqslant i,j\leqslant n} \in M_n(A)$, set $T(a) := \frac{1}{n}\sum_{i=1}^{n} a_{ii}$, so that we have

$$T(a^*a) = \frac{1}{n}\sum_{1\leqslant i,j\leqslant n} a_{ij}^* a_{ij},$$

$T(a^*) = T(a)^*$, and $T(\mathbf{1}_n) = \mathbf{1}$. Let f be in $D(A,\mathbf{1})$. It follows from Lemmas 2.2.5(i) and 2.3.26 that the mapping $f_T := fT$ is an element of P_n. Moreover, for each $a \in M_n(A)$ as above, and all $i,j = 1,\ldots,n$, we have

$$0 \leqslant \frac{1}{n} f(a_{ij}^* a_{ij}) \leqslant \frac{1}{n}\sum_{1\leqslant i,j\leqslant n} f(a_{ij}^* a_{ij}) = f_T(a^*a).$$

Since f is arbitrary in $D(A,\mathbf{1})$, it follows from Proposition 2.3.4 that, for $a \in M_n(A)$ as above, we have

$$\frac{1}{n}\|a_{ij}\|^2 = \frac{1}{n}\|a_{ij}^* a_{ij}\| = \frac{1}{n} v(A,\mathbf{1},a_{ij}^* a_{ij}) \leqslant \|a\|_n^2$$

for all $1 \leqslant i,j \leqslant n$, and hence that

$$\frac{1}{n^2\sqrt{n}}\|a\| \leqslant \|a\|_n.$$

Therefore, $\|\cdot\|_n$ is in fact a complete C^*-norm on $M_n(A)$. Now, since $\|\cdot\|_n$ is an algebra norm, and the natural embedding $(M_n(\mathbb{C}), |\cdot|_n) \hookrightarrow (M_n(A), \|\cdot\|_n)$ is an isometry (because it is a $*$-mapping), assertion (i) follows.

To realize assertion (ii), note that, for $n,m \in \mathbb{N}$, $M_n(A) \times M_m(A)$ is a C^*-algebra under the norm $\max\{\|b\|_n, \|c\|_m\}$, and that the mapping $(b,c) \to b \oplus c$ is an algebra $*$-isomorphism from $M_n(A) \times M_m(A)$ onto the closed subalgebra $M_n(A) \oplus M_m(A)$ of $M_{n+m}(A)$. $\square$

Let A be a C^*-algebra, and let n be in $\mathbb{N}$. After Proposition 2.4.22, $M_n(A)$ will be seen without notice as a new C^*-algebra relative to its canonical involution and the norm $\|\cdot\|_n$. By Lemma 2.4.21, if B is another C^*-algebra, and if F is in $BL(A,B)$, then F_n lies in $BL(M_n(A), M_n(B))$ and moreover, by assertion (ii) in Proposition 2.4.22, we have $\|F\| \leqslant \|F_n\|$.

According to Example 2.3.65, ranges of unit-preserving contractive linear projections on unital C^*-algebras need not be C^*-algebras. Now, we are going to show in Theorem 2.4.24 below that such ranges are C^*-algebras whenever the projections are a little more than contractive.

Lemma 2.4.23 *Let A be an algebra over $\mathbb{K}$. Then A is associative if (and only if) $M_2(A)$ is flexible.*

Proof Let a,b,c be in A. Then we have

$$\left[\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (ab)a & (ab)c \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} a(ba) & a(bc) \\ 0 & 0 \end{pmatrix}.$$

Therefore, if $M_2(A)$ is flexible, then $(ab)c = a(bc)$. □

Let A be a C^*-algebra, let F be in $BL(A)$, and let n be in $\mathbb{N}$. We say that F is *n-contractive* if $F_n : M_n(A) \to M_n(A)$ is contractive.

Theorem 2.4.24 *Let A be a unital C^*-algebra, and let $\pi : A \to A$ be a unit-preserving 2-contractive linear projection. Then $(\pi(A), \odot^\pi)$ is a C^*-algebra.*

Proof Note at first that π is contractive and that $\pi_2(\mathbf{1}_2) = \mathbf{1}_2$. Then, by Lemma 2.2.5 and Proposition 2.3.64, both $(\pi(A), \odot^\pi)$ and $(\pi_2(M_2(A)), \odot^{\pi_2})$ are V-algebras, and hence, by Theorem 2.4.11, they are flexible. Since

$$(\pi_2(M_2(A)), \odot^{\pi_2}) = M_2((\pi(A), \odot^\pi)),$$

and $(\pi_2(M_2(A)), \odot^{\pi_2})$ is flexible, it follows from Lemma 2.4.23 that $(\pi(A), \odot^\pi)$ is associative. Finally, since $(\pi(A), \odot^\pi)$ is a complete associative V-algebra, it follows form Theorem 2.3.32 that $(\pi(A), \odot^\pi)$ is a C^*-algebra. □

Now we are going to get a geometric characterization of unital C^*-algebras among all non-associative normed algebras.

Lemma 2.4.25 *Let (X,u) be a complex numerical-range space such that*

$$X = H(X,u) + iH(X,u),$$

and let $\|\cdot\|_2$ be a norm on $M_2(X)$ satisfying:

(i) $\|\alpha x \beta\|_2 \leqslant |\alpha|_2 \|x\|_2 |\beta|_2$ *for every $x \in M_2(X)$ and all $\alpha, \beta \in M_2(\mathbb{C})$.*
(ii) $\|y \oplus z\|_2 = \max\{\|y\|, \|z\|\}$ *for all $y,z \in X$.*

Then, in the numerical-range space $((M_2(X), \|\cdot\|_2), u \oplus u)$, we have

$$M_2(X) = H(M_2(X), u\oplus u) + iH(M_2(X), u \oplus u).$$

Proof Let h be in $H(X,u)$. It is enough to show that

$$\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & ih \\ -ih & 0 \end{pmatrix}$$

lie in $H(M_2(X), u \oplus u)$.

By condition (ii) and Proposition 2.1.5, we have

$$\begin{aligned} &\max \Re(V(M_2(X), u\oplus u, i(h \oplus 0))) \\ &= \lim_{r \to 0^+} \frac{\|u \oplus u + ir(h \oplus 0)\|_2 - 1}{r} = \lim_{r\to 0^+} \frac{\|(u+irh)\oplus u\|_2 - 1}{r} \\ &= \lim_{r\to 0^+} \frac{\max\{\|u+irh\|, 1\} - 1}{r} = \max\left\{\lim_{r \to 0^+} \frac{\|u+irh\|-1}{r}, 0\right\} = 0, \end{aligned}$$

so that, replacing h with $-h$, we get that $\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} = h \oplus 0 \in H(M_2(X), u \oplus u)$. Analogously, $\begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix} \in H(M_2(X), u \oplus u)$.

Let α be in $M_2(\mathbb{C})$, and let L_α and R_α stand for the operators of left and right multiplication by α on the $M_2(\mathbb{C})$-bimodule $M_2(X)$. By condition (i), the mappings $\alpha \to L_\alpha$ and $\alpha \to R_\alpha$ from $M_2(\mathbb{C})$ to $BL(M_2(A))$ are linear contractions taking the unit $\mathbf{1}$ of $M_2(\mathbb{C})$ to $I_{M_2(A)}$, and hence, by Corollary 2.1.2(i), we have

$$V(BL(M_2(X)), I_{M_2(X)}, L_\alpha) \subseteq V(M_2(\mathbb{C}), \mathbf{1}, \alpha)$$

and

$$V(BL(M_2(X)), I_{M_2(X)}, R_\alpha) \subseteq V(M_2(\mathbb{C}), \mathbf{1}, \alpha).$$

Therefore $L_\alpha - R_\alpha$ lies in $H(BL(M_2(X)), I_{M_2(X)})$ whenever α is in $H(M_2(\mathbb{C}), \mathbf{1})$. Note also that $(L_\alpha - R_\alpha)(u \oplus u) = 0$. It follows from Corollary 2.2.2 that $i(\alpha x - x\alpha)$ lies in $H(M_2(X), u \oplus u)$ whenever α is in $H(M_2(\mathbb{C}), \mathbf{1})$ and x is in $H(M_2(X), u \oplus u)$. By taking

$$\alpha := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in H(M_2(\mathbb{C}), \mathbf{1}) \text{ and } x := \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \in H(M_2(X), u \oplus u)$$

$\left(\text{successively, } \alpha := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } x := \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix}\right)$, the above gives that both $\begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & ih \\ -ih & 0 \end{pmatrix}$ lie in $H(M_2(X), u \oplus u)$, as desired. $\square$

§2.4.26 In §1.2.2, we introduced C^*-algebras as those complete normed associative complex $*$-algebras A satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$. Later, in §2.3.62, we introduced alternative C^*-algebras by simply relaxing associativity to alternativity in the above notion. Moreover, as we will see in Proposition 2.6.8, alternative C^*-algebras need not be associative. Now, we define *non-associative C^*-algebras* by altogether removing in the definition of C^*-algebras any requirement of associativity or alternativity. We note that, by Lemma 2.2.5, a normed unital complex algebra has at most one involution converting it into a non-associative C^*-algebra.

Theorem 2.4.27 *Let $A = (A, \|\cdot\|)$ be a complete normed complex algebra. Then the following conditions are equivalent:*

(i) *A is a unital non-associative C^*-algebra, and the following 'matricial L_∞-property' holds:*
For each $n \in \mathbb{N}$, $M_n(A)$ is equipped with an algebra norm $\|\cdot\|_n$ such that the family of norms satisfies
 (a) *$\|\cdot\|_1 = \|\cdot\|$ on $M_1(A) = A$,*
 (b) *$\|\alpha a \beta\|_n \leqslant |\alpha|_n \|a\|_n |\beta|_n$ for every $a \in M_n(A)$ and all $\alpha, \beta \in M_n(\mathbb{C})$,*
 (c) *$\|b \oplus c\|_{n+m} = \max\{\|b\|_n, \|c\|_m\}$ for all $b \in M_n(A)$ and $c \in M_m(A)$.*

(ii) *A is a V-algebra satisfying the matricial L_∞-property above.*

(iii) *A is a unital non-associative C^*-algebra, and the following 'matricial L^2_∞-property' holds:*

There exists an algebra norm $\|\cdot\|_2$ on $M_2(A)$ satisfying

(a) $\|\alpha a\beta\|_2 \leqslant |\alpha|_2\|a\|_2|\beta|_2$ *for every* $a \in M_2(A)$ *and all* $\alpha,\beta \in M_2(\mathbb{C})$,
(b) $\|b \oplus c\|_2 = \max\{\|b\|,\|c\|\}$ *for all* $b,c \in A$.

(iv) *A is a V-algebra satisfying the matricial L^2_∞-property above.*
(v) *A is a unital C^*-algebra.*

Moreover, if the above conditions are fulfilled, and if, for $n \in \mathbb{N}$, $M_n(A)$ is endowed with the canonical involution corresponding to the involution of the C^*-algebra A, then the norm $\|\cdot\|_n$ provided by the matricial L_∞-property coincides with the unique complete C^*-norm on $M_n(A)$ given by Proposition 2.4.22.

Proof The implications (i)⇒(iii) and (ii)⇒(iv) are clear, whereas the ones (i)⇒(ii) and (iii)⇒(iv) follow from Lemma 2.2.5. On the other hand, the implication (v)⇒(i) follows from Proposition 2.4.22.

(iv)⇒(v) Assume that condition (iv) is fulfilled. Then, by Lemma 2.4.25, $(M_2(A),\|\cdot\|_2)$ is a V-algebra. Therefore, by Theorem 2.4.11 and Lemma 2.4.23, A is associative, and condition (v) holds by invoking Theorem 2.3.32.

Now that we know that conditions (i)–(v) are equivalent, assume that they hold. By condition (v), A is a C^*-algebra. Let n be in $\mathbb{N}$, endow $M_n(A)$ with the canonical involution corresponding to the involution of A, and note that, by condition (i)(c) and Lemma 2.4.21, the norm $\|\cdot\|_n$ in condition (i) is complete. We claim that $\|\cdot\|_{2^{n-1}}$ is a C^*-norm on $M_{2^{n-1}}(A)$. Since, by conditions (i)(a) and (v), the claim is true when $n = 1$, assume that it is true when n equals a given natural number m. Then, keeping in mind condition (i)(c), and invoking the clear equality $M_{2^m}(A) = M_2(M_{2^{m-1}}(A))$, we can apply Lemmas 2.2.5 and 2.4.25, and Theorem 2.3.32, to realize that the claim remains true when n equals $m+1$. Now that the claim has been proved, note that, by condition (i)(c), the mapping $y \to y \oplus 0$ from $(M_n(A),\|\cdot\|_n)$ to $(M_{2^{n-1}}(A),\|\cdot\|_{2^{n-1}})$ becomes an isometric algebra $*$-homomorphism. It follows from the claim that $\|\cdot\|_n$ is a C^*-norm. □

2.4.3 Historical notes and comments

Proposition 2.4.2, as well as its proof, are taken almost verbatim from [694, Lemma 10.9 and Theorem 10.13].

According to [782, pp. 2–3],

> Jordan algebras, named by A. A. Albert in 1946, were first introduced by physicist Pascual Jordan to attempt to introduce an infinite-dimensional algebraic setting for quantum mechanics essentially different from the standard setting of hermitian matrices. A half century after the inception of Jordan theory by Jordan, von Neumann and Wigner [356], there have been remarkable successes in mathematical studies of Jordan algebras.

Today, the theory of Jordan algebras, including a part of its analytic aspect, is nicely contained in several books, for example those of Schafer [808], Braun and Koecher [700], Jacobson [754], Zhevlakov, Slinko, Shestakov, and Shirshov [822], Hanche-Olsen and Størmer [738], Myung [782], Elduque and Myung [728], and McCrimmon [777].

Flexible algebras and Jordan-admissible algebras first appeared in Albert's 1948 paper [12]. Algebras which are both flexible and Jordan-admissible were later called non-commutative Jordan algebras by Schafer [552].

Theorem 2.4.11 is due to Kaidi, Martínez, and Rodríguez [362], who proved it under the assumption (which would be unnecessary today in view of Theorem 2.3.8) that the natural involution of the V-algebra under consideration is an algebra involution. The proof of Theorem 2.4.11 given here, which actually started with Theorem 2.2.9(ii) (cf. the comments immediately before Corollary 2.3.19), is essentially the original one in [362]. As a novelty, we apply Corollary 2.4.3 instead of the Kleinecke–Shirokov theorem (see Proposition 3.6.49 below) and Theorem 2.3.70, as the authors of [362] did.

According to Albert [10], Proposition 2.4.13 is due to Jordan, von Neumann, and Wigner [356]. The nice characterization of flexible algebras, given by Lemma 2.4.15, was first proved in Kaidi's PhD [759], where it is said that the lemma is implicitly contained in [12, 700, 808]. An explicit formulation and proof appeared later in Lemma 1.5(i) of Myung's book [782], where Kaidi's forerunner goes unnoticed. Our proof is essentially that of [782], although the passing through the straightforward but useful Lemma 2.4.14 is emphasized for the first time here.

Power-commutative algebras were introduced by Raffin [497], who proved the most part of Corollary 2.4.16. The actual formulation and proof of Corollary 2.4.16 are taken from [759]. Lemma 2.4.17 and Corollary 2.4.18 are surely folklore, although, with the exception of the implication (i)⇒(ii) of Corollary 2.4.18, we have not found them anywhere. Proposition 2.4.19 is due to Schafer [552].

Proposition 2.4.22 is folklore in the theory of C^*-algebras. The proof given here, avoiding the non-commutative Gelfand–Naimark theorem, could have some methodological interest. The matricial treatment of C^*-algebras originated early in Stinespring's paper [599], where, as the main result, the following theorem is proved.

Theorem 2.4.28 *Let A be a unital C^*-algebra, let H be a complex Hilbert space, and let F be a linear mapping from A to $BL(H)$. Then the following conditions are equivalent:*

(i) *There are a complex Hilbert space K, an operator $V \in BL(H,K)$, and an algebra $*$-homomorphism $\rho : A \to BL(K)$ such that*

$$F(a) = V^*\rho(a)V \text{ for every } a \in A.$$

(ii) *For every $n \in \mathbb{N}$, the mapping $F_n : M_n(A) \to M_n(BL(H))$ takes positive elements of $M_n(A)$ into positive elements of $M_n(BL(H))$.*

Theorem 2.4.24 is originally due to Choi and Effros [169]. The proof given here, through Theorems 2.3.32 and 2.4.11, Proposition 2.3.64, and Lemma 2.4.23, is taken from [515]. Theorem 2.4.24 can be also derived from Hamana's Corollary 2.3.69, by keeping in mind the following 'Schwarz inequality', whose proof can be seen, for example, in [788, Proposition 3.3].

Proposition 2.4.29 *Let A, B be unital C^*-algebras, and let $\phi : A \to B$ be a unit-preserving 2-contractive linear mapping. Then*

$$\phi(a)^*\phi(a) \leqslant \phi(a^*a) \text{ for every } a \in A.$$

The Choi–Effros Theorem 2.4.24 is included in [690, Theorem 1.3.13] and [788, Theorem 15.2]. More results concerning unit-preserving contractive linear projections will be discussed later in Corollaries 3.3.18 and 3.3.19.

The equivalence (ii)$\Leftrightarrow$(v) and the last conclusion in Theorem 2.4.27 are originally due to Blecher, Ruan, and Sinclair [106], who could in fact have also proved the equivalence (i)$\Leftrightarrow$(v) with minor additional effort. The refinement given by the equivalences (iii)$\Leftrightarrow$(v) and (iv)$\Leftrightarrow$(v), as well as the whole proof given here (including Lemma 2.4.25), are new. The Blecher–Ruan–Sinclair proof consists of a straightforward application of the associative Vidav–Palmer theorem (Theorem 2.3.32) and of the main result in their paper [106], stated without proof in Theorem 2.4.30 immediately below. Actually, for the sake of convenience, both the formulation and the proof in [106] of the Blecher–Ruan–Sinclair part of Theorem 2.4.27 involve a dual version of the associative Vidav–Palmer theorem, due to Moore [446], which will be discussed much later (see Corollary 3.3.27 below). Now let us formulate the main result in [106].

Theorem 2.4.30 *Let A be a norm-unital normed complex algebra. Then the following conditions are equivalent:*

(i) *A satisfies the matricial L_∞-property introduced in Theorem 2.4.27.*
(ii) *A is isometrically isomorphic to a unital subalgebra of some C^*-algebra.*

Moreover, if the above conditions are fulfilled, then actually A can be seen as a unital subalgebra of a suitable C^*-algebra B in such a way that, for $n \in \mathbb{N}$, the abstract norm $\|\cdot\|_n$ on $M_n(A)$ provided by the matricial L_∞-property coincides with the restriction to $M_n(A)$ of the unique complete C^*-norm on $M_n(B)$.

We recall that unitary normed algebras were introduced in §2.1.54. As pointed out by Hansen and Kadison [309], it is enough to combine Theorem 2.4.30 and Proposition 2.1.56 to get the following.

Corollary 2.4.31 *Let A be a unitary complete normed complex algebra satisfying the matricial L_∞-property. Then A is a C^*-algebra, and, for $n \in \mathbb{N}$, the abstract norm $\|\cdot\|_n$ on $M_n(A)$ provided by the matricial L_∞-property coincides with the unique complete C^*-norm on $M_n(A)$.*

Through Theorem 2.4.30 above, we have entered the field of so-called *operator algebras*, namely (possibly non-self-adjoint) subalgebras of C^*-algebras. Theorem 2.4.30 remains interesting if the norm-unital normed complex algebra A is assumed to be associative and complete, and even if in addition the last conclusion in the theorem is erased. This is so because of the following two reasons:

(i) Such a restricted version of Theorem 2.4.30 provides us with an abstract characterization of norm-unital complete operator algebras among all norm-unital complete normed associative complex algebras.
(ii) As a matter of fact, ‘most’ norm-unital complete normed associative complex algebras are not (isometrically isomorphic to) operator algebras.

To realize the second reason, we recall that a norm-unital normed power-associative complex algebra A is said *to satisfy the von Neumann inequality* if, for every $a \in \mathbb{B}_A$ and every complex polynomial P, we have

$$\|P(a)\| \leqslant \max\{|P(z)| : z \in \mathbb{B}_{\mathbb{C}}\}.$$

According to the celebrated von Neumann theorem (see for example [796, Section 153]), we have the following.

Theorem 2.4.32 *Unital C^*-algebras satisfy the von Neumann inequality.*

Then, keeping in mind Corollary 1.2.50, we realize that *every norm-unital operator algebra satisfies the von Neumann inequality*. On the other hand, as proved by Foias [264], *if X is a complex Banach space, and if $BL(X)$ satisfies the von Neumann inequality, then X is a Hilbert space*. It follows that, *given a complex non-Hilbert Banach space X, $BL(X)$ is never an operator algebra.*

Remark 2.4.33 For a norm-unital complete normed associative complex algebra A, consider the following conditions:

(i) A is unitary.
(ii) A satisfies the von Neumann inequality.

As shown implicitly by Arazy [25] (see [65, Theorem 5.2 and Remark 5.3(a)] for an explicit formulation and proof), (i)+(ii) is equivalent to the fact that A is a C^*-algebra. On the other hand, clearly, neither (i) nor (ii) implies that A is a C^*-algebra. In the case $A = BL(X)$ for some complex Banach space X, condition (ii) alone is equivalent to the fact that A is a C^*-algebra (by Foias' theorem [264] reviewed above), whereas the same conclusion, with (i) instead of (ii), remains an open problem. Some partial answers to this problem can be found in [80, 63].

Let X be a complex Banach space. A *holomorphic vector field* on $\mathrm{int}(\mathbb{B}_X)$ is nothing other than a holomorphic mapping from $\mathrm{int}(\mathbb{B}_X)$ to X. A holomorphic vector field Λ on $\mathrm{int}(\mathbb{B}_X)$ is said to be *complete* if, for each $x \in \mathrm{int}(\mathbb{B}_X)$, there exists a differentiable function $\varphi : \mathbb{R} \to \mathrm{int}(\mathbb{B}_X)$ satisfying

$$\varphi(0) = x \text{ and } \frac{d}{dt}\varphi(t) = \Lambda(\varphi(t)) \text{ for every } t \in \mathbb{R}.$$

The following theorem is due to Arazy [25].

Theorem 2.4.34 *Let A be a norm-unital complete normed power-associative complex algebra. Then the following conditions are equivalent:*

(i) *A satisfies the von Neumann inequality.*
(ii) *The holomorphic vector field $a \to \mathbf{1} - a^2$ on $\mathrm{int}(\mathbb{B}_A)$ is complete.*

Since condition (ii) in the above theorem has a meaning without requiring the associativity of powers for A, it seems reasonable to us to raise the following.

Problem 2.4.35 Let A be a norm-unital complete normed complex algebra such that the holomorphic vector field $a \to \mathbf{1} - a^2$ on $\mathrm{int}(\mathbb{B}_A)$ is complete. Is A a power-associative algebra?

If the answer to Problem 2.4.35 were affirmative, then Theorem 2.4.34 would assert that, given a norm-unital complete normed complex algebra A, the holomorphic vector field $a \to \mathbf{1} - a^2$ on $\mathrm{int}(\mathbb{B}_A)$ is complete if and only if A is power-associative and satisfies the von Neumann inequality. Anyway, the study of those norm-unital complete normed complex algebras A such that the holomorphic vector field $a \to \mathbf{1} - a^2$ on $\mathrm{int}(\mathbb{B}_A)$ is complete would merit special attention because, by Theorem 2.4.34, such a study would become a general non-associative view of the von Neumann inequality.

Looking back to Theorem 2.4.30, let us comment about its roots. The most direct forerunner is the following abstract characterization of the matricial structure of the so-called *operator spaces* (i.e. subspaces of C^*-algebras), due to Ruan [546].

Theorem 2.4.36 *Let X be a complex normed space such that, for each $n \in \mathbb{N}$, $M_n(X)$ is equipped with a norm $\|\cdot\|_n$ in such a way that the family of norms satisfies*

(i) $\|\cdot\|_1 = \|\cdot\|$ *on* $M_1(X) = X$,
(ii) $\|\alpha x \beta\|_n \leqslant |\alpha|_n \|x\|_n |\beta|_n$ *for every* $x \in M_n(X)$ *and all* $\alpha, \beta \in M_n(\mathbb{C})$,
(iii) $\|y \oplus z\|_{n+m} = \max\{\|y\|_n, \|z\|_m\}$, *for all* $y \in M_n(X)$ *and* $z \in M_m(X)$.

Then X can be seen as a subspace of a suitable C^*-algebra A in such a way that, for $n \in \mathbb{N}$, the abstract norm $\|\cdot\|_n$ on $M_n(X)$ coincides with the restriction to $M_n(X)$ of the unique complete C^*-norm on $M_n(A)$.

In relation to the above theorem, it must be remarked that every complex normed space is linearly isometric to an operator space. Indeed, given a complex normed space X, and taking E equal to the compact Hausdorff topological space $(\mathbb{B}_{X'}, w^*)$, X is linearly isometric to a subspace of the commutative C^*-algebra $C^{\mathbb{C}}(E)$. Therefore, to avoid triviality, the theory of operator spaces involves essentially the C^*-algebras where they are imbedded, and, most precisely, the normed matricial structure that they inherit from those C^*-algebras. To formalize the above idea, consider two C^*-algebras A and B, subspaces $X \subseteq A$ and $Y \subseteq B$, and a bounded linear mapping $F : X \to Y$. Then, running n over $\mathbb{N}$, regarding $M_n(X)$ and $M_n(Y)$ as normed spaces under the restrictions of the unique complete C^*-norms of $M_n(A)$ and $M_n(B)$, respectively, and considering the bounded linear mapping $F_n : M_n(X) \to M_n(Y)$, we are provided with an increasing sequence $(\|F_n\|)_{n \in \mathbb{N}}$ of non-negative real numbers. When this sequence is bounded, F is called *completely bounded*. Now, the theory of operator spaces is nothing other than the study of the category whose objects are the subspaces of C^*-algebras and whose morphisms are the completely bounded mappings. In particular, Theorem 2.4.36 becomes an abstract characterization of operator spaces up to a 'complete isometry'.

For an abstract characterization of the matricial structure of self-adjoint subspaces of C^*-algebras (*operator systems*), in the spirit of Theorem 2.4.36, the reader is referred to the paper of Choi and Effros [169]. Finally, we refer the reader to the books of Blecher and Le Merdy [690], Paulsen [788], and Pisier [791] for a broad view of the theory of operator spaces and operator algebras.

2.5 The Frobenius–Zorn theorem, and the generalized Gelfand–Mazur–Kaplansky theorem

Introduction In this section, we introduce the algebra $\mathbb{H}$ of Hamilton quaternions, and the algebra $\mathbb{O}$ of Cayley numbers. As main results, we prove the Frobenius–Zorn theorem (that $\mathbb{R},\mathbb{C},\mathbb{H}$ and $\mathbb{O}$ are the unique nonzero alternative algebraic real algebras with no nonzero joint divisor of zero) and the generalized Gelfand–Mazur–Kaplansky theorem (that $\mathbb{R},\mathbb{C},\mathbb{H}$ and $\mathbb{O}$ are the unique nonzero normed alternative real algebras with no nonzero joint topological divisor of zero).

2.5.1 Introducing quaternions and octonions

By a *Cayley algebra* over $\mathbb{K}$ we mean a unital algebra A over $\mathbb{K}$ endowed with a linear algebra involution $*$ (called the *standard involution* of A) satisfying

$$a+a^* \in \mathbb{K}\mathbf{1} \text{ and } aa^* \in \mathbb{K}\mathbf{1} \text{ for every } a \in A. \tag{2.5.1}$$

Let A be a Cayley algebra over $\mathbb{K}$. Then we can consider the algebra over $\mathbb{K}$ consisting of the vector space $A\times A$ and the product given by

$$(a_1,a_2)(a_3,a_4) := (a_1a_3 - a_4a_2^*, a_1^*a_4 + a_3a_2).$$

This algebra will be called the *Cayley–Dickson doubling* of A, and will be denoted by $\mathscr{C}\mathscr{D}(A)$. As a matter of fact, $\mathscr{C}\mathscr{D}(A)$ is unital (with $\mathbf{1} = (\mathbf{1},0)$), and actually becomes a new Cayley algebra whose standard involution is given by $(a_1,a_2)^* := (a_1^*,-a_2)$. The passing from A to $\mathscr{C}\mathscr{D}(A)$ is known as the *Cayley–Dickson doubling process*.

Now, let us pay attention to the case $\mathbb{K}=\mathbb{R}$, and consider $\mathbb{R}$ as a real Cayley algebra (with standard involution equal to the identity mapping). Then, clearly, the Cayley algebra $\mathscr{C}\mathscr{D}(\mathbb{R})$ becomes a copy of (the real algebra underlying) $\mathbb{C}$, with standard involution equal to the usual conjugation. By iterating the Cayley–Dickson doubling process twice more, we get the algebra $\mathbb{H}$ of Hamilton's *quaternions*, and the algebra $\mathbb{O}$ of *octonions* (also called *Cayley numbers*). Indeed, they are the Cayley algebras $\mathscr{C}\mathscr{D}(\mathbb{C})$ and $\mathscr{C}\mathscr{D}(\mathbb{H})$, respectively. It is of straightforward verification that the algebra $\mathbb{H}$ is associative but not commutative, whereas the algebra $\mathbb{O}$ is alternative but not associative.

By an *absolute value* on an algebra A over $\mathbb{K}$ we mean a norm $\|\cdot\|$ on the vector space of A satisfying $\|xy\| = \|x\|\|y\|$ for all $x,y \in A$. By an *absolute-valued algebra* we mean a nonzero algebra over $\mathbb{K}$ endowed with an absolute value. It is not difficult to realize that, if we see $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ as $\mathscr{C}\mathscr{D}(\mathbb{A})$, for $\mathbb{A}=\mathbb{R},\mathbb{C}$, and $\mathbb{H}$, respectively, and if, for $a_1,a_2 \in \mathbb{A}$ we set

$$\|(a_1,a_2)\| := \sqrt{\|a_1\|^2 + \|a_2\|^2}, \tag{2.5.2}$$

then $\|\cdot\|$ becomes an absolute value on $\mathscr{C}\mathscr{D}(\mathbb{A})$. Thus, $\mathbb{R},\mathbb{C},\mathbb{H}$ and $\mathbb{O}$ become the basic examples of absolute-valued algebras. It follows from (2.5.2) that *the absolute values of* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$ *come from inner products*.

§**2.5.1** The introduction of $\mathbb{H}$ and $\mathbb{O}$ above is surely the quickest possible one. However, concerning $\mathbb{H}$, there is another more natural approach. Indeed, in the same way as $\mathbb{C}$ can be rediscovered as the subalgebra of the algebra $M_2(\mathbb{R})$ (of all 2×2 matrices over $\mathbb{R}$) given by

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\},$$

$\mathbb{H}$ can be rediscovered as the real subalgebra of $M_2(\mathbb{C})$ given by

$$\left\{ \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} : z, w \in \mathbb{C} \right\}$$

(see for example [727, p. 195]). Regarding $\mathbb{H}$ in this new way, the standard involution of $\mathbb{H}$ corresponds with the operation consisting of the transposition of matrices and taking standard involution in their entries, and the absolute value of an element of $\mathbb{H}$ is nothing other than the non-negative square root of its (automatically non-negative) determinant.

In its turn, $\mathbb{O}$ can be described in terms of 2×2 matrices of scalars and vectors, via the so-called *Zorn's vector matrices*. To be precise, $\mathbb{O}$ can be re-encountered as the real algebra whose vector space is

$$\left\{ \begin{pmatrix} z & \mathbf{w} \\ -\mathbf{w}^* & z^* \end{pmatrix} : z \in \mathbb{C}, \mathbf{w} \in \mathbb{C}^3 \right\},$$

where, for a vector $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, $\mathbf{w}^* := (w_1^*, w_2^*, w_3^*)$, and whose product is defined 'in a natural way' by

$$\begin{pmatrix} z_1 & \mathbf{w}_1 \\ -\mathbf{w}_1^* & z_1^* \end{pmatrix} \begin{pmatrix} z_2 & \mathbf{w}_2 \\ -\mathbf{w}_2^* & z_2^* \end{pmatrix}$$
$$= \begin{pmatrix} z_1 z_2 - (\mathbf{w}_1, \mathbf{w}_2^*) & z_1 \mathbf{w}_2 + z_2^* \mathbf{w}_1 + \mathbf{w}_1^* \times \mathbf{w}_2^* \\ -z_1^* \mathbf{w}_2^* - z_2 \mathbf{w}_1^* - \mathbf{w}_1 \times \mathbf{w}_2 & z_1^* z_2^* - (\mathbf{w}_1^*, \mathbf{w}_2) \end{pmatrix},$$

where, for vectors $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{C}^3$,

$$(\mathbf{u}, \mathbf{v}) := u_1 v_1 + u_2 v_2 + u_3 v_3 \quad \text{(the 'scalar product')}$$

and

$$\mathbf{u} \times \mathbf{v} := (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \quad \text{(the 'vector product')}$$

(see for example [822, p. 46]). Regarding $\mathbb{O}$ in this new way, the standard involution of $\mathbb{O}$ corresponds with the operation consisting of transposing the matrix and taking standard involution in its entries. Moreover, the absolute value of an element of $\mathbb{O}$ is nothing other than the non-negative square root of its (automatically non-negative) 'determinant' $zz^* + (\mathbf{w}^*, \mathbf{w})$.

2.5.2 The Frobenius–Zorn theorem

Now, let us go for a long walk which will lead us to the so-called Frobenius–Zorn theorem, providing us with a first relevant characterization of the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ (see Theorem 2.5.29 below).

§**2.5.2** Let A be a power-associative algebra over $\mathbb{K}$. Then the obvious identities

$$[a^2,a]=0 \quad \text{and} \quad [a^2,a,a]=0$$

can be linearized to get

$$2[a\bullet b,a]+[a^2,b]=0 \tag{2.5.3}$$

and

$$2[a\bullet b,a,a]+[a^2,b,a]+[a^2,a,b]=0. \tag{2.5.4}$$

Again linearizing (2.5.3) and (2.5.4), we find the identities

$$[a\bullet b,c]+[a\bullet c,b]+[b\bullet c,a]=0 \tag{2.5.5}$$

and

$$\begin{aligned}2[a\bullet b,c,a]&+2[a\bullet b,a,c]+2[a\bullet c,b,a]+2[a\bullet c,a,b]\\&+2[b\bullet c,a,a]+[a^2,b,c]+[a^2,c,b]=0.\end{aligned} \tag{2.5.6}$$

Lemma 2.5.3 *Let A be a power-associative algebra over $\mathbb{K}$, and let e be an idempotent in A. Then we have:*

(i) $A=A_1(e)\oplus A_{\frac{1}{2}}(e)\oplus A_0(e)$, *where* $A_k(e):=\{x\in A: e\bullet x=kx\}$ *for* $k=1,\frac{1}{2},0$.
(ii) $A_k(e)=\{x\in A: ex=xe=kx\}$ *for* $k=1,0$.
(iii) $A_k(e)\bullet A_k(e)\subseteq A_k(e)$ *for* $k=1,0$, *and* $A_0(e)\bullet A_1(e)=0$.
(iv) $A_{\frac{1}{2}}(e)\bullet A_{\frac{1}{2}}(e)\subseteq A_1(e)+A_0(e)$.
(v) *The projections* $P_k(e)$ *from* A *onto* $A_k(e)$ $(k=1,\frac{1}{2},0)$ *corresponding to the decomposition* $A=A_1(e)\oplus A_{\frac{1}{2}}(e)\oplus A_0(e)$ *are given by*

$$P_1(e)=L_e^\bullet(2L_e^\bullet-I_A),\quad P_{\frac{1}{2}}(e)=4L_e^\bullet(I_A-L_e^\bullet),$$

and

$$P_0(e)=(L_e^\bullet-I_A)(2L_e^\bullet-I_A),$$

where $L_e^\bullet$ denotes the multiplication operator by e in the algebra A^{sym}.

Proof Since A^{sym} is power-associative (cf. Corollary 2.4.18), to prove assertions (i), (iii), (iv), and (v) we may assume in addition that A is commutative. Then, setting $a=e$ in (2.5.4), we obtain

$$2L_e^3-3L_e^2+L_e=0, \tag{2.5.7}$$

that is $p(L_e)=0$, where $p(\mathbf{x})=\mathbf{x}(\mathbf{x}-1)(\mathbf{x}-\frac{1}{2})$. Therefore assertions (i) and (v) follow from Proposition 1.3.3. In order to prove assertions (iii) and (iv), set $a=e$, $b\in A_k(e)$, and $c\in A_j(e)$ in (2.5.6). This yields

$$2e[e(bc)]+(2k+2j-4)e(bc)+(2k^2+2j^2-8kj+k+j)bc=0.$$

When $k=j=1$ we obtain $0=(L_e^2-I_A)(bc)=(L_e+I_A)(L_e-I_A)(bc)$, and keeping in mind that L_e+I_A belongs to $\mathrm{Inv}(L(A))$, we get $(L_e-I_A)(bc)=0$, that is, $bc\in A_1(e)$, and as a result $A_1(e)$ is a subalgebra of A. Similarly $k=j=0$ yields $0=(L_e^2-2L_e)(bc)=(L_e-2I_A)L_e(bc)$, and keeping in mind that L_e-2I_A belongs to

$\mathrm{Inv}(L(A))$, we get $L_e(bc) = 0$. That is, $bc \in A_0(e)$, and as a result $A_0(e)$ is a subalgebra of A. The values $k = 1$, $j = 0$ yield

$$0 = (2L_e^2 - 2L_e + 3I_A)(bc), \tag{2.5.8}$$

and so $0 = (2L_e^3 - 2L_e^2 + 3L_e)(bc)$. By (2.5.7),

$$0 = (L_e^2 + 2L_e)(bc) = (L_e + 2I_A)L_e(bc),$$

and keeping in mind that $L_e + 2I_A$ belongs to $\mathrm{Inv}(L(A))$, we deduce that $0 = L_e(bc)$, and hence, by (2.5.8), $bc = 0$, and the proof of assertion (iii) is complete. For $k = j = \frac{1}{2}$, we obtain $0 = 2(L_e^2 - L_e)(bc) = -\frac{1}{2}P_{\frac{1}{2}}(e)(bc)$. That is, bc lies in $A_1(e) + A_0(e)$, and the proof of assertion (iv) is complete.

Finally, in order to prove assertion (ii), remove the assumption that A is commutative. Applying the identity (2.5.5) with $a \in A_k(e)$ and $b = c = e$, we obtain $(2k-1)[a,e] = 0$, so $[a,e] = 0$ for $k = 1, 0$, hence $ea = ae = e \bullet a = ka$ for $k = 1, 0$. This concludes the proof. □

The decomposition in Lemma 2.5.3(i) is called the *Peirce decomposition* of A relative to e.

Lemma 2.5.4 *Let A be a power-associative algebra over $\mathbb{K}$, and let e be an idempotent in A such that $A_0(e) = 0$. Then for y in $A_{\frac{1}{2}}(e)$ we have $y^3 = 0$.*

Proof Let y be in $A_{\frac{1}{2}}(e)$. Then $2e \bullet y = ey + ye = y$. Moreover, since $A_{\frac{1}{2}}(e) \bullet A_{\frac{1}{2}}(e) \subseteq A_1(e)$ (by Lemma 2.5.3(iv)), we get that $y^2 = y \bullet y \in A_1(e)$. Now, keeping in mind the identity (2.5.4), we have

$$\begin{aligned} 0 &= 2[e \bullet y, y, y] + [y^2, e, y] + [y^2, y, e] = (y^2e)y - y^2(ey) + y^3e - y^2(ye) \\ &= (y^2e)y + y^3e - y^2(ye + ey) = y^3 + y^3e - y^3 = y^3e, \end{aligned}$$

and hence $y^3e = 0$. Replacing A with the opposite algebra of A, we also obtain that $ey^3 = 0$. Therefore $y^3 \in A_0(e)$, and hence $y^3 = 0$. □

Lemma 2.5.5 *Let A be a power-associative algebra over $\mathbb{K}$ with no nonzero joint divisor of zero, and let e be a nonzero idempotent in A. Then e is a unit for A.*

Proof By Lemma 2.5.3(ii), our assumptions clearly imply that $A_0(e) = 0$, and then, by Lemma 2.5.4, that $A_{\frac{1}{2}}(e) = 0$ also. Therefore, by Lemma 2.5.3(i), we have $A = A_1(e)$, and hence, again by Lemma 2.5.3(ii), e is a unit for A. □

Definition 2.5.6 Let A be an algebra over $\mathbb{K}$, and let I be a $\left\{\begin{matrix}\text{left}\\ \text{right}\\ \text{two-sided}\end{matrix}\right\}$ ideal of A. We say that I is a *minimal* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal* of A if I is nonzero and if the only nonzero $\left\{\begin{matrix}\text{left}\\ \text{right}\\ \text{two-sided}\end{matrix}\right\}$ ideal of A contained in I is I itself.

§**2.5.7** An element a of an algebra is said to be *isotropic* whenever $a \neq 0 = a^2$. We note that *isotropic elements are nonzero joint divisors of zero.*

Lemma 2.5.8 *Let A be a nonzero finite-dimensional associative algebra over $\mathbb{K}$ without isotropic elements. Then A contains a nonzero idempotent.*

Proof Take a minimal left ideal I of A, and a nonzero element $a \in I$. Then we have $0 \neq a^2 \in Ia \subseteq I$, so $Ia = I$, and so there is $b \in I$ such that $ba = a$. Consider the set $J = \{x \in A : xa = 0\}$. It is clear that J is a left ideal of A containing the set $\{x - xb : x \in A\}$. Since $I \cap J$ is a left ideal contained in I and I is not contained in J, the minimality of I gives $I \cap J = 0$. Since $b \in I$, we have $b - b^2 \in I$. Also $b - b^2 \in J$, and so $b - b^2 = 0$. □

§**2.5.9** Let A be an algebra over $\mathbb{K}$. An element $a \in A$ is said to be *algebraic* if the subalgebra of A generated by a is finite-dimensional. We say that A is *algebraic* if all its elements are algebraic. If A is unital and power-associative, and if a is an algebraic element of A, then we can consider the so-called *minimum polynomial* of a, namely the unique non-constant monic polynomial $p(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ such that

$$\{q(\mathbf{x}) \in \mathbb{K}[\mathbf{x}] : q(a) = 0\} = p(\mathbf{x})\mathbb{K}[\mathbf{x}].$$

The algebra A is called *quadratic* if it is unital and, for every $a \in A$, a^2 lies in the linear hull of $\{\mathbf{1}, a\}$. Some authors also require that $A \neq \mathbb{K}\mathbf{1}$, but this is not suitable for our present purpose. *If A is quadratic, then*, for each $a \in A$, the linear hull of $\{\mathbf{1}, a\}$ is a subalgebra of A containing a, and hence A *is algebraic and power-associative.* A relevant partial converse is given by the next proposition.

Proposition 2.5.10 *Let A be a nonzero power-associative algebraic algebra over $\mathbb{K}$ with no nonzero joint divisor of zero. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then A is isomorphic to $\mathbb{C}$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then A is quadratic.*

Proof Take a nonzero element $b \in A$. Then the subalgebra of A generated by b is a finite-dimensional associative algebra with no isotropic element, and hence, by Lemma 2.5.8, has a nonzero idempotent. Therefore, by Lemma 2.5.5, A is unital. Now, let a be any nonzero element of a, and let p stand for the minimum polynomial of A. If p were decomposable (say $p = qs$ with q, s non-constant monic polynomials), then we would have $q(a)s(a) = s(a)q(a) = 0$, and hence $q(a) = 0$ or $s(a) = 0$ (by the absence of nonzero joint divisors of zero), a fact which is not possible because $\max\{\deg(q), \deg(s)\} < \deg(p)$, and p is the minimum polynomial of a. Therefore p is indecomposable. Now, if $\mathbb{K} = \mathbb{C}$, then a lies in $\mathbb{C}\mathbf{1}$, and hence, since a is an arbitrary nonzero element of A, we conclude that A is isomorphic to $\mathbb{C}$. Assume that $\mathbb{K} = \mathbb{R}$. Then we have that either $a \in \mathbb{R}\mathbf{1}$ or $(a - \alpha\mathbf{1})^2 + \beta^2\mathbf{1} = 0$ for suitable $\alpha, \beta \in \mathbb{R}$. In both cases, a^2 belongs to the linear hull of $\{\mathbf{1}, a\}$, and, again by the arbitrariness of $a \in A \setminus \{0\}$, A is quadratic. □

For a variant of assertion (i) in the above proposition, see Lemma 2.6.30 below. Concerning assertion (ii), note that the proof gives more precise information. Indeed, *if A is a nonzero power-associative algebraic real algebra with no nonzero joint divisor of zero, then A is unital, and the subalgebra of A generated by each element and the unit is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

§**2.5.11** Let A be a quadratic algebra over $\mathbb{K}$. By definition, for each $a \in A$, there are elements $t(a)$ and $n(a)$ of $\mathbb{K}$ such that

$$a^2 - t(a)a + n(a)\mathbf{1} = 0. \tag{2.5.9}$$

If a belongs to $A \setminus (\mathbb{K}\mathbf{1})$, then the scalars $t(a)$ and $n(a)$ in (2.5.9) are uniquely determined. Otherwise, we have $a = \alpha\mathbf{1}$ for a unique $\alpha \in \mathbb{K}$, and we set $t(a) := 2\alpha$ and $n(a) := \alpha^2$, so that (2.5.9) is fulfilled. The mappings $a \to t(a)$ and $a \to n(a)$, from A to $\mathbb{K}$, defined in this way are called the *trace function* and the *algebraic norm function* on A, respectively.

Proposition 2.5.12 *Let A be a quadratic algebra over $\mathbb{K}$. Then the trace function is a linear form on A, and the algebraic norm function is a quadratic form on A.*

Proof First we prove that $t(\lambda a) = \lambda t(a)$ for all $\lambda \in \mathbb{K}$ and $a \in A$. If $\lambda = 0$, this is clear. Assume that $\lambda \neq 0$. If $a = \alpha\mathbf{1}$, then we have

$$t(\lambda a) = t(\lambda\alpha\mathbf{1}) = 2\lambda\alpha = \lambda t(a).$$

Assume that $a \notin \mathbb{K}\mathbf{1}$. Then we have

$$\begin{aligned} 0 &= (\lambda a)^2 - t(\lambda a)\lambda a + n(\lambda a)\mathbf{1} \\ &= \lambda^2[t(a)a - n(a)\mathbf{1}] - t(\lambda a)\lambda a + n(\lambda a)\mathbf{1} \\ &= [\lambda^2 t(a) - \lambda t(\lambda a)]a - [\lambda^2 n(a) + n(\lambda a)]\mathbf{1}, \end{aligned}$$

hence $\lambda^2 t(a) - \lambda t(\lambda a) = 0$, and so $t(\lambda a) = \lambda t(a)$.

Now we prove that $t(a+b) = t(a) + t(b)$ for all $a, b \in A$. If a, b are linearly dependent (say $b = \lambda a$), then

$$t(a+b) = t((1+\lambda)a) = (1+\lambda)t(a) = t(a) + t(b).$$

Assume that a, b are linearly independent. If one of them lies in $\mathbb{K}\mathbf{1}$ (say $b = \beta\mathbf{1}$), then we have

$$\begin{aligned} 0 &= (a+\beta\mathbf{1})^2 - t(a+\beta\mathbf{1})(a+\beta\mathbf{1}) + n(a+\beta\mathbf{1})\mathbf{1} \\ &= a^2 + 2\beta a + \beta^2\mathbf{1} - t(a+\beta\mathbf{1})a - \beta t(a+\beta\mathbf{1})\mathbf{1} + n(a+\beta\mathbf{1})\mathbf{1} \\ &= t(a)a - n(a)\mathbf{1} + 2\beta a + \beta^2\mathbf{1} - t(a+\beta\mathbf{1})a - \beta t(a+\beta\mathbf{1})\mathbf{1} + n(a+\beta\mathbf{1})\mathbf{1} \\ &= [t(a) + 2\beta - t(a+\beta\mathbf{1})]a + [-n(a) + \beta^2 - \beta t(a+\beta\mathbf{1}) + n(a+\beta\mathbf{1})]\mathbf{1}, \end{aligned}$$

hence $t(a) + 2\beta - t(a+\beta\mathbf{1}) = 0$, and so $t(a+\beta\mathbf{1}) = t(a) + 2\beta = t(a) + t(b)$. Now, assume additionally that $a, b \notin \mathbb{K}\mathbf{1}$. If $a, b, \mathbf{1}$ are linearly dependent (say $b = \alpha a + \beta\mathbf{1}$), then we have

$$\begin{aligned} t(a+b) &= t((1+\alpha)a + \beta\mathbf{1}) = (1+\alpha)t(a) + 2\beta = t(a) + \alpha t(a) + 2\beta \\ &= t(a) + t(\alpha a + \beta\mathbf{1}) = t(a) + t(b). \end{aligned}$$

If $a, b, \mathbf{1}$ are linearly independent, then we have

$$\begin{aligned}0 &= (a-b)^2 - t(a-b)(a-b) + n(a-b)\mathbf{1} + (a+b)^2 - t(a+b)(a+b) \\ &\quad + n(a+b)\mathbf{1} = 2a^2 + 2b^2 - [t(a-b)+t(a+b)]a + [t(a-b)-t(a+b)]b \\ &\quad + [n(a-b)+n(a+b)]\mathbf{1} = 2[t(a)a - n(a)\mathbf{1}] + 2[t(b)b - n(b)\mathbf{1}] \\ &\quad - [t(a-b)+t(a+b)]a + [t(a-b)-t(a+b)]b + [n(a-b)+n(a+b)]\mathbf{1} \\ &= [2t(a) - t(a-b) - t(a+b)]a + [2t(b) + t(a-b) - t(a+b)]b \\ &\quad + [-2n(a) - 2n(b) + n(a-b) + n(a+b)]\mathbf{1},\end{aligned}$$

hence

$$2t(a) - t(a-b) - t(a+b) = 0 \ \text{ and } \ 2t(b) + t(a-b) - t(a+b) = 0,$$

and adding these equalities we obtain $t(a+b) = t(a) + t(b)$.

Finally, now that we know that t is linear, it is enough to apply t to both members of the equality (2.5.9) to derive

$$n(a) = \frac{1}{2}[t(a)^2 - t(a^2)],$$

which shows ostensibly that n is a quadratic form. □

Let A be a quadratic algebra over $\mathbb{K}$. It follows from the above proposition that $A = \mathbb{K}\mathbf{1} \oplus V$, where $V := \ker(t)$. Thus, each element $a \in A$, can be uniquely written in the form $a = \alpha\mathbf{1} + x$ for $\alpha \in \mathbb{K}$ and $x \in V$. As usual, α is called the 'scalar part' of a, and x is called the 'vector part' of a. If for $x, y \in V$, we denote by $-(x,y)$ and $x \times y$ the scalar and vector parts of xy, respectively, that is to say, if we write

$$xy = -(x,y)\mathbf{1} + x \times y \quad \text{with} \ \ (x,y) \in \mathbb{K} \ \text{ and } \ x \times y \in V,$$

then we are provided with a bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{K}$ and a bilinear mapping $\times : V \times V \to V$. Note that, for $x \in V$, we have $x^2 \in \mathbb{K}\mathbf{1}$ (by (2.5.9)) and also $x^2 = -(x,x)\mathbf{1} + x \times x$, so that, since $x \times x$ lies in V, we get $x \times x = 0$. Therefore $(V, \times)$ is an anticommutative algebra. Keeping in mind these remarks, the following result follows straightforwardly.

Proposition 2.5.13 *The quadratic algebras over $\mathbb{K}$ are the algebras whose vector space is $\mathbb{K}\mathbf{1} \oplus V$, for some vector space V over $\mathbb{K}$ endowed with an anticommutative product $\times$ and a bilinear form $(\cdot,\cdot)$, and whose product is defined by*

$$(\alpha\mathbf{1} + x)(\beta\mathbf{1} + y) := (\alpha\beta - (x,y))\mathbf{1} + (\alpha y + \beta x + x \times y).$$

Moreover, when a quadratic algebra A is regarded in the above manner, then

$$V = \{a \in A : a^2 \in \mathbb{K}\mathbf{1}, a \notin \mathbb{K}\mathbf{1} \setminus \{0\}\}. \tag{2.5.10}$$

§**2.5.14** Given a vector space V over $\mathbb{K}$ endowed with an anticommutative product $\times$ and a bilinear form $(\cdot,\cdot)$, we will denote by $\mathscr{A}(V, \times, (\cdot,\cdot))$ the quadratic algebra over $\mathbb{K}$ which arises in the manner pointed out in the above proposition.

Lemma 2.5.15 *Let $A = \mathscr{A}(V, \times, (\cdot,\cdot))$ be a quadratic real algebra with no nonzero joint divisor of zero. Then $n(x) = (x,x) > 0$ for every $x \in V \setminus \{0\}$.*

Proof Let x be an element in V. If $n(x) \leqslant 0$, then, setting $\alpha := \sqrt{-n(x)}$, we have

$$(\alpha\mathbf{1}+x)(\alpha\mathbf{1}-x) = (\alpha\mathbf{1}-x)(\alpha\mathbf{1}+x) = -(n(x)\mathbf{1}+x^2) = 0,$$

and hence either $\alpha\mathbf{1}+x=0$ or $\alpha\mathbf{1}-x=0$. In any case, we conclude that $x=0$. □

Corollary 2.5.16 *Let A be a commutative quadratic real algebra with no nonzero divisor of zero. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Proof According to Proposition 2.5.13 and the notation in §2.5.14, we can write $A = \mathscr{A}(V, \times, (\cdot,\cdot))$. Since A is commutative, for $x, y \in V$ we have

$$0 = [x,y] = [(y,x)-(x,y)]\mathbf{1} + x\times y - y\times x,$$

which implies that the bilinear form $(\cdot,\cdot)$ on V is symmetric, and that the anti-commutative product $\times$ on V is identically zero. Assume that $\dim(V) \geqslant 2$. Then there exist nonzero elements $x, y \in V$ such that $(x,y) = 0$, so that we have $xy = -(x,y)\mathbf{1} = 0$, contradicting the assumption that A has no nonzero divisor of zero. Therefore we have $\dim(V) \leqslant 1$. If $\dim(V) = 0$, then, clearly, A is isomorphic to $\mathbb{R}$. Otherwise, again by the absence of nonzero divisors of zero in A, Lemma 2.5.15 applies so that there exists $x \in V$ with $x^2 = -\mathbf{1}$, and hence the mapping $r + is \to r\mathbf{1} + sx$ $(r, s \in \mathbb{R})$ becomes a bijective algebra homomorphism from $\mathbb{C}$ to A. □

Applying Proposition 2.5.10(ii) and Corollary 2.5.16, we get the following.

Proposition 2.5.17 *Let A be a nonzero power-associative commutative algebraic real algebra with no nonzero divisor of zero. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

A quadratic form N on an algebra A is said to *admit composition* if the equality $N(ab) = N(a)N(b)$ holds for all $a, b \in A$.

Proposition 2.5.18 *Let $A = \mathscr{A}(V, \times, (\cdot,\cdot))$ be a quadratic algebra over $\mathbb{K}$. Then we have:*

(i) *For every $a = \alpha\mathbf{1} + x \in A$, $t(a) = 2\alpha$ and $n(a) = \alpha^2 + (x,x)$.*
(ii) *A is flexible if and only if $(\cdot,\cdot)$ is symmetric and*

$$(x\times y, z) = (x, y\times z) \quad \textit{for all} \ \ x,y,z \in V.$$

(iii) *n admits composition if and only if A is flexible and*

$$(x\times y, x\times y) = (x,x)(y,y) - (x,y)^2 \quad \textit{for all} \ \ x,y \in V.$$

(iv) *A is alternative if and only if A is flexible and*

$$x\times(x\times y) = (x,y)x - (x,x)y \quad \textit{for all} \ \ x,y \in V.$$

Proof Assertion (i) follows from the fact that, for every $a = \alpha\mathbf{1} + x \in A$, we have

$$a^2 = [\alpha^2 - (x,x)]\mathbf{1} + 2\alpha x = -[\alpha^2 + (x,x)]\mathbf{1} + 2\alpha a.$$

Now, note that, for all $x, y \in V$ we have

$$\begin{aligned}
[x,y,x] &= (xy)x - x(yx) = [-(x,y)\mathbf{1} + x\times y]x - x[-(y,x)\mathbf{1} + y\times x] \\
&= -(x\times y, x)\mathbf{1} - (x,y)x + (x\times y)\times x + (x, y\times x)\mathbf{1} + (y,x)x - x\times(y\times x) \\
&= [-(x\times y, x) + (x, y\times x)]\mathbf{1} + [-(x,y) + (y,x)]x.
\end{aligned}$$

Therefore $[x,y,x]=0$ if and only if

$$(x,y)=(y,x) \text{ and } (x\times y,x)=(x,y\times x),$$

and assertion (ii) follows by linearization of the last equality with respect to x.

Let $a=\alpha\mathbf{1}+x$ and $b=\beta\mathbf{1}+y$ be in A. By assertion (i) we have

$$\begin{aligned} n(ab) &= [\alpha\beta-(x,y)]^2+(\alpha y+\beta x+x\times y,\alpha y+\beta x+x\times y)\\ &= \alpha^2\beta^2-2\alpha\beta(x,y)+(x,y)^2+\alpha^2(y,y)+\alpha\beta(y,x)+\alpha(y,x\times y)\\ &\quad +\alpha\beta(x,y)+\beta^2(x,x)+\beta(x,x\times y)+\alpha(x\times y,y)+\beta(x\times y,x)\\ &\quad +(x\times y,x\times y) \end{aligned}$$

and

$$n(a)n(b)=[\alpha^2+(x,x)][\beta^2+(y,y)]=\alpha^2\beta^2+\alpha^2(y,y)+\beta^2(x,x)+(x,x)(y,y).$$

Therefore

$$\begin{aligned} n(ab)-n(a)n(b) &= \alpha\beta[(y,x)-(x,y)]+\alpha[(y,x\times y)+(x\times y,y)]\\ &\quad +\beta[(x,x\times y)+(x\times y,x)]+(x,y)^2+(x\times y,x\times y)\\ &\quad -(x,x)(y,y). \end{aligned}$$

Now, assertion (iii) follows from assertion (ii), by taking $\alpha,\beta\in\{0,1\}$ for the 'only if' part.

Let x,y be in V. Then we have

$$\begin{aligned} [x,x,y] &= x^2y-x(xy)=-(x,x)y-x[-(x,y)\mathbf{1}+x\times y]\\ &= -(x,x)y+(x,x\times y)\mathbf{1}+(x,y)x-x\times(x\times y), \end{aligned}$$

and hence $[x,x,y]=0$ if and only if

$$(1)\ \ (x,x\times y)=0 \quad\text{and}\quad (2)\ \ x\times(x\times y)=(x,y)x-(x,x)y.$$

Analogously, we have

$$\begin{aligned} [y,x,x] &= (yx)x-yx^2=[-(y,x)\mathbf{1}+y\times x]x+(x,x)y\\ &= -(y\times x,x)\mathbf{1}-(y,x)x+(y\times x)\times x+(x,x)y, \end{aligned}$$

and hence $[y,x,x]=0$ if and only if

$$(1')\ \ (y\times x,x)=0 \quad\text{and}\quad (2')\ \ (y\times x)\times x=(y,x)x-(x,x)y.$$

Since $x\times(x\times y)=(y\times x)\times x$, it follows that (2) and (2′) occur simultaneously if and only if $(\cdot,\cdot)$ is symmetric and (2) occurs. Moreover, in this case, conditions (1) and (1′) agree. Now, assertion (iv) follows from assertion (ii). □

For the proof of the next corollary, assertions (ii), (iii), and (iv) in Proposition 2.5.18 must be kept in mind.

Corollary 2.5.19 *Let $A = \mathscr{A}(V, \times, (\cdot,\cdot))$ be a quadratic algebra over $\mathbb{K}$. We have:*

(i) *If A is alternative, then the algebraic norm function n on A admits composition.*
(ii) *If n admits composition and if $(\cdot,\cdot)$ is nondegenerate, then A is alternative.*

Proof Assume that A is alternative. Then A is flexible, and hence $(\cdot,\cdot)$ is symmetric and satisfies

$$(x \times y, z) = (x, y \times z) \text{ for all } x, y, z \in V.$$

Therefore, for all $x, y \in V$, we have

$$\begin{aligned}(x \times y, x \times y) &= -(y \times x, x \times y) = -(y, x \times (x \times y)) \\ &= -(y, (x,y)x - (x,x)y) = (x,x)(y,y) - (x,y)^2,\end{aligned}$$

hence n admits composition. Thus assertion (i) is proved.

Now, assume that n admits composition, and that $(\cdot,\cdot)$ is nondegenerate. Linearizing the condition

$$(x \times y, x \times y) = (x,x)(y,y) - (x,y)^2$$

with respect to y, we obtain that

$$(x \times y, x \times z) = (x,x)(y,z) - (x,y)(x,z)$$

for all $x, y, z \in V$. Hence

$$(x \times (x \times y) + (x,x)y - (x,y)x, z) = -(x \times y, x \times z) + (x,x)(y,z) - (x,y)(x,z) = 0$$

for all $x, y, z \in V$, and consequently $x \times (x \times y) + (x,x)y - (x,y)x = 0$ for all $x, y \in V$. Thus A is alternative. □

Proposition 2.5.20 *Let A be an algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *A is a Cayley algebra.*
(ii) *A is a quadratic algebra, say $A = \mathscr{A}(V, \times, (\cdot,\cdot))$, and the bilinear form $(\cdot,\cdot)$ is symmetric.*

Moreover, if A is a Cayley algebra, then we have

$$a + a^* = t(a)\mathbf{1} \text{ and } a^*a = aa^* = n(a)\mathbf{1} \text{ for every } a \in A,$$

and $a^* = \alpha\mathbf{1} - x$ whenever $a = \alpha\mathbf{1} + x$ with $\alpha \in \mathbb{K}$ and $x \in V$.

Proof Assume that A is a Cayley algebra. Since for every $a \in A$ we have $a^2 - a(a + a^*) + aa^* = 0$, it follows that A is quadratic and $t(a)\mathbf{1} = a + a^*$ and $n(a)\mathbf{1} = aa^*$ for every $a \in A$. As a consequence of the equality $t(a)\mathbf{1} = a + a^*$ we get that $H(A, *) = \mathbb{K}\mathbf{1}$, and that the space V of the vectors of A is precisely the set of those elements $a \in A$ such that $a^* = -a$. Therefore we have

$$(\alpha\mathbf{1} + x)^* = \alpha\mathbf{1} - x \text{ for all } \alpha \in \mathbb{K}\mathbf{1} \text{ and } x \in V,$$

which implies $a^*a = aa^* = n(a)\mathbf{1}$ for every $a \in A$. On the other hand, since $*$ is an algebra involution, for all $x, y \in V$ we see that

$$0 = (xy)^* - y^*x^* = -(x,y)\mathbf{1} - x \times y + (y,x)\mathbf{1} - y \times x = [(y,x) - (x,y)]\mathbf{1},$$

and hence $(x,y)=(y,x)$. Thus we have proved the implication (i)⇒(ii) in the first conclusion of the proposition, as well as the whole of the second conclusion.

Now, assume that $A=\mathscr{A}(V,\times,(\cdot,\cdot))$, where the bilinear form $(\cdot,\cdot)$ is symmetric. Consider the mapping $*:A\to A$ defined by

$$(\alpha\mathbf{1}+x)^*=\alpha\mathbf{1}-x \text{ for all } \alpha\in\mathbb{K} \text{ and } x\in V.$$

Clearly, $*$ is a vector space involution on A satisfying $a+a^*\in\mathbb{K}\mathbf{1}$ and $aa^*\in\mathbb{K}\mathbf{1}$ for every $a\in A$. Moreover, keeping in mind that the form $(\cdot,\cdot)$ is symmetric, we straightforwardly realize that $*$ is in fact an algebra involution. □

By a *composition algebra* over $\mathbb{K}$ we mean a nonzero algebra over $\mathbb{K}$ endowed with a nondegenerate quadratic form admitting composition. Given a vector space X over $\mathbb{K}$ and a quadratic form q on X, we denote by $q(\cdot,\cdot)$ the unique symmetric bilinear form on X such that $q(x,x)=q(x)$ for every $x\in X$.

Proposition 2.5.21 *Let A be a unital algebra over $\mathbb{K}$. Then there is at most one nondegenerate quadratic form on A admitting composition. Moreover, the following conditions are equivalent:*

(i) *A is a composition algebra.*
(ii) *A is a quadratic algebra, say $A=\mathscr{A}(V,\times,(\cdot,\cdot))$, the bilinear form $(\cdot,\cdot)$ is symmetric and nondegenerate, and the quadratic algebraic norm form n on A admits composition.*
(iii) *A is an alternative Cayley algebra, and the quadratic form N on A determined by $N(a)\mathbf{1}=aa^*$ is nondegenerate.*

Proof The first conclusion will follow as a by-product of the proof of the implication (i)⇒(ii).

(i)⇒(ii) Assume that A is a composition algebra. Let N be the nondegenerate quadratic form on A admitting composition. Then

$$N(ab,ab)=N(a,a)N(b,b) \text{ for all } a,b\in A. \tag{2.5.11}$$

Linearize this relative to a to obtain

$$N(ab,cb)=N(a,c)N(b,b) \text{ for all } a,b,c\in A.$$

Linearizing this relative to b, we have

$$N(ab,cd)+N(ad,cb)=2N(a,c)N(b,d) \text{ for all } a,b,c,d\in A. \tag{2.5.12}$$

Define $T(a):=2N(a,\mathbf{1})$ for every $a\in A$. Note that $N(\mathbf{1})=1$ by (2.5.11) since $N(a)\neq 0$ for some a in A. Then

$$N(\alpha\mathbf{1})=\alpha^2 \text{ and } T(\alpha\mathbf{1})=2\alpha \text{ for every } \alpha \text{ in } \mathbb{K}. \tag{2.5.13}$$

Now (2.5.12) implies

$$N(ab,c)+N(a,cb)=N(a,c)T(b) \text{ for all } a,b,c\in A \tag{2.5.14}$$

and

$$N(ab,d)+N(b,ad)=T(a)N(b,d) \text{ for all } a,b,d\in A. \tag{2.5.15}$$

Now $N(a^2,c)+N(a,ac)=T(a)N(a,c)$ by (2.5.15) and $N(a,ac)=N(a)N(\mathbf{1},c)$ by (2.5.14). Hence $N(a^2-T(a)a+N(a)\mathbf{1},c)=0$ for every c in A. Since N is nondegenerate, we have

$$a^2-T(a)a+N(a)\mathbf{1}=0 \text{ for every } a\in A.$$

Therefore A is a quadratic algebra. By the uniqueness of the algebraic norm in the 'vector part' V of a quadratic algebra, we have $(x,x)=N(x)$ for every $x\in V$. Moreover, by the first equality in (2.5.13) and Proposition 2.5.18(i), we obtain that $n(a)=N(a)$ for every a in A. Thus n, and hence $(\cdot,\cdot)$, is nondegenerate, and n admits composition. Finally, by Proposition 2.5.18(ii)–(iii), $(\cdot,\cdot)$ is symmetric.

(ii)$\Rightarrow$(iii) Assume that (ii) holds. Then, by Corollary 2.5.19, A is alternative. Moreover, by Proposition 2.5.20, A is a Cayley algebra and $n(a)\mathbf{1}=aa^*$ for every $a\in A$. Therefore $2n(a,b)\mathbf{1}=ab^*+ba^*$ for all $a,b\in A$. Since $(\cdot,\cdot)$ is symmetric and nondegenerate, it follows from Proposition 2.5.18(i) that n is nondegenerate.

(iii)$\Rightarrow$(i) Assume that (iii) holds. Then, by Proposition 2.5.20, A is a quadratic algebra, say $A=\mathscr{A}(V,\times,(\cdot,\cdot))$, the bilinear form $(\cdot,\cdot)$ is symmetric, and

$$n(a)\mathbf{1}=aa^* \text{ for every } a\in A.$$

Therefore $n(a,b)\mathbf{1}=\frac{1}{2}(ab^*+ba^*)$ for all $a,b\in A$, and hence, by assumption, n is nondegenerate. Finally, by Corollary 2.5.19, n admits composition. $\square$

Definition 2.5.22 If V is a real vector space endowed with a product $\times$ and an inner product $(\cdot|\cdot)$ satisfying

$$(x\times y|z)=(x|y\times z) \text{ for all } x,y,z\in V, \tag{2.5.16}$$

and

$$\|x\times y\|^2=\|x\|^2\|y\|^2-(x|y)^2 \text{ for all } x,y\in V, \tag{2.5.17}$$

then we will say that $(V,\times,(\cdot|\cdot))$ is a *cross-product algebra*. Note that, if $(V,\times,(\cdot|\cdot))$ is a cross-product algebra, then, by (2.5.17), we have $x\times x=0$ for every $x\in V$, and hence the product $\times$ is anticommutative.

Definition 2.5.23 Let A be a unital alternative algebra, and let a be in A. As in the associative case, we say that a is *invertible* in A if there exists $b\in A$ such that $ab=ba=\mathbf{1}$. Such an element b is called an *inverse* of a.

For elements x,y,z,w in a given algebra, the following equality is straightforwardly realized:

$$[x,y,z]w+x[y,z,w]=[xy,z,w]-[x,yz,w]+[x,y,zw]. \tag{2.5.18}$$

Now let A be a unital alternative algebra, let a be an invertible element of A with inverse b, and let c be in A. Then, setting $(x,y,z,w)=(b,a,a,bc)$ in (2.5.18), we obtain

$$0=-[b,a^2,bc]+[b,a,a(bc)].$$

On the other hand, by the right Moufang identity (2.3.16), we have

$$\begin{aligned}[b,a^2,bc]&=[a^2,bc,b]=-[a^2,b,c]b=[c,b,a^2]b=-[c,ba^2,b]\\&=[b,ba^2,c]=[b,(ba)a,c]=[b,\mathbf{1}a,c]=[b,a,c]\end{aligned}$$

and

$$[b,a,a(bc)] = -[b,a(bc),a] = [b,a,bc]a = -[bc,a,b]a$$
$$= [bc,ab,a] = [bc,\mathbf{1},a] = 0.$$

It follows that

$$[b,a,c] = 0 \text{ for every } c \in A. \tag{2.5.19}$$

Proposition 2.5.24 *Let A be a unital alternative algebra, and let a be an invertible element in A. We have:*

(i) *a has a unique inverse.*
(ii) *L_a (respectively, R_a) is a bijective operator on A with $L_a^{-1} = L_{a^{-1}}$ (respectively, $R_a^{-1} = R_{a^{-1}}$), where a^{-1} denotes the unique inverse of a.*

Proof Suppose that b and c are inverses of a in A. Then, keeping in mind (2.5.19), we see that

$$c = \mathbf{1}c = (ba)c = [b,a,c] + b(ac) = b(ac) = b\mathbf{1} = b.$$

Thus a has a unique inverse. Moreover, if we denote by a^{-1} the unique inverse of a, then (2.5.19) gives that $[a^{-1},a,x] = 0$ for every $x \in A$. Since the associator is an alternating function of its arguments, we obtain that $L_{a^{-1}}L_a = L_aL_{a^{-1}} = I_A$ and $R_{a^{-1}}R_a = R_aR_{a^{-1}} = I_A$, and the proof is complete. □

Definition 2.5.25 Now we can consider the *classical* notion of a *division alternative algebra*, namely a unital alternative algebra A whose nonzero elements are invertible in A.

Proposition 2.5.26 *Let A be a unital real algebra. Then the following conditions are equivalent:*

(i) *A is both a quadratic algebra and a division alternative algebra in the classical sense.*
(ii) *A is an alternative quadratic algebra with no nonzero divisor of zero.*
(iii) *A is an alternative quadratic algebra with no nonzero joint divisor of zero.*
(iv) *$A = \mathscr{A}(V,\times,(\cdot|\cdot))$, where $(V,\times,(\cdot|\cdot))$ is a cross-product algebra.*
(v) *A is a composition algebra whose nondegenerate quadratic form n admitting composition satisfies in fact $n(a) \neq 0$ for every $a \in A \setminus \{0\}$.*

Proof (i)⇒(ii) Let a,b be in A such that $ab = 0$. If $a \neq 0$, then, by the assumption (i), we have $b = L_a^{-1}(L_a(b)) = a^{-1}(ab) = 0$.

(ii)⇒(iii) This implication is clear.

(iii)⇒(iv) Assume that condition (iii) holds. Then, by Proposition 2.5.13, $A = \mathscr{A}(V,\times,(\cdot,\cdot))$, where $(V,\times,(\cdot,\cdot))$ is an anticommutative algebra with a bilinear form. Moreover, since A is alternative, Corollary 2.5.19 applies, so that n admits composition. Finally, by Proposition 2.5.18 and Lemma 2.5.15, we conclude that $(V,\times,(\cdot,\cdot))$ is a cross-product algebra.

(iv)⇒(v) Assume that (iv) holds. Then, by Proposition 2.5.18, n admits composition, and $n(a) = \alpha^2 + \|x\|^2 > 0$ for every nonzero element $a = \alpha\mathbf{1} + x$ in A.

(v)⇒(i) Assume that (v) holds. Then, by Propositions 2.5.21 and 2.5.20, A is an alternative quadratic algebra with an algebra involution $*$ satisfying

$$aa^* = a^*a = n(a)\mathbf{1} \text{ for every } a \in A.$$

Therefore, if $a \neq 0$, then a is invertible in A with $a^{-1} = \frac{1}{n(a)}a^*$. □

Let $(V, \times, (\cdot|\cdot))$ be a cross-product algebra. Then, by Propositions 2.5.26 and 2.5.18, we have

$$x \times (x \times y) = (x|y)x - \|x\|^2 y \quad \text{for all} \quad x, y \in V. \tag{2.5.20}$$

Linearizing with respect to x we obtain

$$x \times (y \times z) - z \times (x \times y) = 2(x|z)y - (x|y)z - (y|z)x \quad \text{for all} \quad x, y, z \in V, \tag{2.5.21}$$

and as a consequence we have

$$x \times (y \times z) = z \times (x \times y) \quad \text{for all pairwise orthogonal} \quad x, y, z \in V. \tag{2.5.22}$$

Lemma 2.5.27 *Let $(V, \times, (\cdot|\cdot))$ be a cross-product algebra. Assume that x, y, z are elements of V and v is a norm-one element of V such that each one of the sets $\{x, y, z, v\}$ and $\{x \times v, v, y \times z\}$ consists of pairwise orthogonal elements. Then*

$$(x \times v) \times (y \times (z \times v)) = x \times (y \times z).$$

Proof By applying (2.5.22) twice we have

$$(x \times v) \times (y \times (z \times v)) = (x \times v) \times (v \times (y \times z)) = (y \times z) \times ((x \times v) \times v).$$

Keeping in mind the anticommutativity of $\times$, by (2.5.20) we realize that $(x \times v) \times v = v \times (v \times x) = -x$, and we conclude that

$$(x \times v) \times (y \times (z \times v)) = x \times (y \times z).$$ □

Now recall that, if $(V, \times, (\cdot|\cdot))$ is a cross-product algebra, then, by Proposition 2.5.20, $\mathscr{A}(V, \times, (\cdot|\cdot))$ is a Cayley algebra, and hence the Cayley–Dickson doubling process is applicable to $\mathscr{A}(V, \times, (\cdot|\cdot))$.

Proposition 2.5.28 *Let $(V, \times, (\cdot|\cdot))$ be a cross-product algebra, let U be a subalgebra of $(V, \times)$ such that $U^\perp \neq 0$, and let v be a norm-one element in $U^\perp$. We have:*

(i) $U \times v \subseteq (U \oplus \mathbb{R}v)^\perp$.
(ii) *If $U^{(v)}$ denotes the subalgebra of $(V, \times)$ generated by U and v, then*

$$U^{(v)} = U \oplus \mathbb{R}v \oplus (U \times v),$$

and the mapping $\alpha\mathbf{1} + u_1 + \beta v + u_2 \times v \to (\alpha\mathbf{1} + u_1, -\beta\mathbf{1} + u_2)$ is an algebra isomorphism from $\mathscr{A}(U^{(v)}, \times, (\cdot|\cdot))$ onto the Cayley–Dickson doubling, $\mathscr{C}\mathscr{D}(\mathscr{A}(U, \times, (\cdot|\cdot)))$, of $\mathscr{A}(U, \times, (\cdot|\cdot))$.
(iii) *If $(U^{(v)})^\perp \neq 0$, then* $\dim(U) \leqslant 1$.

Proof Assertion (i) follows from the fact that, for all $u, u_1, u_2 \in U$ we have

$$(u \times v|v) = (u|v \times v) = 0 \quad \text{and} \quad (u_1 \times v|u_2) = (u_2|u_1 \times v) = (u_2 \times u_1|v) = 0.$$

The equality $U^{(v)} = U \oplus \mathbb{R}v \oplus U \times v$ in assertion (ii) follows from the fact that, for all $u, u_1, u_2 \in U$, by applying (2.5.20) and (2.5.21), we have

$$u_1 \times (u_2 \times v) = -(u_1|u_2)v - (u_1 \times u_2) \times v,$$
$$v \times (u \times v) = -v \times (v \times u) = u,$$

and

$$(u_1 \times v) \times (u_2 \times v) = v \times ((u_1 \times v) \times u_2) = v \times [(u_1|u_2)v - (u_1 \times u_2) \times v]$$
$$= -v \times [(u_1 \times u_2) \times v] = -u_1 \times u_2.$$

Now that we know that $U^{(v)} = U \oplus \mathbb{R}v \oplus (U \times v)$, it is a routine matter to show that the mapping

$$\alpha\mathbf{1} + u_1 + \beta v + u_2 \times v \to (\alpha\mathbf{1} + u_1, -\beta\mathbf{1} + u_2)$$

is an algebra isomorphism from $\mathscr{A}(U^{(v)}, \times, (\cdot|\cdot))$ onto $\mathscr{C}\mathscr{D}(\mathscr{A}(U, \times, (\cdot|\cdot)))$.

In order to prove assertion (iii), suppose, to derive a contradiction, that $(U^{(v)})^\perp \neq 0$ and $\dim(U) \geqslant 2$. Fix norm-one elements $w \in (U^{(v)})^\perp$ and $u_1, u_2 \in U$ such that $(u_1|u_2) = 0$. Set $p := v \times w$. Since $p \in U^{(v)} \times w$, it follows from assertion (i) that $p \in (U^{(v)} \oplus \mathbb{R}w)^\perp$, and then that $U^{(v)} \times p \subseteq (U^{(v)})^\perp$. Therefore each one of the sets $\{u_1, u_2, p, v\}$ and $\{u_1 \times v, v, u_2 \times p\}$ consists of pairwise orthogonal elements. Therefore, by Lemma 2.5.27, we have

$$(u_1 \times v) \times (u_2 \times (p \times v)) = u_1 \times (u_2 \times p).$$

On the other hand, by (2.5.20), we have

$$p \times v = (v \times w) \times v = -v \times (v \times w) = w,$$

and consequently $(u_1 \times v) \times (u_2 \times w) = (u_1 \times v) \times (u_2 \times (p \times v))$. Hence, we can conclude that

$$(u_1 \times v) \times (u_2 \times w) = u_1 \times (u_2 \times p). \tag{2.5.23}$$

Analogously, since $p = -w \times v \in U^{(w)} \times v$, it follows from assertion (i) that $p \in (U^{(w)} \oplus \mathbb{R}v)^\perp$, and then that $U^{(w)} \times p \subseteq (U^{(w)})^\perp$. Therefore each one of the sets $\{u_2, u_1, p, w\}$ and $\{u_2 \times w, w, u_1 \times p\}$ consists of pairwise orthogonal elements. Therefore, by Lemma 2.5.27, we get that

$$(u_2 \times w) \times (u_1 \times (p \times w)) = u_2 \times (u_1 \times p).$$

On the other hand, by (2.5.20), we have $w \times p = -w \times (w \times v) = v$, and consequently $(u_2 \times w) \times (u_1 \times v) = -(u_2 \times w) \times (u_1 \times (p \times w))$. Hence, we obtain that $(u_2 \times w) \times (u_1 \times v) = -u_2 \times (u_1 \times p) = u_2 \times (p \times u_1)$. Since, by (2.5.22), $u_2 \times (p \times u_1) = u_1 \times (u_2 \times p)$, we conclude that

$$(u_2 \times w) \times (u_1 \times v) = u_1 \times (u_2 \times p). \tag{2.5.24}$$

Now (2.5.23) together with (2.5.24) give that

$$(u_1 \times v) \times (u_2 \times w) = u_1 \times (u_2 \times p) = (u_2 \times w) \times (u_1 \times v). \tag{2.5.25}$$

Since v,w are orthogonal, u_2,p are orthogonal, and $u_1,u_2\times p$ are orthogonal, we see, by (2.5.17), that $\|p\|=1$, $\|u_2\times p\|=1$, and $\|u_1\times(u_2\times p)\|=1$. Therefore $u_1\times(u_2\times p)\neq 0$, and consequently (2.5.25) contradicts the anticommutativity of $\times$. $\square$

Theorem 2.5.29 (Frobenius–Zorn) *Let A be a nonzero alternative algebraic real algebra with no nonzero joint divisor of zero. Then A is isomorphic to $\mathbb{R},\mathbb{C},\mathbb{H}$, or $\mathbb{O}$.*

Proof By Proposition 2.5.10, A is quadratic, and then, by Proposition 2.5.26, $A=\mathscr{A}(V,\times,(\cdot|\cdot))$, where $(V,\times,(\cdot|\cdot))$ is a cross-product algebra. Suppose that A is not isomorphic to $\mathbb{R}$. Then $U_0:=0$ is a subalgebra of V with $U_0^{\perp}=V\neq 0$. Choose a norm-one element $v_1\in V$, set $U_1:=U_0^{(v_1)}$, and let A_1 denote the subalgebra of A generated by $\mathbf{1}$ and U_1. It follows from Proposition 2.5.28(ii) that $U_1=\mathbb{R}v_1$ and that A_1 is isomorphic to $\mathbb{C}$. Therefore, if $A_1=A$, then A is isomorphic to $\mathbb{C}$. Suppose that $A_1\neq A$, hence $U_1\neq V$, and consequently $U_1^{\perp}\neq 0$. Choose a norm-one element $v_2\in U_1^{\perp}$ and consider $U_2:=U_1^{(v_2)}$. By Proposition 2.5.28(ii), the subalgebra A_2 of A generated by $\mathbf{1}$ and U_2 is isomorphic to $\mathbb{H}$. Therefore, if $A_2=A$, then A is isomorphic to $\mathbb{H}$. Suppose that $A_2\neq A$, hence $U_2\neq V$, and consequently $U_2^{\perp}\neq 0$. Arguing as above, choose a norm-one element $v_3\in U_2^{\perp}$, set $U_3:=U_2^{(v_3)}$, and let A_3 denote the subalgebra of A generated by $\mathbf{1}$ and U_3. By Proposition 2.5.28(ii), A_3 is isomorphic to $\mathbb{O}$. Since $\dim(U_2)=3$, it follows from Proposition 2.5.28(iii) that $U_3^{\perp}=0$, and hence $A=A_3$ is isomorphic to $\mathbb{O}$. $\square$

Corollary 2.5.30 *Let $(V,\times,(\cdot|\cdot))$ be a nonzero cross-product algebra. Then $(V,\times,(\cdot|\cdot))$ is isomorphic to one of the following cross-product algebras:*

(i) *The Euclidean vector space $\mathbb{R}$ with trivial cross product.*
(ii) *The Euclidean vector space $\mathbb{R}^3$ with cross product equal to the usual vector product.*
(iii) *The Euclidean vector space $\mathbb{R}^7$ (regarded as $\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^3$) with cross product defined by*

$$
\begin{aligned}
(x_1,\alpha_1,y_1)\times(x_2,\alpha_2,y_2):=(&-\alpha_1y_2+\alpha_2y_1+x_1\times x_2-y_1\times y_2,(x_1|y_2)\\
&-(y_1|x_2),\alpha_1x_2-\alpha_2x_1-x_1\times y_2-y_1\times x_2),
\end{aligned}
$$

where for $x,y\in\mathbb{R}^3$ we have denoted by $(x|y)$ and $x\times y$ the usual inner product and vector product in $\mathbb{R}^3$, respectively.

Proof By Proposition 2.5.26 and Theorem 2.5.29, $\mathscr{A}(V,\times,(\cdot|\cdot))$ is isomorphic to $\mathbb{C},\mathbb{H}$, or $\mathbb{O}$, and hence $(V,\times,(\cdot|\cdot))$ is isomorphic to one of the cross-product algebras which appear by considering the vector part of $\mathbb{C},\mathbb{H}$, or $\mathbb{O}$. By keeping in mind the matricial representations of $\mathbb{C},\mathbb{H}$, and $\mathbb{O}$ given in §2.5.1, it turns out that these algebras are nothing other than the cross-product algebras specified in the statement. $\square$

We conclude this subsection with a by-product of Propositions 2.5.10(i) and 2.5.17.

Definition 2.5.31 Let A be an algebra over $\mathbb{K}$. The *centre* of A is defined as the subset of A consisting of those elements $a \in A$ such that

$$[a,A] = [a,A,A] = [A,a,A] = [A,A,a] = 0,$$

and is denoted by $Z(A)$. Elements in $Z(A)$ are called *central elements* of A. We note that $Z(A)$ *becomes an associative and commutative subalgebra of* A.

Lemma 2.5.32 *Let A be an algebra over $\mathbb{K}$ with no nonzero joint divisor of zero. Then A is unital if (and only if) so is $Z(A)$.*

Proof Assume that $Z(A)$ is unital. Let e denote the unit of $Z(A)$. Then, noticing that e is a central idempotent in A, for every $a \in A$ we have

$$e(ae - a) = (ae - a)e = ae^2 - ae = 0$$

and

$$(ea - a)e = e(ea - a) = e^2a - ea = 0,$$

so $ae = ea = a$ because A has no nonzero joint divisor of zero, hence e is a unit for A. □

Proposition 2.5.33 *Let A be an algebraic algebra over $\mathbb{K}$ with no nonzero joint divisor of zero. Then A is unital if (and only if) $Z(A) \neq 0$.*

Proof Assume that $Z(A) \neq 0$. Then, by Propositions 2.5.10(i) and 2.5.17, $Z(A)$ is unital. Now apply Lemma 2.5.32. □

Remark 2.5.34 Let A be an algebra over $\mathbb{K}$. The centre of A is closely related to the centroid, Γ_A, of A (cf. Definition 1.1.10). Indeed, by assigning to each $z \in Z(A)$ the operator of multiplication by z on A, we are provided with a natural algebra homomorphism $\Phi : Z(A) \to \Gamma_A$ such that $\ker(\Phi) = \mathrm{Ann}(A)$ (cf. Definition 1.1.10 again). Moreover Φ is surjective if and only if A has a unit, if and only if Φ is bijective. Thus, when A is unital, A is central if and only if $Z(A) = \mathbb{K}\mathbf{1}$.

2.5.3 The generalized Gelfand–Mazur–Kaplansky theorem

In Definition 1.1.42 we introduced the notion of a division associative algebra in the classical sense, and later, in Definition 2.5.25, we extended this notion to the setting of alternative algebras. Now we will develop other division notions for general non-associative algebras which have appeared in the literature.

Definition 2.5.35 Let A be an algebra. We say that A is a *left-* (respectively, *right-*) *division algebra* if $A \neq 0$ and, for every nonzero element a of A, the operator L_a (respectively, R_a) is bijective. Algebras which are left and right (respectively, left or right) division algebras are called *division algebras* (respectively, *one-sided division algebras*). The algebra A is said to be a *quasi-division algebra* if $A \neq 0$ and, for every nonzero element a of A, at least one of the operators L_a, R_a is bijective. If A is commutative, then, clearly, all the above notions coincide. We note that one-sided division algebras have no nonzero divisor of zero, and that quasi-division algebras have no nonzero two-sided divisor of zero. If A is finite-dimensional, then it is easily realized that A is a division algebra if and only if A is a left-division algebra, if and only if A is a right-division algebra, if and only if A has no nonzero divisor of zero, and that A is a quasi-division algebra if and only if A has no nonzero two-sided divisor of zero. However, even in the finite-dimensional case, quasi-division does not imply one-sided division. Indeed, we have the following.

Example 2.5.36 Let A denote the algebra over $\mathbb{K}$ whose vector space is $\mathbb{K}^2$ and whose product is defined by

$$(\lambda_1,\lambda_2)(\mu_1,\mu_2) := (\lambda_1\mu_2, \lambda_1\mu_1 + \lambda_2\mu_2) .$$

Then A is a quasi-division algebra but not a one-sided division algebra. The verification of this assertion is straightforward.

In the infinite-dimensional setting, one-sided division algebras need not be division algebras. Indeed, we have the following.

Example 2.5.37 Let $\mathbb{K}(\mathbf{x})$ be the algebra of all rational fractions on one indeterminate $\mathbf{x}$ over $\mathbb{K}$. Since $\mathbb{K}(\mathbf{x})$ is infinite-dimensional, there exists an injective non-surjective linear mapping $F : \mathbb{K}(\mathbf{x}) \to \mathbb{K}(\mathbf{x})$. Define a new product $\odot$ on $\mathbb{K}(\mathbf{x})$ by $a \odot b := F(a)b$. With this new product, $\mathbb{K}(\mathbf{x})$ becomes a non-division left-division algebra over $\mathbb{K}$.

It follows from Proposition 2.5.24 that a division alternative algebra in the classical sense (see Definition 2.5.25) becomes a division algebra in the meaning of Definition 2.5.35. As a matter of fact, we have the following.

Proposition 2.5.38 *Let A be an alternative algebra. Then the following conditions are equivalent:*

(i) *A is a quasi-division algebra.*
(ii) *A is a left-division algebra.*
(iii) *A is a right-division algebra.*
(iv) *A is a division algebra.*
(v) *A is a division alternative algebra in the classical sense.*

Proof We already know that condition (v) implies (iv). On the other hand, the implications (iv)$\Rightarrow$(ii)$\Rightarrow$(i) and (iv)$\Rightarrow$(iii)$\Rightarrow$(i) are clear.

(i)$\Rightarrow$(v) Assume that A is a quasi-division algebra. Let b be a nonzero element of A. Then L_b or R_b is a bijective operator (say L_b is bijective). Setting $e := L_b^{-1}(b)$, we have $be = b$, and hence $be = (be)e = be^2$, which implies $e = e^2$. Therefore, e becomes a nonzero idempotent of A, so that, by Lemma 2.5.5, $\mathbf{1} := e$ is a unit for A. Now, let a be in $A\setminus\{0\}$. Then L_a or R_a is bijective (say L_a is bijective), so that there is a unique c in A with $ac = \mathbf{1}$. Since $a(ca - \mathbf{1}) = 0$ (by Theorem 2.3.61), we have also $ca = \mathbf{1}$. It follows that a is invertible in A with $a^{-1} = c$. □

Proposition 2.5.39 *Let A be a normed quasi-division alternative complex algebra. Then A is isomorphic to $\mathbb{C}$.*

Proof By Proposition 2.5.38, A is a division alternative algebra in the classical sense. Let a be in A. By Theorem 1.1.41, there is $\lambda \in \mathrm{sp}(BL(A), L_a)$, which means that $L_{a-\lambda\mathbf{1}} = L_a - \lambda I_A$ is not invertible in $BL(A)$. If $a - \lambda\mathbf{1}$ were not zero, then it would be invertible in A, so, by Proposition 2.5.24, $L_{a-\lambda\mathbf{1}}$ would be a bijective operator on A with $L_{a-\lambda\mathbf{1}}^{-1} = L_{(a-\lambda\mathbf{1})^{-1}} \in BL(A)$, and $L_{a-\lambda\mathbf{1}}^{-1}$ would be an inverse of $L_{a-\lambda\mathbf{1}}$ in $BL(A)$, a contradiction. Therefore $a \in \mathbb{C}\mathbf{1}$. □

Now we prove the so-called *real Gelfand–Mazur theorem*.

Proposition 2.5.40 *Let A be a normed quasi-division associative real algebra. Then A is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

Proof First of all, note that by Proposition 2.5.38, A is a division associative algebra in the classical sense, and hence A has a unit. Let a be in A. By Proposition 1.1.98 and Theorem 1.1.41, there are $\alpha, \beta \in \mathbb{R}$ such that $\alpha + i\beta \in \mathrm{sp}(A_{\mathbb{C}}, a)$. Then, by Proposition 1.1.100, we have that

$$(a - \alpha\mathbf{1})^2 + \beta^2\mathbf{1} \notin \mathrm{Inv}(A).$$

Since A is a division algebra, we deduce that $(a - \alpha\mathbf{1})^2 + \beta^2\mathbf{1} = 0$. Therefore, by the arbitrariness of $a \in A$, we see that A is a quadratic algebra, and Theorem 2.5.29 applies. □

§**2.5.41** Let A be an algebra over $\mathbb{K}$. By a *partially defined centralizer* on A we mean a linear mapping (say f) from a nonzero ideal of A (say $\mathrm{dom}(f)$) to A satisfying $f(xa) = f(x)a$ and $f(ax) = af(x)$ for all $x \in \mathrm{dom}(f)$ and $a \in A$. If f is a partially defined centralizer on A, then it is clear that

$$\ker(f) := \{x \in \mathrm{dom}(f) : f(x) = 0\} \quad \text{and} \quad \mathrm{im}(f) := \{f(x) : x \in \mathrm{dom}(f)\}$$

are ideals of A, and that $\ker(f)\,\mathrm{im}(f) = 0$. The algebra A is said to be *prime* if $A \neq 0$ and the product of two arbitrary nonzero ideals of A is nonzero. If A is prime, then, since $IJ \subseteq I \cap J$ for all ideals I and J of A, it follows that the intersection of two nonzero ideals of A is a nonzero ideal of A.

Lemma 2.5.42 *Let A be a prime algebra over $\mathbb{K}$, and let $\mathscr{C}$ denote the set of all partially defined centralizers on A. Then we have:*

(i) *Every nonzero element f in $\mathscr{C}$ is injective.*
(ii) *If f, g are elements in $\mathscr{C}$ such that $f(x_0) = g(x_0)$ for some nonzero element x_0 in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, then $f(x) = g(x)$ for every $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.*
(iii) *The relation $\cong$ defined on $\mathscr{C}$ by $f \cong g$ if and only if f and g coincide on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ is an equivalence relation.*
(iv) *If for $f, g \in \mathscr{C}$ we define $f + g : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to A$ by $(f+g)(x) := f(x) + g(x)$, then $f + g \in \mathscr{C}$, and $f + g \cong f' + g'$ for all $f', g' \in \mathscr{C}$ with $f \cong f'$ and $g \cong g'$.*
(v) *If for $f, g \in \mathscr{C}$ we define $fg : g^{-1}(\mathrm{dom}(f)) \to A$ by $(fg)(x) := f(g(x))$, then $fg \in \mathscr{C}$, and $fg \cong f'g'$ for all $f', g' \in \mathscr{C}$ with $f \cong f'$ and $g \cong g'$.*

Proof For every $f \in \mathscr{C}$, we know that $\ker(f)$ and $\mathrm{im}(f)$ are ideals of A such that $\ker(f)\,\mathrm{im}(f) = 0$. Since A is prime, it follows that either $f = 0$ or f is injective. Thus assertion (i) is proved. Now, assume that f, g are elements in $\mathscr{C}$ such that $f(x_0) = g(x_0)$ for some nonzero element x_0 in $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. Consider the nonzero ideal of A given by $I := \mathrm{dom}(f) \cap \mathrm{dom}(g)$, and the mapping $h : I \to A$ defined by

$h(x) := f(x) - g(x)$. It is clear that $h \in \mathscr{C}$ and $h(x_0) = 0$. By assertion (i), we conclude that $f(x) = g(x)$ for every $x \in I$, and so assertion (ii) is proved. In order to prove assertion (iii), we note that reflexivity and symmetry being obvious, only transitivity needs proof. To this end, suppose that $f, g, h \in \mathscr{C}$ satisfy $f \cong g \cong h$. Then

$$f(x) = g(x) = h(x) \text{ for every } x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \mathrm{dom}(h),$$

and hence, by assertion (ii), $f \cong h$. In order to prove assertions (iv) and (v), suppose that $f, g \in \mathscr{C}$. Since

$$\mathrm{dom}(f) \cap \mathrm{dom}(g) \text{ and } g^{-1}(\mathrm{dom}(f)) := \{x \in \mathrm{dom}(g) : g(x) \in \mathrm{dom}(f)\}$$

are ideals of A containing $\mathrm{dom}(f)\mathrm{dom}(g)$, it follows that both are nonzero ideals. It is a routine matter to show that $f + g$ and fg (as defined in the statement) are elements of $\mathscr{C}$. Moreover, if $f', g' \in \mathscr{C}$ satisfy $f \cong f'$ and $g \cong g'$, then, for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ and $y \in \mathrm{dom}(g) \cap \mathrm{dom}(g')$, we see that

$$(f + g)(xy) = (f' + g')(xy) \text{ and } (fg)(xy) = (f'g')(xy).$$

Therefore, by assertion (ii), we conclude that $f + g \cong f' + g'$ and $fg \cong f'g'$. $\square$

Definition 2.5.43 Let A be a prime algebra over $\mathbb{K}$. By the above lemma, on the set $\mathscr{C}$ of all partially defined centralizers on A, an equivalence relation $\cong$ is defined, as well as a sum and a composition. Moreover, these operations are compatible with $\cong$. The *extended centroid* C_A of A is defined as the quotient set with the induced operations.

Proposition 2.5.44 *Let A be a prime algebra over $\mathbb{K}$. Then the extended centroid of A is a field extension of $\mathbb{K}$.*

Proof It is straightforward to check that C_A is an associative ring with a unit. We first show that C_A is commutative. Let α, β be in C_A. For $f \in \alpha$ and $g \in \beta$, set $W := g^{-1}(\mathrm{dom}(f)) \cap f^{-1}(\mathrm{dom}(g))$, and choose $x, y \in W$. Then

$$fg(xy) = f(g(x)y) = g(x)f(y) = g(xf(y)) = gf(xy).$$

It follows from Lemma 2.5.42 that $fg \cong gf$, and so $\alpha\beta = \beta\alpha$. Next let α be in $C_A \setminus \{0\}$. Note that, for $f \in \alpha$, we have $f \neq 0$ but $\ker(f) = 0$. Define $g : \mathrm{im}(f) \to A$ by $g(f(x)) = x$ for every $x \in \mathrm{dom}(f)$. The mapping g is well-defined since f is an injection and in fact g is a partially defined centralizer on A. Clearly the equivalent class determined by g is the inverse of α, and hence C_A is a field. Note that for each $\lambda \in \mathbb{K}$, the mapping $x \to \lambda x$ is in fact an everywhere defined centralizer on A, and hence λ determines an element $\hat{\lambda}$ in C_A. Finally, the mapping $\lambda \to \hat{\lambda}$ becomes a field embedding of $\mathbb{K}$ into C_A. $\square$

Remark 2.5.45 Let A be a prime algebra. Thinking about the mapping taking each element f of the centroid of A (cf. Definition 1.1.10) to the equivalence class of $\mathscr{C}$ containing f, we realize that the centroid of A, Γ_A, becomes a subalgebra of C_A. Moreover, if A is actually simple, then, clearly, we have $\Gamma_A = C_A$, and hence, by Proposition 2.5.44, Γ_A is a field extension of $\mathbb{K}$.

Lemma 2.5.46 *Let A be a normed prime complex (respectively, real) algebra, and assume that every partially defined centralizer on A is continuous. Then the extended centroid of A is isomorphic to $\mathbb{C}$ (respectively, to $\mathbb{R}$ or $\mathbb{C}$).*

Proof In view of Corollary 1.1.43 (respectively, Proposition 2.5.40) and Proposition 2.5.44, it is enough to provide C_A with an algebra norm. In fact we will see that, if for α in C_A we define

$$\|\alpha\| := \inf\{\|f\| : f \in \alpha\},$$

then $\|\cdot\|$ becomes an algebra norm on C_A. The property $\|\lambda\alpha\| = |\lambda|\|\alpha\|$, for λ in $\mathbb{C}$ (respectively, in $\mathbb{R}$) and α in C_A, is clear. Let α, β be in C_A. For arbitrary $f \in \alpha$ and $g \in \beta$, consider the partially defined centralizers $f+g$ and fg on A, and note that $f+g \in \alpha+\beta$, $fg \in \alpha\beta$, $\|f+g\| \leqslant \|f\|+\|g\|$, and $\|fg\| \leqslant \|f\|\|g\|$. It follows that

$$\|\alpha+\beta\| \leqslant \|\alpha\|+\|\beta\| \text{ and } \|\alpha\beta\| \leqslant \|\alpha\|\|\beta\|.$$

Now $\|\cdot\|$ is an algebra seminorm on C_A satisfying $\|\mathbf{1}\| = 1$ (where $\mathbf{1}$ denotes the unit of C_A), hence an algebra norm because C_A is a field. □

Proposition 2.5.47 *Let A be a nonzero normed complex (respectively, real) algebra, and assume that every nonzero ideal of A contains an element which is not a joint topological divisor of zero in A. Then A is prime, and the extended centroid of A is isomorphic to $\mathbb{C}$ (respectively, to $\mathbb{R}$ or $\mathbb{C}$).*

Proof Elements x in A satisfying $x^2 = 0$ are joint topological divisors of zero; hence from the assumption on A it follows that, if P is an ideal of A and if $PP = 0$, then $P = 0$. For P and Q ideals of A, from the inclusions

$$QP \subseteq P \cap Q \text{ and } (P\cap Q)(P\cap Q) \subseteq PQ$$

and the above observation it follows that $QP = 0$ whenever $PQ = 0$. Therefore, if $PQ = 0$ and if $P \neq 0$, then every element of Q is a joint topological divisor of zero in A, and so $Q = 0$. Now that we have shown that A is prime, the proof will be concluded by invoking Lemma 2.5.46 and proving that every partially defined centralizer on A is continuous. But, if f is such a partially defined centralizer, then, by Exercise 1.1.88(ii), there exist x in $\mathrm{dom}(f)$ and a positive number m such that

$$m\|y\| \leqslant \|xy\| + \|yx\| \text{ for every } y \in A.$$

Now, for every y in $\mathrm{dom}(f)$, we have

$$m\|f(y)\| \leqslant \|xf(y)\| + \|f(y)x\| = \|f(x)y\| + \|yf(x)\| \leqslant 2\|f(x)\|\|y\|.$$

Hence f is continuous. □

Corollary 2.5.48 *Let A be a nonzero normed complex (respectively, real) algebra with no nonzero joint topological divisor of zero. Then A is prime, and C_A is isomorphic to $\mathbb{C}$ (respectively, to $\mathbb{R}$ or $\mathbb{C}$).*

Proposition 2.5.49 *Let A be a nonzero normed power-associative algebra over $\mathbb{K}$ with no nonzero joint topological divisor of zero. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then A is isomorphic to $\mathbb{C}$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then A is quadratic.*

Proof First assume in addition that A is associative and commutative. Then, identifying each element $a \in A$ with its multiplication operator L_a, elements of A can be regarded as (everywhere defined) centralizers on A. Then the quotient mapping $\mathscr{C} \to C_A$ induces a natural embedding $A \hookrightarrow C_A$. By Corollary 2.5.48, A is finite-dimensional over $\mathbb{K}$.

Now, remove the additional assumption that A is associative and commutative. By the above paragraph, A is algebraic, so the result follows by applying Proposition 2.5.10. □

Since alternative algebras are power-associative (by Theorem 2.3.61), it is enough to combine Proposition 2.5.49(ii) immediately above and Theorem 2.5.29 to derive the following.

Theorem 2.5.50 *Let A be a nonzero normed alternative real algebra with no nonzero joint topological divisor of zero. Then A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

The associative particularization of Theorem 2.5.50 above is known as the *Gelfand–Mazur–Kaplansky theorem*.

Now, we prove the generalization of the real Gelfand–Mazur theorem (Proposition 2.5.40) to alternative algebras.

Corollary 2.5.51 *Let A be a normed quasi-division alternative real algebra. Then A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof In view of Theorem 2.5.50, it is enough to show that A has no nonzero joint topological divisor of zero. Let a be in $A \setminus \{0\}$, and let b_n be a sequence in A such that $ab_n \to 0$. Then, by Propositions 2.5.24 and 2.5.38, we have $b_n = a^{-1}(ab_n) \to 0$. Therefore a is not a joint topological divisor of zero. □

Now that the main results in the section have been proved, we deal with some by-products of the arguments applied in their proofs. Applying Proposition 2.5.49(ii) and Corollary 2.5.16, we get the following.

Proposition 2.5.52 *Let A be a nonzero normed power-associative commutative real algebra with no nonzero topological divisor of zero. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

§2.5.53 Form now on, the following consequence of the Banach isomorphism theorem should be kept in mind: *if A is a nonzero complete normed algebra, and if a is an element of A such that L_a is bijective, then a is not a left topological divisor of zero.*

Noticing that, as a consequence of §2.5.53 above, complete normed quasi-division algebras have no nonzero two-sided topological divisor of zero, Corollaries 2.5.54 and 2.5.55 immediately below follow from Propositions 2.5.49 and 2.5.52, respectively.

Corollary 2.5.54 *Let A be a nonzero complete normed power-associative quasi-division algebra over $\mathbb{K}$. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then A is isomorphic to $\mathbb{C}$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then A is quadratic.*

Assertion (i) in the above corollary will be proved in Corollary 2.7.11 by other methods.

Corollary 2.5.55 *Let A be a nonzero complete normed power-associative division commutative real algebra. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

By a *nearly absolute-valued algebra* we mean a nonzero normed algebra A such that there exists $r > 0$ satisfying $r\|a\|\|b\| \leqslant \|ab\|$ for all $a, b \in A$. Corollaries 2.5.56, 2.5.57, and 2.5.58 immediately below follow straightforwardly from Proposition 2.5.49, Theorem 2.5.50, and Proposition 2.5.52, respectively.

Corollary 2.5.56 *Let A be a nearly absolute-valued power-associative algebra over $\mathbb{K}$. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then A is isomorphic to $\mathbb{C}$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then A is quadratic.*

Corollary 2.5.57 *Let A be a nearly absolute-valued alternative real algebra. Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Corollary 2.5.58 *Let A be a nearly absolute-valued power-associative commutative real algebra. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

Finally we prove the following.

Proposition 2.5.59 *Let A be a normed algebra over $\mathbb{K}$ with no nonzero joint topological divisor of zero. Then A is unital if (and only if) it has nonzero centre.*

Proof Assume that $Z(A) \neq 0$. Then, by Propositions 2.5.49(i) and 2.5.52, $Z(A)$ is unital. Now apply Lemma 2.5.32. □

2.5.4 Historical notes and comments

Quaternions and octonions are widely treated in the books [712], [727], [760], [785], and [801, Chapter 7], as well as in the survey papers [51] and [587]. These works and references therein will provide the reader with a relatively complete panoramic view of the topic. (Other relevant facts concerning quaternions and octonions, not included in the works just quoted, will be covered in Section 2.6 below.) Anyway, let us emphasize the abundance of historical notes and mathematical remarks assembled in [727], and take some excerpts from it. Thus, in a note written for the occasion of the fifteenth birthday of the quaternions, Hamilton says:

> They [the quaternions] started into life, or light, full grown, on the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge.

(see p. 191 of [727]). After the discovery of quaternions, Hamilton devoted the remainder of his life exclusively to their further exploration. An excellent revised version of Hamilton's papers is today available in [737]. It turns out curious to know that Hamilton tried for many years to build a three-dimensional associative real algebra with no nonzero divisors of zero. In fact, shortly before his death in 1865 he wrote to his son:

> Every morning, on my coming down to breakfast, you used to ask me: 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them'.

(see p. 189 of [727]). It is not less curious how, in a very elemental way, one can realize that Hamilton's attempt just quoted could not be successful. Indeed, keeping in mind that every real polynomial of odd degree must have a real root, and arguing as in the solution of Exercise 1.1.86, we realize that *if A is a real algebra of odd finite dimension with no nonzero divisor of zero, then A has dimension* 1*, and hence it is isomorphic to* $\mathbb{R}$.

According to the information included on p. 249 of [727], the octonions were discovered by Graves in December 1843, only two months after the birth of the quaternions. Graves communicated his results to Hamilton in a letter dated 4th January 1844, but they were not published until 1848 [306]. In the meantime, in 1845, the octonions were rediscovered by Cayley, who published his result immediately [164]. For a more detailed history of the discovery of octonions the reader is referred to pp. 146–7 of [51].

According to [765, p. 509] and [104], the description of octonions as pairs of quaternions (the Cayley–Dickson doubling process) can be found in Dickson's paper [211] (1912). The description of octonions as 'vector matrices', as well as the abstract Cayley–Dickson process, are given in a later paper [669] (1933) of Zorn, whereas it was actually Albert (a student of Dickson) who first iterated this construction to produce an infinite family of algebras [7] (1942).

The algebras $\mathbb{A}_n$ $(n \in \mathbb{N} \cup \{0\})$ which appear by applying inductively the Cayley–Dickson doubling process to $\mathbb{A}_0 = \mathbb{R}$ are called *Cayley–Dickson algebras*. As shown by Shafer [551], for each $n > 1$, $\mathbb{A}_n$ is a 2^n-dimensional central simple noncommutative Jordan algebra. But, the fact that the Cayley–Dickson process tends to destroy desirable algebra properties is already visible for the first four Cayley–Dickson algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$. So, for example, $\mathbb{R}$ is the unique Cayley–Dickson algebra with trivial involution, $\mathbb{R}$ and $\mathbb{C}$ are the unique commutative Cayley–Dickson algebras, $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are the unique associative Cayley–Dickson algebras, and $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the unique alternative ones. These first four Cayley–Dickson algebras are classically well-behaved, whereas the larger Cayley–Dickson algebras are considered pathological. Actually, one of the most relevant pathologies is the one given by the following result, due to Moreno [454].

Proposition 2.5.60 *Let $n \geqslant 4$. Then $\mathbb{A}_n$ has nonzero joint divisors of zero. More precisely, $\mathbb{A}_n$ has nonzero divisors of zero, and, for $x, y \in \mathbb{A}_n$, $xy = 0$ if and only if $yx = 0$.*

Over the years, general Cayley–Dickson algebras (with emphasis on the algebra $\mathbb{S} := \mathbb{A}_4$ of *sedenions*) have been considered in several papers by algebraists, as well as by mathematical physicists. Additional references for this matter are [51, 102, 103, 129, 131, 134, 156, 167, 221, 335, 397, 455].

The specialization of Theorem 2.5.29 to the associative setting (that $\mathbb{R}, \mathbb{C}$*, and* $\mathbb{H}$ *are the unique nonzero associative algebraic real algebras with no nonzero joint divisor of zero*) is the celebrated Frobenius' theorem [271] (1878). A new proof of Frobenius' theorem, due to Palais [473] (1968) has attained a certain celebrity, and can be found in Lam [766]. A recent alternative proof of (a generalized version of) Frobenius' theorem has been given by Cuenca [195]. The actual formulation of Theorem 2.5.29 for alternative algebras (called the Frobenius–Zorn theorem) is due to Zorn [668] (1930). A complete proof of the Frobenius–Zorn theorem – where the requirement that the algebra is algebraic is replaced with the apparently stronger one that the algebra is quadratic (see Proposition 2.5.10(ii)) – can be found in [727]. Another complete proof of the Frobenius–Zorn theorem is given in the recent paper by Brešar, Šemrl, and Špenko [131], where a joint abstract characterization of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, and $\mathbb{S}$ is also proved.

Our proof of the Frobenius–Zorn theorem consists of the results from Lemma 2.5.3 to the end of the proof of Theorem 2.5.29. Lemma 2.5.3 is due to Albert [12], and is included in Schafer's book [808, pp. 130–1], whereas, as pointed out in [165], Lemma 2.5.4 is implicitly contained in the proof of [808, Lemma 5.3]. Lemma 2.5.5 is taken from the paper of El-Mallah and Micali [243]. Lemma 2.5.8 is the starting point of Wedderburn's theory (see for example [670, p. 23]). It is far from being trivial that Lemma 2.5.8 remains true if the associativity of the algebra A is altogether removed. Indeed, we have the following theorem, due to Segre [561].

Theorem 2.5.61 *Let A be a nonzero finite-dimensional algebra over $\mathbb{K}$ without isotropic elements. Then A contains a nonzero idempotent.*

Proposition 2.5.10 must be folklore. Its first conclusion, that $\mathbb{C}$ *is the unique nonzero power-associative algebraic complex algebra with no nonzero joint divisor of zero*, can be seen as a refined Frobenius–Zorn type theorem for complex algebras. Proposition 2.5.12 is originally due Dickson [212], and is included in several books. The proof given here is taken from [822]. Proposition 2.5.13 (crucial in the treatment of quadratic algebras) and most of the results from Lemma 2.5.15 to Proposition 2.5.20 are due to Osborn [470]. The remaining part of the results just quoted, as well as Propositions 2.5.21 and 2.5.24, are taken from the books [754, 808, 822]. The core of our proof of the Frobenius–Zorn theorem (consisting of the results from Proposition 2.5.26 to the end of the proof of Theorem 2.5.29), as well as Corollary 2.5.30, is due to Walsh [630] (1967).

When in the definition of cross-product algebras we dispense the requirement that $(x \times y|z) = (x|y \times z)$ for all x, y, z in the algebra, we find the so-called *trigonometric algebras*. Thus, trigonometric algebras are those pre-Hilbert real spaces H endowed with a product $\times$ satisfying $\|x \times y\|^2 = \|x\|^2\|y\|^2 - (x|y)^2$ (or, equivalently, $\|x \times y\| = \|x\|\|y\| \sin\alpha$, where α is the angle between x and y) for all $x, y \in H \setminus \{0\}$. Trigonometric algebras were introduced by Terekhin [610], who showed that *the dimensions of nonzero finite-dimensional trigonometric algebras are precisely* 1, 2, 3, 4, 7, *and* 8. As pointed out in [76, Example 1.1] and [533, p. 143], the existence of complete trigonometric algebras of arbitrary infinite Hilbertian dimension is implicitly known in Remark 1.6 of the Kaidi–Ramírez–Rodríguez paper [369]. Indeed, we have the following.

Example 2.5.62 Let H be an arbitrary infinite-dimensional real Hilbert space (with orthonormal basis $\{u_i : i \in I\}$, say), and let $\times$ stand for the product on H determined by

$$u_i \times u_j := \frac{1}{\sqrt{2}}(u_{\varphi(i,j)} - u_{\varphi(j,i)}),$$

where φ is any prefixed injective mapping from $I \times I$ to I. Then, for $x = \sum_i \lambda_i u_i$ and $y = \sum_i \mu_i u_i$ in H, we have

$$x \times y = \frac{1}{\sqrt{2}} \sum_{i,j} (\lambda_i \mu_j - \mu_i \lambda_j) u_{\varphi(i,j)},$$

and hence

$$\|x \times y\|^2 = \frac{1}{2} \sum_{i,j} (\lambda_i \mu_j - \mu_i \lambda_j)^2 = \|x\|^2 \|y\|^2 - (x|y)^2,$$

so that $(H, \times)$ becomes a trigonometric algebra.

In the Becerra–Rodríguez paper [76], the so-called *super-trigonometric algebras* are considered. These are those real pre-Hilbert spaces H endowed with a product $\times$ satisfying

$$(x \times y | u \times v) = (x|u)(y|v) - (x|v)(y|u) \text{ for all } x, y, u, v \in H.$$

Taking $(u, v) = (x, y)$ in the above equality, we realize that super-trigonometric algebras are trigonometric. Although never previously noticed, it is straightforward to realize that the trigonometric algebra in Example 2.5.62 is in fact super-trigonometric. Therefore *every infinite-dimensional real Hilbert space can be converted into a super-trigonometric algebra under a suitable product*. This fact is derived in [76] from a more general result (see [76, Proposition 2.1]) implying also that *the dimensions of nonzero finite-dimensional super-trigonometric algebras are precisely* 1, 2, *and* 3. As a consequence, by Corollary 2.5.30, cross-product algebras need not be super-trigonometric, and super-trigonometric algebras need not be cross-product algebras. As the main result, it is proved in [76] that all infinite-dimensional trigonometric algebras can be constructed in a transparent way from the absolute-valued real algebras with involution considered in Urbanik's early paper [617].

The notion of a division algebra given in Definition 2.5.35 is an old concept in the theory of general non-associative algebras. It appears for example in Wright's paper [640]. The same happens with the notion of a one-sided division algebra, which appears for example in Kaplansky's paper [377]. On the contrary, the notion of a quasi-division algebra is quite recent. Indeed, we believe it appeared for the first time in [529], where Example 2.5.36 is pointed out. Example 2.5.37 is due to A. Kaidi (recent private communication). Proposition 2.5.38 assures us that, for alternative algebras, all the notions above coincide and, in addition, even coincide with that of a division alternative algebra in the classical sense, previously given in Definition 2.5.25. The equivalences (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v) in Proposition 2.5.38 must be folklore, although we have not found them anywhere. In the particular associative case, these equivalences are proved in [521, p. 936]. Even in the associative case, the crucial implication (i)$\Rightarrow$(v) in Proposition 2.5.38 seems to us to be new.

§**2.5.63** The so-called Gelfand–Mazur theorem consists of Corollary 1.1.43 (that $\mathbb{C}$ *is the unique normed division associative complex algebra*) and Proposition 2.5.40

(that $\mathbb{R}, \mathbb{C}$*, and* $\mathbb{H}$ *are the unique normed division associative real algebras*). According to Zelazko's book [820, p. 18],

> [Proposition 2.5.40] was first announced by S. Mazur in [432]. The first published proof for complex scalars [(the result stated in Corollary 1.1.43)] has been given by I. M. Gelfand in the paper [284] which also contains further basic facts of the theory of commutative Banach algebras.

Between the two above sentences, Zelazko inserts a footnote saying that

> The original manuscript of Mazur's paper contained a complete proof. However, this made the paper too voluminous and the editors of Comptes Rendues required a more concise form. The only sensible way of a shortening was to leave out all the proofs, and the paper was finally so published.

Then, Zelazko reproduces Mazur's original proof in [820, pp. 19–22]. Mazur's proof is also presented by Mazet [430], who took it from the original submission filed in the archives of the Academy of Science in Paris. Mazur's paper [432] contains two other important results (see §§2.6.52 and 2.8.72 below), whose original proofs do not seem to be available, as far as we know.

Remark 2.5.64 We note that the real algebras $\mathbb{H}$ and $\mathbb{O}$ have a one-dimensional centre (namely $\mathbb{R}\mathbf{1}$), and that, consequently, they cannot underlie any complex algebra. Therefore, if $\mathscr{P}$ is a class of real and complex algebras which contains the realifications of its complex members, and whose real members reduce to copies of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, then complex members of $\mathscr{P}$ have to be copies of $\mathbb{C}$. In this way, thinking about the class $\mathscr{P}$ of all normed associative quasi-division real and complex algebras, we realize that the real Gelfand–Mazur theorem (Proposition 2.5.40) contains the complex one (Corollary 1.1.43). Analogously, Corollary 2.5.51 contains Proposition 2.5.39. Similar situations will appear later. Focusing on the most relevant ones, we emphasize that Theorem 2.6.21 contains both Proposition 2.6.17 and the particularization to alternative algebras of Proposition 2.6.2, and that the real parts of Propositions 2.6.25 and 2.6.27 contain their respective complex parts.

The generalization of the Gelfand–Mazur theorem to the case of alternative algebras, given by Proposition 2.5.39 and Corollary 2.5.51, is originally due to Nieto [463] (see also Kaidi [361]).

The specialization of Theorem 2.5.50 to the associative setting (that $\mathbb{R}, \mathbb{C}$*, and* $\mathbb{H}$ *are the unique nonzero normed associative real algebras with no nonzero joint topological divisor of zero*) is due to Kaplansky [375] (1949). In fact, Kaplansky proves the result under the formally stronger requirement of absence of nonzero one-sided topological divisors of zero. The generalization of Kaplansky's theorem to alternative algebras is due to El-Mallah and Micali [243] (1980). The actual formulation of Theorem 2.5.50 is taken from [153]. As far as we know, neither Kaplansky's theorem nor its generalization to alternative algebras have previously been included with a proof in any book. The complex case of Kaplansky's theorem (i.e. the particularization of Proposition 2.5.49 to associative algebras) appears as Theorem 14.4 of Zelazko [820], with a proof taken from [659], whereas the real case is immediately proposed as an exercise. Nevertheless, in both cases, completeness is required, and no reference made to Kaplansky's work.

Our proof of Kaplansky's generalized theorem consists of the Gelfand–Mazur theorem, and the results from Lemma 2.5.42 to the formulation of Theorem 2.5.50 itself. Lemma 2.5.42 and Proposition 2.5.44 are due to Erickson, Martindale III, and Osborn [246], and are included as Theorem 9.2.1 of Beidar–Martindale–Mikhalev [686]. The core of our proof of Kaplansky's generalized theorem (consisting of the results from Lemma 2.5.46 to the formulation of Theorem 2.5.50) is essentially due to Cabrera and Rodríguez [153] (1995). Lemma 2.5.46 follows, with minor variants, some previous ideas of Mathieu [428], later developed in [149]. It is worth mentioning that, although we have invoked the notion introduced in [246] of extended centroid for (possibly non-associative) prime algebras, to arrive at Kaplansky's generalized theorem (and even in germinally more general results, such as Proposition 2.5.49) it is enough to consider extended centroid of associative and commutative prime algebras. Indeed, the proof of Proposition 2.5.49 only involves extended centroid in the associative, commutative and prime setting. We note that the extended centroid of a (possibly non-prime) semiprime associative algebra A is nothing other than the centre of the symmetric Martindale algebra of quotients of A (a topic to be discussed in Volume 2 of this work), and that the extended centroid of a (possibly non-associative non-prime) semiprime algebra is also considered in the literature. The standard references for these topics are the books [702, 686, 794, 819].

The actual organization of the proof of Kaplansky's generalized theorem produced here has generated some new results. These are Propositions 2.5.49, 2.5.52, and 2.5.59, and Corollaries 2.5.54, 2.5.55, 2.5.56, and 2.5.58. The associative forerunner of Corollary 2.5.56(i) (that $\mathbb{C}$ *is the unique nearly absolute-valued associative complex algebra*) is an old result of Shilov [564] (1940). Results related to Propositions 2.5.49 and 2.5.52 will be discussed later (see Theorem 4.1.115).

2.6 Smooth-normed algebras, and absolute-valued unital algebras

Introduction The star result in this section is the so-called non-commutative Urbanik–Wright theorem (a part of Theorem 2.6.21) asserting that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued unital real algebras. We arrive at this result by introducing and describing (in Theorem 2.6.9) smooth-normed algebras, by observing that absolute-valued unital algebras are smooth-normed algebras, and then by selecting from the smooth-normed algebras those which are in fact absolute-valued. As a consequence, we derive Strzelecki's theorem (the remaining part of Theorem 2.6.21) asserting that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique smooth-normed alternative real algebras. Unit-free characterizations of smooth-normed algebras and of absolute-valued unital algebras are also discussed.

2.6.1 Determining smooth-normed algebras and absolute-valued unital algebras

§**2.6.1** Let X be a normed space, and let u be a norm-one element of X. We say that X is *smooth* at u if $D(X,u)$ reduces to a singleton. As an easy consequence of Proposition 2.1.5, X is smooth at u if and only if, for each $x \in X$, there exists $\lim_{r\to 0} \frac{\|u+rx\|-1}{r}$ (the value at x of the *Gâteaux derivative* of the norm of X at u), and, if this is the case, then

$$\lim_{r\to 0} \frac{\|u+rx\|-1}{r} = \Re(f(x)),$$

where f is the unique element of $D(X,u)$.

By a *smooth-normed algebra* we mean a norm-unital normed algebra which is smooth at its unit. As a straightforward consequence of Corollary 2.1.13, we have the following.

Proposition 2.6.2 *$\mathbb{C}$ is the unique smooth-normed complex algebra.*

The determination of smooth-normed real algebras is much more difficult, and will culminate in Theorem 2.6.9 below.

Lemma 2.6.3 *Quadratic algebras are Jordan-admissible. Therefore, flexible quadratic algebras are non-commutative Jordan algebras.*

Proof Let A be a quadratic algebra, and let a,b be in A. Since

$$a^2 = t(a)a - n(a)\mathbf{1},$$

we straightforwardly derive that $(a \bullet b) \bullet a^2 = a \bullet (b \bullet a^2)$, i.e. A is Jordan-admissible. □

Definition 2.6.4 By a *pre-H-algebra* we mean a real pre-Hilbert space E endowed with a bilinear mapping $(x,y) \to x \wedge y$ from $E \times E$ into E (called the product of E) satisfying

$$x \wedge y = -y \wedge x, \ \ \|x \wedge y\| \leqslant \|x\|\|y\|, \ \text{ and } \ (x \wedge y|z) = (x|y \wedge z) \ \text{ for all } \ x,y,z \in E.$$

As an example, every real pre-Hilbert space becomes a pre-H-algebra as soon as we endow it with the zero product.

Let E be an arbitrary pre-H-algebra, consider the real vector space $\mathbb{R} \oplus E$, and define a product on $\mathbb{R} \oplus E$ by

$$(\lambda, x)(\mu, y) := (\lambda\mu - (x|y), \lambda y + \mu x + x \wedge y).$$

In view of Proposition 2.5.18, we obtain in this way a flexible quadratic algebra, which is called the *flexible quadratic algebra of the pre-H-algebra E*, denoted by $\mathscr{A}(E)$. In the application of Proposition 2.5.18, the reader must have noticed that $\mathscr{A}(E)$ is nothing other than $\mathscr{A}(E, \wedge, (\cdot|\cdot))$ in the sense of §2.5.14. In what follows $\mathscr{A}(E) = \mathbb{R} \oplus E$ will be seen endowed with the ℓ_2 norm.

Proposition 2.6.5 *Let E be a pre-H-algebra. Then the flexible quadratic algebra of E is a smooth-normed algebra.*

Proof Clearly $(1,0)$ is a unit for $\mathscr{A}(E)$ and $\|(1,0)\| = 1$. Therefore, since pre-Hilbert spaces are smooth at any norm-one point, it suffices to show that

$$\|(\lambda,x)(\mu,y)\| \leqslant \|(\lambda,x)\|\|(\mu,y)\|.$$

First, we claim that $\|x\wedge y\|^2 \leqslant \|x\|^2\|y\|^2-(x|y)^2$ for all x,y in E. Let x be a nonzero element of E, and write $\alpha=\frac{1}{\|x\|^2}(x|y)$. We have $x\wedge y=x\wedge(y-\alpha x)$ by the anticommutativity of $\wedge$, and hence

$$\|x\wedge y\|^2=\|x\wedge(y-\alpha x)\|^2\leqslant\|x\|^2\|y-\alpha x\|^2=\|x\|^2\|y\|^2-(x|y)^2,$$

as claimed.

Now let (λ,x) and (μ,y) be arbitrary elements in $\mathscr{A}(E)$. Then

$$\begin{aligned}\|(\lambda,x)(\mu,y)\|^2&=\|(\lambda\mu-(x|y),\lambda y+\mu x+x\wedge y)\|^2\\&=(\lambda\mu-(x|y))^2+\|\lambda y+\mu x+x\wedge y\|^2\\&=\lambda^2\mu^2+(x|y)^2+\lambda^2\|y\|^2+\mu^2\|x\|^2+\|x\wedge y\|^2,\end{aligned}$$

where the third equality is true because $(x|x\wedge y)=(x\wedge x|y)=(0|y)=0$ and analogously $(y|x\wedge y)=0$. Finally, by the claim, we have

$$\begin{aligned}\|(\lambda,x)(\mu,y)\|^2&\leqslant\lambda^2\mu^2+(x|y)^2+\lambda^2\|y\|^2+\mu^2\|x\|^2+(\|x\|^2\|y\|^2-(x|y)^2)\\&=(\lambda^2+\|x\|^2)(\mu^2+\|y\|^2)=\|(\lambda,x)\|^2\|(\mu,y)\|^2.\end{aligned}$$ □

Lemma 2.6.6 *Let (X,u) be a real numerical-range space, and let x be in X. Then we have*

$$V(X,u,x)=\Re(V(X_{\mathbb{C}},u,x)).$$

Proof By Lemma 1.1.97, X is a real subspace of the complex normed space $X_{\mathbb{C}}$. Therefore, by Corollary 2.1.2, we have $V(X,u,x)=V((X_{\mathbb{C}})_{\mathbb{R}},u,x)$. It follows from Proposition 2.1.4 that $V(X,u,x)=\Re(V(X_{\mathbb{C}},u,x))$. □

Lemma 2.6.7 *Let A be a smooth-normed real algebra. Then its projective normed complexification $A_{\mathbb{C}}$ is a V-algebra. More precisely, denoting by f the unique element in $D(A,\mathbf{1})$, we have*

$$H(A_{\mathbb{C}},\mathbf{1})=\mathbb{R}\mathbf{1}\oplus i\ker(f).$$

Proof By Proposition 1.1.98, $A_{\mathbb{C}}$ is a norm-unital normed complex algebra. Let f be the unique element in $D(A,\mathbf{1})$, and let x be an arbitrary element in $\ker(f)$. Then we have $V(A,\mathbf{1},x)=0$, and hence, by Lemma 2.6.6, $\Re(V(A_{\mathbb{C}},\mathbf{1},x))=0$ or, equivalently, $ix\in H(A_{\mathbb{C}},\mathbf{1})$. Therefore

$$\mathbb{R}\mathbf{1}\oplus i\ker(f)\subseteq H(A_{\mathbb{C}},\mathbf{1}).$$

Now, since every element $z\in A_{\mathbb{C}}$ can be written as

$$z=\lambda\mathbf{1}+x+i(\mu\mathbf{1}+y)=\lambda\mathbf{1}+iy+i(\mu\mathbf{1}-ix),$$

with λ,μ in $\mathbb{R}$ and x,y in $\ker(f)$, and $\lambda\mathbf{1}+iy$ and $\mu\mathbf{1}-ix$ belong to $H(A_{\mathbb{C}},\mathbf{1})$, we get $A_{\mathbb{C}}=H(A_{\mathbb{C}},\mathbf{1})+iH(A_{\mathbb{C}},\mathbf{1})$, and $A_{\mathbb{C}}$ is indeed a V-algebra. Moreover, if z belongs to $H(A_{\mathbb{C}},\mathbf{1})$, then we have $z-\lambda\mathbf{1}-iy\in H(A_{\mathbb{C}},\mathbf{1})\cap iH(A_{\mathbb{C}},\mathbf{1})=0$, which proves that $H(A_{\mathbb{C}},\mathbf{1})=\mathbb{R}\mathbf{1}\oplus i\ker(f)$. □

The algebra of *complex octonions*, denoted by $C(\mathbb{C})$, can be introduced as the algebra complexification, $\mathbb{C}\otimes\mathbb{O}$, of $\mathbb{O}$. Since the identities defining the variety of alternative algebras are linearizable, $C(\mathbb{C})$ is an alternative complex algebra which is not associative. Although incidental in relation to the main line of the current section, Proposition 2.6.8 immediately below becomes one of the fundamental results in the theory of alternative C^*-algebras.

Proposition 2.6.8 *The algebra of complex octonions $C(\mathbb{C})$, endowed with a suitable norm and a suitable involution, becomes an alternative C^*-algebra.*

Proof Since the absolute value of $\mathbb{O}$ derives from an inner product, $\mathbb{O}$ becomes a smooth-normed real algebra. Therefore, as a by-product of Lemma 2.6.7, $\mathbb{O}_{\mathbb{C}} := \mathbb{C} \otimes_\pi \mathbb{O}$ is a V-algebra. Finally, apply Corollary 2.3.63. □

Now, we retake the main line of the section by proving the following.

Theorem 2.6.9 *Let A be a smooth-normed real algebra. Then there are a pre-H-algebra E and an isometric algebra homomorphism from A onto the flexible quadratic algebra of E.*

Proof Let f be as in Lemma 2.6.7, and let x be in $\ker(f)$. Then, by Lemma 2.6.7, ix is a hermitian element of the V-algebra $A_{\mathbb{C}}$, and, by Theorem 2.3.8, $-x^2 = (ix)^2$ is also a hermitian element of $A_{\mathbb{C}}$. Therefore, a new application of Lemma 2.6.7 gives $-x^2 = \mu\mathbf{1} + iy$ with μ in $\mathbb{R}$ and y in $\ker(f)$. Thus $x^2 + \mu\mathbf{1} \in A \cap iA = 0$, and hence $x^2 = -\mu\mathbf{1}$. Keeping in mind Lemma 2.1.28, and the facts that $V(A, \mathbf{1}, x) = 0$ and $x^2 = -\mu\mathbf{1}$, we conclude that

$$x^2 = -\|x\|^2\mathbf{1}, \tag{2.6.1}$$

and that

$$\|\lambda\mathbf{1} + x\|^2 = \lambda^2 + \|x\|^2 \text{ for every } \lambda \in \mathbb{R}. \tag{2.6.2}$$

It follows clearly from (2.6.1) that $\ker(f)$, endowed with the restriction of the norm of A, is a pre-Hilbert space, the inner product of which will be denoted by $(\cdot|\cdot)$. Moreover, it follows from (2.6.1) and (2.6.2) that

$$a^2 - 2f(a)a + \|a\|^2\mathbf{1} = 0 \text{ for every } a \in A.$$

Therefore A is a quadratic algebra with $t(a) = 2f(a)$ and $n(a) = \|a\|^2$ for every $a \in A$. Thus, by Proposition 2.5.13, we have $A = \mathscr{A}(E, \wedge, (\cdot|\cdot))$, where $E := \ker(f)$, and $\wedge$ is an anticommutative product on E satisfying

$$xy = -(x|y)\mathbf{1} + x \wedge y \text{ for all } x, y \in E. \tag{2.6.3}$$

Since A is flexible (by Theorem 2.4.11), it follows from Proposition 2.5.18(ii) that

$$(x \wedge y|z) = (x|y \wedge z) \text{ for all } x, y, z \in E. \tag{2.6.4}$$

Moreover, since $x \wedge y = \frac{1}{2}(xy - yx)$ (by (2.6.3)), we have

$$\|x \wedge y\| \leqslant \|x\|\|y\|. \tag{2.6.5}$$

From (2.6.4) and (2.6.5) it is clear that the pre-Hilbert space E, endowed with the product $\wedge$ is a pre-H-algebra, and that the mapping $\lambda\mathbf{1} + x \to (\lambda, x)$, from A $(= \mathbb{R}\mathbf{1} \oplus E)$ to the flexible quadratic algebra $\mathscr{A}(E)$ of the pre-H-algebra E, is a bijective algebra homomorphism (by also using (2.6.3)). Therefore, to conclude the proof, it suffices to show that this mapping is an isometry. But this follows from (2.6.2). □

The following corollary exhibits the strength of Theorem 2.6.9.

Corollary 2.6.10 *Let A be a smooth-normed real algebra. Then we have:*

(i) *A is a non-commutative Jordan algebra.*
(ii) *The norm of A comes from an inner product $(\cdot|\cdot)$, and the equality*

$$a^2 - 2(a|\mathbf{1})a + \|a\|^2\mathbf{1} = 0$$

holds for every $a \in A$.
(iii) *for each $a \in A \setminus (\mathbb{R}\mathbf{1})$, the linear hull of $\{\mathbf{1}, a\}$ is a subalgebra of A isometrically isomorphic to $\mathbb{C}$.*
(iv) *A is a Cayley algebra whose standard involution $*$ becomes isometric.*
(v) *For each norm-one element $u \in A$, the operator U_u is a surjective linear isometry with $U_u^{-1} = U_{u^*}$.*

Proof Keeping in mind Theorem 2.6.9, we can write $A = \mathscr{A}(E)$ for some pre-H-algebra E, so the proof of the present corollary becomes almost straightforward. However, we will supply some indications. For example, Lemma 2.6.3 (respectively, Proposition 2.5.20) should be applied in the verification of assertion (i) (respectively, assertion (iv)). Assertion (ii) follows by writing $a = \rho\mathbf{1} + y$ with $\rho \in \mathbb{R}$ and $y \in E$, whereas assertion (iii) follows by writing $a = \lambda_0\mathbf{1} + \mu_0 x$ with $\lambda_0, \mu_0 \in \mathbb{R}$ and $x \in \mathbb{S}_E$, and then by realizing that the mapping $\lambda + i\mu \to \lambda\mathbf{1} + \mu x$ becomes an isometric algebra homomorphism from $\mathbb{C}$ to A whose range is the linear hull of $\{\mathbf{1}, a\}$. The verification of assertion (v) becomes easier if one knows that, since A is flexible, for each $a \in A$ the operator U_a has the same meaning in both A and A^{sym} (cf. Fact 3.3.3 below for details). Then we can assume that A is commutative, and hence that E is a real pre-Hilbert space endowed with the zero product. In this case, it is routine that $U_a(b) = 2(a|b^*)a - \|a\|^2 b^*$ for all $a, b \in A$, a fact from which assertion (v) follows straightforwardly. □

Now we prove a Kadison-type theorem for isometries of smooth-normed algebras.

Proposition 2.6.11 *Let A and B be smooth-normed real algebras. We have:*

(i) *The unit-preserving (possibly non-surjective) linear isometries from A to B are precisely the nonzero Jordan homomorphisms.*
(ii) *If $B \neq \mathbb{R}\mathbf{1}$, then the surjective linear isometries from A to B are precisely the mappings of the form $U_u \circ F$, where u is a norm-one element of B and $F : A \to B$ is a bijective Jordan homomorphism.*

Proof Let $F : A \to B$ be any mapping, and let a be in A. Then, by Corollary 2.6.10(ii), A and B are pre-Hilbert spaces, and we have

$$a^2 - 2(a|\mathbf{1})a + \|a\|^2\mathbf{1} = 0 \tag{2.6.6}$$

and

$$F(a)^2 - 2(F(a)|\mathbf{1})F(a) + \|F(a)\|^2\mathbf{1} = 0. \tag{2.6.7}$$

As a consequence of (2.6.6), if F is linear and preserves units, then

$$F(a^2) - 2(a|\mathbf{1})F(a) + \|a\|^2\mathbf{1} = 0.$$

Now note that, in the case that F is actually a unit-preserving linear isometry, the last equality reads as

$$F(a^2) - 2(F(a)|\mathbf{1})F(a) + \|F(a)\|^2\mathbf{1} = 0.$$

It follows from (2.6.7) that, in this last case, we have $F(a^2) = F(a)^2$, and hence F is a (nonzero) Jordan homomorphism. Conversely, assume that F is a nonzero Jordan homomorphism. Then, $F(\mathbf{1})$ is a nonzero idempotent in B, so that, by Corollary 2.6.10(iii), we have $F(\mathbf{1}) = \mathbf{1}$, and hence $\|F(a)\| = \|a\|$ whenever a belongs to $\mathbb{R}\mathbf{1}$. If $a \notin \mathbb{R}\mathbf{1}$, then, by (2.6.6), we have

$$F(a)^2 - 2(a|\mathbf{1})F(a) + \|a\|^2\mathbf{1} = 0,$$

which, together with (2.6.7), also yields $\|F(a)\| = \|a\|$. Therefore F is a unit-preserving linear isometry, and the proof of assertion (i) is complete.

If u is a norm-one element of B, and if $F : A \to B$ is a bijective Jordan homomorphism, then, by assertion (i) just proved and Corollary 2.6.10(v), $U_u \circ F$ is a surjective linear isometry. Assume that $B \neq \mathbb{R}\mathbf{1}$, and let $G : A \to B$ be any surjective linear isometry. Then $v := G(\mathbf{1})$ is a norm-one element of B, so that, by Corollary 2.6.10(iii), there exists an isometric copy of $\mathbb{C}$ (say C) in B containing v. Take $u \in C$ such that $u^2 = v$, note that u is a norm-one element of B, and set $F := U_{u^*} \circ G$. Then we clearly have $F(\mathbf{1}) = \mathbf{1}$. Moreover, by Corollary 2.6.10(v), F is a surjective linear isometry from A to B and the equality $G = U_u \circ F$ holds. Finally, by assertion (i), F is a bijective Jordan homomorphism. This concludes the proof of assertion (ii). □

Corollary 2.6.12 *Let $\mathbb{A}$ stand for either $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. Then the linear isometries from $\mathbb{A}$ to $\mathbb{A}$ are precisely the mappings of the form $L_u \circ F$, where u is a norm-one element of $\mathbb{A}$ and $F : \mathbb{A} \to \mathbb{A}$ is a bijective Jordan homomorphism.*

Proof First of all, we recall that $\mathbb{A}$ is an alternative Cayley real algebra, as well as an absolute-valued algebra whose norm derives from an inner product, and that these facts imply that $\mathbb{A}$ is a smooth-normed algebra. If u is a norm-one element of B, and if $F : A \to B$ is a bijective Jordan homomorphism, then, by Proposition 2.6.11(i), $L_u \circ F$ is a linear isometry. Conversely, let $G : \mathbb{A} \to \mathbb{A}$ be a linear isometry. Set $u := G(\mathbf{1})$. Then u is a norm-one element of $\mathbb{A}$, so that, by setting $F := L_{u^*} \circ G$, F becomes a surjective linear isometry. Therefore, since $F(\mathbf{1}) = \mathbf{1}$, Proposition 2.6.11(i) applies, so F is a bijective Jordan homomorphism. Finally, the equality $G = L_u \circ F$ is clear. □

Complete pre-H-algebras will be called *H-algebras*. As a straightforward consequence of Proposition 2.6.5 and Theorem 2.6.9, we obtain the following.

Corollary 2.6.13 *Complete smooth-normed real algebras are precisely the flexible quadratic algebras of H-algebras.*

We recall that every real pre-Hilbert space with zero product is a pre-H-algebra. Therefore, by Proposition 2.6.5, every nonzero real pre-Hilbert space X can be structured as a norm-unital normed algebra with arbitrary prefixed unit u in X such that $\|u\| = 1$. (Indeed, take E equal to the orthogonal complement of $\mathbb{R}u$ in X, endow E

with the zero product, and see X as $\mathscr{A}(E)$.) Moreover, the smoothness of the pre-Hilbert space X at u is well known. Conversely, by Corollary 2.6.10(ii), the normed space of a smooth-normed algebra is a pre-Hilbert space. In this way, we get the following characterization of real pre-Hilbert spaces. (For verification of the last sentence, the linearization of Corollary 2.6.10(ii) could be useful.)

Corollary 2.6.14 *Let X be a nonzero real normed space. Then the following conditions are equivalent:*

(i) *X is a pre-Hilbert space.*
(ii) *If u is any norm-one element in X, then X is smooth at u and the set of continuous bilinear mappings $f : X \times X \to X$, satisfying $\|f\| = 1$ and $f(x,u) = f(u,x) = x$ for every x in X, is not empty.*
(iii) *There is a norm-one element u in X such that X is smooth at u, and the set of continuous bilinear mappings $f : X \times X \to X$, satisfying $\|f\| = 1$ and $f(x,u) = f(u,x) = x$ for every x in X, is not empty.*

Moreover, if X is in fact a pre-Hilbert space, and if u is a norm-one element in X, then the mapping

$$(x,y) \to (x|u)y + (y|u)x - (x|y)u$$

is the unique commutative product on X converting X into a norm-unital normed algebra with unit u.

Corollary 2.6.15 *Let A be a norm-unital normed algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *A is a smooth-normed algebra.*
(ii) *$\|\mathbf{1}+a\|\|\mathbf{1}-a\| = \|\mathbf{1}-a^2\|$ for every a in A.*
(iii) *$\|U_a(b)\| = \|a\|^2\|b\|$ for all a, b in A.*

Proof If $\mathbb{K} = \mathbb{C}$, then conditions (ii) and (iii) follow from condition (i) by applying Proposition 2.6.2. If $\mathbb{K} = \mathbb{R}$, then conditions (ii) and (iii) follow from condition (i) by applying Corollary 2.6.10(iii)–(v).

Assume that condition (ii) holds. Let a and r be in A and $\mathbb{R} \setminus \{0\}$, respectively. Then we have

$$\begin{aligned}\frac{\|\mathbf{1}-r^2a^2\|-1}{r} &= \frac{\|\mathbf{1}+ra\|\|\mathbf{1}-ra\|-1}{r}\\ &= \frac{\|\mathbf{1}+ra\|-1}{r}\|\mathbf{1}-ra\| + \frac{\|\mathbf{1}-ra\|-1}{r}.\end{aligned}$$

Therefore, by Proposition 2.1.5, we get

$$0 = \max \Re(V(A,\mathbf{1},a)) - \min \Re(V(A,\mathbf{1},a)). \tag{2.6.8}$$

Thus $V(A,\mathbf{1},a)$ is reduced to a point (in the case $\mathbb{K} = \mathbb{C}$, apply (2.6.8) to both a and ia). By the arbitrariness of $a \in A$, we derive that $D(A,\mathbf{1})$ is reduced to a singleton, and hence that A is smooth.

Now assume that (iii) holds. Let a and r be in A and $\mathbb{R}\setminus\{0\}$, respectively. Since $U_{\mathbf{1}-a}(\mathbf{1}+a)=\mathbf{1}-a-a^2+a^2a$, we have

$$\begin{aligned}\frac{\|\mathbf{1}-ra-r^2a^2+r^3a^2a\|-1}{r} &= \frac{\|\mathbf{1}-ra\|^2\|\mathbf{1}+ra\|-1}{r}\\ &= \frac{\|\mathbf{1}-ra\|^2-1}{r}\|\mathbf{1}+ra\|+\frac{\|\mathbf{1}+ra\|-1}{r}.\end{aligned}$$

Therefore, by Corollary 2.1.6, we get

$$-\min\Re(V(A,\mathbf{1},a)=-2\min\Re(V(A,\mathbf{1},a))+\max\Re(V(A,\mathbf{1},a)),$$

so $\min\Re(V(A,\mathbf{1},a))=\max\Re(V(A,\mathbf{1},a))$, and so A is smooth. □

As a consequence of the implication (ii)⇒(i) in the above corollary, we get the following.

Fact 2.6.16 *Absolute-valued unital algebras are smooth-normed algebras.*

By combining Fact 2.6.16 above and Proposition 2.6.2, we derive the following.

Proposition 2.6.17 *$\mathbb{C}$ is the unique absolute-valued unital complex algebra.*

A better result will be proved later (see Corollary 2.7.17).

The following corollary is contained in both Theorems 2.6.21 and 2.6.41 below. Its placement here has only a methodological interest.

Corollary 2.6.18 *$\mathbb{R}$ and $\mathbb{C}$ are the unique absolute-valued unital commutative real algebras.*

Proof Let A be a unital absolute-valued real algebra. By Fact 2.6.16 and Theorem 2.6.9, A is quadratic. Therefore, if in addition A is commutative, then, by Corollary 2.5.16, A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. The isomorphism has to be an isometry in view of Proposition 2.6.19 immediately below. □

Proposition 2.6.19 *Let A be an algebra over $\mathbb{K}$ endowed with a norm satisfying $\|a^2\|\leqslant\|a\|^2$ for every $a\in A$, let B be an algebra over $\mathbb{K}$ endowed with a norm satisfying $\|b^2\|\geqslant\|b\|^2$ for every $b\in B$, and let $\phi:A\to B$ be a continuous Jordan homomorphism. Then ϕ is in fact contractive.*

Proof Assume to the contrary that ϕ is not contractive. Then we can choose a norm-one element x in A such that $\|\phi(x)\|>1$. Therefore, defining inductively $x_1:=x^2$ and $x_{n+1}:=x_n^2$, we have $\|\phi(x_n)\|\geqslant\|\phi(x)\|^{2^n}\to\infty$. Since $\|x_n\|\leqslant 1$, this contradicts the assumed continuity of ϕ. □

Lemma 2.6.20 *Let E be a pre-H-algebra. Then the following conditions are equivalent:*

(i) *$\mathscr{A}(E)$ is an absolute-valued algebra.*
(ii) *$\mathscr{A}(E)$ is alternative.*

Proof Assume that $\mathscr{A}(E)$ is an absolute-valued algebra. Then, since the norm of $\mathscr{A}(E)$ comes from an inner product, the mapping $n:a\to\|a\|^2$ becomes a non-degenerate quadratic form on $\mathscr{A}(E)$ admitting composition, and satisfying $n(a)>0$

for every $a \in \mathscr{A}(E)$. Therefore, by the implication (v)⇒(iii) in Proposition 2.5.26, $\mathscr{A}(E)$ is alternative.

Now, assume that $\mathscr{A}(E)$ is alternative. Then, noticing that

$$\mathscr{A}(E) = \mathscr{A}(E, \wedge, (\cdot|\cdot))$$

in the sense of §2.5.14, Corollary 2.5.19(i) applies, so that the algebraic norm function n on $\mathscr{A}(E)$ admits composition. Since, by Proposition 2.5.18(i), we have $n(a) = \|a\|^2$ for every $a \in \mathscr{A}(E)$, it follows that $\mathscr{A}(E)$ is indeed an absolute-valued algebra. □

Now we prove the main result in this section.

Theorem 2.6.21 *Let A be a real algebra. Then the following statements are equivalent:*

(i) *A is an absolute-valued unital algebra.*
(ii) *A is an alternative smooth-normed algebra.*
(iii) *$A = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ with its usual modulus as norm.*

Proof The implication (iii)⇒(i) is clear.

The implication (i)⇒(ii) in the present theorem follows from Fact 2.6.16, Theorem 2.6.9, and the implication (i)⇒(ii) in Lemma 2.6.20, whereas the implication (ii)⇒(i) in the present theorem follows from Theorem 2.6.9 and the implication (ii)⇒(i) in Lemma 2.6.20.

Now that we know that statements (i) and (ii) are equivalent, assume that they hold. Then, by (i), A has no nonzero joint divisor of zero and, by (ii) and Theorem 2.6.9, A is alternative and quadratic. Therefore, by the Frobenius–Zorn theorem (Theorem 2.5.29), A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. Finally, by Proposition 2.6.19, the isomorphism has to be an isometry. □

As a straightforward consequence of the implication (ii)⇒(iii) in Theorem 2.6.21, we derive the following.

Corollary 2.6.22 *$\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras whose norm comes from an inner product.*

Definition 2.6.23 Two absolute-valued algebras A and B over $\mathbb{K}$ are said to be *Albert isotopic* if there exist linear isometries ϕ_1, ϕ_2, ϕ_3 from A onto B satisfying $\phi_1(xy) = \phi_2(x)\phi_3(y)$ for all x, y in A.

Corollary 2.6.24 *Let A be an absolute-valued real algebra. Then the following conditions are equivalent:*

(i) *A is finite-dimensional.*
(ii) *A is a division algebra.*
(iii) *There exist $a, b \in A$ such that $aA = Ab = A$.*
(iv) *There exist $a, b \in A$ such that aA and Ab are dense in A.*
(v) *A is Albert isotopic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof The implications (i)⇒(ii)⇒(iii)⇒(iv), as well as (v)⇒(i), are clear.

(iv)⇒(v) Assume that condition (iv) holds. Replacing A with its completion (which is clearly an absolute-valued algebra), we may assume that A is complete. Moreover, we may also assume that $\|a\| = \|b\| = 1$. Then the operators L_a and R_b are linear isometries with dense range on the Banach space of A, and hence they are surjective. Now, defining a new product $\odot$ on A by $x \odot y := R_b^{-1}(x)L_a^{-1}(y)$, we obtain an absolute-valued real algebra, which is Albert isotopic to A, and has a unit (namely, ab). Finally, apply Theorem 2.6.21. □

2.6.2 Unit-free characterizations of smooth-normed algebras, and of absolute-valued unital algebras

Now, we are going to prove unit-free characterizations of smooth-normed algebras, and of absolute-valued unital algebras. The next result follows straightforwardly from Propositions 2.5.59 and 2.6.17, and Theorem 2.6.21.

Proposition 2.6.25 *$\mathbb{C}$ is the unique absolute-valued complex algebra with nonzero centre, whereas $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued real algebras with nonzero centre.*

The following lemma follows straightforwardly from Proposition 2.6.25. Nevertheless, the elementary proof we give here has its own interest.

Lemma 2.6.26 *Let A be an absolute-valued, associative, and commutative algebra over $\mathbb{R}$. Then A is equal to $\mathbb{R}$ or $\mathbb{C}$.*

Proof Since A is an integral domain, we can consider the field of fractions of A (say $\mathbb{F}$), and extend (in the unique possible way) the absolute value of A to an absolute value on $\mathbb{F}$. Now $\mathbb{F}$ is an absolute-valued field extension of $\mathbb{R}$, and hence it is isometrically isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (as a by-product of Corollary 2.6.18). Since A is a subalgebra of $\mathbb{F}$, the result follows. □

Proposition 2.6.27 *Let A be an absolute-valued power-associative real* (*respectively, complex*) *algebra. Then A is equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (respectively, to $\mathbb{C}$).*

Proof Let A be an absolute-valued power-associative algebra over $\mathbb{K}$. By Lemma 2.6.26, there exists a nonzero idempotent in A. Then, by Lemma 2.5.5, A has a unit. Finally, by Theorem 2.6.21 (respectively, by Proposition 2.6.17), A is equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (respectively, to $\mathbb{C}$). □

§**2.6.28** Let A be an algebra over $\mathbb{K}$. We say that A is *algebraic of bounded degree* if there exists a non-negative integer number m such that the subalgebra of A generated by any element of A has dimension $\leqslant m$. If this is the case, then the smallest such number m is called the *degree* of A, and is denoted by $\deg(A)$. Otherwise, we write $\deg(A) = \infty$.

Lemma 2.6.29 *Let A be an algebraic algebra over $\mathbb{K}$ of degree* 1 *and having no isotropic element. Then A is isomorphic to $\mathbb{K}$.*

Proof Since A is of degree 1, for each $a \in A$ we have $a^2 \in \mathbb{K}a$. As a consequence, the unital extension $\mathbb{K}\mathbb{1} \oplus A$ of A is a quadratic algebra, so that by Proposition 2.5.12 its trace function t is linear. Since for $a \in A$ we have $a^2 = t(a)a$, and A is a nonzero algebra without isotropic elements, we deduce that the restriction of t to A is a nonzero linear functional with zero kernel. Therefore A is a one-dimensional algebra over $\mathbb{K}$ with nonzero product, and Exercise 1.1.2 applies. □

Lemma 2.6.30 *Let A be a nonzero algebraic complex algebra with no nonzero divisor of zero. Then A is isomorphic to $\mathbb{C}$.*

Proof By Exercise 1.1.86, for each $a \in A$ we have $a^2 \in \mathbb{C}a$. Therefore A is of degree 1, and Lemma 2.6.29 applies. □

By invoking Proposition 2.6.19, we straightforwardly deduce the following.

Corollary 2.6.31 *Let A be a nearly absolute-valued algebraic complex algebra. Then A is isomorphic to $\mathbb{C}$. If A is in fact absolute-valued, then $A = \mathbb{C}$.*

Lemma 2.6.32 *Let A be an algebraic algebra over $\mathbb{K}$ of bounded degree. Then every family of pairwise orthogonal nonzero idempotents in A is finite, with cardinal $\leqslant \deg(A)$.*

Proof Let $\{e_1, \dots, e_n\}$ be a finite family of pairwise orthogonal nonzero idempotents in A. Choose pairwise different nonzero elements $\lambda_1, \dots, \lambda_n$ in $\mathbb{K}$, and set $x := \sum_{i=1}^n \lambda_i e_i$. Then for $j \in \mathbb{N}$ we have $x^j := \sum_{i=1}^n \lambda_i^j e_i$. Therefore $\{x^1, \dots, x^n\}$ is a linearly independent system of elements of A. Since this system is contained in the subalgebra of A generated by x, we deduce $n \leqslant \deg(A)$. □

Lemma 2.6.33 *Let A be a power-associative algebraic algebra over $\mathbb{K}$ of bounded degree and having no isotropic element. Then A has a unit.*

Proof We may assume $A \neq 0$. Then A has nonzero idempotents. Indeed, the subalgebra of A generated by any nonzero element is a finite-dimensional associative algebra without isotropic elements, and hence, by Lemma 2.5.8, has a nonzero idempotent. Choose a maximal family $\mathscr{F}$ of pairwise orthogonal nonzero idempotents in A. By Lemma 2.6.32 we have $\mathscr{F} = \{e_1, \dots, e_n\}$ for some $n \in \mathbb{N}$. Set $e := \sum_{i=1}^n e_i$, and note that e_i belongs to $A_1(e)$ for $i = 1, \dots, n$.

Now, assume that A is commutative. Then, by Lemma 2.5.3(iii), $A_1(e)$ and $A_0(e)$ are orthogonal subalgebras of A. If $A_0(e) \neq 0$, then, as happened to A, $A_0(e)$ has a nonzero idempotent, which is orthogonal to e_i for $i = 1, \dots, n$, leading to a contradiction. Now that we know that $A_0(e) = 0$, the absence of isotropic elements in A and Lemma 2.5.4 give that $A_{\frac{1}{2}}(e) = 0$, so $A = A_1(e)$, and so e is a unit for A.

Now, remove the assumption that A is commutative. Then, since A^{sym} is a power-associative commutative algebraic algebra over $\mathbb{K}$ of bounded degree and having no isotropic element, the above paragraph applies, so that A^{sym} has a unit (say $\mathbf{1}$), which becomes an idempotent of A. Therefore, since $(A^{\mathrm{sym}})_1(\mathbf{1}) = A_1(\mathbf{1})$ (by Lemma 2.5.3(i)), we have $A = A_1(\mathbf{1})$, that is, $\mathbf{1}$ is a unit for A. □

Lemma 2.6.34 *Let A be a nonzero commutative real algebra endowed with a norm $\|\cdot\|$ which comes from an inner product and satisfies $\|x^2\| = \|x\|^2$ for every $x \in A$.*

Then, for $x,y \in A$, we have

$$|(x|y)| \leqslant \|xy\| \leqslant \|x\|\|y\|, \tag{2.6.9}$$

and hence A is a normed algebra. Moreover, if in addition A is associative, then A is isometrically isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

Proof Let x,y be in A. Then, from

$$\|x+y\|^2 = \|x^2+y^2+2xy\| \leqslant \|x\|^2+\|y\|^2+2\|xy\|$$

we obtain $(x|y) \leqslant \|xy\|$, and, replacing x with $-x$, we get $|(x|y)| \leqslant \|xy\|$, which proves the first inequality in (2.6.9). To prove the second one, we may assume that $\|x\| = \|y\| = 1$. Then we have

$$4\|xy\| = \|(x+y)^2-(x-y)^2\| \leqslant \|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2) = 4,$$

so $\|xy\| \leqslant 1$, as desired. Assume additionally that A is associative. Then, in view of Theorem 2.5.50 and Proposition 2.6.19, to conclude the proof of our lemma it is enough to show that A has no nonzero (joint) topological divisor of zero. Assume to the contrary that there exists a norm-one element $a \in A$, and a sequence x_n in A with $\|x_n\| = 1$ and $ax_n \to 0$. Then, by (2.6.9), we have

$$|(a|x_n)| \leqslant \|ax_n\| \to 0, \tag{2.6.10}$$

and, since A is associative, we also have

$$|(a^2|x_n^2)| \leqslant \|a^2x_n^2\| = \|ax_n\|^2 \to 0. \tag{2.6.11}$$

Therefore, from (2.6.10) and (2.6.11), we derive

$$\|a+x_n\|^4 = (1+2(a|x_n)+1)^2 \to 4$$

and

$$\begin{aligned}\|a+x_n\|^4 &= \|a^2+2ax_n+x_n^2\|^2 \\ &= 1+4\|ax_n\|^2+1+2(a^2|x_n^2)+4(a^2|ax_n)+4(x_n^2|ax_n) \to 2,\end{aligned}$$

respectively, which becomes a contradiction. □

Proposition 2.6.35 *Let A be a nonzero alternative real algebra endowed with a norm which derives from an inner product and satisfies $\|x^2\| = \|x\|^2$ for every $x \in A$. Then A is equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Proof Let a be in $A \setminus \{0\}$. By the last conclusion in Lemma 2.6.34, the subalgebra of A generated by a (say A_a) is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. As a first consequence, by the arbitrariness of $a \in A \setminus \{0\}$, and Lemma 2.6.33, A has a unit $\mathbf{1}$. Let e_a stand for the unit of A_a. Then e_a and $\mathbf{1}-e_a$ are mutually orthogonal idempotents in A, so their linear hull is a commutative subalgebra of A, and so by (2.6.9) in Lemma 2.6.34, we have $(e_a|\mathbf{1}-e_a) = 0$, and hence $\|\mathbf{1}\|^2 = \|e_a\|^2+\|\mathbf{1}-e_a\|^2$. Noticing that nonzero idempotents in A have to be norm-one elements, the above implies $e_a = \mathbf{1}$. Again, by the arbitrariness of $a \in A \setminus \{0\}$, we realize that A is both a quadratic algebra and a division alternative algebra in the classical sense. Therefore, by Proposition 2.5.38 and Theorem 2.5.29, A is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Finally, the isomorphism is isometric thanks to Proposition 2.6.19. □

Corollary 2.6.36 *Let A be a nonzero normed algebra such that there exists a norm-one unit $\mathbf{1}$ for A^{sym}. Then $\mathbf{1}$ is a unit for A.*

Proof By our assumptions, both $L_{\mathbf{1}}$ and $R_{\mathbf{1}}$ lie in the closed unit ball of the norm-unital normed algebra $BL(A)$, and the equality $\frac{1}{2}(L_{\mathbf{1}}+R_{\mathbf{1}})=I_A$ holds. It follows from Corollary 2.1.42 that $L_{\mathbf{1}}=R_{\mathbf{1}}=I_A$, that is, $\mathbf{1}$ is a unit element for A. □

Lemma 2.6.37 *Let A be a nonzero normed algebra satisfying $\|x^2\|=\|x\|^2$ for every $x\in A$, and such that A^{sym} is power-associative and algebraic of bounded degree. Then A has a norm-one unit.*

Proof By Lemma 2.6.33, A^{sym} has a unit element (say $\mathbf{1}$). Moreover, since $\|\mathbf{1}\|=\|\mathbf{1}^2\|=\|\mathbf{1}\|^2$, we have $\|\mathbf{1}\|=1$. Now, apply Corollary 2.6.36. □

Proposition 2.6.38 *Let A be a nonzero normed real algebra. Then the following conditions are equivalent:*

(i) *A is a smooth-normed algebra.*
(ii) *A is power-associative, and the equality $\|U_x(y)\|=\|x\|^2\|y\|$ holds for all $x,y\in A$.*

Proof The implication (i)⇒(ii) follows from Corollary 2.6.10 and the implication (i)⇒(iii) in Corollary 2.6.15. Assume that condition (ii) is fulfilled. Then for x,y in any subalgebra of A generated by a single element, we have

$$\|x\|^2\|y\|=\|U_x(y)\|=\|xyx\|\leqslant\|xy\|\|x\|.$$

Therefore, all subalgebras of A generated by a single element are absolute-valued algebras, and, as a by-product, the equality $\|x^2\|=\|x\|^2$ holds for every $x\in A$. Now, by Lemma 2.6.26, A is algebraic of bounded degree. It follows from Lemma 2.6.37 that A has a norm-one unit. Finally, by the implication (iii)⇒(i) in Corollary 2.6.15, A is a smooth-normed algebra. □

Proposition 2.6.39 *Let A be a nonzero normed real algebra. Then the following conditions are equivalent:*

(i) *A is a smooth-normed algebra.*
(ii) *A is power-associative, the norm of A derives from an inner product, and the equality $\|x^2\|=\|x\|^2$ holds for every $x\in A$.*
(iii) *A^{sym} is power-associative, the norm of A derives from an inner product, and the equality $\|x^2\|=\|x\|^2$ holds for every $x\in A$.*

Proof The implication (i)⇒(ii) follows straightforwardly from Theorem 2.6.9 (see also Corollary 2.6.10), whereas the one (ii)⇒(iii) follows from Corollary 2.4.18. Assume that condition (iii) is fulfilled. Then, by Lemma 2.6.34, the algebra A^{sym} is algebraic of bounded degree. Therefore, by Lemma 2.6.37, A has a norm-one unit. Since pre-Hilbert spaces are smooth at all their norm-one elements, it follows that A is a smooth-normed algebra. □

The next corollary follows straightforwardly from the implication (iii)⇒(i) in Proposition 2.6.39 above, and the implication (i)⇒(iii) in Theorem 2.6.21.

Corollary 2.6.40 *Let A be an absolute-valued real algebra whose norm derives from an inner product, and such that A^{sym} is power-associative. Then A is equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

2.6.3 Historical notes and comments

As mentioned in §2.6.1, the fact that the norm of a normed space is Gâteaux differentiable at a norm-one element if and only if there is a unique state of that point, goes back to Mazur [431].

As a consequence of Proposition 2.6.2, $\mathbb{C}$ *is the unique norm-unital normed complex algebra whose norm comes from an inner product*. The associative forerunner of this result is pointed out in Ingelstam's paper [336], where the vertex property for the unit of norm-unital normed associative real algebras is also explored.

Proposition 2.6.5, Lemma 2.6.7, Theorem 2.6.9, Corollary 2.6.15, and the multiplicative characterization of pre-Hilbert spaces given by Corollary 2.6.14 are due to Rodríguez [515]. For other multiplicative characterizations of pre-Hilbert spaces, see Theorems 2.9.40 and 2.9.70 below, [527], and [73, Section 4]. A slightly less precise version of Theorem 2.6.9 was proved by Strzelecki [606] (see also Nieto [464]) under the additional assumption (unnecessary today, in view of Corollary 2.6.10) of power-associativity. For a proof of Theorem 2.6.9 avoiding Theorem 2.3.8, the reader is referred to [520, Section 2].

Proposition 2.6.8 was predicted by Kaplansky in [762, p. 14] (see also the review of [762] in the preface, p. xiii.), and was proved by Kaidi, Martínez, and Rodríguez [362] (see also Braun [125]). Proposition 2.6.11 is new, whereas Corollary 2.6.12 could be folklore.

The implication (i)⇒(iii) in Theorem 2.6.21 (that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$ *are the unique absolute-valued unital real algebras*) is the celebrated 'non-commutative Urbanik–Wright theorem' [620]. In this memorable paper, whose results were announced in [619], the authors also prove the so-called 'commutative Urbanik–Wright theorem', which reads as follows.

Theorem 2.6.41 $\mathbb{R}$, $\mathbb{C}$, *and* $\overset{*}{\mathbb{C}}$ *are the unique absolute-valued commutative real algebras.*

Here $\overset{*}{\mathbb{C}}$ stands for the so-called *McClay algebra* (see [14]), namely the real absolute-valued algebra obtained by replacing the usual product of $\mathbb{C}$ by the one $(\lambda, \mu) \to \overline{\lambda\mu}$. The proof is based on Schoenberg's theorem [556] (see also Characterization (6.1) in [676, p. 47]) asserting that *a normed space X is a pre-Hilbert space if (and only if) the inequality $4 \leqslant \|x+y\|^2 + \|x-y\|^2$ holds for all norm-one elements $x, y \in X$.* The following proof of Theorem 2.6.41 is essentially the original one in [620], but avoids unnecessary complications.

Proof Let A be an absolute-valued commutative real algebra. Since for all norm-one elements $x, y \in A$ we have

$$4 = 4\|xy\| = \|(x+y)^2 - (x-y)^2\| \leqslant \|x+y\|^2 + \|x-y\|^2,$$

Schoenberg's theorem applies giving that A is a pre-Hilbert space. On the other hand, since $\mathbb{R}$, $\mathbb{C}$, and $\overset{*}{\mathbb{C}}$ are the unique absolute-valued commutative real algebras of dimension $\leqslant 2$ (an easy consequence of the implication (i)$\Rightarrow$(v) in Corollary 2.6.24), it is enough to show that the dimension of A is $\leqslant 2$. Assume to the contrary that we can find pairwise orthogonal norm-one elements u, v, w in A. Then we have $\|u^2 - v^2\| = \|u+v\|\|u-v\| = 2$. Since $\|u^2\| = \|v^2\| = 1$, the parallelogram law implies that $u^2 + v^2 = 0$. Analogously, we obtain $u^2 + w^2 = v^2 + w^2 = 0$. It follows $u^2 = 0$, and hence also $u = 0$, a contradiction. □

The Urbanik–Wright paper [620] also contains a refinement of Proposition 2.6.25 (whose formulation will be stated at the appropriate place in the current subsection), and concludes by exhibiting the first known example of an infinite-dimensional absolute-valued algebra, namely the one corresponding to the case $p = 2$ in Remark 2.7.44 below.

Before continuing our historical review, let us formulate the following consequence of the commutative Urbanik–Wright theorem.

Corollary 2.6.42 *$\mathbb{C}$ is the unique absolute-valued power-commutative complex algebra.*

Proof Let A be an absolute-valued power-commutative complex algebra. Applying Theorem 2.6.41 to the real algebra underlying each subalgebra of A generated by a single element, we see that A is algebraic. Now apply Corollary 2.6.31. □

§**2.6.43** The earliest forerunner of the non-commutative Urbanik–Wright theorem is Hurwitz' classical theorem [334] asserting that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$ *are the unique absolute-valued unital finite-dimensional real algebras whose norms derive from inner products*. Much later, Albert [9] (1949) was able to remove the requirement in Hurwitz' theorem that the norm comes from an inner product. Actually, Albert could have derived his result quickly from Hurwitz' theorem if he had been aware of a result of Auerbach [36] (1935) (see also [800, Theorem 9.5.1]) implying that *finite-dimensional transitive normed spaces are Hilbert spaces*. We recall that a normed space X is said to be *transitive* if, whenever u, v are norm-one elements in X, there is a surjective linear isometry $F : X \to X$ such that $F(u) = v$. The proof of Albert's refinement of Hurwitz' theorem could have been the following.

> Let A be an absolute-valued algebra over $\mathbb{K}$. If A is a division algebra, then the normed space of A is transitive because for all norm-one elements $x, y \in A$ we have $T(x) = y$, where $T := L_{R_x^{-1}(y)}$ is a surjective linear isometry on A. Therefore, when A is finite-dimensional, Auerbach's result applies, giving that the norm of A comes from an inner product. Finally, if $\mathbb{K} = \mathbb{R}$, if A is finite-dimensional, and if A has a unit, then, by Hurwitz' theorem, A is equal to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. □

The argument in the above proof is taken from p. 156 of [51], where no reference is made to the works of Albert and Auerbach. In fact, Albert's refinement of Hurwitz' theorem appears as Theorem 1 of [51] and is directly attributed there to Hurwitz [334], including the above argument as a part of the complete proof of such Hurwitz' theorem. We do not agree with this attribution. Indeed, as far as we know, the observation that absolute-valued division algebras have transitive normed spaces first appears in the proof of Lemma 4 of Wright's paper [640] (55 years after Hurwitz' paper). On the other hand, Auerbach's result, published 36 years after

Hurwitz' paper, seems to us non-obvious. We note that non-separable transitive Banach spaces need not be Hilbert spaces [800, Proposition 9.6.7], and that whether infinite-dimensional separable transitive Banach spaces are Hilbert spaces remains as one of the main open problems in Banach space theory.

The more recent forerunner of the non-commutative Urbanik–Wright theorem is also due to Albert [13], who showed the following.

Fact 2.6.44 *$\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the unique absolute-valued unital algebraic real algebras.*

The original proof in [620] of the non-commutative Urbanik–Wright theorem consists precisely of the verification that absolute-valued unital real algebras are automatically algebraic. The proof we have given here (as any other proof which merits being given today) is close to Kaplansky's prophetic one in [377]. Indeed, the main aim in [377] is to reduce the proof of the non-commutative Urbanik–Wright theorem to the verification of the following.

Fact 2.6.45 *Absolute-valued unital real algebras are pre-Hilbert spaces.*

To clarify Kaplansky's approach, we note at first that, *if A is an absolute-valued real algebra, and if the norm of A comes from an inner product, then A is a composition algebra* (indeed, the mapping $a \to \|a\|^2$ becomes a nondegenerate quadratic form on A admitting composition). On the other hand, Kaplansky's main theorem in [377] (see also [822, Theorem 2.1]) implies that *the unique unital composition real algebras are* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^2$ (with coordinate-wise multiplication), $\mathbb{H}$, $M_2(\mathbb{R})$, $\mathbb{O}$, *and the Cayley–Dickson doubling of* $M_2(\mathbb{R})$ (say $C(\mathbb{R})$). Since $\mathbb{R}^2$, $M_2(\mathbb{R})$, and $C(\mathbb{R})$ cannot be absolute-valued algebras because they have nonzero divisors of zero, the reduction of the proof of the non-commutative Urbanik–Wright theorem to Fact 2.6.45 above is achieved. In the paper [377] just quoted, which was published seven years before that of Urbanik and Wright [620], Kaplansky prophesies both the non-commutative Urbanik–Wright theorem and a proof similar to the one we have just sketched. Even, it seems that he thinks that the non-commutative Urbanik–Wright theorem had already been proved at that time. Thus, he says that:

> Wright [640] succeeded in removing the assumption [in Albert's Fact 2.6.44 above] that the algebra is algebraic.

Since we know that the above assertion is not right, we continue reproducing Kaplansky's words with the appropriate corrections and explanations:

> Wright proceeds by proving that the norm [of a unital absolute-valued *division* algebra] springs from an inner product, and then that the algebra is algebraic. . . . Thus Albert's finite-dimensional theorem [i.e. Fact 2.6.44] can be proved by combining Wright's result with Hurwitz' classical theorem on quadratic forms admitting composition.

Immediately, Kaplansky motivates his work by saying that:

> The main purpose of this paper is to make a similar method possible in the infinite-dimensional case by providing a suitable generalization of Hurwitz' theorem.

After the proof of Fact 2.6.45 given in [515] (which is precisely the one which follows from Corollaries 2.6.15 and 2.6.10), other interesting results implying the same fact have arisen in the literature. This is the case, for example, of El-Mallah's theorem [239] which follows.

Theorem 2.6.46 *Let A be an absolute-valued real algebra containing a nonzero idempotent e which commutes with all elements of A. Then the absolute value of A derives from an inner product $(\cdot|\cdot)$, and the mapping*

$$x \to x^* := 2(x|e)e - x$$

becomes an algebra involution on A satisfying $x^*x = xx^* = \|x\|^2 e$ for every $x \in A$.

Let us say that an element e of an algebra A is a *left unit* for A if $ea = a$ for every $a \in A$. We note that Fact 2.6.45 follows straightforwardly from Corollary 2.7.29 or Proposition 2.7.33 (that *absolute-valued real algebras with a left unit are pre-Hilbert spaces*) below. Proposition 2.7.33 and Theorem 2.6.46, reviewed above, have been unified in the recent paper of Chandid and Rochdi [168], where its authors show that both results are closely related. Indeed, they point out the following for the first time.

Fact 2.6.47 *Let A be an absolute-valued real algebra with a left unit e. Then the normed space of A, endowed with the product $x \odot y := x(ye)$, becomes an absolute-valued algebra (say B), and e becomes an idempotent in B commuting with all elements of B.*

Proof Straightforward. □

It is worth mentioning that all proofs of Fact 2.6.45, quoted until now, simultaneously give rich algebraic information which superimposes the argument in the main result of Kaplansky's paper [377]. In fact, as remarked in [515] and [521], with such additional information in mind, the proof of the non-commutative Urbanik–Wright theorem can be concluded by applying the Frobenius–Zorn theorem (Theorem 2.5.29) instead of Kaplansky's. This already happened in the proof given here (see the formal proof of Theorem 2.6.21). For an additional example, see the alternative proof of the non-commutative Urbanik–Wright theorem given in Subsection 2.7.4 below.

The implication (ii)⇒(iii) in Theorem 2.6.21 (that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$ *are the unique smooth-normed alternative real algebras*) is due to Strzelecki [606] (1966). A new and simpler proof was provided later by Nieto [464]. Strzelecki's original proof is based on his previous result that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the unique smooth-normed algebraic alternative real algebras [605]. As a consequence of Strzelecki's theorem, $\mathbb{R}, \mathbb{C}$, *and* $\mathbb{H}$ *are the unique smooth-normed associative real algebras*. This result is attributed by the associative Banach algebraists to Bonsall and Duncan [116], although the Bonsall–Duncan paper was published two years after Strzelecki's. A slight refinement of the Bonsall–Duncan result, due to Spatz [594], is included in [694, Theorem 5.16].

Corollary 2.6.22 is originally due to Ingelstam [338] (1964). Due to the nice simplicity of its formulation, Ingelstam's corollary (and even its associative specialization [337] that $\mathbb{R}, \mathbb{C}$ *and* $\mathbb{H}$ *are the unique norm-unital normed associative real algebras whose norm comes from an inner product*) has attained considerably more celebrity status than Strzelecki's more general theorem. Thus, Ingelstam's associative result has been re-proved several times (see Smiley [590], Froelich [272], and Kulkarni [394]). Today, results like those of Zalar and Cuenca, formulated in Theorems 2.6.48 and 2.6.49 immediately below, seem more interesting to us than Ingelstam's Corollary 2.6.22.

Theorem 2.6.48 [657] *$\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique unital alternative real algebras A which are pre-Hilbertian spaces satisfying $\|\mathbf{1}\| = 1$ and $\|x^2\| \leqslant \|x\|^2$ for every $x \in A$.*

Theorem 2.6.49 [192, 193] *Let A be a normed real algebra whose norm derives from an inner product. Assume that the equality*

$$(x((xy)x))x = (x^2y)x^2$$

holds for all $x, y \in A$, and that there exists a norm-one idempotent $e \in A$ satisfying $\|ex\| = \|x\|$ for every $x \in A$. Then A is equal to $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, $\overset{*}{\mathbb{O}}$, or $\mathbb{P}$. If in addition the idempotent e above commutes with all elements of A, then A is equal to $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, or $\overset{*}{\mathbb{O}}$.

Here, for $\mathbb{A}$ equal to $\mathbb{H}$ or $\mathbb{O}$, the symbol $\overset{*}{\mathbb{A}}$ stands for the absolute-valued real algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := x^*y^*$, where $*$ means the standard involution, whereas the symbol $\mathbb{P}$ stands for the absolute-valued real algebra of *pseudo-octonions*. Such an algebra was discovered by Okubo [467] (see also [782, pp. 65–71]), and can be described as follows. The vector space of $\mathbb{P}$ is the eight-dimensional real subspace of $M_3(\mathbb{C})$ of those trace-zero elements which remain fixed under the operation consisting of transposing the matrices and of taking conjugates of their entries. The product $\odot$ of $\mathbb{P}$ is defined by choosing a complex number μ satisfying $3\mu(1-\mu) = 1$, and then by setting

$$x \odot y := \mu xy + (1-\mu)yx - \frac{1}{3}t(xy)\mathbf{1},$$

where t denotes the trace function on $M_3(\mathbb{C})$, $\mathbf{1}$ stands for the unit of the associative algebra $M_3(\mathbb{C})$, and, for $x, y \in \mathbb{P}$, xy means the product of x and y as elements of the associative algebra $M_3(\mathbb{C})$. If for $x, y \in \mathbb{P}$ we define $(x|y) := \frac{1}{6}t(xy)$, then $(.|.)$ becomes an inner product on $\mathbb{P}$ whose associated norm is an absolute value.

For a new proof of the obvious associative specialization of Theorem 2.6.48, see [394]. Other results of a similar flavour to that of Theorem 2.6.49 can be found in [192, 193, 194]. To conclude our review of Ingelstam's paper [338], let us say that, according to [338, Remark on p. 234], Ingelstam seems to be aware of Strzelecki's theorem, published two years later.

The implication (i)⇒(v) in Corollary 2.6.24 is due to Albert [9], and has the following as a consequence.

Fact 2.6.50 *Let A be a finite-dimensional absolute-valued real algebra. Then A has dimension 1, 2, 4, or 8, and the norm of A derives from an inner product.*

It is worth mentioning that, some years after Albert's paper [9] (1958), the following more general theorem was proven.

Theorem 2.6.51 *Every nonzero finite-dimensional real algebra with no nonzero divisor of zero has dimension* 1, 2, 4, *or* 8.

The paternity of Theorem 2.6.51 above seems to be in doubt. Indeed, according to [727], [249], and [51], such a theorem was first proved by Kervaire [388] and Milnor [445], Adams [2], and Kervaire [388] and Bott–Milnor [117], respectively. Anyway, in contrast to the case of Albert's result, all known proofs of this theorem are extremely deep (see [727, Chapter 11]).

In relation to Fact 2.6.50, it is also worth citing a recent paper by Garibaldi and Petersson [282], where its authors construct analogues of absolute-valued algebras in all dimensions 2^n ($n \in \mathbb{N} \cup \{0\}$) over arbitrary 2-Henselian fields. We will not give the definition of a 2-Henselian field here. Suffice to say that fields which are complete under a discrete valuation, e.g. formal Laurent series fields over any field of constants, fall into this category, so there are many nontrivial examples.

The implication (ii)⇒(i) in Corollary 2.6.24 (that *absolute-valued division real algebras are finite-dimensional*) is the main result in Wright's paper [640]. The refinement given by the implication (iii)⇒(i) is originally due to Rodríguez [521]. The key tool in the whole proof of Corollary 2.6.24 that we have given (namely, the spectacularly easy deduction of the implication (iii)⇒(v) from the non-commutative Urbanik–Wright theorem) is due to Elduque and Pérez [232]. For the history of the classification of finite-dimensional real algebras and the present status of this topic the reader is referred to [155] and references therein.

As we have already commented, Proposition 2.6.25 is due to Urbanik and Wright [620]. Actually, they prove a finer result, namely that $\mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $\mathbb{O}$ *are the unique absolute-valued real algebras* A *containing some nonzero element* a *satisfying*

$$ax = xa, a(ax) = a^2x, \text{ and } (xa)a = xa^2 \text{ for every } x \in A.$$

According to the information given in pp. 243, 245 of [727], Lemma 2.6.26 and its proof are due to Ostrowski [472] (1918), who seems to have been the first mathematician who considered absolute-valued algebras as abstract objects which were worth studying. Of course, Corollary 2.6.18, invoked in our proof, was not known to him, but he was able to prove what actually he needed to, namely that *every absolute-valued field extension of* $\mathbb{R}$ *is isometrically isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$.

§2.6.52 Proposition 2.6.27 is due to El-Mallah and Micali [243]. An early forerunner is the one of Mazur [432] asserting that $\mathbb{R}, \mathbb{C}$ *and* $\mathbb{H}$ *are the unique absolute-valued associative real algebras*, although, as we commented in §2.5.63, Mazur's proof is not available.

Lemmas 2.6.32 and 2.6.33 are taken from the Cedilnik–Rodríguez paper [165]. Lemma 2.6.34, Proposition 2.6.35, and Corollary 2.6.40 are due to Zalar [656]. The last conclusion in Lemma 2.6.34 has the following non-associative variant, due to Cuenca [194].

Proposition 2.6.53 *Let* A *be a commutative real algebra endowed with a norm* $\|\cdot\|$ *which comes from an inner product and satisfies* $\|x^2\| = \|x\|^2$ *for every* $x \in A$. *Assume additionally that* A *has no nonzero divisor of zero, and that there exists a norm-one idempotent* $e \in A$ *satisfying* $\|ex\| = \|x\|$ *for every* $x \in A$. *Then* A *is equal to* $\mathbb{R}$, $\mathbb{C}$, *or* $\overset{*}{\mathbb{C}}$.

The proof of Proposition 2.6.35 given here has relevant variants on Zalar's original one. For variants and refinements of Proposition 2.6.35, see Theorem 2 in the Cedilnik–Zalar paper [166], Theorem 3.14 in the Moutassim–Rochdi paper [456], and Theorem 2.8 in Cuenca's paper [194]. The unit-free characterizations of smooth-normed real algebras, given by Propositions 2.6.38 and 2.6.39, are due to Benslimane and Merrachi [93] and Rodríguez [533], respectively. Other characterizations of smooth-normed real algebras will be proved later (see Corollary 2.9.41 and Theorems 4.1.96 and 4.2.48). We note that, thanks to Strzelecki's theorem (namely the implication (ii)⇒(iii) in Theorem 2.6.21), each characterization of smooth-normed real algebras yields, when restricted to alternative algebras, a characterization of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ (see Corollaries 2.9.42 and 4.1.100).

Remark 2.6.54 In view of Corollary 2.6.13, a perfect knowledge of complete smooth-normed real algebras depends on the determination of all H-algebras. H-algebras are particular examples of the so-called H^*-algebras. By definition, an H^**-algebra* over $\mathbb{K}$ is a $*$-algebra A over $\mathbb{K}$, which is a Hilbert space relative to an inner product $(\cdot|\cdot)$ satisfying $(xy|z) = (x|zy^*) = (y|x^*z)$ for all $x, y, z \in A$. Let A be an H^*-algebra. Since it is easily realized that the product of A is continuous (see Lemma 2.8.12(i) below), up to multiplication of the inner product by a suitable positive number, A becomes a (complete) normed algebra. However, the requirement

$$\|ab\| \leqslant \|a\|\|b\| \quad (a, b \in A) \tag{2.6.12}$$

is not assumed in the theory because most natural examples of H^*-algebras do not satisfy it. Now, clearly, H-algebras are precisely those anticommutative real H^*-algebras A satisfying (2.6.12), and whose H^*-algebra involution is $-I_A$.

H^*-algebras were first studied in the associative complex case [598, 20], and they were introduced by abstracting the properties of the normed $*$-algebra of all Hilbert–Schmidt operators on a complex Hilbert space [809]. The corresponding study of associative real H^*-algebras was done by Kaplansky [374], and has been rediscovered several times (see [55, 143, 163]). General non-associative real and complex H^*-algebras were first considered in [198, 199, 142] (see also [144, 148, 149, 200, 259, 424, 526, 624]). As a consequence of this general theory, *every H-algebra is the Hilbert sum of closed ideals, one of which is an H-algebra with zero product (so a suitable real Hilbert space equipped with the zero product), and the others are topologically simple H-algebras* (cf. Definition 1.4.31). Moreover, by [142], *if A is a topologically simple H-algebra, then there exists a topologically simple anticommutative complex H^*-algebra B such that $A = \{x \in B : x^* = -x\}$.* Unfortunately, as far as we know, there is no description or classification of topologically simple anticommutative complex H^*-algebras. Precise descriptions of topologically simple Lie (and even Malcev) complex H^*-algebras are, however, known (see [557, 558, 53, 197, 141, 148, 460, 140]).

The theory of associative complex H^*-algebras can be found in [696, Section 34]. Real and complex (possibly non-associative) H^*-algebras are fully surveyed in [525, Section E] (see also [687, Section 7.2]).

2.7 Other Gelfand–Mazur type non-associative theorems

Introduction In this section we prove, as main results, that complete normed quasi-division complex algebras have dimension $\leqslant 2$, and that there are complete absolute-valued left-division infinite-dimensional real algebras.

2.7.1 Focusing on complex algebras

We begin this section by discussing some Gelfand–Mazur type results for general non-associative normed complex algebras.

The fact that elements of normed unital associative complex algebras have non-empty spectra can be reformulated as follows.

Lemma 2.7.1 *Let b_1, b_2 be elements of a normed unital associative complex algebra B. Then there exists $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that*

$$0 \in \mathrm{sp}(B, \lambda_1 b_1 + \lambda_2 b_2).$$

Proof If $0 \in \mathrm{sp}(B, b_1)$, then the result holds with $(\lambda_1, \lambda_2) = (1, 0)$. Otherwise, we can choose $\mu \in \mathrm{sp}(B, b_1^{-1} b_2)$, and consider the equality

$$b_2 - \mu b_1 = b_1(b_1^{-1} b_2 - \mu \mathbf{1}),$$

to obtain that the result is true with $(\lambda_1, \lambda_2) = (-\mu, 1)$. □

Proposition 2.7.2 *Let A be a nonzero normed complex algebra such that, whenever a is any nonzero element of A, L_a is an invertible element of the completion of $BL(A)$. Then A is isomorphic to $\mathbb{C}$.*

Proof Let B denote the completion of $BL(A)$. According to Lemma 2.7.1, whenever x_1 and x_2 are in A we can find $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that

$$0 \in \mathrm{sp}(B, \lambda_1 L_{x_1} + \lambda_2 L_{x_2}) = \mathrm{sp}(B, L_{\lambda_1 x_1 + \lambda_2 x_2}),$$

so we have $\lambda_1 x_1 + \lambda_2 x_2 = 0$, and so the system $\{x_1, x_2\}$ is linearly dependent. Now apply Exercise 1.1.2. □

We recall that, given a bounded linear operator F on a Banach space X, the fact that F is bijective means that F is an invertible element of $BL(X)$ (cf. Example 1.1.12(d)). Therefore the next corollary follows by applying Proposition 2.7.2 to A or to the opposite algebra of A.

Corollary 2.7.3 *Let A be a complete normed one-sided division complex algebra. Then A is isomorphic to $\mathbb{C}$.*

We do not know if the requirement of completeness in Corollary 2.7.3 above can be removed. Actually, even the following more critical problem remains open.

Problem 2.7.4 Is every normed division complex algebra isomorphic to $\mathbb{C}$?

According to Example 2.5.36 and Proposition 1.1.7, Corollary 2.7.3 does not remain true if the assumption that A is a one-sided division algebra is relaxed to the assumption that A is a quasi-division algebra. Nevertheless, we are going to show in Theorem 2.7.7 below that, under this relaxation, Corollary 2.7.3 remains 'almost true'.

If x, y are elements of a complete normed unital associative complex algebra B, then the inclusion

$$\operatorname{sp}(BL(B), L_x - R_y) \subseteq \operatorname{sp}(B, x) - \operatorname{sp}(B, y) \tag{2.7.1}$$

holds. Indeed, since L_x and R_y are commuting elements of the complete normed unital associative complex algebra $BL(B)$, we can apply Corollary 1.1.81(i) to get $\operatorname{sp}(BL(B), L_x - R_y) \subseteq \operatorname{sp}(BL(B), L_x) - \operatorname{sp}(BL(B), R_y)$, and the result follows by keeping in mind that the mapping $z \to L_z$ (respectively, $z \to R_z$) from B to $BL(B)$ is a unit-preserving algebra homomorphism (respectively, algebra antihomomorphism), and consequently the inclusion $\operatorname{sp}(BL(B), L_x) \subseteq \operatorname{sp}(B, x)$ (respectively, $\operatorname{sp}(BL(B), R_y) \subseteq \operatorname{sp}(B, y)$) holds.

Lemma 2.7.5 *Let B be a complete normed unital associative complex algebra, and let a_1, a_2, b_1, b_2 be elements in B. Then there exists a couple $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that*

$$\operatorname{sp}(B, \lambda_1 a_1 + \lambda_2 a_2) \cap \operatorname{sp}(B, \lambda_1 b_1 + \lambda_2 b_2) \neq \emptyset.$$

Proof By Lemma 2.7.1, there is a couple $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that

$$0 \in \operatorname{sp}(BL(B), \lambda_1(L_{a_1} - R_{b_1}) + \lambda_2(L_{a_2} - R_{b_2})).$$

But, by (2.7.1), we have

$$\begin{aligned}
&\operatorname{sp}(BL(B), \lambda_1(L_{a_1} - R_{b_1}) + \lambda_2(L_{a_2} - R_{b_2})) \\
&\quad = \operatorname{sp}(BL(B), L_{\lambda_1 a_1 + \lambda_2 a_2} - R_{\lambda_1 b_1 + \lambda_2 b_2}) \\
&\quad \subseteq \operatorname{sp}(B, \lambda_1 a_1 + \lambda_2 a_2) - \operatorname{sp}(B, \lambda_1 b_1 + \lambda_2 b_2).
\end{aligned}$$

It follows that, for such a couple $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, we have

$$\operatorname{sp}(B, \lambda_1 a_1 + \lambda_2 a_2) \cap \operatorname{sp}(B, \lambda_1 b_1 + \lambda_2 b_2) \neq \emptyset. \qquad \square$$

Proposition 2.7.6 *Let A be a normed complex algebra such that, whenever a is any nonzero element of A, at least one of the operators L_a, R_a is an invertible element of the completion of $BL(A)$. Then $\dim(A) \leqslant 2$.*

Proof Assume that $3 \leqslant \dim(A)$. Let B stand for the completion of $BL(A)$, and denote by Ω_1 (respectively, Ω_2) the set of those elements $a \in A$ such that L_a (respectively, R_a) is an invertible element of B. By Proposition 2.7.2, Ω_1 and Ω_2 are proper subsets of $A \setminus \{0\}$. Since $A \setminus \{0\} = \Omega_1 \cup \Omega_2$, and Ω_1, Ω_2 are open, and $A \setminus \{0\}$ is connected, there must exist some $x \in \Omega_1 \cap \Omega_2$. Take $x_1, x_2 \in A$ such that the system $\{x_1, x_2, x\}$ is linearly independent. Applying Lemma 2.7.5, we find $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $\lambda \in \mathbb{C}$ such that

$$\lambda \in \operatorname{sp}(B, \lambda_1 L_{x_1} L_x^{-1} + \lambda_2 L_{x_2} L_x^{-1}) \cap \operatorname{sp}(B, \lambda_1 R_{x_1} R_x^{-1} + \lambda_2 R_{x_2} R_x^{-1}).$$

Then, setting $y := \lambda_1 x_1 + \lambda_2 x_2 - \lambda x$, y becomes a nonzero element of A such that neither L_y nor R_y are invertible elements of B, a contradiction. $\square$

As a consequence, we have the following.

Theorem 2.7.7 *Every complete normed quasi-division complex algebra has dimension $\leqslant 2$.*

As a straightforward consequence of Proposition 1.1.7 and Theorem 2.7.7, we derive the following.

Corollary 2.7.8 *Every finite-dimensional quasi-division complex algebra has dimension $\leqslant 2$.*

Since unital algebras of dimension $\leqslant 2$ are (associative and) commutative, the following corollary follows from Theorem 2.7.7 and Corollary 2.7.3.

Corollary 2.7.9 *Complete normed unital quasi-division complex algebras are isomorphic to $\mathbb{C}$.*

Corollary 2.7.10 *Let A be a complete normed power-commutative quasi-division complex algebra. Then A is isomorphic to $\mathbb{C}$.*

Proof If A is algebraic of degree 1, then the result follows from Lemma 2.6.29. Assume that A is not algebraic of degree 1. Then, since $\dim(A) \leqslant 2$ (by Theorem 2.7.7), A is algebraic of degree 2, and is generated as an algebra by a single element. Therefore, since A is power-commutative, A is commutative, so A is a division algebra, and so, by Corollary 2.7.3, A is isomorphic to $\mathbb{C}$, contradicting that $\deg(A) = 2$. □

By Corollaries 2.4.16 and 2.4.18, both flexible algebras and power-associative algebras are power-commutative. Therefore, Corollary 2.7.10 immediately above gives rise to the following.

Corollary 2.7.11 *Let A be a complete normed quasi-division complex algebra. If A is flexible or power-associative, then A is isomorphic to $\mathbb{C}$.*

The part of the above corollary corresponding to power-associativity has already been proved in Corollary 2.5.54(i) by other methods.

Now, we are going to discuss completeness-free non-associative versions of the Gelfand–Mazur complex theorem.

Lemma 2.7.12 *Let X be a complex normed space, let F, G be in $BL(X)$, and assume that G is bounded below and has dense range. Then there exists $\lambda \in \mathbb{C}$ such that $F - \lambda G$ is not bounded below.*

Proof Let $\hat{X}$ stand for the completion of X. For $T \in BL(X)$, let $\hat{T}$ denote the unique bounded linear operator on $\hat{X}$ which extends T, and note that, if T is bounded below, then so is $\hat{T}$. By the assumptions on G, we have that $\hat{G} \in \mathrm{Inv}(BL(\hat{X}))$. Therefore, keeping in mind Corollary 1.1.44, we can find λ in the boundary of $\mathrm{sp}(BL(\hat{X}), \hat{G}^{-1}\hat{F})$ relative to $\mathbb{C}$. Then, by Corollary 1.1.95, $\hat{G}^{-1}\hat{F} - \lambda I_{\hat{X}}$ is not bounded below, and hence

$$\widehat{F - \lambda G} = \hat{F} - \lambda \hat{G} = \hat{G}(\hat{G}^{-1}\hat{F} - \lambda I_{\hat{X}})$$

is not bounded below. It follows that $F - \lambda G$ is not bounded below. □

Proposition 2.7.13 *Let A be a nonzero normed complex algebra with no nonzero left topological divisor of zero, and assume that there is $x \in A$ such that L_x has dense range. Then A is isomorphic to $\mathbb{C}$.*

Proof Let y be in A. Noticing that L_x is bounded below and has dense range, we can apply Lemma 2.7.12 to find $\lambda \in \mathbb{C}$ such that $L_{y-\lambda x} = L_y - \lambda L_x$ is not bounded below. Since A has no nonzero left topological divisor of zero, we deduce $y = \lambda x$. Thus, A is one-dimensional, and the proof is concluded by applying Exercise 1.1.2. □

Theorem 2.7.14 *Let A be a normed complex algebra, and assume that at least one of the following conditions holds:*

(i) *A is a division algebra, and has no nonzero two-sided topological divisor of zero.*
(ii) *A is a $\left\{\begin{smallmatrix}\text{left}\\ \text{right}\end{smallmatrix}\right\}$-division algebra, and has no nonzero $\left\{\begin{smallmatrix}\text{left}\\ \text{right}\end{smallmatrix}\right\}$ topological divisor of zero.*
(iii) *A is a quasi-division algebra, and has no nonzero one-sided topological divisor of zero.*

Then A is isomorphic to $\mathbb{C}$.

Proof Assume that condition (i) is fulfilled. Then, for every $a \in A \setminus \{0\}$, the operators L_a and R_a are surjective, and some of them are bounded below. Therefore, whenever a is any nonzero element of A, at least one of the operators L_a, R_a is an invertible element of $BL(A)$, so a fortiori at least one of the operators L_a, R_a is an invertible element of the completion of $BL(A)$. By Proposition 2.7.6, A is finite-dimensional, so that the result follows from Exercise 1.1.86.

Now assume that A is a left-division algebra with no nonzero left topological divisor of zero. Then, for every $a \in A \setminus \{0\}$, the operator L_a is surjective and bounded below. Therefore, whenever a is any nonzero element of A, L_a is an invertible element of $BL(A)$, hence a fortiori it is an invertible element of the completion of $BL(A)$, so that the result follows from Proposition 2.7.2.

Finally, if condition (iii) is fulfilled, then the result follows by applying Proposition 2.7.13 to A or to the opposite algebra of A. □

Noticing that complete normed $\left\{\begin{smallmatrix}\text{left}\\ \text{right}\end{smallmatrix}\right\}$-division algebras have no nonzero $\left\{\begin{smallmatrix}\text{left}\\ \text{right}\end{smallmatrix}\right\}$ topological divisor of zero (courtesy of the Banach isomorphism theorem), we realize that the part of Theorem 2.7.14 above corresponding to assumption (ii) generalizes Corollary 2.7.3.

Combining Propositions 2.5.59 and 2.7.13, we get the following.

Corollary 2.7.15 *Let A be a nonzero normed complex algebra with nonzero centre and with no nonzero $\left\{\begin{smallmatrix}\text{left}\\ \text{right}\end{smallmatrix}\right\}$ topological divisor of zero. Then A is isomorphic to $\mathbb{C}$.*

Proposition 2.7.16 *Let A be a nearly absolute-valued complex algebra. Then the following conditions are equivalent:*

(i) *A has nonzero centre.*
(ii) *A is finite-dimensional.*

(iii) *A is a division algebra.*
(iv) *A is a one-sided division algebra.*
(v) *A is a quasi-division algebra.*
(vi) *There exists $a \in A$ such that L_a or R_a is surjective.*
(vii) *There exists $a \in A$ such that L_a or R_a has dense range in A.*
(viii) *A is isomorphic to $\mathbb{C}$.*

Proof Thinking about the circle of implications (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇒(vii)⇒(viii)⇒(i), only (i)⇒(ii) and (vii)⇒(viii) merit some explanation; but these implications follow from Corollary 2.7.15 and Proposition 2.7.13, respectively. □

We note that the implication (i)⇒(viii) in Proposition 2.7.16 above refines the complex part of Proposition 2.6.25.

Corollary 2.7.17 *Nearly absolute-valued complex algebras with a one-sided unit are isomorphic to $\mathbb{C}$.*

2.7.2 Involving real scalars

Now, we involve connectedness arguments in our development in order to discuss Gelfand–Mazur–Kaplansky-type results for both real and complex algebras.

Lemma 2.7.18 *Let X be a nonzero Banach space over $\mathbb{K}$, and let P be a connected subset of $BL(X)$. Assume that every element of P is either bounded below or surjective. Then either all elements of P are bijective or all elements of P are non-bijective.*

Proof Suppose that the conclusion in the lemma is not true. Then

$$Q := \{F \in P : F \text{ is bijective}\}$$

is a non-empty proper subset of the connected set P, and therefore there must exist some F_0 in the boundary of Q relative to P. Since such a F_0 lies in the boundary of $\mathrm{Inv}(BL(X))$ relative to $BL(X)$, it follows from Corollary 1.1.95 that F_0 is neither bounded below nor surjective, contradicting our assumption. □

Proposition 2.7.19 *Let A be a nonzero complete normed algebra over $\mathbb{K}$ such that L_a is bijective for some $a \in A$. Assume that some of the following conditions hold:*

(i) *A has no nonzero left topological divisor of zero.*
(ii) *For every $x \in A \setminus \{0\}$, L_x is surjective.*

Then A is a left-division algebra.

Proof By Exercise 1.1.2, we can assume that $\dim(A) \geqslant 2$. Then apply Lemma 2.7.18 with $X = A$ and $P = \{L_x : x \in A \setminus \{0\}\}$. □

Combining Propositions 2.5.59 and 2.7.19, we get the next result which, in view of Corollary 2.7.15, has interest only in the case $\mathbb{K} = \mathbb{R}$.

Corollary 2.7.20 *Let A be a nonzero complete normed algebra over $\mathbb{K}$ with nonzero centre and with no nonzero left topological divisor of zero. Then A is a left-division algebra.*

Somehow, the study of complete normed left-division algebras reduces to that of complete normed left-division algebras with a left unit. For, if A is a complete normed left-division algebra over $\mathbb{K}$, and if a is any nonzero element of A, then the vector space of A, endowed with the product $x \odot y := L_a^{-1}(xy)$ and a suitable positive multiple of the norm of A, becomes a complete normed left-division algebra over $\mathbb{K}$ with a as a left unit. Now, keeping in mind §2.5.53, and the fact that the absence of nonzero left topological divisors of zero in a normed algebra is inherited by all subalgebras, Proposition 2.7.19 implies the following.

Corollary 2.7.21 *Let A be a complete normed left-division algebra over $\mathbb{K}$ with a left unit e. Then every closed subalgebra of A containing e is a left-division algebra.*

We note that, in view of Corollary 2.7.3, Corollary 2.7.21 above has interest only if $\mathbb{K} = \mathbb{R}$.

Corollary 2.7.22 *Let A be a complete normed quasi-division algebra over $\mathbb{K}$ which is not a one-sided division algebra. Then A has nonzero left topological divisors of zero as well as nonzero right topological divisors of zero. Moreover, there exists $a \in A$ such that L_a and R_a are bijective.*

Proof By applying Proposition 2.7.19 to A and to the opposite algebra of A, the first conclusion follows. Assume that, for each $a \in A$, one of the operators L_a, R_a is not bijective. Then the sets

$$\Omega_1 := \{a \in A \backslash \{0\} : L_a \text{ is bijective}\} \ \text{ and } \ \Omega_2 := \{a \in A \backslash \{0\} : R_a \text{ is bijective}\}$$

are disjoint non-empty open subsets of $A\backslash\{0\}$ such that $A\backslash\{0\} = \Omega_1 \cup \Omega_2$. Since $A\backslash\{0\}$ is connected (by Exercise 1.1.2), we reach a contradiction. □

We recall that, by §2.5.53, complete normed left-division algebras have no nonzero left topological divisor of zero. A partial relevant converse is given by the following.

Corollary 2.7.23 *Let A be a nonzero complete normed power-associative real algebra with no nonzero left topological divisor of zero. Then A is both a quadratic algebra and a left-division algebra.*

Proof By Proposition 2.5.49(ii), A is quadratic. As a consequence, A has a unit, and hence, by Proposition 2.7.19, A is a left-division algebra. □

Assertions (i) and (ii) in the next corollary follow straightforwardly from Corollaries 2.7.22 and 2.7.23, respectively.

Corollary 2.7.24 *Let A be a complete nearly absolute-valued real algebra. We have:*

(i) *A is a quasi-division algebra (if and) only if it is a one-sided division algebra.*
(ii) *If A is power-associative, then A is both a quadratic algebra and a division algebra.*

Corollary 2.7.25 *Let A be a complete normed algebra over $\mathbb{K}$ with no nonzero left topological divisor of zero and whose norm derives from an inner product $(\cdot|\cdot)$. Assume that there exist $a, a^* \in A \setminus \{0\}$ such that the equality $(ax|y) = (x|a^*y)$ holds for all $x, y \in A$. Then A is a left-division algebra.*

Proof In view of Proposition 2.7.19, it is enough to show that L_a is bijective. Assume to the contrary that L_a is not bijective. Then, since L_a is bounded below, its range is a proper closed subspace of A. Therefore, by the orthogonal projection theorem, there exists $y \in A \setminus \{0\}$ such that $(ax|y) = 0$ for every $x \in A$. Now we have $(x|a^*y) = (ax|y) = 0$ for every $x \in A$, so $a^*y = 0$, and so (since A has no nonzero divisor of zero) either $a^* = 0$ or $y = 0$, a contradiction. □

Combining Proposition 2.7.19 with Corollaries 2.7.3, 2.7.22, and 2.7.25, we derive the following.

Corollary 2.7.26 *Let A be a nonzero complete normed complex algebra, and assume that at least one of the following conditions holds:*

(i) *L_a is injective for some $a \in A$, and L_x is surjective for every $x \in A \setminus \{0\}$.*
(ii) *A is a quasi-division algebra, and either A has no nonzero left topological divisor of zero or A has no nonzero right topological divisor of zero.*
(iii) *A has no nonzero left topological divisor of zero, the norm of A derives from an inner product $(\cdot|\cdot)$, and there exist $a, a^* \in A \setminus \{0\}$ such that the equality $(ax|y) = (x|a^*y)$ holds for all $x, y \in A$.*

Then A is isomorphic to $\mathbb{C}$.

The part of the above corollary corresponding to assumption (ii) can also be derived from Theorem 2.7.7 and Exercises 1.1.88(i) and 1.1.86.

As a consequence of Theorem 2.7.7 and Proposition 2.7.16, *both complete normed quasi-division complex algebras and absolute-valued quasi-division complex algebras are finite-dimensional*. As a matter of fact, a similar result does not hold for real algebras. Indeed, we are going to construct complete absolute-valued one-sided division infinite-dimensional real algebras (see Theorem 2.7.38 below). On the way, we will discover a substantial amount of information about the structure of such algebras.

Lemma 2.7.27 *Let A be a normed real algebra with a left unit e and such that there exists $\rho > 0$ satisfying $\|xy\| \geqslant \rho\|x\|\|y\|$ for all x, y in A. Then, for each two-dimensional subspace M of A with $e \in M$, there is a linear mapping φ from M to the two-dimensional Euclidean real space satisfying*

$$\rho\|m\| \leqslant \|\varphi(m)\| \leqslant \|m\| \textit{ for every } m \in M.$$

Proof By passing to the completion of A if necessary, we may assume that A is complete. Then, since $L_e = I_A$, it follows from Proposition 2.7.19 that A is a left-division algebra. According to Proposition 1.1.98, let us take a complete normed unital associative complex algebra B containing isometrically $BL(A)$ as a closed real subalgebra and whose unit is the same as that of $BL(A)$ (namely, the identity mapping I_A on A). Now, for x in $A\setminus\{0\}$, L_x is an invertible element of B satisfying $\|L_x^{-1}\| \leqslant (\rho\|x\|)^{-1}$. Therefore, for every x in $A \setminus \{0\}$ and every complex number z in $\mathrm{sp}(B, L_x)$, we have that $z^{-1} \in \mathrm{sp}(B, L_x^{-1})$, and hence that

$$\rho\|x\| \leqslant |z| \leqslant \|x\|.$$

Finally, for x in $A\backslash\mathbb{R}e$ and α,β in $\mathbb{R}$, we may choose z in $\mathrm{sp}(B,L_x)$, so that $\alpha+\beta z$ belongs to $\mathrm{sp}(B,L_{\alpha e+\beta x})$, and we have

$$\rho\|\alpha e+\beta x\|\leqslant|\alpha+\beta z|\leqslant\|\alpha e+\beta x\|. \qquad \square$$

Proposition 2.7.28 *Let A be a normed real algebra satisfying the following conditions:*

(i) *There is an element a in A such that aA is dense in A.*
(ii) *There exists $\rho>0$ such that the inequality $\|xy\|\geqslant\rho\|x\|\|y\|$ holds for all x,y in A.*

Then, for each two-dimensional subspace M of A, there is a linear mapping φ from M to the two-dimensional Euclidean real space satisfying

$$\rho^2\|m\|\leqslant\|\varphi(m)\|\leqslant\|m\| \text{ for every } m\in M.$$

Proof We may assume that A is complete. Then the operator L_a is bijective because it is bounded below and has dense range. Therefore, by Proposition 2.7.19, A is a left-division algebra. Let M be a two-dimensional subspace of A. Take a basis $\{u,v\}$ of M such that $\|u\|=\rho^{-1}$. Then, for every x in A, we have

$$\rho\|x\|\leqslant\|L_u^{-1}(x)\|\leqslant\|x\|.$$

Now, consider the real algebra B consisting of the vector space of A and the product $x\square y:=L_u^{-1}(xy)$. Then, for x,y in B, we have

$$\|x\square y\|=\|L_u^{-1}(xy)\|\leqslant\|xy\|\leqslant\|x\|\|y\|,$$

so that $\|.\|$ becomes an algebra norm on B. On the other hand, again for x,y in B, we have

$$\|x\square y\|=\|L_u^{-1}(xy)\|\geqslant\rho\|xy\|\geqslant\rho^2\|x\|\|y\|.$$

Now, since u is a left unit for B, the existence of a linear mapping φ from M to the two-dimensional Euclidean real space satisfying $\rho^2\|m\|\leqslant\|\varphi(m)\|\leqslant\|m\|$ for every m in M follows from Lemma 2.7.27. $\square$

Taking $\rho=1$ in the above proposition, we obtain the following.

Corollary 2.7.29 *Let A be an absolute-valued real algebra, and assume that there exists $a\in A$ such that aA is dense in A. Then A is a pre-Hilbert space.*

We recall that a normed space X is said to be *uniformly non-square* if there exists $0<\sigma<1$ such that the inequality

$$\min\{\|x+y\|,\|x-y\|\}<2\sigma$$

holds for all x,y in the closed unit ball of X. The main significance of this notion relies on the fact that the completion of a uniformly non-square normed space is a superreflexive Banach space [716, Theorem VII.4.4].

As the next example shows, not all nearly absolute-valued real algebras with a left unit are uniformly non-square.

Example 2.7.30 Take A equal to $\mathbb{C}$ (regarded as a real algebra) endowed with the algebra norm $\|.\|$ defined by $\|\alpha + i\beta\| := |\alpha| + |\beta|$. Since $\|1\| = \|i\| = 1$ and $\|1+i\| = \|1-i\| = 2$, certainly A is not uniformly non-square. On the other hand, denoting by $|\cdot|$ the usual module function on $\mathbb{C}$, for x in A we have $|x| \leqslant \|x\| \leqslant 2^{1/2}|x|$. It follows that

$$\|xy\| \geqslant |xy| = |x||y| \geqslant 2^{-1}\|x\|\|y\| \text{ for all } x, y \in A.$$

Corollary 2.7.31 *Let A be a normed real algebra satisfying the following conditions:*

(i) *There is an element a in A such that aA is dense in A.*
(ii) *There exists $\rho > 2^{-1/4}$ such that the inequality $\|xy\| \geqslant \rho\|x\|\|y\|$ holds for all x, y in A.*

Then A is uniformly non-square.

Proof Choose σ with $2^{-1/2}\rho^{-2} < \sigma < 1$, and let x, y be in the closed unit ball of A. By Proposition 2.7.28, there exists a linear mapping φ from the linear hull of $\{x, y\}$ (say M) to an Euclidean real space satisfying

$$\rho^2\|m\| \leqslant \|\varphi(m)\| \leqslant \|m\|$$

for every m in M. It follows that

$$\begin{aligned}
&[\min\{\|x+y\|, \|x-y\|\}]^2 \\
&\quad \leqslant 2^{-1}(\|x+y\|^2 + \|x-y\|^2) \\
&\quad \leqslant 2^{-1}\rho^{-4}(\|\varphi(x+y)\|^2 + \|\varphi(x-y)\|^2) = \rho^{-4}(\|\varphi(x)\|^2 + \|\varphi(y)\|^2) \\
&\quad \leqslant \rho^{-4}(\|x\|^2 + \|y\|^2) \leqslant 2\rho^{-4} < 4\sigma^2.
\end{aligned}$$

□

Lemma 2.7.32 *Let A be an absolute-valued algebra over $\mathbb{K}$ such that the absolute value of A comes from an inner product $(\cdot|\cdot)$, and let x be in A such that there exists $x^* \in A$ satisfying $(xy|z) = (y|x^*z)$ for all $y, z \in A$. Then we have $x^*(xy) = \|x\|^2 y$ for every $y \in A$.*

Proof For $y \in A$, we have $(xy|xy) = \|x\|^2(y|y)$. Then, by linearization, we obtain that the equality $(xz|xy) = \|x\|^2(z|y)$ holds for all $y, z \in A$. Since $(xz|xy) = (z|x^*(xy))$, we deduce $(z|x^*(xy)) = \|x\|^2(z|y)$, which, in view of the arbitrariness of z, yields $x^*(xy) = \|x\|^2 y$. □

Proposition 2.7.33 *Let A be an absolute-valued real algebra with a left unit e. Then the absolute value of A derives from an inner product $(\cdot|\cdot)$, and, setting $x^* := 2(x|e)e - x$, we have $(xy|z) = (y|x^*z)$ and $x^*(xy) = \|x\|^2 y$ for all $x, y, z \in A$.*

Proof By Corollary 2.7.29, the absolute value of A derives from an inner product $(\cdot|\cdot)$. For y, u in A with $(e|u) = 0$, we have

$$\begin{aligned}
(1 + \|u\|^2)\|y\|^2 = \|e+u\|^2\|y\|^2 &= \|(e+u)y\|^2 \\
&= \|y + uy\|^2 = (1 + \|u\|^2)\|y\|^2 + 2(uy|y),
\end{aligned}$$

and hence $(uy|y)=0$. By linearization, we deduce $(uy|z)=-(y|uz)$ for all $u,y,z\in A$ with $(e|u)=0$, or, equivalently, $(xy|z)=(y|x^*z)$ for all $x,y,z\in A$. Finally, apply Lemma 2.7.32. $\square$

Corollary 2.7.34 *Let A be an absolute-valued real algebra. Then A is a left-division algebra if (and only if) there exists $e\in A$ such that $eA=A$.*

Proof Assume that there exists $e\in A$ such that $eA=A$. Then the normed space of A becomes an absolute-valued algebra (say B) with left unit e under the product $x\odot y:=L_e^{-1}(xy)$. Therefore, by Proposition 2.7.33, we have $x\odot(x^*\odot y)=\|x\|^2y$ for all $x,y\in A$ and a suitable involution $*$ on A. This implies that B (and hence A) is a left-division algebra. $\square$

We do not know if the condition $eA=A$ in Corollary 2.7.34 above can be relaxed to the one that eA is dense in A. Anyway, we have the following consequence.

Corollary 2.7.35 *Let A be an absolute-valued real algebra. We have:*

(i) *A is a quasi-division algebra (if and) only if it is a one-sided division algebra.*
(ii) *If A is a left-division algebra, then so is its completion.*

Proof Assertion (i) follows from Corollary 2.7.34 applied to A or to the opposite algebra of A. Assume that A is a left-division algebra. Take a norm-one element $e\in A$, so that we have $eA=A$. Then, since L_e is an isometry, this equality remains true when the completion of A replaces A. Therefore, by Corollary 2.7.34, the completion of A is a left-division algebra. $\square$

We introduced transitive normed spaces in Subsection 2.6.3. The most classical example of a transitive normed space is given by the following.

Lemma 2.7.36 *Every pre-Hilbert space over $\mathbb{K}$ is transitive.*

Proof Let X be a pre-Hilbert space over $\mathbb{K}$. Let u,v be in $\mathbb{S}_X$. If u,v are linearly dependent, then the existence of a surjective linear isometry $F:X\to X$ such that $F(u)=v$ is obvious. Otherwise, we denote by Y the linear hull of $\{u,v\}$, we denote by Z the orthogonal complement of Y in X, we write $X=\mathbb{K}u\oplus\mathbb{K}u'\oplus Z$ and $X=\mathbb{K}v\oplus\mathbb{K}v'\oplus Z$, where u' and v' are norm-one elements of Y orthogonal to u and v, respectively, and we consider the unique linear mapping $F:X\to X$ satisfying $F(u)=v$, $F(u')=v'$, and $F(z)=z$ for every $z\in Z$. Then, clearly, F is a surjective linear isometry such that $F(u)=v$. $\square$

Let H be a nonzero real pre-Hilbert space. Fix a norm-one element $\mathbf{1}\in H$, and define a product on H by

$$xy=(x|\mathbf{1})y+(y|\mathbf{1})x-(x|y)\mathbf{1}\,. \tag{2.7.2}$$

In this way H becomes a quadratic commutative real algebra (say A), which will be called *the quadratic commutative algebra of H*. The algebra A above becomes a Cayley algebra relative to the standard algebra involution

$$x\to x^*:=2(x|\mathbf{1})\mathbf{1}-x. \tag{2.7.3}$$

The notion of the quadratic commutative algebra of a nonzero real pre-Hilbert space is nothing other than a simplification of that of the quadratic flexible algebra of a pre-H-algebra (see Definition 2.6.4) in the particular case where the product of the pre-H-algebra is identically zero. Indeed, denoting by E the orthogonal complement of $\mathbb{R}\mathbf{1}$ in H, and endowing E with the zero product, E becomes a pre-H-algebra, and the quadratic commutative algebra of H, as defined above, coincides with the quadratic flexible algebra of E. Consequently, by Proposition 2.6.5, the quadratic commutative algebra of H is a normed algebra, but this fact does not matter for the current development. We note that, in view of Lemma 2.7.36, the choice of the norm-one element $\mathbf{1}$ above is structurally irrelevant. It follows that quadratic commutative algebras of nonzero real pre-Hilbert spaces, and nonzero real pre-Hilbert spaces themselves, are in a bijective categorical correspondence.

Now, let A be the quadratic commutative algebra of a nonzero real pre-Hilbert space, and let K be a nonzero real pre-Hilbert space (possibly different from the one underlying A). By a *unital $*$-representation* of A on K we mean any unit-preserving Jordan homomorphism $\phi : A \to BL(K)$ satisfying

$$(\phi(x)(\eta)|\zeta) = (\eta|\phi(x^*)(\zeta)) \tag{2.7.4}$$

for every $x \in A$ and all $\eta, \zeta \in K$. Let ϕ be a unital $*$-representation of A on K, and let x be in A. Then, by (2.7.3), we have $\phi(x^*) = 2(x|\mathbf{1})I_K - \phi(x)$, and hence, since $x^2 = 2(x|\mathbf{1})x - \|x\|^2\mathbf{1}$ (as a consequence of (2.7.2)), we get

$$\phi(x^*)\phi(x) = 2(x|\mathbf{1})\phi(x) - \phi(x)^2 = 2(x|\mathbf{1})\phi(x) - \phi(x^2) = \|x\|^2 I_K.$$

Therefore, by (2.7.4), for $\eta \in K$ we have

$$\|\phi(x)(\eta)\|^2 = (\eta|\phi(x^*)\phi(x)(\eta)) = \|x\|^2\|\eta\|^2. \tag{2.7.5}$$

Now, the first assertion in Proposition 2.7.37 immediately below is a straightforward consequence of the equality (2.7.5) above, whereas the second one is simply a reformulation of Proposition 2.7.33.

Proposition 2.7.37 *If A is the quadratic commutative algebra of a nonzero real pre-Hilbert space, and if ϕ is a unital $*$-representation of A on its own pre-Hilbert space, then the normed space of A with the new product $\odot$ defined by $x \odot y := \phi(x)(y)$ becomes an absolute-valued real algebra with a left unit. Moreover, there are no absolute-valued real algebras with a left unit other than those given by the construction method just described.*

Theorem 2.7.38 *Let H denote the linear hull of the canonical Hilbertian basis of the real Hilbert space ℓ_2. Then, under their own norms and suitable products, both H and ℓ_2 become absolute-valued left-division algebras with left units.*

Proof Let $\{u_0, u_1, u_2, \ldots, u_n, \ldots\}$ be the canonical Hilbertian basis of ℓ_2. Let us consider the quadratic commutative algebra of H (say A) by choosing $\mathbf{1} := u_0$ and defining the product according to (2.7.2). Now, for $n \geqslant 0$, let us denote by H_n the subspace of H generated by $\{u_0, \ldots, u_n\}$. Then the quadratic commutative algebra of H_n (say A_n) is a subalgebra of A, and the standard involution of A_n is nothing other than the restriction to A_n of the standard involution of A. For $n \geqslant 0$, let G_n stand for the subspace of H generated by $\{u_0,\dots,u_{2^n-1}\}$. We will prove by induction that, for each $n \geqslant 0$, there exists a unital $*$-representation ϕ_n of A_n on G_n satisfying

$$\phi_n(a)(b) = \phi_m(a)(b) \text{ for all } m < n, a \in A_m, \text{ and } b \in G_m. \tag{2.7.6}$$

Starting from the unique possible unital $*$-representation ϕ_0 from A_0 on G_0, assume that, for a certain $n \geqslant 0$, we have been able to find a unital $*$-representation ϕ_n of A_n on G_n satisfying (2.7.6). Trying to define ϕ_{n+1}, first observe that, since the dimension of G_{n+1} is twice that of G_n, we can choose a linear isometry α_n from G_n onto the orthogonal complement of G_n in G_{n+1}, so that every element of G_{n+1} can be uniquely written as $b+\alpha_n(c)$ for suitable $b,c \in G_n$. Note also that every element of A_{n+1} can be uniquely written as $a+\lambda u_{n+1}$ for suitable $a \in A_n$ and $\lambda \in \mathbb{R}$. We can then define ϕ_{n+1} by

$$\phi_{n+1}(a+\lambda u_{n+1})(b+\alpha_n(c)) := \phi_n(a)(b) - \lambda c + \alpha_n(\phi_n(a^*)(c) + \lambda b),$$

and straightforwardly realize that ϕ_{n+1} becomes a unital $*$-representation of A_{n+1} on G_{n+1} satisfying (2.7.6) with $n+1$ instead of n, concluding in this way the induction argument. Now, for $a \in A$ and $b \in H$, we can choose $n \geqslant 0$ such that $a \in A_n$ and $b \in G_n$, and realize that $\phi_n(a)(b)$ does not depend on the chosen n. Therefore we can define $\phi(a)(b) := \phi_n(a)(b) \in H$, and verify without problems that ϕ becomes a unital $*$-representation of A on H. By Proposition 2.7.37, H $(= A$ as a pre-Hilbert space) becomes an absolute-valued algebra with a left unit, for a suitable product. By Corollaries 2.7.34 and 2.7.35(ii), this absolute-valued algebra and its completion are left-division algebras with a left unit. □

As a consequence of Proposition 2.6.25, infinite-dimensional absolute-valued algebras have zero centre. Nevertheless, invoking Fact 2.6.47, Theorem 2.7.38 above has the following remarkable consequence.

Corollary 2.7.39 *Let H denote the linear hull of the canonical Hilbertian basis of the real Hilbert space ℓ_2. Then, under their own norms and suitable products, both H and ℓ_2 become absolute-valued algebras containing a nonzero idempotent which commutes with all elements in these algebras.*

In view of Corollary 2.7.39 and/or Theorem 2.7.38, results from Proposition 2.7.40 to Corollary 2.7.43 below have their own interests.

Proposition 2.7.40 *Let A be an absolute-valued real algebra containing elements e and a such that $eA = A$, $a \neq 0$, and $[a,A] = 0$. Then A is finite-dimensional.*

Proof Since $eA = A$ and $a \neq 0$, Corollary 2.7.34 applies to obtain that $aA = A$. But then, since $[a,A] = 0$, we have $Aa = A$. Therefore the result follows from the implication (iii)⇒(i) in Corollary 2.6.24. □

Although straightforward, we emphasize the following.

Corollary 2.7.41 *Let A be an absolute-valued real algebra containing both a left unit and a nonzero element commuting with all elements of A. Then A is finite-dimensional.*

Proposition 2.7.42 *Let A be an absolute-valued real algebra with a left unit, and let $|||\cdot|||$ be an algebra norm on A. Then we have $\|\cdot\| \leqslant |||\cdot|||$.*

Proof Let e denote the left unit of A, and let x be in A. According to Proposition 2.7.33, we have $L_{x^*} \circ L_x = \|x\|^2 I_A$, where $x^* := 2(x|e)e - x$. It follows that

$$\|x\|^2 \leqslant |||L_{x^*}||| \, |||L_x||| \leqslant |||x^*||| \, |||x||| \leqslant (2\|x\| \, |||e||| + |||x|||) |||x||| ,$$

so $(1 + |||e|||^2)\|x\|^2 \leqslant (|||x||| + \|x\| \, |||e|||)^2$, and so $(\sqrt{1+|||e|||^2} - |||e|||)\|x\| \leqslant |||x|||$. Now apply Proposition 2.6.19, with ϕ equal to the identity mapping from $(A, |||\cdot|||)$ to $(A, \|\cdot\|)$. □

We do not know if Proposition 2.7.42 remains true whenever the requirement of the existence of a left unit in A is relaxed to the one that A is a left-division algebra. Now note that, as a consequence of Proposition 2.6.19, *finite-dimensional algebras over $\mathbb{K}$ have at most one absolute value.* Since finite-dimensional absolute-valued algebras are of course left-division algebras, the following corollary becomes a generalization of the fact just pointed out.

Corollary 2.7.43 *Let A be a left-division real algebra. Then there exists at most one absolute value on A.*

Proof Let $\|\cdot\|$ and $|||\cdot|||$ be absolute values on A. Fix $e \in A$ with $\|e\| = 1$, and consider the absolute-valued real algebra B consisting of the vector space of A, the norm $\|\cdot\|$, and the product $x \odot y := L_e^{-1}(xy)$. Since B has a left unit, and $|||e|||^{-1}|||\cdot|||$ is an algebra norm on B, Proposition 2.7.42 applies giving that $\|\cdot\| \leqslant |||e|||^{-1}|||\cdot|||$. Then, keeping in mind that $|||\cdot|||$ is an algebra norm on A, and that $\|\cdot\|$ is an absolute value on A, we deduce from Proposition 2.6.19 that $\|\cdot\| \leqslant |||\cdot|||$. By symmetry, we have also $|||\cdot||| \leqslant \|\cdot\|$. □

Remark 2.7.44 Absolute-valued algebras need not have uniqueness of the absolute value. Indeed, the real or complex Banach space ℓ_p $(1 \leqslant p < \infty)$ can be converted into an absolute-valued algebra, by simply taking an injective mapping $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, by defining the product of basic vectors $e_i, e_j \in \ell_p$ by the rule $e_i e_j := e_{\phi(i,j)}$, and then by extending the product by bilinearity and continuity. Now, let $1 \leqslant p < q < \infty$, and consider ℓ_p and ℓ_q as absolute-valued algebras by means of the above construction (with the same injective mapping $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$). Then ℓ_p is a complete absolute-valued algebra which, algebraically, can be seen as a subalgebra of ℓ_q. However the restriction of the natural absolute value of ℓ_q to ℓ_p does not coincide with (nor even is equivalent to) the natural absolute value of ℓ_p. As a consequence, the requirement of the existence of a left unit in Proposition 2.7.42 cannot be altogether removed.

After Theorem 2.7.38, the following problem becomes relevant.

Problem 2.7.45 Is every normed division real algebra finite-dimensional?

In contrast to Problem 2.7.4 (which, thanks to Corollary 2.7.3, only remains open in the non-complete case), a general answer to Problem 2.7.45 is unknown even in the complete case. In this case, the problem reduces to the setting of unital algebras. For, if A is a complete normed division real algebra, and if a is any nonzero element of A, then the vector space of A, endowed with the product $x \odot y := R_a^{-1}(x) L_a^{-1}(y)$

and a suitable positive multiple of the norm of A, becomes a complete normed unital division real algebra (whose unit is precisely a^2). Now the next result has its own interest.

Proposition 2.7.46 *Let A be a complete normed unital division real algebra, and let B be any nonzero closed subalgebra of A. Then B contains the unit of A, and is a division algebra.*

Proof Take a nonzero element $b \in B$. Since $\mathbb{R}\mathbf{1}+B$ is a closed subalgebra of A containing $\mathbf{1}$, Corollary 2.7.21 applies to find $r \in \mathbb{R}$ and $c \in B$ such that $b(r\mathbf{1}+c)=\mathbf{1}$. Therefore $\mathbf{1}=rb+bc \in B$. Now it is enough to apply Corollary 2.7.21 to both A and $A^{(0)}$ to conclude that B is a division algebra. □

We do not know whether there are complete normed infinite-dimensional division real algebras. Anyway, we are going to construct complete normed infinite-dimensional real algebras such that the operators of left and right multiplication by any nonzero element are surjective. Before beginning this construction, let us realize that it is prohibited in the associative setting. Indeed, we have the following.

Proposition 2.7.47 *Let A be a nonzero associative algebra over $\mathbb{K}$ such that R_a is surjective for every nonzero element $a \in A$. Then A is a division algebra. Therefore, if in addition A is a normed algebra, then A is finite-dimensional.*

Proof In view of Proposition 2.5.38, to prove the first conclusion it is enough to show that R_a is injective for every nonzero element $a \in A$. Let a be in A such that there exists $b \in A \setminus \{0\}$ satisfying $ba=0$. Then, since $A=Ab$, we have $Aa=Aba=0$, and hence $a=0$ (since, otherwise, we would have $Aa=A \neq 0$, a contradiction). Now that we know that A is a right-division algebra, the finite dimensionality of A in the normed case follows from Propositions 2.5.39 and 2.5.40. □

As usual, given a bounded linear operator F on a Hilbert space H, we denote by F^* the adjoint of F.

Lemma 2.7.48 *Let A be a complete absolute-valued algebra over $\mathbb{K}$ whose Banach space is a Hilbert space, let $*$ be a conjugate-linear vector space involution on A, and define a new product $\odot$ on A by $a \odot b := (R_{b^*})^*(a)$. Then, for every $b \in A \setminus \{0\}$, the operator $R_b^{\odot}$, of right multiplication by b relative to the product $\odot$, is surjective.*

Proof Note that, for $b \in A$, we have $R_b^{\odot}=(R_{b^*})^*$, so that it is enough to show that $(R_b)^*$ is surjective for every norm-one element $b \in A$. But, since A is an absolute-valued algebra, for such an element b, R_b is a linear isometry, and hence, since A is a Hilbert space, we have $(R_b)^*R_b=I_A$, which implies that $(R_b)^*$ is indeed surjective. □

By Corollary 2.7.26, *if A is a nonzero complete normed complex algebra such that R_a is bijective for some $a \in A$ and R_x is surjective for every $x \in A \setminus \{0\}$, then A is isomorphic to* $\mathbb{C}$. Nevertheless, we have the following.

Corollary 2.7.49 *There exists a complete normed infinite-dimensional complex algebra A such that, for every $x \in A \setminus \{0\}$, the operator R_x is surjective.*

Proof According to Remark 2.7.44, the complex Hilbert space ℓ_2 can be converted into an absolute-valued algebra (say A) under a suitable product. Defining $*$ on the linear hull of the basic vectors of ℓ_2 as the unique conjugate-linear operator on ℓ_2 fixing all basic vectors, and extending $*$ by continuity, $*$ becomes an isometric conjugate-linear vector space involution on A. Now apply Lemma 2.7.48, noticing that, since $*$ is isometric, $(A, \odot)$ is indeed a normed algebra. $\square$

Our next lemma involves standard terminology of H^*-algebras. Following [199, 200, 589], by a *left semi-H^*-algebra* over $\mathbb{K}$ we mean an algebra A over $\mathbb{K}$ which is also a Hilbert space in such a way that the product of A becomes continuous, and is endowed with a conjugate-linear vector space involution satisfying $(ab|c) = (b|a^*c)$ for all $a, b, c \in A$.

Lemma 2.7.50 *Let A be a left semi-H^*-algebra over $\mathbb{K}$ whose involution $*$ is isometric, and define a new product $\odot$ on A by $a \odot b := (R_{b^*})^*(a)$. Then $*$ is an algebra involution on $(A, \odot)$.*

Proof Let a, b, c be in A. By the definition of the product $\odot$, we have $(a \odot b|c) = (a|cb^*)$. Using this fact, together with the axiom of left semi-H^*-algebras $(ab|c) = (b|a^*c)$ and the assumption that the involution $*$ is isometric $(a|b) = (b^*|a^*)$, we obtain

$$\begin{aligned}(b^* \odot a^*|c) &= (b^*|ca) = (c^*b^*|a) = \overline{(a|c^*b^*)}\\ &= \overline{(a \odot b|c^*)} = (c^*|a \odot b) = ((a \odot b)^*|c),\end{aligned}$$

and hence $(a \odot b)^* = b^* \odot a^*$. $\square$

We recall that, by Proposition 2.7.33, absolute-valued real algebras with a left unit are pre-Hilbert spaces.

Proposition 2.7.51 *Let A be a complete absolute-valued real algebra with a left unit e, let $*$ stand for the isometric vector space involution on A given by $a^* := 2(a|e)e - a$, and define a new product $\odot$ on A by $a \odot b := (R_{b^*})^*(a)$. Then we have:*

(i) *$(A, \odot)$ is a complete normed real algebra, and $*$ becomes an algebra involution on $(A, \odot)$.*
(ii) *The operators of left and right multiplication by any nonzero element on $(A, \odot)$ are surjective.*
(iii) *For every $a \in A$, the equalities $a \odot a^* = a^* \odot a = \|a\|^2 e$ hold.*

Proof By Proposition 2.7.33, A, endowed with the involution $*$, becomes a left semi-H^*-algebra. Therefore, by Lemma 2.7.50, $*$ is an algebra involution on $(A, \odot)$. The remaining part of assertion (i) is straightforward.

Assertion (ii) follows from assertion (i) and Lemma 2.7.48.

To prove assertion (iii), note that, for every $a \in A$, we have $(R_a)^* R_a = \|a\|^2 I_A$ (see the proof of Lemma 2.7.48). When applied to e, the above equality gives $a \odot a^* = \|a\|^2 e$. Finally, the equality $a^* \odot a = \|a\|^2 e$ follows from the one just proved because $*$ is an isometry. $\square$

By combining Theorem 2.7.38 and Proposition 2.7.51, we get the following.

Corollary 2.7.52 *There exists a complete normed infinite-dimensional real algebra A such that, for every $a \in A \setminus \{0\}$, the operators L_a and R_a are surjective.*

We do not know if the above corollary remains true for complex algebras.

2.7.3 Discussing the results

Now, we are going to summarize and clarify several results in the current section by discussing the diagram in §2.7.53 immediately below.

§2.7.53 Let A be a nonzero complete normed algebra over $\mathbb{K}$. Then, keeping in mind §2.5.53, the implications in the following diagram hold. In the diagram, the abbreviation *t.d.z.* stands for *topological divisor of zero*.

A is a division algebra	$\Rightarrow$	A is a $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ division algebra	$\Rightarrow$	A is a quasi-division algebra
$\Downarrow$		$\Downarrow$		$\Downarrow$
A has no nonzero one-sided t.d.z.	$\Rightarrow$	A has no nonzero $\left\{\begin{matrix}\text{left}\\ \text{right}\end{matrix}\right\}$ t.d.z.	$\Rightarrow$	A has no nonzero two-sided t.d.z.

We note that the conditions and implications in the top row of the above diagram are of a purely algebraic nature, and hence do not need the algebra A to be normed. On the other hand, conditions and implications in the bottom row of the diagram only need the algebra A to be normed, but not to be complete. However, in general, vertical implications in the diagram need the algebra A to be complete normed (see Example 2.7.54 below). Nevertheless, if the algebra A is alternative, then, in view of Propositions 2.5.24 and 2.5.38, completeness of A can be dispensed for the validity of those vertical implications. Moreover, if the algebra A above is alternative, then, by Theorem 2.5.50 and Proposition 2.5.49(i), the requirement that A has no nonzero joint topological divisor of zero (a condition weaker than the weakest one in the diagram above) implies that A is a division algebra (the strongest condition in the diagram), and hence all conditions in the diagram are equivalent.

The following example shows that the vertical implication in the centre column of this diagram need not be true when completeness of A is removed.

Example 2.7.54 Let H denote the linear hull of the canonical Hilbertian basis $\{e_n : n \in \mathbb{N}\}$ of the real Hilbert space ℓ_2, and, according to Theorem 2.7.38, endow H with a suitable product (denoted by juxtaposition) converting it into an absolute-valued left-division algebra. Let T be the linear operator on H determined by $T(e_n) = \frac{1}{n}e_n$. Then T is bijective and continuous with $\|T\| = 1$. Now, let A be the normed real algebra consisting of the normed space of H and the product $x \odot y := xT(y)$. For $x \in A$ we have $L_x^{\odot} = L_x T$, and hence A is a left division algebra. However, for $x \in A$

we also have $x \odot e_n = \frac{1}{n} x e_n \to 0$, so that all elements in A are left topological divisors of zero.

If A is finite-dimensional, then, by Exercise 1.1.88(i), the six conditions in the diagram in §2.7.53 reduce to two. Indeed, in such a setting, all vertical implications in the diagram are reversible, and one-sided division implies division, whereas, by Example 2.5.36, quasi-division does not imply division. It is worth mentioning that, even in the finite-dimensional case, the weaker condition in the diagram is strictly stronger than the one where A has no nonzero joint topological divisor of zero. Indeed, we have the following.

Example 2.7.55 Let A be the two-dimensional algebra over $\mathbb{K}$ with basis $\{u, v\}$ and multiplication table given by

$$u^2 = uv = -vu = v \quad \text{and} \quad v^2 = u.$$

Then $uA = Au = \mathbb{K}v \neq A$, and hence, since A is finite-dimensional, u becomes a nonzero two-sided divisor of zero. Nevertheless, A has no nonzero joint (topological) divisor of zero. For, if $a = \alpha u + \beta v$ is a nonzero element of A, and if $ab = ba = 0$ for some $b = \lambda u + \mu v$, then we have

$$\beta\mu = 0, \quad \alpha\lambda + \alpha\mu - \beta\lambda = 0, \quad \text{and} \quad \alpha\lambda - \alpha\mu + \beta\lambda = 0,$$

or, equivalently,

$$\beta\mu = 0, \quad \alpha\lambda = 0, \quad \text{and} \quad \alpha\mu - \beta\lambda = 0,$$

which implies, since $(\alpha, \beta) \neq (0, 0)$, that $\lambda = \mu = 0$, and hence $b = 0$.

The discussion of the diagram (in §2.7.53) in the general (possibly non-alternative and infinite-dimensional) case is more complicated. The above finite-dimensional discussion already shows that none of the two horizontal implications on the right of the diagram is reversible. On the other hand, the next example shows that the strongest condition in the bottom row of the diagram does not imply the weakest one in the top row, and hence that none of the vertical implications is reversible.

Example 2.7.56 Let $1 \leqslant p < \infty$, and let A stand for the complete absolute-valued algebra whose Banach space is the real or complex ℓ_p-space, and whose product is constructed according to Remark 2.7.44. As any absolute-valued algebra, A has no nonzero one-sided topological divisor of zero. However, a straightforward calculation shows that, for any basic vector $e_n \in A$, both L_{e_n} and R_{e_n} are not surjective on A, so that A is not a quasi-division algebra.

To continue our discussion, let us consider the following claim, whose proof is straightforward.

Claim 2.7.57 *Let A be a normed algebra over $\mathbb{K}$ with no nonzero left topological divisor of zero, and let F be a norm-one bounded linear operator on A which is injective but not bounded below. Denote by $\odot$ the product of A, and by $B = B(A, F)$ the algebra whose vector space is that of A and whose product is defined by $xy := F(x) \odot y$. Then B (with the same norm as that of A) is a normed algebra* *over $\mathbb{K}$ with no nonzero left topological divisor of zero, but every element in B is a right topological divisor of zero in B.*

Now, the following example shows that the horizontal implication on the left of the bottom row of the diagram in §2.7.53 is not reversible.

Example 2.7.58 Let $1 \leqslant p < \infty$. Then certainly we may find a norm-one bounded linear operator F, on the classical Banach space ℓ_p over $\mathbb{K}$, which is injective but not bounded below. Moreover, by Remark 2.7.44, we can choose a product on the Banach space ℓ_p converting it into an absolute-valued algebra. We denote by A the absolute-valued algebra just constructed, and consider the complete normed algebra $B = B(A,F)$ given by Claim 2.7.57, so that B has no nonzero left topological divisor of zero but every element in B is a right topological divisor of zero in B.

Remark 2.7.59 Taking in the above example $p = 2$ and $\mathbb{K} = \mathbb{R}$, and replacing Remark 2.7.44 with Theorem 2.7.38, we find a complete normed left-division real algebra B (necessarily without nonzero left topological divisors of zero), all elements of which are right topological divisors of zero in B.

It follows from the information collected to this point that the unique implication in the diagram in §2.7.53 that could be reversible is the horizontal one on the left of the top row. If $\mathbb{K} = \mathbb{C}$, then, by Corollary 2.7.3, such an implication is indeed reversible. On the contrary, if $\mathbb{K} = \mathbb{R}$, then, by Theorem 2.7.38 and the implication (ii)⇒(i) in Corollary 2.6.24, we are provided with a complete normed left-division real algebra which is not a division algebra. Summarizing, we have the following.

Proposition 2.7.60 *If $\mathbb{K} = \mathbb{C}$, then none of the implications in the diagram in §2.7.53 is reversible, except the horizontal one on the left of the top row, which is indeed reversible. If $\mathbb{K} = \mathbb{R}$, then, without any exception, none of the implications in the diagram is reversible.*

Now that the discussion of the diagram in §2.7.53 has been formally concluded, we provide the reader with some complementary details. To this end, we consider the following.

Claim 2.7.61 *Let A be a nonzero algebra over $\mathbb{K}$, and let $B = B(A)$ denote the algebra over $\mathbb{K}$ whose vector space is $A \times A$ and whose product is defined by*

$$(x_1,x_2)(y_1,y_2) := (x_1y_2, x_1y_1 + x_2y_2) .$$

Then B has nonzero divisors of zero. Moreover we have:

(i) *B is a quasi-division algebra if and only if A is a division algebra.*
(ii) *If A is in fact a normed algebra, and if we consider B as a normed algebra under the norm $\|(x_1,x_2)\| := \|x_1\| + \|x_2\|$, then B has no nonzero two-sided topological divisor of zero if and only if A has no nonzero one-sided topological divisor of zero.*

Proof Since for x_2 and y_1 in A the equality $(0,x_2)(y_1,0) = 0$ holds, the existence in B of nonzero divisors of zero is not in doubt.

Assume that B is a quasi-division algebra. Let a be in $A\backslash\{0\}$. Since $L_{(0,a)}$ is not bijective on B, the operator $R_{(0,a)}$ must be bijective on B. This means that the mapping $(x_1,x_2)\to(x_1a,x_2a)$ from B to B is bijective. Equivalently, the operator R_a is bijective on A. Analogously, the fact that $R_{(a,0)}$ is not bijective on B allows us to obtain that L_a is bijective on A. Since a is arbitrary in $A\backslash\{0\}$, we get that A is a division algebra. Now assume that A is a division algebra. Let $x=(x_1,x_2)$ be in $B\backslash\{0\}$. If $x_1\neq 0$, then the operator $L_{(x_1,x_2)}$ is bijective on B with inverse mapping given by

$$(y_1,y_2)\longrightarrow(L_{x_1}^{-1}(y_2-L_{x_2}L_{x_1}^{-1}(y_1)),L_{x_1}^{-1}(y_1)).$$

Otherwise, the operator $R_{(x_1,x_2)}$ is bijective on B with inverse mapping given by

$$(y_1,y_2)\longrightarrow(R_{x_2}^{-1}(y_1),R_{x_2}^{-1}(y_2)).$$

Since x is arbitrary in $B\backslash\{0\}$, B is a quasi-division algebra.

In this last paragraph of the proof we suppose that A is actually a normed algebra, and consider B as a normed algebra under the norm

$$\|(x_1,x_2)\|:=\|x_1\|+\|x_2\|.$$

Assume that B has no nonzero two-sided topological divisor of zero. Then, for each nonzero element a in A we have that $(0,a)$ (respectively, $(a,0)$) is not a right (respectively, left) topological divisor of zero in B. This implies that such an a is not a one-sided topological divisor of zero in A. Therefore A has no nonzero one-sided topological divisor of zero. Now assume that A has no nonzero one-sided topological divisor of zero. Let $x=(x_1,x_2)$ be in $B\backslash\{0\}$. First suppose that $x_1\neq 0$. If $\{y_n\}=\{(y_1^n,y_2^n)\}$ is a sequence in B with $\{xy_n\}\to 0$, then we have in A

$$\{x_1y_2^n\}\longrightarrow 0 \;\text{ and }\; \{x_1y_1^n+x_2y_2^n\}\longrightarrow 0,$$

so $\{y_2^n\}\to 0$ and $\{y_1^n\}\to 0$ (since x_1 is not a left topological divisor of zero in A), and so $\{y_n\}\to 0$. Therefore x is not a left topological divisor of zero in B. Now suppose that $x_1=0$ (so that $x_2\neq 0$). Then, for every $y=(y_1,y_2)$ in B we have $yx=(y_1x_2,y_2x_2)$, and we easily realize that x is not a right topological divisor of zero in B (since x_2 is not a right topological divisor of zero in A). $\square$

Now, the following example shows that the horizontal implication on the right of the bottom row of the diagram in §2.7.53 is not reversible, even in the infinite-dimensional case.

Example 2.7.62 Taking the algebra A in Claim 2.7.61 equal to any complete absolute-valued infinite-dimensional algebra over $\mathbb{K}$, we obtain an infinite-dimensional complete normed algebra $B=B(A)$ over $\mathbb{K}$ which, by assertion (ii) in the claim, has no nonzero two-sided topological divisor of zero but has nonzero divisors of zero.

In view of Theorem 2.7.7, a similar example, showing that the horizontal implication on the right of the top row of the diagram in §2.7.53 is not reversible in the infinite-dimensional case, is impossible if $\mathbb{K}=\mathbb{C}$. If $\mathbb{K}=\mathbb{R}$, the existence of such an example is unknown for the moment. Anyway, as a straightforward consequence of Claim 2.7.61, we have the following.

Corollary 2.7.63 *There exist quasi-division real algebras of dimension* 2, 4, 8 *and* 16 *which are not division algebras. Moreover, if there is an infinite-dimensional (complete) normed division algebra, then there is also an infinite-dimensional* (*complete) normed quasi-division algebra which is not a division algebra.*

We continue the complements to the discussion of the diagram in §2.7.53 by considering the case that the complete normed algebra A is commutative. In that case, clearly, all conditions in the top row are pairwise equivalent, and the same can be said for the conditions in the second line. However, conditions in the bottom row do not imply conditions in the top row. Indeed, we have the following (necessarily infinite-dimensional) example.

Example 2.7.64 Let B stand for the complete absolute-valued algebra whose Banach space is the real or complex ℓ_2-space, and whose product is constructed according to Remark 2.7.44, and set $A := B^{\mathrm{sym}}$. For $x = \sum_i \lambda_i e_i$ and $y = \sum_i \mu_i e_i$ in A, we have

$$xy = \sum_{i,j} \lambda_i \mu_j e_{\phi(i,j)} \quad \text{and} \quad yx = \sum_{i,j} \mu_i \lambda_j e_{\phi(i,j)} .$$

Therefore

$$(xy|yx) = \sum_{i,j} \lambda_i \mu_j \overline{\mu_i} \overline{\lambda_j} = \left(\sum_i \lambda_i \overline{\mu_i} \right) \left(\sum_j \mu_j \overline{\lambda_j} \right) = (x|y)(y|x) = |(x|y)|^2 .$$

It follows that

$$\begin{aligned} 4\|x \bullet y\|^2 &= \|xy + yx\|^2 = \|xy\|^2 + \|yx\|^2 + 2\Re((xy|yx)) \\ &= 2(\|x\|^2\|y\|^2 + |(x|y)|^2), \end{aligned}$$

and hence $\|x \bullet y\| \geqslant 2^{-1/2}\|x\|\|y\|$. Thus A is a complete nearly absolute-valued commutative algebra over $\mathbb{K}$ (so it has no nonzero one-sided topological divisor of zero), but, after a straightforward computation, we realize that, for any basic vector e_n, the operator of multiplication by e_n on A is not surjective (so A is not a division algebra).

We conclude the complements to the discussion of the diagram in §2.7.53 by considering the case that the complete normed algebra A is unital. In that case, by Proposition 2.7.19, the vertical implications on the left and middle columns are reversible. Moreover, if in addition A is complex, then, by Corollary 2.7.9, all conditions in the top row of the diagram are equivalent to the one that A is isomorphic to $\mathbb{C}$. Therefore, in the complete normed unital complex case, all conditions in the diagram are equivalent to the one that A is isomorphic to $\mathbb{C}$, unless possibly the one that A has no nonzero two-sided topological divisor of zero. The following proposition shows that this last condition is indeed exceptional.

Proposition 2.7.65 *There exists an infinite-dimensional complete normed unital complex algebra with no nonzero two-sided topological divisor of zero.*

Proof Let $\{e_n : n \in \mathbb{N}\}$ be the canonical Hilbertian basis of the complex Hilbert space ℓ_2, and, according to Remark 2.7.44, convert ℓ_2 into an absolute-valued algebra under the product determined by $e_i e_j = e_{2^i 3^j}$. Let T be the bounded linear operator on ℓ_2 determined by $T(e_n) = \frac{1}{n}e_n$. Then T is injective and compact with $\|T\| = 1$ (cf. the solution of Exercise 1.4.30).

Now, let B be the complete normed complex algebra consisting of the Banach space of ℓ_2 and the product $x \odot y := xT(y)$. We claim that

$$\mathrm{sp}(BL(B), L_x^{\odot}) = \{0\} \text{ for every } x \in B. \tag{2.7.7}$$

To prove the claim, note at first that, since $L_x^{\odot} = L_x T$ and T is compact, $L_x^{\odot}$ is compact, so that, in view of Corollary 1.4.29(i), it is enough to show that $L_x^{\odot}$ has no nonzero eigenvalue. For $x = \sum_i \lambda_i e_i \in B \setminus \{0\}$, set

$$p(x) := \min\{i \in \mathbb{N} : \lambda_i \neq 0\}.$$

Assume that the claim is not true. Then there are $x = \sum_i \lambda_i e_i$ and $y = \sum_i \mu_i e_i$ in $B \setminus \{0\}$ such that $L_x^{\odot}(y) = \alpha y$ for some nonzero complex number α. Then, since $L_x^{\odot}(y) = \sum_{i,j} \frac{\lambda_i \mu_j}{j} e_{2^i 3^j}$, we realize that

$$p(y) = p(\alpha y) = p(L_x^{\odot}(y)) = 2^{p(x)} 3^{p(y)} > p(y),$$

a contradiction.

Now, let A stand for the normed unital extension of B (cf. Proposition 1.1.107). We are going to conclude the proof by showing that A has no nonzero two-sided topological divisor of zero. Let $a = \lambda \mathbb{1} + x$ be a two-sided topological divisor of zero in A. Since a is a left topological divisor of zero, there are sequences λ_n and x_n in $\mathbb{C}$ and B, respectively, such that $|\lambda_n| + \|x_n\| = 1$ for every $n \in \mathbb{N}$, $\lambda \lambda_n \to 0$, and $\lambda x_n + \lambda_n x + x \odot x_n \to 0$. Assume that $\lambda \neq 0$. Then we have $\lambda_n \to 0$, and, consequently, $\lambda x_n + x \odot x_n \to 0$ and $\|x_n\| \to 1$. The two last convergences imply that $-\lambda$ lies in $\mathrm{sp}(BL(B), L_x^{\odot})$, contradicting (2.7.7). Therefore $\lambda = 0$, and hence $a = x \in B$. Now, since x is a right topological divisor of zero in A, there are sequences μ_n and y_n in $\mathbb{C}$ and B, respectively, such that $|\mu_n| + \|y_n\| = 1$ for every $n \in \mathbb{N}$, and $\mu_n x + y_n \odot x \to 0$. By passing to subsequences if necessary, we can assume that μ_n converges to some $\mu \in \mathbb{C}$, and, consequently, that $y_n \odot x \to -\mu x$. Assume that $x \neq 0$. Then, since for $n, m \in \mathbb{N}$ we have

$$\|y_n \odot x - y_m \odot x\| = \|(y_n - y_m) \odot x\| = \|y_n - y_m\| \|T(x)\|,$$

and T is injective, it follows that y_n is a Cauchy sequence in B. By setting $y := \lim_{n\to\infty} y_n$, we have $y \odot x = -\mu x$ (which implies that $-\mu$ lies in $\mathrm{sp}(BL(B), L_y^{\odot})$) and $|\mu| + \|y\| = 1$. The fact that $-\mu \in \mathrm{sp}(BL(B), L_y^{\odot})$ and (2.7.7) yields $\mu = 0$, and consequently, since

$$\|y\| \|T(x)\| = \|y \odot x\| = |\mu| \|x\| = 0,$$

also $y = 0$, which contradicts the equality $|\mu| + \|y\| = 1$. Therefore $a = x = 0$, as desired. $\square$

We conclude this subsection by showing how a purely normed space condition on a normed algebra without nonzero two-sided topological divisors of zero leads to the finite dimensionality.

Proposition 2.7.66 *Let A be a nonzero normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisor of zero, and assume that there is a (possibly non-closed) subspace Y of $BL(A)$ satisfying $BL(A) = \mathbb{K}I_A + Y$, and such that every operator in Y is not bounded below. Then A is isomorphic to $\mathbb{K}$.*

Proof Note at first that, since Y consists of operators which are not bounded below, we must have in fact that $BL(A) = \mathbb{K}I_A \oplus Y$. Let λ be the unique linear functional on $BL(A)$ such that $F - \lambda(F)I_A \in Y$ for every $F \in BL(A)$. Then $\ker(\lambda) = Y$, and hence $\lambda(T) \neq 0$ whenever $T \in BL(A)$ is bounded below. Since L_x or R_x is bounded below whenever x is in $A \setminus \{0\}$, it follows that the linear mapping $x \to (\lambda(L_x), \lambda(R_x))$ from A to $\mathbb{K}^2$ is injective, and hence that $\dim(A) \leqslant 2$. Assume that $\dim(A) = 2$. Then, with suitable identifications, we have $BL(A) = M_2(\mathbb{K}) = \mathbb{K}^4$,

$$BL(A) \setminus \mathrm{Inv}(BL(A)) = \{(x,y,z,t) \in \mathbb{K}^4 : xt - yz = 0\},$$

and $Y = \{(x,y,z,t) \in \mathbb{K}^4 : x + by + cz + dt = 0\}$ for suitable $b, c, d \in \mathbb{K}$. Since

$$Y \subseteq BL(A) \setminus \mathrm{Inv}(BL(A)), \tag{2.7.8}$$

and $(d, 0, 0, -1) \in Y$, we deduce $d = 0$. Then both $(b, -1, 0, 1)$ and $(c, 0, -1, 1)$ lie in Y, and hence, by (2.7.8), $b = c = 0$. Therefore

$$Y = \{(x,y,z,t) \in \mathbb{K}^4 : x = 0\}.$$

Thus $(0, 1, 1, 1) \in Y \cap \mathrm{Inv}(BL(A))$, contradicting (2.7.8). Therefore $\dim(A) = 1$, and the proof is concluded by applying Exercise 1.1.2. □

The actual usefulness of Proposition 2.7.66 will be discussed later (see the proofs of Propositions 2.7.73 and 2.7.74).

2.7.4 Historical notes and comments

Results from Lemma 2.7.1 to Theorem 2.7.7, Lemma 2.7.18, Corollary 2.7.22, and results from Example 2.7.56 to Corollary 2.7.63 are taken from [529]. Some of the original arguments in [529] have been slightly refined (cf. Propositions 2.7.2 and 2.7.6) to be applied in the proof of Theorem 2.7.14. Up to the appropriate correction, Corollary 2.7.3 was previously pointed out by Kaplansky in [762, p. 13] (see our comments at the beginning of the preface). According to the acknowledgements of [529], the existence of two-dimensional quasi-division algebras, given by Example 2.5.36, is due to A. Moreno, who also proved Corollary 2.7.8 (that finite-dimensional quasi-division complex algebras have dimension $\leqslant 2$), thus allowing the author of [529] to conjecture that Theorem 2.7.7 could be true. Example 2.7.55 is also due to Moreno (recent private communication). Lemma 2.7.12, Propositions 2.7.13 and 2.7.65, and the part of Theorem 2.7.14 corresponding to assumption (ii) are taken from the Marcos–Rodríguez–Velasco paper [414]. The remaining parts of Theorem 2.7.14, as well as Corollaries 2.7.10, 2.7.11, 2.7.15, 2.7.20, 2.7.21, and 2.7.23–2.7.26 are new. Propositions 2.7.16 (excluding condition (i)) and 2.7.19, results from Lemma 2.7.27 to Corollary 2.7.31, and Example 2.7.64 are taken from the Kaidi–Ramírez–Rodríguez paper [369]. Results from Lemma 2.7.32 to Proposition 2.7.37

are originally due to Rodríguez [521], with the exception of Lemma 2.7.36, which is folklore.

§**2.7.67** Now, as announced in Subsection 2.6.3, we follow [521, Remark 4(ii)] to provide the reader with an alternative complete proof of the non-commutative Urbanik–Wright theorem, based on Proposition 2.7.33:

Let A be an absolute-valued unital real algebra. Then, as a consequence of Proposition 2.7.33, the norm of A derives from an inner product $(\cdot|\cdot)$ and, for $x, y \in A$ with x orthogonal to $\mathbf{1}$, we have $x(xy) = -\|x\|^2 y$. Taking $y = \mathbf{1}$ in the above equality, we obtain $x^2 = -\|x\|^2\mathbf{1}$ for every $x \in A$ orthogonal to $\mathbf{1}$, which implies that A is a quadratic algebra. Moreover, the same equality now yields $L_{x^2} = L_x^2$ for every $x \in A$ orthogonal to $\mathbf{1}$, and by symmetry we also have $R_{x^2} = R_x^2$ for such an x, so that, by an easy process of linearization, we realize that A is alternative. Now, A is an alternative quadratic real algebra with no nonzero divisor of zero, and hence, by the Frobenius–Zorn theorem (Theorem 2.5.29) and Proposition 2.6.19, A is equal to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. □

We remarked in Subsection 2.6.3 that absolute-valued pre-Hilbert algebras are composition algebras. In [377], Kaplansky proved that composition division algebras are finite-dimensional, and commented on his attempts to show that the same is true when 'division' is relaxed to 'left-division'. After Theorem 2.7.38, it is now clear that Kaplansky's attempts could not be successful. By the way, Kaplansky's result just quoted, together with Corollary 2.7.29, implies Wright's result [640], already stated in Corollary 2.6.24, that absolute-valued division real algebras are finite-dimensional. Theorem 2.7.38 and the proof given here are due to Cuenca [190]. The formalization of the proof, in terms of unital $*$-representations, is taken from [799, pp. 44–6]. For an alternative proof, the reader is referred to [530]. Simultaneously to Cuenca's discovery, Rodríguez [521] applied the possibility of representing the so-called 'canonical anticommutation relations' in quantum mechanics by means of bounded linear operators on complex Hilbert spaces [699, Proposition 5.2.2], to show that the quadratic commutative algebra of every infinite-dimensional real Hilbert space has a 'topologically irreducible' unital $*$-representation on its own Hilbert space. Then he applied Corollary 2.7.34 and Proposition 2.7.37 to get the following.

Theorem 2.7.68 *Every infinite-dimensional real Hilbert space, endowed with a suitable product, becomes an absolute-valued left-division algebra with a left unit and with no nonzero closed proper left ideal.*

More recently, Elduque and Pérez [232] have proved that *every infinite-dimensional real vector space can be endowed with a pre-Hilbertian norm and a product which convert it into an absolute-valued algebra with a left unit*. Since, in the construction of [232], the pre-Hilbertian norm and the product can be chosen in such a way that an arbitrarily prefixed algebraic basis becomes orthonormal, it follows that Theorem 2.7.38 and a part of Rodríguez' generalization, given by Theorem 2.7.68 above, can be derived from the Elduque–Pérez result by passing to the completion. Very recently, relevant progress on the representations of the canonical anticommutation relations on separable real Hilbert spaces has been achieved in the papers of Galina, Kaplan, and Saal [277, 278]. As pointed out by these authors, their results give rise to a classification, up to an Albert isotopy (cf. Definition 2.6.23), of all separable complete absolute-valued left-division real algebras.

Corollary 2.7.39 was first proved by Urbanik [617] by means of a direct construction. Actually, invoking Fact 2.6.47 and Theorem 2.7.68, we realize that *every infinite-dimensional real Hilbert space, endowed with a suitable product, becomes an absolute-valued algebra containing a nonzero idempotent which commutes with all elements in the algebra,* a result also originally due to Urbanik [617]. Proposition 2.7.40 is due to Diankha, Diouf, and Rochdi [210], who, applying a previous classification of all absolute-valued finite-dimensional real algebras with a left unit [508], also provide a classification up to algebra isomorphisms of the algebras appearing in Corollary 2.7.41. Corollary 2.7.41 had been proved earlier by Benslimane and Moutassim [94] using other methods.

Proposition 2.7.42, Corollary 2.7.43, and Remark 2.7.44 are originally due to Rodríguez [799]. Corollary 2.7.43 and Remark 2.7.44 were previously announced in [521, Remark 4(iii)]. The proofs of Proposition 2.7.42 and Corollary 2.7.43 given here are taken from [533]. In relation to Corollary 2.7.43 and Remark 2.7.44, it is worth mentioning the result in [535] asserting that, *if A is a real algebra endowed with a linear algebra involution $*$ such that $a^*a = aa^*$ for every $a \in A$, then there exists at most one absolute value on A*. Absolute-valued real algebras endowed with an isometric involution $*$ as above were introduced and studied by Urbanik [617], and since then their theory has been fully developed by several authors (see [76, 238, 241, 242, 288, 509]). By the way, the main result in [535] asserts that, in Urbanik's axioms, the requirement that the involution $*$ is isometric is superabundant.

A real algebra A is said to be *ordered* if it is provided with a subset A_+ of *positive elements* which is closed with respect to multiplication by positive real numbers and with respect to addition and multiplication in A, and satisfies $A_+ \cap (-A_+) = \emptyset$ and $A_+ \cup (-A_+) = A \setminus \{0\}$. In [618], Urbanik shows that $\mathbb{R}$ *is the unique absolute-valued finite-dimensional ordered real algebra.* Nevertheless, he also proves the following.

Theorem 2.7.69 *Complete absolute-valued infinite-dimensional ordered real algebras do exist.*

As pointed out in [533], a simplification of Urbanik's argument, based on Remark 2.7.44, is the following.

Proof Let $\{e_n : n \in \mathbb{N}\}$ be the canonical Hilbertian basis of the real Hilbert space ℓ_2, and, according to Remark 2.7.44, convert ℓ_2 into an absolute-valued algebra (say A) under the product determined by $e_ie_j = e_{2^i3^j}$ for all $i, j \in \mathbb{N}$. For $x = \sum_i \lambda_i e_i \in A \setminus \{0\}$, set $p(x) := \min\{i \in \mathbb{N} : \lambda_i \neq 0\}$, and define

$$A_+ := \left\{ x = \sum_i \lambda_i e_i \in A \setminus \{0\} : \lambda_{p(x)} > 0 \right\}.$$

Keeping in mind that the mapping $(i,j) \to 2^i 3^j$ is increasing in each one of its variables, it is easily seen that A_+ fulfils the properties required above for the sets of positive elements of ordered real algebras. □

Problem 2.7.45 was first raised by Wright [640]. Propositions 2.7.46 and 2.7.47 could be new. Results from Lemma 2.7.48 to Corollary 2.7.52 are taken from [521]. Example 2.7.54 is new. Proposition 2.7.66 becomes the core of the proof of [68, Proposition 3.8].

As a consequence of Corollary 2.7.29, complete absolute-valued one-sided division algebras are Hilbert spaces. On the other hand, by Corollary 2.7.31, complete normed algebras which are 'very near to being absolute-valued' have superreflexive Banach spaces. Results of the above kind show that, although the finite dimension cannot always be expected in the conclusions of non-associative Gelfand–Naimark–Kaplansky-type theorems for complete normed algebras (see for example Theorems 2.7.38 and 2.7.68, and Proposition 2.7.65), the Banach spaces of such algebras tend to be rather decent. Actually, imposing certain Banach space conditions of closeness to the finite-dimension, we can get the finite dimensionality in the conclusion of some Gelfand–Naimark–Kaplansky-type theorems. One of these Banach space conditions could be that of not being isomorphic to any of its proper subspaces. We note that this condition on a Banach space X is equivalent to the one that bounded linear operators on X which are bounded below become surjective. Therefore, if the Banach space of a complete normed algebra A fulfils this condition, then all vertical implications in the diagram in §2.7.53 are reversible. As a consequence, invoking Corollary 2.6.24 and Theorem 2.7.7, we get Facts 2.7.70 and 2.7.71, respectively, immediately below.

Fact 2.7.70 *Complete absolute-valued algebras over $\mathbb{K}$, whose Banach spaces are not isomorphic to any of their proper subspaces, are in fact finite-dimensional.*

Fact 2.7.71 *If A is a complete normed complex algebra with no nonzero two-sided topological divisor of zero, and if the Banach space of A is not isomorphic to any of its proper subspaces, then* $\dim(A) \leqslant 2$.

Facts 2.7.70 and 2.7.71 above would have no interest if infinite-dimensional Banach spaces which are not isomorphic to any of their proper subspaces did not exist. Let us therefore review how these exotic Banach spaces were born.

A Banach space X is said to be *hereditarily indecomposable* if, for every closed subspace Y of X, the unique complemented subspaces of Y are the finite-dimensional and the closed finite-codimensional ones. According to the celebrated paper of Gowers and Maurey [297], *the existence of infinite-dimensional hereditarily indecomposable separable reflexive Banach spaces over $\mathbb{K}$ is not in doubt*. Moreover, by [297, Corollary 19], *if X is a hereditarily indecomposable Banach space, then X is not isomorphic to any of its proper subspaces*. Therefore, as pointed out in [68], it is enough to apply Fact 2.7.70 to get that *complete absolute-valued algebras over $\mathbb{K}$, whose Banach spaces are hereditarily indecomposable, are in fact finite-dimensional*. In the case $\mathbb{K} = \mathbb{C}$, more can be said. Indeed, as pointed out in [533], it is enough to invoke Fact 2.7.71 to derive the following.

Corollary 2.7.72 *If A is a complete normed complex algebra with no nonzero two-sided topological divisor of zero, and if the Banach space of A is hereditarily indecomposable, then* $\dim(A) \leqslant 2$.

Fact 2.7.71 is better than Corollary 2.7.72. Indeed, there are Banach spaces which are not isomorphic to any of their proper subspaces, but are not hereditarily indecomposable [33].

Let us say that a Banach space X over $\mathbb{K}$ satisfies the *scalar-plus-compact property* if every bounded linear operator on X can be written as $\lambda I_X + T$ with $\lambda \in \mathbb{K}$ and T compact. Recently, Argyros and Haydon [32] have constructed an infinite-dimensional hereditarily indecomposable Banach space over $\mathbb{K}$ satisfying the scalar-plus-compact property, thus solving the so-called scalar-plus-compact problem, which remained open for many years. Let X be a Banach space. *Strictly singular operators* on X are defined as those bounded linear operators on X whose restrictions to any infinite-dimensional subspace of X are not bounded below. The set of all strictly singular operators on X becomes an ideal of $BL(X)$ [769, Proposition 2.c.5] containing all compact operators. Thus, the scalar-plus-compact property is the strongest form of the so-called *scalar-plus-strictly-singular property*, which means that every bounded linear operator on X can be written as $\lambda I_X + T$ with $\lambda \in \mathbb{K}$ and T strictly singular. It is well known that, if X satisfies the scalar-plus-strictly-singular property, then X is not isomorphic to any of its proper subspaces. Indeed, this follows for example from [158, Corollary on p. 63 and Theorem on p. 66].

Now, we can prove the following.

Proposition 2.7.73 *Let A be a complete normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisor of zero, and assume that the Banach space of A satisfies the scalar-plus-strictly-singular property. Then A is finite-dimensional.*

Proof Suppose on the contrary that A is infinite-dimensional. Let Y stand for the space of all strictly singular operators on A. Then, since A satisfies the scalar-plus-strictly-singular property, we have $BL(A) = \mathbb{K}I_A + Y$. On the other hand, since strictly singular operators on an infinite-dimensional Banach space cannot be bounded below, Y consists of operators which are not bounded below. Therefore, since A has no nonzero two-sided topological divisor of zero, it is enough to apply Proposition 2.7.66 to get that $\dim(A) \leqslant 1$, a contradiction. □

The existence of infinite-dimensional Banach spaces fulfilling the scalar-plus-strictly-singular property was known by Gowers and Maurey before the Argyros–Haydon solution to the scalar-plus-compact problem. Indeed, it is proved in [297, Theorem 18] that hereditarily indecomposable complex Banach spaces enjoy the scalar-plus-strictly-singular property. As a consequence, Corollary 2.7.72 can be derived from Proposition 2.7.73 above and Corollary 2.7.8.

Let us say that a Banach space X over $\mathbb{K}$ satisfies the *Shelah–Steprans property* whenever X is not separable and, for every $F \in BL(X)$, there exist

$$\lambda = \lambda(F) \in \mathbb{K} \text{ and } S = S(F) \in BL(X)$$

such that S has separable range and the equality $F = \lambda I_X + S$ holds. In our present discussion, Banach spaces enjoying the Shelah–Steprans property play a role similar to that of infinite-dimensional Banach spaces satisfying the scalar-plus-strictly-singular property. Indeed, *reflexive Banach spaces satisfying the Shelah–Steprans property do exist* [562, 631]. As a matter of fact, it is proved in [68] that *Banach spaces over* $\mathbb{K}$ *fulfilling the Shelah–Steprans property cannot underlie any complete absolute-valued algebra*. As pointed out in [533], the proof of the result of [68] just reviewed can be slightly refined to get the following.

Proposition 2.7.74 *Let X be a Banach space over $\mathbb{K}$ satisfying the Shelah–Steprans property. Then X cannot underlie any complete normed algebra with no nonzero two-sided topological divisor of zero.*

Proof Note that the set Y of all bounded linear operators on X having separable range is a subspace of $BL(X)$, and that, since X satisfies the Shelah–Steprans property, we have $BL(X) = \mathbb{K}I_X + Y$. On the other hand, since X is not separable, Y cannot contain bounded below operators. Now, if X were the Banach space underlying a complete normed algebra with no nonzero two-sided topological divisor of zero, then, by Proposition 2.7.66, we would have $\dim(X) \leqslant 1$, contradicting that X is not separable. □

In §1.4.54 we reported on the recent paper [49] stating the existence of infinite-dimensional Banach spaces in which every injective bounded linear operator is surjective. Here we note that, in the complex case, these spaces cannot underlie any complete normed algebra with no nonzero two-sided divisor of zero. Indeed, as a consequence of Theorem 2.7.7, we have the following.

Proposition 2.7.75 *Let A be a complete normed complex algebra with no nonzero two-sided divisor of zero and such that every injective bounded linear operator from A to itself is surjective. Then* $\dim(A) \leqslant 2$.

For additional information about exotic Banach spaces, the reader is referred to [31, 304, 398, 429, 679].

2.8 Complements on absolute-valued algebras and algebraicity

Introduction In this section, we complement previously reviewed facts related to absolute-valued algebras, semi-H^*-algebras, and normed algebraic algebras. Thus, we prove in Theorem 2.8.4 that algebra homomorphisms from complete normed algebras to absolute-valued algebras are continuous. This applies to the study of absolute values on left semi-H^*-algebras, allowing us to show in Proposition 2.8.13 that semi-H^*-algebras having an absolute value are finite-dimensional. As an interlude, we introduce free non-associative algebras, show that they can be endowed with different absolute values, and apply this to prove that every (complete) normed algebra is the quotient of a suitable (complete) absolute-valued algebra (Corollaries 2.8.20 and 2.8.25). Concerning algebraic algebras, we prove that complete normed algebraic algebras are of bounded degree (Corollary 2.8.33), and that absolute-valued algebraic algebras are finite-dimensional (Theorem 2.8.66).

2.8.1 Continuity of algebra homomorphisms into absolute-valued algebras

Let F be a bounded linear operator on a normed space X. We denote by $k(F)$ the largest non-negative number k satisfying $k\|x\| \leqslant \|F(x)\|$ for every x in X. In this way F is bounded below if and only if $k(F) > 0$.

Lemma 2.8.1 *Let X be a Banach space over $\mathbb{K}$. Then, for F, G in $BL(X)$ we have*

$$|k(F) - k(G)| \leqslant \|F - G\| .$$

Moreover, if $F \in BL(X)$ is bounded below and non-bijective, then the open ball in $BL(X)$ with centre F and radius $k(F)$ consists only of elements which are bounded below and non-bijective.

Proof Let F, G be in $BL(X)$. For all x in X we have

$$(k(F) - \|F - G\|)\|x\| \leqslant \|F(x)\| - \|(F - G)(x)\| \leqslant \|G(x)\|,$$

and hence $k(F) - \|F - G\| \leqslant k(G)$, which proves the first assertion in the lemma. From this first assertion it follows that, if F is bounded below, and if B denotes the open ball in $BL(X)$ with centre F and radius $k(F)$, then B consists only of elements which are bounded below. Therefore, if in addition F is non-bijective, then, by Lemma 2.7.18, all elements of B are non-bijective. □

Lemma 2.8.2 *Let X and Y be Banach spaces over $\mathbb{K}$, let $\Phi : X \to Y$ be a linear mapping with dense range, let F be in $BL(X)$, and let $G : Y \to Y$ be a non-surjective linear isometry such that $\Phi \circ F = G \circ \Phi$. Then $\|F\| \geqslant 1$.*

Proof Let $0 < r < 1$. Then we have $\|G - (G - rI_Y)\| = r < 1 = k(G)$, and hence, by Lemma 2.8.1, the range of $G - rI_Y$ is a proper closed subspace of Y. Since $\Phi \circ (F - rI_X) = (G - rI_Y) \circ \Phi$, and Φ has dense range, it follows that $F - rI_X$ cannot be surjective, so $F - rI_X$ cannot belong to $\mathrm{Inv}(BL(X))$, and so $r \leqslant \|F\|$. Now the proof is concluded by letting $r \to 1$. □

Lemma 2.8.3 *Let X and Y be Banach spaces over $\mathbb{K}$, let $\Phi : X \to Y$ be a surjective linear mapping, and let F and G be in $BL(X)$ and $BL(Y)$, respectively, such that $\Phi \circ F = G \circ \Phi$. Then $\mathfrak{r}(G) \leqslant \mathfrak{r}(F)$.*

Proof First assume that $\mathbb{K} = \mathbb{C}$. Then, by Theorem 1.1.46, there exists $\lambda \in \mathrm{sp}(BL(Y), G)$ such that $|\lambda| = \mathfrak{r}(G)$. Since λ belongs to the boundary of $\mathrm{sp}(BL(Y), G)$ relative to $\mathbb{C}$, it follows from Corollary 1.1.95 that $G - \lambda I_Y$ is not surjective. Since $\Phi \circ (F - \lambda I_X) = (G - \lambda I_Y) \circ \Phi$, and Φ is surjective, it follows that $F - \lambda I_X$ cannot be surjective, so λ belongs to $\mathrm{sp}(BL(X), F)$, and so $\mathfrak{r}(G) = |\lambda| \leqslant \mathfrak{r}(F)$, as required.

Now assume that $\mathbb{K} = \mathbb{R}$. Consider the normed complexifications

$$X_{\mathbb{C}} := \mathbb{C} \otimes_{\pi} X \ \text{ and } \ Y_{\mathbb{C}} := \mathbb{C} \otimes_{\pi} Y$$

of X and Y, respectively, as well as the complex-linear operators

$$\Phi_{\mathbb{C}} := I_{\mathbb{C}} \otimes \Phi : X_{\mathbb{C}} \to Y_{\mathbb{C}}, \quad F_{\mathbb{C}} := I_{\mathbb{C}} \otimes F : X_{\mathbb{C}} \to X_{\mathbb{C}},$$

and

$$G_{\mathbb{C}} := I_{\mathbb{C}} \otimes G : Y_{\mathbb{C}} \to Y_{\mathbb{C}}.$$

Then $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are complex Banach spaces, $\Phi_{\mathbb{C}}$ is surjective, $F_{\mathbb{C}}$ and $G_{\mathbb{C}}$ lie in $BL(X_{\mathbb{C}})$ and $BL(Y_{\mathbb{C}})$, respectively, and the equalities

$$\Phi_{\mathbb{C}} \circ F_{\mathbb{C}} = G_{\mathbb{C}} \circ \Phi_{\mathbb{C}}, \mathfrak{r}(F_{\mathbb{C}}) = \mathfrak{r}(F), \text{ and } \mathfrak{r}(G_{\mathbb{C}}) = \mathfrak{r}(G)$$

hold. It follows from the first paragraph in the proof that $\mathfrak{r}(G) \leqslant \mathfrak{r}(F)$. □

Theorem 2.8.4 *Let A be a complete normed algebra over $\mathbb{K}$, let B be an absolute-valued algebra over $\mathbb{K}$, and let $\Phi : A \to B$ be an algebra homomorphism. Then Φ is continuous with $\|\Phi\| \leqslant 1$.*

Proof Regarding A and B as real algebras if necessary, we may assume that $\mathbb{K} = \mathbb{R}$. Moreover, regarding Φ as a mapping from A to the completion of its range, we may also assume that B is complete and that Φ has dense range.

First assume additionally that Φ is actually surjective. Then, since the equality

$$\Phi \circ L_a = L_{\Phi(a)} \circ \Phi \tag{2.8.1}$$

holds for every $a \in A$, Lemma 2.8.3 applies to obtain

$$\mathfrak{r}(L_{\Phi(a)}) \leqslant \mathfrak{r}(L_a) \leqslant \|L_a\| \leqslant \|a\|.$$

The result in this first considered case now follows from the clear fact that the equality $\mathfrak{r}(L_b) = \|b\|$ holds for every element b in any absolute-valued algebra.

Now, consider the remaining case that Φ is not surjective. Then, since Φ has dense range, B must be infinite-dimensional, and then the implication (iii)⇒(i) in Corollary 2.6.24 shows that, if some left multiplication operator on B is surjective, then all right multiplication operators on B are non-surjective. Therefore, replacing A and B with their corresponding opposite algebras if necessary, we may assume that all left multiplication operators on B are non-surjective. To conclude the proof, it is enough to show that $\|a\| \geqslant 1$ whenever a is in A with $\|\Phi(a)\| = 1$. But, for such an a, $L_{\Phi(a)}$ is a non-surjective linear isometry on B, and hence, by (2.8.1) and Lemma 2.8.2, we have $1 \leqslant \|L_a\| \leqslant \|a\|$, as desired. □

The actual relevance of the conclusion in the above theorem is the automatic continuity of Φ. Indeed, the additional information that $\|\Phi\| \leqslant 1$, although directly given by the proof, would follow from Proposition 2.6.19.

As a straightforward consequence of Theorem 2.8.4, we get the following.

Corollary 2.8.5 *Let A be a complete normed algebra over $\mathbb{K}$. We have:*

(i) *If $|||\cdot|||$ is an absolute value on A, then $|||\cdot||| \leqslant \|\cdot\|$ on A.*
(ii) *Characters on A are contractive, and hence continuous.*

We note that part (ii) of the above corollary was already proved with more elementary techniques (cf. Corollary 1.1.108).

2.8.2 Absolute values on H^*-algebras

In this subsection, we are going to discuss left semi-H^*-algebras having an absolute value. We begin by considering the complex case.

Proposition 2.8.6 *Let A be a complete normed complex algebra whose norm derives from an inner product $(\cdot|\cdot)$ such that there are $a, a^* \in A \setminus \{0\}$ satisfying $(ax|y) = (x|a^*y)$ for all $x, y \in A$. Assume that there exists an absolute value on A. Then A is isomorphic to $\mathbb{C}$.*

Proof By Corollary 2.8.5(i), the topology of A is stronger than that of the absolute value (say $|||\cdot|||$) whose existence has been assumed. Therefore, if aA were not $|||\cdot|||$-dense in A, then aA would not be dense in A for the topology of the Hilbertian norm, and hence, by the orthogonal projection theorem, there would exist $y \in A \setminus \{0\}$ such that $(ax|y) = 0$ for every $x \in A$, and we would have $(x|a^*y) = (ax|y) = 0$ for every $x \in A$, so $a^*y = 0$, and so (since A cannot have nonzero divisors of zero) $a^* = 0$, a contradiction. Therefore, aA is $|||\cdot|||$-dense in A, and the proof is concluded by applying Proposition 2.7.16. □

The next corollary follows straightforwardly from §1.1.3 and Proposition 2.8.6. We recall that left semi-H^*-algebras were incidentally introduced immediately before Lemma 2.7.50. *Right semi-H^*-algebras* and *one-sided semi-H^*-algebras* are defined in an obvious way.

Corollary 2.8.7 *Let A be a nonzero complex one-sided semi-H^*-algebra, and assume that there exists an absolute value on A. Then A is isomorphic to $\mathbb{C}$.*

Proposition 2.8.8 *Let A be a complete normed real algebra whose norm derives from an inner product $(\cdot|\cdot)$ such that there are $a, a^* \in A \setminus \{0\}$ satisfying*

$$(ax|y) = (x|a^*y) \text{ for all } x, y \in A.$$

Assume that there exists an absolute value $|||\cdot|||$ on A. Then we have:

(i) *A is a left-division algebra.*
(ii) *The absolute value $|||\cdot|||$ is continuous and derives from an inner product $\langle\cdot|\cdot\rangle$ satisfying $\langle ax|y\rangle = \langle x|a^*y\rangle$ for all $x, y \in A$.*
(iii) $|||a^*||| = |||a|||$.
(iv) *For every $x \in A$, the equalities $a^*(ax) = |||a|||^2 x$ and $\|ax\| = |||a|||\,\|x\|$ hold.*

Proof As in the proof of Proposition 2.8.6, $|||\cdot|||$ is continuous for the topology of the Hilbertian norm $\|\cdot\|$ on A, and aA is $|||\cdot|||$-dense in A. It follows from Corollary 2.7.29 that there exists a $\|\cdot\|$-continuous inner product $\langle\cdot|\cdot\rangle$ on A satisfying

$$\langle x|x\rangle = |||x|||^2 \text{ for every } x \in A. \tag{2.8.2}$$

Now, keeping in mind the well-known one-to-one correspondence between continuous bilinear forms and bounded linear operators on a real Hilbert space, we get the existence of a positive bounded linear operator T on A satisfying

$$\langle x|y\rangle = (x|T(y)) \text{ for all } x, y \in A. \tag{2.8.3}$$

On the other hand, in view of the equality (2.8.2), the absolute value condition $|||ax|||^2 = |||a|||^2|||x|||^2$ can be linearized to obtain $\langle ax|ay\rangle = |||a|||^2\langle x|y\rangle$ for all $x, y \in A$. Therefore, applying (2.8.3), we have

$$(y|a^*T(ax)) = (ay|T(ax)) = \langle ay|ax\rangle = |||a|||^2\langle y|x\rangle = |||a|||^2(y|T(x))$$

for all $x, y \in A$, and hence

$$a^* T(ax) = |||a|||^2 T(x) \text{ for every } x \in A. \tag{2.8.4}$$

This equality and its consequence

$$|||a^*||| \, |||T(ax)||| = |||a|||^2 |||T(x)|||,$$

together with (2.8.2), give

$$\begin{aligned}
&|||a^*|||^2 |||T(ax) - aT(x)|||^2 \\
&\quad = |||a^*|||^2 (|||T(ax)|||^2 + |||aT(x)|||^2 - 2\langle aT(x)|T(ax)\rangle) \\
&\quad = |||a|||^4 |||T(x)|||^2 + |||a^*|||^2 |||a|||^2 |||T(x)|||^2 - 2|||a^*|||^2 \langle T(x)|a^*T(ax)\rangle \\
&\quad = |||a|||^2 (|||a|||^2 - |||a^*|||^2) |||T(x)|||^2,
\end{aligned}$$

which summarizes as

$$|||a^*|||^2 |||T(ax) - aT(x)|||^2 = |||a|||^2 (|||a|||^2 - |||a^*|||^2) |||T(x)|||^2 \tag{2.8.5}$$

for every $x \in A$. If either $a^* = a$ or $a^* = -a$, it follows from (2.8.5) that $T(ax) = aT(x)$ for every $x \in A$. Otherwise, the couples

$$(b, b^*) := (a + a^*, a + a^*) \quad \text{and} \quad (c, c^*) := (a - a^*, a^* - a)$$

are in the same situation as that of (a, a^*) with $b^* = b$ and $c^* = -c$, and hence, since $a = \frac{1}{2}(b + c)$, we also have $T(ax) = aT(x)$ for every $x \in A$. Now that we know that

$$T(ax) = aT(x) \text{ for every } x \in A, \tag{2.8.6}$$

it follows from (2.8.5) that

$$|||a^*||| = |||a|||. \tag{2.8.7}$$

Now, invoking (2.8.3) and (2.8.6), for $x, y \in A$ we have

$$\langle ax|y\rangle = (ax|T(y)) = (x|a^*T(y)) = (x|T(a^*y)) = \langle x|a^*y\rangle.$$

From (2.8.4), (2.8.6), and (2.8.7), we obtain

$$a(a^*T(y)) = |||a|||^2 T(y) \text{ for every } y \in A,$$

and, since the range of T is $\|\cdot\|$-dense in A (a consequence of (2.8.3) and the orthogonal projection theorem), we actually have

$$a^*(ax) = |||a|||^2 x \text{ for every } x \in A. \tag{2.8.8}$$

This equality implies ostensibly that $a^*A = A$, and hence, by Corollary 2.7.34, that A is a left-division algebra. Finally, invoking (2.8.8), we derive

$$\|ax\|^2 = (ax|ax) = (x|a^*(ax)) = |||a|||^2 (x|x) = |||a|||^2 \|x\|^2. \qquad \square$$

It follows from §1.1.3 and Proposition 2.8.8 that *nonzero real left semi-H^*-algebras having an absolute value are left-division algebras*. Additional information is given by Corollary 2.8.9 immediately below. By an *absolute-valued left semi-H^*-algebra* we mean a nonzero left semi-H^*-algebra whose Hilbertian norm is an absolute value.

Corollary 2.8.9 *Let A be a nonzero real left semi-H^*-algebra. Then A has an absolute value (if and) only if there exists a dense range continuous injective algebra $*$-homomorphism from A to some real absolute-valued left semi-H^*-algebra.*

Proof Assume that there exists an absolute value $|||\cdot|||$ on A. According to Proposition 2.8.8, $|||\cdot|||$ is continuous and derives from an inner product $\langle\cdot|\cdot\rangle$ satisfying $\langle ab|c\rangle = \langle b|a^*c\rangle$ for all $a,b,c \in A$, and moreover the involution $*$ of A is $|||\cdot|||$-isometric. Therefore the completion of A relative to $|||\cdot|||$ becomes a real absolute-valued left semi-H^*-algebra (say B). Finally, the mapping $a \to a$ from A to B is indeed a dense range continuous injective algebra $*$-homomorphism. □

Let A be an algebra over $\mathbb{K}$. By a *left centralizer* on A we mean a linear mapping $T : A \to A$ satisfying $T(ab) = aT(b)$ for all $a,b \in A$. Clearly, the set of all left centralizers on A (say $\Gamma_\ell(A)$) is a subalgebra of $L(A)$ containing I_A and such that T^{-1} lies in $\Gamma_\ell(A)$ whenever T is in $\Gamma_\ell(A) \cap \mathrm{Inv}(L(A))$. Keeping in mind that, for $T \in \Gamma_\ell(A)$, both $\ker(T)$ and $T(A)$ are left ideals of A the following lemma follows.

Lemma 2.8.10 *Let A be a nonzero algebra over $\mathbb{K}$. If A has no nonzero proper left ideal (for instance, if A is a right-division algebra), then $\Gamma_\ell(A)$ is a division algebra.*

Corollary 2.8.11 *Let A be a nonzero finite-dimensional real left semi-H^*-algebra, and assume that there exists an absolute value $|||\cdot|||$ on A. Then the involution of A is an isometry, the absolute value $|||\cdot|||$ is a positive multiple of the norm of A, and, for all $a,b \in A$, we have $a(a^*b) = |||a|||^2 b$.*

Proof It follows from (2.8.2), (2.8.3), and (2.8.6) in the proof of Proposition 2.8.8 that there is $T \in \Gamma_\ell(A)$ such that $\langle a|b\rangle = (a|T(b))$ for all $a,b \in A$, where $\langle\cdot|\cdot\rangle$ is an inner product on A satisfying $\langle a|a\rangle = |||a|||^2$ for every $a \in A$. As a consequence, T is a self-adjoint operator on the finite-dimensional Hilbert space $(A,(\cdot|\cdot))$, and hence the characteristic polynomial of T has only real roots. On the other hand, since A is finite-dimensional and has an absolute value, A is a division algebra, and hence, by Lemma 2.8.10, $\Gamma_\ell(A)$ is a division algebra. It follows that $T \in \mathbb{R}I_A$, and hence that $|||\cdot|||$ on A is a positive multiple of $\|\cdot\|$. Now, by Proposition 2.8.8(iii), $*$ is $\|\cdot\|$-isometric. Finally, the equality $a(a^*b) = |||a|||^2 b$ follows from Proposition 2.8.8(iv). □

By a *(two-sided) semi-H^*-algebra* over $\mathbb{K}$ we mean an algebra A over $\mathbb{K}$ which is also a Hilbert space, and is endowed with a conjugate-linear vector space involution satisfying

$$(ab|c) = (b|a^*c) = (a|cb^*) \text{ for all } a,b,c \in A.$$

Recall that H^*-algebras were already introduced in Remark 2.6.54. Indeed, they are those semi-H^*-algebras whose involution is an algebra involution.

Lemma 2.8.12 *Let A be a semi-H^*-algebra over $\mathbb{K}$. We have:*

(i) *Up to the multiplication of the Hilbertian norm of A by a suitable positive number, A becomes a complete normed algebra.*
(ii) *If the involution of A is isometric, then A is in fact an H^*-algebra.*

Proof In view of Proposition 1.1.9, to prove assertion (i) it is enough to show that the product of A is separately continuous. Let a be in A, and let x_n be a sequence in A converging to zero and such that ax_n converges to some $y \in A$. Then, for every $b \in A$ we have

$$(y|b) \longleftarrow (ax_n|b) = (x_n|a^*b) \longrightarrow 0,$$

so $(y|b) = 0$, and so $y = 0$. It follows from the arbitrariness of $a \in A$ and the closed graph theorem that the product of A is continuous in its second variable. An analogous argument shows that the product of A is continuous in its first variable, as required.

Assume that the involution $*$ of A is isometric. Then, by assertion (i) just proved and Lemma 2.7.50, $*$ is an algebra involution on A, i.e. A is an H^*-algebra. In the application of Lemma 2.7.50, the reader must have noticed that, thanks to the right semi-H^*-algebra condition $(ab|c) = (a|cb^*)$, the product $\odot$ in that lemma coincides with the natural product of A. □

It follows from assertion (i) in the above lemma that *semi-H^*-algebras are both left and right semi-H^*-algebras*, a fact that will be applied in what follows without notice.

As a consequence of Corollary 2.6.24 and Proposition 2.8.8(i), if A is a left semi-H^*-algebra and a right semi-H^*-algebra (possibly for different inner products and involutions), and if there exists an absolute value on A, then A is finite-dimensional. A better result holds in the case that A is in fact a semi-H^*-algebra. Indeed, we have the following.

Proposition 2.8.13 *Let A be a real semi-H^*-algebra, and assume that there exists an absolute value $|||\cdot|||$ on A. Then A is a finite-dimensional H^*-algebra, the involution of A is an isometry, the absolute value $|||\cdot|||$ is a positive multiple of the norm of A, and, for all $a,b \in A$, we have*

$$a(a^*b) = (ba^*)a = |||a|||^2 b. \tag{2.8.9}$$

Proof The finite dimensionality of A follows from the above comments, and then, in view of Corollary 2.8.11 and the symmetry of our present assumptions, it only remains to show that A is in fact an H^*-algebra. But this follows from the fact that the involution of A is isometric and from Lemma 2.8.12(ii). □

The following example shows that, when in the above proposition we replace 'semi-H^*-algebra' with 'one-sided semi-H^*-algebra', the absolute value $|||\cdot|||$ need not be a positive multiple of the natural norm, nor even an equivalent norm.

Example 2.8.14 Let H be the real pre-Hilbert space whose vector space is the space of all quasi-null sequences of real numbers and whose inner product is defined by

$$\langle(\lambda_n)|(\mu_n)\rangle := \sum_{n\in\mathbb{N}} \lambda_n\mu_n.$$

By the proof of Theorem 2.7.38, there exists a unital $*$-representation ϕ of the quadratic commutative algebra of H on H. Now, let G denote the real pre-Hilbert space whose vector space is the space of all quasi-null sequences of elements of H and whose inner product is given by

$$\langle (a_n)|(b_n)\rangle := \sum_{n\in\mathbb{N}} \langle a_n|b_n\rangle$$

for all $(a_n),(b_n)\in G$. Then we can define a unital $*$-representation ψ of the quadratic commutative algebra of H on G by means of the equality

$$\psi(a)((b_n)) := (\phi(a)(b_n))$$

for all $a\in H$ and $(b_n)\in G$. Since the pre-Hilbert spaces H and G are isomorphic, we can choose a surjective linear isometry $u: G\to H$, and then the mapping $\theta: a\to u\psi(a)u^{-1}$ becomes a unital $*$-representation of the quadratic commutative algebra of H (whose unit will be denoted by $\mathbf{1}$) on H. According to Propositions 2.7.33 and 2.7.37, denoting by $|||\cdot|||$ the norm of H and by $\odot$ the product on H given by $a\odot b := \theta(a)(b)$, we have

$$|||a\odot b||| = |||a|||\,|||b||| \quad \text{and} \quad \langle a\odot b\,|\,c\rangle = \langle b\,|\,a^*\odot c\rangle$$

for all $a,b,c\in H$, where $a^* := 2\langle a|\mathbf{1}\rangle - a$. Moreover, if for $n\in\mathbb{N}$ we denote by H_n the image by u of the nth copy of H in G, then H_n becomes a left ideal of $(H,\odot)$, so that, if π_n stands for the natural (orthogonal) projection from H onto H_n, then we have

$$\pi_n(a\odot b) = a\odot \pi_n(b)$$

for all $a,b\in H$. Now define a norm $\|\cdot\|$ on H by

$$\|a\|^2 := \sum_{n\in\mathbb{N}} n|||\pi_n(a)|||^2,$$

which clearly derives from an inner product $(\cdot|\cdot)$ satisfying

$$(a\odot b\,|\,c) = (b\,|\,a^*\odot c)$$

for all $a,b,c\in H$. Then, for $a,b\in H$, we have

$$\begin{aligned}\|a\odot b\| &= \sum_{n\in\mathbb{N}} n|||\pi_n(a\odot b)|||^2 = \sum_{n\in\mathbb{N}} n|||a\odot\pi_n(b)|||^2\\ &= \sum_{n\in\mathbb{N}} n|||a|||^2|||\pi_n(b)|||^2 = |||a|||^2\|b\|^2 \leqslant \|a\|^2\|b\|^2,\end{aligned}$$

and hence $(H,\odot,\|\cdot\|)$ is a normed algebra. In order to extend the vector space involution $*$ to the completion of this normed algebra, we will prove that $*$ is $\|\cdot\|$-continuous. In fact, for $a\in H$, we have

$$\begin{aligned}\|a^*\|^2 &= \sum_{n\in\mathbb{N}} n|||\pi_n(a^*)|||^2 = \sum_{n\in\mathbb{N}} n|||2\langle a|\mathbf{1}\rangle\pi_n(\mathbf{1}) - \pi_n(a)|||^2\\ &= \|a\|^2 + 4\langle a|\mathbf{1}\rangle \sum_{n\in\mathbb{N}} n(\langle a|\mathbf{1}\rangle|||\pi_n(\mathbf{1})|||^2 - \langle\pi_n(\mathbf{1})|\pi_n(a)\rangle).\end{aligned}$$

Since there exists $p\in\mathbb{N}$ such that $\pi_n(\mathbf{1})=0$ for every $n\geqslant p$, the mapping

$$\begin{aligned}a &\to \langle a|\mathbf{1}\rangle \sum_{n\in\mathbb{N}} n(\langle a|\mathbf{1}\rangle|||\pi_n(\mathbf{1})|||^2 - \langle\pi_n(\mathbf{1})|\pi_n(a)\rangle)\\ &= \langle a|\mathbf{1}\rangle \sum_{n=1}^{p} n(\langle a|\mathbf{1}\rangle|||\pi_n(\mathbf{1})|||^2 - \langle\pi_n(\mathbf{1})|\pi_n(a)\rangle)\end{aligned}$$

is a $|||\cdot|||$-continuous quadratic form on H. Therefore, since $|||\cdot||| \leqslant \|\cdot\|$ on H, it follows that there exists a positive number M such that $\|a^*\| \leqslant M\|a\|$ for every $a\in H$. Now we know that $*$ is $\|\cdot\|$-continuous, clearly the completion (say A) of the normed algebra $(H,\odot,\|\cdot\|)$, with the unique continuous extension of $*$, is a left semi-H^*-algebra. Moreover, the unique continuous extension to A of $|||\cdot|||$ is an absolute value on A whose associated topology is strictly weaker than that of the norm $\|\cdot\|$. Indeed, the unique component of the last assertion which could be non-obvious is that, whenever a is in A with $|||a|||=0$, we have $a=0$. To realize this, let K stand for the completion of $(H,|||\cdot|||)$, and regard H (via the isometry u) as the space of all quasi-null sequences of elements of H. Then the completion of $(H,\|\cdot\|)$ is the space of those sequences (k_n) of elements of K such that $\sum_{n=1}^{\infty} n|||k_n||| < +\infty$, and the extension by continuity of $|||\cdot|||$ to $(H,\|\cdot\|)$ is given by $|||(k_n)||| = \sum_{n=1}^{\infty} |||k_n|||$.

A simplification of the argument in the above example allows us to build a (necessarily infinite-dimensional, in view of Corollary 2.8.11) real left semi-H^*-algebra A having an absolute value which is equivalent to the natural norm but is not a positive multiple of it. Indeed, it is enough to take the pre-Hilbert space G in the argument equal to the ℓ_2-sum $H\oplus H$ instead of $G=\oplus_{n=1}^{\infty} H_n$ (with H_n a copy of H for every n), as we did there.

2.8.3 Free non-associative algebras are absolute-valued algebras

We begin this subsection with the following generalization of Remark 2.7.44. The proof is straightforward.

Lemma 2.8.15 *Let U be a non-empty infinite set, let $\vartheta : U\times U \to U$ be any mapping, let $\mathscr{X}=\mathscr{X}(U,\mathbb{K})$ stand for the free vector space over $\mathbb{K}$ generated by U, and let $\mathscr{A}=\mathscr{A}(U,\vartheta,\mathbb{K})$ stand for the algebra over $\mathbb{K}$ whose vector space is $\mathscr{X}$, and whose product is defined as the unique bilinear mapping from $\mathscr{X}\times\mathscr{X}$ to $\mathscr{X}$ which extends ϑ. If ϑ is injective, if $1\leqslant p<\infty$, and if for $x=\sum_{u\in U}\lambda_u u\in\mathscr{A}$ we set*

$$\|x\|_p := \left(\sum_{u\in U} |\lambda_u|^p\right)^{\frac{1}{p}},$$

then $\|\cdot\|_p$ is an absolute value on $\mathscr{A}$.

Let U, ϑ, and $\mathscr{A}$ be as in the above lemma, assume that ϑ is injective, and let $1\leqslant p<\infty$. We denote by $\mathscr{A}_p=\mathscr{A}_p(U,\vartheta,\mathbb{K})$ the absolute-valued algebra over $\mathbb{K}$ obtained by endowing $\mathscr{A}$ with the norm $\|\cdot\|_p$. By considering the completion of $\mathscr{A}_p$, we obtain a complete absolute-valued algebra over $\mathbb{K}$ whose Banach space is nothing other than the familiar space $\ell_p(U,\mathbb{K})$.

§2.8.16 Given a topological space E, we denote by $\mathrm{dens}(E)$ the *density character* of E, namely the smallest cardinal among those of dense subsets of E, and recall that, in the case that E is an infinite-dimensional normed space, $\mathrm{dens}(E)$ coincides with the smallest cardinal among those of subsets of E whose linear hull is dense in E. Thus, if U, ϑ, and $\mathscr{A}$ are as in Lemma 2.8.15, if ϑ is injective, and if $1\leqslant p<\infty$, then we easily realize that $\mathrm{dens}(\mathscr{A}_p(U,\vartheta,\mathbb{K}))$ equals the cardinal of U. For, if D is any dense subset of $\mathscr{A}_p(U,\vartheta,\mathbb{K})$, and if for $u\in U$ we choose $d_u\in D$ with $\|u-d_u\|_p<\frac{1}{2}$, then the mapping $u\to d_u$ from U to D becomes injective.

§2.8.17 Let $\mathbf{X}$ be a non-empty set. *Non-associative words* with characters in $\mathbf{X}$ are defined inductively, according to their (*global*) *degree*, as follows. The non-associative words of degree 1 are just the elements of $\mathbf{X}$. If $n\geqslant 2$, and if we know all the non-associative words of degree $<n$, then the non-associative words of degree n are defined as those of the form $(\mathbf{w}_1)(\mathbf{w}_2)$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are non-associative words with $\deg(\mathbf{w}_1)+\deg(\mathbf{w}_2)=n$. Although the use of brackets is essential in the above definition, some natural simplifications in the writing are usually accepted. For example, brackets covering a word of degree 1 are omitted, and words of the form $(\mathbf{w})(\mathbf{w})$, for some other word $\mathbf{w}$, are written as $(\mathbf{w})^2$. Two non-associative words are taken to be equal only if they are written exactly the same. Thus for example, for $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}$, the non-associative words $(\mathbf{xy})\mathbf{z}$ and $\mathbf{x}(\mathbf{yz})$ are different. The set $\mathscr{M}(\mathbf{X})$ (of all non-associative words with characters in $\mathbf{X}$), endowed with the mapping $(\mathbf{w}_1,\mathbf{w}_2)\to(\mathbf{w}_1)(\mathbf{w}_2)$ from $\mathscr{M}(\mathbf{X})\times\mathscr{M}(\mathbf{X})$ to $\mathscr{M}(\mathbf{X})$, is called the *free (non-associative) monad* generated by $\mathbf{X}$. We recall that *monads* are defined as sets E endowed with a mapping $(x,y)\to xy$ from $E\times E$ to E, which is called the *product* of the monad. The free monad $\mathscr{M}(\mathbf{X})$ constructed above enjoys the following 'universal property': *If E is any monad, and if $\varphi:\mathbf{X}\to E$ is any mapping, then φ extends uniquely to a mapping from $\mathscr{M}(\mathbf{X})$ to E preserving products* (see for example [754, p. 24]).

The free vector space over $\mathbb{K}$ generated by $\mathscr{M}(\mathbf{X})$, with product equal to the extension by bilinearity of the product of $\mathscr{M}(\mathbf{X})$, is called the *free non-associative algebra* over $\mathbb{K}$ generated by $\mathbf{X}$, and is denoted by $\mathscr{F}(\mathbf{X},\mathbb{K})$. The algebra $\mathscr{F}(\mathbf{X},\mathbb{K})$ is characterized up to algebra isomorphisms by the following 'universal property': *If A is any algebra over $\mathbb{K}$, and if $\varphi:\mathbf{X}\to A$ is any mapping, then φ extends uniquely to an algebra homomorphism from $\mathscr{F}(\mathbf{X},\mathbb{K})$ to A* (see for example [822, Theorem 1.1.1]). By the universal property of $\mathscr{M}(\mathbf{X})$, there exists a unique mapping $*:\mathscr{M}(\mathbf{X})\to\mathscr{M}(\mathbf{X})$ fixing the elements of $\mathbf{X}$ and satisfying $(m_1m_2)^*=m_2^*m_1^*$ for all $m_1,m_2\in\mathscr{M}(\mathbf{X})$, and moreover this mapping becomes involutive. Then, extending $*$ to $\mathscr{F}(\mathbf{X},\mathbb{K})$ by conjugate linearity, we are provided with a conjugate-linear algebra involution on $\mathscr{F}(\mathbf{X},\mathbb{K})$, which will be called the *standard involution*.

§2.8.18 Now note that $\mathscr{F}(\mathbf{X},\mathbb{K})$ coincides with the algebra $\mathscr{A}(U,\vartheta,\mathbb{K})$, for $U:=\mathscr{M}(\mathbf{X})$ and ϑ equal to the product of $\mathscr{M}(\mathbf{X})$. Since the mapping ϑ above is injective [822, Proposition 1.1.2], we invoke Lemma 2.8.15 to realize that *there are 'many' absolute-values on $\mathscr{F}(\mathbf{X},\mathbb{K})$*. For $1\leqslant p<\infty$, we denote by $\mathscr{F}_p(\mathbf{X},\mathbb{K})$ the absolute-valued algebra over $\mathbb{K}$ obtained by endowing $\mathscr{F}(\mathbf{X},\mathbb{K})$ $(=\mathscr{A}(U,\vartheta,\mathbb{K})$ for U and ϑ as above) with the absolute value $\|\cdot\|_p$, and note that the standard involution of $\mathscr{F}(\mathbf{X},\mathbb{K})$ becomes an isometry on $\mathscr{F}_p(\mathbf{X},\mathbb{K})$. As we will see in the proof of Proposition 2.8.19 immediately below, the absolute-valued algebra $\mathscr{F}_1(\mathbf{X},\mathbb{K})$ has a special relevance in the general theory of normed algebras.

Proposition 2.8.19 *Let $\mathbf{X}$ be a non-empty set. Then, up to isometric algebra isomorphisms, there exists a unique normed algebra $\mathscr{N}=\mathscr{N}(\mathbf{X},\mathbb{K})$ over $\mathbb{K}$ satisfying* *the following properties:*

(i) $\mathbf{X}$ *is a subset of the closed unit ball of* $\mathscr{N}$.
(ii) *If* A *is any normed algebra over* $\mathbb{K}$*, and if* φ *is any mapping from* $\mathbf{X}$ *into the closed unit ball of* A*, then* φ *extends uniquely to a contractive algebra homomorphism from* $\mathscr{N}$ *to* A.

Moreover, we have:

(iii) *The normed algebra* $\mathscr{N}$ *is in fact an absolute-valued algebra, and has an isometric conjugate-linear algebra involution.*
(iv) *The set* $\mathbf{X}$ *consists only of norm-one elements of* $\mathscr{N}$.
(v) *If* $\mathbf{X}$ *is finite, then* $\mathscr{N}$ *is separable. Otherwise* $\operatorname{dens}(\mathscr{N})$ *equals the cardinal of* $\mathbf{X}$.

Proof Take $\mathscr{N} = \mathscr{F}_1(\mathbf{X},\mathbb{K})$. Then, according to §2.8.18, $\mathscr{N}$ satisfies Properties (i), (iii), and (iv) in the statement. On the other hand, since $\mathscr{M}(\mathbf{X})$ is 'countably generated' by $\mathbf{X}$ (cf. §2.8.17), the cardinal of $\mathscr{M}(\mathbf{X})$ is that of $\mathbb{N}$ if $\mathbf{X}$ is finite, or equal to the cardinal of $\mathbf{X}$ otherwise, so that property (v) follows from §§2.8.16 and 2.8.18. Let A be a normed algebra over $\mathbb{K}$, and let φ be a mapping from $\mathbf{X}$ into the closed unit ball of A. Since, forgetting the norm, $\mathscr{N}$ is nothing other than $\mathscr{F}(\mathbf{X},\mathbb{K})$, the universal property of this last algebra provided us with a unique algebra homomorphism $\psi : \mathscr{F}_1(\mathbf{X},\mathbb{K}) \to A$ which extends φ. Let x be in $\mathscr{N}$. We have $x = \sum_{\mathbf{w}\in\mathscr{M}(\mathbf{X})} x_{\mathbf{w}}\mathbf{w}$, where $\{x_{\mathbf{w}}\}_{\mathbf{w}\in\mathscr{M}(\mathbf{X})}$ stands for the family of coordinates of x relative to $\mathscr{M}(\mathbf{X})$. Therefore

$$\|\psi(x)\| = \left\|\sum_{\mathbf{w}\in\mathscr{M}(\mathbf{X})} x_{\mathbf{w}}\psi(\mathbf{w})\right\| \leqslant \sum_{\mathbf{w}\in\mathscr{M}(\mathbf{X})} |x_{\mathbf{w}}|\|\psi(\mathbf{w})\| \leqslant \sum_{\mathbf{w}\in\mathscr{M}(\mathbf{X})} |x_{\mathbf{w}}| = \|x\|.$$

(Starting from the fact that $\psi(\mathbf{X})$ is contained in the closed unit ball of A, the inequality $\|\psi(\mathbf{w})\| \leqslant 1$ just applied is proved by induction on the degree of $\mathbf{w}\in\mathscr{M}(\mathbf{X})$.) Now that we know that $\mathscr{N}$ also satisfies property (ii), let us conclude the proof by showing that $\mathscr{N}$ is the 'unique' normed algebra over $\mathbb{K}$ satisfying (i) and (ii). Let $\mathscr{N}'$ be a normed algebra over $\mathbb{K}$ satisfying (i) and (ii) with $\mathscr{N}'$ instead of $\mathscr{N}$. Then we are provided with contractive algebra homomorphisms $\phi : \mathscr{N} \to \mathscr{N}'$ and $\phi' : \mathscr{N}' \to \mathscr{N}$ fixing the elements of $\mathbf{X}$. Therefore $\phi'\circ\phi$ and $\phi\circ\phi'$ are contractive algebra endomorphisms of $\mathscr{N}$ and $\mathscr{N}'$, respectively, extending the corresponding inclusions $\mathbf{X}\to\mathscr{N}$ and $\mathbf{X}\to\mathscr{N}'$. By the uniqueness of such extensions, we must have $\phi'\circ\phi = I_{\mathscr{N}}$ and $\phi\circ\phi' = I_{\mathscr{N}'}$. It follows that ϕ is an isometric algebra isomorphism from $\mathscr{N}$ onto $\mathscr{N}'$ respecting the corresponding inclusions of $\mathbf{X}$ in each of the algebras. □

The absolute-valued algebra $\mathscr{N}(\mathbf{X},\mathbb{K})$ in Proposition 2.8.19 has its own right to be called the *free normed non-associative algebra* over $\mathbb{K}$ generated by the set $\mathbf{X}$.

Now, if A is a normed algebra over $\mathbb{K}$, if $\mathbf{X}$ denotes the closed unit ball of A, and if $\Phi : \mathscr{N}(\mathbf{X},\mathbb{K}) \to A$ is the unique contractive algebra homomorphism which is the identity on $\mathbf{X}$, then we easily realize that the induced algebra homomorphism $\mathscr{N}(\mathbf{X},\mathbb{K})/\ker(\Phi) \to A$ is a surjective isometry. Therefore, we have the following.

Corollary 2.8.20 *Every normed algebra over $\mathbb{K}$ is isometrically algebra-isomorphic to a quotient of an absolute-valued algebra over $\mathbb{K}$ having an isometric conjugate-linear algebra involution.*

The proof of the next corollary involves the clear fact that continuous mappings between topological spaces decrease the density character, as well as the well-known fact that, if E is a metrizable topological space, and if F is a subset of E, then $\operatorname{dens}(F) \leqslant \operatorname{dens}(E)$.

Corollary 2.8.21 *Let A be a normed algebra over $\mathbb{K}$, let S be a non-empty subset of A, and let B stand for the closed subalgebra of A generated by S. If S is finite, then B is separable. Otherwise we have* $\operatorname{dens}(B) = \operatorname{dens}(S)$.

Proof We may assume that $S \neq \{0\}$. Then, considering the set

$$T := \left\{ \frac{s}{\|s\|} : s \in S \setminus \{0\} \right\},$$

and noticing that S and T generate the same closed subalgebra of A, that T is finite whenever so is S, and that $\operatorname{dens}(T) \leqslant \operatorname{dens}(S)$, it is enough to replace S with T to realize that we may also assume that S is contained in $\mathbb{B}_A$. Now, take a dense subset $\mathbf{X}$ of S whose cardinal equals $\operatorname{dens}(S)$, and invoke Proposition 2.8.19 to find a continuous algebra homomorphism $\Phi : \mathscr{N}(\mathbf{X}, \mathbb{K}) \to A$ fixing the elements of $\mathbf{X}$ and such that $\overline{\phi(\mathscr{N}(\mathbf{X}, \mathbb{K}))}$ is separable if S is finite and $\operatorname{dens}(\overline{\phi(\mathscr{N}(\mathbf{X}, \mathbb{K}))}) \leqslant \operatorname{dens}(S)$ otherwise. Since $\overline{\phi(\mathscr{N}(\mathbf{X}, \mathbb{K}))}$ is a closed subalgebra of A containing S, it contains B, so the result follows straightforwardly. □

Now we are going to do a first application of Corollary 2.8.21 in Theorem 2.8.23 below. To this end, we also need the following.

Lemma 2.8.22 *Let E be a compact Hausdorff topological space such that there exists a continuous surjective mapping from E to its square. Then the Banach space $C^{\mathbb{K}}(E)$ becomes an absolute-valued algebra over $\mathbb{K}$ for a suitable product.*

Proof Let f be a continuous surjective mapping from E to its square, let $\pi_1, \pi_2 : E \times E \to E$ be the coordinate functions, and define a product $\odot$ on $C^{\mathbb{K}}(E)$ by

$$(x \odot y)(t) := x[\pi_1(f(t))]y[\pi_2(f(t))] \text{ for all } x, y \in C^{\mathbb{K}}(E) \text{ and } t \in E.$$

Let x, y be in $C^{\mathbb{K}}(E)$. Taking $t_1, t_2 \in E$ such that $\|x\| = |x(t_1)|$ and $\|y\| = |y(t_2)|$, and choosing $t_3 \in E$ such that $f(t_3) = (t_1, t_2)$, we have

$$\|x \odot y\| \geqslant |(x \odot y)(t_3)| = |x(t_1)y(t_2)| = \|x\|\|y\|.$$

Since the converse inequality $\|x \odot y\| \leqslant \|x\|\|y\|$ is clear, $(C^{\mathbb{K}}(E), \odot)$ is indeed an absolute-valued algebra over $\mathbb{K}$. □

Theorem 2.8.23 *Let X be a nonzero Banach space over $\mathbb{K}$. Then there exists a complete absolute-valued algebra A over $\mathbb{K}$ satisfying the following properties:*

(i) *X is linearly isometric to a subspace of A.*
(ii) $\operatorname{dens}(A) = \operatorname{dens}(X)$.
(iii) *A is linearly isometric to $C^{\mathbb{K}}(K)$ for some compact Hausdorff topological space K.*

Proof Let F denote the compact Hausdorff topological space consisting of the closed unit ball of X' and the weak* topology. Then X can be seen as a subspace of $C^{\mathbb{K}}(F)$. Let $*$ denote the natural involution of $C^{\mathbb{K}}(K)$ (which, in the case $\mathbb{K}=\mathbb{R}$ is the identity mapping), and let Y stand for the closed subalgebra of $C^{\mathbb{K}}(F)$ generated by $X\cup X^*\cup\{\mathbf{1}\}$. Then, by Corollary 2.8.21, we have

$$\mathrm{dens}(Y)=\mathrm{dens}(X\cup X^*\cup\{\mathbf{1}\})=\mathrm{dens}(X)$$

and, by Theorem 1.2.23 and §1.2.9, we have $Y=C^{\mathbb{K}}(E)$ for some compact Hausdorff topological space E. Set $A:=C^{\mathbb{K}}(E^{\mathbb{N}})$. Since there are continuous surjections from $E^{\mathbb{N}}$ to E (namely the coordinate projections), Y is linearly isometric to a subspace of A (cf. §1.2.27). It follows that X is linearly isometric to a subspace of A. Moreover, since $E^{\mathbb{N}}$ is homeomorphic to its square, Lemma 2.8.22 applies, so that A is an absolute-valued algebra over $\mathbb{K}$ for some product and involution. Therefore, to conclude the proof it is enough to show that $\mathrm{dens}(A)=\mathrm{dens}(Y)$. To this end, for $n\in\mathbb{N}$ denote by π_n the n-coordinate projection from $E^{\mathbb{N}}$ onto E, let $\mathscr{D}$ be a dense subset of Y whose cardinal equals $\mathrm{dens}(Y)$, and consider now A with its natural structure of a unital commutative and associative normed algebra with algebra involution. Then, by Theorem 1.2.10, the unital $*$-subalgebra of A generated by the set

$$S:=\{f\circ\pi_n:(f,n)\in\mathscr{D}\times\mathbb{N}\}$$

is dense in A. Since the cardinal of S equals the one of $\mathscr{D}$, the desired equality $\mathrm{dens}(A)=\mathrm{dens}(Y)$ follows. □

The complete case of Proposition 2.8.19 is also true giving rise to the *free complete normed non-associative algebra* over $\mathbb{K}$ generated by the set $\mathbf{X}$, denoted by $\mathscr{CN}(\mathbf{X},\mathbb{K})$. Indeed, we have the following.

Proposition 2.8.24 *Let $\mathbf{X}$ be a non-empty set. Then, up to isometric algebra isomorphisms, there exists a unique complete normed algebra $\mathscr{CN}=\mathscr{CN}(\mathbf{X},\mathbb{K})$ over $\mathbb{K}$ satisfying the following properties:*

- (i) *$\mathbf{X}$ is a subset of the closed unit ball of $\mathscr{CN}$.*
- (ii) *If A is any complete normed algebra over $\mathbb{K}$, and if φ is any mapping from $\mathbf{X}$ into the closed unit ball of A, then φ extends uniquely to a contractive algebra homomorphism from $\mathscr{CN}$ to A.*

Moreover, we have:

- (iii) *The complete normed algebra $\mathscr{CN}$ is in fact an absolute-valued algebra, and has an isometric conjugate-linear algebra involution.*
- (iv) *The set $\mathbf{X}$ consists only of norm-one elements of $\mathscr{CN}$.*
- (v) *If $\mathbf{X}$ is finite, then $\mathscr{CN}$ is separable. Otherwise $\mathrm{dens}(\mathscr{CN})$ equals the cardinal of $\mathbf{X}$.*

Proof Take $\mathscr{CN}$ equal to the completion of $\mathscr{N}:=\mathscr{N}(\mathbf{X},\mathbb{K})$, note that each contractive algebra homomorphism from $\mathscr{N}$ to a complete normed algebra A over $\mathbb{K}$ extends uniquely to a contractive algebra homomorphism from $\mathscr{CN}$ to A, and apply Proposition 2.8.19. □

Corollary 2.8.25 *Let A be a nonzero complete normed algebra over $\mathbb{K}$. Then there exists a complete absolute-valued algebra $\mathscr{A}$ over $\mathbb{K}$ having an isometric conjugate-linear algebra involution, satisfying* $\operatorname{dens}(\mathscr{A}) = \operatorname{dens}(A)$, *and such that A is isometrically algebra-isomorphic to a quotient of $\mathscr{A}$.*

Proof Choose a dense subset $\mathbf{X}$ of the closed unit ball $\mathbb{B}_A$ of A whose cardinal equals $\operatorname{dens}(A)$. By properties (i), (ii), and (iii) in Proposition 2.8.24, $\mathscr{A} := \mathscr{CN}(\mathbf{X},\mathbb{K})$ is a complete absolute-valued algebra over $\mathbb{K}$ having an isometric conjugate-linear algebra involution, the closed unit ball $\mathbb{B}_{\mathscr{A}}$ of $\mathscr{A}$ contains $\mathbf{X}$, and there exists a contractive algebra homomorphism $\Phi : \mathscr{A} \to A$ fixing the elements of $\mathbf{X}$. Moreover, since $\mathbf{X}$ is infinite, property (v) in Proposition 2.8.24 assures that $\operatorname{dens}(\mathscr{A}) = \operatorname{dens}(A)$. On the other hand, since $\mathbf{X}$ is dense in $\mathbb{B}_A$, we realize that the closure of $\Phi(\mathbb{B}_{\mathscr{A}})$ in A contains $\mathbb{B}_A$. Now, from the main tool in the proof of Banach's open mapping theorem (see for example [689, Lemma 48.3]) we deduce that $\Phi(\mathbb{B}_{\mathscr{A}})$ contains the open unit ball of A. Since $\Phi : \mathscr{A} \to A$ is a contractive algebra homomorphism, it follows from the above that the induced algebra homomorphism $\mathscr{A}/\ker(\Phi) \to A$ is a surjective isometry. □

2.8.4 Complete normed algebraic algebras are of bounded degree

§2.8.26 Let $\mathbf{X}$ be a finite set (say $\mathbf{X} = \{\mathbf{x}_1,\dots,\mathbf{x}_n\}$). As usual, given elements $a_1,\dots,a_n$ in an algebra A over $\mathbb{K}$, and a *non-associative polynomial* $\mathbf{p} \in \mathscr{F}(\mathbf{X},\mathbb{K})$, we denote by $\mathbf{p}(a_1,\dots,a_n)$ the image of $\mathbf{p}$ under the unique algebra homomorphism $\Phi : \mathscr{F}(\mathbf{X},\mathbb{K}) \to A$ satisfying $\Phi(\mathbf{x}_i) = a_i$ for every $i = 1,\dots,n$. We note that, if $a_1,\dots,a_n$ are elements in an algebra A over $\mathbb{K}$, then the set

$$\{\mathbf{p}(a_1,\dots,a_n) : \mathbf{p} \in \mathscr{F}(\mathbf{X},\mathbb{K})\}$$

equals the subalgebra of A generated by $\{a_1,\dots,a_n\}$. We also note that, if A is a normed algebra over $\mathbb{K}$, and if $\mathbf{p}$ is a non-associative polynomial, then the mapping $(a_1,\dots,a_n) \to \mathbf{p}(a_1,\dots,a_n)$ from $A \times \cdots \times A$ to A is continuous. This can be verified by writing $\mathbf{p}$ as a linear combination of elements in $\mathscr{M}(\mathbf{X})$, and then arguing by induction on the degree of such elements.

In what follows, $\mathscr{M}$ (respectively, $\mathscr{F}$) will stand for the free monad (respectively, the free non-associative algebra over $\mathbb{K}$) generated by a singleton. According to §1.1.24, given an element a of an algebra A, we denote by $A(a)$ the subalgebra of A generated by a.

Lemma 2.8.27 *Let A be a normed algebra over $\mathbb{K}$, and let n be a natural number. Then the set $A_n := \{a \in A : \dim(A(a)) \leqslant n\}$ is closed in A.*

Proof Let a be in the closure of A_n in A, so that there exists a sequence a_k in A_n converging to a. Let $x_1,\dots,x_{n+1}$ be in $A(a)$. Then we can find non-associative polynomials $\mathbf{p}_1,\dots,\mathbf{p}_{n+1} \in \mathscr{F}$ satisfying $\mathbf{p}_i(a) = x_i$ for every $i = 1,\dots,n+1$. On the other hand, by the definition of A_n, for each $k \in \mathbb{N}$ the system $\{\mathbf{p}_1(a_k),\dots,\mathbf{p}_{n+1}(a_k)\}$ is linearly dependent, and hence there are numbers $\mu_1^k,\dots,\mu_{n+1}^k \in \mathbb{K}$ satisfying

$$\sum_{i=1}^{n+1} |\mu_i^k| = 1 \ \text{ and } \ \sum_{i=1}^{n+1} \mu_i^k \mathbf{p}_i(a_k) = 0.$$

It follows from the continuity of the functions $\mathbf{p}_i(\cdot)$ and the compactness of the unit sphere of the normed space ℓ_1^{n+1} (of all $(n+1)$-uples with entries in $\mathbb{K}$ endowed with the sum norm) that there exist numbers $\mu_1,\dots,\mu_{n+1}\in\mathbb{K}$ satisfying

$$\sum_{i=1}^{n+1}|\mu_i| = 1 \text{ and } \sum_{i=1}^{n+1}\mu_i x_i = \sum_{i=1}^{n+1}\mu_i\mathbf{p}_i(a) = 0,$$

and hence that the system $\{x_1,\dots,x_{n+1}\}$ is linearly dependent. In view of the arbitrariness of $x_1,\dots,x_{n+1}\in A(a)$, this shows that a lies in A_n. □

Lemma 2.8.27 above must be kept in mind in the proof of the following.

Corollary 2.8.28 *Let A be a normed algebraic algebra of bounded degree. Then we have:*

(i) *The set $\{a\in A:\dim(A(a))=\deg(A)\}$ is open in A.*
(ii) *The completion of A is algebraic of bounded degree equal to that of A.*

Proof We have

$$\{a\in A:\dim(A(a))=\deg(A)\} = A\setminus\{a\in A:\dim(A(a))\leqslant\deg(A)-1\},$$

which proves the first conclusion. On the other hand, denoting by $\hat{A}$ the completion of A, we have

$$A\subseteq\{x\in\hat{A}:\dim(\hat{A}(x))\leqslant\deg(A)\},$$

which proves the second conclusion. □

§2.8.29 Let X be a vector space over $\mathbb{K}$. By a *polynomial function from $\mathbb{K}$ to X* we mean a mapping of the form $\mu\to x_0+\mu x_1+\cdots+\mu^n x_n$, for some non-negative integer n and some fixed elements $x_0,x_1,\dots,x_n\in X$.

Lemma 2.8.30 *Let X be a vector space over $\mathbb{K}$, and let $P_1,\dots,P_m$ be polynomial functions from $\mathbb{K}$ to X. Then the set*

$$D:=\{\mu\in\mathbb{K}:\{P_1(\mu),\dots,P_m(\mu)\}\textit{ is linearly dependent}\}$$

is either finite or equal to the whole $\mathbb{K}$.

Proof First observe that X can be assumed to be finite-dimensional. Therefore, if there is some element $\mu_0\in\mathbb{K}\backslash D$, then we can choose a basis in X whose first m elements are $P_1(\mu_0),\dots,P_m(\mu_0)$. For $x\in X$ and $1\leqslant i\leqslant m$, let $f_i(x)$ denote the ith coordinate of x relative to this basis. Then, for $x_1,\dots,x_m\in X$ with $\{x_1,\dots,x_m\}$ linearly dependent, we have $\det(f_i(x_j))_{1\leqslant i,j\leqslant m}=0$, so D is contained in the set of zeros of the nonzero polynomial

$$\mu\to\det(f_i(P_j(\mu)))_{1\leqslant i,j\leqslant m},$$

and so D is finite. □

For the proof of the next lemma it is convenient to note that, *if A is an algebra over $\mathbb{K}$, if a,b are elements of A, and if $\mathbf{p}$ is in $\mathscr{F}$, then the mapping $\mu\to\mathbf{p}(a+\mu b)$ from $\mathbb{K}$ to A is polynomial*. A formal verification of this fact can be done by reducing to the case that $\mathbf{p}$ lies in $\mathscr{M}$, and then by applying induction on the degree of $\mathbf{p}$.

Lemma 2.8.31 *Let A be a normed algebra over $\mathbb{K}$, and assume that there exists a natural number n such that $A_n := \{a \in A : \dim(A(a)) \leqslant n\}$ has non-empty interior in A. Then $A_n = A$.*

Proof Let b be an interior point of A_n, and let a be an arbitrary element in A. Then there exists a positive number r such that $b + \mu(a - b)$ lies in A_n whenever μ is in $\mathbb{K}$ with $|\mu| < r$. Let $x_1, \ldots, x_{n+1}$ be in $A(a)$. Then there are non-associative polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_{n+1} \in \mathscr{F}$ such that $\mathbf{p}_i(a) = x_i$ for all $i = 1, \ldots, n+1$. If for $i = 1, \ldots, n+1$ we consider the polynomial mapping

$$P_i : \mu \to \mathbf{p}_i(b + \mu(a - b)),$$

then $P_i(\mu)$ lies in $A(b + \mu(a - b))$, and hence, since $b + \mu(a - b)$ belongs to A_n for $|\mu| < r$, we see that $\{P_1(\mu), \ldots, P_{n+1}(\mu)\}$ is linearly dependent for such values of μ. It follows from Lemma 2.8.30 that

$$\{x_1, \ldots, x_{n+1}\} = \{P_1(1), \ldots, P_{n+1}(1)\}$$

is also linearly dependent. Since $x_1, \ldots, x_{n+1}$ are arbitrary elements of $A(a)$, we conclude that a lies in A_n. □

Theorem 2.8.32 *Let A be a complete normed algebra over $\mathbb{K}$ such that there exists a non-empty open set $\Omega \subseteq A$ consisting only of algebraic elements. Then A is algebraic of bounded degree.*

Proof For $n \in \mathbb{N}$, set $\Omega_n := \{a \in \Omega : \dim(A(a)) \leqslant n\}$. Then we have $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$. Since Ω_n is relatively closed in Ω for every n (by Lemma 2.8.27), and Ω is of the second category in itself, there exists n in $\mathbb{N}$ such that Ω_n has non-empty interior in Ω (and hence in A). The proof is concluded by applying Lemma 2.8.31. □

Although obvious, we emphasize the following.

Corollary 2.8.33 *Complete normed algebraic algebras over $\mathbb{K}$ are of bounded degree.*

Completeness cannot be altogether removed in Theorem 2.8.32, nor even in Corollary 2.8.33. Indeed, the normed associative algebra of all finite rank bounded linear operators on an infinite-dimensional Banach space is algebraic, but is not of bounded degree.

By combining Corollaries 2.8.28(ii) and 2.8.33, we get the following.

Corollary 2.8.34 *Let A be a complete normed algebra over $\mathbb{K}$, and let B be a dense subalgebra of A. Then A is algebraic if and only if B is algebraic of bounded degree. Moreover, if this is the case, then A is of bounded degree equal to the one of B.*

Now, note that, as a consequence of Theorem 2.6.51, *nonzero algebraic real algebras with no nonzero divisor of zero are of bounded degree equal to* 1, 2, 4, *or* 8. Therefore, Lemma 2.8.27 implies the following.

Corollary 2.8.35 *Let A be a nonzero normed real algebra containing a dense algebraic subalgebra with no nonzero divisor of zero in itself. Then A is algebraic of bounded degree equal to* 1, 2, 4 *or* 8.

In particular, *the completion of a nonzero normed algebraic real algebra with no nonzero divisor of zero is algebraic of bounded degree equal to* 1, 2, 4 *or* 8.

§2.8.36 Let A be an algebra, and let S be a subset of A. We say that S is *nilpotent* if there exists a natural number n such that, for every $m \geqslant n$, any product of m elements of S, no matter how associated, is zero. If S is nilpotent, then the smallest such n is called the *index of nilpotency* of S. It is easy to realize that, if the subset S is multiplicatively closed (for example, if $S = A$), and if any product of n elements of S, no matter how associated, is zero, then, for every $m \geqslant n$, any product of m elements of S, no matter how associated, is also zero. By identifying elements of A with singletons, the above notions apply to elements. Thus, formalizing somewhat, an element $a \in A$ is nilpotent of index $n \geqslant 2$ if $\mathbf{p}(a) = 0$ for every $\mathbf{p} \in \mathscr{M}$ with $\deg(\mathbf{p}) \geqslant n$, and there exists $\mathbf{q} \in \mathscr{M}$ with $\mathbf{q}(a) \neq 0$ and $\deg(\mathbf{q}) = n-1$. Note that $0 \in A$ is the unique nilpotent element of index 1. As the next example shows, elements $a \in A$ satisfying $\mathbf{p}(a) = 0$ for every $\mathbf{p} \in \mathscr{M}$ with $\deg(\mathbf{p})$ equal to a given natural number need not be nilpotent.

Example 2.8.37 Let A be the two-dimensional algebra over $\mathbb{K}$ with basis $\{u, v\}$, and multiplication table given by $u^2 = v^2 = v$ and $uv = vu = 0$. Then we have $uu^2 = u^2u = 0$, and hence $\mathbf{p}(u) = 0$ for every $\mathbf{p} \in \mathscr{M}$ with $\deg(\mathbf{p}) = 3$. However, u is not nilpotent because $0 \neq v = u^2 = (u^2)^2 = ((u^2)^2)^2 = \cdots$.

Let A be an algebra over $\mathbb{K}$. We say that A is a *nil algebra* if all elements of A are nilpotent. If in fact there is $n \in \mathbb{N}$ such that every element of A is nilpotent of index $\leqslant n$, then we say that A is a *nil algebra of bounded index*, and the minimum such n is called the *(bounded) index* of A.

Fact 2.8.38 *Let A be an algebra over $\mathbb{K}$. Then nilpotent elements of A are algebraic. More precisely, if $a \in A$ is a nilpotent element of index n, then* $\dim(A(a)) \leqslant (n-1)!$.

Proof The key simple idea is that, given $m \in \mathbb{N}$, the set

$$\mathscr{M}_m := \{\mathbf{p} \in \mathscr{M} : \deg(\mathbf{p}) \leqslant m\}$$

is finite. We denote by c_m the cardinal number of $\mathscr{M}_m$. Now, if $a \in A \setminus \{0\}$ is a nilpotent element of index n, then $A(a)$ equals the linear hull of the set $\{\mathbf{p}(a) : \mathbf{p} \in \mathscr{M}_{n-1}\}$, so $A(a)$ is finite-dimensional with $\dim(A(a)) \leqslant c_{n-1}$. Therefore, to conclude the proof it is enough to show that the estimate $c_m \leqslant m!$ holds for every $m \in \mathbb{N}$. To this end, let d_m denote the cardinal number of the set $\{\mathbf{p} \in \mathscr{M} : \deg(\mathbf{p}) = m\}$. We claim that $d_m \leqslant (m-1)!$. The claim is clearly true for $m = 1$, so we assume inductively that $m \geqslant 2$, and that the claim is true when m is replaced with any natural number strictly less than m, and then we prove that the claim is true for the given m. Keeping in mind the definition of non-associative words of degree m, the induction hypothesis, and the obvious inequality $i!j! \leqslant (i+j)!$, we have

$$\begin{aligned} d_m &\leqslant \sum_{i=1}^{m-1} d_i d_{m-i} \leqslant \sum_{i=1}^{m-1} (i-1)!(m-i-1)! \\ &\leqslant \sum_{i=1}^{m-1} (m-2)! = (m-2)!(m-1) = (m-1)!, \end{aligned}$$

as desired. Now that the claim has been proved, for every $m \in \mathbb{N}$, we have

$$c_m = \sum_{i=1}^{m} d_i \leqslant \sum_{i=1}^{m} (i-1)! \leqslant (m-1)!m = m!,$$

which concludes the proof. □

A reasonable converse of Fact 2.8.38 above becomes a little more difficult. We recall that the annihilator $\mathrm{Ann}(A)$ of an algebra A was introduced in Definition 1.1.10, and note that $\mathrm{Ann}(A)$ is an ideal of A.

Lemma 2.8.39 *Let A be an algebra over $\mathbb{K}$, and let a be a nilpotent element in A of index n. Then $n \leqslant 2^{\dim(A(a))}$.*

Proof We may assume that A is generated by a. Then we argue by induction on n. The lemma is clearly true for $n = 1, 2$. Assume inductively that $n > 2$, and that, for every algebra B generated by a nilpotent element b of index $m < n$, we have $m \leqslant 2^{\dim(B)}$. Set $B := A/\mathrm{Ann}(A)$, let $\pi : A \to B$ be the natural quotient algebra homomorphism, and write $b := \pi(a)$. Since A is generated by a we get that

$$B \text{ is generated by } b. \tag{2.8.10}$$

On the other hand, since a is nilpotent of index n, we have that

$$\mathbf{p}(a) = 0 \text{ for every } \mathbf{p} \in \mathscr{M} \text{ with } \deg(\mathbf{p}) \geqslant n, \tag{2.8.11}$$

and

$$\text{there exists } \mathbf{q} \in \mathscr{M} \text{ with } \mathbf{q}(a) \neq 0 \text{ and } \deg(\mathbf{q}) = n-1. \tag{2.8.12}$$

From (2.8.11) we deduce that

$$\mathbf{p}(b) = 0 \text{ for every } \mathbf{p} \in \mathscr{M} \text{ with } \deg(\mathbf{p}) \geqslant n. \tag{2.8.13}$$

Let $\mathbf{r}$ and $\mathbf{s}$ be in $\mathscr{M}$ with $\deg(\mathbf{s}) = n-1$. Then $\mathbf{rs}$ and $\mathbf{sr}$ are elements of $\mathscr{M}$ with $\deg(\mathbf{rs}) = \deg(\mathbf{sr}) \geqslant n$, so that, again by (2.8.11), we have

$$\mathbf{r}(a)\mathbf{s}(a) = \mathbf{s}(a)\mathbf{r}(a) = 0.$$

Since A equals the linear hull of the set $\{\mathbf{r}(a) : \mathbf{r} \in \mathscr{M}\}$, we get

$$x\mathbf{s}(a) = \mathbf{s}(a)x = 0$$

for every $x \in A$, so $\mathbf{s}(a)$ lies in $\mathrm{Ann}(A)$, and so

$$\mathbf{s}(b) = 0 \text{ for every } \mathbf{s} \in \mathscr{M} \text{ with } \deg(\mathbf{s}) = n-1. \tag{2.8.14}$$

By taking $\mathbf{s}$ equal to the non-associative word $\mathbf{q}$ given by (2.8.12), we have $0 \neq \mathbf{q}(a) \in \mathrm{Ann}(A)$, and hence

$$\dim(B) \leqslant \dim(A) - 1. \tag{2.8.15}$$

In fact, it follows from (2.8.13) and (2.8.14) that b is nilpotent of index $m < n$. Invoking (2.8.10) and the induction hypothesis, we obtain

$$m \leqslant 2^{\dim(B)}. \tag{2.8.16}$$

Now note that, since $n > 2$, the monomial $\mathbf{q}$ given by (2.8.12) has degree $n-1 > 1$, and hence it is the product of two elements of $\mathscr{M}$, one of which (say $\mathbf{t}$) must be of degree $\geqslant \frac{n-1}{2}$. Since $\mathbf{q}(a) \neq 0$, it follows that $\mathbf{t}(a) \notin \mathrm{Ann}(A)$, and hence that $\mathbf{t}(b) \neq 0$. Since $\deg(\mathbf{t}) \geqslant \frac{n-1}{2}$, we derive that

$$m \geqslant \frac{n+1}{2}. \tag{2.8.17}$$

Finally, by applying (2.8.17), (2.8.16), and (2.8.15), we obtain

$$n \leqslant n+1 \leqslant 2m \leqslant 2^{\dim(B)+1} \leqslant 2^{\dim(A)}. \qquad \square$$

By combining Fact 2.8.38 and Lemma 2.8.39, we get the following.

Theorem 2.8.40 *Let A be a nil algebra over $\mathbb{K}$. Then A is algebraic. Moreover, A is of bounded index if and only if it is algebraic of bounded degree.*

Invoking Corollary 2.8.33, we straightforwardly derive the following from Theorem 2.8.40 above.

Corollary 2.8.41 *Complete normed nil algebras over $\mathbb{K}$ are of bounded index.*

§**2.8.42** Clearly, *nilpotent algebras over $\mathbb{K}$ are nil algebras of bounded index*. The converse is no longer true. Indeed, every nonzero anticommutative algebra is a nil algebra of index 2. Nevertheless, by the Nagata–Higman theorem (see for example [822, Corollary 6.1]), *associative nil algebras over $\mathbb{K}$ of bounded index are in fact nilpotent*. Therefore, invoking Corollary 2.8.41, we get the following.

Corollary 2.8.43 *Complete normed associative nil algebras over $\mathbb{K}$ are nilpotent.*

Completeness cannot be removed altogether in Corollary 2.8.41, nor even in Corollary 2.8.43. Indeed, we have the following.

Example 2.8.44 Let $M_\infty(\mathbb{K})$ denote the associative algebra of all infinite matrices $(\lambda_{ij})_{i,j\in\mathbb{N}}$ over $\mathbb{K}$ with a finite number of nonzero entries. Then $M_\infty(\mathbb{K})$ becomes a normed algebra under the norm $\|(\lambda_{ij})\| := \sum_{i,j\in\mathbb{N}} |\lambda_{ij}|$. For $n \in \mathbb{N}$, we identify $M_n(\mathbb{K})$ with the subalgebra of $M_\infty(\mathbb{K})$ consisting of those matrices $(\lambda_{ij})_{i,j\in\mathbb{N}}$ satisfying $\lambda_{ij} = 0$ whenever $\min\{i,j\} > n$. Now, let A stand for the subalgebra of $M_\infty(\mathbb{K})$ consisting of the so-called 'strictly triangular matrices', i.e. those matrices $(\lambda_{ij})_{i,j\in\mathbb{N}}$ satisfying $\lambda_{ij} = 0$ whenever $i \geqslant j$. Since $A = \cup_{n\in\mathbb{N}}(A \cap M_n(\mathbb{K}))$, and, for $n \in \mathbb{N}$, $A \cap M_n(\mathbb{K})$ is nilpotent of index n, A becomes a non-nilpotent normed associative nil algebra over $\mathbb{K}$.

To get refined versions of Corollaries 2.8.41 and 2.8.43 in the spirit of Theorem 2.8.32, we need the following additional result.

Lemma 2.8.45 *Let A be a normed algebra over $\mathbb{K}$, and assume that there exists a natural number k such that*

$$A_k := \{a \in A : a \text{ is nilpotent of index} \leqslant k\}$$

has non-empty interior in A. Then $A_k = A$.

Proof Let a be an interior point of A_k, let x be in A, and let $\mathbf{p}$ be in $\mathscr{M}$ with $\deg(\mathbf{p}) \geqslant k$. Keeping in mind that the mapping $P : \mu \to \mathbf{p}(a+\mu x)$ from $\mathbb{K}$ to A is continuous, it follows that $P(\mu) = 0$ for $\mu \in \mathbb{K}$ with $|\mu|$ small enough. Therefore, since P is in fact a polynomial function, we have $P(\mu) = 0$ for every $\mu \in \mathbb{K}$ (by the case $m = 1$ of Lemma 2.8.30). Now, for $0 \neq \mu \in \mathbb{K}$, we have

$$\mathbf{p}\left(\frac{a}{\mu}+x\right) = \frac{1}{\mu^{\deg(\mathbf{p})}}\mathbf{p}(a+\mu x) = \frac{1}{\mu^{\deg(\mathbf{p})}}P(\mu) = 0,$$

and it is enough to let $\mu \to \infty$ to get $\mathbf{p}(x) = 0$. Since $\mathbf{p}$ is any element of $\mathscr{M}$ with $\deg(\mathbf{p}) \geqslant k$, we see that x is nilpotent of index $\leqslant k$. Finally, since x is arbitrary in A, the result follows. □

Proposition 2.8.46 *Let A be a complete normed algebra over $\mathbb{K}$ such that there exists a non-empty open set $\Omega \subseteq A$ consisting only of nilpotent elements. Then A is a nil algebra of bounded index.*

Proof By Fact 2.8.38, Ω consists only of algebraic elements. Therefore, by Theorem 2.8.32, A is algebraic of bounded degree. Now, by Lemma 2.8.39, nilpotent elements of A are of index $\leqslant k := 2^{\deg(A)}$. Since the open set Ω consists only of nilpotent elements, the proof is concluded by applying Lemma 2.8.45. □

Invoking the Nagata–Higman theorem again, we derive the following.

Corollary 2.8.47 *Let A be a complete normed associative algebra over $\mathbb{K}$ such that there exists a non-empty open set $\Omega \subseteq A$ consisting only of nilpotent elements. Then A is nilpotent.*

With the aim of showing that Theorem 2.8.32 (respectively, Corollary 2.8.33) ‘contains’ Proposition 2.8.46 (respectively, Corollary 2.8.41) through purely algebraic results, the proof of Proposition 2.8.46 (respectively, Corollary 2.8.41) given above involved some of the most difficult results in the current subsection, namely Theorem 2.8.32 itself (respectively, Corollary 2.8.33 itself) and Lemma 2.8.39 (respectively, Theorem 2.8.40). It is worth mentioning that a simpler analytic proof of Proposition 2.8.46 (and hence of Corollary 2.8.41) can be given, by invoking only Lemma 2.8.45 and the following.

Lemma 2.8.48 *Let A be a normed algebra over $\mathbb{K}$, and let k be a natural number. Then the set $A_k := \{a \in A : a$ is nilpotent of index $\leqslant k\}$ is closed in A.*

Proof Keeping in mind that

$$A_k = \{a \in A : \mathbf{p}(a) = 0 \text{ for every } \mathbf{p} \in \mathscr{M} \text{ with } \deg(\mathbf{p}) \geqslant k\},$$

the result follows by recalling that, for $\mathbf{p} \in \mathscr{M}$, the mapping $a \to \mathbf{p}(a)$ from A to A is continuous. □

Second proof of Proposition 2.8.46 For $k \in \mathbb{N}$, set

$$\Omega_k := \{a \in \Omega : a \text{ is nilpotent of index } \leqslant k\}.$$

Then we have $\Omega = \cup_{k\in\mathbb{N}}\Omega_k$. Since Ω_k is relatively closed in Ω for every k (by Lemma 2.8.48), and Ω is of the second category in itself, there exists $k \in \mathbb{N}$

such that Ω_k has non-empty interior in Ω (and hence in A). The proof is concluded by applying Lemma 2.8.45. □

Lemma 2.8.48 above has an additional interest. Indeed, combining it with Corollary 2.8.41, we get the following.

Corollary 2.8.49 *Let A be a complete normed algebra over $\mathbb{K}$, and let B be a dense subalgebra of A. Then A is a nil algebra if and only if B is a nil algebra of bounded index. Moreover, if this is the case, then A is of bounded index equal to the one of B.*

The Nagata–Higman theorem does not carry over to alternative algebras. This was proved by Dorofeev [215], who constructed an example of a non-nilpotent alternative nil algebra over $\mathbb{K}$ of bounded index. According to the somewhat altered version of Dorofeev's example given in [822, pp. 127–30], this alternative algebra (say B) is the free vector space generated by a certain monad (say M), with product equal to the extension by bilinearity of the product of M. Therefore, B becomes a normed algebra over $\mathbb{K}$ under the norm

$$\left\| \sum_{m\in M} \lambda_m m \right\| := \sum_{m\in M} |\lambda_m|.$$

By passing to the completion of $(B, \|\cdot\|)$, and applying Corollary 2.8.49, we get the following.

Proposition 2.8.50 *There exists a non-nilpotent complete normed alternative nil algebra over $\mathbb{K}$.*

Thus Corollary 2.8.43 does not carry over to alternative algebras.

Let A be an algebra over $\mathbb{K}$. We denote by A^2 the linear hull of the set $\{ab : a, b \in A\}$. We obtain a *derived series* of subalgebras

$$A^{(1)} \supseteq A^{(2)} \supseteq A^{(3)} \supseteq \cdots$$

by defining $A^{(1)} := A$, $A^{(n+1)} := (A^{(n)})^2$. We say that A is *solvable* if $A^{(n)} = 0$ for some $n \in \mathbb{N}$. The following facts are clear:

(i) If A is nilpotent, then A is solvable.
(ii) If A is power-associative and solvable, then A is a nil algebra of bounded index.
(iii) If A is associative and solvable, then A is nilpotent.

However, although non-nilpotent, the alternative nil algebra of bounded index in Dorofeev's example is actually solvable. This is not casual because, as proved by Zhevlakov [666] (see also [822, Corollary 6.3.1]), *alternative nil-algebras over $\mathbb{K}$ of bounded index are solvable*. Therefore, invoking Proposition 2.8.46, we derive the following.

Corollary 2.8.51 *Let A be a complete normed alternative algebra over $\mathbb{K}$ such that there exists a non-empty open set $\Omega \subseteq A$ consisting only of nilpotent elements. Then A is solvable.*

In particular, we have the following.

Corollary 2.8.52 *Complete normed alternative nil algebras over $\mathbb{K}$ are solvable.*

2.8.5 Absolute-valued algebraic algebras are finite-dimensional

Now, we are going to study absolute-valued algebraic algebras. Since, by Corollary 2.6.31, $\mathbb{C}$ *is the unique absolute-valued algebraic complex algebra*, we focus our attention on the case of real algebras.

As an immediate consequence of Fact 2.6.50, we derive the following.

Corollary 2.8.53 *Let A be an absolute-valued algebraic real algebra. Then A is of bounded degree. More precisely, we have* $\deg(A) = 1, 2, 4,$ *or* 8.

The following claim reduces the question of the finite dimensionality of absolute-valued algebraic algebras to the complete separable case.

Claim 2.8.54 *Assume that every complete separable absolute-valued algebraic real algebra is finite-dimensional. Then every absolute-valued algebraic real algebra is finite-dimensional.*

Proof Let A be an absolute-valued algebraic real algebra. By Corollaries 2.8.53 and 2.8.28(ii), the completion $\hat{A}$ of A is an absolute-valued algebraic algebra. Now, fix a nonzero element a in $\hat{A}$, let b be in $\hat{A}$, and let B denote the closed subalgebra of $\hat{A}$ generated by $\{a,b\}$. By Corollary 2.8.21, B is a complete separable absolute-valued algebraic algebra, hence it is finite-dimensional (by assumption). Therefore, since B is an absolute-valued algebra, there exist c,d in B such that $ca = b$ and $ad = b$. Since b is an arbitrary element in $\hat{A}$, we have $a\hat{A} = \hat{A}a = \hat{A}$, and the implication (iii)$\Rightarrow$(i) in Corollary 2.6.24 applies so that $\hat{A}$ (and hence A) is finite-dimensional. $\square$

§**2.8.55** According to Mazur's theorem (see for example [800, Proposition 9.4.3]), if X is a separable Banach space over $\mathbb{K}$, then the set

$$\{u \in \mathbb{S}_X : X \text{ is smooth at } u\}$$

is dense in the unit sphere $\mathbb{S}_X$ of X. Now, the next corollary follows from Corollaries 2.8.53 and 2.8.28(i).

Corollary 2.8.56 *Let A be a complete separable absolute-valued algebraic real algebra. Then there exists a norm-one element a in A such that A is smooth at a and* $\dim(A(a)) = \deg(A)$.

Lemma 2.8.57 *Let A be an absolute-valued algebraic real algebra, and let a be a norm-one element of A. We have:*

(i) *If* $\dim(A(a)) = \deg(A)$, *and if b is in $A \setminus \{0\}$ with $ab = b$, then $A(b) = A(a)$.*
(ii) *If A is smooth at a, then A is smooth at every norm-one element of $A(a)$.*

Proof Let b be a nonzero element of A. Then $A(b)$ is a finite-dimensional absolute-valued algebra, so it is a division algebra, and so there exists $c \in A(b)$ satisfying $cb = b$. Therefore, if $ab = b$, then we have $(a-c)b = 0$, so $a = c$ lies in $A(b)$, and so $A(a) \subseteq A(b)$. If in addition $\dim(A(a)) = \deg(A)$, then the above inclusion must be an equality, which concludes the proof of assertion (i).

Now, let b be a norm-one element in $A(a)$. Then there exists $c \in A(a)$ such that $cb = a$. Since the left multiplication by c on A is a linear isometry, if A is not smooth at b, then cA is not smooth at $cb = a$, and therefore A is not smooth at a, which proves assertion (ii). □

Now, let us summarize those aspects of the theory of ultraproducts which are needed for our purpose (cf. [318]).

§2.8.58 ***From now until Proposition 2.8.63, I will denote a non-empty set, and $\mathscr{U}$ will stand for an ultrafilter on I.*** Given a family $\{X_i\}_{i\in I}$ of Banach spaces, we may consider the Banach space $\bigoplus_{i\in I}^{\ell_\infty} X_i$ ℓ_∞-sum of this family, consisting of all families $\{x_i\} \in \prod_{i\in I} X_i$ such that

$$\|\{x_i\}\| := \sup\{\|x_i\| : i \in I\} < \infty,$$

and the closed subspace $N_{\mathscr{U}}$ of $\bigoplus_{i\in I}^{\ell_\infty} X_i$ given by

$$N_{\mathscr{U}} := \left\{ \{x_i\} \in \bigoplus_{i\in I}^{\ell_\infty} X_i : \lim_{\mathscr{U}} \|x_i\| = 0 \right\}.$$

The *(Banach) ultraproduct* of the family $\{X_i\}_{i\in I}$ (with respect to the ultrafilter $\mathscr{U}$) is defined as the quotient Banach space $(\bigoplus_{i\in I}^{\ell_\infty} X_i)/N_{\mathscr{U}}$, and is denoted by $(X_i)_{\mathscr{U}}$. If we denote by (x_i) the element in $(X_i)_{\mathscr{U}}$ containing a given family $\{x_i\} \in \bigoplus_{i\in I}^{\ell_\infty} X_i$, then it is easy to verify that $\|(x_i)\| = \lim_{\mathscr{U}} \|x_i\|$. Because of this formula, if Y_i is a closed subspace of X_i for each $i \in I$, then in a natural way we can identify $(Y_i)_{\mathscr{U}}$ with a subspace of $(X_i)_{\mathscr{U}}$. If X_i is equal to a given Banach space X for every $i \in I$, then the ultraproduct $(X_i)_{\mathscr{U}}$ is called the *ultrapower* of X (with respect to $\mathscr{U}$) and is denoted by $X_{\mathscr{U}}$. In this case, the mapping $x \to \hat{x}$ from X into $X_{\mathscr{U}}$, where $\hat{x} = (x_i)$ with $x_i = x$ for all $i \in I$, is an isometric linear imbedding.

Lemma 2.8.59 *Let X be a finite-dimensional normed space. Then $X_{\mathscr{U}} = \hat{X}$.*

Proof Let (x_i) be in $X_{\mathscr{U}}$. Since $\{x_i : i \in I\}$ is contained in a compact subset of X, there exists $x := \lim_{\mathscr{U}} x_i \in X$. Then we have $\hat{x} = (x_i)$. □

Corollary 2.8.60 *Let $\{X_i\}_{i\in I}$ be a family of Hilbert spaces over $\mathbb{K}$ such that there exists a natural number n satisfying $\dim(X_i) \leqslant n$ for every $i \in I$. Then $\dim((X_i)_{\mathscr{U}}) \leqslant n$.*

Proof Let X be a Hilbert space over $\mathbb{K}$ of dimension n. For each $i \in I$ we may find a linear isometry $F_i : X_i \to X$. Then $(x_i) \to (F(x_i))$ is a (well-defined) linear isometry from $(X_i)_{\mathscr{U}}$ to $X_{\mathscr{U}}$. Now apply Lemma 2.8.59. □

§2.8.61 Let $\{A_i\}_{i\in I}$ be a family of complete normed algebras over $\mathbb{K}$. Then the Banach space $(A_i)_{\mathscr{U}}$ will be considered without notice as a new complete normed algebra under the (well-defined) product

$$(a_i)(b_i) := (a_ib_i).$$

In this way, if for each $i \in I$, B_i is a closed subalgebra of A_i, then, up to the natural identification, $(B_i)_{\mathscr{U}}$ becomes a subalgebra of $(A_i)_{\mathscr{U}}$. If, for each $i \in I$, A_i is in fact an absolute-valued algebra, then the equality $\|(a_i)\| = \lim_{\mathscr{U}} \|a_i\|$ shows that $(A_i)_{\mathscr{U}}$ is an absolute-valued algebra too. It follows that the ultrapower $A_{\mathscr{U}}$ of a complete absolute-valued algebra A is an absolute-valued algebra. Let us also note that, if A is any complete normed algebra, then the natural imbedding $A \hookrightarrow A_{\mathscr{U}}$ becomes an algebra homomorphism.

As a straightforward consequence of Fact 2.6.50 and Corollary 2.8.60, we derive the following.

Corollary 2.8.62 *Let $\{A_i\}_{i\in I}$ be a family of finite-dimensional absolute-valued real algebras. Then the absolute-valued algebra $(A_i)_{\mathscr{U}}$ is finite-dimensional with*

$$\dim((A_i)_{\mathscr{U}}) \leqslant \max\{\dim(A_i) : i \in I\} \leqslant 8.$$

Now, we can prove the following.

Proposition 2.8.63 *Let A be a complete absolute-valued algebraic real algebra. Then $A_{\mathscr{U}}$ is an absolute-valued algebraic algebra with*

$$\deg(A_{\mathscr{U}}) = \deg(A).$$

Proof Let (a_i) be in $A_{\mathscr{U}}$. For each $i \in I$, $A(a_i)$ is a finite-dimensional absolute-valued real algebra, and hence, by Corollary 2.8.62, $(A(a_i))_{\mathscr{U}}$ is finite-dimensional as well, and

$$\dim((A(a_i))_{\mathscr{U}}) \leqslant \max\{\dim(A(a_i)) : i \in I\} \leqslant \deg(A).$$

Since, up to the natural identification, $(A(a_i))_{\mathscr{U}}$ is a subalgebra of $A_{\mathscr{U}}$ containing (a_i), it follows that the subalgebra of $A_{\mathscr{U}}$ generated by (a_i) is finite-dimensional with dimension $\leqslant \deg(A)$. Since (a_i) is arbitrary in $A_{\mathscr{U}}$, we obtain that $A_{\mathscr{U}}$ is algebraic with $\deg(A_{\mathscr{U}}) \leqslant \deg(A)$. The converse inequality follows by regarding A as a subalgebra of $A_{\mathscr{U}}$. □

Lemma 2.8.64 *Let X be a Banach space over $\mathbb{K}$, and let $F : X \to X$ be a non-surjective linear isometry. Then $F - I_X$ is neither bounded below nor surjective.*

Proof Let r be a positive real number. If $r < 1$, then we have

$$\|F - (F - rI_X)\| = r < 1 = k(F),$$

and hence, by Lemma 2.8.1, $F - rI_X \notin \mathrm{Inv}(BL(X))$. On the other hand, if $r > 1$, then clearly we have $F - rI_X \in \mathrm{Inv}(BL(X))$. It follows that $F - I_X$ lies in the boundary of $\mathrm{Inv}(BL(X))$ relative to $BL(X)$, and Corollary 1.1.95 applies. □

Lemma 2.8.65 *Let X be a normed space over $\mathbb{K}$, let $F : X \to X$ be a linear contraction, and let M be a finite-dimensional subspace of X. Assume that $F(m) = m$ for every $m \in M$, that the restriction of the norm of X to M derives from an inner product, and that X is smooth at every norm-one element of M. Then there exists a continuous linear projection $\pi : X \to X$ such that $\pi(X) = M$ and* $\ker(\pi)$ *is invariant under F.*

Proof Let $(\cdot|\cdot)$ denote the inner product on M such that $\|m\|^2 = (m|m)$ for every $m \in M$, and let $\{m_1,\dots,m_k\}$ be an orthonormal basis of M relative to $(\cdot|\cdot)$. By the Hahn–Banach theorem, there are norm-one continuous linear functionals $\phi_1,\dots,\phi_k$ on X satisfying $\phi_i(m) = (m|m_i)$ for all $m \in M$ and $i = 1,\dots,k$. Then the mapping $\pi : x \to \sum_{i=1}^k \phi_i(x)m_i$ from X to X is a continuous linear projection on X with $\pi(X) = M$. Let i be in $\{1,\dots,k\}$. Since $\phi_i \in D(X,m_i)$, and F is a linear contraction on X with $F(m) = m$ for every $m \in M$, the mapping $\psi_i := \phi_i F$ belongs to $D(X,m_i)$. It follows from the smoothness of X at every norm-one element of M that $\psi_i = \phi_i$, and therefore $\ker(\phi_i)$ is invariant under F. To conclude the proof, note that $\ker(\pi) = \cap_{i=1}^k \ker(\phi_i)$. □

Theorem 2.8.66 *Absolute-valued algebraic real algebras are finite-dimensional.*

Proof Let A be an absolute-valued algebraic real algebra. We must show that A is finite-dimensional. By Claim 2.8.54, we can assume that A is complete and separable. Then, by Corollary 2.8.56, there exists a norm-one element $a \in A$ such that A is smooth at a and $\dim(A(a)) = \deg(A)$. Now assume that A is infinite-dimensional. Then, by the implication (iii)⇒(i) in Corollary 2.6.24, we cannot have $aA = Aa = A$, so, for example, we must have $aA \neq A$, so that the mapping $F : x \to ax$ from A to A is a non-surjective linear isometry. Let M denote the finite-dimensional subspace of A given by

$$M := \{x \in A(a) : ax = x\}.$$

By Fact 2.6.50, the restriction of the norm of A to M derives from an inner product, and, by Lemma 2.8.57(ii), A is smooth at every norm-one element of M. Now, we are in a position to apply Lemma 2.8.65, so that there exists a continuous linear projection $\pi : A \to A$ such that $\pi(A) = M$ and $\ker(\pi)$ is invariant under F. Keeping in mind that $A = M \oplus \ker(\pi)$, that F is the identity mapping on M, that $\ker(\pi)$ is invariant under F, and that F is a non-surjective linear isometry, it follows that the mapping $y \to F(y) = ay$ from $\ker(\pi)$ to $\ker(\pi)$ is a non-surjective linear isometry. By Lemma 2.8.64, there exists a sequence x_n of norm-one elements in $\ker(\pi)$ such that $ax_n - x_n \to 0$. Choose an ultrafilter $\mathscr{U}$ on $\mathbb{N}$ refining the Fréchet filter (of all cofinite subsets of $\mathbb{N}$), let β denote the norm-one element in $A_{\mathscr{U}}$ given by $\beta := (x_n)$, and consider A as a subalgebra of $A_{\mathscr{U}}$ via the canonical imbedding. Then we have $a\beta = \beta$. But, by Proposition 2.8.63, $A_{\mathscr{U}}$ is an absolute-valued algebraic algebra with $\deg(A_{\mathscr{U}}) = \deg(A)$, hence, since $\dim(A(a)) = \deg(A)$, we have $A_{\mathscr{U}}(a) = A(a)$ and $\dim(A_{\mathscr{U}}(a)) = \deg(A_{\mathscr{U}})$. It follows from Lemma 2.8.57(i) that β lies in $A(a)$ and, more precisely, in M. Now that we know that β is in A, the equality $\beta = (x_n)$ reads as $\lim_{\mathscr{U}} \|x_n - \beta\| = 0$. Hence, since x_n is in $\ker(\pi)$ for all n in $\mathbb{N}$, β is in $\ker(\pi)$ as well. Then $\beta \in M \cap (\ker \pi) = 0$, a contradiction. □

Corollary 2.8.67 *Let A be an absolute-valued power-commutative real algebra. Then A is finite-dimensional, and* $\deg(A) \leqslant 2$.

Proof By the commutative Urbanik–Wright theorem (see Theorem 2.6.41), A is algebraic with $\deg(A) \leqslant 2$. Now apply Theorem 2.8.66. □

Since flexible algebras are power-commutative (by Corollary 2.4.16), we derive the following.

Corollary 2.8.68 *Let A be an absolute-valued flexible real algebra. Then A is finite-dimensional, and* $\deg(A) \leqslant 2$.

Since power-associative algebras are power-commutative, we could also have derived from Corollary 2.8.67 that absolute-valued power-associative real algebras are finite-dimensional and of degree $\leqslant 2$, but this does not merit being emphasized because we have already proved a better result in Proposition 2.6.27. In relation to Corollaries 2.8.67 and 2.8.68, it it worth mentioning that, as we will see in Corollary 2.8.84 below, absolute-valued power-commutative algebras are in fact flexible.

2.8.6 Historical notes and comments

Theorem 2.8.4 and the proof given here are due to Rodríguez [521], although Lemmas 2.8.1 and 2.8.3 are taken from [529] and [516], respectively.

Subsection 2.8.2 is taken from the Cuenca–Rodríguez paper [200], although Lemma 2.8.10 is folklore, and Lemma 2.8.12 was previously known in [199]. Some of the results of [200] developed here have been slightly refined. In particular, Corollary 2.8.11 is new. In Proposition 2.8.88 below, we will formulate the precise description, given in [200], of the algebras considered in Proposition 2.8.13.

In the original proof [618] of Theorem 2.7.69, Urbanik already knew that when the set **X** reduces to a singleton, the free non-associative real algebra generated by **X** becomes an absolute-valued algebra under the norm $\|\cdot\|_2$. The general case of such an observation, stated in §2.8.18, is due to Cabrera and was announced in [149], but it first appeared formulated with the appropriate accuracy in [521]. Results from Proposition 2.8.19 to Corollary 2.8.25 (with the exception of Corollary 2.8.21 and Theorem 2.8.23) are taken from [533]. Corollary 2.8.21 is new, whereas Theorem 2.8.23 is due to Becerra, Moreno, and Rodríguez [68], who acknowledge deep suggestions from Y. Benjamini. Some particular cases of Corollary 2.8.21 seem to be folklore. Indeed, the cases that the algebra A is associative, or that the subset $S \subseteq A$ is finite, were applied without notice in the original proofs of Theorem 2.8.23 (see [68, Theorem 2.11]) or Claim 2.8.54 (see [368, Claim 1]), respectively.

A partial converse of Lemma 2.8.22 and its proof is provided by the following.

Theorem 2.8.69 *Let E be a compact Hausdorff topological space such that $C^{\mathbb{K}}(E)$ becomes an absolute-valued algebra for a certain product $\odot$. Then there exist a closed subset F of E, a continuous function $\phi : E \to \mathbb{S}_{\mathbb{K}}$, and a continuous surjective mapping $f : F \to E \times E$, satisfying*

$$(x \odot y)(t) = \phi(t)x[\pi_1(f(t))]y[\pi_2(f(t))]$$

for all $x, y \in C^{\mathbb{K}}(E)$ and $t \in F$. Here, as in the proof of Lemma 2.8.22, *$\pi_1, \pi_2 : E \times E \to E$ stand for the coordinate projections.*

Theorem 2.8.69 above is due to Moreno and Rodríguez [451], who mimic the proof of the celebrated Holsztynki's theorem [329] asserting that, *if $\Phi : C^{\mathbb{K}}(E) \to C^{\mathbb{K}}(E)$ is a (possibly non-surjective) linear isometry, then there exist a closed subset F of E, a continuous function $\phi : E \to \mathbb{S}_{\mathbb{K}}$, and a continuous surjective mapping $f : F \to E$, satisfying $\Phi(x)(t) = x(f(t))$ for every $x \in C^{\mathbb{K}}(E)$ and*

$t \in F$. Theorem 2.8.69, together with deep suggestions from Y. Benjamini, allowed the authors of [451] to prove the following.

Theorem 2.8.70 *Let E be a compact metrizable topological space not reduced to a point. Then $C^{\mathbb{K}}(E)$ becomes an absolute-valued algebra for some product if and only if E is uncountable.*

Results from Lemma 2.8.27 to Corollary 2.8.34 are due to Cuartero, Galé, Rodríguez, and Slinko [188] (1993). Nevertheless, although published shortly later, Lemma 2.8.30 was previously known by Cuartero and Galé [187], and becomes the key tool in the organization of the material covered here. Actually, Lemma 2.8.30 becomes an algebraic variant of an early analytic result (due to Galé, Ransford, and White [276]), which is formulated in Lemma 2.8.71 immediately below. This lemma was applied in [276] to characterize weakly compact algebra homomorphisms from C^*-algebras. A version of such a characterization in a non-associative setting can be found in [275].

Lemma 2.8.71 *Let X be a complex Banach space, let Ω be a domain in $\mathbb{C}$, and let $f_1, \ldots, f_m$ be holomorphic functions from Ω to X. Then the set*

$$D := \{\mu \in \Omega : \{f_1(\mu), \ldots, f_m(\mu)\} \text{ is linearly dependent}\}$$

is either discrete or equal to Ω.

§**2.8.72** Lemmas 2.8.27 and 2.8.31 refine previous ideas of Cuartero and Galé in [186, 187] for associative and power-associative algebras. Corollary 2.8.33 (that *complete normed algebraic algebras are of bounded degree*) has a long history. It was proved by Mazur [432] (1938) for associative and commutative algebras, by Kaplansky [373] (1947) for (possibly non-commutative) associative algebras, by Slinko [586] (1990) for Jordan algebras, and by Cuartero and Galé [187] (1994) for power-associative algebras. Concerning Mazur's forerunner, let us recall that, as we commented in §2.5.63, the original proof is not available.

The refinement of Corollary 2.8.33, given by Theorem 2.8.32, already appeared in the version of [187] for power-associative algebras. For the refinement, the authors of [187] were inspired by the following result of Aupetit [39] (see also [682, Theorem 3.2.1]).

Theorem 2.8.73 *Let A be a complete normed associative complex algebra such that there exists a non-empty open set $\Omega \subseteq A$ consisting only of elements with finite spectrum. Then A is finite-dimensional modulo its Jacobson radical.* (For the meaning of the Jacobson radical of an associative algebra, the reader is referred to Definition 3.6.12 below.)

The case that $\Omega = A$ in the above theorem is originally due to Kaplansky [378], and was rediscovered much later in [327]. An appropriate version for Jordan algebras, which will be discussed in Volume 2 of our work, can be found in [91]. Thinking about any non-nil complete normed radical associative complex algebra (as, for example, the one in the proof of Proposition 4.4.65(iii) below), we realize that, in its associative complex context, the assumption in Theorem 2.8.73 is ostensibly weaker than the one in Theorem 2.8.32, and that the conclusion in Theorem 2.8.73 is the best that we could hope for. The actual historical interest of the stronger conclusion in Theorem 2.8.32 (that the algebra is algebraic of bounded degree) relies on Theorem 2.8.75 below.

Corollaries 2.8.43 and 2.8.47 are due to Grabiner [298] (see also [696, Theorem 46.3]), although they are germinally contained in [373, p. 537]. For the history of the Nagata–Higman theorem applied in the proof of these corollaries, the reader is referred to [266]. With the exception of Corollaries 2.8.43 and 2.8.47, which have already been commented, results from Fact 2.8.38 to Corollary 2.8.49 could be new. When restricted to power-associative algebras, these results turn out to be much easier, and actually they are known in [187]. Indeed, in the case of power-associative algebras, Fact 2.8.38, Lemma 2.8.39, and Theorem 2.8.40 reduce to the following.

Fact 2.8.74 *If a is a nilpotent element of index n in a power-associative algebra A, then* $\dim(A(a)) = n-1$.

By keeping the above fact in mind, the power-associative version of Proposition 2.8.46 follows straightforwardly from Theorem 2.8.32 by invoking, in addition, only (the power-associative version of) Lemma 2.8.45. We thank A. Moreno for communicating to us the estimate $c_m \leqslant m!$ in the proof of Fact 2.8.38. The proof of Lemma 2.8.39 is inspired by that of [452, Proposition 5.8]. Although easily derivable from previously known facts, results from Proposition 2.8.50 to Corollary 2.8.52 could be also new. We note that Corollaries 2.8.51 and 2.8.52 could have been formulated in [187] by simply referring to Zhevlakov's theorem [666], as we have done.

Let us recall that an algebra A is called *locally finite* if the subalgebra generated by any finite subset of A is finite-dimensional. Now, as announced above, let us focus on the historical interest of algebraic algebras of bounded degree, which in the associative case relies on the following.

Theorem 2.8.75 *Let A be an associative algebraic algebra over $\mathbb{K}$ of bounded degree. Then A is locally finite.*

The above theorem was obtained by Kaplansky [376] by applying Jacobson's structure theory for associative algebraic algebras of bounded degree [347]. Theorem 2.8.75 can also be found in [753, Theorems X.10.1(1) and X.12.1], and, thanks to Corollary 2.8.33, implies the following.

Corollary 2.8.76 *Complete normed associative algebraic algebras over $\mathbb{K}$ are locally finite.*

Theorem 2.8.75 and Corollary 2.8.76 are specially relevant because they provide partial affirmative answers to the so-called *Kurosh's problem* [396] (1941): wondering whether every associative algebraic algebra over $\mathbb{K}$ is locally finite. Kurosh's problem remained open for many years until it was answered negatively by Golod [295] (1964), who constructed an example of an infinite-dimensional associative nil-algebra over $\mathbb{K}$ with three generators. In [744, p. 192] (1968) Herstein pointed out that, in Golod's result, 'we can get by with two generators instead of three'. Thus we have the following.

Theorem 2.8.77 *There exist infinite-dimensional associative nil algebras over $\mathbb{K}$ generated by 2 elements.*

Complete proofs of the above theorem can be found in the books of Andrunakievich and Ryabuhin [677, pp. 60–6], and of Kargapolov and Merzljakov [763, Example 23.2.5], both published in 1979. Theorem 2.8.77 has been rediscovered in [307].

According to Kaplansky [379] (1970), after Golod's negative solution to Kurosh's problem, 'special cases of Kurosh's problem still invite attention. In a draft of this paper I asked whether a primitive algebraic [associative] algebra is necessarily locally finite. In a letter dated July 23, 1969, Amitsur described how to 'surround' the Golod example by a primitive algebraic algebra which is not locally finite. However, the algebra is not itself finitely generated.' Then, Kaplansky asked whether every finitely generated, primitive, algebraic associative algebra over $\mathbb{K}$ is finite-dimensional. This question has also been negatively answered by Bell and Small [84] (2002). (For the meaning of a primitive algebra, the reader is referred to Definition 3.6.12 below.)

In the proof of [615, Theorem 3.7], Turovskii notices that one of the algebras over $\mathbb{K}$ given by Theorem 2.8.77 (just that in [677, 763] already quoted) can be provided with an algebra norm. As a consequence, completeness cannot be altogether removed in Corollary 2.8.76. It is worth mentioning that, according to Dales [203], every associative and commutative nil complex algebra with countable basis can be endowed with an algebra norm. If commutativity could be removed in Dales's result, then we would be provided with a theoretical proof of Turovskii's observation.

Theorem 2.8.75 remains true if the assumption that A is associative is relaxed to the one that A is alternative. This result, due to Shirshov [568], can be seen in [822, Corollary 5.5.1]. Theorem 2.8.75 also remains true if the assumption that A is associative is replaced with the one that A is a Jordan algebra. By keeping in mind [755, Proposition 7.9.1], this Jordan version of Theorem 2.8.75 is due to Zel'manov [661]. Now, applying Theorem 2.8.32, we derive the following.

Corollary 2.8.78 *Let A be a complete normed algebra over $\mathbb{K}$ containing a non-empty, open subset consisting only of algebraic elements. If, further, A is alternative or Jordan, then A is locally finite.*

Although alternative nil algebras of bounded index need not be nilpotent, finite-dimensional alternative nil algebras over $\mathbb{K}$ are indeed nilpotent [808, Theorem III.3.2], and the same is true with Jordan instead of alternative [808, Theorem IV.4.3]. Now, recall that an algebra A is called *locally nilpotent* if the subalgebra generated by any finite subset of A is nilpotent. It follows that *locally finite alternative or Jordan nil algebras over $\mathbb{K}$ are locally nilpotent*. Therefore, invoking Fact 2.8.74, Proposition 2.8.32, and the results in [568] and [661] reviewed above, we obtain the following.

Corollary 2.8.79 *Let A be a complete normed algebra over $\mathbb{K}$ containing a non-empty open subset consisting only of nilpotent elements. If, further, A is alternative or Jordan, then A is locally nilpotent.*

§**2.8.80** *Generalized standard algebras*, introduced by Schafer [553], are defined by a suitable finite set of identities, and, roughly speaking, they compose the minimum class of algebras containing all alternative algebras and all Jordan algebras. In particular, *generalized standard algebras are non-commutative Jordan algebras* [553, Theorem 2]. On the other hand, *finite-dimensional generalized standard nil algebras over* $\mathbb{K}$ *are nilpotent* [553, Theorem 4], and hence *locally finite generalized standard nil algebras over* $\mathbb{K}$ *are locally nilpotent*. It would be interesting to know whether Theorem 2.8.75 remains true when the assumption that A is associative is relaxed to the one that A is generalized standard. If the answer to this question were affirmative, then Corollaries 2.8.78 and 2.8.79 would remain true with 'generalized standard' instead of 'alternative or Jordan'.

In relation to the ideas in the above paragraph, let us mention the celebrated Suttles' example [607] (see also [487, p. 344]) of a five-dimensional power-associative commutative nil algebra which is not nilpotent.

Albert's result (already stated in Fact 2.6.44), that $\mathbb{R},\mathbb{C},\mathbb{H}$, *and* $\mathbb{O}$ *are the unique absolute-valued algebraic unital real algebras*, is now obsolete because of the non-commutative Urbanik–Wright theorem (Theorem 2.6.21). Nevertheless, it has had the merit of encouraging the work on the question of whether every absolute-valued algebraic algebra is finite-dimensional. Since this question has an almost trivial affirmative answer for complex algebras (see Corollary 2.6.31), interest focused on the case of real algebras. Some partial affirmative answers were provided by M. L. El-Mallah. Thus, *an absolute-valued algebraic real algebra A is finite-dimensional whenever there exists a nonzero idempotent in A commuting with every element of A* [235], *or there exists a continuous algebra involution $*$ on A satisfying $xx^* = x^*x$ for every $x \in A$* [238], *or A satisfies the identity $xx^2 = x^2x$* [240]. We note that the result in [235] would later become a consequence of the one in [238] because of Theorem 2.6.46.

Since absolute-valued algebraic real algebras are of degree 1, 2, 4, or 8 (by Corollary 2.8.53), and $\mathbb{R}$ is the unique absolute-valued algebraic algebra of degree 1 (by Lemma 2.6.29), the strategy of studying the cases of degree 2, 4, and 8 separately could seem tempting in order to answer affirmatively the question we are considering. As a matter of fact, such an strategy turned out to be unsuccessful, except for the case of degree 2. Indeed, as proved in [524], we have the following.

Proposition 2.8.81 *The absolute-valued real algebras of degree* 2 *are* $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, ${}^*\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, ${}^*\mathbb{H}$, $\mathbb{H}^*$, $\mathbb{O}$, $\overset{*}{\mathbb{O}}$, ${}^*\mathbb{O}$, $\mathbb{O}^*$, *and* $\mathbb{P}$.

Here, for $\mathbb{A}$ equal to $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, the symbols ${}^*\mathbb{A}$ and $\mathbb{A}^*$ stand for the absolute-valued real algebras obtained by endowing the normed space of $\mathbb{A}$ with the products $x \odot y := x^*y$ and $x \odot y := xy^*$, respectively, where $*$ means the standard involution. We recall that both the symbols $\overset{*}{\mathbb{A}}$ (for $\mathbb{A}$ equal to $\mathbb{C},\mathbb{H}$, or $\mathbb{O}$) and $\mathbb{P}$ have already been introduced in Subsection 2.6.3.

The definitive affirmative answer to the question of whether absolute-valued algebraic algebras are finite-dimensional, given in Theorem 2.8.66, is due to Kaidi, Ramírez, and Rodríguez [367]. The proof given in Subsection 2.8.5 (pp. 270ff.) is essentially the original one in [367], although some relevant simplifications taken from [368] have been incorporated.

According to [533], *Lemma 2.8.65 remains true if the assumption that the restriction of the norm of X to M derives from an inner product is altogether removed*. The proof goes as follows:

> Let X, F, and M be as in Lemma 2.8.65, but remove the assumption that the restriction of the norm of X to M derives from an inner product. By a theorem of Auerbach [36] (see also [800, Lemmas 7.1.6 and 7.1.7]), there are bases $\{m_1,\dots,m_k\}$ and $\{g_1,\dots,g_k\}$ of M and M', respectively, consisting of norm-one elements and satisfying $g_i(m_j) = \delta_{ij}$. Extending each g_i to a norm-one linear functional ϕ_i on X, and arguing as in the conclusion of the proof of Lemma 2.8.65, we realize that the mapping $\pi : x \to \sum_{i=1}^{k} \phi_i(x)m_i$ from X to X becomes a continuous linear projection on X such that $\pi(X) = M$ and $\ker(\pi)$ is invariant under F. □

Corollary 2.8.68, originally due to El-Mallah and Micali [244], is the earliest result among those included in this section. Later, El-Mallah, in a series of papers (see [235, 236, 237, 238, 239]), deeply refines Corollary 2.8.68 by considering absolute-valued algebras satisfying the identity $xx^2 = x^2x$ (which is of course implied by the flexibility), and proving the following.

Theorem 2.8.82 *For an absolute-valued real algebra A, the following conditions are equivalent:*

(i) *A is flexible.*
(ii) *A is a pre-Hilbert space and satisfies the identity $x^2x = xx^2$.*
(iii) *A is finite-dimensional and satisfies the identity $x^2x = xx^2$.*
(iv) *A is equal to $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, $\overset{*}{\mathbb{O}}$, or $\mathbb{P}$.*

A refinement of the above theorem can be found in Corollary 2.7 of Cuenca's paper [191]. In [239], El-Mallah combines Theorems 2.6.46 and 2.8.82 with a result of Urbanik ([617, Theorem 6]) to derive the following.

Corollary 2.8.83 $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, *and* $\overset{*}{\mathbb{O}}$ *are the unique absolute-valued real algebras A satisfying the identity $x^2x = xx^2$ and containing a nonzero idempotent commuting with all elements of A.*

Another relevant consequence of Theorem 2.8.82 is the following.

Corollary 2.8.84 *Absolute-valued algebras over $\mathbb{K}$ are flexible if (and only if) they are power-commutative.*

Proof If $\mathbb{K} = \mathbb{C}$, the result follows from Corollary 2.6.42. Let A be an absolute-valued power-commutative real algebra. By Corollary 2.8.67, A is finite-dimensional. On the other hand, clearly, A satisfies the identity $x^2x = xx^2$. It follows from the implication (iii)⇒(i) in Theorem 2.8.82 that A is flexible. □

Corollary 2.8.84 was first proved in [524] involving Theorem 2.6.41 and Proposition 2.8.81 instead of Corollary 2.8.67 (which was unknown at the time) and Theorem 2.8.82.

In relation to Theorem 2.8.82, it seems to be an open problem whether every absolute-valued real algebra satisfying the identity $x^2x = xx^2$ is finite-dimensional. According to the theorem itself, the answer is affirmative if A is a pre-Hilbert space. The answer is also affirmative if A is algebraic [240], but, in view of Theorem 2.8.66, this result is unsubstantial today.

In relation to the problem just raised, we have the following previously unpublished result, which could help to find an affirmative answer.

Proposition 2.8.85 *Let A be an absolute-valued real algebra satisfying the identity $x^2x = xx^2$, and let M be any two-dimensional subspace of A. Then there exists $m \in M$ such that $-m^2$ is a nonzero idempotent.*

Keeping in mind that *commutative subspaces of an absolute-valued algebra are pre-Hilbert spaces* (indeed, argue as in the proof of Theorem 2.6.41), the proof goes as follows.

Proof Consider the mapping $\Phi : A \to \mathbb{R}$ defined by

$$\Phi(x) := \frac{1}{4}\left(\|x+x^2\|^2 - \|x-x^2\|^2\right).$$

Note that Φ is continuous and satisfies $\Phi(-x) = -\Phi(x)$ for every $x \in A$, and that $\mathbb{S}_M$ is connected and symmetric (i.e. $-\mathbb{S}_M = \mathbb{S}_M$), it follows that $\Phi(\mathbb{S}_M)$ is a connected and symmetric subset of $\mathbb{R}$, and hence we have $0 \in \Phi(\mathbb{S}_M)$. Taking $m \in \mathbb{S}_M$ such that $\Phi(m) = 0$, and noticing that the linear hull of $\{m, m^2\}$ is a Hilbert space for some inner product $(\cdot|\cdot)$, we have $(m|m^2) = \Phi(m) = 0$, and therefore

$$\|m^2 - (m^2)^2\| = \|(m+m^2)(m-m^2)\| = \|m+m^2\|\|m-m^2\| = 2.$$

Since the linear hull of $\{m^2, (m^2)^2\}$ is also a Hilbert space, it follows from the equality $\|m^2 - (m^2)^2\| = 2$ above and the parallelogram law that

$$m^2 + (m^2)^2 = 0.$$

Now $-m^2$ is a nonzero idempotent. □

In recent years, Proposition 2.8.85 has been circulating among people interested in the topic, generating new interesting results. This is the case of the next theorem, due to Cuenca [196].

Theorem 2.8.86 *$\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, and $\overset{*}{\mathbb{O}}$ are the unique absolute-valued real algebras A satisfying the identity $x^2x = xx^2$ and containing a nonzero idempotent which commutes with all idempotents of A.*

Theorem 2.8.86 above refines Corollary 2.8.83, and contains the next result, originally due to Chandid [708].

Corollary 2.8.87 *$\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued real algebras satisfying the identity $x^2x = xx^2$ and whose idempotents pairwise commute. As a consequence, $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique absolute-valued real algebras satisfying the identity $x^2x = xx^2$ and containing only a nonzero idempotent.*

As an example of how Proposition 2.8.85 works, let us introduce a very short argument (courtesy of J. A. Cuenca) reducing the proof of the last conclusion in Corollary 2.8.87 above to El-Mallah's Corollary 2.8.83:

> Let A be an absolute-valued real algebra satisfying the identity $x^2x = xx^2$ and containing only a nonzero idempotent e. Let a be in $A \setminus \mathbb{R}e$. Then, by Proposition 2.8.85, there is $b \in \mathbb{R}e + \mathbb{R}a$ such that $-b^2 = e$. Thus $b \notin \mathbb{R}e$ (which implies that $\mathbb{R}e + \mathbb{R}a = \mathbb{R}e + \mathbb{R}b$) and $[e,b] = 0$. Therefore $[e,a] = 0$, hence $[e,A] = 0$ because of the arbitrariness of $a \in A \setminus \mathbb{R}e$. Now it is enough to invoke Corollary 2.8.83 to conclude that A is equal to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. $\square$

The Cuenca–Rodríguez paper [200] contains a precise determination of the algebras A satisfying the requirements in Proposition 2.8.13. To review such a determination, note at first that, as a consequence of that proposition, we are in fact dealing with an absolute-valued finite-dimensional real algebra A endowed with an isometric algebra involution $*$, whose norm derives from an inner product $(\cdot|\cdot)$ satisfying

$$x^*(xy) = (yx)x^* \text{ and } (xy|z) = (x|zy^*) = (y|x^*z) \text{ for all } x,y,z \in A.$$

Since $*$ is a surjective linear isometry, we can consider the Albert isotope of A (say B) consisting of the normed space of A and the product $x \odot y := x^*y^*$. Now, we trivially realize that the absolute-valued real algebra B is flexible and satisfies $(x \odot y|z) = (x|y \odot z)$ for all $x,y,z \in B$. Then, noticing that $\mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ do not satisfy the last equality, we deduce from Theorem 2.8.82 that B is equal to $\mathbb{R}$, $\overset{*}{\mathbb{C}}$, $\overset{*}{\mathbb{H}}$, $\overset{*}{\mathbb{O}}$, or $\mathbb{P}$. Moreover, $*$ remains an algebra involution on B, and the correspondence $(A,*) \to (B,*)$ is categorical and bijective. After the laborious classification of algebra involutions on $\overset{*}{\mathbb{C}}$, $\overset{*}{\mathbb{H}}$, $\overset{*}{\mathbb{O}}$, and $\mathbb{P}$ done in [200], the determination of the algebras in Proposition 2.8.13 is germinally concluded. In this way, three new distinguished examples of absolute-valued finite-dimensional real algebras appear. These are the natural Albert isotopes of $\mathbb{H}$, $\mathbb{O}$, and $\mathbb{P}$ (denoted respectively by $\hat{\mathbb{H}}$, $\hat{\mathbb{O}}$, and $\tilde{\mathbb{P}}$) built as follows. For every absolute-valued algebra $\mathbb{A}$, and every surjective linear isometry $\psi : \mathbb{A} \to \mathbb{A}$, the *$\psi$-twist* of $\mathbb{A}$ is defined as the absolute-valued algebra consisting of the normed space of $\mathbb{A}$ and the product $x \odot y := \psi(x)\psi(y)$. For $\mathbb{A}$ equal to either $\mathbb{H}$ or $\mathbb{O}$, we define $\hat{A}$ as the ϕ-twist of $\mathbb{A}$, where ϕ stands for the essentially unique involutive automorphism of $\mathbb{A}$ different from the identity operator [349]. On the other hand, the existence of an essentially unique algebra involution σ on $\mathbb{P}$ is proved in [200], a fact which allows us to define $\tilde{\mathbb{P}}$ as the σ-twist of $\mathbb{P}$. Keeping in mind the arguments and notions above, the following theorem follows easily.

Theorem 2.8.88 *The algebras A satisfying the requirements in Proposition 2.8.13 are precisely* $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\hat{\mathbb{H}}$, $\mathbb{O}$, $\hat{\mathbb{O}}$, *and* $\tilde{\mathbb{P}}$.

A slight variant of the proof sketched above, involving Corollary 7 of [468] instead of Theorem 2.8.82, can be seen in Remark 2.9 of [200].

Nearly absolute-valued algebras have appeared incidentally in previous sections. Now we will look at them more closely.

For every nonzero normed algebra A over $\mathbb{K}$, let us define $\rho(A)$ as the largest non-negative real number ρ satisfying $\rho\|x\|\|y\| \leqslant \|xy\|$ for all $x,y \in A$, so that $\rho(A) > 0$

if and only if A is a nearly absolute-valued algebra, and $\rho(A) = 1$ if and only A is an absolute-valued algebra. *Let A be a nonzero normed finite-dimensional algebra over $\mathbb{K}$. Then,* by the compactness of spheres, *A is nearly absolute-valued if and only if A has no nonzero divisor of zero. Moreover, if this is the case, then A is isomorphic to $\mathbb{C}$ when $\mathbb{K} = \mathbb{C}$* (by Exercise 1.1.86)*, and the dimension of A is equal to* 1, 2, 4, *or* 8 *when $\mathbb{K} = \mathbb{R}$* (by Theorem 2.6.51). On the other hand, by Hopf's theorem (see p. 235 of [727]), *nearly absolute-valued finite-dimensional commutative real algebras have dimension $\leqslant 2$.* We note that, since every finite-dimensional algebra over $\mathbb{K}$ can be endowed with an algebra norm, nearly absolute-valued finite-dimensional real algebras and finite-dimensional real algebras with no nonzero divisor of zero essentially coincide.

According to Corollary 2.5.57, *every nearly absolute-valued alternative real algebra is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$,* so that not much more can be said about such an algebra. The consequence that nearly absolute-valued alternative algebras are isomorphic to absolute-valued algebras is no longer true if alternativity is removed (even for finite-dimensional normed algebras A with $\rho(A)$ arbitrarily close to one). For instance, for $0 < \varepsilon < \frac{1}{2}$, consider the normed real algebra A consisting of the normed space of $\mathbb{H}$ and the product

$$x \odot y := (1 - \varepsilon)xy + \varepsilon yx.$$

Then we have $\rho(A) \geqslant 1 - 2\varepsilon$, but A cannot be isomorphic to an absolute-valued algebra. Indeed, A is not associative and has a unit, whereas every absolute-valued four-dimensional unital real algebra is isomorphic to $\mathbb{H}$ (by Theorem 2.6.21).

The above example shows in addition how a theory of nearly absolute-valued algebras parallel to that of absolute-valued algebras cannot be expected. Along the same lines, in contrast to Theorem 2.6.41 and Corollary 2.6.42, *there exist infinite-dimensional complete normed commutative algebras A over $\mathbb{K}$ with $\rho(A) \geqslant 2^{-\frac{1}{2}}$* (see Example 2.7.64).

Despite the above limitations, in the paper of Kaidi, Ramírez, and Rodríguez [369], the authors wondered whether there could be a theory of nearly absolute-valued algebras 'nearly' parallel to that of absolute-valued algebras. More precisely, they raised the following.

Question 2.8.89 Let $\mathscr{P}$ be any one of the purely algebraic properties leading absolute-valued real algebras to the finite dimension. Is there a universal constant $0 \leqslant K_{\mathscr{P}} < 1$ such that every normed real algebra A satisfying $\mathscr{P}$ and $\rho(A) > K_{\mathscr{P}}$ is finite-dimensional?

Applying ultraproduct techniques, the authors of [369] were able to give an affirmative answer to Question 2.8.89 for the most relevant choices of Property $\mathscr{P}$. More precisely, they proved the following.

Theorem 2.8.90 *Question 2.8.89 has an affirmative answer whenever Property $\mathscr{P}$ is equal to the existence of a unit, the commutativity, or the algebraicity. Moreover, for these choices of $\mathscr{P}$, the universal constant $K_{\mathscr{P}}$ can be (uniquely) chosen in such a way that there exists a normed infinite-dimensional real algebra satisfying $\mathscr{P}$ and $\rho(A) = K_{\mathscr{P}}$.*

Now, the essential messages of Theorems 2.6.21, 2.6.41, and 2.8.66 remain 'nearly' true when nearly absolute-valued algebras replace absolute-valued algebras. As a consequence, as shown in [369, Corollary 4.4], we have the following result which generalizes Corollary 2.8.67.

Corollary 2.8.91 *A normed power-commutative real algebra A is finite-dimensional whenever $\rho(A)$ is close enough to one.*

Many other results of the same flavour can be obtained (see for example the variants of our Corollary 2.6.24, Proposition 2.8.13, and Theorem 2.8.4 proved in Corollaries 3.2 and 3.4, and Theorem 3.3 of [369], respectively). Concerning the constant $K_{\mathscr{P}}$ in Theorem 2.8.90, we already know that $K_{\mathscr{P}} \geqslant 2^{-\frac{1}{2}}$ when $\mathscr{P}$ means commutativity. When $\mathscr{P}$ means algebraicity or existence of a unit, we do not know whether or not the equality $K_{\mathscr{P}} = 0$ holds. Note that $K_{\mathscr{P}} = 0$ would mean that every nearly absolute-valued real algebra satisfying $\mathscr{P}$ is finite-dimensional. Anyway, *nearly absolute-valued complex algebras are isomorphic to $\mathbb{C}$ whenever they have a left unit or are algebraic* (see Corollaries 2.7.17 and 2.6.31, respectively).

In relation to Corollary 2.8.91, it is worth mentioning that four-dimensional power-commutative division real algebras have recently been classified in the paper of Darpö and Rochdi [207]. A classification in dimension eight seems to be unknown for the moment. In the introduction of [207], the reader can also find an accurate review of the status of the question concerning the classification of general non-associative finite-dimensional division real algebras.

We conclude our survey on nearly absolute-valued algebras with the following theorem, also proved in [369].

Theorem 2.8.92 *There exists a universal constant $2^{-1} \leqslant K \leqslant 2^{-\frac{1}{4}}$ uniquely determined by the following two properties:*

(i) *There is a normed real algebra B which is not uniformly non-square, has a left unit, and satisfies $\rho(B) = K$.*
(ii) *If A is any normed real algebra with a left unit and satisfying $\rho(A) > K$, then A is uniformly non-square.*

Among other results, the proof of Theorem 2.8.92 involves Example 2.7.30 and Corollary 2.7.31.

2.9 Complements on numerical ranges

Introduction In this section, we resume the study of the geometry of norm-unital normed algebras around their units, which began in Section 2.1 and continued in Sections 2.2, 2.3, 2.4, and 2.6. The outstanding new starting fact is that the duality mapping of any norm-unital normed algebra is norm-norm upper semicontinuous at the unit, in the sense of Giles, Gregory, and Sims [287]. A relevant characterization of the norm-weak upper semicontinuity of the duality mapping of a Banach space at a point of its unit sphere, obtained in Theorem 2.9.8, shows that some results on algebra numerical ranges (such as Lemma 2.3.44), already proved by other methods, are consequences of the starting fact (see Remark 2.9.9(b)). Now, many other consequences of the same kind are obtained in Corollaries 2.9.18, 2.9.32, 2.9.37, 2.9.41, 2.9.42, 2.9.50, and 2.9.51. Corollaries 2.9.18 and 2.9.32 also involve a theorem of Godefroy and Indumathi [291] (see Theorem 2.9.17), asserting that, if X is a Banach space with a complete predual X_*, and if the duality mapping of X is norm-weak upper semicontinuous at a given point $u \in \mathbb{S}_X$, then the equality $V(X,u,x) = \{x(f) : f \in D(X,u) \cap X_*\}^-$ holds for every $x \in X$. In their turn, Corollaries 2.9.37 to 2.9.51 also involve a nontrivial reformulation of the norm-norm upper semicontinuity of the duality mapping (called the strong subdifferentiability of the norm) due to Gregory [299] (see Theorem 2.9.36). As a by-product of our development, we prove a numerical-range characterization of uniformly smooth Banach spaces (see Theorem 2.9.56), already quoted in Subsection 2.1.3.

2.9.1 Involving the upper semicontinuity of the duality mapping

Let σ be a topology on a set E, let τ be a vector space topology on a vector space Y over $\mathbb{K}$, let f be a function from E to the family 2^Y of all subsets of Y (empty values for f are allowed), and let u be in E. We say that f is *σ-τ upper semicontinuous* (in short, *σ-τ usc*) at u if, for every τ-neighbourhood V of zero in Y, there exists a σ-neighbourhood U of u in E such that $f(x) \subseteq f(u) + V$ whenever x lies in U.

Now, let X be a normed space. The *duality mapping* of X is defined as the function $x \to D(X,x)$ from the unit sphere $\mathbb{S}_X$ of X to $2^{X'}$. We note that the norm-norm upper semicontinuity of the duality mapping of X at a given element $u \in \mathbb{S}_X$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever x is in $\mathbb{S}_X$ with $\|x-u\| \leqslant \delta$, and f is in $D(X,x)$, we have $d(f, D(X,u)) \leqslant \varepsilon$.

The above notions are related to the material previously developed because of the following.

Fact 2.9.1 *Let A be a norm-unital normed algebra over $\mathbb{K}$. Then the duality mapping of A is norm-norm usc at $\mathbf{1}$.*

Proof Let $\varepsilon > 0$, let x be in $\mathbb{S}_A$ with $\|x - \mathbf{1}\| \leqslant \varepsilon$, and let f be in $D(A,x)$. Set $g := (L_x)'(f)$. Then g belongs to $D(A, \mathbf{1})$, and we have

$$\|f - g\| = \|(I_A - L_x)'(f)\| \leqslant \|(I_A - L_x)'\| = \|I_A - L_x\| = \|L_{\mathbf{1}-x}\| = \|\mathbf{1} - x\| \leqslant \varepsilon.$$

Therefore $d(f, D(A,\mathbf{1})) \leqslant \varepsilon$. □

Keeping in mind that, for a norm-one element u in a normed space X, we have

$$0 \leqslant n(X,u) \leqslant 1,$$

and that such an element u is geometrically unitary if and only if $n(X,u) > 0$ (by Theorem 2.1.17(i)), the requirement $n(X,u) = 1$ can be read as that u is a geometrically unitary element of the 'best possible type'. Such special behaviour of u gives rise to the following easy consequence of Fact 2.9.1.

Fact 2.9.2 *Let (X,u) be a numerical-range space over $\mathbb{K}$ with $n(X,u) = 1$. Then the duality mapping of X is norm-norm usc at u.*

Proof Consider $D(X,u)$ as a compact Hausdorff topological space under the w^*-topology. Then, for $x \in X$, the function $F(x) : f \to f(x)$ from $D(X,u)$ to $\mathbb{K}$

is continuous, and, since $n(X,u) = 1$, the mapping $F : x \to F(x)$ from X to the norm-unital normed algebra $C^{\mathbb{K}}(D(X,u))$ becomes a linear isometry taking u to the unit $\mathbf{1}$ of $C^{\mathbb{K}}(D(X,u))$. In this way, we can see X as a subspace of $C^{\mathbb{K}}(D(X,u))$ containing $\mathbf{1}$, and the proof is concluded by applying Fact 2.9.1, keeping in mind that the norm-norm upper semicontinuity of the duality mapping at a point goes down to subspaces. □

Norm-weak* upper semicontinuity of the duality mapping of a normed space at a norm-one element is automatic. Indeed, we have the following.

Fact 2.9.3 *Let X be a normed space over $\mathbb{K}$, and let u be a norm-one element of X. Then, for each weak*-open subset G of X' containing $D(X,u)$, we can find $\delta > 0$ such that $D(X,x) \subseteq G$ whenever $x \in \mathbb{S}_X$ and $\|x-u\| \leqslant \delta$. As a consequence, the duality mapping of X is norm-weak* usc at u.*

Proof Assume that the first conclusion does not hold. Then there would exist a w^*-open subset G of X' containing $D(X,u)$ and such that, for each $n \in \mathbb{N}$, we could find $x_n \in \mathbb{S}_X$ and $f_n \in D(X,x_n)$ in such a way that $\|x_n - u\| \leqslant \frac{1}{n}$ and $f_n \notin G$. By taking a w^*-cluster point $f \in X'$ to the sequence f_n, we would have $f \in D(X,u)$ and $f \notin G$, a contradiction. Now that the first conclusion has been proved, the consequence follows by noticing that, for each weak*-open neighbourhood of zero V in X', $D(X,u)+V$ is weak*-open in X'. □

Although straightforward, the following consequence of Fact 2.9.3 above merits being emphasized.

Corollary 2.9.4 *Let X be a reflexive (respectively, finite-dimensional) Banach space over $\mathbb{K}$. Then the duality mapping of X is norm-weak (respectively, norm-norm) usc at every element of $\mathbb{S}_X$.*

Fact 2.9.3 becomes one of the ingredients in the proof of a characterization of the norm-weak upper semicontinuity of the duality mapping of a Banach space at a norm-one element, in terms of numerical ranges, which will be achieved in Theorem 2.9.8. The next folklore lemma, which follows from the Hahn–Banach separation theorem for convex sets (see for example [689, Theorem 34.7 and Exercise 34.12]), becomes another ingredient in that proof.

Lemma 2.9.5 *Let (X,u) be a numerical-range space over $\mathbb{K}$, and let Y be a subspace of X'. Then the following conditions are equivalent:*

(i) *$V(X,u,x) = \{f(x) : f \in D(X,u) \cap Y\}^-$ for every $x \in X$, where $^-$ means closure in $\mathbb{K}$.*
(ii) *$D(X,u)$ is equal to the weak*-closure in X' of the set $D(X,u) \cap Y$.*

Let X be a Banach space with a (possibly non-complete) predual X_*. For $x \in \mathbb{S}_X$ we set

$$D^{w^*}(X,x) := D(X,x) \cap X_*,$$

and define the pre-duality mapping of X as the function $x \to D^{w^*}(X,x)$ from $\mathbb{S}_X$ to 2^{X_*}.

Proposition 2.9.6 *Let X be a Banach space over $\mathbb{K}$ with a predual X_*, and let u be a norm-one element in X. We have:*

(i) *For every $x \in X$ the equality*

$$V(X,u,x) = \bigcap_{\delta>0} \{x(f) : f \in \mathbb{S}_{X_*}, |u(f)-1| < \delta\}^- \tag{2.9.1}$$

holds.

(ii) *If the equality*

$$V(X,u,x) = \{x(f) : f \in D^{w^*}(X,u)\}^- \tag{2.9.2}$$

holds for every $x \in X$, then the pre-duality mapping of X is norm-weak usc at u.

Proof Let x be in X. For each positive number δ, consider the set

$$B_\delta := \{x(f) : f \in \mathbb{S}_{X_*}, |u(f)-1| < \delta\},$$

and let μ be in $\bigcap_{\delta>0} \overline{B_\delta}$, so that for each $\delta > 0$ there is $f_\delta \in \mathbb{S}_{X_*}$ with

$$|u(f_\delta)-1| < \delta \ \text{ and } \ |x(f_\delta)-\mu| < \delta.$$

Then for every $\lambda \in \mathbb{K}$ we have

$$|\mu-\lambda| = |\mu - x(f_\delta) + x(f_\delta) - \lambda u(f_\delta) + \lambda(u(f_\delta)-1)| \leqslant \delta + \|x-\lambda u\| + |\lambda|\delta,$$

and hence, by letting $\delta \to 0$, we get $|\mu-\lambda| \leqslant \|x-\lambda u\|$. Thus, by Proposition 2.1.1, we obtain that

$$\mu \in \bigcap_{\lambda\in\mathbb{K}} B_{\mathbb{K}}(\lambda, \|x-\lambda u\|) = V(X,u,x),$$

which proves the inclusion $\supseteq$ in the equality (2.9.1). In order to prove the converse inclusion, consider the subspace P of X generated by u and x. Since P is w^*-closed in X, the bipolar theorem provides us with a closed subspace H of X_* such that $P = H^\circ$. Consequently, we have $(X_*/H)' \equiv H^\circ = P$ and, since P is reflexive, we also have $X_*/H \equiv P'$ in the natural way. Now let μ be in $V(X,u,x)$. Since $V(X,u,x) = V(P,u,x)$ (by Corollary 2.1.2), there is $f \in X_*$ such that $\|f+H\| = 1$, $u(f) = 1$, and $x(f) = \mu$. Take a sequence h_n in H with $\|f+h_n\| \to 1$, and set $g_n := \frac{f+h_n}{\|f+h_n\|}$. Then g_n becomes a sequence in $\mathbb{S}_{X_*}$ such that $u(g_n) \to 1$ and $x(g_n) \to \mu$. Therefore, for each $\delta > 0$ there exists $k \in \mathbb{N}$ such that $|u(g_{k+n})-1| < \delta$ for every $n \in \mathbb{N}$, and hence $x(g_{k+n})$ lies in B_δ for every $n \in \mathbb{N}$, and then $\mu \in \overline{B_\delta}$. Since δ is an arbitrary positive number, the inclusion $\subseteq$ in the equality (2.9.1) follows. Thus assertion (i) has been proved.

Assume that the equality (2.9.2) holds for every $x \in X$, and let C stand for the weak*-closure of $D^{w^*}(X,u)$ in X'. Then, by Lemma 2.9.5, we have that $C = D(X,u)$. Now let V be a weak-open neighbourhood of zero in X_*. Then there exists a weak*-open neighbourhood of zero W in X' such that $V = X_* \cap W$. Since $D(X,u) = C \subseteq D^{w^*}(X,u) + W$, and $D(X,u) + W$ is weak*-open in X', it follows from Fact 2.9.3 that there is $\delta > 0$ such that $D(X,x) \subseteq D^{w^*}(X,u) + W$ whenever $x \in \mathbb{S}_X$ and $\|x-u\| \leqslant \delta$, and hence

$$D^{w^*}(X,x) \subseteq (D^{w^*}(X,u)+W) \cap X_* = D^{w^*}(X,u) + V$$

whenever $x \in \mathbb{S}_X$ and $\|x-u\| \leqslant \delta$. Since V is an arbitrary weak-open neighbourhood of zero in X_*, we deduce that the pre-duality mapping of X is norm-weak usc at u. This concludes the proof of assertion (ii). □

The last ingredient in the proof of Theorem 2.9.8 below is the celebrated *Bishop–Phelps–Bollobás theorem* [109] (see also [695, Theorem 16.1]), which reads as follows.

Theorem 2.9.7 *Let X be a Banach space over $\mathbb{K}$, let $0<\varepsilon<1$, and let $x \in \mathbb{B}_X$ and $h \in \mathbb{S}_{X'}$ be such that $|h(x)-1| < \frac{\varepsilon^2}{4}$. Then there are $y \in \mathbb{S}_X$ and $f \in D(X,y)$ such that $\|y-x\| < \varepsilon$ and $\|f-h\| < \varepsilon$.*

Theorem 2.9.8 *For a numerical-range space (X,u) over $\mathbb{K}$, consider the following conditions:*

(i) *For every $x'' \in X''$ the equality $V(X'',u,x'') = \{x''(f) : f \in D(X,u)\}^-$ holds.*
(ii) *The duality mapping of X is norm-weak usc at u.*

Then (i)⇒(ii), and (ii)⇒(i) whenever X is complete.

Proof Assume that condition (i) is fulfilled. Then, by Proposition 2.9.6(ii), the pre-duality mapping of X'' is norm-weak usc at u. Therefore, since the duality mapping of X is the restriction to X of the pre-duality mapping of X'', we deduce that the duality mapping of X is norm-weak usc at u.

Now assume that X is complete and that the duality mapping of X is norm-weak usc at u. Let x'' be in X'', and let $0<\varepsilon<1$. Since the set

$$\{h \in X' : |x''(h)| < \varepsilon\}$$

is a weak-neighbourhood of zero in X', and the duality mapping of X is norm-weak usc at u, there exists $0<\delta<\varepsilon$ such that, for each $x \in \mathbb{S}_X$ with $\|x-u\|<\delta$ and each $f \in D(X,x)$, we can find $g \in D(X,u)$ satisfying $|x''(f-g)|<\varepsilon$. Now set

$$B := \left\{ x''(h) : h \in \mathbb{S}_{X'}, |h(u)-1| < \frac{\delta^2}{4} \right\},$$

and let μ be in B, so that there exists $h_\mu \in \mathbb{S}_{X'}$ with $|h_\mu(u)-1| < \frac{\delta^2}{4}$ and $x''(h_\mu)=\mu$. Since X is complete and $|h_\mu(u)-1| < \frac{\delta^2}{4}$, Theorem 2.9.7 applies to find $x_\mu \in \mathbb{S}_X$ and $f_\mu \in D(X,x_\mu)$ satisfying $\|x_\mu - u\| < \delta$ and $\|f_\mu - h_\mu\| < \delta$. By the above, there is also $g_\mu \in D(X,u)$ such that $|x''(f_\mu - g_\mu)| < \varepsilon$. By setting

$$N(x'') := \sup \Re(\{x''(f) : f \in D(X,u)\}),$$

it follows that

$$\begin{aligned} \Re(\mu) = \Re(x''(h_\mu)) &= \Re(x''(h_\mu - f_\mu)) + \Re(x''(f_\mu - g_\mu)) + \Re(x''(g_\mu)) \\ &\leqslant \|x''\|\delta + \varepsilon + N(x'') \leqslant (\|x''\|+1)\varepsilon + N(x''). \end{aligned}$$

Since μ is arbitrary in B, we deduce that

$$B \subseteq \{\mu \in \mathbb{K} : \Re(\mu) \leqslant (\|x''\|+1)\varepsilon + N(x'')\},$$

and hence, by Proposition 2.9.6(i), we have that

$$V(X'', u, x'') \subseteq \bar{B} \subseteq \{\mu \in \mathbb{K} : \Re(\mu) \leqslant (\|x''\| + 1)\varepsilon + N(x'')\},$$

and therefore $\max \Re(V(X'', u, x'')) \leqslant (\|x''\| + 1)\varepsilon + N(x'')$. By letting $\varepsilon \to 0$, we get

$$\max \Re(V(X'', u, x'')) \leqslant N(x'').$$

Since the converse inequality is clear, it is enough to replace x'' with zx'', for $z \in \mathbb{S}_{\mathbb{K}}$, to realize that condition (i) in the theorem is fulfilled. □

Remark 2.9.9 (a) Let X be a normed space over $\mathbb{K}$, let u be in $\mathbb{S}_X$, and let $\hat{X}$ stand for the completion of X. Since X and $\hat{X}$ have the same dual, and the norm-weak upper semicontinuity of the duality mapping goes down to subspaces, it follows that Theorem 2.9.8 can be reformulated by saying that the duality mapping of $\hat{X}$ is norm-weak usc at u if and only if the equality $V(X'', u, x'') = \{x''(f) : f \in D(X, u)\}^-$ holds for every $x'' \in X''$.

(b) Now let A be a norm-unital normed algebra over $\mathbb{K}$. Since the completion $\hat{A}$ of A is also a norm-unital normed algebra, it follows from Fact 2.9.1 (with $\hat{A}$ instead of A) and part (a) of the current remark that for every $a'' \in A''$ we have $V(A'', \mathbf{1}, a'') = \{a''(f) : f \in D(A, \mathbf{1})\}^-$. In this way we have found a new proof of Lemma 2.3.44.

As we will see in Proposition 2.9.11 below, Fact 2.9.2 does not remain true if the assumption that $n(X, u) = 1$ is relaxed to the one that u is a geometrically unitary element of X. To this end, besides Theorem 2.9.8, we need the following.

Lemma 2.9.10 *Let $\{(X_i, u_i) : i \in I\}$ be a family of numerical-range spaces over $\mathbb{K}$, consider the normed space $\bigoplus_{i\in I}^{\ell_\infty} X_i$ ℓ_∞-sum of the family $\{X_i : i \in I\}$, and let X be a subspace of $\bigoplus_{i\in I}^{\ell_\infty} X_i$ containing the element $u := \{u_i\}$ and the elements $\{x_i\}$ with $x_i = 0$ for all but a finite set of i's. Then we have*

$$n(X, u) = \inf\{n(X_i, u_i) : i \in I\}.$$

Proof Write $L := \inf\{n(X_i, u_i) : i \in I\}$, and fix $i_0 \in I$. Since the mapping $x \to x(i_0)$ from X to X_{i_0} is a linear contraction taking u to u_{i_0}, it follows from Corollary 2.1.2(i) that $V(X_{i_0}, u_{i_0}, x(i_0)) \subseteq V(X, u, x)$ for every $x \in X$, and hence

$$v(X, u, x) \geqslant v(X_{i_0}, u_{i_0}, x(i_0)) \geqslant n(X_{i_0}, u_{i_0})\|x(i_0)\| \geqslant L\|x(i_0)\|.$$

Since i_0 was arbitrary in I, we have $v(X, u, x) \geqslant L\|x\|$ for every $x \in X$, so that $n(X, u) \geqslant L$.

Now let i_0 be in I, fix $x_{i_0} \in X_{i_0}$, and let x be in X defined by $x(i) = 0$ for $i \neq i_0$ and $x(i_0) = x_{i_0}$. Then, by Proposition 2.1.5, we have

$$\begin{aligned}
\max \Re(V(X, u, x)) &= \lim_{r\to 0^+} \frac{\|u + rx\| - 1}{r} \\
&= \lim_{r\to 0^+} \max\left\{0, \frac{\|u_{i_0} + rx_{i_0}\| - 1}{r}\right\} \\
&= \max\{0, \max \Re(V(X_{i_0}, u_{i_0}, x_{i_0}))\}.
\end{aligned}$$

Replace x_{i_0} with zx_{i_0} ($z \in \mathbb{S}_{\mathbb{K}}$) in the equality above, we get

$$V(X,u,x) = \mathrm{co}(V(X_{i_0},u_{i_0},x_{i_0}) \cup \{0\})$$

and, consequently,

$$v(X_{i_0},u_{i_0},x_{i_0}) = v(X,u,x) \geqslant n(X,u)\|x\| = n(X,u)\|x_{i_0}\|.$$

Since x_{i_0} was an arbitrary element of X_{i_0}, we have $n(X,u) \leqslant n(X_{i_0},u_{i_0})$, and this is true for every i_0, so that $n(X,u) \leqslant L$. □

Proposition 2.9.11 *Let ρ be any real number with $0 < \rho < 1$. Then there exists a complete numerical-range space (X,u) over $\mathbb{K}$ satisfying $n(X,u) = \rho$ and such that the duality mapping of X is not norm-norm usc at u, nor even norm-weak usc at u.*

Proof Let $0 < \mu < 1$, let Y stand for $\mathbb{K}^2$ endowed with the norm

$$|(\xi,\eta)| := \max\{|\xi| + \rho|\eta|, \mu|\xi| + |\eta|\},$$

and set $w := (1,0) \in Y$. By Proposition 2.1.5, for $(\xi,\eta) \in Y$ we have

$$\max \Re(V(Y,w,(\xi,\eta))) = \Re(\xi) + \rho|\eta|.$$

This implies $v(Y,w,(\xi,\eta)) = |\xi| + \rho|\eta|$ for every $(\xi,\eta) \in Y$. Consequences of the above equality are that $n(Y,w) = \rho$ and that $D(Y,w)$ consists of those linear functionals on Y of the form $(\xi,\eta) \to \xi + b\eta$ for some $b \in \mathbb{K}$ with $|b| \leqslant \rho$.

Now let μ_n be a sequence of real numbers such that $0 < \mu_n < 1$ for every $n \in \mathbb{N}$ and $\mu_n \to 1$. For each $n \in \mathbb{N}$, let X_n stand for $\mathbb{K}^2$ endowed with the norm

$$|(\xi,\eta)|_n := \max\{|\xi| + \rho|\eta|, \mu_n|\xi| + |\eta|\}.$$

Let X denote the Banach space of all convergent sequences $x = ((\xi_n,\eta_n))_{n\in\mathbb{N}}$ in $\mathbb{K}^2$ endowed with the norm

$$\|x\| := \sup\{|(\xi_n,\eta_n)|_n : n \in \mathbb{N}\},$$

and set $u := (u_n)_{n\in\mathbb{N}} \in X$, where $u_n := (1,0)$ for every $n \in \mathbb{N}$. By the above paragraph we have that $n(X_n,u_n) = \rho$ for every $n \in \mathbb{N}$, and hence, by Lemma 2.9.10, that $n(X,u) = \rho$.

A straightforward computation shows that X' is the space of those sequences $f = ((a_n,b_n))_{n\in\mathbb{N}\cup\{0\}}$ in $\mathbb{K}^2$ such that $\sum_{n=1}^{\infty} |(a_n,b_n)|'_n < +\infty$ (where $|(\cdot,\cdot)|'_n$ stands for the dual norm of $|(\cdot,\cdot)|_n$) with the norm

$$\|f\| = \max\{|a_0|,|b_0|\} + \sum_{n=1}^{\infty} |(a_n,b_n)|'_n,$$

the canonical duality being given by

$$f(x) = a_0\xi + b_0\eta + \sum_{n=1}^{\infty} (a_n\xi_n + b_n\eta_n)$$

for $x = ((\xi_n,\eta_n))_{n\in\mathbb{N}} \in X$ and $(\xi,\eta) := \lim((\xi_n,\eta_n))_{n\in\mathbb{N}}$. Then, by the standard procedure, X'' is the space of all bounded sequences $F = ((c_n,d_n))_{n\in\mathbb{N}\cup\{0\}}$ in $\mathbb{K}^2$ with the norm

$$\|F\| = \max\{|c_0| + |d_0|, \sup\{|(c_n,d_n)|_n : n \in \mathbb{N}\}\}.$$

Let F_0 in X'' be given by

$$F_0 := ((0,0),(0,1),(0,1),\ldots,(0,1),\ldots).$$

Then one easily obtains that $V(X'',u,F_0) = \mathbb{B}_{\mathbb{K}}$. However, since

$$D(X,u) = \left\{ ((a_n,b_n))_{n\in\mathbb{N}\cup\{0\}} \in X' : |b_0| \leqslant a_0, |b_n| \leqslant \rho a_n \, (n \in \mathbb{N}), \sum_{n=0}^{\infty} a_n = 1 \right\},$$

we get that

$$\{F_0(f) : f \in D(X,u)\}^- = \rho\mathbb{B}_{\mathbb{K}} \subsetneqq \mathbb{B}_{\mathbb{K}},$$

where $^-$ means closure in $\mathbb{K}$. Therefore, by Theorem 2.9.8, the duality mapping of X is not norm-weak usc at u. □

Now we are going to show that the upper semicontinuity of the duality mapping of a Banach space X at a norm-one element u is more a property of $D(X,u)$ than of u. To this end we prove first the following.

Lemma 2.9.12 *For a numerical-range space (X,u) over $\mathbb{K}$ and a vector space topology τ on X' weaker than or equal to the norm topology, consider the following conditions:*

(i) *For each τ-neighbourhood of zero V in X' there exists $\delta > 0$ such that h belongs to $D(X,u)+V$ whenever h is in $\mathbb{B}_{X'}$ with $|h(u)-1| < \delta$.*
(ii) *The duality mapping of X is norm-τ usc at u.*

Then (i)⇒(ii), and (ii)⇒(i) whenever X is complete.

Proof Assume that condition (i) holds, let V be any τ-neighbourhood of zero in X', and let δ stand for the positive number associated to V by that condition. Then, for each $x \in \mathbb{S}_X$ with $\|x-u\| < \delta$ and each $f \in D(X,x)$, we have $|f(u)-1| = |f(u-x)| \leqslant \|x-u\| < \delta$, and hence $f \in D(X,u)+V$. Thus the duality mapping of X is norm-τ usc at u.

Now assume that the duality mapping of X is norm-τ usc at u and that X is complete, and let V be any τ-neighbourhood of zero in X'. Take a τ-neighbourhood of zero W in X' with $W+W \subseteq V$. Then there is $0 < \eta < 1$ such that $\eta\mathbb{B}_{X'} \subseteq W$ and $D(X,x) \subseteq D(X,u)+W$ whenever x is in $\mathbb{S}_X$ with $\|x-u\| < \eta$. Set $\delta := \frac{\eta^2}{4}$, and let h be in $\mathbb{B}_{X'}$ with $|h(u)-1| < \delta$. Then, by Theorem 2.9.7, there are $x \in \mathbb{S}_X$ and $f \in D(X,x)$ satisfying $\|x-u\| < \eta$ and $\|f-h\| < \eta$. Therefore $h \in f+\eta\mathbb{B}_{X'} \subseteq D(X,u)+W+W \subseteq D(X,u)+V$. Thus assertion (i) is fulfilled. □

Proposition 2.9.13 *Let X be a Banach space over $\mathbb{K}$, let u,v be in $\mathbb{S}_X$ such that $D(X,u) = D(X,v)$, and let τ be a vector space topology on X' weaker than or equal to the norm topology. If the duality mapping of X is norm-τ usc at u, then the duality mapping of X is norm-τ usc at v.*

Proof Assume that the duality mapping of X is norm-τ upper semicontinuous at u but not at v. Then, by Lemma 2.9.12, there exists a τ-neighbourhood V of zero in X'

such that for every $n \in \mathbb{N}$ we can find $f_n \in \mathbb{B}_{X'}$ satisfying

$$|f_n(v) - 1| < \frac{n}{n+1} \tag{2.9.3}$$

and

$$f_n \notin D(X,v) + V, \tag{2.9.4}$$

and there exists $\delta > 0$ such that

$$h \in D(X,u) + V \quad \text{whenever} \quad h \in \mathbb{B}_{X'} \quad \text{and} \quad |h(u) - 1| < \delta. \tag{2.9.5}$$

Take a cluster point f to the sequence f_n in the weak*-topology of X'. Then, by (2.9.3), we have $f \in D(X,v) = D(X,u)$, so $1 = f(u)$ is a cluster point to the sequence $f_n(u)$, and so there is $m \in \mathbb{N}$ with $|f_m(u) - 1| < \delta$. By (2.9.5), f_m belongs to $D(X,u) + V$. But, applying that $D(X,u) = D(X,v)$ again, this contradicts (2.9.4). $\square$

Remark 2.9.14 (a) In Lemma 2.9.12, take τ equal to the norm topology. Then assertion (i) in that lemma is nothing other than '$D(X,u)$ is strongly exposed by u' in the sense of §2.9.35 below. Therefore, as we will prove in Theorem 2.9.36, the implication (ii)⇒(i) remains true in this case if the completeness of X is removed.

(b) Now, take τ equal to the norm topology in Proposition 2.9.13, and look at the proof. It follows from part (a) of the current remark that the proposition remains true in this case if the completeness of X is dispensed.

2.9.2 The upper semicontinuity of the pre-duality mapping

In the previous subsection, we introduced the pre-duality mapping of a dual Banach space as a tool for the proof of Theorem 2.9.8. Now we are going to see how a good behaviour of the pre-duality mapping becomes important in order to compute numerical ranges of elements in dual numerical-range spaces.

Given a numerical-range space (X,u), we denote by $\mathrm{Dis}(X,u)$ the set of all dissipative elements of X (cf. Definition 2.1.8) and for $x \in X$ we set

$$\tau(u,x) := \max \Re(V(X,u,x)).$$

Lemma 2.9.15 *Let (X,u) be a numerical-range space over $\mathbb{K}$ and let f be a linear functional on X. Then the following conditions are equivalent:*

(i) $\Re(f(x)) \leqslant 0$ *for every* $x \in \mathrm{Dis}(X,u)$.
(ii) *There exists* $\lambda \geqslant 0$ *such that* $f \in \lambda D(X,u)$.

Proof The implication (ii)⇒(i) is clear. Assume that condition (i) holds. Then for $x \in X$ we have $x - \tau(u,x)u \in \mathrm{Dis}(X,u)$ so that, since $-u \in \mathrm{Dis}(X,u)$,

$$\Re(f(x)) \leqslant \tau(u,x)\Re(f(u)) \leqslant \|x\|\Re(f(u)).$$

This shows that $\Re \circ f$ is a continuous real-linear functional on X with $\|\Re \circ f\| = \Re(f(u))$, so $f \in X'$ and $\|f\| = \Re(f(u))$. Therefore $f(u) = \|f\|$ and condition (ii) follows by taking $\lambda := f(u)$. $\square$

Lemma 2.9.16 *Let (X,u) be a numerical-range space over $\mathbb{K}$, let τ be a locally convex topology on X, and let X'_τ stand for the space of all τ-continuous linear functionals on X. Then* $\mathrm{Dis}(X,u)$ *is τ-closed in X if and only if the equality $V(X,u,x) = \{f(x) : f \in D(X,u) \cap X'_\tau\}^-$ holds for every $x \in X$.*

Proof The 'if' part is clear. Assume that $\mathrm{Dis}(X,u)$ is τ-closed in X. Let x be in X, and let η be a real number with $\eta < \tau(u,x)$. Then $x - \eta u$ is not dissipative, so by the Hahn–Banach separation theorem for convex sets there is $f \in X'_\tau \setminus \{0\}$ such that $\Re(f(y)) \leqslant \Re(f(x - \eta u))$ for every $y \in \mathrm{Dis}(X,u)$. Since $\mathrm{Dis}(X,u)$ is a cone, we have $\Re(f(y)) \leqslant 0$ for every $y \in \mathrm{Dis}(X,u)$, so that, by Lemma 2.9.15, we can suppose that $f \in D(X,u)$, and then $0 \leqslant \Re(f(x - \eta u)) = \Re(f(x)) - \eta$. Therefore we have that $\eta \leqslant \Re(f(x)) \leqslant \max \Re(H(x))$, where

$$H(x) := \{f(x) : f \in D(X,u) \cap X'_\tau\}^-.$$

By letting $\eta \to \tau(u,x)$, we obtain $\tau(u,x) \leqslant \max \Re(H(x))$. Since the converse inequality is clear, it is enough to replace x with zx, for $z \in \mathbb{S}_{\mathbb{K}}$, to realize that

$$V(X,u,x) = \{f(x) : f \in D(X,u) \cap X'_\tau\}^-.$$

Thus the proof of the 'only if' part is concluded. □

Now we formulate and prove the main result in this subsection.

Theorem 2.9.17 *Let X be a Banach space over $\mathbb{K}$ with a complete predual X_*, and let u be a norm-one element in X. We have:*

(i) *The pre-duality mapping of X is norm-weak usc at u if and only if the equality $V(X,u,x) = \{x(f) : f \in D^{w^*}(X,u)\}^-$ holds for every $x \in X$.*
(ii) *If the duality mapping of X is norm-weak usc at u, then so is the pre-duality mapping of X.*

Proof The 'if' part of assertion (i) follows from Proposition 2.9.6(ii). To prove the 'only if' part, we mimic the proof of the implication (ii)⇒(i) in Theorem 2.9.8, introducing the appropriate changes. Assume that the pre-duality mapping of X is norm-weak usc at u. Let x be in X, and let $0 < \varepsilon < 1$. Since the set $\{h \in X_* : |x(h)| < \varepsilon\}$ is a weak-neighbourhood of zero in X_*, there exists $0 < \delta < \varepsilon$ such that, for each $x \in \mathbb{S}_X$ with $\|x - u\| < \delta$ and each $f \in D^{w^*}(X,x)$, we can find $g \in D^{w^*}(X,u)$ satisfying $|x(f - g)| < \varepsilon$. Now set

$$B := \left\{ x(h) : h \in \mathbb{S}_{X_*}, |u(h) - 1| < \frac{\delta^2}{4} \right\},$$

and let μ be in B, so that there exists $h_\mu \in \mathbb{S}_{X_*}$ with $|u(h_\mu) - 1| < \frac{\delta^2}{4}$ and $x(h_\mu) = \mu$. Since $|u(h_\mu) - 1| < \frac{\delta^2}{4}$, Theorem 2.9.7 applies to find $f_\mu \in \mathbb{S}_{X_*}$ and $x_\mu \in D(X_*, f_\mu)$ satisfying $\|f_\mu - h_\mu\| < \delta$ and $\|x_\mu - u\| < \delta$. Noticing that the conditions $f_\mu \in \mathbb{S}_{X_*}$ and $x_\mu \in D(X_*, f_\mu)$ are equivalent to those $x_\mu \in \mathbb{S}_X$ and $f_\mu \in D^{w^*}(X, x_\mu)$, we derive from the above that there is also $g_\mu \in D^{w^*}(X,u)$ such that $|x(f_\mu - g_\mu)| < \varepsilon$. By setting

$$N(x) := \sup \Re(\{x(f) : f \in D^{w^*}(X,u)\}),$$

it follows that

$$\begin{aligned}\Re(\mu) = \Re(x(h_\mu)) &= \Re(x(h_\mu - f_\mu)) + \Re(x(f_\mu - g_\mu)) + \Re(x(g_\mu)) \\ &\leqslant \|x\|\delta + \varepsilon + N(x) \leqslant (\|x\|+1)\varepsilon + N(x).\end{aligned}$$

Since μ is arbitrary in B, we deduce that

$$B \subseteq \{\mu \in \mathbb{K} : \Re(\mu) \leqslant (\|x\|+1)\varepsilon + N(x)\},$$

and hence, by Proposition 2.9.6(i), we have that

$$V(X,u,x) \subseteq \bar{B} \subseteq \{\mu \in \mathbb{K} : \Re(\mu) \leqslant (\|x\|+1)\varepsilon + N(x)\},$$

and therefore $\tau(u,x) \leqslant (\|x\|+1)\varepsilon + N(x)$. By letting $\varepsilon \to 0$, we obtain that $\tau(u,x) \leqslant N(x)$. Since the converse inequality is clear, it is enough to replace x with zx, for $z \in \mathbb{S}_{\mathbb{K}}$, to realize that

$$V(X,u,x) = \{x(f) : f \in D^{w^*}(X,u)\}^-.$$

Thus the proof of assertion (i) is complete.

Now assume that the duality mapping of X is norm-weak usc at u. Then, by Theorem 2.9.8, we have that $\mathrm{Dis}(X'',u)$ is w^*-closed in X'' and that

$$\mathrm{Dis}(X,u) = \mathrm{Dis}(X'',u) \cap X.$$

Let x_λ be a net in $\mathrm{Dis}(X,u) \cap \mathbb{B}_X$ converging to some $x \in \mathbb{B}_X$ in the w^*-topology of X. Taking a cluster point $x'' \in \mathbb{B}_{X''}$ to the net x_λ in the w^*-topology of X'', it follows that $x'' \in \mathrm{Dis}(X'',u)$. On the other hand, denoting by $P : X'' \to X$ the Dixmier projection (i.e. the transpose of the inclusion $X_* \hookrightarrow X'$), and keeping in mind that P is continuous for the corresponding w^*-topologies, we get $x = P(x'')$. Therefore, since P is a linear contraction with $P(u) = u$, it follows from Corollary 2.1.2(i) that $x \in \mathrm{Dis}(X,u)$. Thus $\mathrm{Dis}(X,u) \cap \mathbb{B}_X$ is w^*-closed in X. By the Krein–Smulian theorem (see for example [726, Theorem V.5.7]), $\mathrm{Dis}(X,u)$ is w^*-closed in X. Finally, by Lemma 2.9.16 (with $\tau = w^*$) and assertion (i) already proved, we get that the pre-duality mapping of X is norm-weak usc at u. This concludes the proof of assertion (ii). □

It is worth remarking that the assumption in assertion (ii) of Theorem 2.9.17 – that the duality mapping of X is norm-weak usc at u – does not involve the predual. Therefore, *if X is a dual Banach space, if u is a norm-one element in X, and if the duality mapping of X is norm-weak usc at u, then the equality $V(X,u,x) = \{x(f) : f \in D(X,u) \cap X_*\}^-$ holds for every complete predual X_* of X and every $x \in X$.*

Concerning norm-unital normed algebras, the main consequence of Theorem 2.9.17 is the next corollary, which follows from that theorem by invoking Fact 2.9.1.

Corollary 2.9.18 *Let A be a norm-unital complete normed algebra over $\mathbb{K}$ with a complete predual. Then, for every $a \in A$, we have*

$$V(A,\mathbf{1},a) = \{a(f) : f \in D^{w^*}(A,\mathbf{1})\}^-.$$

Moreover, when in addition A is associative, the same conclusion holds if we replace $\mathbf{1}$ with any geometrically unitary element of A (cf. §2.1.20).

Keeping in mind that the bidual of a norm-unital normed algebra is a norm-unital normed algebra under the Arens product (see §2.2.11 and Corollary 2.2.13), we realize that the above corollary contains Lemma 2.3.44 (cf. also Remark 2.9.9(b)).

By combining Fact 2.9.2 and Theorem 2.9.17, we find the following.

Corollary 2.9.19 *Let X be a Banach space over $\mathbb{K}$ with a complete predual X_*, and let u be in $\mathbb{S}_X$ such that $n(X,u)=1$. Then the equality*

$$V(X,u,x) = \{x(f) : f \in D^{w^*}(X,u)\}^-$$

holds for every $x \in X$.

Invoking James' theorem (see for example [718, Theorem I.3] or [778, Theorem 2.9.4]), another relevant consequence of Theorem 2.9.17 is the following.

Corollary 2.9.20 *A Banach space X over $\mathbb{K}$ is reflexive if and only if the duality mapping of X' is norm-weak usc at every element of $\mathbb{S}_{X'}$.*

Proof The 'only if' part follows from Corollary 2.9.4. Assume that the duality mapping of X' is norm-weak usc at every element $h \in \mathbb{S}_{X'}$. Then, by Theorem 2.9.17, we have that $\{1\} = V(X',h,h) = \{h(y) : y \in D(X',h) \cap X\}^-$. This implies that $D(X',h) \cap X \neq \emptyset$, which means that h attains its norm at some point of $\mathbb{S}_X$. Now, since h is arbitrary in $\mathbb{S}_{X'}$, the reflexivity of X follows from James' theorem. □

By combining Theorems 2.9.8 and 2.9.17(i), we get the following.

Corollary 2.9.21 *Let X be a Banach space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. If the duality mapping of X is norm-weak usc at u, then so is the pre-duality mapping of X''.*

Now we are going to see how an appropriate behaviour of the preduality mapping of a dual Banach space manages to convert geometrically unitaries into so-called 'w^*-unitaries'.

By a *CS-closed set* in a Banach space X we mean a subset S of X such that, whenever $\sum_{n=1}^{\infty} \alpha_n s_n = x \in X$, with $s_n \in S$, $\alpha_n \geqslant 0$, and $\sum_{n=1}^{\infty} \alpha_n = 1$, we have $x \in S$. The celebrated Banach principle, that quasi-open continuous linear mappings between Banach spaces are open, is codified in Jameson's book [756] as follows.

Lemma 2.9.22 [756, Theorem 22.4] *In a Banach space over $\mathbb{K}$, a CS-closed set and its closure have the same interior.*

Corollary 2.9.23 *Let X be a Banach space over $\mathbb{K}$, and let S be a CS-closed set in X. Then $|\mathrm{co}|(S)$ and $\overline{|\mathrm{co}|}(S)$ have the same interior in X.*

Proof If $\mathbb{K}=\mathbb{R}$, then $|\mathrm{co}|(S) = \mathrm{co}(S \cup -S)$ is a *CS*-closed set (by [756, 22.2 and 22.3]), and the result follows from Lemma 2.9.22. Assume that $\mathbb{K}=\mathbb{C}$. Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $\mathbb{B}_{\mathbb{C}} \subseteq (1+\varepsilon)\mathrm{co}(\{z_1,\ldots,z_n\})$, where $z_1,\ldots,z_n$ are the nth roots of 1 in $\mathbb{C}$. Then we have

$$|\mathrm{co}|(S) = \mathrm{co}(\mathbb{B}_{\mathbb{C}} S) \subseteq (1+\varepsilon)\mathrm{co}(\cup_{i=1}^{n} z_i S) \subseteq (1+\varepsilon)|\mathrm{co}|(S).$$

By keeping in mind that $\mathrm{co}(\cup_{i=1}^{n} z_i S)$ is a *CS*-closed set, and applying Lemma 2.9.22, we deduce that $T \subseteq (1+\varepsilon)|\mathrm{co}|(S)$, where T stands for the interior of $\overline{|\mathrm{co}|}(S)$. Therefore, since T is open, we have $T \subseteq \cup_{\varepsilon>0} \frac{1}{1+\varepsilon} T \subseteq |\mathrm{co}|(S)$. □

§2.9.24 Let X be a Banach space with a (possibly incomplete) predual X_*, and let u be in $\mathbb{S}_X$. If $D^{w^*}(X,u) = \emptyset$, then we set $n^{w^*}(X,u) := 0$. Otherwise, we define $n^{w^*}(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leqslant v^{w^*}(X,x,u) := \sup\{|f(x)| : f \in D^{w^*}(X,u)\}$$

for every $x \in X$. Thus, the set $D^{w^*}(X,u)$ and the number $n^{w^*}(X,u)$ coincide with the set $D^{X_*}(X,u)$ and the number $n^{X_*}(X,u)$, respectively, already introduced in §2.1.14. We say that u is a *w^*-unitary element* of X if the linear hull of $D^{w^*}(X,u)$ equals the whole space X_*, and that u is a *w^*-vertex of* $\mathbb{B}_X$ if $D^{w^*}(X,u)$ separates the points of X. Noticing that X_* is a norming subspace of X', Corollary 2.1.22 applies to get the following.

Corollary 2.9.25 *Let X be a Banach space over $\mathbb{K}$ with a complete predual, and let u be a w^*-unitary element of X. Then u is geometrically unitary.*

Remark 2.9.26 Let X be a normed space, and let u be a norm-one element in X. Noticing that $D(X,u) = D^{w^*}(X'',u)$, it is clear that u is a geometrically unitary element of X if and only if u is a w^*-unitary element of X''. Moreover we have

$$n(X,u) = n^{w^*}(X'',u).$$

Indeed, the inequality $n(X,u) \geqslant n^{w^*}(X'',u)$ is straightforward. To prove the converse inequality, we may assume that $n(X,u) > 0$. Let x'' be in X''. Then we have

$$\begin{aligned} v^{w^*}(X'',u,x'') &= \sup\{|f(x'')| : f \in D(X,u)\} = \sup\{|f(x'')| : f \in \overline{|\mathrm{co}|}(D(X,u))\} \\ &\geqslant n(X,u)\sup\{|f(x'')| : f \in \mathbb{B}_{X'}\} = n(X,u)\|x''\|, \end{aligned}$$

the last inequality being true because of Theorem 2.1.17(ii). Since x'' is arbitrary in X'', we derive $n^{w^*}(X'',u) \geqslant n(X,u)$, as desired.

Proposition 2.9.27 *Let X be a Banach space over $\mathbb{K}$ with a complete predual X_*, and let u be in $\mathbb{S}_X$. We have:*

(i) *u is a w^*-unitary element of X if and only if $n^{w^*}(X,u) > 0$.*
(ii) *If $n^{w^*}(X,u) > 0$, then*

$$\mathrm{int}(\mathbb{B}_{X_*}) \subseteq \frac{1}{n^{w^*}(X,u)}|\mathrm{co}|(D^{w^*}(X,u)). \tag{2.9.6}$$

Proof Assume that u is w^*-unitary. Then, by Lemma 2.1.15, we have

$$n^{w^*}(X,u) > 0.$$

Now assume that $n^{w^*}(X,u) > 0$. Then, in the duality (X,X_*), the set

$$B := \{x \in X : v^{w^*}(X,x,u) \leqslant 1\}$$

is the absolute polar of $D^{w^*}(X,u)$, and the inclusion $B \subseteq \frac{1}{n^{w^*}(X,u)}\mathbb{B}_X$ holds. It follows from the bipolar theorem that $n^{w^*}(X,u)\mathbb{B}_{X_*} \subseteq \overline{|\mathrm{co}|}(D^{w^*}(X,u))$. Therefore, by applying Corollary 2.9.23, the inclusion (2.9.6) follows. Clearly, that inclusion implies that u is w^*-unitary. □

Let X be a Banach space with a complete predual X_*, and let u be a norm-one element of X. It is obvious that w^*-unitaries are w^*-vertices and that w^*-vertices are vertices, whereas it is not so obvious but true that w^*-unitaries are geometric unitaries (by Corollary 2.9.25). On the other hand, if u is a vertex of $\mathbb{B}_X$, and if the pre-duality mapping of X is norm-weak usc at u, then, by Theorem 2.9.17(i), u is a w^*-vertex of $\mathbb{B}_X$. Now, we can complete the picture by proving the following.

Theorem 2.9.28 *Let X be a Banach space over $\mathbb{K}$ with a complete predual X_*, let u be a geometrically unitary element of X, and assume that the pre-duality mapping of X is norm-weak usc at u. Then u is w^*-unitary. More precisely, we have*

$$n(X,u) = n^{w^*}(X,u) > 0 \quad \text{and} \quad n(X,u)\mathrm{int}(\mathbb{B}_{X_*}) \subseteq |\mathrm{co}|(D^{w^*}(X,u)).$$

Proof By Theorem 2.9.17(i), we have $v(X,x,u) = v^{w^*}(X,x,u)$ for every $x \in X$, and, consequently, the equality $n(X,u) = n^{w^*}(X,u)$ holds. Now apply Theorem 2.1.17(i) and Proposition 2.9.27. □

By invoking Fact 2.9.2 and Theorem 2.1.17(i), Theorem 2.9.28 immediately above yields the following.

Corollary 2.9.29 *Let X be a Banach space over $\mathbb{K}$ with a complete predual X_*, and let u be in $\mathbb{S}_X$ such that $n(X,u) = 1$. Then we have*

$$\mathrm{int}(\mathbb{B}_{X_*}) \subseteq |\mathrm{co}|(D^{w^*}(X,u)),$$

and hence u is w^*-unitary. As a consequence,

$$\mathbb{B}_{X_*} = \overline{|\mathrm{co}|}(D^{w^*}(X,u)).$$

§**2.9.30** We note that, when $\mathbb{K} = \mathbb{C}$, the equality $\mathbb{B}_{X_*} = |\mathrm{co}|(D^{w^*}(X,u))$ cannot be expected in Corollary 2.9.29. Indeed, it is enough to look at Example 2.1.18 to realize that the above equality fails by taking X equal to the complex dual Banach space ℓ_∞ and u equal to the sequence constantly equal to 1, the equality $n(X,u) = 1$ being clear because the coordinate projections on X are elements of $D(X,u)$. Note also that, in this case, X is a unital commutative C^*-algebra and u is the unit of X.

A similar counterexample for $\mathbb{K} = \mathbb{R}$ does also exist, but is more involved. Indeed, we have the following.

Example 2.9.31 *First step: construction of the dual numerical-range space (X,u).* Let Y stand for the closed subspace of the real Banach space c_0 given by

$$Y := \left\{ y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} : \lambda_1 + \lambda_2 = \sum_{n=1}^{\infty} \frac{\lambda_{n+2}}{2^n} \right\},$$

and consider the convex subset K of Y defined by

$$K := \{y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} \in Y : \lambda_0 = 1 \text{ and } 0 \leqslant \lambda_n \leqslant 1 \text{ for every } n \in \mathbb{N}\}.$$

Given $y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} \in \mathbb{B}_{(Y,\|\cdot\|_\infty)}$ with $\lambda_0 = \lambda_1 = \lambda_2 = 0$, consider the sequences

$$y^+ := \left\{ 1, \frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_{n+2}^+}{2^n}, \frac{1}{2}\sum_{n=1}^{\infty} \frac{\lambda_{n+2}^+}{2^n}, \lambda_3^+, \lambda_4^+, \ldots \right\}$$

and

$$y^- := \left\{1, \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_{n+2}^-}{2^n}, \frac{1}{2}\sum_{n=1}^{\infty}\frac{\lambda_{n+2}^-}{2^n}, \lambda_3^-, \lambda_4^-, \dots\right\},$$

where $\lambda_n^+ = \max\{\lambda_n, 0\}$ and $\lambda_n^- = \max\{-\lambda_n, 0\}$, and note that $y = y^+ - y^-$ and $y^+, y^- \in K$, to conclude that $y \in 2|\mathrm{co}|(K)$. Now, for an arbitrary element $y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} \in \mathbb{B}_{(Y,\|\cdot\|_\infty)}$, write

$$y = [\lambda_0 - 2(\lambda_1+\lambda_2)]y_0 + (\lambda_1-\lambda_2)y_1 - (\lambda_1-\lambda_2)y_2 + 2(\lambda_1+\lambda_2)y_3 + y_4,$$

where

$$y_0 := \{1, 0, 0, \dots\}, \qquad y_1 := \left\{1, \frac{1}{2}, 0, \frac{2}{3}, \frac{2}{3}, 0, 0, \dots\right\},$$

$$y_2 := \left\{1, 0, \frac{1}{2}, \frac{2}{3}, \frac{2}{3}, 0, 0, \dots\right\}, \qquad y_3 := \left\{1, \frac{1}{4}, \frac{1}{4}, 1, 0, 0, \dots\right\},$$

and

$$y_4 := \{0, 0, 0, \lambda_3 - 2(\lambda_1+\lambda_2), \lambda_4, \lambda_5, \dots\},$$

and note that y_0, y_1, y_2, y_3 belong to K and that y_4 is as studied above, to get that $y \in 15|\mathrm{co}|(K)$. Therefore $\mathbb{B}_{(Y,\|\cdot\|_\infty)} \subseteq 15|\mathrm{co}|(K)$, and hence $|\mathrm{co}|(K)$ is an absorbing subset of Y. On the other hand, since $K \subseteq \mathbb{B}_{(Y,\|\cdot\|_\infty)}$, we have $|\mathrm{co}|(K) \subseteq \mathbb{B}_{(Y,\|\cdot\|_\infty)}$, and so $|\mathrm{co}|(K)$ is a radially bounded subset of Y. It follows that the Minkowski functional of $|\mathrm{co}|(K)$

$$\|y\| := \inf\{\alpha \geqslant 0 : y \in \alpha|\mathrm{co}|(K)\}$$

is a norm on Y equivalent to $\|\cdot\|_\infty$ satisfying $\|\cdot\|_\infty \leqslant \|\cdot\|$, and that

$$\mathrm{int}(\mathbb{B}_{(Y,\|\cdot\|)}) \subseteq |\mathrm{co}|(K) \subseteq \mathbb{B}_{(Y,\|\cdot\|)}. \tag{2.9.7}$$

Set $X_* := (Y, \|\cdot\|)$, let X stand for the dual of X_*, and let u denote the element of X defined by $u(y) = \lambda_0$ for every $y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}} \in X_*$. Keeping in mind that

$$|u(y)| \leqslant \|y\|_\infty \leqslant \|y\| \text{ for every } y \in X_*,$$

that $\{1,0,0,\dots\} \in \mathbb{B}_{X_*}$, and that $u(\{1,0,0,\dots\}) = 1$, we deduce that $\|u\| = 1$.

Second step: proof that $n(X,u) = 1$. Noticing that $u(y) = 1$ for every $y \in K$, and that $K \subseteq \mathbb{B}_{X_*}$, we derive that

$$K \subseteq D^{w^*}(X,u). \tag{2.9.8}$$

Then, since the equality

$$\|x\| = \sup\{|x(y)| : y \in |\mathrm{co}|(K)\} = \sup\{|x(y)| : y \in K\}$$

holds for every $x \in X$ (by (2.9.7)), it follows that $n(X,u) = 1$.

Last step: proof that $\mathbb{B}_{X_*} \neq |\mathrm{co}|(D^{w^*}(X,u))$. Let $y = \{\lambda_n\}_{n\in\mathbb{N}\cup\{0\}}$ be in $D^{w^*}(X,u)$, and let $0 < \alpha < 1$. Then $\lambda_0 = 1$, $|\lambda_n| \leqslant 1$ for every $n \in \mathbb{N}$, and $\alpha y \in |\mathrm{co}|(K)$ (again by (2.9.7)). Therefore there exist $y_1 = \{\mu_n\}_{n\in\mathbb{N}\cup\{0\}}$, $y_2 = \{\nu_n\}_{n\in\mathbb{N}\cup\{0\}} \in K$ and $t \in [0,1]$ such that $\alpha y = ty_1 - (1-t)y_2$, that is to say $\alpha\lambda_n = t\mu_n - (1-t)\nu_n$ for every $n \in \mathbb{N}\cup\{0\}$. In particular, we have $\alpha = \alpha\lambda_0 = t\mu_0 - (1-t)\nu_0 = t - (1-t) = 2t-1$, hence $t = \frac{\alpha+1}{2}$. Now, for each $n \in \mathbb{N}$ we have $\alpha\lambda_n = \frac{\alpha+1}{2}\mu_n - \frac{1-\alpha}{2}\nu_n$, and so

$$\lambda_n = \frac{\alpha+1}{2\alpha}\mu_n - \frac{1-\alpha}{2\alpha}\nu_n \geqslant -\frac{1-\alpha}{2\alpha}.$$

By letting $\alpha \to 1$, we derive that $\lambda_n \geqslant 0$ for every $n \in \mathbb{N}$. Thus, keeping in mind (2.9.8), we have proved that

$$D^{w^*}(X,u) = K.$$

Now to conclude the proof it is enough to show that

$$y = \{0, \tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, \ldots\} \in (\lambda\,|\text{co}|(K)) \backslash |\text{co}|(K) \ \text{ for every } \ \lambda > 1.$$

Suppose to the contrary that $y \in |\text{co}|(K)$, so that there exist $y_1 = \{\mu_n\}_{n \in \mathbb{N}\cup\{0\}}$, $y_2 = \{\nu_n\}_{n \in \mathbb{N}\cup\{0\}} \in K$ and $t \in [0,1]$ such that $y = ty_1 - (1-t)y_2$. Then

$$0 = t\mu_0 - (1-t)\nu_0 = t - (1-t) = 2t - 1,$$

and hence $t = \frac{1}{2}$. Therefore

$$\frac{1}{2} = \frac{1}{2}\mu_1 - \frac{1}{2}\nu_1 \ \text{ and } \ -\frac{1}{2} = \frac{1}{2}\mu_2 - \frac{1}{2}\nu_2,$$

and consequently

$$1 = \frac{1}{2}(\mu_1 - \mu_2) + \frac{1}{2}(\nu_2 - \nu_1).$$

But $\mu_1 - \mu_2 \leqslant 1$, $\nu_2 - \nu_1 \leqslant 1$, and $\mu_i, \nu_i \geqslant 0$ $(i = 1,2)$. It follows that $\mu_1 = \nu_2 = 1$, $\mu_2 = \nu_1 = 0$. But then

$$\sum_{n=1}^{\infty} \frac{\mu_{n+2}}{2^n} = 1 = \sum_{n=1}^{\infty} \frac{\nu_{n+2}}{2^n} \quad \text{and} \quad 0 \leqslant \mu_n, \nu_n \leqslant 1$$

imply $\mu_{n+2} = 1 = \nu_{n+2}$ for every n. Since $y_1, y_2 \in c_0$ this is impossible. Hence $y \notin |\text{co}|(K)$. Let $0 < r < 1$ be given and set $r' = \sum_{n=1}^{N} \frac{1}{2^n}$ where N is sufficiently large so that $r < r'$. Then $ry = \frac{1}{2}(y_1 - y_2)$ where

$$y_1 = \Big\{1, r, 0, \underbrace{\frac{r}{r'}, \ldots, \frac{r}{r'}}_{N}, 0, \ldots\Big\} \quad \text{and} \quad y_2 = \Big\{1, 0, r, \underbrace{\frac{r}{r'}, \ldots, \frac{r}{r'}}_{N}, 0, \ldots\Big\}$$

belong to K. Hence $y \in \lambda\,|\text{co}|(K)$ for every $\lambda = \frac{1}{r} > 1$.

Thinking about positive results for general norm-unital normed algebras, we have the next relevant corollary to Theorem 2.9.28. To derive the corollary, Proposition 2.1.11, Fact 2.9.1, and Theorems 2.1.17(i) and 2.9.17(ii) should also be invoked.

Corollary 2.9.32 *Let A be a norm-unital complete normed complex algebra with a complete predual A_*. Then*

$$\frac{1}{e}\text{int}(\mathbb{B}_{A_*}) \subseteq |\text{co}|(D^{w^*}(A, \mathbf{1})),$$

and hence $\mathbf{1}$ *is a w^*-unitary element of A. Moreover, when in addition A is associative, the same conclusion holds if we replace* $\mathbf{1}$ *with any geometrically unitary element of A.*

In fact, keeping in mind that the norm-norm upper semicontinuity of the duality mapping at an element u goes down to subspaces containing u, and that the numerical index $n(\cdot,u)$ increases when passing to such subspaces, the arguments leading to Corollaries 2.9.18 and 2.9.32 yield the following.

Corollary 2.9.33 *Let A be a norm-unital normed algebra over $\mathbb{K}$, and let X be a subspace of A with a complete predual X_* and such that the unit $\mathbf{1}$ of A lies in X. We have:*

(i) *The equality $V(X,\mathbf{1},x) = \{x(f) : f \in D^{w^*}(X,\mathbf{1})\}^-$ holds for every $x \in X$.*
(ii) *If $\mathbb{K} = \mathbb{C}$, then the inclusion $\frac{1}{e}\mathrm{int}(\mathbb{B}_{X_*}) \subseteq |\mathrm{co}|(D^{w^*}(X,\mathbf{1}))$ holds, and hence $\mathbf{1}$ is a w^*-unitary element of X.*

To conclude the current subsection, let us formulate the following straightforward consequence of Theorem 2.1.17(v) and Corollaries 2.9.19 and 2.9.29.

Corollary 2.9.34 *Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$ such that $n(X,u) = 1$. Then:*

(i) *For every $n \in \mathbb{N}$ and every F in the $2n$-dual $X^{[2n]}$ of X we have*
$$V(X^{[2n]},u,F) = \{F(f) : f \in D(X^{[2(n-1)]},u)\}^-.$$
(ii) *u is w^*-unitary in all even duals of X. More precisely, for every $n \in \mathbb{N}$ we have*
$$\mathrm{int}(\mathbb{B}_{X^{[2n-1]}}) \subseteq |\mathrm{co}|(D(X^{[2(n-1)]},u)).$$

2.9.3 Involving the strong subdifferentiability of the norm

§2.9.35 Let (X,u) be a numerical-range space over $\mathbb{K}$. Then, by Proposition 2.1.5, for each $x \in X$ we have
$$\frac{\|u+rx\|-1}{r} \geqslant \tau(u,x) := \max \Re(V(X,u,x))$$
for every $r > 0$, and
$$\lim_{r\to 0^+} \frac{\|u+rx\|-1}{r} = \tau(u,x).$$
We say that *the norm of X is strongly subdifferentiable at u* if the above limit is uniform when x runs over the unit closed ball of A, equivalently, if $\lim_{r\to 0^+} \varphi(X,u,r) = 0$, where
$$\varphi(X,u,r) := \sup\left\{\frac{\|u+rx\|-1}{r} - \tau(u,x) : x \in \mathbb{B}_X\right\}$$
for every $r > 0$. A subset C of $\mathbb{B}_{X^*}$ is said to be *strongly exposed* by u, if the distance $d(f_n,C)$ tends to zero for any sequence f_n in $\mathbb{B}_{X^*}$ such that $f_n(u) \to 1$. For $K \geqslant 1$ and $r > 0$ let us write
$$\beta_{u,K}(r) := \inf\{1 - \|u+rx\| : x \in K\mathbb{B}_X,\ \tau(u,x) \leqslant -1\}.$$
Then u is said to be a *τ-point of X* if, for any $K \geqslant 1$ there is a $r_K > 0$ such that $\beta_{u,K}(r_K) > 0$.

We can now state one of the main results in this subsection.

Theorem 2.9.36 *Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. Then the following conditions are equivalent:*

(i) *u strongly exposes the set $D(X,u)$.*
(ii) *The duality mapping of X is norm-norm usc at u.*
(iii) *For every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\inf\{\|g-f\| : g \in D(X,x)\,,\, f \in D(X,u)\} < \varepsilon$$

whenever $x \in \mathbb{S}_X$ satisfies $\|x-u\| < \delta$.
(iv) *The norm of X is strongly subdifferentiable at u.*
(v) *u is a τ-point of X.*

Proof (i)⇒(ii) This follows from Lemma 2.9.12.

(ii)⇒(iii) Although this implication is clear, we discuss it to point out that the law $\varepsilon \to \delta$ is the same in both conditions (a fact which will be applied in the verification of Corollary 2.9.37). Indeed, condition (ii) means that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$D(X,x) \subseteq D(X,u) + \varepsilon\,\mathrm{int}(\mathbb{B}_{X'})$$

whenever x is in $\mathbb{S}_X$ with $\|x-u\| < \delta$, whereas condition (iii) means that for every $\varepsilon > 0$ there is $\delta > 0$ such that we have

$$D(X,x) \cap [D(X,u) + \varepsilon\,\mathrm{int}(\mathbb{B}_{X'})] \neq \emptyset \ \text{ for every } \ x \in \mathbb{S}_X \ \text{ with } \ \|x-u\| < \delta.$$

(iii)⇒(iv) Let $\varepsilon > 0$. By assumption (iii), there exists $\rho > 0$ such that, whenever y is in $\mathbb{S}_X$ with $\|y-u\| < \rho$, we can find $g \in D(X,y)$ and $f \in D(X,u)$ such that $\|g-f\| < \varepsilon$. Set $\delta := \min\left\{\frac{\rho}{4}, \frac{1}{2}\right\}$. Then, for $0 < r < \delta$ and $x \in \mathbb{B}_X$, we clearly have $\|u+rx\| \geqslant \frac{1}{2}$ and $1 - \|u+rx\| = \|u\| - \|u+rx\| \leqslant r$, and hence

$$\left\| \frac{u+rx}{\|u+rx\|} - u \right\| \leqslant 2\|(1-\|u+rx\|)u + rx\| \leqslant 4r < \rho.$$

Therefore there are $g \in D(X, \frac{u+rx}{\|u+rx\|})$ and $f \in D(X,u)$ satisfying $\|g-f\| < \varepsilon$. Since

$$\Re(g(x)) = \frac{\Re(g(u+rx)) - \Re(g(u))}{r} = \frac{\|u+rx\| - \Re(g(u))}{r} \geqslant \frac{\|u+rx\| - 1}{r},$$

and $\tau(u,x) \geqslant \Re(f(x))$, we deduce that

$$\frac{\|u+rx\| - 1}{r} - \tau(u,x) \leqslant \Re(g(x)) - \Re(f(x)) \leqslant \|g-f\| < \varepsilon.$$

Thus, by the arbitrariness of $x \in \mathbb{B}_X$, we have $\varphi(X,u,r) \leqslant \varepsilon$ whenever $0 < r < \delta$, i.e. the norm of X is strongly subdifferentiable at u.

(iv)⇒(v) Let $K \geqslant 1$ and $x \in K\mathbb{B}_X$ with $\tau(u,x) \leqslant -1$ be given. By the definition of the function $\varphi(X,u,\cdot)$, for every $r > 0$ we have

$$\begin{aligned}\|u+rx\| = \left\|u + Kr\frac{x}{K}\right\| &\leqslant 1 + Kr\tau\left(u, \frac{x}{K}\right) + Kr\varphi(X,u,Kr) \\ &= 1 + r\tau(u,x) + Kr\varphi(X,u,Kr) \leqslant 1 - r + Kr\varphi(X,u,Kr),\end{aligned}$$

so the arbitrariness of x in $K\mathbb{B}_X$ gives

$$\frac{\beta_{u,K}(r)}{r} \geqslant 1 - K\varphi(X,u,Kr).$$

Now suppose that condition (iv) holds, so that $\lim_{r\to 0^+}\varphi(X,u,Kr) = 0$. It follows that $\beta_{u,K}(r)$ must be positive for small enough r.

(v)⇒(i) Let $f \in \mathbb{B}_{X'}$ be such that $d(f, D(X,u)) > \rho > 0$. By applying the separation theorem to the w^*-compact, convex, disjoint sets $D(X,u)$ and $f + \rho\mathbb{B}_{X'}$ we find an $x \in \mathbb{S}_X$ such that

$$\begin{aligned}\tau(u,x) &= \max\{\Re(g(x)) : g \in D(X,u)\}\\ &\leqslant \min\{\Re(h(x)) : h \in f + \rho\mathbb{B}_{X'}\} = \Re(f(x)) - \rho.\end{aligned}$$

Let us write $y = \rho^{-1}[x - (\tau(u,x)+\rho)u]$, so that we have

$$\tau(u,y) = -1 \text{ and } \|y\| \leqslant 1 + 2\rho^{-1}.$$

For $r > 0$ we have

$$\begin{aligned}\|u+ry\| &\geqslant \Re(f(u+ry))\\ &= \Re(f(u)) + r\rho^{-1}\Re(f(x)) - r\rho^{-1}(\tau(u,x)+\rho)\Re(f(u))\\ &\geqslant \Re(f(u)) + r\rho^{-1}(\tau(u,x)+\rho)(1-\Re(f(u)))\\ &\geqslant \Re(f(u)) + r\rho^{-1}(\rho-1)(1-\Re(f(u)))\end{aligned}$$

where we have used that $\tau(u,x) \geqslant -1$. It follows that

$$\beta_{u,K}(r) \leqslant (1-\Re(f(u)))(1-r\rho^{-1}(\rho-1))$$

where $K = 1+2\rho^{-1}$. Now suppose that condition (i) does not hold, so that there is a sequence f_n in $\mathbb{B}_{X'}$ such that $f_n(u) \to 1$ but $d(f_n, D(X,u)) > \rho$ for some fixed $\rho > 0$ and all n. Then the above argument with f_n in place of f gives $\beta_{u,K}(r) \leqslant 0$ for any $r > 0$, where $K = 1+2\rho^{-1}$. Hence u is not a τ-point. □

Looking at Fact 2.9.1 and Theorem 2.9.36 and their proofs, we get the following.

Corollary 2.9.37 *Let A be a norm-unital normed algebra over $\mathbb{K}$. Then the norm of A is strongly subdifferentiable at $\mathbf{1}$. More precisely, for every $\varepsilon > 0$ we have $\varphi(A,\mathbf{1},r) < \varepsilon$ whenever $0 < r < \min\left\{\frac{\varepsilon}{4}, \frac{1}{2}\right\}$.*

Thus the strong subdifferentiability of the norm of norm-unital normed algebras at their units is ‘uniform’ in the class of such algebras.

Given a normed space X over $\mathbb{K}$ and a non-empty subset U of $\mathbb{S}_X$, we say that *the norm of X is uniformly strongly subdifferentiable on U* if

$$\lim_{r\to 0^+}\sup\{\varphi(X,u,r) : u \in U\} = 0.$$

Proposition 2.9.38 *Let X be a normed space over $\mathbb{K}$, let U be a non-empty subset of $\mathbb{S}_X$, and assume that the norm of X is uniformly strongly subdifferentiable on U. Then we have:*

(i) *The norm of X is uniformly strongly subdifferentiable on the closure of U.*
(ii) *X is smooth at every element of the interior of U relative to $\mathbb{S}_X$.*

Proof Let $\varepsilon > 0$. By the assumption, there is $\delta > 0$ such that we have

$$\frac{\|u+rx\|-1}{r} \leqslant \tau(u,x)+\varepsilon \tag{2.9.9}$$

for every $(u,x,r) \in U \times \mathbb{B}_X \times]0,\delta[$. Now, to prove assertion (i) it is enough to show that the inequality (2.9.9) remains true for every $(u,x,r) \in \bar{U} \times \mathbb{B}_X \times]0,\delta[$. Fix $(u,x,r) \in \bar{U} \times \mathbb{B}_X \times]0,\delta[$, take a sequence u_n in U converging to u, for each $n \in \mathbb{N}$ take $f_n \in D(X,u_n)$ such that $\Re(f_n(x)) = \tau(u_n,x)$, and let $f \in \mathbb{B}_{X'}$ be a cluster point to the sequence f_n in the w^*-topology. Then f lies in $D(X,u)$ and, since

$$\frac{\|u_n+rx\|-1}{r} \leqslant \Re(f_n(x))+\varepsilon,$$

we deduce that $\frac{\|u+rx\|-1}{r} \leqslant \Re(f(x))+\varepsilon$, and hence $\frac{\|u+rx\|-1}{r} \leqslant \tau(u,x)+\varepsilon$, as desired.

To prove assertion (ii), let u be in the interior of U relative to $\mathbb{S}_X$. We want to realize that X is smooth at u. Since the assumption that the norm of X is uniformly strongly subdifferentiable on U appropriately goes down to subspaces, and X is smooth at u if and only if so are all two-dimensional subspaces of X containing u, we can assume without loss of generality that X is two-dimensional. Then, by Mazur's theorem (cf. §2.8.55), we are provided with a sequence u_n in $\mathbb{S}_X$ converging to u and such that X is smooth at u_n for every n. On the other hand, the assumption that the norm of X is uniformly strongly subdifferentiable on U clearly implies that the function $(u,x) \to \tau(u,x)$ from $U \times \mathbb{B}_X$ to $\mathbb{R}$ is (jointly) continuous. As a consequence, for each $x \in X$, the sequence $\tau(u_n,x)$ converges to $\tau(u,x)$. Since the smoothness of X at u_n reads as $\tau(u_n,-x) = -\tau(u_n,x)$ for every $x \in X$ (cf. §2.6.1), it follows that $\tau(u,-x) = -\tau(u,x)$ for every $x \in X$, i.e. X is smooth at u, as desired. □

§2.9.39 A normed space X is said to be *almost transitive* if, for every (equivalently, some) norm-one element $x \in X$, the orbit of x under the group of all surjective linear isometries on X is dense in $\mathbb{S}_X$. The almost transitivity is a weakening of the transitivity already introduced in §2.6.43.

Let E be a topological space, and let S be a subset of E. We recall that S is said to be *nowhere dense in E* if the closure of S in E has empty interior in E. We note that if S is not nowhere dense in E, then we have in fact $S \cap \mathring{\bar{S}} \neq \emptyset$. For

$$S \cap \mathring{\bar{S}} = \emptyset \;\Rightarrow\; S \subseteq E \setminus \mathring{\bar{S}} \;\Rightarrow\; \bar{S} \subseteq E \setminus \mathring{\bar{S}} \;\Rightarrow\; \mathring{\bar{S}} = \bar{S} \cap \mathring{\bar{S}} = \emptyset.$$

Now we can prove the following multiplicative characterization of real pre-Hilbert spaces, which merits being compared with that obtained in Corollary 2.6.14.

Theorem 2.9.40 *Let X be a nonzero real (respectively, complex) normed space. Then the following conditions are equivalent:*

(i) *X is a pre-Hilbert space (respectively, $X = \mathbb{C}$).*
(ii) *X is almost transitive, and there are $v \in \mathbb{S}_X$ and a norm-one continuous bilinear mapping $f : X \times X \to X$ such that $f(x,v) = f(v,x) = x$ for every x in X.*
(iii) *There is a non-nowhere dense subset U of $\mathbb{S}_X$ such that, for each $u \in U$, we can find a norm-one continuous bilinear mapping $f^u : X \times X \to X$ satisfying $f^u(x,u) = f^u(u,x) = x$ for every x in X.*

Proof (i)⇒(ii) By Corollary 2.6.14 and Lemma 2.7.36.

(ii)⇒(iii) Let v and f be given by the assumption (ii), and let U stand for the orbit of v under the group of all surjective linear isometries on X. Then U is a dense (so non-nowhere dense) subset of $\mathbb{S}_X$. Moreover, for each $u \in U$, there is a surjective linear isometry $T^u : X \to X$ such that $T^u(u) = v$, so that it is enough to consider the mapping $f^u : (x,y) \to (T^u)^{-1}[f(T^u(x), T^u(y))]$ from $X \times X$ to X to realize that condition (iii) is fulfilled.

(iii)⇒(i) Let U be the subset of $\mathbb{S}_X$ given by assumption (iii), and let u be in U. Then X, endowed with the product f^u, becomes a norm-unital normed real (respectively, complex) algebra whose unit is u. Since u is arbitrary in U, it follows from Corollary 2.9.37 that the norm of X is uniformly strongly subdifferentiable on U. Therefore, by Proposition 2.9.38, X is smooth at each element of $\mathring{\bar{U}}$. Since U is not nowhere dense in $\mathbb{S}_X$, there exists $w \in U \cap \mathring{\bar{U}}$, so that X is smooth at w, and $f^w : X \times X \to X$ is a norm-one continuous bilinear mapping satisfying

$$f^w(x,w) = f^w(w,x) = x \text{ for every } x \text{ in } X.$$

By Corollary 2.6.14 (respectively, Proposition 2.6.2), X is a pre-Hilbert space (respectively, $X = \mathbb{C}$). □

Recall that smooth-normed algebras were defined as those norm-unital normed algebras which are smooth at their units (cf. §2.6.1).

Corollary 2.9.41 *Let A be a norm-unital normed real (respectively, complex) algebra. Then the following conditions are equivalent:*

(i) *The normed space of A is transitive.*
(ii) *The normed space of A is almost transitive.*
(iii) *A is a smooth-normed algebra (respectively, $A = \mathbb{C}$).*

Proof The implication (i)⇒(ii) is clear, whereas the one (iii)⇒(i) follows from Corollary 2.6.10(ii) and Lemma 2.7.36. Finally, if condition (ii) is fulfilled, then, by the implication (ii)⇒(i) in Theorem 2.9.40, A is a pre-Hilbert space (respectively, $A = \mathbb{C}$), and hence condition (iii) holds. □

Invoking Theorem 2.6.21, Corollary 2.9.41 above yields the following.

Corollary 2.9.42 *Let A be a norm-unital normed alternative real algebra. Then the following conditions are equivalent:*

(i) *The normed space of A is almost transitive.*
(ii) *A is equal to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. By the definition of Fréchet differentiability, the norm of X is Fréchet differentiable at u if there exists a continuous $\mathbb{R}$-linear mapping $g^u : X \to \mathbb{R}$ such that

$$\lim_{\|h\| \to 0} \frac{|\,\|u+h\| - 1 - g^u(h)|}{\|h\|} = 0$$

or, equivalently,

$$\lim_{r \to 0} \frac{\|u+rx\| - 1}{r} = g^u(x) \text{ uniformly when } x \text{ runs over } \mathbb{B}_X.$$

By looking at §§2.6.1 and 2.9.35, we realize that, if the norm of X is Fréchet differentiable at u, then we have $g^u(x) = \tau(u,x)$ for every $x \in X$ and that, consequently, the following fact follows straightforwardly.

Fact 2.9.43 *Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. Then the norm of X is Fréchet differentiable at u if and only if X is smooth at u and the norm of X is strongly subdifferentiable at u.*

Thus strong subdifferentiability of the norm is a non-smooth extension of Fréchet differentiability of the norm.

Again let X be a normed space over $\mathbb{K}$, but now let U be a non-empty subset of $\mathbb{S}_X$. We say that *the norm of X is uniformly Fréchet differentiable on U* if

$$\lim_{\|h\|\to 0} \frac{|\,\|u+h\| - 1 - \tau(u,h)|}{\|h\|} = 0 \text{ uniformly when } u \text{ runs over } U$$

or, equivalently,

$$\lim_{r\to 0} \sup\left\{\tfrac{\|u+rx\|-1}{r} - \tau(u,x) : (u,x) \in U \times \mathbb{B}_X\right\} = 0.$$

The following fact now follows from Fact 2.9.43.

Fact 2.9.44 *Let X be a normed space over $\mathbb{K}$, and let U be a non-empty subset of $\mathbb{S}_X$. Then the norm of X is uniformly Fréchet differentiable on U if and only if X is smooth at every point of U and the norm of X is uniformly strongly subdifferentiable on U.*

Definition 2.9.45 A normed space X is said to be *uniformly smooth* if the norm of X is uniformly Fréchet differentiable on the whole unit sphere $\mathbb{S}_X$ of X.

Now, by keeping Proposition 2.9.38 in mind, we get the following.

Proposition 2.9.46 *A normed space X over $\mathbb{K}$ is uniformly smooth if and only if the norm of X is uniformly strongly subdifferentiable on some dense subset of $\mathbb{S}_X$.*

We note that uniformly smooth Banach spaces are superreflexive, and that, in fact, superreflexivity is the isomorphic side of the uniform smoothness, i.e. every superreflexive Banach space has a uniformly smooth equivalent renorming (see for instance [722, Corollary IV.4.6]).

Now we are going to study numerical ranges in ℓ_∞-sums of numerical-range spaces. We begin with the following fact whose proof is left to the reader.

Fact 2.9.47 *Let I be a non-empty finite set, let $\{X_i\}_{i\in I}$ be a family of normed spaces over $\mathbb{K}$, and, for each i in I, let u_i be a norm-one element in X_i. Denote by Y the ℓ_∞-sum of the family $\{X_i\}_{i\in I}$, and by u the element of $\mathbb{S}_Y$ given by $u(i) := u_i$ for every i in I. Then the equality*

$$V(Y,u,y) = \mathrm{co}\left[\cup_{i\in I} V(X_i, u(i), y(i))\right]$$

holds for every y in Y.

The case in which the set I above is infinite is discussed in the following.

Theorem 2.9.48 *Let I be an infinite set, let $\{X_i\}_{i\in I}$ be a family of normed spaces over $\mathbb{K}$, and, for each i in I, let u_i be a norm-one element in X_i. Denote by Y the ℓ_∞-sum of the family $\{X_i\}_{i\in I}$, and by u the element of $\mathbb{S}_Y$ given by $u(i) := u_i$ for every i in I. Then the following conditions are equivalent:*

(i) *the equality $V(Y,u,y) = \overline{\mathrm{co}}\,[\cup_{i\in I} V(X_i, u(i), y(i))]$ holds for every y in Y.*
(ii) $\lim_{(i,r)\to(\infty,0^+)} \varphi(X_i, u(i), r) = 0$.

In condition (ii) above, the symbol $\lim_{(i,r)\to(\infty,0^+)}$ means the limit along the filter basis on $I\times\mathbb{R}^+$ consisting of all subsets of $I\times\mathbb{R}^+$ of the form $J\times]0,\delta[$, where J is a cofinite subset of I and δ is a positive number.

Proof (ii)$\Rightarrow$(i) Fix y in $\mathbb{S}_Y$, and let ε be a positive number. By assumption (ii), there exists $\delta > 0$, and a cofinite subset J of I, such that the inequality

$$\frac{\|u(j)+rx_j\|-1}{r} - \tau(u(j),x_j) < \varepsilon$$

holds whenever j is in J, x_j is in $\mathbb{B}_{X_j}$ for such a j, and $0<r<\delta$. For k in $I\backslash J$, we can choose $\delta_k > 0$ such that

$$\frac{\|u(k)+ry(k)\|-1}{r} - \tau(u(k),y(k)) < \varepsilon$$

whenever $0<r<\delta_k$. By setting $\rho := \min\{\delta, \min\{\delta_k : k\in I\backslash J\}\}$, it follows that the inequality

$$\frac{\|u(i)+ry(i)\|-1}{r} < \tau(u(i),y(i)) + \varepsilon$$

is true for every i in I and $0<r<\rho$. Therefore, for $0<r<\rho$ we have

$$\frac{\|u+ry\|-1}{r} \leqslant \sup\{\tau(u(i),y(i)) : i\in I\} + \varepsilon,$$

and, by letting $r\to 0$, we obtain

$$\tau(u,y) \leqslant \sup\{\tau(u(i),y(i)) : i\in I\} + \varepsilon.$$

In view of the arbitrariness of ε, we actually have

$$\tau(u,y) \leqslant \sup\{\tau(u(i),y(i)) : i\in I\},$$

so that, replacing y by zy with z in $\mathbb{S}_{\mathbb{K}}$, the inclusion

$$V(Y,u,y) \subseteq \overline{\mathrm{co}}\,[\cup_{i\in I} V(X_i,u(i),y(i))]$$

follows. The reverse inclusion is trivial.

(i)$\Rightarrow$(ii) Assume that condition (ii) is not true. Then there is $\varepsilon > 0$ such that, for every cofinite subset J of I, and for every $\delta > 0$, there exist $0<r<\delta$, $j\in J$, and $x_j\in\mathbb{B}_{X_j}$ such that

$$\frac{\|u(j)+rx_j\|-1}{r} - \tau(u(j),x_j) \geqslant \varepsilon.$$

Take $0<r_1<1$, $i(1)\in I$, and $x_{i(1)}\in\mathbb{B}_{X_{i(1)}}$ such that

$$\frac{\|u(i(1))+r_1x_{i(1)}\|-1}{r_1} - \tau(u(i(1)),x_{i(1)}) \geqslant \varepsilon.$$

Assume that, for some n in $\mathbb{N}$ we have found

$$r_1,\dots,r_n, i(1),\dots,i(n), \text{ and } x_{i(1)},\dots,x_{i(n)}$$

such that $0 < r_m < \frac{1}{m}$, $i(m) \in I$, $x_{i(m)} \in X_{i(m)}$,

$$\frac{\|u(i(m)) + r_m x_{i(m)}\| - 1}{r_m} - \tau(u(i(m)), x_{i(m)}) \geqslant \varepsilon$$

for every $m \in \{1,\dots,n\}$, and $i(m) \neq i(m')$ for all $m, m' \in \{1,\dots,n\}$ with $m \neq m'$. Then we can choose

$$0 < r_{n+1} < \frac{1}{n+1}, i(n+1) \in I\backslash\{i(1),\dots,i(n)\}, \text{ and } x_{i(n+1)} \in \mathbb{B}_{X_{i(n+1)}}$$

satisfying

$$\frac{\|u(i(n+1)) + r_{n+1} x_{i(n+1)}\| - 1}{r_{n+1}} - \tau(u(i(n+1)), x_{i(n+1)}) \geqslant \varepsilon.$$

In this way, we have constructed a sequence $(r_n, i(n), x_{i(n)})$ with $r_n \in \mathbb{R}^+$, $r_n \to 0$, $i(n) \in I$, $i(n) \neq i(m)$ for $n \neq m$, $x_{i(n)} \in \mathbb{B}_{X_{i(n)}}$, and

$$\frac{\|u(i(n)) + r_n x_{i(n)}\| - 1}{r_n} - \tau(u(i(n)), x_{i(n)}) \geqslant \varepsilon.$$

Set $s := \limsup_{n\to\infty} \tau(u(i(n)), x_{i(n)})$, so that, by discarding a finite number of terms in the sequence $(r_n, i(n), x_{i(n)})$, we can assume that, in addition to the above properties, we have

$$\tau(u(i(n)), x_{i(n)}) \leqslant s + \frac{\varepsilon}{2}$$

for every n. Now, consider the element y in Y defined by $y(i) = x_{i(n)}$ if $i = i(n)$ for some n, and $y(i) = su(i)$ otherwise. Then, for every n in $\mathbb{N}$ we have

$$\begin{aligned}\frac{\|u + r_n y\| - 1}{r_n} &\geqslant \frac{\|u(i(n)) + r_n y(i(n))\| - 1}{r_n} \\ &= \frac{\|u(i(n)) + r_n x_{i(n)}\| - 1}{r_n} \geqslant \tau(u(i(n)), x_{i(n)}) + \varepsilon.\end{aligned}$$

By taking upper limits as $n \to \infty$, we obtain

$$\tau(u, y) \geqslant s + \varepsilon.$$

On the other hand, the definition of y yields the inequality

$$\sup\{\tau(u(i), y(i)) : i \in I\} \leqslant s + \frac{\varepsilon}{2}.$$

It follows that

$$\sup\{\tau(u(i), y(i)) : i \in I\} + \frac{\varepsilon}{2} \leqslant \tau(u, y),$$

so that the equality in assertion (i) cannot be true for y. □

§**2.9.49** With the notation in Theorem 2.9.48 above, it is clear that the condition

$$\lim_{r\to 0^+} \sup\{\varphi(X_i, u(i), r) : i \in I\} = 0$$

implies the one

$$\lim_{(i,r)\to(\infty,0^+)} \varphi(X_i, u(i), r) = 0.$$

But, in view of Corollary 2.9.37, the first of these conditions is fulfilled whenever, for each $i \in I$, X_i is a norm-unital normed algebra and u_i is the unit of X_i. Therefore it is enough to invoke Fact 2.9.47 and Theorem 2.9.48 itself to derive the following.

Corollary 2.9.50 *Let $\{A_i\}_{i\in I}$ be any family of norm-unital normed algebras over $\mathbb{K}$, and let B stand for the ℓ_∞-sum of the family $\{A_i\}_{i\in I}$. Then for every $b \in B$ we have*

$$V(B,\mathbf{1},b) = \overline{\mathrm{co}}\,[\cup_{i\in I} V(A_i,\mathbf{1},b(i))],$$

where **1** *stands indistinctly for the unit of B and that of each A_i.*

Given a non-empty set I and a normed space X, we denote by $B(I,X)$ the space of all bounded functions from I to X, endowed with the sup norm. As a straightforward consequence of Corollary 2.9.50, we get the following.

Corollary 2.9.51 *Let I be a non-empty set, and let A be a norm-unital normed algebra over $\mathbb{K}$. Then for every $f \in B(I,A)$ we have*

$$V(B(I,A),\mathbf{1},f) = \overline{\mathrm{co}}\,[\cup_{i\in I} V(A,\mathbf{1},f(i))].$$

Proposition 2.9.52 *Let E be a topological space with no non-empty open finite subset (for instance, every topological T_1-space without isolated points), let X be a normed space, and let $g : E \to \mathbb{S}_X$ be a continuous function. Then the following conditions are equivalent:*

(i) *For every f in $B(E,X)$, the equality*

$$V(B(E,X),g,f) = \overline{\mathrm{co}}\,[\cup_{t\in E} V(X,g(t),f(t))]$$

holds.

(ii) *The norm of X is uniformly strongly subdifferentiable on the range of g.*

Proof (ii)⇒(i) The uniform strong subdifferentiability of the norm of X on the range of g implies $\lim_{(t,r)\to(\infty,0^+)} \varphi(X,g(t),r) = 0$ (cf. §2.9.49). Now, apply Theorem 2.9.48.

(i)⇒(ii) Let $\varepsilon > 0$. By assumption (i) and Theorem 2.9.48, there exist $\delta > 0$ and a cofinite subset J of E such that the inequality

$$\frac{\|g(t)+rx\|-1}{r} - \tau(g(t),x) \leqslant \frac{\varepsilon}{2}$$

is true whenever t is in J, x is in $\mathbb{B}_X$, and $0 < r < \delta$. Then, for $0 < r < \delta$, $0 < s < \delta$, $t \in J$, and $x \in \mathbb{B}_X$, we have

$$\left| \frac{\|g(t)+rx\|-1}{r} - \frac{\|g(t)+sx\|-1}{s} \right| \leqslant \varepsilon.$$

Since J is dense in E and g is continuous, the last inequality remains true for

$$0 < r < \delta, 0 < s < \delta, t \in E, \text{ and } x \in \mathbb{B}_X.$$

By letting $s \to 0$, we obtain

$$\frac{\|g(t)+rx\|-1}{r} - \tau(g(t),x) \leqslant \varepsilon$$

whenever t is in E, x is in $\mathbb{B}_X$, and $0 < r < \delta$. □

If I is an infinite set, and if we endow I with the trivial topology, then I becomes a topological space with no non-empty open finite subset, and the unique continuous functions on I are the constant ones. Therefore Proposition 2.9.52 above applies to get the following.

Corollary 2.9.53 *Let X be a normed space, and let u be in $\mathbb{S}_X$. For every non-empty set I, denote by $\hat{u}$ the element of $B(I,X)$ defined by $\hat{u}(i) = u$ for every i in I. Then the following conditions are equivalent:*

(i) *For every non-empty set I and every f in $B(I,X)$, the equality*

$$V(B(I,X),\hat{u},f) = \overline{\text{co}}\,[\cup_{i\in I} V(X,u,f(i))]$$

holds.

(ii) *There exists an infinite set I such that we have*

$$V(B(I,X),\hat{u},f) = \overline{\text{co}}\,[\cup_{i\in I} V(X,u,f(i))]$$

for every f in $B(I,X)$.

(iii) $V(B(\mathbb{N},X),\hat{u},f) = \overline{\text{co}}\,[\cup_{n\in\mathbb{N}} V(X,u,f(n))]$ *for every f in $B(\mathbb{N},X)$.*

(iv) *The norm of X is strongly subdifferentiable at u.*

We note that, in view of the equivalence (ii)⇔(iv) in Theorem 2.9.36, Corollary 2.9.53 provides us with several reformulations of the norm-norm upper semicontinuity of the duality mapping of a normed space X at a norm-one element u. Then the equivalence (i)⇔(iv) in Corollary 2.9.53 shows that Fact 2.9.1 and Corollary 2.9.51 are 'equivalent' statements. In the same way, Corollary 2.9.54 immediately below becomes a reformulation of the bracketed version of Corollary 2.9.4.

Corollary 2.9.54 *Let I be a non-empty set, let X be a finite-dimensional normed space over $\mathbb{K}$, let u be in $\mathbb{S}_X$, and let f be in $B(I,X)$. Then*

$$V(B(I,X),\hat{u},f) = \overline{\text{co}}[\cup_{i\in I} V(X,u,f(i))],$$

where $\hat{u}$ stands for the mapping constantly equal to u on I.

By combining Propositions 2.9.46 and 2.9.52, we get the following.

Corollary 2.9.55 *Let E be a topological space with no non-empty open finite subset, let X be a normed space, and let g be a continuous function from E onto a dense subset of $\mathbb{S}_X$. Then the following conditions are equivalent:*

(i) *For every f in $B(E,X)$, the equality*

$$V(B(E,X),g,f) = \overline{\text{co}}\,[\cup_{t\in E} V(X,g(t),f(t))]$$

holds.

(ii) *X is uniformly smooth.*

Let X be a normed space over $\mathbb{K}$, and let f be a mapping from $\mathbb{S}_X$ to X. According to §2.1.49, we can consider the *spatial numerical range* $W(f)$ of f, namely

$$W(f) := \cup_{u\in\mathbb{S}_X} V(X,u,f(u)).$$

If the mapping f is bounded, then we can also consider its *intrinsic numerical range*, namely $V(B(\mathbb{S}_X,X),\mathbf{1},f)$, where $\mathbf{1}$ stands for the natural embedding $\mathbb{S}_X \hookrightarrow X$. In this case we have the inclusion

$$\overline{\text{co}}\,(W(f)) \subseteq V(B(\mathbb{S}_X,X),\mathbf{1},f). \tag{2.9.10}$$

(Indeed, for u in $\mathbb{S}_X$ and ϕ in $D(X,u)$, the mapping $g \to \phi(g(u))$ from $B(\mathbb{S}_X,X)$ to $\mathbb{K}$ is an element of $D(B(\mathbb{S}_X,X),\mathbf{1})$.) Now, applying Corollary 2.9.55 with $E = \mathbb{S}_X$ and g equal to the inclusion mapping $\mathbb{S}_X \hookrightarrow X$, we obtain the following numerical-range characterization of uniformly smooth normed spaces.

Theorem 2.9.56 *Let X be a normed space over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *For every bounded function $f : \mathbb{S}_X \to X$, the equality*

$$\overline{\text{co}}(W(f)) = V(B(\mathbb{S}_X,X),\mathbf{1},f)$$

holds.

(ii) *X is uniformly smooth.*

Now let us show an elementary example of a Banach space X, together with a bounded function $f : \mathbb{S}_X \to X$, such that the inclusion (2.9.10) is strict.

Example 2.9.57 Take $X = \mathbb{R}^2$ with norm $\|(\lambda,\mu)\| := \max\{|\lambda|,|\mu|\}$. For x,y in X, set $]x,y[:= \{rx+(1-r)y : 0<r<1\}$, and define a function $f : \mathbb{S}_X \to X$ by

$$f(u) := \begin{cases} (0,0) & \text{if} \quad u \in \{(1,1),(1,-1),(-1,-1),(-1,1)\}, \\ (0,1) & \text{if} \quad u \in](1,1),(1,-1)[\cup](-1,-1),(-1,1)[, \\ (1,0) & \text{if} \quad u \in](1,-1),(-1,-1)[\cup](-1,1),(1,1)[. \end{cases}$$

Then f is bounded with $\|f\| = 1$. Moreover, it is easily checked that the equality $\|\mathbf{1}+rf\| = 1+|r|$ holds for every r in $\mathbb{R}$, and therefore $V(B(\mathbb{S}_X,X),\mathbf{1},f)$ is equal to the closed real interval $[-1,1]$. However, we have $V(X,u,f(u)) = \{0\}$ for every u in $\mathbb{S}_X$, and hence $W(f) = \{0\}$.

§2.9.58 Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. We note that, since the mapping $\tau(u,\cdot) : X \to \mathbb{R}$ is subadditive and satisfies $\tau(u,\cdot) \leqslant \|\cdot\|$, we have

$$|\tau(u,x) - \tau(u,y)| \leqslant \|x-y\| \quad \text{for all} \;\; x,y \in X,$$

and hence $\tau(u,\cdot)$ is continuous. Now let S be a compact subset of X, and for each positive integer n consider the function $\phi_n : S \to \mathbb{R}$ defined by

$$\phi_n(x) = \frac{\|u+\frac{x}{n}\|-1}{\frac{1}{n}} = \|nu+x\|-n.$$

Then $\{\phi_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of continuous functions pointwise converging on S to the continuous function $\tau(u,\cdot)$. Therefore, by Dini's theorem, this convergence is uniform on S, which implies $\lim_{r\to 0^+} \phi_S(r) = 0$, where

$$\phi_S(r) := \sup\left\{ \frac{\|u+rx\|-1}{r} - \tau(u,x) : x \in S \right\}.$$

Incidentally, we note that the above argument shows that the norm of a finite-dimensional normed space is strongly subdifferentiable at any point of the unit sphere, a fact which also follows from Corollary 2.9.4 and Theorem 2.9.36.

Lemma 2.9.59 *Let (X,u) be a numerical-range space over $\mathbb{K}$, let I be a non-empty set, let $f : I \to X$ be a mapping such that $f(I)$ is relatively compact in X, and let $\hat{u}$ stand for the function constantly equal to u on I. Then*

$$V(B(I,X),\hat{u},f) = \overline{\mathrm{co}}[\cup_{i\in I}V(X,u,f(i))].$$

Proof Let S denote the balanced hull of the closure of $f(I)$, so that, by §2.9.58, we have $\lim_{r\to 0^+}\phi_S(r) = 0$. Fix $z \in \mathbb{S}_{\mathbb{K}}$, $r > 0$, and $i \in I$. Since S is balanced we have $zf(i) \in S$ so

$$\frac{\|u+rzf(i)\|-1}{r} \leqslant \tau(u,zf(i)) + \phi_S(r)$$

and we deduce that

$$\frac{\|\hat{u}+rzf\|-1}{r} \leqslant \sup\{\tau(u,zf(i)) : i \in I\} + \phi_S(r) \leqslant \tau(\hat{u},zf) + \phi_S(r).$$

Now let $r \to 0$ to obtain $\tau(\hat{u},zf) = \sup\{\tau(u,zf(i)) : i \in I\}$. This holds for every $z \in \mathbb{S}_{\mathbb{K}}$ and implies that the compact convex sets $V(B(I,X),\hat{u},f)$ and $\overline{\mathrm{co}}[\cup_{i\in I}V(X,u,f(i))]$ coincide. □

Proposition 2.9.60 *Let E be a compact Hausdorff topological space, let F be a dense subset of E, let (X,u) be a numerical-range space over $\mathbb{K}$, and let f be a continuous function from E to X. Then*

$$V(C(E,X),\hat{u},f) = \overline{\mathrm{co}}[\cup_{t\in F}V(X,u,f(t))],$$

where $C(E,X)$ denotes the normed space of all continuous functions from E to X, and $\hat{u}$ stands for the mapping constantly equal to u on E.

Proof By carrying each continuous function from E to X into its restriction to F we obtain an isometric linear embedding $C(E,X) \hookrightarrow B(F,X)$. Therefore, by Corollary 2.1.2(ii) and Lemma 2.9.59, we have

$$V(C(E,X),\hat{u},f) = V(B(F,X),\hat{u}_{|F},f_{|F}) = \overline{\mathrm{co}}[\cup_{t\in F}V(X,u,f(t))]. \qquad \square$$

As a straightforward consequence, we get the following.

Corollary 2.9.61 *Let E be a compact Hausdorff topological space, and let f be in $C^{\mathbb{K}}(E)$. Then*

$$V(C^{\mathbb{K}}(E),\mathbf{1},f) = \mathrm{co}(f(E)).$$

We note that the above corollary also follows from either Corollary 2.3.73, 2.9.51, or 2.9.54.

2.9.4 Historical notes and comments

This section has been elaborated from the papers (cited chronologically) of Cudia [189], Ellis [234], Giles–Gregory–Sims [287], Gregory [299], Martínez–Mena–Payá–Rodríguez [425], Aparicio–Ocaña–Payá–Rodríguez [22], Chonghu [170], Franchetti–Payá [268], Rodríguez [527], Becerra–Rodríguez [70], Rodríguez [531], Bandyopadhyay–Jarosz–Rao [56], Godefroy–Indumathi [291], Godefroy–Rao [293], Dutta–Rao [220], and Acosta-Becerra–Rodríguez [1]. The material has been fully reorganized, so let us comment about the paternity of the results we have collected. (Sometimes these results appear originally formulated in terms different from ours, but, by means of folklore or previously known facts, they become equivalent to those appearing here.)

Cudia's pioneering paper [189] underlies most material included in this section. In particular, Fact 2.9.3 is proved there [189, Theorem 4.3].

In [234], the author formulates Example 2.9.31 (without any proof), and attributes it to L. Asimow. Some arguments in our proof are taken from [681, Example 2.1.6]. An example similar to Asimow's, which is more natural but less elementary, will be discussed below (see Example 2.9.67).

In [287], the authors introduce the different notions of upper semicontinuity of the duality mapping of a normed space, and prove Theorem 2.9.8 and Lemma 2.9.12, which, in the case that X is complete and that τ equals the norm topology, is nothing other than the equivalence (i)⇔(ii) in Theorem 2.9.36. The 'smooth' forerunner of this equivalence (namely, the duality between Fréchet differentiability and strong exposition) goes back to Smulian [593]. The authors of [287] also consider other geometrical aspects of the norm-norm upper semicontinuity of the duality mapping as well as its relation to sufficient conditions for a Banach space to be an Asplund space.

In [299], the author introduces the notion of strong subdifferentiability of the norm of a Banach space at a norm-one element, and proves the crucial equivalence (ii)⇔(iv) in Theorem 2.9.36. It is worth mentioning that, after Gregory's paper [299], most authors have preferred the terminology of strong subdifferentiability of the norm instead of that of norm-norm upper semicontinuity of the duality mapping. This happens in particular in the papers where the points of norm-norm upper semicontinuity for the duality mapping of C^*-algebras, JB^*-triples, and real JB^*-triples are determined (see [180], [74], and [69], respectively).

In [425], the authors prove Fact 2.9.1 (that the duality mapping of any norm-unital normed algebra is norm-norm usc at the unit).

In [22], the authors rediscover the strong subdifferentiability of the norm of a normed space X at a point $u \in \mathbb{S}_X$, although a different terminology is used there. They prove Fact 2.9.2, Lemmas 2.9.5, 2.9.10, 2.9.15, 2.9.16, and 2.9.59, Propositions 2.9.11 and 2.9.60, and Corollaries 2.9.18, 2.9.51, 2.9.53, and 2.9.61, and suggest condition (iii) in Theorem 2.9.36 as an intermediate condition between conditions (ii) and (iv) in that theorem. It is worth mentioning that the authors of [22] prove a light version of Theorem 2.9.17 asserting that, *if X is a Banach space with a complete predual, if u is in $\mathbb{S}_X$, and if the duality mapping of X is norm-norm usc at u, then the equality $V(X,u,x) = \{x(f) : f \in D^{w^*}(X,u)\}^-$ holds for every $x \in X$*. This light version of Theorem 2.9.17 allows them to prove Corollary 2.9.18, recently rediscovered by Magajna [411]. Although straightforwardly derivable from the results in [22], Corollaries 2.9.19, 2.9.33(i) and 2.9.54, are not pointed out there. A particular case of Corollary 2.9.33(i) has been recently rediscovered by Blecher and Magajna [105], who apply it to prove a relevant 'operator system' variant of Sakai's fundamental characterization of von Neumann algebras precisely as the C^*-algebras with a predual [806, Theorem 1.16.7]. We recall that operator systems were incidentally introduced in Subsection 2.4.3. The power-associative forerunner of Corollary 2.9.51 had been proved in [514] by applying Proposition 2.1.7.

In [170], the author proves the implication (iv)⇒(v) in Theorem 2.9.36 (that a point where the norm is strongly subdifferentiable is a τ-point). The notion of a τ-point had been introduced earlier by Franchetti [267].

In [268], the authors give a systematic discussion of strong subdifferentiability of the norm. In particular they prove the actual version of Theorem 2.9.36, together with Propositions 2.9.38(ii) and 2.9.46, and Proposition 2.9.13 in the particular case that τ equals the norm topology. The general case of Proposition 2.9.13 is pointed out in [75]. The relevant contribution of [268] to Theorem 2.9.36 consists of the proof of the implication (v)⇒(i) which was previously unknown, and of the fact that the whole proof allows the authors to avoid completeness in all equivalences in that theorem. The paper [268] also contains a result of G. Godefroy asserting that *Banach spaces with strongly subdifferentiable norm at every point of their unit spheres are Asplund spaces*. In fact, as proved later by Contreras and Payá in [179], *the above result remains true when the strong subdifferentiability of the norm is relaxed to the norm-weak upper semicontinuity of the duality mapping*. Another refinement of Godefroy's result, now with a conclusion of isometric type, appears in the paper of Godefroy–Montesinos–Zizler [292], where it is shown that, *if X is a Banach space whose norm is strongly subdifferentiable at any point of $\mathbb{S}_X$, then X' has no proper closed norming subspace*. To realize how this last result contains Godefroy's one, note that the strong subdifferentiability of the norm of X goes down to subspaces, so that, for every subspace Y of X, Y' has no proper closed norming subspace for Y, which implies, when Y is separable, that Y' is separable too.

In [527], the author proves Corollaries 2.9.37 and 2.9.50, and derives from them that, *if $\{A_i\}_{i\in I}$ is any family of norm-unital complete normed algebras over $\mathbb{K}$, and if $\mathscr{U}$ is an ultrafilter on I, then for every (a_i) in the ultraproduct $(A_i)_{\mathscr{U}}$* (cf. §2.8.61) *we have*

$$V((A_i)_{\mathscr{U}}, \mathbf{1}, (a_i)) \subseteq \overline{\text{co}}[\cup_{i\in I} V(A_i, \mathbf{1}, a_i)],$$

where $\mathbf{1}$ stands indistinctly for the unit of $(A_i)_{\mathscr{U}}$ and that of each A_i. This is applied, together with Theorems 2.5.50 and 2.6.21, to prove that *there exists a universal constant $k > 0$ such that, if A is any norm-unital complete normed alternative real algebra with* $\operatorname{diam}(D(A,\mathbf{1})) < k$, *then A is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$* (a variant of the implication (ii)⇒(iii) in Theorem 2.6.21). The associative forerunner of the fact just formulated is due to Lumer [408].

In [70], the authors prove Theorem 2.9.40 and Corollaries 2.9.41 and 2.9.42. Actually, Theorem 2.9.40 is a consequence of a deeper result, also proved in [70], asserting that, *for a nonzero real (respectively, complex) normed space X, the following conditions are equivalent:*

(i) *X is a pre-Hilbert space (respectively, $X = \mathbb{C}$).*
(ii) *X is almost transitive and there exists $v \in \mathbb{S}_X$ such that we can find a real (respectively, complex) norm-unital normed algebra A and a (possibly non-surjective) linear isometry $T : X \to A$ with $T(v) = \mathbf{1}$.*
(iii) *There exists a non-nowhere dense subset U of $\mathbb{S}_X$ such that for each $u \in U$ we can find a real (respectively, complex) norm-unital normed algebra A_u and a linear isometry $T_u : X \to A_u$ with $T_u(u) = \mathbf{1}$.*

The original proof of the result just formulated depended heavily on the Skorik–Zaidenberg paper [583]. This dependence was avoided later in [73].

In [531], the author proves Propositions 2.9.38(i) and 2.9.52, Theorems 2.9.48 and 2.9.56, and Corollary 2.9.55. Example 2.9.57 is also pointed out there.

In [56], the authors prove Corollary 2.9.25.

In [291], the authors prove Proposition 2.9.6(ii), Theorem 2.9.17, and Corollaries 2.9.20 and 2.9.21. We note that Corollary 2.9.20 contains a result of Diestel [718, Theorem 1, p. 33], which in turn contains the fact, sometimes attributed to Smulian [593] but probably due to Fan and Glicksberg [248], that *a dual Banach space with Fréchet differentiable norm at any point of its unit sphere has to be reflexive.* A detailed discussion of the paternity of this last fact can be seen in [790, 5.5.3.7–8].

In [293], the authors prove the non-quantitative version of Theorem 2.9.28 (i.e. the first conclusion in that theorem). As the main result, they show that, *given any non-reflexive separable Banach space X with a complete predual, there exists $u \in X$ such that, up to a suitable equivalent dual renorming, u becomes geometrically unitary, but is not w^*-unitary, nor even a w^*-vertex (since, in fact, we have $D^{w^*}(X,u) = \emptyset$ in the renorming).* The paper [293] also contains a result related to Proposition 2.9.11. Indeed, *given any non-reflexive Banach space X, and any nonzero element $u \in X$, there exists an equivalent norm on X such that, in the new norm, u is a geometrically unitary element but not a point of norm-weak upper-semicontinuity for the duality mapping.*

In [220], the authors prove Corollary 2.9.34(i).

In [1], the authors prove Corollaries 2.9.23 and 2.9.29, Proposition 2.9.27, and the quantitative version of Theorem 2.9.28. We note that, in the case $\mathbb{K} = \mathbb{R}$, Corollary 2.9.29 can easily be derived from the notions and results in the theory of the so-called unit-order spaces [233] (see also [681, Theorem 2.1.5]). Although straightforwardly derivable from the results in [1], Corollaries 2.9.32, 2.9.33(ii), and 2.9.34(ii) are not pointed out there.

Now that we have concluded our review on the paternity of results included in the section, let us continue with some additional information on the topic. We begin by noticing that, as pointed out in [73], in the case $\mathbb{K} = \mathbb{R}$ Fact 2.9.2 has the following relevant refinement.

Fact 2.9.62 *Let (X,u) be a real numerical-range space with $n(X,u) = 1$. Then we have $D(X,x) \subseteq D(X,u)$ whenever x is in $\mathbb{S}_X$ with $\|x-u\| < 2$.*

Proof Let x be in $\mathbb{S}_X$ with $\|x-u\| < 2$, and let f be in $D(X,x)$. By Theorem 2.1.17(iii), there exist states $g,h \in D(X,u)$ and $0 \leqslant \alpha \leqslant 1$ such that $f = \alpha g - (1-\alpha)h$. Then, to realize that f lies in $D(X,u)$ it is enough to show that $\alpha = 1$. But, if this were not the case, then we would have

$$\begin{aligned} 1 = f(x) = \alpha g(x) - (1-\alpha)h(x) &= \alpha g(x) + (1-\alpha)(h(u-x) - 1) \\ &\leqslant \alpha + (1-\alpha)(\|u-x\| - 1) < \alpha + (1-\alpha)(2-1) = 1, \end{aligned}$$

a contradiction. □

Thus, *given a real numerical-range space (X,u) with $n(X,u) = 1$, the duality mapping of X is norm-τ usc at u for every vector space topology τ on X'.*

Proposition 2.9.6(i) could be new, and implies the following.

Fact 2.9.63 *Let X be a normed space over $\mathbb{K}$, and let u be a norm-one element in X. Then for every $x \in X$ we have*

$$V(X,u,x) = \bigcap_{\delta>0} \{f(x) : f \in \mathbb{S}_{X'}, |f(u)-1| < \delta\}^{-}.$$

Proof Let x be in X. Since $V(X,u,x) = V(X'',u,x)$ (by Corollary 2.1.2), Proposition 2.9.6(i) applies with the couple (X'',X') instead of the couple (X,X_*) appearing there. □

A much deeper result is the following one, due to Lumer and Phillips [409], whose proof will not be discussed here.

Proposition 2.9.64 *Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. Then for every $x \in X$ we have*

$$V(X,u,x) = \bigcap_{\delta>0} \overline{\mathrm{co}}\{\phi(x) : (y,\phi) \in \Gamma, \|y-u\| < \delta\},$$

where Γ stands for any subset of $\Pi(X)$ such that $\pi_1(\Gamma)$ is dense in $\mathbb{S}_X$ (cf. §2.1.30).

Proposition 2.9.64 above is applied in [409] to prove Proposition 2.1.31, and also in [534] to prove the generalization of Proposition 2.1.31 given by Theorem 2.1.50.

Let X be a normed space, and let Y be a subspace of X. We say that Y is a *semi-L-summand* of X if, for each $x \in X$, there exists a unique $y \in Y$ such that $\|x-y\| = \|x+Y\|$, and moreover this y satisfies $\|x\| = \|y\| + \|x-y\|$. This happens in particular if Y is an *L-summand* of X (which means that Y is the range of a linear projection P on X such that $\|x\| = \|P(x)\| + \|x-P(x)\|$ for every $x \in X$). The following result is folklore.

Fact 2.9.65 *Let (X,u) be a numerical-range space over $\mathbb{K}$. If $\mathbb{K}u$ is a semi-L-summand of X, then $n(X,u) = 1$.*

Proof Assume that $\mathbb{K}u$ is a semi-L-summand of X. Let x be in X, and let $\lambda(x)$ stand for the unique element of $\mathbb{K}$ which satisfies $\|x-\lambda(x)u\| = \|x+\mathbb{K}u\|$. Then we have

$$\|x\| = |\lambda(x)| + \|x-\lambda(x)u\|,$$

and hence, since $\lambda(u+zx) = 1+z\lambda(x)$ for every $z \in \mathbb{K}$, Proposition 2.1.5 applies to get $\max \Re(V(X,u,x)) = \Re(\lambda(x)) + \|x-\lambda(x)u\|$. It follows that

$$v(X,u,x) = |\lambda(x)| + \|x-\lambda(x)u\| = \|x\|.$$

Since x is arbitrary in X, we get $n(X,u) = 1$. □

§**2.9.66** In the case $\mathbb{K} = \mathbb{R}$, the converse of the above fact is also true. Indeed, if (X,u) is a real numerical-range space, if $n(X,u) = 1$, and if x is in X, then, denoting by $\lambda(x)$ the mid-point of $V(X,u,x)$, it is straightforward to realize that $\lambda(x)$ is the unique element of $\mathbb{K}$ which satisfies

$$\|x-\lambda(x)u\| = \|x+\mathbb{K}u\|,$$

and that moreover we have $\|x\| = |\lambda(x)| + \|x-\lambda(x)u\|$, so that $\mathbb{R}u$ becomes a semi-L-summand of X.

By a *(semi-)M-ideal* in a normed space X we mean a closed subspace of X whose polar in X' is a (semi-)L-summand of X'. It follows from Fact 2.9.65 that one can build dual numerical-range spaces (X,u) over $\mathbb{K}$ with $n(X,u)=1$ by taking $X := (X_*)'$, where X_* is a Banach space over $\mathbb{K}$ containing a one-codimensional semi-M-ideal M, and taking u equal to any norm-one element of X annihilating M. (We note in passing that, by §2.9.66, in the case $\mathbb{K}=\mathbb{R}$ there are no more such numerical-range spaces.) If, in addition, one wants that $\mathbb{B}_{X_*} \neq |\mathrm{co}|(D^{w^*}(X,u))$ (cf. Corollary 2.9.29 and Example 2.9.31), then it is enough to find X_* with the extra property that there are extreme points of $\mathbb{B}_{X_*}$ which do not belong to $\mathbb{S}_{\mathbb{K}}D^{w^*}(X,u)$.

The philosophy in the above paragraph underlies Example 2.9.67 immediately below. This example is due to Yost [650]. He was not aware at the time he published (but noted shortly afterwards [651]) that Asimow's earlier Example 2.9.31 served the same purpose.

Example 2.9.67 Let A be the so-called *disc algebra*, i.e. the closed subalgebra of $C^{\mathbb{C}}(\mathbb{B}_{\mathbb{C}})$ consisting of those complex-valued continuous functions on $\mathbb{B}_{\mathbb{C}}$ which are holomorphic in its interior, let X_* stand for the closed real subspace of A consisting of those functions $f \in A$ such that $f(1) \in \mathbb{R}$, and let u denote the norm-one element of $X := (X_*)'$ defined by $u(f) := f(1)$ for every $f \in X_*$. We are going to show that $n(X,u)=1$ and that, however, $\mathbb{B}_{X_*} \neq |\mathrm{co}|(D^{w^*}(X,u))$.

By [739, Example 1.4(b)], $M := \{f \in A : f(1)=0\}$ is an M-ideal of A, and this implies that M is an M-ideal of X_*. (Indeed, it is easy to realize that (semi-)M ideals of a complex Banach space Y are (semi-)M ideals of the real Banach space underlying Y, and that (semi-)M ideals of a real or complex Banach space Y remain (semi-)M ideals in any closed subspace of Y containing them.) Therefore, by Fact 2.9.65, we have $n(X,u)=1$.

To show that $\mathbb{B}_{X_*} \neq |\mathrm{co}|(D^{w^*}(X,u))$, define $f \in M$ by $f(z) := w(w(z)^{\frac{1}{2}})$, where

$$w(z) := \frac{i-z}{1-iz},$$

and note that f maps $\mathbb{B}_{\mathbb{C}}$ onto $\{z \in \mathbb{C} : |z| \leqslant 1 \text{ and } \Re(z) \leqslant 0\}$, and that the arc $\{z \in \mathbb{C} : |z|=1 \text{ and } \Re(z) \leqslant 0\}$ is mapped onto itself. It follows [747, pp. 138–9] that f is an extreme point of $\mathbb{B}_A$. Now f is an extreme point of $\mathbb{B}_{X_*}$ and, clearly, f does not belong to $D^{w^*}(X,u) \cup -D^{w^*}(X,u)$. These conditions imply that $f \notin |\mathrm{co}|(D^{w^*}(X,u))$.

We note that the dual real Banach space X in the above example can be converted into the normed space underlying a norm-unital complete normed associative and commutative algebra in such a way that the distinguished element u in the example becomes the unit. Indeed, according to the argument in the example, we can write $X = \mathbb{R}u \oplus^{\ell_1} L$ for a suitable subspace L of X, and then we can regard X as the normed unital extension of L endowed with the zero product (cf. Proposition 1.1.107). On the other hand, looking at the argument in Example 2.9.67, we realize that, if X now stands for the dual of the disc algebra, and if $u \in \mathbb{S}_X$ is defined by $u(f) := f(1)$ for every f in the algebra, then we are provided with a new example of a dual complex numerical-range space (X,u) such that $n(X,u)=1$ and $\mathbb{B}_{X_*} \neq |\mathrm{co}|(D^{w^*}(X,u))$ (cf. §2.9.30).

Now, let u be a nonzero element in a normed space X over $\mathbb{K}$, and note that X can be equivalently renormed in such a way that, in the new norm, u has norm one, and $\mathbb{K}u$ becomes an L-summand of X. It follows from Theorem 2.1.17(i) and Facts 2.9.2 and 2.9.65 that, up to such a renorming, u becomes both a geometrically unitary element and a point of norm-norm upper-semicontinuity for the duality mapping (compare Proposition 2.1 of [293], and the comments following it).

Let X be a real Banach space with a (possibly incomplete) predual X_*, and let u be in $\mathbb{S}_X$. If $n(X,u)=1$, then, by Fact 2.9.62, *we have*

$$D^{w^*}(X,x) \subseteq D^{w^*}(X,u) \ \ \textit{whenever } x \textit{ is in } \mathbb{S}_X \ \ \textit{with} \ \ \|x-u\|<2,$$

and hence the pre-duality mapping of X is norm-τ usc at u for every vector space topology τ on X_*.

Now *let X be a Banach space over $\mathbb{K}$ with a complete predual, and let u be in $\mathbb{S}_X$ such that $\mathbb{K}u$ is a semi-L-summand of X. It follows* from Facts 2.9.2 and 2.9.65, together with Theorems 2.1.17(v) and 2.9.17(ii), *that the pre-duality mapping of X, as well as the pre-duality mapping of any dual of X of even order, is norm-weak usc at u* [220, Proposition 2].

Now we exhibit natural examples of w^*-vertices and w^*-unitaries relative to incomplete preduals.

Example 2.9.68 Let X be a complex Banach space. According to Corollary 2.9.32, $I_{X'}$ is a w^*-unitary element of $BL(X')$ relative to any complete predual, in particular relative to the complete projective tensor product $X\hat{\otimes}_\pi X'$ (the duality being determined by $\langle T, x\otimes f\rangle = T(f)(x)$ for all $T\in BL(X'), x\in X$, and $f\in X'$) [717, Proposition on p. 27]. It is worth mentioning that, thanks to Corollaries 2.1.13 and 2.1.34, $I_{X'}$ is a w^*-vertex of the closed unit ball of $BL(X')$ relative to the (possibly incomplete) predual $X\otimes_\pi X'$. Moreover, if X is actually a complex Hilbert space, then $I_{X'}$ is in fact a w^*-unitary element of $BL(X')$ relative to the predual $X\otimes_\pi X'$. For in this case, denoting by $x\to\bar{x}$ the natural identification $X\equiv X'$, we have $x\otimes\bar{x}\in D^{w^*}(BL(X'),I_{X'})$ whenever x is in $\mathbb{S}_X$, and then the polarization law for the sesquilinear mapping $(x,y)\to x\otimes\bar{y}$ gives that $X\otimes_\pi X'$ is contained in the linear hull of $D^{w^*}(BL(X'),I_{X'})$.

Corollary 2.6.14 and Theorem 2.9.40, as well as the refinement of Theorem 2.9.40 formulated in the review of [70] above, provide us with several multiplicative characterizations of real Hilbert spaces but not of complex ones. In order to get multiplicative characterizations of both real and complex Hilbert spaces, we introduce the following concept. We say that an element u of a Banach space X over $\mathbb{K}$ *acts weakly as a unit* on X whenever u belongs to $\mathbb{S}_X$ and there exists a Banach space Y over $\mathbb{K}$ containing X isometrically, together with a norm-one bounded bilinear mapping $f: X\times X\to Y$, such that the equality $f(u,x)=f(x,u)=x$ holds for every x in X.

Lemma 2.9.69 *For a norm-one element u of a Banach space X over $\mathbb{K}$, the following conditions are equivalent:*

(i) *u acts weakly as a unit on X.*
(ii) *The equality $\|u\otimes x+x\otimes u\|_\pi = 2\|x\|$ holds for every x in X, where $\|\cdot\|_\pi$ means projective tensor norm.*

(iii) *There exist a Banach space Y over $\mathbb{K}$ and a norm-one bounded symmetric bilinear mapping $f : X \times X \to Y$ such that the equality $\|f(u,x)\| = \|x\|$ holds for every x in X.*

Proof (i)⇒(ii) Let Y be a Banach space over $\mathbb{K}$ containing X isometrically and such that there exists a norm-one bounded bilinear mapping $f : X \times X \to Y$ satisfying $f(u,x) = f(x,u) = x$ for every x in X. Then there is a norm-one bounded linear mapping h from the projective tensor product $X \otimes_\pi X$ into Y such that

$$f(x_1,x_2) = h(x_1 \otimes x_2) \text{ for all } x_1,x_2 \text{ in } X.$$

Therefore, for x in X we obtain

$$2\|x\| = \|f(u,x) + f(x,u)\| = \|h(u \otimes x + x \otimes u)\| \leqslant \|u \otimes x + x \otimes u\|_\pi.$$

(ii)⇒(iii) If (ii) is true, then (iii) follows with Y equal to the complete projective tensor product $X \hat{\otimes}_\pi X$, and $f(x_1,x_2) := \frac{1}{2}(x_1 \otimes x_2 + x_2 \otimes x_1)$.

(iii)⇒(i) Assume that (iii) holds. Then the mapping $x \to \hat{x} := f(u,x)$ from X to Y is a linear isometry, and the mapping $\hat{f} : \hat{X} \times \hat{X} \to Y$ defined by $\hat{f}(\hat{x}_1,\hat{x}_2) := f(x_1,x_2)$ is bilinear and bounded with $\|\hat{f}\| = 1$, and satisfies $\hat{f}(\hat{u},\hat{x}) = \hat{f}(\hat{x},\hat{u}) = \hat{x}$ for every $\hat{x}$ in $\hat{X}$. □

Now the desired multiplicative characterization of both real and complex Hilbert spaces reads as follows.

Theorem 2.9.70 *For a Banach space X over $\mathbb{K}$, the following conditions are equivalent:*

(i) *X is a Hilbert space.*
(ii) *Every element of $\mathbb{S}_X$ acts weakly as a unit on X.*
(iii) *There is a dense subset of $\mathbb{S}_X$ consisting of elements which act weakly as units on X.*
(iv) *The equality $\|x_1 \otimes x_2 + x_2 \otimes x_1\|_\pi = 2\|x_1\|\,\|x_2\|$ holds for all $x_1,x_2 \in X$, where $\|.\|_\pi$ means projective tensor norm.*
(v) *There exists a Banach space Y over $\mathbb{K}$, together with a symmetric bilinear mapping $f : X \times X \to Y$, such that the equality $\|f(x_1,x_2)\| = \|x_1\|\,\|x_2\|$ holds for all x_1,x_2 in X.*

Proof Keeping in mind Lemma 2.9.69, in the chain of implications

$$\text{(i)}\Rightarrow\text{(ii)}\Rightarrow\text{(iii)}\Rightarrow\text{(iv)}\Rightarrow\text{(v)}\Rightarrow\text{(i)}$$

only the first and last ones merit a proof.

(i)⇒(ii) Assume that (i) holds. If $\mathbb{K} = \mathbb{R}$, then, by Corollary 2.6.14, condition (ii) is fulfilled. Assume additionally that $\mathbb{K} = \mathbb{C}$. Take a conjugation (i.e. a conjugate-linear involutive isometry) $x \to \bar{x}$ on the complex Hilbert space X, for $x,z \in X$ denote by $x \odot z$ the operator on X defined by

$$(x \odot z)(t) := (t|z)x \text{ for every } t \in X,$$

write $(Y,\|.\|_\tau)$ for the complex Banach space of all trace class operators on X [809, Chapter III], and consider the mapping $f : X \times X \to Y$ given by

$$f(x,z) := \frac{1}{2}(x \odot \bar{z} + z \odot \bar{x}).$$

In view of the implication (iii)⇒(i) of Lemma 2.9.69, to realize that condition (ii) in the theorem is fulfilled it is enough to show that the equality $\|f(x,z)\|_\tau = 1$ holds for all $x, z \in \mathbb{S}_X$. Let x, z be linearly independent elements of $\mathbb{S}_X$. Let T stand for the two-dimensional operator $x \odot \bar{z} + z \odot \bar{x}$. Then we have

$$TT^* = x \odot x + (z|x) x \odot z + (x|z) z \odot x + z \odot z,$$

and hence the nonzero eigenvalues of TT^* are $(1 \pm |(x|z)|)^2$. Since $\|T\|_\tau$ is nothing other than the sum of the eigenvalues of $(TT^*)^{\frac{1}{2}}$ [809, Theorem III.2], we obtain $\|T\|_\tau = 2$.

(v)⇒(i) Assume that (v) holds. Then, for x, z in $\mathbb{S}_X$, we have

$$4 = 4\|f(x,z)\| = \|f(x+z, x+z) - f(x-z, x-z)\| \leqslant \|x+z\|^2 + \|x-z\|^2.$$

Therefore, by Schoenberg's theorem [556] (already applied in the proof of Theorem 2.6.41), X is a Hilbert space. □

Combining Theorem 2.9.70 with Lemma 2.7.36, we get the following.

Corollary 2.9.71 *A Banach space over $\mathbb{K}$ is a Hilbert space if and only if it is almost transitive and has some point acting weakly as a unit.*

We do not know if Theorem 2.9.70 remains true when condition (iii) is replaced with the one where there exists a non-nowhere dense subset of $\mathbb{S}_X$ consisting of elements which act weakly as units. We note that, if a Banach space X over $\mathbb{K}$ satisfies condition (v) in Theorem 2.9.70 with $Y = X$, then, by the commutative Urbanik–Wright theorem (Theorem 2.6.41), X has dimension $\leqslant 2$ over $\mathbb{R}$. Lemma 2.9.69 and Theorem 2.9.70 have been taken from [73].

Let X be a normed space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$ such that the norm of X is strongly subdifferentiable at u. Then it is straightforward that the norm of X is uniformly strongly subdifferentiable on the orbit of u under the group of all surjective linear isometries on X. Therefore it is enough to invoke Proposition 2.9.46 to get the following.

Fact 2.9.72 *Let X be an almost transitive Banach space over $\mathbb{K}$ whose norm is strongly subdifferentiable at some point $u \in \mathbb{S}_X$. Then X is uniformly smooth, and hence superreflexive.*

The class (say $\mathscr{J}$) of almost transitive superreflexive Banach spaces was studied first by Finet [263] (see also [722, Corollary IV.5.7]), who showed that all members of $\mathscr{J}$ are uniformly smooth. Therefore Fact 2.9.72 above can be read as that *members of $\mathscr{J}$ are precisely those almost transitive Banach spaces X whose norm is strongly subdifferentiable at some point $u \in \mathbb{S}_X$*. Among the many known characterizations of members of $\mathscr{J}$, our favourite is that of Becerra [72] asserting that *a Banach space X lies in $\mathscr{J}$ if (and only if) the norm of X is Fréchet differentiable at some 'big point' of X* (cf. §3.1.41 below for the notion of a big point). For other characterizations of almost transitive superreflexive Banach spaces, the reader is referred to [73, Theorem 6.8 and Corollary 6.9] and references therein. More recent results along the same lines can be found in [609, 79].

3

Concluding the proof of the non-associative Vidav–Palmer theorem

3.1 Isometries of *JB*-algebras

Introduction In this section, we introduce one of the fundamental notions in our work, namely that of a *JB*-algebra. *JB*-algebras are complete normed Jordan real algebras which behave like the self-adjoint parts of C^*-algebras endowed with the Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$. *JB*-algebras enjoy a deep and complete structure theory which is comprehensively developed in Hanche-Olsen and Størmer [738]. Since we are unable to organize the basic theory of *JB*-algebras in a better way, we limit ourselves to reviewing those results which are needed for our purpose, and complementing the theory in some geometric aspects (originated by the Wright–Youngson paper [643]) which are not covered by the Hanche-Olsen and Størmer book. Thus, in Theorem 3.1.21 we exhibit several purely algebraic characterizations of surjective linear isometries between *JB*-algebras, along the lines of the Kadison–Paterson–Sinclair Theorem 2.2.19, and derive in Corollary 3.1.32 a purely algebraic characterization of bounded linear operators with zero numerical range on a *JB*-algebra, along the lines of the Paterson–Sinclair Proposition 2.2.26. Then we compute in Corollary 3.1.36 the spatial numerical index of a *JB*-algebra, and prove in Corollary 3.1.51 that *JB*-algebras whose Banach space is convex-transitive are associative.

3.1.1 Isometries of unital *JB*-algebras

JB-algebras are defined as those complete normed Jordan real algebras A satisfying

$$\|x\|^2 \leqslant \|x^2 + y^2\| \tag{3.1.1}$$

for all x, y in A. We note that, if x is any element in a *JB*-algebra, then we have $\|x^2\| = \|x\|^2$ (indeed, take $y = 0$ in (3.1.1)). As mentioned in the introduction, the basic reference for the theory of *JB*-algebras is the book by Hanche-Olsen and Størmer [738].

Lemma 3.1.1 *Let A be a complete V-algebra. Then $H(A,\mathbf{1})$ is a closed real subalgebra of A^{sym} and, endowed with the restriction of the norm of A, becomes a JB-algebra.*

Proof Since the natural involution $*$ of A is a continuous algebra involution (by Lemma 2.3.7(ii) and Theorem 2.3.8), and $H(A,\mathbf{1}) = H(A,*)$, we realize that $H(A,\mathbf{1})$

is indeed a closed real subalgebra of A^{sym}. On the other hand, by Theorem 2.4.11, $H(A,\mathbf{1})$ is a Jordan algebra. Therefore, to conclude the proof it is enough to show that $\|h\|^2 \leqslant \|h^2+k^2\|$ for all $h,k \in H(A,\mathbf{1})$. Let h be in $H(A,\mathbf{1})$, and let B stand for the closed subalgebra of A generated by h and $\mathbf{1}$. By Corollary 2.4.20, B is a unital commutative C^*-algebra. Therefore we have $\|h\|^2 = \|h^2\|$ and, by Gelfand theory (see Theorem 1.1.73) and Proposition 2.3.23, we also have $V(B,\mathbf{1},h^2) \subseteq \mathbb{R}_0^+$, which implies, in view of Corollary 2.1.2, that $V(A,\mathbf{1},h^2) \subseteq \mathbb{R}_0^+$. Now, let k be another hermitian element of A. Since $V(A,\mathbf{1},h^2) \subseteq \mathbb{R}_0^+$ and $V(A,\mathbf{1},k^2) \subseteq \mathbb{R}_0^+$, it follows that $v(h^2) \leqslant v(h^2+k^2)$. Finally, by Proposition 2.3.4, we have

$$\|h\|^2 = \|h^2\| = v(h^2) \leqslant v(h^2+k^2) = \|h^2+k^2\|,$$

as required. □

Corollary 3.1.2 *Let A be a C^*-algebra. Then, the self-adjoint part, $H(A,*)$, of A becomes naturally a JB-algebra.*

Proof If A is unital, then the result follows from Lemmas 2.2.8(iii) and 3.1.1. Otherwise, the result follows from Proposition 1.2.44 and the unital case. □

More examples of JB-algebras can be obtained by considering the so-called *JC-algebras*, namely closed subalgebras of the JB-algebra $H(A,*)$ for some C^*-algebra A.

One of the deepest results in the theory of JB-algebras is the following.

Proposition 3.1.3 [738, 7.2.5] *Let A be a JB-algebra. Then the closed subalgebra of A generated by two elements is a JC-algebra.*

Other relevant results on JB-algebras, needed in what follows, are given in the following.

Proposition 3.1.4 *Let A be a JB-algebra. We have:*

(i) [738, Theorem 3.2.2] *If A is associative, then A is isometrically isomorphic to the JB-algebra $C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E.*
(ii) [738, Proposition 3.4.3] *If B is another JB-algebra, and if $F : A \to B$ is an algebra homomorphism, then F is contractive. Moreover, if the algebra homomorphism F is injective, then F is an isometry.*
(iii) [738, Lemma 1.2.5(ii) and Proposition 3.3.10] *If A is unital, then we have $n(A,\mathbf{1}) = 1$.*

The following corollary follows from Propositions 2.4.13(i) and 3.1.4(i).

Corollary 3.1.5 *Let A be a JB-algebra, and let a be in A. Then the closed subalgebra of A generated by a is isometrically isomorphic to the JB-algebra $C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E. If A is unital, then the closed subalgebra of A generated by a and $\mathbf{1}$ is isometrically isomorphic to the JB-algebra $C^{\mathbb{R}}(E)$ for some compact Hausdorff topological space E.*

It is worth mentioning that assertion (ii) in Proposition 3.1.4 follows easily from assertion (i) in that proposition and Corollary 3.1.5. Indeed, it is enough to restrict the algebra homomorphism F to the closed subalgebra generated by each element, to see this restriction as a mapping with values into the closure of its range (which becomes an associative JB-algebra), and to apply the classical associative fact that assertion (ii) in Proposition 3.1.4 is true when A and B are of the form $C_0^{\mathbb{R}}(E)$ [738, Lemma 1.3.7].

Definition 3.1.6 Let X be a vector space over $\mathbb{K}$, and let S be a subset of X. An element $u \in S$ is said to be a $\mathbb{K}$-*extreme point* of S if, whenever x is in X with $u + \mathbb{B}_{\mathbb{K}} x \subseteq S$, we have $x = 0$. If $\mathbb{K} = \mathbb{R}$, then the notion of a $\mathbb{K}$-extreme point rediscovers the usual one of an extreme point, whereas, if $\mathbb{K} = \mathbb{C}$, then the new notion (of a *complex extreme point*) is weaker than that of an extreme point.

Lemma 3.1.7 *Let E be a locally compact Hausdorff topological space such that there exists a $\mathbb{K}$-extreme point u of the closed unit ball of $C_0^{\mathbb{K}}(E)$. Then E is compact, and $|u(t)| = 1$ for every $t \in E$.*

Proof Let y stand for the element of $C_0^{\mathbb{K}}(E)$ defined by $y(t) := u(t)(1 - |u(t)|)$ for every $t \in E$. Then, for all $t \in E$ and $\lambda \in \mathbb{B}_{\mathbb{K}}$ we have

$$|(u + \lambda y)(t)| \leqslant |u(t)| + |u(t)|(1 - |u(t)|) \leqslant |u(t)| + (1 - |u(t)|) \leqslant 1,$$

and hence $\|u + \lambda y\| \leqslant 1$. Since u is a $\mathbb{K}$-extreme point of $\mathbb{B}_{C_0^{\mathbb{K}}(E)}$, we derive that $y = 0$, and hence that,

$$\text{for every } \; t \in E, \; \text{ we have either } \; u(t) = 0 \; \text{ or } \; |u(t)| = 1. \tag{3.1.2}$$

Set $Z := \{t \in E : u(t) = 0\}$. If Z is empty, then the result follows from (3.1.2). Assume that Z is not empty. Take $t_1 \in Z$, and choose some $v \in \mathbb{S}_{C_0^{\mathbb{R}}(E)}$ such that $v(t_1) \neq 0$. On the other hand, since $Z = \{t \in E : |u(t)| < 1\}$ (by (3.1.2)), Z becomes a non-empty open subset of E, and hence there exists a continuous function h from E to $[0,1]$ vanishing at $E \setminus Z$ and satisfying $h(t_1) = 1$. Setting $w := hv \in C_0^{\mathbb{R}}(E)$, it follows that $\|u + \lambda w\| \leqslant 1$ for every $\lambda \in \mathbb{B}_{\mathbb{K}}$ and that $w(t_1) \neq 0$, which is not possible because u is a $\mathbb{K}$-extreme point of $\mathbb{B}_{C_0^{\mathbb{K}}(E)}$. □

§**3.1.8** Let B and C be JB-algebras. Then, clearly, the algebra direct product $A := B \times C$ becomes a JB-algebra under the sup norm.

An element u of a unital JB-algebra is said to be a *symmetry* if $u^2 = \mathbf{1}$. We note that symmetries of unital JB-algebras are norm-one elements.

Proposition 3.1.9 *Let A be a nonzero JB-algebra. Then A is unital if and only if the closed unit ball of A has extreme points. Moreover, if A is unital, for an element $u \in A$ the following conditions are equivalent:*

(i) *u is a strongly extreme point of $\mathbb{B}_A$.*
(ii) *u is an extreme point of $\mathbb{B}_A$.*
(iii) *u is a symmetry in A.*

Proof If A is unital, then, by Corollary 2.1.42, the unit of A becomes an extreme point of $\mathbb{B}_A$. Conversely, assume that $\mathbb{B}_A$ has some extreme point u. By Corollary 3.1.5, the closed subalgebra of A generated by u is equal to $C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E. Therefore, by Lemma 3.1.7, we have

$$ue = u \text{ and } u^2 = e \text{ for some nonzero idempotent } e \in A. \tag{3.1.3}$$

Now note that, by Lemma 2.5.3, $A_1(e)$ and $A_0(e)$ are closed subalgebras of A satisfying $A_1(e)A_0(e) = 0$ and $A_1(e) \cap A_0(e) = 0$, hence $A_1(e)$ and $A_0(e)$ are JB-algebras, and the mapping $(x,y) \to x+y$, from the algebra direct product $A_1(e) \times A_0(e)$ to A, becomes an injective algebra homomorphism. It follows from §3.1.8 and Proposition 3.1.4(ii) that

$$\|x+y\| = \max\{\|x\|, \|y\|\} \text{ for every } (x,y) \in A_1(e) \times A_0(e).$$

As a consequence, since u lies in $A_1(e)$, we have $\|u \pm y\| = 1$ for every y in the closed unit ball of $A_0(e)$. Since u is an extreme point of $\mathbb{B}_A$, we derive that $A_0(e) = 0$. Now, by applying Lemma 2.5.4, for each $z \in A_{\frac{1}{2}}(e)$ we have $z^3 = 0$, so $\|z\|^4 = \|z^4\| = 0$, and so $A_{\frac{1}{2}}(e) = 0$. Thus $A = A_1(e)$, so that e is a unit for A. Thus the first conclusion in the proposition has been proved. Moreover, regarding (3.1.3), the implication (ii)⇒(iii) has been also proved.

Since the implication (i)⇒(ii) is clear, we are going to conclude with the proof of the implication (iii)⇒(i).

Let u be a symmetry in A. Let $\varepsilon > 0$, and let a be in $\varepsilon\mathbb{S}_A$. By Proposition 3.1.2, the closed subalgebra of A (say B) generated by $\{u,a\}$ is a JC-algebra. Therefore there exists a C^*-algebra C such that B is a closed subalgebra of the JB-algebra $H(C,*)$, and hence, denoting by juxtaposition the product of C and by $\bullet$ the product of B, we have

$$\tfrac{1}{2}(xyz + zyx) = x \bullet (y \bullet z) + z \bullet (y \bullet x) - (x \bullet z) \bullet y \in B \text{ for all } x,y,z \in B.$$

It follows that the normed space of B becomes a norm-unital normed algebra with unit u under the product $x \odot y := \frac{1}{2}(xuy + yux)$. Now, by Corollary 2.1.42 with $(B, \odot)$ instead of A, we have $\max_{\pm} \|u \pm a\| - 1 \geqslant \sqrt{1 + e^{-2}\varepsilon^2} - 1$. Taking infimum in $a \in \varepsilon\mathbb{S}_A$, we derive $\delta_A(u,\varepsilon) \geqslant \sqrt{1 + e^{-2}\varepsilon^2} - 1$. Since ε is an arbitrary positive number, we derive from Lemma 2.1.40 that u is a strongly extreme point of $\mathbb{B}_A$. Thus, the implication (iii)⇒(i) has been proved. □

It is straightforward that the unital extension of a Jordan algebra is a Jordan algebra. This fact will be applied without notice throughout our work, and underlies the proof of the following.

Proposition 3.1.10 [738, Theorem 4.4.3] *Let A be a JB-algebra. Then A'', endowed with the Arens product, becomes a unital JB-algebra.*

Corollary 3.1.11 *Let A be a JB-algebra. Then there is a unique norm on the unital extension $A_{\mathbb{1}}$ of A extending the norm of A and converting $A_{\mathbb{1}}$ into a JB-algebra.*

Proof The uniqueness of the desired norm on $A_{\mathbb{1}}$ follows from Proposition 3.1.4(ii). To prove the existence we distinguish two cases, depending on whether or not A has a unit.

First assume that A has a unit $\mathbf{1}$. Then A and $\mathbb{R}(\mathbb{1}-\mathbf{1})$ are ideals of $A_{\mathbb{1}}$ such that $A_{\mathbb{1}} = A \oplus \mathbb{R}(\mathbb{1}-\mathbf{1})$, and therefore it is enough to apply §§1.1.105 and 3.1.8 to realize that the desired extended norm on $A_{\mathbb{1}}$ is given by

$$\|a+\lambda(\mathbb{1}-\mathbf{1})\| := \max\{\|a\|, |\lambda|\}, \tag{3.1.4}$$

and as a result we have $\|\lambda\mathbb{1}+a\| := \max\{\|a+\lambda\mathbf{1}\|, |\lambda|\}$.

Now, assume that A does not have a unit and denote by $\mathbf{1}$ the unit element of A''. Then the mapping $\lambda\mathbb{1}+a \to \lambda\mathbf{1}+a$ from $A_{\mathbb{1}}$ to A'' becomes an injective algebra homomorphism whose range coincides with the unital closed subalgebra of A'' generated by A. Since such a mapping is also isometric on A, by defining

$$\|\lambda\mathbb{1}+a\| := \|\lambda\mathbf{1}+a\|$$

we get the desired extended norm on $A_{\mathbb{1}}$. □

According to the definition of a *JBW*-algebra in [738, 4.1.1], and [738, Theorem 4.4.16], *JBW-algebras* can be introduced as those *JB*-algebras which are dual Banach spaces. As a consequence of the Banach–Alaoglu and Krein–Milman theorems and Proposition 3.1.9, *nonzero JBW-algebras are unital*. Moreover, by [738, Theorem 4.4.16 and Corollary 4.1.6], if A is a *JBW*-algebra, then the (complete) predual of A is unique, and the product of A is separately w^*-continuous.

Proposition 3.1.12 [738, Proposition 4.2.3] *Let A be a JBW-algebra, let a be in A, and let $\varepsilon > 0$. Then there exist pairwise orthogonal idempotents $e_1,\ldots,e_n \in A$ and real numbers $\lambda_1,\ldots,\lambda_n$ such that $\|a-\sum_{i=1}^n \lambda_i e_i\| < \varepsilon$.*

Proposition 3.1.13 *Let A and B be unital JB-algebras, and let $F : A \to B$ be a mapping. Then the following conditions are equivalent:*

(i) *F is a unit-preserving surjective linear isometry.*
(ii) *F is a bijective algebra homomorphism.*

Proof (i)⇒(ii) Assume that condition (i) holds. To prove that condition (ii) is fulfilled, we can consider the bitranspose of F, and keep in mind Proposition 3.1.10, to realize that there is no loss of generality in assuming that A is in fact a *JBW*-algebra. First note that, by Proposition 3.1.9, F preserves symmetries, and that, since the mapping $u \to \frac{1}{2}(u+\mathbf{1})$ is a bijection from the set of all symmetries of A to the set of all idempotents of A, F also preserves idempotents. As a consequence, F preserves orthogonality of idempotents. Indeed, two idempotents $e_1, e_2 \in A$ are orthogonal if and only if e_1+e_2 is an idempotent. Let $e_1,\ldots,e_n$ be pairwise orthogonal idempotents in A, and set $a := \sum_{i=1}^n \lambda_i e_i$. Since $F(e_1),\ldots,F(e_n)$ are pairwise orthogonal idempotents in B, we have

$$F(a)^2 = \sum_{i=1}^n \lambda_i^2 F(e_i) = F\left(\sum_{i=1}^n \lambda_i^2 e_i\right) = F(a^2).$$

Since the set of elements a as above is dense in A (by Proposition 3.1.12), we have in fact $F(a)^2 = F(a^2)$ for every $a \in A$.

(ii)⇒(i) By Proposition 3.1.4(ii). □

3.1.2 Isometries of non-unital *JB*-algebras

The next lemma follows straightforwardly from assertions (i) and (iii) in Lemma 2.5.3.

Lemma 3.1.14 *Let A be a commutative power-associative algebra over $\mathbb{K}$, and let e be an idempotent in A such that $A_{\frac{1}{2}}(e) = 0$. Then e is central* (cf. Definition 2.5.31).

Given elements a, b, c in a *JB*-algebra, we set

$$\{abc\} := a(bc) + c(ba) - (ac)b.$$

Proposition 3.1.15 *Let A be a unital JB-algebra, and let u be a norm-one element of A. Then the following conditions are equivalent:*

(i) *u is a central symmetry in A.*
(ii) *The Banach space of A, endowed with the product $x \odot y := \{xuy\}$, becomes a JB-algebra with unit u.*
(iii) *The Banach space of A becomes a JB-algebra with unit u, for some product.*
(iv) $n(A, u) = 1$.
(v) *u is a geometrically unitary element of A.*
(vi) *u is a vertex of the closed unit ball of A.*

Proof The implications (ii)⇒(iii) and (v)⇒(vi) are clear, whereas the one (iv)⇒(v) follows from Theorem 2.1.17(i).

(i)⇒(ii) The assumption (i) implies that $L_u^2 = I_A$, so that L_u is an isometry, and also that $\{xuy\} = u(xy)$ for all $x, y \in A$. With these ideas in mind, it is straightforward that L_u becomes an isometric algebra homomorphism from A onto the algebra obtained from A by replacing its product with the new one $x \odot y := \{xuy\}$. Therefore this last algebra is a *JB*-algebra.

(iii)⇒(iv) By Proposition 3.1.4(iii).

(vi)⇒(i) Assume that (vi) holds. Then by Lemma 2.1.25 and Proposition 3.1.9, u is a symmetry, and hence $e := \frac{1}{2}(u + \mathbf{1})$ is an idempotent. Therefore, in view of Lemma 3.1.14, to prove (i) it is enough to show that $A_{\frac{1}{2}}(e) = 0$. Let x be in $A_{\frac{1}{2}}(e)$. Then, since $u = 2e - \mathbf{1}$, we have $ux = 0$. Therefore, for $r \in \mathbb{R}$, we get $\|u + rx\|^2 = \|(u + rx)^2\| = \|\mathbf{1} + r^2x^2\|$, which, in view of Proposition 2.1.5, implies $\max V(A, u, x) = 0$. Replacing x with $-x$, we derive $V(A, u, x) = 0$. Since u is in fact a vertex of $\mathbb{B}_A$, we conclude that $x = 0$, as required. □

We note that conditions (iii)–(vi) in Proposition 3.1.15 become Banach space characterizations of central symmetries in unital *JB*-algebras.

Corollary 3.1.16 *Let A and B be unital JB-algebras. Then the linear isometries from A onto B are the mappings of the form $x \to u\Phi(x)$ where u is a central symmetry in B and Φ is a bijective algebra homomorphism from A to B.*

Proof Let $F : A \to B$ be a surjective linear isometry. By Proposition 3.1.15, $u := F(\mathbf{1})$ is a central symmetry in B. Therefore the mapping $\Phi : A \to B$ given by

$$\Phi(x) := uF(x) \text{ for every} x \in A$$

is a unit-preserving surjective linear isometry, hence by Proposition 3.1.13 it is a bijective algebra homomorphism. □

Now we are going to obtain a non-unital version of Corollary 3.1.16.

Lemma 3.1.17 *Let A and B be normed algebras over $\mathbb{K}$, and let $F : A \to B$ be a continuous algebra homomorphism. Then $F'' : A'' \to B''$ is an algebra homomorphism when A'' and B'' are endowed with their respective Arens products.*

Proof Let x be in A. Then, by Lemma 2.2.12, the set

$$\{y'' \in A'' : F''(xy'') = F(x)F''(y'')\}$$

is a w^*-closed subset of A'' containing A. Therefore, since A is w^*-dense in A'', we have $F''(xy'') = F(x)F''(y'')$ for every $y'' \in A''$.

Now let y'' be in A''. Again by Lemma 2.2.12, the set

$$\{x'' \in A'' : F''(x''y'') = F''(x'')F''(y'')\}$$

is a w^*-closed subset of A'' which, by the above paragraph, contains A. Therefore we have $F''(x''y'') = F''(x'')F''(y'')$ for every $x'' \in A''$. □

From now on, Proposition 3.1.10 will be applied without notice.

Proposition 3.1.18 *Let A be a JB-algebra. Then the set*

$$M(A) := \{a'' \in A'' : a''A \subseteq A\} \tag{3.1.5}$$

becomes a closed subalgebra of A'' containing the unit of A'', and containing A as an ideal.

Proof $M(A)$ is clearly a norm-closed subspace of A'' containing the unit of A''. Let b be in $M(A)$. Taking $c = b$ in the identity (2.4.1) in the proof of Proposition 2.4.13, we get

$$b^2a^2 = b(ba^2) - 2(ab)^2 + 2a(b(ba)) \ \text{ for every } \ a \in A''.$$

Therefore, if a is an element of A, then b^2a^2 is an element of A. Since the set of squares of elements of A linearly generates A (a consequence of Corollary 3.1.5), it follows that b^2 is an element of $M(A)$. Since b is arbitrary in $M(A)$, this implies that $M(A)$ is a subalgebra of A''. Now, clearly, A is an ideal of $M(A)$. □

The set $M(A)$ above is called the *JB-algebra of multipliers* of A. We note that, by Corollary 2.2.13, if A has a unit, then $M(A) = A$.

Lemma 3.1.19 *Let A and B be JB-algebras. Then every linear isometry from A onto B extends to a linear isometry from $M(A)$ onto $M(B)$.*

Proof Let $F : A \to B$ be a surjective linear isometry. By Proposition 3.1.10 and Corollary 3.1.16, there are a central symmetry $u \in B''$ and a bijective algebra homomorphism $\Phi : A'' \to B''$ such that $F''(x) = u\Phi(x)$ for every $x \in A''$. As a consequence we have $F''(x(yz)) = F''(x)(F''(y)F''(z))$ for all $x, y, z \in A''$. Let x and y be in $M(A)$ and A, respectively. Then xy^2 lies in A, and therefore

$$F''(x)(F(y))^2 = F''(x)(F''(y))^2 = F''(xy^2) = F(xy^2) \in B.$$

It follows that $F''(x)b^2$ lies in B for every $b \in B$. Since B is the linear hull of the set $\{b^2 : b \in B\}$ (a consequence of Corollary 3.1.5), we have $F''(x)B \subseteq B$, hence $F''(x) \in M(B)$. In this way we have proved that $F''(M(A)) \subseteq M(B)$. By symmetry, we actually have $F''(M(A)) = M(B)$, and the mapping $x \to F''(x)$ from $M(A)$ to $M(B)$ is a surjective linear isometry extending F. □

Remark 3.1.20 Let A be a JB-algebra, let x be in A, and let T_x denote the bounded linear operator on A given by $T_x(y) := \frac{1}{3}(2x(xy) + x^2 y)$. Then for every natural number n we have

$$\|x\|^{2n+1} = \|x^{2n+1}\| = \|T_x^n(x)\| \leqslant \|T_x^n\|\|x\| \leqslant \|T_x\|^n\|x\| \leqslant \|x\|^{2n+1},$$

and hence $\|T_x^n\| = \|x\|^{2n}$. Therefore $\mathfrak{r}(T_x) = \|x\|^2$.

Theorem 3.1.21 *Let A and B be JB-algebras, and let $F : A \to B$ be a (non necessarily continuous) bijective linear mapping. Then the following conditions are equivalent:*

(i) *F is an isometry.*
(ii) *There are a central symmetry $u \in M(B)$ and a bijective algebra homomorphism $\Phi : A \to B$ such that $F(x) = u\Phi(x)$ for every $x \in A$.*
(iii) *There are two mutually complementary ideals P and Q of A such that $F_{|P}$ and $-F_{|Q}$ are (possibly non-surjective) injective algebra homomorphisms.*
(iv) *$F(x(yz)) = F(x)(F(y)F(z))$ for all $x, y, z \in A$.*
(v) *$F(\{xyz\}) = \{F(x)F(y)F(z)\}$ for all $x, y, z \in A$.*
(vi) *$F(\{xxz\}) = \{F(x)F(x)F(z)\}$ for all $x, z \in A$.*
(vii) *$F(x^3) = (F(x))^3$ for every $x \in A$.*
(viii) *$F(\langle xyz\rangle) = \langle F(x)F(y)F(z)\rangle$ for all $x, y, z \in A$, where for u, v, w, all either in A or in B, $\langle uvw\rangle := \frac{1}{3}(u(vw) + v(uw) + w(uv))$.*
(ix) *$F(\langle xxy\rangle) = \langle F(x)F(x)F(y)\rangle$ for all $x, y \in A$.*

Proof The implications (iv)⇒(v)⇒(vi)⇒(vii) and the one (viii)⇒(ix) are clear.

(i)⇒(ii) Assume that condition (i) holds. Then, by Lemma 3.1.19, F extends to a surjective linear isometry $G : M(A) \to M(B)$, and, by Corollary 3.1.16, there are a central symmetry $u \in M(B)$ and a bijective algebra homomorphism $\Psi : M(A) \to M(B)$ such that $G(x) = u\Psi(x)$ for every $x \in M(A)$. The proof is concluded by observing the almost obvious equality $\Psi(A) = B$.

(ii)⇒(iii) Assume that (ii) holds. Then, by Proposition 3.1.4(ii), Lemma 3.1.17, and the definition of $M(\cdot)$ given by (3.1.5), Φ extends to a bijective algebra homomorphism $\Psi : M(A) \to M(B)$. Now $v := \Psi^{-1}(u)$ is a central symmetry in $M(A)$ such that $F(x) = \Phi(vx)$ for every $x \in A$. Set $e := \frac{1}{2}(\mathbf{1} + v)$. Then e is a central idempotent in $M(A)$, and hence eA and $(1-e)A$ are mutually complementary ideals in A, and we easily see that $F_{|eA} = \Phi_{|eA}$ and $F_{|(\mathbf{1}-e)A} = -\Phi_{|(\mathbf{1}-e)A}$.

(iii)⇒(iv) Assume that (iii) holds. Then, by observing that $F(P)$ and $F(Q)$ are mutually complementary ideals of B, the result follows after a straightforward calculation.

(vii)⇒(viii) Note that the mapping $\langle \cdot\cdot\cdot\rangle$ is symmetric trilinear with $\langle xxx\rangle = x^3$.

(ix)⇒(i) With the notation in Remark 3.1.20, we have $T_u(v) = \langle uuv\rangle$ with u, v both either in A or in B, hence, if we assume (ix), then the equality $FT_xF^{-1} = T_{F(x)}$

holds for every $x \in A$. But the mapping $S \to FSF^{-1}$ from $L(A)$ to $L(B)$ is a bijective algebra homomorphism. By Corollary 1.1.102, we have $\mathfrak{r}(T_{F(x)}) = \mathfrak{r}(T_x)$ and then, by Remark 3.1.20, $\|F(x)\| = \|x\|$ for every $x \in A$. □

As a direct consequence of the implication (i)⇒(ii) in the above theorem, we get the following.

Corollary 3.1.22 *Linearly isometric JB-algebras are (isometrically) isomorphic.*

3.1.3 A metric characterization of derivations of *JB*-algebras

Now we are going to obtain a purely Banach space characterization of derivations of a *JB*-algebra.

Lemma 3.1.23 *Let A be a Jordan algebra, and let a, b be in A. Then the mapping $c \to [b, c, a]$ is a derivation of A.*

Proof Interchanging a and b in the identity (2.4.3) in the proof of Proposition 2.4.13, and subtracting, we obtain

$$[[L_a, L_b], L_c] = L_{[b,c,a]} = L_{[L_a,L_b](c)} \text{ for all } a, b, c \in A.$$

Therefore, by Fact 2.4.7, $[L_a, L_b]$ is a derivation of A. □

Proposition 3.1.24 *A symmetry in a unital JB-algebra A is central if and only if it is an isolated point of the set of all symmetries of A.*

Proof Let S stand for the set of all symmetries in A. If a belongs to S, then $(\mathbf{1} - a)^2 = 2(\mathbf{1} - a)$, and hence $\|\mathbf{1} - a\| = 2$ whenever $a \neq \mathbf{1}$. Therefore $\mathbf{1}$ is an isolated point of S. But, if a is a central symmetry in A, then $\phi : x \to ax$ is a homeomorphism of S onto S with $\phi(\mathbf{1}) = a$. It follows that all central symmetries in A are isolated points of S.

Conversely, let a be an isolated point of S. Then, by Lemma 2.2.21, for every continuous derivation D of A and every real number λ, $\exp(\lambda D)(a)$ lies in S. Since the mapping $f : \lambda \to \exp(\lambda D)(a)$ is continuous with $f(0) = a$, and a is an isolated point of S, it follows the existence of a neighbourhood of zero in $\mathbb{R}$ on which f is constant, and hence $D(a) = f'(0) = 0$. In particular, taking the derivation given by $D(x) := [a, x, y]$ for all $x \in A$, where $y \in A$ (see Lemma 3.1.23), we have $[a, a, y] = 0$ for every $y \in A$. Therefore $[e, e, y] = 0$ for every $y \in A$, where $e := \frac{1}{2}(a + \mathbf{1})$ is an idempotent in A. Thus we have $ey = e(ey)$ for every $y \in A$, which implies $A_{\frac{1}{2}}(e) = 0$. By Lemma 3.1.14, e (hence also a) is central. □

Remark 3.1.25 By combining Propositions 3.1.9 and 3.1.24, we obtain two new Banach space characterizations of central symmetries in a unital *JB*-algebra A. Indeed, *an element $u \in A$ is a central symmetry if and only if it is an isolated point of the set of all extreme points of $\mathbb{B}_A$, if and only if it is an isolated point of the set of all strongly extreme points of $\mathbb{B}_A$.*

Proposition 3.1.26 *Let A be a JB-algebra. Then the connected component of the identity in the group of all linear isometries from A onto A consists only of algebra automorphisms.*

Proof Let C denote the connected component of the identity in the group of all linear isometries from A onto A, and let $\mathbf{1}$ be the unit of A''. Then $\{F''(\mathbf{1}) : F \in C\}$ is a connected subset of A''. On the other hand, by Remark 3.1.25, the set $\{F''(\mathbf{1}) : F \in C\}$ has discrete topology. It follows that $F''(\mathbf{1}) = \mathbf{1}$ for every F in C, and the proof is concluded by applying Proposition 3.1.13. □

§**3.1.27** Let A be a JB-algebra. Then, by Corollary 3.1.11, the unital extension $A_{\mathbb{1}}$ of A can be converted into a unital JB-algebra under a unique norm, and, by Proposition 3.1.4(iii), we have $n(A_{\mathbb{1}}, \mathbb{1}) = 1$. Therefore $\mathbb{1}$ is a vertex of $\mathbb{B}_{A_{\mathbb{1}}}$, and hence, without notice, A will be seen endowed with the order induced by the numerical-range order of $(A_{\mathbb{1}}, \mathbb{1})$ as defined in §2.3.34. In the case that A is unital, the passing to the unital extension is unnecessary. Indeed, in this case the numerical-range order of $(A, \mathbf{1})$ coincides with the order induced on A by the numerical-range order of $(A_{\mathbb{1}}, \mathbb{1})$ because, keeping in mind the explicit expression of the JB-norm on $A_{\mathbb{1}}$ given by (3.1.4), and applying Fact 2.9.47, for $a \in A$ we have $V(A_{\mathbb{1}}, \mathbb{1}, a) = \mathrm{co}(V(A, \mathbf{1}, a) \cup \{0\})$, hence $V(A_{\mathbb{1}}, \mathbb{1}, a) \subseteq \mathbb{R}_0^+$ if and only if $V(A, \mathbf{1}, a) \subseteq \mathbb{R}_0^+$. Anyway, if a, b are in A, and if $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$ (by Fact 2.3.36). Formally, the definition of the order in A just given is not that of [738, 3.3.3]. Nevertheless, as a consequence of assertions (i) and (ii) in Lemma 3.1.29 below, the two definitions agree. We say that an element $a \in A$ is *positive* if $a \geqslant 0$.

Claim 3.1.28 *Let A be a JB-algebra, let B be a closed subalgebra of A, and let b be in B. Then b is a positive element of A if and only if b is a positive element of B. As a consequence, in the case that B is associative (say $B = C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E), b is a positive element of A if and only if $b(t) \geqslant 0$ for every $t \in E$.*

Proof Keeping in mind Proposition 3.1.4(ii), we can see the JB-algebra $B_{\mathbb{1}}$ as a unital closed subalgebra of the JB-algebra $A_{\mathbb{1}}$, so that the first conclusion follows from Corollary 2.1.2. Assume that $B = C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E. Then E can be enlarged to a compact Hausdorff topological space F in such a way that $B_{\mathbb{1}} = C^{\mathbb{R}}(F)$ as JB-algebras and $B = \{x \in C^{\mathbb{R}}(F) : x(F \setminus E) = 0\}$. Therefore, since $B_{\mathbb{1}} = H(C^{\mathbb{C}}(F), *)$, the equivalence

$$b \geqslant 0 \iff b(t) \geqslant 0 \text{ for every } t \in E$$

follows from Proposition 2.3.38 and Example 1.1.32(c). □

Lemma 3.1.29 *Let A be a JB-algebra. We have:*

(i) *$x^2 \geqslant 0$ for every $x \in A$.*
(ii) *Every positive element $x \in A$ has a unique positive square root in A (denoted by $\sqrt{x}$), and $\sqrt{x}$ lies in the closed subalgebra of A generated by x.*
(iii) *If x, y are in A, and if $x \geqslant 0$, then $U_y(x)$ is positive.*

Proof Assertion (i) follows from Corollary 3.1.5 and Claim 3.1.28.

Let x be a positive element of A. By Corollary 3.1.5 and Claim 3.1.28, there is a positive square root of x (say $\sqrt{x}$) in the closed subalgebra of A (say B) generated by x. Let y be any positive square root of x in A. Then the closed subalgebra of A generated by y (say C) is of the form $C_0^{\mathbb{R}}(E)$ for some locally compact Hausdorff topological space E. Moreover, since x lies in C, we have $B \subseteq C$. It follows that $\sqrt{x}$ and y are positive square roots of x in $C_0^{\mathbb{R}}(E)$. By applying Claim 3.1.28 again, we derive that $y = \sqrt{x}$. Thus assertion (ii) has been proved.

It follows from assertions (i) and (ii) that an element of A is positive if and only if it is a square. Therefore, by [738, 3.3.3], our definition of positive elements is in agreement with that of [738]. As a consequence, assertion (iii) follows from [738, Proposition 3.3.6]. □

Proposition 3.1.30 *Let A be a unital JB-algebra, and let $T : A \to A$ be a linear mapping satisfying*

$$T(x^2) + U_x(T(\mathbf{1})) - 2xT(x) \geqslant 0 \tag{3.1.6}$$

for every $x \in A$. Then T is continuous.

Proof Note at first that, for every real α, the linear mapping $\alpha I_A + T$ satisfies condition (3.1.6). Therefore we may assume that $T(\mathbf{1}) \leqslant 0$ by replacing T by $-\|T(\mathbf{1})\| I_A + T$, if necessary. Let x be in A. Then, by Proposition 3.1.4(iii), we have $\|x\| = v(x)$. Therefore we may assume in addition that $\|x\| \in V(x)$. (If $-\|x\| \in V(x)$ only, we may replace x by $-x$ in the subsequent argument.) Take $f \in D(A, \mathbf{1})$ such that $f(x) = \|x\|$. Setting $y := \sqrt{\|x\|\mathbf{1} - x}$ we observe that $f(y^2) = 0$. Since $T(\mathbf{1}) \leqslant 0$, we have $T(x) \leqslant -\|x\|T(\mathbf{1}) + T(x) = -T(y^2)$. From this, the hypothesis on T, and Lemma 3.1.29(iii), we conclude that

$$\begin{aligned} f(T(x)) &\leqslant -f(T(y^2)) \leqslant f(U_y(T(\mathbf{1})) - 2yT(y)) \leqslant -2f(yT(y)) \\ &\leqslant 2|f(yT(y))| \leqslant 2\sqrt{f(y^2)}\sqrt{f(T(y)^2)} = 0, \end{aligned}$$

where the last inequality follows from the fact that, thanks to Lemma 3.1.29(i), the mapping $(u, v) \to f(uv)$ becomes a non-negative symmetric bilinear form on A, and the Cauchy–Schwarz inequality applies. Consequently, for each $\varepsilon > 0$ we have

$$\|x\| = f(x) \leqslant f(x - \varepsilon T(x)) \leqslant \|x - \varepsilon T(x)\|.$$

To deduce the boundedness of T from this inequality it suffices, by the closed graph theorem, to take a sequence x_n in A such that $x_n \to 0$ and $T(x_n) \to y$ for some $y \in A$, and to show that $y = 0$. Let $\varepsilon > 0$. Letting $n \to \infty$ in the estimate

$$\|x_n + \varepsilon y\| \leqslant \|x_n + \varepsilon y - \varepsilon T(x_n + \varepsilon y)\|,$$

we obtain $\|\varepsilon y\| \leqslant \|\varepsilon^2 T(y)\|$. Thus, cancelling ε and then letting $\varepsilon \to 0$, we find that $y = 0$ as claimed. As a result, T is bounded. □

Corollary 3.1.31 *Derivations on JB-algebras are continuous.*

Proof Since every derivation of a JB-algebra can be uniquely extended to a derivation of its unital extension, and a derivation T of a unital JB-algebra satisfies condition (3.1.6) (actually with $=$ in place of $\leqslant$), the result follows from Corollary 3.1.11 and Proposition 3.1.30. □

Corollary 3.1.32 *Let A be a JB-algebra, and let $T : A \to A$ be a linear mapping. Then T is a derivation of A if and only if T is bounded with*

$$V(BL(A), I_A, T) = 0.$$

Proof Assume that T is a derivation of A. Then, by Corollary 3.1.31, T is continuous, so, for every $\lambda \in \mathbb{R}$, we may consider $\exp(\lambda T)$ which is an algebra automorphism of A, and so, by Proposition 3.1.4(ii), we have $\|\exp(\lambda T)\| = 1$. By Corollary 2.1.9(ii), we have $V(BL(A), I_A, T) = 0$.

Conversely, assume that T is bounded with $V(BL(A), I_A, T) = 0$. Then, again by Corollary 2.1.9(ii), the mapping $\lambda \to \exp(\lambda T)$ from $\mathbb{R}$ to $BL(A)$ is a continuous one-parameter group of linear isometries on A, and hence, for all $\lambda \in \mathbb{R}$, the operator $\exp(\lambda T)$ must lie in the connected component of the identity in the group of all isometries from A onto A. By Proposition 3.1.26, $\exp(\lambda T)$ is an algebra automorphism, i.e. for $a, b \in A$ we have

$$\exp(\lambda T)(ab) = \exp(\lambda T)(a)\exp(\lambda T)(b).$$

By computing the derivative at $\lambda = 0$, we get that $T(ab) = aT(b) + T(a)b$, i.e. T is a derivation. □

Now we are going to determine the spatial numerical index of JB-algebras and of preduals of JBW-algebras. Given a locally compact Hausdorff topological space E and a normed space X, we denote by $C_0(E,X)$ the normed space of all X-valued continuous functions on E vanishing at infinity.

Lemma 3.1.33 *Let E be a locally compact Hausdorff topological space, let X be a normed space over $\mathbb{K}$, and let T be in $BL(C_0(E,X))$. Then we have*

$$v(T) = \sup\left\{\left|x'\left(T(f)(t)\right)\right| \ : \ f \in \mathbb{S}_{C_0(E,X)},\ t \in E,\ x' \in \mathbb{S}_{X'},\ x'(f(t)) = 1\right\}.$$

Proof Let f be in $\mathbb{S}_{C_0(E,X)}$. Then there are $t \in E$ and $x' \in \mathbb{S}_{X'}$ such that $x'(f(t)) = 1$. Now, consider the element $x' \otimes \delta_t$ of $D(C_0(E,X), f)$ defined by

$$(x' \otimes \delta_t)(g) := x'(g(t)) \text{ for every } g \in C_0(E,X).$$

It follows from the above that the set

$$\left\{(f, x' \otimes \delta_t) : \ f \in \mathbb{S}_{C_0(E,X)},\ t \in E,\ x' \in \mathbb{S}_{X'},\ x'(f(t)) = 1\right\}$$

is a subset of $\Pi(C_0(E,X))$ whose projection into the first coordinate is the whole unit sphere of $C_0(E,X)$. Now the result follows from Proposition 2.1.31. □

Proposition 3.1.34 *Let E be a locally compact Hausdorff topological space, and let X be a normed space over $\mathbb{K}$. Then $N(C_0(E,X)) = N(X)$.*

Proof To show that $N(C_0(E,X)) \geqslant N(X)$, we fix $T \in BL(C_0(E,X))$ with $\|T\| = 1$ and prove that $v(T) \geqslant N(X)$. Given $\varepsilon > 0$, we may find $f_0 \in C_0(E,X)$ with $\|f_0\| = 1$ and $t_0 \in E$ such that

$$\|T(f_0)(t_0)\| > 1 - \varepsilon. \tag{3.1.7}$$

Denote $y_0 = f_0(t_0)$ and find a continuous function with compact support $\varphi : E \to [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ if $\|f_0(t) - y_0\| \geqslant \varepsilon$. Now write $y_0 = \lambda x_1 + (1-\lambda)x_2$ with $0 \leqslant \lambda \leqslant 1$, $x_1, x_2 \in \mathbb{S}_X$, and consider the functions

$$f_j := (1-\varphi)f_0 + \varphi x_j \in C_0(E,X) \qquad (j = 1,2).$$

Then $\|\varphi f_0 - \varphi y_0\| < \varepsilon$ meaning that $\|f_0 - (\lambda f_1 + (1-\lambda)f_2)\| < \varepsilon$, and, using (3.1.7), we must have

$$\|T(f_1)(t_0)\| > 1-2\varepsilon \ \text{ or } \ \|T(f_2)(t_0)\| > 1-2\varepsilon.$$

By making the right choice of $x_0 = x_1$ or $x_0 = x_2$ we get $x_0 \in \mathbb{S}_X$ such that

$$\|T((1-\varphi)f_0 + \varphi x_0)(t_0)\| > 1-2\varepsilon. \tag{3.1.8}$$

Next we fix $x_0' \in \mathbb{S}_{X'}$ with $x_0'(x_0) = 1$, for $x \in X$ denote

$$\Phi(x) := x_0'(x)(1-\varphi)f_0 + \varphi x \in C_0(E,X),$$

and consider the operator $S \in BL(X)$ given by $S(x) := T(\Phi(x))(t_0)$ for every $x \in X$. Since, by (3.1.8),

$$\|S\| \geqslant \|S(x_0)\| > 1-2\varepsilon,$$

we may find $x \in \mathbb{S}_X$ and $x' \in \mathbb{S}_{X'}$ such that

$$x'(x) = 1 \ \text{ and } \ |x'(S(x))| \geqslant N(X)[1-2\varepsilon].$$

Now, define $g \in \mathbb{S}_{C_0(E,X)}$ by $g := \Phi(x)$, for this x, and consider the functional $g' \in \mathbb{S}_{C_0(E,X)'}$ given by $g'(h) := x'(h(t_0))$ for every $h \in C_0(E,X)$. Since $g(t_0) = x$, we have

$$g'(g) = 1 \ \text{ and } \ |g'(T(g))| = |x'(S(x))| \geqslant N(X)[1-2\varepsilon].$$

Hence $v(T) \geqslant N(X)$, as required.

For the reverse inequality, take an operator $S \in BL(X)$ with $\|S\| = 1$, and define $T \in BL(C_0(E,X))$ by $T(f)(t) := S(f(t))$ for all $t \in E$ and $f \in C_0(E,X)$. Then $\|T\| = 1$, so $v(T) \geqslant N(C_0(E,X))$. By Lemma 3.1.33, given $\varepsilon > 0$, we may find $f \in \mathbb{S}_{C_0(E,X)}$, $x' \in \mathbb{S}_{X'}$, and $t \in E$, such that $x'(f(t)) = 1$ and

$$N(C_0(E,X)) - \varepsilon < \left|x'\left(T(f)(t)\right)\right| = |x'(S(f(t)))|.$$

It clearly follows that $v(S) \geqslant N(C_0(E,X))$, so $N(X) \geqslant N(C_0(E,X))$. □

Actually, for our present goal we only need the consequence of Proposition 3.1.34 that, *if E is a locally compact Hausdorff topological space, then $N(C_0^{\mathbb{K}}(E)) = 1$.*

Proposition 3.1.35 *Let A be a JB-algebra. Then the following conditions are equivalent:*

(i) *A is associative.*
(ii) *$N(A) = 1$.*
(iii) *I_A is a geometrically unitary element of $BL(A)$.*
(iv) *I_A is a vertex of the closed unit ball of $BL(A)$.*

Proof (i)⇒(ii) By Propositions 3.1.4(i) and 3.1.34.

(ii)⇒(iii) By Theorem 2.1.17(i).

(iii)⇒(iv) This is clear.

(iv)⇒(i) Assume that condition (i) does not hold. Then, by Lemma 3.1.23, A has nonzero continuous derivations. Therefore, by Corollary 3.1.32, there are nonzero bounded linear operators on A with zero numerical range, i.e. condition (iv) fails. □

As a consequence of the above proposition, we straightforwardly derive the following.

Corollary 3.1.36 *Let A be a JB-algebra. Then $N(A)$ is equal to* 1 *or* 0 *depending on whether or not A is associative.*

It also follows from Corollary 3.1.32 and Proposition 3.1.35 that *a JB-algebra A is associative* (if and) *only if there is no nonzero derivation of A.*

Lemma 3.1.37 *Bijective algebra homomorphisms between JBW-algebras and derivations of JBW-algebras are w^*-continuous.*

Proof Keeping in mind the uniqueness of the predual of a JBW-algebra, the w^*-continuity of bijective algebra homomorphisms follows from the fact that, by Proposition 3.1.4(ii), such mappings are surjective linear isometries.

Let T be a derivation of a JBW-algebra A. Then, by Corollary 3.1.31, T is norm-continuous, so, for every $\lambda \in \mathbb{R}$, we may consider $\exp(\lambda T)$ which is an algebra automorphism of A, and hence a w^*-continuous mapping (by the above paragraph). Since

$$T = \lim_{\lambda \to 0} \frac{\exp(\lambda T) - I_A}{\lambda}$$

in the norm topology of $BL(A)$, we conclude that T is w^*-continuous. □

Proposition 3.1.38 *Let X be the predual of a JBW-algebra. Then the following conditions are equivalent:*

(i) *X' is associative.*
(ii) $N(X) = 1$.
(iii) *I_X is a geometrically unitary element of $BL(X)$.*
(iv) *I_X is a vertex of the closed unit ball of $BL(X)$.*

Proof (i)⇒(ii) By the inequality (2.1.18) and Corollary 3.1.36.

(ii)⇒(iii) By Theorem 2.1.17(i).

(iii)⇒(iv) This is clear.

(iv)⇒(i) Assume that condition (i) does not hold. Then, by Lemma 3.1.23, X' has a nonzero norm-continuous derivation (say F). By Lemma 3.1.37, we have $F = T'$ for some nonzero $T \in BL(X)$, and then, by Corollaries 2.1.3 and 3.1.32, we have $V(BL(X), I_X, T) = 0$. This implies that condition (iv) fails. □

As a consequence of the above proposition, we derive the following.

Corollary 3.1.39 *Let X be the predual of a JBW-algebra. Then $N(X)$ is equal to* 1 *or* 0 *depending on whether or not X' is associative.*

3.1.4 *JB*-algebras whose Banach spaces are convex-transitive

Given a normed space X over $\mathbb{K}$, we denote by $\mathscr{G} = \mathscr{G}(X)$ the group of all surjective linear isometries from X to X.

Corollary 3.1.40 *Let A be a unital JB-algebra. Then $\mathscr{G}(\mathbf{1})$ coincides with the set of all central symmetries of A.*

Proof As a straightforward consequence of the Banach space characterizations of central symmetries of A (given by Proposition 3.1.15 or Remark 3.1.25), we already know that $\mathscr{G}(\mathbf{1})$ consists only of central symmetries of A. Conversely, to realize that central symmetries of A lie in $\mathscr{G}(\mathbf{1})$, it is enough to keep in mind that multiplications by central symmetries of A are elements of $\mathscr{G}$. □

§**3.1.41** An element u in a normed space X over $\mathbb{K}$ is said to be a *big point* of X if $\|u\| = 1$ and $\overline{\mathrm{co}}(\mathscr{G}(u)) = \mathbb{B}_X$. As an illustrative example, by the Russo–Dye theorem (Lemma 2.3.28), *the unit of any unital C^*-algebra A is a big point of A*. As a straightforward consequence of Corollary 3.1.40, we get the following.

Proposition 3.1.42 *Let A be a unital JB-algebra. If the linear hull of $\mathscr{G}(\mathbf{1})$ is dense in A (for instance, if $\mathbf{1}$ is a big point of A), then A is associative.*

Proposition 3.1.43 *Let A be a JBW-algebra. Then the centre of A is the norm-closed linear hull of $\mathscr{G}(\mathbf{1})$. As a consequence, the following conditions are equivalent:*

(i) *The linear hull of $\mathscr{G}(\mathbf{1})$ is w^*-dense in A.*
(ii) *A is associative.*
(iii) *The linear hull of $\mathscr{G}(\mathbf{1})$ is norm-dense in A.*

Proof Keeping in mind that the centre of A is w^*-closed in A (and hence, a *JBW*-algebra) and Corollary 3.1.40, the first conclusion follows from Proposition 3.1.12 and the fact that $\mathbf{1} - 2e$ is a symmetry whenever e is an idempotent in A. Now, the consequence follows from Proposition 3.1.42. □

A normed space X over $\mathbb{K}$ is said to be *convex-transitive* if every element of $\mathbb{S}_X$ is a big point of X.

Proposition 3.1.44 *Let X be the predual of a JBW-algebra. If X has no nontrivial $\mathscr{G}$-invariant closed subspace (for instance, if X is convex-transitive), then X' is associative.*

Proof Assume that X has no nontrivial $\mathscr{G}$-invariant closed subspace. Then X' has no nontrivial $\mathscr{G}'$-invariant w^*-closed subspace, where $\mathscr{G}' := \{T' : T \in \mathscr{G}\}$. As a consequence, the linear hull of $\mathscr{G}(X')(\mathbf{1})$ is w^*-dense in X', and Proposition 3.1.43 applies. □

As a consequence of Proposition 3.1.42, *convex-transitive unital JB-algebras are associative*. Now, we are going to show that this result remains true in the non-unital case.

Lemma 3.1.45 *Let X be a Banach space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$. Then the following conditions are equivalent:*

(i) *u is a big point of X.*
(ii) *For every positive number δ, the set*

$$\Delta_\delta(u) := \{T'(f) : x \in \mathbb{S}_X, \|x-u\| \leqslant \delta, f \in D(X,x), T \in \mathscr{G}\}$$

is dense in $\mathbb{S}_{X'}$.

Proof (i)⇒(ii) We fix $\delta > 0$ and $g \in \mathbb{S}_{X'}$, and take $0 < \varepsilon < 1$. By the assumption (i), there exists $T^{-1} \in \mathscr{G}$ such that $|g(T^{-1}(u)) - 1| < \frac{\varepsilon'^2}{4}$, where $\varepsilon' := \min\{\varepsilon, \delta\}$. By the Bishop–Phelps–Bollobás theorem (cf. Theorem 2.9.7), there are $x \in \mathbb{S}_X$ and $f \in D(X,x)$ satisfying

$$\|u - x\| < \varepsilon' \leqslant \delta \text{ and } \|g \circ T^{-1} - f\| < \varepsilon' \leqslant \varepsilon.$$

This shows that $g \in \overline{\Delta_\delta(u)}$.

(ii)⇒(i) Assume that (i) is not true, so that there is $z \in \mathbb{B}_X \setminus \overline{\text{co}}(\mathscr{G}(u))$. Then, by the Hahn–Banach theorem, there exists $f \in \mathbb{S}_{X'}$ such that

$$1 \geqslant f(z) > \sup\{f(a) : a \in \overline{\text{co}}(\mathscr{G}(u))\}.$$

By the assumption (ii), for n in $\mathbb{N}$, the set $\Delta_{\frac{1}{n}}(u)$ is dense in $\mathbb{S}_{X'}$, and hence there are $x_n \in \mathbb{S}_X$ with $\|u - x_n\| \leqslant \frac{1}{n}$, $g_n \in D(X, x_n)$, and $T_n \in \mathscr{G}$ such that $\|f - g_n \circ T_n\| \leqslant \frac{1}{n}$. In this way we obtain

$$|f(T_n^{-1}(u)) - 1| \leqslant |f(T_n^{-1}(u)) - g_n(u)| + |g_n(u) - g_n(x_n)| \leqslant \frac{2}{n},$$

which implies $\sup\{f(a) : a \in \mathscr{G}(u)\} = 1$, contrarily to the choice of f. □

Let X be a Banach space over $\mathbb{K}$. An element f in X' is said to be a *w^*-superbig point* of X' if f belongs to $\mathbb{S}_{X'}$ and the convex hull of the set $\{F'(f) : F \in \mathscr{G}\}$ is w^*-dense in $\mathbb{B}_{X'}$. Minor changes to the proof of Lemma 3.1.45 allow us to establish the following.

Lemma 3.1.46 *Let X be a Banach space over $\mathbb{K}$, and let f be an element of $\mathbb{S}_{X'}$. Then the following conditions are equivalent:*

(i) *f is a w^*-superbig point of X'.*
(ii) *For every positive number δ, the set*

$$\Delta'_\delta(f) := \{T(x) : g \in \mathbb{S}_{X'}, \|f - g\| \leqslant \delta, x \in D(X', g) \cap X, T \in \mathscr{G}\}$$

is dense in $\mathbb{S}_X$.

The following corollary follows from Fact 2.9.62 and Lemmas 3.1.45 and 3.1.46.

Corollary 3.1.47 *Let (X, e) be a complete real numerical-range space with $n(X, e) = 1$. Then e is a big point of X if and only if the set*

$$\Delta_0(e) := \{T'(f) : f \in D(X, e), T \in \mathscr{G}\}$$

is dense in $\mathbb{S}_{X'}$. If moreover X is a dual space (with predual X_* say), then e is a w^*-superbig point of X if and only if the set

$$\Delta'_0(e) := \{T(y) : y \in D(X, e) \cap X_*, T \in \mathscr{G}(X_*)\}$$

is dense in $\mathbb{S}_{X_*}$.

Proposition 3.1.48 *Let (X, e) be a complete (respectively, dual) real numerical-range space with $n(X, e) = 1$ such that there exists some big (respectively, w^*-superbig) point u in X satisfying $\|e - u\| < 2$. Then e is a big (respectively, w^*-superbig) point of X.*

Proof Take $0 < \delta < 2 - \|e - u\|$. Then we have $\|x - e\| < 2$ whenever $x \in \mathbb{S}_X$ and $\|x - u\| < \delta$. Therefore, by Fact 2.9.62, we have $\Delta_\delta(u) \subseteq \Delta_0(e)$ (respectively, $\Delta'_\delta(u) \subseteq \Delta'_0(e)$). Since u is a big (respectively, w^*-superbig) point of X, it follows from Lemma 3.1.45 (respectively 3.1.46) and Corollary 3.1.47 that e is a big (respectively, w^*-superbig) point of X. □

Now the next proposition follows from Propositions 3.1.42 and 3.1.48, and the fact that, if A is a unital JB-algebra, then, by Proposition 3.1.4(iii), we have $n(A, \mathbf{1}) = 1$.

Proposition 3.1.49 *Let A be a unital JB-algebra. If there is some big point u of A satisfying $\|\mathbf{1} - u\| < 2$, then A is associative.*

Theorem 3.1.50 *Let A be a JB-algebra such that there are a big point u of A and a positive element p in $2\mathbb{B}_A$ satisfying $\|p - u\| < 1$. Then A is associative.*

Proof We know that A'' is a unital JB-algebra containing A as a subalgebra, and that, consequently, we have $n(A'', \mathbf{1}) = 1$. On the other hand, since p is a positive element in $2\mathbb{B}_A$, we have $\|\mathbf{1} - p\| \leqslant 1$, and hence $\|\mathbf{1} - u\| < 2$ (because $\|p - u\| < 1$). Moreover, u is a w^*-superbig point of A'' (because u is a big point of A). It follows from Proposition 3.1.48 that $\mathbf{1}$ is a w^*-superbig point of A''. Finally, by Proposition 3.1.43, A'' (and hence A) is associative. □

Corollary 3.1.51 *Let A be a nonzero JB-algebra. If some positive element in A is a big point of A, then A is associative. As a consequence, if (the Banach space of) A is convex-transitive, then A is associative.*

Proof The first conclusion follows straightforwardly from Theorem 3.1.50. Assume that A is convex-transitive. Take a norm-one element $x \in A$. Then, by Lemma 3.1.29, x^2 becomes both a positive element and a big point of A, and the first conclusion applies. □

Now, recalling that, by Corollary 3.1.2, the self-adjoint part of any C^*-algebra becomes a JB-algebra in a natural way, some results proved in this section for JB-algebras reflect into results for C^*-algebras. Thus, as a by-product of Corollary 3.1.31, we can now give a proof of Lemma 2.2.25: that *derivations of C^*-algebras are continuous*.

§**3.1.52** *Proof of Lemma 2.2.25* Let A be a C^*-algebra, and let D be a derivation of A. Since D can be written as $D = D_1 + iD_2$ with D_j a derivation of A satisfying $D_j(a^*) = D_j(a)^*$ for every $a \in A$, we can assume that $D(a^*) = D(a)^*$ for every $a \in A$. Then we have $D(H(A,*)) \subseteq H(A,*)$. Since $H(A,*)$ is a JB-algebra (by Corollary 3.1.2), and D, regarded as an operator on $H(A,*)$, becomes a derivation, it follows from Corollary 3.1.31 that D is continuous on $H(A,*)$. Therefore, since the direct sum $A = H(A,*) \oplus iH(A,*)$ is topological, we derive that D, as an operator on A, is continuous. □

Now, consider the following.

Lemma 3.1.53 *Let A be a C^*-algebra. Then the following conditions are equivalent:*

(i) *A is commutative.*
(ii) *The JB-algebra $H(A,*)$ is associative.*
(iii) *The equality $[a,[b,c]] = 0$ holds for all $a,b,c \in A$.*
(iv) *The equality $[a,[a,b]] = 0$ holds for all $a,b \in A$.*

Proof The implications (i)⇒(ii) and (iii)⇒(iv) are clear.

(ii)⇒(iii) Note that, for $a,b,c \in A$, we straightforwardly have

$$4[a,[b,c]] = [b,a,c]^{\mathrm{sym}}, \tag{3.1.9}$$

where $[\cdot,\cdot,\cdot]^{\mathrm{sym}}$ stands for the associator in A^{sym}. Therefore, if condition (ii) is fulfilled, we have $[a,[b,c]] = 0$ whenever a,b,c are in $H(A,*)$, and hence, since $A = H(A,*) + iH(A,*)$, condition (iii) holds.

(iv)⇒(i) Assume that condition (iv) is fulfilled. Since condition (iv) passes to the unital extension of A, in order to prove (i) we can invoke Proposition 1.2.44, and assume without loss of generality that A is unital. Then, by Lemma 2.2.5 and Corollary 2.4.3, we have $[a,b] = 0$ whenever a lies in $H(A,*)$ and b is in A. Since $A = H(A,*) + iH(A,*)$, we conclude that condition (i) holds. □

In view of the above lemma, results involving associativity of JB-algebras (like Propositions 3.1.35, 3.1.38, 3.1.42, 3.1.43, 3.1.44, and 3.1.49; Corollaries 3.1.36, 3.1.39, and 3.1.51; and Theorem 3.1.50) reflect into results involving commutativity of C^*-algebras. As a sample, we formulate the following.

Corollary 3.1.54 *Let A be a C^*-algebra. We have:*

(i) *$N(H(A,*))$ is equal to 1 or 0 depending on whether or not A is commutative.*
(ii) *If the Banach space of $H(A,*)$ is convex-transitive, then A is commutative.*

3.1.5 Historical notes and comments

JC-algebras were first considered and studied by Topping [813] and Størmer [601]. The foundational papers for JB-algebras are those of Alfsen, Shultz, and Størmer [15] (1978) and Shultz [570] (1979), where the theory of unital JB-algebras and JBW-algebras is developed in depth. In particular, Proposition 3.1.12 and the unital forerunners of Propositions 3.1.3, 3.1.4, and 3.1.10 can be found in these papers. The extension of the theory to the non-unital case could be done thanks to Behncke's theorem [82] (1979) that *the unital extension of a non-unital JB-algebra becomes a JB-algebra for a suitable norm*. On the other hand, Hanche-Olsen [308] (1980) achieved a new simpler proof of the unital forerunner of Proposition 3.1.10. All the material roughly reviewed above has been fully re-elaborated in Hanche-Olsen and Størmer [738], which today is the standard reference for the theory.

Lemma 3.1.1 was first proved by Youngson [652] under the additional assumptions (unnecessary today, in view of Theorems 2.3.8 and 2.4.11) that the natural involution of the V-algebra under consideration is an algebra involution, and that the

V-algebra is a non-commutative Jordan algebra. Corollary 3.1.2 is obvious for any C^*-algebraist and, as mentioned in the introduction, becomes the motivation for the theory of JB-algebras.

A Jordan algebra is said to be *special* if it is isomorphic to a subalgebra of A^{sym} for some associative algebra A. A deep and useful result in the theory of Jordan algebras is the following theorem, due to Shirshov [567] and Cohn [178] (see also [738, 2.4.14]).

Theorem 3.1.55 *Any Jordan algebra generated by two elements is special.*

Thus Proposition 3.1.3 becomes a deep analytic variant of the Shirshov–Cohn theorem. A fundamental example of a JB-algebra which is not a JC-algebra is the following.

Example 3.1.56 Consider the algebra $M_3(\mathbb{O})$, of all 3×3 matrices with entries in the algebra $\mathbb{O}$ of Cayley numbers, and endow it with the algebra involution $*$ consisting of transposing the matrix and taking standard involution in each entry. Then, clearly, $H_3(\mathbb{O}) := H(M_3(\mathbb{O}), *)$ is a subalgebra of $M_3(\mathbb{O})^{\mathrm{sym}}$. Moreover $H_3(\mathbb{O})$ is a Jordan algebra, and, endowed with a suitable norm, becomes in fact a JB-algebra [738, Proposition 2.9.2 and Corollary 3.1.7]. Since, as shown by Albert [6] (see also [738, Corollary 2.8.5]), the Jordan algebra $H_3(\mathbb{O})$ is not special, it cannot be a JC-algebra.

Lemma 3.1.7 is folklore. The crucial equivalence (ii)$\Leftrightarrow$(iii) in Proposition 3.1.9 is due to Wright and Youngson [643]. Indeed, they prove that *the extreme points of the positive part (say $\mathbb{B}_A^+$) of the closed unit ball of a unital JB-algebra A are precisely the idempotents of A*. But, since $\mathbb{B}_A$ is affinely isomorphic to $\mathbb{B}_A^+$ via the mapping $x \to \frac{1}{2}(x+\mathbf{1})$, the above fact is equivalent to the equivalence (ii)$\Leftrightarrow$(iii) in Proposition 3.1.9 that *the extreme points of $\mathbb{B}_A$ are precisely the symmetries in A*. The first conclusion in Proposition 3.1.9 (that *a JB-algebra A is unital if and only if $\mathbb{B}_A$ has extreme points*), as well as the implication (iii)$\Rightarrow$(i) in that proposition, remain true in the more general setting of the so-called real JB^*-algebras (see Corollary 4.2.58 below). The whole autonomous proof of Proposition 3.1.9 given here seems to us to be new. Proposition 3.1.13 and the (conclusion of) proof given here are due to Wright and Youngson [643].

Proposition 3.1.15 is due to Leung, Ng, and Wong [399] (2009), although the equivalences (ii)$\Leftrightarrow$(v)$\Leftrightarrow$(vi) were previously known as a consequence of a more general result in [261], which will be included later with proof in Theorem 4.2.53. The next proposition, also proved in [399], follows from Proposition 3.1.15 and the non-quantitative version of Corollary 2.9.29.

Proposition 3.1.57 *Let A be a JBW-algebra, and let u be a norm-one element of A. Then the following conditions are equivalent:*

(i) *u is a central symmetry in A.*
(ii) *The Banach space of A becomes a JBW-algebra with unit u, for some product.*
(iii) $n(A,u) = 1$.
(iv) *u is a w^*-unitary element of A* (cf. §2.9.24).
(v) *u is a w^*-vertex of the closed unit ball of A* (cf. §2.9.24 again).

We note in passing that, in view of [738, Proposition 4.5.3] and the implication (i)⇒(ii) in the above proposition, *if u is a central symmetry in a JBW-algebra A, then we have*

$$\mathbb{B}_{A_*} = |\mathrm{co}|(D^{w^*}(A,u)),$$

which refines the quantitative version of Corollary 2.9.29 in the present case (compare Example 2.9.31).

Corollaries 3.1.16 and 3.1.32, as well as results from Lemma 3.1.19 to Proposition 3.1.26, are taken from the Isidro–Rodríguez paper [342] (1995). The equivalence (i)⇔(v) in Theorem 3.1.21 remains true in the more general setting of real JB^*-algebras (see Theorem 4.2.79 below). The equivalence (i)⇔(vii) in Theorem 3.1.21 remains true in the still more general setting of real JB^*-triples (see §4.2.76 below). With the exception of some auxiliary results (namely Propositions 3.1.18 and 3.1.30, Lemma 3.1.29, and Corollary 3.1.31), the material from Proposition 3.1.15 to Corollary 3.1.32 has been fully re-elaborated. Indeed, Proposition 3.1.15 was not known when Theorem 3.1.21 was first proved in [342], and in fact the original proof of Proposition 3.1.15 in [399] uses Theorem 3.1.21. The rather surprising quick proof of Proposition 3.1.15 we have given has allowed us to replace, in the original proof of Theorem 3.1.21, the Banach space characterizations of central symmetries given by Remark 3.1.25 with those given by Proposition 3.1.15. An additional historical remark on Proposition 3.1.15 can be found in §4.2.78 below.

Proposition 3.1.18 (without which, condition (ii) in Theorem 3.1.21 would not have a right meaning) is due to Edwards [223]. Proposition 3.1.30 is an almost verbatim adaptation of a similar result for C^*-algebras proved by Ara and Mathieu [678, Theorem 4.1.5] in order to get a new simple proof of Sakai's theorem (already stated in Lemma 2.2.25) that derivations of C^*-algebras are continuous. According to the authors of [678], they follow the ideas of Arendt, Chernoff, and Kato in [27]. According to Youngson [654], Corollary 3.1.31 can be proved by modifying Sakai's argument in [806, Lemma 4.1.3] showing the automatic continuity of derivations of C^*-algebras, but, as far as we know, the details of such a modification were never explained. Of course, we have preferred here to modify the simpler Ara–Mathieu argument. By passing to normed complexification, Corollary 3.1.31 follows also from Villena's theorem [625] that *derivations of complete normed J-semisimple Jordan complex algebras are continuous*. (For the meaning of a J-semisimple Jordan algebra, the reader is referred to Definition 4.4.12 below.) However, Villena's result is very deep and difficult since it depends heavily on Zel'manov's classification theorem [662] (see also [777, p. 110]) for prime 'nondegenerate' Jordan algebras, the specialization of Zel'manov's theorem in the J-primitive case [21, 585], and the later analytic treatment in [146]. (For the meaning of a J-primitive Jordan algebra, the reader is referred to §4.4.73 below.)

Lemma 3.1.33 and Proposition 3.1.34 are due to Martín and Payá [421]. The fact actually needed for our development (that $N(C_0^{\mathbb{K}}(E)) = 1$ *for every locally compact Hausdorff topological space* E) is originally due to Duncan, McGregor, Pryce, and White [217].

Although easily derivable from known results, Propositions 3.1.35 and 3.1.38, as well as Corollaries 3.1.36 and 3.1.39, do not appear explicitly in any reference.

Results from Corollary 3.1.40 to Corollary 3.1.51 are taken from the Becerra–Rodríguez paper [73], where previous results in [71] are refined. In particular, Corollary 3.1.51 was known in [71] with a different proof. It follows from Corollary 3.1.51 (respectively, Proposition 3.1.44) and Proposition 3.1.4(i) that the question of convex transitivity for JB-algebras (respectively, for preduals of JBW-algebras) reduces to the consideration of a similar question for the classical Banach spaces $\mathscr{C}_0^{\mathbb{R}}(E)$ (respectively, $L_1^{\mathbb{R}}(\mu)$). As shown in Wood's paper [638] (see also [731, Theorem 12.5.3]), we have the following.

Theorem 3.1.58 *Let E be a locally compact Hausdorff topological space. Then the following conditions are equivalent:*

(i) *The Banach space $C_0^{\mathbb{R}}(E)$ is convex-transitive.*
(ii) *E is totally disconnected and, for every probability measure μ on E and every $t \in E$, there exists a net $\{\gamma_\alpha\}$ of homeomorphisms of E such that the net $\{\mu \circ \gamma_\alpha\}$ is w^*-convergent to δ_t.*

As far as we know, the convex transitivity for $L_1^{\mathbb{R}}(\mu)$-spaces has not been systematically studied.

Transitive and almost transitive normed spaces were introduced in §§2.6.43 and 2.9.39, respectively. Keeping in mind Lemma 2.7.36, we are provided with the following chain of implications between transitivity conditions on a normed space:

$$\boxed{\text{pre-Hilbert}} \Longrightarrow \boxed{\text{transitive}} \Longrightarrow \boxed{\text{almost transitive}} \Longrightarrow \boxed{\text{convex-transitive}}.$$

As shown in the Greim–Rajalopagan paper [300], we have the following.

Theorem 3.1.59 *Let E be a locally compact Hausdorff topological space such that $C_0^{\mathbb{R}}(E)$ is almost transitive. Then E reduces to a point.*

By combining Corollary 3.1.51, Proposition 3.1.4(i), and Theorem 3.1.59, we get the following.

Corollary 3.1.60 *$\mathbb{R}$ is the unique almost transitive JB-algebra.*

The remarkable fact, given by Corollary 3.1.60 above, was first pointed out in [71]. It is worth mentioning that Theorem 3.1.59 does not remain true when $\mathbb{R}$ is replaced with $\mathbb{C}$. Indeed, in 2002, a counterexample was found by Rambla [498] (and in 2003, independently by Kawamura [386]). For additional information about transitivity conditions on normed spaces, the reader is referred to Chapter 12 in Fleming and Jamison [731], Chapter 9 in Rolewicz [800] (excluding Theorems 9.7.3 and 9.7.7 and Corollary 9.7.4 which, according to [73, pp. 17–18], are mistaken), and the survey papers of Cabello [139] and Becerra–Rodríguez [73].

Lemma 3.1.53 is due to Topping [813]. In relation to Corollary 3.1.54(i), it is worth mentioning that *the spatial numerical index $N(A)$ of a C^*-algebra A is equal to* 1 *or* $\frac{1}{2}$ *depending on whether or not A is commutative*, a result of Huruya [333] which will be discussed later (see Corollary 3.5.39(i)). In relation to Corollary 3.1.54(ii), let us mention that *convex-transitive C^*-algebras need not be commutative*. Indeed, *the Calkin algebra* (cf. §1.4.55) *is convex-transitive* [71].

3.2 The unital non-associative Gelfand–Naimark theorem

Introduction In this section, we prove one of the star results in our work. Namely, as a consequence of Theorem 3.2.5, unital non-associative C^*-algebras are 'very nearly' associative: they are indeed alternative. We note that, by Proposition 2.6.8, associativity of such algebras cannot be expected.

3.2.1 The main result

We begin this section by including some useful characterizations of non-commutative Jordan algebras.

Proposition 3.2.1 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *If A is flexible, then the following identities are equivalent:*

$$a(ba^2) = (ab)a^2, \ \ (\text{i.e. } [L_a, R_{a^2}] = 0) \tag{3.2.1}$$

$$(a^2b)a = a^2(ba), \ \ (\text{i.e. } [R_a, L_{a^2}] = 0) \tag{3.2.2}$$

$$(ba)a^2 = (ba^2)a, \ \ (\text{i.e. } [R_a, R_{a^2}] = 0) \tag{3.2.3}$$

$$a^2(ab) = a(a^2b), \ \ (\text{i.e. } [L_a, L_{a^2}] = 0). \tag{3.2.4}$$

(ii) *A is a non-commutative Jordan algebra if and only if A is flexible and satisfies any one of the identities (3.2.1)–(3.2.4).*

Proof Assume that A is flexible. Then, writing the flexibility of A as

$$[R_a, L_a] = 0, \tag{3.2.5}$$

and linearizing, we obtain

$$[R_a, L_b] = [L_a, R_b]. \tag{3.2.6}$$

This with $b = a^2$ gives

$$[R_a, L_{a^2}] = [L_a, R_{a^2}], \tag{3.2.7}$$

hence (3.2.1) and (3.2.2) are equivalent. On the other hand, linearizing the flexible identity $(a,b,a) = 0$ gives $(a,b,c) + (c,b,a) = 0$, and hence

$$L_{ab} - L_aL_b = R_{ba} - R_aR_b.$$

Setting $b = a$ in the above equality we obtain

$$L_{a^2} - L_a^2 = R_{a^2} - R_a^2. \tag{3.2.8}$$

If A satisfies (3.2.3) and (3.2.5), then by (3.2.8) $[R_a, L_{a^2}] = 0$, giving (3.2.2). Similarly, (3.2.2) and (3.2.5) imply (3.2.3), and the equivalence of (3.2.1) and (3.2.4) follows from the same argument. Thus assertion (i) is proved.

Now, remove the assumption that A is flexible, and note that, since

$$a \bullet b = \frac{1}{2}(L_a + R_a)(b),$$

Jordan-admissibility of A is equivalent to the identity

$$[L_a + R_a, L_{a^2} + R_{a^2}] = 0. \tag{3.2.9}$$

Therefore, Jordan-admissibility is assured whenever all identities (3.2.1)–(3.2.4) are fulfilled. Hence, if A is flexible and satisfies any one of (3.2.1)–(3.2.4), then, by assertion (i), A is a non-commutative Jordan algebra. Conversely, if A is a non-commutative Jordan algebra, then identities (3.2.7) and (3.2.8) (respectively, (3.2.9)) hold by flexibility (respectively, by Jordan-admissibility), so

$$\begin{aligned}
0 &= [L_{a^2} + R_{a^2}, L_a] + [L_{a^2} + R_{a^2}, R_a] \\
&= [2R_{a^2} - R_a^2 + L_a^2, L_a] + [2L_{a^2} - L_a^2 + R_a^2, R_a] \\
&= 2[R_{a^2}, L_a] + 2[L_{a^2}, R_a] = 4[R_a, L_{a^2}],
\end{aligned}$$

giving (3.2.2), and hence, by assertion (i), A satisfies any one of (3.2.1)–(3.2.4). $\square$

As a by-product of the above proposition and its proof, we derive the following.

Corollary 3.2.2 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *If A is flexible, then the equality*

$$L_{ab} - L_a L_b = R_{ba} - R_a R_b$$

holds for all $a, b \in A$.

(ii) *If A is a non-commutative Jordan algebra, then, for each $a \in A$, the set $\{L_a, R_a, L_{a^2}, R_{a^2}\}$ is a commutative subset of $L(A)$.*

As usual, every V-algebra will be seen endowed with its natural involution.

Proposition 3.2.3 *Let A and B be complete V-algebras, and let $F : A \to B$ be a bijective linear mapping. Then the following conditions are equivalent:*

(i) *F is a Jordan-$*$-homomorphism.*
(ii) *$V(A, \mathbf{1}, a) = V(B, \mathbf{1}, F(a))$ for every $a \in A$.*

Proof (i)$\Rightarrow$(ii) Assume that F is a Jordan-$*$-homomorphism. Then, clearly, we have $F(H(A,*)) = H(B,*)$. On the other hand, since $H(A,*) = H(A,\mathbf{1})$ and $H(B,*) = H(B,\mathbf{1})$, Lemma 3.1.1, applies, so that $H(A,\mathbf{1})$ and $H(B,\mathbf{1})$ become JB-algebras in a natural way. It follows that F, regarded as a mapping from $H(A,\mathbf{1})$ to $H(B,\mathbf{1})$, becomes a bijective algebra homomorphism, and hence, by Proposition 3.1.4(ii), is an isometry. As a consequence, by Corollary 2.1.2(ii), we have $V(B,\mathbf{1},F(h)) = V(A,\mathbf{1},h)$ for every $h \in H(A,\mathbf{1})$. Now, let a be in A. Writing $a = h + ik$, with $h, k \in H(A,\mathbf{1})$, we have

$$\max \Re(V(F(a))) = \max V(F(h)) = \max V(h) = \max \Re(V(a)).$$

By replacing a by appropriate complex multiples of a, we get $V(F(a)) = V(a)$, as required.

(ii)$\Rightarrow$(i) Assume that (ii) holds. Then, clearly, we have $F(H(A,\mathbf{1})) = H(B,\mathbf{1})$, and, as a consequence, F becomes a $*$-mapping. Moreover, by Proposition 2.3.4, F, regarded as a mapping from $H(A,\mathbf{1})$ to $H(B,\mathbf{1})$, is a surjective linear isometry. Now, keeping in mind Lemma 3.1.1 and the implication (i)⇒(ii) in Proposition 3.1.13, the proof is concluded by showing that $F(\mathbf{1}) = \mathbf{1}$. But, since

$$V(B,\mathbf{1},F(\mathbf{1})) = V(A,\mathbf{1},\mathbf{1}) = \{1\}, \text{ and } V(B,\mathbf{1},\mathbf{1}) = \{1\},$$

we obtain $V(B,\mathbf{1},F(\mathbf{1})-\mathbf{1}) = \{0\}$, and hence, by Corollary 2.1.13, $F(\mathbf{1}) = \mathbf{1}$, as required. □

Lemma 3.2.4 *Let A be a complete normed unital algebra over $\mathbb{K}$, and let $\{F_n\}_{n\geqslant 0}$ be a sequence in $BL(A)$. Assume the existence of a positive number k such that the series $\sum_{n\geqslant 0} \frac{1}{n!} r^n F_n$ converges in $BL(A)$ for every $r \in \mathbb{R}$ with $|r| < k$, and that, for such an r, $G_r := \sum_{n=0}^{\infty} \frac{1}{n!} r^n F_n$ is an algebra automorphism of A. Then we have:*

(i) $F_n(ab) = \sum_{i=0}^{n} \binom{n}{i} F_i(a)F_{n-i}(b)$ *for every $n \geqslant 0$ and all $a,b \in A$.*

(ii) *In particular, if $F_0 = I_A$ and $F_1 = 0$, then F_2 is a derivation of A.*

Proof We have $G_r(ab) = G_r(a)G_r(b)$ for every $r \in \mathbb{R}$ with $|r| < k$ and all $a,b \in A$. Writing the right-hand side of the above equality as a Cauchy product of the corresponding series, and identifying coefficients, assertion (i) is obtained. Then, assertion (ii) becomes clear. □

Let A be a complete normed unital power-associative algebra over $\mathbb{K}$, and let a be in A. Then the closed subalgebra of A generated by $\mathbf{1}$ and a is associative, and so we can consider $\exp(a) \in A$ (cf. §1.1.29).

Theorem 3.2.5 *Let A be a normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a\|^2$ for every $a \in A$. Then A is alternative, and $*$ is an algebra involution on A.*

Proof The assumption that $\|a^*a\| = \|a\|^2$ for every $a \in A$ implies that $*$ is an isometry. Therefore we can pass to the completion of A, and assume without loss of generality that A is complete. By Lemma 2.2.5, A is a V-algebra whose natural involution coincides with $*$, so that, by Theorem 2.3.8, $*$ is an algebra involution on A. Moreover, by Theorem 2.4.11, A is a non-commutative Jordan algebra, and hence, by Proposition 2.4.19, A is power-associative.

Let h be in $H(A,\mathbf{1})$, let r be in $\mathbb{R}$, and let a be in A. Set

$$G_r := L_{\exp(irh)} \exp(-irL_h).$$

Keeping in mind that $\exp(-irL_h)$ is an isometry (by Lemma 2.1.10 and Corollary 2.1.9(iii)), for each $s \in \mathbb{R}$, we have

$$\begin{aligned} \|\mathbf{1}+sa\|^2 &= \|\exp(-irh) + s\exp(-irL_h)(a)\|^2 \\ &= \|[\exp(irh) + s(\exp(-irL_h)(a))^*][\exp(-irh) + s\exp(-irL_h)(a)]\| \\ &= \|\mathbf{1} + s(G_r(a) + (G_r(a))^*) + s^2 \cdots\|. \end{aligned}$$

Taking right derivatives at $s = 0$ (see Corollary 2.1.6), we get

$$2\max \Re(V(A,\mathbf{1},a)) = \max V(A,\mathbf{1},G_r(a) + (G_r(a))^*),$$

and hence $\max \Re(V(A,\mathbf{1},a)) = \max \Re(V(A,\mathbf{1},G_r(a)))$. Thus we have proved that

$$V(A,\mathbf{1},a) = V(A,\mathbf{1},G_r(a)) \text{ for all } a \in A \text{ and } r \in \mathbb{R}. \tag{3.2.10}$$

Since $G_0 = I_A$, and the mapping $r \to G_r$ from $\mathbb{R}$ to $BL(A)$ is continuous, there exists a positive number k such that, for $r \in \mathbb{R}$ with $|r| < k$, G_r is invertible in $BL(A)$. Thus, for $|r| < k$, G_r is a linear bijection satisfying (3.2.10), and hence, by the implication (ii)$\Rightarrow$(i) in Proposition 3.2.3, it is a Jordan-$*$-automorphism. Let $\sum_{n\geqslant 0} \frac{1}{n!} r^n F_n$ be the power series development of G_r. Then we have $F_0 = I_A$, $F_1 = 0$, and

$$F_2 = L_h^2 - L_{h^2}. \tag{3.2.11}$$

It follows from Lemma 3.2.4 that F_2 is a Jordan derivation on A. Moreover, we have $F_2(a^*) = F_2(a)^*$ for every $a \in A$. Indeed, this equality can be derived from (3.2.11) and Corollary 3.2.2(i), or alternatively from the uniqueness of the coefficients in a power series development, and the fact that G_r is a $*$-mapping. Now, for $a \in A$, set $|||a||| := v(A,\mathbf{1},a)$, note that, by Proposition 2.1.11, $|||\cdot|||$ is an equivalent vector space norm on A, and denote also by $|||\cdot|||$ the corresponding operator algebra norm on $BL(A)$. It follows from Lemma 2.2.21 that, for every $t \in \mathbb{R}$, $\exp(tF_2)$ is a Jordan-$*$-automorphism of A, and hence, by the implication (i)$\Rightarrow$(ii) in Proposition 3.2.3, that $|||\exp(tF_2)||| = 1$. Therefore, by Corollary 2.1.9(iii), we have

$$iF_2 \in H((BL(A),|||\cdot|||),I_A), \tag{3.2.12}$$

and then, by Lemma 2.3.21, $\mathrm{sp}(F_2) \subseteq i\mathbb{R}$. On the other hand, by Lemmas 2.1.10 and 2.3.21, we also have $\mathrm{sp}(L_h) \subseteq \mathbb{R}$ and $\mathrm{sp}(L_{h^2}) \subseteq \mathbb{R}$, so that, since L_h and L_{h^2} commute (by Corollary 3.2.2(ii)), Corollary 1.1.81(i) gives

$$\mathrm{sp}(F_2) = \mathrm{sp}(L_h^2 - L_{h^2}) \subseteq \mathrm{sp}(L_h^2) - \mathrm{sp}(L_{h^2}) \subseteq (\mathrm{sp}(L_h))^2 - \mathrm{sp}(L_{h^2}) \subseteq \mathbb{R}.$$

It follows that $\mathrm{sp}(F_2) = \{0\}$. Then, invoking (3.2.12) and Proposition 2.3.22, we have $0 = F_2 = L_h^2 - L_{h^2}$.

Since h is an arbitrary element in $H(A,\mathbf{1})$, we can linearize to obtain

$$L_h \bullet L_k - L_{h\bullet k} = 0 \text{ for all } h,k \in H(A,\mathbf{1}).$$

Since $A = H(A,\mathbf{1}) + iH(A,\mathbf{1})$, we derive $L_a \bullet L_b - L_{a\bullet b} = 0$ for all $a,b \in A$, and in particular $L_a^2 - L_{a^2} = 0$ for every $a \in A$. Since the equality $R_a^2 - R_{a^2} = L_a^2 - L_{a^2}$ holds for every $a \in A$ (by Corollary 3.2.2(i)), we have also $R_a^2 - R_{a^2} = 0$, and A is alternative, as required. □

We call Theorem 3.2.5 above *the unital non-associative Gelfand–Naimark theorem.*

Lemma 3.2.6 *Let A be a normed alternative $*$-algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) $\|a^*a\| = \|a\|^2$ *for every* $a \in A$.
(ii) $\|aa^*a\| = \|a\|^3$ *for every* $a \in A$.

Proof Keep in mind that, by Theorem 2.3.61, for each $a \in A$, the subalgebra of A generated by a and a^* is associative. If $\|aa^*a\| = \|a\|^3$ for every $a \in A$, then we have

$\|a\|^2 \leqslant \|a^*a\|$, and hence $\|a\|^2 = \|a^*a\|$. Conversely, if $\|a^*a\| = \|a\|^2$ for every $a \in A$, then we clearly have $\|a\| = \|a^*\|$, so

$$\|a\|^4 = \|a^*a\|^2 = \|a^*aa^*a\| \leqslant \|a\|\|aa^*a\| \leqslant \|a\|^4,$$

and so $\|a\|^3 = \|aa^*a\|$. □

Now, as announced in Remark 2.2.6, we can prove the following.

Corollary 3.2.7 *Let A be a normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a\|^2$ for every $a \in A$. Then we have $\|U_a(a^*)\| = \|a\|^3$ for every $a \in A$.*

Proof By Theorem 3.2.5, A is alternative and $*$ is an algebra involution. Therefore, by Theorem 2.3.61, for each $a \in A$, the subalgebra of A generated by a and a^* is associative, and consequently $U_a(a^*) = aa^*a$. Then the result follows from Lemma 3.2.6. □

3.2.2 Historical notes and comments

Proposition 3.2.1(i) is originally due to Schafer [552]. The proof given here has been taken almost verbatim from Proposition I.5.1 in Elduque–Myung [728].

Results from Proposition 3.2.3 to Theorem 3.2.5 are due to Rodríguez [514]. Theorem 3.2.5 is attributed erroneously to Braun [125] on p. 280 of Doran–Belfi [725]. Certainly, Braun (both in the introduction and in Theorem 4.5 of [125]) formulates Theorem 3.2.5, but, in both places, he refers the reader to [514] for a proof. The proof of Theorem 3.2.5 given here differs slightly from the original one in [514]. However, some minor changes have been made in order to avoid the unnecessary application done in [514] of the deep results of Youngson [654] and Wright [641]. These results will be discussed later in Theorem 3.3.11 and Proposition 3.4.4, respectively.

3.3 The non-associative Vidav–Palmer theorem

Introduction In this section, we introduce non-commutative JB^*-algebras, and prove as the main result the so-called ‘non-associative Vidav–Palmer theorem’, which identifies complete V-algebras with unital non-commutative JB^*-algebras. In fact, the proof of this theorem already started with Theorems 2.4.11 and 2.3.8, asserting that V-algebras are non-commutative Jordan algebras, and that their natural involutions are algebra involutions. Therefore, we conclude the proof here by showing that the characteristic metric axiom $\|U_a(a^*)\| = \|a\|^3$ of non-commutative JB^*-algebras holds for every element a in any V-algebra. This culminating point of the theorem is due to Youngson [654], and is strongly based on a Jordan refinement of Proposition 2.3.29, due to Wright and Youngson [642]. The section is complemented by proving the so-called ‘dual version of the non-associative Vidav–Palmer theorem’. This result, whose associative forerunner is due to Moore [446], characterizes unital non-commutative JB^*-algebras, among norm-unital complete normed complex algebras, in terms of the behaviour of the dual space.

3.3.1 The main result

Definition 3.3.1 By a *non-commutative JB^*-algebra* we mean a complete normed non-commutative Jordan complex $*$-algebra (say A) such that the equality $\|U_a(a^*)\| = \|a\|^3$ holds for every $a \in A$. Non-commutative JB^*-algebras which are commutative are simply called *JB^*-algebras*.

Without enjoying their name, non-commutative JB^*-algebras have already appeared in Corollary 2.4.12, which shows in addition that, in the case of unital algebras, there are severe redundancies in the conditions required for non-commutative JB^*-algebras in Definition 3.3.1.

We recall that alternative algebras are non-commutative Jordan algebras (by Corollary 2.4.10), and that for elements a, b in an alternative algebra, we have $U_a(b) = aba$ (by Theorem 2.3.61). Therefore, it is enough to invoke Lemma 3.2.6 to derive the following.

Fact 3.3.2 *Alternative C^*-algebras are non-commutative JB^*-algebras. More precisely, alternative C^*-algebras are those non-commutative JB^*-algebras which are alternative.*

The following result is easily verified by applying Corollary 3.2.2(i).

Fact 3.3.3 *Let A be a flexible algebra over $\mathbb{K}$. Then, for $a \in A$, the operator U_a has the same meaning in both A and A^{sym}.*

A straightforward consequence of Fact 3.3.3 is the following.

Fact 3.3.4 *Let A be a complete normed flexible complex $*$-algebra. Then A is a non-commutative JB^*-algebra if and only if A^{sym} is a JB^*-algebra.*

As a by-product of Facts 3.3.2 and 3.3.4 we find that, if A is a C^*-algebra, then closed $*$-subalgebras of A^{sym} are JB^*-algebras. JB^*-algebras arising in this way are called *JC^*-algebras*.

Lemma 3.3.5 *Let A be an associative algebra over $\mathbb{K}$, let a and b be in A, and let m and n be in $\mathbb{N}$. Then*

$$a(ba)^n, \ (ab)^n + (ba)^n, \ \textit{and} \ (ab)^n(ba)^m + (ab)^m(ba)^n$$

belong to the subalgebra of A^{sym} generated by $\{a, b\}$. As a consequence, if A is unital, and if q is in $\mathbb{K}[\mathbf{x}]$, then $q(ab)q(ba)$ lies in the subalgebra of A^{sym} generated by $\{a, b, \mathbf{1}\}$.

Proof Let C denote the subalgebra of A^{sym} generated by $\{a, b\}$. By interchanging the roles of m and n, we may assume that $m \leqslant n$. We argue by induction on n. The lemma is true for $n = 1$ because, by Fact 3.3.3, we have

$$aba = U_a(b), \ ab + ba = 2(a \bullet b), \ \text{and} \ 2abba = 2U_a(b^2).$$

Assume that the lemma is true for some value of n (say p). Then we have

$$a(ba)^{p+1} = U_a[b(ab)^p] \in C,$$

$$(ab)^{p+1} + (ba)^{p+1} = ab(ab)^p + b(ab)^p a = 2a \bullet [b(ab)^p] \in C,$$

and, for $m < p+1$,

$$\begin{aligned}(ab)^{p+1}(ba)^m + (ab)^m(ba)^{p+1} &= (ab)^m\left[(ab)^{p+1-m} + (ba)^{p+1-m}\right](ba)^m \\ &= (U_aU_b)^m\left[(ab)^{p+1-m} + (ba)^{p+1-m}\right] \in C,\end{aligned}$$

whereas, for $m = p+1$,

$$2(ab)^{p+1}(ba)^{p+1} = 2U_a(U_bU_a)^p(b^2) \in C.$$

Now, assume that A is unital, and let $q(t) = \sum_{i=0}^k \lambda_k t^k$ be in $\mathbb{K}[\mathbf{x}]$. Then, since

$$\begin{aligned}q(ab)q(ba) &= \sum_{i,j=0}^{k} \lambda_i\lambda_j(ab)^i(ba)^j \\ &= \sum_{i=0}^{k} \lambda_i^2(ab)^i(ba)^i + \sum_{0\leqslant i<j\leqslant k} \lambda_i\lambda_j((ab)^i(ba)^j + (ab)^j(ba)^i),\end{aligned}$$

it follows from the above paragraph that $q(ab)q(ba)$ lies in the subalgebra of A^{sym} generated by $\{a,b,\mathbf{1}\}$. □

For assertion (i) in the next lemma, Fact 1.1.33(ii), as well as the continuous functional calculus (stated in Theorem 1.2.28 and §1.2.29), should be kept in mind.

Lemma 3.3.6 *Let A be a unital C^*-algebra, and let B be a closed $*$-subalgebra of A^{sym} with $\mathbf{1} \in B$. We have:*

- (i) *If x is in B, and if f is a complex-valued continuous function on the compact set $\mathrm{sp}(A,xx^*) \cup \mathrm{sp}(A,x^*x) \subseteq \mathbb{R}$, then $xf(x^*x)$ and $f(x^*x)f(xx^*)$ lie in B, and we have $f(xx^*)x = xf(x^*x)$.*
- (ii) *If x belongs to $B \cap \mathrm{Inv}(A)$, then x^{-1} lies in B.*
- (iii) *$\mathbb{B}_B = \overline{\mathrm{co}}(U \cap B)$, where U stands for the set of all unitary elements of A.*

Proof Let x be in B, set $K := \mathrm{sp}(A,xx^*) \cup \mathrm{sp}(A,x^*x)$, and let $f : K \to \mathbb{C}$ be a continuous function. Take a sequence q_n in $\mathbb{C}[\mathbf{x}]$ converging to f uniformly on K. Then we have $q_n(xx^*) \to f(xx^*)$ and $q_n(x^*x) \to f(x^*x)$. Since clearly

$$q_n(xx^*)x = xq_n(x^*x) \text{ for every } n \in \mathbb{N},$$

it follows that $f(xx^*)x = xf(x^*x)$. Moreover, since $xq_n(x^*x)$ and $q_n(x^*x)q_n(xx^*)$ lie in B for every $n \in \mathbb{N}$ (by Lemma 3.3.5), we derive that $xf(x^*x)$ and $f(x^*x)f(xx^*)$ also lie in B. This concludes the proof of assertion (i). In what follows, assertion (i) (just proved) will sometimes be applied with x^* instead of x.

Let x be in $B \cap \mathrm{Inv}(A)$. Then $0 \notin \mathrm{sp}(A,xx^*) \cup \mathrm{sp}(A,x^*x)$, and hence, by assertion (i), we have that $x^{-1} = x^*(xx^*)^{-1} \in B$, which proves assertion (ii).

Now, let $x \in B$ with $0 < \|x\| < 1$. For $\lambda \in \mathbb{S}_{\mathbb{C}}$, set

$$F(\lambda) := (\mathbf{1} - xx^*)^{-\frac{1}{2}}(\lambda\mathbf{1} + x)(\mathbf{1} + \lambda x^*)^{-1}(\mathbf{1} - x^*x)^{\frac{1}{2}}.$$

We saw in the proof of Lemma 2.3.28 that, $F(\lambda)$ lies in U for every $\lambda \in \mathbb{S}_{\mathbb{C}}$, and that

$$x = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})\,d\theta.$$

Therefore, to conclude the proof of assertion (iii) it is enough to show that $F(\lambda)$ lies in B for every $\lambda \in \mathbb{S}_{\mathbb{C}}$. To this end, note that, for such a λ, we have

$$\begin{aligned}(\lambda\mathbf{1}+x)(\mathbf{1}+\lambda x^*)^{-1} &= [x(\mathbf{1}+\lambda x^*)+\lambda(\mathbf{1}-xx^*)]\,(\mathbf{1}+\lambda x^*)^{-1}\\ &= x+\lambda(\mathbf{1}-xx^*)(\mathbf{1}+\lambda x^*)^{-1},\end{aligned}$$

and consequently

$$\begin{aligned}F(\lambda) &= (\mathbf{1}-xx^*)^{-\frac{1}{2}}\left[x+\lambda(\mathbf{1}-xx^*)(\mathbf{1}+\lambda x^*)^{-1}\right](\mathbf{1}-x^*x)^{\frac{1}{2}}\\ &= (\mathbf{1}-xx^*)^{-\frac{1}{2}}x(\mathbf{1}-x^*x)^{\frac{1}{2}}+\lambda(\mathbf{1}-xx^*)^{\frac{1}{2}}(\mathbf{1}+\lambda x^*)^{-1}(\mathbf{1}-x^*x)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, by assertion (i), we have

$$(\mathbf{1}-xx^*)^{-\frac{1}{2}}x(\mathbf{1}-x^*x)^{\frac{1}{2}}=x\in B.$$

So it only remains to show that $(\mathbf{1}-xx^*)^{\frac{1}{2}}(\mathbf{1}+\lambda x^*)^{-1}(\mathbf{1}-x^*x)^{\frac{1}{2}}$ is in B. By assertion (ii), it suffices to show that its inverse

$$\begin{aligned}&(\mathbf{1}-x^*x)^{-\frac{1}{2}}(\mathbf{1}+\lambda x^*)(\mathbf{1}-xx^*)^{-\frac{1}{2}}\\ &\quad=(\mathbf{1}-x^*x)^{-\frac{1}{2}}(\mathbf{1}-xx^*)^{-\frac{1}{2}}+\lambda(\mathbf{1}-x^*x)^{-\frac{1}{2}}x^*(\mathbf{1}-xx^*)^{-\frac{1}{2}}\end{aligned}$$

is in B. But this holds because, by assertion (i), we have

$$(\mathbf{1}-x^*x)^{-\frac{1}{2}}(\mathbf{1}-xx^*)^{-\frac{1}{2}}\in B$$

and

$$(\mathbf{1}-x^*x)^{-\frac{1}{2}}x^*(\mathbf{1}-xx^*)^{-\frac{1}{2}}=x^*(1-xx^*)^{-1}\in B.$$

Thus $F(\lambda)$ lies in B for every $\lambda\in\mathbb{S}_{\mathbb{C}}$, as desired. □

Proposition 3.3.7 *Let B be a unital JC^*-algebra. Then $\mathbb{B}_B=\overline{\mathrm{co}}(E)$, where*

$$E:=\{\exp(ih):h\in H(B,*)\}.$$

Proof Let A be a C^*-algebra such that B is a closed $*$-subalgebra of A^{sym}. Then, by Corollary 1.2.50, the unit $\mathbf{1}$ of B becomes a self-adjoint idempotent of A, and hence, by Fact 3.3.3, $\mathbf{1}A\mathbf{1}=U_{\mathbf{1}}(A)$ is a C^*-subalgebra of A containing B. Therefore, replacing A with $\mathbf{1}A\mathbf{1}$, we may assume that $\mathbf{1}$ is a unit for A. Let u be a unitary element of A which lies in B. By Lemma 3.3.6(iii), it is enough to show that $u\in\overline{\mathrm{co}}(E)$. Let C stand for the closed subalgebra of A generated by $\{u,u^*\}$. Then C is a commutative C^*-algebra with $\mathbf{1}\in C$. Since C is a commutative subalgebra of A, the products of A and A^{sym} agree on C, and hence C coincides with the closed subalgebra of A^{sym} generated by $\{u,u^*\}$. Since B is a closed $*$-subalgebra of A^{sym}, and u belongs to B, it follows that $C\subseteq B$. Therefore, by Proposition 2.3.29, we have

$$u\in\overline{\mathrm{co}}(\{\exp(ih):h\in H(C,*)\})\subseteq\overline{\mathrm{co}}(E),$$

as desired. □

As usual, every V-algebra will be seen endowed with its natural involution.

Lemma 3.3.8 *Let A be a complete commutative V-algebra, and let x be in A. Then the closed subalgebra of A generated by $\{\mathbf{1},x,x^*\}$ is $*$-invariant, and is indeed bicontinuously $*$-isomorphic to a JC^*-algebra.*

Proof By Lemma 3.1.1, $H(A,\mathbf{1})$ is a closed real subalgebra of A and, endowed with the restriction of the norm of A, becomes a JB-algebra. Now, write $x = h + ik$, with $h,k \in H(A,\mathbf{1})$, and let J stand for the closed subalgebra of $H(A,\mathbf{1})$ generated by $\{\mathbf{1},h,k\}$. Then, by Proposition 3.1.3, there exists a C^*-algebra B, together with an isometric algebra homomorphism ϕ from J onto a closed subalgebra of the JB-algebra $H(B,*)$. Since the direct sums $A = H(A,\mathbf{1}) \oplus iH(A,\mathbf{1})$ and $B = H(B,*) \oplus iH(B,*)$ are topological, $J + iJ$ and $\phi(J) + i\phi(J)$ are closed $*$-subalgebras of A and B^{sym}, respectively, and the extension of ϕ by complex linearity becomes a bicontinuous algebra $*$-homomorphism from $J + iJ$ onto the JC^*-algebra $\phi(J) + i\phi(J)$. The proof is concluded by noticing that $J + iJ$ is nothing other than the closed subalgebra of A generated by $\{\mathbf{1},x,x^*\}$. $\square$

§**3.3.9** Given an algebra A, a subalgebra B of A, and an element $x \in B$, we denote by U_x^B and L_x^B the operators U_x and L_x regarded as mappings from B to B.

We note that, if A is associative, unital, and complete normed, then, for $x \in A$, $\exp(x)$ has the same meaning in both A and A^{sym} (thanks to Lemma 2.4.17).

Lemma 3.3.10 *Let A be a complete normed unital associative algebra over $\mathbb{K}$, let B be a closed subalgebra of A^{sym} with $\mathbf{1} \in B$, and let x be in B. Then we have $U^B_{\exp(x)} = \exp(2L_x^B)$.*

Proof We may assume that $B = A^{\mathrm{sym}}$. Then, for $a \in A$, L_a^A and R_a^A commute, and we have $L_a^B = \frac{1}{2}(L_a^A + R_a^A)$ and, by Fact 3.3.3, also $U_a^B = U_a^A = L_a^A R_a^A$. Therefore, we get

$$U^B_{\exp(x)} = L^A_{\exp(x)} R^A_{\exp(x)} = \exp(L_x^A)\exp(R_x^A) = \exp(L_x^A + R_x^A) = \exp(2L_x^B). \qquad \square$$

As a by-product of Lemma 2.2.5, unital non-commutative JB^*-algebras are complete V-algebras. Now, we prove the following relevant converse (called in the sequel 'the non-associative Vidav–Palmer theorem').

Theorem 3.3.11 *Let A be a complete V-algebra. Then A, endowed with its natural involution and its own norm, becomes a non-commutative JB^*-algebra.*

Proof Let $*$ denote the natural involution of A. By Theorems 2.3.8 and 2.4.11, $*$ is an algebra involution, and A is a non-commutative Jordan algebra. Therefore, to conclude the proof it is enough to show that the equality $\|U_x(x^*)\| = \|x\|^3$ holds for every $x \in A$. Since, clearly, A^{sym} is a complete commutative V-algebra, we can invoke Fact 3.3.4 to realize that there is no loss of generality in assuming that A is commutative. Then we fix $x \in A$, and note that, by Lemma 2.3.7(ii) and Proposition 1.2.25, the closed subalgebra of A generated by $\{\mathbf{1},x,x^*\}$ is $*$-invariant, and hence becomes a complete commutative V-algebra. Therefore we may assume in addition that A is generated by $\{\mathbf{1},x,x^*\}$ as a normed algebra. Now, we are in a position to apply Lemma 3.3.8 to find an equivalent norm $|||\cdot|||$ on A such that $(A,*,|||\cdot|||)$ becomes a unital JC^*-algebra. Therefore, by Proposition 3.3.7 and Corollary 2.1.9(iii), we have

$$\mathbb{B}_{(A,|||\cdot|||)} = \overline{\mathrm{co}}\{\exp(ih) : h \in H(A,*)\} = \overline{\mathrm{co}}\{\exp(ih) : h \in H(A,\mathbf{1})\} \subseteq \mathbb{B}_A,$$

and hence

$$\|\cdot\| \leqslant |||\cdot||| \ \text{ on } A. \tag{3.3.1}$$

Let h and a be in $H(A,\mathbf{1})$ and A, respectively. Then, since $(A,|||\cdot|||)$ is a closed unital subalgebra of C^{sym} for some complete normed unital associative algebra C, Lemma 3.3.10 applies, so that we have $U_{\exp(ih)}(a) = (\exp(2iL_h))(a)$. On the other hand, by Lemma 2.1.10, L_h belongs to $H(BL(A),I_A)$, and hence, by Corollary 2.1.9(iii), we have $\|\exp(2iL_h)(a)\| = \|a\|$. It follows that

$$\|U_{\exp(ih)}(a)\| = \|a\|. \tag{3.3.2}$$

Since the equality $U_vU_w(v^2) = (U_v(w))^2$ holds for all $v,w \in A$ (indeed, compute this in the associative algebra C above, using Fact 3.3.3), we apply (3.3.2) twice to get

$$\begin{aligned}\|U_a(\exp(ih))\| &= \|U_{\exp(\frac{1}{2}ih)}U_a(\exp(ih))\| = \|(U_{\exp(\frac{1}{2}ih)}(a))^2\| \\ &\leqslant \|U_{\exp(\frac{1}{2}ih)}(a)\|^2 = \|a\|^2.\end{aligned}$$

Since the set $\{b \in A : \|U_a(b)\| \leqslant \|a\|^2\}$ is closed and convex, and h is arbitrary in $H(A,*)$, and $(A,*,|||\cdot|||)$ is a unital JC^*-algebra, it follows from Proposition 3.3.7 that

$$\|U_a(b)\| \leqslant \|a\|^2|||b||| \ \text{ for every } \ b \in A. \tag{3.3.3}$$

Now, let M be the smallest positive number such that $|||\cdot||| \leqslant M\|\cdot\|$ on A. It follows from (3.3.3) that

$$|||a|||^3 = |||U_a(a^*)||| \leqslant M\|U_a(a^*)\| \leqslant M\|a\|^2|||a^*||| = M\|a\|^2|||a|||,$$

and hence that $|||a||| \leqslant \sqrt{M}\|a\|$. By the arbitrariness of $a \in A$, and the definition of M, we get that $M \leqslant 1$. Therefore $|||\cdot||| \leqslant \|\cdot\|$ on A. Thus, by (3.3.1), we actually have $\|\cdot\| = |||\cdot|||$, and as a result A is a JB^*-algebra. □

Via Fact 3.3.2, Theorem 3.3.11 contains the associative Vidav–Palmer theorem (Theorem 2.3.32) and even its generalization to alternative algebras (Corollary 2.3.63).

Looking at the proof of Theorem 3.3.11, we realize that the natural involution of any complete V-algebra is an isometry. We have not emphasized this fact because, by Theorem 3.3.11 itself, complete V-algebras are (unital) non-commutative JB^*-algebras, and, as we will prove in Proposition 3.3.13 below, the involution of any (possibly non-unital) non-commutative JB^*-algebra is isometric.

The following result is a form of Proposition 2.6.19.

Lemma 3.3.12 *Let A and B be algebras over $\mathbb{K}$, each of which is endowed with a conjugate-linear vector space involution $*$ and a norm $\|\cdot\|$ satisfying*

$$\|U_a(a^*)\| \leqslant \|a\|^3 \text{ for every } a \in A \text{ and } \|U_b(b^*)\| = \|b\|^3 \text{ for every } b \in B,$$

and let $\phi : A \to B$ be a continuous real-linear mapping satisfying

$$\phi(U_a(a^*)) = U_{\phi(a)}((\phi(a))^*) \text{ for every } a \in A.$$

Then ϕ is contractive.

Proof Assume to the contrary that ϕ is not contractive. Then we can choose a norm-one element x in A such that $\|\phi(x)\| > 1$. Defining inductively

$$x_1 := x \text{ and } x_{n+1} := U_{x_n}(x_n^*),$$

we have $\|\phi(x_n)\| = \|\phi(x)\|^{3^{n-1}} \to \infty$. Since $\|x_n\| \leqslant 1$, this contradicts the assumed continuity of ϕ. □

Proposition 3.3.13 *Let A be a non-commutative JB^*-algebra. Then the involution of A is isometric.*

Proof Note that, for every $a \in A$, we have $\|a^*\|^3 = \|U_{a^*}(a)\| \leqslant 3\|a^*\|^2\|a\|$, and hence $\|a^*\| \leqslant 3\|a\|$. Therefore the real-linear mapping $\phi : a \to a^*$ from A to A becomes continuous. Moreover, by Fact 3.3.3, for each $a \in A$ we have

$$\phi(U_a(a^*)) = (U_a(a^*))^* = U_{a^*}(a) = U_{\phi(a)}((\phi(a))^*).$$

It follows from Lemma 3.3.12 that ϕ is contractive. But, since $\phi = *$ is an involutive mapping, the result follows. □

Lemma 3.3.14 *Let (X,u) be a complete complex numerical-range space with $n(X,u) > 0$. Then $H(X,u) + iH(X,u)$ is closed in X.*

Proof Let h,k be in $H(X,u)$. Then we have $v(h) \leqslant v(h+ik) \leqslant \|h+ik\|$. Since $\|h\| \leqslant \frac{1}{n(X,u)}v(h)$, we get that $\|h\| \leqslant \frac{1}{n(X,u)}\|h+ik\|$. Set $Y := H(X,u) + iH(X,u)$. It follows from the above that $Y = H(X,u) \oplus iH(X,u)$, and that this sum is topological. Since X is complete, and $H(X,u)$ is closed in X, we deduce that Y is complete, and hence closed in X. □

Keeping in mind Corollary 2.1.2 and Proposition 2.1.11, Lemma 3.3.14 above yields the following.

Corollary 3.3.15 *Let A be a V-algebra. Then the completion of A is a V-algebra whose natural involution extends that of A.*

Finally, combining Corollary 3.3.15, Theorem 3.3.11, and Proposition 3.3.13, we get the following.

Corollary 3.3.16 *Let A be a V-algebra. Then the natural involution of A is isometric.*

After Lemma 2.2.5 (that unital non-commutative JB^*-algebras are V-algebras) and Theorem 3.3.11, some results previously proved for V-algebras attain their definitive form. As a first sample, we formulate the following.

Corollary 3.3.17 (a) *Unit-preserving linear contractions between unital non-commutative JB^*-algebras preserve involutions.*

(b) *If A is a unital non-commutative JB^*-algebra, if B is a norm-unital complete normed complex algebra, and if $T : A \to B$ is a unit-preserving dense range linear contraction, then B is a non-commutative JB^*-algebra, and T preserves involutions.*

Proof Part (a) follows from Lemma 2.2.5 and Corollary 2.1.2(i).

Let A, B, and T be as assumed in part (b). Again by Lemma 2.2.5 and Corollary 2.1.2(i), we have $T(A) \subseteq H(B,\mathbf{1}) + iH(B,\mathbf{1})$, and hence $H(B,\mathbf{1}) + iH(B,\mathbf{1})$ is dense in B. It follows from Proposition 2.1.11, Lemma 3.3.14, and Theorem 3.3.11 that B is a non-commutative JB^*-algebra. □

As another sample, we emphasize in the next corollary the non-commutative JB^*-form of Proposition 2.3.64.

Corollary 3.3.18 *Let A be a unital non-commutative JB^*-algebra, and let $\pi : A \to A$ be a unit-preserving contractive linear projection. Then $\pi(A)$ is a $*$-invariant subspace of A, and $\pi(A)$ becomes a unital non-commutative JB^*-algebra under the norm and the involution of A, and the product*

$$x \odot y := \pi(xy).$$

Invoking Fact 3.3.2, we derive the following.

Corollary 3.3.19 *Let A be a unital C^*-algebra, and let $\pi : A \to A$ be a unit-preserving contractive linear projection. Then $\pi(A)$ is a $*$-invariant subspace of A, and $\pi(A)$ becomes a unital non-commutative JB^*-algebra under the norm and the involution of A, and the product $x \odot y := \pi(xy)$.*

According to Example 2.3.65, not much more can be said in the conclusion of Corollary 3.3.19.

3.3.2 A dual version

§3.3.20 Let (X,u) be a numerical-range space over $\mathbb{K}$. We define the *Kernel*, $K(X,u)$, of (X,u) by

$$K(X,u) := \bigcap_{f \in D(X,u)} \ker(f),$$

so that $K(X,u) = 0$ if and only if u is a vertex of $\mathbb{B}_X$. Clearly, $K(X,u)$ is a closed subspace of X satisfying $\|u + K(X,u)\| = 1$, so that

$$(X/K(X,u)\,,\, u + K(X,u))$$

becomes a new numerical-range space over $\mathbb{K}$. In the duality (X',X), we have clearly $D(X,u) \subseteq (K(X,u))^\circ$, so, keeping in mind the usual identification between $(X/K(X,u))'$ and $(K(X,u))^\circ$, we can write

$$D(X/K(X,u)\,,\, u + K(X,u)) = D(X,u). \tag{3.3.4}$$

From this we obtain

$$V(X/K(X,u)\,,\, u + K(X,u)\,,\, x + K(X,u)) = V(X,u,x) \ \text{ for every } \ x \in X,$$

which implies that $K(X/K(X,u)\,,\, u + K(X,u)) = 0$ and that

$$\left.\begin{array}{l} n(X/K(X,u)\,,\, u + K(X,u)) \text{ is the smallest non-negative number} \\ \alpha \text{ satisfying } \|x + K(X,u)\| \leqslant \alpha v(X,u,x) \text{ for every } x \in X. \end{array}\right\} \tag{3.3.5}$$

For any subset S of a vector space, we denote by $\operatorname{lin}(S)$ the linear hull of S.

Proposition 3.3.21 *Let (X,u) be a complete numerical-range space over $\mathbb{K}$. We have:*

(i) *The following conditions are equivalent:*

 (a) $\mathrm{lin}(D(X,u))$ *is norm-closed in* X'.
 (b) $\mathrm{lin}(D(X,u))$ *is* w^**-closed in* X'.
 (c) $\mathrm{lin}(D(X,u)) = (K(X,u))^\circ$ *in the duality* (X',X).
 (d) $n(X/K(X,u)\,,u+K(X,u)) > 0$.

(ii) *If* $\mathbb{K}=\mathbb{R}$, *and if* $n(X/K(X,u)\,,u+K(X,u)) > 0$, *then the closed unit ball of* $\mathrm{lin}(D(X,u))$ *is contained in*

$$\frac{1}{n(X/K(X,u)\,,u+K(X,u))}|\mathrm{co}|(D(X,u)),$$

and hence, for each $f\in\mathrm{lin}(D(X,u))$, *there are* $\alpha_1,\alpha_2\geqslant 0$ *and* $f_1,f_2\in D(X,u)$ *such that*

$$f=\alpha_1 f_1-\alpha_2 f_2 \quad and \quad \alpha_1+\alpha_2\leqslant\frac{\|f\|}{n(X/K(X,u)\,,u+K(X,u))}.$$

(iii) *If* $\mathbb{K}=\mathbb{C}$, *and if* $n(X/K(X,u)\,,u+K(X,u)) > 0$, *then the open unit ball of* $\mathrm{lin}(D(X,u))$ *is contained in*

$$\frac{1}{n(X/K(X,u)\,,u+K(X,u))}|\mathrm{co}|(D(X,u)),$$

and moreover, for each $f\in\mathrm{lin}(D(X,u))$, *there are* $\alpha_1,\alpha_2,\alpha_3,\alpha_4\geqslant 0$ *and* f_1,f_2,f_3,f_4 *in* $D(X,u)$ *such that*

$$f=\alpha_1 f_1-\alpha_2 f_2+i(\alpha_3 f_3-\alpha_4 f_4)$$

and

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4\leqslant\frac{\sqrt{2}\|f\|}{n((X/K(X,u)\,,u+K(X,u))}.$$

Proof (a)⇔(b) Since $D(X,u)$ is a w^*-compact and convex subset of X', this follows from a well-known result in functional analysis (see for example [726, Corollary V.5.9]).

(b)⇔(c) Since in the duality (X',X) we have $(\mathrm{lin}(D(X,u)))^\circ = K(X,u)$, the bipolar theorem gives that the w^*-closure of $\mathrm{lin}(D(X,u))$ is equal to $(K(X,u))^\circ$.

(c)⇔(d) Keeping in mind (3.3.4) and the usual identification

$$(X/K(X,u))'\equiv(K(X,u))^\circ,$$

condition (c) can be reformulated as

$$\mathrm{lin}(D(X/K(X,u)\,,u+K(X,u))=(X/K(X,u))'.$$

Therefore the present equivalence follows by applying Theorem 2.1.17(i) to the numerical-range space $(X/K(X,u)\,,u+K(X,u))$.

Assertions (ii) and (iii) in the present proposition follow from the above arguments and from assertions (ii), (iii), and (iv) in Theorem 2.1.17. □

§**3.3.22** The above proposition is going to be specially relevant when applied to the real space underlying a complex numerical-range space. Let (X,u) be a complex numerical-range space. Then, by Proposition 2.1.4, we have $K(X_{\mathbb{R}},u) = iH(X,u)$. In particular, iu lies in $K(X_{\mathbb{R}},u)$, and hence $n(X_{\mathbb{R}},u) = 0$. We define the *real numerical index*, $n_{\mathbb{R}}(X,u)$, of (X,u) by

$$n_{\mathbb{R}}(X,u) := n(X_{\mathbb{R}}/K(X_{\mathbb{R}},u)\,,\, u + K(X_{\mathbb{R}},u)).$$

It follows from (3.3.5) and Proposition 2.1.4 that

$$\left.\begin{array}{l} n_{\mathbb{R}}(X,u) \text{ is the smallest non-negative number } \alpha \text{ satisfying} \\ \|x + H(X,u)\| \leqslant \alpha \max\{|\Im(\lambda)| : \lambda \in V(X,u,x)\} \text{ for every } x \in X. \end{array}\right\} \tag{3.3.6}$$

Now, keeping in mind that the mapping $f \to \Re \circ f$ from $(X')_{\mathbb{R}}$ onto $(X_{\mathbb{R}})'$ is a linear homeomorphism for both the norm and w^* topologies taking $D(X,u)$ onto $D(X_{\mathbb{R}},u)$, it is enough to invoke Proposition 3.3.21 to derive the following.

Corollary 3.3.23 *Let (X,u) be a complete complex numerical-range space, and let $\text{lin}_{\mathbb{R}}(D(X,u))$ denote the real linear hull of $D(X,u)$. Then the following conditions are equivalent:*

(i) $\text{lin}_{\mathbb{R}}(D(X,u))$ *is norm-closed in X'.*
(ii) $\text{lin}_{\mathbb{R}}(D(X,u))$ *is w^*-closed in X'.*
(iii) $\text{lin}_{\mathbb{R}}(D(X,u)) = \{f \in X' : f(H(X,u)) \subseteq \mathbb{R}\}$.
(iv) $n_{\mathbb{R}}(X,u) > 0$.

Moreover, if $n_{\mathbb{R}}(X,u) > 0$, then the closed unit ball of $\text{lin}_{\mathbb{R}}(D(X,u))$ is contained in $\frac{1}{n_{\mathbb{R}}(X,u)}|\text{co}|(D(X,u))$, and hence, for each $f \in \text{lin}_{\mathbb{R}}(D(X,u))$, there are $\alpha_1, \alpha_2 \geqslant 0$ and f_1, f_2 in $D(X,u)$ such that

$$f = \alpha_1 f_1 - \alpha_2 f_2 \quad \textit{and} \quad \alpha_1 + \alpha_2 \leqslant \frac{\|f\|}{n_{\mathbb{R}}(X,u)}.$$

Remark 3.3.24 Looking at the proof of Proposition 3.3.21, we realize that, with the only exception of the implication (a)⇒(b), the whole proposition remains true without the requirement of completeness of X. Therefore, with the only exception of the implication (i)⇒(ii), Corollary 3.3.23 remains true without requiring completeness for X.

Theorem 3.3.25 *Let (X,u) be a complete complex numerical-range space such that $n(X,u) > 0$. Then the following conditions are equivalent:*

(i) $\text{lin}_{\mathbb{R}}(D(X,u)) \cap i\,\text{lin}_{\mathbb{R}}(D(X,u)) = 0$.
(ii) $X = H(X,u) + iH(X,u)$.

Moreover, if the above conditions hold, then $\text{lin}_{\mathbb{R}}(D(X,u))$ is w^*-closed in X', and we have

$$X' = \text{lin}_{\mathbb{R}}(D(X,u)) \oplus i\,\text{lin}_{\mathbb{R}}(D(X,u)).$$

Proof The implication (ii)⇒(i) is straightforward.

Assume that (i) holds. Then, by Theorem 2.1.17(iv), we have that

$$X' = \text{lin}_{\mathbb{R}}(D(X,u)) \oplus i\,\text{lin}_{\mathbb{R}}(D(X,u))$$

and that the corresponding real linear projection P onto $i\,\mathrm{lin}_{\mathbb{R}}(D(X,u))$ is norm-continuous (with $\|P\| \leqslant \frac{\sqrt{2}}{n(X,u)}$), so $\mathrm{lin}_{\mathbb{R}}(D(X,u))$ is norm-closed in X'. Therefore, by the implication (i)⇒(ii) in Corollary 3.3.23, $\mathrm{lin}_{\mathbb{R}}(D(X,u))$ is w^*-closed in X'. On the other hand, by the implication (i)⇒(iii) in Corollary 3.3.23, we have $f = 0$ whenever f is in X' with $f(H(X,u) + iH(X,u)) = 0$, and hence $H(X,u) + iH(X,u)$ is dense in X. Finally, since $H(X,u) + iH(X,u)$ is a closed subspace of X (by Lemma 3.3.14), we see that condition (ii) is fulfilled. □

Now, combining Lemma 2.2.5 and Theorems 3.3.11 and 3.3.25, we get the following.

Corollary 3.3.26 *Let A be a norm-unital complete normed complex algebra. Then A is a non-commutative JB^*-algebra (for its own norm and some involution) if and only if*

$$\mathrm{lin}_{\mathbb{R}}(D(A,\mathbf{1})) \cap i\,\mathrm{lin}_{\mathbb{R}}(D(A,\mathbf{1})) = 0.$$

Finally, by invoking Fact 3.3.2, we derive the following.

Corollary 3.3.27 *Let A be a norm-unital complete normed associative complex algebra. Then A is a C^*-algebra (for its own norm and some involution) if and only if*

$$\mathrm{lin}_{\mathbb{R}}(D(A,\mathbf{1})) \cap i\,\mathrm{lin}_{\mathbb{R}}(D(A,\mathbf{1})) = 0.$$

Now that the main goal in this subsection has been achieved, let us involve Subsection 2.9.1 to obtain some interesting by-products.

Proposition 3.3.28 *Let X be a Banach space over $\mathbb{K}$, and let u be in $\mathbb{S}_X$ with*

$$n(X/K(X,u)\,,\,u + K(X,u)) > 0$$

and such that the duality mapping of X is norm-weak usc at u. Then we have:

(i) *$K(X'',u)$ is the w^*-closure of $K(X,u)$ in X''.*
(ii) *$n(X/K(X,u)\,,\,u + K(X,u)) = n(X''/K(X'',u)\,,\,u + K(X'',u))$.*

Proof Since $n(X/K(X,u)\,,\,u + K(X,u)) > 0$, Proposition 3.3.21(i) applies, so that we have $(K(X,u))^\circ = \mathrm{lin}(D(X,u))$ in the duality (X',X). On the other hand, since the duality mapping of X is norm-weak usc at u, Theorem 2.9.8 applies, so that $[\mathrm{lin}(D(X,u))]^\circ = K(X'',u)$ in the duality (X'',X'). It follows from the bipolar theorem that assertion (i) holds.

By the above paragraph, we have that $K(X'',u) = K(X,u)^{\circ\circ}$. Then

$$X''/K(X'',u) = X''/K(X,u)^{\circ\circ} \equiv (X/K(X,u))'',$$

and assertion (ii) follows from Theorem 2.1.17(v). □

Keeping in mind that the norm-weak upper semicontinuity of the duality mapping of a complex normed space X at a point goes down to $X_{\mathbb{R}}$, it is enough to combine §3.3.22 and Proposition 3.3.28 to get the following.

Corollary 3.3.29 *Let X be a complex Banach space, and let u be in $\mathbb{S}_X$ with $n_{\mathbb{R}}(X,u) > 0$ and such that the duality mapping of X is norm-weak usc at u. Then we have:*

(i) *$H(X'',u)$ is the w^*-closure of $H(X,u)$ in X''.*
(ii) *$n_{\mathbb{R}}(X,u) = n_{\mathbb{R}}(X'',u)$.*

Now, by invoking Fact 2.9.1, Corollary 3.3.29 above yields the following

Corollary 3.3.30 *Let A be a norm-unital complete normed associative complex algebra with $n_{\mathbb{R}}(A,\mathbf{1}) > 0$. Then we have:*

(i) *$H(A'',\mathbf{1})$ equals the w^*-closure of $H(A,\mathbf{1})$ in A''.*
(ii) *$n_{\mathbb{R}}(A,\mathbf{1}) = n_{\mathbb{R}}(A'',\mathbf{1})$.*

The result in Corollary 3.3.30(i) is the best possible in general, in the sense that $H(A'',\mathbf{1})$ does not always equal the w^*-closure of $H(A,\mathbf{1})$ in A''.

Example 3.3.31 Let A be the disc algebra (cf. Example 2.9.67). Note first that, since valuations of elements of A at points of $\mathbb{B}_{\mathbb{C}}$ lie in $D(A,\mathbf{1})$, elements of $H(A,\mathbf{1})$ are real-valued functions, and hence we have

$$H(A,\mathbf{1}) = \mathbb{R}\mathbf{1}. \tag{3.3.7}$$

Now, let n be in $\mathbb{N}$, and consider the element x_n of A defined by

$$x_n(z) := \frac{\sqrt[n]{1+z} - \sqrt[n]{1-z}}{\sqrt[n]{1+z} + \sqrt[n]{1-z}}$$

for every $z \in \mathbb{B}_{\mathbb{C}}$, where $\sqrt[n]{\cdot}$ stands for the principal determination of the complex nth root. A straightforward but tedious computation shows that $x_n(\mathbb{B}_{\mathbb{C}})$ is the intersection of the closed discs $D_n^{\pm}$ of centres $c_n^{\pm}$ and radius R_n, where

$$c_n^{\pm} = \pm \frac{\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{1-\cos^2\frac{\pi}{2n}}\, i \ \text{ and } \ R_n = \frac{\sin\frac{\pi}{2n}}{1-\cos^2\frac{\pi}{2n}}.$$

(As a hint, consider the Möbius transform $\phi(z) := \frac{1+z}{1-z}$ on the Riemann sphere, and note that $x_n(z) = \phi^{-1}(\sqrt[n]{\phi(z)})$ for every $z \in \mathbb{B}_{\mathbb{C}}$.) Then, by Corollaries 2.1.2 and 2.9.61, we see that $V(A,\mathbf{1},x_n) = x_n(\mathbb{B}_{\mathbb{C}})$, and consequently

$$|\Im(f(x_n))| \leqslant \frac{\sin\frac{\pi}{2n}}{1+\cos\frac{\pi}{2n}} \ \text{ for every } \ f \in D(A,\mathbf{1}). \tag{3.3.8}$$

Now take a cluster point F to the sequence x_n in the weak* topology of A''. Since $x_n(1) = 1$ and $x_n(-1) = -1$ for every n, we realize that F is not a multiple of $\mathbf{1}$. On the other hand, (3.3.8) implies that $\Im(F(f)) = 0$ for every $f \in D(A,\mathbf{1})$, so that, by Lemma 2.3.44, F lies in $H(A'',\mathbf{1})$. It follows from (3.3.7) that $H(A,\mathbf{1})$ is not w^*-dense in $H(A'',\mathbf{1})$.

§3.3.32 Looking at the roots of Corollary 3.3.30, the above example shows that the equivalent conditions (i)–(iv) in Corollary 3.3.23 need not be automatically fulfilled, even if the complete complex numerical-range space (X,u) satisfies $n(X,u) > 0$, nor even if X is a norm-unital complete normed associative and commutative complex algebra, and u is the unit of X. We note that, in this last case, we have $n(X,u) \geqslant \frac{1}{e}$ (by Proposition 2.1.11), and that, in the case of the example, we have in fact $n(X,u) = 1$.

3.3.3 Historical notes and comments

JB^*-algebras were introduced by Kaplansky in his final lecture to the 1976 St. Andrews Colloquium of the Edinburgh Mathematical Society, pointing out its potential importance. Answering a problem raised by Kaplansky, unital JB^*-algebras were first studied by Wright in [641], who established a strong mutual dependence with unital JB-algebras (see Fact 3.4.9 below). Shortly later, unital JB^*-algebras were reconsidered by Braun, Kaup, and Upmeier [126], who proved that unital JB^*-algebras are JB^*-triples in a natural way (see Theorem 4.1.45 below), and determined those JB^*-triples which underlie unital JB^*-algebras (see Theorem 4.1.55 below). Simultaneously, Wright and Youngson [642, 643] proved a Russo–Dye–Palmer-type theorem for unital JB^*-algebras (a germ of which has been stated in Proposition 3.3.7), and studied unit-preserving surjective linear isometries between unital JB^*-algebras (see Proposition 3.4.25 below). Non-unital JB^*-algebras were considered by Youngson [655], and JB^*-algebras which are dual Banach spaces were considered by Edwards [222].

Non-commutative JB^*-algebras arose in the literature as a necessity for a right formulation of the non-associative Vidav–Palmer theorem. Without enjoying their name, they were first considered by Kaidi, Martínez, and Rodríguez, who, in [362, Theorem 11], proved a forerunner of Theorem 3.3.11. A systematic study of non-commutative JB^*-algebras began with the work of Payá, Pérez, and Rodríguez [481, 482], and continued shortly later in the papers of Braun [124] and Alvermann–Janssen [19].

In the above two paragraphs, we have referred the reader to the pioneering papers on JB^*-algebras and non-commutative JB^*-algebras. Most results in these papers, as well as more recent developments on the topic, will be included in our work at the appropriate point.

Results from Lemma 3.3.5 to Proposition 3.3.7 are due to Wright and Youngson [642]. A definitive version of Proposition 3.3.7 (with 'JB^*-algebra' instead of 'JC^*-algebra') will be discussed later (see Corollary 3.4.7). Theorem 3.3.11 and the proof given here (including Lemmas 3.3.8 and 3.3.10) are due to Youngson [652, 654]. In fact, Youngson proved Lemma 3.3.8 in [652], and Theorem 3.3.11 in [654], under the assumptions (unnecessary today in view of Theorems 2.3.8 and 2.4.11) that the natural involution of A is an algebra involution, and that A is a non-commutative Jordan algebra. An independent proof of Theorem 3.3.11, with the same restrictions, can be seen in [514]. The actual formulation of Theorem 3.3.11 appeared first in [515].

A better version of Lemma 3.3.10, due to Upmeier [814, Corollary 22.8], is proved in Theorem 3.3.38 below.

Definition 3.3.33 Let A be a Jordan algebra. A subalgebra B of A is called a *strongly associative subalgebra* if $[L_a, L_b] = 0$ for all a, b in B. This condition is clearly equivalent to

$$(bx)a - b(xa) = 0 \text{ for all } a, b \text{ in } B \text{ and } x \text{ in } A.$$

In particular, every strongly associative subalgebra of A is associative.

Now Proposition 2.4.13(i)–(ii) reads as follows.

Fact 3.3.34 *The subalgebra generated by each element of a Jordan algebra is strongly associative.*

Lemma 3.3.35 *Let A be a Jordan algebra over $\mathbb{K}$. Then*

$$U_{xz,y} = U_{x,y}L_z + U_{z,y}L_x - L_y U_{x,z} = L_z U_{x,y} + L_x U_{z,y} - U_{x,z}L_y$$

for all x, y, z in A.

Proof By taking $a = x$, $b = z$, and $c = y$ in the identity (2.4.1) in the proof of Proposition 2.4.13, we see that

$$2L_{xy}L_x + L_{x^2}L_y = 2L_x L_{xy} + L_y L_{x^2} \text{ for all } x, y \in A. \tag{3.3.9}$$

On the other hand, by taking $a = c = x$ and $b = y$ in the identity (2.4.3) we obtain

$$L_{x^2}L_y + 2L_{xy}L_x = 2L_x L_y L_x + L_{x^2 y} \text{ for all } x, y \in A. \tag{3.3.10}$$

It follows from (3.3.9) and (3.3.10) that

$$2L_{xy}L_x + L_{x^2}L_y = 2L_x L_y L_x + L_{x^2 y} \text{ for all } x, y \in A. \tag{3.3.11}$$

Now, by using (3.3.10), we deduce that

$$\begin{aligned} U_{x^2,y} &= L_{x^2}L_y + L_y L_{x^2} - L_{x^2 y} \\ &= 2L_x L_y L_x - 2L_{xy}L_x + L_y L_{x^2} \\ &= 2(L_x L_y + L_y L_x - L_{xy})L_x - L_y(2L_x^2 - L_{x^2}) \\ &= 2U_{x,y}L_x - L_y U_x \end{aligned}$$

for all x, y in A, and linearizing in x we conclude that

$$U_{xz,y} = U_{x,y}L_z + U_{z,y}L_x - L_y U_{x,z} \text{ for all } x, y, z \in A.$$

Analogously, by using (3.3.11), we derive that

$$U_{x^2,y} = 2L_x U_{x,y} - U_x L_y \text{ for all } x, y \in A,$$

and linearizing in x we conclude that

$$U_{xz,y} = L_z U_{x,y} + L_x U_{z,y} - U_{x,z}L_y \text{ for all } x, y, z \in A. \qquad \square$$

§3.3.36 Let A be a Jordan algebra over $\mathbb{K}$, and let B be a subalgebra of A. We write $\mathscr{M}(B)^A$ to denote the subalgebra of $L(A)$ generated by I_A and the set $\{L_b : b \in B\}$. Since an associative algebra is commutative if and only if it is generated by a commutative subset, it turns out that B is a strongly associative subalgebra of A if and only if the algebra $\mathscr{M}(B)^A$ is commutative.

Proposition 3.3.37 *Let A be a Jordan algebra over $\mathbb{K}$, let B be a strongly associative subalgebra of A, and let b, c be in B. Then $U_{bc} = U_b U_c$.*

Proof Setting $x = y = b$ and $z = c$ in the first equality of Lemma 3.3.35, and using the commutativity of $\mathscr{M}(B)^A$, we obtain

$$U_{bc,b} = U_b L_c \text{ for all } b, c \in B. \tag{3.3.12}$$

Linearizing (3.3.12) with respect to b in B, we obtain

$$U_{bc,b'} + U_{b'c,b} = 2U_{b,b'}L_c \quad \text{for all} \quad b, b', c \in B. \tag{3.3.13}$$

Now we have

$$\begin{aligned} U_b U_c &= U_b(2L_c^2 - L_{c^2}) \\ &= 2U_{bc,b}L_c - U_{bc^2,b} && \text{(by (3.3.12))} \\ &= U_{(bc)c,b} + U_{bc} - U_{bc^2,b} && \text{(by (3.3.13))} \\ &= U_{bc}, \end{aligned}$$

which proves the statement. □

Lemma 3.3.35 and Proposition 3.3.37 have been taken from [754, pp. 38–9].

Theorem 3.3.38 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, and let x be in A. Then we have $U_{\exp(x)} = \exp(2L_x)$.*

Proof Since the closure of a strongly associative subalgebra of A remains a strongly associative subalgebra of A, it follows from Fact 3.3.34 and Proposition 3.3.37 that the mapping $t \to U_{\exp(tx)}$ from $\mathbb{R}_0^+$ to $BL(A)$ is a continuous one-parameter semigroup in $BL(A)$. Therefore, by Theorem 1.1.31, there exists some $F \in BL(A)$ such that $U_{\exp(tx)} = \exp(tF)$ for every $t \in \mathbb{R}_0^+$, and moreover

$$F = \lim_{t \to 0^+} \frac{U_{\exp(tx)} - I_A}{t}.$$

But, a straightforward computation of the limit in the right-hand side of the above equality shows that $F = 2L_x$, and the result follows. □

The finite-dimensional forerunner of Theorem 3.3.38 above can be seen in [700, Satz XI.2.2].

Lemma 3.3.12 is due to Wright [641], whereas Proposition 3.3.13 (that the involution of a JB^*-algebra is an isometry) is due to Youngson [652]. It is worth mentioning that this last result was a (today redundant) requirement in Kaplansky's definition of JB^*-algebras, and that this requirement was also assumed in [641]. Lemma 3.3.14 is folklore. Indeed, it is nothing other than a straightforward abstract version of [694, Lemma 5.8]. Concerning Corollary 3.3.16 (that the natural involution of a V-algebra is an isometry), let us say that we do not know of any proof avoiding Theorem 3.3.11.

Corollary 3.3.18 is taken from [515]. For a forerunner, see [507, Lemma 1.4] and the comment following it. For other relevant results on contractive projections, the reader is referred to the papers of Bunce–Peralta [136] and Effros–Størmer [229].

The dual version of the associative Vidav–Palmer theorem, given by Corollary 3.3.27, is originally due to Moore [446], and can be found in Bonsall and Duncan [695, pp. 103–9], where a light version of Example 3.3.31 is added (with an argument quite different from ours) to show the content of §3.3.32. Proposition 3.3.21, Corollary 3.3.23, and Theorem 3.3.25 are due to Martínez, Mena, Payá, and Rodríguez [425], and become a non-straightforward abstract version of Moore's argument. The dual version of the non-associative Vidav–Palmer theorem, given by Corollary 3.3.26, as well as Proposition 3.3.28 and Corollaries 3.3.29 and 3.3.30, are also taken from [425]. Without enjoying its name, the real numerical index of a norm-unital normed complex associative algebra A first arises in Smith's paper [591], who defines it as the smallest non-negative number α satisfying $\|a+H(A,\mathbf{1})\| \leqslant \alpha \max\{|\Im(\lambda)| : \lambda \in V(A,\mathbf{1},a)\}$ for every $a \in A$ (compare assertion (3.3.6) in §3.3.22), and establishes Example 3.3.31 (with arguments close to ours) and the associative forerunner of Corollary 3.3.30(i). The associative forerunner of Corollary 3.3.30(ii) is also due to Smith [592].

3.4 Beginning the theory of non-commutative JB^*-algebras

Introduction This section is devoted to establishing the basic theory of non-commutative JB^*-algebras. Since many results can be reduced to the commutative case (via Fact 3.3.4), Wright's theorem [641] (see Fact 3.4.9), asserting that $A \to H(A,*)$ becomes an equivalence from the category of all JB^*-algebras onto that of all JB-algebras, turns out specially relevant. As a consequence, Proposition 3.3.7 attains its definitive form as a Russo–Dye–Palmer-type theorem for unital non-commutative JB^*-algebras, which is due to Wright and Youngson [642] (see Corollary 3.4.7).

In a second subsection, we study surjective linear isometries between unital non-commutative JB^*-algebras, and show in Proposition 3.4.31 and Corollary 3.4.32 that, in the particular case of alternative C^*-algebras, both the Bohnenblust–Karlin Theorem 2.1.27 and Kadison's Theorem 2.2.29 remain true verbatim. However, in the general case no reasonable version of Kadison's theorem can be expected. Indeed, according to Antitheorem 3.4.34, due to Braun, Kaup, and Upmeier [126], there exist linearly isometric unital JC^*-algebras which are not $*$-isomorphic.

In a third subsection, we prove a theorem of Dixmier [723] (see Theorem 3.4.49) implying that continuous algebra automorphisms of a complete normed complex algebra, which are near the identity operator, can be written as exponentials of continuous derivations. Dixmier's arguments are also applied to provide (in Theorem 3.4.42 and Corollary 3.4.44) nontrivial information about the spectrum of continuous derivations and automorphisms of complete normed non-nilpotent complex algebras.

Dixmier's theorem just reviewed becomes one of the ingredients in the proof of a structure theorem for bijective algebra homomorphisms between non-commutative JB^*-algebras (Theorem 3.4.75), which generalizes the classical one by Okayasu [466] for C^*-algebras.

3.4.1 JB-algebras versus JB^*-algebras

We begin this subsection with the following.

Proposition 3.4.1 *Let A be a non-commutative JB^*-algebra. We have:*

(i) *Every closed associative $*$-subalgebra of A is a C^*-algebra.*
(ii) *For every $h \in H(A,*)$, the closed subalgebra of A generated by h (occasionally, by h and $\mathbf{1}$ if A is unital) is $*$-invariant, and is indeed a commutative C^*-algebra.*

Proof Assertion (i) follows from Fact 3.3.2.

Let h be in $H(A,*)$, and let B stand for the closed subalgebra of A generated by h (occasionally, by h and $\mathbf{1}$ if A is unital). By Propositions 3.3.13 and 1.2.25, B is $*$-invariant. On the other hand, by Proposition 2.4.19 and Corollary 2.4.18, B is associative and commutative. It follows from assertion (i) that B is a commutative C^*-algebra. Thus, the proof of assertion (ii) is concluded. □

Now we invoke a new ingredient on JB-algebras to derive fundamental basic results in the theory of non-commutative JB^*-algebras. As stated in Proposition 3.1.4(i), if A is a JB-algebra, then the closed subalgebra of A generated by each element is isometrically isomorphic to $C_0^{\mathbb{R}}(E)$, for a suitable locally compact Hausdorff topological space E. A relevant converse is given by Proposition 3.4.2 immediately below.

Proposition 3.4.2 *Let A be a complete normed Jordan real algebra such that the closed subalgebra of A generated by each element is isometrically isomorphic to $C_0^{\mathbb{R}}(E)$, for a suitable locally compact Hausdorff topological space E. Then A is a JB-algebra.*

The proof of Proposition 3.4.2 is implicitly contained in [738]. Indeed, it is the core of the proof of [738, Proposition 3.8.2]. For the sake of completeness, we reproduce here the argument of Hanche-Olsen and Størmer.

Proof Let $\tilde{A}$ be the unital extension of A. Then $\tilde{A}$ is a complete normed Jordan real algebra [738, 3.3.1], and for all $a \in A$ it follows from the assumption that the closed subalgebra $C(a)$ of A generated by a is isomorphic to a JB-algebra. Thus as in the proof of [738, 3.3.9] $\tilde{A}$ is an order unit space in which $-1 \leqslant a \leqslant 1$ implies $0 \leqslant a^2 \leqslant 1$. By [738, 3.1.6] then, $\tilde{A}$ is a JB-algebra in the order norm, hence so is A. Since the order norm coincides with the given norm on each $C(a)$, A is a JB-algebra in the given norm. □

Invoking Propositions 3.4.2 and 3.4.1(ii), and the commutative Gelfand–Naimark theorem (see Theorem 1.2.4), we derive the following.

Corollary 3.4.3 *Let A be a non-commutative JB^*-algebra. Then $H(A,*)$ becomes a JB-algebra under the product of A^{sym} and the norm of A.*

As a first consequence, we can prove the following generalization of Corollary 1.2.52.

Proposition 3.4.4 *Let A and B be non-commutative JB^*-algebras, and let $F : A \to B$ be an algebra $*$-homomorphism. Then F is contractive. Moreover, if F is injective, then F is an isometry.*

Proof Let $F : A \to B$ be any algebra $*$-homomorphism. In view of Lemma 3.3.12, to prove that F is contractive it is enough to show that it is continuous. To this end, note that, by Corollary 3.4.3, $H(A,*)$ and $H(B,*)$ become JB-algebras in a natural way, and that F induces an algebra homomorphism from $H(A,*)$ to $H(B,*)$. Then, by Proposition 3.1.4(ii), F is continuous on $H(A,*)$. Therefore, since the direct sums $A = H(A,*) \oplus iH(A,*)$ and $B = H(B,*) \oplus iH(B,*)$ are topological (by Proposition 3.3.13), we deduce that F is continuous on A, as desired.

Now, assume that the algebra $*$-homomorphism F above is injective. Then the induced algebra homomorphism from $H(A,*)$ to $H(B,*)$ is injective, and hence, by Proposition 3.1.4(ii), it has closed range in $H(B,*)$. Since the direct sum $B = H(B,*) \oplus iH(B,*)$ is topological, we derive that F has closed range in B, and hence that $F(A)$ is a non-commutative JB^*-algebra. This allows us to assume without loss of generality that F is bijective. Then, the fact that F is isometric follows by applying the first paragraph to F and F^{-1}. $\square$

Remark 3.4.5 Proposition 3.4.4 holds with 'Jordan-$*$-homomorphism' instead of 'algebra $*$-homomorphism'. We have not emphasized this apparent generalization because it follows by applying the proposition itself to the JB^*-algebras A^{sym} and B^{sym} (see Fact 3.3.4)

The proof of the next result is quite similar to that of Lemma 3.3.8.

Proposition 3.4.6 *Let A be a JB^*-algebra, and let x be in A. Then the closed subalgebra of A generated by $\{x,x^*\}$ is $*$-invariant, and is indeed a JC^*-algebra.*

Proof By Corollary 3.4.3, $H(A,*)$ is a closed real subalgebra of A and, endowed with the restriction of the norm of A, becomes a JB-algebra. Now, write $x = h+ik$, with h,k in $H(A,*)$, and let J stand for the closed subalgebra of $H(A,*)$ generated by $\{h,k\}$. Then, by Proposition 3.1.3, there exists a C^*-algebra B, together with an isometric algebra homomorphism ϕ from J onto a closed subalgebra of the JB-algebra $H(B,*)$. Since the direct sums $A = H(A,*) \oplus iH(A,*)$ and $B = H(B,*) \oplus iH(B,*)$ are topological, $J+iJ$ and $\phi(J)+i\phi(J)$ are closed subalgebras of A and B^{sym}, respectively, and the extension of ϕ by complex linearity becomes a bicontinuous algebra $*$-homomorphism from $J+iJ$ onto the JC^*-algebra $\phi(J)+i\phi(J)$. The proof is concluded by noticing that $J+iJ$ is nothing other than the closed subalgebra of A generated by $\{x,x^*\}$, and that, by Proposition 3.4.4, ϕ is an isometry. $\square$

Now, we can formulate and prove a Russo–Dye–Palmer-type theorem for non-commutative JB^*-algebras.

Corollary 3.4.7 *Let A be a unital non-commutative JB^*-algebra. Then $\mathbb{B}_A = \overline{\mathrm{co}}(E)$, where $E := \{\exp(ih) : h \in H(A,*)\}$.*

Proof The inclusion $\mathbb{B}_A \supseteq \overline{\mathrm{co}}(E)$ follows from Lemma 2.2.5 and Corollary 2.1.9(iii).

To prove the converse inclusion, note that, by Lemma 2.4.17 and Fact 3.3.4, we may assume that A is commutative. Let x be in $\mathbb{B}_A$. Then, by Proposition 3.4.6, the closed subalgebra of A (say B) generated by $\{\mathbf{1},x,x^*\}$ is a unital JC^*-algebra. Therefore, by Proposition 3.3.7, we have

$$x \in \overline{\mathrm{co}}(\{\exp(ih) : h \in H(B,*)\}) \subseteq \overline{\mathrm{co}}(E). \qquad \square$$

Theorem 3.4.8 immediately below becomes the main result in the present subsection.

Theorem 3.4.8 *Let B be a JB-algebra. Then there is a unique JB^*-algebra A such that $B = H(A,*)$.*

Proof The uniqueness of A is easy. Indeed, algebraically viewed, A must coincide with the algebra complexification of B, and the JB^*-algebra involution $*$ must be the canonical involution of that complexification. Moreover, if two norms convert $(A,*)$ into a JB^*-algebra, then, by Proposition 3.4.4, they must coincide.

To prove the existence, note first that, since B can be isometrically imbedded into a unital JB-algebra (by Corollary 3.1.11), we may assume that B is unital. Let A denote the projective normed complexification of B, so that $(A,\|\cdot\|_\pi)$ becomes a complete normed unital Jordan complex algebra, and let $*$ stand for the canonical involution of A, so that $*$ becomes an isometric conjugate-linear algebra involution on $(A,\|\cdot\|_\pi)$ satisfying $H(A,*)=B$. Let F denote the closed convex hull in A of the set $\{\exp(ih) : h\in B\}$. Then F is a closed and absolutely convex subset of A. Moreover, for $h\in B$, we have

$$\|\exp(ih)\|_\pi = \|\cos(h)+i\sin(h)\|_\pi \leqslant \|\cos(h)\|+\|\sin(h)\| \leqslant 2$$

(where, for the last inequality, we have applied Corollary 3.1.5), which implies $F\subseteq 2\mathbb{B}_{(A,\|\cdot\|_\pi)}$. Now, let h,k be in B, and let J stand for the closed subalgebra of B generated by $\{\mathbf{1},h,k\}$. Then, by Proposition 3.1.3, there exists a C^*-algebra C, together with an isometric algebra homomorphism ϕ from J onto a closed subalgebra of the JB-algebra $H(C,*)$. Since the direct sums $A=B\oplus iB$ and $C=H(C,*)\oplus iH(C,*)$ are topological, $J+iJ$ and $\phi(J)+i\phi(J)$ are closed $*$-subalgebras of A and C^{sym}, respectively, and the extension of ϕ (say ψ) by complex linearity becomes a bicontinuous algebra $*$-isomorphism from $J+iJ$ onto the JC^*-algebra $\phi(J)+i\phi(J)$. Therefore we have $\|\exp(i\psi(h))\bullet\exp(i\psi(k))\|\leqslant 1$, and hence, invoking Proposition 3.3.7, we derive that

$$\exp(i\psi(h))\bullet\exp(i\psi(k))\in\overline{\mathrm{co}}(\{\exp(il) : l\in\phi(J)\}).$$

By applying ψ^{-1}, we get $\exp(ih)\exp(ik)\in\overline{\mathrm{co}}(\{\exp(il) : l\in J\})$, and hence

$$\exp(ih)\exp(ik) \ \text{ lies in } \ F \ \text{ for all } \ h,k\in B. \tag{3.4.1}$$

Assume that $\|h+ik\|_\pi\leqslant\frac{1}{2}$. Then we have

$$\|\psi(h+ik)\|\leqslant\|\phi(h)\|+\|\phi(k)\|=\|h\|+\|k\|\leqslant 2\|h+ik\|_\pi\leqslant 1,$$

and hence, applying Proposition 3.3.7 again, we derive that

$$\psi(h+ik)\in\overline{\mathrm{co}}(\{\exp(il) : l\in\phi(J)\}).$$

By applying ψ^{-1}, we get $h+ik\in\overline{\mathrm{co}}(\{\exp(il) : l\in J\})$, and hence

$$h+ik \text{ lies in } F \text{ whenever } h,k \text{ are in } B \text{ with } \|h+ik\|_\pi\leqslant\tfrac{1}{2}. \tag{3.4.2}$$

From (3.4.2) we deduce that $\mathbb{B}_{(A,\|\cdot\|_\pi)}\subseteq 2F$. Since we already know that F is a closed and absolutely convex subset of A with $F\subseteq 2\mathbb{B}_{(A,\|\cdot\|_\pi)}$, it follows that F is the closed unit ball for an equivalent vector space norm $|\!|\!|\cdot|\!|\!|$ on A. Moreover, since (3.4.1) implies that $FF\subseteq F$, we realize that $|\!|\!|\cdot|\!|\!|$ is in fact an algebra norm on A satisfying $|\!|\!|\mathbf{1}|\!|\!|=1$. Then, for $h\in B$, we clearly have $|\!|\!|\exp(irh)|\!|\!|=1$ for every $r\in\mathbb{R}$, so h lies in $H((A,|\!|\!|\cdot|\!|\!|),\mathbf{1})$ (by Corollary 2.1.9(iii)). Now, $(A,|\!|\!|\cdot|\!|\!|)$ is a complete commutative V-algebra whose natural involution is $*$, and hence, by Theorem 3.3.11, $(A,|\!|\!|\cdot|\!|\!|,*)$ is a JB^*-algebra. Moreover, by Corollary 3.4.3 and Proposition 3.1.4(ii), we have

$|||h||| = \|h\|$ for every $h \in B$. Thus $H(A, |||\cdot|||, *) = B$ isometrically, and the proof is complete. $\square$

Corollary 3.4.3 and Theorem 3.4.8 provide us with the following.

Fact 3.4.9 *Let $\mathscr{A}$ denote the category whose objects are the JB^*-algebras, and whose morphisms are the algebra $*$-homomorphisms, and let $\mathscr{B}$ stand for the category whose objects are the JB-algebras, and whose morphisms are the algebra homomorphisms. Then $A \to H(A,*)$ establishes a bijective equivalence from $\mathscr{A}$ to $\mathscr{B}$.*

Keeping in mind that every JB-algebra can be enlarged to a unital JB-algebra (cf. Corollary 3.1.11), Fact 3.4.9 above implies the following.

Corollary 3.4.10 *Every JB^*-algebra can be enlarged to a unital JB^*-algebra.*

Corollary 3.4.10 above remains true with 'non-commutative JB^*-algebra' instead of 'JB^*-algebra', but this will be proved later (see Corollary 3.5.36).

§3.4.11 Let B,C be non-commutative JB^*-algebras. Then the algebra direct product $B \times C$ becomes a non-commutative JB^*-algebra under the involution defined coordinate-wise, and the norm $\|(b,c)\| := \max\{\|b\|, \|c\|\}$.

Corollary 3.4.12 *Let A be a JB^*-algebra. Then there are a unique involution and a unique norm on the unital extension $A_{\mathbb{1}}$ of A extending the involution and the norm of A, and converting $A_{\mathbb{1}}$ into a JB^*-algebra.*

Proof As we pointed out when we dealt with C^*-algebras (see the proof of Proposition 1.2.44), the unique conjugate-linear algebra involution $*$ on $A_{\mathbb{1}} = \mathbb{C}\mathbb{1} \oplus A$ extending that of A is given by $(\lambda\mathbb{1} + a)^* := \bar{\lambda}\mathbb{1} + a^*$. Then the uniqueness of the desired norm on $A_{\mathbb{1}}$ follows from Proposition 3.4.4. To prove the existence of such a norm on $A_{\mathbb{1}}$, we distinguish two cases depending on whether or not A has a unit.

First assume that A has a unit $\mathbf{1}$. Then we argue as in the proof of Proposition 1.2.44, with §3.4.11 instead of §1.2.43, to conclude that the desired extended norm on $A_{\mathbb{1}}$ is given by

$$\|a + \lambda(\mathbb{1} - \mathbf{1})\| := \max\{\|a\|, |\lambda|\}.$$

Now, assume that A does not have a unit. By Corollary 3.4.10, there is a unital JB^*-algebra B containing A isometrically as a $*$-subalgebra. By the universal property of $A_{\mathbb{1}}$, the inclusion $A \hookrightarrow B$ induces an injective algebra $*$-homomorphism from $A_{\mathbb{1}}$ to B, so that $A_{\mathbb{1}}$ becomes a JB^*-algebra. $\square$

Now we can prove the following generalization of Proposition 2.3.43.

Proposition 3.4.13 *Let A be a non-commutative JB^*-algebra, and let M be a closed ideal of A. Then M is $*$-invariant, and A/M is a non-commutative JB^*-algebra for the quotient norm and the quotient involution.*

Proof Assume at first that A is commutative. Then, arguing as in the proof of Proposition 2.3.43, with Theorem 3.3.11 instead of Theorem 2.3.32, and Corollary 3.4.12 instead of Proposition 1.2.44, the result follows.

Now, remove the additional assumption that A is commutative. By a part of Fact 3.3.4, A^{sym} is a JB^*-algebra. Since M remains a closed ideal of A^{sym}, it follows from the above paragraph that M is $*$-invariant, and that A^{sym}/M is a JB^*-algebra in the natural way. Finally, since $(A/M)^{\mathrm{sym}} = A^{\mathrm{sym}}/M$, the remaining part of Fact 3.3.4 applies, so that A/M is a non-commutative JB^*-algebra in the natural way. □

Since quotients of alternative algebras are alternative algebras, it is enough to invoke Fact 3.3.2, to derive the following.

Corollary 3.4.14 *Let A be an alternative C^*-algebra, and let M be a closed ideal of A. Then M is $*$-invariant, and A/M is an alternative C^*-algebra for the quotient norm and the quotient involution.*

Now we need to involve in our development a deep result in the theory of Jordan algebras. It is called the *fundamental formula*, and reads as follows.

Proposition 3.4.15 [754, p. 52] *Let A be a Jordan algebra over $\mathbb{K}$, and let a,b be in A. Then we have*

$$U_{U_a(b)} = U_a U_b U_a. \tag{3.4.3}$$

A complete proof of the above proposition, concluding in §4.2.66, will be given in our work (see Remark 3.4.83 below for details).

For any algebra A over $\mathbb{K}$, let $(a,b) \to U_{a,b}$ stand for the unique symmetric bilinear mapping from $A \times A$ to $L(A)$ such that $U_{a,a} = U_a$ for every $a \in A$. In the case that A is a Jordan algebra, the fundamental formula (3.4.3) can be linearized in the variable b to get

$$U_{U_a(b),U_a(c)} = U_a U_{b,c} U_a. \tag{3.4.4}$$

Corollary 3.4.16 *Let A be a Jordan algebra, and let a,b,c be in A. Then we have*

$$U_b U_{a,c}(b^2) = U_b(a) U_b(c).$$

Proof We may assume that A is unital. Then, by (3.4.4), we have

$$U_b U_{a,c}(b^2) = U_b U_{a,c} U_b(\mathbf{1}) = U_{U_b(a),U_b(c)}(\mathbf{1}) = U_b(a) U_b(c). \qquad \square$$

Proposition 3.4.17 *Let A be a non-commutative JB^*-algebra. Then we have*

$$\|U_{a,c}(b)\| \leqslant \|a\|\|b\|\|c\| \text{ for all } a,b,c \in A. \tag{3.4.5}$$

Proof By Facts 3.3.3 and 3.3.4, we may assume that A is commutative. Then, by Corollary 3.4.10, we may additionally assume that A is unital. Let h,k be in $H(A,*)$. Then, by Proposition 3.4.6 (with $x := h+ik$), the closed subalgebra of A generated by $\{\mathbf{1},h,k\}$ is a JC^*-algebra, and hence, working in a C^*-algebra envelope, we have

$$U_{\exp(ik)} U_{\exp(-ik)}(h) = U_{\exp(-ik)} U_{\exp(ik)}(h) = h \text{ for all } h,k \in H(A,*) \tag{3.4.6}$$

and

$$\|U_{\exp(ik)}(\exp(ih))\| = 1 \text{ for all } h,k \in H(A,*). \tag{3.4.7}$$

Now, let k be in $H(A,*)$. Then (3.4.6) gives

$$U_{\exp(ik)}U_{\exp(-ik)} = U_{\exp(-ik)}U_{\exp(ik)} = I_A \text{ for every } k \in H(A,*) \tag{3.4.8}$$

and, since the set $\{a \in A : \|U_{\exp(ik)}(a)\| \leqslant 1\}$ is closed and convex, it is enough to invoke (3.4.7) and Corollary 3.4.7 to conclude that

$$\|U_{\exp(ik)}(a)\| \leqslant \|a\| \text{ for all } k \in H(A,*) \text{ and } a \in A. \tag{3.4.9}$$

It follows from (3.4.8) and (3.4.9) that

$$\|U_{\exp(ik)}(a)\| = \|a\| \text{ for all } k \in H(A,*) \text{ and } a \in A. \tag{3.4.10}$$

Now, let h be in $H(A,*)$, and let a,c be in A. Then, by (3.4.10) and Corollary 3.4.16, we have

$$\begin{aligned}\|U_{a,c}(\exp(ih))\| &= \|U_{\exp(\frac{ih}{2})}U_{a,c}(\exp(ih))\| = \|U_{\exp(\frac{ih}{2})}(a)U_{\exp(\frac{ih}{2})}(c)\| \\ &\leqslant \|U_{\exp(\frac{ih}{2})}(a)\|\|U_{\exp(\frac{ih}{2})}(c)\| = \|a\|\|c\|.\end{aligned}$$

Since h is arbitrary in $H(A,*)$, and the set $\{b \in A : \|U_{a,c}(b)\| \leqslant \|a\|\|c\|\}$ is closed and convex, it is enough to invoke Corollary 3.4.7 again to conclude that the inequality (3.4.5) holds. □

Since every JB-algebra is the self-adjoint part of a suitable JB^*-algebra (by Theorem 3.4.8), Proposition 3.4.17 above implies the following.

Corollary 3.4.18 *Let A be a JB-algebra. Then we have*

$$\|U_{a,c}(b)\| \leqslant \|a\|\|b\|\|c\| \textit{ for all } a,b,c \in A.$$

We conclude this subsection with an entry on normal elements of non-commutative JB^*-algebras. To this end we begin by proving the following.

Lemma 3.4.19 *Let A be a C^*-algebra, and let a be in A. Then a is normal if (and only if) $[[a,a^*],a] = 0$.*

Proof Assume that $[[a,a^*],a] = 0$. Then, writing $a = h + ik$ with $h,k \in H(A,*)$, we have $[[h,k],k] + i[[k,h],h] = 0$, so that, since both $[[h,k],k]$ and $[[k,h],h]$ lie in $H(A,*)$, we deduce that $[[h,k],k] = 0$. Therefore, thinking about the C^*-algebra unital extension of A, and invoking Lemma 2.2.5 and Corollary 2.4.3, we get that $[h,k] = 0$ or, equivalently, $[a,a^*] = 0$. □

Definition 3.4.20 Let A be a $*$-algebra over $\mathbb{K}$, and let a be in A. We say that a is *normal* if the subalgebra of A generated by a and a^* is associative and commutative. In the case that A is associative, we invoke Corollary 1.1.79 to realize that a is normal if and only if $[a,a^*] = 0$. In this way, we are in agreement with the notion of a normal element of a C^*-algebra introduced in Definition 1.2.11.

Actually, keeping in mind Theorem 2.3.61 and Corollary 1.1.79, we obtain the following.

Fact 3.4.21 *Let A be an alternative algebra over $\mathbb{K}$, and let a be in A. Then a is normal if (and only if) $[a,a^*] = 0$.*

Keeping in mind Propositions 3.3.13 and 3.4.1(i), we immediately get the following.

Fact 3.4.22 *Let A be a non-commutative JB^*-algebra, and let a be a normal element of A. Then the closed subalgebra of A generated by $\{a, a^*\}$ (occasionally, by $\{\mathbf{1}, a, a^*\}$ if A is unital) is $*$-invariant, and is indeed a commutative C^*-algebra.*

An independent and deeper result is the following.

Proposition 3.4.23 *Let A be a non-commutative JB^*-algebra, and let a be in A. Then a is normal if (and only if) $[a, a^*] = 0 = [a, a, a^*]$.*

Proof Assume that $[a, a^*] = 0$. Let B denote the closed subalgebra of A generated by $\{a, a^*\}$. Then, by Propositions 3.3.13 and Corollary 2.4.16, B is $*$-invariant and commutative, and hence is a JB^*-algebra. Therefore, by Proposition 3.4.6, there exists a C^*-algebra C such that B can be seen as a closed $*$-subalgebra of C^{sym}. Since B is the closed subalgebra of C^{sym} generated by $\{a, a^*\}$, there is no loss of generality in supposing that C equals the closed subalgebra of C generated by $\{a, a^*\}$. Now assume additionally that $[a, a, a^*] = 0$, an equality which, in terms of the product of C, reads as $[[a, a^*], a] = 0$. It follows from Lemma 3.4.19 that a is a normal element of C, so C is commutative, and so $C^{\text{sym}} = C$ is an associative algebra. Finally, since B was a subalgebra of C^{sym}, we derive that B is associative and commutative. □

Corollary 3.4.24 *Let B be a JB-algebra, and let h, k be in B such that $[h, h, k] = 0 = [k, k, h]$. Then the subalgebra of B generated by $\{h, k\}$ is associative.*

Proof By Theorem 3.4.8, there exists a JB^*-algebra A such that $H(A, *) = B$. Then, writing $a := h + ik \in A$, the assumptions on h and k imply that $[a, a, a^*] = 0$. Therefore, by Proposition 3.4.23, a becomes a normal element of A, and this implies that the subalgebra of B generated by $\{h, k\}$ is associative. □

3.4.2 Isometries of unital non-commutative JB^*-algebras

As the next proposition shows, Kadison's Theorem 2.2.29 survives in the more general setting of non-commutative JB^*-algebras whenever isometries preserve units.

Proposition 3.4.25 *Let A and B be unital non-commutative JB^*-algebras, and let $F : A \to B$ be a bijective linear mapping. Then the following conditions are equivalent:*

(i) *F is a Jordan-$*$-homomorphism.*
(ii) *F is isometric and preserves units.*
(iii) *$V(A, \mathbf{1}, a) = V(B, \mathbf{1}, F(a))$ for every $a \in A$.*

Proof (i)⇒(ii) By Remark 3.4.5.
(ii)⇒(iii) By Corollary 2.1.2(ii).
(iii)⇒(i) By Lemma 2.2.5 and the implication (ii)⇒(i) in Proposition 3.2.3. □

We note that the implication (ii)⇒(i) in Proposition 3.4.25 can also be obtained as a straightforward consequence of Theorem 2.2.9.

Lemma 3.4.26 *Derivations on non-commutative JB^*-algebras are continuous.*

Proof Argue as in the case of C^*-algebras (see §3.1.52) with Corollary 3.4.3 instead of Corollary 3.1.2. □

An apparently more general result is that *Jordan derivations of non-commutative JB^*-algebras are continuous*, but this follows from Lemma 3.4.26, by invoking Fact 3.3.4.

Lemma 3.4.27 *Let A be a non-commutative JB^*-algebra, and let D be a derivation of A such that $D(a^*) = -D(a)^*$ for every $a \in A$. Then D lies in $H(BL(A), I_A)$.*

Proof By Lemmas 3.4.26 and 2.2.21, and Proposition 3.3.13, $\exp(irD)$ is an algebra $*$-automorphism of A for every $r \in \mathbb{R}$. Therefore, by Proposition 3.4.4, we have $\|\exp(irD)\| = 1$ for every $r \in \mathbb{R}$, and D lies in $H(BL(A), I_A)$ by Corollary 2.1.9(iii). □

Proposition 3.4.28 *Let A be a unital non-commutative JB^*-algebra, and let $T: A \to A$ be a mapping. Then the following conditions are equivalent:*

(i) *T belongs to $H(BL(A), I_A)$.*
(ii) *There are $h \in H(A, *)$ and a Jordan derivation D of A such that $T = L_h + D$ and $D(a^*) = -D(a)^*$ for every $a \in A$.*

Proof (i)⇒(ii) By Theorem 2.2.9(ii).

(ii)⇒(i) Assume that condition (ii) holds. Then, by Lemmas 2.2.5 and 2.1.10, we have $L_h \in H(BL(A), I_A)$. On the other hand, by Lemma 3.4.27, we also have $D \in H(BL(A), I_A)$. It follows that $T \in H(BL(A), I_A)$. □

For elements x, y, z in a given algebra, the following equality is straightforwardly realized:

$$[xy, z] - x[y, z] - [x, z]y = [x, y, z] - [x, z, y] + [z, x, y]. \tag{3.4.11}$$

Remark 3.4.29 If A is a C^*-algebra, then, by Lemma 2.2.23, the Jordan derivation D arising in assertion (ii) of Proposition 3.4.28 is in fact a derivation. The same conclusion holds (in this case, in a trivial way) if A is a JB^*-algebra. However, in the general case of non-commutative JB^*-algebras, D need not be a derivation. The reason is of a purely algebraic nature. Indeed, let B be any alternative algebra over $\mathbb{K}$, whose Jordan derivations are derivations. Then, since for alternative algebras the equality (3.4.11) takes the form

$$[xy, z] - x[y, z] - [x, z]y = 3[x, y, z], \tag{3.4.12}$$

and the operator $[\cdot, z]$ is a Jordan derivation (by Lemma 2.4.15), we derive that B is associative. Now, let A be an alternative C^*-algebra, and assume that Proposition 3.4.28 is true with 'derivation' instead of 'Jordan derivation'. Then every Jordan derivation D of A satisfying $D(a^*) = -D(a)^*$ for every $a \in A$ is a derivation, and this implies that every Jordan derivation of A is a derivation, hence A is associative. But we already know the existence of alternative C^*-algebras which are not associative (cf. Proposition 2.6.8).

We recall that the notions of an invertible element and that of the inverse of such an element work for unital alternative algebras as if they were associative (see Proposition 2.5.24 for details). Therefore, as in the associative case (cf. §2.1.20), we can define *algebraically unitary elements* of a norm-unital alternative normed algebra A as those invertible elements $u \in A$ satisfying $\|u\| = \|u^{-1}\| = 1$.

Lemma 3.4.30 *Let A be a norm-unital alternative normed algebra over $\mathbb{K}$, and let u be an algebraically unitary element of A. Then L_u and R_u are surjective linear isometries on A. As a consequence, u is a geometrically unitary element of A.*

Proof Since L_u and R_u are bijective operators with $L_u^{-1} = L_{u^{-1}}$ and $R_u^{-1} = R_{u^{-1}}$ (see Proposition 2.5.24 again), the first conclusion follows. Since $u = L_u(\mathbf{1})$, the consequence follows from the first conclusion and Corollary 2.1.19. □

As in the associative case, *unitary elements* of a unital alternative C^*-algebra A are defined as those elements $u \in A$ satisfying $uu^* = u^*u = \mathbf{1}$ (equivalently, as those invertible elements $u \in A$ such that $u^{-1} = u^*$). Now we can formulate and prove the following generalization of Theorem 2.1.27.

Proposition 3.4.31 *Let A be a unital alternative C^*-algebra, and let u be in A. Then the following conditions are equivalent:*

(i) *u is unitary.*
(ii) *u is algebraically unitary.*
(iii) *u is geometrically unitary.*
(iv) *u is a vertex of the closed unit ball of A.*

Proof The implications (i)⇒(ii) and (iii)⇒(iv) are clear, whereas the one (ii)⇒(iii) follows from Lemma 3.4.30.

(iv)⇒(i) Assume that u is a vertex of $\mathbb{B}_A$. Then u remains a vertex of the closed unit ball of the closed subalgebra of A generated by $\{\mathbf{1}, u, u^*\}$. Since this last algebra is an (associative) C^*-algebra, it follows from the implication (iv)⇒(i) in Theorem 2.1.27 that $uu^* = u^*u = \mathbf{1}$, i.e. u is unitary in A. □

Corollary 3.4.32 *Let A be a norm-unital normed complex algebra, let B be a unital alternative C^*-algebra, and let $F : A \to B$ be a mapping. Then the following conditions are equivalent:*

(i) *F is a surjective linear isometry.*
(ii) *A is a non-commutative JB^*-algebra for some involution, and there exists a Jordan-$*$-isomorphism $G : A \to B$, together with a unitary element u in B, satisfying $F = L_uG$.*

Proof (i)⇒(ii) Assume that F is a surjective linear isometry. Set $u := F(\mathbf{1})$. By Corollary 2.1.13 and Proposition 3.4.31, u is a unitary element of B. Write $G := L_{u^*}F$. Then, by Lemma 3.4.30, G is a surjective linear isometry. Therefore, since $G(\mathbf{1}) = \mathbf{1}$, and B is a non-commutative JB^*-algebra (by Fact 3.3.2), Corollary 3.3.17(b) applies (with G^{-1} instead of T), so that A is a non-commutative JB^*-algebra. Then, by Proposition 3.4.25, G is a Jordan-$*$-isomorphism. Finally, the equality $F = L_uG$ is clear.

(ii)⇒(i) By Remark 3.4.5 and Lemma 3.4.30. □

The following straightforward corollary has its own interest.

Corollary 3.4.33 *Linearly isometric unital alternative C^*-algebras are Jordan-$*$-isomorphic.*

Corollary 3.4.33 above does not remain true if we replace 'alternative C^*-algebras' with 'non-commutative JB^*-algebras'. As a matter of fact, nor does it even survive in the setting of unital JC^*-algebras. Indeed, we have the following.

Antitheorem 3.4.34 *There exist linearly isometric unital JC^*-algebras which are not $*$-isomorphic.*

Proof Consider the unital C^*-algebra C of all continuous functions from $\mathbb{T}$ to $M_2(\mathbb{C})$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, let A stand for the closed $*$-subalgebra of C^{sym} consisting of those functions $x \in C$ such that $x(z)$ is a symmetric matrix (relative to the transposition) for every $z \in \mathbb{T}$, and note that the unit $\mathbf{1}$ of C lies in A. Let u be the unitary element of C given by $u(z) := \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ for every $z \in \mathbb{T}$, so that the Banach space of C, endowed with the product $x \odot y := xu^*y$ and the involution $x^\sharp := ux^*u$, becomes a unital C^*-algebra (say D) whose unit is u. Now, note that the Banach space of A becomes a closed $\sharp$-invariant subalgebra (say B) of D^{sym}, and that the unit of D lies in B. It follows that A and B are linearly isometric unital JC^*-algebras. However, as a matter of fact, A and B are not $*$-isomorphic.

To realize the last assertion, note that A contains the JC^*-algebra of all 2×2 complex symmetric matrices, just as the constant functions, and hence the matrices $e := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $f := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are elements of A satisfying $e^* = e$, $f^* = f$, $e^2 = f^2 = \mathbf{1}$, and $e \bullet f = 0$. Now we argue by contradiction, so we assume that A and B are $*$-isomorphic. Then, there must exist $x, y \in B$ such that $x^\sharp = x$, $y^\sharp = y$, $x \odot x = y \odot y = u$, and $x \odot y + y \odot x = 0$. Keeping in mind the definition of $\sharp$ and $\odot$, the above conditions are equivalent to the ones $ux^* = xu^*$, $uy^* = yu^*$, $x^*x = y^*y = \mathbf{1}$, and $x^*y = -y^*x$. Therefore, writing $x := \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}$ and $y := \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}$, with $x_{ij}, y_{ij} \in C^{\mathbb{C}}(\mathbb{T})$, and denoting by v the inclusion of $\mathbb{T}$ into $\mathbb{C}$, we derive that

$$v\overline{x_{12}} = x_{12}, \quad v\overline{y_{12}} = y_{12}, \tag{3.4.13}$$

$$x_{22} \text{ and } y_{22} \text{ are real-valued functions,} \tag{3.4.14}$$

$$|x_{12}|^2 + x_{22}^2 = |y_{12}|^2 + y_{22}^2 = 1, \tag{3.4.15}$$

$$\rho := i(\overline{x_{12}}y_{12} + x_{22}y_{22}) \text{ is a real-valued function.} \tag{3.4.16}$$

Invoking (3.4.14) and (3.4.16), we get

$$|x_{12}|^2|y_{12}|^2 = x_{22}^2 y_{22}^2 + \rho^2. \tag{3.4.17}$$

On the other hand, (3.4.16) implies

$$\overline{x_{12}}^2 y_{12}^2 = x_{22}^2 y_{22}^2 - \rho^2 + 2i\rho x_{22} y_{22},$$

which, together with

$$\overline{x_{12}}^2 y_{12}^2 = |x_{12}|^2 \bar{v} v |y_{12}|^2 = |x_{12}|^2 |y_{12}|^2$$

(a consequence of (3.4.13)), gives

$$|x_{12}|^2|y_{12}|^2 = x_{22}^2 y_{22}^2 - \rho^2 + 2i\rho x_{22} y_{22}. \tag{3.4.18}$$

Comparing (3.4.17) and (3.4.18), and invoking (3.4.14) and (3.4.16) again, we obtain $x_{22}^2 y_{22}^2 + \rho^2 = x_{22}^2 y_{22}^2 - \rho^2$, so $\rho = 0$, and so

$$\overline{x_{12}} y_{12} = -x_{22} y_{22}. \tag{3.4.19}$$

Multiplying (3.4.19) by v, and applying (3.4.13), we get

$$x_{12} y_{12} = -v x_{22} y_{22}. \tag{3.4.20}$$

It follows from (3.4.19) and (3.4.15) that

$$|x_{12}|^2 |y_{12}|^2 = (1 - |x_{12}|^2)(1 - |y_{12}|^2) = 1 - |x_{12}|^2 - |y_{12}|^2 + |x_{12}|^2 |y_{12}|^2,$$

so $|x_{12}|^2 + |y_{12}|^2 = 1$, and so, keeping in mind (3.4.15),

$$x_{22}^2 + y_{22}^2 = 1. \tag{3.4.21}$$

Finally, we have

$$\begin{aligned}
(y_{22} x_{12} - x_{22} y_{12})^2 &= y_{22}^2 x_{12}^2 + x_{22}^2 y_{12}^2 - 2 y_{22} x_{12} x_{22} y_{12} \\
&= (y_{22}^2 |x_{12}|^2 + x_{22}^2 |y_{12}|^2 + 2 y_{22}^2 x_{22}^2) v \\
&= [y_{22}^2 (|x_{12}|^2 + x_{22}^2) + (|y_{12}|^2 + y_{22}^2) x_{22}^2] v \\
&= (y_{22}^2 + x_{22}^2) v = v,
\end{aligned}$$

where we have applied (3.4.13) and (3.4.20) for the second equality, (3.4.15) for the penultimate one, and (3.4.21) for the last one. Thus $y_{22} x_{12} - x_{22} y_{12}$ becomes a continuous square root of the function $z \to z$ on $\mathbb{T}$, the desired contradiction. □

We note that Antitheorem 3.4.34 prohibits any reasonable version of Kadison's Theorem 2.2.29 for non-commutative JB^*-algebras.

3.4.3 An interlude: derivations and automorphisms of normed algebras

§3.4.35 Let X be a normed space over $\mathbb{K}$, and let n be in $\mathbb{N}$. We denote by X_n the normed space over $\mathbb{K}$ of all continuous n-linear mappings from X^n to X. For $F \in BL(X)$, we define $F_0, F_1, \ldots, F_n \in BL(X_n)$ by

$$F_0(f)(x_1, \ldots, x_n) := F(f(x_1, \ldots, x_n))$$

and

$$F_i(f)(x_1, \ldots, x_n) := f(x_1, \ldots, x_{i-1}, F(x_i), x_{i+1}, \ldots, x_n)$$

for all $f \in X_n$, $(x_1, \ldots, x_n) \in X^n$, and $1 \leqslant i \leqslant n$. We have the following fact, whose routine verification is left to the reader.

Fact 3.4.36 *With the notation above, we have:*

(i) $\|F_i\| \leqslant \|F\|$ *for all* $0 \leqslant i \leqslant n$.
(ii) *The mapping* $F \to F_0$ *from* $BL(X)$ *to* $BL(X_n)$ *is a unit-preserving algebra homomorphism.*
(iii) *For* $1 \leqslant i \leqslant n$, *the mapping* $F \to F_i$ *from* $BL(X)$ *to* $BL(X_n)$ *is a unit-preserving algebra antihomomorphism.*
(iv) *If* $0 \leqslant i \neq j \leqslant n$, *and if* $F, G \in BL(X)$, *then* F_i *and* G_j *commute.*

Lemma 3.4.37 *Let X be a complex Banach space, let H be in $BL(X)$, and let n be in $\mathbb{N}$. We have:*

(i) $\mathrm{sp}(BL(X_n), H_1 + \cdots + H_n - H_0)$

$$\subseteq \overbrace{\mathrm{sp}(BL(X),H) + \cdots + \mathrm{sp}(BL(X),H)}^{n \text{ summands}} - \mathrm{sp}(BL(X),H).$$

(ii) *If H is bijective, then* $\mathrm{sp}(BL(X_n), H_1 H_2 \cdots H_n H_0^{-1})$

$$\subseteq \overbrace{\mathrm{sp}(BL(X),H) \cdots \mathrm{sp}(BL(X),H)}^{n \text{ factors}} (\mathrm{sp}(BL(X),H))^{-1}.$$

Proof Since $H_0, H_1, \ldots, H_n$ are mutually commuting elements of $BL(X_n)$ (by Fact 3.4.36(iv)), it follows from Corollary 1.1.81 that

$$\begin{aligned} &\mathrm{sp}(BL(X_n), H_1 + \cdots + H_n - H_0) \\ &\quad \subseteq \mathrm{sp}(BL(X_n),H_1) + \cdots + \mathrm{sp}(BL(X_n),H_n) - \mathrm{sp}(BL(X_n),H_0). \end{aligned}$$

Moreover, by Fact 3.4.36(ii)–(iii) and Lemma 1.1.34, we have

$$\mathrm{sp}(BL(X_n),H_i) \subseteq \mathrm{sp}(BL(X),H) \ \text{ for every } \ i = 0, \ldots, n,$$

and assertion (i) is proved. By keeping in mind Lemma 1.2.16, assertion (ii) is verified in an analogous manner. □

Let n be in $\mathbb{N}$. By an *n-algebra over* $\mathbb{K}$ we mean a couple (X,p), where X is a vector space over $\mathbb{K}$, and p is a fixed n-linear mapping from X^n to X (called the *product* of the n-algebra X). Let (X,p) be an n-algebra over $\mathbb{K}$. By a *derivation* of X we mean a linear mapping $D : X \to X$ satisfying

$$\begin{aligned} D(p(x_1,\ldots,x_n)) = p(D(x_1),x_2,\ldots,x_n) + p(x_1,D(x_2),\ldots,x_n) + \cdots \\ \cdots + p(x_1,x_2,\ldots,D(x_n)) \end{aligned}$$

for all $x_1,\ldots,x_n \in X$. By an *automorphism* of X we mean a bijective linear mapping $\Phi : X \to X$ satisfying $\Phi(p(x_1,\ldots,x_n)) = p(\Phi(x_1),\ldots,\Phi(x_n))$ for all $x_1,\ldots,x_n \in X$. When X is in fact a normed space over $\mathbb{K}$, and the n-linear product p is continuous, we say that (X,p) is a *normed n-algebra*. Since we will not deal with geometric properties of normed n-algebras, we do not assume that $\|p\| \leqslant 1$ in the above definition. Given a normed n-algebra (X,p), and $H \in BL(X)$, with the notation in §3.4.35 we realize that H is a derivation (respectively, an automorphism) if and only if

$$H_0(p) = H_1(p) + H_2(p) + \cdots + H_n(p)$$

(respectively, H is bijective and $H_0(p) = (H_1 H_2 \cdots H_n)(p)$).

Proposition 3.4.38 *Let X be a complete normed complex n-algebra with nonzero product. Then we have:*

(i) *For each continuous derivation D of X, there exist*

$$\lambda_1, \ldots, \lambda_n \in \mathrm{sp}(BL(X),D)$$

such that $\lambda_1 + \cdots + \lambda_n \in \mathrm{sp}(BL(X),D)$.

(ii) *For each continuous automorphism* Φ *of* X*, there exist*

$$\lambda_1, \lambda_2, \ldots, \lambda_n \in \operatorname{sp}(BL(X), \Phi)$$

such that $\lambda_1 \lambda_2 \cdots \lambda_n \in \operatorname{sp}(BL(X), \Phi)$.

Proof In what follows, p stands for the product of X. Let D be a continuous derivation of X. Since

$$(D_1 + \cdots + D_n - D_0)(p) = 0 \text{ and } p \neq 0,$$

we have that $0 \in \operatorname{sp}(BL(X_n), D_1 + \cdots + D_n - D_0)$. Therefore, invoking Lemma 3.4.37(i), we derive that

$$0 \in \overbrace{\operatorname{sp}(BL(X), D) + \cdots + \operatorname{sp}(BL(X), D)}^{n \text{ summands}} - \operatorname{sp}(BL(X), D),$$

and assertion (i) follows. By invoking Lemma 3.4.37(ii) instead of Lemma 3.4.37(i), assertion (ii) follows in a similar way. □

As a particular case of Proposition 3.4.38 we get the following.

Corollary 3.4.39 *Let* A *be a complete normed complex algebra with nonzero product. Then we have:*

(i) *For each continuous derivation* D *of* A*, there exist* λ_1, λ_2 *in* $\operatorname{sp}(BL(A), D)$ *such that* $\lambda_1 + \lambda_2 \in \operatorname{sp}(BL(A), D)$.
(ii) *For each continuous algebra automorphism* Φ *of* A*, there exist* λ_1, λ_2 *in* $\operatorname{sp}(BL(A), \Phi)$ *such that* $\lambda_1 \lambda_2 \in \operatorname{sp}(BL(A), \Phi)$.

As the following example shows, the information given by Corollary 3.4.39 is the best possible, even if A is associative and commutative.

Example 3.4.40 Let K be a compact subset of $\mathbb{C}$ such that there are $\lambda_1, \lambda_2 \in K$ with $\lambda_1 + \lambda_2 \in K$ (respectively, such that $0 \notin K$ and there are $\lambda_1, \lambda_2 \in K$ with $\lambda_1 \lambda_2 \in K$). We are going to show that there exists a complete normed associative and commutative complex algebra A with nonzero product, together with a continuous derivation (respectively, a continuous algebra automorphism) of A, whose spectrum relative to $BL(A)$ equals K.

Let B stand for the three-dimensional associative and commutative complex algebra with basis $\{u_1, u_2, u_3\}$ and multiplication table given by

$$u_1 u_2 = u_2 u_1 = u_3 \text{ and } u_i u_j = 0 \text{ if } (i, j) \neq (1,2), (2,1),$$

and consider the linear operator D (respectively, F) from B to B determined by the conditions

$$D(u_1) = \lambda_1 u_1, \;\; D(u_2) = \lambda_2 u_2, \;\text{ and }\; D(u_3) = (\lambda_1 + \lambda_2) u_3$$

$$(\text{respectively, } F(u_1) = \lambda_1 u_1, \; F(u_2) = \lambda_2 u_2, \text{ and } F(u_3) = \lambda_1 \lambda_2 u_3).$$

It is routine to verify that D is a derivation (respectively, F is an algebra automorphism) of B, and that $\operatorname{sp}(L(B), D) = \{\lambda_1, \lambda_2, \lambda_1 + \lambda_2\}$ (respectively, $\operatorname{sp}(L(B), F) = \{\lambda_1, \lambda_2, \lambda_1 \lambda_2\}$).

Now, invoke Exercise 1.1.45 to be provided with a complex Banach space C, together with an operator $T \in BL(C)$, such that $\mathrm{sp}(BL(C),T) = K$. Endowing C with the zero product, C becomes a complete normed associative and commutative complex algebra, and every linear operator on C is a derivation (respectively, an algebra automorphism, if the operator is bijective) of C.

Now, endow B with any algebra norm, and let A denote the algebra $B \times C$ with coordinate-wise operations and the sup norm. Then A is a complete normed associative and commutative complex algebra with nonzero product, and the operator on A given by

$$(x,y) \to (D(x),T(y)) \qquad \big(\text{respectively, } (x,y) \to \big(F(x),T(y)\big)\big)$$

is a continuous derivation (respectively, a continuous algebra automorphism) of A whose spectrum relative to $BL(A)$ equals K.

§3.4.41 In §2.8.36, we introduced the notion of a nilpotent algebra in a rather intuitive way. Now, we need to formalize that notion somewhat. To this end, we recall that, in §§2.8.17 and 2.8.26, we introduced non-associative words with characters on an arbitrary set of indeterminates, the (global) degree of such a word, and the meaning of $\mathbf{p}(a_1,\ldots,a_n)$ when $\mathbf{p}$ is a non-associative word with characters in a set of n indeterminates, and $a_1,\ldots,a_n$ are elements of a given algebra. Given a non-associative word $\mathbf{p}$ with characters in a set $\mathbf{X}$ of indeterminates, and given $\mathbf{x} \in \mathbf{X}$, we define the *degree of* $\mathbf{p}$ *in* $\mathbf{x}$ by induction on the global degree of $\mathbf{p}$, as follows. If the global degree of $\mathbf{p}$ is 1, then the degree of $\mathbf{p}$ in $\mathbf{x}$ is 1 or 0, depending on whether or not $\mathbf{p} = \mathbf{x}$, whereas, if the global degree of $\mathbf{p}$ is > 1, then the degree of $\mathbf{p}$ in $\mathbf{x}$ is defined as $m_1 + m_2$, where m_1 and m_2 are the degrees of $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathbf{x}$, respectively, $\mathbf{p}_1$,$\mathbf{p}_2$ being the unique non-associative words such that $\mathbf{p} = \mathbf{p}_1\mathbf{p}_2$. A non-associative word in n indeterminates which is of degree 1 in each of them will be called an *n-linear non-associative word*. The name is justified because, if A is any algebra, and if $\mathbf{p}$ is an n-linear non-associative word, then the mapping

$$\mathbf{p}_A : (a_1,\ldots,a_n) \to \mathbf{p}(a_1,\ldots,a_n),$$

from A^n to A, is n-linear. This can be easily realized by induction on the global degree n of $\mathbf{p}$. With such an induction argument in mind, the following facts, involving a given n-linear non-associative word $\mathbf{p}$, are easily verified:

(i) If A is any algebra, and if H is a derivation (respectively, an algebra automorphism) of A, then H remains a derivation (respectively, an automorphism) of the n-algebra $(A,\mathbf{p}_A)$.

(ii) If A is a normed algebra, then the mapping $\mathbf{p}_A$ is continuous.

On the other hand, according to §2.8.36, an algebra A is nilpotent if and only if, for some $n \in \mathbb{N}$, and every n-linear non-associative word $\mathbf{p}$, we have $\mathbf{p}_A = 0$.

The above facts will be applied without notice in the proof of the following.

Theorem 3.4.42 *Let A be a complete normed non-nilpotent complex algebra. Then we have:*

(i) *For each continuous derivation D of A, and each natural number n, there exist $\lambda_1, \dots, \lambda_n$ in $\mathrm{sp}(BL(A), D)$ such that*

$$\lambda_1 + \cdots + \lambda_n \in \mathrm{sp}(BL(A), D).$$

(ii) *For each continuous algebra automorphism Φ of A, and each natural number n, there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathrm{sp}(BL(A), \Phi)$ such that*

$$\lambda_1 \lambda_2 \cdots \lambda_n \in \mathrm{sp}(BL(A), \Phi).$$

Proof Let n be in $\mathbb{N}$. Since A is not nilpotent, there exists an n-linear non-associative word $\mathbf{p}_n$ in n indeterminates, and elements $x_1, \dots, x_n \in A$ such that $\mathbf{p}_n(x_1, \dots, x_n) \neq 0$. Now, consider the continuous n-linear mapping $p_n : A^n \to A$ defined by $p_n(a_1, \dots, a_n) := \mathbf{p}_n(a_1, \dots, a_n)$ for every $(a_1, \dots, a_n) \in A^n$. Then the Banach space of A, endowed with the product p_n, becomes a complete normed complex n-algebra with nonzero product. Moreover, derivations (respectively, algebra automorphisms) of A, become derivations (respectively, automorphisms) of the n-algebra (A, p_n). Now, the proof is concluded by applying Proposition 3.4.38. □

Remark 3.4.43 With the argument in the above proof, it can be shown that, *if A is a complete normed nilpotent complex algebra of index $m \geqslant 2$, then, for each continuous derivation D of A (respectively, each continuous algebra automorphism Φ of A), and for each natural number $n \leqslant m-1$, there exist $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathrm{sp}(BL(A), D)$ (respectively, in $\mathrm{sp}(BL(A), \Phi)$) such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n \in \mathrm{sp}(BL(A), D)$ (respectively, $\lambda_1 \lambda_2 \cdots \lambda_n \in \mathrm{sp}(BL(A), \Phi)$).* We note that the case $m = 3$ of this result also follows from Corollary 3.4.39.

Corollary 3.4.44 *Let A be a complete normed non-nilpotent complex algebra. Then:*

(i) *For each continuous derivation D of A, we have $0 \in \mathrm{co}(\mathrm{sp}(BL(A), D))$.*
(ii) *For each continuous algebra automorphism Φ of A, there are λ, μ in $\mathrm{sp}(BL(A), \Phi)$ such that $|\lambda| \leqslant 1 \leqslant |\mu|$.*

Proof Let D be a continuous derivation of A. By Corollary 3.4.42, for each $n \in \mathbb{N}$, there exist $\lambda_1, \dots, \lambda_n \in \mathrm{sp}(BL(A), D)$ such that

$$\lambda_1 + \cdots + \lambda_n \in \mathrm{sp}(BL(A), D),$$

and hence $\mu_n := \frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \in \mathrm{co}(\mathrm{sp}(BL(A), D))$. Since $|\mu_n| \leqslant \frac{1}{n}\mathfrak{r}(D) \to 0$, it follows that $0 \in \overline{\mathrm{co}}(\mathrm{sp}(BL(A), D))$. By Carathéodory theorem, we conclude that $0 \in \mathrm{co}(\mathrm{sp}(BL(A), D))$.

Let Φ be a continuous algebra automorphism of A. Suppose that

$$1 < k := \min\{|\lambda| : \lambda \in \mathrm{sp}(BL(A), \Phi)\}.$$

By Corollary 3.4.42, for each $n \in \mathbb{N}$, there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathrm{sp}(BL(A), \Phi)$ such that $\lambda_1 \lambda_2 \cdots \lambda_n \in \mathrm{sp}(BL(A), \Phi)$, and hence $k^n \leqslant |\lambda_1 \lambda_2 \cdots \lambda_n| \leqslant \mathfrak{r}(\Phi)$, a contradiction. Therefore $k \leqslant 1$, and hence there exists $\lambda \in \mathrm{sp}(BL(A), \Phi)$ such that $|\lambda| \leqslant 1$. Since Φ^{-1} is a continuous algebra automorphism of A, and $\mathrm{sp}(BL(A),\Phi^{-1}) = \mathrm{sp}(BL(A),\Phi)^{-1}$ (by Lemma 1.2.16 and Example 1.1.32(d)), it follows from the above that there exists $\mu \in \mathrm{sp}(BL(A),\Phi)$ such that $|\mu^{-1}| \leqslant 1$, and the proof is complete. □

Lemma 3.4.45 *Let X be a complex Banach space, let T be in $BL(X)$, and let x be in X. We have:*

(i) *If $T(x) = 0$, then $\exp(T)(x) = x$.*
(ii) *If $\exp(T)(x) = x$, and if $\mathrm{sp}(BL(X),T) \cap \{\pm 2n\pi i : n \in \mathbb{N}\} = \emptyset$, then $T(x) = 0$.*

Proof Consider the entire function f defined by $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$. Note that

$$f(z)z = zf(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} = \exp(z) - 1 \tag{3.4.22}$$

vanishes on the set $2\pi i\mathbb{Z}$. Since $f(0) = 1$, it follows that the zero set of f is

$$Z(f) = \{\pm 2n\pi i : n \in \mathbb{N}\}.$$

On the other hand, keeping in mind Theorem 1.3.13, it follows from (3.4.22) that

$$f(T)T = Tf(T) = \exp(T) - I_X.$$

If $x \in X$ is such that $T(x) = 0$, then we have $0 = f(T)T(x) = (\exp(T) - I_X)(x)$, and hence $\exp(T)(x) = x$. Now assume that $\mathrm{sp}(BL(X),T) \cap Z(f) = \emptyset$ and that $x \in X$ is such that $\exp(T)(x) = x$. By Theorem 1.3.13, we have

$$0 \notin f(\mathrm{sp}(BL(X),T)) = \mathrm{sp}(BL(X), f(T)),$$

and hence $f(T) \in \mathrm{Inv}(BL(X))$. Therefore

$$T(x) = f(T)^{-1} f(T)T(x) = f(T)^{-1}(\exp(T) - I_X)(x) = 0,$$

and the proof is complete. □

The proof of the next lemma is left to the reader.

Lemma 3.4.46 *Let A, B be complete normed unital power-associative algebras, and let ϕ be a continuous unit-preserving homomorphism or antihomomorphism from A to B. Then $\phi(\exp(a)) = \exp(\phi(a))$ for every $a \in A$.*

Lemma 3.4.47 *Let X be a complete normed complex n-algebra, and let D be in $BL(X)$. Then we have:*

(i) *If D is a derivation of X, then $\exp(D)$ is an automorphism of X.*
(ii) *If $\exp(D)$ is an automorphism of X, and if*

$$\mathrm{sp}(BL(X),D) \subseteq \left\{ z \in \mathbb{C} : |\Im(z)| < \frac{2\pi}{n+1} \right\},$$

then D is a derivation of X.

Proof Let p denote the product of the n-algebra X. We claim that

$$\exp(D_1+\cdots+D_n-D_0)(p)=p \tag{3.4.23}$$

if and only if $\exp(D)$ is an automorphism of X. Indeed, keeping in mind Fact 3.4.36(iv) and Exercise 1.1.30, we see that the equality (3.4.23) can be rewritten as $\exp(D_1)\cdots\exp(D_n)\exp(D_0)^{-1}(p)=p$, or equivalently

$$\exp(D_1)\cdots\exp(D_n)(p)=\exp(D_0)(p). \tag{3.4.24}$$

On the other hand, it follows from Fact 3.4.36(ii)–(iii) and Lemma 3.4.46 that

$$\exp(D_i)=\exp(D)_i \text{ for every } i=0,\dots,n.$$

Therefore, the equality (3.4.24) can be rewritten as

$$\exp(D)_1\cdots\exp(D)_n(p)=\exp(D)_0(p),$$

which, in its turn, is equivalent to the fact that $\exp(D)$ is an automorphism of X. This concludes the verification of the claim.

Assume that D is a derivation of X. Then we know that

$$(D_1+\cdots+D_n-D_0)(p)=0.$$

By Lemma 3.4.45(i), we deduce that $\exp(D_1+\cdots+D_n-D_0)(p)=p$, and consequently $\exp(D)$ is an automorphism of X because of the claim above.

Now assume that $\exp(D)$ is an automorphism of X and that

$$\mathrm{sp}(BL(X),D)\subseteq\left\{z\in\mathbb{C}:|\Im(z)|<\frac{2\pi}{n+1}\right\}.$$

By Lemma 3.4.37, for each $\lambda\in\mathrm{sp}(BL(X_n),D_1+\cdots+D_n-D_0)$ there exist λ_0, $\lambda_1,\dots,\lambda_n$ in $\mathrm{sp}(BL(X),D)$ such that $\lambda=\lambda_1+\cdots+\lambda_n-\lambda_0$, and consequently

$$|\Im(\lambda)|\leqslant|\Im(\lambda_1)|+\cdots+|\Im(\lambda_n)|+|\Im(\lambda_0)|<2\pi.$$

Therefore $\mathrm{sp}(BL(X_n),D_1+\cdots+D_n-D_0)\cap\{\pm2n\pi i:n\in\mathbb{N}\}=\emptyset$. Now, since

$$\exp(D_1+\cdots+D_n-D_0)(p)=p$$

(by the claim), Lemma 3.4.45(ii) applies, so that $(D_1+\cdots+D_n-D_0)(p)=0$, and so D is a derivation of X. □

Assertion (i) in the above lemma remains true for complete normed real n-algebras. This can be realized by arguing as in Lemma 2.2.21 or, preferably, by passing to 'complexification' and then applying the proved complex result.

Proposition 3.4.48 *Let X be a complete normed complex n-algebra, and let Φ be a continuous automorphism of X such that*

$$\mathrm{sp}(BL(X),\Phi)\subseteq\left\{z\in\mathbb{C}\setminus\{0\}:|\arg(z)|<\frac{2\pi}{n+1}\right\}.$$

Then $\log(\Phi)$ *is a derivation of X, where* log *stands for the principal determination of the complex logarithm.*

Proof Define $D := \log(\Phi)$. By Theorem 1.3.13, $\exp(D) = \Phi$, and

$$\operatorname{sp}(BL(X), D) = \log(\operatorname{sp}(BL(X), \Phi)) \subseteq \log\left(\left\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \frac{2\pi}{n+1}\right\}\right)$$
$$= \left\{z \in \mathbb{C} : |\Im(z)| < \frac{2\pi}{n+1}\right\}.$$

Now, by Lemma 3.4.47(ii), D is a derivation of X. □

Although obvious, we emphasize the case $n = 2$ of the above proposition in the following.

Theorem 3.4.49 *Let A be a complete normed complex algebra, and let Φ be a continuous algebra automorphism of A such that*

$$\operatorname{sp}(BL(A), \Phi) \subseteq \left\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| < \frac{2\pi}{3}\right\}.$$

Then $\log(\Phi)$ *is a derivation of A.*

Proposition 3.4.50 *Let A be a normed algebra over $\mathbb{K}$ such that there exists $M > 0$ satisfying $\|a\|^2 \leqslant M\|a^2\|$ for every $a \in A$. We have:*

(i) *If B is a normed algebra over $\mathbb{K}$, and if $\phi : B \to A$ is a continuous Jordan homomorphism, then $\|\phi\| \leqslant M$.*
(ii) *If $\mathbb{K} = \mathbb{C}$, then A has no nonzero continuous Jordan derivations.*

Proof Noticing that $|||\cdot||| := \frac{1}{M}\|\cdot\|$ is an equivalent (possibly non-submultiplicative) norm on A satisfying $|||a|||^2 \leqslant |||a^2|||$, assertion (i) follows from Proposition 2.6.19.

To prove assertion (ii), we may assume that A is complete. Let D be a continuous Jordan derivation of A, and let z be in $\mathbb{C}$. Then $\exp(zD)$ is a continuous Jordan automorphism of A. Therefore, by assertion (i), we have $\|\exp(zD)\| \leqslant M$, and hence $D = 0$ because of Liouville's theorem. □

By combining Lemma 2.2.25 and Proposition 3.4.50(ii), we derive the following.

Corollary 3.4.51 *Commutative C^*-algebras have no nonzero derivations.*

Now we can realize that, in Theorem 3.4.49, the number $\frac{2\pi}{3}$ cannot be replaced by a larger one. Indeed, we have the following.

Example 3.4.52 Let A stand for the commutative C^*-algebra $\mathbb{C}^3$, and let Φ denote the automorphism of A defined by $\Phi(\lambda_1, \lambda_2, \lambda_3) := (\lambda_3, \lambda_1, \lambda_2)$. Then we have

$$\operatorname{sp}(BL(A), \Phi) = \{z \in \mathbb{C} : z^3 = 1\} \subseteq \left\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| \leqslant \frac{2\pi}{3}\right\}.$$

However, in view of Corollary 3.4.51, $\log(\Phi)$ cannot be a derivation.

A relevant consequence of Theorem 3.4.49 is the following.

Corollary 3.4.53 *Let A be a complete normed algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *A has no nonzero continuous derivation.*
(ii) *The set of all continuous algebra automorphisms of A is discrete (for the norm topology).*

Proof Let $\mathscr{A} = \mathscr{A}(A)$ stand for the set of all continuous algebra automorphisms of A.

(ii)⇒(i) Let D be a continuous derivation of A, so that $\exp(\lambda D)$ lies in $\mathscr{A}$ for every real number λ. Suppose that $\mathscr{A}$ is discrete. Since the mapping $f : \lambda \to \exp(\lambda D)$ is continuous with $f(0) = I_A$, and I_A is an isolated point of $\mathscr{A}$, it follows that there exists a neighbourhood of zero in $\mathbb{R}$ on which f is constant. Therefore $D = f'(0) = 0$.

(i)⇒(ii) Suppose that A has no nonzero continuous derivation. Assume that $\mathbb{K} = \mathbb{C}$. Since $\mathscr{A}$ is a subgroup of $\mathrm{Inv}(BL(A))$, and $\mathrm{Inv}(BL(A))$ is a topological group (by Proposition 1.1.15), $\mathscr{A}$ is a topological group too, and hence, to prove that $\mathscr{A}$ is discrete, it is enough to show that I_A is an isolated point of $\mathscr{A}$. Let Φ be in $\mathscr{A}$ with $\|I_A - \Phi\| < 1$. Then we have

$$\mathrm{sp}(BL(A), \Phi) \subseteq \{z \in \mathbb{C} : |1 - z| < 1\} \subseteq \left\{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \frac{2\pi}{3}\right\},$$

and hence, by Theorem 3.4.49, $\log(\Phi)$ is a derivation of A. Since A has no nonzero continuous derivation (by the assumption), we obtain $\log(\Phi) = 0$, so $\Phi = I_A$. Thus I_A is an isolated point of $\mathscr{A}$, as desired. Now assume that $\mathbb{K} = \mathbb{R}$. Then $A_{\mathbb{C}} := \mathbb{C} \otimes_\pi A$ is a complete normed complex algebra with no nonzero continuous derivation. (Indeed, involving the canonical conjugate-linear automorphism of $A_{\mathbb{C}}$, we easily realize that every continuous derivation D of $A_{\mathbb{C}}$ can be written as $D = D_1 + iD_2$ with D_i a derivation of $A_{\mathbb{C}}$ satisfying $D_i(A) \subseteq A$, so that D_i can be regarded as a continuous derivation of A, and the assumption that A has no nonzero continuous derivation applies.) Since $I_{\mathbb{C}} \otimes \Phi$ lies in $\mathscr{A}(A_{\mathbb{C}})$ whenever Φ is in $\mathscr{A}(A)$, and the mapping $\Phi \to I_{\mathbb{C}} \otimes \Phi$ from $\mathscr{A}(A)$ to $\mathscr{A}(A_{\mathbb{C}})$ is a topological imbedding (it is in fact an isometry), and $\mathscr{A}(A_{\mathbb{C}})$ is discrete (by the complex case already discussed), we see that $\mathscr{A}(A)$ is discrete. □

Another consequence of Theorem 3.4.49 is the following.

Corollary 3.4.54 *Let A be a complete normed associative semiprime complex algebra. Then the set of all continuous algebra automorphisms of A is relatively open in the set of all continuous Jordan automorphisms of A.*

Proof Let $\mathscr{A}$ (respectively, $\mathscr{J}$) stand for the set of all continuous algebra automorphisms (respectively, Jordan automorphisms) of A. Since $\mathscr{J}$ is a topological group, and $\mathscr{A}$ is a subgroup of $\mathscr{J}$, it is enough to show that I_A lies in the interior of $\mathscr{A}$ relative to $\mathscr{J}$. Let Φ be in $\mathscr{J}$ with $\|I_A - \Phi\| < 1$. Then we have

$$\mathrm{sp}(BL(A), \Phi) \subseteq \{z \in \mathbb{C} : |1 - z| < 1\} \subseteq \left\{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \frac{2\pi}{3}\right\},$$

and hence, by applying Theorem 3.4.49 with A^{sym} instead of A, $\log(\Phi)$ is a Jordan derivation of A. Now, by Lemma 2.2.23, $\log(\Phi)$ is a derivation, and hence Φ lies in $\mathscr{A}$. □

Now, we are going to take advantage of Proposition 3.4.50(ii) to compute numerical indexes of unital non-commutative JB^*-algebras (see Theorem 3.4.59 below).

Proposition 3.4.55 *Let A be a normed non-commutative Jordan complex algebra such that there exists $M > 0$ satisfying $\|a\|^2 \leqslant M\|a^2\|$ for every $a \in A$. Then A is associative and commutative.*

Proof Since A is flexible, Lemma 2.4.15 applies, so that for every $a \in A$, the mapping $b \to [a,b]$ is a (continuous) Jordan derivation of A. Therefore, by applying Proposition 3.4.50(ii), we get that A is commutative, i.e. A is in fact a Jordan algebra. Then Lemma 3.1.23, together with a new application of Proposition 3.4.50(ii), gives that A is associative. □

Lemma 3.4.56 *Let A be a normed associative algebra over $\mathbb{K}$, and let a,b be in A. Then*

$$\mathfrak{r}(a+b) \leqslant \max\{\|a\|,\|b\|\} + \sqrt{\|ab\|}.$$

Proof By Propositions 1.1.98 and 1.1.107, and Definition 1.1.113, we may suppose that A is complex, norm-unital, and complete. Then the conclusion in the lemma is equivalent to

$$|\lambda| \leqslant \max\{\|a\|,\|b\|\} + \sqrt{\|ab\|} \ \text{ for every } \ \lambda \in \mathrm{sp}(a+b). \tag{3.4.25}$$

Let λ be in $\mathrm{sp}(a+b)$. Since the inequality (3.4.25) becomes evident whenever $|\lambda| \leqslant \max\{\|a\|,\|b\|\}$, let us suppose that $|\lambda| > \max\{\|a\|,\|b\|\}$. Then $\lambda\mathbf{1}-a$ and $\lambda\mathbf{1}-b$ are invertible and, by Corollary 1.1.14, we have

$$\|(\lambda\mathbf{1}-a)^{-1}\| \leqslant (|\lambda|-\|a\|)^{-1} \ \text{ and } \ \|(\lambda\mathbf{1}-b)^{-1}\| \leqslant (|\lambda|-\|b\|)^{-1}.$$

Hence

$$\|((\lambda\mathbf{1}-a)(\lambda\mathbf{1}-b))^{-1}\| \leqslant (|\lambda|-\max\{\|a\|,\|b\|\})^{-2}. \tag{3.4.26}$$

Moreover, since

$$(\lambda\mathbf{1}-a)(\lambda\mathbf{1}-b)-\lambda(\lambda\mathbf{1}-(a+b)) = ab,$$

and $\lambda(\lambda\mathbf{1}-(a+b)) \notin \mathrm{Inv}(A)$, it follows from Corollary 1.1.21 that

$$\|((\lambda\mathbf{1}-a)(\lambda\mathbf{1}-b))^{-1}\|^{-1} \leqslant \|(\lambda\mathbf{1}-a)(\lambda\mathbf{1}-b)-\lambda(\lambda\mathbf{1}-(a+b))\| = \|ab\|.$$

This and the inequality (3.4.26) complete the proof. □

Lemma 3.4.57 *Let A be a non-commutative JB^*-algebra, and let a be in A. Then*

$$\|a^* \bullet a\| \leqslant \frac{1}{2}\left(\|a\|^2 + \sqrt{\|U_{a^*}(a^2)\|}\right).$$

Proof By Fact 3.3.4 and Proposition 3.4.6, there exist a C^*-algebra B and an isometric algebra $*$-homomorphism ϕ from the closed subalgebra of A^{sym} generated by $\{a,a^*\}$ onto a closed $*$-subalgebra of B^{sym}. Since for $b \in B$ we have

$$\|b^*b+bb^*\| \leqslant \|b\|^2 + \sqrt{\|b^*b^2b^*\|}$$

(by Lemma 3.4.56), we derive

$$\begin{aligned}\|a^* \bullet a\| &= \|\phi(a)^* \bullet \phi(a)\| \\ &\leqslant \frac{1}{2}\left(\|\phi(a)\|^2 + \sqrt{\|\phi(a)^*\phi(a)^2\phi(a)^*\|}\right) \\ &= \frac{1}{2}\left(\|a\|^2 + \sqrt{\|U_{a^*}(a^2)\|}\right).\end{aligned}$$

□

Results such as Proposition 3.3.13 (that the involution of a non-commutative JB^*-algebra is isometric) or Corollary 3.4.3 (that the self-adjoint part of a non-commutative JB^*-algebra is a JB-algebra) must be omnipresent when we are dealing with non-commutative JB^*-algebras. This is the case in the proof of the following.

Proposition 3.4.58 *Let A be a unital non-commutative JB^*-algebra, and let a be in A. Then*

$$\frac{1}{2}\|a\| \leqslant v(A,\mathbf{1},a) \leqslant \frac{1}{2}\left(\|a\| + \sqrt{\|a^2\|}\right).$$

Proof The first inequality follows from Lemmas 2.2.5 and 2.3.7(i). For the second one, by multiplying a by a suitable unimodular number (a fact that does not affect the conclusion), we may suppose that $v(a) = \max \Re(V(A,\mathbf{1},a))$. Set $h := \frac{1}{2}(a+a^*)$. Then, by Lemma 2.2.5, we have $v(a) = v(h)$, and hence

$$(v(a))^2 \leqslant \|h\|^2 = \|h^2\| = \frac{1}{4}\|a^2 + a^{*2} + a^*a + aa^*\| \leqslant \frac{1}{2}(\|a^2\| + \|a^* \bullet a\|).$$

On the other hand, by Lemma 3.4.57 and Proposition 3.4.17, we have

$$\|a^* \bullet a\| \leqslant \frac{1}{2}\left(\|a\|^2 + \|a\|\sqrt{\|a^2\|}\right).$$

It follows that

$$\begin{aligned}(v(a))^2 &\leqslant \frac{1}{4}\left(2\|a^2\| + \|a\|\sqrt{\|a^2\|} + \|a\|^2\right) \\ &= \frac{1}{4}\left(\|a^2\| + \sqrt{\|a^2\|}\sqrt{\|a^2\|} + \|a\|\sqrt{\|a^2\|} + \|a\|^2\right) \\ &\leqslant \frac{1}{4}\left(\|a^2\| + \|a\|\sqrt{\|a^2\|} + \|a\|\sqrt{\|a^2\|} + \|a\|^2\right) \\ &= \frac{1}{4}\left(\sqrt{\|a^2\|} + \|a\|\right)^2,\end{aligned}$$

and this completes the proof. □

Theorem 3.4.59 *Let A be a unital non-commutative JB^*-algebra. Then $n(A,\mathbf{1})$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is associative and commutative.*

Proof By the first inequality in Proposition 3.4.58, we have $n(A,\mathbf{1}) \geqslant \frac{1}{2}$. On the other hand, if A is associative and commutative, then, by Fact 3.3.2, A is a commutative C^*-algebra, and hence, by Theorem 1.2.23, we have $A = C^{\mathbb{C}}(E)$ for a suitable compact Hausdorff topological space E, and therefore $n(A,\mathbf{1}) = 1$ (indeed, the valuations of elements of A at points of E lie in $D(A,\mathbf{1})$). Now, to conclude the proof it is enough to show that, if $n(A,\mathbf{1}) > \frac{1}{2}$, then A is associative and commutative. But, if $n(A,\mathbf{1}) > \frac{1}{2}$, then, by the second inequality in Proposition 3.4.58, we have

$$(2n(A,\mathbf{1}) - 1)^2 \|a\|^2 \leqslant \|a^2\| \text{ for every } a \in A,$$

and A is associative and commutative because of Proposition 3.4.55. □

Keeping in mind Fact 3.3.2 and the fact that commutative alternative algebras over $\mathbb{K}$ are associative (by identity (3.4.12)), we derive the following.

Corollary 3.4.60 *Let A be a unital alternative C^*-algebra. Then $n(A,\mathbf{1})$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is commutative.*

3.4.4 The structure theorem of isomorphisms of non-commutative JB^*-algebras

§**3.4.61** If a is an element of a normed algebra generating an associative subalgebra, then we can consider the *spectral radius* $\mathfrak{r}(a)$ of a, defined as in the associative case by

$$\mathfrak{r}(a) := \inf\left\{ \|a^n\|^{\frac{1}{n}} : n \in \mathbb{N} \right\}$$

(cf. §1.1.16). We note that, in view of Corollary 1.1.18(i), for such an a we have

$$\mathfrak{r}(a) = \lim \|a^n\|^{\frac{1}{n}}.$$

Now the simple argument in the proof of Corollary 1.1.19 yields the following.

Fact 3.4.62 *Let A be a normed power-associative algebra over $\mathbb{K}$, let B be a normed algebra over $\mathbb{K}$, let $F : A \to B$ be a continuous algebra homomorphism, and let a be in A. Then $F(a)$ generates an associative subalgebra of B, and the inequality $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$ holds. As a consequence, every equivalent algebra norm on A gives rise to the same spectral radius on A.*

The following outstanding variant of Fact 3.4.62 above generalizes Corollary 1.1.114.

Proposition 3.4.63 *Let A be a complete normed power-associative algebra over $\mathbb{K}$, let B be a normed algebra over $\mathbb{K}$, let $F : A \to B$ be an algebra homomorphism, and let a be in A. Then $F(a)$ generates an associative subalgebra of B, and the inequality $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$ holds.*

Proof Let C denote the closed subalgebra of A generated by a. Then C is a complete normed associative algebra over $\mathbb{K}$, and, as a consequence, $F(C)$ is an associative subalgebra of B containing $F(a)$. Therefore, regarding F as an algebra homomorphism from C to $F(C)$, and applying Corollary 1.1.114, we get $\mathfrak{r}(F(a)) \leqslant \mathfrak{r}(a)$. □

As an immediate consequence, we derive the following.

Corollary 3.4.64 *Two complete algebra norms on a power-associative algebra A over $\mathbb{K}$ give rise to the same spectral radius on A.*

Lemma 3.4.65 *Let A be a non-commutative JB^*-algebra, and let x be in A. Then*

$$\|x^* \bullet x\| \geqslant \frac{1}{2}\|x\|^2.$$

Proof By Fact 3.3.4 and Proposition 3.4.6, there exist a C^*-algebra B and an isometric algebra $*$-homomorphism ϕ from the closed subalgebra of A^{sym} generated by $\{x, x^*\}$ onto a closed $*$-subalgebra of B^{sym}. Now in B we have

$$\phi(x)^* \bullet \phi(x) = \frac{1}{2}\left(\phi(x)^*\phi(x) + \phi(x)\phi(x)^*\right),$$

and hence

$$\phi(x)^* \bullet \phi(x) \geqslant \frac{1}{2}\phi(x)^*\phi(x) \geqslant 0.$$

Therefore, we have

$$\|x^* \bullet x\| = \|\phi(x)^* \bullet \phi(x)\| \geqslant \frac{1}{2}\|\phi(x)^*\phi(x)\| = \frac{1}{2}\|\phi(x)\|^2 = \frac{1}{2}\|x\|^2. \qquad \square$$

The inequality $\|x^* \bullet x\| \geqslant \frac{1}{2}\|x\|^2$ in the above Lemma is sharp (see Example 2.3.65).

Proposition 3.4.66 *All complete algebra norms on a non-commutative JB^*-algebra A are equivalent.*

Proof Let a be in A. Since $H(A, *)$ is a JB-algebra (by Corollary 3.4.3), and $a^* \bullet a$ belongs to $H(A, *)$, we see that $\|a^* \bullet a\| = \mathfrak{r}(a^* \bullet a)$. Now, let $|||\cdot|||$ be any complete algebra norm on A. Keeping in mind Lemma 3.4.65, and that the spectral radius is the same for $\|\cdot\|$ and $|||\cdot|||$ (by Corollary 3.4.64), we have

$$\frac{1}{2}\|a\|^2 \leqslant \|a^* \bullet a\| = \mathfrak{r}(a^* \bullet a) \leqslant |||a^* \bullet a||| \leqslant |||a^*|||\,|||a|||. \tag{3.4.27}$$

Now we shall show that the algebra involution $*$ on A is $|||\cdot|||$-continuous. If $|||x_n||| \to 0$ and $|||x_n^* - y||| \to 0$, then, by (3.4.27),

$$\frac{1}{2}\|x_n\|^2 \leqslant |||x_n^*|||\,|||x_n||| \longrightarrow 0.$$

Hence $\|x_n\| \to 0$. On the other hand,

$$\frac{1}{2}\|x_n - y^*\|^2 \leqslant |||x_n^* - y|||\,|||x_n - y^*||| \longrightarrow 0.$$

Hence $\|x_n^* - y\| \to 0$ and so $y = 0$. By the closed graph theorem, the algebra involution $*$ on A is $|||\cdot|||$-continuous. It follows from (3.4.27) that $\|\cdot\|$ and $|||\cdot|||$ are comparable, and hence equivalent. $\square$

Corollary 3.4.67 *Let A and B be non-commutative JB^*-algebras, and let $F : A \to B$ be a bijective algebra homomorphism. Then F is continuous.*

Proof Define a complete algebra norm $|||\cdot|||$ on A by $|||a||| := \|F(a)\|$, and apply Proposition 3.4.66. $\square$

§3.4.68 Let A be a non-commutative JB^*-algebra. In view of Corollary 3.4.3, $H(A,*)$ becomes a JB-algebra in a natural way, and hence it will be seen endowed with the order introduced in §3.1.27. Thus, according to assertions (i) and (ii) in Lemma 3.1.29, an element h of $H(A,*)$ is positive if and only if $h = k^2$ for some $k \in H(A,*)$. Note that, for $a \in A$, we have $a^* \bullet a \geqslant 0$. Indeed, writing $a = h + ik$, with $h,k \in H(A,*)$, we have $a^* \bullet a = h^2 + k^2$. Actually we have $a^*a \geqslant 0$, but this will be proved in the appropriate place of Volume 2 of this work. Anyway, according to Claim 3.1.28, if B is any closed $*$-subalgebra of A, then the order of $H(B,*)$ (as the self-adjoint part of the non-commutative JB^*-algebra B) coincides with the order induced by that of $H(A,*)$. We note that, if A is in fact a C^*-algebra, then, by Proposition 2.3.39(i), the order in $H(A,*)$ just introduced coincides with the (more usual) order introduced in §1.2.47.

The next fact becomes the unit-free version of Corollary 2.3.67.

Fact 3.4.69 *Let A be an alternative C^*-algebra. Then $a^*a \geqslant 0$ for every $a \in A$.*

Proof Let a be in A, and let B stand for the closed subalgebra of A generated by $\{a, a^*\}$. Then, since $*$ is continuous, B is $*$-invariant (cf. Proposition 1.2.25) and is indeed an associative C^*-algebra (cf. Theorem 2.3.61). Therefore the result follows from Proposition 2.3.39(i). □

Lemma 3.4.70 *Let A be a non-commutative JB^*-algebra, let a be a positive element of $H(A,*)$, let b be in A, let B denote the closed subalgebra of A generated by b, let C stand for the unital extension of B, and let z be in $\mathrm{sp}(C,b)$. Then $\|a-b\| \geqslant d(z,\mathbb{R}_0^+)$.*

Proof We may assume that $z \neq 0$. On the other hand, by Fact 3.3.4 and Lemma 2.4.17, we may assume that A is commutative. Then, by Corollary 3.4.10, we may also assume that A is unital. Let $\mathbb{1}$ and $\mathbf{1}$ stand for the units of C and A, respectively, and note that, since B is an associative algebra containing b, and C is the unital extension of B, and $0 \neq z \in \mathrm{sp}(C,b)$, we have $z \in \mathrm{sp}(\mathbb{C}\mathbf{1}+B,b)$. (Indeed, if $\mathbf{1} \in B$, this follows from Proposition 1.1.106; otherwise, the mapping $\phi : \lambda\mathbb{1}+x \to \lambda\mathbf{1}+x$ from $C = \mathbb{C}\mathbb{1} \oplus B$ to $\mathbb{C}\mathbf{1}+B$ is a bijective algebra homomorphism with $\phi(b) = b$.) Now, applying Lemma 2.3.21 and Corollary 2.1.2, we get

$$z \in V(\mathbb{C}\mathbf{1}+B,\mathbf{1},b) = V(A,\mathbf{1},b).$$

Let f be in $D(A,\mathbf{1})$ such that $f(b) = z$. Since the restriction of f to $H(A,*)$ belongs to $D(H(A,*),\mathbf{1})$ (by Lemma 2.2.5), and a is a positive element of the JB-algebra $H(A,*)$, we have $f(a) \geqslant 0$. It follows that

$$\|a-b\| \geqslant |f(a-b)| = |f(a)-z| \geqslant d(z,\mathbb{R}_0^+). \qquad \square$$

§3.4.71 Let X and Y be complex vector spaces, each of which is endowed with a conjugate-linear involution $*$. For $T \in L(X,Y)$, we consider the element $T^* \in L(X,Y)$ defined by $T^*(x) := (T(x^*))^*$, and note that the mapping $T \to T^*$, from $L(X,Y)$ to itself, becomes a conjugate-linear involution on $L(X,Y)$. Moreover, if Z is another complex vector space endowed with a conjugate-linear involution $*$, then, for $F \in L(X,Y)$ and $G \in L(Y,Z)$, we have $(GF)^* = G^*F^*$. As a consequence, if T is a bijective operator in $L(X,Y)$, then T^* is bijective, and $(T^*)^{-1} = (T^{-1})^*$, which in its turn implies that, if T is in $L(X)$, then

$$\mathrm{sp}(L(X),T^*) = \{\bar{\lambda} : \lambda \in \mathrm{sp}(L(X),T)\}. \tag{3.4.28}$$

If X and Y are in fact complex $*$-algebras, and if $T : X \to Y$ is an algebra homomorphism, then so is T^*.

Now, let A and B be non-commutative JB^*-algebras. We denote by $\mathrm{Aut}(A,B)$ the set of all bijective algebra homomorphisms from A to B. For $F \in \mathrm{Aut}(A,B)$, we define $F^\bullet \in \mathrm{Aut}(B,A)$ by $F^\bullet = (F^*)^{-1}$. When $B = A$, we write $\mathrm{Aut}(A)$ instead of $\mathrm{Aut}(A,A)$. The set of those $F \in \mathrm{Aut}(A)$ such that $F^\bullet = F$ and $\mathrm{sp}(L(A),F) \subseteq \mathbb{R}_0^+$ will be denoted by $\mathrm{Aut}^+(A)$. The set of those derivations D of A such that $D^* = -D$ will be denoted by $\mathrm{Der}^*(A)$. We recall that bijective algebra homomorphisms from A to B, as well as derivations of A, are automatically continuous (see Corollary 3.4.67 and Lemma 3.4.26), and that elements of $\mathrm{Der}^*(A)$ are in fact hermitian operators on A (see Lemma 3.4.27).

Lemma 3.4.72 *Let A be a non-commutative JB^*-algebra. Then the mapping $D \to \exp(D)$ is a homeomorphism from $\mathrm{Der}^*(A)$ onto $\mathrm{Aut}^+(A)$.*

Proof By the holomorphic functional calculus, the mapping $F \to \exp(F)$ is a homeomorphism from the set of those elements in $BL(A)$ whose spectrum is included in the complex band $\{z \in \mathbb{C} : -\pi < \Im(z) < \pi\}$ onto the set of those elements in $BL(A)$ whose spectrum is included in $\mathbb{C} \setminus \mathbb{R}_0^-$. On the other hand, as noted above, elements of $\mathrm{Der}^*(A)$ are hermitian operators on A. Let D be in $\mathrm{Der}^*(A)$. Then we have

$$\mathrm{sp}(BL(A),D) \subseteq V(BL(A),I_A,D) \subseteq \mathbb{R},$$

and hence, by the spectral mapping theorem, $\mathrm{sp}(BL(A),\exp(D)) \subseteq \mathbb{R}^+$. Since clearly $(\exp(D))^\bullet = \exp(D)$, we have in fact that $\exp(D) \in \mathrm{Aut}^+(A)$. Now let F be in $\mathrm{Aut}^+(A)$. Then there exists a unique element $D \in BL(A)$ such that $\mathrm{sp}(BL(A),D) \subseteq \mathbb{R}$ and $F = \exp(D)$. Moreover, since

$$\exp(D^*) = F^* = F^{-1} = \exp(-D),$$

and $\mathrm{sp}(BL(A),D^*) \subseteq \mathbb{R}$ (by the equality (3.4.28) and Example 1.1.32(d)), we get $D^* = -D$. Since D is a derivation of A (by Theorem 3.4.49), we conclude that D lies in $\mathrm{Der}^*(A)$. □

Corollary 3.4.73 *Let A be a non-commutative JB^*-algebra, and let F be in $\mathrm{Aut}^+(A)$. Then there exists a unique $G \in \mathrm{Aut}^+(A)$ such that $G^2 = F$. (This G will be denoted by $\sqrt{F}$).*

Proof With Lemma 3.4.72 in mind, we have $F = \exp(D)$ for a unique $D \in \mathrm{Der}^*(A)$. Then $G := \exp(\frac{1}{2}D)$ is the unique element of $\mathrm{Aut}^+(A)$ such that $G^2 = F$. □

The next lemma is crucial to state the structure theorem we are after.

Lemma 3.4.74 *Let A and B be non-commutative JB^*-algebras, and let F be in $\mathrm{Aut}(A,B)$. Then $F^\bullet F$ lies in $\mathrm{Aut}^+(A)$.*

Proof Let a be in A, and let z be in $\mathbb{C}$. Then, keeping in mind Proposition 3.3.13 and its consequence (that $\|(F^\bullet)^{-1}\| = \|F\|$), we have

$$\begin{aligned}\|a\|\,\|F^\bullet F(a) - za\| &\geqslant \|a^* \bullet (F^\bullet F(a) - za)\| \\ &= \|F^\bullet (F(a)^* \bullet F(a) - zF^*(a^* \bullet a))\| \\ &\geqslant \|F\|^{-1}\|F(a)^* \bullet F(a) - zF^*(a^* \bullet a)\|.\end{aligned}$$

Set $\alpha := \|a^* \bullet a\|$, let C denote the closed subalgebra of A generated by $a^* \bullet a$, and let D stand for the unital extension of C. Then, since C is a C^*-algebra (by Lemma 3.4.1(ii)), and $a^* \bullet a$ is a positive element of C, we have

$$\alpha \in \mathrm{sp}(D, a^* \bullet a).$$

Now F^* is a continuous algebra isomorphism, and hence $F^*(C)$ is the closed subalgebra of B generated by $F^*(a^* \bullet a)$, and $\alpha z \in \mathrm{sp}(E, zF^*(a^* \bullet a))$, where E stands for the unital extension of $F^*(D)$. On the other hand, $F(a)^* \bullet F(a)$ is a positive element in B. It follows from Lemma 3.4.70 that

$$\|F(a)^* \bullet F(a) - zF^*(a^* \bullet a)\| \geqslant d(\alpha z, \mathbb{R}_0^+) = \alpha d(z, \mathbb{R}_0^+).$$

Now by Lemma 3.4.65 we have

$$\|F(a)^* \bullet F(a) - zF^*(a^* \bullet a)\| \geqslant \frac{1}{2}\|a\|^2 d(z, \mathbb{R}_0^+),$$

and hence

$$\|F^\bullet F(a) - za\| \geqslant \frac{1}{2}\|F\|^{-1}\|a\| d(z, \mathbb{R}_0^+).$$

Since a is arbitrary in A, we see that $F^\bullet F - zI_A$ is bounded below whenever z is in $\mathbb{C} \setminus \mathbb{R}_0^+$. Therefore, by Proposition 1.1.94(i) and Corollary 1.1.91, the boundary of $\mathrm{sp}(BL(A), F^\bullet F)$ relative to $\mathbb{C}$ is contained in $\mathbb{R}_0^+$, and hence $\mathrm{sp}(BL(A), F^\bullet F) \subseteq \mathbb{R}_0^+$. Since $F^\bullet F \in \mathrm{Aut}(A)$, and $(F^\bullet F)^\bullet = F^\bullet F$, the proof is complete. □

Theorem 3.4.75 *Let A and B be non-commutative JB*-algebras, and let F be in* $\mathrm{Aut}(A,B)$. *Then F can be written in a unique way as*

$$F = G\exp(D), \tag{3.4.29}$$

where D is in $\mathrm{Der}^*(A)$ *and $G : A \to B$ is a bijective algebra $*$-homomorphism.*

Proof Assume that there is a decomposition such as that in (3.4.29). Then we have $F^\bullet = \exp(D)G^{-1}$, so $F^\bullet F = \exp(2D)$, and so, by Lemma 3.4.72 and Corollary 3.4.73, $F^\bullet F$ lies in $\mathrm{Aut}^+(A)$ and D must be the unique element in $\mathrm{Der}^*(A)$ such that $\exp(D) = \sqrt{F^\bullet F}$. Then, obligatorily, $G = F\exp(-D)$, and the uniqueness of the decomposition (3.4.29) follows.

To show the existence of a decomposition such as that in (3.4.29), use Lemma 3.4.74 and Corollary 3.4.73 to find $D \in \mathrm{Der}^*(A)$ such that $\exp(D) = \sqrt{F^\bullet F}$, and define $G := F\exp(-D)$. Then the proof is concluded by showing that $G^* = G$. But, since

$$(G^*)^{-1}G = \exp(-D)F^\bullet F\exp(-D) = \exp(-D)\exp(2D)\exp(-D) = I_A$$

and

$$G(G^*)^{-1} = F\exp(-D)\exp(-D)F^\bullet = F\exp(-2D)F^\bullet = F(F^\bullet F)^{-1}F^\bullet = I_B,$$

we get $G^* = G$, as desired. □

The following relevant corollary is known as *the essential uniqueness of the JB^*-structure in non-commutative JB^*-algebras*.

Corollary 3.4.76 *If two non-commutative JB^*-algebras are isomorphic, then they are isometrically $*$-isomorphic.*

Proof Combine Theorem 3.4.75 and Proposition 3.4.4. □

Since C^*-algebras are non-commutative JB^*-algebras (by Fact 3.3.2), and commutative C^*-algebras have no nonzero derivations (by Corollary 3.4.51), it follows from Theorem 3.4.75 that bijective algebra homomorphisms between commutative C^*-algebras are $*$-homomorphisms, and hence are automatically isometric (by Corollary 1.2.14). This was already proved in Proposition 1.2.44 with more elementary methods.

As the reader may have noticed, there is some redundancy in Corollary 3.4.76 because, by Proposition 3.4.4, bijective algebra $*$-homomorphisms between non-commutative JB^*-algebras are automatically isometric (in other words, the involution of a non-commutative JB^*-algebra determines the norm). Now we are going to show that, conversely, the norm of a non-commutative JB^*-algebra determines the involution. To this end, we prove the following.

Lemma 3.4.77 *Let A be a non-commutative JB^*-algebra, and let D be in* $\mathrm{Der}^*(A)$. *Then*

$$\|\exp(D)\| = e^{\|D\|}.$$

Proof By Lemma 3.4.27, D lies in $H(BL(A), I_A)$. Then, by Lemma 2.3.21 and Proposition 2.3.22, $\{\|D\|, -\|D\|\} \cap \mathrm{sp}(BL(A), D) \neq \emptyset$. Since $D^* = -D$, it follows from the equality (3.4.28) and Example 1.1.32(d) that

$$\{\|D\|, -\|D\|\} \subseteq \mathrm{sp}(BL(A), D).$$

Thus $e^{\|D\|}$ lies in $\mathrm{sp}(BL(A), \exp(D))$, and hence

$$e^{\|D\|} \leqslant \|\exp(D)\| \leqslant e^{\|D\|}.$$ □

Proposition 3.4.78 *Bijective algebra homomorphisms between non-commutative JB^*-algebras are isometric (if and) only if they preserve involutions.*

Proof Let A and B be non-commutative JB^*-algebras, and let $F : A \to B$ be an isometric surjective algebra homomorphism. Then, writing $F = G\exp(D)$ as in Theorem 3.4.75, and applying Lemma 3.4.77, we have

$$1 = \|F\| = \|\exp(D)\| = e^{\|D\|},$$

so $D = 0$, and so $F = G$ preserves involutions. □

The following corollary includes several by-products of Theorem 3.4.75 and of the arguments in its proof. Given non-commutative JB^*-algebras A and B, we denote by $\mathrm{Aut}^*(A,B)$ the set of all bijective algebra $*$-homomorphisms from A to B.

Corollary 3.4.79 *Let A and B be non-commutative JB^*-algebras. We have:*

(i) *If F is in $\mathrm{Aut}(A)$, and if $F^\bullet = F$, then $\mathrm{sp}(BL(A),F) \subseteq \mathbb{R}$.*
(ii) *If F is in $\mathrm{Aut}(A,B)$, then $\|F^\bullet F\| = \|F\|^2$.*
(iii) *If F is in $\mathrm{Aut}(A,B)$, then $\|F^\bullet\| = \|F^{-1}\| = \|F\|$.*
(iv) *If F is in $\mathrm{Aut}(A)$ with $F^\bullet F = FF^\bullet$ (equivalently, with $F^*F = FF^*$), then $\mathfrak{r}(F) = \|F\|$.*
(v) *An algebra automorphism F of A is a $*$-automorphism (i.e. $F^\bullet = F^{-1}$) if and only if $F^\bullet F = FF^\bullet$ and $\mathrm{sp}(BL(A),F) \subseteq \mathbb{S}_{\mathbb{C}}$.*
(vi) *If D is in $\mathrm{Der}^*(A)$, then $\|\exp(D) - I_A\| = \|\exp(D)\| - 1$.*
(vii) *If F is in $\mathrm{Aut}(A,B)$, and if $F = G\exp(D)$, with G and D as in Theorem 3.4.75, then $\|F - G\| = d(F, \mathrm{Aut}^*(A,B)) = \|F\| - 1$.*
(viii) *$\mathrm{Aut}(A,B)$ is closed in $BL(A,B)$.*

Proof Let F be in $\mathrm{Aut}(A)$ with $F^\bullet = F$. Then, by Lemma 3.4.74, we have $\mathrm{sp}(BL(A),F^2) \subseteq \mathbb{R}_0^+$, and hence $\mathrm{sp}(BL(A),F) \subseteq \mathbb{R}$. This proves assertion (i).

Let F be in $\mathrm{Aut}(A,B)$. By Theorem 3.4.75, F can be written as $F = G\exp(D)$, where D is in $\mathrm{Der}^*(A)$ and $G : A \to B$ is a bijective algebra $*$-homomorphism. Then we have $F^\bullet F = \exp(2D)$, and hence, invoking Lemma 3.4.77 twice, and keeping in mind that G is a surjective linear isometry, we get

$$\|F^\bullet F\| = \|\exp(2D)\| = e^{2\|D\|} = (e^{\|D\|})^2 = \|\exp(D)\|^2 = \|G\exp(D)\|^2 = \|F\|^2.$$

This proves assertion (ii).

Since the involutions of A and B are isometric (cf. Proposition 3.3.13), assertion (iii) follows from assertion (ii).

Keeping in mind assertion (ii), assertion (iv) is proved by mimicking the proof of Lemma 1.2.12.

Since $*$-automorphisms of A are surjective linear isometries, the 'only if' part of assertion (v) becomes straightforward. The 'if' part follows by applying assertion (iv) to both F and F^{-1}, and then by invoking Proposition 3.4.78.

Let D be in $\mathrm{Der}^*(A)$. Then we have

$$\begin{aligned}\|\exp(D)\| - 1 \leqslant \|\exp(D) - I_A\| &= \left\| \sum_{n=1}^{\infty} \frac{D^n}{n!} \right\| \\ &\leqslant \sum_{n=1}^{\infty} \frac{\|D^n\|}{n!} = e^{\|D\|} - 1 = \|\exp(D)\| - 1,\end{aligned}$$

where, for the last equality, we have applied Lemma 3.4.77. This proves assertion (vi).

Let F be in $\mathrm{Aut}(A,B)$. Then, by Proposition 3.4.4, for $H \in \mathrm{Aut}^*(A,B)$ we have $\|H\| = 1$, so $\|F - H\| \geqslant \|F\| - \|H\| = \|F\| - 1$, and so

$$d(F, \mathrm{Aut}^*(A,B)) \geqslant \|F\| - 1.$$

Now write $F = G\exp(D)$, with $G \in \mathrm{Aut}^*(A,B)$ and $D \in \mathrm{Der}^*(A)$. Then, by assertion (vi), we have

$$\begin{aligned}\|F - G\| &= \|G(\exp(D) - I_A)\| = \|\exp(D) - I_A\| \\ &= \|\exp(D)\| - 1 = \|G\exp(D)\| - 1 = \|F\| - 1.\end{aligned}$$

This proves assertion (vii).

Let F_n be a sequence in $\mathrm{Aut}(A,B)$ converging to some $F \in BL(A,B)$. Then, clearly, F is an algebra homomorphism from A to B. On the other hand, by assertion (iii), we have $\|F_n^{-1}\| = \|F_n\|$ for every n, so the sequence F_n^{-1} is bounded, and so, since $\|F_n^{-1} - F_m^{-1}\| \leqslant \|F_n^{-1}\|\|F_m^{-1}\|\|F_n - F_m\|$ for all n,m, we see that F_n^{-1} is a Cauchy sequence in $BL(B,A)$. By taking $L := \lim_{n\to\infty} F_n^{-1}$, we have $LF = I_A$ and $FL = I_B$, which shows that F is bijective. This concludes the proof of assertion (viii). □

For the formulation and proof of the next corollary, recall that, if A is a C^*-algebra, then by Facts 3.3.2 and 3.3.4, A is a non-commutative JB^*-algebra, and A^{sym} is a JB^*-algebra.

Corollary 3.4.80 *Let A and B be C^*-algebras. We have:*

(i) *If $F : A \to B$ is a bijective Jordan homomorphism, then F can be written in a unique way as $F = G\exp(D)$, where D is in $\mathrm{Der}^*(A)$ and $G : A \to B$ is a bijective Jordan-$*$-homomorphism.*
(ii) $\mathrm{Aut}(A^{\mathrm{sym}}, B^{\mathrm{sym}}) \setminus \mathrm{Aut}(A,B)$ *is closed in* $BL(A,B)$.

Proof Assertion (i) follows from Theorem 3.4.75 and Lemma 2.2.23.

Now note that, if $\mathrm{Aut}(A,B) = \emptyset$, then, by Corollary 3.4.79(viii), assertion (ii) holds. Therefore, to conclude the proof of assertion (ii), we may assume that $\mathrm{Aut}(A,B) \neq \emptyset$. In this case, choosing $H \in \mathrm{Aut}(A,B)$, the mapping $F \to HF$ becomes a homeomorphism from $BL(A)$ onto $BL(A,B)$ taking $\mathrm{Aut}(A^{\mathrm{sym}})$ onto $\mathrm{Aut}(A^{\mathrm{sym}}, B^{\mathrm{sym}})$, and taking $\mathrm{Aut}(A)$ onto $\mathrm{Aut}(A,B)$. Hence it is enough to show that $\mathrm{Aut}(A^{\mathrm{sym}}) \setminus \mathrm{Aut}(A)$ is closed in $BL(A)$. But, by Corollary 3.4.54, $\mathrm{Aut}(A^{\mathrm{sym}}) \setminus \mathrm{Aut}(A)$ is relatively closed in $\mathrm{Aut}(A^{\mathrm{sym}})$. Since $\mathrm{Aut}(A^{\mathrm{sym}})$ is closed in $BL(A)$ (by Corollary 3.4.79(viii)), we derive that $\mathrm{Aut}(A^{\mathrm{sym}}) \setminus \mathrm{Aut}(A)$ is closed in $BL(A)$, as desired. □

3.4.5 Historical notes and comments

Propositions 3.4.1, 3.4.4, 3.4.6, and 3.4.13, Corollary 3.4.3, and Theorem 3.4.8 are due to Wright [641] in the unital case. Wright's proof of Proposition 3.4.13 is different from ours. The actual unit-free version of Corollary 3.4.3 was first proved by Youngson [655], by applying the main result in Behncke's paper [82] (see Proposition 4.5.24 below) and a generalization of Bonsall's facts [112, 114], obtained in [653], which was later refined in [48] (see Corollary 4.5.32 below). Corollary 3.4.3 is explicitly stated in Proposition 3.8.2 of the Hanche-Olsen and Størmer book [738], with a different proof from that of Youngson. The core of the proof in [738] is formulated in our Proposition 3.4.2. Youngson's paternity of Corollary 3.4.3 affects the paternity of the actual unit-free versions of Propositions 3.4.4, 3.4.6, and 3.4.13, and Theorem 3.4.8. A relevant consequence of Proposition 3.4.6 is the following.

Corollary 3.4.81 *Let A be a unital non-commutative JB^*-algebra. Then A satisfies the von Neumann inequality.*

Proof Since powers in A and in A^{sym} coincide (by Lemma 2.4.17), and A^{sym} is a JB^*-algebra (by Fact 3.3.4), we may assume that A is commutative. Let a be in $\mathbb{B}_A$. Then, by Proposition 3.4.6, the closed subalgebra of A generated by $\{\mathbf{1}, a, a^*\}$ is a subalgebra of B^{sym}, for some unital C^*-algebra B. Since powers in B and in B^{sym} coincide, it follows from Theorem 2.4.32 that, for every complex polynomial P, we have $\|P(a)\| \leqslant \max\{|P(z)| : z \in \mathbb{B}_{\mathbb{C}}\}$. $\square$

Wright's ideas in [641] underlie the proof of Theorem 3.4.8 given here, although the concluding application of the non-associative Vidav–Palmer theorem (Theorem 3.3.11) seems to us to be new. Corollary 3.4.7 and Proposition 3.4.17 are due to Wright and Youngson [642], whereas Corollary 3.4.10 is due to Youngson [655]. The Russo–Dye-type theorem for non-commutative JB^*-algebras underlying Corollary 3.4.7, as well as some refinements given in [574], will be discussed later (see Subsection 4.2.3).

Historically, the fundamental formula $U_{U_a(b)} = U_a U_b U_a$ for Jordan algebras, stated in Proposition 3.4.15, was derived from the so-called *Macdonald's theorem* [410]. This theorem is one of the deepest results in the theory of Jordan algebras. In order to formulate it, we recall that, in §§2.8.17 and 2.8.26, we introduced non-associative words with characters on an arbitrary set of indeterminates, the (global) degree of such a word, the notion of a non-associative polynomial on a finite set of indeterminates, and the meaning of $\mathbf{p}(a_1, \ldots, a_n)$ where $\mathbf{p}$ is a non-associative polynomial in n indeterminates, and $a_1, \ldots, a_n$ are elements of a given algebra. Later, in §3.4.41, we introduced the degree of a non-associative word in one of the indeterminates. Now, we invite the reader to imagine the notion of 'degree of a non-associative polynomial in one of the indeterminates'. We also recall that special Jordan algebras were introduced immediately before Theorem 3.1.55. According to the formulation in [822, Corollary 3.3.2], Macdonald's theorem reads as follows.

Theorem 3.4.82 *Let $\mathbf{p}$ be a non-associative polynomial in three indeterminates, of degree $\leqslant 1$ in one of them. If $\mathbf{p}(a_1, a_2, a_3) = 0$ for all a_1, a_2, a_3 in every special Jordan algebra, then $\mathbf{p}(a_1, a_2, a_3) = 0$ for all a_1, a_2, a_3 in every Jordan algebra.*

We note that, when the degree of $\mathbf{p}$ is zero in one of the indeterminates, the conclusion in Theorem 3.4.82 follows from the Shirshov–Cohn Theorem 3.1.55.

Remark 3.4.83 A complete proof of the fundamental formula for Jordan algebras, which avoids Macdonald's theorem, will be given in our work. The proof begins with the linearizations of the Jordan identity pointed out in the proof of Proposition 2.4.13, continues with Lemmas 3.1.23, 3.3.35, and 4.1.33, and Propositions 4.1.34 and 4.2.16, concluding finally in §4.2.66.

Until Corollary 3.4.18, JB^*-algebras have parasitized JB-algebras. Now, Corollary 3.4.18 becomes the first sample showing how the relation JB-algebra to JB^*-algebra is in fact symbiotic. Since the inequality $\|U_{a,c}(b)\| \leqslant \|a\|\|b\|\|c\|$ is straightforward for elements a, b, c in any JC-algebra, the theory of JB-algebras developed in [738] reduces the proof of Corollary 3.4.18 to the case of the exceptional JB-algebra $H_3(\mathbb{O})$ in Example 3.1.56. However, we do not know of a direct proof in this case.

Proposition 3.4.23 could be new. Via Fact 3.4.9, the commutative version of Proposition 3.4.23 is equivalent to Corollary 3.4.24, which in its turn, via Proposition 3.1.3, is equivalent to a result of Topping [813, Proposition 1].

Proposition 3.4.25 was first proved by Wright and Youngson [643], with an argument different from that given here. Actually, the core of the proof in [643] consisted of Corollary 3.4.3 and Proposition 3.1.13. Lemmas 3.4.26 and 3.4.27, and Proposition 3.4.28 are due to Youngson [654]. Keeping in mind that non-commutative JB^*-algebras are JB^*-triples in a natural way (see Theorem 4.1.45 below), the result of Barton and Friedman [60], asserting the automatic continuity of derivations of JB^*-triples, generalizes Lemma 3.4.26. Proposition 3.4.31 is the particularization to alternative C^*-algebras of a more general result for non-commutative JB^*-algebras proved in [126, 365] (see Theorem 4.2.28 below). Corollary 3.4.32 is taken from [366].

Antitheorem 3.4.34 was first proved by Braun, Kaup, and Upmeier [126, Section 5] with an argument quite different from ours. In fact, with the notation in our proof, they showed that there is no surjective linear isometry on (the Banach space of) A taking $\mathbf{1}$ to u. But, since the JC^*-algebras A and B coincide as Banach spaces, and $\mathbf{1}$ and u are the units of A and B, respectively, this is equivalent to the fact proved here that A and B cannot be $*$-isomorphic (by Proposition 3.4.25).

Subsection 3.4.3 is chaired by Theorem 3.4.49. This theorem is the first deep result in the theory of associative normed algebras which remains true in the non-associative setting without needing any change in its proof. It appears as Lemma III.9.9 of Dixmier's book [723], and, according to [723, p. 315], it is essentially due to J.-P. Serre. The clever ideas of Dixmier–Serre are included in Fact 3.4.36, and Lemmas 3.4.37, 3.4.45, and 3.4.47. In fact, the original results were the case $n = 2$ of ours, but our generalizations are obvious, and some of them (namely the actual formulations of Fact 3.4.36 and Lemma 3.4.37) become essential for the proof of Theorem 3.4.42. Results from Proposition 3.4.38 to Corollary 3.4.44 are essentially due to Aparicio and Rodríguez [23], although some of them (including Theorem 3.4.42) are presented here in a refined way. The information about the spectrum of derivations and automorphisms given by Theorem 3.4.42 and Corollary 3.4.44 could seem rather poor. However, the complete normed complex algebras in those results are not assumed to be associative, and the unique additional assumption, that they are not nilpotent, is really smooth. For better algebras, the information about the spectrum of automorphisms is much more precise. This is the case of the following classical theorem of Kamowitz and Scheinberg [371].

Theorem 3.4.84 *Let A be a complete normed semisimple associative and commutative complex algebra, and let Φ be an algebra automorphism of A. Then either $\Phi^n = I_A$ for some $n \in \mathbb{N}$, or $\mathrm{sp}(BL(A), \Phi)$ contains the whole unit sphere $\mathbb{S}_{\mathbb{C}}$ of $\mathbb{C}$.*

The actual formulation of Lemma 3.4.47 appears in the paper of Castellón and Cuenca [162], who also emphasize the case $n = 3$ in the following.

Corollary 3.4.85 *Let A be a complete normed complex 3-algebra, and let Φ be a continuous automorphism of A such that*

$$\mathrm{sp}(BL(A),\Phi) \subseteq \{z \in \mathbb{C} : \Re(z) > 0\}.$$

Then $\log(\Phi)$ *is a derivation of A.*

A classical associative forerunner of Theorem 3.4.49 is due to Zeller-Meier [660] (see also [696, Theorem 18.15]), who proved that *if A is a complete normed associative complex algebra, and if Φ is a continuous algebra automorphism of A such that* $\mathrm{sp}(BL(A),\Phi) \subseteq \{z \in \mathbb{C} : \Re(z) > 0\}$, *then* $\log(\Phi)$ *is a derivation of A.* We note that even the particularization of Theorem 3.4.49 to associative algebras becomes a better result, and that Zeller-Meier's proof needs associativity. In relation to Proposition 3.4.50, we note that the unique assumption on the normed algebra A (that there exists $M > 0$ such that $\|a\|^2 \leqslant M\|a^2\|$ for every $a \in A$) is automatically fulfilled when A is a nearly absolute-valued algebra. Therefore, as a by-product, we get the following.

Corollary 3.4.86 *Nearly absolute-valued complex algebras have no nonzero continuous derivation.*

In particular, as pointed out in [533, Proposition 3.7], *absolute-valued complex algebras have no nonzero continuous derivation.*

Corollary 3.4.51 is due to Sakai [806]. A relevant generalization, due to Johnson [354] (see also [696, Theorem 18.21]), assures that *complete normed semisimple associative and commutative complex algebras have no nonzero derivation.* One of the ingredients in Johnson's proof is the celebrated Singer–Wermer theorem [582] asserting that *the range of a continuous derivation of a complete normed associative and commutative complex algebra is contained in the radical.* The question of whether the continuity can be removed in the Singer–Wermer theorem (known as the Singer–Wermer conjecture) was affirmatively answered by Thomas [611] (see also [715, Theorem 5.2.48]). A more modest generalization of the Singer–Wermer theorem will be proved later in Proposition 3.6.51.

Corollary 3.4.53 could be new. Corollary 3.4.54 is due to Sinclair [577], who, being unaware of Theorem 3.4.49, shows in a previous lemma how Zeller-Meier's proof can be adapted to cover the case of continuous Jordan automorphisms of complete normed associative complex algebras.

Results from Proposition 3.4.55 to Corollary 3.4.60 are taken from [514]. The associative forerunner of Corollary 3.4.60 is originally due to Crabb, Duncan, and McGregor [183].

§**3.4.87** Corollary 3.4.64 is originally due to Balachandran and Rema [54] with the following clever proof:

> Let $\|\cdot\|$ and $|||\cdot|||$ be two complete algebra norms on a power-associative algebra A, and let a be in A. Choose a maximal associative subalgebra B of A containing a. Then B is both $\|\cdot\|$- and $|||\cdot|||$-closed in A. Therefore $\|\cdot\|$ and $|||\cdot|||$ are complete algebra norms on the associative algebra B, and the well-known associative result applies (see Corollary 1.1.109) to get that the spectral radius of a is the same for both $\|\cdot\|$ and $|||\cdot|||$. □

The refinement of Corollary 3.4.64, given by Proposition 3.4.63, was pointed out first in [520].

Results from Lemma 3.4.65 to Corollary 3.4.76 are due to Payá, Pérez, and Rodríguez [481]. Some arguments from [806, p. 162] (respectively, [512]), invoked by the authors of [481] in the proof of Proposition 3.4.66 (respectively, Lemma 3.4.74 and Theorem 3.4.75), have been incorporated. The associative forerunners of Theorem 3.4.75 and Corollary 3.4.76 are originally due to Okayasu [466] and Gardner [280], respectively. Both Okayasu's and Gardner's results are included in Sakai's book [806, Corollary 4.1.21 and Theorem 4.1.20].

Lemma 3.4.77 and Corollary 3.4.79 are new, although they are adaptations of previous arguments in the study of algebra automorphisms and Jordan automorphisms of C^*-algebras, done in [798] and [512], respectively. Actually, the whole of Subsection 3.4.4 is tributary to the new proof of Okayasu's theorem given in [798]. The associative forerunner of assertion (ii) in Corollary 3.4.79 is proved in [798] as follows:

Let A and B be C^*-algebras, and let F be in $\mathrm{Aut}(A,B)$. Then, for $a \in A$ we have

$$\begin{aligned}\|F(a)\|^2 &= \|F(a)^*F(a)\| = \mathfrak{r}(F(a)^*F(a)) = \mathfrak{r}(F^{-1}(F(a)^*F(a)))\\ &\leqslant \|F^{-1}(F(a)^*F(a))\| = \|F^{-1}(F(a)^*)a\| \leqslant \|F^{-1}(F(a)^*)\|\|a\|\\ &= \|(F^\bullet F(a))^*\|\|a\| = \|F^\bullet F(a)\|\|a\| \leqslant \|F^\bullet F\|\|a\|^2,\end{aligned}$$

and hence $\|F\|^2 \leqslant \|F^\bullet F\|$. That this inequality is in fact an equality, is easily derived. □

The simple proof above does not work in the actual setting of Corollary 3.4.79, where A and B are non-commutative JB^*-algebras, so that the proof of the equality $\|F^\bullet F\| = \|F\|^2$ given there is actually new. The associative forerunner of assertion (viii) in Corollary 3.4.79 is originally due to Sakai [806, Proposition 4.1.13].

Proposition 3.4.78 (that isometric surjective algebra homomorphisms between non-commutative JB^*-algebras preserve involutions) is folklore in the theory, although the proof given here is new. Usually, it is verified by passing to biduals via Lemma 3.1.17, by applying that biduals of non-commutative JB^*-algebras are unital non-commutative JB^*-algebras (a result which will be proved in Theorem 3.5.34 below), and then by invoking Corollary 3.3.17(a).

Corollary 3.4.80 is taken from [512].

3.5 The Gelfand–Naimark axiom $\|a^*a\| = \|a^*\|\|a\|$, and the non-unital non-associative Gelfand–Naimark theorem

Introduction In Theorem 3.5.5 we will give a precise description of quadratic non-commutative JB^*-algebras. This becomes an auxiliary tool to prove in Theorem 3.5.15 that, if A is a complete normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ such that $\mathbf{1}^* = \mathbf{1}$ and $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$, and if $\dim(A) \geqslant 3$, then A is an alternative C^*-algebra. We show that, if A is a nonzero non-commutative JB^*-algebra, then A has an approximate unit bounded by 1 (Proposition 3.5.23), and the bidual of A is a unital non-commutative JB^*-algebra in a natural manner (Theorem 3.5.34). These results are applied to prove (in Theorem 3.5.49) that Theorem 3.5.15 reviewed above remains true if the assumption that A is unital with $\mathbf{1}^* = \mathbf{1}$ is relaxed to that of the existence of an approximate unit bounded by 1 and consisting of $*$-invariant elements. As a consequence, we derive the so-called *non-unital non-associative Gelfand–Naimark theorem*, which asserts that alternative C^*-algebras are precisely those non-associative C^*-algebras having an approximate unit bounded by 1 (Theorem 3.5.53). Moreover, we exhibit abundant examples of non-associative C^*-algebras which are not alternative. The section concludes by proving in Theorem 3.5.66 that complete normed non-commutative Jordan complex $*$-algebras A satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$ are in fact alternative C^*-algebras. This generalizes Vowden's celebrated theorem [629] asserting that complete normed associative complex $*$-algebras A satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$ are C^*-algebras.

3.5.1 Quadratic non-commutative JB^*-algebras

Let X,Y be vector spaces over $\mathbb{K}$, and let Z be a subspace of Y. Then $X \otimes Z$ imbeds naturally into $X \otimes Y$. Moreover, if X and Y are in fact normed spaces, it is straightforward to realize that the natural embedding $X \otimes_\pi Z \hookrightarrow X \otimes_\pi Y$ becomes contractive, i.e. for $\alpha \in X \otimes Z$, we have

$$\|\alpha\|_{X \otimes_\pi Y} \leqslant \|\alpha\|_{X \otimes_\pi Z}. \tag{3.5.1}$$

In the case that Z is the range of a contractive linear projection on Y, a better result holds. Indeed, we have the following.

Lemma 3.5.1 *Let X,Y be normed spaces over $\mathbb{K}$, and let Z be the range of a contractive linear projection on Y. Then the natural embedding*

$$X \otimes_\pi Z \hookrightarrow X \otimes_\pi Y$$

is an isometry.

Proof Let P be a contractive linear projection on Y with $P(Y) = Z$, and note that the operator $I_X \otimes P$ is a linear projection on $X \otimes Y$ which becomes the identity on $X \otimes Z$. Now, let α be in $X \otimes Z$, and write

$$\alpha = \sum_{i=1}^{n} x_i \otimes y_i \tag{3.5.2}$$

for suitable $n \in \mathbb{N}$, $x_i \in X$, and $y_i \in Y$. Then, applying $I_X \otimes P$ to both members of the equality (3.5.2), we obtain $\alpha = \sum_{i=1}^{n} x_i \otimes P(y_i)$, and hence

$$\|\alpha\|_{X \otimes_\pi Z} \leqslant \sum_{i=1}^{n} \|x_i\|\|P(y_i)\| \leqslant \sum_{i=1}^{n} \|x_i\|\|y_i\|.$$

By considering the different ways of writing α in (3.5.2), we get

$$\|\alpha\|_{X \otimes_\pi Z} \leqslant \|\alpha\|_{X \otimes_\pi Y},$$

which, together with (3.5.1), concludes the proof. $\square$

Proposition 3.5.2 *Let H be a real pre-Hilbert space, and let $\|\cdot\|_\pi$ stand for the projective tensor norm on $\mathbb{C}\otimes H = H\oplus iH$. Then we have*

$$\|x+iy\|_\pi^2 = \|x\|^2+\|y\|^2+2\sqrt{\|x\|^2\|y\|^2-(x|y)^2} \tag{3.5.3}$$

for all $x,y\in H$.

Proof Let x,y be in H. If x,y are linearly dependent, then the equality (3.5.3) becomes clear. Assume that x,y are linearly independent. Then, keeping in mind the orthogonal projection theorem and Lemma 3.5.1, there is no loss of generality in assuming that $H=\text{lin}\{x,y\}$.

As a first step, we assume additionally that $\|x\|=\|y\|=1$ and $(x|y)=0$. Then by considering the product on H determined by $x^2=x$, $xy=yx=y$, and $y^2=-x$, H becomes an isometric copy of $\mathbb{C}$, regarded as a real normed algebra. Since this algebra is a smooth-normed algebra, Lemma 2.6.7 applies, so that $\mathbb{C}\otimes_\pi H$ is a V-algebra whose natural involution $*$ is determined by $x^*=x$ and $y^*=-y$. Then, we straightforwardly realize that the mapping

$$\lambda x+\mu y \longrightarrow (\lambda+i\mu,\lambda-i\mu)\ \ (\lambda,\mu\in\mathbb{C}),$$

is an algebra $*$-isomorphism from $\mathbb{C}\otimes_\pi H$ onto the C^*-algebra $\mathbb{C}^2$. Since $(\mathbb{C}\otimes_\pi H,*)$ is also a C^*-algebra (by Theorem 2.3.32), it follows that the above isomorphism is an isometry, i.e. we have

$$\|\lambda x+\mu y\|_\pi = \max\{|\lambda+i\mu|,|\lambda-i\mu|\} \tag{3.5.4}$$

for all $\lambda,\mu\in\mathbb{C}$.

Now, remove the additional assumptions on x,y done in the above paragraph. Then, setting

$$x_0 := \frac{x}{\|x\|} \ \text{ and } \ y_0 := \frac{y-(x_0|y)x_0}{\|y-(x_0|y)x_0\|},$$

x_0,y_0 satisfy the conditions required for x,y in the above paragraph. Therefore, taking

$$\lambda_0 := \|x\|+i(x_0|y) \ \text{ and } \ \mu_0 := i\|y-(x_0|y)x_0\|,$$

and applying (3.5.4) with $(x_0,y_0,\lambda_0,\mu_0)$ instead of (x,y,λ,μ), we get (3.5.3). □

Lemma 3.5.3 *Let A be a unital algebra over a field extension $\mathbb{F}$ of $\mathbb{K}$, and assume that A (regarded as an algebra over $\mathbb{K}$) is a normed algebra. Then we have:*

(i) $\mathbb{F}=\mathbb{C}$ *if* $\mathbb{K}=\mathbb{C}$, *and* $\mathbb{F}=\mathbb{R}$ *or* $\mathbb{C}$ *if* $\mathbb{K}=\mathbb{R}$.
(ii) *Up to an equivalent renorming, A becomes a normed algebra over $\mathbb{F}$.*

Proof Assertion (i) is an immediate consequence of the Gelfand–Mazur theorem.

Assertion (ii) becomes obvious in the case that $\mathbb{F}=\mathbb{K}$. Assume that $\mathbb{F}\neq\mathbb{K}$. Then, by assertion (i), we have that $\mathbb{F}=\mathbb{C}$ and $\mathbb{K}=\mathbb{R}$. Note that, for $\theta\in\mathbb{R}$ and $a\in A$, we have

$$\|e^{i\theta}a\| = \|\cos(\theta)a+\sin(\theta)(i\mathbf{1})a\| \leqslant k\|a\|,$$

where $k := 1 + \|i\mathbf{1}\|$. Therefore, by defining $|||a||| := \sup\{\|e^{i\theta}a\| : \theta \in \mathbb{R}\}$ for every $a \in A$, we realize that $|||\cdot|||$ becomes a complex vector space norm on A satisfying $\|\cdot\| \leqslant |||\cdot||| \leqslant k\|\cdot\|$. Moreover, for $a, b \in A$ we have

$$|||ab||| = \sup\{\|e^{i\theta}ab\| : \theta \in \mathbb{R}\} \leqslant \sup\{\|e^{i\theta}a\| : \theta \in \mathbb{R}\}\|b\| \leqslant |||a|||\,|||b|||.$$

It follows that $|||\cdot|||$ is a complex algebra norm on A equivalent to the given norm $\|\cdot\|$. □

Let $A = \mathscr{A}(V, \times, (\cdot,\cdot))$ be the quadratic algebra over $\mathbb{K}$ associated to a vector space V over $\mathbb{K}$ endowed with an anticommutative product $\times$ and a bilinear form $(\cdot,\cdot)$ (see §2.5.14). Assume that V is endowed with a norm $\|\cdot\|$ making $(\cdot,\cdot)$ and $\times$ continuous. Then A becomes a normed algebra under a suitable norm whose restriction to V is equivalent to $\|\cdot\|$, and such that the direct sum $A = \mathbb{K}\mathbf{1} \oplus V$ becomes topological. Indeed, clearly the product of A becomes continuous for the norm $\alpha\mathbf{1} + x \to |\alpha| + \|x\|$, and §1.1.3 applies.

The following result provides us with a relevant converse to the above comment.

Proposition 3.5.4 *Let A be a quadratic algebra over a field extension $\mathbb{F}$ of $\mathbb{K}$, and assume that A (regarded as an algebra over $\mathbb{K}$) is a normed algebra. Then we have:*

(i) $\mathbb{F} = \mathbb{C}$ *if* $\mathbb{K} = \mathbb{C}$, *and* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *if* $\mathbb{K} = \mathbb{R}$. *(Hence,* according to Proposition 2.5.13, *we can write $A = \mathscr{A}(V, \times, (\cdot,\cdot))$, where V is a vector space over $\mathbb{F}$, $(\cdot,\cdot) : V \times V \to \mathbb{F}$ is an $\mathbb{F}$-bilinear form, and $\times$ is an $\mathbb{F}$-bilinear anticommutative product on V.)*

(ii) *Up to an equivalent renorming, A becomes a normed algebra over $\mathbb{F}$.*

(iii) *The direct sum $A = \mathbb{F}\mathbf{1} \oplus V$ is topological.*

(iv) *The bilinear form $(\cdot,\cdot)$ and the anticommutative product $\times$ are continuous on $V \times V$.*

Proof Assertions (i) and (ii) follow from Lemma 3.5.3, and allow us to assume that $\mathbb{F} = \mathbb{K}$. Then, to prove (iii) it is enough to show that V is closed in A. To this end, we note that, for $x, y \in V$, we have

$$xy = -(x,y)\mathbf{1} + x \times y, \tag{3.5.5}$$

and that we may assume that $V \neq 0$. Let x_n be a sequence in V converging to $a = \alpha\mathbf{1} + x$ with $\alpha \in \mathbb{K}$ and $x \in V$. Then, taking a nonzero element $y \in V$, and invoking (3.5.5), we have

$$-\frac{(x_n,y)+(y,x_n)}{2}\mathbf{1} = x_n \bullet y \to (\alpha\mathbf{1}+x)\bullet y = -\frac{(x,y)+(y,x)}{2}\mathbf{1} + \alpha y.$$

It follows that $\alpha y = 0$, so $\alpha = 0$, and so a lies in V. Now that assertion (iii) has been proved, it is enough to invoke (3.5.5) again to conclude that $(\cdot,\cdot)$ and $\times$ are continuous on $V \times V$, i.e. assertion (iv) holds. □

Now we are going to determine those non-commutative JB^*-algebras which are quadratic. Indeed, they can be constructed from H-algebras (introduced immediately before Corollary 2.6.13) as follows.

Theorem 3.5.5 *Let E be an H-algebra, and let $\mathscr{A}(E)$ stand for the flexible quadratic algebra of E* (see Definition 2.6.4). *Then the complexification of $\mathscr{A}(E)$, with involution $*$ defined by*

$$[(\alpha,x)+i(\beta,y)]^* := (\alpha,-x)-i(\beta,-y) \tag{3.5.6}$$

for all $(\alpha,x),(\beta,y)\in\mathscr{A}(E)=\mathbb{R}\oplus E$, and norm given by

$$\|b+ic\|^2 := \|b\|^2+\|c\|^2+2\sqrt{\|b\|^2\|c\|^2-(b|c)^2} \tag{3.5.7}$$

for all $b,c\in\mathscr{A}(E)$, becomes a quadratic non-commutative JB^*-algebra.

Conversely, every quadratic non-commutative JB^*-algebra comes from a suitable H-algebra E in the way described in the above paragraph.

Proof Let E be an H-algebra. Since $\mathscr{A}(E)$ is quadratic over $\mathbb{R}$, the complexification of $\mathscr{A}(E)$ is quadratic over $\mathbb{C}$ (an easy consequence of Proposition 2.5.13). On the other hand, by Proposition 2.6.5, $\mathscr{A}(E)$ is a complete smooth-normed algebra. Therefore, by Lemma 2.6.7, its projective normed complexification, $\mathbb{C}\otimes_\pi\mathscr{A}(E)$, is a complete V-algebra whose natural involution $*$ is given by (3.5.6). Since $\mathscr{A}(E)$ is a Hilbert space, it is enough to invoke Proposition 3.5.2 to derive that the norm of $\mathbb{C}\otimes_\pi\mathscr{A}(E)$ must be given by (3.5.7). Moreover, since $\mathbb{C}\otimes_\pi\mathscr{A}(E)$ is a complete V-algebra, it follows from Theorem 3.3.11 that $(\mathbb{C}\otimes_\pi\mathscr{A}(E),*)$ is a non-commutative JB^*-algebra. This concludes the proof of the first paragraph of the theorem.

Now, let A be any quadratic non-commutative JB^*-algebra. According to Proposition 2.5.13 and §2.5.14, write $A=\mathscr{A}(V,\times,(\cdot,\cdot))$, and note that

(i) $A=\mathbb{C}\mathbf{1}\oplus V$;
(ii) for $x,y\in V$, we have $xy=-(x,y)\mathbf{1}+x\times y$;
(iii) the form $(\cdot,\cdot)$ is symmetric (since A is flexible, and Proposition 2.5.18(ii) applies);
(iv) V is $*$-invariant (a consequence of the equality (2.5.10) in Proposition 2.5.13).

By Proposition 3.5.4, V becomes a closed subspace of A. Set $M:=V\cap iH(A,*)$ and $B:=\mathbb{R}\mathbf{1}+M$. Then B is a closed real subspace of A. Moreover, for $x,y\in M$ we have that $[x,y]\in M$ and $x\bullet y\in(\mathbb{C}\mathbf{1})\cup H(A,*)$, which implies that $x\bullet y\in\mathbb{R}\mathbf{1}$ and $xy\in B$. It follows that B is a real subalgebra of A. More precisely, B is a norm-unital complete normed real algebra. Moreover, keeping in mind the straightforward equality $A=B\oplus iB$, we realize that, algebraically regarded, A is a copy of the complexification of B. Now observe that every state of B (relative to $\mathbf{1}$) is the restriction to B of the real part of a state of A. Since states of A are real-valued on $H(A,*)$ (by Lemma 2.2.5), it follows that states of B vanish on M, so $D(B,\mathbf{1})$ reduces to a singleton, and so B is a complete smooth-normed real algebra. By Theorem 2.6.9, there exists an H-algebra E such that B equals the flexible quadratic algebra, $\mathscr{A}(E)$, of E. Let C stand for the non-commutative JB^*-algebra built from E according to the first paragraph in the theorem. Then it is routine to verify that the natural identification of $A=\mathscr{A}(E)\oplus i\mathscr{A}(E)$ with C becomes an algebra $*$-isomorphism. By Proposition 3.4.4, this identification is an isometry. This concludes the proof of the second paragraph of the theorem. □

We recall that the algebra of complex octonions $C(\mathbb{C})$ was introduced immediately before Proposition 2.6.8, where it was proved that $C(\mathbb{C})$ becomes an alternative C^*-algebra for suitable norm and involution. We also recall that, by Corollary 3.4.76, the C^*-algebra structure of an alternative C^*-algebra is essentially unique.

Corollary 3.5.6 *The quadratic alternative C^*-algebras are $\mathbb{C}, \mathbb{C}^2, M_2(\mathbb{C})$ and $C(\mathbb{C})$.*

Proof Let A be a quadratic alternative C^*-algebra. By Theorem 3.5.5, there exists an H-algebra E such that A equals the complexification of the flexible quadratic algebra $\mathscr{A}(E)$, with involution and norm given by (3.5.6) and (3.5.7), respectively. Moreover, since A is alternative, so is $\mathscr{A}(E)$, and hence, by Corollary 2.6.13 and Theorem 2.6.21, we get that $\mathscr{A}(E) = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. Then, easily, we have $A = \mathbb{C}, \mathbb{C}^2, M_2(\mathbb{C})$, or $C(\mathbb{C})$. □

In the case of (commutative) JB^*-algebras, Theorem 3.5.5 attains a simpler form. Indeed, in this case the parameter H-algebra E in the theorem must have zero product. With this idea in mind, we derive the next corollary, and leave the details of proof to the reader. We recall that a *conjugation* on a complex Hilbert space H is an isometric conjugate-linear involutive operator on H.

Corollary 3.5.7 *Let H be a complex Hilbert space, let σ be a conjugation on H, let $(\cdot|\cdot)$ stand for the inner product on $\mathbb{C}\mathbf{1} \oplus H$ regarded as the ℓ_2-sum of $\mathbb{C}$ and H, and, for $a = \lambda\mathbf{1} + x \in \mathbb{C}\mathbf{1} \oplus H$, set $a^\sharp := \bar{\lambda}\mathbf{1} - \sigma(x)$. Then the vector space $\mathbb{C}\mathbf{1} \oplus H$, with product, involution $*$, and norm $\|\cdot\|$ defined by*

$$(\lambda\mathbf{1} + x)(\mu\mathbf{1} + y) := [\lambda\mu + (x|\sigma(y))]\mathbf{1} + \lambda y + \mu x,$$

$$(\lambda\mathbf{1} + x)^* := \bar{\lambda}\mathbf{1} + \sigma(x),$$

and

$$\|a\|^2 := (a|a) + \sqrt{(a|a)^2 - |(a|a^\sharp)|^2},$$

respectively, becomes a quadratic JB^*-algebra.

Conversely, every quadratic JB^*-algebra comes from a suitable complex Hilbert space H, with conjugation σ, in the way described in the above paragraph.

3.5.2 The axiom $\|a^*a\| = \|a^*\|\|a\|$ on unital algebras

We begin this subsection by realizing that there exist unusual involutions $\sharp$ on the C^*-algebra $\mathbb{C}^2$ satisfying

$$\mathbf{1}^\sharp = \mathbf{1} \text{ and } \|a^\sharp a\| = \|a^\sharp\|\|a\| \text{ for every } a \text{ in } A. \tag{3.5.8}$$

Example 3.5.8 Let A denote the associative and commutative algebra $\mathbb{C}^2$ with its unique structure of C^*-algebra given by

$$\|(\lambda, \mu)\| := \max\{|\lambda|, |\mu|\} \text{ and } (\lambda, \mu)^* := (\bar{\lambda}, \bar{\mu}).$$

Let f be any linear form on A such that $f(\mathbf{1}) = 0$ and $f(a^*) = -\overline{f(a)}$ for every $a \in A$, and define a mapping $\sharp$ from A to A by

$$a^\sharp := a^* + \overline{f(a)}\mathbf{1}.$$

Then $\sharp$ is a conjugate-linear vector space involution on A satisfying (3.5.8). Indeed, all the assertions are clear except possibly the last one in (3.5.8). We can easily see that there is α in $\mathbb{R}$ such that for every $a=(\lambda,\mu)$ in A we have $f(\lambda,\mu)=i\alpha(\lambda-\mu)$. Thus, the involution $\sharp$ implemented by f is determined by

$$(\lambda,\mu)^\sharp=(\overline{\lambda+i\alpha(\lambda-\mu)},\overline{\mu+i\alpha(\lambda-\mu)}),$$

and therefore, we can write the desired equality $\|a^\sharp a\|=\|a^\sharp\|\|a\|$ in the form

$$\begin{aligned}&\max\{|\lambda+i\alpha(\lambda-\mu)|\,|\lambda|,|\mu+i\alpha(\lambda-\mu)|\,|\mu|\}\\&\quad=\max\{|\lambda+i\alpha(\lambda-\mu)|,|\mu+i\alpha(\lambda-\mu)|\}\max\{|\lambda|,|\mu|\}.\end{aligned}$$

To verify this equality, note that for all λ and μ in $\mathbb{C}$ we have

$$|\lambda+i\alpha(\lambda-\mu)|^2-|\lambda|^2=|\mu+i\alpha(\lambda-\mu)|^2-|\mu|^2,$$

and therefore $|\lambda|\leqslant|\mu|$ if and only if $|\lambda+i\alpha(\lambda-\mu)|\leqslant|\mu+i\alpha(\lambda-\mu)|$.

It will be crucial for our development that things behave in a totally different way for the C^*-algebra $\mathbb{C}^3$. This is shown in the following lemma.

Lemma 3.5.9 *Let f be a linear form on the C^*-algebra $A:=\mathbb{C}^3$ such that $f(\mathbf{1})=0$ and $f(a^*)=-\overline{f(a)}$ for every a in A, define a mapping $\sharp$ from A to A by $a^\sharp:=a^*+\overline{f(a)}\mathbf{1}$, and assume the equality $\|a^\sharp a\|=\|a^\sharp\|\|a\|$ holds for every a in A. Then $f=0$.*

Proof There exist α and β in $\mathbb{R}$ such that for every $a=(\lambda,\mu,\nu)$ in A we have

$$f(\lambda,\mu,\nu)=i\alpha(\lambda-\mu)+i\beta(\lambda-\nu).$$

Taking in particular $(\lambda,\mu,\nu)=(i\sigma\rho,-\rho,\rho+1)$, where ρ is an arbitrary positive number and σ denotes the sign of α, we have

$$f(\lambda,\mu,\nu)=-\sigma(\alpha+\beta)\rho+i[\rho(\alpha-\beta)-\beta],$$

so

$$|\lambda+f(\lambda,\mu,\nu)|^2-|\nu+f(\lambda,\mu,\nu)|^2=4|\alpha|\rho^2+2(|\alpha|-1)\rho-1.$$

Therefore, if α were not zero, then for ρ large enough we would have

$$|\lambda+f(\lambda,\mu,\nu)|>|\nu+f(\lambda,\mu,\nu)|. \tag{3.5.9}$$

Noticing that $|\lambda|=|\mu|<|\nu|$, and that the condition $\|a^\sharp a\|=\|a^\sharp\|\|a\|$ now reads as

$$\begin{aligned}&\max\{|\lambda+f(\lambda,\mu,\nu)|\,|\lambda|,|\mu+f(\lambda,\mu,\nu)|\,|\mu|,|\nu+f(\lambda,\mu,\nu)|\,|\nu|\}\\&\quad=\max\{|\lambda+f(\lambda,\mu,\nu)|,|\mu+f(\lambda,\mu,\nu)|,|\nu+f(\lambda,\mu,\nu)|\}\max\{|\lambda|,|\mu|,|\nu|\},\end{aligned}$$

the inequality (3.5.9) leads to a contradiction. Hence $\alpha=0$. Since a change of the variables μ and ν in $f(\lambda,\mu,\nu)$ corresponds to an interchange of the roles of α and β, then $\beta=0$ also. □

Lemma 3.5.10 *Let A be an associative and commutative unital C^*-algebra having an idempotent different from 0 and $\mathbf{1}$, let f be a linear form on A, define $\sharp$ on A by $a^\sharp:=a^*+\overline{f(a)}\mathbf{1}$, and assume that the equality $\|a^\sharp a\|=\|a^\sharp\|\|a\|$ holds for every $a\in A$. Then f is continuous.*

Proof We may assume $A = B \times C$, where B and C are associative and commutative unital C^*-algebras. Then there are linear forms g and h on B and C, respectively, such that for every (b,c) in A we have $f((b,c)) = g(b) + h(c)$, and the assumption $\|a^\sharp a\| = \|a^\sharp\|\|a\|$ for every $a \in A$ reads as

$$\begin{aligned}&\max\{\|b^*[b+(g(b)+h(c))\mathbf{1}]\|, \|c^*[c+(g(b)+h(c))\mathbf{1}]\|\}\\ &\quad = \max\{\|b+(g(b)+h(c))\mathbf{1}\|, \|c+(g(b)+h(c))\mathbf{1}\|\}\max\{\|b\|,\|c\|\},\end{aligned} \tag{3.5.10}$$

for all $b \in B$ and $c \in C$, where we have denoted by $\mathbf{1}$ the unit elements of B and C. Assume that g is not continuous. Then, for fixed c in C and z in $\mathbb{C}$, there is a null sequence $\{b_n\}$ in B such that $g(b_n) = z - h(c)$ for every n in $\mathbb{N}$. Setting $b := b_n$ and taking limits in the equality (3.5.10), we obtain

$$\|c^*(c+z\mathbf{1})\| = \max\{|z|, \|c+z\mathbf{1}\|\}\|c\|.$$

Taking $z = -1$ and $c = \mathbf{1}$, we have $0 = 1$, a contradiction. Therefore g is continuous. Analogously, h is also continuous, hence so is f. □

Let E be a topological space. We recall that E is said to be *totally disconnected* if the unique connected subsets of E are the singletons. It is easily realized that, if E is totally disconnected, compact, and Hausdorff, and if x, y are distinct points of E, then there exists a clopen subset F of E such that $x \in F$ and $y \in E \setminus F$. Therefore, the characteristic functions of clopen subsets of a totally disconnected compact Hausdorff topological space separate the points.

Proposition 3.5.11 *Let E be a compact Hausdorff topological space. Then $C^{\mathbb{C}}(E)$ is the closed linear hull of its idempotents if and only if E is totally disconnected.*

Proof It is clear that the idempotents of $C^{\mathbb{C}}(E)$ are the characteristic functions of the clopen subsets of E. We denote by B the linear hull of the idempotents in $C^{\mathbb{C}}(E)$.

Assume that B is dense in $C^{\mathbb{C}}(E)$. Since $C^{\mathbb{C}}(E)$ separates the points of E, so does B. It follows that if x and y are distinct points of E, then there exists an idempotent χ in $C^{\mathbb{C}}(E)$ such that $\chi(x) \neq \chi(y)$. In others words, there exists a clopen subset F of E such that $x \in F$ and $y \in E \setminus F$. Therefore, any subset of E of cardinality $\geqslant 2$ is disconnected. Thus E is totally disconnected.

In order to prove the converse, assume that E is totally disconnected. Since B is a $*$-subalgebra of $C^{\mathbb{C}}(E)$, with $\mathbf{1} \in B$, and B separates the points of E, it is enough to invoke the Stone–Weierstrass theorem (Theorem 1.2.10) to conclude that B is dense in $C^{\mathbb{C}}(E)$. □

Lemma 3.5.12 *Let K be a compact subset of $\mathbb{R}$ with at least three points, and let f be a linear form on the C^*-algebra $C^{\mathbb{C}}(K)$ such that $f(\mathbf{1}) = 0$ and $f(a^*) = -\overline{f(a)}$ for every a in $C^{\mathbb{C}}(K)$. Define a mapping $\sharp$ from $C^{\mathbb{C}}(K)$ to $C^{\mathbb{C}}(K)$ by $a^\sharp := a^* + \overline{f(a)}\mathbf{1}$, and assume that the equality $\|a^\sharp a\| = \|a^\sharp\|\|a\|$ holds for every a in $C^{\mathbb{C}}(K)$. Then $f(x) = 0$, where x denotes the inclusion of K in $\mathbb{C}$.*

Proof Suppose first that K is not totally disconnected, so that K contains a non-trivial closed interval $[\alpha, \beta]$. We may also assume that $K \subseteq [-1,1]$, and then we can consider the element y of $C^{\mathbb{C}}(K)$ given by $y := x + i(1-x^2)^{\frac{1}{2}}$. Writing $\Delta := y(K)$, for w in $\mathbb{C}$ we have

$$\begin{aligned}\max\{|z+w+f(y)|\,|z+w| : z \in \Delta\}\\ &= \|(y+w\mathbf{1})^\sharp(y+w\mathbf{1})\| = \|(y+w\mathbf{1})^\sharp\|\,\|y+w\mathbf{1}\|\\ &= \max\{|z+w+f(y)| : z\in\Delta\}\max\{|z+w| : z\in\Delta\}.\end{aligned}$$

This equality implies that the mappings $z \to |z+w+f(y)|$ and $z \to |z+w|$ from Δ into $\mathbb{R}$ attain their respective maxima at a common point of Δ. Now consider the arc $\Gamma := \{y(t) : \alpha < t < \beta\}$, and the open angular region

$$\Omega := \left\{\lambda \in \mathbb{C}\backslash\{0\} : \frac{\lambda}{|\lambda|} \in \Gamma\right\},$$

and note that, for λ in Ω, the mapping $z \to |z+\lambda|$ from Δ into $\mathbb{R}$ only attains its maximum at the point $z := \frac{\lambda}{|\lambda|}$. It follows that $\frac{w+f(y)}{|w+f(y)|} = \frac{w}{|w|}$ for every w in the non-empty open set $\Omega \cap (\Omega - f(y))$, which implies that $f(y) = 0$, since otherwise $\Omega \cap (\Omega - f(y))$ would be contained in the line $\mathbb{R}f(y)$. Also $f(y^*) = -\overline{f(y)} = 0$, so $f(x) = 0$ because $2x = y + y^*$.

Now suppose that K is totally disconnected, so that, by Lemma 3.5.10, f is continuous, and therefore, since $f(\mathbf{1}) = 0$ and $C^{\mathbb{C}}(K)$ is the closed linear hull of its idempotents (by Proposition 3.5.11), it is enough to prove $f(p) = 0$ for every non-trivial idempotent p in $C^{\mathbb{C}}(K)$. Changing p by $\mathbf{1} - p$, if necessary, we may assume that the clopen subset $\{t \in K : p(t) = 0\}$ is not connected. Then there are nontrivial idempotents q and r in $C^{\mathbb{C}}(K)$ satisfying $pq = pr = qr = 0$ and $p+q+r = \mathbf{1}$. In this way the linear hull of $\{p,q,r\}$ is a $\sharp$-invariant C^*-subalgebra of $C^{\mathbb{C}}(K)$ isomorphic to $\mathbb{C}^3$. From Lemma 3.5.9 we obtain $f(p) = 0$, as desired. □

In the context of the above lemma we actually have $f = 0$. This is a direct consequence of the following result.

Lemma 3.5.13 *Let A be a unital non-commutative JB*-algebra such that there exists a nonzero linear form f on A satisfying*

$$f(\mathbf{1}) = 0,\ f(a^*) = -\overline{f(a)},\ \text{and } \|a^\sharp a\| = \|a^\sharp\|\,\|a\| \text{ for every } a \text{ in } A,$$

where $a^\sharp := a^* + \overline{f(a)}\mathbf{1}$. Then A is (isometrically) isomorphic to the C^*-algebra $\mathbb{C}^2$.

Proof Consider the subspace T of $H(A,*)$ given by

$$T := \{h \in H(A,*) : f(h) = 0\}.$$

By Proposition 3.4.1(ii), the closed subalgebra $A(h)$ of A generated by $\mathbf{1}$ and any element h of $H(A,*)$ is a unital commutative C^*-algebra, hence there is a compact subset K of $\mathbb{R}$ such that $A(h)$ can be seen as $C^{\mathbb{C}}(K)$ and h can be seen as the inclusion of K in $\mathbb{C}$. Moreover $A(h)$ is $\sharp$-invariant. It follows from Lemma 3.5.12 that, if h is actually in $H(A,*)\backslash T$, then K has at most two points, so there are real numbers α_i $(i = 1,2,3)$ satisfying $\sum|\alpha_i| = 1$ and $\alpha_1 h^2 + \alpha_2 h + \alpha_3\mathbf{1} = 0$. Since $H(A,*)\backslash T$ is dense in $H(A,*)$ (T is a proper subspace of $H(A,*)$), and the set

$$\{h \in H(A,*) : \exists\, \alpha_1,\alpha_2,\alpha_3 \in \mathbb{R} \ \text{ with } \ \textstyle\sum|\alpha_i| = 1 \ \text{ and } \ \alpha_1 h^2 + \alpha_2 h + \alpha_3\mathbf{1} = 0\}$$

is closed in $H(A,*)$, we have that $H(A,*)$ with product $h \bullet k := \frac{1}{2}(hk+kh)$ is a quadratic real algebra, and hence A is a (complex) quadratic algebra.

If f was discontinuous, then we would have $\|a^*a\| = \|a^*\|\,\|a\|$ for every a in the dense subset $\ker(f)$, hence, since $*$ is isometric on A (by Proposition 3.3.13), we would actually have $\|a^*a\| = \|a\|^2$ for every a in A. By Theorem 3.2.5, A would be an alternative algebra. Since, by Corollary 3.5.6, A would be finite-dimensional, f would be a discontinuous linear form on a finite-dimensional normed space. Therefore f is continuous.

Since a quadratic non-commutative JB^*-algebra of dimension 2 is (isometrically) isomorphic to the C^*-algebra $\mathbb{C}^2$, to conclude the proof it is enough to show that the assumption $\dim(A) \geqslant 3$ leads to a contradiction.

By Theorem 3.5.5, there exists an H-algebra E such that A equals the complexification of the flexible quadratic algebra $\mathscr{A}(E)$ with involution $*$ given by (3.5.6) and norm given by (3.5.7). Since clearly $0 \neq f(E) \subseteq \mathbb{R}$, by the continuity of f and the Riesz–Fréchet representation theorem, there exists a nonzero element u in E such that $f(x) = (x|u)$ for every x in E. Since we assume $\dim(A) \geqslant 3$, we can take a norm-one element x in E with $(x|u) = 0$. Now, writing $a := u + i\rho x$ (where ρ is an arbitrary positive number) and $\varepsilon := \|u\|$, we have:

$$\begin{aligned}
a^\sharp &= \varepsilon^2 \mathbf{1} - u + i\rho x,\\
a^\sharp a &= \left(\varepsilon^2+\rho^2\right)\mathbf{1} + \varepsilon^2 u + i\rho\left(\varepsilon^2 x + 2x\wedge u\right),\\
\|a\| &= \varepsilon + \rho,\\
\|a^\sharp\| &= \varepsilon\left(\varepsilon^2+1\right)^{\frac{1}{2}} + \rho,\\
\|a^\sharp a\| &= \left[\left(\varepsilon^2+\rho^2\right)^2 + \varepsilon^6\right]^{\frac{1}{2}} + \rho\left(\varepsilon^4 + 4\|x\wedge u\|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\|a^\sharp a\| = \|a^\sharp\|\|a\|$ and $\|x\wedge u\| \leqslant \|x\|\,\|u\| = \varepsilon$, we obtain

$$(\varepsilon+\rho)\left[\varepsilon\left(\varepsilon^2+1\right)^{\frac{1}{2}} + \rho\right] \leqslant \left[\left(\varepsilon^2+\rho^2\right)^2 + \varepsilon^6\right]^{\frac{1}{2}} + \varepsilon\rho(\varepsilon^2+4)^{\frac{1}{2}},$$

or equivalently

$$(\varepsilon+\rho)\left[\varepsilon\left(\varepsilon^2+1\right)^{\frac{1}{2}} + \rho\right] - \varepsilon\rho\left(\varepsilon^2+4\right)^{\frac{1}{2}} \leqslant \left[\left(\varepsilon^2+\rho^2\right)^2 + \varepsilon^6\right]^{\frac{1}{2}}.$$

Taking ρ large enough to make positive the left-hand side of the above inequality, we may rationalize the variable ρ to obtain

$$2\varepsilon\left[1 + \left(\varepsilon^2+1\right)^{\frac{1}{2}} - \left(\varepsilon^2+4\right)^{\frac{1}{2}}\right]\rho^3 + M\rho^2 + N\rho \leqslant 0,$$

where M and N are suitable functions of ε. Dividing by ρ^3 and letting $\rho \to +\infty$, it follows that

$$1 + \left(\varepsilon^2+1\right)^{\frac{1}{2}} - \left(\varepsilon^2+4\right)^{\frac{1}{2}} \leqslant 0.$$

Finally, from this last inequality we derive $\varepsilon^2 \leqslant 0$, a contradiction. □

Lemma 3.5.14 *Let A be a complete normed unital complex algebra endowed with a conjugate-linear vector space involution $\sharp$ satisfying $\mathbf{1}^\sharp = \mathbf{1}$ and $\|a^\sharp a\| = \|a^\sharp\|\|a\|$ for every a in A. Then A is norm-unital, and the inclusion $H(A,\sharp) \subseteq H(A,\mathbf{1}) + i\mathbb{R}\mathbf{1}$ holds.*

Proof That A is norm-unital is straightforward. Let h be in $H(A,\sharp)$, and let r be a positive real number. Then, since $(\mathbf{1}+irh)^\sharp = \mathbf{1}-irh$, we have

$$\|\mathbf{1}-irh\|\|\mathbf{1}+irh\| = \left\|\mathbf{1}+r^2h^2\right\| \leqslant 1+r^2\left\|h^2\right\|,$$

and hence

$$\frac{\|\mathbf{1}-irh\|\|\mathbf{1}+irh\|-1}{r} \leqslant r\left\|h^2\right\|.$$

Writing

$$\frac{\|\mathbf{1}-irh\|\|\mathbf{1}+irh\|-1}{r} = \|\mathbf{1}-irh\|\frac{\|\mathbf{1}+irh\|-1}{r} + \frac{\|\mathbf{1}-irh\|-1}{r},$$

letting $r \to 0^+$, and applying Proposition 2.1.5, we deduce that

$$\max \Re(V(A,\mathbf{1},ih)) - \min \Re(V(A,\mathbf{1},ih)) \leqslant 0.$$

Therefore there exists a real number λ_h such that $\Re(f(ih)) = \lambda_h$ for every $f \in D(A,\mathbf{1})$, and hence $h+i\lambda_h\mathbf{1} \in H(A,\mathbf{1})$. Now, since h is an arbitrary $\sharp$-invariant element of A, we derive that $H(A,\sharp) \subseteq H(A,\mathbf{1}) + i\mathbb{R}\mathbf{1}$. □

Theorem 3.5.15 *Let A be a complete normed unital complex algebra endowed with a conjugate-linear vector space involution $\sharp$ satisfying $\mathbf{1}^\sharp = \mathbf{1}$ and*

$$\|a^\sharp a\| = \|a^\sharp\|\|a\|$$

for every a in A. Then A is an alternative C^*-algebra for its own norm and a suitable involution $*$. Moreover, except possibly in the case that $(A,*)$ is the C^*-algebra $\mathbb{C}^2$, we have $\sharp = *$. In the exceptional case that $(A,*) = \mathbb{C}^2$, the involutions $\sharp$ satisfying the above requirements are exactly the mappings of the form $a \to a^* + \overline{f(a)}\mathbf{1}$, where f is any linear form on $\mathbb{C}^2$ such that $f(\mathbf{1}) = 0$ and $f(a^*) = -\overline{f(a)}$ for every a in $\mathbb{C}^2$.

Proof By Lemma 3.5.14, A is norm-unital and, for a in A with $a^\sharp = a$, we have $a = h + ir\mathbf{1}$ for suitable h in $H(A,\mathbf{1})$ and r in $\mathbb{R}$. It follows that A is a V-algebra and that, for each element $a \in A$, there exists a unique complex number $f(a)$ such that

$$a^\sharp = a^* + \overline{f(a)}\mathbf{1}, \tag{3.5.11}$$

where $*$ stands for the natural involution of the V-algebra A. From the fact that both $\sharp$ and $*$ are conjugate-linear vector space involutions on A fixing the unit element $\mathbf{1}$, it follows that the mapping $f : a \to f(a)$ is a linear form on A satisfying $f(\mathbf{1}) = 0$ and $f(a^*) = -\overline{f(a)}$ for every a in A. Moreover, by Theorem 3.3.11, $(A, \|\cdot\|, *)$ is a non-commutative JB^*-algebra. Now the result follows from (3.5.11), Lemma 3.5.13, Example 3.5.8, and Theorem 3.2.5 (recalling that, by Proposition 3.3.13, $*$ is isometric). □

Corollary 3.5.16 *Let A be a complete normed unital complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying*

$$(a^2)^* = (a^*)^2 \ \textit{ and } \ \|a^*a\| = \|a^*\|\|a\| \ \textit{ for every } \ a \in A.$$

Then A is an alternative C^*-algebra.

Although Corollary 3.5.16 is a straightforward consequence of Theorem 3.5.15, we feel it appropriate to add the following.

§3.5.17 *Autonomous proof of Corollary 3.5.16* With the notation in §3.4.71, the assumption $(a^*)^2 = (a^2)^*$ for every $a \in A$ (equivalent to the fact that $*$ is an algebra involution on A^{sym}) is also equivalent to $(T_a)^* = T_{a^*}$ for every $a \in A$, where T_a is defined by $T_a(b) := a \bullet b$. It follows from the equality (3.4.28) and Example 1.1.32(d) that

$$\mathrm{sp}(BL(A), T_{a^*}) = \overline{\mathrm{sp}(BL(A), T_a)}.$$

As a consequence, if $h \in H(A, \mathbf{1})$, then, by Lemmas 2.1.10 and 2.3.21, we have

$$\mathrm{sp}(T_{h^*}) = \mathrm{sp}(T_h) \subseteq \mathbb{R}.$$

Let a be in $H(A, *)$. By Lemma 3.5.14, $a = h + ir\mathbf{1}$ with $h \in H(A, \mathbf{1})$ and $r \in \mathbb{R}$. Then $T_a = T_h + irI_A$ and $T_a = T_{a^*} = T_{h^*} - irI_A$. Hence

$$\mathrm{sp}(T_a) \subseteq (\mathbb{R} + ir) \cap (\mathbb{R} - ir),$$

which implies that $r = 0$. In this way, we have shown that $H(A, *) \subseteq H(A, \mathbf{1})$. Therefore, A is a V-algebra whose natural involution is the given one $*$. Then, by Corollary 3.3.16, $*$ is isometric, so the equality $\|a^*a\| = \|a\|^2$ holds for every $a \in A$, and so, by Theorem 3.2.5, A is an alternative C^*-algebra, as required. □

Corollary 3.5.18 *Let A be a complete normed unital associative complex $*$-algebra satisfying $\|a^*a\| = \|a^*\| \|a\|$ for every a in A. Then A is a C^*-algebra.*

Another straightforward consequence of Theorem 3.5.15 is the following.

Corollary 3.5.19 *Let A be a complete normed associative unital complex algebra with a conjugate-linear vector space involution $*$ satisfying*

$$\mathbf{1}^* = \mathbf{1} \textit{ and } \|a^*a\| = \|a^*\| \|a\| \textit{ for every } a \textit{ in } A.$$

If $\dim(A) \geqslant 3$, *then A is a C^*-algebra.*

Remark 3.5.20 If one is only interested in the associative specialization of Theorem 3.5.15, given by Corollary 3.5.19, then all non-associative references used in the proof can be avoided. For instance, Theorem 3.2.5 becomes unnecessary, whereas the non-associative Vidav–Palmer theorem (Theorem 3.3.11) can be replaced with its earlier associative version (Theorem 2.3.32). Moreover, since $\mathbb{C}, \mathbb{C}^2$, and $M_2(\mathbb{C})$ are the unique quadratic associative C^*-algebras, the determination of quadratic non-commutative JB^*-algebras in Theorem 3.5.5 can be replaced by the easy computation of the norm on the C^*-algebra $M_2(\mathbb{C})$ given by Exercise 1.2.15.

3.5.3 An interlude: the bidual and the spacial numerical index of a non-commutative JB^*-algebra

By an *approximate unit* in a normed algebra A we mean a net $\{a_\lambda\}_{\lambda\in\Lambda}$ in A such that

$$\lim_\lambda a_\lambda a = \lim_\lambda aa_\lambda = a \text{ for every } a\in A.$$

Lemma 3.5.21 *Let A be a normed flexible algebra over $\mathbb{K}$ such that A equals the linear hull of the set $\{b^2 : b\in A\}$. Then each approximate unit in A^{sym} remains an approximate unit in A.*

Proof Let a,b be in A. By Lemma 2.4.15 we have $[a,b^2]=2b\bullet[a,b]$, and then, by Lemma 2.4.14, the equality $[a,b^2]=2[a\bullet b,b]$ holds. Therefore, if $\{a_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit in A^{sym}, then we have $\lim_\lambda[a_\lambda,b^2]=0$ for every $b\in A$, and hence $\lim_\lambda[a_\lambda,a]=0$ for every $a\in A$ because A equals the linear hull of the set $\{b^2:b\in A\}$. Now we have

$$\lim_\lambda a_\lambda a = \lim_\lambda\left(a_\lambda\bullet a+\frac{1}{2}[a_\lambda,a]\right)=a$$

and

$$\lim_\lambda aa_\lambda = \lim_\lambda\left(a_\lambda\bullet a-\frac{1}{2}[a_\lambda,a]\right)=a. \qquad\square$$

Now we invoke Hanche-Olsen and Størmer [738] once more to prove the following.

Lemma 3.5.22 *Let A be a JB-algebra. Then A has an approximate unit a_λ bounded by 1 and such that a_λ^2 is also an approximate unit in A.*

Proof Let Λ stand for the set of all positive elements in the open unit ball of A. Then, according to [738, Proposition 3.5.4] and its proof, Λ is a directed set for the natural order of A (cf. §3.1.27), and the net $\{\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for A. Now, for $\lambda\in\Lambda$, set $a_\lambda:=\sqrt{\lambda}$ (cf. Lemma 3.1.29(ii)), and note that $\lambda\leqslant\sqrt{\lambda}\in\Lambda$. It follows that $\{a_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit for A. $\square$

We recall that, by Corollary 3.4.3, the self-adjoint part of a non-commutative JB^*-algebra becomes a JB-algebra in a natural way.

Proposition 3.5.23 *Let A be a non-commutative JB^*-algebra. Then each approximate unit in the JB-algebra $H(A,*)$ remains an approximate unit in A. Therefore A has an approximate unit a_λ bounded by 1, consisting of $*$-invariant elements and such that a_λ^2 is also an approximate unit in A.*

Proof The second conclusion follows from the former because of Lemma 3.5.22. Now note that, by Propositions 3.4.1(ii) and 1.2.48, A equals the linear hull of the set $\{b^2:b\in A\}$. Since the first conclusion in the present proposition is clear if A is commutative, it follows from Fact 3.3.4 and Lemma 3.5.21 that it remains true in the general case. $\square$

Recall that the bidual of any normed algebra is a normed algebra for the Arens product.

Lemma 3.5.24 *Let A be an Arens regular normed algebra over $\mathbb{K}$. We have:*

(i) *If A is commutative, then so is A''.*
(ii) *If A has a bounded approximate unit, then A'' has a unit.*

Proof Assume that A is commutative, so that with the notation in Definition 2.2.14, we have $m^r = m$, where m stands for the product of A. Therefore, involving also the notation in §2.2.11, the Arens regularity of A gives $m^{tr} = m^{rt} = m^t$, showing that A'' is commutative.

Now, assume that A has a bounded approximate unit a_λ. Let $\mathbf{1}$ be a cluster point to a_λ in A'' relative to the w^*-topology, and let a be in A. Since A is Arens regular, the product of A'' is separately w^*-continuous (by Lemma 2.3.51), and hence $\mathbf{1}a$ and $a\mathbf{1}$ are cluster points in A'' of $a_\lambda a$ and aa_λ, respectively, for the w^*-topology. Since both $a_\lambda a$ and aa_λ converge to a in the norm topology, we deduce that $\mathbf{1}a = a\mathbf{1} = a$. Applying the separate w^*-continuity of the product of A'' again, we realize that $\mathbf{1}$ is a unit for A''. □

For later reference, we state here the following straightforward consequence of Theorem 2.2.15 and Lemma 3.5.24(i).

Corollary 3.5.25 *Let A be a commutative C^*-algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital commutative C^*-algebra.*

The proof of the next proposition is quite similar to that of Theorem 2.2.15 given in §2.3.53.

Proposition 3.5.26 *Let A be a nonzero JB^*-algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital JB^*-algebra. As a consequence, A is Arens regular.*

Proof Assume at first that A is unital. Then, by Lemma 2.2.5, A is a V-algebra. Therefore, by Proposition 2.3.48, and Theorem 3.3.11, A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital non-commutative JB^*-algebra, and A is Arens regular. Moreover, by Lemma 3.5.24(i), A'' is commutative.

To conclude the proof, we must show that the same conclusions hold if A is not unital. In this case, we consider the JB^*-algebra unital extension $A_{\mathbb{1}}$ of A (see Corollary 3.4.12), so that, by the above paragraph, the desired conclusion holds with $A_{\mathbb{1}}$ instead of A. Therefore, since A is a $*$-subalgebra of $A_{\mathbb{1}}$, Lemmas 2.3.50 and 2.3.52 apply to get that the bipolar $A^{\circ\circ}$ of A in $(A_{\mathbb{1}})''$ is a $*$-subalgebra of the JB^*-algebra $(A_{\mathbb{1}})''$, that the natural Banach space identification $\Phi : A^{\circ\circ} \to A''$ becomes a bijective algebra $*$-homomorphism, and that A is Arens regular. Now, since Φ is an isometry becoming the identity on A, it turns out that A'' is clearly a JB^*-algebra in the natural way. Finally, since A has a bounded approximate unit (by Proposition 3.5.23), it follows from Lemma 3.5.24(ii) that A'' is unital. □

§3.5.27 Let X be a normed space over $\mathbb{K}$. We denote by $P(X)$ the normed space over $\mathbb{K}$ of all continuous bilinear mappings from $X \times X$ to X. Given $u \in \mathbb{S}_X$, $U(X,u)$

will stand for the set of those elements $f \in \mathbb{B}_{P(X)}$ such that $f(x,u) = f(u,x) = x$ for every $x \in X$.

Lemma 3.5.28 *Let (X,u) be a numerical-range space over $\mathbb{K}$. Then $U(X,u)$ is a (possibly empty) face of $\mathbb{B}_{P(X)}$.*

Proof Let $0 \leqslant \lambda \leqslant 1$, and let f, g be in $\mathbb{B}_{P(X)}$ such that

$$\lambda f + (1-\lambda)g \in U(X,u).$$

Consider the bounded linear operators S and T on X defined by $S(x) := f(x,u)$ and $T(x) := g(x,u)$ for every $x \in X$. Then S and T lie in the closed unit ball of $BL(X)$ and we have $\lambda S + (1-\lambda)T = I_X$. Since I_X is an extreme point of $\mathbb{B}_{BL(X)}$ (by Corollary 2.1.13 and Lemma 2.1.25), we deduce that $f(x,u) = g(x,u) = x$ for every $x \in X$. A similar argument shows that $f(u,x) = g(u,x) = x$ for every $x \in X$. □

Definition 3.5.29 By a *JB^*-admissible algebra* we mean a normed complex algebra A such that A^{sym} is a JB^*-algebra for the norm of A and some involution $*$. We note that we do not assume $*$ to be an algebra involution on A, and that, by Fact 3.3.4, non-commutative JB^*-algebras are JB^*-admissible algebras.

Lemma 3.5.30 *Let A be a nonzero JB^*-admissible algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital non-commutative JB^*-algebra. As a consequence, A is Arens regular.*

Proof Set $A = (X,m)$, where X and m stand for the Banach space of A and the product of A, respectively. Then we have:

$$A^{\mathrm{sym}} = (X, \tfrac{1}{2}(m+m^r)), \ \ A'' = (X'', m^t), \ \text{and} \ (A^{\mathrm{sym}})'' = (X'', \tfrac{1}{2}(m^t + m^{rt})).$$

Since A^{sym} is a JB^*-algebra, Proposition 3.5.26 applies, so that $(A^{\mathrm{sym}})''$, endowed with its Arens product $\frac{1}{2}(m^t + m^{rt})$ and the bitranspose of the involution of A, becomes a unital JB^*-algebra. Let $\mathbf{1}$ stand for the unit of $(A^{\mathrm{sym}})''$. We clearly have $\frac{1}{2}(m^t + m^{rt}) \in U(X'', \mathbf{1})$. Therefore, by Lemma 3.5.28, $m^t \in U(X'', \mathbf{1})$, i.e. A'' is a unital normed algebra with the same unit $\mathbf{1}$. Now, since $(A^{\mathrm{sym}})''$ is a V-algebra (by Lemma 2.2.5) whose natural involution is (the bitranspose of the given involution) $*$, and $\mathbf{1}$ is a common unit for both $(A^{\mathrm{sym}})''$ and A'', we see that A'' is a V-algebra. Therefore, by Theorem 3.3.11, A'' is a unital non-commutative JB^*-algebra for the involution $*$. Finally, the Arens regularity of A follows from Corollary 2.3.47. □

A first straightforward consequence of Lemma 3.5.30 is the following.

Proposition 3.5.31 *JB^*-admissible algebras are nothing other than non-commutative JB^*-algebras.*

§3.5.32 By an *identity* over $\mathbb{K}$ we mean a nonzero non-associative polynomial over $\mathbb{K}$ on a finite set of indeterminates (cf. §2.8.26). Given an algebra A over $\mathbb{K}$, we say that A satisfies the identity $\mathbf{p} = \mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ if the equality $\mathbf{p}(a_1, \ldots, a_n) = 0$ holds for all $a_1, \ldots, a_n \in A$. An identity is said to be *multilinear* if there exists $n \in \mathbb{N}$ such that it can be written as a linear combination of n-linear non-associative words on the indeterminates $\mathbf{x}_1, \ldots, \mathbf{x}_n$ (cf. §3.4.41).

Lemma 3.5.33 *Let A be an Arens regular normed algebra over $\mathbb{K}$. Then A'' satisfies all multilinear identities satisfied by A.*

Proof Note that, by Lemma 2.3.51, the product of A'' is separately w^*-continuous. Then we realize that, given a multilinear identity $\mathbf{p} = \mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ over $\mathbb{K}$, the mapping $(a_1, \ldots, a_n) \to \mathbf{p}(a_1, \ldots, a_n)$ from $A'' \times \cdots \times A''$ to A'' is separately w^*-continuous. Indeed, this can be verified by writing $\mathbf{p}$ as a linear combination of n-linear non-associative words, and then arguing by induction on the degree of such words. Therefore, if A satisfies the identity $\mathbf{p}$, it is enough to apply the w^*-density of A in A'' to conclude that A'' also satisfies $\mathbf{p}$. $\square$

The following theorem follows from Lemmas 3.5.30 and 3.5.33.

Theorem 3.5.34 *Let A be a nonzero non-commutative JB^*-algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital non-commutative JB^*-algebra satisfying all multilinear identities satisfied by A.*

Since the alternative identities are 3-linearizable, it is enough to invoke Fact 3.3.2 to get the following.

Corollary 3.5.35 *Let A be a nonzero alternative C^*-algebra. Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital alternative C^*-algebra.*

It follows from Theorem 3.5.34 (respectively, Corollary 3.5.35) that every non-commutative JB^*-algebra (respectively, every alternative C^*-algebra) can be enlarged to a unital non-commutative JB^*-algebra (respectively, to a unital alternative C^*-algebra). Replacing Corollary 3.4.10 with the fact just pointed out, and arguing as in the proof of Corollary 3.4.12, we get the following.

Corollary 3.5.36 *Let A be a nonzero non-commutative JB^*-algebra (respectively, an alternative C^*-algebra). Then there are a unique involution and a unique norm on the unital extension $A_{\mathbb{1}}$ of A extending the involution and the norm of A and converting $A_{\mathbb{1}}$ into a non-commutative JB^*-algebra (respectively, an alternative C^*-algebra).*

Let A be an algebra over $\mathbb{K}$. If A is unital, we set $A_1 := A$. Otherwise, A_1 will stand for the unital extension of A. For $z \in A_1$, we denote by T_z the linear operator on A defined by $T_z(x) := xz$ for every $x \in A$, and remark that T_z is continuous when A is normed. Now, in the case that A is a non-commutative JB^*-algebra, the JB^*-norm on A_1 given by Corollary 3.5.36 can be computed as follows.

Lemma 3.5.37 *Let A be a nonzero non-commutative JB^*-algebra. Then the unique JB^*-norm on A_1 which extends that of A is given by $\|z\| = \|T_z\|$ for every $z \in A_1$.*

Proof By Theorem 3.5.34, we can see A_1 as the norm-closed $*$-subalgebra of A'' consisting of those elements z in A'' which can be written in the form $\lambda\mathbf{1} + x$ for some $\lambda \in \mathbb{C}$ and $x \in A$, where $\mathbf{1}$ stands for the unit of A''. In this way A_1 is a non-commutative JB^*-algebra containing A isometrically so that, to conclude the proof, it is enough to show that the equality $\|z\| = \|T_z\|$ holds for every $z \in A_1$ (here $\|z\|$

means the norm of z as an element of A''). Let z be in A_1. Then the operator $(T_z)'' : A'' \to A''$ and the operator of right multiplication by z on A'', say $R_z^{A''}$, coincide on A and are weak*-continuous (the second one, by Lemma 2.2.12(i)). It follows from the weak*-density of A in A'' that $(T_z)'' = R_z^{A''}$, and hence $\|T_z\| = \|(T_z)''\| = \|R_z^{A''}\|$. Since $\|z\| = \|R_z^{A''}\|$, because A'' is a norm-unital normed algebra, we obtain $\|z\| = \|T_z\|$, as required. □

We recall that, given a non-empty set I and a normed space X over $\mathbb{K}$, $B(I,X)$ stands for the normed space over $\mathbb{K}$ of all bounded functions from I to X. On the other hand, we note that, given a normed algebra A over $\mathbb{K}$, the normed space $P(A)$ (cf. §3.5.27) has a natural distinguished element, namely the product of A, which will be denoted by p_A. In this case we always have $\|p_A\| \leqslant 1$, and $\|p_A\| = 1$ if A is actually a nonzero non-commutative JB^*-algebra.

Theorem 3.5.38 *Let A be a nonzero alternative C^*-algebra. Then $n(P(A),p_A)$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is commutative.*

Proof Recall that A'' is a unital alternative C^*-algebra in a natural way (cf. Corollary 3.5.35), and consider the chain of linear mappings

$$A_1 \xrightarrow{F_1} BL(A) \xrightarrow{F_2} P(A) \xrightarrow{F_3} P(A'') \xrightarrow{F_4} B(H(A'',*) \times H(A'',*), A''),$$

where $F_1(z) := T_z$ for every $z \in A_1$, $F_2(T)(x,y) := T(xy)$ for every $T \in BL(A)$ and all $x, y \in A$, $F_3(f) := f'''$ (the third Arens adjoint of f) for every $f \in P(A)$, and

$$F_4(g)(h,k) := \exp(-ih)[g(\exp(ih), \exp(ik)) \exp(-ik)]$$

for every $g \in P(A'')$ and all $h, k \in H(A'',*)$. Then F_1, F_2, and F_3 are isometries (by Lemma 3.5.37, Proposition 3.5.23, and §2.2.11, respectively). But F_4 is also an isometry. Indeed, keep in mind Corollary 3.4.7 together with the fact that left and right multiplications by unitary elements on A'' are isometries (by Lemma 3.4.30 and Proposition 3.4.31). On the other hand, the equalities $F_1(\mathbf{1}) = I_A$, $F_2(I_A) = p_A$, $F_3(p_A) = p_{A''}$ are clear, whereas the one $F_4(p_{A''}) = \hat{\mathbf{1}}$ (the mapping constantly equal to $\mathbf{1}$ on $H(A'',*) \times H(A'',*)$) follows from Artin's theorem (cf. Theorem 2.3.61). Now let δ denote either 1 or $\frac{1}{2}$ depending on whether or not A is commutative. Since A_1 and $B(H(A'',*) \times H(A'',*), A'')$ are unital alternative C^*-algebras (by Corollary 3.5.36), and they are commutative if and only if so is A (by Theorem 3.5.34), it follows from Corollary 3.4.60 that

$$\begin{aligned}\delta = n(A_1, \mathbf{1}) &\geqslant n(BL(A), I_A) \geqslant n(P(A), p_A) \geqslant n(P(A''), p_{A''}) \\ &\geqslant n[B(H(A'',*) \times H(A'',*), A''), \hat{\mathbf{1}}] = \delta.\end{aligned}$$ □

Looking at the above proof, and invoking Corollaries 2.1.2(ii) and 2.9.51, Theorem 2.1.17(i), and Fact 2.9.1, we get the following.

Corollary 3.5.39 *Let A be a nonzero alternative C^*-algebra. We have:*

(i) *$N(A)$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is commutative.*
(ii) *p_A is both a geometrically unitary element of $P(A)$ and a point of norm-norm upper semicontinuity of the duality mapping of $P(A)$.*

(iii) *Numerical ranges in* $(P(A), p_A)$ *can be computed in terms of numerical ranges in* $(A'', \mathbf{1})$. *Indeed, for every* $f \in P(A)$ *we have*

$$V(P(A), p_A, f) = \overline{\mathrm{co}}\left[\bigcup_{(h,k)\in H(A'',*)\times H(A'',*)} V(A'', \mathbf{1}, \Phi_f(h,k))\right],$$

where $\Phi_f(h,k) := \exp(-ih)[f'''(\exp(ih), \exp(ik))\exp(-ik)]$, *and* f''' *stands for the third Arens adjoint of* f.

If the alternative algebra A in Theorem 3.5.38 is unital, then the passing to the bidual in the proof is unnecessary. Therefore assertion (iii) in Corollary 3.5.39 is also true with A instead of A'' and f instead of f'''.

Remark 3.5.40 Let X be a Banach space with a complete predual X_*. Then, since

$$P(X) \equiv BL(X\hat{\otimes}_\pi X, X) \equiv (X\hat{\otimes}_\pi X\hat{\otimes}_\pi X_*)',$$

the complete projective tensor product $X\hat{\otimes}_\pi X\hat{\otimes}_\pi X_*$ becomes a natural predual for $P(X)$, the duality being determined by $\langle f, x\otimes y\otimes x_*\rangle = f(x,y)(x_*)$ for all $f \in P(X)$, $x, y \in X$, and $x_* \in X_*$. Now let A be an *alternative* W^**-algebra* (i.e. an alternative C^*-algebra with a complete predual). It follows from Corollary 3.5.39(ii) and Theorems 2.9.28 and 2.9.17 that p_A is a w^*-unitary element of $P(A)$, and that the equality $V(P(A), p_A, f) = \{f(g) : g \in D^{w^*}(P(A), p_A)\}^-$ holds for every $f \in P(A)$.

§3.5.41 Thinking about Theorem 3.5.38, one could suspect that, if A is a nonzero non-commutative JB^*-algebra, then $n(P(A), p_A)$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is associative and commutative, which would imply (by Theorem 2.1.17(i)) that p_A is a geometrically unitary element of $P(A)$. As a matter of fact, this suspicion is very far from being right. Indeed, it is very easy to provide a non-commutative JB^*-algebra A whose product p_A is not an extreme point of $\mathbb{B}_{P(A)}$. (We note that, if this is the case, then, by Lemma 2.1.25, p_A cannot be a vertex of the closed unit ball of $P(A)$, much less a geometrically unitary element of $P(A)$, and so $n(P(A), p_A) = 0$.) For instance, if B is a C^*-algebra which fails to be commutative, if λ is a real number with $0 < \lambda < 1$, and if we replace the product xy of B by the one

$$(x,y) \to \lambda xy + (1-\lambda)yx,$$

then we obtain a non-commutative JB^*-algebra (say A) whose product is not an extreme point of $\mathbb{B}_{P(A)}$. With $\lambda = 1/2$ in the above construction we even obtain a (commutative) JB^*-algebra with such a pathology.

For the moment, we do not know of any non-commutative JB^*-algebras A whose products are geometrically unitary elements of $P(A)$ other than alternative C^*-algebras. Therefore we raise the following.

Problem 3.5.42 Let A be a non-commutative JB^*-algebra such that the product of A is a geometrically unitary element of $P(A)$. Is A alternative?

Concerning the above problem, the only remarkable fact that we know is that, if we relax the condition that p_A is a geometrically unitary element of $P(A)$ to the one that p_A is an extreme point of $P(A)$, then the answer is negative. This is shown in the following.

Example 3.5.43 Let A be the (commutative) JB^*-algebra whose Banach space is the $*$-invariant subspace of the C^*-algebra $M_2(\mathbb{C})$ given by

$$\left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} : \alpha,\beta,\gamma \in \mathbb{C}\right\},$$

and whose product is the one $\bullet$ defined by

$$x \bullet y := \tfrac{1}{2}(xy+yx) \quad \text{for all } x,y \text{ in } A.$$

Since A is commutative and fails to be associative, it follows that A is not alternative. We are going to prove that p_A is an extreme point of $\mathbb{B}_{P(A)}$. To this end we take f,g in $\mathbb{B}_{P(A)}$ and $0<\lambda<1$ such that $\lambda f+(1-\lambda)g = p_A$, and we proceed to show that $f = p_A$. If $\mathbf{1}$ denotes the unit of A then, by Lemma 3.5.28, we have $f(x,\mathbf{1}) = f(\mathbf{1},x) = x$ for every x in A. In this way $\mathbf{1}$ is a unit for the normed complex algebras B and B^{sym}, where B consists of the normed space of A endowed with the product f. Since A is a V-algebra (by Lemma 2.2.5) and Vidav's requirement involves only the Banach space and the unit, B and B^{sym} are also V-algebras whose natural involutions coincide with the JB^*-involution of A. Since A and B^{sym} are commutative V-algebras, and the mapping $F : x \to x$ from A to B^{sym} is a surjective linear isometry preserving the units, Corollary 2.3.19 applies, so that F is an algebra isomorphism, and hence $B^{\mathrm{sym}} = A$. Now, by Theorem 2.4.11 and Fact 3.3.4, B is a non-commutative JB^*-algebra and, since A is quadratic and squares in B and B^{sym} coincide, B is also quadratic. Then, by Theorem 3.5.5, B is the complexification of the flexible quadratic algebra $\mathscr{A}(E)$ for a suitable H-algebra E. Since A is three-dimensional over $\mathbb{C}$, E must be two-dimensional over $\mathbb{R}$. Let $\{u,v\}$ be a basis of E. Then we have $(u|u\wedge v) = (v|u\wedge v) = 0$, so $u\wedge v = 0$ and hence $\wedge$ is identically zero on E. Therefore, B is commutative, so that $B = B^{\mathrm{sym}}$. Since we know that $B^{\mathrm{sym}} = A$, we obtain $B = A$. This means $f = p_A$ as required.

Let A stand for the JB^*-algebra in the above example. We do not know if p_A is actually a vertex of $P(A)$. If this were the case, then, by the finite dimensionality of $P(A)$, p_A would be a geometrically unitary element of $P(A)$, and the answer to Problem 3.5.42 would be negative.

Despite what we mentioned in §3.5.41, some technical aspects in the proof of Theorem 3.5.38 can be used to get the following generalization of Corollary 3.5.39(i) to the setting of non-commutative JB^*-algebras.

Proposition 3.5.44 *Let A be a nonzero non-commutative JB^*-algebra. Then $N(A)$ is equal to 1 or $\frac{1}{2}$ depending on whether or not A is associative and commutative.*

Proof Recall that A'' is a unital non-commutative JB^*-algebra in a natural way (cf. Theorem 3.5.34), and consider the chain of linear mappings

$$A_1 \xrightarrow{G_1} BL(A) \xrightarrow{G_2} BL(A'') \xrightarrow{G_3} B(H(A'',*),A''),$$

where $G_1(z) := T_z$ for every $z \in A_1$, $G_2(T) := T''$ for every $T \in BL(A)$, and

$$G_3(S)(h) := \exp(-iL_h)S(\exp(ih)) \quad \text{for all } S \in BL(A'') \text{ and } h \in H(A'',*).$$

Then G_1 and G_2 are isometries (the former by Lemma 3.5.37). Moreover, by Lemmas 2.1.10 and 2.2.5 and Corollaries 2.1.9(iii) and 3.4.7, G_3 is also an isometry. On the other hand, the equalities $G_1(\mathbf{1}) = I_A$, $G_2(I_A) = I_{A''}$ are clear, whereas the one $G_3(I_{A''}) = \hat{\mathbf{1}}$ (the mapping constantly equal to $\mathbf{1}$ on $H(A'',*)$) follows by noticing that $\exp(ih) = \exp(iL_h)(\mathbf{1})$ for every $h \in H(A'',*)$. Now let δ denote either 1 or $\frac{1}{2}$ depending on whether or not A is associative and commutative. Since A_1 and $B(H(A'',*),A'')$ are unital non-commutative JB^*-algebras (by Corollary 3.5.36), and they are associative and commutative if and only if so is A (by Theorem 3.5.34 again), it follows from Theorem 3.4.59 that

$$\delta = n(A_1, \mathbf{1}) \geqslant n(BL(A), I_A) \geqslant n(BL(A''), I_{A''}) \geqslant n[B(H(A'',*),A''), \hat{\mathbf{1}}] = \delta. \qquad \square$$

Looking at the above proof, and invoking Corollaries 2.1.2(ii) and 2.9.51, we get the following.

Corollary 3.5.45 *Let A be a nonzero non-commutative JB^*-algebra. Then numerical ranges in $(BL(A), I_A)$ can be computed in terms of numerical ranges in $(A'', \mathbf{1})$. Indeed, for every $T \in BL(A)$ we have*

$$V(BL(A), I_A, T) = \overline{\mathrm{co}}\left[\bigcup_{h \in H(A'',*)} V[A'', \mathbf{1}, \exp(-iL_h)T''(\exp(ih))]\right].$$

Moreover, if A is unital, then we have in fact

$$V(BL(A), I_A, T) = \overline{\mathrm{co}}\left[\bigcup_{h \in H(A,*)} V[A, \mathbf{1}, \exp(-iL_h)T(\exp(ih))]\right]$$

for every $T \in BL(A)$.

3.5.4 The axiom $\|a^*a\| = \|a^*\|\|a\|$ on non-unital algebras

The main result in this section is built from several previously proved results, and the following lemmata.

Lemma 3.5.46 *Let A be a complete normed complex algebra, and let a_λ be an approximate unit bounded by 1 in A. Consider A'' as a complete normed algebra under the Arens product, and let $\mathbf{1}$ be a w^*-cluster point in A'' to the net a_λ. Then we have $\mathbf{1}a = a$ for every $a \in A$, and $a\mathbf{1} = a$ for every $a \in A''$. Consequently, $B := \mathbb{C}\mathbf{1} + A$ becomes a subalgebra of A'', and $\mathbf{1}$ is a unit for B. Moreover, for $\alpha \in \mathbb{C}$ and $a \in A$, we have $\|\alpha\mathbf{1} + a\| = \lim\|\alpha a_\lambda + a\|$.*

Proof Arguing as in the proof of Lemma 3.5.24(ii), the first conclusion follows from the w^*-density of A in A'', and the fact that the Arens product on A'' is w^*-continuous in its first variable and w^*-continuous in the second one when the first variable is fixed in A (by Lemma 2.2.12(iii)). Now, let α and a be in $\mathbb{C}$ and A, respectively. Then, for b in the closed unit ball of A, we have

$$\|b(\alpha\mathbf{1} + a)\| = \|\alpha b + ba\| = \lim\|b(\alpha a_\lambda + a)\| \leqslant \liminf\|\alpha a_\lambda + a\|.$$

Since the closed unit ball of A is w^*-dense in the closed unit ball of A'', and the operator $b \to b(\alpha\mathbf{1} + a)$ from A'' to A'' is w^*-continuous, and closed balls in A'' are w^*-closed, we actually have $\|b(\alpha\mathbf{1} + a)\| \leqslant \liminf\|\alpha a_\lambda + a\|$ for every b in the closed unit ball of A''. In particular, by taking $b = \mathbf{1}$, we derive $\|\alpha\mathbf{1}+a\| \leqslant \liminf \|\alpha a_\lambda + a\|$. On the other hand, for every λ we have

$$\begin{aligned}\|\alpha a_\lambda + a\| &\leqslant \|\alpha a_\lambda + a - (\alpha\mathbf{1}+a)a_\lambda\| + \|(\alpha\mathbf{1}+a)a_\lambda\| \\ &\leqslant \|a - aa_\lambda\| + \|\alpha\mathbf{1}+a\|,\end{aligned}$$

and hence $\limsup \|\alpha a_\lambda + a\| \leqslant \|\alpha\mathbf{1}+a\|$. □

Corollary 3.5.47 *Let A be a complete normed complex algebra such that A'', endowed with the Arens product, has a left unit $\mathbf{1}$, and let a_λ be any approximate unit bounded by 1 in A. Then, for $\alpha \in \mathbb{C}$ and $a \in A$, we have*

$$\|\alpha\mathbf{1}+a\| = \lim \|\alpha a_\lambda + a\|.$$

Proof Let $\mathbf{1}'$ be a w^*-cluster point to the net a_λ in A''. By the first conclusion in Lemma 3.5.46, we have $\mathbf{1}' = \mathbf{1}\mathbf{1}' = \mathbf{1}$. Now, apply the last conclusion in Lemma 3.5.46. □

Lemma 3.5.48 *Let A be a non-commutative JB*-algebra such that the equality $\|a^*a\| = \|a\|^2$ holds for every $a \in A$. Then A is alternative.*

Proof By Proposition 3.5.23, A has an approximate unit a_λ bounded by 1, consisting of $*$-invariant elements, and such that a_λ^2 is also an approximate unit in A. Therefore, invoking Theorem 3.5.34 and Corollary 3.5.47, for $\alpha \in \mathbb{C}$ and $a \in A$, we have

$$\begin{aligned}\|\alpha\mathbf{1}+a\|^2 &= \lim \|\alpha a_\lambda + a\|^2 = \lim \|(\alpha a_\lambda + a)^*(\alpha a_\lambda + a)\| \\ &= \lim \||\alpha|^2 a_\lambda^2 + \bar{\alpha} a_\lambda a + \alpha a^* a_\lambda + a^*a\| \\ &= \lim \||\alpha|^2 a_\lambda^2 + \bar{\alpha} a + \alpha a^* + a^*a\| \\ &= \||\alpha|^2\mathbf{1} + \bar{\alpha} a + \alpha a^* + a^*a\| = \|(\alpha\mathbf{1}+a)^*(\alpha\mathbf{1}+a)\|.\end{aligned}$$

By Theorem 3.2.5, the algebra $\mathbb{C}\mathbf{1}+A$ (and hence A) is alternative. □

Now we state and prove the main result in the present subsection.

Theorem 3.5.49 *Let A be a complete normed complex algebra, let $\sharp$ be a conjugate-linear vector space involution on A such that*

$$\|a^\sharp a\| = \|a^\sharp\|\|a\|$$

for every $a \in A$, and assume that A has an approximate unit bounded by 1 consisting of $\sharp$-invariant elements. Then A is an alternative C*-algebra for a suitable involution $*$. Moreover, except possibly in the case that $(A,*)$ is the C*-algebra $\mathbb{C}^2$, we have $\sharp = *$. In the exceptional case that $(A,*) = \mathbb{C}^2$, the involutions $\sharp$ satisfying the above requirements are exactly the mappings of the form $a \mapsto a^* + \overline{f(a)}\mathbf{1}$, where $\mathbf{1}$ stands for the unit of A, and f is any linear form on A such that $f(\mathbf{1}) = 0$ and $f(a^*) = -\overline{f(a)}$ for every $a \in A$.

Proof We may assume that $A \neq 0$. Let a_λ be the approximate unit of A whose existence is assumed, let $\mathbf{1}$ and B be as in Lemma 3.5.46, and note that $\|\mathbf{1}\| = 1$.

Let h be a $\sharp$-invariant element of A, and let r be a positive real number. Then, since $(a_\lambda + irh)^\sharp = a_\lambda - irh$, we have

$$\begin{aligned}\|\mathbf{1} - irh\|\|\mathbf{1} + irh\| &= \lim \|a_\lambda - irh\|\|a_\lambda + irh\| = \lim \|(a_\lambda - irh)(a_\lambda + irh)\| \\ &= \lim \|a_\lambda^2 + ira_\lambda h - irha_\lambda + r^2h^2\| \\ &= \lim \|a_\lambda^2 + r^2h^2\| \leqslant 1 + r^2\|h^2\|,\end{aligned}$$

and hence

$$\frac{\|\mathbf{1} - irh\|\|\mathbf{1} + irh\| - 1}{r} \leqslant r\|h^2\|.$$

Writing

$$\frac{\|\mathbf{1} - irh\|\|\mathbf{1} + irh\| - 1}{r} = \|\mathbf{1} - irh\| \frac{\|\mathbf{1} + irh\| - 1}{r} + \frac{\|\mathbf{1} - irh\| - 1}{r},$$

letting $r \to 0^+$, and applying Proposition 2.1.5, we deduce that

$$\max\{\Re(f(ih)) : f \in D(B, \mathbf{1})\} - \min\{\Re(f(ih)) : f \in D(B, \mathbf{1})\} \leqslant 0.$$

Therefore there exists a real number λ_h such that $\Re(f(ih)) = \lambda_h$ for every $f \in D(B, \mathbf{1})$, and hence

$$h + i\lambda_h\mathbf{1} \in H(B, \mathbf{1}). \tag{3.5.12}$$

Now, since h is arbitrary in $H(A, \sharp)$, we derive that $B = H(B, \mathbf{1}) + iH(B, \mathbf{1})$. Therefore, by the non-associative Vidav–Palmer theorem (Theorem 3.3.11), we have in fact $B = H(B, \mathbf{1}) \oplus iH(B, \mathbf{1})$, and B becomes a non-commutative JB^*-algebra for the involution $*$ whose $*$-invariant elements are precisely the elements in $H(B, \mathbf{1})$. Now note that, as a consequence of (3.5.12), $h + i\lambda_h\mathbf{1}$ is $*$-invariant, and consequently we have

$$h + i\lambda_h\mathbf{1} = h^* - i\lambda_h\mathbf{1}. \tag{3.5.13}$$

Assume that $\mathbf{1} \notin A$. Then, since A is a closed ideal of B, and $(B, *)$ is a non-commutative JB^*-algebra, it follows from Proposition 3.4.13 that A is $*$-invariant. Therefore h^* lies in A, so that, by (3.5.13), we have

$$2i\lambda_h\mathbf{1} = h^* - h \in A,$$

which implies that $\lambda_h = 0$, and hence that h is $*$-invariant. Now, by the arbitrariness of $h^\sharp = h \in A$, we obtain that $\sharp = *$. Since $*$ is isometric (by Proposition 3.3.13), the proof in this case is concluded by applying Lemma 3.5.48.

Now, assume that $\mathbf{1} \in A$. Then, writing $\mathbf{1} = h + ik$ for suitable $\sharp$-invariant elements $h, k \in A$, and keeping in mind (3.5.13), we have

$$\begin{aligned}\mathbf{1}^\sharp &= h - ik = h^* - 2i\lambda_h\mathbf{1} - i(k^* - 2i\lambda_k\mathbf{1}) = h^* - ik^* - 2(\lambda_k + i\lambda_h)\mathbf{1} \\ &= \mathbf{1}^* - 2(\lambda_k + i\lambda_h)\mathbf{1} = \mathbf{1} - 2(\lambda_k + i\lambda_h)\mathbf{1},\end{aligned}$$

and hence $\mathbf{1}^\sharp = \alpha\mathbf{1}$ for a suitable complex number α. Moreover, since

$$\mathbf{1} = \mathbf{1}^{\sharp\sharp} = (\alpha\mathbf{1})^\sharp = \bar{\alpha}\mathbf{1}^\sharp = \bar{\alpha}\alpha\mathbf{1} = |\alpha|^2\mathbf{1},$$

we get $|\alpha| = 1$. Now, for $a \in A$, set $a^{\triangle} := \bar{\alpha} a^{\sharp}$. Then $\triangle$ becomes a conjugate-linear vector space involution on A satisfying

$$\mathbf{1}^{\triangle} = \mathbf{1} \text{ and } \|a^{\triangle} a\| = \|a^{\triangle}\| \|a\| \text{ for every } a \in A.$$

By applying Theorem 3.5.15, we obtain that the conclusion of our theorem is true with $\triangle$ instead of $\sharp$. Therefore we have $\triangle = *$, except possibly in the case that $\dim(A) = 2$. As a consequence, since $*$ is an isometry, $\triangle$ (and hence $\sharp$) is continuous without any exception. Finally, since $\mathbf{1} = \lim a_\lambda \mathbf{1} = \lim a_\lambda$, and $a_\lambda^{\sharp} = a_\lambda$ for every λ, we get $\mathbf{1}^{\sharp} = \mathbf{1}$, so $\alpha = 1$, and so $\sharp = \triangle$, which completes the proof. □

A direct consequence of Theorem 3.5.49 is the following.

Corollary 3.5.50 *Let A be a complete normed complex algebra endowed with a conjugate-linear vector space involution $*$ satisfying*

$$(a^2)^* = (a^*)^2 \textit{ and } \|a^* a\| = \|a^*\| \|a\| \textit{ for every } a \in A,$$

and assume that A has an approximate unit bounded by 1 and consisting of $*$-invariant elements. Then A is an alternative C^*-algebra.

3.5.5 The non-unital non-associative Gelfand–Naimark theorem

We begin this subsection with another direct consequence of Theorem 3.5.49; namely, the following.

Proposition 3.5.51 *Let A be a complete normed complex algebra endowed with a conjugate-linear vector space involution $*$ such that $\|a^* a\| = \|a\|^2$ for every $a \in A$, and assume that A has an approximate unit bounded by 1 and consisting of $*$-invariant elements. Then A is an alternative C^*-algebra.*

Certainly, Proposition 3.5.51 follows from Theorem 3.5.49 in a straightforward way. However, Theorem 3.5.49 depends heavily on Theorem 3.5.15, whose proof is quite involved. Therefore we feel it appropriate to provide the reader with the following.

§3.5.52 *Autonomous proof of Proposition 3.5.51* We may assume that $A \neq 0$. Let a_λ be the approximate unit of A whose existence is assumed, let $\mathbf{1}$ and B be as in Lemma 3.5.46, and note that $\|\mathbf{1}\| = 1$. Let h be a $*$-invariant element of A, and let r be a positive real number. Then we have

$$\begin{aligned}\|\mathbf{1} + irh\|^2 &= \lim \|a_\lambda + irh\|^2 = \lim \|(a_\lambda + irh)^*(a_\lambda + irh)\| \\ &= \lim \|a_\lambda^2 + ira_\lambda h - irha_\lambda + r^2 h^2\| \\ &= \lim \|a_\lambda^2 + r^2 h^2\| \leqslant 1 + r^2 \|h^2\|,\end{aligned}$$

and hence

$$\lim_{r \to 0^+} \frac{\|\mathbf{1} + irh\| - 1}{r} \leqslant \lim_{r \to 0^+} \frac{\sqrt{1 + r^2 \|h^2\|} - 1}{r} = 0.$$

Replacing h with $-h$, and applying Proposition 2.1.5, we deduce that h belongs to $H(B, \mathbf{1})$, where B stands for the complete normed complex algebra $\mathbb{C}\mathbf{1} + A$. Now, since h is an arbitrary $*$-invariant element of A, we derive that $B = H(B, \mathbf{1}) + iH(B, \mathbf{1})$.

Therefore, by the non-associative Vidav–Palmer theorem (Theorem 3.3.11), B is a non-commutative JB^*-algebra for the involution

$$(\alpha\mathbf{1} + a)^{*'} := \bar{\alpha}\mathbf{1} + a^*. \tag{3.5.14}$$

(We note that, in the dangerous case that $\mathbf{1}$ lies in A, writing $\mathbf{1} = h + ik$ with h and k $*$-invariant elements of A, and recalling that $*$-invariant elements of A belong to $H(A,\mathbf{1})$, we get $\mathbf{1} - h \in H(A,\mathbf{1}) \cap iH(A,\mathbf{1}) = 0$, so $\mathbf{1}^* = \mathbf{1}$, and so (3.5.14) becomes a good definition.) Now, clearly, A is a non-commutative JB^*-algebra for the involution $*$, and the proof is concluded by applying Lemma 3.5.48. □

According to §2.4.26, by a non-associative C^*-algebra we mean a complete normed complex $*$-algebra A satisfying $\|a^*a\| = \|a\|^2$ for every $a \in A$. Now we can formulate and prove the main result in this section, namely the so-called *non-unital non-associative Gelfand–Naimark theorem*.

Theorem 3.5.53 *Let A be a non-associative C^*-algebra. Then A is alternative if and only if A has a bounded approximate unit bounded by* 1.

Proof Assume that there exists an approximate unit a_λ bounded by 1 in A. Then, since $*$ is an isometric algebra involution on A, the net $\frac{1}{2}(a_\lambda + a_\lambda^*)$ becomes a bounded approximate unit bounded by 1 and consisting of $*$-invariant elements. Therefore, by Proposition 3.5.51, A is alternative.

Now, assume that A is alternative. Then, by Fact 3.3.2 and Proposition 3.5.23, A has an approximate unit bounded by 1. □

Since alternative commutative complex algebras are associative (a consequence of the equality (3.4.12)), we derive the following.

Corollary 3.5.54 *Commutative* (associative) *C^*-algebras are nothing other than commutative non-associative C^*-algebras having an approximate unit bounded by* 1.

The remaining part of this subsection is devoted to show the abundance of non-associative C^*-algebras which are not alternative. To this end, we begin by remarking that non-associative C^*-algebras have an 'isotopy type' stability property. Let A be a non-associative C^*-algebra. Given surjective linear isometries $F, G : A \to A$ preserving the involution of A, the Banach space of A, endowed with the new product $a \odot b := F(G(a)G(b))$, becomes a new non-associative C^*-algebra for the same involution. Non-associative C^*-algebras obtained in this way are called *(non-associative) C^*-isotopes of A*. As a matter of fact, C^*-isotopes of alternative C^*-algebras need not be alternative. As an example, take $A := \mathbb{C}^2$, and define a product $\odot$ on A by

$$(\lambda_1, \lambda_2) \odot (\mu_1, \mu_2) := (\lambda_2\mu_2, \lambda_1\mu_1).$$

Then $(A, \odot)$ becomes a C^*-isotope of A which is not alternative. Actually, most alternative C^*-algebras have C^*-isotopes which are not alternative. Indeed, we have the following.

Proposition 3.5.55 *Let A be an alternative C^*-algebra. Then the following conditions are equivalent:*

(i) *All C^*-isotopes of A are alternative.*
(ii) *The identity mapping on A is the unique Jordan-$*$-automorphism of A.*

(iii) $A = C_0^{\mathbb{C}}(E)$ *for some locally compact Hausdorff topological space E such that there is no homeomorphism from E onto E other than the identity.*

Proof (i)$\Rightarrow$(ii) Let F be a Jordan-$*$-automorphism of A. By Fact 3.3.2 and Remark 3.4.5, F is a surjective linear isometry preserving $*$, and hence, by setting $a \odot b := F(ab)$, $(A, \odot)$ becomes a C^*-isotope of A. Therefore, by the assumption (i), $(A, \odot)$ is an alternative algebra. This means that

$$F(a^2)b = aF(ab) \text{ and } bF(a^2) = F(ba)a$$

for all $a, b \in A$. Since A has an approximate unit a_λ (by Proposition 3.5.23), it is enough to set $b := a_\lambda$ in the equalities above and take limits in λ to deduce

$$F(a^2) = aF(a) = F(a)a \text{ for every } a \in A.$$

Linearizing, we get

$$2F(a \bullet b) = aF(b) + bF(a) \text{ and } 2F(a \bullet b) = F(a)b + F(b)a$$

for all $a, b \in A$, which implies (with $b := a_\lambda$ as above)

$$F(a) = \lim_\lambda aF(a_\lambda) \text{ and } F(a) = \lim_\lambda F(a_\lambda)a$$

for every $a \in A$, and hence, since F is a Jordan automorphism,

$$F(a) = \lim_\lambda a \bullet F(a_\lambda) = \lim_\lambda F(F^{-1}(a) \bullet a_\lambda) = F(F^{-1}(a)) = a$$

for every $a \in A$. Thus F is the identity mapping on A.

(ii)$\Rightarrow$(iii) Let D be a continuous Jordan derivation of A preserving $*$. Then, for every $r \in \mathbb{R}$, $\exp(rD)$ is a Jordan-$*$-automorphism of A, and hence, by the assumption (ii), we have $\exp(rD) = I_A$, so $D = 0$ (by taking derivatives at $r = 0$). Now let h be in $H(A, *)$. Then, by Lemma 2.4.15, $D := i[h, \cdot]$ is a continuous Jordan derivation of A preserving $*$. Therefore, by the above, we have $[h, \cdot] = 0$. Since h is arbitrary in $H(A, *)$, we get that A is commutative. By the equality (3.4.12), A is also associative. Now, by Theorem 1.2.4, we have $A = C_0^{\mathbb{C}}(E)$ for some locally compact Hausdorff topological space E. Finally, since different homeomorphisms from E onto E give rise to different $*$-automorphisms of A, a new application of the assumption (ii) assures that condition (iii) holds.

(iii)$\Rightarrow$(i) Assume that condition (iii) holds. Then, by Corollary 1.1.77, A has no $*$-automorphism other than the identity mapping. Then, by Theorem 2.2.19, the linear isometries from A onto A preserving $*$ are precisely the mappings of the form $a \to ua$, where u runs over the set of all self-adjoint unitary elements in the C^*-algebra of multipliers $M(A)$ of A. Therefore, since $M(A)$ is commutative (as a consequence of Corollary 3.5.25), the product $\odot$ of any C^*-isotope of A is of the form $a \odot b = uab$, with u as above, and such a product is clearly associative (and hence alternative). □

By an *absolute-valued C^*-algebra* we mean a complete absolute-valued complex algebra A endowed with a conjugate-linear algebra involution $*$. *We note that,* since conjugate-linear algebra involutions on complete absolute-valued algebras are isometric (a consequence of Theorem 2.8.4), *absolute-valued C^*-algebras are non-associative C^*-algebras, and that,* by Proposition 2.6.27, *none of them, with the exception of $\mathbb{C}$, is alternative*, nor even power-associative. *As a consequence,* since

C^*-isotopes of absolute-valued C^*-algebras are absolute-valued C^*-algebras, *no C^*-isotope of an alternative C^*-algebra different from $\mathbb{C}$ can be an absolute-valued C^*-algebra.*

Now, let us recall some basic facts about the Hilbert tensor product of two Hilbert spaces (see [723, pp. 21–2] for details) which will be applied without notice in the proof of the next proposition. Let H_1, H_2 be Hilbert spaces over $\mathbb{K}$. Then the algebraic tensor product $H_1 \otimes H_2$ becomes a pre-Hilbert space under the inner product $(\cdot|\cdot)$ determined on elementary tensors by

$$(x \otimes y | u \otimes v) = (x|u)(y|v).$$

The completion of the pre-Hilbert space $(H_1 \otimes H_2, (\cdot|\cdot))$ is called the *Hilbert tensor product* of H_1 and H_2, and is denoted by $H_1 \hat{\otimes} H_2$. The Hilbertian dimension of $H_1 \hat{\otimes} H_2$ is equal to the product of the Hilbertian dimensions of H_1 and H_2. On the other hand, the natural identification $H_1 \otimes H_2 \equiv H_2 \otimes H_1$ becomes an isometry, and hence extends uniquely to a surjective linear isometry $G : H_1 \hat{\otimes} H_2 \to H_2 \hat{\otimes} H_1$. Moreover, if H_3, H_4 are Hilbert spaces over $\mathbb{K}$, and if a and b are in $BL(H_1, H_3)$ and $BL(H_2, H_4)$, respectively, then there exists a unique operator

$$a \otimes b \in BL(H_1 \hat{\otimes} H_2, H_3 \hat{\otimes} H_4)$$

satisfying $(a \otimes b)(x \otimes y) = ax \otimes by$ for all $x \in H_1$ and $y \in H_2$, and we have $\|a \otimes b\| = \|a\|\|b\|$ and $(a \otimes b)^* = a^* \otimes b^*$. Finally, note that the mapping $(a, b) \to a \otimes b$ from $BL(H_1, H_3) \times BL(H_2, H_4)$ to $BL(H_1 \hat{\otimes} H_2, H_3 \hat{\otimes} H_4)$ is bilinear.

Proposition 3.5.56 *Let H be an infinite-dimensional complex Hilbert space, and let A stand for the C^*-algebra of all bounded linear operators on H. Then A becomes an absolute-valued C^*-algebra for its natural involution and a suitable product.*

Proof Consider the Hilbert tensor product $H \hat{\otimes} H$, take a surjective linear isometry $F : H \hat{\otimes} H \to H$, and let $G : H \hat{\otimes} H \to H \hat{\otimes} H$ be the unique surjective linear isometry such that $G(x \otimes y) = y \otimes x$ for all $x, y \in H$. Now, define a product $\odot$ on A by $a \odot b := F \circ G \circ (a \otimes b) \circ F^*$. Then we have $\|a \odot b\| = \|a\|\|b\|$ for all $a, b \in A$. Moreover, noticing that $G^* = G$, and that for $a, b \in A$ we have $G(a \otimes b) = (b \otimes a)G$, we straightforwardly realize that $(a \odot b)^* = b^* \odot a^*$. □

Theorem 3.5.57 *Let X be a nonzero complex Banach space. Then there exists an absolute-valued C^*-algebra A with* $\mathrm{dens}(A) = \mathrm{dens}(X)$ *and such that X is linearly isometric to a subspace of A.*

Proof Let E denote the compact Hausdorff topological space consisting of the closed unit ball of X' and the weak* topology. Then X can be seen isometrically as a subspace of $C^{\mathbb{C}}(E)$. Now, by Exercise 1.2.5, X can be seen isometrically as a subspace of $BL(H)$ for some infinite-dimensional complex Hilbert space H, and, by Proposition 3.5.56, $BL(H)$ becomes an absolute-valued C^*-algebra for its natural involution $*$ and a suitable product $\odot$. Let A stand for the closed subalgebra of $(BL(H), \odot)$ generated by $X \cup X^*$. Then, by Proposition 1.2.25, A is $*$-invariant, and hence it is an absolute-valued C^*-algebra containing X isometrically. Moreover, by Corollary 2.8.21, we have $\mathrm{dens}(A) = \mathrm{dens}(X \cup X^*)$. But, clearly, $\mathrm{dens}(X \cup X^*) = \mathrm{dens}(X)$. □

To conclude the present subsection, let us note that the case $\mathbb{K}=\mathbb{C}$ of Corollary 2.8.25 can now be reformulated as follows

Fact 3.5.58 *Let A be a complete normed complex algebra. Then there exists an absolute-valued C^*-algebra $\mathscr{A}$ with $\operatorname{dens}(\mathscr{A})=\operatorname{dens}(A)$ and such that A is isometrically algebra-isomorphic to a quotient of $\mathscr{A}$.*

3.5.6 Vowden's theorem

The aim of this subsection is to prove a reasonable non-associative generalization of Vowden's theorem that complete normed associative complex $*$-algebras A satisfying $\|a^*a\|=\|a^*\|\|a\|$ for every $a\in A$ are in fact C^*-algebras. We begin by proving a particular case of this last result.

Lemma 3.5.59 *Let A be a complete normed associative and commutative complex $*$-algebra satisfying $\|a^*a\|=\|a^*\|\|a\|$ for every $a\in A$. Then A is a C^*-algebra.*

Proof Given a $*$-invariant element h in A, for every $n\in\mathbb{N}$, h^{2^n} remains $*$-invariant, and hence, by induction, we get $\|h^{2^n}\|=\|h\|^{2^n}$, which implies $\mathfrak{r}(h)=\|h\|$. In particular, we have $\mathfrak{r}(a^*a)=\|a^*a\|$ for every $a\in A$. On the other hand, given $a\in A$ and $n\in\mathbb{N}$, we see that $(a^*a)^n=(a^n)^*a^n$, so

$$\|(a^*a)^n\|^{\frac{1}{n}}=\|(a^n)^*a^n\|^{\frac{1}{n}}=\|(a^*)^n\|^{\frac{1}{n}}\|a^n\|^{\frac{1}{n}},$$

and so $\mathfrak{r}(a^*a)=\mathfrak{r}(a^*)\mathfrak{r}(a)$. Since $\mathfrak{r}(a^*)=\mathfrak{r}(a)$ (by Corollary 1.1.109), it follows that

$$\|a^*\|\|a\|=\|a^*a\|=\mathfrak{r}(a^*a)=\mathfrak{r}(a^*)\mathfrak{r}(a)=\mathfrak{r}(a)^2\leqslant\|a\|^2,$$

and so $\|a^*\|\leqslant\|a\|$. Replacing a by a^*, we also have $\|a\|\leqslant\|a^*\|$. Thus $\|a^*\|=\|a\|$, and hence $\|a^*a\|=\|a\|^2$. □

Definition 3.5.60 Let A be a $*$-algebra over $\mathbb{K}$, and let S be a non-empty subset of A. We say that S is *normal* if the subalgebra $A(S\cup S^*)$ of A generated by $S\cup S^*$ is associative and commutative. In the case that A is associative, we invoke Corollary 1.1.79 to realize that S is normal if and only if $S\cup S^*$ is a commutative subset of A. Anyway, an element $a\in A$ is normal in the sense of Definition 3.4.20 if and only if the singleton $\{a\}$ is a normal subset of A.

Proposition 3.5.61 *Let A be a $*$-algebra over $\mathbb{K}$. We have:*

(i) *Each normal subset of A is contained in a maximal normal subset of A.*
(ii) *Maximal normal subsets of A are $*$-subalgebras of A.*
(iii) *If A is normed, then maximal normal subsets of A are closed in A.*

Proof In view of Zorn's lemma, to prove assertion (i) it is enough to show that the family $\mathscr{F}$ of all normal subsets of A is inductive for the inclusion order. Let $\mathscr{C}$ be a chain of $\mathscr{F}$. Then the set $\{A(S\cup S^*) : S\in\mathscr{C}\}$ is a chain of associative and commutative subalgebras of A (relative to the inclusion), and hence

$$\cup_{S\in\mathscr{C}}A(S\cup S^*)$$

is an associative and commutative subalgebra of A. Since

$$\cup_{S\in\mathscr{C}}A(S\cup S^*) \supseteq A((\cup_{S\in\mathscr{C}}S)\cup(\cup_{S\in\mathscr{C}}S)^*),$$

we get that $\cup_{S\in\mathscr{C}}S$ is an upper bound for $\mathscr{C}$ in $\mathscr{F}$.

Let S be a normal subset of A. Then, by Proposition 1.2.25, $A(S\cup S^*)$ is an associative and commutative $*$-subalgebra of A, and as a consequence, it is a normal subset of A. Therefore, in the case that S is in fact a maximal normal subset of A, we have $S = A(S\cup S^*)$, which proves assertion (ii).

Assume that A is a normed algebra, and let B be a maximal normal subset of A. The fact that B is an associative and commutative $*$-subalgebra of A (assured by assertion (ii) just proved) must be omnipresent along the argument which follows. Let x be an element in $\bar{B}$. Then $A(B\cup\{x\}) \subseteq \bar{B}$ so that, since $\bar{B}$ is an associative and commutative subalgebra of A, the same is true for $\overline{A(B\cup\{x\})}$. Therefore, since $A(B\cup\{x^*\}) = (A(B\cup\{x\}))^*$, we deduce that $A(B\cup\{x^*\})$ (and hence $\overline{A(B\cup\{x^*\})}$) is an associative and commutative subalgebra of A. Since $x\in\bar{B}\subseteq\overline{A(B\cup\{x^*\})}$, it follows that

$$A(B\cup\{x,x^*\}) \subseteq \overline{A(B\cup\{x^*\})},$$

hence $A(B\cup\{x,x^*\})$ is an associative and commutative subalgebra of A, so $B\cup\{x\}$ is a normal subset of A, and so $x\in B$ by maximality of B. Thus B is closed in A because of the arbitrariness of $x\in\bar{B}$. □

Let A be a (normed) algebra over $\mathbb{K}$ endowed with a (continuous) conjugate-linear algebra involution $*$, and let S be a non-empty subset of A. Then, as a consequence of Proposition 1.2.25, the (closed) subalgebra of A generated by $S\cup S^*$ is the smallest (closed) $*$-subalgebra of A containing S. Thus, when we are dealing with possibly discontinuous involutions on normed algebras, the following definition becomes useful.

Definition 3.5.62 Let A be a normed $*$-algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Since the intersection of any family of closed $*$-subalgebras of A is again a closed $*$-subalgebra of A, it follows that the intersection of all closed $*$-subalgebras of A containing S is the smallest closed $*$-subalgebra of A containing S. This subalgebra is called *the closed $*$-subalgebra of A generated by S*.

Now, as a straightforward consequence of Proposition 3.5.61, we derive the following.

Corollary 3.5.63 *Let A be a normed $*$-algebra over $\mathbb{K}$, and let S be a normal subset of A. Then the closed $*$-subalgebra of A generated by S is associative and commutative.*

If A is a power-associative algebra over $\mathbb{K}$, then, by Corollary 2.4.18, A is power-commutative, and hence, if in addition A is a $*$-algebra, then, for every $*$-invariant element h in A, the set $\{h\}$ (occasionally, $\{h,\mathbf{1}\}$ if A is unital) is a normal subset of A. Therefore, invoking Corollary 3.5.63, we get the following.

Corollary 3.5.64 *Let A be a normed power-associative $*$-algebra over $\mathbb{K}$, and let h be in $H(A,*)$. Then the closed $*$-subalgebra of A generated by h is associative and* *commutative. Moreover, if A is unital, then the closed $*$-subalgebra of A generated by $\{h,\mathbf{1}\}$ is associative and commutative.*

Proposition 3.5.65 *Let A be a complete normed power-associative complex $*$-algebra satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$. Then $*$ is continuous.*

Proof Let x be any $*$-invariant element of A, and let B^x stand for the closed $*$-subalgebra of A generated by x. According to Corollary 3.5.64, B^x is associative and commutative. Therefore, by Lemma 3.5.59, B^x is a C^*-algebra, and hence, by Proposition 1.2.44, the unital extension $B^x_{\mathbb{1}}$ is a C^*-algebra containing B^x as a closed $*$-subalgebra. Now, we will show that $\|x^2 - x^4\| \leqslant \|x^2\|$ whenever $\|x\| \leqslant 1$. For such an x, Proposition 1.3.4(ii) gives that

$$\mathrm{sp}(B^x_{\mathbb{1}}, x^2 - x^4) = \left\{\lambda^2 - \lambda^4 : \lambda \in \mathrm{sp}(B^x_{\mathbb{1}}, x)\right\}.$$

Since $\|\cdot\|$ and $\mathfrak{r}(\cdot)$ coincide on $B^x_{\mathbb{1}}$ (by Lemma 1.2.12) and $\mathrm{sp}(B^x_{\mathbb{1}}, x) \subseteq \mathbb{R}$ (by Proposition 1.2.20(ii)), Theorem 1.1.46 applies, so that we have

$$\|x^2 - x^4\| = \sup\left\{\lambda^2 - \lambda^4 : \lambda \in \mathrm{sp}(B^x_{\mathbb{1}}, x)\right\} \leqslant \sup\left\{\lambda^2 : \lambda \in \mathrm{sp}(B^x_{\mathbb{1}}, x)\right\} = \|x^2\|.$$

Now, we will show that the set $H(A,*)$ is closed in A. Let h_n be a convergent sequence in $H(A,*)$ whose limit is $h + ik$, with $h,k \in H(A,*)$. Since $h_n - h$ converges to ik, we may assume (by setting h_n for $h_n - h$) that h_n converges to ik, and also that $\|h_n\| \leqslant 1$. Then, by the first paragraph in the proof, we have $\|h_n^2 - h_n^4\| \leqslant \|h_n^2\|$ for every $n \in \mathbb{N}$. Letting $n \to \infty$ we obtain

$$\left\|-k^2 - k^4\right\| \leqslant \|k^2\|.$$

Therefore, suitably repeating the arguments in the first paragraph of the proof, we see that

$$\sup\left\{\lambda^2 + \lambda^4 : \lambda \in \mathrm{sp}(B^k_{\mathbb{1}}, k)\right\} \leqslant \sup\left\{\lambda^2 : \lambda \in \mathrm{sp}(B^k_{\mathbb{1}}, k)\right\}.$$

Choose $\mu \in \mathrm{sp}(B^k_{\mathbb{1}}, k)$ such that $\mu^2 = \sup\{\lambda^2 : \lambda \in \mathrm{sp}(B^k_{\mathbb{1}}, k)\}$. Then we have $\mu^2 + \mu^4 \leqslant \mu^2$, so $\mu = 0$. It follows that $\|k\| = \mathfrak{r}(k) = 0$ and so $k = 0$. This shows that $H(A,*)$ is closed.

Finally, since $H(A,*)$ is closed in A (by the above paragraph), and A is complete, the direct sum $A = H(A,*) \oplus iH(A,*)$ is topological, and hence $* : h + ik \to h - ik$ $(h,k \in H(A,*))$ is continuous. □

Now we can prove the main result in the present subsection.

Theorem 3.5.66 *Let A be a complete normed non-commutative Jordan complex $*$-algebra satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$. Then A is an alternative C^*-algebra.*

Proof By Proposition 3.5.65, $H(A,*)$ is a closed real subalgebra of A^{sym}, and hence it is a complete normed Jordan real algebra. Let h be any element of $H(A,*)$, and let B stand for the closed subalgebra of A generated by h. Then, since $*$ is continuous (again by Proposition 3.5.65), B is $*$-invariant. On the other hand, by Corollary 2.4.18, B is associative and commutative. It follows from Lemma 3.5.59 that B is a C^*-algebra in the natural way, and hence, by Theorem 1.2.4, we have $B = C_0^{\mathbb{C}}(E)$ for some locally compact Hausdorff topological space E. Therefore, by Lemma 2.4.17, the closed subalgebra of A^{sym} generated by h equals $C_0^{\mathbb{C}}(E)$, which implies that the closed subalgebra of the normed Jordan real algebra $H(A,*)$ generated by h equals $C_0^{\mathbb{R}}(E)$. Since h is an arbitrary element of $H(A,*)$, Proposition 3.4.2 applies, so that $H(A,*)$ is a JB-algebra. Then, by Lemma 3.5.22, $H(A,*)$ has an approximate unit a_λ bounded by 1, which is clearly an approximate unit for A^{sym}. Since A equals the linear hull of the set $\{b^2 : b \in A\}$ (because closed subalgebras generated by self-adjoint elements are $C_0^{\mathbb{C}}(E)$ algebras), Lemma 3.5.21 gives us that a_λ is an approximate unit for A. Finally, the result follows from Corollary 3.5.50. □

The following unit-free version of Corollary 3.5.18 follows straightfor- wardly from Theorem 3.5.66 above.

Corollary 3.5.67 (Vowden theorem) *Let A be a complete normed associative complex $*$-algebra satisfying $\|a^*a\| = \|a^*\|\,\|a\|$ for every a in A. Then A is a C^*-algebra.*

Now we conclude with a theorem summarizing the main results in the present section.

Theorem 3.5.68 *Let A be a complete normed complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A has an approximate unit bounded by 1 and consisting of $*$-invariant elements, and the equality $\|a^*a\| = \|a^*\|\|a\|$ holds for every $a \in A$.*
(ii) *A has an approximate unit bounded by 1, and the equality $\|a^*a\| = \|a\|^2$ holds for every $a \in A$.*
(iii) *A is alternative, and the equality $\|a^*a\| = \|a^*\|\|a\|$ holds for every $a \in A$.*
(iv) *A is a non-commutative Jordan algebra, and the equality $\|a^*a\| = \|a^*\|\|a\|$ holds for every $a \in A$.*
(v) *A is alternative, and the equality $\|a^*a\| = \|a\|^2$ holds for every $a \in A$.*

Proof (ii)⇔(v) By Theorem 3.5.53.

The implication (iv)⇒(iii) follows from Theorem 3.5.66, whereas the one (iii)⇒(iv) follows from Corollary 2.4.10.

(i)⇒(v) as a consequence of Corollary 3.5.50, whereas the implication (v)⇒(i) follows from Fact 3.3.2 and Proposition 3.5.23.

The implication (v)⇒(iii) is clear, whereas the one (iii)⇒(v) follows from Corollary 2.4.10 and Theorem 3.5.66. □

3.5.7 Historical notes and comments

Lemma 3.5.1 is folklore (see for example [717, Proposition 3.9(1)]). Proposition 3.5.2 and Theorem 3.5.5 are due to Payá, Pérez, and Rodríguez [482]. According to Theorem 3.5.5, a perfect knowledge of quadratic non-commutative JB^*-algebras depends on the determination of all H-algebras. A similar situation happened when studying complete smooth-normed real algebras (see Remark 2.6.54).

Proposition 3.5.4 is a refined version of Proposition 2.3 in the Cabrera–Moreno–Rodríguez paper [145], where Lemma 3.5.3 is also proved, pointing out that the renorming argument in the proof is taken from Theorem 1.3.3 in Rickart's book [795]. With more or less precision in the formulation and proof, the content of Proposition 3.5.4 often arises in the literature, mainly in the case that $\mathbb{K} = \mathbb{C}$ (see for example [91] and [152]). Restricting again to the case $\mathbb{K} = \mathbb{C}$, the germ for the proof we have provided can be found at the beginning of the proof of Theorem 3.1 of [482]. Corollary 3.5.6 can be proved without involving Theorem 3.5.5. Indeed, it is enough to note that, by Corollary 2.5.19(i), the algebraic norm function on a quadratic alternative algebra admits composition, that the algebraic norm function on a quadratic alternative C^*-algebra is nondegenerate, and that consequently, by [808, Theorem 3.25], $\mathbb{C}, \mathbb{C}^2, M_2(\mathbb{C})$ and $C(\mathbb{C})$ are the unique quadratic alternative C^*-algebras.

With the exception of Proposition 3.5.11 which is taken from Rickart's book [795, p. 293], results from Example 3.5.8 to Theorem 3.5.15 are due to Cabrera and Rodríguez [150]. Corollary 3.5.16, as well as the autonomous proof given in §3.5.17, are earlier [514]. The associative forerunner, given by Corollary 3.5.18, is due to Glimm and Kadison [290], and is included in [694, Theorem 7.2] (with a proof rather similar to that in §3.5.17) and in [725, Sections 15 and 16] (with a proof close to that of Glimm and Kadison).

Lemma 3.5.21 and the fact that non-commutative JB^*-algebras have approximate units bounded by 1 (a consequence of Proposition 3.5.23) were proved first by Villena in his PhD thesis [816]. The first published proof of this fact appeared in [365] as a consequence of a deeper result. Lemma 3.5.22 and the actual formulation of Proposition 3.5.23 are taken from [537]. Lemma 3.5.24 is folklore, whereas Proposition 3.5.26 is due to Youngson [655] with a different proof. The unital forerunner of Proposition 3.5.26 is due to Edwards [222]. Results from Lemma 3.5.30 to Corollary 3.5.36 are built from [481] and [520]. It is worth mentioning that Proposition 3.5.31 answered affirmatively a question implicitly raised by Alvermann and Janssen [19], who also re-proved Theorem 3.5.34 as a consequence of a more general result.

With the exception of Remark 3.5.40 which is taken from [536], results from Lemma 3.5.37 to Proposition 3.5.44 are due to Kaidi, Morales, and Rodríguez [364]. Corollary 3.5.45 is earlier [514]. The associative (respectively, the unital non-associative) forerunner of Proposition 3.5.44 is due to Huruya [333] (respectively, to Rodríguez [514]). Twelve years before the paper [364] was published, the general statement of Proposition 3.5.44 was formulated in [340] as a straightforward consequence of its unital forerunner, of Theorem 3.5.34, and of the claim in [216] that the equality $N(X') = N(X)$ holds for every normed space X. However this argument had a gap because, as we commented in §2.1.47, the claim of [216] turned out to be false. By the way, now that we are provided with a correct proof of Proposition 3.5.44, it is enough to invoke in addition Theorem 3.5.34, together with the fact that the chain of inequalities $N(X) \geqslant N(X') \geqslant N(X'')$ is true for every normed space X, to get the following result, also pointed out in [364].

Corollary 3.5.69 *The equality $N(A') = N(A)$ holds for every non-commutative JB^*-algebra A.*

The results and methods in [364] inspired the following theorem of Becerra, Cowell, Rodríguez, and Wood [65].

Theorem 3.5.70 *Let A be a norm-unital complete normed associative algebra over $\mathbb{K}$, and let U stand for the set of all algebraically unitary elements of A* (cf. §2.1.20). *Then the following conditions are equivalent:*

(i) *A is unitary* (cf. §2.1.54).
(ii) *For every $f \in P(A)$ we have*

$$V(P(A), p_A, f) = \overline{\mathrm{co}}\left[\bigcup_{(u,v)\in U\times U} V(A, \mathbf{1}, u^{-1}f(u,v)v^{-1})\right].$$

(iii) *For every $T \in BL(A)$ we have*

$$V(BL(A), I_A, T) = \overline{\mathrm{co}}\left[\bigcup_{u\in U} V(A, \mathbf{1}, u^{-1}T(u))\right].$$

Moreover, if A is unitary, then p_A is both a strongly extreme point of $\mathbb{B}_{P(A)}$ and a point where the duality mapping of $P(A)$ is norm-norm upper semicontinuous, and if in addition $\mathbb{K} = \mathbb{C}$, then we have in fact $n(P(A), p_A) \geqslant \frac{1}{e}$, and hence p_A is a geometrically unitary element of $P(A)$ (compare Proposition 2.1.41).

Lemmas 3.5.46 and 3.5.48, as well as Corollary 3.5.47, are taken from [537]. Theorem 3.5.49 and Corollary 3.5.50 are due to Cabrera and Rodríguez [154].

Proposition 3.5.51 and the autonomous proof following it, as well as Theorem 3.5.53, Corollary 3.5.54, and Proposition 3.5.56, are due to Rodríguez [537]. Proposition 3.5.55 is a refined version of [537, Proposition 4.10]. Proposition 3.5.55 leads to the question of the existence of locally compact Hausdorff topological spaces not reduced to a point, and having no homeomorphism onto itself other than the identity mapping. According to [731, p. 196], the first example of such a (even compact) space is due to A. Pelczyński, who communicated it to W. J. Davis to be included at the end of [208]. For recent progresses on this topic, the reader is referred to [392]. Theorem 3.5.57 is claimed in [537, Remark 4.12(b)]. However, the hints for the proof given there are incorrect because they rely on [537, Remark 4.12(a)] (which has an unsolvable mistake) and on the proof of Theorem 2.8.23. Thus Theorem 3.5.57 and the proof given here are new.

Subsection 3.5.6 is a non-associative reading of Vowden's paper [629], incorporating and/or refining those results in Rickart's book [795] which were applied there. Thus, Lemma 3.5.59 is a part of [795, Theorem 4.2.2], whereas Proposition 3.5.61 becomes a non-associative version of [795, Theorem 4.1.3]. Proposition 3.5.61 and Corollary 3.5.64 are new. With Corollary 3.5.64 and Lemma 3.5.59 in mind, the proof of Proposition 3.5.65 is essentially that of [629, Lemma 3.1] (see also [725, Proposition 10.1]). Proposition 3.5.65 was proved first by Cabrera and Rodríguez [154] using different methods. A refinement of Proposition 3.5.65 will be obtained later in Proposition 4.4.70. Corollary 3.5.67 is the main result in [629], and is known as *Vowden's theorem*. Both its non-associative generalization, stated in Theorem 3.5.66, and the (conclusion of) proof given here are new.

For the long history of Vowden's theorem, the reader is referred to pages 6–8 of Doran–Belfi [725]. This book also includes in detail Vowden's arguments. Indeed, Vowden's theorem is derived in [725, Theorem 16.1] from the Glim–Kadison Corollary 3.5.18 by applying a suitable refinement of our Proposition 1.2.44 (see [725, Proposition 14.1] for details), also due to Vowden. Let us note that, in [725], the term C^*-algebra means a complete normed associative complex $*$-algebra satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every element a in the algebra.

Given a set I of identities over $\mathbb{K}$ (cf. §3.5.32), we define the *variety* of algebras over $\mathbb{K}$ determined by I as the class of all algebras over $\mathbb{K}$ satisfying all identities in I. Now let $\mathscr{V}$ be a variety of algebras over $\mathbb{K}$, let A be an algebra over $\mathbb{K}$ endowed with an algebra involution $*$, and let S be a non-empty subset of A. We say that S is $\mathscr{V}$*-normal* if the subalgebra of A generated by $S \cup S^*$ lies in $\mathscr{V}$. (Thus, normal subsets, as introduced in Definition 3.5.60, are nothing other than $\mathscr{V}$-normal subsets when $\mathscr{V}$ equals the variety of all associative and commutative algebras.) An element $a \in A$ is said to be $\mathscr{V}$*-normal* if $\{a\}$ is a $\mathscr{V}$-normal subset of A.

Let $\mathscr{V}$ be a variety of algebras. Looking at the proof of Proposition 3.5.61, we realize that it remains true whenever we replace 'normal subset' with '$\mathscr{V}$-normal subset' everywhere in its formulation and proof and, accordingly, we replace '[a certain algebra] is associative and commutative' with '[a certain algebra] lies in $\mathscr{V}$' in the proof. Therefore we have the following.

Proposition 3.5.71 *Let $\mathscr{V}$ be a variety of algebras over $\mathbb{K}$, and let A be a $*$-algebra over $\mathbb{K}$. We have:*

(i) *Each $\mathscr{V}$-normal subset of A is contained in a maximal $\mathscr{V}$-normal subset of A.*
(ii) *Maximal $\mathscr{V}$-normal subsets of A are $*$-subalgebras of A.*
(iii) *If A is normed, then maximal $\mathscr{V}$-normal subsets of A are closed.*

The above proposition implies the following.

Corollary 3.5.72 *Let $\mathscr{V}$ be a variety of algebras over $\mathbb{K}$, let A be a normed $*$-algebra over $\mathbb{K}$, and let S be a $\mathscr{V}$-normal subset of A. Then the closed $*$-subalgebra of A generated by S lies in $\mathscr{V}$.*

Interesting consequences along the lines of Corollary 3.5.64 are given in the following.

Corollary 3.5.73 *Let A be a normed $*$-algebra over $\mathbb{K}$. We have:*

(i) *If A is power-commutative, and if a is in $H(A,*)$, then the closed $*$-subalgebra of A generated by a is commutative.*
(ii) *If A is alternative, and if a is in A, then the closed $*$-subalgebra of A generated by a is associative.*

Proof Assume that A is power-commutative (respectively, alternative), and let a be in $H(A,*)$ (respectively, in A). Take $\mathscr{V}$ equal to the variety of all commutative (respectively, associative) algebras over $\mathbb{K}$, and note that the element a is $\mathscr{V}$-normal in a clear way (respectively, by Theorem 2.3.61). Then, by Corollary 3.5.72, the closed $*$-subalgebra of A generated by a is commutative (respectively, associative). □

The implication (iii)⇒(v) in Theorem 3.5.68 was proved first in [154] by involving Vowden's associative forerunner (stated in Corollary 3.5.67). Now, with the aid of Corollary 3.5.73 just proved, an easier proof of the same kind can be given, by arguing as follows:

> Let A be a complete normed alternative complex $*$-algebra satisfying $\|x^*x\| = \|x^*\|\|x\|$ for every $x \in A$, and let a be in A. By Corollary 3.5.73(ii), the closed $*$-subalgebra of A generated by a is associative. Finally, by Corollary 3.5.67, we have $\|a^*a\| = \|a\|^2$. □

Since there are quotients of special Jordan algebras which are not special (see for example [822, Theorem 3.2.5]), the class of all special Jordan algebras over $\mathbb{K}$ is not a variety of algebras over $\mathbb{K}$. Nevertheless, it can be enlarged to a minimum variety, namely that of the so-called *i-special Jordan algebras* over $\mathbb{K}$. These are defined as those Jordan algebras over $\mathbb{K}$ which satisfy all identities satisfied by all special Jordan algebras over $\mathbb{K}$. It is worth remarking that not all Jordan algebras are i-special. More precisely, the Albert Jordan algebra in Example 3.1.56 is not i-special (see [754, p. 51] or [777, Theorem B.5.3]). Now, as a last consequence of Corollary 3.5.72, we can prove the following.

Corollary 3.5.74 *Let A be a normed Jordan $*$-algebra over $\mathbb{K}$, and let a be in A. Then the closed $*$-subalgebra of A generated by a is i-special.*

Proof Take $\mathscr{V}$ equal to the variety of all i-special algebras over $\mathbb{K}$, and note that, by Theorem 3.1.55, the element a is $\mathscr{V}$-normal. Then, by Corollary 3.5.72, the closed $*$-subalgebra of A generated by a lies in $\mathscr{V}$. □

3.6 Jordan axioms for C^*-algebras

Introduction We begin Subsection 3.6.1 by introducing modular ideals and the unit-free version of the strong radical of an algebra, and by proving in Theorems 3.6.7 and 3.6.9 the unit-free versions of Rickart's dense-range-homomorphism and Gelfand's homomorphism theorems. (The unital forerunners of these results were proved in Theorems 1.1.62 and 1.1.75.) Then we introduce primitive ideals and the radical (called the Jacobson radical in the associative setting) of an algebra. According to Proposition 3.6.14, the strong radical is included in the radical, and the inclusion becomes an equality in the commutative setting. The subsection is concluded by introducing quasi-invertible elements and quasi-invertible subsets of an alternative algebra, and by proving in Theorem 3.6.21 that the Jacobson radical of any associative algebra A is the largest quasi-invertible ideal of A.

Subsection 3.6.2 contains the main result in the section. Indeed, we prove in Theorem 3.6.30 that a norm and a conjugate-linear involution on the vector space of a complex associative algebra A convert A into a C^*-algebra if (and only if) they convert the Jordan algebra A^{sym} into a JB^*-algebra. We note that the 'only if' part of the statement just formulated follows from Facts 3.3.2 and 3.3.4.

Subsection 3.6.3 contains some complements to Jacobson's representation theory of associative algebras, to be applied later. Our development includes several applications of Jacobson's representation theory to normed algebras. To this end, we introduce associative normed Q-algebras, which are (possibly non-complete) normed associative algebras that behave spectrally like complete normed associative algebras. As we will realize in Subsection 4.4.4, as well as at the appropriate place in Volume 2 (where the results of [523] will be developed), associative normed Q-algebras become crucial in the treatment of some relevant topics in the theory of complete normed non-associative algebras.

3.6.1 Jacobson's representation theory: preliminaries

Let A be an algebra over $\mathbb{K}$. Given $a \in A$, we consider the subsets $A(\mathbb{1}-a)$ and $(\mathbb{1}-a)A$ of A defined by

$$A(\mathbb{1}-a) := \{x - xa : x \in A\} \quad \text{and} \quad (\mathbb{1}-a)A := \{x - ax : x \in A\}.$$

(The notation subliminally uses the unital extension $A_{\mathbb{1}}$ of A.) An element $e \in A$ is called a *left* (respectively, *right*) *modular unit* for a subspace X of A if $(\mathbb{1}-e)A \subseteq X$ (respectively, $A(\mathbb{1}-e) \subseteq X$). A left ideal I of A is called a *modular left ideal* of A if there exists a right modular unit for I in A. Analogously, a right ideal I of A is called a *modular right ideal* of A if there exists a left modular unit for I in A. A (two-sided) ideal I of A is said to be a *modular ideal* of A if it is both a modular left ideal and a modular right ideal of A. If I is a modular ideal of A, then left and right modular units for I in A are the same, and therefore they are called (two-sided) *modular units* for I. (Indeed, if e is a left modular unit for the ideal I, and if f is a right modular unit for I, then $f - ef \in I$ and $e - ef \in I$, hence $e - f \in I$, so that for every $x \in A$ we have $x - xe = (x - xf) + x(f - e) \in I + I \subseteq I$ and $x - fx = (x - ex) + (e - f)x \in I + I \subseteq I$.)

The next result is straightforward.

Fact 3.6.1 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *If A is unital, then every* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal I of A is modular, and* $\mathbf{1}$ *is a* $\left\{\begin{matrix}\textit{right}\\ \textit{left}\\ \textit{two-sided}\end{matrix}\right\}$ *modular unit for I.*

(ii) *If e is a* $\left\{\begin{matrix}\textit{right}\\ \textit{left}\\ \textit{two-sided}\end{matrix}\right\}$ *modular unit for a* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal I of A, then it is a* $\left\{\begin{matrix}\textit{right}\\ \textit{left}\\ \textit{two-sided}\end{matrix}\right\}$ *modular unit for any* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal of A containing I.*

(iii) *If e is a* $\left\{\begin{matrix}\textit{right}\\ \textit{left}\\ \textit{two-sided}\end{matrix}\right\}$ *modular unit for a proper* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal I of A, then $e \notin I$.*

(iv) *A proper ideal I of A is modular if and only if the quotient algebra A/I is unital.*

The next result generalizes Proposition 1.1.49.

Proposition 3.6.2 *Let A be a complete normed algebra over $\mathbb{K}$. Then the closure of a proper modular* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal of A is a proper modular* $\left\{\begin{matrix}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{matrix}\right\}$ *ideal of A.*

Proof Assume for example that I is a modular left ideal of A. Then, by Exercise 1.1.48 and Fact 3.6.1(ii), $\bar{I}$ is a modular left ideal of A. Now suppose that $\bar{I} = A$. Take a right modular unit e for I, and choose $b \in I$ such that $\|e-b\| < 1$. Since

$$\|R_e - R_b\| = \|R_{e-b}\| \leqslant \|e-b\| < 1,$$

it follows from Corollary 1.1.21(i) that $I_A - (R_e - R_b)$ is invertible in $BL(A)$, hence $I_A - (R_e - R_b)$ is bijective, and so

$$A = [I_A - (R_e - R_b)](A) \subseteq (I_A - R_e)(A) + R_b(A) = A(\mathbb{1} - e) + Ab \subseteq I. \qquad \square$$

Let A be an algebra over $\mathbb{K}$. By a *maximal modular* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal* of A we mean a modular $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A which is a maximal $\left\{\begin{array}{c}\text{left}\\ \text{right}\\ \text{two-sided}\end{array}\right\}$ ideal of A (cf. Definition 1.1.51). Keeping in mind Fact 3.6.1(ii)–(iii), an easy application of Zorn's lemma yields the following.

Fact 3.6.3 *Let A be an algebra over $\mathbb{K}$. Then every proper modular* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A is contained in a maximal modular* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A.*

As a straightforward consequence of Proposition 3.6.2, we get the following.

Corollary 3.6.4 *Let A be a complete normed algebra over $\mathbb{K}$. Then every maximal modular* $\left\{\begin{array}{c}\textit{left}\\ \textit{right}\\ \textit{two-sided}\end{array}\right\}$ *ideal of A is closed.*

The next lemma, which generalizes Lemma 1.1.54, follows straightfor- wardly from Fact 3.6.1(iv).

Lemma 3.6.5 *Let A be an algebra over $\mathbb{K}$, and let M be a maximal modular ideal of A. Then the quotient algebra A/M is a unital simple algebra.*

In Definition 1.1.61, we introduced the strong radical of any unital algebra. Now, keeping in mind that in the unital case maximal ideals and maximal modular ideals are the same, we can extend the notion of strong radical to (possibly non-unital) algebras as follows.

Definition 3.6.6 Let A be an algebra over $\mathbb{K}$. The *strong radical* of A, denoted by s-Rad(A), is defined as the intersection of all maximal modular ideals of A (with the usual convention that s-Rad$(A) = A$ if maximal modular ideals of A do not exist). The algebra A is called *strongly semisimple* if s-Rad$(A) = 0$. We note that, in view of Fact 3.6.3, s-Rad$(A) \neq A$ if and only if there exists some proper modular ideal of A.

Now, arguing as in the proof of Theorem 1.1.62, with 'maximal modular ideal' instead of 'maximal ideal', Lemma 3.6.5 instead of Lemma 1.1.54, and Corollary 3.6.4 instead of Corollary 1.1.53, we obtain the following non-unital version of Rickart's dense range homomorphism theorem.

Theorem 3.6.7 *Dense range algebra homomorphisms, from complete normed complex algebras to complete normed strongly semisimple associative complex algebras, are automatically continuous.*

A wide generalization of the above theorem will be proved later (see Theorem 4.1.19).

Arguing as in the proof of Corollary 1.1.63, with Theorem 3.6.7 instead of Theorem 1.1.62, we get the following non-unital version of Rickart's uniqueness-of-norm theorem.

Corollary 3.6.8 *Strongly semisimple associative complex algebras have at most one complete algebra norm topology.*

Let A be an algebra over $\mathbb{K}$. Then the kernels of characters on A are maximal modular ideals of A. Indeed, if φ is a character on A, and if we take $e \in A$ such that $\varphi(e) = 1$, then e becomes a modular unit for $\ker(\varphi)$. Now assume that $\mathbb{K} = \mathbb{C}$, and that A is in fact a complete normed associative and commutative algebra. Then, keeping in mind Corollary 3.6.4 and Lemma 3.6.5, minor changes to the arguments leading to Proposition 1.1.68(i) allow us to realize that, actually, the mapping $\varphi \to \ker(\varphi)$ defines a bijection from the set of all characters on A onto the set of all maximal modular ideals of A. As a consequence, the strong radical of A equals the intersection of kernels of characters on A. This last fact will be applied without notice in the proof of the following non-unital version of Theorem 1.1.75.

Theorem 3.6.9 (Gelfand homomorphism theorem) *Let A be a complete normed complex algebra, let B be a complete normed strongly semisimple associative and commutative complex algebra, and let $\Phi : A \to B$ be an algebra homomorphism. Then Φ is continuous.*

Proof Let a_n be a sequence in A with $a_n \to 0$ and $\Phi(a_n) \to b \in B$. In view of the closed graph theorem, to prove that Φ is continuous it is enough to show that $b = 0$. Let φ be a character on B. Then, by Corollary 1.1.108, both $\varphi : B \to \mathbb{C}$ and $\varphi \circ \Phi : A \to \mathbb{C}$ are continuous mappings. Therefore we have $0 \leftarrow (\varphi \circ \Phi)(a_n) = \varphi(\Phi(a_n)) \to \varphi(b)$, and hence $\varphi(b) = 0$. Since φ is an arbitrary character on B, and B is strongly semisimple, we conclude that $b = 0$, as desired. □

Now we are going to show how the study of the strong radical of a non-commutative JB^*-algebra can be reduced to the commutative case.

Lemma 3.6.10 *Let A be a flexible algebra over $\mathbb{K}$ such that A^{sym} is unital. Then A is unital.*

Proof Let $\mathbf{1}$ stand for the unit of A^{sym}, and let a be in A. Since the mapping $x \to [a,x]$ is a derivation of A^{sym} (by Lemma 2.4.15), we have $[a,\mathbf{1}] = 0$. Therefore

$$a\mathbf{1} = a \bullet \mathbf{1} + \frac{1}{2}[a,\mathbf{1}] = a \text{ and } \mathbf{1}a = \mathbf{1} \bullet a - \frac{1}{2}[a,\mathbf{1}] = a. \qquad \square$$

Proposition 3.6.11 *Let A be a non-commutative JB^*-algebra. Then we have:*

(i) *Closed ideals of A and closed ideals of A^{sym} are the same.*

(ii) *Maximal modular ideals of A and maximal modular ideals of A^{sym} are the same, hence*

$$\text{s-Rad}(A) = \text{s-Rad}(A^{\mathrm{sym}}).$$

Proof Let I be a closed ideal of A^{sym}. Then, by Fact 3.3.4 and Proposition 3.4.13, I is a JB^*-algebra. Therefore, by Propositions 3.4.1(ii) and 1.2.48, I is the linear hull of the set $\{x^2 : x \in I\}$, hence $I \bullet I = I$, and so I remains invariant under every derivation of A^{sym}. It follows from Lemma 2.4.15 that $[A,I] \subseteq I$, and consequently $AI + IA \subseteq I$, i.e. I is an ideal of A. Since, clearly, closed ideals of A are closed ideals of A^{sym}, assertion (i) has been proved.

Now let I be a maximal modular ideal of A. Then, clearly, I is a modular ideal of A^{sym}. Therefore, by Fact 3.6.3, there exists a maximal modular ideal J of A^{sym} containing I, and, by Corollary 3.6.4 and assertion (i) just proved, J is an ideal of A. Therefore $J = I$ by maximality of I, and hence I is a maximal modular ideal of A^{sym}. Conversely, let I be a maximal modular ideal of A^{sym}. Then, by Corollary 3.6.4 and assertion (i), I is an ideal of A. Keeping in mind Fact 3.6.1(iv), that $A^{\mathrm{sym}}/I = (A/I)^{\mathrm{sym}}$, and that A/I is a flexible algebra, it follows from Lemma 3.6.10 that A/I is a unital algebra, and hence that I is a modular ideal of A. Therefore, according to Fact 3.6.3, there exists a maximal modular ideal J of A containing I. Since J is a proper ideal of A^{sym}, we deduce that $J = I$ by maximality of I, and hence that I is a maximal modular ideal of A. Thus the proof of assertion (ii) is complete. □

In the above proof, Lemma 3.6.10 could have been replaced with Proposition 3.4.13 and Corollary 2.6.36.

Definition 3.6.12 Let A be an algebra over $\mathbb{K}$. Given a subspace X of A, the *core* of X (in A) is defined as the largest ideal of A contained in X. *Primitive ideals* of A are defined as the cores of maximal modular left ideals of A. The *radical* of A, denoted by $\mathrm{Rad}(A)$, is defined as the intersection of all primitive ideals of A (with the usual convention that $\mathrm{Rad}(A) = A$ if primitive ideals of A do not exist). The algebra A is called *semisimple* if $\mathrm{Rad}(A) = 0$, and *radical* if $\mathrm{Rad}(A) = A$. We note that, in view of Fact 3.6.3, A is non-radical if and only if there exists some proper modular left ideal of A. An outstanding case of semisimplicity occurs when A is a *primitive algebra*, which means that $A \neq 0$ and that zero is a primitive ideal of A.

Most radicals for algebras are named after the mathematicians who introduced them and/or showed their usefulness. Thus, for example, the strong radical is called the Brown–McCoy radical. Nevertheless, when we deal with a (possibly non-associative) algebra A, the radical of A (as defined above) will not enjoy any author name in our work, and will be called the *Jacobson radical* of A only when A is associative. Several reasons justify this convention. Indeed, for alternative algebras, our unnamed radical is known in the literature as the Kleinfeld radical (see §3.6.58 below). Moreover, for Jordan-admissible algebras, a 'Jacobson radical' different from our unnamed radical, but coinciding with it in the associative case, is well understood in the literature (see Definition 4.4.12 below).

As a straightforward consequence of Corollary 3.6.4, we derive the following.

Corollary 3.6.13 *Let A be a complete normed algebra over $\mathbb{K}$. Then primitive ideals of A, the strong radical of A, and the radical of A are closed subsets of A.*

Proposition 3.6.14 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *Every maximal modular ideal of A is a primitive ideal of A, and hence* $\mathrm{Rad}(A) \subseteq \text{s-Rad}(A)$.
(ii) *If A is commutative, then every primitive ideal of A is a maximal modular ideal of A, and hence* $\mathrm{Rad}(A) = \text{s-Rad}(A)$.

Proof Let I be a maximal modular ideal of A. Then, by Fact 3.6.3, there exists a maximal modular left ideal J of A containing I, and the core of J in A must be equal to I by maximality of I as an ideal. This proves assertion (i).

Assume that A is commutative. Then, since left ideals of A are ideals of A, the core of each left ideal I must coincide with the whole I. Therefore, primitive ideals of A are maximal modular ideals of A. This proves assertion (ii). □

In view of assertion (ii) in the above proposition, strongly semisimple commutative algebras will be called simply semisimple (commutative) algebras.

§3.6.15 Let A be an algebra over $\mathbb{K}$. By a *semiprime ideal* of A we mean an ideal I of A such that, whenever J is any ideal of A with $JJ \subseteq I$, we have $J \subseteq I$. Thus semiprime ideals of A are precisely those ideals I of A such that A/I is a semiprime algebra (cf. §2.2.22), and consequently A is a semiprime algebra if and only if zero is a semiprime ideal of A. By a *prime ideal* of A we mean an ideal P of A such that, whenever I and J are ideals of A with $IJ \subseteq P$, we have $I \subseteq P$ or $J \subseteq P$. Thus prime ideals of A are precisely those ideals P of A such that A/P is a prime algebra (cf. §2.5.41), and consequently A is a prime algebra if and only if zero is a prime ideal of A.

Proposition 3.6.16 *Let A be an algebra over $\mathbb{K}$. Then we have:*

(i) *Primitive ideals of A are prime ideals of A.*
(ii) $\mathrm{Rad}(A)$ *is a semiprime ideal of A.*

Proof Let P be a primitive ideal of A, let I and J be ideals of A with $IJ \subseteq P$, and assume that $J \nsubseteq P$. To prove assertion (i) we must show that $I \subseteq P$. Take a maximal modular left ideal M of A such that P is the core of M in A. Since J is not contained in P, J cannot be contained in M, and hence $J + M = A$ by maximality of M. Take $x \in J$ and $y \in M$ such that $x + y$ is a right modular unit for M. Then x is also a right modular unit for M, and hence for every $z \in I$ we have $z = (z - zx) + zx \in M + IJ \subseteq M + P \subseteq M$. Thus $I \subseteq M$, which implies $I \subseteq P$, as desired.

Since prime ideals of A are semiprime ideals of A, and the intersection of any family of semiprime ideals of A is a semiprime ideal of A, assertion (ii) follows from assertion (i) and the definition of $\mathrm{Rad}(A)$. □

Lemma 3.6.17 *Let A be an algebra over $\mathbb{K}$, and let I be a left ideal of A such that for each $x \in I$ there is $a \in A$ satisfying $x + a - ax = 0$. Then I is contained in every maximal modular left ideal of A.*

Proof Assume, to derive a contradiction, that M is a maximal modular left ideal of A such that I is not contained in M. Then $I + M = A$ by maximality of M. Take

$x \in I$ and $y \in M$ such that $x+y$ is a right modular unit for M. Then x is also a right modular unit for M. Let a be in A such that $x+a-ax=0$. We have $x=ax-a \in M$, contradicting Fact 3.6.1(iii). □

The following proposition follows from Lemma 3.6.17 just proved and the definition of the radical.

Proposition 3.6.18 *Let A be an algebra over $\mathbb{K}$, and let I be an ideal of A such that for each $x \in I$ there is $a \in A$ satisfying $x+a-ax=0$. Then $I \subseteq \mathrm{Rad}(A)$.*

It is straightforward that the unital extension of an alternative algebra is an alternative algebra. This fact will be applied without notice.

Definition 3.6.19 Let A be an alternative algebra over $\mathbb{K}$, and let a be in A. An element $b \in A$ is said to be a *quasi-inverse* of a in A if

$$a+b-ab=a+b-ba=0.$$

It is straightforward that b is a quasi-inverse of a in A if and only if $\mathbb{1}-b$ is an inverse of $\mathbb{1}-a$ in the unital extension $A_{\mathbb{1}}$ of A. Therefore a has at most one quasi-inverse in A (cf. Proposition 2.5.24(i)). If a has a quasi-inverse in A, then we say that a is a *quasi-invertible* element of A, and its unique quasi-inverse will be denoted by $a^{\diamond}$. It is also straightforward that, if A is unital, then a is quasi-invertible in A if and only if $\mathbf{1}-a$ is invertible in A, and that, if this is the case, then we have $(\mathbf{1}-a)^{-1}=\mathbf{1}-a^{\diamond}$. A subset S of A is said to be *quasi-invertible* if all elements of S are quasi-invertible in A.

Lemma 3.6.20 *Let A be an associative algebra over $\mathbb{K}$, and let I be a left ideal of A such that for each $x \in I$ there is $a \in A$ satisfying $x+a-ax=0$. Then I is a quasi-invertible subset of A.*

Proof Let x be in I. Take $a \in A$ such that $x+a-ax=0$. Then $a=ax-x$ lies in I, and hence there exists $b \in A$ such that $a+b-ba=0$. Now in $A_{\mathbb{1}}$ we have $(\mathbb{1}-a)(\mathbb{1}-x)=\mathbb{1}$ and $(\mathbb{1}-b)(\mathbb{1}-a)=\mathbb{1}$. Therefore, by Lemma 1.1.59, $\mathbb{1}-a$ is invertible in $A_{\mathbb{1}}$. But then $\mathbb{1}-x=(\mathbb{1}-a)^{-1}$ is also invertible in $A_{\mathbb{1}}$, and hence x is quasi-invertible in A. Since x is arbitrary in I, the result follows. □

Theorem 3.6.21 *Let A be an associative algebra over $\mathbb{K}$. Then* $\mathrm{Rad}(A)$ *is the largest quasi-invertible ideal of A.*

Proof By Proposition 3.6.18, all quasi-invertible ideals of A are contained in $\mathrm{Rad}(A)$. Therefore, since $\mathrm{Rad}(A)$ is an ideal of A, to conclude the proof it is enough to show that $\mathrm{Rad}(A)$ is a quasi-invertible subset of A. Let y be in A such that there is no $a \in A$ satisfying $y+a-ay=0$. Then $y \notin A(\mathbb{1}-y)$ and y becomes a right modular unit for $A(\mathbb{1}-y)$. On the other hand, since A is associative, $A(\mathbb{1}-y)$ is a left ideal of A. It follows from Facts 3.6.3 and 3.6.1(ii)–(iii) that there exists a maximal modular left ideal M of A such that $y \notin M$. Then, clearly, y cannot belong to the core of M in A, and hence $y \notin \mathrm{Rad}(A)$ (cf. Definition 3.6.12). Therefore, for each $x \in \mathrm{Rad}(A)$, there must exist $a \in A$ such that $x+a-ax=0$. But, by Lemma 3.6.20, this implies that $\mathrm{Rad}(A)$ is a quasi-invertible subset of A, as desired. □

Let A be an algebra over $\mathbb{K}$. There is a lack of symmetry in the definition of $\mathrm{Rad}(A)$ in that it is defined as the intersection of cores of maximal modular *left* ideals of A. Thus, in general, we can have $\mathrm{Rad}(A^{(0)}) \neq \mathrm{Rad}(A)$, where $A^{(0)}$ stands for the opposite algebra of A (cf. §1.1.36). Even the extreme situation that A is a radical algebra and $A^{(0)}$ is a primitive algebra may happen, as one can easily realize by choosing A equal to the real algebra obtained from $\mathbb{C}$ after replacing its usual product with the one $\odot$ defined by $\lambda \odot \mu := \bar{\lambda}\mu$ (see Corollary 3.6.60 below for details). Nevertheless, algebras such as those in the above example cannot be associative. Indeed, noticing that (for any choice of A) ideals of A and of $A^{(0)}$ are the same, and that in the associative case quasi-invertible elements (and hence quasi-invertible subsets) of A and of $A^{(0)}$ are the same, Theorem 3.6.21 just proved implies the following.

Corollary 3.6.22 *Let A be an associative algebra over $\mathbb{K}$. Then*

$$\mathrm{Rad}(A^{(0)}) = \mathrm{Rad}(A).$$

Another relevant consequence of Theorem 3.6.21 is the following.

Corollary 3.6.23 *Let A be a normed associative algebra over $\mathbb{K}$, and let x be in* $\mathrm{Rad}(A)$. *Then* $\mathfrak{r}(x) = 0$.

Proof Suppose first that $\mathbb{K} = \mathbb{C}$. Then, by Theorem 3.6.21, $\frac{x}{\lambda}$ is quasi-invertible in A for every $\lambda \in \mathbb{C} \setminus \{0\}$, and hence $\mathrm{sp}(A_{\mathbb{1}}, x) \subseteq \{0\}$. A fortiori, the spectrum of x relative to the completion of $A_{\mathbb{1}}$ has no nonzero element, and therefore, by Theorem 1.1.46, we have $\mathfrak{r}(x) = 0$, as desired.

Suppose now that $\mathbb{K} = \mathbb{R}$, and let α, β be in $\mathbb{R}$ such that $\lambda := \alpha + i\beta \neq 0$. Then, since $\frac{2\alpha x - x^2}{|\lambda|^2}$ lies in $\mathrm{Rad}(A)$, Theorem 3.6.21 applies again to obtain that $\frac{2\alpha x - x^2}{|\lambda|^2}$ is quasi-invertible in A, and hence that

$$(x - \alpha\mathbb{1})^2 + \beta^2\mathbb{1} \ \ (= |\lambda|^2\mathbb{1} - 2\alpha x + x^2)$$

is invertible in $A_{\mathbb{1}}$ (so also in the completion of $A_{\mathbb{1}}$). It follows from Corollary 1.1.101 that $\mathfrak{r}(x) = 0$. □

We stop our development of Jacobson's representation theory for the moment because we are already provided with enough information about it in order to attack the proof of the main result in this section.

3.6.2 The main result

Let A be an associative complex algebra, and let $*$ and $\|\cdot\|$ be a conjugate-linear involution and a norm, respectively, on (the vector space of) A. If one wants $(A, *, \|\cdot\|)$ to be a C^*-algebra, then, according to the current axioms for C^*-algebras, both $*$ and $\|\cdot\|$ have to satisfy requirements involving the associative product of A. As the main result, we will prove in Theorem 3.6.30 how the fact that $(A, *, \|\cdot\|)$ being a C^*-algebra can be characterized in terms involving only the behaviour of $*$ and $\|\cdot\|$ on the Jordan algebra A^{sym}.

Lemma 3.6.24 *Let A be a non-commutative JB^*-algebra, and let h be in $H(A, *)$. Then L_h and R_h belong to $H(BL(A), I_A)$.*

Proof Since $A_{\mathbb{1}}$ is a non-commutative JB^*-algebra in a natural manner (cf. Corollary 3.5.36), and $h \in H(A_{\mathbb{1}}, *)$, we have $L_h^{A_{\mathbb{1}}}, R_h^{A_{\mathbb{1}}} \in H(BL(A_{\mathbb{1}}), I_{A_{\mathbb{1}}})$ (by Lemmas 2.2.8(iii) and 2.1.10). Therefore, by Lemma 2.2.24, L_h^A and R_h^A lie in $H(BL(A), I_A)$. □

Given an algebra A over $\mathbb{K}$ and a subset S of A, we set

$$L_S := \{L_x : x \in S\} \text{ and } R_S := \{R_x : x \in S\}.$$

Theorem 3.6.25 *Let A be an alternative complex $*$-algebra, and assume that A^{sym} is a JB^*-algebra for a suitable norm $\|\cdot\|$ and the given involution $*$. Then A, with the norm $\|\cdot\|$ and the involution $*$, is an alternative C^*-algebra.*

Proof In view of Facts 3.3.2 and 3.3.4, it is enough to show that $\|\cdot\|$ is an algebra norm on A. For $a \in A$, let D_a denote the linear operator on A defined by $D_a(b) := [a,b]$, and let X stand for the complex Banach space consisting of the vector space of A (equal to that of A^{sym}) and the norm $\|\cdot\|$. Let h be in $H(A,*)$. Then, since A is a flexible algebra, and $*$ is a conjugate-linear algebra involution on A, it follows from Lemma 2.4.15 that D_h is a derivation of A^{sym} satisfying $D_h(b^*) = -D_h(b)^*$ for every $b \in A$. Therefore, since $(A^{\mathrm{sym}}, \|\cdot\|, *)$ is a JB^*-algebra, Lemma 3.4.27 applies, so that we have $D_h \in H(BL(X), I_X)$. On the other hand, by Lemma 3.6.24, we also have $L_h^{A^{\mathrm{sym}}} \in H(BL(X), I_X)$. It follows that

$$L_h = L_h^{A^{\mathrm{sym}}} + \frac{1}{2} D_h \in H(BL(X), I_X). \tag{3.6.1}$$

As a by-product, since $A = H(A,*) + iH(A,*)$, we have that $L_a \in BL(X)$ for every $a \in A$. Now, since A is an alternative algebra, the mapping $a \to L_a$ from A to $BL(X)$ is a Jordan homomorphism, and so the closure (say J) in $BL(X)$ of the set $\mathbb{C}I_X + L_A$ is a closed subalgebra of $BL(X)^{\mathrm{sym}}$. Moreover, by Corollary 2.1.2 and (3.6.1) we have $H(J, I_X) \supseteq \mathbb{R}I_X + L_{H(A,*)}$, so

$$H(J, I_X) + iH(J, I_X) \supseteq \mathbb{C}I_X + L_A,$$

and so $H(J, I_X) + iH(J, I_X)$ is dense in J. But, by Proposition 2.1.11 and Lemma 3.3.14, $H(J, I_X) + iH(J, I_X)$ is closed in J, so actually we have

$$J = H(J, I_X) + iH(J, I_X).$$

Therefore, by the non-associative Vidav–Palmer theorem (Theorem 3.3.11), J is a JB^*-algebra in a natural way. Now, by (3.6.1), the mapping $a \to L_a$ is an algebra $*$-homomorphism from the JB^*-algebra $(A^{\mathrm{sym}}, \|\cdot\|, *)$ to the JB^*-algebra J, and hence, by Proposition 3.4.4, we have $\|L_a\| \leqslant \|a\|$ for every $a \in A$ or, equivalently, $\|ab\| \leqslant \|a\|\|b\|$ for all $a, b \in A$. Thus $\|\cdot\|$ is an algebra norm on A, as desired. □

Now we are going to refine Theorem 3.6.25 above in the case where the alternative algebra A is actually associative. To this end, we begin by revisiting Lemma 2.3.71 and Proposition 3.4.55 in Lemmas 3.6.26 and 3.6.27 immediately below.

Lemma 3.6.26 *Let A be a norm-unital normed associative complex algebra, and let a be in A such that $a = h + ik$ for commuting elements $h, k \in H(A, \mathbf{1})$. Then $\|a\| \leqslant 2\mathfrak{r}(a)$.*

Proof Since h and k remain hermitian in the completion of A (cf. Corollary 2.1.2), we may assume that A is complete. Then, by Lemma 2.3.71, we have $V(a) = \mathrm{co}(\mathrm{sp}(a))$, and hence, by Theorem 1.1.46, $v(a) = \mathfrak{r}(a)$. Now the result follows from Corollary 2.3.5(iii). $\square$

Lemma 3.6.27 *Let A be a normed non-commutative Jordan complex algebra such that there exists $M > 0$ satisfying $\|a\| \leqslant M\mathfrak{r}(a)$ for every $a \in A$. Then A is associative and commutative.*

Proof For every $a \in A$ we have $\|a\|^2 \leqslant M^2\mathfrak{r}(a)^2 = M^2\mathfrak{r}(a^2) \leqslant M^2\|a^2\|$. Now apply Proposition 3.4.55. $\square$

Lemma 3.6.28 *Let A be an associative complex algebra endowed with a conjugate-linear involution $*$ satisfying $(ab)^* = a^*b^*$ for all $a, b \in A$, and assume that A^{sym} is a JB^*-algebra for some norm $\|\cdot\|$ and the given involution $*$. Then A is commutative.*

Proof With the notation and arguments at the beginning of the proof of Theorem 3.6.25, for $h \in H(A, *)$ we have (as there) that $L_h = L_h^{A^{\mathrm{sym}}} + \frac{1}{2}D_h$ and $L_h^{A^{\mathrm{sym}}} \in H(BL(X), I_X)$, but now $iD_h \in H(BL(X), I_X)$. Let a be in A, and write $a = h + ik$ with $h, k \in H(A, *)$. Then we have $L_a = L_h + iL_k = F + iG$, where

$$F := L_h^{A^{\mathrm{sym}}} + \tfrac{i}{2}D_k \in H(BL(X), I_X) \ \text{ and } \ G := L_k^{A^{\mathrm{sym}}} - \tfrac{i}{2}D_h \in H(BL(X), I_X).$$

Moreover, since $F = \frac{1}{2}(L_a + R_{a^*})$, and $G = \frac{1}{2i}(L_a - R_{a^*})$, and A is associative, we get that $FG = GF$, and hence, by Lemma 3.6.26, we have $\|L_a\| \leqslant 2\mathfrak{r}(L_a)$. On the other hand, the associativity of A implies that L_A is a subalgebra of $BL(X)$. Since a is arbitrary in A, it follows from Lemma 3.6.27 that L_A is a commutative algebra. Now, since the mapping $x \to L_x$ is an algebra homomorphism from A to L_A, the desired commutativity of A will follow as soon as we prove that this homomorphism has zero kernel. But, if x is in A with $L_x = 0$, then $U_x^{A^{\mathrm{sym}}}(x^*) = xx^*x = L_x(x^*x) = 0$, so $\|x\|^3 = \|U_x^{A^{\mathrm{sym}}}(x^*)\| = 0$ because $(A^{\mathrm{sym}}, \|\cdot\|, *)$ is a JB^*-algebra, and so $x = 0$. $\square$

The last ingredient for the proof of the associative refinement of Theorem 3.6.25 is the following celebrated result of Herstein [320, Theorem H] (included also in [743, Theorem 3.1]), whose proof will not be discussed here.

Proposition 3.6.29 *Surjective Jordan homomorphisms from associative algebras over $\mathbb{K}$ to associative prime algebras over $\mathbb{K}$ are either algebra homomorphisms or algebra antihomomorphisms.*

Now the main result in this subsection reads as follows.

Theorem 3.6.30 *Let A be an associative complex algebra, and assume that A^{sym} is a JB^*-algebra for a suitable norm $\|\cdot\|$ and a suitable involution $*$. Then A, endowed with the norm $\|\cdot\|$ and the involution $*$, becomes a C^*-algebra.*

Proof In view of Theorem 3.6.25, it is enough to show that $*$ is an algebra involution on A.

Assume first that A is prime. Then, since $*$ is a conjugate-linear algebra involution on A^{sym}, the mapping $x \to x^*$ from $A_{\mathbb{R}}$ to $A_{\mathbb{R}}$ is a surjective Jordan homomorphism. Therefore, by Proposition 3.6.29, it is either an algebra homomorphism or an algebra antihomomorphism. In both cases, keeping in mind Lemma 3.6.28, $*$ becomes an algebra involution on A, as desired.

Now remove the assumption that A is prime. Let a be in A. With the notation in the proof of Theorem 3.6.25, we have

$$L_a = L_a^{A^{\mathrm{sym}}} + \tfrac{1}{2}D_a \ \text{ and } \ R_a = L_a^{A^{\mathrm{sym}}} - \tfrac{1}{2}D_a.$$

On the other hand, since D_a is a derivation of A^{sym}, Lemma 3.4.26 applies, so that we have $D_a \in BL(X)$. Since clearly $L_a^{A^{\mathrm{sym}}} \in BL(X)$, it follows that $L_a, R_a \in BL(X)$. Since a is arbitrary in A, we realize that the product of A is separately $\|\cdot\|$-continuous, and hence, by Proposition 1.1.9, there exists $M > 0$ such that $M\|\cdot\|$ becomes a complete algebra norm on A. Let x be in $\mathrm{Rad}(A)$. Then, $x^* \bullet x$ lies in $\mathrm{Rad}(A)$, and hence, by Corollaries 3.6.23 and 1.1.18(i), we have $\lim M^{\frac{1}{n}}\|(x^* \bullet x)^n\|^{\frac{1}{n}} = 0$, so $\lim \|(x^* \bullet x)^n\|^{\frac{1}{n}} = 0$. But, on the other hand, we have $\|(x^* \bullet x)^n\| = \|x^* \bullet x\|^n$ for every $n \in \mathbb{N}$ because $x^* \bullet x$ is a self-adjoint element of the JB^*-algebra A^{sym}, and Proposition 3.4.1(ii) applies. Therefore $x^* \bullet x = 0$, and hence, by (the commutative version of) Lemma 3.4.65, $x = 0$. Since x is arbitrary in $\mathrm{Rad}(A)$, the above shows that A is semisimple. Now let P be any primitive ideal of A. Since $M\|\cdot\|$ is a complete algebra norm on A, Corollary 3.6.13 applies, so that P is $\|\cdot\|$-closed in A, hence P is a closed ideal of the JB^*-algebra $(A^{\mathrm{sym}}, \|\cdot\|, *)$. By Proposition 3.4.13, P is $*$-invariant, and A^{sym}/P is a JB^*-algebra for the quotient norm and the quotient involution. Thus, since $(A/P)^{\mathrm{sym}} = A^{\mathrm{sym}}/P$, we realize that, replacing $\|\cdot\|$ and $*$ with the corresponding quotient norm and quotient involution, A/P satisfies the properties required for A in the statement of the theorem, with the advantage now, thanks to Proposition 3.6.16(i), that A/P is a prime algebra. According to the prime case already considered, the quotient involution on A/P is an algebra involution, i.e. we have $(ab)^* - b^*a^* \in P$ for all $a, b \in A$. Since P is an arbitrary primitive ideal of A, and A is semisimple, we conclude that $*$ is an algebra involution on A, as desired. □

Remark 3.6.31 The associative refinement of Theorem 3.6.25 given by Theorem 3.6.30 does not remain true for general alternative algebras. Indeed, let B be an alternative C^*-algebra, let F be a Jordan automorphism of B, and let A stand for the (alternative) complex algebra consisting of the vector space of B and the product $x \odot y := F^{-1}(F(x)F(y))$. Then A^{sym} $(= B^{\mathrm{sym}})$ is a JB^*-algebra and therefore, if Theorem 3.6.30 were true for alternative algebras, then we would have $\|F^{-1}(F(x)F(y))\| = \|x \odot y\| \leqslant \|x\|\|y\|$ for all $x, y \in B$. In particular, we could take $F := \exp(zD)$, with $z \in \mathbb{C}$ and D a Jordan derivation of B, and apply Liouville's theorem to get

$$\exp(-zD)(\exp(zD)(x)\exp(zD)(y)) = xy$$

for all $x, y \in B$, obtaining in this way that $\exp(zD)$ would be an algebra automorphism of B, and hence that D would be a derivation of B. Thus all Jordan derivations of B would be derivations of B, and hence B would be associative (cf. Remark 3.4.29). But we know the existence of non-associative choices of B (cf. Proposition 2.6.8).

3.6.3 Jacobson's representation theory continued

Now we retake our development of Jacobson's representation theory in order to prove some fundamental results which will be applied later, mainly in Subsection 4.4.4.

From now until Lemma 3.6.52, A will stand for an associative algebra over $\mathbb{K}$. The assumed associativity of A is crucial in the proof of the following.

Proposition 3.6.32 *We have:*

(i) *If I is a modular left ideal of A, then the core of I in A coincides with the set of those elements $a \in A$ such that $aA \subseteq I$.*
(ii) *If a is in A, and if $Aa \subseteq \mathrm{Rad}(A)$, then a lies in $\mathrm{Rad}(A)$.*

Proof Let I be a modular left ideal of A, and set $J := \{a \in A : aA \subseteq I\}$. Then J is an ideal of A containing all ideals of A contained in I. Therefore, to conclude the proof of assertion (i) it is enough to show that $J \subseteq I$. But this follows by taking a right modular unit $e \in A$ for I because then, for every $a \in J$, we have $a = (a - ae) + ae \in I + I \subseteq I$.

Let a be in A such that $Aa \subseteq \mathrm{Rad}(A)$. Then, by Corollary 3.6.22, we have $aA \subseteq \mathrm{Rad}(A)$, and hence aA is contained in every primitive ideal of A, so also in every maximal modular left ideal of A. It follows from assertion (i) just proved that a belongs to every primitive ideal of A, i.e. a lies in $\mathrm{Rad}(A)$. Thus assertion (ii) has been proved. □

Assertion (ii) in the above proposition can also be derived from the fact that $\mathrm{Rad}(A)$ is a semiprime ideal of A (cf. Proposition 3.6.16(ii)). Indeed, if I is a semiprime ideal of A, then, since the set $J := \{a \in A : Aa \subseteq I\}$ is an ideal of A, and $JJ \subseteq I$, we have $J \subseteq I$.

As usual, by a *left A-module* we mean a vector space X over $\mathbb{K}$ endowed with a bilinear mapping $(a,x) \to ax$ from $A \times X$ to X (called the *module multiplication* of X) satisfying $a(bx) = (ab)x$ for all $a, b \in A$ and $x \in X$.

Definition 3.6.33 Let X be a vector space over $\mathbb{K}$. By a *representation* of A on X we mean an algebra homomorphism from A to $L(X)$. Given a representation π of A on X, the *corresponding left A-module* is the vector space X with the module multiplication defined by

$$ax = \pi(a)(x) \;\text{ for all }\; a \in A \;\text{ and }\; x \in X. \tag{3.6.2}$$

Conversely, given a left A-module X, the *corresponding representation* of A on X is the algebra homomorphism $\pi : A \to L(X)$ defined by (3.6.2).

We shall make frequent use of the simple observation that the kernel of a representation π is given in terms of the corresponding left A-module X by $\ker(\pi) = \{a \in A : aX = 0\}$.

Let X be a left A-module. By an *A-submodule* of X we mean a subspace Y of X such that $AY \subseteq Y$. We remark that, given an A-submodule Y of X, the quotient vector space X/Y naturally becomes a left A-module under the (well-defined) module multiplication given by $a(x+Y) := ax + Y$.

Example 3.6.34 If we take $X = A$ as a vector space over $\mathbb{K}$, then X becomes a left A-module under the module product ax as defined in the algebra A. This left A-module is called the *regular left A-module*. Clearly, the A-submodules of the regular left A-module are precisely the left ideals of A. Therefore, given a left ideal I of A, the quotient vector space A/I becomes naturally a left A-module. We will refer to it as the *standard left A-module A/I*. The corresponding representation is called the *left* *standard representation* of A on A/I. It is clear that the kernel of this representation is the set $\{a \in A : aA \subseteq I\}$.

Definition 3.6.35 By an *irreducible left A-module* we mean a left A-module X with no nonzero proper A-submodule and such that $AX \neq 0$. A representation of A is *irreducible* if the corresponding left A-module is irreducible.

Let x_0 be an element of a left A-module X. We denote by $\ker(x_0)$ the left ideal of A given by $\ker(x_0) := \{a \in A : ax_0 = 0\}$. We say that x_0 is a *cyclic vector* if $Ax_0 = X$. We denote by $\mathrm{id}(x_0)$ the subset of A given by

$$\mathrm{id}(x_0) := \{e \in A : ex_0 = x_0\}.$$

Propositions 3.6.36 and 3.6.37 immediately below show how irreducible representations are related to maximal modular left ideals.

Proposition 3.6.36 *Let X be a left A-module and let x_0 be in $X \setminus \{0\}$. We have:*

(i) *Each element of* $\mathrm{id}(x_0)$ *is a right modular unit for the left ideal* $\ker(x_0)$.
(ii) *If x_0 is a cyclic vector, then* $\mathrm{id}(x_0)$ *is non-empty and* $\ker(x_0)$ *is a modular left ideal of A.*
(iii) *If X is irreducible, then x_0 is a cyclic vector and* $\ker(x_0)$ *is a maximal modular left ideal of A.*

Proof Assertion (i) is straightforward.

Assume that x_0 is cyclic. Then, since $Ax_0 = X$, there exists $e \in A$ such that $ex_0 = x_0$. Therefore, by assertion (i), e becomes a right modular unit for $\ker(x_0)$. This proves assertion (ii).

Assume that X is irreducible. Then, since Ax_0 is an A-submodule of X, and X has no nonzero proper A-submodules, we have either $Ax_0 = 0$ or $Ax_0 = X$. If $Ax_0 = 0$, then $Y := \{x \in X : Ax = 0\}$ is a nonzero A-submodule of X, and so $Y = X$. But this is impossible because $AX \neq 0$. Therefore $Ax_0 = X$, and hence x_0 is cyclic. By assertion (ii), $\ker(x_0)$ is a modular left ideal of A. Let I be a left ideal of A with $\ker(x_0) \subsetneqq I$. Then Ix_0 is an A-submodule of X, and $Ix_0 \neq 0$. Therefore $Ix_0 = X$, and hence there exists $e \in I \cap \mathrm{id}(x_0)$. By assertion (i), e is a right modular unit for $\ker(x_0)$ and therefore also for I (cf. Fact 3.6.1(ii)). But then, since $e \in I$, we have $I = A$ (cf. Fact 3.6.1(iii)). This proves assertion (iii). □

Proposition 3.6.37 *Let I be a proper modular left ideal of A, let $e \in A$ be a right modular unit for I, let X denote the standard left A-module A/I* (cf. Example 3.6.34)*, and set $x_0 := e + I \in X$. We have:*

(i) *x_0 is a cyclic vector and* $\ker(x_0) = I$.
(ii) *If I is actually a maximal modular left ideal, then X is irreducible.*

Proof Let $\phi : A \to X := A/I$ stand for the natural quotient mapping. For every $a \in A$ we have $a - ae \in I$, so $\phi(a) - ax_0 = \phi(a - ae) = 0$, and so $X = Ax_0$ because ϕ is surjective. If $a \in \ker(x_0)$, then $\phi(ae) = ax_0 = 0$, and so $ae \in I$. But then $a = (a - ae) + ae \in I + I \subseteq I$. Therefore $\ker(x_0) \subseteq I$. Conversely, if $a \in I$, then $ax_0 = ae + I = a - (a - ae) + I = a + I = 0$, hence $a \in \ker(x_0)$. Thus assertion (i) is proved.

Let Y be a nonzero A-submodule of X, and set $J := \{a \in A : \phi(a) \in Y\}$. Then J is a left ideal of A with $I \subsetneqq J$. Therefore, if I is maximal, then we have $J = A$, so $\phi(a) \in Y$ for every $a \in A$, and so $Y = X$. Since $AX = X$ (by assertion (i)) and $X \neq 0$, the above proves assertion (ii). □

Now we can prove the main result in this subsection.

Theorem 3.6.38 *We have:*

(i) *Primitive ideals of A coincide with kernels of irreducible representations of A.*

(ii) *Each primitive ideal of A is the intersection of the maximal modular left ideals of A containing it.*

(iii) $\mathrm{Rad}(A)$ *is the intersection of the maximal modular left ideals of A.*

(iv) $\mathrm{Rad}(A)$ *is the intersection of the maximal modular right ideals of A.*

(v) $\mathrm{Rad}(A)$ *is the largest quasi-invertible left ideal of A.*

(vi) $\mathrm{Rad}(A)$ *is the largest quasi-invertible right ideal of A.*

(vii) $\mathrm{Rad}(A)$ *coincides with the set of those elements $a \in A$ such that Aa is a quasi-invertible subset of A.*

(viii) $\mathrm{Rad}(A)$ *coincides with the set of those elements $a \in A$ such that aA is a quasi-invertible subset of A.*

Proof Let P be a primitive ideal of A. By definition, there exists a maximal modular left ideal I of A such that P is the core of I in A. Therefore, by Proposition 3.6.32(i), we have that $P = \{a \in A : aA \subseteq I\}$. But, as proved in Example 3.6.34, $\{a \in A : aA \subseteq I\}$ is the kernel of the left standard representation of A on A/I, and moreover, by Proposition 3.6.37(ii), this representation is irreducible. Thus P becomes the kernel of an irreducible representation of A. Conversely, let now P be the kernel of some irreducible representation π of A. Let X stand for the corresponding irreducible left A-module, and take $x_0 \in X \setminus \{0\}$. Then, by Proposition 3.6.36(iii), we have that $Ax_0 = X$ and that $I := \ker(x_0)$ is a maximal modular left ideal of A. Therefore, since

$$aX = 0 \Longleftrightarrow aAx_0 = 0 \Longleftrightarrow aA \subseteq \ker(x_0) = I,$$

and $P = \ker(\pi) = \{a \in A : aX = 0\}$, we derive from Proposition 3.6.32(i) that P is the core of the maximal modular left ideal I, and hence that P is a primitive ideal of A. Thus assertion (i) has been proved.

Let P be a primitive ideal of A. By assertion (i), there exists an irreducible left A-module X such that $P = \{a \in A : aX = 0\}$. This shows that

$$P = \bigcap\{\ker(x) : x \in X \setminus \{0\}\},$$

and, by Proposition 3.6.36(iii), each $\ker(x)$ with $x \in X \setminus \{0\}$ is a maximal modular left ideal of A. This proves assertion (ii).

Assertion (iii) follows from assertion (ii) and the definition of $\mathrm{Rad}(A)$.

By Theorem 3.6.21, $\mathrm{Rad}(A)$ is a quasi-invertible left ideal of A. Therefore, to conclude the proof of assertion (v) it is enough to show that every quasi-invertible left ideal of A is contained in $\mathrm{Rad}(A)$. But this follows from assertion (iii) by invoking Lemma 3.6.17.

Let a be in $\mathrm{Rad}(A)$. Then $Aa \subseteq \mathrm{Rad}(A)$, and hence, by Theorem 3.6.21, Aa is a quasi-invertible subset of A. Conversely, let a be in A such that Aa is a quasi-invertible subset of A. Then, since Aa is a left ideal of A, it follows from assertion (v) that $Aa \subseteq \mathrm{Rad}(A)$. But then, by Proposition 3.6.32(ii), we have $a \in \mathrm{Rad}(A)$. Thus assertion (vii) has been proved.

Finally, assertions (iv), (vi), and (viii) follow from assertions (iii), (v), and (vii), respectively, by invoking the symmetry of the radical given by Corollary 3.6.22. □

Let X and Y be left A-modules. By a *module homomorphism* from X to Y we mean a linear mapping $\Phi : X \to Y$ satisfying $\Phi(ax) = a\Phi(x)$ for all $a \in A$ and $x \in X$. We say that X and Y are *isomorphic* if there is a bijective module homomorphism from X to Y. The *centralizer set* for the left A-module X is defined as the set of all module homomorphisms from X to X. Clearly, the centralizer set for X is a full subalgebra of $L(X)$ containing I_X.

Proposition 3.6.39 *Let X be an irreducible left A-module. Then we have:*

(i) *The centralizer set for X is a division algebra.*
(ii) *X is isomorphic to the standard left A-module A/I for some maximal modular left ideal I of A.*

Proof To prove assertion (i) it is enough to show that every nonzero operator in the centralizer set for X is bijective. But, if T is such an operator, then $\ker(T)$ and $T(X)$ are A-submodules of X with $\ker(T) \neq X$ and $T(X) \neq 0$, so that $\ker(T) = 0$ and $T(X) = X$ because X is irreducible.

Take $x_0 \in X \setminus \{0\}$, and set $I := \ker(x_0)$. By Proposition 3.6.36(iii), we have that $Ax_0 = X$ and that I is a maximal modular left ideal of A. If a, b are in A with $a - b \in I$, then $ax_0 = bx_0$ by the definition of I. Therefore $\Phi : a + I \to ax_0$ becomes a well-defined mapping from A/I to X, and it is easily realized that Φ is in fact a surjective module homomorphism from the standard left A-module A/I to X. Moreover Φ is injective because $\ker(\Phi)$ is a submodule of X, $\ker(\Phi) \neq X$, and X is irreducible. Thus assertion (ii) has been proved. □

Now we will deal with some applications of the material included so far in this subsection to the case of normed algebras. We begin with the following.

Example 3.6.40 Let X be a nonzero normed space over $\mathbb{K}$, and let $\mathfrak{A}$ be a subalgebra of $L(X)$ containing the space $\mathfrak{F}(X)$ of all finite-rank operators on X (cf. §1.4.12). We claim that the inclusion $\mathfrak{A} \hookrightarrow L(X)$ is an irreducible representation of $\mathfrak{A}$. Indeed, let Y be a nonzero submodule of the left $\mathfrak{A}$-module corresponding to the representation $\mathfrak{A} \hookrightarrow L(X)$. By taking $y \in Y$ and $f \in X'$ with $f(y) = 1$, for every $x \in X$ we have $x \otimes f \in \mathfrak{A}$ and then $x = (x \otimes f)(y) \in Y$, hence $Y = X$. Now that the claim has been proved, we note that, by Theorem 3.6.38(i), $\mathfrak{A}$ is a primitive algebra. Therefore, by letting $\mathfrak{A}$ run over the family of all subalgebras of $BL(X)$ containing $\mathfrak{F}(X)$, we are provided with abundant examples of normed primitive algebras over $\mathbb{K}$. In particular, $BL(X)$, $\mathfrak{F}(X)$, and the algebra $\mathscr{K}(X)$ (of all compact operators on X) become normed primitive algebras over $\mathbb{K}$.

§3.6.41 We denote by q-Inv(A) the set of all quasi-invertible elements of A. A subalgebra B of A is said to be a *quasi-full subalgebra* of A if, whenever b is in $B \cap$ q-Inv(A), the quasi-inverse of b in A lies in B. It is easy to realize that a subalgebra B of A is a quasi-full subalgebra of A if and only if $B_{\mathbb{1}}$ is a full subalgebra of $A_{\mathbb{1}}$ in the sense of Definition 1.1.72. We say that A is a *normed Q-algebra* if A is normed and q-Inv(A) is a neighbourhood of zero in A. If A is normed and unital, then, clearly, A is a normed Q-algebra if and only if Inv(A) is a neighbourhood of $\mathbf{1}$ in A.

Example 3.6.42 Assume that A is complete normed. Then A is a normed Q-algebra. Actually A satisfies the apparently stronger condition that q-Inv(A) is open in A. For q-Inv(A) is the inverse image of Inv$(A_{\mathbb{1}})$ under the continuous mapping $a \to \mathbb{1} - a$ from A to $A_{\mathbb{1}}$, and Inv$(A_{\mathbb{1}})$ is open in $A_{\mathbb{1}}$ in view of Theorem 1.1.23.

Proposition 3.6.43 *Assume that A is normed. Then the following conditions are equivalent:*

(i) *A is a normed Q-algebra.*
(ii) *Every element $a \in A$ with $\mathfrak{r}(a) < 1$ is quasi-invertible in A.*
(iii) *Every element $a \in A$ with $\|a\| < 1$ is quasi-invertible in A.*
(iv) *Maximal modular left ideals of A are closed in A.*
(v) *The closure in A of each proper modular left ideal of A is a proper left ideal of A.*
(vi) *A is a quasi-full subalgebra of its completion.*
(vii) *A is a quasi-full subalgebra of some complete normed associative algebra over $\mathbb{K}$.*
(viii) q-Inv(A) *is open in A.*

Moreover, if $\mathbb{K} = \mathbb{C}$, then the above conditions are equivalent to

(ix) $\mathfrak{r}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(A_{\mathbb{1}}, a)\}$ *for every $a \in A$.*

Proof The implications (ii)⇒(iii), (vi)⇒(vii), and (viii)⇒(i) are clear.

(i)⇒(ii) By assumption (i), there exists $\delta > 0$ such that x is quasi-invertible in A whenever x is in A with $\|x\| < \delta$. Let a be in A such that $\mathfrak{r}(a) < 1$. Then $a^n \to 0$ (cf. Corollary 1.1.18(ii)), and hence there exists $n \in \mathbb{N}$ such that $\|a^n\| < \delta$, which implies that a^n is quasi-invertible in A. Now we have $1 \notin \mathrm{sp}(A_{\mathbb{1}}, a^n)$, which implies (in view of Proposition 1.3.4(i)) that $1 \notin \mathrm{sp}(A_{\mathbb{1}}, a)$. Thus a is quasi-invertible in A.

(iii)⇒(iv) Assume that condition (iv) is not fulfilled. Then there exists a maximal modular left ideal I of A such that $\bar{I} = A$. Therefore, taking a right modular unit $e \in A$ for I, there is $x \in I$ with $\|e - x\| < 1$. Thus, if condition (iii) were fulfilled, $e - x$ would have a quasi-inverse y in A, and we would have

$$e = x - y + y(e - x) = x - yx - (y - ye) \in I,$$

contradicting Fact 3.6.1(iii).

(iv)⇒(v) Let I be a proper modular left ideal of A. By Fact 3.6.3, there is a maximal modular left ideal J of A containing I, and, by the assumption (iv), J is closed in A. Therefore $\bar{I} \subseteq J$, which implies that $\bar{I}$ is proper.

(v)⇒(vi) Let $\hat{A}$ stand for the completion of A, and let x be in $A \cap$ q-Inv$(\hat{A})$. Then $\mathbb{1} - x$ belongs to Inv$(\hat{A}_{\mathbb{1}})$, so the operator of right multiplication by $\mathbb{1} - x$ on $\hat{A}_{\mathbb{1}}$ is a homeomorphism, and so $A_{\mathbb{1}}(\mathbb{1}-x)$ is dense in $\hat{A}_{\mathbb{1}}$. Therefore, given $z \in A$, there are sequences α_n and z_n in $\mathbb{K}$ and A, respectively, such that $\lim(\alpha_n \mathbb{1} + z_n)(\mathbb{1}-x) = z$. But, since $(\alpha_n \mathbb{1} + z_n)(\mathbb{1}-x) = \alpha_n \mathbb{1} + z_n - \alpha_n x - z_n x$ and $z_n - \alpha_n x - z_n x \in A$ for every $n \in \mathbb{N}$, and the sum $A_{\mathbb{1}} = \mathbb{K}\mathbb{1} \oplus A$ is topological, we get that $\alpha_n \to 0$ and then that

$$z_n(\mathbb{1}-x) = z_n - z_n x \longrightarrow z.$$

Hence $A(\mathbb{1}-x)$ is dense in A. Since $A(\mathbb{1}-x)$ is a modular left ideal of A, it follows from assumption (v) that $A(\mathbb{1}-x) = A$, and hence there is $y \in A$ satisfying $y(\mathbb{1}-x) = -x$. Now, if $x^\diamond$ denotes the quasi-inverse of x in $\hat{A}$, we have in $\hat{A}_{\mathbb{1}}$ that

$$x^\diamond = -x + xx^\diamond = -x(\mathbb{1}-x^\diamond) = -x(\mathbb{1}-x)^{-1} = y \in A.$$

(vii)⇒(viii) Assume that there is a complete normed associative algebra B over $\mathbb{K}$ containing A as a quasi-full subalgebra. Then q-Inv$(A) = A \cap$ q-Inv(B) and, as pointed out in Example 3.6.42, q-Inv(B) is open in B. Therefore q-Inv(A) is open in A.

Now that the equivalence of conditions (i)–(viii) has been proved, suppose that $\mathbb{K} = \mathbb{C}$.

(vii)⇒(ix) Assume that there is a complete normed associative algebra B over $\mathbb{K}$ containing A as a quasi-full subalgebra, and let a be in A. Then $A_{\mathbb{1}}$ is a full subalgebra of $B_{\mathbb{1}}$, and hence $\mathrm{sp}(A_{\mathbb{1}}, a) = \mathrm{sp}(B_{\mathbb{1}}, a)$. On the other hand, we have $\mathfrak{r}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(B_{\mathbb{1}}, a)\}$ thanks to Theorem 1.1.46. It follows that $\mathfrak{r}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(A_{\mathbb{1}}, a)\}$.

(ix)⇒(ii) Let a be in A with $\mathfrak{r}(a) < 1$. By the assumption (ix), we have that $1 \notin \mathrm{sp}(A_{\mathbb{1}}, a)$, so $\mathbb{1} - a \in \mathrm{Inv}(A_{\mathbb{1}})$, and so $a \in$ q-Inv(A).

Now the proof of the proposition is complete. □

Corollary 3.6.44 *Assume that A is a normed Q-algebra, and let X be an irreducible left A-module. Then we have:*

(i) *X can be endowed with a norm $\|\cdot\|$ satisfying $\|ax\| \leqslant \|a\|\|x\|$ for all $a \in A$ and $x \in X$, and such that the centralizer set for X consists only of $\|\cdot\|$-continuous operators.*
(ii) *The centralizer set for X is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ if $\mathbb{K} = \mathbb{R}$, and to $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$.*

Proof In view of Proposition 3.6.39(ii), we may assume that X equals the standard left A-module A/I for some maximal modular left ideal I of A. Then, by the implication (i)⇒(iv) in Proposition 3.6.43, I is closed in A, so that we can consider the quotient norm $\|\cdot\|$ on A/I, and we easily realize that the inequality $\|a(b+I)\| \leqslant \|a\|\|b+I\|$ holds for all $a, b \in A$. Let T be in the centralizer set for A/I. Then, taking a right modular unit $e \in A$ for I, for every $a \in A$ we have $a + I = a(e+I)$, and hence

$$\|T(a+I)\| = \|T(a(e+I))\| = \|aT(e+I)\| \leqslant \|a\|\|T(e+I)\|,$$

which implies that $\|T(a+I)\| \leqslant \|a+I\|\|T(e+I)\|$, and hence that T is $\|\cdot\|$-continuous. This concludes the proof of assertion (i). Assertion (ii) follows from assertion (i), Proposition 3.6.39(i), and the Gelfand–Mazur theorem (cf. Proposition 2.5.40 and Corollary 1.1.43). □

Keeping in mind Example 3.6.42, the proof of the above corollary yields the following.

Fact 3.6.45 *Assume that A is complete normed, and let X be an irreducible left A-module. Then X can be endowed with a complete norm $\|\cdot\|$ satisfying*

$$\|ax\| \leqslant \|a\|\|x\| \textit{ for all } a \in A \textit{ and } x \in X,$$

and such that the centralizer set for X consists only of $\|\cdot\|$-continuous operators.

Corollary 3.6.46 *Assume that A is a normed Q-algebra. Then*

$$\begin{aligned}\operatorname{Rad}(A) &= \{x \in A : \mathfrak{r}(ax) = 0 \textit{ for every } a \in A\} \\ &= \{x \in A : \mathfrak{r}(xa) = 0 \textit{ for every } a \in A\}.\end{aligned}$$

Proof Let x be in $\operatorname{Rad}(A)$. Then, for every $a \in A$, both ax and xa lie in $\operatorname{Rad}(A)$, and hence, by Corollary 3.6.23, we have $\mathfrak{r}(ax) = \mathfrak{r}(xa) = 0$. Conversely, let x be in A such that $\mathfrak{r}(ax) = 0$ (respectively, $\mathfrak{r}(xa) = 0$) for every $a \in A$. Then, by the implication (i)⇒(ii) in Proposition 3.6.43, ax (respectively, xa) is quasi-invertible in A for every $a \in A$. It follows from assertion (vii) (respectively, assertion (viii)) in Theorem 3.6.38 that x lies in $\operatorname{Rad}(A)$. □

The next result follows straightforwardly from Corollaries 1.1.115, 3.6.23, and 3.6.46.

Corollary 3.6.47 *Assume that A is a commutative normed Q-algebra. Then*

$$\operatorname{Rad}(A) = \{x \in A : \mathfrak{r}(x) = 0\}.$$

Now we are going to prove the Kleinecke–Shirokov and Singer–Wermer theorems.

Lemma 3.6.48 *Let a be in A, let D be a derivation of A such that $D^2(a) = 0$, and let n be in $\mathbb{N}$. Then we have:*

(i) $D^{n+1}(a^n) = 0$.
(ii) $D^n(a^n) = n!(D(a))^n$.

Proof Assertion (i) is true for $n = 1$ because of the assumption $D^2(a) = 0$. Assume that it is true for some n. Then, by the Leibnitz rule and the assumption $D^2(a) = 0$, we have

$$D^{n+2}(a^{n+1}) = D^{n+2}(aa^n) = aD^{n+2}(a^n) + (n+2)D(a)D^{n+1}(a^n) = 0.$$

Assertion (ii) is clearly true for $n = 1$. Assume that it is true for some n. Then, by the Leibnitz rule, the assumption $D^2(a) = 0$, and assertion (i), we have

$$D^{n+1}(a^{n+1}) = aD^{n+1}(a^n) + (n+1)D(a)D^n(a^n) = (n+1)!(D(a))^{n+1}.$$ □

Proposition 3.6.49 (Kleinecke–Shirokov) *Assume that A is normed, and let a, b be in A such that $[b,[b,a]] = 0$. Then $\mathfrak{r}([b,a]) = 0$.*

Proof Let D stand for the continuous derivation of A defined by $D(x) := [b,x]$. By the assumption, we have $D^2(a) = 0$. Therefore, by Lemma 3.6.48(ii), for $n \in \mathbb{N}$ we have $D^n(a^n) = n!(D(a))^n$, and hence

$$\|[b,a]^n\| = \|(D(a))^n\| \leqslant \frac{\|D\|^n \|a\|^n}{n!}.$$

The result now follows by taking nth roots and letting $n \to \infty$. □

Corollary 3.6.50 *Assume that A is normed, let a be in A, and let D be a continuous Jordan derivation of A such that $[a,D(a)]$ belongs to the centre of A. Then $\mathfrak{r}(D(a)) = 0$.*

Proof For $x \in A$, let $L_x^\bullet$ stand for the multiplication operator by x on A^{sym}. Then, by Lemma 2.4.7, we have $[D, L_a^\bullet] = L_{D(a)}^\bullet$. On the other hand, in view of the identity (3.1.9) in the proof of Lemma 3.1.53, the assumption that $[a,D(a)]$ belongs to the centre of A reads as that $[L_a^\bullet, L_{D(a)}^\bullet] = 0$. It follows that $[L_a^\bullet,[L_a^\bullet,D]] = 0$ in the normed algebra $BL(A)$. Therefore, by Proposition 3.6.49, we have $\mathfrak{r}(L_{D(a)}^\bullet) = \mathfrak{r}([D,L_a^\bullet]) = 0$. Now the proof is concluded by noticing that, for every $x \in A$, the inequality $\mathfrak{r}(x) \leqslant \mathfrak{r}(L_x^\bullet)$ holds. Indeed, for $n \in \mathbb{N}$, we have $x^{n+1} = (L_x^\bullet)^n(x)$, and hence

$$\mathfrak{r}(x) = \lim \|x^{n+1}\|^{\frac{1}{n}} \leqslant \lim \|(L_x^\bullet)^n\|^{\frac{1}{n}} \|x\|^{\frac{1}{n}} = \mathfrak{r}(L_x^\bullet).$$ □

The next result follows straightforwardly from Corollaries 3.6.47 and 3.6.50.

Proposition 3.6.51 (Singer–Wermer) *Assume that A is a commutative normed Q-algebra, and let D be a continuous derivation of A. Then the range of D is contained in* $\mathrm{Rad}(A)$.

Lemma 3.6.52 *Let X be a left A-module. Assume that A is normed, that there is a norm $\|\cdot\|$ on X satisfying $\|ax\| \leqslant \|a\|\|x\|$ for all $a \in A$ and $x \in X$, and that there exists a cyclic vector x_0 in X. Then an algebra norm can be built on the centralizer set for X.*

Proof Let $\mathscr{D}$ stand for the centralizer set for X, for $F \in \mathscr{D}$ set $|F| := \|F(x_0)\|$, and note that, if $|F| = 0$, then $F(x_0) = 0$, so $F(X) = F(Ax_0) = AF(x_0) = 0$, hence $F = 0$. Then it becomes clear that $|\cdot|$ becomes a norm on the vector space of $\mathscr{D}$. Since for any fixed element $F \in \mathscr{D}$ there exists $a \in A$ with $ax_0 = F(x_0)$, for every G in $\mathscr{D}$ we have

$$|G \circ F| = \|(G \circ F)(x_0)\| = \|G(ax_0)\| = \|aG(x_0)\| \leqslant \|a\|\|G(x_0)\| = \|a\||G|,$$

and hence the mapping $\Theta_F : G \to G \circ F$ is a bounded linear operator on $(\mathscr{D}, |\cdot|)$. If for F in $\mathscr{D}$ we define $|||F|||$ to be the operator norm of Θ_F as a bounded linear operator on $(\mathscr{D}, |\cdot|)$, then $|||\cdot|||$ becomes an algebra norm on $\mathscr{D}$, as required. □

§3.6.53 Let A be a (possibly non-associative) algebra over $\mathbb{K}$. The subalgebra of $L(A)$ generated by the set of all left and right multiplication operators on A is called the *multiplication ideal* of A, and is denoted by $\mathscr{M}^\sharp(A)$. Thinking about the representation $\mathscr{M}^\sharp(A) \hookrightarrow L(A)$ of $\mathscr{M}^\sharp(A)$ on the vector space of A, we can see A as a left $\mathscr{M}^\sharp(A)$-module in a natural way. Then, clearly, the ideals of A are precisely the $\mathscr{M}^\sharp(A)$-submodules of A, and the centroid of A coincides with the centralizer set (say $\mathscr{D}$) for the $\mathscr{M}^\sharp(A)$-module A. Therefore, if A has zero annihilator, then, by Proposition 1.1.11(i), $\mathscr{D}$ is a commutative algebra. Moreover, if A is actually semiprime, then the minimal ideals of A (cf. Definition 2.5.6) are precisely the irreducible $\mathscr{M}^\sharp(A)$-submodules of A.

Lemma 3.6.54 *Let A be a prime algebra over $\mathbb{K}$, and let X be a minimal ideal of A. Then X is the smallest nonzero ideal of A. Moreover, regarding X as a left $\mathscr{M}^\sharp(A)$-module, the centralizer set for X is isomorphic to the extended centroid of A.*

Proof Let I be a nonzero ideal of A. Then $X \cap I$ is an ideal of A contained in X, so either $X \cap I = X$ or $X \cap I = 0$. Since the latter possibility is prohibited by the primeness of A, we have that $X \subseteq I$. Thus X becomes the smallest nonzero ideal of A.

Let $\mathscr{C}$ and $\mathscr{D}$ stand for the set of all partially defined centralizers on A and for the centralizer set for X, respectively. It is clear that each $F \in \mathscr{D}$ can be regarded as an element of $\mathscr{C}$. Moreover, denoting by $\hat{F}$ the equivalence class in $\mathscr{C}$ containing F (cf. Lemma 2.5.42), we realize that the mapping $F \to \hat{F}$ becomes an injective algebra homomorphism from $\mathscr{D}$ to the extended centroid of A. Let f be a nonzero partially defined centralizer on A. Then, by the first paragraph of the proof, we have $X \subseteq \mathrm{dom}(f)$ and $X \subseteq f(X)$. The inclusion $X \subseteq \mathrm{dom}(f)$ allows us to consider the mapping $x \to f(x)$, from the left $\mathscr{M}^\sharp(A)$-module X onto the left $\mathscr{M}^\sharp(A)$-module $f(X)$, which becomes a nonzero module homomorphism, and hence a module isomorphism because X is an irreducible left $\mathscr{M}^\sharp(A)$-module. As a consequence, $f(X)$ is an irreducible left $\mathscr{M}^\sharp(A)$-module, so we have $X = f(X)$ because of the inclusion $X \subseteq f(X)$. Therefore, the restriction of f to X (say F) can be regarded as an element of $\mathscr{D}$, which clearly verifies $f \in \hat{F}$. Since f is an arbitrary nonzero partially defined centralizer, we conclude that the mapping $F \to \hat{F}$ is surjective, and so the proof is complete. □

Proposition 3.6.55 *Let A be a normed prime algebra over $\mathbb{K}$ with a minimal ideal. Then the extended centroid of A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$, and to $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$.*

Proof Let X stand for the minimal ideal of A whose existence has been assumed. Then X becomes an irreducible left $\mathscr{M}^\sharp(A)$-module. Therefore, keeping in mind Proposition 3.6.36(iii) and that $\mathscr{M}^\sharp(A)$ is a normed algebra (as a subalgebra of $BL(A)$), it follows from Lemma 3.6.52 that an algebra norm can be built on the centralizer set (say $\mathscr{D}$) for the left $\mathscr{M}^\sharp(A)$-module X. Since $\mathscr{D}$ is isomorphic to C_A (by Lemma 3.6.54) and C_A is a field extension of $\mathbb{K}$ (by Proposition 2.5.44), we derive from the Gelfand–Mazur theorem (cf. Proposition 2.5.40 and Corollary 1.1.43) that C_A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$, and to $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$. □

Keeping in mind Remark 2.5.45, Proposition 3.6.55 above yields the following.

Corollary 3.6.56 *Let A be a normed simple algebra over $\mathbb{K}$. Then the centroid of A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ if $\mathbb{K} = \mathbb{R}$, and to $\mathbb{C}$ if $\mathbb{K} = \mathbb{C}$.*

3.6.4 Historical notes and comments

Proposition 3.6.2 and Corollary 3.6.13 first appear (somewhat implicitly) in [165, Lemmas 1 and 4] with a different proof. The paternity of results from Theorems 3.6.7–3.6.9 is the same as that of their unital forerunners (cf. §§1.1.118

and 1.1.119). The associative forerunner of assertion (i) in Proposition 3.6.11 (that closed 'Jordan' ideals of C^*-algebras are ideals) is due to Civin and Yood [175], whereas the actual version is taken from [481]. Assertion (ii) in that proposition is new.

According to Lam [766, p. 48],

> The notion of the radical was a direct outgrowth of the notion of semisimplicity. It may be somewhat surprising, however, to remark that the radical was studied first in the context of nonassociative rings (namely finite-dimensional Lie algebras) rather than associative rings. In the work of E. Cartan [[159]], the radical of a finite-dimensional Lie algebra A (say over $\mathbb{C}$) is defined to be the maximal solvable ideal of A. The Lie algebra A is semisimple iff its radical is zero, i.e. iff it has no nonzero solvable ideals. Cartan characterized the semisimplicity of a Lie algebra in terms of the nondegeneracy of its Killing form, and showed that any semisimple Lie algebra is a finite direct sum of simple Lie algebras. Moreover, he classified the finite-dimensional simple Lie algebras (over $\mathbb{C}$). Therefore, the structure theory of finite-dimensional semisimple Lie algebras is completely determined.

For a detailed history of radicals for associative algebras, we refer the reader to Palmer's book [786, 4.8.1]. Concerning the Jacobson radical, we reproduce Palmer's words in [786, p. 506]:

> Sam Perlis [490] showed that the nil radical of a finite-dimensional algebra is the largest quasi-regular ideal [quasi-invertible ideal in our terminology] of the algebra. Baer [50] introduced the term 'quasi-regular ideal' and showed that in any ring the sum of all quasi-regular ideals is a quasi-regular ideal, and that the quotient modulo this ideal has no nonzero quasi-regular ideals. However, Nathan Jacobson [345], in a paper the great importance of which was immediately recognized, showed that a significant theory could be based on the use of this ideal as a radical. Indeed, except for the role played by modular ideals, Jacobson's original paper, together with his paper [346], contains the complete theory very much as it is known today. Jacobson noted that his radical agrees with the nil-radical in rings satisfying the descending chain condition on left ideals, and agrees with the Gelfand radical [the strong radical in our terminology] for commutative Banach algebras.
>
> Several of the results on the Jacobson radical were obtained independently and simultaneously by Max Zorn and Einar Hille, cf. [745, pp. 475–6]. Hille has access to preliminary versions of Irving E. Segal's paper [560] in which modular ideals were first defined (with the name regular ideals). He seems to have been the first mathematician to recognize the important connections between modular ideals and the theory of the Jacobson radical. Most of these connections are presented in his book [745]. However, an exposition of the theory containing all elements of our present day theory does not seem to have been given until the book of Jacobson [753].

Results from Proposition 3.6.14 to Corollary 3.6.22 and from Propositions 3.6.32 to 3.6.39 are essentially due to Jacobson [344, 345], although we have been helped by the development done in the Bonsall–Duncan book [696, Sections 24–5]. The actual (non-associative) versions of Propositions 3.6.14 and 3.6.18, as well as the (non-associative) argument in the proof of Lemma 3.6.17, are taken from [516], whereas the actual (non-associative) version of Proposition 3.6.16 seems to us to be new. Corollary 3.6.23 has been taken from [696, Proposition 25.1(i) and Remark (1) that follows]. Assertion (i) in Proposition 3.6.39 is known as *Schur's lemma* [559]. Although elementary, it becomes an extremely useful statement in representation theory of groups and algebras. An outstanding sample of its usefulness is the so-called *Jacobson density theorem* [344, Theorem 6] (see also [753, p. 28]), which reads as follows.

Theorem 3.6.57 *Let A be an associative algebra over $\mathbb{K}$, let X be an irreducible left A-module, and let $\mathscr{D}$ stand for the centralizer set for X* (which, by Schur's lemma, is a division algebra over $\mathbb{K}$). *If $x_1,\dots,x_n$ are linearly $\mathscr{D}$-independent vectors of X and if $y_1,\dots,y_n$ are arbitrary vectors in X, then there exists $a\in A$ such that $ax_i=y_i$ for every $i=1,\dots,n$.*

Proof (Rowen [545]) We proceed by induction. The theorem is true if $n=1$(cf. Proposition 3.6.36(iii)); assume the theorem holds for $n-1$, and let $x_1,\dots,x_n$ be linearly $\mathscr{D}$-independent vectors of X. Note that, by the universal property of the unital extensions, X becomes an irreducible left $A_{\mathbb{1}}$-module for the module action defined by $(\lambda\mathbb{1}+a)x=\lambda x+ax$, and that the centralizer set for X as a left $A_{\mathbb{1}}$-module coincides with $\mathscr{D}$. Let X^{n-1} stand for the left $A_{\mathbb{1}}$-module direct product of $n-1$ copies of X. By the induction hypothesis, we have $X^{n-1}=A(x_1,\dots,x_{n-1})$ so, a fortiori, $X^{n-1}=A_{\mathbb{1}}(x_1,\dots,x_{n-1})$. We claim that there exists $b_n\in A_{\mathbb{1}}$ such that $b_nx_n\neq 0$ and $b_nx_i=0$ for all $i\neq n$. Otherwise the mapping $\phi:A_{\mathbb{1}}(x_1,\dots,x_{n-1})\to X$ given by

$$\phi(b(x_1,\dots,x_{n-1}))=bx_n \tag{3.6.3}$$

would be a well-defined module homomorphism from X^{n-1} to X. But it is clear that, if ψ is any module homomorphism from X^{n-1} to X, and if I_i denotes the natural imbedding from X into the ith coordinate of X^{n-1}, then $\psi_i:=\psi\circ I_i$ is an element of $\mathscr{D}$, and that for each $(y_1,\dots,y_{n-1})\in X^{n-1}$ we have $\psi(y_1,\dots,y_{n-1})=\sum_{i=1}^{n-1}\psi_i(y_i)$. Therefore, by taking $b=\mathbb{1}$ in (3.6.3), we would have $x_n=\phi(x_1,\dots,x_{n-1})=\sum_{i=1}^{n-1}\phi_i(x_i)$, contrary to the assumption that $x_1,\dots,x_n$ are linearly $\mathscr{D}$-independent. Now that the claim is proved, by symmetry, for each j with $1\leqslant j\leqslant n$ there exists $b_j\in A_{\mathbb{1}}$ such that $b_jx_j\neq 0$ and $b_jx_i=0$ for all $i\neq j$. Since X is irreducible, given arbitrary vectors $y_1,\dots,y_n$ in X, for each j with $1\leqslant j\leqslant n$ there is $a_j\in A$ such that $a_jb_jx_j=y_j$. By setting $a:=\sum_{j=1}^{n}a_jb_j\in A$, we have $ax_i=a_ib_ix_i=y_i$ for each i. $\square$

§**3.6.58** Theorem 3.6.21 (that the radical of any associative algebra is the largest quasi-invertible ideal) remains true for alternative algebras. The history of this result is the following. Let A be an alternative algebra over $\mathbb{K}$. In 1948, Smiley [588] showed that there exists a quasi-invertible ideal of A which contains all quasi-invertible one-sided ideals of A (compare assertions (v) and (vi) in Theorem 3.6.38). This largest quasi-invertible ideal is known in the literature as the *Smiley radical* of A. Later, in 1955, Kleinfeld [390] showed that $\mathrm{Rad}(A)$ is the intersection of the maximal modular left ideals of A, as well as the intersection of the maximal modular right ideals of A (compare assertions (iii) and (iv) in Theorem 3.6.38), that the Smiley radical of A is contained in $\mathrm{Rad}(A)$ (a consequence of Proposition 3.6.18), and that the quotient algebra $A/\mathrm{Rad}(A)$ is a subdirect sum of primitive associative algebras and of eight-dimensional unital composition algebras over a field extension of $\mathbb{K}$ (cf. Proposition 2.5.21). Because of the relevance of the results in [390], in the alternative setting we are considering, $\mathrm{Rad}(A)$ is known as the *Kleinfeld radical* of A. Finally, in 1969, Zhevlakov [667] showed that the inclusion of the Smiley radical in the Kleinfeld radical is in fact an equality. All these results are included with proof in [822, Chapter 10], whose authors propose the name *Zhevlakov's radical* to refer to the common value of the Kleinfeld and Smiley radicals.

The asymmetry of the radical for general non-associative algebras, discussed immediately before Corollary 3.6.22, is clarified in what follows.

Fact 3.6.59 *Let A be a nonzero (complete normed) algebra over $\mathbb{K}$. We have:*

(i) *If A is non-radical and has no nonzero (closed) proper left ideal, then A has a right unit.*
(ii) *If A has a right unit and has no nonzero (closed) proper ideal, then A is a primitive algebra.*
(iii) *If A has a left unit, has no right unit, and has no nonzero (closed) proper left ideal, then A is a radical algebra, whereas $A^{(0)}$ is a primitive algebra.*

Proof Assume that A is non-radical. Then there exists a maximal modular left ideal of A, (which, by Corollary 3.6.4, must be closed in A). Therefore, if in addition A has no nonzero (closed) proper left ideal, then zero is a modular left ideal of A, and each right modular unit for zero in A becomes a right unit for A. Thus assertion (i) has been proved.

Assume that A has a right unit e. Then e becomes a right modular unit for every subspace of A, in particular for zero. As a consequence, since zero is a left ideal of A, Fact 3.6.3 applies, so that there exists a maximal modular left ideal M of A, (the core of which must be closed in A in view of Corollary 3.6.13). Therefore, if in addition A has no nonzero (closed) proper ideal, then zero is the core of M, and hence a primitive ideal of A. Thus assertion (ii) has been proved.

Assertion (iii) follows from assertions (i) and (ii), the latter with $A^{(0)}$ instead of A. □

Invoking assertion (iii) in Fact 3.6.59 just proved, and the non-commutative Urbanik–Wright theorem (the implication (i)⇒(iii) in Theorem 2.6.21), we derive the following.

Corollary 3.6.60 *The situation that A is a radical algebra and that $A^{(0)}$ is a primitive algebra occurs if A is equal to ${}^*\mathbb{C}$, ${}^*\mathbb{H}$, or ${}^*\mathbb{O}$* (in the sense of the paragraph immediately after Proposition 2.8.81)*, as well as if A is any complete absolute-valued infinite-dimensional real algebra with a left unit and with no nonzero closed proper left ideal* (the existence of which is assured by Theorem 2.7.68).

By the way, combining assertion (i) in Fact 3.6.59 and the non-commutative Urbanik–Wright theorem, we obtain that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$ *are the unique (complete) absolute-valued real algebras with no nonzero (closed) proper one-sided ideal and such that both they and their opposite algebras are non-radical*.

Theorems 3.6.25 and 3.6.30, as well as Remark 3.6.31, are due to Rodríguez [518], whose arguments have been reproduced almost verbatim. Theorem 3.6.25 had been previously proved by Braun [125] under the additional assumptions that the algebra A in the theorem is unital and that the product of A is continuous. Anyway, Braun's arguments in [125] were different from ours. Lemma 3.6.26 is taken from [695, Lemma 25.1], whereas Lemma 3.6.27 is taken from [514]. The associative forerunner of Lemma 3.6.27 (which in fact is the version applied in our development) can be seen in [696, Corollary 15.7]. Lemma 3.6.28 contains an argument in the proof of [482, Proposition 2.3].

§**3.6.61** Associative normed Q-algebras are nothing other than particular cases of certain topological rings, namely the so-called 'Q-rings', whose study goes back to Kaplansky [372]. Since then, associative normed Q-algebras have seldom been studied. Exceptions are Yood's relevant papers [646, 648], and Palmer's approach in terms of the so-called 'spectral algebras' (see [478] and [786, Chapter 2]). With Palmer's terminology, Proposition 3.6.43 can be found in [786, Theorem 2.2.5, Corollary 2.4.8, and Proposition 2.5.15]. Our proof follows the Yood–Palmer arguments, together with the adaptation to the associative setting of some ideas in the non-associative approach to normed Q-algebras done in [488] (see §4.4.71 below). A nice characterization of associative normed Q-algebras, not contained in Proposition 3.6.43 , is that of Fuster and Marquina [274] (see also [786, Proposition 2.2.7]) asserting that *a normed associative algebra A is a normed Q-algebra if and only if, whenever a is in A with $\|a\| < 1$, the geometric series $\sum_{n\in\mathbb{N}} a^n$ converges in A.*

The implication (i)⇒(vii) in Proposition 3.6.43 becomes an affirmative answer to the so-called Wilansky's conjecture [635] (that every associative normed Q-algebra is a quasi-full subalgebra of some complete normed associative algebra). Curiously enough, as pointed out by Beddaa and Oudadess in [81], Wilansky's conjecture had been solved ten years earlier by Arosio in [35, Theorem 2], by proving that every associative normed Q-algebra is a quasi-full subalgebra of its completion.

Corollary 3.6.44 is due to Palmer [478] (see also [786, Theorems 4.2.7 and 4.2.11]). Its complete normed forerunner (including Fact 3.6.45) is due to Rickart [501], and is included in both [795, Theorem 2.2.6 and Lemma 2.4.4] and [696, Lemma 25.2 to Corollary 25.4]. Actually, the sources just quoted contain the (occasionally complete normed forerunner of the) next fact, which follows by combining Corollary 3.6.44(ii) and Theorem 3.6.57.

Fact 3.6.62 *Let A be an associative complex normed Q-algebra, and let X be an irreducible left A-module. If $x_1,\dots,x_n$ are linearly independent vectors of X and if $y_1,\dots,y_n$ are arbitrary vectors in X, then there exists $a \in A$ such that $ax_i = y_i$ for every $i = 1,\dots,n$.*

It is worth emphasizing that Jacobson cares about analytic characterizations of his radical in the complete normed (associative) case. This is done in [345, Section 7], where the complete normed complex forerunner of Corollary 3.6.46 is proven. The complete normed (real or complex) forerunner of Corollary 3.6.46 can be found in [696, Proposition 25.1].

Proposition 3.6.49 states the celebrated Kleinecke–Shirokov theorem [389, 566] (see also [696, Proposition 18.13]). The proof of Proposition 3.6.49 actually shows that, *if A is a normed associative algebra over $\mathbb{K}$, if a is in A, and if D is a continuous derivation of A such that $D^2(a) = 0$, then $\mathfrak{r}(D(a)) = 0$.* According to the main theorem in Thomas' paper [612], the result just formulated remains true if the requirement of continuity for D is removed altogether. Corollary 3.6.50 is taken from [511]. As discussed in Subsection 3.4.5, the complete normed complex forerunner of Proposition 3.6.51 is due to Singer and Wermer [582].

Now, let us follow [442] to construct, in Lemma 3.6.63 and Proposition 3.6.64 immediately below, normed Q-algebras of bounded linear operators on any (possibly non-complete) normed space.

Lemma 3.6.63 *Let X be a normed space, and let B be a subalgebra of $BL(X)$ such that:*

(i) *The identity mapping I_X lies in B.*
(ii) *If F belongs to B and is bijective, then F^{-1} belongs to B.*
(iii) *An operator $F \in B$ is bijective if and only if so is F'.*

Then B is a normed Q-algebra.

Proof For F in B, we have that F is invertible in B if and only if F is bijective (by the assumption (ii)), if and only if F' is invertible in the Banach algebra $BL(X')$ (by the assumption (iii) and the Banach isomorphism theorem). Now, let us consider the mapping h from B into $BL(X')$ defined by $h(F) = F'$. It follows that $\mathrm{Inv}(B) = h^{-1}(\mathrm{Inv}(BL(X')))$. Since h is continuous, and $\mathrm{Inv}(BL(X'))$ is open in $BL(X')$, we conclude that $\mathrm{Inv}(B)$ is open in B, and hence that B is a normed Q-algebra (cf. §3.6.41). □

As a matter of fact, there are not many natural examples of non-complete normed Q-algebras. Fortunately, Lemma 3.6.63 above, together with some results proved in Subsection 1.4.2, provide us with two of them. Indeed, we have the following.

Proposition 3.6.64 *Let X be a normed space over $\mathbb{K}$. Then both $\mathfrak{K}(X) + \mathbb{K}I_X$ and $\mathfrak{W}(X) + \mathbb{K}I_X$ are normed Q-algebras.*

Here, as in Section 1.4, $\mathfrak{K}(X)$ (respectively, $\mathfrak{W}(X)$) stands for the ideal of $BL(X)$ consisting of all compact (respectively, weakly compact) operators on X.

Proof Let B stand for either $\mathfrak{K}(X) + \mathbb{K}I_X$ or $\mathfrak{W}(X) + \mathbb{K}I_X$, so that assumption (i) in Lemma 3.6.63 is fulfilled by B. On the other hand, by Corollary 1.4.26 (respectively, Corollary 1.4.25), assumption (ii) (respectively, assumption (iii)) in Lemma 3.6.63 is also fulfilled by B. □

Results from Lemma 3.6.52 to Corollary 3.6.56 have been taken from [148, 525]. For a forerunner of Proposition 3.6.55, see [177].

4

Jordan spectral theory

4.1 Involving the Jordan inverse

Introduction In Subsection 4.1.1, we introduce the notion of a Jordan-invertible (in short J-invertible) element of a unital Jordan algebra, as well as that of the J-inverse of such an element. As we notice in §4.1.1, these notions generalize the classical ones of an invertible element and of the inverse of such an element in a unital alternative algebra A (cf. Definition 2.5.23), when A is seen as a Jordan algebra by passing to A^{sym}. By considering the J-spectrum linked in the obvious way to the notion of a J-invertible element, we develop a spectral theory for normed unital Jordan algebras in parallel to that developed in the associative case (cf. Subsection 1.1.1). It is worth emphasizing the Jordan variant of Corollary 1.1.38 given by Theorem 4.1.10, as well as the Gelfand–Beurling formula stated in Theorem 4.1.17. We conclude the subsection by proving a wide non-associative generalization of Rickart's dense-range-homomorphism theorem in Theorem 4.1.19. Indeed, dense-range algebra homomorphisms from complete normed algebras to complete normed Jordan-admissible strongly semisimple algebras are automatically continuous (compare Theorem 3.6.7).

In Subsection 4.1.2, we introduce the notion of a topological J-divisor of zero in a normed Jordan algebra, noticing in Proposition 4.1.23 that this notion generalizes that of a one-sided topological divisor of zero in a normed alternative algebra A (cf. Definition 1.1.87), when A is seen as a normed Jordan algebra by passing to A^{sym}. Then we develop a Jordan variant of Subsection 1.1.4, by considering topological J-divisors of zero instead of one-sided topological divisors of zero. Most results in the subsection are applied to prove in the concluding Corollary 4.1.31 that, if b is an element of a closed subalgebra B of a complete normed Jordan complex algebra A, then the spectra of the multiplication operators by b on B and A are essentially the same.

Corollary 4.1.31 just reviewed becomes one of the main tools in Subsection 4.1.3. Indeed, when suitably translated to the setting of Banach Jordan $*$-triples (see Lemma 4.1.42), it is applied to prove that non-commutative JB^*-algebras are JB^*-triples in a natural way, and that there is a natural bijective correspondence between unital JB^*-algebras and nonzero JB^*-triples with a distinguished unitary element (see Theorems 4.1.45 and 4.1.55, respectively). We recall that JB^*-triples were introduced in §2.2.27. Other main tools in the subsection are Proposition 3.4.6 and Theorem 3.4.8.

In Subsection 4.1.4, we extend the notions of J-invertible element, J-inverse, and J-spectrum to the setting of a unital Jordan-admissible algebra A, by simply translating to A the corresponding ones in the Jordan algebra A^{sym}. As a first relevant application of the translation of results in Subsection 4.1.1 to this more general setting, we prove in Theorem 4.1.63 that normed non-commutative Jordan quasi-division complex algebras are isomorphic to $\mathbb{C}$, thus providing a partial affirmative answer to Problem 2.7.4. We introduce the notion of a J-full subalgebra B of a unital Jordan-admissible algebra A, meaning that $\mathbf{1}$ lies in B and that the equality J-sp$(A,x) =$ J-sp(B,x) holds for every $x \in B$. As the main result, we prove in Theorem 4.1.71 that closed unital $*$-subalgebras of unital non-commutative JB^*-algebras are J-full subalgebras. This allows us to build in Corollary 4.1.72 a continuous functional calculus at each normal element of any unital non-commutative JB^*-algebra, which generalizes the one built in Theorem 1.2.28 in the particular case of C^*-algebras.

We begin Subsection 4.1.5 by proving that each element of a normed unital non-commutative Jordan algebra is contained in a closed associative and commutative J-full subalgebra (see Proposition 4.1.86). This allows us to develop in Theorems 4.1.88 and 4.1.93 a holomorphic functional calculus for a single element of a complete normed unital non-commutative Jordan complex algebra, in parallel to that done in Theorems 1.3.13 and 1.3.21, where the particular case of a complete normed unital associative complex algebra was considered.

Finally, in Subsection 4.1.6, we characterize smooth-normed algebras as those norm-unital normed Jordan-admissible algebras A such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every a in some non-empty open subset of A consisting only of J-invertible elements (see Theorem 4.1.96). As a consequence, $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital normed alternative real algebras such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every a in some non-empty open subset of A consisting only of invertible elements (see Corollary 4.1.100).

4.1.1 Basic spectral theory for normed Jordan algebras

§**4.1.1** Let A be a unital alternative algebra over $\mathbb{K}$, and suppose that a and b are inverses in A (cf. Definition 2.5.23), i.e. $ab = \mathbf{1} = ba$. Then we have the Jordan relations

$$a \bullet b = \tfrac{1}{2}(ab + ba) = \mathbf{1} \ \text{ and } \ a^2 \bullet b = \tfrac{1}{2}(a^2 b + ba^2) = \tfrac{1}{2}(a(ab) + (ba)a) = a.$$

Conversely, assume a and b are elements which satisfy the Jordan relations

$$a \bullet b = \mathbf{1} \ \text{ and } \ a^2 \bullet b = a.$$

Then we have $ab + ba = 2\mathbf{1}$ and $a^2 b + ba^2 = 2a$. Hence, by Theorem 2.3.61,

$$a^2 b + aba = 2a = aba + ba^2 \ \text{ and } \ a^2 b = ba^2.$$

Then $2a = 2a^2 b = 2ba^2$ so $a = a^2 b = ba^2$ and $ab = ba^2 b = ba$. This and $ab + ba = 2\mathbf{1}$ imply that $ab = \mathbf{1} = ba$, so a and b are inverses. These considerations lead to the following.

Definition 4.1.2 Let A be a unital Jordan algebra over $\mathbb{K}$. An element $a \in A$ is called *J-invertible* with b as a *J-inverse* if the equalities $ab = \mathbf{1}$ and $a^2 b = a$ hold.

Let A be a unital associative and commutative algebra over $\mathbb{K}$, and let a be in A. Then A is a Jordan algebra and, without involving §4.1.1, we straightforwardly realize that *a is J-invertible with b as a J-inverse if and only if a is invertible with b as an inverse.* This will be applied without notice.

Let A be a Jordan algebra over $\mathbb{K}$. We note that the identity (2.4.2) in the proof of Proposition 2.4.13 can be read as

$$[L_{ac}, L_d] + [L_{ad}, L_c] + [L_{cd}, L_a] = 0 \text{ for all } a, c, d \in A. \tag{4.1.1}$$

Theorem 4.1.3 *Let A be a unital Jordan algebra over $\mathbb{K}$. Then:*

(i) *If a is J-invertible in A with b as J-inverse, then b is J-invertible with a as J-inverse.*
(ii) *For $a \in A$, the following three conditions are equivalent:*
  (a) *a is J-invertible;*
  (b) *$\mathbf{1}$ is in the range of U_a;*
  (c) *U_a is a bijective operator on A.*
(iii) *If a is J-invertible in A, then it has a unique J-inverse and, in fact, this is the element $b = U_a^{-1}(a)$.*
(iv) *If a and b are J-inverses in A, then $U_b = U_a^{-1}$ and $L_b = U_a^{-1} L_a$.*
(v) *If a and b are J-inverses in A, then $[L_{a^k}, L_{b^\ell}] = 0$ for all $k, \ell > 0$. Also, if we define $a^{-n} := b^n$ for $n > 0$ and $a^0 := \mathbf{1}$, then $a^k a^\ell = a^{k+\ell}$ for all integer numbers k, ℓ.*
(vi) *Given $a, b \in A$, both a and b are J-invertible if and only if $U_a(b)$ is J-invertible.*

Proof If a is J-invertible with b as a J-inverse, then the identity (4.1.1) gives $[L_{a^2}, L_b] + [L_{\mathbf{1}}, L_a] + [L_{\mathbf{1}}, L_a] = 0$. Since $L_{\mathbf{1}} = I_A$ we have $[L_{a^2}, L_b] = 0$. Then

$$a^2 b^2 = L_{a^2} L_b(b) = L_b L_{a^2}(b) = b(a^2 b) = ba = \mathbf{1}$$

and $b^2 a = b^2(a^2 b) = (b^2 a^2) b = b$. This shows that a satisfies the conditions for a J-inverse of b; hence assertion (i) is proved.

Now, we proceed to prove assertion (ii). If a is J-invertible with b as a J-inverse then $U_a(b^2) = 2a(ab^2) - a^2 b^2 = 2ab - \mathbf{1} = \mathbf{1}$, using the result $a^2 b^2 = \mathbf{1}$ established in the proof of assertion (i). Hence $\mathbf{1}$ is in the range of U_a. Hence (a)⇒(b). Next assume $\mathbf{1} \in U_a(A)$ so we have a c in A such that $\mathbf{1} = U_a(c)$. Then the operator $I_A = U_{\mathbf{1}} = U_{U_a(c)} = U_a U_c U_a$ by the fundamental formula (3.4.3) in Proposition 3.4.15. Hence U_a^{-1} exists and so (b)⇒(c). Next assume (c) and let $b = U_a^{-1}(a)$. Then $a = U_a(b)$ and

$$U_a(\mathbf{1}) = a^2 = L_a(a) = L_a U_a(b) = U_a L_a(b).$$

Since U_a^{-1} exists this gives $\mathbf{1} = L_a(b) = ab$. Similarly,

$$U_a(a) = a^3 = L_{a^2}(a) = L_{a^2} U_a(b) = U_a L_{a^2}(b).$$

Hence $a = a^2 b$. Hence a is J-invertible with b as J-inverse and we have proved that (c)⇒(a).

If b is a J-inverse of a then $U_a(b) = 2a(ab) - a^2 b = a$. Since U_a^{-l} exists by assertion (ii), this gives $b = U_a^{-1}(a)$ and b is unique. Thus the proof of assertion (iii) is complete.

Assume that a and b are J-inverses. Then we have $U_a(b) = a$ so, by the fundamental formula (3.4.3), $U_a = U_{U_a(b)} = U_a U_b U_a$. Since U_a^{-1} exists this gives $U_a U_b = I_A = U_b U_a$ and so $U_b = U_a^{-1}$. We had $[L_{a^2}, L_b] = 0$ in the proof of assertion (i). Also $[U_a, L_b] = [U_b^{-1}, L_b] = 0$. Since $U_a = 2L_a^2 - L_{a^2}$ these two relations give $[L_a^2, L_b] = 0$. We now note that, by the identity (2.4.3) in the proof of Proposition 2.4.13, we have

$$L_{a^2}L_b - L_a^2 L_b + L_{ab}L_a - L_{a(ba)} + L_{ab}L_a - L_b L_a^2 = 0,$$

and hence $L_{a^2}L_b - L_a^2 L_b + L_a - L_a + L_a - L_b L_a^2 = 0$. Since $L_a^2 L_b = L_b L_a^2$ we get $L_a = (2L_a^2 - L_{a^2})L_b$, so $L_a = U_a L_b$ and $L_b = U_a^{-1} L_a$. This proves assertion (iv).

Assume again that a and b are J-inverses. Then it follows from assertion (iv) that the operators L_a, U_a, L_b and U_b pairwise commute. On the other hand, by Proposition 2.4.13(iii), for $k > 0$, L_{a^k} (respectively, L_{b^k}) belongs to the subalgebra of $L(A)$ generated by $\{L_a, U_a\}$ (respectively, by $\{L_b, U_b\}$). It follows that $[L_{a^k}, L_{b^\ell}] = 0$ for all $k, \ell \geqslant 0$. Now if $k > 0$ then

$$a^k a^{-1} = L_{a^{-1}} L_{a^{k-1}}(a) = L_{a^{k-1}} L_{a^{-1}}(a) = L_{a^{k-1}}(\mathbf{1}) = a^{k-1}.$$

Assuming $a^k a^{-(\ell-1)} = a^{k-\ell+1}$ for $\ell > 1$ we get from $[L_{a^k}, L_b] = 0$ that

$$a^k a^{-\ell} = a^k \left(a^{-(\ell-1)} a^{-1}\right) = \left(a^k a^{-(\ell-1)}\right) a^{-1} = a^{k-\ell+1} a^{-1}.$$

If $k - \ell + 1 > 0$ this gives $a^k a^{-\ell} = a^{k-\ell}$ by what we proved first and the result is clear if $k - \ell + 1 = 0$ or < 0 by power associativity (cf. Proposition 2.4.13(i)). The case $a^k a^\ell$ where k and ℓ are of the same sign is covered by power associativity. Hence assertion (v) holds.

Given $a, b \in A$, the fundamental formula (3.4.3) gives $U_{U_a(b)} = U_a U_b U_a$. Therefore the linear operator $U_{U_a(b)}$ is invertible in $L(A)$ if and only if U_a and U_b are invertible. Hence assertion (vi) follows from assertion (ii). □

According to assertion (iii) in the above theorem, if a is a J-invertible element in a unital Jordan algebra A, then the element $b = U_a^{-1}(a)$ is called *the J-inverse* of a, and is denoted by a^{-1}. We denote by J-Inv(A) the set of all J-invertible elements of A.

For any commutative algebra A over $\mathbb{K}$, and for all a, b in A, note that

$$U_{a,b} = L_a L_b + L_b L_a - L_{ab}.$$

Lemma 4.1.4 *Let A be a normed unital Jordan algebra over $\mathbb{K}$, and let a and b be in* J-Inv(A). *Then:*

(i) $\|a^{-1} - b^{-1}\| \leqslant 3\|a^{-1}\|\,\|b^{-1}\|\,\|a - b\|$.

(ii) $\left|\frac{1}{\|a^{-1}\|} - \frac{1}{\|b^{-1}\|}\right| \leqslant 3\|a - b\|$.

(iii) *If* $\|a - b\| < \frac{1}{3\|a^{-1}\|}$, *then* $\|b^{-1}\| \leqslant \frac{\|a^{-1}\|}{1 - 3\|a^{-1}\|\,\|a-b\|}$.

Proof We have

$$
\begin{aligned}
&U_{a^{-1},b^{-1}}(b-a)\\
&\quad= a^{-1}\left(\mathbf{1}-b^{-1}a\right)+b^{-1}\left(a^{-1}b-\mathbf{1}\right)-\left(a^{-1}b^{-1}\right)(b-a)\\
&\quad= a^{-1}-a^{-1}\left(b^{-1}a\right)+b^{-1}\left(a^{-1}b\right)-b^{-1}-\left(a^{-1}b^{-1}\right)b+\left(a^{-1}b^{-1}\right)a\\
&\quad= a^{-1}+[L_a,L_{a^{-1}}]\left(b^{-1}\right)+[L_{b^{-1}},L_b]\left(a^{-1}\right)-b^{-1}.
\end{aligned}
$$

Now, keeping in mind Theorem 4.1.3(v), we see that

$$U_{a^{-1},b^{-1}}(b-a)=a^{-1}-b^{-1}. \tag{4.1.2}$$

Therefore

$$\left\|a^{-1}-b^{-1}\right\|=\left\|U_{a^{-1},b^{-1}}(b-a)\right\|\leqslant 3\left\|a^{-1}\right\|\left\|b^{-1}\right\|\|a-b\|,$$

which proves assertion (i). Now, keeping in mind that

$$\left\|b^{-1}\right\|-\left\|a^{-1}\right\|\leqslant\left\|a^{-1}-b^{-1}\right\|,$$

it follows from assertion (i) that

$$\frac{1}{\|a^{-1}\|}-\frac{1}{\|b^{-1}\|}\leqslant 3\|a-b\|. \tag{4.1.3}$$

The proof of assertion (ii) is concluded by combining the inequality (4.1.3) with the one obtained by interchanging the roles of a and b. On the other hand, it follows from the inequality (4.1.3) that

$$\frac{1-3\left\|a^{-1}\right\|\|a-b\|}{\|a^{-1}\|}=\frac{1}{\|a^{-1}\|}-3\|a-b\|\leqslant\frac{1}{\|b^{-1}\|}. \tag{4.1.4}$$

Since the condition $\|a-b\|<\frac{1}{3\|a^{-1}\|}$ leads to $1-3\|a^{-1}\|\|a-b\|>0$, assertion (iii) follows from (4.1.4). □

Corollary 4.1.5 *Let A be a normed unital Jordan algebra over $\mathbb{K}$, let a be in A, and let z be in $\mathbb{K}$ such that $a-z\mathbf{1}\in$ J-Inv(A) and $|z|>3\|\mathbf{1}\|\|a\|$. Then*

$$\left\|(a-z\mathbf{1})^{-1}\right\|\leqslant\frac{\|\mathbf{1}\|}{|z|-3\|\mathbf{1}\|\|a\|}.$$

Proof We have

$$\|-z\mathbf{1}-(a-z\mathbf{1})\|=\|a\|<\frac{1}{3\left\|(z\mathbf{1})^{-1}\right\|},$$

and hence, by Lemma 4.1.4(iii),

$$\left\|(a-z\mathbf{1})^{-1}\right\|\leqslant\frac{\left\|(z\mathbf{1})^{-1}\right\|}{1-3\left\|(z\mathbf{1})^{-1}\right\|\|a\|}=\frac{\|\mathbf{1}\|}{|z|-3\|\mathbf{1}\|\|a\|}.$$ □

Given a normed unital Jordan algebra A over $\mathbb{K}$, we can argue as in the proof of Proposition 1.1.15, with Lemma 4.1.4 instead of Lemma 1.1.13, to obtain the continuity of the mapping $x\to x^{-1}$ from J-Inv(A) to A. Next, we give an alternative proof of this fact.

Proposition 4.1.6 *Let A be a normed unital Jordan algebra over $\mathbb{K}$. Then:*

(i) *An element $a \in A$ is J-invertible in A if and only if $U_a \in \mathrm{Inv}(BL(A))$.*
(ii) *The mapping $x \to x^{-1}$ from* J-Inv(A) *to A is continuous.*

Proof Let a be in A. If $a \in$ J-Inv(A), then, by Theorem 4.1.3(ii)–(iv),

$$U_a \in \mathrm{Inv}(L(A)) \text{ and } U_a^{-1} = U_{a^{-1}} \in BL(A),$$

hence $U_a \in \mathrm{Inv}(BL(A))$. Conversely, if $U_a \in \mathrm{Inv}(BL(A))$, then $U_a \in \mathrm{Inv}(L(A))$, and hence, by Theorem 4.1.3(ii), $a \in$ J-Inv(A). Thus assertion (i) is proved.

Keeping in mind that the mappings $x \to U_x$ from A to $BL(A)$, $F \to F^{-1}$ from $\mathrm{Inv}(BL(A))$ to $BL(A)$, and $(F,x) \to F(x)$ from $BL(A) \times A$ to A are continuous (the second one by Proposition 1.1.15), assertion (ii) follows from assertion (i) by noticing that the equality $x^{-1} = U_x^{-1}(x)$ holds for every $x \in$ J-Inv(A) (cf. Theorem 4.1.3(iii)). □

Theorem 4.1.7 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$. Then* J-Inv(A) *is open in A, and the mapping $x \to x^{-1}$ from* J-Inv(A) *to A is differentiable at any point $a \in$* J-Inv(A)*, with derivative equal to the mapping $-U_{a^{-1}}$.*

Proof By Proposition 4.1.6(i), we have J-Inv$(A) = f^{-1}(\mathrm{Inv}(BL(A)))$, where $f : A \to BL(A)$ is the mapping defined by $f(x) = U_x$. Since f is continuous, and $\mathrm{Inv}(BL(A))$ is open in $BL(A)$ (by Theorem 1.1.23), it follows that J-Inv(A) is open in A.

Keeping in mind Theorem 4.1.3, for $a, x \in$ J-Inv(A), we have

$$\begin{aligned}
&U_a(x^{-1} - a^{-1} + U_{a^{-1}}(x-a)) \\
&\quad = U_a\left(x^{-1}\right) - U_a\left(a^{-1}\right) + x - a = U_a\left(x^{-1}\right) + x - 2a \\
&\quad = U_a\left(x^{-1}\right) + U_x\left(x^{-1}\right) - 2U_{x,a}\left(x^{-1}\right) = U_{x-a}\left(x^{-1}\right),
\end{aligned}$$

and hence

$$x^{-1} - a^{-1} + U_{a^{-1}}(x-a) = (U_{a^{-1}} \circ U_{x-a})\left(x^{-1}\right). \tag{4.1.5}$$

It follows that

$$\begin{aligned}
\left\|x^{-1} - a^{-1} + U_{a^{-1}}(x-a)\right\| &= \left\|(U_{a^{-1}} \circ U_{x-a})\left(x^{-1}\right)\right\| \\
&\leqslant \left\|U_{a^{-1}}\right\| \left\|U_{x-a}\right\| \left\|x^{-1}\right\| \\
&\leqslant 3\left\|U_{a^{-1}}\right\| \left\|x-a\right\|^2 \left\|x^{-1}\right\|,
\end{aligned}$$

and hence, keeping in mind Proposition 4.1.6(ii), we conclude that

$$\lim_{\substack{x \to a \\ x \in \text{J-Inv}(A) \setminus \{a\}}} \frac{\left\|x^{-1} - a^{-1} + U_{a^{-1}}(x-a)\right\|}{\|x-a\|} = 0.$$

Therefore, the mapping $x \to x^{-1}$ is differentiable at a with derivative equal to $-U_{a^{-1}} \in BL(A)$. □

Given an element a of a unital Jordan algebra A over $\mathbb{K}$, the *J-spectrum* of a relative to A is the subset $\text{J-sp}(A,a)$ of $\mathbb{K}$ defined by

$$\text{J-sp}(A,a) := \{\lambda \in \mathbb{K} : a - \lambda\mathbf{1} \text{ is not J-invertible in } A\}.$$

Corollary 4.1.8 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, and let a be in A. Then $\text{J-sp}(A,a)$ is a closed subset of $\mathbb{K}$, and the mapping $f : \mathbb{K} \setminus \text{J-sp}(A,a) \to A$ given by $f(\lambda) = (a - \lambda\mathbf{1})^{-1}$ is differentiable with derivative $f'(\lambda) = (a - \lambda\mathbf{1})^{-2}$ for every $\lambda \in \mathbb{K} \setminus \text{J-sp}(A,a)$.*

Proof Note that $\text{J-sp}(A,a) = g^{-1}(A \setminus \text{J-Inv}(A))$, where $g : \mathbb{K} \to A$ is given by $g(\lambda) = a - \lambda\mathbf{1}$. Since g is continuous and $\text{J-Inv}(A)$ is an open subset of A (by Theorem 4.1.7), it follows that $\text{J-sp}(A,a)$ is a closed subset of $\mathbb{K}$. Moreover, since g is a differentiable function with derivative $g'(\lambda) = -\mathbf{1}$ for every $\lambda \in \mathbb{K}$, and $h : x \to x^{-1}$ is differentiable in $\text{J-Inv}(A)$ with derivative $h'(x) = -U_{x^{-1}}$ for every $x \in \text{J-Inv}(A)$, it follows from the chain rule that $f = h \circ g_{|\mathbb{K} \setminus \text{J-sp}(A,a)}$ is differentiable with derivative

$$f'(\lambda) = h'(g(\lambda))(g'(\lambda)) = -U_{(a-\lambda\mathbf{1})^{-1}}(-\mathbf{1}) = (a - \lambda\mathbf{1})^{-2}$$

for every $\lambda \in \mathbb{K} \setminus \text{J-sp}(A,a)$. □

Proposition 4.1.9 *Let A be a unital Jordan algebra over $\mathbb{K}$, and let a be in A. If L_a is invertible in $L(A)$, then a is J-invertible in A, and $L_{a^{-1}}$ is invertible in $L(A)$.*

Proof Assume that L_a is invertible in $L(A)$, and set $b = L_a^{-1}(\mathbf{1})$. Then $ab = \mathbf{1}$, and hence $L_a(a) = a^2 = a^2\mathbf{1} = a^2(ab) = a(a^2b) = L_a(a^2b)$, where we have applied the Jordan identity for the penultimate equality. Since L_a is injective, it follows that $a = a^2b$, and so a is J-invertible in A with $a^{-1} = b$. Finally, since $L_{a^{-1}} = U_a^{-1}L_a$ (by Theorem 4.1.3(iv)), $L_{a^{-1}}$ is invertible in $L(A)$. □

Theorem 4.1.10 *Let A be a complete normed unital Jordan complex algebra, and let a be in A. Then*

$$\text{J-sp}(A,a) \subseteq \text{sp}(BL(A),L_a) \subseteq \frac{1}{2}\left(\text{J-sp}(A,a) + \text{J-sp}(A,a)\right)$$

and

$$\text{sp}(BL(A),U_a) \subseteq \text{J-sp}(A,a)\,\text{J-sp}(A,a).$$

Proof If $\lambda \in \text{J-sp}(A,a)$, then $a - \lambda\mathbf{1}$ is not J-invertible in A, and hence, by Proposition 4.1.9, $L_{a-\lambda\mathbf{1}} = L_a - \lambda I_A$ is not invertible in $L(A)$. Therefore $L_a - \lambda I_A$ is not invertible in $BL(A)$, and so $\lambda \in \text{sp}(BL(A),L_a)$. Thus

$$\text{J-sp}(A,a) \subseteq \text{sp}(BL(A),L_a).$$

Since $S := \{L_a, L_{a^2}\}$ is a commutative subset of $BL(A)$, Proposition 1.1.78 applies so that the bicommutant S^{cc} of S in $BL(A)$ is a closed commutative subalgebra of $BL(A)$ containing $S \cup \{I_A\}$. We remark that

$$U_{a-z\mathbf{1}} = U_a - 2zL_a + z^2 I_A \in S^{cc} \quad \text{for every} \quad z \in \mathbb{C}.$$

Let φ be a character on S^{cc}, let z_1^φ, z_2^φ stand for the roots of the equation

$$\varphi(U_a) - 2z\varphi(L_a) + z^2 = 0,$$

and note that

$$\varphi(L_a) = \frac{1}{2}(z_1^\varphi + z_2^\varphi) \text{ and } \varphi(U_a) = z_1^\varphi z_2^\varphi.$$

Since $\varphi(U_{a-z_i^\varphi \mathbf{1}}) = 0$ $(i = 1,2)$, it follows that $U_{a-z_i^\varphi \mathbf{1}}$ is not invertible in S^{cc}, hence is not invertible in $BL(A)$ (by Lemma 1.1.80). Therefore, by Proposition 4.1.6(i), $a - z_i^\varphi \mathbf{1}$ is not J-invertible in A. Thus $z_1^\varphi, z_2^\varphi \in$ J-sp(A,a).

Let λ be in sp$(BL(A), L_a)$. Then, clearly, λ lies in sp(S^{cc}, L_a) so, by Proposition 1.1.68(ii), there exists a character φ on S^{cc} such that $\lambda = \varphi(L_a)$, hence

$$\lambda = \frac{1}{2}\left(z_1^\varphi + z_2^\varphi\right) \in \frac{1}{2}\left(\text{J-sp}(A,a) + \text{J-sp}(A,a)\right).$$

Analogously, if $\lambda \in$ sp$(BL(A), U_a)$, then $\lambda = \varphi(U_a)$ for some character φ on S^{cc}, and hence

$$\lambda = z_1^\varphi z_2^\varphi \in \text{J-sp}(A,a)\,\text{J-sp}(A,a).$$

□

Corollary 4.1.11 *Let A be a complete normed unital Jordan complex algebra, and let a be in A. Then*

$$\text{co}(\text{J-sp}(A,a)) = \text{co}(\text{sp}(BL(A), L_a)).$$

Theorem 4.1.12 *Let A be a normed unital Jordan complex algebra, and let a be in A. Then* J-sp$(A,a) \neq \emptyset$.

Proof Let $\hat{A}$ stand for the completion algebra of A. Since $\hat{A}$ is a unital Jordan algebra and J-Inv$(A) \subseteq$ J-Inv$(\hat{A})$, we have J-sp$(\hat{A},a) \subseteq$ J-sp(A,a). Therefore, we can assume that A is complete. Then the result follows from Corollary 4.1.11 and Theorem 1.1.41. (Alternatively, we can argue as in the proof of Theorem 1.1.41 with Corollary 4.1.5 instead of Corollary 1.1.14.) □

Definition 4.1.13 Let A be a Jordan algebra over $\mathbb{K}$. We say that A is a *J-division* algebra if A is unital and J-Inv$(A) = A \setminus \{0\}$.

Arguing as in the proof of Corollary 1.1.43, with Theorem 4.1.12 instead of Theorem 1.1.41, we get the following.

Corollary 4.1.14 *Let A be a normed J-division Jordan complex algebra. Then A is isomorphic to $\mathbb{C}$.*

The next result generalizes von Neumann's Lemma 1.1.20. For a good understanding of its formulation and of some subsequent results, §3.4.61 could be kept in mind.

Lemma 4.1.15 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, and let a be in A with $\mathfrak{r}(a) < 1$. Then $\mathbf{1} - a \in$ J-Inv(A) and*

$$(\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof Consider the closed subalgebra B of A generated by $\mathbf{1}$ and a. Then B is a complete normed unital associative and commutative algebra, and a is an element of B with $\mathfrak{r}(a) < 1$. By Lemma 1.1.20, $\mathbf{1} - a$ is invertible in B with inverse $c := \sum_{n=0}^{\infty} a^n$. Thus, clearly, $\mathbf{1} - a$ is a J-invertible element of A with J-inverse c. □

Lemma 4.1.16 *Let A be a normed power-associative algebra over $\mathbb{K}$, and let a be in A. Then $\mathfrak{r}(a) \leqslant \mathfrak{r}(L_a)$.*

Proof For each $n \in \mathbb{N}$ we have $\|a^{n+1}\| = \|L_a^n(a)\| \leqslant \|L_a^n\| \|a\|$. Therefore, by taking nth roots, and letting $n \to \infty$, the result follows. □

The next theorem generalizes the Gelfand–Beurling formula stated in Theorem 1.1.46.

Theorem 4.1.17 *Let A be a complete normed unital Jordan complex algebra, and let a be in A. Then* J-sp(A,a) *is a non-empty compact subset of $\mathbb{C}$, and we have*

$$\mathfrak{r}(a) = \mathfrak{r}(L_a) = \max\{|\lambda| : \lambda \in \text{J-sp}(A,a)\}.$$

Proof By Theorem 4.1.12 and Corollary 4.1.8, J-sp(A,a) is a non-empty closed subset of $\mathbb{C}$. On the other hand, if λ is a nonzero complex number such that $\lambda \in$ J-sp(A,a), then $a - \lambda\mathbf{1} \notin$ J-Inv(A), hence $\mathbf{1} - \lambda^{-1}a \notin$ J-Inv(A), so $\mathfrak{r}(\lambda^{-1}a) \geqslant 1$ (by Lemma 4.1.15), and so $|\lambda| \leqslant \mathfrak{r}(a)$. Thus J-sp$(A,a)$ is a compact subset of $\mathbb{C}$ contained in $\mathfrak{r}(a)\mathbb{B}_{\mathbb{C}}$.

Set $\rho(a) := \max\{|\lambda| : \lambda \in \text{J-sp}(A,a)\}$. It follows from the above that $\rho(a) \leqslant \mathfrak{r}(a)$. On the other hand, by Corollary 4.1.11, we have

$$\rho(a) = \max\{|\lambda| : \lambda \in \text{sp}(BL(A), L_a)\}.$$

Therefore, invoking Lemma 4.1.16 and Theorem 1.1.46, we realize that $\mathfrak{r}(a) \leqslant \mathfrak{r}(L_a) = \rho(a)$, and the proof is complete. □

We conclude this subsection by proving a wide non-associative generalization of Rickart's dense-range-homomorphism theorem. The core of the proof is the following Jordan version of Proposition 1.1.60.

Proposition 4.1.18 *Let A be a complete normed algebra over $\mathbb{K}$, let B be a normed unital Jordan algebra over $\mathbb{K}$, let $\Phi : A \to B$ be an algebra homomorphism, and let x be in A. Then*

$$\mathfrak{r}(\Phi(x)) \leqslant 3\|x\|. \tag{4.1.6}$$

As a consequence,

$$1 \leqslant 3\|x\| + \|\mathbf{1} - \Phi(x)\|. \tag{4.1.7}$$

Proof Regarding Φ as an algebra homomorphism from A to the completion of B, there is no loss of generality in assuming that B is complete. Moreover, in the case $\mathbb{K} = \mathbb{R}$, passing to projective normed complexifications (cf. Proposition 1.1.98), and extending Φ by complex linearity, we may additionally assume that in any case we have $\mathbb{K} = \mathbb{C}$. Then, in view of Theorem 4.1.17, to prove (4.1.6) it is enough to show that $\mathbf{1} - \Phi(x)$ is J-invertible in B whenever x is in A and satisfies $\|x\| < \frac{1}{3}$. Let x be in A with $\|x\| < \frac{1}{3}$. Then we have

$$\left\|2L_x + L_{x^2} - 2(L_x)^2\right\| \leqslant 2\|x\| + 3\|x\|^2 < 1,$$

so that by Lemma 1.1.20 and Example 1.1.12(d), $I_A - (2L_x + L_{x^2} - 2(L_x)^2)$ is a bijective operator on A. Therefore there exists some $y \in A$ satisfying

$$\left[I_A - \left(2L_x + L_{x^2} - 2(L_x)^2\right)\right](y) = x^2 - 2x$$

or, equivalently, $x^2 - 2x - y + 2xy + x^2y - 2x(xy) = 0$. Now we have the equality

$$\Phi(x)^2 - 2\Phi(x) - \Phi(y) + 2\Phi(x)\Phi(y) + \Phi(x)^2\Phi(y) - 2\Phi(x)(\Phi(x)\Phi(y)) = 0,$$

which can be reformulated as $U_{\mathbf{1}-\Phi(x)}(\mathbf{1} - \Phi(y)) = \mathbf{1}$. Therefore, by Theorem 4.1.3(ii), $\mathbf{1} - \Phi(x)$ is J-invertible in B, as desired. Now that (4.1.6) has been proved, note that the closed subalgebra of B generated by $\mathbf{1}$ and $\Phi(x)$ is associative. It follows from Corollary 1.1.81(ii) that

$$1 = \mathfrak{r}(\mathbf{1}) \leqslant \mathfrak{r}(\Phi(x)) + \mathfrak{r}(\mathbf{1} - \Phi(x)) \leqslant 3\|x\| + \|\mathbf{1} - \Phi(x)\|,$$

and the proof of the proposition is complete. □

Theorem 4.1.19 *Let A be a complete normed algebra over $\mathbb{K}$, let B be a complete normed Jordan-admissible strongly semisimple algebra over $\mathbb{K}$, and let $\Phi : A \to B$ be an algebra homomorphism with dense range. Then Φ is continuous.*

Proof Assume at first that B is actually unital and simple. By Lemma 1.1.58, $\mathfrak{S}(\Phi)$ is an ideal of B, and hence $\mathfrak{S}(\Phi) = 0$ or B because of the simplicity of B. Suppose that $\mathfrak{S}(\Phi) = B$. Then we can choose a sequence a_n in A such that $a_n \to 0$ and $\Phi(a_n) \to \mathbf{1}$, so that, regarding Φ as an algebra homomorphism from A^{sym} to B^{sym}, the inequality (4.1.7) in Proposition 4.1.18 applies to get $1 \leqslant 3\|a_n\| + \|\mathbf{1} - \Phi(a_n)\| \to 0$, a contradiction. Therefore $\mathfrak{S}(\Phi) = 0$, and Φ is continuous in view of Fact 1.1.56.

Now remove the assumption that B is unital and simple. Let a_n be a sequence in A with $a_n \to 0$ and $\Phi(a_n) \to b \in B$. In view of the closed graph theorem, to prove that Φ is continuous it is enough to show that $b = 0$. Let M be a maximal modular ideal of B. By Lemma 3.6.5, the quotient algebra B/M is a simple unital Jordan-admissible algebra over $\mathbb{K}$. Moreover, by Corollary 3.6.4, M is closed in B, and hence, by §1.1.55, B/M is a complete normed algebra. Since the quotient mapping $\pi : A \to A/M$ is a continuous surjective algebra homomorphism, it follows that $\pi \circ \Phi$ is an algebra homomorphism from A to B/M with dense range. Therefore, by the first paragraph in the proof, $\pi \circ \Phi$ is continuous. As a consequence, we have $(\pi \circ \Phi)(a_n) \to 0$. But, on the other hand, we deduce from the continuity of π that $(\pi \circ \Phi)(a_n) = \pi(\Phi(a_n)) \to \pi(b)$. Therefore $\pi(b) = 0$, and so $b \in M$. Now, keeping in mind the arbitrariness of the maximal modular ideal M, and the strong semisimplicity of B, we conclude that $b = 0$, as desired. □

Corollary 4.1.20 *Let A be a strongly semisimple Jordan-admissible algebra over $\mathbb{K}$. Then A has at most one complete algebra norm topology.*

Remark 4.1.21 We do not know any generalizations of Theorem 4.1.19. However, Corollary 4.1.20 will be significantly refined later. Indeed, remaining in the strongly semisimple setting, the requirement of Jordan-admissibility in Corollary 4.1.20 can be removed altogether (a consequence of Corollary 4.4.60). On the other hand, remaining in the Jordan-admissible setting, the requirement of strong semisimplicity in Corollary 4.1.20 can be relaxed to that of 'Jacobson semisimplicity' (see Definition 4.4.12 and Corollary 4.4.14) which, in the particular associative setting, is nothing other than the usual semisimplicity (cf. Definition 3.6.12 and Theorem 3.6.21). Actually, via Proposition 4.4.59, these refinements are by-products of the general uniqueness-of-norm theorem provided by Theorem 4.4.43.

4.1.2 Topological J-divisors of zero

Let A be an alternative algebra over $\mathbb{K}$, and let a be an element of A. Then a is a one-sided divisor of zero in A if and only if there exists $b \in A \setminus \{0\}$ such that $aba = 0$ (equivalently, $U_a(b) = 0$ in the Jordan algebra A^{sym}). This consideration leads to the following.

Definition 4.1.22 Let A be a Jordan algebra over $\mathbb{K}$. An element a of A is said to be a *J-divisor of zero* in A if there exists $b \in A \setminus \{0\}$ such that $U_a(b) = 0$. In this way, J-divisors of zero in A are nothing other than those elements a of A such that the operator U_a is not injective.

Proposition 4.1.23 *Let A be a normed alternative algebra over $\mathbb{K}$, and let a be an element of A. Then the following conditions are equivalent:*

(i) *a is a one-sided topological divisor of zero in A.*
(ii) *There exists a sequence x_n in $\mathbb{S}_A$ satisfying $\lim ax_na = 0$ (equivalently, $\lim U_a(x_n) = 0$ in the normed Jordan algebra A^{sym}).*

Proof The implication (i)⇒(ii) is clear. To prove (ii)⇒(i), assume the existence of a sequence x_n in $\mathbb{S}_A$ such that $\lim ax_na = 0$ and $\lim ax_n \neq 0$. By passing to a subsequence if necessary, we may assume the existence of a positive number ε such that $\|ax_n\| \geqslant \varepsilon$ for every $n \in \mathbb{N}$. Setting $y_n := \frac{ax_n}{\|ax_n\|}$, we realize that y_n is a sequence in $\mathbb{S}_A$ such that $\lim y_na = 0$. □

Proposition 4.1.23 above leads to the following.

Definition 4.1.24 Let A be a normed Jordan algebra over $\mathbb{K}$. An element a of A is said to be a *topological J-divisor of zero* in A if there exists a sequence x_n in $\mathbb{S}_A$ satisfying $\lim U_a(x_n) = 0$. In this way, topological J-divisors of zero in A are nothing other than those elements a of A such that the operator U_a is not bounded below. As a consequence, equivalent algebra norms on A give rise to the same topological J-divisors of zero. On the other hand, it is clear that every J-divisor of zero in A is a topological J-divisor of zero in A, and that the converse is true whenever A is finite-dimensional.

Proposition 4.1.25 *Let A be a normed Jordan algebra over $\mathbb{K}$, and let a be an element of A. We have:*

(i) *a is a topological J-divisor of zero in A if and only if U_a is a left topological divisor of zero in $BL(A)$.*
(ii) *If A is unital, and if a is a topological J-divisor of zero in A, then $a \notin$ J-Inv(A).*

Proof Assertion (i) follows from Proposition 1.1.94(i). Assume that A is unital, and that a is a topological J-divisor of zero in A. Then, by assertion (i) and Exercise 1.1.88(iii), $U_a \notin \mathrm{Inv}(BL(A))$, and hence, by Proposition 4.1.6(i), $a \notin$ J-Inv(A). □

The next result becomes the Jordan version of Proposition 1.1.90.

Proposition 4.1.26 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, and let a be in the boundary of* J-Inv(A) *relative to A. Then a is a topological J-divisor of zero in A.*

Proof Since J-Inv(A) is open in A (by Theorem 4.1.7) and a lies in the boundary of J-Inv(A) relative to A, it follows that $a \notin$ J-Inv(A) and there exists a sequence a_n in J-Inv(A) with a_n converging to a. By Proposition 4.1.6(i), we realize that $U_a \notin \mathrm{Inv}(BL(A))$, but $U_{a_n} \in \mathrm{Inv}(BL(A))$ for every n. Moreover, it follows from the continuity of the mapping $x \to U_x$ that U_{a_n} converges to U_a. Therefore U_a lies in the boundary of $\mathrm{Inv}(BL(A))$ relative to $BL(A)$. Finally, by Propositions 1.1.90 and 4.1.25(i), we conclude that a is a topological J-divisor of zero in A. $\square$

Corollary 4.1.27 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, let a be an element of A, and let λ be in the boundary of* J-sp(A,a) *relative to $\mathbb{K}$. Then $a-\lambda\mathbf{1}$ is a topological J-divisor of zero in A.*

Proposition 4.1.28 *Let A be a complete normed unital Jordan algebra over $\mathbb{K}$, and let B be a closed subalgebra of A containing the unit of A. Then we have:*

(i) J-Inv(B) *is a clopen subset of* J-Inv$(A) \cap B$, *and the boundary of* J-Inv(B) *relative to B is contained in the boundary of* J-Inv(A) *relative to A.*
(ii) *For each $b \in B$, we have* J-sp$(A,b) \subseteq$ J-sp(B,b), *and the boundary of* J-sp(B,b) *relative to $\mathbb{K}$ is contained in the boundary of* J-sp(A,b) *relative to $\mathbb{K}$. As a consequence,* co(J-sp$(B,b)) =$ co(J-sp(A,b)).
(iii) *If b is in B, and if* J-sp(A,b) *has no hole in $\mathbb{K}$, then* J-sp$(A,b) =$ J-sp(B,b).

Proof Argue as in the proof of Proposition 1.1.93, with Theorem 4.1.7 and Proposition 4.1.26 instead of Theorem 1.1.23 and Proposition 1.1.90, respectively. $\square$

The notation in §3.3.9 is involved in the formulations and proofs of the next results.

Corollary 4.1.29 *Let A be a complete normed unital Jordan complex algebra, let B be a closed subalgebra of A containing the unit of A, and let b be an element of B. Then*

$$\mathrm{co}\left(\mathrm{sp}\left(BL(B), L_b^B\right)\right) = \mathrm{co}\left(\mathrm{sp}\left(BL(A), L_b^A\right)\right).$$

Proof By Corollary 4.1.11 and Proposition 4.1.28(ii), we have

$$\mathrm{co}\left(\mathrm{sp}\left(BL(B), L_b^B\right)\right) = \mathrm{co}\left(\text{J-sp}(B,b)\right) = \mathrm{co}(\text{J-sp}(A,b)) = \mathrm{co}\left(\mathrm{sp}\left(BL(A), L_b^A\right)\right),$$

as desired. $\square$

Now, by passing to unital extensions (cf. §1.1.104), we are going to take advantage of Corollary 4.1.29 in the non-unital case.

Proposition 4.1.30 *Let A be an algebra over $\mathbb{K}$, and let a be in A. Then:*

(i) *For $\lambda \in \mathbb{K} \setminus \{0\}$, we have that $L_a^A - \lambda I_A \in \mathrm{Inv}(L(A))$ if and only if $L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}} \in \mathrm{Inv}(L(A_{\mathbb{1}}))$.*
(ii) $\mathrm{sp}(L(A_{\mathbb{1}}), L_a^{A_{\mathbb{1}}}) = \{0\} \cup \mathrm{sp}(L(A), L_a^A)$.

Proof Let $\lambda \in \mathbb{K} \setminus \{0\}$. Assume that $L_a^A - \lambda I_A$ is invertible in $L(A)$ with inverse F. It is easily realized that the linear mapping $T : A_{\mathbb{1}} \to A_{\mathbb{1}}$ defined by

$$T(\alpha \mathbb{1} + x) := F(x) + \lambda^{-1}\alpha(F(a) - \mathbb{1})$$

satisfies

$$\left(L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}}\right) T = T\left(L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}}\right) = I_{A_{\mathbb{1}}},$$

hence $L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}} \in \mathrm{Inv}(L(A_{\mathbb{1}}))$. Conversely, assume that $L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}}$ is invertible in $L(A_{\mathbb{1}})$ with inverse T. Then, for each $x \in A$ we have

$$\left(L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}}\right) T(x) = T\left(L_a^{A_{\mathbb{1}}} - \lambda I_{A_{\mathbb{1}}}\right)(x) = I_{A_{\mathbb{1}}}(x),$$

hence

$$aT(x) - \lambda T(x) = T(ax - \lambda x) = x. \tag{4.1.8}$$

In particular, we have $T(x) = \lambda^{-1}(aT(x) - x) \in A$. Therefore we can consider the linear mapping $F : A \to A$ defined by $F(x) = T(x)$. Now, note that (4.1.8) can be rewritten as follows

$$\left(L_a^A - \lambda I_A\right) F = F\left(L_a^A - \lambda I_A\right) = I_A.$$

Thus $L_a^A - \lambda I_A \in \mathrm{Inv}(L(A))$, and the proof of assertion (i) is complete.

It follows from assertion (i) that

$$\mathrm{sp}\left(L(A_{\mathbb{1}}), L_a^{A_{\mathbb{1}}}\right) \setminus \{0\} = \mathrm{sp}\left(L(A), L_a^A\right) \setminus \{0\}.$$

Since A is an ideal of $A_{\mathbb{1}}$, we realize that $L_a^{A_{\mathbb{1}}}(A_{\mathbb{1}}) \subseteq A$, so $L_a^{A_{\mathbb{1}}} \notin \mathrm{Inv}(L(A_{\mathbb{1}}))$, hence $0 \notin \mathrm{sp}(L(A_{\mathbb{1}}), L_a^{A_{\mathbb{1}}})$. Thus assertion (ii) follows. □

Corollary 4.1.31 *Let A be a complete normed Jordan complex algebra, let B be a closed subalgebra of A, and let b be an element of B. Then*

$$\mathrm{co}\left(\{0\} \cup \mathrm{sp}\left(BL(B), L_b^B\right)\right) = \mathrm{co}\left(\{0\} \cup \mathrm{sp}(BL(A), L_b^A)\right).$$

Proof Note that $A_{\mathbb{1}}$ is a complete normed unital Jordan complex algebra, and that $B_{\mathbb{1}}$ can be seen as a closed subalgebra of $A_{\mathbb{1}}$ containing the unit of $A_{\mathbb{1}}$. Therefore, by Corollary 4.1.29, we have

$$\mathrm{co}\left(\mathrm{sp}\left(BL(B_{\mathbb{1}}), L_b^{B_{\mathbb{1}}}\right)\right) = \mathrm{co}\left(\mathrm{sp}\left(BL(A_{\mathbb{1}}), L_b^{A_{\mathbb{1}}}\right)\right). \tag{4.1.9}$$

On the other hand, by Example 1.1.32(d), we know that

$$\mathrm{sp}\left(BL(A), L_b^A\right) = \mathrm{sp}\left(L(A), L_b^A\right) \text{ and } \mathrm{sp}\left(BL(A_{\mathbb{1}}), L_b^{A_{\mathbb{1}}}\right) = \mathrm{sp}\left(L(A_{\mathbb{1}}), L_b^{A_{\mathbb{1}}}\right),$$

so that, by Proposition 4.1.30, we find that

$$\mathrm{sp}\left(BL(A_{\mathbb{1}}), L_b^{A_{\mathbb{1}}}\right) = \{0\} \cup \mathrm{sp}\left(BL(A), L_b^A\right) \tag{4.1.10}$$

and

$$\mathrm{sp}\left(BL(B_{\mathbb{1}}), L_b^{B_{\mathbb{1}}}\right) = \{0\} \cup \mathrm{sp}\left(BL(B), L_b^B\right). \tag{4.1.11}$$

By combining (4.1.9), (4.1.10), and (4.1.11), the result follows. □

4.1.3 Non-commutative JB^*-algebras are JB^*-triples

Definition 4.1.32 By a *Jordan $*$-triple* over $\mathbb{K}$ we mean a vector space J over $\mathbb{K}$ endowed with a mapping $\{\cdots\} : J \times J \times J \to J$ (called the *triple product* of J), which is linear in the outer variables and conjugate-linear in the middle variable, and satisfies the 'commutative' condition

$$\{xyz\} = \{zyx\} \tag{4.1.12}$$

and the *Jordan triple identity*

$$\{uv\{xyz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\} + \{xy\{uvz\}\}. \tag{4.1.13}$$

Note that, by taking $z = x$ in (4.1.13), and keeping in mind (4.1.12), we obtain

$$\{uv\{xyx\}\} = 2\{\{uvx\}yx\} - \{x\{vuy\}x\}. \tag{4.1.14}$$

By taking $u = x$ in (4.1.14), keeping in mind (4.1.12), and applying (4.1.14) again, we see that

$$\begin{aligned}\{xv\{xyx\}\} &= 2\{\{xvx\}yx\} - \{x\{vxy\}x\} = 2\{xy\{xvx\}\} - \{x\{vxy\}x\} \\ &= 2(2\{\{xyx\}vx\} - \{x\{yxv\}x\}) - \{x\{vxy\}x\} \\ &= 4\{\{xyx\}vx\} - 3\{x\{vxy\}x\},\end{aligned}$$

hence $3\{xv\{xyx\}\} = 3\{x\{vxy\}x\}$, and so

$$\{xv\{xyx\}\} = \{x\{vxy\}x\} = \{\{xvx\}yx\}. \tag{4.1.15}$$

When necessary, we write $\{\cdot,\cdot,\cdot\}$ instead of $\{\cdots\}$.

Jordan algebras and Jordan $*$-triples are closely related as we shall see in the results that follow.

Lemma 4.1.33 *Let A be a Jordan algebra over $\mathbb{K}$. Then*

$$U_{ab,c} + U_{a,bc} = L_b U_{a,c} + U_{a,c} L_b \ \text{ for all } \ a,b,c \in A.$$

Proof By Lemma 3.3.35, for all x,y,z in A we have the following double writing to the operator $U_{xz,y}$

$$U_{xz,y} = U_{x,y}L_z + U_{z,y}L_x - L_y U_{x,z} = L_z U_{x,y} + L_x U_{z,y} - U_{x,z}L_y. \tag{4.1.16}$$

For a,b,c given in A, replacing (x,y,z) with (a,c,b) in the first writing of $U_{xz,y}$ in (4.1.16), we obtain

$$U_{ab,c} = U_{a,c}L_b + U_{b,c}L_a - L_c U_{a,b}, \tag{4.1.17}$$

whereas, replacing (x,y,z) with (c,a,b) in the second writing of $U_{xz,y}$ in (4.1.16), we get

$$U_{a,bc} = L_b U_{a,c} + L_c U_{a,b} - U_{b,c} L_a. \tag{4.1.18}$$

Adding identities (4.1.17) and (4.1.18), the result follows. □

Proposition 4.1.34 *Let A be a Jordan $*$-algebra over $\mathbb{K}$. Then A is a Jordan $*$-triple over $\mathbb{K}$ for the triple product $\{abc\} := U_{a,c}(b^*)$.*

Proof It is clear that the triple product $\{\cdots\}$ just defined is linear in the outer variables and conjugate-linear in the middle variable, and satisfies the commutative condition (4.1.12). In order to verify the Jordan triple identity (4.1.13), consider the trilinear triple product $[abc] := U_{a,c}(b)$, and note that

$$[abc] = ([L_a, L_b] + L_{ab})(c). \tag{4.1.19}$$

It is immediate to realize that, for every derivation D of A, we have

$$DU_{a,c} = U_{D(a),c} + U_{a,c}D + U_{a,D(c)} \tag{4.1.20}$$

for all $a, c \in A$. Since $[L_u, L_v]$ is a derivation of A for all $u, v \in A$ (by Lemma 3.1.23), it follows from (4.1.20) that

$$[L_u, L_v]U_{a,c} = U_{[L_u,L_v](a),c} + U_{a,c}[L_u, L_v] + U_{a,[L_u,L_v](c)} \tag{4.1.21}$$

for all $a, c \in A$. Moreover, by Lemma 4.1.33 we have

$$L_{uv}U_{a,c} = U_{L_{uv}(a),c} - U_{a,c}L_{uv} + U_{a,L_{uv}(c)} \tag{4.1.22}$$

for all $a, c \in A$. Now, since clearly $[L_u, L_v] - L_{uv} = -([L_v, L_u] + L_{vu})$, it follows from (4.1.19), (4.1.21), and (4.1.22) that

$$[uv[abc]] = [[uva]bc] - [a[vub]c] + [ab[uvc]] \tag{4.1.23}$$

for all $u, v, a, b, c \in A$. Finally, keeping in mind that

$$\{abc\} = [ab^*c] \text{ and } [abc]^* = [a^*b^*c^*],$$

and consequently $\{abc\}^* = [a^*bc^*]$, by replacing v and b with v^* and b^*, respectively, in (4.1.23) we obtain

$$\{uv\{abc\}\} = \{\{uva\}bc\} - \{a\{vub\}c\} + \{ab\{uvc\}\}$$

for all $u, v, a, b, c \in A$, and the proof concludes. □

Now we will see that every Jordan $*$-triple can be regarded as a family of Jordan algebra structures.

Proposition 4.1.35 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be in J. Then J becomes a Jordan algebra over $\mathbb{K}$ under the product $xy = \{xey\}$.*

Proof The product just defined is commutative by (4.1.12). Let x, y be in J. Setting $z := xy$, (4.1.14) implies

$$(xy)x^2 = \{ze\{xex\}\} = 2\{\{zex\}ex\} - \{x\{eze\}x\}$$

and

$$2\{zex\} = 2\{\{yex\}ex\} = \{ye\{xex\}\} + \{x\{eye\}x\}.$$

Applying (4.1.15) twice, we get

$$\{x\{eze\}x\} = \{x\{e\{xey\}e\}x\} = \{x\{ex\{eye\}\}x\} = \{xe\{x\{eye\}x\}\}.$$

Combining these identities, it follows that

$$\begin{aligned}(xy)x^2 &= \{\{ye\{xex\}\}ex\} + \{\{x\{eye\}x\}ex\} - \{xe\{x\{eye\}x\}\} \\ &= \{\{ye\{xex\}\}ex\} = \{xe\{ye\{xex\}\}\} = x(yx^2).\end{aligned}$$

Hence J is a Jordan algebra. □

Definition 4.1.36 Given a Jordan $*$-triple J over $\mathbb{K}$ and an element e in J, the Jordan algebra over $\mathbb{K}$ obtained from e via Proposition 4.1.35 above is called the *e-homotope algebra* of J, and is denoted by $J^{(e)}$.

Given elements x,y in a Jordan $*$-triple J, we denote by $L(x,y)$ the linear operator on J defined by $L(x,y)(z) := \{xyz\}$.

Definition 4.1.37 By a *Banach Jordan $*$-triple* over $\mathbb{K}$ we mean a Jordan $*$-triple over $\mathbb{K}$ endowed with a complete norm making the triple product continuous. Let J be a Banach Jordan $*$-triple over $\mathbb{K}$. Then clearly $L(x,y)$ belongs to $BL(J)$ for all $x,y \in J$. We say that J is *hermitian* if $\mathbb{K} = \mathbb{C}$ and if, for every $x \in J$, $L(x,x)$ lies in $H(BL(J), I_J)$. If in addition the inclusion

$$\mathrm{sp}(BL(J), L(x,x)) \subseteq \mathbb{R}_0^+$$

holds for every $x \in J$, then we say that J is *positive hermitian*. We will apply without notice that, thanks to Proposition 2.3.23, *the complex Banach Jordan $*$-triple J is positive hermitian if and only if $V(BL(J), I_J, L(x,x)) \subseteq \mathbb{R}_0^+$ for every $x \in J$.*

By a *subtriple* of a Jordan $*$-triple J we mean a subspace M of J such that $\{xyz\}$ lies in M whenever x,y,z are in M. Clearly, subtriples of a Jordan $*$-triple are Jordan $*$-triples.

As a straightforward consequence of Lemma 2.2.24, we get the following.

Lemma 4.1.38 *Let J be a hermitian (respectively, positive hermitian) Banach Jordan $*$-triple, and let M be a closed subtriple of J. Then the Banach Jordan $*$-triple M is hermitian (respectively, positive hermitian).*

§4.1.39 Now, the reader is invited to recall the definition of a JB^*-triple, given in §2.2.27, and to note that, according to Definitions 4.1.32 and 4.1.37, JB^*-triples are nothing other than those positive hermitian Banach Jordan $*$-triples J such that the equality

$$\|\{xxx\}\| = \|x\|^3 \tag{4.1.24}$$

holds for every $x \in J$. Therefore, as a first consequence of Lemma 4.1.38 above, we get the following.

Fact 4.1.40 *Every closed subtriple of a JB^*-triple is a JB^*-triple.*

Now, as announced in Subsection 2.2.3, we can prove the following.

Fact 4.1.41 *Let A be a C^*-algebra. Then A becomes a JB^*-triple under its own norm and the triple product*

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x). \tag{4.1.25}$$

Proof We straightforwardly realize that $(A, \{\cdots\})$ is a complex Banach Jordan $*$-triple (perhaps note that A^{sym} is a Jordan algebra, and apply Proposition 4.1.34). On the other hand, the equality (4.1.24) follows from Lemma 3.2.6. Therefore it only remains to show that $(A, \{\cdots\})$ is positive hermitian. To this end, note that, by

Proposition 1.2.44 and Lemma 4.1.38, we may assume that A is unital. Let x be in A. By Lemmas 2.2.5 and 2.3.26, we have

$$V(A,\mathbf{1},xx^*)\cup V(A,\mathbf{1},x^*x)\subseteq\mathbb{R}_0^+,$$

and hence, by Lemma 2.1.10, we get

$$V(BL(A),I_A,L_{xx^*}+R_{x^*x})\subseteq\mathbb{R}_0^+.$$

Noticing that the Jordan $*$-triple structure and the $*$-algebra structure of A are related by means of the equality $L(x,x)=\frac{1}{2}(L_{xx^*}+R_{x^*x})$, the proof is concluded. $\square$

The next lemma becomes the crucial link between the present subsection and the preceding one. It will be applied in the proofs of Lemma 4.1.43 and Proposition 4.1.50.

Lemma 4.1.42 *Let J be a complex Banach Jordan $*$-triple, let M be a closed subtriple of J, and let x,y be in M. Then we have*

$$\operatorname{co}\left(\{0\}\cup\operatorname{sp}\left(BL(M),L^M(x,y)\right)\right)=\operatorname{co}\left(\{0\}\cup\operatorname{sp}\left(BL(J),L^J(x,y)\right)\right).$$

Proof By Proposition 4.1.35, J becomes a complex Jordan algebra (say A) under the product $uv:=\{uyv\}$. Moreover, up to multiplication of the norm of J by a suitable positive number, A is a complete normed algebra, and the equality $L_x^A=L^J(x,y)$ holds. Noticing that M becomes a closed subalgebra (say B) of A, and that the equality $L_x^B=L^M(x,y)$ holds, the proof is concluded by applying Corollary 4.1.31. $\square$

The next lemma is a converse form of Lemma 4.1.38.

Lemma 4.1.43 *Let J be a hermitian Banach Jordan $*$-triple such that for each $x\in J$ there exists a positive hermitian closed subtriple of J containing x. Then J is positive hermitian.*

Proof Let us fix $x\in J$ and, according to the assumption, take a positive hermitian closed subtriple M of J with $x\in M$. Since both M and J are hermitian Banach Jordan $*$-triples, Proposition 2.3.23 applies, so that we have

$$V(BL(M),I_M,L^M(x,x))=\operatorname{co}\left(\operatorname{sp}(BL(M),L^M(x,x))\right)$$

and

$$V(BL(J),I_J,L^J(x,x))=\operatorname{co}\left(\operatorname{sp}(BL(J),L^J(x,x))\right).$$

It follows from Lemma 4.1.42 that

$$\operatorname{co}(\{0\}\cup V(BL(M),I_M,L^M(x,x)))=\operatorname{co}\left(\{0\}\cup V(BL(J),I_J,L^J(x,x))\right).$$

Therefore, since $V(BL(M),I_M,L^M(x,x))\subseteq\mathbb{R}_0^+$ (because M is positive hermitian), we get $V(BL(J),I_J,L^J(x,x))\subseteq\mathbb{R}_0^+$. Finally, since x is arbitrary in J, we conclude that J is positive hermitian. $\square$

Let J be a (Banach) Jordan $*$-triple over $\mathbb{K}$, and let S be a non-empty subset of J. Since the intersection of any family of (closed) subtriples of J is a (closed) subtriple of J, it follows that the intersection of all (closed) subtriples of J containing S is the smallest (closed) subtriple of J containing S. This subtriple is called *the (closed)* *subtriple of J generated by S*. In the case where J is a Banach Jordan $*$-triple, the closure in J of any subtriple of J is a subtriple too, and hence the closed subtriple of J generated by S coincides with the closure in J of the subtriple generated by S.

Combining Lemmas 4.1.38 and 4.1.43, we obtain the following.

Corollary 4.1.44 *A hermitian Banach Jordan $*$-triple J is positive hermitian (respectively, a JB^*-triple) if and only if the closed subtriple generated by each of its elements is positive hermitian (respectively, a JB^*-triple).*

Now we can prove the main result in this subsection.

Theorem 4.1.45 *Let A be a non-commutative JB^*-algebra. Then A becomes a JB^*-triple under its own norm and the triple product*

$$\{xyz\} := U_{x,z}(y^*). \tag{4.1.26}$$

Proof By Facts 3.3.3 and 3.3.4, we may assume that A is a (commutative) JB^*-algebra. Then, since $*$ is continuous on A (by Proposition 3.3.13), it is enough to apply Proposition 4.1.34 to realize that A becomes a complex Banach Jordan $*$-triple (say J) under the product (4.1.26). Moreover, for $x \in J$, the equality $\|\{xxx\}\| = \|x\|^3$ follows from the axiom $\|U_x(x^*)\| = \|x\|^3$ of JB^*-algebras. Therefore, it only remains to show that J is positive hermitian.

We begin by proving that J is hermitian. To this end, we note that, by Corollary 3.4.10 and Lemma 4.1.38, we may additionally assume that the JB^*-algebra A is unital. Let x be in J. Then, since the Jordan $*$-triple structure of J is related to the $*$-algebra structure of A by means of the equality

$$L^J(x,x) = L^A_{xx^*} + \left[L^A_x, L^A_{x^*}\right],$$

and $D := [L^A_x, L^A_{x^*}]$ is a derivation of A (by Lemma 3.1.23) satisfying

$$D(y^*) = -D(y)^* \ \text{ for every } \ y \in A,$$

it is enough to apply Proposition 3.4.28 to get that $L^J(x,x)$ lies in $H(BL(J), I_J)$. Since x is arbitrary in J, we conclude that J is hermitian, as desired.

Now, let us fix $x \in J$. Then, by Proposition 3.4.6, there exists a C^*-algebra B such that the closed subalgebra of A generated by $\{x, x^*\}$ (say C) can be seen as a closed $*$-subalgebra of B^{sym}. Since B becomes a positive hermitian Banach Jordan $*$-triple under the triple product (4.1.25) (cf. Fact 4.1.41), and the restriction to C of this triple product coincides with the restriction to C of the triple product (4.1.26) of J, it follows from Lemma 4.1.38 that C becomes a positive hermitian closed Jordan subtriple of J containing x. Finally, since J is hermitian (by the above paragraph), and x is arbitrary in J, it is enough to invoke Lemma 4.1.43 to conclude that J is positive hermitian. Thus the proof is complete. □

Now we are going to determine unital JB^*-algebras among JB^*-triples. To this end, we begin with the following.

Proposition 4.1.46 *Let A be a unital Jordan real algebra endowed with a vector space complete norm $\|\cdot\|$ satisfying $\|x^2\| = \|x\|^2 \leqslant \|x^2 + y^2\|$ for all $x, y \in A$. Then A is a JB-algebra.*

Proof For $a,b \in \mathbb{S}_A$ we have

$$\begin{aligned}\|ab\| &= \frac{1}{4}\left\|(a+b)^2-(a-b)^2\right\| \leqslant \frac{1}{4}\left(\left\|(a+b)^2\right\|+\left\|(a-b)^2\right\|\right)\\ &= \frac{1}{4}\left(\|a+b\|^2+\|a-b\|^2\right) \leqslant \frac{1}{2}\left(\|a\|+\|b\|\right)^2 = 2.\end{aligned}$$

The above implies $\|ab\| \leqslant 2\|a\|\|b\|$ for all $a,b \in A$, and hence the product of A is continuous. Let us fix an element x in A, denote by B the closed subalgebra of A generated by $\{\mathbf{1},x\}$, and consider the smallest positive number M such that

$$\|uv\| \leqslant M\|u\|\|v\| \text{ for all } u,v \text{ in } B.$$

Since B is a commutative and associative algebra, for all $u,v \in B$ we have

$$\|uv\|^2 = \left\|(uv)^2\right\| = \left\|u^2v^2\right\| \leqslant M\left\|u^2\right\|\left\|v^2\right\| = M\|u\|^2\|v\|^2,$$

and hence $\|uv\| \leqslant M^{\frac{1}{2}}\|u\|\|v\|$. Therefore $M \leqslant M^{\frac{1}{2}}$, so $M \leqslant 1$, and so the norm of B is an algebra norm. Thus B is a *JB*-algebra. Therefore, as a consequence of Proposition 3.1.4(iii), we have $v(B,\mathbf{1},x) = \|x\|$, and, by Lemma 3.1.29(i), $f(x^2) \geqslant 0$ whenever f is in $D(B,\mathbf{1})$. Since elements of $D(B,\mathbf{1})$ are nothing other than the restrictions to B of the elements of $D(A,\mathbf{1})$, it follows from the arbitrariness of x in A that, for each $a \in A$, there exists $f \in D(A,\mathbf{1})$ such that $\|a\| = |f(a)|$, and that, for each $f \in D(A,\mathbf{1})$, the mapping $(a,b) \to f(ab)$ from $A \times A$ to $\mathbb{R}$ becomes a non-negative symmetric bilinear form. Now, for $a,b \in A$, choose $f \in D(A,\mathbf{1})$ such that $\|ab\| = |f(ab)|$. Then, by the Cauchy–Schwarz inequality we have

$$\|ab\| = |f(ab)| \leqslant \sqrt{f(a^2)f(b^2)} \leqslant \sqrt{\|a^2\|\|b^2\|} = \|a\|\|b\|.$$

As a result, A is a *JB*-algebra. □

§**4.1.47** Given an element x of a Jordan $*$-triple J over $\mathbb{K}$, the *triple powers* of x are inductively defined by

$$x^{(1)} = x \text{ and } x^{(2n+3)} = \left\{xx^{(2n+1)}x\right\} \text{ for every } n \in \mathbb{N}\cup\{0\},$$

so that $x^{(3)} = \{xxx\}$.

A Jordan $*$-triple J is called *abelian* if the identity

$$\{uv\{xyz\}\} = \{\{uvx\}yz\} \tag{4.1.27}$$

is satisfied. It is immediate to verify that an equivalent condition to (4.1.27) is that $L(J,J) := \{L(x,y) : x,y \in J\}$ is a commutative set of linear operators on J. Moreover, in view of the Jordan triple identity (4.1.13), it is clear that (4.1.27) is also equivalent to the identity

$$\{uv\{xyz\}\} = \{u\{vxy\}z\}. \tag{4.1.28}$$

It is easy to show that, for elements x,y,z in an abelian Jordan $*$-triple J, we have

$$\{xyz\}^{(3)} = \left\{x^{(3)}y^{(3)}z^{(3)}\right\} \text{ and } L(x^{(3)},x^{(3)}) = (L(x,x))^3.$$

Lemma 4.1.48 *Let J be an abelian Banach Jordan $*$-triple over $\mathbb{K}$.*

(i) *If for each $x \in J$ the equality $\|\{xxx\}\| = \|x\|^3$ holds, then we have*
$$\|\{xyz\}\| \leqslant \|x\|\|y\|\|z\| \text{ for all } x,y,z \in J,$$
and hence $\|L(x,x)\| = \|x\|^2$ for every $x \in J$.

(ii) *If for each $x \in J$ the equality $\mathfrak{r}(L(x,x)) = \|x\|^2$ holds, then we have*
$$\|\{xxx\}\| = \|x\|^3 \text{ for every } x \in J.$$

Proof Assume that for each $x \in J$ the equality $\|\{xxx\}\| = \|x\|^3$ holds. Let c be the smallest positive constant such that $\|\{xyz\}\| \leqslant c\|x\|\|y\|\|z\|$ for all $x,y,z \in J$. Since J is abelian, it follows that
$$\begin{aligned}\|\{xyz\}\|^3 &= \left\|\{xyz\}^{(3)}\right\| = \left\|\left\{x^{(3)}y^{(3)}z^{(3)}\right\}\right\| \\ &\leqslant c\left\|x^{(3)}\right\|\left\|y^{(3)}\right\|\left\|z^{(3)}\right\| = c\left(\|x\|\,\|y\|\|z\|\right)^3.\end{aligned}$$

Hence $c \leqslant c^{\frac{1}{3}}$, so $c \leqslant 1$, and so $\|\{xyz\}\| \leqslant \|x\|\|y\|\|z\|$ for all $x,y,z \in J$. As a consequence, we deduce that $\|L(x,x)\| \leqslant \|x\|^2$ for every $x \in J$. On the other hand, we have
$$\|x\|^3 = \|\{xxx\}\| = \|L(x,x)(x)\| \leqslant \|L(x,x)\|\|x\|,$$
and hence $\|x\|^2 \leqslant \|L(x,x)\|$. Thus $\|L(x,x)\| = \|x\|^2$ for every $x \in J$.

Now assume that for each $x \in J$ the equality $\mathfrak{r}(L(x,x)) = \|x\|^2$ holds. Then, since J is abelian, for $x \in J$ we have
$$\begin{aligned}\|\{xxx\}\|^2 &= \left\|x^{(3)}\right\|^2 = \mathfrak{r}\left(L\left(x^{(3)},x^{(3)}\right)\right) = \mathfrak{r}\left((L(x,x))^3\right) \\ &= (\mathfrak{r}(L(x,x)))^3 = \|x\|^6,\end{aligned}$$
and hence $\|\{xxx\}\| = \|x\|^3$. □

Lemma 4.1.49 *Let J be a Jordan $*$-triple over $\mathbb{K}$, let z be an element of J, and let M stand for the subtriple of J generated by z. Then we have*
$$\left\{z^{(2m+1)}z^{(2k+1)}z^{(2n+1)}\right\} = z^{(2(m+k+n+1)+1)} \text{ for all } m,k,n \in \mathbb{N}\cup\{0\}. \tag{4.1.29}$$
Consequently, M coincides with the linear hull of all triple powers of z, and is abelian. Moreover, the Jordan algebra $M^{(z)}$ (cf. Definition 4.1.36) *is the subalgebra of $J^{(z)}$ generated by z.*

Proof We recall that $J^{(z)}$ is the Jordan algebra obtained by endowing J with the product $xy := \{xzy\}$, and that, as any Jordan algebra, $J^{(z)}$ is power-associative (cf. Proposition 2.4.13(i)). First of all, we note that the triple powers of z in J are closely related to the powers of z in $J^{(z)}$. More precisely,
$$z^{(2n+1)} = z^{n+1} \text{ for every } n \in \mathbb{N}\cup\{0\}. \tag{4.1.30}$$

We argue by induction on n. Clearly $z^{(1)} = z = z^1$. Assume that $p \in \mathbb{N}$ is such that (4.1.30) is true for every value n with $n \leqslant p$. Then we have
$$z^{(2p+3)} = \left\{zz^{(2p+1)}z\right\} = \left\{zz^{p+1}z\right\} = \left\{z\left\{z^pzz\right\}z\right\},$$

and applying (4.1.15) we obtain

$$z^{(2p+3)} = \{\{zz^p z\} zz\} = \left\{\left\{zz^{(2p-1)}z\right\} zz\right\} = \left\{z^{(2p+1)}zz\right\} = \left\{z^{p+1}zz\right\} = z^{p+2}.$$

Now that (4.1.30) has been established, we are going to prove (4.1.29). Indeed, if $k=0$, then

$$\left\{z^{(2m+1)}z^{(1)}z^{(2n+1)}\right\} = \left\{z^{m+1}zz^{n+1}\right\} = z^{m+1}z^{n+1} = z^{m+n+2} = z^{(2(m+n+1)+1)},$$

whereas, if $k \in \mathbb{N}$, then

$$\begin{aligned}\left\{z^{(2m+1)}z^{(2k+1)}z^{(2n+1)}\right\} &= \left\{z^{(2m+1)}\left\{zz^{(2k-1)}z\right\} z^{(2n+1)}\right\}\\ &= \left\{z^{m+1}\left\{zz^k z\right\} z^{n+1}\right\},\end{aligned}$$

and applying (4.1.13)

$$\begin{aligned}&\left\{z^{(2m+1)}z^{(2k+1)}z^{(2n+1)}\right\}\\ &= \left\{\left\{z^k zz^{m+1}\right\} zz^{n+1}\right\} + \left\{z^{m+1}z\left\{z^k zz^{n+1}\right\}\right\} - \left\{z^k z\left\{z^{m+1}zz^{n+1}\right\}\right\}\\ &= \left(z^k z^{m+1}\right) z^{n+1} + z^{m+1}\left(z^k z^{n+1}\right) - z^k\left(z^{m+1}z^{n+1}\right)\\ &= z^{m+k+n+2} = z^{(2(m+k+n+1)+1)}.\end{aligned}$$

It follows from (4.1.29) that the linear hull of all triple powers of z is an abelian Jordan $*$-triple, which clearly coincides with the subtriple M of J generated by z. Moreover, it follows from (4.1.30) that $M^{(z)}$ coincides with the subalgebra of $J^{(z)}$ generated by z. □

Proposition 4.1.50 *Let J be a hermitian Banach Jordan $*$-triple. Then the following conditions are equivalent:*

(i) *The equality $\|\{xxx\}\| = \|x\|^3$ holds for every $x \in J$.*
(ii) *The equality $\|L(x,x)\| = \|x\|^2$ holds for every $x \in J$.*

Moreover, if the above conditions are fulfilled, then we have $\mathfrak{r}(L(x,x)) = \|x\|^2$ for every $x \in J$.

Proof Let y be in J, and let M stand for the closed subtriple of J generated by y. Then, as a consequence of Lemma 4.1.42, we have

$$\mathfrak{r}\left(L^M(z,z)\right) = \mathfrak{r}\left(L^J(z,z)\right) \text{ for every } z \in M.$$

Therefore, since both M and J are hermitian Banach Jordan $*$-triples, Proposition 2.3.22 applies to get

$$\left\|L^M(z,z)\right\| = \mathfrak{r}\left(L^M(z,z)\right) = \mathfrak{r}\left(L^J(z,z)\right) = \left\|L^J(z,z)\right\| \text{ for every } z \in M. \tag{4.1.31}$$

Note also that, by Lemma 4.1.49, M is abelian.

Assume that condition (i) is true for J. Then it remains true for M, and hence, by Lemma 4.1.48(i), we have $\|L^M(y,y)\| = \|y\|^2$. Therefore, invoking (4.1.31), we get $\mathfrak{r}(L^J(y,y)) = \|L^J(y,y)\| = \|y\|^2$. Since y is arbitrary in J, we realize that condition (ii), as well as the last conclusion in the proposition, are true for J.

Now assume that condition (ii) is true for J. Then, by (4.1.31), it remains true for M, and moreover for each $z \in M$ the equality $\mathfrak{r}(L^M(z,z)) = \|z\|^2$ holds. Therefore, by Lemma 4.1.48(ii), we have $\|\{yyy\}\| = \|y\|^3$. Since y is arbitrary in J, we realize that condition (i) is true for J. $\square$

The following corollary follows straightforwardly from §4.1.39 and Proposition 4.1.50.

Corollary 4.1.51 *JB^*-triples are precisely those positive hermitian Banach Jordan $*$-triples J satisfying $\|L(x,x)\| = \|x\|^2$ for every $x \in J$. If J is a JB^*-triple, then the equality $\mathfrak{r}(L(x,x)) = \|x\|^2$ holds for every $x \in J$.*

A mapping F between Jordan $*$-triples J and K over $\mathbb{K}$ is called a *triple homomorphism* if it is linear and preserves triple products, i.e.

$$F(\{xyz\}) = \{F(x)F(y)F(z)\} \text{ for all } x,y,z \in J.$$

Proposition 4.1.52 *Let J and K be JB^*-triples, and let $F : J \to K$ be a bijective triple homomorphism. Then F is an isometry.*

Proof Argue as in the proof of the implication (iii)⇒(i) in Theorem 2.2.19 with the equality $\mathfrak{r}(L(x,x)) = \|x\|^2$ in Corollary 4.1.51 instead of Lemma 2.2.10. $\square$

Replacing Corollary 4.1.51 with Proposition 4.1.50 in the above proof, we realize that Proposition 4.1.52 remains true if we relax 'JB^*-triples' to 'hermitian Banach Jordan $*$-triples satisfying $\|\{xxx\}\| = \|x\|^3$ for every element x'.

Definition 4.1.53 An element e of a Jordan $*$-triple J over $\mathbb{K}$ is called *unitary* if $\{eex\} = x$ for every $x \in J$.

Proposition 4.1.54 *There is a bijective correspondence between unital Jordan $*$-algebras over $\mathbb{K}$ and nonzero Jordan $*$-triples over $\mathbb{K}$ with a distinguished unitary element. More precisely, if A is a Jordan $*$-algebra over $\mathbb{K}$ with unit element $e \neq 0$, then e is a unitary element with respect to the Jordan $*$-triple product on A given by $\{abc\} := U_{a,c}(b^*)$. Conversely, if J is a nonzero Jordan $*$-triple over $\mathbb{K}$, and if e is a unitary element of J, then J endowed with the product $x \odot y := \{xey\}$ and the involution $x^* := \{exe\}$ becomes a Jordan $*$-algebra over $\mathbb{K}$ with unit element $e \neq 0$. These two constructions are mutually inverse.*

Proof If A is a Jordan $*$-algebra with unit element $e \neq 0$, then $e^* = e$, and therefore e is a unitary element for the Jordan $*$-triple associated to A via Proposition 4.1.34. Further, the Jordan product $\odot$ associated to the Jordan $*$-triple A with distinguished unitary element e via Proposition 4.1.35 coincides with the original product of A since

$$a \odot b = \{aeb\} = U_{a,b}(e) = a(be) + b(ae) - (ab)e = ab.$$

Conversely, suppose J is a nonzero Jordan $*$-triple with unitary element e. By Proposition 4.1.35, J is a Jordan algebra with unit element e for the product $x \odot y = \{xey\}$. By (4.1.14), we have

$$\{e\{exe\}e\} = 2\{\{xee\}ee\} - \{xe\{eee\}\} = 2\{xee\} - \{xee\} = 2x - x = x$$

for every $x \in J$, showing that the conjugate-linear mapping $x \to x^* := \{exe\}$ is involutive. Further, (4.1.15) implies

$$(x \odot y)^* = \{e\{xey\}e\} = \{ex\{eye\}\} = \{exy^*\},$$

whereas (4.1.13) implies

$$x^* \odot y^* = \{\{exe\}ey^*\} = \{ex\{eey^*\}\} - \{ee\{exy^*\}\} + \{e\{xee\}y^*\} = \{exy^*\}.$$

Hence $*$ is an algebra involution on the Jordan algebra J. Further, the Jordan $*$-triple product associated to the involutive unital Jordan algebra J via Proposition 4.1.34 coincides with the original Jordan $*$-triple product, since (4.1.13) implies

$$\begin{aligned} U_{x,z}(y^*) &= x \odot (y^* \odot z) + z \odot (x \odot y^*) - (x \odot z) \odot y^* \\ &= \{xe\{y^*ez\}\} + \{ze\{xey^*\}\} - \{\{xez\}ey^*\} = \{x\{ey^*e\}z\} = \{xyz\}. \end{aligned}$$

□

Theorem 4.1.55 *There is a bijective correspondence between unital JB^*-algebras and nonzero JB^*-triples with a distinguished unitary element. More precisely, if A is a JB^*-algebra with unit element $e \neq 0$, then A endowed with the triple product $\{abc\} := U_{a,c}(b^*)$ becomes a JB^*-triple with unitary element e. Conversely, if J is a nonzero JB^*-triple, and if e is a unitary element of J, then J endowed with the product $xy := \{xey\}$ and the involution $x^* := \{exe\}$ becomes a JB^*-algebra with unit element $e \neq 0$. These two constructions are mutually inverse.*

Proof Let A be a JB^*-algebra with unit element $e \neq 0$. Then, by Theorem 4.1.45, A becomes a JB^*-triple (say J) with respect to the Jordan $*$-triple product on A given by $\{abc\} := U_{a,c}(b^*)$, and, clearly, e becomes a unitary element of J.

Now, let J be a nonzero JB^*-triple, and let e be a unitary element of J. Then, by Proposition 4.1.54, J endowed with the product $xy := \{xey\}$ and the involution $x^* := \{exe\}$ becomes a Jordan $*$-algebra (say A) with unit element $e \neq 0$. Moreover, A is endowed with a complete norm $\|\cdot\|$ (namely, the one of J) making the product and the involution of A continuous. Let h be in $H(A,*)$. Then the closed subalgebra of A generated by h (say B_h) is $*$-invariant and associative. Moreover, by Proposition 4.1.54, B_h is a closed subtriple of J (say M), and, since B_h is an associative and commutative algebra, the Jordan $*$-triple M is abelian. Therefore, since $\|e\| = 1$ (as $\|e\| = \|\{eee\}\| = \|e\|^3$), Lemma 4.1.48(i) gives $\|xy\| = \|\{xey\}\| \leqslant \|x\|\|y\|$ for all $x, y \in B_h$. Thus, since

$$\|U_x(x^*)\| = \|\{xxx\}\| = \|x\|^3 \text{ for every } x \in B_h,$$

we realize that B_h is a unital JB^*-algebra. By Proposition 3.4.1(i), B_h is in fact a unital C^*-algebra. Therefore we have $\|h^2\| = \|h\|^2$ and $f(h^2) \geqslant 0$ whenever f is in $D(B_h, e)$. Now, let k be another element of $H(A,*)$, take $g \in D(B_h, e)$ such that $g(h^2) = \|h\|^2$, extend g to an element $s \in D(A,e)$, and note that the restriction of s to B_k is an element of $D(B_k, e)$. Then we have

$$\|h^2\| = \|h\|^2 = s(h^2) \leqslant s(h^2 + k^2) \leqslant \|h^2 + k^2\|.$$

Since h and k are arbitrary elements of $H(A,*)$, it follows from Proposition 4.1.46 that $H(A,*)$ is a JB-algebra. Therefore, by Theorem 3.4.8, there exists a norm $|||\cdot|||$

on A such that $(A,*,|||\cdot|||)$ becomes a JB^*-algebra. Then, by Theorem 4.1.45 (or the first paragraph of the present proof), $(A,*,|||\cdot|||)$ can be seen as a JB^*-triple in such a way that the mapping $x \to x$ from $(A,*,|||\cdot|||)$ to J becomes a triple homomorphism. Finally, by Proposition 4.1.52, we have $\|\cdot\| = |||\cdot|||$, and $(A,*,\|\cdot\|)$ is indeed a JB^*-algebra.

It then follows from Proposition 4.1.54 that the two constructions in the statement are mutually inverse. □

4.1.4 Extending the Jordan spectral theory to Jordan-admissible algebras

Definition 4.1.56 Let A be a unital Jordan-admissible algebra over $\mathbb{K}$. We say that an element a in A is *J-invertible* in A with *J-inverse* b if a is J-invertible in the Jordan algebra A^{sym} with J-inverse b, i.e. if the equalities $a \bullet b = \mathbf{1}$ and $a^2 \bullet b = a$ hold (cf. Definition 4.1.2). Thus, when a is a J-invertible element of A, we keep the notation a^{-1} for the (unique) J-inverse of a in A (cf. Theorem 4.1.3(iii)). We now proceed to study these notions in the particular case where A is a non-commutative Jordan algebra. First of all, we note that, in the still more particular case where A is alternative, §4.1.1 reads as follows.

Fact 4.1.57 *Let A be an alternative algebra over $\mathbb{K}$, and let a be in A. Then a is J-invertible in A with J-inverse b if and only if a is invertible in A with inverse b.*

The next result contains a nice intrinsic characterization of J-invertible elements of unital non-commutative Jordan algebras.

Proposition 4.1.58 *Let A be a unital non-commutative Jordan algebra over $\mathbb{K}$, and let a be an element of A. Then the following conditions are equivalent:*

(i) *a is J-invertible in A.*
(ii) *There exists $b \in A$ such that $ab = ba = \mathbf{1}$ and $a^2b = ba^2 = a$.*

Moreover, in the case where the above conditions are fulfilled, the element b appearing in condition (ii) is unique and coincides with the J-inverse of a, and we have:

(iii) *U_a is a bijective operator on A, and*

$$(U_a)^{-1} = U_{a^{-1}},\ \ L_aL_{a^{-1}} = R_aR_{a^{-1}},\ \ L_{a^{-1}} = U_{a^{-1}}R_a,\ \ R_{a^{-1}} = U_{a^{-1}}L_a.$$

(iv) *For $i,j,k,\ell \in \mathbb{N}$, the operators $L_{a^i}, R_{a^j}, L_{a^{-k}}$, and $R_{a^{-\ell}}$ pairwise commute.*

Proof (i)⇒(ii) Assume that a is J-invertible in A. It follows from Corollary 3.2.2 that

$$\begin{aligned}(L_{a^2} - R_{a^2})(a^{-1}) &= (L_a^2 - R_a^2)(a^{-1}) = (L_a - R_a)(L_a + R_a)(a^{-1}) \\ &= 2(L_a - R_a)(a \bullet a^{-1}) = 2(L_a - R_a)(\mathbf{1}) = 0,\end{aligned}$$

so $a^2a^{-1} = a^{-1}a^2$, and so, since $a^2 \bullet a^{-1} = a$, we get that $a^2a^{-1} = a^{-1}a^2 = a$. Then $a^{-1}a = a^{-1}(a^2a^{-1}) = (a^{-1}a^2)a^{-1} = aa^{-1}$, and hence, since $a \bullet a^{-1} = \mathbf{1}$, we conclude that $aa^{-1} = a^{-1}a = \mathbf{1}$. Thus condition (ii) is fulfilled with $b = a^{-1}$.

(ii)⇒(i) Assume the existence of an element b in A satisfying $ab = ba = \mathbf{1}$ and $a^2b = ba^2 = a$. Then it is immediate to verify that $a \bullet b = \mathbf{1}$ and $a^2 \bullet b = a$. Thus a is J-invertible in A with J-inverse b, and the uniqueness of b follows.

From now on, we suppose that conditions (i) and (ii) are fulfilled.

In order to prove assertion (iii) we begin by noticing that, by Fact 3.3.3 and Theorem 4.1.3(iv), U_a is a bijective operator on A and $(U_a)^{-1} = U_{a^{-1}}$. Moreover, as a consequence of the identity (3.2.6) in the proof of Proposition 3.2.1 we have $[L_a, R_{a^{-1}}] = [R_a, L_{a^{-1}}]$, and as consequence of Corollary 3.2.2(i) we obtain

$$L_aL_{a^{-1}} = R_aR_{a^{-1}} \quad \text{and} \quad L_{a^{-1}}L_a = R_{a^{-1}}R_a, \tag{4.1.32}$$

and so $[L_a, L_{a^{-1}}] = [R_a, R_{a^{-1}}]$. Therefore

$$[L_a + R_a, L_{a^{-1}}] = [L_a + R_a, R_{a^{-1}}] = \frac{1}{2}[L_a + R_a, L_{a^{-1}} + R_{a^{-1}}] = 2\left[L_a^{A^{\mathrm{sym}}}, L_{a^{-1}}^{A^{\mathrm{sym}}}\right],$$

and hence, invoking Theorem 4.1.3(v), we have

$$[L_a + R_a, L_{a^{-1}}] = [L_a + R_a, R_{a^{-1}}] = 0. \tag{4.1.33}$$

On the other hand, replacing a by $a + \lambda c$ in the identity (3.2.2) of Proposition 3.2.1, and equalizing the coefficients of λ we obtain

$$((ac + ca)b)a + (a^2b)c = (ac + ca)(ba) + a^2(bc).$$

This relation becomes

$$R_aR_b(L_a + R_a) + L_{a^2b} = R_{ba}(L_a + R_a) + L_{a^2}L_b. \tag{4.1.34}$$

Hence setting $b = a^{-1}$ we get

$$R_a = R_aR_{a^{-1}}(L_a + R_a) - L_{a^2}L_{a^{-1}}.$$

Now, using (4.1.32) and (4.1.33), and recalling the definition of U_a (cf. §2.2.4), we have

$$R_a = [L_a(L_a + R_a) - L_{a^2}]L_{a^{-1}} = U_aL_{a^{-1}}.$$

By interchanging the role of a and a^{-1}, we conclude that $R_{a^{-1}} = U_{a^{-1}}L_a$. The equality $L_{a^{-1}} = U_{a^{-1}}R_a$ follows by duality. Thus assertion (iii) is proved.

In order to prove assertion (iv) we begin by recalling that, by Corollary 3.2.2(ii), the set $S_1 := \{L_a, R_a, L_{a^2}, R_{a^2}\}$ is a commutative subset of $L(A)$. Keeping in mind the definition of U_a, it is then clear that $U_a \in S_1^c$ (the commutator of S_1 in $L(A)$), and hence, by Lemma 1.1.80, $U_{a^{-1}} = U_a^{-1} \in S_1^c$. Now, it follows from assertion (iii) that $L_{a^{-1}}, R_{a^{-1}} \in S_1^c$, and, by the definition of $U_{a^{-1}}$, also $L_{a^{-2}}, R_{a^{-2}} \in S_1^c$. Therefore, setting $S_2 := \{L_{a^{-1}}, R_{a^{-1}}, L_{a^{-2}}, R_{a^{-2}}\}$, we realize that $S := S_1 \cup S_2$ is a commutative subset of $L(A)$ because S_1 and S_2 are commutative and each element of S_1 commutes with every element of S_2. Let k be in $\mathbb{N}$. Keeping in mind (4.1.34) and its dual, an induction argument shows that L_{x^k} and R_{x^k} belong to the subalgebra of $L(A)$ generated by S_1. Analogously, $L_{a^{-k}}, R_{a^{-k}}$ belong to the subalgebra generated by S_2. Therefore $L_{a^k}, R_{a^k}, L_{a^{-k}}, R_{a^{-k}}$ belong to the subalgebra generated by S. Since S is a commutative set, assertion (iv) follows from Corollary 1.1.79. □

Let A be a unital Jordan-admissible algebra over $\mathbb{K}$. According to the concept of a J-invertible element of A introduced in Definition 4.1.56, we set $\text{J-Inv}(A) := \text{J-Inv}(A^{\text{sym}})$, and we say that A is a *J-division* algebra whenever $\text{J-Inv}(A) = A \setminus \{0\}$. With this convention, Corollary 4.1.14 reads as follows.

Corollary 4.1.59 *Let A be a normed J-division Jordan-admissible complex algebra. Then A is isomorphic to $\mathbb{C}$.*

The next result becomes the non-commutative generalization of Proposition 4.1.9.

Proposition 4.1.60 *Let A be a unital non-commutative Jordan algebra over $\mathbb{K}$, and let a be in A such that L_a is bijective. Then a is J-invertible in A, and $R_{a^{-1}}$ is bijective.*

Proof Set $b := L_a^{-1}(\mathbf{1})$. Then, clearly, $ab = \mathbf{1}$. Moreover, since R_a, L_{a^2}, and R_{a^2} commute with L_a (by Proposition 3.2.1), we have

$$ba = R_a(b) = R_a L_a^{-1}(\mathbf{1}) = L_a^{-1} R_a(\mathbf{1}) = L_a^{-1}(a) = L_a^{-1} L_a(\mathbf{1}) = \mathbf{1},$$

$$a^2 b = L_{a^2}(b) = L_{a^2} L_a^{-1}(\mathbf{1}) = L_a^{-1} L_{a^2}(\mathbf{1}) = L_a^{-1}(a^2) = L_a^{-1} L_a(a) = a,$$

and

$$ba^2 = R_{a^2}(b) = R_{a^2} L_a^{-1}(\mathbf{1}) = L_a^{-1} R_{a^2}(\mathbf{1}) = L_a^{-1}(a^2) = L_a^{-1} L_a(a) = a.$$

Therefore, by Proposition 4.1.58, a is J-invertible in A, and $R_{a^{-1}} = U_{a^{-1}} L_a$ is bijective. □

To take the maximum profit of Corollary 4.1.59 and Proposition 4.1.60 above, some additional auxiliary results have to be proved.

Lemma 4.1.61 *Let A be a non-commutative Jordan algebra over $\mathbb{K}$, let a be in A such that L_a is bijective, and set $e := L_a^{-1}(a)$. If at least one of the operators L_{a^2}, R_{a^2} is injective, then e is an idempotent.*

Proof We have clearly $ae = a$ (I). Therefore, since A is flexible, we obtain

$$0 = (ae)a - a(ea) = a^2 - a(ea) = a(a - ea),$$

which, by the injectivity of L_a, implies that $ea = a$ (II). Moreover, by (I) and the identities (3.2.1) and (3.2.4) in Proposition 3.2.1, we get that

$$0 = (ae)a^2 - a(ea^2) = a^3 - a(ea^2) = a(a^2 - ea^2)$$

and

$$0 = a^2(ae) - a(a^2 e) = a^3 - a(a^2 e) = a(a^2 - a^2 e),$$

which, again by the injectivity of L_a, imply $a^2 = ea^2$ (III) and $a^2 = a^2 e$ (IV). Now note that the identity (3.2.1) can be linearized as

$$[z, y, x^2] + [x, y, xz + zx] = 0 \ \text{ for all } \ x, y, z \in A,$$

and that consequently, by (I), (II), and (III), we have

$$\begin{aligned} 0 &= [e, e, a^2] + [a, e, ae + ea] = [e, e, a^2] + 2[a, e, a] \\ &= [e, e, a^2] = e^2 a^2 - e(ea^2) = e^2 a^2 - ea^2 = e^2 a^2 - a^2, \end{aligned}$$

which together with (III) again gives $(e^2-e)a^2=0$ (V). An analogous argument, involving (IV) and the identity (3.2.2) instead of (III) and the identity (3.2.1), gives that $a^2(e^2-e)=0$ (VI). Now, if L_{a^2} is injective, then $e^2=e$ (by (VI)), whereas if R_{a^2} is injective, then $e^2=e$ as well (by (V)). □

We recall that an algebra A is said to be a quasi-division algebra if $A\neq 0$ and, for every $a\in A\setminus\{0\}$, at least one of the operators L_a,R_a is bijective (cf. Definition 2.5.35).

Proposition 4.1.62 *Let A be a non-commutative Jordan quasi-division algebra over $\mathbb{K}$. Then A is a J-division algebra.*

Proof Replacing A with the opposite algebra $A^{(0)}$ of A if necessary, we may assume that there exists $a\in A\setminus\{0\}$ such that L_a is bijective. Then we have $a^2=L_a(a)\neq 0$, and hence certainly at least one of the operators L_{a^2},R_{a^2} is bijective. Therefore, by Lemma 4.1.61, $e:=L_a^{-1}(a)$ is a nonzero idempotent, and hence, since A cannot have nonzero joint divisors of zero, Lemma 2.5.5 applies to get that A is unital. Finally, applying Proposition 4.1.60 to both A and $A^{(0)}$, we realize that every nonzero element of A is J-invertible in A. □

Now the following partial affirmative answer to Problem 2.7.4 follows from Corollary 4.1.59 and Proposition 4.1.62.

Theorem 4.1.63 *Let A be a normed non-commutative Jordan quasi-division complex algebra. Then A is isomorphic to $\mathbb{C}$.*

As a by-product of Proposition 4.1.62, non-commutative Jordan one-sided (say left) division algebras over $\mathbb{K}$ are unital. Therefore it is enough to invoke Proposition 4.1.60 to get the following.

Corollary 4.1.64 *Non-commutative Jordan one-sided division algebras over $\mathbb{K}$ are in fact division algebras.*

Definition 4.1.65 Let A be a unital Jordan-admissible algebra over $\mathbb{K}$. A subalgebra B of A is called a *J-full subalgebra* of A if B contains the unit element of A and the J-inverses of those of their elements which are J-invertible in A. According to the concept of invertibility in A introduced in Definition 4.1.56, for $a\in A$ we set $\text{J-sp}(A,a):=\text{J-sp}(A^{\text{sym}},a)$. It is then clear that a subalgebra B of A is a J-full subalgebra if and only if it contains the unit of A and the equality $\text{J-sp}(A,x)=\text{J-sp}(B,x)$ holds for every $x\in B$.

Unital finite-dimensional subalgebras of unital non-commutative Jordan algebras become examples of J-full subalgebras. Indeed, this follows from the next more general result.

Proposition 4.1.66 *Let A be a unital non-commutative Jordan algebra over $\mathbb{K}$, and let B be an algebraic subalgebra of A containing* $\mathbf{1}$. *Then B is a J-full subalgebra of A.*

Proof Let b be in $B\setminus\text{J-Inv}(B)$. Let C stand for the subalgebra of B generated by $\{b,\mathbf{1}\}$. By Proposition 2.4.19, C is a unital associative algebra. Let $p(\mathbf{x})$ be the minimum polynomial of b (cf. §2.5.9). Since $b \notin \text{J-Inv}(B)$, it follows that $b \notin \text{Inv}(C)$, hence $0 \in \text{sp}(C,b)$, and so, by Corollary 1.3.5, $p(0) = 0$. Therefore $p(\mathbf{x}) = \mathbf{x}q(\mathbf{x})$ for a suitable polynomial $q(\mathbf{x})$. Hence $0 = p(b) = bq(b)$, and consequently $U_b(q(b)) = bq(b)b = 0$. Since $q(b) \neq 0$, it follows that U_b is not injective, and hence that b is not J-invertible in A (by Proposition 4.1.58(iii)). Thus B is a J-full subalgebra of A. □

Assertion (i) in the next result generalizes Proposition 1.2.20(ii), whereas, in the particular case of unital C^*-algebras, assertion (ii) makes the usual order in their self-adjoint parts (cf. §1.2.41) agree with the order introduced in §3.4.68.

Fact 4.1.67 *Let A be a unital non-commutative JB^*-algebra, and let a be in $H(A,*)$. Then we have:*

(i) $\text{J-sp}(A,a) \subseteq \mathbb{R}$.
(ii) $a \geqslant 0$ (in the sense of §3.4.68) *if and only if* $\text{J-sp}(A,a) \subseteq \mathbb{R}_0^+$.

Proof Let B stand for the closed subalgebra of A generated by $\{\mathbf{1},a\}$. Then, by Propositions 3.4.1(ii) and 4.1.28(ii), we have that

$$\left.\begin{array}{r} B \text{ is a unital commutative } C^*\text{-algebra} \\ \text{and } \operatorname{co}(\operatorname{sp}(B,a)) = \operatorname{co}(\text{J-sp}(A,a)). \end{array}\right\} \tag{4.1.35}$$

It follows from (4.1.35) and Proposition 1.2.20(ii) that $\text{J-sp}(A,a) \subseteq \mathbb{R}$, and assertion (i) is proved.

Now we prove assertion (ii). By Claim 3.1.28, a is a positive element of A if and only if a is a positive element of B. Therefore the result follows by invoking (4.1.35). □

Now we are going to prove that closed $*$-subalgebras of unital non-commutative JB^*-algebras are J-full subalgebras.

Lemma 4.1.68 *Let C be a unital C^*-algebra, let x be a non-invertible element of C, and for $n \in \mathbb{N}$ set*

$$x_n := \frac{(nx^*x + i\mathbf{1})^{-\frac{1}{2}}(nxx^* + i\mathbf{1})^{-\frac{1}{2}}}{\|(nx^*x + i\mathbf{1})^{-\frac{1}{2}}(nxx^* + i\mathbf{1})^{-\frac{1}{2}}\|}.$$

Then the sequence xx_nx converges to zero in C.

Proof We may assume that $x \neq 0$. Let n be in $\mathbb{N}$. Then we have

$$\begin{aligned}
\left\|x(nx^*x + i\mathbf{1})^{-\frac{1}{2}}\right\|^2 &= \left\|(nx^*x - i\mathbf{1})^{-\frac{1}{2}}x^*x(nx^*x + i\mathbf{1})^{-\frac{1}{2}}\right\| \\
&= \left\|x^*x(n^2(x^*x)^2 + \mathbf{1})^{-\frac{1}{2}}\right\| \\
&\leqslant \max\left\{\frac{t}{\sqrt{n^2t^2+1}} : t \in \left[0, \|x\|^2\right]\right\} = \frac{\|x\|^2}{\sqrt{n^2\|x\|^4+1}},
\end{aligned}$$

and analogously

$$\left\|(nxx^* + i\mathbf{1})^{-\frac{1}{2}}x\right\|^2 \leqslant \frac{\|x\|^2}{\sqrt{n^2\|x\|^4+1}}.$$

Therefore we get

$$\left\| x(nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}}x \right\| \leqslant \frac{\|x\|^2}{\sqrt{n^2\|x\|^4+1}}.$$

Now we invoke the assumption that x is not invertible in C, so that, by Lemma 1.1.99(i), x^*x or xx^* must be non-invertible in C. If x^*x is not invertible, then we have

$$\begin{aligned} 1 \leqslant \left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}} \right\| &= \left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{\frac{1}{2}} \right\| \\ &\leqslant \left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}} \right\| \left\| (nxx^*+i\mathbf{1})^{\frac{1}{2}} \right\| \\ &\leqslant \left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}} \right\| \sqrt[4]{n^2\|x\|^4+1}, \end{aligned}$$

and the same conclusion

$$1 \leqslant \left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}} \right\| \sqrt[4]{n^2\|x\|^4+1}$$

holds analogously if xx^* is not invertible. It follows that

$$\|xx_nx\| = \frac{\left\| x(nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}}x \right\|}{\left\| (nx^*x+i\mathbf{1})^{-\frac{1}{2}}(nxx^*+i\mathbf{1})^{-\frac{1}{2}} \right\|} \leqslant \frac{\frac{\|x\|^2}{\sqrt{n^2\|x\|^4+1}}}{\frac{1}{\sqrt[4]{n^2\|x\|^4+1}}} = \frac{\|x\|^2}{\sqrt[4]{n^2\|x\|^4+1}},$$

hence $xx_nx \to 0$, as desired. □

Definition 4.1.69 Let A be a (normed) Jordan-admissible algebra A. An element $a \in A$ is said to be a (*topological*) *J-divisor of zero* in A if it is a (topological) J-divisor of zero in the (normed) Jordan algebra A^{sym} (cf. Definitions 4.1.22 and 4.1.24). We note that, according to Definition 4.1.56 and Proposition 4.1.25(ii), if A is unital, then (topological) J-divisors of zero in A cannot be J-invertible.

First of all, we note that, in the particular case where A is alternative, Proposition 4.1.23 reads as follows.

Fact 4.1.70 *Let A be a normed alternative algebra over $\mathbb{K}$, and let a be in A. Then a is a topological J-divisor of zero in A if and only if a is a one-sided topological divisor of zero in A.*

Now, the following theorem generalizes Proposition 1.2.24.

Theorem 4.1.71 *Let A be a unital non-commutative JB^*-algebra. Then we have:*

(i) *Each non-J-invertible element of A is a topological J-divisor of zero in A.*
(ii) *Each closed $*$-subalgebra of A containing $\mathbf{1}$ is a J-full subalgebra of A.*

Proof By Fact 3.3.4, we may assume that A is commutative.

Let x be in $A \setminus \text{J-Inv}(A)$. Then x is not J-invertible in the closed subalgebra B of A generated by $\{\mathbf{1}, x, x^*\}$. But, by Proposition 3.4.6, there exists a C^*-algebra C such that $\mathbf{1}$ is a unit for C, and B can be seen as a closed $*$-subalgebra of C^{sym}. Moreover, by Lemma 3.3.6(ii), x is not invertible in C, and then, by Lemma 3.3.6(i), for each $n \in \mathbb{N}$, the element x_n of C given by Lemma 4.1.68 lies in B. It follows from

Lemma 4.1.68 that $U_x(x_n) \to 0$, with $\|x_n\| = 1$ and $x_n \in A$, which shows that x is a topological J-divisor of zero in A, thus concluding the proof of assertion (i).

Now, let B be any closed $*$-subalgebra of A containing $\mathbf{1}$, and let b be an element of $B \setminus \text{J-Inv}(B)$. By assertion (i) just proved, b is a topological J-divisor of zero in B, so it is a topological J-divisor of zero in A, and so, by Proposition 4.1.25(ii), b is not J-invertible in A. Thus B is a J-full subalgebra of A, and the proof of assertion (ii) is concluded. □

Now we can generalize the continuous functional calculus stated in Theorem 1.2.28, §1.2.29, and Proposition 1.2.34. We recall that an element a of a $*$-algebra A over $\mathbb{K}$ is said to be normal if the subalgebra of A generated by $\{a, a^*\}$ is associative and commutative (cf. Definition 3.4.20). Also we recall that J-spectra of elements of a complete normed unital Jordan-admissible complex algebra are non-empty compact subsets of $\mathbb{C}$ (cf. Theorem 4.1.17).

Corollary 4.1.72 *Let A be a unital non-commutative JB^*-algebra, let a be a normal element of A, and denote by u the inclusion mapping* $\text{J-sp}(A,a) \hookrightarrow \mathbb{C}$. *Then there exists a unique unit-preserving algebra $*$-homomorphism $f \to f(a)$ from* $C^{\mathbb{C}}(\text{J-sp}(A,a))$ *to A taking u to a. Moreover, we have:*

(i) *The $*$-homomorphism above is isometric, and its range coincides with the closed subalgebra of A generated by $\{\mathbf{1}, a, a^*\}$.*

(ii) *If f is in* $C^{\mathbb{C}}(\text{J-sp}(A,a))$, *then* $\text{J-sp}(A, f(a)) = f(\text{J-sp}(A,a))$ ('spectral mapping theorem').

(iii) *If f is in* $C^{\mathbb{C}}(\text{J-sp}(A,a))$, *and if g is in* $C^{\mathbb{C}}(\text{J-sp}(A, f(a)))$, *then the equality* $(g \circ f)(a) = g(f(a))$ *holds.*

Proof Let B stand for the closed subalgebra of A generated by $\{\mathbf{1}, a, a^*\}$, and let $\Phi : C^{\mathbb{C}}(\text{J-sp}(A,a)) \to A$ be a unit-preserving algebra $*$-homomorphism such that $\Phi(u) = a$. Then, since B is $*$-invariant (by Propositions 3.3.13 and 1.2.25), and Φ is continuous (by Proposition 3.4.4), and the closed subalgebra of $C^{\mathbb{C}}(\text{J-sp}(A,a))$ generated by $\{\mathbf{1}, u, u^*\}$ equals $C^{\mathbb{C}}(\text{J-sp}(A,a))$ (by the Stone–Weierstrass theorem), it follows from Lemma 1.1.82(ii) that

$$\Phi(C^{\mathbb{C}}(\text{J-sp}(A,a))) \subseteq B.$$

This allows us to see Φ as a unit-preserving algebra $*$-homomorphism from $C^{\mathbb{C}}(\text{J-sp}(A,a))$ to B taking u to a. On the other hand, since a is normal, B is associative and commutative, and hence it is a commutative C^*-algebra (by Proposition 3.4.1(i)). Moreover, by Theorem 4.1.71, B is a J-full subalgebra of A, which implies $\text{J-sp}(A,x) = \text{J-sp}(B,x) = \text{sp}(B,x)$ for every x in B. Now, replacing A with B, we may assume that A is a C^*-algebra, so that the result follows from Theorem 1.2.28, §1.2.29, and Proposition 1.2.34. □

Given a normal element a of a unital non-commutative JB^*-algebra A, the algebra $*$-homomorphism $f \to f(a)$ from $C^{\mathbb{C}}(\text{J-sp}(A,a))$ to A, given by Corollary 4.1.72, will be called the *continuous functional calculus* at a.

In view of Fact 4.1.57 and Definition 4.1.65, and to be in agreement with the associative case (cf. Definition 1.1.72), J-full subalgebras of a unital alternative algebra

A will be called simply *full subalgebras* of A. With this convention in mind, it is enough to invoke Facts 3.3.2 and 4.1.70 to get the following.

Corollary 4.1.73 *Let A be a unital alternative C^*-algebra. Then we have:*

(i) *Each non-invertible element of A is a one-sided topological divisor of zero in A.*
(ii) *Each closed $*$-subalgebra of A containing $\mathbf{1}$ is a full subalgebra of A.*

If A is a unital Jordan-admissible complex $*$-algebra, then, by the uniqueness of the J-inverse, $H(A,*)$ becomes a J-full real subalgebra of A^{sym}. With this idea in mind, it is enough to combine Theorem 3.4.8, Fact 4.1.67, and Corollary 4.1.72, to get the following continuous functional calculus for a single element of a unital JB-algebra.

Corollary 4.1.74 *Let A be a unital JB-algebra, let a be in A, and denote by u the inclusion mapping* $\text{J-sp}(A,a) \hookrightarrow \mathbb{R}$. *Then there exists a unique unit-preserving algebra homomorphism $f \to f(a)$ from $C^{\mathbb{R}}(\text{J-sp}(A,a))$ to A taking u to a. Moreover, we have:*

(i) *The homomorphism above is isometric, and its range coincides with the closed subalgebra of A generated by $\{\mathbf{1},a\}$.*
(ii) *If f is in $C^{\mathbb{R}}(\text{J-sp}(A,a))$, then* $\text{J-sp}(A,f(a)) = f(\text{J-sp}(A,a))$.
(iii) *Whenever f is in $C^{\mathbb{R}}(\text{J-sp}(A,a))$ and g is in $C^{\mathbb{R}}(\text{J-sp}(A,f(a)))$, we have $(g \circ f)(a) = g(f(a))$.*

4.1.5 The holomorphic functional calculus for complete normed unital non-commutative Jordan complex algebras

The main goal of this subsection is to prove that unital non-commutative Jordan algebras over $\mathbb{K}$ are 'locally spectrally' associative, and to take advantage of this fact.

Lemma 4.1.75 *Let A be a Jordan algebra over $\mathbb{K}$, let G be a non-empty subset of A, and let B stand for the subalgebra of A generated by G. Then $\mathscr{M}(B)^A$* (in the sense of §3.3.36) *coincides with the subalgebra of $L(A)$ generated by the set $\{L_a, L_{ab} : a,b \in G\}$.*

Proof It is obvious that $\mathscr{M}(B)^A$ is generated by operators of the form $L_{\bar{\mathbf{w}}}$, where $\mathbf{w} = \mathbf{w}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is some non-associative word and $\bar{\mathbf{w}} = \mathbf{w}(a_1,\ldots,a_n)$ with $a_i \in G$ (cf. §§2.8.17 and 2.8.26). Therefore it suffices to show that, for any non-associative word $\mathbf{w}$, the operator $L_{\bar{\mathbf{w}}}$ lies in the subalgebra (say $\mathscr{C}$) of $L(A)$ generated by the set $\{L_a, L_{ab} : a,b \in G\}$. We proceed by induction on the degree of the non-associative word $\mathbf{w}$. If $\deg(\mathbf{w}) \leqslant 2$, then it is obvious that $L_{\bar{\mathbf{w}}} \in \mathscr{C}$. If $\deg(\mathbf{w}) \geqslant 3$, then either $\mathbf{w} = (\mathbf{w}_1\mathbf{w}_2)\mathbf{w}_3$ or $\mathbf{w} = \mathbf{w}_3(\mathbf{w}_2\mathbf{w}_1)$, where $\mathbf{w}_i$ are non-associative words of lesser degree. Since A is commutative, we have $(\overline{\mathbf{w}_1}\,\overline{\mathbf{w}_2})\overline{\mathbf{w}_3} = \overline{\mathbf{w}_3}(\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_1})$, and hence the identity (2.4.3) in the proof of Proposition 2.4.13 gives

$$L_{\bar{\mathbf{w}}} = L_{\overline{\mathbf{w}_1}\,\overline{\mathbf{w}_2}} L_{\overline{\mathbf{w}_3}} + L_{\overline{\mathbf{w}_1}\,\overline{\mathbf{w}_3}} L_{\overline{\mathbf{w}_2}} + L_{\overline{\mathbf{w}_2}\,\overline{\mathbf{w}_3}} L_{\overline{\mathbf{w}_1}} - L_{\overline{\mathbf{w}_1}} L_{\overline{\mathbf{w}_3}} L_{\overline{\mathbf{w}_2}} - L_{\overline{\mathbf{w}_2}} L_{\overline{\mathbf{w}_3}} L_{\overline{\mathbf{w}_1}}.$$

By the induction assumption, $L_{\bar{\mathbf{w}}_i}, L_{\bar{\mathbf{w}}_i\bar{\mathbf{w}}_j} \in \mathscr{C}$, and therefore $L_{\bar{\mathbf{w}}} \in \mathscr{C}$ also. This proves the statement. □

Corollary 4.1.76 *Let A be a unital Jordan algebra over $\mathbb{K}$, let G be a subset of A containing $\mathbf{1}$, and let B stand for the subalgebra of A generated by G. Then $\mathscr{M}(B)^A$ coincides with the subalgebra of $L(A)$ generated by the set*

$$\{U_{a,b} : a,b \in G\}.$$

Proof Let $\mathscr{C}$ (respectively, $\mathscr{D}$) stand for the subalgebra of $L(A)$ generated by $\{L_a, L_{ab} : a,b \in G\}$ (respectively, $\{U_{a,b} : a,b \in G\}$). It is clear that $U_{a,b} \in \mathscr{C}$ for all $a,b \in G$, and hence $\mathscr{D} \subseteq \mathscr{C}$. Moreover, it follows from the equalities $L_a = U_{a,\mathbf{1}}$ and $L_{ab} = L_a L_b + L_b L_a - U_{a,b}$ that $\{L_a, L_{ab} : a,b \in G\} \subseteq \mathscr{D}$, and hence $\mathscr{C} \subseteq \mathscr{D}$. Thus $\mathscr{C} = \mathscr{D}$, and the result follows from Lemma 4.1.75. □

The notion of a strongly associative subalgebra of a Jordan algebra was introduced in Definition 3.3.33.

Proposition 4.1.77 *Let A be a unital Jordan algebra over $\mathbb{K}$, let B be a strongly associative subalgebra of A containing $\mathbf{1}$, let C be a subset of $B \cap \text{J-Inv}(A)$. Then the subalgebra of A generated by $B \cup \{z^{-1} : z \in C\}$ is strongly associative.*

Proof Let D stand for the subalgebra of A generated by $B \cup \{z^{-1} : z \in C\}$. By Corollary 4.1.76 above, $\mathscr{M}(D)^A$ is equal to the subalgebra of $L(A)$ generated by

$$\mathscr{G} := \left\{U_{x,y} : x,y \in B \cup \{z^{-1} : z \in C\}\right\}.$$

Therefore, in view of §3.3.36, it is enough to show that $\mathscr{G}$ is a commutative subset of $L(A)$. Let b and c be in B and C, respectively. It follows from the identity (3.4.4) before Corollary 3.4.16 that

$$U_c U_{c^{-1},b} U_c = U_{U_c(c^{-1}),U_c(b)} = U_{c,U_c(b)},$$

so that, since U_c is bijective (by Theorem 4.1.3(ii)), we have

$$U_{c^{-1},b} = U_c^{-1} U_{c,U_c(b)} U_c^{-1}.$$

Now let c,d be in C. Keeping in mind the identity (3.4.4) and the fact that $\mathscr{M}(B)^A$ is commutative (cf. again §3.3.36), we have

$$\begin{aligned} U_d U_c U_{c^{-1},d^{-1}} U_c U_d &= U_d U_{U_c(c^{-1}),U_c(d^{-1})} U_d = U_d U_{c,U_c(d^{-1})} U_d \\ &= U_{U_d(c),U_d U_c(d^{-1})} = U_{U_d(c),U_c U_d(d^{-1})} = U_{U_d(c),U_c(d)}, \end{aligned}$$

and hence

$$U_{c^{-1},d^{-1}} = U_c^{-1} U_d^{-1} U_{U_d(c),U_c(d)} U_d^{-1} U_c^{-1}.$$

Therefore $U_{c^{-1},b}$ as well as $U_{c^{-1},d^{-1}}$ are products of an element in $\mathscr{M}(B)^A$ by inverses in $L(A)$ of elements of this algebra. Since b is arbitrary in B and c,d are arbitrary in C, this shows that $\mathscr{G}$ is contained in the double commutator of $\mathscr{M}(B)^A$ in $L(A)$ (cf. Proposition 1.1.78(iii) and Lemma 1.1.80). Finally, since $\mathscr{M}(B)^A$ is commutative, it follows from Proposition 1.1.78(iv) that $\mathscr{G}$ is a commutative subset of $L(A)$, as desired. □

The next corollary follows straightforwardly from Proposition 4.1.77.

Corollary 4.1.78 *Let A be a unital Jordan algebra over $\mathbb{K}$. Then each maximal strongly associative subalgebra of A is a J-full subalgebra of A.*

Let A be a Jordan algebra over $\mathbb{K}$. We already know that the subalgebra of A generated by each element of A is strongly associative (cf. Fact 3.3.34). On the other hand, the set of all strongly associative subalgebras of A is inductively ordered by inclusion. Therefore each strongly associative subalgebra of A is contained in a maximal strongly associative subalgebra of A. These arguments, together with Corollary 4.1.78 above, lead to the following.

Corollary 4.1.79 *Let A be a Jordan algebra over $\mathbb{K}$. For each a in A there is a maximal strongly associative subalgebra B of A containing a. If in addition A is unital, then B is a J-full subalgebra of A.*

As a first consequence of the above corollary, we have the following.

Proposition 4.1.80 *Let A be a normed J-division Jordan-admissible real algebra. Then A is quadratic.*

Proof By passing to A^{sym} if necessary, we may assume that A is a Jordan algebra. Let a be in A, and note that, by Corollary 4.1.79, there is a J-full associative subalgebra B of A containing a. Since B is a normed associative real algebra, Proposition 1.1.98, Theorem 1.1.41, and Proposition 1.1.100 apply to obtain the existence of $\alpha, \beta \in \mathbb{R}$ such that $(a-\alpha\mathbf{1})^2+\beta^2\mathbf{1} \notin \mathrm{Inv}(B)$. Then, since B is a J-full subalgebra of A, we derive that $(a-\alpha\mathbf{1})^2+\beta^2\mathbf{1} \notin \text{J-Inv}(A)$. Finally, since A is a J-division algebra, we get that $(a-\alpha\mathbf{1})^2+\beta^2\mathbf{1} = 0$. □

Now it is enough to invoke Proposition 4.1.62 to get the following.

Corollary 4.1.81 *Normed non-commutative Jordan quasi-division real algebras are quadratic.*

Then, invoking Corollary 2.5.16, we obtain the following partial affirmative answer to Problem 2.7.45.

Proposition 4.1.82 *Normed Jordan division real algebras are isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

The subproblem of Problem 2.7.45, whether (even complete) normed non-commutative Jordan division real algebras are finite-dimensional, remains open. In view of Corollary 4.1.81, the answer would be affirmative if flexible quadratic division real algebras were finite-dimensional.

Lemma 4.1.83 *Let A be a unital non-commutative Jordan algebra over $\mathbb{K}$, let a be a J-invertible element of A, and let D be a derivation of A. Then*

$$D(a) = -U_a(D(a^{-1})).$$

Proof It is clear that D is a derivation of A^{sym}, and hence, by Fact 3.3.3, we may assume that A is commutative. Note that

$$0 = D(\mathbf{1}) = D(aa^{-1}) = aD(a^{-1}) + D(a)a^{-1},$$

and consequently

$$0 = 2a(aD(a^{-1})) + 2a(D(a)a^{-1}). \tag{4.1.36}$$

Moreover, we have $D(a) = D(a^2a^{-1}) = a^2D(a^{-1}) + D(a^2)a^{-1}$, and hence

$$D(a) = a^2D(a^{-1}) + 2(aD(a))a^{-1}. \tag{4.1.37}$$

By subtracting (4.1.37) from (4.1.36) we obtain

$$-D(a) = U_a(D(a^{-1})) + 2[L_a, L_{a^{-1}}](D(a)),$$

and, keeping in mind Theorem 4.1.3(v), the result follows. □

Proposition 4.1.84 *Let A be a unital non-commutative Jordan algebra over $\mathbb{K}$. Then each maximal commutative subset of A is a J-full subalgebra of A.*

Proof Let B be a maximal commutative subset of A. By Corollary 2.4.16, B is a subalgebra of A, and clearly $\mathbf{1}$ lies in B. Let a be in B, and assume that a is J-invertible in A. Then, since the mapping $y \to [x, y]$ is a derivation of A^{sym} (by Lemma 2.4.15), Fact 3.3.3 and Lemma 4.1.83 apply, so that we have

$$[x, a^{-1}] = -U_{a^{-1}}([x, a]) = 0 \ \text{ for every } \ x \in B.$$

Hence by maximality of B, a^{-1} lies in B. □

Definition 4.1.85 Let A be a (normed) Jordan-admissible algebra over $\mathbb{K}$, and let S be a non-empty subset of A. Since the intersection of any family of (closed) J-full subalgebras of A is a (closed) J-full subalgebra of A, it follows that the intersection of all (closed) J-full subalgebras of A containing S is the smallest (closed) J-full subalgebra of A containing S. This subalgebra is called *the (closed) J-full subalgebra of A generated by S.*

If A is a normed algebra over $\mathbb{K}$, then clearly each maximal commutative subset of A is closed in A. Moreover, if in addition A is a Jordan algebra, then each maximal strongly associative subalgebra of A is closed in A.

Proposition 4.1.86 *Let A be a (normed) unital non-commutative Jordan algebra over $\mathbb{K}$, and let a be in A. Then the (closed) J-full subalgebra of A generated by a is associative and commutative.*

Proof By Zorn's lemma, there exists a maximal commutative subset B of A containing a. By Proposition 4.1.84, B is a J-full subalgebra of A. Furthermore, if A is normed, then B is closed in A. Now, since B is a Jordan algebra, Corollary 4.1.79 applies, so that there is a maximal strongly associative (hence associative) subalgebra C of B containing a, and C is a J-full subalgebra of B. Moreover, if A is normed, then C is closed in B. Since the relation 'to be a J-full subalgebra of' is transitive, we get that C is a J-full subalgebra of A. Moreover, if A is normed, then C is closed in A because B is closed in A and C is closed in B. Finally note that the (closed) J-full subalgebra of A generated by a is contained in C. □

The proof of the following lemma is straightforward.

Lemma 4.1.87 *Let A and B be unital Jordan-admissible algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be a unit-preserving algebra homomorphism. If x is J-invertible in A, then $\Phi(x)$ is J-invertible in B with $\Phi(x)^{-1} = \Phi(x^{-1})$. As a consequence, if C is a J-full subalgebra of B, then $\Phi^{-1}(C)$ is a J-full subalgebra of A.*

The next theorem generalizes Theorem 1.3.13 and becomes the main result in the present subsection. We recall that, given a non-empty open subset Ω of $\mathbb{C}$, the complex algebra $\mathscr{H}(\Omega)$ (of all complex-valued holomorphic functions on Ω) is canonically endowed with the topology of the uniform convergence on compact subsets of Ω.

Theorem 4.1.88 *Let A be a complete normed unital non-commutative Jordan complex algebra, let a be in A, let Ω be an open subset of $\mathbb{C}$ containing* J-sp(A,a), *and let u stand for the inclusion mapping $\Omega \hookrightarrow \mathbb{C}$. Then there is a unique continuous unit-preserving algebra homomorphism $f \to f(a)$ from $\mathscr{H}(\Omega)$ into A taking u to a. Furthermore, we have:*

(i) $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma f(z)(z\mathbf{1}-a)^{-1}dz$ *for every $f \in \mathscr{H}(\Omega)$, where Γ is any contour surrounding* J-sp(A,a) *in Ω.*

(ii) $f(a) = \displaystyle\sum_{n=0}^{\infty} c_n(a - z_0\mathbf{1})^n$ *when Ω is the open disc of centre z_0 and radius $R \leqslant \infty$, and $f \in \mathscr{H}(\Omega)$ is represented by the power series* $\displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$.

(iii) *For each $f \in \mathscr{H}(\Omega)$, $f(a)$ belongs to the closed J-full subalgebra of A generated by a.*

(iv) (Spectral mapping theorem) *For each $f \in \mathscr{H}(\Omega)$ we have*
$$\text{J-sp}(A, f(a)) = f(\text{J-sp}(A,a)).$$

(v) *If $f \in \mathscr{H}(\Omega)$ and if Ω_1 is an open set in $\mathbb{C}$ such that $f(\Omega) \subseteq \Omega_1$, then*
$$(g \circ f)(a) = g(f(a)) \text{ for every } g \in \mathscr{H}(\Omega_1).$$

(vi) *If $f \in \mathscr{H}(\Omega)$, then for each natural number k we have*
$$f^{(k)}(a) = \frac{k!}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a)^{-k-1}\,dz,$$
where Γ is any contour surrounding J-sp(A,a) *in Ω.*

Proof Let B stand for the closed J-full subalgebra of A generated by a, and let $\Phi : \mathscr{H}(\Omega) \to A$ be a continuous unit-preserving algebra homomorphism such that $\Phi(u) = a$. Then, by Lemma 4.1.87, $\Phi^{-1}(B)$ is a closed subalgebra of $\mathscr{H}(\Omega)$ containing all rational functions with poles outside Ω, and hence, by Runge's theorem, we have $\Phi^{-1}(B) = \mathscr{H}(\Omega)$ or, equivalently, $\Phi(\mathscr{H}(\Omega)) \subseteq B$. This allows us to see Φ as a continuous unit-preserving algebra homomorphism from $\mathscr{H}(\Omega)$ to B taking u to a. Moreover, by Proposition 4.1.86, B is a complete normed associative and commutative complex algebra, and, since B is a J-full subalgebra of A, we have J-sp$(A,x) =$ J-sp$(B,x) =$ sp(B,x) for every x in B. Now, replacing A with B, we may assume that A is associative, so that the result follows from Theorem 1.3.13 and Proposition 1.3.15. □

As in the associative case, the algebra homomorphism $f \to f(a)$ in Theorem 4.1.88 will be called the *holomorphic functional calculus* at a.

Arguing as in the proof of Proposition 1.3.23, with Theorem 4.1.88 above instead of Theorem 1.3.13, we get the following.

Proposition 4.1.89 *Let A be a complete normed unital non-commutative Jordan complex algebra. If the J-spectrum of some element of A is not connected, then A contains an idempotent different from 0 and $\mathbf{1}$. More precisely, if $a \in A$ is such that $\text{J-sp}(A,a) = F \cup G$ for some disjoint non-empty closed subsets F, G of $\text{J-sp}(A,a)$, then there is an idempotent $e \neq 0, \mathbf{1}$ in the closed J-full subalgebra of A generated by a satisfying:*

(i) *If $a_1 := ae$ and $a_2 := a(\mathbf{1}-e)$, then $a = a_1 + a_2$ and $a_1 a_2 = a_2 a_1 = 0$.*
(ii) $\text{J-sp}(A,a_1) = F \cup \{0\}$ *and* $\text{J-sp}(A,a_2) = G \cup \{0\}$.

Fact 4.1.90 *Let a be a normal element of a unital non-commutative JB*-algebra A, let Ω be an open subset of $\mathbb{C}$ such that $\text{J-sp}(A,a) \subseteq \Omega$, and let f be in $\mathscr{H}(\Omega)$. Then $f(a)$ has the same meaning in both continuous functional calculus and holomorphic functional calculus at a.*

Proof Argue as in the proof of Fact 1.3.17 with Theorem 4.1.88 and Corollary 4.1.72 instead of Theorems 1.3.13 and 1.2.28, respectively. □

Proposition 4.1.91 *Let A be a complete normed unital Jordan-admissible complex algebra, let a be in A, and let Ω be an open subset of $\mathbb{C}$ containing $\text{J-sp}(A,a)$. Then there exists $\varepsilon > 0$ such that $\text{J-sp}(A,b) \subseteq \Omega$ for every $b \in A$ with $\|b-a\| < \varepsilon$.*

Proof Reduce to the commutative case, and then argue as in the proof of Proposition 1.3.20 with Theorem 4.1.7 instead of Theorem 1.1.23. □

Theorem 4.1.3 will be applied without notice in the proof of the following.

Proposition 4.1.92 *Let A be a complete normed unital Jordan-admissible algebra over $\mathbb{K}$, let a be in $\text{J-Inv}(A)$, and let b be in A such that*

$$\|a-b\| < \sqrt{\frac{1}{3^2\|a^{-1}\|^2} + \|a\|^2} - \|a\|. \tag{4.1.38}$$

Then $b \in \text{J-Inv}(A)$.

Proof We may assume that A is commutative. Noticing that

$$\|U_a - U_b\| = \|U_{a-b,a+b}\| \leqslant 3\|a-b\|\|a+b\|$$

and that $\|a+b\| = \|2a-(a-b)\| \leqslant 2\|a\| + \|a-b\|$, we obtain

$$\|U_a - U_b\| \leqslant 3\|a-b\|^2 + 6\|a\|\|a-b\|.$$

Therefore, invoking the assumption (4.1.38), and recalling that $U_{a^{-1}} = U_a^{-1}$, we obtain that

$$\|U_a - U_b\| \leqslant 3\left[(\|a-b\| + \|a\|)^2 - \|a\|^2\right] < \frac{1}{3\|a^{-1}\|^2} \leqslant \frac{1}{\|U_a^{-1}\|}.$$

Now, since $\|U_a - U_b\| < \frac{1}{\|U_a^{-1}\|}$, it follows from Corollary 1.1.21(ii) that U_b is a bijective operator, i.e. $b \in \text{J-Inv}(A)$. $\square$

The next result generalizes Theorem 1.3.21.

Theorem 4.1.93 *Let A be a complete normed unital non-commutative Jordan complex algebra, let Ω be a non-empty open set in $\mathbb{C}$, and let f be in $\mathscr{H}(\Omega)$. Then*

$$A_\Omega := \{x \in A : \text{J-sp}(A,x) \subseteq \Omega\}$$

is a non-empty open subset of A, and the mapping $\tilde{f} : x \to f(x)$ from A_Ω to A is holomorphic. Moreover, for each $a \in A_\Omega$, the Fréchet derivative of $\tilde{f}$ at a is represented by the following $BL(A)$-valued integral

$$D\tilde{f}(a) = \frac{1}{2\pi i} \int_\Gamma f(z)\, U_{(z\mathbf{1}-a)^{-1}}\, dz,$$

where Γ is any contour that surrounds J-sp(A,a) *in Ω.*

Proof Clearly $\Omega\mathbf{1} \subseteq A_\Omega$, and hence A_Ω is not empty. Moreover, by Proposition 4.1.91, A_Ω is an open subset of A. Let a be in A_Ω and let Γ be a contour surrounding J-sp(A,a) in Ω. Since the mapping $x \to x^{-1}$ is continuous on J-Inv(A), the integrand $f(z)\,U_{(z\mathbf{1}-a)^{-1}}$ is continuous on Γ, so that the integral

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, U_{(z\mathbf{1}-a)^{-1}}\, dz$$

exists and defines an element (say T) of $BL(A)$. Moreover, as a consequence of Corollary 4.1.8, we see that the integrand is actually a holomorphic $BL(A)$-valued function on $\Omega \setminus \text{J-sp}(A,a)$. The Cauchy theorem implies therefore that the integral is independent of the choice of Γ, provided only that Γ surrounds J-sp(A,a) in Ω. For each $h \in A$, the valuation of elements of $BL(A)$ at h becomes a continuous linear mapping from $BL(A)$ to A, and hence

$$T(h) = \left(\frac{1}{2\pi i} \int_\Gamma f(z)\, U_{(z\mathbf{1}-a)^{-1}}\, dz\right)(h) = \frac{1}{2\pi i} \int_\Gamma f(z)\, U_{(z\mathbf{1}-a)^{-1}}(h)\, dz.$$

Since Γ is a compact subset of $\mathbb{C} \setminus \text{J-sp}(A,a)$, there exist positive numbers M and N such that $\|z\mathbf{1} - a\| < M$ and $\|(z\mathbf{1}-a)^{-1}\| \leqslant \frac{N}{3}$ for every $z \in \Gamma$. Let z be in Γ, and let h be in A with $\|h\| < \sqrt{\frac{1}{N^2} + M^2} - M$. Keeping in mind that the function $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $g(t,s) := \sqrt{\frac{1}{t^2} + s^2} - s$ is strictly decreasing in each of its variables, we see that

$$\|(z\mathbf{1}-a) - (z\mathbf{1}-a-h)\| = \|h\| < \sqrt{\frac{1}{3^2\|(z\mathbf{1}-a)^{-1}\|^2} + \|z\mathbf{1}-a\|^2} - \|z\mathbf{1}-a\|,$$

and hence, by Proposition 4.1.92, we can assert that $z\mathbf{1} - a - h \in \text{J-Inv}(A)$. Moreover, since

$$\sqrt{\frac{1}{3^2\|(z\mathbf{1}-a)^{-1}\|^2} + \|z\mathbf{1}-a\|^2} - \|z\mathbf{1}-a\| < \frac{1}{3\|(z\mathbf{1}-a)^{-1}\|},$$

it follows from Lemma 4.1.4(iii) that

$$\|(z\mathbf{1}-a-h)^{-1}\| \leqslant \frac{\|(z\mathbf{1}-a)^{-1}\|}{1-3\|(z\mathbf{1}-a)^{-1}\|\|h\|} \leqslant \frac{N}{3(1-N\|h\|)}.$$

Therefore, since $\|h\| < \sqrt{\frac{1}{N^2}+M^2}-M$, we have that

$$\left\|(z\mathbf{1}-a-h)^{-1}\right\| < \frac{N}{3[1-N(\sqrt{\frac{1}{N^2}+M^2}-M)]} = \frac{N}{3(1+MN-\sqrt{1+(MN)^2})}.$$

Now, since

$$(z\mathbf{1}-a-h)^{-1}-(z\mathbf{1}-a)^{-1}-U_{(z\mathbf{1}-a)^{-1}}(h) = (U_{(z\mathbf{1}-a)^{-1}}\circ U_h)((z\mathbf{1}-a-h)^{-1})$$

(by (4.1.5)), we get that

$$\begin{aligned}&\left\|(z\mathbf{1}-a-h)^{-1}-(z\mathbf{1}-a)^{-1}-U_{(z\mathbf{1}-a)^{-1}}(h)\right\|\\ &\qquad \leqslant 9\left\|(z\mathbf{1}-a)^{-1}\right\|^2\|h\|^2\left\|(z\mathbf{1}-a-h)^{-1}\right\| < \frac{N^3\|h\|^2}{3(1+MN-\sqrt{1+(MN)^2})}.\end{aligned}$$

On the other hand, arguing as in the proof of Theorem 1.3.21, with Proposition 4.1.91 instead of Proposition 1.3.20, we find $\varepsilon > 0$ such that, for $\|h\| < \varepsilon$, the inclusion J-sp$(A, a+h) \subseteq \Omega$ holds and Γ surrounds J-sp$(A, a+h)$ in Ω. Therefore, if in addition we assume that $\|h\| < \varepsilon$, then we have

$$\begin{aligned}&f(a+h)-f(a)-T(h)\\ &\qquad = \frac{1}{2\pi i}\int_\Gamma f(z)\left[(z\mathbf{1}-a-h)^{-1}-(z\mathbf{1}-a)^{-1}-U_{(z\mathbf{1}-a)^{-1}}(h)\right]dz.\end{aligned}$$

It follows that

$$\begin{aligned}&\|f(a+h)-f(a)-T(h)\|\\ &\qquad \leqslant \frac{1}{2\pi}\ell(\Gamma)\max\{|f(z)| : z\in\Gamma\}\frac{N^3\|h\|^2}{3(1+MN-\sqrt{1+(MN)^2})},\end{aligned}$$

where $\ell(\Gamma)$ denotes the length of Γ. Hence

$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-T(h)\|}{\|h\|} = 0. \qquad \square$$

4.1.6 A characterization of smooth-normed algebras

We recall that, according to Corollary 2.6.10(i), smooth-normed algebras are non-commutative Jordan algebras. Thus one may wonder about the behaviour of the Jordan inverse in these algebras. In fact, we are going to characterize smooth-normed algebras, among norm-unital normed non-commutative Jordan algebras, in terms of such a behaviour.

Lemma 4.1.94 *Let A be a normed unital Jordan algebra over $\mathbb{K}$, and let u and x be in A. Assume that there exists $\delta > 0$ such that $u + rx \in$ J-Inv(A) whenever r is in*

$\mathbb{R}$ *with* $|r| \leqslant \delta$. *Then*

$$\lim_{r\to 0} \frac{(u+rx)^{-1} - u^{-1}}{r} = -U_{u^{-1}}(x).$$

Proof By the equality (4.1.5), for $r \in \mathbb{R}$ with $|r| \leqslant \delta$ we have

$$(u+rx)^{-1} - u^{-1} + rU_{u^{-1}}(x) = (U_{u^{-1}} \circ U_{rx})((u+rx)^{-1}),$$

and hence

$$\|(u+rx)^{-1} - u^{-1} + rU_{u^{-1}}(x)\| \leqslant r^2 \|U_{u^{-1}}\| \|U_x\| \|(u+rx)^{-1}\|.$$

Keeping in mind Proposition 4.1.6(ii), we deduce that

$$\lim_{r\to 0} \frac{\|(u+rx)^{-1} - u^{-1} + rU_{u^{-1}}(x)\|}{r} = 0. \qquad \square$$

Lemma 4.1.95 *Let A be a norm-unital normed associative real algebra such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every element a in some non-empty open subset W consisting only of invertible elements. Then A is (isometrically isomorphic to) $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.*

Proof First note that, for $w \in W$ and $b \in A$, we have $\|w^{-1}b\| = \|w^{-1}\|\|b\|$ and $\|bw\| = \|b\|\|w\|$. Therefore, choosing $w \in W$, we realize that $w^{-1}W$ enjoys all properties satisfied by W, and hence, replacing W with $w^{-1}W$, we may assume that $\mathbf{1} \in W$. Let a be in A. Then, for $r \in \mathbb{R}$ with $|r|$ small enough, we have $\mathbf{1} + ra \in W$, and consequently

$$\|\mathbf{1} + ra\|\,\|(\mathbf{1} + ra)^{-1}\| = 1. \tag{4.1.39}$$

Since $\lim_{r\to 0^+} \frac{(\mathbf{1}+ra)^{-1} - \mathbf{1}}{r} = -a$ (by Lemma 4.1.94 applied to A^{sym}), it is enough to invoke Corollary 2.1.6 and take right derivatives at $r = 0$ in (4.1.39) to get $\min V(A, \mathbf{1}, a) = \max V(A, \mathbf{1}, a)$. Since a is arbitrary in A, we deduce that A is smooth, and the result follows from the implication (ii)$\Rightarrow$(iii) in Theorem 2.6.21. $\square$

Now we can prove the main result in the present subsection.

Theorem 4.1.96 *Let A be a norm-unital normed real algebra. Then the following conditions are equivalent:*

(i) *A is a smooth-normed algebra.*
(ii) *A is a J-division non-commutative Jordan algebra, and moreover the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every nonzero element $a \in A$.*
(iii) *A is a J-division Jordan-admissible algebra, and the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every nonzero element $a \in A$.*
(iv) *A is a Jordan-admissible algebra, and there exists a non-empty open subset W of A contained in* J-Inv(A) *and such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every $a \in W$.*

Proof (i)$\Rightarrow$(ii) Assume that (i) holds. Then, by Corollary 2.6.10(i), A is a non-commutative Jordan algebra. Let a be a nonzero element of A. By Corollary 2.6.10(iii), the subalgebra of A generated by $\{\mathbf{1}, a\}$ is an isometric copy of $\mathbb{R}$ or $\mathbb{C}$, so that the inverse a^{-1} of a in this subalgebra satisfies $\|a\|\,\|a^{-1}\| = 1$. But, clearly, a^{-1} is the J-inverse of a in A. Since a is arbitrary in $A \setminus \{0\}$, condition (ii) holds.

The implications (ii)⇒(iii) and (iii)⇒(iv) are clear.

(iv)⇒(i) Suppose that (iv) holds. In order to prove (i), we can replace A with A^{sym}, so that we may assume that A is a Jordan algebra. Moreover, replacing W with the interior of the set $\{x \in \text{J-Inv}(A) : \|x\|\,\|x^{-1}\| = 1\}$, we can additionally assume that W is an open subset of A satisfying $(\mathbb{R} \setminus \{0\})W \subseteq W$ and such that $x^{-1} \in W$ whenever $x \in W$ (cf. Proposition 4.1.6(ii)). We claim that there exists $v \in W$ such that L_v is bijective. This is clear if $\mathbf{1} \in W$. Assume that $\mathbf{1} \notin W$. Let $v \in W$. By Proposition 4.1.86, there exists a closed J-full associative and commutative subalgebra B of A containing v. Then, clearly, $W' := W \cap B$ is a non-empty open subset of B contained in $\mathrm{Inv}(B)$, and the equality $\|x\|\,\|x^{-1}\| = 1$ holds for every $x \in W'$. By Lemma 4.1.95, B is (isometrically isomorphic to) $\mathbb{R}$ or $\mathbb{C}$. In particular, there exist $\alpha, \beta \in \mathbb{R}$ such that $v^2 = \alpha\mathbf{1} + \beta v$. Note that $\alpha \neq 0$ because otherwise we would have $v^2 = \beta v$, which would imply $v = \beta\mathbf{1}$, contradicting the assumption that $\mathbf{1} \notin W$. Moreover, replacing if necessary v with $v + \delta\mathbf{1}$ for δ small enough, we can assume that $\beta \neq 0$. On the other hand, taking $a = v$ and $n = 2$ in the identity (2.4.4) in the proof of Proposition 2.4.13, we get that $L_{v^3} + 2L_v^3 - 3L_{v^2}L_v = 0$. Since $v^2 = \alpha\mathbf{1} + \beta v$ and

$$v^3 = v(\alpha\mathbf{1} + \beta v) = \alpha v + \beta(\alpha\mathbf{1} + \beta v) = \alpha\beta\mathbf{1} + (\alpha + \beta^2)v,$$

we find that

$$2L_v^3 - 3\beta L_v^2 + (-2\alpha + \beta^2)L_v + \alpha\beta I_A = 0.$$

Therefore $L_v F = F L_v = I_A$, where

$$F = -\alpha^{-1}\beta^{-1}\left[2L_v^2 - 3\beta L_v + (-2\alpha + \beta^2)I_A\right],$$

and hence L_v is bijective. Now that the claim has been proved, let us fix $v \in W$ such that L_v is bijective, and set $u := \|v\|v^{-1}$. Then $u \in W$, $\|u\| = \|u^{-1}\| = 1$, and $L_{u^{-1}}$ is bijective. Let x be in A. Then, for $r \in \mathbb{R}$ with $|r|$ small enough, we have $u + rx \in W$, and consequently

$$\|u + rx\|\,\left\|(u + rx)^{-1}\right\| = 1. \tag{4.1.40}$$

Now, keeping in mind Lemma 4.1.94 and Corollary 2.1.6, it is enough to take right derivatives at $r = 0$ in (4.1.40) to deduce that

$$\max V(A, u, x) = \min V(A, u^{-1}, U_{u^{-1}}(x)). \tag{4.1.41}$$

Replacing x with $-x$ in (4.1.41), we find that

$$\min V(A, u, x) = \max V(A, u^{-1}, U_{u^{-1}}(x)),$$

which together with (4.1.41) implies that $V(A, u, x)$ consists of a single point. Since $L_{u^{-1}}$ is a surjective linear contraction taking u to $\mathbf{1}$, and x is arbitrary in A, it follows from Corollary 2.1.2(i) that, for every $a \in A$, $V(A, \mathbf{1}, a)$ is reduced to a point. Thus $D(A, \mathbf{1})$ is reduced to a singleton, and hence A is smooth. □

Since the set of all J-invertible elements of a complete normed unital Jordan-admissible algebra is open (by Theorem 4.1.7), the next result follows from Theorem 4.1.96.

Corollary 4.1.97 *Let A be a norm-unital complete normed real algebra. Then A is smooth if and only if A is Jordan-admissible and the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every $a \in \text{J-Inv}(A)$.*

Since the real normed algebra underlying a smooth-normed complex algebra is a smooth-normed real algebra, it is enough to invoke Theorem 4.1.96 and Proposition 2.6.2 to get the following.

Corollary 4.1.98 *Let A be a norm-unital normed Jordan-admissible complex algebra such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every element a in some non-empty open subset consisting only of J-invertible elements. Then A is (isometrically isomorphic to) $\mathbb{C}$.*

With Theorem 4.1.7 in mind, Corollary 4.1.98 above yields the following.

Corollary 4.1.99 *$\mathbb{C}$ is the unique norm-unital complete normed Jordan-admissible complex algebra satisfying $\|x\|\|x^{-1}\| = 1$ for every J-invertible element x.*

The following corollary follows straightforwardly from Theorem 4.1.96, the implication (ii)⇒(iii) in Theorem 2.6.21, and Fact 4.1.57 (cf. also Proposition 2.5.38).

Corollary 4.1.100 *Let A be a norm-unital normed alternative real algebra. Then the following conditions are equivalent:*

(i) *A is a division algebra, and the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every nonzero element $a \in A$.*
(ii) *There exists a non-empty open subset W of A contained in $\text{Inv}(A)$ and such that the equality $\|a\|\,\|a^{-1}\| = 1$ holds for every $a \in W$.*
(iii) *A is (isometrically isomorphic to) $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Keeping in mind Fact 4.1.57 and Theorem 4.1.7, Corollary 4.1.100 above yields the following.

Corollary 4.1.101 *$\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the unique norm-unital complete normed alternative real algebras satisfying $\|x\|\|x^{-1}\| = 1$ for every invertible element x.*

4.1.7 Historical notes and comments

Theorem 4.1.3 and its proof have been taken from [754, pp. 52–3]. According to [754, footnote on p. 53],

> The notion of [J-]inverse [of an element in a unital Jordan algebra] was introduced by N. Jacobson in [348], Theorem 4.1.3 was proved in this paper but the proof was considerably more complicated than the present one since it did not use [the fundamental formula stated in] Proposition 3.4.15 (which was conjectured in this paper). The present simple proof is due to K. McCrimmon [433].

The pioneering paper dealing with spectral theory for normed Jordan algebras is that of Viola Devapakkiam [628], where Proposition 4.1.6, the first assertion in Theorem 4.1.7, Corollaries 4.1.8 and 4.1.14, and Theorems 4.1.12 and 4.1.17 are proved. Viola Devapakkiam also introduces the notion of a topological J-divisor of zero, and proves Propositions 4.1.25, 4.1.26, and 4.1.28(ii).

Our approach to results from Lemma 4.1.4 to Theorem 4.1.17 (respectively, from Proposition 4.1.23 to Proposition 4.1.28) is close to that of Martínez' PhD thesis [775] (respectively, Kaidi's PhD thesis [759]). The second assertion in Theorem 4.1.7 is due to Martínez [775]. In the original proof, the identity (4.1.5) was derived straightforwardly from the following.

Theorem 4.1.102 *Any unital Jordan algebra generated by two J-invertible elements and their inverses is special.*

The above theorem, due to McCrimmon [434], becomes a variant of the Shirshov–Cohn theorem (Theorem 3.1.55), and is known as *the Shirshov–Cohn theorem with inverses*. Proposition 4.1.9 is due to Kaidi [759]. Theorem 4.1.10 is due to Rodríguez. With his permission, it was included in Martínez' PhD thesis [775], and was published later in [422]. Theorem 4.1.10 is applied in [775, 422] to prove that the natural involution of a complete Jordan V-algebra is an algebra involution. Although this result is now obsolete (in view of Theorem 2.3.8), Martínez' argument remains useful. Indeed, as pointed out by Bensebah [85], it can be adapted to prove the following.

Theorem 4.1.103 *Let A be a complete normed Jordan complex algebra with zero annihilator* (cf. Definition 1.1.10)*, and let $*$ be a conjugate-linear vector space involution on A such that L_a lies in $H(BL(A),I_A)$ whenever a is in $H(A,*)$. Then $*$ is an algebra involution on A.*

Proof It is enough to show that a^2 lies in $H(A,*)$ whenever a is in $H(A,*)$. Let a be in $H(A,*)$, and write $a^2 = b + ic$ with $b,c \in H(A,*)$. Then the Jordan identity yields

$$0 = [L_a, L_{a^2}] = [L_a, L_b] + i[L_a, L_c].$$

Since $i[L_a,L_b]$ and $i[L_a,L_c]$ lie in $H(BL(A),I_A)$ (by Lemma 2.3.1), it follows from Corollary 2.1.13 that $[L_a,L_b] = [L_a,L_c] = 0$. Then, since

$$[L_b,L_c](a) = [L_a,L_c](b) - [L_a,L_b](c)$$

(because A is commutative), we obtain that $[L_b,L_c](a) = 0$. Therefore, invoking Lemma 3.1.23 and Fact 2.4.7, we have

$$[[L_b,L_c],L_b] + i[[L_b,L_c],L_c] = [[L_b,L_c],L_{a^2}] = L_{[L_b,L_c](a^2)} = 2L_{a[L_b,L_c](a)} = 0.$$

Since $[[L_b,L_c],L_b]$ and $[[L_b,L_c],L_c]$ lie in $H(BL(A),I_A)$ (again by Lemma 2.3.1), Corollary 2.1.13 applies again to get $[[L_b,L_c],L_b] = 0$. Now it is enough to invoke Corollary 2.4.3 to derive $[L_b,L_c] = 0$, and then, by Lemma 2.3.71, we have

$$V(BL(A),I_A,L_{a^2}) = \mathrm{co}(\mathrm{sp}(BL(A),L_{a^2})). \tag{4.1.42}$$

On the other hand, since $\mathrm{sp}(BL(A),L_a) \subseteq \mathbb{R}$ (by Lemma 2.3.21), it follows from Proposition 4.1.30(ii) that $\mathrm{sp}(BL(A_{\mathbb{1}}),L_a^{A_{\mathbb{1}}}) \subseteq \mathbb{R}$, so that, by Corollary 4.1.11, we have J-sp$(A_{\mathbb{1}},a) \subseteq \mathbb{R}$. Then Theorem 4.1.88(iv) assures that J-sp$(A_{\mathbb{1}},a^2) \subseteq \mathbb{R}_0^+$, and, by applying Corollary 4.1.11 again, we realize that $\mathrm{sp}(BL(A_{\mathbb{1}}),L_{a^2}^{A_{\mathbb{1}}}) \subseteq \mathbb{R}_0^+$. Again Proposition 4.1.30(ii) applies to obtain that $\mathrm{sp}(BL(A),L_{a^2}) \subseteq \mathbb{R}_0^+$. It follows from

(4.1.42) that L_{a^2} lies in $H(BL(A), I_A)$. Finally, since $L_{a^2} = L_b + iL_c$, and L_{a^2}, L_b, L_c are in $H(BL(A), I_A)$, it follows from Corollary 2.1.13 that $L_c = 0$, so $c = 0$ because A has zero annihilator, and so $a^2 = b \in H(A, *)$, as desired. □

The next corollary is also pointed out in [85].

Corollary 4.1.104 *Jordan complex semi-H^*-algebras with zero annihilator are H^*-algebras.*

Proof Let A be a Jordan complex semi-H^*-algebra with zero annihilator. By Lemma 2.8.12, we may assume that the norm of A is an algebra norm, and hence that A is a complete normed algebra. On the other hand, for each $a \in H(A, *)$, L_a is a self-adjoint element of the C^*-algebra of all bounded linear operators on (the Hilbert space of) A, and hence, by Lemma 2.2.5, we have $L_a \in H(BL(A), I_A)$. Therefore Theorem 4.1.103 applies, so that the involution $*$ of A is an algebra involution, i.e. A is an H^*-algebra. □

The natural variant of Theorem 4.1.103 for associative algebras is also true. Indeed, we have the following.

Proposition 4.1.105 *Let A be a complete normed associative complex algebra with zero annihilator, and let $*$ be a conjugate-linear vector space involution on A such that L_a and R_a lie in $H(BL(A), I_A)$ whenever a is in $H(A, *)$. Then there is a contractive and injective algebra $*$-homomorphism from A into some C^*-algebra. As a consequence, $*$ is an algebra involution on A.*

Proof By associativity, the set

$$L := \{\lambda I_A + L_a : (\lambda, a) \in \mathbb{C} \times A\} \text{ (respectively, } R := \{\lambda I_A + R_a : (\lambda, a) \in \mathbb{C} \times A\})$$

is a subalgebra of $BL(A)$, and the mapping $a \to L_a$ (respectively, $a \to R_a$) from A to L (respectively, R) is a contractive algebra homomorphism (respectively, anti-homomorphism). Moreover, by the assumption on $*$, both L and R are V-algebras. Therefore, by Corollary 3.3.15 and Theorem 2.3.32, the closures of L and R in $BL(A)$ are C^*-algebras, and the mappings $a \to L_a$ and $a \to R_a$ from A to the C^*-algebras $\bar{L}$ and $\bar{R}$, respectively, are $*$-mappings. Now consider the C^*-algebra $\bar{R}^{(0)}$ opposite of $\bar{R}$ (cf. §1.1.36), and the C^*-algebra $\bar{L} \times \bar{R}^{(0)}$ direct product of $\bar{L}$ and $\bar{R}^{(0)}$ (cf. §1.2.43). Since A has zero annihilator, it follows that the mapping $a \to (L_a, R_a)$ from A to $\bar{L} \times \bar{R}^{(0)}$ is a contractive and injective algebra $*$-homomorphism. □

Arguing as in the proof of Corollary 4.1.104, with Proposition 4.1.105 instead of Theorem 4.1.103, we realize that *associative complex semi-H^*-algebras with zero annihilator are H^*-algebras*. However, as a matter of fact, this result has the following easier proof, which also works in the case of real algebras:

Let A be an associative real or complex semi-H^*-algebra with zero annihilator. Keeping in mind that A is a Hilbert space, for each $T \in BL(A)$ we denote as usual by $T^* \in BL(A)$ the adjoint operator of T. Then, since A is an associative semi H^*-algebra, for $a, b \in A$ we have

$$L_{(ab)^*} = (L_{ab})^* = (L_a L_b)^* = (L_b)^*(L_a)^* = L_{b^*} L_{a^*} = L_{b^* a^*},$$

and analogously $R_{(ab)^*} = R_{b^*a^*}$. Since A has zero annihilator, we deduce that $(ab)^* = b^*a^*$. □

We do not know if Proposition 4.1.105 remains true when 'associative' is replaced with 'alternative'. Anyway, as proved by Cuenca in [714, Proposition 1.2.25], *alternative complex semi-H^*-algebras with zero annihilator are H^*-algebras*.

Proposition 4.1.18 and Theorem 4.1.19 are due to Rodríguez and Velasco [540]. Proposition 4.1.108 below states a variant of Theorem 4.1.19, also proved in [540]. The formulation and proof of this result involve the definition and the fact which follow.

Definition 4.1.106 An algebra A over $\mathbb{K}$ is said to *admit power-associativity* if the algebra A^{sym} is power-associative. Without enjoying their name, algebras admitting power-associativity have already appeared in our development (cf. Corollaries 2.4.18 and 2.6.40, Lemma 2.6.37, and Proposition 2.6.39). We note that, by Corollary 2.4.18, *power-associative algebras admit power-associativity*. We also note that, since Jordan algebras are power-associative, *Jordan-admissible algebras admit power-associativity*.

Fact 4.1.107 [520] *Let A be a complete normed power-associative algebra over $\mathbb{K}$, let B be a normed unital algebra over $\mathbb{K}$, let $\Phi : A \to B$ be an algebra homomorphism, and let x be in A. Then $1 \leqslant \|x\| + \|\mathbf{1} - \Phi(x)\|$.*

Proof By Proposition 3.4.63, the subalgebra of B generated by $\mathbf{1}$ and $\Phi(x)$ is associative and the inequality $\mathfrak{r}(\Phi(x)) \leqslant \mathfrak{r}(x)$ holds. Therefore, by Corollary 1.1.115, we have

$$\begin{aligned} 1 = \mathfrak{r}(\mathbf{1}) &\leqslant \mathfrak{r}(\Phi(x)) + \mathfrak{r}(\mathbf{1} - \Phi(x)) \\ &\leqslant \mathfrak{r}(x) + \mathfrak{r}(\mathbf{1} - \Phi(x)) \leqslant \|x\| + \|\mathbf{1} - \Phi(x)\|. \end{aligned}$$ □

Proposition 4.1.108 *Let A be a complete normed algebra over $\mathbb{K}$ admitting power-associativity, let B be a complete normed strongly semisimple algebra over $\mathbb{K}$, and let $\Phi : A \to B$ be an algebra homomorphism with dense range. Then Φ is continuous.*

Proof Argue as in the proof of Theorem 4.1.19, with Fact 4.1.107 instead of the inequality (4.1.7) in Proposition 4.1.18. □

The uniqueness-of-norm consequence of the above proposition (that *strongly semisimple algebras admitting power-associativity have at most a complete algebra norm topology*) is better than Corollary 4.1.20, but, as commented in Remark 4.1.21, still better results are known. We note in passing that the (commutative) Jordan forerunner of Corollary 4.1.20 goes back to Balachandran and Rema [54]. Both Theorem 4.1.19 and Proposition 4.1.108 contain Rickart's classical dense-range-homomorphism theorem, as well as the result of Putter and Yood [495] that *dense range algebra homomorphisms from complete normed Jordan algebras to complete normed* (automatically Jordan) *strongly semisimple algebras are continuous.*

It is easily realized that, if B is a normed algebra, and if there exists a dense range algebra homomorphism from some algebra admitting power-associativity to B, then B admits power-associativity. Therefore, since Jordan-admissible algebras admit power-associativity, a proof of the next conjecture would provide us with a result containing both Theorem 4.1.19 and Proposition 4.1.108.

Conjecture 4.1.109 *Let B be a complete normed strongly semisimple algebra over $\mathbb{K}$ admitting power-associativity. Then dense range algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to B are automatically continuous.*

Actually, a proof of the above conjecture could become only a partial affirmative answer to the following.

Problem 4.1.110 Let B be a complete normed strongly semisimple algebra over $\mathbb{K}$. Are dense range algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to B automatically continuous?

Looking at the second paragraph in the proof of Theorem 4.1.19, we realize that both Conjecture 4.1.109 and Problem 4.1.110 reduce to the particular case where the complete normed algebra B (occasionally admitting power-associativity) is unital and simple. Interesting attempts to establish Conjecture 4.1.109 and to solve Problem 4.1.110 can be found in [174, 416, 621]. Problem 4.1.110 has an affirmative answer if the algebra B is algebraic [165].

Proposition 4.1.30 is due to Bensebah [85]. Corollaries 4.1.29 and 4.1.31 are new. Together with the consequence given later in Lemma 4.1.42, they become a codification of an argument of Kaup in the proof of [381, Theorem 3.3] (see also [814, Theorem 20.1]).

To conclude our comments on Subsections 4.1.1 and 4.1.2, let us review the following Jordan version of Theorem 1.1.116, due to Loos [403].

Theorem 4.1.111 *Let A be a complete normed unital Jordan complex algebra. Then the connected component of the unit in the set of all invertible elements of A is the set*

$$\{U_{\exp(a_1)} \cdots U_{\exp(a_n)}(\mathbf{1}) : n \in \mathbb{N}; a_1, \ldots, a_n \in A\}.$$

The purely algebraic results contained in Lemmas 4.1.33, 4.1.48, and 4.1.49, as well as in Propositions 4.1.34, 4.1.35, and 4.1.54, are taken from Upmeier [814]. According to Upmeier's note in [814, p. 392],

> Jordan [$*$]-triples are particular cases of Jordan pairs which form the most satisfactory category from an algebraic point of view. For a systematic account of the theory of Jordan pairs, cf. [771].

Lemma 4.1.38 and Facts 4.1.40 and 4.1.41 are folklore, whereas Lemma 4.1.43 is due to Kaup [381]. Theorem 4.1.45 is due to Braun, Kaup, and Upmeier [126] in the unital case, and to Youngson [655] in the actual unit-free case. Our proof is close to that of [814, Proposition 20.35], where the unital version of Theorem 4.1.45 is stated.

The celebrated Araki–Elliott theorem [24] (see also [725, Theorem 37.1]) asserts that the axiom $\|ab\| \leqslant \|a\|\|b\|$ in the definition of a C^*-algebra is redundant. The Araki–Elliott type theorem for unital JB-algebras, stated in Proposition 4.1.46, is due to Shultz and was announced in [672, pp. 111–12]. It was later proved by

Alvermann [17], who also proved the following Araki–Elliott-type theorem for unital JB^*-algebras.

Theorem 4.1.112 *Let A be a unital Jordan complex algebra endowed with a complete norm $\|\cdot\|$ and a $\|\cdot\|$-continuous conjugate-linear algebra involution $*$ satisfying $\|U_a(a^*)\| = \|a\|^3$ for every $a \in A$. Then A is a JB^*-algebra.*

Actually, Alvermann requires the involution to be $\|\cdot\|$-isometric, but this follows from our weaker requirement of continuity by keeping in mind Lemma 3.3.12, and then arguing as in the proof of Proposition 3.3.13.

Lemma 4.1.48 is taken from Upmeier [814, Lemmas 20.8 and 20.32]. Proposition 4.1.50 and Corollary 4.1.51 are due to Kaup [380, 381]. In view of Corollary 4.1.51, JB^*-triples could have been defined as those positive hermitian Banach Jordan $*$-triples J satisfying $\|L(x,x)\| = \|x\|^2$ for every $x \in J$. Historically, this is the first definition of a JB^*-triple. It is introduced in [380] (under the name of a C^*-triple system), and is the one taken in Chu [710, Definition 2.5.25] and Upmeier [814, Definition 20.7] with its current name.

Proposition 4.1.52 is half of Theorem 2.2.28, reviewed at that time without proof. We referred there to Kaup's paper [381] for a proof. However, keeping Proposition 4.1.50 in mind, we realize that Theorem 2.2.28 is six years older (see [380, Proposition 5.4]). Theorem 4.1.55 is due to Braun, Kaup, and Upmeier [126]. We note that, keeping Proposition 4.1.54 in mind, Theorem 4.1.55 follows straightforwardly from Theorems 4.1.45 and 4.1.112. Thus Theorem 4.1.112, whose proof has not been discussed, could have been the key tool for a complete proof of Theorem 4.1.55. Anyway, the proof of Theorem 4.1.55 we have given is close to both the original one in [126] and Alvermann's proof of Theorem 4.1.112. An alternative proof of Theorem 4.1.55 is reviewed in what follows.

In [270], Friedman and Russo apply Horn's determination of the so-called Type I JBW^*-triple factors [330] to prove the following converse form of Theorem 4.1.45.

Theorem 4.1.113 *Every JB^*-triple is isometrically triple-isomorphic to a closed subtriple of (the JB^*-triple underlying) a suitable JB^*-algebra.*

Then, as pointed out in [270], Theorem 4.1.113 above and Proposition 3.4.17 imply the following.

Corollary 4.1.114 *Let J be a JB^*-triple. Then we have*

$$\|\{xyz\}\| \leqslant \|x\|\|y\|\|z\| \text{ for all } x,y,z \in J.$$

Now it is worth noticing that Theorem 4.1.55 follows straightforwardly from Theorem 4.1.45, Proposition 4.1.54, and Corollary 4.1.114 immediately above. Although quite involved as a whole, the proof of Theorem 4.1.55 just suggested has today become the current one, mainly in the mind of the youngest JB^*-algebraists.

Proposition 4.1.58 is due to McCrimmon [433], whereas Proposition 4.1.60 is due to Kaidi [759]. With the exception of Facts 4.1.67 and 4.1.70 (which are folklore), the results from Lemma 4.1.61 to Corollary 4.1.73 are new. We note that Corollary 4.1.73 could have been derived almost straightforwardly from its associative forerunner (Proposition 1.2.24) and the fact that, if A is a unital alternative

C^*-algebra, and if x is in A, then the closed subalgebra of A generated by $\{\mathbf{1}, x, x^*\}$ is an (associative) C^*-algebra. Corollary 4.1.74 is well-known in the theory of JB-algebras (see [738, Theorem 3.2.4]).

Outside the setting of (normed) Jordan-admissible algebras, (topological) J-divisors of zero have seldom been considered in the literature. Nevertheless, they can be defined without any problem in the setting of general (normed) algebras. Indeed, let A be an algebra over $\mathbb{K}$, and let a be in A. We say that a is a *J-divisor of zero* in A if the operator $U_a^\bullet := U_a^{A^{\mathrm{sym}}}$ is not injective. In the case where A is normed, we say that a is a *topological J-divisor of zero* in A if the operator $U_a^\bullet$ is not bounded below. With these notions in mind, we are provided with the following.

Theorem 4.1.115 *Let A be a nonzero normed power-associative algebra over $\mathbb{K}$ with no nonzero topological J-divisor of zero. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then A is isomorphic to $\mathbb{C}$.*
(ii) *If $\mathbb{K} = \mathbb{R}$, then A is both a quadratic* (hence Jordan-admissible) *algebra and a J-division algebra.*

Proof We note that, since A has no nonzero J-divisor of zero, and for $a \in A$ the equality $U_a^\bullet(a) = a^3$ holds, A has no isotropic element. Now let a be an arbitrary nonzero element of A, and let B be the subalgebra of A generated by a. Then B is a nonzero normed associative and commutative algebra over $\mathbb{K}$ with no nonzero topological J-divisor of zero, and hence with no nonzero one-sided topological divisor of zero (by Proposition 4.1.23).

Assume that $\mathbb{K} = \mathbb{C}$. Then, by Proposition 2.5.49(i), B is isomorphic to $\mathbb{C}$. Therefore A is algebraic of degree 1, and hence, by Lemma 2.6.29, A is isomorphic to $\mathbb{C}$.

Finally, assume that $\mathbb{K} = \mathbb{R}$. Then, by Theorem 2.5.50, B is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Therefore A is algebraic of bounded degree, and then, by Lemma 2.6.33, A is unital. Now, denoting by C the subalgebra of A generated by $\{\mathbf{1}, a\}$, and arguing as above (with C instead of B), we obtain that C is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Therefore A is quadratic. Moreover, since a is invertible in C, it follows that a is J-invertible in A. Therefore A is a J-division algebra. □

The above theorem is a refined version, proved in [93], of the main result in [370]. It is worth mentioning that, in the proof of Theorem 4.1.115(ii), the use of Lemma 2.6.33 can be avoided. Indeed, the algebra A has a nonzero idempotent e (because it contains copies of $\mathbb{R}$ or $\mathbb{C}$), and we have $U_e^\bullet(A_0(e)) = U_e^\bullet(A_{\frac{1}{2}}(e)) = 0$ (cf. Lemma 2.5.3); hence $A_0(e) = A_{\frac{1}{2}}(e) = 0$ (because A has no nonzero J-divisor of zero), and e becomes a unit for A.

Lemma 4.1.75 has been taken from [822, Proposition 2, p. 67], whereas its consequence, stated in Corollary 4.1.76, appears as Lemma 1 in [754, p. 42]. Proposition 4.1.77 is the key tool in Subsection 4.1.5. It is due to N. Jacobson, who communicated it to Martínez to be included in his PhD thesis [775] and later published it in [423]. By the indication of a referee, Jacobson's argument was incorporated as a part of the proof of [95, Theorem 1]. Proposition 4.1.80 is due to Viola Devapakkiam [628], whereas Corollary 4.1.81 and Proposition 4.1.82 are new. It is worth mentioning that, reducing to the commutative case (as we have done in our proof), and invoking Proposition 4.1.25(ii), Proposition 4.1.80 follows from Theorem 4.1.115 above.

Lemma 4.1.83 is raised as an exercise in [754, p. 54]. Propositions 4.1.84 and 4.1.86 are due to Kaidi [759, 361]. They also appear incorporated with a proof in [423, 95]. Theorem 4.1.88 is the main result in Martínez' paper [423]. Its commutative forerunner is also due to him [775]. Theorem 4.1.93 is new.

In relation to Definition 4.1.85, the following result, taken from [759], should be noted.

Proposition 4.1.116 *Let A be a complete normed unital Jordan-admissible algebra over $\mathbb{K}$, and let B be a J-full subalgebra of A. Then the closure of B in A is a J-full subalgebra of A. As a consequence, for any non-empty subset S of A, the closed J-full subalgebra of A generated by S coincides with the closure in A of the J-full subalgebra of A generated by S.*

Proof Let x be in $\bar{B} \cap \text{J-Inv}(A)$. Then $x = \lim b_n$ for a suitable sequence b_n in B, and, since $\text{J-Inv}(A)$ is open in A (cf. Theorem 4.1.7), we may assume that $b_n \in \text{J-Inv}(A)$ for every $n \in \mathbb{N}$. But, since B is a J-full subalgebra of A, we have $b_n^{-1} \in B$ for every $n \in \mathbb{N}$. Therefore, invoking Proposition 4.1.6(ii), we deduce that $x^{-1} = \lim b_n^{-1} \in \bar{B}$. □

The associative forerunner of Proposition 4.1.116 can be found in [697, p. 18], where the assumption of completeness for A is forgotten.

Lemma 4.1.95 is due to Aupetit [682, Corollary 1.2.3], and generalizes Edwards' forerunner [227] asserting that $\mathbb{R}, \mathbb{C}$, *and* $\mathbb{H}$ *are the only norm-unital complete normed associative real algebras satisfying* $\|x\|\|x^{-1}\| = 1$ *for every invertible element* x. Theorem 4.1.96 is the main result in the paper of Benslimane and Merrachi [92]. Although easily deducible from results known at that time, Corollaries 4.1.97–4.1.101 went unnoticed in [92]. It is worth mentioning that, keeping in mind Proposition 2.5.24 and the fact that the product of invertible elements a, b in a unital alternative algebra is invertible with $(ab)^{-1} = b^{-1}a^{-1}$ [822, Lemma 10.9], Corollary 4.1.100 can be proved without any resource to Theorem 4.1.96. Indeed, it is enough to mimic the proof of Lemma 4.1.95.

4.2 Unitaries in JB^*-triples and in non-commutative JB^*-algebras

Introduction In Subsection 4.2.1, we prove Kaup's theorems that abelian JB^*-triples are closed subtriples of commutative C^*-algebras [381], and that JB^*-triples generated by a single element are in fact commutative C^*-algebras [380] (see Theorems 4.2.7 and 4.2.9, respectively). Either directly or by means of its consequence stated in Corollary 4.2.12, the second of these theorems becomes one of the key tools in the remaining part of the present section.

Subsection 4.2.2 contains the main results in the section. After developing the Peirce arithmetics for a tripotent of a Jordan $*$-triple [771], we prove the Braun–Kaup–Upmeier theorems [126] that vertices of the closed unit ball of a JB^*-triple J are precisely the unitary elements of J, and that vertices of the closed unit ball of a non-commutative JB^*-algebra A are precisely the 'J-unitary elements' of A

(see Theorems 4.2.24 and 4.2.28, respectively). These theorems generalize the Bohnenblust–Karlin Theorem 2.1.27. Refining an argument of Peralta [486], we provide a simple proof of a result by Friedman–Russo [269] and Upmeier [814] (see Proposition 4.2.32), and derive from it that the extreme points of the closed unit ball of a JB^*-triple J are precisely the complete tripotents of J (see Theorem 4.2.34). Then we conclude the subsection by proving Youngson's theorem [655] that non-commutative JB^*-algebras are unital if and only if their closed unit balls have extreme points (see Theorem 4.2.36).

We begin Subsection 4.2.3 by establishing the Wright–Youngson generalization of the Russo–Dye theorem to the setting of non-commutative JB^*-algebras [642] (see Fact 4.2.39). Then we prove Siddiqui's refinement [574] that each element in the open unit ball of a unital non-commutative JB^*-algebra is an arithmetic mean of a finite number of J-unitary elements (see Theorem 4.2.43). This generalizes associative forerunners of Gardner [281] and Kadison–Pedersen [360].

In Subsection 4.2.4, we do a quick incursion into the world of real non-commutative JB^*-algebras [19] and of real JB^*-triples [341]. Real non-commutative JB^*-algebras constitute an important unifying class of algebras because they include (complex) non-commutative JB^*-algebras (regarded as real algebras), real JB^*-algebras [18] (so also JB-algebras), and real alternative C^*-algebras [762] (so also real C^*-algebras [735]). They also include smooth-normed real algebras, introduced in §2.6.1. Actually, smooth-normed real algebras are precisely those real non-commutative JB^*-algebras which are J-division algebras (see Theorem 4.2.48). As in the complex case, a real non-commutative JB^*-algebra is unital if and only if its closed unit ball has extreme points (see Corollary 4.2.58), and biduals of real non-commutative JB^*-algebras are real non-commutative JB^*-algebras in a natural way (see Proposition 4.2.62). Real JB^*-triples contain (complex) JB^*-triples (regarded as real spaces) and real non-commutative JB^*-algebras (under a natural triple product). Algebraic characterizations of vertices [261] and of extreme points [341] of the closed unit ball of a real JB^*-triple are given (see Theorems 4.2.53 and 4.2.57, respectively).

4.2.1 A commutative Gelfand–Naimark theorem for JB^*-triples

We recall that C^*-algebras are JB^*-triples in a natural way, and that closed subtriples of JB^*-triples are JB^*-triples (cf. Facts 4.1.41 and 4.1.40, respectively). Now let E be a subset of a Hausdorff locally convex complex space such that $0 \notin E$, $E \cup \{0\}$ is compact, and $\mathbb{T}E \subseteq E$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Then E is a locally compact Hausdorff topological space, we set

$$C_0^{\mathbb{T}}(E) := \{x \in C_0^{\mathbb{C}}(E) : x(zt) = zx(t) \text{ for every } (z,t) \in \mathbb{T} \times E\},$$

and note that $C_0^{\mathbb{T}}(E)$ is a closed subtriple of the C^*-algebra $C_0^{\mathbb{C}}(E)$, so that $C_0^{\mathbb{T}}(E)$ becomes an abelian JB^*-triple. As the main result in this subsection, we will show in Theorem 4.2.7 that there are no more abelian JB^*-triples other than those given by the construction just done. To this end, we begin by proving the following Stone–Weierstrass type theorem.

Proposition 4.2.1 *Let E be as above, and let J be a subtriple of $C_0^{\mathbb{T}}(E)$ satisfying*

(i) *for each $t \in E$ there is $x \in J$ such that $x(t) \neq 0$;*
(ii) *J separates the points of E.*

Then J is dense in $C_0^{\mathbb{T}}(E)$.

Proof For $x \in C_0^{\mathbb{C}}(E)$, let $P(x)$ denote the complex-valued function on E defined by

$$P(x)(t) := \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta}x(e^{i\theta}t)d\theta$$

for every $t \in E$. Then it is easily realized that $P(x)$ lies in $C_0^{\mathbb{T}}(E)$ for every $x \in C_0^{\mathbb{C}}(E)$, and that $P(x) = x$ whenever x is in $C_0^{\mathbb{T}}(E)$. Therefore the mapping $P : x \to P(x)$ becomes a contractive linear projection on $C_0^{\mathbb{C}}(E)$ whose range is $C_0^{\mathbb{T}}(E)$.

Now let M be a subtriple of $C_0^{\mathbb{T}}(E)$, and let $x_1,\dots,x_n,y_1,\dots,y_m$ be in M. Then for every $(z,t) \in \mathbb{T}\times E$ we have

$$(x_1x_2\cdots x_ny_1^*y_2^*\cdots y_m^*)(zt) = z^{n-m}(x_1x_2\cdots x_ny_1^*y_2^*\cdots y_m^*)(t),$$

hence $P(x_1x_2\cdots x_ny_1^*y_2^*\cdots y_m^*) = 0$ whenever $n \neq m+1$. But, if $n = m+1$, then a straightforward induction argument on m shows that $x_1x_2\cdots x_{m+1}y_1^*y_2^*\cdots y_m^*$ lies in M, hence

$$P(x_1x_2\cdots x_{m+1}y_1^*y_2^*\cdots y_m^*) = x_1x_2\cdots x_{m+1}y_1^*y_2^*\cdots y_m^* \in M.$$

Therefore $P(x_1x_2\cdots x_ny_1^*y_2^*\cdots y_m^*) \in M$ for all values of n and m, which implies that the image under P of the subalgebra of $C_0^{\mathbb{C}}(E)$ generated by $M \cup M^*$ is contained in M.

To conclude the proof, note that, by the assumptions on J and the Stone–Weierstrass theorem (cf. Corollary 1.2.53), the subalgebra of $C_0^{\mathbb{C}}(E)$ (say A) generated by $J \cup J^*$ is dense in $C_0^{\mathbb{C}}(E)$. Therefore, since $P(A) \subseteq J$, and P is continuous, and P is the identity on $C_0^{\mathbb{T}}(E)$, it follows that J is dense in $C_0^{\mathbb{T}}(E)$. □

Lemma 4.2.2 *Let J be a complex Banach Jordan $*$-triple, and let $\lambda : J \to \mathbb{C}$ be a triple homomorphism. Then λ is continuous and $\|\lambda\| \leqslant \sqrt{N}$, where*

$$N := \sup\{\|\{xyz\}\| : x,y,z \in \mathbb{S}_J\}.$$

Proof Otherwise, there would be a z in $\mathbb{S}_J$ with $|\lambda(z)|^2 > N$. Setting $x := \frac{\overline{\lambda(z)}}{|\lambda(z)|^2}z$, we see that $\lambda(x) = 1$ and $N\|x\|^2 < 1$. By induction we have $\|x^{(2n+1)}\| \leqslant (N\|x\|^2)^n\|x\|$ for every natural number n, and therefore the series $\sum_{n\geqslant 0}\|x^{(2n+1)}\|$ converges. It follows from the completeness of J that the series $\sum_{n\geqslant 0}x^{(2n+1)}$ is convergent in J. Set $y := \sum_{n=0}^{\infty}x^{(2n+1)}$. Since for each n we have $x+\{x,\sum_{k=0}^{n}x^{(2k+1)},x\} = \sum_{k=0}^{n+1}x^{(2k+1)}$, it follows that $x+\{xyx\} = y$, and consequently $1+\overline{\lambda(y)} = \lambda(y)$, a contradiction. □

Let J be a complex Banach Jordan $*$-triple, and denote by $\Lambda = \Lambda_J$ the set of all nonzero triple homomorphisms from J to $\mathbb{C}$. It follows from Lemma 4.2.2 that Λ is contained in J'. Thus Λ can and will be endowed with the relative w^*-topology. Note that Λ may be empty. Anyway, note that $\mathbb{T}\Lambda \subseteq \Lambda$, so that the relation $\lambda \in \mathbb{T}\mu$ becomes an equivalence relation on Λ. We denote by $\Lambda/\mathbb{T}$ the corresponding quotient topological space.

Proposition 4.2.3 *If J is a complex Banach Jordan $*$-triple, then $\Lambda \cup \{0\}$ is a w^*-compact subset of J'. As a consequence, Λ is a locally compact Hausdorff topological space for the relative w^*-topology.*

Proof It follows from Lemma 4.2.2 that Λ is in fact contained in $r\mathbb{B}_{J'}$ for suitable $r > 0$. Since, by the Banach–Alaoglu theorem, $\mathbb{B}_{J'}$ is compact in the w^*-topology, it suffices to show that $\Lambda \cup \{0\}$ is w^*-closed in J'. Note that, for each $x, y, z \in J$, the mapping $h_{x,y,z} : f \to f(\{xyz\}) - f(x)\overline{f(y)}f(z)$ from J' to $\mathbb{C}$ is w^*-continuous, and hence the set $H_{x,y,z} := \{f \in J' : h_{x,y,z}(f) = 0\}$ is w^*-closed. Since $\Lambda \cup \{0\} = \bigcap_{(x,y,z)\in J\times J\times J} H_{x,y,z}$, it follows that $\Lambda \cup \{0\}$ is w^*-closed in J'. □

Proposition 4.2.4 *Let J be a complex Banach Jordan $*$-triple. Then we have:*

(i) *For each $x \in J$, the complex-valued function $G(x)$ on Λ, defined by*

$$G(x)(\lambda) := \lambda(x) \text{ for every } \lambda \in \Lambda,$$

lies in $C_0^{\mathbb{T}}(\Lambda)$.

(ii) *The mapping $G : x \to G(x)$ from J to $C_0^{\mathbb{T}}(\Lambda)$ becomes a dense range continuous triple homomorphism.*

Proof Let x be in J. Then, clearly, $G(x)$ is a continuous function satisfying $G(x)(z\lambda) = zG(x)(\lambda)$ for every $(z, \lambda) \in \mathbb{T} \times \Lambda$. Moreover, since $\Lambda \cup \{0\}$ is compact in the w^*-topology of J', and for every $\varepsilon > 0$ we have

$$\{\lambda \in \Lambda : |G(x)(\lambda)| \geqslant \varepsilon\} = \{\lambda \in \Lambda : |\lambda(x)| \geqslant \varepsilon\} = \{\lambda \in \Lambda \cup \{0\} : |\lambda(x)| \geqslant \varepsilon\},$$

we conclude that $G(x)$ lies indeed in $C_0^{\mathbb{T}}(\Lambda)$. Thus assertion (i) has been proved.

Assertion (ii) follows straightforwardly from Lemma 4.2.2 and Proposition 4.2.1 applied to the subtriple $G(J)$ of $C_0^{\mathbb{T}}(\Lambda)$. □

Given a complex Banach Jordan $*$-triple J, the mapping G in the above proposition is called the *Gelfand representation* of J.

Proposition 4.2.5 *Let J be a nonzero abelian hermitian Banach Jordan $*$-triple. Then the closed linear hull $\mathfrak{A}$ of $L(J,J) \cup \{I_J\}$ in $BL(J)$ is a unital commutative C^*-algebra for the involution determined by $L(x,y)^* = L(y,x)$.*

Proof Since J is abelian we have

$$L(u,v)L(x,y) = L(x,y)L(u,v) = L(\{x,y,u\},v) \tag{4.2.1}$$

for all $u, v, x, y \in J$. Therefore $L(J,J) \cup \{I_J\}$ is a commutative and multiplicatively closed subset of $BL(J)$, and hence the closed linear hull $\mathfrak{A}$ of $L(J,J) \cup \{I_J\}$ in $BL(J)$ is a unital commutative subalgebra of $BL(J)$. Since

$$\begin{aligned} 4L(x,y) = {} & L(x+y,x+y) - L(x-y,x-y) \\ & + i\left(L(x+iy,x+iy) - L(x-iy,x-iy)\right), \end{aligned} \tag{4.2.2}$$

and by Corollary 2.1.2

$$L(x+y,x+y) - L(x-y,x-y) \text{ and } L(x+iy,x+iy) - L(x-iy,x-iy)$$

lie in $H(\mathfrak{A}, I_J)$, it follows that $\mathfrak{A}$ is a commutative and associative complete V-algebra. Hence $\mathfrak{A}$ becomes a commutative C^*-algebra for its natural involution $*$ because of the associative Vidad–Palmer theorem (cf. Theorem 2.3.32). Moreover, by (4.2.2), we have $L(x,y)^* = L(y,x)$ for all $x,y \in J$. $\square$

Let J be an abelian hermitian Banach Jordan $*$-triple, and let Δ stand for the carrier space (cf. Proposition 1.1.71) of the unital commutative C^*-algebra $\mathfrak{A}$ in Proposition 4.2.5. Then, by Corollary 1.2.22, we have $\varphi(L(x,x)) \in \mathbb{R}$ for all $\varphi \in \Delta$ and $x \in J$. We set

$$\Delta_0 := \{\phi \in \Delta : \phi(L(x,x)) \neq 0 \text{ for some } x \in J\},$$

and, for $\sigma = \pm$, we set

$$\Delta_0^\sigma := \{\varphi \in \Delta_0 : \sigma\varphi(L(x,x)) \geqslant 0 \text{ for every } x \in J\},$$

which is clearly a closed subset of Δ_0. We note that, since Δ is compact and Hausdorff, and $\Delta \setminus \Delta_0$ is either empty or a singleton (depending on whether or not I_J lies in the closed linear hull of $L(J,J)$ in $BL(J)$), Δ_0 becomes a locally compact Hausdorff topological space.

Proposition 4.2.6 *Let J be a nonzero abelian hermitian Banach Jordan $*$-triple. Then we have:*

(i) *Δ_0 is the disjoint union of Δ_0^+ and Δ_0^-. In particular, Δ_0^+ and Δ_0^- are clopen in Δ_0.*
(ii) *For every $\lambda \in \Lambda$ there exists a unique $\tilde{\lambda} \in \Delta$ such that $\tilde{\lambda}(L(x,y)) = \lambda(x)\overline{\lambda(y)}$ for all $x,y \in J$, and moreover $\tilde{\lambda}$ lies in Δ_0^+.*
(iii) *The mapping $\lambda \to \tilde{\lambda}$ from Λ to Δ_0^+ is continuous and surjective, and induces a continuous bijective mapping from $\Lambda/\mathbb{T}$ to Δ_0^+.*

Proof By polarization law (cf. (4.2.2)), we have $\Delta_0^+ \cap \Delta_0^- = \emptyset$. Now let φ be in $\Delta_0 \setminus \Delta_0^-$. Then $\varphi(L(x,x)) = 1$ for some $x \in J$, and then, applying (4.2.1), for each $u \in J$ we have

$$\begin{aligned}\varphi(L(u,u)) &= \varphi(L(x,x))\varphi(L(u,u)) = \varphi(L(x,x)L(u,u)) = \varphi(L(\{xxu\},u))\\ &= \varphi(L(\{uxx\},u)) = \varphi(L(u,x)L(x,u)) = \varphi(L(u,x)L(u,x)^*)\\ &= \varphi(L(u,x))\varphi(L(u,x)^*) = \varphi(L(u,x))\overline{\varphi(L(u,x))} \geqslant 0.\end{aligned}$$

Thus $\varphi \in \Delta_0^+$, and we have proved assertion (i).

Let $\lambda \in \Lambda$ be given and fix $x \in J$ with $\lambda(x) = 1$. Consider the mapping $\tilde{\lambda} : \mathfrak{A} \to \mathbb{C}$ defined by

$$\tilde{\lambda}(F) := \lambda(F(x)) \quad \text{for every} \quad F \in \mathfrak{A}.$$

Keeping in mind Lemma 4.2.2, it turns out clear that $\tilde{\lambda}$ is a continuous linear mapping satisfying $\tilde{\lambda}(I_J) = 1$ and $\tilde{\lambda}(L(x,x)) = 1 \neq 0$. Moreover, for each $u,v \in J$ we have

$$\tilde{\lambda}(L(u,v)) = \lambda(\{uvx\}) = \lambda(u)\overline{\lambda(v)}\lambda(x),$$

and hence

$$\tilde{\lambda}(L(u,v)) = \lambda(u)\overline{\lambda(v)}. \tag{4.2.3}$$

Therefore

$$\begin{aligned}\tilde{\lambda}(L(u_1,v_1)L(u_2,v_2)) &= \tilde{\lambda}(L(\{u_1v_1u_2\}),v_2) = \lambda(\{u_1v_1u_2\})\overline{\lambda(v_2)} \\ &= \lambda(u_1)\overline{\lambda(v_1)}\lambda(u_2)\overline{\lambda(v_2)} = \tilde{\lambda}(L(u_1,v_1))\tilde{\lambda}(L(u_2,v_2))\end{aligned}$$

for all $u_1,u_2,v_1,v_2 \in J$. Keeping in mind the description of $\mathfrak{A}$, we easily deduce that $\tilde{\lambda}(F \circ G) = \tilde{\lambda}(F)\tilde{\lambda}(G)$ for all $F,G \in \mathfrak{A}$, that is to say $\tilde{\lambda} \in \Delta_0$. Moreover, it follows from (4.2.3) that $\tilde{\lambda}(L(u,u)) \geqslant 0$ for every $u \in J$, and so we have in fact that $\tilde{\lambda} \in \Delta_0^+$. Assume that $\psi \in \Delta$ satisfies $\psi(L(u,v)) = \lambda(u)\overline{\lambda(v)}$ for all $u,v \in J$. Then $\psi(I_J) = \tilde{\lambda}(I_J)$ and $\psi(L(u,v)) = \tilde{\lambda}(L(u,v))$ for all $u,v \in J$, and consequently $\psi = \tilde{\lambda}$ because of the continuity of ψ (cf. Corollary 1.1.64) and the description of $\mathfrak{A}$. This completes the proof of assertion (ii).

Let $\varphi \in \Delta_0^+$ be given and fix $x \in J$ such that $\varphi(L(x,x)) = 1$. Then consider the mapping $\lambda : J \to \mathbb{C}$ defined by

$$\lambda(u) := \varphi(L(u,x)) \text{ for every } u \in J.$$

It is clear that λ is a nonzero linear mapping. Since, for all $u,v,w \in J$ we have

$$\{\{uxx\}vw\} = \{\{xxu\}vw\} = \{xx\{uvw\}\},$$

it follows that

$$\begin{aligned}L(x,x)L(\{uvw\},x) &= L(\{xx\{uvw\}\},x) = L(\{\{uxx\}vw\},x) \\ &= L(\{uxx\},v)L(w,x) = L(u,x)L(x,v)L(w,x),\end{aligned}$$

and hence

$$\begin{aligned}\lambda(\{uvw\}) &= \varphi(L(\{uvw\},x)) = \varphi(L(x,x))\varphi(L(\{uvw\},x)) \\ &= \varphi(L(x,x)L(\{uvw\},x)) = \varphi(L(u,x)L(x,v)L(w,x)) \\ &= \varphi(L(u,x))\varphi(L(v,x)^*)\varphi(L(w,x)) = \lambda(u)\overline{\lambda(v)}\lambda(w).\end{aligned}$$

Therefore $\lambda \in \Lambda$. Moreover, for all $u,v \in J$ we have

$$\begin{aligned}\varphi(L(u,v)) &= \varphi(L(x,x))\varphi(L(u,v)) = \varphi(L(x,x)L(u,v)) \\ &= \varphi(L(\{xxu\},v)) = \varphi(L(\{uxx\},v)) = \varphi(L(u,x)L(x,v)) \\ &= \varphi(L(u,x))\varphi(L(v,x)^*) = \varphi(L(u,x))\overline{\varphi(L(v,x))} = \lambda(u)\overline{\lambda(v)},\end{aligned}$$

and consequently, by assertion (ii), $\varphi = \tilde{\lambda}$. Thus we have proved that the mapping $\lambda \to \tilde{\lambda}$ from Λ to Δ_0^+ is surjective. If λ_i is a net on Λ such that $\lambda_i \to \lambda \in \Lambda$ and $\tilde{\lambda}_i \to \varphi \in \Delta$, then for all $x,y \in J$ we see that $\lambda_i(x) \to \lambda(x)$, $\overline{\lambda_i(y)} \to \overline{\lambda(y)}$, and

$$\lambda(x)\overline{\lambda(y)} \longleftarrow \lambda_i(x)\overline{\lambda_i(y)} = \tilde{\lambda}_i(L(x,y)) \longrightarrow \varphi(L(x,y)).$$

Therefore, by assertion (ii), $\varphi = \tilde{\lambda}$. Hence $\lambda \to \tilde{\lambda}$, regarded as a mapping from Λ to Δ, has closed graph. Therefore, since Δ is compact, that mapping is continuous by the closed graph theorem [674, Theorem 2.58]. For $\lambda \in \Lambda$ and $z \in \mathbb{T}$, it is clear that $\widetilde{z\lambda} = \tilde{\lambda}$. Assume that $\lambda, \mu \in \Lambda$ satisfy $\tilde{\lambda} = \tilde{\mu}$. Then we have $\lambda(x)\overline{\lambda(y)} = \mu(x)\overline{\mu(y)}$ for all $x,y \in J$, and choosing $y \in J$ such that $\lambda(y) = 1$ we see that $\lambda(x) = \mu(x)\overline{\mu(y)}$ for every $x \in J$, and in particular $1 = |\mu(y)|^2$. Therefore $\lambda = \overline{\mu(y)}\mu$ and $|\overline{\mu(y)}| = 1$.

Thus the mapping $\tilde{}$ induces a bijective mapping from $\Lambda/\mathbb{T}$ onto Δ_0^+. Moreover, this mapping is continuous from $\Lambda/\mathbb{T}$ endowed with the quotient topology onto Δ_0^+ [818, Theorem 6.5.4]. □

Now we can prove the main result in this subsection.

Theorem 4.2.7 *Let J be a nonzero abelian JB^*-triple. Then J is isometrically triple-isomorphic to a subtriple of a commutative C^*-algebra. More precisely, the Gelfand representation*

$$G : J \longrightarrow C_0^{\mathbb{T}}(\Lambda)$$

becomes an isometric surjective triple homomorphism.

Proof In view of Proposition 4.2.4, it is enough to show that G is an isometry. Let x be in $J \setminus \{0\}$. Then, by Corollary 4.1.51 and Propositions 1.1.68(ii) and 4.2.6(i), there exists $\varphi \in \Delta_0^+$ such that $\varphi(L(x,x)) = \|x\|^2$. Moreover, by Proposition 4.2.6(ii)–(iii), there is $\lambda \in \Lambda$ satisfying $\varphi(L(x,x)) = |\lambda(x)|^2$. Therefore we have $\|x\|^2 = |\lambda(x)|^2 = |G(x)(\lambda)|^2 \leqslant \|G(x)\|^2$. Now, if G were not an isometry, then there would exist $y \in J$ with $\|y\| < \|G(y)\| = 1$, so that, by Lemma 4.1.48(i), the sequence $y^{(2n+1)}$ would converge to zero in J, whereas $G(y^{(2n+1)})$ would not converge to zero in $C_0^{\mathbb{T}}(\Lambda)$, a contradiction. □

Now we are going to show that JB^*-triples generated by a single element are in fact commutative C^*-algebras (regarded as JB^*-triples). To this end, we begin by proving the following lemma, which follows by applying the Stone–Weierstrass theorem twice.

Lemma 4.2.8 *Let E be a subset of $\mathbb{R}^+$ such that $E \cup \{0\}$ is compact, and let u stand for the inclusion mapping $E \hookrightarrow \mathbb{C}$. Then the set $\{p(u^2)u : p \in \mathbb{C}[\mathbf{x}]\}$ is dense in $C_0^{\mathbb{C}}(E)$.*

Proof By Corollary 1.2.53, the set $\{f(u)u : f \in C^{\mathbb{C}}(E)\}$ is dense in $C_0^{\mathbb{C}}(E)$. But, by Theorem 1.2.10, the set $\{p(u^2) : p \in \mathbb{C}[\mathbf{x}]\}$ is dense in $C^{\mathbb{C}}(E)$. □

Theorem 4.2.9 *Let J be a JB^*-triple, let x be a nonzero element of J, and let M stand for the closed subtriple of J generated by x. Then there exists a unique couple (E,Φ), where E is a subset of $\mathbb{R}^+$ such that $E \cup \{0\}$ is compact, and Φ is an isometric triple homomorphism from the C^*-algebra $C_0^{\mathbb{C}}(E)$ to J satisfying $\Phi(C_0^{\mathbb{C}}(E)) = M$ and $\Phi(u) = x$, where u denotes the inclusion mapping $E \hookrightarrow \mathbb{C}$.*

Proof First we prove the existence of the couple (E,Φ). By Fact 4.1.40, Lemma 4.1.49, and Theorem 4.2.7, we may assume that M is a closed subtriple of $C_0^{\mathbb{C}}(F)$ for some locally compact Hausdorff topological space F. Then, by induction, we get $|x|^{2n}x = x^{(2n+1)}$ for every $n \in \mathbb{N} \cup \{0\}$ (cf. §4.1.47), and hence $p(|x|^2)x$ lies in M for every $p \in \mathbb{C}[\mathbf{x}]$. Therefore

$$N := \left\{p(|x|^2)x : p \in \mathbb{C}[\mathbf{x}]\right\}$$

is a subtriple of M containing x, so N is dense in M because M is generated by x as a Banach Jordan $*$-triple. Define $E := \{|x(t)| : t \in F\} \setminus \{0\}$, so that E is a subset of $\mathbb{R}^+$

such that $E \cup \{0\}$ is compact, let u denote the inclusion mapping $E \hookrightarrow \mathbb{C}$, and note that, by Lemma 4.2.8,

$$L := \{p(u^2)u : p \in \mathbb{C}[\mathbf{x}]\}$$

is a dense subtriple of $C_0^{\mathbb{C}}(E)$. Since $\|p(u^2)u\| = \|p(|x|^2)x\|$ for every $p \in \mathbb{C}[\mathbf{x}]$, the correspondence $p(u^2)u \to p(|x|^2)x$ becomes a well-defined isometric dense range triple homomorphism from L to M taking u to x. Extending by continuity this triple homomorphism, and composing this extension with the inclusion mapping $M \hookrightarrow J$, we get the desired isometric triple homomorphism $\Phi : C_0^{\mathbb{C}}(E) \to J$ satisfying $\Phi(C_0^{\mathbb{C}}(E)) = M$ and $\Phi(u) = x$.

Now we show the uniqueness of the couple (E, Φ). Let (E_1, Φ_1) be a couple with the same properties as (E, Φ), and, accordingly, let u_1 stand for the inclusion mapping $E_1 \hookrightarrow \mathbb{C}$. Then, regarding Φ and Φ_1 as surjective M-valued mappings, $\Phi^{-1} \circ \Phi_1$ becomes a surjective linear isometry from $C_0^{\mathbb{C}}(E_1)$ to $C_0^{\mathbb{C}}(E)$ taking u_1 to u. Therefore, by the Banach–Stone theorem (cf. Theorem 2.3.58), there is a $\mathbb{T}$-valued continuous function v on E, together with a homeomorphism $\tau : E \to E_1$, such that

$$(\Phi^{-1} \circ \Phi_1)(g)(t) = v(t)g(\tau(t)) \text{ for every } (g,t) \in C_0^{\mathbb{C}}(E_1) \times E. \tag{4.2.4}$$

By taking $g = u_1$ in (4.2.4), we get $t = v(t)\tau(t)$ for every $t \in E$, which implies that $v(t) = 1$ for every $t \in E$, so $\tau(t) = t$ for every $t \in E$, and so $E_1 = E$. Finally, invoking (4.2.4) again, we get that $\Phi_1 = \Phi$. □

Definition 4.2.10 Let J and x be as in the above theorem. Then the locally compact subset E of $\mathbb{R}^+$, given by the theorem, will be called the *triple spectrum* of x (relative to J), and will be denoted by $\sigma(x)$. We note that $\|x\| = \max \sigma(x)$.

The isometric triple homomorphism Φ in Theorem 4.2.9 does not have a specific name in the literature. Indeed, by setting $f(x) := \Phi(f)$ for every $f \in C_0^{\mathbb{C}}(\sigma(x))$, Φ is nothing other than the mapping $f \to f(x)$ from $C_0^{\mathbb{C}}(\sigma(x))$ to J, which suggests a 'continuous triple functional calculus' in the element x of J.

Corollary 4.2.11 *Let J be a JB^*-triple, and let x be in J. Then there exists a unique $y \in J$ such that $\{yyy\} = x$.*

Proof Assume at first that J is equal to (the JB^*-triple underlying) $C_0^{\mathbb{C}}(E)$ for some locally compact Hausdorff topological space E. Then the mapping $y : E \to \mathbb{C}$, defined by $y(t) = \frac{x(t)}{\sqrt[3]{|x(t)|^2}}$ if $x(t) \neq 0$ and $y(t) = 0$ otherwise, is the unique element in J such that $\{yyy\} = x$.

Now let J be an arbitrary JB^*-triple, let x be in $J \setminus \{0\}$, and let M stand for the closed subtriple of J generated by x. Then, according to Theorem 4.2.9 and a half of the above paragraph there exists $y \in M$ such that $\{yyy\} = x$. Let z be any element of J satisfying $\{zzz\} = x$, and let N stand for the closed subtriple of J generated by z. Then $M \subseteq N$, and hence $y, z \in N$. It follows from Theorem 4.2.9 again and the other half of the above paragraph that $z = y$. □

Corollary 4.2.12 *Let J be a JB^*-triple, and let x and φ be norm-one elements of J and J', respectively, such that $\varphi(x) = 1$. Then the restriction of φ to the closed subtriple of J generated by x is a triple homomorphism. In particular, $\varphi(\{xxx\}) = 1$.*

Proof In view of Lemma 4.1.49, it is enough to show that $\varphi(x^{(2n+1)}) = 1$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$, and let r be in $\mathbb{R}$ with $0 < r \leqslant \frac{1}{2n+1}$. Then the mean value theorem shows that the function $t \to t - rt^{2n+1}$ is increasing in $[0,1]$. Since $1 = \|x\| = \max \sigma(x)$, it follows from Theorem 4.2.9 that

$$\left\| x - rx^{(2n+1)} \right\| = \max \left\{ t - rt^{2n+1} : t \in \sigma(x) \right\} = 1 - r.$$

Therefore, by Proposition 2.1.5, we have $\min \Re(V(J, x, x^{(2n+1)})) = 1$. On the other hand, since

$$\left\| x^{(2n+1)} \right\| = \max \left\{ t^{2n+1} : t \in \sigma(x) \right\} = 1,$$

we have $V(J, x, x^{(2n+1)}) \subseteq \mathbb{B}_{\mathbb{C}}$. It follows that $V(J, x, x^{(2n+1)}) = \{1\}$. Therefore, since $\varphi(x^{(2n+1)}) \in V(J, x, x^{(2n+1)})$, we see that $\varphi(x^{(2n+1)}) = 1$, as desired. □

4.2.2 The main results

Definition 4.2.13 An element e of a Jordan $*$-triple over $\mathbb{K}$ is said to be a *tripotent* ('triple idempotent') if it satisfies $\{eee\} = e$.

The so-called *Peirce decomposition* of a Jordan $*$-triple relative to a tripotent follows easily from the corresponding Pierce decomposition of a Jordan algebra relative to an idempotent. Indeed, we have the following.

Fact 4.2.14 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be a tripotent in J. Then:*

(i) *$J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e)$, where $J_k(e) := \{x \in J : \{eex\} = kx\}$ for $k \in \{1, \frac{1}{2}, 0\}$.*

(ii) *The projections $P_k(e)$ from J onto $J_k(e)$ $(k = 1, \frac{1}{2}, 0)$ corresponding to the decomposition $J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e)$ are given by*

$$P_1(e) = L(e,e)(2L(e,e) - I_J),\ P_{\frac{1}{2}}(e) = 4L(e,e)(I_J - L(e,e)),$$

and

$$P_0(e) = (L(e,e) - I_J)(2L(e,e) - I_J).$$

Proof The tripotent e becomes an idempotent in the e-homotope Jordan algebra $J^{(e)}$ of J (cf. Definition 4.1.36) in such a way that $L(e,e)$ becomes the multiplication operator by e in $J^{(e)}$. Therefore the result follows from assertions (i) and (v) in Lemma 2.5.3. □

Proposition 4.2.15 *Let J be a hermitian Banach Jordan $*$-triple, and let e be a tripotent in J. Then for every $(x,y,z) \in J_1(e) \times J_{\frac{1}{2}}(e) \times J_0(e)$ we have*

$$\|\lambda^2 x + \lambda y + z\| = \|x + y + z\| \ \text{for every}\ \lambda \in \mathbb{S}_{\mathbb{C}} \tag{4.2.5}$$

and

$$\max\{\|x\|, \|y\|, \|z\|\} \leqslant \|x + y + z\|. \tag{4.2.6}$$

Proof Let k be in $\{1, \frac{1}{2}, 0\}$, and let u be in $J_k(e)$. Then we have $L(e,e)(u) = ku$, so $L(e,e)^n(u) = k^n u$ for every $n \in \mathbb{N}$, hence

$$\exp(\mu L(e,e))(u) = e^{k\mu} u \ \text{ for every }\ \mu \in \mathbb{C}.$$

Now let (x,y,z) be in $J_1(e)\times J_{\frac{1}{2}}(e)\times J_0(e)$, let λ be in $\mathbb{S}_{\mathbb{C}}$, and write $\lambda = e^{i\theta}$ for some $\theta\in\mathbb{R}$. Then, since $\exp(2i\theta L(e,e))$ is an isometry (cf. Corollary 2.1.9(iii)), it follows that

$$\begin{aligned}\|\lambda^2 x+\lambda y+z\| &= \|e^{2i\theta}x+e^{i\theta}y+z\|\\ &= \|\exp(2i\theta L(e,e))(x+y+z)\| = \|x+y+z\|,\end{aligned}$$

which proves (4.2.5). Keeping in mind (4.2.5) with $\lambda=-1$, we get

$$\begin{aligned}\|y\| &= \frac{1}{2}\|x+y+z-(x-y+z)\|\\ &\leqslant \frac{1}{2}(\|x+y+z\|+\|x-y+z\|)\\ &= \|x+y+z\| \end{aligned}\tag{4.2.7}$$

and

$$\begin{aligned}\|x+z\| &= \frac{1}{2}\|x+y+z+(x-y+z)\|\\ &\leqslant \frac{1}{2}(\|x+y+z\|+\|x-y+z\|)\\ &= \|x+y+z\|. \end{aligned}\tag{4.2.8}$$

On the other hand, taking $y=0$ and $\lambda=i$ in (4.2.5), we obtain

$$\|x\| = \frac{1}{2}\|x+z-(-x+z)\| \leqslant \frac{1}{2}(\|x+z\|+\|-x+z\|) = \|x+z\| \tag{4.2.9}$$

and

$$\|z\| = \frac{1}{2}\|x+z+(-x+z)\| \leqslant \frac{1}{2}(\|x+z\|+\|-x+z\|) = \|x+z\|. \tag{4.2.10}$$

Keeping in mind (4.2.7), and combining (4.2.8) with (4.2.9) and (4.2.10), the inequality (4.2.6) follows. □

Let J be a Jordan $*$-triple over $\mathbb{K}$. Note that, in terms of the operator $L(x,y)$, the Jordan triple identity (4.1.13) can be reformulated as

$$[L(u,v),L(x,y)] = L(\{uvx\},y)-L(x,\{vuy\}). \tag{4.2.11}$$

Apart from the operator $L(x,y)$, there are two important basic operators on J, namely the so-called quadratic operator and the Bergmann operator. The *quadratic operator* $Q_x : J\to J$, induced by $x\in J$, is defined by

$$Q_x(y) = \{xyx\} \text{ for every } y\in J.$$

It follows from the Jordan triple identity (4.2.11) that

$$0 = [L(x,u),L(x,u)] = L(Q_x(u),u)-L(x,Q_u(x)),$$

and hence we have

$$L(Q_x(u),u) = L(x,Q_u(x)), \tag{4.2.12}$$

that is to say

$$\{\{xux\}uz\} = \{x\{uxu\}z\}. \tag{4.2.13}$$

Note that linearizing (4.2.13) we get the identities

$$2\{\{xuy\}uz\} = \{x\{uyu\}z\} + \{y\{uxu\}z\} \tag{4.2.14}$$

and

$$2\{x\{uxv\}z\} = \{\{xux\}vz\} + \{\{xvx\}uz\}), \tag{4.2.15}$$

which can be reformulated as

$$2L(\{xuy\},u) = L(x,Q_u(y)) + L(y,Q_u(x)) \tag{4.2.16}$$

and

$$2L(x,\{uxv\}) = L(Q_x(u),v) + L(Q_x(v),u), \tag{4.2.17}$$

respectively. Given $x,y \in J$, let us define the mapping $Q_{x,z} : J \to J$ by

$$Q_{x,z}(y) = \{xyz\} \text{ for every } y \in J.$$

It is clear that $Q_x = Q_{x,x}$ for every $x \in J$, and that

$$Q_{x,z} = \tfrac{1}{2}(Q_{x+z} - Q_x - Q_z) \text{ for all } x,z \in J. \tag{4.2.18}$$

In terms of quadratic operators, the identity (4.1.14) has the following three readings

$$2L(x,y)Q_{x,u} = Q_{u,Q_x(y)} + Q_x L(y,u), \tag{4.2.19}$$

$$2L(x,y)L(x,v) = L(Q_x(y),v) + Q_x Q_{y,v}, \tag{4.2.20}$$

and

$$2Q_{\{uvx\},x} = L(u,v)Q_x + Q_x L(v,u). \tag{4.2.21}$$

Note that the identity (4.1.15) can be reformulated as

$$L(x,u)Q_x = Q_x L(u,x) = Q_{Q_x(u),x}. \tag{4.2.22}$$

Proposition 4.2.16 *Given u,x,y in a Jordan $*$-triple over $\mathbb{K}$, we have*

$$\{\{xyx\}u\{xyx\}\} = \{x\{y\{xux\}y\}x\}. \tag{4.2.23}$$

Proof Since the left-hand side of (4.2.11) changes sign if we interchange (x,y) and (u,v), so does the right-hand side. This implies

$$L(u,\{yxv\}) - L(\{xyu\},v) = L(\{uvx\},y) - L(x,\{vuy\}),$$

that is to say

$$\{u\{yxv\}z\} - \{\{xyu\}vz\} = \{\{uvx\}yz\} - \{x\{vuy\}z\}.$$

In particular, setting $v = y$ and $z = x$, we have

$$\{u\{yxy\}x\} = 2\{xy\{xyu\}\} - \{x\{yuy\}x\}. \tag{4.2.24}$$

On the other hand, the Jordan triple identity (4.1.13) gives

$$\{\{xyx\}u\{xyx\}\} = 2\{\{\{xyx\}ux\}yx\} - \{x\{u\{xyx\}y\}x\}.$$

Now, writing $\{u\{xyx\}y\}$ according to (4.2.24), we obtain

$$\{\{xyx\}u\{xyx\}\} = 2\{\{\{xyx\}ux\}yx\} - 2\{x\{yx\{yxu\}\}x\} + \{x\{y\{xux\}y\}x\}. \tag{4.2.25}$$

But, applying (4.2.22), we have

$$\begin{aligned}\{\{\{xyx\}ux\}yx\} &= L(x,y)L(x,u)Q_x(y)\\ &= Q_xL(y,x)L(u,x)(y) = \{x\{yx\{uxy\}\}x\}.\end{aligned} \tag{4.2.26}$$

Therefore the identity (4.2.23) follows by combining (4.2.25) and (4.2.26). □

Let J be a Jordan $*$-triple over $\mathbb{K}$. The identity (4.2.23), proved in Proposition 4.2.16, is called the *fundamental formula*, and can be written as

$$Q_{Q_x(y)} = Q_xQ_yQ_x. \tag{4.2.27}$$

By applying the fundamental formula twice, we get

$$Q_{Q_xQ_y(u)} = Q_xQ_yQ_uQ_yQ_x. \tag{4.2.28}$$

Note also that the identity (4.2.15) can be reformulated as

$$2Q_{x,z}L(u,x) = Q_{Q_x(u),z} + L(z,u)Q_x. \tag{4.2.29}$$

It is easy to verify that

$$Q_{u+tx} = Q_u + 2tQ_{u,x} + t^2Q_x$$

for all scalars t. Hence we have

$$\begin{aligned}Q_{Q_{u+tx}(v)} &= Q_{Q_u(v)+2tQ_{u,x}(v)+t^2Q_x(v)}\\ &= Q_{Q_u(v)} + 4tQ_{Q_u(v),Q_{u,x}(v)} + 2t^2(Q_{Q_u(v),Q_x(v)} + 2Q_{Q_{u,x}(v)})\\ &\quad + 4t^3Q_{Q_{u,x}(v),Q_x(v)} + t^4Q_{Q_x(v)}.\end{aligned}$$

On the other hand, from the fundamental formula (4.2.27),

$$Q_{Q_{u+tx}(v)} = Q_{u+tx}Q_vQ_{u+tx} = (Q_u + 2tQ_{u,x} + t^2Q_x)Q_v(Q_u + 2tQ_{u,x} + t^2Q_x).$$

Comparing the coefficients of t in both expressions, we obtain

$$2Q_{Q_u(v),\{uvx\}} = Q_uQ_vQ_{u,x} + Q_{u,x}Q_vQ_u. \tag{4.2.30}$$

Also, comparing the coefficients of t^2 in both expressions, we obtain

$$2Q_{Q_u(v),Q_x(v)} + 4Q_{\{uvx\}} = Q_xQ_vQ_u + Q_uQ_vQ_x + 4Q_{u,x}Q_vQ_{u,x}. \tag{4.2.31}$$

Lemma 4.2.17 *For u,w,x in a Jordan $*$-triple J over $\mathbb{K}$, we have*

$$2Q_{Q_uQ_w(x),\{uwx\}} = Q_uQ_wQ_xL(w,u) + L(u,w)Q_xQ_wQ_u.$$

Proof Linearizing (4.2.30) with respect to v, we get

$$Q_{Q_u(v),\{uwx\}} + Q_{Q_u(w),\{uvx\}} = Q_uQ_{v,w}Q_{u,x} + Q_{u,x}Q_{v,w}Q_u. \tag{4.2.32}$$

Setting $v = Q_w(x)$ in (4.2.32), we have

$$Q_{Q_uQ_w(x),\{uwx\}} = Q_uQ_{Q_w(x),w}Q_{u,x} + Q_{u,x}Q_{Q_w(x),w}Q_u - Q_{Q_u(w),\{uQ_w(x)x\}}. \tag{4.2.33}$$

Note that, by (4.2.22) and (4.2.19), we have

$$\begin{aligned} 2Q_uQ_{Q_w(x),w}Q_{u,x} &= 2Q_uQ_wL(x,w)Q_{u,x} \\ &= Q_uQ_wQ_{Q_x(w),u} + Q_uQ_wQ_xL(w,u), \end{aligned} \tag{4.2.34}$$

and similarly, by (4.2.22) and (4.2.29), we have

$$\begin{aligned} 2Q_{u,x}Q_{Q_w(x),w}Q_u &= 2Q_{u,x}L(w,x)Q_wQ_u \\ &= Q_{Q_x(w),u}Q_wQ_u + L(u,w)Q_xQ_wQ_u. \end{aligned} \tag{4.2.35}$$

On the other hand, invoking (4.2.12) and (4.2.30), we see that

$$\begin{aligned} 2Q_{Q_u(w),\{uQ_w(x)x\}} &= 2Q_{Q_u(w),\{uwQ_x(w)\}} \\ &= Q_uQ_wQ_{Q_x(w),u} + Q_{Q_x(w),u}Q_wQ_u. \end{aligned} \tag{4.2.36}$$

It follows from (4.2.34), (4.2.35), and (4.2.36) that

$$\begin{aligned} &2(Q_uQ_{Q_w(x),w}Q_{u,x} + Q_{u,x}Q_{Q_w(x),w}Q_u - Q_{Q_u(w),\{uQ_w(x)x\}}) \\ &\quad = Q_uQ_wQ_xL(w,u) + L(u,w)Q_xQ_wQ_u, \end{aligned}$$

so the statement follows from (4.2.33). □

Lemma 4.2.18 *For u,v,x in a Jordan $*$-triple J over $\mathbb{K}$, we have*

$$2Q_{x,Q_uQ_v(x)} + 4Q_{\{uvx\}} = Q_uQ_vQ_x + Q_xQ_vQ_u + 4L(u,v)Q_xL(v,u). \tag{4.2.37}$$

Proof If we compare (4.2.37) with (4.2.31) we see that we have to show

$$2L(u,v)Q_xL(v,u) = Q_{x,Q_uQ_v(x)} + 2Q_{x,u}Q_vQ_{x,u} - Q_{Q_u(v),Q_x(v)}. \tag{4.2.38}$$

By (4.2.29), we have

$$2L(u,v)Q_xL(v,u) = 4Q_{x,u}L(v,x)L(v,u) - 2Q_{Q_x(v),u}L(v,u),$$

and applying (4.2.20), we obtain

$$2L(u,v)Q_xL(v,u) = 2Q_{x,u}L(Q_v(x),u) + 2Q_{x,u}Q_vQ_{x,u} - 2Q_{Q_x(v),u}L(v,u).$$

Now, using (4.2.29) twice more, we have

$$\begin{aligned} 2L(u,v)Q_xL(v,u) = {}& Q_{x,Q_uQ_v(x)} + L(x,Q_v(x))Q_u + 2Q_{x,u}Q_vQ_{x,u} \\ & - Q_{Q_u(v),Q_x(v)} - L(Q_x(v),v)Q_u, \end{aligned}$$

and applying (4.2.12), we get the identity (4.2.38), as desired. □

Let J be a Jordan $*$-triple over $\mathbb{K}$. The *Bergmann operator* $B(x,y) : J \to J$, induced by $x,y \in J$, is defined by

$$B(x,y) = I_J - 2L(x,y) + Q_xQ_y. \tag{4.2.39}$$

Obviously, we have $B(tx,y) = B(x,ty)$ for every $t \in \mathbb{R}$.

Lemma 4.2.19 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let x,y,z be in J. The Bergmann operator satisfies*

$$Q_{B(x,y)(z)} = B(x,y)Q_zB(y,x). \tag{4.2.40}$$

Proof The identity (4.2.40) can be proved by comparing the expansions of both sides. Indeed, the left-hand side is equal to

$$Q_z - 4Q_{z,\{xyz\}} + 2Q_{z,Q_xQ_y(z)} + 4Q_{\{x,y,z\}} - 4Q_{Q_xQ_y(z),\{xyz\}} + Q_{Q_xQ_y(z)},$$

whereas the right-hand side equals

$$\begin{aligned} &Q_z - 2L(x,y)Q_z - 2Q_zL(y,x) + Q_xQ_yQ_z + Q_zQ_yQ_x + 4L(x,y)Q_zL(y,x) \\ &\quad - 2Q_xQ_yQ_zL(y,x) - 2L(x,y)Q_zQ_yQ_x + Q_xQ_yQ_zQ_yQ_x, \end{aligned}$$

which is identical to the left-hand side by the identity (4.2.21), Lemmas 4.2.18 and 4.2.17, and the identity (4.2.28). □

Lemma 4.2.20 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be a tripotent in J. Then the projections $P_k(e)$ from J onto $J_k(e)$ $(k = 1, \frac{1}{2}, 0)$ corresponding to the decomposition*

$$J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e) \quad \text{(cf. Fact 4.2.14(i))}$$

are given by

$$P_1(e) = Q_e^2, \;\; P_{\frac{1}{2}}(e) = 2(L(e,e) - Q_e^2), \;\; \text{and} \;\; P_0(e) = B(e,e).$$

Proof Keeping in mind Fact 4.2.14(ii), it follows from (4.2.14) that

$$\begin{aligned} P_1(e)(u) &= 2L(e,e)^2(u) - L(e,e)(u) = 2\{ee\{eeu\}\} - \{eeu\} \\ &= \{e\{eue\}e\} = Q_e^2(u) \end{aligned}$$

for every $u \in J$, and hence

$$P_{\frac{1}{2}}(e) = 4(L(e,e) - L(e,e)^2) = 2(L(e,e) - P_1(e)) = 2(L(e,e) - Q_e^2).$$

As a consequence,

$$\begin{aligned} P_0(e) &= I_J - P_{\frac{1}{2}}(e) - P_1(e) = I_J - 2(L(e,e) - Q_e^2) - Q_e^2 \\ &= I_J - 2L(e,e) + Q_e^2 = B(e,e). \end{aligned}$$ □

After simple computations, the next result follows from Lemma 4.2.20 above and the definition of the Bergmann operator.

Corollary 4.2.21 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be a tripotent in J. Then*

(i) *$B(e,(1-t)e) = t^2P_1(e) + tP_{\frac{1}{2}}(e) + P_0(e)$ for every $t \in \mathbb{R}$.*
(ii) *For each $t \in \mathbb{R} \setminus \{0\}$, $B(e,(1-t)e)$ is bijective with*

$$B(e,(1-t)e)^{-1} = B(e,(1-t^{-1})e).$$

(iii) *Given $k \in \{1, \frac{1}{2}, 0\}$, the elements of $J_k(e)$ are precisely those elements $x \in J$ such that $B(e,(1-t)e)(x) = t^{2k}x$ for every $t \in \mathbb{R} \setminus \{0\}$.*

Proposition 4.2.22 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be a tripotent in J. Then we have*

$$\begin{aligned}&\{J_i(e)J_j(e)J_k(e)\} \subseteq J_{i-j+k}(e) \ \text{ and}\\ &\{J_1(e)J_0(e)J\} = 0 = \{J_0(e)J_1(e)J\},\end{aligned} \tag{4.2.41}$$

where $J_\ell(e) := 0$ whenever $\ell \notin \{1, \frac{1}{2}, 0\}$.

Proof Let t be a nonzero real number. It follows from Lemma 4.2.19 and Corollary 4.2.21(ii) that

$$\begin{aligned}&B(e,(1-t)e)(\{xyx\})\\ &\quad = B(e,(1-t)e)Q_xB(e,(1-t)e)B(e,(1-t^{-1})e)(y)\\ &\quad = Q_{B(e,(1-t)e)(x)}B(e,(1-t^{-1})e)(y)\\ &\quad = \{B(e,(1-t)e)(x), B(e,(1-t^{-1})e)(y), B(e,(1-t)e)(x)\}\end{aligned}$$

for all $x,y \in J$. Linearizing in x, we have

$$B(e,(1-t)e)(\{xyz\}) = \{B(e,(1-t)e)(x), B(e,(1-t^{-1})e)(y), B(e,(1-t)e)(z)\}$$

for all $x,y,z \in J$. Keeping in mind Corollary 4.2.21(iii), for $v_\alpha \in J_\alpha(e)$, the last equality implies

$$B(e,(1-t)e)(\{v_iv_jv_k\}) = \{(t^{2i}v_i)(t^{-2j}v_j)(t^{2k}v_k)\} = t^{2(i-j+k)}\{v_iv_jv_k\},$$

and consequently $\{v_iv_jv_k\} \in J_{i-j+k}(e)$ whenever $i-j+k \in \{1, \frac{1}{2}, 0\}$, and $\{v_iv_jv_k\} = 0$ otherwise.

To show the second part of (4.2.41), let us fix $x \in J_1(e)$ and $z \in J_0(e)$. We first observe that $Q_e(z) = 0$ because of the first part of (4.2.41). Hence, by (4.2.16), we have

$$L(z,e) = L(z,Q_e(e)) = 2L(\{eez\},e) - L(e,Q_e(z)) = 0.$$

Therefore (4.2.17) implies

$$L(x,z) = L(Q_e^2(x),z) = 2L(e,\{Q_e(x)ez\}) - L(Q_e(z),Q_e(x)) = 0.$$

Likewise, another application of (4.2.16) gives

$$L(z,x) = L(z,Q_e^2(x)) = 2L(\{zeQ_e(x)\},e) - L(Q_e(x),Q_e(z)) = 0. \qquad \square$$

We recall that the Kernel $K(X,u)$ of a numerical-range space (X,u) over $\mathbb{K}$ was defined as the subspace of X consisting of those elements $x \in X$ such that $V(X,u,x) = \{0\}$ (cf. §3.3.20).

Lemma 4.2.23 *Let J be a Banach Jordan $*$-triple over $\mathbb{K}$ such that the equaliy $\|\{aaa\}\| = \|a\|^3$ holds for every $a \in J$, and let e be a nonzero tripotent in J. Then*

$$J_{\frac{1}{2}}(e) \oplus J_0(e) \subseteq K(J,e). \tag{4.2.42}$$

Proof Let u be in $\mathbb{S}_J$. Then, given $a \in J$ and $r \in \mathbb{R}$, we can write

$$\{u+ra, u+ra, u+ra\} = \{uuu\} + r(2\{uua\} + \{uau\}) + r^2b + r^3c,$$

where b and c are elements of J which do not depend on r, and hence, invoking Corollary 2.1.6, we have

$$\begin{aligned} 3\max \Re(V(J,u,a)) &= \lim_{r\to 0^+} (\|u+ra\|^2 + \|u+ra\| + 1)\frac{\|u+ra\|-1}{r} \\ &= \lim_{r\to 0^+} \frac{\|u+ra\|^3-1}{r} \\ &= \lim_{r\to 0^+} \frac{\|\{u+ra,u+ra,u+ra\}\|-1}{r} \\ &= \lim_{r\to 0^+} \frac{\|\{uuu\}+r(2\{uua\}+\{uau\})+r^2b+r^3c\|-1}{r} \\ &= \max \Re(V(J,\{uuu\},2\{uua\}+\{uau\})). \end{aligned}$$

Now take u equal to the nonzero tripotent e in the statement. If a lies in $J_k(e)$ for $k \in \{0, \frac{1}{2}\}$, then $\{eea\} = ka$ and, by Proposition 4.2.22, $\{eae\} = 0$, so that we have

$$3\max \Re(V(J,e,a)) = \max \Re(V(J,e,2ka)) = 2k\max \Re(V(J,e,a)),$$

which implies $V(J,e,a) = \{0\}$. Thus the inclusion (4.2.42) has been proved. □

Theorem 4.2.24 *Let J be a JB^*-triple, and let u be a norm-one element of J. Then the following conditions are equivalent:*

(i) *u is a unitary element of J* (cf. Definition 4.1.53).
(ii) *J is the JB^*-triple underlying a JB^*-algebra with unit u* (cf. Theorem 4.1.45).
(iii) *The Banach space of J becomes a non-commutative JB^*-algebra with unit u, for some product and involution.*
(iv) *The Banach space of J becomes a norm-unital normed algebra with unit u, for some product.*
(v) *$n(J,u) = 1$ or $\frac{1}{2}$ depending on whether or not J is abelian.*
(vi) *u is a geometrically unitary element of J* (cf. Definition 2.1.16).
(vii) *u is a vertex of the closed unit ball of J* (cf. Definition 2.1.12).

Proof The implications (ii)⇒(iii), (iii)⇒(iv), and (vi)⇒(vii) are clear.

(i)⇒(ii) By assumption (i) and Theorem 4.1.55, the Banach space of J, endowed with the product $ab := \{aub\}$ and the involution $x^* := \{uxu\}$, becomes a JB^*-algebra (say A) with unit u. Now, if we think about the JB^*-triple underlying A, then, by Proposition 4.1.54, we re-encounter J.

(ii)⇒(v) By Theorems 3.4.59 and 4.1.55.

(iv)⇒(vi) By Corollary 2.1.19.

(v)⇒(vi) By Theorem 2.1.17(i).

(vii)⇒(i) By Corollary 4.2.12, we have $V(J,u,u-\{uuu\}) = \{0\}$. Therefore, by assumption (vii), u becomes a nonzero tripotent. Then, invoking assumption (vii) again and Lemma 4.2.23, we see that $J_{\frac{1}{2}}(u) = 0 = J_0(u)$, so $J = J_1(u)$, and so u is a unitary element of J. □

Definition 4.2.25 Let A be a unital Jordan-admissible $*$-algebra over $\mathbb{K}$, and let a be in A. We say that a is *J-unitary* if a is J-invertible in A with $a^{-1} = a^*$.

Fact 4.2.26 *Let A be a unital Jordan-admissible $*$-algebra over $\mathbb{K}$, and let a be in A. Then the following conditions are equivalent:*

(i) *a is J-unitary in A.*
(ii) *a^* is J-unitary in A.*
(iii) *a is J-invertible in A and $U^\bullet_a(a^*) = a$, where $U^\bullet$ denotes the operator U on A^{sym}.*

Moreover, when A is in fact a non-commutative Jordan algebra, we have:

(iv) *a is J-unitary in A if and only if a is J-invertible in A and $U_a(a^*) = a$.*
(v) *If a is J-unitary, then a is normal* (cf. Definition 3.4.20).

Proof The first conclusion follows from Theorem 4.1.3, whereas assertion (iv) follows from Fact 3.3.3 and the equivalence (i)⇔(iii) above. Finally, assertion (v) follows from the definition of a J-unitary element and the bracket-free version of Proposition 4.1.86. □

Definition 4.2.27 Now let A be a norm-unital normed Jordan-admissible algebra over $\mathbb{K}$, and let a be in A. We say that a is *algebraically J-unitary* if a is J-invertible in A and $\|a\| = \|a^{-1}\| = 1$.

According to Theorem 4.2.24, the existence of a geometrically unitary element in a JB^*-triple J is equivalent to the fact that J is isometrically triple-isomorphic to a unital JB^*-algebra. Therefore the study of geometrically unitary elements in JB^*-triples is concluded with the following.

Theorem 4.2.28 *Let A be a unital non-commutative JB^*-algebra, and let u be in A. Then the following conditions are equivalent:*

(i) *u is J-unitary.*
(ii) *u is algebraically J-unitary.*
(iii) *The Banach space of A becomes a norm-unital normed algebra with unit u, for some product.*
(iv) *u is geometrically unitary.*
(v) *u is a vertex of the closed unit ball of A.*
(vi) *u is a unitary element of the JB^*-triple underlying A* (cf. Theorem 4.1.45).
(vii) *U_u is a surjective isometry.*

Proof Since all conditions mean the same in A and in A^{sym}, we may assume that A is commutative.

(i)⇒(ii) By Fact 4.2.26, assumption (i) can be read as that u is J-invertiblein A and $U_u(u^*) = u$. Therefore we have $\|u\|^3 = \|U_u(u^*)\| = \|u\|$, hence$\|u\| = 1$. Then, by assumption (i) again and Proposition 3.3.13, $\|u^{-1}\| = \|u^*\| = \|u\| = 1$.

(ii)⇒(iii) Assume that u is algebraically J-unitary. By Theorem 4.1.3(v), we have $[L_u, L_{u^{-1}}] = 0$, and hence

$$U_{u,a}(u^{-1}) = u(au^{-1}) + a(uu^{-1}) - u^{-1}(ua) = [L_u, L_{u^{-1}}](a) + a = a.$$

Therefore A becomes a unital complex algebra with unit u under the product $a \odot b := U_{a,b}(u^{-1})$. But, by Proposition 3.4.17, we have $\|a \odot b\| \leqslant \|a\|\|b\|$ for all $a, b \in A$, and hence $(A, \odot, \|\cdot\|)$ becomes a norm-unital normed complex algebra with unit u.

(iii)⇒(iv) By Corollary 2.1.19.

(iv)⇒(v) This is clear.

(v)⇒(vi) By the implication (vii)⇒(i) in Theorem 4.2.24.

(vi)⇒(i) By assumption (vi) and the identity (4.2.14) with $x = z = u$, we have that $Q_u^2 = I_A$, which in terms of the product and the involution of A reads as $U_u U_{u^*} = I_A$. Therefore, by Theorem 4.1.3, u is J-invertible in A with $u^{-1} = u^*$.

(ii)⇒(vii) Assume that u is algebraically J-unitary. By Theorem 4.1.3, we have $U_u U_{u^{-1}} = I_A$. But, by Proposition 3.4.17, $\|U_u\| \leqslant \|u\|^2 = 1$ and $\|U_{u^{-1}}\| \leqslant \|u^{-1}\|^2 = 1$. It follows that U_u is a surjective isometry.

(vii)⇒(ii) Assume that U_u is a surjective isometry. Then, by Theorem 4.1.3, u is J-invertible in A and $U_u^{-1} = U_{u^{-1}}$. Therefore, by Proposition 3.3.13,for v equal to u or u^{-1} we have $\|v\|^3 = \|U_v(v^*)\| = \|v^*\| = \|v\|$, and hence $\|v\| = 1$. □

The above theorem generalizes Proposition 3.4.31, which in its turn generalizes Theorem 2.1.27.

Given Jordan $*$-triples J and K over $\mathbb{K}$, the vector space $J \times K$ will be seen without notice as a new Jordan $*$-triple over $\mathbb{K}$ under the triple product defined coordinate-wise.

Fact 4.2.29 *Let J and K be JB*-triples. Then the direct product $J \times K$ becomes a JB*-triple under the sup norm.*

Proof It only merits to be proved that

$$V[BL(J \times K), I_{J\times K}, L((x,y),(x,y))] \subseteq \mathbb{R}_0^+ \text{ for every } (x,y) \in J \times K.$$

To realize this, endow the vector space $BL(J) \times BL(K)$ with the sup norm, define a mapping $\varphi : BL(J) \times BL(K) \to BL(J \times K)$ by

$$\varphi(F,G)(x,y) := (F(x), G(y))$$

for all $(F,G) \in BL(J) \times BL(K)$ and $(x,y) \in J \times K$, and note that φ becomes a linear isometry satisfying $\varphi(I_J, I_K) = I_{J\times K}$ and

$$\varphi(L(x,x), L(y,y)) = L((x,y),(x,y)) \text{ for every } (x,y) \in J \times K.$$

Then, by Corollary 2.1.2(ii) and Fact 2.9.47, for every $(x,y) \in J \times K$ we have

$$\begin{aligned} &V[BL(J \times K), I_{J\times K}, L((x,y),(x,y))] \\ &\quad = V[BL(J) \times BL(K), (I_J, I_K), (L(x,x), L(y,y))] \\ &\quad = \text{co}[V(BL(J), I_J, L(x,x)) \cup V(BL(K), I_K, L(y,y))] \subseteq \mathbb{R}_0^+. \end{aligned}$$ □

Let J be a Jordan $*$-triple over $\mathbb{K}$. Two subtriples M and N of J are said to be *orthogonal* if

$$\{MNN\} = \{NMM\} = \{MNM\} = \{NMN\} = 0.$$

If M and N are orthogonal subtriples of J, then, clearly, $M + N$ is a subtriple of J, and the mapping $(x,y) \to x + y$ from $M \times N$ to J becomes a triple homomorphism.

Corollary 4.2.30 *Let J be a Jordan $*$-triple over $\mathbb{K}$, and let e be a tripotent in J. We have:*

(i) *$J_{\frac{1}{2}}(e)$ is a subtriple of J, and $J_1(e)$ and $J_0(e)$ are orthogonal subtriples of J.*
(ii) *If J is in fact a Banach Jordan $*$-triple, then the direct sum*

$$J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e)$$

is topological, hence $J_k(e)$ $(k \in \{0, \frac{1}{2}, 1\})$ and $J_1(e) \oplus J_0(e)$ are closed subtriples of J.
(iii) *If J is actually a JB^*-triple, then:*

(a) $\|x+z\| = \max\{\|x\|, \|z\|\}$ *for every* $(x,z) \in J_1(e) \times J_0(e)$.
(b) *The normed space of $J_1(e)$, endowed with the product $ab := \{aeb\}$ and the involution $x^* := \{exe\}$, becomes a JB^*-algebra with unit e.*

Proof Assertion (i) follows straightforwardly from Proposition 4.2.22.

Assume that J is a Banach Jordan $*$-triple. Then, by the definition of the subspaces $J_k(e)$ $(k \in \{0, \frac{1}{2}, 1\})$, they are closed in J, so the direct sum $J = J_1(e) \oplus J_{\frac{1}{2}}(e) \oplus J_0(e)$ is topological, and so $J_1(e) \oplus J_0(e)$ is closed in J. That $J_k(e)$ $(k \in \{0, \frac{1}{2}, 1\})$ and $J_1(e) \oplus J_0(e)$ are subtriples of J follows from assertion (i).

Finally, assume that J is a JB^*-triple. Then, by assertion (ii) and Fact 4.1.40, $J_k(e)$ $(k \in \{0, \frac{1}{2}, 1\})$ and $J_1(e) \oplus J_0(e)$ become JB^*-triples. Moreover, by assertion (i), the mapping $(x,z) \to x+z$ from $J_1(e) \times J_0(e)$ to $J_1(e) \oplus J_0(e)$ is a bijective triple homomorphism. It follows from Fact 4.2.29 and Proposition 4.1.52 that $\|x+z\| = \max\{\|x\|, \|z\|\}$ for every $(x,z) \in J_1(e) \times J_0(e)$. This proves part (a) of assertion (iii). Since e clearly becomes a unitary element of the JB^*-triple $J_1(e)$ (cf. Definition 4.1.53), part (b) follows from Theorem 4.1.55. □

The next corollary will not be applied later, but has its own interest.

Corollary 4.2.31 *Let J be a JB^*-triple, and let e be a nonzero tripotent in J. Then*

$$J_{\frac{1}{2}}(e) \oplus J_0(e) = K(J,e).$$

Proof Since the inclusion $J_{\frac{1}{2}}(e) \oplus J_0(e) \subseteq K(J,e)$ was already proved in Lemma 4.2.23 under weaker assumptions, now it is enough to show that $K(J,e) \cap J_1(e) = 0$. But, since

$$K(J,e) \cap J_1(e) = K(J_1(e), e)$$

(cf. Corollary 2.1.2), this follows from Corollaries 4.2.30(iii)(b) and 2.1.13. □

Proposition 4.2.32 *Let J be a JB^*-triple, and let e be a nonzero tripotent in J. Then $\{xye\}$ lies in the JB^*-algebra $J_1(e)$* (cf. Corollary 4.2.30(iii)(b)) *whenever x and y belong to $J_{\frac{1}{2}}(e)$, and the sesquilinear mapping*

$$(x,y) \to [x|y] := \{xye\} \text{ from } J_{\frac{1}{2}}(e) \times J_{\frac{1}{2}}(e) \text{ to } J_1(e)$$

satisfies the following properties:

(i) $[x|y]^* = [y|x]$ *for all* $x, y \in J_{\frac{1}{2}}(e)$.
(ii) $[x|x] \geqslant 0$ *for every* $x \in J_{\frac{1}{2}}(e)$.
(iii) *If* x *is in* $J_{\frac{1}{2}}(e)$, *and if* $[x|x] = 0$, *then* $x = 0$.

Proof That $[x|y] := \{xye\}$ lies in $J_1(e)$ whenever x and y belong to $J_{\frac{1}{2}}(e)$ follows from Proposition 4.2.22. Since the mapping $T \to T(e)$ from $BL(J)$ to J is a linear contraction taking I_J to e, it follows from Corollary 2.1.2 that, for every $x \in J_{\frac{1}{2}}(e)$, we have

$$V(J_1(e), e, [x|x]) = V(J, e, [x|x]) = V(J, e, L(x,x)(e))$$
$$\subseteq V(BL(J), I_J, L(x,x)) \subseteq \mathbb{R}_0^+,$$

hence $[x|x]$ lies in $H(J_1(e), *)$ (cf. Lemma 2.2.5), and then actually $[x|x] \geqslant 0$ (cf. §3.4.68). Thus property (ii) has been proved. Property (i) follows from the equality $[x|x]^* = [x|x]$ already proved, by polarization law. The proof of property (iii) is more ingenious than involved. Let x be in $J_{\frac{1}{2}}(e)$, and suppose that the equality $\{xxe\} = 0$ holds. Take $\psi \in D(J_1(e), e)$. Then, by the properties already proved, the mapping $(a,b) \to \psi(\{abe\})$ becomes a non-negative hermitian sesquilinear form on $J_{\frac{1}{2}}(e)$. Therefore, by the assumption on x and the Cauchy–Schwarz inequality, we have $\psi(\{x^{(3)}xe\}) = 0$. By the arbitrariness of $\psi \in D(J_1(e), e)$ and Corollary 2.1.13, we deduce that $\{x^{(3)}xe\} = 0$, hence

$$0 = \{x^{(3)}xe\} = 2\{\{xxe\}xx\} - \{x\{xex\}x\} = -Q_x^2(e)$$

because of the Jordan triple identity (4.1.13) and the assumption on x again. Now, the fundamental formula (4.2.27) yields

$$\{\{xex\}\{xex\}\{xex\}\} = Q_{Q_x(e)}Q_x(e) = Q_xQ_eQ_x^2(e) = 0,$$

so $\{xex\} = 0$, and so

$$0 = \{ex\{xex\}\} = 2\{\{exx\}ex\} - \{x\{xee\}x\} = -\frac{1}{2}\{xxx\}.$$

Therefore $x = 0$. □

Lemma 4.2.33 *Let J be a JB*-triple, let e be a nonzero tripotent in J, and let y be in $J_{\frac{1}{2}}(e)$ such that $\|e+y+z\| \leqslant 1$ for some z in $J_0(e)$. Then $y = 0$.*

Proof By the equality (4.2.5) in Proposition 4.2.15 we have

$$\|-e+iy+z\| = \|e+y+z\| \leqslant 1,$$

so that, if we set

$$u := \frac{1}{2}[(e+y+z) - (-e+iy+z)],$$

then $\|u\| \leqslant 1$, and hence $\|\{uuu\}\| = \|u\|^3 \leqslant 1$. Noticing that $u = e + \alpha y$, where $\alpha = \frac{1}{2}(1-i)$, and keeping in mind Proposition 4.2.22, we see that $\{uuu\} = e + 2|\alpha|^2\{yye\} + 2\alpha\{yee\} + |\alpha|^2\alpha\{yyy\} + \alpha^2\{yey\}$ and that

$$e + 2|\alpha|^2\{yye\} \in J_1(e), \quad 2\alpha\{yee\} + |\alpha|^2\alpha\{yyy\} \in J_{\frac{1}{2}}(e), \quad \alpha^2\{yey\} \in J_0(e).$$

Therefore we derive from the inequality (4.2.6) in Proposition 4.2.15 that $\|e+2|\alpha|^2\{yye\}\| \leqslant \|\{uuu\}\|$, and hence $\|e+2|\alpha|^2\{yye\}\| \leqslant 1$. On the other hand, by Proposition 4.2.32(ii), we have

$$\|e+2|\alpha|^2\{yye\}\| = 1+2|\alpha|^2\|\{yye\}\|.$$

It follows that $\{yye\}=0$, and then, by Proposition 4.2.32(iii), $y=0$. □

A tripotent e in a Jordan $*$-triple J is said to be *complete* if $J_0(e)=0$.

Theorem 4.2.34 *Let J be a nonzero JB*-triple, and let u be in J. Then the following conditions are equivalent:*

(i) *u is a complete tripotent in J.*
(ii) *u is an extreme point of $\mathbb{B}_J$.*
(iii) *u is a complex extreme point of $\mathbb{B}_J$* (cf. Definition 3.1.6).

Proof (i)⇒(ii) Assume that u is a complete tripotent of J. Let v be in J such that $\|u\pm v\| \leqslant 1$, and write $v=x+y$ with $x\in J_1(u)$ and $y\in J_{\frac{1}{2}}(u)$. Since $u\pm x\in J_1(u)$, it follows from the inequality (4.2.6) in Proposition 4.2.15 that $\|u\pm x\| \leqslant 1$. Now, keeping in mind that $J_1(u)$ is a JB^*-algebra with unit u (cf. Corollary 4.2.30(iii)(b)), it follows from Corollary 2.1.42 that $x=0$. Therefore $v=y$, hence $\|u\pm y\| \leqslant 1$, and so, by Lemma 4.2.33, $y=0$. Thus $v=0$, and hence u is an extreme point of $\mathbb{B}_J$.

(ii)⇒(iii) This is clear.

(iii)⇒(i) Assume that u is a complex extreme point of $\mathbb{B}_J$. By Theorem 4.2.9, the closed subtriple of J generated by u is equal to the JB^*-triple underlying $C_0^{\mathbb{C}}(E)$ for some locally compact Hausdorff topological space E. Therefore, by Lemma 3.1.7, u becomes a tripotent in $C_0^{\mathbb{C}}(E)$, and hence in J. Now, since $u\in J_1(u)$, it follows from Corollary 4.2.30(iii)(a) that $\|u\pm\lambda z\|=1$ for every z in the closed unit ball of $J_0(u)$, and every $\lambda\in\mathbb{B}_{\mathbb{C}}$. Since u is a complex extreme point of $\mathbb{B}_J$, we derive that $J_0(u)=0$, i.e. u is a complete tripotent in J. □

Fact 4.2.35 *An element u in a JB*-triple J is a complete tripotent if and only if $B(u,u)=0$.*

Proof Let J be a JB^*-triple, and let u be in J. If u is a complete tripotent in J, then, by definition, $J_0(u)=0$, and hence $B(u,u)=0$ because of the equality $P_0(u)=B(u,u)$ in Lemma 4.2.20. Conversely, assume that $B(u,u)=0$. By Theorem 4.2.9, the closed subtriple of J generated by u can be regarded as the JB^*-triple underlying the C^*-algebra $C_0^{\mathbb{C}}(\sigma(u))$ in such a way that $u(t)=t$ for every $t\in\sigma(u)$, so that, since $B(u,u)(u)=0$, we have that $t(1-t^2)^2=0$ for every $t\in\sigma(u)$, and hence $\sigma(u)=\{1\}$. Therefore u is a tripotent in $C_0^{\mathbb{C}}(\sigma(u))$, and hence in J. Since $B(u,u)=0$, the equality $P_0(u)=B(u,u)$ in Lemma 4.2.20 applies again to get that u is a complete tripotent. □

Theorem 4.2.36 *Let A be a nonzero non-commutative JB*-algebra. Then the extreme points of $\mathbb{B}_A$ are precisely those elements $u\in A$ satisfying*

$$a-2U_{u,a}(u^*)+U_uU_{u^*}(a)=0 \tag{4.2.43}$$

for every $a \in A$. Moreover the following conditions are equivalent:

(i) *A is unital.*
(ii) *A has geometrically unitary elements.*
(iii) *$\mathbb{B}_A$ has vertices.*
(iv) *$\mathbb{B}_A$ has extreme points.*

Proof The first conclusion follows from Theorems 4.1.45 and 4.2.34, and Fact 4.2.35, by expressing the Bergmann operator $B(u,u)$ in terms of the product and the involution of A. The implication (ii)$\Rightarrow$(iii) is clear, whereas the ones (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv) follow from Corollary 2.1.19 and Lemma 2.1.25, respectively. Therefore to conclude the proof it is enough to show that (iv)$\Rightarrow$(i). Assume that $\mathbb{B}_A$ has an extreme point (say u), so that, by the first conclusion, the equality (4.2.43) holds for every $a \in A$. Since A'' is a unital non-commutative JB^*-algebra containing A as a $*$-subalgebra, and the product of A'' is separately w^*-continuous (cf. Theorem 3.5.34, Corollary 2.3.47, and Lemma 2.3.51), and A is w^*-dense in A'', we deduce that the equality (4.2.43) also holds for every $a \in A''$. Then, by taking in (4.2.43) a equal to the unit $\mathbf{1}$ of A'', we obtain that

$$\mathbf{1} - 2u \bullet u^* + U_u((u^*)^2) = 0,$$

and hence $\mathbf{1} = 2u \bullet u^* - U_u((u^*)^2) \in A$. Thus A is unital. □

Lemma 2.1.26, which was assumed without proof at the time, now becomes almost obvious. Indeed, we have the following.

§**4.2.37** *Proof of Lemma 2.1.26* Let A be a nonzero C^*-algebra. Then, by Fact 3.3.2, A is a non-commutative JB^*-algebra. Therefore, by the equivalence (i)$\Leftrightarrow$(iv) in Theorem 4.2.36, $\mathbb{B}_A$ has extreme points if and only if A is unital. Moreover, keeping in mind that, for a,b,c in any associative algebra, the equality $2U_{a,b}(c) = acb + bca$ holds, it follows from the first conclusion in Theorem 4.2.36 that, if A is unital, then the extreme points of $\mathbb{B}_A$ are precisely those elements $u \in A$ such that $(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0$. □

Remark 4.2.38 According to Theorem 4.2.36, a non-commutative JB^*-algebra A has geometrically unitary elements if and only if $\mathbb{B}_A$ has vertices, if and only if $\mathbb{B}_A$ has extreme points. A similar situation need not hold for general JB^*-triples. Indeed, every complex Hilbert space H becomes a JB^*-triple under the triple product $\{xyz\} := \frac{(x|y)z+(z|y)x}{2}$. However, every element in $\mathbb{S}_H$ is an extreme point of $\mathbb{B}_H$ whereas, if $\dim(H) \geqslant 2$, $\mathbb{B}_H$ has no vertex.

4.2.3 Russo–Dye type theorems for non-commutative JB^*-algebras

Let A be a unital non-commutative JB^*-algebra. Then clearly $\exp(ih)$ is a J-unitary element of A whenever h is in $H(A,*)$. On the other hand, as a by-product of Theorem 4.2.28, every J-unitary element of A lies in $\mathbb{S}_A$. Therefore, invoking Corollary 3.4.7, we get the following generalization of the classical Russo–Dye theorem (cf. Lemma 2.3.28).

Fact 4.2.39 *Let A be a unital non-commutative JB^*-algebra, and let U stand for the set of all J-unitary elements of A. Then $\mathbb{B}_A = \overline{\text{co}}(U)$.*

Let J be a nonzero JB^*-triple having a unitary element. Then, according to the implication (i)⇒(ii) in Theorem 4.2.24, J is the JB^*-triple underlying some unital JB^*-algebra A. But, in view of the equivalence (i)⇔(vi) in Theorem 4.2.28, the unitaries of J are precisely the J-unitaries of A. Therefore it is enough to invoke Fact 4.2.39 above to get the following.

Corollary 4.2.40 *If a JB^*-triple J has a unitary element, then $\mathbb{B}_J$ equals the closed convex hull of the set of all unitaries in J.*

Certainly the Russo–Dye–Palmer-type theorem contained in Corollary 3.4.7 is better than its consequence established in Fact 4.2.39. Now we are going to refine Fact 4.2.39 in a different direction.

Lemma 4.2.41 *Let A be a unital JB^*-algebra, and let u be a J-unitary element of A. Then we have:*

(i) *The Banach space of A becomes a unital JB^*-algebra with unit u for the product $\odot$ defined by $x \odot y := U_{x,y}(u^*)$ and the involution $\sharp$ defined by $x^\sharp := U_u(x^*)$.* (This JB^*-algebra is called the *u-isotope* of A, and is denoted by $A^{(u)}$.)

(ii) *The J-unitary elements of A and of $A^{(u)}$ are the same.*

Proof By Theorem 4.1.45, A has an underlying JB^*-triple (say J) which consists of the Banach space of A and the triple product $\{xyz\} := U_{x,z}(y^*)$, and, by the implication (i)⇒(vi) in Theorem 4.2.28, u becomes a unitary element of J. Therefore, assertion (i) follows from the third sentence in Theorem 4.1.55. Moreover, by the last sentence in Theorem 4.1.55, the JB^*-triple underlying $A^{(u)}$ coincides with J. Now, since the JB^*-algebras A and $A^{(u)}$ have the same underlying JB^*-triple, assertion (ii) follows from the equivalence (i)⇔(vi) in Theorem 4.2.28. □

Lemma 4.2.42 *Let A be a unital JB^*-algebra, let u be a J-unitary element of A, and let x be in A with $\|x\| < 1$. Then there are J-unitary elements u_1 and v_1 of A such that $x + u = u_1 + v_1$.*

Proof By Lemma 4.2.41, we may assume that $u = \mathbf{1}$. Let B stand for the closed $*$-subalgebra of A generated by x and $\mathbf{1}$, and recall that B can be seen as a $*$-subalgebra of C^{sym} for some unital C^*-algebra C (cf. Proposition 3.4.6). Since $\|x\| < 1$, both $\mathbf{1}+x$ and $\mathbf{1}+x^*$ are invertible elements of C (cf. Lemma 1.1.20). Set $y := \mathbf{1}+x$ and $v := y(y^*y)^{-\frac{1}{2}} \in C$. Then v becomes a unitary element of C. Since $\|y\| \leqslant 2$, it follows from §1.2.37 that $\sqrt{y^*y} = w + w^*$, where $w := \frac{\sqrt{y^*y}}{2} + i\sqrt{\mathbf{1} - \frac{y^*y}{4}}$ is a unitary element of C. Therefore, if we set $u_1 := vw$ and $v_1 := vw^*$, then u_1 and v_1 are unitary elements of C such that $x + \mathbf{1} = u_1 + v_1$. But, since

$$u_1 = y(y^*y)^{-\frac{1}{2}} \left(\frac{\sqrt{y^*y}}{2} + i\sqrt{\mathbf{1} - \frac{y^*y}{4}} \right)$$

and

$$v_1 = y(y^*y)^{-\frac{1}{2}}\left(\frac{\sqrt{y^*y}}{2} - i\sqrt{\mathbf{1} - \frac{y^*y}{4}}\right),$$

and y belongs to B, and B is a closed $*$-subalgebra of C^{sym}, it follows from Lemma 3.3.6(i) that u_1 and v_1 lie in B. Therefore u_1 and v_1 are J-unitary elements of A satisfying $x + \mathbf{1} = u_1 + v_1$. □

Theorem 4.2.43 *Let A be a unital non-commutative JB^*-algebra, let n be in $\mathbb{N}$ with $n > 2$, and let a be in A such that $\|a\| < 1 - \frac{2}{n}$. Then there are J-unitary elements $u_1, u_2, \ldots, u_n \in A$ satisfying*

$$a = \frac{1}{n}(u_1 + u_2 + \cdots + u_n).$$

Proof Since A^{sym} is a unital JB^*-algebra, and the J-unitary elements of A and of A^{sym} are the same, we may assume that A is commutative. Set $x := (n-1)^{-1}(na - \mathbf{1})$, so that $\|x\| < 1$ and $na = (n-1)x + \mathbf{1}$. Applying Lemma 4.2.42 $n-1$ times, we obtain J-unitary elements $v_1, v_2, \ldots, v_{n-1}$ and $u_1, u_2, \ldots, u_n$ (where $u_n = v_{n-1}$) such that

$$\begin{aligned}
na &= (n-1)x + \mathbf{1} = (n-2)x + (x + \mathbf{1}) = (n-2)x + (v_1 + u_1) \\
&= (n-3)x + (x + v_1) + u_1 = (n-3)x + (v_2 + u_2) + u_1 \\
&= (n-4)x + (x + v_2) + u_2 + u_1 = (n-4)x + (v_3 + u_3) + u_2 + u_1 = \cdots \\
&= (x + v_{n-2}) + u_{n-2} + \cdots + u_1 = u_n + u_{n-1} + u_{n-2} + \cdots + u_1.
\end{aligned}$$ □

As a consequence of Theorem 4.2.43, the open unit ball of a unital non-commutative JB^*-algebra is contained in the convex hull of the set of all J-unitary elements. Therefore, as announced above, Theorem 4.2.43 refines Fact 4.2.39.

According to Theorem 4.1.45 and Proposition 4.1.52, linear bijections between non-commutative JB^*-algebras are isometries as soon as they preserve the triple products $\{xyz\} := U_{x,z}(y^*)$. Now, with the help of Lemma 4.2.41, we can prove the converse.

Proposition 4.2.44 *Surjective linear isometries between non-commutative JB^*-algebras preserve triple products.*

Proof Let A and B be non-commutative JB^*-algebras, and let $F : A \to B$ be a surjective linear isometry. To prove that F preserves triple products, we may assume that A and B are commutative. Moreover, thinking about the double transpose of F, and applying Proposition 3.5.26, we may also assume that A and B are unital. Then, by the equivalence (i)⇔(iv) (or (i)⇔(v)) in Theorem 4.2.28, $u := F(\mathbf{1})$ must be a J-unitary element of B. Now, by Lemma 4.2.41(i), F can be seen as a unit-preserving surjective linear isometry from A to the JB^*-algebra $B^{(u)} = (B, \odot, \sharp)$, and hence, by the implication (ii)⇒(i) in Proposition 3.4.25, for all $x, y \in A$ we have $F(xy) = F(x) \odot F(y)$ and $F(x^*) = F(x)^\sharp$. Therefore F, regarded as a mapping from the JB^*-triple underlying A to the JB^*-triple underlying $B^{(u)}$, becomes a triple homomorphism. But, as we saw in the proof of Lemma 4.2.41(ii), B and $B^{(u)}$ have the same underlying JB^*-triple. □

Somehow, Proposition 4.2.44 mitigates Antitheorem 3.4.34.

4.2.4 A touch of real JB^*-triples and of real non-commutative JB^*-algebras

Definition 4.2.45 By a *real non-commutative JB^*-algebra* (respectively, a *real JB^*-algebra*, a *real alternative C^*-algebra*, or a *real C^*-algebra*) we mean a closed real $*$-subalgebra of a (complex) non-commutative JB^*-algebra (respectively, JB^*-algebra, alternative C^*-algebra, or C^*-algebra).

Example 4.2.46 (a) Every non-commutative JB^*-algebra (respectively, JB^*-algebra, alternative C^*-algebra, or C^*-algebra) becomes a real non-commutative JB^*-algebra (respectively, real JB^*-algebra, real alternative C^*-algebra, or real C^*-algebra) when it is regarded as a real algebra.

(b) According to Lemma 2.6.7 and Theorem 3.3.11, every complete smooth-normed real algebra, endowed with the involution $a^* := 2f(a)\mathbf{1} - a$ (where f stands for the unique element in $D(A,\mathbf{1})$), becomes a unital real non-commutative JB^*-algebra. In particular, $\mathbb{H}$ is a real C^*-algebra, and $\mathbb{O}$ is a real alternative C^*-algebra, when they are endowed with their standard involutions.

(c) According to Theorem 3.4.8, every JB-algebra, with the identity mapping as involution, becomes a real JB^*-algebra.

Now, invoking Proposition 3.4.1(i) and Corollary 3.4.3, we get the following.

Fact 4.2.47 *Let A be a real non-commutative JB^*-algebra. Then:*

(i) *Every closed associative $*$-subalgebra of A is a real C^*-algebra.*
(ii) *$H(A,*)$ becomes a JB-algebra under the product of A^{sym} and the norm of A.*

Theorem 4.2.48 *Let A be a real algebra. Then the following conditions are equivalent:*

(i) *A is a complete smooth-normed real algebra.*
(ii) *A is a J-division real non-commutative JB^*-algebra.*
(iii) *A is a unital real non-commutative JB^*-algebra such that $H(A,*) = \mathbb{R}\mathbf{1}$.*

Proof (i)⇒(ii) Assume that condition (i) is fulfilled. Then, as shown in Example 4.2.46(b), A is a unital real non-commutative JB^*-algebra. Moreover, by the implication (i)⇒(ii) in Theorem 4.1.96, A is a J-division algebra.

(ii)⇒(iii) Assume that A is a unital real non-commutative JB^*-algebra. Then, by Fact 4.2.47(ii), $H(A,*)$ is a JB-algebra in a natural way. Now suppose that condition (iii) does not hold, so that there exists $x \in H(A,*) \setminus \mathbb{R}\mathbf{1}$. Then, by Proposition 3.1.4(i), the closed subalgebra of $H(A,*)$ generated by x and $\mathbf{1}$ is of the form $C^{\mathbb{R}}(E)$ for some compact Hausdorff topological space E, which must have at least two points. Take $y,z \in C^{\mathbb{R}}(E) \setminus \{0\}$ such that $yz = 0$. Then we have $U_y(z) = 0$ in A, so y is a nonzero J-divisor of zero in A (cf. Definition 4.1.69), and so y is a nonzero non-J-invertible element of A, hence A is not a J-division algebra, i.e. condition (ii) is not fulfilled.

(iii)⇒(i) Set $S(A,*) := \{a \in A : a^* = -a\}$. Since $z \bullet t$ lies in $H(A,*)$ whenever z,t are in $S(A,*)$, the assumption (iii) provides us with a symmetric bilinear form $(\cdot|\cdot)$ on $S(A,*)$ satisfying $z \bullet t = -(z|t)\mathbf{1}$ for all $z,t \in S(A,*)$. Since $A = \mathbb{R}\mathbf{1} \oplus S(A,*)$, we can extend $(\cdot|\cdot)$ to a symmetric bilinear form on A by defining $(\mathbf{1}|\mathbf{1}) := 1$ and

$(\mathbf{1}|z) := 0$ for every $z \in S(A,*)$. Let a be in A, and write $a = \lambda\mathbf{1} + z$ with $\lambda \in \mathbb{R}$ and $z \in S(A,*)$. Then a belongs to the closed (automatically $*$-invariant) subalgebra of A generated by z and $\mathbf{1}$, which is a unital commutative real C^*-algebra (say B) because of Fact 4.2.47(i), and we have

$$a^*a = (\lambda\mathbf{1} - z)(\lambda\mathbf{1} + z) = [\lambda^2 + (z|z)]\mathbf{1} = (a|a)\mathbf{1}.$$

Regarding B as a closed real $*$-subalgebra of some unital commutative C^*-algebra, and keeping in mind Theorem 1.2.23, it follows that $(a|a) \geqslant 0$, and hence that $(\cdot|\cdot)$ is an inner product on A satisfying $(a|a) = \|a\|^2$. Thus the Banach space of A is a Hilbert space, hence A is smooth at $\mathbf{1}$, i.e. A is a smooth-normed algebra. □

In relation to the above theorem, we recall that the structure of smooth-normed real algebras was described in Theorem 2.6.9.

Lemma 4.2.49 *Let A be a unital real non-commutative JB^*-algebra whose unit is a vertex of $\mathbb{B}_A$. Then A is a JB-algebra, and $*$ is the identity on A.*

Proof Set $S(A,*) := \{a \in A : a^* = -a\}$, and let x and r be in $S(A,*)$ and $\mathbb{R}$, respectively. Then, by Fact 4.2.47(i), we have

$$\|\mathbf{1} \pm rx\|^2 = \|(\mathbf{1} \pm rx)^*(\mathbf{1} \pm rx)\| = \|(\mathbf{1} \mp rx)(\mathbf{1} \pm rx)\| = \|\mathbf{1} - r^2x^2\| \leqslant 1 + r^2\|x^2\|,$$

and hence $\lim_{r\to 0^+} \frac{\|\mathbf{1} \pm rx\| - 1}{r} \leqslant 0$. It follows from Proposition 2.1.5 that

$$0 \leqslant \min V(A,\mathbf{1},x) \leqslant \max V(A,\mathbf{1},x) \leqslant 0,$$

so $V(A,\mathbf{1},x) = \{0\}$, and therefore $x = 0$ because $\mathbf{1}$ is a vertex of $\mathbb{B}_A$. Thus $S(A,*) = 0$, and hence $A = H(A,*)$ because $A = H(A,*) \oplus S(A,*)$. Now $*$ is the identity mapping on A, so A is commutative, and so, by Fact 4.2.47(ii), A is a JB-algebra. □

Definition 4.2.50 By a *real JB^*-triple* we mean a closed real subtriple of a (complex) JB^*-triple.

Example 4.2.51 (a) Every JB^*-triple becomes a real JB^*-triple when it is regarded as a real space.

(b) By Theorem 4.1.45, every real non-commutative JB^*-algebra becomes a real JB^*-triple under its own norm and the triple product $\{xyz\} := U_{x,z}(y^*)$.

(c) Every real Hilbert space X becomes a real JB^*-triple under its own norm and the triple product $\{xyz\} := \frac{(x|y)z + (z|y)x}{2}$. Indeed, this follows by considering the Hilbert space complexification of X, and then by invoking Remark 4.2.38.

Lemma 4.2.52 *Let J be a real JB^*-triple. We have:*

(i) *If x and φ are norm-one elements of J and J', respectively, such that $\varphi(x) = 1$, then the restriction of φ to the closed subtriple of J generated by x is a triple homomorphism, so in particular we have $\varphi(\{xxx\}) = 1$.*
(ii) *If e is a tripotent in J, then $J_1(e)$, endowed with the product $ab := \{aeb\}$ and the involution $x^* := \{exe\}$, becomes a real JB^*-algebra with unit e.*

Proof Let X be any JB^*-triple containing J as a closed real subtriple.

Let x and φ be norm-one elements of J and J', respectively, such that $\varphi(x) = 1$. By the Hahn–Banach theorem, there exists $\psi \in D(X,x)$ such that the restriction to J of

$\Re \circ \psi$ equals φ. But then, by Corollary 4.2.12, for every $n \in \mathbb{N}$ we have $\psi(x^{(2n+1)}) = 1$, and hence $\varphi(x^{(2n+1)}) = 1$. Therefore, by Lemma 4.1.49, the restriction of φ to the closed subtriple of J generated by x is a triple homomorphism.

Now let e be a tripotent in J. Then, by Corollary 4.2.30(iii)(b), $X_1(e)$, endowed with the product $ab := \{aeb\}$ and the involution $x^* := \{exe\}$, becomes a JB^*-algebra with unit e. Since $J_1(e)$ is a closed real subtriple of $X_1(e)$, it follows that $J_1(e)$ becomes a closed real $*$-subalgebra of the JB^*-algebra $X_1(e)$ containing e. □

Theorem 4.2.53 *Let J be a real JB^*-triple, and let e be a norm-one element of J. Then the following conditions are equivalent:*

(i) *e is a geometrically unitary element of J.*
(ii) *e is a vertex of the closed unit ball of J.*
(iii) *J is the real JB^*-triple underlying a JB-algebra with unit e* (cf. Examples 4.2.46(c) and 4.2.51(b)).

Proof (i)⇒(ii) This is clear.

(ii)⇒(iii) By Lemma 4.2.52(i), we have $V(J,e,e-\{eee\}) = \{0\}$. Therefore, by assumption (ii), e becomes a tripotent. Then, invoking assumption (ii) again and Lemma 4.2.23, we see that $J_{\frac{1}{2}}(e) = 0 = J_0(e)$, and hence $J = J_1(e)$. Therefore, by Lemma 4.2.52(ii), the Banach space of J, endowed with the product $ab := \{aeb\}$ and the involution $x^* := \{exe\}$, becomes a real JB^*-algebra (say A) with unit e. Now, by the assumption once more and Lemma 4.2.49, A is a JB-algebra and $*$ is the identity on A. Finally, if we think about the real JB^*-triple underlying A, then, by Proposition 4.1.54, we re-encounter J.

(iii)⇒(i) By Proposition 3.1.4(iii) and Theorem 2.1.17(i). □

It follows from the above theorem that the existence of a geometrically unitary element in a nonzero real JB^*-triple J is equivalent to the fact that J is isometrically triple-isomorphic to a unital JB-algebra. Therefore, the study of geometrically unitary elements in real JB^*-triples is concluded with Proposition 3.1.15, which was proved much earlier.

If J is a real JB^*-triple, then its vector space complexification has a natural triple product, namely the one obtained by extending that of J by complex linearity in the outer variables, and by conjugate linearity in the middle variable. Analogously, if A is a real non-commutative JB^*-algebra, we can extend both the product of A (by complex bilinearity) and the involution of A (by conjugate-linearity) to the complexification of A. Actually, we have the following.

Proposition 4.2.54 *Let $(X,\|\cdot\|)$ be a JB^*-triple (respectively, a non-commutative JB^*-algebra, a JB^*-algebra, an alternative C^*-algebra, or a C^*-algebra), and let A be a closed real subtriple (respectively, a closed real $*$-subalgebra) of X. Then the complexification $A \oplus iA$ of A, endowed with its natural triple product (respectively, with its natural product and involution) and the norm $|||a \oplus ib||| := \max\{\|a+ib\|, \|a-ib\|\}$, becomes a JB^*-triple (respectively, a non-commutative JB^*-algebra, a JB^*-algebra, an alternative C^*-algebra, or a C^*-algebra).*

Proof Consider a set copy $\hat{X}$ of X with sum, product by scalars, triple product (respectively, product and involution), and norm defined by $\hat{x}+\hat{y} := \widehat{x+y}$, $\lambda\hat{x} := \widehat{\bar{\lambda}x}$, $\{\hat{x}\hat{y}\hat{z}\} := \widehat{\{xyz\}}$ (respectively, $\hat{x}\hat{y} := \widehat{xy}$ and $\hat{x}^* := \widehat{x^*}$), and $\|\hat{x}\| := \|x\|$, respectively. Then $\hat{X}$ becomes a JB^*-triple (respectively, a non-commutative JB^*-algebra, a JB^*-algebra, an alternative C^*-algebra, or a C^*-algebra), and hence, by Fact 4.2.29 (respectively, in an obvious way), $X \times \hat{X}$ is also a JB^*-triple (respectively, a non-commutative JB^*-algebra, a JB^*-algebra, an alternative C^*-algebra, or a C^*-algebra) under the norm

$$\|(x,\hat{y})\| := \max\{\|x\|, \|y\|\}.$$

Now, the mapping

$$a \oplus ib \to (a,\hat{a}) + i(b,\hat{b}) = (a+ib, \widehat{a-ib})$$

becomes a bijective triple homomorphism (respectively, a bijective algebra $*$-homomorphism) from the complexification of A onto a closed subtriple (respectively, a closed $*$-subalgebra) of $X \times \hat{X}$. □

In more intrinsic terms, Proposition 4.2.54 above can be reformulated as follows.

Fact 4.2.55 *Let A be a real JB^*-triple (respectively, a real non-commutative JB^*-algebra, a real JB^*-algebra, a real alternative C^*-algebra, or a real C^*-algebra). Then the complexification $A \oplus iA$ of A, endowed with its natural triple product (respectively, with its natural product and involution), becomes a JB^*-triple (respectively, a non-commutative JB^*-algebra, a JB^*-algebra, an alternative C^*-algebra, or a C^*-algebra) in a unique norm. Moreover this norm extends the norm of A, and converts the canonical involution of $A \oplus iA$ ($x+iy \to x-iy$) into an isometry.*

Proof The existence of the required norm on $A \oplus iA$ follows from Proposition 4.2.54, which also assures that this norm extends the norm of A and converts the canonical involution of $A \oplus iA$ into an isometry. The uniqueness of the required norm on $A \oplus iA$ follows from Proposition 4.1.52 (respectively, Proposition 3.4.4). □

Definition 4.2.56 Given a real JB^*-triple (respectively, a real non-commutative JB^*-algebra, a real JB^*-algebra, a real alternative C^*-algebra, or a real C^*-algebra) A, the complexification of A, as described in Fact 4.2.55 above, will be called the *JB^*-triple complexification* (respectively, the *non-commutative JB^*-complexification*, the *JB^*-complexification*, the *alternative C^*-complexification*, or the *C^*-complexification*) of A.

Theorem 4.2.57 *Let J be a nonzero real JB^*-triple. Then the extreme points of $\mathbb{B}_J$ are precisely the complete tripotents of J.*

Proof Let X stand for the JB^*-triple complexification of J, and note that, since the canonical involution of X is an isometry, we have

$$\|x\| \leqslant \|x+iy\| \quad \text{for all } x,y \in J. \tag{4.2.44}$$

Let e be a complete tripotent in J. If x and y are in J with $x+iy \in X_0(e)$, then we have $0 = \{e,e,x+iy\} = \{eex\} + i\{eey\}$, so $\{eex\} = \{eey\} = 0$ because $\{eex\}$ and $\{eey\}$ lie in J, so $x = y = 0$ because e is a complete tripotent in J, hence $x+iy = 0$.

Thus e becomes a complete tripotent in X. Therefore, by the implication (i)⇒(ii) in Theorem 4.2.34, e is an extreme point of $\mathbb{B}_X$, and hence of $\mathbb{B}_J$.

Conversely, let now e be an extreme point of $\mathbb{B}_J$. If x and y are in J, and if

$$\|e+\lambda(x+iy)\| \leqslant 1 \text{ for every } \lambda \in \mathbb{S}_{\mathbb{C}},$$

then we have

$$\|e \pm x\| \leqslant \|e \pm x \pm iy\| \leqslant 1 \text{ and } \|e \pm y\| \leqslant \|e \pm y \mp ix\| \leqslant 1$$

because of (4.2.44), so $x = y = 0$ because e is an extreme point of $\mathbb{B}_J$, hence $x + iy = 0$. Thus e becomes a complex extreme point of $\mathbb{B}_X$. Therefore, by the implication (iii)⇒(i) in Theorem 4.2.34, e is a complete tripotent in X, and hence in J. □

Corollary 4.2.58 *Let A be a nonzero real non-commutative JB^*-algebra. Then the extreme points of $\mathbb{B}_A$ are precisely those elements $u \in A$ satisfying*

$$a - 2U_{u,a}(u^*) + U_u U_{u^*}(a) = 0 \tag{4.2.45}$$

for every $a \in A$. Moreover the following conditions are equivalent:

(i) *A is unital.*
(ii) *$\mathbb{B}_A$ has strongly extreme points.*
(iii) *$\mathbb{B}_A$ has extreme points.*

Proof Let X stand for the non-commutative JB^*-complexification of A. Let $u \in A$ be an extreme point of $\mathbb{B}_A$. Then, as we have shown in the proof of Theorem 4.2.57, u is a complex extreme point of $\mathbb{B}_X$. Therefore, by the implication (iii)⇒(i) in Theorem 4.2.34, u is a complete tripotent in X. Therefore, keeping in mind Fact 4.2.35, and expressing the Bergmann operator $B(u,u)$ in terms of the product and the involution of A, the equality (4.2.45) holds for every $a \in X$, so in particular for every $a \in A$. Conversely, let u be in A such that the equality (4.2.45) holds for every $a \in A$. Then that equality holds for every $a \in X$ because $X = A \oplus iA$, so u is an extreme point of $\mathbb{B}_X$ (by the first conclusion in Theorem 4.2.36), and so u is an extreme point of $\mathbb{B}_A$. Thus the first conclusion in the corollary has been proved. The implication (ii)⇒(iii) is clear, whereas the one (i)⇒(ii) follows from Corollary 2.1.42. Therefore to conclude the proof it is enough to show that (iii)⇒(i). Assume that $\mathbb{B}_A$ has an extreme point (say u). Then, as we already know, u is a complex extreme point of $\mathbb{B}_X$, and hence, by the implication (iii)⇒(ii) in Theorem 4.2.34, u is an extreme point of $\mathbb{B}_X$. Therefore, by the implication (iv)⇒(i) in Theorem 4.2.36, X is unital. But the unit of X has to lie in A because the canonical involution of X is a conjugate-linear algebra automorphism. □

Now, arguing as in §4.2.37, with Corollary 4.2.58 above instead of Theorem 4.2.36, we get the following real version of Lemma 2.1.26.

Corollary 4.2.59 *The closed unit ball of a nonzero real C^*-algebra A has extreme points if and only if A is unital. In this case, the extreme points of $\mathbb{B}_A$ are precisely the elements $u \in A$ such that $(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0$.*

§4.2.60 Let X be a complex normed space, and let $\sharp$ be an isometric conjugate-linear involution on X. As we pointed out in §2.3.45, the involution of X can be transposed to the successive duals of X, giving rise to isometric conjugate-linear involutions on them, and moreover the bitranspose involution on X'' extends the given one on X. As there, these successive transposed mappings will be denoted with the same symbol $\sharp$. It is easy to realize that the mapping $f \to f_{|H(X,\sharp)}$ from $H(X',\sharp)$ to $H(X,\sharp)'$ becomes a surjective linear isometry. In short, we have $H(X,\sharp)' \equiv H(X',\sharp)$ naturally as dual real Banach spaces. Therefore $H(X,\sharp)''$ identifies with $H(X'',\sharp)$ in such a way that this identification becomes the identity on $H(X,\sharp)$.

Assume additionally that X is endowed with a (possibly different) continuous conjugate-linear involution $*$ commuting with $\sharp$. Then the following assertions are straightforwardly verified:

(i) $H(X,\sharp)$ is $*$-invariant, and hence the involution $*$ of $H(X,\sharp)$ can be transposed to the successive duals of $H(X,\sharp)$.
(ii) The bitransposes of $*$ and $\sharp$ commute, hence $H(X'',\sharp)$ is a $*$-invariant subset of X''.
(iii) The natural identification $H(X,\sharp)'' \equiv H(X'',\sharp)$ becomes a $*$-mapping.

Lemma 4.2.61 *Let X be a normed complex algebra, and let $\sharp$ be an isometric involutive conjugate-linear algebra automorphism of X. We have:*

(i) *$H(X,\sharp)$ is a real subalgebra of X, which contains the unit of X if X is unital.*
(ii) *If we endow X'' with the Arens product, then the bitranspose of $\sharp$ becomes a conjugate-linear algebra automorphism of X'', hence $H(X'',\sharp)$ is a real subalgebra of X'', which contains the unit of X'' if X'' is unital.*
(iii) *If in addition we endow $H(X,\sharp)''$ with the Arens product, then the natural identification $H(X,\sharp)'' \equiv H(X'',\sharp)$ as Banach spaces becomes an algebra homomorphism.*

Proof Assertion (i) is clear, whereas assertion (ii) follows from the appropriate conjugate-linear variant of Lemma 3.1.17. The verification of assertion (iii) consists of an adaptation of the argument in the proof of Lemma 3.1.17 just applied. Indeed, let $F : H(X,\sharp)'' \to H(X'',\sharp)$ denote the natural identification, and let x be in $H(X,\sharp)$. Then, since F is w^*-continuous, Lemma 2.2.12 applies so that the set

$$\{y'' \in H(X,\sharp)'' : F(xy'') = F(x)F(y'')\}$$

is w^*-closed in $H(X,\sharp)''$, and contains $H(X,\sharp)$ because F is the identity on $H(X,\sharp)$. Therefore, since $H(X,\sharp)$ is w^*-dense in $H(X,\sharp)''$, we have

$$F(xy'') = F(x)F(y'') \text{ for every } y'' \in H(X,\sharp)''. \tag{4.2.46}$$

Now let y'' be in $H(X,\sharp)''$. Again by Lemma 2.2.12, the set

$$\{x'' \in H(X,\sharp)'' : F(x''y'') = F(x'')F(y'')\}$$

is a w^*-closed subset of $H(X,\sharp)''$, which contains $H(X,\sharp)$ because of (4.2.46). Therefore we have $F(x''y'') = F(x'')F(y'')$ for every $x'' \in H(X,\sharp)''$. □

Proposition 4.2.62 *Let A be a nonzero real non-commutative JB^*-algebra (respectively, real JB^*-algebra, real alternative C^*-algebra, or real C^*-algebra). Then A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital real non-commutative JB^*-algebra (respectively, real JB^*-algebra, real alternative C^*-algebra, or real C^*-algebra). As a consequence, A is Arens regular.*

Proof Let X stand for the non-commutative JB^*-complexification (respectively, JB^*-complexification, alternative C^*-complexification, or C^*-complexification) of A, and let $\sharp$ denote the canonical involution of X. Then $\sharp$ becomes an isometric conjugate-linear algebra automorphism of X commuting with $*$, and hence we have $A = H(X, \sharp)$ as real non-commutative JB^*-algebras. On the other hand, by Theorem 3.5.34 (respectively, Proposition 3.5.26, Corollary 3.5.35, or Theorem 2.2.15), the Banach space X'', endowed with the Arens product and the bitranspose of $*$, becomes a unital non-commutative JB^*-algebra (respectively, JB^*-algebra, alternative C^*-algebra, or C^*-algebra). Therefore, by §4.2.60(ii) and Lemma 4.2.61(ii), $H(X'', \sharp)$ is a real $*$-subalgebra of X'' containing the unit of X'', and hence is a unital real non-commutative JB^*-algebra (respectively, real JB^*-algebra, real alternative C^*-algebra, or real C^*-algebra). Since $A = H(X, \sharp)$, it follows from §4.2.60(iii) and Lemma 4.2.61(iii) that A'', endowed with the Arens product and the bitranspose of the involution of A, becomes a unital real non-commutative JB^*-algebra. Finally, the Arens regularity of A follows from Corollary 2.3.47. □

Corollary 4.2.63 *Let A be a nonzero JB-algebra, let B be a real non-commutative JB^*-algebra, and let $\Phi : A \to B$ be a surjective linear isometry. Then Φ is a triple homomorphism* (cf. Examples 4.2.46(c) and 4.2.51(b)).

Proof By Proposition 3.1.10, A'' is a unital JB-algebra containing A as a subalgebra, and, by Proposition 4.2.62, B'' is a real non-commutative JB^*-algebra containing B as a $*$-subalgebra. Therefore, thinking about the bitranspose mapping $\Phi'' : A'' \to B''$, there is no loss of generality in assuming that A is unital. Then, by Proposition 3.1.4(iii) and Theorem 2.1.17(i), $\Phi(\mathbf{1})$ is a geometrically unitary element of B. Therefore, by the implication (i)⇒(iii) in Theorem 4.2.53, there exists a unital JB-algebra C such that $B = C$ as real JB^*-triples. Therefore, as a consequence of Corollary 3.1.16, Φ is a triple homomorphism. □

Surjective linear isometries between real JB^*-triples need not be triple homomorphisms. Even more, as the next example shows, unitary elements of real JB^*-triples cannot be characterized in terms of their underlying Banach spaces.

Example 4.2.64 Let X stand for the two-dimensional real Euclidean space. Then we can identify X with $\mathbb{C}$, and consider it as a real JB^*-triple under the triple product $\{\lambda\mu\rho\} := \lambda\bar{\mu}\rho$ (cf. Example 4.2.51(a)), so that all elements in $\mathbb{S}_X$ become unitary elements. However, as it happens with any real Hilbert space, we can also consider X as a real JB^*-triple under the triple product $\{xyz\} := \frac{(x|y)z+(z|y)x}{2}$ (cf. Example 4.2.51(c)), so that no element in $\mathbb{S}_X$ becomes unitary.

4.2.5 Historical notes and comments

Results from Proposition 4.2.1 to Theorem 4.2.7 are due to Kaup [381], although the proof of Proposition 4.2.1 is only sketched there. It is worth mentioning that, since abelian JB^*-triples are particular cases of *ternary C^*-rings* in the meaning of Zettl's paper [665] (published the same year as [381]), some of Kaup's arguments overlap with Zettl's. In particular, Proposition 4.2.5 can be derived from [665, Proposition 3.2]. In relation to Theorem 4.2.7, it should be remarked that, according to

[381, Corollary 1.13] and the example following it, abelian JB^*-triples need not be triple isomorphic to commutative C^*-algebras. Theorem 4.2.9 is also due to Kaup, but is earlier [380]. We thought that our proof of Theorem 4.2.9 was new. However, after having been kindly notified by A. M. Peralta, we must acknowledge that our argument is close to one suggested in [137, p. 80]. Kaup's original proof in [380] is included in Chu's book [710, pp. 177–81].

Corollary 4.2.11 (that *every element of a JB^*-triple has a unique cubic root*) is folklore regarding the existence, but could be new regarding the uniqueness.

For people who understand the geometry of $C_0^{\mathbb{C}}(E)$-spaces well, Corollary 4.2.12 is a straightforward consequence of Kaup's Theorem 4.2.9. Indeed, note at first that, if a normed space X is smooth at an element $x \in \mathbb{S}_X$, then the unique state of X relative to x is an extreme point of $\mathbb{B}_{X'}$. Therefore, if one knows that the extreme points of the closed unit ball of $C_0^{\mathbb{C}}(E)'$ are precisely the valuations at points of E after multiplication by unimodular complex numbers (see for example [730, Theorem 2.3.5]), then it turns out clear that $C_0^{\mathbb{C}}(E)$ is smooth at a given norm-one element x if and only if there is a unique $t_0 \in E$ such that $|x(t_0)| = 1$. Now let J be a JB^*-triple, let x be in $\mathbb{S}_J$, let M stand for the closed subtriple of J generated by x, and invoke Theorem 4.2.9 and Definition 4.2.10 to identify M with $C_0^{\mathbb{C}}(\sigma(x))$ in such a way that $x(t) = t$ for every $t \in \sigma(x)$. It follows that M is smooth at x, and hence that the valuation at $1 \in \sigma(x)$ is the unique state of M relative to x. But the valuation at 1 is surely a triple homomorphism, which proves Corollary 4.2.12.

Despite the above comments, it seems to us that Corollary 4.2.12 could have gone unnoticed before the writing of our work (see also Peralta's paper [486]). Besides the applications of Corollary 4.2.12 we have done (indeed, it is one of the ingredients in our proofs of Theorems 4.2.24 and 4.2.53), we are going to point out in Proposition 4.2.65 immediately below an outstanding new application.

JBW^*-triples are defined as those JB^*-triples which are dual Banach spaces. Let J be a JBW^*-triple, let J_* stand for the (complete) predual of J, let φ be in $\mathbb{S}_{J_*}$, and take $z \in \mathbb{S}_J$ with $\varphi(z) = 1$. Since the mapping $T \to \varphi(T(z))$ is a state of $BL(J)$ relative to I_J, for every $x \in J$ we have

$$\varphi(\{xxz\}) = \varphi(L(x,x)(z)) \in V(BL(J), I_J, L(x,x)) \subseteq \mathbb{R}_0^+.$$

Therefore the mapping $(x,y) \to \varphi(\{xyz\})$ becomes a non-negative hermitian sesquilinear form on J, and hence the mapping $x \to \|x\|_\varphi := \sqrt{\varphi(\{xxz\})}$ is a seminorm on J. The symbol $\|\cdot\|_\varphi$ is appropriate because, as proved by Barton and Friedman [60], this seminorm does not depend on the chosen support z for φ.

Proposition 4.2.65 *Let J be a JBW^*-triple. Then for every $x \in J$ we have*

$$\sup\{\|x\|_\varphi : \varphi \in \mathbb{S}_{J_*}\} = \|x\|.$$

Proof For $x \in J$ set $|||x||| := \sup\{\|x\|_\varphi : \varphi \in \mathbb{S}_{J_*}\}$. The inequality $|||\cdot||| \leqslant \|\cdot\|$ follows from Corollary 4.1.51, and hence $|||\cdot|||$ becomes a continuous seminorm on J. Now, in view of the Bishop–Phelps theorem, to conclude the proof it is enough to show that $|||x||| \geqslant 1$ for every $x \in \mathbb{S}_J$ which attains its norm at some $\varphi \in \mathbb{S}_{J_*}$. Let therefore x and φ be in $\mathbb{S}_J$ and $\mathbb{S}_{J_*}$, respectively, such that $\varphi(x) = 1$. Then, by Corollary 4.2.12, we have $|||x||| \geqslant \|x\|_\varphi = \sqrt{\varphi(\{xxx\})} = 1$, as desired. $\square$

Fact 4.2.14, as well as the remaining purely algebraic material in Subsection 4.2.2, are originally due to Loos [771, Chapter I] and Meyberg [443, 779]. Our approach to this material follows the one of Loos, together with those of Upmeier [814, Sections 18 and 21] and Chu [710, Section 1.2].

§**4.2.66** Now the fundamental formula for Jordan algebras, given in Proposition 3.4.15, can be proved as follows.

Let A be a Jordan algebra over $\mathbb{K}$, and let a,b be in A. Regarding A as a real Jordan algebra with conjugate-linear involution $*$ equal to the identity operator, it is enough to apply Proposition 4.1.34 and the fundamental formula for Jordan $*$-triples (cf. the reformulation in (4.2.27) of Proposition 4.2.16) to get that $U_{U_a(b)} = U_a U_b U_a$. □

Proposition 4.2.15, assertions (i) and (ii) in Proposition 4.2.32, and Lemma 4.2.33 are due to Friedman and Russo [269]. The beginning of our proof of Proposition 4.2.15 follows [710, proof of Lemma 3.2.1], whereas the remaining part is the original one in [269]. Assertion (iii) in Proposition 4.2.32 was first proved by Upmeier [814, Theorem 21.25] by means of a deep and long argument of a holomorphic nature. The short proof we have given is a refinement of that provided recently by Peralta in [486]. Invoking the structure theory of JB^*-triples [330, 270], assertion (iii) in Proposition 4.2.32 can be deeply refined. Indeed, as proved by Bunce, Fernández-Polo, Martínez, and Peralta [135, Proposition 2.4], we have the following.

Proposition 4.2.67 *Let J, e, and $[\cdot|\cdot] : J_{\frac{1}{2}}(e) \times J_{\frac{1}{2}}(e) \to J_1(e)$ be as in Proposition 4.2.32. Then $\|x\|^2 \leqslant 4\|[x|x]\|$ for every $x \in J_{\frac{1}{2}}(e)$.*

By keeping in mind Corollary 4.1.114, and quantifying the proof of assertion (iii) in Proposition 4.2.32, we are able to prove a less accurate version of the above proposition. Indeed, the constant 4 should be replaced with the much bigger one $2^6(2+\sqrt[3]{4})^6$.

Proof Let x be a norm-one element in $J_{\frac{1}{2}}(e)$, and set $M := \|\{xxe\}\| \leqslant 1$. Then for every $\psi \in D(J_1(e), e)$ we have

$$\left|\psi\left(\{x^{(3)}xe\}\right)\right| \leqslant \sqrt{\psi\left(\{x^{(3)}x^{(3)}e\}\right)\psi(\{xxe\})} \leqslant \sqrt{M},$$

so

$$v\left(J_1(e), e, \{x^{(3)}xe\}\right) \leqslant \sqrt{M},$$

and so $\|\{x^{(3)}xe\}\| \leqslant 2\sqrt{M}$ because of Lemmas 2.2.5 and 2.3.7(i). Therefore

$$\left\|Q_x^2(e)\right\| = \left\|2\{\{xxe\}xx\} - \{x^{(3)}xe\}\right\| \leqslant 2M + 2\sqrt{M} \leqslant 4\sqrt{M}.$$

Now we have

$$\|\{xex\}\|^3 = \|\{\{xex\}\{xex\}\{xex\}\}\| = \left\|Q_{Q_x(e)}Q_x(e)\right\| = \left\|Q_x Q_e Q_x^2(e)\right\| \leqslant 4\sqrt{M},$$

and hence

$$\frac{1}{2}=\frac{1}{2}\|\{xxx\}\|=\|\{x\{xee\}x\}\|=\|2\{\{exx\}ex\}-\{ex\{xex\}\}\|$$
$$\leqslant 2M+\sqrt[3]{4\sqrt{M}}\leqslant (2+\sqrt[3]{4})\sqrt[6]{M}.$$

Therefore $\|x\|^2=1\leqslant 2^6(2+\sqrt[3]{4})^6M=2^6(2+\sqrt[3]{4})^6\|[x|x]\|$. □

Proposition 4.2.67 above is better understood if we are aware of the Minkowski inequality for JB^*-algebra-valued non-negative hermitian sesquilinear forms, which is proved in Fact 4.2.68 immediately below. Indeed, Proposition 4.2.67 then asserts that the mapping $x\to\sqrt{\|[x|x]\|}$ from $J_{\frac{1}{2}}(e)$ to $\mathbb{R}$ becomes in fact a norm equivalent to the restriction to $J_{\frac{1}{2}}(e)$ of the norm of J.

Fact 4.2.68 *Let X be a complex vector space, let A be a non-commutative JB^*-algebra, and let $[\cdot|\cdot]:X\times X\to A$ be a sesquilinear mapping such that $[x|y]^*=[y|x]$ and $[x|x]\geqslant 0$ for all $x,y\in X$. Then the mapping $x\to\sqrt{\|[x|x]\|}$ becomes a seminorm on X.*

Proof [798, pp. 126–9] Let x,y be in X, and let a,b be positive numbers. Then we have

$$ab[x+y|x+y]\leqslant ab[x+y|x+y]+[bx-ay|bx-ay]=(a+b)(b[x|x]+a[y|y]),$$

hence, dividing by $ab(a+b)$, we obtain

$$0\leqslant\frac{[x+y|x+y]}{a+b}\leqslant\frac{[x|x]}{a}+\frac{[y|y]}{b},\tag{4.2.47}$$

and therefore

$$\frac{\|[x+y|x+y]\|}{a+b}\leqslant\left(\left\|\frac{[x|x]}{a}+\frac{[y|y]}{b}\right\|\leqslant\right)\frac{\|[x|x]\|}{a}+\frac{\|[y|y]\|}{b}.\tag{4.2.48}$$

Now, if both $\|[x|x]\|$ and $\|[y|y]\|$ are nonzero, then the inequality

$$\sqrt{\|[x+y|x+y]\|}\leqslant\sqrt{\|[x|x]\|}+\sqrt{\|[y|y]\|}$$

follows by taking $a=\sqrt{\|[x|x]\|}$ and $b=\sqrt{\|[y|y]\|}$ in (4.2.48). If for example $\|[x|x]\|=0$, then, taking $b=1$ and letting $a\to 0$ in (4.2.48), we get

$$\sqrt{\|[x+y|x+y]\|}\leqslant\sqrt{\|[y|y]\|}=\sqrt{\|[x|x]\|}+\sqrt{\|[y|y]\|},$$

which concludes the proof. □

Remark 4.2.69 (a) Let X be a vector space over $\mathbb{K}$, and let p be a mapping from X to $\mathbb{R}_0^+$. Then, as pointed out in [798, pp. 127-128], p is a seminorm on X if and only if, whenever x,y are in X, we have

$$p(\lambda x)=|\lambda|p(x)\ \text{ for every }\ \lambda\in\mathbb{K},\ \text{ and}$$
$$\frac{p(x+y)^2}{a+b}\leqslant\frac{p(x)^2}{a}+\frac{p(y)^2}{b}\ \text{ for all }\ a,b\in\mathbb{R}^+.\tag{4.2.49}$$

(b) Now let X,A, and $[\cdot|\cdot]:X\times X\to A$ be as in Fact 4.2.68. One can wonder whether the non-negative JB-algebra-valued mapping $p:x\to\sqrt{[x|x]}$ is a 'seminorm' on X, with a meaning to be determined. Certainly the inequality $p(x+y) \leqslant p(x)+p(y)$ cannot be expected. (Indeed, as pointed out in [798, pp. 131–3], this inequality fails by taking X equal to any complex Hilbert space of dimension $\geqslant 2$ and A equal to the C^*-algebra $BL(X)$, by defining $[u|v](w) := (w|v)u$ for all $u,v,w \in X$, and then by picking $x,y \in X$ such that $\|x\| = \|y\| = 1$ and $(x|y) = 0$.) Nevertheless, according to the inequality (4.2.47) in the proof of Fact 4.2.68, p satisfies (4.2.49). It follows from part (a) of the present remark that p could be called a seminorm on X in a very precise sense which coincides with the usual one in the case $A = \mathbb{C}$.

Theorems 4.2.24 and 4.2.28 are due to Braun, Kaup, and Upmeier [126]. In this paper, the notion of a vertex of the closed unit ball of a normed space appeared with an apparently different meaning, and with a different name. Indeed, let X be a normed space over $\mathbb{K}$, let u be in X, and let Q be a subset of X. The authors of [126] defined the *tangent cone to Q at u*, $T_u(Q)$, as the set of those $x \in X$ such that $x = \lim \frac{x_n - u}{t_n}$ for some sequence x_n in Q with $\lim x_n = u$ and some sequence t_n of positive real numbers. When $\mathbb{K} = \mathbb{C}$, they defined the *holomorphic tangent cone to Q at u*, $\hat{T}_u(Q)$, as $\hat{T}_u(Q) := \cap_{\lambda \in \mathbb{C}\setminus\{0\}} \lambda T_u(Q)$, and proved Theorems 4.2.24 and 4.2.28 with '$\hat{T}_u(\mathrm{int}(\mathbb{B}_X)) = 0$' instead of '$u$ is a vertex of $\mathbb{B}_X$' (for X equal to J and A, respectively). Much later, Kaidi, Morales, and Rodríguez [365] proved the next fact, which gave rise to the actual formulations of Theorems 4.2.24 and 4.2.28.

Fact 4.2.70 *Let X be a nonzero complex normed space, and let u be in $\mathbb{S}_X$. Then*

$$\hat{T}_u(\mathrm{int}(\mathbb{B}_X)) = K(X,u) \quad (\text{cf. } \S 3.3.20).$$

As a consequence, u is a vertex of $\mathbb{B}_X$ if and only if $\hat{T}_u(\mathrm{int}(\mathbb{B}_X)) = 0$.

Non-commutative JBW-algebras* are defined as those non-commutative JB^*-algebras which are dual Banach spaces. Propositions 4.2.71 and 4.2.72 immediately below follow from Theorems 4.2.24 and 4.2.28, respectively, and from the non-quantitative version of Corollary 2.9.32.

Proposition 4.2.71 *Let J be a JBW*-triple, and let u be a norm-one element of J. Then the following conditions are equivalent:*

(i) *u is a unitary element of J* (cf. Definition 4.1.53).
(ii) *u is a w^*-unitary element of J* (cf. §2.9.24).
(iii) *u is a w^*-vertex of the closed unit ball of J* (cf. §2.9.24 again).

Proposition 4.2.72 *Let A be a unital non-commutative JBW*-algebra, and let u be in A. Then the following conditions are equivalent:*

(i) *u is J-unitary* (cf. Definition 4.2.25).
(ii) *u is w^*-unitary.*
(iii) *u is a w^*-vertex of the closed unit ball of A.*

The associative forerunner of Proposition 4.2.72 above is due to Akemann and Weaver [5].

The equivalence (i)⇔(iii) in Theorem 4.2.34 (that *complex extreme points of the closed unit ball of a JB*-triple are precisely the complete tripotents*) is originally due to Kaup and Upmeier [385] (1977), who raised the question whether complete tripotents in a JB^*-triple are in fact extreme points of the closed unit ball. Partial affirmative answers to this question were known by Harris [314, Theorem 11] (1974) (for closed subtriples of C^*-algebras) and Loos [772, Theorem 5.6] (1977) (for finite-dimensional JB^*-triples), and another partial affirmative answer was given later by Braun, Kaup, and Upmeier [126, Lemma 4.1] (1978) (for unital JB^*-algebras). It seems to be difficult to settle the paternity of the definitive affirmative answer to the Kaup–Upmeier question, given by the implication (i)⇒(ii) in Theorem 4.2.34. According to a clever argument in Chu [710, p. 187], such an affirmative answer can easily be derived from the Kaup–Upmeier implication (i)⇒(iii) in Theorem 4.2.34 (1977), the Braun–Kaup–Upmeier Corollary 4.2.30(iii)(b) (1978), and the Friedman–Russo Proposition 4.2.15 (1985). Chu's argument goes as follows:

> Let J be a JB^*-triple, let u be a complete tripotent in J, let v be in J such that $\|u \pm v\| \leqslant 1$, and write $v = x + y$ with $x \in J_1(u)$ and $y \in J_{\frac{1}{2}}(u)$. Then, by the equality (4.2.5) in Proposition 4.2.15, we have $\|\lambda^2(u \pm x) \pm \lambda y\| \leqslant 1$ for every $\lambda \in \mathbb{S}_{\mathbb{C}}$, and hence, by the inequality (4.2.6) in Proposition 4.2.15, $\|u \pm x\| \leqslant 1$. But, in view of Corollaries 4.2.30(iii)(b) and 2.1.42, the last inequality implies $x = 0$, so $\|u + \mu y\| \leqslant 1$ for every $\mu \in \mathbb{S}_{\mathbb{C}}$, and so $\|u + \mu y\| \leqslant 1$ for every $\mu \in \mathbb{B}_{\mathbb{C}}$ by convexity. On the other hand, by the implication (i)⇒(iii) in Theorem 4.2.34, u is a complex extreme point of $\mathbb{B}_J$. It follows that $v = y = 0$, and u is indeed an extreme point of $\mathbb{B}_J$. □

Theorem 4.2.34, just mentioned, could be seen as the starting point for a good understanding of norm-closed faces of closed unit balls of JB^*-triples, as achieved in the paper of Edwards, Fernández-Polo, Hoskin, and Peralta [224]. The C^*-algebra forerunner of [224] is due to Akemann and Pedersen [4].

§**4.2.73** As a consequence of [213] and [225, Theorem 4.6], *tripotents in a JB^*-triple J are precisely those elements in J which are centres of symmetry of some w^*-closed face of the closed unit ball of J''.*

A more recent Banach space characterization of tripotents in JB^*-triples is due to Fernández-Polo, Martínez, and Peralta [261], who prove the following.

Theorem 4.2.74 *A norm-one element u of a JB^*-triple J is a tripotent if and only if the sets*

$$\{x \in J : \text{ there exists } \alpha > 0 \text{ with } \|u + \alpha x\| = \|u - \alpha x\| = 1\}$$

and

$$\{x \in J : \|u + \beta x\| = \max\{1, \|\beta x\|\} \text{ for all } \beta \in \mathbb{C}\}$$

coincide.

The above theorem generalizes the associative forerunner of Akemann and Weaver [5, Theorem 1], already reviewed in Theorem 2.1.46. As the authors of [261] show, Theorem 4.2.34 can easily be derived from Theorem 4.2.74 above.

Fact 4.2.35 is due to Kaup and Upmeier [385], whereas Theorem 4.2.36 is due to Youngson [655].

Fact 4.2.39 is due to Wright and Youngson [642]. Results from Lemma 4.2.41 to Theorem 4.2.43 are due to Siddiqui [571, 573, 574], and become nontrivial generalizations of associative forerunners due to Gardner [281], and Kadison and Pedersen [360] (see also [789, p. 98]). Our proof of Lemma 4.2.42 introduces relevant simplifications on Siddiqui's original proof. For further associative developments on the topic, some of which were inspired in [34], the reader is referred to [302, 360, 469, 484, 541]. Jordan versions of these developments can be found in [572, 575, 576].

Proposition 4.2.44 is a straightforward consequence of Theorem 4.1.45 and of the part of Kaup's Theorem 2.2.28 which has still not been proved in our work. (The other part of Theorem 2.2.28 was already proved in Proposition 4.1.52.) Nevertheless, our proof of Proposition 4.2.44 could be interesting because it provides an autonomous argument to Theorem 2.2.28 in the particular case of non-commutative JB^*-algebras.

Real C^*-algebras can be characterized by means of intrinsic axioms. This approach seems to go back to Arens and Kaplansky [30] (who studied the commutative case), and continued in the general case with the contributions of Ingelstam [339] and Palmer [477]. The standard references for real C^*-algebras are the books of Goodearl [735] and Li [768]. The reader is also referred to pp. 274–5 of Doran–Belfi [725] for a short survey on the topic. Among recent works on the matter, we emphasize that of Chu, Dang, Russo, and Ventura [171], where, as the main result, the following theorem is proved.

Theorem 4.2.75 *Surjective linear isometries between real C^*-algebras are precisely the bijective triple homomorphisms.*

As far as we know, to date, real alternative C^*-algebras have been considered only by Kaplansky [762] (see the review of [762] in the preface of our work).

In the case of the existence of a unit, real JB^*-algebras were introduced (under the name of J^*B-algebras) by Alvermann [18], who provided a system of intrinsic axioms for them, and classified them in the finite-dimensional case.

To conclude our comments on Definition 4.2.45, let us say that, although strongly involved in our development, real non-commutative JB^*-algebras (which contain real JB^*-algebras, real alternative C^*-algebras, and hence real C^*-algebras) have been considered only by Alvermann and Janssen [19], as particular cases of their 'real weakly admissible algebras'. Nevertheless, in most cases, only minor changes have had to be made to transfer the results known for real (commutative) JB^*-algebras to the possibly non-commutative setting. This should be kept in mind in relation to our next comments about the paternity of results.

Theorem 4.2.48 is due to Becerra, López, Peralta, and Rodríguez [66].

§**4.2.76** After a comprehensive finite-dimensional forerunner, due to Loos [772, Chapter 11], real JB^*-triples (in the sense of Definition 4.2.50) were introduced by Isidro, Kaup, and Rodríguez [341], who proved the bracket-free versions of Proposition 4.2.54 and Fact 4.2.55, Theorem 4.2.57, and Corollary 4.2.63. As the main result, they showed that *surjective linear isometries between real JB^*-triples are precisely those bijective linear mappings preserving the cubes*. We recall that, according to Example 4.2.64, *surjective linear isometries between real JB^*-triples need not preserve triple products*. Nevertheless, if J is a (complex) JB^*-triple, then the cube mapping determines the triple product because of the straightforward equality

$$4\{xyx\} = (y+x)^{(3)} + (y-x)^{(3)} - (y+ix)^{(3)} - (y-ix)^{(3)},$$

which holds for all $x,y \in J$. Therefore the main result in [341], reviewed above, contains Kaup's celebrated theorem on (complex-linear) surjective isometries of JB^*-triples (cf. Theorem 2.2.28). For real-linear isometries of JB^*-triples, the reader is referred to Dang's paper [205]. Other fundamental papers on real JB^*-triples are [66, 226, 260, 383, 426].

For later reference, we formulate here the following result, also proved in [341].

Proposition 4.2.77 *Let A be a nonzero JB-algebra, let e be in A, and,* according to Propositions 4.1.34 and 4.1.35*, consider A as a real Jordan algebra with product $x \odot y := U_{x,y}(e)$. Then $(A, \odot)$ is a JB-algebra with unit e if and only if e is an isolated point of the set of all extreme points of $\mathbb{B}_A$.*

Several other notions of JB^*-triples over the real field (different from that in Definition 4.2.50) appear in the literature, all of them attempting to find an intrinsic axiomatic definition of real JB^*-triples.

The attempt due to Upmeier [814, Definition 20.7] appeared in 1985. Nine years later, Dang and Russo [206, Definition 1.3] extracted the following axiomatic definition of a JB^*-triple over the real field. By a *J^*B-triple* they mean a real Banach space J equipped with a structure of a real Banach Jordan $*$-triple which satisfies

(J^*B1) $\|\{xxx\}\| = \|x\|^3$ for every $x \in J$;

(J^*B2) $\|\{xyz\}\| \leqslant \|x\|\,\|y\|\,\|z\|$ for all $x,y,z \in J$;

(J^*B3) $\mathrm{sp}(L(J)_{\mathbb{C}}, L(x,x)) \subset [0,+\infty)$ for every $x \in J$;

(J^*B4) $\mathrm{sp}(L(J)_{\mathbb{C}}, L(x,y) - L(y,x)) \subset i\mathbb{R}$ for all $x,y \in J$.

Every closed subtriple of a J^*B-triple is a J^*B-triple [206, Remark 1.5]. The class of J^*B-triples includes all real JB^*-triples (in the sense of Definition 4.2.50). Moreover, complex JB^*-triples are precisely those complex Banach Jordan $*$-triples whose underlying real Banach space is a J^*B-triple [206, Proposition 1.4]. According to [206, Theorem 3.11], the complexification of every abelian J^*B-triple J is a complex JB^*-triple in some norm extending the norm on J, in other words, every abelian J^*B-triple is a real JB^*-triple.

In [485], Peralta introduced the so-called numerically positive J^*B-triples. By a *numerically positive J^*B-triple* he means a J^*B-triple J satisfying the following axiom:

(J^*B5) $V(BL(J), I_J, L(x,x)) \subset [0,+\infty)$ for every $x \in J$.

Every real JB^*-triple is a numerically positive J^*B-triple. Moreover, a J^*B-triple J having a unitary element is a real JB^*-triple if and only if it is a numerically positive J^*B-triple, equivalently, J is a real JB^*-algebra [485, Theorem 2.6]. It remains as an open problem if every numerically positive J^*B-triple is a real JB^*-triple [485, p. 105].

The above discussion about axiomatic approaches to real JB^*-triples has been taken from [137, pp. 72–3].

§**4.2.78** Theorem 4.2.53 is due to Fernández-Polo, Martínez, and Peralta [261]. It is worth mentioning that, combining Remark 3.1.25 ([342] 1995), Proposition 4.2.77 ([341] 1995), and Theorem 4.2.53 ([261] 2004), the Banach space characterizations of central symmetries in unital JB-algebras, provided by Proposition 3.1.15 ([399] 2009), follow straightforwardly. Nevertheless, as far as we know, we have been the first to notice this incidence.

The bracketed versions of Proposition 4.2.54 and Fact 4.2.55, Corollary 4.2.58, and Proposition 4.2.62 have been taken from [536], where, with more or less precision, the indications for the corresponding proofs are given. Proposition 4.2.62 can be also obtained from a more general result of Alvermann and Janssen [19, Theorem 6.1]. Corollary 4.2.59 is pointed out in [343, Remark 1.10].

As we mentioned above, surjective linear isometries between real JB^*-triples are precisely those bijective linear mappings preserving the cubes. As a consequence, bijective triple homomorphisms between real JB^*-triples are isometries. In the particular case of real non-commutative JB^*-algebras, we have the following relevant converse, which generalizes Proposition 4.2.44, Corollary 4.2.63, and Theorem 4.2.75.

Theorem 4.2.79 *Surjective linear isometries between real non-commutative JB^*-algebras are in fact triple homomorphisms.*

The above theorem was proved by Fernández-Polo, Martínez, and Peralta, under the additional assumption that the real non-commutative JB^*-algebras under consideration are unital (see [260, Corollary 3.4]). But, as pointed out in [536], this additional assumption can be removed by passing to biduals, by keeping in mind Proposition 4.2.62, and then by applying the original result in [260] to the bitranspose mapping of the given isometry.

An independent generalization of Corollary 4.2.63 is that *surjective linear isometries from JB-algebras to real JB^*-triples are triple homomorphisms* [341, Proposition 4.11]. The proof of this result is the same as that of Corollary 4.2.63 whenever one knows that *the bidual of a real JB^*-triple J becomes a JB^*-triple under a triple product extending the one of J* [341, Lemma 4.2].

Unitary elements of a real JB^*-triple J are strongly extreme points of $\mathbb{B}_J$ because, if u is a unitary element of J, then J becomes a real JB^*-algebra with unit u for a suitable product and a suitable involution (cf. Lemma 4.2.52(ii)), and Corollary 2.1.42 applies. Keeping in mind Proposition 4.2.54, Corollary 4.2.58, and Theorem 4.2.28, we easily realize that, if a real non-commutative JB^*-algebra A has a unitary element in the Jordan $*$-triple sense, then A has a unit, and unitary elements of A in the Jordan $*$-triple sense coincide with J-unitary elements, as well as with algebraically J-unitary elements. In particular, if a JB-algebra A has a unitary element in the Jordan $*$-triple sense, then A has a unit, and unitary elements in the Jordan $*$-triple sense coincide with symmetries in the usual JB-algebra meaning.

Keeping in mind the above paragraph and Theorem 4.2.79, we see that, in the setting of unital real non-commutative JB^*-algebras, the notion of a J-unitary element is actually a Banach space notion. In the particular case of unital JB-algebras, explicit Banach space characterizations of J-unitaries are provided by Proposition 3.1.9. For a general real non-commutative JB^*-algebra, we do not know any explicit Banach space characterization of its J-unitary elements. Could they coincide with the strongly extreme points of its closed unit ball?

To conclude our comments on Subsection 4.2.4, let us mention that, as a consequence of [341, Lemma 4.2] and [226, Theorem 3.9], the Banach space characterization of tripotents of (complex) JB^*-triples, formulated in §4.2.73, remains true verbatim in the case of real JB^*-triples. In turn, the natural real variant of the geometric characterization of tripotents of (complex) JB^*-triples, reviewed in Theorem 4.2.74, determines tripotents in real JB^*-triples [261, Theorem 2.3].

4.3 C^*- and JB^*-algebras generated by a non-self-adjoint idempotent

Introduction Let A be a C^*-algebra generated (as a normed $*$-algebra) by a non-self-adjoint idempotent e, and set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$. As a consequence of Proposition 1.2.49, K is a compact subset of $[1,\infty[$ whose maximum element is greater than 1. Moreover, as we show in Proposition 4.3.3, in general no more can be said about K. We prove that, if 1 does not belong to K, then A is $*$-isomorphic to the C^*-algebra $C(K, M_2(\mathbb{C}))$ of all continuous functions from K to the C^*-algebra $M_2(\mathbb{C})$ (see Theorem 4.3.11), and that, if 1 belongs to K, then A is $*$-isomorphic to a distinguished proper closed $*$-subalgebra of $C(K, M_2(\mathbb{C}))$ (see §4.3.14 and Theorem 4.3.16). Replacing C^*-algebra with JB^*-algebra, $\mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$ with the triple spectrum $\sigma(e)$ of e (cf. Theorem 4.1.45 and Definition 4.2.10), and $M_2(\mathbb{C})$ with the three-dimensional spin factor C_3, similar results are obtained (see Theorems 4.3.29 and 4.3.32).

Several relevant consequences of the results reviewed above are derived. Among them, we emphasize Spitkovsky's theorem [595], that C^*-algebras generated by a non-self-adjoint idempotent are generated by two self-adjoint idempotents (see Corollary 4.3.17), as well as the appropriate variant of this theorem for JB^*-algebras (see Corollary 4.3.34). On the other hand, after proving in Theorem 4.3.47 that the centre of any non-commutative JB^*-algebra A coincides with the centre of the JB^*-algebra A^{sym} [19], we deduce that a non-commutative JB^*-algebra has a non-self-adjoint idempotent if and only if it has a non-central self-adjoint idempotent (see Proposition 4.3.49).

4.3.1 The case of C^*-algebras

As in other occasions, $M_2(\mathbb{C})$ will stand for the C^*-algebra of all 2×2 matrices with entries in $\mathbb{C}$.

Lemma 4.3.1 *Let e be an idempotent in $M_2(\mathbb{C})$ different from 0 and $\mathbf{1}$, and set $e_{11} := \|e\|^{-2}e^*e$, $e_{12} := \|e\|^{-1}e^*$, $e_{21} := \|e\|^{-1}e$, and $e_{22} := \|e\|^{-2}ee^*$. Then, for*

$i,j,k,l \in \{1,2\}$, we have $e_{ij}^* = e_{ji}$, $e_{ij}e_{kl} = e_{il}$ if $j = k$, and $e_{ij}e_{kl} = \|e\|^{-1}e_{il}$ if $j \neq k$. As a consequence, $ee_{11} = e_{22}e = e$.

Proof The equality $e_{ij}^* = e_{ji}$ is clear. On the other hand, by Lemma 1.2.12, we have $\mathrm{sp}(M_2(\mathbb{C}), e^*e) = \{0, \|e\|^2\}$, and hence $(e^*e - \|e\|^2\mathbf{1})e^*e = 0$, which reads as $e_{11}^2 = e_{11}$. Analogously, $e_{22}^2 = e_{22}$. Now we have

$$(ee^*e - \|e\|^2e)^*(ee^*e - \|e\|^2e) = (e^*ee^* - \|e\|^2e^*)(ee^*e - \|e\|^2e)$$
$$= (e^*e - \|e\|^2\mathbf{1})e^*e(e^*e - \|e\|^2\mathbf{1}) = 0,$$

and hence $ee^*e - \|e\|^2e = 0$, which reads as both $e_{21}e_{11} = e_{21}$ and $e_{22}e_{21} = e_{21}$. By taking adjoints, we deduce $e_{11}e_{12} = e_{12}$ and $e_{12}e_{22} = e_{12}$. The remaining assertions in the lemma are either obvious or easily deducible from the above computations. □

The mapping $\eta : [1,\infty[\to M_2(\mathbb{C})$, which is introduced in Lemma 4.3.2 immediately below, will play a crucial role through this section.

Lemma 4.3.2 *Let t be in $[1,\infty[$, and let $\eta(t)$ denote the element of $M_2(\mathbb{C})$ defined by*

$$\eta(t) := \frac{1}{2}\begin{pmatrix} 1 & t + \sqrt{t^2-1} \\ t - \sqrt{t^2-1} & 1 \end{pmatrix}.$$

Then $\eta(t)$ is an idempotent satisfying $\|\eta(t)\| = t$. Therefore, setting

$$\eta_{11}(t) := t^{-2}\eta(t)^*\eta(t), \;\; \eta_{12}(t) := t^{-1}\eta(t)^*, \;\; \eta_{21}(t) := t^{-1}\eta(t),$$

and

$$\eta_{22}(t) := t^{-2}\eta(t)\eta(t)^*,$$

we have

$$\eta_{ij}(t)^* = \eta_{ji}(t), \;\; \eta_{ij}(t)\eta_{kl}(t) = \eta_{il}(t) \;\text{ if }\; j = k,$$

and

$$\eta_{ij}(t)\eta_{kl}(t) = t^{-1}\eta_{il}(t) \;\text{ if }\; j \neq k.$$

As a consequence, $\eta(t)\eta_{11}(t) = \eta_{22}(t)\eta(t) = \eta(t)$.

Proof That $\eta(t)$ is an idempotent in $M_2(\mathbb{C})$ is straightforward. Moreover, computing its norm according to Exercise 1.2.15, we have $\|\eta(t)\| = t$. Now the remaining assertions in the statement follow from Lemma 4.3.1. □

Let K be a subset of $[1,\infty[$. We denote by η_K the restriction to K of the continuous mapping $t \to \eta(t)$ from $[1,\infty[$ to $M_2(\mathbb{C})$, given by Lemma 4.3.2. Moreover, for $i,j \in \{1,2\}$, we denote by η_{ij}^K the restriction to K of the continuous mapping $t \to \eta_{ij}(t)$ from $[1,\infty[$ to $M_2(\mathbb{C})$, given by that lemma.

Now, let K be a compact subset of $[1,\infty[$. Let u stand for the element of $C^{\mathbb{C}}(K)$ defined by $u(t) := t$ for every $t \in K$. We denote by $\mathscr{A}(K)$ the complete normed associative complex $*$-algebra whose vector space is that of all 2×2 matrices with entries in $C^{\mathbb{C}}(K)$, whose (bilinear) product is determined by the equalities

$$(f[ij])(g[kl]) := (fg)[il] \text{ if } j = k \text{ and } (f[ij])(g[kl]) := (u^{-1}fg)[il] \text{ if } j \neq k,$$

whose norm is given by $\|(f_{ij})\| := \|f_{11}\| + \|f_{12}\| + \|f_{21}\| + \|f_{22}\|$, and whose (conjugate-linear) involution $*$ is determined by $(f[ij])^* := \bar{f}[ji]$. Here, as usual, for

$f \in C^{\mathbb{C}}(K)$ and $i, j \in \{1,2\}$, $f[ij]$ means the matrix having f in the (i,j)-position and 0's elsewhere. It is useful to see $\mathscr{A}(K)$ as a $C^{\mathbb{C}}(K)$-module in the natural manner, namely by defining the product of a function $f \in C^{\mathbb{C}}(K)$ and a matrix $(f_{ij}) \in \mathscr{A}(K)$ by $f(f_{ij}) := (ff_{ij})$. In this regard, we straightforwardly realize that $\mathscr{A}(K)$ becomes in fact an algebra over $C^{\mathbb{C}}(K)$, i.e. the operators of left and right multiplication by arbitrary elements of $\mathscr{A}(K)$ are $C^{\mathbb{C}}(K)$-module homomorphisms. Moreover, the symbol $f[ij]$ can now be read as the product of the function $f \in C^{\mathbb{C}}(K)$ and the matrix $[ij] \in \mathscr{A}(K)$, where, for $i, j \in \{1,2\}$, $[ij]$ stands for the matrix having the constant function equal to one in the (i,j)-position and 0's elsewhere.

According to Example 1.1.4(d), for K as above, we denote by $C(K, M_2(\mathbb{C}))$ the algebra of all continuous functions from K to $M_2(\mathbb{C})$, and remark that $C(K, M_2(\mathbb{C}))$ becomes naturally a C^*-algebra. We will see $C(K, M_2(\mathbb{C}))$ as a $C^{\mathbb{C}}(K)$-module in the natural manner. From now on in this section, u will always stand for the element of $C^{\mathbb{C}}(K)$ defined by $u(t) := t$ for every $t \in K$.

Proposition 4.3.3 *Let K be a compact subset of $[1,\infty[$ whose maximum element is greater than 1. Then η_K is a non-self-adjoint idempotent in $C(K, M_2(\mathbb{C}))$ satisfying*

$$\mathrm{sp}(C(K, M_2(\mathbb{C})), \sqrt{\eta_K^* \eta_K}) \setminus \{0\} = K,$$

and the mapping $\mathscr{F}$ from $\mathscr{A}(K)$ to $C(K, M_2(\mathbb{C}))$, defined by

$$\mathscr{F}((f_{ij})) := \sum_{i,j \in \{1,2\}} f_{ij} \eta_{ij}^K,$$

becomes a continuous algebra $*$-homomorphism satisfying $\mathscr{F}(u[21]) = \eta_K$.

Proof By the first part of Lemma 4.3.2, for $t \in K$, $\eta(t)$ is an idempotent in $M_2(\mathbb{C})$ satisfying $\|\eta(t)\| = t$, which implies

$$\mathrm{sp}(M_2(\mathbb{C}), \sqrt{\eta(t)^* \eta(t)}) \setminus \{0\} = \{t\}.$$

It follows that η_K is a non-self-adjoint idempotent of $C(K, M_2(\mathbb{C}))$ satisfying $\mathrm{sp}(C(K, M_2(\mathbb{C})), \sqrt{\eta_K^* \eta_K}) \setminus \{0\} = K$. On the other hand, the mapping $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$ is an algebra $*$-homomorphism if (and only if), for every $t \in K$, the composition of $\mathscr{F}$ with the valuation at t is an algebra $*$-homomorphism from $\mathscr{A}(K)$ to $M_2(\mathbb{C})$. But this last fact follows from the definition of the operations on $\mathscr{A}(K)$, and the second part of Lemma 4.3.2.Finally, both the continuity of $\mathscr{F}$ (it is in fact contractive) and that $\mathscr{F}(u[21]) = \eta_K$ become obvious facts. $\square$

Given a normed $*$-algebra A over $\mathbb{K}$ and a subset S of A, we say that A is generated by S as a normed $*$-algebra if the closed $*$-subalgebra of A generated by S is equal to the whole algebra A.

Lemma 4.3.4 *Let K be a compact subset of $[1,\infty[$. Then $\mathscr{A}(K)$ is generated by $u[21]$ as a normed $*$-algebra.*

Proof Set $p := u[21]$, and let C denote the closed $*$-subalgebra of $\mathscr{A}(K)$ generated by p. We have $u^2[11] = p^* p \in C$. Therefore, since $C^{\mathbb{C}}(K)$ is bicontinuously isomorphic to $C^{\mathbb{C}}(K)[11]$ by means of the mapping $f \to f[11]$, and $C^{\mathbb{C}}(K)$ is generated by u^2 as a normed algebra, we obtain that $C^{\mathbb{C}}(K)[11] \subseteq C$, and hence that

$$C^{\mathbb{C}}(K)[21] = uC^{\mathbb{C}}(K)[21] = (u[21])(C^{\mathbb{C}}(K)[11]) = p(C^{\mathbb{C}}(K)[11]) \subseteq C.$$

Starting with the fact that $u^2[22] = pp^* \in C$, a similar argument shows that $C^{\mathbb{C}}(K)[22]$ and $C^{\mathbb{C}}(K)[12]$ are contained in C. It follows that $\mathscr{A}(K) = C$. $\square$

Let A be a C^*-algebra. We recall that the unital extension $A_{\mathbb{1}}$ becomes naturally a C^*-algebra containing A isometrically as a $*$-subalgebra (cf. Proposition 1.2.44), and that A becomes naturally a JB^*-triple (cf. Fact 4.1.41).

Lemma 4.3.5 *Let A be a C^*-algebra, let a be a nonzero element in A, set*

$$E := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{a^*a}) \setminus \{0\},$$

and let v stand for the inclusion mapping $E \hookrightarrow \mathbb{C}$. Then:

(i) *There exists a unique continuous triple homomorphism $\Phi : C_0^{\mathbb{C}}(E) \to A$ taking v to a.*

(ii) *Φ is an isometry, and the range of Φ equals the closed subtriple of A generated by a.*

(iii) *For $f, g \in C_0^{\mathbb{C}}(E)$, we have*

$$\Phi(f)\Phi(\bar{g})^* = f(\sqrt{aa^*})g(\sqrt{aa^*}), \ \ \Phi(\bar{f})^*\Phi(g) = f(\sqrt{a^*a})g(\sqrt{a^*a}),$$

$$\Phi(f)g(\sqrt{a^*a}) = \Phi(fg), \ \ \textit{and} \ \ g(\sqrt{aa^*})\Phi(f) = \Phi(gf).$$

(When necessary, elements of $C_0^{\mathbb{C}}(E)$ should be identified with those complex-valued continuous mappings on $\mathrm{sp}(A_{\mathbb{1}}, \sqrt{a^*a}) = \mathrm{sp}(A_{\mathbb{1}}, \sqrt{aa^*})$ vanishing at 0.)

Proof Let M denote the closed subtriple of A generated by a. By induction we get $a(a^*a)^n = a^{(2n+1)}$ for every $n \in \mathbb{N} \cup \{0\}$ (cf. §4.1.47), and hence $ap(a^*a)$ belongs to M for every $p \in \mathbb{C}[\mathbf{x}]$. Therefore the set $N := \{ap(a^*a) : p \in \mathbb{C}[\mathbf{x}]\}$ is a subtriple of M containing a, so N is dense in M because M is generated by a as a Banach Jordan $*$-triple. Now note that, by Lemma 4.2.8, $L := \{vp(v^2) : p \in \mathbb{C}[\mathbf{x}]\}$ is a dense subtriple of $C_0^{\mathbb{C}}(E)$, and that for every $p \in \mathbb{C}[\mathbf{x}]$ we have

$$\|ap(a^*a)\|^2 = \|\bar{p}(a^*a)a^*ap(a^*a)\| = \sup\{t^2|p(t^2)|^2 : t \in E\} = \|vp(v^2)\|^2.$$

It follows that the correspondence $vp(v^2) \to ap(a^*a)$ becomes a well-defined isometric dense range triple homomorphism from L to M taking v to a. Extending by continuity this triple homomorphism, and composing this extension with the inclusion mapping $M \hookrightarrow A$, we get an isometric triple homomorphism $\Phi : C_0^{\mathbb{C}}(E) \to A$ satisfying $\Phi(C_0^{\mathbb{C}}(E)) = M$ and $\Phi(v) = a$. If $\Psi : C_0^{\mathbb{C}}(E) \to A$ is any continuous triple homomorphism with $\Psi(v) = a$, then $\ker(\Phi - \Psi)$ is a closed subtriple of $C_0^{\mathbb{C}}(E)$ containing L, so $\ker(\Phi - \Psi) = C_0^{\mathbb{C}}(E)$, hence $\Psi = \Phi$.

To prove the equality $\Phi(f)\Phi(\bar{g})^* = f(\sqrt{aa^*})g(\sqrt{aa^*})$ for $f, g \in C_0^{\mathbb{C}}(E)$, we can assume that f and g are of the form $vp(v^2)$ and $vq(v^2)$, for suitable p and q in $\mathbb{C}[\mathbf{x}]$, respectively. Then we have

$$\Phi(f)\Phi(\bar{g})^* = ap(a^*a)q(a^*a)a^* = aa^*p(aa^*)q(aa^*) = f(\sqrt{aa^*})g(\sqrt{aa^*}),$$

as desired. The proof of the equality $\Phi(\bar{f})^*\Phi(g) = f(\sqrt{a^*a})g(\sqrt{a^*a})$ is similar. To realize that $\Phi(f)g(\sqrt{a^*a}) = \Phi(fg)$ and $g(\sqrt{aa^*})\Phi(f) = \Phi(gf)$, take f of the form $vp(v^2)$ with $p \in \mathbb{C}[\mathbf{x}]$, and g of the form $q(v^2)$ for some $q \in \mathbb{C}[\mathbf{x}]$ with $q(0) = 0$. $\square$

Proposition 4.3.6 *Let A be a C^*-algebra, and let e be a non-self-adjoint idempotent in A. We have:*

(i) *$K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$ is a compact subset of $[1, \infty[$ whose maximum element (namely $\|e\|$) is greater than 1.*
(ii) *There exists a unique continuous algebra $*$-homomorphism $F : \mathscr{A}(K) \to A$ such that $F(u[21]) = e$.*
(iii) *The closure in A of the range of F coincides with the closed $*$-subalgebra of A generated by e.*
(iv) *F is injective if and only if either 1 does not belong to K or 1 is an accumulation point of K.*
(v) *If 1 is an isolated point of K, then $\ker(F)$ consists precisely of those matrices $(f_{ij}) \in \mathscr{A}(K)$ which vanish at every $t \in K \setminus \{1\}$ and satisfy*

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

Proof Assertion (i) follows from Propositions 1.1.40 and 1.2.49.

According to assertion (i), we write $C^{\mathbb{C}}(K)$ instead of $C_0^{\mathbb{C}}(K)$. Nevertheless, at a certain moment of the proof, elements of $C^{\mathbb{C}}(K)$ should be identified with those complex-valued continuous mappings on $\mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) = \mathrm{sp}(A_{\mathbb{1}}, \sqrt{ee^*})$ vanishing at 0. Let $\Phi : C^{\mathbb{C}}(K) \to A$ be the linear isometry given by Lemma 4.3.5 when we take $K = E$ (so $v = u$) and $a = e$ in that lemma. For $i, j \in \{1, 2\}$, consider the linear isometry $\Phi_{ij} : C^{\mathbb{C}}(K) \to A$ defined, for $f \in C^{\mathbb{C}}(K)$, by

$$\Phi_{11}(f) := f(\sqrt{e^*e}), \quad \Phi_{22}(f) := f(\sqrt{ee^*}),$$
$$\Phi_{21}(f) := \Phi(f), \quad \Phi_{12}(f) := \Phi(\bar{f})^*.$$

We claim that, for $f, g \in C^{\mathbb{C}}(K)$ and $i, j, k, l \in \{1, 2\}$, we have

$$\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(fg) \ \text{ if } \ j = k,$$

and

$$\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(u^{-1}fg) \ \text{ if } \ j \neq k.$$

Indeed, the equality $\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(fg)$ for $j = k$ follows from Lemma 4.3.5(iii). To realize that $\Phi_{ij}(f)\Phi_{kl}(g) = \Phi_{il}(u^{-1}fg)$ if $j \neq k$, keep in mind that e is an idempotent, and take f (respectively, g) of the form $p(u^2)$ for some $p \in \mathbb{C}[\mathbf{x}]$ with $p(0) = 0$, if $i = j$ (respectively, $k = l$), and of the form $up(u^2)$ for some $p \in \mathbb{C}[\mathbf{x}]$, otherwise. Now that the claim is proved, it is clear that the mapping $F : \mathscr{A}(K) \to A$ defined by

$$F((f_{ij})) := \Phi_{11}(f_{11}) + \Phi_{12}(f_{12}) + \Phi_{21}(f_{21}) + \Phi_{22}(f_{22})$$

becomes a continuous algebra $*$-homomorphism satisfying $F(u[21]) = e$. Moreover, both the uniqueness of F under the above conditions, and that the closure in A of the range of F coincides with the closed $*$-subalgebra of A generated by e, follow from Lemma 4.3.4. Thus assertions (ii) and (iii) have been proved.

Assertions (iv) and (v) are proved together in the next two paragraphs.

Let (f_{ij}) be in $\mathscr{A}(K)$. Then we have:

$$[11](f_{ij})[11] = (f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + u^{-2}f_{22})[11],$$
$$[12](f_{ij})[12] = (u^{-1}f_{11} + u^{-2}f_{12} + f_{21} + u^{-1}f_{22})[12],$$
$$[21](f_{ij})[21] = (u^{-1}f_{11} + f_{12} + u^{-2}f_{21} + u^{-1}f_{22})[21],$$
$$[22](f_{ij})[22] = (u^{-2}f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + f_{22})[22].$$

Assume that (f_{ij}) is in $\ker(F)$. Then, since $\ker(F)$ is an ideal of $\mathscr{A}(K)$, and, for all $i, j \in \{1,2\}$, the restriction of F to $C^{\mathbb{C}}(K)[ij]$ is an isometry, we deduce:

$$f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + u^{-2}f_{22} = 0,$$
$$u^{-1}f_{11} + u^{-2}f_{12} + f_{21} + u^{-1}f_{22} = 0,$$
$$u^{-1}f_{11} + f_{12} + u^{-2}f_{21} + u^{-1}f_{22} = 0,$$
$$u^{-2}f_{11} + u^{-1}f_{12} + u^{-1}f_{21} + f_{22} = 0.$$

Therefore, for every $t \in K$ we have:

$$t^2 f_{11}(t) + t f_{12}(t) + t f_{21}(t) + f_{22}(t) = 0,$$
$$t f_{11}(t) + f_{12}(t) + t^2 f_{21}(t) + t f_{22}(t) = 0,$$
$$t f_{11}(t) + t^2 f_{12}(t) + f_{21}(t) + t f_{22}(t) = 0,$$
$$f_{11}(t) + t f_{12}(t) + t f_{21}(t) + t^2 f_{22}(t) = 0.$$

As a first consequence, if 1 belongs to K, then

$$f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1) = 0.$$

On the other hand, keeping in mind that, for $t \in K \setminus \{1\}$, we have

$$\begin{vmatrix} t^2 & t & t & 1 \\ t & 1 & t^2 & t \\ t & t^2 & 1 & t \\ 1 & t & t & t^2 \end{vmatrix} = -(t^2 - 1)^4 \neq 0,$$

for such a t we deduce

$$f_{11}(t) = f_{12}(t) = f_{21}(t) = f_{22}(t) = 0.$$

Therefore, if either 1 does not belong to K, or 1 is an accumulation point of K, then

$$f_{11} = f_{12} = f_{21} = f_{22} = 0,$$

and hence F is injective.

Assume that 1 is an isolated point of K. Then the function $\chi : K \to \mathbb{C}$, defined by $\chi(1) := 1$ and $\chi(t) := 0$ for $t \in K \setminus \{1\}$, is continuous. Set

$$p := F(\chi[11]), \;\; q := F(\chi[22]), \;\text{ and }\; r := F(\chi[12]).$$

Since, for $i, j, k, l \in \{1,2\}$ the equalities

$$(\chi[ij])^* = \chi[ji] \;\text{ and }\; (\chi[ij])(\chi[kl]) = \chi[il]$$

hold, we have that p and q are self-adjoint idempotents of A satisfying $pqp=p$ and $qpq=q$, and that $pq=r$. But the equality $pqp=p$ implies $[(\mathbb{1}-q)p]^*[(\mathbb{1}-q)p]=p(\mathbb{1}-q)p=0$, and hence $p=qp=pq$. Analogously $q=pq=qp$. It follows $p=q=r=r^*$. Let (f_{ij}) be in $\mathscr{A}(K)$ vanishing at every $t\in K\setminus\{1\}$ and such that

$$f_{11}(1)+f_{12}(1)+f_{21}(1)+f_{22}(1)=0.$$

Then we have

$$(f_{ij})=f_{11}(1)(\chi[11])+f_{12}(1)(\chi[12])+f_{21}(1)(\chi[21])+f_{22}(1)(\chi[22]),$$

and hence $F((f_{ij}))=(f_{11}(1)+f_{12}(1)+f_{21}(1)+f_{22}(1))p=0$. □

As an immediate consequence of Propositions 4.3.3 and 4.3.6, we obtain the following.

Corollary 4.3.7 *Let K be a compact subset of $[1,\infty[$ whose maximum element is greater than 1, and let $\mathscr{F}:\mathscr{A}(K)\to C(K,M_2(\mathbb{C}))$ be the algebra $*$-homomorphism given by Proposition 4.3.3. Then we have:*

(i) *The closure in $C(K,M_2(\mathbb{C}))$ of the range of $\mathscr{F}$ coincides with the closed $*$-subalgebra of $C(K,M_2(\mathbb{C}))$ generated by η_K.*
(ii) *$\mathscr{F}$ is injective if and only if either 1 does not belong to K or 1 is an accumulation point of K.*
(iii) *If 1 is an isolated point of K, then $\ker(\mathscr{F})$ consists precisely of those matrices $(f_{ij})\in\mathscr{A}(K)$ which vanish at every $t\in K\setminus\{1\}$ and satisfy*

$$f_{11}(1)+f_{12}(1)+f_{21}(1)+f_{22}(1)=0.$$

The next fact follows straightforwardly from Wedderburn's theory. Nevertheless, for the sake of completeness, we give here an alternative proof.

Fact 4.3.8 *Let A be a C^*-algebra of dimension $\leqslant 3$. Then A is commutative.*

Proof As any finite-dimensional C^*-algebra, A has a unit $\mathbf{1}$. (Thinking about complete arguments, the most elementary proof of this result could be to apply Lemma 2.6.33 to the Jordan real algebra $H(A,*)$, and then to invoke Lemma 3.6.10.) Now, since $\dim(A)\leqslant 3$, there are $h,k\in H(A,*)$ such that A equals the linear hull of $\{\mathbf{1},h,k\}$. Writing $[h,k]=\alpha\mathbf{1}+\beta h+\gamma k$ with $\alpha,\beta,\gamma\in\mathbb{C}$, we have $[[h,k],h]=-\gamma[h,k]$, and hence $[[h,k],[[h,k],h]]=0$. Therefore, by applying both Proposition 3.6.49 and Lemma 1.2.12 twice, we obtain that $[h,k]=0$, which implies the commutativity of A. □

Lemma 4.3.9 *Let X be a complex normed space, let E be a compact Hausdorff topological space, and let f be a function from E to $\mathbb{C}$ such that there are continuous mappings $\alpha,\beta:E\to X$ satisfying $\beta(t)\neq 0$ and $\alpha(t)=f(t)\beta(t)$ for every $t\in E$. Then f is continuous.*

Proof Set $M:=\max\{\|\alpha(t)\|:t\in E\}$ and $m:=\min\{\|\beta(t)\|:t\in E\}$. Then we have $m>0$, and hence $|f(t)|\leqslant m^{-1}M$ for every $t\in E$, so that f is bounded. Let t be in E, and let t_λ be a net in E converging to t. Take a cluster point z of the net $f(t_\lambda)$ in $\mathbb{C}$. Then $(z,\beta(t))$ is a cluster point of the net $(f(t_\lambda),\beta(t_\lambda))$ in $\mathbb{C}\times X$, hence $z\beta(t)$ is a cluster point of the net $f(t_\lambda)\beta(t_\lambda)$ in X, and therefore we have $\alpha(t) = z\beta(t)$, which implies (since $\beta(t) \neq 0$) $z = f(t)$. In this way we have shown that $f(t)$ is the unique cluster point of $f(t_\lambda)$ in $\mathbb{C}$. Since $f(t_\lambda)$ is bounded, we deduce that $f(t_\lambda)$ converges to $f(t)$. □

Lemma 4.3.10 *Let K be a compact subset of $]1,\infty[$. Then the $*$-homomorphism $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$, given by Proposition 4.3.3, is surjective. As a consequence, $C(K, M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra.*

Proof Let us fix $t \in K$. In view of Lemma 4.3.2, the linear hull of the set $\{\eta_{ij}(t) : i, j \in \{1,2\}\}$ is a $*$-subalgebra of $M_2(\mathbb{C})$. Moreover, since $t \in]1,\infty[$, such a subalgebra is not commutative (indeed, $\eta_{12}(t)$ does not commute with $\eta_{21}(t)$). It follows from Fact 4.3.8 that such a subalgebra is the whole algebra $M_2(\mathbb{C})$, and, consequently, that the set $\{\eta_{ij}(t) : i, j \in \{1,2\}\}$ becomes a basis of $M_2(\mathbb{C})$.

Let α be in $C(K, M_2(\mathbb{C}))$. It follows from the above that, for each $t \in K$, there are complex numbers $f_{11}(t), f_{12}(t), f_{21}(t), f_{22}(t)$ uniquely determined by the condition

$$\alpha(t) = f_{11}(t)\eta_{11}(t) + f_{12}(t)\eta_{12}(t) + f_{21}(t)\eta_{21}(t) + f_{22}(t)\eta_{22}(t). \tag{4.3.1}$$

Moreover, applying Lemma 4.3.2 again, for every $t \in K$ we have:

$$\begin{aligned}
\eta_{11}(t)\alpha(t)\eta_{11}(t) &= (f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + t^{-2}f_{22}(t))\eta_{11}(t),\\
\eta_{12}(t)\alpha(t)\eta_{12}(t) &= (t^{-1}f_{11}(t) + t^{-2}f_{12}(t) + f_{21}(t) + t^{-1}f_{22}(t))\eta_{12}(t),\\
\eta_{21}(t)\alpha(t)\eta_{21}(t) &= (t^{-1}f_{11}(t) + f_{12}(t) + t^{-2}f_{21}(t) + t^{-1}f_{22}(t))\eta_{21}(t),\\
\eta_{22}(t)\alpha(t)\eta_{22}(t) &= (t^{-2}f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + f_{22}(t))\eta_{22}(t).
\end{aligned}$$

Since, for $i, j \in \{1,2\}$, $\eta_{ij}^K\alpha\eta_{ij}^K$ and η_{ij}^K are continuous functions on K, and $\eta_{ij}(t) \neq 0$ for every $t \in K$, it follows from Lemma 4.3.9 that the mappings

$$\begin{aligned}
&t \to f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + t^{-2}f_{22}(t),\\
&t \to t^{-1}f_{11}(t) + t^{-2}f_{12}(t) + f_{21}(t) + t^{-1}f_{22}(t),\\
&t \to t^{-1}f_{11}(t) + f_{12}(t) + t^{-2}f_{21}(t) + t^{-1}f_{22}(t),\\
&t \to t^{-2}f_{11}(t) + t^{-1}f_{12}(t) + t^{-1}f_{21}(t) + f_{22}(t)
\end{aligned}$$

from K to $\mathbb{C}$ are continuous. Since, for $t \in K$ we have

$$\begin{vmatrix} 1 & t^{-1} & t^{-1} & t^{-2}\\ t^{-1} & t^{-2} & 1 & t^{-1}\\ t^{-1} & 1 & t^{-2} & t^{-1}\\ t^{-2} & t^{-1} & t^{-1} & 1 \end{vmatrix} = t^{-8}\begin{vmatrix} t^2 & t & t & 1\\ t & 1 & t^2 & t\\ t & t^2 & 1 & t\\ 1 & t & t & t^2 \end{vmatrix} = -t^{-8}(t^2-1)^4 \neq 0,$$

we deduce that, for all $i, j \in \{1,2\}$, the function $f_{ij} : t \to f_{ij}(t)$ from K to $\mathbb{C}$ is continuous. Therefore, we can consider the element (f_{ij}) of $\mathscr{A}(K)$, which, in view of (4.3.1), satisfies $\mathscr{F}((f_{ij})) = \alpha$. Since α is arbitrary in $C(K, M_2(\mathbb{C}))$, the surjectivity of $\mathscr{F}$ is proved. Now, it follows from assertion (i) in Corollary 4.3.7 that $C(K, M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra. □

Now we are ready to prove the first main result in this subsection.

Theorem 4.3.11 *Let A be a C^*-algebra, and let e be a non-self-adjoint idempotent in A. Set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$* (which, in view of Proposition 4.3.6, is a compact subset of $[1,\infty[$ whose maximum element is greater than 1)*, and assume that* 1 *does not belong to K. Then the closed $*$-subalgebra of A generated by e is $*$-isomorphic to $C(K, M_2(\mathbb{C}))$. More precisely, we have:*

(i) *There exists a unique algebra $*$-homomorphism $\Phi : C(K, M_2(\mathbb{C})) \to A$ such that $\Phi(\eta_K) = e$.*
(ii) *Such an algebra $*$-homomorphism is isometric, and its range coincides with the closed $*$-subalgebra of A generated by e.*

Proof Let $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$ and $F : \mathscr{A}(K) \to A$ be the algebra $*$-homomorphisms given by Propositions 4.3.3 and 4.3.6, respectively. By assertion (ii) in Corollary 4.3.7 (respectively, assertion (iv) in Proposition 4.3.6), $\mathscr{F}$ (respectively, F) is injective. On the other hand, by the first conclusion in Lemma 4.3.10, $\mathscr{F}$ is surjective. It follows that $\Phi := F \circ \mathscr{F}^{-1}$ is an injective algebra $*$-homomorphism from $C(K, M_2(\mathbb{C}))$ to A satisfying $\Phi(\eta_K) = e$. As any injective algebra $*$-homomorphism between C^*-algebras, Φ is isometric (cf. Corollary 1.2.52), and hence has closed range. Now, that Φ is the unique algebra $*$-homomorphism from $C(K, M_2(\mathbb{C}))$ to A satisfying $\Phi(\eta_K) = e$, and that the range of Φ coincides with the closed $*$-subalgebra of A generated by e, follow from the fact (given also by Lemma 4.3.10) that $C(K, M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra. $\square$

The following fact follows straightforwardly from Lemma 1.1.35 and the definition of quasi-invertible elements of an associative algebra (cf. Definition 3.6.19).

Fact 4.3.12 *Let A be an associative algebra over $\mathbb{K}$, and let a be in A. Then a is quasi-invertible in A if and only if there exists a unique $b \in A$ such that $a + b - ab = 0$.*

Lemma 4.3.13 *Let K be a compact subset of $[1,\infty[$ whose maximum element is greater than* 1*, and let $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$ be the algebra $*$-homomorphism given by Proposition 4.3.3. Then an element $x \in \mathscr{A}(K)$ is quasi-invertible in $\mathscr{A}(K)$ if and only if $\mathscr{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$.*

Proof Let $x = (f_{ij})$ be in $\mathscr{A}(K)$. We claim that x is quasi-invertible in $\mathscr{A}(K)$ if and only if $\lambda_x(t) \neq 0$ for every $t \in K$, where

$$\begin{aligned}\lambda_x(t) := {} & \frac{t^2 - 1}{t^2}(f_{11}(t)f_{22}(t) - f_{12}(t)f_{21}(t)) \\ & - \frac{1}{t}(f_{12}(t) + f_{21}(t)) - f_{11}(t) - f_{22}(t) + 1.\end{aligned}$$

Assume that x is quasi-invertible in $\mathscr{A}(K)$. Let us fix $t \in K$, and identify complex-valued continuous functions on $\{t\}$ with complex numbers. Then, since the restriction mapping $\mathscr{A}(K) \to \mathscr{A}(\{t\})$ is an algebra homomorphism, $(f_{ij}(t))$ is a quasi-invertible element of $\mathscr{A}(\{t\})$, and hence, by Fact 4.3.12, there are complex numbers $g_{11}(t), g_{12}(t), g_{21}(t), g_{22}(t)$ uniquely determined by the condition $(f_{ij}(t))(g_{ij}(t)) - (f_{ij}(t)) - (g_{ij}(t)) = 0$. This means that the linear system in the indeterminates $x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{C}$

$$\left.\begin{aligned}(f_{11}(t)+t^{-1}f_{12}(t)-1)x_{11}+(f_{12}(t)+t^{-1}f_{11}(t))x_{21} &= f_{11}(t)\\ (f_{11}(t)+t^{-1}f_{12}(t)-1)x_{12}+(f_{12}(t)+t^{-1}f_{11}(t))x_{22} &= f_{12}(t)\\ (f_{21}(t)+t^{-1}f_{22}(t))x_{11}+(f_{22}(t)+t^{-1}f_{21}(t)-1)x_{21} &= f_{21}(t)\\ (f_{21}(t)+t^{-1}f_{22}(t))x_{12}+(f_{22}(t)+t^{-1}f_{21}(t)-1)x_{22} &= f_{22}(t)\end{aligned}\right\} \tag{4.3.2}$$

has a unique solution (namely $x_{ij} = g_{ij}(t)$), and hence that the principal determinant of the system (by the way, equal to $\lambda_x(t)^2$) is nonzero. Conversely, assume that $\lambda_x(t) \neq 0$ for every $t \in K$. Then, for each $t \in K$, the system (4.3.2) has a unique solution $x_{ij} = g_{ij}(t)$, and, since the function $t \to \lambda_x(t)$ from K to $\mathbb{C}$ is continuous, the functions $g_{ij} : t \to g_{ij}(t)$ from K to $\mathbb{C}$ are continuous. Then we easily realize that $y := (g_{ij}) \in \mathscr{A}(K)$ is the unique element of $\mathscr{A}(K)$ satisfying $xy - x - y = 0$, which implies that x is quasi-invertible in $\mathscr{A}(K)$ because of Fact 4.3.12. Now, the claim is proved.

On the other hand, $\mathscr{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$ if and only if $\mathbf{1} - \mathscr{F}(x)$ is invertible in $C(K, M_2(\mathbb{C}))$, if and only if $\mathbf{1} - \mathscr{F}(x)(t)$ is invertible in $M_2(\mathbb{C})$ for every $t \in K$, if and only if $\det(\mathbf{1} - \mathscr{F}(x)(t)) \neq 0$ for every $t \in K$, where $\det(\cdot)$ means determinant. But, for $t \in K$, a straightforward but tedious computation shows that $\det(\mathbf{1} - \mathscr{F}(x)(t)) = \lambda_x(t)$. Therefore, $\mathscr{F}(x)$ is quasi-invertible in $C(K, M_2(\mathbb{C}))$ if and only if $\lambda_x(t) \neq 0$. By invoking the claim proved in the preceding paragraph, the result follows. □

§4.3.14 Let K be a compact subset of $[1, \infty[$ with $1 \in K$, and let p be a self-adjoint idempotent in $M_2(\mathbb{C})$, different from 0 and $\mathbf{1}$. Then $\mathbb{C}p$ is a self-adjoint subalgebra of $M_2(\mathbb{C})$, and hence

$$C_p(K, M_2(\mathbb{C})) := \{\alpha \in C(K, M_2(\mathbb{C})) : \alpha(1) \in \mathbb{C}p\}$$

is a proper closed $*$-subalgebra of $C(K, M_2(\mathbb{C}))$. We note that, in the construction of the C^*-algebra $C_p(K, M_2(\mathbb{C}))$, the choice of the idempotent p is structurally irrelevant. Indeed, if, for $i \in \{1, 2\}$, p_i is a self-adjoint idempotent in $M_2(\mathbb{C})$, different from 0 and $\mathbf{1}$, then there exists a norm-one element χ_i in the Hilbert space $\mathbb{C}^2$ such that p_i is the operator $\chi \to (\chi|\chi_i)\chi_i$ on $\mathbb{C}^2$, and hence, since there exists a unitary element $v \in M_2(\mathbb{C})$ with $v(\chi_1) = \chi_2$ (by Lemma 2.7.36 and Exercise 1.2.19), the mapping $\alpha \to v\alpha v^*$ becomes an algebra $*$-automorphism of $C(K, M_2(\mathbb{C}))$ sending $C_{p_1}(K, M_2(\mathbb{C}))$ onto $C_{p_2}(K, M_2(\mathbb{C}))$. We also note that, if we take $p = \eta(1)$, then $C_p(K, M_2(\mathbb{C}))$ contains η_K.

Lemma 4.3.15 *Let K be a compact subset of $[1, \infty[$ with $1 \in K$, and whose maximum element is greater than 1, and let $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$ be the algebra $*$-homomorphism given by Proposition 4.3.3. Then the closure in $C(K, M_2(\mathbb{C}))$ of the range of $\mathscr{F}$ coincides with $C_{\eta(1)}(K, M_2(\mathbb{C}))$. As a consequence, $C_{\eta(1)}(K, M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra.*

Proof For $x = (f_{ij})$ in $\mathscr{A}(K)$, we have

$$\mathscr{F}(x)(1) = (f_{11}(1) + f_{12}(1) + f_{21}(1) + f_{22}(1))\eta(1) \in \mathbb{C}\eta(1),$$

and therefore $\mathscr{F}(x)$ lies in $C_{\eta(1)}(K, M_2(\mathbb{C}))$. This shows that the range of $\mathscr{F}$ (say B) is contained in $C_{\eta(1)}(K, M_2(\mathbb{C}))$.

To continue our argument, we identify $C(K, M_2(\mathbb{C}))$ with $C^{\mathbb{C}}(K) \otimes M_2(\mathbb{C})$ in the natural manner. Then we have:

$$2(\mathbf{1} \otimes \eta(1)) = \mathbf{1} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = (\mathbf{1} + u)^{-1} u(\eta_{11}^K + \eta_{12}^K + \eta_{21}^K + \eta_{22}^K) \in B, \tag{4.3.3}$$

$$\sqrt{u^2 - \mathbf{1}} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = u(\eta_{21}^K - \eta_{12}^K) \in B, \tag{4.3.4}$$

$$\sqrt{u^2 - \mathbf{1}} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = u(\eta_{22}^K - \eta_{11}^K) \in B, \tag{4.3.5}$$

$$(u^2 - \mathbf{1}) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u^2(\eta_{22}^K + \eta_{11}^K) - u(\eta_{12}^K + \eta_{21}^K) \in B. \tag{4.3.6}$$

Now, keep in mind that B is a $C^{\mathbb{C}}(K)$-submodule of $C(K, M_2(\mathbb{C}))$, and denote by $C_1(K)$ the closed ideal of $C^{\mathbb{C}}(K)$ consisting of those complex-valued continuous functions on K vanishing at 1. It follows from (4.3.3) that

$$C^{\mathbb{C}}(K) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \subseteq B,$$

and, by invoking the Stone–Weierstrass theorem, it follows from (4.3.4), (4.3.5), and (4.3.6) that

$$C_1(K) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \subseteq \bar{B}, \quad C_1(K) \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \subseteq \bar{B},$$

and

$$C_1(K) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \subseteq \bar{B}.$$

Since

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{C})$, we deduce that $C_1(K) \otimes M_2(\mathbb{C}) \subseteq \bar{B}$. Since

$$C_{\eta(1)}(K, M_2(\mathbb{C})) = [\mathbb{C}\mathbf{1} \otimes \eta(1)] \oplus [C_1(K) \otimes M_2(\mathbb{C})],$$

and $\mathbb{C}\mathbf{1} \otimes \eta(1) \subseteq B$ (by (4.3.3)), we obtain $C_{\eta(1)}(K, M_2(\mathbb{C})) \subseteq \bar{B}$. By invoking the first paragraph in the present proof, we have $C_{\eta(1)}(K, M_2(\mathbb{C})) = \bar{B}$.

Now, it follows from assertion (i) in Corollary 4.3.7 that $C_{\eta(1)}(K, M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra. □

Now we are ready to prove the second main result in this subsection.

Theorem 4.3.16 *Let A be a C^*-algebra, and let e be a non-self-adjoint idempotent in A. Set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^* e}) \setminus \{0\}$* (which, in view of Proposition 4.3.6, is a compact subset of $[1, \infty[$ whose maximum element is greater than 1)*, and assume that 1 belongs to K. Then the closed $*$-subalgebra of A generated by e is $*$-isomorphic* *to $C_p(K,M_2(\mathbb{C}))$ for any self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from* 0 *and* $\mathbf{1}$. *More precisely, we have:*

(i) *There exists a unique algebra $*$-homomorphism $\Phi : C_{\eta(1)}(K,M_2(\mathbb{C})) \to A$ such that $\Phi(\eta_K) = e$.*
(ii) *Such an algebra $*$-homomorphism is isometric, and its range coincides with the closed $*$-subalgebra of A generated by e.*

Proof Let $\mathscr{F} : \mathscr{A}(K) \to C(K,M_2(\mathbb{C}))$ and $F : \mathscr{A}(K) \to A$ be the algebra $*$-homomorphisms given by Propositions 4.3.3 and 4.3.6, respectively, and let x be in $\mathscr{A}(K)$. Then, by Lemma 1.2.12 and Corollary 1.1.114, we have

$$\|F(x)\|^2 = \|F(x)^*F(x)\| = \mathfrak{r}(F(x^*x)) \leqslant \mathfrak{r}(x^*x).$$

On the other hand, by Lemma 4.3.13, we have

$$\mathrm{sp}(\mathscr{A}(K)_{\mathbb{1}}, x) = \mathrm{sp}(C(K,M_2(\mathbb{C}))_{\mathbb{1}}, \mathscr{F}(x)),$$

so $\mathfrak{r}(x) = \mathfrak{r}(\mathscr{F}(x))$ (by Theorem 1.1.46), and so

$$\mathfrak{r}(x^*x) = \mathfrak{r}(\mathscr{F}(x^*x)) = \mathfrak{r}(\mathscr{F}(x)^*\mathscr{F}(x)) = \|\mathscr{F}(x)^*\mathscr{F}(x)\| = \|\mathscr{F}(x)\|^2.$$

It follows $\|F(x)\| \leqslant \|\mathscr{F}(x)\|$. Therefore $\mathscr{F}(x) \to F(x)$ ($x \in \mathscr{A}(K)$) becomes a (well-defined) continuous algebra $*$-homomorphism from the range of $\mathscr{F}$ to A. Then, by the first conclusion in Lemma 4.3.15, such an algebra $*$-homomorphism extends by continuity to an algebra $*$-homomorphism

$$\Phi : C_{\eta(1)}(K,M_2(\mathbb{C})) \longrightarrow A$$

satisfying $\Phi \circ \mathscr{F} = F$, and hence $\Phi(\eta_K) = e$. Now, that Φ is the unique algebra $*$-homomorphism from $C_{\eta(1)}(K,M_2(\mathbb{C}))$ to A satisfying $\Phi(\eta_K) = e$, and that the range of Φ coincides with the closed $*$-subalgebra of A generated by e, follow from the fact (also given by Lemma 4.3.15) that $C_{\eta(1)}(K,M_2(\mathbb{C}))$ is generated by η_K as a normed $*$-algebra.

To conclude the proof, it is enough to show that Φ is injective. Let α be in $\ker(\Phi)$. Then, by the first conclusion in Lemma 4.3.15, there exists a sequence $x_n = (f_{ij}^n)$ in $\mathscr{A}(K)$ such that $\mathscr{F}(x_n) \to \alpha$. For $n \in \mathbb{N}$ and $i,j \in \{1,2\}$, define $g_{ij}^n \in C^{\mathbb{C}}(K)$ by

$$\begin{aligned}
g_{11}^n &:= f_{11}^n + u^{-1}f_{12}^n + u^{-1}f_{21}^n + u^{-2}f_{22}^n,\\
g_{12}^n &:= u^{-1}f_{11}^n + u^{-2}f_{12}^n + f_{21}^n + u^{-1}f_{22}^n,\\
g_{21}^n &:= u^{-1}f_{11}^n + f_{12}^n + u^{-2}f_{21}^n + u^{-1}f_{22}^n,\\
g_{22}^n &:= u^{-2}f_{11}^n + u^{-1}f_{12}^n + u^{-1}f_{21}^n + f_{22}^n.
\end{aligned}$$

Then we have $[ij]x_n[ij] = g_{ij}^n[ij]$. Now, since the restriction of F to $C^{\mathbb{C}}(K)[ij]$ is an isometry (by the proof of Proposition 4.3.6), we deduce

$$\begin{aligned}
\|g_{ij}^n\| = \|g_{ij}^n[ij]\| = \|F(g_{ij}^n[ij])\| &= \|F([ij]x_n[ij])\| = \|F([ij])F(x_n)F([ij])\|\\
&= \|F([ij])\Phi(\mathscr{F}(x_n))F([ij])\| \to \|F([ij])\Phi(\alpha)F([ij])\| = 0.
\end{aligned}$$

As a consequence, $g_{ij}^n(t) \to 0$ for every $t \in K$. Since for $t \in K \setminus \{1\}$, we have

$$\begin{vmatrix} 1 & t^{-1} & t^{-1} & t^{-2} \\ t^{-1} & t^{-2} & 1 & t^{-1} \\ t^{-1} & 1 & t^{-2} & t^{-1} \\ t^{-2} & t^{-1} & t^{-1} & 1 \end{vmatrix} = -t^{-8}(t^2-1)^4 \neq 0,$$

it follows from the definition of g_{ij}^n that $f_{ij}^n(t) \to 0$ for every $t \in K \setminus \{1\}$. Now, since for $t \in K \setminus \{1\}$ we have $\mathscr{F}(x_n)(t) \to \alpha(t)$ and

$$\mathscr{F}(x_n)(t) = \sum_{i,j \in \{1,2\}} f_{ij}^n(t)\eta_{ij}(t) \to 0,$$

for such a t we obtain $\alpha(t) = 0$. Therefore, if 1 is an accumulation point of K, then $\alpha = 0$, as desired. Assume that 1 is an isolated point of K. Then the function

$$\chi : K \to \mathbb{C}, \text{ defined by } \chi(1) := 1 \text{ and } \chi(t) := 0 \text{ for} t \in K \setminus \{1\},$$

is continuous, and, since there exists $\lambda \in \mathbb{C}$ such that $\alpha(1) = \lambda\eta(1)$, for such a λ we have $\alpha = \lambda\chi\eta_{21}^K = \mathscr{F}(\lambda\chi[21])$. Therefore

$$0 = \Phi(\alpha) = \Phi(\mathscr{F}(\lambda\chi[21])) = F(\lambda\chi[21])),$$

which, in view of assertion (v) in Proposition 4.3.6, implies $\lambda = 0$, and hence $\alpha = 0$. □

Now we combine Theorems 4.3.11 and 4.3.16 to derive some attractive consequences.

Corollary 4.3.17 *Let A be a C^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent e. Then A is generated, as a normed algebra, by two self-adjoint idempotents p_1 and p_2. Moreover p_1 and p_2 can be chosen so as to satisfy*

$$ep_1 = p_2 e = e, \ \ p_1 e = p_1, \ \ \text{and} \ \ ep_2 = p_2. \tag{4.3.7}$$

Proof By Theorems 4.3.11 and 4.3.16, we may assume that A is of the form $C(K, M_2(\mathbb{C}))$ or $C_{\eta(1)}(K, M_2(\mathbb{C}))$ (where, in the first case, K is a compact subset of $]1,\infty[$ and, in the second case, K is a compact subset of $[1,\infty[$ not reduced to a point and such that $1 \in K$) and that $e = \eta_K$. Set $p_1 := \eta_{11}^K$ and $p_2 := \eta_{22}^K$, and note that, in the case $1 \in K$, p_1 and p_2 lie in $C_{\eta(1)}(K, M_2(\mathbb{C}))$. According to Lemma 4.3.2, p_1 and p_2 are self-adjoint idempotents, and for every $t \in K$ we have $p_2(t)p_1(t) = t^{-2}e(t)$, so $(p_2(t)p_1(t))^m = t^{-2m}e(t)$ for every $m \in \mathbb{N}$, and so $q(p_2(t)p_1(t)) = q(t^{-2})e(t)$ for every $q \in \mathbb{C}[\mathbf{x}]$ with $q(0) = 0$. Now invoke the Stone–Weierstrass theorem (cf. Corollary 1.2.53) to find a sequence q_n in $\mathbb{C}[\mathbf{x}]$ with $q_n(0) = 0$ for every n, and $q_n(t^{-2}) \to 1$ uniformly on K. Then we have

$$\begin{aligned} \|q_n(p_2 p_1) - e\| &= \max\{\|q_n(p_2(t)p_1(t)) - e(t)\| : t \in K\} \\ &= \max\{|q_n(t^{-2}) - 1|\|e(t)\| : t \in K\} \\ &\leqslant \max\{|q_n(t^{-2}) - 1| : t \in K\}\|e\| \longrightarrow 0. \end{aligned}$$

Therefore e belongs to the closed (automatically $*$-invariant) subalgebra of A generated by p_1 and p_2. Since A is generated by e as a normed $*$-algebra, the first conclusion in the corollary follows. Finally, the equalities (4.3.7) follow from Lemma 4.3.2. □

Corollary 4.3.18 *Let A be a C^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent e, and set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$. If 1 is an isolated point of the compact set K, then A is $*$-isomorphic to the C^*-algebra $\mathbb{C} \times C(K \setminus \{1\}, M_2(\mathbb{C}))$.*

Proof If 1 belongs to K, then, for each $\alpha \in C_{\eta(1)}(K, M_2(\mathbb{C}))$, there exists a unique complex number $\lambda(\alpha)$ such that $\alpha(1) = \lambda(\alpha)\eta(1)$, and the mapping $\alpha \to (\lambda(\alpha), \alpha_{|K\setminus\{1\}})$ becomes an injective algebra $*$-homomorphism from $C_{\eta(1)}(K, M_2(\mathbb{C}))$ to $\mathbb{C} \times C_b(K \setminus \{1\}, M_2(\mathbb{C}))$, where $C_b(K \setminus \{1\}, M_2(\mathbb{C}))$ stands for the C^*-algebra of all bounded continuous functions from $K \setminus \{1\}$ to $M_2(\mathbb{C})$. Moreover, if 1 is in fact an isolated point of K, then we have that

$$C_b(K \setminus \{1\}, M_2(\mathbb{C})) = C(K \setminus \{1\}, M_2(\mathbb{C})),$$

and that the above algebra $*$-homomorphism is surjective. Finally, apply Theorem 4.3.16. □

Corollary 4.3.19 *Let A be a C^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent e, and set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$. Then A has a unit if and only if either* 1 *does not belong to K or* 1 *is an isolated point of K.*

Proof In view of Theorems 4.3.11 and 4.3.16, and Corollary 4.3.18, it is enough to show that, if 1 is an accumulation point of K, then $C_{\eta(1)}(K, M_2(\mathbb{C}))$ does not have a unit. Assume that $1 \in K$. We claim that, given $t_0 \in K \setminus \{1\}$, the valuation at t_0 (as a mapping from $C_{\eta(1)}(K, M_2(\mathbb{C}))$ to $M_2(\mathbb{C})$) is surjective. Indeed, if $a = (\lambda_{ij})$ is an arbitrary element of $M_2(\mathbb{C})$, then, for $i, j \in \{1, 2\}$, there exists $f_{ij} \in C^{\mathbb{C}}(K)$ such that $f_{ij}(1) = 0$ and $f_{ij}(t_0) = \lambda_{ij}$, and hence the element α of $C(K, M_2(\mathbb{C}))$, defined by $\alpha(t) := (f_{ij}(t))$ for every $t \in K$, belongs to $C_{\eta(1)}(K, M_2(\mathbb{C}))$ and satisfies $\alpha(t_0) = a$. Assume in addition that $C_{\eta(1)}(K, M_2(\mathbb{C}))$ has a unit $\mathbf{1}$. Then, by the claim just proved, for every $t \in K \setminus \{1\}$, $\mathbf{1}(t)$ must be equal to the unit of $M_2(\mathbb{C})$. Now, if 1 is in fact an accumulation point of K, then $\mathbf{1}(1)$ is the unit of $M_2(\mathbb{C})$, which is not possible because $\mathbf{1}(1)$ is a complex multiple of $\eta(1)$. □

Corollary 4.3.20 *Let A be a C^*-algebra. Then A has a non-self-adjoint idempotent (if and) only if it contains (as a closed $*$-subalgebra) a copy of either $M_2(\mathbb{C})$ or $C_p([1,2], M_2(\mathbb{C}))$ for any self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from* 0 *and* $\mathbf{1}$.

Proof In order to prove the ‘only if’ part, we assume that A has a non-self-adjoint idempotent e, and note that we may suppose in addition that A is generated by e as a normed $*$-algebra. Set $K := \mathrm{sp}(A_{\mathbb{1}}, \sqrt{e^*e}) \setminus \{0\}$. If 1 does not belong to K, then, by Theorem 4.3.11, A contains a copy of $M_2(\mathbb{C})$. Assume that 1 belongs to K, and that K is disconnected. Take a clopen proper subset U of K with $1 \in U$. Then, arguing as in the proof of Corollary 4.3.18, we realize that A is $*$-isomorphic to $C_p(U, M_2(\mathbb{C})) \times C(K \setminus U, M_2(\mathbb{C}))$, for some self-adjoint idempotent $p \in M_2(\mathbb{C})$ different from 0 and $\mathbf{1}$, and hence it contains a copy of $M_2(\mathbb{C})$. Finally, assume that

1 belongs to K, and that K is connected. Then we have $K = [1, \|e\|]$, and therefore, by Theorem 4.3.16, A is isomorphic to $C_p([1, \|e\|], M_2(\mathbb{C}))$, for some p as above. But, taking a homeomorphism ϕ from $[1, \|e\|]$ onto $[1,2]$ with $\phi(1) = 1$, ϕ induces a $*$-isomorphism from $C([1, \|e\|], M_2(\mathbb{C}))$ onto $C([1,2], M_2(\mathbb{C}))$ sending $C_p([1, \|e\|], M_2(\mathbb{C}))$ onto $C_p([1,2], M_2(\mathbb{C}))$. □

§**4.3.21** We remark that $C_p([1,2], M_2(\mathbb{C}))$ does not contain any copy of $M_2(\mathbb{C})$. To realize this, we argue by contradiction. Assume that $C_p([1,2], M_2(\mathbb{C}))$ contains a copy (say B) of $M_2(\mathbb{C})$, and, for $\alpha \in C_p([1,2], M_2(\mathbb{C}))$, let $\lambda(\alpha)$ stand for the unique complex number satisfying $\alpha(1) = \lambda(\alpha)p$. Then, since the mapping $\lambda : C_p([1,2], M_2(\mathbb{C})) \to \mathbb{C}$ is an algebra homomorphism, by the simplicity of B we have $\lambda(B) = 0$. Therefore B is contained in the ideal (say M) of $C([1,2], M_2(\mathbb{C}))$ consisting of those continuous functions from $[1,2]$ to $M_2(\mathbb{C})$ vanishing at 1. Now, since M has no nonzero idempotent (by Fact 1.1.6), and the unit of B is a nonzero idempotent of M, the contradiction is clear.

In relation to Proposition 4.3.22 immediately below, we note that, as a by-product of Theorems 4.3.11 and 4.3.16, non-self-adjoint idempotents in a C^*-algebra are non-normal, hence non-central. Indeed, for every $t \in]1, \infty[$ we have $[\eta(t), \eta(t)^*] \neq 0$ in $M_2(\mathbb{C})$.

Proposition 4.3.22 *Let A be a C^*-algebra containing a non-central idempotent e. Then there exists a continuous mapping $r \to e_r$ from $[1, \infty[$ to the set of idempotents of A satisfying $e_{\|e\|} = e$ and $\|e_r\| = r$ for every $r \in [1, \infty[$.*

Proof First assume that e is not self-adjoint. Then, by Theorems 4.3.11 and 4.3.16, we may assume that A is of the form $C(K, M_2(\mathbb{C}))$ or $C_{\eta(1)}(K, M_2(\mathbb{C}))$ (where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ not reduced to a point and such that $1 \in K$) and that $e = \eta_K$. In any case, set $\rho := \max K > 1$. Let r be in $[1, \infty[$, let e_r denote the element of $C(K, M_2(\mathbb{C}))$ defined by

$$e_r(t) := \eta\left(1 + \frac{(r-1)(t-1)}{\rho - 1}\right)$$

for every $t \in K$, and note that $e_\rho = \eta_K$. Noticing that, in the case where 1 belongs to K, e_r lies in $C_{\eta(1)}(K, M_2(\mathbb{C}))$, it turns out that in any case e_r is an element of A. Moreover, keeping in mind Lemma 4.3.2, we easily realize that e_r is an idempotent, and that $\|e_r\| = r$. On the other hand, since $\|\eta_K\| = \rho$, we have $e_{\|\eta_K\|} = \eta_K$. Now it only remains to show that the mapping $r \to e_r$ is continuous. Fix $r \in [1, \infty[$ and $\varepsilon > 0$, and take $\delta > 0$ such that $\|\eta(s) - \eta(r)\| < \varepsilon$ whenever s is in $[1, \infty[$ with $|s - r| < \delta$. Then, for $s \in [1, \infty[$ with $|s - r| < \delta$, we have

$$\left| \left[1 + \frac{(s-1)(t-1)}{\rho - 1}\right] - \left[1 + \frac{(r-1)(t-1)}{\rho - 1}\right] \right| = \frac{|s-r|(t-1)}{\rho - 1} \leqslant |s - r| < \delta$$

for every $t \in K$, so $\|e_s(t) - e_r(t)\| < \varepsilon$ for every $t \in K$, and so $\|e_s - e_r\| \leqslant \varepsilon$.

Now assume that e is self-adjoint. Since e is non-central, we may choose a self-adjoint element $a \in A$ with $ea - ae \neq 0$. Then the mapping $D : A \to A$ defined by $D(b) := ba - ab$ for every $b \in A$ becomes a continuous derivation satisfying $D(e) \neq 0$

and $D(b^*) = -D(b)^*$ for every $b \in A$. Therefore, for $s \in \mathbb{R}$, $\exp(sD)$ is a continuous algebra automorphism of A satisfying

$$[\exp(sD)(b)]^* = \exp(-sD)(b^*)$$

for every $b \in A$, and consequently $g(s) := \exp(sD)(e)$ is a nonzero idempotent in A, and we have

$$g(s)^* = g(-s). \tag{4.3.8}$$

Now, let $f : \mathbb{R} \to [1,\infty[$ be the continuous mapping defined by $f(s) := \|g(s)\|$. By (4.3.8), we have

$$f(-s) = f(s) \tag{4.3.9}$$

for every $s \in \mathbb{R}$. Let r,s be in $\mathbb{R}$. Then, keeping in mind (4.3.8), (4.3.9), and that $\exp\left(\frac{s-r}{2}D\right)$ is an automorphism of A, we have

$$\begin{aligned}
f\left(\frac{r+s}{2}\right)^2 &= \left\|g\left(\frac{r+s}{2}\right)\right\|^2 = \left\|g\left(\frac{r+s}{2}\right)^* g\left(\frac{r+s}{2}\right)\right\| \\
&= \mathfrak{r}\left(g\left(\frac{r+s}{2}\right)^* g\left(\frac{r+s}{2}\right)\right) = \mathfrak{r}\left(g\left(-\frac{r+s}{2}\right) g\left(\frac{r+s}{2}\right)\right) \\
&= \mathfrak{r}\left[\exp\left(\frac{s-r}{2}D\right)\left(g\left(-\frac{r+s}{2}\right) g\left(\frac{r+s}{2}\right)\right)\right] \\
&= \mathfrak{r}\left[\left[\exp\left(\frac{s-r}{2}D\right)\left(g\left(-\frac{r+s}{2}\right)\right)\right]\left[\exp\left(\frac{s-r}{2}D\right)\left(g\left(\frac{r+s}{2}\right)\right)\right]\right] \\
&= \mathfrak{r}(g(-r)g(s)) \leqslant \|g(-r)\|\|g(s)\| = f(-r)f(s) = f(r)f(s),
\end{aligned}$$

and therefore

$$f\left(\frac{r+s}{2}\right) \leqslant \sqrt{f(r)f(s)} \leqslant \frac{f(r)+f(s)}{2}.$$

In this way we have shown that f is convex. Assume that $f(r) = 1$ for some $r \in]0,\infty[$. Then, by (4.3.9) and the convexity of f, we have $f(s) = 1$ for every $s \in [-r,r]$. Therefore, for $s \in [-r,r]$, the idempotent $g(s)$ has norm equal to 1, so it is self-adjoint, and so, by (4.3.8) the equality $g(s) = g(-s)$ holds. Since g is differentiable at 0 with $g'(0) = D(e)$, the above implies $D(e) = 0$, which is a contradiction. Thus, $f(r) > 1$ for every $r \in]0,\infty[$. Now, let $0 < r < s$. Noticing that $f(0) = 1$ and that then, by the convexity of f, the mapping $t \to \frac{f(t)-1}{t}$ is increasing, we have

$$0 < f(r) - 1 < \frac{s}{r}(f(r) - 1) \leqslant f(s) - 1.$$

In this way, we have shown that $f_{|[0,\infty[}$ is strictly increasing and non-bounded. As a consequence, the range of $f_{|[0,\infty[}$ is $[1,\infty[$, and the inverse mapping $h : [1,\infty[\to [0,\infty[$ is continuous. Now, for $r \in [1,\infty[$, let e_r be the idempotent of A defined by $e_r := g(h(r))$. Then, clearly, the mapping $r \to e_r$ is continuous, and we have $e_1 = e$. Moreover, by the definition of g and h, we also have that $\|e_r\| = f(h(r)) = r$ for every $r \in [1,\infty[$. □

We recall that *partial isometries* in a C^*-algebra A are defined as those elements $a \in A$ satisfying $aa^*a = a$. Thus partial isometries in C^*-algebras are nothing other than the tripotents of their underlying JB^*-triples.

Lemma 4.3.23 *Let A be a C^*-algebra, and let a be a partial isometry in A such that both a^*a and aa^* lie in the centre of A. Then a is normal.*

Proof Since a^*a and aa^* lie in the centre of A, we have $[a^*a, a] = 0$ and $[aa^*, a] = 0$, which reads as $a^*a^2 = a$ and $a^2a^* = a$, respectively. The two last equalities, together with the one $aa^*a = a$, and those obtained by taking adjoints, imply $[[a, a^*], a] = 0$. Therefore, by Lemma 3.4.19, a is normal. □

Let A denote the C^*-algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space H, let $b : H \to H$ be any non-surjective linear isometry, and set $a := b$ (respectively $a := b^*$). Then a is a non-normal partial isometry in A such that a^*a (respectively, aa^*) lies in the centre of A.

Corollary 4.3.24 *Let A be a C^*-algebra. Then the following conditions are equivalent:*

(i) *A contains a non-self-adjoint idempotent.*
(ii) *There exists a non-normal partial isometry $a \in A$ such that a belongs to a^2Aa^2.*
(iii) *A contains a non-normal partial isometry.*
(iv) *A contains a non-central self-adjoint idempotent.*

Proof (i)⇒(ii) By the assumption (i) and Theorems 4.3.11 and 4.3.16, we may assume that A is of the form $C(K, M_2(\mathbb{C}))$ or $C_{\eta(1)}(K, M_2(\mathbb{C}))$, where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1 and such that $1 \in K$. In any case, by Lemma 4.3.2, η_{21}^K is a non-normal partial isometry in A, and we have $\eta_{21}^K = (\eta_{21}^K)^2(u^2\eta_{12}^K)(\eta_{21}^K)^2$.

(ii)⇒(iii) This is clear.

(iii)⇒(iv) Let a be the partial isometry whose existence is assumed in (iii). Then, keeping in mind that both a^*a and aa^* are self-adjoint idempotents, it follows from Lemma 4.3.23 that A contains a non-central self-adjoint idempotent.

(iv)⇒(i) Since nonzero self-adjoint idempotents of A are norm-one elements, this implication follows from Proposition 4.3.22. □

Set $A := M_2(\mathbb{C})$ and $a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then a is a non-normal partial isometry in A, which does not belong to a^2Aa^2.

4.3.2 The case of JB^*-algebras

Let K be a compact subset of $[1, \infty[$. Let $\Theta : \mathscr{A}(K) \to \mathscr{A}(K)$ be the linear mapping determined by

$$\Theta(f[ij]) := f[ij] \text{ if } i \neq j, \quad \Theta(f[11]) := f[22], \quad \Theta(f[22]) := f[11]$$

for every $f \in C^{\mathbb{C}}(K)$. Then Θ becomes an isometric involutive algebra $*$-antiautomorphism of $\mathscr{A}(K)$. Therefore, the set of fixed elements for Θ is a closed

$*$-subalgebra of $\mathscr{A}(K)^{\mathrm{sym}}$, and hence a complete normed Jordan complex $*$-algebra. Such a normed Jordan complex $*$-algebra will be denoted by $\mathscr{B}(K)$. Note that elements of $\mathscr{B}(K)$ are precisely those matrices $(f_{ij}) \in \mathscr{A}(K)$ satisfying $f_{11} = f_{22}$, or equivalently, those elements of $\mathscr{A}(K)$ of the form $f([11]+[22]) + g[12] + h[21]$ with $f, g, h \in C^{\mathbb{C}}(K)$.

Lemma 4.3.25 *Let K be a compact subset of $[1, \infty[$. Then $\mathscr{B}(K)$ is generated by $u[21]$ as a normed $*$-algebra.*

Proof Set $p := u[21] \in \mathscr{B}(K)$, and let B denote the closed $*$-subalgebra of $\mathscr{B}(K)$ generated by p. We have $u^2[11] = p^*p$ and $u^2[22] = pp^*$, which, in view of Lemma 3.3.5, implies for $n \in \mathbb{N}$ that

$$u^{2n+1}[21] = p(p^*p)^n \in B, \quad u^{2n+1}[12] = p^*(pp^*)^n \in B,$$

and

$$u^{2n}([11]+[22]) = (p^*p)^n + (pp^*)^n \in B.$$

Therefore, for every $q \in \mathbb{C}[\mathbf{x}]$, $uq(u^2)[21]$ and $uq(u^2)[12]$ lie in B, and, if $q(0) = 0$, then $q(u^2)([11]+[22])$ also lies in B. It follows that $C^{\mathbb{C}}(K)[21] \subseteq B$, $C^{\mathbb{C}}(K)[12] \subseteq B$, and $C^{\mathbb{C}}(K)([11]+[22]) \subseteq B$. This implies $\mathscr{B}(K) = B$. □

We recall that, if A is a C^*-algebra, then A^{sym} is a JB^*-algebra (cf. Facts 3.3.2 and 3.3.4).

Now consider the mapping

$$\vartheta : \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda_{22} & \lambda_{12} \\ \lambda_{21} & \lambda_{11} \end{pmatrix}$$

from $M_2(\mathbb{C})$ to itself. Then ϑ is an involutive algebra $*$-antiautomorphism of $M_2(\mathbb{C})$. Therefore, the set of fixed elements for ϑ is a $*$-subalgebra of the JB^*-algebra $M_2(\mathbb{C})^{\mathrm{sym}}$, and hence a JB^*-algebra. Such a JB^*-algebra is called the *three-dimensional spin factor*, and is denoted by $\mathscr{C}_3$. Without enjoying its name, the three-dimensional spin factor has already appeared in our development (cf. Examples 2.3.65 and 3.5.43).

Let K be a compact subset of $[1, \infty[$. According to Example 1.1.4(d), we denote by $C(K, \mathscr{C}_3)$ the algebra of all continuous functions from K to $\mathscr{C}_3$, and remark that $C(K, \mathscr{C}_3)$ naturally becomes a JB^*-algebra. We will identify $C(K, \mathscr{C}_3)$ with the closed $*$-subalgebra of $C(K, M_2(\mathbb{C}))^{\mathrm{sym}}$ consisting of those continuous functions from K to $M_2(\mathbb{C})$ whose range is contained in $\mathscr{C}_3$.

Lemma 4.3.26 *Let K be a compact subset of $[1, \infty[$ whose maximum element is greater than 1, let $\mathscr{F} : \mathscr{A}(K) \to C(K, M_2(\mathbb{C}))$ be the algebra $*$-homomorphism given by Proposition 4.3.3, and let $\mathscr{G}$ denote the restriction to $\mathscr{B}(K)$ of $\mathscr{F}$. Then $\mathscr{G}$ is an algebra $*$-homomorphism from $\mathscr{B}(K)$ to $C(K, M_2(\mathbb{C}))^{\mathrm{sym}}$, and the closure in $C(K, M_2(\mathbb{C}))$ of the range of $\mathscr{G}$ coincides with the closed $*$-subalgebra of $C(K, \mathscr{C}_3)$ generated by η_K.*

Proof Noticing that $\mathscr{G}(u[21]) = \eta_K$, and keeping in mind Lemma 4.3.25, it is enough to show that the range of $\mathscr{G}$ is contained in $C(K, \mathscr{C}_3)$. But this follows from the fact that η_K actually belongs to $C(K,\mathscr{C}_3)$, and from a new application of Lemma 4.3.25. $\square$

Lemma 4.3.27 *Let K be a compact subset of $]1,\infty[$. Then $C(K,\mathscr{C}_3)$ is generated by η_K as a normed $*$-algebra.*

Proof Identifying $C(K,M_2(\mathbb{C}))$ with $C^{\mathbb{C}}(K)\otimes M_2(\mathbb{C})$ in the natural manner, the operator $\hat{\vartheta} := I_{C^{\mathbb{C}}(K)}\otimes\vartheta$ becomes an involutive $*$-antiautomorphism of $C(K,M_2(\mathbb{C}))$, whose set of fixed points is precisely $C(K,\mathscr{C}_3)$. Moreover, since $\mathscr{A}(K)$ is generated by $u[21]$ as a normed $*$-algebra (by Lemma 4.3.4), and $\mathscr{F}(\Theta(u[21])) = \hat{\vartheta}(\mathscr{F}(u[21]))$, we have $\mathscr{F}\circ\Theta = \hat{\vartheta}\circ\mathscr{F}$. On the other hand, by Lemma 4.3.10, $\mathscr{F} : \mathscr{A}(K)\to C(K,M_2(\mathbb{C}))$ is surjective. Since $\mathscr{B}(K)$ is the set of fixed points for Θ, and $C(K,\mathscr{C}_3)$ is the set of fixed points for $\hat{\vartheta}$, and $\mathscr{G}$ is the restriction to $\mathscr{B}(K)$ of $\mathscr{F}$, it follows that $\mathscr{G}$ (as a mapping from $\mathscr{B}(K)\to C(K,\mathscr{C}_3)$) is surjective. Now, apply Lemma 4.3.26. $\square$

According to Definition 4.2.10, each nonzero element x of a JB^*-triple J has a triple spectrum $\sigma(x)\subseteq\mathbb{R}^+$ determined by Theorem 4.2.9. We note that $\sigma(x)$ does not change when we replace J with any closed subtriple of J containing x. Now, recalling that C^*-algebras are JB^*-triples in a natural way, and invoking assertions (i) and (ii) in Lemma 4.3.5, we get the following.

Corollary 4.3.28 *Let A be a C^*-algebra, and let a be a nonzero element in A. Then we have $\sigma(a) = \mathrm{sp}(A_{\mathbb{1}},\sqrt{a^*a})\setminus\{0\}$.*

We recall that JB^*-algebras become naturally JB^*-triples (cf. Theorem 4.1.45), and that, if A is a C^*-algebra, then A and the JB^*-algebra A^{sym} have the same underlying JB^*-triple.

Theorem 4.3.29 *Let B be a JB^*-algebra, and let e be a non-self-adjoint idempotent in B. Set $K := \sigma(e)$, and assume that 1 does not belong to K. Then K is a compact subset of $]1,\infty[$, and the closed $*$-subalgebra of B generated by e is $*$-isomorphic to $C(K,\mathscr{C}_3)$. More precisely, we have:*

(i) *There exists a unique algebra $*$-homomorphism $\Psi : C(K,\mathscr{C}_3)\to B$ such that $\Psi(\eta_K) = e$.*
(ii) *Such an algebra $*$-homomorphism is isometric, and its range coincides with the closed $*$-subalgebra of B generated by e.*

Proof Let B_e denote the closed $*$-subalgebra of B generated by e. By Proposition 3.4.6, there exists a C^*-algebra A such that B_e can be seen as a closed $*$-subalgebra of A^{sym}. Therefore, by Corollary 4.3.28 and Proposition 4.3.6, $K :=\sigma(e)$ is a compact subset of $]1,\infty[$. By Theorem 4.3.11, there exists an isometric algebra $*$-homomorphism $\Phi : C(K,M_2(\mathbb{C}))\to A$ such that $\Phi(\eta_K) = e$. Let Ψ stand for the restriction of Φ to $C(K,\mathscr{C}_3)$. Then, clearly, Ψ is an isometric algebra $*$-homomorphism from $C(K,\mathscr{C}_3)$ to A^{sym}, which satisfies $\Psi(\eta_K) = e$. Noticing that the closed $*$-subalgebras of A^{sym} and B generated by e coincide, it follows from Lemma 4.3.27 that the range of Ψ is B_e. This last fact allows us to see Ψ as an algebra $*$-homomorphism from $C(K,\mathscr{C}_3)$ to B. That Ψ is the unique (automatically continuous because of Proposition 3.4.4) algebra $*$-homomorphism from $C(K,\mathscr{C}_3)$ to B with $\Psi(\eta_K) = e$ follows from a new application of Lemma 4.3.27. □

§**4.3.30** Let K be a compact subset of $[1,\infty[$ with $1 \in K$, and let p be a self-adjoint idempotent in $\mathscr{C}_3$, different from 0 and $\mathbf{1}$. Then

$$C_p(K,\mathscr{C}_3) := \{\alpha \in C(K,\mathscr{C}_3) : \alpha(1) \in \mathbb{C}p\}$$

is a proper closed $*$-subalgebra of $C(K,\mathscr{C}_3)$. As in the case of the C^*-algebra $C_p(K,M_2(\mathbb{C}))$, the JB^*-algebra $C_p(K,\mathscr{C}_3)$ does not depend structurally on p. Indeed, it is straightforward that self-adjoint idempotents in $\mathscr{C}_3$, different from 0 and $\mathbf{1}$, are precisely the matrices $\frac{1}{2}\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix}$ with $\theta \in \mathbb{R}$. Therefore if, for $j \in \{1,2\}$, p_j is a self-adjoint idempotent in $\mathscr{C}_3$, different from 0 and $\mathbf{1}$, say $p_j = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta_j} \\ e^{-i\theta_j} & 1 \end{pmatrix}$ with $\theta_j \in \mathbb{R}$, and if we set $v := \begin{pmatrix} 0 & e^{i\theta_1} \\ e^{-i\theta_2} & 0 \end{pmatrix}$, then v becomes a unitary element of $M_2(\mathbb{C})$ satisfying $vp_1v^* = p_2$ and $v\mathscr{C}_3v^* = \mathscr{C}_3$, and hence the mapping $\alpha \to v\alpha v^*$ becomes an algebra $*$-automorphism of $C(K,\mathscr{C}_3)$ sending $C_{p_1}(K,\mathscr{C}_3)$ onto $C_{p_2}(K,\mathscr{C}_3)$.

Lemma 4.3.31 *Let K be a compact subset of $[1,\infty[$ with $1 \in K$, whose maximum element is greater than 1. Then $C_{\eta(1)}(K,\mathscr{C}_3)$ is generated by η_K as a normed $*$-algebra.*

Proof Argue as in the proof of Lemma 4.3.27, invoking Lemma 4.3.15 instead of Lemma 4.3.10. □

By invoking Theorem 4.3.16 and Lemma 4.3.31 instead of Theorem 4.3.11 and Lemma 4.3.27, respectively, the proof of the following theorem is similar to that of Theorem 4.3.29, and hence is omitted.

Theorem 4.3.32 *Let B be a JB^*-algebra, and let e be a non-self-adjoint idempotent in B. Set $K := \sigma(e)$, and assume that 1 belongs to K. Then K is a compact subset of $[1,\infty[$ whose maximum element is greater than 1, and the closed $*$-subalgebra of B generated by e is $*$-isomorphic to $C_p(K,\mathscr{C}_3)$ for any self-adjoint idempotent $p \in \mathscr{C}_3$ different from 0 and $\mathbf{1}$. More precisely, we have:*

(i) *There exists a unique algebra $*$-homomorphism $\Psi : C_{\eta(1)}(K,\mathscr{C}_3) \to B$ such that $\Psi(\eta_K) = e$.*
(ii) *Such an algebra $*$-homomorphism is isometric, and its range coincides with the closed $*$-subalgebra of B generated by e.*

Now we deal with the main corollaries to Theorems 4.3.29 and 4.3.32.

Lemma 4.3.33 *Let E be a locally compact Hausdorff topological space, let x be in $C_0^{\mathbb{R}}(E)$ such that $x(t) > 0$ for every $t \in E$, and for $n \in \mathbb{N}$ let x_n denote the element of $C_0^{\mathbb{R}}(E)$ defined by $x_n(t) := \frac{nx(t)}{1+nx(t)}$. Then the sequence x_n becomes an approximate unit for $C_0^{\mathbb{C}}(E)$.*

Proof Let y be in $C_0^{\mathbb{C}}(E)$, let $\varepsilon > 0$, and note that $|y(t) - y(t)x_n(t)| = \frac{|y(t)|}{1+nx(t)}$ for all $(t,n) \in E \times \mathbb{N}$. Then there exists a compact subset F of E such that $|y(t)| \leqslant \varepsilon$

whenever t lies in $E \setminus F$, and hence $|y(t) - y(t)x_n(t)| \leqslant \varepsilon$ for all $(t,n) \in (E \setminus F) \times \mathbb{N}$. Now set

$$M := \min\{x(t) : t \in F\} > 0,$$

and take $m \in \mathbb{N}$ such that $\frac{\|y\|}{1+nM} \leqslant \varepsilon$ whenever $n \geqslant m$. It follows that $\|y - yx_n\| \leqslant \varepsilon$ whenever $n \geqslant m$. $\square$

Corollary 4.3.34 *Let B be a JB^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent. Then B is generated, as a normed algebra, by two self-adjoint idempotents.*

Proof By Theorems 4.3.29 and 4.3.32, we may assume that B is of the form $C(K, \mathscr{C}_3)$ or $C_{\eta(1)}(K, \mathscr{C}_3)$, where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ not reduced to a point and such that $1 \in K$. Let $|K|$ stand for the maximum of K, and let $\theta : K \to \mathbb{R}$ be defined by $\theta(t) = \frac{\pi}{2|K|}(t-1)$. We note that $\cos\theta$ and $\sin\theta$ are injective functions on K, that $\cos\theta(t) > 0$ for every $t \in K$, and that $\sin\theta(t) > 0$ for every $t \in K \setminus \{1\}$. Now let p_1 and p_2 be the continuous functions from K to $\mathscr{C}_3$ defined by

$$p_1(t) := \eta(1) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } p_2(t) := \frac{1}{2}\begin{pmatrix} 1 & e^{2i\theta(t)} \\ e^{-2i\theta(t)} & 1 \end{pmatrix},$$

and note that, in the case $1 \in K$, p_1 and p_2 lie in $C_{\eta(1)}(K, \mathscr{C}_3)$, so that p_1 and p_2 become self-adjoint idempotents of B. In what follows C will denote the closed subalgebra of B generated by p_1 and p_2, and D will stand for either $\{f \in C^{\mathbb{C}}(K) : f(1) = 0\} \equiv C_0^{\mathbb{C}}(K \setminus \{1\})$ or $C^{\mathbb{C}}(K)$ depending on whether or not 1 lies in K.

Let (j,k) stand for either $(1,2)$ or $(2,1)$. Then we straightforwardly realize that, for every $t \in K$, we have $U_{p_j(t)}(p_k(t)) = \cos^2\theta(t)p_j(t)$, so

$$[U_{p_j(t)}(p_k(t))]^m = \cos^{2m}\theta(t)p_j(t) \text{ for every } m \in \mathbb{N},$$

and so $q[U_{p_j(t)}(p_k(t))] = q(\cos^2\theta(t))p_j(t)$ for every $q \in \mathbb{C}[\mathbf{x}]$ with $q(0) = 0$. Now let f be an arbitrary element of $C^{\mathbb{C}}(K)$, and invoke the Stone–Weierstrass theorem (cf. Corollary 1.2.53) to find a sequence q_n in $\mathbb{C}[\mathbf{x}]$ with $q_n(0) = 0$ for every n, and $q_n(\cos^2\theta(t)) \to f(t)$ uniformly on K. Then we have

$$\begin{aligned} \|q_n(U_{p_j}(p_k)) - fp_j\| &= \max\{\|q_n[U_{p_j(t)}(p_k(t))] - f(t)p_j(t)\| : t \in K\} \\ &= \max\left\{\left|q_n(\cos^2\theta(t)) - f(t)\right| \|p_j(t)\| : t \in K\right\} \\ &\leqslant \max\left\{\left|q_n(\cos^2\theta(t)) - f(t)\right| : t \in K\right\} \|p_j\| \to 0. \end{aligned}$$

Therefore fp_j belongs to C. Thus $C^{\mathbb{C}}(K)p_j \subseteq C$.

On the other hand, for every $t \in K$ we have $(p_1(t) - p_2(t))^2 = \sin^2\theta(t)\mathbf{1}$ in $\mathscr{C}_3$, and hence $q[(p_1(t) - p_2(t))^2] = q(\sin^2\theta(t))\mathbf{1}$ for every $q \in \mathbb{C}[\mathbf{x}]$ with $q(0) = 0$. As above, it follows from the Stone–Weierstrass theorem that $D\mathbf{1} \subseteq C$.

It follows from the above paragraphs that the $C^{\mathbb{C}}(K)$-module

$$E := D\mathbf{1} + C^{\mathbb{C}}(K)p_1 + C^{\mathbb{C}}(K)p_2$$

is contained in C. Let $b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{pmatrix}$ be an arbitrary element of B. Then we straightforwardly realize that $\sin\theta b = f\mathbf{1} + gp_1 + hp_2$, where

$$f := \sin\theta \left(b_{11} - \frac{b_{12}e^{-i\theta} + b_{21}e^{i\theta}}{2\cos\theta} \right) \in D,$$

$$g := \frac{i(b_{12}e^{-2i\theta} - b_{21}e^{2i\theta})}{2\cos\theta} \in C^{\mathbb{C}}(K),$$

$$h := \frac{i(b_{21} - b_{12})}{2\cos\theta} \in C^{\mathbb{C}}(K),$$

and hence $\sin\theta b \in E$. As a first consequence, if $1 \notin K$, then $\frac{f}{\sin\theta} \in D$, $\frac{g}{\sin\theta} \in C^{\mathbb{C}}(K)$, and $\frac{h}{\sin\theta} \in C^{\mathbb{C}}(K)$, hence $b = \frac{f}{\sin\theta}\mathbf{1} + \frac{g}{\sin\theta}p_1 + \frac{h}{\sin\theta}p_2 \in E \subseteq C$, which concludes the proof in this case. Assume that $1 \in K$. Then for every $n \in \mathbb{N}$ we have

$$\frac{n\sin\theta}{1+n\sin\theta} b \in C^{\mathbb{C}}(K)\sin\theta b \subseteq C^{\mathbb{C}}(K)E \subseteq E \subseteq C.$$

Suppose that $b(1) = 0$. Then the entries b_{11}, b_{12}, and b_{21} belong to D, and hence $b = \lim \frac{n\sin\theta}{1+n\sin\theta} b \in C$ because, thanks to Lemma 4.3.33, the sequence $\frac{n\sin\theta}{1+n\sin\theta}$ is an approximate unit for D. For arbitrary $b \in B$, take $\mu \in \mathbb{C}$ such that $b(1) = \mu\eta(1)$, so that $b - \mu\eta(1)$ belongs to B and vanishes at 1, and hence lies in C. It follows that $b = (b - \mu\eta(1)) + \mu\eta(1) \in C + \mu C \subseteq C$, and the proof is complete. □

Corollary 4.3.35 *Let B be a JB^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent e, and set $K := \sigma(e)$. If 1 is an isolated point of K, then B is $*$-isomorphic to the JB^*-algebra $\mathbb{C} \times C(K \setminus \{1\}, \mathscr{C}_3)$.*

Proof Argue as in the proof of Corollary 4.3.18, invoking Theorem 4.3.32 instead of Theorem 4.3.16. □

Corollary 4.3.36 *Let B be a JB^*-algebra generated, as a normed $*$-algebra, by a non-self-adjoint idempotent e, and set $K := \sigma(e)$. Then B has a unit if and only if either* 1 *does not belong to K or* 1 *is an isolated point of K.*

Proof Argue as in the proof of Corollary 4.3.19, invoking Theorems 4.3.29 and 4.3.32, and Corollary 4.3.35 instead of Theorems 4.3.11 and 4.3.16, and Corollary 4.3.18, respectively. □

Corollary 4.3.37 *Let B be a JB^*-algebra. Then B has a non-self-adjoint idempotent (if and) only if it contains (as a closed $*$-subalgebra) a copy of either $\mathscr{C}_3$ or $C_p([1,2], \mathscr{C}_3)$ for any self-adjoint idempotent $p \in \mathscr{C}_3$ different from* 0 *and* $\mathbf{1}$.

Proof Argue as in the proof of Corollary 4.3.20, invoking Theorems 4.3.29 and 4.3.32, and Corollary 4.3.35 instead of Theorems 4.3.11 and 4.3.16, and Corollary 4.3.18, respectively. □

Arguing as in §4.3.21, one can realize that, for any self-adjoint idempotent $p \in \mathscr{C}_3$ different from 0 and $\mathbf{1}$, the JB^*-algebra $C_p([1,2], \mathscr{C}_3)$ does not contain any copy of $\mathscr{C}_3$.

§**4.3.38** In relation to Proposition 4.3.39 immediately below, we note that, as a by-product of Theorems 4.3.29 and 4.3.32, non-self-adjoint idempotents in a JB^*-algebra are non-normal (cf. Definition 3.4.20), hence non-central. Indeed, for every $t \in]1,\infty[$ we have $[\eta(t),\eta(t),\eta(t)^*] \neq 0$ in $\mathscr{C}_3$.

Proposition 4.3.39 *Let B be a JB^*-algebra containing a non-central idempotent e. Then there exists a continuous mapping $r \to e_r$ from $[1,\infty[$ to the set of idempotents of B satisfying $e_{\|e\|} = e$ and $\|e_r\| = r$ for every $r \in [1,\infty[$.*

Proof First assume that e is not self-adjoint. Then, invoking Theorems 4.3.29 and 4.3.32 instead of Theorems 4.3.11 and 4.3.16, respectively, and keeping in mind that, for every $t \in [1,\infty[$, $\eta(t)$ lies in $\mathscr{C}_3$, the first part of the proof of Proposition 4.3.22 works verbatim.

Now assume that e is self-adjoint. Since e is non-central, we may apply Lemma 3.1.14 to find $c \in H(B,*)\setminus\{0\}$ such that $ec = \frac{1}{2}c$, which implies $[e,e,c] = \frac{1}{4}c \neq 0$. There is no loss of generality in assuming that B is generated by $\{e,c\}$ as a normed algebra. Then, by Proposition 3.4.6, there exists a C^*-algebra A such that B becomes a closed $*$-subalgebra of A^{sym}. Set $a := i(ec-ce) \in A$, and consider the mapping $D : A \to A$ defined by

$$D(b) := ba - ab \ \text{ for every } \ b \in A.$$

Then D becomes a continuous derivation of A satisfying $D(b^*) = -D(b)^*$ for every $b \in A$ (since a is self-adjoint). Moreover, in view of the identity (3.1.9) in the proof of Lemma 3.1.53, we have

$$D(b) = 4i[e,b,c] \in B \ \text{ for every } \ b \in B, \tag{4.3.10}$$

and consequently $D(e) \neq 0$. By the second part of the proof of Proposition 4.3.22, there exists a continuous function $h : [1,\infty[\to \mathbb{R}$ such that the continuous mapping $e \to e_r := \exp(h(r)D)(e)$, from $[1,\infty[$ to the set of idempotents of A, satisfies $e_1 = e$ and $\|e_r\| = r$ for every $r \in [1,\infty[$. Therefore, the proof is concluded by realizing that, for every $r \in [1,\infty[$, e_r lies in B. But this follows from the fact that, by (4.3.10), B is invariant under D. ☐

Lemma 4.3.40 *Let B be a Jordan algebra, and let a be in B such that a belongs to $U_{a^2}(B)$. Then there exists an idempotent e in B such that $U_a(B) = U_e(B)$.*

Proof Throughout the proof, the fundamental formula, stated in Proposition 3.4.15, will be applied (sometimes to the unital extension of B) without notice. Set $C := U_a(B)$, and note that $U_C(B) \subseteq C$. Since $a \in U_{a^2}(B) = U_a^2(B)$, there is $v \in C$ such that $a = U_a(v)$. For an arbitrary element $x = U_a(z)$ of C we have $U_aU_v(x) = U_aU_vU_a(z) = U_{U_a(v)}(z) = U_a(z) = x$, so U_aU_v is the identity on C. In particular, $v = U_aU_v(v) = U_a(v^3)$, and hence for $x \in C$ we have

$$\begin{aligned} x = U_aU_v(x) &= U_aU_{U_a(v^3)}(x) = U_a^2U_v^3U_a(x) = U_aU_aU_vU_v^2U_a(x) \\ &= U_aU_v^2U_a(x) \ \ (\text{since } U_v^2U_a(x) \in U_C(B) \subseteq C) = U_aU_vU_vU_a(x) \\ &= U_vU_a(x) \ \ (\text{since } U_vU_a(x) \in U_C(B) \subseteq C). \end{aligned}$$

Thus $(U_a)_{|C}$, regarded as a mapping from C to itself, is bijective with inverse $(U_v)_{|C}$. Let $e = U_a(v^2) \in U_a(B) = C$. Then $U_e = U_a U_v^2 U_a = U_a U_v U_v U_a$ is the identity on C, so

$$C = U_e(C) \subseteq U_e(B) \subseteq U_C(B) \subseteq C$$

shows $C = U_e(B)$. To conclude the proof, we show that e is an idempotent. Indeed, keeping in mind that $U_v(a^2) \in U_C(B) \subseteq C$, we get that

$$e^2 = (U_a(v^2))^2 = U_a U_{v^2}(a^2) = U_a U_v U_v(a^2) = U_v(a^2) \in C.$$

Therefore,

$$e^2 = U_e(e^2) = (e^2)^2 = (U_v(a^2))^2 = U_v U_a^2(v^2) = U_v U_a U_a(v^2) = U_v U_a(e) = e. \quad \square$$

In relation to Corollary 4.3.41 immediately below, we note that tripotents in (the JB^*-triple underlying) a JB^*-algebra are precisely those elements a satisfying $U_a(a^*) = a$.

Corollary 4.3.41 *Let B be a JB^*-algebra. Then the following conditions are equivalent:*

(i) *B contains a non-self-adjoint idempotent.*
(ii) *There exists a non-normal tripotent $a \in B$ such that a belongs to $U_{a^2}(B)$.*
(iii) *B contains a non-central self-adjoint idempotent.*

Proof (i)⇒(ii) By the assumption (i) and Theorems 4.3.29 and 4.3.32, we may assume that B is of the form $C(K, \mathscr{C}_3)$ or $C_{\eta(1)}(K, \mathscr{C}_3)$, where, in the first case, K is a compact subset of $]1, \infty[$ and, in the second case, K is a compact subset of $[1, \infty[$ whose maximum element is greater than 1 and such that $1 \in K$. In any case, by Lemma 4.3.2, η_{21}^K is a non-normal tripotent in B, and we have $\eta_{21}^K = U_{(\eta_{21}^K)^2}(u^2 \eta_{12}^K)$, with $u^2 \eta_{12}^K \in B$.

(ii)⇒(iii) Assume that condition (ii) is fulfilled. We may suppose that B is generated by a as a normed $*$-algebra. Since a lies in $U_{a^2}(B)$, Lemma 4.3.40 applies, giving the existence of an idempotent $e \in B$ such that $U_a(B) = U_e(B)$. Note that $U_e = L_e(2L_e - I_B)$, so that, by Lemma 2.5.3, $U_e(B)$ becomes a subalgebra of B, and e is a unit for such a subalgebra. Suppose, to derive a contradiction, that e is self-adjoint. Then $U_e(B)$ is a closed $*$-subalgebra of B, and hence, since $a = U_a(a^*) \in U_a(B) = U_e(B)$, and B is generated by a as a normed $*$-algebra, we deduce that $U_a(B) = B$ and that e is a unit for B. It follows from Theorem 4.1.3 that a is J-invertible in B and that $a^{-1} = a^*$, i.e. a is a J-unitary element of B (cf. Definition 4.2.25), and this implies that a is normal (cf. Fact 4.2.26(v)), contrarily to the assumption. Therefore the idempotent e is not self-adjoint, hence non-central (cf. §4.3.38).

(iii)⇒(i) Since nonzero self-adjoint idempotents of B are norm-one elements, this implication follows from Proposition 4.3.39. $\square$

Comparing Corollary 4.3.41 with Corollary 4.3.24, one is tempted to conjecture that the equivalent conditions (i)–(iii) in Corollary 4.3.41 are also equivalent to the following:

(iv) B contains a non-normal tripotent.

As a matter of fact, we have been unable to prove or disprove the conjecture just formulated. Actually, an eventual establishment of such a conjecture would provide in particular an affirmative answer to the following unsolved question.

Problem 4.3.42 Let B be a JB^*-algebra containing a nonzero tripotent. Does B contain a nonzero self-adjoint idempotent?

We conclude this subsection with an application to the theory of JB-algebras. By Corollary 3.4.3, the self-adjoint part of the three-dimensional (complex) spin factor $\mathscr{C}_3$ is a JB-algebra, which is called the *three-dimensional real spin factor*, and is denoted by $\mathscr{S}_3$. According to Example 1.1.4(d), we denote by $C([1,2],\mathscr{S}_3)$ the algebra of all continuous functions from $[1,2]$ to $\mathscr{S}_3$, and remark that $C([1,2],\mathscr{S}_3)$ becomes naturally a JB-algebra. Now let p be an idempotent in $\mathscr{S}_3$ different from 0 and $\mathbf{1}$. Then

$$C_p([1,2],\mathscr{S}_3) := \{\alpha \in C([1,2],\mathscr{S}_3) : \alpha(1) \in \mathbb{R}p\}$$

is a proper closed subalgebra of $C([1,2],\mathscr{S}_3)$. We note that, according to the argument in §4.3.30, $C_p([1,2],\mathscr{S}_3)$ does not depend structurally on p.

Now we have the following.

Corollary 4.3.43 *Let B be a JB-algebra. Then B has a non-central idempotent (if and) only if it contains (as a closed subalgebra) a copy of either $\mathscr{S}_3$ or $C_p([1,2],\mathscr{S}_3)$ for any idempotent $p \in \mathscr{S}_3$ different from* 0 *and* $\mathbf{1}$.

Proof By Theorem 3.4.8, there exists a JB^*-algebra whose self-adjoint part is equal to B. Now apply Corollary 4.3.37 and the equivalence (i)⇔(iii) in Corollary 4.3.41. □

4.3.3 An application to non-commutative JB^*-algebras

§4.3.44 Let A be an algebra over $\mathbb{K}$. We have clearly

$$Z(A) \subseteq \{a \in A : [a,A] = 0\}, \tag{4.3.11}$$

where $Z(A)$ stands for the centre of A (cf. Definition 2.5.31). Now denote by A^{ant} the algebra consisting of the vector space of A and the product $[a,b] = ab - ba$. Then, according to Definition 1.1.10, the right-hand side of the inclusion (4.3.11) is nothing other than $\text{Ann}(A^{\text{ant}})$. (We note in passing that, since the algebra A^{ant} is anticommutative, we have $\text{Ann}(A^{\text{ant}}) = Z(A^{\text{ant}})$.) Since for $x,y,z \in A$ we straightforwardly have

$$[x,y,z]^{\text{ant}} + 4[x,y,z]^{\text{sym}} = 2[x,y,z] - 2[z,y,x], \tag{4.3.12}$$

where $[\cdot,\cdot,\cdot]^{\text{ant}}$ and $[\cdot,\cdot,\cdot]^{\text{sym}}$ denote the associator in A^{ant} and A^{sym} respectively, it follows that

$$Z(A) \subseteq Z(A^{\text{sym}}) \cap \text{Ann}(A^{\text{ant}}). \tag{4.3.13}$$

Proposition 4.3.45 *Let A be a flexible algebra. Then*

$$Z(A) = Z(A^{\text{sym}}) \cap \text{Ann}(A^{\text{ant}}).$$

Proof Linearizing the flexibility condition $[x,y,x]=0$ in the variable x, we get $[x,y,z]=-[z,y,x]$. Therefore, the equality (4.3.12) yields

$$[x,y,z]^{\mathrm{ant}}+4[x,y,z]^{\mathrm{sym}}=4[x,y,z],$$

which implies the converse inclusion in (4.3.13). □

Fact 4.3.46 *Let A be an algebra, and let D be a derivation of A.*

$$Then\ D(Z(A))\subseteq Z(A).$$

Proof Noticing that, for $x,y,z\in A$, we have $D([x,y])=[D(x),y]+[x,D(y)]$ and $D([x,y,z])=[D(x),y,z]+[x,D(y),z]+[x,y,D(z)]$, the conclusion is evident. □

Theorem 4.3.47 *Let A be a non-commutative JB^*-algebra. Then*

$$Z(A)=Z(A^{\mathrm{sym}}).$$

Proof In view of Proposition 4.3.45, it is enough to show that $Z(A^{\mathrm{sym}})$ is contained in $\mathrm{Ann}(A^{\mathrm{ant}})$. Clearly, $Z(A^{\mathrm{sym}})$ is a closed associative $*$-subalgebra of A^{sym}, and hence, by Fact 3.3.4 and Proposition 3.4.1(i), $Z(A^{\mathrm{sym}})$ is a commutative C^*-algebra. On the other hand, by Lemma 2.4.15 and Fact 4.3.46, for each $a\in A$, the mapping $D_a : x\to[a,x]$ is a derivation of $Z(A^{\mathrm{sym}})$. It follows from Corollary 3.4.51 that $D_a(Z(A^{\mathrm{sym}}))=0$ for every $a\in A$. Therefore $[A,Z(A^{\mathrm{sym}})]=0$ or, equivalently, $Z(A^{\mathrm{sym}})\subseteq\mathrm{Ann}(A^{\mathrm{ant}})$, as desired. □

Another interesting application of Proposition 4.3.45 is the following.

Corollary 4.3.48 *Let A be a flexible power-associative algebra over $\mathbb{K}$. Then central idempotents in A and in A^{sym} are the same.*

Proof In view of (4.3.13), it is enough to show that central idempotents in A^{sym} are central elements of A. Let e be a central idempotent in A^{sym}. It follows from the definition itself of $A_{\frac{1}{2}}(e)$ (cf. Lemma 2.5.3(i)) that $A_{\frac{1}{2}}(e)=0$, and then, by Lemma 2.5.3(i)–(ii), $e\in\mathrm{Ann}(A^{\mathrm{ant}})$. Therefore, by Proposition 4.3.45, $e\in Z(A)$. □

Now we can generalize and unify Propositions 4.3.22 and 4.3.39, as well as (partially) Corollaries 4.3.24 and 4.3.41.

Proposition 4.3.49 *Let A be a non-commutative JB^*-algebra. We have:*

(i) *If A contains a non-central idempotent e, then there exists a continuous mapping $r\to e_r$ from $[1,\infty[$ to the set of idempotents of A satisfying $e_{\|e\|}=e$ and $\|e_r\|=r$ for every $r\in[1,\infty[$.*
(ii) *A contains a non-self-adjoint idempotent if and only if A contains a non-central self-adjoint idempotent.*

Proof Noticing that A^{sym} is a JB^*-algebra, assertion (i) (respectively, assertion (ii)) follows from Proposition 4.3.39 (respectively, Corollary 4.3.41) by applying either Theorem 4.3.47 or Corollary 4.3.48. □

Remark 4.3.50 Since alternative C^*-algebras are non-commutative JB^*-algebras (cf. Fact 3.3.2), Proposition 4.3.49 above remains true with ‘alternative C^*-algebra’ instead of ‘non-commutative JB^*-algebra’. Actually, since the closed $*$-subalgebra generated by a single element of an alternative C^*-algebra is associative, most results in Subsection 4.3.1 remain true with 'alternative C^*-algebra' instead of 'C^*-algebra'. In particular, this is the case with Theorems 4.3.11 and 4.3.16.

4.3.4 Historical notes and comments

Most results in Subsections 4.3.1 and 4.3.2 are due to Becerra and Rodríguez [77, 78].

Corollary 4.3.28 (or, equivalently, assertions (i) and (ii) in Lemma 4.3.5) are due to Iochum, Loupias, and Rodríguez [340, Lemma 8 and Remark 1], whereas Lemma 4.3.40 is due to Loos [401, Lemma 1.1]. Corollary 4.3.17 goes unnoticed in [77, 78], and becomes a unit-free version of a theorem by Spitkovsky [595], which has been rediscovered in [393, Theorem 6] (see also [83, 317] for related results). Corollary 4.3.34 is new.

In view of Corollary 4.3.17, C^*-algebras generated by a non-self-adjoint idempotent are closely related to C^*-algebras generated by two self-adjoint idempotents. These last algebras have been studied by several authors since Pedersen's paper [483]. According to the formulation of Pedersen's results done by Raeburn and Sinclair [496], we are provided with the following.

Theorem 4.3.51 *There exists an essentially unique triple $(\mathscr{A}, \mathbf{p}_1, \mathbf{p}_2)$, where $\mathscr{A}$ is a unital C^*-algebra and $\mathbf{p}_1$ and $\mathbf{p}_2$ are self-adjoint idempotents in $\mathscr{A}$, satisfying the following universal property:*

(i) *If A is any unital C^*-algebra with two self-adjoint idempotents p_1 and p_2, then there is a unique unit-preserving algebra $*$-homomorphism from $\mathscr{A}$ to A taking $\mathbf{p}_1$ and $\mathbf{p}_2$ to p_1 and p_2, respectively.*

Moreover, we have:

(ii) *$\mathscr{A}$ is generated by $\{\mathbf{1}, \mathbf{p}_1, \mathbf{p}_2\}$ as a normed algebra.*

(iii) *The C^*-algebra $\mathscr{A}$ can be materialized as the closed $*$-subalgebra of the C^*-algebra $C([0,1], M_2(\mathbb{C}))$ consisting of those functions $\alpha \in C([0,1], M_2(\mathbb{C}))$ such that $\alpha(0)$ and $\alpha(1)$ are diagonal matrices, and in this materialization we have*

$$\mathbf{p}_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{p}_2(t) = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$$

for every $t \in [0,1]$.

For additional information about normed associative algebras generated by two idempotents the reader is referred to the survey paper by Böttcher and Spitkovsky [118], the book by Roch, Santos, and Silbermann [797], and the references in these two works.

Now from Theorem 4.3.51 above, let us derive the following non-associative variant.

Theorem 4.3.52 *There exists an essentially unique triple $(\mathscr{B}, \mathbf{e}_1, \mathbf{e}_2)$, where $\mathscr{B}$ is a unital JB^*-algebra and $\mathbf{e}_1$ and $\mathbf{e}_2$ are self-adjoint idempotents in $\mathscr{B}$, satisfying the* *following universal property:*

(i) *If B is any unital JB^*-algebra with two self-adjoint idempotents e_1 and e_2, then there is a unique unit-preserving algebra $*$-homomorphism from $\mathscr{B}$ to B taking $\mathbf{e}_1$ and $\mathbf{e}_2$ to e_1 and e_2, respectively.*

Moreover, we have:

(ii) *$\mathscr{B}$ is generated by $\{\mathbf{1},\mathbf{e}_1,\mathbf{e}_2\}$ as a normed algebra.*
(iii) *The JB^*-algebra $\mathscr{B}$ can be materialized as the closed $*$-subalgebra of the JB^*-algebra $C([0,1],\mathscr{C}_3)$ consisting of those functions $\beta \in C([0,1],\mathscr{C}_3)$ such that $\beta(0)$ and $\beta(1)$ lie in the linear hull of $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$, and in this materialization we have*

$$\mathbf{e}_1(t)=\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \ \text{and}\ \mathbf{e}_2(t)=\frac{1}{2}\begin{pmatrix}1 & e^{2i\theta(t)}\\ e^{-2i\theta(t)} & 1\end{pmatrix}$$

for every $t\in[0,1]$, where $\theta(t):=\arccos\sqrt{t}$.

Proof The essential uniqueness of the triple $(\mathscr{B},\mathbf{e}_1,\mathbf{e}_2)$ under condition (i) is easy. Indeed, let $\mathscr{C}$ be a unital JB^*-algebra, and let $\mathbf{f}_1$ and $\mathbf{f}_2$ be self-adjoint idempotents in $\mathscr{C}$ such that condition (i) holds with $(\mathscr{C},\mathbf{f}_1,\mathbf{f}_2)$ instead of $(\mathscr{B},\mathbf{e}_1,\mathbf{e}_2)$. Then we are provided with unit-preserving algebra $*$-homomorphisms

$$\phi:\mathscr{B}\to\mathscr{C} \text{ and } \psi:\mathscr{C}\to\mathscr{B}$$

such that $\phi(\mathbf{e}_j)=\mathbf{f}_j$ and $\psi(\mathbf{f}_j)=\mathbf{e}_j$. Therefore $\psi\circ\phi$ and $\phi\circ\psi$ are unit-preserving algebra $*$-endomorphisms of $\mathscr{B}$ and $\mathscr{C}$ fixing $\mathbf{e}_j$ and $\mathbf{f}_j$, respectively. By the uniqueness of such homomorphisms, we must have $\psi\circ\phi=I_{\mathscr{B}}$ and $\phi\circ\psi=I_{\mathscr{C}}$, and hence ϕ is an algebra $*$-isomorphism from $\mathscr{B}$ onto $\mathscr{C}$ taking $\mathbf{e}_j$ to $\mathbf{f}_j$.

Let $(\mathscr{B},\mathbf{e}_1,\mathbf{e}_2)$ be as in assertion (iii) of the present theorem. We are going to show that $\mathscr{B}$ is generated by $\{\mathbf{1},\mathbf{e}_1,\mathbf{e}_2\}$ as a normed algebra. Let $\mathfrak{C}$ denote the closed subalgebra of $\mathscr{B}$ generated by $\{\mathbf{1},\mathbf{e}_1,\mathbf{e}_2\}$, let $\mathscr{F}^0$ and $\mathscr{F}^1$ stand for the ideals of $C^{\mathbb{C}}([0,1])$ defined by

$$\mathscr{F}^0:=\{x\in C^{\mathbb{C}}([0,1]):x(0)=0\} \ \text{ and } \ \mathscr{F}^1:=\{x\in C^{\mathbb{C}}([0,1]):x(1)=0\},$$

and note that $\mathscr{F}^0\equiv C_0^{\mathbb{C}}(]0,1])$, $\mathscr{F}^1\equiv C_0^{\mathbb{C}}([0,1[)$, and $\mathscr{F}^0\cap\mathscr{F}^1\equiv C_0^{\mathbb{C}}(]0,1[)$. Note also that, for $t\in[0,1]$, we have $\cos\theta(t)=\sqrt{t}$ and $\sin\theta(t)=\sqrt{1-t}$, so that $\cos\theta$ and $\sin\theta$ are injective functions on $[0,1]$ satisfying $\cos\theta(t)>0$ for every $t\in]0,1]$ and $\sin\theta(t)>0$ for every $t\in[0,1[$. Now, as in the proof of Corollary 4.3.34, for every $t\in[0,1]$ we have

$$(\mathbf{e}_1(t)-\mathbf{e}_2(t))^2=\sin^2\theta(t)\mathbf{1}=(1-t)\mathbf{1}$$

and

$$U_{\mathbf{e}_j(t)}(\mathbf{e}_k(t))=\cos^2\theta(t)\mathbf{e}_j(t)=t\mathbf{e}_j(t)$$

whenever (j,k) equals either $(1,2)$ or $(2,1)$. Therefore, resuming the argument in the proof of Corollary 4.3.34, we obtain that $\mathscr{F}^1\mathbf{1}\subseteq\mathfrak{C}$ and $\mathscr{F}^0\mathbf{e}_j\subseteq\mathfrak{C}$, and hence that the $C^{\mathbb{C}}([0,1])$-module $E:=\mathscr{F}^1\mathbf{1}+\mathscr{F}^0\mathbf{e}_1+\mathscr{F}^0\mathbf{e}_2$ is contained in $\mathfrak{C}$. Let $\beta=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{11}\end{pmatrix}$ be an arbitrary element of $\mathscr{B}$. Then we have

$$\sin(2\theta)\beta = 2\sin\theta\cos\theta\beta = f\mathbf{1} + g\mathbf{e}_1 + h\mathbf{e}_2,$$

where

$$f := \sin\theta\left(2\cos\theta b_{11} - b_{12}e^{-i\theta} - b_{21}e^{i\theta}\right) \in \mathscr{F}^1,$$
$$g := i(b_{12}e^{-2i\theta} - b_{21}e^{2i\theta}) \in \mathscr{F}^0,$$
$$h := i(b_{21} - b_{12}) \in \mathscr{F}^0,$$

so $\sin(2\theta)\beta \in E$, and so for every $n \in \mathbb{N}$ we have

$$\frac{n\sin(2\theta)}{1+n\sin(2\theta)}\beta \in C^{\mathbb{C}}([0,1])\sin(2\theta)\beta \subseteq C^{\mathbb{C}}([0,1])E \subseteq E \subseteq \mathfrak{C}.$$

Suppose that $\beta(0) = \beta(1) = 0$. Then b_{11}, b_{12}, and b_{21} lie in $\mathscr{F}^0 \cap \mathscr{F}^1$, and hence $\beta = \lim \frac{n\sin(2\theta)}{1+n\sin(2\theta)}\beta \in \mathfrak{C}$ because, thanks to Lemma 4.3.33, the sequence $\frac{n\sin(2\theta)}{1+n\sin(2\theta)}$ is an approximate unit for $\mathscr{F}^0 \cap \mathscr{F}^1$. Now remove the assumption that $\beta(0) = \beta(1) = 0$. Let γ be the element of $\mathfrak{C}$ defined by

$$\gamma := \lambda\mathbf{1} + \mu\mathbf{e}_1 + \nu\mathbf{e}_2 + \eta\mathbf{e}_1 \bullet \mathbf{e}_2,$$

where

$$\lambda := b_{11}(1) - b_{12}(1),\ \ \mu := b_{11}(0) - b_{11}(1) + b_{12}(0) + b_{12}(1),$$
$$\nu := b_{11}(0) - b_{11}(1) - b_{12}(0) + b_{12}(1),\ \text{ and }\ \eta := 2(b_{11}(1) - b_{11}(0)).$$

Then we straightforwardly realize that $(\beta-\gamma)(0) = (\beta-\gamma)(1) = 0$. It follows that $\beta = (\beta-\gamma)+\gamma \in \mathfrak{C}+\mathfrak{C} \subseteq \mathfrak{C}$. Since β is arbitrary in $\mathscr{B}$, and $\mathfrak{C}$ is the closed subalgebra of $\mathscr{B}$ generated by $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$, $\mathscr{B}$ is certainly generated by $\{\mathbf{1}, \mathbf{e}_1, \mathbf{e}_2\}$ as a normed algebra.

Now let $(\mathscr{A}, \mathbf{p}_1, \mathbf{p}_2)$ be as in assertion (iii) of Theorem 4.3.51, and let $(\mathscr{B}, \mathbf{e}_1, \mathbf{e}_2)$ be as in assertion (iii) of the present theorem. Set

$$v := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix},$$

and note that v is a unitary element of $M_2(\mathbb{C})$ such that $v\mathscr{A}v^*$ equals the closed $*$-subalgebra of $C([0,1], M_2(\mathbb{C}))$ (say $\mathscr{D}$) consisting of those functions $\beta \in C([0,1], M_2(\mathbb{C}))$ such that $\beta(0)$ and $\beta(1)$ lie in the linear hull of $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$, that $v\mathbf{p}_jv^* = \mathbf{e}_j$, and that $\mathscr{B}$ becomes a closed $*$-subalgebra of $\mathscr{D}^{\mathrm{sym}}$. Let B be any unital JB^*-algebra with two self-adjoint idempotents e_1 and e_2, and let C stand for the closed $*$-subalgebra of B generated by $\{\mathbf{1}, e_1, e_2\}$. By Proposition 3.4.6, there is a unital C^*-algebra A such that C can be seen as a closed $*$-subalgebra of A^{sym}. By Theorem 4.3.51, there exists a unit-preserving algebra $*$-homomorphism $\Phi : \mathscr{A} \to A$ such that $\Phi(\mathbf{p}_j) = e_j$. It follows that the mapping $\Psi : \beta \to \Phi(v^*\beta v)$ from $\mathscr{D}$ to A becomes a unit-preserving algebra $*$-homomorphism satisfying $\Psi(\mathbf{e}_j) = e_j$. Moreover, invoking the preceding paragraph, we realize that $\Psi(\mathscr{B}) \subseteq C$. Regarding Ψ as a mapping from $\mathscr{B}$ to C, composing this mapping with the inclusion $C \hookrightarrow B$, and invoking again the preceding paragraph, we finally find a unique unit-preserving algebra $*$-homomorphism from $\mathscr{B}$ to B taking $\mathbf{e}_j$ to e_j. □

The arguments leading to Proposition 4.3.45 are due to Skosyrskii [584]. Theorem 4.3.47 is originally due to Alvermann and Janssen [19, Theorem 7.3]. Our proof of Theorem 4.3.47, as well as Proposition 4.3.49, are new.

4.4 Algebra norms on non-commutative JB^*-algebras

Introduction In 1967, Johnson [353] proved one of the more outstanding results in the theory of normed associative algebras, namely that surjective algebra homomorphisms from complete normed associative algebras to complete normed semisimple associative algebras are automatically continuous. The consequence, that semisimple associative algebras have at most one complete algebra norm topology, has become known as 'Johnson's uniqueness-of-norm theorem'.

In Subsection 4.4.1, we follow McCrimmon [435, 436] to introduce the notions of Jacobson radical (in short J-radical) and of Jacobson semisimplicity (in short J-semisimplicity) of a Jordan-admissible algebra. These notions generalize the usual ones of radical and semisimplicity in the particular associative case. Then we prove Aupetit's generalization of Johnson's theorem to the setting of Jordan-admissible algebras [40] (see Theorem 4.4.13). Our proof of Aupetit's theorem is not at all the original one, but relies on a non-associative adaptation of 'T. J. Ransford's three circles theorem' [499] (see Proposition 4.4.3), which simplifies Aupetit's techniques.

In Subsection 4.4.2, we extend the notion of a normed Q-algebra (introduced in §3.6.41 for associative algebras) to the more general case of Jordan-admissible algebras. Then we refine the associative arguments in [204, 519] to prove the automatic continuity of surjective algebra homomorphisms from Jordan-admissible normed Q-algebras to complete normed J-semisimple Jordan-admissible algebras having minimality of norm topology (see Theorem 4.4.23).

In Subsection 4.4.3, we apply Theorem 4.4.23 just reviewed to derive that non-commutative JB^*-algebras have minimum norm topology (see Theorem 4.4.29), a result which generalizes Cleveland's associative theorem [176]. We also prove that non-commutative JB^*-algebras have minimality of norm (see Proposition 4.4.34).

In Subsection 4.4.4, we follow [516] to introduce the notions of weak radical and ultra-weak radical of an arbitrary algebra. The weak radical is always contained in the ultra-weak radical, and the ultra-weak radical is contained in most classical radicals which are relevant in the automatic continuity theory (see Proposition 4.4.59). In particular, the ultra-weak radical of an associative (or even Jordan-admissible) algebra is contained in the Jacobson radical. As main results we prove that algebras with zero weak radical have at most one complete algebra norm topology (see Theorem 4.4.43), and that surjective algebra homomorphisms, from complete normed algebras to complete normed algebras with zero ultra-weak radical, are automatically continuous (see Theorem 4.4.45). Several applications of Theorems 4.4.43 and 4.4.45 are done. In particular, the Johnson–Aupetit–Ransford Theorem 4.4.13 is deeply refined (see Corollary 4.4.62). The subsection also contains an example, due to Haïly [303], of an (unfortunately non-normed) associative algebra with zero weak radical and nonzero ultra-weak radical.

4.4.1 The Johnson–Aupetit–Ransford uniqueness-of-norm theorem

We begin this subsection with the following slight variant of Dini's theorem.

Lemma 4.4.1 *Let E be a compact Hausdorff topological space, and let $f_n : E \to \mathbb{R}_0^+$ be a decreasing sequence of continuous functions. Then*

$$\lim_{n\to\infty} \sup_{t\in E} f_n(t) = \sup_{t\in E} \lim_{n\to\infty} f_n(t).$$

Proof Since f_n is a decreasing sequence of non-negative functions, we can consider the function $f : E \to \mathbb{R}_0^+$ defined by $f(t) := \lim_{n\to\infty} f_n(t)$. Moreover, it is clear that $\sup_{t\in E} f_n(t)$ is a decreasing sequence of non-negative numbers such that $\sup_{t\in E} f(t) \leqslant \sup_{t\in E} f_n(t)$ for every $n \in \mathbb{N}$, and hence

$$\sup_{t\in E} f(t) \leqslant L := \lim_{n\to\infty} \sup_{t\in E} f_n(t).$$

For each $n \in \mathbb{N}$ consider the set $F_n := \{t \in E : f_n(t) \geqslant L\}$. It follows from the continuity of the functions f_n that F_n is a decreasing sequence of non-empty closed subsets of E. Since E is compact, it follows that $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$. For any $t_0 \in \cap_{n\in\mathbb{N}} F_n$ we realize that

$$f(t_0) = \lim_{n\to\infty} f_n(t_0) \geqslant L,$$

and so $\sup_{t\in E} f(t) \geqslant L$. This shows that

$$\sup_{t\in E} f(t) = L = \lim_{n\to\infty} \sup_{t\in E} f_n(t). \qquad \square$$

§**4.4.2** Let A be an algebra over $\mathbb{K}$, and let a be in A. Following [812, p. 23], we define inductively the *plenary powers* of a by

$$a^{[0]} := a \ \text{ and } \ a^{[n+1]} := (a^{[n]})^2.$$

Without enjoying their name, plenary powers have already appeared in the proof of Proposition 2.6.19. Assume in what follows that A is normed. Then we have $\|a^{[n+1]}\| \leqslant \|a^{[n]}\|^2$, and hence the sequence $\|a^{[n]}\|^{\frac{1}{2^n}}$ is decreasing. Therefore we can define the number

$$\mathfrak{s}(a) := \inf\left\{ \left\|a^{[n]}\right\|^{\frac{1}{2^n}} : n \in \mathbb{N} \cup \{0\} \right\} = \lim \left\|a^{[n]}\right\|^{\frac{1}{2^n}}.$$

We note that, if a generates an associative subalgebra of A, then we have $a^{[n]} = a^{2^n}$ for every n, and hence the number $\mathfrak{s}(a)$ just introduced is nothing other than the usual spectral radius $\mathfrak{r}(a)$ of a (cf. §3.4.61). In this case, we will always use without notice the symbol $\mathfrak{r}(a)$ instead of $\mathfrak{s}(a)$.

Proposition 4.4.3 *Let A be a normed complex algebra, let $p : \mathbb{C} \to A$ be a polynomial function* (cf. §2.8.29)*, and let $R > 0$. Then*

$$\sup_{|w|=1} \mathfrak{s}(p(w))^2 \leqslant \sup_{|z|=R} \mathfrak{s}(p(z)) \sup_{|z|=\frac{1}{R}} \mathfrak{s}(p(z)).$$

Proof Let w be an arbitrary complex number of modulus 1. Choose a norm-one continuous linear functional f on A satisfying $f(p(w)) = \|p(w)\|$. Set $q(z) = f(p(z))$, so that $q(z) = \sum_{k=0}^{d} \alpha_k z^k$ say, where d is the degree of p. Then the Cauchy–Schwarz inequality gives:

$$\begin{aligned}
\|p(w)\|^2 = |f(p(w))|^2 = |q(w)|^2 &= \left|\sum_{k=0}^{d} \alpha_k w^k\right|^2 \leqslant \left(\sum_{k=0}^{d} |w^k|^2\right)\left(\sum_{k=0}^{d} |\alpha_k|^2\right) \\
&= (d+1)\left(\sum_{k=0}^{d} \alpha_k R^k \bar{\alpha}_k R^{-k}\right) \\
&\leqslant (d+1)\left(\sum_{k=0}^{d} |\alpha_k|^2 R^{2k}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{d} |\alpha_k|^2 R^{-2k}\right)^{\frac{1}{2}} \\
&= (d+1)\left(\frac{1}{2\pi}\int_0^{2\pi} |q(Re^{it})|^2 dt\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi} |q(R^{-1}e^{it})|^2 dt\right)^{\frac{1}{2}} \\
&\leqslant (d+1) \sup_{|z|=R} \|p(z)\| \sup_{|z|=\frac{1}{R}} \|p(z)\|.
\end{aligned}$$

Repeating with $p(z)$ replaced by $p(z)^{[n]}$, for an arbitrary natural number n, and taking 2^nth roots, we obtain

$$\left\|p(w)^{[n]}\right\|^{\frac{2}{2^n}} \leqslant (2^n d+1)^{\frac{1}{2^n}} \sup_{|z|=R} \left\|p(z)^{[n]}\right\|^{\frac{1}{2^n}} \sup_{|z|=\frac{1}{R}} \left\|p(z)^{[n]}\right\|^{\frac{1}{2^n}}. \tag{4.4.1}$$

Keeping in mind that the convergence $\|p(\cdot)^{[n]}\|^{\frac{1}{2^n}} \to \mathfrak{s}(p(\cdot))$ is monotone decreasing, the proof is concluded by letting $n \to \infty$ in (4.4.1) and applying Lemma 4.4.1. □

Corollary 4.4.4 *Let X be a complex normed space, let B be a normed complex algebra, and let Φ be a linear mapping from X to B. Suppose that there exists $M > 0$ such that $\mathfrak{s}(\Phi(x)) \leqslant M\|x\|$ for every $x \in X$. Then*

$$\mathfrak{s}(\Phi(x))^2 \leqslant M\|x\|\mathrm{d}(\Phi(x), \mathfrak{S}(\Phi)) \ \text{ for every } \ x \in X.$$

Accordingly, $\mathfrak{s}(b) = 0$ for every $b \in \mathfrak{S}(\Phi) \cap \Phi(X)$.

Proof For x, y given in X, consider the polynomial $p(z) = \Phi(x) + \Phi(y)z$, and note that, for $R > 0$ we have

$$\mathfrak{s}(p(z)) = \mathfrak{s}(\Phi(x+yz)) \leqslant M\|x+yz\| \leqslant M(\|x\| + \|y\|R) \ \text{ whenever } \ |z| = R,$$

and

$$\mathfrak{s}(p(z)) \leqslant \|p(z)\| \leqslant \|\Phi(x)\| + \|\Phi(y)\|R^{-1} \ \text{ whenever } \ |z| = R^{-1}.$$

It follows from Proposition 4.4.3 that

$$\mathfrak{s}(\Phi(x+y))^2 = \mathfrak{s}(p(1))^2 \leqslant M(\|x\| + \|y\|R)(\|\Phi(x)\| + \|\Phi(y)\|R^{-1}). \tag{4.4.2}$$

Now, fix $x \in X$ and $b \in \mathfrak{S}(\Phi)$, and choose a sequence x_n in X such that $x_n \to 0$ and $\Phi(x_n) \to b$. By (4.4.2), for $R > 0$ and for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathfrak{s}(\Phi(x))^2 &= \mathfrak{s}(\Phi((x - x_n) + x_n))^2 \\
&\leqslant M(\|x_n - x\| + \|x_n\|R)(\|\Phi(x - x_n)\| + \|\Phi(x_n)\|R^{-1}).
\end{aligned}$$

Letting $n \to \infty$, we deduce that

$$\mathfrak{s}(\Phi(x))^2 \leqslant M\|x\|(\|\Phi(x) - b\| + \|b\|R^{-1}),$$

and now, letting $R \to \infty$, we find that

$$\mathfrak{s}(\Phi(x))^2 \leqslant M\|x\|\|\Phi(x) - b\|.$$

It follows from the arbitrariness of b in $\mathfrak{S}(\Phi)$ that

$$\mathfrak{s}(\Phi(x))^2 \leqslant M\|x\|\mathrm{d}(\Phi(x), \mathfrak{S}(\Phi)). \qquad \square$$

At this point, we are interested in the following consequence of Corollary 4.4.4.

Lemma 4.4.5 *Let A and B be normed complex algebras, and let $\Phi : A \to B$ be a linear mapping such that there exists $M > 0$ satisfying $\mathfrak{s}(\Phi(a)) \leqslant M\mathfrak{s}(a)$ for every $a \in A$. Then $\mathfrak{s}(b) = 0$ for every $b \in \mathfrak{S}(\Phi) \cap \Phi(A)$.*

§4.4.6 Let A be a normed algebra over $\mathbb{K}$, and let a be in A. Keeping in mind that squares are the same in A and A^{sym}, we realize that the number $\mathfrak{s}(a)$ has the same meaning in A and A^{sym}. Therefore, if A admits power-associativity (cf. Definition 4.1.106), then, by §4.4.2, $\mathfrak{s}(a)$ coincides with the spectral radius of a in A^{sym}. Invoking Proposition 3.4.63 and Lemma 4.4.5, the fact just formulated yields the following.

Proposition 4.4.7 *Let A be a complete normed complex algebra admitting power-associativity, let B be a normed complex algebra, and let $\Phi : A \to B$ be a Jordan homomorphism. Then $\mathfrak{s}(b) = 0$ for every $b \in \mathfrak{S}(\Phi) \cap \Phi(A)$.*

§4.4.8 Let A be a Jordan-admissible algebra. Then, since $(A_{\mathbb{1}})^{\mathrm{sym}} = (A^{\mathrm{sym}})_{\mathbb{1}}$ and $(A^{\mathrm{sym}})_{\mathbb{1}}$ is a Jordan algebra, it follows that $A_{\mathbb{1}}$ is a Jordan-admissible algebra. An element a of A is said to be *quasi-J-invertible* in A if $\mathbb{1} - a$ is J-invertible in the unital extension $A_{\mathbb{1}}$ of A, and a subset of A is called *quasi-J-invertible* if all its elements are quasi-J-invertible in A. Keeping in mind the definition of a J-invertible element in a unital Jordan-admissible algebra (cf. Definition 4.1.56) and the fact already pointed out that $(A_{\mathbb{1}})^{\mathrm{sym}} = (A^{\mathrm{sym}})_{\mathbb{1}}$, we realize that quasi-J-invertible elements, and consequently quasi-J-invertible subsets, are the same in A and A^{sym}. It is worth remarking that, in view of Fact 4.1.57, if A is alternative, then quasi-J-invertible elements (respectively, quasi-J-invertible subsets) of A are precisely quasi-invertible elements (respectively, quasi-invertible subsets) of A in the sense of Definition 3.6.19.

Keeping in mind §4.4.6 and the fact that the unital extension of a complete normed algebra is a complete normed algebra (cf. Proposition 1.1.107), Lemma 4.1.15 implies the following.

Fact 4.4.9 *Let A be a complete normed Jordan-admissible algebra over $\mathbb{K}$, and let a be in A with $\mathfrak{s}(a) < 1$. Then a is quasi-J-invertible in A.*

Theorem 4.1.3 must be omnipresent along the proof of the following.

Lemma 4.4.10 *Let A be a Jordan-admissible algebra over $\mathbb{K}$, let a be a quasi-J-invertible element of A, and let I be a quasi-J-invertible ideal of A. Then $a + I$ is a quasi-J-invertible subset of A.*

Proof Since ideals of A are ideals of A^{sym}, we may assume that A is commutative. Set $b := \mathbb{1} - a$ and let x be in I. Then, since a is quasi-J-invertible in A, b is J-invertible in $A_{\mathbb{1}}$, and hence

$$\begin{aligned} U_b^{-1}((b-x)^2) &= U_b^{-1}(b^2 - 2bx + x^2) \\ &= U_b^{-1}(b^2) - U_b^{-1}(2bx - x^2) \\ &= \mathbb{1} - U_b^{-1}(2bx - x^2) \end{aligned}$$

is J-invertible in $A_{\mathbb{1}}$ because $U_b^{-1}(2bx - x^2) = U_{b^{-1}}(2bx - x^2) \in I$ and I is a quasi-invertible subset of A. Therefore

$$U_{b^{-1}} U_{b-x}(\mathbb{1}) = U_{b^{-1}}((b-x)^2) = U_b^{-1}((b-x)^2) \in \text{J-Inv}(A_{\mathbb{1}}),$$

so $U_{b-x}(\mathbb{1}) \in \text{J-Inv}(A_{\mathbb{1}})$, hence $\mathbb{1} - (a+x) = b - x \in \text{J-Inv}(A_{\mathbb{1}})$, and finally $a+x$ is quasi-J-invertible in A. □

Proposition 4.4.11 *Let A be a Jordan-admissible algebra over $\mathbb{K}$. Then we have:*

(i) *There exists a largest quasi-J-invertible ideal of A (say* J-Rad(A)*).*
(ii) J-Rad(A) *is the core of* J-Rad(A^{sym}) *in A.*

Proof To prove assertion (i) it is enough to show that the sum of all quasi-J-invertible ideals of A is a quasi-J-invertible subset of A. Assume, to derive a contradiction, that this is not true. Then there exists a minimum natural number n such that there are quasi-J-invertible ideals $I_1, \ldots, I_n$ of A and elements $x_1 \in I_1, \ldots, x_n \in I_n$ in such a way that $\sum_{i=1}^n x_i$ is not quasi-J-invertible in A. Then, clearly $n \geqslant 2$ and $\sum_{i=1}^{n-1} x_i$ must be quasi-J-invertible in A, so that, by Lemma 4.4.10, $\sum_{i=1}^{n-1} x_i + I_n$ is a quasi-J-invertible subset of A, which implies that $\sum_{i=1}^n x_i$ is quasi-J-invertible in A, the desired contradiction.

Now we proceed to prove assertion (ii). Let I stand for the core of J-Rad(A^{sym}) in A. Then, clearly, I is a quasi-J-invertible ideal of A. On the other hand, every quasi-J-invertible ideal of A is a quasi-J-invertible ideal of A^{sym}, so it is an ideal of A contained in J-Rad(A^{sym}), hence it must be contained in I. It follows that $I = \text{J-Rad}(A)$. □

Definition 4.4.12 Let A be a Jordan-admissible algebra over $\mathbb{K}$. The ideal J-Rad(A) given by the above proposition will be called the *Jacobson radical* of A. The Jordan-admissible algebra A is said to be *J-semisimple* when J-Rad$(A) = 0$. We remark that, according to Theorem 3.6.21, we have J-Rad$(A) = \text{Rad}(A)$ whenever A is actually associative, and that, consequently, J-semisimplicity is nothing other than semisimplicity when we deal with associative algebras. This remark will be frequently used, mostly without notice. As we commented in §3.6.58, the equality J-Rad$(A) = \text{Rad}(A)$ actually holds in the more general case where A is alternative, but this will only be applied incidentally with the appropriate notice.

Theorem 4.4.13 *Let A be a complete normed complex algebra admitting power-associativity, let B be a complete normed Jordan-admissible complex algebra, and let $\Phi : A \to B$ be a surjective algebra homomorphism. If B is J-semisimple, then Φ is continuous.*

Proof It is enough to show that $\mathfrak{S}(\Phi) \subseteq \text{J-Rad}(B)$. By Proposition 4.4.7, for every $b \in \mathfrak{S}(\Phi)$, we have $\mathfrak{s}(b) = 0$, and hence, by Fact 4.4.9, b is quasi-J-invertible in B. Thus $\mathfrak{S}(\Phi)$ is a quasi-J-invertible subset of B. Since $\mathfrak{S}(\Phi)$ is an ideal of B (by Lemma 1.1.58), it follows that $\mathfrak{S}(\Phi) \subseteq \text{J-Rad}(B)$, as desired. □

We call Theorem 4.4.13 above the *Johnson–Aupetit–Ransford theorem*.

Arguing as in the proof of Corollary 1.1.63, with Theorem 4.4.13 instead of Theorem 1.1.62, we get the following.

Corollary 4.4.14 *Let A be a J-semisimple Jordan-admissible complex algebra. Then A has at most one complete algebra norm topology.*

When the algebra B in Theorem 4.4.13 is in fact a non-commutative Jordan algebra, we can relax the assumption of Φ being an algebra homomorphism to that of Φ being merely a Jordan homomorphism. This will be established in Corollary 4.4.18. To this end, we begin by pointing out the next fact, which follows by arguing as in Example 3.6.42, with Theorem 4.1.7 instead of Theorem 1.1.23.

Fact 4.4.15 *Let A be a complete normed Jordan-admissible algebra over $\mathbb{K}$. Then the set of all quasi-J-invertible elements of A is open in A.*

Lemma 4.4.16 *Let A be a Jordan-admissible algebra over $\mathbb{K}$, let I be an ideal of A, and let S stand for the set of all quasi-J-invertible elements of A. Then the following conditions are equivalent:*

(i) $I \subseteq S$.
(ii) $S + I \subseteq S$.
(iii) $(A \setminus S) + I \subseteq A \setminus S$.

Proof The equivalence (i)⇔(ii) follows from Lemma 4.4.10 and the fact that $0 \in S$, whereas the one (ii)⇔(iii) follows from the fact that, for $a \in A$ and $x \in I$, we have $a = (a + x) + (-x)$. □

Proposition 4.4.17 *Let A be a complete normed Jordan-admissible algebra over $\mathbb{K}$. We have:*

(i) $\text{J-Rad}(A)$ *is closed in A.*
(ii) $\text{J-Rad}(A)$ *is invariant under any continuous derivation of A.*
(iii) *If A is in fact a non-commutative Jordan algebra, then*

$$\text{J-Rad}(A) = \text{J-Rad}(A^{\text{sym}}).$$

Proof Let S stand for the set of all quasi-J-invertible elements of A. By Lemma 4.4.16, we have $(A \setminus S) + \text{J-Rad}(A) \subseteq A \setminus S$. Since $A \setminus S$ is closed in A (by Fact 4.4.15), it follows that $(A \setminus S) + \overline{\text{J-Rad}(A)} \subseteq A \setminus S$. Since $\overline{\text{J-Rad}(A)}$ is an ideal of A, Lemma 4.4.16 applies again to derive that $\overline{\text{J-Rad}(A)} \subseteq \text{J-Rad}(A)$, and the proof of assertion (i) is complete.

Let D be a continuous derivation of A. Then, for $r \in \mathbb{R}$, $\exp(rD)$ is a bijective algebra endomorphism on A (cf. Lemma 2.2.21), and hence

$$\exp(rD)(\text{J-Rad}(A)) = \text{J-Rad}(A).$$

Therefore, by assertion (i) just proved, for $x \in \text{J-Rad}(A)$ we have

$$D(x) = \lim_{r \to 0} \frac{\exp(rD)(x) - x}{r} \in \text{J-Rad}(A),$$

which proves assertion (ii).

Assume that A is actually a non-commutative Jordan algebra. Then, by Lemma 2.4.15 and (the commutative case of) assertion (ii) just proved, we have $[A, \text{J-Rad}(A^{\text{sym}})] \subseteq \text{J-Rad}(A^{\text{sym}})$. Therefore

$$\begin{aligned} A(\text{J-Rad}(A^{\text{sym}})) &\subseteq A \bullet (\text{J-Rad}(A^{\text{sym}})) + [A, \text{J-Rad}(A^{\text{sym}})] \\ &\subseteq \text{J-Rad}(A^{\text{sym}}) + \text{J-Rad}(A^{\text{sym}}) \subseteq \text{J-Rad}(A^{\text{sym}}), \end{aligned}$$

and analogously $(\text{J-Rad}(A^{\text{sym}}))A \subseteq \text{J-Rad}(A^{\text{sym}})$. Thus $\text{J-Rad}(A^{\text{sym}})$ is an ideal of A. Since $\text{J-Rad}(A)$ is the core of $\text{J-Rad}(A^{\text{sym}})$ in A, it follows that $\text{J-Rad}(A) = \text{J-Rad}(A^{\text{sym}})$, which proves assertion (iii). □

By combining Proposition 4.4.17(iii) and (the commutative particularization of) Theorem 4.4.13, we get the following variant of Theorem 4.4.13.

Corollary 4.4.18 *Let A be a complete normed complex algebra admitting power-associativity, let B be a complete normed J-semisimple non-commutative Jordan complex algebra, and let $\Phi : A \to B$ be a surjective Jordan homomorphism. Then Φ is continuous.*

4.4.2 A non-complete variant

Now we are going to prove an interesting non-complete variant of Theorem 4.4.13. To this end, we begin by establishing the following.

Lemma 4.4.19 *Let X and Y be Banach spaces over $\mathbb{K}$, and let $\Phi : X \to Y$ be a linear mapping. Then $\mathfrak{S}(\Phi)$ is a closed subspace of Y, and the mapping $x \to \Phi(x) + \mathfrak{S}(\Phi)$ from X to $Y/\mathfrak{S}(\Phi)$ is continuous.*

Proof We already know that $\mathfrak{S}(\Phi)$ is a closed subspace of Y (cf. Lemma 1.1.57). Let x_n be a sequence converging to 0 in X and such that $\Phi(x_n) + \mathfrak{S}(\Phi)$ converges to $y + \mathfrak{S}(\Phi)$ for some $y \in Y$. Then there exists a sequence y_n in $\mathfrak{S}(\Phi)$ such that $\Phi(x_n) - y + y_n$ converges to 0 in Y. For each $n \in \mathbb{N}$, take $z_n \in X$ with $\|z_n\| < \frac{1}{n}$ and $\|y_n - \Phi(z_n)\| < \frac{1}{n}$. It follows that $x_n + z_n \to 0$ and $\Phi(x_n + z_n) \to y$, so that y lies in $\mathfrak{S}(\Phi)$, and hence $y + \mathfrak{S}(\Phi) = 0$. The proof is concluded by applying the closed graph theorem. □

Lemma 4.4.20 *Let A and B be Jordan-admissible algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be an algebra homomorphism. We have:*

(i) *If x is quasi-J-invertible in A, then $\Phi(x)$ is quasi-J-invertible in B.*
(ii) *If Φ is surjective, then $\Phi(\text{J-Rad}(A)) \subseteq \text{J-Rad}(B)$.*

Proof Assertion (i) follows straightforwardly from Lemma 4.1.87 and the universal property of unital extensions.

Assume that Φ is surjective. Then, by assertion (i), $\Phi(\text{J-Rad}(A))$ is a quasi-J-invertible ideal of B, and hence it is contained in $\text{J-Rad}(B)$. □

In §3.6.41, we introduced the notion of a normed Q-algebra in the associative setting. Now we extend this notion to the more general setting of Jordan-admissible algebras. Thus, by a (Jordan-admissible) *normed Q-algebra over* $\mathbb{K}$ we mean a normed Jordan-admissible algebra A over $\mathbb{K}$ such that the set of all quasi-J-invertible elements of A is a neighbourhood of zero in A. According to Fact 4.4.15, *every complete normed Jordan-admissible algebra over* $\mathbb{K}$ *is a normed Q-algebra over* $\mathbb{K}$.

Lemma 4.4.21 *Let A be a normed Q-algebra over $\mathbb{K}$. We have:*

(i) *If Φ is a surjective algebra homomorphism from A to a J-semisimple Jordan-admissible algebra over $\mathbb{K}$, then* $\ker(\Phi)$ *is closed in A.*
(ii) *If M is a closed ideal of A, then A/M is a normed Q-algebra over $\mathbb{K}$.*
(iii) *If x is in A with $\mathfrak{s}(x) < 1$, then x is quasi-J-invertible in A.*

Proof Let B be a J-semisimple Jordan-admissible algebra over $\mathbb{K}$, and let $\Phi : A \to B$ be a surjective algebra homomorphism. Take $\delta > 0$ such that a is quasi-J-invertible in A whenever a is in A with $\|a\| < \delta$, and let b be in $\Phi(\overline{\ker(\Phi)})$. Then there are $x \in A$ and $y \in \ker(\Phi)$ such that $b = \Phi(x)$ and $\|x - y\| < \delta$. Since $b = \Phi(x - y)$, and $x - y$ is quasi-J-invertible in A, it follows from Lemma 4.4.20(i) that b is quasi-J-invertible in B. Thus $\Phi(\overline{\ker(\Phi)})$ is a quasi-J-invertible ideal of B, and hence $\Phi(\overline{\ker(\Phi)}) = 0$ because B is J-semisimple. Therefore $\overline{\ker(\Phi)} \subseteq \ker(\Phi)$, and $\ker(\Phi)$ is closed in A, which proves assertion (i).

Assertion (ii) follows from Lemma 4.4.20(i) and the fact that, for any closed ideal M of A, the natural quotient homomorphism $A \to A/M$ is open.

In view of §4.4.6, to prove assertion (iii) we may assume that A is commutative. Choose $\delta > 0$ such that a is quasi-J-invertible in A whenever a is in A with $\|a\| < \delta$, and let x be in A with $\mathfrak{r}(x) < 1$. Then $x^n \to 0$ (cf. Corollary 1.1.18(ii)), and hence there exists $n \in \mathbb{N}$ such that $\|x^n\| < \delta$, which implies by the choice of δ that x^n is quasi-J-invertible in A. Now we have $1 \notin \text{J-sp}(A_{\mathbb{1}}, x^n)$, which implies in view of Corollary 4.1.79 and Proposition 1.3.4(i) that $1 \notin \text{J-sp}(A_{\mathbb{1}}, x)$. Thus x is quasi-J-invertible in A, and the proof of assertion (iii) is complete. □

Definition 4.4.22 We say that a normed algebra $(A, \|\cdot\|)$ *has minimality of norm topology* if every continuous algebra norm on A is equivalent to $\|\cdot\|$.

Theorem 4.4.23 *Let A be a normed complex Q-algebra, let B be a complete normed J-semisimple Jordan-admissible complex algebra having minimality of norm topology, and let $\Phi : A \to B$ be a surjective algebra homomorphism. Then Φ is continuous.*

Proof By assertions (i) and (ii) in Lemma 4.4.21, $\ker(\Phi)$ is a closed ideal of A, and the quotient algebra $A/\ker(\Phi)$ is a normed complex Q-algebra. Therefore, if $\hat{\Phi} : A/\ker(\Phi) \to B$ denotes the algebra isomorphism induced by Φ, and if for $b \in B$ we define $|b| := \|\hat{\Phi}^{-1}(b)\|$, then $(B, |\cdot|)$ becomes a normed complex Q-algebra. Now the desired continuity of Φ is equivalent to the existence of a positive number K such that $\|b\| \leqslant K|b|$ for every $b \in B$. To prove this, let $(C, |\cdot|)$ denote the completion of $(B, |\cdot|)$, and let F stand for the mapping $b \to b$ from $(B, \|\cdot\|)$ to $(C, |\cdot|)$. Since F is an algebra homomorphism with dense range, $\mathfrak{S}(F)$ is a closed ideal of $(C, |\cdot|)$ (by Lemma 1.1.58). Moreover, by Lemma 4.4.19, the mapping $b \to b + \mathfrak{S}(F)$ from $(B, \|\cdot\|)$ to $(C/\mathfrak{S}(F), |\cdot|)$ is continuous, i.e. there is a positive number M such that

$|b+\mathfrak{S}(F)| \leqslant M\|b\|$ for every $b \in B$. On the other hand, for $b \in B \cap \mathfrak{S}(F)$ we have in view of Proposition 4.4.7 that $\mathfrak{s}_{|\cdot|}(b) = 0$ and, since $(B, |\cdot|)$ is a normed complex Q-algebra, this implies that b is quasi-J-invertible in B (by Lemma 4.4.21(iii)). Now $B \cap \mathfrak{S}(F)$ is a quasi-J-invertible ideal of B, and hence $B \cap \mathfrak{S}(F) = 0$ because B is J-semisimple. It follows that, if for $b \in B$ we define $|||b||| := |b+\mathfrak{S}(F)|$, then $|||\cdot|||$ becomes an algebra norm on B satisfying $|||\cdot||| \leqslant M\|\cdot\|$. Since B has minimality of norm topology, $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on B, so there is a positive number K such that

$$\|b\| \leqslant K|||b||| = K|b+\mathfrak{S}(F)| \leqslant K|b|$$

for every $b \in B$, and the proof is concluded. □

4.4.3 The main results

According to Proposition 1.2.51, every algebra norm on a commutative C^*-algebra is greater than the natural norm. This result does not remain true if commutativity is removed. Indeed, we have the following.

Fact 4.4.24 *Let A be a complete normed complex algebra such that the inequality $\|\cdot\| \leqslant K_A |||\cdot|||$ holds for some positive constant K_A and every equivalent algebra norm $|||\cdot|||$ on A. Then A has no nonzero continuous derivation. Therefore, if in addition A is associative, then A is commutative.*

Proof Let D be a continuous derivation of A. Then, by Lemma 2.2.21, for each $z \in \mathbb{C}$ the mapping $a \to \|\exp(zD)(a)\|$ is an equivalent algebra norm on A, and hence we have $\|a\| \leqslant K_A\|\exp(zD)(a)\|$ for every $a \in A$. With $\exp(-zD)(a)$ instead of a, the last inequality reads as $\|\exp(-zD)(a)\| \leqslant K_A\|a\|$, so that, by Liouville's theorem, we get $D = 0$. □

In view of the above fact, the best non-commutative generalization of Proposition 1.2.51 one can expect is that the topology of the norm of any C^*-algebra A be smaller than the topology of any algebra norm on A. This expectation will become true as a consequence of Theorem 4.4.29 below.

Fact 4.4.25 *Let A be a complete normed unital Jordan-admissible algebra over $\mathbb{K}$, and let a be in A. We have:*

(i) *If $\mathbb{K} = \mathbb{C}$, then* $\mathfrak{s}(a) = \max\{|\lambda| : \lambda \in \text{J-sp}(A,a)\}$.
(ii) *If $\mathbb{K} = \mathbb{R}$, then*

$$\mathfrak{s}(a) = \max\{|\alpha + i\beta| : \alpha, \beta \in \mathbb{R} \ \textit{with} \ (a-\alpha\mathbf{1})^2 + \beta^2\mathbf{1} \notin \text{J-Inv}(A)\}.$$

Proof Since squares of elements, J-invertible elements, J-spectra of elements, and $\mathfrak{s}(\cdot)$ mean the same in A and in A^{sym}, we may assume that A is commutative. Then assertion (i) follows from Theorem 4.1.17, whereas assertion (ii) follows from Proposition 4.1.86 and Corollary 1.1.101. □

The next lemma generalizes Corollary 3.6.23.

Lemma 4.4.26 *Let A be a normed Jordan-admissible algebra over $\mathbb{K}$, and let x be in* J-Rad(A). *Then $\mathfrak{s}(x) = 0$.*

Proof Suppose first that $\mathbb{K} = \mathbb{C}$. Then, $\frac{x}{\lambda}$ is quasi-J-invertible in A for every $\lambda \in \mathbb{C} \setminus \{0\}$, and hence $\text{J-sp}(A_{\mathbb{1}}, x) \subseteq \{0\}$. A fortiori, the J-spectrum of x relative to the completion of $A_{\mathbb{1}}$ has no nonzero element, and therefore, by Fact 4.4.25(i), we have $\mathfrak{s}(x) = 0$, as desired.

Suppose now that $\mathbb{K} = \mathbb{R}$. Let α, β be in $\mathbb{R}$ such that $\lambda := \alpha + i\beta \neq 0$. Then, since $\frac{2\alpha x - x^2}{|\lambda|^2}$ lies in $\text{J-Rad}(A)$, $\frac{2\alpha x - x^2}{|\lambda|^2}$ is quasi-J-invertible in A, and hence

$$(x - \alpha\mathbb{1})^2 + \beta^2\mathbb{1} \; (= |\lambda|^2\mathbb{1} - 2\alpha x + x^2)$$

is J-invertible in $A_{\mathbb{1}}$ (so also in the completion of $A_{\mathbb{1}}$). It follows from Fact 4.4.25(ii) that $\mathfrak{s}(x) = 0$. □

Since for every element x in a JB-algebra we have $\mathfrak{s}(x) = \mathfrak{r}(x) = \|x\|$, Lemma 4.4.26 immediately above implies the following.

Corollary 4.4.27 *Every JB-algebra is J-semisimple.*

Lemma 4.4.28 *Let A be a non-commutative JB^*-algebra. We have:*

(i) *If $|||\cdot|||$ is any algebra norm on A, then*

  (a) $\|x\|^2 \leqslant 2|||x^*||| \, |||x|||$ *for every $x \in A$,*

  (b) $(A, |||\cdot|||)$ *is a normed Q-algebra.*

(ii) *A has minimality of norm topology.*

(iii) *A is J-semisimple.*

Proof Let $|||\cdot|||$ be an algebra norm on A, and let x be in A. By Proposition 3.4.1(ii), the closed subalgebra of A generated by $x^* \bullet x$ is a commutative C^*-algebra, and hence, by Proposition 1.2.51, we have

$$\|x^* \bullet x\| \leqslant |||x^* \bullet x||| \leqslant |||x^*||| \, |||x|||.$$

Since $\|x\|^2 \leqslant 2\|x^* \bullet x\|$ (cf. Lemma 3.4.65), we deduce that $\|x\|^2 \leqslant 2|||x^*||| \, |||x|||$. Hence $\|x^n\|^2 \leqslant 2|||(x^*)^n||| \, |||x^n|||$ for every natural number n, which implies that $(\mathfrak{r}_{\|\cdot\|}(x))^2 \leqslant \mathfrak{r}_{|||\cdot|||}(x^*)\mathfrak{r}_{|||\cdot|||}(x)$. Since $\mathfrak{r}_{|||\cdot|||}(x^*) \leqslant \mathfrak{r}_{\|\cdot\|}(x)$ (by Proposition 3.4.63), it follows that $\mathfrak{r}_{\|\cdot\|}(x) \leqslant \mathfrak{r}_{|||\cdot|||}(x)$. Therefore, if $|||x||| < 1$, then $\mathfrak{r}_{\|\cdot\|}(x) < 1$, so $\mathbb{1} - x$ is J-invertible in $A_{\mathbb{1}}$ (by Lemma 4.1.15), and so x is quasi-J-invertible in A. This shows that $(A, |||\cdot|||)$ is a normed Q-algebra, and the proof of assertion (i) is concluded.

Let $|||\cdot|||$ be a continuous algebra norm on A (say $|||\cdot||| \leqslant M\|\cdot\|$ for some $M > 0$). By assertion (i)(a) already proved and Proposition 3.3.13, for every $x \in A$ we have

$$\|x\|^2 \leqslant 2M\|x^*\| \, |||x||| = 2M\|x\| \, |||x|||,$$

so $\|x\| \leqslant 2M|||x|||$. Thus $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on A, hence A has minimality of norm topology, and assertion (ii) is proved.

Let x be in $\text{J-Rad}(A)$. Then $x^* \bullet x$ lies in $\text{J-Rad}(A)$, so $\mathfrak{r}(x^* \bullet x) = 0$ (by Lemma 4.4.26), and so $x^* \bullet x = 0$ (by Proposition 3.4.1(ii) and Lemma 1.2.12). Since $\|x\|^2 \leqslant 2\|x^* \bullet x\|$ (cf. Lemma 3.4.65 again), we deduce that $x = 0$. This proves assertion (iii). □

Let A be an alternative C^*-algebra. Arguing as in the proof of Lemma 2.3.27, we realize that, *if $|||\cdot|||$ is any algebra norm on A, then we have $\|x\|^2 \leqslant |||x^*||| |||x|||$ for every $x \in A$.* In our present particular case, this becomes a refinement of assertion (i)(a) in the above lemma. The associative forerunner of the result just formulated can be found in [795, Theorem 4.8.3(ii)].

Theorem 4.4.29 *Let A be a non-commutative JB^*-algebra. Then the topology of any algebra norm on A is stronger than that of the natural norm.*

Proof Let $|||\cdot|||$ be an algebra norm on A. By Lemma 4.4.28(i)(b), $(A, |||\cdot|||)$ is a normed complex Q-algebra, whereas, by Lemma 4.4.28(ii)–(iii), $(A, \|\cdot\|)$ is a complete normed J-semisimple Jordan-admissible complex algebra having minimality of norm topology. Therefore the result follows by applying Theorem 4.4.23 to the surjective algebra homomorphism $\Phi : x \to x$ from $(A, |||\cdot|||)$ to $(A, \|\cdot\|)$. □

Theorem 4.4.29 above contains one of the ingredients in its proof, namely that (as assured by Lemma 4.4.28(ii)) non-commutative JB^*-algebras have minimality of norm topology. But Lemma 4.4.28(ii) also has other interesting consequences, which are discussed in the rest of this subsection. The most direct one is the following.

Corollary 4.4.30 *Let A be a non-commutative JB^*-algebra, let B be a normed complex algebra, and let $\phi : A \to B$ be a continuous algebra homomorphism. Then the range of ϕ is bicontinuously isomorphic to a non-commutative JB^*-algebra. As a consequence, $\phi(A)$ is closed in B.*

Proof By Proposition 3.4.13, $C := A/\ker(\phi)$ is a non-commutative JB^*-algebra. Let $\hat{\phi} : C \to B$ stand for the induced (continuous and injective) algebra homomorphism. Since $c \to \|\hat{\phi}(c)\|$ is a continuous algebra norm on C, it follows from Lemma 4.4.28(ii) that there exists $M > 0$ such that $\|c\| \leqslant M\|\hat{\phi}(c)\|$ for every $c \in C$. Thus $\hat{\phi}$, regarded as a mapping from C to $\phi(A)$, becomes a bicontinuous algebra isomorphism. □

Invoking Fact 3.3.4, it follows from Corollary 4.4.30 just proved that, *if A is a non-commutative JB^*-algebra, if B is a normed complex algebra, and if $\phi : A \to B$ is a continuous Jordan homomorphism, then $\phi(A)$ is closed in B.*

Corollary 4.4.31 *Let A be a unital non-commutative JB^*-algebra, let B be a normed unital Jordan-admissible complex algebra, and let $\phi : A \to B$ be a continuous unit-preserving algebra homomorphism. Then $\phi(A)$ is a J-full subalgebra of B.*

Proof Let b be in $\phi(A) \setminus$ J-Inv$(\phi(A))$. By Corollary 4.4.30 and Theorem 4.1.71(i), b is a topological J-divisor of zero in $\phi(A)$, so it is a topological J-divisor of zero in B (cf. Definition 4.1.69). Therefore, by Definition 4.1.56 and Proposition 4.1.25(ii), b is not J-invertible in B. □

Corollary 4.4.32 *Let B be a normed associative complex algebra, and assume that B is the range of a continuous Jordan homomorphism from some non-commutative JB^*-algebra. Then B is bicontinuously isomorphic to a C^*-algebra.*

Proof By the assumption, there is a non-commutative JB^*-algebra A, and a continuous surjective Jordan homomorphism $\phi : A \to B$. Since A^{sym} is a JB^*-algebra (cf. Fact 3.3.4), and ϕ becomes a Jordan homomorphism from A^{sym} to B, we may assume that A is commutative. Then $\ker(\phi)$ is a closed ideal of A, and hence, by Proposition 3.4.13, $A/\ker(\phi)$ is a JB^*-algebra, so there is no loss of generality in assuming additionally that ϕ is bijective. Then, since $a \to \|\phi(a)\|$ is a continuous algebra norm on the JB^*-algebra A, we can apply Lemma 4.4.28(ii) to realize that ϕ is bicontinuous. Let C denote the associative complex algebra consisting of the vector space of A and the product $x \odot y := \phi^{-1}(\phi(x)\phi(y))$. Then C^{sym} $(= A)$ is a JB^*-algebra under the norm and the involution of A, and hence, by Theorem 3.6.30, C becomes a C^* algebra under the same norm and involution. Finally, since

$$\phi(x \odot y) = \phi(x)\phi(y) \text{ for all } x, y \in C,$$

ϕ becomes a bicontinuous algebra isomorphism from the C^*-algebra C onto B. □

Definition 4.4.33 Let $(A, \|\cdot\|)$ be a normed algebra. We say that $(A, \|\cdot\|)$ *has minimality of norm* if, whenever $|||\cdot|||$ is an algebra norm on A with $|||\cdot||| \leqslant \|\cdot\|$, we have in fact $|||\cdot||| = \|\cdot\|$.

The next result becomes a relevant geometric complement to Theorem 4.4.29. Instead of Theorem 4.4.23, the key tool here is the non-associative Vidav–Palmer theorem.

Proposition 4.4.34 *Every non-commutative JB^*-algebra has minimality of norm.*

Proof Let A be a nonzero non-commutative JB^*-algebra, and let $|||\cdot|||$ be an algebra norm on A such that $|||\cdot||| \leqslant \|\cdot\|$. By Lemma 4.4.28(ii), $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on A.

Assume at first that A is unital. Then, according to §1.1.5, we have $1 \leqslant |||\mathbf{1}||| \leqslant \|\mathbf{1}\| = 1$, so $|||\mathbf{1}||| = 1$. Let $|||\cdot|||$ and $\|\cdot\|$ stand for the corresponding dual norms of $|||\cdot|||$ and $\|\cdot\|$. It follows that for $f \in A'$ we have $\|f\| \leqslant |||f|||$, so $D((A, |||\cdot|||), \mathbf{1}) \subseteq D((A, \|\cdot\|), \mathbf{1})$, and so

$$V((A, |||\cdot|||), \mathbf{1}, a) \subseteq V((A, \|\cdot\|), \mathbf{1}, a)$$

for every $a \in A$. Since $(A, \|\cdot\|)$ is a V-algebra (by Lemma 2.2.5), we deduce that $(A, |||\cdot|||)$ is a complete V-algebra whose natural involution coincides with the JB^*-involution of A. Consequently, $(A, |||\cdot|||)$ is a non-commutative JB^*-algebra (by Theorem 3.3.11), the identity mapping $(A, |||\cdot|||) \to (A, \|\cdot\|)$ is an algebra $*$-homomorphism, and hence, by Proposition 3.4.4, we have that $|||\cdot||| = \|\cdot\|$.

Now remove the assumption that A is unital. Let $|||\cdot|||$ and $\|\cdot\|$ stand for the corresponding bidual norms of $|||\cdot|||$ and $\|\cdot\|$. By Theorem 3.5.34, $(A'', \|\cdot\|)$ becomes naturally a unital non-commutative JB^*-algebra. Since $|||\cdot||| \leqslant \|\cdot\|$ on A'', we deduce from the above paragraph that $|||\cdot||| = \|\cdot\|$ on A'', and hence on A. □

Corollary 4.4.35 *Let B be a normed complex algebra, and assume that B is the range of a contractive Jordan homomorphism from some non-commutative JB^*-algebra. Then B is isometrically isomorphic to a non-commutative JB^*-algebra.*

Proof By the assumption, there is a non-commutative JB^*-algebra A, and a contractive surjective Jordan homomorphism $\phi : A \to B$. As in the proof of Corollary 4.4.32, we may assume that A is commutative and that ϕ is bijective. Then, since

$a \to |||a||| := \|\phi(a)\|$ is an algebra norm on the JB^*-algebra A, and $|||\cdot||| \leqslant \|\cdot\|$ on A, it follows from Proposition 4.4.34 that $|||\cdot||| = \|\cdot\|$ on A, and hence that ϕ is an isometry. Let C denote the complex algebra consisting of the vector space of A and the product $x \odot y := \phi^{-1}(\phi(x)\phi(y))$, and note that, since ϕ is an isometry and B is a normed algebra, the norm of A becomes an algebra norm on C. Then, since C^{sym} $(= A)$ is a JB^*-algebra under the norm and the involution of A, C becomes a JB^*-admissible algebra in the sense of Definition 3.5.29. Therefore, by Proposition 3.5.31, C is in fact a non-commutative JB^*-algebra under the same norm and involution. Finally, since $\phi(x \odot y) = \phi(x)\phi(y)$ for all $x, y \in C$, ϕ becomes an isometric algebra isomorphism from the non-commutative JB^*-algebra C onto B. □

Combining Fact 3.3.2 and Corollary 4.4.35, we get the following.

Corollary 4.4.36 *Let B be a normed alternative complex algebra, and assume that B is the range of a contractive Jordan homomorphism from some non-commutative JB^*-algebra. Then B is isometrically isomorphic to an alternative C^*-algebra.*

4.4.4 The uniqueness-of-norm theorem for general non-associative algebras

In this subsection, we apply the associative particularization of Lemma 4.4.5 to prove a powerful uniqueness-of-norm theorem for general non-associative algebras.

Proposition 4.4.37 *Let A and B be normed associative complex algebras, and let $\Phi : A \to B$ be an algebra homomorphism. We have:*

(i) *If A is a normed Q-algebra, then the inequality $\mathfrak{r}(\Phi(a)) \leqslant \mathfrak{r}(a)$ holds for every $a \in A$.*
(ii) *If both A and B are normed Q-algebras, and if Φ is surjective, then $\mathfrak{S}(\Phi)$ is contained in the Jacobson radical of B.*

Proof Assume that A is a normed Q-algebra. To prove assertion (i), we can see Φ as an algebra homomorphism from A to the completion of B, and hence we may assume that B is complete. Then, extending Φ to an algebra homomorphism from $A_{\mathbb{1}}$ to $B_{\mathbb{1}}$, and applying Lemma 1.1.34(ii), we derive that for every $a \in A$ we have $\text{sp}(B_{\mathbb{1}}, \Phi(a)) \subseteq \text{sp}(A_{\mathbb{1}}, a)$. Therefore, by Theorem 1.1.46 and the implication (i)⇒(ix) in Proposition 3.6.43, we have

$$\mathfrak{r}(\Phi(a)) = \max\{|\lambda| : \lambda \in \text{sp}(B_{\mathbb{1}}, \Phi(a))\} \leqslant \max\{|\lambda| : \lambda \in \text{sp}(A_{\mathbb{1}}, a)\} = \mathfrak{r}(a)$$

for every $a \in A$.

Now assume that both A and B are normed Q-algebras and that Φ is surjective. Then, by assertion (i) just proved and Lemma 4.4.5, we have $\mathfrak{r}(b) = 0$ whenever b is in $\mathfrak{S}(\Phi)$. But, for $b \in B$, the condition $\mathfrak{r}(b) < 1$ implies that b is quasi-invertible in B (by the implication (i)⇒(ii) in Proposition 3.6.43). Therefore $\mathfrak{S}(\Phi)$ becomes a quasi-invertible ideal of B, hence $\mathfrak{S}(\Phi) \subseteq \text{Rad}(B)$ (by Theorem 3.6.21), which proves assertion (ii). □

§**4.4.38** Let A be an associative algebra over $\mathbb{K}$. Recall that, according to §3.6.41, a subalgebra B of A is said to be a quasi-full subalgebra of A if, whenever b belongs to

B and is quasi-invertible in A, the quasi-inverse of b lies in B. The following routines will be applied without notice.

(i) A subalgebra B of A is a quasi-full subalgebra of A if and only if $B_{\mathbb{1}}$ is a full subalgebra of $A_{\mathbb{1}}$ (cf. Definition 1.1.72), if and only if $\mathrm{sp}(A_{\mathbb{1}}, b) = \mathrm{sp}(B_{\mathbb{1}}, b)$ for every $b \in B$.
(ii) If B is a quasi-full subalgebra of A, then B remains quasi-full in any subalgebra of A containing it.
(iii) The intersection of any family of quasi-full subalgebras of A is a quasi-full subalgebra of A.
(iv) One-sided ideals of A are quasi-full subalgebras of A.
(v) If A is unital, then full subalgebras of A are quasi-full subalgebras of A.
(vi) The relation 'to be a quasi-full subalgebra of' is transitive.

Definition 4.4.39 (a) Let A be an associative algebra over $\mathbb{K}$. Since the intersection of any family of quasi-full subalgebras of A is a quasi-full subalgebra of A, it follows that, for any non-empty subset S of A, there is a smallest quasi-full subalgebra of A containing S. This subalgebra will be called *the quasi-full subalgebra of A generated by S*.

(b) Now let A be an arbitrary algebra over $\mathbb{K}$. The quasi-full subalgebra of $L(A)$ generated by $L_A \cup R_A$ will be called *the quasi-full multiplication algebra of* A, and will be denoted by $\mathscr{QFM}(A)$.

(c) Again, let A be an arbitrary algebra over $\mathbb{K}$. Then the set $\mathscr{W}(A)$, of those elements $a \in A$ such that L_a and R_a belong to the Jacobson radical of $\mathscr{QFM}(A)$, is a subspace of A, and hence contains a largest subspace of A invariant under the algebra of operators $\mathscr{QFM}(A)$. This last subspace of A, which is clearly an ideal of A, will be called the *weak radical* of A, and will be denoted by w-Rad(A).

Lemma 4.4.40 *Let A be a real algebra. Then*

$$\text{w-Rad}(A_{\mathbb{C}}) \cap A \subseteq \text{w-Rad}(A).$$

Proof The real algebra $L(A)$ can and will be identified in an obvious way with the full real subalgebra of $L(A_{\mathbb{C}})$ consisting of those complex-linear operators on $A_{\mathbb{C}}$ which leave A invariant. For $a \in A$, the multiplication operators by a on A are then identified with the multiplication operators by a on $A_{\mathbb{C}}$, so the symbols L_a and R_a have unambiguous meanings. Now, clearly, $\mathscr{QFM}(A_{\mathbb{C}}) \cap L(A)$ is a quasi-full subalgebra of $L(A)$ containing $L_A \cup R_A$, and hence we have

$$\mathscr{QFM}(A) \subseteq \mathscr{QFM}(A_{\mathbb{C}}) \cap L(A). \tag{4.4.3}$$

As a first consequence of (4.4.3), $\mathscr{QFM}(A)$ becomes a quasi-full real subalgebra of $\mathscr{QFM}(A_{\mathbb{C}})$, so $\mathrm{Rad}(\mathscr{QFM}(A_{\mathbb{C}})) \cap \mathscr{QFM}(A)$ is a quasi-invertible ideal of $\mathscr{QFM}(A)$, and so

$$\mathrm{Rad}(\mathscr{QFM}(A_{\mathbb{C}})) \cap \mathscr{QFM}(A) \subseteq \mathrm{Rad}(\mathscr{QFM}(A)).$$

This inclusion, together with the definition of the weak radical, gives

$$\text{w-Rad}(A_{\mathbb{C}}) \cap A \subseteq \mathscr{W}(A).$$

To conclude the proof, it only remains to show that w-Rad$(A_{\mathbb{C}})\cap A$ is invariant under $\mathscr{QFM}(A)$. But this follows from the inclusion (4.4.3) and the fact that w-Rad$(A_{\mathbb{C}})$ is invariant under $\mathscr{QFM}(A_{\mathbb{C}})$. □

Remark 4.4.41 Let A be complete normed algebra over $\mathbb{K}$. Since $BL(A)$ is a full subalgebra of $L(A)$ (by the Banach isomorphism theorem) containing $L_A \cup R_A$, and $\mathscr{QFM}(A)$ is the smallest quasi-full subalgebra of $L(A)$ containing $L_A \cup R_A$, it follows that $\mathscr{QFM}(A) \subseteq BL(A)$, and that $\mathscr{QFM}(A)$ is actually a quasi-full subalgebra of $BL(A)$.

Proposition 4.4.42 *Let A and B be complete normed algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be a bijective algebra homomorphism. Then $\mathfrak{S}(\Phi)$ is contained in the weak radical of B.*

Proof Assume at first that $\mathbb{K} = \mathbb{C}$. Consider the bijective algebra homomorphism $\hat{\Phi} : L(A) \to L(B)$ defined by $\hat{\Phi}(F) := \Phi F \Phi^{-1}$ for every $F \in L(A)$. Since

$$\hat{\Phi}(L_a) = L_{\phi(a)} \text{ and } \hat{\Phi}(R_a) = R_{\phi(a)} \text{ for every } a \in A, \tag{4.4.4}$$

$\hat{\Phi}$ maps $L_A \cup R_A$ onto $L_B \cup R_B$, so $\mathscr{QFM}(A)$ onto $\mathscr{QFM}(B)$. Now $\mathscr{QFM}(A)$ and $\mathscr{QFM}(B)$ are quasi-full subalgebras of the complete normed associative complex algebras $BL(A)$ and $BL(B)$, respectively (cf. Remark 4.4.41 above), and the mapping $\tilde{\Phi} : F \to \hat{\Phi}(F)$ from $\mathscr{QFM}(A)$ to $\mathscr{QFM}(B)$ is a bijective algebra homomorphism. Therefore, keeping in mind Proposition 4.4.37(ii) and the implication (vii)⇒(i) in Proposition 3.6.43, we have

$$\mathfrak{S}(\tilde{\Phi}) \subseteq \mathrm{Rad}(\mathscr{QFM}(B)).$$

On the other hand, keeping in mind (4.4.4), we easily realize that L_b and R_b belong to $\mathfrak{S}(\tilde{\Phi})$ whenever b is in $\mathfrak{S}(\Phi)$. It follows that L_b and R_b belong to $\mathrm{Rad}(\mathscr{QFM}(B))$ whenever b is in $\mathfrak{S}(\Phi)$. Thus $\mathfrak{S}(\Phi) \subseteq \mathscr{W}(B)$, and the proof will be concluded by showing that $\mathfrak{S}(\Phi)$ is invariant under $\mathscr{QFM}(B)$. Let b be in $\mathfrak{S}(\Phi)$, and let G be in $\mathscr{QFM}(B)$. Take a sequence a_n in A with $a_n \to 0$ and $\Phi(a_n) \to b$, and take $F \in \mathscr{QFM}(A)$ such that $\tilde{\Phi}(F) = G$. Then $F(a_n) \to 0$ and $\Phi(F(a_n)) = G(\Phi(a_n)) \to G(b)$, so $G(b) \in \mathfrak{S}(\Phi)$, as desired.

Now assume that $\mathbb{K} = \mathbb{R}$. Then Φ can be extended to a bijective algebra homomorphism $\Psi := I_{\mathbb{C}} \otimes \Phi$ from $A_{\mathbb{C}}$ to $B_{\mathbb{C}}$. Since $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$ are complete normed complex algebras (cf. Proposition 1.1.98), the above paragraph applies, so that $\mathfrak{S}(\Psi) \subseteq$ w-Rad$(B_{\mathbb{C}})$. But, clearly, $\mathfrak{S}(\Phi) \subseteq \mathfrak{S}(\Psi) \cap B$, so $\mathfrak{S}(\Phi) \subseteq$ w-Rad(B) because of Lemma 4.4.40. □

As a consequence of the above proposition, *bijective algebra homomorphisms between complete normed algebras with zero weak radical are automatically continuous*. Thus, in equivalent terms, we have established the following.

Theorem 4.4.43 *Let A be an algebra over $\mathbb{K}$ with zero weak radical. Then A has at most one complete algebra norm topology.*

To prove an automatic continuity theorem for surjective algebra homomorphisms between general complete normed non-associative algebras we need to slightly enlarge the weak radical, as follows.

Definition 4.4.44 Let A be an algebra over $\mathbb{K}$. Let $\mathfrak{A}$ be any subalgebra of $L(A)$ containing $L_A \cup R_A$. As in the definition of the weak radical, we can consider the largest $\mathfrak{A}$-invariant subspace of A consisting of elements a such that L_a and R_a lie in the Jacobson radical of $\mathfrak{A}$. This subspace will be called the $\mathfrak{A}$-*radical* of A, and will be denoted by $\mathfrak{A}$-Rad(A). The *ultra-weak radical* of A (denoted by uw-Rad(A)) is defined as the sum of all $\mathfrak{A}$-radicals of A when $\mathfrak{A}$ runs over the set of all subalgebras of $L(A)$ satisfying

$$L_A \cup R_A \subseteq \mathfrak{A} \subseteq \mathscr{QFM}(A).$$

Since the weak radical of A is precisely the $\mathscr{QFM}(A)$-radical of A, it follows that

$$\text{w-Rad}(A) \subseteq \text{uw-Rad}(A). \tag{4.4.5}$$

Theorem 4.4.45 *Let A and B be complete normed algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be a surjective algebra homomorphism. Assume that B has zero ultra-weak radical. Then Φ is continuous.*

Proof By (4.4.5), the extra assumption on B, and Theorem 4.4.43, B has a unique complete algebra norm topology. Therefore it is enough to show that $\ker(\Phi)$ is closed in A. Consider the couples (F, G) with $F \in \mathscr{QFM}(A)$, $G \in \mathscr{QFM}(B)$, and $\Phi F = G\Phi$. Let $\mathfrak{A}$ (respectively, $\mathfrak{B}$) be the set of all F (respectively, G) which appear in these couples. It is easy to realize that $\mathfrak{A}$ (respectively, $\mathfrak{B}$) is a subalgebra of $\mathscr{QFM}(A)$ (respectively, $\mathscr{QFM}(B)$) containing $L_A \cup R_A$ (respectively $L_B \cup R_B$) and that $\hat{\Phi} : F \to G$ becomes a surjective algebra homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ satisfying

$$\hat{\Phi}(L_a) = L_{\phi(a)} \text{ and } \hat{\Phi}(R_a) = R_{\phi(a)} \text{ for every } a \in A. \tag{4.4.6}$$

Let $\mathfrak{C}$ stand for the closure of $\ker(\hat{\Phi})$ in $\mathfrak{A}$ (recall that, by Remark 4.4.41, $\mathscr{QFM}(A)$, so also $\mathfrak{A}$, is an algebra of continuous operators on A), and let G be in $\hat{\Phi}(\mathfrak{C})$. Then there are $H \in \mathfrak{A}$ and $I \in \ker(\hat{\Phi})$ such that $G = \hat{\Phi}(H)$ and $\|H - I\| < 1$. By setting $F := H - I \in \mathfrak{A}$, we have

$$\|F\| < 1 \text{ and } \Phi F = G\Phi. \tag{4.4.7}$$

As a first consequence, it is enough to invoke Lemma 2.8.3 to deduce that $\mathfrak{r}(G) \leqslant \mathfrak{r}(F) < 1$, and hence F (respectively, G) has a quasi-inverse $F^\diamond$ (respectively, $G^\diamond$) in $BL(A)$ (respectively, $BL(B)$). Since $\mathscr{QFM}(A)$ and $\mathscr{QFM}(B)$ are quasi-full subalgebras of $BL(A)$ and $BL(B)$, respectively, $F^\diamond$ and $G^\diamond$ lie in $\mathscr{QFM}(A)$ and $\mathscr{QFM}(B)$, respectively. On the other hand, the equality $\Phi F = G\Phi$ in (4.4.7) implies that $\Phi F^\diamond = G^\diamond \Phi$. It follows that $G^\diamond$ belongs to $\mathfrak{B}$. Since G is arbitrary in $\hat{\Phi}(\mathfrak{C})$, and $\hat{\Phi}(\mathfrak{C})$ is an ideal of $\mathfrak{B}$, we derive that $\hat{\Phi}(\mathfrak{C})$ is a quasi-invertible ideal of $\mathfrak{B}$, and hence $\hat{\Phi}(\mathfrak{C}) \subseteq \text{Rad}(\mathfrak{B})$. If a is any element in the closure of $\ker(\Phi)$ in A, then, keeping in mind (4.4.6), we realize that L_a and R_a belong to $\mathfrak{C}$, so $L_{\Phi(a)}$ $(= \hat{\Phi}(L_a))$ and $R_{\Phi(a)}$ $(= \hat{\Phi}(R_a))$ belong to $\hat{\Phi}(\mathfrak{C})$. Thus $\Phi(\overline{\ker(\Phi)})$ is a subspace of B, any element b of which satisfies that L_b and R_b belong to $\text{Rad}(\mathfrak{B})$. This, together with the invariance of $\Phi(\overline{\ker(\Phi)})$ under $\mathfrak{B}$ (which is easy to see), shows that

$$\Phi(\overline{\ker(\Phi)}) \subseteq \mathfrak{B}\text{-Rad}(B) \subseteq \text{uw-Rad}(B) = 0.$$

Hence $\ker(\Phi)$ is closed in A, as desired. $\square$

Since both the weak and the ultra-weak radical are not classical notions, we devote the remaining part of this subsection to derive relevant consequences from Theorems 4.4.43 and 4.4.45 which can be formulated in classical terms. In particular, the most relevant results in Subsection 4.4.1 will be fully refined (see Corollaries 4.4.62 and 4.4.64 below). A detailed discussion of the results obtained in this process will also be included.

We begin by deriving some remarkable consequences of Theorem 4.4.43.

Exercise 4.4.46 Let A be a $*$-algebra over $\mathbb{K}$. Then both the weak and the ultra-weak radical of A are $*$-invariant subsets.

Hint With the notation in §3.4.71, the mapping $\Theta : T \to T^*$ becomes a conjugate-linear algebra automorphism of $L(A)$ leaving the set $L_A \cup R_A$ invariant. Therefore Θ leaves $\mathscr{Q}\mathscr{F}\mathscr{M}(A)$ invariant, and hence induces an involutive map on the set of all subalgebras $\mathfrak{A}$ of $L(A)$ with

$$L_A \cup R_A \subseteq \mathfrak{A} \subseteq \mathscr{Q}\mathscr{F}\mathscr{M}(A). \qquad \square$$

By an *algebra with hermitian multiplication* we mean a complete normed complex $*$-algebra A such that L_a and R_a lie in $H(BL(A), I_A)$ whenever a is in $H(A,*)$. Without enjoying their name, algebras with hermitian multiplication already appeared in Theorem 4.1.103 and Proposition 4.1.105. The annihilator $\mathrm{Ann}(A)$ of an algebra A was introduced in Definition 1.1.10.

Proposition 4.4.47 *Let A be an algebra with hermitian multiplication. Then*

$$\text{w-Rad}(A) \subseteq \mathrm{Ann}(A). \tag{4.4.8}$$

As a consequence, if A has zero annihilator, then A has a unique complete algebra norm topology.

Proof In view of Exercise 4.4.46, to prove (4.4.8) it is enough to show that $\text{w-Rad}(A) \cap H(A,*) \subseteq \mathrm{Ann}(A)$. Let a be in $\text{w-Rad}(A) \cap H(A,*)$. Since a lies in $\text{w-Rad}(A)$, we invoke Remark 4.4.41 and Corollary 3.6.23 to obtain that $\mathfrak{r}(L_a) = \mathfrak{r}(R_a) = 0$. Then, since L_a and R_a belong to $H(BL(A), I_A)$, it follows from Proposition 2.3.22 that $L_a = R_a = 0$. Hence $a \in \mathrm{Ann}(A)$, as desired. The consequence now follows from Theorem 4.4.43. $\square$

In view of Proposition 4.4.59(i) below, the inclusion $\text{w-Rad}(A) \subseteq \mathrm{Ann}(A)$ in the above proposition is in fact an equality.

Lie algebras are defined as those anticommutative algebras A satisfying the Jacobi identity $(ab)c + (bc)a + (ca)b = 0$ for all $a, b, c \in A$. All associative algebras become Lie algebras under the commutator product

$$(a, b) \to ab - ba,$$

and, by the *Birkhoff–Witt theorem*, there are no Lie algebras others than the subalgebras of the Lie algebras obtained by this procedure (see [752, pp. 159–62]). To formalize the above fact, we recall that, given any algebra A over $\mathbb{K}$, we introduced the symbol A^{ant} to mean the algebra consisting of the vector space of A and the commutator product $[a, b] = ab - ba$ (cf. §4.3.44). If A is in fact a normed algebra under the norm $\|\cdot\|$, then A^{ant} will be seen without notice as a normed algebra under the norm $2\|\cdot\|$.

Remark 4.4.48 (a) Let A be a non-commutative JB^*-algebra. Since L_a and R_a belong to $H(BL(A),I_A)$ for every $a\in H(A,*)$ (by Lemma 3.6.24), and clearly $\mathrm{Ann}(A)=0$, it follows that A is an algebra with hermitian multiplication and zero annihilator. Therefore, by applying Proposition 4.4.47, we re-encounter Proposition 3.4.66 (that A has a unique complete algebra norm topology).

(b) Now let A be an associative algebra with hermitian multiplication and zero centre. Then, clearly, A^{ant} is a Lie algebra with hermitian multiplication and zero annihilator. Therefore A^{ant} has a unique complete algebra norm topology.

(c) Let A be a C^*-algebra with zero centre (for example, the C^*-algebra of all compact operators on an infinite-dimensional complex Hilbert space). It follows from parts (a) and (b) of the present remark that the Lie algebra A^{ant} has a unique complete algebra norm topology.

(d) Again, let A be a non-commutative JB^*-algebra. Since every derivation of A is continuous (cf. Lemma 3.4.26), the set $\mathfrak{D}$ of all derivations of A has a natural complete normed Lie algebra structure as a closed subalgebra of the complete normed Lie algebra $(BL(A))^{\mathrm{ant}}$. It is straightforward that the mapping $D\to D^\sharp:=-D^*$ (cf. §3.4.71) becomes a conjugate-linear algebra involution on $\mathfrak{D}$. Moreover, if D is in $H(\mathfrak{D},\sharp)$, then D belongs to $H(BL(A),I_A)$ (by Lemma 3.4.27), hence $L_D^{BL(A)^{\mathrm{ant}}}=L_D^{BL(A)}-R_D^{BL(A)}$ lies in $H(BL(BL(A)),I_{BL(A)})$ (by Lemma 2.1.10), so $R_D^{\mathfrak{D}}=-L_D^{\mathfrak{D}}\in H(BL(\mathfrak{D}),I_{\mathfrak{D}})$ (by Lemma 2.2.24). Thus $\mathfrak{D}$ is a Lie algebra with hermitian multiplication.

(e) Let A be a C^*-algebra. Let a be in A, and let D be in $\mathrm{Ann}(\mathfrak{D})$. Since $L_a-R_a\in\mathfrak{D}$, Fact 2.4.7 yields $L_{D(a)}-R_{D(a)}=[D,L_a-R_a]=0$, and hence $D(a)$ belongs to the centre of A. Therefore we have $[[D,L_a],L_a]=[L_{D(a)},L_a]=0$. Now, if $a=a^*$, then, by part (a) of the present remark, L_a lies in $H(BL(A),I_A)$, so $[D,L_a]=0$ (by Corollary 2.4.3), hence $L_{D(a)}=[D,L_a]=0$, and finally, by the arbitrariness of $a\in H(A,*)$, $D=0$. Thus $\mathrm{Ann}(\mathfrak{D})=0$. Since $\mathfrak{D}$ is a Lie algebra with hermitian multiplication (by part (d) of the present remark), it follows that $\mathfrak{D}$ has a unique complete algebra norm topology.

(f) Finally, let A be a complex H^*-algebra (cf. Remark 2.6.54). Then, for $a\in H(A,*)$, L_a and R_a are self-adjoint elements of the unital C^*-algebra $BL(A)$, and hence L_a and R_a lie in $H(BL(A),I_A)$. Therefore A is an algebra with hermitian multiplication. As a consequence, if $\mathrm{Ann}(A)=0$, then A has a unique complete algebra norm topology.

Now we deal with the main consequences of Theorem 4.4.45.

Proposition 4.4.49 *Let A be a complete normed algebra over $\mathbb{K}$ with no nonzero two-sided topological divisor of zero. Then* $\text{uw-Rad}(A)=0$. *As a consequence, surjective algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to A are continuous.*

Proof Let x be in $\text{uw-Rad}(A)$. Then, by the definition of the ultra-weak radical and Remark 4.4.41, there are subalgebras $\mathfrak{A}_1,\dots,\mathfrak{A}_m$ of $BL(A)$ containing $L_A\cup R_A$, and elements $x_1,\dots,x_m\in A$ such that L_{x_i} and R_{x_i} belong to $\mathrm{Rad}(\mathfrak{A}_i)$ for every $i=1,\dots,m$, and $x=\sum_{i=1}^m x_i$. By Corollary 3.6.23 and Exercise 1.1.89, for $i=1,\ldots,m$, L_{x_i} and R_{x_i} are left topological divisors of zero in $\mathfrak{A}_i$ (so also in $BL(A)$), and therefore, by Proposition 1.1.94(i), x_i is a two-sided topological divisor of zero in A. Thus $x=0$ because A has no nonzero two-sided topological divisor of zero, and, since x is arbitrary in uw-Rad(A), the first conclusion in the proposition holds. The consequence now follows from Theorem 4.4.45. □

Let A be an algebra over $\mathbb{K}$. We say that A satisfies the *descending chain condition* (DCC) *on ideals* if every descending chain

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

of ideals of A stabilizes. It is easy to realize that A satisfies the DCC on ideals if and only if every non-empty family of nonzero ideals of A has a minimal element. Now let S be a subset of A. Then the intersection of the family of all ideals of A containing S is the smallest ideal of A containing S. This ideal is called *the ideal of* A *generated by* S. It is straightforward that, if A is associative, then the ideal of A generated by a single element $a\in A$ is precisely

$$\mathbb{K}a+Aa+aA+\left\{\sum_{k=1}^n x_k a y_k : n\in\mathbb{N};\ x_1,\ldots,x_n,y_1,\ldots,y_n\in A\right\}.$$

Lemma 4.4.50 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *If* uw-Rad$(A)=0$*, then A is semiprime.*
(ii) *If A is semiprime and satisfies the DCC on ideals, then* uw-Rad$(A)=0$.
(iii) *If A is simple, then* uw-Rad$(A)=0$.

Proof Let P be an ideal of A such that $PP=0$, and let x be in P. Then we have

$$L_x^2 = L_x F L_x = 0 \ \text{ for every } \ F\in\mathscr{M}^\sharp(A)$$

(cf. §3.6.53 for notation), which implies that the ideal of $\mathscr{M}^\sharp(A)$ generated by L_x (say $\mathscr{P}$) satisfies $\mathscr{P}\mathscr{P}=0$, and hence is a quasi-invertible ideal. Thus $L_x\in\mathrm{Rad}(\mathscr{M}^\sharp(A))$ and, analogously, $R_x\in\mathrm{Rad}(\mathscr{M}^\sharp(A))$. Since x is arbitrary in P, we deduce that $P\subseteq\mathscr{M}^\sharp(A)$-Rad$(A)$. Now, if uw-Rad$(A)=0$, then $\mathscr{M}^\sharp(A)$-Rad$(A)=0$, so $P=0$, hence A is semiprime. This proves assertion (i).

Assume that A satisfies the DCC on ideals, and that uw-Rad$(A)\neq 0$. Then we have $\mathfrak{A}$-Rad$(A)\neq 0$ for some subalgebra $\mathfrak{A}$ of $L(A)$ containing $L_A\cup R_A$. Since $\mathfrak{A}$-invariant subspaces of A are ideals of A, and A satisfies the DCC on ideals, there is a minimal $\mathfrak{A}$-invariant subspace P of A with $P\subseteq\mathfrak{A}$-Rad(A). If $PA=0$, then $PP=0$, and hence A is not semiprime. Otherwise, P is an irreducible $\mathfrak{A}$-module (cf. Definition 3.6.35). But then, for $x\in\mathfrak{A}$-Rad(A) we have that $L_x(P)=0$ because $L_x\in\mathrm{Rad}(\mathfrak{A})$ (cf. Theorem 3.6.38(i)), hence $\mathfrak{A}$-Rad$(A)P=0$, so $PP=0$, and finally A is again not semiprime. This proves assertion (ii).

Assume that A is simple. Then A is a semiprime algebra with DCC on ideals. Therefore, by assertion (ii) just proved, we have uw-Rad$(A)=0$. □

By combining Theorem 4.4.45 and Lemma 4.4.50(iii), we get the following.

Corollary 4.4.51 *Surjective algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to complete normed simple algebras over $\mathbb{K}$ are continuous.*

Now we are going to discuss the weak and the ultra-weak radical in the finite-dimensional setting.

Proposition 4.4.52 *Let A be a finite-dimensional algebra over $\mathbb{K}$. We have:*

(i) $\mathscr{M}^\sharp(A) = \mathscr{QFM}(A)$.
(ii) $\mathscr{M}^\sharp(A)\text{-Rad}(A) = \text{uw-Rad}(A) = \text{w-Rad}(A)$.
(iii) *A is semiprime if and only if* $\text{uw-Rad}(A) = 0$.

Proof Since $\mathscr{M}^\sharp(A)$ is a finite-dimensional subalgebra of $L(A)$, the unital extension of $\mathscr{M}^\sharp(A)$ is a finite-dimensional subalgebra of the unital extension of $L(A)$, and hence, by the associative particularization of Proposition 4.1.66, $\mathscr{M}^\sharp(A)$ is a quasi-full subalgebra of $L(A)$. This proves assertion (i).

By assertion (i) just proved, for every subalgebra $\mathfrak{A}$ of $L(A)$ such that

$$L_A \cup R_A \subseteq \mathfrak{A} \subseteq \mathscr{QFM}(A),$$

we must have $\mathscr{M}^\sharp(A) = \mathfrak{A} = \mathscr{QFM}(A)$, and hence

$$\mathscr{M}^\sharp(A)\text{-Rad}(A) = \mathfrak{A}\text{-Rad}(A) = \text{w-Rad}(A).$$

This proves assertion (ii).

Asume that $\text{uw-Rad}(A) = 0$. Then, by Lemma 4.4.50(i), A is semiprime. This proves the 'if' part of assertion (iii).

Assume that A is semiprime. Since A has the DCC on ideals, it follows from Lemma 4.4.50(ii) that $\text{uw-Rad}(A) = 0$. This proves the 'only if' part of assertion (iii). $\square$

We recall that the unique Hausdorff vector space topology of a given finite-dimensional algebra over $\mathbb{K}$ comes from an algebra norm (cf. Proposition 1.1.7). Therefore, when no geometric considerations are involved, to say that a finite-dimensional algebra over $\mathbb{K}$ is normed becomes redundant. The following corollary follows straightforwardly from Theorem 4.4.45 and Proposition 4.4.52(iii).

Corollary 4.4.53 *Surjective algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to finite-dimensional semiprime algebras over $\mathbb{K}$ are continuous.*

Now we can settle the question of the automatic continuity of algebra homomorphisms into finite-dimensional algebras. The unique new ingredient is the following.

Fact 4.4.54 *Let A be a normed algebra over $\mathbb{K}$ having isotropic elements* (cf. §2.5.7)*, then there exists a discontinuous algebra homomorphism from some complete normed, associative, and commutative algebra over $\mathbb{K}$ into A.*

Proof Let B be the complete normed, associative, and commutative algebra consisting of any infinite-dimensional Banach space over $\mathbb{K}$, and the zero product. Take a discontinuous linear functional f on B, and an isotropic element $x \in A$. Then the mapping $b \to f(b)x$ becomes a discontinuous algebra homomorphism from B into A. $\square$

Corollary 4.4.55 *Let A be a finite-dimensional algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *Algebra homomorphisms from complete normed algebras over $\mathbb{K}$ into A are continuous.*
(ii) *Algebra homomorphisms from complete normed, associative, and commutative algebras over $\mathbb{K}$ into A are continuous.*
(iii) *A has no isotropic element.*

Proof In view of Fact 4.4.54, it is enough to show that condition (iii) implies condition (i). Suppose that A has no isotropic element. Let B be a complete normed algebra over $\mathbb{K}$, and let $\Phi : B \to A$ be an algebra homomorphism. Since both the finite dimensionality and the absence of isotropic elements are inherited by all subalgebras of A, to prove that Φ is continuous there is no loss of generality in assuming that Φ is surjective. Then, since the absence of isotropic elements implies semiprimeness, the continuity of Φ follows from Corollary 4.4.53. □

Now we are going to relate the weak and ultra-weak radicals with other classical radicals.

Lemma 4.4.56 *Let A be an algebra over $\mathbb{K}$, let $\mathfrak{A}$ be a subalgebra of $L(A)$ containing $L_A \cup R_A$, and let x be in $\mathfrak{A}$-$\mathrm{Rad}(A)$. Then there exists $a \in A$ such that $x + a - ax = 0$.*

Proof By the definition of $\mathfrak{A}$-$\mathrm{Rad}(A)$, R_x has a quasi-inverse (say T) in $\mathfrak{A}$, so that, writing $a := T(x) - x$, we have

$$x + a - ax = T(x) - T(x)x + x^2 = (T + R_x - R_xT)(x) = 0. \qquad \square$$

Lemma 4.4.57 *Let A be a non-commutative Jordan algebra over $\mathbb{K}$, and let x be in A such that L_x is quasi-invertible in $L(A)$. Then x is quasi-J-invertible in A.*

Proof Since L_x is quasi-invertible in $L(A)$, $I_A - L_x$ is invertible in $L(A)$, hence $L^{A_{\mathbb{1}}}_{\mathbb{1}-x} = I_{A_{\mathbb{1}}} - L^{A_{\mathbb{1}}}_x$ is invertible in $L(A_{\mathbb{1}})$ (by Proposition 4.1.30(i)), so $\mathbb{1} - x$ is J-invertible in $A_{\mathbb{1}}$ (by Proposition 4.1.60), and finally x is quasi-J-invertible in A, as desired. □

Definition 4.4.58 Let A be a Jordan-admissible algebra over $\mathbb{K}$, and let x be a quasi-J-invertible element of A. Then $\mathbb{1} - x$ is J-invertible in $A_{\mathbb{1}}$, so there is a unique element $x^\diamond \in A$ such that $\mathbb{1} - x^\diamond = (\mathbb{1} - x)^{-1}$. The element $x^\diamond$ above is called the *quasi-J-inverse* of x in A. We note that, since

$$(\mathbb{1} - x^\diamond) \bullet (\mathbb{1} - x) = (\mathbb{1} - x)^{-1} \bullet (\mathbb{1} - x) = \mathbb{1},$$

we have $x + x^\diamond - x^\diamond \bullet x = 0$. As a consequence, $x^\diamond$ belongs to any ideal of A containing x.

The strong radical, s-$\mathrm{Rad}(A)$, and the radical, $\mathrm{Rad}(A)$, of a (possibly non-associative) algebra A were introduced in Definitions 3.6.6 and 3.6.12, respectively.

Proposition 4.4.59 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) $\mathrm{Ann}(A) \subseteq \text{w-Rad}(A) \subseteq \text{uw-Rad}(A) \subseteq \mathrm{Rad}(A) \subseteq \text{s-Rad}(A)$.
(ii) *If A is Jordan-admissible, then* $\text{uw-Rad}(A) \subseteq \text{J-Rad}(A)$.
(iii) *If A is non-commutative Jordan, then* $\text{uw-Rad}(A) \subseteq \text{J-Rad}(A) \subseteq \mathrm{Rad}(A)$.

Proof The inclusion $\mathrm{Ann}(A) \subseteq \mathscr{W}(A)$ is clear. Therefore, to conclude that $\mathrm{Ann}(A) \subseteq \text{w-Rad}(A)$ it is enough to show that $\mathrm{Ann}(A)$ is invariant under $\mathscr{QFM}(A)$. But the set

$$\mathfrak{M} := \{F \in L(A) : F(\mathrm{Ann}(A)) = 0\}$$

is a left ideal (so a quasi-full subalgebra) of $L(A)$ containing $L_A \cup R_A$. Therefore $\mathscr{QFM}(A) \subseteq \mathfrak{M}$, which means that $\mathscr{QFM}(A)(\mathrm{Ann}(A)) = 0$, and this is more than the invariance of $\mathrm{Ann}(A)$ under $\mathscr{QFM}(A)$.

The inclusion $\text{w-Rad}(A) \subseteq \text{uw-Rad}(A)$ has already been pointed out in the statement (4.4.5) of Definition 4.4.44.

By the definition of $\text{uw-Rad}(A)$, to prove the inclusion $\text{uw-Rad}(A) \subseteq \mathrm{Rad}(A)$ it is enough to show that $\mathfrak{A}\text{-Rad}(A) \subseteq \mathrm{Rad}(A)$ for every subalgebra $\mathfrak{A}$ of $L(A)$ containing $L_A \cup R_A$. But this follows from Lemma 4.4.56 and Proposition 3.6.18.

The inclusion $\mathrm{Rad}(A) \subseteq \text{s-Rad}(A)$ was already proved (cf. Proposition 3.6.14(i)).

Now, assume that A is Jordan-admissible. In order to prove the inclusion $\text{uw-Rad}(A) \subseteq \text{J-Rad}(A)$ it is enough to show that $\mathfrak{A}\text{-Rad}(A) \subseteq \text{J-Rad}(A)$ for every subalgebra $\mathfrak{A}$ of $L(A)$ containing $L_A \cup R_A$. Let $\mathfrak{A}$ be such a subalgebra, and let x be in $\mathfrak{A}\text{-Rad}(A)$. Then $L_x^{A^{\mathrm{sym}}} = \frac{1}{2}(L_x + R_x)$ is quasi-invertible in $\mathfrak{A}$, so also in $L(A)$. Therefore, since A^{sym} is a Jordan algebra, it follows from Lemma 4.4.57 that x is quasi-J-invertible in A^{sym}, and hence in A (cf. §4.4.8). Thus $\mathfrak{A}\text{-Rad}(A)$ becomes a quasi-J-invertible ideal of A, so $\mathfrak{A}\text{-Rad}(A) \subseteq \text{J-Rad}(A)$, as desired.

Finally, assume that A is non-commutative Jordan, and let x be in $\text{J-Rad}(A)$. Then x is quasi-J-invertible in A (with quasi-J-inverse $x^\diamond$, say). Since

$$(\mathbb{1} - x^\diamond)(\mathbb{1} - x) = (\mathbb{1} - x)^{-1}(\mathbb{1} - x) = \mathbb{1}$$

(by Proposition 4.1.58), we deduce that $x + x^\diamond - x^\diamond x = 0$. Since x is arbitrary in $\text{J-Rad}(A)$, it follows from Proposition 3.6.18 that $\text{J-Rad}(A) \subseteq \mathrm{Rad}(A)$. □

The next result is the weaker we can obtain from Theorem 4.4.45 and from the chain of inclusions in Proposition 4.4.59(i).

Corollary 4.4.60 *Surjective algebra homomorphisms, from complete normed algebras over $\mathbb{K}$ to complete normed strongly semisimple algebras over $\mathbb{K}$, are continuous.*

Let A be an algebra over $\mathbb{K}$ and, according to §1.1.36, let $A^{(0)}$ stand for the opposite algebra of A. Noticing that $\text{uw-Rad}(A) = \text{uw-Rad}(A^{(0)})$, and selecting the appropriate inclusions in Proposition 4.4.59(i), we get that

$$\text{uw-Rad}(A) \subseteq \mathrm{Rad}(A) \cap \mathrm{Rad}(A^{(0)}) \subseteq \text{s-Rad}(A).$$

Therefore, invoking Theorem 4.4.45, we obtain the following refinement of Corollary 4.4.60 immediately above.

Corollary 4.4.61 *Let A and B be complete normed algebras over $\mathbb{K}$, and let $\Phi : A \to B$ be a surjective algebra homomorphism. If $\mathrm{Rad}(B) \cap \mathrm{Rad}(B^{(0)}) = 0$, then Φ is continuous.*

By combining Theorem 4.4.45 and Proposition 4.4.59(ii), we get the next refinement of Theorem 4.4.13.

Corollary 4.4.62 *Surjective algebra homomorphisms from complete normed algebras over $\mathbb{K}$ to complete normed J-semisimple Jordan-admissible algebras over $\mathbb{K}$ are continuous.*

By combining Lemma 4.4.28(iii) and Corollary 4.4.62, we obtain the following refinement of Proposition 3.4.66

Corollary 4.4.63 *Surjective algebra homomorphisms from complete normed complex algebras to non-commutative JB*-algebras are continuous.*

Invoking Proposition 4.4.17(iii) and the commutative particularization of Corollary 4.4.62, we obtain the following refinement of Corollary 4.4.18.

Corollary 4.4.64 *Surjective Jordan homomorphisms from complete normed algebras over $\mathbb{K}$ to complete normed J-semisimple non-commutative Jordan algebras over $\mathbb{K}$ are continuous.*

Let A be a non-commutative Jordan algebra over $\mathbb{K}$. Keeping in mind Proposition 4.4.59(iii), and noticing that $\text{J-Rad}(A) = \text{J-Rad}(A^{(0)})$, we get that

$$\text{J-Rad}(A) \subseteq \mathrm{Rad}(A) \cap \mathrm{Rad}(A^{(0)}).$$

Therefore Corollary 4.4.64 above also refines Corollary 4.4.61 in the particular non-commutative Jordan setting.

Now we begin a detailed discussion of Proposition 4.4.59 and of its consequences. Throughout the discussion, the facts that $\mathrm{Rad}(A) = \text{s-Rad}(A)$ when A is a commutative algebra (cf. Proposition 3.6.14(ii)) and that $\text{J-Rad}(A) = \mathrm{Rad}(A)$ when A is an associative algebra (cf. Theorem 3.6.21) will be applied without notice. First we consider the associative and commutative case.

Proposition 4.4.65 *Let A be an associative and commutative algebra over $\mathbb{K}$. Then we have:*

(i) $L_A = R_A = \mathscr{M}^\sharp(A) = \mathscr{QFM}(A)$.
(ii) $\mathrm{Ann}(A) \subseteq \text{w-Rad}(A) = \text{uw-Rad}(A) = \text{J-Rad}(A) = \mathrm{Rad}(A) = \text{s-Rad}(A)$.
(iii) *The situation $0 = \mathrm{Ann}(A) \subsetneqq \text{w-Rad}(A) = A$ can be exemplified even in the complete normed complex case.*

Proof Since L_A is a subalgebra of $L(A)$ (by associativity), and $L_A = R_A$ (by commutativity), we get that $L_A = R_A = \mathscr{M}^\sharp(A)$. But L_A is actually a quasi-full subalgebra. Indeed, if a is in A such that L_a is quasi-invertible in $L(A)$, then, by Lemma 4.4.57, a is quasi-invertible in A (with quasi-inverse $a^\diamond$, say), and hence $(L_a)^\diamond = L_{a^\diamond} \in L_A$ because the mapping $x \to L_x$ is an algebra homomorphism. Now it is clear that $\mathscr{QFM}(A) = L_A$, which concludes the proof of assertion (i).

In view of Proposition 4.4.59, to prove assertion (ii) only the inclusion $\mathrm{Rad}(A) \subseteq \text{w-Rad}(A)$ must be settled. But, by assertion (i) just proved, $\mathrm{Rad}(A)$ is a $\mathscr{QFM}(A)$-invariant subspace of A. Therefore, to conclude, we must only show that, for every $a \in \mathrm{Rad}(A)$, L_a $(= R_a)$ lies in $\mathrm{Rad}(\mathscr{QFM}(A))$. But this follows from the fact that $x \to L_x$ is now an algebra homomorphism from A onto $\mathscr{QFM}(A)$, so that Lemma 4.4.20(ii) applies.

To prove assertion (iii), fix a sequence α_n of positive real numbers satisfying $\alpha_{n+m} \leqslant \alpha_n \alpha_m$ and $\lim \sqrt[n]{\alpha_n} = 0$, and take A equal to the complete normed complex algebra consisting of those formal power series $a = \sum_{n\in\mathbb{N}} \lambda_n \mathbf{x}^n$ such that $\|a\| := \sum_{n=1}^{\infty} \alpha_n |\lambda_n| < +\infty$. Then, clearly, $\mathrm{Ann}(A) = 0$. Moreover, since $\mathfrak{r}(\mathbf{x}) = \lim \sqrt[n]{\|\mathbf{x}^n\|} = \lim \sqrt[n]{\alpha_n} = 0$, and the set $\{a \in A : \mathfrak{r}(a) = 0\}$ is a closed subalgebra of A (by Proposition 1.1.107 and Theorem 1.1.73), and A is generated by $\mathbf{x}$ as a normed algebra, we derive from Lemma 1.1.20, that every element of A is quasi-invertible in A, i.e. $\mathrm{Rad}(A) = A$. □

As a result, the particularization of Theorem 4.4.45 to the case where the range algebra is associative and commutative gives rise to a result weaker than Gelfand's Theorem 3.6.9.

The discussion of Proposition 4.4.59 in the non-commutative associative case is more involved. Indeed, besides some classical results, it needs the construction of a non-semiprime associative algebra over $\mathbb{K}$ with zero weak radical (compare Lemma 4.4.50(i)). The starting point of the construction is the so-called *Weil algebra* over $\mathbb{K}$. This algebra (say W) is nothing other than the free unital associative algebra over $\mathbb{K}$ generated by two indeterminates $\mathbf{x}, \mathbf{y}$ subjected to the condition $[\mathbf{y}, \mathbf{x}] = 1$. The algebra W has the universal property that, whenever A is any unital associative algebra over $\mathbb{K}$, and a, b are in A with $[b, a] = \mathbf{1}$, there exists a unique (automatically unit-preseving) algebra homomorphism $\Phi : W \to A$ such that $\Phi(\mathbf{x}) = a$ and $\Phi(\mathbf{y}) = b$. Given $f = f(\mathbf{x}, \mathbf{y}) \in W$, there exists a unique quasi-null sequence $\{p_n\}_{n\in\mathbb{N}\cup\{0\}}$ in $\mathbb{K}[\mathbf{x}]$ such that $f = \sum_{n=0}^{\infty} p_n(\mathbf{x})\mathbf{y}^n$. As a consequence, the mapping

$$(p, f) \to p(\mathbf{x}) + f(\mathbf{x}, \mathbf{y})\mathbf{y}$$

from $\mathbb{K}[\mathbf{x}] \times W$ to W becomes a linear bijection, a fact that will be codified by writing $W = \mathbb{K}[\mathbf{x}] \oplus W\mathbf{y}$. It it easily realized that for $f \in W$ we have $[f, \mathbf{x}] = \frac{\partial f}{\partial \mathbf{y}}$, where $\frac{\partial f}{\partial \mathbf{y}}$ stands for the formal partial derivative of f with respect to $\mathbf{y}$. Moreover it is well known that W is a simple algebra [802, Corollary 1.6.34].

Lemma 4.4.66 *Let W stand for the Weil algebra over $\mathbb{K}$. Then there exists a unique algebra homomorphism $\rho : W \to L(W)$ such that*

$$\rho(\mathbf{x}) = L_{\mathbf{x}} \text{ and } \rho(\mathbf{y}) = L_{\mathbf{x}} - R_{\mathbf{x}} + L_{\mathbf{y}} - R_{\mathbf{y}},$$

and a unique algebra antihomomorphism $\theta : W \to L(W)$ such that

$$\theta(\mathbf{x}) = L_{\mathbf{x}} + R_{\mathbf{y}} \text{ and } \theta(\mathbf{y}) = -R_{\mathbf{x}} - R_{\mathbf{y}}.$$

Moreover, both ρ and θ preserve units, and we have

$$[\rho(f), \theta(g)] = 0 \text{ for all } f, g \in W. \tag{4.4.9}$$

Proof The existence and uniqueness of ρ and θ under the requirements in the statement, as well as the fact that ρ and θ preserve units, follow from the universal property of W, once we notice that, for the prefixed values of ρ and θ at $\mathbf{x}$ and $\mathbf{y}$, we have $[\rho(\mathbf{y}), \rho(\mathbf{x})] = [\theta(\mathbf{x}), \theta(\mathbf{y})] = I_W$. By noticing that $[\rho(\mathbf{x}), \theta(\mathbf{x})] = [\rho(\mathbf{x}), \theta(\mathbf{y})] = [\rho(\mathbf{y}), \theta(\mathbf{x})] = [\rho(\mathbf{y}), \theta(\mathbf{y})] = 0$, and that W is generated by $\{\mathbf{x}, \mathbf{y}\}$, the fact that $[\rho(f), \theta(g)] = 0$ for all $f, g \in W$ follows almost straightforwardly. □

Theorem 4.4.67 *Let W stand for the Weil algebra over $\mathbb{K}$, let ρ and θ be the linear mappings from W to $L(W)$ given by Lemma 4.4.66, and let A denote the algebra over $\mathbb{K}$ consisting of the vector space $W \times W$ and the product*

$$(f_1, g_1)(f_2, g_2) := (f_1 f_2, \rho(f_1)(g_2) + \theta(f_2)(g_1)).$$

Then we have:

(i) *A becomes a unital associative algebra.*
(ii) *$J := \{0\} \times W$ is an ideal of A with $JJ = 0$, hence A is not semiprime.*
(iii) *J is the unique nonzero proper ideal of A, hence A satisfies the DCC on ideals.*
(iv) $0 = \mathrm{Ann}(A) = \text{w-Rad}(A) \subsetneqq \text{uw-Rad}(A) = \mathrm{Rad}(A) = \text{s-Rad}(A) = J.$

Proof Assertion (i) follows straightforwardly from the associativity of W and the fact that ρ (respectively, θ) is a unit-preserving algebra homomorphism (respectively, antihomomorphism) from W to $L(W)$ satisfying (4.4.9). Moreover, it is also straightforward that $W \times \{0\}$ is a subalgebra of A isomorphic to W via the mapping $(f, 0) \to f$, that (as assured in assertion (ii)) J is a nonzero ideal of A with $JJ = 0$, and that the natural vector space isomorphism $A/J \cong W$ becomes an algebra isomorphism. The prefixed values of ρ and θ at $\mathbf{x}$ and $\mathbf{y}$ can be forgotten at these points of the proof, but not in those which follow.

We are now going to prove assertion (iii). First, let I be a nonzero ideal of A contained in J, so that $I = \{0\} \times L$ for some nonzero subspace L of W. Then, for every $f \in L$ we have:

(a) $(\mathbf{x}, 0)(0, f) = (0, \rho(\mathbf{x})(f)) = (0, \mathbf{x}f) \in I$, so $\mathbf{x}f \in L$;
(b) $(0, f)(\mathbf{x}, 0) = (0, \theta(\mathbf{x})(f)) = (0, \mathbf{x}f + f\mathbf{y}) \in I$, so $\mathbf{x}f + f\mathbf{y} \in L$, and so $f\mathbf{y} \in L$;
(c) $(0, f)(\mathbf{y}, 0) = (0, \theta(\mathbf{y})(f)) = (0, -f\mathbf{x} - f\mathbf{y}) \in I$, so $f\mathbf{x} + f\mathbf{y} \in L$, and so $f\mathbf{x} \in L$;
(d) $(\mathbf{y}, 0)(0, f) = (0, \rho(\mathbf{y})(f)) = (0, \mathbf{x}f - f\mathbf{x} + \mathbf{y}f - f\mathbf{y}) \in I$, so $\mathbf{x}f - f\mathbf{x} + \mathbf{y}f - f\mathbf{y} \in L$, and so $\mathbf{y}f \in L$;

and, summarizing, $\mathbf{x}f, f\mathbf{y}, f\mathbf{x}, \mathbf{y}f \in L$. Thus L is invariant under left and right multiplication by the generators of W, and is therefore a nonzero ideal of W. Since W is simple, we obtain $L = W$ or, equivalently, $I = J$. Since I is an arbitrary nonzero ideal of A contained in J, we conclude that J is a minimal ideal of A. But, on the other hand, the algebra isomorphism $A/J \cong W$ and the simplicity of W imply that J is a maximal ideal of A. Now let I be any nonzero proper ideal of A. By the minimality of J, either $I \cap J = 0$ or $I \cap J = J$. The first of these possibilities cannot happen because it would imply that $I \oplus J = A$ (by the maximality of J), and then, writing $(1, 0) = u + v$ with $u \in I$ and $v \in J$, v would be a nonzero idempotent in J, contradicting that $JJ = 0$. Therefore, $I \cap J = J$, so $J \subseteq I$, and so, again by the maximality of J, we finally have $I = J$. Thus assertion (iii) has been proved.

Set $F := R_{(\mathbf{x},1)} - L_{(\mathbf{x},0)} \in \mathscr{M}^{\sharp}(A)$. We are going to show that F is a bijective operator on A. To this end, note that for every $(f,g) \in A$ we have

$$\begin{aligned} F(f,g) &= (f,g)(\mathbf{x},1) - (\mathbf{x},0)(f,g) = (f\mathbf{x}, \rho(f)(1) + \theta(\mathbf{x})(g)) - (\mathbf{x}f, \rho(\mathbf{x})(g)) \\ &= (f\mathbf{x}, \rho(f)(1) + \mathbf{x}g + g\mathbf{y}) - (\mathbf{x}f, \mathbf{x}g) = ([f,\mathbf{x}], g\mathbf{y} + \rho(f)(1)). \end{aligned}$$

Let (f,g) be in A such that $F(f,g) = 0$. Then we have

$$f\mathbf{x} - \mathbf{x}f = 0 \ \text{ and } \ g\mathbf{y} + \rho(f)(1) = 0. \tag{4.4.10}$$

The first equality in (4.4.10) implies that $\frac{\partial f}{\partial \mathbf{y}} = 0$, so $f = p(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$, hence

$$\rho(f) = \rho(p(\mathbf{x})) = p(\rho(\mathbf{x})) = p(L_{\mathbf{x}}),$$

and finally $\rho(f)(1) = p(\mathbf{x}) = f$. Therefore, by the second equality in (4.4.10), we get $g\mathbf{y} + f = 0$, which implies $f = g = 0$ because $W = \mathbb{K}[\mathbf{x}] \oplus W\mathbf{y}$. Thus F is injective. To prove that F is surjective, let now (f,g) be arbitrary in A. Since the mapping $\frac{\partial}{\partial \mathbf{y}} : W \to W$ is surjective, there exists $h \in W$ such that $[h,\mathbf{x}] = \frac{\partial h}{\partial \mathbf{y}} = f$. Write $h = h_1 + h_2\mathbf{y}$ and $g = g_1 + g_2\mathbf{y}$ with $h_1, g_1 \in \mathbb{K}[\mathbf{x}]$ and $h_2, g_2 \in W$, and set $e := g_1 + h_2\mathbf{y}$. Then we have

$$F(e, g_2) = ([e,\mathbf{x}], g_2\mathbf{y} + \rho(e)(1)).$$

But $e - h = g_1 - h_1 \in \mathbb{K}[\mathbf{x}]$, so $[e,\mathbf{x}] = \frac{\partial e}{\partial \mathbf{y}} = \frac{\partial h}{\partial \mathbf{y}} = f$. On the other hand, since

$$\rho(e) = \rho(g_1) + \rho(h_2\mathbf{y}) = \rho(g_1) + \rho(h_2)\rho(\mathbf{y}),$$

and $\rho(\mathbf{y})(1) = 0$, we derive that $\rho(e)(1) = \rho(g_1)(1) = g_1$, and hence that $g_2\mathbf{y} + \rho(e)(1) = g_2\mathbf{y} + g_1 = g$. It follows that $F(e, g_2) = (f,g)$. Thus F is surjective.

Since the linear operator $F : A \to A$ in the above paragraph is bijective, it is invertible in $L(A)$ and moreover, since $F \in \mathscr{QFM}(A)$, and $\mathscr{QFM}(A)$ is a quasi-full subalgebra of $L(A)$ containing the unit I_A of $L(A)$ (because A is unital), F^{-1} lies in $\mathscr{QFM}(A)$. Since $F^{-1}(0,1) = (1,0)$ (because $F(1,0) = (0,1)$), and $(0,1)$ belongs to J, and $(1,0)$ does not belong to J, it follows that J is not invariant under $\mathscr{QFM}(A)$. Now, invoking the definitions of the weak and strong radicals, Lemma 4.4.50(i), Proposition 4.4.59, and assertions (ii) and (iii) in the present theorem, assertion (iv) follows. □

In Remark 4.4.68 immediately below, we conclude our discussion of Proposition 4.4.59.

Remark 4.4.68 (a) Let A be an associative algebra over $\mathbb{K}$. We already know that the inclusion $\mathrm{Ann}(A) \subseteq$ w-Rad(A) in Proposition 4.4.59(i) may be strict (cf. Proposition 4.4.65(iii)). Moreover, according to Theorem 4.4.67, the inclusion w-Rad$(A) \subseteq$ uw-Rad(A) may also be strict. We note however that the algebra A in Theorem 4.4.67 cannot be endowed with any algebra norm. (Indeed, the algebra A in Theorem 4.4.67 contains a a copy of the Weil algebra, and hence there are $a, b \in A$ such that $[b,a] = \mathbf{1}$; therefore, if there were an algebra norm $\|\cdot\|$ on A, then, by Proposition 3.6.49, we would have $\mathfrak{r}_{\|\cdot\|}(\mathbf{1}) = 0$, a contradiction.) On the other hand, the inclusion uw-Rad$(A) \subseteq$ Rad(A) may be strict too. Indeed, since simple algebras have zero ultra-weak radical (by Lemma 4.4.50(iii)), the celebrated Sasiada's construction [550] (see also [743, pp. 125–31]) of a simple radical associative algebra provides us with an example where in fact we have $0 = \text{uw-Rad}(A) \subsetneqq \text{Rad}(A) = A$. Nevertheless, the question: does the equality $\text{uw-Rad}(A) = \text{Rad}(A)$ hold whenever the associative algebra A is complete normed and complex, remains open to date. A counterexample of such an algebra A with $0 = \text{uw-Rad}(A) \subsetneqq \text{Rad}(A)$ would show that, even in the associative setting, Theorem 4.4.45 is a strict refinement of Johnson's theorem. To conclude the discussion of Proposition 4.4.59 in the associative setting, let us note that the inclusion $\text{Rad}(A) \subseteq \text{s-Rad}(A)$ may be strict, even in the unital C^*-algebra case. Indeed, let H be any infinite-dimensional complex Hilbert space, and let A stand for the C^*-algebra $BL(H)$. Then the space $\mathfrak{K}(H)$, of all compact operators on H, is a nonzero proper ideal of A (which implies that zero is not a maximal ideal of A). Moreover, by Corollary 1.4.33, $\mathfrak{K}(H)$ is contained in every nonzero closed ideal of A. It follows from Corollary 1.1.53 that $\text{s-Rad}(A) \supseteq \mathfrak{K}(H)$, and hence, keeping in mind Example 3.6.40, we get that $0 = \text{Rad}(A) \subsetneqq \text{s-Rad}(A)$. Note that, since A is unital, the equality $\text{s-Rad}(A) = A$ cannot be expected (cf. Fact 1.1.52). Now, let H be as above, and let A stand for the C^*-algebra $\mathfrak{K}(H)$ (cf. Proposition 2.3.43). Then, since A is not unital, we realize that zero is not a modular ideal of A. Moreover, by Corollary 1.4.33, A is topologically simple. It follows from Corollary 3.6.4 that $\text{s-Rad}(A) = A$, and hence, keeping in mind Example 3.6.40 again, we get that $0 = \text{Rad}(A) \subsetneqq \text{s-Rad}(A) = A$.

(b) Now, let A be an anticommutative algebra over $\mathbb{K}$. Then, clearly, A is a non-commutative Jordan algebra. Moreover we have $\text{J-Rad}(A) = A$ because, for each $x \in A$, $-x$ is the quasi-J-inverse of x (cf. Definition 4.4.58). Therefore, by assertions (i) and (iii) in Proposition 4.4.59, we have

$$\text{J-Rad}(A) = \text{Rad}(A) = \text{s-Rad}(A) = A.$$

Nevertheless, since there are choices of A such that A is simple (for example the subalgebra of the Lie algebra $(M_2(\mathbb{K}))^{\text{ant}}$ consisting of all trace-zero matrices), it follows from Lemma 4.4.50(iii) that, for such choices of A, we have

$$0 = \text{uw-Rad}(A) \subsetneqq \text{J-Rad}(A) = A \ \text{ and } \ 0 = \text{uw-Rad}(A) \subsetneqq \text{Rad}(A) = A.$$

This last strict inclusion clarifies the penultimate inclusion in Proposition 4.4.59(i), whereas the former clarifies Proposition 4.4.59(ii) and the first inclusion in Proposition 4.4.59(iii). When we were in the associative setting (cf. part (a) of the present remark), the inclusion $\text{uw-Rad}(A) \subseteq \text{Rad}(A)$ was already clarified. But the algebra A appearing there was not finite-dimensional, nor even complete normed.

(c) The inclusion $\text{J-Rad}(A) \subseteq \text{Rad}(A)$ in Proposition 4.4.59(iii) may be strict, even in the unital JB^*-algebra case. Indeed, let H be any infinite-dimensional complex Hilbert space, and let A stand for the JB^*-algebra $(BL(H))^{\text{sym}}$. By Lemma 4.4.28(iii), we have $\text{J-Rad}(A) = 0$. On the other hand, since A is commutative, we have $\text{Rad}(A) = \text{s-Rad}(A)$. Moreover, by Proposition 3.6.11(ii) and part (a) of the present remark, we have

$$\text{s-Rad}(A) = \text{s-Rad}(BL(H)) \supseteq \mathfrak{K}(H).$$

It follows that $0 = \text{J-Rad}(A) \subsetneqq \text{Rad}(A)$. Now, let H be as above, and let A stand for the JB^*-algebra $(\mathfrak{K}(H))^{\text{sym}}$. Then, by Lemma 3.6.10, Corollary 1.4.33, and

Proposition 3.6.11(i), A is non-unital and topologically simple. It follows from Lemma 4.4.28(iii) and Corollary 3.6.4 that $0 = \text{J-Rad}(A) \subsetneqq \text{Rad}(A) = A$.

4.4.5 Historical notes and comments

In [353], Johnson proved that *every irreducible representation of a complete normed associative algebra on a normed space, whose range consists only of bounded operators, is automatically continuous*. Then he derived his celebrated uniqueness-of-norm theorem (the associative particularizations of Theorem 4.4.13 and Corollary 4.4.14). Johnson's uniqueness-of-norm theorem, which is included with its classical proof in [696, Section 25], has inspired Subsections 4.4.1, 4.4.2, and 4.4.4.

According to Allan [16], Lemma 4.4.1 was proved by Zemánek in his PhD thesis [821], as a step in the proof of his celebrated characterization of the radical of a complete normed associative complex algebra B as the set of those elements $b \in B$ satisfying $\mathfrak{r}(b+x) = 0$ for every $x \in B$ such that $\mathfrak{r}(x) = 0$ (see also [683, Theorem 5.3.1]); but, in the published version in [664], the lemma does not appear, since a different method of proof was used. Both Proposition 4.4.3, which (as done by Palmer in [786, p. 227]) we call *Ransford's three circles theorem*, and the proof given here, are due to Ransford [499]. Corollary 4.4.4 was pointed out by Villena [626], and becomes an abstract version of Ransford's simple argument in [499] to derive Johnson's uniqueness-of-norm theorem from Ransford's three circles theorem. Allan's book [675, pp. 248–59] contains a proof of Lemma 4.4.1, as well as relevant generalizations of Ransford's three circles theorem. These are applied to give some very elementary proofs of results of Aupetit, Ransford and others on the variations of the spectral radius of a holomorphic family of elements in a complete normed associative complex algebra (see also [16] again).

Lemma 4.4.10 and Proposition 4.4.11 are due to McCrimmon, who in [435] proved the commutative case of assertion (i) in Proposition 4.4.11 (see also [822, Section 14.3] and [777, Section III.1.3]), and later completed the proof of the proposition with the argument reproduced in the last paragraph of our proof (see [436, Section 6]). The remaning results from Lemma 4.4.5 to Corollary 4.4.14 are due to Aupetit [40] (1982), and predate Ransford's work [499] (1989). In fact, a revolution in the theory of complete normed Jordan algebras arose when Aupetit jordanized the associative methods in his books [682] and [683] in order to solve several Jordan problems that seemed intractable using classical tools. Thus in Aupetit's paper [40] just quoted, at the same time that a first Jacobson-representation-theory-free proof of Johnson's uniqueness-of-norm theorem was given, the generalization of Johnson's theorem to the setting of Jordan-admissible algebras (as stated in Theorem 4.4.13) was proved. The new tool in Aupetit's original proof of Proposition 4.4.7 was the subharmonicity of the spectral radius in complete normed Jordan complex algebras (see Theorem 4.5.14 below), generalizing the corresponding associative forerunner, due to Vesentini [622]. Following on from this explanation, the reader will understand why we call Corollary 4.4.14 (or Theorem 4.4.13 itself) *the Johnson–Aupetit–Ransford uniqueness-of-norm theorem*.

§**4.4.69** Our versions of results from Proposition 4.4.3 to Corollary 4.4.14 are more general than those in the literature previously cited. The changes to the original proofs, although clever, are however not particularly deep. Indeed, the idea of introducing the function $\mathfrak{s}(\cdot)$ on a general non-associative normed algebra as a succedaneous of the associative spectral radius $\mathfrak{r}(\cdot)$, and that of formulating the Aupetit–Ransford techniques involving $\mathfrak{s}(\cdot)$ instead of $\mathfrak{r}(\cdot)$, are due to A. Cedilnik (old private communication). On the other hand, according to [786, p. 227], the idea of removing completeness nontrivially in the Aupetit–Ransford techniques first appears in [516]. It is also worth mentioning that the actual versions of Proposition 4.4.7, Theorem 4.4.13, and Corollary 4.4.18 have been possible thanks to the refinement of the ideas of Balachandran–Rema [54] (cf. §3.4.87) stated in Proposition 3.4.63. Finally, let us say that Ransford deals only with associative algebras, that Aupetit deals only with associative or Jordan algebras, but that the unification of Aupetit's results in terms of Jordan-admissible algebras is almost straightforward.

Now, let us apply Proposition 4.4.7 to prove the following refinement of Proposition 3.5.65.

Proposition 4.4.70 *Let A be a complete normed complex $*$-algebra admitting power-associativity and satisfying $\|a^*a\| = \|a^*\|\|a\|$ for every $a \in A$. Then $*$ is continuous.*

Proof Given $h \in H(A,*)$, we have $h^2 \in H(A,*)$ and $\|h^2\| = \|h\|^2$, so for every $n \in \mathbb{N}$ we have $h^{[n]} \in H(A,*)$ (cf. §4.4.2) and $\|h^{[n]}\| = \|h\|^{2^n}$, which implies $\mathfrak{s}(h) = \|h\|$. Moreover, denoting by B the normed complex algebra obtained from A by only replacing the product by scalars λa with $\bar{\lambda} a$, and noticing that $*$ can be seen as a (linear) Jordan homomorphism from A to B, Proposition 4.4.7 applies, so that the equality $\mathfrak{s}(x) = 0$ holds whenever x lies in $\mathfrak{S}(*)$. On the other hand, for $x \in \mathfrak{S}(*)$ we know that $x^*x \in \mathfrak{S}(*) \cap H(A,*)$. It follows that, for $x \in \mathfrak{S}(*)$ we have $\|x^*\|\|x\| = \|x^*x\| = \mathfrak{s}(x^*x) = 0$, and so $x = 0$. Therefore $\mathfrak{S}(*) = 0$. □

Assertion (i) in Proposition 4.4.17 is taken from Martínez' PhD thesis [775] (see also [250, Lemma 6.5] for a different proof), whereas the remaining assertions in that proposition are taken from Cuenca's PhD thesis [714], and also appear in [259, Lemma 16]. Although easily derivable from previously known results, Corollary 4.4.18 could be new. The associative forerunner of this corollary (that surjective Jordan homomorphism, from complete normed associative algebras to complete normed semisimple associative algebras, are continuous) is due to Sinclair (see the introduction of [577]).

Lemma 4.4.19 is folklore in automatic continuity theory (see for example [810, Lemma 1.3]).

Without enjoying their name, Jordan normed Q-algebras first appeared in Viola Devapakkiam's paper [628], and were considered later, in an incidental way, by Bensebah in [86].

§4.4.71 Normed Q-algebras (in our Jordan-admissible meaning) were introduced and studied in depth by Pérez, Rico, and Rodríguez [488], following Yood's and Palmer's associative forerunners (cf. §3.6.61). We summarize the main results of [488] on normed Q-algebras in Theorem 4.4.72 immediately below. Some definitions are suitable. Let A be a Jordan algebra over $\mathbb{K}$. A vector subspace M of A such that $U_m(A) \subseteq M$ for every $m \in M$ is called an *inner ideal* of A. If in addition M is also a subalgebra of A, then it is called a *strict inner ideal* of A. Given $x \in A$, a strict inner ideal M of A is called *x-modular* when the following three conditions are satisfied:

(i) $U_{\mathbb{1}-x}(A) \subseteq M$.
(ii) $U_{\mathbb{1}-x,m}(z) \in M$ for all $z \in A_{\mathbb{1}}$ and $m \in M$.
(iii) $x^2 - x \in M$.

If there exists $x \in A$ such that M is a maximal element (relative to the inclusion) of the family of all proper x-modular strict inner ideals of A, then we say that M is a *maximal modular inner ideal* of A. Now, let A be a Jordan-admissible algebra over $\mathbb{K}$. The *maximal modular inner ideals* of A are, by definition, the maximal modular inner ideals of the Jordan algebra A^{sym}. In agreement with §4.4.38, a subalgebra B of A is said to be a *quasi-J-full subalgebra* of A if, whenever b belongs to B and is quasi-J-invertible in A, the quasi-J-inverse of b lies in B.

Theorem 4.4.72 *Let A be a normed Jordan-admissible algebra over $\mathbb{K}$. Then the following conditions are equivalent:*

(i) *The set of all quasi-J-invertible elements of A is open.*
(ii) *A is a Q-algebra* (in our meaning, that the set of all quasi-J-invertible elements of A is a neighbourhood of zero in A).
(iii) *The set of all quasi-J-invertible elements of A has some interior point.*
(iv) $\mathfrak{s}(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}((A_{\mathbb{1}})_{\mathbb{C}}, a)\}$ *for every $a \in A$.*
(v) $\sup\{|\lambda| : \lambda \in \mathrm{sp}((A_{\mathbb{1}})_{\mathbb{C}}, a)\} \leqslant \|a\|$ *for every $a \in A$.*
(vi) *A is a quasi-J-full subalgebra of its completion.*
(vii) *A is a quasi-J-full subalgebra of some complete normed Jordan-admissible algebra over $\mathbb{K}$.*
(viii) *Every element $a \in A$ with $\|a\| < 1$ is quasi-J-invertible in A.*
(ix) *Maximal modular inner ideals of A are closed in A.*

We will not discuss here the proof of Theorem 4.4.72 above. Let us say however that the implication (ii)⇒(viii) in the theorem follows from Lemma 4.4.21(iii), and that, in slightly different terms, the equivalence (iv)⇔(vi) will be proved later (see Proposition 4.6.11). Let us also recall that, in the particular associative setting, the most part of Theorem 4.4.72 was already proved (cf. Proposition 3.6.43).

§**4.4.73** The notion of inner ideal in the Jordan algebra setting becomes a succedaneous of that of one-sided ideal in the associative setting (see [777, pp. 86–7] for details). Accordingly, given a Jordan algebra A, *J-primitive ideals* of A are defined as the cores in A of maximal modular inner ideals of A, and A is said to be a *J-primitive algebra* if zero is a J-primitive ideal of A (compare Definition 3.6.12). The concept of modularity for inner ideals of a Jordan algebra is due to Hogben and McCrimmon [328], who prove, as the main result, that *the Jacobson radical of a Jordan algebra A coincides with the intersection of all J-primitive ideals of A* (see also [777, Theorem III.5.3.1]). Thus, a Jordan algebra is J-semisimple if and only if it is a subdirect product of J-primitive Jordan algebras. The reader is referred to [257] for the non-commutative Jordan version of the result of [328] just quoted.

The associative forerunner of Theorem 4.4.23 is due to Rodríguez [519], and its actual version is proved independently in [86, 488].

As pointed out by Yood in [646], the associative forerunner of Lemma 4.4.28(i)(b) (that a C^*-algebra is a normed Q-algebra under any algebra norm) is due to Rickart [502] (see also [795, Theorem 4.8.3]). The associative forerunner of Theorem 4.4.29 is due to Cleveland [176], and is included in [786, Theorem 6.1.16]. Its actual non-associative version is due independently to Bensebah [86] and Pérez-Rico-Rodríguez [488]. The proof of Theorem 4.4.29 given in [86, 488] (based on the Aupetit–Ransford techniques) is essentially the same in both papers, and has been incorporated in our text almost verbatim. According to Palmer [786, pp. 560, 567], in the associative setting, proofs similar to the one we have given had been previously discovered by Dales [204] and Rodríguez [519]. Theorem 4.4.29 has been re-proved by Hejazian and Niknam [319] by jordanizing Cleveland's original arguments.

It turns out rather curious that, via Fact 3.3.4, the commutative particularization of Theorem 4.4.29 is better than the theorem itself. Thus, for example, if A is a C^*-algebra, then, applying Theorem 4.4.29 to the JB^*-algebra A^{sym}, we get that the topology of any algebra norm on A^{sym} is stronger than that of the natural norm. Since algebra norms on any algebra A are algebra norms on A^{sym}, this fact refines Cleveland's classical theorem. A similar comment can be made concerning Proposition 4.4.34.

The appropriate variant of Cleveland's theorem for JB^*-triples is also true. Indeed, as proved by Fernández-Polo, Garcés, and Peralta [262], we have the following.

Theorem 4.4.74 *Let J be a JB^*-triple, and let $|||\cdot|||$ be any norm on J making continuous the triple product of J. Then the topology of $|||\cdot|||$ is stronger than that of the natural norm.*

As a consequence, *if J is a JB^*-triple, and if $|||\cdot|||$ is a continuous norm on J making continuous the triple product of J, then $|||\cdot|||$ is equivalent to the natural norm of J.* This particular case of Theorem 4.4.74 had been previously proved by Bouhya and Fernández [121], who also proved the following variant of Proposition 4.4.34.

Proposition 4.4.75 *Let J be a JB^*-triple, and let $|||\cdot|||$ be any norm on J such that $|||\cdot|||\leqslant\|\cdot\|$ and $|||\{xyz\}|||\leqslant|||x|||\,|||y|||\,|||z|||$ for all $x,y,z\in J$. Then $|||\cdot|||=\|\cdot\|$.*

It is worth remarking that, although C^*-algebras are JB^*-triples, Cleveland's theorem is not straightforwardly derivable from Theorem 4.4.74. Actually Theorems 4.4.29 and 4.4.74 are mutually independent results.

Although easily derivable from the results in [86, 488], Corollary 4.4.30 went unnoticed there. Corollary 4.4.31 is new. Its associative forerunner is due to Barnes [59]. Proposition 4.4.34 has been taken from [488]. The associative forerunner of Proposition 4.4.34 (that C^*-algebras have minimality of norm) had already been proved in Lemma 2.3.27 without involving any version of the Vidav–Palmer theorem. As we commented in Subsection 2.3.5, it is due to Bonsall [111].

Corollaries 4.4.32, 4.4.35, and 4.4.36 become refinements of the following fact, proved in [488].

Fact 4.4.76 *Let B be a normed associative complex algebra, and assume that B is the range of a continuous (respectively, contractive) Jordan homomorphism from some C^*-algebra. Then B is bicontinuously (respectively, isometrically) isomorphic to a C^*-algebra.*

Let A be a normed algebra over $\mathbb{K}$. We say that A has *minimum norm* (respectively, *minimum norm topology*) if it satisfies the conclusion in Proposition 1.2.51 (respectively, Theorem 4.4.29). These notions are related to those previously introduced in Definitions 4.4.22 and 4.4.33 by means of the following diagram:

$$\begin{array}{ccc}
\boxed{A \text{ has minimum norm}} & \Longrightarrow & \boxed{A \text{ has minimum norm topology}} \\
\Downarrow & & \Downarrow \\
\boxed{A \text{ has minimality of norm}} & \overset{A \text{ associative}}{\Longrightarrow} & \boxed{A \text{ has minimality of norm topology}}
\end{array}$$

§**4.4.77** All implications in the diagram are clear, except for the one involving the additional requirement that A be associative. The proof of this implication, taken from [519], goes as follows:

Assume that the normed associative algebra $(A, \|\cdot\|)$ has minimality of norm, and let $|||\cdot|||$ be a continuous algebra norm on A. Then the closed unit ball of A for the norm $\|\cdot\|$ is a $|||\cdot|||$-bounded and multiplicatively closed subset of A. Therefore, by a theorem of Rota and Strang [544] (see also [696, Theorem 4.1]) which will be proved later in Proposition 4.5.2(i), there exists an algebra norm p on A which is equivalent to $|||\cdot|||$ and satisfies $p(a) \leqslant 1$ whenever a is in A with $\|a\| \leqslant 1$. Since $p(\cdot) \leqslant \|\cdot\|$ and A has minimality of norm, we have $p(\cdot) = \|\cdot\|$. Therefore $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on A. $\square$

Even in the case of normed alternative algebras, we do not know if minimality of norm implies minimality of norm topology. For a full non-associative discussion of the Rota–Strang associative theorem, applied in the above proof, the reader is referred to [452, 453]. The property of having minimum norm is enjoyed by some normed algebras which are far from being associative. Indeed, according to Proposition 2.7.42, every absolute-valued real algebra with a left unit has minimum norm (compare Proposition 2.6.27 and Theorem 2.7.38).

In the paper [111] already quoted, Bonsall proves that, *if X is a normed space over $\mathbb{K}$, and if B is any subalgebra of $BL(X)$ containing all finite-rank operators, then B* (endowed with the operator norm) *has minimality of norm* (and hence, also has minimality of norm topology). Therefore, by Example 3.6.40 and Theorem 4.4.23, *if X is a complex Banach space, and if B is any closed subalgebra of $BL(X)$ containing all finite-rank operators, then surjective algebra homomorphisms from normed complex Q-algebras to B are automatically continuous.* Actually, as proved by Dales in [204], *if X is a Banach space over $\mathbb{K}$, and if B is any subalgebra of $BL(X)$ containing all* *finite-rank operators, then B has minimum norm topology.* According to [204], this last fact is a very old result of Eidelheit [230], which may even go back to S. Mazur before 1939. Generalizations of the results of Bonsall and Dales quoted above can be seen in the paper of Esterle [247].

Most notions and results in Subsection 4.4.4 are taken from Rodríguez' paper [516]. Exceptions are Proposition 4.4.49 (which is taken from [529]), assertions (i) and (ii) in Lemma 4.4.50, Proposition 4.4.52, Corollary 4.4.53, and assertions (i) and (ii) in Proposition 4.4.65 (all of which are taken from [520]), Fact 4.4.54 (which is folklore), Corollary 4.4.55 (which was proved in [165] by other methods), Corollary 4.4.64 (which, as mentioned in relation to its forerunner stated in Corollary 4.4.18, could be new), assertion (iii) in Proposition 4.4.65 (which is folklore, see for example [696, Corollary 1.24]), and Lemma 4.4.66 and Theorem 4.4.67 (due to Haïly [303]).

Before the publication of [516], the problem of the uniqueness of the complete algebra norm topology in general non-associative algebras had not been solved. Even more, this problem had been not formally raised at that time because, for general non-associative algebras, there was no notion of 'radical' such that: (i) it had the same algebraic relevance as that of Jacobson radical in the associative case; and (ii) one could expect that a non-associative algebra with zero 'radical' to have at most one complete algebra norm topology. The proof of Theorem 4.4.43 leads in a natural way to a suitable notion of such a 'radical', namely the weak radical as introduced in Definition 4.4.39. Analogously, in order to solve the more ambitious problem of the automatic continuity of surjective algebra homomorphisms, the proof of Theorem 4.4.45 leads in a natural way to the notion of ultra-weak radical as introduced in Definition 4.4.44.

The existence of infinite-dimensional Lie algebras with a unique complete algebra norm topology, shown in Remark 4.4.48, encouraged some authors to produce deeper results along these lines. In order to review these results, note that, given a complete normed associative algebra B, the complete normed Lie algebra B^{ant} will have a unique complete algebra norm topology (respectively, surjective algebra homomorphisms from complete normed algebras to B^{ant} will be automatically continuous) as soon as $\text{w-Rad}(B^{\mathrm{ant}})=0$ (respectively, $\text{uw-Rad}(B^{\mathrm{ant}})=0$). But, what does it mean that $\text{w-Rad}(B^{\mathrm{ant}})=0$ (respectively, that $\text{uw-Rad}(B^{\mathrm{ant}})=0$) in terms of the classical associative properties of B? The pioneering paper trying to answer these questions is that of Berenguer and Villena [98]. The definitive solution is given by Aupetit and Mathieu [47].

Let B be an associative algebra. The *centre modulo the radical*, $\mathscr{Z}(B)$, of B is defined as the subspace of B consisting of those elements $b\in B$ such that $[b,x]\in \mathrm{Rad}(B)$ for every $x\in B$ (see [683, p. 92]). Clearly, we have that $\mathrm{Rad}(B)\subseteq\mathscr{Z}(B)$, so that $\mathscr{Z}(B)=0$ if and only if B is semisimple and has zero centre. For $b\in B$, we denote by D_b the inner derivation of B defined by $D_b(x):=[b,x]$ for every $x\in B$ (note that $D_b=L_b^{B^{\mathrm{ant}}}=-R_b^{B^{\mathrm{ant}}}$). As a key tool in the whole paper [47], Aupetit and Mathieu prove the following.

Proposition 4.4.78 *Let B be a complete normed associative complex algebra. Then $b\in B$ belongs to $\mathscr{Z}(B)$ if and only if $\mathfrak{r}(D_{[b,x]})=0$ for every $x\in B$.*

Now, with Definition 4.4.44 being omnipresent in the whole argument, we can prove the following.

Corollary 4.4.79 *Let B be a complete normed associative complex algebra. Then* $\text{uw-Rad}(B^{\text{ant}}) \subseteq \mathscr{Z}(B)$.

Proof In view of Remark 4.4.41, it is enough to show that

$$\mathfrak{A}\text{-Rad}(B^{\text{ant}}) \subseteq \mathscr{Z}(B) \tag{4.4.11}$$

for every subalgebra $\mathfrak{A}$ of $BL(B)$ containing D_B. Let $\mathfrak{A}$ be such a subalgebra, and let b be in $\mathfrak{A}\text{-Rad}(B^{\text{ant}})$. Then, for every $x \in B$, $[b,x]$ lies in $\mathfrak{A}\text{-Rad}(B^{\text{ant}})$ (because $\mathfrak{A}\text{-Rad}(B^{\text{ant}})$ is an ideal of B^{ant}), so $D_{[b,x]}$ belongs to $\text{Rad}(\mathfrak{A})$, and so $\mathfrak{r}(D_{[b,x]}) = 0$ (by Corollary 3.6.23). Therefore, by Proposition 4.4.78, b belongs to $\mathscr{Z}(B)$. Thus the desired inclusion (4.4.11) has been proved. □

Combining Theorem 4.4.45 and Corollary 4.4.79, we get the following.

Theorem 4.4.80 *Let B be a complete normed semisimple associative complex algebra with zero centre. Then surjective algebra homomorphisms from complete normed complex algebras to the complete normed Lie algebra B^{ant} are automatically continuous. As a consequence B^{ant} has a unique complete algebra norm topology.*

The above theorem is the main result in the Aupetit–Mathieu paper [47]. Our discussion shows that it is somehow contained in Theorem 4.4.45. Indeed, it is contained in Theorem 4.4.45, but modulo Proposition 4.4.78 (as we have already commented) becomes the key tool in [47]. Theorem 4.4.80, with ‘bijective’ instead of ‘surjective’, is the main result in the Berenguer–Villena forerunner [98] already quoted. A variant along the same lines, which is not contained in Theorem 4.4.80, can be seen in [97]. A generalization of Theorem 4.4.80 to the setting of Jordan algebras, due to Brešar, Cabrera, Fosner, and Villena [130], will be completely proved in the second volume of our book.

In relation to Proposition 4.4.78, we note that, as was proved earlier by Harris and Kadison [316], *the centre modulo the radical of a complete normed associative complex algebra B can* also *be characterized as the set of those elements $b \in B$ such that* $\text{sp}(B_{\mathbb{1}}, b+x) \subseteq \text{sp}(B_{\mathbb{1}}, b) + \text{sp}(B_{\mathbb{1}}, x)$ *for every $x \in B$.*

In the same way as Theorem 4.4.80 is the natural generalization of Remark 4.4.48(c), the following theorem, due to Villena [627], becomes the natural generalization of Remark 4.4.48(e). We recall that derivations of a complete normed semisimple associative algebra are continuous [355], so that the set of all derivations of such an algebra B becomes a closed subalgebra of the complete normed Lie algebra $(BL(B))^{\text{ant}}$.

Theorem 4.4.81 *Let B be a complete normed semisimple associative complex algebra, and let $\mathfrak{D}$ stand for the complete normed Lie complex algebra of all derivations of B. Then surjective algebra homomorphisms from complete normed Lie complex algebras to $\mathfrak{D}$ are automatically continuous. As a consequence, $\mathfrak{D}$ has a unique complete algebra norm topology.*

Remark 4.4.48(f) has also been significantly refined. Indeed, as proved in [526], *dense range algebra homomorphisms from complete normed algebras to H^*-algebras with zero annihilator are automatically continuous*.

In the complex case, Proposition 4.4.49 has a relevant refinement. Indeed, as proved in [529], *if A is a complete normed complex algebra with no nonzero two-sided topological divisor of zero, then* (possibly non-surjective) *algebra homomorphisms from complete normed complex algebras to A are continuous*. The question of whether a similar result is true for real algebras remains unanswered to date, even if the assumption that A has no nonzero two-sided topological divisor of zero is strengthened to the one that A has no nonzero one-sided topological divisor of zero. An affirmative answer to this question depends on an affirmative answer to the complete version of Problem 2.7.45 (see [529] for details).

Corollary 4.4.60 contains Rickart's classical uniqueness-of-norm theorem (Corollary 3.6.8) and even its generalization to Jordan-admissible algebras (Corollary 4.1.20), but does not contain non-associative dense-range-homomorphism theorems like Theorem 4.1.19 or Proposition 4.1.108, nor even Rickart's associative forerunner. Corollary 4.4.61 has recently been rediscovered in [415]. It is worth mentioning that, in view of Remark 4.4.68(c), Corollary 4.4.61 does not contain Corollary 4.4.62 and 4.4.63, nor even Aupetit's forerunner stated in Theorem 4.4.13.

As pointed out in [303], the weak radical does not behave 'too well' as a radical. Indeed, if A stands for the associative algebra in Haïly's Theorem 4.4.67, then we have $\text{w-Rad}(A) = 0$, but, if J denotes the unique nonzero proper ideal of A, then, since $JJ = 0$, we get $\text{w-Rad}(J) = J \neq 0$. This pathology resembles that of the annihilator. Indeed, if A is any nonzero algebra with zero product, then its unital extension $A_{\mathbb{1}}$ has zero annihilator, but A is an ideal of $A_{\mathbb{1}}$ with $\text{Ann}(A) = A \neq 0$. It is also proved in Haïly's paper [303] that *the weak radical of any finite-dimensional Jordan algebra becomes the largest nilpotent ideal of A*. The variant of this result with 'alternative' instead of 'Jordan', as well as related results for generalized standard algebras (cf. §2.8.80), had been proved earlier in [520].

§4.4.82 To conclude our comments on Subsection 4.4.4, let us relate the weak and ultra-weak radicals to other classical radicals which need not be relevant in the automatic continuity theory. Let A be an algebra over $\mathbb{K}$. Since A is a semiprime ideal of A (cf. §3.6.15), and the intersection of any family of semiprime ideals of A is a semiprime ideal of A, it follows that there exists a smallest semiprime ideal of A. This ideal is called the *Baer radical* of A. Since $\text{Rad}(A)$ is a semiprime ideal of A (by Proposition 3.6.16(ii)), the Baer radical of A is contained in $\text{Rad}(A)$. Following [728, Section I.1] and [819, Section 21], we define the *Albert radical* of A as the intersection of the family of those maximal ideals M of A such that $AA \not\subseteq M$ (with the usual convention that the Albert radical of A equals the whole algebra A if such maximal ideals do not exist). Clearly, *the Albert radical of A is contained in the strong radical of A, and is equal to the strong radical of A whenever A is unital.* In the finite-dimensional setting, the Albert radical was introduced by Albert [8] (see also Jacobson [350, pp. 1090–1]), who proved that, *if A is a finite-dimensional algebra, then the Albert radical of A is the smallest ideal R of A such that A/R is zero or a finite direct sum of simple ideals.*

Assertions (i) and (ii) in the following lemma are well known (see for example [819, 21.9]). Assertion (iii) is due to Albert [8].

Lemma 4.4.83 *Let A be an algebra over $\mathbb{K}$. We have:*

(i) *If M is a maximal ideal of A such that $AA \not\subseteq M$, then M is a semiprime ideal of A.*
(ii) *The Albert radical of A is a semiprime ideal of A, and hence contains the Baer radical of A.*
(iii) *The situation that the Baer radical of A is equal to zero, but the Albert radical of A is isomorphic to $\mathbb{K}$, can be exemplified even in the unital finite-dimensional case.*

Proof Let M be a maximal ideal of A such that $AA \not\subseteq M$. Then the quotient algebra A/M is simple, and hence semiprime. Therefore M is a semiprime ideal of A. This proves assertion (i).

Assertion (ii) follows from assertion (i) by keeping in mind that the intersection of any family of semiprime ideals of A is a semiprime ideal of A.

Let A be the three-dimensional unital algebra over $\mathbb{K}$ with basis $\{\mathbf{1}, u, v\}$ and multiplication table given by $u^2 = 1$, $uv = v^2 = v$, $vu = 0$. The reader may easily verify that the unique nonzero proper ideals of A are $I_1 := \mathbb{K}v$, $I_2 := \mathbb{K}v + \mathbb{K}(\mathbf{1}+u)$, and $I_3 := \mathbb{K}v + \mathbb{K}(\mathbf{1}-u)$. It follows that I_1 is isomorphic to $\mathbb{K}$, and that A is a prime algebra (hence the Baer radical of A equals zero). On the other hand, since I_2 and I_3 are the unique maximal ideals of A, and $AA = A$, and $I_1 = I_2 \cap I_3$, we deduce that I_1 equals the Albert radical of A. □

Proposition 4.4.84 *Let A be an algebra over $\mathbb{K}$. We have*

(i) *$\mathscr{M}^\sharp(A)$-$\mathrm{Rad}(A)$ is contained in the Albert radical of A.*
(ii) *The situation that $\mathscr{M}^\sharp(A)$-$\mathrm{Rad}(A) = 0$ and that A has nonzero Albert radical can be exemplified even in the unital finite-dimensional case.*

Proof To prove assertion (i), it is enough to show that $R := \mathscr{M}^\sharp(A)$-$\mathrm{Rad}(A)$ is contained in every maximal ideal M of A such that $AA \not\subseteq M$. Let M be such an ideal. Regard A as a left $\mathscr{M}^\sharp(A)$-module in the natural way, and note that the ideals of A are nothing other than the $\mathscr{M}^\sharp(A)$-submodules of A. Since M is a maximal ideal of A with $AA \not\subseteq M$, it follows that the quotient $\mathscr{M}^\sharp(A)$-module A/M is irreducible. Therefore, for every $x \in R$ we have $L_x(A/M) = 0$ (because $L_x \in \mathrm{Rad}(\mathscr{M}^\sharp(A))$) or, equivalently, $xA \subseteq M$. Thus $RA \subseteq M$ and, in particular $RR \subseteq M$. Since M is a semiprime ideal of A (by Lemma 4.4.83(i)), we derive that $R \subseteq M$, as desired.

To prove assertion (ii), let A stand for the unital finite-dimensional algebra over $\mathbb{K}$ given by Lemma 4.4.83(iii), so that the Albert radical of A equals $\mathbb{K}v$ for some nonzero idempotent v. If v were in $\mathscr{M}^\sharp(A)$-$\mathrm{Rad}(A)$, then, by Lemma 4.4.56, there would exist $a \in A$ such that $a = va - v \in \mathbb{K}v$, and hence, since v is the unit of $\mathbb{K}v$, we would have $v = 0$, a contradiction. Therefore $v \notin M^\sharp(A)$-$\mathrm{Rad}(A)$. Since, by assertion (i), $M^\sharp(A)$-$\mathrm{Rad}(A)$ is contained in $\mathbb{K}v$, we conclude that $M^\sharp(A)$-$\mathrm{Rad}(A) = 0$. □

Remark 4.4.85 Let A be a finite-dimensional algebra over $\mathbb{K}$. It follows from Propositions 4.4.52(ii) and 4.4.84 that, as pointed out in [516], uw-Rad(A) is contained in the Albert radical of A, and that there are choices of A where uw-Rad$(A) = 0$ and A has nonzero Albert radical.

Let A be an algebra over $\mathbb{K}$, and let I be an ideal of A. We denote by $\tilde{I}$ the ideal of $\mathscr{M}^{\sharp}(A)$ generated by the set $\{L_x, R_x : x \in I\}$, and we say that I is a *multiplicatively nil ideal* of A if $\tilde{I}$ is a nil algebra. Now assume that A is associative. For $a, b \in A_{\mathbb{1}}$, we will denote by $M_{a,b}$ the two-sided multiplication operator on A defined by $M_{a,b}(x) = axb$. Note that $M_{\mathbb{1},\mathbb{1}} = I_A$, and that, for each $a \in A$, $M_{a,\mathbb{1}} = L_a^A$ and $M_{\mathbb{1},a} = R_a^A$. Moreover, we have $M_{a,b}M_{c,d} = M_{ac,db}$ for all $a,b,c,d \in A_{\mathbb{1}}$. Then it is easy to realize that $\tilde{I}$ consists precisely of those elements of $\mathscr{M}^{\sharp}(A)$ which can be written as $\sum_{i=1}^{n} M_{a_i,b_i}$ for suitable $n \in \mathbb{N}$ and $a_i, b_i \in A_{\mathbb{1}}$ $(1 \leqslant i \leqslant n)$ such that, for each i, at least one of a_i, b_i lies in I.

Lemma 4.4.86 *Let A be an associative algebra over $\mathbb{K}$, let I be an ideal of A, and let $\mathscr{S}(I)$ stand for the sum of all ideals J of A satisfying $JJ \subseteq I$. If I is a multiplicatively nil ideal of A, then so is $\mathscr{S}(I)$.*

Proof Note that each element of $\mathscr{S}(I)$ can be written as $\sum_{i=1}^{m} c_i$ for suitable $m \in \mathbb{N}$ and $c_i \in A$ $(1 \leqslant i \leqslant m)$ such that $c_i A_{\mathbb{1}} c_i \subseteq I$. Let F be in $\widetilde{\mathscr{S}(I)}$. It follows that $F = \sum_{i=1}^{n} M_{a_i,b_i}$ for suitable $n \in \mathbb{N}$ and $a_i, b_i \in A_{\mathbb{1}}$ $(1 \leqslant i \leqslant n)$ such that, for each i, at least one of $a_i A_{\mathbb{1}} a_i$, $b_i A_{\mathbb{1}} b_i$ is contained in I. Since

$$F^{n+1} = \sum_{1 \leqslant i_1, i_2, \ldots, i_{n+1} \leqslant n} M_{a_{i_1} a_{i_2} \cdots a_{i_{n+1}}, b_{i_{n+1}} \cdots b_{i_2} b_{i_1}},$$

and for each choice of $i_1, i_2, \ldots, i_{n+1} \in \{1, \ldots, n\}$ there exist $j, k \in \{1, \ldots, n+1\}$ such that $i_j = i_k$, we deduce that at least one of $a_{i_1} a_{i_2} \cdots a_{i_{n+1}}$, $b_{i_{n+1}} \cdots b_{i_2} b_{i_1}$ lies in I, and consequently $F^{n+1} \in \tilde{I}$. Now, if I is a multiplicatively nil ideal of A, then F^{n+1} is nilpotent, so F is nilpotent, and so $\mathscr{S}(I)$ is a multiplicatively nil ideal of A because F is arbitrary in $\widetilde{\mathscr{S}(I)}$. □

Let A be any algebra over $\mathbb{K}$. Following [822, p.161], we define by transfinite induction the *Baer chain* of A as the chain

$$\mathscr{B}_0(A) \subseteq \mathscr{B}_1(A) \subseteq \cdots \subseteq \mathscr{B}_{\beta}(A) \subseteq \cdots$$

of ideals of A, indexed by ordinal numbers β, as follows. We set $\mathscr{B}_0(A) := 0$. If $\beta > 0$ is not a limit ordinal, then $\mathscr{B}_{\beta}(A)$ denotes the sum of all ideals J of A such that $JJ \subseteq \mathscr{B}_{\beta-1}(A)$. Finally, if β is a limit ordinal, then we set $\mathscr{B}_{\beta}(A) := \cup_{\alpha<\beta} \mathscr{B}_{\alpha}(A)$. The Baer chain stabilizes at some ordinal γ (for example when γ is greater than the cardinality of the algebra A), and $\mathscr{B}(A) := \mathscr{B}_{\gamma}(A)$ coincides with the Baer radical of A as defined in §4.4.82 (see [822, Proposition 8.2.4]).

We already know that both the Baer radical of A and the ultra-weak radical of A are contained in Rad(A). In the associative case we have the following clarifying result.

Proposition 4.4.87 *Let A be an associative algebra over $\mathbb{K}$. Then we have:*

(i) *The Baer radical of A is a multiplicatively nil ideal of A.*
(ii) *The Baer radical of A is contained in $\mathscr{M}^{\sharp}(A)$-*Rad(A)*, and hence in* uw-Rad(A).

Proof To prove assertion (i) it is enough to show that $\mathscr{B}_\alpha(A)$ is a multiplicatively nil ideal of A for every ordinal α. Suppose the contrary, and let β stand for the minimum ordinal such that $\mathscr{B}_\beta(A)$ is not a multiplicatively nil ideal of A. It is clear that $\beta > 0$. In the case where β is not a limit ordinal, with the terminology of Lemma 4.4.86 we have $\mathscr{B}_\beta(A) = \mathscr{S}(\mathscr{B}_{\beta-1}(A))$, and hence, by that lemma, $\mathscr{B}_\beta(A)$ is a multiplicatively nil ideal of A, which is a contradiction. Now, assume that β is a limit ordinal. Let F be in $\widetilde{\mathscr{B}_\beta(A)}$. Write $F = \sum_{i=1}^n M_{a_i,b_i}$ for suitable $n \in \mathbb{N}$ and $a_i, b_i \in A_{\mathbb{1}}$ $(1 \leqslant i \leqslant n)$ such that, for each i, at least one of a_i, b_i lies in $\mathscr{B}_\beta(A)$, and fix $\alpha < \beta$ such that, for each i, at least one of a_i, b_i lies in $\mathscr{B}_\alpha(A)$. Then $F \in \widetilde{\mathscr{B}_\alpha(A)}$, and consequently F is nilpotent. It follows from the arbitrariness of $F \in \widetilde{\mathscr{B}_\beta(A)}$ that $\mathscr{B}_\beta(A)$ is again a multiplicatively nil ideal of A. This contradiction concludes the proof of assertion (i).

Assertion (ii) follows from assertion (i) and the fact that nil-ideals of $\mathscr{M}^\sharp(A)$ are quasi-invertible ideals of $\mathscr{M}^\sharp(A)$. $\square$

Let A be an algebra over $\mathbb{K}$, and let I be an ideal of A. We denote by $(I : A)$ the ideal of A consisting of those $x \in A$ such that $xA + Ax \subseteq I$. We always have $I \subseteq (I : A)$. If the above inclusion becomes an equality, then we say that I is a *zero-annihilator ideal* (in short, *z-ideal*) of A. Thus I is a z-ideal of A if and only if the quotient algebra A/I has zero-annihilator. Since A is a z-ideal of A, and the intersection of any family of z-ideals of A is a z-ideal of A, it follows the existence of a smallest z-ideal of A. This ideal will be called the *zero-annihilator radical* (in short, *z-radical*) of A. Clearly, the z-radical of A contains the annihilator of A and is contained in the Baer radical of A.

Let A be any algebra over $\mathbb{K}$. Define by transfinite induction a chain

$$\mathscr{C}_0(A) \subseteq \mathscr{C}_1(A) \subseteq \cdots \subseteq \mathscr{C}_\beta(A) \subseteq \cdots \tag{4.4.12}$$

of ideals of A, indexed by ordinal numbers β, as follows. Set $\mathscr{C}_0(A) := 0$, $\mathscr{C}_\beta(A) := (\mathscr{C}_{\beta-1}(A) : A)$ if $\beta > 0$ is not a limit ordinal, and $\mathscr{C}_\beta(A) := \cup_{\alpha<\beta}\mathscr{C}_\alpha(A)$ otherwise. This chain stabilizes at some ordinal γ (for example whenever γ is greater than the cardinality of the algebra A), and accordingly we set$\mathscr{C}(A) := \mathscr{C}_\gamma(A)$.

Now we can mimic the proof of [822, Proposition 8.2.4] to prove the following.

Proposition 4.4.88 *Let A be an algebra over $\mathbb{K}$. Then $\mathscr{C}(A)$* (as defined immediately above) *coincides with the z-radical of A.*

Proof Let R stand for the z-radical of A. By the definition of $\mathscr{C}(A)$, we have $\mathscr{C}(A) = (\mathscr{C}(A) : A)$, so $\mathscr{C}(A)$ is a z-ideal of A, and so $R \subseteq \mathscr{C}(A)$. Assume that this inclusion is strict. Then we have $\mathscr{C}(A) \nsubseteq R$, and hence there exists a minimum ordinal δ such that $\mathscr{C}_\delta(A) \nsubseteq R$. It is clear that δ cannot be a limit ordinal, and that $\mathscr{C}_{\delta-1}(A) \subseteq R$. But then

$$\mathscr{C}_\delta(A) = (\mathscr{C}_{\delta-1}(A) : A) \subseteq (R : A) = R,$$

a contradiction. $\square$

Lemma 4.4.89 *Let A be an algebra over $\mathbb{K}$, and let I be a $\mathscr{QFM}(A)$-invariant subspace of A. Then we have:*

(i) $\mathscr{QFM}(A)((I : A)) \subseteq I$.
(ii) $(I : A)$ *is* $\mathscr{QFM}(A)$*-invariant.*

Proof The set $\{F \in \mathcal{QFM}(A) : F((I:A)) \subseteq I\}$ is a left ideal (hence a quasi-full subalgebra) of $\mathcal{QFM}(A)$ containing $L_A \cup R_A$, and therefore it contains $\mathcal{QFM}(A)$. This proves assertion (i). Assertion (ii) follows from assertion (i) and the fact that $I \subseteq (I:A)$. □

Corollary 4.4.90 *Let A be an algebra over $\mathbb{K}$. Then we have:*

(i) *Every link in the chain (4.4.12) is $\mathcal{QFM}(A)$-invariant.*
(ii) *The z-radical of A is $\mathcal{QFM}(A)$-invariant.*

Proof Assume that assertion (i) does not hold. Then there exists a minimum ordinal δ such that $\mathcal{C}_\delta$ is not $\mathcal{QFM}(A)$-invariant. Clearly $\delta > 0$ and δ cannot be an ordinal limit. Therefore, by Lemma 4.4.89(ii), $\mathcal{C}_{\delta-1}$ is not $\mathcal{QFM}(A)$-invariant, a contradiction.

Assertion (ii) follows from assertion (i) and Proposition 4.4.88. □

Proposition 4.4.91 *Let A be an algebra over $\mathbb{K}$, and let a be in A. Then the following conditions are equivalent:*

(i) *a lies in the z-radical of A.*
(ii) *Every sequence a_n in A such that $a_1 = a$ and $a_{n+1} \in \mathcal{QFM}(A)(a_n)$ is quasi-null.*
(iii) *Every sequence a_n in A such that $a_1 = a$ and $a_{n+1} \in (L_A \cup R_A)(a_n)$ is quasi-null.*

Proof (i)⇒(ii) Assume that (i) holds. Let a_n be a sequence in A with the properties required in (ii). Then, by Corollary 4.4.90(ii), a_n lies in the z-radical of A for every $n \in \mathbb{N}$. Therefore, by Proposition 4.4.88, for each $n \in \mathbb{N}$, there exists a minimum ordinal number α_n such that $a_n \in \mathcal{C}_{\alpha_n}(A)$. Note that α_n cannot be an ordinal limit, and set $\alpha_m := \min\{\alpha_n : n \in \mathbb{N}\}$. Assume that $\alpha_m > 0$. Then $\alpha_m = \beta + 1$, so that, since $a_m \in \mathcal{C}_{\alpha_m}(A)$, Corollary 4.4.90(i) and Lemma 4.4.89(i) apply to get that $a_{m+1} \in \mathcal{C}_\beta(A)$. This implies $\alpha_{m+1} \leqslant \beta < \alpha_m$, which contradicts the definition of α_m. Therefore $\alpha_m = 0$, and hence $a_m = 0$.

(ii)⇒(iii) This is clear.

(iii)⇒(i) Let R stand for the z-radical of A. Assume that a does not belong to R, and set $a_1 := a$. Then, since R is a z-ideal of A, we have $a_1 \notin (R:A)$, and hence there exists $x_1 \in A$ such that $a_1x_1 \notin R$ or $x_1a_1 \notin R$. Let us set $a_2 := a_1x_1$ if $a_1x_1 \notin R$ and $a_2 := x_1a_1$ otherwise, so that $a_2 \notin R$. Proceeding by induction, we get a non-quasi-null sequence a_n in A with the properties required in (iii). □

As a straightforward consequence of the implication (iii)⇒(i) in the above proposition, we derive the following.

Corollary 4.4.92 *Let A be a nilpotent algebra over $\mathbb{K}$* (cf. §2.8.36). *Then the z-radical of A is the whole algebra A.*

A linear operator F on a vector space X over $\mathbb{K}$ is said to be *pointwise nilpotent* if for each $x \in X$ there exists $n \in \mathbb{N}$ such that $F^n(x) = 0$.

Fact 4.4.93 *Let X be a vector space over $\mathbb{K}$, and let F be a pointwise nilpotent linear operator on X. Then F is quasi-invertible in $L(X)$.*

Proof Let $G : X \to X$ stand for the mapping defined by $G(x) := -\sum_{n\in\mathbb{N}} F^n(x)$. It is routine to verify that G belongs to $L(X)$ and that G becomes the quasi-inverse of F in $L(X)$. □

The next theorem refines the first inclusion in Proposition 4.4.59.

Theorem 4.4.94 *Let A be an algebra over $\mathbb{K}$. Then the z-radical of A is contained in* w-Rad(A).

Proof Let R stand for the z-radical of A. In view of Corollary 4.4.90(ii) and the definition of the weak radical, it is enough to show that L_y, R_y belong to $\mathrm{Rad}(\mathscr{QFM}(A))$ whenever y is in R. Let (y, x, F) be in $R \times A \times \mathscr{QFM}(A)$, for each natural number n set $a_n := (FL_y)^n(x)$, and note that, since $y \in R$, $a_1 = FL_y(x) = FR_x(y) \in R$ (by Corollary 4.4.90(ii)). Since $a_{n+1} = FL_y(a_n)$, it follows from the implication (i)⇒(ii) in Proposition 4.4.91 that there exists $m \in \mathbb{N}$ such that $(FL_y)^m(x) = a_m = 0$. Since x is arbitrary in A, this shows that the operator FL_y is pointwise nilpotent, hence it is quasi-invertible in $L(A)$ (by Fact 4.4.93), so also in $\mathscr{QFM}(A)$ because $\mathscr{QFM}(A)$ is a quasi-full subalgebra of $L(A)$. Now, since F is arbitrary in $\mathscr{QFM}(A)$, we derive from Theorem 3.6.38(vii) that $L_y \in \mathrm{Rad}(\mathscr{QFM}(A))$. An analogous argument shows that $R_y \in \mathrm{Rad}(\mathscr{QFM}(A))$. Since y is arbitrary in R, the proof is complete. □

By combining Corollary 4.4.92 and Theorem 4.4.94, we get the following.

Corollary 4.4.95 *Let A be a nilpotent algebra over $\mathbb{K}$. Then $A =$* w-Rad(A).

The above corollary can be proved straightforwardly. Indeed, if $A \neq 0$ is a nilpotent algebra of index m, then $\mathscr{M}^\sharp(A)$ is a nilpotent algebra of index $\leqslant m-1$, so $\mathscr{M}^\sharp(A) = \mathscr{QFM}(A)$ is a radical algebra, and so w-Rad$(A) = A$.

Following [786, p. 515], we say that a normed associative algebra A is *topologically nilpotent* if

$$\lim_{n\to\infty} \sup\{\|a_1 a_2 \cdots a_n\|^{\frac{1}{n}} : a_i \in \mathbb{B}_A \text{ for } i = 1, 2, \ldots, n\} = 0. \tag{4.4.13}$$

As pointed out in [453], the notion of a topologically nilpotent normed algebra can be generalized to the non-associative setting by simply replacing $a_1 a_2 \cdots a_n$ in (4.4.13) with the values at $(a_1, a_2, \ldots, a_n)$ of all n-linear non-associative words on a set of n indeterminates (cf. §3.4.41). Clearly, normed nilpotent algebras are topologically nilpotent, but the converse is far from being true [786, p. 516]. The following variant of Corollary 4.4.95 is proved in [453]: *if A is a topologically nilpotent complete normed algebra, then $A =$* w-Rad(A).

4.5 JB^*-representations and alternative C^*-representations of hermitian algebras

Introduction In this section we deal with local characterizations, as well as with characterizations up to equivalent renormings, of non-commutative JB^*-algebras. As an outstanding auxiliary tool, we introduce hermitian Jordan-admissible complex

$*$-algebras (see Definition 4.5.18), and discuss algebra $*$-homomorphisms from them to non-commutative JB^*-algebras.

We begin Subsection 4.5.1 by proving in Theorem 4.5.1 that a complete normed unital complex $*$-algebra is a non-commutative JB^*-algebra (in its own norm) if and only if so is the closed $*$-subalgebra generated by each of its self-adjoint elements. The question of whether this result is true when the requirement of the existence of a unit is dispensed seems to remain open to date, even if the algebra is associative and commutative. (We note that non-commutative JB^*-algebras which are associative and commutative are nothing other than commutative C^*-algebras.) Then we prove Barnes' theorem [57] that a complete normed (possibly non-unital) associative and commutative complex $*$-algebra is a C^*-algebra in an equivalent norm if and only if so is the closed $*$-subalgebra generated by each of its self-adjoint elements (see Corollary 4.5.7). Actually, the requirement of commutativity in Barnes' theorem can be removed altogether. This is the content of a celebrated result of Cuntz [201], formulated later without proof in Theorem 4.5.54. We conclude Subsection 4.5.1 by proving in Proposition 4.5.11 that algebra $*$-homomorphisms from complete normed Jordan-admissible complex $*$-algebras to non-commutative JB^*-algebras are continuous, a result whose associative forerunner goes back to Rickart [501].

Subsection 4.5.2 contains the main results in the section, and begins with a general non-associative version of Vesentini's theorem [622] that the spectral radius is a subharmonic function (see Theorem 4.5.14). Then we develop Behncke's theory of complete normed hermitian Jordan complex $*$-algebras [82] following later arguments of Aupetit and Youngson [48], which rely on Vesentini's theorem. Theorem 4.5.29 and Corollary 4.5.30 summarize Behncke's theory in a somewhat new way. As done in [48], we apply Theorem 4.5.29 to derive that a complete normed unital Jordan complex $*$-algebra A is a JB^*-algebra in an equivalent norm if and only the set $\{\exp(ih) : h \in H(A,*)\}$ is bounded (see Corollary 4.5.32). With Theorem 3.6.25 as a key tool, we also derive from Theorem 4.5.29 a full theory of complete normed hermitian alternative complex $*$-algebras, which contains its classical associative forerunner [493, 565] as stated in Bonsall–Duncan [696, Section 41] (see Theorem 4.5.37 and Corollaries 4.5.38 and 4.5.39). A characterization of normed unital complex $*$-algebras which are alternative C^*-algebras in an equivalent norm, similar to that proved in the Jordan setting, is also obtained (see Corollary 4.5.42). As shown in Proposition 4.5.45, the requirement in Corollaries 4.5.32 and 4.5.42 that the algebra be unital is not that essential. Nevertheless, even in the unital case, a verbatim unification of Corollaries 4.5.32 and 4.5.42 in the natural common setting of non-commutative Jordan algebras is far from being true (see Example 4.5.43).

As a matter of fact, we do not know any reasonable characterization of normed unital complex $*$-algebras which are non-commutative JB^*-algebras in an equivalent norm. In Subsection 4.5.3, we formulate a conjecture which, if were proved, would overcome this gap.

4.5.1 Preliminary results

As an aperitif, we begin this subsection with the following theorem, which provides us with several local characterizations of non-commutative JB^*-algebras, and which could have been proved much earlier. It is indeed a by-product of the non-associative Vidav–Palmer theorem.

Theorem 4.5.1 *Let A be a complete normed unital complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is a non-commutative JB^*-algebra (in its own norm).*
(ii) *For each $h \in H(A,*)$, the closed $*$-subalgebra of A generated by h is a non-commutative JB^*-algebra (in the norm of A).*
(iii) *For each $h \in H(A,*)$, the closed $*$-subalgebra of A generated by h is a commutative C^*-algebra (in the norm of A).*
(iv) *A is a power-associative algebra, and the equality $\|\exp(ih)\| = 1$ holds for every $h \in H(A,*)$.*

Proof (i)$\Rightarrow$(ii) This is clear.

(ii)$\Rightarrow$(iii) Let h be in $H(A,*)$. By assumption (ii) and Proposition 3.4.1(ii), the closed $*$-subalgebra of A generated by h is indeed a commutative C^*-algebra.

(iii)$\Rightarrow$(i) As a first consequence of assumption (iii), A is norm-unital. Let h be in $H(A,*)$, and note that, as a new consequence of assumption (iii), the subalgebra of A generated by $\{h, \mathbf{1}\}$ (say B) is associative. Let M be in $\mathbb{R}$ such that $\|h\| < M$, set $k := \mathbf{1} - \frac{h}{M} \in B \cap H(A,*)$, let C stand for the closed $*$-subalgebra of A generated by k, and note that, again by assumption (iii), C is a C^*-algebra. Since $\|\mathbf{1} - k\| < 1$, we have $(\mathbf{1} - k)^n \to 0$ as $n \to \infty$, and hence $\mathbf{1}$ lies in C because $\mathbf{1} - (\mathbf{1} - k)^n \in C$ for every $n \in \mathbb{N}$ and $\mathbf{1} - (\mathbf{1} - k)^n \to \mathbf{1}$ as $n \to \infty$. Since C is now a unital C^*-algebra, it follows from §1.2.18 that $\|\exp(irk)\| = 1$ for every $r \in \mathbb{R}$, and therefore, by Corollaries 2.1.2 and 2.1.9(iii), k lies in $H(A, \mathbf{1})$. Then $h = M(\mathbf{1} - k)$ also lies in $H(A, \mathbf{1})$. Since h is arbitrary in $H(A,*)$, we get that $H(A,*) \subseteq H(A, \mathbf{1})$, so A is a V-algebra whose natural involution coincides with $*$, and so A is a non-commutative JB^*-algebra because of the non-associative Vidav–Palmer theorem (Theorem 3.3.11).

(i)$\Rightarrow$(iv) This follows from Proposition 2.4.19, Lemma 2.2.5, and Corollary 2.1.9(iii).

(iv)$\Rightarrow$(i) If condition (iv) is fulfilled, then, by Corollaries 2.1.2 and 2.1.9(iii), we have $H(A,*) \subseteq H(A, \mathbf{1})$, so A is a V-algebra whose natural involution coincides with $*$, and so A is a non-commutative JB^*-algebra because of Theorem 3.3.11. □

Let A be a complete normed unital complex $*$-algebra. As a consequence of the implication (i)$\Rightarrow$(iv) in the above theorem, if A is a non-commutative JB^*-algebra in an equivalent norm, then there exists a constant $M \in \mathbb{R}$ such that $\|\exp(ih)\| \leqslant M$ for every $h \in H(A,*)$. However, as we will show in Example 4.5.43, the converse assertion is not true even if A is a finite-dimensional non-commutative Jordan algebra. We will prove in Corollary 4.5.32 (respectively, Corollary 4.5.42) that such a converse assertion is true if A is a Jordan (respectively, alternative) algebra. For the moment, we limit ourselves to consider the case where A is an associative and commutative algebra (see Theorem 4.5.5 below). To this end, we begin by proving the following.

Proposition 4.5.2 *Let A be a normed associative algebra over $\mathbb{K}$, and let S be a bounded and multiplicatively closed subset of A. We have:*

(i) *There exists an algebra norm $|||\cdot|||$ on A satisfying*

$$|||\cdot||| \leqslant 1 \text{ on } S \text{ and } \frac{1}{M}\|\cdot\| \leqslant |||\cdot||| \leqslant M\|\cdot\| \text{ on } A,$$

where $M := \max\{1, \sup\{\|s\| : s \in S\}\}$.

(ii) *If A is unital, then there exists an algebra norm $|||\cdot|||$ on A satisfying $|||\mathbf{1}||| = 1$, $|||\cdot||| \leqslant 1$ on S, and $\frac{1}{K}\|\cdot\| \leqslant |||\cdot||| \leqslant K\|\cdot\|$ on A,*

$$\text{where } K := \max\{\|\mathbf{1}\|, \sup\{\|s\| : s \in S\}\}.$$

Proof We begin by proving assertion (ii), so that we assume that A is unital. Then, clearly, $T := S \cup \{\mathbf{1}\}$ is a multiplicatively closed subset of A contained in $K\mathbb{B}_A$. For $a \in A$, set

$$p(a) := \sup\{\|sa\| : s \in T\}.$$

Then $p(\cdot) \leqslant K\|\cdot\|$ on A. Since $\mathbf{1} \in T$, we have $\|\cdot\| \leqslant p(\cdot)$, and so, for $a, b \in A$, we have

$$p(ab) = \sup\{\|sab\| : s \in T\} \leqslant \sup\{\|sa\| : s \in T\}\|b\| = p(a)\|b\| \leqslant p(a)p(b).$$

It is now clear that p is an algebra norm on A equivalent to $\|\cdot\|$. Therefore, defining $|||\cdot|||$ on A by $|||a||| := p(L_a)$ (here $p(\cdot)$ stands for the operator norm on $BL(A)$ corresponding to p), and keeping in mind §1.1.122, we see that $|||\cdot|||$ becomes an algebra norm on A satisfying $|||\mathbf{1}||| = 1$ and $|||\cdot||| \leqslant p(\cdot) \leqslant p(\mathbf{1})\,|||\cdot|||$ on A. Then, since $p(\mathbf{1}) \leqslant K$ (by the definitions of K and p), and $\|\cdot\| \leqslant p(\cdot) \leqslant K\|\cdot\|$ on A, we derive that $\frac{1}{K}\|\cdot\| \leqslant |||\cdot||| \leqslant K\|\cdot\|$ on A. Also, for $t \in T$ and $x \in A$, we have

$$p(tx) = \sup\{\|stx\| : s \in T\} \leqslant p(x),$$

since $st \in T$. Thus $|||t||| \leqslant 1$ for every $t \in T$, and hence $|||\cdot||| \leqslant 1$ on S.

Assertion (i) follows from assertion (ii) just proved by regarding A as a subalgebra of its normed unital extension (cf. Proposition 1.1.107). □

Corollary 4.5.3 *Let A be a normed associative algebra over $\mathbb{K}$, and let a be in A. Then*

$$\mathfrak{r}(a) = \inf\{|||a||| : |||.||| \in \mathrm{En}(A)\},$$

where $\mathrm{En}(A)$ denotes the set of all equivalent algebra norms on A.

Proof By the definition of the spectral radius and Corollary 1.1.19, we have $\mathfrak{r}(a) \leqslant |||a|||$ for every $|||\cdot||| \in \mathrm{En}(A)$. Let $\varepsilon > 0$. Setting $b := \frac{1}{\mathfrak{r}(a)+\varepsilon}a$, we see that $\mathfrak{r}(b) < 1$, and consequently, by Corollary 1.1.18(ii), $\{b^n : n \in \mathbb{N}\}$ is a bounded and multiplicatively closed subset of A. Therefore, by Proposition 4.5.2(i), there exists $|||.||| \in \mathrm{En}(A)$ such that $|||b||| \leqslant 1$, so $|||a||| \leqslant \mathfrak{r}(a) + \varepsilon$, and the result follows. □

Corollary 4.5.4 *Let A be a complete normed unital power-associative complex algebra, and let h be in A. Assume the existence of $\kappa > 0$ such that $\|\exp(ith)\| \leqslant \kappa$ for every $t \in \mathbb{R}$. Then $\mathfrak{r}(h) \geqslant \kappa^{-1}\|h\|$.*

Proof Let B stand for the closed subalgebra of A generated by h and $\mathbf{1}$. It is clear that B is a complete normed unital associative complex algebra and that

$S := \{\exp(ith) : t \in \mathbb{R}\}$ is a multiplicatively closed subset of B containing $\mathbf{1}$ (cf. Exercise 1.1.30). Since S is bounded by κ, it follows from Proposition 4.5.2(ii) that there exists an equivalent algebra norm $|||\cdot|||$ on B such that $|||\mathbf{1}||| = 1$, $|||\exp(ith)||| \leqslant 1$ for every $t \in \mathbb{R}$, and $\kappa^{-1}\|x\| \leqslant |||x|||$ for every $x \in B$. But, since $|||\cdot|||$ is an algebra norm on B, the conditions $|||\mathbf{1}||| = 1$ and $|||\exp(ith)||| \leqslant 1$ for every $t \in \mathbb{R}$ imply $|||\exp(ith)||| = 1$ for every $t \in \mathbb{R}$. Thus, by Corollary 2.1.9(iii), h is a hermitian element of the norm-unital complete normed complex algebra $(B, |||\cdot|||)$. Therefore, by Corollary 1.1.19 and Proposition 2.3.22, we have $\mathfrak{r}(h) = |||h||| \geqslant \kappa^{-1}\|h\|$. □

Now, as the main result in this subsection, we prove the following theorem, which becomes an associative-commutative variant of Theorem 4.5.1.

Theorem 4.5.5 *Let A be a complete normed unital associative and commutative complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is a C^*-algebra in an equivalent norm.*
(ii) *For each $h \in H(A,*)$, the closed $*$-subalgebra of A generated by h and $\mathbf{1}$ is a C^*-algebra in an equivalent norm.*
(iii) *There exists $M > 0$ such that $\|\exp(ih)\| \leqslant M$ for every $h \in H(A,*)$.*

Proof (i)⇒(ii) This is clear.

(ii)⇒(iii) By the assumption (ii) and §1.2.18, for each $h \in H(A,*)$, there exists a constant κ depending on h such that $\|\exp(ith)\| \leqslant \kappa$ for every $t \in \mathbb{R}$. Let x be in s-Rad(A). Then $h := \frac{1}{2}(x + x^*)$ and $k := \frac{1}{2i}(x - x^*)$ lie in s-Rad$(A) \cap H(A,*)$ because s-Rad(A) is $*$-invariant. Therefore, by Theorem 1.1.73(v), we have $\mathfrak{r}(h) = \mathfrak{r}(k) = 0$, so, by Corollary 4.5.4, $h = k = 0$, and so $x = h + ik = 0$. Thus A is (strongly) semisimple, so that, by Corollary 1.1.63, $*$ is continuous on A, and consequently $H(A,*)$ is closed in A. For $n \in \mathbb{N}$, set

$$H_n := \{h \in H(A,*) : \|\exp(ith)\| \leqslant n \text{ for every } t \in \mathbb{R}\}.$$

Then H_n is closed in $H(A,*)$ and $H(A,*) = \cup_{n\in\mathbb{N}} H_n$. By Baire's theorem, there exists $m \in \mathbb{N}$ such that H_m has non-empty interior in $H(A,*)$. Fix h_0 and $\delta > 0$ such that the ball in $H(A,*)$ with centre h_0 and radius δ is contained in H_m. If $h \in H(A,*)$ with $\|h\| < \delta$, then

$$\begin{aligned}\|\exp(ith)\| &= \|\exp(it(h+h_0))\exp(-ith_0)\| \\ &\leqslant \|\exp(it(h+h_0))\|\|\exp(-ith_0)\| \leqslant m^2\end{aligned}$$

for every $t \in \mathbb{R}$. Hence $\|\exp(ih)\| \leqslant m^2$ for every $h \in H(A,*)$.

(iii)⇒(i) By assumption (iii), the set $\{\exp(ih) : h \in H(A,*)\}$ is a bounded and multiplicatively closed subset of A. Therefore, in view of Proposition 4.5.2(ii), there exists an equivalent algebra norm $|||\cdot|||$ on A such that $|||\exp(ih)||| \leqslant 1$ for every $h \in H(A,*)$. Thus, the normed algebra $(A, |||\cdot|||)$ is norm-unital and we have in fact $|||\exp(ih)||| = 1$ for every $h \in H(A,*)$. Now, by Corollary 2.1.9(iii), we have $H(A,*) \subseteq H((A, |||\cdot|||), \mathbf{1})$. Therefore $(A, |||\cdot|||)$ is a V-algebra whose natural involution coincides with $*$, and so $(A, |||\cdot|||)$ is a C^*-algebra because of the associative Vidav–Palmer theorem (Theorem 2.3.32). □

§**4.5.6** The assumption in the above theorem that the algebra be unital is not that essential. To realize this, some elementary facts have to be kept in mind. Let A be a normed algebra over $\mathbb{K}$. By a *normed unital extension* of A we mean the unital extension $A_{\mathbb{1}}$ of A (cf. §1.1.104) endowed with an algebra norm extending the norm of A and making the direct sum $A_{\mathbb{1}} = \mathbb{K}\mathbb{1} \oplus A$ topological. We note that, if A is complete, the last requirement (that the direct sum $A_{\mathbb{1}} = \mathbb{K}\mathbb{1} \oplus A$ is topological) is superfluous. Anyway, the norms of two normed unital extensions of A are equivalent. Moreover, we are always provided with a distinguished normed unital extension of A, namely the one given by Proposition 1.1.107, which will be called the *standard normed unital extension* of A. On the other hand, if A is actually a C^*-algebra (or, more generally, a non-commutative JB^*-algebra), then we are provided with the unique *C^*-unital extension* (or *non-commutative JB^*-unital extension*) of A given by Proposition 1.2.44 (or Corollary 3.5.36). Now assume that A is complete and power-associative, and for $a \in A$ set

$$(\exp -1)(a) := \sum_{n=1}^{\infty} \frac{a^n}{n!} \in A.$$

Then in any normed unital extension of A we have $\exp(a) = \mathbb{1} + (\exp -1)(a)$, and hence

$$\|\exp(a)\| \leqslant \|\mathbb{1}\| + \|(\exp -1)(a)\| \quad \text{and} \quad \|(\exp -1)(a)\| \leqslant \|\mathbb{1}\| + \|\exp(a)\|.$$

Now we can prove the following unit-free version of Theorem 4.5.5.

Corollary 4.5.7 *Let A be a complete normed associative and commutative complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is a C^*-algebra in an equivalent norm.*
(ii) *For each $h \in H(A,*)$, the closed $*$-subalgebra of A generated by h is a C^*-algebra in an equivalent norm.*
(iii) *There exists a constant $K > 0$ such that $\|(\exp -1)(ih)\| \leqslant K$ for every $h \in H(A,*)$.*

Proof (i)$\Rightarrow$(ii) This is clear.

In the remaining parts of the proof, $(A_{\mathbb{1}}, \|\cdot\|)$ will denote the standard normed unital extension of A endowed with the unique conjugate-linear algebra involution which extends $*$, $r\mathbb{1} + k$ ($r \in \mathbb{R}$ and $k \in H(A,*)$) will be an arbitrary element in $H(A_{\mathbb{1}},*)$, and B will stand for the closed $*$-subalgebra of A generated by k. Then we note that $\mathbb{C}\mathbb{1} + B$ is the closed $*$-subalgebra of $A_{\mathbb{1}}$ generated by $r\mathbb{1} + k$ and $\mathbb{1}$, and that $\mathbb{C}\mathbb{1} + B$, endowed with the norm $\|\cdot\|$, can be seen as the standard normed unital extension $(B_{\mathbb{1}}, \|\cdot\|)$ of B.

(ii)$\Rightarrow$(iii) By assumption (ii), we can consider the C^*-unital extension $(B_{\mathbb{1}}, |||\cdot|||)$ of B, so that $B_{\mathbb{1}}$ is a C^*-algebra in an equivalent norm. Since $r\mathbb{1} + k$ is an arbitrary element of $H(A_{\mathbb{1}},*)$, it follows from the implication (ii)$\Rightarrow$(iii) in Theorem 4.5.5 that there exists a constant $M \in \mathbb{R}$ such that $\|\exp(ih)\| \leqslant M$ for every $h \in H(A_{\mathbb{1}},*)$, so in particular for every $h \in H(A,*)$. Thus

$$\|(\exp -1)(ih)\| \leqslant 1 + \|\exp(ih)\| \leqslant 1 + M \quad \text{for every} \quad h \in H(A,*).$$

(iii)$\Rightarrow$(i) Assume that condition (iii) is fulfilled. Then we have

$$\|\exp(i(r\mathbb{1} + k))\| = \|e^{ir}\exp(ik)\| = \|\exp(ik)\| \leqslant 1 + \|(\exp -1)(ik)\| \leqslant 1 + K.$$

Since $r\mathbb{1}+k$ is an arbitrary element of $H(A_{\mathbb{1}},*)$, it follows from the implication (iii)⇒(i) in Theorem 4.5.5 that $A_{\mathbb{1}}$ (and hence A) is a C^*-algebra in an equivalent norm. □

The remaining part of this subsection will be devoted to discussing basic results concerning algebra $*$-homomorphisms into non-commutative JB^*-algebras.

Definition 4.5.8 Let A be a complex $*$-algebra. By a *non-commutative JB^*-representation* (respectively, a *JB^*-representation*, an *alternative C^*-representation*, or a *C^*-representation*) of A we mean an algebra $*$-homomorphism from A to some non-commutative JB^*-algebra (respectively, JB^*-algebra, alternative C^*-algebra, or C^*-algebra).

Fact 4.5.9 *Let A be a complete normed complex $*$-algebra admitting power-associativity, let Φ be a non-commutative JB^*-representation of A, and let h be in $H(A,*)$. Then $\|\Phi(h)\| \leqslant \mathfrak{s}(h)$.*

Proof Let B stand for the non-commutative JB^*-algebra where the range of Φ lives. Since $\Phi(h) \in H(B,*)$, we have $\|\Phi(h)\| = \mathfrak{r}(\Phi(h))$ (cf. Proposition 3.4.1(ii) and Lemma 1.2.12). Therefore, regarding Φ as an algebra homomorphism from A^{sym} to B^{sym}, and keeping in mind §4.4.6, the result follows from Proposition 3.4.63. □

Let A be an algebra over $\mathbb{K}$, and let a be in A. We denote by $U_a^{A^{\mathrm{sym}}}$ the operator U_a in the algebra A^{sym}, and recall that, if A is Jordan-admissible (respectively, unital and Jordan-admissible), then a is quasi-J-invertible (respectively, J-invertible) in A if and only if a is quasi-J-invertible (respectively, J-invertible) in A^{sym} (cf. Definition 4.1.56 and §4.4.8).

Proposition 4.5.10 *Let A be a Jordan-admissible algebra over $\mathbb{K}$. Then $A/$J-Rad(A) is J-semisimple.*

Proof Consider the quotient mapping $a \to \bar{a}$ from A to $\bar{A} := A/$J-Rad(A), and note that $A_{\mathbb{1}}/$J-Rad$(A) \equiv (\bar{A})_{\mathbb{1}}$. Suppose that $a \in A$ is such that $\bar{a} \in$ J-Rad$(\bar{A})$. Then $\bar{a}$ is quasi-J-invertible in $\bar{A}$, and hence $\bar{\mathbb{1}} - \bar{a}$ is J-invertible in $(\bar{A})_{\mathbb{1}}$. Therefore, by Theorem 4.1.3(ii), $\bar{\mathbb{1}}$ is in the range of the operator $U_{\bar{\mathbb{1}}-\bar{a}}^{((\bar{A})_{\mathbb{1}})^{\mathrm{sym}}}$, so that there exist $\lambda \in \mathbb{K}$ and $b \in A$ such that $\bar{\mathbb{1}} = U_{\bar{\mathbb{1}}-\bar{a}}^{((\bar{A})_{\mathbb{1}})^{\mathrm{sym}}}(\lambda\bar{\mathbb{1}} + \bar{b})$, and hence there exists $c \in$ J-Rad(A) such that $\mathbb{1} = U_{\mathbb{1}-a}^{(A_{\mathbb{1}})^{\mathrm{sym}}}(\lambda\mathbb{1} + b) + c$. Since c is quasi-J-invertible in A, it follows that $\mathbb{1} - c = U_{\mathbb{1}-a}^{(A_{\mathbb{1}})^{\mathrm{sym}}}(\lambda\mathbb{1} + b)$ is J-invertible in $A_{\mathbb{1}}$. Therefore, by Theorem 4.1.3(vi), $\mathbb{1} - a$ is J-invertible in $A_{\mathbb{1}}$, and as a result a is quasi-J-invertible in A. Thus the set $Q := \{a \in A : \bar{a} \in \text{J-Rad}(\bar{A})\}$ is a quasi-J-invertible ideal of A. Therefore $Q =$ J-Rad(A), and consequently $\bar{A}$ is J-semisimple. □

Proposition 4.5.11 *Let A be a complete normed Jordan-admissible complex $*$-algebra, and let Φ be a non-commutative JB^*-representation of A. Then:*

(i) J-Rad$(A) \subseteq \ker(\Phi)$.
(ii) *Φ is continuous.*

Proof Let x be in J-Rad(A). Then $h := \frac{1}{2}(x + x^*)$ and $k := \frac{1}{2i}(x - x^*)$ lie in J-Rad$(A) \bigcap H(A,*)$ because J-Rad(A) is $*$-invariant. Therefore, by Lemma 4.4.26, we have $\mathfrak{s}(h) = \mathfrak{s}(k) = 0$, so $\Phi(h) = \Phi(k) = 0$ (by Fact 4.5.9), and so $\Phi(x) = \Phi(h+ik) = 0$. Thus $\text{J-Rad}(A) \subseteq \ker(\Phi)$, which proves assertion (i).

Now we proceed to prove assertion (ii).

Assume at first that A is J-semisimple. Then, by Corollary 4.4.14, $*$ is continuous on A, and hence the direct sum $A = H(A,*) \oplus iH(A,*)$ is topological. On the other hand, by Fact 4.5.9, we have $\|\Phi(h)\| \leqslant \|h\|$ for every $h \in H(A,*)$, and hence Φ is continuous on $H(A,*)$. It follows that Φ is continuous on A.

To conclude the proof, remove the assumption in the above paragraph that A is J-semisimple. Then, by Propositions 4.4.17(i) and 4.5.10, $A/\text{J-Rad}(A)$ is a complete normed J-semisimple Jordan-admissible complex algebra, and is endowed with the quotient involution (again because $\text{J-Rad}(A)$ is $*$-invariant). By keeping in mind the above paragraph, it follows from assertion (i) that

$$\Psi : a + \text{J-Rad}(A) \longrightarrow \Phi(a)$$

becomes a well-defined continuous non-commutative JB^*-representation of the $*$-algebra $A/\text{J-Rad}(A)$. Since $\Phi = \Psi \circ \pi$, where $\pi : A \to A/\text{J-Rad}(A)$ stands for the quotient mapping, we derive that Φ is continuous. $\square$

Remark 4.5.12 Let A be a complex $*$-algebra. Then, as usual, $H(A,*)$ will be seen as a real algebra, precisely as a real subalgebra of A^{sym}.

Assume that A is Jordan-admissible. Then $H(A,*)$ is a real Jordan algebra, and we have

$$\text{J-Rad}(A) \subseteq \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*)). \tag{4.5.1}$$

Indeed, $I := \text{J-Rad}(A) \cap H(A,*)$ is an ideal of $H(A,*)$, each element of which has a quasi-J-inverse in A, which must lie in $H(A,*)$ because of the uniqueness of the quasi-J-inverse. Therefore I is a quasi-J-invertible ideal of $H(A,*)$, and hence $I \subseteq \text{J-Rad}(H(A,*))$. But $\text{J-Rad}(A) = I + iI$ because $\text{J-Rad}(A)$ is $*$-invariant, so the inclusion (4.5.1) follows.

Now assume that A is complete normed and Jordan-admissible, and let Φ be a non-commutative JB^*-representation of A. Then $H(A,*)$ is a (possibly non-complete) normed Jordan real algebra, and we have the following refinement of Proposition 4.5.11(i):

$$\text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*)) \subseteq \ker(\Phi). \tag{4.5.2}$$

Indeed, if h is in $\text{J-Rad}(H(A,*))$, then, by Lemma 4.4.26, we have $\mathfrak{s}(h) = 0$, so that, by Fact 4.5.9, $\Phi(h) = 0$. Thus $\text{J-Rad}(H(A,*)) \subseteq \ker(\Phi)$, and the inclusion (4.5.2) follows.

4.5.2 The main results

Let Ω be an open subset of $\mathbb{C}$. A function $u : \Omega \to [-\infty, +\infty[$ is called *subharmonic* if it is upper semicontinuous and satisfies the submean inequality, i.e. whenever a closed disc $B_{\mathbb{C}}(w,r)$ in $\mathbb{C}$ is contained in Ω, we have

$$u(w) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})\, dt.$$

Lemma 4.5.13 *Let Ω be an open subset of $\mathbb{C}$, let X be a complex normed space, and let $f : \Omega \to X$ be a holomorphic function. Then $z \to \log\|f(z)\|$ is a subharmonic function on Ω.*

Proof Certainly $\log\|f(z)\|$ is upper semicontinuous on Ω, indeed even continuous, so we only need to check the submean inequality. Let $B_{\mathbb{C}}(w,r)$ be a closed disc in $\mathbb{C}$ contained in Ω. By the Hahn–Banach theorem, there exists $\phi \in \mathbb{S}_{X'}$ such that $\phi(f(w)) = \|f(w)\|$. Then $\phi \circ f$ is a complex-valued holomorphic function on Ω, so $\log|\phi \circ f|$ is subharmonic on Ω. Consequently, we have

$$\log\|f(w)\| = \log|\phi(f(w))| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|\phi(f(w+re^{it}))|\,dt$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log\|f(w+re^{it})\|\,dt,$$

so $\log\|f\|$ does indeed satisfy the submean inequality. □

Theorem 4.5.14 *Let Ω be an open subset of $\mathbb{C}$, let A be a normed complex algebra, and let $f : \Omega \to A$ be a holomorphic function. Then both functions $z \to \log\mathfrak{s}(f(z))$ and $z \to \mathfrak{s}(f(z))$ are subharmonic on Ω.*

Proof For each $n \in \mathbb{N} \cup \{0\}$, consider the holomorphic function f_n on Ω defined by

$$f_n(z) := (f(z))^{[n]}$$

in the sense of §4.4.2, set

$$u_n(z) := \tfrac{1}{2^n}\log\|f_n(z)\| \quad (z \in \Omega),$$

and note that, by Lemma 4.5.13, u_n is subharmonic on Ω. Moreover, keeping in mind §4.4.2 again, we see that the sequence u_n is decreasing and converges to the function $\log\mathfrak{s}(f(z))$. Hence $\log\mathfrak{s}(f(z))$ is subharmonic on Ω. Since the real exponential function is increasing and convex, it follows that $\mathfrak{s}(f(z))$ is also subharmonic on Ω. □

Let S be a subset of $\mathbb{C}$ and let $\zeta \in \mathbb{C}$. We say that S is *non-thin at* ζ if ζ belongs to the closure of $S \setminus \{\zeta\}$ in $\mathbb{C}$ and if, for every subharmonic function u defined on a neighbourhood of ζ, we have

$$\limsup_{\substack{z\to\zeta \\ z\in S\setminus\{\zeta\}}} u(z) = u(\zeta).$$

For the proof of the following proposition, the reader is referred to Ransford's book [793, Theorem 3.8.3]. A particular case, enough for our later application, can be seen in Aupetit's book [682, Théorème II.11], where it is attributed to K. Oka and W. Rothstein.

Proposition 4.5.15 *A connected subset of $\mathbb{C}$ containing more than one point is non-thin at every point of its closure.*

Lemma 4.5.16 *Let A be a complete normed unital Jordan complex algebra, let Ω be an open subset of $\mathbb{C}$ containing $[0,1]$, and let $f : \Omega \to A$ be a holomorphic* *function. If $a < b < c < d$ are real numbers such that*

$$\text{J-sp}(A, f(t)) \subseteq [a,b] \cup [c,d] \text{ for every } t \in [0,1] \text{ and } \text{J-sp}(A, f(0)) \subseteq [a,b],$$

then

$$\text{J-sp}(A, f(t)) \subseteq [a,b] \text{ for every } t \in [0,1].$$

Proof We may assume that $a > 0$. Let $E := \{t \in [0,1] : \text{J-sp}(A, f(t)) \subseteq [a,b]\}$. We first show that E is open in $[0,1]$. Assume to the contrary that there exist $t_0 \in E$ and a sequence t_n in [0,1] converging to t_0 with $t_n \notin E$ for every $n \in \mathbb{N}$. Hence there exists $\alpha_n \in \text{J-sp}(A, f(t_n)) \cap [c,d]$ for every $n \in \mathbb{N}$. By passing to a subsequence if necessary, we may assume that α_n converges to some $\alpha \in [c,d]$. Thus $f(t_0) - \alpha\mathbf{1}$ is a J-invertible element which is the limit of a sequence $f(t_n) - \alpha_n\mathbf{1}$ of non- J-invertible elements. This contradicts the fact that the set of J-invertible elements is open in A (cf. Theorem 4.1.7). Hence E is open in $[0,1]$. Let $\gamma := \sup\{r \in [0,1] : [0,r] \subseteq E\}$. Since E is open, we have $0 < \gamma \leqslant 1$. Furthermore $[0,\gamma[\subseteq E$, and because $z \to \mathfrak{r}(f(z))$ is subharmonic (by Theorem 4.5.14) and $[0,\gamma[$ is non-thin at γ (by Proposition 4.5.15), we have

$$\mathfrak{r}(f(\gamma)) = \limsup_{\substack{r \to \gamma \\ r < \gamma}} \mathfrak{r}(f(r)) \leqslant b,$$

hence $\gamma \in E$ and so as E is open in $[0,1]$, we have $\gamma = 1$ and the result follows. □

Let B and C be unital Jordan algebras over $\mathbb{K}$. Then the algebra direct product $B \times C$ is a unital Jordan algebra, and, keeping in mind Definition 4.1.2, we see that

$$\text{J-Inv}(B \times C) = \text{J-Inv}(B) \times \text{J-Inv}(C). \tag{4.5.3}$$

As a consequence, for $(b,c) \in B \times C$ we have

$$\text{J-sp}(B \times C, (b,c)) = \text{J-sp}(B,b) \cup \text{J-sp}(C,c). \tag{4.5.4}$$

Proposition 4.5.17 *Let A be a unital Jordan-admissible algebra over $\mathbb{K}$, and let a be in A. Then*

(i) *a is quasi-J-invertible in A if and only if $\mathbf{1} - a$ is J-invertible in A.*
(ii) $\text{J-sp}(A_{\mathbb{1}}, a) = \text{J-sp}(A, a) \cup \{0\}$.

Proof Since $(A_{\mathbb{1}})^{\text{sym}} = (A^{\text{sym}})_{\mathbb{1}}$, it follows from the definitions of quasi-J-invertibility (cf. §4.4.8) and of the J-spectrum (cf. Definition 4.1.65) that we may assume that A is commutative. Noticing that A and $\mathbb{K}(\mathbb{1} - \mathbf{1})$ are ideals of $A_{\mathbb{1}}$ such that

$$A_{\mathbb{1}} = A \oplus \mathbb{K}(\mathbb{1} - \mathbf{1}),$$

and that A and $\mathbb{K}(\mathbb{1} - \mathbf{1})$ are unital algebras, assertion (i) (respectively, assertion (ii)) follows from §1.1.105 and the equality (4.5.3) (respectively, the equality (4.5.4)). □

Definition 4.5.18 Let A be a Jordan-admissible complex $*$-algebra. We denote by A^+ the set of those elements $h \in H(A,*)$ such that $\text{J-sp}(A_{\mathbb{1}}, h) \subseteq \mathbb{R}_0^+$, and say that A is *hermitian* if $\text{J-sp}(A_{\mathbb{1}}, h) \subseteq \mathbb{R}$ for every $h \in H(A,*)$. It follows from the definition of

J-spectrum (cf. Definition 4.1.65) that $A^+ = (A^{\mathrm{sym}})^+$, and that A is hermitian if and only if A^{sym} is hermitian. If in addition A is unital, then, by Proposition 4.5.17(ii),

$$A^+ = \{h \in H(A,*) : \text{J-sp}(A,h) \subseteq \mathbb{R}_0^+\},$$

and A is hermitian if and only if $\text{J-sp}(A,h) \subseteq \mathbb{R}$ for every $h \in H(A,*)$. As a consequence, in any case we have

$$(A_{\mathbb{1}})^+ = \{\alpha\mathbb{1} + h \in A_{\mathbb{1}} : \alpha \in \mathbb{R}, h \in H(A,*), \text{ and } \alpha + \text{J-sp}(A_{\mathbb{1}},h) \subseteq \mathbb{R}_0^+\}$$

(hence $A \cap (A_{\mathbb{1}})^+ = A^+$), and A is hermitian if and only if $A_{\mathbb{1}}$ is hermitian.

The next fact follows from Corollary 3.5.36 and Fact 4.1.67.

Fact 4.5.19 *Let A be a non-commutative JB^*-algebra. Then A is hermitian. Moreover, an element $a \in A$ lies in A^+ if and only if $a \geqslant 0$ in the order of $H(A,*)$ introduced in §3.4.68.*

Lemma 4.5.20 *Let A be a complete normed power-associative $*$-algebra over $\mathbb{K}$, and let a be in A. Then $\mathfrak{r}(a) = \mathfrak{r}(a^*)$.*

Proof For $x \in A$, set $|||x||| := \|x^*\|$. Then $|||\cdot|||$ becomes a complete algebra norm on A. Therefore, by Corollary 3.4.64, we have

$$\mathfrak{r}(a) = \mathfrak{r}_{|||\cdot|||}(a) = \inf\{|||a^n|||^{\frac{1}{n}} : n \in \mathbb{N}\} = \inf\{\|(a^*)^n\|^{\frac{1}{n}} : n \in \mathbb{N}\} = \mathfrak{r}(a^*). \qquad \square$$

Proposition 4.5.21 *Let A be a complete normed algebra over $\mathbb{K}$. We have:*

(i) *If a is in A with $\|a\| < 1$, then there exists a unique $x \in A$ such that $2x - x^2 = a$ and $\|x\| < 1$.*
(ii) *If A is power-associative, and if a is in A with $\mathfrak{r}(a) < 1$, then there exists a unique $x \in A$ such that $2x - x^2 = a$ and $\mathfrak{r}(x) < 1$.*
(iii) *If A is a power-associative $*$-algebra, and if h is in $H(A,*)$ with $\mathfrak{r}(h) < 1$, then there exists a unique $x \in H(A,*)$ such that $2x - x^2 = h$ and $\mathfrak{r}(x) < 1$.*
(iv) *If $\mathbb{K} = \mathbb{C}$, if A is a unital non-commutative Jordan $*$-algebra, and if h is in $A^+ \cap \text{J-Inv}(A)$, then there exists a unique $a \in A^+ \cap \text{J-Inv}(A)$ such that $a^2 = h$.*

Proof Let a be in A with $\|a\| < 1$, set $E := \{u \in A : \|u\| \leqslant \|a\|\}$, and let T be the mapping from E to E defined by $T(u) := \frac{1}{2}(a + u^2)$. Then we have

$$\begin{aligned} \|T(u) - T(v)\| &= \frac{1}{2}\|u^2 - v^2\| = \frac{1}{2}\|(u+v) \bullet (u-v)\| \\ &\leqslant \frac{1}{2}\|u+v\|\|u-v\| \leqslant \|a\|\|u-v\| \end{aligned}$$

for all $u, v \in E$. Therefore T is a contraction mapping. By the contraction mapping principle (see for example [709, Théorème V.3.21.1]), there exists $x \in E$ with $T(x) = x$, i.e. with $2x - x^2 = a$, and we have $\|x\| \leqslant \|a\| < 1$. Suppose now that $y \in A$, $2y - y^2 = a$, and $\|y\| < 1$. Then we have

$$\|x - y\| = \frac{1}{2}\|x^2 - y^2\| \leqslant \frac{1}{2}\|x+y\|\|x-y\| \leqslant \frac{\|x\| + \|y\|}{2}\|x - y\|.$$

Since $\frac{\|x\|+\|y\|}{2} < 1$, the above implies that $y = x$, which concludes the proof of assertion (i).

Assume that A is power-associative, and let a be in A with $\mathfrak{r}(a) < 1$. Let B stand for the closed subalgebra of A generated by a. Then B is associative, and hence, by Corollary 4.5.3, there exists an equivalent algebra norm $|||\cdot|||$ on B such that $|||a||| < 1$. Therefore, by assertion (i), there exists $x \in B$ such that $2x - x^2 = a$ and $\mathfrak{r}(x) \leqslant |||x||| < 1$. Suppose now that $y \in A$, $2y - y^2 = a$, and $\mathfrak{r}(y) < 1$. Let C stand for the closed subalgebra of A generated by y, and note that, as a lies in C, the inclusion $B \subseteq C$ holds. Since C is associative and commutative, and $x \in B$, it follows from Corollary 1.1.115 that

$$\mathfrak{r}(x+y) \leqslant \mathfrak{r}(x) + \mathfrak{r}(y) < 2,$$

and hence $\mathfrak{r}(\frac{1}{2}(x+y)) < 1$. It follows from Definition 3.6.19 and Lemma 1.1.20 that $\frac{1}{2}(x+y)$ is quasi-invertible in C, and hence there exists $w \in C$ such that $w + \frac{1}{2}(x+y) - \frac{1}{2}w(x+y) = 0$. Since $2x - x^2 = 2y - y^2$, we have

$$\begin{aligned} x - y = \frac{1}{2}(x^2 - y^2) &= \frac{1}{2}(x+y)(x-y) = \left[\frac{1}{2}w(x+y) - w\right](x-y) \\ &= w\left[\frac{1}{2}(x^2 - y^2) - (x-y)\right] = 0, \end{aligned}$$

so $x = y$, and the proof of assertion (ii) is complete.

Now, assume that A is a power-associative $*$-algebra, and that h is in $H(A,*)$ with $\mathfrak{r}(h) < 1$. Then, by assertion (ii), there exists a unique $x \in A$ such that $2x - x^2 = h$ and $\mathfrak{r}(x) < 1$. But $h = h^* = (2x - x^2)^* = 2x^* - (x^*)^2$. Moreover, by Lemma 4.5.20, we have $\mathfrak{r}(x^*) = \mathfrak{r}(x) < 1$. Therefore by the uniqueness of x, we conclude that $x = x^*$, and so assertion (iii) is proved.

Finally, assume that $\mathbb{K} = \mathbb{C}$, that A is a unital non-commutative Jordan $*$-algebra, and that h is in $A^+ \cap \text{J-Inv}(A)$ with $\|h\| = 1$. Then, according to Theorem 4.1.17, we have $\mathfrak{r}(\mathbf{1} - h) < 1$, and hence, by assertion (iii), there exists a unique $x \in H(A,*)$ such that $2x - x^2 = \mathbf{1} - h$ and $\mathfrak{r}(x) < 1$. By taking $a := \mathbf{1} - x \in H(A,*)$, we get $a^2 = \mathbf{1} - 2x + x^2 = h$. Then, by Propositions 4.1.86 and 1.3.4(ii), we have $\text{J-sp}(A,h) = \{\lambda^2 : \lambda \in \text{J-sp}(A,a)\}$. Therefore, since $\text{J-sp}(A,h) \subseteq \mathbb{R}^+$ and $\mathfrak{r}(\mathbf{1} - a) = \mathfrak{r}(x) < 1$, we derive that

$$\text{J-sp}(A,a) \subseteq \mathbb{R}^+.$$

Thus a belongs to $A^+ \cap \text{J-Inv}(A)$. Suppose now that $b \in A^+ \cap \text{J-Inv}(A)$ and $b^2 = h$. Then $y := \mathbf{1} - b \in H(A,*)$, $2y - y^2 = \mathbf{1} - h$, and $\mathfrak{r}(y) < 1$, i.e. y satisfies the conditions determining the element x above. Therefore $y = x$, and hence $b = a$. Now, up to an obvious normalization, assertion (iv) has been proved. □

In what follows, the Jordan versions of the Gelfand–Beurling formula (given in Theorem 4.1.17) and of the spectral mapping theorem (stated in Theorem 4.1.88(iv)) will be applied without notice.

Theorem 4.5.22 *Let A be a complete normed hermitian Jordan complex $*$-algebra. Then $A^+ + A^+ \subseteq A^+$.*

Proof In light of the comments in Definition 4.5.18, we may assume that, in addition, A is unital. Suppose, to obtain a contradiction, that there exist $h, k \in A^+$ such that $h + k$ does not belong to A^+. Then there exists $\gamma > 0$ such that $h + k + \gamma\mathbf{1}$ is not

J-invertible. Since $h+\frac{1}{2}\gamma\mathbf{1}$ lies in $A^+\cap\text{J-Inv}(A)$, Proposition 4.5.21(iv) applies, so that there exists $a\in A^+\cap\text{J-Inv}(A)$ such that $a^2=h+\frac{1}{2}\gamma\mathbf{1}$. Now, setting $c:=k+\frac{1}{2}\gamma\mathbf{1}$, we see that

$$U_a(\mathbf{1}+U_{a^{-1}}(c))=a^2+c=h+k+\gamma\mathbf{1}$$

is not J-invertible in A while a is J-invertible. Hence $\mathbf{1}+U_{a^{-1}}(c)$ is not J-invertible (by Theorem 4.1.3(vi)) which means that $-1\in\text{J-sp}(A,U_{a^{-1}}(c))$. Let $f:\mathbb{C}\to A$ be the holomorphic function defined by

$$f(z):=-U_{a^{-1}}(zc+(1-z)\mathbf{1}).$$

Note that, for $0\leqslant t\leqslant 1$, $tc+(1-t)\mathbf{1}=tk+(1-t)\mathbf{1}+\frac{1}{2}\gamma\mathbf{1}$ is J-invertible, and hence so is $f(t)$ (cf. Theorem 4.1.3(vi) again). Set

$$\alpha:=(\max\{\|f(t)^{-1}\|:t\in[0,1]\})^{-1},$$

and take $\beta>\max\{\|f(t)\|:t\in[0,1]\}$. Then

$$1\leqslant\|\mathbf{1}\|=\|f(0)f(0)^{-1}\|\leqslant\|f(0)\|\|f(0)^{-1}\|<\beta\alpha^{-1},$$

and hence $0<\alpha<\beta$. On the other hand, for every $t\in[0,1]$ we have

$$\text{J-sp}(A,f(t))\subseteq\beta\mathbb{B}_{\mathbb{C}}\ \text{ and }\ [\text{J-sp}(A,f(t))]^{-1}=\text{J-sp}(A,f(t)^{-1})\subseteq\alpha^{-1}\mathbb{B}_{\mathbb{C}}.$$

Since $f(t)\in H(A,*)$, it follows that

$$\text{J-sp}(A,f(t))\subseteq[-\beta,-\alpha]\cup[\alpha,\beta]\ \text{ for every }\ t\in[0,1].$$

However $\text{J-sp}(A,f(0))=\text{J-sp}(A,-a^{-2})\subseteq[-\beta,-\alpha]$ whereas $1\in\text{J-sp}(A,f(1))$, which contradicts Lemma 4.5.16. $\square$

Corollary 4.5.23 *Let A be a complete normed Jordan complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is hermitian.*
(ii) *$x^*x\in A^+$ for every $x\in A$.*

Moreover, if A is unital, then the above conditions are equivalent to

(iii) *$H(A,*)=A^+-A^+$ and $A^++A^+\subseteq A^+$.*

Proof (i)⇒(ii) For each $h\in H(A,*)$, we have $\text{J-sp}(A_{\mathbb{1}},h)\subseteq\mathbb{R}$, and so $\text{J-sp}(A_{\mathbb{1}},h^2)\subseteq\mathbb{R}_0^+$, that is $h^2\in A^+$. Now, given $x\in A$, writing $x=h+ik$ with $h,k\in H(A,*)$, we see that $x^*x=h^2+k^2$, and, by Theorem 4.5.22, we conclude that $x^*x\in A^+$.

(ii)⇒(i) For each $h\in H(A,*)$, the inclusion $\text{J-sp}(A_{\mathbb{1}},h^2)\subseteq\mathbb{R}_0^+$ gives the inclusion $\text{J-sp}(A_{\mathbb{1}},h)\subseteq\mathbb{R}$.

Now that the equivalence of conditions (i) and (ii) has been proved, suppose that A is unital.

(i)⇒(iii) For each $h\in H(A,*)$, note that $\mathfrak{r}(h)\mathbf{1}$ and $\mathfrak{r}(h)\mathbf{1}-h$ lie in A^+ and

$$h=\mathfrak{r}(h)\mathbf{1}-(\mathfrak{r}(h)\mathbf{1}-h).$$

Therefore $H(A,*)=A^+-A^+$. The inclusion $A^++A^+\subseteq A^+$ is provided by Theorem 4.5.22.

(iii)⇒(i) Let $h \in H(A,*)$. Write $h = u - v$ with $u, v \in A^+$. Then we have $h + \mathfrak{r}(v)\mathbf{1} = u + (\mathfrak{r}(v)\mathbf{1} - v) \in A^+ + A^+ \subseteq A^+$. Thus J-sp$(A, h + \mathfrak{r}(v)\mathbf{1}) \subseteq \mathbb{R}_0^+$, and hence J-sp$(A,h) \subseteq \mathbb{R}$. □

Proposition 4.5.24 *Let A be a complete normed hermitian Jordan complex $*$-algebra. Then:*

(i) *The set A^+ is a closed convex cone in $H(A,*)$. Moreover, if in addition A is unital, then*

$$A^+ = \{h \in H(A,*) : \mathfrak{r}(\mathfrak{r}(h)\mathbf{1} - h) \leqslant \mathfrak{r}(h)\}. \tag{4.5.5}$$

(ii) *For all $h,k \in H(A,*)$ we have:*

(a) $\mathfrak{r}(h+k) \leqslant \mathfrak{r}(h) + \mathfrak{r}(k)$.
(b) $\mathfrak{r}(hk) \leqslant \mathfrak{r}(h)\mathfrak{r}(k)$.
(c) $\mathfrak{r}(h)^2 \leqslant \mathfrak{r}(h^2 + k^2)$.

Proof In view of the comments in Definition 4.5.18, we may assume that A is unital. Given $h,k \in H(A,*)$, it turns out clear that $\mathfrak{r}(h)\mathbf{1} \pm h, \mathfrak{r}(k)\mathbf{1} \pm k \in A^+$, so $(\mathfrak{r}(h) + \mathfrak{r}(k))\mathbf{1} \pm (h+k) \in A^+$ (by Theorem 4.5.22), and therefore we have $\mathfrak{r}(h+k) \leqslant \mathfrak{r}(h) + \mathfrak{r}(k)$, which proves assertion (ii)(a).

It is clear that $\mathbb{R}_0^+ A^+ \subseteq A^+$, so that, applying Theorem 4.5.22 again, we realize that A^+ is a convex cone. Moreover, it follows immediately from assertion (ii)(a) proved above that

$$|\mathfrak{r}(h) - \mathfrak{r}(k)| \leqslant \mathfrak{r}(h-k) \text{ for all } h,k \in H(A,*). \tag{4.5.6}$$

On the other hand, if $h \in A^+$, then the inclusion J-sp$(A,h) \subseteq [0, \mathfrak{r}(h)]$ holds, hence J-sp$(A, \mathfrak{r}(h)\mathbf{1} - h) \subseteq [0, \mathfrak{r}(h)]$, and so $\mathfrak{r}(\mathfrak{r}(h)\mathbf{1} - h) \leqslant \mathfrak{r}(h)$. Conversely, if $h \in H(A,*)$ with $\mathfrak{r}(\mathfrak{r}(h)\mathbf{1} - h) \leqslant \mathfrak{r}(h)$, then J-sp$(A, \mathfrak{r}(h)\mathbf{1} - h) \subseteq [-\mathfrak{r}(h), \mathfrak{r}(h)]$, so that J-sp$(A, h - \mathfrak{r}(h)\mathbf{1}) \subseteq [-\mathfrak{r}(h), \mathfrak{r}(h)]$, hence J-sp$(A,h) \subseteq [0, 2\mathfrak{r}(h)]$, and in particular $h \in A^+$. Therefore, we have proved the equality (4.5.5). Since $\mathfrak{r}(\cdot) \leqslant \|\cdot\|$, it follows from (4.5.6) and (4.5.5) that A^+ is closed in $H(A,*)$. Thus the proof of assertion (i) is complete.

Now we proceed to prove assertion (ii)(b). Let h,k be in $H(A,*)$ and let t be in $\mathbb{R}$. Then $(h-tk)^2 \in A^+$. Since $\mathfrak{r}(h^2)\mathbf{1} - h^2$ and $\mathfrak{r}(k^2)\mathbf{1} - k^2$ are both in A^+, and A^+ is a convex cone, it follows that

$$(\mathfrak{r}(h^2) + t^2\mathfrak{r}(k^2))\mathbf{1} - 2thk = (h-tk)^2 + (\mathfrak{r}(h^2)\mathbf{1} - h^2) + t^2(\mathfrak{r}(k^2)\mathbf{1} - k^2) \in A^+.$$

Hence, if $\lambda \in$ J-sp(A,hk), we have $\mathfrak{r}(h^2) + t^2\mathfrak{r}(k^2) - 2\lambda t \in \mathbb{R}_0^+$. Since this holds for every $t \in \mathbb{R}$, it follows that $\lambda^2 \leqslant \mathfrak{r}(h^2)\mathfrak{r}(k^2)$, and hence $\mathfrak{r}(hk) \leqslant \mathfrak{r}(h)\mathfrak{r}(k)$.

Finally, we prove assertion (ii)(c). Let h,k be in $H(A,*)$. Then, since both $\mathfrak{r}(h^2+k^2)\mathbf{1} - (h^2+k^2)$ and k^2 lie in A^+, and $A^+ + A^+ \subseteq A^+$, we have

$$\mathfrak{r}(h^2+k^2)\mathbf{1} - h^2 = [\mathfrak{r}(h^2+k^2)\mathbf{1} - (h^2+k^2)] + k^2 \in A^+,$$

and so $\mathfrak{r}(h)^2 = \mathfrak{r}(h^2) \leqslant \mathfrak{r}(h^2+k^2)$ because $h^2 \in A^+$. □

Lemma 4.5.25 *Let A be a complete normed Jordan-admissible complex $*$-algebra such that $h = 0$ whenever h is in $H(A,*)$ with $\mathfrak{s}(h) = 0$. Then $*$ is continuous.*

Proof Let x be in J-Rad(A). Then $h := \frac{1}{2}(x+x^*)$ and $k := \frac{1}{2i}(x-x^*)$ lie in J-Rad$(A)\bigcap H(A,*)$ because J-Rad(A) is $*$-invariant. Therefore, by Lemma 4.4.26, we have $\mathfrak{s}(h) = \mathfrak{s}(k) = 0$, so $h = k = 0$, and so $x = h + ik = 0$. Thus A is J-semisimple, so that the continuity of $*$ follows from Corollary 4.4.14. □

Fact 4.5.26 *Let X be a complex vector space endowed with a conjugate-linear involution $*$, let Y be complex normed space endowed with a continuous conjugate-linear involution $*$, and let $\Phi : X \to Y$ be a dense range linear $*$-mapping. Then $\Phi(H(X,*))$ is dense in $H(Y,*)$.*

Proof Noticing that the direct sum $Y = H(Y,*) \oplus iH(Y,*)$ is topological, that $\Phi(H(X,*)) \subseteq H(Y,*)$, and that $\Phi(H(X,*)) + i\Phi(H(X,*))$ is dense in Y, the result follows. □

Definition 4.5.27 Let A be a complex $*$-algebra. Let $\Phi_1 : A \to B_1$ and $\Phi_2 : A \to B_2$ be non-commutative JB^*-representations of A (cf. Definition 4.5.8). We say that Φ_1 and Φ_2 are *equivalent* if there exists an automatically isometric (cf. Proposition 3.4.4) bijective algebra $*$-homomorphism F from B_1 to B_2 such that $\Phi_2 = F \circ \Phi_1$. If the mapping F above is merely an automatically contractive (cf. again Proposition 3.4.4) algebra $*$-homomorphism, then we say that Φ_2 *factors through* Φ_1. We note that, if Φ_1 and Φ_2 are equivalent, and if one of them is actually a JB^*-representation, an alternative C^*-representation, or a C^*-representation, then so is the other.

Fact 4.5.28 *Two dense range non-commutative JB^*-representations of a complex $*$-algebra are equivalent if (and only if) each of them factorizes through the other.*

Proof Let A be a complex $*$-algebra, and let $\Phi_1 : A \to B_1$ and $\Phi_2 : A \to B_2$ be non-commutative JB^*-representations of A such that there exist algebra $*$-homomorphisms $F : B_1 \to B_2$ and $G : B_2 \to B_1$ satisfying $\Phi_2 = F \circ \Phi_1$ and $\Phi_1 = G \circ \Phi_2$. Then we have $\Phi_2 = F \circ G \circ \Phi_2$ and $\Phi_1 = G \circ F \circ \Phi_1$. Now assume that Φ_1 and Φ_2 have dense ranges. Then $F \circ G = I_{B_2}$ and $G \circ F = I_{B_1}$ because $F \circ G$ and $G \circ F$ are continuous, so F is bijective, hence Φ_1 and Φ_2 are equivalent. □

Theorem 4.5.29 *Let A be a complete normed hermitian Jordan complex $*$-algebra. Then, up to equivalence, there exists a unique dense range JB^*-representation Φ of A satisfying $\|\Phi(h)\| = \mathfrak{r}(h)$ for every $h \in H(A,*)$. Moreover we have:*

(i) *Every JB^*-representation of A factors through Φ.*
(ii) $\ker(\Phi) = \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*))$.
(iii) *Φ is bijective if and only if there exists $\kappa > 0$ such that $\mathfrak{r}(h) \geqslant \kappa\|h\|$ for every $h \in H(A,*)$.*
(iv) $A^+ = \Phi^{-1}(B^+) \cap H(A,*)$, *where B stands for the JB^*-algebra where the range of Φ lives.*

Proof First of all, we prove the existence of Φ. According to Proposition 4.5.24(ii)(a)–(b), $\mathfrak{r}(\cdot)$ is a submultiplicative seminorm on $H(A,*)$. Therefore $M := \{h \in H(A,*) : \mathfrak{r}(h) = 0\}$ is an ideal of $H(A,*)$ and $|h+M| := \mathfrak{r}(h)$ is an algebra norm on the Jordan algebra $H(A,*)/M$, which, in view of Proposition 4.5.24(ii)(c), satisfies $|x|^2 \leqslant |x^2+y^2|$ for all $x,y \in H(A,*)/M$. Therefore the completion of $(H(A,*)/M, |\cdot|)$ becomes a JB-algebra (say $(D, \|\cdot\|)$), and hence, by Theorem 3.4.8, there exists a JB^*-algebra $(B, \|\cdot\|)$ such that $(H(B,*), \|\cdot\|) = (D, \|\cdot\|)$. Moreover, clearly, the mapping

$$\Phi : h + ik \to (h+M) + i(k+M) \ \ (h,k \in H(A,*))$$

from A to B becomes a dense range JB^*-representation of A, and we have

$$\mathfrak{r}(h) = |h+M| = \|\Phi(h)\| \text{ for every } h \in H(A,*).$$

In what follows, B will stand for the JB^*-algebra where the range of Φ lives.

Now we prove assertion (i). Let $\Psi : A \to C$ be any JB^*-representation of A. Then, by Fact 4.5.9, we have $\|\Psi(h)\| \leqslant \mathfrak{r}(h) = \|\Phi(h)\|$ for every $h \in H(A,*)$, so that $\Phi(h) \to \Psi(h)$ $(h \in H(A,*))$ becomes a well-defined contractive algebra homomorphism from $\Phi(H(A,*))$ to $\Psi(H(A,*))$. Therefore, since $\Phi(H(A,*))$ is dense in $H(B,*)$ (by Proposition 3.3.13 and Fact 4.5.26), and $H(B,*)$ is complete, there exists a contractive algebra homomorphism G from $H(B,*)$ to $H(C,*)$ satisfying $G \circ \Phi = \Psi$ on $H(A,*)$. By extending G by complex linearity, we get an algebra $*$-homomorphism F from B to C satisfying $F \circ \Phi = \Psi$ on A. Thus Ψ factors through Φ.

Noticing that the proof of assertion (i) just done involves only the abstract properties of Φ, the essential uniqueness of Φ follows from assertion (i) and Fact 4.5.28.

Now we prove assertion (ii). The inclusion

$$\text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*)) \subseteq \ker(\Phi)$$

follows from Remark 4.5.12. If h is in $\ker(\Phi) \cap H(A,*)$, then we have $\mathfrak{r}(h) = \|\Phi(h)\| = 0$, and hence, by Lemma 4.1.15, h has a quasi-J-inverse in A, which must lie in $\ker(\Phi) \cap H(A,*)$ because $\ker(\Phi)$ is an ideal of A and the quasi-J-inverse is unique. Therefore $\ker(\Phi) \cap H(A,*)$ becomes a quasi-J-invertible ideal of $H(A,*)$, and hence $\ker(\Phi) \cap H(A,*) \subseteq \text{J-Rad}(H(A,*))$. Since $\ker(\Phi)$ is $*$-invariant, the inclusion

$$\ker(\Phi) \subseteq \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*))$$

follows.

Now we prove assertion (iii). Assume that Φ is bijective. Then, since Φ is continuous (by Proposition 4.5.11(ii)), it follows from the Banach isomorphism theorem that there exists $\kappa > 0$ such that $\|\Phi(a)\| \geqslant \kappa\|a\|$ for every $a \in A$, and this implies $\mathfrak{r}(h) = \|\Phi(h)\| \geqslant \kappa\|h\|$ for every $h \in H(A,*)$. Conversely, assume that there exists $\kappa > 0$ such that $\mathfrak{r}(h) \geqslant \kappa\|h\|$ for every $h \in H(A,*)$. Then, by Lemma 4.5.25, $H(A,*)$ is closed in A. On the other hand, Φ, regarded as a mapping from $H(A,*)$ to $H(B,*)$, has dense range, and we have $\kappa\|\cdot\| \leqslant \mathfrak{r}(\cdot) = \|\Phi(\cdot)\| = \mathfrak{r}(\cdot) \leqslant \|\cdot\|$ on $H(A,*)$. It follows that Φ, regarded as a mapping from $H(A,*)$ to $H(B,*)$, is bijective, and this implies that Φ is bijective.

Finally, we prove assertion (iv). Assume at first that A is unital. Then, since Φ has dense range, B is unital and $\Phi(\mathbf{1}) = \mathbf{1}$. Moreover, keeping in mind that $\mathfrak{r}(\cdot) = \|\Phi(\cdot)\| = \mathfrak{r}(\Phi(\cdot))$ on $H(A,*)$, given $h \in H(A,*)$, we realize that

$$\mathfrak{r}(\mathfrak{r}(h)\mathbf{1} - h) \leqslant \mathfrak{r}(h) \ \text{ if and only if } \ \mathfrak{r}(\mathfrak{r}(\Phi(h))\mathbf{1} - \Phi(h)) \leqslant \mathfrak{r}(\Phi(h)).$$

Therefore, by Proposition 4.5.24(i), $h \in A^+$ if and only if $\Phi(h) \in B^+$, as desired. To conclude the proof, remove the assumption above that A is unital. Let $A_{\mathbb{1}}$ stand for the standard normed unital extension of A. Then $A_{\mathbb{1}}$ becomes a complete normed Jordan complex $*$-algebra, and is hermitian. Therefore, by the first conclusion in the theorem, there exists a dense range JB^*-representation $\Phi_{\mathbb{1}} : A_{\mathbb{1}} \to D$ satisfying $\|\Phi_{\mathbb{1}}(h)\| = \mathfrak{r}(h)$ for every $h \in H(A_{\mathbb{1}}, *)$, and the restriction of $\Phi_{\mathbb{1}}$ to A, regarded as a mapping from A to the closure of $\Phi_{\mathbb{1}}(A)$ in D, is a JB^*-representation of A equivalent to Φ. Therefore, by the previously considered unital case, for $h \in H(A,*)$ we have

$$h \in A^+ \Longleftrightarrow h \in (A_{\mathbb{1}})^+ \Longleftrightarrow \Phi_{\mathbb{1}}(h) \in D^+ \Longleftrightarrow \Phi(h) \in B^+. \qquad \square$$

Corollary 4.5.30 *Let A be a complete normed hermitian Jordan complex $*$-algebra. We have:*

(i) *$U_h(A^+) \subseteq A^+$ for every $h \in H(A,*)$.*
(ii) *If $H(A,*)$ is J-semisimple, then the closed $*$-subalgebra of A generated by each element of A is special.*

Proof Throughout the proof, $\Phi : A \to B$ will denote the JB^*-representation of A given by Theorem 4.5.29.

Let h be in $H(A,*)$, and let k be in A^+. In view of Theorem 4.5.29(iv), to prove assertion (i) it is enough to show that $U_{\Phi(h)}(\Phi(k))$ lies in B^+. But $\Phi(h) \in H(B,*)$ and $\Phi(k) \in B^+$, so that we have indeed $U_{\Phi(h)}(\Phi(k)) \in B^+$ because $H(B,*)$ is a JB-algebra (cf. Corollary 3.4.3) and Lemma 3.1.29(iii) applies.

Let a be in A, and let C stand for the closed $*$-subalgebra of A generated by a. Then, since Φ is continuous (by Proposition 4.5.11(ii)), $\Phi(C)$ is contained in the closed $*$-subalgebra of B generated by $\Phi(a)$. But, in view of Proposition 3.4.6, this last subalgebra is a JC^*-algebra, and hence is special. Therefore $\Phi(C)$ is a special Jordan algebra. Now assume that $H(A,*)$ is J-semisimple. Then, by Theorem 4.5.29(ii), Φ is injective, and hence $C \approx \Phi(C)$ is special. Thus assertion (ii) has been proved. $\square$

If B is any non-commutative JB^*-algebra, then B is hermitian (cf. Fact 4.5.19), and the Jordan algebra $H(B,*)$ is J-semisimple (cf. Corollaries 3.4.3 and 4.4.27). Therefore the next corollary generalizes Proposition 4.5.11(ii).

Corollary 4.5.31 *Let A and B be complete normed Jordan-admissible complex $*$-algebras, assume that B is hermitian and that the Jordan algebra $H(B,*)$ is J-semisimple, and let $\Phi : A \to B$ be an algebra $*$-homomorphism. Then Φ is continuous.*

Proof By Theorem 4.5.29, there exists an injective JB^*-representation (say Ψ) of B^{sym}, which is continuous by Proposition 4.5.11(ii). Then $\Psi \circ \Phi$ becomes a JB^*-representation of A^{sym}, and hence, again by Proposition 4.5.11(ii), it is continuous. Therefore, if a_n is any sequence in A such that $a_n \to 0$ and $\Phi(a_n) \to b \in B$, then we have $0 \leftarrow \Psi(\Phi(a_n)) \to \Psi(b)$, so $b = 0$ because Ψ is injective. Now the continuity of Φ follows from the closed graph theorem. $\square$

Corollary 4.5.32 *Let A be a complete normed unital Jordan complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is a JB^*-algebra in an equivalent norm.*
(ii) *There exists $\kappa_1 > 0$ satisfying $\|a\|^2 \leqslant \kappa_1 \|a^*a\|$ for every $a \in A$.*

(iii) *There exists* $\kappa_2 > 0$ *such that* $\|\exp(ih)\| \leqslant \kappa_2$ *for every* $h \in H(A,*)$.
(iv) A *is hermitian and there exists* $\kappa_3 > 0$ *such that* $\mathfrak{r}(h) \geqslant \kappa_3\|h\|$ *for every* $h \in H(A,*)$.

Proof (i)⇒(ii) By the assumption (i), there is an equivalent norm $|||\cdot|||$ on A converting A into a JB^*-algebra. Let m and M be positive numbers such that $m|||\cdot||| \leqslant \|\cdot\| \leqslant M|||\cdot|||$. Then, by Lemma 3.4.65, for every $a \in A$ we have

$$\|a\|^2 \leqslant M^2|||a|||^2 \leqslant 2M^2|||a^*a||| \leqslant \frac{2M^2}{m}\|a^*a\|.$$

(ii)⇒(iii) Assume that condition (ii) is fulfilled. Let h be in $H(A,*)$. Then, by induction, for every $n \in \mathbb{N}$ we have $\|h\|^{2^n} \leqslant \kappa_1^{2^n-1}\|h^{2^n}\|$, and hence $\|h\| \leqslant \kappa_1\mathfrak{r}(h)$. Since h is arbitrary in $H(A,*)$, it follows from Lemma 4.5.25 that $*$ is continuous. Therefore, for every $h \in H(A,*)$ we have

$$\|\exp(ih)\|^2 \leqslant \kappa_1\|(\exp(ih))^*\exp(ih)\| = \kappa_1\|\exp(-ih)\exp(ih)\| = \kappa_1\|\mathbf{1}\|.$$

(iii)⇒(iv) Assume that condition (iii) is fulfilled. Then, by Corollary 4.5.4, $\mathfrak{r}(h) \geqslant \kappa_2^{-1}\|h\|$ for every $h \in H(A,*)$. Moreover, for $h \in H(A,*)$ we have $\|\exp(inh)\| \leqslant \kappa_2$ for every $n \in \mathbb{Z}$, so $\mathfrak{r}(\exp(ih)) = \mathfrak{r}(\exp(-ih)) = 1$, and so J-sp$(A,h) \subseteq \mathbb{R}$. Thus A is hermitian.

(iv)⇒(i) Assume that condition (iv) is fulfilled. Let Φ be the JB^*-representation of A given by Theorem 4.5.29. Then, by assertion (iii) in that theorem, Φ is bijective, and hence bicontinuous (by Proposition 4.5.11(ii) and the Banach isomorphism theorem). Therefore A is a JB^*-algebra in the equivalent norm $|||\cdot||| := \|\Phi(\cdot)\|$. □

Now we are going to prove the appropriate variants of Theorem 4.5.29 and Corollary 4.5.32 for alternative algebras (see Theorem 4.5.37 and Corollary 4.5.42, respectively).

Lemma 4.5.33 *Let A be a complete normed hermitian Jordan-admissible complex $*$-algebra, and let B be a closed $*$-subalgebra of A. Then B is hermitian.*

Proof Since A is hermitian, so is $A_{\mathbb{1}}$. Therefore, regarding $B_{\mathbb{1}}$ as a closed $*$-subalgebra of $A_{\mathbb{1}}$, it follows from Proposition 4.1.28(iii) that

$$\text{J-sp}(B_{\mathbb{1}},h) = \text{J-sp}(A_{\mathbb{1}},h) \subseteq \mathbb{R} \quad \text{for every} \quad h \in H(B,*),$$

and B is indeed hermitian. □

Proposition 4.5.34 *Let A be a complete normed hermitian alternative complex $*$-algebra, and let a be in A. Then $\mathfrak{r}(a)^2 \leqslant \mathfrak{r}(a^*a)$.*

Proof Thinking about the closed $*$-subalgebra of A generated by a, and invoking Corollary 3.5.73(ii) and Lemma 4.5.33, we may assume that A is associative. Then it turns out clear that we may assume in addition that A is unital. The Gelfand–Beurling formula (cf. Theorem 1.1.46) implies that the inequality $\mathfrak{r}(a)^2 \leqslant \mathfrak{r}(a^*a)$ holds if and only if, given $\lambda \in \mathbb{C}$ with $|\lambda|^2 > \mathfrak{r}(a^*a)$, the element $a - \lambda\mathbf{1}$ is invertible in A. Therefore, replacing a with $\lambda^{-1}a$, it is sufficient to show that $1 \notin \mathrm{sp}(A,a)$ whenever $\mathfrak{r}(a^*a) < 1$. Suppose, then, that $\mathfrak{r}(a^*a) < 1$. Then $\mathbf{1} - a^*a$ lies in $A^+ \cap \mathrm{Inv}(A)$. Therefore, by Proposition 4.5.21(iv), there exists $h \in A^+ \cap \mathrm{Inv}(A)$ such that $h^2 = \mathbf{1} - a^*a$. Note that

$$(\mathbf{1}+a^*)(\mathbf{1}-a)=\mathbf{1}+a^*-a-a^*a=h^2+a^*-a=h\left[\mathbf{1}+h^{-1}(a^*-a)h^{-1}\right]h.$$

Since $ih^{-1}(a^*-a)h^{-1}\in H(A,*)$, it has real spectrum. Thus $\mathbf{1}+h^{-1}(a^*-a)h^{-1}$ is invertible in A, and hence $\mathbf{1}-a$ is left invertible. A similar argument applied to $(\mathbf{1}-a)(\mathbf{1}+a^*)$ shows that $\mathbf{1}-a$ is right invertible. Thus, by Lemma 1.1.59, $1\notin \mathrm{sp}(A,a)$. □

By keeping in mind Artin's theorem (Theorem 2.3.61), the following fact becomes straightforward.

Fact 4.5.35 *Let A be an alternative algebra over $\mathbb{K}$, and let a,b be in A. Then*

$$[a,b]^2=2\left(a^2\bullet b^2-a\bullet(a\bullet b^2)\right)+4\left(a\bullet(a\bullet b)-a^2\bullet b\right)\bullet b.$$

Let A be a complex $*$-algebra, and note that $*$ remains a conjugate-linear algebra involution on A^{sym}. If $\Phi:A\to B$ is a non-commutative JB^*-representation of A, then Φ, regarded as a mapping from A^{sym} to B^{sym}, clearly becomes a JB^*-representation of A^{sym}. According to assertion (ii) in the next proposition, when A is alternative, all dense range JB^*-representations of A^{sym} are obtained by the procedure just described.

Proposition 4.5.36 *Let A be an alternative complex $*$-algebra, and let*

$$\Phi:A^{\mathrm{sym}}\to C$$

be a JB^*-representation of A^{sym}. Then:

(i) $\ker(\Phi)$ *is an ideal of A, and the alternative complex algebra $A/\ker(\Phi)$ becomes a normed algebra in the norm* $|||\,a+\ker(\Phi)\,|||:=(1+4\sqrt{3})\|\Phi(a)\|$.

(ii) *If Φ has dense range, then there exists an alternative C^*-algebra B such that*

  (a) *$B^{\mathrm{sym}}=C$ as JB^*-algebras, and*

  (b) *Φ, regarded as a mapping from A to B, becomes an alternative C^*-representation of A.*

Proof Let h,k be in $H(A,*)$. Then, by Fact 4.5.35, we have

$$\begin{aligned}\Phi\left([h,k]^2\right)&=2\left(\Phi(h)^2\bullet\Phi(k)^2-\Phi(h)\bullet(\Phi(h)\bullet\Phi(k)^2)\right)\\&\quad+4\left(\Phi(h)\bullet(\Phi(h)\bullet\Phi(k))-\Phi(h)^2\bullet\phi(k)\right)\bullet\Phi(k).\end{aligned}$$

Therefore, since $i\Phi([h,k])\in H(C,*)$, and $H(C,*)$ is a JB-algebra, we have

$$\|\Phi([h,k])\|^2=\|\Phi([h,k])^2\|=\|\Phi([h,k]^2)\|\leqslant 12\|\Phi(h)\|^2\|\Phi(k)\|^2,$$

and hence

$$\|\Phi([h,k])\|\leqslant 2\sqrt{3}\|\Phi(h)\|\|\Phi(k)\|. \tag{4.5.7}$$

Now let a_1,a_2 be in A, and write $a_j=h_j+ik_j$ with $h_j,k_j\in H(A,*)$ $(j=1,2)$. It follows from (4.5.7) that

$$\|\Phi([a_1,a_2])\|\leqslant 2\sqrt{3}(\|\Phi(h_1)\|+\|\Phi(k_1)\|)(\|\Phi(h_2)\|+\|\Phi(k_2)\|).$$

Therefore, since the involution of C is an isometry (cf. Proposition 3.3.13), we derive that $\|\Phi([a_1,a_2])\| \leqslant 8\sqrt{3}\|\Phi(a_1)\|\|\Phi(a_2)\|$, and hence

$$\begin{aligned}\|\Phi(a_1a_2)\| &\leqslant \|\Phi(a_1 \bullet a_2)\| + \frac{1}{2}\|\Phi([a_1,a_2])\| \\ &= \|\Phi(a_1)\bullet\Phi(a_2)\| + \frac{1}{2}\|\Phi([a_1,a_2])\| \\ &\leqslant \|\Phi(a_1)\|\|\Phi(a_2)\| + 4\sqrt{3}\|\Phi(a_1)\|\|\Phi(a_2)\| \\ &= (1+4\sqrt{3})\|\Phi(a_1)\|\|\Phi(a_2)\|.\end{aligned}$$

Since a_1,a_2 are arbitrary in A, the inequality

$$\|\Phi(a_1a_2)\| \leqslant (1+4\sqrt{3})\|\Phi(a_1)\|\|\Phi(a_2)\|$$

just proved makes the proof of assertion (i) straightforward.

Now that assertion (i) has been proved, keep it in mind and note that, since $\ker(\Phi)$ is $*$-invariant, the quotient involution (also denoted by $*$) becomes a $|||\cdot|||$-isometric conjugate-linear algebra involution on $A/\ker(\Phi)$, and hence we can consider the completion (say D) of $(A/\ker(\Phi), |||\cdot|||, *)$, which becomes an alternative complex $*$-algebra. Now assume that Φ has dense range. Then, since the mapping $a+\ker(\Phi) \to \Phi(a)$ from $A/\ker(\Phi)$ to C is both a multiple of an isometry and a Jordan-$*$-homomorphism, it extends to a bijective Jordan-$*$-homomorphism $\Psi : D \to C$ satisfying

$$\Phi = \Psi \circ \pi, \text{ where } \pi \text{ stands for the quotient mapping } A \to A/\ker(\Phi). \tag{4.5.8}$$

Let B denote the alternative complex algebra consisting of the vector space of C and the product

$$x \odot y := \Psi(\Psi^{-1}(x)\Psi^{-1}(y)). \tag{4.5.9}$$

It follows from (4.5.8) and (4.5.9) that Φ, regarded as a mapping from A to B, becomes an algebra homomorphism. Moreover we easily realize that $B^{\text{sym}} = C$ as Jordan algebras, and that the involution $*$ of C becomes a conjugate-linear algebra involution on B. Since C is a JB^*-algebra, it follows from Theorem 3.6.25 that B is an alternative C^*-algebra for the norm and the involution of C, and then clearly Φ, regarded as a mapping from A to B, becomes an alternative C^*-representation of A. Thus assertion (ii) has been proved. □

We note that assertion (ii) in the above proposition refines one of the ingredients in its proof, namely Theorem 3.6.25.

Theorem 4.5.37 *Let A be a complete normed hermitian alternative complex $*$-algebra. Then, up to equivalence, there exists a unique dense range alternative C^*-representation Φ of A satisfying $\|\Phi(a)\|^2 = \mathfrak{r}(a^*a)$ for every $a \in A$. Moreover we have:*

(i) *Every alternative C^*-representation of A factors through Φ.*
(ii) $\ker(\Phi) = \operatorname{Rad}(A)$.

(iii) *Φ is bijective if and only if there exists $\kappa > 0$ such that $\mathfrak{r}(h) \geqslant \kappa\|h\|$ for every $h \in H(A,*)$.*

(iv) *$A^+ = \Phi^{-1}(B^+) \cap H(A,*)$, where B stands for the alternative C^*-algebra where the range of Φ lives.*

Proof First of all, we prove the existence of Φ. Since A^{sym} is a complete normed Jordan complex algebra, and $*$ remains an algebra involution on A^{sym}, and A^{sym} is hermitian, we can apply Theorem 4.5.29 to A^{sym}, and consider the dense range JB^*-representation $\Phi : A^{\mathrm{sym}} \to C$ given by that theorem, which satisfies

$$\|\Phi(h)\| = \mathfrak{r}(h) \text{ for every } h \in H(A,*).$$

Then, according to Proposition 4.5.36(ii), Φ is in fact an alternative C^*-representation of A into an alternative C^*-algebra B such that $B^{\mathrm{sym}} = C$. Therefore, for every $a \in A$ we have

$$\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a^*a)\| = \mathfrak{r}(a^*a).$$

Thus the existence of the representation Φ in the statement has been proved.

In what follows, B will stand for the alternative C^*-algebra where the range of Φ lives.

Now we prove assertion (i). Let $\Psi : A \to D$ be any alternative C^*-representation of A. Then, by Fact 4.5.9, for every $a \in A$ we have

$$\|\Psi(a)\|^2 = \|\Psi(a)^*\Psi(a)\| = \|\Psi(a^*a)\| \leqslant \mathfrak{r}(a^*a) = \|\Phi(a)\|^2,$$

so that $\Phi(a) \to \Psi(a)$ $(a \in A)$ becomes a well-defined contractive algebra $*$-homomorphism from $\Phi(A)$ to $\Psi(A)$. Therefore, since $\Phi(A)$ is dense in B, there exists a contractive algebra $*$-homomorphism G from B to D satisfying $G \circ \Phi = \Psi$. Thus Ψ factors through Φ.

Noticing that the proof of assertion (i) just done involves only the abstract properties of Φ, the essential uniqueness of Φ follows from assertion (i) and Fact 4.5.28.

Now we prove assertion (ii). Keeping in mind that $\mathrm{Rad}(A) = \text{J-Rad}(A)$ (cf. §3.6.58), we must show that $\ker(\Phi) = \text{J-Rad}(A)$. The inclusion $\text{J-Rad}(A) \subseteq \ker(\Phi)$ follows from Proposition 4.5.11(i). Conversely, if x is in $\ker(\Phi)$, then, by Proposition 4.5.34, we have

$$\mathfrak{r}(x)^2 \leqslant \mathfrak{r}(x^*x) = \|\Phi(x)\|^2 = 0,$$

so $\mathfrak{r}(x) = 0$, and so x is quasi-invertible in A (cf. Lemma 4.1.15, Definition 3.6.19, and Fact 4.1.57). Thus $\ker(\Phi)$ is a quasi-invertible ideal of A, and hence $\ker(\Phi) \subseteq \text{J-Rad}(A)$.

Finally, assertions (iii) and (iv) follow straightforwardly from the first paragraph in the proof and the corresponding assertions in Theorem 4.5.29. □

Since C^*-representations are alternative C^*-representations, and dense range alternative C^*-representations are C^*-representations whenever the starting algebra is associative, the next corollary follows straightforwardly from Theorem 4.5.37 above.

Corollary 4.5.38 *Let A be a complete normed hermitian associative complex $*$-algebra. Then, up to equivalence, there exists a unique dense range C^*-representation Φ of A satisfying $\|\Phi(a)\|^2 = \mathfrak{r}(a^*a)$ for every $a \in A$. Moreover we have:*

(i) *Every C^*-representation of A factors through Φ.*
(ii) $\ker(\Phi) = \mathrm{Rad}(A)$.
(iii) *Φ is bijective if and only if there exists $\kappa > 0$ such that $\mathfrak{r}(h) \geqslant \kappa\|h\|$ for every $h \in H(A,*)$.*
(iv) *$A^+ = \Phi^{-1}(B^+) \cap H(A,*)$, where B stands for the C^*-algebra where the range of Φ lives.*

Corollary 4.5.39 *Let A be a complete normed hermitian alternative complex $*$-algebra. Then:*

(i) *The mapping $a \to \sqrt{\mathfrak{r}(a^*a)}$ is a C^*-seminorm on A* (cf. Definition 2.3.30) *whose kernel is equal to the radical of A.*
(ii) *$a^*a \in A^+$ for every $a \in A$.*
(iii) $\mathrm{Rad}(A) = \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*))$.

Proof Let $\Phi : A \to B$ be the alternative C^*-representation of A given by Theorem 4.5.37.

By the first conclusion in that theorem, we have $\sqrt{\mathfrak{r}(a^*a)} = \|\Phi(a)\|$ for every $a \in A$, so that, keeping in mind that B is an alternative C^*-algebra, the mapping $a \to \sqrt{\mathfrak{r}(a^*a)}$ becomes indeed a C^*-seminorm on A. Moreover, by Theorem 4.5.37(ii), we have

$$\{a \in A : \sqrt{\mathfrak{r}(a^*a)} = 0\} = \mathrm{Rad}(A).$$

Thus assertion (i) has been proved.

Let a be in A. In view of Theorem 4.5.37(iv), to prove assertion (ii) it is enough to show that $\Phi(a)^*\Phi(a) \in B^+$. But this is indeed true because of Facts 4.5.19 and 3.4.69.

Invoking §3.6.58, Remark 4.5.12, and Theorem 4.5.29(ii), we have

$$\mathrm{Rad}(A) = \text{J-Rad}(A) \subseteq \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*)) \subseteq \ker(\Phi) = \mathrm{Rad}(A),$$

and assertion (iii) follows. □

With Theorem 4.5.37 instead of Theorem 4.5.29, arguments like those in the proof of Corollary 4.5.31 yield the following.

Corollary 4.5.40 *Let A be a complete normed Jordan-admissible complex $*$-algebra, let B be a complete normed semisimple hermitian alternative complex $*$-algebra, and let $\Phi : A \to B$ be an algebra $*$-homomorphism. Then Φ is continuous.*

Recalling that, for an alternative algebra A, we have $\mathrm{Rad}(A) = \text{J-Rad}(A)$, and looking at Corollary 4.5.39(iii), the next problem arises naturally (see also the inclusion (4.5.1) in Remark 4.5.12).

Problem 4.5.41 Let A be a complete normed hermitian non-commutative Jordan $*$-algebra. Does the equality $\text{J-Rad}(A) = \text{J-Rad}(H(A,*)) + i\text{J-Rad}(H(A,*))$ hold?

We note that the above problem reduces to the commutative case. For, if A is as in Problem 4.5.41, then so is A^{sym}, and moreover $H(A,*) = H(A^{\mathrm{sym}},*)$ as Jordan real algebras, and $\text{J-Rad}(A) = \text{J-Rad}(A^{\mathrm{sym}})$ thanks to Proposition 4.4.17(iii). We also note that, if Problem 4.5.41 had an affirmative answer, then Theorem 4.5.29(ii) and Corollaries 4.5.30(ii) and 4.5.31 would have a better formulation. Keeping in mind §3.6.58, such a better formulation of Corollary 4.5.31 would contain Corollary 4.5.40.

Corollary 4.5.42 *Let A be a complete normed unital alternative complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is an alternative C^*-algebra in an equivalent norm.*
(ii) *There exists $\kappa_1 > 0$ satisfying $\|a\|^2 \leqslant \kappa_1 \|a^* a\|$ for every $a \in A$.*
(iii) *There exists $\kappa_2 > 0$ such that $\|\exp(ih)\| \leqslant \kappa_2$ for every $h \in H(A,*)$.*
(iv) *A is hermitian and there exists $\kappa_3 > 0$ such that $\mathfrak{r}(h) \geqslant \kappa_3 \|h\|$ for every $h \in H(A,*)$.*

Proof (i)$\Rightarrow$(ii) By the assumption (i), there is an equivalent norm $|||\cdot|||$ on A converting A into an alternative C^*-algebra. Let $m, M > 0$ be such that $m|||\cdot||| \leqslant \|\cdot\| \leqslant M|||\cdot|||$. Then for every $a \in A$ we have

$$\|a\|^2 \leqslant M^2 |||a|||^2 = M^2 |||a^* a||| \leqslant \frac{M^2}{m} \|a^* a\|.$$

(ii)$\Rightarrow$(iii) Argue verbatim as in the proof of the implication (ii)$\Rightarrow$(iii) in Corollary 4.5.32.

(iii)$\Rightarrow$(iv) Noticing that A^{sym} is a complete normed unital Jordan complex algebra, that $*$ remains a conjugate-linear algebra involution on A^{sym}, that A is hermitian if and only if so is A^{sym}, and that both $\exp(\cdot)$ and $\mathfrak{r}(\cdot)$ have the same meaning in A and A^{sym}, the present implication follows from the implication (iii)$\Rightarrow$(iv) in Corollary 4.5.32.

(iv)$\Rightarrow$(i) Assume that condition (iv) is fulfilled. Let Φ be the alternative C^*-representation of A given by Theorem 4.5.37. Then, by assertion (iii) in that theorem, Φ is bijective, and hence bicontinuous (by Proposition 4.5.11(ii) and the Banach isomorphism theorem). Therefore A is an alternative C^*-algebra in the equivalent norm $|||\cdot||| := \|\Phi(\cdot)\|$. □

Corollary 4.5.32 (respectively, Corollary 4.5.42) does not remain true if we replace 'Jordan algebra' and 'JB^*-algebra' (respectively, 'alternative algebra' and 'alternative C^*-algebra') with 'non-commutative Jordan algebra' and 'non-commutative JB^*-algebra'. Indeed, we have the following.

Example 4.5.43 Let λ be a real number, set $M := |\lambda| + |1-\lambda|$, let $|||\cdot|||$ denote the usual C^*-norm on $M_2(\mathbb{C})$, and let A stand for the complete normed unital complex algebra consisting of the vector space of $M_2(\mathbb{C})$, the product $a \odot b := \lambda ab + (1-\lambda)ba$, and the norm $\|\cdot\| := M|||\cdot|||$. Note that A is a non-commutative Jordan algebra, and that the usual involution $*$ of $M_2(\mathbb{C})$ remains a conjugate-linear algebra involution on A.

We are going to realize that A satisfies conditions (ii)–(iv) in both Corollary 4.5.32 and Corollary 4.5.42. Indeed, we have clearly $\|\exp(ih)\| = M$ for every $h \in H(A,*)$, so A satisfies condition (iii) with $\kappa_2 = M$. On the other hand, A is hermitian because

$A^{\mathrm{sym}} = M_2(\mathbb{C})^{\mathrm{sym}}$ as algebras with involution, and clearly $M\mathfrak{r}(h) = \|h\|$ for every $h \in H(A,*)$, so A satisfies condition (iv) with $\kappa_3 = \frac{1}{M}$. Now, let a be in A. Then, noticing that

$$|||\, a^* \odot a\, ||| \geqslant |\, |\lambda| - |1-\lambda|\, |\; |||\, a\, |||^2,$$

we have $\|a^* \odot a\| \geqslant \frac{|\,|\lambda|-|1-\lambda|\,|}{M}\|a\|^2$. Therefore, if $\lambda \neq \frac{1}{2}$, then A satisfies condition (ii) with $\kappa_1 = \frac{M}{|\,|\lambda|-|1-\lambda|\,|}$. But, if $\lambda = \frac{1}{2}$, then in the order of $M_2(\mathbb{C})$ we have $a^* \odot a \geqslant \frac{1}{2}a^*a$, so $|||\, a^* \odot a\, ||| \geqslant \frac{1}{2}|||\, a\, |||^2$, and so $\|a^* \odot a\| \geqslant \frac{1}{2M}\|a\|^2$.

Assume that A is a non-commutative JB^*-algebra in some norm $|\cdot|$. Then $|\cdot| = |||\cdot|||$ because the mapping $x \to x$ from the JB^*-algebra $(A^{\mathrm{sym}}, |\cdot|)$ to the JB^*-algebra $(M_2(\mathbb{C})^{\mathrm{sym}}, |||\cdot|||)$ is an algebra $*$-homomorphism and Proposition 3.4.4 applies. Therefore we have

$$|||\, \lambda ab + (1-\lambda)ba\, ||| = |||\, a \odot b\, ||| = |a \odot b| \leqslant |a||b| = |||\, a\, |||\, |||\, b\, |||$$

for all $a,b \in M_2(\mathbb{C})$. Set $a := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, note that $|||\, a\, ||| = 1$ and $a^2 = 0$, take unimodular complex numbers α, β such that $\alpha\lambda = |\lambda|$ and $\beta(1-\lambda) = |1-\lambda|$, and write $b := \alpha a^* a + \beta a a^*$. Then $|||\, b\, ||| = 1$ and

$$1 = |||\, a\, |||\, |||\, b\, ||| \geqslant |||\, \lambda ab + (1-\lambda)ba\, ||| = (|\lambda| + |1-\lambda|)|||\, aa^*a\, ||| = |\lambda| + |1-\lambda|,$$

and hence $0 \leqslant \lambda \leqslant 1$.

Now, when $\lambda \notin [0,1]$, the corresponding complete normed unital non-commutative Jordan complex $*$-algebra A satisfies conditions (ii)–(iv) in both Corollary 4.5.32 and Corollary 4.5.42, but is not a non-commutative JB^*-algebra in any norm.

Remark 4.5.44 (a) Let λ be in $\mathbb{R} \setminus [0,1]$, and let A be the corresponding algebra in Example 4.5.43 above, so that A is a complete normed hermitian non-commutative Jordan complex $*$-algebra, but A is not a non-commutative JB^*-algebra in any norm. Then, since $M_2(\mathbb{C})$ is simple (by Proposition 1.4.32), and

$$ab = \frac{\lambda}{2\lambda - 1}\, a \odot b + \frac{\lambda - 1}{2\lambda - 1} b \odot a$$

for all $a,b \in M_2(\mathbb{C})$ (which implies that ideals of A are ideals of $M_2(\mathbb{C})$), A is simple. As a consequence, every non-commutative JB^*-representation of A is either zero or injective. But, if there were an injective non-commutative JB^*-representation of A (say $\Phi : A \to B$), then A would be a non-commutative JB^*-algebra in the norm $|\cdot| := \|\Phi(\cdot)\|$, which is impossible. Therefore A has no nonzero non-commutative JB^*-representation. It follows that Theorem 4.5.29 (respectively, Theorem 4.5.37) does not remain true if we replace 'Jordan algebra' and 'JB^*-representation' (respectively, 'alternative algebra' and 'alternative C^*-representation') with 'non-commutative Jordan algebra' and 'non-commutative JB^*-representation'. Note also that, since both $M_2(\mathbb{C})$ and A are unital and simple, we have $\mathrm{Rad}(M_2(\mathbb{C})) = 0$ and $\mathrm{Rad}(A) = 0$, and then, since $\mathrm{J\text{-}Rad}(H(M_2(\mathbb{C}),*)) = 0$ (by Corollary 4.5.39(iii)) and $H(M_2(\mathbb{C}),*) = H(A,*)$ as Jordan real algebras, we realize that $\mathrm{J\text{-}Rad}(H(A,*)) = 0$.

(b) Let λ be in $\mathbb{R} \setminus [0,1]$, let A be the corresponding algebra in Example 4.5.43 above (the norm $\|\cdot\|$ of A can be forgotten), and let $|||\cdot|||$ be the usual C^*-norm on $M_2(\mathbb{C})$. Then A is a non-commutative Jordan complex $*$-algebra such that A^{sym} $(= M_2(\mathbb{C})^{\mathrm{sym}})$ is a JB^*-algebra for the norm $|||\cdot|||$ and the involution $*$. However, according to the third paragraph in the example, A cannot be a non-commutative JB^*-algebra for the norm $|||\cdot|||$ and the involution $*$. Therefore Theorem 3.6.25 does not remain true if we replace 'alternative algebra' and 'alternative C^*-algebra' with 'non-commutative Jordan algebra' and 'non-commutative JB^*-algebra', respectively.

As we show in Proposition 4.5.45 immediately below, the assumption in Corollary 4.5.32 (respectively, Corollary 4.5.42) that the algebra be unital is not that essential. For the formulation and proof, §4.5.6 should be kept in mind.

Proposition 4.5.45 *Let A be a complete normed Jordan (respectively, alternative) complex $*$-algebra. Then the following conditions are equivalent:*

(i) *A is a JB^*-algebra (respectively, an alternative C^*-algebra) in an equivalent norm.*
(ii) *There exists $\kappa_1 > 0$ satisfying $\|a\|^2 \leqslant \kappa_1 \|a^*a\|$ for every $a \in A$.*
(iii) *There exists $\kappa_2 > 0$ such that $\|(\exp -1)(ih)\| \leqslant \kappa_2$ for every $h \in H(A,*)$.*
(iv) *A is hermitian and there exists $\kappa_3 > 0$ such that $\mathfrak{r}(h) \geqslant \kappa_3 \|h\|$ for every $h \in H(A,*)$.*

Proof The proofs of the implications (i)⇒(ii) and (iv)⇒(i) in the proposition do not need changing from those of the corresponding implications in Corollary 4.5.32 (respectively, Corollary 4.5.42) because the existence of a unit was never applied there.

(ii)⇒(iii) Assume that condition (ii) is fulfilled. Then, arguing as in the beginning of the proof of the implication (ii)⇒(iii) in Corollary 4.5.32, we obtain that $*$ is continuous. Let h be in $H(A,*)$. Then the closed subalgebra B of A generated by h is associative, commutative, and $*$-invariant. Working in the standard normed unital extension $B_{\mathbb{1}}$ of B, and keeping in mind Exercises 1.1.30 and 1.2.17, we see that

$$\begin{aligned}\mathbb{1} &= \exp(-ih)\exp(ih) = \exp(ih)^*\exp(ih)\\ &= (\mathbb{1} + (\exp -1)(ih))^*(\mathbb{1} + (\exp -1)(ih))\\ &= [\mathbb{1} + ((\exp -1)(ih))^*](\mathbb{1} + (\exp -1)(ih))\\ &= \mathbb{1} + ((\exp -1)(ih))^* + (\exp -1)(ih) + ((\exp -1)(ih))^*(\exp -1)(ih),\end{aligned}$$

and hence

$$((\exp -1)(ih))^*(\exp -1)(ih) = -((\exp -1)(ih))^* - (\exp -1)(ih).$$

Therefore, we have

$$\begin{aligned}\|(\exp -1)(ih)\|^2 &\leqslant \kappa_1 \|((\exp -1)(ih))^*(\exp -1)(ih)\|\\ &= \kappa_1 \|((\exp -1)(ih))^* + (\exp -1)(ih)\|\\ &\leqslant \kappa_1 (\|*\| + 1)\|(\exp -1)(ih)\|,\end{aligned}$$

and hence

$$\|(\exp -1)(ih)\| \leqslant \kappa_1 (\|*\| + 1).$$

(iii)⇒(iv) Assume that condition (iii) is fulfilled. Let $A_{\mathbb{1}}$ be the standard normed unital extension of A. Then, arguing as in the proof of the implication (iii)⇒(i) in

Corollary 4.5.7, we get that $\|\exp(ih)\| \leqslant 1+\kappa_2$ for every $h \in H(A_{\mathbb{1}}, *)$. Now, by the implication (iii)⇒(iv) in Corollary 4.5.32 (respectively, Corollary 4.5.42) we conclude that $A_{\mathbb{1}}$ is hermitian and that there exists a constant $\kappa_3 > 0$ such that

$$\mathfrak{r}(h) \geqslant \kappa_3 \|h\| \text{ for every } h \in H(A_{\mathbb{1}}, *),$$

so that in particular A is hermitian and $\mathfrak{r}(h) \geqslant \kappa_3\|h\|$ for every $h \in H(A, *)$. □

Corollary 4.5.46 *Let B be a complete normed Jordan real algebra. Then the following conditions are equivalent:*

(i) *B is a JB-algebra in an equivalent norm.*
(ii) *There exists $\kappa > 0$ satisfying $\|x\|^2 \leqslant \kappa\|x^2+y^2\|$ for all $x, y \in B$.*

Proof (i)⇒(ii) By the assumption (i), there is an equivalent norm $|||\cdot|||$ on B converting B into a JB-algebra. Let $m, M > 0$ be such that

$$m\,|||\cdot||| \leqslant \|\cdot\| \leqslant M\,|||\cdot|||.$$

Then for all $x, y \in B$ we have

$$\|x\|^2 \leqslant M^2|||x|||^2 \leqslant M^2|||x^2+y^2||| \leqslant \frac{M^2}{m}\|x^2+y^2\|.$$

(ii)⇒(i) Let A denote the projective normed complexification of B, so that $(A, \|\cdot\|_\pi)$ becomes a complete normed Jordan complex algebra, and let $*$ stand for the canonical involution of A, so that $*$ becomes a conjugate-linear algebra involution on $(A, \|\cdot\|_\pi)$ satisfying $H(A, *) = B$. Then, by the assumption (ii), for $a = x + iy \in A$ with $x, y \in B$, we have

$$\begin{aligned}\|a\|_\pi^2 &\leqslant (\|x\|+\|y\|)^2 \leqslant 4\max\{\|x\|^2, \|y\|^2\}\\ &\leqslant 4\kappa\|x^2+y^2\| = 4\kappa\|a^*a\| = 4\kappa\|a^*a\|_\pi.\end{aligned}$$

Therefore, by the implication (ii)⇒(i) in Proposition 4.5.45, A is a JB^*-algebra in an equivalent norm. Finally, by Corollary 3.4.3, B $(= H(A, *))$ is a JB-algebra in an equivalent norm. □

Let A be a complete normed power-associative complex $*$-algebra, let h be in $H(A, *)$, and let B stand for the closed $*$-subalgebra of A generated by h. Then, according to Corollary 3.5.64, B is associative (and commutative). Therefore, to say that B is a C^*-algebra in the norm of A is equivalent to simply saying that the norm of A becomes a C^*-norm on B.

Corollary 4.5.47 *Let A be a complete normed Jordan (respectively, alternative) complex $*$-algebra, and assume that, for every $h \in H(A, *)$, the closed $*$-subalgebra of A generated by h is a C^*-algebra in the norm of A. Then A is a JB^*-algebra (respectively, an alternative C^*-algebra) in an equivalent norm.*

Proof Let h be in $H(A, *)$, and let B stand for the closed $*$-subalgebra of A generated by h. Then, thinking about the C^*-unital extension of B, we realize that $\|(\exp - 1)(ih)\| \leqslant 2$. Since h is arbitrary in A, the result follows from Proposition 4.5.45. □

Of course, we could expect a better result than the one given by the above corollary. Therefore we raise the following.

Problem 4.5.48 Let A be a complete normed Jordan (respectively, alternative) complex $*$-algebra, and assume that, for every $h \in H(A,*)$, the closed $*$-subalgebra of A generated by h is a C^*-algebra in the norm of A. Is A a JB^*-algebra (respectively, an alternative C^*-algebra) in its own norm?

As far as we know, the above problem remains open to date even if the algebra A is associative and commutative. (Note that an algebra over $\mathbb{K}$ is associative and commutative if and only if it is Jordan and alternative.) Nevertheless, as a by-product of Theorem 4.5.1, we have the following.

Corollary 4.5.49 *Problem 4.5.48 has an affirmative answer whenever the algebra A is unital.*

We note that, in view of Theorem 3.6.25, an affirmative answer to the Jordan version of Problem 4.5.48 would imply an affirmative answer to its alternative version. On the other hand, the alternative version of Problem 4.5.48 can be reduced to its associative version. Indeed, we have the following.

Fact 4.5.50 *Assume that Problem 4.5.48 has an affirmative answer for any associative algebra A. Then it has an affirmative answer for any alternative algebra A.*

Proof Let A be a complete normed alternative complex $*$-algebra such that the closed $*$-subalgebra of A generated by each $h \in H(A,*)$ is a C^*-algebra in the norm of A. Let a be in A, and let B stand for the closed $*$-subalgebra of A generated by a. Then, clearly, the closed $*$-subalgebra of B generated by each $h \in H(B,*)$ is a C^*-algebra. On the other hand, by Corollary 3.5.73(ii), B is associative. It follows from the assumption that B is a C^*-algebra, and hence $\|a^*a\| = \|a\|^2$. Thus A is an alternative C^*-algebra because of the arbitrariness of $a \in A$. □

4.5.3 A conjecture on non-commutative JB^*-equivalent algebras

Let A be a complete normed non-commutative Jordan complex $*$-algebra. If A is a non-commutative JB^*-algebra in some equivalent norm, then, by Fact 3.3.4, A^{sym} is a JB^*-algebra in an equivalent norm, and hence, by Proposition 4.5.45, *the set* $\{(\exp -1)(ih) : h \in H(A,*)\}$ *is bounded.* However, in view of Example 4.5.43, the converse of the assertion just formulated is far from being true. Unfortunately, we do not know any reasonable characterization of normed complex $*$-algebras which are non-commutative JB^*-algebras in an equivalent norm. Nevertheless, looking for such a characterization, we have found the following fact, which could be useful.

Fact 4.5.51 *Let A be a non-commutative JB^*-algebra, and let a be a normal element of A* (cf. Definition 3.4.20). *Then* $\mathfrak{r}(L_a) = \mathfrak{r}(R_a) = \mathfrak{r}(a)$.

Proof By Fact 3.4.22, the closed $*$-subalgebra of A generated by a is a commutative C^*-algebra. Therefore, by Lemma 1.2.12, we have

$$\mathfrak{r}(L_a) \leqslant \|L_a\| \leqslant \|a\| = \mathfrak{r}(a).$$

But in view of Lemma 4.1.16, we have in fact $\mathfrak{r}(L_a) = \mathfrak{r}(a)$. Analogously, $\mathfrak{r}(R_a) = \mathfrak{r}(a)$. □

Noticing that the normality of an element of a $*$-algebra is a purely algebraic property, and that the conclusion in the above fact remains true under equivalent algebra renormings of the non-commutative JB^*-algebra A (by Fact 3.4.62), we cherish the following.

Conjecture 4.5.52 *Let A be a complete normed non-commutative Jordan complex $*$-algebra. Then A is a non-commutative JB^*-algebra in an equivalent norm if (and only if) the set $\{(\exp-1)(ih) : h \in H(A,*)\}$ is bounded and, for every normal element $a \in A$, we have $\mathfrak{r}(L_a) = \mathfrak{r}(R_a) = \mathfrak{r}(a)$.*

In the next remark, the reader can find several facts endorsing the above conjecture.

Remark 4.5.53 (a) Conjecture 4.5.52 is in agreement with Proposition 4.5.45 because, as pointed out in [452, 6.4], *if A is a normed Jordan or alternative algebra over $\mathbb{K}$, then the equalities $\mathfrak{r}(L_a) = \mathfrak{r}(R_a) = \mathfrak{r}(a)$ hold automatically for every $a \in A$.* Indeed, in the Jordan case this follows by passing to complexification, unital extension, and completion if necessary, and then by applying Theorem 4.1.17 and Proposition 4.1.30(ii), whereas in the alternative case it is enough to invoke Lemma 4.1.16, to note that the mappings $a \to L_a$ and $a \to R_a$ from A to $BL(A)$ are continuous Jordan homomorphisms, and to apply Fact 3.4.62.

(b) Let λ be in $\mathbb{R}\setminus[0,1]$, and let A be the corresponding algebra in Example 4.5.43. Then, as we already know, A is a complete normed non-commutative Jordan complex $*$-algebra such that the set $\{(\exp-1)(ih) : h \in H(A,*)\}$ is bounded, but A is not a non-commutative JB^*-algebra in any norm. Therefore, if Conjecture 4.5.52 were right, then there would exist a normal element $a \in A$ such that either $\mathfrak{r}(L_a) \neq \mathfrak{r}(a)$ or $\mathfrak{r}(R_a) \neq \mathfrak{r}(a)$. Actually, the existence of such a normal element is easily realized. Indeed, take $a := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and let b stand for $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ if $\lambda > 1$ (respectively, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ if $\lambda < 0$). Then a is a self-adjoint idempotent of A and we have $a \odot b = \lambda b$ (respectively, $a \odot b = (1-\lambda)b$), hence

$$\lambda \in \operatorname{sp}(BL(A), L_a) \ \ (\text{respectively, } \ 1-\lambda \in \operatorname{sp}(BL(A), L_a)),$$

and finally $\mathfrak{r}(L_a) > 1 = \mathfrak{r}(a)$. Thus Conjecture 4.5.52 is also in agreement with Example 4.5.43.

(c) Let A be a complete normed non-commutative Jordan complex $*$-algebra such that

(i) the set $\{(\exp-1)(ih) : h \in H(A,*)\}$ is bounded;
(ii) for every normal element $a \in A$, the equalities $\mathfrak{r}(L_a) = \mathfrak{r}(R_a) = \mathfrak{r}(a)$ hold.

Then, by the requirement (i) and Proposition 4.5.45, there exists an equivalent norm $|||\cdot|||$ on A converting A^{sym} into a JB^*-algebra. Therefore, in view of Fact 3.3.4, Conjecture 4.5.52 would be right as soon as we were able to prove that the inequality $|||ab||| \leqslant |||a|||\,|||b|||$ holds for all $a, b \in A$. Unfortunately, we are able to show that this inequality holds only when at least one of the elements a, b is self-adjoint. Assume for example that $a \in H(A,*)$. Then, by Lemmas 2.4.15, 3.4.27 and 3.6.24, the operators $b \to [a,b]$ and $b \to a \bullet b$ lie in $H(BL(A,|||\cdot|||), I_A)$, and hence also $L_a \in H(BL(A, |||\cdot|||), I_A)$. Therefore, by Proposition 2.3.22, we have $\mathfrak{r}(L_a) = |||L_a|||$, so that, by requirement (ii), we get $|||L_a||| = \mathfrak{r}(a) \leqslant |||a|||$, hence $|||ab||| \leqslant |||a||| |||b|||$ for every $b \in A$, as desired. We note that, in the above argument, requirement (ii) has been applied only in the particular case where the element a is self-adjoint. Actually, if someone were able to prove the inequality $|||ab||| \leqslant |||a||| |||b|||$ for normal a and arbitrary b, then he/she would have established Conjecture 4.5.52 in the case where A is unital. Indeed, if A is unital, then $\exp(ih)$ is normal and $|||\exp(ih)||| = 1$ whenever h is in $H(A, *)$, so that, if the inequality $|||ab||| \leqslant |||a||| |||b|||$ were true for normal a and arbitrary b, we would have $|||\exp(ih)b||| \leqslant |||b|||$ for every b, so $|||ab||| \leqslant |||b|||$ for every a with $|||a||| \leqslant 1$ and arbitrary b (by Corollary 3.4.7), and so $|||ab||| \leqslant |||a||| |||b|||$ for arbitrary a, b.

4.5.4 Historical notes and comments

Theorem 4.5.1 is new. Even the associative particularization of the implication (ii)⇒(i) (that *a complete normed unital associative complex $*$-algebra is a C^*-algebra whenever so is the closed $*$-subalgebra generated by each self-adjoint element*) could be new. The associative forerunner of the equivalence (i)⇔(iv) in that theorem is originally due to Glickfeld [289, Theorem 2.4], and is included by Pták [494, Theorem 10.1] in a long list of characterizations of C^*-algebras, which in its turn is reproduced in [725, Theorem 36.1].

As we already pointed out in §4.4.77, Proposition 4.5.2 is due to Rota and Strang [544], who also prove (a generalization of) Corollary 4.5.3. These results appear as Theorem 4.1 and Corollary 4.2, respectively, of Bonsall–Duncan [696]. Corollary 4.5.4 is taken from the proof of [653, Theorem 11].

In [57, 58] Barnes attempted to characterize *C^*-equivalent algebras* (i.e. normed complex $*$-algebras which are C^*-algebras in an equivalent norm) in terms of the closed $*$-subalgebras generated by their self-adjoint elements. To this end, he introduced the concept of *locally C^*-equivalent algebras* (i.e. complete normed associative complex $*$-algebras such that the closed $*$-subalgebra generated by each self-adjoint element is C^*-equivalent), established their basic properties, and asked whether they were necessarily C^*-equivalent. Actually, he answered the question affirmatively in several interesting special cases. For example, he proved in [57, Proposition 2.2] the outstanding equivalence (i)⇔(ii) in Theorem 4.5.5. The simple proof given here is due to Wichmann, who announced it in [633] but did not publish it. Wichmann's proof is included in the books by Aupetit [682, pp. 126–7] and Doran–Belfi [725, Proposition 47.5]. The equivalence (i)⇔(iii) in Theorem 4.5.5 had been proved previously by Glickfeld [289, Theorem 1.8]. In 1976, Cuntz [201], with an ingenious proof, answered Barnes' question affirmatively. Indeed, he proved the following.

Theorem 4.5.54 *Locally C^*-equivalent algebras are C^*-equivalent.*

Cuntz's proof of the above theorem is included in Chapter 9 of Doran–Belfi [725]. According to Palmer [787, p. 1185], 'No one seems to have improved Cuntz's original proof… An easier, more direct proof would be of considerable interest.' It would also be quite interesting to know whether the natural Jordan and alternative generalizations of Cuntz's theorem remain true. Thus, keeping in mind Corollary 3.5.64 and Fact 3.3.2, we raise the following.

Problem 4.5.55 Let A be a complete normed Jordan (respectively, alternative) complex $*$-algebra, and assume that the closed $*$-subalgebra of A generated by each self-adjoint element of A is C^*-equivalent. Is A a JB^*-algebra (respectively, an alternative C^*-algebra) in an equivalent norm?

We note that arguments like those in the proof of Corollary 4.5.7 enable the reduction of the above problem to the case where A is unital. We also note that, in view of Theorem 3.6.25, an affirmative answer to the Jordan version of Problem 4.5.55 would imply an affirmative answer to its alternative version, and hence would contain Cuntz' associative theorem. Nevertheless, with certain reservations, we could agree with the concluding remark in Youngson's paper [653] that Cuntz' methods do not generalize [straightforwardly] to the case of Jordan algebras.

To conclude our comments on Cuntz' theorem, let us review the following weaker form, proved by Wu [644].

Theorem 4.5.56 *Let A be a complete normed associative complex $*$-algebra such that the subalgebra of A generated by each self-adjoint element of A can be endowed with a* (possibly non-complete) *equivalent C^*-norm. Then A is C^*-equivalent.*

It is easy to realize that, if Problem 4.5.55 had an affirmative answer, then Theorem 4.5.56 itself would imply its natural Jordan (respectively, alternative) generalization.

Fact 4.5.9, Proposition 4.5.11(i), and Remark 4.5.12 could be new. The associative forerunner of Proposition 4.5.11(ii) is due to Rickart [501], who includes it in his book [795, Theorem 4.1.20] (see also [696, Theorem 37.3] and [725, Theorem 23.11]). Actually, Rickart's theorem is more general than the associative forerunner of Proposition 4.5.11(ii), and was later refined by Yood [645]. The following Jordan variant of Yood's theorem has been proved by Putter and Yood [495].

Theorem 4.5.57 *Let A be a complete normed Jordan complex algebra, let B be a complete normed Jordan complex $*$-algebra such that there exists a vector space norm $|\cdot|$ on $H(B,*)$ satisfying $|h| \leqslant \mathfrak{r}(h)$ for every $h \in H(B,*)$, and let $\Phi : A \to B$ be an algebra homomorphism such that $\Phi(A) = \Phi(A)^*$. Then Φ is continuous.*

The above theorem is indeed better than Yood's associative variant because it implies the following fact which, on the other hand, contains Proposition 4.5.11(ii).

Fact 4.5.58 *Let A be a complete normed Jordan-admissible complex algebra, let B be a complete normed Jordan-admissible complex $*$-algebra such that there exists a vector space norm $|\cdot|$ on $H(B,*)$ satisfying $|h| \leqslant \mathfrak{s}(h)$ for every $h \in H(B,*)$, and let $\Phi : A \to B$ be an algebra homomorphism such that $\Phi(A) = \Phi(A)^*$. Then Φ is continuous.*

Proof Regard Φ as an algebra homomorphism from A^{sym} to B^{sym}, note that $H(B,*) = H(B^{\mathrm{sym}},*)$ as real vector spaces, keep in mind §4.4.6, and apply Theorem 4.5.57. □

Proposition 4.5.10 is due to McCrimonn, who proved it first in the commutative case [435] (see also [822, Theorem 14.8] and [777, Theorem III.1.7.1]), and pointed out its actual version later [436].

Lemma 4.5.13 is taken from Ransford's book [793, Lemma 6.4.1]. With a slightly different argument, it can also be found as part (a) of the proof of Theorem 1.2.1 of Aupetit's book [682]. Theorem 4.5.14 is originally due to Vesentini [622], and is included in both [682, Théorème 1.2.1] and [793, Theorem 6.4.2]. Our versions of Lemma 4.5.13 and Theorem 4.5.14 are more general than those which have previously been cited in the literature. The paternity of these (not particularly deep) generalizations is the same as that of other similar generalizations already mentioned in §4.4.69.

In relation to Proposition 4.5.15, we remark that the notion of thinness was introduced by Brelot in [127, 128]. Thinness can be characterized in terms of the so-called *fine topology* introduced by Cartan in [160], namely the weakest topology on $\mathbb{C}$ with respect to which all subharmonic functions are continuous. In fact a subset S of $\mathbb{C}$ is non-thin at ζ precisely when ζ is a fine limit point of S. For more information we refer to Brelot [701].

Lemma 4.5.16, as well as the proof of Theorem 4.5.22 given here, are due to Aupetit and Youngson [48]. For a comprehensive account of the applications of potential theory and of analytic multifunctions to the theory of normed Jordan algebras, including Aupetit's Theorem 4.4.13, the reader is referred to Aupetit's survey papers [41] and [42, Chapter 8].

Proposition 4.5.21 is a non-associative version of the so-called *Ford's lemma*. Our arguments to prove assertions (i)–(iii) in that proposition become appropriate non-associative adaptations of those in Propositions 8.13 and 12.11 of Bonsall–Duncan [696]. The associative forerunner of assertions (iii) and (iv) can be found in Doran–Belfi [725, Lemma 22.5], where it is commented that 'The key to the study of Banach algebras with arbitrary involutions is the following "square-root" lemma due to J. W. M. Ford [265]. This lemma generalizes a classical square-root lemma which required that the involution be continuous.' Surely assertion (iv) in Proposition 4.5.21 was known by Aupetit and Youngson in [48] (otherwise their proof of Theorem 4.5.22 would not be correct). Nevertheless, no hint or indication is given there.

Results from Theorem 4.5.22 to Corollary 4.5.30 are originally due to Behncke [82], although some auxiliary tools (like Lemma 4.5.25 and Fact 4.5.26), as well as the formal statement of Theorem 4.5.29, could be new. Concerning hermitian Jordan algebras, Behncke's paper advantageously overlaps those of Putter–Yood [495] and Youngson [653], published almost at the same time. The proof of Proposition 4.5.24 has been taken from [653].

The next corollary refines Corollary 4.5.31.

Corollary 4.5.59 *Let A be a complete normed Jordan-admissible complex algebra, let B be a complete normed hermitian Jordan-admissible complex $*$-algebra such that the Jordan algebra $H(B,*)$ is J-semisimple, and let $\Phi : A \to B$ be an algebra homomorphism such that $\Phi(A) = \Phi(A)^*$. Then Φ is continuous.*

Proof Applying Theorem 4.5.29 to B^{sym}, and keeping in mind §4.4.6, we realize that $\mathfrak{s}(\cdot)$ is a vector space norm on $H(B,*)$. Therefore the result follows from Fact 4.5.58. □

The associative forerunner of Proposition 4.5.34 is due to Pták [493], whereas its current version is new. After reducing to the associative case, our proof has been taken from [696, Lemma 41.2]. Although straightforward, Fact 4.5.35 (which seems to go back to Jacobson and Rickart [351]) becomes a classical tool in the study of Jordan and Lie structures in associative algebras.

Results from Proposition 4.5.36 to Corollary 4.5.42 are new, although the newness of Corollary 4.5.38 is only of the formal type. Actually, Corollary 4.5.38 and the associative forerunners of Proposition 4.5.34 and of assertions (i) and (ii) in Corollary 4.5.39 gather the best known important results in the theory of complete normed hermitian associative complex $*$-algebras (mainly due to Pták [493] and Shirali and Ford [565]), as included in Bonsall–Duncan [696, Section 41]. Additional relevant results on hermitian associative complex $*$-algebras, due to Aupetit [37, 38], Doran [214], Harris [313], Pták [494], and Wichmann [632, 634], can be found in [682, Chapitre 4] and [725, Chapter 6]. The associative particularization of assertion (iii) in Corollary 4.5.39 is not recorded anywhere. The reason could be that the self-adjoint part of an associative $*$-algebra need not be an associative algebra, but only a Jordan algebra, and that the Jacobson radical of a Jordan algebra could turn out uninteresting (maybe even unknown) for the associative algebraist.

Proposition 4.5.34, as well as assertions (i) and (ii) in Corollary 4.5.39, are proved in [440] under the additional assumption of the continuity of the involution.

With Theorem 4.5.37 instead of Theorem 4.5.29, arguments like those in the proof of Corollary 4.5.59 yield the following refinement of Corollary 4.5.40.

Corollary 4.5.60 *Let A be a complete normed Jordan-admissible complex algebra, let B be a complete normed semisimple hermitian alternative complex $*$-algebra, and let $\Phi : A \to B$ be an algebra homomorphism such that $\Phi(A) = \Phi(A)^*$. Then Φ is continuous.*

We note that, if Problem 4.5.41 had an affirmative answer, then Corollary 4.5.59 would attain a better formulation which would contain Corollary 4.5.60 immediately above.

The associative forerunner of the equivalence (i)⇔(iii) in Corollary 4.5.42 is due Pták [494, Theorem 8.4]. The associative forerunners of the equivalences (i)⇔(ii) and (i)⇔(iv) in Corollary 4.5.42 (which were conjectured by Kaplansky [375]) had been proved earlier by Yood [647, 649], and are also included in [494, Theorem 8.4]. The associative forerunner of the equivalence (i)⇔(ii) in Corollary 4.5.42 can also be found in Palmer [787, Theorem 11.2.8].

Example 4.5.43 and Remark 4.5.44 are new, although, concerning Example 4.5.43, some arguments taken from the papers [19, 517] have been incorporated. Let λ be in $\mathbb{R} \setminus [0,1]$, and let A be the corresponding algebra in Example 4.5.43. Then A is a complete normed non-commutative Jordan complex $*$-algebra such that the closed $*$-subalgebra of A generated by each self-adjoint element of A is a C^*-algebra in an equivalent norm, but A is not a non-commutative JB^*-algebra in any equivalent norm. Therefore Problem 4.5.55 does not merit to be considered outside its actual version.

Corollary 4.5.46 is new. It becomes a new sample of the symbiotic relationship between JB-algebras and JB^*-algebras (cf. Corollary 3.4.18 for a similar situation).

4.6 Domains of closed derivations

Introduction In Subsection 4.6.1, we prove that the domain of a closed densely defined derivation of a complete normed unital non-commutative Jordan complex algebra A is a J-full subalgebra of A (see Proposition 4.6.14) and that, more generally, if a is an element in the domain of such a derivation, then also $f(a)$ (in the sense of the holomorphic functional calculus) lies in the domain, where f is any complex-valued holomorphic function on an open neighbourhood of J-sp(A,a) (see Theorem 4.6.15).

We begin Subsection 4.6.2 by constructing Bollobás' extremal algebra $\mathrm{Ea}(K)$ (of a given compact convex subset K of $\mathbb{C}$) [110] in a new simple way. Then we prove in Fact 4.6.31 that $\mathrm{Ea}(K)$ can be seen as a norm-unital complete normed algebra of complex-valued continuous functions on K, so that the universal property of $\mathrm{Ea}(K)$ gives rise to a functional calculus at a single element a of any norm-unital complete normed power-associative complex algebra, provided the numerical range of a is contained in K (see Proposition 4.6.32). Since this functional calculus depends on the numerical range instead of on the spectrum, we call it *the geometric functional calculus*. As a first outstanding result, we show in Theorem 4.6.39 how the geometric functional calculus provides us with natural bounds for Fréchet derivatives of functions defined by the holomorphic functional calculus (cf. Theorem 4.1.93). The key tool in the proof of Theorem 4.6.39 (namely Proposition 4.6.46) is also applied to prove in Proposition 4.6.56 that domains of closed densely defined derivations of any norm-unital complete normed power-associative complex algebra are closed under the geometric functional calculus, as soon as functions are 'derivable' in a natural sense (cf. Definition 4.6.42). As a consequence, we derive in Corollaries 4.6.64 and 4.6.69 the theorems of Sakai [807] and Bratteli–Robinson [123], respectively, that domains of closed densely defined derivations of any unital C^*-algebra are closed under the continuous functional calculus at self-adjoint elements, as soon as functions are reasonably regular.

4.6.1 Stability under the holomorphic functional calculus

Let A be an algebra over $\mathbb{K}$. By an *A-bimodule* we mean a vector space X over $\mathbb{K}$ endowed with two bilinear mappings $(a,x) \to ax$ and $(a,x) \to xa$ from $A \times X$ to X. If X is an A-bimodule, then the product vector space $A \times X$ can and will be seen as an algebra over $\mathbb{K}$ under the product defined by

$$(a,x)(b,y) := (ab, ay + xb).$$

The algebra $A \times X$ just introduced is called the *split null X-extension* of A. It is clear that the natural imbeddings of A and X into $A \times X$ allow us to see A as a subalgebra of $A \times X$, and X as an ideal of $A \times X$ such that $XX = 0$. We note that, since A and

X are subsets of a common algebra, expressions like $U_a(x) = a(ax + xa) - a^2x$, for $a \in A$ and $x \in X$, have a meaning (cf. §2.2.4). Nevertheless, to avoid any confusion, given $a \in A$, we denote by L_a^X, R_a^X, and U_a^X the mappings from X to X given by

$$L_a^X(x) = ax, \quad R_a^X(x) = xa \quad \text{and} \quad U_a^X(x) = a(ax + xa) - a^2x.$$

It is worth noticing that, given $a \in A$, the operators $L_{(a,0)}$, $R_{(a,0)}$, and $U_{(a,0)}$ on $A \times X$ are diagonal. Indeed, for every $(b,x) \in A \times X$ we have

$$L_{(a,0)}(b,x) = (L_a(b), L_a^X(x)), \quad R_{(a,0)}(b,x) = (R_a(b), R_a^X(x)), \tag{4.6.1}$$

and

$$U_{(a,0)}(b,x) = (U_a(b), U_a^X(x)). \tag{4.6.2}$$

In the case where A is unital, we will say that X is a *unital A-bimodule* whenever $\mathbf{1}x = x\mathbf{1} = x$ for every x in X. It is clear that in this case, the split null X-extension of A is unital with unit element $(\mathbf{1}, 0)$.

Let A be a (non-commutative) Jordan algebra over $\mathbb{K}$. Following [754, p. 80], an A-bimodule X is said to be a *(non-commutative) Jordan A-bimodule* if the split null X-extension of A is a (non-commutative) Jordan algebra.

Lemma 4.6.1 *Let A be a unital non-commutative Jordan algebra, let X be a unital non-commutative Jordan A-bimodule, and let a be a J-invertible element in A. Then we have:*

(i) $L_a^X L_{a^{-1}}^X = R_a^X R_{a^{-1}}^X$, $L_{a^{-1}}^X = U_{a^{-1}}^X R_a^X$, *and* $R_{a^{-1}}^X = U_{a^{-1}}^X L_a^X$.

(ii) *For $i, j, k, \ell \in \mathbb{N}$, the operators $L_{a^i}^X, R_{a^j}^X, L_{a^{-k}}^X$, and $R_{a^{-\ell}}^X$ pairwise commute.*

Proof Keeping in mind (4.6.1) and (4.6.2), assertions (i) and (ii) in the present lemma follow from assertions (iii) and (iv) in Proposition 4.1.58. □

For the proof of the next result, Proposition 4.1.58 should be kept in mind.

Proposition 4.6.2 *Let A be a unital non-commutative Jordan algebra, and let X be a unital non-commutative Jordan A-bimodule. We have:*

(i) *An element (a,x) is J-invertible in the split null X-extension of A if and only if a is J-invertible in A. Moreover, in this case we have*

$$(a,x)^{-1} = (a^{-1}, -U_{a^{-1}}(x)).$$

(ii) *For every element (a,x) in the split null X-extension of A, we have*

$$\text{J-sp}(A \times X, (a,x)) = \text{J-sp}(A, a).$$

Proof Suppose that (a,x) is J-invertible in the split null X-extension of A with J-inverse (b,y). Then we have

$$(a,x)(b,y) = (b,y)(a,x) = (\mathbf{1}, 0) \quad \text{and} \quad (a,x)^2(b,y) = (b,y)(a,x)^2 = (a,x),$$

hence

$$(ab, ay + xb) = (ba, bx + ya) = (\mathbf{1}, 0)$$

and

$$(a^2b, a^2y + (ax + xa)b) = (ba^2, b(ax + xa) + ya^2) = (a,x),$$

and so $ab = ba = \mathbf{1}$ and $a^2b = ba^2 = a$. Thus a is J-invertible in A. Conversely, suppose that a is J-invertible in A. Then, by Lemma 4.6.1, we see that

$$L_a^X U_{a^{-1}}^X = U_{a^{-1}}^X L_a^X = R_{a^{-1}}^X \tag{4.6.3}$$

and

$$L_{a^2}^X U_{a^{-1}}^X = U_{a^{-1}}^X L_{a^2}^X = U_{a^{-1}}^X L_a^X (L_a^X + R_a^X) - U_{a^{-1}}^X U_a^X = R_{a^{-1}}^X (L_a^X + R_a^X) - I_X,$$

and hence

$$L_{a^2}^X U_{a^{-1}}^X = (L_a^X + R_a^X) R_{a^{-1}}^X - I_X. \tag{4.6.4}$$

Therefore, by (4.6.3), we have

$$\begin{aligned}(a,x)(a^{-1}, -U_{a^{-1}}(x)) &= (\mathbf{1}, -aU_{a^{-1}}(x) + xa^{-1}) \\ &= (\mathbf{1}, (-L_a U_{a^{-1}} + R_{a^{-1}})(x)) = (\mathbf{1}, 0),\end{aligned}$$

and, by (4.6.4), we have

$$\begin{aligned}(a,x)^2(a^{-1}, -U_{a^{-1}}(x)) &= (a^2, ax + xa)(a^{-1}, -U_{a^{-1}}(x)) \\ &= (a, -a^2 U_{a^{-1}}(x) + (ax + xa)a^{-1}) \\ &= (a, (-L_{a^2} U_{a^{-1}} + (L_a + R_a) R_{a^{-1}})(x)) \\ &= (a, x).\end{aligned}$$

Arguing similarly we can also show that

$$(a^{-1}, -U_{a^{-1}}(x))(a,x) = (\mathbf{1}, 0) \text{ and } (a^{-1}, -U_{a^{-1}}(x))(a,x)^2 = (a,x).$$

Summarizing, we have shown that $(a,x) \in \text{J-Inv}(A \times X)$ and that

$$(a,x)^{-1} = (a^{-1}, -U_{a^{-1}}(x)).$$

Thus assertion (i) is proved. Finally, assertion (ii) follows from assertion (i) and the fact that $(a,x) - \lambda(\mathbf{1}, 0) = (a - \lambda\mathbf{1}, x)$ for all $(a,x) \in A \times X$ and $\lambda \in \mathbb{K}$. □

§**4.6.3** Let A be a (complete) normed algebra. If X is an A-bimodule endowed with a (complete) norm satisfying

$$\|ax\| \leqslant \|a\| \, \|x\| \quad \text{and} \quad \|xa\| \leqslant \|x\| \, \|a\| \quad \text{for all } a \text{ in } A \text{ and } x \text{ in } X,$$

then we say that X is a *(complete) normed A-bimodule*. In this case, the split null X-extension of A can and will be regarded as a new (complete) normed algebra under the norm

$$\|(a,x)\| := \|a\| + \|x\|.$$

Proposition 4.6.4 *Let A be a complete normed unital non-commutative Jordan complex algebra, let a be in A, let Ω be an open subset of $\mathbb{C}$ containing* J-sp(A,a), *let f be in $\mathscr{H}(\Omega)$, and let X be a complete normed unital non-commutative Jordan A-bimodule. Then, for every $x \in X$ we have*

$$\text{J-sp}(A \times X, (a,x)) = \text{J-sp}(A, a)$$

and

$$f((a,x)) = (f(a), \frac{1}{2\pi i}\int_\Gamma f(z)U_{(z\mathbf{1}-a)^{-1}}(x)\,dz),$$

where Γ *is any contour that surrounds* J-sp(A,a) *in* Ω.

Proof Let x be an element of X, and let Γ be a contour that surrounds J-sp(A,a) in Ω. It follows from Proposition 4.6.2(ii) that

$$\text{J-sp}(A\times X,(a,x)) = \text{J-sp}(A,a),$$

and hence Γ is a contour that surrounds J-sp$(A\times X,(a,x))$ in Ω. It follows from Theorem 4.1.88(i) that

$$f((a,x)) = \frac{1}{2\pi i}\int_\Gamma f(z)(z(\mathbf{1},0)-(a,x))^{-1}\,dz = \frac{1}{2\pi i}\int_\Gamma f(z)(z\mathbf{1}-a,-x)^{-1}\,dz.$$

Now, applying Proposition 4.6.2(i), we obtain that

$$f((a,x)) = \frac{1}{2\pi i}\int_\Gamma (f(z)(z\mathbf{1}-a)^{-1}, f(z)U_{(z\mathbf{1}-a)^{-1}}(x))\,dz.$$

The proof concludes by using the facts that the natural projections from $A\times X$ onto A and X, respectively, are continuous linear mappings, and that the integral commutes with such mappings. □

Let A be an algebra. If we take $X = A$ as a vector space, and if, given $(a,x)\in A\times X$, the products ax and xa stand for the products as defined in the algebra A, then X becomes an A-bimodule. This bimodule is called the *regular A-bimodule*, and the split null X-extension of A (with X as above) will be called the *split null A-extension of A*. It is clear that the algebra A is unital if and only if the regular A-bimodule is unital. On the other hand, if the algebra A is (complete) normed, then clearly the regular A-bimodule is (complete) normed, and hence, according to §4.6.3, the split null A-extension of A can and will be seen as a (complete) normed algebra.

Fact 4.6.5 *If A is a non-commutative Jordan algebra, then the regular A-bimodule is a non-commutative Jordan A-bimodule.*

Proof By Proposition 3.2.1, it is enough to show that the split null A-extension of A satisfies the flexible and Jordan identities. But, flexible and Jordan identities can be linearized to get

$$[c,b,a]+[a,b,c]=0 \quad \text{and} \quad [c,b,a^2]+[a,b,ac+ca]=0,$$

respectively. Therefore, given elements $(a,x),(b,y)$ in the split null A-extension of A, we see that

$$[(a,x),(b,y),(a,x)] = ([a,b,a],[x,b,a]+[a,y,a]+[a,b,x]) = (0,0)$$

and

$$\begin{aligned}[(a,x),(b,y),(a,x)^2] &= [(a,x),(b,y),(a^2,ax+xa)]\\ &= ([a,b,a^2],[x,b,a^2]+[a,y,a^2]+[a,b,ax+xa]) = (0,0).\end{aligned}$$

Thus, the split null A-extension of A is a non-commutative Jordan algebra. □

Now, invoking Theorem 4.1.93 and the notation introduced in its formulation, it is enough to apply Proposition 4.6.4 to get the following.

Corollary 4.6.6 *Let A be a complete normed unital non-commutative Jordan complex algebra, let a be in A, let Ω be an open subset of $\mathbb{C}$ containing* J-sp(A,a)*, and let f be in $\mathscr{H}(\Omega)$. Then, for every $b \in A$, in the split null A-extension $A \times A$ of A, we have*

$$\text{J-sp}(A \times A, (a,b)) = \text{J-sp}(A,a)$$

and

$$f((a,b)) = (f(a), D\tilde{f}(a)(b)),$$

where $D\tilde{f}(a)$ denotes the Fréchet derivative of $\tilde{f} : A_\Omega \to A$ at a.

Let X_1 and X_2 be vector spaces over $\mathbb{K}$, and let T be an X_2-valued linear mapping on a subspace of X_1 (called the domain of T and denoted by $\text{dom}(T)$). Then we say that T is a *partially defined linear operator* from X_1 to X_2. Now, let A be an algebra over $\mathbb{K}$, let X be an A-bimodule, and let D be a partially defined linear operator from A to X whose domain is a subalgebra of A and satisfies

$$D(ab) = aD(b) + D(a)b \text{ for all } a,b \text{ in } \text{dom}(D).$$

Then we say that D is an *X-valued partially defined derivation* of A.

In the next statement the reader can find two properties of direct verification which show how the structure of the split null X-extension of A is the appropriate one for the study of X-valued derivations of A.

Lemma 4.6.7 *Let A be an algebra, and let X be an A-bimodule. We have:*

(i) *A mapping from a subset of A into X is an X-valued derivation of A if and only if its graph is a subalgebra of the split null X-extension of A.*
(ii) *If D is an X-valued derivation of A, then the mapping $a \to (a, D(a))$ is an algebra isomorphism from* $\text{dom}(D)$ *onto the graph of D.*

§**4.6.8** Let A be a unital algebra, and let X be a unital A-bimodule. When $\mathbf{1}$ belongs to the domain of an X-valued derivation D of A, it is clear that $D(\mathbf{1}) = 0$. On the other hand, if $\mathbf{1}$ is not in $\text{dom}(D)$, then it is straightforward to see that the mapping $D_1 : \lambda\mathbf{1} + a \to D(a)$ is an X-valued derivation of A whose domain (the subalgebra $\mathbb{K}\mathbf{1} \oplus \text{dom}(D)$) contains the unit of A. So we have the following.

Fact 4.6.9 *If A is a unital algebra, and if X is a unital A-bimodule, then every X-valued derivation of A can be extended to an X-valued derivation of A whose domain contains the unit of A. This extension vanishes at the unit.*

Proposition 4.6.10 *Let A be a unital non-commutative Jordan algebra, let X be a unital non-commutative Jordan A-bimodule, and let D be an X-valued derivation of*

A such that $\mathbf{1} \in \mathrm{dom}(D)$. *We have:*

(i) *If* a *is a J-invertible element in* A *such that* a *and* a^{-1} *belong to* $\mathrm{dom}(D)$, *then* $D(a) = -U_a(D(a^{-1}))$.

(ii) $\mathrm{dom}(D)$ *is a J-full subalgebra of* A *if and only if the graph of* D *is a J-full subalgebra of the split null* X*-extension of* A.

Proof Assertion (i) is a generalization of Lemma 4.1.83, and is proved by arguing verbatim as there. On the other hand, keeping in mind Proposition 4.6.2(i), assertion (ii) follows from assertion (i). □

Let A be a normed unital non-commutative Jordan complex algebra, and let a be in A. The 'algebraic' J-spectral radius $\rho_A(a)$ of a relative to A is defined by

$$\rho_A(a) := \sup\{|z| : z \in \text{J-sp}(A,a)\}.$$

We note that, thanks to Fact 4.4.25(i), we have

$$\rho_A(a) = \mathfrak{r}(a) \text{ whenever } A \text{ is complete.} \tag{4.6.5}$$

Proposition 4.6.11 *Let* A *be a complete normed unital non-commutative Jordan complex algebra, and let* B *be a dense subalgebra of* A *containing the unit of* A. *Then* B *is a J-full subalgebra of* A *if and only if* $\rho_A(b) = \rho_B(b)$ *for every* b *in* B.

Proof We may assume that A, and hence B, is a Jordan algebra. The 'only if' part is clear. For the 'if' part, let a be in B such that a is J-invertible in A. Since B is dense in A, and $\mathbf{1}$ lies in the range of U_a (by Theorem 4.1.3(ii)), it follows the existence of b in B such that $\|U_a(b) - \mathbf{1}\| < 1$, and, by (4.6.5), we have $\rho_A(U_a(b) - \mathbf{1}) < 1$. By the assumption, ρ_A and ρ_B agree in B. Hence $\rho_B(U_a(b) - \mathbf{1}) < 1$, so 1 does not lie in $\text{J-sp}(B, U_a(b) - \mathbf{1})$, and so $U_a(b)$ is J-invertible in B. Now, by Theorem 4.1.3(vi), a is J-invertible in B, and as a result B is a J-full subalgebra of A. □

The next result is easily proved by induction.

Lemma 4.6.12 *Let* A *be a normed power-associative algebra, let* X *be a normed* A*-bimodule, let* D *be an* X*-valued derivation of* A, *and let* n *be a natural number. Then* $\|D(a^n)\| \leqslant n\|a\|^{n-1}\|D(a)\|$ *for every* a *in* $\mathrm{dom}(D)$.

Let X_1 and X_2 be normed spaces over $\mathbb{K}$, and let T be a partially defined linear operator from X_1 to X_2. We recall that T is said to be *densely defined* if $\mathrm{dom}(T)$ is dense in X_1, and that T is said to be a *closed operator* if its graph is closed in the normed direct product space $X_1 \times X_2$.

Lemma 4.6.13 *Let* A *be a complete normed unital power-associative complex algebra, let* X *be a complete normed unital* A*-bimodule, and let* D *be a closed densely defined* X*-valued derivation of* A. *Then* $\mathbf{1}$ *lies in* $\mathrm{dom}(D)$.

Proof By the density of $\mathrm{dom}(D)$ in A there exists a in $\mathrm{dom}(D)$ such that $\|\mathbf{1} - a\| < 1$. For each natural number n, let us write $a_n := \mathbf{1} - (\mathbf{1} - a)^n$. Clearly a_n lies in $\mathrm{dom}(D)$ and $a_n \to \mathbf{1}$ as $n \to \infty$. Let D_1 denote the extension of D assured by Fact 4.6.9. Then $D(a_n) = -D_1((\mathbf{1} - a)^n)$ so, by Lemma 4.6.12,

$$\|D(a_n)\| \leqslant n\|\mathbf{1} - a\|^{n-1}\|D(a)\|,$$

and so $D(a_n) \to 0$. Therefore $\mathbf{1}$ lies in $\mathrm{dom}(D)$ because D is closed. □

Proposition 4.6.14 *Let A be a complete normed unital non-commutative Jordan complex algebra, let X be a complete normed unital non-commutative Jordan A-bimodule, and let D be a closed densely defined X-valued derivation of A. Then* $\mathrm{dom}(D)$ *is a J-full subalgebra of A.*

Proof Keeping in mind Lemma 4.6.13 and Proposition 4.6.11, in order to prove that $\mathrm{dom}(D)$ is a J-full subalgebra of A it is enough to show that $\rho_A(a) = \rho_{\mathrm{dom}(D)}(a)$ for every a in $\mathrm{dom}(D)$. But $\rho_A(a) = \rho_{A\times X}((a,D(a)))$ (by Proposition 4.6.2(ii)), and, if G denotes the graph of D, then we have $\rho_{A\times X}((a,D(a))) = \rho_G((a,D(a)))$ because G is a closed subalgebra of the complete normed unital non-commutative Jordan algebra $A\times X$ containing the unit of $A\times X$, and (4.6.5) applies. Also $\rho_G((a,D(a))) = \rho_{\mathrm{dom}(D)}(a)$ (by Lemma 4.6.7(ii)). Therefore $\rho_A(a) = \rho_{\mathrm{dom}(D)}(a)$, as required. □

Theorem 4.6.15 *Let A be a complete normed unital non-commutative Jordan complex algebra, let X be a complete normed unital non-commutative Jordan A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in* $\mathrm{dom}(D)$*, let Ω be an open subset of $\mathbb{C}$ containing* $\text{J-sp}(A,a)$*, and let f be in $\mathscr{H}(\Omega)$. Then $f(a) \in \mathrm{dom}(D)$, and*

$$D(f(a)) = \frac{1}{2\pi i}\int_\Gamma f(z)U_{(z\mathbf{1}-a)^{-1}}(D(a))\,dz,$$

where Γ is any contour that surrounds $\text{J-sp}(A,a)$ *in Ω.*

Proof Let Γ be a contour that surrounds $\text{J-sp}(A,a)$ in Ω. Then, by Proposition 4.6.4, $\text{J-sp}(A\times X,(a,D(a))) = \text{J-sp}(A,a)$ and

$$f((a,D(a))) = \left(f(a), \frac{1}{2\pi i}\int_\Gamma f(z)U_{(z\mathbf{1}-a)^{-1}}(D(a))\,dz\right).$$

Since, by Proposition 4.6.14, $\mathrm{dom}(D)$ is a J-full subalgebra of A, and hence, by Proposition 4.6.10(ii), the graph (say G) of D is a (closed) J-full subalgebra of $A\times X$, Theorem 4.1.88(iii) applies, so that $f((a,D(a))) \in G$. Therefore $f(a) \in \mathrm{dom}(D)$ and

$$D(f(a)) = \frac{1}{2\pi i}\int_\Gamma f(z)U_{(z\mathbf{1}-a)^{-1}}(D(a))\,dz.$$

Thus the proof is complete. □

Let A be an algebra. By a *(partially defined) derivation* of A we mean a linear mapping D from a subalgebra $\mathrm{dom}(D)$ of A to A satisfying

$$D(ab) = aD(b) + D(a)b$$

for all $a,b \in \mathrm{dom}(D)$. Clearly, (partially defined) derivations of A are nothing other than X-valued (partially defined) derivations of A, when X equals the regular A-bimodule. Now, recalling Theorem 4.1.93 and the notation introduced in its formulation, it is enough to invoke Fact 4.6.5 and Theorem 4.6.15 above to obtain the following.

Corollary 4.6.16 *Let A be a complete normed unital non-commutative Jordan complex algebra, let D be a closed densely defined derivation of A, let a be in*

$\mathrm{dom}(D)$, *let* Ω *be an open subset of* $\mathbb{C}$ *containing* $\text{J-sp}(A,a)$, *and let* f *be in* $\mathscr{H}(\Omega)$. *Then* $f(a) \in \mathrm{dom}(D)$ *and*

$$D(f(a)) = D\tilde{f}(a)(D(a)),$$

where $D\tilde{f}(a)$ *denotes the Fréchet derivative of* $\tilde{f} : A_\Omega \to A$ *at* a.

The following corollary shows in particular that when A splits into two direct summands, all closed densely defined derivations of A arise in a natural way from the knowledge of closed densely defined derivations of the given direct summands.

Corollary 4.6.17 *Let* A *be a complete normed unital non-commutative Jordan complex algebra, let* X *be a complete normed unital non-commutative Jordan* A*-bimodule, and let* D *be a closed densely defined* X*-valued derivation of* A. *Then, every central idempotent in* A *lies in* $\mathrm{dom}(D)$.

Proof In view of Lemma 4.6.13, 0 and $\mathbf{1}$ belong to $\mathrm{dom}(D)$. Let e be a central idempotent in A different from 0 and $\mathbf{1}$. Then the subalgebra B of A generated by $\{\mathbf{1}, e\}$ is isomorphic to $\mathbb{C}^2$ (via the mapping $(\lambda,\mu) \to \lambda e + \mu(\mathbf{1}-e)$), and hence, by Proposition 4.1.66, B is a closed J-full subalgebra of A containing e. As a consequence, we have $\text{J-sp}(A,x) = \mathrm{sp}(B,x)$ for every $x \in B$ (which implies $\text{J-sp}(A,e) = \{0,1\}$), and

$$x = 0 \ \text{ whenever } \ x \text{ is in } B \ \text{ with } \ \text{J-sp}(A,x) = \{0\}. \tag{4.6.6}$$

Now, consider the open subset Ω of $\mathbb{C}$ given by $\Omega := B(0,\frac{1}{2}) \cup B(1,\frac{1}{2})$ (where $B(z,r)$ denotes the open ball in $\mathbb{C}$ with centre z and radius r), and the complex-valued holomorphic mapping f on Ω defined by $f(z) = 0$ if $z \in B(0,\frac{1}{2})$ and $f(z) = 1$ if $z \in B(1,\frac{1}{2})$. Since A_Ω is an open subset of A containing e (by Proposition 4.1.91), and the mapping $a \to f(a)$ from A_Ω into A is continuous (by Theorem 4.1.93), the density of $\mathrm{dom}(D)$ in A gives the existence of an element a in $\mathrm{dom}(D) \cap A_\Omega$ such that $\|f(e) - f(a)\| < 1$. But, according to Theorem 4.1.88(iii), $f(e)$ lies in B and, by Theorem 4.1.88(iv), we have $\text{J-sp}(A, f(e) - e) = \{0\}$, which implies $f(e) = e$ because of (4.6.6). On the other hand, since $f(a)$ is an idempotent and e is a central idempotent, we have $e - f(a) = (e - f(a))^3$, and hence $\|e - f(a)\| \leqslant \|e - f(a)\|^3$. It follows that $e = f(a)$ because $\|e - f(a)\| < 1$, and then e lies in $\mathrm{dom}(D)$ by Theorem 4.6.15. $\square$

The notion of a non-commutative Jordan bimodule over a non-commutative Jordan algebra is a particular case of that of a bimodule over an algebra relative to a variety. Indeed, given a variety $\mathscr{V}$ and an algebra A in $\mathscr{V}$, the A*-bimodules relative to* $\mathscr{V}$ are defined as those A-bimodules X such that the algebra split null X-extension of A remains a member of $\mathscr{V}$. When $\mathscr{V}$ is the variety of all associative (respectively, alternative) algebras, the bimodules relative to $\mathscr{V}$ are called *associative* (respectively, *alternative*) *bimodules*.

In view of Fact 4.1.57 and Definition 4.1.65, given an element a of a unital alternative algebra A over $\mathbb{K}$, we write $\mathrm{sp}(A,a)$ instead of $\text{J-sp}(A,a)$. Now, since alternative bimodules are non-commutative Jordan bimodules, and for elements x,y in an alternative algebra we have $U_x(y) = xyx$, Theorem 4.6.15 has the following remarkable consequence.

Corollary 4.6.18 *Let A be a complete normed unital alternative complex algebra, let X be a complete normed unital alternative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in* $\mathrm{dom}(D)$, *let Ω be an open subset of $\mathbb{C}$ containing* $\mathrm{sp}(A,a)$, *and let f be in $\mathscr{H}(\Omega)$. Then $f(a) \in \mathrm{dom}(D)$, and*

$$D(f(a)) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(z\mathbf{1}-a)^{-1}D(a)(z\mathbf{1}-a)^{-1}\,dz,$$

where Γ is any contour that surrounds $\mathrm{sp}(A,a)$ *in Ω.*

§**4.6.19** When $\mathscr{V}$ is the variety of all associative and commutative algebras, the bimodules relative to $\mathscr{V}$ are called *associative and commutative bimodules*. Combining Corollary 4.6.18 above and Proposition 1.3.15 we get the following.

Corollary 4.6.20 *Let A be a complete normed unital associative and commutative complex algebra, let X be a complete normed unital associative and commutative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in* $\mathrm{dom}(D)$, *let Ω be an open subset of $\mathbb{C}$ containing* $\mathrm{sp}(A,a)$, *and let f be in $\mathscr{H}(\Omega)$. Then $f(a) \in \mathrm{dom}(D)$, and*

$$D(f(a)) = f'(a)D(a).$$

4.6.2 Stability under the geometric functional calculus

This subsection relies on the following.

Theorem 4.6.21 *Let K be a non-empty compact and convex subset of $\mathbb{C}$. Then, up to isometric algebra isomorphisms preserving distinguished ele- ments, there exists a unique norm-unital complete normed associative complex algebra $\mathscr{A}$ with a distinguished element u, satisfying the following properties:*

(i) $V(\mathscr{A},\mathbf{1},u) = K$.
(ii) *If A is any norm-unital complete normed associative complex algebra, and if a is in A with $V(A,\mathbf{1},a) \subseteq K$, then there is a unique contractive unit-preserving algebra homomorphism from $\mathscr{A}$ to A taking u to a.*

Moreover, we have:

(iii) *$\mathscr{A}$ is generated by $\{\mathbf{1},u\}$ as a normed algebra.*
(iv) $\mathrm{sp}(\mathscr{A},u) = K$.

§**4.6.22** In the case where K is reduced to a point (say $K = \{\lambda\}$ for some $\lambda \in \mathbb{C}$), the above theorem becomes trivial. Indeed, if a is an element of any norm-unital normed complex algebra, and if $V(a) = \{\lambda\}$, then, by Corollary 2.1.13, we have $a = \lambda\mathbf{1}$. Thus, in this case, Theorem 4.6.21 follows straightforwardly by taking $\mathscr{A} = \mathbb{C}$ and $u = \lambda$.

§**4.6.23** The essential uniqueness of the couple $(\mathscr{A},u)$ under the conditions (i) and (ii) in Theorem 4.6.21 follows from a categorical argument like the one in the first paragraph of the proof of Theorem 4.3.52.

Assertions (iii) and (iv) in Theorem 4.6.21 are easily realized from the abstract characterization of the couple $(\mathscr{A}, u)$ given by the theorem (see §§4.6.71 and 4.6.72 below). We do not include the arguments here because, actually, assertions (iii) and (iv) will follow almost straightforwardly from the construction of the couple $(\mathscr{A}, u)$ which is done in the following.

From now until Proposition 4.6.36, K will stand for a non-empty compact and convex subset of $\mathbb{C}$. For $z \in \mathbb{C}$ we set

$$\omega_K(z) := \max\{|e^{zw}| : w \in K\} = e^{\max \Re(zK)}.$$

The next lemma is straightforward, and will be applied without notice.

Lemma 4.6.24 *The following assertions hold:*

(i) $\omega_K(z_1 + z_2) \leqslant \omega_K(z_1)\omega_K(z_2)$ *for all* $z_1, z_2 \in \mathbb{C}$.
(ii) $\omega_K(z) \leqslant e^{|K||z|}$ *for every* $z \in \mathbb{C}$, *where* $|K|$ *stands for* $\max\{|w| : w \in K\}$.

§4.6.25 Let us denote by $D(K)$ the set of all entire functions g such that

$$\|g\| := \sup\{(\omega_K(z))^{-1}|g(z)| : z \in \mathbb{C}\} < +\infty,$$

so that, given $g \in D(K)$, $\|g\|$ is the smallest non-negative number M satisfying $|g(z)| \leqslant M\omega_K(z)$ for every $z \in \mathbb{C}$, and in particular we have

$$|g(z)| \leqslant \|g\|\omega_K(z) \ \text{ for every } \ z \in \mathbb{C}. \tag{4.6.7}$$

Now, noticing that $\|\cdot\|$ is a norm on $D(K)$, and that convergence of sequences in $(D(K), \|\cdot\|)$ implies uniform convergence on compact subsets of $\mathbb{C}$, it is easily realized that $(D(K), \|\cdot\|)$ is a complex Banach space. It is also worth mentioning that, as a consequence of (4.6.7), we have

$$|g(0)| \leqslant \|g\| \ \text{ whenever } \ g \ \text{ is in } \ D(K). \tag{4.6.8}$$

Lemma 4.6.26 *Let g be in $D(K)$. Then g' lies in $D(K)$ and we have*

$$\|g'\| \leqslant e|K|\|g\|.$$

Proof Let z and r be in $\mathbb{C}$ and $\mathbb{R}^+$, respectively. Since

$$g'(z) = \frac{1}{2\pi i}\int_{|\xi - z| = r} \frac{g(\xi)}{(\xi - z)^2} d\xi,$$

and

$$|g(\xi)| \leqslant \|g\|\omega_K(\xi) \leqslant \|g\|\omega_K(\xi - z)\omega_K(z) \leqslant \|g\|e^{r|K|}\omega_K(z)$$

whenever $|\xi - z| = r$, we deduce that

$$|g'(z)| \leqslant \frac{1}{2\pi} 2\pi r \frac{1}{r^2}\|g\|e^{r|K|}\omega_K(z) = \frac{1}{r}\|g\|e^{r|K|}\omega_K(z).$$

By taking $r := \frac{1}{|K|}$, we get $|g'(z)| \leqslant e|K|\|g\|\omega_K(z)$. Since z is arbitrary in $\mathbb{C}$, the result follows. □

§**4.6.27** According to Lemma 4.6.26 just proved, the linear operator $u : g \to g'$ on the complex Banach space $D(K)$ is continuous. Therefore we can consider the closed subalgebra $\mathscr{A}$ of $BL(D(K))$ generated by $\mathbf{1} := I_{D(K)}$ and u. Clearly, *$\mathscr{A}$ is a norm-unital complete normed associative complex algebra, u is an element of $\mathscr{A}$, and the couple $(\mathscr{A}, u)$ satisfies property (iii) in Theorem 4.6.21*.

We are going to show that the couple $(\mathscr{A}, u)$ in §4.6.27 above satisfies conditions (i) and (ii) in Theorem 4.6.21, as well as property (iv) in that theorem. To this end, the following reformulation of the first equality in Proposition 2.1.7 becomes crucial.

Fact 4.6.28 *Let A be a norm-unital complete normed associative complex algebra, and let a be in A. Then $V(A, \mathbf{1}, a) \subseteq K$ if and only if $\|\exp(za)\| \leqslant \omega_K(z)$ for every $z \in \mathbb{C}$.*

§**4.6.29** *Let u and $\mathscr{A}$ be as in §4.6.27*. We are going to prove that *the couple $(\mathscr{A}, u)$ satisfies condition (ii) in Theorem 4.6.21*. Indeed, let A be any a norm-unital complete normed associative complex algebra, let a be in A such that $V(A, \mathbf{1}, a) \subseteq K$, and let θ be in A' with $\|\theta\| \leqslant 1$. Then the mapping $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) := \theta(\exp(za))$ is an entire function satisfying $g^{(k)}(0) = \theta(a^k)$ for every non-negative entire number k. Moreover, by Fact 4.6.28, we have $|g(z)| \leqslant \omega_K(z)$ for every $z \in \mathbb{C}$, i.e. g belongs to $D(K)$ and $\|g\| \leqslant 1$. Now, let $p(\mathbf{x}) = \sum_{k=0}^{n} \lambda_k \mathbf{x}^k$ be in $\mathbb{C}[\mathbf{x}]$. Then we have

$$\begin{aligned}\theta(p(a)) &= \sum_{k=0}^{n} \lambda_k \theta(a^k) = \sum_{k=0}^{n} \lambda_k g^{(k)}(0) = \sum_{k=0}^{n} \lambda_k \left[u^k(g)\right](0) \\ &= \left[\left(\sum_{k=0}^{n} \lambda_k u^k\right)(g)\right](0) = [p(u)(g)](0),\end{aligned}$$

and hence, invoking (4.6.8), we get

$$|\theta(p(a))| = |[p(u)(g)](0)| \leqslant \|p(u)(g)\| \leqslant \|p(u)\|\|g\| \leqslant \|p(u)\|.$$

Since θ is an arbitrary element of the closed unit ball of A', we deduce that $\|p(a)\| \leqslant \|p(u)\|$. Therefore $p(u) \to p(a)$ becomes a well-defined contractive algebra homomorphism from the dense subalgebra $\mathscr{B} := \{p(u) : p \in \mathbb{C}[\mathbf{x}]\}$ of $\mathscr{A}$ to A. By extending this homomorphism by continuity, we find a contractive unit-preserving algebra homomorphism from $\mathscr{A}$ to A taking u to a. It follows from Lemma 1.1.82(i) that there is no other such homomorphism. This shows that the couple $(\mathscr{A}, u)$ satisfies condition (ii) in Theorem 4.6.21, as desired.

§**4.6.30** *Let u and $\mathscr{A}$ be as in §4.6.27*. We are going to prove that *the couple $(\mathscr{A}, u)$ satisfies condition (i) in Theorem 4.6.21, as well as property (iv) in that theorem.* Let λ be in K. Then we straightforwardly realize that the complex-valued function g_λ on $\mathbb{C}$ defined by $g_\lambda(z) := e^{\lambda z}$ lies in $D(K)$, and that $u(g_\lambda) = \lambda g_\lambda$. Thus λ is an eigenvalue of u, and hence

$$\lambda \in \mathrm{sp}(L(D(K)), u) \subseteq \mathrm{sp}(\mathscr{A}, u).$$

Since λ is arbitrary in K, we conclude that

$$K \subseteq \mathrm{sp}(\mathscr{A}, u). \tag{4.6.9}$$

Now, let w be in $\mathbb{C}$ and let g be in $D(K)$. Then for every $z \in \mathbb{C}$ we have

$$[\exp(wu)(g)](z) = \sum_{n=0}^{\infty} \frac{w^n}{n!} g^{(n)}(z) = g(w+z), \tag{4.6.10}$$

so, applying (4.6.7), we get

$$|[\exp(wu)(g)](z)| = |g(w+z)| \leqslant \|g\|\omega_K(w+z) \leqslant \|g\|\omega_K(w)\omega_K(z),$$

hence $\|\exp(wu)(g)\| \leqslant \|g\|\omega_K(w)$. Since g is arbitrary in $D(K)$, we derive that

$$\|\exp(wu)\| \leqslant \omega_K(w).$$

Then, since w is arbitrary in $\mathbb{C}$, Fact 4.6.28 applies to get

$$V(\mathscr{A}, \mathbf{1}, u) \subseteq K. \tag{4.6.11}$$

Finally, since $\mathrm{sp}(\mathscr{A}, u) \subseteq V(\mathscr{A}, \mathbf{1}, u)$ (by Lemma 2.3.21), it follows from (4.6.9) and (4.6.11) that

$$K = \mathrm{sp}(\mathscr{A}, u) = V(\mathscr{A}, \mathbf{1}, u).$$

Keeping in mind §§4.6.23, 4.6.25, 4.6.27, 4.6.29, and 4.6.30, *the proof of Theorem 4.6.21 is complete.*

Now that Theorem 4.6.21 has been proved, the essentially unique norm-unital complete normed associative complex algebra $\mathscr{A}$ with distinguished element u, associated to K via the theorem, will be called *the extremal algebra of* K and will be denoted by $\mathrm{Ea}(K)$, whereas, according to assertion (iii) in the theorem, the distinguished element u will be called *the generator of* $\mathrm{Ea}(K)$. The algebra $\mathrm{Ea}(K)$ fulfils a relevant additional property. Indeed, we have the following.

Fact 4.6.31 $\mathrm{Ea}(K)$ *is contractively isomorphic, in a unique manner, to a full subalgebra of* $C^{\mathbb{C}}(K)$ *in such a way that the generator* u *becomes the inclusion mapping* $K \hookrightarrow \mathbb{C}$.

Proof In view of properties (iii) and (iv) in Theorem 4.6.21 (which will be applied without notice at some future point of the argument) and of Theorem 1.1.83, it is enough to show that $\mathrm{Ea}(K)$ is (strongly) semisimple. Moreover, by §4.6.22, we can assume that K has more than one point. Let g be in $D(K) \setminus \{0\}$. Then, by the inequality (4.6.7) and Lemma 4.6.24(ii), we have $|g(z)| \leqslant \|g\|e^{|K||z|}$ for every $z \in \mathbb{C}$, and hence g is an entire function of exponential type. By the main theorem in the theory of such functions (see for example [107, pp. 839–40]), there exists $\tilde{g} \in \mathscr{H}(\mathbb{C} \setminus |K|\mathbb{B}_{\mathbb{C}})$ (the so-called Borel transform of g) such that, for $r > |K|$, we have

$$g(z) = \int_{|w|=r} e^{wz}\tilde{g}(w)dw \ \text{ for every } \ z \in \mathbb{C}. \tag{4.6.12}$$

Moreover, by the well-known determination of $\tilde{g}$ on each open half-plane tangent to $|K|\mathbb{B}_{\mathbb{C}}$ (say $\tilde{g}(w) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{-wt}g(t)dt$ for $\Re(w) > |K|$) , we have

$$|\tilde{g}(w)| \leqslant \frac{\|g\|}{2\pi(|w| - |K|)} \ \text{ for every } \ w \in \mathbb{C} \setminus |K|\mathbb{B}_{\mathbb{C}}. \tag{4.6.13}$$

Since $\tilde{g}$ depends linearly on g, we derive from (4.6.13) that, if $g_n \to g$ in $D(K)$, then $\tilde{g}_n \to \tilde{g}$ uniformly on compact subsets of $\mathbb{C} \setminus |K|\mathbb{B}_{\mathbb{C}}$. On the other hand, keeping in mind our construction of $\mathrm{Ea}(K)$ in §4.6.27, it follows from (4.6.12) that, for $p \in \mathbb{C}[\mathbf{x}]$, and $r > |K|$, we have

$$[p(u)(g)](z) = \int_{|w|=r} e^{wz} p(w)\tilde{g}(w)dw \ \text{ for every } \ z \in \mathbb{C}, \tag{4.6.14}$$

i.e. the Borel transform of $p(u)(g)$ is the function $w \to p(w)\tilde{g}(w)$. Now, let T be arbitrary in $\mathrm{Ea}(K)$, and let $r > |K|$ be such that the circumference $\Gamma := \{w \in \mathbb{C} : |w| = r\}$ avoids the zeros of $\tilde{g}$. Take a sequence p_n in $\mathbb{C}[\mathbf{x}]$ such that $p_n(u) \to T$ in $\mathrm{Ea}(K)$, and note that $p_n(u)(g) \to T(g)$ in $D(K)$. It follows that $p_n(\cdot)\tilde{g}(\cdot) \to \widetilde{T(g)}(\cdot)$ uniformly on Γ, and hence (since $\tilde{g}$ does not vanish on Γ) that the sequence $p_n(\cdot)$ is uniformly Cauchy on Γ, so also on $r\mathbb{B}_{\mathbb{C}}$ (by the maximum modulus principle). Let h stand for the uniform limit of $p_n(\cdot)$ on $r\mathbb{B}_{\mathbb{C}}$. Then, by (4.6.14), we have that

$$[T(g)](z) = \int_{|w|=r} e^{wz} h(w)\tilde{g}(w)dw \ \text{ for every } \ z \in \mathbb{C}. \tag{4.6.15}$$

Now, assume that T lies in the strong radical of $\mathrm{Ea}(K)$. Then, noticing that $p_n(\cdot) \to 0$ on K (by Theorem 1.1.83), we derive that $h = 0$ on K. Since h is continuous on $r\mathbb{B}_{\mathbb{C}}$ and holomorphic on its interior (say Ω), and K is an infinite subset of Ω, we get that $h = 0$ on $r\mathbb{B}_{\mathbb{C}}$. By invoking (4.6.15), we obtain $T(g) = 0$. Since g is arbitrary in $D(K) \setminus \{0\}$, we get $T = 0$. Finally, since T is arbitrary in the strong radical of $\mathrm{Ea}(K)$, we realize that $\mathrm{Ea}(K)$ is strongly semisimple, as desired. □

In view of Fact 4.6.31, elements of $\mathrm{Ea}(K)$ will often be seen as complex-valued continuous functions on K. When this is the case, given $f \in \mathrm{Ea}(K)$, $\|f\|$ will mean the norm of f as an element of $\mathrm{Ea}(K)$, whereas $\|f\|_\infty$ will mean the norm of f as an element of $C^{\mathbb{C}}(K)$. The display of elements of $\mathrm{Ea}(K)$ as complex-valued functions on K gives rise to the following *geometric functional calculus* at a single element of a norm-unital complete normed power-associative complex algebra.

Proposition 4.6.32 *Let A be a norm-unital complete normed power-associative complex algebra, and let a be in A with $V(A,\mathbf{1},a) \subseteq K$. Then there exists a unique contractive unit-preserving algebra homomorphism $f \to f(a)$ from* $\mathrm{Ea}(K)$ *to A taking the generator u to a. As a consequence, for $f \in$* $\mathrm{Ea}(K)$ *we have*

$$V(A,\mathbf{1},f(a)) \subseteq V(\mathrm{Ea}(K),\mathbf{1},f).$$

Proof Let B stand for the closed subalgebra of A generated by $\{\mathbf{1},a\}$, and let $\Phi : \mathrm{Ea}(K) \to A$ be a contractive unit-preserving algebra homomorphism such that $\Phi(u) = a$. Then, by Theorem 4.6.21(iii) and Lemma 1.1.82(ii), we have

$$\Phi(\mathrm{Ea}(K)) \subseteq B.$$

On the other hand, since A is power-associative, B is associative. This allows us to see Φ as a contractive unit-preserving algebra homomorphism from $\mathrm{Ea}(K)$ to B taking u to a. Moreover, according to Corollary 2.1.2, we have $V(A,\mathbf{1},a) = V(B,\mathbf{1},a)$. Therefore, replacing A with B, we may assume that A is associative, so that the first conclusion in the proposition follows from Theorem 4.6.21(ii). Now that the first conclusion has been proved, the consequence follows from Corollary 2.1.2(i). □

Remark 4.6.33 Let γ, δ be in $\mathbb{C}$ with $\gamma \neq 0$, and set $L := \gamma K + \delta$. In view of Theorem 4.6.21, it is a tautology that $\mathrm{Ea}(K)$ and $\mathrm{Ea}(L)$ are isometrically isomorphic.

More precisely, denoting by u and v the generators of $\mathrm{Ea}(K)$ and $\mathrm{Ea}(L)$, respectively, and arguing as in §4.6.23, we realize that there exists a unique continuous algebra homomorphism $\phi : \mathrm{Ea}(L) \to \mathrm{Ea}(K)$ taking v to $\gamma u + \delta$, and that moreover such an algebra homomorphism ϕ is a surjective isometry. Now, denoting by η the function $z \to \gamma z + \delta$ from K to L, it is enough to invoke Proposition 4.6.32 to get that $f \circ \eta$ lies in $\mathrm{Ea}(K)$ whenever f is in $\mathrm{Ea}(L)$, and that the surjective isometric algebra homomorphism $\phi : \mathrm{Ea}(L) \to \mathrm{Ea}(K)$ above is nothing other than the mapping $f \to f \circ \eta$.

§**4.6.34** Let L be a non-empty compact and convex subset of $\mathbb{C}$ with $L \subseteq K$. By Theorem 4.6.21 and Fact 4.6.31, there is a contractive unit-preserving algebra homomorphism $\phi : \mathrm{Ea}(K) \to \mathrm{Ea}(L)$ taking $K \hookrightarrow \mathbb{C}$ to its restriction $L \hookrightarrow \mathbb{C}$. Then, given $f \in \mathrm{Ea}(K)$, it is enough to valuate $\phi(f)$ at points of L to realize that $\phi(f) = f_{|L}$. Therefore $f_{|L}$ lies in $\mathrm{Ea}(L)$ and $\|f_{|L}\| \leqslant \|f\|$ whenever f belongs to $\mathrm{Ea}(K)$. Now let A be a norm-unital complete normed power-associative complex algebra, let a be in A with $V(A, \mathbf{1}, a) \subseteq L$, and let f be in $\mathrm{Ea}(K)$. Then it follows from the above and Proposition 4.6.32 that $f(a) = f_{|L}(a)$. In other words, the element $f(a)$ in the sense of Proposition 4.6.32 does not depend on K, so we could have taken $K = V(A, \mathbf{1}, a)$. The more general formulation we have done is interesting only when f is fixed and a moves.

§**4.6.35** Let A be a norm-unital normed complex algebra. We denote by A^K the closed and convex subset of A given by

$$A^K := \{x \in A : V(A, \mathbf{1}, x) \subseteq K\}.$$

Proposition 4.6.36 *Let f be in* $\mathrm{Ea}(K)$, *and let A be a norm-unital complete normed power-associative complex algebra. Then the mapping $x \to f(x)$ from A^K to A is continuous.*

Proof Take a sequence p_n of polynomial functions on K with $p_n \to f$ in $\mathrm{Ea}(K)$. Then $p_n(x) \to f(x)$ uniformly in $x \in A^K$. □

As one could easily have expected, we have the following.

Fact 4.6.37 *Let a be a normal element of a unital non-commutative JB^*-algebra A, and set $K :=$* $\mathrm{co}(\text{J-sp}(A, a))$. *Then we have*

$$V(A, \mathbf{1}, a) = K.$$

Moreover, for $f \in$ $\mathrm{Ea}(K)$, *$f(a)$ has the same meaning in both continuous functional calculus* (cf. Corollary 4.1.72) *and geometric functional calculus at a.*

Proof Let B stand for the closed $*$-subalgebra of A generated by a. Then, by Proposition 4.1.28(ii), we have $\mathrm{co}(\text{J-sp}(B, a)) = \mathrm{co}(\text{J-sp}(A, a)) = K$. Moreover, by Fact 3.4.22, B is a commutative C^*-algebra, and then Corollaries 2.1.2 and 2.3.73 apply to get

$$V(A, \mathbf{1}, a) = V(B, \mathbf{1}, a) = \mathrm{co}(\mathrm{sp}(B, a)) = \mathrm{co}(\text{J-sp}(B, a)).$$

It follows that $V(A, \mathbf{1}, a) = K$, which proves the first conclusion. Now, the second conclusion follows because the mappings $f \to f_{|\text{J-sp}(A,a)}$ from $\mathrm{Ea}(K)$ to

$C^{\mathbb{C}}(\text{J-sp}(A,a))$, and $g \to g(a)$ from $C^{\mathbb{C}}(\text{J-sp}(A,a))$ to A, are contractive unit-preserving algebra homomorphisms, so the mapping $f \to f_{|\text{J-sp}(A,a)}(a)$ from $\text{Ea}(K)$ to A is a contractive unit-preserving algebra homomorphism taking the generator u to a, and so we must have $f_{|\text{J-sp}(A,a)}(a) = f(a)$ for every $f \in \text{Ea}(K)$, in view of Proposition 4.6.32. □

Let K be a non-empty compact and convex subset of $\mathbb{C}$. Then regarding $\text{Ea}(K)$ as an algebra of complex-valued continuous functions on $\mathbb{K}$ as assured by Fact 4.6.31, we have the following.

Fact 4.6.38 $\text{Ea}(K)$ *contains the restriction to K of any complex-valued holomorphic function on some open neighbourhood of K.*

Proof Let Ω be an open neighbourhood of K. Since $\text{sp}(\text{Ea}(K),u) = K$ (by Theorem 4.6.21(iv)), for $f \in \mathscr{H}(\Omega)$, we can consider the element $f(u) \in \text{Ea}(K)$ (in the sense of the holomorphic functional calculus). Then, since the mapping $g \to g$ from $\text{Ea}(K)$ to $C^{\mathbb{C}}(K)$ is a continuous unit-preserving algebra homomorphism, it is enough to apply Corollaries 1.3.16 and 1.3.18 to get

$$f_{|K} = f(u) \in \text{Ea}(K) \tag{4.6.16}$$

for every $f \in \mathscr{H}(\Omega)$. □

Now recall that, according to Theorem 4.1.93, if A is a complete normed unital non-commutative Jordan complex algebra, if Ω is a non-empty open set in $\mathbb{C}$, and if f is in $\mathscr{H}(\Omega)$, then $A_\Omega := \{x \in A : \text{J-sp}(A,x) \subseteq \Omega\}$ is a non-empty open subset of A, and the mapping $\tilde{f} : x \to f(x)$ from A_Ω to A (where $f(x)$ should be understood in the sense of the holomorphic functional calculus stated in Theorem 4.1.88) is holomorphic. With Fact 4.6.38 and these ideas in mind, we can formulate one of the main results in the present subsection.

Theorem 4.6.39 *Let A be a norm-unital complete normed non-commutative Jordan complex algebra, let a be in A, and set $K := V(A,\mathbf{1},a)$. We have:*

(i) $\text{J-sp}(A,a) \subseteq K$.

(ii) *If Ω is an open subset of $\mathbb{C}$ containing K, and if f is in $\mathscr{H}(\Omega)$, then:*

  (a) *$f(a)$ has the same meaning in both holomorphic functional calculus and geometric functional calculus* (cf. Proposition 4.6.32).

  (b) *$\|D\tilde{f}(a)\| \leqslant \|f'{}_{|K}\|$, where $D\tilde{f}(a)$ denotes the Fréchet derivative of the mapping $\tilde{f} : A_\Omega \to A$ at a, and the norm of $f'{}_{|K}$ is taken in* $\text{Ea}(K)$.

§**4.6.40** Assertion (i) in Theorem 4.6.39 follows by arguing as in the proof of Lemma 2.3.21, with Fact 4.4.25(i) instead of Proposition 1.1.40.

To prove assertion (ii)(a), note that, in view of (4.6.16) and Proposition 4.6.32, the mappings $f \to f_{|K}$ from $\mathscr{H}(\Omega)$ to $\text{Ea}(K)$, and $g \to g(a)$ from $\text{Ea}(K)$ to A, are continuous unit-preserving algebra homomorphisms. Therefore the mapping $f \to f_{|K}(a)$ from $\mathscr{H}(\Omega)$ to A is a continuous unit-preserving algebra homomorphism taking the inclusion $\Omega \hookrightarrow \mathbb{C}$ to a, and hence we must have $f_{|K}(a) = f(a)$ for every $f \in \mathscr{H}(\Omega)$, in view of Theorem 4.1.88.

Assertion (ii)(b) is much deeper, and needs several auxiliary results.

Let X and Y be normed spaces over $\mathbb{K}$, and let T be a partially defined linear operator from X to Y. We recall that T is said to be *closeable* if, whenever x_n is a sequence in $\mathrm{dom}(T)$ with $x_n \to 0$ and $T(x_n) \to y \in Y$, we have $y = 0$. If T is closeable, then the closure of the graph of T in $X \times Y$ becomes the graph of a closed partially defined linear operator from X to Y, which is called the *closure* of T and is denoted by $\bar{T}$. Thus, when T is closeable, $\mathrm{dom}(\bar{T})$ consists precisely of those elements $x \in X$ such that $x = \lim x_n$ for some sequence x_n in $\mathrm{dom}(T)$ such that the sequence $T(x_n)$ is convergent in Y, and for such an x we have $\bar{T}(x) = \lim T(x_n)$.

From now until Proposition 4.6.56, K will stand for a compact and convex subset of $\mathbb{C}$ with more than one point.

Proposition 4.6.41 *Let u stand* (as always in this subsection) *for the generator of* $\mathrm{Ea}(K)$. *Then the mapping $p(u) \to p'(u)$ from $\mathscr{B} := \{p(u) : p \in \mathbb{C}[\mathbf{x}]\}$ to* $\mathrm{Ea}(K)$ *becomes a (well-defined) closeable densely defined derivation of* $\mathrm{Ea}(K)$.

Proof Keeping in mind properties (iii) and (iv) in Theorem 4.6.21, only closeability merits a proof. Let p_n be a sequence in $\mathbb{C}[\mathbf{x}]$ with $p_n(u) \to 0$ and ${p_n}'(u) \to f \in \mathrm{Ea}(K)$, both convergences being regarded in $\mathrm{Ea}(K)$. Let z, w be different elements of K, and consider the complex-valued continuous functions g_n and h on the real interval $[0,1]$ defined by

$$g_n(t) := p_n(z + t(w - z)) \ \text{ and } \ h(t) := (w - z) f(z + t(w - z)),$$

respectively. Then, by Fact 4.6.31, we have $g_n \to 0$ and ${g_n}' \to h$ uniformly on $[0,1]$. Therefore, for $0 \leqslant r \leqslant 1$ we obtain that

$$0 \leftarrow g_n(r) - g_n(0) = \int_0^r {g_n}'(t)\,dt \to \int_0^r h(t)\,dt\,.$$

Hence $h = 0$, so $(w - z) f(z) = h(0) = 0$, and so $f = 0$ because z is arbitrary in K. □

Definition 4.6.42 The closure of the closeable densely defined derivation of $\mathrm{Ea}(K)$ in the above proposition is clearly a closed densely defined derivation of $\mathrm{Ea}(K)$, which will be called *the canonical derivation* of $\mathrm{Ea}(K)$. Its domain will be denoted by $\mathrm{Ea}^1(K)$, and, for $f \in \mathrm{Ea}^1(K)$, f' will denote the image of f under the canonical derivation. This notation does not involve any confusion because, as we will prove later, every function $f \in \mathrm{Ea}^1(K)$ is of class C^1 on K, and its derivative in the classical sense coincides with the image of f under the canonical derivation of $\mathrm{Ea}(K)$ (see Remark 4.6.53(c)).

The following fact is straightforward.

Fact 4.6.43 $\mathrm{Ea}^1(K)$ *endowed with the norm $\|f\|^1 := \|f\| + \|f'\|$ becomes a norm-unital complete normed algebra containing densely the algebra of all polynomial functions on K.*

Let A be a power-associative algebra. An A-bimodule X is said to be a *power-associative A-bimodule* if the split null X-extension of A is a power-associative algebra.

The next lemma is straightforwardly realized by induction.

Lemma 4.6.44 *Let A be a power-associative algebra, let X be a power-associative A-bimodule, let a be in A, let n be in $\mathbb{N}$, and let π_1 and π_2 stand for the natural projections from the split null X-extension $A\times X$ of A to A and X, respectively. We have:*

(i) $\pi_1[(a,x)^n]=a^n$ *for every* $x\in X$.
(ii) *The mapping* $x\to\pi_2[(a,x)^n]$ *from X to X is linear.*

Lemma 4.6.45 *Let A be a norm-unital complete normed power-associative complex algebra, let X be a complete normed unital power-associative A-bimodule, let (a,x) be in the complete normed split null X-extension $A\times X$ of A, and let π_1 and π_2 stand for the natural projections from $A\times X$ to A and X, respectively. Then we have:*

(i) $\|\pi_2[(a,x)^n]\|\leqslant n\|a\|^{n-1}\|x\|$ *for every* $n\in\mathbb{N}$.
(ii) $\|\pi_2[\exp(a,x)]\|\leqslant e^{\max\Re(V(A,\mathbf{1},a))}\|x\|$.
(iii) *If* $V(A,\mathbf{1},a)\subseteq K$, *then* $\|\pi_2[\exp(z(a,x))]\|\leqslant|z|\|x\|\omega_K(z)$ *for every* $z\in\mathbb{C}$.

Proof Assertion (i) follows easily by induction.

As a consequence of assertion (i), we get that $\|\pi_2[\exp(a,x)]\|\leqslant e^{\|a\|}\|x\|$. Taking $r\in\mathbb{R}\setminus\{0\}$, setting $a+\frac{1}{r}\mathbf{1}$ instead of a in the last inequality, and keeping in mind that

$$\left(a+\frac{1}{r}\mathbf{1},x\right)=(a,x)+\frac{1}{r}(\mathbf{1},0)$$

(which implies that $\exp(a+\frac{1}{r}\mathbf{1},x)=e^{\frac{1}{r}}\exp(a,x)$), we derive that

$$\|\pi_2[\exp(a,x)]\|\leqslant e^{\frac{\|\mathbf{1}+ra\|-1}{r}}\|x\|.$$

Hence, letting $r\to 0^+$, it is enough to invoke Proposition 2.1.5 to obtain the inequality in assertion (ii).

Assume that $V(A,\mathbf{1},a)\subseteq K$. Then, by assertion (ii), for $z\in\mathbb{C}$ we have

$$\begin{aligned}\|\pi_2[\exp(z(a,x))]\|&\leqslant|z|\|x\|e^{\max\{\Re(wz):w\in V(A,\mathbf{1},a)\}}\\&\leqslant|z|\|x\|e^{\max\{\Re(wz):w\in K\}}=|z|\|x\|\omega_K(z),\end{aligned}$$

which proves assertion (iii). $\square$

Now we prove the key tool in the proof of Theorem 4.6.39.

Proposition 4.6.46 *Let A be a norm-unital complete normed power-associative complex algebra, let a be in A^K, and let X be a complete normed unital power-associative A-bimodule. Then, for each $x\in X$, there exists a unique continuous unit-preserving algebra homomorphism $f\to f(a,x)$, from $(\mathrm{Ea}^1(K),\|\cdot\|^1)$ to the complete normed split null X-extension $A\times X$ of A, taking the generator u to (a,x). Moreover, denoting by π_1 and π_2 the natural projections from $A\times X$ to A and X, respectively, for $f\in\mathrm{Ea}^1(K)$ we have:*

(i) $\pi_1(f(a,x))=f(a)$ *for every $x\in X$, where $f(a)$ should be understood in the sense of the geometric functional calculus* (cf. Proposition 4.6.32).
(ii) *The mapping* $d\hat{f}(a):x\to\pi_2(f(a,x))$ *from X to X is linear and continuous with* $\|d\hat{f}(a)\|\leqslant\|f'\|$.

Proof Let x be in X, and let θ be in X' with $\|\theta\| \leqslant 1$. Then, the mapping

$$h : z \to \theta(\pi_2[\exp(z(a,x))])$$

is an entire function vanishing at zero, and hence, by Lemma 4.6.45(iii), the mapping $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(0) := h'(0) \ \text{ and } \ g(z) := \frac{h(z)}{z} \ \text{ for } \ z \neq 0$$

belongs to $D(K)$ and $\|g\| \leqslant \|x\|$. Moreover, we have that

$$g^{(k-1)}(0) = \theta\left[\pi_2\left(\frac{(a,x)^k}{k}\right)\right] \ \text{ for every } \ k \in \mathbb{N}.$$

Now, let $p(\mathbf{x}) = \sum_{k=0}^{n} \lambda_k \mathbf{x}^k$ be in $\mathbb{C}[\mathbf{x}]$. Then, regarding $\mathrm{Ea}(K)$ as in §4.6.27, we have

$$\begin{aligned}\theta[\pi_2(p(a,x))] &= \sum_{k=1}^{n} \lambda_k k \theta\left[\pi_2\left(\frac{(a,x)^k}{k}\right)\right] = \sum_{k=1}^{n} \lambda_k k g^{(k-1)}(0) \\ &= \sum_{k=1}^{n} \lambda_k k [u^{k-1}(g)](0) = \left[\left(\sum_{k=1}^{n} \lambda_k k u^{k-1}\right)(g)\right](0) \\ &= \left[p'(u)(g)\right](0),\end{aligned}$$

and hence, invoking (4.6.8), we get

$$|\theta[\pi_2(p(a,x))]| = \left|[p'(u)(g)](0)\right| \leqslant \left\|p'(u)(g)\right\| \leqslant \left\|p'(u)\right\| \|g\| \leqslant \left\|p'(u)\right\| \|x\|.$$

Since θ is an arbitrary element of the closed unit ball of X', we deduce that

$$\|\pi_2(p(a,x))\| \leqslant \left\|p'(u)\right\| \|x\|. \tag{4.6.17}$$

Therefore, invoking Lemma 4.6.44(i), we obtain

$$\begin{aligned}\|p(a,x)\| &= \|\pi_1(p(a,x))\| + \|\pi_2(p(a,x))\| \\ &= \|p(a)\| + \|\pi_2(p(a,x))\| \leqslant \|p(u)\| + \left\|p'(u)\right\| \|x\| \\ &\leqslant \max\{1, \|x\|\}\left(\|p(u)\| + \left\|p'(u)\right\|\right) = \max\{1, \|x\|\}\|p(u)\|^1.\end{aligned}$$

This shows that the algebra homomorphism $p(u) \to p(a,x)$, from the dense subalgebra $\mathscr{B} := \{p(u) : p \in \mathbb{C}[\mathbf{x}]\}$ of $(\mathrm{Ea}^1(K), \|\cdot\|^1)$ to A, is continuous. By extending this homomorphism by continuity, we find a continuous unit-preserving algebra homomorphism from $(\mathrm{Ea}^1(K), \|\cdot\|^1)$ to A taking u to (a,x). It follows from Lemma 1.1.82(i) that there is no other such homomorphism. Thus the first conclusion in the proposition has been proved.

Let x be in X, and let f be in $\mathrm{Ea}^1(K)$. Take a sequence p_n in $\mathbb{C}[\mathbf{x}]$ with $p_n(u) \to f$ in $(\mathrm{Ea}^1(K), \|\cdot\|^1)$. Then

$$p_n(u) \to f \text{ in } \mathrm{Ea}(K), {p_n}'(u) \to f' \text{ in } \mathrm{Ea}(K), \text{ and } p_n(a,x) \to f(a,x) \text{ in } A \times X.$$

Therefore, invoking Lemma 4.6.44(i), Proposition 4.6.32, and (4.6.17), we get

$$\pi_1(f(a,x)) = \lim \pi_1(p_n(a,x)) = \lim p_n(a) = f(a)$$

and

$$\|\pi_2(f(a,x))\| = \lim \|\pi_2(p_n(a,x))\| \leqslant \lim \left\|{p_n}'(u)\right\| \|x\| = \|f'\|\|x\|.$$

Thus the second assertion in the proposition will hold as soon as we realize that $\pi_2(f(a,x))$ depends linearly on x. But this follows from the fact that, by Lemma 4.6.44(ii), $\pi_2(p_n(a,x))$ depends linearly on x for every n. □

Te following refinement of Fact 4.6.38 will be useful.

Fact 4.6.47 *Let Ω be an open neighbourhood of K in $\mathbb{C}$, and let f be in $\mathscr{H}(\Omega)$. Then $f_{|K}$ belongs to* $\mathrm{Ea}^1(K)$ *and $f_{|K}{}' = f'{}_{|K}$.*

Proof Since the canonical derivation $g \to g'$ of $\mathrm{Ea}(K)$ is densely defined and closed, Corollary 4.6.20 applies, so that we have that $f(u)$ lies in $\mathrm{Ea}^1(K)$ and that $(f(u))' = f'(u)$, both $f(u)$ and $f'(u)$ being understood in the sense of the holomorphic functional calculus. Since $f(u) = f_{|K}$ and $f'(u) = f'{}_{|K}$ (by (4.6.16)), the result follows. □

End of the proof of Theorem 4.6.39 In view of §4.6.40, it is enough to prove assertion (ii)(b) in the theorem. Let A, a, and

$$K := V(A, \mathbf{1}, a) \supseteq \text{J-sp}(A, a)$$

be as in the theorem. Let Ω be an open neighbourhood of K in $\mathbb{C}$, and let f be in $\mathscr{H}(\Omega)$. We must show that $\|D\tilde{f}(a)\| \leqslant \|f'{}_{|K}\|$. If K reduces to a point (say λ), then, by Corollary 2.1.11, we have that $a = \lambda\mathbf{1}$, and hence, by Theorem 4.1.93, we obtain $D\tilde{f}(a) = f'(\lambda)I_A$, which proves the result in this case. Assume that K is not reduced to a point. Let b be in A, and let us consider the element (a,b) of the complete normed split null A-extension of A. Keeping in mind Fact 4.6.47, we can argue as in §4.6.40 to realize that the symbol $f(a,b)$ has the same meaning in both the holomorphic functional calculus (cf. Proposition 4.6.2) and the functional calculus suggested by Proposition 4.6.46. But, by Theorem 4.1.93 and Proposition 4.6.4, in the meaning of the holomorphic functional calculus we have $\pi_2(f(a,b)) = D\tilde{f}(a)(b)$. Therefore, by Proposition 4.6.46(ii), we have $D\tilde{f}(a)(b) = d\hat{f}(a)(b)$ and then $\|D\tilde{f}(a)\| = \|d\hat{f}(a)\| \leqslant \|f'{}_{|K}\|$ because b is arbitrary in A. □

Now that Theorem 4.6.39 has been proved, we proceed to discuss other relevant consequences of Proposition 4.6.46. For the first one, Fact 4.6.48 immediately below becomes crucial. To prove this fact, we recall that *an algebra A over $\mathbb{K}$ is power-associative if (and only if) it is third and fourth power associative; that is, A satisfies the identities*

$$[a,a,a] = 0 \quad \textit{and} \quad [a^2,a,a] = 0. \tag{4.6.18}$$

This result, whose proof can be found in [11] or [471], should be kept in mind for the proof of the following.

Fact 4.6.48 *Let A be a power-associative algebra over $\mathbb{K}$. Then the regular A-bimodule is a power-associative A-bimodule.*

Proof As already commented in §2.5.2, identities (4.6.18) can be linearized to get

$$[b,a,a] + [a,b,a] + [a,a,b] = 0 \tag{4.6.19}$$

and

$$[ab+ba,a,a]+[a^2,b,a]+[a^2,a,b]=0. \tag{4.6.20}$$

In view of the identities (4.6.18), (4.6.19), and (4.6.20), for (a,x) in the split null A-extension of A we see that

$$[(a,x),(a,x),(a,x)] = ([a,a,a],[x,a,a]+[a,x,a]+[a,a,x]) = (0,0)$$

and

$$\begin{aligned}[(a,x)^2,(a,x),(a,x)] &= [(a^2,ax+xa),(a,x),(a,x)]\\ &= ([a^2,a,a],[ax+xa,a,a]+[a^2,x,a]+[a^2,a,x]) = (0,0).\end{aligned}$$

Therefore, the split null A-extension of A is a power-associative algebra, i.e. the regular A-bimodule is a power-associative A-bimodule. □

The above fact should be kept in mind for the formulation and proof of the following.

Lemma 4.6.49 *Let A be a normed power-associative unital algebra over $\mathbb{K}$, let a be in A, and let p be in $\mathbb{K}[\mathbf{x}]$. Then the mapping $x \to p(x)$ from A to A is differentiable at a with Fréchet derivative equal to the mapping $x \to \pi_2(p(a,x))$, where π_2 stands for the natural projection from the split null A-extension of A onto the regular A-module.*

Proof We may assume that $p(\mathbf{x}) = \mathbf{x}^n$ for some $n \in \mathbb{N}$, and then we argue by induction on n. Let x be in A, and set $F_n(x) := \pi_2((a,x)^n)$. Then, by Lemma 4.6.44, F depends linearly on x, and we have

$$\begin{aligned}\left(a^{n+1},F_{n+1}(x)\right) &= (a,x)^{n+1} = (a,x)^n(a,x)\\ &= (a^n,F_n(x))\,(a,x) = \left(a^{n+1},a^nx+F_n(x)a\right),\end{aligned}$$

i.e. $F_{n+1}(x) = a^nx + F_n(x)a$. Replacing x with $x-a$, the last equality can be written as

$$x^{n+1}-a^{n+1}-F_{n+1}(x-a) = (x^n-a^n)(x-a)+[x^n-a^n-F_n(x-a)]a.$$

Now let n be in $\mathbb{N}$ such that

$$\lim_{x\to a}\frac{\|x^n-a^n-F_n(x-a)\|}{\|x-a\|}=0.$$

Then, since the mapping $x \to x^n$ is continuous, we derive that

$$\begin{aligned}0 &\leqslant \lim_{x\to a}\frac{\|x^{n+1}-a^{n+1}-F_{n+1}(x-a)\|}{\|x-a\|}\\ &\leqslant \lim_{x\to a}\left[\|x^n-a^n\|+\|a\|\frac{\|x^n-a^n-F_n(x-a)\|}{\|x-a\|}\right]=0.\end{aligned}$$

Since the case $n=1$ is clear, the proof is complete. □

Let A be a norm-unital normed complex algebra. We note that, since K has two distinct points (say λ,μ), the convex subset A^K of A introduced in §4.6.35 has two distinct elements (namely $\lambda\mathbf{1},\mu\mathbf{1}$), and hence A^K is a perfect subset of A. On the other hand, if in addition A is power-associative and complete, then, in view of Fact 4.6.48, Proposition 4.6.46 and its notation are applicable when X equals the regular A-bimodule. Keeping these ideas in mind, we have the following.

Proposition 4.6.50 *Let A be a norm-unital complete normed power-associative complex algebra, let a be in A^K, and let f be in* $\mathrm{Ea}^1(K)$. *Then*

$$\lim_{\substack{x\to a\\ x\in A^K\setminus\{a\}}} \frac{\|f(x)-f(a)-d\hat{f}(a)(x-a)\|}{\|x-a\|}=0,$$

where $f(\cdot)$ should be understood in the sense of the geometric functional calculus (cf. Proposition 4.6.32).

Proof According to Fact 4.6.43, take a sequence p_n of polynomial functions on K converging to f in $(\mathrm{Ea}^1(K),\|\cdot\|^1)$. Then $p_n(x)\to f(x)$ in A for every $x\in A^K$, whereas ${p_n}'\to f'$ in $\mathrm{Ea}(K)$, and, by Proposition 4.6.46, $d\hat{p}_n(x)\to d\hat{f}(x)$ in $BL(A)$ for every $x\in A^K$. Let n,m be in $\mathbb{N}$, and let x be in A^K. Then, by Lemma 4.6.49, the function

$$\Phi : t\to [p_n-p_m-(d\hat{p}_n(a)-d\hat{p}_m(a))]\,(a+t(x-a))$$

from $[0,1]$ to A is derivable in $[0,1]$ with

$$\Phi'(t)=[d\hat{p}_n(a+t(x-a))-d\hat{p}_m(a+t(x-a))-(d\hat{p}_n(a)-d\hat{p}_m(a))]\,(x-a).$$

Therefore, by the classical mean value theorem and Proposition 4.6.46(ii), we have

$$\begin{aligned}
&\|p_n(x)-p_n(a)-(p_m(x)-p_m(a))-(d\hat{p}_n(a)-d\hat{p}_m(a))(x-a)\|\\
&\qquad=\|\Phi(1)-\Phi(0)\|\leqslant 2\|x-a\|\sup_{0\leqslant t\leqslant 1}\|d(\hat{p}_n-\hat{p}_m)(a+t(x-a))\|\\
&\qquad\leqslant 2\|x-a\|\|{p_n}'-{p_m}'\|.
\end{aligned}$$

By letting $n\to\infty$, we derive that

$$\begin{aligned}
&\frac{\|f(x)-f(a)-d\hat{f}(a)(x-a)\|}{\|x-a\|}\\
&\qquad\leqslant 2\|f'-{p_m}'\|+\frac{\|p_m(x)-p_m(a)-d\hat{p}_m(a)(x-a)\|}{\|x-a\|}
\end{aligned}$$

whenever $x\neq a$. Now let $\varepsilon>0$. We may choose m such that $\|f'-{p_m}'\|<\frac{\varepsilon}{3}$, and then, according to Lemma 4.6.49, there is $\delta>0$ such that

$$\frac{\|p_m(x)-p_m(a)-d\hat{p}_m(a)(x-a)\|}{\|x-a\|}<\frac{\varepsilon}{3}$$

whenever $0<\|x-a\|<\delta$. It follows that

$$\frac{\|f(x)-f(a)-d\hat{f}(a)(x-a)\|}{\|x-a\|}<\varepsilon$$

whenever $0<\|x-a\|<\delta$. □

Proposition 4.6.50 just proved can be seen as a geometric variant of Theorem 4.1.93.

Corollary 4.6.51 *Let A be a norm-unital complete normed power-associative complex algebra, let a,b be in A^K, and let f be in* $\mathrm{Ea}^1(K)$. *Then*

$$\|f(b)-f(a)\| \leqslant \|f'\|\|b-a\|.$$

Proof In view of Proposition 4.6.50, the mapping $\phi : [0,1] \to A$ defined by $\phi(t) := f(a+t(b-a))$ is derivable with $\phi'(t) = d\hat{f}(a+t(b-a))(b-a)$. Therefore, by the classical mean value theorem and Proposition 4.6.46(ii), we have

$$\|f(b)-f(a)\| = \|\phi(1)-\phi(0)\| \leqslant \|f'\|\|b-a\|. \qquad \square$$

As a by-product of Lemma 4.6.49, we have the following.

Corollary 4.6.52 *Assume that K has non-empty interior in $\mathbb{C}$, let f be in* $\mathrm{Ea}(K)$, *let A be a norm-unital complete normed power-associative complex algebra, and let U stand for the interior of A^K in A. Then U is not empty, and the mapping $x \to f(x)$ from U to A is holomorphic.*

Proof It is plain that $\lambda\mathbf{1}$ lies in U whenever λ is in $\overset{\circ}{K}$ (indeed, if $\lambda + \varepsilon\mathbb{B}_{\mathbb{C}} \subseteq K$, then we have $\lambda\mathbf{1} + \varepsilon\mathbb{B}_A \subseteq A^K$). Thus U is not empty. Take a sequence p_n of polynomial functions on K with $p_n \to f$ in $\mathrm{Ea}(K)$. Then $p_n(x) \to f(x)$ uniformly in $x \in A^K$. Since the mapping $x \to p_n(x)$ is holomorphic on U (by Lemma 4.6.49), the result follows from the generalized Weierstrass convergence theorem (see for example [707, Theorem 16.12]). $\square$

Remark 4.6.53 (a) Let f be in $\mathrm{Ea}^1(K)$, let A be a norm-unital complete normed associative and commutative complex algebra, let a be in A^K, and let X be a complete normed unital associative and commutative A-bimodule (cf. §4.6.19). Then the mapping

$$d\hat{f}(a) : X \to X$$

in Proposition 4.6.46(ii) is nothing other than the operator of left (= right) multiplication by $f'(a)$ on X. Indeed, given $x \in X$, prove by induction that in the split null X-extension of A we have $(a,x)^n = (a^n, na^{n-1}x)$ for every $n \in \mathbb{N}$, and then apply Fact 4.6.43 to get that $f(a,x) = (f(a), f'(a)x)$, which implies that $d\hat{f}(a) = \pi_2(f(a,x)) = f'(a)x$.

(b) Let f be in $\mathrm{Ea}^1(K)$, let A be a norm-unital complete normed associative and commutative complex algebra, and let a be in A^K. Keeping in mind that the regular A-bimodule is associative and commutative, it follows from Proposition 4.6.50, and part (a) of the present remark, that

$$\lim_{\substack{x\to a \\ x\in A^K\setminus\{a\}}} \frac{\|f(x)-f(a)-f'(a)(x-a)\|}{\|x-a\|} = 0.$$

(c) Let f be in $\mathrm{Ea}^1(K)$. By taking $A := \mathbb{C}$ and $a := \lambda \in K$ in part (b) of the present remark, we realize that f is of class C^1 on K, and that the derivative of f in the classical meaning coincides with the image of f under the canonical derivation of $\mathrm{Ea}(K)$.

As in the case of complex-valued continuous functions on an interval, each function in $\mathrm{Ea}(K)$ has a 'primitive' (now relative to the canonical derivation of $\mathrm{Ea}(K)$). Indeed, we have the following.

Proposition 4.6.54 *The kernel of the canonical derivation of* $\mathrm{Ea}(K)$ *reduces to* $\mathbb{C}\mathbf{1}$, *and its range is the whole algebra* $\mathrm{Ea}(K)$. *More precisely, given* $\lambda \in K$ *and* $f \in \mathrm{Ea}(K)$, *there exists a unique* $g = \mathscr{I}_\lambda(f) \in \mathrm{Ea}^1(K)$ *such that* $g' = f$ *and* $g(\lambda) = 0$. *Moreover the operator* $\mathscr{I}_\lambda : \mathrm{Ea}(K) \to \mathrm{Ea}(K)$ *is linear and continuous, and we have*

$$\|\mathscr{I}_\lambda\| = \|u - \lambda\mathbf{1}\|,$$

where u *stands for the generator of* $\mathrm{Ea}(K)$.

Proof Let $f \in \mathrm{Ea}^1(K)$ be in the kernel of the canonical derivation of $\mathrm{Ea}(K)$. It follows from Remark 4.6.53(c) and the convexity of K that f is a constant function. Thus the kernel of the canonical derivation of $\mathrm{Ea}(K)$ reduces to $\mathbb{C}\mathbf{1}$.

Let λ be in K, and let f be in $\mathrm{Ea}(K)$. Following on from the above paragraph, the uniqueness of $g \in \mathrm{Ea}^1(K)$ such that $g' = f$ and $g(\lambda) = 0$ is clear. To prove the existence of such a function g, let us begin by considering the case where $f = p$ on K for some $p \in \mathbb{C}[\mathbf{x}]$. Let q be the unique polynomial in $\mathbb{C}[\mathbf{x}]$ such that $q' = p$ and $q(\lambda) = 0$. Since $V(\mathrm{Ea}(K), \mathbf{1}, \lambda\mathbf{1} + t(u - \lambda\mathbf{1})) \subseteq K$ for every $t \in [0, 1]$, Theorem 4.6.21 applies, so that we have $\|p(\lambda\mathbf{1} + t(u - \lambda\mathbf{1}))\| \leqslant \|p(u)\|$. Therefore, since

$$\begin{aligned}(u - \lambda\mathbf{1}) \int_0^1 p(\lambda\mathbf{1} + t(u - \lambda\mathbf{1}))dt &= \Big[q(\lambda\mathbf{1} + t(u - \lambda\mathbf{1}))\Big]_0^1 \\ &= q(u) - q(\lambda\mathbf{1}) = q(u) - q(\lambda)\mathbf{1} = q(u),\end{aligned}$$

we obtain that $\|q(u)\| \leqslant \|p(u)\|\|u - \lambda\mathbf{1}\|$. Therefore the linear mapping $\mathscr{I}_\lambda : p(u) \to q(u)$ from the dense subalgebra of $\mathrm{Ea}(K)$ consisting of all polynomial functions on K into $\mathrm{Ea}(K)$ is continuous, and hence we can consider its continuous extension to $\mathrm{Ea}(K)$, which is also denoted by $\mathscr{I}_\lambda$. Clearly, for each $f \in \mathrm{Ea}(K)$ we have $\mathscr{I}_\lambda(f) \in \mathrm{Ea}^1(K)$, $\mathscr{I}_\lambda(f)' = f$, $\mathscr{I}_\lambda(f)(\lambda) = 0$, and $\|\mathscr{I}_\lambda(f)\| \leqslant \|u - \lambda\mathbf{1}\|\|f\|$. Since $\mathscr{I}_\lambda(\mathbf{1}) = u - \lambda\mathbf{1}$, we have in fact the equality $\|\mathscr{I}_\lambda\| = \|u - \lambda\mathbf{1}\|$. □

Corollary 4.6.55 *A function* $f : K \to \mathbb{C}$ *lies in* $\mathrm{Ea}^1(K)$ *if and only if it is of class* C^1 *on* K *and its classical derivative* f' *belongs to* $\mathrm{Ea}(K)$. *If this is the case, then* f' *coincides with the image of* f *under the canonical derivation of* $\mathrm{Ea}(K)$.

Proof The 'only if' part follows from Remark 4.6.53(c).

Assume that f is of class C^1 on K and that $f' \in \mathrm{Ea}(K)$. Then, by the last requirement and Proposition 4.6.54, there exists $g \in \mathrm{Ea}^1(K)$ such that $g' = f'$, where g' stands for the image of g under the canonical derivation of $\mathrm{Ea}(K)$. Moreover, by Remark 4.6.53(c), g is of class C^1 on K, and g' has an unambiguous meaning. Therefore, since f is of class C^1 on K, $f - g$ is a function of class C^1 on K with zero derivative, which implies that $f - g$ is constant on K because K is convex. Finally, since $g \in \mathrm{Ea}^1(K)$, we derive that $f \in \mathrm{Ea}^1(K)$. □

The following application of Proposition 4.6.46 becomes a geometric variant of Theorem 4.6.15.

Proposition 4.6.56 *Let A be a norm-unital complete normed power-associative complex algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in* $\mathrm{dom}(D) \cap A^K$, *and let f be in* $\mathrm{Ea}^1(K)$. *Then $f(a)$ (in the sense of the geometric functional calculus) lies in* $\mathrm{dom}(D)$. *Moreover the equality*

$$D(f(a)) = d\hat{f}(a)(D(a))$$

holds, and hence $\|D(f(a))\| \leqslant \|f'\|\|D(a)\|$, where the norm of f' is taken in $\mathrm{Ea}(K)$.

Proof The graph (say G) of D is a closed subalgebra of the complete normed split null X-extension $A \times X$ of A. Moreover, by Lemma 4.6.13, G contains the unit of $A \times X$. Therefore, since $\mathrm{Ea}^1(K)$ is generated by $\{\mathbf{1}, u\}$ as a normed algebra (by Fact 4.6.43) and $(a, D(a))$ belongs to G, it follows from the first conclusion in Proposition 4.6.46 and Lemma 1.1.82(ii) that $f(a, D(a))$ lies in G. Therefore, by the whole second conclusion in Proposition 4.6.46, we have

$$f(a) \in \mathrm{dom}(D), \ \ D(f(a)) = d\hat{f}(a)(D(a)), \ \text{ and } \ \|D(f(a))\| \leqslant \|f'\|\|D(a)\|,$$

as desired. □

By combining Proposition 4.6.56 and Fact 4.6.37, we get the following.

Corollary 4.6.57 *Let A be a unital non-commutative JB^*-algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be a normal element in* $\mathrm{dom}(D) \setminus \mathbb{C}\mathbf{1}$, *set* $K := \mathrm{co}(\text{J-sp}(A, a))$, *and let f be in* $\mathrm{Ea}^1(K)$. *Then $f(a)$ (in the sense of the continuous functional calculus) lies in* $\mathrm{dom}(D)$ *and we have $\|D(f(a))\| \leqslant \|f'\|\|D(a)\|$, where the norm of f' is taken in* $\mathrm{Ea}(K)$.

To progress with the remaining part of the present subsection, let us prove the following.

Proposition 4.6.58 *Let A be a unital commutative C^*-algebra, let D be a closed densely defined derivation of A, and let a be in* $\mathrm{dom}(D) \setminus \mathbb{C}\mathbf{1}$. *Set* $K := \mathrm{co}(\mathrm{sp}(A, a))$, *and let $f : K \to \mathbb{C}$ be a function of class C^1 on K. Then $f(a)$ (in the sense of the continuous functional calculus) lies in* $\mathrm{dom}(D)$ *and $D(f(a)) = f'(a)D(a)$.*

Proof Fix z_0 in K. Since f' is continuous on K and holomorphic on $\overset{\circ}{K}$, it follows from Mergelyan's theorem (see for example [803, Theorem 20.3.1]) that we can find a sequence p_n of polynomial functions on K such that $p_n(z_0) = 0$ and $p_n{}' \to f'$ uniformly on K. Therefore, since for $z \in K$ we have

$$f(z) - p_n(z) = (z - z_0) \int_0^1 \left[f'(z_0 + t(z - z_0)) - p_n{}'(z_0 + t(z - z_0)) \right] dt \,,$$

we derive that

$$\|f - p_n\|_\infty \leqslant \max\{|z - z_0| : z \in K\} \|f' - p_n{}'\|_\infty,$$

and hence that $p_n \to f$ uniformly on K. Now, by Lemma 4.6.13, we have that $p_n(a) \in \mathrm{dom}(D)$ and, by the above,

$$p_n(a) \to f(a) \ \text{ and } \ D(p_n(a)) = p_n{}'(a)D(a) \to f'(a)D(a).$$

Since D is closed, the result follows. □

Let A be a unital C^*-algebra, let D be a closed densely defined derivation of A, let a be a normal element in $\mathrm{dom}(D)$, let f be in $C^{\mathbb{C}}(\mathrm{sp}(A,a))$, and consider $f(a)$ in the sense of the continuous functional calculus. If a is in $\mathbb{C}\mathbf{1}$, then so is $f(a)$, and hence, by Lemma 4.6.13, $f(a) \in \mathrm{dom}(D)$. Assume that $a \notin \mathbb{C}\mathbf{1}$. Then, in general, $f(a)$ need not lie in $\mathrm{dom}(D)$. (Indeed, take $A = C^{\mathbb{C}}([-1,1])$, D equal to the usual derivation of A, $a(t) = t$, and $f(t) = |t|$.) Therefore, one can wonder about conditions on f implying that $f(a) \in \mathrm{dom}(D)$. If A is commutative, then, by Proposition 4.6.58, one such condition is that f be (the restriction to $\mathrm{sp}(A,a)$ of) a function of class C^1 on K, where $K := \mathrm{co}(\mathrm{sp}(A,a))$. Even dispensing the commutativity of A, Corollary 4.6.57 provides us with another such condition. Indeed, if the mapping f is (the restriction to $\mathrm{sp}(A,a)$ of) a function in $\mathrm{Ea}^1(K)$, then certainly $f(a)$ lies in $\mathrm{dom}(D)$. But this naturally raises another question, namely, which complex-valued continuous functions on K lie in $\mathrm{Ea}^1(K)$? A partial answer is provided by Fact 4.6.47 above. In the case where a is self-adjoint, other answers, independent of Fact 4.6.47, are given by Theorem 4.6.60 and Lemma 4.6.66 below.

Fact 4.6.59 *Let K be a closed and bounded interval of $\mathbb{R}$ not reduced to a point, and let $g : K \to \mathbb{C}$ be of class C^1 on K. Then there exists a polynomial $p \in \mathbb{C}[\mathbf{x}]$ of degree $\leqslant 1$, and a continuous function $f : \mathbb{R} \to \mathbb{C}$ whose support is contained in K, and such that the Fourier transform of f lies in $L^1(\mathbb{R})$ and $f + p = g$ on K.*

Proof Write $K = [\alpha, \beta]$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. Take $p \in \mathbb{C}[\mathbf{x}]$ of degree $\leqslant 1$ such that $(g-p)(\alpha) = (g-p)(\beta) = 0$. Set $f = g - p$ on K, and $f = 0$ on $\mathbb{R} \setminus K$. Then, denoting by $\hat{f}$ the Fourier transform of f, for $0 \neq t \in \mathbb{R}$ we have

$$\begin{aligned}
\hat{f}(t) &= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(r) e^{-itr} dr = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\alpha}^{\beta} f(r) e^{-itr} dr \\
&= \frac{1}{(2\pi)^{\frac{1}{2}}} \left[f(r) \frac{e^{-itr}}{-it} \right]_{\alpha}^{\beta} + \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\alpha}^{\beta} f'(r) \frac{e^{-itr}}{it} dr \\
&= \frac{1}{(2\pi)^{\frac{1}{2}} it} \int_{\alpha}^{\beta} f'(r) e^{-itr} dr,
\end{aligned}$$

and hence

$$it\hat{f}(t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\alpha}^{\beta} f'(r) e^{-itr} dr. \tag{4.6.21}$$

Since $f' \in L^2(\mathbb{R})$, it follows from (4.6.21) that $it\hat{f}(t) \in L^2(\mathbb{R})$ (see [803, Theorem 9.13(a)]) and so

$$\begin{aligned}
\int_{-\infty}^{\infty} |\hat{f}(r)| dr &= \int_{-1}^{1} |\hat{f}(r)| dr + \int_{1}^{\infty} |\hat{f}(r)| dr + \int_{-\infty}^{-1} |\hat{f}(r)| dr \\
&= \int_{-1}^{1} |\hat{f}(r)| dr + \int_{1}^{\infty} \left| \frac{1}{r} r\hat{f}(r) \right| dr + \int_{-\infty}^{-1} \left| \frac{1}{r} r\hat{f}(r) \right| dr \\
&\leqslant \int_{-1}^{1} |\hat{f}(r)| dr + \left(\int_{1}^{\infty} \frac{dr}{r^2} \right)^{\frac{1}{2}} \left(\int_{1}^{\infty} |r\hat{f}(r)|^2 dr \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{-\infty}^{-1} \frac{dr}{r^2} \right)^{\frac{1}{2}} \left(\int_{-\infty}^{-1} |r\hat{f}(r)|^2 dr \right)^{\frac{1}{2}} < +\infty.
\end{aligned}$$

□

Theorem 4.6.60 *Let K be a closed and bounded interval of $\mathbb{R}$ not reduced to a point, and for $n \in \mathbb{N}$ let $C^n(K)$ stand for the space of all complex-valued functions on K which are of class C^n on K. Then we have*

$$C^2(K) \subseteq \mathrm{Ea}^1(K) \subseteq C^1(K) \subseteq \mathrm{Ea}(K).$$

Moreover, all inclusions in the above chain are continuous when we consider the topology of the natural complete norm on each link of the chain (cf. Fact 4.6.43 for the case of $\mathrm{Ea}^1(K)$).

Proof The second inclusion has been already proved (cf. Remark 4.6.53(c)).

Now let us prove the last inclusion. Let g be in $C^1(K)$. Consider the polynomial $p \in \mathbb{C}[\mathbf{x}]$ and the function $f : \mathbb{R} \to \mathbb{C}$ given by Fact 4.6.59, let $\hat{f}$ denote the Fourier transform of f, and let u stand for the generator of $\mathrm{Ea}(K)$. Then, keeping in mind Corollary 2.1.9(iii), we have

$$\int_{-\infty}^{+\infty} \|\hat{f}(r)\exp(-iru)\|\, dr = \int_{-\infty}^{+\infty} |\hat{f}(r)|\, dr < +\infty.$$

Therefore, since $\hat{f}$ is continuous, we realize that

$$h := \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \hat{f}(r)\exp(-iru)\, dr \tag{4.6.22}$$

defines an element of $\mathrm{Ea}(K)$. Since $f(t) := \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \hat{f}(r)e^{-tr}dr$ for every $t \in \mathbb{R}$, it is enough to valuate (4.6.22) at points of K, to get that $f = h$ on K. Therefore, since $h \in \mathrm{Ea}(K)$ and $g = f + p$ on K, we conclude that $g \in \mathrm{Ea}(K)$.

Now let us prove the first inclusion. Let f be in $C^2(K)$. By the inclusion $C^1(K) \subseteq \mathrm{Ea}(K)$ just proved, we have that $f' \in \mathrm{Ea}(K)$. It follows from Corollary 4.6.55 that $f \in \mathrm{Ea}^1(K)$.

Now that all inclusions have been proved, the continuity of these inclusions follows from the closed graph theorem by keeping in mind that convergence in the natural norm of each link of the chain implies pointwise convergence. (The continuity of the second inclusion is indeed straightforward because $\|\cdot\| \geqslant \|\cdot\|_\infty$ on $\mathrm{Ea}(K)$.) ☐

The continuity of the inclusion $C^1(K) \subseteq \mathrm{Ea}(K)$ in the above theorem can be specified as follows.

Fact 4.6.61 *Let K be a closed and bounded interval of $\mathbb{R}$ not reduced to a point, and let f be in $C^1(K)$. Then f lies in $\mathrm{Ea}(K)$ and we have $\|f\| \leqslant M(\|f\|_\infty + \ell(K)\|f'\|_\infty)$, where $M > 0$ is a universal constant, and $\ell(K)$ denotes the length of K.*

Proof According to Theorem 4.6.60, let $M > 0$ denote the norm of the continuous inclusion $C^1([0,1]) \subseteq \mathrm{Ea}([0,1])$, so that we have

$$\|g\| \leqslant M(\|g\|_\infty + \|g'\|_\infty) \ \text{ for every } \ g \in C^1([0,1]). \tag{4.6.23}$$

Write $K = [\alpha, \beta]$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$, and consider the mapping $\eta : t \to (\beta - \alpha)t + \alpha$ from $[0,1]$ onto K. Since $f \in \mathrm{Ea}(K)$ (by the last inclusion in Theorem 4.6.60), we derive from Remark 4.6.33 that $f \circ \eta \in \mathrm{Ea}([0,1])$ and that $\|f \circ \eta\| = \|f\|$. On the other hand, the equalities $\|f \circ \eta\|_\infty = \|f\|_\infty$ and

$\|(f \circ \eta)'\|_\infty = (\beta - \alpha)\|f'\|_\infty$ are clear. It follows from the inequality (4.6.23) that $\|f\| \leqslant M(\|f\|_\infty + \ell(K)\|f'\|_\infty)$. □

Through the geometric functional calculus built in Proposition 4.6.32, the inclusion $C^1(K) \subseteq \mathrm{Ea}(K)$ in Theorem 4.6.60 shows that hermitian elements of a norm-unital complete normed power-associative complex algebra admit C^1-functional calculus. This fact must be kept in mind for the formulation of the following.

Corollary 4.6.62 *Let A be a norm-unital complete normed power-associative complex algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in $(\mathrm{dom}(D) \cap H(A, \mathbf{1})) \setminus \mathbb{R}\mathbf{1}$, set $K := V(A, \mathbf{1}, a)$, and let $f : K \to \mathbb{C}$ be of class C^2 on K. Then $f(a)$ (in the sense of the geometric functional calculus) lies in $\mathrm{dom}(D)$. Moreover we have*

$$\|D(f(a))\| \leqslant M\left(\left\|f'\right\|_\infty + \ell(K)\left\|f''\right\|_\infty\right)\|D(a)\|,$$

where $M > 0$ is a universal constant, and $\ell(K)$ denotes the length of K.

Proof The first conclusion in the present corollary follows from the first conclusion in Proposition 4.6.56 and the inclusion $C^2(K) \subseteq \mathrm{Ea}^1(K)$ in Theorem 4.6.60. But Proposition 4.6.56 also assures that $\|D(f(a))\| \leqslant \|f'\|\|D(a)\|$, where the norm of f' is taken in $\mathrm{Ea}(K)$. Therefore, since f' is of class C^1 on K, the second conclusion follows from Fact 4.6.61. □

Now, combining Facts 4.1.67 and 4.6.37, and Corollary 4.6.62, we get the following.

Theorem 4.6.63 *Let A be a unital non-commutative JB^*-algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in $(\mathrm{dom}(D) \cap H(A, *)) \setminus \mathbb{R}\mathbf{1}$, set $K := \mathrm{co}(\text{J-sp}(A, a))$, and let f be of class C^2 on K. Then $f(a)$ (in the sense of the continuous functional calculus) lies in $\mathrm{dom}(D)$, and we have*

$$\|D(f(a))\| \leqslant M\left(\left\|f'\right\|_\infty + \ell(K)\left\|f''\right\|_\infty\right)\|D(a)\|,$$

where $M > 0$ is a universal constant, and $\ell(K)$ denotes the length of K.

When we take in Theorem 4.6.63 A equal to any C^*-algebra (cf. Fact 3.3.2), and X equal to the regular A-bimodule, we obtain the following.

Corollary 4.6.64 *Let A be a unital C^*-algebra, let D be a closed densely defined derivation of A, let a be in $(\mathrm{dom}(D) \cap H(A, *)) \setminus \mathbb{R}\mathbf{1}$, set $K := \mathrm{co}(\mathrm{sp}(A, a))$, and let f be of class C^2 on K. Then $f(a)$ (in the sense of the continuous functional calculus) lies in $\mathrm{dom}(D)$.*

Now we are going to prove interesting variants of results from Corollary 4.6.62 to Corollary 4.6.64.

§**4.6.65** In what follows, f will stand for a complex-valued function on $\mathbb{R}$ of the form

$$f(t) = \int_{-\infty}^{+\infty} e^{irt} h(r)\, dr \tag{4.6.24}$$

for some complex-valued continuous function h on $\mathbb{R}$ such that

$$\int_{-\infty}^{+\infty} |rh(r)|\,dr < +\infty. \tag{4.6.25}$$

We note that such a function f is derivable with

$$f'(t) = i\int_{-\infty}^{+\infty} re^{irt}h(r)\,dr. \tag{4.6.26}$$

Now, since the integral in (4.6.25) becomes a natural upper bound for modules of values of f', we denote it by $|||\, f'\,|||$.

Lemma 4.6.66 *Let K be a closed and bounded interval of $\mathbb{R}$ not reduced to a point, and let f be as in §4.6.65. Then $f_{|K}$ belongs to* $\mathrm{Ea}^1(K)$ *and $f_{|K}{}' = f'{}_{|K}$, where $f_{|K}{}'$ denotes the image of $f_{|K}$ under the canonical derivation of* $\mathrm{Ea}(K)$*. Moreover, we have $\|f_{|K}{}'\| \leqslant |||\, f'\,|||$.*

Proof Recall that the canonical derivation of $\mathrm{Ea}(K)$ is densely defined and closed (cf. §4.6.42). Let u stand for the generator of $\mathrm{Ea}(K)$, and let r be in $\mathbb{R}$. By Corollary 4.6.20 (or Fact 4.6.47), $\exp(iru)$ lies in $\mathrm{Ea}^1(K)$ and $[\exp(iru)]' = ir\exp(iru)$. On the other hand, since u lies in $H(\mathrm{Ea}(K),\mathbf{1})$ (by Theorem 4.6.21(i)), Corollary 2.1.9(iii) applies to obtain that,

$$\int_{-\infty}^{+\infty} \|h(r)\exp(iru)\|dr = \int_{-\infty}^{+\infty} |h(r)|\,dr < +\infty$$

and

$$\int_{-\infty}^{+\infty} \|h(r)ir\exp(iru)\|dr = \int_{-\infty}^{+\infty} |rh(r)|\,dr < +\infty.$$

It follows that

$$g := \int_{-\infty}^{+\infty} h(r)\exp(iru)\,dr$$

lies in $\mathrm{Ea}^1(K)$, and that

$$g' = i\int_{-\infty}^{+\infty} rh(r)\exp(iru)\,dr.$$

Valuating the above expressions of g and g' at each point of K, and invoking (4.6.24) and (4.6.26), respectively, we get that $g = f_{|K}$ and $g' = f'{}_{|K}$. Hence $f_{|K}$ lies in $\mathrm{Ea}^1(K)$ and $f_{|K}{}' = f'{}_{|K}$, which proves the first conclusion in the lemma. Finally, since

$$\|f_{|K}{}'\| = \|g'\| \leqslant \int_{-\infty}^{+\infty} |rh(r)|\|\exp(iru)\|\,dr = \int_{-\infty}^{+\infty} |rh(r)|\,dr = |||\, f'\,|||,$$

the second conclusion certainly holds. □

Corollary 4.6.67 *Let A be a norm-unital complete normed power-associative complex algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in* $\mathrm{dom}(D) \cap H(A,\mathbf{1})$*, and let f be as in §4.6.65. Then $f(a)$ (in the sense of the geometric functional calculus) lies in* $\mathrm{dom}(D)$ *and we have $\|D(f(a))\| \leqslant |||\, f'\,|||\|D(a)\|$.*

Proof If a is in $\mathbb{R}\mathbf{1}$, then $f(a)$ belongs to $\mathbb{C}\mathbf{1}$, and hence, by Lemma 4.6.13, $f(a)$ lies in $\mathrm{dom}(D)$ and then clearly $D(a) = 0$, so that the result follows. Otherwise, take $K = V(A, \mathbf{1}, a)$, note that K is a closed and bounded interval of $\mathbb{R}$ not reduced to a point, and apply Proposition 4.6.56 and Lemma 4.6.66. □

Now, combining Lemma 2.2.5, Fact 4.6.37, and Corollary 4.6.67, we get the following.

Proposition 4.6.68 *Let A be a unital non-commutative JB*-algebra, let X be a complete normed unital power-associative A-bimodule, let D be a closed densely defined X-valued derivation of A, let a be in $\mathrm{dom}(D) \cap H(A,*)$, and let f be as in §4.6.65. Then $f(a)$ (in the sense of the continuous functional calculus) lies in $\mathrm{dom}(D)$ and we have $\|D(f(a))\| \leqslant \| f' \|\|D(a)\|$.*

When in Proposition 4.6.68 we take A equal to any C^*-algebra, and X equal to the regular A-bimodule, then we obtain the following.

Corollary 4.6.69 *Let A be a unital C^*-algebra, let D be a closed densely defined derivation of A, let a be in $\mathrm{dom}(D) \cap H(A,*)$, and let f be as in §4.6.65. Then $f(a)$ (in the sense of the continuous functional calculus) lies in $\mathrm{dom}(D)$ and we have $\|D(f(a))\| \leqslant \| f' \|\|D(a)\|$.*

Keeping in mind Lemma 4.6.13, and looking at the proofs of Propositions 4.6.56 and 4.6.58, we realize that Proposition 4.6.56 (as well as its consequences given in Proposition 4.6.68, Theorem 4.6.63, and Corollaries 4.6.57, 4.6.62, 4.6.64, 4.6.67, and 4.6.69) and Proposition 4.6.58 remain true if we relax the assumption that D is densely defined to the one that $\mathbf{1}$ lies in $\mathrm{dom}(D)$. As is proved in Proposition 4.6.70 immediately below, even this last assumption is not too restrictive.

Proposition 4.6.70 *Let A be a complete normed unital power-associative algebra over $\mathbb{K}$, let X be a complete normed unital A-bimodule, and let D be a closed partially defined X-valued derivation of A with $\mathbf{1} \notin \mathrm{dom}(D)$. Then the mapping $\hat{D} : \lambda\mathbf{1} + a \to D(a)$ from $\mathbb{K}\mathbf{1} + \mathrm{dom}(D)$ to X is a closed partially defined X-valued derivation of A. Obviously $\hat{D}$ extends D, and $\mathbf{1}$ lies in $\mathrm{dom}(\hat{D})$.*

Proof In view of §4.6.8, only the assertion that $\hat{D}$ is closed merits a proof. Assume that $\|\mathbf{1} + \mathrm{dom}(D)\| < 1$. Then, arguing as in the proof of Lemma 4.6.13, we obtain that $\mathbf{1} \in \mathrm{dom}(D)$, contrarily to the assumption. Therefore $\|\mathbf{1} + \mathrm{dom}(D)\| \geqslant 1$. Let $\lambda_n\mathbf{1} + a_n$ be a sequence in $\mathbb{K}\mathbf{1} + \mathrm{dom}(D)$ such that

$$\lambda_n\mathbf{1} + a_n \to a \in A \text{ and } \hat{D}(\lambda_n\mathbf{1} + a_n) = D(a_n) \to x \in X.$$

Then

$$|\lambda_n - \lambda_m| \leqslant \|(\lambda_n - \lambda_m)\mathbf{1} + \mathrm{dom}(D)\| \leqslant \|(\lambda_n\mathbf{1} + a_n) - (\lambda_m\mathbf{1} + a_m)\|$$

for all $n, m \in \mathbb{N}$, and so λ_n is a Cauchy sequence in $\mathbb{K}$. If λ is the limit of λ_n, then a_n converges to $-\lambda\mathbf{1} + a$. Since D is closed, we get that $-\lambda\mathbf{1} + a$ lies in $\mathrm{dom}(D)$ and $x = D(-\lambda\mathbf{1} + a)$. Therefore $a = \lambda\mathbf{1} + (-\lambda\mathbf{1} + a) \in \mathbb{K}\mathbf{1} + \mathrm{dom}(D)$ and $x = \hat{D}(a)$, which concludes the proof. □

4.6.3 Historical notes and comments

Subsection 4.6.1 is a non-associative reading of Rodríguez' paper [522], following the indications in the concluding remark of that paper. Actually, a few indications were somewhat daring, and should be read as predictions. This is the case for Corollaries 4.6.6 and 4.6.16 which, for both the formulations and the proofs, need a result unknown at that time, namely Theorem 4.1.93.

According to [754, footnote on p. 80], the notion of a bimodule relative to a given variety of algebras is due to Eilenberg [231]. In [754, Theorem II.9], the reader can find an intrinsic condition, on a variety $\mathscr{V}$, which is equivalent to the fact that, for every algebra A in $\mathscr{V}$, the regular A-bimodule is an A-bimodule relative to $\mathscr{V}$. Facts 4.6.5 and 4.6.48 could have been obtained as corollaries of the result in [754] just quoted.

Proposition 4.6.11 is due to Pérez, Rico, and Rodríguez [488]. Lemmas 4.6.12 and 4.6.13 (and consequently, Proposition 4.6.70) remain true if the assumption that the algebra A is power-associative is removed altogether. Indeed, it is enough to understand powers of an element $a \in A$ as *left powers* of a, defined inductively by $a^1 := a$ and $a^{n+1} := aa^n$. The associative forerunner of Corollary 4.6.17 was suggested to the author of [522] by a referee of *Publicacions Matemàtiques*.

Let A be a complete normed unital associative and commutative complex algebra, let X be a complete normed unital non-commutative Jordan A-bimodule, and let D be a closed densely defined X-valued derivation of A. Since $\mathrm{dom}(D)$ is isomorphic to the graph of D (via Lemma 4.6.7), and the graph of D is a closed subalgebra of the complete normed algebra $A \times X$, we realize that $\mathrm{dom}(D)$ is a complete normed complex algebra under the norm $\|a\|_1 := \|a\| + \|D(a)\|$. Since $\mathrm{dom}(D)$ is a full subalgebra of A (by Proposition 4.6.14), we obtain that the pair of complete normed unital associative and commutative complex algebras $(A, \mathrm{dom}(D))$, with norms $\|\cdot\|$ and $\|\cdot\|_1$ respectively, becomes a *Wiener pair* as defined in [783, 1.7]. But it is easy to see that, if (R, R_1) is a Wiener pair of complete normed unital associative and commutative complex algebras with R_1 dense in R, then the mapping $\phi \to \phi_{|R_1}$ is a homeomorphism from the carrier space of R onto the carrier space of R_1 (see [783, 11.7.V] for details). Therefore, as pointed out in [522], *the mapping $\phi \to \phi_{|\mathrm{dom}(D)}$ is a homeomorphism from the carrier space of A onto the carrier space of* $\mathrm{dom}(D)$. When X equals the regular A-module, the result just proved is originally due to Loy [406], who seems to have been the first author interested in domains of closed densely defined derivations.

Theorem 4.6.21 is due to Bollobás [110], who constructed the extremal algebra $\mathrm{Ea}(K)$ (of a compact and convex set $K \subseteq \mathbb{C}$) as the completion of a certain normed algebra of formal power series.

The argument suggested in §4.6.23 to show the essential uniqueness of the couple $(\mathscr{A}, u)$ in Theorem 4.6.21, as well as those given in §§4.6.71 and 4.6.72 immediately below, are folklore.

§4.6.71 Property (iii) in Theorem 4.6.21 can be derived from the intrinsic characterization of the couple $(\mathscr{A}, u)$ provided by the first conclusion in the theorem. Indeed, let $\mathscr{B}$ stand for the closed subalgebra of $\mathscr{A}$ generated by $\{\mathbf{1}, u\}$. Let A be any norm-unital complete normed associative complex algebra, let a be in A

with $V(A,\mathbf{1},a) \subseteq K$, and let ψ be the unique contractive unit-preserving algebra homomorphism from $\mathscr{A}$ to A taking u to a, given by condition (ii). Then $\psi_{|\mathscr{B}}$ is a contractive unit-preserving algebra homomorphism from $\mathscr{B}$ to A taking u to a. On the other hand, if η is any continuous unit-preserving algebra homomorphism from $\mathscr{B}$ to A taking u to a, then, by Lemma 1.1.82(i), we have $\eta = \psi_{|\mathscr{B}}$. Since $V(\mathscr{B},\mathbf{1},u) = V(\mathscr{A},\mathbf{1},u) = K$, it follows that the couple $(\mathscr{B},u)$ fulfils conditions (i) and (ii) in the theorem with $(\mathscr{B},u)$ instead of $(\mathscr{A},u)$. Since $\mathscr{B}$ is generated by $\{\mathbf{1},u\}$ as a normed algebra, the essential uniqueness of such couples (given by §4.6.23) applies to get that $\mathscr{A}$ is generated by $\{\mathbf{1},u\}$ as a normed algebra, i.e. property (iii) in Theorem 4.6.21 holds.

§4.6.72 Property (iv) in Theorem 4.6.21 is also easily realized from the intrinsic characterization of the couple $(\mathscr{A},u)$ provided by the first conclusion in the theorem. Indeed, by condition (i) and Lemma 2.3.21, we have $\mathrm{sp}(\mathscr{A},u) \subseteq K$. To prove the converse inclusion let λ be in K. By taking $A = \mathbb{C}$ and $a = \lambda$ in condition (ii), we find a character ϕ on $\mathscr{A}$ such that $\phi(u) = \lambda$, and hence, by Corollary 1.1.67, we have that λ lies in $\mathrm{sp}(\mathscr{A},u)$.

The Banach space of entire functions $D(K)$, considered in §4.6.25, seems to have been introduced first in [182]. The construction of $\mathrm{Ea}(K)$ we have given (through results from Lemma 4.6.26 to §4.6.30), as an algebra of bounded linear operators on $D(K)$ endowed with the operator norm, could be new. Nevertheless, in the case where K equals the closed unit disc, this construction is clearly suggested by some ideas of Browder [132] included in [695, Remark (2), p. 64]. In the particular case where K is a line segment, our construction is well known in the literature (see [133, 184, 185, 581]).

Fact 4.6.31 is well known thanks to a more involved construction of $\mathrm{Ea}(K)$, done by Crabb, Duncan, and McGregor in the paper [182] quoted above. This construction is included in detail in Section 24 of the Bonsall–Duncan book [695], and will not be discussed here. Briefly, in the Crabb–Duncan–McGregor construction, $\mathrm{Ea}(K)$ arises directly as a suitable norm-unital complete normed algebra of complex-valued continuous functions on K in such a way that the generator u becomes the inclusion mapping $K \hookrightarrow \mathbb{C}$. Regarding $\mathrm{Ea}(K)$ in this way, and keeping in mind that valuations at points of K are then characters on $\mathrm{Ea}(K)$, it is clear that the inclusion $\mathrm{Ea}(K) \hookrightarrow C^{\mathbb{C}}(K)$ becomes a contractive unit-preserving injective algebra homomorphism taking the generator to $K \hookrightarrow \mathbb{C}$. By keeping in mind that properties (iii) and (iv) of $\mathrm{Ea}(K)$ in Theorem 4.6.21 can be proved independently of any construction (cf. §§4.6.71 and 4.6.72), the additional property in Fact 4.6.31 that $\mathrm{Ea}(K)$ is a full subalgebra of $C^{\mathbb{C}}(K)$ then follows from Theorem 1.1.83. Thus, in the Crabb–Duncan–McGregor construction, Fact 4.6.31 becomes almost obvious. By the way, the fact that $\mathrm{Ea}(K)$ is a full subalgebra of $C^{\mathbb{C}}(K)$ is explicitly established in [182, Corollary 1.4].

The independent proof of Fact 4.6.31 we have given, based on our own construction, is new. In relation to this proof, it is worth mentioning that, if g is in $D(K)$, then $\mathbb{C} \setminus K$ becomes a domain of holomorphy for the Borel transform $\tilde{g}$ of g, and the equality $g(z) = \int_\Gamma e^{wz}\tilde{g}(w)dw$ holds for every $z \in \mathbb{C}$, where Γ is any contour surrounding K in $\mathbb{C}$. Indeed, this follows easily from [693, Theorem 5.3.5]. It is also worth mentioning that the embedding $\mathrm{Ea}(K) \hookrightarrow C^{\mathbb{C}}(K)$ can be explicitly given in terms of our construction. Indeed, as we pointed out in §4.6.30, the function $g_\lambda : \mathbb{C} \to \mathbb{C}$ defined by $g_\lambda(z) := e^{\lambda z}$ belongs to $D(K)$ whenever λ is in K. Then, given $T \in \mathrm{Ea}(K) \subseteq BL(D(K))$, it is easily realized that $\hat{T}(\lambda) = T(g_\lambda)(0)$ for every $\lambda \in K$, where $\hat{T}$ stands for T regarded as a complex-valued continuous function on K.

The next remarkable theorem is due to Crabb–Duncan–McGregor [182] (see also [695, Theorem 24.5]). In terms of our construction of $\mathrm{Ea}(K)$, as an algebra of bounded linear operators on $D(K)$, the theorem has the following formulation and proof.

Theorem 4.6.73 *$D(K)$ is the dual space of* $\mathrm{Ea}(K)$. *More precisely, we have:*

(i) *For each $g \in D(K)$, the mapping $\phi(g) : T \to T(g)(0)$ from* $\mathrm{Ea}(K)$ *to $\mathbb{C}$ is linear and continuous, i.e. $\phi(g)$ lies in* $(\mathrm{Ea}(K))'$.
(ii) *The mapping $\phi : g \to \phi(g)$ from $D(K)$ to* $(\mathrm{Ea}(K))'$ *is a surjective linear isometry.*

Proof Assertion (i) follows from the inequality (4.6.8) in §4.6.25, which also gives that $\|\phi(g)\| \leqslant \|g\|$ for every $g \in D(K)$. Now, clearly, the mapping ϕ in assertion (ii) is a linear contraction. On the other hand, taking $z = 0$ in the equality (4.6.10) of §4.6.30, we obtain that

$$g(w) = \phi(g)(\exp(wu)) \text{ for all } w \in \mathbb{C} \text{ and } g \in D(K). \tag{4.6.27}$$

As a first consequence of (4.6.27), ϕ is injective. Let θ be in $(\mathrm{Ea}(K))'$, and consider the mapping $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \theta(\exp(zu))$. Then, by Fact 4.6.28, g lies in $D(K)$ and $\|g\| \leqslant \|\theta\|$. Moreover, again by (4.6.27), we have $\phi(g) = \theta$. Since ϕ is injective, and θ is arbitrary in $(\mathrm{Ea}(K))'$, this shows that ϕ is bijective and that $\|\phi^{-1}(\theta)\| \leqslant \|\theta\|$ for every $\theta \in (\mathrm{Ea}(K))'$. Finally, since ϕ is contractive, certainly it is an isometry. □

A less explicit but more intrinsic formulation of Theorem 4.6.73 is that *there is a surjective linear isometry $\phi : D(K) \to (\mathrm{Ea}(K))'$ determined by the condition that $g(w) = \phi(g)(\exp(wu))$ for all $w \in \mathbb{C}$ and $g \in D(K)$,* the determination being assured by Remark 1.1.84. This formulation is close to that in [182].

The authors of [182] also introduce the Banach space $D_0(K)$ as the closed subspace of $D(K)$ consisting of those entire functions $g : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z \to \infty} (\omega_K(z))^{-1} |g(z)| = 0,$$

and prove that, *in the duality* $(\mathrm{Ea}(K), D(K))$ *given by Theorem 4.6.73,* $\mathrm{Ea}(K)$ *becomes the dual space of $D_0(K)$*. More explicitly, we have the following.

Theorem 4.6.74 *For each $T \in \mathrm{Ea}(K)$, the mapping $\psi(T) : g \to T(g)(0)$ from $D_0(K)$ to $\mathbb{C}$ is linear and continuous, and the mapping $\psi : T \to \psi(T)$ from* $\mathrm{Ea}(K)$ *to $(D_0(K))'$ is a surjective linear isometry.*

Results from Proposition 4.6.32 to Proposition 4.6.56, as well as Proposition 4.6.70, become a non-associative reading of Rodríguez' paper [513]. Actually, these results can be considered as new even in their associative particularization. The reason is that, although the paper [513] was carefully written in Spanish, its published version is not easily available, and in addition the editors of the publication did not send the author the proofs for correction. As a result, the published version of [513] is almost unreadable, even for native Spanish speakers.

In view of Proposition 4.6.32, many questions involving the numerical range (say K) of an element of a norm-unital complete normed power-associative complex algebra can be answered by looking only at the algebra $\mathrm{Ea}(K)$. The crucial advantage is that $\mathrm{Ea}(K)$ can be regarded as an algebra of complex-valued functions on K. As a sample of the application of this procedure, let us prove a non-associative version of the main result in [427]. For $0 \leqslant \alpha \leqslant \frac{\pi}{2}$, we denote by K_α the closed angular region in $\mathbb{C}$ with vertex at zero, angle 2α, and bisected by the non-negative part of the real axis, i.e.

$$K_\alpha := \{0\} \cup \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leqslant \alpha\}.$$

Theorem 4.6.75 *Let A be a norm-unital complete normed power-associative complex algebra, and let a be in A such that $V(A, \mathbf{1}, a) \subseteq K_\alpha$ for some $0 \leqslant \alpha < \frac{\pi}{2}$. Then there exists an A-valued holomorphic semigroup $z \to a^z$, with parameter in the right open half-plane of $\mathbb{C}$, satisfying $a^1 = a$. As a consequence, a has a square root in A.*

Proof Set $K := V(A, \mathbf{1}, a)$. Since continuous algebra homomorphisms preserve both holomorphy and the product, we invoke Theorem 4.6.21(i) and Proposition 4.6.32 to realize that it is enough to prove the theorem in the particular case where $A = \mathrm{Ea}(K)$ and $a = u$ (the generator of $\mathrm{Ea}(K)$). Then, since $V(\mathrm{Ea}(K), \mathbf{1}, u) = K \subseteq K_\alpha$, we have

$$\min \Re(V(\mathrm{Ea}(K), \mathbf{1}, zu)) = \min \Re(zK) \geqslant 0 \quad \text{for} \quad z \in K_{\frac{\pi}{2}-\alpha},$$

and hence $\|\exp(-zu)\| \leqslant 1$ for such a z (by Corollary 2.1.9(i)). Let $t > 0$, consider the closed disc in $\mathbb{C}$ with centre at t and radius $t\cos\alpha$, which is contained in $K_{\frac{\pi}{2}-\alpha}$ where the boundedness given above is valid, and let n be in $\mathbb{N}$. It follows from the Cauchy inequalities for the entire function $z \to \exp(-zu)$ that $\|u^n \exp(-tu)\| \leqslant \frac{n!}{t^n(\cos\alpha)^n}$. Now, let z be in $\mathbb{C}$ with $0 < \Re(z) < n$. Since

$$\|t^{n-z-1}u^n \exp(-tu)\| \leqslant \min\left\{t^{n-\Re(z)-1}\|u^n\|, t^{-(\Re(z)+1)}\frac{n!}{(\cos\alpha)^n}\right\},$$

the integral $\int_0^{+\infty} t^{n-z-1}u^n \exp(-tu)dt$ is absolutely convergent, and the convergence is uniform on compact subsets of $\Omega_n := \{z \in \mathbb{C} : 0 < \Re(z) < n\}$. Therefore the mapping

$$z \to \frac{1}{\Gamma(n-z)} \int_0^{+\infty} t^{n-z-1}u^n \exp(-tu)\, dt$$

from Ω_n to $\mathrm{Ea}(K)$ is holomorphic (here Γ stands for Euler's gamma function). Finally, since for $\lambda \in K$ and $z \in \Omega_n$ we have

$$\int_0^{+\infty} t^{n-z-1}\lambda^n \exp(-t\lambda)\, dt = \lambda^z \int_0^{+\infty} (\lambda t)^{n-z-1} \exp(-t\lambda)\lambda\, dt = \lambda^z\Gamma(n-z),$$

and n is arbitrary in $\mathbb{N}$, it is enough to invoke Fact 4.6.31 to conclude that, for each $z \in \mathbb{C}$ with $\Re(z) > 0$, the function $u^z : \lambda \to \lambda^z$ from K to $\mathbb{C}$ lies in $\mathrm{Ea}(K)$, and that the mapping $z \to u^z$ becomes an $\mathrm{Ea}(K)$-valued holomorphic semigroup, with parameter in the right open half-plane of $\mathbb{C}$, satisfying $u^1 = u$. □

Let A be a power-associative algebra, let a be in A, let X be a power-associative A-bimodule, and let π_2 stand for the natural projection from the split null X-extension $A\times X$ of A to X. Then it is easily realized by induction that, for $n\in\mathbb{N}$, the linear mapping $x\to\pi_2[(a,x)^n]$ from X to X considered in Lemma 4.6.44(ii) is precisely the operator $\sum_{k=0}^{n-1}(R_a^X)^k L_{a^{n-k-1}}^X$.

Let K be a closed and convex subset of $\mathbb{C}$ with more than one point. It is proved in [513] that, for each $\lambda\in K$, the 'integral' operator $\mathscr{I}_\lambda:\mathrm{Ea}(K)\to\mathrm{Ea}(K)$, given by Proposition 4.6.54, satisfies

$$\|\mathscr{I}_\lambda^n\|=\frac{\|(u-\lambda\mathbf{1})^n\|}{n!}$$

for every $n\in\mathbb{N}$, and hence $\mathfrak{r}(\mathscr{I}_\lambda)=0$. Therefore, given $g\in\mathrm{Ea}(K)$, $\lambda\in K$, and $0\neq z\in\mathbb{C}$, there exists a unique $f\in\mathrm{Ea}^1(K)$ such that $f-zf'=g$ and $f(\lambda)=0$ (indeed, take $f=\mathscr{I}_\lambda(\mathscr{I}_\lambda-zI_{\mathrm{Ea}(K)})^{-1}(g)$). For $n\in\mathbb{N}$ let $\mathrm{Ea}^n(K)$ stand for the domain of the nth iteration of the canonical derivation of $\mathrm{Ea}(K)$, and for $f\in\mathrm{Ea}^n(K)$ let $f^{(n)}$ denote the image of f under that nth iteration. Proposition 4.6.54 is applied in [513] to show that $\mathrm{Ea}^n(K)$ becomes a complete normed algebra under the norm

$$\|f\|_n:=\sum_{k=0}^n\frac{\|f^{(k)}\|}{k!}.$$

Now assume additionally that K has empty interior. Then the arguments in the proof of Theorem 4.6.60, subjected to an easy induction, show that

$$C(K)\supseteq\mathrm{Ea}(K)\supseteq C^1(K)\supseteq\mathrm{Ea}^1(K)\supseteq C^2(K)\supseteq\cdots$$
$$\cdots\supseteq\mathrm{Ea}^n(K)\supseteq C^{n+1}(K)\supseteq\mathrm{Ea}^{n+1}(K)\supseteq\cdots,$$

and hence that

$$C^\infty(K)=\mathrm{Ea}^\infty(K):=\bigcap_{n\in\mathbb{N}}\mathrm{Ea}^n(K).$$

Proposition 4.6.58 could be new. The particular case, when the element $a\in\mathrm{dom}(D)\setminus\mathbb{C}\mathbf{1}$ is assumed to be self-adjoint, can be found in Bratteli's book [698, Example 1.6.1]. In relation to this last result, it is worth mentioning a celebrated counterexample of McIntosh [439] exhibiting a (necessarily non-commutative) unital C^*-algebra A, a closed densely defined derivation D of A, a self-adjoint element $a\in\mathrm{dom}(D)\setminus\mathbb{R}\mathbf{1}$, and a complex-valued C^1-function on $\mathrm{co}(\mathrm{sp}(A,a))$, such that $f(a)\notin\mathrm{dom}(D)$. This gives special relevance to Corollaries 4.6.64 and 4.6.69, which are due to Sakai [807, Theorem 3.3.7] and Bratteli–Robinson [123], respectively. The Bratteli–Robinson result is included with proof in both [698, Theorem 1.6.2] and [807, Proposition 3.3.6]. The proof of Fact 4.6.59 becomes a simplification of Sakai's argument in [807] to derive Corollary 4.6.64 from Corollary 4.6.69. For additional information about closed derivations of C^*-algebras, the reader is referred to [698, 807] and references therein. We note that, in view of Corollary 4.6.57, McIntosh's counter-example implies that the inclusion $\mathrm{Ea}^1(K)\subseteq C^1(K)$ in Theorem 4.6.60 is strict. The inclusion $C^1(K)\subseteq\mathrm{Ea}(K)$ in Theorem 4.6.60 is due to Baillet [52]. Actually, Baillet proves the following much more general result.

Theorem 4.6.76 *Let A be a norm-unital complete normed associative complex algebra, let a be in $H(A,\mathbf{1})$, let Ω be an open subset of $\mathbb{R}$ containing $\operatorname{sp}(A,a)$, and let $\imath$ stand for the inclusion mapping $\Omega \hookrightarrow \mathbb{R}$. Then there exists a unique continuous unit-preserving algebra homomorphism $f \to f(a)$ from $C^1(\Omega)$ to A taking $\imath$ to a, and such that $f(a)=0$ for every function $f \in C^1(\Omega)$ vanishing in some neighbourhood of $\operatorname{sp}(A,a)$ contained in Ω.*

In the above theorem, $C^1(\Omega)$ is endowed with the topology of uniform convergence of functions and their derivatives on compact subsets of Ω. Now, if K is a closed interval of $\mathbb{R}$ not reduced to a point, and if we take $A=\mathrm{Ea}(K)$ and $a=u$ (the generator of $\mathrm{Ea}(K)$), then we have $a \in H(A,\mathbf{1})$ and $\operatorname{sp}(A,a)=K$ (cf. Theorem 4.6.21), so that, keeping in mind Fact 4.6.31, the inclusion $C^1(K) \subseteq \mathrm{Ea}(K)$ in Theorem 4.6.60 follows straightforwardly from Baillet's Theorem 4.6.76.

Results from Corollary 4.6.57 to Proposition 4.6.68 not previously quoted in this subsection are new.

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Symbol index

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