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Analytic Hilbert Modules

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Preface

These notes deal with the aspects of analytic Hilbert modules in several complex variables that have been the focus of considerable attention from the authors in recent years. This volume has been inspired primarily by the work initiated on the topic by Douglas and Paulsen in *Hilbert Modules over Function Algebras* [DP]. Briefly, the Hilbert module theory provides an appropriate framework for the multi-variable operator theory. In this module context, operator theory has proved mutually enriching for other areas of mathematics, including algebra, geometry, homology theory and complex analysis in one and several variables.

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Chapter 1

Introduction

These notes arose out of a series of research papers completed by the authors and others. This volume is devoted to recent developments in some topics of analytic Hilbert modules.

In the last decades, the study of nonself-adjoint operator algebras has enjoyed considerable success, but the development has largely excluded spectral theory. Multi-variable spectral theory can be viewed by analogy as "noncommutative algebra geometry," and such a development was the goal of the module approach to the multi-variable operator theory presented in *Hilbert Modules over Function Algebras* by Douglas and Paulsen [DP].

The theory of Hilbert modules over function algebras, largely due to R. Douglas, provides an appropriate framework to the multi-variable operator theory. In this module context, function algebras play important roles rather than "n-tuples," and one considers Hilbert spaces as modules over it. The change of viewpoint has some remarkable consequences. At the very least, it facilitates the introduction of techniques and methods drawn from algebraic geometry, homology theory and complex analysis in one variable and several variables, etc.

Now we will simply describe the background and developments of analytic Hilbert modules; of course, the description is never complete. Let $\mathbb{T}$ be the unit circle in the complex plane and let $L^2(\mathbb{T})$ be the Hilbert space of square integrable functions, with respect to arc-length measure. Recall that the Hardy space $H^2(\mathbb{D})$ over the open unit disk $\mathbb{D}$ is the closed subspace of $L^2(\mathbb{T})$ spanned by the nonnegative powers of the coordinate function z. In the language of the Hilbert module [DP], the Hardy space $H^2(\mathbb{D})$ is a module over the disk algebra $A(\mathbb{D})$ with multiplication defined pointwise on $\mathbb{T}$. If M is a submodule of $H^2(\mathbb{D})$ (which means that M is invariant for the multiplication by polynomials), then Beurling's theorem [Beu] says that there exists an inner function η such that $M = \eta\, H^2(\mathbb{D})$. Therefore, each submodule M of the Hardy module $H^2(\mathbb{D})$ has the form $M = \eta\, H^2(\mathbb{D})$. For the Hardy module $H^2(\mathbb{D}^n)$ in the multi-variable version, a natural problem is to consider the structure of submodules. However, one quickly sees that a Beurling-like characterization is impossible [Ru1]. Perhaps the earliest results showing that a straightforward generalization of Beurling's theorem fails in $H^2(\mathbb{D}^n)$ involve considering cyclic vectors and are due to Ahern and Clark [AC].

Theorem 1.0.1 (Ahern-Clark) *If $f_1, ..., f_k$ are in $H^2(\mathbb{D}^n)$ with $k < n$, then the submodule generated by $f_1, ..., f_k$ is either all of $H^2(\mathbb{D}^n)$ or is of infinite codimension.*

Therefore, in particular, no submodule of finite codimension in $H^2(\mathbb{D}^n)$ can be of the form $\eta H^2(\mathbb{D}^n)$. However, there are infinitely many submodules of finite codimension. To describe these we let $\mathcal{C}$ denote the ring of complex polynomials in several variables. For an ideal I in $\mathcal{C}$, we set $Z(I) = \{z \in \mathbb{C}^n : p(z) = 0, \ \forall p \in I\}$, the zero variety of I. We also let $[I]$ be the closure of I in $H^2(\mathbb{D}^n)$.

Theorem 1.0.2 (Ahern-Clark) *Suppose M is a submodule of $H^2(\mathbb{D}^n)$ of codimension $k < \infty$. Then $\mathcal{C} \cap M$ is an ideal in the ring $\mathcal{C}$ such that*

(1) $\mathcal{C} \cap M$ *is dense in* M;

(2) $\dim \mathcal{C}/\mathcal{C} \cap M = k$;

(3) $Z(\mathcal{C} \cap M)$ *is finite and lies in* $\mathbb{D}^n$.

Conversely, if I is an ideal of the ring $\mathcal{C}$ with $Z(I)$ being finite, and $Z(I) \subset \mathbb{D}^n$, then $[I]$ is a submodule of $H^2(\mathbb{D}^n)$ with the same codimension as I in $\mathcal{C}$ and $[I] \cap \mathcal{C} = I$.

The above theorem shows that the description of finite codimensional submodules is, at least conceptually, an algebraic matter. We also get our first glimpse of the interplay between the algebra and analysis. "Algebraic reduction," the starting point of this topic, is Theorem 1.0.2. Along these lines, many authors, including Douglas, Paulsen, Sah and Yan [DP, DPSY], Axler and Bourdon [AB], Bercovici, Foias and Pearcy [BFP], Agrawal and Salinas [AS], Guo [Guo1, Guo2, Guo4, Guo12], Putinar and Salinas [PS], Putinar [Pu1, Pu2], and Chen and Douglas [CD1, CD2], have developed a series of techniques to investigate algebraic reduction for analytic Hilbert modules over some domains.

For Hilbert modules in several variables, most of them arise from reproducing analytic function spaces. This leads us to introduce the notion of analytic Hilbert modules. Let Ω be a bounded nonempty open subset of $\mathbb{C}^n$, $Hol(\Omega)$ denote the ring of analytic functions on Ω, and X be a Banach space contained in $Hol(\Omega)$. We call X to be a reproducing Ω-space if X contains 1 and if for each $\lambda \in \Omega$ the evaluation functional, $E_\lambda(f) = f(\lambda)$, is a continuous linear functional on X. We say X is a reproducing $\mathcal{C}$-module on Ω if X is a reproducing Ω-space, and for each $p \in \mathcal{C}$ and each $x \in X$, px is in X. Note that, by a simple application of the closed graph theorem, the operator T_p defined to be multiplication by p is bounded on X. Note also that $\mathcal{C} \subset X$ follows from the fact that 1 is in X. For $\lambda \in \mathbb{C}^n$, one says that λ is a virtual point of X provided that the homomorphism $p \mapsto p(\lambda)$ defined on $\mathcal{C}$ extends to a bounded linear functional on X. Since X is a reproducing Ω-space, every point in Ω is a virtual point. We use $vp(X)$ to denote the collection of virtual points, then $vp(X) \supseteq \Omega$. Finally we say that X is an analytic Hilbert module on Ω if the following conditions are satisfied:

(1) X is a reproducing $\mathcal{C}$-module on Ω;

(2) $\mathcal{C}$ is dense in X;

(3) $vp(X) = \Omega$.

In Chapter 2, we give a rather extended consideration to algebraic reduction of analytic Hilbert modules by using the characteristic space theory. The characteristic space theory in several variables, developed by Guo [Guo1, Guo2, Guo4, Guo12], is a natural generalization of the fundamental theorem of algebra. This theory allows us to treat algebraic reduction of analytic Hilbert modules in several variables (see Theorem 2.2.5, Corollary 2.2.6) uniformly. In fact, for the topics discussed in Chapters 2 through 5, our main technique is the characteristic space theory. Let X be an analytic Hilbert module over a domain Ω. Recall that a submodule M of X is an AF-cosubmodule if M is equal to the intersection of all finite codimensional submodules that contain M. We found that AF-cosubmodules enjoy many properties of submodules of finite codimension. Therefore, an algebraic reduction theorem for AF-cosubmodules was proved (see Theorem 2.4.2).

It is well known that "rank" is one of the important invariants of Hilbert modules. Submodules with finite rank enjoy many algebraic properties, for example, the Hilbert-Samuel polynomials introduced by Douglas and Yan [DY2]. In addition, we proved that if M is a submodule of the analytic Hilbert module X over the domain Ω generated by $f_1, ..., f_k$, then for each $f \in M$, there are analytic functions $g_1, ...g_k$ on Ω such that $f = \sum_{i=1}^{k} f_i g_i$ (see Theorem 2.3.3). This result will be applied to study equivalence of Hardy submodules over the polydisk in Chapter 4.

Chapter 3 explores the topic of "rigidity" of analytic Hilbert modules in several variables. By rigidity we mean that an analytic Hilbert module in several variables exhibits a much more rigid structure than is expected from the single-variable theory. This section starts by introducing the following definitions.

Definition. Let M_1, M_2 be two submodules of X on Ω. We say that

1. they are unitarily equivalent if there exists a unitary module map $X : M_1 \to M_2$, that is, X is a unitary operator and for any polynomial p, $X(p\,h) = p\,X(h), \ \forall h \in M_1$;

2. they are similar if there exists an invertible module map $X : \ M_1 \to M_2$;

3. they are quasi-similar if there exist module maps $X : \ M_1 \to M_2$ and $Y : \ M_2 \to M_1$ with dense ranges.

By Beurling's theorem, submodules of $H^2(\mathbb{D})$ are all unitarily equivalent. However, for the higher dimensional Hardy module $H^2(\mathbb{D}^n)$, an earlier result on non-unitarily equivalent submodules, due to Berger, Coburn and Lebow [BCL], started to indicate just how much rigidity there is among those submodules with finite codimension. They considered the restriction of multiplication by the coordinate functions to submodules obtained as the closure of certain ideals of polynomials in several variables having the origin as zero set. By applying their results on commuting isometries, they showed that different ideals yield inequivalent submodules. Almost at the same time Hastings [Ha2] even showed that $[z-w]$ is never quasi-similar to $H^2(\mathbb{D}^2)$, where $[z-w]$ denotes the submodule generated by $z-w$. The rigidity theorem for finite codimensional submodules of $H^2(\mathbb{D}^n)$ is due to Agrawal, Clark and Douglas [ACD]. This theorem showed that two submodules of finite codimension are unitarily equivalent only if they are equal. By using localization techniques, Douglas, Paulsen, Sah and Yan [DPSY] gave a complete generalization of the rigidity theorem above. Actually they obtained a stronger result [DPSY, Corollary 2.8] than stated here.

Theorem 1.0.3 (DPSY) *Let X be an analytic Hilbert module over the domain Ω, and let I_1 and I_2 be ideals of the polynomials ring $\mathcal{C}$ in n-complex variables. Suppose that* $\dim Z(I_i) \leq n-2$, *and each algebraic component of $Z(I_i)$ intersects Ω for $i = 1, 2$. Then $[I_1]$ and $[I_2]$ are quasi-similar only if $I_1 = I_2$.*

The condition $\dim Z(I) \leq n-2$ is sharp because there exists a unitary equivalence between the submodules $[z]$ and $[w]$ of $H^2(\mathbb{D}^2)$ but they are never equal. Note that this condition is also equivalent to the fact that the ideal I has the greatest common divisor 1 (see Chapter 3). Therefore, Theorem 1.0.3 provides more examples for nonquasi-similar submodules. A simple example is that the submodules $[z_1, z_2]$ and $[z_1, z_3]$ of the Hardy module $H^2(\mathbb{D}^3)$ are not quasi-similar. By using the characteristic space theory, Guo [Guo1] obtained a generalization for the rigidity theorem above (see Theorem 3.1.6).

Another kind of rigidity theorem closely related to the preceding consideration (but different, since the submodules involved do not possess an algebraic model) was obtained by Douglas and Yan [DY1]. Given a submodule M of $H^2(\mathbb{D}^n)$, let $Z(M) = \{z \in \mathbb{D}^n : f(z) = 0, \forall f \in M\}$. For f in $H^2(\mathbb{D}^n)$, let σ_f be the unique singular measure on n-torus $\mathbb{T}^n$ for which $P_z(\log|f| - \sigma_f)$ is the least harmonic majorant of $\log|f|$, where P_z denotes the Poisson integral.

Theorem 1.0.4 (Douglas-Yan) *Let M_1 and M_2 be submodules of $H^2(\mathbb{D}^n)$ such that*

(1) the Hausdorff dimension of $Z(M_i)$ is at most $n-2$, $i = 1, 2$;

(2) $\inf\{\sigma_f : f \in M_1\} = \inf\{\sigma_f : f \in M_2\}$.

Then M_1 and M_2 are quasi-similar if and only if $M_1 = M_2$.

The above theorem relies heavily on the function theory of $H^2(\mathbb{D}^n)$, and although some generalizations are known [Guo8, or see Section 3.3], it is not clear yet what a generalization of this result is to other analytic Hilbert modules.

Chapter 3 is devoted to these remarkable rigidity features of analytic Hilbert modules in several variables. From an analytic point of view, appearance of this feature is natural because of the Hartogs phenomenon in several variables. From an algebraic point of view, the reason may be that submodules are not singly generated.

The equivalence problem is, in general, quite difficult. On one hand, under the hypotheses of Theorem 1.0.3, the equivalence necessarily preserves the ideal. On the other hand, for the submodules $[z]$ and $[w]$ of $H^2(\mathbb{D}^2)$ the unitary equivalence $U : [z] \to [w]$ defined by multiplication $\bar{z}w$ does not even preserve the zero sets of the ideals. Yan has examined the case of homogeneous principal ideals [Yan1].

Theorem 1.0.5 (Yan) *Let p; q be two homogeneous polynomials. Then $[p]$ and $[q]$, as the submodules of $H^2(\mathbb{D}^n)$, are unitarily equivalent if and only if there exists a constant c such that $|p| = c|q|$ on $\mathbb{T}^n$; similar if and only if the quotient $|p|/|q|$ is bounded above and below on $\mathbb{T}^n$.*

While in the case of the unit ball $\mathbb{B}^n$, Chen and Douglas [CD1] proved that $[p]$ and $[q]$ are quasi-similar only if $p = cq$ for some constant c. From these facts one finds that the equivalence problem depends heavily on the geometric properties of domains.

Chapter 4 considers the equivalence problem of Hardy submodules generated by polynomials in the cases of both the polydisk and the unit ball. Let I be an ideal of the polynomials ring $\mathcal{C}$. Since $\mathcal{C}$ is a Noetherian ring [AM, ZS], the ideal I is generated by finitely many polynomials. This implies that I has a greatest common divisor p. Thus, I can be uniquely written as $I = p\,L$, which is called the Beurling form of I. The following classification theorems were proved in [Guo2](see Theorem 4.2.1, Corollary 4.2.2 and Theorem 4.4.2 in this volume).

Theorem 1.0.6 (Guo) *Let I_1 and I_2 be two ideals of polynomials and let $I_1 = p_1 L_1,\ I_2 = p_2 L_2$ be their Beurling forms. Then*

(1) on the polydisk $\mathbb{D}^n$, $[I_1]$ and $[I_2]$, as submodules of $H^2(\mathbb{D}^n)$, are unitarily equivalent if and only if there exist two polynomials q_1, q_2 with $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on $\mathbb{T}^n$, and $[p_1 L_1] = [p_1 L_2]$.

(2) on the unit ball $\mathbb{B}_n$, $[I_1]$ and $[I_2]$, as submodules of $H^2(\mathbb{B}_n)$, are unitarily equivalent only if $[I_1] = [I_2]$.

As an immediate consequence, we have

Corollary 1.0.7 (Guo) *Let p_1, p_2 be two polynomials. Then on the polydisk, principal submodules $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there exist two polynomials q_1, q_2 with $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on $\mathbb{T}^n$.*

Chapter 4 is devoted to the necessary background to understand these equivalence problems, and to proofs of the above theorems.

In Chapter 5 we return to considering the case of reproducing function spaces on $\mathbb{C}^n$, and exploring the structure of such spaces. First let us examine the Segal-Bargmann space, or the so-called Fock space. It is the analog of the Bergman space in the context of the complex n-space $\mathbb{C}^n$. Let

$$d\mu(z) = e^{-|z|^2/2} dv(z) (2\pi)^{-n}$$

be the Gaussian measure on $\mathbb{C}^n$ (dv is the usual Lebesgue measure). The Fock space $L_a^2(\mathbb{C}^n, d\mu)$ (in short, $L_a^2(\mathbb{C}^n)$), by definition, is the space of all μ-square-integrable entire functions on $\mathbb{C}^n$. It is easy to check that $L_a^2(\mathbb{C}^n)$ is a closed subspace of $L^2(\mathbb{C}^n)$ with the reproducing kernel functions $K_\lambda(z) = e^{\bar{\lambda} z/2}$. The Fock space is important because of the relationship between the operator theory on it and the Weyl quantization [Be]. As is well known, the Fock space preserves many properties of the Bergman space. However, the Fock space exhibits many new features since its underlying domain is unbounded. Just as we have shown in Proposition 5.0.1, such a space has no nontrivial invariant subspace for the coordinate functions. Thus, an appropriate substitute for invariant subspace, the so-called quasi-invariant subspace is needed. Namely, a (closed) subspace M of $L_a^2(\mathbb{C}^n)$ is called quasi-invariant if $p\,M \cap X \subset M$ for each polynomial p. Quasi-invariant subspaces enjoy many properties of invariant subspaces which are shown in Chapter 5. However, unlike the Bergman space $L_a^2(\mathbb{D})$ (or the Hardy space $H^2(\mathbb{D})$), Beurling's theorem fails in general for the Fock space. Recall that on the Bergman space $L_a^2(\mathbb{D})$, Beurling's theorem [ARS] says that $M \ominus zM$ is a generating set of M for each invariant subspace M. The next example shows that there are infinitely many quasi-invariant subspaces for which Beurling's theorem fails.

Example 1.0.8 *Let $\alpha \neq 0$, and let $[z-\alpha]$ be the quasi-invariant subspace generated by $z-\alpha$. Then $[z-\alpha] \ominus [z(z-\alpha)]$ is not a generating set of $[z-\alpha]$.*

In fact, it is easy to check that the subspace $[z-\alpha] \ominus [z(z-\alpha)]$ is one dimensional, and the function $f_\alpha(z) = e^{|\alpha|^2/2} - e^{\bar{\alpha} z/2}$ is in it. Since the function $f_\alpha(z)$ has infinitely many zero points $\{\alpha + 4n\pi i/\bar{\alpha} : n \text{ range over all integer}\}$ in $\mathbb{C}$, but $z-\alpha$ has a unique zero point α, this implies that $f_\alpha(z)$ does not generate the quasi-invariant subspace $[z-\alpha]$.

The preceding example exhibits an important difference between the Fock space and the Bergman space on the unit disk. Just as one has seen, the differences arise out of the underlying domain of the Fock space being unbounded.

Since the Fock space and reproducing function spaces enjoy many common properties, Chapter 5 will explore the structure of general reproducing function spaces on $\mathbb{C}^n$. This requires us to introduce the following notion. Let $Hol(\mathbb{C}^n)$ denote the ring of all entire functions on the complex n-space $\mathbb{C}^n$, and let X be a Banach space contained in $Hol(\mathbb{C}^n)$. We call X a reproducing Banach space on $\mathbb{C}^n$ if X satisfies:

(a) the polynomial ring $\mathcal{C}$ is dense in X;

(b) the evaluation linear functional $E_\lambda(f) = f(\lambda)$ is continuous on X for each $\lambda \in \mathbb{C}^n$.

The basic example is the Fock space mentioned above. Chapter 5 will mainly be concerned with the following:

1. the structure of quasi-invariant subspace,
2. algebraic reduction of quasi-invariant subspaces, and
3. the equivalence problem of quasi-invariant subspaces.

We refer the reader to Chapter 5 for more details on these questions.

To generalize the operator-theoretic aspects of function theory on the unit disk to multivariable operator theory, Arveson began a systematic study for the theory of d-contractions [Arv1, Arv2, Arv3]. This theory depends essentially on a special function space on the unit ball, namely, the Arveson space H_d^2 which is defined by the reproducing kernel $1/\langle z, \lambda\rangle$. It is not difficult to verify that the Arveson space is an analytic Hilbert module on the unit ball. In Chapter 6, we collect some basic results from [Arv1, Arv2] which are related to preceding Chapters. In the preceding Chapters we are mainly concerned with Hardy modules and Bergman modules. However, the Arveson module, unlike Hardy modules and Bergman modules associated with some measures on underlying domains, is not associated with any measure on $\mathbb{C}^d$ and it is distinguished among all analytic Hilbert modules on the unit ball, which have some natural property by being the largest Hilbert norm (cf. [Arv1]). Hence the Arveson module is included in each other analytic Hilbert module on the unit ball which has the above required property. Since the Arveson module is never given by some measure on $\mathbb{C}^d$, this leads to the fact that the d-duple $(M_{z_1}, \cdots, M_{z_d})$ of the coordinate multipliers on the Arveson space (called the d-shift) is not subnormal. Just as we will see, the d-shift plays an essential role in this modification of dilation theory, especially in the theory of d-contractions [Arv1, Arv2]. By the d-shift an appropriate version of Von Neumann's inequality was obtained by Arveson (cf. Theorem 6.1.7). The Toeplitz algebra $\mathcal{T}^d$ $(d > 1)$ on the Arveson space, unlike the Toeplitz algebra on the Hardy space $H^2(\mathbb{B}_d)$, enjoys many interesting properties, one of which is that the identity representation of the $\mathcal{T}^d$ is a boundary representation for $d+1$-dimensional space $span\{I, S_1, \cdots, S_d\}$ (cf. [Arv1] or see Corollary 6.2.4). From this one finds that there is no nontrivial isometry that commutes with d-shift (cf. Corollary 6.2.5). This yields that there is not a nontrivial submodule that is unitarily equivalent to the Arveson module itself. In fact, we see from Section 6.4 that Arveson submodules have stronger rigidity than Hardy submodules. We call the reader's attention to Chapter 6 for detailed results on the Arveson submodule.

Chapter 7 turns to the extension theory of Hilbert modules over function algebras. In view of Hilbert modules, the theory of function algebras is emphasized since it plays the analogous role of ring theory as in the context of algebraic modules. In the purely algebraic setting there are two different methods of constructing the Ext-functor. One is as a derived functor of Hom and the other is the Yoneda approach (see [HS]), which realizes Ext as equivalence classes of resolutions. Let $\mathbb{A}$ be a function algebra, and let $\mathcal{H}(\mathbb{A})$ be the category of Hilbert modules over $\mathbb{A}$ together with Hilbert module maps. What seems to make things most difficult is, in general, that the category $\mathcal{H}(\mathbb{A})$ lacks enough projective and injective objects, and hence it is not possible to define the functor Ext as the derived functor of Hom as in [HS]. However, the Yoneda construction applies to our situation, as done by Carlson and Clark in [CC1]. Since extensions have some interesting applications to operator theory and analytic Hilbert modules, Chapter 6 is mainly concerned with a situation of extensions (i.e., 1-extensions), where we only consider 1-extensions, but not n-extensions ($n > 1$). We refer the interested reader to [Pau3, HS] for some results concerning n-extensions.

In studying Hilbert modules, as in studying any algebraic structure, the standard procedure is to look at submodules and associated quotient modules. The extension problem then appears quite naturally: given two Hilbert modules H, K, what module J may be constructed with submodule H and associated quotient module K, i.e., $K \cong J/H (= J \ominus H)$? We then have a short exact sequence

$$E: \ 0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$$

of Hilbert $\mathbb{A}$-modules, where α, β are Hilbert module maps. Such a sequence is called an extension of K by H, or simply J is called an extension of K by H. By definition $Ext_{\mathbb{A}}(K, H)$ is the extension group whose elements are equivalence classes of K by H (defined in Chapter 7). For the purposes of this volume we omit the superscript "1" on Ext because we shall only consider the 1-extensions. For the analyst, the homological invariant $Ext(-,-)$ is important because it is closely related to function theory and operator theory. To illustrate this connection, let us mention results obtained by Carlson and Clark [CC1, CC2].

Theorem 1.0.9 (Carlson-Clark) *The following are true:*

(1) $Ext_{A(\mathbb{D})}(H^2(\mathbb{D}), H^2(\mathbb{D})) = 0$;

(2) $Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), H^2(\mathbb{D}^n)) \neq 0$,*if* $n > 1$.

By a simple analysis, all extensions of the Hardy modules on the unit disk are trivial because the resulting function theory is one dimensional and the module is isometric. However, for nonvanishing of the extension group of the Hardy module on the polydisk, the reason may be based on the Hartogs phenomenon in several complex variables. Actually, Guo and Chen [GC] obtained an explicit representation for the extension group of the Hardy module by the restricted BMO function space introduced in [CS] (see Example 7.2.13). We mention work of Ferguson [Fe1, Fe2], where the extension theory is applied to operator theory. To see the connection between the extension theory and operator theory, recall that "the Halmos problem" is to ask if each polynomially bounded operator is similar to a contraction. Pisier has produced examples of polynomially bounded operators which are not similar to contractions [Pi2]. We say that a Hilbert module over $A(\mathbb{D})$ is cramped if it is similar to a contractive Hilbert module. In [CCFW], Carlson, Clark, Foias and Williams introduced the category $\mathfrak{C}$ of all cramped Hilbert modules over $A(\mathbb{D})$, and proved that in the category $\mathfrak{C}$, a module is projective if and only if it is similar to an isometric module. Hence, the Hardy module $H^2(\mathbb{D})$ is projective in the category $\mathfrak{C}$. A natural problem is: is $H^2(\mathbb{D})$ projective in the category $\mathcal{H}(A(\mathbb{D}))$? Although this problem remains open, a negative answer for this problem will lead to negation of "the Halmos problem." Applications of extension theory to rigidity of analytic Hilbert modules are presented in [CG, GC, Guo3] (see Chapter 7 for more details).

Finally, we should mention the book of Helemskii [Hel], and the paper of Paulsen [Pau3], which reported a systematic study of the homology of Banach and topological algebras, and of operator spaces, respectively. The ideals of the homology in [Hel, Pau3], of course, can be adapted to study Hilbert modules over function algebras.

This volume is mainly based on our work. Therefore there are many important works in Hilbert modules that are not included here, for example, the geometric invariants on quotient modules by Douglas, Misra and Varughese [DM1, DM2, DMV], and curvature invariants on Hilbert modules by Arveson [Arv1, Arv2, Arv3]. We refer the interested reader to these works for more topics related to Hilbert modules.

Chapter 2

Characteristic spaces and algebraic reduction

In this chapter, we first introduce a notion of characteristic spaces for ideals of the polynomial ring, and we use the characteristic space theory to study the structure of ideals. Then we extend the method to the case of analytic Hilbert modules. This allows us to obtain results on algebraic reduction of analytic Hilbert modules. The characteristic space theory is essential for the study of rigidity of analytic Hilbert modules. This will be discussed in Chapter 3.

2.1 Characteristic spaces for ideals of polynomials

Algebra fundamental theorem says that a polynomial p in one variable is completely determined by its zeros (counting multiplicities). We state this fact in the following manner. Set $I = p\mathcal{C}$, where $\mathcal{C}$ is the polynomial ring in one variable. For $\lambda \in \mathbb{C}$, define

$$I_\lambda = \{q \in \mathcal{C} : \ q(D)r|_\lambda = 0, \ \forall r \in I\},$$

where $q(D) = \sum_{i=1}^n a_i \frac{d^i}{dz^i}$ if $q = \sum_{i=1}^n a_i z^i$. Notice that for any analytic function f in some neighborhood of λ, one easily verifies the equality

$$q(D)(zf)|_\lambda = \lambda q(D)f|_\lambda + \frac{dq}{dz}(D)f|_\lambda.$$

Therefore, the space I_λ is invariant under $\frac{d}{dz}$. Furthermore, we have the following observations:

1. λ is the zero of p if and only if $\dim I_\lambda > 0$, and in this case;
2. multiplicity of λ, as a zero point of p, is equal to $\dim I_\lambda$.

Based on the above observations, one finds that the space I_λ carries key information of the multiplicity of p at λ.

We will generalize this idea to an ideal of the polynomial ring in several variables by introducing the characteristic spaces of the ideal.

We also let $\mathcal{C}$ denote the ring of all polynomials on $\mathbb{C}^n$, and let

$$q = \sum a_{m_1 \dots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$$

be a polynomial. We use $q(D)$ to denote the linear partial differential operator

$$q(D) = \sum a_{m_1 \cdots m_n} \frac{\partial^{m_1+m_2+\cdots+m_n}}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}}.$$

Let I be an ideal of $\mathcal{C}$, and $Z(I)$ be the zero variety of I, that is,

$$Z(I) = \{\lambda \in \mathbb{C}^n : \, q(\lambda) = 0, \forall q \in I\}.$$

For $\lambda \in \mathbb{C}^n$, set

$$I_\lambda = \{q \in \mathcal{C} : \; q(D)p|_\lambda = 0, \; \forall p \in I\},$$

where $q(D)p|_\lambda$ denotes $(q(D)p)(\lambda)$. Just as in the case of one variable, a careful verification shows that for any polynomial q and analytic function f,

$$q(D)(z_j f)|_\lambda = \lambda_j q(D) f|_\lambda + \frac{\partial q}{\partial z_j}(D) f|_\lambda, \qquad j = 1, 2, \cdots, n.$$

Therefore I_λ is invariant under the action of the basic partial differential operators $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \cdots, \frac{\partial}{\partial z_n}\}$, and $I_\lambda \neq 0$ if and only if $\lambda \in Z(I)$. We call I_λ the characteristic space of I at λ. The envelope, I_λ^e, of I at λ is defined as

$$I_\lambda^e \; = \; \{q \in \mathcal{C} : \; p(D)q|_\lambda \; = \; 0, \; \forall p \in I_\lambda\}.$$

The preceding equalities imply that I_λ^e is an ideal of $\mathcal{C}$ and it contains I.

For the next results we need to recall some concepts from commutative algebra. General references on commutative algebra are [AM, ZS]. For the polynomial ring $\mathcal{C}$, a basic fact is that it is Noetherian. This means that each ideal of $\mathcal{C}$ is finitely generated. An ideal Q of $\mathcal{C}$ is called *primary* if the conditions $p, q \in \mathcal{C}$, $pq \in Q$ and $p \notin Q$ imply the existence of an integer m such that $q^m \in Q$. For a primary ideal Q, its radical ideal P, which is defined by

$$P = \{p \in \mathcal{C} : \; p^l \in Q \; \text{ for some positive integer } l\},$$

is prime, and it is called the associated prime ideal of Q, while Q is said to be P-primary. In general, for an ideal of $\mathcal{C}$, the most important thing is the following Lasker-Noether decomposition theorem[ZS]. For this theorem, the following definition is needed: a primary decomposition $I = \cap_{j=1}^m I_j$ of an ideal I, as an intersection of finitely many primary ideals I_j, is said to be irredundant if it satisfies the following conditions:

(a) no I_j contains the intersection of the other ones;

(b) the I_j have distinct associated prime ideals.

We can now state the Lasker-Noether decomposition theorem for the polynomial ring: let I be an ideal of polynomial ring on $\mathbb{C}^n$, then I has an irredundant primary decomposition,

$$I = \bigcap_{j=1}^{m} I_j,$$

where each I_j is P_j-primary for some prime ideal P_j. While the set $\{I_j : 1 \leqq j \leqq m\}$ is not uniquely determined by I, the set $\{P_j : 1 \leqq j \leqq m\}$ is, and these are the associated primes of I. Note that

$$Z(I) = \bigcup_{j=1}^{m} Z(P_j),$$

and we will call each $Z(P_j) = Z(I_j)$ an algebraic component of I.

Now let $\mathcal{A}$ be an ideal of $\mathcal{C}$. Then one can define the $\mathcal{A}$–adic topology on $\mathcal{C}$ by the powers of $\mathcal{A}$. This means that the closure of a subset E of $\mathcal{C}$ in the $\mathcal{A}$-adic topology is

$$\bar{E} = \bigcap_{j \geq 1} (E + \mathcal{A}^j).$$

By Krull's theorem (see [ZS, Vol(I), p. 217, Theorem 13]), if $I = \bigcap_{j=1}^{m} I_j$ is an irredundant primary decomposition, then the closure of I in the $\mathcal{A}$-adic topology is equal to the intersection of those I_j with $I_j + \mathcal{A} \neq \mathcal{C}$. This fact will be used in the proof of the following theorem.

Motivated by polynomials in one variable, Guo proved the following theorem in [Guo1].

Theorem 2.1.1 *Let $I = \bigcap_{j=1}^{m} I_j$ be an irredundant primary decomposition of the ideal I, and $\lambda \in Z(I)$. Setting $\sum_\lambda = \{j : \lambda \in Z(I_j)\}$, we then have that*

$$I_\lambda^e = \bigcap_{j \in \sum_\lambda} I_j.$$

Proof. For $\lambda \in Z(I)$, let $\mathcal{U}_\lambda$ denote the maximal ideal of polynomials that vanish at λ, that is, $\mathcal{U}_\lambda = \{q \in \mathcal{C} : q(\lambda) = 0\}$. We claim

$$I_\lambda^e = \bigcap_{j \geq 1} (I + \mathcal{U}_\lambda^j).$$

In fact, the inclusion $\supseteq$ is straightforward and the reverse inclusion goes as follows: for each polynomial $p \in \mathcal{C}$, let $L_\lambda(p)$ denote the linear functional on $\mathcal{C}$ defined by

$$\langle q, L_\lambda(p) \rangle = q(D)p|_\lambda.$$

Fix $j \geq 1$ and let $\hat{p} \in I_\lambda^e$. To see that $\hat{p} \in I + \mathcal{U}_\lambda^j$, we need to find a polynomial $p \in I$ such that the linear functional $L_\lambda(\hat{p})$ agrees with $L_\lambda(p)$ on $\mathcal{P}_j$, the space of polynomials of degree less than j. Since $\hat{p}$ is in I_λ^e, $I_\lambda \subseteq Ker(L_\lambda(\hat{p}))$ and so

$$I_\lambda \cap \mathcal{P}_j \subseteq Ker(L_\lambda(\hat{p})) \cap \mathcal{P}_j = Ker(L_\lambda(\hat{p})|\mathcal{P}_j).$$

By definition, $I_\lambda \cap \mathcal{P}_j = \cap Ker(L_\lambda(p)|\mathcal{P}_j)$ where the intersection is over all $p \in I$. Now the map defined on I by $p \mapsto L_\lambda(p)|\mathcal{P}_j$ has finite dimensional range and so there exist polynomials $p_1, \cdots, p_l$ contained in I, such that

$$\bigcap_{p \in I} Ker(L_\lambda(p)|\mathcal{P}_j) = \bigcap_{k=1}^{l} Ker(L_\lambda(p_k)|\mathcal{P}_j).$$

Hence,

$$\bigcap_{k=1}^{l} Ker(L_\lambda(p_k)|\mathcal{P}_j) \subseteq Ker(L_\lambda(\hat{p})|\mathcal{P}_j).$$

It follows that there exist constants α_k such that

$$L_\lambda(\hat{p})|\mathcal{P}_j = \sum_k \alpha_k L_\lambda(p_k)|\mathcal{P}_j = L_\lambda(\sum_k \alpha_k p_k)|\mathcal{P}_j.$$

Since the polynomial $p = \sum_k \alpha_k p_k$ is in I, the claim follows immediately.

Note that the above claim means that the envelope of I at λ, I_λ^e, equals the closure of I in the $\mathcal{U}_\lambda$–adic topology. The preceding statements immediately imply that

$$I_\lambda^e = \bigcap_{j \in \sum_\lambda} I_j$$

and hence Theorem 2.1.1 follows.

Applying Theorem 2.1.1 gives the following.

Corollary 2.1.2 *Let Ω be a subset of $\mathbb{C}^n$. If each algebraic component of the ideal I meets Ω nontrivially, then*

$$I = \bigcap_{\lambda \in \Omega} I_\lambda^e.$$

In particular, if P is a primary ideal and $\lambda \in Z(P)$, then $P = P_\lambda^e$.

The next corollary generalizes Theorem 2.7 in [DPSY], and it will be used in the study of algebraic reduction of analytic Hilbert modules.

Corollary 2.1.3 *Let Ω be a subset of $\mathbb{C}^n$, and I, J be two ideals of $\mathcal{C}$. If each algebraic component of I meets Ω nontrivially, and for each $\lambda \in \Omega$, $I_\lambda \subseteq J_\lambda$, then $J \subseteq I$.*

Proof. Write $J = J_1 \cap J_2$ such that each algebraic component of J_1 meets Ω nontrivially, and each of J_2 does not. From $J_1 J_2 \subseteq J \subseteq J_1$, one obtains that $J_\lambda = J_{1\lambda}$ for every $\lambda \in \Omega$. By Corollary 2.1.2, it follows that

$$I = \bigcap_{\lambda\in\Omega} I_\lambda^e \supseteq \bigcap_{\lambda\in\Omega} J_\lambda^e = \bigcap_{\lambda\in\Omega} J_{1\lambda}^e = J_1 \supseteq J,$$

which completes the proof of Corollary 2.1.3.

Remark 2.1.4 *Corollary 2.1.2 shows that every primary ideal is completely determined by its characteristic space at any zero point. This is a very useful localization property which says that if P_1, P_2 are primary ideals such that $Z(P_1) \cap Z(P_2) \neq \emptyset$, then $P_1 \neq P_2$ if and only if for each $\lambda \in Z(P_1) \cap Z(P_2)$, $P_{1\lambda} \neq P_{2\lambda}$, if and only if for some $\lambda \in Z(P_1) \cap Z(P_2)$ $P_{1\lambda} \neq P_{2\lambda}$. In an attempt to generalize this result to an ideal of polynomial ring, we introduce terminology about the characteristic set of the ideal. For a finite set Ω of $\mathbb{C}^n$ such that each algebraic component of the ideal I meets Ω nontrivially, we use $|\Omega|$ to denote the cardinality of Ω. The minimum cardinality of such a set is called the characteristic cardinality of I and is denoted by $C(I)$. For such a set Ω with $|\Omega| = C(I)$, we call Ω a characteristic set of I. It is easy to see that the characteristic cardinality $C(I)$ of I is less than the cardinality of the algebraic components of I unless the algebraic components of I do not mutually intersect. Moreover, for two ideals I_1 and I_2, we have that*

$$C(I_1 \cap I_2) \leqq C(I_1) + C(I_2).$$

For the ideal I, Corollary 2.1.2 implies that I is completely determined by a characteristic set.

From Remark 2.1.4, one sees that a prime ideal P is completely determined by characteristic space at any zero point. Then we want to know how characteristic space of P at any zero point behaves.

Theorem 2.1.5 *Let P be a prime ideal. Then the following statements are equivalent:*

(1) for $\lambda \in Z(P)$, the ideal P has the same characteristic space at any zero point in some neighborhood of λ;

(2) the ideal P is generated by polynomials with degree 1, i.e., by linear polynomials;

(3) the ideal P has the same characteristic space at any point in $Z(P)$.

Before proving Theorem 2.1.5, let us translate the theorem into geometric language. Let V be an algebraic variety and $\lambda \in V$. The characteristic space of V at λ is defined as that of $I(V)$ at λ, here $I(V) = \{p \in \mathcal{C} : p|_V = 0\}$.

Hence, the theorem says that for an irreducible algebraic variety V, V has the same characteristic space at any point if and only if V is a linear variety.
Proof. For simplicity, all proofs are sketched for the prime ideal of polynomials in two variables, while the conclusions hold in several variables.
(1)$\Rightarrow$(2). Without loss of generality we may assume that $\lambda = 0$ and $\mathcal{O}$ is a neighborhood of 0 such that for any $\mu \in \mathcal{O} \cap Z(P)$, $P_\mu = P_0$. By Corollary 2.1.2, we have that

$$P = P_0^e = P_\mu^e.$$

From the definition of the envelope and the equality $P = P_0^e = P_\mu^e$, one immediately obtains that

$$P = \{p(z - \mu_1, w - \mu_2) : p \in P\} \quad \text{for} \quad \mu = (\mu_1, \mu_2) \in \mathcal{O} \cap Z(P).$$

For any $q \in P$, since there exists a polynomial $p \in P$ such that $q(z, w) = p(z - \mu_1, w - \mu_2)$, it follows that for each natural number n, $n\mu = (n\mu_1, n\mu_2)$ is in $Z(P)$. Let $F \in P$, and let $F = F_0 + F_1 + \cdots + F_d$ be the decomposition of F into homogeneous polynomials F_k of degree k. Note that

$$F(n\mu) = \sum_{i=0}^{d} F_i(n\mu) = \sum_{i=0}^{d} n^i F_i(\mu) = 0, \;\; n = 1, 2, \cdots .$$

Thus, for any $\mu \in \mathcal{O} \cap Z(P)$,

$$F_i(\mu) = 0, \quad i = 0, 1, \cdots, d.$$

Since P is prime, applying [Ken, Theorem 2.11] gives that F_i is in P for $i = 0, 1, \cdots, d$. The above discussion thus shows that the ideal P is generated by homogeneous polynomials. Since any homogeneous polynomial p in z and w has the decomposition

$$p(z, w) = \prod_k (\alpha_k z + \beta_k w),$$

and P is prime, we conclude that the ideal P is generated by polynomials with degree 1, i.e., by linear polynomials.
(2)$\Rightarrow$(3). We may assume that P is generated by $\alpha_1 z + \beta_1 w + \gamma_1,\ \alpha_2 z + \beta_2 w + \gamma_2$. One immediately finds that for any $\mu \in Z(P)$, $P_\mu = I_0$, where I is the ideal generated by $\alpha_1 z + \beta_1 w,\ \alpha_2 z + \beta_2 w$.
(3)$\Rightarrow$(1). Trivially.

We now return to the comparison between ideals of polynomials by using characteristic space theory. Actually, arising from the observation for polynomials in one variable, for an ideal I of polynomial ring in several variables we define the multiplicity of I at $\lambda \in \mathbb{C}^n$ by the dimension of characteristic space, $\dim I_\lambda$. Of course, we allow the case that the multiplicity is infinite. Let I_1, I_2 be ideals of polynomial ring, and $\lambda \in \mathbb{C}^n$. We say that I_1 and I_2

have the same multiplicity at λ if $I_{1\lambda} = I_{2\lambda}$, and use the symbol $Z(I_2)\backslash Z(I_1)$ to denote the set $\{\lambda \in Z(I_2) : I_{2\lambda} \neq I_{1\lambda}\}$. If $I_1 \supseteq I_2$ and $\lambda \in Z(I_2)\backslash Z(I_1)$, the multiplicity of I_2 relative to I_1 at λ is defined by dim $I_{2\lambda}/I_{1\lambda}$. In this way, the cardinality, card $(Z(I_2)\backslash Z(I_1))$ of $Z(I_2)\backslash Z(I_1)$, is defined by the equality

$$card\,(Z(I_2)\backslash Z(I_1)) = \sum_{\lambda \in Z(I_2)\backslash Z(I_1)} \dim\, I_{2\lambda}/I_{1\lambda}.$$

The following theorem was proved originally in [Guo4]. Here, its proof is based on characteristic space theory.

Theorem 2.1.6 *Let I_1, I_2 be ideals in $\mathcal{C}$, and $I_1 \supseteq I_2$. If the set $Z(I_2)\backslash Z(I_1)$ is bounded, then* dim $I_1/I_2 < \infty$, *that is, I_2, is finite codimensional in I_1.*

Proof. First note that the Lasker-Noether decomposition theorem implies that any ideal J can be uniquely decomposed into $J_1 \cap J_2$ such that $Z(J_1)$ is bounded, and each algebraic component of J_2 is unbounded. Let

$$I_j = I_j' \cap I_j'', \quad j = 1, 2$$

be the decompositions of I_j as mentioned above. Set

$$\Sigma = Z(I_1') \cup Z(I_2') \cup (Z(I_2)\backslash Z(I_1)).$$

Since I_j' meets $\mathbb{C}^n\backslash\Sigma$ trivially, and

$$I_j' I_j'' \subset I_j \subset I_j'' \quad \text{for} \quad j = 1, 2,$$

one obtains that

$$I_{j\lambda} = I''_{j\lambda} \quad \text{for each} \quad \lambda \in \mathbb{C}^n\backslash\Sigma.$$

By the assumption and Corollary 2.1.2, we have

$$I_1'' = \bigcap_{\lambda\in\mathbb{C}^n\backslash\Sigma} I''^e_{1\lambda} = \bigcap_{\lambda\in\mathbb{C}^n\backslash\Sigma} I^e_{1\lambda} = \bigcap_{\lambda\in\mathbb{C}^n\backslash\Sigma} I^e_{2\lambda} = \bigcap_{\lambda\in\mathbb{C}^n\backslash\Sigma} I''^e_{2\lambda} = I_2''.$$

Let $L = I_1'' = I_2''$. Therefore

$$I_1 = I_1' \cap L, \quad I_2 = I_2' \cap L.$$

Since an ideal is of finite codimension in $\mathcal{C}$ if and only if its zero variety is bounded (see [AC] or [ZS]), I_1', I_2' are finite codimensional. Define a linear map σ by

$$\sigma : I_1/I_2 \to (I_1' + I_2')/I_2'; \;\; \sigma(p + I_2) = p + I_2', \; \forall p \in I_1.$$

By the above equalities, it is easy to check that σ is injective. Because I_2' is of finite codimension in $\mathcal{C}$, hence I_2' is of finite codimension in $I_1' + I_2'$. This implies that I_2 is of finite codimension in I_1, that is, dim $I_1/I_2 < \infty$. The proof is complete.

Now suppose that $I_1 \supseteq I_2$. Let $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$ be the n-tuple of operators that are defined on the quotient module I_1/I_2 over the ring $\mathcal{C}$ by

$$M_{z_i}\tilde{f} = (\widetilde{z_i f}), \quad i = 1, \cdots, n.$$

We use $\sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$ to denote the joint eigenvalues for the tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$. This means that

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$$

if and only if there exists a $q \in I_1$ but $q \notin I_2$ such that

$$(z_i - \lambda_i)q \in I_2, \quad i = 1, 2, \cdots, n.$$

We will now give the following theorem due to Guo [Guo4].

Theorem 2.1.7 *Let I_1, I_2 be ideals in $\mathcal{C}$, $I_1 \supseteq I_2$ and* dim $I_1/I_2 = k < \infty$. *Then we have*

(1) $Z(I_2)\backslash Z(I_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$;

(2) $I_2 = \{q \in I_1 : \ p(D)q|_\lambda = 0, \ p \in I_{2\lambda}, \ \lambda \in Z(I_2)\backslash Z(I_1)\}$;

(3) dim $I_1/I_2 = \sum_{\lambda \in Z(I_2)\backslash Z(I_1)} dim\, I_{2\lambda}/I_{1\lambda} = card(Z(I_2)\backslash Z(I_1))$.

It is worth noting that (3) of Theorem 2.1.7 says the codimension dim I_1/I_2 of I_2 in I_1 is equal to the cardinality of zeros of I_2 relative to I_1. In this way, the equality (3) is an interesting codimension formula whose left side is an algebraic invariant, but right side is geometric invariant.

Proof. (1) Write

$$I_1 = I_2 \dot{+} R$$

where R is a linear space of polynomials with dim $R =$ dim I_1/I_2. We may consider the n-tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$ being defined on R by

$$M_{z_i}q = \text{decomposition of } \ z_i q$$

on R for any $q \in R$. By [Cur1], they can be simultaneously triangularized as

$$M_{z_i} = \begin{pmatrix} \lambda_i^{(1)} & & \star \\ & \ddots & \\ & & \lambda_i^{(k)} \end{pmatrix}.$$

Here $i = 1, 2, \cdots, n$ and $k =$ dim I_1/I_2, so that $\sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$ is equal to $\{\lambda^{(1)}, \cdots, \lambda^{(k)}\}$. Then we have

$$\mathcal{U}_{\lambda^{(k)}} \cdots \mathcal{U}_{\lambda^{(2)}} \mathcal{U}_{\lambda^{(1)}}\, I_1 \subseteq I_2 \subseteq I_1$$

This implies

$$Z(I_2)\backslash Z(I_1) \subseteq \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}).$$

Let $\lambda \in \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$, that is, λ is a joint eigenvalue of the operator tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$. This means that there is a polynomial q in R such that $q\mathcal{U}_\lambda \subseteq I_2$. Setting $I_2^\dagger$ to be the ideal generated by I_2 and q, then for any $\lambda' \neq \lambda$, it holds that

$$(I_2^\dagger)_{\lambda'} = I_{2\lambda'}.$$

Therefore, by Corollary 2.1.2, we have that

$$(I_2^\dagger)_\lambda \subsetneqq I_{2\lambda}.$$

Since $(I_2^\dagger)_\lambda \supseteq I_{1\lambda}$, this yields that

$$I_{1\lambda} \subsetneqq I_{2\lambda},$$

that is, λ is in $Z(I_2)\backslash Z(I_1)$. The above discussion gives that

$$Z(I_2)\backslash Z(I_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}).$$

(2) Set

$$I_2^\natural = \{q \in I_1 : p(D)q|_\lambda = 0,\ p \in I_{2\lambda},\ \lambda \in Z(I_2)\backslash Z(I_1)\}.$$

Then $I_2^\natural$ is an ideal which contains I_2. It follows that

$$(I_2^\natural)_\lambda \subseteq I_{2\lambda}, \quad \text{for all} \quad \lambda \in \mathbb{C}^n.$$

By the representation of $I^\natural$, we have that

$$(I_2^\natural)_\lambda \supseteq I_{2\lambda}, \quad \text{for all} \quad \lambda \in \mathbb{C}^n.$$

According to Corollary 2.1.2, we obtain that $I_2^\natural = I_2$. The proof of (2) is complete.

(3) The proof is by induction on numbers of points in $Z(I_2)\backslash Z(I_1)$. If $Z(I_2)\backslash Z(I_1)$ contains only a point λ, then by (2), I_2 can be written as

$$I_2 = \{q \in I_1 : p(D)q|_\lambda = 0,\ p \in I_{2\lambda}\}.$$

We define the pairing

$$[-,-] :\ I_{2\lambda}/I_{1\lambda} \times I_1/I_2 \to \mathbb{C}$$

by $[\tilde{p}, \tilde{q}] = p(D)q|_\lambda$. Clearly, this is well defined. From this pairing and the representation of I_2, it is not difficult to see that

$$\dim\, I_1/I_2 = \dim\, I_{2\lambda}/I_{1\lambda} = \text{card}\,(Z(I_2)\backslash Z(I_1)).$$

Now let $l > 1$, and assume that (3) has been proved for $Z(I_2)\backslash Z(I_1)$ containing different points less than l. Let $Z(I_2)\backslash Z(I_1) = \{\lambda_1, \cdots, \lambda_l\}$, here $\lambda_i \neq \lambda_j$ for $i \neq j$. Writing

$$I_2^\star = \{q \in I_1 : p(D)q|_{\lambda_1} = 0, \, p \in I_{2\lambda_1}\},$$

then $I_2^\star$ is an ideal, and $(I_2^\star)_{\lambda_1} = I_{2\lambda_1}$. Just as the above proof, we have that

$$\dim I_1/I_2^\star = \dim I_{2\lambda_1}/I_{1\lambda_1}$$

Set

$$I_{2\lambda_1} = I_{1\lambda_1} \dot{+} R, \quad \text{where} \quad \dim R = \dim I_{2\lambda_1}/I_{1\lambda_1}.$$

Let $\hat{R}$ denote linear space of polynomials generated by R that is invariant under the action of $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. Put

$$\mathcal{Q}_{\mathcal{R}} = \{p \in \mathcal{C} : \; q(D)p|_{\lambda_1} = 0, \; q \in \hat{R}\}.$$

Then it is easily verified that $\mathcal{Q}_{\mathcal{R}}$ is a finite codimensional ideal of $\mathcal{C}$ with only zero point λ_1 because $\hat{R}$ is finite dimensional. Thus

$$\mathcal{Q}_{\mathcal{R}} I_1 \subseteq I_2^\star \subseteq I_1.$$

From the above inclusions, we see that

$$I_{1\lambda} = (I_2^\star)_\lambda \quad \text{if} \quad \lambda \neq \lambda_1,$$

and hence

$$Z(I_2)\backslash Z(I_2^\star) = \{\lambda_2, \cdots, \lambda_l\}.$$

By the induction hypothesis, we have

$$\dim I_2^\star/I_2 = \sum_{j=2}^{l} \dim I_{2\lambda_j}/(I_2^\star)_{\lambda_j} = \sum_{j=2}^{l} \dim I_{2\lambda_j}/I_{1\lambda_j}.$$

This gives that

$$\begin{aligned}\dim I_1/I_2 &= \dim I_1/I_2^\star + \dim I_2^\star/I_2 \\ &= \sum_{j=1}^{l} \dim I_{2\lambda_j}/I_{1\lambda_j} \\ &= \mathrm{card}(Z(I_2)\backslash Z(I_1)).\end{aligned}$$

The proof is thus completed.

Combining Theorem 2.1.6 and Theorem 2.1.7 gives the following corollary.

Corollary 2.1.8 *Let I_1, I_2 be ideals in $\mathcal{C}$, and $I_1 \supseteq I_2$. Then the following are equivalent:*

(1) I_2 is finite codimensional in I_1,

(2) $Z(I_2)\backslash Z(I_1)$ is a bounded subset of $\mathbb{C}^n$,

(3) $Z(I_2)\backslash Z(I_1)$ is a finite set,

and in the above case, we have

$$\dim\, I_1/I_2 = \mathrm{card}\,(Z(I_2)\backslash Z(I_1)).$$

In particular, we have the following.

Corollary 2.1.9 *Let I be an ideal in $\mathcal{C}$. Then the following are equivalent:*

(1) I is finite codimensional,

(2) $Z(I)$ is a bounded subset of $\mathbb{C}^n$,

(3) $Z(I)$ is a finite set,

and in the above case, we have

$$\mathrm{codim}\, I = \dim \mathcal{C}/I = \mathrm{card}\,(Z(I)).$$

As an application of Corollary 2.1.9, it will be shown that for a finite codimensional ideal I of the polynomial ring $\mathcal{C}$ on $\mathbb{C}^n$, the cardinality of zeros of I^k is of polynomial growth as k is large enough. We assume that $I = \bigcap_{j=1}^m I_j$ is an irredundant primary decomposition of I, where I_j is primary for a maximal ideal of evaluation at some point λ_j. Since $I_i + I_j = \mathcal{C}$ for $i \neq j$, it follows that

$$I = \prod_{j=1}^m I_j.$$

This gives that for any natural number k,

$$I^k = \prod_{j=1}^m I_j^k = \bigcap_{j=1}^m I_j^k.$$

Thus,

$$\mathrm{codim}\, I^k = \sum_{j=1}^m \mathrm{codim}\, I_j^k.$$

It is well known that for large integer k, $\mathrm{codim}(I_j^k)$ is a polynomial of k with the degree n which is called the Hilbert-Samuel polynomial of I_j (see [AM, ZS] or [DY2]). Therefore, for each finite codimensional ideal I, the codimension of I^k in $\mathcal{C}$, $\mathrm{codim}(I^k)$, is a polynomial of k with the degree n, and denoted by $\mathcal{P}_I(k)$. The leading coefficient of $\mathcal{P}_I(k)$ thus is directly related to the number of zeros of I. By Corollary 2.1.9, it holds that

$$\mathrm{card}\big(Z(I)\big) = \mathrm{codim}\, I.$$

This gives the following.

Proposition 2.1.10 *Let I be a finite codimensional ideal of $\mathcal{C}$. Then for large integer k, the cardinality of zeros of I^k, $\mathrm{card}\big(Z(I^k)\big)$, is a polynomial of k with the degree n; more precisely,*

$$\mathrm{card}\big(Z(I^k)\big) = \mathcal{P}_I(k).$$

2.2 Algebraic reduction for analytic Hilbert modules

Let $\mathbb{T}$ be the unit circle in the complex plane, and let $L^2(\mathbb{T})$ be the Hilbert space of square integrable functions with respect to arc-length measure. Recall that the Hardy space $H^2(\mathbb{D})$ over the open unit disk $\mathbb{D}$ is the closed subspace of $L^2(\mathbb{T})$ spanned by the nonnegative powers of the coordinate function z. If M is a (closed) subspace of $H^2(\mathbb{D})$ that is invariant for the multiplication operator M_z, then Beurling's theorem [Beu] says that there exists an inner function η such that $M = \eta H^2(\mathbb{D})$. In terms of the language of Douglas's Hilbert module [DP], each submodule M of Hardy module $H^2(\mathbb{D})$ over the disk algebra $A(\mathbb{D})$ has the form $M = \eta H^2(\mathbb{D})$. For the Hardy module $H^2(\mathbb{D}^n)$ in the multi-variable version, a natural problem is to consider the structure of submodules of $H^2(\mathbb{D}^n)$. However, one quickly sees that a Beurling-like characterization is not possible (see [AC, Ru1]). Now we again return to the case of the Hardy module $H^2(\mathbb{D})$ and consider finite codimensional submodules. A routine verifying shows that M is a finite codimensional submodule if and only if there is a polynomial p whose zeros lie in $\mathbb{D}$ such that $M = p H^2(\mathbb{D})$, and the codimension of M in $H^2(\mathbb{D})$ equals the degree of p. At this point, Ahern and Clark [AC] first obtained the following characterization of submodules of $H^2(\mathbb{D}^n)$ having finite codimension.

Theorem 2.2.1 (Ahern-Clark) *Suppose M is a submodule of $H^2(\mathbb{D}^n)$ of codimension $k < \infty$. Then $\mathcal{C} \cap M$ is an ideal in the ring $\mathcal{C}$ such that*

(1) $\mathcal{C} \cap M$ is dense in M;

(2) $\dim \mathcal{C}/\mathcal{C} \cap M = k$;

(3) $Z(\mathcal{C} \cap M)$ is finite and lies in $\mathbb{D}^n$.

Conversely, if I is an ideal of the ring $\mathcal{C}$ with $Z(I)$ being finite, and $Z(I) \subset \mathbb{D}^n$, then $[I]$, the closure of I in $H^2(\mathbb{D}^n)$, is a submodule of $H^2(\mathbb{D}^n)$ with the same codimension as I in $\mathcal{C}$, and $[I] \cap \mathcal{C} = I$.

The above theorem shows that the description of finite codimensional submodules is, at least conceptually, an algebraic matter. Along this line, many authors, including Douglas, Paulsen, Sah and Yan [DP, DPSY], Axler and Bourdon [AB], Bercovici, Foias and Pearcy [BFP], Agrawal and Salinas [AS], Guo [Guo1, Guo2, Guo4], Putinar and Salinas [PS], Putinar [Pu1, Pu2] and Chen and Douglas [CD1, CD2], have developed a series of techniques to investigate algebraic reduction for Hilbert modules consisting of analytic functions on domains.

To provide an appropriate framework for study of algebraic reduction of analytic Hilbert modules on domains, we introduce some terminologies and notations that will be used throughout this chapter and the next chapter.

Let Ω be a bounded nonempty open subset of $\mathbb{C}^n$, $Hol(\Omega)$ denote the ring of analytic functions on Ω, and X be Banach space contained in $Hol(\Omega)$. We call X a reproducing Ω-space if X contains 1 and if for each $\lambda \in \Omega$ the evaluation functional, $E_\lambda(f) = f(\lambda)$, is a continuous linear functional on X. We say X is a reproducing $\mathcal{C}$-module on Ω if X is a reproducing Ω-space, and for each $p \in \mathcal{C}$ and each $x \in X$, px is in X. Note that, by a simple application of the closed graph theorem, the operator T_p defined to be multiplication by p is bounded on X. Note also that $\mathcal{C} \subset X$ follows from the fact that 1 is in X. For $\lambda \in \mathbb{C}^n$, one says that λ is a virtual point of X provided that the homomorphism $p \mapsto p(\lambda)$ defined on $\mathcal{C}$ extends to a bounded linear functional on X. Since X is a reproducing Ω-space, every point in Ω is a virtual point. We use $vp(X)$ to denote the collection of virtual points, then $vp(X) \supseteq \Omega$. Finally we say that X is an analytic Hilbert module on Ω if the following conditions are satisfied:

(1) X is a reproducing $\mathcal{C}$-module on Ω;

(2) $\mathcal{C}$ is dense in X;

(3) $vp(X) = \Omega$.

Remark 2.2.2 *Note that the above conditions (2) and (3) are equivalent to the following statement: for each $\lambda \notin \Omega$, $\mathcal{U}_\lambda$, the maximal ideal of $\mathcal{C}$ that vanishes at λ, is dense in X. In fact, if for each $\lambda \notin \Omega$, $\mathcal{U}_\lambda$ is dense in X. Then condition (2) is immediate. If there is a $\lambda_0 \notin \Omega$, while $\lambda_0 \in vp(X)$, then there exists a constant c_0 such that for any polynomial p, $|p(\lambda_0)| \le c_0 \|p\|$. By the assumptions, the maximal ideal $\mathcal{U}_{\lambda_0}$ is dense in X. It follows that there exists a sequence $\{p_n\}(\subset \mathcal{U}_{\lambda_0})$ converges to 1 in the norm of X. Note the inequality*

$$1 = |p_n(\lambda_0) - 1| \le c_0 \|p_n - 1\|.$$

This contradiction says that $vp(X) = \Omega$. In the opposite direction, suppose that there is a $\lambda_0 \notin \Omega$ such that $\mathcal{U}_{\lambda_0}$ is not dense in X. Then there exists a bounded linear functional x^* on X that annihilates $\mathcal{U}_{\lambda_0}$. Therefore for any polynomial p,

$$x^*(p) = x^*(p - p(\lambda_0)) + p(\lambda_0)x^*(1) = p(\lambda_0)x^*(1).$$

By the condition (2), $x^*(1) \neq 0$. This ensures that there exists a constant c_0 such that $|p(\lambda_0)| \le c_0 \|p\|$ for each polynomial p. This is contradictory to condition (3). We thus achieve the opposite direction.

For most "natural" reproducing Ω-spaces, they are analytic Hilbert modules on Ω. The basic examples are the Hardy modules, the Bergman modules and the Ditchless modules on both the ball and the polydisk. Especially, we will also study the Arveson module on the unit ball in Chapter 6. In the following we will use submodule to mean a closed subspace of X that is invariant under the multiplications of polynomials. For an ideal I of polynomials, we let $[I]$ denote the closure of I in X; then $[I]$ is a submodule of X. Let M be a submodule of analytic Hilbert module X on Ω. The zero variety of M is defined by

$$Z(M) = \{z \in \Omega : f(z) = 0, \forall f \in M\}.$$

We are now in a position to give a complete generalization of Ahern-Clark's result. The next theorem is due to Douglas, Paulsen, Sah and Yan [DPSY]. Actually they obtained a stronger result in [DPSY, Corollary 2.8] than stated here.

Theorem 2.2.3 (DPSY) *Let X be an analytic Hilbert module on Ω. Then the maps $I \mapsto [I]$ and $M \mapsto M \cap \mathcal{C}$ define bijective correspondence between the ideal I of $\mathcal{C}$ of finite codimension with $Z(I) \subset \Omega$ and the submodule M of X of finite codimension. These maps are mutually inverse and preserve codimension.*

Given submodules $M_1,\ M_2$ of X, and $M_1 \supseteq M_2$, we can define a canonical module homomorphism over the ring $\mathcal{C}$,

$$\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2,$$

by $\tau(\tilde{p}) = \tilde{p}$. If $M_1 \cap \mathcal{C}$ is dense in M_1, and M_2 is finite codimensional in M_1, then it is not difficult to verify the following proposition.

Proposition 2.2.4 *Under the above assumption, we have*

(1) $M_2 \cap \mathcal{C}$ is dense in M_2,

(2) the canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ is an isomorphism.

To obtain further generalization for Theorem 2.2.3, we will be concerned with the following problems:

1. How do we describe the structure of $Z(M_2)$ relative to $Z(M_1)$?
2. How is the submodule M_2 represented by M_1 and the zeros of M_2 via considering multiplicity?
3. How is codimension dim M_1/M_2 relative to the zeros and their multiplicities of $M_1,\ M_2$?
4. Conversely, if $I_1,\ I_2$ are ideals of $\mathcal{C}$, $I_1 \supseteq I_2$ and the set $Z(I_2)\backslash Z(I_1)$ is bounded. We want to know how $[I_2]$ is related to $[I_1]$.

Based on the characteristic space theory of analytic Hilbert modules, we will completely answer the problems mentioned above. First, as in the case of polynomials, we introduce the concepts of characteristic space and envelope of a submodule.

Let M be a submodule of X on Ω. For $\lambda \in \Omega$, set

$$M_\lambda = \{q \in \mathcal{C} : q(D)f|_\lambda = 0, \ \forall f \in M\}.$$

Then as in the case of polynomials, M_λ is invariant under the action by the basic partial differential operators $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \cdots, \frac{\partial}{\partial z_n}\}$, and M_λ is called to be the characteristic space of M at λ. The envelope of M at λ, M_λ^e, is defined by

$$M_\lambda^e = \{f \in X : q(D)f|_\lambda = 0, \ \forall q \in M_\lambda\}.$$

The same reasoning as in the case of polynomials shows that M_λ^e is a submodule of X, and $M_\lambda^e \supseteq M$. We now define the multiplicity of M at λ to be the dimension of characteristic space, dim M_λ. Let M_1, M_2 be submodules, and $\lambda \in \Omega$. We say that M_1 and M_2 have the same multiplicity at λ if $M_{1\lambda} = M_{2\lambda}$. The symbol $Z(M_2)\backslash Z(M_1)$ denotes the set $\{\lambda \in Z(M_2) : M_{2\lambda} \neq M_{1\lambda}\}$. If $M_1 \supseteq M_2$ and $\lambda \in Z(M_2)\backslash Z(M_1)$, the multiplicity of M_2 relative to M_1 at λ is defined by dim $M_{2\lambda}/M_{1\lambda}$. In this way, the cardinality, card $(Z(M_2)\backslash Z(M_1))$, of $Z(M_2)\backslash Z(M_1)$ is defined by the equality

$$\text{card}\,(Z(M_2)\backslash Z(M_1)) = \sum_{\lambda \in Z(M_2)\backslash Z(M_1)} \dim\, M_{2\lambda}/M_{1\lambda}.$$

The next theorem strengthens the conclusions of Theorem 3.1 in [Guo4]. In [Guo4], we deal only with the case of the Hardy module on the polydisk.

Theorem 2.2.5 *Let X be an analytic Hilbert module on Ω. Suppose M_2 is finite codimensional in M_1, and $M_1 \cap \mathcal{C}$ is dense in M_1. Then we have*

(1) $M_2 \cap \mathcal{C}$ *is dense in* M_2,

(2) the canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ *is an isomorphism,*

(3) $Z(M_2 \cap \mathcal{C})\backslash Z(M_1 \cap \mathcal{C}) = Z(M_2)\backslash Z(M_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega$,

(4) $M_2 = \{f \in M_1 : p(D)f|_\lambda = 0, \ p \in M_{2\lambda}, \ \lambda \in Z(M_2)\backslash Z(M_1)\}$,

(5) $\dim\, M_1/M_2 = \dim\, M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} = \text{card}(Z(M_2 \cap \mathcal{C})\backslash Z(M_1 \cap \mathcal{C})) = \text{card}(Z(M_2)\backslash Z(M_1))$.

Conversely, if I_1, I_2 are ideals in $\mathcal{C}$, $I_1 \supseteq I_2$ and $Z(I_2)\backslash Z(I_1) \subset \Omega$. Then

$$\dim\,[I_1]/[I_2] = \dim\, I_1/I_2,$$

that is, the canonical homomorphism $\tau : I_1/I_2 \to [I_1]/[I_2]$ is an isomorphism.

Notice that (5) of the theorem says that the codimension of M_2 in M_1 equals the cardinality of zeros of M_2 relative to M_1 by counting multiplicities. This is an interesting codimension formula whose left side is an algebraic invariant, while the right side is a geometric invariant.

Proof. We first claim that M_1 can be expressed as

$$M_1 = M_2 \dot{+} R$$

where R is a linear space of polynomials with dim $R =$ dim M_1/M_2. In fact, since $M_1 \cap \mathcal{C}$ is dense in M_1, there exists a polynomial q in $M_1 \cap \mathcal{C}$, $q \notin M_2$. Let Σ be the collection

$$\{L : \ L \text{ is linear space of polynomials, } L \subseteq M_1 \cap \mathcal{C}, \text{ and } L \cap M_2 = \{0\}\}.$$

Then Σ is not empty. If

$$\cdots \subseteq \Phi_\alpha \subseteq \Phi_\beta \subseteq \Phi_\gamma \subseteq \cdots$$

is an ascending chain in Σ, then $\bigcup_\alpha \Phi_\alpha$ is a linear space of polynomials, and $\bigcup_\alpha \Phi_\alpha$ is in Σ. It follows that there exists a maximal element R in Σ such that $M_2 \cap R = \{0\}$. Since $M_2 \dot{+} R \subseteq M_1$, and M_2 is finite codimensional in M_1, R is finite dimensional, and hence $M_2 \dot{+} R$ is closed. If $M_2 \dot{+} R \neq M_1$, then there is a polynomial $p \in M_1$, but $p \notin M_2 \dot{+} R$. This yields that the linear space $\{R, \ p\}$ spanned by R and p satisfies $\{R, \ p\} \cap M_2 = \{0\}$. This is impossible, and it follows that $M_1 = M_2 \dot{+} R$ with dim $R =$ dim M_1/M_2. From this assertion we immediately obtain

$$M_1 \cap \mathcal{C} = M_2 \cap \mathcal{C} \dot{+} R.$$

By the above equality, $M_2 \cap \mathcal{C}$ is dense in M_2, and the canonical homomorphism $\tau : M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C} \to M_1/M_2$ is an isomorphism. This completes the proofs of (1) and (2).

To prove (3), pick any $\lambda \in Z(M_1 \cap \mathcal{C}) \backslash Z(M_2 \cap \mathcal{C})$. By Theorem 2.1.7 (1), we see that $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a joint eigenvalue of $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$ on $M_1 \cap \mathcal{C}/M_2 \cap \mathcal{C}$, that is, there is $q \in M_1 \cap \mathcal{C}$, but $q \notin M_2 \cap \mathcal{C}$ such that $(z_i - \lambda_i)q \in M_2 \cap \mathcal{C}$ for $i = 1, 2, \cdots, n$. If $\lambda \notin \Omega$, then by Remark 2.2.2, $\mathcal{U}_\lambda$ is dense in X. This implies that there exists a sequence of polynomials, $\{\sum_{i=1}^n (z_i - \lambda_i) p_m^{(i)}\}_m$, such that $\{\sum_{i=1}^n (z_i - \lambda_i) p_m^{(i)}\}_m$ converges to 1 in the norm of X as $m \to \infty$, and therefore $\{\sum_{i=1}^n (z_i - \lambda_i) q \, p_m^{(i)}\}_m$ converges to q. Therefore,

$$q \in \overline{M_2 \cap \mathcal{C}} = M_2,$$

and hence $q \in M_2 \cap \mathcal{C}$. This contradiction shows that $\lambda \in \Omega$. Thus,

$$Z(M_2 \cap \mathcal{C}) \backslash Z(M_1 \cap \mathcal{C}) \subset \Omega.$$

By (1),(2) and Theorem 2.1.7 (1), we immediately obtain that

$$Z(M_2 \cap \mathcal{C}) \backslash Z(M_1 \cap \mathcal{C}) = Z(M_2) \backslash Z(M_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega.$$

We notice that (4) is from (1) and Theorem 2.1.7 (2), and (5) is from (2), (3) and Theorem 2.1.7 (3).

Conversely, assume that $I_1,\ I_2$ are ideals in $\mathcal{C}$, $I_1 \supseteq I_2$ and $Z(I_2)\backslash Z(I_1) \subset \Omega$. Then by Corollary 2.1.8, I_2 is finite codimensional in I_1, and by Theorem 2.1.7 (2),

$$I_2 = \{q \in I_1 :\ p(D)q|_\lambda = 0,\ p \in I_{2\lambda},\ \lambda \in Z(I_2)\backslash Z(I_1) \subset \Omega\}.$$

Now write

$$I_1 = I_2 \dot{+} R$$

where R is a linear space of polynomials with dim $R =$ dim I_1/I_2. Since each function f in $[I_2]$ satisfies that

$$p(D)f|_\lambda = 0,\ p \in I_{2\lambda},\ \lambda \in Z(I_2)\backslash Z(I_1),$$

this ensures that

$$[I_2] \cap R = \{0\}.$$

By the fact that $[I_2]\dot{+}R$ contains I_1, and $[I_2]\dot{+}R$ is closed, we obtain that

$$[I_2]\dot{+}R = [I_1].$$

Therefore, it holds that

$$I_1/I_2 \cong [I_1]/[I_2] \cong R.$$

The proof of Theorem 2.2.5 is complete.

Applying Theorem 2.2.5 and Corollary 2.1.9, we have

Corollary 2.2.6 *Let M be a finite codimensional submodule of X. Then we have the following.*

(1) $Z(M) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega$,

(2) $M = \bigcap_{\lambda \in Z(M)} M_\lambda^e$,

(3) $\operatorname{codim} M = \operatorname{card}\big(Z(M)\big) \hat{=} \sum_{\lambda \in Z(M)} \dim M_\lambda$.

Remark 2.2.7 *Notice that (3) of Corollary 2.2.6 shows that the codimension* codim M *of M in X equals the cardinality of zeros of M by counting multiplicities. Corollary 2.2.6 can be considered as a generalization of the case of the plane. In fact, let $L_a^2(\mathbb{D})$ be the Bergman module. Then M is a finite codimensional submodule of $L_a^2(\mathbb{D})$ if and only if there exists a polynomial p with $Z(p) \subset \mathbb{D}$ such that $M = p\,L_a^2(\mathbb{D})$. It is easy to check that the codimension of M equals the number of zeros of p (counting multiplicities). We refer the reader to the case of domains on the plane considered by Axler and Bourdon [AB].*

Recall that the most simple submodules of $H^2(\mathbb{D})$ are $p\,H^2(\mathbb{D})$, where p are polynomials with $Z(p) \subset \mathbb{D}$. Since each ideal I of polynomials in one variable is principal, this implies that $[I] = p\,H^2(\mathbb{D})$ for some polynomial p with $Z(p) \subset \mathbb{D}$. The above view suggests that we consider submodules of analytic Hilbert module generated by ideals of polynomials. Equivalently, can we characterize submodule M which has the property that the ideal $M \cap \mathcal{C}$ is dense in M [DPSY, Pau1]? Note that if M is as above and we let $I = M \cap \mathcal{C}$, then $[I] \cap \mathcal{C} = I$. Thus the problem is to characterize those ideals I for which $[I] \cap \mathcal{C} = I$. For the ideal I, we call I contracted if $[I] \cap \mathcal{C} = I$.

The next theorem is due to Douglas and Paulsen [DP]. We refer the reader to [DPSY] for considering a general case.

Theorem 2.2.8 (Douglas-Paulsen) *Let X be an analytic Hilbert module on Ω. If every algebraic component of $Z(I)$ has a nonempty intersection with Ω, then I is contracted.*

Proof. If $J = [I] \cap \mathcal{C}$, then J is an ideal in $\mathcal{C}$ which contains I. For each $\lambda \in \Omega$, since $I_\lambda = [I]_\lambda \subset J_\lambda$, Corollary 2.1.3 implies that $J \subset I$, and hence $J = I$. The proof is complete.

For various analytic Hilbert modules it would be interesting to classify the contracted ideals. In particular, for $H^2(\mathbb{D}^n)$, Douglas and Paulsen [DP, DPSY, Pau1] conjectured that the contracted ideals are precisely the ideals given by Theorem 2.2.8, that is, those ideals for which every algebraic component of their zero varieties meets $\mathbb{D}^n$ nontrivially. We record this conjecture here.

Douglas-Paulsen Conjecture: For the Hardy module $H^2(\mathbb{D}^n)$, if an ideal I of $\mathcal{C}$ is contracted, then each algebraic component of its zero variety $Z(I)$ intersects $\mathbb{D}^n$.

Clearly, for the Hardy module $H^2(\mathbb{D})$, the problem is trivial. In the case of $n = 2$, Gelca [Ge1] affirmatively answered this problem. However, the problem remains unknown for $n \geq 3$.

Now let I be an ideal of the ring $\mathcal{C}$. Since $\mathcal{C}$ is a Noetherian ring [AM, ZS], the ideal I is generated by finitely many polynomials. This implies that I has a greatest common divisor p. Thus, I can be uniquely written as

$$I = p\,L,$$

which is called the Beurling form of I.

Before going on let us present several preliminary lemmas. First we give a lemma, due to Yang [Ya1].

Lemma 2.2.9 (Yang) *Let I be an ideal of polynomials in two variables, and let $I = p\,L$ be the Beurling form of I. Then the ideal L is of finite codimension in $\mathcal{C}$.*

Proof. First of all, the ring $\mathcal{C}$ in two variables is a unique factorization domain of Krull dimension 2. We claim: if $p_1, p_2, \cdots, p_k$ are polynomials in two variables such that the greatest common divisor $GCD\{p_1, p_2, \cdots, p_k\} = 1$, then the ideal $(p_1, p_2, \cdots, p_k)$ generated by $p_1, p_2, \cdots, p_k$ is of finite codimension. Let

$$(p_1, p_2, \cdots, p_k) = \cap_{s=1}^{n} I_s$$

be an irredundant primary decomposition with the associated prime ideals $J_1, J_2, \cdots, J_n$. Note that each J_s is prime and it is either maximal or minimal since the Krull dimension of the ring of polynomials in two variables is 2. In a unique factorization domain, every minimal prime ideal is principal [ZS, Vol(I) p. 238]. Since $GCD\{p_1, p_2, \cdots, p_k\} = 1$, the associated prime ideals $J_1, J_2, \cdots, J_n$ must all be maximal, and hence each J_s must have the form $(z - z_s, w - w_s)$ with $(z_s, w_s) \in \mathbb{C}^2$, $s = 1, 2, \cdots, n$ mutually different. Therefore we can choose an integer, say m, sufficiently large such that

$$J_s^m = (z - z_s, w - w_s)^m \subset I_s$$

for each s. Then

$$\cap_{s=1}^{n} J_s^m \subset \cap_{s=1}^{n} I_s = (p_1, p_2, \cdots, p_k).$$

Obviously, the ideal $\cap_{s=1}^{n} J_s^m$ is of finite codimension, and hence $(p_1, p_2, \cdots, p_k)$ is of finite codimension. The claim is proved. Now for the ideal L, since it is generated by finitely many polynomials, say $p_1, p_2, \cdots, p_k$. Then clearly $GCD\{p_1, p_2, \cdots, p_k\} = 1$. The preceding claim implies that L is finite codimensional ideal. This completes the proof.

The next three lemmas are from [Ge1].

Lemma 2.2.10 *If* $|\beta| \leq |\alpha| \neq 0$ *and* $0 < r < 1$, *then*

$$|\beta - \alpha| / |r\beta - \alpha| \leq 1/r.$$

Proof. Lemma follows from $|\beta - \alpha| \leq |\beta - \alpha/r|$. The last inequalities are obvious since the triangles formed by the points β, α/r and α and the angles at α are obtuse.

From Lemma 2.2.10, we immediately obtain

Lemma 2.2.11 *Let* $p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n)$. *If* $1/2 < r < 1$ *then for any* z *with* $|z| \leq min\{|z_1|, |z_2|, \cdots, |z_n|\}$, $|p(z)/p(rz)| \leq 2^n$. *Consequently, for a polynomial* p *with the degree* n, *if* p *has no zero points in a disk* V *with the center* 0, *then for any* $z \in V$, *and* $1/2 < r < 1$, $|p(z)/p(rz)| \leq 2^n$.

Let Ω be a bounded complete Reinhardt domain (i.e., a bounded domain with the property that for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Omega$, if $|z_i| \leq 1$, $i = 1, 2 \cdots, n$, then $(z_1\lambda_1, z_2\lambda_2, \cdots, z_n\lambda_n) \in \Omega$). For a polynomial p, we denote by $d(p)$ the sum of the degrees of p in each variable.

Lemma 2.2.12 *Let p be a polynomial having no zeros in Ω. Then for every $(z_1, z_2, \cdots, z_n) \in \Omega$ and $1/2 < r < 1$,*

$$|p(z_1, z_2, \cdots, z_n)/p(rz_1, rz_2, \cdots, rz_n)| \le 2^{d(p)}.$$

Proof. The proof will be done by induction on the number of variables. By Lemma 2.2.11, Lemma 2.2.12 is true for $n = 1$.

Fix $(z_1, z_2, \cdots, z_n) \in \Omega$. Let $U = \{z : (z, z_2, \cdots, z_n) \in \Omega\}$. Then U is a disk with the center 0 and $p(z, z_2, \cdots, z_n)$ has no zero in U. Thus

$$|p(z_1, z_2, \cdots, z_n)/p(rz_1, z_2, \cdots, z_n)| \le 2^{d_1},$$

where we denote by d_1 the degree of p in the variable z_1. Applying the induction hypothesis to $p(z_1, \cdots)$ we obtain that

$$|p(rz_1, z_2, \cdots, z_n)/p(rz_1, rz_2, \cdots, rz_n)| \le 2^{d_2+\cdots+d_n},$$

that is,

$$|p(z_1, z_2, \cdots, z_n)/p(rz_1, rz_2, \cdots, rz_n)| \le 2^{d(p)}.$$

The following result was proved by Gelca [Ge1] for $n = 2$. Here the proof comes from [Guo2].

Proposition 2.2.13 *Let p be a polynomial having no zero in $\mathbb{D}^n$. Then $p\mathcal{C}$ is dense in $H^2(\mathbb{D}^n)$.*

Proof. Set

$$f_r(z_1, z_2, \cdots, z_n) = p(z_1, z_2, \cdots, z_n)/p(rz_1, rz_2, \cdots, rz_n).$$

Then for $1/2 < r < 1$, $f_r \in pH^2(\mathbb{D}^n)$. Let $\{r_n\}_n$ be a sequence of positive real numbers such that $r_n \to 1$ as $n \to \infty$. Now by Lemma 2.2.12, the family f_{r_n} is uniformly bounded, and $f_{r_n} \to 1$ almost everywhere as $n \to \infty$. By the Dominated Convergence theorem, f_{r_n} converges to 1 in the norm of $H^2(\mathbb{D}^n)$. The conclusion follows by using the fact that the ring of polynomials is dense in $H^2(\mathbb{D}^n)$.

Remark 2.2.14 *Using the proof of Proposition 2.2.13, we see that if $Z(p) \cap \mathbb{B}_n = \emptyset$, then $p\mathcal{C}$ is dense in the Hardy space $H^2(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$. Furthermore, if Ω is a bounded complete Reinhardt domain and $Z(p) \cap \Omega = \emptyset$, then $p\mathcal{C}$ is dense in the Bergman space $L^2_a(\Omega)$ on the domain Ω (see [Ge2]).*

The next theorem comes from [Ge1].

Theorem 2.2.15 (Gelca) *Let I be an ideal of polynomials in two variables. Then I is contracted in $H^2(\mathbb{D}^2)$ if and only if every algebraic component of $Z(I)$ has a nonempty intersection with $\mathbb{D}^2$.*

Proof. We shall merely indicate that if I is contracted in $H^2(\mathbb{D}^2)$, then every algebraic component of $Z(I)$ has a nonempty intersection with $\mathbb{D}^2$. Let $I = p\,L$ be the Beurling form of I. We decompose p as the product of the powers of irreducible polynomials, say,

$$p = p_1^{k_1} \cdots p_l^{k_l}.$$

By Lemma 2.2.9, L is of finite codimension and, hence, L has finitely many zeros, say, $\lambda_1, \cdots, \lambda_s$. Therefore the algebraic components of I are $Z(p_j)$ for $j = 1, \cdots, l$, and zero points $\lambda_1, \cdots, \lambda_s$ of L. Now we apply Proposition 2.2.13 to obtain the desired conclusion. The proof is complete.

Theorem 2.2.15 relies heavily on the ideal structure of polynomials in two variables. Hence it extends with essentially no change to the Hardy module $H^2(\mathbb{B}_2)$ and Bergman modules on bounded complete Reinhardt domains [Ge2]. However, for general analytic Bergman modules, little is known about the classification of contracted ideals. Some related results have been obtained by Putinar and Salinas [PS]. However, we notice that Theorem 3.1 and Corollary 3.3 in [PS] are not valid in general for strongly pseudoconvex domains because such domains are not necessarily so-called α-domains. An example is the unit ball $\mathbb{B}_2$. Let $p(z, w) = z^2 + w^2 - 1$; then $Z(p) \cap \mathbb{B}_2 = \emptyset$, but $Z(p) \cap \partial\mathbb{B}_2 (\supseteq \{(\cos\theta, \sin\theta)\})$ is a infinite set, and hence by the definition of α-domains, $\mathbb{B}_2$ is not a α-domain.

Finally, we wish to record here the following result.

Proposition 2.2.16 *Let I be an ideal of polynomials on $\mathbb{C}^2$. If $Z(I) \cap \mathbb{D}^2 = \emptyset$, then there exists a polynomial $q \in I$ such that $Z(q) \cap \mathbb{D}^2 = \emptyset$.*

Proof. Let $I = p\,L$ be the Beurling form of I. Obviously, $Z(p) \cap \mathbb{D}^2 = \emptyset$. To complete the proof, we will find a polynomial h in L such that $Z(h) \cap \mathbb{D}^2 = \emptyset$. By Lemma 2.2.9, L is of finite codimension and $Z(L) \cap \mathbb{D}^2 = \emptyset$. Thus there exist finitely many zeros $\lambda_1, \lambda_2, \cdots, \lambda_k$ of L for which they lie in $\mathbb{C}^2 \setminus \mathbb{D}^2$. It follows that L is decomposed as

$$L = \cap_{i=1}^{k} L_k$$

where L_i are $\mathcal{U}_{\lambda_i}$-primary for $i = 1, 2, \cdots, k$. From [AM] or [ZS], we see that there is a positive integer number m such that

$$\mathcal{U}_{\lambda_i}^m \subset L_i, \;\; i = 1, 2, \cdots, k.$$

Note that $\lambda_i = (\lambda_i', \lambda_i'') \notin \mathbb{D}^2$ for $i = 1, \cdots, k$. This means that for each i, $|\lambda_i'| \geq 1$ or $|\lambda_i''| \geq 1$. Now for each i, we may suppose that $|\lambda_i'| \geq 1$. This yields that $\prod_{i=1}^k (z - \lambda_i')^m$ has no zero in $\mathbb{D}^2$, and obviously $\prod_{i=1}^k (z - \lambda_i')^m \in L$. Set

$$q = p \prod_{i=1}^{k} (z - \lambda_i')^m,$$

which gives the desired conclusion.

However, for $n > 2$, we do not know if the same conclusion is true.

Conjecture 2.2.17 *Let $n > 2$ and I be an ideal of polynomials on $\mathbb{C}^n$. If $Z(I) \cap \mathbb{D}^n = \emptyset$, then there exists a polynomial q in I such that $Z(q) \cap \mathbb{D}^n = \emptyset$.*

Note that Conjecture 2.2.17 is equivalent to: for each prime ideal P, if $Z(P) \cap \mathbb{D}^n = \emptyset$, does there exist a polynomial q in P such that $Z(q) \cap \mathbb{D}^n = \emptyset$? It is also equivalent to: for each irreducible variety V, if $V \cap \mathbb{D}^n = \emptyset$, does there exist an irreducible variety V' such that $V' \supsetneqq V$ and $V' \cap \mathbb{D}^n = \emptyset$? Of course, Conjecture 2.2.17 can be stated for general domains. Let I be an ideal of polynomials on $\mathbb{C}^n$. If $Z(I) \cap \Omega = \emptyset$, does there exist a polynomial q in I such that $Z(q) \cap \Omega = \emptyset$?

Remark 2.2.18 *If we assume Conjecture 2.2.17, then Douglas-Paulsen's conjecture is immediate. That is, if I is contracted in $H^2(\mathbb{D}^n)$ (i.e., $[I] \cap \mathcal{C} = I$), then each algebraic component of I meets $\mathbb{D}^n$ nontrivially. In fact, first let $I = \cap_{j=1}^m I_j$ be an irredundant primary decomposition of I. We may suppose that there are $I_1, I_2 \cdots, I_k$ such that $Z(I_j) \cap \mathbb{D}^n = \emptyset$ for $j = 1, 2, \cdots, k$, and $Z(I_j) \cap \mathbb{D}^n \neq \emptyset$ for $j = k+1, \cdots, m$. By Conjecture 2.2.17, there exists a polynomial $q \in I_1 I_2 \cdots I_k$ such that $Z(q) \cap \mathbb{D}^n = \emptyset$. Therefore, by Proposition 2.2.13,*

$$H^2(\mathbb{D}^n) = [q\mathcal{C}] = [I_1 I_2 \cdots I_k].$$

Note that

$$I_1 I_2 \cdots I_k \,(\cap_{j=k+1}^m I_j) \subset I \subset \cap_{j=k+1}^m I_j.$$

Thus,

$$[I] = [\cap_{j=k+1}^m I_j].$$

By Theorem 2.2.8, we see that

$$[I] \cap \mathcal{C} = \cap_{j=k+1}^m I_j.$$

This contradicts the contractness of I. From the above discussion, we see that if one assumes Conjecture 2.2.17, then Douglas-Paulsen's conjecture can be answered affirmatively.

2.3 Submodules with finite rank

It is well known that "rank" is one of the important invariants of Hilbert modules. Recall that a submodule M of the analytic Hilbert module X is finitely generated if there exists a finite set of vectors $x_1, x_2, \cdots, x_n$ in M such that $\mathcal{C}\, x_1 + \mathcal{C}\, x_2 + \cdots + \mathcal{C}\, x_n$ is dense in M. The minimum cardinality of such a set is called the rank of M and denoted by rank(M). If rank$(M) = 1$, we call M a principal submodule. By Beurling's theorem, each submodule of $H^2(\mathbb{D})$ is principal. Even more generally, for any analytic Hilbert module X on a domain in the complex plane $\mathbb{C}$, a submodule generated by some polynomials is principal because every ideal of polynomials in one variable is principal. However for the Hardy module $H^2(\mathbb{D}^n)$ in several variables, there exist submodules with any rank [Ru1, p. 72].

Let $\lambda \in \mathbb{C}^n$. We denote by $\mathcal{O}_\lambda$ the ring of all germs of analytic functions at λ. For detailed information about $\mathcal{O}_\lambda$ we refer the reader to [DY1, Kr]. We summarize some properties of $\mathcal{O}_\lambda$. First $\mathcal{O}_\lambda$ is a unique factorization domain (UFD), and the units of $\mathcal{O}_\lambda$ are those germs that are nonvanishing at λ. Second, $\mathcal{O}_\lambda$ is a Noetherian local ring of Krull dimension n.

Let I be an ideal of $\mathcal{O}_\lambda$. As in the case of analytic submodules, we define the characteristic space of I by

$$I_\lambda = \{q \in \mathcal{C} : q(D)f|_\lambda = 0, \ \forall f \in I\}.$$

The envelope of I, I^e_λ, is defined by

$$I^e_\lambda = \{f \in \mathcal{O}_\lambda : q(D)f|_\lambda = 0, \ \forall q \in I_\lambda\}.$$

It is easy to check that I^e_λ is an ideal of $\mathcal{O}_\lambda$, and $I^e_\lambda \supseteq I$. Furthermore, by the reasoning as in the proof of Theorem 2.1.1, we have

$$I^e_\lambda = \bigcap_{j \geq 1} (I + \mathcal{M}^j_\lambda),$$

where $\mathcal{M}_\lambda$ is the maximal ideal of $\mathcal{O}_\lambda$, that is, $\mathcal{M}_\lambda = \{f \in \mathcal{O}_\lambda : f(\lambda) = 0\}$. By using Krull's theorem [ZS, Vol(I), p. 217, Theorem 12'] or Lemma 2.11 in [DPSY], we have the following proposition [Guo2].

Proposition 2.3.1 *Let I be an ideal of $\mathcal{O}_\lambda$. Then*

$$I = I^e_\lambda.$$

Hence I is completely determined by its characteristic space.

Let X be an analytic Hilbert module on $\Omega (\subset \mathbb{C}^n)$, and let $\lambda \in \Omega$. For $f \in X$ we denote by f_λ the element of $\mathcal{O}_\lambda$ defined by the restriction of f to a neighborhood of λ. For a submodule M of X, we denote by $M^{(\lambda)}$ the ideal of $\mathcal{O}_\lambda$ generated by $\{f_\lambda : f \in M\}$. Let $f_1, f_2, \cdots, f_m$ be in X. Write $[f_1, f_2, \cdots, f_m]$ for the submodule generated by $f_1, f_2, \cdots, f_m$.

Lemma 2.3.2 *Let $\lambda \in \Omega$. Then*

$$[f_1, f_2, \cdots, f_m]^{(\lambda)} = f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda.$$

Proof. From the inclusion

$$f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda \subset [f_1, f_2, \cdots, f_m]^{(\lambda)},$$

we see that

$$\{[f_1, f_2, \cdots, f_m]^{(\lambda)}\}_\lambda \subset (f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda.$$

For $f \in [f_1, f_2, \cdots, f_m]$, there exist polynomials $p_n^{(1)}, p_n^{(2)}, \cdots, p_n^{(m)}$ such that

$$f = \lim_{n\to\infty}(p_n^{(1)} f_1 + p_n^{(2)} f_2 + \cdots + p_n^{(m)} f_m)$$

in the norm of X. It is easy to check that for every polynomial q and each $w \in \Omega$, the linear functional on X, $f \mapsto q(D)f|_w$, is continuous. Let q be in $(f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda$. Since

$$q(D)(p_n^{(1)} f_1 + p_n^{(2)} f_2 + \cdots + p_n^{(m)} f_m)|_\lambda = 0,$$

this gives that

$$q(D)f|_\lambda = 0.$$

It follows that

$$(f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda \subset [f_1, f_2, \cdots, f_m]_\lambda.$$

It is easy to see that

$$[f_1, f_2, \cdots, f_m]_\lambda = \{[f_1, f_2, \cdots, f_m]^{(\lambda)}\}_\lambda,$$

and hence

$$\{[f_1, f_2, \cdots, f_m]^{(\lambda)}\}_\lambda = (f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda)_\lambda.$$

Applying Proposition 2.3.1, we obtain that

$$[f_1, f_2, \cdots, f_m]^{(\lambda)} = f_{1\lambda}\mathcal{O}_\lambda + f_{2\lambda}\mathcal{O}_\lambda + \cdots + f_{m\lambda}\mathcal{O}_\lambda.$$

The next result states an important property of submodules with finite rank, due to Guo [Guo2].

Theorem 2.3.3 *Let M be the submodule of X generated by $f_1, f_2, \cdots, f_m$. Then for each $f \in M$, there are $g_1, g_2, \cdots, g_m$ in $Hol(\Omega)$ such that*

$$f = f_1 g_1 + f_2 g_2 + \cdots + f_m g_m.$$

Proof. The proof of Theorem 2.3.3 is based on the sheaf theory (see [Kr, Chapters 6, 7]). Let $\mathcal{O}$ denote the sheaf of germs of analytic functions on Ω. The sheaf $\mathcal{F} = \mathcal{F}(M)$ generated by M is defined as follows: for $\lambda \in \Omega$,

$\mathcal{F}_\lambda = M^{(\lambda)}$. From Lemma 2.3.2, we see that $\mathcal{F}$ is the subsheaf of $\mathcal{O}$ generated by $f_1, f_2, \cdots, f_m$. Consider the exact sequence of sheafs

$$0 \longrightarrow \mathcal{R} \xrightarrow{i} \mathcal{O}^m \xrightarrow{\alpha} \mathcal{F} \longrightarrow 0$$

where $\alpha(g_{1\lambda}, g_{2\lambda}, \cdots, g_{m\lambda}) = \sum_{i=1}^m f_{i\lambda} g_{i\lambda}$, $\mathcal{R}$ is the kernel sheaf and i is the inclusion. The Oka Coherence theorem [Kr, Theorem 7.1.8] implies that $\mathcal{R}$ is coherent, so by Theorem B of Cartan [Kr, Theorem 7.1.7], $H^1(\Omega, \mathcal{R}) = 0$. Now the long exact cohomology sequence [Kr, Theorem 6.2.22] gives

$$H^0(\Omega, \mathcal{O}^m) \xrightarrow{\alpha_*} H^0(\Omega, \mathcal{F}) \xrightarrow{\delta_*} H^1(\Omega, \mathcal{R}) = 0.$$

Thus, α_* is a surjective map. This shows that for every $f \in M$, there exist $g_1, g_2, \cdots, g_m$ in $Hol(\Omega)$ such that

$$f = f_1 g_1 + f_2 g_2 + \cdots + f_m g_m.$$

The proof is complete.

In Douglas-Paulsen's book [DP, p. 42, Problem 2.23], it is asked when a principal submodule of $H^2(\mathbb{D}^n)$ is the closure of an ideal of polynomials. Combining Theorem 2.3.3 with the characteristic space theory of ideals of polynomials, Guo [Guo2] characterized that a principal submodule is generated by polynomials.

Theorem 2.3.4 *Let $I = pL$ be the Beurling form of I. If $[I]$ is principal, then $Z(L) \cap \Omega = \emptyset$. Equivalently, if $Z(L) \cap \Omega \neq \emptyset$, then $rank([I]) \geq 2$.*

Proof. Let $\{p_1, p_2, \cdots, p_k\}$ be a set of generators of L. Then the greatest common divisor $GCD\{p_1, p_2, \cdots, p_k\} = 1$. Now suppose that there exists $\lambda \in \Omega$ such that $p_i(\lambda) = 0$ for $i = 1, 2, \cdots, k$. Decompose $p_i = p_i' p_i''$ such that each prime factor of p_i' vanishes at λ, and $p_i''(\lambda) \neq 0$ and $p = q_1 q_2$ such that each prime factor of q_1 vanishes at λ and $q_2(\lambda) \neq 0$. Since $[I]$ is principal, this says that there exists some f in X such that

$$[f] = [pp_1, pp_2, \cdots, pp_k].$$

By Theorem 2.3.3, there exist analytic functions on Ω, $g_1, g_2, \cdots, g_k$ and $h_1, h_2, \cdots, h_k$ such that

$$pp_1 = fg_1,\ pp_2 = fg_2, \cdots,\ pp_k = fg_k, \text{ and } f = \sum_{i=1}^k h_i pp_i.$$

Therefore,

$$\sum_{i=1}^k h_i g_i = 1.$$

Therefore, the functions $g_1, g_2, \cdots, g_k$ have no common zero in Ω. This implies that there is some g_s such that $g_s(\lambda) \neq 0$. From the equality $pp_s = fg_s$ one has

$$[pp_s]_\lambda = [f]_\lambda.$$

According to Theorem 2.1.1,

$$[f]^e_\lambda = [pp_s]^e_\lambda = q_1 p'_s \mathcal{C}.$$

However, for each i, $[f]_\lambda \subset [pp_i]_\lambda$ and hence for every i,

$$[f]^e_\lambda \supset [pp_i]^e_\lambda = q_1 p'_i \mathcal{C}.$$

Thus, for every i,

$$q_1 p'_s \mathcal{C} \supset q_1 p'_i \mathcal{C}.$$

Thus each p'_i is divisible by p'_s. Thus, every p_i is divisible by p'_s. This is impossible. Therefore, $p_1, p_2, \cdots, p_k$ have no common zero in Ω, that is,

$$Z(L) \cap \Omega = \emptyset.$$

The proof is complete.

Corollary 2.3.5 *Let $I = pL$ be the Beurling form of the ideal I. If every algebraic component of $Z(I)$ has a nonempty intersection with Ω, then $[I]$ is principal if and only if $I = p\mathcal{C}$. In particular, if I is prime, and $Z(I) \cap \Omega \neq \emptyset$, then $[I]$ is principal if and only if $I = p\mathcal{C}$ and p is prime.*

Proof. If $[I]$ is principal, then by Theorem 2.3.4,

$$Z(L) \cap \Omega = \emptyset.$$

So for each $\lambda \in \Omega$,

$$[I]_\lambda = [p\mathcal{C}]_\lambda.$$

From Corollary 2.1.3 we see that $I \supset p\mathcal{C}$, and hence $I = p\mathcal{C}$. The opposite direction is obvious.

When $n = 2$, one can obtain a more detailed result.

Corollary 2.3.6 *Let X be an analytic Hilbert module on $\Omega(\subset C^2)$, and let $I = pL$ be the Beurling form of I. Then $[I]$ is principal if and only if $[I] = [p]$.*

Proof. Let $[I]$ be principal. By Theorem 2.3.4, $Z(L) \cap \Omega = \emptyset$. Using Lemma 2.2.9, we see $[L] = X$. Thus $[I] = [p]$.

In commutative algebra it is shown in the presence of certain finiteness hypotheses that the Hilbert-Samuel polynomial can be defined for a module, thus providing an invariant for the module. However, the story does not stop here. One seeks to interpret the coefficients and degree of the polynomial in terms of algebraic and geometric properties of the modules and algebra. In the context of Hilbert modules, Douglas and Yan [DY2] established the existence of a Hilbert-Samuel polynomial for a Hilbert module and showed the power of the polynomial for the study of Hilbert modules. For analytic Hilbert modules, we will establish the existence of a Hilbert-Samuel polynomial that is a special version of Hilbert-Samuel polynomials in [DY2].

We need the following lemma which first was proved by Douglas and Yang [DYa1], where proof is different to that of [DYa1].

Lemma 2.3.7 *Let M be a submodule of an analytic Hilbert module X with finite rank and let I be a finite codimensional ideal of $\mathcal{C}$. Then*

$$\dim\, M/[IM] \,\le\, \operatorname{codim}(I)\operatorname{rank}(M).$$

Proof. We assume $\operatorname{rank}(M) = m$ with a generating set $\{f_1, f_2, \cdots, f_m\}$. Now let $P: M \to M \ominus [IM]$ be the orthogonal projection. We write $\tilde{f}_i = Pf_i$ for $i = 1, 2, \cdots, m$. Since I annihilates the quotient module $M/[IM]$, one can consider $M/[IM]$ as a $\mathcal{C}/I$-module with a generating set $\{\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_m\}$. Since $\mathcal{C}/I$ is finite dimensional, this ensures $\dim\, M/[IM] < \infty$, and actually

$$\dim\, M/[IM] \le \operatorname{codim}(I)\operatorname{rank}(M).$$

Theorem 2.3.8 *Let M be a finitely generated submodule of an analytic Hilbert module X, and let I be a finite codimensional ideal of $\mathcal{C}$. Then there exists a polynomial $p_{I,M}$ with rational coefficients such that*

$$p_{I,M}(k) = \dim\, M/[I^k M]$$

for large integer k. Moreover, the degree of $p_{I,M}$ is less than or equal to rank(I).

Proof. Set

$$gr\, I = \bigoplus_{k\ge 0} I^k/I^{k+1}, \quad \text{and} \quad gr\, M = \bigoplus_{k\ge 0} [I^k M]/[I^{k+1}M].$$

From Lemma 2.3.7, we see that $gr\, M$ is a finitely generated graded module on Noether graded ring $gr\, I$. Let $\operatorname{rank}(I) = l$ and let $\{p_1, p_2 \cdots, p_l\}$ be a generating set of I. Then $gr\, I$, as a $\mathcal{C}/I$-algebra, is generated by the images $\tilde{p}_i$ of p_i in I/I^2 for $i = 1, 2, \cdots, l$. Since $\deg(\tilde{p}_i) = 1$ for $i = 1, 2, \cdots, l$, Hilbert's original results (cf. [AM, Chapter 11]) imply that for large integer k, $\dim\, [I^k M]/[I^{k+1}M]$ is a polynomial of k with rational coefficients. Moreover, the degree of the polynomial is less than or equal to $\operatorname{rank}(I) - 1$. Since

$$\dim\, M/[IM] = \sum_{i=0}^{k} \dim\, [I^i M]/[I^{i+1}M],$$

this implies that there exists a polynomial $p_{I,M}$ with rational coefficients such that

$$p_{I,M}(k) = \dim\ M/[I^k M] \quad \text{and} \quad \deg\ p_{I,M} \leq \operatorname{rank}(I).$$

For an ideal I, recall that the radical ideal $\sqrt{I}$ of I is

$$\sqrt{I} = \{p \in \mathcal{C} :\ p^m \in I \ \text{ for some positive integer } m\}.$$

If I is finite codimensional, then by [AM], there exists a positive integer m such that

$$(\sqrt{I})^m \subseteq I \subseteq \sqrt{I}.$$

Corollary 2.3.9 *Let I_1, I_2 be finite codimensional ideals with $\sqrt{I_1} = \sqrt{I_2}$. Then*

$$\deg\ p_{I_1,M} = \deg\ p_{I_2,M}.$$

Proof. Obviously, we only need to prove that for any finite codimensional ideal I, $\deg\ p_{I,M} = \deg\ p_{\sqrt{I},M}$. Since there exists a positive integer m such that

$$(\sqrt{I})^m \subseteq I \subseteq \sqrt{I},$$

we have

$$p_{\sqrt{I},M}(k) \leq p_{I,M}(k) \leq p_{(\sqrt{I})^m,M}(k) = p_{\sqrt{I},M}(mk)$$

for large integer k. From the above inequalities, the desired result follows.

2.4 AF-cosubmodules

Recall that each submodule $BH^2(\mathbb{D})$ (where B is a Blaschke product) of the Hardy module $H^2(\mathbb{D})$ is equal to the intersection of all finite codimensional submodules that contain $BH^2(\mathbb{D})$. Also, for any submodule M of $H^2(\mathbb{D})$, it is easy to check that M can be uniquely decomposed into

$$M = \eta H^2(\mathbb{D}) \bigcap BH^2(\mathbb{D}),$$

where η is a singular inner function and B a Blaschke product. One can show that $BH^2(\mathbb{D})$ is just equal to the intersection of all finite codimensional submodules that contain M. Motivated by these observations we introduce the following notions. Let X be an analytic Hilbert module on Ω, and let M be a submodule of X. We call M approximately finite codimensional (in short, an AF-cosubmodule) if M is equal to the intersection of all finite codimensional submodules which contain M. Therefore when M is an AF-cosubmodule, M is just the limit of decreasing net ($\supseteq$) of all finite codimensional submodules containing M. For a submodule M, the AF-envelope of M is defined by the intersection of all finite codimensional submodules containing M, and denoted by M^e. Clearly, by the definition, the envelope of a submodule M is an AF-cosubmodule.

The following theorem is from [Guo1].

Theorem 2.4.1 *Let X be an analytic Hilbert module and M a submodule of X. Then we have*

(1) if $Z(M) = \emptyset$, then $M^e = X$;

(2) if $Z(M) \neq \emptyset$, then $M \subseteq M^e \neq X$, $(M^e)^e = M^e$, and $Z(M) = Z(M^e)$;

(3) if $Z(M) \neq \emptyset$, then $M^e = \bigcap_{\lambda \in Z(M)} M^e_\lambda$.

In particular, if M_1, M_2 are two submodules of X, then $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$, and for every $\lambda \in Z(M_1)$, $M_{1\lambda} = M_{2\lambda}$.

Note that (3) says that the AF-envelope of M is equal to the intersection of envelopes of M at all zero points.

Proof. Clearly, both (1) and (2) are true. We give the proof of (3). We define the degree of a monomial $z_1^{m_1} \cdots z_n^{m_n}$ to be $m_1 + \cdots + m_n$, and the degree of a polynomial p to be maximum of the degrees of the monomials that occur in p with nonzero coefficients, and denoted by $\deg(p)$. For every $\lambda \in Z(M)$ and each natural number k, set

$$M_\lambda^{(k)} = \{f \in X : \ p(D)f|_\lambda = 0, \, p \in M_\lambda \text{ and } \deg(p) \leq k\}.$$

Then $M_\lambda^{(k)}$ is a finite codimensional submodule and it contains M. Since

$$M_\lambda^e = \{f \in X : \ p(D)f|_\lambda = 0, \, \forall p \in M_\lambda\} = \bigcap_{k \geqq 1} M_\lambda^{(k)},$$

this gives

$$M^e \subseteq \bigcap_{\lambda \in Z(M)} M_\lambda^e.$$

Let N be a finite codimensional submodule of X, and $N \supseteq M$. Then by Corollary 2.2.6, it is easily verified that

$$N = \bigcap_{\lambda \in Z(N)} N_\lambda^e.$$

Let $\lambda \in Z(N)$. Then λ also is in $Z(M)$. From the inclusion $N \supseteq M$, we have that $N_\lambda \subseteq M_\lambda$. Consequently, $M_\lambda^e \subseteq N_\lambda^e$. Therefore,

$$\bigcap_{\lambda \in Z(N)} M_\lambda^e \subseteq N.$$

This deduces the inclusion

$$\bigcap_{\lambda \in Z(M)} M_\lambda^e \subseteq N$$

and hence

$$\bigcap_{\lambda \in Z(M)} M_\lambda^e = M^e.$$

For each $\lambda \in Z(M)$, it is not difficult to verify that $M_\lambda = (M^e)_\lambda$. From (2) and (3), one easily deduces that for any two submodules M_1, M_2, $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$ and $M_{1\lambda} = M_{2\lambda}$ for each $\lambda \in Z(M_1)$. This completes the proof of Theorem 2.4.1.

Now let $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$ be the n-tuple of operators that are defined on the quotient module M_1/M_2 by $M_{z_i}\tilde{f} = \widetilde{(z_i f)}$ for $i = 1, \cdots, n$, and $\sigma_p(M_{z_1}, \cdots, M_{z_n})$ denotes the joint eigenvalues of $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$.

We will now generalize Theorem 2.2.5 to AF-cosubmodules. By modifying the proof of Theorem 2.2.5, we have the following.

Theorem 2.4.2 *Let M_1, M_2 be submodules of X such that $M_1 \supseteq M_2$ and* $\dim M_1/M_2 = k < \infty$. *If M_2 is an AF-cosubmodule, then we have*

(1) $Z(M_2)\backslash Z(M_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega$,

(2) $M_2 = \{h \in M_1 : \ p(D)h|_\lambda = 0, \ p \in M_{2\lambda}, \ \lambda \in Z(M_2)\backslash Z(M_1)\}$,

(3) $\dim M_1/M_2 = \sum_{\lambda \in Z(M_2)\backslash Z(M_1)} \dim M_{2\lambda}/M_{1\lambda} = card\big(Z(M_2)\backslash Z(M_1)\big)$.

Remark 2.4.3 *It is worth noticing that the condition is necessary in the above theorem that M_2 is approximately finite codimensional. In fact, by [Hed] we know that there exists a submodule M of the Bergman module $L_a^2(\mathbb{D})$ such that* $\dim M/zM = 2$, *but* $Z(zM)\backslash Z(M) = \{0\}$ *and* $\dim (zM)_0/M_0 = 1$.

Proof. (1) Write

$$M_1 = M_2 \oplus R$$

and restrict $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$ on R. By [Cur1], they can be simultaneously triangularized as

$$M_{z_i} = \begin{pmatrix} \lambda_i^{(1)} & & \star \\ & \ddots & \\ & & \lambda_i^{(k)} \end{pmatrix},$$

where $i = 1, 2, \cdots, n$, and $k = \dim M_1/M_2$, so that $\sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$ is equal to $\{\lambda^{(1)}, \cdots, \lambda^{(k)}\}$. Similarly to the proof of Theorem 2.2.5, we have that

$$\sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega.$$

Writing

$$\mathcal{U}_{\lambda^{(i)}} = \{p : \text{ p are polynomials, and } \ p(\lambda^{(i)}) = 0\}$$

$i = 1, \cdots, k$, then

$$\mathcal{U}_{\lambda^{(k)}} \cdots \mathcal{U}_{\lambda^{(2)}} \mathcal{U}_{\lambda^{(1)}} M_1 \subseteq M_2 \subseteq M_1.$$

Therefore, for $\lambda \in \Omega$, but $\lambda \notin \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$, one sees that $M_{1\lambda} = M_{2\lambda}$. This implies that

$$Z(M_2) \backslash Z(M_1) \subseteq \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}).$$

Let $\lambda \in \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n})$. Since λ is a joint eigenvalue of the operator tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_n}\}$, there is a function h in M_1 such that

$$\mathcal{U}_\lambda h \subseteq M_2.$$

Setting $M_2^\dagger$ to be the submodule generated by M_2 and h, then for $\lambda' \neq \lambda$ and $\lambda' \in \Omega$ it holds that

$$(M_2^\dagger)_{\lambda'} = M_{2\lambda'}.$$

If $(M_2^\dagger)_\lambda = M_{2\lambda}$, then by Theorem 2.4.1(4) we have that

$$M_2^{\dagger e} = M_2^e = M_2.$$

This is impossible. Hence,

$$M_{2\lambda} \supsetneqq (M_2^\dagger)_\lambda \supseteq M_{1\lambda}.$$

It follows that λ belongs to $Z(M_2) \backslash Z(M_1)$. We thus conclude that

$$Z(M_2) \backslash Z(M_1) = \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega.$$

(2) Set

$$M_2^\natural = \{h \in M_1 : p(D)h|_\lambda = 0, \ p \in M_{2\lambda}, \ \lambda \in Z(M_2) \backslash Z(M_1)\}.$$

Then $M_2^\natural$ is a submodule that contains M_2. It is easy to see that for every $\lambda \in \Omega$,

$$(M_2^\natural)_\lambda = M_{2\lambda}.$$

By Theorem 2.4.1(4), we have that

$$(M_2^\natural)^e = M_2^e = M_2.$$

This implies that $M_2^\natural = M_2$. The proof of (2) is complete.

(3) The proof is by induction on numbers of points in $Z(M_2) \backslash Z(M_1)$. If $Z(M_2) \backslash Z(M_1)$ contains only a point λ, then by (2), M_2 can be written as

$$M_2 = \{h \in M_1 : p(D)h|_\lambda = 0, \ p \in M_{2\lambda}\}.$$

We define the pairing

$$[-,-]: \ M_{2\lambda}/M_{1\lambda} \times M_1/M_2 \to C$$

by $[\tilde{p}, \tilde{h}] = p(D)h|_\lambda$. Clearly, this is well defined. From this pairing and the representation of M_2, it is not difficult to see that

$$\dim \ M_1/M_2 = \dim \ M_{2\lambda}/M_{1\lambda}.$$

Now let $l > 1$, and assume that (3) has been proved for $Z(M_2)\backslash Z(M_1)$ containing points less than l. Let $Z(M_2)\backslash Z(M_1) = \{\lambda_1, \cdots, \lambda_l\}$ where $\lambda_i \neq \lambda_j$ for $i \neq j$. Writing

$$M_2^\star = \{h \in M_1 : \ p(D)h|_{\lambda_1} = 0, \, p \in M_{2\lambda_1}\},$$

then

$$(M_2^\star)_{\lambda_1} = M_{2\lambda_1}.$$

Just as in the above proof, we have

$$\dim \ M_1/M_2^\star = \dim \ M_{2\lambda_1}/M_{1\lambda_1}.$$

Write $M_{2\lambda_1} = M_{1\lambda_1} \dot{+} R$ with $\dim \ R = \dim \ M_{2\lambda_1}/M_{1\lambda_1}$. Let $\sharp R$ denote linear space of polynomials generated by R which is invariant under the action by $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. Put

$$\mathcal{Q}_{\mathcal{R}} = \{p \in \mathcal{C} : \ q(D)p|_{\lambda_1} = 0, \, q \in \sharp R\}.$$

Then it is easily verified that $\mathcal{Q}_{\mathcal{R}}$ is a finite codimensional ideal of $\mathcal{C}$ with only zero point λ_1 because $\sharp R$ is finite dimensional. Thus

$$\mathcal{Q}_{\mathcal{R}} M_1 \subseteq M_2^\star \subseteq M_1.$$

From the above inclusions, we see that for $\lambda \neq \lambda_1$, $M_{1\lambda} = (M_2^\star)_\lambda$. Therefore

$$Z(M_2)\backslash Z(M_2^\star) = \{\lambda_2, \cdots, \lambda_l\}.$$

By the induction hypothesis, we have that

$$\dim \ M_2^\star/M_2 = \sum_{j=2}^{l} \dim \ M_{2\lambda_j}/(M_2^\star)_{\lambda_j} = \sum_{j=2}^{l} \dim \ M_{2\lambda_j}/M_{1\lambda_j}.$$

We thus obtain that

$$\begin{aligned} \dim \ M_1/M_2 &= \dim \ M_1/M_2^\star + \dim \ M_2^\star/M_2 \\ &= \sum_{j=1}^{l} \dim \ M_{2\lambda_j}/M_{1\lambda_j} \\ &= \operatorname{card}(Z(M_2)\backslash Z(M_1)). \end{aligned}$$

The proof of the theorem is thus completed.

Moreover, the same argument as the above proof enables us to obtain the following.

Theorem 2.4.4 *Let both M_1 and M_2 be submodules of X. If $M_1 \supseteq M_2$ and* $\dim M_1/M_2 = k < \infty$, *then we have*

(1) $Z(M_2)\backslash Z(M_1) \subseteq \sigma_p(M_{z_1}, M_{z_2}, \cdots, M_{z_n}) \subset \Omega$,

(2) $\dim M_1/M_2 \geq \sum_{\lambda \in Z(M_2)\backslash Z(M_1)} \dim M_{2\lambda}/M_{1\lambda} = \mathrm{card}(Z(M_2)\backslash Z(M_1))$,
and "equal" if and only if M_2 is an AF-cosubmodule.

We end with two examples.

Example 2.4.5 *Let f be analytic on a neighborhood of the origin in $\mathbb{C}^n$, and let $f(z) = \sum_m f_m(z)$ be the homogeneous expansion of f at the origin. Then there is the smallest m such that f_m is not the zero-polynomial. This m is called the order of the zero which f has at the origin.*

Recall that Rudin's submodule M [Ru1, p. 71] of $H^2(\mathbb{D}^2)$ over the bidisk is defined as the collection of all functions in $H^2(\mathbb{D}^2)$ that have a zero of order greater than or equal to n at $(0, 1 - n^{-3})$ for $n = 1, 2, \cdots$. Douglas and Yang [DYa1] showed that $M \ominus (zM + wM)$ is finite dimensional, while $M \ominus (zM + wM)$ is not a generating set of M. A natural problem is what $\dim\big(M \ominus (zM + wM)\big)$ *is equal to. It is easy to check that both M and $\overline{zM + wM}$ are AF-cosubmodules, and*

$$Z(\overline{zM + wM}) \setminus Z(M) = \{(0,0)\} \quad \textit{and} \quad \mathrm{card}\big(Z(\overline{zM + wM}) \setminus Z(M)\big) = 2.$$

By Theorem 2.4.2, we get immediately that

$$\dim\big(M \ominus (zM + wM)\big) = 2.$$

Example 2.4.6 *Let M be a submodule of the Bergman module $L^2_a(\mathbb{D})$. By Aleman, Richter and Sundberg's work [ARS], we know that $M \ominus zM$ is a generating set for M. By Lemma 2.3.7,*

$$\dim(M \ominus zM) \leq \mathrm{rank}(M),$$

and it follows that

$$\dim(M \ominus zM) = \mathrm{rank}(M).$$

Notice that for any natural number n, unlike the Hardy module $H^2(\mathbb{D})$, the Bergman module $L^2_a(\mathbb{D})$ has a submodule M with rank n [Hed]. Now let M be an AF-cosubmodule of $L^2_a(\mathbb{D})$ with rank$(M) < \infty$. Then zM also is an AF-cosubmodule and $\mathrm{rank}(zM) = \mathrm{rank}(M)$. *It is easy to check that*

$$\mathrm{card}\big(Z(zM) \setminus Z(M)\big) = 1,$$

and hence by Theorem 2.4.2, $\dim(M \ominus zM) = 1$. *Thus,* $\mathrm{rank}(M) = 1$. *This shows that every AF-cosubmodule of the Bergman module with finite rank is generated by a single function.*

2.5 Finite codimensional submodules of Bergman modules

In this section, we will let domain Ω be the unit ball $\mathbb{B}_n$ or the unit polydisk $\mathbb{D}^n$. For a submodule M of Bergman module $L_a^2(\Omega)$, our interest is to study the structure of $M^\perp$ when M is of finite codimension. In [GZ], Guo and Zheng completely characterized orthogonal complements of finite codimensional submodules of Bergman modules on bounded symmetric domains. Here we present a special case of this general result, but the idea is completely the same.

Let Ω be the unit ball or the unit polydisk in $\mathbb{C}^n$ and let $Aut(\Omega)$ be the automorphism group of Ω (all biholomorphic mappings of Ω onto Ω). Here we collect some properties of the automorphism group of Ω. First we can canonically define (cf. [Ru2]) for each $\lambda \in \Omega$, an automorphism ϕ_λ in $Aut(\Omega)$, such that

1. $\phi_\lambda \circ \phi_\lambda(z) = z$;
2. $\phi_\lambda(0) = \lambda, \ \ \phi_\lambda(\lambda) = 0$;
3. ϕ_λ has a unique fixed point in Ω;
4. if $\phi \in Aut(\Omega)$ and $\lambda = \phi^{-1}(0)$, then there is a unique operator U on $\mathbb{C}^n$ such that $\phi = U\phi_\lambda$.

Let M be a finite codimensional submodule of $L_a^2(\Omega)$. Then by Corollary 2.2.6, the submodule M has only finitely many zero points $\lambda_1, \lambda_2, \cdots, \lambda_l$ in Ω, such that M can be uniquely represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where each M_i is a finite codimensional submodule having a unique zero λ_i.

Since Ω is circular, the functions z^α, α ranging over all nonnegative multi-indices, are orthogonal in $L_a^2(\Omega)$. Also, the uniqueness of the Taylor expansion implies that $\{z^\alpha : \alpha \geq 0\}$ is a complete set. Thus $\{z^\alpha/\|z^\alpha\| : \alpha \geq 0\}$ is an orthonormal basis for $L_a^2(\Omega)$. This implies that for any polynomial p, the Toeplitz operator $T_{\bar{p}}$ maps $\mathcal{C}$ to $\mathcal{C}$. Now let $\mathcal{P}$ be a linear space consisting of polynomials. We say that $\mathcal{P}$ is an invariant polynomial space, if for any polynomial p, $\mathcal{P}$ is invariant under the action of $T_{\bar{p}}$. It is easy to see that in the case of $n = 1$, an invariant polynomial space with the dimension m $(1 \leq m \leq \infty)$ is the linear space with the basis $\{1, z^1, \cdots, z^m\}$.

Lemma 2.5.1 *Let M be a finite codimensional submodule with a unique zero $\lambda = 0$. Then $M^\perp$ is a finite dimensional invariant polynomial space.*

Proof. Since M is a submodule of $L_a^2(\Omega)$, M is invariant under the action of the Toeplitz operator T_p for any polynomial p. Hence $M^\perp$ is invariant under the action of $T_{\bar{p}}$. To complete the proof, we need the characteristic space theory. Let M_0 be the characteristic space of M at $\lambda = 0$, that is, $M_0 = \{p \in \mathcal{C} : p(D)f|_{\lambda=0} = 0, \forall f \in M\}$. Since $\{z^\alpha/\|z^\alpha\| : \alpha \geq 0\}$ is the orthonormal basis for $L_a^2(\Omega)$, the Taylor expansion gives that

$$[\partial^\alpha f](0) = \frac{\alpha!}{\|z^\alpha\|^2}\langle f, z^\alpha\rangle,$$

for each $f \in L_a^2$, where $\alpha! = \prod_{i=1}^n \alpha_i!$. Let $p(z) = \sum a_\alpha z^\alpha$ be a polynomial. Then

$$\begin{aligned} p(D)f|_{\lambda=0} &= \sum a_\alpha [\partial^\alpha f](0) \\ &= \sum a_\alpha \frac{\alpha!}{\|z^\alpha\|^2}\langle f, z^\alpha\rangle \\ &= \langle f, \sum \bar{a}_\alpha \frac{\alpha!}{\|z^\alpha\|^2} z^\alpha\rangle. \end{aligned}$$

Define the conjugate linear map $\gamma : \mathcal{C} \to \mathcal{C}$ by

$$\gamma(\sum a_\alpha z^\alpha) = \sum \bar{a}_\alpha \frac{\alpha!}{\|z^\alpha\|^2} z^\alpha.$$

It is easy to verify that γ is one to one, and onto. Thus the image of M_0 under the conjugate linear operator γ is a subspace of $M^\perp$. By Corollary 2.2.6,

$$\operatorname{codim} M = \dim M_0.$$

Therefore, we get that $M^\perp = \gamma(M_0)$. Hence $M^\perp$ is a finite dimensional invariant polynomial space, completing the proof.

Let M be a finite codimensional submodule of $L_a^2(\Omega)$. Then M has finitely many zero points $\lambda_1, \lambda_2, \cdots, \lambda_l$ in Ω such that M can be uniquely represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where M_i is a finite codimensional submodule and has a unique zero λ_i.

For $\lambda \in \Omega$, let $K_\lambda(z)$ be the reproducing kernel of $L_a^2(\Omega)$ at λ. This means that $f(\lambda) = \langle f, K_\lambda\rangle$ for each $f \in L_a^2(\Omega)$. Let $k_\lambda(z)$ be the normalized reproducing kernel, that is, $k_\lambda = K_\lambda/\|K_\lambda\|$. On the Bergman space $L_a^2(\Omega)$, we define the operator U_λ by

$$U_\lambda f = f \circ \phi_\lambda k_\lambda,$$

where ϕ_λ is an element in $Aut(\Omega)$. Note that $\det[\partial\phi_\lambda(z)] = (-1)^n k_\lambda(z)$. Hence U_λ is a unitary operator from $L_a^2(\Omega)$ onto $L_a^2(\Omega)$.

Theorem 2.5.2 *Under the above assumption, there are invariant polynomial spaces* $\mathcal{P}_i, i = 1, 2, \cdots, l$, *such that*

$$M^\perp = \sum_{i=1}^{l} \mathcal{P}_i \circ \phi_{\lambda_i}\, k_{\lambda_i}.$$

Proof. Let N be a submodule. We claim that $U_\lambda N = \{f \circ \phi_\lambda k_\lambda : f \in N\}$ is also a submodule. It is not difficult to verify $k_\lambda\, k_\lambda \circ \phi_\lambda = 1$. We thus get

$$U_\lambda N = \{\frac{f}{k_\lambda} \circ \phi_\lambda : f \in N\} = \{f \circ \phi_\lambda : f \in N\}.$$

From the equation

$$\phi_\lambda \circ \phi_\lambda(z) = z,$$

we see that each coordinate function $z_i = \phi_\lambda^{(i)} \circ \phi_\lambda(z)$, where $\phi_\lambda^{(i)}$ is the *ith* argument of ϕ_λ. This ensures that $U_\lambda N$ is invariant under the multiplication by all polynomials, and hence $U_\lambda N$ is a submodule. Now let N be a submodule of finite codimension with a unique zero point λ. Since

$$L_a^2 = N \oplus N^\perp = U_\lambda N \oplus U_\lambda N^\perp,$$

the submodule $U_\lambda N$ is of the same codimension as N. Note that $U_\lambda N$ has only the zero point 0. Thus from Lemma 2.5.1, $U_\lambda N^\perp$ is a finite dimensional invariant polynomial space. We denote $U_\lambda N^\perp$ by $\mathcal{P}$, thus,

$$N^\perp = U_\lambda \mathcal{P} = \mathcal{P} \circ \phi_\lambda\, k_\lambda.$$

Since $M = \cap_{i=1}^{l} M_i$, we have

$$M^\perp = \sum_{i=1}^{k} M_i{}^\perp.$$

Thus there are finite dimensional invariant polynomial spaces $\mathcal{P}_i, i = 1, 2, \cdots, l$ such that

$$M^\perp = \sum_{i=1}^{l} \mathcal{P}_i \circ \phi_{\lambda_i}\, k_{\lambda_i}.$$

This completes the proof of the theorem.

Let $A(\Omega)$ be the so-called Ω-algebra, that is, $A(\Omega)$ consists of all functions f that are analytic on Ω and continuous on the closure $\overline{\Omega}$ of Ω.

Corollary 2.5.3 *Let* M *be a submodule of* $L_a^2(\Omega)$. *Then* M *is of finite codimension if and only if* $M^\perp \subset A(\Omega)$.

Proof. The necessity is given by Theorem 2.5.2. The sufficiency of the corollary comes essentially from the proof of Theorem 5.2 in [Ru5]. Indeed, by the assumption $M^\perp \subset A(\Omega)$ each $f \in M^\perp$ is a bounded analytic function on Ω. From the inequality $\|f\|_2 \leq \|f\|_\infty$ it is easy to verify that $M^\perp$ is complete under the L^∞–norm. By the open mapping theorem, there exists a constant γ such that $\|f\|_\infty \leq \gamma \|f\|_2$ for each $f \in M^\perp$. Let $\{f_1, f_2, \cdots, f_l\}$ be a unit orthogonal set in $M^\perp$. On $\Omega \backslash \cap_{k=1}^l Z(f_k)$, we define functions $r_k(z)$ by

$$r_k(z) = \frac{\overline{f_k(z)}}{(\sum_{i=1}^l |f_i(z)|^2)^{1/2}}, \quad k = 1, 2, \cdots, l,$$

where $Z(f) = \{\lambda \in \Omega : \ f(\lambda) = 0\}$. For every $w \in \Omega \backslash \cap_{k=1}^l Z(f_k)$, since

$$\| \sum_{k=1}^l r_k(w) f_k \|_2 = 1,$$

we get

$$| \sum_{k=1}^l r_k(w) f_k(z) | \leq \gamma, \quad \forall z \in \Omega.$$

Consequently, for each $w \in \Omega \backslash \cap_{k=1}^l Z(f_k)$,

$$| \sum_{k=1}^l r_k(w) f_k(w) | = (\sum_{k=1}^l |f_k(w)|^2)^{1/2} \leq \gamma.$$

Thus,

$$\sum_{k=1}^l |f_k(w)|^2 \leq \gamma^2, \quad \forall\, w \in \Omega \backslash \cap_{k=1}^l Z(f_k).$$

Since $\cap_{k=1}^l Z(f_k)$ is a null-measurable set, integrating the above inequality gives $l \leq \gamma^2$. We conclude that dim $M^\perp$ is finite. This completes the proof of the corollary.

In the cases of both the unit ball $\mathbb{B}_n$ and the unit polydisk $\mathbb{D}^n$, their automorphisms and reproducing kernels are rational functions. This gives the following result.

Corollary 2.5.4 *Let Ω be the unit ball $\mathbb{B}_n$ or the unit polydisk $\mathbb{D}^n$, and let M be a submodule of $L_a^2(\Omega)$. Then M is of finite codimension if and only if $M^\perp$ consists of rational functions.*

Remark 2.5.5 *As the reader has seen, the results in this section are easily generalized to the case of the Hardy modules on the unit ball and the unit polydisk.*

2.6 Remarks on Chapter 2

This chapter is mainly based on Guo's papers [Guo1, Guo2, Guo4, Guo6]. The starting point for algebraic reduction is Theorem 2.2.1 of Ahern-Clark, which characterized submodules of $H^2(\mathbb{D}^n)$ of finite codimension as closures of ideals. Significant generalization and simplification of this result to reproducing Banach modules were done by Douglas, Paulsen et al. [DP, DPSY, Pau1]. There are numerous references concerning this topic. Here we have made no attempt to compile a comprehensive list of references. We call the reader's attention to [AB, AS, Ber, CD2, CDo2, DPSY, DY3, JLS, Ri2] for more information. We also refer the reader to the references [Cur1, Cur2, Guo11, **?**, Jew] for index theory of the tuple of the coordinate multipliers associated with an analytic Hilbert module.

For the research of analytic Hilbert modules, part of the difficulty lies in the analysis of zero varieties of submodules. Douglas's localization technique has always played an important role (see Douglas's survey paper [Dou]). Another technique developed by Guo, so-called "characteristic space theory for analytic Hilbert modules," has exhibited its power, just as we have seen in this chapter. In fact, Chapters 2 through 5 in this monograph are mainly devoted to analytic Hilbert modules and reproducing analytic Hilbert spaces on $\mathbb{C}^n$ by using characteristic space theory. We also notice that Bercovici have even used a similar method to the characteristic space theory to deal with finite codimensional invariant subspaces of Bergman spaces [Ber].

In [Guo1], Guo introduced the notion of analytic Hilbert modules and established the characteristic space theory for analytic Hilbert modules. Theorems 2.1.1, 2.1.5 and 2.4.1 and Corollaries 2.1.2 and 2.1.3 were proved in [Guo1]. For the notion of multiplicity of zero variety of an analytic submodule, it was introduced by Guo [Guo4]. In that paper he obtained Theorems 2.1.6, 2.1.7 and 2.2.5 and Corollaries 2.1.8 and 2.2.6. The extension of Theorem 2.2.5 to AF-cosubmodules was considered in [Guo6]. Theorems 2.4.2 and 2.4.4 and Examples 2.4.5 and 2.4.6 appeared in [Guo6]. In the case of Hilbert modules with finite rank, Douglas and Yan [DY2] first established the existence of a Hilbert-Samuel polynomial for such a module. We presented Theorem 2.3.8 for analytic submodules as a special version of the result of Douglas and Yan. Related to the research of submodules with finite rank, Theorems 2.3.3 and 2.3.4 and Proposition 2.3.1 are due to Guo [Guo2]. Concerning orthogonal complements of finite codimensional submodules of Bergman modules, Lemma 2.5.1, Theorem 2.5.2 and Corollaries 2.5.3 and 2.5.4 appeared in [GZ].

Chapter 3

Rigidity for analytic Hilbert modules

The classic paper of Beurling [Beu] led to a spate of research in operator theory, H^p-theory and other areas which continues to the present. His explicit characterization for all submodules of the Hardy module $H^2(\mathbb{D})$, in terms of the inner–outer factorization of analytic functions, has had a major impact. Since the Hardy module over the unit disk is the primary nontrivial example for so many different areas, it is not surprising that this characterization has proved so important.

Almost everyone who has thought about this topic must have considered the corresponding problem for higher dimensional Hardy module $H^2(\mathbb{D}^n)$. Although the existence of inner functions in the context is obvious, one quickly sees that a Beurling-like characterization is not possible [Ru1], and hence this directs one's attention to investigating equivalence classes of submodules of analytic Hilbert modules in reasonable sense.

Let M_1, M_2 be two submodules of the analytic Hilbert module X. We say that $A: M_1 \to M_2$ is a module map if A is a bounded linear operator, and for any polynomial p, $A(ph) = pA(h)$, $h \in M_1$. Furthermore, we say that

1. they are unitarily equivalent if there exists a unitary module map $A: M_1 \to M_2$, that is, A is both a unitary operator and a module map;
2. they are similar if there exists an invertible module map $A: M_1 \to M_2$;
3. they are quasi-similar if there exist module maps $A: M_1 \to M_2$ and $B: M_2 \to M_1$ with dense ranges.

From the Beurling theorem, any two submodules of the Hardy module $H^2(\mathbb{D})$ are unitarily equivalent. However, for higher dimensional Hardy module $H^2(\mathbb{D}^n)$, an earlier result on nonunitarily equivalent submodules is due to Berger, Coburn and Lebow [BCL], who considered the restriction of multiplication by the coordinate functions to invariant subspaces obtained as the closure of certain ideals of polynomials in two variables having the origin as zero set. By applying their results on commuting isometries, they showed that different ideals yield inequivalent submodules. Almost at the same time Hastings [Ha2] even showed that $[z - w]$ is never quasi-similar to $H^2(\mathbb{D}^2)$. In [ACD], Agrawal, Clark and Douglas introduced the concept of unitary equivalence of Hardy-submodules. In particular, they showed that two submodules of finite codimension are unitarily equivalent if and only if they are equal. Furthermore, Douglas and Yan [DY1] proved that under some technical hypotheses on the zero varieties, two submodules that are quasi-similar must be equal. More recently, Douglas, Paulsen, Sah and Yan [DPSY] showed that under mild restrictions, submodules obtained from the closure of ideals are quasi-similar if and only if the ideals coincide. From an analytic point of view, appearance of this phenomenon, the so-called "rigidity phenomenon," is natural because of the Hartogs phenomenon in several variables. From an algebraic point of view, the reason may be that the submodules are not singly generated.

The present chapter is devoted to these remarkable features of analytic Hilbert modules in several variables to understand the connection between function theory, operator theory and some related fields.

3.1 Rigidity for analytic Hilbert modules

In this section, we will use the characteristic space theory to obtain general results of rigidity. This section is mainly based on Guo's paper [Guo1]. Now we suppose that X is an analytic Hilbert module over Ω in $\mathbb{C}^n$ $(n > 1)$. Let M_1 and M_2 be two submodules of X, and $\sigma : M_1 \to M_2$ be a module map. We say that the map σ is canonical if

$$\sigma(M_1^{(z)}) \subseteq \sigma(M_2^{(z)}), \quad \text{for any} \quad z \in \Omega \setminus Z(M_1),$$

where $M_i^{(z)} = \{h \in M_i : h(z) = 0\}$ for $i = 1, 2$. An equivalent description is given by the following proposition.

Proposition 3.1.1 *Let M_1 and M_2 be submodules of X, and $\sigma : M_1 \to M_2$ be a module map. Then the map σ is canonical if and only if there exists an analytic function ϕ on $\Omega \setminus Z(M_1)$ such that for any $h \in M_1$ and $z \in \Omega \setminus Z(M_1)$, $\sigma(h)(z) = \phi(z)h(z)$.*

Proof. For $h_1 \in M_1$, we define an analytic function on $\Omega \setminus Z(h_1)$ by

$$\phi_{h_1}(z) = \frac{\sigma(h_1)(z)}{h_1(z)}, \quad \forall z \in \Omega \setminus Z(h_1).$$

For another $h_2 \in M_1$, we also define an analytic function on $\Omega \setminus Z(h_2)$ by

$$\phi_{h_2}(z) = \frac{\sigma(h_2)(z)}{h_2(z)}, \quad \forall z \in \Omega \setminus Z(h_2).$$

Since σ is canonical, we have that

$$\phi_{h_1}(z) = \phi_{h_2}(z), \quad \forall z \in \Omega \setminus (Z(h_1) \cup Z(h_2)).$$

The above argument shows that for any $z \in \Omega \setminus Z(M_1)$, we can define

$$\phi(z) = \frac{\sigma(h)(z)}{h(z)}$$

for any $h \in M_1$ with $h(z) \neq 0$. Clearly, ϕ is independent of h and is analytic on $\Omega \setminus Z(M_1)$. It follows that

$$\sigma(h)(z) = \phi(z)h(z)$$

for any $h \in M_1$ and $z \in \Omega \setminus Z(M_1)$, completing the proof of necessity. Sufficiency is obvious.

Using the proof of Proposition 3.1.1, we can prove the following.

Corollary 3.1.2 *If there exists an open set $\mathcal{O}$ contained in $\Omega \setminus Z(M_1)$ such that for each $z \in \Omega \setminus Z(M_1)$, $\sigma(M_1^{(z)}) \subseteq (M_2^{(z)})$, then the map σ is canonical. Equivalently, if there exists an analytic function ϕ on $\mathcal{O}$ such that for any $h \in M_1$ and $z \in \mathcal{O}$, $\sigma(h)(z) = \phi(z)h(z)$, then the map σ is canonical.*

For most "natural" function spaces, module maps are canonical.

Example 3.1.3 *Let X be the Hardy module $H^2(\mathbb{D}^n)$ (or $H^2(\Omega)$ where Ω is a strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary). Then each module map $\sigma : M_1 \to M_2$ is canonical.*

In fact, from [DY1], one knows that there exists a function $\phi \in L^\infty(\mathbb{T}^n)$ (or $\phi \in L^\infty(\partial\Omega)$) such that for each $h \in M_1$, $\sigma(h) = \phi h$. One can extend ϕ to a meromorphic function $\tilde{\phi}$ on $\mathbb{D}^n$ (or Ω) and $\tilde{\phi}$ is analytic on $\mathbb{D}^n \setminus Z(M_1)$ (or $\Omega \setminus Z(M_1)$) such that for each $z \in \mathbb{D}^n \setminus Z(M_1)$ (or $z \in \Omega \setminus Z(M_1)$), $\sigma(h)(z) = \tilde{\phi}(z)\, h(z)$. From the above discussion one sees that the map σ is canonical.

Example 3.1.4 *Recall that a submodule M of X is said to have the codimension 1 property if $\dim M/[\mathcal{U}_z M] = 1$ for each $z \in \Omega \setminus Z(M)$, where $\mathcal{U}_z = \{p \in \mathcal{C} : p(z) = 0\}$. It is easy to check that if a submodule M contains a dense linear manifold Λ with the Gleason property, then M has the codimension 1 property, where the Gleason property means that for each $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Omega \setminus Z(M)$ if $h(\lambda) = 0$ (here $h \in \Lambda$), then there exist $h_1, \cdots, h_n \in \Lambda$ such that $h = \sum_{i=1}^n (z_i - \lambda_i) h_i$. Let $\sigma : M_1 \to M_2$ be a module map and M_1 have the codimension 1 property. Then it is easy to verify that the map σ is canonical.*

For the analytic Hilbert module X, we let $A(\Omega)$ denote the closure of all polynomials in the operator norm. Then $A(\Omega)$ is a Banach algebra consisting of analytic functions in X. It is easy to know that each $\mathcal{C}$-module map extends uniquely to an $A(\Omega)$-module map.

Example 3.1.5 *Let M_1, M_2 be submodules of X, and $M_1 \cap A(\Omega) \neq \{0\}$. If $\sigma : M_1 \to M_2$ is a module map, then the map σ is canonical.*

In fact, taking a nonzero $p \in M_1 \cap A(\Omega)$, we define an analytic function $\tilde{\phi}$ on $\Omega \setminus Z(p)$ by

$$\tilde{\phi}(z) = \frac{\sigma(p)(z)}{p(z)}, \quad \forall z \in \Omega \setminus Z(p).$$

For any $h \in M_1$, there exists a sequence of polynomials $\{p_n\}$ such that $\{p_n\}$ converges to h in the norm of X. It follows that $pp_n \to ph$ in the norm of X. This implies that for each $z \in \Omega \setminus Z(p)$,

$$\sigma(pp_n)(z) = p_n(z)\sigma(p)(z) \to \sigma(ph)(z) = p(z)\sigma(h)(z).$$

Since $p_n(z) \to h(z)$, one concludes easily that for any $h \in M_1$ and every $z \in \Omega \setminus Z(p)$,

$$\sigma(h)(z) = \tilde{\phi}(z)h(z).$$

Now let z be any point in $\Omega \setminus Z(M_1)$. Then there exists a function $h \in M_1$ such that $h(z) \neq 0$. We can define an analytic function $\tilde{\psi}$ on $\Omega \setminus Z(h)$ by

$$\tilde{\psi}(z) = \frac{\sigma(h)(z)}{h(z)}.$$

The preceding discussion implies that

$$\tilde{\phi}(z) = \tilde{\psi}(z), \quad z \in \Omega \setminus (Z(p) \cup Z(h)).$$

This means that $\tilde{\phi}$ analytically extends to $\Omega \setminus Z(M_1)$, and for each $h \in M_1$

$$\sigma(h)(z) = \tilde{\phi}(z)h(z), \quad \forall z \in \Omega \setminus Z(M_1).$$

Therefore, by Proposition 3.1.1, the map σ is canonical.

For two submodules M_1 and M_2 of X, we say that M_1 and M_2 are subsimilar if there exist two canonical module maps $\sigma_1 : M_1 \to M_2$ and $\sigma_2 : M_2 \to M_1$ with dense ranges.

The following is the main result in this section.

Theorem 3.1.6 *Let M_1 and M_2 be two submodules of X. If M_1 and M_2 are subsimilar, and $h_{2n-2}(Z(M_1)) = h_{2n-2}(Z(M_2)) = 0$, then $M_1^e = M_2^e$. In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$.*

Remark 3.1.7 *For a submodule M of X, the condition $h_{2n-2}(Z(M)) = 0$ simply says that the complex dimension of the analytic variety is less than $n-1$, or equivalently that the codimension of this variety is at least 2. It is well known that analytic varieties in $\mathbb{C}^n$ of codimension at least 2 are removable singularities for analytic functions [KK]. We use the notion of the Hausdorff measure to avoid the dimension theory of analytic varieties.*

The proof of Theorem 3.1.6 From Proposition 3.1.1, one sees that there exist analytic functions ϕ_1 on $\Omega \setminus Z(M_1)$ and ϕ_2 on $\Omega \setminus Z(M_2)$ such that for each $f \in M_1$ and each $g \in M_2$,

$$\sigma_1(f)(z) = \phi_1(z)f(z),\ z \in \Omega \setminus Z(M_1),\ \text{and}\ \sigma_2(g) = \phi_2(z)g(z),\ z \in \Omega \setminus Z(M_2).$$

Since

$$h_{2n-2}(Z(M_1)) = h_{2n-2}(Z(M_2)) = 0,$$

the theorem [KK] on the removability of singularities implies that ϕ_1 and ϕ_2 can be analytically continued onto Ω. Therefore, on Ω, we have that

$$\sigma_1(f) = \phi_1 f \quad \text{and} \quad \sigma_2(g) = \phi_2 g$$

for any $f \in M_1$ and $g \in M_2$. From the above equalities and Leibnitz's rule, it is easy to check that $Z(M_1) = Z(M_2)$, and for each $\lambda \in Z(M_1)$

$$M_{1\lambda} = M_{2\lambda}.$$

Hence, by Theorem 2.4.1, it holds that

$$M_1^e = M_2^e.$$

In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$. This completes the proof of the theorem.

We are now in a position to give some examples to illustrate applications of Theorem 3.1.6.

Example 3.1.8 *Let X be the Hardy module $H^2(\mathbb{D}^n)$ (or $H^2(\Omega)$, where Ω is a strongly pseudomonades domain in $\mathbb{C}^n$ with a smooth boundary). If the submodules M_1 and M_2 are quasi-similar and $h_{2n-2}(Z(M_1)) = h_{2n-2}(Z(M_2)) = 0$, then $M_1^e = M_2^e$. In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$.*

Example 3.1.9 *Let the submodules M_1 and M_2 of X have the codimension 1 property. If they are quasi-similar and $h_{2n-2}(Z(M_1)) = h_{2n-2}(Z(M_2)) = 0$, then $M_1^e = M_2^e$. In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$.*

Example 3.1.10 *Assume that the submodules M_1 and M_2 of X have the properties*

$$M_1 \cap A(\Omega) \neq 0,\ M_2 \cap A(\Omega) \neq 0,\ \textit{and}\ h_{2n-2}(Z(M_1)) = h_{2n-2}(Z(M_2)) = 0.$$

Then their quasi-similarity implies $M_1^e = M_2^e$. In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$.

To obtain the next example let us first state a theorem on ideals of polynomials whose proof will be placed in the end of this section.

Let P be a prime ideal. The height of P, denoted by height(P), is defined as the maximal length l of any properly increasing chain of prime ideals

$$0 = P_0 \subset P_1 \cdots \subset P_l = P.$$

Since the polynomial ring $\mathcal{C}$ is Noetherian, every prime ideal has finite height and the height of an arbitrary ideal is defined as the minimum of the heights of its associated prime ideals. For an ideal I, one has

$$\dim_C Z(I) = n - l,$$

where $l = height(I)$ is the height of I, and $\dim_{\mathbb{C}} Z(I)$ the complex dimension of the zero variety of I (cf. [DPSY, KK] or [ZS]). When ideals are a height of at least 2, this condition guarantees that the zero variety of the ideal is a h_{2n-2} null set, and hence removable singularity for analytic functions [KK].

The following theorem was proved by Guo.

Theorem 3.1.11 *Let I be an ideal of the polynomial ring $\mathcal{C}$ and $I = pL$ be its Beurling form. If $p = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ is the product of its prime factors, and $L = \cap_{i=1}^{l} L_i$ a primary decomposition with their associated primes $P_1, \cdots, P_l$, then there exists an irredundant primary decomposition of I,*

$$I = \cap_{j=1}^{t+l} I_j,$$

such that $\sqrt{I_k} = p_k \mathcal{C}$ for $k = 1, 2, \cdots t$, and $\sqrt{I_k} = P_k$ for $k > t$.

Note that height of a prime ideal P equals 1 if and only if P is principal. Theorem 3.1.11 thus implies the following.

Corollary 3.1.12 *Let $I = pL$ be its Beurling form. Then $height(I) = 1$ if and only if p is not a constant. Equivalently, $height(I) \geq 2$ if and only if the greatest common divisor $GCD(I) = 1$. In particular, $height(L) \geq 2$.*

Remark. From Corollary 3.1.12, we see that $Z(I)$ is a removable singularity for analytic functions if and only if $GCD(I) = 1$.

The following example will show that one can reobtain the main result in [DPSY] by using the characteristic space theory.

Example 3.1.13 *Let I_1 and I_2 be ideals of polynomials with $GCD(I_1) = GCD(I_2) = 1$. If each algebraic component of their zero varieties meets Ω nontrivially, then $[I_1]$ and $[I_2]$ are quasi-similar if and only if $I_1 = I_2$.*

In fact, by Corollary 3.1.12, we see that the ideals are a height of at least 2. Note that the algebraic condition on the height of the ideals just guarantees that the zero varieties of the ideals are h_{2n-2} null sets. From Example 3.1.10, we see that

$$[I_1]^e = [I_2]^e.$$

Setting

$$J = [I_1]^e \cap \mathcal{C},$$

then for each $\lambda \in \Omega$,

$$I_{1\lambda} = [I_1]_\lambda = \big([I_1]^e\big)_\lambda \subseteq \big([I_1]^e \cap \mathcal{C}\big)_\lambda = J_\lambda.$$

By Corollary 2.1.3, $J \subseteq I_1$. This implies that $I_1 = J$. The same reasoning shows that $I_2 = J$, and hence $I_1 = I_2$, completing the proof.

The proof of Theorem 3.1.11 Let $I = \cap_{j=1}^{N} I_j$ be an irredundant primary decomposition of I. For each $k(1 \leq k \leq t)$, we claim that there is some positive integer m such that every element of I_m is divisible by p_k. In fact, if for each m there exists some element q_m of I_m that is not divisible by p_k, then $q = q_1 q_2 \cdots q_N \in I$ is not divisible by p_k. This is not possible, and therefore there is some m such that every element of I_m is divisible by p_k. Note that p_k is a prime polynomial, and hence each element of $\sqrt{I_m}$ is divisible by p_k. Since $\sqrt{I_m}$ is prime, this ensures that

$$\sqrt{I_m} = p_k \mathcal{C}.$$

Now we may rearrange the order of $I_1, I_2, \cdots, I_N$ such that

$$\sqrt{I_k} = p_k \mathcal{C}, \quad k = 1, 2, \cdots t.$$

Now suppose that there is some $k(> t)$ such that $\sqrt{I_k}$ is principal. Since $\sqrt{I_k}$ is prime, there is a prime polynomial p_0 such that $\sqrt{I_k} = p_0 \mathcal{C}$. It follows that each element of I_k is divisible by p_0, and hence every element of I is divisible by p_0. Since $GCD(L) = 1$, this means that p is divisible by p_0. Hence there exists an i such that $p_0 = p_i$. This contradicts the fact that the primary decomposition $I = \cap_{j=1}^{N} I_j$ is irredundant. Therefore, $\sqrt{I_k}$ is not principal if $k > t$. Let J be an ideal of polynomials. Then by [ZS, Vol(I), p. 210, Theorem 6] or [AM, Theorem 4.5], a prime ideal P is the associated prime ideal of J if and only if there exists an element x of $\mathcal{C}$ such that $P = \sqrt{J : x}$. Now for each associated prime ideal P_i of L, there exists an element $x_i \in \mathcal{C}$ such that $P_i = \sqrt{L : x_i}$. From the equality

$$\sqrt{L : x_i} = \sqrt{pL : px_i},$$

P_i are the associated prime ideals of I for $i = 1, \cdots, l$. Furthermore, assume that Q is an associated prime ideal of I, and Q is not principal; then there exists an element $x \in \mathcal{C}$ such that

$$Q = \sqrt{I : x} = \sqrt{pL : x}.$$

Since Q is not principal, there is a set $\{q_1, \cdots, q_r\}$ of generators of Q such that $GCD\{q_1, \cdots, q_r\} = 1$ and $r \geq 2$. Consequently, for each q_i, there is a positive integer k_i such that $q_i^{k_i} x \in pL$ for $i = 1, 2, \cdots, r$. Since

$$GCD(q_1^{k_1}, \cdots, q_r^{k_r}) = 1,$$

this ensures that x is divisible by p. Thus, there exists an element $y \in \mathcal{C}$ such that $x = py$. We thus have that

$$Q = \sqrt{pL : x} = \sqrt{pL : py} = \sqrt{L : y}.$$

Therefore, Q is equal to some P_i. From the above discussion, there exists an irredundant primary decomposition of I,

$$I = \cap_{j=1}^{t+l} I_j,$$

such that $\sqrt{I_k} = p_k \mathcal{C}$ for $k = 1, 2, \cdots t$, and $\sqrt{I_k} = P_k$ for $k > t$. The proof of the theorem is complete.

3.2 Rigidity for quotient modules

While the study of canonical models for contraction operators on Hilbert space has been fruitful and useful [NF] this can be said for multi-variate analogies. Besides the troubling existence problem, there is also the nonuniqueness of models when they do exist. Douglas and Paulsen's module theoretic approach [DP] provided, in part, multi-variate analogies. In this section, we will examine some special cases in which the canonical model is unique. Using module theoretic language, this means that quotient modules are necessarily equal if they are unitarily equivalent (similar or quasi-similar).

Let M_1 and M_2 be two submodules of analytic Hilbert module X, and $M_1 \supseteq M_2$. Then we form the quotient module M_1/M_2 on the ring $\mathcal{C}$. The module action is given by $p \cdot (h + M_2) = ph + M_2$. It is easy to see that this action is bounded. In the following we will give a coordinate free method to deal with rigidity of quotient modules. The next theorem is due to Guo [Guo1].

Theorem 3.2.1 *Let M_1 and M_2 be two submodules of X. If X/M_1 and X/M_2 are quasi-similar, then $M_1^e = M_2^e$. In particular, if M_1 and M_2 are AF-cosubmodules, then $M_1 = M_2$.*

Proof. Let $\{M_1^\alpha\}_{\alpha \in \Lambda}$ be the collection of all finite codimensional submodules containing M_1. If $\sigma : X/M_1 \to X/M_2$ is a $\mathcal{C}$-module quasi-similarity, then there exists a collection of submodules of X, $\{M_2^\alpha\}_{\alpha\in\Lambda}$, each of which contains M_2, so that σ induces the following quasi-similarity:

$$\sigma: M_1^\alpha/M_1 \to M_2^\alpha/M_2, \quad \alpha\in\Lambda.$$

This implies that σ induces again a $\mathcal{C}$-module quasi-similarity for each α:

$$\sigma_\alpha: X/M_1^\alpha \to X/M_2^\alpha.$$

Since M_1^α is of finite codimension, M_2^α is finite codimensional.

For a submodule M of X, let $Ann(X/M)$ denote the annihilator of X/M, that is,

$$Ann(X/M) = \{p \in \mathcal{C}: p\cdot\tilde{x} = 0, \forall \tilde{x}\in X/M\}.$$

Then $Ann(X/M)$ is an ideal of $\mathcal{C}$. In particular, if M is of finite codimension, then by Theorem 2.2.3 $Ann(X/M)$ is of the same codimension as M, and $[Ann(X/M)] = M$.

Since each σ_α is quasi-similar, this gives

$$Ann(X/M_1^\alpha) = Ann(X/M_2^\alpha),$$

and hence

$$M_1^\alpha = M_2^\alpha, \quad \alpha\in\Lambda.$$

This implies that $M_1^e \supset M_2^e$. The same reasoning shows that $M_2^e \supset M_1^e$, and hence we have $M_1^e = M_2^e$.

In particular, if M_1 and M_2 are AF-cosubmodules, then clearly $M_1 = M_2$. This completes the proof of the theorem.

By Theorem 3.2.1 and the proof of Example 3.1.13, we immediately get the following.

Corollary 3.2.2 *Let I_1 and I_2 be ideals of polynomials such that each algebraic component of their zero varieties meets Ω nontrivially. If $X/[I_1]$ and $X/[I_2]$ are quasi-similar, then $I_1 = I_2$.*

For the analytic Hilbert module X, we define the multiplier ring $M(\Omega)$ of X by

$$M(\Omega) = \{f\in Hol(\Omega): f\,x\in X, \forall\, x\in X\}.$$

It is easily checked that

$$X \supseteq M(\Omega) \supseteq A(\Omega) \supseteq \mathcal{C}.$$

Theorem 3.2.3 *Let M_1 and M_2 be two $M(\Omega)$-submodules of X generated by multipliers. If X/M_1 and X/M_2 are quasi-similar over $M(\Omega)$, then $M_1 = M_2$.*

Proof. By quasi-similarity, the equality $Ann(X/M_1) = Ann(X/M_2)$ is immediate, where $Ann(X/M_i)$ denote annihilator of X/M_i, that is,

$$Ann(X/M_i) = \{f \in M(\Omega) : f \cdot \tilde{x} = 0, \ \forall \tilde{x} \in X/M_i\}, \ i = 1, 2.$$

By the assumption it follows easily that

$$M_1 = [Ann(X/M_1)] = [Ann(X/M_2)] = M_2.$$

The proof is thus completed.

Let M_1 and M_2 be two submodules of $H^2(\mathbb{D}^n)$. Douglas and Foias [DF] proved that $H^2(\mathbb{D}^n)/M_1$ and $H^2(\mathbb{D}^n)/M_2$ are unitarily equivalent only if M_1 and M_2 are equal. The techniques used in the proof of this result are strictly restricted to the polydisk since they depend on the fact that the coordinate functions are inner, which implies that the operators defined by the coordinate multipliers are isometries. Up to similarity or quasi-similarity, uniqueness of Hardy-quotient module remains unknown. Moreover, although there exist a lot of inner functions in the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ [Ru4], it is never obvious to extend Douglas-Foias's result to $H^2(\mathbb{B}_n)$.

Below we present Douglas and Foias's result.

Theorem 3.2.4 (Douglas-Foias) *Let M_1 and M_2 be submodules of $H^2(\mathbb{D}^n)$. Then H^2/M_1 and H^2/M_2 are unitarily equivalent only if $M_1 = M_2$.*

Proof. For a submodule M of H^2, we identify H^2/M with $H^2 \ominus M$. Then the action of the ring $\mathcal{C}$ on $H^2 \ominus M$ is given by

$$p \cdot h = P_{H^2 \ominus M} ph,$$

where $P_{H^2 \ominus M}$ is the orthogonal projection from H^2 onto $H^2 \ominus M$.

Set

$$N_1 = H^2 \ominus M_1, \quad N_2 = H^2 \ominus M_2.$$

We let $V : N_1 \to N_2$ be a unitary equivalence. To obtain the desired result we will proceed by induction on n recalling first the proof for the case $n = 1$. If we set $K_1 = \vee_m z^m N_1$, then we can extend V to K_1 by defining

$$V'(\sum z^m h_m) = \sum z^m V(h_m).$$

This is the fundamental construction [NF]. It is clear that V' takes K_1 onto K_2, the corresponding space defined using N_2, and both K_1 and K_2 are reducing subspaces for M_z since

$$M_z^*(\sum z^m h_m) = M_z^* h_0 + \sum z^m h_{m+1},$$

which is in K_1 because $M_z^* h_0$ is in $N_1 \subset K_1$. The analogous argument holds for K_2. Moreover, it follows that V' is a unitary operator on $H^2(\mathbb{D})$ which extends V. Finally, since

$$(V'M_z)M_z^m h = V'M_z^{m+1}h = M_z^{m+1}Vh$$
$$= M_z(M_z^m Vh) = (M_z V')M_z^m h$$

for h in N_1, we see that

$$V'M_z = M_z V'$$

and hence $V' = \mu I$ for a complex number μ, $|\mu| = 1$, and $M_1 = M_2$.

Now suppose $n = 2$ and we have extended V to V' , a unitary operator from K_1 onto K_2. This time K_1 and K_2 need not be all of $H^2(\mathbb{D}^2)$. Again K_1 and K_2 are reducing subspaces for M_z. Moreover, K_1 and K_2 are invariant subspaces for M_w^*. This follows for K_1 because

$$M_w^*(\sum M_z^m h_m) = \sum M_z^m M_w^* h_m$$

is in K_1 since $M_z M_w^* = M_w^* M_z$ and N_1 is invariant for M_w^*. The proof for K_2 is the same. We now repeat the previous construction to extend $V' : K_1 \to K_2$ to a unitary operator V'' on $H^2(\mathbb{D}^2)$ by setting

$$V''(\sum M_w^m h_m) = \sum M_w^m V' h_m$$

for $\{h_m\} \subset K_1$ for $m = 0, 1, \cdots$.

Since $\vee M_w^m K_1$ and $\vee M_w^m K_2$ are reducing subspaces for both M_z and M_w, it follows that each is equal to $H^2(\mathbb{D}^2)$. That V'' extends V is obvious. Finally we have

$$(V''M_w)M_w^m h = V''M_w^{m+1}h = M_w^{m+1}V'h$$
$$= M_w(M_w^m V'h) = (M_w V'')M_w^m h$$

and

$$(V''M_z)M_w^m h = V''M_w^m M_z h = M_w^m V' M_z h$$
$$= M_w^m M_z V'h = M_z M_w^m V'h = (M_z V'')M_w^m h$$

for h in K_1. Therefore, we see that V'' commutes with both M_z and M_w, which implies that $V'' = \mu I$ for some complex number μ with $|\mu| = 1$, and that $M_1 = M_2$. This completes the proof for the case $n = 2$.

The preceding is the induction step necessary to establish the result for all n. To see that, suppose V has been extended to $V^{(k)}$ from the reducing subspaces for k-tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_k}\}$ generated by N_1 to the corresponding space generated by N_2. Then we repeat the above construction to obtain $V^{(k+1)}$, where the k-tuple $\{M_{z_1}, M_{z_2}, \cdots, M_{z_k}\}$ is replaced by $\{M_{z_1}, M_{z_2}, \cdots, M_{z_k}, M_{z_{k+1}}\}$. This proves the theorem.

3.3 Rigidity for Hardy submodules on the polydisk

In the preceding sections, we studied rigidity of general analytic submodules. In this section we will concentrate attention on the case of Hardy submodules on the polydisk. As we have previously seen, rigidity depends heavily on properties of zero varieties. When zero varieties meet domains nontrivially, then the problem can be solved by using characteristic space theory. However, when the common "zeros" of submodules lie on boundary of domain, the problem becomes quite complex. Results in this section are restricted to Hardy submodules on the polydisk. By the same reasoning, the results are also valid for Hardy submodules on the ball. However, for general domains, the difficulty lies with the lack of corresponding results on the boundary behavior for analytic functions.

First we begin with the following observation. By the Beurling theorem, each nontrivial submodule M of the Hardy module $H^2(\mathbb{D})$ has a unique representation

$$M = \eta B H^2(\mathbb{D}) = \eta H^2(\mathbb{D}) \cap BH^2(\mathbb{D}),$$

where η is the singular inner function and B the Blaschke product. The "zeros" of the submodule lying on the boundary $\mathbb{T}$ depend on the singular component $\eta H^2(\mathbb{D})$. From [Gar] or [Hof], we see that the singular inner function η is determined by a singular measure σ_η on $\mathbb{T}$. Although higher dimensional Hardy submodules have no Beurling form, we can define the relevant analogue of the singular component of submodules.

Let $N(\mathbb{D}^n)$ denote the Nevanlinna class defined in [Ru1]. Then for each $f \in N(\mathbb{D}^n)$, f has radial limits f^* on $\mathbb{T}^n$ a.e. Moreover, there is a real singular measure $d\sigma_f$ on $\mathbb{T}^n$ determined by f such that the least harmonic majorant $u(f)$ of $\log|f|$ is given by

$$u(f) = P_z(\log|f^*| + d\sigma_f),$$

where P_z denotes Poisson integration. Use $N_*(\mathbb{D}^n)$ to denote the class of all $f \in N(\mathbb{D}^n)$ for which the functions $\log^+|f_r|$ form a uniformly integrable family. Therefore

$$H^p(\mathbb{D}^n) \subset N_*(\mathbb{D}^n)$$

for any $p > 0$. Furthermore, by [Ru1, Theorem 3.3.5],

$$N_*(\mathbb{D}^n) = \{f \in N(\mathbb{D}^n) : \ d\sigma_f \leq 0\}.$$

For M a submodule of $H^2(\mathbb{D}^n)$, let $Z_\partial(M)$ be the singular component of M on $\mathbb{T}^n$, defined by

$$Z_\partial(M) = inf\{-d\sigma_f : \ f \in M, \, f \neq 0\}.$$

As shown by Douglas and Yan [DY1], the singular component Z_∂ is directly related to rigidity of Hardy submodules.

We recall that a submodule M of $H^2(\mathbb{D}^n)$ satisfies the condition $(\star)$, if

(1) $h_{2n-2}(Z(M)) = 0$ and

(2) $Z_\partial(M) = 0$.

Remark 3.3.1 *From [Ru1, Theorem 3.3.6], we see that for a function $f \in H^2(\mathbb{D}^n)$, $d\sigma_f = 0$ if and only if for almost all $w \in \mathbb{T}^n$, the inner factor of the slice function $f_w(z) = f(zw)$ is a Blaschke product. Hence if there is such a function in M, then $Z_\partial(M) = 0$. In particular, if $M \cap A(\mathbb{D}^n) \neq \emptyset$, then $Z_\partial(M) = 0$, where $A(\mathbb{D}^n)$ is the polydisk algebra.*

We can now state Douglas and Yan's technical result [DY1], whose proof can be found in [DY1].

Theorem 3.3.2 (Douglas-Yan) *If M satisfies the condition (1) in $(\star)$, and $\phi \in L^\infty(\mathbb{T}^n)$, then $\phi M \subseteq H^2(\mathbb{D}^n)$ if and only if $\phi \in N(\mathbb{D}^n) \cap L^\infty(\mathbb{T}^n)$ and $d\sigma_\phi \leq Z_\partial(M)$. Moreover, if M satisfies $(\star)$, then $\phi M \subseteq H^2(\mathbb{D}^n)$ if and only if $\phi \in H^\infty(\mathbb{D}^n)$.*

Before giving applications of Theorem 3.3.2, we need a lemma.

Lemma 3.3.3 *If M_1 and M_2 are submodules of $H^2(\mathbb{D}^n)$ and $A : M_1 \to M_2$ is a module map, then there exists a function $\phi \in L^\infty(\mathbb{T}^n)$ such that $A(h) = \phi h$.*

Proof. Set

$$\mathcal{D} = \{\bar{\eta} h : \eta \text{ are inner functions and } h \in M_1\}.$$

It is easy to check that $\mathcal{D}$ is a dense linear subspace of $L^2(\mathbb{T}^n)$. We define a map

$$\hat{A} : \mathcal{D} \to L^2(\mathbb{T}^n)$$

by

$$\hat{A}(\bar{\eta} h) = \bar{\eta} A(h).$$

Since A is a module map, the definition is well defined. From the relation

$$\|\hat{A}(\bar{\eta} h)\| = \|A(h)\| \leq \|A\| \|\bar{\eta} h\|,$$

we can extend $\hat{A}$ to a bounded map from $L^2(\mathbb{T}^n)$ to $L^2(\mathbb{T}^n)$. Also, we denote this extension by $\hat{A}$. It is obvious that $\hat{A}$ satisfies

$$\hat{A} M_g = M_g \hat{A}$$

for any $g \in L^\infty(\mathbb{T}^n)$, and hence there exists a function $\phi \in L^\infty(\mathbb{T}^n)$ such that

$$\hat{A} = M_\phi.$$

This ensures that $A(h) = \phi h$ for any $h \in M_1$.

As direct consequences of Theorem 3.3.2, Douglas and Yan [DY1] obtained the following.

Corollary 3.3.4 *If M and N are quasi-similar and M satisfies $(\star)$, then $N \subseteq M$. Furthermore, If M and N satisfy $(\star)$, then M and N are quasi-similar only if $M = N$.*

Proof. Since M and N are quasi-similar, there exists a function $\phi \in L^\infty(\mathbb{T}^n)$ such that ϕM is dense in N. By Theorem 3.3.2, $\phi \in H^\infty(\mathbb{D}^n)$. It follows that

$$N = \overline{\phi M} \subseteq M.$$

The remaining case is obvious.

Corollary 3.3.5 *Let M, N be submodules of $H^2(\mathbb{D}^n)$ satisfying*

(1) $h_{2n-2}(Z(M)) = h_{2n-2}(Z(N)) = 0$;

(2) $Z_\partial(M) = Z_\partial(N)$.

If M and N are quasi-similar, then $M = N$.

Proof. From Theorem 3.3.2, there exists a function ϕ in $N(\mathbb{D}^n) \cap L^\infty(\mathbb{T}^n)$ such that ϕM is dense in N. For any $f \in M$, $\phi f = g$ in N, and hence we have

$$d\sigma_\phi + d\sigma_f = d\sigma_g,$$

that is,

$$d\sigma_\phi + (-d\sigma_g) = -d\sigma_f.$$

It follows easily that

$$d\sigma_\phi + Z_\partial(N) \leq Z_\partial(M).$$

This implies that $d\sigma_\phi \leq 0$. From [Ru1, Theorem 3.3.5], it follows that

$$\log|\phi(z)| \leq P_z(\log|\phi^*|).$$

This yields $\phi \in H^\infty(\mathbb{D}^n)$. Thus,

$$N = \overline{\phi M} \subseteq M.$$

Using the same reasoning we conclude $M \subseteq N$, and hence $M = N$, completing the proof.

From Corollary 3.3.4, one sees that when M satisfies $(\star)$, each submodule which is quasi-similar to M is necessarily contained in M. Hence before continuing we will introduce some concepts for submodules. Let M be a submodule of $H^2(\mathbb{D}^n)$. We recall that

(1) M is podal if each submodule that is unitarily equivalent to M is a submodule of M;

(2) M is s-podal if each submodule that is similar to M is a submodule of M;

(3) M is quasi-podal if each submodule that is quasi-similar to M is a submodule of M.

By the definitions, podal submodule, s-podal submodule and quasi-podal submodule are the maximum in their unitary orbit, similarity orbit and quasi-similarity orbit, respectively. The reader easily verifies that the following inclusions are true:

$$\{\text{all quasi-podals}\} \subset \{\text{all s-podals}\} \subset \{\text{all podals}\}.$$

Now by Corollary 3.3.4, one sees that if M satisfies $(\star)$, then M is quasi-podal, that is, M is the maximum in its quasi-similarity orbit. Although each podal submodule M is the maximum in its unitary orbit $orb_u(M)$, the next example will show that not every unitary orbit has the maximum.

Example 3.3.6 *Considering the Hardy module $H^2(\mathbb{D}^2)$, it is easy to verify that the submodules $[z+\frac{1}{2}w]$ and $[w+\frac{1}{2}z]$ are unitarily equivalent because*

$$|z+\frac{1}{2}w| = |w+\frac{1}{2}z|$$

on $\mathbb{T}^2$. Assume that there is the podal submodule M in the unitary orbit $orb_u([z+\frac{1}{2}w])$. Then

$$[z+\frac{1}{2}w] \subseteq M \quad \textit{and} \quad [w+\frac{1}{2}z] \subseteq M$$

and therefore the functions z and w are in M. Thus,

$$M \supseteq M_0.$$

where $M_0 = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\}$. This means that $M = M_0$, or $M = H^2(\mathbb{D}^2)$. Note that $\operatorname{rank}([z+\frac{1}{2}w]) = 1$, *but* $\operatorname{rank}(M_0) = 2$. *This implies that $M \neq M_0$. Since both $[z+\frac{1}{2}w]$ and $H^2(\mathbb{D}^2)$ are homogeneous principal submodules, by Theorem 2 in [Yan1] or Corollary 4.2.7 in Chapter 4, we see that $[z+\frac{1}{2}w]$ and $H^2(\mathbb{D}^2)$ are not unitarily equivalent, and hence $M \neq H^2(\mathbb{D}^2)$. This shows that the unitary orbit $orb_u([z+\frac{1}{2}w])$ has no maximum. Notice the inclusion*

$$orb_u([z+\frac{1}{2}w]) \subset orb_s([z+\frac{1}{2}w]),$$

where orb_s denotes the similarity orbit. This example also shows that the inclusion is strict because $H^2(\mathbb{D}^2)$ is the s-podal point in $orb_s([z+\frac{1}{2}w])$.

Proposition 3.3.7 *Let M be a submodule of $H^2(\mathbb{D}^n)$, and $\phi \in L^\infty(\mathbb{T}^n)$. If $\phi M \subseteq M$, then $\phi \in H^\infty(\mathbb{D}^n)$.*

Proof. By using an idea of Schneider [Sch], as done in the proof of Proposition 3 in [ACD], one can show that $\phi \in H^\infty(\mathbb{D}^n)$. Here we give a proof that is slightly different to that. First we may assume $|\phi| \leq 1$ and pick a nonzero $f \in M$. For every natural number k, set $g_k = \phi^k f$. Then we extend ϕ^k to those z in $\mathbb{D}^n$ where $f(z) \neq 0$ by defining

$$\phi^k(z) = g_k(z)/f(z).$$

Since

$$g_{k+1}/g_k = g_k/g_{k-1} = \cdots = g_2/g_1 = g_1/f = \phi,$$

this implies that for any $z \notin Z(f)$,

$$\phi^k(z) = (\phi^1(z))^k.$$

Since

$$(\phi^k f)(z) = \phi^k(z)f(z) = P_z[\phi^k f]$$

for $z \notin Z(f)$, one sees that

$$|(\phi^1(z))^k f(z)| \leq P_z[|\phi^k f|] \leq P_z[|f|]$$

for each natural number k. This implies that $|\phi^1(z)| \leq 1$ for each $z \notin Z(f)$. Now the fact that $\phi(z) = \phi^1(z)$ is analytic in $\mathbb{D}^n$ follows from Hartogs' theorem, and hence ϕ is a bounded analytic function in $\mathbb{D}^n$. The proof is completed.

Applying Proposition 3.3.7, we can modify an example given in [DY1] to show that not every quasi-similarity orbit has a quasi-podal point.

Example 3.3.8 *From [Ru3], there exist two functions f, g in $H^2(\mathbb{D}^2)$ such that*

(1) $|f| = |g|$ a.e. on $\mathbb{T}^2$;

(2) f/g is not the quotient of two H^∞ functions.

Then, obviously, $[f]$ and $[g]$ are quasi-similar. However, there exists no maximum in the quasi-similarity orbit $orb_q([f])$. If there were one, say M, then M is principal. This means that there is a function $h \in H^2(\mathbb{D}^2)$ such that $M = [h]$. Now applying Proposition 3.3.7, there exist functions $\phi_1; \phi_2 \in H^\infty(\mathbb{D}^2)$ such that

$$\overline{\phi_1[h]} = [\phi_1 h] = [f] \quad \textit{and} \quad \overline{\phi_2[h]} = [\phi_2 h] = [g].$$

Since

$$\phi_2[\phi_1 h] = \phi_2[f] \quad \textit{and} \quad \phi_1[\phi_2 h] = \phi_1[g],$$

this gives

$$[\phi_1\phi_2 h] = [\phi_2 f] = [\phi_1 g].$$

Then by Proposition 4.4.1 in Chapter 4, or [Yan1], there is a Nevanlinna class function r without zeros in $\mathbb{D}^2$ such that

$$\phi_2 f = \phi_1 g\, r.$$

Because $\frac{1}{r}\phi_2 f = \phi_1 g$ *this shows that* $1/r$ *is in the Nevanlinna class. Since* $r\,\frac{1}{r} = 1$, *this means that*

$$d\sigma_r \leq 0 \quad or \quad d\sigma_{\frac{1}{r}} \leq 0.$$

Therefore we may assume $d\sigma_r \leq 0$. *Now combining the equalities* $|f| = |g|$, $\phi_2 f = \phi_1 g r$ *and Proposition 2 in [DY1] give that* $\phi_1 r$ *is in* H^∞. *This thus implies that* f/g *is the quotient of two* H^∞ *functions. This contradiction shows that the quasi-similarity orbit* $orb_q([f])$ *has no maximum, that is, there exists no quasi-podal submodule in* $orb_q([f])$.

Before going on, recall that a function f in the Hardy module $H^2(\mathbb{D})$ is outer if and only if $|g| \leq |f|$ a.e. on $\mathbb{T}$, then $f|g$, that is, there exists a function $h \in H^2(\mathbb{D})$ such that $g = fh$. Equivalently, f is outer if and only if for any $\phi \in L^\infty(\mathbb{T})$, the relation $\phi f \in H^2(\mathbb{D})$ implies $\phi \in H^\infty(\mathbb{D})$. Motivated by this observation, we say that a function $f \in H^2(\mathbb{D}^n)$ is quasi-outer if $|g| \leq |f|$ a.e. on $\mathbb{T}^n$, then $f|g$, that is, there exists a function $h \in H^2(\mathbb{D}^n)$ such that $g = fh$. Then by the definition, f is quasi-outer if and only if for any $\phi \in L^\infty(\mathbb{T}^n)$, the relation $\phi f \in H^2(\mathbb{D}^n)$ implies $\phi \in H^\infty(\mathbb{D})$.

This is different from the concept of outer function in the sense of Rudin [Ru1]. A function f is outer in the sense of Rudin if

$$\log|f(z)| = P_z[\log|f|]$$

for some $z \in \mathbb{D}^n$ and hence for all $z \in \mathbb{D}^n$ (we call it R-outer). For convenience, we say that a function $f \in H^2(\mathbb{D}^n)$ is outer separately if for each i, $f(z_1, \cdots, z_{i-1}, z, z_{i+1}, \cdots, z_n)$ is outer in $H^2(\mathbb{D})$ for $a.e.$ fixed

$$(z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n) \in \mathbb{T}^{n-1}.$$

Obviously, in the case of $n = 1$, these concepts are identical. In the case of $n > 1$, from the proof of Theorem 2 in [Izu], one sees that every R-outer function is outer separately. In fact, R-outer functions are actually a sub-class of separately outer functions; for example, $z + w$ is outer separately, but not R-outer. More general, by Sarason [Sar], for two inner functions η_1, η_2 in the unit disk, $\eta_1(z) + \eta_2(w)$ is outer separately, but, in general, not R-outer.

The following proposition says that if a function is outer separately, then it necessarily is quasi-outer.

Proposition 3.3.9 *Let* f *in* $H^2(\mathbb{D}^n)$ *be outer separately. Then* f *is quasi-outer.*

Proof. For simplicity, we only prove the case in two variables. Let $\phi \in L^\infty(\mathbb{T}^2)$ such that $\phi f \in H^2(\mathbb{D}^2)$. For $a.e.$ fixed $w \in \mathbb{T}$, since $f(\cdot, w)$ is outer in $H^2(\mathbb{D})$ and $\phi(\cdot, w)f(\cdot, w) \in H^2(\mathbb{D})$, it follows that $\phi(\cdot, w)$ is in $H^\infty(\mathbb{D})$. By the same argument, for $a.e.$ fixed $z \in \mathbb{T}$, $\phi(z, \cdot)$ is in $H^\infty(\mathbb{D})$. Therefore, for any $\bar{z}^n w^m$ $(n > 0)$ and $\bar{w}^n z^m$ $(n > 0)$,

$$\begin{aligned}\langle \phi, \bar{z}^n w^m \rangle &= \frac{1}{(2\pi)^2} \iint_{\mathbb{T}^2} \phi z^n \bar{w}^m d\theta_1 d\theta_2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}} \bar{w}^m d\theta_2 \int_{\mathbb{T}} \phi(z,w) z^n d\theta_1 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\langle \phi, \bar{w}^n z^m \rangle &= \frac{1}{(2\pi)^2} \iint_{\mathbb{T}^2} \phi w^n \bar{z}^m d\theta_1 d\theta_2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{T}} \bar{z}^m d\theta_1 \int_{\mathbb{T}} \phi(z,w) w^n d\theta_2 \\ &= 0.\end{aligned}$$

It follows that $\phi \in H^2(\mathbb{D}^2)$ and hence $\phi \in H^\infty(\mathbb{D}^2)$. This shows that f is quasi-outer, completing the proof.

Proposition 3.3.10 *Let p be a polynomial in two variables. Then p is outer separately if and only if* $Z(p) \cap (\mathbb{D} \times \mathbb{T} \cup \mathbb{T} \times \mathbb{D}) = \emptyset$.

Proof. In fact, if the condition is satisfied by p, then obviously p is outer separately. Now assume that p is outer separately. If there is some point $(z_0, w_0) \in \mathbb{D} \times \mathbb{T}$ such that $p(z_0, w_0) = 0$, by slightly perturbing w_0 on $\mathbb{T}$ and using the continuous dependence of the roots on the coefficients, one deduces that for each w near w_0, there exists $z \in \mathbb{D}$ such that $p(z,w) = 0$. This shows that p is not outer separately. Therefore,

$$Z(p) \cap \mathbb{D} \times \mathbb{T} = \emptyset.$$

The same reasoning shows that

$$Z(p) \cap \mathbb{T} \times \mathbb{D} = \emptyset.$$

This completes the proof.

Below we will see that quasi-outer functions play an important role in the study of rigidity of Hardy submodules.

Theorem 3.3.11 *Assume that a submodule M satisfies the condition* $(\star)$. *Then the submodule* $[fM]$ *is quasi-podal if and only if* f *is quasi-outer.*

Proof. First assume that f is quasi-outer, and a submodule N is quasi-similar to $[fM]$. Then there exists a function $\phi \in L^\infty(\mathbb{T}^n)$ such that $\phi[fM]$ is dense in N. Then by Theorem 3.3.2, $\phi f \in H^2(\mathbb{D}^n)$. Thus, $\phi \in H^\infty(\mathbb{D}^n)$. This ensures that N is a submodule of $[fM]$. We thus conclude that if f is quasi-outer, then $[fM]$ is quasi-podal.

Now let $[fM]$ be quasi-podal, and $\phi \in L^\infty(\mathbb{T}^n)$ such that $\phi f \in H^2(\mathbb{D}^n)$. Our aim is to prove that ϕ is in $H^\infty(\mathbb{D}^n)$. We may assume that $1 \le |\phi|$ on $\mathbb{T}^n$; otherwise, we replace ϕ by ϕ adding a large positive constant. One can thus establish a quasi-similarity:

$$M_\phi : \ [fM] \to [\phi f M]; \quad Y : M_{\phi^{-1}} : \ [\phi f M] \to [fM].$$

Since $[fM]$ is quasi-podal, we see that

$$\phi[fM] \subseteq [fM].$$

Applying Proposition 3.3.7 gives that ϕ is in $H^\infty(\mathbb{D}^n)$. This proves that f is quasi-outer.

Corollary 3.3.12 *A principal submodule $[f]$ is quasi-podal if and only if f is quasi-outer.*

Combining Theorem 3.3.11 with Corollary 3.1.12, we have the following.

Corollary 3.3.13 *Let I be an ideal of polynomials and $I = pL$ be its Beurling form. Then $[I]$ is quasi-podal if and only if p is quasi-outer. In particular, if $GCD(I) = 1$, then $[I]$ is quasi-podal.*

Corollary 3.3.14 *Let polynomial p be quasi-outer, and let the zero variety of each prime factor of p meet $\mathbb{D}^n$ nontrivially. If $[p]$ and $[q]$ are quasi-similar, then $p|q$. Furthermore, if the polynomial q also satisfies these conditions, then there exists a constant c such that $p = cq$.*

Similar to the proof of Theorem 3.3.11, we have the following.

Proposition 3.3.15 *Let M be a submodule of $H^2(\mathbb{D}^n)$. If M contains a quasi-outer function, then M is quasi-podal.*

Before ending this section, we will look at the connection between Hankel operators and s-podal submodules. For $\phi \in L^\infty(\mathbb{T}^n)$, the Hankel operator $H_\phi : H^2(\mathbb{D}^n) \to L^2(\mathbb{T}^n) \ominus H^2(\mathbb{D}^n)$ with symbol ϕ is defined by

$$H_\phi f = (I - P)\phi f, \quad \forall f \in H^2(\mathbb{D}^n),$$

where P is the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. It is easy to see that the kernel, $\ker H_\phi$, of H_ϕ is a submodule. By the Beurling theorem [Beu], each submodule of $H^2(\mathbb{D})$ is a kernel of some Hankel operator. However, in the case of $n > 1$, there is a submodule that is not contained in the kernel of any Hankel operator. An example is to pick the s-podal submodule $M = [z+w]$, then M is not contained in the kernel of any Hankel operator. In fact, if there exists some Hankel operator H_ϕ such that $M \subseteq \ker H_\phi$, then

$$\phi(z + w) \in H^2(\mathbb{D}^2).$$

Since $z + w$ is quasi-outer, ϕ is in the $H^\infty(\mathbb{D}^2)$, and hence $H_\phi = 0$. This means that the s-podal submodule $[z + w]$ is not contained in the kernel of any Hankel operator.

This example, in fact, gives a general conclusion.

Proposition 3.3.16 *Let M be a submodule. Then M is s-podal if and only if M is not contained in the kernel of any Hankel operator.*

Proof. Assume that M is s-podal, and $M \subset \ker H_\phi$. We may suppose that $|\phi| \geq 1$ on $\mathbb{T}^n$; otherwise, we replace ϕ by ϕ adding a large positive constant. Then ϕM is similar to M. Since M is s-podal, $\phi M \subseteq M$. Applying Proposition 3.3.7 gives $\phi \in H^\infty(\mathbb{D}^n)$, and hence $H_\phi = 0$. Thus, M is not contained in the kernel of any Hankel operator.

To prove the opposite direction we may assume that M is not contained in the kernel of any Hankel operator, and a submodule N is similar to M. Then there exists an invertible function $\phi \in L^\infty(\mathbb{T}^n)$ such that $N = \phi M$. This deduces that $M \subset \ker H_\phi$, and hence $H_\phi = 0$. Therefore, ϕ is in $H^\infty(\mathbb{D}^n)$. We thus obtain that $N \subset M$. This shows that M is s-podal.

Corollary 3.3.17 *Let $M = \ker H_\phi$. If the similarity orbit $orb_s(M)$ of M contains the s-podal point N, then $N = H^2(\mathbb{D}^n)$.*

Proof. Let N be the s-podal point in the $orb_s(M)$. Then there is an invertible function ψ in $L^\infty(\mathbb{T}^n)$ such that $M = \psi N$. Since $M \subset N$, using Proposition 3.3.7 gives $\psi \in H^\infty(\mathbb{D}^n)$. Note that $N = \ker H_{\phi\psi}$. By Proposition 3.3.16, we see that $\phi\psi \in H^\infty(\mathbb{D}^n)$. The relation $\phi\psi \in H^\infty(\mathbb{D}^n)$ implies that $\psi \in \ker H_\phi$, that is, $\psi \in M$. This ensures $1 \in N$, and hence $N = H^2(\mathbb{D}^n)$.

The next example shows that not each similarity orbit has the maximum.

Example 3.3.18 *Let $f; g$ be as in Example 3.3.8, that is, $|f| = |g|$ a.e. on $\mathbb{T}^2$, and f/g is not the quotient of two H^∞ functions. Take $M = \ker H_{f/g}$. Then $orb_s(M)$ contains no s-podal point. If there is the s-podal point N in $orb_s(M)$, then by Corollary 3.3.17, $N = H^2(\mathbb{D}^2)$. Since M is similar to $H^2(\mathbb{D}^2)$, there is a function $\phi \in L^\infty(\mathbb{T}^n)$ such that $M = \phi H^2(\mathbb{D}^2)$, and hence $\phi \in M$. Thus,*

$$\frac{f}{g}\phi \in H^\infty(\mathbb{D}^2).$$

This contradicts the fact that f/g is not the quotient of two H^∞ functions. Therefore, $orb_s(M)$ contains no s-podal point. By a similar reason, one sees that $orb_u(M)$ contains no podal point.

3.4 Rigidity for Bergman modules

Let Ω be a bounded domain in $\mathbb{C}^n$. Recall that the Hilbert space $L^2(\Omega)$ consists of all square integrable functions on Ω with respect to the volume measure. The Bergman module $L_a^2(\Omega)$ on Ω is the closed subspace of $L^2(\Omega)$ which is spanned by all square integrable analytic functions.

For a bounded domain Ω on the complex plane, Richter proved that two submodules of the Bergman module $L_a^2(\Omega)$ are unitarily equivalent only if they are equal (in fact, Richter proved a stronger conclusion than stated here [Ri1]). Putinar proved an analogue of this result in the case of bounded pseudoconvex domains in $\mathbb{C}^n$ [Pu2]. In fact, this result is easily generalized to general bounded domains in $\mathbb{C}^n$.

Theorem 3.4.1 *Let M_1 and M_2 be two submodules of $L_a^2(\Omega)$. Then M_1 is unitarily equivalent to M_2 only if $M_1 = M_2$.*

Proof. Let $U : M_1 \to M_2$ be a unitary equivalence. Suppose $f \in M_1$. We have to show that $f \in M_2$. This will imply that $M_1 \subset M_2$. By symmetry we shall then have proved that $M_1 = M_2$.

We may assume that $f \neq 0$. Set $g = Uf$. Then we have that

$$\|qf\|_2 = \|qg\|_2$$

for any polynomial q, and it follows that

$$\int_\Omega |q(z)|^2(|f(z)|^2 - |g(z)|^2)dv(z) = 0.$$

By the equality

$$p\bar{q} = \frac{1}{2}(|p+q|^2 + i|p+iq|^2 - (i+1)|p|^2 - (i+1)|q|^2)$$

we get

$$\int_\Omega p(z)\overline{q(z)}(|f(z)|^2 - |g(z)|^2)dv(z) = 0$$

for any polynomials p and q. Note that $(|f(z)|^2 - |g(z)|^2)dv(z)$ is a regular Borel measure on Ω, and it annihilates $p(z)\overline{q(z)}$ for any polynomials p and q. It follows from the Stone-Weierstrass theorem that it annihilates $C(\overline{\Omega})$, the algebra of all continuous functions on $\overline{\Omega}$. Now applying the Riesz Representation theorem gives that

$$(|f(z)|^2 - |g(z)|^2)dv(z) = 0,$$

and thus $f(z) = \gamma g(z)$ for some constant γ with $|\gamma| = 1$. This gives the desired result.

From Theorem 3.4.1, one sees that there is not a multiplier ϕ of $L_a^2(\Omega)$ such that $M_\phi^* M_\phi = I$ (except constants). Indeed, if there is a multiplier ϕ satisfying the condition mentioned above, then it is easy to see that $\phi L_a^2(\Omega)$ and $L_a(\Omega)$ are unitarily equivalent, and hence $L_a^2(\Omega) = \phi L_a^2(\Omega)$. From this it is easy to deduce that ϕ is an invertible function in $H^\infty(\Omega)$. By the condition mentioned above we have the equality $M_\phi^* = M_{\phi^{-1}}$, and hence

$$\overline{\phi(\lambda)} K_\lambda = M_\phi^* K_\lambda = M_{\phi^{-1}} K_\lambda = \phi^{-1} K_\lambda$$

for each $\lambda \in \Omega$, where K_λ is the reproducing kernel of $L_a^2(\Omega)$. The above equality ensures that ϕ is a constant. However, in the case of the Hardy space $H^2(\mathbb{B}_n)$ we have $M_\eta^* M_\eta = I$ for each inner function η.

In fact, for the Bergman space $L_a^2(\Omega)$, we have a more general conclusion.

Proposition 3.4.2 *If $\phi_1, \phi_2, \cdots$ is a finite or infinite sequence of multipliers of $L_a^2(\Omega)$ satisfying*

$$M_{\phi_1}^* M_{\phi_1} + M_{\phi_2}^* M_{\phi_2} + \cdots = I,$$

then each ϕ_k is a scalar constant.

Proof. For each $\lambda \in \Omega$, we have $(\phi_k - \phi_k(\lambda))k_\lambda \perp \phi_k(\lambda)k_\lambda$ for all k, where $k_\lambda = K_\lambda / \|K_\lambda\|$ is the normalized reproducing kernel of $L_a^2(\Omega)$. Consequently,

$$\|M_{\phi_k} k_\lambda\|^2 = \|(\phi_k - \phi_k(\lambda))k_\lambda\|^2 + |\phi_k(\lambda)|^2 \geq |\phi_k(\lambda)|^2.$$

We thus conclude that

$$\sum_k |\phi_k(\lambda)|^2 \leq \sum_k \|M_{\phi_k} k_\lambda\|^2 = 1.$$

By the assumption we have

$$\langle 1, 1 \rangle = \int_\Omega dV = \sum_k \langle M_{\phi_k}^* M_{\phi_k} 1, 1 \rangle = \int_\Omega \sum_k |\phi_k(z)|^2 \, dV,$$

and hence

$$\int_\Omega (1 - \sum_k |\phi_k(z)|^2) dV = 0.$$

The preceding reasoning implies that $\sum_k |\phi_k(z)|^2 = 1$ for each $z \in \Omega$. Thus, for $i = 1, \cdots, n$ we have

$$\frac{\partial^2 |\phi_1(z)|^2}{\partial z_i \partial \bar{z}_i} + \frac{\partial^2 |\phi_2(z)|^2}{\partial z_i \partial \bar{z}_i} + \cdots = |\frac{\partial \phi_1(z)}{\partial z_i}|^2 + |\frac{\partial \phi_2(z)}{\partial z_i}|^2 + \cdots = 0$$

for $z \in \Omega$. This means that each ϕ_k is a scalar constant on Ω, completing the proof.

This proposition is sharp in some sense. To show this we consider the Hardy space $H^2(\mathbb{B}_n)$, as one knows, on this space,

$$M_{z_1}^* M_{z_1} + \cdots + M_{z_i}^* M_{z_i} + \cdots + M_{z_n}^* M_{z_n} = I.$$

In general, characterizing the similarity of submodules seems a little difficult. Now we proceed in the case of the classical Bergman module $L_a^2(\mathbb{D})$. Let M be a submodule of $L_a^2(\mathbb{D})$. If M contains a Nevanlinna function, then from Theorem 13 [Zhu3], M is generated by an inner function. This means that there is an inner function ψ such that

$$M = \overline{\psi H^\infty(\mathbb{D})}.$$

Now we assume that M contains a Nevanlinna function, and N is a submodule that is contained in M.

Proposition 3.4.3 *If M and N are similar, then under the above assumption, there exists an inner function η such that $N = \eta M$.*

Proof. Let $A: M \to N$ be a similarity. Note that there is an inner function ψ such that

$$M = \overline{\psi H^\infty(\mathbb{D})}.$$

Set $G = A\psi$. Then it is easy to see that

$$Af = \frac{G}{\psi} f, \quad \text{for all} \quad f \in M.$$

Set $G' = G/\psi$. Then

$$N = G'M.$$

For $\lambda \in \mathbb{D}$, we denote the reproducing kernel of M at λ by $K_\lambda^{(M)}$, and the normalized reproducing kernel by $k_\lambda^{(M)}$. For $\lambda \notin Z(\psi)$, Since

$$\langle G' k_\lambda^{(M)}, k_\lambda^{(M)} \rangle = G'(\lambda),$$

we get

$$|G'(\lambda)| \leq \|A\|.$$

This implies that G' is analytic and bounded on $\mathbb{D}$. We decompose $G' = \eta F$, where η is the inner part of G', and F, the outer part of G'. It follows that there exists a positive constant μ such that

$$\mu \|f\| \leq \|G' f\| \leq \|Ff\| \quad \text{for every} \quad f \in M.$$

This says that the multiplication operator M_F is bounded below on M, and hence the space FM is closed. Since F is outer, there exists a sequence of polynomials, $\{p_n\}$, such that, in the norm of $H^2(\mathbb{D})$,

$$Fp_n \to 1, \quad as \quad n \to \infty.$$

Also note that the inequality

$$\|f\| \leq \|f\|'$$

for every $f \in H^2(\mathbb{D})$, where $\|\cdot\|'$ is the norm of $H^2(\mathbb{D})$. Therefore, in the norm of $L_a^2(\mathbb{D})$,

$$Fp_n\psi f \to \psi f, \quad \textit{as} \quad n \to \infty,$$

for each $f \in H^\infty(\mathbb{D})$. Thus,

$$FM = M.$$

From this it is easy to derive that F is an invertible function in $H^\infty(\mathbb{D})$. The above discussion gives that

$$N = \eta M.$$

Combining Proposition 3.4.3 and Proposition 22 in [MSu], we can deduce the following result which first appeared in [Bou].

Corollary 3.4.4 *Let M be a submodule of $L_a^2(\mathbb{D})$. Then M is similar to $L_a^2(\mathbb{D})$ if and only if there exists the product of finitely many interpolating Blaschke products, B, such that $M = BL_a^2(\mathbb{D})$.*

3.5 Remarks on Chapter 3

"Rigidity phenomenon" appear in the study for analytic Hilbert modules in several variables. From an analytic point of view, appearance of these phenomenon is natural because of the Hartogs phenomenon in several variables. From an algebraic point of view, the reason may be that the submodules are not singly generated.

Two earlier results on inequivalent submodules of the higher dimensional Hardy module appeared in [BCL] and [Ha2]. Agrawal, Clark and Douglas introduced the notion of unitary equivalence of Hardy submodules [ACD] and made a deep investigation. There is an extensive literature on "rigidity"; cf. [AS, CD1, CDo2, Dou, DP, DPSY, DPY, DY1, Guo1, Guo2, Guo4, Guo8, HG, HKZ, Pu2, Yan2].

Section 3.1 is mainly based on Guo's paper [Guo1], except Theorem 3.1.11 and Corollary 3.1.12. Theorem 3.1.11 is due to K. Y. Guo (an unpublished result). Corollary 3.1.12 was proved in [Guo8]. Concerning Example 3.1.13, it was first obtained by Douglas, Paulsen, Sah and Yan [DPSY], where its proof is based on the characteristic space theory (see [Guo1]).

Theorem 3.2.1, Corollary 3.2.2 and Theorem 3.2.3 are all from [Guo1]. For Theorem 3.2.3, Yang proved the case of the Hardy module over the polydisk [Ya5]. Theorem 3.2.4, including its proof, comes from [DF].

Theorem 3.3.2, Corollary 3.3.4 and Corollary 3.3.5 appeared first in [DY1]. For Lemma 3.3.3 and Proposition 3.3.7, there are several different proofs; here, the proofs are given by us. Example 3.3.6 is due to Guo [Guo8]. Proposition 3.3.9 appeared in [Guo1]. Proposition 3.3.10 seems to be new. Theorem 3.3.11 and Corollaries 3.3.12, 3.3.13 and 3.3.14 were obtained in [Guo8]. In [Guo8], Propositions 3.3.15, 3.3.16 and Corollary 3.3.17 were also proved. Example 3.3.8 (Example 3.3.18) is a modification of an example given in [DY1] to show that not each quasi-similarity (similarity) orbit contains the maximum.

In the case of domains in the complex plane, Theorem 3.4.1 is a special case of the rigidity theorem by Richter [Ri1]. In the case of bounded pseudoconvex domains, this theorem appeared in [Pu2]. A more general rigidity theorem for Bergman submodules was proved in [GHX]. Proposition 3.4.2 is a special case of a general result in [GHX]. Although Proposition 3.4.3 is new, its proof comes essentially from [Bou]. Corollary 3.4.4 was given in [Bou]. For a different proof of Corollary 3.4.4, see [Zhu2]. We refer the reader to [Zhu1] for operator theory in function spaces.

Chapter 4

Equivalence of Hardy submodules

In this chapter we will give a complete classification under unitary equivalence for Hardy submodules on the polydisk which are generated by polynomials. Furthermore, we consider classification under similarity for homogeneous submodules. We note that the techniques used for the polydisk are also available for the ball. This chapter is mainly based on Guo's paper [Guo2].

Hastings's example shows that the submodule $[z+w]$ is not quasi-similar $H^2(\mathbb{D}^2)$ [Ha2]. Therefore, deciding when the principal ideals $\{p\} = p\mathcal{C}$ and $\{q\} = q\mathcal{C}$ give equivalent submodules must involve more analysis since all principal ideals are isomorphic as modules. Douglas, Paulsen and Yan also point out that the equivalence problem is, in general, quite difficult [DPY].

Let us recall Yan's work [Yan1]. Let $p; q$ be two homogeneous polynomials. Yan proved that $[p]; [q]$, as the submodules of $H^2(\mathbb{D}^n)$, are unitarily equivalent if and only if there exists a constant c such that $|p| = c|q|$ on $\mathbb{T}^n$; similar if and only if the quotient $|p|/|q|$ is bounded above and below on $\mathbb{T}^n$. While in the case of the unit ball $\mathbb{B}^n$, Chen and Douglas [CD1] proved that $[p]$ and $[q]$ are quasi-similar only if $p = cq$ for some constant c. From the fact mentioned above one finds that the classification of submodules depends heavily on the geometric properties of domains.

To exhibit main results let us begin with preliminaries associated with function theory in polydisks.

4.1 Preliminaries

The Nevanlinna class $N(\mathbb{D}^n)$ consists of all $f \in Hol(\mathbb{D}^n)$ that satisfy the growth condition

$$\sup_{0<r<1} \int_{\mathbb{T}^n} \log^+ |f_r| dm_n < \infty,$$

where, as usual, $f_r(w) = f(rw)$ for $w \in \mathbb{T}^n$. That is, the functions $\log^+ |f_r|$ are required to lie in a bounded subset of $L^1(\mathbb{T}^n)$. Recall that the function $\log^+ x = \log x$ if $x \geq 1$, and $\log^+ x = 0$ if $x < 1$.

The Smirnov class $N_*(\mathbb{D}^n)$ consists of all $f \in N(\mathbb{D}^n)$ for which the functions $\log^+ |f_r|$ form a uniformly integrable family. This means that each $\epsilon > 0$

should correspond with a $\delta > 0$ such that $m_n(E) < \delta$ implies

$$\sup_{0<r<1} \int_E \log^+ |f_r| dm_n < \epsilon.$$

As shown by [Ru1], for each $f \in N(\mathbb{D}^n)$ and $f \neq 0$, $\log |f|$ has a least n-harmonic majorant which will be denoted by $u(f)$. From [Ru1, Theorem 3.3.5], every f in $N(\mathbb{D}^n)$ has radial limits f^* defined on $\mathbb{T}^n$ a.e. Moreover, there is a real singular measure σ_f on $\mathbb{T}^n$ determined by f such that the least harmonic majorant $u(f)$ is given by

$$u(f)(z) = P_z(\log |f^*| + d\sigma_f),$$

where P_z denotes Poisson integration. In particular, $f \in N_*(\mathbb{D}^n)$ if and only if

$$d\sigma_f \leq 0.$$

Proposition 4.1.1 *Let $f; g \in H^2(\mathbb{D}^n)$, and $f \neq 0$. If there is an analytic function h such that $f = gh$, then $h \in N(\mathbb{D}^n)$. In particular, if $g \in A(\mathbb{D}^n)$, then $h \in N_*(\mathbb{D}^n)$.*

Proof. Since $f, g \in H^2(\mathbb{D}^n)$, we see that $\log |f_r|$ and $\log |g_r|$ lie in a bounded set in $L^1(\mathbb{T}^n)$ for $0 < r < 1$. From the equalities

$$\log |f_r| = \log |g_r| + \log |h_r|,$$

one derives

$$|\log |h_r|| \leq |\log |f_r|| + |\log |g_r||.$$

Thus,

$$\sup_{0<r<1} \int_{\mathbb{T}^n} |\log |h_r|| dm_n < \infty,$$

and therefore, $h \in N(\mathbb{D}^n)$.

When $g \in A(\mathbb{D}^n)$, then the slice function g_w has no singular inner factor, and hence from Theorem 3.3.6 [Ru1], we have that $d\sigma_g = 0$. Since

$$d\sigma_f = d\sigma_g + d\sigma_h,$$

this ensures that

$$d\sigma_h = d\sigma_f \leq 0.$$

Applying Theorem 3.3.5 (2) in [Ru1], we see that $h \in N_*(\mathbb{D}^n)$. The proof is complete.

Let $f \in N_*(\mathbb{D}^n)$. We say that f is R-outer (in Rudin's sense) if

$$\log |f(0)| = \int_{\mathbb{T}^n} \log |f^*| dm_n.$$

From the definition, f is R-outer if and only if

$$\log|f(z)| = P_z(\log|f^*|).$$

The importance of R-outer functions is based on the following proposition.

Proposition 4.1.2 *Let $f, g \in N_*(\mathbb{D}^n)$, and g be R-outer. If $|f| \le |g|$ a.e. on $\mathbb{T}^n$, then $|f(z)| \le |g(z)|$ for each $z \in \mathbb{D}^n$.*

Proof. Applying inequalities

$$\begin{aligned}\log|f(z)| &\le u(f) \le P_z(\log|f|)\\ &\le P_z(\log|g|) = \log|g(z)|,\end{aligned}$$

the proposition follows.

Proposition 4.1.3 *Let $f \in A(\mathbb{D}^n)$. If g is a cyclic vector in $[f]$, the submodule of $H^2(\mathbb{D}^n)$ generated by f, then there is R-outer functions r_1 and r_2 such that*

$$g = r_1 f, \quad f = r_2 g.$$

Proof. Since the submodule $[f]$ is also generated by g, that is, $[f] = [g]$, then by Theorem 2.3.3, there are analytic functions r_1 and r_2 such that

$$g = r_1 f, \quad f = r_2 g.$$

By Proposition 4.1.1, $r_1 \in N_*(\mathbb{D}^n)$. Defining the functional, Δ, on $H^2(\mathbb{D}^n)$ by

$$\Delta(h) = \int_{\mathbb{T}^n} \log|h^*|dm_n - \log|h(0)|,$$

then

$$\Delta(hp) = \Delta(h) + \Delta(p) \ge \Delta(h)$$

for each polynomial p. If $h \in [f]$, then there exists a sequence $\{p_m\}$ of polynomials such that $h = \lim f p_m$ in the norm of $H^2(\mathbb{D}^n)$. According to Lemma 4.4.5 in [Ru1], Δ is upper semicontinuous on $H^2(\mathbb{D}^n)$, and hence

$$\Delta(h) \ge \Delta(f)$$

for each $h \in [f]$. Therefore,

$$\Delta(g) \ge \Delta(f).$$

The same reasoning shows

$$\Delta(f) \ge \Delta(g).$$

Therefore we get that

$$\Delta(f) = \Delta(g).$$

Note that

$$\Delta(g) = \Delta(f) + \int_{\mathbb{T}^n} \log |r_1^*| dm_n - \log |r_1(0)|.$$

This gives that

$$\log |r_1(0)| = \int_{\mathbb{T}^n} \log |r_1^*| dm_n,$$

and hence r_1 is R-outer.

Applying Proposition 4.1.1, we see that $r_2 \in N(\mathbb{D}^n)$. From the equality $r_1(z)r_2(z) = 1$ and the fact $d\sigma_{r_1} = 0$, one gets

$$d\sigma_{r_2} = 0,$$

and hence $r_2 \in N_*(\mathbb{D}^n)$. Since r_1 is R-outer, it is easy to derive that r_2 is R-outer from the equality $r_1 r_2 = 1$.

The following theorem comes from [Guo2].

Theorem 4.1.4 *Let $f = p/q$ be a rational function on a domain $\Omega(\subset \mathbb{C}^n)$, where p and q are without common factors. If f is analytic on Ω, then we have $Z(q) \cap \Omega = \emptyset$.*

Proof. In fact, if there is a $\lambda \in \Omega$ such that $q(\lambda) = 0$, then

$$p(\lambda) = f(\lambda)q(\lambda) = 0.$$

Let $p_i^{m_i}$ be the primary factors of p with $\lambda \in Z(p_i)$, $i = 1, 2, \cdots, s$. Then by Theorem 2.1.1 and Theorem 3.1.11, we see

$$\{p\}_\lambda^e = p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i} \cdots p_s^{m_s} \mathcal{C},$$

where $\{p\}$ is the ideal of $\mathcal{C}$ generated by p, and $\{p\}_\lambda^e$ is the envelope of $\{p\}$ at λ. Let $\{p\}_\lambda$ and $\{fp\}_\lambda$ be the characteristic spaces of $\{p\}$ and $\{fp\}$ at λ, respectively, where $\{fp\}$ is the ideal of $Hol(\mathbb{D}^n)$ generated by fp. Since

$$\{p\}_\lambda = \{fq\}_\lambda \supset \{q\}_\lambda,$$

we have

$$\{p\}_\lambda^e \subset \{q\}_\lambda^e.$$

Thus,

$$p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i} \cdots p_s^{m_s} \mathcal{C} \subset q_1^{n_1} q_2^{n_2} \cdots q_i^{n_i} \cdots q_t^{n_t} \mathcal{C},$$

where $q_i^{n_i}$ are the primary factors of q with $\lambda \in Z(q_i)$, $i = 1, 2, \cdots, t$. Therefore there exists a polynomial r such that

$$p_1^{m_1} p_2^{m_2} \cdots p_i^{m_i} \cdots p_s^{m_s} = r\, q_1^{n_1} q_2^{n_2} \cdots q_i^{n_i} \cdots q_t^{n_t}.$$

This is contradictory to our assumption. This completes the proof.

Let $f \in Hol(\mathbb{D}^n)$. For each $w \in \mathbb{T}^n$, the slice function f_w on $\mathbb{D}$ is defined by $f_w(z) = f(zw)$, $\forall z \in \mathbb{D}$.

The next theorem appeared in [Guo2]. It is a modification of Theorem 5.2.2 in Rudin's book [Ru1].

Theorem 4.1.5 *Let f be in the Nevanlinna class on $\mathbb{D}^n$, and let the slice functions f_w be rational (in one variable) for almost all $w \in \mathbb{T}^n$. Then f is a rational function (in n variables).*

Proof. First of all, for a rational function $r = p/q$ in one variable, we define the degree of r to be the maximum of deg p, deg q, provided that the common factors of p, q have first been cancelled. If for almost all $w \in \mathbb{T}^n$, deg $f_w = 0$, it is easy to verify that $f = c$ for some constant c. Thus, without a loss of generality, we assume that there exist a subset E of $\mathbb{T}^n$ with $m_n(E) > 0$ and a natural number k such that deg $f_w = k$ for each $w \in E$. This means that $f_w(z)$ is uniquely written as

$$f_w(z) = \frac{\beta_k(w)z^k + \beta_{k-1}(w)z^{k-1} + \cdots + \beta_0(w)}{\alpha_k(w)z^k + \alpha_{k-1}(w)z^{k-1} + \cdots + 1}$$

for every $w \in E$. Let $f = \sum_{i=0}^{\infty} F_i$ be the homogeneous expansion of f. Hence for every $w \in E$, the infinite system of linear equations

$$F_m(w) + F_{m-1}(w)x_1 + \cdots + F_{m-k}(w)x_k = 0, \ \ (m > k)$$

has a unique solution $(\alpha_1(w), \alpha_2(w), \cdots, \alpha_k(w))$. This uniqueness ensures that the vectors

$$v_m(w) = (F_{m-1}(w), \cdots, F_{m-k}(w)), \ \ (m > k)$$

span all of $\mathbb{C}^k$ for each $w \in E$. Now let $w_0 \in E$. It follows that there exist vectors $v_{m_1}(w_0), v_{m_2}(w_0), \cdots, v_{m_k}(w_0)$ which are linearly independent. Consider the determinant $r(w)$ of these k vectors $v_{m_1}(w), v_{m_2}(w), \cdots, v_{m_k}(w)$. Then $r(w)$ is a polynomial and $r(w_0) \neq 0$. Notice that $Z(r) \cap \mathbb{T}^n$ is a null-measurable subset of $\mathbb{T}^n$. If we write E' for $E - Z(r) \cap \mathbb{T}^n$, then

$$m_n(E') = m_n(E) > 0.$$

On E', we can use the corresponding k equations

$$F_{m_t}(w) + F_{m_t-1}(w)x_1 + \cdots + F_{m_t-k}(w)x_k = 0, \ \ (t = 1, 2, \cdots, k),$$

to solve for the α_i. By Cramer's rule, there are rational functions $h_1, h_2, \cdots, h_k$, whose denominators have no zero on E', such that $\alpha_i(w) = h_i(w)$ for all $w \in E'$, $i = 1, 2, \cdots, k$. The equalities

$$h'_i = F_i + F_{i-1}h_1 + \cdots + F_0 h_i, \ \ (i = 0, 1, \cdots, k),$$

then define rational functions $h'_0, h'_1, \cdots, h'_k$, whose denominators have no zero on E', such that

$$f_w(z) = f(zw) = \frac{h'_0(w) + h'_1(w)z + \cdots + h'_k(w)z^k}{1 + h_1(w)z + \cdots + h_k(w)z^k}$$

for $w \in E'$. Since f is in the Nevanlinna class, $f(w) = \lim_{r\to 1} f(rw)$ exist for almost all $w \in \mathbb{T}^n$. It follows that there exists a subset E'' of E' such that $m_n(E'') > 0$, and on E'',

$$f(w)(1 + h_1(w) + \cdots + h_k(w)) = h_0'(w) + h_1'(w) + \cdots + h_k'(w).$$

Since h_i, h_j' are rational functions for all i and j, we multiply the two sides of the above equality by a polynomial p so that the functions $p(w)(1 + h_1(w) + \cdots + h_k(w))$ and $p(w)(h_0'(w) + h_1'(w) + \cdots + h_k'(w))$ become polynomials. Therefore, there exist polynomials q_1 and q_2 such that on E'',

$$f(w)q_1(w) = q_2(w).$$

By [Ru1, Theorem 3.3.5], we see that E'' is a determining set for Nevanlinna functions. Therefore, for almost all $w \in \mathbb{T}^n$,

$$f(w)q_1(w) = q_2(w).$$

Thus, for every $z \in \mathbb{D}^n$, we have

$$f(z)q_1(z) = q_2(z).$$

Now assume that the common factors of q_1 and q_2 have been cancelled, and hence by Theorem 4.1.4, $f(z) = q_2(z)/q_1(z)$ is a rational function and the intersection $Z(q_1) \cap \mathbb{D}^n = \emptyset$, completing the proof.

The following is the polynomial version of Theorem 4.1.5; it was first proved in [Guo1] (see also [Guo2]).

Theorem 4.1.6 *Let $f \in Hol(\mathbb{D}^n)$. If for almost all $w \in \mathbb{T}^n$, the slice function $f_w(z) = f(zw)$ are polynomials, then f is a polynomial.*

Proof. Let $f = F_0 + F_1 + \cdots$ be f's homogeneous expression. For almost all $w \in \mathbb{T}^n$, since

$$f_w(z) = \sum_{n \geqq 0} F_n(zw) = \sum_{n \geqq 0} F_n(w)z^n,$$

there exists a measurable subset E of $\mathbb{T}^n$ with $m_n(E) = 1$ such that, for each $w \in E$, there is a natural number $n(w)$ which satisfies $F_n(w) = 0$ if $n \geqq n(w)$. Assume that there exist infinitely many $F_{k_1}, \cdots, F_{k_n}, \cdots$ that are not zero. Since

$$E \subseteq \bigcup_{i=1}^{\infty} \big(Z(F_{k_i}) \cap \mathbb{T}^n\big)$$

and each $Z(F_{k_i}) \cap \mathbb{T}^n$ is null-measurable on $\mathbb{T}^n$, this leads to a contradiction. We therefore conclude that there exist only finitely many $F_i's$ with $F_i \neq 0$, that is, f is a polynomial. This completes the proof.

Remark 4.1.7 *Let E be a subset of $\mathbb{T}^n$ with positive measure. Then for Theorems 4.1.5 and 4.1.6, we can change condition "for almost all $w \in \mathbb{T}^n$" into "for almost all $w \in E$"; the conclusions remain true.*

4.2 Unitary equivalence of Hardy submodules on the unit polydisk

In this section we will prove the following classification theorem that was given by Guo [Guo2].

Theorem 4.2.1 *Let I_1 and I_2 be two ideals of polynomials, and let $I_1 = p_1L_1$, $I_2 = p_2L_2$ be their Burling forms. Then $[I_1]$ and $[I_2]$, as submodules of $H^2(\mathbb{D}^n)$, are unitarily equivalent if and only if there exist two polynomials q_1, q_2 satisfying $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1q_1| = |p_2q_2|$ on $\mathbb{T}^n$, and $[p_1L_1] = [p_1L_2]$.*

As an immediate consequence of Theorem 4.2.1, we have the following.

Corollary 4.2.2 *Let p_1, p_2 be two polynomials. Then principal submodules $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there exist two polynomials q_1, q_2 satisfying $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1q_1| = |p_2q_2|$ on $\mathbb{T}^n$.*

To prove Theorem 4.2.1, we need a lemma.

Lemma 4.2.3 *Let $I = p\,L$ be the Beurling form of the ideal I. If there is a function $\varphi \in L^\infty(\mathbb{T}^n)$ such that $\varphi\, I \subset H^2(\mathbb{D}^n)$, then φp belongs to $H^\infty(\mathbb{D}^n)$.*

Proof. By Corollary 3.1.12, we see that the submodule $[L]$ satisfies the condition $(\star)$ in Section 3.3. Since $\varphi p[L] \subset H^2(\mathbb{D}^n)$, using Theorem 3.3.2 gives that $\varphi p \in H^\infty(\mathbb{D}^n)$.

The proof of Theorem 4.2.1.
$(\Rightarrow)$ By assumption, there is a unimodular function η such that

$$p_1q_1 = \eta p_2q_2.$$

Since each q_i is a generator of $H^2(\mathbb{D}^n)$ (see Proposition 2.2.13), we have

$$\begin{aligned}[I_1] = [p_1L_1] = [p_1L_2] = [p_1q_1L_2] = \eta[p_2q_2L_2] \\ = \eta[p_2L_2] = \eta[I_2]\end{aligned}$$

and hence $[I_1]$ and $[I_2]$ are unitarily equivalent.

$(\Leftarrow)$ Suppose that $[I_1]$ and $[I_2]$ are unitarily equivalent. Then by Lemma 3.3.3, there exists a unimodular function η such that

$$\eta[I_1] = [I_2].$$

Let $\{p_2q_1, \cdots, p_2q_k\}$ be a set of generators of I_2, and hence a set of generators of $[I_2]$. By Theorem 2.3.3, every function $g(z)$ in $[I_2]$ has the form

$$g(z) = p_2(z)\gamma(z),$$

where $\gamma(z)$ is analytic on $\mathbb{D}^n$. From Lemma 4.2.3, $\eta p_1 \in H^\infty(\mathbb{D}^n)$. This implies that for each $f \in L_1$, there is a unique analytic function h_f on $\mathbb{D}^n$ such that

$$(\eta p_1)(z) f(z) = p_2(z) h_f(z).$$

Now for $f_1 \in L_1$, we define an analytic function on $\mathbb{D}^n \setminus Z(f_1)$ by

$$\phi_{f_1}(z) = \frac{h_{f_1}(z)}{f_1(z)}.$$

For another $f_2 \in L_1$, we also define an analytic function on $\mathbb{D}^n \setminus Z(f_2)$ by

$$\phi_{f_2}(z) = \frac{h_{f_2}(z)}{f_2(z)}.$$

Since

$$(\eta p_1)(z) f_1(z) f_2(z) = p_2(z) h_{f_1}(z) f_2(z) = p_2(z) h_{f_2}(z) f_1(z),\ \forall z \in \mathbb{D}^n,$$

we have that

$$\phi_{f_1}(z) = \phi_{f_2}(z),\ \forall z \in \mathbb{D}^n \setminus Z(f_1) \cup Z(f_2).$$

The above argument shows that for any $z \in \mathbb{D}^n \setminus Z(L_1)$, we can define

$$\phi(z) = \frac{h_f(z)}{f(z)}$$

for any $f \in L_1$ with $f(z) \neq 0$ and ϕ is independent of f, and $\phi(z)$ is analytic on $\mathbb{D}^n \setminus Z(L_1)$.

From Corollary 3.1.12, *height* $L_1 \geq 2$, and hence by [KK], the zero variety $\mathbb{D}^n \cap Z(L_1)$ is a removable singularity for analytic functions. This shows that $\phi(z)$ extends to an analytic function on all of $\mathbb{D}^n$. Now we regard $\phi(z)$ as an analytic function on $\mathbb{D}^n$, and notice that for $f \in L_1$,

$$(\eta p_1)(z) f(z) = p_2(z) \phi(z) f(z).$$

This yields that

$$(\eta p_1)(z) = p_2(z) \phi(z).$$

Also notice that

$$\bar{\eta}[I_2] = [I_1].$$

Just as in the above discussion, there is an analytic function $\psi(z)$ on $\mathbb{D}^n$ such that

$$(\bar{\eta} p_2)(z) = p_1(z) \psi(z).$$

Applying Proposition 4.1.1 gives that both functions ϕ and ψ belong to $N_*(\mathbb{D})$. By the equality

$$(\eta p_1)(\bar{\eta} p_2) = p_2 p_1 \phi \psi,$$

we have that $\phi\psi = 1$, and hence both ϕ and ψ are R-outer.

Now let $p(z,w)$ be the Poisson kernel for $\mathbb{D}^n$. Then

$$(\eta p_1)(z) = \int_{\mathbb{T}^n} p(z,w)(\eta p_1)(w) dm_n(w).$$

This implies that

$$|(\eta p_1)(z)| = |\int_{\mathbb{T}^n} p(z,w)(\eta p_1)(w) dm_n(w)| \leq \int_{\mathbb{T}^n} p(z,w)|p_1(w)| dm_n(w).$$

Set

$$\tilde{p_1}(z) = \int_{\mathbb{T}^n} p(z,w)|p_1(w)| dm_n(w).$$

Then $\tilde{p_1}(z)$ extends to a continuous function on the closure $\overline{\mathbb{D}^n}$ of $\mathbb{D}^n$, and

$$\tilde{p_1}(w) = |p_1(w)|$$

on $\mathbb{T}^n$. For each $w \in \mathbb{T}^n$, let $(\eta p_1)^*_w$ be the radial limit of the slice function $(\eta p_1)_w(z)$. Thus by the above inequality, one has that

$$|(\eta p_1)^*_w(e^{i\theta})| \leq |p_{1w}(e^{i\theta})|.$$

Now by the equality

$$(\eta p_1)(z) = p_2(z)\phi(z),$$

we get that

$$|\phi_w(e^{i\theta})| \leq \frac{|p_{1w}(e^{i\theta})|}{|p_{2w}(e^{i\theta})|}.$$

Just as in the above discussion, by the equality

$$(\bar{\eta} p_2)(z) = p_1(z)\psi(z),$$

one has that

$$|\psi_w(e^{i\theta})| \leq \frac{|p_{2w}(e^{i\theta})|}{|p_{1w}(e^{i\theta})|}.$$

Then from the equality $\phi\psi = 1$, the following is immediate:

$$|\phi_w(e^{i\theta})| = \frac{|p_{1w}(e^{i\theta})|}{|p_{2w}(e^{i\theta})|}.$$

Let us observe the fact that for a polynomial q in one variable, then the outer factor q' of q is also a polynomial, and $|q'| = |q|$ on $\mathbb{T}$. Now by Rudin [Ru1, Lemma 4.4.4], the slice functions ϕ_w are outer for almost all $w \in \mathbb{T}^n$. Combining the preceding observation and Proposition 4.1.2, there is a constant c_w such that

$$\phi_w(z) = c_w \frac{p'_{1w}(z)}{p'_{2w}(z)}$$

for almost each $w \in \mathbb{T}^n$, where p'_{1w}, p'_{2w} are the outer factors of p_{1w}, p_{2w}, respectively (and hence are polynomials). This means that ϕ_w is a rational function in one variable for almost each $w \in \mathbb{T}^n$. Applying Theorems 4.1.4 and 4.1.5, there exist polynomials $q_1(z), q_2(z)$ with $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that

$$\phi(z) = \frac{q_2(z)}{q_1(z)}.$$

Since $\eta p_1 = \phi p_2$, it follows that

$$|p_1 q_1| = |p_2 q_2|$$

on $\mathbb{T}^n$.

Next we will show that $[p_1 L_1] = [p_1 L_2]$. By the equality

$$|p_1 q_1| = |p_2 q_2|,$$

there is a unimodular function η', such that $p_2 q_2 = \eta' p_1 q_1$. Because each q_i is a generator of $H^2(\mathbb{D}^n)$ for $i = 1, 2$ we have

$$[I_2] = [p_2 L_2] = [p_2 q_2 L_2] = \eta' [p_1 q_1 L_2] = \eta' [p_1 L_2].$$

This implies that $[I_2]$ and $[p_1 L_2]$ are unitarily equivalent, and hence $[p_1 L_1]$ and $[p_1 L_2]$ are unitarily equivalent. Therefore, there is a unimodular function η such that

$$\eta [p_1 L_1] = [p_1 L_2].$$

Just as in the preceding proof, we see that there exists an R-outer function $\phi(z)$ such that

$$\eta p_1 = \phi p_1,$$

and hence $\eta = \phi$. Note that

$$\log |\phi(z)| = P_z(\log |\phi|) = P_z(\log |\eta|) = 0$$

because ϕ is R-outer. This means that $\phi(z)$ is a unimodular constant, and therefore η is a constant. This gives that $[p_1 L_1] = [p_1 L_2]$. The proof is completed.

Below is an example.

Example 4.2.4 *Let*

$$p(z, w) = z + w + 2zw, \quad q(z, w) = z + w - 2zw$$

be two polynomials on $\mathbb{C}^2$. Since on $\mathbb{T}^2$,

$$|(z + w + 2zw)(z + w - 2)| = |(z + w - 2zw)(z + w + 2)|,$$

it follows that $[p]$ and $[q]$ are unitarily equivalent. In fact, they are all unitarily equivalent to $H^2(\mathbb{D}^2)$ because $|z + w + 2zw| = |z + w + 2|$ on $\mathbb{T}^2$ and $z + w + 2$ generates $H^2(\mathbb{D}^2)$.

Corollary 4.2.5 *Let p be a polynomial and let q be a homogeneous polynomial. Then $[p]$ and $[q]$ are unitarily equivalent if and only if there exists a polynomial r with $Z(r) \cap \mathbb{D}^n = \emptyset$ such that $|p| = |rq|$ on $\mathbb{T}^n$. In particular, if p, q are homogeneous, then $[p]$ and $[q]$ are unitarily equivalent if and only if there exists a constant c such that $|p| = c|q|$ on $\mathbb{T}^n$.*

Proof. By Corollary 4.2.2, the sufficiency is obvious. If $[p]$ and $[q]$ are unitarily equivalent, then by Corollary 4.2.2, there exist two polynomials q_1, q_2 with $Z(q_1) \cap \mathbb{D}^n = Z(q_1) \cap \mathbb{D}^n = \emptyset$ such that on $\mathbb{T}^n$,

$$|qq_1| = |pq_2|.$$

For any $z \in \mathbb{D}^n$, set

$$r(z) = \frac{q_1(z)}{q_2(z)}.$$

Since the slice functions q_{1w}, q_{2w} are outer in $H^2(\mathbb{D})$, there exists a unimodular constant c_w such that for any $z \in \mathbb{D}$,

$$q_{2w}(z)p^{(w)}(z) = c_w q_{1w}(z)q(w),$$

where $p^{(w)}(z)$ is the outer factor of $p_w(z)$. Thus,

$$r_w(z) = \frac{q_{1w}(z)}{q_{2w}(z)} = \bar{c}_w \frac{p^{(w)}(z)}{q(w)}$$

for almost all $w \in \mathbb{T}^n$. Note that $p^{(w)}(z)$ is a polynomial. Theorem 4.1.6 thus implies that $r(z)$ is a polynomial, and $Z(r) \cap \mathbb{D}^n = \emptyset$. The remaining case is obvious. This completes the proof.

One can compare the next example with Example 3.3.6.

Example 4.2.6 *In this example we will look at whether the unitary orbit of the submodules $[z + \alpha w]$ has the podal point, where α is a constant.*

If $\alpha = 0$, then obviously $[z]$ is unitarily equivalent to $H^2(\mathbb{D}^2)$. If $|\alpha| = 1$, then the function $z + \alpha w$ is outer separately, and hence by Corollary 3.3.12, the submodule $[z + \alpha w]$ is podal in its unitary orbit.

Now assume $|\alpha| \neq 0, 1$; then it is easy to verify that the submodules $[z+\alpha w]$ and $[w + \bar{\alpha} z]$ are unitarily equivalent because

$$|z + \alpha w| = |w + \bar{\alpha} z|$$

on $\mathbb{T}^2$. For each such α, if there was the podal submodule M in the unitary orbit $orb_u([z + \alpha w])$, then

$$[z + \alpha w] \subseteq M \quad \text{and} \quad [w + \bar{\alpha} z] \subseteq M.$$

Therefore, the functions z and w are in M. This shows

$$M \supseteq M_0,$$

where $M_0 = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\}$. *Hence* $M = M_0$, *or* $H^2(\mathbb{D}^2)$. *Notice* $\operatorname{rank}([z + \alpha w]) = 1$ *and* $\operatorname{rank}(M_0) = 2$. *This implies that* $M \neq M_0$. *Because both* $[z + \alpha w]$ *and* $H^2(\mathbb{D}^2) = [1]$ *are homogeneous, then by Corollary 4.2.5, we see that* $[z+\alpha w]$ *and* $H^2(\mathbb{D}^2)$ *are not unitarily equivalent, and hence* $M \neq H^2(\mathbb{D}^2)$. *This shows that for each such* α, *the unitary orbit* $orb_u([z+\alpha w])$ *has no maximum, that is, there exists no podal point.*

Before giving the next corollary let us recall a theorem due to Rudin [Ru1, Theorem 5.2.6]. Set

$$V_n = \{z \in C^n : |z_i| > 1, \text{ for } i = 1, 2, \cdots, n\}.$$

Then by Theorem 5.2.6 in [Ru1], a polynomial p is the numerator of a rational inner function in $\mathbb{D}^n$ if and only if p has no zero in V_n.

Let p be a polynomial with $Z(p) \cap \mathbb{D}^n \neq \emptyset$. We decompose $p = p_1 p_2$ such that zero set of each prime factor of p_1 meets $\mathbb{D}^n$ nontrivially, and $Z(p_2) \cap \mathbb{D}^n = \emptyset$. Now define $L(p)$ on $\mathcal{C}$ as follows: $L(p) = 1$ if $Z(p) \cap \mathbb{D}^n = \emptyset$; $L(p) = p_1$ if $Z(p) \cap \mathbb{D}^n \neq \emptyset$.

Corollary 4.2.7 *Let* $I = pL$ *be the Beurling form of the ideal* I. *Then* $[I]$ *and* $H^2(\mathbb{D}^n)$ *are unitarily equivalent if and only if* $L(p)$ *has no zero in* V_n *and* L *is dense in* $H^2(\mathbb{D}^n)$.

In particular, when $n = 2$, *then* $[I]$ *and* $H^2(\mathbb{D}^2)$ *are unitarily equivalent if and only if* $L(p)$ *has no zero in* V_2 *and* L *has no zero in* $\mathbb{D}^2$.

Proof. From Proposition 2.2.13, each polynomial without zero point in $\mathbb{D}^n$ is a generator of $H^2(\mathbb{D}^n)$. This implies that

$$[pL] = [L(p)L].$$

Combining Theorem 4.2.1 with Corollary 4.2.5, we see that $[L(p)L]$ and $H^2(\mathbb{D}^n)$ are unitarily equivalent if and only if there exists a polynomial r with $Z(r) \cap \mathbb{D}^n = \emptyset$ such that $|L(p)| = |r|$ on $\mathbb{T}^n$ and L is dense in $H^2(\mathbb{D}^n)$. Since $Z(r) \cap \mathbb{D}^n = \emptyset$, r is a generator of $H^2(\mathbb{D}^n)$, and hence r is R-outer. Then by Proposition 4.1.2, $|L(p)(z)| \leq |r(z)|$ for every $z \in \mathbb{D}^n$. This means that the function $L(p)/r$ is a rational inner function, and therefore by Rudin's theorem mentioned above, $L(p)$ has no zero in V_n.

Conversely, if $L(p)$ has no zero in V_n, then Rudin's theorem implies that $L(p)$ is the numerator of a rational inner function. This means that there is a polynomial r with $Z(r) \cap \mathbb{D}^n = \emptyset$ such that the function $L(p)/r$ is inner, and hence $|L(p)| = |r|$ on $\mathbb{T}^n$.

In particular when $n = 2$, by Lemma 2.2.9, the following two statements are equivalent: (1) L has no zero in $\mathbb{D}^2$ and (2) L is dense in $H^2(\mathbb{D}^2)$. This completes the proof.

We have given several examples related to homogeneous principal submodules. The next example will exhibit a nonhomogeneous case that seems more interesting.

Example 4.2.8 *We consider the submodule* $[z_1+z_2+\alpha]$ *of* $H^2(\mathbb{D}^2)$*, where* α *is constant. If* $\alpha=0$*, then* z_1+z_2 *is outer separately, and hence by Corollary 3.3.12* $[z_1+z_2]$ *is podal. If* $|\alpha|\geq 2$*, then by Proposition 2.2.13, we have* $[z_1+z_2+\alpha]=H^2(\mathbb{D}^2)$*.*

Below we will show that for each α*,* $0<|\alpha|<2$*, there exists no podal point in the unitary orbit* $orb_u([z_1+z_2+\alpha])$*. For such an* α*, assume that there is a podal point, say,* M*, in the unitary orbit* $orb_u([z_1+z_2+\alpha])$*. Since on* $\mathbb{T}^2$*,*

$$|z_1+z_2+\alpha|=|z_1+z_2+\bar{\alpha}z_1z_2|,$$

this ensures that $[z_1+z_2+\alpha]$ *and* $[z_1+z_2+\bar{\alpha}z_1z_2]$ *are unitarily equivalent. From the inclusions*

$$[z_1+z_2+\alpha]\subset M, \quad [z_1+z_2+\bar{\alpha}z_1z_2]\subset M,$$

the function $z_1z_2-\alpha/\bar{\alpha}$ *belongs to* M*. By Proposition 2.2.13,* $M=H^2(\mathbb{D}^2)$ *because* $Z(z_1z_2-\alpha/\bar{\alpha})\cap\mathbb{D}^2=\emptyset$*. By unitary equivalence of* $[z_1+z_2+\alpha]$ *and* $H^2(\mathbb{D}^2)$*, applying Corollary 4.2.7 gives that the function* $z_1+z_2+\alpha$ *has no zero in* V_2*. Clearly, this is impossible, and hence* $[z_1+z_2+\alpha]$ *is never unitarily equivalent to* $H^2(\mathbb{D}^2)$*. We conclude that the unitary orbit* $orb_u([z_1+z_2+\alpha])$ *contains no podal point if* $0<|\alpha|<2$*.*

An ideal I is said to be homogeneous if the relation $p\in I$ implies that all homogeneous components of p are in I. Equivalently, an ideal I is homogeneous if and only if I is generated by homogeneous polynomials.

Let I be homogeneous, and $I=qL$ be the Beurling form of I. Then it is easy to check that both q and L are homogeneous.

The next corollary generalizes Theorem 2 in [Yan1].

Corollary 4.2.9 *Let* I_1*,* I_2 *be homogeneous, and* $I_1=p_1L_1$*,* $I_2=p_2L_2$ *be their Beurling forms. Then* $[I_1]$ *and* $[I_2]$ *are unitarily equivalent if and only if there exists a constant* c *such that* $|p_1|=c|p_2|$ *on* $\mathbb{T}^n$ *and* $L_1=L_2$*.*

Proof. From Theorem 4.2.1, the sufficiency is immediate. Now assume that $[I_1]$ and $[I_2]$ are unitarily equivalent. Then as in the proof of Corollary 4.2.5, one sees that there exists a constant c such that $|p_1|=c\,|p_2|$ on $\mathbb{T}^n$. Note that both p_1L_1 and p_1L_2 are homogeneous, and hence by [ZS, Vol(II), p. 153, Theorem 9 and its corollary], each of their associated prime ideals is homogeneous. Now combining Theorem 2.2.8 with Theorem 4.2.1, the equality $[p_1L_1]=[p_1L_2]$ implies that $p_1L_1=p_1L_2$, and therefore $L_1=L_2$, completing the proof.

Now let us endow the ring $\mathcal{C}$ with the topology induced by the Hardy space $H^2(\mathbb{D}^n)$. It is easy to see that studying the unitary equivalence of submodules generated by ideals and by their closures is the same thing. For an ideal I, we write $\bar{I}$ for the closure of I under the Hardy topology.

The following theorem is an equivalent form of Theorem 4.2.1.

Theorem 4.2.10 *Let I_1,I_2 be two ideals of polynomials, and let $\bar{I_1} = p_1 L_1$, $\bar{I_2} = p_2 L_2$ be the Beurling forms of their closures. Then $[I_1]$ and $[I_2]$ are unitarily equivalent if and only if there exist polynomials q_1, q_2 satisfying $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on $\mathbb{T}^n$, and $L_1 = L_2$.*

Proof. It is easy to see that the sufficiency is obvious. Now suppose that $[I_1]$ and $[I_2]$ are unitarily equivalent. This means that $[\bar{I_1}]$ and $[\bar{I_2}]$ are unitarily equivalent. By Theorem 4.2.1, there exist polynomials q_1, q_2 which satisfy $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on $\mathbb{T}^n$, and

$$[p_1 L_1] = [p_1 L_2], \quad [p_2 L_2] = [p_2 L_1].$$

Since $p_1 L_1$, $p_2 L_2$ are closed ideals, we have

$$p_1 L_1 \supseteq p_1 L_2, \quad p_2 L_2 \supseteq p_2 L_1.$$

The above inclusions imply that $L_1 = L_2$.

When $n = 2$ we can give simpler conditions for $[I_1]$ and $[I_2]$ to be unitarily equivalent (cf.[Guo2]).

Theorem 4.2.11 *Let I_1 and I_2 be ideals in two variables, and let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be their Beurling forms. Then the submodules $[I_1]$, $[I_2]$ are unitarily equivalent if and only if there exist polynomials q_1, q_2 that satisfy $Z(q_1) \cap \mathbb{D}^2 = Z(q_2) \cap \mathbb{D}^2 = \emptyset$ such that $|p_1 q_1| = |p_2 q_2|$ on $\mathbb{T}^2$, and $[L_1] = [L_2]$.*

Proof. It is easy to see that the sufficiency is obvious. Now we decompose p_1 as the product of p_1' and p_1'' such that the zero set of each of the prime factors of p_1' meets $\mathbb{D}^2$ nontrivially, and each of p_1'' does not. Then by Proposition 2.2.13,

$$[p_1 L_1] = [p_1' L_1], \quad [p_1 L_2] = [p_1' L_2].$$

From Theorem 4.2.1, one has

$$[p_1' L_1] = [p_1' L_2].$$

Note that both L_1 and L_2 are finite codimensional by Lemma 2.2.9. Now combining Theorem 3.1.11 with Theorem 2.2.8, we see that $p_1'[L_1] = p_1'[L_2]$ and hence $[L_1] = [L_2]$.

4.3 Similarity of Hardy submodules on the unit polydisk

In this section, we will consider the similarity problem.

Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be two ideals of polynomials, and let $[I_1]$ and $[I_2]$ be quasi-similar. This means that there exist module maps $X : [I_1] \to [I_2]$

and $Y : [I_2] \to [I_1]$ having dense ranges. From Lemma 3.3.3, there exist $\phi, \psi \in L^\infty(\mathbb{T}^n)$ such that $X = M_\phi$, $Y = M_\psi$. Moreover, by Lemma 4.2.3, both $f = \phi p_1$ and $g = \psi p_2$ are in $H^\infty(\mathbb{D}^n)$. As in the proof of Theorem 4.2.1, there exist analytic functions r_1 and r_2 such that $f = r_1 p_2$, $g = r_2 p_1$.

Lemma 4.3.1 *Under the above statements, both r_1 and r_2 are in the class $N_*(\mathbb{D}^n)$ and they have no zero in $\mathbb{D}^n$.*

Proof. The fact that both r_1 and r_2 are in the class $N_*(\mathbb{D}^n)$ comes from Proposition 4.1.1. Now suppose that there exists a point $z_0 \in \mathbb{D}^n$ such that $r_1(z_0) = 0$. From Lemma 2.3.2, we obtain that

$$[fL_1]^{(z_0)} = [r_1 p_2 L_1]^{(z_0)} = r_{1\,z_0} p_{2\,z_0} L_1^{(z_0)} = [p_2 L_2]^{(z_0)} = p_{2\,z_0} L_2^{(z_0)}$$

and

$$[gL_2]^{(z_0)} = [r_2 p_1 L_2]^{(z_0)} = r_{2\,z_0} p_{1\,z_0} L_2^{(z_0)} = [p_1 L_1]^{(z_0)} = p_{1\,z_0} L_1^{(z_0)},$$

where $L_i^{(z_0)}$ denote the ideals of $\mathcal{O}_{z_0}$ generated by $\{p_{z_0} : p \in L_i\}$, and $r_{i\,z_0}, p_{i\,z_0}$ denote the elements of $\mathcal{O}_{z_0}$ defined by the restriction of r_i, p_i to neighborhoods of z_0, $i = 1, 2$. By the above equalities, we see that

$$L_2^{(z_0)} = r_{1\,z_0} r_{2\,z_0} L_2^{(z_0)}.$$

This is impossible by Nakayama's lemma (see [AM, Proposition 2.6]), and hence r_1 has no zeros in D^n. Using the same reasoning, r_2 has no zero in $\mathbb{D}^n$.

Now we can strengthen Theorem 3.3 in [DPSY] as follows (cf. [Guo2]).

Theorem 4.3.2 *Let $I_1 = p_1 L_1,\ I_2 = p_2 L_2$ be two ideals of polynomials such that each of their algebraic components meets $\mathbb{D}^n$ nontrivially. If $[I_1]$ and $[I_2]$ are quasi-similar, then $L_1 = L_2$.*

Proof. By the assumptions, there exist module maps $X : [I_1] \to [I_2]$ and $Y : [I_2] \to [I_1]$ with dense ranges. Thus by Lemma 3.3.3, there exist functions $\phi, \psi \in L^\infty(\mathbb{T}^n)$ such that $X = M_\phi$, $Y = M_\psi$. Then by Lemma 4.3.1, there are analytic functions r_1 and r_2, each of which has no zero in $\mathbb{D}^n$ such that

$$\phi p_1 = r_1 p_2, \quad \psi p_2 = r_2 p_1.$$

Note that both ϕp_1 and ψp_2 are in $H^\infty(\mathbb{D}^n)$. It follows that for $\lambda \in \mathbb{D}^n$,

$$\{p_2 L_2\}_\lambda = [p_2 L_2]_\lambda = [\phi p_1 L_1]_\lambda = [r_1 p_2 L_1]_\lambda = [p_2 L_1]_\lambda = \{p_2 L_1\}_\lambda$$

and

$$\{p_1 L_1\}_\lambda = [p_1 L_1]_\lambda = [\psi p_2 L_2]_\lambda = [r_2 p_1 L_2]_\lambda = [p_1 L_2]_\lambda = \{p_1 L_2\}_\lambda,$$

where $\{p_1L_1\}_\lambda$ and $\{p_2L_2\}_\lambda$ are the characteristic spaces of the ideals p_1L_1 and p_2L_2 at λ, respectively (cf. Section 2.1). Using Corollary 2.1.3, we see that

$$p_2L_2 \supset p_2L_1, \quad p_1L_1 \supset p_1L_2$$

and hence $L_1 = L_2$, which proves the assertion.

The following theorem strengthens [Yan1, Theorem 1] which was given in [Guo2].

Theorem 4.3.3 *Let $I_1 = p_1L_1,\ I_2 = p_2L_2$ be homogeneous ideals. Then the following are equivalent:*

(1) $[I_1]$ and $[I_2]$ are similar;

(2) $[I_1]$ and $[I_2]$ are quasi-similar;

(3) there exists a constant c such that $c < \frac{|p_1|}{|p_2|} < c^{-1}$ on $\mathbb{T}^n$, and $L_1 = L_2$.

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3). Because p_1L_1 and p_2L_2 are homogeneous ideals, each of their associated prime ideals is homogeneous (cf. [ZS, Vol(II), p. 153, Theorem 9 and its corollary]). We apply Theorem 4.3.2 to obtain $L_1 = L_2$. Now we use some techniques in [Yan1]. Let both module maps $X : [I_1] \to [I_2]$ and $Y : [I_2] \to [I_1]$ have dense ranges. Then there exist $\phi,\ \psi \in L^\infty(\mathbb{T}^n)$ such that $X = M_\phi,\ Y = M_\psi$. Applying Lemma 4.2.3 gives that $f = \phi p_1,\ g = \psi p_2$ are in $H^\infty(\mathbb{D}^n)$. Since p_1 is homogeneous, from the equality $f = \phi p_1$, we have that

$$|f(w)| \le \|\phi\|_\infty |p_1(w)| \quad a.e. \text{ on } \quad \mathbb{T}^n.$$

Therefore,

$$|f_w(e^{i\theta})| \le \|\phi\|_\infty |p_1(w)|$$

for almost all $w \in \mathbb{T}^n$. This leads to the inequality

$$|f_w(z)| \le \|\phi\|_\infty |p_1(w)| \quad \forall z \in \mathbb{D}$$

for almost all $w \in \mathbb{T}^n$, and hence for all $w \in \mathbb{T}^n$, we have the inequality

$$|f_w(z)| \le \|\phi\|_\infty |p_1(w)| \quad \forall z \in \mathbb{D}.$$

Suppose there is a sequence $w_n \to w_0$ such that

$$\frac{|p_1(w_n)|}{|p_2(w_n)|} \to 0.$$

From Lemma 4.3.1, $f = r_1p_2$. Now for every fixed $z \ne 0, z \in \mathbb{D}$, we have

$$|r_1(zw_n)| = \frac{|f(zw_n)|}{|p_2(zw_n)|} \le \frac{\|\phi\|_\infty |p_1(w_n)|}{|p_2(w_n)||z|^k} \to 0,$$

where $k = \deg p_2$. This implies that $r_1(zw_0) = 0$. This contradicts Lemma 4.3.1, and hence there exists a positive constant c' such that

$$c' < \frac{|p_1|}{|p_2|}$$

on $\mathbb{T}^n$. Similarly, there exists a positive constant c'' such that

$$c'' < \frac{|p_2|}{|p_1|}$$

on $\mathbb{T}^n$. Thus, there exists a constant c such that

$$c < \frac{|p_1|}{|p_2|} < c^{-1}$$

on $\mathbb{T}^n$.
(3) $\Rightarrow$ (1). Set $\phi = p_2/p_1$ and $\psi = p_1/p_2$. Then module maps

$$M_\phi : [I_1] \to [I_2], \;\; M_\psi : [I_2] \to [I_1]$$

give similarity between $[I_1]$ and $[I_2]$.

When $n = 2$, we can give more detailed conditions for two homogeneous submodules to be quasi-similar. For a homogeneous polynomial p in z and w, there is the decomposition:

$$p(z, w) = c\, w^k \prod_{|\alpha_i| \neq 1} (z - \alpha_i w) \prod_{|\beta_i| = 1} (z - \beta_i w),$$

where c, k, α_i, β_i are determined by p. Since only the last factor $\prod_{|\beta_i|=1}(z - \beta_i w)$ can vanish on $\mathbb{T}^n$, we denote this factor by $F(p)$.

By Theorem 4.3.3, we have the following.

Corollary 4.3.4 *Let $I_1 = p_1 L_1$, $I_2 = p_2 L_2$ be homogeneous ideals on $\mathbb{C}^2$. Then the following are equivalent:*

(1) $[I_1]$ and $[I_2]$ are similar;

(2) $[I_1]$ and $[I_2]$ are quasi-similar;

(3) $F(p_1)L_1 = F(p_2)L_2$.

From Theorem 4.3.3, one sees that under the conditions of Theorem 4.3.3, quasi-similarity implies similarity. We do not know if there exists an example of two submodules generated by polynomials that are quasi-similar, but not similar.

Furthermore, from Theorems 4.3.2 and 4.3.3, one sees that question about the similarity of submodules can be reduced to question about the similarity of principal submodules. Then one wants to know when two principal submodules are similar.

Proposition 4.3.5 *Let p_1 and p_2 be two polynomials. If there exist polynomials q_1, q_2 with $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $c < \frac{|p_1 q_1|}{|p_2 q_2|} < c^{-1}$ for some constant c, then $[p_1]$ and $[p_2]$ are similar.*

Proof. Set $\phi = \frac{p_1 q_1}{p_2 q_2}$. Then $c < |\phi| < c^{-1}$. Since $p_i q_i$ is a generator of $[p_i]$ for $i = 1, 2$,

$$\phi[p_2] \subset [p_1] \text{ and } \phi^{-1}[p_1] \subset [p_2].$$

It follows that the maps defined by

$$X_\phi : [p_2] \to [p_1],\ X_\phi f = \phi f; \quad X_{\phi^{-1}} : [p_1] \to [p_2],\ X_{\phi^{-1}} f = \phi^{-1} f$$

are module maps. It is easy to see that

$$X_\phi X_{\phi^{-1}} = 1,\ X_{\phi^{-1}} X_\phi = 1.$$

This implies that $[p_1]$ and $[p_2]$ are similar.

Proposition 4.3.5 and Corollary 4.2.2 thus suggest the following conjecture.

Conjecture. If $[p_1]$ and $[p_2]$ are similar, then there exist polynomials q_1, q_2 with $Z(q_1) \cap \mathbb{D}^n = Z(q_2) \cap \mathbb{D}^n = \emptyset$ such that $c < \frac{|p_1 q_1|}{|p_2 q_2|} < c^{-1}$ for some constant c. That is, the conditions in Proposition 4.3.5 are also necessary.

4.4 Equivalence of Hardy submodules on the unit ball

In this section we look at the case of Hardy submodules on the unit ball $\mathbb{B}_n$. In [CD1], Chen and Douglas proved that two homogeneous principal submodules of $H^2(\mathbb{B}_n)$ are quasi-similar if and only if the corresponding homogeneous polynomials are equal, except a constant factor. Combining this fact with our preceding discussion, one finds that the classification of submodules depends heavily on the geometric properties of domains.

Based on function theory in the unit ball of $\mathbb{C}^n$ from Rudin [Ru2], especially [Ru4], and techniques used in the polydisk case, we can obtain corresponding results in the case of Hardy submodules on the unit ball $\mathbb{B}_n$. However, on the unit ball $\mathbb{B}_n$, one has a special conclusion for polynomials.

Proposition 4.4.1 *Let p_1, p_2 be two polynomials on $\mathbb{C}^n (n > 1)$. If $|p_1| = |p_2|$ on $\partial \mathbb{B}_n$ (the boundary of $\mathbb{B}_n$), then there is a unimodular constant c such that $p_1 = c\, p_2$.*

The proof of Proposition 4.4.1 is based on the remarkable Theorem 14.3.3 in Rudin's book [Ru2]. Assume $n > 1$. Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A(\Omega) = C(\bar{\Omega}) \cap Hol(\Omega)$ be the so-called Ω-algebra. If $f \in A(\Omega),\ g \in A(\Omega)$, and $|f(\lambda)| \leq |g(\lambda)|$ for each boundary point λ of Ω, then $|f(z)| \leq |g(z)|$ for every $z \in \Omega$. Also notice that if p is a polynomial and $Z(p) \cap \mathbb{B}_n = \emptyset$, then p is a generator of $H^2(\mathbb{B}_n)$ (see Remark 2.2.14 following Proposition 2.2.13). Thus, combining Proposition 4.4.1 with the techniques used in Section 4.2, in the case of Hardy submodules on the unit ball $\mathbb{B}_n$, we can prove the following [Guo2].

Theorem 4.4.2 *Let $I_1 = p_1 L_1,\ I_2 = p_2 L_2$ be the Beurling forms of I_1 and I_2, respectively. Then the following are equivalent:*

(1) $[I_1]$ and $[I_2]$ are unitarily equivalent,

(2) there exist polynomials q_1 and q_2 satisfying $Z(q_1) \cap \mathbb{B}_n = Z(q_2) \cap \mathbb{B}_n = \emptyset$ such that $p_1 q_1 = p_2 q_2$ and $[p_1 L_1] = [p_1 L_2]$,

(3) $[I_1] = [I_2]$.

Corollary 4.4.3 *Let p_1, p_2 be two polynomials. Then $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there exist polynomials q_1 and q_2 that satisfy $Z(q_1) \cap \mathbb{B}_n = Z(q_2) \cap \mathbb{B}_n = \emptyset$ such that $p_1 q_1 = p_2 q_2$. In particular, if the zero set of each of the prime factors of p_i meets $\mathbb{B}_n$ nontrivially for $i = 1, 2$, then $[p_1]$ and $[p_2]$ are unitarily equivalent if and only if there is a constant c such that $p_1 = c\, p_2$.*

From Theorem 4.4.2, one sees that there is more rigidity among submodules of $H^2(\mathbb{B}_n)$ than in the case of $H^2(\mathbb{D}^n)$. Furthermore, from the proof of Theorem 4.3.3 and Proposition 4.4.1, the following is immediate.

Theorem 4.4.4 *Let I_1, I_2 be homogeneous ideals. Then the following are equivalent:*

(1) $[I_1]$ and $[I_2]$ are unitarily equivalent,

(2) $[I_1]$ and $[I_2]$ are similar,

(3) $[I_1]$ and $[I_2]$ are quasi-similar,

(4) $I_1 = I_2$.

Based on Theorems 4.4.2 and 4.4.4, one thus conjectures the following.

Conjecture. Let I_1 and I_2 be two ideals of polynomials. Then $[I_1]$, $[I_2]$, as submodules of $H^2(\mathbb{B}_n)$, are quasi-similar only if $[I_1] = [I_2]$.

4.5 Remarks on Chapter 4

In this chapter we obtain a complete classification under unitary equivalence for Hardy submodules on the polydisk (and on the unit ball) which are generated by polynomials. Furthermore, we give a complete similar classification for submodules generated by homogeneous ideals. The chapter is mainly based on Guo's paper [Guo2]. The reader may see that the characteristic space theory plays a crucial role.

Section 4.1 provides some necessary preliminaries. Propositions 4.1.1, 4.1.2 and 4.1.3 are probably well known. Theorems 4.1.4, 4.1.5 and 4.1.6 are due to Guo [Guo2].

In [Guo2], Guo proved Theorem 4.2.1 and Corollaries 4.2.2, 4.2.5 and 4.2.9. Theorems 4.2.10, 4.2.11 and Example 4.2.4 appeared also in [Guo2]. Corollary 4.2.7 and Examples 4.2.6 and 4.2.8 are new.

Lemma 4.3.1 and Theorems 4.3.2 and 4.3.3 come from [Guo2]. Corollary 4.3.4 and Proposition 4.3.5 were also proved in [Guo2].

Proposition 4.4.1 is an immediate result of Theorem 14.3.3 in Rudin's book [Ru2]. As we have seen, there exists more rigidity among Hardy submodules on the unit ball $\mathbb{B}_n$. Theorems 4.4.2, 4.4.4 and Corollary 4.4.3 appeared in [Guo2].

Concerning the Hardy module over the bidisk we call the reader's attention to the following references, where operator theory and function theory on the bidisk are presented [GM, INS, IY, JOS, Nak, Man, Ya1, Ya2, Ya3, Ya4, Ya5, Ya6, Ya7].

Chapter 5

Reproducing function spaces on the complex n-space

The Fock space, or the so-called Segal-Bargmann space, is the analog of the Bergman space in the context of the complex n-space $\mathbb{C}^n$. It is a Hilbert space consisting of entire functions in $\mathbb{C}^n$. Let

$$d\mu(z) = e^{-|z|^2/2} dv(z)(2\pi)^{-n}$$

be the Gaussian measure on $\mathbb{C}^n$ (dv is the ordinary Lebesgue measure). The Fock space $L^2_a(\mathbb{C}^n, d\mu)$ (in short, $L^2_a(\mathbb{C}^n)$), by definition, is the space of all μ-square-integrable entire functions on $\mathbb{C}^n$. It is easy to check that $L^2_a(\mathbb{C}^n)$ is a closed subspace of $L^2(\mathbb{C}^n)$ with the reproducing kernel

$$K_\lambda(z) = e^{\langle z,\lambda\rangle/2}, \quad \text{here } \langle z, \lambda\rangle = \sum_{i=1}^{n} \bar{\lambda}_i z_i.$$

The Fock space is important because of the relationship between the operator theory on it and the Weyl quantization [Be].

In this chapter we will study structure of reproducing function spaces on $\mathbb{C}^n$, and especially the structure of quasi-invariant subspaces. We first establish some terminology which will be used throughout this chapter.

Let $Hol(\mathbb{C}^n)$ denote the ring of all entire functions on the complex n-space $\mathbb{C}^n$, and let X be a Banach space contained in $Hol(\mathbb{C}^n)$. We call X a reproducing Banach space on $\mathbb{C}^n$ if X satisfies:

(a) the polynomial ring $\mathcal{C}$ is dense in X;

(b) the evaluation linear functional $E_\lambda(f) = f(\lambda)$ is continuous on X for each $\lambda \in \mathbb{C}^n$.

The basic example is the Fock space mentioned above.

For a reproducing Banach space X on $\mathbb{C}^n$, the following proposition shows that there exists no nontrivial invariant subspace for polynomials.

Proposition 5.0.1 *Let X be a reproducing Banach space and let $M \neq \{0\}$ be a (closed) subspace of X. If f is an entire function on $\mathbb{C}^n$ such that $fM \subset M$, then f is a constant.*

In fact, by the assumption and the closed graph theorem, it is easy to see the multiplication by f on M, denoted by M_f, is a bounded linear operator. Therefore, M_f^* is bounded on M^*, the conjugate space of M. Use $\hat{K}_\lambda$ to denote the reproducing kernel functions associated with M, and $\hat{k}_\lambda$ the normalized reproducing kernel functions, where both $\hat{K}_\lambda$ and $\hat{k}_\lambda$ belong to M^*. Since

$$\|\hat{k}_\lambda\| = \sup_{g\in M;\, \|g\|=1} |\langle \hat{k}_\lambda, g\rangle| = 1,$$

there exists a function g_λ in M with $\|g_\lambda\| = 1$ such that

$$1/2 \leq |\langle \hat{k}_\lambda, g_\lambda\rangle| \leq 1.$$

From inequalities

$$|\langle M_f^*\hat{k}_\lambda, g_\lambda\rangle| = |\langle \hat{k}_\lambda, fg_\lambda\rangle| = |f(\lambda)\langle \hat{k}_\lambda, g_\lambda\rangle| \geq |f(\lambda)|/2$$

and

$$|\langle M_f^*\hat{k}_\lambda, g_\lambda\rangle| \leq \|M_f\|\|\hat{k}_\lambda\|\|g_\lambda\| = \|M_f\|,$$

we have

$$|f(\lambda)| \leq 2\|M_f\|.$$

Hence f is a bounded entire function on $\mathbb{C}^n$. Thus by Liouville's theorem f is a constant.

According to Proposition 5.0.1, there exists no nontrivial invariant subspace for polynomials. Thus, an appropriate substitute for invariant subspace, the so-called quasi-invariant subspace is needed. Namely, a (closed) subspace M of X is called quasi-invariant if $pM \cap X \subset M$ for each polynomial p. Equivalently, M is quasi-invariant if the relation $pf \in X$ implies $pf \in M$ for any $f \in M$ and any polynomial p.

5.1 Algebraic reduction for quasi-invariant subspaces

It is difficult to completely characterize quasi-invariant subspaces. However, using the characteristic space theory, we can obtain algebraic reduction for finite codimensional quasi-invariant subspaces. This can be viewed as a generalization of algebraic reduction theorem of analytic Hilbert modules on bounded domains (see Chapter 2). This section comes from Guo and Zheng's paper [GZ].

Let us begin with a lemma.

Lemma 5.1.1 *Let I be an ideal in the polynomial ring $\mathcal{C}$ and let $[I]$ be the closure of I in X. Then $[I] \cap \mathcal{C} = I$.*

Proof. Let $Z(I)$ be the zero variety of I, i.e.,

$$Z(I) = \{z \in \mathbb{C}^n : p(z) = 0, \forall p \in I\}.$$

For $\lambda \in Z(I)$, I_λ denotes the characteristic space of I at λ, and I_λ^e, the envelope of I at λ. Then by Corollary 2.1.2,

$$I = \bigcap_{\lambda \in Z(I)} I_\lambda^e.$$

Clearly, for each $p \in [I] \cap \mathcal{C}$ and $\lambda \in Z(I)$,

$$q(D)p|_\lambda = 0, \quad \forall q \in I_\lambda,$$

and hence $p \in I_\lambda^e$. This implies that $p \in I$, and hence $[I] \cap \mathcal{C} \subset I$. Obviously, $I \subset [I] \cap \mathcal{C}$. The desired conclusion follows.

Now we can establish a canonical linear map

$$\tau : \mathcal{C}/I \to X/[I]$$

by $\tau(p + I) = p + [I]$. By Lemma 5.1.1, the map τ is injective.

Lemma 5.1.2 *Let I be an ideal of finite codimension. Then $[I]$ is a quasi-invariant subspace of X of finite codimension. Furthermore, the canonical map*

$$\tau : \mathcal{C}/I \to X/[I]$$

is an isomorphism.

Proof. We express $\mathcal{C}$ as

$$\mathcal{C} = I \dot{+} R$$

where R is a linear space of polynomials with dim $R =$ dim $\mathcal{C}/I$. Since the polynomial ring $\mathcal{C}$ is dense in X, and $[I] + R$ is closed, we have that

$$[I] + R = X.$$

By the equality

$$[I] \cap R = [I] \cap \mathcal{C} \cap R,$$

and Lemma 5.1.1, we get

$$[I] \cap R = \{0\}.$$

Thus,

$$X/[I] = ([I] + R)/[I] \cong R/([I] \cap R) = R/\{0\} = R \cong \mathcal{C}/I.$$

It follows that

$$\tau : \mathcal{C}/I \to X/[I]$$

is an isomorphism.

Suppose that $f \in [I]$ and p is a polynomial satisfying $pf \in X$. It is enough to show $pf \in [I]$. Since $X = [I]\dot{+}R$, pf can be expressed as

$$pf = g + h;$$

here $g \in [I]$ and $h \in R$. Note that for each $\lambda \in Z(I)$ and any $q \in I_\lambda$,

$$q(D)f|_\lambda = 0 \text{ and } q(D)g|_\lambda = 0.$$

Since I_λ is invariant under the action by the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \cdots, \partial/\partial z_n\}$, it follows that

$$q(D)pf|_\lambda = 0.$$

This gives that

$$q(D)h|_\lambda = 0$$

for each $\lambda \in Z(I)$ and any $q \in I_\lambda$. By Corollary 2.1.2, $h \in I$, and hence $h = 0$. It follows that $pf \in [I]$. We conclude that $[I]$ is quasi-invariant, completing the proof.

Lemma 5.1.3 *Let M be a quasi-invariant subspace of finite codimension in X. Then $M \cap \mathcal{C}$ is an ideal of $\mathcal{C}$, and $M \cap \mathcal{C}$ is dense in M. Furthermore, the canonical map*

$$\tau' : \mathcal{C}/M \cap \mathcal{C} \to X/M$$

is an isomorphism, where $\tau'(p + M \cap \mathcal{C}) = p + M$.

Proof. Obviously, $M \cap \mathcal{C}$ is an ideal of $\mathcal{C}$ because M is quasi-invariant. It is easy to see that the map τ' is injective, and hence the ideal $M \cap \mathcal{C}$ is of finite codimension and

$$\dim\, \mathcal{C}/M \cap \mathcal{C} \leq \dim\, X/M.$$

By Lemma 5.1.2,

$$\dim\, \mathcal{C}/M \cap \mathcal{C} = \dim\, X/[M \cap \mathcal{C}].$$

Since

$$[M \cap \mathcal{C}] \subset M,$$

we have

$$\dim\, X/[M \cap \mathcal{C}] \geq \dim\, X/M.$$

Therefore,

$$\dim\, X/[M \cap \mathcal{C}] = \dim\, X/M.$$

This implies

$$[M \cap \mathcal{C}] = M,$$

that is, $M \cap \mathcal{C}$ is dense in M. Applying Lemma 5.1.2, the map τ' is an isomorphism.

From Lemmas 5.1.1, 5.1.2 and 5.1.3, we obtain the following algebraic reduction theorem for finite codimensional quasi-invariant subspaces.

Theorem 5.1.4 *Let M be a quasi-invariant subspace of finite codimension. Then $\mathcal{C} \cap M$ is an ideal in the ring $\mathcal{C}$, and*

(1) $\mathcal{C} \cap M$ is dense in M;

(2) the canonical map $\tau : \mathcal{C}/M \cap \mathcal{C} \to X/M$ is an isomorphism, where $\tau(p + M \cap \mathcal{C}) = p + M$.

Conversely, if I is an ideal in $\mathcal{C}$ of finite codimension, then $[I]$ is a quasi-invariant subspace of the same codimension and $[I] \cap \mathcal{C} = I$.

Remark 5.1.5 *For bounded domains Ω in the complex plane, which satisfy certain technical hypotheses, Axler and Bourdon [AB] proved that each finite codimensional invariant subspace M of $L_a^2(\Omega)$ has the form $M = p\, L_a^2(\Omega)$, where p is a polynomial with its zeros in Ω. M. Putinar [Pu1] extended this result to some bounded pseudoconvex domains in $\mathbb{C}^n$. Namely, for such a domain Ω, Putinar proved that every finite codimensional invariant subspace M has the form*

$$M = \sum_{i=1}^{k} p_i L_a^2(\Omega),$$

where p_i are polynomials having a finite number of common zeros, all contained in Ω. However, from Proposition 5.0.1, we see that each finite codimensional quasi-invariant subspace does not have the above form. This may be an essential difference between analytic Hilbert spaces on bounded domains and those on unbounded domains.

5.2 Some basic properties of reproducing Hilbert spaces on the complex plane

This section is based on [CGH]. In this section we will develop some basic properties of reproducing Hilbert spaces over the complex plane to serve the next section.

Let X be a reproducing Hilbert space over the complex plane $\mathbb{C}$. For each $\lambda \in \mathbb{C}$, we let K_λ be the reproducing kernel of X at λ, and k_λ be the normalized reproducing kernel. This means that for every $f \in X$, we have

$$f(\lambda) = \langle f, K_\lambda \rangle.$$

Proposition 5.2.1 *For each $f \in X$, we have*

$$\lim_{|\lambda| \to \infty} \langle f, k_\lambda \rangle = 0,$$

that is, k_λ converges weakly to zero as $|\lambda| \to \infty$.

Proof. Since the polynomial ring $\mathcal{C}$ is dense in X, it suffices to show that for each polynomial p,

$$\lim_{|\lambda|\to\infty} \langle p, k_\lambda \rangle = 0.$$

Indeed, noticing that the equalities

$$\|K_\lambda\| = \sup_{\|f\|=1} |\langle f, K_\lambda \rangle|| = \sup_{\|f\|=1} |f(\lambda)|,$$

and taking $f = z^n/\|z^n\|$, the above equality implies that

$$\|K_\lambda\| \geq |\lambda|^n/\|z^n\|$$

for each positive integer n. Now for any polynomial p, since

$$\langle p, k_\lambda \rangle = \frac{1}{\|K_\lambda\|}\langle p, K_\lambda \rangle = \frac{1}{\|K_\lambda\|}p(\lambda),$$

combining this with a previous inequality gives

$$\lim_{|\lambda|\to\infty} \langle p, k_\lambda \rangle = 0$$

for each polynomial p. This ensures the desired conclusion.

For a closed subspace M of X, we let $K_\lambda^{(M)}$ be the reproducing kernel function of M. Then it is easy to see that $K_\lambda^{(M)} = P_M K_\lambda$, where P_M is the orthogonal projection from X onto M.

Corollary 5.2.2 *If M is of finite codimension, then*

$$\lim_{|\lambda|\to\infty} \|P_M k_\lambda\| = 1.$$

Proof. Notice that

$$\|P_M k_\lambda\| = \frac{\|K_\lambda^{(M)}\|}{\|K_\lambda\|},$$

and we let $f_1, f_2, \cdots, f_n$ be an orthonormal basis of $M^\perp = X \ominus M$. It is easy to check that the reproducing kernel function $K_\lambda^{(M^\perp)}$ is given by

$$K_\lambda^{(M^\perp)}(z) = \sum_{k=1}^{n} \overline{f_k(\lambda)} f_k(z).$$

Thus,

$$\|K_\lambda^{(M^\perp)}\|^2 = \sum_{k=1}^{n} |f_k(\lambda)|^2.$$

Since

$$\frac{\sum_{k=1}^n |f_k(\lambda)|^2}{\|K_\lambda\|^2} = \sum_{k=1}^n |\langle f_k, K_\lambda/\|K_\lambda\|\rangle|^2$$
$$= \sum_{k=1}^n |\langle f_k, k_\lambda\rangle|^2,$$

By Proposition 5.2.1,

$$\lim_{|\lambda|\to\infty} \frac{\|K_\lambda^{(M^\perp)}\|}{\|K_\lambda\|} = 0.$$

From the equality

$$\|K_\lambda^{(M)}\|^2 = \|K_\lambda\|^2 - \|K_\lambda^{(M^\perp)}\|^2,$$

the required result is deduced.

Remark. The classical Beurling's theorem [Beu] says that for each submodule M of the Hardy module $H^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ there is an inner function η such that $M = \eta H^2(\mathbb{D})$. Equivalently, the orthogonal projection P_M from $H^2(\mathbb{D})$ onto M has the form

$$P_M = M_\eta M_\eta^*.$$

It follows that

$$\|P_M k_\lambda\| = \frac{\|P_M K_\lambda\|}{\|K_\lambda\|} = \frac{\|K_\lambda^M\|}{\|K_\lambda\|} = |\eta(\lambda)|,$$

and hence, for almost all $z \in \mathbb{T}$ we have

$$\lim_{\lambda\to z} \|P_M k_\lambda\| = 1,$$

where K_λ and K_λ^M be the reproducing kernels of $H^2(\mathbb{D})$ and M, respectively, and $k_\lambda = K_\lambda/\|K_\lambda\|$. From Corollary 5.2.2, we see that if M is a finite codimensional quasi-invariant subspace of an analytic Hilbert space X over $\mathbb{C}$, then a modification of Beurling's theorem is true for M. Considering the Fock space $L_a^2(\mathbb{C})$, then the operator M_z defined in the Fock space is unbounded. Therefore, by the closed graph theorem there exists a function $f \in L_a^2(\mathbb{C})$ such that $zf \notin L_a^2(\mathbb{C})$. It is easy to check that the one-dimensional subspace $\mathcal{M}_f = \{cf : c \in \mathbb{C}\}$ is quasi-invariant. However, by Corollary 5.2.2, we have

$$\lim_{|\lambda|\to\infty} \|P_{\mathcal{M}_f} k_\lambda\| = 0.$$

If M is an infinite dimensional quasi-invariant subspace of $L_a^2(\mathbb{C})$, does

$$\lim_{|\lambda|\to\infty} \|P_M k_\lambda\| = 1\,?$$

Now we return to the problem of when a Hilbert space consisting of entire functions is an analytic Hilbert space over $\mathbb{C}$.

Example 5.2.3 *Let H be a Hilbert space consisting of entire functions. If $\{z^n : n = 0, 1, 2, \cdots\}$ is contained in H and they are mutually orthogonal, then H is a reproducing Hilbert space over $\mathbb{C}$.*

By the assumption it is easy to see that the set $\{z^n/\|z^n\| : n = 0, 1, \cdots\}$ forms an orthonormal basis of H. Note that $f(z) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{z^n}{\|z^n\|} \in H$, and hence f is an entire function. From the formula of the radius of convergence of a power series, we have

$$\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \infty.$$

Therefore, for each $\lambda \in \mathbb{C}$, there exists a natural number N such that

$$\sqrt[n]{\|z^n\|} \geq 2|\lambda|$$

if $n \geq N$. Now let $g = \sum_{n=0}^{\infty} a_n \frac{z^n}{\|z^n\|}$ be in H. Then

$$\begin{aligned} |g(\lambda)| &\leq \sum_{n=0}^{N} |a_n| \frac{|\lambda|^n}{\|z^n\|} + \sum_{n=N+1}^{\infty} |a_n| \frac{|\lambda|^n}{\|z^n\|} \\ &\leq \sum_{n=0}^{N} |a_n| \frac{|\lambda|^n}{\|z^n\|} + (\sum_{n=N+1}^{\infty} |a_n|^2)^{\frac{1}{2}} (\sum_{n=N+1}^{\infty} \frac{|\lambda|^{2n}}{\|z^n\|^2})^{\frac{1}{2}} \\ &\leq \sum_{n=0}^{N} |a_n| \frac{|\lambda|^n}{\|z^n\|} + \|g\|. \end{aligned}$$

Since

$$a_n = \langle g, z^n/\|z^n\| \rangle, \quad n = 0, 1, \cdots, N,$$

there is a positive constant c' depending only on λ such that

$$|a_n| \leq c' \, \|g\|$$

for $n = 0, 1, \cdots, N$. Combining this with previous inequalities implies that there is some positive constant c depending only on λ such that

$$|g(\lambda)| \leq c \, \|g\|, \quad \textit{for each } \ g \in H.$$

This means that the evaluation linear functional $E_\lambda(f) = f(\lambda)$ is continuous on H for each $\lambda \in \mathbb{C}$. Obviously, the polynomial ring $\mathcal{C}$ is dense in H. We thus conclude that H is a reproducing Hilbert space over the complex plane $\mathbb{C}$, and its reproducing kernel K_λ is given by

$$K_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n \bar{\lambda}^n}{\|z^n\|^2}.$$

Remark 5.2.4 *From the proof of Example 5.2.3, we see that if H is a reproducing Hilbert space over $\mathbb{C}$ with the orthonormal basis $\{z^n/\|z^n\| : n = 0, 1, \cdots\}$, then*

$$\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \infty.$$

Now let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of positive numbers, and $\sqrt[n]{\gamma_n} \to \infty$ as $n \to \infty$. We set

$$\|z^n\| = \gamma_n,\ n = 0, 1, \cdots .$$

Then the completion H of the polynomial ring $\mathcal{C}$, under the orthonormal basis $\{z^n/\gamma_n\ n = 0, 1, \cdots\}$, is a reproducing Hilbert space over the complex plane. In fact, if $f(z) = \sum_{n=0}^{\infty} a_n z^n/\gamma_n \in H$, then by the formula of the radius of convergence of a power series, we see that

$$\frac{1}{R} = \overline{\lim} \sqrt[n]{\frac{|a_n|}{\gamma_n}} = 0.$$

Thus, $R = \infty$. This says that $f(z)$ is an entire function, and hence H consists of entire functions. By Example 5.2.3, such an H is a reproducing Hilbert space over the complex plane.

By Remark 5.2.4, if we take $\gamma_n = \sqrt{2^n n!}$, then we get the Fock space $L^2_a(\mathbb{C}, d\mu)$ with the reproducing kernel $K_\lambda(z) = e^{z\bar{\lambda}/2}$, here $d\mu = e^{-|z|^2/2} dv/2\pi$.

Let f, g be two entire functions over $\mathbb{C}$; we say $f \preceq g$ if there exist positive constants r and M such that

$$\frac{|f(z)|}{|g(z)|} \leq M, \quad \text{whenever} \quad |z| \geq r.$$

A simple observation shows that $\preceq$ is a partial order on the ring of all entire functions. The next proposition characterizes this partial order.

Proposition 5.2.5 *Let f and g be entire functions. Then $f \preceq g$ if and only if there exist polynomials p and q with* $\deg p \leq \deg q$ *such that*

$$\frac{f}{g} = \frac{p}{q},$$

where "deg" *denotes the degree of polynomial.*

Proof. "$\Rightarrow$" Without a loss of generality, we may assume that f and g have no common zeros. Let r, $M > 0$ be positive constants such that

$$|f(z)|/|g(z)| \leq M$$

if $|z| > r$. Then the function $f(z)/g(z)$ is analytic on $\{z \in \mathbb{C} : |z| > r\}$. Now let $\lambda_1, \ldots, \lambda_n$ be zeros of g in $\{z \in \mathbb{C} : |z| \leq r\}$ with corresponding degrees

$k_1, \ldots, k_n$. It follows that $\phi(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_n)^{k_n} f(z)/g(z)$ is an entire function on $\mathbb{C}$. Hence we have

$$|\phi(z)| \leq M|(z-\lambda_1)^{k_1} \cdots (z-\lambda_n)^{k_n}|$$

if $|z| > r$. Set $l = k_1 + \cdots + k_n$. Then there exist positive constants r' and M' such that

$$|\phi(z)|/|z^l| \leq M'$$

for $|z| > r'$. Write

$$\phi(z) = a_0 + a_1 z + \cdots + a_{l-1} z^{l-1} + z^l h,$$

where h is an entire function. From the above inequality, we see that h is bounded. By Liouville's theorem, h is a constant and hence $\phi(z)$ is a polynomial. Set $p = \phi$, and $q = (z-\lambda_1)^{k_1} \cdots (z-\lambda_n)^{k_n}$. Then deg $p \leq$ deg q, and

$$\frac{f(z)}{g(z)} = \frac{p(z)}{q(z)}.$$

This gives the desired conclusion.

The opposite direction is obvious, completing the proof.

Given a reproducing Hilbert space X over $\mathbb{C}$, we say that X is an ordered reproducing Hilbert space (or a reproducing Hilbert space with order) if $g \in X$ and $f \preceq g$, then $f \in X$. A simple reasoning shows that the Fock space $L_a^2(\mathbb{C})$ is an ordered reproducing Hilbert space.

Proposition 5.2.6 *Let X be a reproducing Hilbert space over $\mathbb{C}$. Then X is an ordered reproducing Hilbert space if and only if for any polynomial p and entire function f, the relation $pf \in X$ implies $f \in X$.*

Proof. To prove that X is of the desired order, we let $f \preceq g$ and $g \in X$. Then by Proposition 5.2.5, there exist polynomials p_1 and p_2 without common zeros, which satisfy deg $p_1 \leq$ deg p_2, such that

$$f/p_1 = g/p_2.$$

Set $h = f/p_1$. Then h is an entire function and $g = p_2 h$. Decomposing p_2 and using the assumption gives $z^l h \in X$, for $l = 0, 1, 2, \cdots,$ deg p_2. Note that $f = p_1 h$, and deg $p_1 \leq$ deg p_2. One thus obtains $f = p_1 h \in X$.

Conversely, suppose that X is an ordered reproducing Hilbert space, and $pf \in X$ (where p is a polynomial and f is an entire function). Applying Proposition 5.2.5 gives

$$f \preceq pf$$

and hence $f \in X$.

The next example shows that there exist reproducing Hilbert spaces over $\mathbb{C}$ without the desired order.

Example 5.2.7 *We set*

$$\|1\| = 1,\ \|z^{2n}\| = \sqrt{(2n-2)!},\ \|z^{2n+1}\| = \sqrt{(2n+1)!}$$

for $n = 1, 2, \cdots$. *Then by Example 5.2.3 and Remark 5.2.4, the Hilbert space* H *with the orthonormal basis* $\{z^n/\|z^n\|\}$ *is a reproducing Hilbert space over* $\mathbb{C}$. *However,* H *is not of the desired order. In fact, it is easy to check that the function* $\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{z^{2n+1}}{\|z^{2n}\|}$ *is an entire function, and* $z \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{z^{2n+1}}{\|z^{2n}\|} \in H$. *But* $\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{z^{2n+1}}{\|z^{2n}\|} \notin H$ *because*

$$\sum_{n=1}^{\infty} [\frac{\|z^{2n+1}\|}{n^2\|z^{2n}\|}]^2 = \infty.$$

Applying Proposition 5.2.6, one sees that H *is not of the desired order.*

5.3 Equivalence of quasi-invariant subspaces on the complex plane

Let X be a reproducing Hilbert space on the complex plane, and let M_1 and M_2 be two quasi-invariant subspaces of X. For a bounded linear operator $A : M_1 \to M_2$, we call A a quasi-module map if $A(p\,f) = p\,A(f)$ whenever $p\,f \in M_1$ (here p is any polynomial and $f \in M_1$). Thus by the definition, if A is a quasi-module map, then the relation $p\,f \in M_1$ forces $p\,A(f) \in M_2$. Furthermore, we say that

1. they are unitarily equivalent if there exists a unitary quasi-module map $A : M_1 \to M_2$ such that $A^{-1} : M_2 \to M_1$ is also a quasi-module map;
2. they are similar if there exist an invertible quasi-module map $A : M_1 \to M_2$ such that $A^{-1} : M_2 \to M_1$ is also a quasi-module map;
3. they are quasi-similar if there exist quasi-module maps $A : M_1 \to M_2$ and $B : M_2 \to M_1$ with dense ranges.

It is easy to check that unitary equivalence, similarity and quasi-similarity are equivalence relations.

This section comes from [CGH]. First we recall that the polynomial ring in one complex variable is a principal ideal domain. This means that each ideal is principal, i.e., generated by a polynomial. Let X be the reproducing Hilbert space on the complex plane $\mathbb{C}$. Then by Theorem 5.1.4, each finite codimensional quasi-invariant subspace M is of the form $[p]$(here p is a polynomial, and $[p]$ is quasi-invariant subspace generated by p). Moreover, for every polynomial p, $[p]$ is a finite codimensional quasi-invariant subspace with codimension $\deg p$.

Theorem 5.3.1 *Let M be quasi-invariant. If M is similar to a finite codimensional quasi-invariant subspace $[p]$, then there is a polynomial q with* $\deg q = \deg p$ *such that $M = [q]$.*

Proof. Let $A : [p] \to M$ be a similarity. Set $q = A\,p$. Since p generates $[p]$, q generates M, that is, $M = [q]$. Let $K_\lambda^{(p)}$ and $K_\lambda^{(q)}$ be reproducing kernels of $[p]$ and $[q]$, respectively. Since $A : [p] \to [q]$ is a similarity, it is easy to verify that for any $h \in [p]$,

$$A\,h = \frac{q}{p} h.$$

Thus, we have

$$\langle A\,h, K_\lambda^{(q)} \rangle = \frac{q(\lambda)}{p(\lambda)} h(\lambda)$$

for each $\lambda \notin Z(p)$, where $Z(p)$ is the set of all zeros of p. From the inequality

$$|\langle A\,h, K_\lambda^{(q)} \rangle| \le \|A\| \, \|h\| \, \|K_\lambda^{(q)}\|,$$

we have

$$\frac{|q(\lambda)|}{|p(\lambda)|} \sup_{h \in [p];\ \|h\|=1} |h(\lambda)| \le \|A\| \, \|K_\lambda^{(q)}\|.$$

By the equality

$$\|K_\lambda^{(p)}\| = \sup_{h \in [p];\ \|h\|=1} |h(\lambda)|,$$

it follows that

$$\frac{|q(\lambda)|}{|p(\lambda)|} \|K_\lambda^{(p)}\| \le \|A\| \, \|K_\lambda^{(q)}\|.$$

Note that $[p]$ is of finite codimension. Applying Corollary 5.2.2 gives that

$$\lim_{|\lambda| \to \infty} \frac{\|K_\lambda^{(p)}\|}{\|K_\lambda\|} = 1.$$

By the inequality

$$\|K_\lambda^{(q)}\| \le \|K_\lambda\|,$$

there exist positive constants r and M such that

$$\frac{|q(\lambda)|}{|p(\lambda)|} \le M$$

if $|\lambda| \ge r$. From Proposition 5.2.5 there exist polynomials p_1 and p_2 with $\deg p_1 \le \deg p_2$ such that

$$q/p = p_1/p_2$$

and hence $q = pp_1/p_2$. Since q is an entire function, q is a polynomial and $\deg q \le \deg p$.

Note that $A^{-1} : [q] \to [p]$ is also a similarity. The same reasoning shows that $\deg p \le \deg q$. It follows that $\deg p = \deg q$.

The next example shows that there is a reproducing Hilbert space H over $\mathbb{C}$ such that $\deg p = \deg q$, but $[p]$ and $[q]$ are not similar.

Example 5.3.2 *Let H be the reproducing Hilbert space given in Example 5.2.7. We claim that $[z^2]$ and $[z^2 + z]$ are not similar. Assume that they are similar, and let $A : [z^2] \to [z^2 + z]$ be a similarity. Then by the proof of Theorem 5.3.1, there exists a constant γ such that*

$$Az^2 = \gamma\,(z^2 + z).$$

So for any natural number m, we have

$$Az^2 z^{2m} = \gamma\, z^{2m}(z^2 + z) = \gamma\,(z^{2m+2} + z^{2m+1}).$$

From the equalities

$$\|z^{2m+2} + z^{2m+1})\|^2 = (2m)! + (2m+1)!, \quad \text{and} \quad \|(z^{2m+2})\|^2 = (2m)!,$$

we see that

$$\gamma^2((2m)! + (2m+1)!) \le \|A\|^2\,(2m)!$$

for any natural number m. This clearly is impossible and hence $[z^2]$ and $[z^2 + z]$ are not similar.

Corollary 5.3.3 *Let X be a reproducing Hilbert space over $\mathbb{C}$. If a quasi-invariant subspace M is similar to X, then $M = X$.*

Remark 5.3.4 *From Corollary 5.3.3, one sees that the structure of reproducing Hilbert modules over the complex plane is completely different from that of analytic Hilbert spaces over bounded domains. For example, for any two submodules of the Bergman space over the unit disk, if they are of finite codimension, then they are necessarily similar.*

Corollary 5.3.5 *Let X be an ordered reproducing Hilbert space. Then finite codimensional quasi-invariant subspaces $[p]$ and $[q]$ are similar if and only if* $\deg p = \deg q$.

Proof. Let $\deg p = \deg q$. For $f \in [p]$, it is easy to see that f/p is an entire function, and hence $\frac{q}{p} f$ is an entire function. Since $\deg p = \deg q$, applying Proposition 5.2.5 gives $\frac{q}{p} f \preceq f$, and hence by the definition $\frac{q}{p} f$ is in X for each $f \in [p]$. Thus we can define an operator $A : [p] \to X$ by $A f = \frac{q}{p} f$ for $f \in [p]$. By a simple application of the closed graph theorem, the operator A is bounded. Since A maps $p\mathcal{C}$ onto $q\mathcal{C}$, this deduces that A maps $[p]$ to $[q]$. Therefore, $A : [p] \to [q]$ is a bounded operator. Clearly, $A(rf) = rA(f)$ if $rf \in [p]$ (here $f \in [p]$ and r is a polynomial). Similarly, we can show that the operator $B : [q] \to [p]$ defined by $B\,f = \frac{p}{q} f$ for $f \in [q]$ is bounded. It is easy to see that $A\,B$ and $B\,A$ are the identity operators on $[q]$ and $[p]$, respectively. We thus conclude that $[p]$ and $[q]$ are similar if $\deg p = \deg q$.

The opposite direction is achieved by applying Theorem 5.3.1. This completes the proof.

Theorem 5.3.6 *Let X be a reproducing Hilbert space over $\mathbb{C}$ with the orthonormal basis $\{z^n/\|z^n\|\}$. Then finite codimensional quasi-invariant subspaces $[p]$ and $[q]$ are unitarily equivalent only if $p = c\,q$ for some nonzero constant c, and hence only if $[p] = [q]$.*

Before proving the theorem, let us note that if $A : [p] \to [q]$ is a unitary equivalence, then from the proof of Theorem 5.3.1, there is a constant γ such that $A(p) = \gamma\,q$ and $\deg p = \deg q$. In what follows we always assume that $A(p) = q$ if $A : [p] \to [q]$ is a unitary equivalence. To prove the theorem, we need two lemmas.

Lemma 5.3.7 *Let $p = \sum_{k=0}^{l} a_l z^l$ and $q = \sum_{k=0}^{l} b_l z^l$, and let both a_l and b_l be nonzero. If $A : [p] \to [q]$ is a unitary equivalence, and $A(p) = q$, then $|a_l| = |b_l|$.*

Proof. Let $K_\lambda^{(p)}$ and $K_\lambda^{(q)}$ be reproducing kernels of $[p]$ and $[q]$, respectively. Since $A : [p] \to [q]$ is a unitary equivalence, it is easy to verify that for any $h \in [p]$,

$$A\,h = \frac{q}{p} h.$$

Thus, we have

$$\langle A\,h, K_\lambda^{(q)} \rangle = \frac{q(\lambda)}{p(\lambda)} h(\lambda)$$

for each $\lambda \notin Z(p)$. By

$$|\langle A\,h, K_\lambda^{(q)} \rangle| \leq \|h\|\,\|K_\lambda^{(q)}\|,$$

this deduces

$$\frac{|q(\lambda)|}{|p(\lambda)|} \sup_{h \in [p];\ \|h\|=1} |h(\lambda)| \leq \|K_\lambda^{(q)}\|.$$

From the equality

$$\|K_\lambda^{(p)}\| = \sup_{h \in [p];\ \|h\|=1} |h(\lambda)|,$$

we obtain that

$$\frac{|q(\lambda)|}{|p(\lambda)|} \|K_\lambda^{(p)}\| \leq \|K_\lambda^{(q)}\|.$$

Thus,

$$|q(\lambda)|/|p(\lambda)| \leq \|K_\lambda^{(q)}\|/\|K_\lambda^{(p)}\|.$$

Since $A^{-1} : [q] \to [p]$ is a unitary equivalence, the same reasoning shows that

$$|p(\lambda)|/|q(\lambda)| \leq \|K_\lambda^{(p)}\|/\|K_\lambda^{(q)}\|,$$

and therefore

$$|p(\lambda)|/|q(\lambda)| = \|K_\lambda^{(p)}\|/\|K_\lambda^{(q)}\|$$

for $\lambda \notin Z(q)$. Since both $[p]$ and $[q]$ are of finite codimension, applying Corollary 5.2.2 gives that

$$\lim_{|\lambda|\to\infty} \frac{|p(\lambda)|}{|q(\lambda)|} = \lim_{|\lambda|\to\infty} \frac{\|K_\lambda^{(p)}\|}{\|K_\lambda^{(q)}\|} = 1.$$

By the equality

$$\lim_{|\lambda|\to\infty} \frac{|p(\lambda)|}{|q(\lambda)|} = \frac{|a_l|}{|b_l|},$$

we see that $|a_l| = |b_l|$.

Lemma 5.3.8 *For each fixed natural number m, the sequence $\{\|z^{n+m}\|/\|z^n\|\}$ has a subsequence $\{\|z^{n_k+m}\|/\|z^{n_k}\|\}$ such that*

$$\lim_{k\to\infty} \frac{\|z^{n_k+m}\|}{\|z^{n_k}\|} = \infty.$$

Proof. Assume that there exists a constant γ such that

$$\|z^{n+m}\|/\|z^n\| \leq \gamma$$

for all n. Let $n = k_n m + s_n$, where s_n is a nonnegative integer such that $0 \leq s_n < m$. This implies that

$$\|z^n\| \leq \gamma^{k_n} \max\{\|1\|, \|z\|, \cdots, \|z^m\|\}$$

for all n, and hence $\sqrt[n]{\|z^n\|} \leq 2\gamma$ if n is large enough. However, by Remark 5.2.4,

$$\lim_{n\to\infty} \sqrt[n]{\|z^n\|} = \infty.$$

This contradiction says that there exists a subsequence $\{\|z^{n_k+m}\|/\|z^{n_k}\|\}$ such that

$$\lim_{k\to\infty} \frac{\|z^{n_k+m}\|}{\|z^{n_k}\|} = \infty.$$

Proof of Theorem 5.3.6. By Lemma 5.3.7, we may assume that

$$p = z^l + a_{l-1}z^{l-1} + \cdots + a_0, \quad q = z^l + b_{l-1}z^{l-1} + \cdots + b_0$$

and $A(p) = q$. Since A is a unitary equivalence of quasi-invariant subspaces, we have

$$\langle p\, z^m, p\, z^n\rangle = \langle q\, z^m, q\, z^n\rangle$$

for any nonnegative integer m, n. Let s_1, s_2 be the nonnegative integer such that

$$a_{s_1} \neq 0, \quad a_t = 0\ (t < s_1) \text{ and } b_{s_2} \neq 0, \quad b_t = 0\ (t < s_2).$$

We claim that $s_1 = s_2$ and $a_{s_1} = b_{s_2}$. In fact, since

$$\langle p, z^{l-s_1} p \rangle = \overline{a_{s_1}} \|z^l\|^2$$

and

$$\langle q, z^{l-s_1} q \rangle = \overline{b_{s_1}} \|z^l\|^2 + b_{l-1} \overline{b_{s_1-1}} \|z^{l-1}\|^2 + \cdots,$$

we see that $s_2 \leq s_1$. The same reasoning implies that $s_1 \leq s_2$ and hence $s_1 = s_2$. Set $s = s_1 = s_2$. Since

$$\langle p, z^{l-s} p \rangle = \overline{a_s} \|z^l\|^2 \text{ and } \langle q, z^{l-s} q \rangle = \overline{b_s} \|z^l\|^2,$$

this gives that $a_s = b_s$. Now by the equality

$$\langle z^n p, z^{n+l-(s+1)} p \rangle = \langle z^n q, z^{n+l-(s+1)} q \rangle,$$

we have that

$$\overline{a_{s+1}} \|z^{n+l}\|^2 + a_{l-1} \overline{a_s} \|z^{n+l-1}\|^2 = \overline{b_{s+1}} \|z^{n+l}\|^2 + b_{l-1} \overline{b_s} \|z^{n+l-1}\|^2$$

for any natural number n. Dividing the above equality by $\|z^{n+l}\|^2$ and using Lemma 5.3.8 gives that $a_{s+1} = b_{s+1}$, and hence also leads to the equality $a_{l-1} = b_{l-1}$. Furthermore, from the equality

$$\langle z^n p, z^{n+l-(s+2)} p \rangle = \langle z^n q, z^{n+l-(s+2)} q \rangle,$$

we obtain that

$$\begin{aligned} &\overline{a_{s+2}} \|z^{n+l}\|^2 + a_{l-1} \overline{a_{s+1}} \|z^{n+l-1}\|^2 + a_{l-2} \overline{a_s} \|z^{n+l-2}\|^2 \\ = &\overline{b_{s+2}} \|z^{n+l}\|^2 + b_{l-1} \overline{b_{s+1}} \|z^{n+l-1}\|^2 + b_{l-2} \overline{b_s} \|z^{n+l-2}\|^2 \end{aligned}$$

and hence

$$\overline{a_{s+2}} \|z^{n+l}\|^2 + a_{l-2} \overline{a_s} \|z^{n+l-2}\|^2 = \overline{b_{s+2}} \|z^{n+l}\|^2 + b_{l-2} \overline{b_s} \|z^{n+l-2}\|^2$$

for any natural number n. Dividing the above equality by $\|z^{n+l}\|^2$and applying Lemma 5.3.8, we get that $a_{s+2} = b_{s+2}$ and hence also leads to the equality $a_{l-2} = b_{l-2}$. By repeating the preceding process one gets that

$$a_0 = b_0,\ a_1 = b_1, \cdots, a_{l-1} = b_{l-1},$$

and hence $p = q$. This completes the proof.

From Theorems 5.3.1 and 5.3.6, we immediately obtain the following.

Corollary 5.3.9 *Let X be a reproducing Hilbert space over $\mathbb{C}$ with the orthonormal basis $\{z^n/\|z^n\|\}$, and let M be a quasi-invariant subspace of X. If M is unitarily equivalent to $[p]$ for some polynomial p, then $M = [p]$.*

Remark 5.3.10 *From Proposition 5.0.1, the operator M_z is unbounded on the Fock space $L^2_a(\mathbb{C})$, and hence there is a function $f \in L^2_a(\mathbb{C})$ such that $z f \notin L^2_a(\mathbb{C})$. It is easy to check that one-dimensional subspace $\{c f : c \in \mathbb{C}\}$ is quasi-invariant. We choose two such functions f, g with $f/g \neq$ constant, then quasi-invariant subspaces $\{c f : c \in \mathbb{C}\}$ and $\{c g : c \in \mathbb{C}\}$ are unitarily equivalent, but they are never equal.*

5.4 Similarity of quasi-invariant subspaces on the complex n-space

Let X be a reproducing Hilbert space on the complex n-space $\mathbb{C}^n$. By Theorem 5.1.4, each finite codimensional quasi-invariant subspace M has the form

$$M = [I],$$

where I is a finite codimensional ideal with the same codimension as M.

The next theorem comes essentially from [GZ].

Theorem 5.4.1 *Let M, N be quasi-invariant subspaces of X on $\mathbb{C}^n$ $(n > 1)$, and let $M = [I]$ be finite codimensional. If there exists a quasi-module map $A : M \to N$, then $M \subset N$ and $A = \gamma$ for some constant γ.*

Proof. Since for any p, $q \in I$, we have that

$$A(p\,q) = pA(q) = qA(p),$$

and hence

$$\frac{A(p)}{p} = \frac{A(q)}{q}.$$

Thus we can define an analytic function on $\mathbb{C}^n \setminus Z(I)$ by

$$\phi(z) = \frac{A(p)(z)}{p(z)}$$

for any $p \in I$ with $p(z) \neq 0$. Clearly, ϕ is independent of p and is analytic on $\mathbb{C}^n \setminus Z(I)$. Since I is finite codimensional, $Z(I)$ is a finite set. By Hartogs' extension theorem, $\phi(z)$ extends to an analytic function on $\mathbb{C}^n$, that is, $\phi(z)$ is an entire function. It follows that

$$A(p) = \phi\, p$$

for any $p \in I$. Because I is dense in M, we conclude that

$$A(h) = \phi h$$

for any $h \in M$. Below we claim that $\phi h \in M$ if $h \in M$. From the proof of Lemma 5.1.3, there is a finite dimensional space R consisting of some polynomials such that

$$X = [I] \dot{+} R.$$

For $h \in [I]$, ϕh can be expressed as

$$\phi h = g_1 + g_2,$$

where $g_1 \in [I]$ and $g_2 \in R$. Note that for each $\lambda \in Z(I)$ and any $q \in I_\lambda$,

$$q(D)h|_\lambda = 0 \text{ and } q(D)g_1|_\lambda = 0.$$

Since I_λ is invariant under the action by the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, \cdots, \partial/\partial z_n\}$, it follows that

$$q(D)rh|_\lambda = 0$$

for any polynomial r. We choose polynomials $\{r_n\}$ such that r_n uniformly converge to ϕ on some bounded neighborhood $\mathcal{O}$ of λ, as $n \to \infty$. Thus we have that

$$0 = \lim_{n\to\infty} q(D)r_n h|_\lambda = q(D)\phi h|_\lambda.$$

This yields that

$$q(D)g_2|_\lambda = 0$$

for each $\lambda \in Z(I)$ and any $q \in I_\lambda$. By Corollary 2.1.2, $g_2 \in I$, and hence $g_2 = 0$. Thus, $\phi h \in [I]$. Consequently, $\phi h \in M$ if $h \in M$. Now by Proposition 5.0.1, ϕ is a constant γ, and hence $A = \gamma$ and $M \subset N$. This completes the proof of the theorem.

Corollary 5.4.2 *Let M, N be quasi-invariant subspaces of X on $\mathbb{C}^n$ $(n > 1)$, and let $M = [I]$ be finite codimensional. If M and N are quasi-similar, then $M = N$.*

Remark 5.4.3 *On the bidisk $\mathbb{D}^2$, it is easy to check that $L_a^2(\mathbb{D}^2)$ is similar to $z_1 L_a^2(\mathbb{D}^2)$. Hence Theorem 5.4.1 again points out the difference between the Bergman spaces and the Fock spaces.*

5.5 Quasi-invariant subspaces of the Fock space

In this section we will concentrate attention on the Fock space $L_a^2(\mathbb{C}^n)$ on $\mathbb{C}^n$. This section comes mainly from Guo's paper [Guo7]. It is well known that for each analytic Hilbert module X on a bounded domain Ω, the closure $[I]$ of an ideal I of the polynomial ring is always a submodule of X. However, it is never obvious if the closure $[I]$ of I in the norm of the Fock space $L_a^2(\mathbb{C}^n)$ is quasi-invariant.

Here we give a proposition which shows that the closure of a homogeneous ideal is quasi-invariant. Recall that an ideal I is homogeneous if the relation $p \in I$ implies that all homogeneous components of p are in I. Equivalently, an ideal I is homogeneous if and only if I is generated by homogeneous polynomials.

Proposition 5.5.1 *Let I be a homogeneous ideal. Then the closure $[I]$ of I in the Fock space is quasi-invariant.*

Proof. Let $f \in [I]$, and $f = \sum_{k=0}^{\infty} f_k$ be f's homogeneous expression. We claim that every f_k is in I. To prove the claim, we let $\mathcal{I}_k$ consist of all those $p \in I$ with homogeneous degree of p being at most k. Then $\mathcal{I}_k$ is of finite dimension. From the relation $f \in [I]$, there is a sequence $\{p_n\}$ in I such that $p_n \to f$ as $n \to \infty$. This implies that $p_n^{(k)} \to f_k$, where $p_n^{(k)}$ denote homogeneous component of degree k of p_n. Since I is homogeneous, $p_n^{(k)}$ belong to I, and hence they are in $\mathcal{I}_k$. Because $\mathcal{I}_k$ is finite dimensional, and hence closed, this forces $f_k \in I$.

Assume that $qf \in L_a^2(\mathbb{C}^n)$ for some polynomial q. Let $q = \sum_{i=0}^{m} q_i$ be the homogeneous expression of q. Then the homogeneous expression of qf is given by

$$qf = \sum_{n=0}^{\infty}\Big(\sum_{i+j=n} q_i f_j\Big).$$

Now it is easy to derive that $qf \in [I]$ by the above homogeneous expression of qf. It follows that $[I]$ is quasi-invariant.

Theorem 5.5.2 *Let I_1 and I_2 be homogeneous ideals. Then $[I_1]$ and $[I_2]$ are similar if and only if $I_1 = I_2$.*

To prove this theorem, we need some preliminaries. First we give a result of Hardy submodules on the unit ball $\mathbb{B}_n$.

Proposition 5.5.3 *If M_1 and M_2 are submodules of $H^2(\mathbb{B}_n)$ and $A: M_1 \to M_2$ is a module map, then there exists a bounded function ϕ on $\partial\mathbb{B}_n$ such that $A(h) = \phi h$ for any $h \in M_1$.*

Proof. From Rudin [Ru2], we see that all inner functions on the $\mathbb{B}_n$ and their adjoints generate $L^\infty(\partial\mathbb{B}_n)$ in the weak*-topology. Set

$$\mathcal{D} = \{\bar{\eta}h : \eta \text{ are inner functions, and } h \in M_1\}.$$

Then $\mathcal{D}$ is a dense linear subspace of $L^2(\partial\mathbb{B}_n)$. We define a map

$$\hat{A} : \mathcal{D} \to L^2(\partial\mathbb{B}_n)$$

by

$$\hat{A}(\bar{\eta}h) = \bar{\eta}A(h).$$

Since A is a module map, the above definition is well defined. From the relation

$$\|\hat{A}(\bar{\eta}h)\| = \|A(h)\| \leq \|A\|\|\bar{\eta}h\|,$$

$\hat{A}$ extends to a bounded map from $L^2(\partial\mathbb{B}_n)$ to $L^2(\partial\mathbb{B}_n)$, still denoted by $\hat{A}$. Then, obviously, $\hat{A}$ satisfies

$$\hat{A}M_g = M_g\hat{A}$$

for any $g \in L^\infty(\partial\mathbb{B}_n)$, and hence there exists a function $\phi \in L^\infty(\partial\mathbb{B}_n)$ such that

$$\hat{A} = M_\phi.$$

This ensures that $A(h) = \phi h$ for any $h \in M_1$.

Let $A(\mathbb{B}_n)$ be the ball algebra, that is, $A(\mathbb{B}_n)$ consists of all functions f, which are analytic on $\mathbb{B}_n$ and continuous on the closure $\overline{\mathbb{B}_n}$ of $\mathbb{B}_n$.

Lemma 5.5.4 *Assume $n > 1$. Let $f; g \in A(\mathbb{B}_n)$, and $|f(\xi)| \leq |g(\xi)|$ for each $\xi \in \partial\mathbb{B}_n$; then also $|f(z)| \leq |g(z)|$ for every $z \in \mathbb{B}_n$.*

Proof. See [Ru1, Theorem 14.3.3].

For a polynomial p, if p has its homogeneous expression $p = p_0 + p_1 + \cdots + p_l$ with $p_l \neq 0$, we say that p is of homogeneous degree l, and denoted by $deg_h\, p$.

Proposition 5.5.5 *Let both ϕ_1 and ϕ_2 be homogeneous polynomials, and $\deg_h \phi_1 \geq \deg_h \phi_2$. If there exists a constant γ such that on the Fock space $L^2_a(\mathbb{C}^n)$, $\|\phi_1\,\phi\| \leq \gamma\,\|\phi_2\,\phi\|$ for any homogeneous polynomial ϕ, then we have*

$$\phi_1 = c\,\phi_2$$

for some constant c.

Proof. In the case of $n = 1$, a straightforward verification shows that the conclusion is true. Below we may assume $n > 1$. First we recall that integration in polar coordinates (corresponding the volume measure) is given by [Ru1, p. 13]:

$$\int_{\mathbb{C}^n} f dv = \frac{2n\pi^n}{n!} \int_0^\infty r^{2n-1} dr \int_{\partial\mathbb{B}_n} f(r\xi) d\sigma(\xi).$$

Then by the inequality

$$\|\phi_1\,\phi\|^2 \leq \gamma^2\,\|\phi_2\,\phi\|^2,$$

one has that

$$\begin{aligned}
\|\phi_1\phi\|^2 &= \int_{\mathbb{C}^n} |\phi_1|^2|\phi|^2 e^{-\frac{|z|^2}{2}}\, dv \\
&= \frac{2n\pi^n}{n!}\int_0^\infty r^{2(n+k_1+l)-1} e^{-\frac{|r|^2}{2}}\, dr \int_{\partial\mathbb{B}_n} |\phi_1(\xi)\phi(\xi)|^2 d\sigma(\xi) \\
&\le \gamma^2 \int_{\mathbb{C}^n} |\phi_2|^2|\phi|^2 e^{-\frac{|z|^2}{2}}\, dv \\
&= \frac{2n\pi^n\gamma^2}{n!}\int_0^\infty r^{2(n+k_2+l)-1} e^{-\frac{|r|^2}{2}}\, dr \int_{\partial\mathbb{B}_n} |\phi_2(\xi)\phi(\xi)|^2 d\sigma(\xi),
\end{aligned}$$

where $k_1,\ k_2,\ l$ are the homogeneous degrees of homogeneous polynomials $\phi_1,\ \phi_2,\ \phi$, respectively. From the formula

$$\int_0^\infty r^{2m+1} e^{-\frac{|r|^2}{2}}\, dr = 2^m\, m!$$

and $k_1 \ge k_2$, we obtain that

$$\int_{\partial\mathbb{B}_n} |\phi_1(\xi)\phi(\xi)|^2 d\sigma(\xi) \le \gamma^2 \int_{\partial\mathbb{B}_n} |\phi_2(\xi)\phi(\xi)|^2 d\sigma(\xi).$$

Now let r be any polynomial with its homogeneous expression

$$r = r_0 + r_1 + \cdots r_t.$$

Then on the Hardy space $H^2(\mathbb{B}_n)$,

$$\phi_1\, r_i \perp \phi_1\, r_j\, ; \quad \phi_2\, r_i \perp \phi_2\, r_j$$

if $i \ne j$. This gives that

$$\begin{aligned}
\int_{\partial\mathbb{B}_n} |\phi_1(\xi)r(\xi)|^2 d\sigma(\xi) &= \int_{\partial\mathbb{B}_n} |\sum_{i=0}^t |\phi_1(\xi)r_i(\xi)|^2 d\sigma(\xi) \\
&= \sum_{i=0}^t \int_{\partial\mathbb{B}_n} |\phi_1(\xi)r_i(\xi)|^2 d\sigma(\xi) \\
&\le \gamma^2 \sum_{i=0}^t \int_{\partial\mathbb{B}_n} |\phi_2(\xi)r_i(\xi)|^2 d\sigma(\xi) \\
&= \gamma^2 \int_{\partial\mathbb{B}_n} |\phi_2(\xi) \sum_{i=0}^t r_i(\xi)|^2 d\sigma(\xi) \\
&= \gamma^2 \int_{\partial\mathbb{B}_n} |\phi_2(\xi)r(\xi)|^2 d\sigma(\xi).
\end{aligned}$$

Let $[\phi_1]_n$ and $[\phi_2]_n$ be submodules of $H^2(\mathbb{B}_n)$ generated by ϕ_1, ϕ_2, respectively. Then applying the preceding inequalities yields the following bounded module map:

$$B : [\phi_2]_n \to [\phi_1]_n, \quad B\phi_2 r = \phi_1 r.$$

By Proposition 5.5.3, there is a bounded function f on $\partial\mathbb{B}_n$ such that $B = M_f$. This implies that $f\phi_2 = \phi_1$ on $\partial\mathbb{B}_n$. Thus,

$$|\phi_1(\xi)| \leq \|f\|_\infty |\phi_2(\xi)|$$

for every $\xi \in \partial\mathbb{B}_n$. By Lemma 5.5.4,

$$|\phi_1(z)| \leq \|f\|_\infty |\phi_2(z)|$$

for each $z \in \mathbb{B}_n$. Set

$$\psi(z) = \phi_1(z)/\phi_2(z).$$

Then $\psi(z)$ is bounded on $\mathbb{B}_n$, and hence analytic. Since

$$\phi_1(z) = \psi(z)\,\phi_2(z)$$

and ϕ_1, ϕ_2 are homogeneous, the function $\psi(z)$ is a homogeneous polynomial.

Now by the inequality $\|\phi_1\phi\| \leq \gamma\,\|\phi_2\phi\|$ and the fact that on the Fock space,

$$\phi_1\,\psi_1 \perp \phi_1\,\psi_2; \quad \phi_2\,\psi_1 \perp \phi_2\,\psi_2,$$

for any homogeneous polynomials ψ_1, ψ_2 with $\deg_h\,\psi_1 \neq \deg_h\,\psi_2$, we have that

$$\|\phi_1 r\| \leq \gamma\|\phi_2 r\|$$

for each polynomial r, and hence for every entire function f with the property $\phi_2 f \in L^2_a(\mathbb{C}^n)$. This means that the relation $\phi_2 f \in L^2_a(\mathbb{C}^n)$ implies that $\phi_1 f$ belongs to $L^2_a(\mathbb{C}^n)$. It is not difficult to check that the quasi-invariant subspaces $[\phi_i]$, the closures of $\phi_i\mathcal{C}$ on the Fock space, are given by

$$[\phi_i] = \{\phi_i f \in L^2_a(\mathbb{C}^n) : f \in Hol(\mathbb{C}^n)\}$$

for $i = 1, 2$. By the equality $\phi_1 = \psi\phi_2$ and a simple application of the closed graph theorem, the multiplication operator $M_\psi : [\phi_2] \to [\phi_1]$ is bounded. Since $[\phi_1] \subset [\phi_2]$, Proposition 5.0.1 gives that ψ is a constant. This completes the proof.

Proof of Theorem 5.5.2. Let $A : [I_1] \to [I_2]$ be a similarity. For a homogeneous polynomial p in I_1, set $q = A(p)$. If $q = \sum_{i=0}^{\infty} q_i$ is the homogeneous expression of q, then

$$\|q\phi\|^2 = \sum_{i=0}^{\infty} \|q_i\phi\|^2 \leq \|A\|^2 \|p\phi\|^2$$

for every homogeneous polynomial ϕ. We thus have that

$$\|q_i\phi\|^2 \leq \|A\|^2\|p\phi\|^2$$

for $i \geq \deg_h p$. Applying Proposition 5.5.5 gives that $q_i = cp$ if $i = \deg_h p$, $q_i = 0$ if $i > \deg_h p$. By Lemma 5.1.1, $q = q_0 + q_1 + \cdots + q_{k-1} + cp$ is in I_2, where $k = \deg_h p$. First we claim that the constant c is never zero. In fact, since I_2 is homogeneous, this implies that q_i are in I_2 for $i = 1, 2, \cdots k-1$. Based on the same reasoning as above, one sees that $A^{-1}(q_0 + q_1 + \cdots + q_{k-1})$ is a polynomial, and its homogeneous degree is at most $k-1$. Thus,

$$p - A^{-1}(q_0 + q_1 + \cdots + q_{k-1}) \neq 0.$$

This ensures that $c \neq 0$. According to the claim, $I_1 \subset I_2$. The same reasoning shows that $I_2 \subset I_1$, and hence $I_1 = I_2$.

We note the following proposition. For a quasi-invariant subspace M, it is easy to see that $M \cap \mathcal{C}$ is an ideal.

Proposition 5.5.6 *Let $A : M_1 \to M_2$ be a quasi-module map. Then A maps $M_1 \cap \mathcal{C}$ to $M_2 \cap \mathcal{C}$.*

Proof. We may assume that M_1 contains a nonzero polynomial p. Set $q = A(p)$. We claim that $\deg_i q \leq \deg_i p$ for $i = 1, 2, \cdots, n$, where $\deg_i p$ denotes degree of p in the variable z_i. Suppose that there is some i, say, 1, such that $\deg_1 q > \deg_1 p$. Then we expand p and q in the variable z_1 by

$$p = p_0 + p_1 z_1 + \cdots + p_l z_1^l, \quad q = q_0 + q_1 z_1 + \cdots .$$

Since $\deg_1 q > \deg_1 p$, there exists a positive integer $s(> l)$ such that $q_s \neq 0$. From the equality

$$\|A(z_1^k p)\|^2 = \|z_1^k q\|^2 = \sum_{i=0}^{\infty} \|z_1^{k+i} q_i\|^2,$$

we have that

$$\|z_1^{k+s} q_s\|^2 \leq \|A\|^2 \sum_{i=0}^{l} \|z_1^{k+i} p_i\|^2.$$

Since

$$\|z_1^{k+s} q_s\|^2 = \|z_1^{k+s}\|^2 \|q_s\|^2 = 2^{k+s}(k+s)! \|q_s\|^2$$

and

$$\|z_1^{k+i} p_i\|^2 = 2^{k+i}(k+i)! \|p_i\|^2, \quad i = 0, 1, 2, \cdots l,$$

for any natural number k, we get that $q_s = 0$. This yields the desired contradiction, and hence Proposition 5.5.6 follows.

We endow the ring $\mathcal{C}$ with the topology induced by the Fock space $L_a^2(\mathbb{C}^n)$. For an ideal I, we regard I as module over the ring $\mathcal{C}$.

Corollary 5.5.7 *Let $A : M_1 \to M_2$ be a similarity. Then A induces a continuous module isomorphism from $M_1 \cap \mathcal{C}$ onto $M_2 \cap \mathcal{C}$.*

From Theorem 5.5.2, one sees that for homogeneous quasi-invariant subspaces, similarity only appears in the case of equality. Therefore, a natural problem is to determine the similarity orbit of quasi-invariant subspaces. Let M be a quasi-invariant subspace. Then the similarity orbit, $orb_s(M)$, of M consists of all quasi-invariant subspaces which are similar to M. There is no doubt that the problem is difficult. Here we will exhibit an example to show how to determine similarity orbits.

For a polynomial p, we let $[p]$ denote the closure of $p\mathcal{C}$ on the Fock space. Using sheaf theory or by Theorem 2.3.3, one can verify that for each $g \in [p]$, there exists an entire function f such that $g = p\,f$. Moreover, if p is homogeneous, then $[p]$ is quasi-invariant.

Theorem 5.5.8 *On the Fock space $L^2_a(\mathbb{C}^2)$, the similarity orbit $orb_s([z^n])$ of $[z^n]$ consists of $[p(z)]$, where $p(z)$ range over all polynomials in the variable z with* $\deg p = n$.

Proof. We first claim: if $(z+\alpha)f \in L^2_a(\mathbb{C}^2)$, then $zf; f \in L^2_a(\mathbb{C}^2)$, where α is a constant. Let

$$f = \sum_{n\geq 0} f_n(z)w^n$$

be the expansion of f relative to the variable w. Then

$$(z+\alpha)f = \sum_{n\geq 0}(z+\alpha)f_n(z)w^n.$$

By an easy verification, there exists positive constants C_α, C'_α such that

$$C_\alpha\|zg(z)\| \leq \|(z+\alpha)g(z)\| \leq C'_\alpha\|zg(z)\|,$$

for any entire function $g(z)$ on the complex plane $\mathbb{C}$. Since

$$\|(z+\alpha)f\|^2 = \sum_{n\geq 0}\|(z+\alpha)f_n(z)\|^2\|w^n\|^2 < \infty,$$

this gives that and hence $zf \in L^2_a(\mathbb{C}^2)$. Let $f = \sum_{n\geq 0}\phi_n(w)z^n$ be the expansion of f relative to the variable z. Then $zf = \sum_{n\geq 0}\phi_n(w)z^{n+1}$, and hence

$$\|zf\|^2 = \sum_{n\geq 0}\|\phi_n(w)\|^2\|z^{n+1}\|^2 \geq \sum_{n\geq 0}\|\phi_n(w)\|^2\|z^n\|^2 = \|f\|^2.$$

This means $f \in L^2_a(\mathbb{C}^n)$ and, therefore, the claim follows.

Combining the above claim with induction, we see that for a polynomial $p(z)$ in the variable z, the relation $pf \in L^2_a(\mathbb{C}^2)$ implies that $z^n f, z^{n-1}f, \cdots, f$ are in $L^2_a(\mathbb{C}^2)$, where $n = \deg p$.

Let $p(z)$ be a polynomial in the variable z with $\deg p = n$. Then we can establish an inequality

$$C_1\|z^n f\| \le \|p(z)f\| \le C_2\|z^n f\|,$$

where $C_1; C_2$ are positive constants only depending on $p(z)$.

In fact, Let

$$f = \sum_{n\ge 0} f_n(z)w^n$$

be the expansion of f relative to the variable w. Then

$$p(z)f = \sum_{n\ge 0} p(z)f_n(z)w^n.$$

It is easy to prove that there exists positive constants $C_1; C_2$, which depend only on $p(z)$ such that

$$C_1\|z^n g(z)\| \le \|p(z)g(z)\| \le C_2\|z^n g(z)\|$$

for any entire function $g(z)$ on the complex plane. Now by the equality

$$\|p(z)f\|^2 = \sum_{n\ge 0} \|p(z)f_n(z)\|^2\|w^n\|^2,$$

we thus have the inequality

$$C_1\|z^n f\| \le \|p(z)f\| \le C_2\|z^n f\|.$$

Note that the homogeneous quasi-invariant subspace $[z^n]$ is given by

$$[z^n] = \{z^n f \in L^2_a(\mathbb{C}^2) \,:\, f \text{ are entire functions}\}.$$

The above inequality implies that

$$[p(z)] = \{p(z)f \in L^2_a(\mathbb{C}^2) \,:\, f \text{ are entire functions}\},$$

and hence $[p(z)]$ is quasi-invariant.

Now we establish a map

$$A : [z^n] \to [p(z)], \quad z^n f \mapsto p(z)f.$$

Then by the preceding discussion and the closed graph theorem, A is continuous. Obviously, A is injective, surjective, and both A and A^{-1} are quasi-module maps. Thus, A is a similarity.

On the other hand, we let M be quasi-invariant and $A : [z^n] \to M$ be a similarity. Set $q = A(z^n)$. We claim that q is a polynomial in the variable z, and $\deg q = n$. To prove the claim, we expand q relative to the variables w, by

$$q = q_0(z) + wq_1(z) + w^2 q_2(z) \cdots .$$

Assume that $\deg_w q > \deg_w z^n = 0$, where $\deg_w q$ denotes degree of q in the variable w (allowed as ∞). Then there exists a positive integer s such that $q_s(z) \neq 0$. Since

$$\|A(w^k z^n)\|^2 = \|w^k q\|^2 = \sum_{i=0}^{\infty} \|w^{k+i} q_i(z)\|^2,$$

this implies that

$$\|w^{k+s} q_s(z)\|^2 \leq \|A\|^2 \|w^k z^n\|^2$$

for any positive integer k. Since

$$\|w^{k+s} q_s(z)\|^2 = 2^{k+s}\,(k+s)! \|q_s(z)\|^2 \quad \text{and} \quad \|w^k z^n\|^2 = 2^{k+n}\,k!\,n!$$

for any positive integer k, this deduces $q_s = 0$. This contradicts the assumption, and hence $\deg_w q = 0$. Thus, q depends only on the variable z. Now we expand q in the variable z by

$$q(z) = \sum_{m \geq 0} a_m z^m.$$

If there is a positive integer l, and $l > n$ such that $a_l \neq 0$, then the equality $A(z^k z^n) = \sum_{m \geq 0} a_m z^{k+m}$ implies that

$$|a_l|^2 \|z^{k+l}\|^2 \leq \|A\|^2 \|z^{k+n}\|^2.$$

This leads to the following:

$$2^{k+l}\,|a_l|^2\,(k+l)! \leq 2^{k+n}\,\|A\|^2\,(k+n)!$$

for any positive integer k. This clearly is impossible, and hence $q(z)$ is a polynomial in the variable z with $\deg q \leq n$.

Set $s = \deg q$. Then $M = [q]$ is similar to $[z^s]$ by the preceding discussion, and hence $[z^s]$ is similar to $[z^n]$. An easy verification shows $s = n$. This means that $q(z)$ is a polynomial in the variable z with degree n.

Based on the above discussion, we conclude that the similarity orbit $orb_s([z^n])$ of $[z^n]$ consists of $[p(z)]$, where $p(z)$ range over all polynomials in the variable z with $\deg p = n$.

Remark 5.5.9 *From the proof of Theorem 5.5.8, it is not difficult to see that Theorem 5.5.8 remains true in the case of the Fock space $L^2_a(\mathbb{C}^n)$ (all $n \geq 1$).*

Let $p = \sum_{i \leq m, j \leq n} a_{ij} z^i w^j$ be a polynomial in two variables. If $a_{mn} \neq 0$, we say that p has a nonzero leading term $a_{mn} z^m w^n$.

We leave the following proposition as an exercise.

Proposition 5.5.10 *If $p(z,w)$ has a nonzero leading term. Then $[p]$ is a quasi-invariant subspace of $L_a^2(\mathbb{C}^2)$. Furthermore, the similarity orbit of the quasi-invariant subspace $[z^m w^n]$ consists of all $[p(z,w)]$, where $p(z,w)$ have nonzero leading term $z^m w^n$.*

Proposition 5.5.10 has been generalized to a more general case. We refer the reader to reference [GH].

5.6 Finite codimensional quasi-invariant subspaces

As shown in Section 5.1, there is a simple algebraic reduction theorem for finite codimensional quasi-invariant subspaces. In this section we look at the structure of $M^{\perp}$ if M is finite codimensional quasi-invariant subspace of the Fock space.

Let M be finite codimensional quasi-invariant subspace of $L_a^2(\mathbb{C}^n)$. Then by Theorem 5.1.4, there is a unique finite codimensional ideal I such that $M=[I]$. Since I is of finite codimension, $Z(I)$ is a finite set, say, $Z(I)=\{\lambda_1,\lambda_2,\cdots,\lambda_l\}$. The ideal I can then be uniquely decomposed as

$$I=\bigcap_{k=1}^{l} I_k,$$

where each I_k is an ideal with a unique zero λ_k. Clearly,

$$[I]\subset \cap_{k=1}^{k}[I_k].$$

Note that $\cap_{k=1}^{k}[I_k]$ is quasi-invariant and is of finite codimension. From Lemmas 5.1.1 and 5.1.3,

$$\mathcal{C}\cap(\cap_{k=1}^{k}[I_k])=\cap_{k=1}^{l}I_k=I,$$

and I is dense in $\cap_{k=1}^{k}[I_k]$. Thus,

$$[I]=\cap_{k=1}^{k}[I_k].$$

The above reasoning shows that each finite codimensional quasi-invariant subspace M can be uniquely decomposed as

$$M=\cap_{k=1}^{l}M_k,$$

where each M_k is quasi-invariant and is of a unique zero point.

Because

$$\{z^\alpha/\|z^\alpha\| : \alpha \text{ ranging over all nonnegative indices}\}$$

is the orthonormal basis of $L_a^2(\mathbb{C}^n)$, this means that although for each polynomial p, the Toeplitz operator $T_{\bar{p}}$ is unbounded on $L_a^2(\mathbb{C}^n)$, $T_{\bar{p}}$ maps $\mathcal{C}$ to $\mathcal{C}$. Now let $\mathcal{P}$ be a linear space consisting of some polynomials. We say that $\mathcal{P}$ is an invariant polynomial space, if for any polynomial p, $\mathcal{P}$ is invariant under the action by $T_{\bar{p}}$.

Just as in the proof of Lemma 2.5.1, we have the following result.

Lemma 5.6.1 *Let M be a finite codimensional quasi-invariant subspace with a unique zero $\lambda = 0$. Then $M^{\perp}$ is a finite dimensional invariant polynomial space.*

Consider the transformations on $\mathbb{C}^n$, $\gamma_\lambda(z) = \lambda - z$. These maps determine unitary operators on $L^2(\mathbb{C}^n)$ given by

$$V_\lambda f = f \circ \gamma_\lambda \, k_\lambda,$$

where $k_\lambda(z)$ is the normalized reproducing kernels of the $L_a^2(\mathbb{C}^n)$, i.e.,

$$k_\lambda(z) = e^{\bar{\lambda} z/2 - |\lambda|^2/4}.$$

It is easy to verify that V_λ commute with the Fock projection P (here P is the orthogonal projection from $L^2(\mathbb{C}^n)$ onto $L_a^2(\mathbb{C}^n)$), and $V_\lambda^2 = I$.

Let M be a finite codimensional quasi-invariant subspace. Then M has finitely many zero points $\lambda_1, \lambda_2, \cdots, \lambda_l$ such that M can uniquely be represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where each M_i is quasi-invariant and has a unique zero λ_i.

The following theorem was contributed by Guo and Zheng [GZ].

Theorem 5.6.2 *Under the above assumption, there are finite dimensional invariant polynomial spaces $\mathcal{P}_i$ with* $\dim \mathcal{P}_i = \operatorname{codim} M_i$ *for* $i = 1, 2, \cdots, l$ *such that*

$$M^{\perp} = \sum_{i=1}^{l} \dot{+} \, \mathcal{P}_i \circ \gamma_{\lambda_i} \, k_{\lambda_i}.$$

Furthermore, we have

$$\operatorname{codim} M = \dim M^{\perp} = \sum_{i=1}^{l} \dim \mathcal{P}_i.$$

Proof. Let N be a quasi-invariant subspace of finite codimension with a unique zero point λ. We claim that $V_\lambda N = \{f \circ \gamma_\lambda \, k_\lambda : \ f \in N\}$ is quasi-invariant. In fact, if there is a polynomial q and some $f \in N$ such that $q(z) V_\lambda f(z) \in L_a^2(\mathbb{C}^n)$, then $q(z) f(\lambda - z) k_\lambda(z) \in L_a^2(\mathbb{C}^n)$. It is easy to see that

$q(z)$ can be written as $q(z) = p(\lambda - z) + c$ for some polynomial p and some constant c. Since

$$q(z)f(\lambda - z)k_\lambda(z) = V_\lambda((p + c)f)(z),$$

we have

$$(p + c)f = V_\lambda V_\lambda((p + c)f) \in L^2_a(\mathbb{C}^n).$$

Hence $pf \in L^2_a(\mathbb{C}^n)$. Since N is quasi-invariant, $pf \in N$. It follows that $(p + c)f \in N$, and hence

$$q(z)f(\lambda - z)k_\lambda(z) = V_\lambda((p + c)f)(z) \in V_\lambda N.$$

This ensures that $V_\lambda N$ is quasi-invariant. From the equality,

$$L^2_a(\mathbb{C}^n) = N \oplus N^\perp = V_\lambda N \oplus V_\lambda N^\perp,$$

$V_\lambda N$ is a finite codimensional quasi-invariant subspace, and $V_\lambda N$ is of the same codimension as N. Note that $V_\lambda N$ has only zero point 0. Thus from Lemma 5.6.1, $V_\lambda N^\perp$ is a finite dimensional invariant polynomial space. We denote $V_\lambda N^\perp$ by $\mathcal{P}$. Thus,

$$N^\perp = V_\lambda \mathcal{P} = \mathcal{P} \circ \gamma_\lambda\, k_\lambda.$$

Since $M = \cap_{i=1}^l M_i$, this gives that

$$M^\perp = \sum_{i=1}^{l} \dot{+}\, {M_i}^\perp.$$

Thus, there are finite dimensional invariant polynomial spaces $\mathcal{P}_i, i = 1, 2, \cdots, l$ such that

$$M^\perp = \sum_{i=1}^{l} \dot{+}\, \mathcal{P}_i \circ \gamma_{\lambda_i}\, k_{\lambda_i}.$$

The rest of the theorem comes from Corollary 2.2.6. This completes the proof of the theorem.

In the case of the Fock space $L^2_a(\mathbb{C})$, Theorem 5.6.2 has a simple corollary. To see this, we let π_n denote the linear space spanned by $\{1, z, ..., z^n\}$ for each nonnegative integer n. Then it is easy to verify that $\mathcal{P}$ is an invariant polynomial space in the variable z with dimension $n+1$ if and only if $\mathcal{P} = \pi_n$. Note that each finite codimensional quasi-invariant subspace M of $L^2_a(\mathbb{C})$ has its unique form $M = [p]$, where p is a polynomial with deg $p = \text{codim} M$. Let $p(z) = (z - \lambda_1)^{k_1+1}(z - \lambda_2)^{k_2+1} \cdots (z - \lambda_m)^{k_m+1}$ be the irreducible decomposition of p. Then we have the following.

Corollary 5.6.3 *Under the above assumption, it holds that*

$$[p]^\perp = \sum_{i=1}^{l} \dot{+}\, \pi_{k_i}\, e^{\bar{\lambda_i} z/2}.$$

5.7 Beurling's phenomenon on the Fock space

Beurling's classical theorem [Beu] says that each submodule M of the Hardy module $H^2(\mathbb{D})$ has the form $\eta\, H^2(\mathbb{D})$, where η is an inner function. Equivalently, the subspace $M \ominus zM = \mathbb{C}\eta$ is one dimensional, and it generates M, i.e., $M = [\eta] = \eta H^2(\mathbb{D})$. Beurling's theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. However, despite the great development in these fields over the past fifty years, it is only recently that progress has been made in proving the analogue for the Bergman module $L^2_a(\mathbb{D})$ on the unit disk. The analogue of Beurling's theorem on the Bergman module $L^2_a(\mathbb{D})$ was made by Aleman, Richter and Sundberg [ARS]. They proved that any submodule M of the Bergman module $L^2_a(\mathbb{D})$ also has the form $M = [M \ominus zM]$, that is, M, as a submodule of $L^2_a(\mathbb{D})$, is generated by $M \ominus zM$. However, unlike the Bergman space (or the Hardy space), an analogue of Beurling's theorem fails in general for the Fock space. The next example shows that there are infinitely many quasi-invariant subspaces for which Beurling's theorem fails.

Example 5.7.1 *Let $\alpha \neq 0$, and let $[z-\alpha]$ be the quasi-invariant subspace generated by $z-\alpha$. Then $[z-\alpha] \ominus [z(z-\alpha)]$ is not a generating set of $[z-\alpha]$.*

In fact, it is easy to check that the subspace $[z-\alpha] \ominus [z(z-\alpha)]$ is one dimensional, and the function $f_\alpha(z) = e^{|\alpha|^2/2} - e^{\bar{\alpha} z/2}$ is in it. Since the function $f_\alpha(z)$ has infinitely many zero points $\{\alpha + 4n\pi i/\bar{\alpha} : n \text{ range over all integer}\}$ in $\mathbb{C}$, but $z-\alpha$ has a unique zero point α, this implies that $f_\alpha(z)$ does not generate the quasi-invariant subspace $[z-\alpha]$.

More generally, we have the following.

Theorem 5.7.2 *Let $p(z)$ be a polynomial with degree m. Then for the Fock space $L^2_a(\mathbb{C})$, $[p] \ominus [zp]$ is a generating set of $[p]$ if and only if there is a constant γ such that $p(z) = \gamma\, z^m$.*

This theorem was proved in [CH]. This section will give a more simple proof of the theorem which is distinct from [CH]. For the theorem, we need two lemmas.

Lemma 5.7.3 *Let $q_1(z), \cdots, q_l(z)$ be polynomials. If $f(z) = \sum_{k=1}^{l} q_k(z)\, e^{\mu_k z}$ has finitely many zeros on the complex plane $\mathbb{C}$, say, $\lambda_1, \lambda_2, \cdots, \lambda_m$ (counting multiplicities), then there are constants a, b such that*

$$f(z) = a\,(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)e^{bz}.$$

Proof. Set

$$r(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m).$$

Then the function $f(z)/r(z)$ is holomorphic on the complex plane and it has no zero on $\mathbb{C}$. It follows that there exists an entire function $g(z)$ such that $f(z) = r(z)e^{g(z)}$. Since

$$|f(z)| \leq \sum_{k=1}^{l} |q_k(z)| e^{|\mu_k||z|},$$

there exists a positive constant R_0 such that

$$|f(z)| \leq e^{(\mu+1)|z|}$$

if $|z| \geq R_0$ (here $\mu = \max\{|\mu_1|, \cdots, |\mu_l|\}$), and hence there exists a large positive constant R such that

$$|e^{g(z)}| \leq e^{(\mu+1)|z|}$$

if $|z| \geq R$. Thus, we can find a positive constant d such that

$$|e^{g(z)}| \leq e^{(\mu+1)|z|+d}$$

for any $z \in \mathbb{C}$. Write $g(z) = g(0) + g'(0)z + z^2 h(z)$, where $h(z)$ is an entire function. Then there are positive constants $d_1 > 0$ and $d_2 > 0$ such that

$$|e^{z^2 h(z)}| \leq e^{d_1|z|+d_2}$$

for each $z \in \mathbb{C}$. This gives $Re(z^2 h(z)) \leq d_1|z| + d_2$, and hence

$$[Re(z^2 h(z))]^2 + [Im(z^2 h(z))]^2 \leq [2(d_1 r + d_2) - Re(z^2 h(z))]^2 + [Im(z^2 h(z))]^2$$

if $|z| \leq r$, that is,

$$|z^2 h(z)| \leq |2(d_1 r + d_2) - z^2 h(z)|$$

if $|z| \leq r$. The function

$$h_r(z) = r^2 h(z)/[2(d_1 r + d_2) - z^2 h(z)]$$

is holomorphic on $\{z : |z| \leq 2r\}$, and $|h_r(z)| \leq 1$ if $|z| = r$. Applying the maximum modulus theorem yields

$$|h_r(z)| \leq 1$$

if $|z| \leq r$. Fixing z and letting $r \to \infty$ give $h(z) = 0$. It follows that there exist constant a, b such that

$$f(z) = a\,(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)e^{bz},$$

completing the proof.

By induction, the following lemma is easily proved.

Lemma 5.7.4 *Let $\mu_i \neq \mu_j$ $(i \neq j)$, and let $q_1(z), \cdots, q_l(z)$ be polynomials. Then $\sum_{k=1}^{l} q_k(z)\, e^{\mu_k z} = 0$ only if $q_1(z) = q_2(z) = \cdots = q_l(z) = 0$.*

The proof of Theorem 5.7.2. Note that finite codimensional quasi-invariant subspaces $[p]$ and $[zp]$ have codimension m and $m+1$, respectively, and hence the subspace $[p] \ominus [zp]$ is one dimensional. Take a nonzero ϕ in $[p] \ominus [zp]$. If $[p] \ominus [zp]$ is a generating set of $[p]$, then ϕ generates $[p]$, and therefore ϕ can be decomposed as $\phi(z) = p(z)\psi(z)$ such that $\psi(z)$ has no zero on the complex plane $\mathbb{C}$. Since $\phi \in [zp]^{\perp}$, applying Corollary 5.6.3 and Lemma 5.7.3 gives

$$\phi(z) = a\, p(z) e^{bz},$$

where a, b are constants and $a \neq 0$. Combining Corollary 5.6.3, Lemma 5.7.4 and the equality $[p] \ominus [zp] = [zp]^{\perp} \ominus [p]^{\perp}$, we obtain $b = 0$. Since $[zp]^{\perp}$ is invariant for $T_{\bar{z}}$, and $p \in [zp]^{\perp}$, we deduce $z^k \in [zp]^{\perp}$ for $k = 0, 1, \cdots, m$. It follows that there exists a constant γ such that $p(z) = \gamma\, z^m$ because $\dim [zp]^{\perp} = \deg p + 1 = m + 1$. This completes the proof.

5.8 Remarks on Chapter 5

The concept of quasi-invariant subspace was first introduced by Guo and Zheng [GZ]. Proposition 5.0.1 and results in Sections 5.1, 5.4 and 5.6 were basically contributed by Guo and Zheng [GZ]. Sections 5.2 and 5.3 are mainly based on the paper [CGH] by Chen, Guo and Hou. Section 5.5 comes basically from Guo's paper [Guo7]. Proposition 5.5.10 appeared in [GH]. The Beurling problem for the Fock space was raised by Zheng. Theorem 5.7.2 was proved by Chen and Hou [CH]. The present proof was given by Guo and it is distinct from [CH]. We also mention a work [GY] by Guo and Yan, where they studied reproducing Hilbert spaces with $\mathcal{U}$-invariant kernels on the complex plane. Concerning reproducing kernels, a general theory is presented in [Ar].

Chapter 6

The Arveson module

To generalize the operator-theoretic aspects of function theory on the unit disk to multivariable operator theory, Arveson investigated a new function space H_d^2 on the unit ball $\mathbb{B}_d$ in the d-dimensional complex space $\mathbb{C}^d$ (cf. [Arv1, Arv2, Arv3, Arv6]) (for convenience, we call H_d^2 the Arveson space). Indeed, the Arveson space plays an important role in the multi-variable operator theory as shown by Arveson [Arv1, Arv2, Arv3]. Recall that the Arveson space H_d^2 on the unit ball $\mathbb{B}_d$ is defined by the reproducing kernel $K_\lambda(z) = 1/(1 - \langle z, \lambda \rangle)$, where $\langle z, \lambda \rangle = \sum_{j=1}^d z_j \bar{\lambda}_j$, and it is easy to verify that the space is invariant under multiplication by polynomials. Therefore, we will regard the Arveson space H_d^2 as a module over the polynomial ring $\mathcal{C}$. Unlike Hardy modules and Bergman modules, the Arveson module $H_d^2 (d \geq 2)$ is never associated with some measure on $\mathbb{C}^d$ as shown by [Arv1]. Therefore, the Arveson module enjoys many properties distinct from those of the Hardy module and the Bergman module.

6.1 Some basic results on the Arveson module

First we give the following proposition.

Proposition 6.1.1 *The Arveson space has a canonical orthonormal basis*

$$\{ \, [\frac{(j_1 + j_2 + \cdots + j_d)!}{j_1! j_2! \cdots j_d!}]^{1/2} z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d} \, \},$$

where $J = (j_1, j_2, \cdots, j_d)$ *run over all multi-indexes of nonnegative integers.*

Proof. Let H be the Hilbert space with an orthonormal basis given by this proposition. For convenience, we set $e_J = [\frac{(j_1+\cdots+j_d)!}{j_1!\cdots j_d!}]^{1/2} z_1^{j_1} \cdots z_d^{j_d}$. Then for any $f \in H$, f has the expression $f = \sum_J a_J e_J$ with $\sum_J |a_J|^2 < \infty$. For any $\lambda \in \mathbb{B}_d$, the evaluation $E_\lambda(f)$ of f at λ satisfies

$$
\begin{aligned}
|E_\lambda(f)| &= |\sum_J a_J e_J(\lambda)| \le (\sum_J |a_J|^2)^{1/2} (\sum_J |e_J(\lambda)|^2)^{1/2} \\
&= (\sum_J |a_J|^2)^{1/2} / (1-|\lambda|^2)^{1/2} = \|f\| / (1-|\lambda|^2)^{1/2}.
\end{aligned}
$$

This means that E_λ is continuous; we thus think of E_λ as an element in H. Then it is easy to see that $E_\lambda = \sum_J \overline{e_J(\lambda)} e_J$. Furthermore, if $E_\lambda(f) = 0$ for each $\lambda \in \mathbb{B}_d$, then $f = 0$. Indeed, from the following expression

$$
\frac{\partial^{j_1+\cdots+j_d} f}{\partial {\lambda_1}^{j_1} \cdots \partial {\lambda_d}^{j_d}} |_{\lambda=0} = 0,
$$

the equality $a_J = 0$ is easily deduced, and hence $f = 0$. This shows that the set of all E_λ is dense in H. Notice that for any $\lambda, \mu \in \mathbb{B}_d$, we have

$$
\langle E_\lambda, E_\mu \rangle_H = \langle K_\lambda, K_\mu \rangle_{H_d^2} = 1/(1 - \langle \mu, \lambda \rangle).
$$

The proposition follows.

By a multiplier of H_d^2 we mean a complex-valued function f on $\mathbb{B}_d$ with the property $f\, H_d^2 \subset H_d^2$. It is easy to see that a multiplication operator M_f on H_d^2 defined by a multiplier f is bounded. From Proposition 6.1.1, for any polynomial p, $pH_d^2 \subset H_d^2$. Therefore, we will regard the Arveson space H_d^2 as a module over the polynomial ring $\mathcal{C}$.

In fact, the Arveson space is an analytic Hilbert module on the unit ball $\mathbb{B}_d$ (for the definition, see Section 2.2 of Chapter 2). For this assertion it is enough to show that each point $\mu, |\mu| \ge 1$ is not a virtual point of H_d^2, where $|\mu| = (\sum_{i=1}^d |\mu_i|^2)^{1/2}$. Suppose that there is a virtual point μ such that $|\mu| \ge 1$. Then there exists a constant C_u such that for any $\lambda, |\lambda| < 1/|\mu|$ we have

$$
|\frac{1}{1 - \langle \lambda, \mu \rangle}|^2 \le C_u \|K_\lambda\|^2 = \frac{C_u}{1 - |\lambda|^2}.
$$

Taking $\lambda = t\mu$ with $0 \le t < 1/|\mu|^2$ we then obtain

$$
\frac{1 - t^2|\mu|^2}{(1 - t|\mu|^2)^2} \le C_u.
$$

When $|\mu| = 1$, the left side of the above inequality tends to $+\infty$ as $t \to 1-$. Similarly, when $|\mu| > 1$, the left side of the inequality tends to $+\infty$ as $t \to 1/|\mu|^2-$. This contradiction shows that μ is not the virtual point of H_d^2 if $|\mu| \ge 1$. Therefore, the Arveson space is an analytic Hilbert module on the unit ball.

Before going on we give the following proposition (see [Arv1]).

Proposition 6.1.2 *Let M_{z_i} be the multiplications given by the coordinate multipliers z_i for $i = 1, 2, \cdots, d$. Then*

$$M_{z_1}M_{z_1}^* + \cdots + M_{z_d}M_{z_d}^* = I - 1 \otimes 1.$$

Proof. Since H_d^2 is spanned by $\{K_\lambda : \lambda \in \mathbb{B}_d\}$, it is enough to show that for all $\lambda, \mu \in \mathbb{B}_d$ we have

$$\langle M_{z_1}M_{z_1}^* K_\lambda, K_\mu\rangle + \cdots + \langle M_{z_d}M_{z_d}^* K_\lambda, K_\mu\rangle = \langle K_\lambda, K_\mu\rangle - \langle (1 \otimes 1)K_\lambda, K_\mu\rangle.$$

Since

$$M_{z_i}^* K_\lambda = \overline{\lambda_i} K_\lambda,\ i = 1, 2, \cdots, d$$

and

$$\langle (1 \otimes 1)K_\lambda, K_\mu\rangle = \langle K_\lambda, 1\rangle\langle 1, K_\mu\rangle = 1,$$

a simple computation gives the desired conclusion.

Arveson [Arv1] called the d-duple of operators $(M_{z_1}, \cdots, M_{z_d})$ the d-shift. For convenience, we will write $(S_1, \cdots, S_d)$ for the d-shift $(M_{z_1}, \cdots, M_{z_d})$. The d-shift plays the important role in the theory of d-contraction (cf. [Arv1]). A commuting d-tuple $\bar{T} = (T_1, \cdots, T_d)$ acting on a common Hilbert space H is called a d-contraction if the following condition is satisfied:

$$\|T_1\xi_1 + \cdots + T_d\xi_d\|^2 \leq \|\xi_1\|^2 + \cdots + \|\xi_d\|^2.$$

It is easy to verify that a commuting d-tuple $\bar{T}$ is a d-contraction if and only if

$$T_1T_1^* + \cdots + T_dT_d^* \leq 1.$$

To see this we let $d \cdot H$ denote the direct sum of d copies of H, and let $\bar{T} \in B(d \cdot H, H)$ be the operator defined by $\bar{T}(\xi_1, \cdots, \xi_d) = T_1\xi_1 + \cdots + T_d\xi_d$. Then the adjoint of $\bar{T}$, $\bar{T}^* : H \to d \cdot H$ is given by

$$\bar{T}^*\xi = (T_1^*\xi_1, \cdots, T_d^*\xi_d).$$

This implies that

$$\bar{T}\bar{T}^* = T_1T_1^* + \cdots T_dT_d^*,$$

and hence the equivalence follows.

A simple verification shows that the d-tuples $(M_{z_1}, \cdots, M_{z_d})$ of coordinate multipliers on both the Hardy module and the Bergman module of the unit ball are d-contractions. The next theorem will show that the Arveson module H_d^2 is distinguished among all analytic Hilbert modules on the unit ball by being the largest Hilbert norm. As a consequence, the Arveson module is contained in every other analytic Hilbert module on the unit ball that has some natural properties.

The following theorem is due to Arveson [Arv1].

Theorem 6.1.3 *Let H be an analytic Hilbert module on the unit ball $\mathbb{B}_d$ such that $1 \perp H_0$, where $H_0 = \{f \in H : f(0) = 0\}$. If the d-tuple of the coordinate multipliers $\bar{M} = (M_{z_1}, \cdots, M_{z_d})$ is a d-contraction, then for each polynomial p we have*

$$\|p\| \leq \|1\| \, \|p\|_{H_d^2}.$$

The theorem shows that the H_d^2 norm is the largest Hilbert norm among all Hilbert modules on $\mathbb{B}_d$ which have the properties desired as the theorem. Furthermore, applying the closed graph theorem gives that if an analytic Hilbert module H on the $\mathbb{B}_d$ has the properties desired as the theorem, then H contains H_d^2, and the inclusion map of H_d^2 to H is a bounded operator.

For the proof of Theorem 6.1.3, let us first recall some operator theoretic results which are important in themselves. Each d-contraction $\bar{T} = (T_1, \cdots, T_d)$ in $\mathcal{B}(H)$ gives a completely positive map P on $\mathcal{B}(H)$ by way of

$$P(A) = T_1 A T_1^* + \cdots + T_d A T_d^*, \quad A \in \mathcal{B}(H).$$

(We refer the interested reader to an appendix at the end of this section for the definition and some basic properties of completely positive maps.) Notice that since $P(I) = T_1 T_1^* + \cdots + T_d T_d^* \leq I$ we have

$$\|P\| = \|P(I)\| \leq 1,$$

and hence the sequence of positive operators

$$I \geq P(I) \geq P^2(I) \geq \cdots$$

converges strongly to a positive operator $T_\infty = \lim P^n(I)$. Obviously,

$$0 \leq T_\infty \leq I.$$

A d-contraction is called null if $T_\infty = 0$. For a d-contraction $\bar{T} = (T_1, \cdots, T_d)$, its defect operator Δ is defined as

$$\Delta = (I - T_1 T_1^* - \cdots - T_d T_d^*)^{1/2}.$$

The next dilation theorem, due to Arveson [Arv1], will serve us to prove theorem 6.1.3.

Theorem 6.1.4 *Let $\bar{T} = (T_1, \cdots, T_d)$ be a d-contraction on a Hilbert space H. Then there is a unique bounded linear operator $L : H_d^2 \otimes \overline{\Delta H} \to H$ satisfying*

$$L(p \otimes \xi) = p(\bar{T}) \, \Delta\xi,$$

and we have

$$LL^* = I - T_\infty.$$

In particular, the operator L is a contraction.

Proof. For each $\eta \in H$, define $A\eta$ as a sequence of vectors $(\xi_0, \xi_1 \cdots)$ where

$$\xi_n = \sum_{j_1+\cdots+j_d=n} \frac{n!}{j_1!\cdots j_d!} z_1^{j_1} \cdots j_d^{j_d} \otimes \Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta.$$

Then we have

$$\begin{aligned}
\|\xi_n\|^2 &= \sum_{j_1+\cdots+j_d=n} [\frac{n!}{j_1!\cdots j_d!}]^2 \|z_1^{j_1} \cdots j_d^{j_d}\|^2 \|\Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta\|^2 \\
&= \sum_{j_1+\cdots+j_d=n} \frac{n!}{j_1!\cdots j_d!} \|\Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta\|^2 \\
&= \sum_{j_1+\cdots+j_d=n} \frac{n!}{j_1!\cdots j_d!} \langle T_1^{j_1} \cdots T_d^{j_d} \Delta^2 T_1^{j_1 *} \cdots T_d^{j_d *} \eta, \eta\rangle \\
&= \langle P^n(\Delta^2)\eta, \eta\rangle \\
&= \langle (P^n(I) - P^{n+1}(I))\eta, \eta\rangle.
\end{aligned}$$

Therefore we obtain

$$\|A\eta\|^2 = \sum_{n=0}^{\infty} \|\xi_n\|^2 = \|\eta\|^2 - \langle T_\infty \eta, \eta\rangle.$$

Now the linear operator $A : H \to H_d^2 \otimes \overline{\Delta H}$ given by the above reasoning is well defined and bounded. Notice that for any $\xi \in \overline{\Delta H}$ and $\eta \in H$,

$$\begin{aligned}
&\langle z_1^{j_1} \cdots z_d^{j_d} \otimes \xi, A\eta\rangle \\
&= \frac{(j_1 + \cdots + j_d)!}{j_1!\cdots j_d!} \langle z_1^{j_1} \cdots z_d^{j_d} \otimes \xi, z_1^{j_1} \cdots z_d^{j_d} \otimes \Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta\rangle \\
&= \frac{(j_1 + \cdots + j_d)!}{j_1!\cdots j_d!} \|z_1^{j_1} \cdots z_d^{j_d}\|^2 \langle \xi, \Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta\rangle \\
&= \langle \xi, \Delta T_1^{j_1 *} \cdots T_d^{j_d *} \eta\rangle = \langle T_1^{j_1} \cdots T_d^{j_d} \Delta \xi, \eta\rangle \\
&= \langle L(z_1^{j_1} \cdots z_d^{j_d} \otimes \xi), \eta\rangle.
\end{aligned}$$

From the above reasoning we see that the desired operator L is given by A^*, and it satisfies

$$LL^* = A^* A = I - T_\infty.$$

Obviously, such an operator L is unique, completing the proof.

The proof of Theorem 6.1.3. Let $\Delta = (I - M_{z_1} M_{z_1}^* - \cdots - M_{z_d} M_{z_d}^*)^{1/2}$ be the defect operator for the d-tuple of the coordinate multipliers $(M_{z_1}, \cdots, M_{z_d})$. We claim that $\Delta 1 = 1$. In fact, since $1 \perp H_0$, we obtain that $M_{z_k}^* 1 = 0$ for $k = 1, 2, \cdots, d$. It follows that

$$\|\Delta 1\|^2 = \langle \Delta^2 1, 1\rangle = \|1\|^2,$$

and hence $\Delta 1 = 1$ because $0 \leq \Delta \leq I$. In particular, $1 = \Delta 1 \in \overline{\Delta H}$. Applying Theorem 6.1.4 we see that $L(p \otimes 1) = p$ for any polynomial p. Since L is a contraction we have

$$\|p\| \leq \|1\| \, \|p\|_{H_d^2}$$

for any polynomial p, completing the proof.

Another application of Theorem 6.1.4 is that an appropriate version of Von Neumann's inequality for d-contraction can been deduced from it. Recall the following classical von Neumann's inequality satisfied by any contraction T on a Hilbert space H.

Proposition 6.1.5 (Von Neumann's inequality) *For any contraction T (i.e., $\|T\| \leq 1$) on a Hilbert space H and for any polynomial p in one variable, we have*

$$\|p(T)\| \leq \|p\|_\infty,$$

where $\|p\|_\infty$ is defined by $\|p\|_\infty = \sup_{z \in \mathbb{D}} |p(z)|$.

We refer the interested reader to [Pi1] for many different proofs of the von Neumann's inequality. Perhaps most natural generalization of von Neumann's inequality for d-contraction would make the following assertion. Let $\bar{T} = (T_1, \cdots, T_d)$ be a d-contraction, and let p be a polynomial in d-complex variables. Then

$$\|p(T_1, \cdots, T_d)\| \leq \|p\|_\infty \overset{\triangle}{=} \sup_{z \in \mathbb{B}_d} |p(z)|.$$

The next example shows that this inequality fails for d-shift, in that there is no constant K for which

$$\|p(S_1, , \cdots, S_d)\| \leq K \sup_{z \in \mathbb{B}_d} |p(z)|,$$

for any polynomial p.

Example 6.1.6 *This example comes from [Arv1]. Assume $d \geq 2$. Let $\{c_0, c_1, \cdots\}$ be a sequence of complex numbers with properties*

$$\sum_n |c_n| = 1, \quad \sum_n |c_n|^2 n^{(d-1)/2} = \infty.$$

The existence of such a sequence is obvious; for example, take $c_n = c/n^{(d-1)/4}$ if $n = m^8$ for some positive integer m and $c_n = 0$ otherwise, where $c = 1/\sum_{m=1}^{\infty} \frac{1}{m^{2(d-1)}}$. We define a sequence of polynomials, $p_0, p_1, \cdots$, as follows:

$$p_N = \sum_{n=0}^{N} \frac{c_n}{s^n} (z_1 \cdots z_d)^n,$$

where $s = \sup_{z \in \mathbb{B}_d} |z_1 \cdots z_d| = d^{-d/2}$. Then we have $\|p_N\|_\infty \leq 1$ for every N, and the sequence $\{p_N\}$ converges uniformly on the closed unit ball to a *function f satisfying $\|f\|_\infty \leq 1$; but $f \notin H_d^2$, where $f = \sum_{n=0}^{\infty} \frac{c_n}{s^n}(z_1 \cdots z_d)^n$. Furthermore, we have*

$$\lim_{N\to\infty} \|p_N(S_1, \cdots, S_d)\| = \infty.$$

In fact, notice that

$$\|p_N\|_\infty \leq \sum_{n=0}^{N} \frac{|c_n|}{s^n} \|(z_1 \cdots z_d)^n\|_\infty = \sum_{n=0}^{N} |c_n|,$$

and hence by the assumption we obtain that $\|p_N\|_\infty \leq 1$, and the sequence $\{p_N\}$ converges uniformly on the closed unit ball to a function f satisfying $\|f\|_\infty \leq 1$. By Stirling's formula $n! \backsim \sqrt{2\pi} n^{n+1/2} e^{-n}$ and Proposition 6.1.1, there exists a positive constant C such that

$$\|(z_1 \cdots z_d)^n\| \geq C\, d^{-dn} n^{(d-1)/2}, \quad n = 1, 2, \cdots.$$

This implies that

$$\|p_N\|^2 = \sum_{n=0}^{N} \frac{|c_n|^2}{s^{2n}} \|(z_1 \cdots z_d)^n\|^2 \geq C \sum_{n=0}^{N} |c_n|^2 n^{(d-1)/2} \to \infty$$

as $N \to \infty$. It follows that $f \notin H_d^2$. Since

$$\|p_N(S_1, \cdots, S_d)\| \geq \|p_N(S_1, \cdots, S_d)1\| = \|p_N\|,$$

this gives

$$\lim_{N\to\infty} \|p_N(S_1, \cdots, S_d)\| = \infty.$$

However, using Theorem 6.1.4, Arveson obtained an appropriate version of von Neumann's inequality for d-contraction as follows (cf. [Arv1]).

Theorem 6.1.7 *Let $\bar{T} = (T_1, \cdots, T_d)$ be a d-contraction on a Hilbert space H. Then for every polynomial p in d-complex variables we have*

$$\|p(T_1, \cdots, T_d)\| \leq \|p(S_1, \cdots, S_d)\|.$$

Proof. Let $\Delta = (I - T_1T_1^* - \cdots - T_dT_d^*)^{1/2}$ be the defect operator for d-tuple $\bar{T}$, and let $K = \overline{\Delta H}$. By Theorem 6.1.4, we see that for any polynomial p,

$$L\, p(S_1, \cdots, S_d) \otimes I_K = p(T_1, \cdots, T_d) L.$$

We first show that for each null d-contraction $\bar{T}$, the theorem is true. Indeed, if $\bar{T}$ is a null d-contraction, then Theorem 6.1.4 implies that L is a coisometry, that is, $LL^* = I$. In this case, we have

$$L\, p(S_1, \cdots, S_d) \otimes I_K\, L^* = p(T_1, \cdots, T_d),$$

and hence

$$\|p(T_1, \cdots, T_d)\| \le \|p(S_1, \cdots, S_d)\|.$$

The general case is deduced from this by simple reasoning. Let $\bar{T}$ be any d-contraction. For $0 < r < 1$, set $\bar{T}_r = (rT_1, \cdots, rT_d)$. Then $\bar{T}_r$ is a null d-contraction. Applying the preceding conclusion to the tuple $\bar{T}_r$ gives

$$\|p(rT_1, \cdots, rT_d)\| \le \|p(S_1, \cdots, S_d)\|.$$

Since

$$\|p(T_1, \cdots, T_d)\| = \lim_{r \to 1} \|p(rT_1, \cdots, rT_d)\|,$$

the desired conclusion follows.

The two most common analytic Hilbert modules on the unit ball are the Hardy module $H^2(\mathbb{B}_d)$ and the Bergman module $L_a^2(\mathbb{B}_d)$. Their multiplier algebras are all $H^\infty(\mathbb{B}_d)$. However, Example 6.1.6 shows that there exist functions in the ball algebras $A(\mathbb{B}_d)$ which are not multipliers of H_d^2. Furthermore, the following is deduced from Example 6.1.6. (cf. [Arv1]).

Proposition 6.1.8 *There is no positive measure μ on $\mathbb{C}^d$, $d \ge 2$, such that on the Arveson module H_d^2*

$$\|f\|^2 = \int_{\mathbb{C}^d} |f(z)|^2 d\mu$$

for all polynomial f.

Proof. Suppose that such a measure μ did exist; then μ must be a probability measure because $\|1\| = 1$. For any $e = (e_1, \cdots, e_d) \in \partial\mathbb{B}_d$, set $f_e = \overline{e_1} z_1 + \cdots + \overline{e_d} z_d$. Then by Proposition 6.1.1, $\|f_e^n\| = 1$ for each natural number n. Therefore

$$\int_{\mathbb{C}^d} |f_e(z)|^{2n} d\mu = 1.$$

Let X be the closed support of the measure μ. We find that

$$\sup_{z \in X} |f_e(z)| = \lim_{n \to \infty} \left(\int_{\mathbb{C}^d} |f_e(z)|^{2n} d\mu\right)^{1/2n} = 1.$$

This proves that for each $z \in X$ and e in the unit sphere of $\mathbb{B}_d$ we have

$$|\langle z, e \rangle| \le 1,$$

and it follows that X must be contained in the closed unit ball of $\mathbb{C}^d$. Since the polynomial ring is dense in the ball algebra $A(\mathbb{B}_d)$ in the sup norm, this forces $A(\mathbb{B}_d) \subset H_d^2$. However, Example 6.1.6 shows that this is impossible, and hence the proposition follows.

As a consequence of Proposition 6.1.8, we will see that the d-shift is not a subnormal d-tuple. More earlier examples for non-subnormal commuting

d-tuple appeared in [Lu]. Recall that a d-tuple of commuting operators $\bar{T} = (T_1, \cdots, T_d)$ on a Hilbert space H is said to be subnormal if there is a commuting d-tuple of normal operators $\bar{N} = (N_1, \cdots, N_d)$ on a larger Hilbert space $K \supset H$ such that $T_k = N_k|_H$ for $k = 1, 2, \cdots, d$.

The following corollary appeared in [Arv1]; here the proof is similar to that of [CSa, Theorem 2.1], which is different from the proof in [Arv1]. We also point out a recent work of Arazy and Zhang [AZ], which gives an exact condition for the d-tuple being non-subnormal concerning bounded symmetric domains.

Corollary 6.1.9 *For each $d \geq 2$ the d-shift is not subnormal.*

Proof. Suppose that the d-shift is subnormal. Then there is a normal extension $\bar{N}$ for the d-shift such that 1 is a cyclic vector for the commuting C^*-algebra $C^*(\bar{N})$ generated by I and $N_1, \cdots, N_d$. Notice that there is a unique $*$-isomorphism $\tau : C^*(\bar{N}) \to C(\sigma(\bar{N})), \tau(N_i) = z_i$ for $i = 1, \cdots, d$, where $\sigma(\bar{N})$ is the Taylor spectrum for the d-tuple $\bar{N} = (N_1, \cdots, N_d)$. A functional $\phi : C(\sigma(\bar{N})) \to \mathbb{C}$ is given by

$$\phi(f) = \langle f(\bar{N})1, 1\rangle.$$

It is easy to see that ϕ is positive and $\phi(1) = 1$, hence there is a probability measure μ supported on $\sigma(\bar{N})$ such that

$$\phi(f) = \int_{\sigma(\bar{N})} f d\mu.$$

From this it is deduced that for any polynomial p

$$\phi(|p|^2) = \|p\|^2_{H^2_d} = \int_{\sigma(\bar{N})} |p(z)|^2 d\mu.$$

This contradicts Proposition 6.1.8, and hence the corollary follows.

Appendix. In this section we have used the notion of completely positive maps, and we will use this notion in subsequent sections. In this appendix we will give a brief remark on completely positive maps for the reader's convenience. We refer the interested reader to Paulsen's book [Pau4] or Pisier's book [Pi1] for the more detailed materials.

For a C^*-algebra $\mathcal{A}$ we form a new $*$-algebra $M_n(\mathcal{A})$, the algebra of all $n \times n$ matrices with entries in $\mathcal{A}$, and the adjoint of an element of $M_n(\mathcal{A})$ is defined as the transpose of the matrix whose entries are the adjoints of the original entries. To define the norm on $M_n(\mathcal{A})$ we take a faithful representation $\rho : \mathcal{A} \to \mathcal{B}(H)$, and define $\rho_n : M_n(\mathcal{A}) \to \mathcal{B}(n \cdot H)$ by $\rho_n([a_{ij}]) = [\rho(a_{ij})]$, where $n \cdot H$ denotes the direct sum of n copies of H. Then it is easy to check that ρ_n is a $*$-isomorphism from the $*$-algebra $M_n(\mathcal{A})$ to $\mathcal{B}(n \cdot H)$, and

$\rho_n(M_n(\mathcal{A})) = M_n(\rho(\mathcal{A}))$ is closed in $\mathcal{B}(n \cdot H)$. Now defining the norm on $M_n(\mathcal{A})$ by

$$\|[a_{ij}]\| = \|\rho_n([a_{ij}])\|,$$

$M_n(\mathcal{A})$ is then a C^*-algebra in this norm. By the uniqueness of C^*-norm (see [Co, Corollary 1.8]), one sees that the definition of the norm does not depend on the choice of representation, and the above norm on $M_n(\mathcal{A})$ is the unique norm that makes $M_n(\mathcal{A})$ into a C^*-algebra.

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. We call that the map ϕ is positive if ϕ maps positive elements to positive elements. Define the linear maps $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$ for $n = 1, 2, \cdots$. We say

1. the map ϕ is completely bounded (c.b.) if each ϕ_n is bounded and $\|\phi\|_{cb} = \sup_n \|\phi_n\| < \infty$, in this case, $\|\phi\|_{cb}$, is called the completely bounded norm of ϕ;
2. the map ϕ is completely positive (c.p.) if each ϕ_n is positive;
3. the map ϕ is completely isometric (c.i.) if ϕ_n is isometric for each $n \geq 1$;
4. the map ϕ is completely contractive (c.c.) if ϕ_n is contractive for each $n \geq 1$.

For completely positive maps, a basic result is the following Stinespring's theorem whose proof can be found in [Arv4, Pau4, Pi1].

Theorem 6.1.10 (Stinespring's theorem) *Let $\mathcal{A}$ be a C^*-algebra, and let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be a completely positive map for some Hilbert space H. Then there exists a Hilbert space K, a representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ and a bounded linear operator $V : H \to K$ such that*

$$\phi(a) = V^*\pi(a)V, \quad a \in \mathcal{A}.$$

Moreover, if $\mathcal{A}$ is unital and $\phi(1) = 1$, then V is an isometry and we can assume, identifying H with its image VH in K, that $H \subset K$, $V = P_H$.

Because each C^*-algebra has a faithful representation the following properties for a completely positive map are easily deduced from Stinespring's theorem.

Proposition 6.1.11 *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a completely positive map. Then we have*

1. *the map is completely bounded and*

$$\|\phi\|_{cb} = \|\phi\| = \|\phi_n\|, \quad n = 1, 2, \cdots ;$$

2. *if $\mathcal{A}$ has a unit, then $\|\phi\| = \|\phi(1)\|$.*

The following proposition is frequently used in this chapter.

Proposition 6.1.12 *Let $T_1, T_2, T_3, \cdots$ be a (finite or infinite) sequence of bounded linear operators on a Hilbert space H which satisfies that sum $\sum_k T_k^* T_k$ converges in the strongly operator topology. If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$, then the map $\phi : \mathcal{A} \to \mathcal{B}(H)$ defined by $\phi(A) = \sum_k T_k^* A T_k$ is completely positive.*

Proof. Indeed, for each $A \in \mathcal{B}(H)$, the sum $\sum_k T_k^* A T_k$ converges in the strongly operator topology because the sum strongly converges for each positive A. Let l be the number of operators in the sequence $T_1, T_2, \cdots$ and let V be the linear operator from H to $l \cdot H$ defined by

$$Vh = (T_1 h, T_2 h, \cdots).$$

By the assumption the operator V is bounded and $V^* V = \sum_k T_k^* T_k$. Letting π be the representation of $\mathcal{B}(H)$ on $l \cdot H$ defined by

$$\pi(A) = A \oplus A \oplus \cdots ,$$

then it is easy to check that

$$\phi(A) = V^* \pi(A) V.$$

Now observe that for every $n \geq 1$,

$$\phi_n([A_{[ij]}]) = V_n^* \pi_n([A_{ij}]) V_n,$$

where $V_n : n \cdot H \to nl \cdot H$ is defined as $V_n(h_1, \cdots, h_n) = (Vh_1, \cdots, Vh_n)$. From the above observation we see that each ϕ_n is positive, and hence ϕ is completely positive.

Concerning completely positive maps, another very useful result is the Arveson extension theorem. For the proof of this theorem the reader is advised to consult [Arv4].

Theorem 6.1.13 (Arveson extension theorem) *Let $\mathcal{B}$ be a C^*-algebra with identity and let $\mathcal{A}$ be a self-adjoint closed subspace of $\mathcal{B}$ containing the identity. If $\phi : \mathcal{A} \to \mathcal{B}(H)$ is a completely positive map, then there exists a completely positive map $\psi : \mathcal{B} \to \mathcal{B}(H)$ such that $\phi = \psi|_{\mathcal{A}}$.*

6.2 The Toeplitz algebra on the Arveson space

The Toeplitz algebra $\mathcal{T}_d$ on the Arveson space is defined as

$$\mathcal{T}_d = C^*\{M_f : f \text{ belong to the polynomial ring } \mathcal{C}\}.$$

However, the next proposition will show that the Toeplitz algebra $\mathcal{T}_d$ is generated by the d-shift $(S_1, \cdots, S_d)$. Let E_n denote the space of all homogeneous polynomials with degree n. Define the number operator N in H_d^2 as follows:

$$N = \sum_{n \geq 0} n E_n,$$

where E_n also denote the orthogonal projection from H_d^2 onto E_n. Then we have the following proposition (cf. [Arv1]).

Proposition 6.2.1 *Let $d \geq 1$ and let $(S_1, \cdots, S_d)$ be the d-shift. Then for all $i, j = 1, \cdots, d$ we have*

1. $S_1^* S_1 + \cdots + S_d^* S_d = (d + N)(1 + N)^{-1}$;
2. $S_i^* S_j - S_j S_i^* = (1 + N)^{-1}(\delta_{ij} I - S_j S_i^*)$.

Proof. By Proposition 6.1.1, for each monomial $z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, an easy computation shows that

$$S_i^* z_1^{\alpha_1} \cdots z_d^{\alpha_d} = 0$$

if $\alpha_i = 0$, and

$$S_i^* z_1^{\alpha_1} \cdots z_d^{\alpha_d} = \frac{\alpha_i}{\alpha_1 + \cdots + \alpha_d} z_1^{\alpha_1} \cdots z_i^{\alpha_i - 1} \cdots z_d^{\alpha_d}$$

if $\alpha_i \neq 0$. Also notice that

$$(d + N)(1 + N)^{-1} = \sum_{n \geq 0} \frac{d + n}{1 + n} E_n, \quad (1 + N)^{-1} = \sum_{n \geq 0} \frac{1}{1 + n} E_n.$$

By considering two sides of equalities acting on monomials, a simple computation gives the desired conclusion.

Observe that the operator $(d + N)(1 + N)^{-1}$ is invertible, and its inverse is given by

$$(d + N)^{-1}(1 + N) = \sum_{n \geq 0} \frac{1 + n}{d + n} E_n.$$

We therefore conclude that the Toeplitz algebra $\mathcal{T}_d$ is generated by the d-shift $(S_1, \cdots, S_d)$.

The following theorem contains some basic information of the Toeplitz algebra. The theorem and its proof come from [Arv1].

Theorem 6.2.2 *Let $\mathcal{K}$ be all compact operators on H_d^2. Then we have a short exact sequence of C^*-algebras*

$$0 \to \mathcal{K} \hookrightarrow \mathcal{T}_d \xrightarrow{\pi} C(\partial \mathbb{B}_d) \to 0$$

where π is the unital $*$-homomorphism defined by $\pi(S_j) = z_j$ for $j = 1, \cdots, d$.

Proof. By Proposition 6.1.2, the rank operator $1 \otimes 1 = I - \sum_{j=1}^{d} S_j S_j^*$ belongs to $\mathcal{T}_d$, and hence for any polynomials p, q, the rank one operator $p \otimes q = M_p(1 \otimes 1)M_q^*$ is in $\mathcal{T}_d$. This implies that $\mathcal{T}_d$ contains all compact operators. By Proposition 6.2.1 (2), we see that the quotient $\mathcal{T}_d/\mathcal{K}$ is a commutative C^*-algebra. It is obvious that the quotient $\mathcal{T}_d/\mathcal{K}$ is generated by commuting normal elements $Z_j = \pi(S_j), j = 1, \cdots, d$ satisfying

$$Z_1 Z_1^* + \cdots + Z_d Z_d^* = 1.$$

Let X be the Taylor spectrum of the d-tuple $(Z_1, \cdots, Z_d)$. Then X is a nonvoid subset of the unit sphere ∂B_d (see [Cur1]). We claim $X = \partial B_d$. In fact, since the unitary group $\mathbf{U}(\mathbb{C}^d)$ acts transitively on the unit sphere it is enough to show that for every unitary $d \times d$ matrix $u = (u_{ij})$, there exists a $*$-automorphism θ_u such that

$$\theta_u(Z_i) = \sum_{j=1}^{d} \bar{u}_{ji} Z_j.$$

To achieve this we consider the unitary operator U acting on $\mathbb{C}^d$ by $Ue_i = \sum_{j=1}^{d} \bar{u}_{ji} e_j$, where $e_1, \cdots, e_d$ are the coordinate vectors of $\mathbb{C}^d$. Then $\Gamma(U)$ is a unitary operator on H_d^2 which is defined by $\Gamma(U)f(z) = f(U^{-1}z)$, and it satisfies

$$\Gamma(U) S_i \Gamma(U)^* = \sum_{j=1}^{d} \bar{u}_{ji} S_j.$$

It follows that θ_u can be obtained by promoting the spatial automorphism $T \mapsto \Gamma(U) T \Gamma(U)^*$ of $\mathcal{T}_d$ to the quotient $\mathcal{T}_d/\mathcal{K}$. Now the identification of $\mathcal{T}_d/\mathcal{K}$ with $C(\partial \mathbb{B}_d)$ asserted by $\pi(S_i) = z_i, i = 1, \cdots, d$ is obvious, completing the proof.

The Toeplitz algebra on the Arveson space enjoys many important properties. One of the properties is that the identity representation of the Toeplitz algebra $\mathcal{T}_d$ is a boundary representation for the $d+1$-dimensional operator space $span\{I, S_1, \cdots, S_d\}$. For this assertion we recall some facts from the theory of boundary representations [Arv4, Arv5]. Let $\mathcal{B}$ be a C^*-algebra with the identity and let $\mathcal{S}$ be a linear subspace of C^*-algebra $\mathcal{B}$, which contains the identity of $\mathcal{B}$ and generates $\mathcal{B}$ as a C^*-algebra, $\mathcal{B} = C^*(\mathcal{S})$. An irreducible representation $\pi : \mathcal{B} \to \mathcal{B}(H)$ is said to be a boundary representation for $\mathcal{S}$ if $\pi|_{\mathcal{S}}$ has a unique completely positive linear extension to $\mathcal{B}$, namely, π itself.

In the theory of boundary representations, a basic result is the following Boundary theorem, due to Arveson [Arv5, Theorem 2.1.1].

Theorem 6.2.3 *Let $\mathcal{S}$ be a irreducible linear subspace of operators on a Hilbert space H, such that $\mathcal{S}$ contains the identity and $C^*(\mathcal{S})$ contains the ideal $\mathcal{K}(H)$ of all compact operators on H. Then the identity representation of $C^*(\mathcal{S})$ is a boundary representation for $\mathcal{S}$ if and only if the quotient map*

$Q : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ is not completely isometric on the linear span of $\mathcal{S} \cup \mathcal{S}^*$.

The following corollary, including the proof, appeared in [Arv1].

Corollary 6.2.4 *Suppose $d \geq 2$. Then the identity representation of the Toeplitz algebra $\mathcal{T}_d$ is a boundary representation for the $d+1$-dimensional operator space $\mathcal{S} = span\{I, S_1, \cdots, S_d\}$.*

Proof. By Theorem 6.2.3, it suffices to show that the Calkin map is not isometric when promoted to the space $M_d \otimes \mathcal{S}$ of $d \times d$ matrices over $\mathcal{S}$. Considering the operator $A \in M_d \otimes \mathcal{S}$ defined by

$$A = \begin{pmatrix} S_1 & 0 & \dots & 0 \\ S_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ S_d & 0 & \dots & 0 \end{pmatrix},$$

then by Proposition 6.2.1, we have

$$\|A^* A\| = \|S_1^* S_1 + \cdots + S_d^* S_d\| = d,$$

and hence $\|A\| = \sqrt{d}$. On the other hand, by Theorem 6.2.2, the image of A under the Calkin map is the matrix function on the unit sphere

$$F(z) = \begin{pmatrix} z_1 & 0 & \dots & 0 \\ z_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ z_d & 0 & \dots & 0 \end{pmatrix}.$$

It is easy to see that $\sup_{z \in \partial \mathbb{B}_d} \|F(z)\| = 1 < \sqrt{d} = \|A\|$. Applying the Boundary theorem gives the desired conclusion.

As a consequence of the above corollary, we obtain the following whose proof is different from the original one [Arv1].

Corollary 6.2.5 *Let $\phi_1, \phi_2, \cdots$ be a finite or infinite sequence of multipliers on $H_d^2, d \geq 2$, which satisfy*

$$M_{\phi_1}^* M_{\phi_1} + M_{\phi_2}^* M_{\phi_2} + \cdots = 1.$$

Then each ϕ_j is a scalar constant.

Proof. Consider the completely positive map Φ defined on the Toeplitz algebra $\mathcal{T}_d$ by

$$\Phi(A) = M_{\phi_1}^* A M_{\phi_1} + M_{\phi_2}^* A M_{\phi_2} + \cdots .$$

Clearly, the sum converges strongly for each operator A because for each positive operator B we have

$$\sum_k M^*_{\phi_k} B M_{\phi_k} \leq \|B\| \, I.$$

Obviously, we have $\Phi(S_i) = S_i$ for $i = 1, \cdots, d$. It follows that $\Phi(A) = A$ for each $A \in \mathcal{T}_d$ by Corollary 6.2.4. Since $\mathcal{T}_d$ contains all compact operators, this means that for any $\lambda \in \mathbb{B}_d$, we have

$$M^*_{\phi_1}(K_\lambda \otimes K_\lambda)M_{\phi_1} + M^*_{\phi_2}(K_\lambda \otimes K_\lambda)M_{\phi_2} + \cdots = K_\lambda \otimes K_\lambda.$$

A simple reasoning gives that

$$|\phi_1(\lambda)|^2 + |\phi_2(\lambda)|^2 + \cdots = 1.$$

Thus, for $i = 1, \cdots, d$ we have

$$\frac{\partial^2 |\phi_1(\lambda)|^2}{\partial \lambda_i \partial \bar{\lambda}_i} + \frac{\partial^2 |\phi_2(\lambda)|^2}{\partial \lambda_i \partial \bar{\lambda}_i} + \cdots = |\frac{\partial \phi_1(\lambda)}{\partial \lambda_i}|^2 + |\frac{\partial \phi_2(\lambda)}{\partial \lambda_i}|^2 + \cdots = 0,$$

and hence each ϕ_j is a constant.

Concerning the Hardy space $H^2(\mathbb{B}_d)$ one has

$$T^*_{z_1} T_{z_1} + \cdots + T^*_{z_d} T_{z_d} = I.$$

Applying the reasoning as in Corollary 6.2.5, we see that the identity representation of the Toeplitz algebra on the Hardy space $H^2(\mathbb{B}_d)$ is not the boundary representation for the $d+1$-dimensional space $span\{I, M_{z_1}, \cdots, M_{z_d}\}$ generated by the coordinate multipliers.

As an immediate consequence of Corollary 6.2.5, we can verify that if a submodule M of $H^2_d (d \geq 2)$ is unitarily equivalent to H^2_d, then $M = H^2_d$. Indeed, if $U : H^2_d \to M$ is a unitary equivalence, then it is easy to see that there is a multiplier ϕ on H^2_d such that $U = M_\phi$. Since U is unitary, this implies that $M^*_\phi M_\phi = I$, and hence ϕ is a nonzero constant. The assertion is proved.

As one of applications of Theorem 6.2.2, a conclusion is presented at the end of this section.

Proposition 6.2.6 *Let r be a rational function with pole points off $\overline{\mathbb{B}_d}$. Then r is a multiplier of H^2_d.*

Proof. It suffices to show that for every polynomial p with $Z(p) \cap \overline{\mathbb{B}_d} = \emptyset$, the function $1/p$ is a multiplier of H^2_d. Considering the multiplication operator M_p on the Arveson space, then M_p is Fredholm by Theorem 6.2.2. Therefore, the space pH^2_d is closed and it is finite codimensional in H^2_d. Since the space pH^2_d is a submodule, applying Theorem 2.2.3 gives that $pH^2_d = H^2_d$, and hence $\frac{1}{p} H^2_d = H^2_d$. The desired result follows.

6.3 Submodules of the Arveson module

By a submodule M of H_d^2 we mean that M is closed and invariant under action of the d-shift. The next proposition shows that a submodule is invariant under action of multipliers of H_d^2 (cf.[GRS]).

Proposition 6.3.1 *Let M be a submodule, and let ϕ be a multiplier of H_d^2. Then $\phi M \subset M$.*

Proof. Let

$$F_n(e^{it}) = \frac{\sin^2 nt/2}{n \sin^2 t/2}$$

for $n = 1, 2, \cdots$, be the Fejér kernel. Define $p_n(z) = \int_0^{2\pi} \phi(e^{it}z) F_n(e^{it}) dt$; then from the well-known properties of the Fejér kernel [Hof], each p_n is a polynomial, and $p_n(\lambda) \to \phi(\lambda)$ for each $\lambda \in \mathbb{B}_d$. We claim

$$\|p_n f\| \le \|M_\phi\| \, \|f\|, \quad f \in H_d^2.$$

For any real number t and $f \in H_d^2$, set $f_t(z) = f(e^{it}z)$. Then $f_t \in H_d^2$, $\|f_t\| = \|f\|$ and $f_t \to f_{t_0}$ in the norm as $t \to t_0$. The reader easily checks that for any real number t, ϕ_t is a multiplier of H_d^2, and it satisfies that $\|M_{\phi_t}\| = \|M_\phi\|$ and $\phi_t f \to \phi_{t_0} f$ for any $f \in H_d^2$. Therefore, for each natural n the integral $p_n f = \int_0^{2\pi} \phi_t f F_n(e^{it}) dt$ converges in the norm of H_d^2, and it follows that

$$\|p_n f\| \le \|M_\phi\| \, \|f\|.$$

Since $\{M_{p_n}\}$ is uniformly bounded, this implies that there exists a subnet of $\{M_{p_n}\}$, $\{M_{p_{n_\mu}}\}$, which converges in the weak operator topology. It is easy to see that such a subnet converges to M_ϕ in the weak operator topology. Now for any $f \in M$ and any $g \in M^\perp$, we have

$$0 = \langle p_{n_\mu} f, g \rangle \to \langle \phi f, g \rangle.$$

The desired conclusion follows.

Let M be a submodule of H_d^2, and let $S_M = (S_1|_M, \cdots, S_d|_M)$ be the restriction of the d-shift to M. It is obvious that the d-tuple S_M is a d-contraction, and hence

$$S_1 P_M S_1^* + \cdots + S_d P_M S_d^* \le P_M,$$

where P_M is the orthogonal projection from H_d^2 onto M. Furthermore, one easily verifies that the d-shift $(S_1, \cdots, S_d)$ is a null d-contraction, and it follows that the d-tuple S_M is null for any submodule M. For a submodule M, its defect operator Δ_M is defined as

$$\Delta_M = [P_M - S_1 P_M S_1^* - \cdots - S_d P_M S_d^*]^{1/2}.$$

A simple reasoning gives that

$$P_M K_\lambda = K_\lambda \, \Delta_M^2 K_\lambda.$$

Hence a submodule is uniquely determined by its defector operator. This means that if $\Delta_M = \Delta_N$, then $M = N$. When $d = 1$, H_d^2 is the classical Hardy module $H^2(\mathbb{D})$, and in this case there is an inner function η such that $P_M = M_\eta M_\eta^*$, and hence

$$\Delta_M = [P_M - M_z P_M M_z^*]^{1/2} = \eta \otimes \eta.$$

This shows that in the case of the classical Hardy module, the inner function can be recovered by the defect operator.

The following Theorems 6.3.2 and 6.3.4 can be regarded as an analogue of Beurling's theorem in the case of Arveson submodules.

Theorem 6.3.2 *Let M be a submodule of H_d^2. Then there exists a (finite or infinite) sequence of multipliers, $\{\phi_n\} \subset \Delta_M H_d^2$ such that*

$$P_M = M_{\phi_1} M_{\phi_1}^* + M_{\phi_2} M_{\phi_2} + \cdots \ (SOT) \qquad (\star)$$

and M is generated by $\{\phi_n\}$.

This theorem appeared in [Arv1]. In a more general setting, for example, spaces of vector-valued analytic functions with completely Nevanlinna-Pick kernels, McCullough and Trent [MT] proved an analogous theorem.

The proof of Theorem 6.3.2. Since the tuple $S_M = (S_1|_M, \cdots, S_d|_M)$ is a null d-contraction, we apply Theorem 6.1.4 to the d-tuple S_M. Then there is a unique bounded linear operator $L: H_d^2 \otimes \overline{\Delta_M H_d^2} \to H_d^2$ satisfying

$$L(p \otimes \xi) = p \, \Delta_M \xi,$$

and $LL^* = P_M$. Let $e_1, e_2, \cdots$ be an orthogonal basis for $\overline{\Delta_M H_d^2}$ and define $\phi_k = L(1 \otimes e_k)$ for $k = 1, 2 \cdots$. Since L is a homomorphism of Hilbert modules of norm at most 1 we find that the ϕ_k are multipliers of H_d^2, and obviously, $\phi_k \in \Delta_M H_d^2$. It is easy to see that there are bounded linear operator $A_k : H_d^2 \to H_d^2 \otimes \overline{\Delta_M H_d^2}$ for $k = 1, 2, \cdots$ such that

$$L^* f = \sum_k A_k f \otimes e_k.$$

Notice that

$$\sum_k \|A_k f\|^2 = \|L^* f\|^2 \le \|f\|^2, \quad f \in H_d^2,$$

and it follows that the sum $\sum_k A_k^* A_k$ converges in the strongly operator topology. Furthermore, for any polynomial p and any $f \in H_d^2$ we have

$$\begin{aligned} \langle p, A_k f \rangle &= \langle p \otimes e_k, L^* f \rangle \\ &= \langle L(p \otimes e_k), f \rangle = \langle M_{\phi_k} p, f \rangle \\ &= \langle p, M_{\phi_k}^* f \rangle \end{aligned}$$

Thus, $A_k = M^*_{\phi_k}$ for $k = 1, 2, \cdots$. Now we have

$$P_M f = LL^* f = \sum_k L(M^*_{\phi_k} f \otimes e_k) = (\sum_k M_{\phi_k} M^*_{\phi_k})f, \quad f \in H^2_d.$$

We therefore conclude that $P_M = \sum_k M_{\phi_k} M^*_{\phi_k}$ (SOT). To prove that M is generated by $\{\phi_k\}$, we write N for the submodule generated by $\{\phi_k\}$, then $N \subset M$. Now we assume $\xi \in M$, $\xi \perp N$. Then we have $M^*_{\phi_k}\xi = 0$ for $k = 1, 2, \cdots$, and hence

$$P_M \xi = \sum_k M_{\phi_k} M^*_{\phi_k} \xi = 0.$$

Thus, $\xi = 0$, and it follows that $M = N$. This completes the proof.

In dimension $d = 1$ the Beurling's theorem implies that there is a single inner function ϕ satisfying equation $(\star)$, $P_M = M_\phi M_\phi$. However, when dimension $d > 1$ we have the following proposition.

Proposition 6.3.3 *Let M be a proper submodule of H^2_d, $d \geq 2$. The cardinal number of $\{\phi_n\}$ which satisfies $(\star)$ is at least* 2.

Proof. Clearly, the cardinal number card$\{\phi_n\} \geq 1$. If card$\{\phi_n\} = 1$, then we have $P_M = M_\phi M^*_\phi$. Thus, $M = \phi H^2_d$. For each $f \in H^2_d$, since

$$M_\phi M^*_\phi \phi f = P_M \phi f = \phi f,$$

this gives

$$M^*_\phi M_\phi = \text{the identity operator.}$$

Now by Corollary 6.2.5, we see that ϕ is a constant, and hence $M = H^2_d$. This contradiction leads to the desired conclusion.

In fact, the sequence of $(\star)$ in Theorem 6.3.2 is typically infinite, and we will turn to this problem later in this section.

For $\lambda \in \mathbb{B}_d$ we define the normalized reproducing kernel k_λ of H^2_d at λ as follows:

$$k_\lambda = K_\lambda / \|K_\lambda\| = (1 - |\lambda|^2)^{1/2} K_\lambda.$$

Considering $(\star)$ we have

$$\|P_M k_\lambda\|^2 = \sum_k |\phi_k(\lambda)|^2 \leq 1.$$

Because each ϕ_k belongs to $H^\infty(\mathbb{B}_d)$ we will identify ϕ_k with its boundary function on $\partial\mathbb{B}_d$. Therefore we have

$$\sum_k |\phi_k(z)|^2 \leq 1, \quad z \in \partial\mathbb{B}_d$$

almost everywhere with respect to the natural normalized measure σ on $\partial\mathbb{B}_d$.

Theorem 6.3.4 *Let M be a submodule of H_d^2, and let $\{\phi_k\}$ satisfy ($\star$) in Theorem 6.3.2. Then we have*

$$\sum_k |\phi_k(z)|^2 = 1, \quad z \in \partial\mathbb{B}_d$$

almost everywhere with respect to the measure σ. Equivalently, for almost all $z \in \partial\mathbb{B}_n$, $\|P_M k_\lambda\| \to 1$ as $\lambda \to z$ nontangentially.

When M contains a nonzero polynomial, Arveson [Arv2] proved this theorem. The general case was proved by Greene, Richter and Sundberg [GRS]. We will present the proof that appeared in [GRS]. The following two propositions are needed.

Proposition 6.3.5 *For any polynomial p and each $z \in \partial\mathbb{B}_d$ we have*

$$\lim_{\lambda \to z} \|pk_\lambda\| = |p(z)|.$$

Proof. First we claim that $k_\lambda \xrightarrow{w} 0$ as $|\lambda| \to 1$. In fact, since the polynomial ring $\mathcal{C}$ is dense in H_d^2, it is enough to show that for each polynomial q,

$$\lim_{|\lambda| \to 1} \langle q, k_\lambda \rangle = 0.$$

Indeed, note that

$$\langle q, k_\lambda \rangle = (1 - |\lambda|^2)^{1/2} q(\lambda) \to 0$$

as $|\lambda| \to 1$. The claim follows.

To reach our goal we consider

$$\begin{aligned} \|pk_\lambda\|^2 &= \langle M_p^* M_p k_\lambda, k_\lambda \rangle \\ &= \langle (M_p^* M_p - M_p M_p^*) k_\lambda, k_\lambda \rangle + |p(\lambda)|^2. \end{aligned}$$

By Theorem 6.2.2, the operator $M_p^* M_p - M_p M_p^*$ is compact, and hence using the claim gives the desired conclusion.

The next proposition, due to Greene, Richter and Sundberg [GRS], is essential for completing the proof of the theorem.

Proposition 6.3.6 *Let f be a multiplier of H_d^2. Then we have*

1. $|f(\lambda)|^2 \le \|f\, k_\lambda\|^2 \le 2\, Re\langle f, f K_\lambda \rangle - \|f\|^2$;
2. *for almost all $z \in \partial\mathbb{B}_n$, $\|f k_\lambda\| \to |f(z)|$ as $\lambda \to z$ nontangentially.*

In fact, Greene, Richter and Sundberg [GRS] proved that (2) of the proposition is true for any $f \in H_d^2$. We will not need this stronger conclusion.

Proof. (1) Noticing that $f(\lambda)k_\lambda$ and $fk_\lambda - f(\lambda)k_\lambda$ are orthogonal, we see that

$$\begin{aligned} \|fk_\lambda\|^2 &= \|(fk_\lambda - f(\lambda)k_\lambda) + f(\lambda)k_\lambda\|^2 \\ &= \|fk_\lambda - f(\lambda)k_\lambda\|^2 + |f(\lambda)|^2 \ge |f(\lambda)|^2, \end{aligned}$$

and hence reach the left side of (1).

To achieve the right inequality of (1) we define $H_\lambda = \{f \in H_d^2 : f(\lambda) = 0\}$ for each $\lambda \in \mathbb{B}_d$. Then H_λ is a submodule of H_d^2. Write P_λ for P_{H_λ}. Observing the proof of Theorem 6.3.2 we see that

$$P_\lambda = \sum_k M_{\phi_k} M_{\phi_k}^*,$$

where $\phi_k = L(1 \otimes e_k)$, and $\{e_k\}$ is an orthogonal basis for $\overline{\Delta_{H_\lambda} H_d^2}$. Therefore, for any $f \in H_d^2$ we have

$$\begin{aligned}
\|(P_\lambda 1) f\|^2 &= \| \sum_k \overline{\phi_k(0)} \phi_k f\|^2 \\
&= \| \sum_k L(\overline{\phi_k(0)} f \otimes e_k)\|^2 = \|L(\sum_k \overline{\phi_k(0)} f \otimes e_k)\|^2 \\
&\le \sum_k |\phi_k(0)|^2 \, \|f\|^2 = (P_\lambda 1)(0) \|f\|^2.
\end{aligned}$$

Since $P_\lambda = I - k_\lambda \otimes k_\lambda$, this gives that

$$P_\lambda 1 = 1 - K_\lambda / \|K_\lambda\|^2.$$

We therefore conclude that when $\lambda \neq 0$, the function

$$\psi_\lambda = \frac{P_\lambda 1}{\sqrt{(P_\lambda 1)(0)}} = \frac{\|K_\lambda\|^2 - K_\lambda}{\|K_\lambda\| \sqrt{\|K_\lambda\|^2 - 1}}$$

is a contractive multiplier of H_d^2, that is, $\|\psi_\lambda f\|^2 \le \|f\|^2$ for any $f \in H_d^2$. Using this inequality, a short calculation leads to the right inequality of (1).

(2) For $\alpha > 1$ and $z \in \partial \mathbb{B}_d$ we define

$$\Omega_\alpha(z) = \{\lambda \in \mathbb{B}_d : |1 - \langle \lambda, z \rangle| < \frac{\alpha}{2}(1 - |\lambda|^2)\}.$$

Recall that a function $h : \mathbb{B}_d \to \mathbb{C}$ has a limit A as $\lambda \to z$ nontangentially if for any $\alpha > 1$ and for each sequence $\{\lambda_n\} \subset \Omega_\alpha(z)$ that converges to z, we have $h(\lambda_n) \to A$ as $n \to \infty$. Now we turn to the proof of (2). For $\alpha > 1$ we define the maximal function

$$M_\alpha f(z) = \sup\{\|f k_\lambda\| : \lambda \in \Omega_\alpha(z)\}.$$

Noticing that the right-hand side of (1), the function

$$2Re\langle f, f K_\lambda \rangle - \|f\|^2 = 2Re M_f^* f(\lambda) - \|f\|^2,$$

is positive and the real part of an analytic function, and hence it can be represented as the invariant Poisson integral of a positive measure μ on $\partial \mathbb{B}_d$ (see [Ru2]),

$$P\mu(\lambda) = 2Re\langle f, f K_\lambda \rangle - \|f\|^2.$$

Furthermore, we observe that

$$\|\mu\| = P\mu(0) = \|f\|^2.$$

Now combining Theorem 5.2.4 and Theorem 5.4.5 in Rudin's book [Ru2] we see that for all $\alpha > 1$ the Ω_α-maximal function of $P\mu$ satisfies a weak-type estimate with constant C_α, that is, for any $\epsilon > 0$,

$$\sigma(\{z \in \partial\mathbb{B}_d : \sup_{\lambda \in \Omega_\alpha(z)} P\mu(\lambda) > \epsilon\}) \le C_\alpha \frac{\|\mu\|}{\epsilon} = C_\alpha \frac{\|f\|^2}{\epsilon}.$$

By the right-hand side of (1) we obtain the weak-type estimate

$$\sigma(\{z \in \partial\mathbb{B}_d : M_\alpha f(z) > \epsilon\}) \le C_\alpha \frac{\|f\|^2}{\epsilon}.$$

By the left-hand side of (1), for any polynomial p we have

$$\begin{aligned} &\|(f - f(\lambda))k_\lambda\| \le \|(f-p)k_\lambda\| + \|(p - p(\lambda))k_\lambda\| + |f(\lambda) - p(\lambda)| \\ &\le 2\|(f-p)k_\lambda\| + \|(p-p(\lambda))k_\lambda\|. \end{aligned}$$

Noticing the equality

$$\|(p - p(\lambda))k_\lambda\|^2 = \|pk_\lambda\|^2 - |p(\lambda)|^2,$$

applying Proposition 6.3.5 leads to the following:

$$\limsup_{\lambda \in \Omega_\alpha(z);\, \lambda \to z} \|(f - f(\lambda))k_\lambda\| \le 2M_\alpha(f-p)(z).$$

Therefore, the weak-type estimate implies that for any $\epsilon > 0$ and each polynomial p we have

$$\sigma(\{z \in \partial\mathbb{B}_d : \limsup_{\lambda \in \Omega_\alpha(z);\, \lambda \to z} \|(f - f(\lambda))k_\lambda\| > \epsilon\}) \le 2C_\alpha \frac{\|f-p\|^2}{\epsilon}.$$

Since the polynomial ring $\mathcal{C}$ is dense in H_d^2 we obtain

$$\sigma(\{z \in \partial\mathbb{B}_d : \limsup_{\lambda \in \Omega_\alpha(z);\, \lambda \to z} \|(f - f(\lambda))k_\lambda\| > \epsilon\}) = 0.$$

Next, a simple reasoning gives that

$$\sigma(\{z \in \partial\mathbb{B}_d : \limsup_{\lambda \in \Omega_\alpha(z);\, \lambda \to z} \|(f - f(\lambda))k_\lambda\| = 0\}) = 1.$$

Since $\|(f - f(\lambda))k_\lambda\|^2 = \|fk_\lambda\|^2 - |f(\lambda)|^2$ and $f \in H^\infty(\mathbb{B}_d)$, this leads to the desired conclusion.

The proof of Theorem 6.3.4. By Theorem 6.3.2 there exists a nonzero multiplier ϕ in M. We let $[\phi]$ be the submodule generated by ϕ, then $[\phi] \subset M$.

To complete the proof, it is enough to show that for almost all $z \in \partial\mathbb{B}_d$, $\|P_{[\phi]}k_\lambda\| \to 1$ as $\lambda \to z$ nontangentially because

$$1 \geq \|P_M k_\lambda\| \geq \|P_{[\phi]}k_\lambda\|.$$

First one checks that

$$\begin{aligned}
1 \geq \|P_{[\phi]}k_\lambda\|^2 &= \frac{\|P_{[\phi]}K_\lambda\|^2}{\|K_\lambda\|^2} \\
&= \frac{1}{\|K_\lambda\|^2} \sup_{\|p\phi\|\leq 1} |\langle P_{[\phi]}K_\lambda, p\phi\rangle|^2 \\
&\geq \frac{1}{\|K_\lambda\|^2} |\langle P_{[\phi]}K_\lambda, \ \frac{\phi k_\lambda}{\|\phi k_\lambda\|}\rangle|^2 \\
&= \frac{1}{\|K_\lambda\|^2} |\langle K_\lambda, \ \frac{\phi k_\lambda}{\|\phi k_\lambda\|}\rangle|^2 \\
&= \frac{|\phi(\lambda)|^2}{\|\phi k_\lambda\|^2}.
\end{aligned}$$

Using Proposition 6.3.6 (2) leads to the desired conclusion.

Now we return to the problem of when the sequence appearing in Theorem 6.3.2 ($\star$) is finite. In the case of the classical Hardy module $H^2(\mathbb{D})$, each submodule M corresponds to an inner function ϕ such that $P_M = M_\phi M_\phi^*$. When $d \geq 2$, Proposition 6.3.3 says that the cardinal number of the sequence is at least 2. In fact, we have the following proposition.

Proposition 6.3.7 *For a submodule M of $H_d^2, d \geq 2$, the sequence associated with M in Theorem 6.3.2 is finite if and only if the defect operator Δ_M is of finite rank, and in this case the cardinal number of the sequence equals* rankΔ_M.

Proof. First assume that the sequence $\{\phi_k\}_{k=1}^l$ associated with M is finite, and $P_M = \sum_{k=1}^l M_{\phi_k} M_{\phi_k}^*$. Then we get

$$\begin{aligned}
\Delta_M^2 &= P_M - \sum_{s=1}^d M_{z_s} P_M M_{z_s}^* \\
&= \sum_{k=1}^l M_{\phi_k} M_{\phi_k}^* - \sum_{k=1}^l \sum_{s=1}^d M_{z_s} M_{\phi_k} M_{\phi_k}^* M_{z_s}^* \\
&= \sum_{k=1}^l M_{\phi_k} (I - \sum_{s=1}^d M_{z_s} M_{z_s}^*) M_{\phi_k}^* \\
&= \sum_{k=1}^l M_{\phi_k} (1 \otimes 1) M_{\phi_k}^* = \sum_{k=1}^l \phi_k \otimes \phi_k.
\end{aligned}$$

Since Δ_M is positive, this means that Δ_M is of finite rank, and

$$\text{rank}\Delta_M = \text{rank}\Delta_M^2.$$

To reach the opposite direction we first claim that each function from $\Delta_M^2 H_d^2$ is a multiplier of H_d^2. Indeed, we recall that the d-tuple $S_M = (S_1|_M, \cdots, S_d|_M)$ is a null d-contraction, and hence Theorem 6.1.4 can be applied in this case. This means that there exists a unique bounded module homomorphism

$$L: \ H_d^2 \otimes \overline{\Delta_M H_d^2} \to M$$

satisfying

$$L(p \otimes \xi) = p\,\Delta_M \xi, \ \ p \in \mathcal{C}, \ \xi \in \overline{\Delta_M H_d^2}.$$

For each $h \in H_d^2$, noticing $\Delta_M h \in \overline{\Delta_M H_d^2}$ we have

$$L(1 \otimes \Delta_M h) = \Delta_M^2 h.$$

Setting $\phi_h = \Delta_M^2 h$ we get that for any polynomial p,

$$\|\phi_h p\| = \|L(p \otimes \Delta_M h)\| \leq \|\Delta_M h\| \|p\|.$$

This implies that ϕ_h is a multiplier of H_d^2 because the polynomial ring is dense in H_d^2. The claim follows.

Now assume that Δ_M is of finite rank; then Δ_M^2 is also of finite rank, and in this case we have

$$\Delta_M H_d^2 = \Delta_M^2 H_d^2.$$

Letting $l = \dim \Delta_M^2 H_d^2$, then one easily checks that there are $\phi_1, \cdots, \phi_l$ in $\Delta_M^2 H_d^2$ such that

$$\Delta_M^2 = \sum_{k=1}^{l} \phi_k \otimes \phi_k.$$

This leads to the fact

$$\langle \Delta_M^2 K_\lambda, K_\mu \rangle = \langle (P_M - \sum_{i=1}^{d} S_i P_M S_i^*) K_\lambda, K_\mu \rangle$$

$$= (1 - \sum_{i=1}^{d} \overline{\lambda_i} \mu_i) \langle P_M K_\lambda, K_\mu \rangle = \langle (\sum_{k=1}^{l} \phi_k \otimes \phi_k) K_\lambda, K_\mu \rangle$$

$$= \sum_{k=1}^{l} \overline{\phi_k(\lambda)} \phi_k(\mu),$$

and hence we obtain that

$$\langle P_M K_\lambda, K_\mu \rangle = \sum_{k=1}^{l} \overline{\phi_k(\lambda)} \phi_k(\mu)\, K_\lambda(\mu).$$

By the preceding claim that each ϕ_k is a multiplier of H_d^2, the above equality can be represented as

$$\langle P_M K_\lambda, K_\mu \rangle = \langle \sum_{k=1}^{l} M_{\phi_k} M_{\phi_k}^* K_\lambda, K_\mu \rangle.$$

We are therefore led to the desired conclusion

$$P_M = \sum_{k=1}^{l} M_{\phi_k} M_{\phi_k}^*.$$

The remains of the proof is obvious.

To study when the sequence appeared in Theorem 6.3.2 is finite, we are naturally concerned with studying when the corresponding defect operator is of finite rank. Arveson [Arv2] proved the following.

Theorem 6.3.8 *Suppose that M is a homogeneous submodule of H_d^2. Then its defect operator Δ_M is of finite rank if and only if M has finite codimension in H_d^2.*

Proof. Notice that a submodule is homogeneous if and only if the submodule is generated by homogeneous polynomials. By the fact that M is homogeneous, one easily verifies that M has an orthonormal basis consisting of homogeneous polynomials, and hence P_M maps polynomials to polynomials. This implies that the operator Δ_M^2 maps polynomials to polynomials. If Δ_M is of finite rank, then Δ_M^2 is of finite rank. This ensures that there exist polynomials $p_1, \cdots, p_l$ such that

$$\Delta_M^2 = \sum_{k=1}^{l} p_k \otimes p_k.$$

This leads to the following

$$K_\lambda \, \Delta_M^2 K_\lambda = \sum_{k=1}^{l} M_{p_k} M_{p_k}^* K_\lambda.$$

Because of the equality

$$P_M K_\lambda = K_\lambda \, \Delta_M^2 K_\lambda,$$

we get that

$$P_M = \sum_{k=1}^{l} M_{p_k} M_{p_k}^*,$$

and hence

$$P_{H_d^2 \ominus M} = I - \sum_{k=1}^{l} M_{p_k} M_{p_k}^*.$$

From Theorem 6.3.4 we have

$$\sum_{k=1}^{l} |p(\xi)|^2 = 1, \quad \xi \in \partial\mathbb{B}_d.$$

Applying Theorem 6.2.2 shows that the projection $P_{H_d^2 \ominus M}$ is compact, and hence M is of finite codimension in H_d^2. The opposite direction is obvious.

Because defect operators of Arveson submodules play an important role in multi-variable operator theory, a natural problem was asked by Arveson in [Arv2, Arv6].

Arveson's problem. In dimension $d \geq 2$, the defect operator for a nonzero submodule of H_d^2 is of finite rank only if the submodule is of finite codimension in H_d^2.

Recently, Guo made some progress in this direction [Guo10] and proved that if the defect operator for a submodule generated by polynomials is of finite rank, then this submodule has only finitely many zeros in the unit ball $\mathbb{B}_d$. In particular, in the case of two variables, the defect operator for a submodule generated by polynomials is of finite rank if and only if this submodule is finite codimensional. We also point out some recent work of Guo and Yang [GYa] relating to defect operators for Hardy submodules over the bidisk, and some work of Yang and Zhu [YZ] concerning defect operators for Bergman submodules.

6.4 Rigidity for Arveson submodules

As shown in Section 6.2, if a submodule of $H_d^2, d \geq 2$ is unitarily equivalent to H_d^2, then such a submodule must be H_d^2 itself. In fact, in this section we will see that Arveson submodules have stronger rigidity than Hardy submodules on the unit ball. This section comes mainly from the paper by Guo, Hu and Xu [GHX]. First we give the following proposition.

Proposition 6.4.1 *Let M be a submodule of H_d^2. Suppose that the operator Δ_M^2 can be represented as*

$$\Delta_M^2 = \sum_k \phi_k \otimes \phi_k.$$

Then $\phi_k \in H^\infty(\mathbb{B}_d) \cap M$, and $\sum_k |\phi_k(\xi)|^2 = 1$ for almost all $\xi \in \partial\mathbb{B}_d$.

Proof. Noticing that

$$\|P_M k_\lambda\|^2 = (1 - |\lambda|^2)\|P_M K_\lambda\|^2$$
$$= \langle \Delta_M^2 K_\lambda, K_\lambda \rangle = \sum_k |\phi_k(\lambda)|^2,$$

Theorem 6.3.4 implies that $\sum_k |\phi_k(\xi)|^2 = 1$ for almost all $\xi \in \partial\mathbb{B}_d$ and, clearly, each $\phi_k \in H^\infty(\mathbb{B}_d)$. Since $\phi_k \otimes \phi_k \le \Delta_M^2$, this ensures that

$$\begin{aligned}\mathbb{C}\,\phi_k &= \operatorname{range} \phi_k \otimes \phi_k \\ &= (\ker \phi_k \otimes \phi_k)^\perp \subseteq (\ker \Delta_M^2)^\perp = \overline{\operatorname{range} \Delta_M^2} \subseteq M.\end{aligned}$$

This completes the proof.

As we have seen in previous sections, $H_d^2 \subseteq H^2(\mathbb{B}_d)$. Thus, we will frequently identify functions from H_d^2 with their boundary functions on the unit sphere $\partial\mathbb{B}_d$.

Proposition 6.4.2 *Let M and N be two submodules of H_d^2. If $U : M \to N$ is a unitary equivalence, then there is a function $\eta \in L^\infty(\partial\mathbb{B}_d)$ that satisfies $|\eta(\xi)| = 1$ for almost all $\xi \in \partial\mathbb{B}_d$ such that $Uf = \eta f$ for any $f \in M$.*

Proof. From Theorem 6.3.2 there is a sequence of multipliers of H_d^2, $\{\phi_k\}$ such that

1. $P_M = \sum_k M_{\phi_k} M_{\phi_k}^*$;
2. the submodule M is generated by $\{\phi_k\}$.

By the proof of Proposition 6.3.1, for each multiplier ϕ of H_d^2 there exists a net of polynomials $\{p_\alpha\}$ such that M_{p_α} converges to M_ϕ in the weak operator topology. It follows that for any $f \in M$ we have $U\phi_k f = \phi_k U f$ for each k; especially, we have $U\phi_k\phi_l = \phi_k U\phi_l = \phi_l U\phi_k$ for any k, l. This ensures that

$$\eta = \frac{U\phi_k}{\phi_k} = \frac{U\phi_l}{\phi_l}, \quad k; l = 1, 2, \cdots .$$

Just as shown in the proof of Proposition 6.3.7, from the equality

$$P_M = \sum_k M_{\phi_k} M_{\phi_k}^*,$$

the following is easily deduced:

$$\Delta_M^2 = \sum_k \phi_k \otimes \phi_k.$$

Since U is a unitary module homomorphism this implies that

$$U\Delta_M^2 U^* = \Delta_N^2 = \sum_k U\phi_k \otimes U\phi_k = \sum_k \eta\phi_k \otimes \eta\phi_k.$$

Applying Proposition 6.4.1 and Theorem 6.3.4 gives $\eta \in L^\infty(\partial\mathbb{B}_d)$ satisfying $|\eta(\xi)| = 1$ for almost all $\xi \in \partial\mathbb{B}_d$. Since the submodule M is generated by $\{\phi_k\}$, this says that the space $\{\sum_{\text{finite sum}} p_s\phi_s : p_s \text{ are polynomials}\}$ is dense in M. Notice that for each g in this space we have $Ug = \eta g$. Now for any $f \in M$, there exists a sequence $\{g_n\}$ contained in this space that converges to f in the norm of H_d^2. By Theorem 6.1.3 we get

$$\|\eta g_n - Uf\|_{H^2(\mathbb{B}_d)} \leq \|Ug_n - Uf\|_{H_d^2} = \|g_n - f\|_{H_d^2} \to 0$$

as $n \to \infty$. Combining the above with the fact that $\|g_n - f\|_{H^2(\mathbb{B}_d)} \to 0$ as $n \to \infty$, we have that

$$Uf = \eta f, \quad f \in M.$$

Lemma 6.4.3 *Let M be a submodule of the Hardy module $H^2(\mathbb{B}_d)$, and let ϕ be in $L^\infty(\partial\mathbb{B}_d)$. If $\phi M \subset M$, then ϕ belongs to $H^\infty(\mathbb{B}_d)$.*

Proof. Follow the proof of Proposition 3.3.7, or see [Sch].

For an ideal I of polynomials we let $[I]_a$ denote the submodule of H_d^2 $(d \geq 2)$ generated by I.

Theorem 6.4.4 *Let I, J be ideals of polynomials. If $[I]_a$ is unitarily equivalent to $[J]_a$, then $[I]_a = [J]_a$.*

Proof. Applying Proposition 6.4.2 there is a function $\eta \in L^\infty(\partial\mathbb{B}_d)$ satisfying $|\eta(\xi)| = 1$ for almost all $\xi \in \partial\mathbb{B}_d$ such that

$$\eta[I]_a = [J]_a.$$

Notice that both $[I]_a$ and $[J]_a$ are included in the Hardy module $H^2(\mathbb{B}_d)$ as sets. Taking the closures for two sides of the above equality in the Hardy module we have

$$\eta[I] = [J],$$

where $[I]$ and $[J]$ denote submodules of $H^2(\mathbb{B}_d)$ generated by I and J, respectively. We apply Theorem 4.4.2 to obtain $[I] = [J]$. Furthermore, by the above lemma, both η and $\bar{\eta}$ are inner. It follows from this that η is a constant, and hence $[I]_a = [J]_a$, completing the proof.

The main result of this section is the following.

Theorem 6.4.5 *Let M, N be submodules of $H_d^2, d \geq 2$, and let $M \supseteq N$. If they are unitarily equivalent, then $M = N$.*

From this theorem we see that Arveson submodules have stronger rigidity than Hardy submodules on the unit ball. Because there exist a lot of inner functions on the unit ball [Ru4] this ensures that for any submodule M of $H^2(\mathbb{B}_d)$ and any inner function η, ηM, M, as submodules of $H^2(\mathbb{B}_d)$, are unitarily equivalent, but $\eta M \subsetneqq M$.

From Theorems 6.4.4 and 6.4.5 we are therefore led to ask the following question.

Question: If two submodules of $H_d^2, d \geq 2$ are unitarily equivalent, must they be equal?

For Theorem 6.4.5 we need the following proposition.

Let M be a submodule M of H_d^2. We write the d-tuple $S^M = (S_1^M, \cdots, S_d^M)$ for the restriction $(S_1|_M, \cdots, S_d|_M)$ of the d-shift $(S_1, \cdots, S_d)$ to M. Let $\mathcal{T}_d^M$ denote the C^*-algebra generated by the $d+1$-dimensional operator space $span\{I, S_1^M, \cdots, S_d^M\}$, that is, $\mathcal{T}_d^M$ is the Toeplitz algebra on the submodule M.

Proposition 6.4.6 *Let M be submodule of $H_d^2, d \geq 2$. Then we have*

1. *the algebra $\mathcal{T}_d^M$ is irreducible, and $\mathcal{T}_d^M$ contains all compact operators on M;*
2. *the identity representation of the algebra $\mathcal{T}_d^M$ is a boundary representation for the $d+1$-dimensional operator space $span\{I, S_1^M, \cdots, S_d^M\}$.*

Proof. (1) To obtain a contradiction we assume that $\mathcal{T}_d^M$ is reducible. This means that there exists a nontrivial projection Q on M such that $QS_i^M = S_i^M Q$ for $i = 1, \cdots, d$. From this it is easily deduced that M can be decomposed as direct sum of two submodules, that is, there are two submodules M_1, M_2 such that $M = M_1 \oplus M_2$. It follows that

$$\|P_M k_\lambda\|^2 = \|P_{M_1} k_\lambda\|^2 + \|P_{M_2} k_\lambda\|^2$$

for any $\lambda \in \mathbb{B}_d$. Theorem 6.3.4 shows that this is impossible as λ nontangentially tends to the boundary of the unit ball. This contradiction shows that the algebra $\mathcal{T}_d^M$ is irreducible. It remains only to show that $\mathcal{T}_d^M$ contains a nonzero compact operator, and hence from [Arv7], $\mathcal{T}_d^M$ contains all compact operators on M. Indeed, considering the operator $A = \sum_{i=1}^d S_i^{M*} S_i^M - I_M$, then $A = P_M(\sum_{i=1}^d S_i^* S_i - I)P_M$ is compact. We claim that $A \neq 0$. In fact, by Proposition 6.2.1,

$$A = (d-1)P_M(I+N)^{-1}P_M.$$

Notice that $\ker(I+N)^{-1} = \{0\}$ and $(I+N)^{-1} \geq 0$ to show $A \neq 0$. The claim follows, and hence the proof of (1) is completed.

(2) To complete the proof of (2) let us follow the proof of Corollary 6.2.4. By Theorem 6.2.3, it suffices to show that the Calkin map is not isometric when promoted to the space $M_d \otimes \mathcal{S}^M$ of $d \times d$ matrices over $\mathcal{S}^M$, where $\mathcal{S}^M$ denotes $d+1$-dimensional operator space $span\{I, S_1^M, \cdots, S_d^M\}$. Considering the operator $A \in M_d \otimes \mathcal{S}^M$ defined by taking the first column of A as $(S_1^M, \cdots, S_d^M)^T$ and others as zeros, then by Proposition 6.2.1 we have

$$\|A\|^2 = \|A^*A\| = \|P_M(S_1^*S_1 + \cdots + S_d^*S_d)P_M\|$$

$$= \|(S_1^*S_1 + \cdots + S_d^*S_d)^{1/2}P_M\|^2 = \|[\sum_{n=0}^{\infty}(\frac{n+d}{n+1})^{1/2}E_n]P_M\|^2,$$

and hence

$$\|A\| = \|[\sum_{n=0}^{\infty}(\frac{n+d}{n+1})^{1/2}E_n]P_M\|.$$

For any $f \in M$, $\|f\| = 1$, decomposing f as $f = \sum_{n=0}^{\infty} f_n$, where $f_n = E_n f$ is the n-th homogeneous component of f, we have

$$\|[\sum_{n=0}^{\infty}(\frac{n+d}{n+1})^{1/2}E_n]P_M f\|^2 = \sum_{n=0}^{\infty}\frac{n+d}{n+1}\|f_n\|^2 > \sum_{n=0}^{\infty}\|f_n\|^2 = 1.$$

From the above reasoning it is easily deduced that $\|A\| > 1$. On the other hand, by Theorem 6.2.2, the essential norm of A, $\|A\|_e$, satisfies $\|A\|_e \leq 1$, and hence $\|A\|_e < \|A\|$. Applying the Boundary Theorem 6.2.3 gives the required conclusion.

Corollary 6.4.7 *Let M be a submodule of $H_d^2, d \geq 2$, and let $T_1, T_2, \cdots$ be a finite or infinite sequence of operators on M. Suppose that each T_k commutes with the d-tuple $(S_1^M, \cdots, S_d^M)$, and the sequence satisfies*

$$T_1^* T_1 + T_2^* T_2 + \cdots = I.$$

Then each T_k is a scalar multiple of the identity operator.

Proof. Following the proof of [Arv1, Proposition 8.13], we consider the completely positive map Φ defined on the algebra $\mathcal{T}_d^M$ by

$$\Phi(A) = T_1^* A T_1 + T_2^* A T_2 + \cdots .$$

Clearly the sum converges strongly for each operator A because for each positive operator B we have

$$\sum_k T_k^* B T_k \leq \|B\| \, I.$$

Obviously, we have $\Phi(S_i^M) = S_i^M$ for $i = 1, \cdots, d$, and $\Phi(I) = I$. By Proposition 6.4.6 (2) it follows that $\Phi(A) = A$ for each $A \in \mathcal{T}_d^M$. Let l be the number of operators in the sequence $T_1, T_2, \cdots$ and let V be the linear operator from M to $l \cdot M$ defined by

$$Vh = (T_1 h, T_2 h, \cdots).$$

By the assumption it is easy to see that V is an isometry. Letting π be the representation of $\mathcal{T}_d^M$ on $l \cdot M$ defined by

$$\pi(A) = A \oplus A \oplus \cdots ,$$

then it is easy to check that

$$\Phi(A) = V^* \pi(A) V.$$

Since

$$(VA - \pi(A)V)^*(VA - \pi(A)V) = A^*A - \Phi(A^*)A - A^*\Phi(A) + \phi(A^*A) = 0$$

for each $A \in \mathcal{T}_d^M$, we conclude that $VA = \pi(A)V$. By examining the components of this operator equation one sees that $AT_k = T_kA$ for each k and every $A \in \mathcal{T}_d^M$. Since $\mathcal{T}_d^M$ is irreducible it follows that each T_k must be a scalar multiple of the identity operator.

The proof of Theorem 6.4.5. Let $U : M \to N$ be a unitary equivalence. Since $N \subset M$ then $U : M \to M$ is an isometry, and it satisfies $US_i^M = S_i^M U$ for $i = 1, \cdots, d$. Now applying Corollary 6.4.7 gives that there is a constant γ, $|\gamma| = 1$ satisfying $U = \gamma I$, and hence $N = UM = M$, completing the proof.

6.5 Remarks on Chapter 6

After the appearance of the dilation theory for 1-contractions [NF], dilation theory became very important in operator theory. There are a number of positive results on dilation theories for commutative or non-commutative d-tuples. In the case of commutative d-tuples, there are many references concerning dilation theories [AE, AEM, Ag1, Ag2, AM1, AM2, Arv1, Arv2, Arv3, Arv4, Arv5, Arv6, At1, At2, BTV, CV, DP, MP, MV]. In particular, the framework of the module developed by Douglas and Paulsen [DP] can be applied to this case, and this module context has some remarkable consequences. At the very least, it facilitates the introduction of techniques and methods drawn from algebraic geometry, homology theory and complex analysis in one variable and several variables. If a commutative d-tuple $(T_1, \cdots, T_d)$ is a d-contraction (i.e., $T_1T_1^* + \cdots + T_dT_d^* \leq I$), then the theory of d-contractions developed by Arveson [Arv1, Arv2, Arv3] yields some remarkable results, and this theory parallels some principal assertions of 1-contractions by Nagy and Foias [NF]. In the case of noncommutative d-tuples, we refer the reader to references [Bun, Fr, Po1, Po2, Po3, Po4, Po5, Po6, Po7, Po8] for the related dilation theories. Popescu has clarified that a noncommutative d-tuple can be obtained by compressing certain natural d-tuple of isometries acting on the full Fock space $\mathcal{F}(\mathbb{C}^d)$ over $\mathbb{C}^d$ (the left creation operators) to a co-invariant subspace [Po2, Po3, Po4, Po5, Po6, Po7, Po8]. Concerning the full Fock space, recently Davidson and Pitts [DP1, DP2, DP3] have studied the Toeplitz algebra generated by the left creation operators on the full Fock space $\mathcal{F}(\mathbb{C}^d)$.

Turning to some recent work of Arveson, to generalize the operator-theoretic aspects of function theory on the unit disk to multi-variable operator theory, Arveson began a systematic study for the theory of d-contractions [Arv1, Arv2, Arv3]. This theory depends essentially on a special function space on the unit ball, namely, the Arveson space H_d^2. We refer the reader to [Arv1, Arv2] for function theory and operator theory on the Arveson space. There also have been attempts to study interpolation and model theory relating to the Arveson space [Ag1, AM1, AM2, BTV, GRS, MT]. We call the reader's attention to work of Paulsen, Popescu and Singh [PPS], where one can find Bohr inequality and Bohr set related to the Arveson space. We also mention Xu's work [Xu], which has made some progress in the study of composition operators on the Arveson space.

For a d-contraction $T = (T_1, \cdots, T_d)$ on a Hilbert space H, one makes H into a Hilbert module over the polynomial ring $\mathcal{C}$ by $p \cdot h = p(T_1, \cdots, T_d)h$, $h \in H$, $p \in \mathcal{C}$. For such a Hilbert module H, the curvature invariant $K(H)$ was introduced by Arveson [Arv2], which is analogous to the integral of the Gaussian curvature over a compact oriented even-dimensional Riemann manifold. The importance of the curvature invariant is based on the fact that this invariant has significant operator-theoretic implications. For a Hilbert module, determining the best method of calculating the curvature of the module is often difficult. We refer the reader to [Arv3, Lev, Par] for curvature formulas of some special Hilbert modules. We wish to compare the curvature theory developed by Cowen and Douglas [CDo1, CDo2] with that developed by Arveson. Intuitively, the former is based on local properties of modules while the latter is based on global properties of modules. The connection between the former and latter is not completely understood, but the two approaches are fundamentally different. Concerning the curvature theory of Cowen and Douglas we call the reader's attention to references [BM1, BM2, CD2, DM1, DM2, DMV, Mc, MP, MSa, Zhu4].

We now turn to this chapter. In this chapter we collect some basic results from [Arv1, Arv2] which are related to preceding chapters. In the preceding chapters we are mainly concerned with Hardy modules and Bergman modules. However, the Arveson module, unlike the Hardy and Bergman modules associated with some measures on underlying domains, is not associated with any measure on $\mathbb{C}^d$ (cf. Proposition 6.1.8), and it is distinguished among all analytic Hilbert modules on the unit ball which have some natural property by being the largest Hilbert norm (cf. Theorem 6.1.3). Hence the Arveson module is included in each other analytic Hilbert module on the unit ball which has the above required property. Since the Arveson module is never given by some measure on $\mathbb{C}^d$, this leads to the fact that the d-duple $(M_{z_1}, \cdots, M_{z_d})$ of the coordinate multipliers on the Arveson space (called the d-shift) is not subnormal. Just as we have seen, the d-shift plays an essential role in the theory of d-contractions. By the d-shift an appropriate version of Von Neumann's inequality was obtained by Arveson (cf. Theorem 6.1.7). In fact, numerous attempts were made to prove appropriate versions of Von Neumann's inequality for commuting or noncommuting d-tuples. An earlier version of Von Neumann's inequality for the unit ball was given by Drury [Dr]. For noncommutative d-tuples of operators Popescu has establish various versions of Von Neumann's inequality [Po1, Po3, Po6]. We refer the reader to [Pi1, Chapter 1] for more comments and references on Von Neumann's inequality.

The Toeplitz algebra $\mathcal{T}^d (d > 1)$ on the Arveson space, unlike the Toeplitz algebra on the Hardy space $H^2(\mathbb{B}_d)$, enjoys many interesting properties, one of which is that the identity representation of the $\mathcal{T}^d$ is a boundary representation for $d+1$-dimensional space $span\{I, S_1, \cdots, S_d\}$ (cf. Corollary 6.2.4). From this one finds that there are nontrivial isometries that commute with d-shift (cf. Corollary 6.2.5). This yields that there is not a nontrivial submodule that is unitarily equivalent to the Arveson module itself. In fact, we see from Section 4 that Arveson submodules have stronger rigidity than Hardy submodules. Just as shown in Section 4, for Arveson submodules M and N, if they are unitarily equivalent, and $M \supset N$, then $M = N$ [GHX]. Because there are a lot of inner functions on the unit ball [Ru4], this ensures that for any Hardy submodule M and any nonconstant inner function η, the submodules M and ηM are unitarily equivalent and $M \supset \eta M$, but $M \neq \eta M$.

Chapter 7

Extensions of Hilbert modules

Let H be a Hilbert space and $\mathbb{A}$ a function algebra. We say that H is a Hilbert module over $\mathbb{A}$ if there is a multiplication $(a, f) \to af$ from $\mathbb{A} \times H$ to H, making H into $\mathbb{A}$-module, and if, in addition, the action is jointly continuous in the sup-norm on $\mathbb{A}$ and the Hilbert space norm on H. The framework of the module developed by Douglas and Paulsen was systematically presented in [DP]. In this module context, they began the study of applications of homological theory in the categories of Hilbert modules. In view of Hilbert modules, the theory of function algebras is emphasized since it plays the analogous role of ring theory in the context of algebraic modules. Therefore, in studying Hilbert modules, as in studying any algebraic structure, the standard procedure is to look at submodules and associated quotient modules. The extension problem then appears quite naturally: given two Hilbert modules H, K, what module J may be constructed with submodule H and associated quotient module K, i.e., $K \cong J/H (= J \ominus H)$? We then have a short exact sequence

$$E: \ 0 \longrightarrow H \xrightarrow{\alpha} J \xrightarrow{\beta} K \longrightarrow 0$$

of Hilbert $\mathbb{A}$-modules, where α, β are Hilbert module maps. Such a sequence is called an extension of K by H, or simply J is called an extension of K by H. The set of equivalence classes of extensions of K by H (this will be defined in the next section), denoted by $Ext_{\mathbb{A}}(K, H)$, may then be given a natural $\mathbb{A}$-module structure. The homological invariant $Ext_{\mathbb{A}}(-,-)$ is important for algebraists because in general the category of all Hilbert modules over $\mathbb{A}$ and maps is nonabelian. For analysts, we expect Ext-groups to be a fruitful object of study and useful tool in operator theory.

7.1 The basic theory of extensions

This section is mainly based on [CC1]. Our aim is to quote necessary homological constructions and analogous results as in the pure algebraic case that was presented in [CC1](also see [Hel]). Given a function algebra $\mathbb{A}$, if H is a Hilbert module over $\mathbb{A}$, we write $T_f : H \to H$ for the multiplication by $f \in \mathbb{A}$, that is, $T_f h = fh$. Let H_1, H_2 be two Hilbert modules over $\mathbb{A}$, and $A : H_1 \to H_2$ be a bounded linear operator. We recall that A is a module map if $AT_f = T_f A$ for any $f \in \mathbb{A}$. The symbol $Hom_{\mathbb{A}}(H_1, H_2)$ will denote all module maps from H_1 to H_2. We will also use $\mathcal{H}(\mathbb{A})$ to denote the category of all Hilbert modules over $\mathbb{A}$ together with module maps. If H is a Hilbert module over $\mathbb{A}$, then one may consider H as a Hilbert module over $\bar{\mathbb{A}}$ (adjoints of elements in $\mathbb{A}$) by setting $\bar{f}h = T_f^* h$. For emphasis, we denote this $\bar{\mathbb{A}}$-module by H_*. Note that if $\alpha : H \to K$ is a module map over $\mathbb{A}$, then naturally α induces its adjoint map $\alpha^* : K_* \to H_*$ such that $\alpha^* T_f^* = T_f^* \alpha^*$ for each $f \in \mathbb{A}$. This means that the opposite category of $\mathcal{H}(\mathbb{A})$ is naturally identified with $\mathcal{H}(\bar{\mathbb{A}})$ under the above rule.

In general the category $\mathcal{H}(\mathbb{A})$ is not abelian. What seems to make things most difficult is the category lacks enough projective or injective objects. Hence it is not possible to define the functor Ext as the derived functor of Hom as in [HS]. However, the standard homological construction from homological algebra enables us to define Ext-groups as the following.

Let H, K be Hilbert modules over $\mathbb{A}$ and let $S(K, H)$ be the set of all short exact sequences

$$E : 0 \longrightarrow H \stackrel{\alpha}{\longrightarrow} J \stackrel{\beta}{\longrightarrow} K \longrightarrow 0,$$

where α, β are Hilbert module maps.

We say that two elements $E, E' \in S(K, H)$ are equivalent if there exists a Hilbert module map θ from the middle term of E to that of E' such that the diagram

$$\begin{array}{ccccccccc} E: & 0 \longrightarrow & H & \stackrel{\alpha}{\longrightarrow} & J & \stackrel{\beta}{\longrightarrow} & K & \longrightarrow 0 \\ & & \| & & \downarrow \theta & & \| & \\ E': & 0 \longrightarrow & H & \stackrel{\alpha'}{\longrightarrow} & J' & \stackrel{\beta'}{\longrightarrow} & K & \longrightarrow 0 \end{array}$$

commutes. The set of equivalence classes of $S(K, H)$ under this relation is defined to be the extension group of K by H and denoted by $Ext_{\mathbb{A}}(K, H)$. The group structure on the $Ext_{\mathbb{A}}(K, H)$ is naturally given in the way described by Carlson and Clark in [CC1]. The zero element of $Ext_{\mathbb{A}}(K, H)$ is the split extension

$$0 \longrightarrow H \longrightarrow H \oplus K \longrightarrow K \longrightarrow 0,$$

where the middle term is the (orthogonal) direct sum of the two modules. Our aim is to show that $Ext_{\mathbb{A}}$ is a functor; we therefore have to define induced maps. This will be done by showing the existence of pullbacks and pushouts in the category $\mathcal{H}(\mathbb{A})$.

A pullback diagram in $\mathcal{H}(\mathbb{A})$ is a diagram of Hilbert $\mathbb{A}$-modules and maps of the form

$$H_1 \stackrel{\alpha_1}{\longrightarrow} K \stackrel{\alpha_2}{\longleftarrow} H_2.$$

The pullback of the diagram is a Hilbert module J together with a pair of Hilbert module maps $\beta_1 : J \to H_1$ and $\beta_2 : J \to H_2$ such that

$$\alpha_1 \beta_1 = \alpha_2 \beta_2,$$

and if, in addition, J' is another Hilbert module with maps $\gamma_i : J' \to H_i$ such that $\alpha_1\gamma_1 = \alpha_2\gamma_2$, then there exists a unique map $\theta : J' \to J$ with $\beta_i\theta = \gamma_i$ for $i = 1, 2$.

The definition of a pushout is the dual statement, obtained by reversing all of the arrows.

The following proposition appeared in [CC1].

Proposition 7.1.1 *Pullbacks and pushouts exist in the category $\mathcal{H}(\mathbb{A})$.*

Proof. For a pullback diagram

$$H_1 \xrightarrow{\alpha_1} K \xleftarrow{\alpha_2} H_2,$$

the pullback J is obtained by setting

$$J = \{(h_1, h_2) \in H_1 \oplus H_2 \, : \alpha_1(h_1) = \alpha_2(h_2)\},$$

and $\beta_1 : J \to H_1$ by $\beta_1(h_1, h_2) = h_1$, and $\beta_2 : J \to H_2$ by $\beta_2(h_1, h_2) = h_2$. The module structure on J is derived from that on the direct sum $H_1 \oplus H_2$. The reader easily checks that $\{J, \beta_1, \beta_2\}$ forms the pullback for the given diagram.

As mentioned above the opposite category of $\mathcal{H}(\mathbb{A})$ is naturally identified with $\mathcal{H}(\bar{\mathbb{A}})$. Similarly, one can verify that pullbacks exist in the category $\mathcal{H}(\bar{\mathbb{A}})$. By duality we see that pushouts exist in the category $\mathcal{H}(\mathbb{A})$.

Following Proposition 7.1.1, one can establish the functoriality of $Ext_{\mathbb{A}}$. Let $[E]$ be the equivalence class of a short exact sequence

$$E : \; 0 \longrightarrow H \xrightarrow{\gamma} J \xrightarrow{\delta} K \longrightarrow 0,$$

and if $\alpha : K' \to K$, then $\alpha^*[E]$ is defined as the equivalence class of $E\alpha$ which is the upper row of the following diagram:

$$\begin{array}{lccccccccc} E\alpha : & 0 & \longrightarrow & H & \xrightarrow{\gamma'} & J' & \xrightarrow{\delta'} & K' & \longrightarrow & 0 \\ & & & \| & & \downarrow \psi & & \downarrow \alpha & & \\ E : & 0 & \longrightarrow & H & \xrightarrow{\gamma} & J & \xrightarrow{\delta} & K & \longrightarrow & 0 \end{array}$$

where $\{J', \delta', \psi\}$ is the pullback of the diagram in the lower right corner and γ' is given by $\gamma'(h) = (\gamma(h), 0), \; for \; h \in H.$

Likewise, if $\beta : H \to H'$, then βE is obtained by taking the pushout of E along β. Set $\beta_*[E]$ to be the equivalence class of βE. As done in [CC1], the reader should verify that if E and E' are equivalent extensions of K by H, then $E\alpha$ and $E'\alpha$ are equivalent, and βE and $\beta E'$ are equivalent. Furthermore, we have induced homomorphisms

$$\beta_* : \; Ext_{\mathbb{A}}(K, H) \to Ext_{\mathbb{A}}(K, H')$$
$$\alpha^* : \; Ext_{\mathbb{A}}(K, H) \to Ext_{\mathbb{A}}(K', H)$$

where $\beta_*[E] = [\beta E]$ and $\alpha^*[E] = [E\alpha]$.

Using the standard methods from homological algebra (cf. [HS]), one can prove that the induced maps α^*, β_* satisfy the following conditions:

$$\alpha^*\beta_* = \beta_*\alpha^*$$

and if $\alpha' : K'' \to K'$ and $\beta' : H' \to H''$ are Hilbert modules maps, then

$$(\alpha\alpha')^* = \alpha'^*\alpha^* \quad \text{and} \quad (\beta'\beta)_* = \beta'_*\beta_*.$$

Moreover, one naturally makes $Ext_{\mathbb{A}}(K, H)$ into $\mathbb{A}$-module. In fact, if $K_f : K \to K$ and $H_f : H \to H$ are multiplications by $f \in \mathbb{A}$, then we have the homomorphisms K_f^* and H_{f*} from $Ext_{\mathbb{A}}(K, H)$ to itself. It is easy to verify that both actions of $\mathbb{A}$ are the same and that they are compatible with the addition on $Ext_{\mathbb{A}}(K, H)$. We have therefore shown the following [CC1].

Proposition 7.1.2 *$Ext_{\mathbb{A}}(-,-)$ is a bifunctor from $\mathcal{H}(\mathbb{A})$ to the category of $\mathbb{A}$-modules. It is contravariant in the first and covariant in the second variable.*

Let $B(K, H)$ be all bounded linear operators from K to H. Suppose a bounded linear map $D : \mathbb{A} \to B(K, H)$ satisfies

$$D(fg) = D(f)T_g + T_f D(g), \ \forall f, g \in \mathbb{A};$$

we say that D is a derivation from $\mathbb{A}$ to $B(K, H)$. For a derivation D, if there exists a bounded linear operator $T : K \to H$ such that

$$D(f) = D_T(f) = TT_f - T_f T$$

for any $f \in \mathbb{A}$, we say that D is an inner derivation. Let $Der(K, H)$ denote all derivations from $\mathbb{A}$ to $B(K, H)$ and $Inn(K, H)$ all inner derivations. As shown in [CC1], extensions are closely related to the derivation problem. For an extension of K by H

$$E : \ 0 \longrightarrow H \stackrel{\alpha}{\longrightarrow} J \stackrel{\beta}{\longrightarrow} K \longrightarrow 0,$$

there exists a derivation $D \in Der(K, H)$ such that E is equivalent to the following extension E_D defined by D:

$$E_D : 0 \longrightarrow H \stackrel{i}{\longrightarrow} H\tilde{\oplus}K \stackrel{\pi}{\longrightarrow} K \longrightarrow 0,$$

where $H\tilde{\oplus}K$ is Hilbert space direct sum of H and K with the $\mathbb{A}$-module structure defined by $f(h, k) = (fh + D(f)k, fk)$. Finally, two extensions E_{D_1} and E_{D_2} are equivalent if and only if $D_1 - D_2$ is inner.

The next proposition is well known in the purely algebraic setting. In the context of Hilbert modules it was first proved in [CC1].

Proposition 7.1.3 *Let H, K be Hilbert modules over $\mathbb{A}$. Then*

$$Ext_{\mathbb{A}}(K, H) = Der(K, H)/Inn(K, H).$$

We note the following facts which will be useful. If $\alpha : K' \to K$ and $\beta : H \to H'$ are Hilbert module maps, then extensions $\alpha^*[E_D]$ and $\beta_*[E_D]$ are defined by derivations $D\alpha \in Der(K', H)$ and $\beta D \in Der(K, H')$, respectively, where $D\alpha(f) = D(f) \circ \alpha$ and $\beta D(f) = \beta \circ D(f)$ for $f \in \mathbb{A}$.

Using Propositions 7.1.2, 7.1.3 and the facts mentioned above one can establish the following Hom-Ext-sequences [CC1]. This will be our basic tool of computing extension groups.

Proposition 7.1.4 *Let*

$$E: \ 0 \longrightarrow H_1 \xrightarrow{\alpha} H_2 \xrightarrow{\beta} H_3 \longrightarrow 0$$

be an exact sequence of Hilbert modules over $\mathbb{A}$. Then for a Hilbert module H over $\mathbb{A}$, we have the following Hom-Ext-sequences:

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{A}}(H, H_1) &\xrightarrow{\alpha_*} Hom_{\mathbb{A}}(H, H_2) \xrightarrow{\beta_*} Hom_{\mathbb{A}}(H, H_3) \\ \xrightarrow{\delta} \ Ext_{\mathbb{A}}(H, H_1) &\xrightarrow{\alpha_*} Ext_{\mathbb{A}}(H, H_2) \xrightarrow{\beta_*} Ext_{\mathbb{A}}(H, H_3), \end{aligned}$$

where δ is the connecting homomorphism and is given by $\delta(\theta) = [E\theta]$ for $\theta : \ H \to H_3$, and

$$\begin{aligned} 0 \longrightarrow Hom_{\mathbb{A}}(H_3, H) &\xrightarrow{\beta^*} Hom_{\mathbb{A}}(H_2, H) \xrightarrow{\alpha^*} Hom_{\mathbb{A}}(H_1, H) \\ \xrightarrow{\delta} \ Ext_{\mathbb{A}}(H_3, H) &\xrightarrow{\beta^*} Ext_{\mathbb{A}}(H_2, H) \xrightarrow{\alpha^*} Ext_{\mathbb{A}}(H_1, H), \end{aligned}$$

where $\delta(\theta) = [\theta E]$ for $\theta : \ H_1 \to H$.

7.2 Extensions of hypo-Šilov modules over unit modulus algebras

This section is mainly based on [CG]. Let H_1, H_2 be two Hilbert modules over $\mathbb{A}$, and set

$$Hom^*_{\mathbb{A}}(H_1, H_2) = \{\alpha^* : \ \alpha \in Hom_{\mathbb{A}}(H_1, H_2)\}.$$

We have the following.

Proposition 7.2.1 *Under the above assumptions we have*

(1) $Hom^*_{\mathbb{A}}(H_1, H_2) = Hom_{\bar{\mathbb{A}}}(H_{2*}, H_{1*})$;

(2) *the extension group* $Ext_{\mathbb{A}}(H_1, H_2)$ *is naturally isomorphic to* $Ext_{\bar{\mathbb{A}}}(H_{2*}, H_{1*})$ *by the corresponding* $[E_D] \mapsto [E_{\bar{D}}]$*, where the derivation* $\bar{D} : \bar{\mathbb{A}} \to B(H_{2*}, H_{1*})$ *is defined by* $\bar{D}(\bar{f}) = D(f)^*$*.*

Proof. Obviously, (1) is true. For $D \in Der(H_1, H_2)$, we define a map $\bar{D} : \bar{\mathbb{A}} \to B(H_{2*}, H_{1*})$ by $\bar{D}(\bar{f}) = D(f)^*$, it is easy to verify that $\bar{D}$ is a derivation from $\bar{\mathbb{A}}$ to $B(H_{2*}, H_{1*})$ and the corresponding $D \mapsto \bar{D}$ is one to one and onto. By Proposition 7.1.3, the extension group $Ext_{\mathbb{A}}(H_1, H_2)$ is naturally isomorphic to $Ext_{\bar{\mathbb{A}}}(H_{2*}, H_{1*})$ by the corresponding $[E_D] \mapsto [E_{\bar{D}}]$.

Let $\mathbb{A}$ be a function algebra, $\mathbb{A} \subseteq C(\partial\mathbb{A})$, where $\partial\mathbb{A}$ is the Šilov boundary of $\mathbb{A}$. If $\mathcal{U}$ is a Hilbert module over $C(\partial\mathbb{A})$, a closed subspace $M \subseteq \mathcal{U}$ which is invariant for $\mathbb{A}$ is called a hypo-Šilov module over $\mathbb{A}$ and $\mathcal{U}$ is called a $C(\partial\mathbb{A})$-extension of M. A hypo-Šilov module over $\mathbb{A}$ is reductive if it is invariant for $C(\partial\mathbb{A})$ and pure if no nonzero subspace of it is reductive. Furthermore, if $\mathcal{U}$ is contractive over $C(\partial\mathbb{A})$, then we also call M a Šilov module over $\mathbb{A}$ (here by contraction we mean $\sup_{f\in\mathbb{A},\, \|f\|=1} \|fu\| \leq \|u\|$ for each $u \in \mathcal{U}$).

A function algebra $\mathbb{A} \subseteq C(\partial\mathbb{A})$ is called a unit modulus algebra if the set $\{f \in \mathbb{A} : |f(x)| = 1, \quad \text{all } x \in \partial\mathbb{A}\}$ generate $\mathbb{A}$. The example is the polydisk algebra $A(\mathbb{D}^n)$ on $\mathbb{T}^n$. In the following we write $\mathbb{A}_U$ for a fixed unit modulus algebra.

In the present section, our interest is to study extensions of Hilbert modules over the unit modulus algebra $\mathbb{A}_U$. Let $\mathcal{U}_0$ be a hypo-Šilov module over $\mathbb{A}_U$ and $\mathcal{U}$ be a $C(\partial\mathbb{A}_U)$- extension of $\mathcal{U}_0$. It follows that we have an exact sequence of Hilbert modules:

$$E_{\mathcal{U}_0} :\ 0 \longrightarrow \mathcal{U}_0 \stackrel{i}{\longrightarrow} \mathcal{U} \stackrel{\pi}{\longrightarrow} \mathcal{U} \ominus \mathcal{U}_0 \longrightarrow 0$$

where i is the inclusion map and π the quotient map, that is, π is the orthogonal projection $P_{\mathcal{U}\ominus\mathcal{U}_0}$ from $\mathcal{U}$ onto $\mathcal{U} \ominus \mathcal{U}_0$. As usual, the action of $\mathbb{A}_U$ on $\mathcal{U} \ominus \mathcal{U}_0$ is given by the formula $f \cdot h = P_{\mathcal{U}\ominus\mathcal{U}_0} T_f h$.

To further develop the properties of hypo-Šilov modules, we need the following terminology. Let $\mathcal{U}$ be any $C(\partial\mathbb{A}_U)$-extension of hypo-Šilov module $\mathcal{U}_0$ over $\mathbb{A}_U$. We call $\mathcal{U}$ a minimal $C(\partial\mathbb{A}_U)$-extension of $\mathcal{U}_0$, if $C(\partial\mathbb{A}_U) \cdot \mathcal{U}_0$ is dense in $\mathcal{U}$. The next proposition shows that minimal $C(\partial\mathbb{A}_U)$-extensions of a hypo-Šilov module over $\mathbb{A}_U$ are similar as $C(\partial\mathbb{A}_U)$-Hilbert modules.

Proposition 7.2.2 *Let* M_i *be hypo-Šilov modules over* $\mathbb{A}_U$ *and* $\mathcal{U}_i$ *be* $C(\partial\mathbb{A}_U)$*-extension of* $M_i, i = 1, 2$*. Then for each* $\theta \in Hom_{\mathbb{A}_U}(M_1, M_2)$*, it can lift to a* $C(\partial\mathbb{A}_U)$*-module map* $\theta' : \mathcal{U}_1 \to \mathcal{U}_2$*. Furthermore, if* $\mathcal{U}_1$ *is a minimal* $C(\partial\mathbb{A}_U)$*-extension of* M_1*, then lifting is unique.*

Proof. By Proposition 2.19 in [DP], the unit modulus algebra $\mathbb{A}_U$ is convexly approximating in modulus on $\partial\mathbb{A}_U$. Applying Theorems 1.9 and 2.20 in [DP] gives the proof of the proposition.

From Proposition 7.2.2, one finds that it is independent on the choice of $C(\partial\mathbb{A}_U)$-extensions of $\mathcal{U}_0$ that a hypo-Šilov module $\mathcal{U}_0$ over $\mathbb{A}_U$ is reductive or pure.

Let us again consider the short exact sequence

$$E_{\mathcal{U}_0}:\ 0 \longrightarrow \mathcal{U}_0 \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U} \ominus \mathcal{U}_0 \longrightarrow 0$$

in the category $\mathcal{H}(\mathbb{A}_U)$. By duality, this induces the exact sequence

$$E^*_{\mathcal{U}_0}:\ 0 \longrightarrow (\mathcal{U} \ominus \mathcal{U}_0)_* \xrightarrow{\pi^*} \mathcal{U}_* \xrightarrow{i^*} \mathcal{U}_{0*} \longrightarrow 0$$

in the category $\mathcal{H}(\bar{\mathbb{A}}_U)$. Thus $(\mathcal{U} \ominus \mathcal{U}_0)_*$ is a hypo-Šilov module over $\bar{\mathbb{A}}_U$. We may thus call $\mathcal{U} \ominus \mathcal{U}_0$ a cohypo-Šilov module over $\mathbb{A}_U$. From Proposition 7.2.2 one sees that a hypo-Šilov module over $\mathbb{A}_U$ is cohypo-Šilov if and only if it is reductive.

To prove the main theorem in this section, we need the following notation. Let G be a semigroup. An invariant mean of G is a state μ on $l^\infty(G)$ such that $\mu(F) = \mu({}_gF)$ for all $g \in G$ and $F \in l^\infty(G)$, where ${}_gF(g') = F(gg')$(recall that a state on a C^*-algebra is a positive linear functional of norm 1). A basic fact is that every abelian semigroup has an invariant mean (see [Pa]).

Theorem 7.2.3 *Let $\mathcal{U}$ be a $C(\partial\mathbb{A}_U)$-Hilbert module. Then for every Hilbert module K over $\mathbb{A}_U$, $Ext_{\mathbb{A}_U}(K,\mathcal{U}) = 0$ and $Ext_{\mathbb{A}_U}(\mathcal{U}, K) = 0$, where $\mathcal{U}$ is viewed as a Hilbert module over $\mathbb{A}_U$.*

Proof. By Proposition 7.1.3, it is enough to show that each $D \in Der(K,\mathcal{U})$ is inner. Equivalently, one must prove that there exists a bounded linear operator $T : K \to \mathcal{U}$ such that $D(f) = TT_f - T_fT$. To do this, we write $B_1(\mathcal{U}, K)$ for all trace class operators from $\mathcal{U}$ to K, $B(K,\mathcal{U})$ for all bounded linear operators from K to $\mathcal{U}$, and identify $B(K,\mathcal{U})$ with $B_1^*(\mathcal{U}, K)$ by setting

$$\langle T, C\rangle = tr(TC), \quad T \in B(K,\mathcal{U}),\ C \in B_1(\mathcal{U}, K).$$

Let μ be an invariant mean of the multiplication semigroup $U_{\mathbb{A}_U}$, where $U_{\mathbb{A}_U}$ is $\{\eta \in \mathbb{A}_U : |\eta(x)| = 1, \text{ all } x \in \partial\mathbb{A}_U\}$. We define $T \in B(K,\mathcal{U}) = B_1^*(\mathcal{U}, K)$ by

$$\langle T, C\rangle = \mu_\eta(\langle T^{(\mathcal{U})}_{\bar{\eta}} D(\eta), C\rangle),$$

that is, $\langle T, C\rangle$ is the mean of the bounded complex function

$$\eta \mapsto \langle T^{(\mathcal{U})}_{\bar{\eta}} D(\eta), C\rangle,$$

where $T^{(\mathcal{U})}_f$ denotes the multiplication by f on $\mathcal{U}$ for $f \in C(\partial\mathbb{A}_U)$. For each $\eta' \in U_{\mathbb{A}_U}$, we have

$$\begin{aligned}
&\langle T_{\eta'}^{(\mathcal{U})}T - TT_{\eta'}^{(K)}, C\rangle = \langle T, CT_{\eta'}^{(\mathcal{U})} - T_{\eta'}^{(K)}C\rangle \\
&= \mu_\eta(\langle T_{\bar\eta}^{(\mathcal{U})}D(\eta), CT_{\eta'}^{(\mathcal{U})} - T_{\eta'}^{(K)}C\rangle) \\
&= \mu_\eta(\langle T_{\bar\eta\eta'}^{(\mathcal{U})}D(\eta) - T_{\bar\eta}^{(\mathcal{U})}D(\eta)T_{\eta'}^{(K)}, C\rangle) \\
&= \mu_\eta(\langle T_{\bar\eta\eta'}^{(\mathcal{U})}D(\eta) - T_{\bar\eta}^{(\mathcal{U})}(D(\eta\eta') - T_{\eta}^{(\mathcal{U})}D(\eta')), C\rangle) \\
&= \mu_\eta(\langle D(\eta'), C\rangle) + \mu_\eta(\langle T_{\bar\eta\eta'}^{(\mathcal{U})}D(\eta) - T_{\bar\eta}^{(\mathcal{U})}D(\eta\eta'), C\rangle) \\
&= \langle D(\eta'), C\rangle + \mu_\eta(\langle T_{\bar\eta\eta'}^{(\mathcal{U})}D(\eta), C\rangle) - \mu_{\eta\eta'}(\langle T_{\overline{\eta\eta'}\eta'}^{(\mathcal{U})}D(\eta\eta'), C\rangle) \\
&= \langle D(\eta'), C\rangle
\end{aligned}$$

for all $C \in B_1(\mathcal{U}, K)$, so that $D(\eta') = T_{\eta'}^{(\mathcal{U})}T - TT_{\eta'}^{(K)}$ for each $\eta' \in U_{\mathbb{A}_U}$. Since $U_{\mathbb{A}_U}$ generate $\mathbb{A}_U$, this implies $D = D_T$, that is, D is an inner derivation. Thus $Ext_{\mathbb{A}_U}(K,\mathcal{U}) = \{0\}$.

Since $\mathbb{A}_U$ is the unit modulus algebra, the adjoint algebra $\bar{\mathbb{A}}_U$ is also unit modulus. Now applying Proposition 7.2.1 (2) gives that $Ext_{\mathbb{A}_U}(\mathcal{U}, K) = \{0\}$.

As is well known, projective modules form the cornerstone for the study of general modules in homological algebra. Here we introduce projective modules in the context of Hilbert modules [DP]. Let $\mathbb{A}$ be a function algebra. A Hilbert module H over $\mathbb{A}$ is projective if for every pair of Hilbert module H_1, H_2 over $\mathbb{A}$ and every pair module maps $\psi : H \to H_2$ and $\phi : H_1 \to H_2$ with ϕ onto, there exists a module map $\tilde\psi$ such that $\phi\tilde\psi = \psi$. Dually, a Hilbert module H over $\mathbb{A}$ is called injective if for every pair H_1, H_2 over $\mathbb{A}$ and every pair of module maps $\psi : H_1 \to H$ and $\phi : H_1 \to H_2$, with ϕ one-to-one and having closed range, there exists a module map $\tilde\psi : H_2 \to H$ such that $\psi = \tilde\psi\phi$.

The following corollary comes immediately from Theorem 7.2.3 and Proposition 7.1.4.

Corollary 7.2.4 *Let $\mathbb{A}_U \subseteq C(\partial\mathbb{A}_U)$ be the unit modulus algebra, and $\mathcal{U}$ be a $C(\partial\mathbb{A}_U)$-Hilbert module. Then $\mathcal{U}$, viewed as $\mathbb{A}_U$-Hilbert module, is projective and injective.*

Remark 7.2.5 *(1) Douglas and Paulsen [DP] asked whether there is any function algebra, other than $C(X)$, with any (nonzero) projective module (see Problem 4.6 in [DP]). From Corollary 7.2.4, we see that there exist nonzero projective modules over every unit modulus algebra. In the cases of the disk algebra $A(\mathbb{D})$ and polydisk algebra $A(\mathbb{D}^n)$, these results were obtained in [CCFW, CC2].*

(2) In the purely algebraic setting, one knows from [HS] that there is not a nonzero module that is projective and injective over every principle ideal domain (other than a field). Hence, Corollary 7.2.4 shows that Hilbert modules are distinct from modules in the purely algebraic setting.

Corollary 7.2.6 *Let $\mathcal{U}_0$ be a hypo-Šilov module over $\mathbb{A}_U$ and $\mathcal{U}$ be any $C(\partial\mathbb{A}_U)$-extension of $\mathcal{U}_0$. Then the following statements are equivalent:*

(1) $\mathcal{U}_0$ is injective,

(2) $\mathcal{U}\ominus\mathcal{U}_0$ is projective,

(3) $\mathcal{U}_0$ is reductive,

(4) the short exact sequence

$$E_{\mathcal{U}_0}: \quad 0 \longrightarrow \mathcal{U}_0 \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U}\ominus\mathcal{U}_0 \longrightarrow 0$$

is split.

Proof. Since $\mathcal{U}$ is projective and injective, this implies that (1), (2) and (4) are equivalent. From Corollary 7.2.4, it is easy to see that (3) leads to (1). If $E_{\mathcal{U}_0}$ is split, then there is a split map $\sigma : \mathcal{U}\ominus\mathcal{U}_0 \to \mathcal{U}$ such that

$$\pi\sigma = I_{\mathcal{U}\ominus\mathcal{U}_0}.$$

Taking any $\xi \in \mathcal{U}_0$ and each $\eta \in \mathbb{A}_U$ with unit modulus, we write

$$T_{\bar{\eta}}^{(\mathcal{U})}\xi = \xi_1 + \xi_2,\ \xi_1 \in \mathcal{U}_0,\ \xi_2 \in \mathcal{U}\ominus\mathcal{U}_0.$$

Hence

$$\xi = T_{\eta}^{(\mathcal{U})}\xi_1 + T_{\eta}^{(\mathcal{U})}\xi_2.$$

This implies the following:

$$\pi(T_{\eta}^{(\mathcal{U})}\xi_2) = T_{\eta}^{(\mathcal{U}\ominus\mathcal{U}_0)}\xi_2 = 0.$$

We have

$$\sigma(T_{\eta}^{(\mathcal{U}\ominus\mathcal{U}_0)}\xi_2) = T_{\eta}^{(\mathcal{U})}\sigma(\xi_2) = 0,$$

i.e., $\sigma(\xi_2) = 0$. Since σ is an injective Hilbert module map, it follows easily that $\xi_2 = 0$. Thus, $\mathcal{U}_0$ is reductive. This completes the proof of Corollary 7.2.6.

According to Corollary 7.2.4, the following "Hom-Isomorphism" theorem will result which states that there exists a natural isomorphism between Hom of hypo-Šilov modules and that of the corresponding cohypo-Šilov modules.

Theorem 7.2.7 *Let M_1, M_2 be hypo-Šilov modules over $\mathbb{A}_U$, with M_2 being pure. If $\mathcal{U}_1$ is the minimal $C(\partial\mathbb{A}_U)$-extension of M_1, and $\mathcal{U}_2$ a $C(\partial\mathbb{A}_U)$-extension of M_2, then the following are isomorphic as $\mathbb{A}_U$-module:*

$$Hom_{\mathbb{A}_U}(M_1, M_2) \cong Hom_{\mathbb{A}_U}(\mathcal{U}_1\ominus M_1, \mathcal{U}_2\ominus M_2).$$

The isomorphism is given by $\beta(\theta) = P_{\mathcal{U}_2\ominus M_2}\theta' \mid_{\mathcal{U}_1\ominus M_1}$ for $\theta \in Hom_{\mathbb{A}_U}(M_1, M_2)$, where θ' is that uniquely determined by θ using Proposition 7.2.2.

Proof. Theorem 7.2.7 can be reduced to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{i_1} & \mathcal{U}_1 & \xrightarrow{\pi_1} & \mathcal{U}_1 \ominus M_1 & \longrightarrow & 0 \\ & & \downarrow \theta & & \downarrow \theta' & & \downarrow \beta(\theta) & & \\ 0 & \longrightarrow & M_2 & \xrightarrow{i_2} & \mathcal{U}_2 & \xrightarrow{\pi_2} & \mathcal{U}_2 \ominus M_2 & \longrightarrow & 0 \end{array}$$

where i_1, i_2 are the inclusion maps and π_1, π_2 the quotient Hilbert module maps. By Proposition 7.2.2, it is easy to see that

$$\beta : \, Hom_{\mathbb{A}_U}(M_1, M_2) \to Hom_{\mathbb{A}_U}(\mathcal{U}_1 \ominus M_1, \mathcal{U}_2 \ominus M_2)$$

is a $\mathbb{A}_U$-module map, where the module structure of $Hom_{\mathbb{A}_U}(M_1, M_2)$ is given by $(f \cdot \theta)(h) = \theta(f \cdot h)$ for $f \in \mathbb{A}_U, h \in M_1$; the definition of the module structure of $Hom_{\mathbb{A}_U}(\mathcal{U}_1 \ominus M_1, \mathcal{U}_2 \ominus M_2)$ is similar to that for $Hom_{\mathbb{A}_U}(M_1, M_2)$. Since M_2 is pure, Proposition 7.2.2 implies that β is injective. Since $\mathcal{U}_1$ is projective, this ensures that β is surjective. The proof is complete.

Note that the polydisk algebra $A(\mathbb{D}^n)$ is a unit modulus algebra. Let us turn to extensions of Hilbert modules over the polydisk algebra. To do this, let Γ be a subset of $L^2(\mathbb{T}^n, m_n)$ (in short, $L^2(\mathbb{T}^n)$), where m_n is the measure $1/(2\pi)^n d\theta_1 d\theta_2 \cdots d\theta_n$ on $\mathbb{T}^n$). We say that a Borel set $E \subseteq \mathbb{T}^n$ is the support of Γ, and denoted by $S(\Gamma)$, if each function from Γ vanishes on $\mathbb{T}^n - E$, and for any Borel subset E' of E with $m_n(E') > 0$, there exists a function $f \in \Gamma$ such that $f\,|_{E'} \neq 0$. For a submodule M of $L^2(\mathbb{T}^n)$ over $A(\mathbb{D}^n)$, it is not difficult to prove that $\chi_{S(M)} L^2(\mathbb{T}^n)$ is its minimal $C(\mathbb{T}^n)$-extension, where $\chi_{S(M)}$ is the characteristic function of $S(M)$. We also note that a submodule N of $L^2(\mathbb{T}^n)$ is pure if and only if $m_n(S(N^\perp)) = 1$. Then from Theorem 7.2.7 we derive the following.

Corollary 7.2.8 *Let M_1 and M_2 be submodules of $L^2(\mathbb{T}^n)$, and $m_n(S(M_1)) = m_n(S(M_2^\perp)) = 1$. Then*

$$Hom_{A(\mathbb{D}^n)}(M_1, M_2) \cong Hom_{A(\mathbb{D}^n)}(L^2(\mathbb{T}^n) \ominus M_1, L^2(\mathbb{T}^n) \ominus M_2).$$

The isomorphism is given by $\varphi \mapsto H_\varphi^{[M_2]}\,|_{L^2(\mathbb{T}^n) \ominus M_1}$, where $H_\varphi^{[M_2]}$ is defined by $H_\varphi^{[M_2]} f = P_{L^2(\mathbb{T}^n) \ominus M_2}(\varphi f)$ for all $f \in L^2(\mathbb{T}^n)$.

Example 7.2.9 *Let $H^2(\mathbb{D}^n)$ be the usual Hardy module, and $H^2(\mathbb{D}^n)^\perp$ be the corresponding quotient module $L^2(\mathbb{T}^n) \ominus H^2(\mathbb{D}^n)$. Then by Corollary 7.2.8, we have*

$$Hom_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n)^\perp, H^2(\mathbb{D}^n)^\perp) \cong H^\infty(\mathbb{D}^n).$$

If we define a Hankel-type operator A_f for $f \in L^\infty(\mathbb{T}^n)$ by

$$A_f(h) = P_{H^2(\mathbb{D}^n)^\perp}(fh), \quad h \in H^2(\mathbb{D}^n)^\perp,$$

then the commutators of $\{A_{z_1}, \cdots, A_{z_n}\}$ are equal to $\{A_f : f \in H^\infty(\mathbb{D}^n)\}$.

Before going on we deduce a homological formula to compute extension groups over unit modulus algebras. Let $\mathcal{U}_0$ be a hypo-Šilov module over $\mathbb{A}_U$ and $\mathcal{U}$ be any $C(\partial\mathbb{A}_U)$-extension of $\mathcal{U}_0$. If M is a Hilbert module over $\mathbb{A}_U$ and $\theta \in Hom_{\mathbb{A}_U}(M, \mathcal{U} \ominus \mathcal{U}_0)$, then we define a derivation $D_\theta \in Der(M, \mathcal{U}_0)$ by

$$D_\theta(f) = P_{\mathcal{U}_0} T_f^{(\mathcal{U})} \theta,$$

where $P_{\mathcal{U}_0}$ is the orthogonal projection from $\mathcal{U}$ onto $\mathcal{U}_0$. Furthermore, for every $\theta \in Hom_{\mathbb{A}_U}(\mathcal{U}_0, M)$, we define a derivation $D^\theta \in Der(\mathcal{U} \ominus \mathcal{U}_0, M)$ by

$$D^\theta(f) = \theta P_{\mathcal{U}_0} T_f^{(\mathcal{U})}.$$

Combining the exact sequence

$$E_{\mathcal{U}_0}: \quad 0 \longrightarrow \mathcal{U}_0 \xrightarrow{i} \mathcal{U} \xrightarrow{\pi} \mathcal{U} \ominus \mathcal{U}_0 \longrightarrow 0$$

with the Hom-Ext-sequence (see Proposition 7.1.4), we see that the connecting homomorphism

$$\delta_1 : Hom_{\mathbb{A}_U}(M, \mathcal{U} \ominus \mathcal{U}_0) \to Ext_{\mathbb{A}_U}(M, \mathcal{U}_0)$$

is given by $\delta_1(\theta) = [D_\theta]$, and

$$\delta_2 : Hom_{\mathbb{A}_U}(\mathcal{U}_0, M) \to Ext_{\mathbb{A}_U}(\mathcal{U} \ominus \mathcal{U}_0, M)$$

is given by $\delta_2(\theta) = [D^\theta]$. Since $\mathcal{U}$ is projective, applying Proposition 7.1.4 gives the following.

Theorem 7.2.10 *We let $\mathcal{U}_0$ be a hypo-Šilov module and $\mathcal{U}$ any $C(\partial\mathbb{A}_U)$-extension of $\mathcal{U}_0$. Then for each Hilbert module M over $\mathbb{A}_U$, we have*

(1) $Ext_{\mathbb{A}_U}(M, \mathcal{U}_0) \cong coker(\pi_* : Hom_{\mathbb{A}_U}(M, \mathcal{U}) \to Hom_{\mathbb{A}_U}(M, \mathcal{U} \ominus \mathcal{U}_0))$. The correspondence is given by $\delta_1(\theta) = [D_\theta]$ for $\theta \in Hom_{\mathbb{A}_U}(M, \mathcal{U} \ominus \mathcal{U}_0)$.

(2) $Ext_{\mathbb{A}_U}(\mathcal{U} \ominus \mathcal{U}_0, M) \cong coker(i^* : Hom_{\mathbb{A}_U}(\mathcal{U}, M) \to Hom_{\mathbb{A}_U}(\mathcal{U}_0, M))$. The correspondence is given by $\delta_2(\theta) = [D^\theta]$ for $\theta \in Hom_{\mathbb{A}_U}(\mathcal{U}_0, M)$.

Remark 7.2.11 *Theorem 7.2.10 provides us a very valid method to calculate extension groups of Hilbert modules over $\mathbb{A}_U$. In particular, if one of M, $\mathcal{U}_0$ or $\mathcal{U} \ominus \mathcal{U}_0$ is cyclic or co-cyclic, then the characterization of $Ext_{\mathbb{A}_U}$-groups may be summed up as the actions of module maps on cyclic vectors, or co-cyclic vectors, where we use the concept of co-cyclic Hilbert modules, which means that M is a co-cyclic Hilbert module over $\mathbb{A}$ if and only if M_* is a cyclic Hilbert module over $\bar{\mathbb{A}}$.*

Below are several examples to show applications of Theorem 7.2.10.

Example 7.2.12 *Let* $H^2(\mathbb{D})$ *be the usual Hardy module over* $A(\mathbb{D})$*. Then for any Hilbert module* K *over* $A(\mathbb{D})$*,* $Ext_{A(\mathbb{D})}(K, H^2(\mathbb{D}))$ *is characterized as an* $A(\mathbb{D})$*-module by the following:*

$$Ext_{A(\mathbb{D})}(K, H^2(\mathbb{D})) \cong K_1/K_0,$$

where K_1 *is given by* $\{\theta^*(\bar{z}) : \theta \in Hom_{A(\mathbb{D})}(K, L^2(\mathbb{T}) \ominus H^2(\mathbb{D}))\}$*, and* $K_0 = \{L^*(\bar{z}) : L \in Hom_{A(\mathbb{D})}(K, L^2(\mathbb{T}^n))\}$*. The action of* $A(\mathbb{D})$ *on* K_1 *is given by* $f \cdot \theta^*(\bar{z}) \triangleq \theta^*(\overline{fz})$*; the action of* $A(\mathbb{D})$ *on* K_0 *is similar to that of* $A(\mathbb{D})$ *on* K_1*.*

Remark. This example first appeared in [CC1]. If K is a weighted Hardy module on the unit disk, Ferguson obtained an explicit characterization for the extension group $Ext_{A(\mathbb{D})}(K, H^2(\mathbb{D}))$ [Fe1].

In what follows we will verify this example. First notice that every derivation $D : A(\mathbb{D}) \to B(K, H^2(\mathbb{D}))$ is completely determined by $D(z)$. For each $\theta \in Hom_{A(\mathbb{D})}(K, L^2(\mathbb{T}^n) \ominus H^2(\mathbb{D}))$, one sees that $\delta_1(\theta)$ is the extension determined by D_θ. Note

$$D_\theta(z) = P_{H^2(\mathbb{D})} T_z^{(L^2(\mathbb{T}^n))} \theta = \langle \cdot, \theta^*(\bar{z}) \rangle.$$

Suppose that there exists a Hilbert module map $L : K \to L^2(\mathbb{T}^n)$ such that $\theta^*(\bar{z}) = L^*(\bar{z})$. Then it is easy to see that D_θ is inner. Conversely, if D_θ is inner, then there is a bounded linear operator $A : K \to H^2(\mathbb{D})$ such that

$$D_\theta(f) = P_{H^2(\mathbb{D})} T_f^{(L^2(\mathbb{T}^n))} \theta = A T_f^{(K)} - T_f^{(H^2(\mathbb{D}))} A.$$

Define an operator $L : K \to L^2(\mathbb{T}^n)$ by $Lk = Ak + \theta k,\ k \in K$. Since

$$\begin{aligned} fLk &= T_f^{(H^2(\mathbb{D}))} Ak + T_f^{(L^2(\mathbb{T}^n))} \theta k \\ &= T_f^{(H^2(\mathbb{D}))} Ak + P_{H^2(\mathbb{D})} T_f^{(L^2(\mathbb{T}^n))} \theta k + P_{H^2(\mathbb{D})^\perp} T_f^{(L^2(\mathbb{T}^n))} \theta k \\ &= A T_f^{(K)} k + \theta(fk), \end{aligned}$$

it follows that L is a Hilbert module map from K to $L^2(\mathbb{T}^n)$. Moreover, for any $k \in K$, we see

$$\langle L(k), \bar{z} \rangle = \langle \theta(k), \bar{z} \rangle.$$

This leads to $\theta^*(\bar{z}) = L^*(\bar{z})$, and thus the example is verified.

Let M be a submodule of $L^2(\mathbb{T}^n)$, and $\varphi \in L^2(\mathbb{T}^n)$. We define a Hankel operator $H_\varphi^{(M)} : H^2(\mathbb{D}^n) \to L^2(\mathbb{T}^n) \ominus M$ with symbol ϕ by

$$H_\varphi^{(M)} f = P_{L^2(\mathbb{T}^n) \ominus M}(\varphi f),\ f \in A(\mathbb{D}^n).$$

Then Hankel operator $H_\varphi^{(M)}$ is densely defined. We write $B(M)$ for the set of all $\varphi \in L^2(\mathbb{T}^n)$ such that Hankel operator $H_\varphi^{(M)}$ can be continuously extended onto $H^2(\mathbb{D}^n)$. It is easy to check that for every $\varphi \in B(M)$,

$H_\varphi^{(M)}$ is a module map from $H^2(\mathbb{D}^n)$ to $L^2(\mathbb{T}^n) \ominus M$, and each module map $\beta : H^2(\mathbb{D}^n) \to L^2(\mathbb{T}^n) \ominus M$ has such a form, that is, there exists an $\varphi \in B(M)$ such that $\beta = H_\varphi^{(M)}$. In particular, if $M = H^2(\mathbb{D}^n)$, then $B(H^2(\mathbb{D}^n))$ is equal to $BMO_r + H^2(\mathbb{D}^n)$, where BMO_r is the restricted BMO space introduced in [CS]. Furthermore, for a nonzero Hardy submodule $M_0 (\subset H^2(\mathbb{D}^n))$, we define a function space $B(M_0, M)$ by $\varphi \in B(M_0, M)$ if $\varphi \in B(M)$ and $\ker H_\varphi^{(M)} \supseteq M_0$.

Example 7.2.13 *Applying Theorem 7.2.10, we have the following.*

(1) $Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), M) \cong B(M)/(L^\infty(\mathbb{T}^n) + M)$,

(2) $Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), H^2(\mathbb{D}^n)) \cong (BMO_r + H^2(\mathbb{D}^n))/(L^\infty(\mathbb{T}^n) + H^2(\mathbb{D}^n))$,

(3) $Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_0, M) \cong B(M_0, M)/M$.

Remark 7.2.14 *Note that in the case* $n = 1$,

$$BMO_r + H^2(\mathbb{D}) = L^\infty(\mathbb{T}) + H^2(\mathbb{D}).$$

This implies that $Ext_{A(\mathbb{D})}(H^2(\mathbb{D}), H^2(\mathbb{D})) = 0$. *This result was first obtained in [CC1]. However, for* $n > 1$, *by [CS], it is easy to check that*

$$BMO_r + H^2(\mathbb{D}^n) \underset{\neq}{\supset} L^\infty(\mathbb{T}^n) + H^2(\mathbb{D}^n),$$

and hence

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n), H^2(\mathbb{D}^n)) \neq 0.$$

This result first appeared in [CC2].

If two Hardy submodules M_1, M_2 satisfy

$$0 \neq M_1 \subseteq M_2 \underset{\neq}{\subsetneq} H^2(\mathbb{D}^n),$$

then $1 \in B(M_1, M_2)$. This implies that

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) \neq 0$$

by Example 7.2.13. Therefore, for Hardy submodules M_1, M_2 and $M_1 \neq 0$, if $Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) = 0$, then there is not a proper Hardy submodule M_3 such that $M_3 \supseteq M_1$, and M_3 is similar to M_2.

The next example recaptures some information of rigidity of Hardy submodules.

Example 7.2.15 *Let* $F \in H^\infty(\mathbb{D}^n)$ *be quasi-outer. Then*

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^2) \ominus [F], H^2(\mathbb{D}^n)) = 0.$$

Hence there is no proper Hardy submodule M *such that* $[F] \subseteq M$ *and* M *is similar to* $H^2(\mathbb{D}^n)$.

Let $\varphi \in B([F], H^2(\mathbb{D}^n))$. Then $\varphi F \in H^2(\mathbb{D}^n)$. Since F is quasi-outer, the relation $\varphi F \in H^2(\mathbb{D}^n)$ implies $\varphi \in H^2(\mathbb{D}^n)$. Thus by Example 7.2.13 (3),

$$Ext_{A(\mathbb{D}^2)}(H^2(\mathbb{D}^2) \ominus [F], H^2(\mathbb{D}^2)) = 0.$$

Moreover, we shall prove the following by using the techniques in [ACD].

Proposition 7.2.16 *For $n > 1$, let M_1 be of finite codimension in $H^2(\mathbb{D}^n)$ and $M_2 \subseteq H^2(\mathbb{D}^n)$. Then*

$$M_2 \subseteq B(M_1, M_2) \subseteq H^2(\mathbb{D}^n).$$

In particular, if $M_1 \subseteq M_2$, then $B(M_1, M_2) = H^2(\mathbb{D}^n)$.

Proof. For any $\varphi \in B(M_1, M_2)$, write

$$\varphi = \sum_{s=-\infty}^{+\infty} f_s(z_2, \cdots, z_n) z_1^s.$$

Since M_1 has finite codimension in $H^2(\mathbb{D}^n)$, for sufficiently large integer l, some nonzero linear combination $\sum_{j=0}^{l} c_j z_2^j$ of the functions $1, z_2, z_2^2, \cdots, z_2^l$ belongs to M_1. Thus, $\varphi \sum_{j=0}^{l} c_j z_2^j$ is in M_2. Since

$$\varphi \sum_{j=0}^{l} c_j z_2^j = \sum_{s=-\infty}^{+\infty} [f_s \sum_{j=0}^{l} c_j z_2^j] z_1^s \in M_2,$$

this yields $f_s = 0$ for $s < 0$. Treating the expansion of φ relative to the other variables $z_2, z_3, \cdots, z_n$ shows that φ is in $H^2(\mathbb{D}^n)$. Note that the inclusion $M_2 \subseteq B(M_1, M_2)$ is clear. Thus we have

$$M_2 \subseteq B(M_1, M_2) \subseteq H^2(\mathbb{D}^n).$$

In particular, if $M_1 \subseteq M_2$, then $H^\infty(\mathbb{D}^n) \subseteq B(M_1, M_2)$. Since M_2 is of finite codimension in $H^2(\mathbb{D}^n)$, by Remark 2.5.5 $H^2(\mathbb{D}^n) \ominus M_2$ is contained in $H^\infty(\mathbb{D}^n)$. We thus conclude that $B(M_1, M_2) = H^2(\mathbb{D}^n)$.

Corollary 7.2.17 *Let M_1 be of finite codimension in $H^2(\mathbb{D}^n)$ and $M_1 \subseteq M_2 \subseteq H^2(\mathbb{D}^n)$. If $n > 1$, then*

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M_1, M_2) \cong H^2(\mathbb{D}^n) \ominus M_2.$$

From Corollary 7.2.17, one finds that for $n > 1$, a finite codimensional Hardy submodule $M(\neq H^2(\mathbb{D}^n))$ is never similar to $H^2(\mathbb{D}^n)$. The reason is

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M, H^2(\mathbb{D}^n)) = 0,$$

but

$$Ext_{A(\mathbb{D}^n)}(H^2(\mathbb{D}^n) \ominus M, M) \cong H^2(\mathbb{D}^n) \ominus M.$$

Of course, this is not new observation; we may compare it with that in [ACD].

7.3 Extensions of Hilbert modules and Hankel operators

In this section we will concentrate attention on studying extensions of the Hardy module $H^2(\mathbb{D})$ and the Bergman module $L_a^2(\mathbb{D})$ on the disk algebra $A(\mathbb{D})$. This section is based mainly on Guo's paper [Guo5]. When one studies the Hardy module and Bergman module over the disk algebra, Hankel operators will play an important role in this context. In almost all cases, one must investigate the symbol spaces of corresponding Hankel operators. On one hand, by using such symbol spaces, one can give the explicit expressions of *Ext*-groups. On the other hand, by the *Hom-Ext* sequences, one can determine when a Hankel operator is bounded. We say that a Hilbert module H over the disk algebra $A(\mathbb{D})$ is cramped if the multiplication by z on H is similar to a contraction. Let $\mathcal{H}(A(\mathbb{D}))$ be the category of all Hilbert modules over $A(\mathbb{D})$ together with module maps, and $\mathfrak{C}$ the category of all cramped Hilbert modules over $A(\mathbb{D})$ and module maps. It is easy to check that $\mathfrak{C}$ is a subcategory of $\mathcal{H}(A(\mathbb{D}))$ and is full in $\mathcal{H}(A(\mathbb{D}))$. This means that for H_1, $H_2 \in \mathfrak{C}$, the set of maps from H_1 to H_2 is the same as in $\mathcal{H}(A(\mathbb{D}))$, i.e., $Hom_{\mathfrak{C}}(H_1, H_2) = Hom_{A(\mathbb{D})}(H_1, H_2)$.

As we have seen, what seems to make things most difficult is that the category $\mathcal{H}(A(\mathbb{D}))$ lacks enough projective or injective objects. If one replaces $\mathcal{H}(A(\mathbb{D}))$ by the category $\mathfrak{C}$, Carlson, Clark, Foias and Williams [CCFW] proved that the category $\mathfrak{C}$ has enough projective and injective objects, and the Šilov resolution [DP] of a contractive module gives a projective resolution. From [CCFW, DP], it is not difficult to see that a Hilbert module in $\mathfrak{C}$ is projective if and only if it is similar to isometric Hilbert module; a Hilbert module is both projective and injective if and only if it is similar to a unitary Hilbert module; by an isometric (unitary) Hilbert module H we mean that the operator of multiplication by z on H is an isometry (a unitary operator). In [CCFW], the concept of extension in the category $\mathfrak{C}$ was introduced, and this concept is completely analogous to that in the category $\mathcal{H}(A(\mathbb{D}))$. It is easy to see that for H_1 and H_2 in $\mathfrak{C}$, there is a canonically injective $A(\mathbb{D})$-module map

$$i : Ext_{\mathfrak{C}}(H_1, H_2) \to Ext_{A(\mathbb{D})}(H_1, H_2).$$

One thus often works in the category $\mathfrak{C}$ instead of the category $\mathcal{H}(A(\mathbb{D})$. In the category $\mathfrak{C}$, it is easy to prove the following proposition [CCFW].

Proposition 7.3.1 $Ext_{\mathfrak{C}}(-,-)$ *is a bifunctor from* $\mathfrak{C}$ *to the category of* A-*modules. It is contravariant in the first and covariant in the second variable.*

Similar to the category $\mathcal{H}(A(\mathbb{D}))$, in the category $\mathfrak{C}$, we have *Hom-Ext* sequences [CCFW].

Proposition 7.3.2 *Let*

$$E : 0 \longrightarrow H_1 \xrightarrow{\alpha} H_2 \xrightarrow{\beta} H_3 \longrightarrow 0$$

be an exact sequence of cramped Hilbert modules. Then for each cramped Hilbert module H we have the following Hom-Ext sequences:

$$\begin{aligned} 0 \longrightarrow Hom(H,H_1) &\xrightarrow{\alpha_*} Hom(H,H_2) \xrightarrow{\beta_*} Hom(H,H_3) \\ \xrightarrow{\delta} Ext_{\mathfrak{C}}(H,H_1) &\xrightarrow{\alpha_*} Ext_{\mathfrak{C}}(H,H_2) \xrightarrow{\beta_*} Ext_{\mathfrak{C}}(H,H_3), \end{aligned}$$

where δ is the connecting homomorphism and is given by $\delta(\theta) = [E\theta]$ for $\theta:\ H \to H_3$, and

$$\begin{aligned} 0 \longrightarrow Hom(H_3,H) &\xrightarrow{\beta^*} Hom(H_2,H) \xrightarrow{\alpha^*} Hom(H_1,H) \\ \xrightarrow{\delta} Ext_{\mathfrak{C}}(H_3,H) &\xrightarrow{\beta^*} Ext_{\mathfrak{C}}(H_2,H) \xrightarrow{\alpha^*} Ext_{\mathfrak{C}}(H_1,H), \end{aligned}$$

where $\delta(\theta) = [\theta E]$ for $\theta:\ H_1 \to H$.

From the above *Hom-Ext* sequences, we immediately obtain the following proposition which generalizes Proposition 3.2.6 in [CC1]. The next proposition was first proved in [Fe2].

Proposition 7.3.3 *If H_1, H_2 are similar to isometric Hilbert modules, then*

$$Ext_{\mathfrak{C}}(H_1,H_2) = Ext_{A(\mathbb{D})}(H_1,H_2) = 0.$$

Proof. By the Wold decomposition, we may suppose $H_2 = H^2(H)$ for some finite or infinite dimensional Hilbert space H. One thus has the exact sequence of Hilbert modules:

$$E:\ 0 \to H^2(H) \xrightarrow{i} L^2(H) \xrightarrow{\pi} H^2(H)^{\perp} \to 0.$$

Since H_1 is projective in $\mathfrak{C}$, this forces the sequence

$$0 \to Hom(H_1,H^2(H)) \xrightarrow{i_*} Hom(H_1,L^2(H)) \xrightarrow{\pi_*} Hom(H_1,H^2(H)^{\perp}) \to 0$$

to be exact. From this fact and $L^2(H)$ being projective in $\mathcal{H}(A(\mathbb{D}))$, applying the *Hom-Ext* sequences gives

$$Ext_{A(\mathbb{D})}(H_1,H^2(H)) = 0, \quad \text{and hence} \quad Ext_{\mathfrak{C}}(H_1,H^2(H)) = 0.$$

The desired conclusion follows, completing the proof.

In studying extensions of the Hardy and the Bergman modules, we will be concerned with the following four kinds of Hankel operators: from $H^2(\mathbb{D})$ to $H^2(\mathbb{D})^{\perp}$; from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})^{\perp}$; from $L^2_a(\mathbb{D})$ to $H^2(\mathbb{D})^{\perp}$; from $L^2_a(\mathbb{D})$ to $L^2_a(\mathbb{D})^{\perp}$. For their definitions, we only see the case from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})^{\perp}$. Let ϕ be in $L^2(\mathbb{D})$. A densely defined Hankel operator $H_\phi:\ H^2(\mathbb{D}) \to L^2_a(\mathbb{D})^{\perp}$ is defined by $H_\phi h = (I-P)\phi h,\ h \in A(\mathbb{D})$, where P is the orthogonal projection from $L^2(\mathbb{D})$ to $L^2_a(\mathbb{D})$. Therefore, Hankel operator H_ϕ is bounded if and only if H_ϕ in $H^2(\mathbb{D})$ can be continuously extended onto $H^2(\mathbb{D})$.

Let μ be a positive finite measure on $\mathbb{D}$. The measure μ is called an i-th ($i = 1, 2$) Carleson measure if there is a constant c such that $\mu(S) \leq c\,h^i$ for each Carleson square $S = \{z = re^{i\theta} |\, 1-h \leq r \leq 1;\, \theta_0 \leq \theta \leq \theta_0 + h\}$. For $\phi \in L^2(\mathbb{D})$, we say that ϕ is an i-th Carleson function if $|\phi|^2 dA$ is an i-th Carleson measure ($i = 1, 2$), where dA is the usual normalized area measure on the unit disk. Write $S_i(\mathbb{D})$ for the set of all i-th Carleson functions ($i = 1, 2$). Then it is easily seen that $S_i(\mathbb{D})$ are $A(\mathbb{D})$–modules. It is well known that a Hankel operator H_ϕ from $H^2(\mathbb{D})$ to $H^2(\mathbb{D})^\perp$ is bounded if and only if $\phi \in L^\infty(\mathbb{T}) + H^2(\mathbb{D})$. This fact, translated into homological language, is equivalent to the exactness of the following sequence:

$$0 \to Hom(H^2(\mathbb{D}), H^2(\mathbb{D})) \xrightarrow{i_*} Hom(H^2(\mathbb{D}), L^2(\mathbb{T})) \xrightarrow{\pi_*} Hom(H^2(\mathbb{D}), H^2(\mathbb{D})^\perp) \to 0.$$

It is easily verified that each bounded Hankel operator H_ϕ from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})^\perp$ is a Hilbert module map, and every Hilbert module map from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})^\perp$ is a bounded Hankel operator. Since $H^2(\mathbb{D})$ is projective in the category $\mathfrak{C}$, one thus obtains

Proposition 7.3.4 *A Hankel operator H_ϕ from $H^2(\mathbb{D})$ to $L^2_a(\mathbb{D})^\perp$ is bounded if and only if $\phi \in S_1(\mathbb{D}) + L^2_a(\mathbb{D})$.*

Proof. Since $H^2(\mathbb{D})$ is projective in the category $\mathfrak{C}$ (see [CCFW]), we have the following exact Hom sequence:

$$0 \to Hom(H^2(\mathbb{D}), L^2_a(\mathbb{D})) \xrightarrow{i_*} Hom(H^2(\mathbb{D}), L^2(\mathbb{D})) \xrightarrow{\pi_*} Hom(H^2(\mathbb{D}), L^2_a(\mathbb{D})^\perp) \to 0.$$

Assume that H_ϕ is bounded. Hence, the above exact sequence ensures that there is an $\alpha \in Hom(H^2(\mathbb{D}), L^2(\mathbb{D}))$ such that $H_\phi = \pi_*(\alpha)$. It is easily seen that there exists a $\phi_0 \in L^2(\mathbb{D})$ such that for any $f \in H^2(\mathbb{D})$, $\alpha(f) = \phi_0 f$. This induces that ϕ_0 is a 1-th Carleson function by Theorem 9.3 in [Dur]. We conclude thus that $\phi - \phi_0 \in L^2_a(\mathbb{D})$. The opposite direction is achieved by considering the above exact sequence and Theorem 9.3 in [Dur], completing the proof.

"The Halmos problem" asks: whether each polynomially bounded operator is similar to a contraction [Hal]. Pisier's recent example [Pi2] gives a negative answer for "the Halmos problem." Equivalently, the category $\mathfrak{C}$ is a proper subcategory of $\mathcal{H}(A(\mathbb{D}))$. An interesting question is whether the category $\mathcal{H}(A(\mathbb{D}))$ has enough projectives in the sense that every object in $\mathcal{H}(A(\mathbb{D}))$ is a quotient of some projective module.

Theorem 7.3.5 *Let H be similar to a nonunitarily isometric module. Then*

$$Ext_{\mathfrak{C}}(L^2_a(\mathbb{D}), H) = Ext_{A(\mathbb{D})}(L^2_a(\mathbb{D}), H) \neq 0.$$

Proof. If H is similar to a unitary module, then H is injective in $\mathcal{H}(A(\mathbb{D}))$ by Corollary 7.2.4, and therefore

$$Ext_{\mathfrak{C}}(L^2_a(\mathbb{D}), H) = Ext_{A(\mathbb{D})}(L^2_a(\mathbb{D}), H) = 0.$$

Now by the Wold decomposition of an isometry, we only need to prove

$$Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) = Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(H)) \neq 0$$

for some finite or infinite dimensional Hilbert space H. Consider the exact sequence

$$E: \ 0 \to H^2(H) \xrightarrow{i} L^2(H) \xrightarrow{\pi} H^2(H)^{\perp} \to 0.$$

Since $L^2(H)$ is projective in $\mathcal{H}(A(\mathbb{D}))$ (also in $\mathfrak{C}$), applying the *Hom-Ext* sequences gives the following:

$$\begin{aligned} 0 &\to Hom(L_a^2(\mathbb{D}), H^2(H)) \xrightarrow{i_*} Hom(L_a^2(\mathbb{D}), L^2(H)) \\ &\xrightarrow{\pi_*} Hom(L_a^2(\mathbb{D}), H^2(H)^{\perp}) \xrightarrow{\delta} Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(H)) \to 0 \\ 0 &\to Hom(L_a^2(\mathbb{D}), H^2(H)) \xrightarrow{i_*} Hom(L_a^2(\mathbb{D}), L^2(H)) \\ &\xrightarrow{\pi_*} Hom(L_a^2(\mathbb{D}), H^2(H)^{\perp}) \xrightarrow{\delta} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) \to 0. \end{aligned}$$

We claim

$$Hom(L_a^2(\mathbb{D}), L^2(H)) = 0.$$

In fact, for any $\alpha \in Hom(L_a^2(\mathbb{D}), L^2(H))$, since

$$\|\alpha(z^n)\| \ = \ \|z^n \alpha(1)\| \ = \ \|\alpha(1)\| \ \leq \ \|\alpha\| [\int |z^n|^2 dA]^{\frac{1}{2}} \to 0$$

as $n \ \to \ \infty$, it follows that $\alpha(1) \ = \ 0$. This means that

$$\alpha(f) = f\alpha(1) = 0, \ \forall f \in A(\mathbb{D}).$$

The claim follows. Combining the above exact sequences with the claim immediately shows that

$$Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) = Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(H)) = Hom(L_a^2(\mathbb{D}), H^2(H)^{\perp}).$$

For $\phi \in L^2(H)$, a densely defined Hankel operator $H_\phi : L_a^2(\mathbb{D}) \to H^2(H)^{\perp}$ is defined by $H_\phi h = (I - P)\phi h; \ h \in A(\mathbb{D})$, where P is the orthogonal projection from $L^2(H)$ onto $H^2(H)^{\perp}$. If the densely defined operator H_ϕ in $L_a^2(\mathbb{D})$ can be continuously extended onto $L_a^2(\mathbb{D})$, then it is easy to check that H_ϕ is a Hilbert module map from $L_a^2(\mathbb{D})$ to $H^2(H)^{\perp}$, and each Hilbert module map from $L_a^2(\mathbb{D})$ to $H^2(H)^{\perp}$ has such a form. Writing $\mathcal{S}(L_a^2(\mathbb{D}), H^2(H))$ for the set of all such ϕ, then

$$\begin{aligned} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) &= Ext_{(A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(H)) \\ &= \mathcal{S}(L_a^2(\mathbb{D}), H^2(H))/H^2(H) \end{aligned}.$$

Taking any $h \in H$ with $\|h\| = 1$, it is easy to prove that $\phi = \bar{z}h$ is in $\mathcal{S}(L_a^2(\mathbb{D}), H^2(H))$, but ϕ is not in $H^2(H)$. This gives the desired conclusion, completing the proof.

Remark 7.3.6 *From the proof of Theorem 7.3.5, one can deduce the following explicit formula for* $Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) = Ext_{\mathcal{H}(A(\mathbb{D}))}(L_a^2(\mathbb{D}), H^2(H))$, *that is,*

$$\begin{aligned} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(H)) &= Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(H)) \\ &= \mathcal{S}(L_a^2(\mathbb{D}), H^2(H))/H^2(H) \end{aligned}.$$

Let $C_H(\mathbb{T})$ be the set of all continuous functions ϕ on the unit circle $\mathbb{T}$ such that Hankel operator $H_\phi : L_a^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ is bounded. Obviously, $C_H(\mathbb{T})$ is an $A(\mathbb{D})$–module.

Corollary 7.3.7 *We have*

$$Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(\mathbb{D})) = Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(\mathbb{D})) = C_H(\mathbb{T})/A(\mathbb{D}).$$

Proof. Let ϕ be in $L^2(\mathbb{T})$ and $H_\phi : L_a^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ be a bounded Hankel operator. Since $H^2(\mathbb{D})$ is contractively contained in $L_a^2(\mathbb{D})$, one concludes that $H_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ is a bounded Hankel operator. This shows that there is a $\phi_0 \in L^\infty(\mathbb{T})$ such that $\phi - \phi_0 \in H^2(\mathbb{D})$. Suppose that $\{h_k\} \subset H^2(\mathbb{D})$, and $\{h_k\}$ weakly converge to 0 in $H^2(\mathbb{D})$. Let

$$h_k = \sum_{n\geq 0} a_n^{(k)} z^n$$

be $h_k' s$ power series expansion. Since $\{\|h_k\|_{H^2(\mathbb{D})}\}$ is bounded, there exists a constant c_1 such that $\sum_{n\geq 0} |a_n^{(k)}|^2 < c_1$ for all k. On the other hand, since $H_\phi : L_a^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ is bounded, there exists a constant c_2 such that

$$\|H_\phi h_k\|_{L^2(\mathbb{T})}^2 = \|H_{\phi_0} h_k\|_{L^2(\mathbb{T})}^2 \leq c_2 \int_{\mathbb{D}} |h_k|^2 dA = c_2 [\sum_{n\geq 0} \frac{|a_n^{(k)}|^2}{n+1}].$$

For each fixed n, since $a_n^{(k)} \to 0$ as $k \to \infty$, the above discussion implies that $\|H_{\phi_0} h_k\|_{L^2(\mathbb{T})}^2 \to 0$ as $k \to \infty$, and therefore H_{ϕ_0} is a compact Hankel operator from $H^2(\mathbb{D})$ to $H^2(\mathbb{D})^\perp$. Thus ϕ_0 is in $H^\infty(\mathbb{D}) + C(\mathbb{T})$, where $C(\mathbb{T})$ is the set of all continuous functions on $\mathbb{T}$. It now follows that the operator $H_\phi : L_a^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ is bounded if and only if there is a $\psi \in C_H(\mathbb{T})$ such that $\phi - \psi \in H^2(\mathbb{D})$. Hence from Remark 7.3.6, we have

$$\begin{aligned} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), H^2(\mathbb{D})) &= Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H^2(\mathbb{D})) \\ &\cong [C_H(\mathbb{T}) + H^2(\mathbb{D})]/H^2(\mathbb{D}) \\ &\cong C_H(\mathbb{T})/A(\mathbb{D}). \end{aligned}$$

This completes the proof.

Remark 7.3.8 *From the proof of Corollary 7.3.7, we observe that a Hankel operator*

$$H_\phi : L_a^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$$

is bounded if and only if $\phi \in C_H(\mathbb{T}) + H^2(\mathbb{D})$.

From the *Hom-Ext* sequences, it is easy to see that a Hilbert module H in $\mathcal{H}(A(\mathbb{D}))$ (resp. in $\mathfrak{C}$) is injective if and only if $Ext_{A(\mathbb{D})}(\tilde{H}, H) = 0$ (resp. $Ext_{\mathfrak{C}}(\tilde{H}, H) = 0$) for each Hilbert module $\tilde{H}$ in $\mathcal{H}(A(\mathbb{D}))$ (resp. in $\mathfrak{C}$). The question naturally arises as to whether it is necessary to use all Hilbert modules $\tilde{H}$ in $Ext(\tilde{H}, H)$ to test whether H is injective; might it not happen that there exists a small family of Hilbert modules H_λ such that if $Ext(H_\lambda, H) = 0$ for each H_λ in the family, then H is injective? By Theorem 7.3.5, when H is similar to an isometric module, then H is injective in $\mathcal{H}(A(\mathbb{D})$ if and only if $Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), H) = 0$. Hence the Bergman module $L_a^2(\mathbb{D})$ seems to play a role to test injective modules. In the following we shall show that $Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \neq 0$, and therefore $Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \neq 0$.

Consider the exact sequence

$$E: \ 0 \to L_a^2(\mathbb{D}) \xrightarrow{i} L^2(\mathbb{D}) \xrightarrow{\pi} L_a^2(\mathbb{D})^\perp \to 0.$$

Hence it induces the following *Hom-Ext* sequence:

$$\begin{aligned} 0 &\to Hom(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \xrightarrow{i_*} Hom(L_a^2(\mathbb{D}), L^2(\mathbb{D})) \xrightarrow{\pi_*} Hom(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})^\perp) \\ &\xrightarrow{\delta} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \xrightarrow{i_*} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L^2(\mathbb{D})) \xrightarrow{\pi_*} Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})^\perp). \end{aligned}$$

To prove $Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \neq 0$, it suffices to show that the map

$$\pi_*: \ Hom(L_a^2(\mathbb{D}), L^2(\mathbb{D})) \to Hom(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})^\perp)$$

is not surjective. To get the desired conclusion, we first give the descriptions of $Hom(L_a^2(\mathbb{D}), L^2(\mathbb{D}))$ and $Hom(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})^\perp)$. Let $\alpha \in Hom(L_a^2(\mathbb{D}), L^2(\mathbb{D}))$. It is easy to check that α is a multiplier from $L_a^2(\mathbb{D})$ to $L^2(\mathbb{D})$, that is, there is a $\phi \in L^2(\mathbb{D})$ such that $\alpha = M_\phi$. According to [Ha1], M_ϕ is a multiplier from $L_a^2(\mathbb{D})$ to $L^2(\mathbb{D})$ if and only if ϕ is a 2-th Carleson function. We thus have

$$Hom(L_a^2(\mathbb{D}), L^2(\mathbb{D})) = S_2(\mathbb{D}).$$

Write $B(\mathbb{D})$ for all those $\phi \in L^2(\mathbb{D})$ such that $H_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})^\perp$ is bounded. Clearly, $B(\mathbb{D})$ is an $A(\mathbb{D})$-module, and

$$Hom(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})^\perp) \cong B(\mathbb{D})/L_a^2(\mathbb{D}).$$

The above observations lead to the conclusion

$$coker\pi_* = B(\mathbb{D})/[S_2(\mathbb{D}) + L_a^2(\mathbb{D})].$$

Theorem 7.3.9 $B(\mathbb{D}) \supsetneqq S_2(\mathbb{D}) + L_a^2(\mathbb{D})$

Before proving this theorem we give the following corollary and remark.

Corollary 7.3.10 *The following are true:*

(1) $Ext_{\mathfrak{C}}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \neq 0$, *and*

(2) $Ext_{A(\mathbb{D})}(L_a^2(\mathbb{D}), L_a^2(\mathbb{D})) \neq 0$.

Remark 7.3.11 *Let $\mathcal{B}$ be a Banach $A(\mathbb{D})$-bimodule. A derivation δ from $A(\mathbb{D})$ to $\mathcal{B}$, by definition, is a continuously linear map such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all a, $b \in A(\mathbb{D})$. Let $\mathcal{B}(L_a^2(\mathbb{D}))$ (resp., $\mathcal{B}(H^2(\mathbb{D}))$) denote the set of all bounded linear operators on $L_a^2(\mathbb{D})$ (resp., $H^2(\mathbb{D})$) with natural $A(\mathbb{D})$-bimodule structure. By Remark 7.2.14, we see that each derivation from $A(\mathbb{D})$ to $\mathcal{B}(H^2(\mathbb{D}))$ is inner. However, Corollary 7.3.10 indicates that there is some derivation from $A(\mathbb{D})$ to $\mathcal{B}(L_a^2(\mathbb{D}))$ that is not inner. This may be an essential difference between the Hardy module and the Bergman module over $A(\mathbb{D})$.*

The proof of Theorem 7.3.9. To prove this theorem, it is enough to find a function b such that $b \in B(\mathbb{D})$, but $b \notin S_2(\mathbb{D}) + L_a^2(\mathbb{D})$. First note that the analytic function $f(z) = \sum_{n=0}^{\infty} z^{2^n}$ is in $L_a^2(\mathbb{D})$. We claim that $f(z)$ is in the Bloch space. This claim is easily from the theorem on Madamard power series [Dur]. For the reader's convenience, we give the proof's details here. That is, we must prove the following:

$$\sup\{(1-|z|^2)|f'(z)| \,:\, z \in \mathbb{D}\} < +\infty.$$

Since $f'(z) = \sum_{n=0}^{\infty} 2^n z^{2^n - 1}$, one needs to show

$$\sum_{n=0}^{\infty} 2^n r^{2^n-1} = O(\frac{1}{1-r})$$

for $0 < r < 1$. This is equivalent to estimating the integral

$$\int_0^{\infty} 2^x r^{2^x-1} dx = \frac{1}{\ln 2} \int_1^{\infty} r^{t-1} dt \;\; (t = 2^x).$$

Since $\ln r < r - 1$, the second integral is majorized by

$$\int_1^{\infty} r^{t-1} dt = \int_1^{\infty} e^{(t-1)\ln r} dt \leq \int_1^{\infty} e^{(r-1)(t-1)} dt = \frac{1}{1-r}$$

which proves the claim, and hence $f(z)$ is in the Bloch space. It follows by [Axl, Theorem 6] that Hankel operator $H_{\bar{f}}$ is bounded. The rest of the proof is to check $\bar{f} \notin S_2(\mathbb{D}) + L_a^2(\mathbb{D})$.

For any $z \in \mathbb{D}$, let k_z be the normalized Bergman-reproducing kernel at z (i.e., $\|k_z\| = 1$). If there is a $\phi \in S_2(\mathbb{D})$ and a $\psi \in L_a^2(\mathbb{D})$ such that $\phi = \bar{f} + \psi$, then

$$|\bar{f}(z) + \psi(z)| = |\int (\bar{f} + \psi)|k_z|^2 dA| = \langle \phi k_z, k_z \rangle \leq \|M_\phi\|.$$

This implies that $\phi(z) = \bar{f}(z) + \psi(z)$ is a bounded harmonic function. Since $\bar{f}(re^{i\theta}) - \bar{f}(0)$ and $\psi(re^{i\theta}) + \bar{f}(0)$ are orthogonal in $L^2(\mathbb{T}^n)$, it follows that

$$\|\phi(re^{i\theta})\|^2_{L^2(\mathbb{T}^n)} = \|f(re^{i\theta}) - f(0)\|^2_{H^2(\mathbb{D})} + \|\psi(re^{i\theta}) + \bar{f}(0)\|^2_{H^2(\mathbb{D})}.$$

Since ϕ is bounded, the above equality shows that there exists some positive constant c such that $\|f(re^{i\theta}) - f(0)\|_{H^2(\mathbb{D})} \leq c$ for any r. Thus, f is in $H^2(\mathbb{D})$. This is impossible and hence $S_2(\mathbb{D}) + L^2_a(\mathbb{D})$ is a proper subset of $B(\mathbb{D})$, completing the proof.

7.4 Extensions of normal Hilbert module over the ball algebra

Let $\mathbb{B}_n$ be the unit ball of $\mathbb{C}^n$ and $A(\mathbb{B}_n)$ be the so-called ball algebra, i.e., the set of all functions continuous on the closure $\overline{\mathbb{B}_n}$ of $\mathbb{B}_n$ and holomorphic on $\mathbb{B}_n$. A Hilbert module H over $A(\mathbb{B}_n)$ is called normal if for every $h \in H$, the map $f \mapsto fh$ is continuous from weak*-topology of $L^\infty(\partial\mathbb{B}_n, d\sigma)$ restricted to $A(\mathbb{B}_n)$ to the weak-topology on H, where $d\sigma$ is the normalized sphere area measure on the boundary $\partial\mathbb{B}_n$ ($\{\xi \in \mathbb{C}^n : \|\xi\| = 1\}$) with $\sigma(\partial\mathbb{B}_n) = 1$. Some obvious facts are that the category $\mathcal{N}$ of all normal Hilbert modules over $A(\mathbb{B}_n)$ is a proper subcategory of the category $\mathcal{H}(A(\mathbb{B}_n))$ of all Hilbert modules, and $\mathcal{N}$ is full in $\mathcal{H}(A(\mathbb{B}_n))$ which means that if $N_1, N_2 \in \mathcal{N}$, then the set of all module maps from N_1 to N_2 in $\mathcal{N}$ is the same as in $\mathcal{H}(A(\mathbb{B}_n))$, that is, $Hom_{\mathcal{N}}(N_1, N_2) = Hom_{A(\mathbb{B}_n)}(N_1, N_2)$.

The present section comes from [Guo3]. In this section, we shall work in the category $\mathcal{N}$. For N_1, N_2 in $\mathcal{N}$, a normal extension of N_2 by N_1 is an exact sequence of normal Hilbert modules, which begins by N_1 and ends by N_2:

$$E:\ 0 \longrightarrow N_1 \xrightarrow{\alpha} N \xrightarrow{\beta} N_2 \longrightarrow 0,$$

where α, β are Hilbert module maps and N is normal. Under the equivalence relation described as in Section 7.1, the set of equivalence classes of all normal extensions of N_2 by N_1, denoted by $Ext_{\mathcal{N}}(N_2, N_1)$, is called normal extension group of N_2 by N_1. Similar to extensions in Section 7.1, we can verify the following proposition.

Proposition 7.4.1 *$Ext_{\mathcal{N}}(-,-)$ is a bifunctor from $\mathcal{N}$ to the category of $A(\mathbb{B}_n)$-modules. It is contravariant in the first and covariant in the second variable.*

Let $B(N_2, N_1)$ be all bounded linear operators from N_2 to N_1. If a bounded derivation $D : A(\mathbb{B}_n) \to B(N_2, N_1)$ satisfies that for each $h \in N_2$, the map $f \mapsto D(f)h$ is continuous from the weak*-topology of $L^\infty(\partial\mathbb{B}_n, d\sigma)$ restricted to $A(\mathbb{B}_n)$ to the weak-topology on N_1, then we call D normal. Let $Dern(N_2, N_1)$ denote all normal derivations from $A(\mathbb{B}_n)$ to $B(N_2, N_1)$ and $Inn(N_2, N_1)$ all inner derivations. Just as in Section 7.1, normal extensions are closely related to the derivation problem. For a normal extension of N_2 by N_1

$$E: \ 0 \longrightarrow N_1 \xrightarrow{\alpha} N \xrightarrow{\beta} N_2 \longrightarrow 0,$$

there exists a normal derivation $D \in Dern(N_2, N_1)$ such that E is equivalent to the following extension E_D defined by D:

$$E_D: \ 0 \longrightarrow N_1 \xrightarrow{i} N_1 \tilde{\oplus} N_2 \xrightarrow{\pi} N_2 \longrightarrow 0,$$

where $N_1 \tilde{\oplus} N_2$ is Hilbert space direct sum of N_1 and N_2 with the $A(\mathbb{B}_n)$-module structure defined by $f(h_1, h_2) = (fh_1 + D(f)h_2, fh_2)$. It is easy to check that such a module is normal. Finally, two extensions E_{D_1} and E_{D_2} are equivalent if and only if $D_1 - D_2$ is inner.

Proposition 7.4.2 *Let N_1, N_2 be normal. Then*

$$Ext_{\mathcal{N}}(N_2, N_1) = Dern(N_2, N_1)/Inn(N_2, N_1).$$

Moreover, if $\alpha: N_1 \to N_1'$ and $\beta: N_2' \to N_2$ are Hilbert module maps, with regard to pullbacks and pushouts in the category $\mathcal{N}$, the extensions αE_D and $E_D\beta$ are defined by derivations $\alpha D \in Dern(N_2, N_1')$ and $D\beta \in Dern(N_2', N_1)$, respectively, where $\alpha D(f) = \alpha(D(f))$ and $D\beta(f) = D(f)\beta$ for $f \in A(\mathbb{B}_n)$.

Now it is time to establish the following *Hom-Ext* sequences in $\mathcal{N}$ whose proof is similar to the preceding sections. This will be our basic tool of computing normal extension groups.

Proposition 7.4.3 *Let*

$$E: \ 0 \longrightarrow N_1 \xrightarrow{\alpha} N_2 \xrightarrow{\beta} N_3 \longrightarrow 0$$

be an exact sequence of normal Hilbert modules over $A(\mathbb{B}_n)$. Then for each normal Hilbert module N, we have the following Hom-Ext sequences:

$$\begin{aligned} 0 \longrightarrow \ & Hom(N, N_1) \xrightarrow{\alpha_*} Hom(N, N_2) \xrightarrow{\beta_*} Hom(N, N_3) \\ \xrightarrow{\delta} \ & Ext_{\mathcal{N}}(N, N_1) \xrightarrow{\alpha_*} Ext_{\mathcal{N}}(N, N_2) \xrightarrow{\beta_*} Ext_A(N, N_3), \end{aligned}$$

where δ is the connecting homomorphism and is given by $\delta(\theta) = [E\theta]$ for $\theta: \ N \to N_3$, and

$$\begin{aligned} 0 \longrightarrow \ & Hom(N_3, N) \xrightarrow{\beta^*} Hom(N_2, N) \xrightarrow{\alpha^*} Hom(N_1, N) \\ \xrightarrow{\delta} \ & Ext_{\mathcal{N}}(N_3, N) \xrightarrow{\beta^*} Ext\mathcal{N}(N_2, N) \xrightarrow{\alpha^*} Ext_{\mathcal{N}}(N_1, N), \end{aligned}$$

where $\delta(\theta) = [\theta E]$ for $\theta: \ N_1 \to N$.

Let H be a normal Hilbert module over $C(\partial\mathbb{B}_n)$ ($\subset L^\infty(\partial\mathbb{B}_n, d\sigma)$), where $C(\partial\mathbb{B}_n)$ is the algebra consisting of all continuous functions on $\partial\mathbb{B}_n$. By Kaplansky's density theorem [Arv7] and a simple continuity argument, H can be extended into a normal Hilbert module over $L^\infty(\partial\mathbb{B}_n, d\sigma)$ without change of module bound. Moreover, from [DP] one sees that H is similar to a normally contractive Hilbert module over $C(\partial\mathbb{B}_n)$. Hence we concentrate attention on normally contractive Hilbert modules over $C(\partial\mathbb{B}_n)$. Let N be a normally contractive Hilbert module over $C(\partial\mathbb{B}_n)$. A closed subspace $M \subseteq N$ which is invariant for $A(\mathbb{B}_n)$ is called a normal Šilov module for $A(\mathbb{B}_n)$ and N is called a normally contractive $C(\partial\mathbb{B}_n)$-extension of M. A normal Šilov module for $A(\mathbb{B}_n)$ is reductive if it is invariant for $C(\partial\mathbb{B}_n)$ and pure if no nonzero subspace of it is reductive.

The following proposition is basic for our analysis. It can help us to calculate $Ext_{\mathcal{N}}$-groups of some normal Hilbert modules over $A(\mathbb{B}_n)$.

Proposition 7.4.4 *For any N in $\mathcal{N}$, the module action of $A(\mathbb{B}_n)$ on N can be uniquely extended to $H^\infty(\mathbb{B}_n)$ without change of the module bound of N, making N into a normal $H^\infty(\mathbb{B}_n)$-Hilbert module, where $H^\infty(\mathbb{B}_n)$ denotes the set of all bounded and holomorphic functions on $\mathbb{B}_n$.*

Proof. By [CGa, Corollary 2.3], the unit ball of $A(\mathbb{B}_n)$ is weak*-dense in the unit ball of $H^\infty(\mathbb{B}_n)$. Also, since $H^\infty(\mathbb{B}_n)$ is weak*-closed and the unit ball of $H^\infty(\mathbb{B}_n)$ is weak*-compact and weak*-metrizable, a simple continuity argument implies that Proposition 7.4.4 is true.

According to Proposition 7.4.4, it is easily seen that the category $\mathcal{N}$ is essentially the same as the category $\mathcal{N}^\infty$, where $\mathcal{N}^\infty$ is the category of all normal Hilbert modules over $H^\infty(\mathbb{B}_n)$. This thus implies that Propositions 7.4.1, 7.4.2 and 7.4.3 are valid in the category $\mathcal{N}^\infty$. For $N_1, N_2 \in \mathcal{N}$, since $Ext_{\mathcal{N}}(N_2, N_1)$ is isomorphic to $Ext_{\mathcal{N}^\infty}(N_2, N_1)$ as $A(\mathbb{B}_n)$-modules, this ensures that $Ext_{\mathcal{N}}(N_2, N_1)$ can be extended into $H^\infty(\mathbb{B}_n)$-module. Note that all inner functions generate $H^\infty(\mathbb{B}_n)$ in the weak*-topology [Ru4]. Hence just as the proof of Theorem 7.2.3, if N is a normal $C(\partial\mathbb{B}_n)$-Hilbert module, then for every normal Hilbert module K over $A(\mathbb{B}_n)$, we have

$$Ext_{\mathcal{N}^\infty}(K, N) = 0, \quad Ext_{\mathcal{N}^\infty}(N, K) = 0.$$

Thus, we get the following theorem.

Theorem 7.4.5 *Let N be a normal $C(\partial\mathbb{B}_n)$-Hilbert module (of course, $N \in \mathcal{N}$). Then for every normal Hilbert module K over $A(\mathbb{B}_n)$, we have*

$$Ext_{\mathcal{N}}(K, N) = 0, \quad Ext_{\mathcal{N}}(N, K) = 0.$$

For $P \in \mathcal{N}$, we say that P is normally projective if for each pair $N_1, N_2 \in \mathcal{N}$, and every pair Hilbert module maps $\psi : P \to N_2$ and $\phi : N_1 \to N_2$ with ϕ onto, there exists a Hilbert module map $\tilde{\psi} : P \to N_1$ such that $\psi = \phi\tilde{\psi}$. Also, for $I \in \mathcal{N}$, I is called normally injective if for every pair $N_1, N_2 \in \mathcal{N}$ and every pair Hilbert module maps $\psi : N_1 \to I$, and $\phi : N_1 \to N_2$ with ϕ one-to-one and having closed range, there exists a Hilbert module map $\tilde{\psi} : N_2 \to I$ such that $\psi = \tilde{\psi}\phi$.

Now applying Theorem 7.4.5 we have the following.

Corollary 7.4.6 *Let N be a normal Hilbert module over $C(\partial\mathbb{B}_n)$. Then N (viewed as $A(\mathbb{B}_n)$-Hilbert module) is both normally projective and normally injective.*

Remark 7.4.7 *Although we do not know if there is any nonzero projective module over the ball algebra $A(\mathbb{B}_n)(n > 1)$, Corollary 7.4.6 guarantees that there exist normal projective modules and normal injective modules.*

Similar to the proof of Corollary 7.2.6, we have the following.

Corollary 7.4.8 *Let N_0 be a normal Šilov module over $A(\mathbb{B}_n)$ and N be a normally contractive $C(\partial\mathbb{B}_n)$-extension of N_0. Then the following statements are equivalent:*

(1) N_0 is normally injective,

(2) $N \ominus N_0$ is normally projective,

(3) N_0 is reductive,

(4) the short exact sequence

$$E_{N_0}: \quad 0 \longrightarrow N_0 \xrightarrow{i} N \xrightarrow{\pi} N \ominus N_0 \longrightarrow 0$$

is split,

where i is the inclusion map and π the quotient map, that is, π is the orthogonal projection $P_{N\ominus N_0}$ from N onto $N \ominus N_0$. As usual, the action of $A(\mathbb{B}_n)$ on $N \ominus N_0$ is given by the formula $f \cdot h = P_{N\ominus N_0} T_f^{(N)} h$ for $f \in A(\mathbb{B}_n)$ and $h \in N \ominus N_0$.

For a normal Šilov module M, let N be any normally contractive $C(\partial\mathbb{B}_n)$ -extension of M. We say that N is minimal if $C(\partial\mathbb{B}_n) \cdot M$ is dense in N. From [DP, Corollary 2.14], one can show that the minimal normally contractive $C(\partial\mathbb{B}_n)$-extension of M is essentially unique.

Lemma 7.4.9 *Let M_i be normal Šilov modules over $A(\mathbb{B}_n)$ and N_i be normally contractive $C(\partial\mathbb{B}_n)$-extensions of M_i, $i = 1, 2$. Then for each $\theta \in Hom(M_1, M_2)$, it can lift to a $C(\partial\mathbb{B}_n)$-Hilbert module map $\theta' : N_1 \to N_2$. Furthermore, if N_1 is minimal, then lifting is unique.*

Proof. By Proposition 7.4.4, we may regard M_i as normally contractive Hilbert modules over $H^\infty(\mathbb{B}_n)$ and N_i as normally contractive Hilbert modules over $L^\infty(\partial\mathbb{B}_n, d\sigma)$ for $i = 1, 2$. Set

$$D = \{\bar{\eta}h \ : \ h \in M_1, \, \eta \text{ are inner functions}\}.$$

It is easy to check that D is a linear subspace of N_1. Because all inner functions and their adjoints generate $L^\infty(\partial\mathbb{B}_n, d\sigma)$ in the weak*-topology (see [Ru4]), it follows that the closure $\overline{D}$ of D is an $L^\infty(\partial\mathbb{B}_n, d\sigma)$-Hilbert submodule of N_1. If we set $\theta''(\bar{\eta}h) = \bar{\eta}\theta(h)$ for inner function η and $h \in M_1$, then it is easy to check that θ'' is well defined and θ'' can be continuously extended into an $L^\infty(\partial\mathbb{B}_n, d\sigma)$-Hilbert module map from $\overline{D}$ to N_2. Hence, if we use θ' to denote an $L^\infty(\partial\mathbb{B}_n, d\sigma)$-Hilbert module map from N_1 to N_2 by setting $\theta'(h) = \theta''(h), h \in \overline{D}$; $\theta'(h) = 0, h \in N_1 \ominus \overline{D}$, then the map θ' is a $C(\partial\mathbb{B}_n)$-lifting of θ. In particular, if N_1 is minimal, then lifting is unique.

Applying Lemma 7.4.9 and the proof of Theorem 7.2.7 gives the following.

Theorem 7.4.10 *Let M_1, M_2 be normal Šilov modules over $A(\mathbb{B}_n)$, and N_1 be a minimal normally contractive $C(\partial\mathbb{B}_n)$-extension of M_1. If N_2 is a normally contractive $C(\partial\mathbb{B}_n)$-extension of M_2 and M_2 is pure, then the following are isomorphic as $A(\mathbb{B}_n)$-modules:*

$$Hom_{\mathcal{N}}(M_1, M_2) \cong Hom_{\mathcal{N}}(N_1 \ominus M_1, N_2 \ominus M_2).$$

The isomorphism is given by

$$\beta(\theta) = P_{N_2 \ominus M_2}\theta' \mid_{N_1 \ominus M_1}, \ \theta \in Hom_{\mathcal{N}}(M_1, M_2),$$

where θ' is one given by Lemma 7.4.9, and $P_{N_2 \ominus M_2}$ is the orthogonal projection from N_2 onto $N_2 \ominus M_2$.

For Hardy submodules, Theorem 7.4.10 has a natural form. To see this, let Γ be a subset of $L^2(\partial\mathbb{B}_n, d\sigma)$. A Borel set $E \subseteq \partial\mathbb{B}_n$ is said to be the support of Γ (denoted by $S(\Gamma)$) if each function from Γ vanishes on $\partial\mathbb{B}_n - E$; and for any Borel subset E' of E with $\sigma(E') > 0$, there exists a function $f \in \Gamma$ such that $f \mid_{E'} \neq 0$. For a submodule M of $L^2(\partial\mathbb{B}_n, d\sigma)$ over $A(\mathbb{B}_n)$, it is not difficult to prove that $\chi_{S(M)}L^2(\partial\mathbb{B}_n, d\sigma)$ is its minimal $C(\partial\mathbb{B}_n)$-extension, where $\chi_{S(M)}$ is the characteristic function of $S(M)$. We also note that a submodule N of $L^2(\partial\mathbb{B}_n, d\sigma)$ is pure if and only if $\sigma(S(N^\perp)) = 1$. By Theorem 7.4.10 we immediately obtain the following.

Corollary 7.4.11 *Let M_1 and M_2 be submodules of $L^2(\partial\mathbb{B}_n, d\sigma)$ over $A(\mathbb{B}_n)$, and $\sigma(S(M_1)) = \sigma(S(M_2^\perp)) = 1$. Then*

$$Hom_{\mathcal{N}}(M_1, M_2) \cong Hom_{\mathcal{N}}(L^2(\partial\mathbb{B}_n, d\sigma) \ominus M_1, L^2(\partial\mathbb{B}_n, d\sigma) \ominus M_2).$$

The isomorphism is given by $\varphi \mapsto H_\varphi^{[M_2]} \mid_{L^2(\partial\mathbb{B}_n, d\sigma) \ominus M_1}$, where $H_\varphi^{[M_2]}$ is defined by $H_\varphi^{[M_2]} f = P_{L^2(\partial\mathbb{B}_n, d\sigma) \ominus M_2}(\varphi f)$ for all $f \in L^2(\partial\mathbb{B}_n, d\sigma)$.

Example 7.4.12 *Let* $H^2(\mathbb{B}_n)$ *be the Hardy module over* $A(\mathbb{B}_n)$ *and* $H^2(\mathbb{B}_n)^\perp(= L^2(\partial\mathbb{B}_n, d\sigma) \ominus H^2(\mathbb{B}_n))$ *be the corresponding quotient module. By Corollary 7.4.11, one obtains that*

$$Hom_{\mathcal{N}}(H^2(\mathbb{B}_n)^\perp, H^2(\mathbb{B}_n)^\perp) \cong H^\infty(\mathbb{B}_n).$$

If we define a Hankel-type operator A_f *for* f *in* $L^\infty(\partial\mathbb{B}_n, d\sigma)$ *by*

$$A_f(h) = P_{L^2(\partial\mathbb{B}_n,d\sigma)\ominus H^2(\mathbb{B}_n)}(fh),\ h \in L^2(\partial\mathbb{B}_n, d\sigma) \ominus H^2(\mathbb{B}_n),$$

then the commutator of $\{A_{z_1}, \dots, A_{z_n}\}$ *is equal to* $\{A_f : f \in H^\infty(\mathbb{B}_n)\}$.

Now let N_0 be a normal Šilov module and N be any normally contractive $C(\partial\mathbb{B}_n)$-extension of N_0. It follows that we have a normally injective presentation of N_0:

$$E_{N_0} :\ 0 \longrightarrow N_0 \overset{i}{\longrightarrow} N \overset{\pi}{\longrightarrow} N \ominus N_0 \longrightarrow 0.$$

By Proposition 7.4.3 and Theorem 7.4.5, we have the following proposition.

Proposition 7.4.13 *The following are true:*

(1) $Ext_{\mathcal{N}}(M, N_0) \cong coker(\pi_* : Hom_{\mathcal{N}}(M, N) \to Hom_{\mathcal{N}}(M, N \ominus N_0))$. *The correspondence is given by* $\delta_1(\theta) = [D_\theta]$ *for* $\theta \in Hom(M, N \ominus N_0)$, *where* D_θ *is the normal derivation defined by* $D_\theta(f) = P_{N_0} T_f^{(N)}\theta$*; also* P_{N_0} *is the orthogonal projection from* N *to* N_0.

(2) $Ext_{\mathcal{N}}(N \ominus N_0, M) \cong coker(i^* : Hom_{\mathcal{N}}(N, M) \to Hom_{\mathcal{N}}(N_0, M))$. *The correspondence is given by* $\delta_2(\theta) = [D^\theta]$, *where* D^θ *is the normal derivation defined by* $D^\theta(f) = \theta P_{N_0} T_f^{(N)}$.

Proposition 7.4.13 provides a very valid method to calculate normal extension groups of Hilbert modules in $\mathcal{N}$. In particular, if M (or N_0) is cyclic, then the characterizations of $Ext_{\mathcal{N}}(M, N_0)$ ($Ext_{\mathcal{N}}(N \ominus N_0, M)$) may be summed up as the action of module maps on cyclic vectors.

We now turn to the calculating of $Ext_{\mathcal{N}}$-groups of Hardy submodules over the ball algebra. For a submodule N of $L^2(\partial\mathbb{B}_n, d\sigma)$, we define a function space $B(N)$ in the following manner. For $\varphi \in L^2(\partial\mathbb{B}_n, d\sigma)$, the function φ is said to be in $B(N)$ if densely defined Hankel operator $H_\varphi^{(N)}$: $H^2(\mathbb{B}_n) \to L^2(\partial\mathbb{B}_n, d\sigma) \ominus N$ can be continuously extended onto $H^2(\mathbb{B}_n)$, where $H_\varphi^{(N)} f = P_{L^2(\partial\mathbb{B}_n,d\sigma)\ominus N}(\varphi f)$, $f \in A(\mathbb{B}_n)$. It is easy to check that for every $\varphi \in B(N)$, $H_\varphi^{(N)}$ is a module map from $H^2(\mathbb{B}_n)$ to $L^2(\partial\mathbb{B}_n, d\sigma) \ominus N$, and each module map β from $H^2(\mathbb{B}_n)$ to $L^2(\partial\mathbb{B}_n, d\sigma) \ominus N$ has such a form, that is, there exists an $\varphi \in B(N)$ such that $\beta = H_\varphi^{(N)}$. Furthermore, for a nonzero Hardy submodule $N_0(\subseteq H^2(\mathbb{B}_n))$, another function space $B(N_0, N)$ is defined by $\varphi \in B(N_0, N)$ if $\varphi \in B(N)$ and $\ker H_\varphi^{(N)} \supseteq N_0$. From Proposition 7.4.13 and Lemma 7.4.9, the following are immediate.

Proposition 7.4.14 *The following hold:*

(1) $Ext_{\mathcal{N}}(H^2(\mathbb{B}_n), N) \cong B(N)/(L^\infty(\partial\mathbb{B}_n, d\sigma) + N)$;

(2) $Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N_0, N) \cong B(N_0, N)/N$.

Remark 7.4.15 *From Proposition 7.4.14, if $n > 1$, then one can check that $Ext_{\mathcal{N}}(H^2(\mathbb{B}_n), H^2(\mathbb{B}_n)) \neq 0$. This says that $H^2(\mathbb{B}_n)$ is never normally projective. In the case of $n = 1$, it holds that*

$$Ext_{\mathcal{N}}(H^2(\mathbb{D}), H^2(\mathbb{D})) = 0$$

from Remark 7.2.14. However, we do not know if $H^2(\mathbb{D})$ is normally projective.

Let Hardy submodules N_1, N_2 satisfy

$$0 \neq N_1 \subseteq N_2 \subsetneqq H^2(\mathbb{B}_n).$$

Then 1 is in $B(N_1, N_2)$ and hence

$$Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N_1, N_2) \neq \{0\}.$$

This indicates that for Hardy submodules N_1 and N_2, if $N_1 \neq 0$ and

$$Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N_1, N_2) = 0,$$

then there is not a proper Hardy submodule N_3 such that $N_3 \supseteq N_1$, and N_3 is similar to N_2. The next proposition may give us some information of the rigidity of Hardy submodules.

Proposition 7.4.16 *Let N be of finite codimension in $H^2(\mathbb{B}_n)$, $n > 1$. Then the following is true:*

$$Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N, H^2(\mathbb{B}_n)) = 0.$$

Proof. By Theorem 2.2.3 we see that $\mathcal{C} \cap N$ is dense in N, and the set of common zeros (in $\mathbb{C}^n$) of the members of $\mathcal{C} \cap N$ is finite and lies in $\mathbb{B}_n$, where $\mathcal{C}$ is the ring of all polynomials on $\mathbb{C}^n$. For $\phi \in B(N, H^2(\mathbb{B}_n))$, since

$$\phi\,(\mathcal{C} \cap N) \subseteq H^2(\mathbb{B}_n),$$

using the harmonic extension of ϕ and the removable singularities theorem [KK] gives $\phi \in H^2(\mathbb{B}_n)$. Proposition 7.4.14 thus implies

$$Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N, H^2(\mathbb{B}_n)) = \{0\}.$$

The proof is complete.

Let N be a submodule of $H^2(\mathbb{B}_n)$, $n > 1$, and let N be of finite codimension in $H^2(\mathbb{B}_n)$. Since

$$Ext_{\mathcal{N}}(H^2(\mathbb{B}_n) \ominus N, N) \neq \{0\},$$

it follows that N is never similar to $H^2(\mathbb{B}_n)$ by Proposition 7.4.16. We refer the reader to [CD1] for a further consideration of the rigidity of Hardy submodules over the ball algebra.

Remark 7.4.17 *The main results in the present section also are valid for the Hardy modules on strongly pseudoconvex domains with smooth boundary because inner functions generate H^∞ in the weak*-topology on these domains [Eri].*

7.5 Remarks on Chapter 7

The extension theory of Hilbert modules over function algebras was first explored by Carlson and Clark [CC1]. Section 7.1 is mainly based on [CC1]. Section 7.2 basically comes from the paper by Chen and Guo [CG]. Section 7.3 basically arises from Guo [Guo5]. The results in Section 7.4 were contributed by Guo [Guo3]. For applications of the extension theory to operator theory we refer the reader to [Fe1, Fe2] and the references therein. In [Guo9], Guo developed extension theory of Hilbert modules over semigroups, and obtained applications to operator theory and representations of semigroups. We refer the reader to [Hel, MS, Pau2, Pau3] for more topics along this line.

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