

The Order of Presentation of the Formulas

The question of the most expedient order in which to give the formulas, in particular, in what division to include particular formulas such as the definite integrals, turned out to be quite complicated. The thought naturally occurs to set up an order analogous to that of a dictionary. However, it is almost impossible to create such a system for the formulas of integral calculus. Indeed, in an arbitrary formula of the form

$$\int_a^b f(x) dx = A$$

one may make a large number of substitutions of the form $x = \varphi(t)$ and thus obtain a number of "synonyms" of the given formula.

We must point out that the table of definite integrals by Bierens de Haan and the earlier editions of the present reference both sin in the plethora of such "synonyms" and formulas of complicated form. In the present edition, we have tried to keep only the simplest of the "synonym" formulas. Basically, we judged the simplicity of a formula from the standpoint of the simplicity of the arguments of the "outer" functions that appear in the integrand. Where possible, we have replaced a complicated formula with a simpler one.

Sometimes, several complicated formulas were thereby reduced to a single simpler one. We then kept only the simplest formula. As a result of such substitutions, we sometimes obtained an integral that could be evaluated by use of the formulas of chapter two and the Newton--Leibniz formula, or to an integral of the form

$$\int_{-a}^a f(x) dx,$$

where $f(x)$ is an odd function. In such cases the complicated integrals have been omitted.

Let us give an example using the expression

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg} x - 1)^{p-1}}{\sin^2 x} \ln \operatorname{tg} x dx = -\frac{\pi}{p} \operatorname{cosec} p\pi. \quad (1)$$

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By making the natural substitution $\operatorname{ctg} x - 1 = u$, we obtain

$$\int_0^\infty u^{p-1} \ln(1+u) du = \frac{\pi}{p} \operatorname{cosec} p\pi. \quad (2)$$

Integrals similar to formula (1) are omitted in this new edition. Instead, we have formula (2) and the formula obtained from the integral (1) by making the substitution $\operatorname{ctg} x = v$.

As a second example, let us take

$$I = \int_0^{\frac{\pi}{2}} \ln \operatorname{tg}^p x + \operatorname{ctg}^p x \ln \operatorname{tg} x \, dx = 0.$$

The substitution $\operatorname{tg} x = u$ yields

$$I = \int_0^{\infty} \frac{\ln(u^p + u^{-p}) \ln u}{1 + u^2} \, du.$$

If we now set $v = \ln u$, we obtain

$$I = \int_{-\infty}^{\infty} \frac{v e^v}{1 + e^{2v}} \ln(e^{pv} + e^{-pv}) \, dv = \int_{-\infty}^{\infty} v \frac{\ln 2 \operatorname{ch} pv}{2 \operatorname{ch} v} \, dv.$$

The integrand is odd and, consequently, the integral is equal to 0.

Thus, before looking for an integral in the tables, the user should simplify as much as possible the arguments (the "inner" functions) of the functions in the integrand.

The functions are ordered as follows:

First we have the elementary functions:

1. The function $f(x) = x$.
2. The exponential function.
3. The hyperbolic functions.
4. The trigonometric functions.
5. The logarithmic function.
6. The inverse hyperbolic functions. (These are replaced with the corresponding logarithms in the formulas containing definite

integrals.)

7.The inverse trigonometric functions.

Then follow the special functions:

8.Elliptic integrals.

9.Elliptic functions.

10.The logarithm integral, the exponential integral, the sine integral, and the cosine integral functions.

11.Probability integrals and Fresnel's integrals.

12.The gamma function and related functions.

13.Bessel functions.

14.Mathieu functions.

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15.Legendre functions.

16.Orthogonal polynomials.

17.Hypergeometric functions.

18.Confluent hypergeometric functions.

19.Parabolic cylinder functions.

20.Meijer's and MacRobert's functions.

21.Riemann's zeta function.

The integrals are arranged in order of outer function according to the above scheme: the farther down in the list a function occurs, (i.e., the more complex it is) the later will the corresponding formula appear in the tables. Suppose that several expressions have the same outer function. For example, consider $\sin e^x$, $\sin x$, $\sin \ln x$. Here, the outer function is the sine in all three cases. Such expressions are then arranged in order of the inner function. In the present work, these functions are therefore arranged in the following order: $\sin x$, $\sin e^x$, $\sin \ln x$.

Our list does not include polynomials, rational functions, powers, or other algebraic functions. An algebraic function that is included

in tables of definite integrals can usually be reduced to a finite combination of roots of rational power. Therefore, for classifying our formulas, we can conditionally treat a power function as a generalization of an algebraic and, consequently, of a rational function.* For any natural number n , the involution $(a + bx)^n$ of the binomial $a + bx$ is a polynomial. If n is a negative integer, $(a + bx)^n$ is a rational function. If n is irrational, the function $(a + bx)^n$ is not even an algebraic function.

We shall distinguish between all these functions and those listed above and we shall treat them as operators. Thus, in the expression $\sin^2 e^x$, we shall think of the squaring operator as applied to the outer function, namely, the sine. In the expression $\frac{\sin x + \cos x}{\sin x - \cos x}$, we shall think of the rational operator as applied to the trigonometric functions sine and cosine. We shall arrange the operators according to the following order:

1. Polynomials (listed in order of their degree).
2. Rational operators.
3. Algebraic Operators (Expressions Of The Form $A^{p/q}$, where q and p are rational, and $q > 0$; these are listed according to the size of q).
4. Power operators.

Expressions with the same outer and inner functions are arranged in the order of complexity of the operators. For example, the following functions (whose outer functions are all trigonometric, and whose inner functions are all $f(x) = x$) are arranged in the order shown:

$$\sin x, \quad \sin x \cos x, \quad \frac{1}{\sin x} = \operatorname{cosec} x, \quad \frac{\sin x}{\cos x} = \operatorname{tg} x, \quad \frac{\sin x + \cos x}{\sin x - \cos x}, \quad \sin^m x, \quad \sin^m x \cos x.$$

Furthermore, if two outer functions $\varphi_1(x)$ and $\varphi_2(x)$, where $\varphi_1(x)$ is more complex than $\varphi_2(x)$, appear in an integrand and if any of the operations mentioned are performed on them, the corresponding integral will appear (in the order determined by the position of $\varphi_2(x)$ in the list) after all integrals containing only the function $\varphi_1(x)$. Thus, following the trigonometric functions

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are the trigonometric and power functions (that is, $\varphi_2(x) = x$). Then come

combinations of trigonometric and exponential functions,

combinations of trigonometric functions, exponential functions, and powers, etc.,

combinations of trigonometric and hyperbolic functions, etc.

Integrals containing two functions $\varphi_1(x)$ and $\varphi_2(x)$ are located in the division and order corresponding to the more complicated function of the two. However, if the positions of several integrals coincide because they contain the same complicated function, these integrals are put in the position defined by the complexity of the second function.

To these rules of a general nature, we need to add certain particular considerations that will be easily understood from the tables. For example, according to the above remarks, the function e^1 comes after e^x as regards complexity, but $\ln x$ and $\ln \frac{1}{x}$ are equally complex since $\ln \frac{1}{x} = -\ln x$. In the section on "powers and algebraic functions", polynomials, rational functions, and powers of powers are formed from power functions of the form $(a + bx)^n$ and $(\alpha + \beta x)^\nu$.

Use of the Tables

Prepared by Alan Jeffrey for the English language edition.

For the effective use of the tables contained in this book it is necessary that the user should first become familiar with the classification system for integrals devised by the authors Ryzhik and Gradshteyn. This classification is described in detail in the section entitled The Order of Presentation of the Formulas and essentially involves the separation of the integrand into *inner* and *outer* functions. The principal function involved in the integrand is called the *outer* function and its argument, which is itself usually another function, is called the *inner* function. Thus, if the integrand comprised the expression $\ln \sin x$, the *outer* function would be the logarithmic function while its argument, the *inner* function, would be the trigonometric function $\sin x$. The desired integral would then be found in the section dealing with logarithmic functions, its position within that section being determined by the position of the *inner* function (here a trigonometric function) in Ryzhik and Gradshteyn's list of functional forms.

It is inevitable that some duplication of symbols will occur within such a large collection of integrals and this happens most frequently in the first part of the book dealing with algebraic and trigonometric integrands. The symbols most frequently involved are

$\alpha, \beta, \gamma, \delta, t, u, z, z_k,$ and Δ . The expressions associated with these symbols are used consistently within each section and are defined at the start of each new section in which they occur. Consequently, reference should be made to the beginning of the section being used in order to verify the meaning of the substitutions involved.

Integrals of algebraic functions are expressed as combinations of roots with rational power indices, and definite integrals of such functions are frequently expressed in terms of the Legendre elliptic integrals $F(\phi, k), E(\phi, k)$ and $\Pi(\phi, n, k)$, respectively, of the first, second and third kinds.

An abbreviated notation is used for the trigonometric and hyperbolic functions $\tan x, \cot x, \sinh x, \cosh x, \tanh x$ and $\coth x$ which are denoted, respectively, by $\text{tg } x, \text{ctg } x, \text{sh } x, \text{ch } x, \text{th } x$ and $\text{cth } x$. Also the four inverse hyperbolic functions $\text{Arsh } z, \text{Arch } z, \text{Arth } z$ and $\text{Arcth } z$ are introduced through the definitions

$$\begin{aligned}\arcsin z &= \frac{1}{i} \text{Arsh } (iz) \\ \arccos z &= \frac{1}{i} \text{Arch } (z) \\ \text{arctg } z &= \frac{1}{i} \text{Arth } (iz) \\ \text{arcctg } z &= i \text{Arcth } (iz)\end{aligned}$$

or,

$$\begin{aligned}\text{Arsh } z &= \frac{1}{i} \arcsin (iz) \\ \text{Arch } z &= i \arccos z \\ \text{Arth } z &= \frac{1}{i} \text{arctg } (iz) \\ \text{Arcth } z &= \frac{1}{i} \text{arcctg } (-iz)\end{aligned}$$

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The numerical constants **C** and **G** which often appear in the definite integrals denote Euler's constant and Catalan's constant, respectively. Euler's constant **C** is defined by the limit

$$\mathbf{C} = \lim_{s \rightarrow \infty} \left(\sum_{m=1}^s \frac{1}{m} - \ln s \right) = 0.577215 \dots$$

On occasions other writers denote Euler's constant by the symbol γ , but this is also often used instead to denote the constant

Catalan's constant \mathbf{G} is related to the complete elliptic integral

$$\mathbf{K} \equiv \mathbf{K}(k) \equiv \int_0^{\pi/2} \frac{da}{\sqrt{1 - k^2 \sin^2 a}}$$

by the expression

$$\mathbf{G} = \frac{1}{2} \int_0^1 \mathbf{K} dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = 0.915965 \dots$$

Since the notations and definitions for higher transcendental functions that are used by different authors are by no means uniform, it is advisable to check the definitions of the functions that occur in these tables. This can be done by identifying the required function by symbol and name in the Index of Special Functions and Notations that appears at the front of the book, and by then referring to the defining formula or section number listed there. We now present a brief

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discussion of some of the most commonly used alternative notations and definitions for higher transcendental functions.

Bernoulli and Euler Polynomials and Numbers

Extensive use is made throughout the book of the Bernoulli and Euler numbers B_n and E_n that are defined in terms of the Bernoulli and Euler polynomials of order n , $B_n(x)$ and $E_n(x)$, respectively. These polynomials are defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{for } |t| < 2\pi$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \text{for } |t| < \pi.$$

The Bernoulli numbers are always denoted by B_n and are defined by the relation

$$B_n = B_n(0) \quad \text{for } n = 0, 1, \dots,$$

when

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \dots$$

The Euler numbers E_n are defined by setting

$$E_n = 2^n E_n \left(\frac{1}{2} \right) \quad \text{for } n = 0, 1, \dots$$

The E_n are all integral and $E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \dots$

An alternative definition of Bernoulli numbers, which we shall denote by the symbol B_n^* , uses the same generating function but identifies the B_n^* differently in the following manner:

$$\frac{t}{e^t - 1} = 1 - \frac{1}{2}t + B_1^* \frac{t^2}{2!} - B_2^* \frac{t^4}{4!} + \dots$$

This definition then gives rise to the alternative set of Bernoulli numbers

$$B_1^* = 1/6, \quad B_2^* = 1/30, \quad B_3^* = 1/42, \quad B_4^* = 1/30, \quad B_5^* = 5/66, \\ B_6^* = 691/2730, \quad B_7^* = 7/6, \quad B_8^* = 3617/510, \dots$$

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These differences in notation must also be taken into account when using the following relationships that exist between the Bernoulli and Euler polynomials:

$$B_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} B_{n-k} E_k(2x) \quad n = 0, 1, \dots \\ E_{n-1}(x) = \frac{2^n}{n} \left\{ B_n \left(\frac{x+1}{2} \right) - B_n \left(\frac{x}{2} \right) \right\}$$

or

$$E_{n-1}(x) = \frac{2}{n} \left\{ B_n(x) - 2^n B_n \left(\frac{x}{2} \right) \right\} \quad n = 1, 2, \dots$$

and

$$E_{n-2}(x) = 2 \binom{n}{k}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k} B_k(x) \quad n = 2, 3, \dots$$

There are also alternative definitions of the Euler polynomial of order n , and it should be noted that some authors, using a modification of the third expression above, call

$$\left(\frac{2}{n+1} \right) \left\{ B_n(x) - 2^n B_n \left(\frac{x}{2} \right) \right\}$$

the Euler polynomial of order n .

Elliptic Functions and Elliptic Integrals

The following notations are often used in connection with the inverse elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$:

$$\begin{array}{lll} \operatorname{ns} u = \frac{1}{\operatorname{sn} u} & \operatorname{nc} u = \frac{1}{\operatorname{cn} u} & \operatorname{nd} u = \frac{1}{\operatorname{dn} u} \\ \operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u} & \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} & \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u} \\ \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u} & \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u} & \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u} \end{array}$$

The following elliptic integral of the third kind is defined by Ryzhik and Gradshteyn to be

$$\begin{aligned} \Pi(\varphi, n^2, k) &= \int_0^\varphi \frac{da}{(1 - n^2 \sin^2 a) \sqrt{1 - k^2 \sin^2 a}} \\ &= \int_0^{\sin \varphi} \frac{dx}{(1 - n^2 x^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}} \quad (-\infty < n^2 < \infty). \end{aligned}$$

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The Jacobi Zeta Function and Theta Functions

The Jacobi zeta function $\text{zn}(u, k)$, frequently written $Z(u)$, is defined by the relation

$$\text{zn}(u, k) = Z(u) = \int_0^u \left\{ \text{dn}^2 v - \frac{\mathbf{E}}{\mathbf{K}} \right\} dv = \mathbf{E}(u) - \frac{\mathbf{E}}{\mathbf{K}} u.$$

This is related to the theta functions by the relationship

$$\text{zn}(u, k) = \frac{\partial}{\partial u} \ln \Theta(u)$$

giving

$$\begin{aligned} \text{(i)} \quad \text{zn}(u, k) &= \frac{\pi}{2\mathbf{K}} \frac{\vartheta_1' \left(\frac{\pi u}{2\mathbf{K}} \right)}{\vartheta_1 \left(\frac{\pi u}{2\mathbf{K}} \right)} - \frac{\text{cn } u \text{ dn } u}{\text{sn } u} \\ \text{(ii)} \quad \text{zn}(u, k) &= \frac{\pi}{2\mathbf{K}} \frac{\vartheta_2' \left(\frac{\pi u}{2\mathbf{K}} \right)}{\vartheta_2 \left(\frac{\pi u}{2\mathbf{K}} \right)} + \frac{\text{dn } u \text{ sn } u}{\text{cn } u} \\ \text{(iii)} \quad \text{zn}(u, k) &= \frac{\pi}{2\mathbf{K}} \frac{\vartheta_3' \left(\frac{\pi u}{2\mathbf{K}} \right)}{\vartheta_3 \left(\frac{\pi u}{2\mathbf{K}} \right)} - k^2 \frac{\text{sn } u \text{ cn } u}{\text{dn } u} \\ \text{(iv)} \quad \text{zn}(u, k) &= \frac{\pi}{2\mathbf{K}} \frac{\vartheta_4' \left(\frac{\pi u}{2\mathbf{K}} \right)}{\vartheta_4 \left(\frac{\pi u}{2\mathbf{K}} \right)}. \end{aligned}$$

Many different notations for the theta function are in current use. The most common variants are the replacement of the argument u by the argument u/π and, occasionally, a permutation of the identification of the functions ϑ_1 to ϑ_4 with the function ϑ_4 replaced by ϑ .

The Factorial (Gamma) Function

In older reference texts the gamma function $\Gamma(z)$, defined by the Euler integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

$$\Gamma(z)$$

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt,$$

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is sometimes expressed in the alternative notation

$$\Gamma(1+z) = z! = \Pi(z).$$

On occasions the related derivative of the logarithmic factorial function $\Psi(z)$ is used where

$$\frac{d(\ln z!)}{dz} = \frac{(z!)'}{z!} = \Psi(z).$$

This function satisfies the recurrence relation

$$\Psi(z) = \Psi(z-1) + \frac{1}{z-1}$$

and is defined by the series

$$\Psi(z) = -\mathbf{C} + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z+n} \right).$$

The derivative $\Psi'(z)$ satisfies the recurrence relation

$$-\Psi'(z-1) = -\Psi'(z) + \frac{1}{z^2}$$

and is defined by the series

$$\Psi'(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

Exponential and Related Integrals

The exponential integrals $E_n(z)$ have been defined by Schloemilch using the integral

$$E_n(z) = \int_1^{\infty} e^{-zt} t^{-n} dt \quad (n = 0, 1, \dots, \operatorname{Re} z > 0).$$

They should not be confused with the Euler polynomials already mentioned. The function $E_1(z)$ is related to the exponential integral $Ei(z)$ and to the logarithmic integral $li(z)$ through the expressions

$$E_1(z) = -Ei(-z) = \int_z^\infty e^{-t} t^{-1} dt$$

and

$$li(z) = \int_0^z \frac{dt}{\ln t} = Ei(\ln z) \quad (z > 1).$$

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The functions $E_n(z)$ satisfy the recurrence relations

$$E_n(z) = \frac{1}{n-1} \{e^{-z} - zE_{n-1}(z)\}, \quad (n > 1)$$

and

$$E'_n(z) = -E_{n-1}(z)$$

with

$$E_0(z) = e^{-z}/z.$$

The function $E_n(z)$ has the asymptotic expansion

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \dots \right\} \quad [|\arg z| < 3\pi/2],$$

while for large n ,

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n, x) \right\},$$

where

$$-0.36n^{-4} \leq R(n, x) \leq \left(1 + \frac{1}{x+n-1}\right) n^{-4} \quad (x > 0).$$

The sine and cosine integrals $\text{si}(x)$ and $\text{ci}(x)$ are related to the functions $\text{Si}(x)$ and $\text{Ci}(x)$ by the integrals

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt = \text{si}(x) + \pi/2$$

and

$$\text{Ci}(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cos t - 1)}{t} dt.$$

The hyperbolic sine and cosine integrals $\text{Shi}(x)$ and $\text{Chi}(x)$ are defined by the relations

$$\text{Shi}(x) = \int_0^x \frac{\sinh t}{t} dt$$

and

$$\text{Chi}(x) = \mathbf{C} + \ln x + \int_0^x \frac{(\cosh t - 1)}{t} dt.$$

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Some authors write

$$\text{Cin}(x) = \int_0^x \frac{(1 - \cos t)}{t} dt$$

when

$$\text{Cin}(x) = -\text{Ci}(x) + \ln x + \mathbf{C}.$$

The error function $\text{erf}(x)$ is defined by the relation

$$\text{erf}(x) = \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and the complementary error function $\text{erfc}(x)$ is related to the error function $\text{erf}(x)$ and to $\Phi(x)$ by the expression

$$\text{erfc}(x) = 1 - \text{erf}(x).$$

The Fresnel integrals $S(x)$ and $C(x)$ are defined by Ryzhik and Gradshteyn as

$S(x)$ $C(x)$

$$S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt$$

and

$$C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt.$$

Other definitions that are in use are

$$S_1(x) = \int_0^x \sin \frac{\pi t^2}{2} dt, \quad C_1(x) = \int_0^x \cos \frac{\pi t^2}{2} dt$$

and

$$S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt$$

These are related by the expressions

$$S(x) = S_1 \left(x \sqrt{\frac{2}{\pi}} \right) = S_2(x^2)$$

and

$$C(x) = C_1 \left(x \sqrt{\frac{2}{\pi}} \right) = C_2(x^2)$$

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Hermite and Chebyshev Orthogonal Polynomials

The Hermite polynomials $H_n(x)$ are related to the Hermite polynomials $\text{He}_n(x)$ by the relations

$$\text{He}_n(x) = 2^{-n/2} H_n \left(\frac{x}{\sqrt{2}} \right)$$

and

These functions satisfy the differential equations

$$\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2nH_n = 0$$

and

$$\frac{d^2 \text{He}_n}{dx^2} - x \frac{d\text{He}_n}{dx} + n \text{He}_n = 0.$$

They obey the recurrence relations

$$H_{n+1} = 2xH_n - 2nH_{n-1}$$

and

$$\text{He}_{n+1} = x \text{He}_n - n \text{He}_{n-1}.$$

The first six orthogonal polynomials He_n are

$$\text{He}_0 = 1, \quad \text{He}_1 = x, \quad \text{He}_2 = x^2 - 1, \quad \text{He}_3 = x^3 - 3x, \quad \text{He}_4 = x^4 - 6x^2 + 3, \quad \text{He}_5 = x^5 - 10x^3 + 15x.$$

Sometimes the Chebyshev polynomial $U_n(x)$ of the second kind is defined as a solution of the equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + n(n + 2)y = 0.$$

Bessel Functions

A variety of different notations for Bessel functions are in use. Some common ones involves the replacement of $Y_n(z)$ by $N_n(z)$ and the introduction of the symbol

$$\Lambda_n(z) = \left(\frac{1}{2}z\right)^{-n} \Gamma(n + 1)J_n(z).$$

In the book by Gray, Mathews and MacRobert the symbol $Y_n(z)$ is used to denote $\frac{1}{2}\pi Y_n(z) + (\ln 2 - \mathbf{C})J_n(z)$ while Neumann uses the symbol $Y^{(n)}(z)$ for the identical quantity.

The Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are sometimes denoted by $H_{s_\nu}(z)$ and $H_{i_\nu}(z)$ and some authors write $G_\nu(z) = \left(\frac{1}{2}\right) \pi i H_\nu^{(1)}(z)$.

The Neumann polynomial $O_n(t)$ is a polynomial of degree $n + 1$ in $1/t$, with $O_0(t) = 1/t$. The polynomials $O_n(t)$ are defined by the generating function

$$\frac{1}{t-z} = J_0(z)O_0(t) + 2\sum_{k=1}^{\infty} J_k(z)O_k(t),$$

giving

$$O_n(t) = \frac{1}{4} \sum_{k=0}^{[n/2]} \frac{n(n-k-1)!}{k!} \left(\frac{2}{t}\right)^{n-2k+1} \quad \text{for } n = 1, 2, \dots,$$

where $[\frac{1}{2}n]$ signifies the integral part of $\frac{1}{2}n$. The following relationship holds between three successive polynomials:

$$(n-1)O_{n+1}(t) + (n+1)O_{n-1}(t) - \frac{2(n^2-1)}{t}O_n(t) = \frac{2n}{t} \sin^2 \frac{n\pi}{2}.$$

The Airy functions $\text{Ai}(z)$ and $\text{Bi}(z)$ are independent solutions of the equation

$$\frac{d^2u}{dz^2} - zu = 0.$$

The solutions can be represented in terms of Bessel functions by the expressions

$$\begin{aligned} \text{Ai}(z) &= \frac{1}{3}\sqrt{z} \left\{ I_{-1/3} \left(\frac{2}{3}z^{3/2} \right) - I_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\} = \frac{1}{\pi} \sqrt{\frac{z}{3}} K_{\frac{1}{3}} \left(\frac{2}{3}z^{3/2} \right), \\ \text{Ai}(-z) &= \frac{1}{3}\sqrt{z} \left\{ J_{1/3} \left(\frac{2}{3}z^{3/2} \right) + J_{-1/3} \left(\frac{2}{3}z^{3/2} \right) \right\} \end{aligned}$$

and by

$$\begin{aligned} \text{Bi}(z) &= \sqrt{\frac{z}{3}} \left\{ I_{-1/3} \left(\frac{2}{3}z^{3/2} \right) + I_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\}, \\ \text{Bi}(-z) &= \sqrt{\frac{z}{3}} \left\{ I_{-1/3} \left(\frac{2}{3}z^{3/2} \right) - I_{1/3} \left(\frac{2}{3}z^{3/2} \right) \right\} \end{aligned}$$

The differential equation

$$\frac{d^2y}{dz^2} + (az^2 + bz + c)y = 0$$

has associated with it the two equations

$$\frac{d^2y}{dz^2} + \left(\frac{1}{4}z^2 + a\right)y = 0 \quad \text{and} \quad \frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0$$

the solutions of which are parabolic cylinder functions. The first equation can be derived from the second by replacing z by $ze^{i\pi/4}$ and a by $-ia$.

The solutions of the equation

$$\frac{d^2y}{dz^2} - \left(\frac{1}{4}z^2 + a\right)y = 0$$

are sometimes written $U(a, z)$ and $V(a, z)$. These solutions are related to Whittaker's function $D_p(z)$ by the expressions

$$U(a, z) = D_{-a-\frac{1}{2}}(z)$$

and

$$V(a, z) = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + a\right) \{D_{-a-\frac{1}{2}}(-z) + (\sin \pi a)D_{-a-\frac{1}{2}}(z)\}.$$

Mathieu Functions

There are several accepted notations for Mathieu functions and for their associated parameters. The defining equation used by Ryzhik and Gradshteyn is

$$\frac{d^2y}{dz^2} + (a - 2k^2 \cos 2z)y = 0 \quad \text{with} \quad k^2 = q.$$

Different notations involve the replacement of a and q in this equation by h and θ , λ and h^2 and b and $c = 2\sqrt{q}$, respectively. The

$$a \quad q \quad h \quad \theta \quad \lambda \quad h^2 \quad b \quad c = 2\sqrt{q}$$

periodic solutions $se_n(z, q)$ and $ce_n(z, q)$ and the modified periodic solutions $Se_n(z, q)$ and $Ce_n(z, q)$ are suitably altered and, sometimes, re-normalized. A description of these relationships together with the normalizing factors is contained in: Tables relating to Mathieu functions. National Bureau of Standards, Columbia University Press, New York, 1951.

Index of Special Functions and Notations

Notation	Name of the function	formula no.
$am(u, k)$	Amplitude (of an elliptic function)	8.141
B_n	Bernoulli numbers	9.61, 9.71
$B_n(x)$	Bernoulli polynomials	9.620
$B(x, y)$	Beta functions	8.38
$B_x(p, q)$	Incomplete beta functions	8.39
$\beta(x)$		8.37
$bei_\nu(z), ber_\nu(z)$	Thomson's functions	8.56
\mathbf{C}	Euler's constant	9.73, 8.367
$C(x)$	Fresnel's cosine-integral	8.25
$C_\nu(a)$	Young's function	3.76
$C_n^\lambda(t)$	Gegenbauer's polynomials	8.93
$C_\nu^\lambda(t)$	Gegenbauer's function	8.932 1
$ce_{2n}(z, q), ce_{2n+1}(z, q)$	Periodic Mathieu functions (Mathieu functions of the first kind)	8.61
$Ce_{2n}(z, q), Ce_{2n+1}(z, q)$	Associated (modified) Mathieu functions of the first kind	8.63
$chi(x)$	Hyperbolic-cosine-integral function	8.22
$ci(x)$	Cosine-integral	8.23
$cn(u)$	Cosine-amplitude	8.14
$D(k) \equiv D$		8.112
$D(\varphi, k)$		8.111
$D_n(z), D_p(z)$	Parabolic cylinder functions	9.24–9.25
$dn u$	Delta amplitude	8.14
e_1, e_2, e_3		8.162
E_n	Euler numbers	9.63, 9.72
$E(\varphi, k)$	Elliptic integral of the second kind	8.11–8.12
$\mathbf{E}(k) = \mathbf{E} \left. \begin{array}{l} \mathbf{E}(k') = \mathbf{E} \end{array} \right\}$	Complete elliptic integral of the second kind	8.11–8.12

xlii

Notation	Name of the function	formula no.
$E(p; a_r; q; \varrho_s; x)$	MacRobert's function	9.4
$\mathbf{E}_\nu(z)$	Weber's function	8.58
$Ei(z)$	Exponential-integral function	8.21
$\bar{E}i(z)$	Related exponential integral	8.21
$\Phi(x)$	Probability integral	8.25
$\text{erf}(x) = \Phi(x)$	Error function	8.25

Notation	Name of the function	formula no.
$H_n(z)$	Hermite polynomials	8.95
$\mathbf{H}_\nu(z)$	Struve functions	8.55
$I_\nu(z)$	Modified Bessel functions	8.406, 8.43
$I_x(p, q)$	Incomplete beta function	8.39
$J_\nu(z)$	Bessel function	8.402, 8.41
$\mathbf{J}_\nu(z)$	Anger's function	8.58
$k_\nu(x)$	Bateman's function	9.21
$\mathbf{K}(k) = \mathbf{K}, \quad \mathbf{K}(k') = \mathbf{K}'$	Complete elliptic integral of the first kind	8.11–8.12
$K_\nu(z)$	Modified Bessel functions	8.407, 8.43
$\text{kei}(z), \quad \text{ker}(z)$	Thomson's functions	8.56
$\xi(s)$		9.56
$L(x)$	Lobachevskiy's function	8.26
$\mathbf{L}_\nu(z)$	Modified Struve function	8.55
$L_n^\alpha(z)$	Laguerre polynomials	8.97
$\text{li}(x)$	Logarithm-integral	8.24
$\lambda(x, y)$		9.640
$M_{\lambda, \mu}(z)$	Whittaker's functions	9.22, 9.23
$\mu(x, \beta), \quad \mu(x, \beta, \alpha)$		9.640
$N_\nu(z)$	Bessel functions of the second kind (Neumann functions)	8.403, 8.41
$\nu(x)$		9.640
$\nu(x), \nu(x, \alpha)$		9.640
$O_n(x)$	Neumann's polynomials	8.59
$\wp(u)$	Weierstrass elliptic function	8.16
$P_\nu^\mu(z), \quad P_\nu^\mu(x)$	Associated Legendre functions of the first kind	8.7, 8.8
$P_\nu(z), \quad P_n(x)$	Legendre functions and polynomials	8.82, 8.83, 8.91
$P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' \end{matrix} \right\}$	Riemann's differential equation (diagram)	9.160
$P_n^{(\alpha, \beta)}(x)$	Jacobi's polynomials	8.96
$\Pi(x)$	Lobachevskiy's angle of parallelism	1.48
$\Pi(\varphi, n, k)$	Elliptic integral of the third kind	8.11
$\Phi(x)$	Probability integral	8.25
$\Phi(z, s, v)$	Lerch function	9.55

xliv

Notation	Name of the function	formula no.
$\Phi(\alpha, \gamma; x) = {}_1F_1(\alpha; \gamma; x)$		9.21
$\Phi_1(\alpha, \beta, \gamma, x, y), \quad \Phi_2(\beta, \beta', \gamma, x, y),$ $\Phi_3(\beta, \gamma, x, y)$	Confluent hypergeometric series in two variables	9.26
$\psi(x)$	Euler's psi function	8.36
$\Psi(a, c; x)$	Confluent hypergeometric function	9.27
$Q_\nu^\mu(z), Q_\nu^\mu(x)$	Ass. Legendre func. of the 2nd kind	8.7, 8.8

The letter k (when not used as an index of summation) denotes a number in the interval $[0, 1]$. This notation is used in integrals that lead to elliptic integrals. In such a connection, the number $\sqrt{1 - k^2}$ is denoted by k' .

$R(x)$	A rational function
$\operatorname{Re} z \equiv x, \operatorname{Im} z \equiv y$	The real and imaginary parts of the complex number $z = x + iy$.
$\bar{z} = x - iy$	The complex conjugate of $z = x + iy$.
$\arg z$	The argument of the complex number $z = x + iy$.
$\operatorname{sign} x$	The sign (signum) of the real number x ; $\operatorname{sign} x = +1$ for $x > 0$; $\operatorname{sign} x = -1$ for $x < 0$.
$[x]$	The integral part of the real number x .
$\int_a^{(b+)} \int_a^{(b-)}$	Contour integrals; the path of integration starting at the point a extends to the point b (along a straight line unless there is an indication to the contrary), encircles the point b along a small circle in the positive (negative) direction, and returns to the point a , proceeding along the original path in the opposite direction.
\int_C	Line integral along the curve C .
$n!$	$= 1 \cdot 2 \cdot 3 \dots n, \quad 0! = 1.$
$(2n + 1)!!$	$= 1 \cdot 3 \dots (2n + 1).$
$(2n)!!$	$= 2 \cdot 4 \dots (2n).$
$\binom{p}{n}$	$= \frac{p(p-1)\dots(p-n+1)}{1 \cdot 2 \dots n} = \frac{p!}{n!(p-n)!}, \quad \binom{p}{0} = 1. \quad [n, p = 0, 1, \dots; p \geq n]$
$x(x-1)\dots(x-n+1)/n!$	$[n = 0, 1, \dots]$
$(a)_n$	$= a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$
$\sum_{k=m}^n u_k$	$= u_m + u_{m+1} + \dots + u_n.$ If $n < m$, we define $\sum_{k=m}^n u_k = 0.$
$\sum'_n, \sum'_{m,n}$	Summation over all integral values of n excluding $n = 0$, and summation over all integral values of n and m excluding $m = n = 0$, respectively.
$O(f(z))$	The order of the function $f(z)$. Suppose that the point z approaches z_0 . If there exists an $M > 0$ such that $ g(z) \leq M f(z) $ in some sufficiently small neighborhood of the point z_0 , we write $g(z) = O(f(z))$.
$0!! = 1$	
$(-1)!! = 1$	(cf. 3.372 for $n = 0$.)
$0^0 = 1$	(cf. 0.112 and 0.113 for $q = 0$.)
$\sum_{k \in \phi} (\dots) = 0$	An empty \sum has the value 0
$\prod_{k \in \phi} (\dots) = 1$	An empty \prod has the value 1
$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$	Kronecker
$\delta_{ij} = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$	Heaviside step function

$$\begin{aligned}\sum_{k=1}^n k^q &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{1}{2} \binom{q}{1} B_2 n^{q-1} + \frac{1}{4} \binom{q}{3} B_4 n^{q-3} + \frac{1}{6} \binom{q}{5} B_6 n^{q-5} + \dots = \\ &= \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \frac{qn^{q-1}}{12} - \frac{q(q-1)(q-2)}{720} n^{q-3} + \frac{q(q-1)(q-2)(q-3)(q-4)}{30,240} n^{q-5} - \dots \\ &\quad \text{[last term contains either } n \text{ or } n^2].\end{aligned}$$

CE 332

2

$$1. \quad \sum_{k=1}^n k = \frac{n(n+1)}{2}.$$

CE 333

$$2. \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

CE333

$$3. \quad \sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

CE 333

$$4. \quad \sum_{k=1}^n k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1).$$

CE 333

$$5. \quad \sum_{k=1}^n k^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1).$$

CE 333

$$6. \quad \sum_{k=1}^n k^6 = \frac{1}{42} n(n+1)(2n+1)(3n^4+6n^3-3n+1).$$

CE 333

0.122

$$\sum_{k=1}^n (2k-1)^q = \frac{2^q}{q+1} n^{q+1} - \frac{1}{2} \binom{q}{1} 2^{q-1} B_2 n^{q-1} - \frac{1}{4} \binom{q}{3} 2^{q-3} (2^3-1) B_4 n^{q-3} - \dots$$

[last term contains either n or n^2 .]

$$1. \quad \sum_{k=1}^n (2k-1) = n^2.$$

$$2. \quad \sum_{k=1}^n (2k-1)^2 = \frac{1}{3} n(4n^2-1).$$

JO (32a)

$$3. \quad \sum_{k=1}^n (2k-1)^3 = n^2(2n^2-1).$$

JO (32b)

$$4^*. \quad \sum_{k=1}^n (mk-1) = \frac{1}{2} [m(n+1) - 2].$$

$$5^*. \quad \sum_{k=1}^n (mk-1)^2 = \frac{1}{6} n [m^2(n+1)(2n+1) - 6m(n+1) + 6].$$

$$6^*. \quad \sum_{k=1}^n (mk-1)^3 = \frac{1}{4} n [m^3 n(n+1)^2 - 2m^2(n+1)(2n+1) + 6m(n+1) - 4].$$

0.123

$$\sum_{k=1}^n k(k+1)^2 = \frac{1}{12} n(n+1)(n+2)(3n+5).$$

0.124

$$1. \quad \sum_{k=1}^q k(n^2 - k^2) = \frac{1}{4} q(q+1)(2n^2 - q^2 - q) \quad [q = 1, 2, \dots].$$

$$2^* \cdot \sum_{k=1}^n k(k+1)^3 = \frac{1}{60}n(n+1)(12n^3 + 63n^2 + 107n + 58).$$

3

0.125

$$\sum_{k=1}^n k! \cdot k = (n+1)! - 1.$$

AD (188.1)

0.126

$$\sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} = \sqrt{\frac{e}{\pi}} K_{n+\frac{1}{2}} \left(\frac{1}{2} \right).$$

WA 94

0.13 Sums of reciprocals of natural numbers

0.131

$$\sum_{k=1}^n \frac{1}{k} = C + \ln n + \frac{1}{2n} - \sum_{k=2}^{\infty} \frac{A_k}{n(n+1)\dots(n+k-1)},$$

where

$$A_k = \frac{1}{k} \int_0^1 x(1-x)(2-x)(3-x)\dots(k-1-x) dx.$$

$$A_2 = \frac{1}{12}, \quad A_3 = \frac{1}{12},$$

$$A_4 = \frac{19}{120}, \quad A_5 = \frac{9}{20},$$

0.132

$$\sum_{k=1}^n \frac{1}{2k-1} = \frac{1}{2} (C + \ln n) + \ln 2 + \frac{B_2}{8n^2} + \frac{(2^3-1)B_4}{64n^4} + \dots \quad [n \rightarrow \infty]$$

JO (71a)a

0.133

$$\sum_{k=2}^n \frac{1}{k^2-1} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}.$$

JO (184f)

0.14 Sums of products of reciprocals of natural numbers

0.141

$$1. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q](p+kq)} = \frac{n}{p(p+nq)}.$$

GI III (64)a

$$2. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q](p+kq)[p+(k+1)q]} = \frac{n(2p+nq+q)}{2p(p+q)(p+nq)[p+(n+1)q]}.$$

GI III (65)a

$$3. \quad \sum_{k=1}^n \frac{1}{[p+(k-1)q](p+kq)\dots[p+(k+l)q]} = \\ = \frac{1}{(l+1)q} \left\{ \frac{1}{p(p+q)\dots(p+lq)} - \frac{1}{(p+nq)[p+(n+1)q]\dots[p+(n+l)q]} \right\}.$$

AD (1856)a

$$4. \quad \sum_{k=1}^n \frac{1}{[1+(k-1)q][1+(k-l)q+p]} = \frac{1}{p} \left[\sum_{k=1}^n \frac{1}{1+(k-1)q} - \sum_{k=1}^n \frac{1}{1+(k-1)q+p} \right].$$

GI III (66)a

4

0.142

$$\sum_{k=1}^n \frac{k^2+k-1}{(k+2)!} = \frac{1}{2} - \frac{n+1}{(n+2)!}.$$

0.15 Sums of the binomial coefficients

0.151

$$1. \sum_{k=0}^m \binom{n+k}{n} = \binom{n+m+1}{n+1}.$$

KR 64 (70.1)

$$2. 1 + \binom{n}{2} + \binom{n}{4} + \dots = 2^{n-1}.$$

KR 62 (58.1)

$$3. \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots = 2^{n-1}.$$

KR 62 (58.1)

$$4. \sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \quad [n \geq 1].$$

KR 64 (70.2)

0.152

$$1. \binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{n\pi}{3} \right).$$

KR 62 (59.1)

$$2. \binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-2)\pi}{3} \right).$$

KR 62 (59.2)

$$3. \binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \dots = \frac{1}{3} \left(2^n + 2 \cos \frac{(n-4)\pi}{3} \right).$$

KR 62 (59.3)

0.153

0.153

$$1. \binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right).$$

KR 63 (60.1)

$$2. \binom{n}{1} + \binom{n}{5} + \binom{n}{9} + \dots = \frac{1}{2} \left(2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right).$$

KR 63 (60.2)

$$3. \binom{n}{2} + \binom{n}{6} + \binom{n}{10} + \dots = \frac{1}{2} \left(2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right).$$

KR 63 (60.3)

$$4. \binom{n}{3} + \binom{n}{7} + \binom{n}{11} + \dots = \frac{1}{2} \left(2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right).$$

KR 63 (60.4)

0.154

$$1. \sum_{k=0}^n (k+1) \binom{n}{k} = 2^{n-1} (n+2) \quad [n \geq 0].$$

KR 63 (66.1)

$$2. \sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} = 0 \quad [n \geq 2].$$

KR 63 (66.2)

5

$$3. \sum_{k=0}^N (-1)^k \binom{N}{k} k^{n-1} = 0 \quad [N \geq n \geq 1; \quad 0^0 \equiv 1].$$

$$4. \sum_{k=0}^n (-1)^k \binom{n}{k} k^n = (-1)^n n! \quad [n \geq 0; \quad 0^0 \equiv 1].$$

$$5. \sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha + k)^n = (-1)^n n! \quad [n \geq 0; \quad 0^0 \equiv 1].$$

$$6. \sum_{k=0}^N (-1)^k \binom{N}{k} (\alpha + k)^{n-1} = 0 \quad [N \geq n \geq 1, \quad 0^0 \equiv 1 \quad N, n \in N^+].$$

0.155

$$1. \sum_{k=1}^n \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = \frac{n}{n+1}.$$

KR 63 (67)

$$2. \sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}.$$

KR 63 (68.1)

$$3. \sum_{k=0}^n \frac{\alpha^{k+1}}{k+1} \binom{n}{k} = \frac{(\alpha+1)^{n+1} - 1}{n+1}.$$

KR 63 (68.2)

$$4. \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{m=1}^n \frac{1}{m}.$$

KR 64 (69)

0.156

$$1. \sum_{k=0}^p \binom{n}{k} \binom{m}{p-k} = \binom{n+m}{p} \quad [m \text{ is a natural number}].$$

KR 64 (71.1)

$$2. \sum_{k=0}^{n-p} \binom{n}{k} \binom{n}{p+k} = \frac{(2n)!}{(n-p)!(n+p)!}.$$

KR 64 (71.2)

0.157

$$1. \sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

KR 64 (72.1)

$$2. \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}.$$

KR 64 (72.2)

$$3. \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0.$$

KR 64 (72.3)

$$4. \sum_{k=1}^n k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2}.$$

KR 64 (72.4)

0.158

$$1. \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k = 4^n - \binom{2n}{n}.$$

$$2. \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^2 = 4^n - \binom{2n}{n} 3 \cdot 4^n.$$

6

$$3. \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^3 = (6n+13)4^n - 18n \binom{2n}{n}.$$

$$4. \sum_{k=1}^n \left[2^k \binom{2n-k}{n-k} - 2^{k+1} \binom{2n-k-1}{n-k-1} \right] k^4 = (32n^2+104n) \binom{2n}{n} - (60n+75)4^n.$$

0.159

$$1. \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k = \frac{1}{2} \left[4^n - \binom{2n}{n} \right].$$

$$2. \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^2 = \frac{1}{2} \left[(2n+1) \binom{2n}{n} - 4^n \right].$$

$$3. \sum_{k=0}^n \left[\binom{2n}{n-k} - \binom{2n}{n-k-1} \right] k^3 = \frac{(3n+2)}{4} \cdot 4^n - \frac{1}{2} \binom{2n}{n} (3n+1).$$

0.160

$$1. \sum_{k=n+1}^{2n} \binom{2n}{k} \alpha^k + \frac{1}{2} \binom{2n}{n} \alpha^n + \frac{(1+\alpha)^{2n-1} (1-\alpha)}{2} \sum_{k=0}^{n-1} \binom{2k}{k} \left[\frac{\alpha}{(1+\alpha)^2} \right]^k = \frac{1}{2} (1+\alpha)^{2n}.$$

0.2 Numerical Series and Infinite Products

0.21 The convergence of numerical series

The series

0.211

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

is said to *converge absolutely* if the series

0.212

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \dots,$$

composed of the absolute values of its terms converges. If the series 0.211 converges and the series 0.212 diverges, the series 0.211 is said to *converge conditionally*. Every absolutely convergent series converges.

0.22 Convergence tests

0.221

Suppose that

$$\lim_{k \rightarrow \infty} |u_k|^{\frac{1}{k}} = q.$$

If $q < 1$, the series 0.211 converges absolutely. On the other hand, if $q > 1$, the series 0.211 diverges. (Cauchy)

7

0.222

Suppose that

$$\lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = q.$$

Here, if $q < 1$, the series 0.211 converges absolutely. If $q > 1$, the series 0.211 diverges. If $\left| \frac{u_{k+1}}{u_k} \right|$ approaches 1 but remains greater than unity, the series 0.211 diverges. (d'Alembert)

0.223

Suppose that

$$\lim_{k \rightarrow \infty} k \left\{ \left| \frac{u_k}{u_{k+1}} \right| - 1 \right\} = q.$$

Here, if $q > 1$, the series 0.211 converges absolutely. If $q < 1$, the series 0.211 diverges. (Raabe)

0.224

Suppose that $f(x)$ is a positive decreasing function and that

$$\lim_{k \rightarrow \infty} \frac{e^k f(e^k)}{f(k)} = q.$$

for natural k . If $q < 1$, the series $\sum_{k=1}^{\infty} f(k)$

$$q > 1$$

0.225

Suppose that

$$\left| \frac{u_k}{u_{k+1}} \right| = 1 + \frac{q}{k} + \frac{|v_k|}{k^p},$$

where $p > 1$ and the $|v_k|$ are bounded, that is, the $|v_k|$ are all less than some M , which is independent of k . Here, if $q > 1$, the series 0.211 converges absolutely. If $q \leq 1$, this series diverges. (Gauss)

0.226

Suppose that a function $f(x)$ defined for $x \geq q \geq 1$ is continuous, positive, and decreasing. Under these conditions, the series

$$\sum_{k=1}^{\infty} f(k)$$

converges or diverges according as the integral

$$\int_q^{\infty} f(x) dx.$$

converges or diverges (the Cauchy integral test.)

8

0.227

Suppose that all terms of a sequence u_1, u_2, \dots, u_n are positive. In such a case, the series

1.
$$\sum_{k=1}^{\infty} (-1)^{k+1} u_k = u_1 - u_2 + u_3 - \dots$$

is called an *alternating series*.

If the terms of an alternating series decrease monotonically in absolute value and approach zero, that is, if

$$2. \quad u_{k+1} < u_k \quad \text{and} \quad \lim_{k \rightarrow \infty} u_k = 0,$$

the series 0.227 1. converges. Here, the remainder of the series is

$$3. \sum_{k=n+1}^{\infty} (-1)^{k-n+1} u_k = \left| \sum_{k=1}^{\infty} (-1)^{k+1} u_k - \sum_{k=1}^n (-1)^{k+1} u_k \right| < u_{n+1}. \quad (\text{Leibnitz})$$

0.228

If the series

$$1. \quad \sum_{k=1}^{\infty} v_k = v_1 + v_2 + \dots + v_k + \dots$$

converges and the numbers u_k form a monotonic bounded sequence, that is, if $|u_k| < M$ for some number M and for all k , the series

$$2. \quad \sum_{k=1}^{\infty} u_k v_k = u_1 v_1 + u_2 v_2 + \dots + u_k v_k + \dots$$

converges. (Abel)

0.229

If the partial sums of the series 0.228 1. are bounded and if the numbers u_k constitute a monotonic sequence that approaches zero, that is, if

$$\left| \sum_{k=1}^n v_k \right| < M \quad [n = 1, 2, \dots] \quad \text{and} \quad \lim_{k \rightarrow \infty} u_k = 0,$$

then the series 0.228 2. converges. (Dirichlet)

0.23-0.24 Examples of numerical series

0.231

Progressions

$$1. \sum_{k=0}^{\infty} aq^k = \frac{a}{1-q} \quad [|q| < 1].$$

$$2. \sum_{k=0}^{\infty} (a + kr)q^k = \frac{a}{1-q} + \frac{rq}{(1-q)^2} \quad [|q| < 1]$$

0.113

9

0.232

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = \ln 2$$

1.511

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{\pi}{4}$$

1.643

0.233

$$1. \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots = \zeta(p) \quad [\operatorname{Re} p > 1]$$

WH

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^p} = (1 - 2^{1-p})\zeta(p) \quad [\operatorname{Re} p > 0]$$

$$3^* \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} |B_{2n}|, \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

FI II 721

$$4. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2n}} = \frac{(2^{2n-1} - 1) \pi^{2n}}{(2n)!} |B_{2n}|.$$

JO (165)

$$5. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n} - 1) \pi^{2n}}{2 \cdot (2n)!} |B_{2n}|.$$

JO (184b)

$$6. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2} (2n)!} |E_{2n}|.$$

JO (184d)

0.234

$$1. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12}.$$

EU

$$2. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

EU

$$3. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \mathbf{G}.$$

FI II 482

$$4. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

$$5. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{96}.$$

EU

10

$$6. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^5} = \frac{5\pi^5}{1536}.$$

EU

$$7. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{(k+1)^2} = \frac{\pi^2}{12} - \ln 2.$$

$$8^6. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)} = 2 - 2 \ln 2.$$

0.235

$$S_n = \sum_{k=1}^{\infty} \frac{1}{(4k^2-1)^n},$$

$$S_1 = \frac{1}{2}, \quad S_2 = \frac{\pi^2 - 8}{16}, \quad S_3 = \frac{32 - 3\pi^2}{64}, \quad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}$$

JO (186)

0.236

$$1. \sum_{k=1}^{\infty} \frac{1}{k(4k^2-1)} = 2 \ln 2 - 1.$$

BR 51a

$$2. \sum_{k=1}^{\infty} \frac{1}{k(9k^2-1)} = \frac{3}{2} (\ln 3 - 1).$$

BR 51a

$$3. \sum_{k=1}^{\infty} \frac{1}{k(36k^2-1)} = -3 + \frac{3}{2} \ln 3 + 2 \ln 2.$$

$$4. \sum_{k=1}^{\infty} \frac{k}{(4k^2 - 1)^2} = \frac{1}{8}.$$

BR 52

$$5. \sum_{k=1}^{\infty} \frac{1}{k(4k^2 - 1)^2} = \frac{3}{2} - 2 \ln 2.$$

BR 52

$$6. \sum_{k=1}^{\infty} \frac{12k^2 - 1}{k(4k^2 - 1)^2} = 2 \ln 2.$$

AD (6917.3), BR 52

$$7^6. \sum_{k=1}^{\infty} \frac{1}{k(2k+1)^2} = 4 - \frac{\pi^2}{4} - 2 \ln 2.$$

0.237

$$1. \sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}.$$

AD (6917.2), BR 52

$$2. \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+1)} = \frac{1}{2} - \frac{\pi}{8}.$$

$$3. \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+1)} = \frac{3}{4}$$

0.133

11

$$4. \sum'_{k=1, k \neq m}^{\infty} \frac{1}{(m+k)(m-k)} = -\frac{3}{4m^2} \quad [m \text{ is an integer}].$$

AD (6916.1)

0.238

$$1. \sum_{k=1}^{\infty} \frac{1}{(2k-1)2k(2k+1)} = \ln 2 - \frac{1}{2}.$$

GI III (93)

$$2. \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)2k(2k+1)} = \frac{1}{2}(1 - \ln 2).$$

GI III (94)a

$$3. \sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+2)(3k+3)(3k+4)} = \frac{1}{6} - \frac{1}{4} \ln 3 + \frac{\pi}{12\sqrt{3}}.$$

GI III (95)

0.239

$$1. \sum_{k=1}^{\infty} (-1)^{n+1} \frac{1}{3k-2} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} + \ln 2 \right)$$

GI III (85), BR* 161 (1)

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{3k-1} = \frac{1}{3} \left(\frac{\pi}{\sqrt{3}} - \ln 2 \right).$$

BR* 161 (1)}

$$3. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{4k-3} = \frac{1}{4\sqrt{2}} [\pi + 2 \ln(\sqrt{2} + 1)].$$

BR* 161 (1)

$$4. \sum_{k=1}^{\infty} (-1)^{\lfloor \frac{k+3}{2} \rfloor} \frac{1}{k} = \frac{\pi}{4} + \frac{1}{2} \ln 2.$$

GI III (87)

$$5. \sum_{k=1}^{\infty} (-1)^{\lfloor \frac{k+3}{2} \rfloor} \frac{1}{2k-1} = \frac{\pi}{2\sqrt{2}}.$$

$$6. \sum_{k=1}^{\infty} (-1)^{\lfloor \frac{k+5}{3} \rfloor} \frac{1}{2k-1} = \frac{5\pi}{12}.$$

GI III (88)

$$7. \sum_{k=1}^{\infty} \frac{1}{(8k-1)(8k+1)} = \frac{1}{2} - \frac{\pi}{16}(\sqrt{2}+1).$$

0.241

$$1. \sum_{k=1}^{\infty} \frac{1}{2^k k} = \ln 2.$$

JO (172g)

$$2. \sum_{k=1}^{\infty} \frac{1}{2^k k^2} = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2.$$

JO (174)

12

$$3^*. \sum_{n=0}^{\infty} \binom{2n}{n} p^n = \frac{1 - \sqrt{1-4p}}{\sqrt{1-4p} - (1-4p)} \quad \left[0 \leq p < \frac{1}{4} \right].$$

$$4^*. \sum_{n=1}^{\infty} \frac{p^n}{n^2} = \frac{\pi^2}{6} - \int_1^{1-p} \frac{\ln(1-x)}{x} dx \quad [0 \leq p \leq 1].$$

$$5^*. \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j = 4^i - \binom{2i}{i} \quad \left[\binom{n}{m} = 0, \quad m < 0 \right].$$

$$6^*. \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^2 = 4i \binom{2i}{i} - 3 \cdot 4^i \quad \left[\binom{n}{m} = 0, \quad m < 0 \right].$$

$$7^*. \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^3 = (6i+13)4^i - 18i \binom{2i}{i} \quad \left[\binom{n}{m} = 0, \quad m < 0 \right].$$

$$8^* \cdot \sum_{j=1}^i \left[2^j \binom{2i-j}{i-j} - 2^{j+1} \binom{2i-(j+1)}{i-(j+1)} \right] j^4 = (32i^2+104i) \binom{2i}{i} - (60i+75)4^i$$

$$\left[\binom{n}{m} = 0, \quad m < 0 \right].$$

$$9^* \cdot \sum_{j=n+1}^{2n} \binom{2n}{j} k^j + \frac{1}{2} \binom{2n}{n} k^n + \frac{(1+k)^{2n-1}(1-k)}{2} \sum_{i=0}^{n-1} \binom{2i}{i} \left[\frac{k}{(1+k)^2} \right]^i = \frac{1}{2}(1+k)^{2n}.$$

$$10^* \cdot \sum_{k=0}^i \binom{i+k}{k} 2^{i-k} = 4^i.$$

$$11^* \cdot \sum_{k=0}^i \binom{i+k}{k}^{i-k} k = (i+1)4^i - (2i+1) \binom{2i}{i}.$$

$$12^* \cdot \sum_{k=0}^i \binom{2i}{k} = \frac{1}{2} \left(4^i + \binom{2i}{i} \right).$$

$$13^* \cdot \sum_{k=0}^i \binom{2i}{k} k = \frac{i}{2} 4^i.$$

$$14^* \cdot \sum_{k=0}^i \binom{2i}{k} k^2 = (2i+1)i4^{i-1} - \frac{i^2}{2} \binom{2i}{i}.$$

0.242

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n^{2k}} = \frac{n^2}{n^2+1} \quad (n > 1).$$

0.243

$$1. \quad \sum_{k=1}^{\infty} \frac{1}{[p+(k-1)q](p+kq) \dots [p+(k+l)q]} = \frac{1}{(l+1)q} \frac{1}{p(p+q) \dots (p+lq)}$$

$$2.7 \sum_{k=1}^{\infty} \frac{x^{k-1}}{[p + (k-1)q][p + (k-1)q + 1][p + (k-1)q + 2] \dots [p + (k-1)q + l]} =$$

$$= \frac{1}{l!} \int_0^1 \frac{t^{p-1}(1-t)^t}{1-xt^q} dt \quad [q > 0, \quad x^2 \leq 1].$$

BR* 161 (2), AD (6.704)

$$3. \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \operatorname{th} \left[\frac{(2k+1)\pi x}{2} \right] + xt \operatorname{h} \left[\frac{(2k+1)\pi}{2x} \right] \right] = \frac{\pi^3}{16}.$$

0.244

$$1. \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \int_0^1 \frac{x^p - x^q}{1-x} dx \quad [p > -1, \quad q > -1, \quad p \neq q].$$

GI III (90)

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{p + (k-1)q} = \int_0^1 \frac{t^{p-1}}{1+t^q} dt \quad [p > 0, \quad q > 0].$$

$$3^*. \sum_{k=1}^{\infty} \frac{1}{(k+p)(k+q)} = \frac{1}{q-p} \sum_{m=p+1}^q \frac{1}{m} \quad [q > p > -1, \quad p \text{ and } q \text{ integers}].$$

BR* 161 (1)

Summations of reciprocals of factorials

0.245

$$1. \sum_{k=0}^{\infty} \frac{1}{k!} = e = 2.71828 \dots$$

$$2. \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e} = 0.36787 \dots$$

$$3. 2 \sum_{k=1}^{\infty} \frac{k}{(2k+1)!} = \frac{1}{e} = 0.36787 \dots$$

$$4. \sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$$

$$5. \sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2} \left(e + \frac{1}{e} \right) = 1.54308 \dots$$

$$6. \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2} \left(e - \frac{1}{e} \right) = 1.17520 \dots$$

$$7. \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} = \cos 1 = 0.54030 \dots$$

$$8. \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} = \sin 1 = 0.84147 \dots$$

14

0.246

$$1. \sum_{k=0}^{\infty} \frac{1}{(k!)^2} = I_0(2) = 2.27958530 \dots$$

$$2. \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} = I_1(2) = 1.590636855 \dots$$

$$3. \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} = I_n(2).$$

$$4. \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} = J_0(2) = 0.22389078 \dots$$

0.247

$$\sum_{k=1}^{\infty} \frac{k!}{(n+k-1)!} = \frac{1}{(n-2) \cdot (n-1)!}.$$

0.248

$$\sum_{k=1}^{\infty} \frac{k^n}{k!} = S_n,$$

$$\begin{array}{llll} S_1 = e, & S_2 = 2e, & S_3 = 5e, & S_4 = 15e, \\ S_5 = 52e, & S_6 = 203e, & S_7 = 877e, & S_8 = 4140e. \end{array}$$

0.249⁸

$$\sum_{k=0}^{\infty} \frac{(k+1)^3}{k!} = 15e - 1.$$

0.25 Infinite products

0.250

Suppose that a sequence of numbers $a_1, a_2, \dots, a_k, \dots$ is given. If the limit $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k)$ exists, whether finite or infinite (but of definite sign), this limit is called the value of the *infinite product* $\prod_{k=1}^{\infty} (1 + a_k)$ and we write

$$1. \quad \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + a_k) = \prod_{k=1}^{\infty} (1 + a_k).$$

If an infinite product has a finite *nonzero* value, it is said to converge. Otherwise, the infinite product is said to diverge. We assume that no a_k is equal to -1 .

FI II 400

15

0.251

For the infinite product $\prod_{k=1}^{\infty} a_k$ to converge, it is necessary that $\lim_{k \rightarrow \infty} a_k = 0$.

0.252

If $a_k > 0$ or $a_k < 0$ for all values of the index k starting with some particular value, then, for the product $\prod_{k=1}^{\infty} a_k$ to converge, it is necessary and sufficient that the series $\sum_{k=1}^{\infty} a_k$ converges.

0.253

The product $\prod_{k=1}^{\infty} (1 + a_k)$ is said to converge absolutely if the product $\prod_{k=1}^{\infty} (1 + |a_k|)$ converges.

FI II 403

0.254

Absolute convergence of an infinite product implies its convergence.

0.255

The product $\prod_{k=1}^{\infty} (1 + a_k)$ converges absolutely if, and only if, the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

FI II 406

0.26 Examples of infinite products

0.261

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1} \right) = \sqrt{2}.$$

EU
FI II 403

0.262

$$1. \quad \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2} \right) = \frac{1}{2}.$$

FI II 401

$$2. \quad \prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2} \right) = \frac{2}{\pi}.$$

FI II 401

0.263

$$\frac{2}{1} \cdot \left(\frac{4}{3}\right)^{\frac{1}{2}} \left(\frac{6 \cdot 8}{5 \cdot 7}\right)^{\frac{1}{4}} \left(\frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{\frac{1}{8}} \cdots = e.$$

0.264

$$\prod_{k=1}^{\infty} \frac{\sqrt[k]{e}}{1 + \frac{1}{k}} = e^C.$$

FI II 402

0.265

$$\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots = \frac{2}{\pi}.$$

FI II 402

0.266

$$\prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x} \quad [|x| < 1].$$

FI II 401

16

0.3 Functional Series

0.30 Definitions and theorems

0.301

The series

$$1. \quad \sum_{k=1}^{\infty} f_k(x),$$

0.302

A series that converges for all values of x in a region M is said to *converge uniformly* in that region if, for every $\varepsilon \geq 0$, there exists a number N such that, for $n > N$, the inequality

$$\left| \sum_{k=n+1}^{\infty} f_k(x) \right| < \varepsilon$$

holds for *all* x in M .

0.303

If the terms of the functional series 0.301 1. satisfy the inequalities

$$|f_k(x)| < u_k \quad (k = 1, 2, 3, \dots),$$

throughout the region M , where the u_k are the terms of some *convergent* numerical series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots,$$

the series 0.301 1. converges uniformly in M . (Weierstrass)

0.304

Suppose that the series 0.301 1. converges uniformly in a region M and that a set of functions $g_k(x)$ constitutes (for each x) a monotonic sequence, and that these functions are uniformly bounded, that is, suppose that a number L exists such that the inequalities

$$1. \quad |g_n(x)| \leq L$$

hold for all n and x . Then, the series

$$2. \quad \sum_{k=1}^{\infty} f_k(x)g_k(x)$$

converges uniformly in the region M . (Abel)

Suppose that the partial sums of the series 0.301 1. are uniformly bounded; that is, suppose that, for some L and for all n and x in M , the inequalities

$$\left| \sum_{k=1}^n f_k(x) \right| < L$$

FI II 451

hold. Suppose also that for each x the functions $g_n(x)$ constitute a monotonic sequence that approaches zero uniformly in the region M . Then, the series 0.304 2. converges uniformly in the region M . (Dirichlet)

0.306⁶

If the functions $f_k(x)$ ($k = 1, 2, 3, \dots$) are integrable on the interval $[a, b]$ and if the series 0.301 1. made up of these functions converges uniformly on that interval, this series may be integrated *termwise*; that is,

$$\int_a^b \left(\sum_{k=1}^{\infty} f_k(x) \right) dx = \sum_{k=1}^{\infty} \int_a^b f_k(x) dx \quad [a \leq x \leq b].$$

FI II 459

FI II 451

0.307

Suppose that the functions $f_k(x)$ (for $k = 1, 2, 3, \dots$) have continuous derivatives $f'_k(x)$ on the interval $[a, b]$. If the series 0.301 1. converges on this interval and if the series $\sum_{k=1}^{\infty} f'_k(x)$ of these derivatives converges uniformly, the series 0.301 1. may be differentiated termwise; that is,

$$\left\{ \sum_{k=1}^{\infty} f_k(x) \right\}' = \sum_{k=1}^{\infty} f'_k(x).$$

FI II 460

0.31 Power series

0.311

A functional series of the form

$$1. \quad \sum_{k=0}^{\infty} a_k(x - \xi)^k = a_0 + a_1(x - \xi) + a_2(x - \xi)^2 + \dots$$

is called a *power series*. The following is true of any power series: if it is not everywhere convergent, the region of convergence is a circle with its center at the point ξ and a radius equal to R ; at every interior point of this circle, the power series 0.311 1. converges absolutely and outside this circle, it diverges. This circle is called the *circle of convergence* and its radius is called the *radius of convergence*. If the series converges at all points of the complex plane, we say that the radius of convergence is infinite ($R = +\infty$).

0.312

Power series may be integrated and differentiated termwise inside the circle of convergence; that is,

$$\int_{\xi}^x \left\{ \sum_{k=0}^{\infty} a_k(x - \xi)^k \right\} dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - \xi)^{k+1},$$

$$\frac{d}{dx} \left\{ \sum_{k=0}^{\infty} a_k(x - \xi)^k \right\} = \sum_{k=1}^{\infty} k a_k (x - \xi)^{k-1}.$$

18

The radius of convergence of a series that is obtained from termwise integration or differentiation of another power series coincides with the radius of convergence of the original series.

Operations on power series

0.313

Division of power series.

$$\frac{\sum_{k=0}^{\infty} b_k x^k}{\sum_{k=0}^{\infty} a_k x^k} = \frac{1}{a_0} \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_n + \frac{1}{a_0} \sum_{k=1}^n c_{n-k} a_k - b_n = 0,$$

or

$$c_n = \frac{(-1)^n}{a_0^n} \begin{vmatrix} a_1 b_0 - a_0 b_1 & a_0 & 0 & \cdots & 0 \\ a_2 b_0 - a_0 b_2 & a_1 & a_0 & \cdots & 0 \\ a_3 b_0 - a_0 b_3 & a_2 & a_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdot \\ \cdots & \cdots & \cdots & & \cdot \\ a_{n-1} b_0 - a_0 b_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n b_0 - a_0 b_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}. \quad \text{AD (6360)}$$

0.314

Power series raised to powers.

$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k,$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{m a_0} \sum_{k=1}^m (kn - m + k) a_k c_{m-k} \quad \text{for } m \geq 1 \quad [n \text{ is a natural number}].$$

AD (6361)

0.315

The substitution of one series into another.

$$\begin{aligned} \sum_{k=1}^{\infty} b_k y^k &= \sum_{k=1}^{\infty} c_k x^k & y &= \sum_{k=1}^{\infty} a_k x^k; \\ c_1 &= a_1 b_1, \quad c_2 = a_2 b_1 + a_1^2 b_2, \quad c_3 = a_3 b_1 + 2a_1 a_2 b_2 + a_1^3 b_3, \\ c_4 &= a_4 b_1 + a_2^2 b_2 + 2a_1 a_3 b_2 + 3a_1^2 a_2 b_3 + a_1^4 b_4, \quad \dots \end{aligned}$$

AD (6362)

19

0.316

Multiplication of power series

$$\sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} c_k x^k; \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

FI II 372

Taylor series

0.317

If a function $f(x)$ has derivatives of all orders throughout a neighborhood of a point ξ , then we may write the series

$$1. \quad f(\xi) + \frac{(x - \xi)}{1!} f'(\xi) + \frac{(x - \xi)^2}{2!} f''(\xi) + \frac{(x - \xi)^3}{3!} f'''(\xi) + \dots,$$

which is known as the Taylor series of the function $f(x)$.

The Taylor series converges to the function $f(x)$ if the remainder

$$2. \quad R_n(x) = f(x) - f(\xi) - \sum_{k=1}^n \frac{(x - \xi)^k}{k!} f^{(k)}(\xi)$$

approaches zero as $n \rightarrow \infty$.

The following are different forms for the remainder of a Taylor series.

$$3. \quad R_n(x) = \frac{(x - \xi)^{n+1}}{(n + 1)!} f^{(n+1)}(\xi + \theta(x - \xi)) \quad [0 < \theta < 1]. \quad (\text{Lagrange})$$

$$4. \quad R_n(x) = \frac{(x - \xi)^{n+1}}{n!} (1 - \theta)^n f^{(n+1)}(\xi + \theta(x - \xi)) \quad [0 < \theta < 1]. \quad (\text{Cauchy})$$

$$5. \quad R_n(x) = \frac{\psi(x - \xi) - \psi(0)}{\psi'[(x - \xi)(1 - \theta)]} \frac{(x - \xi)^n (1 - \theta)^n}{n!} f^{(n+1)}(\xi + \theta(x - \xi)) \quad [0 < \theta < 1], (\text{Schl\"{o}milch})$$

where $\psi(x)$ is an arbitrary function satisfying the following two conditions: 1) It and its derivative $\psi'(x)$ are continuous in the interval $(0, x - \xi)$; 2) the derivative $\psi'(x)$ does not change sign in that interval. If we set $\psi(x) = x^{p+1}$, we obtain the following form for the remainder:

$$R_n(x) = \frac{(x - \xi)^{n+1} (1 - \theta)^{n-p-1}}{(p + 1)n!} f^{(n+1)}(\xi + \theta(x - \xi)) \quad [0 < p \leq n; \quad 0 < \theta < 1]. \quad (\text{Rouch\'{e}})$$

$$6. \quad R_n(x) = \frac{1}{n!} \int_{\xi}^x f^{(n+1)}(t)(x-t)^n dt.$$

20

0.318

Other forms in which a Taylor series may be written:

$$1. \quad f(a+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(a) = f(a) + \frac{x}{1!} f'(a) + \frac{x^2}{2!} f''(a) + \dots$$

$$2. \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad [\text{Maclaurin series}]$$

0.319

The Taylor series of functions of several variables:

$$f(x, y) = f(\xi, \eta) + (x - \xi) \frac{\partial f(\xi, \eta)}{\partial x} + (y - \eta) \frac{\partial f(\xi, \eta)}{\partial y} + \frac{1}{2!} \left\{ (x - \xi)^2 \frac{\partial^2 f(\xi, \eta)}{\partial x^2} + 2(x - \xi)(y - \eta) \frac{\partial^2 f(\xi, \eta)}{\partial x \partial y} + (y - \eta)^2 \frac{\partial^2 f(\xi, \eta)}{\partial y^2} \right\} + \dots$$

0.32 Fourier series

0.320

Suppose that $f(x)$ is a *periodic* function of period $2l$ and that it is absolutely integrable (possibly improperly) over the interval $(-l, l)$. The following trigonometric series is called the *Fourier series* of $f(x)$:

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos \left\{ \frac{k\pi x}{l} \right\} + b_k \sin \left\{ \frac{k\pi x}{l} \right\} \right),$$

$$2. \quad a_k = \frac{1}{l} \int_{-l}^l f(t) \cos \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \cos \frac{k\pi t}{l} dt \quad (k = 0, 1, 2, \dots),$$

$$3.7 \quad b_k = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{k\pi t}{l} dt = \frac{1}{l} \int_{\alpha}^{\alpha+2l} f(t) \sin \frac{k\pi t}{l} dt \quad (k = 1, 2, \dots).$$

Convergence tests

0.321

The Fourier series of a function $f(x)$ at a point x_0 converges to the number

$$\frac{f(x_0 + 0) + f(x_0 - 0)}{2},$$

if, for some $h > 0$, the integral

$$\int_0^h \frac{|f(x_0 + t) + f(x_0 - t) - f(x_0 + 0) - f(x_0 - 0)|}{t} dt$$

21

exists. Here, it is assumed that the function $f(x)$ either is continuous at the point x_0 or has a discontinuity of the first kind (a *saltus*) at that point and that both one-sided limits $f(x_0 + 0)$ and $f(x_0 - 0)$ exist. (Dini)

0.322

The Fourier series of a periodic function $f(x)$ that satisfies the Dirichlet conditions on the interval $[a, b]$ converges at every point x_0 to the value $\frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$. (Dirichlet)

We say that a function $f(x)$ satisfies the Dirichlet conditions on the interval $[a, b]$ if it is bounded on that interval and if the interval $[a, b]$ can be partitioned into a finite number of subintervals inside each of which the function $f(x)$ is continuous and monotonic.

0.323

The Fourier series of a function $f(x)$ at a point x_0 converges to $\frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$ if $f(x)$ is of bounded variation in some interval $(x_0 - h, x_0 + h)$ with center at x_0 . (Jordan-Dirichlet)

FI III 528

The definition of a function of bounded variation. Suppose that a function $f(x)$ is defined on some interval $[a, b]$, where $a < b$. Let us

partition this interval in an arbitrary manner into subintervals with the dividing points

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

FI III 524

and let us form the sum

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})|.$$

Different partitions of the interval $[a, b]$ (that is, different choices of points of division x_i) yield, generally speaking, different sums.

If the set of these sums is bounded above, we say that the function $f(x)$ is of *bounded variation* on the interval $[a, b]$. The least upper bound of these sums is called the *total variation* of the function $f(x)$ on the interval $[a, b]$.

0.324

Suppose that a function $f(x)$ is piecewise-continuous on the interval $[a, b]$ and that in each interval of continuity it has a piecewise-continuous derivative. Then, at every point x_0 of the interval $[a, b]$, the Fourier series of the function $f(x)$ converges to $\frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$.

0.325

A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a cosine series of the form

$$1. \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

where

$$2. \quad a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l} dt.$$

22

0.326

A function $f(x)$ defined in the interval $(0, l)$ can be expanded in a sine series of the form

$$1. \quad \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{l},$$

where

$$2. \quad b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l} dt.$$

The convergence tests for the series 0.325 1. and 0.326 1. are analogous to the convergence tests for the series 0.320 1. (see 0.321-0.324).

0.327

The Fourier coefficients a_k and b_k (given by formulas 0.320 2. and 0.320 3.) of an absolutely integrable function approach zero as $k \rightarrow \infty$.

If a function $f(x)$ is square-integrable on the interval $(-l, l)$, the equation of closure is satisfied:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{l} \int_{-l}^l f^2(x) dx. \quad (\text{A.M. Lyapunov})$$

FI III 705

0.328

Suppose that $f(x)$ and $\varphi(x)$ are two functions that are square-integrable on the interval $(-l, l)$ and that a_k, b_k and α_k, β_k are their Fourier coefficients. For such functions, the generalized equation of closure (Parseval's equation) holds:

$$\frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty} (a_k\alpha_k + b_k\beta_k) = \frac{1}{l} \int_{-l}^l f(x)\varphi(x) dx.$$

FI III 709

For examples of Fourier series, see 1.44-1.45

0.33 Asymptotic series

0.330

Included in the collection of all divergent series is the broad class of series known as *asymptotic* or *semiconvergent* series. *Despite the fact that these series diverge*, the values of the functions that they represent can be calculated with a high degree of accuracy if we take the sum of a suitable number of terms of such series. In the case of alternating asymptotic series, we obtain greatest accuracy if

we break off the series in question at whatever term is of lowest absolute value. In this case, the error (in absolute value) does not exceed the absolute value of the first of the discarded terms (cf. 0.227 3.).

Asymptotic series have many properties that are analogous to the properties of convergent series and, for that reason, they play a significant role in analysis.

The asymptotic expansion of a function is denoted as follows:

$$f(z) \sim \sum_{n=0}^{\infty} A_n z^{-n}.$$

23

The definition of an asymptotic expansion. The divergent series $\sum_{n=0}^{\infty} \frac{A_n}{z^n}$ is called the *asymptotic expansion* of a function $f(z)$ in a given region of values of $\arg z$ if the expression $R_n(z) = z^n[f(z) - S_n(z)]$, where $S_n(z) = \sum_{k=0}^n \frac{A_k}{z^k}$, satisfies the condition

$$\lim_{|z| \rightarrow \infty} R_n(z) = 0 \text{ for fixed } n.$$

A divergent series that represents the asymptotic expansion of some function is called an *asymptotic series*.

FI II 820

0.331

Properties of asymptotic series

1. The operations of addition, subtraction, multiplication, and raising to a power can be performed on asymptotic series just as on absolutely convergent series. The series obtained as a result of these operations will also be asymptotic.

2. One asymptotic series can be divided by another provided that the first term A_0 of the divisor is not equal to zero. The series obtained as a result of division will also be asymptotic.

FI II 823-825

3. An asymptotic series can be integrated termwise, and the resultant series will also be asymptotic. In contrast, differentiation of an asymptotic series is, in general, not permissible.

FI II 824

4. A single asymptotic expansion can represent different functions. On the other hand, a given function can be expanded in an

asymptotic series in only one manner.

WH

0.4 Certain Formulas from Differential Calculus

0.41 Differentiation of a definite integral with respect to a parameter

0.410

$$\frac{d}{da} \int_{\psi(a)}^{\varphi(a)} f(x, a) dx = f(\varphi(a), a) \frac{d\varphi(a)}{da} - f(\psi(a), a) \frac{d\psi(a)}{da} + \int_{\psi(a)}^{\varphi(a)} \frac{d}{da} f(x, a) dx.$$

FI II 680
FI II 820

0.411

In particular,

$$1. \quad \frac{d}{da} \int_b^a f(x) dx = f(a).$$

$$2. \quad \frac{d}{db} \int_b^a f(x) dx = -f(b).$$

0.42 The n th derivative of a product

(Leibniz rule)

Suppose that u and v are n -times-differentiable functions of x . Then,

$$\frac{d^n(uv)}{dx^n} = u \frac{d^n v}{dx^n} + \binom{n}{1} \frac{du}{dx} \frac{d^{n-1} v}{dx^{n-1}} + \binom{n}{2} \frac{d^2 u}{dx^2} \frac{d^{n-2} v}{dx^{n-2}} + \binom{n}{3} \frac{d^3 u}{dx^3} \frac{d^{n-3} v}{dx^{n-3}} + \dots + v \frac{d^n u}{dx^n}$$

or, symbolically,

$$\frac{d^n(uv)}{dx^n} = (u + v)^{(n)}.$$

0.43 The n th derivative of a composite function

0.430

If $j(x)=F(y)$ and $y=\varphi(x)$, then

$$1. \quad \frac{d^n}{dx^n} f(x) = \frac{U_1}{1!} F'(y) + \frac{U_2}{2!} F''(y) + \frac{U_3}{3!} F'''(y) + \dots + \frac{U_n}{n!} F^{(n)}(y),$$

where

$$U_k = \frac{d^n}{dx^n} y^k - \frac{k}{1!} y \frac{d^n}{dx^n} y^{k-1} + \frac{k(k-1)}{2!} y^2 \frac{d^n}{dx^n} y^{k-2} - \dots + (-1)^{k-1} k y^{k-1} \frac{d^n y}{dx^n}.$$

AD (7361) GO

$$2. \quad \frac{d^n}{dx^n} f(x) = \sum \frac{n!}{i!j!h!\dots k!} \frac{d^m F}{dy^m} \left(\frac{y'}{1!}\right)^i \left(\frac{y''}{2!}\right)^j \left(\frac{y'''}{3!}\right)^h \dots \left(\frac{y^{(l)}}{l!}\right)^k,$$

Here, the symbol \sum indicates summation over all solutions in non negative integers of the equation $i + 2j + 3h + \dots + lk = n$ and $m = i + j + h + \dots + k$.

0.431

$$1. \quad (-1)^n \frac{d^n}{dx^n} F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}} F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}} \frac{n}{1!} F^{(n-1)}\left(\frac{1}{x}\right) + \\ + \frac{(n-1)(n-2)}{x^{2n-2}} \frac{n(n-1)}{2!} F^{(n-2)}\left(\frac{1}{x}\right) + \dots$$

AD (7362.1)

$$2. \quad (-1)^n \frac{d^n}{dx^n} e^{\frac{a}{x}} = \frac{1}{x^n} e^{\frac{a}{x}} \left\{ \left(\frac{a}{x}\right)^n + (n-1) \binom{n}{1} \left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2) \binom{n}{2} \left(\frac{a}{x}\right)^{n-2} + \dots \right. \\ \left. + (n-1)(n-2)(n-3) \binom{n}{3} \left(\frac{a}{x}\right)^{n-3} + \dots \right\}.$$

AD (7362.2)

0.432

$$1. \frac{d^n}{dx^n} F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!} (2x)^{n-2} F^{(n-1)}(x^2) + \\ + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} F^{(n-2)}(x^2) + \\ + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!} (2x)^{n-6} F^{(n-3)}(x^2) + \dots$$

AD (7363.1)

$$2. \frac{d^n}{dx^n} e^{ax^2} = (2ax)^n e^{ax^2} \left\{ 1 + \frac{n(n-1)}{1!(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!(4ax^2)^2} + \right. \\ \left. + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!(4ax^2)^3} + \dots \right\}.$$

AD (7363.2)

25

$$3. \frac{d^n}{dx^n} (1 + ax^2)^p = \frac{p(p-1)(p-2)\dots(p-n+1)(2ax)^n}{(1+ax^2)^{n-p}} \times \\ \times \left\{ 1 + \frac{n(n-1)}{1!(p-n+1)} \frac{1+ax^2}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(p-n+1)(p-n+2)} \left(\frac{1+ax^2}{4ax^2} \right)^2 + \dots \right\},$$

AD (7363.3)

$$4. \frac{d^{m-1}}{dx^{m-1}} (1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1} \frac{(2m-1)!!}{m} \sin(m \arccos x).$$

AD (7363.4)

0.433

$$1. \frac{d^n}{dx^n} F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!} \frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!} \frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots$$

AD (7364.1)

$$2. \frac{d^n}{dx^n} (1 + a\sqrt{x})^{2n-1} = \frac{(2n-1)!!}{2^n} \frac{a}{\sqrt{x}} \left(a^2 - \frac{1}{x} \right)^{n-1}.$$

AD (7364.2)

0.434

$$\frac{d^n}{dx^n} y^p = p \binom{n-p}{n} \left\{ -\binom{n}{1} \frac{1}{p-1} y^{p-1} \frac{d^n y}{dx^n} + \binom{n}{2} \frac{1}{p-2} y^{p-2} \frac{d^n y^2}{dx^n} - \dots \right\}.$$

AD (737.1)

0.435

$$\frac{d^n}{dx^n} \ln y = \left\{ \binom{n}{1} \frac{1}{1 \cdot y} \frac{d^n y}{dx^n} - \binom{n}{2} \frac{1}{2 \cdot y^2} \frac{d^n y^2}{dx^n} + \binom{n}{3} \frac{1}{3 \cdot y^3} \frac{d^n y^3}{dx^n} - \dots \right\}.$$

AD (737.2)

0.44 Integration by substitution

0.440

Let $f(x)$ and $g(x)$ be continuous in $[a, b]$. Further, let $g'(x)$ exist and be continuous there. Then

$$\int_a^b f[g(x)]g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

1. Elementary Functions

1.1 Power of Binomials

1.11 Power series

1.110

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \dots + \frac{q(q-1)\dots(q-k+1)}{k!}x^k + \dots$$

If q is neither a natural number nor zero, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, the series converges for $q > -1$ and diverges for $q \leq -1$. For $x = -1$, the series converges absolutely for $q > 0$. For $x = -1$, it converges absolutely for $q > 0$ and diverges for $q < 0$. If $q = n$ is a natural number, the series 1.110 is reduced to the finite sum 1.111.

FI II 425

1.111

$$(a+x)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}.$$

1.112

$$1. \quad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1}$$

1.121

$$2. \quad (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{k=1}^{\infty} (-1)^{k-1} kx^{k-1}.$$

$$3. \quad (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1 \cdot 1}{2 \cdot 4}x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \dots$$

$$4. \quad (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \dots$$

1.113

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k \quad [x^2 < 1].$$

1.114

$$1. \quad (1 + \sqrt{1+x})^q = 2^q \left\{ 1 + \frac{q}{1!} \left(\frac{x}{4}\right) + \frac{q(q-3)}{2!} \left(\frac{x}{4}\right)^2 + \frac{q(q-4)(q-5)}{3!} \left(\frac{x}{4}\right)^3 + \dots \right\} \quad [x^2 < 1, q \text{ is a real number}].$$

AD (6351.1)

27

$$2. \quad (x + \sqrt{1+x^2})^q = 1 + \sum_{k=0}^{\infty} \frac{q^2(q^2-2^2)(q^2-4^2)\dots[q^2-(2k)^2]x^{2k+2}}{(2k+2)!} +$$

$$\sum_{k=1}^{\infty} \frac{(q^2-1^2)(q^2-3^2)\dots[q^2-(2k-1)^2]x^{2k+1}}{(2k+1)!} \quad [x^2 < 1, q \text{ is a real number}].$$

1.12 Series of rational fractions

1.121

$$1. \quad \frac{x}{1-x} = \sum_{k=1}^{\infty} \frac{2^{k-1} x^{2^{k-1}}}{1+x^{2^{k-1}}} = \sum_{k=1}^{\infty} \frac{x^{2^{k-1}}}{1-x^{2^k}} \quad [x^2 < 1].$$

AD (6350.3)

$$2. \quad \frac{1}{x-1} = \sum_{k=1}^{\infty} \frac{2^{k-1}}{x^{2^{k-1}}+1} \quad [x^2 > 1].$$

AD (6350.3)

1.2 The Exponential Function

1.21 Series representations

1.211

$$1.^7 \quad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$2. \quad a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!}.$$

$$3. \quad e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}.$$

1.212

$$e^x(1+x) = \sum_{k=0}^{\infty} \frac{x^k(k+1)}{k!}.$$

1.213

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k)!} \quad [x < 2\pi].$$

1.214

$$e^{e^x} = e \left(1 + x + \frac{2x^2}{2!} + \frac{5x^3}{3!} + \frac{15x^4}{4!} + \dots \right).$$

AD (6460.3)

28

1.215

$$1. \quad e^{\sin x} = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} - \frac{8x^5}{5!} - \frac{3x^6}{6!} + \frac{56x^7}{7!} + \dots$$

AD (6460.4)

$$2. \quad e^{\cos x} = e \left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \dots \right).$$

AD (6460.5)

$$3. \quad e^{\operatorname{tg} x} = 1 + x + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{9x^4}{4!} + \frac{37x^5}{5!} + \dots$$

AD (6460.6)

1.216

$$1. \quad e^{\arcsin x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \dots$$

AD (6460.7)

$$2. \quad e^{\operatorname{arctg} x} = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{7x^4}{4!} + \dots$$

AD (6460.8)

1.217

$$1. \quad \pi \frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \text{ (cf. 1.421 3.)}.$$

AD (6707.1)

$$2. \frac{2\pi}{e^{\pi x} - e^{-\pi x}} = \frac{1}{x} + 2x \sum_{k=1}^{\infty} (-1)^k \frac{1}{x^2 + k^2} \quad (\text{cf. 1.422 3.}).$$

AD (6707.2)

1.22 Functional relations

1.221

$$1. a^x = e^{x \ln a}.$$

$$2. a^{\log_a x} = a^{\frac{1}{\log_x a}} = x.$$

1.222

$$1. e^x = \text{ch } x + \text{sh } x.$$

$$2. e^{ix} = \cos x + i \sin x.$$

1.223

$$e^{ax} - e^{bx} = (a - b)x \exp \left[\frac{1}{2}(a + b)x \right] \prod_{k=1}^{\infty} \left[1 + \frac{(a - b)^2 x^2}{2k^2 \pi^2} \right].$$

MO 216

1.23 Series of exponentials

1.231

$$\sum_{k=0}^{\infty} a^{kx} = \frac{1}{1 - a^x} \quad [a > 1 \text{ and } x < 0 \text{ or } 0 < a < 1 \text{ and } x > 0].$$

$$1. \quad \operatorname{th} x = 1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-2kx} \quad [x > 0].$$

$$2. \quad \operatorname{sech} x = 2 \sum_{k=0}^{\infty} (-1)^k e^{-(2k+1)x} \quad [x > 0].$$

$$3. \quad \operatorname{cosech} x = 2 \sum_{k=0}^{\infty} e^{-(2k+1)x} \quad [x > 0].$$

1.3-1.4 Trigonometric and Hyperbolic Functions

1.30 Introduction

The trigonometric and hyperbolic sines are related by the identities

$$\operatorname{sh} x = \frac{1}{i} \sin(ix), \quad \sin x = \frac{1}{i} \operatorname{sh}(ix).$$

The trigonometric and hyperbolic cosines are related by the identities

$$\operatorname{ch} x = \cos(ix), \quad \cos x = \operatorname{ch}(ix).$$

Because of this duality, every relation involving trigonometric functions has its formal counterpart involving the corresponding hyperbolic functions, and vice-versa. In many (though not all) cases, both pairs of relationships are meaningful.

The idea of matching the relationships is carried out in the list of formulas given below. However, not all the meaningful "pairs" are included in the list.

1.31 The basic functional relations

1.311

$$\begin{array}{ll}
 1. \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}); & 2. \quad \operatorname{sh} x = \frac{1}{2}(e^x - e^{-x}); \\
 \quad \quad = -i \operatorname{sh}(ix). & \quad \quad = -i \sin(ix). \\
 3. \quad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}); & 4. \quad \operatorname{ch} x = \frac{1}{2}(e^x + e^{-x}); \\
 \quad \quad = \operatorname{ch}(ix). & \quad \quad = \cos(ix). \\
 5. \quad \operatorname{th} x = \frac{\sin x}{\cos x} = \frac{1}{i} \operatorname{th}(ix) & 6. \quad \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{1}{i} \operatorname{th}(ix)
 \end{array}$$

1.312

$$1. \quad \cos^2 x + \sin^2 x = 1. \qquad 2. \quad \operatorname{ch}^2 x - \operatorname{sh}^2 x = 1.$$

1.313

$$1. \quad \sin(x \pm y) = \sin x \cos y \pm \sin y \cos x.$$

$$2. \quad \operatorname{sh}(x \pm y) = \operatorname{sh} x \operatorname{ch} y \pm \operatorname{sh} y \operatorname{ch} x.$$

$$3. \quad \sin(x \pm iy) = \sin x \operatorname{ch} y \pm i \operatorname{sh} y \cos x.$$

$$4. \quad \operatorname{sh}(x \pm iy) = \operatorname{sh} x \cos y \pm i \sin y \operatorname{ch} x.$$

$$5. \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.$$

$$6. \quad \operatorname{ch}(x \pm y) = \operatorname{ch} x \operatorname{ch} y \pm \operatorname{sh} x \operatorname{sh} y.$$

$$7. \quad \cos(x \pm iy) = \cos x \operatorname{ch} y \mp i \sin x \operatorname{sh} y.$$

$$8. \quad \operatorname{ch}(x \pm iy) = \operatorname{ch} x \cos y \pm i \operatorname{sh} x \sin y.$$

$$9. \quad \operatorname{tg}(x \pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1 \mp \operatorname{tg} x \operatorname{tg} y}.$$

$$10. \quad \operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x \operatorname{th} y}.$$

1.314

$$1. \quad \sin x \pm \sin y = 2 \sin \frac{1}{2}(x \pm y) \cos \frac{1}{2}(x \mp y).$$

$$2. \quad \operatorname{sh} x \pm \operatorname{sh} y = 2 \operatorname{sh} \frac{1}{2}(x \pm y) \operatorname{ch} \frac{1}{2}(x \mp y).$$

$$3. \quad \cos x + \cos y = 2 \cos \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y).$$

$$4. \quad \operatorname{ch} x + \operatorname{ch} y = 2 \operatorname{ch} \frac{1}{2}(x + y) \operatorname{ch} \frac{1}{2}(x - y).$$

$$5. \quad \cos x - \cos y = 2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(y - x).$$

$$6. \quad \operatorname{ch} x - \operatorname{ch} y = 2 \operatorname{sh} \frac{1}{2}(x + y) \operatorname{sh} \frac{1}{2}(x - y).$$

$$7. \quad \operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}. \quad 8. \quad \operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x \operatorname{ch} y}.$$

1.315

$$1. \quad \sin^2 x - \sin^2 y = \sin(x + y) \sin(x - y) = \cos^2 y - \cos^2 x.$$

$$2. \quad \operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{sh}(x + y) \operatorname{sh}(x - y) = \operatorname{ch}^2 x - \operatorname{ch}^2 y.$$

$$3. \quad \cos^2 x - \sin^2 y = \cos(x + y) \cos(x - y) = \cos^2 y - \sin^2 x.$$

$$4. \quad \text{sh}^2 x + \text{ch}^2 y = \text{ch}(x+y) \text{ch}(x-y) = \text{ch}^2 x + \text{sh}^2 y.$$

31

1.316

$$1. \quad (\cos x + i \sin x)^n = \cos nx + i \sin nx. \quad 2. \quad (\text{ch } x + \text{sh } x)^n = \text{sh } nx + \text{ch } nx$$

[n is an integer].

1.317

$$1. \quad \sin \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 - \cos x)}. \quad 2. \quad \text{sh} \frac{x}{2} = \pm \sqrt{\frac{1}{2}(\text{ch } x - 1)}.$$

$$3. \quad \cos \frac{x}{2} = \pm \sqrt{\frac{1}{2}(1 + \cos x)}. \quad 4. \quad \text{ch} \frac{x}{2} = \sqrt{\frac{1}{2}(\text{ch } x + 1)}.$$

$$5. \quad \text{tg} \frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}. \quad 6. \quad \text{th} \frac{x}{2} = \frac{\text{ch } x - 1}{\text{sh } x} = \frac{\text{sh } x}{\text{ch } x + 1}.$$

The signs in front of the radical in formulas 1.317 1., 1.317 2., and 1.317 3. are taken so as to agree with the signs of the left hand members. The sign of the left hand members depends in turn on the value of x .

1.32 The representation of powers of trigonometric and hyperbolic functions in terms of functions of multiples of the argument (angle)

1.320

$$1. \quad \sin^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}.$$

KR 56 (10, 2)

$$2. \quad \text{sh}^{2n} x = \frac{(-1)^n}{2^{2n}} \left\{ \sum_{k=0}^{n-1} (-1)^{n-k} 2 \binom{2n}{k} \text{ch } 2(n-k)x + \binom{2n}{n} \right\}.$$

$$3. \quad \sin^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \sin(2n-2k-1)x$$

$$4. \quad \operatorname{sh}^{2n-1} x = \frac{(-1)^{n-1}}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^{n+k-1} \binom{2n-1}{k} \operatorname{sh} (2n-2k-1)x.$$

$$5. \quad \cos^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n} \right\}.$$

KR 56 (10, 1)

$$6. \quad \operatorname{ch}^{2n} x = \frac{1}{2^{2n}} \left\{ \sum_{k=0}^{n-1} 2 \binom{2n}{k} \operatorname{ch} 2(n-k)x + \binom{2n}{n} \right\}.$$

$$7. \quad \cos^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \cos (2n-2k-1)x.$$

KR 56 (10, 3)

$$8. \quad \operatorname{ch}^{2n-1} x = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} \binom{2n-1}{k} \operatorname{ch} (2n-2k-1)x.$$

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Special cases

1.321

$$1. \quad \sin^2 x = \frac{1}{2}(-\cos 2x + 1).$$

$$2. \quad \sin^3 x = \frac{1}{4}(-\sin 3x + 3 \sin x).$$

$$3. \quad \sin^4 x = \frac{1}{8}(\cos 4x - 4 \cos 2x + 3).$$

$$4. \quad \sin^5 x = \frac{1}{16}(\sin 5x - 5 \sin 3x + 10 \sin x).$$

$$5. \quad \sin^6 x = \frac{1}{32}(-\cos 6x + 6 \cos 4x - 15 \cos 2x + 10).$$

$$6. \sin^7 x = \frac{1}{64}(-\sin 7x + 7 \sin 5x - 21 \sin 3x + 35 \sin x).$$

1.322

$$1. \operatorname{sh}^2 x = \frac{1}{2}(\operatorname{ch} 2x - 1).$$

$$2. \operatorname{sh}^3 x = \frac{1}{4}(\operatorname{sh} 3x - 3 \operatorname{sh} x).$$

$$3. \operatorname{sh}^4 x = \frac{1}{8}(\operatorname{ch} 4x - 4 \operatorname{ch} 2x + 3).$$

$$4. \operatorname{sh}^5 x = \frac{1}{16}(\operatorname{sh} 5x - 5 \operatorname{sh} 3x + 10 \operatorname{sh} x).$$

$$5. \operatorname{sh}^6 x = \frac{1}{32}(\operatorname{ch} 6x - 6 \operatorname{ch} 4x + 15 \operatorname{ch} 2x + 10).$$

$$6. \operatorname{sh}^7 x = \frac{1}{64}(\operatorname{sh} 7x - 7 \operatorname{sh} 5x + 21 \operatorname{sh} 3x + 35 \operatorname{sh} x).$$

1.323

$$1. \cos^2 x = \frac{1}{2}(\cos 2x + 1).$$

$$2. \cos^3 x = \frac{1}{4}(\cos 3x + 3 \cos x).$$

$$3. \cos^4 x = \frac{1}{8}(\cos 4x + 4 \cos 2x + 3).$$

$$4. \quad \cos^5 x = \frac{1}{16}(\cos 5x + 5 \cos 3x + 10 \cos x).$$

$$5. \quad \cos^6 x = \frac{1}{32}(\cos 6x + 6 \cos 4x + 15 \cos 2x + 10).$$

$$6. \quad \cos^7 x = \frac{1}{64}(\cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x).$$

1.324

$$1. \quad \operatorname{ch}^2 x = \frac{1}{2}(\operatorname{ch} 2x + 1).$$

$$2. \quad \operatorname{ch}^3 x = \frac{1}{4}(\operatorname{ch} 3x + 3 \operatorname{ch} x).$$

$$3. \quad \operatorname{ch}^4 x = \frac{1}{8}(\operatorname{ch} 4x + 4 \operatorname{ch} 2x + 3).$$

$$4. \quad \operatorname{ch}^5 x = \frac{1}{16}(\operatorname{ch} 5x + 5 \operatorname{ch} 3x + 10 \operatorname{ch} x).$$

$$5. \quad \operatorname{ch}^6 x = \frac{1}{32}(\operatorname{ch} 6x + 6 \operatorname{ch} 4x + 15 \operatorname{ch} 2x + 10).$$

$$6. \quad \operatorname{ch}^7 x = \frac{1}{64}(\operatorname{ch} 7x + 7 \operatorname{ch} 5x + 21 \operatorname{ch} 3x + 35 \operatorname{ch} x).$$

1.33 *The representation of trigonometric and hyperbolic functions of multiples of the argument (angle) in terms of powers of these functions*

1.331

$$\begin{aligned}
1^3. \sin nx &= n \cos^{n-1} x \sin x + \binom{n}{3} \cos^{n-3} x \sin^3 x + \\
&\quad \binom{n}{5} \cos^{n-5} x \sin^5 x - \dots ; \\
&= \sin x \left\{ 2^{n-1} \cos^{n-1} x - \binom{n-2}{1} 2^{n-3} \cos^{n-3} x + \right. \\
&\quad \left. + \binom{n-3}{2} 2^{n-5} \cos^{n-5} x - \binom{n-4}{3} 2^{n-7} \cos^{n-7} x + \dots \right\}.
\end{aligned}$$

AD (3.175)

$$\begin{aligned}
2. \operatorname{sh} nx &= x \sum_{k=1}^{[(n+1)/2]} \binom{n}{2k-1} \operatorname{sh}^{2k-2} x \operatorname{ch}^{n-2k+1} x; \\
&= \operatorname{sh} x \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n-k-1}{k} 2^{n-2k-1} \operatorname{ch}^{n-2k-1} x.
\end{aligned}$$

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$$\begin{aligned}
3. \cos nx &= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \dots ; \\
&= 2^{n-1} \cos^n x - \frac{n}{1} 2^{n-3} \cos^{n-2} x + \frac{n}{2} \binom{n-3}{1} 2^{n-5} \cos^{n-4} x - \\
&\quad - \frac{n}{3} \binom{n-4}{2} 2^{n-7} \cos^{n-6} x + \dots .
\end{aligned}$$

AD (3.175)

$$4^3. \operatorname{ch} nx = \sum_{k=0}^{[n/2]} \binom{n}{2k} \operatorname{sh}^{2k} x \operatorname{ch}^{n-2k} x = 2^{n-1} \operatorname{ch}^n x + n \sum_{k=1}^{[n/2]} (-1)^k \frac{1}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \operatorname{ch}^{n-2k} x.$$

1.332

$$\begin{aligned}
1. \sin 2nx &= 2n \cos x \left\{ \sin x - \frac{4n^2 - 2^2}{3!} \sin^3 x + \frac{(4n^2 - 2^2)(4n^2 - 4^2)}{5!} \sin^5 x - \dots \right\}; \\
&= (-1)^{n-1} \cos x \left\{ 2^{2n-1} \sin^{2n-1} x - \frac{2n-2}{1!} 2^{2n-3} \sin^{2n-3} x + \right. \\
&\quad \left. + \frac{(2n-3)(2n-4)}{2!} 2^{2n-5} \sin^{2n-5} x - \right. \\
&\quad \left. - \frac{(2n-4)(2n-5)(2n-6)}{3!} 2^{2n-7} \sin^{2n-7} x + \dots \right\}.
\end{aligned}$$

$$\begin{aligned}
2. \quad \sin(2n-1)x &= (2n-1) \left\{ \sin x - \frac{(2n-1)^2 - 1^2}{3!} \sin^3 x + \right. \\
&\quad \left. + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{5!} \sin^5 x - \dots \right\}; \\
&= (-1)^{n-1} \left\{ 2^{2n-2} \sin^{2n-1} x - \frac{2n-1}{1!} 2^{2n-4} \sin^{2n-3} x + \right. \\
&\quad \left. + \frac{(2n-1)(2n-4)}{2!} 2^{2n-6} \sin^{2n-5} x - \right. \\
&\quad \left. - \frac{(2n-1)(2n-5)(2n-6)}{3!} 2^{2n-8} \sin^{2n-7} x + \dots \right\}.
\end{aligned}$$

AD (3.174)

AD (3.172)

$$\begin{aligned}
3. \quad \cos 2nx &= 1 - \frac{4n^2}{2!} \sin^2 x + \frac{4n^2(4n^2 - 2^2)}{4!} \sin^4 x - \frac{4n^2(4n^2 - 2)(4n^2 - 4^2)}{6!} \sin^6 x + \dots; \\
&= (-1)^n \left\{ 2^{2n-1} \sin^{2n} x - \frac{2n}{1!} 2^{2n-3} \sin^{2n-2} x + \right. \\
&\quad \left. + \frac{2n(2n-3)}{2!} 2^{2n-5} \sin^{2n-4} x - \frac{2n(2n-4)(2n-5)}{3!} 2^{2n-7} \sin^{2n-6} x + \dots \right\}.
\end{aligned}$$

AD (3.173)a

AD (3.171)

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$$\begin{aligned}
4. \quad \cos(2n-1)x &= \cos x \left\{ 1 - \frac{(2n-1)^2 - 1^2}{2!} \sin^2 x + \right. \\
&\quad \left. + \frac{[(2n-1)^2 - 1^2][(2n-1)^2 - 3^2]}{4!} \sin^4 x - \dots \right\}; \\
&= (-1)^{n-1} \cos x \left\{ 2^{2n-2} \sin^{2n-2} x - \frac{2n-3}{1!} 2^{2n-4} \sin^{2n-4} x + \right. \\
&\quad \left. + \frac{(2n-4)(2n-5)}{2!} 2^{2n-6} \sin^{2n-6} x - \right. \\
&\quad \left. - \frac{(2n-5)(2n-6)(2n-7)}{3!} 2^{2n-8} \sin^{2n-8} x + \dots \right\}.
\end{aligned}$$

AD (3.174)

AD (3.172)

By using the formulas and values of 1.30, we can write formulas for $\text{sh } 2nx$, $\text{sh } (2n-1)x$, $\text{ch } 2nx$, and $\text{ch } (2n-1)x$ that are analogous to those of 1.332, just as was done in the formulas in 1.331.

Special cases

1.333

$$1. \quad \sin 2x = 2 \sin x \cos x.$$

2. $\sin 3x = 3 \sin x - 4 \sin^3 x$.

3. $\sin 4x = \cos x(4 \sin x - 8 \sin^3 x)$.

4. $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$.

5. $\sin 6x = \cos x(6 \sin x - 32 \sin^3 x + 32 \sin^5 x)$.

6. $\sin 7x = 7 \sin x - 56 \sin^3 x + 112 \sin^5 x - 64 \sin^7 x$.

1.334

1. $\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x$.

2. $\operatorname{sh} 3x = 3 \operatorname{sh} x + 4 \operatorname{sh}^3 x$.

3. $\operatorname{sh} 4x = \operatorname{ch} x(4 \operatorname{sh} x + 8 \operatorname{sh}^3 x)$.

4. $\operatorname{sh} 5x = 5 \operatorname{sh} x + 20 \operatorname{sh}^3 x + 16 \operatorname{sh}^5 x$.

5. $\operatorname{sh} 6x = \operatorname{ch} x(6 \operatorname{sh} x + 32 \operatorname{sh}^3 x + 32 \operatorname{sh}^5 x)$.

6. $\operatorname{sh} 7x = 7 \operatorname{sh} x + 56 \operatorname{sh}^3 x + 112 \operatorname{sh}^5 x + 64 \operatorname{sh}^7 x$.

1.335

1. $\cos 2x = 2 \cos^2 x - 1$.

2. $\cos 3x = 4 \cos^3 x - 3 \cos x.$

3. $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.$

4. $\cos 5x = 16 \cos^5 x - 20 \cos^3 x + 5 \cos x.$

5. $\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1.$

6. $\cos 7x = 64 \cos^7 x - 112 \cos^5 x + 56 \cos^3 x - 7 \cos x.$

1.336

1. $\operatorname{ch} 2x = 2 \operatorname{ch}^2 x - 1.$

2. $\operatorname{ch} 3x = 4 \operatorname{ch}^3 x - 3 \operatorname{ch} x.$

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3. $\operatorname{ch} 4x = 8 \operatorname{ch}^4 x - 8 \operatorname{ch}^2 x + 1.$

4. $\operatorname{ch} 5x = 16 \operatorname{ch}^5 x - 20 \operatorname{ch}^3 x + 5 \operatorname{ch} x.$

5. $\operatorname{ch} 6x = 32 \operatorname{ch}^6 x - 48 \operatorname{ch}^4 x + 18 \operatorname{ch}^2 x - 1.$

6. $\operatorname{ch} 7x = 64 \operatorname{ch}^7 x - 112 \operatorname{ch}^5 x + 56 \operatorname{ch}^3 x - 7 \operatorname{ch} x.$

1.34 Certain sums of trigonometric and hyperbolic functions

1.341

$$1. \sum_{k=0}^{n-1} \sin(x+ky) = \sin\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2}.$$

AD (361.8)

$$2. \sum_{k=0}^{n-1} \operatorname{sh}(x+ky) = \operatorname{sh}\left(x + \frac{n-1}{2}y\right) \operatorname{sh} \frac{ny}{2} \frac{1}{\operatorname{sh} \frac{y}{2}}.$$

$$3. \sum_{k=0}^{n-1} \cos(x+ky) = \cos\left(x + \frac{n-1}{2}y\right) \sin \frac{ny}{2} \operatorname{cosec} \frac{y}{2}.$$

AD (361.9)

$$4. \sum_{k=0}^{n-1} \operatorname{ch}(x+ky) = \operatorname{ch}\left(x + \frac{n-1}{2}y\right) \operatorname{sh} \frac{ny}{2} \frac{1}{\operatorname{sh} \frac{y}{2}}.$$

$$5. \sum_{k=0}^{2n-1} (-1)^k \cos(x+ky) = \sin\left(x + \frac{2n-1}{2}y\right) \sin ny \sec \frac{y}{2}.$$

JO (202)

$$6. \sum_{k=0}^{n-1} (-1)^k \sin(x+ky) = \sin\left\{x + \frac{n-1}{2}(y+\pi)\right\} \sin \frac{n(y+\pi)}{2} \sec \frac{y}{2}.$$

AD (202a)

Special cases

1.342

$$1. \sum_{k=1}^n \sin kx = \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}.$$

AD (361.1)

$$2^*. \sum_{k=0}^n \cos kx = \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + 1 = \cos \frac{nx}{2} \sin \frac{n+1}{2}x \operatorname{cosec} \frac{x}{2}$$

$$= \frac{1}{2} \left[\frac{\sin\left(n + \frac{1}{2}\right)x}{\sin \frac{1}{2}x} \right]$$

$$3. \sum_{k=1}^n \sin(2k-1)x = \sin^2 nx \operatorname{cosec} x.$$

AD (361.7)

$$4. \sum_{k=1}^n \cos(2k-1)x = \frac{1}{2} \sin 2nx \operatorname{cosec} x.$$

JO (207)

37

1.343

$$1. \sum_{k=1}^n (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos\left(\frac{2n+1}{2}x\right)}{2 \cos \frac{x}{2}}.$$

AD (361.11)

$$2. \sum_{k=1}^n (-1)^{k+1} \sin(2k-1)x = (-1)^{n+1} \frac{\sin 2nx}{2 \cos x}.$$

AD (361.10)

$$3. \sum_{k=1}^n \cos(4k-3)x + \sum_{k=1}^n \sin(4k-1)x = \sin 2nx (\cos 2nx + \sin 2nx) (\cos x + \sin x) \operatorname{cosec} 2x.$$

JO (208)

1.344

$$1. \sum_{k=1}^{n-1} \sin \frac{\pi k}{n} = \operatorname{ctg} \frac{\pi}{2n}.$$

AD (361.19)

$$2. \sum_{k=1}^{n-1} \sin \frac{2\pi k^2}{n} = \frac{\sqrt{n}}{2} \left(1 + \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2}\right).$$

AD (361.18)

1.35 Sums of powers of trigonometric functions of multiple angles

1.351

$$1. \quad \sum_{k=1}^n \sin^2 kx = \frac{1}{4}[(2n+1)\sin x - \sin(2n+1)x]\operatorname{cosec} x;$$

$$= \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x}.$$

AD (361.3)

$$2. \quad \sum_{k=1}^n \cos^2 kx = \frac{n-1}{2} + \frac{1}{2} \cos nx \sin(n+1)x \operatorname{cosec} x;$$

$$= \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x}.$$

AD (361.4)a

$$3. \quad \sum_{k=1}^n \sin^3 kx = \frac{3}{4} \sin \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} - \frac{1}{4} \sin \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}.$$

JO (210)

$$4. \quad \sum_{k=1}^n \cos^3 kx = \frac{3}{4} \cos \frac{n+1}{2}x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2} + \frac{1}{4} \cos \frac{3(n+1)x}{2} \sin \frac{3nx}{2} \operatorname{cosec} \frac{3x}{2}.$$

JO (211)a

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$$5. \quad \sum_{k=1}^n \sin^4 kx = \frac{1}{8}[3n - 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x].$$

JO (212)

$$6. \quad \sum_{k=1}^n \cos^4 kx = \frac{1}{8}[3n + 4 \cos(n+1)x \sin nx \operatorname{cosec} x + \cos 2(n+1)x \sin 2nx \operatorname{cosec} 2x].$$

JO (213)

1.352

AD (361.5)

$$2. \sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \frac{2n-1}{2}x}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}.$$

AD (361.6)

1.353

$$1. \sum_{k=1}^{n-1} p^k \sin kx = \frac{p \sin x - p^n \sin nx + p^{n+1} \sin (n-1)x}{1 - 2p \cos x + p^2}.$$

AD (361.12)a

$$2. \sum_{k=1}^{n-1} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x - p^n \operatorname{sh} nx + p^{n+1} \operatorname{sh} (n-1)x}{1 - 2p \operatorname{ch} x + p^2}.$$

$$3. \sum_{k=0}^{n-1} p^k \cos kx = \frac{1 - p \cos x - p^n \cos nx + p^{n+1} \cos (n-1)x}{1 - 2p \cos x + p^2}.$$

AD (361.13)a

$$4. \sum_{k=0}^{n-1} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x - p^n \operatorname{ch} nx + p^{n+1} \operatorname{ch} (n-1)x}{1 - 2p \operatorname{ch} x + p^2}.$$

JO (396)

1.36 Sums of products of trigonometric functions of multiple angles

1.361

$$1. \sum_{k=1}^n \sin kx \sin (k+1)x = \frac{1}{4} [(n+1) \sin 2x - \sin 2(n+1)x] \operatorname{cosec} x.$$

JO (214)

$$2. \sum_{k=1}^n \sin kx \sin (k+2)x = \frac{n}{2} \cos 2x - \frac{1}{2} \cos (n+3)x \sin nx \operatorname{cosec} x.$$

$$3. \quad 2 \sum_{k=1}^n \sin kx \cos(2k-1)y = \sin \left\{ ny + \frac{n+1}{2}x \right\} \sin \frac{n(x+2y)}{2} \operatorname{cosec} \frac{x+2y}{2} - \\ - \sin \left\{ ny - \frac{n+1}{2}x \right\} \sin \frac{n(2y-x)}{2} \operatorname{cosec} \frac{2y-x}{2}.$$

JO (217)

39

1.362

$$1. \quad \sum_{k=1}^n \left(2^k \sin^2 \frac{x}{2^k} \right)^2 = \left(2^n \sin \frac{x}{2^n} \right)^2 - \sin^2 x.$$

AD (361.15)

$$2. \quad \sum_{k=1}^n \left(\frac{1}{2^k} \sec \frac{x}{2^k} \right)^2 = \operatorname{cosec}^2 x - \left(\frac{1}{2^n} \operatorname{cosec} \frac{x}{2^n} \right)^2.$$

AD (361.14)

1.37 Sums of tangents of multiple angles

1.371

$$1. \quad \sum_{k=0}^n \frac{1}{2^k} \operatorname{tg} \frac{x}{2^k} = \frac{1}{2^n} \operatorname{ctg} \frac{x}{2^n} - 2 \operatorname{ctg} 2x.$$

AD (361.16)

$$2. \quad \sum_{k=0}^n \frac{1}{2^{2k}} \operatorname{tg}^2 \frac{x}{2^k} = \frac{2^{2n+2} - 1}{3 \cdot 2^{2n-1}} + 4 \operatorname{ctg}^2 2x - \frac{1}{2^{2n}} \operatorname{ctg}^2 \frac{x}{2^n}.$$

AD (361.20)

1.38 Sums leading to hyperbolic tangents and cotangents

1.381

$$1. \quad \sum_{k=0}^{n-1} \frac{\operatorname{th} x \frac{1}{n \sin^2 \frac{2k+1}{4n} \pi}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{2k+1}{4n} \pi}} = \operatorname{th} 2nx.$$

$$2. \sum_{k=1}^{n-1} \frac{\operatorname{th} x \frac{1}{n \sin^2 \frac{k\pi}{2n}}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{k\pi}{2n}}} = \operatorname{cth} 2nx - \frac{1}{2n}(\operatorname{th} x + \operatorname{cth} x).$$

JO (403)

$$3. \sum_{k=0}^{n-1} \frac{\operatorname{th} x \frac{2}{(2n+1) \sin^2 \frac{2k+1}{2(2n+1)} \pi}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{2k+1}{2(2n+1)} \pi}} = \operatorname{th} (2n+1)x - \frac{\operatorname{th} x}{2n+1}.$$

JO (404)

40

$$4. \sum_{k=1}^n \frac{\operatorname{th} x \frac{2}{(2n+1) \sin^2 \frac{k\pi}{2(2n+1)}}}{1 + \frac{\operatorname{th}^2 x}{\operatorname{tg}^2 \frac{k\pi}{(2n+1)}}} = \operatorname{cth} (2n+1)x - \frac{\operatorname{cth} x}{2n+1}.$$

JO (405)

1.382

$$1. \sum_{k=0}^{n-1} \frac{1}{\frac{\sin^2 \frac{2k+1}{4n} \pi}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = 2n \operatorname{th} nx.$$

JO (406)

$$2. \sum_{k=1}^{n-1} \frac{1}{\frac{\sin^2 \frac{k\pi}{2n}}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = 2n \operatorname{cth} nx - 2 \operatorname{cth} x.$$

JO (407)

$$3. \sum_{k=0}^{n-1} \frac{1}{\frac{\sin^2 \frac{2k+1}{2(2n+1)} \pi}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = (2n+1) \operatorname{th} \frac{(2n+1)x}{2} - \operatorname{th} \frac{x}{2}.$$

$$4. \sum_{k=1}^n \frac{1}{\frac{\sin^2 \frac{k\pi}{2n+1}}{\operatorname{sh} x} + \frac{1}{2} \operatorname{th} \frac{x}{2}} = (2n+1) \operatorname{cth} \frac{(2n+1)x}{2} - \operatorname{cth} \frac{x}{2}.$$

JO (409)

1.39 The representation of cosines and sines of multiples of the angle as finite products

1.391

$$1. \sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n - \text{even}].$$

JO (568)

$$2. \cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n - \text{even}].$$

JO (569)

41

$$3. \sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}} \right) \quad [n - \text{odd}].$$

JO (570)

$$4. \cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{(2k-1)\pi}{2n}} \right) \quad [n - \text{odd}].$$

JO (571)a

1.392

$$1. \sin nx = 2^{n-1} \prod_{k=0}^{n-1} \sin \left(x + \frac{k\pi}{n} \right).$$

JO (548)

JO (549)

1.393

$$1. \quad \prod_{k=0}^{n-1} \cos \left(x + \frac{2k}{n} \pi \right) = \frac{1}{2^{n-1}} \cos nx \quad [n - \text{odd}],$$

$$= \frac{1}{2^{n-1}} \left[(-1)^{\frac{n}{2}} - \cos nx \right] \quad [n - \text{even}].$$

JO (543)

$$2. \quad \prod_{k=0}^{n-1} \sin \left(x + \frac{2k}{n} \pi \right) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sin nx \quad [n - \text{odd}],$$

$$= \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} (1 - \cos nx) \quad [n - \text{even}].$$

JO (544)

1.394

$$\prod_{k=0}^{n-1} \left\{ x^2 - 2xy \cos \left(\alpha + \frac{2k\pi}{n} \right) + y^2 \right\} = x^{2n} - 2x^n y^n \cos n\alpha + y^{2n}.$$

JO (573)

1.395

$$1. \quad \cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \cos x - \cos \left(y + \frac{2k\pi}{n} \right) \right\}.$$

JO (573)

$$2. \quad \operatorname{ch} nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{ \operatorname{ch} x - \cos \left(y + \frac{2k\pi}{n} \right) \right\}.$$

JO (538)

42

1.396

$$1. \quad \prod_{k=1}^{n-1} \left(x^2 - 2x \cos \frac{k\pi}{n} + 1 \right) = \frac{x^{2n} - 1}{x^2 - 1}.$$

$$2. \prod_{k=1}^n \left(x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x - 1}.$$

KR 58 (28.2)

$$3. \prod_{k=1}^n \left(x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1 \right) = \frac{x^{2n+1} - 1}{x + 1}.$$

KR 58 (28.3)

$$4. \prod_{k=0}^{n-1} \left(x^2 - 2x \cos \frac{(2k+1)\pi}{2n} + 1 \right) = x^{2n} + 1.$$

KR 58 (28.4)

1.41 The expansion of trigonometric and hyperbolic functions in power series

1.411

$$1. \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad 2. \operatorname{sh} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

$$3. \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad 4. \operatorname{ch} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

$$5. \operatorname{tg} x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} |B_{2k}| x^{2k-1} \quad \left[x^2 < \frac{\pi^2}{4} \right].$$

FI II 523

$$6. \operatorname{th} x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315}x^7 + \dots = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k} x^{2k-1} \quad \left[x^2 < \frac{\pi^2}{4} \right].$$

$$7. \operatorname{ctg} x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2].$$

FI II 523a

$$8. \operatorname{cth} x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2].$$

$$9. \quad \sec x = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{(2k)!} x^{2k} \left[x^2 < \frac{\pi^2}{4} \right].$$

CE 330a

$$10. \quad \operatorname{sech} x = 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \dots = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad \left[x^2 < \frac{\pi^2}{4} \right]$$

CE 330

$$11. \quad \operatorname{cosec} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)|B_{2k}|x^{2k-1}}{(2k)!} \quad [x^2 < \pi^2]$$

CE 329a

$$12. \quad \operatorname{cosech} x = \frac{1}{x} - \frac{1}{6}x + \frac{7x^3}{360} - \frac{31x^5}{15120} + \dots = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1)B_{2k}}{(2k)!} x^{2k-1} \quad [x^2 < \pi^2].$$

JO (418)

43

1.412

$$1. \quad \sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}.$$

JO (452)a

$$2. \quad \cos^2 x = 1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{2k}}{(2k)!}.$$

JO (443)

$$3. \quad \sin^3 x = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^{2k+1} - 3}{(2k+1)!} x^{2k+1}.$$

JO (452a)a

$$4. \quad \cos^3 x = \frac{1}{4} \sum_{k=0}^{\infty} (-1)^k \frac{(3^{2k} + 3)x^{2k}}{(2k)!}.$$

1.413

$$1. \quad \operatorname{sh} x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1} x^{4k-2}}{(4k-1)!}.$$

JO (508)

$$2. \quad \operatorname{ch} x = \sec x + \sec x \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{4k}}{(4k)!}.$$

JO (507)

$$3. \quad \operatorname{sh} x = \sec x \sum_{k=1}^{\infty} (-1)^{[k/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}.$$

JO (510)

$$4. \quad \operatorname{ch} x = \operatorname{cosec} x \sum_{k=1}^{\infty} (-1)^{[(k-1)/2]} \frac{2^{k-1} x^{2k-1}}{(2k-1)!}.$$

JO (509)

1.414

$$1. \quad \cos [n \ln(x + \sqrt{1+x^2})] = 1 - \sum_{k=0}^{\infty} (-1)^k \frac{(n^2 + 0^2)(n^2 + 2^2) \dots [n^2 + (2k)^2]}{(2k+2)!} x^{2k+2} \quad [x^2 < 1].$$

AD (6456.1)

$$2. \quad \sin [n \ln(x + \sqrt{1+x^2})] = nx - n \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(n^2 + 1^2)(n^2 + 3^2) \dots [n^2 + (2k-1)^2] x^{2k+1}}{(2k+1)!} \quad [x^2 < 1].$$

AD (6456.2)

Power series for $\ln \sin x$, $\ln \cos x$ and $\ln \operatorname{tg} x$ see 1.518.

1.42 Expansion in series of simple fractions

1.421

$$1. \quad \operatorname{tg} \frac{\pi x}{2} = \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 - x^2}.$$

$$2. \quad \operatorname{th} \frac{\pi x}{2} = \frac{4\pi}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 + x^2}.$$

$$3. \quad \operatorname{ctg} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2} = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{\substack{h=-\infty \\ h \neq 0}}^{\infty} \frac{1}{k(x-k)}.$$

AD (6495.2), JO (450a)

$$4. \quad \operatorname{cth} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2} \quad (\text{cf. 1.2171}).$$

1.217

$$5. \quad \operatorname{tg}^2 \frac{\pi x}{2} = x^2 \sum_{k=1}^{\infty} \frac{2(2k-1)^2 - x^2}{(1^2 - x^2)^2 (3^2 - x^2)^2 \dots [(2k-1)^2 - x^2]^2}.$$

JO (450)

1.422

$$1. \quad \sec \frac{\pi x}{2} = \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k-1}{(2k-1)^2 - x^2}.$$

AD (6495.3)a

$$2. \quad \sec^2 \frac{\pi x}{2} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k-1-x)^2} + \frac{1}{(2k-1+x)^2} \right\}.$$

JO (451)a

$$3. \quad \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 - k^2} \quad (\text{see also 1.217 2.}).$$

AD (6495.4)a

$$4. \operatorname{cosec}^2 \pi x = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2} = \frac{1}{\pi^2 x^2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{x^2 + k^2}{(x^2 - k^2)^2}.$$

JO (446)

$$5. \frac{1 + x \operatorname{cosec} x}{2x^2} = \frac{1}{x^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(x^2 - k^2 \pi^2)}.$$

JO (449)

$$6. \operatorname{cosec} \pi x = \frac{1}{\pi x} + \frac{1}{\pi} \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{1}{x-k} + \frac{1}{k} \right).$$

JO (450b)

1.423

$$\frac{\pi^2}{4m^2} \operatorname{cosec}^2 \frac{\pi}{m} + \frac{\pi}{4m} \operatorname{ctg} \frac{\pi}{m} - \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{(1 - k^2 m^2)^2}.$$

JO (477)

1.43 Representation in the form of an infinite product

1.431

$$1. \sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2} \right).$$

EU

$$2. \operatorname{sh} x = x \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2 \pi^2} \right).$$

EU

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$$3. \cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2 \pi^2} \right).$$

$$4. \quad \operatorname{ch} x = \prod_{k=0}^{\infty} \left(1 + \frac{4x^2}{(2k+1)^2\pi^2} \right).$$

EU

1.432

$$1. \quad \cos x - \cos y = 2 \left(1 - \frac{x^2}{y^2} \right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{(2k\pi + y)^2} \right) \left(1 - \frac{x^2}{(2k\pi - y)^2} \right).$$

AD (653.2)

$$2. \quad \operatorname{ch} x - \cos y = 2 \left(1 + \frac{x^2}{y^2} \right) \sin^2 \frac{y}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{(2k\pi + y)^2} \right) \left(1 + \frac{x^2}{(2k\pi - y)^2} \right).$$

AD (653.1)

1.433

$$\cos \frac{\pi x}{4} - \sin \frac{\pi x}{4} = \prod_{k=1}^{\infty} \left[1 + \frac{(-1)^k x}{2k-1} \right].$$

BR* 189

1.434

$$\cos^2 x = \frac{1}{4} (\pi + 2x)^2 \prod_{k=1}^{\infty} \left[1 - \left(\frac{\pi + 2x}{2k\pi} \right)^2 \right]^2.$$

MO 216

1.435

$$\frac{\sin \pi(x+a)}{\sin \pi a} = \frac{x+a}{a} \prod_{k=1}^{\infty} \left(1 - \frac{x}{k-a} \right) \left(1 + \frac{x}{k+a} \right).$$

MO 216

1.436

$$1 - \frac{\sin^2 \pi x}{\sin^2 \pi a} = \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{x}{k-a} \right)^2 \right].$$

MO 216

1.437

1.437

$$\frac{\sin 3x}{\sin x} = - \prod_{k=-\infty}^{\infty} \left[1 - \left(\frac{2x}{x + k\pi} \right)^2 \right].$$

MO 216

1.438

$$\frac{\operatorname{ch} x - \cos a}{1 - \cos a} = \prod_{k=-\infty}^{\infty} \left[1 + \left(\frac{x}{2k\pi + a} \right)^2 \right].$$

MO 216

1.439

$$1. \quad \sin x = x \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \quad [|x| < 1].$$

AD (615), MO 216

$$2. \quad \frac{\sin x}{x} = \prod_{k=1}^{\infty} \left[1 - \frac{4}{3} \sin^2 \left(\frac{x}{3^k} \right) \right].$$

MO 216

1.44-1.45 Trigonometric (Fourier) series

1.441

$$1. \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \quad [0 < x < 2\pi].$$

FI III 539

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$$2. \quad \sum_{k=1}^{\infty} \frac{\cos kx}{k} = \frac{1}{2} \ln \frac{1}{2(1 - \cos x)} \quad [0 < x < 2\pi].$$

FI III 530a, AD (6814)

$$3. \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin kx}{k} = \frac{x}{2} \quad [-\pi < x < \pi].$$

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k} = \ln \left(2 \cos \frac{x}{2} \right) \quad [-\pi < x < \pi].$$

FI III 550

1.442

$$1. \sum_{k=1}^{\infty} \frac{\sin (2k-1)x}{2k-1} = \frac{\pi}{x} \quad [0 < x < 2\pi].$$

FI III 541

$$2. \sum_{k=1}^{\infty} \frac{\cos (2k-1)x}{2k-1} = \frac{1}{2} \ln \operatorname{ctg} \frac{x}{2} \quad [0 < x < \pi].$$

BR* 168, JO (266), GI III(195)

$$3. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin (2k-1)x}{2k-1} = \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \quad \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right].$$

BR* 168, JO (268)a

$$4^*. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos (2k-1)x}{2k-1} = \begin{cases} \frac{\pi}{4} \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right] \\ -\frac{\pi}{4} \left[\frac{\pi}{2} < x < \frac{3\pi}{2} \right] \end{cases}.$$

BR* 168, JO (269)

1.443

$$\begin{aligned} 1. \sum_{k=1}^{\infty} \frac{\cos k\pi x}{k^{2n}} &= (-1)^{n-1} 2^{2n-1} \frac{\pi^{2n}}{(2n)!} \sum_{k=0}^{2n} \binom{2n}{k} B_{2n-k} \varrho^k; \\ &= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \left(\frac{x}{2} \right) \quad \left[0 < x < 1, \quad \varrho = \frac{x}{2} - [x/2] \right]. \end{aligned}$$

CE 340, GE 71

$$\begin{aligned} 2. \sum_{k=1}^{\infty} \frac{\sin k\pi x}{k^{2n+1}} &= (-1)^{n-1} 2^{2n} \frac{\pi^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} B_{2n-k+1} \rho^k; \\ &= (-1)^{n-1} \frac{1}{2} \frac{(2\pi)^{2n+1}}{(2n+1)!} B_{2n+1} \left(\frac{x}{2} \right) \quad \left[0 < x < 1; \quad \rho = \frac{x}{2} - [x/2] \right]. \end{aligned}$$

$$3. \sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \quad [0 \leq x \leq 2\pi].$$

FI III 547

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos kx}{k^2} = \frac{\pi^2}{12} - \frac{x^2}{4} \quad [-\pi \leq x \leq \pi].$$

FI III 544

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$$\left. \begin{aligned} 5. \sum_{k=1}^{\infty} \frac{\sin kx}{k^3} &= \frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} \\ 6. \sum_{k=1}^{\infty} \frac{\cos kx}{k^4} &= \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} \\ 7. \sum_{k=1}^{\infty} \frac{\sin kx}{k^5} &= \frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} \end{aligned} \right\} [0 \leq x \leq 2\pi]$$

AD (6818)

AD (6617)

1.444

$$1. \sum_{k=1}^{\infty} \frac{\sin 2(k+1)x}{k(k+1)} = \sin 2x - (\pi - 2x) \sin^2 x - \sin x \cos x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi].$$

BR* 168, GI III (190)

$$2. \sum_{k=1}^{\infty} \frac{\cos 2(k+1)x}{k(k+1)} = \cos 2x - \left(\frac{\pi}{2} - x\right) \sin 2x + \sin^2 x \ln(4 \sin^2 x) \quad [0 \leq x \leq \pi].$$

BR* 168

$$3. \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k+1)x}{k(k+1)} = \sin x - \frac{x}{2}(1 + \cos x) - \sin x \ln \left| 2 \cos \frac{x}{2} \right|.$$

MO 213

$$4. \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k+1)x}{k(k+1)} = \cos x - \frac{x}{2} \sin x - (1 + \cos x) \ln \left| 2 \cos \frac{x}{2} \right|$$

MO 213

$$6^6. \sum_{k=1}^{\infty} \frac{\cos (2k-1)x}{(2k-1)^2} = \frac{\pi}{4} \left(\frac{\pi}{2} - |x| \right) \quad [-\pi \leq x \leq \pi].$$

FI III 546

$$7. \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k-1)(2k+1)} = \frac{1}{2} - \frac{\pi}{4} \sin x \quad \left[0 \leq x \leq \frac{\pi}{2} \right].$$

JO (591)

1.445

$$1. \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi \operatorname{sh} \alpha(\pi - x)}{2 \operatorname{sh} \alpha\pi} \quad [0 < x < 2\pi].$$

BR* 157, JO (411)

$$2. \sum_{k=1}^{\infty} \frac{\cos kx}{k^2 + \alpha^2} = \frac{\pi \operatorname{ch} \alpha(\pi - x)}{2\alpha \operatorname{sh} \alpha\pi} - \frac{1}{2\alpha^2} \quad [0 \leq x \leq 2\pi].$$

BR* 257, JO (410)

$$3. \sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2 + \alpha^2} = \frac{\pi \operatorname{ch} \alpha x}{2\alpha \operatorname{sh} \alpha\pi} - \frac{1}{2\alpha^2} \quad [-\pi \leq x \leq \pi].$$

FI III 546

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k \sin kx}{k^2 + \alpha^2} = \frac{\pi \operatorname{sh} \alpha x}{2 \operatorname{sh} \alpha\pi} \quad [-\pi < x < \pi].$$

FI III, 546

48

$$5. \sum_{k=1}^{\infty} \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin \{ \alpha[(2m+1)\pi - x] \}}{2 \sin \alpha\pi} \quad \left[\text{when } x = 2m\pi, \sum \dots = 0 \right].$$

$$[2m\pi < x < (2m+2)\pi, \alpha \text{—not an integer}].$$

MO 213

$$7. \sum_{k=1}^{\infty} (-1)^k \frac{k \sin kx}{k^2 - \alpha^2} = \pi \frac{\sin [\alpha(2m\pi - x)]}{2 \sin \alpha\pi} \quad [\text{when } x = (2m+1)\pi, \quad \sum \dots = 0]$$

$$[(2m-1)\pi < x < (2m+1)\pi, \alpha \text{—not an integer}].$$

FI III 545a

$$8. \sum_{k=1}^{\infty} (-1)^k \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi \cos [\alpha(2m\pi - x)]}{2 \alpha \sin \alpha\pi}$$

$$[(2m-1)\pi \leq x \leq (2m+1)\pi, \alpha \text{—not an integer}].$$

FI III 545a

1.446

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos (2k+1)x}{(2k-1)(2k+1)(2k+3)} = \frac{\pi}{8} \cos^2 x - \frac{1}{3} \cos x \quad \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right].$$

BR* 256, GI III (189)

1.447

$$\left. \begin{aligned} 1. \sum_{k=1}^{\infty} p^k \sin kx &= \frac{p \sin x}{1 - 2p \cos x + p^2} \\ 2. \sum_{k=0}^{\infty} p^k \cos kx &= \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \\ 3. 1 + 2 \sum_{k=1}^{\infty} p^k \cos kx &= \frac{1 - p^2}{1 - 2p \cos x + p^2} \end{aligned} \right\} \quad [|p| < 1].$$

FI II 559a, MO 213

FI II 559

FI II 559

1.448

$$\left. \begin{aligned} 1. \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k} &= \arctg \frac{p \sin x}{1 - p \cos x} \\ 2. \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k} &= \ln \frac{1}{\sqrt{1 - 2p \cos x + p^2}} \\ 3. \sum_{k=1}^{\infty} \frac{p^{2k-1} \sin (2k-1)x}{2k-1} &= \frac{1}{2} \arctg \frac{2p \sin x}{1 - p^2} \\ 4. \sum_{k=1}^{\infty} \frac{p^{2k-1} \cos (2k-1)x}{2k-1} &= \frac{1}{4} \ln \frac{1 + 2p \cos x + p^2}{1 - 2p \cos x + p^2} \end{aligned} \right\} \quad [0 < x < 2\pi, \quad p^2 \leq 1].$$

$$\left. \begin{aligned}
 5. \quad & \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \sin(2k-1)x}{2k-1} = \\
 & = \frac{1}{4} \ln \frac{1+2p \sin x + p^2}{1-2p \sin x + p^2} \\
 6. \quad & \sum_{k=1}^{\infty} \frac{(-1)^{k-1} p^{2k-1} \cos(2k-1)x}{2k-1} = \\
 & = \frac{1}{2} \operatorname{arctg} \frac{2p \cos x}{1-p^2}
 \end{aligned} \right\} [0 < x < \pi, \quad p^2 \leq 1].$$

JO (597)

JO (261)

1.449

$$\left. \begin{aligned}
 1. \quad & \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k!} = e^{p \cos x} \sin(p \sin x) \\
 2. \quad & \sum_{k=0}^{\infty} \frac{p^k \cos kx}{k!} = e^{p \cos x} \cos(p \sin x)
 \end{aligned} \right\} [p^2 \leq 1].$$

JO (485)

JO (486)

Fourier expansions of hyperbolic functions

1.451

$$1. \quad \operatorname{sh} x = \cos x \sum_{k=0}^{\infty} \frac{(1^2 + 0^2)(1^2 + 2^2) \dots [1^2 + (2k)^2]}{(2k+1)!} \sin^{2k+1} x.$$

JO (504)

$$2. \quad \operatorname{ch} x = \cos x + \cos x \sum_{k=1}^{\infty} \frac{(1^2 + 1^2)(1^2 + 3^2) \dots [1^2 + (2k-1)^2]}{(2k)!} \sin^{2k} x.$$

JO (503)

1.452

$$\left. \begin{aligned}
 1. \quad & \operatorname{sh}(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \cos(2k+1)\theta}{(2k+1)!} \\
 2. \quad & \operatorname{ch}(x \cos \theta) = \sec(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k} \cos 2k\theta}{(2k)!}
 \end{aligned} \right\} [x^2 < 1].$$

JO (390)

JO (390)

JO (391)

$$\left. \begin{aligned} 3. \quad \operatorname{sh}(x \cos \theta) &= \operatorname{cosec}(x \sin \theta) \sum_{k=1}^{\infty} \frac{x^{2k} \sin 2k\theta}{(2k)!} \\ 4. \quad \operatorname{ch}(x \cos \theta) &= \operatorname{cosec}(x \sin \theta) \sum_{k=0}^{\infty} \frac{x^{2k+1} \sin (2k+1)\theta}{(2k+1)!} \end{aligned} \right\} [x^2 < 1], \quad [x \sin \theta \neq 0].$$

JO (392)

JO (393)

50

1.46 Series of products of exponential and trigonometric functions

1.461

$$1. \quad \sum_{k=0}^{\infty} e^{-kt} \sin kx = \frac{1}{2} \frac{\sin x}{\operatorname{ch} t - \cos x} \quad [t > 0].$$

MO 213

$$2. \quad 1 + 2 \sum_{k=1}^{\infty} e^{-kt} \cos kx = \frac{\operatorname{sh} t}{\operatorname{ch} t - \cos x} \quad [t > 0].$$

MO 213

1.462

$$\sum_{k=1}^{\infty} \frac{\sin kx \sin ky}{k} e^{-2k|t|} = \frac{1}{4} \ln \left[\frac{\sin \frac{x+y}{2} + \operatorname{sh}^2 t}{\sin^2 \frac{x-y}{2} + \operatorname{sh}^2 t} \right].$$

MO 214

1.463

$$1. \quad e^{x \cos \varphi} \cos(x \sin \varphi) = \sum_{n=0}^{\infty} \frac{x^n \cos n\varphi}{n!} \quad [x^2 < 1].$$

AD (6476.1)

$$2. \quad e^{x \cos \varphi} \sin(x \sin \varphi) = \sum_{n=1}^{\infty} \frac{x^n \sin n\varphi}{n!} \quad [x^2 < 1].$$

1.47 Series of hyperbolic functions

1.471

$$1. \quad \sum_{k=1}^{\infty} \frac{\operatorname{sh} kx}{k!} = e^{\operatorname{ch} x} \operatorname{sh}(\operatorname{sh} x).$$

JO (395)

$$2. \quad \sum_{k=0}^{\infty} \frac{\operatorname{ch} kx}{k!} = e^{\operatorname{ch} x} \operatorname{ch}(\operatorname{sh} x).$$

JO (394)

$$3.* \quad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \left[\frac{1}{x} \operatorname{th} \frac{(2m+1)\pi x}{2} + x \operatorname{th} \frac{(2m+1)\pi}{2x} \right] = \frac{\pi^3}{16}.$$

1.472

$$1. \quad \sum_{k=1}^{\infty} p^k \operatorname{sh} kx = \frac{p \operatorname{sh} x}{1 - 2p \operatorname{ch} x + p^2} \quad [p^2 < 1].$$

JO (396)

$$2. \quad \sum_{k=0}^{\infty} p^k \operatorname{ch} kx = \frac{1 - p \operatorname{ch} x}{1 - 2p \operatorname{ch} x + p^2} \quad [p^2 < 1].$$

JO (397)a

1.48 Lobachevskiy's "Angle of parallelism" $\Pi(x)$

1.480

Definition.

$$1. \quad \Pi(x) = 2 \operatorname{arctg} e^x = 2 \operatorname{arctg} e^{-x} \quad [x \geq 0].$$

$$2. \quad \Pi(x) = \pi - \Pi(-x) \quad [x < 0].$$

LO III 183, LOI 193

LO III 297, LOI 120

Functional relations

$$1. \quad \sin \Pi(x) = \frac{1}{\operatorname{ch} x}.$$

LO III 297

$$2. \quad \cos \Pi(x) = \operatorname{th} x.$$

LO III 297

$$3. \quad \operatorname{tg} \Pi(x) = \frac{1}{\operatorname{sh} x}.$$

LO III 297

$$4. \quad \operatorname{ctg} \Pi(x) = \operatorname{sh} x.$$

LO III 297

$$5. \quad \sin \Pi(x + y) = \frac{\sin \Pi(x) \sin \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}.$$

LO III 297

$$6. \quad \cos \Pi(x + y) = \frac{\cos \Pi(x) + \cos \Pi(y)}{1 + \cos \Pi(x) \cos \Pi(y)}.$$

LO III 183

1.482

Connection with the gudermannian.

$$\operatorname{gd}(-x) = \Pi(x) - \frac{\pi}{2}.$$

(Definite) integral of the angle of parallelism; cf. 4.581 and 4.561.

1.49 The hyperbolic amplitude (the Gudermannian) $\operatorname{gd} x$

1.490

Definition.

$$1. \quad \operatorname{gd} x = \int_0^x \frac{dt}{\operatorname{ch} t} = 2 \operatorname{arctg} e^x - \frac{\pi}{2}.$$

JA

$$2. \quad x = \int_0^{\operatorname{gd} x} \frac{dt}{\cos t} = \ln \operatorname{tg} \left(\frac{\operatorname{gd} x}{2} + \frac{\pi}{4} \right).$$

JA

1.491

Functional relations.

$$1. \quad \operatorname{ch} x = \sec(\operatorname{gd} x).$$

AD (343.1), JA

$$2. \quad \operatorname{sh} x = \operatorname{tg}(\operatorname{gd} x).$$

AD (343.2), JA

$$3. \quad e^x = \sec(\operatorname{gd} x) + \operatorname{tg}(\operatorname{gd} x) = \operatorname{tg} \left(\frac{\pi}{4} + \frac{\operatorname{gd} x}{2} \right) = \frac{1 + \sin(\operatorname{gd} x)}{\cos(\operatorname{gd} x)}.$$

AD (343.5), JA

$$4. \quad \operatorname{th} x = \sin(\operatorname{gd} x).$$

AD (343.3), JA

$$5. \quad \operatorname{th} \frac{x}{2} = \operatorname{tg} \left(\frac{1}{2} \operatorname{gd} x \right).$$

$$6. \operatorname{arctg}(\operatorname{th} x) = \frac{1}{2} \operatorname{gd} 2x.$$

AD (343.6a)

1.492

If $\gamma = \operatorname{gd} x$, then $ix = \operatorname{gd} i\gamma$.

JA

52

1.493

Series expansion.

$$1. \frac{\operatorname{gd} x}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{th}^{2k+1} \frac{x}{2}.$$

JA

$$2. \frac{x}{2} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \operatorname{tg}^{2k+1} \left(\frac{1}{2} \operatorname{gd} x \right).$$

JA

$$3. \operatorname{gd} x = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{61x^7}{5040} + \dots$$

JA

$$4. x = \operatorname{gd} x + \frac{(\operatorname{gd} x)^3}{6} + \frac{(\operatorname{gd} x)^5}{24} + \frac{61(\operatorname{gd} x)^7}{5040} + \dots \quad \left[\operatorname{gd} x < \frac{\pi}{2} \right].$$

JA

1.5 The Logarithm

1.51 Series representation

1.511

1.512

$$1. \quad \ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k} \quad [0 < x \leq 2].$$

$$2. \quad \ln x = 2 \left[\frac{x-1}{x+1} + \frac{1}{3} \left(\frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left(\frac{x-1}{x+1} \right)^5 + \dots \right] = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{x-1}{x+1} \right)^{2k-1} \quad [0 < x].$$

$$3. \quad \ln x = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \dots = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x-1}{x} \right)^k \quad \left[x \geq \frac{1}{2} \right].$$

AD (644.6)

1.513

$$1. \quad \ln \frac{1+x}{1-x} = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} x^{2k-1} \quad [x^2 < 1].$$

FI II 421

$$2. \quad \ln \frac{x+1}{x-1} = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)x^{2k-1}} \quad [x^2 > 1].$$

AD (644.9)

$$3. \quad \ln \frac{x}{x-1} = \sum_{k=1}^{\infty} \frac{1}{kx^k} \quad [x \leq -1 \text{ or } x > 1].$$

JO (88a)

$$4. \quad \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad [-1 \leq x < 1].$$

JO (88b)

53

$$5. \quad \frac{1-x}{x} \ln \frac{1}{1-x} = 1 - \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} \quad [-1 \leq x < 1].$$

$$6. \quad \frac{1}{1-x} \ln \frac{1}{1-x} = \sum_{k=1}^{\infty} x^k \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1].$$

JO (88e)

$$7. \quad \frac{(1-x)^2}{2x^3} \ln \frac{1}{1-x} = \frac{1}{2x^2} - \frac{3}{4x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{k(k+1)(k+2)} \quad [-1 \leq x < 1].$$

AD (6445.1)

1.514

$$\ln(1 - 2x \cos \varphi + x^2) = -2 \sum_{k=1}^{\infty} \frac{\cos k\varphi}{k} x^k; \quad \ln(x + \sqrt{1+x^2}) = \operatorname{Arsh} x.$$

see **1.631**, **1.641**, **1.642**, **1.646** $[x^2 \leq 1, \quad x \cos \varphi \neq 1].$

MO 98, FI II 485

1.631

1.641

1.642

1.646

1.515

$$1. \quad \ln(1 + \sqrt{1+x^2}) = \ln 2 + \frac{1 \cdot 1}{2 \cdot 2} x^2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4} x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6} x^6 - \dots;$$

$$= \ln 2 - \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k} (k!)^2} x^{2k} \quad [x^2 \leq 1].$$

JO (91)

$$2. \quad \ln(1 + \sqrt{1+x^2}) = \ln x + \frac{1}{x} - \frac{1}{2 \cdot 3x^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots;$$

$$= \ln x + \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} \cdot k!(k-1)!(2k+1)x^{2k+1}} \quad [x^2 \geq 1].$$

AD (644.4)

$$3. \quad \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) = x - \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1} (k-1)! k!}{(2k+1)!} x^{2k+1} \quad [x^2 \leq 1].$$

JO (93)

$$4. \frac{\ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} (k!)^2}{(2k+1)!} x^{2k+1} \quad [x^2 \leq 1].$$

JO (94)

1.516

$$1. \frac{1}{2} \{\ln(1 \pm x)\}^2 = \sum_{k=1}^{\infty} \frac{(\mp 1)^{k+1} x^{k+1}}{k+1} \sum_{n=1}^k \frac{1}{n} \quad [x^2 < 1].$$

JO (86), JO (85)

$$2. \frac{1}{6} \{\ln(1+x)\}^3 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+2}}{k+2} \sum_{n=1}^k \frac{1}{n+1} \sum_{m=1}^n \frac{1}{m} \quad [x^2 < 1].$$

AD (644.14)

$$3. -\ln(1+x) \cdot \ln(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n+1}}{n} \quad [x^2 < 1].$$

JO (87)

54

$$4. \frac{1}{4x} \left\{ \frac{1+x}{\sqrt{x}} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}} + 2 \ln(1-x) \right\} = \frac{1}{2x} + \sum_{k=1}^{\infty} \frac{x^{k-1}}{(2k-1)2k(2k+1)} \quad [0 < x < 1].$$

AD (6445.2)

1.517

$$1.^6 \frac{1}{2x} \left\{ 1 - \ln(1+x) - \frac{1-x}{\sqrt{x}} \operatorname{arctg} \sqrt{x} \right\} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{(2k-1)2k(2k+1)} \quad [0 < x \leq 1].$$

AD (6445.3)

$$2. \frac{1}{2} \operatorname{arctg} x \ln \frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{x^{4k-2}}{2k-1} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{2n-1} \quad [x^2 < 1].$$

BR* 163

1.518

$$\begin{aligned}
 1. \quad \ln \sin x &= \ln x - \frac{x^2}{6} - \frac{x^4}{180} - \frac{x^6}{2835} - \dots : \\
 &= \ln x + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!} \quad [0 < x < \pi].
 \end{aligned}$$

AD (643.1)a

$$\begin{aligned}
 2.^3 \quad \ln \cos x &= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \frac{17x^8}{2520} - \dots, \\
 &= -\sum_{k=1}^{\infty} \frac{2^{2k-1} (2^{2k} - 1) |B_{2k}|}{k(2k)!} x^{2k} = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{\sin^{2k} x}{k} \quad \left[x^2 < \frac{\pi^2}{4} \right].
 \end{aligned}$$

FI II 524

$$\begin{aligned}
 3. \quad \ln \operatorname{tg} x &= \ln x + \frac{x^2}{3} + \frac{7}{90} x^4 + \frac{62}{2835} x^6 + \frac{127}{18,900} x^8 + \dots ; \\
 &= \ln x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2^{2k-1} - 1) 2^{2k} B_{2k} x^{2k}}{k(2k)!} \quad \left[0 < x < \frac{\pi}{2} \right].
 \end{aligned}$$

AD (643.3)a

Power series for $\frac{\cos}{\sin} \{n \ln(x + \sqrt{1-x^2})\}$ cf. 1.414.

1.52 Series of logarithms

1.431

1.521

$$1. \quad \sum_{k=1}^{\infty} \ln \left(1 - \frac{4x^2}{(2k-1)^2 \pi^2} \right) = \ln \cos x \quad \left[-\frac{\pi}{2} < x < \frac{\pi}{2} \right].$$

$$2. \quad \sum_{k=1}^{\infty} \ln \left(1 - \frac{x^2}{k^2 \pi^2} \right) = \ln \sin x - \ln x \quad [0 < x < \pi].$$

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1.6 The Inverse Trigonometric and Hyperbolic Functions

1.61 The domain of definition

The principal values of the inverse trigonometric functions are defined by the inequalities:

$$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}; \quad 0 \leq \arccos x \leq \pi \quad [-1 \leq x \leq 1].$$

FI II 553

$$-\frac{\pi}{2} < \operatorname{arctg} x < \frac{\pi}{2}; \quad 0 < \operatorname{arcctg} x < \pi \quad [-\infty < x < +\infty].$$

FI II 552

1.62 - 1.63 Functional relations

1.621

The relationship between the inverse and the direct trigonometric functions.

$$\begin{aligned} 1. \quad \arcsin(\sin x) &= x - 2n\pi && \left[2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2}\right]; \\ &= -x + (2n+1)\pi && \left[(2n+1)\pi - \frac{\pi}{2} \leq x \leq (2n+1)\pi + \frac{\pi}{2}\right]. \end{aligned}$$

$$\begin{aligned} 2. \quad \arccos(\cos x) &= x - 2n\pi && [2n\pi \leq x \leq (2n+1)\pi]; \\ &= -x + 2(n+1)\pi && [(2n+1)\pi \leq x \leq 2(n+1)\pi]. \end{aligned}$$

$$3. \quad \operatorname{arctg}(\operatorname{tg} x) = x - n\pi \quad \left[n\pi - \frac{\pi}{2} < x < n\pi + \frac{\pi}{2}\right].$$

$$4. \quad \operatorname{arcctg}(\operatorname{ctg} x) = x - n\pi \quad [n\pi < x < (n+1)\pi].$$

1.622

The relationship between the inverse trigonometric functions, the inverse hyperbolic functions, and the logarithm.

$$1. \quad \arcsin z = \frac{1}{i} \ln (iz + \sqrt{1 - z^2}) = \frac{1}{i} \operatorname{Arsh} (iz)$$

$$2. \quad \arccos z = \frac{1}{i} \ln (z + \sqrt{z^2 - 1}) = \frac{1}{i} \operatorname{Arch} z.$$

$$3. \quad \operatorname{arctg} z = \frac{1}{2i} \ln \frac{1 + iz}{1 - iz} = \frac{1}{i} \operatorname{Arth} (iz).$$

$$4. \quad \operatorname{arcctg} z = \frac{1}{2i} \ln \frac{iz - 1}{iz + 1} = i \operatorname{Arcth} (iz).$$

$$5. \quad \operatorname{Arsh} z = \ln (z + \sqrt{z^2 + 1}) = \frac{1}{i} \arcsin (iz).$$

$$6. \quad \operatorname{Arch} z = \ln (z + \sqrt{z^2 - 1}) = i \arccos z.$$

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$$7. \quad \operatorname{Arth} z = \frac{1}{2} \ln \frac{1 + z}{1 - z} = \frac{1}{i} \operatorname{arctg} (iz).$$

$$8. \quad \operatorname{Arcth} z = \frac{1}{2} \ln \frac{z + 1}{z - 1} = \frac{1}{i} \operatorname{arcctg} (-iz).$$

Relations between different inverse trigonometric functions

1.623

$$1. \quad \arcsin x + \arccos x = \frac{\pi}{2}.$$

$$2. \operatorname{arctg} x + \operatorname{arcctg} x = \frac{\pi}{2}.$$

NV 43

1.624

$$1. \operatorname{arcsin} x = \arccos \sqrt{1-x^2} \quad [0 \leq x \leq 1];$$

$$= -\arccos \sqrt{1-x^2} \quad [-1 \leq x \leq 0].$$

NV 47 (5)

$$2. \operatorname{arcsin} x = \operatorname{arctg} \frac{x}{\sqrt{1-x^2}} \quad [x^2 < 1].$$

NV 46 (2)

$$3. \operatorname{arcsin} x = \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1];$$

$$= \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} - \pi \quad [-1 \leq x < 0].$$

NV 49 (10)

$$4. \arccos x = \operatorname{arcsin} \sqrt{1-x^2} \quad [0 \leq x \leq 1];$$

$$= \pi - \operatorname{arcsin} \sqrt{1-x^2} \quad [-1 \leq x \leq 0].$$

NV 48 (6)

$$5. \arccos x = \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [0 < x \leq 1];$$

$$= \pi + \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \quad [-1 \leq x < 0].$$

NV 48 (8)

$$6. \arccos x = \operatorname{arctg} \frac{x}{\sqrt{1-x^2}} \quad [-1 \leq x < 1].$$

NV 46 (4)

$$7. \operatorname{arctg} x = \operatorname{arcsin} \frac{x}{\sqrt{1+x^2}}.$$

$$\begin{aligned}
8. \quad \operatorname{arctg} x &= \arccos \frac{1}{\sqrt{1+x^2}} & [x \geq 0]; \\
&= -\arccos \frac{1}{\sqrt{1+x^2}} & [x \leq 0].
\end{aligned}$$

NV 48 (7)

57

$$\begin{aligned}
9. \quad \operatorname{arctg} x &= \operatorname{arctg} \frac{1}{x} & [x > 0], \\
&= -\operatorname{arctg} \frac{1}{x} - \pi & [x < 0].
\end{aligned}$$

NV 49 (9)

$$\begin{aligned}
10. \quad \operatorname{arctg} x &= \arcsin \frac{1}{\sqrt{1+x^2}} & [x > 0]; \\
&= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} & [x < 0].
\end{aligned}$$

NV 49 (11)

$$11. \quad \operatorname{arctg} x = \arccos \frac{x}{\sqrt{1+x^2}}.$$

NV 46 (4)

$$\begin{aligned}
12. \quad \operatorname{arctg} x &= \operatorname{arctg} \frac{1}{x} & [x > 0]; \\
&= \pi + \operatorname{arctg} \frac{1}{x} & [x < 0].
\end{aligned}$$

NV 49 (12)

1.625

$$\begin{aligned}
1. \quad \arcsin x + \arcsin y &= \arcsin (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & [xy \leq 0 \text{ or } x^2 + y^2 \leq 1]; \\
&= \pi - \arcsin (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & [x > 0, y > 0 \text{ and } x^2 + y^2 > 1]; \\
&= -\pi - \arcsin (x\sqrt{1-y^2} + y\sqrt{1-x^2}) & [x < 0, y < 0, \text{ and } x^2 + y^2 > 1].
\end{aligned}$$

NV 54(1), GI I (880)

$$\begin{aligned}
2. \quad \arcsin x + \arcsin y &= \arccos (\sqrt{1-x^2}\sqrt{1-y^2} - xy) & [x \geq 0, y \geq 0]; \\
&= -\arccos (\sqrt{1-x^2}\sqrt{1-y^2} - xy) & [x < 0, y < 0].
\end{aligned}$$

$$\begin{aligned}
3. \quad \arcsin x + \arcsin y &= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} && [xy \leq 0 \text{ or } x^2 + y^2 < 1]; \\
&= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} + \pi && [x > 0, \quad y > 0 \text{ and } x^2 + y^2 > 1]; \\
&= \operatorname{arctg} \frac{x\sqrt{1-y^2} + y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2} - xy} - \pi && [x < 0, \quad y < 0 \text{ and } x^2 + y^2 > 1].
\end{aligned}$$

NV 56

$$\begin{aligned}
4. \quad \arcsin x - \arcsin y &= \arcsin (x\sqrt{1-y^2} - y\sqrt{1-x^2}) && [xy \geq 0 \text{ or } x^2 + y^2 \leq 1]; \\
&= \pi - \arcsin (x\sqrt{1-y^2} - y\sqrt{1-x^2}) && [x > 0, \quad y < 0 \text{ and } x^2 + y^2 > 1]; \\
&= -\pi - \arcsin (x\sqrt{1-y^2} - y\sqrt{1-x^2}) && [x < 0, \quad y > 0 \text{ and } x^2 + y^2 > 1].
\end{aligned}$$

NV 55(2)

58

$$\begin{aligned}
5. \quad \arcsin x - \arcsin y &= \arccos (x\sqrt{1-x^2}\sqrt{1-y^2} + xy) && [xy > y]; \\
&= -\arccos (\sqrt{1-x^2}\sqrt{1-y^2} + xy) && [x < y].
\end{aligned}$$

NV 56

$$\begin{aligned}
6. \quad \arccos x + \arccos y &= \arccos (xy - \sqrt{1-x^2}\sqrt{1-y^2}) && [x + y \geq 0]; \\
&= 2\pi - \arccos (xy - \sqrt{1-x^2}\sqrt{1-y^2}) && [x + y < 0].
\end{aligned}$$

NV 57 (3)

$$\begin{aligned}
7. \quad \arccos x - \arccos y &= \arccos (xy + \sqrt{1-x^2}\sqrt{1-y^2}) && [x \geq y]; \\
&= \arccos (xy + \sqrt{1-x^2}\sqrt{1-y^2}) && [x < y].
\end{aligned}$$

NV 57 (4)

$$\begin{aligned}
8. \quad \operatorname{arctg} x + \operatorname{arctg} y &= \operatorname{arctg} \frac{x+y}{1-xy} && [xy < 1]; \\
&= \pi + \operatorname{arctg} \frac{x+y}{1-xy} && [x > 0, \quad xy > 1]; \\
&= -\pi + \operatorname{arctg} \frac{x+y}{1-xy} && [x < 0, \quad xy > 1].
\end{aligned}$$

$$\begin{aligned}
9. \quad \operatorname{arctg} x - \operatorname{arctg} y &= \operatorname{arctg} \frac{x-y}{1+xy} \quad [xy > -1] \\
&= \pi + \operatorname{arctg} \frac{x-y}{1+xy} \quad [x > 0, \quad xy < -1]; \\
&= -\pi + \operatorname{arctg} \frac{x-y}{1+xy} \quad [x < 0, \quad xy < -1].
\end{aligned}$$

NV 59(6)

1.626

$$\begin{aligned}
1. \quad 2 \arcsin x &= \arcsin (2x\sqrt{1-x^2}) \quad \left[|x| \leq \frac{1}{\sqrt{2}} \right]; \\
&= \pi - \arcsin (2x\sqrt{1-x^2}) \quad \left[\frac{1}{\sqrt{2}} < x \leq 1 \right]; \\
&= -\pi - \arcsin (2x\sqrt{1-x^2}) \quad \left[-1 \leq x < -\frac{1}{\sqrt{2}} \right].
\end{aligned}$$

NV 61 (7)

$$\begin{aligned}
2. \quad 2 \arccos x &= \arccos (2x^2 - 1) \quad [0 \leq x \leq 1]; \\
&= 2\pi - \arccos (2x^2 - 1) \quad [-1 \leq x < 0].
\end{aligned}$$

NV 61 (8)

$$\begin{aligned}
3. \quad 2 \operatorname{arctg} x &= \operatorname{arctg} \frac{2x}{1-x^2} \quad [|x| < 1]; \\
&= \operatorname{arctg} \frac{2x}{1-x^2} + \pi \quad [x > 1]; \\
&= \operatorname{arctg} \frac{2x}{1-x^2} - \pi \quad [x < -1].
\end{aligned}$$

NV 61 (9)

1.627

$$\begin{aligned}
1. \quad \operatorname{arctg} x + \operatorname{arctg} \frac{1}{x} &= \frac{\pi}{2} \quad [x > 0]; \\
&= -\frac{\pi}{2} \quad [x < 0].
\end{aligned}$$

GI I (878)

59

$$\begin{aligned}
2. \quad \operatorname{arctg} x + \operatorname{arctg} \frac{1-x}{1+x} &= \frac{\pi}{4} \quad [x > -1]; \\
&= -\frac{3}{4}\pi \quad [x < -1].
\end{aligned}$$

$$\begin{aligned}
 1. \quad \arcsin \frac{2x}{1+x^2} &= -\pi - 2 \operatorname{arctg} x \quad [x \leq -1]; \\
 &= 2 \operatorname{arctg} x \quad [-1 \leq x \leq 1]; \\
 &= \pi - 2 \operatorname{arctg} x \quad [x \geq 1].
 \end{aligned}$$

NV 65

$$\begin{aligned}
 2. \quad \arccos \frac{1-x^2}{1+x^2} &= 2 \operatorname{arctg} x \quad [x \geq 0]; \\
 &= -2 \operatorname{arctg} x \quad [x \leq 0].
 \end{aligned}$$

NV 66

1.629

$$\frac{2x-1}{2} - \frac{1}{\pi} \operatorname{arctg} \left(\operatorname{tg} \frac{2x-1}{2} \pi \right) = E(x).$$

GI (886)

1.631

Relations between the inverse hyperbolic functions.

$$1. \quad \operatorname{Arsh} x = \operatorname{Arch} \sqrt{x^2 + 1} = \operatorname{Arth} \frac{x}{\sqrt{x^2 + 1}}.$$

JA

$$2. \quad \operatorname{Arch} x = \operatorname{Arsh} \sqrt{x^2 - 1} = \operatorname{Arth} \frac{\sqrt{x^2 - 1}}{x}.$$

JA

$$3. \quad \operatorname{Arth} x = \operatorname{Arsh} \frac{x}{\sqrt{1-x^2}} = \operatorname{Arch} \frac{1}{\sqrt{1-x^2}} = \operatorname{Arcth} \frac{1}{x}.$$

JA

$$4. \quad \operatorname{Arsh} x \pm \operatorname{Arsh} y = \operatorname{Arsh} (x\sqrt{1+y^2} \pm y\sqrt{1+x^2}).$$

JA

$$5. \quad \operatorname{Arch} x \pm \operatorname{Arch} y = \operatorname{Arch} (xy \pm \sqrt{(x^2-1)(y^2-1)}).$$

$$6. \operatorname{Arth} x \pm \operatorname{Arth} y = \operatorname{Arth} \frac{x \pm y}{1 \pm xy}.$$

JA

1.64 Series representations

1.641

$$\begin{aligned} 1. \quad \arcsin x &= \frac{\pi}{2} - \arccos x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots; \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1} = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) \quad [x^2 \leq 1]. \end{aligned}$$

FI II 479

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$$\begin{aligned} 2. \quad \operatorname{Arsh} x &= x - \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 - \dots; \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} x^{2k+1}; \\ &= xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -x^2\right) \quad [x^2 \leq 1]. \end{aligned}$$

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1.642

$$\begin{aligned} 1. \quad \operatorname{Arsh} x &= \ln 2x + \frac{1}{2} \frac{1}{2x^2} - \frac{1 \cdot 3}{2 \cdot 4} \frac{1}{4x^4} + \dots; \\ &= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \quad [x \geq 1]. \end{aligned}$$

AD (6480.2)a

$$2. \quad \operatorname{Arch} x = \ln 2x - \sum_{k=1}^{\infty} \frac{(2k)! x^{-2k}}{2^{2k} (k!)^2 2k} \quad [x \geq 1].$$

AD (6480.3)a

1.643

$$\begin{aligned} 1. \quad \operatorname{arctg} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots; \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \quad [x^2 \leq 1]. \end{aligned}$$

$$2. \operatorname{Arth} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \quad [x^2 < 1].$$

AD (6480.4)

1.644

$$\begin{aligned} 1. \operatorname{arctg} x &= \frac{x}{\sqrt{1+x^2}} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{x^2}{1+x^2} \right)^k \\ &= \frac{x}{\sqrt{1+x^2}} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{1+x^2}\right) \quad [x^2 < \infty] \end{aligned}$$

AD (641.3)

$$\begin{aligned} 2. \operatorname{arctg} x &= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots \\ &= \frac{\pi}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)x^{2k+1}} \quad [x^2 \geq 1] \quad (\text{see also } \mathbf{1.643}). \end{aligned}$$

AD (641.4)

1.643

1.645

$$\begin{aligned} 1. \operatorname{arcsec} x &= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{2 \cdot 3x^3} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 5x^5} - \dots = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!x^{-(2k+1)}}{(k!)^2 2^{2k}(2k+1)}; \\ &= \frac{\pi}{2} - \frac{1}{x} F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{x^2}\right) \quad [x^2 > 1]. \end{aligned}$$

AD (641.5)

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$$2. (\arcsin x)^2 = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 x^{2k+2}}{(2k+1)!(k+1)} \quad [x^2 \leq 1].$$

AD (642.2), GI III (152)a

$$3. (\arcsin x)^3 = x^3 + \frac{3!}{5!} 3^2 \left(1 + \frac{1}{3^2}\right) x^5 + \frac{3!}{7!} 3^2 \cdot 5^2 \left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right) x^7 + \dots \quad [x^2 \leq 1].$$

BR* 188, AD (642.2), GI III (153)a

1.646

$$1. \quad \operatorname{Arsh} \frac{1}{x} = \operatorname{Arcosech} x = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k} (k!)^2 (2k+1)} x^{-2k-1} \quad [x^2 \geq 1].$$

AD (6480.5)

$$2. \quad \operatorname{Arch} \frac{1}{x} = \operatorname{Arsech} x = \ln \frac{2}{x} - \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 2k} x^{2k} \quad [0 < x \leq 1].$$

AD (6480.6)

$$3. \quad \operatorname{Arsh} \frac{1}{x} = \operatorname{Arcosech} x = \ln \frac{2}{x} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)!}{2^{2k} (k!)^2 2k} x^{2k} \quad [0 < x \leq 1].$$

AD (6480.7)a

$$4. \quad \operatorname{Arth} \frac{1}{x} = \operatorname{Arcth} x = \sum_{k=0}^{\infty} \frac{x^{-(2k+1)}}{2k+1} \quad [x^2 > 1].$$

AD (6480.8)

1.647* Due to T. Harrett.

$$1. \quad T_n = \sum_{k=1}^{\infty} \frac{\tanh (2k-1)(\pi/2)}{(2k-1)^{4n+3}} =$$

$$= \frac{\pi^{4n+3}}{2} \left(2 \sum_{j=1}^n \frac{(-1)^{j-1} (2^{2j}-1) (2^{4n-2j+4}-1) B_{2j-1}^* B_{4n-2j+3}^*}{(2j)!(4n-2j+4)!} + \right.$$

$$\left. + \frac{(-1)^n (2^{2n+2}-1)^2 B_{2n+1}^{*2}}{[(2n+2)!]^2} \right), \quad n = 0, 1, 2, \dots,$$

$$2. \quad S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \operatorname{sech} (2k-1)(\pi/2)}{(2k-1)^{4n+1}} =$$

$$= \frac{\pi^{4n+1}}{2^{4n+3}} \left(2 \sum_{j=1}^{n-1} \frac{(-1)^j B_{2j}^* B_{4n-2j}^*}{(2j)!(4n-2j)!} + \frac{2B_{4n}^*}{(4n)!} + \frac{(-1)^n B_{2n}^{*2}}{[(2n)!]^2} \right), \quad n = 1, 2, \dots$$

(Summation term on right to be omitted for $n=1$.) (See page xxix for definition of B_r^* .)

2. Indefinite Integrals of Elementary Functions

2.0 Introduction

2.00 General remarks

We omit the constant of integration in all the formulas of this chapter. Therefore, the equality sign (=) means that the functions on the left and right of this symbol differ by a constant. For example (see 2.01 15.), we write

$$\int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arcctg} x,$$

although

$$\operatorname{arctg} x = -\operatorname{arcctg} x + \frac{\pi}{2}.$$

When we integrate certain functions, we obtain the logarithm of the absolute value (for example, $\int \frac{dx}{\sqrt{1+x^2}} = \ln|x + \sqrt{1+x^2}|$). In such formulas, the absolute-value bars in the argument of the logarithm are omitted for simplicity in writing.

In certain cases, it is important to give the complete form of the primitive function. Such primitive functions, written in the form of definite integrals, are given in Chapter 2 and in other chapters.

Closely related to these formulas are formulas in which the limits of integration and the integrand depend on the same parameter.

A number of formulas lose their meaning for certain values of the constants (parameters) or for certain relationships between these constants (for example, formula 2.02 8. for $n = -1$ or formula 2.02 15. for $a = b$). These values of the constants and the relationships between them are for the most part completely clear from the very structure of the right hand member of the formula (the one not containing an integral sign). Therefore, throughout the chapter, we omit remarks to this effect. However, if the value of the integral is given by means of some other formula for those values of the parameters for which the formula in question loses meaning, we accompany this second formula with the appropriate explanation.

The letters x, y, t, \dots denote independent variables; f, g, φ, \dots denote functions of x, y, t, \dots ;

$f', g', \varphi', \dots, f'', g'', \varphi'', \dots$ denote their first, second, etc., derivatives; a, b, m, p, \dots denote constants, by which we

x, y, t, \dots f, g, φ, \dots x, y, t, \dots $f', g', \varphi', \dots, f'', g'', \varphi'', \dots$ a, b, m, p, \dots

generally mean arbitrary real numbers. If a particular formula is valid only for certain values of the constants (for example, only for positive numbers or only for integers), an appropriate remark is

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made provided the restriction that we make does not follow from the form of the formula itself. Thus, in formulas 2.148 4. and 2.424 6., we make no remark since it is clear from the form of these formulas themselves that n must be a natural number (that is, a positive integer).

2.01 The basic integrals

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1).$$

For $n = -1$

$$2. \int \frac{dx}{x} = \ln x.$$

$$3. \int e^x dx = e^x.$$

$$4. \int a^x dx = \frac{a^x}{\ln a}.$$

$$5. \int \sin x dx = -\cos x.$$

$$6. \int \cos x dx = \sin x.$$

$$7. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$8. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$9. \int \frac{\sin x}{\cos^2 x} dx = \sec x.$$

$$10. \int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cosec} x.$$

$$11. \int \operatorname{tg} x dx = -\ln \cos x.$$

$$12. \int \operatorname{ctg} x dx = \ln \sin x.$$

20. $\int \operatorname{sh} x \, dx = \operatorname{ch} x.$

22. $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$

24. $\int \operatorname{th} x \, dx = \ln \operatorname{ch} x.$

26. $\int \frac{dx}{\operatorname{sh} x} = \ln \operatorname{th} \frac{x}{2}.$

21. $\int \operatorname{ch} x \, dx = \operatorname{sh} x.$

23. $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$

25. $\int \operatorname{cth} x \, dx = \ln \operatorname{sh} x.$

2.02 General formulas

1. $\int a f \, dx = a \int f \, dx.$

2. $\int [a f \pm b \varphi \pm c \psi \pm \dots] \, dx = a \int f \, dx \pm b \int \varphi \, dx \pm c \int \psi \, dx \pm \dots$

3. $\frac{d}{dx} \int f \, dx = f.$

4. $\int f' \, dx = f.$

5. $\int f' \varphi \, dx = f \varphi - \int f \varphi' \, dx$ [integration by parts].

6. $\int f^{(n+1)} \varphi \, dx = \varphi f^{(n)} - \varphi' f^{(n-1)} + \varphi'' f^{(n-2)} - \dots + (-1)^n \varphi^{(n)} f + (-1)^{n+1} \int \varphi^{(n+1)} f \, dx.$

$$7. \int f(x) dx = \int f[\varphi(y)]\varphi'(y) dy \quad [x = \varphi(y)] \quad [\text{change of variable}].$$

$$8. \int (f)^n f' dx = \frac{(f)^{n+1}}{n+1}.$$

For $n = -1$

$$\int \frac{f' dx}{f} = \ln f.$$

$$9. \int (af + b)^n f' dx = \frac{(af + b)^{n+1}}{a(n+1)}.$$

$$10. \int \frac{f' dx}{\sqrt{af + b}} = \frac{2\sqrt{af + b}}{a}.$$

$$11. \int \frac{f'\varphi - \varphi'f}{\varphi^2} dx = \frac{f}{\varphi}.$$

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$$12. \int \frac{f'\varphi - \varphi'f}{f\varphi} dx = \ln \frac{f}{\varphi}.$$

$$13. \int \frac{dx}{f(f \pm \varphi)} = \pm \int \frac{dx}{f\varphi} \mp \int \frac{dx}{\varphi(f \pm \varphi)}.$$

$$14. \int \frac{f' dx}{\sqrt{f^2 + a}} = \ln(f + \sqrt{f^2 + a}).$$

For $a = b$

$$\int \frac{f dx}{(f+a)^2} = \int \frac{dx}{f+a} - a \int \frac{dx}{(f+a)^2}.$$

$$16. \int \frac{f dx}{(f+\varphi)^n} = \int \frac{dx}{(f+\varphi)^{n-1}} - \int \frac{\varphi dx}{(f+\varphi)^n}.$$

$$17. \int \frac{f' dx}{p^2 + q^2 f^2} = \frac{1}{pq} \operatorname{arctg} \frac{qf}{p}.$$

$$18. \int \frac{f' dx}{q^2 f^2 - p^2} = \frac{1}{2pq} \ln \frac{qf-p}{qf+p}.$$

$$19. \int \frac{f dx}{1-f} = -x + \int \frac{dx}{1-f}.$$

$$20. \int \frac{f^2 dx}{f^2 - a^2} = \frac{1}{2} \int \frac{f dx}{f-a} + \frac{1}{2} \int \frac{f dx}{f+a}.$$

$$21. \int \frac{f' dx}{\sqrt{a^2 - f^2}} = \arcsin \frac{f}{a}.$$

$$22. \int \frac{f' dx}{af^2 + bf} = \frac{1}{b} \ln \frac{f}{af+b}.$$

$$23. \int \frac{f' dx}{f\sqrt{f^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{f}{a}.$$

$$24. \int \frac{(f'\varphi - f\varphi') dx}{f^2 + \varphi^2} = \operatorname{arctg} \frac{f}{\varphi}.$$

$$25. \int \frac{(f'\varphi - f\varphi') dx}{f^2 - \varphi^2} = \frac{1}{2} \ln \frac{f - \varphi}{f + \varphi}.$$

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2.1 Rational Functions

2.10 General integration rules

2.101

To integrate an arbitrary rational function $\frac{F(x)}{f(x)}$, where $F(x)$ and $f(x)$ are polynomials with no common factors, we first need to separate out the integral part $E(x)$ (where $E(x)$ is a polynomial), if there is an integral part, and then to integrate separately the integral part and the remainder, thus:

$$\int \frac{F(x) dx}{f(x)} = \int E(x) dx + \int \frac{\varphi(x)}{f(x)} dx.$$

Integration of the remainder, which is then a proper rational function (that is, one in which the degree of the numerator is less than the degree of the denominator) is based on the decomposition of the fraction into elementary fractions, the so-called *partial fractions*

2.102

If a, b, c, \dots, m are roots of the equation $f(x) = 0$ and if $\alpha, \beta, \gamma, \dots, \mu$ are their corresponding multiplicities, so that $f(x) = (x - a)^\alpha (x - b)^\beta \dots (x - m)^\mu$ then, $\frac{\varphi(x)}{f(x)}$ can be decomposed into the following partial fractions:

$$\begin{aligned} \frac{\varphi(x)}{f(x)} &= \frac{A_\alpha}{(x - a)^\alpha} + \frac{A_{\alpha-1}}{(x - a)^{\alpha-1}} + \dots + \frac{A_1}{x - a} + \frac{B_\beta}{(x - b)^\beta} + \frac{B_{\beta-1}}{(x - b)^{\beta-1}} + \dots + \frac{B_1}{x - b} + \dots + \\ &+ \frac{M_\mu}{(x - m)^\mu} + \frac{M_{\mu-1}}{(x - m)^{\mu-1}} + \dots + \frac{M_1}{x - m}, \end{aligned}$$

where the numerators of the individual fractions are determined by the following formulas:

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \quad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \dots, \quad M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$A_{\alpha-k+1} = \frac{\psi_1^{(k-1)}(a)}{(k-1)!}, \quad B_{\beta-k+1} = \frac{\psi_2^{(k-1)}(b)}{(k-1)!}, \dots, \quad M_{\mu-k+1} = \frac{\psi_m^{(k-1)}(m)}{(k-1)!},$$

$$\psi_1(x) = \frac{\varphi(x)(x-a)^\alpha}{f(x)}, \quad \psi_2(x) = \frac{\varphi(x)(x-b)^\beta}{f(x)}, \dots, \quad \psi_m(x) = \frac{\varphi(x)(x-m)^\mu}{f(x)}.$$

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If a, b, \dots, m are simple roots, that is, if $\alpha = \beta = \dots = \mu = 1$, then

$$\frac{\varphi(x)}{f(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots + \frac{M}{x-m},$$

where

$$A = \frac{\varphi(a)}{f'(a)}, \quad B = \frac{\varphi(b)}{f'(b)}, \quad \dots, \quad M = \frac{\varphi(m)}{f'(m)}.$$

67

If some of the roots of the equation $f(x) = 0$ are imaginary, we group together the fractions that represent conjugate roots of the equation. Then, after certain manipulations, we represent the corresponding pairs of fractions in the form of real fractions of the form

$$\frac{M_1x + N_1}{x^2 + 2Bx + C} + \frac{M_2x + N_2}{(x^2 + 2Bx + C)^2} + \dots + \frac{M_px + N_p}{(x^2 + 2Bx + C)^p}.$$

2.103

Thus, the integration of a proper rational fraction $\frac{\varphi(x)}{f(x)}$ reduces to integrals of the form $\int \frac{g dx}{(x-a)^\alpha}$ or $\int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx$.

Fractions of the first form yield rational functions for $\alpha > 1$ and logarithms for $\alpha = 1$. Fractions of the second form yield rational functions and logarithms or arctangents:

$$1. \quad \int \frac{g dx}{(x-a)^\alpha} = g \int \frac{d(x-a)}{(x-a)^\alpha} = -\frac{g}{(\alpha-1)(x-a)^{\alpha-1}}.$$

$$2. \quad \int \frac{g dx}{x-a} = g \int \frac{d(x-a)}{x-a} = g \ln |x-a|.$$

$$3. \quad \int \frac{Mx+N}{(A+2Bx+Cx^2)^p} dx = \frac{NB-MA+(NC-MB)x}{2(p-1)(AC-B^2)(A+2Bx+Cx^2)^{p-1}} + \frac{(2p-3)(NC-MB)}{2(p-1)(AC-B^2)} \int \frac{dx}{(A+2Bx+Cx^2)^{p-1}}$$

$$4. \int \frac{dx}{A + 2Bx + Cx^2} = \frac{1}{\sqrt{AC - B^2}} \operatorname{arctg} \frac{Cx + B}{\sqrt{AC - B^2}} \quad [AC > B^2];$$

$$= \frac{1}{2\sqrt{B^2 - AC}} \ln \left| \frac{Cx + B - \sqrt{B^2 - AC}}{Cx + B + \sqrt{B^2 - AC}} \right| \quad [AC < B^2].$$

$$5. \int \frac{(Mx + N) dx}{A + 2Bx + Cx^2} = \frac{M}{2C} \ln |A + 2Bx + Cx^2| + \frac{NC - MB}{C\sqrt{AC - B^2}} \operatorname{arctg} \frac{Cx + B}{\sqrt{AC - B^2}} \quad [AC > B^2];$$

$$= \frac{M}{2C} \ln |A + 2Bx + Cx^2| + \frac{NC - MB}{2C\sqrt{B^2 - AC}} \ln \left| \frac{Cx + B - \sqrt{B^2 - AC}}{Cx + B + \sqrt{B^2 - AC}} \right| \quad [AC < B^2].$$

The ostrogradskiy-hermite method

2.104

By means of the Ostrogradskiy-Hermite method, we can find the rational part of $\int \frac{\varphi(x)}{f(x)} dx$ without finding the roots of the equation $f(x) = 0$ and without decomposing the integrand into partial fractions:

$$\int \frac{\varphi(x)}{f(x)} dx = \frac{M}{D} + \int \frac{N dx}{Q}.$$

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Here, M , N , D , and Q are rational functions of x . Specifically, D is the greatest common divisor of the function $f(x)$ and its derivative $f'(x)$; $Q = \frac{f'(x)}{D}$; M is a polynomial of degree no higher than $m - 1$, where m is the degree of the polynomial D ; N is a polynomial of degree no higher than $n - 1$, where n is the degree of the polynomial Q . The coefficients of the polynomials M and N are determined by equating the coefficients of like powers of x in the following identity:

$$\varphi(x) = M'Q - M(T - Q') + ND$$

where $T = \frac{f'(x)}{D}$ and M' and Q' are the derivatives of the polynomials M and Q .

2.11-2.13 Forms containing the binomial $a + bx^k$

2.110

Reduction formulas for $z = a + bx^k$

LA 126(4)

$$2. \int x^n z_k^m dx = \frac{-x^{n+1} z_k^{m+1}}{ak(m+1)} + \frac{km+k+n+1}{ak(m+1)} \int x^n z_k^{m+1} dx.$$

LA 126 (6)

$$3. \int x^n z_k^m dx = \frac{x^{n+1} z_k^m}{n+1} - \frac{bkm}{n+1} \int x^{n+k} z_k^{m-1} dx.$$

LA 125 (1)

$$4. \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{bk(m+1)} - \frac{n+1-k}{bk(m+1)} \int x^{n-k} z_k^{m+1} dx.$$

LA 125 (2)

$$5. \int x^n z_k^m dx = \frac{x^{n+1-k} z_k^{m+1}}{b(km+n+1)} - \frac{a(n+1-k)}{b(km+n+1)} \int x^{n-k} z_k^m dx.$$

LA 126 (3)

$$6. \int x^n z_k^m dx = \frac{x^{n+1} z_k^{m+1}}{a(n+1)} - \frac{b(km+k+n+1)}{a(n+1)} \int x^{n+k} z_k^m dx.$$

LA 126 (5)

Forms containing the binomial $z_1 = a + bx$

2.111

$$1. \int z_1^m dx = \frac{z_1^{m+1}}{b(m+1)}.$$

For $m = -1$

$$\int \frac{dx}{z_1} = \frac{1}{b} \ln z_1.$$

$$2. \int \frac{x^n dx}{z_1^m} = \frac{x^n}{z_1^{m-1}(n+1-m)b} - \frac{na}{(n+1-m)b} \int \frac{x^{n-1} dx}{z_1^m}.$$

For $n = m - 1$, we may use the formula

$$3.8 \int \frac{x^{m-1} dx}{z_1^m} = -\frac{x^{m-1}}{z_1^{m-1}(m-1)b} + \frac{1}{b} \int \frac{x^{m-2} dx}{z_1^{m-1}}.$$

For $m = 1$

$$\int \frac{x^n dx}{z_1} = \frac{x^n}{nb} - \frac{ax^{n-1}}{(n-1)b^2} + \frac{a^2x^{n-2}}{(n-2)b^3} - \dots + (-1)^{n-1} \frac{a^{n-1}x}{1 \cdot b^n} + \frac{(-1)^n a^n}{b^{n+1}} \ln z_1.$$

$$4. \int \frac{x^n dx}{z_1^2} = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{ka^{k-1}x^{n-k}}{(n-k)b^{k+1}} + (-1)^{n-1} \frac{a^n}{b^{n+1}z_1} + (-1)^{n+1} \frac{na^{n-1}}{b^{n+1}} \ln z_1.$$

2.112

$$1. \int \frac{x dx}{z_1} = \frac{x}{b} - \frac{a}{b^2} \ln z_1.$$

$$2. \int \frac{x^2 dx}{z_1} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \ln z_1.$$

2.113

$$1. \int \frac{dx}{z_1^2} = -\frac{1}{bz_1}.$$

$$2. \int \frac{x dx}{z_1^2} = -\frac{x}{bz_1} + \frac{1}{b^2} \ln z_1 = \frac{a}{b^2 z_1} + \frac{1}{b^2} \ln z_1.$$

$$3. \int \frac{x^2 dx}{z_1^2} = \frac{x}{b^2} - \frac{a^2}{b^3 z_1} - \frac{2a}{b^3} \ln z_1.$$

2.114

$$1. \int \frac{dx}{z_1^3} = -\frac{1}{2bz_1^2}.$$

$$2. \int \frac{x dx}{z_1^3} = -\left[\frac{x}{b} + \frac{a}{2b^2}\right] \frac{1}{z_1^2}.$$

$$3. \int \frac{x^2 dx}{z_1^3} = \left[\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right] \frac{1}{z_1^2} + \frac{1}{b^3} \ln z_1.$$

$$4. \int \frac{x^3 dx}{z_1^3} = \left[\frac{x^3}{b} + 2\frac{a}{b^2}x^2 - 2\frac{a^2}{b^3}x - \frac{5a^3}{2b^4}\right] \frac{1}{z_1^2} - 3\frac{a}{b^4} \ln z_1.$$

70

2.115

$$1. \int \frac{dx}{z_1^4} = -\frac{1}{3bz_1^3}.$$

$$2. \int \frac{x dx}{z_1^4} = -\left[\frac{x}{2b} + \frac{a}{6b^2}\right] \frac{1}{z_1^3}.$$

$$3. \int \frac{x^2 dx}{z_1^4} = -\left[\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right] \frac{1}{z_1^3}.$$

$$4. \int \frac{x^3 dx}{z_1^4} = \left[\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^2} + \frac{11a^3}{6b^4}\right] \frac{1}{z_1^3} + \frac{1}{b^4} \ln z_1.$$

$$1. \int \frac{dx}{z_1^5} = -\frac{1}{4bz_1^4}.$$

$$2. \int \frac{x dx}{z_1^5} = -\left[\frac{x}{3b} + \frac{a}{12b^2}\right] \frac{1}{z_1^4}.$$

$$3. \int \frac{x^2 dx}{z_1^5} = -\left[\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right] \frac{1}{z_1^4}.$$

$$4. \int \frac{x^3 dx}{z_1^5} = -\left[\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right] \frac{1}{z_1^4}.$$

2.117

$$1. \int \frac{dx}{x^n z_1^m} = \frac{-1}{(n-1)ax^{n-1}z_1^{m-1}} + \frac{b(2-n-m)}{a(n-1)} \int \frac{dx}{x^{n-1}z_1^m}.$$

$$2. \int \frac{dx}{z_1^m} = -\frac{1}{(m-1)bz_1^{m-1}}.$$

$$3. \int \frac{dx}{xz_1^m} = \frac{1}{z_1^{m-1}a(m-1)} + \frac{1}{a} \int \frac{dx}{xz_1^{m-1}}.$$

$$4. \int \frac{dx}{x^n z_1} = \sum_{k=1}^{n-1} \frac{(-1)^k b^{k-1}}{(n-k)a^k x^{n-k}} + \frac{(-1)^n b^{n-1}}{a^n} \ln \frac{z_1}{x}.$$

2.118

$$1. \int \frac{dx}{xz_1} = -\frac{1}{a} \ln \frac{z_1}{x},$$

2.119

$$1. \int \frac{dx}{xz_1^2} = \frac{1}{az_1} - \frac{1}{a^2} \ln \frac{z_1}{x}.$$

$$2. \int \frac{dx}{x^2z_1^2} = - \left[\frac{1}{ax} + \frac{2b}{a^2} \right] \frac{1}{z_1} + \frac{2b}{a^3} \ln \frac{z_1}{x}.$$

$$3. \int \frac{dx}{x^3z_1^2} = \left[-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3} \right] \frac{1}{z_1} - \frac{3b^2}{a^4} \ln \frac{z_1}{x}.$$

2.121

$$1. \int \frac{dx}{xz_1^3} = \left[\frac{3}{2a} + \frac{bx}{a^2} \right] \frac{1}{z_1^2} - \frac{1}{a^3} \ln \frac{z_1}{x}.$$

$$2. \int \frac{dx}{x^2z_1^3} = - \left[\frac{1}{ax} + \frac{9b}{2a^2} + \frac{3b^2x}{a^3} \right] \frac{1}{z_1^2} + \frac{3b}{a^4} \ln \frac{z_1}{x}.$$

$$3. \int \frac{dx}{x^3z_1^3} = \left[-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4} \right] \frac{1}{z_1^2} - \frac{6b^2}{a^5} \ln \frac{z_1}{x}.$$

2.122

$$1. \int \frac{dx}{xz_1^4} = \left[\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3} \right] \frac{1}{z_1^3} - \frac{1}{a^4} \ln \frac{z_1}{x}.$$

$$2. \int \frac{dx}{x^2z_1^4} = - \left[\frac{1}{ax} + \frac{22b}{3a^2} + \frac{10b^2x}{a^3} + \frac{4b^3x^2}{a^4} \right] \frac{1}{z_1^3} + \frac{4b}{a^5} \ln \frac{z_1}{x}.$$

$$3. \int \frac{dx}{x^3z_1^4} = \left[-\frac{1}{2ax^2} + \frac{5b}{2a^2x} + \frac{55b^2}{3a^3} + \frac{25b^3x}{a^4} + \frac{10b^4x^2}{a^5} \right] \frac{1}{z_1^3} - \frac{10b^2}{a^6} \ln \frac{z_1}{x}.$$

$$1. \int \frac{dx}{xz^5} = \left[\frac{25}{12a} + \frac{13bx}{3a^2} + \frac{7b^2x^2}{2a^3} + \frac{b^3x^3}{a^4} \right] \frac{1}{z_1^4} - \frac{1}{a^5} \ln \frac{z_1}{x}.$$

72

$$2. \int \frac{dx}{x^2z_1^5} = \left[-\frac{1}{ax} - \frac{125b}{12a^2} - \frac{65b^2x}{3a^3} - \frac{35b^3x^2}{2a^4} - \frac{5b^4x^3}{a^5} \right] \frac{1}{z_1^4} + \frac{5b}{a^6} \ln \frac{z_1}{x}.$$

$$3. \int \frac{dx}{x^3z_1^5} = \left[-\frac{1}{2ax^2} + \frac{3b}{a^2x} + \frac{125b^2}{4a^3} + \frac{65b^3x}{a^4} + \frac{105b^4x^2}{2a^5} + \frac{15b^5x^3}{a^6} \right] \frac{1}{z_1^4} - \frac{15b^2}{a^7} \ln \frac{z_1}{x}.$$

2.124

Forms containing the binomial $z_2 = a + bx^2$

$$1. \int \frac{dx}{z_2} = \frac{1}{\sqrt{ab}} \operatorname{arctg} x \sqrt{\frac{b}{a}} \quad [ab > 0] \quad (\text{see also } \mathbf{2.141} \text{ 2.});$$

$$= \frac{1}{2i\sqrt{ab}} \ln \frac{a + xi\sqrt{ab}}{a - xi\sqrt{ab}} \quad [ab < 0] \quad (\text{see also } \mathbf{2.143} \text{ 2. and } \mathbf{2.143} \text{ 3.}).$$

2.143

2.141

$$2. \int \frac{x dx}{z_2^m} = -\frac{1}{2b(m-1)z_2^{m-1}} \quad (\text{see also } \mathbf{2.145} \text{ 2., } \mathbf{2.145} \text{ 6. and } \mathbf{2.18}).$$

2.18

2.145

Forms containing the binomial $z_3 = a + bx^3$

$$\text{Notation: } \alpha = \sqrt[3]{\frac{a}{b}}$$

2.125

$$1. \int \frac{x^n dx}{z_3^m} = \frac{x^{n-2}}{z_3^{m-1}(n+1-3m)b} - \frac{(n-2)a}{b(n+1-3m)} \int \frac{x^{n-3} dx}{z_3^m}.$$

$$2. \int \frac{x^n dx}{z_3^m} = \frac{x^{n+1}}{3a(m-1)z_3^{m-1}} - \frac{n+4-3m}{3a(m-1)} \int \frac{x^n dx}{z_3^{m-1}}.$$

LA 133 (1)

2.126

$$1. \int \frac{dx}{z_3} = \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \operatorname{arctg} \frac{x\sqrt{3}}{2\alpha - x} \right\};$$

$$= \frac{\alpha}{3a} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} + \sqrt{3} \operatorname{arctg} \frac{2x - \alpha}{\alpha\sqrt{3}} \right\} \quad (\text{see also } \mathbf{2.141} \text{ 3. and } \mathbf{2.143} \text{ 4. }).$$

2.143

2.141

$$2. \int \frac{x dx}{z_3} = -\frac{1}{3b\alpha} \left\{ \frac{1}{2} \ln \frac{(x+\alpha)^2}{x^2 - \alpha x + \alpha^2} - \sqrt{3} \operatorname{arctg} \frac{2x - \alpha}{\alpha\sqrt{3}} \right\} \quad (\text{see also } \mathbf{2.145} \text{ 3. and } \mathbf{2.145} \text{ 7. }).$$

2.145

2.145

$$3. \int \frac{x^2 dx}{z_3} = \frac{1}{3b} \ln(1 + x^3 \alpha^{-3}) = \frac{1}{3b} \ln z_3.$$

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$$4. \int \frac{x^3 dx}{z_3} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1.}).$$

$$5. \int \frac{x^4 dx}{z_3} = \frac{x^2}{2b} - \frac{a}{b} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2.}).$$

2.127

$$1. \int \frac{dx}{z_3^2} = \frac{x}{3az_3} + \frac{2}{3a} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1.}).$$

$$2. \int \frac{x dx}{z_3^2} = \frac{x^2}{3az_3} + \frac{1}{3a} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2.}).$$

$$3. \int \frac{x^2 dx}{z_3^2} = -\frac{1}{3bz_3}.$$

$$4. \int \frac{x^3 dx}{z_3^2} = -\frac{x}{3bz_3} + \frac{1}{3b} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1.}).$$

2.128

$$1. \int \frac{dx}{x^n z_3^m} = -\frac{1}{(n-1)ax^{n-1}z_3^{m-1}} - \frac{b(3m+n-4)}{a(n-1)} \int \frac{dx}{x^{n-3}z_3^m}.$$

$$2. \int \frac{dx}{x^n z_3^m} = \frac{1}{3a(m-1)x^{n-1}z_3^{m-1}} + \frac{n+3m-4}{3a(m-1)} \int \frac{dx}{x^n z_3^{m-1}}.$$

2.129

$$1. \int \frac{dx}{xz_3} = \frac{1}{3a} \ln \frac{x^3}{z_3}.$$

$$2. \int \frac{dx}{x^2 z_3} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2.}).$$

2.126

$$3. \int \frac{dx}{x^3 z_3} = -\frac{1}{2ax^2} - \frac{b}{a} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1.}).$$

2.126

2.131

$$1. \int \frac{dx}{x^2 z_3^2} = \frac{1}{3az_3} + \frac{1}{3a^2} \ln \frac{x^3}{z_3}$$

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$$2. \int \frac{dx}{x^2 z_3^2} = -\left[\frac{1}{ax} + \frac{4bx^2}{3a^2} \right] \frac{1}{z_3} - \frac{4b}{3a^2} \int \frac{x dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 2.}).$$

2.126

$$3. \int \frac{dx}{x^3 z_3^2} = -\left[\frac{1}{2ax^2} + \frac{5bx}{6a^2} \right] \frac{1}{z_3} - \frac{5b}{3a^2} \int \frac{dx}{z_3} \quad (\text{see } \mathbf{2.126} \text{ 1.}).$$

2.126

Forms containing the binomial $z_4 = A + Bx^4$

$$\text{Notations: } \alpha = \sqrt[4]{\frac{a}{b}} \quad \alpha' = \sqrt[4]{\frac{-a}{b}}$$

$$\begin{aligned}
 1.^8 \int \frac{dx}{z_4} &= \frac{\alpha}{4a\sqrt{2}} \left\{ \ln \frac{x^2 + \alpha x\sqrt{2} + \alpha^2}{x^2 - \alpha x\sqrt{2} + \alpha^2} + 2 \operatorname{arctg} \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\} \quad [ab > 0] \quad (\text{see also } \mathbf{2.141} \text{ 4.}). \\
 &= \frac{\alpha'}{4a} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} + 2 \operatorname{arctg} \frac{x}{\alpha'} \right\} \quad [ab < 0] \quad (\text{see also } \mathbf{2.143} \text{ 5.}).
 \end{aligned}$$

2.143

2.141

$$\begin{aligned}
 2. \int \frac{x dx}{z} &= \frac{1}{2\sqrt{ab}} \operatorname{arctg} x^2 \sqrt{\frac{b}{a}} \quad [ab > 0] \quad (\text{see also } \mathbf{2.145} \text{ 4.}). \\
 &= \frac{1}{4i\sqrt{ab}} \ln \frac{a + x^2 i\sqrt{ab}}{a - x^2 i\sqrt{ab}} \quad [ab < 0] \quad (\text{see also } \mathbf{2.145} \text{ 8.}).
 \end{aligned}$$

2.145

$$\begin{aligned}
 3. \int \frac{x^2 dx}{z_4} &= \frac{1}{4b\alpha\sqrt{2}} \left\{ \ln \frac{x^2 - \alpha x\sqrt{2} + \alpha^2}{x^2 + \alpha x\sqrt{2} + \alpha^2} + 2 \operatorname{arctg} \frac{\alpha x\sqrt{2}}{\alpha^2 - x^2} \right\} \quad [ab > 0]; \\
 &= -\frac{1}{4b\alpha'} \left\{ \ln \frac{x + \alpha'}{x - \alpha'} - 2 \operatorname{arctg} \frac{x}{\alpha'} \right\} \quad [ab < 0].
 \end{aligned}$$

$$4. \int \frac{x^3 dx}{z_4} = \frac{1}{4b} \ln z_4$$

2.133

$$1. \int \frac{x^n dx}{z_4^m} = \frac{x^{n+1}}{4a(m-1)z_4^{m-1}} + \frac{4m-n-5}{4a(m-1)} \int \frac{x^n dx}{z_4^{m-1}}$$

LA 134 (1)

$$2. \int \frac{x^n dx}{z_4^m} = \frac{x^{n-3}}{z_4^{m-1}(n+1-4m)b} - \frac{(n-3)a}{b(n+1-4m)} \int \frac{x^{n-4} dx}{z_4^m}.$$

2.134

$$2. \int \frac{x dx}{z_4^2} = \frac{x^2}{4az_4} + \frac{1}{2a} \int \frac{x dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 2.}).$$

2.132

$$3. \int \frac{x^2 dx}{z_4^2} = \frac{x^3}{4az_4} + \frac{1}{4a} \int \frac{x^2 dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 3.}).$$

2.132

$$4. \int \frac{x^3 dx}{z_4^2} = \frac{x^4}{4az_4} = -\frac{1}{4bz_4}.$$

2.135

$$\int \frac{dx}{x^n z_4^m} = -\frac{1}{(n-1)ax^{n-1}z_4^{m-1}} - \frac{b(4m+n-5)}{(n-1)a} \int \frac{dx}{x^{n-4}z_4^m}.$$

For $n = 1$

$$\int \frac{dx}{xz_4^m} = \frac{1}{a} \int \frac{dx}{xz_4^{m-1}} - \frac{b}{a} \int \frac{dx}{x^{-3}z_4^m}.$$

2.136

$$1. \int \frac{dx}{xz_4} = \frac{\ln x}{a} - \frac{\ln z_4}{4a} = \frac{1}{4a} \ln \frac{x^4}{z_4}.$$

$$2. \int \frac{dx}{x^2 z_4} = -\frac{1}{ax} - \frac{b}{a} \int \frac{x^2 dx}{z_4} \quad (\text{see } \mathbf{2.132} \text{ 3.}).$$

2.14 Forms containing the binomial $1 \pm x^n$

2.141

$$1. \int \frac{dx}{1+x} = \ln(1+x).$$

$$2. \int \frac{dx}{1+x^2} = \operatorname{arctg} x = -\operatorname{arcctg} x \quad (\text{see also } \mathbf{2.124} \text{ 1.}).$$

2.124

$$3. \int \frac{dx}{1+x^3} = \frac{1}{3} \ln \frac{1+x}{\sqrt{1-x+x^2}} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{2-x} \quad (\text{see also } \mathbf{2.126} \text{ 1.}).$$

2.126

$$4. \int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \operatorname{arctg} \frac{x\sqrt{2}}{1-x^2} \quad (\text{see also } \mathbf{2.132} \text{ 1.}).$$

2.132

2.142

$$\int \frac{dx}{1+x^n} = -\frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n}{2}-1} Q_k \sin \frac{2k+1}{n} \pi \quad [n-\text{a positive even number}];$$

$$= \frac{1}{n} \ln(1+x) - \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi$$

$$[n-\text{a positive odd number}].$$

TI (45)
TI (43)a

2.143

$$1. \int \frac{dx}{1-x} = -\ln(1-x).$$

$$2. \int \frac{dx}{1-x^2} = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{Arth} x \quad [-1 < x < 1] \quad (\text{see also } \mathbf{2.141} \text{ 1.}).$$

2.141

$$3. \int \frac{dx}{x^2-1} = \frac{1}{2} \ln \frac{x-1}{x+1} = -\operatorname{Arcth} x \quad [x > 1, \quad x < -1].$$

$$4. \int \frac{dx}{1-x^3} = \frac{1}{3} \ln \frac{\sqrt{1+x+x^2}}{1-x} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x\sqrt{3}}{2+x} \quad (\text{see also } \mathbf{2.126} \text{ 1.}).$$

2.126

$$5. \int \frac{dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x}{1-x} + \frac{1}{2} \operatorname{arctg} x = \frac{1}{2} (\operatorname{Arth} x + \operatorname{arctg} x) \quad (\text{see also } \mathbf{2.132} \text{ 1.}).$$

2.132

2.144

$$1. \int \frac{dx}{1-x^n} = \frac{1}{n} \ln \frac{1+x}{1-x} - \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} P_k \cos \frac{2k}{n} \pi + \frac{2}{n} \sum_{k=1}^{\frac{n}{2}-1} Q_k \sin \frac{2k}{n} \pi [n-\text{a positive even number}].$$

$$P_k = \frac{1}{2} \ln \left(x^2 - 2x \cos \frac{2k}{n} \pi + 1 \right), \quad Q_k = \operatorname{arctg} \frac{x - \cos \frac{2k}{n} \pi}{\sin \frac{2k}{n} \pi}.$$

$$2. \int \frac{dx}{1-x^n} = -\frac{1}{n} \ln(1-x) + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} P_k \cos \frac{2k+1}{n} \pi + \frac{2}{n} \sum_{k=0}^{\frac{n-3}{2}} Q_k \sin \frac{2k+1}{n} \pi [n-\text{a positive odd number}].$$

$$P_k = \frac{1}{2} \ln \left(x^2 + 2x \cos \frac{2k+1}{n} \pi + 1 \right), \quad Q_k = \operatorname{arctg} \frac{x + \cos \frac{2k+1}{n} \pi}{\sin \frac{2k+1}{n} \pi}.$$

$$1. \int \frac{x dx}{1+x} = x - \ln(1+x).$$

$$2. \int \frac{x dx}{1+x^2} = \frac{1}{2} \ln(1+x^2).$$

$$3. \int \frac{x dx}{1+x^3} = -\frac{1}{6} \ln \frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} \quad (\text{see also } \mathbf{2.126} \text{ 2.}).$$

2.126

$$4. \int \frac{x dx}{1+x^4} = \frac{1}{2} \operatorname{arctg} x^2.$$

$$5. \int \frac{x dx}{1-x} = -\ln(1-x) - x.$$

$$6. \int \frac{x dx}{1-x^2} = -\frac{1}{2} \ln(1-x^2).$$

$$7. \int \frac{x dx}{1-x^3} = -\frac{1}{6} \ln \frac{(1-x)^2}{1+x+x^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \quad (\text{see also } \mathbf{2.126} \text{ 2.}).$$

2.126

$$8. \int \frac{x dx}{1-x^4} = \frac{1}{4} \ln \frac{1+x^2}{1-x^2} \quad (\text{see also } \mathbf{2.132} \text{ 2.}).$$

2.132

2.146

For m and n —natural numbers.

$$1. \int \frac{x^{m-1} dx}{1+x^{2n}} = -\frac{1}{2n} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n} \pi + x^2 \right\} + \\ + \frac{1}{n} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n} \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n} \pi}{\sin \frac{2k-1}{2n} \pi} \quad [m < 2n].$$

$$2. \int \frac{x^{m-1} dx}{1+x^{2n+1}} = (-1)^{m+1} \frac{\ln(1+x)}{2n+1} - \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left\{ 1 - 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right\} + \\ + \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \operatorname{arctg} \frac{x - \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \quad [m \leq 2n].$$

TI (46)a

$$3. \int \frac{x^{m-1} dx}{1-x^{2n}} = \frac{1}{2n} \{(-1)^{m+1} \ln(1+x) - \ln(1-x)\} - \frac{1}{2n} \sum_{k=1}^{n-1} \cos \frac{km\pi}{n} \ln \left(1 - 2x \cos \frac{k\pi}{n} + x^2 \right) + \\ + \frac{1}{n} \sum_{k=1}^{n-1} \sin \frac{km\pi}{n} \operatorname{arctg} \frac{x - \cos \frac{k\pi}{n}}{\sin \frac{k\pi}{n}} \quad [m < 2n].$$

TI (48)

78

$$4. \int \frac{x^{m-1} dx}{1-x^{2n+1}} = -\frac{1}{2n+1} \ln(1-x) + \\ + (-1)^{m+1} \frac{1}{2n+1} \sum_{k=1}^n \cos \frac{m\pi(2k-1)}{2n+1} \ln \left(1 + 2x \cos \frac{2k-1}{2n+1} \pi + x^2 \right) + \\ + (-1)^{m+1} \frac{2}{2n+1} \sum_{k=1}^n \sin \frac{m\pi(2k-1)}{2n+1} \operatorname{arctg} \frac{x + \cos \frac{2k-1}{2n+1} \pi}{\sin \frac{2k-1}{2n+1} \pi} \quad [m \leq 2n].$$

TI (50)

2.147

$$1. \int \frac{x^m dx}{1-x^{2n}} = \frac{1}{2} \int \frac{x^m dx}{1-x^n} + \frac{1}{2} \int \frac{x^m dx}{1+x^n}.$$

$$2. \int \frac{x^m dx}{(1+x^2)^n} = -\frac{1}{2n-m-1} \cdot \frac{x^{m-1}}{(1+x^2)^{n-1}} + \frac{m-1}{2n-m-1} \int \frac{x^{m-2} dx}{(1+x^2)^n}.$$

LA 139 (28)

$$3. \int \frac{x^m}{1+x^2} dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{1+x^2} dx.$$

$$5. \int \frac{x^m dx}{1-x^2} = -\frac{x^{m-1}}{m-1} + \int \frac{x^{m-2} dx}{1-x^2}.$$

2.148

$$1. \int \frac{dx}{x^m(1+x^2)^n} = -\frac{1}{m-1} \frac{1}{x^{m-1}(1+x^2)^{n-1}} - \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2}(1+x^2)^n}.$$

For $m = 1$

$$\int \frac{dx}{x(1+x^2)^n} = \frac{1}{2n-2} \frac{1}{(1+x^2)^{n-1}} + \int \frac{dx}{x(1+x^2)^{n-1}}.$$

For $m = 1$ and $n = 1$

$$\int \frac{dx}{x(1+x^2)} = \ln \frac{x}{\sqrt{1+x^2}}.$$

LA 139 (31)

LA 139 (29)

$$2. \int \frac{dx}{x^m(1+x^2)} = -\frac{1}{(m-1)x^{m-1}} - \int \frac{dx}{x^{m-2}(1+x^2)}.$$

79

$$3. \int \frac{dx}{(1+x^2)^n} = \frac{1}{2n-2} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}}.$$

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$$4. \int \frac{dx}{(1+x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1+x^2)^{n-k}} + \frac{(2n-3)!!}{2^{n-1}(n-1)!} \operatorname{arctg} x.$$

TI (91)

2.149

$$1. \int \frac{dx}{x^m(1-x^2)^n} = -\frac{1}{(m-1)x^{m-1}(1-x^2)^{n-1}} + \frac{2n+m-3}{m-1} \int \frac{dx}{x^{m-2}(1-x^2)^n}.$$

For $m = 1$

$$\int \frac{dx}{x(1-x^2)^n} = \frac{1}{2(n-1)(1-x^2)^{n-1}} + \int \frac{dx}{x(1-x^2)^{n-1}}.$$

For $m = 1$ and $n = 1$

$$\int \frac{dx}{x(1-x^2)} = \frac{x}{\sqrt{1-x^2}}.$$

$$2. \int \frac{dx}{(1-x^2)^n} = \frac{1}{2n-2} \frac{x}{(1-x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1-x^2)^{n-1}}.$$

LA 139 (35)

$$3. \int \frac{dx}{(1-x^2)^n} = \frac{x}{2n-1} \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)(2n-5)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)(1-x^2)^{n-k}} + \frac{(2n-3)!!}{2^n \cdot (n-1)!} \ln \frac{1+x}{1-x}.$$

TI (91)

2.15 Forms containing pairs of binomials: $a + bx$ and $\alpha + \beta x$

Notations: $z = a + bx$; $t = \alpha + \beta x$; $\Delta = a\beta - \alpha b$

2.151

$$\int z^n t^m dx = \frac{z^{n+1} t^m}{(m+n+1)b} - \frac{m\Delta}{(m+n+1)b} \int z^n t^{m-1} dx.$$

2.152

$$1. \int \frac{z}{t} dx = \frac{bx}{\beta} + \frac{\Delta}{\beta^2} \ln t.$$

$$2. \int \frac{t}{z} dx = \frac{\beta x}{b} - \frac{\Delta}{b^2} \ln z.$$

80

2.153

$$\begin{aligned} \int \frac{t^m dx}{z^n} &= \frac{1}{(m-n+1)b} \frac{t^m}{z^{n-1}} - \frac{m\Delta}{(m-n+1)b} \int \frac{t^{m-1} dx}{z^n}; \\ &= \frac{1}{(n-1)\Delta} \frac{t^{m+1}}{z^{n-1}} - \frac{(m-n+2)\beta}{(n-1)\Delta} \int \frac{t^m dx}{z^{n-1}}; \\ &= -\frac{1}{(n-1)b} \frac{t^m}{z^{n-1}} + \frac{m\beta}{(n-1)b} \int \frac{t^{m-1}}{z^{n-1}} dx. \end{aligned}$$

2.154

$$\int \frac{dx}{zt} = \frac{1}{\Delta} \ln \frac{t}{z}.$$

$$\int \frac{dx}{z^n t^m} = -\frac{1}{(m-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} - \frac{(m+n-2)b}{(m-1)\Delta} \int \frac{dx}{t^{m-1} z^n};$$

$$= \frac{1}{(n-1)\Delta} \frac{1}{t^{m-1} z^{n-1}} + \frac{(m+n-2)\beta}{(n-1)\Delta} \int \frac{dx}{t^m z^{n-1}}.$$

2.156

$$\int \frac{x dx}{zt} = \frac{1}{\Delta} \left(\frac{a}{b} \ln z - \frac{\alpha}{\beta} \ln t \right).$$

2.16 Forms containing the trinomial $a + bx^k + cx^{2k}$

2.160

Reduction formulas for $R_k a + bx^k + cx^{2k}$

$$1. \int x^{m-1} R_k^n dx = \frac{x^m R_k^{n+1}}{ma} - \frac{(m+k+nk)b}{ma} \int x^{m+k-1} R_k^n dx - \frac{(m+2k+2kn)c}{ma} \int x^{m+2k-1} R_k^n dx.$$

$$2. \int x^{m-1} R_k^n dx = \frac{x^m R_k^n}{m} - \frac{bkn}{m} \int x^{m+k-1} R_k^{n-1} dx - \frac{2ckn}{m} \int x^{m+2k-1} R_k^{n-1} dx.$$

$$3. \int x^{m-1} R_k^n dx = \frac{x^{m-2k} R_k^{n+1}}{(m+2kn)c} - \frac{(m-2k)a}{(m+2kn)c} \int x^{m-2k-1} R_k^n dx - \frac{(m-k+kn)b}{(m+2kn)c} \int x^{m-k-1} R_k^n dx;$$

$$= \frac{x^m R_k^n}{m+2kn} + \frac{2kna}{m+2kn} \int x^{m-1} R_k^{n-1} dx + \frac{bkn}{m+2kn} \int x^{m+k-1} R_k^{n-1} dx.$$

2.161

Forms containing the trinomial $R_2 = a + bx^2 + cx^4$

Notations: $f = \frac{b}{2} - \frac{1}{2}\sqrt{b^2 - 4ac}$, $g = \frac{b}{2} + \frac{1}{2}\sqrt{b^2 - 4ac}$,

$$h = \sqrt{b^2 - 4ac}, \quad q = \sqrt[4]{\frac{a}{c}}, \quad l = 2a(n-1)(b^2 - 4ac), \quad \cos \alpha = -\frac{b}{2\sqrt{ac}}.$$

$$1. \int \frac{dx}{R_2} = \frac{c}{h} \left\{ \int \frac{dx}{cx^2 + f} - \int \frac{dx}{cx^2 + g} \right\} \quad [h^2 > 0];$$

$$= \frac{1}{4cq^3 \sin \alpha} \left\{ \sin \frac{\alpha}{2} \ln \frac{x^2 + 2qx \cos \frac{\alpha}{2} + q^2}{x^2 - 2qx \cos \frac{\alpha}{2} + q^2} + 2 \cos \frac{\alpha}{2} \operatorname{arctg} \frac{x^2 - q^2}{2qx \sin \frac{\alpha}{2}} \right\} \quad [h^2 < 0].$$

$$2. \int \frac{x dx}{R_2} = \frac{1}{2h} \ln \frac{cx^2 + f}{cx^2 + g} \quad [h^2 > 0];$$

$$= \frac{1}{2cq^2 \sin \alpha} \operatorname{arctg} \frac{x^2 - q^2 \cos \alpha}{q^2 \sin \alpha} \quad [h^2 < 0].$$

LA 146 (9)a

LA 146 (6)

$$3. \int \frac{x^2 dx}{R_2} = \frac{g}{h} \int \frac{dx}{cx^2 + g} - \frac{f}{h} \int \frac{dx}{cx^2 + f} \quad [h^2 > 0].$$

LA 146 (7)

$$4. \int \frac{dx}{R_2^2} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2} + \frac{b^2 - 6ac}{l} \int \frac{dx}{R_2} + \frac{bc}{l} \int \frac{x^2 dx}{R_2}.$$

$$5. \int \frac{dx}{R_2^n} = \frac{bcx^3 + (b^2 - 2ac)x}{lR_2^{n-1}} + \frac{(4n-7)bc}{l} \int \frac{x^2 dx}{R_2^{n-1}} + \frac{2(n-1)h^2 + 2ac - b^2}{l} \int \frac{dx}{R_2^{n-1}}$$

[n > 1].

LA 146

$$6. \int \frac{dx}{x^m R_2^n} = -\frac{1}{(m-1)ax^{m-1}R_2^{n-1}} - \frac{(m+2n-3)b}{(m-1)a} \int \frac{dx}{x^{m-2}R_2^n} - \frac{(m+4n-5)b}{(m-1)a} \int \frac{dx}{x^{m-4}R_2^n}.$$

LA 147 (12)a

2.17 Forms containing the quadratic trinomial $a + bx + cx^2$ and powers of x

Notations: $R = a + bx + cx^2$; $\Delta = 4ac - b^2$

2.171

$$1. \int x^{m+1} R^n dx = \frac{x^m R^{n+1}}{c(m+2n+2)} - \frac{am}{c(m+2n+2)} \int x^{m-1} R^n dx - \frac{b(m+n+1)}{c(m+2n+2)} \int x^m R^n dx.$$

TI (97)

$$3. \int \frac{dx}{R^{n+1}} = \frac{b+2cx}{n\Delta R^n} + \frac{(4n-2)c}{n\Delta} \int \frac{dx}{R^n}$$

TI (94)a

$$4. \int \frac{dx}{R^{n+1}} = \frac{(2cx+b)}{2n+1} \sum_{k=0}^{n-1} \frac{2k(2n+1)(2n-1)(2n-3)\dots(2n-2k+1)c^k}{n(n-1)\dots(n-k)\Delta^{k+1}R^{n-k}} + \\ + 2^n \frac{(2n-1)!!c^n}{n!\Delta^n} \int \frac{dx}{R}.$$

TI (96)a

2.172³

$$\int \frac{dx}{R} = \frac{1}{\sqrt{-\Delta}} \ln \frac{\sqrt{-\Delta} - (b+2cx)}{(b+2cx) + \sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}} \operatorname{Arth} \frac{b+2cx}{\sqrt{-\Delta}} \quad [\Delta < 0]; \\ = \frac{-2}{b+2cx} \quad [\Delta = 0]; \\ = \frac{2}{\sqrt{\Delta}} \operatorname{arctg} \frac{b+2cx}{\sqrt{\Delta}} \quad [\Delta > 0].$$

82

2.173

$$1. \int \frac{dx}{R^2} = \frac{b+2cx}{\Delta R} + \frac{2c}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$2. \int \frac{dx}{R^3} = \frac{b+2cx}{\Delta} \left\{ \frac{1}{2R^2} + \frac{3c}{\Delta R} \right\} + \frac{6c^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

2.174

$$1. \int \frac{x^m dx}{R^n} = -\frac{x^{m-1}}{(2n-m-1)cR^{n-1}} - \frac{(n-m)b}{(2n-m-1)c} \int \frac{x^{m-1} dx}{R^n} + \frac{(m-1)a}{(2n-m-1)c} \int \frac{x^{m-2} dx}{R^n}.$$

$$m = 2n - 1$$

$$2. \int \frac{x^{2n-1} dx}{R^n} = \frac{1}{c} \int \frac{x^{2n-3} dx}{R^{n-1}} - \frac{a}{c} \int \frac{x^{2n-3} dx}{R^n} - \frac{b}{c} \int \frac{x^{2n-2} dx}{R^n}.$$

2.175

$$1. \int \frac{x dx}{R} = \frac{1}{2c} \ln R - \frac{b}{2c} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$2. \int \frac{x dx}{R^2} = -\frac{2a + bx}{\Delta R} - \frac{b}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$3. \int \frac{x dx}{R^3} = -\frac{2a + bx}{2\Delta R^2} - \frac{3b(b + 2cx)}{2\Delta^2 R} - \frac{3bc}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$4. \int \frac{x^2 dx}{R} = \frac{x}{c} - \frac{b}{2c^2} \ln R + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$5. \int \frac{x^2 dx}{R^2} = \frac{ab + (b^2 - 2ac)x}{c\Delta R} + \frac{2a}{\Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$6. \int \frac{x^2 dx}{R^3} = \frac{ab + (b^2 - 2ac)x}{2c\Delta R^2} + \frac{(2ac + b^2)(b + 2cx)}{2c\Delta^2 R} + \frac{2ac + b^2}{\Delta^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

$$7. \int \frac{x^3 dx}{R} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2 - ac}{2c^3} \ln R - \frac{b(b^2 - 3ac)}{2c^3} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$8. \int \frac{x^3 dx}{R^2} = \frac{1}{2c^2} \ln R + \frac{a(2ac - b^2) + b(3ac - b^2)x}{c^2 \Delta R} - \frac{b(6ac - b^2)}{2c^2 \Delta} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$9. \int \frac{x^3 dx}{R^3} = - \left(\frac{x^2}{c} + \frac{abx}{c\Delta} + \frac{2a^2}{c\Delta} \right) \frac{1}{2R^2} - \frac{3ab}{2c\Delta} \int \frac{dx}{R^2} \quad (\text{see } \mathbf{2.173} \text{ 1.}).$$

2.173

2.176

$$\int \frac{dx}{x^m R^n} = \frac{-1}{(m-1)ax^{m-1}R^{n-1}} - \frac{b(m+n-2)}{a(m-1)} \int \frac{dx}{x^{m-1}R^n} - \frac{c(m+2n-3)}{a(m-1)} \int \frac{dx}{x^{m-2}R^n}.$$

83

2.177

$$1. \int \frac{dx}{xR} = \frac{1}{2a} \ln \frac{x^2}{R} - \frac{b}{2a} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$2. \int \frac{dx}{xR^2} = \frac{1}{2a^2} \ln \frac{x^2}{R} + \frac{1}{2aR} \left\{ 1 - \frac{b(b+2cx)}{\Delta} \right\} - \frac{b}{2a^2} \left(1 + \frac{2ac}{\Delta} \right) \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

2.173

2.172

$$4. \int \frac{dx}{x^2 R} = -\frac{b}{2a^2} \ln \frac{x^2}{R} - \frac{1}{ax} + \frac{b^2 - 2ac}{2a^2} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$5. \int \frac{dx}{x^2 R^2} = -\frac{b}{a^3} \ln \frac{x^2}{R} - \frac{a + bx}{a^2 x R} + \frac{(b^2 - 3ac)(b + 2cx)}{a^2 \Delta R} - \frac{1}{\Delta} \left(\frac{b^4}{a^3} - \frac{6b^2 c}{a^2} + \frac{6c^2}{a} \right) \int \frac{dx}{R}$$

(see **2.172**).

2.172

$$6. \int \frac{dx}{x^2 R^3} = -\frac{1}{axR^2} - \frac{3b}{a} \int \frac{dx}{xR^3} - \frac{5c}{a} \int \frac{dx}{R^3} \quad (\text{see } \mathbf{2.173} \text{ and } \mathbf{2.177} \text{ 3.}).$$

2.177

2.173

$$7. \int \frac{dx}{x^3 R} = -\frac{ac - b^2}{2a^3} \ln \frac{x^2}{R} + \frac{b}{a^2 x} - \frac{1}{2ax^2} + \frac{b(3ac - b^2)}{2a^3} \int \frac{dx}{R} \quad (\text{see } \mathbf{2.172}).$$

2.172

$$8. \int \frac{dx}{x^3 R^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2 x} \right) \frac{1}{R} + \left(\frac{3b^2}{a^2} - \frac{2c}{a} \right) \int \frac{dx}{xR^2} + \frac{9bc}{2a^2} \int \frac{dx}{R^2}$$

(see **2.173** 1. and **2.177** 2.).

2.177

$$9. \int \frac{dx}{x^3 R^3} = \left(\frac{-1}{2ax^2} + \frac{2b}{a^2 x} \right) \frac{1}{R^2} + \left(\frac{6b^2}{a^2} - \frac{3c}{a} \right) \int \frac{dx}{x R^3} + \frac{10bc}{a^2} \int \frac{dx}{R^3} \quad (\text{see } \mathbf{2.173} \text{ 2., } \mathbf{2.177} \text{ 3.}).$$

2.177

2.173

2.18 Forms containing the quadratic trinomial $a + bx + cx^2$ and the binomial $\alpha + \beta x$

Notations:

$$R = a + bx + cx^2; \quad z = \alpha + \beta x; \quad A = \alpha\beta^2 - ab\beta + c\alpha^2;$$

$$B = b\beta - 2c\alpha; \quad \Delta = 4ac - b^2.$$

$$1. \int z^m R^n dx = \frac{\beta z^{m-1} R^{n+1}}{(m+2n+1)c} - \frac{(m+n)B}{(m+2n+1)c} \int z^{m-1} R^n dx - \frac{(m-1)A}{(m+2n+1)c} \int z^{m-2} R^n dx.$$

$$\begin{aligned} 2. \int \frac{R^n dx}{z^m} &= -\frac{1}{(m-2n-1)\beta} \frac{R^n}{z^{m-1}} - \frac{2nA}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^m} - \\ &\quad - \frac{nB}{(m-2n-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}}; \\ &= \frac{-\beta}{(m-1)A} \frac{R^{n+1}}{z^{m-1}} - \frac{(m-n-2)B}{(m-1)A} \int \frac{R^n dx}{z^{m-1}} - \frac{(m-2n-3)c}{(m-1)A} \int \frac{R^n dx}{z^{m-2}}; \\ &= -\frac{1}{(m-1)\beta} \frac{R^n}{z^{m-1}} + \frac{nB}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-1}} + \frac{2nc}{(m-1)\beta^2} \int \frac{R^{n-1} dx}{z^{m-2}} \end{aligned}$$

LA 418 (6)

LA 148 (5)

LA 184 (4)a

$$\begin{aligned} 3. \int \frac{z^m dx}{R^n} &= \frac{\beta}{(m-2n+1)c} \frac{z^{m-1}}{R^{n-1}} - \frac{(m-n)B}{(m-2n+1)c} \int \frac{z^{m-1} dx}{R^n} - \frac{(m-1)A}{(m-2n+1)c} \int \frac{z^{m-2} dx}{R^n}; \\ &= \frac{b+2cx}{(n-1)\Delta} \frac{z^m}{R^{n-1}} - \frac{2(m-2n+3)c}{(n-1)\Delta} \int \frac{z^m dx}{R^{n-1}} - \frac{Bm}{(n-1)\Delta} \int \frac{z^{m-1} dx}{R^{n-1}} \end{aligned}$$

$$\begin{aligned}
4^3. \int \frac{dx}{z^m R^n} &= -\frac{\beta}{(m-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{(m+n-2)B}{(m-1)A} \int \frac{dx}{z^{m-1} R^n} - \frac{(m+2n-3)c}{(m-1)A} \int \frac{dx}{z^{m-2} R^n}; \\
&= \frac{\beta}{2(n-1)A} \frac{1}{z^{m-1} R^{n-1}} - \frac{B}{2A} \int \frac{dx}{z^{m-1} R^n} + \frac{(m+2n-3)\beta^2}{2(n-1)A} \int \frac{dx}{z^m R^{n-1}}.
\end{aligned}$$

LA 148 (8)
LA 148 (7)

For $m = 1$ and $n = 1$

$$\int \frac{dx}{zR} = \frac{\beta}{2A} \ln \frac{z^2}{R} - \frac{B}{2A} \int \frac{dx}{R}.$$

For $A = 0$

$$\int \frac{dx}{z^m R^n} = -\frac{\beta}{(m+n-1)B} \frac{1}{z^m R^{n-1}} - \frac{(m+2n-2)c}{(m+n-1)B} \int \frac{dx}{z^{m-1} R^n}.$$

LA 148 (9)

2.2 Algebraic Functions

2.20 Introduction

2.201

The integrals $\int R\left(x, \left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)^r, \left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)^s, \dots\right) dx$, where r, s, \dots are rational numbers, can be reduced to integrals of rational functions by means of the substitution

$$\frac{\alpha x + \beta}{\gamma x + \delta} = t^m,$$

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where m is the common denominator of the fractions r, s, \dots

2.202

Integrals of the form $\int x^m (a + bx^n)^p dx$, * Transl. The authors term such integrals "integrals of binomial differentials".

$$\int x^m(a + bx^n)^p dx$$

where m , n , and p are rational numbers, can be expressed in terms of elementary functions only in the following

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(a) When p is an integer; then, this integral takes the form of the sum of the integrals shown in 2.201;

(b) When $\frac{m+1}{n}$ is an integer: by means of the substitution $x^n = z$, this integral can be transformed to the form $\frac{1}{n} \int (a + bz)^p z^{\frac{m+1}{n} - 1} dz$, which we considered in 2.201;

(c) When $\frac{m+1}{n} + p$ is an integer; by means of the same substitution $x^n = z$, this integral can be reduced to an integral of the form $\frac{1}{n} \int \left(\frac{a+bz}{z}\right)^p z^{\frac{m+1}{n} + p - 1} dz$, considered in 2.201;

For reduction formulas for integrals of binomial differentials, see 2.110.

2.21 Forms containing the binomial $a + bx^k$ and \sqrt{x}

Notation: $z_1 = A + Bx$.

2.211

$$\int \frac{dx}{z_1 \sqrt{x}} = \frac{2}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{bx}{a}} \quad [ab > 0];$$

$$= \frac{1}{i\sqrt{ab}} \ln \frac{a - bx + 2i\sqrt{xab}}{z_1} \quad [ab < 0].$$

2.212

$$\int \frac{x^m \sqrt{x}}{z_1} dx = 2\sqrt{x} \sum_{k=0}^m \frac{(-1)^k a^k x^{m-k}}{(2m - 2k + 1)b^{k+1}} + (-1)^{m+1} \frac{a^{m+1}}{b^{m+1}} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

2.213

$$1. \int \frac{\sqrt{x} dx}{z_1} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$2. \int \frac{x\sqrt{x} dx}{z_1} = \left(\frac{x}{3b} - \frac{a}{b^2}\right) 2\sqrt{x} + \frac{a^2}{b^2} \int \frac{dx}{z_1\sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$3. \int \frac{x^2\sqrt{x} dx}{z_1} = \left(\frac{x^2}{5b} - \frac{xa}{3b^2} + \frac{a^2}{b^3}\right) 2\sqrt{x} - \frac{a^3}{b^3} \int \frac{dx}{z_1\sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$4. \int \frac{dx}{z_1^2\sqrt{x}} = \frac{\sqrt{x}}{az_1} + \frac{1}{2a} \int \frac{dx}{z_1\sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$5. \int \frac{\sqrt{x} dx}{z_1^2} = -\frac{\sqrt{x}}{bz_1} + \frac{1}{2b} \int \frac{dx}{z_1\sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$6. \int \frac{x\sqrt{x} dx}{z_1^2} = \frac{2x\sqrt{x}}{bz_1} - \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see } \mathbf{2.213} \text{ 5.}).$$

2.213

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$$7. \int \frac{x^2\sqrt{x} dx}{z_1^2} = \left(\frac{x^2}{3b} - \frac{5ax}{3b^2}\right) \frac{2\sqrt{x}}{z_1} + \frac{5a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^2} \quad (\text{see } \mathbf{2.213} \text{ 5.}).$$

$$8. \int \frac{dx}{z_1^3 \sqrt{x}} = \left(\frac{1}{2az_1^2} + \frac{3}{4a^2 z_1} \right) \sqrt{x} + \frac{3}{8a^2} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$9. \int \frac{\sqrt{x} dx}{z_1^3} = \left(-\frac{1}{2bz_1^2} + \frac{1}{4abz_1} \right) \sqrt{x} + \frac{1}{8ab} \int \frac{dx}{z_1 \sqrt{x}} \quad (\text{see } \mathbf{2.211}).$$

2.211

$$10. \int \frac{x\sqrt{x} dx}{z_1^3} = -\frac{2x\sqrt{x}}{bz_1^2} + \frac{3a}{b} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see } \mathbf{2.213} \text{ 9.}).$$

2.213

$$11. \int \frac{x^2 \sqrt{x} dx}{z_1^3} = \left(\frac{x^2}{b} + \frac{5ax}{b^2} \right) \frac{2\sqrt{x}}{z_1^2} - \frac{15a^2}{b^2} \int \frac{\sqrt{x} dx}{z_1^3} \quad (\text{see } \mathbf{2.213} \text{ 9.}).$$

2.213

Notations: $z_2 = a + bx^2$, $\alpha = \sqrt[4]{\frac{a}{b}}$, $\alpha' = \sqrt[4]{-\frac{a}{b}}$.

2.214

$$\begin{aligned} \int \frac{dx}{z_2 \sqrt{x}} &= \frac{1}{b\alpha^3 \sqrt{2}} \left[\ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \operatorname{arctg} \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] \quad \left[\frac{a}{b} > 0 \right]; \\ &= \frac{1}{2b\alpha^3} \left(\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} - 2 \operatorname{arctg} \frac{\sqrt{x}}{\alpha'} \right) \quad \left[\frac{a}{b} < 0 \right]. \end{aligned}$$

2.215

$$\begin{aligned} \int \frac{\sqrt{x} dx}{z_2} &= \frac{1}{b\alpha\sqrt{2}} \left[-\ln \frac{x + \alpha\sqrt{2x} + \alpha^2}{\sqrt{z_2}} + \operatorname{arctg} \frac{\alpha\sqrt{2x}}{\alpha^2 - x} \right] \quad \left[\frac{a}{b} > 0 \right]; \\ &= \frac{1}{2b\alpha'} \left[\ln \frac{\alpha' - \sqrt{x}}{\alpha' + \sqrt{x}} + 2 \operatorname{arctg} \frac{\sqrt{x}}{\alpha'} \right] \quad \left[\frac{a}{b} < 0 \right]. \end{aligned}$$

$$1. \int \frac{x\sqrt{x} dx}{z_2} = \frac{2\sqrt{x}}{b} - \frac{a}{b} \int \frac{dx}{z_2\sqrt{x}} \quad (\text{see } \mathbf{2.214}).$$

2.214

$$2. \int \frac{x^2\sqrt{x} dx}{z_2} = \frac{2x\sqrt{x}}{3b} - \frac{a}{b} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{see } \mathbf{2.215}).$$

2.215

$$3. \int \frac{x}{z_2^2\sqrt{x}} = \frac{\sqrt{x}}{2az_2} + \frac{3}{4a} \int \frac{dx}{z_2\sqrt{x}} \quad (\text{see } \mathbf{2.214}).$$

2.214

$$4. \int \frac{\sqrt{x} dx}{z_2^2} = \frac{x\sqrt{x}}{2az_2} + \frac{1}{4a} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{see } \mathbf{2.215}).$$

2.215

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$$5. \int \frac{x\sqrt{x} dx}{z_2^2} = -\frac{\sqrt{x}}{2bz_2} + \frac{1}{4b} \int \frac{dx}{z_2\sqrt{x}} \quad (\text{see } \mathbf{2.214}).$$

2.214

$$6. \int \frac{x^2\sqrt{x} dx}{z_2^2} = -\frac{x\sqrt{x}}{2bz_2} + \frac{3}{4b} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{see } \mathbf{2.215}).$$

2.215

2.214

$$8. \int \frac{\sqrt{x} dx}{z_2^3} = \left(\frac{1}{4az_2^2} + \frac{5}{16a^2z_2} \right) x\sqrt{x} + \frac{5}{32a^2} \int \frac{\sqrt{x} dx}{z_2} \quad (\text{see } \mathbf{2.215}).$$

2.215

$$9. \int \frac{x\sqrt{x} dx}{z_2^3} = \frac{(bx^2 - 3a)\sqrt{x}}{16abz_2^2} + \frac{3}{32ab} \int \frac{dx}{z_2\sqrt{x}} \quad (\text{see } \mathbf{2.214}).$$

2.214

$$10. \int \frac{x^2\sqrt{x} dx}{z_2^3} = -\frac{2x\sqrt{x}}{5bz_2^2} + \frac{3a}{5b} \int \frac{\sqrt{x} dx}{z_2^3} \quad (\text{see } \mathbf{2.216} \text{ 8.}).$$

2.216

2.22-2.23 Forms containing $\sqrt[n]{(a + bx)^k}$

Notation: $z = A + Bx$.

2.220

$$\int x^n \sqrt[l]{z^{lm+f}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{ln - lk + l(m+1) + f} \right\} \frac{l \sqrt[l]{z^{l(m+1)+f}}}{b^{n+1}}.$$

The square root

2.221

$$\int x^n \sqrt{z^{2m-1}} dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{2n - 2k + 2m + 1} \right\} \frac{2\sqrt{z^{2m+1}}}{b^{n+1}}.$$

$$1. \int \frac{dx}{\sqrt{z}} = \frac{2}{b} \sqrt{z}.$$

$$2. \int \frac{x dx}{\sqrt{z}} = \left(\frac{1}{3} z - a \right) \frac{2\sqrt{z}}{b^2}.$$

$$3. \int \frac{x^2 dx}{\sqrt{z}} = \left(\frac{1}{5} z^2 - \frac{2}{3} az + a^2 \right) \frac{2\sqrt{z}}{b^3}.$$

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2.223

$$1. \int \frac{dx}{\sqrt{z^3}} = -\frac{2}{b\sqrt{z}}.$$

$$2. \int \frac{x dx}{\sqrt{z^3}} = (z + a) \frac{2}{b^2 \sqrt{z}}.$$

$$3. \int \frac{x^2 dx}{\sqrt{z^3}} = \left(\frac{z^2}{3} - 2az - a^2 \right) \frac{2}{b^3 \sqrt{z}}.$$

2.224

$$1. \int \frac{z^m dx}{x^n \sqrt{z}} = -\frac{z^m \sqrt{z}}{(n-1)ax^{n-1}} + \frac{2m-2n+3}{2(n-1)} \frac{b}{a} \int \frac{z^m dx}{x^{n-1} \sqrt{z}}.$$

$$2. \int \frac{z^m dx}{x^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1)ax^{n-1}} + \sum_{k=1}^{n-2} \frac{(2m-2n+3)(2m-2n+5) \dots (2m-2n+2k+1)}{2^k (n-1)(n-2) \dots (n-k-1) x^{n-k-1}} \frac{b^k}{a^{k+1}} \right\} + \frac{(2m-2n+3)(2m-2n+5) \dots (2m-3)(2m-1)}{2^{n-1} (n-1)!} \frac{b^{n-1}}{a^{n-1}} \int \frac{z^m dx}{x \sqrt{z}}.$$

For $n=1$

2.225

1. $\int \frac{\sqrt{z} dx}{x} = 2\sqrt{z} + a \int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

2. $\int \frac{\sqrt{z} dx}{x^2} = -\frac{\sqrt{z}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

3. $\int \frac{\sqrt{z} dx}{x^3} = -\frac{\sqrt{z^3}}{2ax^2} + \frac{b\sqrt{z}}{4ax} - \frac{b^2}{8a} \int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

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2.226

1. $\int \frac{\sqrt{z^3} dx}{x} = \left(\frac{z}{3} + a\right) 2\sqrt{z} + a^2 \int \frac{dx}{x\sqrt{z}}$ (see **2.224** 4.).

2.224

2. $\int \frac{\sqrt{z^3} dx}{x^2} = -\frac{\sqrt{z^5}}{ax} + \frac{3b}{2a} \int \frac{\sqrt{z^3} dx}{x}$ (see **2.226** 1.).

2.226

3. $\int \frac{\sqrt{z^3} dx}{x^3} = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right) \sqrt{z^5} + \frac{3b^2}{8a^2} \int \frac{\sqrt{z^3} dx}{x}$ (see **2.226** 1.).

2.227

$$\int \frac{dx}{xz^m\sqrt{z}} = \sum_{k=0}^{m-1} \frac{2}{(2k+1)a^{m-k}z^k\sqrt{z}} + \frac{1}{a^m} \int \frac{dx}{x\sqrt{z}}. \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

2.224

2.228

$$1. \int \frac{dx}{x^2\sqrt{z}} = -\frac{\sqrt{z}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

2.224

$$2. \int \frac{dx}{x^3\sqrt{z}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right)\sqrt{z} + \frac{3b^2}{8a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

2.224

2.229

$$1. \int \frac{dx}{x\sqrt{z^3}} = \frac{2}{a\sqrt{z}} + \frac{1}{a} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

2.224

$$2. \int \frac{dx}{x^2\sqrt{z^3}} = \left(-\frac{1}{ax} - \frac{3b}{a^2}\right)\frac{1}{\sqrt{z}} - \frac{3b}{2a^2} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

2.224

$$3. \int \frac{dx}{x^3\sqrt{z^3}} = \left(-\frac{1}{2ax^2} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3}\right)\frac{1}{\sqrt{z}} + \frac{15b^2}{8a^3} \int \frac{dx}{x\sqrt{z}} \quad (\text{see } \mathbf{2.224} \text{ 4.}).$$

Cube root

2.231

$$1. \int \sqrt[3]{z^{3m+1}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 1} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+1}}}{b^{n+1}}.$$

$$2. \int \frac{x^n dx}{\sqrt[3]{z^{3m+2}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 2} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+2}}}.$$

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$$3. \int \sqrt[3]{z^{3m+2}} x^n dx = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k + 3(m+1) + 2} \right\} \frac{3 \sqrt[3]{z^{3(m+1)+2}}}{b^{n+1}}.$$

$$4. \int \frac{x^n dx}{\sqrt[3]{z^{3m+1}}} = \left\{ \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} z^{n-k} a^k}{3n - 3k - 3(m-1) - 1} \right\} \frac{3}{b^{n+1} \sqrt[3]{z^{3(m-1)+1}}}.$$

$$5. \int \frac{z^n dx}{x^m \sqrt[3]{x^2}} = -\frac{z^{n+\frac{1}{3}}}{(m-1)ax^{m-1}} + \frac{3n-3m+4}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z^2}}.$$

For $m = 1$

$$\int \frac{z^n dx}{x \sqrt[3]{z^2}} = \frac{3z^n}{(3n-2)\sqrt[3]{z^2}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z^2}}.$$

2.232

$$\int \frac{dx}{x\sqrt[3]{z^2}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} - \sqrt{3} \operatorname{arctg} \frac{\sqrt{3}\sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}.$$

2.233

$$1. \int \frac{\sqrt[3]{z} dx}{x} = 3\sqrt[3]{z} + a \int \frac{dx}{x\sqrt[3]{z^2}} \quad (\text{see } \mathbf{2.232}).$$

2.232

$$2. \int \frac{\sqrt[3]{z} dx}{x^2} = -\frac{z\sqrt[3]{z}}{ax} + \frac{b}{a}\sqrt[3]{z} + \frac{b}{3} \int \frac{dx}{x\sqrt[3]{z^2}} \quad (\text{see } \mathbf{2.232}).$$

2.232

$$3. \int \frac{\sqrt[3]{z} dx}{x^3} = \left(-\frac{1}{2ax^2} + \frac{b}{3a^2x} \right) z\sqrt[3]{z} - \frac{b^2}{3a^2}\sqrt[3]{z} - \frac{b^2}{9a} \int \frac{dx}{x\sqrt[3]{z^2}} \quad (\text{see } \mathbf{2.232}).$$

2.232

$$4. \int \frac{dx}{x^2\sqrt[3]{z^2}} = -\frac{\sqrt[3]{z}}{ax} - \frac{2b}{3a} \int \frac{dx}{x\sqrt[3]{z^2}} \quad (\text{see } \mathbf{2.232}).$$

2.232

$$5. \int \frac{dx}{x^3\sqrt[3]{z^2}} = \left[-\frac{1}{2ax^2} + \frac{5b}{6a^2x} \right] \sqrt[3]{z} + \frac{5b^2}{9a^2} \int \frac{dx}{x\sqrt[3]{z^2}} \quad (\text{see } \mathbf{2.232}).$$

2.232

2.234

$$1. \int \frac{z^n dx}{x^m \sqrt[3]{z^2}} = -\frac{z^n \sqrt[3]{z^2}}{(m-1)ax^{m-1}} + \frac{3n-3m+5}{3(m-1)} \frac{b}{a} \int \frac{z^n dx}{x^{m-1} \sqrt[3]{z}}.$$

For $m = 1$:

$$2. \int \frac{z^n dx}{x \sqrt[3]{z}} = \frac{3z^n}{(3n-1)\sqrt[3]{z}} + a \int \frac{z^{n-1} dx}{x \sqrt[3]{z}}.$$

$$3. \int \frac{dx}{xz^n \sqrt[3]{z}} = \frac{3\sqrt[3]{z^2}}{(3n-2)az^n} + \frac{1}{a} \int \frac{\sqrt[3]{z^2} dx}{xz^n}.$$

2.235

$$\int \frac{dx}{x \sqrt[3]{z}} = \frac{1}{\sqrt[3]{a^2}} \left\{ \frac{3}{2} \ln \frac{\sqrt[3]{z} - \sqrt[3]{a}}{\sqrt[3]{x}} + \sqrt{3} \operatorname{arctg} \frac{\sqrt{3} \sqrt[3]{z}}{\sqrt[3]{z} + 2\sqrt[3]{a}} \right\}.$$

2.236

$$1. \int \frac{\sqrt[3]{z^2} dx}{x} = \frac{3}{2} \sqrt[3]{z^2} + a \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } \mathbf{2.235}).$$

2.235

$$2. \int \frac{\sqrt[3]{z^2} dx}{x^2} = -\frac{\sqrt[3]{z^5}}{ax} + \frac{b}{a} \sqrt[3]{z^2} + \frac{2b}{3} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } \mathbf{2.235}).$$

2.235

$$3. \int \frac{\sqrt[3]{z^2} dx}{x^3} = \left[-\frac{1}{2ax^2} + \frac{b}{6a^2x} \right] z^{5/3} - \frac{b^2}{6a^2} \sqrt[3]{z^2} - \frac{b^2}{9a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } \mathbf{2.235}).$$

$$4. \int \frac{dx}{x^2 \sqrt[3]{z}} = -\frac{\sqrt[3]{z^2}}{ax} - \frac{b}{3a} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } \mathbf{2.235}).$$

$$5. \int \frac{dx}{x^3 \sqrt[3]{z}} = \left[-\frac{1}{2ax^2} + \frac{2b}{3a^2x} \right] \sqrt[3]{z} + \frac{2b^2}{9a^2} \int \frac{dx}{x \sqrt[3]{z}} \quad (\text{see } \mathbf{2.235}).$$

2.24 Forms containing $\sqrt{a+bx}$ and the binomial $\alpha + \beta x$

Notation: $z = a + bx$, $t = \alpha + \beta x$, $\Delta = a\beta - b\alpha$.

2.241

$$1. \int \frac{z^m t^n dx}{\sqrt{z}} = \frac{2}{(2n+2m+1)\beta} t^{n+1} z^{m-1} \sqrt{z} + \frac{(2m-1)\Delta}{(2n+2m+1)\beta} \int \frac{z^{m-1} t^n dx}{\sqrt{z}}.$$

LA 176 (1)

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$$2. \int \frac{t^n z^m dx}{\sqrt{z}} = 2\sqrt{z^{2m+1}} \sum_{k=0}^n \binom{n}{k} \frac{\alpha^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p+2m+1}.$$

2.242

$$1. \int \frac{t dx}{\sqrt{z}} = \frac{2a\sqrt{z}}{b} + \beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2}.$$

$$2. \int \frac{t^2 dx}{\sqrt{z}} = \frac{2\alpha^2 \sqrt{z}}{b} + 2\alpha\beta \left(\frac{z}{3} - a \right) \frac{2\sqrt{z}}{b^2} + \beta^2 \left(\frac{z^2}{5} - \frac{2}{3}za + a^2 \right) \frac{2\sqrt{z}}{b^3}.$$

$$12. \int \frac{t^3 z^3 dx}{\sqrt{z}} = \frac{2\alpha^3 \sqrt{z^7}}{7b} + 3\alpha^2 \beta \left(\frac{z}{9} - \frac{a}{7} \right) \frac{2\sqrt{z^7}}{b^2} + 3\alpha\beta^2 \left(\frac{z^2}{11} - \frac{2za}{9} + \frac{a^2}{7} \right) \frac{2\sqrt{z^7}}{b^3} + \beta^3 \left(\frac{z^3}{13} - \frac{3z^2a}{11} + \frac{3za^2}{9} - \frac{a^3}{7} \right) \frac{2\sqrt{z^7}}{b^4}.$$

2.243

$$1. \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{t^{n+1}}{z^m} \sqrt{z} - \frac{(2n-2m+3)\beta}{(2m-1)\Delta} \int \frac{t^n dx}{z^{m-1} \sqrt{z}};$$

$$= -\frac{2}{(2m-1)b} \frac{t^n}{z^m} \sqrt{z} + \frac{2n\beta}{(2m-1)b} \int \frac{t^{n-1} dx}{z^{m-1} \sqrt{z}}.$$

LA 176 (2)

$$2. \int \frac{t^n dx}{z^m \sqrt{z}} = \frac{2}{\sqrt{z^{2m-1}}} \sum_{k=0}^n \binom{n}{k} \frac{a^{n-k} \beta^k}{b^{k+1}} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{z^{k-p} a^p}{2k-2p-2m+1}.$$

2.244

$$1. \int \frac{t dx}{z \sqrt{z}} = -\frac{2a}{b\sqrt{z}} + \frac{2\beta(z+a)}{b^2 \sqrt{z}}.$$

$$2. \int \frac{t^2 dx}{z \sqrt{z}} = -\frac{2\alpha^2}{b\sqrt{z}} + \frac{4\alpha\beta(z+a)}{b^2 \sqrt{z}} + \frac{2\beta^2 \left(\frac{z^2}{3} - 2za - a^2 \right)}{b^3 \sqrt{z}}.$$

$$3. \int \frac{t^3 dx}{z \sqrt{z}} = -\frac{2\alpha^3}{b\sqrt{z}} + \frac{6\alpha^2 \beta(z+a)}{b^2 \sqrt{z}} + \frac{6\alpha\beta^2 \left(\frac{z^2}{3} - 2za - a^2 \right)}{b^3 \sqrt{z}} + \frac{2\beta^3 \left(\frac{z^3}{5} - z^2a + 3za^2 + a^3 \right)}{b^4 \sqrt{z}}.$$

$$4. \int \frac{t dx}{z^2 \sqrt{z}} = -\frac{2a}{3b\sqrt{z^3}} - \frac{2\beta \left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}}.$$

$$5. \int \frac{t^2 dx}{z^2 \sqrt{z}} = -\frac{2a^2}{3b\sqrt{z^3}} - \frac{4\alpha\beta \left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}} + \frac{2\beta^2 \left(z^2 + 2az - \frac{a^2}{3}\right)}{b^3 \sqrt{z^3}}.$$

$$6. \int \frac{t^3 dx}{z^2 \sqrt{z}} = -\frac{2\alpha^3}{3b\sqrt{z^3}} - \frac{6\alpha^2\beta \left(z - \frac{a}{3}\right)}{b^2 \sqrt{z^3}} + \frac{6\alpha\beta^2 \left(z^2 + 2za - \frac{a^2}{3}\right)}{b^3 \sqrt{z^3}} + \frac{2\beta^3 \left(\frac{z^3}{3} - 3z^2a - 3za^2 + \frac{a^3}{3}\right)}{b^4 \sqrt{z^3}}.$$

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$$7. \int \frac{t dx}{z^3 \sqrt{z}} = -\frac{2\alpha}{5b\sqrt{z^5}} - \frac{2\beta \left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}}.$$

$$8. \int \frac{t^2 dx}{z^3 \sqrt{z}} = -\frac{2\alpha^2}{5b\sqrt{z^5}} - \frac{4\alpha\beta \left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}} - \frac{2\beta^2 \left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3 \sqrt{z^5}}.$$

$$9. \int \frac{t^3 dx}{z^3 \sqrt{z}} = -\frac{2\alpha^3}{5b\sqrt{z^5}} - \frac{6\alpha^2\beta \left(\frac{z}{3} - \frac{a}{5}\right)}{b^2 \sqrt{z^5}} - \frac{6\alpha\beta^2 \left(z^2 - \frac{2za}{3} + \frac{a^2}{5}\right)}{b^3 \sqrt{z^5}} + \frac{2\beta^3 \left(z^3 + 3z^2a - za^2 + \frac{a^3}{5}\right)}{b^4 \sqrt{z^5}}.$$

2.245

$$\begin{aligned} 1. \int \frac{z^m dx}{t^n \sqrt{z}} &= -\frac{2}{(2n-2m-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} - \frac{(2m-1)\Delta}{(2n-2m-1)\beta} \int \frac{z^{m-1} dx}{t^n \sqrt{z}}; \\ &= -\frac{1}{(n-1)\beta} \frac{z^{m-1}}{t^{n-1}} \sqrt{z} + \frac{(2m-1)b}{2(n-1)\beta} \int \frac{z^{m-1}}{t^{n-1} \sqrt{z}} dx; \\ &= -\frac{1}{(n-1)\Delta} \frac{z^m}{t^{n-1}} \sqrt{z} - \frac{(2n-2m-3)b}{2(n-1)\Delta} \int \frac{z^m dx}{t^{n-1} \sqrt{z}}. \end{aligned}$$

$$2. \int \frac{z^m dz}{t^n \sqrt{z}} = -z^m \sqrt{z} \left\{ \frac{1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \sum_{k=2}^{n-1} \frac{(2n-2m-3)(2n-2m-5)\dots(2n-2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k} \frac{1}{t^{n-k}} \right\} - \frac{(2n-2m-3)(2n-2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1} \cdot (n-1)!\Delta^n} \int \frac{z^m dx}{t\sqrt{z}}.$$

For $n = 1$

$$3. \int \frac{z^m dx}{t\sqrt{z}} = \frac{2}{(2m-1)\beta} \frac{z^m}{\sqrt{z}} + \frac{\Delta}{\beta} \int \frac{z^{m-1} dx}{t\sqrt{z}}.$$

$$4. \int \frac{z^m dx}{t\sqrt{z}} = 2 \sum_{k=0}^{m-1} \frac{\Delta^k}{(2m-2k-1)\beta^{k+1}} \frac{z^{m-k}}{\sqrt{z}} + \frac{\Delta^m}{\beta^m} \int \frac{dx}{t\sqrt{z}}.$$

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2.246

$$\begin{aligned} \int \frac{dx}{t\sqrt{z}} &= \frac{1}{\sqrt{\beta\Delta}} \ln \frac{\beta\sqrt{z} - \sqrt{\beta\Delta}}{\beta\sqrt{z} + \sqrt{\beta\Delta}} & [\beta\Delta > 0]; \\ &= \frac{2}{\sqrt{-\beta\Delta}} \operatorname{arctg} \frac{\beta\sqrt{z}}{\sqrt{-\beta\Delta}} & [\beta\Delta < 0]; \\ &= -\frac{2\sqrt{z}}{bt} & [\Delta = 0]. \end{aligned}$$

2.247

$$\int \frac{dx}{tz^m \sqrt{z}} = \frac{2}{z^{m-1} \sqrt{z}} + \sum_{k=1}^m \frac{\beta^{k-1} z^k}{\Delta^k (2m-2k+1)} + \frac{\beta^m}{\Delta^m} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

2.248

$$1. \int \frac{dx}{tz\sqrt{z}} = \frac{2}{\Delta\sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$3. \int \frac{dx}{tz^3\sqrt{z}} = \frac{2}{5\Delta z^2\sqrt{z}} + \frac{2\beta}{3\Delta^2 z\sqrt{z}} + \frac{2\beta^2}{\Delta^3\sqrt{z}} + \frac{\beta^3}{\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$4. \int \frac{dx}{t^2 z\sqrt{z}} = -\frac{\sqrt{z}}{\Delta t} - \frac{b}{2\Delta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$5. \int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t\sqrt{z}} - \frac{3b}{\Delta^2\sqrt{z}} - \frac{3b\beta}{2\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$6. \int \frac{dx}{t^2 z^2\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{5b}{3\Delta^2 z\sqrt{z}} - \frac{5b\beta}{\Delta^3\sqrt{z}} - \frac{5b\beta^2}{2\Delta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$7. \int \frac{dx}{t^2 z^3\sqrt{z}} = -\frac{1}{\Delta t z^2\sqrt{z}} - \frac{7b}{5\Delta^2 z^2\sqrt{z}} - \frac{7b\beta}{3\Delta^3 z\sqrt{z}} - \frac{7b\beta^2}{\Delta^4\sqrt{z}} - \frac{7b\beta^3}{2\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$8. \int \frac{dx}{t^3\sqrt{z}} = -\frac{\sqrt{z}}{2\Delta t^2} + \frac{3b\sqrt{z}}{4\Delta^2 t} + \frac{3b^2}{8\Delta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$10. \int \frac{dx}{t^3 z^2 \sqrt{z}} = -\frac{1}{2\Delta t^2 z \sqrt{z}} + \frac{7b\sqrt{z}}{4\Delta^2 t z \sqrt{z}} + \frac{35b^2}{12\Delta^2 z \sqrt{z}} + \frac{35b^2 \beta}{4\Delta^4 \sqrt{z}} + \frac{35b^2 \beta^2}{8\Delta^4} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$11. \int \frac{dx}{t^3 z^3 \sqrt{z}} = -\frac{1}{2\Delta t^2 z^2 \sqrt{z}} + \frac{9b}{4\Delta^2 t z^2 \sqrt{z}} + \frac{63b^2}{20\Delta^3 z^2 \sqrt{z}} + \frac{21b^2 \beta}{4\Delta^4 z \sqrt{z}} + \frac{63b^2 \beta^2}{4\Delta^5 \sqrt{z}} + \frac{63b^2 \beta^3}{8\Delta^5} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

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$$12. \int \frac{z dx}{t\sqrt{z}} = \frac{2\sqrt{z}}{\beta} + \frac{\Delta}{\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$13. \int \frac{z^2 dx}{t\sqrt{z}} = \frac{2z\sqrt{z}}{3\beta} + \frac{2\Delta\sqrt{z}}{\beta^2} + \frac{\Delta^2}{\beta^2} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$14. \int \frac{z^3 dx}{t\sqrt{z}} = \frac{2z^2\sqrt{z}}{5\beta} + \frac{2\Delta z\sqrt{z}}{3\beta^2} + \frac{2\Delta^2\sqrt{z}}{\beta^3} + \frac{\Delta^3}{\beta^3} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.246

$$15. \int \frac{z dx}{t^2 \sqrt{z}} = -\frac{z\sqrt{z}}{\Delta t} + \frac{b\sqrt{z}}{\beta\Delta} + \frac{b}{2\beta} \int \frac{dx}{t\sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$16. \int \frac{z^2 dx}{t^2 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{\Delta t} + \frac{bz \sqrt{z}}{\beta \Delta} + \frac{3b \sqrt{z}}{\beta^2} + \frac{3b \Delta}{2\beta^2} \int \frac{dx}{t \sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$17. \int \frac{z^3 dx}{t^2 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{\Delta t} + \frac{bz^2 \sqrt{z}}{\beta \Delta} + \frac{5bz \sqrt{z}}{3\beta^2} + \frac{5b \Delta \sqrt{z}}{\beta^3} + \frac{5\Delta^2 b}{2\beta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$18^3. \int \frac{z dx}{t^3 \sqrt{z}} = -\frac{z \sqrt{z}}{2\Delta t^2} + \frac{bz \sqrt{z}}{4\Delta^2 t} - \frac{b^2 \sqrt{z}}{4\beta \Delta^2} + \frac{b^2}{8\beta \Delta} \int \frac{dx}{t \sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$19. \int \frac{z^2 dx}{t^3 \sqrt{z}} = -\frac{z^2 \sqrt{z}}{2\Delta t^2} + \frac{bz^2 \sqrt{z}}{4\Delta^2 t} + \frac{b^2 z \sqrt{z}}{4\beta \Delta^2} + \frac{3b^2 \sqrt{z}}{4\beta^2 \Delta} + \frac{3b^2}{8\beta^2} \int \frac{dx}{t \sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

$$20. \int \frac{z^3 dx}{t^3 \sqrt{z}} = -\frac{z^3 \sqrt{z}}{2\Delta t^2} + \frac{3bz^3 \sqrt{z}}{\Delta^2 t} + \frac{3b^2 z^2 \sqrt{z}}{4\beta \Delta^2} + \frac{5b^2 z \sqrt{z}}{4\beta^2 \Delta} + \frac{15b^2 \sqrt{z}}{4\beta^3} + \frac{15b^2 \Delta}{8\beta^3} \int \frac{dx}{t \sqrt{z}} \quad (\text{see } \mathbf{2.246}).$$

2.249

$$1. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{\sqrt{z}}{t^{n-1} z^m} + \frac{(2n+2m-3)\beta}{(2m-1)\Delta} \int \frac{dx}{t^n z^{m-1} \sqrt{z}}; \\ = -\frac{1}{(n-1)\Delta} \frac{\sqrt{z}}{z^m t^{n-1}} - \frac{(2n+2m-3)b}{2(n-1)\Delta} \int \frac{dx}{t^{n-1} z^m \sqrt{z}}.$$

$$2. \int \frac{dx}{z^m t^n \sqrt{z}} = \frac{\sqrt{z}}{z^m} \left\{ \frac{-1}{(n-1)\Delta} \frac{1}{t^{n-1}} + \sum_{k=2}^{n-1} (-1)^k \frac{(2n+2m-3)(2n+2m-5)\dots(2n+2m-2k+1)b^{k-1}}{2^{k-1}(n-1)(n-2)\dots(n-k)\Delta^k} \cdot \frac{1}{t^{n-k}} \right\} + (-1)^{n-1} \frac{(2n+2m-3)(2n+2m-5)\dots(-2m+3)(-2m+1)b^{n-1}}{2^{n-1}(n-1)!\Delta^{n-1}} \int \frac{dx}{tz^m \sqrt{z}}.$$

For $n = 1$

$$\int \frac{dx}{z^m t \sqrt{z}} = \frac{2}{(2m-1)\Delta} \frac{1}{z^{m-1} \sqrt{z}} + \frac{\beta}{\Delta} \int \frac{dx}{tz^{m-1} \sqrt{z}}.$$

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2.25 Forms containing $\sqrt{a + bx + cx^2}$

Integration techniques

2.251

It is possible to rationalize the integrand in integrals of the form $\int R(x, \sqrt{a + bx + cx^2}) dx$ by using one or more of the following three substitutions, known as the "Euler substitutions".

1) $\sqrt{a + bx + cx^2} = xt \pm \sqrt{a}$ for $a > 0$;

2) $\sqrt{a + bx + cx^2} = t \pm x\sqrt{c}$ for $c > 0$;

3) $\sqrt{c(x - x_1)(x - x_2)} = t(x - x_1)$ when x_1 and x_2 are real roots of the equation $a + bx + cx^2 = 0$.

2.252

Besides the Euler substitutions, there is also the following method of calculating integrals of the form $\int R(x, \sqrt{a + bx + cx^2}) dx$.

By removing the irrational expressions in the denominator and performing simple algebraic operations, we can reduce the integrand to the sum of some rational function of x and an expression of the form $\frac{P_1(x)}{P_2(x)\sqrt{a+bx+cx^2}}$, where $P_1(x)$ and $P_2(x)$ are both polynomials. By separating the integral portion of the rational function $\frac{P_1(x)}{P_2(x)}$ from the remainder and decomposing the latter into partial fractions, we can reduce the integral of these partial fractions to the sum of integrals each of which is in one of the following three forms:

I. $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$, where $P(x)$ is a polynomial of some degree r ;

II. $\int \frac{dx}{(x+p)^k \sqrt{a+bx+cx^2}}$;

III. $\int \frac{(Mx+N) dx}{(a+\beta x+x^2)^m \sqrt{c(a_1+b_1x+x^2)}}$, $(a_1 = \frac{a}{c}, \quad b_1 = \frac{b}{c})$.

I. $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}} = Q(x)\sqrt{a+bx+cx^2} + \lambda \int \frac{dx}{\sqrt{a+bx+cx^2}}$, where $Q(x)$ is a polynomial of degree $(r-1)$. Its coefficients, and also the number λ , can be calculated by the method of undetermined coefficients from the identity

$$P(x) = Q'(x)(a+bx+cx^2) + \frac{1}{2}Q(x)(b+2cx) + \lambda.$$

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Integrals of the form $\int \frac{P(x) dx}{\sqrt{a+bx+cx^2}}$ (where $r \leq 3$) can also be calculated by use of formulas 2.26.

II. Integrals of the form $\int \frac{P(x) dx}{(x+p)^k \sqrt{a+bx+cx^2}}$, where the degree n of the polynomial $P(x)$ is

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lower than k can, by means of the substitution $t = \frac{1}{x+p}$, be reduced to an integral of the form $\int \frac{P(t) dt}{\sqrt{a+\beta t+\gamma t^2}}$. (See also 2.281).

III. Integrals of the form $\int \frac{(Mx+N) dx}{(\alpha+\beta x+x^2)^m \sqrt{c(a_1+b_1x+x^2)}}$ can be calculated by the following procedure.

If $b_1 \neq \beta$, by using the substitution

$$x = \frac{a_1 - \alpha}{\beta - b_1} + \frac{t - 1}{t + 1} \frac{\sqrt{(a_1 - \alpha)^2 - (\alpha b_1 - a_1 \beta)(\beta - b_1)}}{\beta - b_1}$$

we can reduce this integral to an integral of the form $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{c(t^2+q)}}$, where $P(t)$ is a polynomial of degree no higher than

$2m-1$. The integral $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{t^2+q}}$ can be reduced to the sum of integrals of the forms $\int \frac{t dt}{(t^2+p)^k \sqrt{t^2+q}}$ and $\int \frac{dt}{(t^2+p)^k \sqrt{t^2+q}}$.

If $b_1 = \beta$, we can reduce it to integrals of the form $\int \frac{P(t) dt}{(t^2+p)^m \sqrt{c(t^2+q)}}$ by means of the substitution $t = x + \frac{b_1}{2}$.

The integral $\int \frac{t dt}{(t^2+p)^k \sqrt{c(t^2+q)}}$ can be evaluated by means of the substitution $t^2 + q = u^2$.

The integral $\int \frac{dt}{(t^2+p)^k \sqrt{c(t^2+q)}}$ can be evaluated by means of the substitution $\frac{t}{\sqrt{t^2+q}} = v$ (see also 2.283).

2.26 Forms containing $\sqrt{a + bx + cx^2}$ and integral powers of x

$$\text{Notation: } R = a + bx + cx^2, \quad \Delta = 4ac - b^2$$

Simplified formulas for the case $b = 0$. See 2.27.

2.260

$$1. \int x^m \sqrt{R^{2n+1}} dx = \frac{x^{m-1} \sqrt{R^{2n+3}}}{(m+2n+2)c} - \frac{(2m+2n+1)b}{2(m+2n+2)c} \int x^{m-1} \sqrt{R^{2n+1}} dx - \frac{(m-1)a}{(m+2n+2)c} \int x^{m-2} \sqrt{R^{2n+1}} dx.$$

TI (192)a
FI II 78-82

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$$2. \int \sqrt{R^{2n+1}} dx = \frac{2cx + b}{4(n+1)c} \sqrt{R^{2n+1}} + \frac{2n+1}{8(n+1)} \frac{\Delta}{c} \int \sqrt{R^{2n-1}} dx.$$

TI (188)

$$3. \int \sqrt{R^{2n+1}} dx = \frac{(2cx + b)\sqrt{R}}{4(n+1)c} \left\{ R^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\dots(2n-2k+1)}{8^{k+1}n(n-1)\dots(n-k)} \left(\frac{\Delta}{c}\right)^{k+1} R^{n-k-1} \right\} + \frac{(2n+1)!!}{8^{n+1}(n+1)!} \left(\frac{\Delta}{c}\right)^{n+1} \int \frac{dx}{\sqrt{R}}.$$

TI (190)

2.261

⁶ For $n = -1$

$$\begin{aligned} \int \frac{dx}{\sqrt{R}} &= \frac{1}{\sqrt{c}} \ln(2\sqrt{cR} + 2cx + b) \quad [c > 0]; \\ &= \frac{1}{\sqrt{c}} \operatorname{Arsh} \frac{2cx + b}{\sqrt{\Delta}} \quad [c > 0, \quad \Delta > 0]; \\ &= \frac{-1}{\sqrt{-c}} \arcsin \frac{2cx + b}{\sqrt{-\Delta}} \quad [c < 0, \quad \Delta < 0]; \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sqrt{c}} \ln(2cx + b) \quad [c > 0, \quad \Delta = 0]. \\ &\text{Printed from Table of Integrals, Series, and Products CD-ROM. Copyright } \textcircled{c} \text{ 1996 by Academic Press, Inc. All rights reserved.} \end{aligned}$$

2.262

$$1. \int \sqrt{R} dx = \frac{(2cx + b)\sqrt{R}}{4c} + \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$2. \int x\sqrt{R} dx = \frac{\sqrt{R^3}}{3c} - \frac{(2cx + b)b}{8c^2} \sqrt{R} - \frac{b\Delta}{16c^2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$3. \int x^2 \sqrt{R} dx = \left(\frac{x}{4c} - \frac{5b}{24c^2} \right) \sqrt{R^3} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c} \right) \frac{(2cx + b)\sqrt{R}}{4c} + \left(\frac{5b^2}{16c^2} - \frac{a}{4c} \right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$4. \int x^3 \sqrt{R} dx = \left(\frac{x^2}{5c} - \frac{7bx}{40c^2} + \frac{7b^2}{48c^3} - \frac{2a}{15c^2} \right) \sqrt{R^3} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2} \right) \frac{(2cx + b)\sqrt{R}}{4c} - \left(\frac{7b^3}{32c^3} - \frac{3ab}{8c^2} \right) \frac{\Delta}{8c} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$5. \int \sqrt{R^3} dx = \left(\frac{R}{8c} + \frac{3\Delta}{64c^2} \right) (2cx + b)\sqrt{R} + \frac{3\Delta^2}{128c^2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

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$$6. \int x\sqrt{R^3} dx = \frac{\sqrt{R^5}}{5c} - (2cx + b) \left(\frac{b}{16c^2} \sqrt{R^3} + \frac{3\Delta b}{128c^3} \sqrt{R} \right) - \frac{3\Delta^2 b}{256c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

$$7. \int x^2 \sqrt{R^3} dx = \left(\frac{x}{6c} - \frac{7b}{60c^2} \right) \sqrt{R^5} + \left(\frac{7b^2}{24c^2} - \frac{a}{6c} \right) \left(2x + \frac{b}{c} \right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R} \right) + \left(\frac{7b^2}{4c} - a \right) \frac{\Delta^2}{256c^3} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

$$8. \int x^3 \sqrt{R^3} dx = \left(\frac{x^2}{7c} - \frac{3bx}{28c^2} + \frac{3b^2}{40c^3} - \frac{2a}{35c^2} \right) \sqrt{R^5} - \left(\frac{3b^3}{16c^3} - \frac{ab}{4c^2} \right) \left(2x + \frac{b}{c} \right) \left(\frac{\sqrt{R^3}}{8} + \frac{3\Delta}{64c} \sqrt{R} \right) - \left(\frac{3b^2}{4c} - a \right) \frac{3\Delta^2 b}{512c^4} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.263

$$1. \int \frac{x^m dx}{\sqrt{R^{2n+1}}} = \frac{x^{m-1}}{(m-2n)c\sqrt{R^{2n-1}}} - \frac{(2m-2n-1)b}{2(m-2n)c} \int \frac{x^{m-1} dx}{\sqrt{R^{2n+1}}} - \frac{(m-1)a}{(m-2n)c} \int \frac{x^{m-2} dx}{\sqrt{R^{2n+1}}}.$$

TI (193)a

For $m = 2n$

$$2. \int \frac{x^{2n} dx}{\sqrt{R^{2n+1}}} = -\frac{x^{2n-1}}{(2n-1)c\sqrt{R^{2n-1}}} - \frac{b}{2c} \int \frac{x^{2n-1} dx}{\sqrt{R^{2n+1}}} + \frac{1}{c} \int \frac{x^{2n-2} dx}{\sqrt{R^{2n-1}}}.$$

TI (194)a

$$3. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} + \frac{8(n-1)c}{(2n-1)\Delta} \int \frac{dx}{\sqrt{R^{2n-1}}}.$$

TI (189)

$$4. \int \frac{dx}{\sqrt{R^{2n+1}}} = \frac{2(2cx+b)}{(2n-1)\Delta\sqrt{R^{2n-1}}} \left\{ 1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\dots(n-k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \frac{c^k}{\Delta^k} R^k \right\}$$

[$n \geq 1$].

2.264

1. $\int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

2. $\int \frac{x dx}{\sqrt{R}} = \frac{\sqrt{R}}{c} - \frac{b}{2c} \int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

3. $\int \frac{x^2 dx}{\sqrt{R}} = \left(\frac{x}{2c} - \frac{3b}{4c^2} \right) \sqrt{R} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c} \right) \int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

4. $\int \frac{x^3 dx}{\sqrt{R}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2} \right) \sqrt{R} - \left(\frac{5b^3}{16c^3} - \frac{3ab}{4c^2} \right) \int \frac{dx}{\sqrt{R}}$ (see **2.261**).

2.261

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5. $\int \frac{dx}{\sqrt{R^3}} = \frac{2(2cx + b)}{\Delta \sqrt{R}}$.

6. $\int \frac{x dx}{\sqrt{R^3}} = -\frac{2(2a + bx)}{\Delta \sqrt{R}}$.

7. $\int \frac{x^2 dx}{\sqrt{R^3}} = -\frac{(\Delta - b^2)x - 2ab}{c\Delta \sqrt{R}} + \frac{1}{c} \int \frac{dx}{\sqrt{R}}$ (see **2.261**).

$$8. \int \frac{x^3 dx}{\sqrt{R^3}} = \frac{c\Delta x^2 + b(10ac - 3b^2)x + a(8ac - 3b^2)}{c^2\Delta\sqrt{R}} - \frac{3b}{2c^2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261}).$$

2.261

2.265

$$\int \frac{\sqrt{R^{2n+1}}}{x^m} dx = -\frac{\sqrt{R^{2n+3}}}{(m-1)ax^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-1}} dx + \frac{(2n-m+4)c}{(m-1)a} \int \frac{\sqrt{R^{2n+1}}}{x^{m-2}} dx.$$

TI (195)

For $m = 1$

$$\int \frac{\sqrt{R^{2n+1}}}{x} dx = \frac{\sqrt{R^{2n+1}}}{2n+1} + \frac{b}{2} \int \sqrt{R^{2n-1}} dx + a \int \frac{\sqrt{R^{2n-1}}}{x} dx.$$

TI (198)

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^m} dx = \frac{2\sqrt{(bx+cx^2)^{2n+3}}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)c}{(2n-2m+3)b} \int \frac{\sqrt{(bx+cx^2)^{2n+1}}}{x^{m-1}} dx.$$

LA 169 (3)

For $m = 0$ see 2.260 2. and 2.260 3.

For $n = -1$ and $m = 1$:

2.266⁸

$$\begin{aligned} \int \frac{dx}{x\sqrt{R}} &= -\frac{1}{\sqrt{a}} \ln \frac{2a+bx+2\sqrt{aR}}{x} \quad [a > 0]; \\ &= \frac{1}{\sqrt{-a}} \arcsin \frac{2a+bx}{x\sqrt{b^2-4ac}} \quad [a < 0, \quad \Delta < 0]; \\ &= \frac{1}{\sqrt{-a}} \operatorname{arctg} \frac{2a+bx}{2\sqrt{-a}\sqrt{R}} \quad [a < 0]; \\ &= -\frac{1}{\sqrt{a}} \operatorname{Arsh} \frac{2a+bx}{x\sqrt{\Delta}} \quad [a > 0, \quad \Delta > 0]; \\ &= -\frac{1}{\sqrt{a}} \operatorname{Arth} \frac{2a+bx}{2\sqrt{a}\sqrt{R}} \quad [a > 0]; \\ &= \frac{1}{\sqrt{a}} \ln \frac{x}{2a+bx} \quad [a > 0, \quad \Delta = 0]; \\ &= -\frac{2\sqrt{bx+cx^2}}{bx} \quad [a = 0, \quad b \neq 0]. \\ &= -\frac{1}{\sqrt{a}} \operatorname{Arch} \left(\frac{2a+bx}{x\sqrt{\Delta}} \right) \quad [a > 0, \quad \Delta < 0]. \end{aligned}$$

2.267

$$1. \int \frac{\sqrt{R} dx}{x} = \sqrt{R} + a \int \frac{dx}{x\sqrt{R}} + \frac{b}{2} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266}).$$

2.266

2.261

$$2. \int \frac{\sqrt{R} dx}{x^2} = -\frac{\sqrt{R}}{x} + \frac{b}{2} \int \frac{dx}{x\sqrt{R}} + c \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266}).$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{bx + cx^2}}{x^2} dx = -\frac{2\sqrt{bx + cx^2}}{x} + c \int \frac{dx}{\sqrt{bx + cx^2}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$3. \int \frac{\sqrt{R} dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4ax}\right) \sqrt{R} - \left(\frac{b^2}{8a} - \frac{c}{2}\right) \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266}).$$

2.266

For $a = 0$

$$\int \frac{\sqrt{bx + cx^2}}{x^3} dx = -\frac{2\sqrt{(bx + cx^2)^3}}{3bx^3}$$

2.266

2.261

$$5. \int \frac{\sqrt{R^3}}{x^2} dx = -\frac{\sqrt{R^5}}{ax} + \frac{cx+b}{a}\sqrt{R^3} + \frac{3}{4}(2cx+3b)\sqrt{R} + \frac{3}{2}ab \int \frac{dx}{x\sqrt{R}} + \frac{3(4ac+b^2)}{8} \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266}).$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^2} = \frac{\sqrt{(bx+cx^2)^3}}{2x} + \frac{3b}{4}\sqrt{bx+cx^2} + \frac{3b^2}{8} \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{see } \mathbf{2.261}).$$

2.261

$$6. \int \frac{\sqrt{R^3}}{x^3} dx = -\left(\frac{1}{2ax^2} + \frac{b}{4a^2x}\right)\sqrt{R^5} + \frac{bcx+2ac+b^2}{4a^2}\sqrt{R^3} + \frac{3(bc x+2ac+b^2)}{4a}\sqrt{R} + \frac{3}{8}(4ac+b^2) \int \frac{dx}{x\sqrt{R}} + \frac{3}{2}bc \int \frac{dx}{\sqrt{R}} \quad (\text{see } \mathbf{2.261} \text{ and } \mathbf{2.266}).$$

2.266

2.261

For $a = 0$

$$\int \frac{\sqrt{(bx+cx^2)^3}}{x^3} dx = \left(c - \frac{2b}{x}\right)\sqrt{bx+cx^2} + \frac{3bc}{2} \int \frac{dx}{\sqrt{bx+cx^2}} \quad (\text{see } \mathbf{2.261}).$$

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2.268

$$\int \frac{dx}{x^m \sqrt{R^{2n+1}}} = -\frac{1}{(m-1)ax^{m-1}\sqrt{R^{2n-1}}} - \frac{(2n+2m-3)b}{2(m-1)a} \int \frac{dx}{x^{m-1}\sqrt{R^{2n+1}}} - \frac{(2n+m-2)c}{(m-1)a} \int \frac{dx}{x^{m-2}\sqrt{R^{2n+1}}}.$$

2.268
TI (196)

For $m = 1$

$$\int \frac{dx}{x\sqrt{R^{2n+1}}} = \frac{1}{(2n-1)a\sqrt{R^{2n-1}}} - \frac{b}{2a} \int \frac{dx}{\sqrt{R^{2n+1}}} + \frac{1}{a} \int \frac{dx}{x\sqrt{R^{2n-1}}}.$$

TI (199)

For $a = 0$

$$\int \frac{dx}{x^m \sqrt{(bx+cx^2)^{2n+1}}} = -\frac{2}{(2n+2m-1)bx^m \sqrt{(bx+cx^2)^{2n-1}}} - \frac{(4n+2m-2)c}{(2n+2m-1)b} \int \frac{dx}{x^{m-1} \sqrt{(bx+cx^2)^{2n+1}}} \quad (\text{cf. 2.265})$$

2.265

2.269

$$1. \int \frac{dx}{x\sqrt{R}} \quad (\text{see 2.266}).$$

2.266

$$2. \int \frac{dx}{x^2\sqrt{R}} = -\frac{\sqrt{R}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{R}} \quad (\text{see 2.266})$$

For $a = 0$

$$\int \frac{dx}{x^2 \sqrt{bx + cx^2}} = \frac{2}{3} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x} \right) \sqrt{bx + cx^2}.$$

$$3. \int \frac{dx}{x^3 \sqrt{R}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x} \right) \sqrt{R} + \left(\frac{3b^2}{8a^2} - \frac{c}{2a} \right) \int \frac{dx}{x \sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

$$\int \frac{dx}{x^3 \sqrt{bx + cx^2}} = \frac{2}{5} \left(-\frac{1}{bx^3} + \frac{4c}{3b^2x^2} - \frac{8c^2}{3b^3x} \right) \sqrt{bx + cx^2}.$$

$$4. \int \frac{dx}{x \sqrt{R^3}} = -\frac{2(bc x - 2ac + b^2)}{a \Delta \sqrt{R}} + \frac{1}{a} \int \frac{dx}{x \sqrt{R}} \quad (\text{see } \mathbf{2.266})$$

For $a = 0$

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$$\int \frac{dx}{x \sqrt{(bx + cx^2)^3}} = \frac{2}{3} \left(-\frac{1}{bx} + \frac{4c}{b^2} + \frac{8c^2x}{b^3} \right) \frac{1}{\sqrt{bx + cx^2}}.$$

$$5. \int \frac{dx}{x^2 \sqrt{R^3}} = -\frac{A}{\sqrt{R}} + \frac{1}{a^2 \Delta \sqrt{R}} [(3b^2 - 8ac)cx + (3b^2 - 10ac)b] - \frac{3b}{2a^2} \int \frac{dx}{x \sqrt{R}}$$

where $A = \left(-\frac{1}{ax} - \frac{b(10ac - 3b^2)}{a^2 \Delta} - \frac{c(8ac - 3b^2)x}{a^2 \Delta} \right)$. (see **2.266**).

For $a = 0$

$$\int \frac{dx}{x^2 \sqrt{(bx + cx^2)^3}} = \frac{2}{5} \left(-\frac{1}{bx^2} + \frac{2c}{b^2x} - \frac{8c^2}{b^3} - \frac{16c^3x}{b^4} \right) \frac{1}{\sqrt{bx + cx^2}}.$$

$$6. \int \frac{dx}{x^3 \sqrt{R^3}} = \left(-\frac{1}{ax^2} + \frac{5b}{2a^2x} - \frac{15b^4 - 62acb^2 + 24a^2c^2}{2a^3\Delta} - \frac{bc(15b^2 - 52ac)x}{2a^3\Delta} \right) \frac{1}{2\sqrt{R}} + \frac{15b^2 - 12ac}{8a^3} \int \frac{dx}{x\sqrt{R}} \quad (\text{see } \mathbf{2.266}).$$

For $a = 0$

$$\int \frac{dx}{x^3 \sqrt{(bx + cx^2)^3}} = \frac{2}{7} \left(-\frac{1}{bx^3} + \frac{8c}{5b^2x^2} - \frac{16c^2}{5b^3x} + \frac{64c^3}{5b^4} + \frac{128c^4x}{5b^5} \right) \frac{1}{\sqrt{bx + cx^2}}.$$

2.27 Forms containing $\sqrt{a + cx^2}$ and integral powers of x

Notations: $u = A + Cx^2$.

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{c}} \ln(x\sqrt{c} + u) \quad [c > 0]; \\ &= \frac{1}{\sqrt{-c}} \arcsin x \sqrt{-\frac{c}{a}} \quad [c < 0 \text{ and } a > 0]. \\ I_2 &= \frac{1}{2\sqrt{a}} \ln \frac{u - \sqrt{a}}{u + \sqrt{a}} \quad [a > 0 \text{ and } c > 0]; \\ &= \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a} - u}{\sqrt{a} + u} \quad [a > 0 \text{ and } c < 0]; \\ &= \frac{1}{\sqrt{-a}} \operatorname{arcsec} x \sqrt{-\frac{c}{a}} = \frac{1}{\sqrt{-a}} \arccos \frac{1}{x} \sqrt{-\frac{a}{c}} \quad [a < 0 \text{ and } c > 0]. \end{aligned}$$

2.271

$$1. \int u^5 dx = \frac{1}{6}xu^5 + \frac{5}{24}axu^3 + \frac{5}{16}a^2xu + \frac{5}{16}a^3I_1.$$

$$2. \int u^3 dx = \frac{1}{4}xu^3 + \frac{3}{8}axu + \frac{3}{8}a^2I_1.$$

DW

$$3. \int u dx = \frac{1}{2}xu + \frac{1}{2}aI_1.$$

DW

$$4. \int \frac{dx}{u} = I_1.$$

DW

$$5. \int \frac{dx}{u^3} = \frac{1}{a} \frac{x}{u}$$

DW

$$6. \int \frac{dx}{u^{2n+1}} = \frac{1}{a^n} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} \binom{n-1}{k} \frac{c^k x^{2k+1}}{u^{2k+1}}.$$

$$7. \int \frac{x dx}{u^{2n+1}} = -\frac{1}{(2n-1)cu^{2n-1}}.$$

DW

2.272

$$1. \int x^2 u^3 dx = \frac{1}{6} \frac{xu^5}{c} - \frac{1}{24} \frac{axu^3}{c} - \frac{1}{16} \frac{a^2xu}{c} - \frac{1}{16} \frac{a^3}{c} I_1.$$

DW

$$2. \int x^2 u dx = \frac{1}{4} \frac{xu^3}{c} - \frac{1}{8} \frac{axu}{c} - \frac{1}{8} \frac{a^2}{c} I_1.$$

$$3. \int \frac{x^2}{u} dx = \frac{1}{2} \frac{xu}{c} - \frac{1}{2} \frac{a}{c} I_1.$$

DW

$$4. \int \frac{x^2}{u^3} dx = -\frac{x}{cu} + \frac{1}{c} I_1.$$

DW

$$5. \int \frac{x^2}{u^5} dx = \frac{1}{3} \frac{x^3}{au^3}.$$

DW

$$6. \int \frac{x^2 dx}{u^{2n+1}} = \frac{1}{a^{n-1}} \sum_{k=0}^{n-2} \frac{(-1)^k}{2k+3} \binom{n-2}{k} \frac{c^k x^{2k+3}}{u^{2k+3}}.$$

$$7. \int \frac{x^3 dx}{u^{2n+1}} = -\frac{1}{(2n-3)c^2 u^{2n-3}} + \frac{a}{(2n-1)c^2 u^{2n-1}}.$$

DW

2.273

$$1. \int x^4 u^3 dx = \frac{1}{8} \frac{x^3 u^5}{c} - \frac{axu^5}{16c^2} + \frac{a^2 x u^3}{64c^2} + \frac{3a^3 x u}{128c^2} + \frac{3a^4}{128c^2} I_1.$$

DW

$$2. \int x^4 u dx = \frac{1}{6} \frac{x^3 u^3}{c} - \frac{axu^3}{8c^2} + \frac{a^2 x u}{16c^2} + \frac{a^3}{16c^2} I_1.$$

DW

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$$3. \int \frac{x^4}{u} dx = \frac{1}{4} \frac{x^3 u}{c} - \frac{3}{8} \frac{axu}{c^2} + \frac{3}{8} \frac{a^2}{c^2} I_1.$$

DW

DW

$$5. \int \frac{x^4}{u^5} dx = -\frac{x}{c^2 u} - \frac{1}{3} \frac{x^3}{c u^3} + \frac{1}{c^2} I_1.$$

DW

$$6. \int \frac{x^4}{u^7} dx = \frac{1}{5} \frac{x^5}{a u^5}.$$

DW

$$7. \int \frac{x^4 dx}{u^{2n+1}} = \frac{1}{a^{n-2}} \sum_{k=0}^{n-3} \frac{(-1)^k}{2k+5} \binom{n-3}{k} \frac{c^k x^{2k+5}}{u^{2k+5}}.$$

$$8. \int \frac{x^5 dx}{u^{2n+1}} = -\frac{1}{(2n-5)c^3 u^{2n-5}} + \frac{2a}{(2n-3)c^3 u^{2n-3}} - \frac{a^2}{(2n-1)c^3 u^{2n-1}}.$$

DW

2.274

$$1. \int x^6 u^3 dx = \frac{1}{10} \frac{x^5 u^5}{c} - \frac{a x^3 u^5}{16 c^2} + \frac{a^2 x u^5}{32 c^3} - \frac{a^3 x u^3}{128 c^3} - \frac{3 a^4 x u}{256 c^3} - \frac{3}{256} \frac{a^5}{c^3} I_1.$$

$$2. \int x^6 u dx = \frac{1}{8} \frac{x^5 u^3}{c} - \frac{5}{48} \frac{a x^3 u^3}{c^2} + \frac{5 a^2 x u^3}{64 c^3} - \frac{5 a^3 x u}{128 c^3} - \frac{5}{128} \frac{a^4}{c^3} I_1.$$

$$3. \int \frac{x^6}{u} dx = \frac{1}{6} \frac{x^5 u}{c} - \frac{5}{24} \frac{a x^3 u}{c^2} + \frac{5}{16} \frac{a^2 x u}{c^3} - \frac{5}{16} \frac{a^3}{c^3} I_1.$$

DW

$$4. \int \frac{x^6}{u^3} dx = \frac{1}{4} \frac{x^5}{c u} - \frac{5}{8} \frac{a x^3}{c^2 u} - \frac{15}{8} \frac{a^2 x}{c^3 u} + \frac{15}{8} \frac{a^2}{c^3} I_1.$$

DW

DW

$$6. \int \frac{x^6}{u^7} dx = -\frac{23}{15} \frac{x^5}{cu^5} - \frac{7}{3} \frac{ax^3}{c^2u^5} - \frac{a^2x}{c^3u^5} + \frac{1}{c^3} I_1.$$

DW

$$7. \int \frac{x^6}{u^9} dx = \frac{1}{7} \frac{x^7}{au^7}.$$

DW

$$8. \int \frac{x^6 dx}{u^{2n+1}} = \frac{1}{a^{n-3}} \sum_{k=0}^{n-4} \frac{(-1)^k}{2k+7} \binom{n-4}{k} \frac{c^k x^{2k+7}}{u^{2k+7}}.$$

$$9. \int \frac{x^7 dx}{u^{2n+1}} = -\frac{1}{(2n-7)c^4u^{2n-7}} + \frac{3a}{(2n-5)c^4u^{2n-5}} - \frac{3a^2}{(2n-3)c^4u^{2n-3}} + \frac{a^3}{(2n-1)c^4u^{2n-1}}.$$

DW

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2.275

$$1. \int \frac{u^5}{x} dx = \frac{u^5}{5} + \frac{1}{3} au^3 + a^2u + a^3 I_2$$

DW

$$2. \int \frac{u^3}{x} dx = \frac{u^3}{3} + au + a^2 I_2.$$

DW

$$3. \int \frac{u}{x} dx = u + a I_2.$$

DW

$$4. \int \frac{dx}{xu} = I_2.$$

$$5. \int \frac{dx}{xu^{2n+1}} = \frac{1}{a^n} I_2 + \sum_{k=0}^{n-1} \frac{1}{(2k+1)a^{n-k}u^{2k+1}}.$$

$$6. \int \frac{u^5}{x^2} dx = -\frac{u^5}{x} + \frac{5}{4}cxu^3 + \frac{15}{8}acxu + \frac{15}{8}a^2I_1.$$

DW

$$7. \int \frac{u^3}{x^2} dx = -\frac{u^3}{x} + \frac{3}{2}cxu + \frac{3}{2}aI_1.$$

DW

$$8. \int \frac{u}{x^2} dx = -\frac{u}{x} + cI_1.$$

DW

$$9. \int \frac{dx}{x^2u^{2n+1}} = -\frac{1}{a^{n+1}} \left\{ \frac{u}{x} + \sum_{k=1}^n \frac{(-1)^{k+1}}{2k-1} \binom{n}{k} c^k \left(\frac{x}{u}\right)^{2k-1} \right\}.$$

2.276

$$1. \int \frac{u^5}{x^3} dx = -\frac{u^5}{2x^2} + \frac{5}{6}cu^3 + \frac{5}{2}acu + \frac{5}{2}a^2cI_2.$$

DW

$$2. \int \frac{u^3}{x^3} dx = -\frac{u^3}{2x^2} + \frac{3}{2}cu + \frac{3}{2}acI_2.$$

DW

$$3. \int \frac{u}{x^3} dx = -\frac{u}{2x^2} + \frac{c}{2}I_2.$$

DW

$$4. \int \frac{dx}{x^3u} = -\frac{u}{2ax^2} - \frac{c}{2a}I_2.$$

$$5. \int \frac{dx}{x^3 u^3} = -\frac{1}{2ax^2 u} - \frac{3c}{2a^2 u} - \frac{3c}{2a^2} I_2.$$

DW

$$6. \int \frac{dx}{x^3 u^5} = -\frac{1}{2ax^2 u^3} - \frac{5c}{6a^2 u^3} - \frac{5c}{2a^3 u} - \frac{5c}{2a^3} I_2.$$

DW

$$7. \int \frac{u^5}{x^4} dx = -\frac{au^3}{3x^3} - \frac{2acu}{x} + \frac{c^2 xu}{2} + \frac{5}{2} ac I_1.$$

DW

108

$$8. \int \frac{u^3}{x^4} dx = -\frac{u^3}{3x^3} - \frac{cu}{x} + c I_1.$$

DW

$$9. \int \frac{u}{x^4} dx = -\frac{u^3}{3ax^3}.$$

DW

$$10. \int \frac{dx}{x^4 u^{2n+1}} = \frac{1}{a^{n+2}} \left\{ -\frac{u^3}{3x^3} + (n+1) \frac{cu}{x} + \sum_{k=2}^{n+1} \frac{(-1)^k}{2k-3} \binom{n+1}{k} c^k \left(\frac{x}{u}\right)^{2k-3} \right\}.$$

2.277

$$1. \int \frac{u^3}{x^5} dx = -\frac{u^3}{4x^4} - \frac{3cu^3}{8ax^2} + \frac{3c^2 u}{8a} + \frac{3}{8} c^2 I_2.$$

DW

$$2. \int \frac{u}{x^5} dx = -\frac{u}{4x^4} - \frac{1}{8} \frac{cu}{ax^2} - \frac{1}{8} \frac{c^2}{a} I_2.$$

$$3. \int \frac{dx}{x^5 u} = -\frac{u}{4ax^4} + \frac{3}{8} \frac{cu}{a^2 x^2} + \frac{3}{8} \frac{c^2}{a^2} I_2.$$

DW

$$4. \int \frac{dx}{x^5 u^3} = -\frac{1}{4ax^4 u} + \frac{5}{8} \frac{c}{a^2 x^2 u} + \frac{15}{8} \frac{c^2}{a^3 u} + \frac{15}{8} \frac{c^2}{a^3} I_2.$$

DW

2.278

$$1. \int \frac{u^3}{x^6} dx = -\frac{u^5}{5ax^5}.$$

DW

$$2. \int \frac{u}{x^6} dx = -\frac{u^3}{5ax^5} + \frac{2}{15} \frac{cu^3}{a^2 x^3}.$$

DW

$$3. \int \frac{dx}{x^6 u} = \frac{1}{a^3} \left(-\frac{u^5}{5x^5} + \frac{2}{3} \frac{cu^3}{x^3} - \frac{c^2 u}{x} \right).$$

DW

$$4. \int \frac{dx}{x^6 u^{2n+1}} = \frac{1}{a^{n+3}} \left\{ -\frac{u^5}{5x^5} + \frac{1}{3} \binom{n+2}{1} \frac{cu^3}{x^3} - \binom{n+2}{2} \frac{c^2 u}{x} + \sum_{k=3}^{n+2} \frac{(-1)^k}{2k-5} \binom{n+2}{k} c^k \left(\frac{x}{u} \right)^{2k-5} \right\}.$$

2.28 Forms containing $\sqrt{a + bx + cx^2}$ and first- and second-degree polynomials

Notation: $R = A + Bx + Cx^2$

See also 2.252.

2.281³

$$\int \frac{dx}{(x+p)^n \sqrt{R}} = -\int \frac{t^{n-1} dt}{\sqrt{c + (b-2pc)t + (a-bp+cp^2)t^2}} \quad \left[t = \frac{1}{x+p} > 0 \right].$$

$$1.^3 \int \frac{\sqrt{R} dx}{x+p} = c \int \frac{x dx}{\sqrt{R}} + (b-cp) \int \frac{dx}{\sqrt{R}} + (a-bp+cp^2) \int \frac{dx}{(x+p)\sqrt{R}} \quad [x+p > 0].$$

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$$2. \int \frac{dx}{(x+p)(x+q)\sqrt{R}} = \frac{1}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{1}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}.$$

$$3. \int \frac{\sqrt{R} dx}{(x+p)(x+q)} = \frac{1}{q-p} \int \frac{\sqrt{R} dx}{x+p} + \frac{1}{p-q} \int \frac{\sqrt{R} dx}{x+q}.$$

$$4. \int \frac{(x+p)\sqrt{R} dx}{x+q} = \int \sqrt{R} dx + (p-q) \int \frac{\sqrt{R} dx}{x+q}.$$

$$5. \int \frac{(rx+s) dx}{(x+p)(x+q)\sqrt{R}} = \frac{s-pr}{q-p} \int \frac{dx}{(x+p)\sqrt{R}} + \frac{s-qr}{p-q} \int \frac{dx}{(x+q)\sqrt{R}}.$$

2.283

$$\int \frac{(Ax+B) dx}{(p+R)^n \sqrt{R}} = \frac{A}{c} \int \frac{du}{(p+u^2)^n} + \frac{2Bc-Ab}{2c} \int \frac{(1-cv^2)^{n-1} dv}{\left[p+a-\frac{b^2}{4c}-cpv^2 \right]^n},$$

where $u = \sqrt{R}$ and $v = \frac{b+2cx}{2c\sqrt{R}}$.

2.284

$$\int \frac{Ax+B}{(p+R)\sqrt{R}} dx = \frac{A}{c} I_1 + \frac{2Bc-Ab}{\sqrt{c^2 p [b^2 - 4(a+p)c]}} I_2,$$

where

$$I_1 = \frac{1}{\sqrt{p}} \operatorname{arctg} \sqrt{\frac{R}{p}} \quad [p > 0]; \quad = \frac{1}{2\sqrt{-p}} \ln \frac{\sqrt{-p} - \sqrt{R}}{\sqrt{-p} + \sqrt{R}} \quad [p < 0].$$

$$\begin{aligned}
I_2 &= \operatorname{arctg} \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} \quad [p\{b^2 - 4(a+p)c\} > 0, \quad p < 0]; \\
&= -\operatorname{arctg} \sqrt{\frac{p}{b^2 - 4(a+p)c}} \frac{b + 2cx}{\sqrt{R}} \quad [p\{b^2 - 4(a+p)c\} > 0, \quad p > 0]; \\
&= \frac{1}{2i} \ln \frac{\sqrt{4(a+p)c - b^2} \sqrt{R} + \sqrt{p}(b + 2cx)}{\sqrt{4(a+p)c - b^2} \sqrt{R} - \sqrt{p}(b + 2cx)} \quad [p\{b^2 - 4(a+p)c\} < 0, \quad p > 0]; \\
&= \frac{1}{2i} \ln \frac{\sqrt{b^2 - 4(a+p)c} \sqrt{R} - \sqrt{-p}(b + 2cx)}{\sqrt{b^2 - 4(a+p)c} \sqrt{R} + \sqrt{-p}(b + 2cx)} \quad [p\{b^2 - 4(a+p)c\} < 0, \quad p < 0].
\end{aligned}$$

2.29 Integrals that can be reduced to elliptic or pseudo-elliptic integrals

2.290

Integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial can, by means of algebraic transformations, be reduced to a sum of integrals expressed in terms of elementary functions and elliptic integrals (see 8.11). Since the substitutions that transform the given integral into an elliptic integral in the normal Legendre form are different for different intervals of integration, the corresponding formulas are given in the chapter on definite integrals (see 3.13-3.17).

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2.291

Certain integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $k \geq 2$ and $P_n(x)$ is a polynomial of not more than fourth degree, can be reduced to integrals of the form $\int R(x, \sqrt[k]{P_n(x)}) dx$. Below are examples of this procedure.

$$1. \int \frac{dx}{\sqrt{1-x^6}} = -\int \frac{dz}{\sqrt{3+3z^2+z^4}} \quad \left[x^2 = \frac{1}{1+z^2} \right].$$

$$2. \int \frac{dx}{\sqrt{a+bx^2+cx^4+dx^6}} = \frac{1}{2} \int \frac{dz}{\sqrt{az+bz^2+cz^3+dz^4}} \quad [x^2 = z].$$

$$3. \int (a+2bx+cx^2+gx^3)^{\pm 1/3} dx = \frac{3}{2} \int \frac{z^2 A^{\pm 1/3} dz}{B}$$

$$\left[a+2bx+cx^2 = z^3, \quad A = g \left(\frac{-b + \sqrt{b^2 + (z^3 - a)c}}{c} \right)^3 + z^3, \quad B = \sqrt{b^2 + (z^3 - a)c} \right]$$

$$\begin{aligned}
4. \int \frac{dx}{\sqrt{a+bx+cx^2+dx^3+cx^4+bx^5+ax^6}} &= \\
&= -\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(z+1)p}} - \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x = z + \sqrt{z^2-1}]; \\
&= -\frac{1}{\sqrt{2}} \int \frac{d}{\sqrt{(z+1)p}} + \frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [x = z - \sqrt{z^2-1}],
\end{aligned}$$

where

$$p = 2a(4z^3 - 3z) + 2b(2z^2 - 1) + 2cz + d.$$

$$\begin{aligned}
5. \int \frac{dx}{\sqrt{a+bx^2+cx^4+bx^6+ax^8}} &= \frac{1}{2} \int \frac{dy}{\sqrt{y}\sqrt{a+by+cy^2+by^3+ay^4}} \quad [x = \sqrt{y}]; \\
&= -\frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} + \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [y = z + \sqrt{z^2-1}]; \\
&= \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z+1)p}} - \frac{1}{2\sqrt{2}} \int \frac{dz}{\sqrt{(z-1)p}} \quad [y = z - \sqrt{z^2-1}],
\end{aligned}$$

where $p = 2a(2z^2 - 1) + 2bz + c$.

$$\begin{aligned}
6. \int \frac{dx}{\sqrt{a+bx^4+cx^8}} &= \frac{1}{2} \sqrt[8]{\frac{a}{c}} \int \frac{dt}{\sqrt{t}\sqrt{a+b_1t^2+at^4}} \quad \left[x = \sqrt[8]{\frac{a}{c}} \sqrt[4]{t} \right]; \\
&= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} - \int \frac{dz}{\sqrt{(z-1)p}} \right\} \quad [t = z + \sqrt{z^2-1}]; \\
&= -\frac{1}{2\sqrt{2}} \sqrt[8]{\frac{a}{c}} \left\{ \int \frac{dz}{\sqrt{(z+1)p}} + \int \frac{dz}{\sqrt{(z-1)p}} \right\} \quad [t = z - \sqrt{z^2-1}],
\end{aligned}$$

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where $p = 2a(2z^2 - 1) + b_1$; $b_1 = b\sqrt[8]{\frac{a}{c}}$.

$$7. \int \frac{x dx}{\sqrt[4]{a+bx^2+cx^4}} = 2 \int \frac{z^2 dz}{\sqrt{A+Bz^4}} \quad [a+bx^2+cx^4 = z^4, \quad A = b^2-4ac, \quad B = 4c].$$

$$8. \int \frac{dx}{\sqrt[4]{a+2bx^2+cx^4}} = \int \frac{\sqrt{b^2-a(c-z^4)}+b}{(c-z^4)\sqrt{b^2-a(c-z^4)}} z^2 dz = \int R_1(z^4)z^2 dz + \int \frac{R_2(z^4)z^2 dz}{\sqrt{b^2-a(c-z^4)}},$$

where $R_1(z^4)$ and $R_2(z^4)$ are rational functions of z^4 ; $a + 2bx^2 + cx^4 = x^4 z^4$.

2.292

In certain cases, integrals of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be expressed in terms of elementary functions. Such integrals are called *pseudo-elliptic* integrals.

Thus, if the relations

$$f_1(x) = -f_1\left(\frac{1}{k^2 x}\right), \quad f_2(x) = -f_2\left(\frac{1 - k^2 x}{k^2(1 - x)}\right), \quad f_3(x) = -f_3\left(\frac{1 - x}{1 - k^2 x}\right),$$

hold, then

$$1. \int \frac{f_1(x) dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_1(z) dz \quad [zx = \sqrt{x(1-x)(1-k^2x)}];$$

$$2. \int \frac{f_2(x) dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_2(z) dz \quad \left[z = \frac{\sqrt{x(1-k^2x)}}{\sqrt{1-x}} \right];$$

$$3. \int \frac{f_3(x) dx}{\sqrt{x(1-x)(1-k^2x)}} = \int R_3(z) dz \quad \left[z = \frac{\sqrt{x(1-x)}}{\sqrt{1-k^2x}} \right],$$

where $R_1(z)$, $R_2(z)$, and $R_3(z)$ are rational functions of z .

2.3 The Exponential Function

2.31 Forms containing e^{ax}

2.311

2.312

a^x in the integrands should be replaced with $e^{x \ln a} = a^x$.

112

2.313

$$1. \int \frac{dx}{a + be^{mx}} = \frac{1}{am} [mx - \ln(a + be^{mx})].$$

PE (410)

$$2. \int \frac{dx}{1 + e^x} = \ln \frac{e^x}{1 + e^x} = x - \ln(1 + e^x).$$

PE (409)

2.314

$$\begin{aligned} \int \frac{dx}{ae^{mx} + be^{-mx}} &= \frac{1}{m\sqrt{ab}} \operatorname{arctg} \left(e^{mx} \sqrt{\frac{a}{b}} \right) \quad [ab > 0]; \\ &= \frac{1}{2m\sqrt{-ab}} \ln \frac{b + e^{mx} \sqrt{-ab}}{b - e^{mx} \sqrt{-ab}} \quad [ab < 0]. \end{aligned}$$

PE (411)

2.315

$$\begin{aligned} \int \frac{dx}{\sqrt{a + be^{mx}}} &= \frac{1}{m\sqrt{a}} \ln \frac{\sqrt{a + be^{mx}} - \sqrt{a}}{\sqrt{a + be^{mx}} + \sqrt{a}} \quad [a > 0]; \\ &= \frac{2}{m\sqrt{-a}} \operatorname{arctg} \frac{\sqrt{a + be^{mx}}}{\sqrt{-a}} \quad [a < 0]. \end{aligned}$$

2.32 The exponential combined with rational functions of x

2.321

$$1. \int x^m e^{ax} dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} dx.$$

$$2. \int x^n e^{ax} dx = e^{ax} \left(\frac{x^n}{a} + \sum_{k=1}^n (-1)^k \frac{n(n-1)\dots(n-k+1)}{a^{k+1}} x^{n-k} \right).$$

2.322

$$1. \int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right).$$

$$2. \int x^2 e^{ax} dx = e^{ax} \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right).$$

$$3. \int x^3 e^{ax} dx = e^{ax} \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4} \right).$$

$$4^*. \int x^4 e^{ax} dx = e^{ax} \left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{24}{a^5} \right).$$

2.323

$$\int P_m(x) e^{ax} dx = \frac{e^{ax}}{a} \sum_{k=0}^m (-1)^k \frac{P^{(k)}(x)}{a^k},$$

where $P_m(x)$ is a polynomial in x of degree m and $P^{(k)}(x)$ is the k -th derivative of $P_m(x)$ with respect to x .

113

2.324

$$1. \int \frac{e^{ax} dx}{x^m} = \frac{1}{m-1} \left[-\frac{e^{ax}}{x^{m-1}} + a \int \frac{e^{ax} dx}{x^{m-1}} \right].$$

$$2. \int \frac{e^{ax}}{x^n} dx = -e^{ax} \sum_{k=1}^{n-1} \frac{a^{k-1}}{(n-1)(n-2)\dots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!} \text{Ei}(ax).$$

2.325

$$1. \int \frac{e^{ax}}{x} dx = \text{Ei}(ax).$$

2.326

$$\int \frac{x e^{ax} dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$$

2.33⁸

$$\int e^{-(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-ac}{a}\right) \operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) \quad [a \neq 0]$$

2.4 Hyperbolic Functions

2.41-2.43 Powers of $\operatorname{sh} x$, $\operatorname{ch} x$, $\operatorname{th} x$ and $\operatorname{cth} x$

2.411

$$\begin{aligned} \int \operatorname{sh}^p x \operatorname{ch}^q x dx &= \\ &= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \operatorname{sh}^p x \operatorname{ch}^{q-2} x dx; \\ &= \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{p+q} - \frac{p-1}{p+q} \int \operatorname{sh}^{p-2} x \operatorname{ch}^q x dx; \\ &= \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{q+1} - \frac{p-1}{q+1} \int \operatorname{sh}^{p-2} x \operatorname{ch}^{q+2} x dx; \\ &= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+1} - \frac{q-1}{p+1} \int \operatorname{sh}^{p+2} x \operatorname{ch}^{q-2} x dx; \\ &= \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{p+1} - \frac{p+q+2}{p+1} \int \operatorname{sh}^{p+2} x \operatorname{ch}^q x dx; \\ &= -\frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \operatorname{sh}^p x \operatorname{ch}^{q+2} x dx. \end{aligned}$$

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2.412

$$\begin{aligned} 1. \int \operatorname{sh}^p x \operatorname{ch}^{2n} x dx &= \frac{\operatorname{sh}^{p+1} x}{2n+p} \left\{ \operatorname{ch}^{2n-1} x + \right. \\ &+ \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \operatorname{ch}^{2n-2k-1} x \left. \right\} + \\ &+ \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \operatorname{sh}^p x dx \end{aligned}$$

number and $n = 0$, we have

$$2. \int \text{sh}^{2m} x \, dx = (-1)^m \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \frac{\text{sh}(2m-2k)x}{2m-2k}.$$

TI (543)

$$3. \int \text{sh}^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \frac{\text{ch}(2m-2k+1)x}{2m-2k+1};$$

$$= (-1)^n \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\text{ch}^{2k+1} x}{2k+1}.$$

GU ((351)) (5)

TI (544)

$$4. \int \text{sh}^p x \text{ch}^{2n+1} x \, dx = \frac{\text{sh}^{p+1} x}{2n+p+1} \left\{ \text{ch}^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1) \dots (n-k+1) \text{ch}^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.413

$$1. \int \text{ch}^p x \text{sh}^{2n} x \, dx = \frac{\text{ch}^{p+1} x}{2n+p} \left\{ \text{sh}^{2n-1} x + \sum_{k=1}^{n-1} (-1)^k \frac{(2n-1)(2n-3) \dots (2n-2k+1) \text{sh}^{2n-2k-1} x}{(2n+p-2)(2n+p-4) \dots (2n+p-2k)} \right\} +$$

$$+ (-1)^n \frac{(2n-1)!!}{(2n+p)(2n+p-2) \dots (p+2)} \int \text{ch}^p x \, dx.$$

This formula is applicable for arbitrary real p except for the following negative even integers: $-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

$$2. \int \text{ch}^{2m} x \, dx = \binom{2m}{m} \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \frac{\text{sh}(2m-2k)x}{2m-2k}.$$

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$$3. \int \text{ch}^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \frac{\text{sh}(2m-2k+1)x}{2m-2k+1};$$

$$= \sum_{k=0}^m \binom{m}{k} \frac{\text{sh}^{2k+1} x}{2k+1}.$$

$$4. \int \operatorname{ch}^p x \operatorname{sh}^{2n+1} x \, dx = \frac{\operatorname{ch}^{p+1} x}{2n+p+1} \left\{ \operatorname{sh}^{2n} x + \sum_{k=1}^n (-1)^k \frac{2^k n(n-1) \dots (n-k+1) \operatorname{sh}^{2n-2k} x}{(2n+p-1)(2n+p-3) \dots (2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.414

$$1. \int \operatorname{sh} ax \, dx = \frac{1}{a} \operatorname{ch} ax.$$

$$2. \int \operatorname{sh}^2 ax \, dx = \frac{1}{4a} \operatorname{sh} 2ax - \frac{x}{2}.$$

$$3. \int \operatorname{sh}^3 x \, dx = -\frac{3}{4} \operatorname{ch} x + \frac{1}{12} \operatorname{ch} 3x = \frac{1}{3} \operatorname{ch}^3 x - \operatorname{ch} x.$$

$$4. \int \operatorname{sh}^4 x \, dx = \frac{3}{8} x - \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x = \frac{3}{8} x - \frac{3}{8} \operatorname{sh} x \operatorname{ch} x + \frac{1}{4} \operatorname{sh}^3 x \operatorname{ch} x.$$

$$5. \int \operatorname{sh}^5 x \, dx = \frac{5}{8} \operatorname{ch} x - \frac{5}{48} \operatorname{ch} 3x + \frac{1}{80} \operatorname{ch} 5x; \\ = \frac{4}{5} \operatorname{ch} x + \frac{1}{5} \operatorname{sh}^4 x \operatorname{ch} x - \frac{4}{15} \operatorname{ch}^3 x.$$

$$6. \int \operatorname{sh}^6 x \, dx = -\frac{5}{16} x + \frac{15}{64} \operatorname{sh} 2x - \frac{3}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x; \\ = -\frac{5}{16} x + \frac{1}{6} \operatorname{sh}^5 x \operatorname{ch} x - \frac{5}{24} \operatorname{sh}^3 x \operatorname{ch} x + \frac{5}{16} \operatorname{sh} x \operatorname{ch} x.$$

$$7. \int \operatorname{sh}^7 x \, dx = -\frac{35}{64} \operatorname{ch} x + \frac{7}{64} \operatorname{ch} 3x - \frac{7}{320} \operatorname{ch} 5x + \frac{1}{448} \operatorname{ch} 7x; \\ = -\frac{24}{35} \operatorname{ch} x + \frac{8}{35} \operatorname{ch}^3 x - \frac{6}{35} \operatorname{ch} x \operatorname{sh}^4 x + \frac{1}{7} \operatorname{ch} x \operatorname{sh}^6 x.$$

$$8. \int \operatorname{ch} ax \, dx = \frac{1}{a} \operatorname{sh} ax.$$

$$9. \int \operatorname{ch}^2 ax \, dx = \frac{x}{2} + \frac{1}{4a} \operatorname{sh} 2ax.$$

$$10. \int \operatorname{ch}^3 x \, dx = \frac{3}{4} \operatorname{sh} x + \frac{1}{12} \operatorname{sh} 3x = \operatorname{sh} x + \frac{1}{3} \operatorname{sh}^3 x.$$

$$11. \int \operatorname{ch}^4 x \, dx = \frac{3}{8}x + \frac{1}{4} \operatorname{sh} 2x + \frac{1}{32} \operatorname{sh} 4x = \frac{3}{8}x + \frac{3}{8} \operatorname{sh} x \operatorname{ch} x + \frac{1}{4} \operatorname{sh} x \operatorname{ch}^3 x.$$

$$12. \int \operatorname{ch}^5 x \, dx = \frac{5}{8} \operatorname{sh} x + \frac{5}{48} \operatorname{sh} 3x + \frac{1}{80} \operatorname{sh} 5x;$$

$$= \frac{4}{5} \operatorname{sh} x + \frac{1}{5} \operatorname{ch}^4 x \operatorname{sh} x + \frac{4}{15} \operatorname{sh}^3 x.$$

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$$13. \int \operatorname{ch}^6 x \, dx = \frac{5}{16}x + \frac{15}{64} \operatorname{sh} 2x + \frac{3}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x;$$

$$= \frac{5}{16}x + \frac{5}{16} \operatorname{sh} x \operatorname{ch} x + \frac{5}{24} \operatorname{sh} x \operatorname{ch}^3 x + \frac{1}{6} \operatorname{sh} x \operatorname{ch}^5 x.$$

$$14. \int \operatorname{ch}^7 x \, dx = \frac{35}{64} \operatorname{sh} x + \frac{7}{64} \operatorname{sh} 3x + \frac{7}{320} \operatorname{sh} 5x + \frac{1}{448} \operatorname{sh} 7x;$$

$$= \frac{24}{35} \operatorname{sh} x + \frac{8}{35} \operatorname{sh}^3 x + \frac{6}{35} \operatorname{sh} x \operatorname{ch}^4 x + \frac{1}{7} \operatorname{sh} x \operatorname{ch}^6 x.$$

2.415

$$1. \int \operatorname{sh} ax \operatorname{ch} bx \, dx = \frac{\operatorname{ch}(a+b)x}{2(a+b)} + \frac{\operatorname{ch}(a-b)x}{2(a-b)}.$$

$$2. \int \operatorname{sh} ax \operatorname{ch} ax \, dx = \frac{1}{4a} \operatorname{ch} 2ax.$$

$$3. \int \operatorname{sh}^2 x \operatorname{ch} x \, dx = \frac{1}{3} \operatorname{sh}^3 x.$$

$$4. \int \operatorname{sh}^3 x \operatorname{ch} x \, dx = \frac{1}{4} \operatorname{sh}^4 x.$$

$$5. \int \operatorname{sh}^4 x \operatorname{ch} x \, dx = \frac{1}{5} \operatorname{sh}^5 x.$$

$$6. \int \operatorname{sh} x \operatorname{ch}^2 x \, dx = \frac{1}{3} \operatorname{ch}^3 x.$$

$$7. \int \operatorname{sh}^2 x \operatorname{ch}^2 x \, dx = -\frac{x}{8} + \frac{1}{32} \operatorname{sh} 4x.$$

$$8. \int \operatorname{sh}^3 x \operatorname{ch}^2 x \, dx = \frac{1}{5} \left(\operatorname{sh}^2 x - \frac{2}{3} \right) \operatorname{ch}^3 x.$$

$$9. \int \operatorname{sh}^4 x \operatorname{ch}^2 x \, dx = \frac{x}{16} - \frac{1}{64} \operatorname{sh} 2x - \frac{1}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x.$$

$$10. \int \operatorname{sh} x \operatorname{ch}^3 x \, dx = \frac{1}{4} \operatorname{ch}^4 x.$$

$$11. \int \operatorname{sh}^2 x \operatorname{ch}^3 x \, dx = \frac{1}{5} \left(\operatorname{ch}^2 x + \frac{2}{3} \right) \operatorname{sh}^3 x.$$

$$12. \int \operatorname{sh}^3 x \operatorname{ch}^3 x \, dx = -\frac{3}{64} \operatorname{ch} 2x + \frac{1}{192} \operatorname{ch} 6x = \frac{1}{48} \operatorname{ch}^3 2x - \frac{1}{16} \operatorname{ch} 2x;$$

$$= \frac{\operatorname{sh}^6 x}{6} + \frac{\operatorname{sh}^4 x}{4} = \frac{\operatorname{ch}^6 x}{6} - \frac{\operatorname{ch}^4 x}{4}.$$

$$14. \int \operatorname{sh} x \operatorname{ch}^4 x \, dx = \frac{1}{5} \operatorname{ch}^5 x.$$

$$15. \int \operatorname{sh}^2 x \operatorname{ch}^4 x \, dx = -\frac{x}{16} - \frac{1}{64} \operatorname{sh} 2x + \frac{1}{64} \operatorname{sh} 4x + \frac{1}{192} \operatorname{sh} 6x.$$

$$16. \int \operatorname{sh}^3 x \operatorname{ch}^4 x \, dx = \frac{1}{7} \operatorname{ch}^3 x \left(\operatorname{sh}^4 x + \frac{3}{5} \operatorname{sh}^2 x - \frac{2}{5} \right) = \frac{1}{7} \left(\operatorname{sh}^2 x - \frac{2}{5} \right) \operatorname{ch}^5 x.$$

$$17. \int \operatorname{sh}^4 x \operatorname{ch}^4 x \, dx = \frac{3x}{128} - \frac{1}{128} \operatorname{sh} 4x + \frac{1}{1024} \operatorname{sh} 8x.$$

2.416

$$1. \int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{2n} x} \, dx = \frac{\operatorname{sh}^{p+1} x}{2n-1} \left\{ \operatorname{sech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{sech}^{2n-2k-1} x \right\} + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \operatorname{sh}^p x \, dx.$$

This formula is applicable for arbitrary real p . For $\int \operatorname{sh}^p x \, dx$, where p is a natural number, see 2.412 2. and 2.412 3. For $n = 0$ and p a negative integer, we have for this integral:

$$2. \int \frac{dx}{\operatorname{sh}^{2m} x} = \frac{\operatorname{ch} x}{2m-1} \left\{ -\operatorname{cosech}^{2m-1} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{2^k (m-1)(m-2)\dots(m-k)}{(2m-3)(2m-5)\dots(2m-2k-1)} \operatorname{cosech}^{2m-2k-1} x \right\}.$$

$$3. \int \frac{dx}{\operatorname{sh}^{2m+1} x} = \frac{\operatorname{ch} x}{2m} \left\{ -\operatorname{cosech}^{2m} x + \sum_{k=1}^{m-1} (-1)^{k-1} \cdot \frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k (m-1)(m-2)\dots(m-k)} \operatorname{cosech}^{2m-2k} x \right\} + \frac{(-1)^m (2m-1)!!}{(2m)!!} \ln \frac{x}{2}.$$

$$1. \int \frac{\text{sh}^p x}{\text{ch}^{2n+1} x} dx = \frac{\text{sh}^{p+1} x}{2n} \left\{ \text{sech}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \text{sech}^{2n-2k} x \right\} + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!} \int \frac{\text{sh}^p x}{\text{ch} x} dx.$$

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This formula is applicable for arbitrary real p . For $n = 0$ and p integral, we have

$$2. \int \frac{\text{sh}^{2m+1} x}{\text{ch} x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \text{sh}^{2k} x + (-1)^m \ln \text{ch} x; \\ = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k} \binom{m}{k} \text{ch}^{2k} x + (-1)^m \ln \text{ch} x \quad [m \geq 1].$$

$$3. \int \frac{\text{sh}^{2m} x}{\text{ch} x} dx = \sum_{k=1}^m \frac{(-1)^{m+k}}{2k-1} \text{sh}^{2k-1} x + (-1)^m \text{arctg}(\text{sh} x) \quad [m \geq 1].$$

$$4. \int \frac{dx}{\text{sh}^{2m+1} x \text{ch} x} = \sum_{k=1}^m \frac{(-1)^k \text{cosech}^{2m-2k+2} x}{2m-2k+2} + (-1)^m \ln \text{th} x.$$

$$5. \int \frac{dx}{\text{sh}^{2m} x \text{ch} x} = \sum_{k=1}^m \frac{(-1)^k \text{cosech}^{2m-2k+2} x}{2m-2k+1} + (-1)^m \text{arctg} \text{sh} x.$$

2.418

$$1. \int \frac{\text{ch}^p x}{\text{sh}^{2n} x} dx = -\frac{\text{ch}^{p+1} x}{2n-1} \left\{ \text{cosech}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \text{cosech}^{2n-2k-1} x \right\} + \frac{(-1)^n (2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \text{ch}^p x dx.$$

$$2. \int \frac{dx}{\text{ch}^{2m} x} = \frac{\text{sh } x}{2m-1} \left\{ \text{sech}^{2m-1} x + \sum_{k=1}^{m-1} \frac{2^k (m-1)(m-2) \dots (m-k)}{(2m-3)(2m-5) \dots (2m-2k-1)} \text{sech}^{2m-2k-1} x \right\}.$$

$$3. \int \frac{dx}{\text{ch}^{2m+1} x} = \frac{\text{sh } x}{2m} \left\{ \text{sech}^{2m} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3) \dots (2m-2k+1)}{2^k (m-1)(m-2) \dots (m-k)} \text{sech}^{2m-2k} x \right\} + \frac{(2m-1)!!}{(2m)!!} \text{arctg sh } x.$$

2.419

$$1. \int \frac{\text{ch}^p x}{\text{sh}^{2n+1} x} dx = -\frac{\text{ch}^{p+1} x}{2n} \left\{ \text{cosech}^{2n} x + \sum_{k=1}^{n-1} \frac{(-1)^k (2n-p-1)(2n-p-3) \dots (2n-p-2k+1)}{2^k (n-1)(n-2) \dots (n-k)} \text{cosech}^{2n-2k} x \right\} + \frac{(-1)^n (2n-p-1)(2n-p-3) \dots (3-p)(1-p)}{2^n n!} \int \frac{\text{ch}^p x}{\text{sh } x} dx.$$

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This formula is applicable for arbitrary real p . For $n = 0$ and p an integer

$$2. \int \frac{\text{ch}^{2m} x}{\text{sh } x} dx = \sum_{k=1}^m \frac{\text{ch}^{2k-1} x}{2k-1} + \ln \text{th} \frac{x}{2}.$$

$$3. \int \frac{\text{ch}^{2m+1} x}{\text{sh } x} dx = \sum_{k=1}^m \frac{\text{ch}^{2k} x}{2k} + \ln \text{sh } x; \\ = \sum_{k=1}^m \binom{m}{k} \frac{\text{sh}^{2k} x}{2k} + \ln \text{sh } x.$$

$$4. \int \frac{dx}{\text{sh } x \text{ch}^{2m} x} = \sum_{k=1}^m \frac{\text{sech}^{2m-2k+1} x}{2m-2k+1} + \ln \text{th} \frac{x}{2}.$$

2.421

$$1. \int \frac{\operatorname{sh}^{2n+1} x}{\operatorname{ch}^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{n+k} \binom{n}{k} \frac{\operatorname{ch}^{2k-m+1} x}{2k-1+1} + s(-1)^{n+\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \operatorname{ch} x.$$

$$2. \int \frac{\operatorname{ch}^{2n+1} x}{\operatorname{sh}^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n \binom{n}{k} \frac{\operatorname{sh}^{2k-m+1} x}{2k-m+1} + s \binom{n}{\frac{m-1}{2}} \ln \operatorname{sh} x.$$

[In formulas 2.421 1. and 2.421 2., $s = 1$ for m odd and $m < 2n + 1$; in all other cases, $s = 0$.]

2.422

$$1. \int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^{k+1}}{2m-2k-1} \binom{m+n-1}{k} \operatorname{th}^{2k-2m+1} x.$$

GI ((351))(11, 13)

$$2. \int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} = \sum_{\substack{k=0 \\ k \neq n}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} \binom{m+n}{k} \operatorname{th}^{2k-2m} x + (-1)^m \binom{m+n}{m} \ln \operatorname{th} x.$$

GI ((351))(15)

2.423

$$1. \int \frac{dx}{\operatorname{sh} x} = \ln \operatorname{th} \frac{x}{2} = \frac{1}{2} \ln \frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}.$$

$$2. \int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$$

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$$3. \int \frac{dx}{\operatorname{sh}^3 x} = -\frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{1}{2} \ln \operatorname{th} \frac{x}{2}.$$

$$4. \int \frac{dx}{\operatorname{sh}^4 x} = -\frac{\operatorname{ch} x}{3 \operatorname{sh}^3 x} + \frac{2}{3} \operatorname{cth} x = -\frac{1}{3} \operatorname{cth}^3 x + \operatorname{cth} x.$$

$$5. \int \frac{dx}{\operatorname{sh}^5 x} = -\frac{\operatorname{ch} x}{4 \operatorname{sh}^4 x} + \frac{3}{8} \frac{\operatorname{ch} x}{\operatorname{sh}^2 x} + \frac{3}{8} \ln \operatorname{th} \frac{x}{2}.$$

$$6. \int \frac{dx}{\operatorname{sh}^6 x} = -\frac{\operatorname{ch} x}{5 \operatorname{sh}^5 x} + \frac{4}{15} \operatorname{cth}^3 x - \frac{4}{5} \operatorname{cth} x;$$

$$= -\frac{1}{5} \operatorname{cth}^5 x + \frac{2}{3} \operatorname{cth}^3 x - \operatorname{cth} x.$$

$$7. \int \frac{dx}{\operatorname{sh}^7 x} = -\frac{\operatorname{ch} x}{6 \operatorname{sh}^2 x} \left(\frac{1}{\operatorname{sh}^4 x} - \frac{5}{4 \operatorname{sh}^2 x} + \frac{15}{8} \right) - \frac{5}{16} \ln \operatorname{th} \frac{x}{2}.$$

$$8. \int \frac{dx}{\operatorname{sh}^8 x} = \operatorname{cth} x - \operatorname{cth}^3 x + \frac{3}{5} \operatorname{cth}^5 x - \frac{1}{7} \operatorname{cth}^7 x.$$

$$9. \int \frac{dx}{\operatorname{ch} x} = \operatorname{arctg}(\operatorname{sh} x) = 2 \operatorname{arctg}(e^x);$$

$$= \operatorname{arcsin}(\operatorname{th} x);$$

$$= \operatorname{gd} x.$$

$$10. \int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$$

$$11. \int \frac{dx}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{1}{2} \operatorname{arctg}(\operatorname{sh} x).$$

$$12. \int \frac{dx}{\operatorname{ch}^4 x} = \frac{\operatorname{sh} x}{3 \operatorname{ch}^3 x} + \frac{2}{3} \operatorname{th} x;$$

$$= -\frac{1}{3} \operatorname{th}^3 x + \operatorname{th} x.$$

$$13. \int \frac{dx}{\operatorname{ch}^5 x} = \frac{\operatorname{sh} x}{4 \operatorname{ch}^4 x} + \frac{3}{8} \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} + \frac{3}{8} \operatorname{arctg}(\operatorname{sh} x).$$

$$14. \int \frac{dx}{\operatorname{ch}^6 x} = \frac{\operatorname{sh} x}{5 \operatorname{ch}^5 x} - \frac{4}{15} \operatorname{th}^3 x + \frac{4}{5} \operatorname{th} x;$$

$$= \frac{1}{5} \operatorname{th}^5 x - \frac{2}{3} \operatorname{th}^3 x + \operatorname{th} x.$$

$$15.^6 \int \frac{dx}{\operatorname{ch}^7 x} = \frac{\operatorname{sh} x}{6 \operatorname{ch}^2 x} \left(\frac{1}{\operatorname{ch}^4 x} + \frac{5}{4 \operatorname{ch}^2 x} + \frac{15}{8} \right) + \frac{5}{16} \operatorname{arctg}(\operatorname{sh} x).$$

$$16. \int \frac{dx}{\operatorname{ch}^8 x} = -\frac{1}{7} \operatorname{th}^7 x + \frac{3}{5} \operatorname{th}^5 x - \operatorname{th}^3 x + \operatorname{th} x.$$

$$17. \int \frac{\operatorname{sh} x}{\operatorname{ch} x} dx = \ln \operatorname{ch} x.$$

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$$18. \int \frac{\operatorname{sh}^2 x}{\operatorname{ch} x} dx = \operatorname{sh} x - \operatorname{arctg}(\operatorname{sh} x).$$

$$19. \int \frac{\operatorname{sh}^3 x}{\operatorname{ch} x} dx = \frac{1}{2} \operatorname{sh}^2 x - \ln \operatorname{ch} x;$$

$$= \frac{1}{2} \operatorname{ch}^2 x - \ln \operatorname{ch} x.$$

$$20. \int \frac{\operatorname{sh}^4 x}{\operatorname{ch} x} dx = \frac{1}{3} \operatorname{sh}^3 x - \operatorname{sh} x + \operatorname{arctg}(\operatorname{sh} x).$$

$$21. \int \frac{\operatorname{sh} x}{\operatorname{ch}^2 x} dx = -\frac{1}{\operatorname{ch} x}.$$

$$22. \int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x} dx = x - \operatorname{th} x.$$

$$33. \int \frac{\operatorname{ch} x}{\operatorname{sh} x} dx = \ln \operatorname{sh} x.$$

$$34. \int \frac{\operatorname{ch}^2 x}{\operatorname{sh} x} dx = \operatorname{ch} x + \ln \operatorname{th} \frac{x}{2}.$$

$$35. \int \frac{\operatorname{ch}^3 x}{\operatorname{sh} x} dx = \frac{1}{2} \operatorname{ch}^2 x + \ln \operatorname{sh} x.$$

$$36. \int \frac{\operatorname{ch}^4 x}{\operatorname{sh} x} dx = \frac{1}{3} \operatorname{ch}^3 x + \operatorname{ch} x + \ln \operatorname{th} \frac{x}{2}.$$

$$37. \int \frac{\operatorname{ch} x}{\operatorname{sh}^2 x} dx = -\frac{1}{\operatorname{sh} x}.$$

$$38. \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x} dx = x - \operatorname{cth} x.$$

$$39. \int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^2 x} dx = \operatorname{sh} x - \frac{1}{\operatorname{sh} x}.$$

$$40. \int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^2 x} dx = \frac{3}{2} x + \frac{1}{4} \operatorname{sh} 2x - \operatorname{cth} x.$$

$$41. \int \frac{\operatorname{ch} x}{\operatorname{sh}^3 x} dx = -\frac{1}{2\operatorname{sh}^2 x};$$

$$= -\frac{1}{2}\operatorname{cth}^2 x.$$

$$42. \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^3 x} dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \ln \operatorname{th} \frac{x}{2}.$$

$$43. \int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^3 x} dx = -\frac{1}{2\operatorname{sh}^2 x} + \ln \operatorname{sh} x;$$

$$= -\frac{1}{2}\operatorname{cth}^2 x + \ln \operatorname{sh} x.$$

$$44. \int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^3 x} dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \operatorname{ch} x + \frac{3}{2} \ln \operatorname{th} \frac{x}{2}.$$

$$45. \int \frac{\operatorname{ch} x}{\operatorname{sh}^4 x} dx = -\frac{1}{3\operatorname{sh}^3 x}.$$

$$46. \int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^4 x} dx = -\frac{1}{3}\operatorname{cth}^3 x.$$

$$47. \int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^4 x} dx = -\frac{1}{\operatorname{sh} x} - \frac{1}{3\operatorname{sh}^3 x}.$$

$$48. \int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^4 x} dx = -\frac{1}{3}\operatorname{cth}^3 x - \operatorname{cth} x + x.$$

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$$49. \int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln \operatorname{th} x.$$

50. $\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^2 x} = \frac{1}{\operatorname{ch} x} + \ln \operatorname{th} \frac{x}{2}.$
51. $\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^3 x} = \frac{1}{2 \operatorname{ch}^2 x} + \ln \operatorname{th} x;$
 $= -\frac{1}{2} \operatorname{th}^2 x + \ln \operatorname{th} x.$
52. $\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^4 x} = \frac{1}{\operatorname{ch} x} + \frac{1}{3 \operatorname{ch}^3 x} + \ln \operatorname{th} \frac{x}{2}.$
53. $\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch} x} = -\frac{1}{\operatorname{sh} x} - \operatorname{arctg} \operatorname{sh} x.$
54. $\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2 \operatorname{cth} 2x.$
55. $\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^3 x} = -\frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} - \frac{1}{\operatorname{sh} x} - \frac{3}{2} \operatorname{arctg} \operatorname{sh} x.$
56. $\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^4 x} = \frac{1}{3 \operatorname{sh} x \operatorname{ch}^3 x} - \frac{8}{3} \operatorname{cth} 2x.$
57. $\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch} x} = -\frac{1}{2 \operatorname{sh}^2 x} - \ln \operatorname{th} x;$
 $= -\frac{1}{2} \operatorname{cth}^2 x + \ln \operatorname{cth} x.$
58. $\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^2 x} = -\frac{1}{\operatorname{ch} x} - \frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{3}{2} \ln \operatorname{th} \frac{x}{2}.$
59. $\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^3 x} = -\frac{2 \operatorname{ch} 2x}{\operatorname{sh}^2 2x} - 2 \ln \operatorname{th} x;$
 $= \frac{1}{2} \operatorname{th}^2 x - \frac{1}{2} \operatorname{cth}^2 x - 2 \ln \operatorname{th} x.$

$$60. \int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^4 x} = -\frac{2}{\operatorname{ch} x} - \frac{1}{3 \operatorname{ch}^2 x} - \frac{\operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{5}{2} \ln \operatorname{th} \frac{x}{2}.$$

$$61. \int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch} x} = \frac{1}{\operatorname{sh} x} - \frac{1}{3 \operatorname{sh}^3 x} + \operatorname{arctg} \operatorname{sh} x.$$

$$62. \int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^2 x} = -\frac{1}{3 \operatorname{ch} x \operatorname{sh}^3 x} + \frac{8}{3} \operatorname{cth} 2x.$$

$$63. \int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^3 x} = \frac{2}{\operatorname{sh} x} - \frac{1}{3 \operatorname{sh}^3 x} + \frac{\operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{5}{2} \operatorname{arctg} \operatorname{sh} x.$$

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$$64. \int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^4 x} = 8 \operatorname{cth} 2x - \frac{8}{3} \operatorname{cth}^3 2x.$$

2.424

$$1. \int \operatorname{th}^p x \, dx = -\frac{\operatorname{th}^{p-1} x}{p-1} + \int \operatorname{th}^{p-2} x \, dx \quad [p \neq 1].$$

$$2. \int \operatorname{th}^{2n+1} x \, dx = \sum_{k=1}^n \frac{(-1)^{k-1}}{2k} \binom{n}{k} \frac{1}{\operatorname{ch}^{2k} x} + \ln \operatorname{ch} x;$$

$$= -\sum_{k=1}^n \frac{\operatorname{th}^{2n-2k+2} x}{2n-2k+2} + \ln \operatorname{ch} x.$$

$$3. \int \operatorname{th}^{2n} x \, dx = -\sum_{k=1}^n \frac{\operatorname{th}^{2n-2k+1} x}{2n-2k+1} + x.$$

GU ((351))(12)

$$4. \int \operatorname{cth}^p x \, dx = -\frac{\operatorname{cth}^{p-1} x}{p-1} + \int \operatorname{cth}^{p-2} x \, dx \quad [p \neq 1].$$

$$5. \int \operatorname{cth}^{2n+1} x \, dx = -\sum_{k=1}^n \frac{1}{2n} \binom{n}{k} \frac{1}{\operatorname{sh}^{2k} x} + \ln \operatorname{sh} x;$$

$$6. \int \operatorname{cth}^{2n} x \, dx = - \sum_{k=1}^n \frac{\operatorname{cth}^{2n-2k+1} x}{2n-2k+1} + x.$$

GU ((351))(14)

For formulas containing powers of $\operatorname{th} x$ and $\operatorname{cth} x$ equal to $n = 1, 2, 3, 4$, see 2.423 17., 2.423 22., 2.423 27., 2.423 32., 2.423 33., 2.423 38., 2.423 43., 2.423 48.

Powers of hyperbolic functions and hyperbolic functions of linear functions of the argument

2.425

$$1. \int \operatorname{sh}(ax+b) \operatorname{sh}(cx+d) \, dx = \frac{1}{2(a+c)} \operatorname{sh}[(a+c)x+b+d] - \frac{1}{2(a-c)} \operatorname{sh}[(a-c)x+b-d] \quad [a^2 \neq c^2].$$

GU ((352))(2a)

$$2. \int \operatorname{sh}(ax+b) \operatorname{ch}(cx+d) \, dx = \frac{1}{2(a+c)} \operatorname{ch}[(a+c)x+b+d] + \frac{1}{2(a-c)} \operatorname{ch}[(a-c)x+b-d] \quad [a^2 \neq c^2].$$

GU ((352))(2c)

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$$3. \int \operatorname{ch}(ax+b) \operatorname{ch}(cx+d) \, dx = \frac{1}{2(a+c)} \operatorname{sh}[(a+c)x+b+d] + \frac{1}{2(a-c)} \operatorname{sh}[(a-c)x+b-d] \quad [a^2 \neq c^2].$$

GU ((352))(2b)

When $a = c$:

$$4. \int \operatorname{sh}(ax+b) \operatorname{sh}(ax+d) \, dx = -\frac{x}{2} \operatorname{ch}(b-d) + \frac{1}{4a} \operatorname{sh}(2ax+b+d).$$

GU ((352))(3a)

GU ((352))(3c)

$$6. \int \operatorname{ch}(ax + b) \operatorname{ch}(ax + d) dx = \frac{x}{2} \operatorname{ch}(b - d) + \frac{1}{4a} \operatorname{sh}(2ax + b + d).$$

GU ((352))(3b)

2.426

$$1. \int \operatorname{sh} ax \operatorname{sh} bx \operatorname{sh} cx dx = \frac{\operatorname{ch}(a + b + c)x}{4(a + b + c)} - \frac{\operatorname{ch}(-a + b + c)x}{4(-a + b + c)} - \frac{\operatorname{ch}(a - b + c)x}{4(a - b + c)} - \frac{\operatorname{ch}(a + b - c)x}{4(a + b - c)}.$$

GU ((352))(4a)

$$2. \int \operatorname{sh} ax \operatorname{sh} bx \operatorname{ch} cx dx = \frac{\operatorname{sh}(a + b + c)x}{4(a + b + c)} - \frac{\operatorname{sh}(-a + b + c)x}{4(-a + b + c)} - \frac{\operatorname{sh}(a - b + c)x}{4(a - b + c)} + \frac{\operatorname{sh}(a + b - c)x}{4(a + b - c)}.$$

GU ((352))(4b)

$$3. \int \operatorname{sh} ax \operatorname{ch} bx \operatorname{ch} cx dx = \frac{\operatorname{ch}(a + b + c)x}{4(a + b + c)} - \frac{\operatorname{ch}(-a + b + c)x}{4(-a + b + c)} + \frac{\operatorname{ch}(a - b + c)x}{4(a - b + c)} + \frac{\operatorname{ch}(a + b - c)x}{4(a + b - c)}.$$

GU ((352))(4c)

$$4. \int \operatorname{ch} ax \operatorname{ch} bx \operatorname{ch} cx dx = \frac{\operatorname{sh}(a + b + c)x}{4(a + b + c)} + \frac{\operatorname{sh}(-a + b + c)x}{4(-a + b + c)} + \frac{\operatorname{sh}(a - b + c)x}{4(a - b + c)} + \frac{\operatorname{sh}(a + b - c)x}{4(a + b - c)}.$$

GU ((352))(4d)

2.427

$$1. \int \operatorname{sh}^p x \operatorname{sh} ax dx = \frac{1}{p + a} \left\{ \operatorname{sh}^p x \operatorname{ch} ax - p \int \operatorname{sh}^{p-1} x \operatorname{ch}(a - 1)x dx \right\}.$$

$$\begin{aligned}
2. \quad \int \operatorname{sh}^p x \operatorname{sh}(2n+1)x \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \times \\
&\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \operatorname{sh}^{p-2k} x \operatorname{ch}(2n-2k+1)x - \right. \right. \\
&\quad \left. \left. - \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \operatorname{sh}^{p-2k-1} x \operatorname{sh}(2n-2k)x \right] + \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p+1-2n)} \int \operatorname{sh}^{p-2n} x \operatorname{sh} x \, dx \right\} \quad [p \text{ is not a negative integer}].
\end{aligned}$$

$$\begin{aligned}
3. \quad \int \operatorname{sh}^p x \operatorname{sh} 2nx \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \times \\
&\times \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \operatorname{sh}^{p-2k} x \operatorname{ch}(2n-2k)x - \right. \\
&\quad \left. - \frac{\Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \operatorname{sh}^{p-2k-1} x \operatorname{sh}(2n-2k-1)x \right] \\
&\quad [p \text{ is not a negative integer}].
\end{aligned}$$

GU ((352))(5)a

2.428

$$1. \quad \int \operatorname{sh}^p x \operatorname{ch} ax \, dx = \frac{1}{p+a} \left\{ \operatorname{sh}^p x \operatorname{sh} ax - p \int \operatorname{sh}^{p-1} x \operatorname{sh}(a-1)x \, dx \right\}.$$

$$\begin{aligned}
2. \quad \int \operatorname{sh}^p x \operatorname{ch}(2n+1)x \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \times \\
&\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \operatorname{sh}^{p-2k} x \operatorname{sh}(2n-2k+1)x - \right. \right. \\
&\quad \left. \left. - \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \operatorname{sh}^{p-2k-1} x \operatorname{ch}(2n-2k)x \right] + \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p+1-2n)} \int \operatorname{sh}^{p-2n} x \operatorname{ch} x \, dx \right\} \quad [p \text{ is not a negative integer}].
\end{aligned}$$

$$\begin{aligned}
3. \quad \int \operatorname{sh}^p x \operatorname{ch} 2nx \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \times \\
&\times \left\{ \sum_{k=0}^{n-1} \left[\frac{\Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1}\Gamma(p-2k+1)} \operatorname{sh}^{p-2k} x \operatorname{sh}(2n-2k)x - \right. \right. \\
&\quad \left. \left. - \frac{\Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2}\Gamma(p-2k)} \operatorname{sh}^{p-2k-1} x \operatorname{ch}(2n-2k-1)x \right] + \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n}\Gamma(p+1-2n)} \int \operatorname{sh}^{p-2n} x \, dx \right\} \quad [p \text{ is not a negative integer}].
\end{aligned}$$

GU ((352))(6)a

2.429

$$\begin{aligned}
1. \quad \int \operatorname{ch}^p x \operatorname{sh} ax \, dx &= \frac{1}{p+a} \left\{ \operatorname{ch}^p x \operatorname{ch} ax + p \int \operatorname{ch}^{p-1} x \operatorname{sh}(a-1)x \, dx \right\}. \\
2. \quad \int \operatorname{ch}^p x \operatorname{sh}(2n+1)x \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1}\Gamma(p-k+1)} \operatorname{ch}^{p-k} x \operatorname{ch}(2n-k+1)x + \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n\Gamma(p-n+1)} \int \operatorname{ch}^{p-n} x \operatorname{sh}(n+1)x \, dx \right\} \\
&\quad [p \text{ is not a negative integer}].
\end{aligned}$$

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$$\begin{aligned}
3. \quad \int \operatorname{ch}^p x \operatorname{sh} 2nx \, dx &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1}\Gamma(p-k+1)} \operatorname{ch}^{p-k} x \operatorname{ch}(2n-k)x + \right. \\
&\quad \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n\Gamma(p-n+1)} \int \operatorname{ch}^{p-n} x \operatorname{sh} nx \, dx \right\} \quad [p \text{ is not a negative integer}].
\end{aligned}$$

GU ((352))(7)a

2.431

$$1. \int \operatorname{ch}^p x \operatorname{ch} ax \, dx = \frac{1}{p+a} \left\{ \operatorname{ch}^p x \operatorname{sh} ax + p \int \operatorname{ch}^{p-1} x \operatorname{ch}(a-1)x \, dx \right\}.$$

$$2. \int \operatorname{ch}^p x \operatorname{ch}(2n+1)x \, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1}\Gamma(p-k+1)} \operatorname{ch}^{p-k} x \operatorname{sh}(2n-k+1)x + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \operatorname{ch}^{p-n} x \operatorname{ch}(n+1)x \, dx \right\}$$

[p is not a negative integer].

$$3. \int \operatorname{ch}^p x \operatorname{ch} 2nx \, dx = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1}\Gamma(p-k+1)} \operatorname{ch}^{p-k} x \operatorname{sh}(2n-k)x + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \operatorname{ch}^{p-n} x \operatorname{ch} nx \, dx \right\} \quad [p \text{ is not a negative integer }].$$

GU ((352))(8)a

2.432

$$1. \int \operatorname{sh}(n+1)x \operatorname{sh}^{n-1} x \, dx = \frac{1}{n} \operatorname{sh}^n x \operatorname{sh} nx.$$

$$2. \int \operatorname{sh}(n+1)x \operatorname{ch}^{n-1} x \, dx = \frac{1}{n} \operatorname{ch}^n x \operatorname{ch} nx.$$

$$3. \int \operatorname{ch}(n+1)x \operatorname{sh}^{n-1} x \, dx = \frac{1}{n} \operatorname{sh}^n x \operatorname{ch} nx.$$

$$4. \int \operatorname{ch}(n+1)x \operatorname{ch}^{n-1} x \, dx = \frac{1}{n} \operatorname{ch}^n x \operatorname{sh} nx.$$

$$1. \int \frac{\text{sh}(2n+1)x}{\text{sh } x} dx = 2 \sum_{k=0}^{n-1} \frac{\text{sh}(2n-2k)x}{2n-2k} + x.$$

$$2. \int \frac{\text{sh } 2nx}{\text{sh } x} dx = 2 \sum_{k=0}^{n-1} \frac{\text{sh}(2n-2k-1)x}{2n-2k-1}.$$

GU ((352))(5d)

$$3. \int \frac{\text{ch}(2n+1)x}{\text{sh } x} dx = 2 \sum_{k=0}^{n-1} \frac{\text{ch}(2n-2k)x}{2n-2k} + \ln \text{sh } x.$$

$$4. \int \frac{\text{ch } 2nx}{\text{sh } x} dx = 2 \sum_{k=0}^{n-1} \frac{\text{ch}(2n-2k-1)x}{2n-2k-1} + \ln \text{th} \frac{x}{2}.$$

GU ((352))(6d)

$$5. \int \frac{\text{sh}(2n+1)x}{\text{ch } x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\text{ch}(2n-2k)x}{2n-2k} + (-1)^n \ln \text{ch } x.$$

$$6. \int \frac{\text{sh } 2nx}{\text{ch } x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\text{ch}(2n-2k-1)x}{2n-2k-1}.$$

GU ((352))(7d)

$$7. \int \frac{\text{ch}(2n+1)x}{\text{ch } x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\text{sh}(2n-2k)x}{2n-2k} + (-1)^n x.$$

$$8. \int \frac{\text{ch } 2nx}{\text{ch } x} dx = 2 \sum_{k=0}^{n-1} (-1)^k \frac{\text{sh}(2n-2k-1)x}{2n-2k-1} + (-1)^n \arcsin(\text{th } x).$$

GU ((352))(8d)

$$9. \int \frac{\text{sh } 2x}{\text{sh}^n x} dx = -\frac{2}{(n-2)\text{sh}^{n-2} x}.$$

$$n = 2$$

$$10. \int \frac{\text{sh } 2x}{\text{sh}^2 x} dx = 2 \ln \text{sh } x.$$

$$11. \int \frac{\text{sh } 2x dx}{\text{ch}^n x} = \frac{2}{(2-n)\text{ch}^{n-2} x}.$$

For $n = 2$:

$$12. \int \frac{\text{sh } 2x}{\text{ch}^2 x} dx = 2 \ln \text{ch } x.$$

$$13. \int \frac{\text{ch } 2x}{\text{sh } x} dx = 2 \text{ch } x + \ln \text{th} \frac{x}{2}.$$

$$14. \int \frac{\text{ch } 2x}{\text{sh}^2 x} dx = -\text{cth } x + 2x.$$

$$15. \int \frac{\text{ch } 2x}{\text{sh}^3 x} dx = -\frac{\text{ch } x}{2 \text{sh}^2 x} + \frac{3}{2} \ln \text{th} \frac{x}{2}.$$

$$16. \int \frac{\text{ch } 2x}{\text{ch } x} dx = 2 \text{sh } x - \arcsin(\text{th } x).$$

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$$17. \int \frac{\text{ch } 2x}{\text{ch}^2 x} dx = -\text{th } x + 2x.$$

$$18. \int \frac{\text{ch } 2x}{\text{ch}^3 x} dx = -\frac{\text{sh } x}{2 \text{ch}^2 x} + \frac{3}{2} \arcsin(\text{th } x).$$

$$19. \int \frac{\text{sh } 3x}{\text{sh } x} dx = x + \text{sh } 2x.$$

$$20. \int \frac{\text{sh } 3x}{\text{sh}^2 x} dx = 3 \ln \text{th} \frac{x}{2} + 4 \text{ch } x.$$

$$21. \int \frac{\text{sh } 3x}{\text{sh}^3 x} dx = -3 \text{cth } x + 4x.$$

$$22. \int \frac{\text{sh } 3x}{\text{ch}^n x} dx = \frac{4}{(3-n)\text{ch}^{n-3} x} - \frac{1}{(1-n)\text{ch}^{n-1} x}.$$

For $n = 1$ and $n = 3$:

$$23. \int \frac{\text{sh } 3x}{\text{ch } x} dx = 2 \text{sh}^2 x - \ln \text{ch } x.$$

$$24. \int \frac{\text{sh } 3x}{\text{ch}^3 x} dx = \frac{1}{2 \text{ch}^2 x} + 4 \ln \text{ch } x.$$

$$25. \int \frac{\text{ch } 3x}{\text{sh}^n x} dx = \frac{4}{(3-n)\text{sh}^{n-3} x} + \frac{1}{(1-n)\text{sh}^{n-1} x}.$$

For $n = 1$ and $n = 3$:

$$26. \int \frac{\text{ch } 3x}{\text{sh } x} dx = 2 \text{sh}^2 x + \ln \text{sh } x.$$

$$27. \int \frac{\text{ch } 3x}{\text{sh}^3 x} dx = -\frac{1}{2 \text{sh}^2 x} + 4 \ln \text{sh } x.$$

$$28. \int \frac{\text{ch } 3x}{\text{ch } x} dx = \text{sh } 2x - x.$$

2.44-2.45 Rational functions of hyperbolic functions

2.441

$$1. \int \frac{A + B \operatorname{sh} x}{(a + b \operatorname{sh} x)^n} dx = \frac{aB - bA}{(n-1)(a^2 + b^2)} \cdot \frac{\operatorname{ch} x}{(a + b \operatorname{sh} x)^{n-1}} + \\ + \frac{1}{(n-1)(a^2 + b^2)} \int \frac{(n-1)(aA + bB) + (n-2)(aB - bA)\operatorname{sh} x}{(a + b \operatorname{sh} x)^{n-1}} dx.$$

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For $n = 1$:

$$2. \int \frac{A + B \operatorname{sh} x}{a + b \operatorname{sh} x} dx = \frac{B}{b} x - \frac{aB - bA}{b} \int \frac{dx}{a + b \operatorname{sh} x} \quad (\text{see } \mathbf{2.441} \text{ 3.}).$$

2.441

$$3. \int \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a \operatorname{th} \frac{x}{2} - b + \sqrt{a^2 + b^2}}{a \operatorname{th} \frac{x}{2} - b - \sqrt{a^2 + b^2}}; \\ = \frac{2}{\sqrt{a^2 + b^2}} \operatorname{Arth} \frac{a \operatorname{th} \frac{x}{2} - b}{\sqrt{a^2 + b^2}}.$$

2.442

$$1. \int \frac{A + B \operatorname{ch} x}{(a + b \operatorname{sh} x)^n} dx = -\frac{B}{(n-1)b(a + b \operatorname{sh} x)^{n-1}} + A \int \frac{dx}{(a + b \operatorname{sh} x)^n}.$$

For $n = 1$:

$$2. \int \frac{A + B \operatorname{ch} x}{a + b \operatorname{sh} x} dx = \frac{B}{b} \ln(a + b \operatorname{sh} x) + A \int \frac{dx}{a + b \operatorname{sh} x} \quad (\text{see } \mathbf{2.441} \text{ 3.}).$$

2.443

$$1. \int \frac{A + B \operatorname{ch} x}{(a + b \operatorname{ch} x)^n} dx = \frac{aB - bA}{(n-1)(a^2 - b^2)} \cdot \frac{\operatorname{sh} x}{(a + b \operatorname{ch} x)^{n-1}} + \\ + \frac{1}{(n-1)(a^2 - b^2)} \int \frac{(n-1)(aA - bB) + (n-2)(aB - bA) \operatorname{ch} x}{(a + b \operatorname{ch} x)^{n-1}} dx.$$

For $n = 1$:

$$2. \int \frac{A + B \operatorname{ch} x}{a + b \operatorname{ch} x} dx = \frac{B}{b} x - \frac{aB - bA}{b} \int \frac{dx}{a + b \operatorname{ch} x} \quad (\text{see } \mathbf{2.443} \text{ } 3.).$$

2.443

$$3. \int \frac{dx}{a + b \operatorname{ch} x} = \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \quad [b^2 > a^2, \quad x < 0]; \\ - \frac{1}{\sqrt{b^2 - a^2}} \arcsin \frac{b + a \operatorname{ch} x}{a + b \operatorname{ch} x} \quad [b^2 > a^2, \quad x > 0]; \\ = \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}}{a + b - \sqrt{a^2 - b^2} \operatorname{th} \frac{x}{2}} \quad [a^2 > b^2].$$

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2.444

$$1. \int \frac{dx}{\operatorname{ch} a + \operatorname{ch} x} = \operatorname{cosech} a \left[\ln \operatorname{ch} \frac{x+a}{2} - \ln \operatorname{ch} \frac{x-a}{2} \right]; \\ = 2 \operatorname{cosech} a \operatorname{Arth} \left(\operatorname{th} \frac{x}{2} \operatorname{th} \frac{a}{2} \right).$$

$$2. \int \frac{dx}{\cos a + \operatorname{ch} x} = 2 \cos a \operatorname{arctg} \left(\operatorname{th} \frac{x}{2} \operatorname{tg} \frac{a}{2} \right).$$

2.445

$$1. \int \frac{A + B \operatorname{sh} x}{(a + b \operatorname{ch} x)^n} dx = -\frac{B}{(n-1)b(a + b \operatorname{ch} x)^{n-1}} + A \int \frac{dx}{(a + b \operatorname{ch} x)^n}.$$

For $n = 1$:

$$2. \int \frac{A + B \operatorname{sh} x}{a + b \operatorname{ch} x} dx = \frac{B}{b} \ln(a + b \operatorname{ch} x) + A \int \frac{dx}{a + b \operatorname{ch} x}$$

2.443

In evaluating definite integrals by use of formulas 2.441 2.442 2.443 and 2.445, one may not take the integral over points at which the integrand becomes infinite, that is, over the points

$$x = \operatorname{Arsh} \left(-\frac{a}{b} \right)$$

in formulas 2.441 or 2.442 or over the points

$$x = \operatorname{Arch} \left(-\frac{a}{b} \right)$$

in formulas 2.443 or 2.445. Formulas 2.443 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

2.446

$$1. \int \frac{A + B \operatorname{ch} x}{(\varepsilon + \operatorname{ch} x)^n} dx = \frac{B \operatorname{sh} x}{(1-n)(\varepsilon + \operatorname{ch} x)^n} + \left(\varepsilon A + \frac{n}{n-1} B \right) \frac{(n-1)!}{(2n-1)!!} \operatorname{sh} x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!} \cdot \frac{\varepsilon^k}{(\varepsilon + \operatorname{ch} x)^{n-k}} \quad [\varepsilon = \pm 1, \quad n > 1].$$

For $n = 1$:

$$2. \int \frac{A + B \operatorname{ch} x}{\varepsilon + \operatorname{ch} x} dx = Bx + (\varepsilon A - B) \frac{\operatorname{ch} x - \varepsilon}{\operatorname{sh} x} \quad [\varepsilon = \pm 1].$$

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2.447

$$1. \int \frac{\operatorname{sh} x dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{a \ln \operatorname{ch} \left(x + \operatorname{Arth} \frac{b}{a} \right) - bx}{a^2 - b^2} \quad [a > |b|];$$

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$$= \frac{bx - a \ln \operatorname{sh} \left(x + \operatorname{Arth} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|].$$

For $a = b = 1$:

$$2. \int \frac{\operatorname{sh} x \, dx}{\operatorname{ch} x + \operatorname{sh} x} = \frac{x}{2} + \frac{1}{4}e^{-2x}.$$

For $a = -b = 1$:

$$3. \int \frac{\operatorname{sh} x \, dx}{\operatorname{ch} x - \operatorname{sh} x} = -\frac{x}{2} + \frac{1}{4}e^{2x}.$$

MZ 215

2.448

$$1. \int \frac{\operatorname{ch} x \, dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{ax - b \ln \operatorname{ch} \left(x + \operatorname{Arth} \frac{b}{a} \right)}{a^2 - b^2} \quad [a > |b|];$$
$$= \frac{-ax + b \ln \operatorname{sh} \left(x + \operatorname{Arth} \frac{a}{b} \right)}{b^2 - a^2} \quad [b > |a|].$$

For $a = b = 1$:

$$2. \int \frac{\operatorname{ch} x \, dx}{\operatorname{ch} x + \operatorname{sh} x} = \frac{x}{2} - \frac{1}{4}e^{-2x}.$$

For $a = -b = 1$:

$$3. \int \frac{\operatorname{ch} x \, dx}{\operatorname{ch} x - \operatorname{sh} x} = \frac{x}{2} + \frac{1}{4}e^{2x}.$$

MZ 214, 215

2.449

$$1.^6 \int \frac{dx}{(a \operatorname{ch} x + b \operatorname{sh} x)^n} = \frac{1}{\sqrt{(a^2 - b^2)^n}} \int \frac{dx}{\operatorname{sh}^n \left(x + \operatorname{Arth} \frac{b}{a} \right)} \quad [a > |b|];$$
$$= \frac{1}{\sqrt{(b^2 - a^2)^n}} \int \frac{dx}{\operatorname{ch}^n \left(x + \operatorname{Arth} \frac{a}{b} \right)} \quad [b > |a|].$$

$$n = 1$$

$$2. \int \frac{dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 - b^2}} \operatorname{arctg} \left| \operatorname{sh} \left(x + \operatorname{Arth} \frac{b}{a} \right) \right| \quad [a > |b|];$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \left| \operatorname{th} \frac{x + \operatorname{Arth} \frac{a}{b}}{2} \right| \quad [b > |a|].$$

For $a = b = 1$:

$$3. \int \frac{ax}{\operatorname{ch} x + \operatorname{sh} x} = -e^{-x} = \operatorname{sh} x - \operatorname{ch} x.$$

For $a = -b = 1$:

$$4. \int \frac{dx}{\operatorname{ch} x - \operatorname{sh} x} = e^x = \operatorname{sh} x + \operatorname{ch} x.$$

MZ 214

2.451

$$1. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} dx$$

$$= \frac{Bc - Cb + (Ac - Ca) \operatorname{ch} x + (Ab - Ba) \operatorname{sh} x}{(1-n)(a^2 - b^2 + c^2)(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba) \operatorname{ch} x - (n-2)(Ac - Ca) \operatorname{sh} x}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-1}} dx$$

$$[a^2 + c^2 \neq b^2];$$

$$= \frac{Bc - Cb - Ca \operatorname{ch} x - Ba \operatorname{sh} x}{(n-1)a(a + b \operatorname{ch} x + c \operatorname{sh} x)^n} + \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2} \right] (c \operatorname{ch} x + b \operatorname{sh} x) \frac{(n-1)!}{(2n-1)!} \times$$

$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)! a^k} \frac{1}{(a + b \operatorname{ch} x + c \operatorname{sh} x)^{n-k}} \quad [a^2 + c^2 = b^2].$$

$$2. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x + c \operatorname{sh} x} dx$$

$$= \frac{Cb - Bc}{b^2 - c^2} \ln(a + b \operatorname{ch} x + c \operatorname{sh} x) +$$

$$+ \frac{Bb - Cc}{b^2 - c^2} x + \left(A - a \frac{Bb - Cc}{b^2 - c^2} \right) \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} \quad [b^2 \neq c^2] \quad (\text{see 2.4514}).$$

$$3. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{a + b \operatorname{ch} x \pm b \operatorname{sh} x} dx = \frac{C \mp B}{2a} (\operatorname{ch} x \mp \operatorname{sh} x) + \left[\frac{A}{a} - \frac{(B \mp C)b}{2a^2} \right] x + \left[\frac{C \pm B}{2b} \pm \frac{A}{a} - \frac{(C \mp B)b}{2a^2} \right] \ln(a + b \operatorname{ch} x \pm b \operatorname{sh} x) \quad [ab \neq 0].$$

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$$4. \int \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \operatorname{arctg} \frac{(b-a) \operatorname{th} \frac{x}{2} + c}{\sqrt{b^2 - a^2 - c^2}} \quad [b^2 > a^2 + c^2 \text{ and } a \neq b];$$

$$= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{(a-b) \operatorname{th} \frac{x}{2} - c + \sqrt{a^2 - b^2 + c^2}}{(a-b) \operatorname{th} \frac{x}{2} - c - \sqrt{a^2 - b^2 + c^2}} \quad [b^2 < a^2 + c^2 \text{ and } a \neq b];$$

$$= \frac{1}{c} \ln \left(a + c \operatorname{th} \frac{x}{2} \right) \quad [a = b, \quad c \neq 0];$$

$$= \frac{2}{(a-b) \operatorname{th} \frac{x}{2} + c} \quad [b^2 = a^2 + c^2].$$

GU ((351))(18)

2.452

$$1. \int \frac{A + B \operatorname{ch} x + C \operatorname{sh} x}{(a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x)(a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x)} dx = A_0 \ln \frac{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x} + A_1 \int \frac{dx}{a_1 + b_1 \operatorname{ch} x + c_1 \operatorname{sh} x} + A_2 \int \frac{dx}{a_2 + b_2 \operatorname{ch} x + c_2 \operatorname{sh} x}$$

where

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ b_1 & c_1 & A \\ B & C & A \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ C & B & A \\ c_2 & b_2 & A \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \right].$$

GU ((351))(19)

$$\begin{aligned}
2. \quad & \int \frac{A \operatorname{ch}^2 x + 2B \operatorname{sh} x \operatorname{ch} x + C \operatorname{sh}^2 x}{a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x} dx \\
&= \frac{1}{4b^2 - (a+c)^2} \{ [4Bb - (A+C)(a+c)]x + \\
&\quad + [(A+C)b - B(a+c)] \ln(a \operatorname{ch}^2 x + 2b \operatorname{sh} x \operatorname{ch} x + c \operatorname{sh}^2 x) + \\
&\quad + [2(A-C)b^2 - 2Bb(a-c) + (Ca - Ac)(a+c)] f(x) \},
\end{aligned}$$

where

$$\begin{aligned}
f(x) &= \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{th} x + b - \sqrt{b^2 - ac}}{c \operatorname{th} x + b + \sqrt{b^2 - ac}} & [b^2 > ac]; \\
&= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{th} x + b}{\sqrt{ac - b^2}} & [b^2 < ac]; \\
&= -\frac{1}{c \operatorname{th} x + b} & [b^2 = ac].
\end{aligned}$$

GU ((351))(24)

2.453

$$1. \quad \int \frac{(A + B \operatorname{sh} x) dx}{\operatorname{sh} x(a + b \operatorname{sh} x)} = \frac{1}{a} \left[A \ln \left| \operatorname{th} \frac{x}{2} \right| + (aB - bA) \int \frac{dx}{a + b \operatorname{sh} x} \right] \quad (\text{see } \mathbf{2.441} \text{ 3.}).$$

2.441

$$2. \quad \int \frac{(A + B \operatorname{sh} x) dx}{\operatorname{sh} x(a + b \operatorname{ch} x)} = \frac{A}{a^2 - b^2} \left(a \ln \left| \operatorname{th} \frac{x}{2} \right| + b \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{sh} x} \right| \right) + B \int \frac{dx}{a + b \operatorname{ch} x} \quad (\text{see } \mathbf{2.443} \text{ 3.}).$$

2.443

For $a^2 = b^2 (= 1)$:

$$3. \quad \int \frac{(A + B \operatorname{sh} x) dx}{\operatorname{sh} x(1 + \operatorname{ch} x)} = \frac{A}{2} \left(\ln \left| \operatorname{th} \frac{x}{2} \right| - \frac{1}{2} \operatorname{th}^2 \frac{x}{2} \right) + B \operatorname{th} \frac{x}{2}.$$

$$4. \quad \int \frac{(A + B \operatorname{sh} x) dx}{\operatorname{sh} x(1 - \operatorname{ch} x)} = \frac{A}{2} \left(-\ln \left| \operatorname{cth} \frac{x}{2} \right| + \frac{1}{2} \operatorname{cth}^2 \frac{x}{2} \right) + B \operatorname{cth} \frac{x}{2}.$$

2.454

1.
$$\int \frac{(A + B \operatorname{sh} x) dx}{\operatorname{ch} x(a + b \operatorname{sh} x)} = \frac{1}{a^2 + b^2} \left[(Aa + Bb) \operatorname{arctg}(\operatorname{sh} x) + (Ab - Ba) \ln \left| \frac{a + b \operatorname{sh} x}{\operatorname{ch} x} \right| \right].$$
2.
$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{sh} x(a + b \operatorname{sh} x)} = \frac{1}{a} \left(A \ln \left| \operatorname{th} \frac{x}{2} \right| + B \ln \left| \frac{\operatorname{sh} x}{a + b \operatorname{sh} x} \right| - Ab \int \frac{dx}{a + b \operatorname{sh} x} \right) \quad (\text{see } \mathbf{2.441} \text{ 3.}).$$

2.441

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2.455

1.
$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{sh} x(a + b \operatorname{ch} x)} = \frac{1}{a^2 - b^2} \left[(Aa + Bb) \ln \left| \operatorname{th} \frac{x}{2} \right| + (Ab - Ba) \ln \left| \frac{a + b \operatorname{ch} x}{\operatorname{sh} x} \right| \right].$$

For $a^2 = b^2 (= 1)$:

2.
$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{sh} x(1 + \operatorname{ch} x)} = \frac{A + B}{2} \ln \left| \operatorname{th} \frac{x}{2} \right| - \frac{A - B}{4} \operatorname{th}^2 \frac{x}{2}.$$
3.
$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{sh} x(1 - \operatorname{ch} x)} = \frac{A + B}{4} \operatorname{cth}^2 \frac{x}{2} - \frac{A - B}{2} \ln \operatorname{cth} \frac{x}{2}.$$

2.456

$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{ch} x(a + b \operatorname{sh} x)} = \frac{A}{a^2 + b^2} \left[a \operatorname{arctg}(\operatorname{sh} x) + b \ln \left| \frac{a + b \operatorname{sh} x}{\operatorname{ch} x} \right| \right] + B \int \frac{dx}{a + b \operatorname{sh} x} \quad (\text{see } \mathbf{2.441} \text{ 3.}).$$

2.441

2.457

$$\int \frac{(A + B \operatorname{ch} x) dx}{\operatorname{ch} x(a + b \operatorname{ch} x)} = \frac{1}{a} \left[A \operatorname{arctg} \operatorname{sh} x - (Ab - Ba) \int \frac{dx}{a + b \operatorname{ch} x} \right] \quad (\text{see } \mathbf{2.443} \text{ 3.}).$$

2.458

$$\begin{aligned}
1. \quad \int \frac{dx}{a + b \operatorname{sh}^2 x} &= \frac{1}{\sqrt{a(b-a)}} \operatorname{arctg} \left(\sqrt{\frac{b}{a} - 1} \operatorname{th} x \right) \quad \left[\frac{b}{a} > 1 \right] \\
&= \frac{1}{\sqrt{a(a-b)}} \operatorname{Arth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \quad \left[0 < \frac{b}{a} < 1 \text{ or } \frac{b}{a} < 0 \text{ and } \operatorname{sh}^2 x < -\frac{a}{b} \right]; \\
&= \frac{1}{\sqrt{a(a-b)}} \operatorname{Arcth} \left(\sqrt{1 - \frac{b}{a}} \operatorname{th} x \right) \quad \left[\frac{b}{a} < 0 \text{ and } \operatorname{sh}^2 x > -\frac{a}{b} \right].
\end{aligned}$$

MZ 195

$$\begin{aligned}
2. \quad \int \frac{dx}{a + b \operatorname{ch}^2 x} &= \frac{1}{\sqrt{-a(a+b)}} \operatorname{arctg} \left(\sqrt{-\left(1 + \frac{b}{a}\right)} \operatorname{cth} x \right) \quad \left[\frac{b}{a} < -1 \right]; \\
&= \frac{1}{\sqrt{a(a+b)}} \operatorname{Arth} \left(\sqrt{1 + \frac{b}{a}} \operatorname{cth} x \right) \quad \left[-1 < \frac{b}{a} < 0 \text{ and } \operatorname{ch}^2 x > -\frac{a}{b} \right]; \\
&= \frac{1}{\sqrt{a(a+b)}} \operatorname{Arcth} \left(\sqrt{1 + \frac{b}{a}} \operatorname{cth} x \right) \\
&\quad \left[\frac{b}{a} > 0 \text{ or } -1 < \frac{b}{a} < 0 \text{ and } \operatorname{ch}^2 x < -\frac{a}{b} \right].
\end{aligned}$$

MZ 202

For $a^2 = b^2 = 1$:

$$3. \quad \int \frac{dx}{1 + \operatorname{sh}^2 x} = \operatorname{th} x.$$

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$$\begin{aligned}
4. \quad \int \frac{dx}{1 - \operatorname{sh}^2 x} &= \frac{1}{\sqrt{2}} \operatorname{Arth}(\sqrt{2} \operatorname{th} x) \quad [\operatorname{sh}^2 x < 1]; \\
&= \frac{1}{\sqrt{2}} \operatorname{Arcth}(\sqrt{2} \operatorname{th} x) \quad [\operatorname{sh}^2 x > 1].
\end{aligned}$$

$$5. \quad \int \frac{dx}{1 + \operatorname{ch}^2 x} = \frac{1}{\sqrt{2}} \operatorname{Arcth}(\sqrt{2} \operatorname{cth} x).$$

$$6. \int \frac{dx}{1 - \operatorname{ch}^2 x} = \operatorname{cth} x.$$

2.459

$$1. \int \frac{dx}{(a + b \operatorname{sh}^2 x)^2} = \frac{1}{2a(b-a)} \left[\frac{b \operatorname{sh} x \operatorname{ch} x}{a + b \operatorname{sh}^2 x} + (b-2a) \int \frac{dx}{a + b \operatorname{sh}^2 x} \right] \quad (\text{see } \mathbf{2.458} \text{ 1.}).$$

2.458
MZ 196

$$2. \int \frac{dx}{(a + b \operatorname{ch}^2 x)^2} = \frac{1}{2a(a+b)} \left[-\frac{b \operatorname{sh} x \operatorname{ch} x}{a + b \operatorname{ch}^2 x} + (2a+b) \int \frac{dx}{a + b \operatorname{ch}^2 x} \right] \quad (\text{see } \mathbf{2.458} \text{ 2.}).$$

2.458
MZ 203

$$\begin{aligned} 3. \int \frac{dx}{(a + b \operatorname{sh}^2 x)^3} &= \frac{1}{8pa^3} \left[\left(3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{th} x) + \left(3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{th} x}{1 + p^2 \operatorname{th}^2 x} + \right. \\ &\quad \left. + \left(1 + \frac{2}{p^2} - \frac{1}{p^2} \operatorname{th}^2 x \right) \frac{2p \operatorname{th} x}{(1 + p^2 \operatorname{th}^2 x)^2} \right] \quad \left[p^2 = \frac{b}{a} - 1 > 0 \right]; \\ &= \frac{1}{8qa^3} \left[\left(3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{th} x) + \left(3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{th} x}{1 - q^2 \operatorname{th}^2 x} + \right. \\ &\quad \left. + \left(1 - \frac{2}{q^2} + \frac{1}{q^2} \operatorname{th}^2 x \right) \frac{2q \operatorname{th} x}{(1 - q^2 \operatorname{th}^2 x)^2} \right] \quad \left[q^2 = 1 - \frac{b}{a} > 0 \right]. \end{aligned}$$

MZ 196

* If $\frac{b}{a} < 0$ and $\operatorname{ch}^2 x > -\frac{a}{b}$, then

$\varphi(x) = \operatorname{Arth}(q \operatorname{cth} x)$. If $\frac{b}{a} < 0$, but $\operatorname{ch}^2 x < -\frac{a}{b}$, or if $\frac{b}{a} > 0$, then $\varphi(x) = \operatorname{Arcth}(q \operatorname{cth} x)$.

$$\begin{aligned} 4. \int \frac{dx}{(a + b \operatorname{ch}^2 x)^3} &= \frac{1}{8pa^3} \left[\left(3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{cth} x) + \left(3 - \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{cth} x}{1 + p^2 \operatorname{cth}^2 x} + \right. \\ &\quad \left. + \left(1 + \frac{2}{p^2} - \frac{1}{p^2} \operatorname{cth}^2 x \right) \frac{2p \operatorname{cth} x}{(1 + p^2 \operatorname{cth}^2 x)^2} \right] \quad \left[p^2 = -1 - \frac{b}{a} > 0 \right]; \\ &= \frac{1}{8qa^3} \left[\left(3 + \frac{2}{q^2} + \frac{3}{q^4} \right) \varphi(x)^* + \left(3 + \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{cth} x}{1 - q^2 \operatorname{cth}^2 x} + \right. \\ &\quad \left. + \left(1 - \frac{2}{q^2} + \frac{1}{q^2} \operatorname{cth}^2 x \right) \frac{2q \operatorname{cth} x}{(1 - q^2 \operatorname{cth}^2 x)^2} \right] \quad \left[q^2 = 1 + \frac{b}{a} > 0 \right]. \end{aligned}$$

$$1. \int \sqrt{\text{th } x} dx = \text{Arth} \sqrt{\text{th } x} - \text{arctg} \sqrt{\text{th } x}.$$

MZ 221

$$2. \int \sqrt{\text{cth } x} dx = \text{Arcth} \sqrt{\text{cth } x} - \text{arctg} \sqrt{\text{cth } x}.$$

MZ 222

2.462

$$1. \int \frac{\text{sh } x dx}{\sqrt{a^2 + \text{sh}^2 x}} = \text{Arsh} \frac{\text{ch } x}{\sqrt{a^2 - 1}} = \ln(\text{ch } x + \sqrt{a^2 + \text{sh}^2 x}) \quad [a^2 > 1];$$

$$= \text{Arch} \frac{\text{ch } x}{\sqrt{1 - a^2}} = \ln(\text{ch } x + \sqrt{a^2 + \text{sh}^2 x}) \quad [a^2 < 1];$$

$$= \ln \text{ch } x \quad [a^2 = 1].$$

$$2. \int \frac{\text{sh } x dx}{\sqrt{a^2 - \text{sh}^2 x}} = \arcsin \frac{\text{ch } x}{\sqrt{a^2 + 1}} \quad [\text{sh}^2 x < a^2].$$

$$3. \int \frac{\text{sh } x dx}{\sqrt{\text{sh}^2 x - a^2}} = \text{Arch} \frac{\text{ch } x}{\sqrt{a^2 + 1}} = \ln(\text{ch } x + \sqrt{\text{sh}^2 x - a^2}) \quad [\text{sh}^2 x > a^2].$$

MZ 199

$$4. \int \frac{\text{ch } x dx}{\sqrt{a^2 + \text{sh}^2 x}} = \text{Arsh} \frac{\text{sh } x}{a} = \ln(\text{sh } x + \sqrt{a^2 + \text{sh}^2 x}).$$

$$5. \int \frac{\text{ch } x dx}{\sqrt{a^2 - \text{sh}^2 x}} = \arcsin \frac{\text{sh } x}{a} \quad [\text{sh}^2 x < a^2].$$

$$6. \int \frac{\text{ch } x dx}{\sqrt{\text{sh}^2 x - a^2}} = \text{Arch} \frac{\text{sh } x}{a} = \ln(\text{sh } x + \sqrt{\text{sh}^2 x - a^2}) \quad [\text{sh}^2 x > a^2].$$

$$7. \int \frac{\text{sh } x dx}{\sqrt{a^2 + \text{ch}^2 x}} = \text{Arsh} \frac{\text{ch } x}{a} = \ln(\text{ch } x + \sqrt{a^2 + \text{ch}^2 x}).$$

$$8. \int \frac{\operatorname{sh} x \, dx}{\sqrt{a^2 - \operatorname{ch}^2 x}} = \arcsin \frac{\operatorname{ch} x}{a} \quad [\operatorname{ch}^2 x < a^2].$$

$$9. \int \frac{\operatorname{sh} x \, dx}{\sqrt{\operatorname{ch}^2 x - a^2}} = \operatorname{Arch} \frac{\operatorname{ch} x}{a} = \ln(\operatorname{ch} x + \sqrt{\operatorname{ch}^2 x - a^2}) \quad [\operatorname{ch}^2 x > a^2].$$

MZ 215, 216

$$10. \int \frac{\operatorname{ch} x \, dx}{\sqrt{a^2 + \operatorname{ch}^2 x}} = \operatorname{Arsh} \frac{\operatorname{sh} x}{\sqrt{a^2 + 1}} = \ln(\operatorname{sh} x + \sqrt{a^2 + \operatorname{ch}^2 x}).$$

$$11. \int \frac{\operatorname{ch} x \, dx}{\sqrt{a^2 - \operatorname{ch}^2 x}} = \arcsin \frac{\operatorname{sh} x}{\sqrt{a^2 - 1}} \quad [\operatorname{ch}^2 x < a^2].$$

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$$12. \int \frac{\operatorname{ch} x \, dx}{\sqrt{\operatorname{ch}^2 x - a^2}} = \operatorname{Arch} \frac{\operatorname{sh} x}{\sqrt{a^2 - 1}} \quad [a^2 > 1];$$

$$= \ln \operatorname{sh} x \quad [a^2 = 1].$$

MZ 206

$$13. \int \frac{\operatorname{cth} x \, dx}{\sqrt{a + b \operatorname{sh} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \quad [b \operatorname{sh} x > 0, \quad a > 0];$$

$$= 2\sqrt{a} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{sh} x} \quad [b \operatorname{sh} x < 0, \quad a > 0];$$

$$= 2\sqrt{-a} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{sh} x\right)} \quad a < 0.$$

$$14. \int \frac{\operatorname{th} x \, dx}{\sqrt{a + b \operatorname{ch} x}} = 2\sqrt{a} \operatorname{Arcth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x} \quad [b \operatorname{ch} x > 0, \quad a > 0];$$

$$= 2\sqrt{a} \operatorname{Arth} \sqrt{1 + \frac{b}{a} \operatorname{ch} x} \quad [b \operatorname{ch} x < 0, \quad a > 0];$$

$$= 2\sqrt{-a} \operatorname{Arth} \sqrt{-\left(1 + \frac{b}{a} \operatorname{ch} x\right)} \quad [a < 0].$$

MZ 220, 221

$$\begin{aligned}
1. \int \frac{\operatorname{sh} x \sqrt{a + b \operatorname{ch} x}}{p + q \operatorname{ch} x} dx &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{ch} x)}{aq - bp}} \quad \left[b \operatorname{ch} x > 0, \frac{aq - bp}{q} > 0 \right]; \\
&= 2\sqrt{\frac{aq - bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{ch} x)}{aq - bp}} \quad \left[b \operatorname{ch} x < 0, \frac{aq - bp}{q} > 0 \right]; \\
&= 2\sqrt{\frac{bp - aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{ch} x)}{bp - aq}} \quad \left[\frac{aq - bp}{q} < 0 \right].
\end{aligned}$$

MZ 220

$$\begin{aligned}
2. \int \frac{\operatorname{ch} x \sqrt{a + b \operatorname{sh} x}}{p + q \operatorname{sh} x} dx &= 2\sqrt{\frac{aq - bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{sh} x)}{aq - bp}} \quad \left[b \operatorname{sh} x > 0, \frac{aq - bp}{q} > 0 \right]; \\
&= 2\sqrt{\frac{aq - bp}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{sh} x)}{aq - bp}} \quad \left[b \operatorname{sh} x < 0, \frac{aq - bp}{q} > 0 \right]; \\
&= 2\sqrt{\frac{bp - aq}{q}} \operatorname{Arth} \sqrt{\frac{q(a + b \operatorname{sh} x)}{bp - aq}} \quad \left[\frac{aq - bp}{q} < 0 \right].
\end{aligned}$$

MZ 221

$$1. \int \frac{dx}{\sqrt{k^2 + k'^2 \operatorname{ch}^2 x}} = \int \frac{dx}{\sqrt{1 + k'^2 \operatorname{sh}^2 x}} = F(\operatorname{arcsin}(\operatorname{th} x), k) \quad [x > 0].$$

BY (295.00)(295.10)

$$2. \int \frac{dx}{\sqrt{\operatorname{ch}^2 x - k^2}} = \int \frac{dx}{\sqrt{\operatorname{sh}^2 x + k'^2}} = F\left(\operatorname{arcsin}\left(\frac{1}{\operatorname{ch} x}\right), k\right) \quad [x > 0].$$

BY (295.40)(295.30)

$$3. \int \frac{dx}{\sqrt{1 - k'^2 \operatorname{ch}^2 x}} = F\left(\operatorname{arcsin}\left(\frac{\operatorname{th} x}{k}\right), k\right) \quad \left[0 < x < \operatorname{Arch} \frac{1}{k'}\right].$$

BY (295.20)

In 2.464 4.-2.464 8., we set $\alpha = \operatorname{arccos} \frac{1 - \operatorname{sh} 2ax}{1 + \operatorname{sh} 2ax}$, $r = \frac{1}{\sqrt{2}}$ $[ax > 0]$:

BY (296.50)

$$5. \int \sqrt{\text{sh } 2ax} \, dx = \frac{1}{2a} [F(\alpha, r) - 2E(\alpha, r)] + \frac{1}{a} \frac{\sqrt{\text{sh } 2ax(1 + \text{sh}^2 2ax)}}{1 + \text{sh } 2ax}.$$

BY (296.53)

$$6. \int \frac{\text{ch}^2 2ax \, dx}{(1 + \text{sh } 2ax)^2 \sqrt{\text{sh } 2ax}} = \frac{1}{2a} E(\alpha, r).$$

BY (296.51)

$$7. \int \frac{(1 - \text{sh } 2ax)^2 \, dx}{(1 + \text{sh } 2ax)^2 \sqrt{\text{sh } 2ax}} = \frac{1}{2a} [2E(\alpha, r) - F(\alpha, r)].$$

BY (296.55)

$$8. \int \frac{\sqrt{\text{sh } 2ax} \, dx}{(1 + \text{sh } 2ax)^2} = \frac{1}{4a} [F(\alpha, r) - E(\alpha, r)].$$

BY (296.54)

In 2.464 9.-2.464 15., we set $\alpha = \arcsin \sqrt{\frac{\text{ch } 2ax - 1}{\text{ch } 2ax}}$, $r = \frac{1}{\sqrt{2}} [x \neq 0]$:

$$9. \int \frac{dx}{\sqrt{\text{ch } 2ax}} = \frac{1}{a\sqrt{2}} F(\alpha, r).$$

BY (296.00)

$$10. \int \sqrt{\text{ch } 2ax} \, dx = \frac{1}{a\sqrt{2}} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\text{sh } 2ax}{a\sqrt{\text{ch } 2ax}}.$$

BY (296.03)

$$11. \int \frac{dx}{\sqrt{\text{ch}^3 2ax}} = \frac{1}{a\sqrt{2}} [2E(\alpha, r) - F(\alpha, r)].$$

BY (296.04)

BY (296.04)

$$13. \int \frac{\operatorname{sh}^2 2ax \, dx}{\sqrt{\operatorname{ch} 2ax}} = -\frac{\sqrt{2}}{3a} F(\alpha, r) + \frac{1}{3a} \operatorname{sh} 2ax \sqrt{\operatorname{ch} 2ax}.$$

BY (296.07)

$$14. \int \frac{\operatorname{th}^2 2ax \, dx}{\sqrt{\operatorname{ch} 2ax}} = \frac{\sqrt{2}}{3a} F(\alpha, r) - \frac{\operatorname{th} 2ax}{3a\sqrt{\operatorname{ch} 2ax}}.$$

BY (296.05)

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$$15. \int \frac{\sqrt{\operatorname{ch} 2ax} \, dx}{sp^2 + (1-p^2)\operatorname{ch} 2ax} = \frac{1}{a\sqrt{2}} \Pi(\alpha, p^2, r).$$

BY (296.02)

In 2.464 16.-2.464 20., we set

$$\alpha = \arccos \frac{\sqrt{a^2 + b^2} - a - b \operatorname{sh} x}{\sqrt{a^2 + b^2} + a + b \operatorname{sh} x},$$

$$r = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2\sqrt{a^2 + b^2}}} \quad \left[a > 0, \quad b > 0, \quad x > -\operatorname{Arsh} \frac{a}{b} \right] :$$

$$16. \int \frac{dx}{\sqrt{a + b \operatorname{sh} x}} = \frac{1}{\sqrt[4]{a^2 + b^2}} F(\alpha, r).$$

BY (298.00)

$$17. \int \sqrt{a + b \operatorname{sh} x} \, dx = \sqrt[4]{a^2 + b^2} [F(\alpha, r) - 2E(\alpha, r)] + \frac{2b \operatorname{ch} x \sqrt{a + b \operatorname{sh} x}}{\sqrt{a^2 + b^2} + a + b \operatorname{sh} x}.$$

BY (298.02)

BY (298.03)

$$19. \int \frac{\operatorname{ch}^2 x \, dx}{[\sqrt{a^2 + b^2} + a + b \operatorname{sh} x]^2 \sqrt{a + b \operatorname{sh} x}} = \frac{1}{b^2 \sqrt[4]{a^2 + b^2}} E(\alpha, r).$$

BY (298.01)

$$20. \int \frac{\sqrt{a + b \operatorname{sh} x} \, dx}{[\sqrt{a^2 + b^2} - a - b \operatorname{sh} x]^2} = -\frac{1}{\sqrt[4]{a^2 + b^2}(\sqrt{a^2 + b^2} - a)} E(\alpha, r) + \frac{b}{\sqrt{a^2 + b^2} - a} \cdot \frac{\operatorname{ch} x \sqrt{a + b \operatorname{sh} x}}{a^2 + b^2 - (a + b \operatorname{sh} x)^2}.$$

BY (298.04)

In 2.464 21.-2.464 31., we set $\alpha = \arcsin(\operatorname{th} \frac{x}{2})$, $r = \sqrt{\frac{a-b}{a+b}}$ [$0 < b < a$, $x > 0$]:

$$21. \int \frac{dx}{\sqrt{a + b \operatorname{ch} x}} = \frac{2}{\sqrt{a + b}} F(\alpha, r).$$

BY (297.25)

$$22. \int \sqrt{a + b \operatorname{ch} x} \, dx = 2\sqrt{a + b} [F(\alpha, r) - E(\alpha, r)] + 2 \operatorname{th} \frac{x}{2} \sqrt{a + b \operatorname{ch} x}.$$

BY (297.29)

$$23. \int \frac{\operatorname{ch} x \, dx}{\sqrt{a + b \operatorname{ch} x}} = \frac{2}{\sqrt{a + b}} F(\alpha, r) - \frac{2\sqrt{a + b}}{b} E(\alpha, r) + \frac{2}{b} \operatorname{th} \frac{x}{2} \sqrt{a + b \operatorname{ch} x}.$$

BY (297.33)

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$$24. \int \frac{\operatorname{th}^2 \frac{x}{2}}{\sqrt{a + b \operatorname{ch} x}} \, dx = \frac{2\sqrt{a + b}}{a - b} [F(\alpha, r) - E(\alpha, r)].$$

BY (297.28)

$$25. \int \frac{\operatorname{th}^4 \frac{x}{2}}{\sqrt{a + b \operatorname{ch} x}} \, dx = \frac{2\sqrt{a + b}}{3(a - b)^2} [(3a + b)F(\alpha, r) - 4aE(\alpha, r)] + \frac{2}{3(a - b)} \frac{\operatorname{sh} \frac{x}{2} \sqrt{a + b \operatorname{ch} x}}{\operatorname{ch}^3 \frac{x}{2}}.$$

$$26. \int \frac{\operatorname{ch} x - 1}{\sqrt{a + b \operatorname{ch} x}} dx = \frac{2}{b} \left[\operatorname{th} \frac{x}{2} \sqrt{a + b \operatorname{ch} x} - \sqrt{a + b} E(\alpha, r) \right].$$

BY (297.31)

$$27. \int \frac{(\operatorname{ch} x - 1)^2}{\sqrt{a + b \operatorname{ch} x}} dx = \frac{4\sqrt{a+b}}{3b^2} [(a+3b)E(\alpha, r) - bF(\alpha, r)] + \\ + \frac{4}{3b^2} \left[b \operatorname{ch}^2 \frac{x}{2} - (a+3b) \right] \operatorname{th} \frac{x}{2} \sqrt{a + b \operatorname{ch} x}.$$

BY (297.31)

$$28. \int \frac{\sqrt{a + b \operatorname{ch} x}}{\operatorname{ch} x + 1} dx = \sqrt{a + b} E(\alpha, r).$$

BY (297.26)

$$29. \int \frac{dx}{(\operatorname{ch} x + 1)\sqrt{a + b \operatorname{ch} x}} = \frac{\sqrt{a+b}}{a-b} E(\alpha, r) - \frac{2b}{(a-b)\sqrt{a+b}} F(\alpha, r).$$

BY (297.30)

$$30. \int \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{a + b \operatorname{ch} x}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} [b(5b-a)F(\alpha, r) + \\ + (a-3b)(a+b)E(\alpha, r)] + \frac{1}{6(a-b)} \cdot \frac{\operatorname{sh} \frac{x}{2}}{\operatorname{ch}^3 \frac{x}{2}} \sqrt{a + b \operatorname{ch} x}.$$

BY (297.30)

$$31. \int \frac{(1 + \operatorname{ch} x) dx}{[1 + p^2 + (1 - p^2)\operatorname{ch} x]\sqrt{a + b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} \Pi(\alpha, p^2, r).$$

BY (297.27)

In 2.464 32.-2.464 40., we set $\alpha = \arcsin \sqrt{\frac{a-b \operatorname{ch} x}{a-b}}$, $r = \sqrt{\frac{a-b}{a+b}}$ [$0 < b < a$, $0 < x < \operatorname{Arch} \frac{a}{b}$]:

$$32. \int \frac{dx}{\sqrt{a - b \operatorname{ch} x}} = \frac{2}{\sqrt{a+b}} F(\alpha, r).$$

$$33. \int \sqrt{a-b \operatorname{ch} x} dx = 2\sqrt{a+b} [F(\alpha, r) - E(\alpha, r)].$$

BY (297.54)

$$34. \int \frac{\operatorname{ch} x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r) - \frac{2}{\sqrt{a+b}} F(\alpha, r).$$

BY (297.56)

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$$35. \int \frac{\operatorname{ch}^2 x dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2(b-2a)}{3b\sqrt{a+b}} F(\alpha, r) + \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) + \frac{2}{3b} \operatorname{sh} x \sqrt{a-b \operatorname{ch} x}.$$

BY (297.56)

$$36. \int \frac{(1+\operatorname{ch} x) dx}{\sqrt{a-b \operatorname{ch} x}} = \frac{2\sqrt{a+b}}{b} E(\alpha, r).$$

BY (297.51)

$$37. \int \frac{dx}{\operatorname{ch} x \sqrt{a-b \operatorname{ch} x}} = \frac{2b}{a\sqrt{a+b}} \Pi\left(\alpha, \frac{a-b}{a}, r\right).$$

BY (297.57)

$$38. \int \frac{dx}{(1+\operatorname{ch} x)\sqrt{a-b \operatorname{ch} x}} = \frac{1}{\sqrt{a+b}} E(\alpha, r) - \frac{1}{a+b} \operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}.$$

BY (297.58)

$$39. \int \frac{dx}{(1+\operatorname{ch} x)^2 \sqrt{a-b \operatorname{ch} x}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+3b)E(\alpha, r) - bF(\alpha, r)] - \frac{1}{3(a+b)^2} \frac{\operatorname{th} \frac{x}{2} \sqrt{a-b \operatorname{ch} x}}{\operatorname{ch} x + 1} [2a+4b+(a+3b)\operatorname{ch} x].$$

BY (297.58)

$$40. \int \frac{dx}{(a-b-ap^2+bp^2 \operatorname{ch} x)\sqrt{a-b \operatorname{ch} x}} = \frac{2}{(a-b)\sqrt{a+b}} \Pi(\alpha, p^2, r).$$

In 2.464 41.-2.464 47., we set $\alpha = \arcsin \sqrt{\frac{b(\operatorname{ch} x - 1)}{b \operatorname{ch} x - a}}$, $r = \sqrt{\frac{a+b}{2b}}$ [$0 < a < b$, $x > 0$]:

$$41. \int \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} F(\alpha, r).$$

BY (297.00)

$$42. \int \sqrt{b \operatorname{ch} x - a} dx = (b-a) \sqrt{\frac{2}{b}} F(\alpha, r) - 2\sqrt{2b} E(\alpha, r) + \frac{2b \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

BY (297.05)

$$43. \int \frac{dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{1}{b^2 - a^2} \cdot \sqrt{\frac{2}{b}} [2bE(\alpha, r) - (b-a)F(\alpha, r)].$$

BY (297.06)

$$44. \int \frac{dx}{\sqrt{(b \operatorname{ch} x - a)^5}} = \frac{1}{3(b^2 - a^2)^2} \sqrt{\frac{2}{b}} [(b-3a)(b-a)F(\alpha, r) + 8abE(\alpha, r)] + \frac{2b}{3(b^2 - a^2)} \cdot \frac{\operatorname{sh} x}{\sqrt{(b \operatorname{ch} x - a)^3}}.$$

BY (297.06)

$$45. \int \frac{\operatorname{ch} x dx}{\sqrt{b \operatorname{ch} x - a}} = \sqrt{\frac{2}{b}} [F(\alpha, r) - 2E(\alpha, r)] + \frac{2 \operatorname{sh} x}{\sqrt{b \operatorname{ch} x - a}}.$$

BY (297.03)

$$46. \int \frac{(\operatorname{ch} x + 1) dx}{\sqrt{(b \operatorname{ch} x - a)^3}} = \frac{2}{b-a} \sqrt{\frac{2}{b}} E(\alpha, r).$$

BY (297.01)

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$$47. \int \frac{\sqrt{b \operatorname{ch} x - a} dx}{p^2 b - a + b(1 - p^2) \operatorname{ch} x} = \sqrt{\frac{2}{b}} \Pi(\alpha, p^2, r).$$

BY (297.02)

In 2.464 48.-2.464 55., we set $\alpha = \arcsin \sqrt{\frac{b \operatorname{ch} x - a}{b(\operatorname{ch} x - 1)}}$, $r = \sqrt{\frac{2b}{a+b}}$ $[0 < b < a, x > \operatorname{Arch} \frac{a}{b}]$:

$$48. \int \frac{dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2}{\sqrt{a+b}} F(\alpha, r).$$

BY (297.75)

$$49. \int \sqrt{b \operatorname{ch} x - a} dx = -2\sqrt{a+b} E(\alpha, r) + 2 \operatorname{cth} \frac{x}{2} \sqrt{b \operatorname{ch} x - a}.$$

BY (297.79)

$$50. \int \frac{\operatorname{cth}^2 \frac{x}{2} dx}{\sqrt{b \operatorname{ch} x - a}} = \frac{2\sqrt{a+b}}{a-b} E(\alpha, r).$$

BY (297.76)

$$51. \int \frac{\sqrt{b \operatorname{ch} x - a}}{\operatorname{ch} x - 1} dx = \sqrt{a+b} [F(\alpha, r) - E(\alpha, r)].$$

BY (297.77)

$$52. \int \frac{dx}{(\operatorname{ch} x - 1)\sqrt{b \operatorname{ch} x - a}} = \frac{\sqrt{a+b}}{a-b} E(\alpha, r) - \frac{1}{\sqrt{a+b}} F(\alpha, r).$$

BY (297.78)

$$53. \int \frac{dx}{(\operatorname{ch} x - 1)^2 \sqrt{b \operatorname{ch} x - a}} = \frac{1}{3(a-b)^2 \sqrt{a+b}} [(a-2b)(a-b)F(\alpha, r) + \\ + (3a-b)(a+b)E(\alpha, r)] + \frac{a+b}{6b(a-b)} \cdot \frac{\operatorname{ch} \frac{x}{2}}{\operatorname{sh}^3 \frac{x}{2}} \sqrt{b \operatorname{ch} x - a}.$$

BY (297.78)

$$54. \int \frac{dx}{(\operatorname{ch} x + 1)\sqrt{b \operatorname{ch} x - a}} = \frac{1}{\sqrt{a+b}} [F(\alpha, r) - E(\alpha, r)] + \frac{2\sqrt{b \operatorname{ch} x - a}}{(a+b)\operatorname{sh} x}.$$

BY (297.80)

BY (297.80)

$$55. \int \frac{dx}{(\operatorname{ch} x + 1)^2 \sqrt{b \operatorname{ch} x - a}} = \frac{1}{3\sqrt{(a+b)^3}} [(a+b)F(\alpha, r) - (a+3b)E(\alpha, r)] + \frac{\sqrt{b \operatorname{ch} x - a}}{3(a+b)\operatorname{sh} x} \left(2\frac{a+3b}{a+b} - \operatorname{th}^2 \frac{x}{2} \right).$$

BY (297.80)

In 2.464 56.-2.464 60., we set $\alpha = \arccos \frac{\sqrt[4]{b^2 - a^2}}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}$, $r = \frac{1}{\sqrt{2}} \left[0 < a < b, -\operatorname{Arsh} \frac{a}{\sqrt{b^2 - a^2}} < x \right]$:

$$56. \int \frac{dx}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}} = \sqrt[4]{\frac{4}{b^2 - a^2}} F(\alpha, r).$$

BY (299.00)

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$$57. \int \sqrt{a \operatorname{sh} x + b \operatorname{ch} x} dx = \sqrt[4]{4(b^2 - a^2)} [F(\alpha, r) - 2E(\alpha, r)] + \frac{2(a \operatorname{ch} x + b \operatorname{sh} x)}{\sqrt{a \operatorname{sh} x + b \operatorname{ch} x}}.$$

BY (299.02)

$$58. \int \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = \sqrt[4]{\frac{4}{(b^2 - a^2)^3}} [2E(\alpha, r) - F(\alpha, r)].$$

BY (299.03)

$$59. \int \frac{dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^5}} = \frac{1}{3} \sqrt[4]{\frac{4}{(b^2 - a^2)^5}} F(\alpha, r) + \frac{2}{3(b^2 - a^2)} \cdot \frac{a \operatorname{ch} x + b \operatorname{sh} x}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}}.$$

BY (299.03)

$$60. \int \frac{(\sqrt{b^2 - a^2} + a \operatorname{sh} x + b \operatorname{ch} x) dx}{\sqrt{(a \operatorname{sh} x + b \operatorname{ch} x)^3}} = 2 \sqrt[4]{\frac{4}{b^2 - a^2}} E(\alpha, r).$$

BY (299.01)

2.47 Combinations of hyperbolic functions and powers

$$\begin{aligned}
1. \quad \int x^r \operatorname{sh}^p x \operatorname{ch}^q x \, dx &= \frac{1}{(p+q)^2} \left[(p+q)x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x - \right. \\
&\quad \left. - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r+1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x \, dx + \right. \\
&\quad \left. + r p \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x \, dx + (q-1)(p+q) \int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x \, dx \right]; \\
&= \frac{1}{(p+q)^2} \left[(p+q)x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x - \right. \\
&\quad \left. - r x^{r-1} \operatorname{sh}^p x \operatorname{ch}^q x + r(r-1) \int x^{r-2} \operatorname{sh}^p x \operatorname{ch}^q x \, dx - \right. \\
&\quad \left. - r q \int x^{r-1} \operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x \, dx - (p-1)(p+q) \int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x \, dx \right].
\end{aligned}$$

GU ((353))(1)

$$2. \quad \int x^n \operatorname{sh}^{2m} x \, dx = (-1)^m \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \operatorname{ch}(2m-2k)x \, dx.$$

$$3. \quad \int x^n \operatorname{sh}^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \operatorname{sh}(2m-2k+1)x \, dx.$$

$$4. \quad \int x^n \operatorname{ch}^{2m} x \, dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \operatorname{ch}(2m-2k)x \, dx.$$

$$5. \quad \int x^n \operatorname{ch}^{2m+1} x \, dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \operatorname{ch}(2m-2k+1)x \, dx.$$

$$\begin{aligned}
1. \quad \int x^n \operatorname{sh} x \, dx &= x^n \operatorname{ch} x - n \int x^{n-1} \operatorname{ch} x \, dx = \\
&= x^n \operatorname{ch} x - n x^{n-1} \operatorname{sh} x + n(n-1) \int x^{n-2} \operatorname{sh} x \, dx.
\end{aligned}$$

2.473

Notation: $z_1 = a + bx$

$$1. \int z_1 \operatorname{sh} kx \, dx = \frac{1}{k} z_1 \operatorname{ch} kx - \frac{b}{k^2} \operatorname{sh} kx.$$

$$2. \int z_1 \operatorname{ch} kx \, dx = \frac{1}{k} z_1 \operatorname{sh} kx - \frac{b}{k^2} \operatorname{ch} kx.$$

$$3. \int z_1^2 \operatorname{sh} kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{ch} kx - \frac{2bz_1}{k^2} \operatorname{sh} kx.$$

$$4. \int z_1^2 \operatorname{ch} kx \, dx = \frac{1}{k} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{sh} kx - \frac{2bz_1}{k^2} \operatorname{ch} kx.$$

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$$5. \int z_1^3 \operatorname{sh} kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{ch} kx - \frac{3b}{k^2} \left(z_1^2 + \frac{2b^2}{k^2} \right) \operatorname{sh} kx.$$

$$6. \int z_1^3 \operatorname{ch} kx \, dx = \frac{z_1}{k} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{sh} kx - \frac{3b}{k^2} \left(z_1^3 + \frac{2b^2}{k^2} \right) \operatorname{ch} kx.$$

$$7. \int z_1^4 \operatorname{sh} kx \, dx = \frac{1}{k} \left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \operatorname{ch} kx - \frac{4bz_1}{k^2} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{sh} kx.$$

$$8. \int z_1^4 \operatorname{ch} kx \, dx = \frac{1}{k} \left(z_1^4 + \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \operatorname{sh} kx - \frac{4bz_1}{k^2} \left(z_1^2 + \frac{6b^2}{k^2} \right) \operatorname{ch} kx.$$

$$9. \int z_1^5 \operatorname{sh} kx \, dx = \frac{z_1}{k} \left(z_1^4 + \frac{20b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \operatorname{ch} kx - \frac{5b}{k^2} \left(z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \operatorname{sh} kx.$$

$$10. \int z_1^5 \operatorname{ch} kx \, dx = \frac{z_1}{k} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \operatorname{sh} kx - \frac{5b}{k^2} \left(z_1^4 + 12 \frac{b^2}{k^2} z_1^2 + 24 \frac{b^4}{k^4} \right) \operatorname{ch} kx.$$

$$11. \int z_1^6 \operatorname{sh} kx \, dx = \frac{1}{k} \left(z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \operatorname{ch} kx - \frac{6bz_1}{k^2} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \operatorname{sh} kx.$$

$$12. \int z_1^6 \operatorname{ch} kx \, dx = \frac{1}{k} \left(z_1^6 + 30 \frac{b^2}{k^2} z_1^4 + 360 \frac{b^4}{k^4} z_1^2 + 720 \frac{b^6}{k^6} \right) \operatorname{sh} kx - \frac{6bz_1}{k^2} \left(z_1^4 + 20 \frac{b^2}{k^2} z_1^2 + 120 \frac{b^4}{k^4} \right) \operatorname{ch} kx.$$

2.474

$$1. \int x^n \operatorname{sh}^2 x \, dx = -\frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{[n/2]} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x \right\}.$$

GU ((353))(2b)

$$2. \int x^n \operatorname{ch}^2 x \, dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \sum_{k=0}^{[n/2]} \left\{ \frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x \right\}.$$

GU ((353))(3e)

$$3. \int x \operatorname{sh}^2 x \, dx = \frac{1}{4} x \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x - \frac{x^2}{4}.$$

$$4. \int x^2 \operatorname{sh}^2 x \, dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \operatorname{sh} 2x - \frac{x}{4} \operatorname{ch} 2x - \frac{x^3}{6}.$$

$$5. \int x \operatorname{ch}^2 x \, dx = \frac{x}{4} \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x + \frac{x^2}{4}.$$

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$$6. \int x^2 \operatorname{ch}^2 x \, dx = \frac{1}{4} \left(x^2 + \frac{1}{2} \right) \operatorname{sh} 2x - \frac{x}{4} \operatorname{ch} 2x + \frac{x^3}{6}.$$

MZ 261

$$7. \int x^n \operatorname{sh}^3 x \, dx = \frac{n!}{4} \sum_{k=0}^{[n/2]} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left(\frac{\operatorname{ch} 3x}{3^{2k+1}} - 3 \operatorname{ch} x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\operatorname{sh} 3x}{3^{2k+2}} - 3 \operatorname{sh} x \right) \right\}.$$

GU ((353))(2f)

$$8. \int x^n \operatorname{ch}^3 x \, dx = \frac{n!}{4} \sum_{k=0}^{[n/2]} \left\{ \frac{x^{n-2k}}{(n-2k)!} \left(\frac{\operatorname{sh} 3x}{3^{2k+1}} + 3 \operatorname{sh} x \right) - \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\operatorname{ch} 3x}{3^{2k+2}} + 3 \operatorname{ch} x \right) \right\}.$$

GU ((353))(3f)

$$9. \int x \operatorname{sh}^3 x \, dx = \frac{3}{4} \operatorname{sh} x - \frac{1}{36} \operatorname{sh} 3x - \frac{3}{4} x \operatorname{ch} x - \frac{x}{12} \operatorname{ch} 3x.$$

$$10. \int x^2 \operatorname{sh}^3 x \, dx = - \left(\frac{3x^2}{4} + \frac{3}{2} \right) \operatorname{ch} x + \left(\frac{x^2}{12} + \frac{1}{54} \right) \operatorname{ch} 3x + \frac{3x}{2} \operatorname{sh} x - \frac{x}{18} \operatorname{sh} 3x.$$

MZ 257

$$11. \int x \operatorname{ch}^3 x \, dx = -\frac{3}{4} \operatorname{ch} x - \frac{1}{36} \operatorname{ch} 3x + \frac{3}{4} x \operatorname{sh} x + \frac{x}{12} \operatorname{sh} 3x.$$

$$12. \int x^2 \operatorname{ch}^3 x \, dx = \left(\frac{3}{4} x^2 + \frac{3}{2} \right) \operatorname{sh} x + \left(\frac{x^2}{12} + \frac{1}{54} \right) \operatorname{sh} 3x - \frac{3}{2} x \operatorname{ch} x - \frac{x}{18} \operatorname{ch} 3x.$$

MZ 262

2.475

GU ((353))(6a)

$$2. \int \frac{\text{ch}^q x}{x^p} dx = -\frac{(p-2)\text{ch}^q x + q x \text{ch}^{q-1} x \text{sh} x}{(p-1)(p-2)x^{p-1}} - \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\text{ch}^{q-2} x}{x^{p-2}} dx + \frac{q^2}{(p-1)(p-2)} \int \frac{\text{ch}^q x}{x^{p-2}} dx \quad [p > 2].$$

GU ((353))(7a)

$$3. \int \frac{\text{sh} x}{x^{2n}} dx = -\frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \text{ch} x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \text{sh} x \right\} + \frac{1}{(2n-1)!} \text{chi}(x).$$

GU ((353))(6b)

$$4. \int \frac{\text{sh} x}{x^{2n+1}} dx = -\frac{1}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \text{ch} x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \text{sh} x \right\} + \frac{1}{(2n)!} \text{shi}(x).$$

GU ((353))(6b)

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$$5. \int \frac{\text{ch} x}{x^{2n}} dx = -\frac{1}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(2k+1)!}{x^{2k+1}} \text{ch} x + \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \text{ch} x \right\} + \frac{1}{(2n-1)!} \text{shi}(x).$$

GU ((353))(7b)

$$6. \int \frac{\text{ch} x}{x^{2n+1}} dx = -\frac{1}{(2n)!x} \left\{ \sum_{k=0}^{n-1} \frac{(2k)!}{x^{2k}} \text{sh} x + \sum_{k=0}^{n-1} \frac{(2k+1)!}{x^{2k+1}} \text{ch} x \right\} + \frac{1}{(2n)!} \text{chi}(x).$$

GU ((353))(7b)

$$7. \int \frac{\text{sh}^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \text{chi}(2m-2k)x + \frac{(-1)^m}{2^{2m}} \binom{2m}{m} \ln x.$$

GU ((353))(6c)

$$8. \int \frac{\text{sh}^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \text{shi}(2m-2k+1)x.$$

$$9. \int \frac{\text{ch}^{2m} x}{x} dx = \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \text{chi}(2m-2k)x + \frac{1}{2^{2m}} \binom{2m}{m} \ln x.$$

GU ((353))(7c)

$$10. \int \frac{\text{ch}^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \text{chi}(2m-2k+1)x.$$

GU ((353))(7c)

$$11. \int \frac{\text{sh}^{2m} x}{x^2} dx = \frac{(-1)^{m-1}}{2^{2m} x} \binom{2m}{m} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\text{ch}(2m-2k)x}{x} - (2m-2k) \text{shi}(2m-2k)x \right\}.$$

$$12. \int \frac{\text{sh}^{2m+1} x}{x^2} dx = \frac{1}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \times \left\{ \frac{\text{sh}(2m-2k+1)x}{x} - (2m-2k+1) \text{chi}(2m-2k+1)x \right\}.$$

$$13. \int \frac{\text{ch}^{2m} x}{x^2} dx = -\frac{1}{2^{2m} x} \binom{2m}{m} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\text{ch}(2m-2k)x}{x} - (2m-2k) \text{shi}(2m-2k)x \right\}.$$

$$14. \int \frac{\text{ch}^{2m+1} x}{x^2} dx = -\frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\text{ch}(2m-2k+1)x}{x} - (2m-2k+1) \text{shi}(2m-2k+1)x \right\}.$$

2.476

$$1. \int \frac{\text{sh} kx}{a+bx} dx = \frac{1}{b} \left[\text{ch} \frac{ka}{b} \text{shi}(u) - \text{sh} \frac{ka}{b} \text{chi}(u) \right]; \\ = \frac{1}{2b} \left[\exp\left(-\frac{ka}{b}\right) \text{Ei}(u) - \exp\left(\frac{ka}{b}\right) \text{Ei}(-u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right].$$

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$$2. \int \frac{\text{ch} kx}{a+bx} dx = \frac{1}{b} \left[\text{ch} \frac{ka}{b} \text{chi}(u) - \text{sh} \frac{ka}{b} \text{shi}(u) \right]; \\ = \frac{1}{2b} \left[\exp\left(-\frac{ka}{b}\right) \text{Ei}(u) + \exp\left(\frac{ka}{b}\right) \text{Ei}(-u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right].$$

$$3. \int \frac{\text{sh } kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\text{sh } kx}{a+bx} + \frac{k}{b} \int \frac{\text{ch } kx}{a+bx} dx \quad \text{see 2.476 2.).}$$

2.476

$$4. \int \frac{\text{ch } kx}{(a+bx)^2} dx = -\frac{1}{b} \cdot \frac{\text{ch } kx}{a+bx} + \frac{k}{b} \int \frac{\text{sh } kx}{a+bx} dx \quad \text{see 2.476 1.).}$$

2.476

$$5. \int \frac{\text{sh } kx}{(a+bx)^3} dx = -\frac{\text{sh } kx}{2b(a+bx)^2} - \frac{k \text{ ch } kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\text{sh } kx}{a+bx} dx \text{ see 2.476 1.).}$$

2.476

$$6. \int \frac{\text{ch } kx}{(a+bx)^3} dx = -\frac{\text{ch } kx}{2b(a+bx)^2} - \frac{k \text{ sh } kx}{2b^2(a+bx)} + \frac{k^2}{2b^2} \int \frac{\text{ch } kx}{a+bx} dx \text{ see 2.476 2.).}$$

2.476

$$7. \int \frac{\text{sh } kx}{(a+bx)^4} dx = -\frac{\text{sh } kx}{3b(a+bx)^3} - \frac{k \text{ ch } kx}{6b^2(a+bx)^2} - \frac{k^2 \text{ sh } kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\text{ch } kx}{a+bx} dx \text{ see 2.476 2.).}$$

2.476

$$8. \int \frac{\text{ch } kx}{(a+bx)^4} dx = -\frac{\text{ch } kx}{3b(a+bx)^3} - \frac{k \text{ sh } kx}{6b^2(a+bx)^2} - \frac{k^2 \text{ ch } kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\text{sh } kx}{a+bx} dx \text{ see 2.476 1.).}$$

2.476

$$9. \int \frac{\text{sh } kx}{(a+bx)^5} dx = -\frac{\text{sh } kx}{4b(a+bx)^4} - \frac{k \text{ ch } kx}{12b^2(a+bx)^3} - \frac{k^2 \text{ sh } kx}{24b^3(a+bx)^2} - \frac{k^3 \text{ ch } kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\text{sh } kx}{a+bx} dx \quad \text{see 2.476 1.).}$$

$$10. \int \frac{\operatorname{ch} kx}{(a+bx)^5} dx = -\frac{\operatorname{ch} kx}{4b(a+bx)^4} - \frac{k \operatorname{sh} kx}{12b^2(a+bx)^3} - \frac{k^2 \operatorname{ch} kx}{24b^3(a+bx)^2} - \frac{k^3 \operatorname{sh} kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\operatorname{ch} kx}{a+bx} dx \quad \text{see 2.476 2.}.$$

2.476

$$11. \int \frac{\operatorname{sh} kx}{(a+bx)^6} dx = -\frac{\operatorname{sh} kx}{5b(a+bx)^5} - \frac{k \operatorname{ch} kx}{20b^2(a+bx)^4} - \frac{k^2 \operatorname{sh} kx}{60b^3(a+bx)^3} - \frac{k^3 \operatorname{ch} kx}{120b^4(a+bx)^2} - \frac{k^4 \operatorname{sh} kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\operatorname{ch} kx}{a+bx} dx \quad \text{see 2.476 2.}.$$

2.476

$$12. \int \frac{\operatorname{ch} kx}{(a+bx)^6} dx = -\frac{\operatorname{ch} kx}{5b(a+bx)^5} - \frac{k \operatorname{sh} kx}{20b^2(a+bx)^4} - \frac{k^2 \operatorname{ch} kx}{60b^3(a+bx)^3} - \frac{k^3 \operatorname{sh} kx}{120b^4(a+bx)^2} - \frac{k^4 \operatorname{ch} kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\operatorname{sh} kx}{a+bx} dx \quad \text{see 2.476 1.}.$$

2.476

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2.477

$$1. \int \frac{x^p dx}{\operatorname{sh}^q x} = \frac{-px^{p-1} \operatorname{sh} x - (q-2)x^p \operatorname{ch} x}{(q-1)(q-2)\operatorname{sh}^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2}}{\operatorname{sh}^{q-2} x} dx - \frac{q-2}{q-1} \int \frac{x^p dx}{\operatorname{sh}^{q-2} x} \quad [q > 2].$$

GU ((353))(8a)

$$2. \int \frac{x^p dx}{\operatorname{ch}^q x} = \frac{px^{p-1} \operatorname{ch} x + (q-2)x^p \operatorname{sh} x}{(q-1)(q-2)\operatorname{ch}^{q-1} x} - \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\operatorname{ch}^{q-2} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\operatorname{ch}^{q-2} x} \quad [q > 2].$$

GU ((353))(10a)

GU((353))(8b)

$$4. \int \frac{x^n}{\operatorname{ch} x} dx = \sum_{k=0}^{\infty} \frac{E_{2k} x^{n+2k+1}}{(n+2k+1)(2k)!} \quad \left[|x| < \frac{\pi}{2}, \quad n \geq 0 \right].$$

GU ((353))(10b)

$$5. \int \frac{dx}{x^n \operatorname{sh} x} = -[1 + (-1)^n] \frac{2^{n-1} - 1}{n!} B_n \ln x + \\ + \sum_{\substack{k=0 \\ k \neq \frac{n}{2}}}^{\infty} \frac{2 - 2^{2k}}{(2k-n)(2k)!} B_{2k} x^{2k-n} \quad [|x| < \pi, \quad n \geq 1].$$

GU ((353))(9b)

$$6. \int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{E_{2k}}{(2k-n+1)(2k)!} x^{2k-n+1} + \frac{1}{2} [1 - (-1)^{n-1}] + \\ + \frac{E_{n-1}}{(n-1)!} \ln x \quad \left[|x| < \frac{\pi}{2} \right].$$

GU ((353))(11b)

$$7. \int \frac{x^n}{\operatorname{sh}^2 x} dx = -x^n \operatorname{cth} x + n \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1} \quad [n > 1, \quad |x| < \pi].$$

GU ((353))(8c)

$$8. \int \frac{x^n}{\operatorname{ch}^2 x} dx = x^n \operatorname{th} x - n \sum_{k=1}^{\infty} \frac{2^{2k} (2^{2k} - 1) B_{2k}}{(n+2k-1)(2k)!} x^{n+2k-1} \quad \left[n > 1, \quad |x| < \frac{\pi}{2} \right].$$

GU ((353))(10c)

$$9. \int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1 - (-1)^n] \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{B_{2k}}{(2k-n-1)(2k)!} (2x)^{2k} \quad [|x| < \pi].$$

GU ((353))(9c)

$$10. \int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + [1 - (-1)^n] - \frac{2n(2^{n+1} - 1)n}{(n+1)!} B_{n+1} \ln x + \\ + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}} \frac{(2^{2k} - 1)B_{2k}}{(2k - n - 1)(2k)!} (2x)^{2k} \quad \left[|x| < \frac{\pi}{2} \right].$$

GU ((353))(11c)

$$11. \int \frac{x}{\operatorname{sh}^{2n} x} dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \\ \times \left\{ \frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k+1} x} + \frac{1}{(2n-2k)\operatorname{sh}^{2n-2k} x} \right\} + (-1)^{n-1} \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\operatorname{sh}^2 x} \\ \text{(see 2.477 17.)}$$

2.477

GU ((353))(8e)

$$12. \int \frac{x}{\operatorname{sh}^{2n-1} x} dx = \sum_{k=1}^{n-1} (-1)^k \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \\ \times \left\{ \frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k} x} + \frac{1}{(2n-2k-1)\operatorname{sh}^{2n-2k-1} x} \right\} + (-1)^{n-1} \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\operatorname{sh} x} \\ \text{(see 2.477 15.)}$$

2.477

GU ((353))(8e)

$$13. \int \frac{x}{\operatorname{ch}^{2n} x} dx = \sum_{k=1}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+1)} \times \\ \times \left\{ \frac{x \operatorname{sh} x}{\operatorname{ch}^{2n-2k+1} x} + \frac{1}{(2n-2k)\operatorname{ch}^{2n-2k} x} \right\} + \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x dx}{\operatorname{ch}^2 x} \\ \text{(see 2.477 18.)}$$

2.477

GU ((353))(10e)

$$14. \int \frac{x}{\operatorname{ch}^{2n-1} x} dx = \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\dots(2n-2k+1)}{(2n-2)(2n-4)\dots(2n-2k)} \times \\ \times \left\{ \frac{x \operatorname{sh} x}{\operatorname{ch}^{2n-2k} x} + \frac{1}{(2n-2k-1)\operatorname{ch}^{2n-2k-1} x} \right\} + \frac{(2n-3)!!}{(2n-2)!!} \int \frac{x dx}{\operatorname{ch} x} \\ \text{(see 2.477 16.)}$$

2.477
GU ((353))(10e)

$$15. \int \frac{x dx}{\operatorname{sh} x} = \sum_{k=0}^{\infty} \frac{2 - 2^{2k}}{(2k+1)(2k)!} B_{2k} x^{2k+1} \quad |x| < \pi.$$

GU ((353))(8b)a

$$16. \int \frac{x dx}{\operatorname{ch} x} = \sum_{k=0}^{\infty} \frac{E_{2k} x^{2k+2}}{(2k+2)(2k)!} \quad |x| < \frac{\pi}{2}.$$

GU ((353))(10b)a

$$17. \int \frac{x dx}{\operatorname{sh}^2 x} = -x \operatorname{cth} x + \ln \operatorname{sh} x.$$

MZ 257

$$18. \int \frac{x dx}{\operatorname{ch}^2 x} = x \operatorname{th} x - \ln \operatorname{ch} x.$$

MZ 262

$$19. \int \frac{x dx}{\operatorname{sh}^3 x} = -\frac{x \operatorname{ch} x}{2 \operatorname{sh}^2 x} - \frac{1}{2 \operatorname{sh} x} - \frac{1}{2} \int \frac{x dx}{\operatorname{sh} x} \quad (\text{see } \mathbf{2.477} \text{ 15.}).$$

2.477
MZ 257

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$$20. \int \frac{x dx}{\operatorname{ch}^3 x} = \frac{x \operatorname{sh} x}{2 \operatorname{ch}^2 x} + \frac{1}{2 \operatorname{ch} x} + \frac{1}{2} \int \frac{x dx}{\operatorname{ch} x} \quad (\text{see } \mathbf{2.477} \text{ 16.}).$$

2.477
MZ 262

$$21. \int \frac{x dx}{\operatorname{sh}^4 x} = -\frac{x \operatorname{ch} x}{3 \operatorname{sh}^3 x} - \frac{1}{6 \operatorname{sh}^2 x} + \frac{2}{3} x \operatorname{cth} x - \frac{2}{3} \ln \operatorname{sh} x.$$

$$22. \int \frac{x dx}{\operatorname{ch}^4 x} = \frac{x \operatorname{sh} x}{3 \operatorname{ch}^3 x} + \frac{1}{6 \operatorname{ch}^2 x} + \frac{2}{3} x \operatorname{th} x - \frac{2}{3} \ln \operatorname{ch} x.$$

MZ 262

$$23. \int \frac{x dx}{\operatorname{sh}^5 x} = -\frac{x \operatorname{ch} x}{4 \operatorname{sh}^4 x} - \frac{1}{12 \operatorname{sh}^3 x} + \frac{3x \operatorname{ch} x}{8 \operatorname{sh}^2 x} + \frac{3}{8 \operatorname{sh} x} + \frac{3}{8} \int \frac{x dx}{\operatorname{sh} x} \quad (\text{see } \mathbf{2.477} \text{ 15.}).$$

2.477
MZ 258

$$24. \int \frac{x dx}{\operatorname{ch}^5 x} = \frac{x \operatorname{sh} x}{4 \operatorname{ch}^4 x} + \frac{1}{12 \operatorname{ch}^3 x} + \frac{3x \operatorname{sh} x}{8 \operatorname{ch}^2 x} + \frac{3}{8 \operatorname{ch} x} + \frac{3}{8} \int \frac{x dx}{\operatorname{ch} x} \quad (\text{see } \mathbf{2.477} \text{ 16.}).$$

2.477
MZ 262

2.478

$$1. \int \frac{x^n \operatorname{ch} x dx}{(a + b \operatorname{sh} x)^m} = -\frac{x^n}{(m-1)b(a + b \operatorname{sh} x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \operatorname{sh} x)^{m-1}} \quad [m \neq 1].$$

MZ 263

$$2. \int \frac{x^n \operatorname{sh} x dx}{(a + b \operatorname{ch} x)^m} = -\frac{x^n}{(m-1)b(a + b \operatorname{ch} x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a + b \operatorname{ch} x)^{m-1}} \quad [m \neq 1].$$

MZ 263

$$3. \int \frac{x dx}{1 + \operatorname{ch} x} = x \operatorname{th} \frac{x}{2} - 2 \ln \operatorname{ch} \frac{x}{2}.$$

$$4. \int \frac{x dx}{1 - \operatorname{ch} x} = x \operatorname{cth} \frac{x}{2} - 2 \ln \operatorname{sh} \frac{x}{2}.$$

$$5. \int \frac{x \operatorname{sh} x dx}{(1 + \operatorname{ch} x)^2} = -\frac{x}{1 + \operatorname{ch} x} + \operatorname{th} \frac{x}{2}.$$

$$6. \int \frac{x \operatorname{sh} x \, dx}{(1 - \operatorname{ch} x)^2} = \frac{x}{1 - \operatorname{ch} x} - \operatorname{cth} \frac{x}{2}.$$

MZ 262-264

$$7. \int \frac{x \, dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2 \sin 2t} [L(u+t) - L(u-t) - 2L(t)]$$

$$[u = \operatorname{arctg}(\operatorname{th} x \operatorname{ctg} t), \quad t \neq \pm n\pi].$$

LO III 402

$$8. \int \frac{x \operatorname{ch} x \, dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2 \sin t} \left[L\left(\frac{u+t}{2}\right) - L\left(\frac{u-t}{2}\right) + L\left(\pi - \frac{v+t}{2}\right) + \right. \\ \left. + L\left(\frac{v-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right]$$

$$\left[u = 2 \operatorname{arctg}\left(\operatorname{th} \frac{x}{2} \cdot \operatorname{ctg} \frac{t}{2}\right), \quad v = 2 \operatorname{arctg}\left(\operatorname{cth} \frac{x}{2} \cdot \operatorname{ctg} \frac{t}{2}\right); \quad t \neq \pm n\pi \right].$$

LO III 403

2.479

$$1. \int x^p \frac{\operatorname{sh}^{2m} x}{\operatorname{ch}^n x} \, dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int \frac{x^p \, dx}{\operatorname{ch}^{n-2k} x} \quad (\text{see } \mathbf{2.477} \text{ 2.}).$$

2.477

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$$2. \int x^p \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch}^n x} \, dx = \sum_{k=0}^m (-1)^{m+k} \binom{m}{k} \int x^p \frac{\operatorname{sh} x}{\operatorname{ch}^{n-2k} x} \, dx \quad [n > 1] \quad (\text{see } \mathbf{2.479} \text{ 3.})$$

2.479

$$3. \int x^p \frac{\operatorname{sh} x}{\operatorname{ch}^n x} \, dx = -\frac{x^p}{(n-1)\operatorname{ch}^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} \, dx}{\operatorname{ch}^{n-1} x} \quad [n > 1] \quad (\text{see } \mathbf{2.477} \text{ 2.}).$$

2.477
GU ((353))(12)

2.477

$$5. \int x^p \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh}^n x} dx = \sum_{k=0}^m \binom{m}{k} \int \frac{x^p \operatorname{ch} x}{\operatorname{sh}^{n-2k} x} dx \quad (\text{see } \mathbf{2.479} \text{ 6.}).$$

2.479

$$6. \int x^p \frac{\operatorname{ch} x}{\operatorname{sh}^n x} dx = -\frac{x^p}{(n-1)\operatorname{sh}^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\operatorname{sh}^{n-1} x} \quad [n > 1] \quad (\text{see } \mathbf{2.477} \text{ 1.}).$$

2.477

GU ((353))(13c)

$$7. \int x^p \operatorname{th} x dx = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k+p)(2k)!} x^{p+2k} \quad \left[p > -1, \quad |x| < \frac{\pi}{2} \right].$$

GU ((353))(12d)

$$8. \int x^p \operatorname{cth} x dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq +1, \quad |x| < \pi].$$

GU ((353))(13d)

$$9. \int \frac{x \operatorname{ch} x}{\operatorname{sh}^2 x} dx = \ln \operatorname{th} \frac{x}{2} - \frac{x}{\operatorname{sh} x}.$$

$$10. \int \frac{x \operatorname{sh} x}{\operatorname{ch}^2 x} dx = -\frac{x}{\operatorname{ch} x} + \operatorname{arctg}(\operatorname{sh} x).$$

MZ 263

2.48 Combinations of hyperbolic functions, exponentials, and powers

2.481

For $a^2 = b^2$:

$$3. \int e^{ax} \operatorname{sh}(ax + c) dx = -\frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}.$$

$$4. \int e^{-ax} \operatorname{sh}(ax + c) dx = \frac{1}{2}xe^c + \frac{1}{4a}e^{-(2ax+c)}.$$

$$5. \int e^{ax} \operatorname{ch}(ax + c) dx = \frac{1}{2}xe^{-c} + \frac{1}{4a}e^{2ax+c}.$$

$$6. \int e^{-ax} \operatorname{ch}(ax + c) dx = \frac{1}{2}xe^c - \frac{1}{4a}e^{-(2ax+c)}.$$

MZ 275-277

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2.482

$$1. \int x^p e^{ax} \operatorname{sh} bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx - \int x^p e^{(a-b)x} dx \right\} \quad [a^2 \neq b^2] \text{ see } \mathbf{2.321}.$$

2.321

$$2. \int x^p e^{ax} \operatorname{ch} bx dx = \frac{1}{2} \left\{ \int x^p e^{(a+b)x} dx + \int x^p e^{(a-b)x} dx \right\} \quad [a^2 \neq b^2] \text{ see } \mathbf{2.321}.$$

2.321

For $a^2 = b^2$:

$$3. \int x^p e^{ax} \operatorname{sh} ax dx = \frac{1}{2} \int x^p e^{2ax} dx - \frac{x^{p+1}}{2(p+1)} \quad (\text{see } \mathbf{2.321}).$$

$$4. \int x^p e^{-ax} \operatorname{sh} ax \, dx = \frac{x^{p+1}}{2(p+1)} - \frac{1}{2} \int x^p e^{-2ax} \, dx \quad (\text{see } \mathbf{2.321}).$$

2.321

$$5. \int x^p e^{ax} \operatorname{ch} ax \, dx = \frac{x^{p+1}}{2(p+1)} + \frac{1}{2} \int x^p e^{2ax} \, dx \quad (\text{see } \mathbf{2.321})$$

2.321

MZ 276, 278

2.483

$$1. \int x e^{ax} \operatorname{sh} bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \operatorname{sh} bx - \left(bx - \frac{2ab}{a^2 - b^2} \right) \operatorname{ch} bx \right] \quad [a^2 \neq b^2].$$

$$2. \int x e^{ax} \operatorname{ch} bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left[\left(ax - \frac{a^2 + b^2}{a^2 - b^2} \right) \operatorname{ch} bx - \left(bx - \frac{2ab}{a^2 - b^2} \right) \operatorname{sh} bx \right] \quad [a^2 \neq b^2].$$

$$3. \int x^2 e^{ax} \operatorname{sh} bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2} x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \operatorname{sh} bx - \left[bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \operatorname{ch} x \right\} \quad [a^2 \neq b^2].$$

$$4. \int x^2 e^{ax} \operatorname{ch} bx \, dx = \frac{e^{ax}}{a^2 - b^2} \left\{ \left[ax^2 - \frac{2(a^2 + b^2)}{a^2 - b^2} x + \frac{2a(a^2 + 3b^2)}{(a^2 - b^2)^2} \right] \operatorname{ch} bx - \left[bx^2 - \frac{4ab}{a^2 - b^2} x + \frac{2b(3a^2 + b^2)}{(a^2 - b^2)^2} \right] \operatorname{sh} x \right\} \quad [a^2 \neq b^2].$$

For $a^2 = b^2$:

$$5. \int x e^{ax} \operatorname{sh} ax \, dx = \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right) - \frac{x^2}{4}.$$

$$6. \int x e^{-ax} \operatorname{sh} ax \, dx = \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right) + \frac{x^2}{4}.$$

MZ 276, 278

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$$7. \int x e^{ax} \operatorname{ch} ax \, dx = \frac{x^2}{4} + \frac{e^{2ax}}{4a} \left(x - \frac{1}{2a} \right).$$

$$8. \int x e^{-ax} \operatorname{ch} ax \, dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a} \left(x + \frac{1}{2a} \right).$$

$$9. \int x^2 e^{ax} \operatorname{sh} ax \, dx = \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right) - \frac{x^3}{6}.$$

$$10. \int x^2 e^{-ax} \operatorname{sh} ax \, dx = \frac{e^{-2ax}}{4a} \left(x^2 + \frac{x}{a} + \frac{1}{2a^2} \right) + \frac{x^3}{6}.$$

$$11. \int x^2 e^{ax} \operatorname{ch} ax \, dx = \frac{x^3}{6} + \frac{e^{2ax}}{4a} \left(x^2 - \frac{x}{a} + \frac{1}{2a^2} \right).$$

2.484

$$1. \int e^{ax} \operatorname{sh} bx \frac{dx}{x} = \frac{1}{2} \{ \operatorname{Ei}[(a+b)x] - \operatorname{Ei}[(a-b)x] \} \quad [a^2 \neq b^2].$$

$$2. \int e^{ax} \operatorname{ch} bx \frac{dx}{x} = \frac{1}{2} \{ \operatorname{Ei}[(a+b)x] + \operatorname{Ei}[(a-b)x] \} \quad [a^2 \neq b^2].$$

$$3. \int e^{ax} \operatorname{sh} bx \frac{dx}{x^2} = -\frac{e^{ax} \operatorname{sh} bx}{2x} + \frac{1}{2} \{ (a+b) \operatorname{Ei}[(a+b)x] - (a-b) \operatorname{Ei}[(a-b)x] \} \quad [a^2 \neq b^2].$$

$$4. \int e^{ax} \operatorname{ch} bx \frac{dx}{x^2} = -\frac{e^{ax} \operatorname{ch} bx}{2x} + \frac{1}{2} \{ (a+b) \operatorname{Ei}[(a+b)x] + (a-b) \operatorname{Ei}[(a-b)x] \} \quad [a^2 \neq b^2].$$

For $a^2 = b^2$:

$$5. \int e^{ax} \operatorname{sh} ax \frac{dx}{x} = \frac{1}{2} [\operatorname{Ei}(2ax) - \ln x].$$

$$6. \int e^{-ax} \operatorname{sh} ax \frac{dx}{x} = \frac{1}{2} [\ln x - \operatorname{Ei}(-2ax)].$$

$$7. \int e^{ax} \operatorname{ch} ax \frac{dx}{x} = \frac{1}{2} [\ln x + \operatorname{Ei}(2ax)].$$

$$8. \int e^{ax} \operatorname{sh} ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} - 1) + a \operatorname{Ei}(2ax).$$

$$9. \int e^{-ax} \operatorname{sh} ax \frac{dx}{x^2} = -\frac{1}{2x} (1 - e^{-2ax}) + a \operatorname{Ei}(-2ax).$$

$$10. \int e^{ax} \operatorname{ch} ax \frac{dx}{x^2} = -\frac{1}{2x} (e^{2ax} + 1) + a \operatorname{Ei}(2ax).$$

MZ 276, 278

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2.5-2.6 Trigonometric Functions

2.50 Introduction

2.501

Integrals of the form $\int R(\sin x, \cos x) dx$ can always be reduced to integrals of rational functions by means of the substitution

$$t = \operatorname{tg} \frac{x}{2}.$$

2.502

If $R(\sin x, \cos x)$ satisfies the relation

$$R(\sin x, \cos x) = -R(-\sin x, \cos x),$$

it is convenient to make the substitution $t = \cos x$.

2.503

If this function satisfies the relation

$$R(\sin x, \cos x) = -R(\sin x, -\cos x),$$

it is convenient to make the substitution $t = \sin x$.

2.504

If this function satisfies the relation

$$R(\sin x, \cos x) = R(-\sin x, -\cos x),$$

it is convenient to make the substitution $t = \operatorname{tg} x$.

2.51-2.52 Powers of trigonometric functions

2.510

$$\begin{aligned} \int \sin^p x \cos^q x \, dx &= -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^{q+2} x \, dx; \\ &= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx; \\ &= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1} \int \sin^{p+2} x \cos^q x \, dx; \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1} \int \sin^{p+2} x \cos^{q-2} x \, dx; \\ &= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} \int \sin^p x \cos^{q-2} x \, dx; \\ &= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1} \int \sin^p x \cos^{q+2} x \, dx; \\ &= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q} \left\{ \sin^2 x - \frac{q-1}{p+q-2} \right\} + \\ &\quad + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x \, dx. \end{aligned}$$

FI II 89, TI 214

$$1. \int \sin^p x \cos^{2n} x dx = \frac{\sin^{p+1} x}{2n+p} \left\{ \cos^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1) \cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right\} + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \sin^p x dx.$$

This formula is applicable for arbitrary real p except for the following negative even integers:

$-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have:

$$2. \int \sin^{2l} x dx = -\frac{\cos x}{2l} \left\{ \sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sin^{2l-2k-1} x \right\} + \frac{(2l-1)!!}{2^l l!} x \quad (\text{see also } \mathbf{2.513} \text{ 1.}).$$

2.513
TI (232)

$$3. \int \sin^{2l+1} x dx = -\frac{\cos x}{2l+1} \left\{ \sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \sin^{2l-2k-2} x \right\} \quad (\text{see also } \mathbf{2.513} \text{ 2.}).$$

2.513
TI (233)

$$4. \int \sin^p x \cos^{2n+1} x dx = \frac{\sin^{p+1} x}{2n+p+1} \left\{ \cos^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.512

$$1. \int \cos^p x \sin^{2n} x dx = -\frac{\cos^{p+1} x}{2n+p} \left\{ \sin^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1) \sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\dots(2n+p-2k)} \right\} + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\dots(p+2)} \int \cos^p x dx.$$

$-2, -4, \dots, -2n$. If p is a natural number and $n = 0$, we have

$$2. \int \cos^{2l} x \, dx = \frac{\sin x}{2l} \left\{ \cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \cos^{2l-2k-1} x \right\} + \frac{(2l-1)!!}{2^l l!} x$$

TI (230)

$$3. \int \cos^{2l+1} x \, dx = \frac{\sin x}{2l+1} \left\{ \cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)}{(2l-1)(2l-3)\dots(2l-2k-1)} \cos^{2l-2k-2} x \right\}$$

2.513
TI (231)

$$4. \int \cos^p x \sin^{2n+1} x \, dx = -\frac{\cos^{p+1} x}{2n+p+1} \left\{ \sin^{2n} x + \sum_{k=1}^n \frac{2^k n(n-1)\dots(n-k+1) \sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\dots(2n+p-2k+1)} \right\}.$$

This formula is applicable for arbitrary real p except for the following negative odd integers: $-1, -3, \dots, -(2n+1)$.

2.513

$$1. \int \sin^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

2.511
TI (226)

$$2. \int \sin^{2n+1} x \, dx = \frac{1}{2^{2n}} (-1)^{n+1} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \frac{\cos(2n+1-2k)x}{2n+1-2k}$$

$$3. \int \cos^{2n} x \, dx = \frac{1}{2^{2n}} \binom{2n}{n} x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin(2n-2k)x}{2n-2k}$$

$$4. \int \cos^{2n+1} x \, dx = \frac{1}{2^{2n}} \sum_{k=0}^n \binom{2n+1}{k} \frac{\sin(2n-2k+1)x}{2n-2k+1} \quad (\text{see also } \mathbf{2.512} \text{ 3.}).$$

$$5. \int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x = -\frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$6. \int \sin^3 x \, dx = \frac{1}{12} \cos 3x - \frac{3}{4} \cos x = \frac{1}{3} \cos^3 x - \cos x.$$

$$7. \int \sin^4 x \, dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} = -\frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^3 x \cos x + \frac{3}{8} x.$$

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$$8. \int \sin^5 x \, dx = -\frac{5}{8} \cos x + \frac{5}{48} \cos 3x - \frac{1}{80} \cos 5x = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{15} \cos^3 x - \frac{4}{5} \cos x.$$

$$9. \int \sin^6 x \, dx = \frac{5}{16} x - \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{192} \sin 6x = -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x.$$

$$\begin{aligned}
 10. \quad \int \sin^7 x \, dx &= -\frac{35}{64} \cos x + \frac{7}{64} \cos 3x - \frac{7}{320} \cos 5x + \frac{1}{448} \cos 7x = \\
 &= -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x + \frac{8}{35} \cos^3 x - \frac{24}{35} \cos x.
 \end{aligned}$$

$$11. \quad \int \cos^2 x \, dx = \frac{1}{4} \sin 2x + \frac{x}{2} = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$$

$$12. \quad \int \cos^3 x \, dx = \frac{1}{12} \sin 3x + \frac{3}{4} \sin x = \sin x - \frac{1}{3} \sin^3 x.$$

$$13. \quad \int \cos^4 x \, dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x = \frac{3}{8} x + \frac{3}{8} \sin x \cos x + \frac{1}{4} \sin x \cos^3 x.$$

$$14. \quad \int \cos^5 x \, dx = \frac{5}{8} \sin x + \frac{5}{48} \sin 3x + \frac{1}{80} \sin 5x = \frac{4}{5} \sin x - \frac{4}{15} \sin^3 x + \frac{1}{5} \cos^4 x \sin x.$$

$$\begin{aligned}
 15. \quad \int \cos^6 x \, dx &= \frac{5}{16} x + \frac{15}{64} \sin 2x + \frac{3}{64} \sin 4x + \frac{1}{192} \sin 6x = \\
 &= \frac{5}{16} x + \frac{5}{16} \sin x \cos x + \frac{5}{24} \sin x \cos^3 x + \frac{1}{6} \sin x \cos^5 x.
 \end{aligned}$$

$$\begin{aligned}
 16. \quad \int \cos^7 x \, dx &= \frac{35}{64} \sin x + \frac{7}{64} \sin 3x + \frac{7}{320} \sin 5x + \frac{1}{448} \sin 7x = \\
 &= \frac{24}{35} \sin x - \frac{8}{35} \sin^3 x + \frac{6}{35} \sin x \cos^4 x + \frac{1}{7} \sin x \cos^6 x.
 \end{aligned}$$

$$17. \quad \int \sin x \cos^2 x \, dx = -\frac{1}{4} \left\{ \frac{1}{3} \cos 3x + \cos x \right\} = -\frac{\cos^3 x}{3}.$$

$$18. \quad \int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}.$$

$$19. \quad \int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5}.$$

$$20. \int \sin^2 x \cos x \, dx = -\frac{1}{4} \left\{ \frac{1}{3} \sin 3x - \sin x \right\} = \frac{\sin^3 x}{3}.$$

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$$21. \int \sin^2 x \cos^2 x \, dx = -\frac{1}{8} \left\{ \frac{1}{4} \sin 4x - x \right\}.$$

$$\begin{aligned} 22. \int \sin^2 x \cos^3 x \, dx &= -\frac{1}{16} \left\{ \frac{1}{5} \sin 5x + \frac{1}{3} \sin 3x - 2 \sin x \right\} = \\ &= \frac{\sin^3 x}{5} \left(\cos^2 x + \frac{2}{3} \right) = \frac{\sin^3 x}{5} \left(\frac{5}{3} - \sin^2 x \right). \end{aligned}$$

$$23. \int \sin^2 x \cos^4 x \, dx = \frac{x}{16} + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x.$$

$$24. \int \sin^3 x \cos x \, dx = \frac{1}{8} \left(\frac{1}{4} \cos 4x - \cos 2x \right) = \frac{\sin^4 x}{4}.$$

$$\begin{aligned} 25. \int \sin^3 x \cos^2 x \, dx &= \frac{1}{16} \left(\frac{1}{5} \cos 5x - \frac{1}{3} \cos 3x - 2 \cos x \right) = \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x. \end{aligned}$$

$$26. \int \sin^3 x \cos^3 x \, dx = \frac{1}{32} \left(\frac{1}{6} \cos 6x - \frac{3}{2} \cos 2x \right).$$

$$27. \int \sin^3 x \cos^4 x \, dx = \frac{1}{7} \cos^3 x \left(-\frac{2}{5} - \frac{3}{5} \sin^2 x + \sin^4 x \right).$$

$$28. \int \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5}.$$

$$29. \int \sin^4 x \cos^2 x dx = \frac{1}{16}x - \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x + \frac{1}{192} \sin 6x.$$

$$30. \int \sin^4 x \cos^3 x dx = \frac{1}{7} \sin^3 x \left(\frac{2}{5} + \frac{3}{5} \cos^2 x - \cos^4 x \right).$$

$$31. \int \sin^4 x \cos^4 x dx = \frac{3}{128}x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x.$$

2.514

$$\begin{aligned} \int \frac{\sin^p x}{\cos^{2n} x} dx &= \\ &= \frac{\sin^{p+1} x}{2n-1} \left\{ \sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \sec^{2n-2k-1} x \right\} + \\ &+ \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!} \int \sin^p x dx. \end{aligned}$$

This formula is applicable for arbitrary real p . For $\int \sin^p x dx$, where p is a natural number, see 2.511 2., 3. and 2.513 1., 2. If $n = 0$ and p is a negative integer, we have for this integral:

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2.515

$$1. \int \frac{dx}{\sin^{2l} x} = -\frac{\cos x}{2l-1} \left\{ \operatorname{cosec}^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k(l-1)(l-2)\dots(l-k)}{(2l-3)(2l-5)\dots(2l-2k-1)} \operatorname{cosec}^{2l-2k-1} x \right\}.$$

TI (242)

$$\begin{aligned} 2. \int \frac{dx}{\sin^{2l+1} x} &= -\frac{\cos x}{2l} \left\{ \operatorname{cosec}^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \operatorname{cosec}^{2l-2k} x \right\} + \\ &+ \frac{(2l-1)!!}{2^l l!} \ln \operatorname{tg} \frac{x}{2}. \end{aligned}$$

TI (243)

2.516

$$\begin{aligned} 1. \int \frac{\sin^p x dx}{\cos^{2n+1} x} &= \\ &= \frac{\sin^{p+1} x}{2n} \left\{ \sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \sec^{2n-2k} x \right\} + \\ &+ \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{(2n)!!} \int \frac{\sin^p x}{\cos^{2n} x} dx. \end{aligned}$$

$$p \quad n = 0 \quad p$$

$$2. \int \frac{\sin^{2l+1} x dx}{\cos x} = - \sum_{k=1}^l \frac{\sin^{2k} x}{2k} - \ln \cos x.$$

$$3. \int \frac{\sin^{2l} x dx}{\cos x} = - \sum_{k=1}^l \frac{\sin^{2k-1} x}{2k-1} + \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

2.517

$$1. \int \frac{dx}{\sin^{2m+1} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+2) \sin^{2m-2k+2} x} + \ln \operatorname{tg} x.$$

$$2. \int \frac{dx}{\sin^{2m} x \cos x} = - \sum_{k=1}^m \frac{1}{(2m-2k+1) \sin^{2m-2k+1} x} + \ln \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

2.518

$$1. \int \frac{\sin^p x}{\cos^2 x} dx = \frac{\sin^{p-1} x}{\cos x} - (p-1) \int \sin^{p-2} x dx.$$

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$$2. \int \frac{\cos^p x dx}{\sin^{2n} x} = - \frac{\cos^{p+1} x}{2n-1} \left\{ \operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4) \dots (2n-p-2k)}{(2n-3)(2n-5) \dots (2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x \right\} + \frac{(2n-p-2)(2n-p-4) \dots (2-p)(-p)}{(2n-1)!!} \int \cos^p x dx.$$

This formula is applicable for arbitrary real p . For $\int \cos^p x dx$ where p is a natural number, see 2.512 2., 3. and 2.513 3., 4. If $n = 0$ and p is a negative integer, we have for this integral:

2.519

$$1. \int \frac{dx}{\cos^{2l} x} = \frac{\sin x}{2l-1} \left\{ \sec^{2l-1} x + \sum_{k=1}^{l-1} \frac{2^k (l-1)(l-2) \dots (l-k)}{(2l-3)(2l-5) \dots (2l-2k-1)} \sec^{2l-2k-1} x \right\}.$$

$$2. \int \frac{dx}{\cos^{2l+1} x} = \frac{\sin x}{2l} \left\{ \sec^{2l} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)}{2^k(l-1)(l-2)\dots(l-k)} \sec^{2l-2k} x \right\} + \frac{(2l-1)!!}{2^l l!} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

TI (241)

2.521

$$1. \int \frac{\cos^p x dx}{\sin^{2n+1} x} = -\frac{\cos^{p+1} x}{2n} \left\{ \operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \operatorname{cosec}^{2n-2k} x \right\} + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n \cdot n!} \int \frac{\cos^p x}{\sin x} dx.$$

This formula is applicable for arbitrary real p . For $n = 0$ and p a natural number, we have

$$2. \int \frac{\cos^{2l+1} x dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k} x}{2k} + \ln \sin x.$$

$$3. \int \frac{\cos^{2l} x dx}{\sin x} = \sum_{k=1}^l \frac{\cos^{2k-1} x}{2k-1} + \ln \operatorname{tg} \frac{x}{2}.$$

2.522

$$1. \int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^m \frac{1}{(2m-2k+2) \cos^{2m-2k+2} x} + \ln \operatorname{tg} x.$$

$$2. \int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^m \frac{1}{(2m-2k+1) \cos^{2m-2k+1} x} + \ln \operatorname{tg} \frac{x}{2}.$$

GW ((331))(15)

165

2.523

$$\int \frac{\cos^m x}{\sin^2 x} dx = -\frac{\cos^{m-1} x}{\sin x} - (m-1) \int \cos^{m-2} x dx.$$

2.524

$$1. \int \frac{\sin^{2n+1} x}{\cos^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^{k+1} \binom{n}{k} \frac{\cos^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m+1}{2}} \binom{n}{\frac{m-1}{2}} \ln \cos x.$$

GU ((331))(11d)

$$2. \int \frac{\cos^{2n+1} x}{\sin^m x} dx = \sum_{\substack{k=0 \\ k \neq \frac{m-1}{2}}}^n (-1)^k \binom{n}{k} \frac{\sin^{2k-m+1} x}{2k-m+1} + s(-1)^{\frac{m-1}{2}} \binom{n}{\frac{m-1}{2}} \ln \sin x.$$

[In formulas 2.524 1. and 2.524 2., $s = 1$ for m odd and $m < 2n + 1$; in other cases, $s = 0$.]

2.525

$$1. \int \frac{dx}{\sin^{2m} x \cos^{2n} x} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} \frac{\operatorname{tg}^{2k-2m+1} x}{2k-2m+1}.$$

TI (267)

$$2. \int \frac{dx}{\sin^{2m+1} x \cos^{2n+1} x} = \sum_{k=0}^{m+n} \binom{m+n}{k} \frac{\operatorname{tg}^{2k-2m} x}{2k-2m} + \binom{m+n}{m} \ln \operatorname{tg} x.$$

TI (268), GU ((331))(15f)

2.526

$$1. \int \frac{dx}{\sin x} = \ln \operatorname{tg} \frac{x}{2}.$$

$$2. \int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

$$3. \int \frac{dx}{\sin^3 x} = -\frac{1}{2} \frac{\cos x}{\sin^2 x} + \frac{1}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$4. \int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3 \sin^3 x} - \frac{2}{3} \operatorname{ctg} x = -\frac{1}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

$$5. \int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4 \sin^4 x} - \frac{3 \cos x}{8 \sin^2 x} + \frac{3}{8} \ln \operatorname{tg} \frac{x}{2}.$$

$$6. \int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5 \sin^5 x} - \frac{4}{15} \operatorname{ctg}^3 x - \frac{4}{5} \operatorname{ctg} x;$$

$$= -\frac{1}{5} \operatorname{ctg}^5 x - \frac{2}{3} \operatorname{ctg}^3 x - \operatorname{ctg} x.$$

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$$7. \int \frac{dx}{\sin^7 x} = -\frac{\cos x}{6 \sin^2 x} \left(\frac{1}{\sin^4 x} + \frac{5}{4 \sin^2 x} + \frac{15}{8} \right) + \frac{5}{16} \ln \operatorname{tg} \frac{x}{2}.$$

$$8. \int \frac{dx}{\sin^8 x} = -\left(\frac{1}{7} \operatorname{ctg}^7 x + \frac{3}{5} \operatorname{ctg}^5 x + \operatorname{ctg}^3 x + \operatorname{ctg} x \right).$$

$$9. \int \frac{dx}{\cos x} = \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) = \ln \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) = \ln \sqrt{\frac{1 + \sin x}{1 - \sin x}}.$$

$$10. \int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

$$11. \int \frac{dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$12. \int \frac{dx}{\cos^4 x} = \frac{\sin x}{3 \cos^3 x} + \frac{2}{3} \operatorname{tg} x = \frac{1}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

$$13. \int \frac{dx}{\cos^5 x} = \frac{\sin x}{4 \cos^4 x} + \frac{3 \sin x}{8 \cos^2 x} + \frac{3}{8} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$14. \int \frac{dx}{\cos^6 x} = \frac{\sin x}{5 \cos^5 x} + \frac{4}{15} \operatorname{tg}^3 x + \frac{4}{5} \operatorname{tg} x = \frac{1}{5} \operatorname{tg}^5 x + \frac{2}{3} \operatorname{tg}^3 x + \operatorname{tg} x.$$

$$15. \int \frac{dx}{\cos^7 x} = \frac{\sin x}{6 \cos^6 x} + \frac{5 \sin x}{24 \cos^4 x} + \frac{5 \sin x}{16 \cos^2 x} + \frac{5}{16} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$16. \int \frac{dx}{\cos^8 x} = \frac{1}{7} \operatorname{tg}^7 x + \frac{3}{5} \operatorname{tg}^5 x + \operatorname{tg}^3 x + \operatorname{tg} x.$$

$$17. \int \frac{\sin x}{\cos x} dx = -\ln \cos x.$$

$$18. \int \frac{\sin^2 x}{\cos x} dx = -\sin x + \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$19. \int \frac{\sin^3 x}{\cos x} dx = -\frac{\sin^2 x}{2} - \ln \cos x = \frac{1}{2} \cos^2 x - \ln \cos x.$$

$$20. \int \frac{\sin^4 x}{\cos x} dx = -\frac{1}{3} \sin^3 x - \sin x + \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$21. \int \frac{\sin^2 x dx}{\cos^2 x} = \frac{1}{\cos x}.$$

$$22. \int \frac{\sin^2 x dx}{\cos^2 x} = \operatorname{tg} x - x.$$

$$23. \int \frac{\sin^3 x dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}.$$

$$24. \int \frac{\sin^4 x dx}{\cos^2 x} = \operatorname{tg} x + \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$25. \int \frac{\sin x \, dx}{\cos^3 x} = \frac{1}{2 \cos^2 x} = \frac{1}{2} \operatorname{tg}^2 x.$$

$$26. \int \frac{\sin^2 x \, dx}{\cos^3 x} = \frac{\sin x}{2 \cos^2 x} - \frac{1}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$27. \int \frac{\sin^3 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \ln \cos x.$$

$$28. \int \frac{\sin^4 x \, dx}{\cos^3 x} = \frac{1}{2} \frac{\sin x}{\cos^2 x} + \sin x - \frac{3}{2} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$29. \int \frac{\sin x \, dx}{\cos^4 x} = \frac{1}{3 \cos^3 x}.$$

$$30. \int \frac{\sin^2 x \, dx}{\cos^4 x} = \frac{1}{3} \operatorname{tg}^3 x.$$

$$31. \int \frac{\sin^3 x \, dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3 \cos^3 x}.$$

$$32. \int \frac{\sin^4 x \, dx}{\cos^4 x} = \frac{1}{3} \operatorname{tg}^3 x - \operatorname{tg} x + x.$$

$$33. \int \frac{\cos x \, dx}{\sin x} = \ln \sin x.$$

$$34. \int \frac{\cos^2 x \, dx}{\sin x} = \cos x + \ln \operatorname{tg} \frac{x}{2}.$$

$$35. \int \frac{\cos^3 x \, dx}{\sin x} = \frac{\cos^2 x}{2} + \ln \sin x.$$

$$36. \int \frac{\cos^4 x \, dx}{\sin x} = \frac{1}{3} \cos^3 x + \cos x + \ln \operatorname{tg} \left(\frac{x}{2} \right).$$

$$37. \int \frac{\cos x}{\sin^2 x} \, dx = -\frac{1}{\sin x}.$$

$$38. \int \frac{\cos^2 x}{\sin^2 x} \, dx = -\operatorname{ctg} x - x.$$

$$39. \int \frac{\cos^3 x}{\sin^2 x} \, dx = -\sin x - \frac{1}{\sin x}.$$

$$40. \int \frac{\cos^4 x}{\sin^2 x} \, dx = -\operatorname{ctg} x - \frac{1}{2} \sin x \cos x - \frac{3}{2} x.$$

$$41. \int \frac{\cos x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x}.$$

$$42. \int \frac{\cos^2 x}{\sin^3 x} \, dx = -\frac{\cos x}{2 \sin^2 x} - \frac{1}{2} \ln \operatorname{tg} \frac{x}{2}.$$

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$$43. \int \frac{\cos^3 x}{\sin^3 x} \, dx = -\frac{1}{2 \sin^2 x} - \ln \sin x.$$

$$44. \int \frac{\cos^4 x}{\sin^3 x} \, dx = -\frac{1}{2} \frac{\cos x}{\sin^2 x} - \cos x - \frac{3}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$62. \int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3 \cos x \sin^3 x} - \frac{8}{3} \operatorname{ctg} 2x.$$

$$63. \int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3 \sin^3 x} + \frac{\sin x}{2 \cos^2 x} + \frac{5}{2} \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$64. \int \frac{dx}{\sin^4 x \cos^4 x} = -8 \operatorname{ctg} 2x - \frac{8}{3} \operatorname{ctg}^3 2x.$$

2.527

$$1. \int \operatorname{tg}^p x \, dx = \frac{\operatorname{tg}^{p-1} x}{p-1} - \int \operatorname{tg}^{p-2} x \, dx \quad [p \neq 1].$$

$$\begin{aligned} 2. \int \operatorname{tg}^{2n+1} x \, dx &= \sum_{k=1}^n (-1)^{n+k} \binom{n}{k} \frac{1}{2k \cos^{2k} x} - (-1)^n \ln \cos x = \\ &= \sum_{k=1}^n \frac{(-1)^{k-1} \operatorname{tg}^{2n-2k+2} x}{2n-2k+2} - (-1)^n \ln \cos x. \end{aligned}$$

$$3. \int \operatorname{tg}^{2n} x \, dx = \sum_{k=1}^n (-1)^{k-1} \frac{\operatorname{tg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x.$$

$$4. \int \operatorname{ctg}^p x \, dx = -\frac{\operatorname{ctg}^{p-1} x}{p-1} - \int \operatorname{ctg}^{p-2} x \, dx \quad [p \neq 1].$$

$$\begin{aligned} 5. \int \operatorname{ctg}^{2n+1} x \, dx &= \sum_{k=1}^n (-1)^{n+k+1} \binom{n}{k} \frac{1}{2k \sin^{2k} x} + (-1)^n \ln \sin x = \\ &= \sum_{k=1}^n (-1)^k \frac{\operatorname{ctg}^{2n-2k+2} x}{2n-2k+2} + (-1)^n \ln \sin x. \end{aligned}$$

$$6. \int \operatorname{ctg}^{2n} x \, dx = \sum_{k=1}^n (-1)^k \frac{\operatorname{ctg}^{2n-2k+1} x}{2n-2k+1} + (-1)^n x.$$

GU ((331))(14)

For special formulas for $p = 1, 2, 3, 4$, see 2.526 17., 2.526 33., 2.526 22., 2.526 38., 2.526 27., 2.526 43., 2.526 32., 2.526 48.

2.53-2.54 Sines and cosines of multiple angles and of linear and more complicated functions of the argument

2.531

$$1. \int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b).$$

$$2. \int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b).$$

2.532

$$1. \int \sin(ax+b) \sin(cx+d) \, dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} - \frac{\sin[(a+c)x+b+d]}{2(a+c)} \quad [a^2 \neq c^2].$$

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$$2. \int \sin(ax+b) \cos(cx+d) \, dx = -\frac{\cos[(a-c)x+b-d]}{2(a-c)} - \frac{\cos[(a+c)x+b+d]}{2(a+c)} \quad [a^2 \neq c^2].$$

$$3. \int \cos(ax+b) \cos(cx+d) dx = \frac{\sin[(a-c)x+b-d]}{2(a-c)} + \frac{\sin[(a+c)x+b+d]}{2(a+c)} \quad [a^2 \neq c^2].$$

For $c = a$:

$$4. \int \sin(ax+b) \sin(ax+d) dx = \frac{x}{2} \cos(b-d) - \frac{\sin(2ax+b+d)}{4a}.$$

$$5. \int \sin(ax+b) \cos(ax+d) dx = \frac{x}{2} \sin(b-d) - \frac{\cos(2ax+b+d)}{4a}.$$

$$6. \int \cos(ax+b) \cos(ax+d) dx = \frac{x}{2} \cos(b-d) + \frac{\sin(2ax+b+d)}{4a}.$$

GU ((332))(3)

2.533

$$1.^8 \int \sin ax \cos bx dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} \quad [a^2 \neq b^2].$$

$$2.^8 \int \sin ax \sin bx \sin cx dx = -\frac{1}{4} \left\{ \frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c} \right\}.$$

PE (376)

$$3. \int \sin ax \cos bx \cos cx dx = -\frac{1}{4} \left\{ \frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b} \right\}.$$

PE (378)

$$4. \int \cos ax \sin bx \sin cx dx = \frac{1}{4} \left\{ \frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a} \right\}.$$

$$5. \int \cos ax \cos bx \cos cx \, dx = \frac{1}{4} \left\{ \frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c} \right\}.$$

PE (377)

2.534

$$\left. \begin{aligned} 1. \int \frac{\cos px + i \sin px}{\sin nx} \, dx &= -2 \int \frac{z^{p+n-1}}{1-z^{2n}} \, dz \\ 2. \int \frac{\cos px + i \sin px}{\cos nx} \, dx &= -2i \int \frac{z^{p+n-1}}{1-z^{2n}} \, dz \end{aligned} \right\} \quad [z = \cos x + i \sin x].$$

PE (373)

PE (374)

171

2.535

$$1. \int \sin^p x \sin ax \, dx = \frac{1}{p+a} \left\{ -\sin^p x \cos ax + p \int \sin^{p-1} x \cos(a-1)x \, dx \right\}.$$

GU ((332))(5a)

$$\begin{aligned} 2. \int \sin^p x \sin(2n+1)x \, dx &= \\ &= (2n+1) \left\{ \int \sin^{p+1} x \, dx + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \times \right. \\ &\quad \left. \times \int \sin^{2k+p+1} x \, dx \right\}; \\ &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^{k-1} \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \cos(2n-2k+1)x + \right. \right. \\ &\quad \left. \left. + (-1)^k \frac{\Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \sin(2n-2k)x \right] + \right. \\ &\quad \left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n+1} x \, dx \right\}. \end{aligned}$$

$$1. \int \sin^p x \cos ax \, dx = \frac{1}{p+1} \left\{ \sin^p x \sin ax - p \int \sin^{p-1} x \sin (a-1)x \, dx \right\}.$$

GU ((332))(6a)

$$\begin{aligned}
 2. \int \sin^p x \cos (2n+1)x \, dx &= \\
 &= \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \times \\
 &\quad \times \sin^{2k+p+1} x; \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k \Gamma\left(\frac{p+1}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k+1)x + \right. \right. \\
 &\quad \left. \left. + \frac{(-1)^k \Gamma\left(\frac{p-1}{2} + n - 2k\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k)x \right] + \right. \\
 &\quad \left. + \frac{(-1)^n \Gamma\left(\frac{p+3}{2} - n\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n} x \cos x \, dx \right\}; \\
 &\quad [p \text{ is not equal to } -3, -5, \dots, -(2n+1)].
 \end{aligned}$$

GU ((332))(6c)
TI (301)

$$\begin{aligned}
 3. \int \sin^p x \cos 2nx \, dx &= \int \sin^p x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2 \cdot (4n^2 - 2^2) \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \sin^{2k+p} x \, dx; \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \left[\frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k\right)}{2^{2k+1} \Gamma(p-2k+1)} \sin^{p-2k} x \sin(2n-2k)x + \right. \right. \\
 &\quad \left. \left. + \frac{(-1)^k \Gamma\left(\frac{p}{2} + n - 2k - 1\right)}{2^{2k+2} \Gamma(p-2k)} \sin^{p-2k-1} x \cos(2n-2k-1)x \right] + \right. \\
 &\quad \left. + \frac{(-1)^n \Gamma\left(\frac{p}{2} - n + 1\right)}{2^{2n} \Gamma(p-2n+1)} \int \sin^{p-2n} x \, dx \right\}.
 \end{aligned}$$

GU ((332))(6c)
TI (300)

$$1. \int \cos^p x \sin ax \, dx = \frac{1}{p+a} \left\{ -\cos^p x \cos ax + p \int \cos^{p-1} x \sin (a-1)x \, dx \right\} .$$

GU ((332))(7a)

$$\begin{aligned}
 2. \int \cos^p x \sin (2n+1)x \, dx &= \\
 &= (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} + \right. \\
 &+ \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k)!(2k+p+1)} \cos^{2k+p+1} x \left. \right\}; \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{2k+1} \Gamma(p-2k+1)} \cos^{p-k} x \cos (2n-k+1)x + \right. \\
 &+ \left. \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin (n+1)x \, dx \right\}; \\
 &\quad [p \text{ is not equal to } -3, -5, \dots, -(2n+1)].
 \end{aligned}$$

GU ((332))(7b)a
TI (295)

$$\begin{aligned}
 3. \int \cos^p x \sin 2nx \, dx &= (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \right. \\
 &+ \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2 - 2^2)(4n^2 - 4^2) \dots [4n^2 - (2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \left. \right\}; \\
 &= \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ - \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \cos (2n-k)x + \right. \\
 &+ \left. \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \sin nx \, dx \right\}; \\
 &\quad [p \text{ is not equal to } -2, -4, \dots, -2n].
 \end{aligned}$$

GU ((332))(7b)a
TI (297)

2.538

$$1. \int \cos^p x \cos ax \, dx = \frac{1}{p+a} \left\{ \cos^p x \sin ax + p \int \cos^{p-1} x \cos (a-1)x \, dx \right\} .$$

$$\begin{aligned}
2. \quad & \int \cos^p x \cos(2n+1)x \, dx = \\
& = (-1)^n (2n+1) \left\{ \int \cos^{p+1} x \, dx + \right. \\
& \quad \left. + \sum_{k=1}^n (-1)^k \frac{[(2n+1)^2 - 1^2][(2n+1)^2 - 3^2] \dots [(2n+1)^2 - (2k-1)^2]}{(2k+1)!} \times \right. \\
& \quad \left. \times \int \cos^{2k+p+1} x \, dx \right\}; \\
& = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p+3}{2} + n\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p+1}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k+1)x + \right. \\
& \quad \left. + \frac{\Gamma\left(\frac{p+3}{2}\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos(n+1)x \, dx \right\}.
\end{aligned}$$

GU ((332))(8b)a
TI (293)

$$\begin{aligned}
3. \quad & \int \cos^p x \cos 2nx \, dx = \\
& = (-1)^n \left\{ \int \cos^p x \, dx + \sum_{k=1}^n (-1)^k \frac{4n^2[4n^2 - 2^2] \dots [4n^2 - (2k-2)^2]}{(2k)!} \int \cos^{2k+p} x \, dx \right\}; \\
& = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2} + n + 1\right)} \left\{ \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{p}{2} + n - k\right)}{2^{k+1} \Gamma(p-k+1)} \cos^{p-k} x \sin(2n-k)x + \right. \\
& \quad \left. + \frac{\Gamma\left(\frac{p}{2} + 1\right)}{2^n \Gamma(p-n+1)} \int \cos^{p-n} x \cos nx \, dx \right\}.
\end{aligned}$$

GU ((332))(8b)a
TI (294)

2.539

$$1. \quad \int \frac{\sin(2n+1)x}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\sin 2kx}{2k} + x.$$

$$2. \quad \int \frac{\sin 2nx}{\sin x} \, dx = 2 \sum_{k=1}^n \frac{\sin(2k-1)x}{2k-1}.$$

$$3. \int \frac{\cos(2n+1)x}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos 2kx}{2k} + \ln \sin x.$$

$$4. \int \frac{\cos 2nx}{\sin x} dx = 2 \sum_{k=1}^n \frac{\cos(2k-1)x}{2k-1} + \ln \operatorname{tg} \frac{x}{2}.$$

GI ((332))(6e)

$$5. \int \frac{\sin(2n+1)x}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos 2kx}{2k} + (-1)^{n+1} \ln \cos x.$$

$$6. \int \frac{\sin 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k+1} \frac{\cos(2k-1)x}{2k-1}.$$

GU ((332))(7d)

$$7. \int \frac{\cos(2n+1)x}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin 2kx}{2k} + (-1)^n x.$$

$$8. \int \frac{\cos 2nx}{\cos x} dx = 2 \sum_{k=1}^n (-1)^{n-k} \frac{\sin(2k-1)x}{2k-1} + (-1)^n \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

GU ((332))(8d)

2.541

$$1. \int \sin(n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \sin nx.$$

BI ((71))(1)a

$$2. \int \sin(n+1)x \cos^{n-1} x dx = -\frac{1}{n} \cos^n x \cos nx.$$

BI ((71))(2)a

$$3. \int \cos(n+1)x \sin^{n-1} x dx = \frac{1}{n} \sin^n x \cos nx.$$

$$4. \int \cos(n+1)x \cos^{n-1}x \, dx = \frac{1}{n} \cos^n x \sin nx.$$

BI ((71))(4)a

$$5. \int \sin \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1}x \, dx = \frac{1}{n} \sin^n x \cos n \left(\frac{\pi}{2} - x \right).$$

BI ((71))(5)a

$$6. \int \cos \left[(n+1) \left(\frac{\pi}{2} - x \right) \right] \sin^{n-1}x \, dx = -\frac{1}{n} \sin^n x \sin n \left(\frac{\pi}{2} - x \right).$$

BI ((71))(6)a

2.542

$$1. \int \frac{\sin 2x}{\sin^n x} \, dx = -\frac{2}{(n-2) \sin^{n-2} x}.$$

For $n = 2$:

$$2. \int \frac{\sin 2x}{\sin^2 x} \, dx = 2 \ln \sin x.$$

2.543

$$1. \int \frac{\sin 2x \, dx}{\cos^n x} = \frac{2}{(n-2) \cos^{n-2} x}.$$

176

For $n = 2$:

$$2. \int \frac{\sin 2x}{\cos^2 x} \, dx = -2 \ln \cos x.$$

2.544

$$1. \int \frac{\cos 2x \, dx}{\sin x} = 2 \cos x + \ln \operatorname{tg} \frac{x}{2}.$$

$$2. \int \frac{\cos 2x \, dx}{\sin^2 x} = -\operatorname{ctg} x - 2x.$$

$$3. \int \frac{\cos 2x \, dx}{\sin^3 x} = -\frac{\cos x}{2 \sin^2 x} - \frac{3}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$4. \int \frac{\cos 2x \, dx}{\cos x} = 2 \sin x - \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$5. \int \frac{\cos 2x \, dx}{\cos^2 x} = 2x - \operatorname{tg} x.$$

$$6. \int \frac{\cos 2x \, dx}{\cos^3 x} = -\frac{\sin x}{2 \cos^2 x} + \frac{3}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right).$$

$$7. \int \frac{\sin 3x \, dx}{\sin x} = x + \sin 2x.$$

$$8. \int \frac{\sin 3x}{\sin^2 x} \, dx = 3 \ln \operatorname{tg} \frac{x}{2} + 4 \cos x.$$

$$9. \int \frac{\sin 3x}{\sin^3 x} \, dx = -3 \operatorname{ctg} x - 4x.$$

2.545

$$1. \int \frac{\sin 3x}{\cos^n x} \, dx = \frac{4}{(n-3) \cos^{n-3} x} - \frac{1}{(n-1) \cos^{n-1} x}.$$

For $n = 1$ and $n = 3$:

$$2. \int \frac{\sin 3x}{\cos x} \, dx = 2 \sin^2 x + \ln \cos x.$$

$$3. \int \frac{\sin 3x}{\cos^3 x} dx = -\frac{1}{2 \cos^2 x} - 4 \ln |\cos x|.$$

2.546

$$1. \int \frac{\cos 3x}{\sin^n x} dx = \frac{4}{(n-3) \sin^{n-3} x} - \frac{1}{(n-1) \sin^{n-1} x}.$$

177

For $n = 1$ and $n = 3$:

$$2. \int \frac{\cos 3x}{\sin x} dx = -2 \sin^2 x + \ln |\sin x|.$$

$$3. \int \frac{\cos 3x}{\sin^3 x} dx = -\frac{1}{2 \sin^2 x} - 4 \ln |\sin x|.$$

2.547

$$1. \int \frac{\sin nx}{\cos^p x} dx = 2 \int \frac{\sin(n-1)x dx}{\cos^{p-1} x} - \int \frac{\sin(n-2)x dx}{\cos^p x}.$$

$$2. \int \frac{\cos 3x}{\cos x} dx = \sin 2x - x.$$

$$3. \int \frac{\cos 3x}{\cos^2 x} dx = 4 \sin x - 3 \ln \left| \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \right|.$$

$$4. \int \frac{\cos 3x}{\cos^3 x} dx = 4x - 3 \operatorname{tg} x.$$

2.548

$$1. \int \frac{\sin^m x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \sum_{k=0}^{2n} (-1)^{n+k} \cos^m \left[\frac{2k+1}{2(2n+1)} \pi \right] \ln \frac{\sin \left[\frac{(k-n)\pi}{2(2n+1)} + \frac{x}{2} \right]}{\sin \left[\frac{k+n+1}{2(2n+1)} \pi - \frac{x}{2} \right]}$$

$$2. \int \frac{\sin^{2m} x dx}{\sin 2nx} = \frac{(-1)^n}{2n} \left\{ \ln \cos x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[m – a natural number $\leq n$].

TI (379)

$$3. \int \frac{\sin^{2m+1} x}{\sin 2nx} dx = \frac{(-1)^n}{2n} \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\operatorname{tg} \left(\frac{n+k}{4n} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{n-k}{4n} \pi - \frac{x}{2} \right) \right] \right\}$$

[m – a natural number $< n$].

TI (380)

$$4. \int \frac{\sin^{2m} x dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + \sum_{k=1}^n (-1)^k \times \right.$$

$$\left. \times \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\operatorname{tg} \left(\frac{2n+2k+1}{4(2n+1)} \pi - \frac{x}{2} \right) \operatorname{tg} \left(\frac{2n-2k+1}{2(2n+1)} \pi - \frac{x}{2} \right) \right] \right\}$$

[m – a natural number $\leq n$].

TI (381)

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$$5. \int \frac{\sin^{2m+1} x dx}{\cos(2n+1)x} = \frac{(-1)^{n+1}}{2n+1} \left\{ \ln \cos x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\cos^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m – a natural number $\leq n$].

TI (382)a

$$6. \int \frac{\sin^m x dx}{\cos 2nx} = \frac{1}{2n} \sum_{k=0}^{2n-1} (-1)^{n+k} \cos^m \left[\frac{2k+1}{4n} \pi \right] \ln \frac{\sin \left[\frac{2k-2n+1}{8n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+2n+1}{8n} \pi - \frac{x}{2} \right]}$$

[m – a natural number $< 2n$].

TI (377)

$$7. \int \frac{\cos^{2m+1} x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \sin x + \sum_{k=1}^n (-1)^k \cos^{2m+1} \frac{k\pi}{2n+1} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n+1} \right) \right\}$$

[m – a natural number $\leq n$].

$$8. \int \frac{\cos^{2m} x dx}{\sin(2n+1)x} = \frac{1}{2n+1} \left\{ \ln \operatorname{tg} \frac{x}{2} + \sum_{k=1}^n (-1)^k \cos^{2m} \frac{k\pi}{2n+1} \ln \left[\operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4n+2} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4n+2} \right) \right] \right\}$$

[m - a natural number $\leq n$].

TI (375)

$$9. \int \frac{\cos^{2m+1} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln \operatorname{tg} \frac{x}{2} + \sum_{k=1}^{n-1} (-1)^k \cos^{2m+1} \frac{k\pi}{2n} \ln \left[\operatorname{tg} \left(\frac{x}{2} + \frac{k\pi}{4} \right) \operatorname{tg} \left(\frac{x}{2} - \frac{k\pi}{4} \right) \right] \right\}$$

[m - a natural number $< n$].

TI (374)

$$10. \int \frac{\cos^{2m} x}{\sin 2nx} dx = \frac{1}{2n} \left\{ \ln \sin x + \sum_{k=1}^{n-1} (-1)^k \cos^{2m} \frac{k\pi}{2n} \ln \left(\sin^2 x - \sin^2 \frac{k\pi}{2n} \right) \right\}$$

[m - a natural number $\leq n$].

TI (373)

$$11. \int \frac{\cos^m x}{\cos nx} dx = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \cos^m \frac{2k+1}{2n} \pi \ln \frac{\sin \left[\frac{2k+1}{4n} \pi + \frac{x}{2} \right]}{\sin \left[\frac{2k+1}{4n} \pi - \frac{x}{2} \right]}$$

[m - a natural number $\leq n$].

TI (372)

2.549

$$1. \int \sin x^2 dx = \sqrt{\frac{\pi}{2}} S(x).$$

$$2. \int \cos x^2 dx = \sqrt{\frac{\pi}{2}} C(x).$$

$$3. \int \sin(ax^2+2bx+c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{ac-b^2}{a} S \left(\frac{ax+b}{\sqrt{a}} \right) + \sin \frac{ac-b^2}{a} C \left(\frac{ax+b}{\sqrt{a}} \right) \right\}.$$

PE (444)

$$6. \int \cos \ln x \, dx = \frac{x}{2} (\sin \ln x + \cos \ln x).$$

PE (445)

2.55-2.56 Rational functions of the sine and cosine

2.551

$$1. \int \frac{A + B \sin x}{(a + b \sin x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(Ab - aB) \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(Aa - Bb)(n-1) + (aB - bA)(n-2) \sin x}{(a + b \sin x)^{n-1}} dx \right].$$

TI (358)a

For $n = 1$:

$$2. \int \frac{A + B \sin x}{a + b \sin x} dx = \frac{B}{b} x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \sin x} \quad (\text{see 2.551 3.}).$$

2.551

TI (342)

$$3. \int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{a \operatorname{tg} \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \quad [a^2 > b^2];$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{a \operatorname{tg} \frac{x}{2} + b - \sqrt{b^2 - a^2}}{a \operatorname{tg} \frac{x}{2} + b + \sqrt{b^2 - a^2}} \quad [a^2 < b^2].$$

2.552

$$1. \int \frac{A + B \cos x}{(a + b \sin x)^n} dx = -\frac{B}{(n-1)b(a + b \sin x)^{n-1}} + A \int \frac{dx}{(a + b \sin x)^n} \quad (\text{see 2.552 3.}).$$

2.552

For $n = 1$:

$$2. \int \frac{A + B \cos x}{a + b \sin x} dx = \frac{B}{b} \ln(a + b \sin x) + A \int \frac{dx}{a + b \sin x} \quad (\text{see } \mathbf{2.551} \text{ 3.}).$$

2.551
TI (344)

$$3. \int \frac{dx}{(a + b \sin x)^n} = \frac{1}{(n-1)(a^2 - b^2)} \left\{ \frac{b \cos x}{(a + b \sin x)^{n-1}} + \int \frac{(n-1)a - (n-2)b \sin x}{(a + b \sin x)^{n-1}} dx \right\} \quad (\text{see } \mathbf{2.551} \text{ 1.}).$$

2.551
TI (359)

2.553

$$1. \int \frac{A + B \sin x}{(a + b \cos x)^n} dx = \frac{B}{(n-1)b(a + b \cos x)^{n-1}} + A \int \frac{dx}{(a + b \cos x)^n} \quad (\text{see } \mathbf{2.554} \text{ 3.}).$$

2.554
TI (355)

180

For $n = 1$:

$$2. \int \frac{A + B \sin x}{a + b \cos x} dx = -\frac{B}{b} \ln(a + b \cos x) + A \int \frac{dx}{a + b \cos x} \quad (\text{see } \mathbf{2.553} \text{ 3.}).$$

2.553
TI (343)

$$3. \int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \operatorname{arctg} \frac{\sqrt{a^2 - b^2} \operatorname{tg} \frac{x}{2}}{a + b} \quad [a^2 > b^2];$$

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$$= \frac{1}{\sqrt{b^2 - a^2}} \ln \frac{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} + a + b}{\sqrt{b^2 - a^2} \operatorname{tg} \frac{x}{2} - a - b} \quad [a^2 < b^2].$$

2.554

$$1. \int \frac{A + B \cos x}{(a + b \cos x)^n} dx = \frac{1}{(n-1)(a^2 - b^2)} \left[\frac{(aB - Ab) \sin x}{(a + b \cos x)^{n-1}} + \int \frac{(Aa - bB)(n-1) + (n-2)(aB - bA) \cos x}{(a + b \cos x)^{n-1}} dx \right].$$

TI (353)

For $n = 1$:

$$2. \int \frac{A + B \cos x}{a + b \cos x} dx = \frac{B}{b}x + \frac{Ab - aB}{b} \int \frac{dx}{a + b \cos x} \quad (\text{see 2.5533}).$$

2.553
TI (341)

$$3. \int \frac{dx}{(a + b \cos x)^n} = -\frac{1}{(n-1)(a^2 - b^2)} \left\{ \frac{b \sin x}{(a + b \cos x)^{n-1}} - \int \frac{(n-1)a - (n-2)b \cos x}{(a + b \cos x)^{n-1}} dx \right\} \quad (\text{see 2.5541}).$$

2.554
TI (354)

In integrating the functions in formulas 2.551 3. and 2.553 3, we may not take the integration over points at which the integrand becomes infinite, that is, over the points $x = \arcsin\left(-\frac{a}{b}\right)$ in formula 2.551 3. or over the points $x = \arccos\left(-\frac{a}{b}\right)$ in formula 2.553 3.

2.555

Formulas 2.551 3. and 2.553 3 are not applicable for $a^2 = b^2$. Instead, we may use the following formulas in these cases:

$$1. \int \frac{A + B \sin x}{(1 \pm \sin x)^n} dx = -\frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left(\frac{\pi}{4} \mp \frac{x}{2} \right)}{2k+1} \right\}.$$

$$2. \int \frac{A + B \cos x}{(1 \pm \cos x)^n} dx = \frac{1}{2^{n-1}} \left\{ 2B \sum_{k=0}^{n-2} \binom{n-2}{k} \frac{\operatorname{tg}^{2k+1} \left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2} \right) \right]}{2k+1} \pm \right. \\ \left. \pm (A \mp B) \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\operatorname{tg}^{2k+1} \left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2} \right) \right]}{2k+1} \right\}.$$

TI (356)

For $n = 1$:

$$3. \int \frac{A + B \sin x}{1 \pm \sin x} dx = \pm Bx + (A \mp B) \operatorname{tg} \left(\frac{\pi}{4} \mp \frac{x}{2} \right).$$

TI (250)

$$4. \int \frac{A + B \cos x}{1 \pm \cos x} dx = \pm Bx \pm (A \mp B) \operatorname{tg} \left[\frac{\pi}{4} \mp \left(\frac{\pi}{4} - \frac{x}{2} \right) \right].$$

TI (248)

2.556

$$1. \int \frac{(1 - a^2) dx}{1 - 2a \cos x + a^2} = 2 \operatorname{arctg} \left(\frac{1+a}{1-a} \operatorname{tg} \frac{x}{2} \right) \quad [0 < a < 1, \quad |x| < \pi].$$

FI II 93

$$2. \int \frac{(1 - a \cos x) dx}{1 - 2a \cos x + a^2} = \frac{x}{2} + \operatorname{arctg} \left(\frac{1+a}{1-a} \operatorname{tg} \frac{x}{2} \right) \quad [0 < a < 1, \quad |x| < \pi].$$

FI II 93

2.557

$$1. \int \frac{dx}{(a \cos x + b \sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}} \int \frac{dx}{\sin^n \left(x + \operatorname{arctg} \frac{a}{b} \right)} \quad (\text{see 2.515}).$$

2.515
MZ 173a

$$2.6 \int \frac{\sin x dx}{a \sin x + b \cos x} = \frac{ax - b \ln \sin \left(x + \operatorname{arctg} \frac{b}{a} \right)}{a^2 + b^2}.$$

$$3. \int \frac{\cos x \, dx}{a \cos x + b \sin x} = \frac{ax + b \ln \sin \left(x + \operatorname{arctg} \frac{a}{b} \right)}{a^2 + b^2}.$$

MZ 174a

$$4. \int \frac{dx}{a \cos x + b \sin x} = \frac{\ln \operatorname{tg} \left[\frac{1}{2} \left(x + \operatorname{arctg} \frac{a}{b} \right) \right]}{\sqrt{a^2 + b^2}}.$$

$$5. \int \frac{dx}{(a \cos x + b \sin x)^2} = -\frac{\operatorname{ctg} \left(x + \operatorname{arctg} \frac{a}{b} \right)}{a^2 + b^2} = +\frac{1}{a^2 + b^2} \cdot \frac{a \sin x - b \cos x}{a \cos x + b \sin x}.$$

MZ 174a

182
2.558

$$\begin{aligned} 1. \int \frac{A + B \cos x + C \sin x}{(a + b \cos x + c \sin x)^n} dx &= \\ &= \frac{(Bc - Cb) + (Ac - Ca) \cos x - (Ab - Ba) \sin x}{(n-1)(a^2 - b^2 - c^2)(a + b \cos x + c \sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times \\ &\quad \times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba) \cos x - (Ac - Ca) \sin x]}{(a + b \cos x + c \sin x)^{n-1}} dx \\ &\quad [n \neq 1, \quad a^2 \neq b^2 + c^2]; \\ &= \frac{Cb - Bc + Ca \cos x - Ba \sin x}{(n-1)a(a + b \cos x + c \sin x)^n} + \left(\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2} \right) (-c \cos x + b \sin x) \times \\ &\quad \times \frac{(n-1)!}{(2n-1)!!} \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!a^k} \cdot \frac{1}{(a + b \cos x + c \sin x)^{n-k}} [n \neq 1, \quad a^2 = b^2 + c^2]. \end{aligned}$$

For $n = 1$:

$$\begin{aligned} 2. \int \frac{A + B \cos x + C \sin x}{a + b \cos x + c \sin x} dx &= \frac{Bc - Cb}{b^2 + c^2} \ln(a + b \cos x + c \sin x) + \frac{Bb + Cc}{b^2 + c^2} x + \\ &\quad + \left(A - \frac{Bb + Cc}{B^2 + c^2} a \right) \int \frac{dx}{a + b \cos x + c \sin x} \quad (\text{see } \mathbf{2.558} \text{ 4}). \end{aligned}$$

2.558
GU ((331))(18)

$$3. \int \frac{dx}{(a + b \cos x + c \sin x)^n} = \int \frac{d(x - \alpha)}{[a + r \cos(x - \alpha)]^n},$$

where $b = r \cos \alpha$, $c = r \sin \alpha$ (see 2.554 3.).

$$4. \int \frac{dx}{a + b \cos x + c \sin x} =$$

$$= \frac{2}{\sqrt{a^2 - b^2 - c^2}} \operatorname{arctg} \frac{(a - b) \operatorname{tg} \frac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}} \quad [a^2 > b^2 + c^2];$$

$$= \frac{1}{\sqrt{b^2 + c^2 - a^2}} \ln \frac{(a - b) \operatorname{tg} \frac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a - b) \operatorname{tg} \frac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}} \quad [a^2 < b^2 + c^2];$$

$$= \frac{1}{c} \ln \left(a + c \cdot \operatorname{tg} \frac{x}{2} \right) \quad [a = b];$$

$$= \frac{-2}{c + (a - b) \operatorname{tg} \frac{x}{2}} \quad [a^2 = b^2 + c^2].$$

TI (253)a
 TI (253)a
 TI (253), FI II 94

2.559

$$1. \int \frac{dx}{[a(1 + \cos x) + c \sin x]^2} = \frac{1}{c^3} \left[\frac{c(a \sin x - c \cos x)}{a(1 + \cos x) + c \sin x} - a \ln \left(a + c \operatorname{tg} \frac{x}{2} \right) \right].$$

183

$$2. \int \frac{A + B \cos x + C \sin x}{(a_1 + b_1 \cos x + c_1 \sin x)(a_2 + b_2 \cos x + c_2 \sin x)} dx =$$

$$= A_0 \ln \frac{a_1 + b_1 \cos x + c_1 \sin x}{a_2 + b_2 \cos x + c_2 \sin x} + A_1 \int \frac{dx}{a_1 + b_1 \cos x + c_1 \sin x} + A_2 \int \frac{dx}{a_2 + b_2 \cos x + c_2 \sin x},$$

where

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2}, \quad A_1 = \frac{\begin{vmatrix} \begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix} & \begin{vmatrix} A & C \\ a_1 & c_1 \end{vmatrix} & \begin{vmatrix} B & A \\ b_1 & a_1 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2};$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \right] \quad (\text{see } \mathbf{2.558} \text{ 4.}).$$

2.558
GU ((331))(19)

$$\begin{aligned} 3. \int \frac{A \cos^2 x + 2B \sin x \cos x + C \sin^2 x}{a \cos^2 x + 2b \sin x \cos x + c \sin^2 x} dx &= \\ &= \frac{1}{4b^2 + (a-c)^2} \{ [4Bb + (A-C)(a-c)]x + [(A-C)b - B(a-c)] \times \\ &\quad \times \ln(a \cos^2 x + 2b \sin x \cos x + c \sin^2 x) + \\ &\quad + [2(A+C)b^2 - 2Bb(a+c) + (aC - Ac)(a-c)]f(x) \}, \end{aligned}$$

where

$$\begin{aligned} f(x) &= \frac{1}{2\sqrt{b^2 - ac}} \ln \frac{c \operatorname{tg} x + b - \sqrt{b^2 - ac}}{c \operatorname{tg} x + b + \sqrt{b^2 - ac}} \quad [b^2 > ac]; \\ &= \frac{1}{\sqrt{ac - b^2}} \operatorname{arctg} \frac{c \operatorname{tg} x + b}{\sqrt{ac - b^2}} \quad [b^2 < ac]; \\ &= -\frac{1}{c \operatorname{tg} x + b} \quad [b^2 = ac]. \end{aligned}$$

GU ((331))(24)

184
2.561

$$1. \int \frac{(A + B \sin x) dx}{\sin x(a + b \sin x)} = \frac{A}{a} \ln \operatorname{tg} \frac{x}{2} + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \sin x} \quad (\text{see } 2.5513.).$$

2.551
TI (348)

$$\begin{aligned} 2. \int \frac{(A + B \sin x) dx}{\sin x(a + b \cos x)} &= \frac{A}{a^2 - b^2} \left\{ a \ln \operatorname{tg} \frac{x}{2} + b \ln \frac{a + b \cos x}{\sin x} \right\} + \\ &\quad + B \int \frac{dx}{a + b \cos x} \quad (\text{see } 2.5533.). \end{aligned}$$

For $a^2 = b^2 (= 1)$:

$$3. \int \frac{(A + B \sin x) dx}{\sin x(a + b \cos x)} = \frac{A}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} + \frac{1}{1 + \cos x} \right\} + B \operatorname{tg} \frac{x}{2}.$$

$$4. \int \frac{(A + B \sin x) dx}{\sin x(1 - \cos x)} = \frac{A}{2} \left\{ \ln \operatorname{tg} \frac{x}{2} - \frac{1}{1 - \cos x} \right\} - B \operatorname{ctg} \frac{x}{2}.$$

$$5. \int \frac{(A + B \sin x) dx}{\cos x(a + b \sin x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - (Ab - aB) \ln \frac{a + b \sin x}{\cos x} \right\}.$$

TI (346)

For $a^2 = b^2 (= 1)$:

$$6. \int \frac{(A + B \sin x) dx}{\cos x(1 \pm \sin x)} = \frac{A \pm B}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}.$$

$$7. \int \frac{(A + B \sin x) dx}{\cos x(a + b \cos x)} = \frac{A}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{B}{a} \ln \frac{a + b \cos x}{\cos x} - \frac{Ab}{a} \int \frac{dx}{a + b \cos x} \quad (\text{see } \mathbf{2.553} \text{ 3}).$$

2.551
TI (351)a

$$8. \int \frac{(A + B \cos x) dx}{\sin x(a + b \sin x)} = \frac{A}{a} \ln \operatorname{tg} \frac{x}{2} - \frac{B}{a} \ln \frac{a + b \sin x}{\sin x} - \frac{Ab}{a} \int \frac{dx}{a + b \sin x} \quad (\text{see } \mathbf{2.5513}).$$

2.551
TI (352)

$$9. \int \frac{(A + B \cos x) dx}{\sin x(a + b \cos x)} = \frac{1}{a^2 - b^2} \left\{ (Aa - Bb) \ln \operatorname{tg} \frac{x}{2} + (Ab - Ba) \ln \frac{a + b \cos x}{\sin x} \right\}.$$

For $a^2 = b^2 (= 1)$:

$$10. \int \frac{(A + B \cos x) dx}{\sin x(1 \pm \cos x)} = \pm \frac{A \mp B}{2(1 \pm \cos x)} + \frac{A \pm B}{2} \ln \operatorname{tg} \frac{x}{2}.$$

$$11. \int \frac{(A + B \cos x) dx}{\cos x(a + b \sin x)} = \frac{A}{a^2 - b^2} \left\{ a \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) - b \ln \frac{a + b \sin x}{\cos x} \right\} + B \int \frac{dx}{a + b \sin x}$$

(see **2.551** 3.).

2.551
TI (350)

185

For $a^2 = b^2 (= 1)$:

$$12. \int \frac{(A + B \sin x) dx}{\cos x(1 \pm \sin x)} = \frac{A \pm B}{2} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) \mp \frac{A \mp B}{2(1 \pm \sin x)}.$$

$$13. \int \frac{(A + B \cos x) dx}{\cos x(a + b \cos x)} = \frac{A}{a} \ln \operatorname{tg} \left(\frac{\pi}{4} + \frac{x}{2} \right) + \frac{Ba - Ab}{a} \int \frac{dx}{a + b \cos x}$$

(see **2.553** 3.).

2.553
TI (347)

2.562

$$1. \int \frac{dx}{a + b \sin^2 x} = \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg} \left(\sqrt{\frac{a+b}{a}} \operatorname{tg} x \right) \quad \left[\frac{b}{a} > -1 \right];$$

$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth} \left(\sqrt{-\frac{a+b}{a}} \operatorname{tg} x \right) \quad \left[\frac{b}{a} < -1, \quad \sin^2 x < -\frac{a}{b} \right];$$

$$= \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth} \left(\sqrt{-\frac{a+b}{a}} \operatorname{tg} x \right) \quad \left[\frac{b}{a} < -1, \quad \sin^2 x > -\frac{a}{b} \right].$$

$$\begin{aligned}
2. \int \frac{dx}{a+b \cos^2 x} &= \frac{-\operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg} \left(\sqrt{\frac{a+b}{a}} \operatorname{ctg} x \right) \quad \left[\frac{b}{a} > -1 \right]; \\
&= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth} \left(\sqrt{\frac{-a+b}{a}} \operatorname{ctg} x \right) \quad \left[\frac{b}{a} < -1, \quad \cos^2 x < -\frac{a}{b} \right]; \\
&= \frac{-\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth} \left(\sqrt{\frac{-a+b}{a}} \operatorname{ctg} x \right) \quad \left[\frac{b}{a} < -1, \quad \cos^2 x > -\frac{a}{b} \right].
\end{aligned}$$

MZ 162

$$3. \int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{tg} x).$$

$$4. \int \frac{dx}{1-\sin^2 x} = \operatorname{tg} x.$$

$$5. \int \frac{dx}{1+\cos^2 x} = -\frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2} \operatorname{ctg} x).$$

$$6. \int \frac{dx}{1-\cos^2 x} = -\operatorname{ctg} x.$$

2.563

$$1. \int \frac{dx}{(a+b \sin^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a+b \sin^2 x} + \frac{b \sin x \cos x}{a+b \sin^2 x} \right] \quad (\text{see 2.5621}).$$

2.562
MZ 155

186

$$2. \int \frac{dx}{(a+b \cos^2 x)^2} = \frac{1}{2a(a+b)} \left[(2a+b) \int \frac{dx}{a+b \cos^2 x} - \frac{b \sin x \cos x}{a+b \cos^2 x} \right] \quad (\text{see 2.5622}).$$

2.562
MZ 163

$$\begin{aligned}
3. \int \frac{dx}{(a+b \sin^2 x)^3} &= \frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{tg} x) + \right. \\
&\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{tg} x}{1+p^2 \operatorname{tg}^2 x} + \right. \\
&\quad \left. + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{tg}^2 x \right) \frac{2p \operatorname{tg} x}{(1+p^2 \operatorname{tg}^2 x)^2} \right] \quad \left[p^2 = 1 + \frac{b}{a} > 0 \right]; \\
&= \frac{1}{8pa^3} \left[\left(3 - \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{Arth}(a \operatorname{tg} x) + \right.
\end{aligned}$$

$$\begin{aligned}
4. \int \frac{dx}{(a + b \cos^2 x)^3} &= -\frac{1}{8pa^3} \left[\left(3 + \frac{2}{p^2} + \frac{3}{p^4} \right) \operatorname{arctg}(p \operatorname{ctg} x) + \right. \\
&\quad \left. + \left(3 + \frac{2}{p^2} - \frac{3}{p^4} \right) \frac{p \operatorname{ctg} x}{1 + p^2 \operatorname{ctg}^2 x} + \left(1 - \frac{2}{p^2} - \frac{1}{p^2} \operatorname{ctg}^2 x \right) \frac{2p \operatorname{ctg} x}{(1 + p^2 \operatorname{ctg}^2 x)^2} \right] \\
&\quad \left[p^2 = 1 + \frac{b}{a} > 0 \right]; \\
&= -\frac{1}{8qa^3} \left[\left(3 - \frac{2}{q^2} + \frac{3}{q^4} \right) \operatorname{Arth}(q \operatorname{ctg} x) + \right. \\
&\quad \left. + \left(3 - \frac{2}{q^2} - \frac{3}{q^4} \right) \frac{q \operatorname{ctg} x}{1 - q^2 \operatorname{ctg}^2 x} + \left(1 + \frac{2}{q^2} + \frac{1}{q^2} \operatorname{ctg}^2 x \right) \frac{2p \operatorname{ctg} x}{(1 - q^2 \operatorname{ctg}^2 x)^2} \right] \\
\left[q^2 = -1 - \frac{b}{a} < 0, \quad \cos^2 x < -\frac{a}{b}, \quad \text{for } \cos^2 x > -\frac{a}{b}, \quad \text{one should replace} \right. \\
&\quad \left. \operatorname{Arth}(q \operatorname{ctg} x) \quad \text{with} \quad \operatorname{Arcth}(q \operatorname{ctg} x) \right]
\end{aligned}$$

MZ 163a

2.564

$$1. \int \frac{\operatorname{tg} x \, dx}{1 + m^2 \operatorname{tg}^2 x} = \frac{\ln(\cos^2 x + m^2 \sin^2 x)}{2(m^2 - 1)}.$$

LA 210 (10)

187

$$2. \int \frac{\operatorname{tg} \alpha - \operatorname{tg} x}{\operatorname{tg} \alpha + \operatorname{tg} x} dx = \sin 2\alpha \ln \sin(x + \alpha) - x \cos 2\alpha.$$

LA 210 (11)a

$$3. \int \frac{\operatorname{tg} x \, dx}{a + b \operatorname{tg} x} = \frac{1}{a^2 + b^2} \{bx - a \ln(a \cos x + b \sin x)\}.$$

PE (335)

$$4. \int \frac{dx}{a + b \operatorname{tg}^2 x} = \frac{1}{a - b} \left[x - \sqrt{\frac{b}{a}} \operatorname{arctg} \left(\sqrt{\frac{b}{a}} \operatorname{tg} x \right) \right].$$

PE (334)

2.57 Forms containing $\sqrt{a \pm b \sin x}$ or $\sqrt{a \pm b \cos x}$ and forms reducible to such expressions

Notations: $\alpha = \arcsin \sqrt{\frac{1-\sin x}{2}}$, $\beta = \arcsin \sqrt{\frac{b(1-\sin x)}{a+b}}$,

$\gamma = \arcsin \sqrt{\frac{b(1-\cos x)}{a+b}}$, $\delta = \arcsin \sqrt{\frac{(a+b)(1-\cos x)}{2(a-b \cos x)}}$, $r = \sqrt{\frac{2b}{a+b}}$.

2.571

$$1. \int \frac{dx}{\sqrt{a+b \sin x}} = \frac{-2}{\sqrt{a+b}} F(\alpha, r) \quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right];$$

$$= -\sqrt{\frac{2}{b}} F\left(\beta, \frac{1}{r}\right) \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

BY (288.00, 288.50)

$$2. \int \frac{\sin x \, dx}{\sqrt{a+b \sin x}} = \frac{2a}{b\sqrt{a+b}} F(\alpha, r) - \frac{2\sqrt{a+b}}{b} E(\alpha, r) \quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right];$$

$$= \sqrt{\frac{2}{b}} \left\{ F\left(\beta, \frac{1}{r}\right) - 2E\left(\beta, \frac{1}{r}\right) \right\} \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

BY (288.54)

BY (288.03)

$$3. \int \frac{\sin^2 x \, dx}{\sqrt{a+b \sin x}} = \frac{4a\sqrt{a+b}}{3b^2} E(\alpha, r) - \frac{2(2a^2+b^2)}{3b^2\sqrt{a+b}} F(\alpha, r) -$$

$$-\frac{2}{3b} \cos x \sqrt{a+b \sin x} \quad \left[a > b > 0, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right];$$

$$= \sqrt{\frac{2}{b}} \left\{ \frac{4a}{3b} E\left(\beta, \frac{1}{r}\right) - \frac{2a+b}{3b} F\left(\beta, \frac{1}{r}\right) \right\} - \frac{2}{3b} \cos x \sqrt{a+b \sin x}$$

$$\left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

BY (288.03, 288.54)

188

$$4. \int \frac{dx}{\sqrt{a+b \cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2}, r\right) \quad [a > b > 0, \quad 0 \leq x \leq \pi];$$

$$= \sqrt{\frac{2}{b}} F\left(\gamma, \frac{1}{r}\right)$$

$$\left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

BY (290.00)

BY (289.00)

$$5. \int \frac{dx}{\sqrt{a-b \cos x}} = \frac{2}{\sqrt{a+b}} F(\delta, r) \quad [a > b > 0, \quad 0 \leq x \leq \pi].$$

BY (291.00)

$$6. \int \frac{\cos x dx}{\sqrt{a+b \cos x}} = \frac{2}{b\sqrt{a+b}} \left\{ (a+b)E\left(\frac{x}{2}, r\right) - aF\left(\frac{x}{2}, r\right) \right\} \quad [a > b > 0, \quad 0 \leq x \leq \pi];$$

$$= \sqrt{\frac{2}{b}} \left\{ 2E\left(\gamma, \frac{1}{r}\right) - F\left(\gamma, \frac{1}{r}\right) \right\} \quad \left[b > |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

BY (290.04)

BY (289.03)

$$7.^6 \int \frac{\cos x dx}{\sqrt{a-b \cos x}} = \frac{2}{b\sqrt{a+b}} \{ (b-a)\Pi(\delta, r^2, r) + aF(\delta, r) \} \quad [a > b > 0, \quad 0 \leq x \leq \pi].$$

BY (291.03)

$$8. \int \frac{\cos^2 x dx}{\sqrt{a+b \cos x}} = \frac{2}{3b^2\sqrt{a+b}} \left\{ (2a^2 + b^2)F\left(\frac{x}{2}, r\right) - 2a(a+b)E\left(\frac{x}{2}, r\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x}$$

$$[a > b > 0, \quad 0 \leq x \leq \pi];$$

$$= \frac{1}{3b} \sqrt{\frac{2}{b}} \left\{ (2a+b)F\left(\gamma, \frac{1}{r}\right) - 4aE\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2}{3b} \sin x \sqrt{a+b \cos x}$$

$$\left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].$$

BY (290.04)

BY (289.03)

$$9. \int \frac{\cos^2 x dx}{\sqrt{a-b \cos x}} = \frac{2}{3b^2\sqrt{a+b}} \{ (2a^2 + b^2)F(\delta, r) - 2a(a+b)E(\delta, r) \} +$$

$$+ \frac{2}{3b} \sin x \frac{a+b \cos x}{\sqrt{a-b \cos x}} [a > b > 0, \quad 0 \leq x < \pi].$$

BY (291.04)a

2.572

$$3. \int \frac{\operatorname{tg}^2 x dx}{\sqrt{a+b \sin x}} = \frac{1}{\sqrt{a+b}} F(\alpha, r) + \frac{a}{(a-b)\sqrt{a+b}} E(\alpha, r) -$$

$$- \frac{b-a \sin x}{(a^2-b^2) \cos x} \sqrt{a+b \sin x} \quad \left[0 < b < a, \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \right];$$

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$$= \sqrt{\frac{2}{b}} \left\{ \frac{2a+b}{2(a+b)} F\left(\beta, \frac{1}{r}\right) + \frac{a}{a^2-b^2} E\left(\beta, \frac{1}{r}\right) \right\} -$$

$$\frac{b-a \sin x}{\sqrt{a+b \sin x}} \quad \left[\frac{a}{a^2-b^2} \right]$$

$$1. \int \frac{1 - \sin x}{1 + \sin x} \cdot \frac{dx}{\sqrt{a + b \sin x}} = \frac{2}{a - b} \left\{ \sqrt{a + b} E(\alpha, r) - \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \sqrt{a + b \sin x} \right\} \quad \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

BY (288.07)

$$2. \int \frac{1 - \cos x}{1 + \cos x} \frac{dx}{\sqrt{a + b \cos x}} = \frac{2}{a - b} \operatorname{tg} \frac{x}{2} \sqrt{a + b \cos x} - \frac{2\sqrt{a + b}}{a - b} E \left(\frac{x}{2}, r \right) \quad [a > b > 0, \quad 0 \leq x < \pi].$$

BY (289.07)

2.574

$$1. \int \frac{dx}{(2 - p^2 + p^2 \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a + b} \Pi(\alpha, p^2, r) \quad \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right].$$

BY (288.02)

$$2. \int \frac{dx}{(a + b - p^2 b + p^2 b \sin x) \sqrt{a + b \sin x}} = -\frac{1}{a + b} \sqrt{\frac{2}{b}} \Pi \left(\beta, p^2, \frac{1}{r} \right) \quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].$$

BY (288.52)

$$3. \int \frac{dx}{(2 - p^2 + p^2 \cos x) \sqrt{a + b \cos x}} = \frac{1}{\sqrt{a + b}} \Pi \left(\frac{x}{2}, p^2, r \right) \quad [a > b > 0, \quad 0 \leq x < \pi].$$

BY (289.02)

$$4. \int \frac{dx}{(a + b - p^2 b + p^2 b \cos x) \sqrt{a + b \cos x}} = \frac{\sqrt{2}}{(a + b) \sqrt{b}} \Pi \left(\gamma, p^2, \frac{1}{r} \right) \quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos \left(-\frac{a}{b} \right) \right].$$

BY (290.02)

$$\begin{aligned}
 1. \int \frac{dx}{\sqrt{(a+b \sin x)^3}} &= \frac{2b \cos x}{(a^2 - b^2)\sqrt{a+b \sin x}} - \frac{2}{(a-b)\sqrt{a+b}} E(\alpha, r) \\
 &\quad \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right]; \\
 &= \sqrt{\frac{2}{b}} \left\{ \frac{2b}{b^2 - a^2} E\left(\beta, \frac{1}{r}\right) - \frac{1}{a+b} F\left(\beta, \frac{1}{r}\right) \right\} + \frac{2b}{b^2 - a^2} \cdot \frac{\cos x}{\sqrt{a+b \sin x}} \\
 &\quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].
 \end{aligned}$$

BY (288.56)
BY (288.05)

$$\begin{aligned}
 2. \int \frac{dx}{\sqrt{(a+b \sin x)^5}} &= \frac{2}{3(a^2 - b^2)^2 \sqrt{a+b}} \{ (a^2 - b^2)F(\alpha, r) - 4a(a+b)E(\alpha, r) \} + \\
 &\quad + \frac{2b(5a^2 - b^2 + 4ab \sin x)}{3(a^2 - b^2)^2 \sqrt{(a+b \sin x)^3}} \cos x \\
 &\quad \left[0 < b < a, \quad -\frac{\pi}{2} \leq x < \frac{\pi}{2} \right]; \\
 &= -\frac{1}{3(a^2 - b^2)^2} \sqrt{\frac{2}{b}} \left\{ (3a-b)(a-b)F\left(\beta, \frac{1}{r}\right) + \right. \\
 &\quad \left. + 8abE\left(\beta, \frac{1}{r}\right) \right\} + \frac{2b[a^2 - b^2 + 4a(a+b \sin x)]}{3(a^2 - b^2)^2 \sqrt{(a+b \sin x)^3}} \cos x \\
 &\quad \left[0 < |a| < b, \quad -\arcsin \frac{a}{b} < x < \frac{\pi}{2} \right].
 \end{aligned}$$

BY (288.56)
BY (288.05)

$$\begin{aligned}
 3. \int \frac{dx}{\sqrt{(a+b \cos x)^3}} &= \frac{2}{(a-b)\sqrt{a+b}} E\left(\frac{x}{2}, r\right) - \frac{2b}{a^2 - b^2} \cdot \frac{\sin x}{\sqrt{a+b \cos x}} \\
 &\quad [a > b > 0, \quad 0 \leq x \leq \pi]; \\
 &= \frac{1}{a^2 - b^2} \sqrt{\frac{2}{b}} \left\{ (a-b)F\left(\gamma, \frac{1}{r}\right) + 2bE\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2b}{b^2 - a^2} \cdot \frac{\sin x}{\sqrt{a+b \cos x}} \\
 &\quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].
 \end{aligned}$$

BY (290.06)
BY (289.05)

$$4. \int \frac{dx}{\sqrt{(a-b \cos x)^3}} = \frac{2}{(a-b)\sqrt{a+b}} E(\delta, r) \quad [a > b > 0, \quad 0 \leq x \leq \pi].$$

$$\begin{aligned}
5. \int \frac{dx}{\sqrt{(a+b \cos x)^5}} &= \frac{2\sqrt{a+b}}{3(a^2-b^2)^2} \left\{ 4aE\left(\frac{x}{2}, r\right) - (a-b)F\left(\frac{x}{2}, r\right) \right\} - \\
&\quad - \frac{2b}{3(a^2-b^2)^2} \cdot \frac{5a^2-b^2+4ab \cos x}{\sqrt{(a+b \cos x)^3}} \sin x \quad [a > b > 0, \quad 0 \leq x \leq \pi]; \\
&= \frac{1}{3(a^2-b^2)^2} \sqrt{\frac{2}{b}} \left\{ (a-b)(3a-b)F\left(\gamma, \frac{1}{r}\right) + \right. \\
&\quad \left. + 8abE\left(\gamma, \frac{1}{r}\right) \right\} + \frac{2b(5a^2-b^2+4ab \cos x) \sin x}{3(a^2-b^2)^2 \sqrt{(a+b \cos x)^3}} \\
&\quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].
\end{aligned}$$

BY (290.06)

BY (289.05)

191

2.576

$$\begin{aligned}
1. \int \sqrt{a+b \cos x} dx &= 2\sqrt{a+b}E\left(\frac{x}{2}, r\right) \quad [a > b > 0, \quad 0 \leq x \leq \pi]; \\
&= \sqrt{\frac{2}{b}} \left\{ (a-b)F\left(\gamma, \frac{1}{r}\right) + 2bE\left(\gamma, \frac{1}{r}\right) \right\} \\
&\quad \left[b \geq |a| > 0, \quad 0 \leq x < \arccos\left(-\frac{a}{b}\right) \right].
\end{aligned}$$

BY (290.03)

BY (289.01)

$$2. \int \sqrt{a-b \cos x} dx = 2\sqrt{a+b}E(\delta, r) - \frac{2b \sin x}{\sqrt{a-b \cos x}} \quad [a > b > 0, \quad 0 \leq x \leq \pi].$$

BY (291.05)

2.577

$$\begin{aligned}
1.^3 \int \frac{\sqrt{a-b \cos x}}{1+p \cos x} dx &= \frac{2(a-b)}{(1+p)\sqrt{a+b}} \Pi\left(\delta, \frac{2ap}{(a+b)(1+p)}, r\right) \\
&\quad [a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1].
\end{aligned}$$

BY (291.02)

$$\begin{aligned}
2.^3 \int \sqrt{\frac{a-b \cos x}{1+p \cos x}} dx &= \frac{2(a-b)}{\sqrt{(1+p)(a+b)}} \Pi\left(\delta, -r^2, \sqrt{\frac{2(ap+b)}{(1+p)(a+b)}}\right) \\
&\quad [a > b > 0, \quad 0 \leq x \leq \pi, \quad p \neq -1].
\end{aligned}$$

2.578

$$\int \frac{\operatorname{tg} x \, dx}{\sqrt{a + b \operatorname{tg}^2 x}} = \frac{1}{\sqrt{b-a}} \arccos \left(\frac{\sqrt{b-a}}{\sqrt{b}} \cos x \right) \quad [b > a, \quad b > 0].$$

PE (333)

2.58-2.62 Integrals reducible to elliptic and pseudo-elliptic integrals

2.580

$$1. \int \frac{d\varphi}{\sqrt{a + b \cos \varphi + c \sin \varphi}} = 2 \int \frac{d\psi}{\sqrt{a - p + 2p \cos^2 \psi}} \quad \left[\varphi = 2\psi + \alpha, \quad \operatorname{tg} \alpha = \frac{c}{b}, \quad p = \sqrt{b^2 + c^2} \right]$$

$$2. \int \frac{d\varphi}{\sqrt{a + b \cos \varphi + c \sin \varphi + d \cos^2 \varphi + e \sin \varphi \cos \varphi + f \sin^2 \varphi}} = 2 \int \frac{dx}{\sqrt{A + Bx + Cx^2 - Dx^3 + Ex^4}}$$

$$\left[\operatorname{tg} \frac{\varphi}{2} = x, \quad A = a + b + d, \quad B = 2c + 2e, \quad C = 2a - 2d + 4f, \quad D = 2c - 2e, \quad E = a - b + d \right]$$

Forms containing $\sqrt{1 - k^2 \sin^2 x}$; Notations: $\Delta = \sqrt{1 - k^2}$, $\sin^2 x$,

$$k' = \sqrt{1 - k^2}$$

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2.581

$$1. \int \sin^m x \cos^n x \Delta^r dx =$$

$$= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m-3} x \cos^{n+1} x \Delta^{r+2} + [(m+n-2) + (m+r-1)k^2] \times \right.$$

$$\left. \times \int \sin^{m-2} x \cos^n x \Delta^r dx - (m-3) \int \sin^{m-4} x \cos^n x \Delta^r dx \right\} =$$

$$= \frac{1}{(m+n+r)k^2} \left\{ \sin^{m+1} x \cos^{n-3} x \Delta^{r+2} + [(n+r-1)k^2 - (m+n-2)k'^2] \times \right.$$

$$\left. \times \int \sin^m x \cos^{n-2} x \Delta^r dx + (n-3)k'^2 \int \sin^m x \cos^{n-4} x \Delta^r dx \right\} [m+n+r \neq 0]$$

For $r = -3$ and $r = -5$:

$$2. \int \frac{\sin^m x \cos^n x}{\Delta^3} dx = \frac{\sin^{m-1} x \cos^{n-1} x}{k^2 \Delta} -$$

$$- \frac{m-1}{k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta} dx + \frac{n-1}{k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta} dx.$$

$$3. \int \frac{\sin^m x \cos^n x}{\Delta^5} dx = \frac{\sin^{m-1} x \cos^{n-1} x}{3k^2 \Delta^3} - \frac{m-1}{3k^2} \int \frac{\sin^{m-2} x \cos^n x}{\Delta^3} dx + \frac{n-1}{3k^2} \int \frac{\sin^m x \cos^{n-2} x}{\Delta^3} dx.$$

For $m = 1$ or $n = 1$:

$$4. \int \sin x \cos^n x \Delta^r dx = -\frac{\cos^{n-1} x \Delta^{r+2}}{(n+r+1)k^2} - \frac{(n-1)k'^2}{(n+r+1)k^2} \int \cos^{n-2} x \sin x \Delta^r dx.$$

$$5. \int \sin^m x \cos x \Delta^r dx = -\frac{\sin^{m-1} x \Delta^{r+2}}{(m+r+1)k^2} + \frac{m-1}{(m+r+1)k^2} \int \sin^{m-2} x \cos x \Delta^r dx.$$

For $m = 3$ or $n = 3$:

$$6. \int \sin^3 x \cos^n x \Delta^r dx = \frac{(n+r+1)k^2 \cos^2 x - [(r+2)k^2 + n+1]}{(n+r+1)(n+r+3)k^4} \cos^{n-1} x \Delta^{r+2} - \frac{[(r+2)k^2 + n+1](n-1)k'^2}{(n+r+1)(n+r+3)k^4} \int \cos^{n-2} x \sin x \Delta^r dx.$$

$$7. \int \sin^m x \cos^3 x \Delta^r dx = \frac{(m+r+1)k^2 \sin^2 x - [(r+2)k^2 - (m+1)k'^2]}{(m+r+1)(m+r+3)k^4} \times \sin^{m-1} x \Delta^{r+2} + \frac{[(r+2)k^2 - (m-1)k'^2](m-1)}{(m+r+1)(m+r+3)k^4} \int \sin^{m-2} x \cos x \Delta^r dx.$$

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2.582

$$1. \int \Delta^n dx = \frac{n-1}{n} (2-k^2) \int \Delta^{n-2} dx - \frac{n-2}{n} (1-k^2) \int \Delta^{n-4} dx + \frac{k^2}{n} \sin x \cos x \cdot \Delta^{n-2}.$$

LA (316)(1)a

$$2. \int \frac{dx}{\Delta^{n+1}} = -\frac{k^2 \sin x \cos x}{(n-1)k'^2 \Delta^{n-1}} + \frac{n-2}{n-1} \frac{2-k^2}{k'^2} \int \frac{dx}{\Delta^{n-1}} - \frac{n-3}{n-1} \frac{1}{k'^2} \int \frac{dx}{\Delta^{n-3}}$$

$$3. \int \frac{\sin^n x}{\Delta} dx = \frac{\sin^{n-3} x}{(n-1)k^2} \cos x \cdot \Delta + \frac{n-2}{n-1} \frac{1+k^2}{k^2} \int \frac{\sin^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)k^2} \int \frac{\sin^{n-4} x}{\Delta} dx.$$

LA 316(1)a

$$4. \int \frac{\cos^n x}{\Delta} dx = \frac{\cos^{n-3} x}{(n-1)k^2} \sin x \cdot \Delta + \frac{n-2}{n-1} \frac{2k^2-1}{k^2} \int \frac{\cos^{n-2} x}{\Delta} dx + \frac{n-3}{n-1} \frac{k'^2}{k^2} \int \frac{\cos^{n-4} x}{\Delta} dx.$$

LA 316(2)a

$$5. \int \frac{\operatorname{tg}^n x}{\Delta} dx = \frac{\operatorname{tg}^{n-3} x}{(n-1)k'^2} \frac{\Delta}{\cos^2 x} - \frac{(n-2)(2-k^2)}{(n-1)k'^2} \int \frac{\operatorname{tg}^{n-2} x}{\Delta} dx - \frac{n-3}{(n-1)k'^2} \int \frac{\operatorname{tg}^{n-4} x}{\Delta} dx.$$

LA 317(3)

$$6. \int \frac{\operatorname{ctg}^n x}{\Delta} dx = -\frac{\operatorname{ctg}^{n-1} x}{n-1} \frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1} (2-k^2) \int \frac{\operatorname{ctg}^{n-2} x}{\Delta} dx - \frac{n-3}{n-1} k'^2 \int \frac{\operatorname{ctg}^{n-4} x}{\Delta} dx.$$

LA 317(6)

2.583

$$1. \int \Delta dx = E(x, k).$$

$$2. \int \Delta \sin x dx = -\frac{\Delta \cos x}{2} - \frac{k'^2}{2k} \ln(k \cos x + \Delta).$$

$$3. \int \Delta \cos x dx = \frac{\Delta \sin x}{2} + \frac{1}{2k} \arcsin(k \sin x).$$

6.
$$\int \Delta \cos^2 x \, dx = \frac{\Delta}{3} \sin x \cos x - \frac{k'^2}{3k^2} F(x, k) + \frac{k^2 + 1}{3k^2} E(x, k).$$
7.
$$\int \Delta \sin^3 x \, dx = -\frac{2k^2 \sin^2 x + 3k^2 - 1}{8k^2} \Delta \cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k \cos x + \Delta).$$
8.
$$\int \Delta \sin^2 x \cos x \, dx = \frac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \frac{1}{8k^3} \arcsin(k \sin x).$$
9.
$$\int \Delta \sin x \cos^2 x \, dx = -\frac{2k^2 \cos^2 x + k'^2}{8k^2} \Delta \cos x + \frac{k'^4}{8k^3} \ln(k \cos x + \Delta).$$
10.
$$\int \Delta \cos^3 x \, dx = \frac{2k^2 \cos^2 x + 2k^2 + 1}{8k^2} \Delta \sin x + \frac{4k^2 - 1}{8k^3} \arcsin(k \sin x).$$
11.
$$\int \Delta \sin^4 x \, dx = -\frac{3k^2 \sin^2 x + 4k^2 - 1}{15k^2} \Delta \sin x \cos x - \frac{2(2k^4 - k^2 - 1)}{15k^4} F(x, k) + \frac{8k^4 - 3k^2 - 2}{15k^4} E(x, k).$$
12.
$$\int \Delta \sin^3 x \cos x \, dx = \frac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$$
13.
$$\int \Delta \sin^2 x \cos^2 x \, dx = -\frac{3k^2 \cos^2 x - 2k^2 + 1}{15k^2} \Delta \sin x \cos x - \frac{k'^2(1 + k'^2)}{15k^4} F(x, k) + \frac{2(k^4 - k^2 + 1)}{15k^4} E(x, k).$$
14.
$$\int \Delta \sin x \cos^3 x \, dx = -\frac{3k^4 \sin^4 x - k^2(5k^2 + 1) \sin^2 x + 5k^2 - 2}{15k^4} \Delta.$$

$$15. \int \Delta \cos^4 x dx = \frac{3k^2 \cos^2 x + 3k^2 + 1}{15k^2} \Delta \sin x \cos x + \\ + \frac{2k'^2(k'^2 - 2k^2)}{15k^4} F(x, k) + \frac{3k^4 + 7k^2 - 2}{15k^4} E(x, k).$$

$$16. \int \Delta \sin^5 x dx = \frac{-8k^4 \sin^4 x - 2k^2(5k^2 - 1) \sin^2 x - 15k^4 + 4k^2 + 3}{48k^4} \Delta \cos x + \\ + \frac{5k^6 - 3k^4 - k^2 - 1}{16k^5} \ln(k \cos x + \Delta).$$

$$17. \int \Delta \sin^4 x \cos x dx = \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin(k \sin x).$$

$$18. \int \Delta \sin^3 x \cos^2 x dx = \frac{8k^4 \sin^4 x - 2k^2(k^2 + 1) \sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \Delta \cos x + \\ + \frac{k'^4(k^2 + 1)}{16k^5} \ln(k \cos x + \Delta).$$

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$$19. \int \Delta \sin^2 x \cos^3 x dx = \frac{-8k^4 \sin^4 x + 2k^2(6k^2 + 1) \sin^2 x - 6k^2 + 3}{48k^4} \Delta \sin x + \\ + \frac{2k^2 - 1}{16k^5} \arcsin(k \sin x).$$

$$20. \int \Delta \sin x \cos^4 x dx = \frac{-8k^4 \sin^4 x + 2k^2(7k^2 + 1) \sin^2 x - 3k^4 - 8k^2 + 3}{48k^4} \Delta \cos x - \\ - \frac{k'^6}{16k^5} \ln(k \cos x + \Delta).$$

$$21. \int \Delta \cos^5 x dx = \frac{8k^4 \sin^4 x - 2k^2(12k^2 + 1) \sin^2 x + 24k^4 + 12k^2 - 3}{48k^4} \Delta \sin x + \\ + \frac{8k^4 - 4k^2 + 1}{16k^5} \arcsin(k \sin x).$$

$$22. \int \Delta^3 dx = \frac{2}{3}(1 + k'^2)E(x, k) - \frac{k'^2}{3}F(x, F) + \frac{k^2}{3}\Delta \sin x \cos x.$$

$$23. \int \Delta^3 \sin x \, dx = \frac{2k^2 \sin^2 x + 3k^2 - 5}{8} \Delta \cos x - \frac{3k'^4}{8k} \ln(k \cos x + \Delta).$$

$$24. \int \Delta^3 \cos x \, dx = \frac{-2k^2 \sin^2 x + 5}{8} \Delta \sin x + \frac{3}{8k} \arcsin(k \sin x).$$

$$25. \int \Delta^3 \sin^2 x \, dx = \frac{3k^2 \sin^2 x + 4k^2 - 6}{15} \Delta \sin x \cos x + \frac{k'^2(3 - 4k^2)}{15k^2} F(x, k) - \frac{8k^4 - 13k^2 + 3}{15k^2} E(x, k).$$

$$26. \int \Delta^3 \sin x \cos x \, dx = -\frac{\Delta^5}{5k^2}.$$

$$27. \int \Delta^3 \cos^2 x \, dx = \frac{-3k^2 \sin^2 x + k^2 + 5}{15} \Delta \sin x \cos x - \frac{k'^2(k^2 + 3)}{15k^2} F(x, k) - \frac{2k^4 - 7k^2 - 3}{15k^2} E(x, k).$$

$$28. \int \Delta^3 \sin^3 x \, dx = \frac{8k^4 \sin^4 x + 2k^2(5k^2 - 7) \sin^2 x + 15k^4 - 22k^2 + 3}{48k^2} \Delta \cos x - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3} \ln(k \cos x + \Delta).$$

$$29. \int \Delta^3 \sin^2 x \cos x \, dx = \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + \frac{1}{16k^3} \arcsin(k \sin x).$$

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$$30. \int \Delta^3 \sin x \cos^2 x \, dx = \frac{-8k^4 \sin^4 x + 2k^2(k^2 + 7) \sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \times \Delta \cos x + \frac{k'^6}{16k^3} \ln(k \cos x + \Delta).$$

$$31. \int \Delta^3 \cos^3 x \, dx = \frac{8k^4 \sin^4 x - 2k^2(6k^2 + 7) \sin^2 x + 30k^2 + 3}{48k^2} \Delta \sin x + \frac{6k^2 - 1}{16k^3} \arcsin(k \sin x).$$

$$32. \int \frac{\Delta \, dx}{\sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + k \ln k(k \cos x + \Delta).$$

$$33. \int \frac{\Delta \, dx}{\cos x} = \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} + k \arcsin(k \sin x).$$

$$34. \int \frac{\Delta \, dx}{\sin^2 x} = k'^2 F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$$

$$35. \int \frac{\Delta \, dx}{\sin x \cos x} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$36. \int \frac{\Delta \, dx}{\cos^2 x} = F(x, k) - E(x, k) + \Delta \operatorname{tg} x.$$

$$37. \int \frac{\sin x}{\cos x} \Delta \, dx = \int \Delta \operatorname{tg} x \, dx = -\Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$38. \int \frac{\cos x}{\sin x} \Delta \, dx = \int \Delta \operatorname{ctg} x \, dx = \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}.$$

$$39. \int \frac{\Delta \, dx}{\sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$40. \int \frac{\Delta \, dx}{\sin^2 x \cos x} = \frac{-\Delta}{\sin x} - \frac{1 + k^2}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$41. \int \frac{\Delta \, dx}{\sin x \cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$42. \int \frac{\Delta dx}{\cos^3 x} = \frac{\Delta \sin x}{2 \cos^2 x} + \frac{1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$43. \int \frac{\Delta \sin x dx}{\cos^2 x} = \frac{\Delta}{\cos x} - k \ln(k \cos x + \Delta).$$

$$44. \int \frac{\Delta \cos x dx}{\sin^2 x} = -\frac{\Delta}{\sin x} - k \arcsin(k \sin x).$$

$$45. \int \frac{\Delta \sin^2 x dx}{\cos x} = -\frac{\Delta \sin x}{2} + \frac{2k^2 - 1}{2k} \arcsin(k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

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$$46. \int \frac{\Delta \cos^2 x dx}{\sin x} = \frac{\Delta \cos x}{2} + \frac{k^2 + 1}{2k} \ln(k \cos x + \Delta) + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$47. \int \frac{\Delta dx}{\sin^4 x} = \frac{1}{3} \{-\Delta \operatorname{ctg}^3 x + (k^2 - 3)\Delta \operatorname{ctg} x + 2k'^2 F(x, k) + (k^2 - 2)E(x, k)\}.$$

$$48. \int \frac{\Delta dx}{\sin^3 x \cos x} = -\frac{\Delta}{2 \sin^2 x} + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'} + \frac{k^2 - 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$$

$$49. \int \frac{\Delta dx}{\sin^2 x \cos^2 x} = \left(\frac{1}{k'^2} \operatorname{tg} x - \operatorname{ctg} x \right) \Delta + 2F(x, k) - \frac{1 + k'^2}{k'^2} E(x, k).$$

$$50. \int \frac{\Delta dx}{\sin x \cos^3 x} = \frac{\Delta}{2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$51. \int \frac{\Delta dx}{\cos^4 x} = \frac{1}{3k'^2} \{[k'^2 \operatorname{tg}^2 x - (2k^2 - 3) \operatorname{tg} x] \Delta + 2k'^2 F(x, k) + (k^2 - 2)E(x, k)\}.$$

$$52. \int \frac{\sin dx}{\cos^3 x} \Delta dx = \frac{\Delta}{2 \cos^2 x} + \frac{k^2}{4k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$53. \int \frac{\cos x}{\sin^3 x} \Delta dx = -\frac{\Delta}{2 \sin^2 x} + \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$$

$$54. \int \frac{\sin^2 x}{\cos^2 x} \Delta dx = \int \operatorname{tg}^2 x \Delta dx = \Delta \operatorname{tg} x + F(x, k) - 2E(x, k).$$

$$55. \int \frac{\cos^2 x}{\sin^2 x} \Delta dx = \int \operatorname{ctg}^2 x \Delta dx = -\Delta \operatorname{ctg} x + k'^2 F(x, k) - 2E(x, k).$$

$$56. \int \frac{\sin^3 x}{\cos x} \Delta dx = -\frac{k^2 \sin^2 x + 3k^2 - 1}{3k^2} \Delta + \frac{k'}{2} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$57. \int \frac{\cos^3 x}{\sin x} \Delta dx = -\frac{k^2 \sin^2 x - 3k^2 - 1}{3k^2} \Delta + \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}.$$

$$58. \int \frac{\Delta dx}{\sin^5 x} = \frac{(k^2 - 3) \sin^2 x + 2}{8 \sin^4 x} \cos x \Delta + \frac{k'^2(k^2 + 3)}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$59. \int \frac{\Delta dx}{\sin^4 x \cos x} = -\frac{(3 - k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{k'}{2} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$60. \int \frac{\Delta dx}{\sin^3 x \cos^2 x} = \frac{3 \sin^2 x - 1}{2 \sin^2 x \cos x} \Delta + \frac{k^2 - 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$$

$$61. \int \frac{\Delta dx}{\sin^2 x \cos^3 x} = \frac{3 \sin^2 x - 2}{2 \sin x \cos^2 x} \Delta - \frac{2k^2 - 3}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$64. \int \frac{\sin x}{\cos^4 x} \Delta dx = \frac{-(2k^2 + 1)k^2 \sin^2 x + 3k^4 - k^2 + 1}{3k'^2 \cos^3 x} \Delta.$$

$$65. \int \frac{\cos x}{\sin^4 x} \Delta dx = -\frac{\Delta^3}{3 \sin^3 x}.$$

$$66. \int \frac{\sin^2 x}{\cos^3 x} \Delta dx = \frac{\sin x}{2 \cos^2 x} \Delta + \frac{2k^2 - 1}{4k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - k \arcsin(k \sin x).$$

$$67. \int \frac{\cos^2 x}{\sin^3 x} \Delta dx = -\frac{\cos x}{2 \sin^2 x} \Delta - \frac{k^2 + 1}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x} - k \ln(k \cos x + \Delta).$$

$$68. \int \frac{\sin^3 x}{\cos^2 x} \Delta dx = -\frac{\sin^2 x - 3}{2 \cos x} \Delta - \frac{3k^2 - 1}{2k} \ln(k \cos x + \Delta).$$

$$69. \int \frac{\cos^3 x}{\sin^2 x} \Delta dx = -\frac{\sin^2 x + 2}{2 \sin x} \Delta - \frac{2k^2 + 1}{2k} \arcsin(k \sin x).$$

$$70. \int \frac{\sin^4 x}{\cos x} \Delta dx = -\frac{2k^2 \sin^2 x + 4k^2 - 1}{8k^2} \sin x \Delta + \frac{8k^4 - 4k^2 - 1}{8k^3} \arcsin(k \sin x) + \frac{k'}{2} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$71. \int \frac{\cos^4 x}{\sin x} \Delta dx = \frac{-2k^2 \sin^2 x + 5k^2 + 1}{8k^2} \cos x \Delta + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^4 + 6k^2 - 1}{8k^3} \ln(k \cos x + \Delta).$$

8.
$$\int \frac{\sin^2 x \cos x dx}{\Delta} = -\frac{\sin x \Delta}{2k^2} + \frac{\arcsin(k \sin x)}{2k^3}.$$
9.
$$\int \frac{\sin x \cos^2 x dx}{\Delta} = -\frac{\cos x \Delta}{2k^2} + \frac{k'^2}{2k^3} \ln(k \cos x + \Delta).$$
10.
$$\int \frac{\cos^3 x dx}{\Delta} = \frac{\sin x \Delta}{2k^2} + \frac{2k^2 - 1}{2k^3} \arcsin(k \sin x).$$
11.
$$\int \frac{\sin^4 x dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{2 + k^2}{3k^4} F(x, k) - \frac{2(1 + k^2)}{3k^4} E(x, k).$$
12.
$$\int \frac{\sin^3 x \cos x dx}{\Delta} = -\frac{1}{3k^4} (2 + k^2 \sin^2 x) \Delta.$$
13.
$$\int \frac{\sin^2 x \cos^2 x dx}{\Delta} = -\frac{\sin x \cos x \Delta}{3k^2} + \frac{2 - k^2}{3k^4} E(x, k) + \frac{2k^2 - 2}{3k^4} F(x, k).$$
14.
$$\int \frac{\sin x \cos^3 x dx}{\Delta} = -\frac{1}{3k^4} (k^2 \cos^2 x - 2k'^2) \Delta.$$
15.
$$\int \frac{\cos^4 x dx}{\Delta} = \frac{\sin x \cos x \Delta}{3k^2} + \frac{4k^2 - 2}{3k^4} E(x, k) + \frac{3k^4 - 5k^2 + 2}{3k^4} F(x, k).$$

$$16. \int \frac{\sin^5 x dx}{\Delta} = \frac{2k^2 \sin^2 x + 3k^2 + 3}{8k^4} \cos x \Delta - \frac{3 + 2k^2 + 3k^4}{8k^5} \ln(k \cos x + \Delta).$$

$$17. \int \frac{\sin^4 x \cos x dx}{\Delta} = -\frac{2k^2 \sin^2 x + 3}{8k^4} \sin x \Delta + \frac{3}{8k^5} \arcsin(k \sin x).$$

$$18. \int \frac{\sin^3 x \cos x dx}{\Delta} = \frac{2k^2 \cos^2 x - k^2 - 3}{8k^4} \cos x \Delta - \frac{k^4 + 2k^2 - 3}{8k^5} \ln(k \cos x + \Delta).$$

$$19. \int \frac{\sin^2 x \cos^3 x dx}{\Delta} = -\frac{2k^2 \cos^2 x + 2k^2 - 3}{8k^4} \sin x \Delta + \frac{4k^2 - 3}{8k^5} \arcsin(k \sin x).$$

$$20. \int \frac{\sin x \cos^4 x dx}{\Delta} = \frac{3 - 5k^2 + 2k^2 \sin^2 x}{8k^4} \cos x \Delta - \frac{3k^4 - 6k^2 + 3}{8k^5} \ln(k \cos x + \Delta).$$

$$21. \int \frac{\cos^5 x dx}{\Delta} = \frac{2k^2 \cos^2 x + 6k^2 - 3}{8k^4} \sin x \Delta + \frac{8k^4 - 8k^2 + 3}{8k^5} \arcsin(k \sin x).$$

$$22. \int \frac{\sin^6 x dx}{\Delta} = \frac{3k^2 \sin^2 x + 4k^2 + 4}{15k^4} \sin x \cos x \Delta + \\ + \frac{4k^4 + 3k^2 + 8}{15k^6} F(x, k) - \frac{8k^4 + 7k^2 + 8}{15k^6} E(x, k).$$

$$23. \int \frac{\sin^5 x \cos x dx}{\Delta} = -\frac{3k^4 \sin^4 x + 4k^2 \sin^2 x + 8}{15k^6} \Delta.$$

$$24. \int \frac{\sin^4 x \cos x dx}{\Delta} = \frac{3k^2 \cos^2 x - 2k^2 - 4}{15k^4} \sin x \cos x \Delta + \\ + \frac{k^4 + 7k^2 - 8}{15k^6} F(x, k) - \frac{2k^4 + 3k^2 - 8}{15k^6} E(x, k).$$

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$$25. \int \frac{\sin^3 x \cos^3 x dx}{\Delta} = \frac{3k^4 \sin^4 x - (5k^4 - 4k^2) \sin^2 x - 10k^2 + 8}{15k^6} \Delta.$$

26.
$$\int \frac{\sin^2 x \cos^4 x dx}{\Delta} = -\frac{3k^2 \cos^2 x + 3k^2 - 4}{15k^4} \sin x \cos x \Delta +$$

$$+ \frac{9k^4 - 17k^2 + 8}{15k^6} F(x, k) - \frac{3k^4 - 13k^2 + 8}{15k^6} E(x, k).$$
27.
$$\int \frac{\sin x \cos^5 x dx}{\Delta} = \frac{-3k^4 \cos^4 x + 4k^2 k'^2 \cos^2 x - 8k^4 + 16k^2 - 8}{15k^6} \Delta.$$
28.
$$\int \frac{\cos^6 x dx}{\Delta} = \frac{3k^2 \cos^2 x + 8k^2 - 4}{15k^4} \sin x \cos x \Delta +$$

$$+ \frac{15k^6 - 34k^4 + 27k^2 - 8}{15k^6} F(x, k) + \frac{23k^4 - 23k^2 + 8}{15k^6} E(x, k).$$
29.
$$\int \frac{\sin^7 x dx}{\Delta} = \frac{8k^4 \sin^4 x + 10k^2(k^2 + 1) \sin^2 x + 15k^4 + 14k^2 + 15}{48k^6} \cos x \Delta -$$

$$- \frac{(5k^4 - 2k^2 + 5)(k^2 + 1)}{16k^7} \ln(k \cos x + \Delta).$$
30.
$$\int \frac{\sin^6 x \cos x dx}{\Delta} = -\frac{8k^4 \sin^4 x + 10k^2 \sin^2 x + 15}{48k^6} \sin x \Delta + \frac{5}{16k^7} \arcsin(k \sin x).$$
31.
$$\int \frac{\sin^5 x \cos^2 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2(k^2 - 5) \sin^2 x + 3k^4 + 4k^2 - 15}{48k^6} \cos x \Delta -$$

$$- \frac{k^6 + k^4 + 3k^2 - 5}{16k^7} \ln(k \cos x + \Delta).$$
32.
$$\int \frac{\sin^4 x \cos^3 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(6k^2 - 5) \sin^2 x - 18k^2 + 15}{48k^6} \sin x \Delta +$$

$$+ \frac{6k^2 - 5}{16k^7} \arcsin(k \sin x).$$
33.
$$\int \frac{\sin^3 x \cos^4 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(6k^2 - 5) \sin^2 x + 3k^4 - 22k^2 + 15}{48k^6} \cos x \Delta -$$

$$- \frac{k^6 + 3k^4 - 9k^2 + 5}{16k^7} \ln(k \cos x + \Delta).$$

$$34. \int \frac{\sin^2 x \cos^5 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2(12k^2 - 5) \sin^2 x - 24k^4 + 36k^2 - 15}{48k^6} \sin x \Delta + \\ + \frac{8k^4 - 12k^2 + 5}{16k^7} \arcsin(k \sin x).$$

$$35. \int \frac{\sin x \cos^6 x dx}{\Delta} = \frac{-8k^4 \sin^4 x + 2k^2(13k^2 - 5) \sin^2 x - 33k^4 + 40k^2 - 15}{48k^6} \cos x \Delta + \\ + \frac{5k'^6}{16k^7} \ln(k \cos x + \Delta).$$

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$$36. \int \frac{\cos^7 x dx}{\Delta} = \frac{8k^4 \sin^4 x - 2k^2(18k^2 - 5) \sin^2 x + 72k^4 - 54k^2 + 15}{48k^6} \sin x \Delta + \\ + \frac{16k^6 - 24k^4 + 18k^2 - 5}{16k^7} \arcsin(k \sin x).$$

$$37. \int \frac{dx}{\Delta^3} = \frac{1}{k'^2} E(x, k) - \frac{k^2 \sin x \cos x}{k'^2 \Delta}.$$

$$38. \int \frac{\sin x dx}{\Delta^3} = -\frac{\cos x}{k'^2 \Delta}.$$

$$39. \int \frac{\cos x dx}{\Delta^3} = \frac{\sin x}{\Delta}.$$

$$40. \int \frac{\sin x dx}{\Delta^3} = \frac{1}{k'^2 k^2} E(x, k) - \frac{1}{k^2} F(x, k) - \frac{1}{k'^2} \frac{\sin x \cos x}{\Delta}.$$

$$41. \int \frac{\sin x \cos x dx}{\Delta^3} = \frac{1}{k^2 \Delta}.$$

$$42. \int \frac{\cos^2 x dx}{\Delta^3} = \frac{1}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{\sin x \cos x}{\Delta}.$$

$$53. \int \frac{\sin^4 x \cos x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \sin x - \frac{3}{2k^5} \arcsin(k \sin x).$$

$$54. \int \frac{\sin^3 x \cos^2 x dx}{\Delta} = \frac{-k^2 \sin^2 x + 3}{2k^4 \Delta} \cos x + \frac{k^2 - 3}{2k^5} \ln(k \cos x + \Delta).$$

$$55. \int \frac{\sin^2 x \cos^3 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \sin x - \frac{2k^2 - 3}{2k^5} \arcsin(k \sin x).$$

$$56. \int \frac{\sin x \cos^4 x dx}{\Delta^3} = \frac{k^2 \sin^2 x + 2k^2 - 3}{2k^4 \Delta} \cos x + \frac{3k'^2}{2k^5} \ln(k \cos x + \Delta).$$

$$57. \int \frac{\cos^5 x dx}{\Delta^3} = \frac{-k^2 \sin^2 x + 2k^4 - 4k^2 + 3}{2k^4 \Delta} \sin x + \frac{4k^2 - 3}{2k^5} \arcsin(k \sin x).$$

$$58. \int \frac{dx}{\Delta^5} = \frac{-k^2 \sin x \cos x}{3k'^2 \Delta^3} - \frac{2k^2(k'^2 + 1) \sin x \cos x}{3k'^4 \Delta} - \frac{1}{3k'^2} F(x, k) + \frac{2(k'^2 + 1)}{3k'^4} E(x, k).$$

$$59. \int \frac{\sin x dx}{\Delta^5} = \frac{2k^2 \sin^2 x + k^2 - 3}{3k'^4 \Delta^3} \cos x.$$

$$60. \int \frac{\cos x \, dx}{\Delta^5} = \frac{-2k^2 \sin^2 x + 3}{3\Delta^3} \sin x.$$

$$61. \int \frac{\sin^2 x \, dx}{\Delta^5} = \frac{k^2 + 1}{3k'^4 k^2} E(x, k) - \frac{1}{3k'^2 k^2} F(x, k) + \frac{k^2(k^2 + 1) \sin^2 x - 2}{3k'^4 \Delta^3} \sin x \cos x.$$

$$62. \int \frac{\sin x \cos x \, dx}{\Delta^5} = \frac{1}{3k^2 \Delta^3}.$$

$$63. \int \frac{\cos^2 x \, dx}{\Delta^5} = \frac{1}{3k^2} F(x, k) + \frac{2k^2 - 1}{3k^2 k'^2} E(x, k) + \frac{k^2(2k^2 - 1) \sin^2 x - 3k^2 + 2}{2k'^2 \Delta} \sin x \cos x.$$

$$64. \int \frac{\sin^3 x \, dx}{\Delta^5} = \frac{(3k^2 - 1) \sin^2 x - 2}{3k'^4 \Delta^3} \cos x.$$

$$65. \int \frac{\sin^2 x \cos x \, dx}{\Delta^5} = \frac{\sin^3 x}{3\Delta^3}.$$

$$66. \int \frac{\sin x \cos^2 x \, dx}{\Delta^5} = -\frac{\cos^3 x}{3k'^2 \Delta^3}.$$

$$67. \int \frac{\cos^3 x \, dx}{\Delta^5} = \frac{-(2k^2 + 1) \sin^2 x + 3}{3\Delta^3} \sin x.$$

$$68. \int \frac{dx}{\Delta \sin x} = -\frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

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$$69. \int \frac{dx}{\Delta \cos x} = -\frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$70. \int \frac{dx}{\Delta \sin^2 x} = \int \frac{1 + \operatorname{ctg}^2 x}{\Delta} dx = F(x, k) - E(x, k) - \Delta \operatorname{ctg} x.$$

$$71. \int \frac{dx}{\Delta \sin x \cos x} = \int (\operatorname{tg} x + \operatorname{ctg} x) \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$72. \int \frac{dx}{\Delta \cos^2 x} = \int (1 + \operatorname{tg}^2 x) \frac{dx}{\Delta} = F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{1}{k'^2} \Delta \operatorname{tg} x.$$

$$73. \int \frac{\sin x \, dx}{\cos x \, \Delta} = \int \operatorname{tg} x \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$74. \int \frac{\cos x \, dx}{\sin x \, \Delta} = \int \operatorname{ctg} x \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta}.$$

$$75. \int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta \cos x}{2 \sin^2 x} - \frac{1 + k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$76. \int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$77. \int \frac{dx}{\Delta \sin x \cos^2 x} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$$

$$78. \int \frac{dx}{\Delta \cos^3 x} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} + \frac{2k^2 - 1}{4k'^3} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$79. \int \frac{\sin x \, dx}{\cos^2 x \, \Delta} = \frac{\Delta}{k'^2 \cos x}.$$

$$80. \int \frac{\cos x \, dx}{\sin^2 x \, \Delta} = -\frac{\Delta}{\sin x}.$$

$$81. \int \frac{\sin^2 x}{\cos x} \frac{dx}{\Delta} = \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{1}{k} \arcsin(k \sin x).$$

$$82. \int \frac{\cos^2 x}{\sin x} \frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$83. \int \frac{dx}{\Delta \sin^4 x} = \frac{1}{3} \{-\Delta \operatorname{ctg}^3 x - \Delta(2k^2 + 3) \operatorname{ctg} x + (k^2 + 2)F(x, k) - 2(k^2 + 1)E(x, k)\}.$$

$$84. \int \frac{dx}{\Delta \sin^3 x \cos x} = \int (\operatorname{tg} x + 2 \operatorname{ctg} x + \operatorname{ctg}^3 x) \frac{dx}{\Delta} =$$

$$= -\frac{\Delta}{2 \sin^2 x} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} - \frac{k^2 + 2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$$

$$85. \int \frac{dx}{\Delta \sin^2 x \cos^2 x} = \int (\operatorname{tg}^2 x + 2 + \operatorname{ctg}^2 x) \frac{dx}{\Delta} =$$

$$= \left(\frac{\operatorname{tg} x}{k'^2} - \operatorname{ctg} x \right) \Delta + \frac{k^2 - 2}{k'^2} E(x, k) + 2F(x, k).$$

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$$86. \int \frac{dx}{\Delta \sin x \cos^3 x} = \int (\operatorname{ctg} x + 2 \operatorname{tg} x + \operatorname{tg}^3 x) \frac{dx}{\Delta} =$$

$$= -\frac{\Delta}{2k'^2 \cos^2 x} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta} + \frac{2 - 3k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$87. \int \frac{dx}{\Delta \cos^4 x} = \frac{1}{3k'^2} \left\{ \Delta \operatorname{tg}^3 x - \frac{5k^2 - 3}{k'^2} \Delta \operatorname{tg} x - (3k^2 - 2)F(x, k) + \frac{2(2k^2 - 1)}{k'^2} E(x, k) \right\}.$$

$$88. \int \frac{\sin x}{\cos^3 x} \frac{dx}{\Delta} = \int \operatorname{tg} x (1 + \operatorname{tg}^2 x) \frac{dx}{\Delta} = \frac{\Delta}{2k'^2 \cos^2 x} - \frac{k^2}{4k'^3} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$89. \int \frac{\cos x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta}{2 \sin^2 x} - \frac{k^2}{4} \ln \frac{1 + \Delta}{1 - \Delta}.$$

$$90. \int \frac{\sin^2 x}{\cos^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{tg}^2 x}{\Delta} dx = \frac{\Delta}{k'^2} \operatorname{tg} x - \frac{1}{k'^2} E(x, k).$$

$$91. \int \frac{\cos^2 x}{\sin^2 x} \frac{dx}{\Delta} = \int \frac{\operatorname{ctg}^2 x}{\Delta} dx = -\Delta \operatorname{ctg} x - E(x, k).$$

$$92. \int \frac{\sin^3 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$93. \int \frac{\cos^3 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1 + \Delta}{1 - \Delta}.$$

$$94. \int \frac{dx}{\Delta \sin^5 x} = -\frac{[3(1 + k^2) \sin^2 x + 2]}{8 \sin^2 x} \Delta \cos x + \frac{3k^4 + 2k^2 + 3}{16} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$95. \int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3 + 2k^2) \sin^2 x + 1}{3 \sin^3 x} \Delta - \frac{1}{2k'} \ln \frac{\Delta - k' \sin x}{\Delta + k' \sin x}.$$

$$96. \int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3 - k^2) \sin^2 x - k'^2}{2k'^2 \sin^2 x \cos x} \Delta + \frac{k^2 + 3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$$

$$97. \int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{(3 - 2k^2) \sin^2 x - 2k'^2}{2k'^2 \sin x \cos^2 x} \Delta - \frac{4k^2 - 3}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$98. \int \frac{dx}{\Delta \sin x \cos^4 x} = \frac{(5k^2 - 3) \sin^2 x - 6k^2 + 4}{3k'^4 \cos^3 x} \Delta - \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$99. \int \frac{dx}{\Delta \cos^5 x} = \frac{3(2k^2 - 1) \sin^2 x - 8k^2 + 5}{8k'^4 \cos^4 x} \Delta \sin x + \frac{8k^4 - 8k^2 + 3}{16k'^5} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$100. \int \frac{\sin x}{\cos^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \cos^2 x - k'^2}{2k'^4 \cos^3 x} \Delta.$$

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$$101. \int \frac{\cos x}{\sin^4 x} \frac{dx}{\Delta} = -\frac{2k^2 \sin^2 x + 1}{3 \sin^3 x} \Delta.$$

$$102. \int \frac{\sin^2 x}{\cos^3 x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k'^2 \cos^2 x} - \frac{1}{4k'^3} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x}.$$

$$103. \int \frac{\cos^3 x}{\sin^3 x} \frac{dx}{\Delta} = -\frac{\Delta \cos x}{2 \sin^2 x} + \frac{k'^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$$

$$104. \int \frac{\sin^3 x}{\cos^2 x} \frac{dx}{\Delta} = \frac{\Delta}{k'^2 \cos x} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$105. \int \frac{\cos^3 x}{\sin^2 x} \frac{dx}{\Delta} = \frac{-\Delta}{\sin x} - \frac{1}{k} \arcsin(k \sin x).$$

$$106. \int \frac{\sin^4 x}{\cos x} \frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} + \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} - \frac{2k^2 + 1}{2k^3} \arcsin(k \sin x).$$

$$107. \int \frac{\cos^4 x}{\sin x} \frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2 - 1}{2k^3} \ln(k \cos x + \Delta).$$

2.585

$$1. \int \frac{(a + \sin x)^{p+3} dx}{\Delta} =$$

$$= \frac{1}{(p+2)k^2} \left[(a + \sin x)^p \cos x \Delta + \right.$$

$$+ 2(2p+3)ak^2 \int \frac{(a + \sin x)^{p+2} dx}{\Delta} + (p+1)(1+k^2-6a^2k^2) \int \frac{(a + \sin x)^{p+1} dx}{\Delta} -$$

$$- a(2p+1)(1+k^2-2a^2k^2) \int \frac{(a + b \sin x)^p dx}{\Delta} -$$

$$\left. - p(1-a^2)(1-a^2k^2) \int \frac{(a + \sin x)^p dx}{\Delta} \right] \quad \left[p \neq -2, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right].$$

$$p = n$$

$$2. \int \frac{a + \sin x}{\Delta} dx = aF(x, k) + \frac{1}{2k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$3. \int \frac{(a + \sin x)^2}{\Delta} dx = \frac{1 + k^2 a^2}{k^2} F(x, k) - \frac{1}{k^2} E(x, k) + \frac{a}{k} \ln \frac{\Delta - k \cos x}{\Delta + k \cos x}.$$

$$4.^6 \int \frac{dx}{(a + \sin x)\Delta} = \frac{1}{a} \Pi \left(x, \frac{1}{a^2}, k \right) - \int \frac{\sin x dx}{(a^2 - \sin^2 x)\Delta},$$

where

$$5. \int \frac{\sin x dx}{(a^2 - \sin^2 x)\Delta} = \frac{-1}{2\sqrt{(1-a^2)(1-a^2k^2)}} \ln \frac{\sqrt{1-a^2}\Delta - \sqrt{1-k^2a^2} \cos x}{\sqrt{1-a^2}\Delta + \sqrt{1-k^2a^2} \cos x}.$$

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2.586

$$1. \int \frac{dx}{(a + \sin x)^n \Delta} = \frac{1}{(n-1)(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^{n-1}} - \right. \\ \left. - (2n-3)(1+k^2-2a^2k^2)a \int \frac{dx}{(a + \sin x)^{n-1} \Delta} - \right. \\ \left. - (n-2)(6a^2k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x)^{n-2} \Delta} - \right. \\ \left. - (10-4n)ak^2 \int \frac{dx}{(a + \sin x)^{n-3} \Delta} - (n-3)k^2 \int \frac{dx}{(a + \sin x)^{n-4} \Delta} \right] \\ \left[n \neq 1, \quad a \neq \pm 1, \quad a \neq \pm \frac{1}{k} \right].$$

This integral can be reduced to the integrals:

$$2. \int \frac{dx}{(a + \sin x)^2 \Delta} = \frac{1}{(1-a^2)(1-a^2k^2)} \left[-\frac{\cos x \Delta}{a + \sin x} - \right. \\ \left. - a(1+k^2-2a^2k^2) \int \frac{dx}{(a + \sin x)\Delta} - 2ak^2 \int \frac{(a + \sin x) dx}{\Delta} + \right. \\ \left. + k^2 \int \frac{(a + \sin x)^2 dx}{\Delta} \right] \quad (\text{see } \mathbf{2.585} \text{ 2., 3., 4.}).$$

$$3. \int \frac{dx}{(a + \sin x)^3 \Delta} = \frac{1}{2(1 - a^2)(1 - a^2 k^2)} \left[-\frac{\cos x \Delta}{(a + \sin x)^2} - 3a(1 + k^2 - 2a^2 k^2) \int \frac{dx}{(a + \sin x)^2 \Delta} - (6a^2 k^2 - k^2 - 1) \int \frac{dx}{(a + \sin x) \Delta} + 2ak^2 F(x, k) \right]$$

(see **2.585** 4. and **2.586** 2.).

2.586

2.585

For $a = \pm 1$, we have:

$$4. \int \frac{dx}{(1 \pm \sin x)^n \Delta} = \frac{1}{(2n - 1)k'^2} \left[\mp \frac{\cos x \Delta}{(1 \pm \sin x)^n} + (n - 1)(1 - 5k^2) \int \frac{dx}{(1 \pm \sin x)^{n-1} \Delta} + 2(2n - 3)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-2} \Delta} - (n - 2)k^2 \int \frac{dx}{(1 \pm \sin x)^{n-3} \Delta} \right].$$

GU ((241))(6a)

This integral can be reduced to the integrals

$$5. \int \frac{dx}{(1 \pm \sin x) \Delta} = \frac{\mp \cos x \Delta}{k'^2 (1 \pm \sin x)} + F(x, k) - \frac{1}{k'^2} E(x, k).$$

GU ((241))(6c)

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$$6. \int \frac{dx}{(1 \pm \sin x)^2 \Delta} = \frac{1}{3k'^4} \left[\mp \frac{k'^2 \cos x \Delta}{(1 \pm \sin x)^2} \mp \frac{(1 - 5k^2) \cos x \Delta}{1 \pm \sin x} + (1 - 3k^2)k'^2 F(x, k) - (1 - 5k^2)E(x, k) \right].$$

GU ((241))(6b)

For $a = \pm \frac{1}{k}$, we have

This integral can be reduced to the integrals

$$8. \int \frac{dx}{(1 \pm k \sin x)\Delta} = \pm \frac{k \cos x \Delta}{k'^2(1 \pm k \sin x)} + \frac{1}{k'^2} E(x, k).$$

GU ((241))(7b)

$$9. \int \frac{dx}{(1 \pm k \sin x)^2 \Delta} = \frac{1}{3k'^4} \left[\pm \frac{kk'^2 \cos x \Delta}{(1 \pm k \sin x)^2} \pm \frac{k(5 - k^2) \cos x \Delta}{1 \pm k \sin x} - 2k'^2 F(x, k) + (5 - k^2) E(x, k) \right].$$

GU((241))(7c)

2.587

$$1. \int \frac{(b + \cos x)^{p+3} dx}{\Delta} = \frac{1}{(p+2)k^2} \left[(b + \cos x)^p \sin x \Delta + 2(2p+3)bk^2 \int \frac{(b + \cos x)^{p+2} dx}{\Delta} - (p+1)(k'^2 - k^2 + 6b^2k^2) \int \frac{(b + \cos x)^{p+1} dx}{\Delta} + (2p+1)b(k'^2 - k^2 + b^2k^2) \int \frac{(b + \cos x)^p dx}{\Delta} + p(1 - b^2)(k'^2 + k^2b^2) \int \frac{(b + \cos x)^{p-1} dx}{\Delta} \right] \left[p \neq -2, \quad b \neq \pm 1, \quad b \neq \frac{ik'}{k} \right].$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

$$2. \int \frac{b + \cos x}{\Delta} dx = bF(x, k) + \frac{1}{k} \arcsin(k \sin x).$$

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$$3. \int \frac{(b + \cos x)^2}{\Delta} dx = \frac{b^2k^2 - k'^2}{k^2} F(x, k) + \frac{1}{k^2} E(x, k) + \frac{2b}{k} \arcsin(k \sin x).$$

$$4. \int \frac{dx}{(b + \cos x)\Delta} = \frac{b}{b^2 - 1} \Pi \left(x, \frac{1}{b^2 - 1}, k \right) + \int \frac{\cos x dx}{(1 - b^2 - \sin^2 x)\Delta},$$

where

$$5. \int \frac{\cos x dx}{(1 - b^2 - \sin^2 x)\Delta} = \frac{1}{2\sqrt{(1 - b^2)(k'^2 + k^2b^2)}} \ln \frac{\sqrt{1 - b^2}\Delta + k\sqrt{k'^2 + k^2b^2} \sin x}{\sqrt{1 - b^2}\Delta - k\sqrt{k'^2 + k^2b^2} \sin x}.$$

2.588

$$1. \int \frac{dx}{(b + \cos x)^n \Delta} = \frac{1}{(n - 1)(1 - b^2)(k'^2 + b^2k^2)} \left[\frac{-k'^2 \sin x \Delta}{(b + \cos x)^{-1}} - \right. \\ \left. - (2n - 3)(1 - 2k^2 + 2b^2k^2)b \int \frac{dx}{(b + \cos x)^{n-1} \Delta} - \right. \\ \left. - (n - 2)(2k^2 - 1 - 6b^2k^2) \int \frac{dx}{(b + \cos x)^{n-2} \Delta} - \right. \\ \left. - (4n - 10)bk^2 \int \frac{dx}{(b + \cos x)^{n-3} \Delta} + (n - 3)k^2 \int \frac{dx}{(b + \cos x)^{n-4} \Delta} \right] \\ \left[n \neq 1, \quad b \neq \pm 1, \quad b \neq \pm \frac{ik'}{k} \right].$$

This integral can be reduced to the following integrals:

$$2. \int \frac{dx}{(b + \cos x)^2 \Delta} = \\ = \frac{1}{(1 - b^2)(k'^2 + b^2k^2)} \left[\frac{-k'^2 \sin x \Delta}{b + \cos x} - (1 - 2k^2 + 2b^2k^2)b \int \frac{dx}{(b + \cos x)\Delta} + \right. \\ \left. + 2bk^2 \int \frac{b + \cos x}{\Delta} dx - k^2 \int \frac{(b + \cos x)^2}{\Delta} dx \right] \quad (\text{see } \mathbf{2.587} \text{ 2., 3., 4.}).$$

2.587

$$3. \int \frac{dx}{(b + \cos x)^3 \Delta} = \frac{1}{2(1 - b^2)(k'^2 + b^2k^2)} \left[\frac{-k'^2 \sin x \Delta}{(b + \cos x)^2} - \right. \\ \left. - 3b(1 - 2k^2 + 2k^2b^2) \int \frac{dx}{(b + \cos x)^2 \Delta} - \right. \\ \left. - (2k^2 - 1 - 6b^2k^2) \int \frac{dx}{(b + \cos x)\Delta} - 2bk^2 F(x, k) \right] \\ (\text{see } \mathbf{2.588} \text{ 2. and } \mathbf{2.587} \text{ 4.}).$$

$$\begin{aligned}
 1. \int \frac{(c + \operatorname{tg} x)^{p+3} dx}{\Delta} &= \\
 &= \frac{1}{(p+2)k'^2} \left[\frac{(c + \operatorname{tg} x)^p \Delta}{\cos^2 x} + 2(2n+3)ck'^2 \int \frac{(c + \operatorname{tg} x)^{p+2} dx}{\Delta} - \right. \\
 &\quad - (p+1)(1+k'^2+6c^2k'^2) \int \frac{(c + \operatorname{tg} x)^{p+1} dx}{\Delta} + \\
 &\quad + (2p+1)c(1+k'^2+2c^2k'^2) \int \frac{(c + \operatorname{tg} x)^p dx}{\Delta} - \\
 &\quad \left. - p(1+c^2)(1+k'^2c^2) \int \frac{(c + \operatorname{tg} x)^{p-1} dx}{\Delta} \right] \quad [p \neq -2].
 \end{aligned}$$

For $p = n$ a natural number, this integral can be reduced to the following three integrals:

$$2. \int \frac{c + \operatorname{tg} x}{\Delta} dx = cF(x, k) + \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$3. \int \frac{(c + \operatorname{tg} x)^2}{\Delta} dx = \frac{1}{k'^2} \operatorname{tg} x \Delta + c^2 F(x, k) - \frac{1}{k'^2} E(x, k) + \frac{c}{k'} \ln \frac{\Delta + k'}{\Delta - k'}.$$

$$\begin{aligned}
 4. \int \frac{dx}{(c + \operatorname{tg} x)\Delta} &= \frac{c}{1+c^2} F(x, k) + \frac{1}{c(1+c^2)} \Pi \left(x, -\frac{1+c^2}{c^2}, k \right) - \\
 &\quad - \int \frac{\sin x \cos x dx}{[c^2 - (1+c^2) \sin^2 x] \Delta},
 \end{aligned}$$

where

$$5. \int \frac{\sin x \cos x dx}{[c^2 - (1+c^2) \sin^2 x] \Delta} = \frac{1}{2\sqrt{(1+c^2)(1+c^2k'^2)}} \ln \frac{\sqrt{1+c^2k'^2} + \sqrt{1+c^2}\Delta}{\sqrt{1+c^2k'^2} - \sqrt{1+c^2}\Delta}.$$

2.591

$$\begin{aligned}
 1. \int \frac{dx}{(c + \operatorname{tg} x)^n \Delta} &= \frac{1}{(n-1)(1+c^2)(1+k'^2c^2)} \left[-\frac{\Delta}{(c + \operatorname{tg} x)^{n-1} \cos^2 x} + \right. \\
 &\quad + (2n-3)c(1+k'^2+2c^2k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^{n-1} \Delta} - \\
 &\quad - (n-2)(1+k'^2+6c^2k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^{n-2} \Delta} + \\
 &\quad \left. + (4n-10)ck'^2 \int \frac{dx}{(c + \operatorname{tg} x)^{n-3} \Delta} - (n-3)k'^2 \int \frac{dx}{(c + \operatorname{tg} x)^{n-4} \Delta} \right].
 \end{aligned}$$

This integral can be reduced to the integrals:

$$2. \int \frac{dx}{(c + \operatorname{tg} x)^2 \Delta} = \frac{1}{(1 + c^2)(1 + k'^2 c^2)} \left[\frac{-\Delta}{(c + \operatorname{tg} x) \cos^2 x} + \right. \\ \left. + c(1 + k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x) \Delta} - \right. \\ \left. - 2ck'^2 \int \frac{c + \operatorname{tg} x}{\Delta} dx + k'^2 \int \frac{(c + \operatorname{tg} x)^2}{\Delta} dx \right] \text{ (see 2.589 2., 3., 4.).}$$

2.589

$$3. \int \frac{dx}{(c + \operatorname{tg} x)^3 \Delta} = \frac{1}{2(1 + c^2)(1 + k'^2 c^2)} \left[\frac{-\Delta}{(c + \operatorname{tg} x)^2 \cos^2 x} + \right. \\ \left. + 3c(1 + k'^2 + 2c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x)^2 \Delta} - \right. \\ \left. - (1 + k'^2 + 6c^2 k'^2) \int \frac{dx}{(c + \operatorname{tg} x) \Delta} + 2ck'^2 F(x, k) \right] \\ \text{(see 2.591 2. and 2.589 4.).}$$

2.589

2.591

2.592

$$1. P_n = \int \frac{(a + \sin^2 x)^n}{\Delta} dx.$$

The recursion formula

$$P_{n+2} = \frac{1}{(2n+3)k^2} \{ (a + \sin^2 x)^n \sin x \cos x \Delta + (2n+2)(1+k^2+3ak^2)P_{n+1} - \\ - (2n+1)[1+2a(1+k^2)+3a^2k^2]P_n + 2na(1+a)(1+k^2a)P_{n-1} \}$$

reduces this integral (for n an integer) to the integrals

$$2. P_1 \text{ see 2.584 1. and 2.584 4.}$$

3. P_0 see **2.584** 1.

$$4. P_{-1} = \int \frac{dx}{(a + \sin^2 x)\Delta} = \frac{1}{a} \Pi \left(x, \frac{1}{a}, k \right).$$

For $a = 0$

$$5. \int \frac{dx}{\sin^2 x \Delta} \quad \text{see } \mathbf{2.584} \text{ 70.}$$

$$6. T_n = \int \frac{dx}{(h + g \sin^2 x)^n \Delta}$$

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can be calculated by means of the recursion formula:

$$T_{n-3} = \frac{1}{(2n-5)k^2} \left\{ \frac{-g^2 \sin x \cos x \Delta}{(h + g \sin^2 x)^{n-1}} + 2(n-2)[g(1+k^2) + 3hk^2]T_{n-2} - \right. \\ \left. - (2n-3)[g^2 + 2hg(1+k^2) + 3h^2k^2]T_{n-1} + 2(n-1)h(g+h)(g+hk^2)T_n \right\}.$$

2.593

$$1. Q_n = \int \frac{(b + \cos^2 x)^n}{\Delta} dx.$$

The recursion formula

$$Q_{n+2} = \frac{1}{(2n+3)k^2} \left\{ (b + \cos^2 x)^n \sin x \cos x \Delta - (2n+2)(1-2k^2-3bk^2)Q_{n+1} + (2n+1)[k'^2 + 2b(k'^2 - k^2) - 3b^2k^2]Q_n - 2nb(1-b)(k'^2 - k^2b)Q_{n-1} \right\}$$

reduces this integral (for n an integer) to the integrals:

2. Q_1 see **2.584** 1. and **2.584** 6.

2.584

2.584

3. Q_0 see **2.584** 1.

2.584

4. $Q_{-1} = \int \frac{dx}{(b + \cos^2 x)\Delta} = \frac{1}{b+1} \Pi \left(x, -\frac{1}{b+1}, k \right).$

For $b = 0$

5. $\int \frac{dx}{\cos^2 x \Delta}$ see **2.584** 72.

2.584
ZH (123)

2.594

1. $R_n = \int \frac{(c + \operatorname{tg}^2 x)^n dx}{\Delta}.$

$$R_{n+2} = \frac{1}{(2n+3)k'^2} \left\{ \frac{(c + \operatorname{tg}^2 x)^n \operatorname{tg} x \Delta}{\cos^2 x} - (2n+2)(1+k'^2 - 3ck'^2)R_{n+1} + \right. \\ \left. + (2n-1)[1 - 2c(1+k'^2) + 3c^2k'^2]R_n + 2nc(1-c)(1-k'^2c)R_{n-1} \right\}$$

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reduces this integral (for n an integer) to the integrals:

2. R_1 see **2.584** 1. and **2.584** 90.

2.584

2.584

3. R_0 see **2.584** 1.

2.584

$$4. R_{-1} = \int \frac{dx}{(c + \operatorname{tg}^2 x)\Delta} = \frac{1}{c-1} F(x, k) + \frac{1}{c(1-c)} \Pi \left(x, \frac{1-c}{c}, k \right).$$

For $c = 0$ see 2.582 5.

2.595

Integrals of the type $\int R(\sin x, \cos x, \sqrt{1-p^2 \sin^2 x}) dx$ for $p^2 > 1$.

Notation: $\alpha = \arcsin(p \sin x)$.

Basic formulas

BY (283.00)

$$2. \int \sqrt{1 - p^2 \sin^2 x} \, dx = pE\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1].$$

BY (283.03)

$$3. \int \frac{dx}{(1 - r^2 \sin^2 x)\sqrt{1 - p^2 \sin^2 x}} = \frac{1}{p} \Pi\left(\alpha, \frac{r^2}{p^2}, \frac{1}{p}\right) \quad [p^2 > 1].$$

BY (283.02)

To evaluate integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}) \, dx$ for $p^2 > 1$, we may use formulas 2.583 and 2.584, making the following modifications in them.

We replace (1) k with p , (2) k'^2 with $1 - p^2$, (3) $F(x, k)$ with $\frac{1}{p} F\left(\alpha, \frac{1}{p}\right)$, and (4) $E(x, k)$ with $pE\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right)$.

For example (see 2.584 15.):

2.596

$$\begin{aligned} 1. \frac{\cos^4 x \, dx}{\sqrt{1 - p^2 \sin^2 x}} &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} + \frac{4p^2 - 2}{3p^4} \left[pE\left(\alpha, \frac{1}{p}\right) - \right. \\ &\quad \left. - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] + \frac{2 - 5p^2 + 3p^4}{3p^4} \cdot \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) = \\ &= \frac{\sin x \cos x \sqrt{1 - p^2 \sin^2 x}}{3p^2} - \frac{p^2 - 1}{3p^3} F\left(\alpha, \frac{1}{p}\right) + \frac{4p^2 - 2}{3p^3} E\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1]; \end{aligned}$$

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(see 2.583 36.):

$$\begin{aligned} 2. \int \frac{\sqrt{1 - p^2 \sin^2 x}}{\cos^2 x} \, dx &= \operatorname{tg} x \sqrt{1 - p^2 \sin^2 x} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \\ &\quad - \left[pE\left(\alpha, \frac{1}{p}\right) - \frac{p^2 - 1}{p} F\left(\alpha, \frac{1}{p}\right) \right] = \\ &= p \left[F\left(\alpha, \frac{1}{p}\right) - E\left(\alpha, \frac{1}{p}\right) \right] + \operatorname{tg} x \sqrt{1 - p^2 \sin^2 x} \quad [p^2 > 1]; \end{aligned}$$

$$3. \int \frac{dx}{\sqrt{(1-p^2 \sin^2 x)^3}} = \frac{-1}{p^2-1} \left[pE\left(\alpha, \frac{1}{p}\right) - \frac{p^2-1}{p} F\left(\alpha, \frac{1}{p}\right) \right] -$$

$$-\frac{p^2}{1-p^2} \cdot \frac{\sin x \cos x}{\sqrt{1-p^2 \sin^2 x}} = \frac{p^2}{p^2-1} \cdot \frac{\sin x \cos x}{\sqrt{1-p^2 \sin^2 x}} + \frac{1}{p} F\left(\alpha, \frac{1}{p}\right) - \frac{p}{p^2-1} E\left(\alpha, \frac{1}{p}\right) \quad [p^2 > 1].$$

2.597

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$.

$$\text{Notation: } \alpha = \arcsin\left(\frac{\sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x}}\right).$$

Basic formulas

$$1. \int \frac{dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{\sqrt{1+p^2}} F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right).$$

BY (282.00)

$$2. \int \sqrt{1+p^2 \sin^2 x} dx = \sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}}.$$

BY (282.03)

$$3. \frac{\sqrt{1+p^2 \sin^2 x} dx}{1+(p^2-r^2 p^2-r^2) \sin^2 x} = \frac{1}{\sqrt{1+p^2}} \Pi\left(\alpha, r^2, \frac{p}{\sqrt{1+p^2}}\right).$$

BY (282.02)

$$4. \int \frac{\sin x dx}{\sqrt{1+p^2 \sin^2 x}} = -\frac{1}{p} \arcsin\left(\frac{p \cos x}{\sqrt{1+p^2}}\right).$$

$$5. \int \frac{\cos x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

$$6. \int \frac{dx}{\sin x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{\sqrt{1+p^2 \sin^2 x} - \cos x}{\sqrt{1+p^2 \sin^2 x} + \cos x}.$$

$$7. \int \frac{dx}{\cos x \sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2} \sin x}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2} \sin x}.$$

$$8. \int \frac{\operatorname{tg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2\sqrt{1+p^2}} \ln \frac{\sqrt{1+p^2 \sin^2 x} + \sqrt{1+p^2}}{\sqrt{1+p^2 \sin^2 x} - \sqrt{1+p^2}}.$$

$$9. \int \frac{\operatorname{ctg} x dx}{\sqrt{1+p^2 \sin^2 x}} = \frac{1}{2} \ln \frac{1 - \sqrt{1+p^2 \sin^2 x}}{1 + \sqrt{1+p^2 \sin^2 x}}.$$

2.598

To calculate integrals of the form $\int R(\sin x, \cos x, \sqrt{1+p^2 \sin^2 x}) dx$, we may use formulas 2.583 and 2.584, making the following modifications in them:

We replace 1) k^2 with $-p^2$; 2) k'^2 with $1+p^2$;

$$3) F(x, k) \text{ with } \frac{1}{\sqrt{1+p^2}} F\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right);$$

$$4) E(x, k) \text{ with } \sqrt{1+p^2} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right) - p^2 \frac{\sin x \cos x}{\sqrt{1+p^2 \sin^2 x}};$$

$$5) \frac{1}{k} \ln(k \cos x + \Delta) \text{ with } \frac{1}{p} \arcsin \frac{p \cos x}{\sqrt{1+p^2}};$$

$$6) \frac{1}{k} \arcsin(k \sin x) \text{ with } \frac{1}{p} \ln(p \sin x + \sqrt{1+p^2 \sin^2 x}).$$

For example (see 2.584 90.):

(see 2.584 37.):

$$2. \int \frac{dx}{\sqrt{(1+p^2 \sin^2 x)^3}} = \frac{1}{\sqrt{1+p^2}} E\left(\alpha, \frac{p}{\sqrt{1+p^2}}\right)$$

2.599

Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) dx \quad [a^2 > 1]$

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Notation: $\alpha = \arcsin\left(\frac{\alpha \cos x}{\sqrt{a^2 - 1}}\right)$.

Basic formulas:

$$1. \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1].$$

BY (285.00)a

$$2. \int \sqrt{a^2 \sin^2 x - 1} dx = \frac{1}{a} F\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) - aE\left(\alpha, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1].$$

BY (285.06)a

$$3. \int \frac{dx}{(1-r^2 \sin^2 x)\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a(r^2 - 1)} \Pi\left(\alpha, \frac{r^2(a^2 - 1)}{a^2(r^2 - 1)}, \frac{\sqrt{a^2 - 1}}{a}\right) \quad [a^2 > 1, r^2 > 1].$$

BY (285.02)a

$$4. \int \frac{\sin x dx}{\sqrt{a^2 \sin^2 x - 1}} = -\frac{\alpha}{a} \quad [a^2 > 1].$$

$$5. \int \frac{\cos x \, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a} \ln(a \sin x + \sqrt{a^2 \sin^2 x - 1}) \quad [a^2 > 1].$$

$$6. \int \frac{dx}{\sin x \sqrt{a^2 \sin^2 x - 1}} = -\operatorname{arctg} \frac{\cos x}{\sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1].$$

$$7. \int \frac{dx}{\cos x \sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 2}} \ln \frac{\sqrt{a^2 - 1} \sin x + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} \sin x - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1].$$

$$8. \int \frac{\operatorname{tg} x \, dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{2\sqrt{a^2 - 1}} \ln \frac{\sqrt{a^2 - 1} + \sqrt{a^2 \sin^2 x - 1}}{\sqrt{a^2 - 1} - \sqrt{a^2 \sin^2 x - 1}} \quad [a^2 > 1].$$

$$9. \int \frac{\operatorname{ctg} x \, dx}{\sqrt{a^2 \sin^2 x - 1}} = -\arcsin \left(\frac{1}{a \sin x} \right) \quad [a^2 > 1].$$

2.611

To calculate integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2 \sin^2 x - 1}) \, dx$ (for $a^2 > 1$), we may use formulas 2.583 and 2.584. In doing so, we should follow the procedure outlined below:

1) In the right members of these formulas, the following functions should be replaced with integrals equal to them:

$$\begin{array}{ll} F(x, k) & \text{should be replaced with } \int \frac{dx}{\Delta}, \\ E(x, k) & \text{should be replaced with } \int \Delta dx, \\ -\frac{1}{k} \ln(k \cos x + \Delta) & \text{should be replaced with } \int \frac{\sin x \, dx}{\Delta}, \\ \frac{1}{k} \arcsin(k \sin x) & \text{should be replaced with } \int \frac{\cos x \, dx}{\Delta}, \\ \frac{1}{2} \ln \frac{\Delta - \cos x}{\Delta + \cos x} & \text{should be replaced with } \int \frac{dx}{\Delta \sin x}, \\ \frac{1}{2k'} \ln \frac{\Delta + k' \sin x}{\Delta - k' \sin x} & \text{should be replaced with } \int \frac{dx}{\Delta \cos x}, \\ \frac{1}{2k'} \ln \frac{\Delta + k'}{\Delta - k'} & \text{should be replaced with } \int \frac{\operatorname{tg} x}{\Delta} dx, \\ \frac{1}{2} \ln \frac{1 - \Delta}{1 + \Delta} & \text{should be replaced with } \int \frac{\operatorname{ctg} x}{\Delta} dx. \end{array}$$

1. We rewrite equation 2.584 4. in the form

$$\int \frac{\sin^2 x}{i\sqrt{a^2 \sin^2 x - 1}} dx = \frac{1}{a^2} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{1}{a^2} \int i\sqrt{a^2 \sin^2 x - 1} dx,$$

from which we get

$$\int \frac{\sin^2 x dx}{\sqrt{a^2 \sin^2 x - 1}} = \frac{1}{a^2} \left\{ \int \frac{dx}{\sqrt{a^2 \sin^2 x - 1}} + \int \sqrt{a^2 \sin^2 x - 1} dx \right\} = -\frac{1}{a} E \left(\alpha, \frac{\sqrt{a^2 - 1}}{a} \right) \quad [a^2 > 1].$$

2. We rewrite equation 2.584 58. as follows:

$$\begin{aligned} \int \frac{dx}{i^5 \sqrt{(a^2 \sin^2 x - 1)^5}} &= -\frac{2a^4(a^2 - 2) \sin^2 x - (3a^2 - 5)a^2}{3(1 - a^2)^2 i^3 \sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x - \\ &- \frac{1}{3(1 - a^2)} \int \frac{dx}{i\sqrt{a^2 \sin^2 x - 1}} - \frac{2a^2 - 4}{3(1 - a^2)^2} \int i\sqrt{a^2 \sin^2 x - 1} dx, \end{aligned}$$

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from which we obtain

$$\begin{aligned} \int \frac{dx}{\sqrt{(a^2 \sin^2 x - 1)^5}} &= \frac{2a^4(a^2 - 2) \sin^2 x - (3a^2 - 5)a^2}{3(1 - a^2)^2 \sqrt{(a^2 \sin^2 x - 1)^3}} \sin x \cos x + \frac{1}{3(1 - a^2)^2 a} \times \\ &\times \left\{ (a^2 - 3)F \left(\alpha, \frac{\sqrt{a^2 - 1}}{a} \right) - 2a^2(a^2 - 2) E \left(\alpha, \frac{\sqrt{a^2 - 1}}{a} \right) \right\} \quad [a^2 > 1]. \end{aligned}$$

3. We rewrite equation 2.584 71. in the form

$$\int \frac{dx}{\sin x \cos x i\sqrt{a^2 \sin^2 x - 1}} = \int \frac{\operatorname{ctg} x dx}{i\sqrt{a^2 \sin^2 x - 1}} + \int \frac{\operatorname{tg} x dx}{i\sqrt{a^2 \sin^2 x - 1}},$$

from which we obtain

2.612

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$

To find integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx$ we make the substitution $x = \frac{\pi}{2} - y$, which yields

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = - \int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy.$$

The integrals $\int R(\cos y, \sin y, \sqrt{1 - k^2 \sin^2 y}) dy$ are found from formulas 2.583 and 2.584. As a result of the use of these formulas (where it is assumed that the original integral can be reduced only to integrals of the first and second Legendre forms), when we replace the functions $F(x, k)$ and $E(x, k)$ with the corresponding integrals, we obtain an expression of the form

$$-g(\cos y, \sin y) - A \int \frac{dy}{\sqrt{1 - k^2 \sin^2 y}} - B \int \sqrt{1 - k^2 \sin^2 y} dy.$$

Returning now to the original variable x , we obtain

$$\int R(\sin x, \cos x, \sqrt{1 - k^2 \cos^2 x}) dx = -g(\sin x, \cos x) - A \int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} - B \int \sqrt{1 - k^2 \cos^2 x} dx.$$

The integrals appearing in this expression are found from the formulas

$$1. \int \frac{dx}{\sqrt{1 - k^2 \cos^2 x}} = F\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right).$$

$$2. \int \sqrt{1 - k^2 \cos^2 x} dx = E\left(\arcsin\left(\frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}}\right), k\right) - \frac{k^2 \sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}}.$$

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2.613

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx$ $[p > 1]$

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1 - p^2 \cos^2 x}) dx$, where $[p > 1]$, we proceed as in 2.612. Here, we use the formulas

$$1. \int \frac{dx}{\sqrt{1 - p^2 \cos^2 x}} = -\frac{1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) \quad [p > 1].$$

$$2. \int \sqrt{1 - p^2 \cos^2 x} dx = \frac{p^2 - 1}{p} F\left(\arcsin(p \cos x), \frac{1}{p}\right) - pE\left(\arcsin(p \cos x), \frac{1}{p}\right).$$

2.614

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 + p^2 \cos^2 x}) dx$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{1 + p^2 \cos^2 x}) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{1 + p^2 \cos^2 x}) dx = -\int R(\cos y, \sin y, \sqrt{1 + p^2 \sin^2 y}) dy.$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{1 + p^2 \sin^2 y}) dy$, we need to use first what was said in 2.598 and 2.612 and then, after returning to the variable x , the formulas

$$1. \int \frac{dx}{\sqrt{1 + p^2 \cos^2 x}} = \frac{1}{\sqrt{1 + p^2}} F\left(x, \frac{p}{\sqrt{1 + p^2}}\right).$$

$$2. \int \sqrt{1 + p^2 \cos^2 x} dx = \sqrt{1 + p^2} E\left(x, \frac{p}{\sqrt{1 + p^2}}\right).$$

2.615

Integrals of the form $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx$ $[a > 1]$.

To find integrals of the type $\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx$, we need to make the substitution $x = \frac{\pi}{2} - y$. This yields

$$\int R(\sin x, \cos x, \sqrt{a^2 \cos^2 x - 1}) dx = - \int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy.$$

To calculate the integrals $-\int R(\cos y, \sin y, \sqrt{a^2 \sin^2 y - 1}) dy$, we use what was said in 2.611 and then,

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after returning to the variable x , we use the formulas

$$1. \int \frac{dx}{\sqrt{a^2 \cos^2 x - 1}} = \frac{1}{a} F \left(\arcsin \left(\frac{a \sin x}{\sqrt{a^2 - 1}} \right), \frac{\sqrt{a^2 - 1}}{a} \right) \quad [a > 1].$$

$$2. \int \sqrt{a^2 \cos^2 x - 1} dx = aE \left(\arcsin \left(\frac{a \sin x}{\sqrt{a^2 - 1}} \right), \frac{\sqrt{a^2 - 1}}{a} \right) - \frac{1}{a} F \left(\arcsin \left(\frac{a \sin x}{\sqrt{a^2 - 1}} \right), \frac{\sqrt{a^2 - 1}}{a} \right) \quad [a > 1].$$

2.616

Integrals of the form $\int R(\sin x, \cos x, \sqrt{1 - p^2 \sin^2 x}, \sqrt{1 - q^2 \sin^2 x}) dx$.

$$\text{Notation: } \alpha = \arcsin \left(\frac{\sqrt{1 - p^2} \sin x}{\sqrt{1 - p^2 \sin^2 x}} \right).$$

$$1. \int \frac{dx}{\sqrt{(1 - p^2 \sin^2 x)(1 - q^2 \sin^2 x)}} = \frac{1}{\sqrt{1 - p^2}} F \left(\alpha, \sqrt{\frac{q^2 - p^2}{1 - p^2}} \right) \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right].$$

BY (284.00)

$$\begin{aligned}
3. \quad & \int \frac{\operatorname{tg}^4 x \, dx}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} = \\
& = \frac{1}{3(1-q^2)^2(1-p^2)^{\frac{3}{2}}} \times \left[2(2-p^2-q^2)E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - (1-q^2)F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right] + \\
& + \frac{2p^2+q^2-3+\sin^2 x(4-3p^2-2q^2+p^2q^2)}{3(1-p^2)(1-q^2)^2} \frac{\sin x}{\cos^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-p^2 \sin^2 x}} \\
& \quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right].
\end{aligned}$$

BY (284.07)

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$$\begin{aligned}
4. \quad & \int \frac{\sin^2 x \, dx}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)^3}} = \\
& = \frac{\sqrt{1-p^2}}{(1-q^2)(q^2-p^2)} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1}{(q^2-p^2)\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \\
& - \frac{\sin x \cos x}{(1-q^2)\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} \quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right].
\end{aligned}$$

BY (284.06)

$$\begin{aligned}
5. \quad & \int \frac{\cos^2 x \, dx}{\sqrt{(1-p^2 \sin^2 x)^3(1-q^2 \sin^2 x)}} = \\
& = \frac{\sqrt{1-p^2}}{q^2-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{1-q^2}{(q^2-p^2)\sqrt{1-p^2}} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\
& \quad \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2} \right].
\end{aligned}$$

BY (284.05)

$$\begin{aligned}
6. \quad & \int \frac{\cos^4 x \, dx}{\sqrt{(1-p^2 \sin^2 x)^5(1-q^2 \sin^2 x)}} = \\
& = \frac{(1-p^2)^{\frac{3}{2}}}{3(q^2-p^2)^2} \left[\frac{(2+p^2-3q^2)(1-q^2)}{(1-p^2)^2} F\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) + \right. \\
& \left. + 2\frac{2q^2-p^2-1}{1-p^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \right] + \frac{(1-p^2) \sin x \cos x \sqrt{1-q^2 \sin^2 x}}{3(q^2-p^2)\sqrt{(1-p^2 \sin^2 x)^3}}
\end{aligned}$$

$$7. \int \frac{dx}{1-p^2 \sin^2 x} \sqrt{\frac{1-q^2 \sin^2 x}{1-p^2 \sin^2 x}} x = \frac{1}{\sqrt{1-p^2}} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2}\right]$$

BY (284.01)

$$8. \int \sqrt{\frac{1-p^2 \sin^2 x}{(1-q^2 \sin^2 x)^3}} dx = \frac{\sqrt{1-p^2}}{1-q^2} E\left(\alpha, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) - \frac{q^2-p^2}{1-q^2} \frac{\sin x \cos x}{\sqrt{(1-p^2 \sin^2 x)(1-q^2 \sin^2 x)}} \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2}\right].$$

BY (284.04)

$$9. \int \frac{dx}{1+(p^2 r^2 - p^2 - r^2) \sin^2 x} \sqrt{\frac{1-p^2 \sin^2 x}{1-q^2 \sin^2 x}} = \frac{1}{\sqrt{1-p^2}} \Pi\left(\alpha, r^2, \sqrt{\frac{q^2-p^2}{1-p^2}}\right) \\ \left[0 < p^2 < q^2 < 1, \quad 0 < x \leq \frac{\pi}{2}\right].$$

BY (284.02)

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2.617

$$\text{Notation: } \alpha = \arcsin \sqrt{\frac{\sqrt{b^2+c^2}-b \sin x - c \cos x}{2\sqrt{b^2+c^2}}}, \quad r = \sqrt{\frac{2\sqrt{b^2+c^2}}{a+\sqrt{b^2+c^2}}}$$

$$1. \int \frac{dx}{\sqrt{a+b \sin x + c \cos x}} = -\frac{2}{\sqrt{a+\sqrt{b^2+c^2}}} F(\alpha, r) \\ \left[0 < \sqrt{b^2+c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}}\right]; \\ = -\frac{\sqrt{2}}{\sqrt[4]{b^2+c^2}} F(\alpha, r) \\ \left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}}\right].$$

BY (293.00)

BY (294.00)

$$2. \int \frac{\sin x dx}{\sqrt{a+b \sin x + c \cos x}} = -\frac{\sqrt{2b}}{\sqrt[4]{(b^2+c^2)^3}} \{2E(\alpha, r) - F(\alpha, r)\} + \\ + \frac{2c}{b^2+c^2} \sqrt{a+b \sin x + c \cos x} \\ \left[0 < |a| < \sqrt{b^2+c^2}, \quad \arcsin \frac{b}{\sqrt{b^2+c^2}} - \arccos\left(-\frac{a}{\sqrt{b^2+c^2}}\right) \leq x < \arcsin \frac{b}{\sqrt{b^2+c^2}}\right].$$

$$3. \int \frac{(b \cos x - c \sin x) dx}{\sqrt{a + b \sin x + c \cos x}} = 2\sqrt{a + b \sin x + c \cos x}.$$

$$4. \int \frac{\sqrt{b^2 + c^2} + b \sin x + c \cos x}{\sqrt{a + b \sin x + c \cos x}} dx =$$

$$= -2\sqrt{a + \sqrt{b^2 + c^2}} E(\alpha, r) + \frac{2(a - \sqrt{b^2 + c^2})}{\sqrt{a + \sqrt{b^2 + c^2}}} F(\alpha, r)$$

$$\left[0 < \sqrt{b^2 + c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right];$$

$$= -2\sqrt{2} \sqrt[4]{b^2 + c^2} E(\alpha, r)$$

$$\left[0 < |a| < \sqrt{b^2 + c^2}, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \arccos \left(-\frac{a}{\sqrt{b^2 + c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right].$$

BY (293.01)

BY (294.04)

222

$$5. \int \sqrt{a + b \sin x + c \cos x} dx = -2\sqrt{a + \sqrt{b^2 + c^2}} E(\alpha, r)$$

$$\left[0 < \sqrt{b^2 + c^2} < a, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \pi \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right];$$

$$= -2\sqrt{2} \sqrt[4]{b^2 + c^2} E(\alpha, r) + \frac{\sqrt{2}(\sqrt{b^2 + c^2} - a)}{\sqrt[4]{b^2 + c^2}} F(\alpha, r)$$

$$\left[0 < |a| < \sqrt{b^2 + c^2}, \quad \arcsin \frac{b}{\sqrt{b^2 + c^2}} - \arccos \left(\frac{-a}{\sqrt{b^2 + c^2}} \right) \leq x < \arcsin \frac{b}{\sqrt{b^2 + c^2}} \right].$$

BY (293.03)

BY (294.01)

2.618

Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) dx = \frac{1}{a} \int R(\sin t, \cos t, \sqrt{12 \sin^2 t}) dt (t = ax)$.

Notation: $\alpha = \arcsin(\sqrt{2} \sin ax)$

The integrals $\int R(\sin ax, \cos ax, \sqrt{\cos 2ax}) dx$ are special cases of the integrals 2.595. for $(p = 2)$. We give some formulas:

$$1. \int \frac{dx}{\sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$2. \int \frac{\cos^2 ax}{\sqrt{\cos 2ax}} dx = \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$3. \int \frac{dx}{\cos^2 ax \sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\operatorname{tg} x}{a} \sqrt{\cos 2ax} \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$4. \int \frac{dx}{\cos^4 ax \sqrt{\cos 2ax}} = \frac{2\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{3a} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{(6 \cos^2 ax + 1) \sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax} \quad \left[0 < x \leq \frac{\pi}{4}\right].$$

$$5. \int \frac{\operatorname{tg}^2 ax dx}{\sqrt{\cos 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax} \quad \left[0 < x \leq \frac{\pi}{2}\right].$$

$$6. \int \frac{\operatorname{tg}^4 ax dx}{\sqrt{\cos 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin ax}{3a \cos^3 ax} \sqrt{\cos 2ax} \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$7. \int \frac{dx}{(1 - 2r^2 \sin^2 ax) \sqrt{\cos 2ax}} = \frac{1}{a\sqrt{2}} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

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$$8. \int \frac{dx}{\sqrt{\cos^3 2ax}} = \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{a\sqrt{\cos 2ax}} \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$9. \int \frac{\sin^2 ax dx}{\sqrt{\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{\cos 2ax}} - \frac{1}{a\sqrt{2}} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$10. \int \frac{dx}{\sqrt{\cos^5 2ax}} = \frac{1}{3a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) + \frac{\sin 2ax}{3a\sqrt{\cos^3 2ax}} \quad \left[0 < ax \leq \frac{\pi}{4}\right].$$

$$11. \int \sqrt{\cos 2ax} dx = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad \left[0 < ax \leq \frac{\pi}{4}\right]$$

$$12. \int \frac{\sqrt{\cos 2ax}}{\cos^2 ax} dx = \frac{\sqrt{2}}{a} \left\{ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right\} + \frac{1}{a} \operatorname{tg} ax \sqrt{\cos 2ax} \quad \left[0 < x \leq \frac{\pi}{4}\right].$$

Integrals of the form

$$\int R(\sin ax, \cos ax, \sqrt{-\cos 2ax}) dx = \frac{1}{a} \int R(\sin x, \cos x, \sqrt{2 \sin^2 x - 1}) dx.$$

$$\text{Notation: } \alpha = \arcsin(\sqrt{2} \cos ax).$$

The integrals $\int R(\sin x, \cos x, \sqrt{2 \sin^2 x - 1}) dx$ are special cases of the integrals 2.599 and 2.611 for ($a = 2$). We give some formulas:

$$1. \int \frac{dx}{\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right).$$

$$2. \int \frac{\cos^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right) \right].$$

$$3. \int \frac{\cos^4 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{3a\sqrt{2}} \left[3F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{5}{2}E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] - \frac{1}{12a} \sin 2ax \sqrt{-\cos 2ax}.$$

$$4. \int \frac{dx}{\sin^2 ax \sqrt{-\cos 2ax}} = \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax} - \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right).$$

$$5. \int \frac{dx}{\sin^4 ax \sqrt{-\cos 2ax}} = \frac{2}{3a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 6E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{3a} \frac{\cos ax}{\sin^3 ax} (6 \sin^2 ax + 1) \sqrt{-\cos 2ax}.$$

$$6. \int \frac{\operatorname{ctg}^2 ax dx}{\sqrt{-\cos 2ax}} = \frac{1}{a\sqrt{2}} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{1}{a} \operatorname{ctg} ax \sqrt{-\cos 2ax}.$$

$$7. \int \frac{dx}{(1 - 2r^2 \cos^2 ax)\sqrt{-\cos 2ax}} = -\frac{1}{a\sqrt{2}}\Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right).$$

$$8. \int \frac{dx}{\sqrt{-\cos^3 2ax}} = \frac{1}{a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right] + \frac{\sin 2ax}{a\sqrt{-\cos 2ax}}.$$

$$9. \int \frac{\cos^2 ax dx}{\sqrt{-\cos^3 2ax}} = \frac{\sin 2ax}{2a\sqrt{-\cos 2ax}} - \frac{1}{a\sqrt{2}}E\left(\alpha, \frac{1}{\sqrt{2}}\right).$$

$$10. \int \frac{dx}{\sqrt{-\cos^5 2ax}} = -\frac{1}{3a\sqrt{2}}F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{\sin 2ax}{3a\sqrt{-\cos^3 2ax}}.$$

$$11. \int \sqrt{-\cos 2ax} dx = \frac{1}{a\sqrt{2}}\left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right].$$

Integrals of the form $\int R(\sin ax, \cos ax, \sqrt{\sin 2ax}) dx$.

$$\text{Notation: } \alpha = \arcsin \sqrt{\frac{2 \sin ax}{1 + \sin ax + \cos ax}}.$$

2.621

$$1. \int \frac{dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}F\left(\alpha, \frac{1}{\sqrt{2}}\right).$$

BY (287.50)

$$2. \int \frac{\sin ax dx}{\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a}\left\{\frac{1+i}{2}\Pi\left(\alpha, \frac{1+i}{2}, \frac{1}{\sqrt{2}}\right) + \frac{1-i}{2}\Pi\left(\alpha, \frac{1-i}{2}, \frac{1}{\sqrt{2}}\right) + F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right)\right\}.$$

BY (287.57)

BY (287.54)

$$4. \int \frac{\sin ax \, dx}{(1 - \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ \sqrt{\operatorname{tg} ax} - E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right\} \quad \left[ax \neq \frac{\pi}{2} \right].$$

BY (287.55)

$$5. \int \frac{(1 + \cos ax) \, dx}{(1 + \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} E\left(\alpha, \frac{1}{\sqrt{2}}\right).$$

BY (287.51)

$$6. \int \frac{(1 + \cos ax) \, dx}{(1 - \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ F\left(\alpha, \frac{1}{\sqrt{2}}\right) - E\left(\alpha, \frac{1}{\sqrt{2}}\right) + \sqrt{\operatorname{tg} ax} \right\} \quad \left[ax \neq \frac{\pi}{2} \right].$$

BY (287.56)

$$7. \int \frac{(1 - \sin ax + \cos ax) \, dx}{(1 + \sin ax + \cos ax)\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \left\{ 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) - F\left(\alpha, \frac{1}{\sqrt{2}}\right) \right\}.$$

BY (287.53)

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$$8. \int \frac{(1 + \sin ax + \cos ax) \, dx}{[1 + \cos ax + (1 - 2r^2) \sin ax]\sqrt{\sin 2ax}} = \frac{\sqrt{2}}{a} \Pi\left(\alpha, r^2, \frac{1}{\sqrt{2}}\right).$$

BY (287.52)

2.63-2.65 Products of trigonometric functions and powers

2.631

$$1. \int x^r \sin^p x \cos^q x \, dx = \frac{1}{(p+q)^2} \left[(p+q)x^r \sin^{p+1} x \cos^{q-1} x + \right. \\ \left. + rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx - \right. \\ \left. - rp \int x^{r-1} \sin^{p-1} x \cos^{q-1} x \, dx + (q-1)(p+q) \int x^r \sin^p x \cos^{q-2} x \, dx \right]; \\ = \frac{1}{(p+q)^2} \left[-(p+q)x^r \sin^{p-1} x \cos^{q+1} x + \right. \\ \left. + rx^{r-1} \sin^p x \cos^q x - r(r-1) \int x^{r-2} \sin^p x \cos^q x \, dx + \right. \\ \left. + (p-1)(p+q) \int x^r \sin^{p-2} x \cos^q x \, dx \right].$$

$$2. \int x^m \sin^n x dx = \frac{x^{m-1} \sin^{n-1} x}{n^2} \{m \sin x - nx \cos x\} + \\ + \frac{n-1}{n} \int x^m \sin^{n-2} x dx - \frac{m(m-1)}{n^2} \int x^{m-2} \sin^n x dx.$$

$$3. \int x^m \cos^n x dx = \frac{x^{m-1} \cos^{n-1} x}{n^2} \{m \cos x + nx \sin x\} + \\ + \frac{n-1}{n} \int x^m \cos^{n-2} x dx - \frac{m(m-1)}{n^2} \int x^{m-2} \cos^n x dx.$$

$$4. \int x^n \sin^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \\ + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \int x^n \cos(2m-2k)x dx \quad (\text{see } \mathbf{2.633} \text{ 2}).$$

2.633
TI 333

$$5. \int x^n \sin^{2m+1} x dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \int x^n \sin(2m-2k+1)x dx \\ (\text{see } \mathbf{2.633} \text{ 1}).$$

2.633
TI 333

$$6. \int x^n \cos^{2m} x dx = \binom{2m}{m} \frac{x^{n+1}}{2^{2m}(n+1)} + \\ + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \int x^n \cos(2m-2k)x dx \quad (\text{see } \mathbf{2.6332}).$$

2.633
TI 333

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$$7. \int x^n \cos^{2m+1} x dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \int x^n \cos(2m-2k+1)x dx. \quad (\text{see } \mathbf{2.6332}).$$

2.633
TI 333

2.632

$$1. \int x^{\mu-1} \sin \beta x dx = \frac{i}{2}(i\beta)^{-\mu} \gamma(\mu, i\beta x) - \frac{i}{2}(-i\beta)^{-\mu} \gamma(\mu, -i\beta x). \quad [\operatorname{Re} \mu > -1, \quad x > 0].$$

ET I 317(2)

$$2. \int x^{\mu-1} \sin ax dx = -\frac{1}{2a^\mu} \left\{ \exp \left[\frac{\pi i}{2}(\mu-1) \right] \Gamma(\mu, -iax) + \exp \left[\frac{\pi i}{2}(1-\mu) \right] \Gamma(\mu, iax) \right\} \quad [\operatorname{Re} \mu < 1, \quad a > 0, \quad x > 0].$$

ET I 317(3)

$$3. \int x^{\mu-1} \cos \beta x dx = \frac{1}{2} \{ (i\beta)^{-\mu} \gamma(\mu, i\beta x) + (-i\beta)^{-\mu} \gamma(\mu, -i\beta x) \} \\ [\operatorname{Re} \mu > 0, \quad x > 0].$$

ET I 319(22)

$$4. \int x^{\mu-1} \cos ax dx = -\frac{1}{2a^\mu} \left\{ \exp \left(i\mu \frac{\pi}{2} \right) \Gamma(\mu, -iax) + \exp \left(-i\mu \frac{\pi}{2} \right) \Gamma(\mu, iax) \right\}.$$

ET I 319(23)

2.633

$$1. \int x^n \sin ax dx = -\sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \cos \left(ax + \frac{1}{2}k\pi \right).$$

TI (487)

$$2.^8 \int x^n \cos ax dx = \sum_{k=0}^n k! \binom{n}{k} \frac{x^{n-k}}{a^{k+1}} \sin \left(ax + \frac{1}{2}k\pi \right).$$

TI (486)

$$3. \int x^{2n} \sin x dx = (2n)! \left\{ \sum_{k=0}^n \frac{(-1)^{k+1} x^{2n-2k}}{(2n-2k)!} \cos x + \sum_{k=0}^{n-1} \frac{(-1)^k x^{2n-2k-1}}{(2n-2k-1)!} \sin x \right\}.$$

$$4. \int x^{2n+1} \sin x dx = (2n+1)! \left\{ \sum_{k=0}^n \frac{(-1)^{k+1} x^{2n-2k+1}}{(2n-2k+1)!} \cos x + \sum_{k=0}^n \frac{(-1)^k x^{2n-2k}}{(2n-2k)!} \sin x \right\}.$$

$$5. \int x^{2n} \cos x \, dx = (2n)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k}}{(2n-2k)!} \sin x + \sum_{k=0}^{n-1} (-1)^k \frac{x^{2n-2k-1}}{(2n-2k-1)!} \cos x \right\}.$$

$$6. \int x^{2n+1} \cos x \, dx = (2n+1)! \left\{ \sum_{k=0}^n (-1)^k \frac{x^{2n-2k+1}}{(2n-2k+1)!} \sin x + \sum_{k=0}^n \frac{x^{2n-2k}}{(2n-2k)!} \cos x \right\}.$$

2.634

$$1. \int P_n(x) \sin mx \, dx = -\frac{\cos mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\sin mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

$$2. \int P_n(x) \cos mx \, dx = \frac{\sin mx}{m} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{P_n^{(2k)}(x)}{m^{2k}} + \frac{\cos mx}{m} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(x)}{m^{2k-1}}.$$

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In formulas 2.634, $P_n(x)$ is an n th-degree polynomial and $P_n^{(k)}(x)$ is its k th derivative with respect to x .

Notation: $z_1 = A + Bx$.

2.635

$$1. \int z_1 \sin kx \, dx = -\frac{1}{k} z_1 \cos kx + \frac{b}{k^2} \sin kx.$$

$$2. \int z_1 \cos kx \, dx = \frac{1}{k} z_1 \sin kx + \frac{b}{k^2} \cos kx.$$

$$3. \int z_1^2 \sin kx \, dx = \frac{1}{k} \left(\frac{2b^2}{k^2} - z_1^2 \right) \cos kx + \frac{2bz_1}{k^2} \sin kx.$$

$$4. \int z_1^2 \cos kx \, dx = \frac{1}{k} \left(z_1^2 - \frac{2b^2}{k^2} \right) \sin kx + \frac{2bz_1}{k^2} \cos kx.$$

$$5. \int z_1^3 \sin kx \, dx = \frac{z_1}{k} \left(\frac{6b^2}{k^2} - z_1^2 \right) \cos kx + \frac{3b}{k^2} \left(z_1^2 - \frac{2b^2}{k^2} \right) \sin kx.$$

$$6. \int z_1^3 \cos kx \, dx = \frac{z_1}{k} \left(z_1^2 - \frac{6b^2}{k^2} \right) \sin kx + \frac{3b}{k^2} \left(z_1^2 - \frac{2b^2}{k^2} \right) \cos kx.$$

$$7. \int z_1^4 \sin kx \, dx = -\frac{1}{k} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{4bz_1}{k^2} \left(z_1^2 - \frac{6b^2}{k^2} \right) \sin kx.$$

$$8. \int z_1^4 \cos kx \, dx = \frac{1}{k} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx + \frac{4bz_1}{k^2} \left(z_1^2 - \frac{6b^2}{k^2} \right) \cos kx.$$

$$9. \int z_1^5 \sin kx \, dx = \frac{5b}{k^2} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \sin kx - \frac{z_1}{k} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx.$$

$$10. \int z_1^5 \cos kx \, dx = \frac{5b}{k^2} \left(z_1^4 - \frac{12b^2}{k^2} z_1^2 + \frac{24b^4}{k^4} \right) \cos kx + \frac{z_1}{k} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx.$$

$$11. \int z_1^6 \sin kx \, dx = \frac{6bz_1}{k^2} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \sin kx - \frac{1}{k} \left(z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \cos kx.$$

$$12. \int z_1^6 \cos kx \, dx = \frac{6bz_1}{k^2} \left(z_1^4 - \frac{20b^2}{k^2} z_1^2 + \frac{120b^4}{k^4} \right) \cos kx + \frac{1}{k} \left(z_1^6 - \frac{30b^2}{k^2} z_1^4 + \frac{360b^4}{k^4} z_1^2 - \frac{720b^6}{k^6} \right) \sin kx.$$

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2.636

$$1. \int x^n \sin^2 x \, dx = \frac{x^{n+1}}{2(n+1)} + \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k} (n-2k)!} \sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1} (n-2k-1)!} \cos 2x \right\}.$$

$$2. \int x^n \cos^2 x dx = \frac{x^{n+1}}{2(n+1)} - \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^{k+1} x^{n-2k}}{2^{2k}(n-2k)!} \sin 2x + \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^{k+1} x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \cos 2x \right\}.$$

GU((333))(3e)

$$3. \int x \sin^2 x dx = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x.$$

$$4. \int x^2 \sin^2 x dx = \frac{x^3}{6} - \frac{x}{4} \cos 2x - \frac{1}{4} \left(x^2 - \frac{1}{2} \right) \sin 2x.$$

MZ 241

$$5. \int x \cos^2 x dx = \frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x.$$

$$6. \int x^2 \cos^2 x dx = \frac{x^3}{6} + \frac{x}{4} \cos 2x + \frac{1}{4} \left(x^2 - \frac{1}{2} \right) \sin 2x.$$

MZ 245

2.637

$$1. \int x^n \sin^3 x dx = \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left(\frac{\cos 3x}{3^{2k+1}} - 3 \cos x \right) - \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\sin 3x}{2^{2k+2}} - 3 \sin x \right) \right\}.$$

GU((333))(2f)

$$2. \int x^n \cos^3 x dx = \frac{n!}{4} \left\{ \sum_{k=0}^{[n/2]} \frac{(-1)^k x^{n-2k}}{(n-2k)!} \left(\frac{\sin 3x}{3^{2k+1}} + 3 \sin x \right) + \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{x^{n-2k-1}}{(n-2k-1)!} \left(\frac{\cos 3x}{3^{2k+2}} + 3 \cos x \right) \right\}.$$

GU((333))(3f)

$$3. \int x \sin^3 x dx = \frac{3}{4} \sin x - \frac{1}{36} \sin 3x - \frac{3}{4} x \cos x + \frac{x}{12} \cos 3x$$

$$4. \int x^2 \sin^3 x \, dx = -\left(\frac{3}{4}x^2 + \frac{3}{2}\right) \cos x + \left(\frac{x^2}{12} + \frac{1}{54}\right) \cos 3x + \frac{3}{2}x \sin x - \frac{x}{18} \sin 3x.$$

MZ 241

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$$5. \int x \cos^3 x \, dx = \frac{3}{4} \cos x + \frac{1}{36} \cos 3x + \frac{3}{4}x \sin x + \frac{x}{12} \sin 3x.$$

$$6. \int x^2 \cos^3 x \, dx = \left(\frac{3}{4}x^2 - \frac{3}{2}\right) \sin x + \left(\frac{x^2}{12} - \frac{1}{54}\right) \sin 3x + \frac{3}{2}x \cos x + \frac{x}{18} \cos 3x.$$

MZ 245, 246

2.638

$$1. \int \frac{\sin^q x}{x^p} \, dx = -\frac{\sin^{q-1} x [(p-2) \sin x + qx \cos x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)} \int \frac{\sin^q x \, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sin^{q-2} x \, dx}{x^{p-2}} \quad [p \neq 1, \quad p \neq 2].$$

TI (496)

$$2. \int \frac{\cos^q x}{x^p} \, dx = -\frac{\cos^{q-1} x [(p-2) \cos x - qx \sin x]}{(p-1)(p-2)x^{p-1}} - \frac{q^2}{(p-1)(p-2)} \int \frac{\cos^q x \, dx}{x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\cos^{q-2} x \, dx}{x^{p-2}} \quad [p \neq 1, \quad p \neq 2].$$

TI (495)

$$3.6 \int \frac{\sin x \, dx}{x^p} = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{\cos x \, dx}{x^{p-1}}; \\ = -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\sin x \, dx}{x^{p-2}} \quad (p > 2).$$

TI (492)

$$4.6 \int \frac{\cos x \, dx}{x^p} = -\frac{\cos x}{(p-1)x^{p-1}} - \frac{1}{p-1} \int \frac{\sin x \, dx}{x^{p-1}}; \\ = -\frac{\cos x}{(p-1)x^{p-1}} + \frac{\sin x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\cos x \, dx}{x^{p-2}} \quad (p > 2).$$

2.639

$$1. \int \frac{\sin x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \cos x + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^{n+1}}{(2n-1)!} \text{ci}(x).$$

GU ((333))(6b)a

$$2. \int \frac{\sin x}{x^{2n+1}} dx = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \text{si}(x).$$

GU ((333))(6b)a

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$$3. \int \frac{\cos x \, dx}{x^{2n}} = \frac{(-1)^{n+1}}{x(2n-1)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \cos x - \sum_{k=0}^{n-2} \frac{(-1)^k (2k+1)!}{x^{2k+1}} \sin x \right\} + \frac{(-1)^n}{(2n-1)!} \text{si}(x).$$

GU ((333))(7b)

$$4. \int \frac{\cos x \, dx}{x^{2n+1}} = \frac{(-1)^{n+1}}{x(2n)!} \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k+1)!}{x^{2k+1}} \cos x - \sum_{k=0}^{n-1} \frac{(-1)^{k+1} (2k)!}{x^{2k}} \sin x \right\} + \frac{(-1)^n}{(2n)!} \text{ci}(x).$$

GU ((333))(7b)

2.641

$$1. \int \frac{\sin kx}{a+bx} dx = \frac{1}{b} \left[\cos \frac{ka}{b} \text{si}(u) - \sin \frac{ka}{b} \text{ci}(u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right].$$

$$2. \int \frac{\cos kx}{a+bx} dx = \frac{1}{b} \left[\cos \frac{ka}{b} \text{ci}(u) + \sin \frac{ka}{b} \text{si}(u) \right] \quad \left[u = \frac{k}{b}(a+bx) \right].$$

$$3. \int \frac{\sin kx}{(a+bx)^2} dx = -\frac{1}{b} \frac{\sin kx}{a+bx} + \frac{k}{b} \int \frac{\cos kx}{a+bx} dx \quad (\text{see 2.641.2}).$$

$$4. \int \frac{\cos kx}{(a+bx)^2} dx = -\frac{1}{b} \frac{\cos kx}{a+bx} - \frac{k}{b} \int \frac{\sin kx}{a+bx} dx \quad (\text{see 2.6411.}).$$

2.641

$$5. \int \frac{\sin kx}{(a+bx)^3} dx = -\frac{\sin kx}{2b(a+bx)^2} - \frac{k \cos kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\sin kx}{a+bx} dx \quad (\text{see 2.6411.}).$$

2.641

$$6. \int \frac{\cos kx}{(a+bx)^3} dx = -\frac{\cos kx}{2b(a+bx)^2} + \frac{k \sin kx}{2b^2(a+bx)} - \frac{k^2}{2b^2} \int \frac{\cos kx}{a+bx} dx \quad (\text{see 2.6412.}).$$

2.641

$$7. \int \frac{\sin kx}{(a+bx)^4} dx = -\frac{\sin kx}{3b(a+bx)^3} - \frac{k \cos kx}{6b^2(a+bx)^2} + \frac{k^2 \sin kx}{6b^2(a+bx)} - \frac{k^3}{6b^3} \int \frac{\cos kx}{a+bx} dx \quad (\text{see 2.6412.}).$$

2.641

$$8. \int \frac{\cos kx}{(a+bx)^4} dx = -\frac{\cos kx}{3b(a+bx)^3} + \frac{k \sin kx}{6b^2(a+bx)^2} + \frac{k^2 \cos kx}{6b^3(a+bx)} + \frac{k^3}{6b^3} \int \frac{\sin kx}{a+bx} dx \quad (\text{see 2.6411.}).$$

2.641

$$9. \int \frac{\sin kx}{(a+bx)^5} dx = -\frac{\sin kx}{4b(a+bx)^4} - \frac{k \cos kx}{12b^2(a+bx)^3} + \frac{k^2 \sin kx}{24b^3(a+bx)^2} + \frac{k^3 \cos kx}{24b^4(a+bx)} - \frac{k^4}{24b^4} \int \frac{\sin kx}{a+bx} dx \quad (\text{see 2.6411.}).$$

$$10. \int \frac{\cos kx}{(a+bx)^5} dx = -\frac{\cos kx}{4b(a+bx)^4} + \frac{k \sin kx}{12b^2(a+bx)^3} + \frac{k^2 \cos kx}{24b^3(a+bx)^2} - \frac{k^3 \sin kx}{24b^4(a+bx)} + \frac{k^4}{24b^4} \int \frac{\cos kx}{a+bx} dx \quad (\text{see 2.6412}).$$

2.641

$$11. \int \frac{\sin kx}{(a+bx)^6} dx = -\frac{\sin kx}{5b(a+bx)^5} - \frac{k \cos kx}{20b^2(a+bx)^4} + \frac{k^2 \sin kx}{60b^3(a+bx)^3} + \frac{k^3 \cos kx}{120b^4(a+bx)^2} - \frac{k^4 \sin kx}{120b^5(a+bx)} + \frac{k^5}{120b^5} \int \frac{\cos kx}{a+bx} dx \quad (\text{see 2.6412}).$$

2.641

$$12. \int \frac{\cos kx}{(a+bx)^6} dx = -\frac{\cos kx}{5b(a+bx)^5} + \frac{k \sin kx}{20b^2(a+bx)^4} + \frac{k^2 \cos kx}{60b^3(a+bx)^3} - \frac{k^3 \sin kx}{120b^4(a+bx)^2} - \frac{k^4 \cos kx}{120b^5(a+bx)} - \frac{k^5}{120b^5} \int \frac{\sin kx}{a+bx} dx \quad (\text{see 2.6411}).$$

2.641

2.642

$$1. \int \frac{\sin^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \text{ci} [(2m-2k)x].$$

$$2. \int \frac{\sin^{2m+1} x}{x} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \text{si} [(2m-2k+1)x].$$

$$3. \int \frac{\cos^{2m} x}{x} dx = \binom{2m}{m} \frac{\ln x}{2^{2m}} + \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \text{ci} [(2m-2k)x].$$

$$4. \int \frac{\cos^{2m+1} x}{x} dx = \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \text{ci} [(2m-2k+1)x].$$

$$5. \int \frac{\sin^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} + \frac{(-1)^m}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\}.$$

$$6. \int \frac{\sin^{2m+1} x}{x^2} dx = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^{k+1} \binom{2m+1}{k} \times \left\{ \frac{\sin(2m-2k+1)x}{x} - (2m-2k+1) \operatorname{ci}[(2m-2k+1)x] \right\}.$$

$$7. \int \frac{\cos^{2m} x}{x^2} dx = - \binom{2m}{m} \frac{1}{2^{2m} x} - \frac{1}{2^{2m-1}} \sum_{k=0}^{m-1} \binom{2m}{k} \left\{ \frac{\cos(2m-2k)x}{x} + (2m-2k) \operatorname{si}[(2m-2k)x] \right\}.$$

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$$8. \int \frac{\cos^{2m+1} x}{x^2} dx = - \frac{1}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{k} \left\{ \frac{\cos(2m-2k+1)x}{x} + (2m-2k+1) \operatorname{si}[(2m-2k+1)x] \right\}.$$

2.643

$$1. \int \frac{x^p dx}{\sin^q x} = - \frac{x^{p-1} [p \sin x + (q-2)x \cos x]}{(q-1)(q-2) \sin^{q-1} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\sin^{q-2} x}.$$

$$2. \int \frac{x^p dx}{\cos^q x} = - \frac{x^{p-1} [p \cos x - (q-2)x \sin x]}{(q-1)(q-2) \cos^{q-1} x} + \frac{q-2}{q-1} \int \frac{x^p dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2} dx}{\cos^{q-2} x}.$$

$$3.^4 \int \frac{x^n}{\sin x} dx = \frac{x^n}{n} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1} - 1)}{(n+2k)(2k)!} B_{2k} x^{n+2k} \quad [|x| < \pi, \quad n > 0].$$

$$4. \int \frac{dx}{x^n \sin x} = -\frac{1}{nx^n} - [1 + (-1)^n](-1) \frac{n 2^{n-1} - 1}{2 n!} B_n \ln x - \sum_{\substack{k=1 \\ k \neq \frac{n}{2}}}^{\infty} (-1)^k \frac{2(2^{2k} - 1)}{(2k - n) \cdot (2k)!} B_{2k} x^{2k-n} \quad [n > 1, \quad |x| > \pi].$$

GU ((333))(9b)

$$5.8 \int \frac{x^n dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{n+2k+1}}{(n+2k+1)(2k)!} \quad \left[|x| < \frac{\pi}{2}, \quad n > 0 \right].$$

GU ((333))(10b)

$$6. \int \frac{dx}{x^n \cos x} = \frac{1}{2} [1 - (-1)^n] \frac{|E_{n-1}|}{(n-1)!} \ln x + \sum_{\substack{k=0 \\ k \neq \frac{n-1}{2}}}^{\infty} \frac{|E_{2k}| x^{2k-n+1}}{(2k-n+1) \cdot (2k)!} \quad \left[|x| < \frac{\pi}{2} \right].$$

GU ((333))(11b)

$$7. \int \frac{x^n dx}{\sin^2 x} = -x^n \operatorname{ctg} x + \frac{n}{n-1} x^{n-1} + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} x^{n+2k-1}}{(n+2k-1)(2k)!} B_{2k} \quad [|x| < \pi, \quad n > 1].$$

GU ((333))(8c)

$$8. \int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} - [1 - (-1)^n](-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} B_{n+1} \ln x - \frac{n}{2^{n+1}} \sum_{\substack{k=1 \\ k \neq \frac{n+1}{2}}}^{\infty} \frac{(-1)^k (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \quad [|x| < \pi].$$

GU ((333))(9c)

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$$9. \int \frac{x^n dx}{\cos^2 x} = x^n \operatorname{tg} x + n \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k} (2^{2k} - 1) x^{n+2k-1}}{(n+2k-1) \cdot (2k)!} B_{2k} \quad \left[n > 1, \quad |x| < \frac{\pi}{2} \right].$$

GU ((333))(10c)

$$10. \int \frac{dx}{x^n \cos^2 x} = \frac{\operatorname{tg} x}{x^n} - [1 - (-1)^n](-1)^{\frac{n+1}{2}} \frac{2^n n}{(n+1)!} (2^{n+1} - 1) B_{n+1} \ln x - \frac{n}{x^{n+1}} \sum_{k=1}^{\infty} \frac{(-1)^k (2^{2k} - 1) (2x)^{2k}}{(2k-n-1)(2k)!} B_{2k} \quad \left[|x| < \frac{\pi}{2} \right].$$

2.644

$$1. \int \frac{x dx}{\sin^{2n} x} = - \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{\sin x + (2n-2k)x \cos x}{(2n-2k+1)(2n-2k) \sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (\ln \sin x - x \operatorname{ctg} x).$$

$$2. \int \frac{x dx}{\sin^{2n+1} x} = - \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{\sin x + (2n-2k-1)x \cos x}{(2n-2k)(2n-2k-1) \sin^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\sin x} \quad (\text{see 2.6445}).$$

2.644

$$3. \int \frac{x dx}{\cos^{2n} x} = \sum_{k=0}^{n-1} \frac{(2n-2)(2n-4)\dots(2n-2k+2)}{(2n-1)(2n-3)\dots(2n-2k+3)} \frac{(2n-2k)x \sin x - \cos x}{(2n-2k+1)(2n-2k) \cos^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!} (x \operatorname{tg} x + \ln \cos x).$$

$$4. \int \frac{x dx}{\cos^{2n+1} x} = \sum_{k=0}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2n(2n-2)\dots(2n-2k+2)} \frac{(2n-2k+1)x \sin x - \cos x}{(2n-2k)(2n-2k-1) \cos^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!} \int \frac{x dx}{\cos x} \quad (\text{see 2.6446}).$$

2.644

$$5. \int \frac{x dx}{\sin x} = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2(2^{2k-1} - 1)}{(2k+1)!} B_{2k} x^{2k+1}.$$

$$6. \int \frac{x dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}| x^{2k+2}}{(2k+2)(2k)!}.$$

$$7. \int \frac{x dx}{\sin^2 x} = -x \operatorname{ctg} x + \ln \sin x.$$

$$8. \int \frac{x dx}{\cos^2 x} = x \operatorname{tg} x + \ln \cos x.$$

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$$9. \int \frac{x dx}{\sin^3 x} = -\frac{\sin x + x \cos x}{2 \sin^2 x} + \frac{1}{2} \int \frac{x}{\sin x} dx \quad (\text{see 2.6445.}).$$

2.644

$$10. \int \frac{x dx}{\cos^3 x} = \frac{x \sin x - \cos x}{2 \cos^2 x} + \frac{1}{2} \int \frac{x dx}{\cos x} \quad (\text{see 2.6446.}).$$

2.644

$$11. \int \frac{x dx}{\sin^4 x} = -\frac{x \cos x}{3 \sin^3 x} - \frac{1}{6 \sin^2 x} - \frac{2}{3} x \operatorname{ctg} x + \frac{2}{3} \ln(\sin x).$$

$$12. \int \frac{x dx}{\cos^4 x} = \frac{x \sin x}{3 \cos^3 x} - \frac{1}{6 \cos^2 x} + \frac{2}{3} x \operatorname{tg} x - \frac{2}{3} \ln(\cos x).$$

$$13. \int \frac{x dx}{\sin^5 x} = -\frac{x \cos x}{4 \sin^4 x} - \frac{1}{12 \sin^3 x} - \frac{3x \cos x}{8 \sin^2 x} - \frac{3}{8 \sin x} + \frac{3}{8} \int \frac{x dx}{\sin x} \quad (\text{see 2.6445.}).$$

2.644

$$14. \int \frac{x dx}{\cos^5 x} = \frac{x \sin x}{4 \cos^4 x} - \frac{1}{12 \cos^3 x} + \frac{3x \sin x}{8 \cos^2 x} - \frac{3}{8 \cos x} + \frac{3}{8} \int \frac{x dx}{\cos x} \quad (\text{see 2.6446.}).$$

2.644

2.645

$$1. \int x^p \frac{\sin^{2m} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\cos^{n-2k} x} \quad (\text{see 2.6432.}).$$

2.643

$$2. \int x^p \frac{\sin^{2m+1} x}{\cos^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \sin x}{\cos^{n-2k} x} dx \quad (\text{see 2.6453}).$$

2.645

$$3. \int x^p \frac{\sin x dx}{\cos^n x} = \frac{x^p}{(n-1) \cos^{n-1} x} - \frac{p}{n-1} \int \frac{x^{p-1}}{\cos^{n-1} x} dx$$

[$n > 1$] (see **2.643** 2.).

2.643

GU ((333))(12)

$$4. \int x^p \frac{\cos^{2m} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p dx}{\sin^{n-2k} x} \quad (\text{see 2.6431}).$$

2.643

$$5. \int x^p \frac{\cos^{2m+1} x}{\sin^n x} dx = \sum_{k=0}^m (-1)^k \binom{m}{k} \int \frac{x^p \cos x}{\sin^{n-2k} x} dx \quad (\text{see 2.6456}).$$

2.645

$$6. \int x^p \frac{\cos x}{\sin^n x} = -\frac{x^p}{(n-1) \sin^{n-1} x} + \frac{p}{n-1} \int \frac{x^{p-1} dx}{\sin^{n-1} x}$$

[$n > 1$] (see **2.643** 1.).

2.643

GU ((333))(13)

$$7. \int \frac{x \cos x}{\sin^2 x} dx = -\frac{x}{\sin x} + \ln \operatorname{tg} \frac{x}{2}.$$

$$8. \int \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x} - \ln \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

2.646

$$1. \int x^p \operatorname{tg} x dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k}(2^{2k-1} - 1)}{(p+2k) \cdot (2k)!} B_{2k} x^{p+2k} \quad \left[p \geq -1, \quad |x| < \frac{\pi}{2} \right].$$

GU ((333))(12d)

235

$$2. \int x^p \operatorname{ctg} x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(p+2k)(2k)!} x^{p+2k} \quad [p \geq 1, \quad |x| < \pi].$$

GU ((333))(13d)

$$3. \int x^p \operatorname{tg}^2 x dx = x \operatorname{tg} x + \ln \cos x - \frac{x^2}{2}.$$

$$4. \int x \operatorname{ctg}^2 x dx = -x \operatorname{ctg} x + \ln \sin x - \frac{x^2}{2}.$$

2.647

$$1. \int \frac{x^n \cos x dx}{(a+b \sin x)^m} = -\frac{x^n}{(m-1)b(a+b \sin x)^{m-1}} + \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \sin x)^{m-1}} \quad [m \neq 1].$$

MZ 247

$$2. \int \frac{x^n \sin x dx}{(a+b \cos x)^m} = \frac{x^n}{(m-1)b(a+b \cos x)^{m-1}} - \frac{n}{(m-1)b} \int \frac{x^{n-1} dx}{(a+b \cos x)^{m-1}} \quad [m \neq 1].$$

MZ 247

$$3. \int \frac{x dx}{1 + \sin x} = -x \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \cos \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

$$4. \int \frac{x dx}{1 - \sin x} = x \operatorname{ctg} \left(\frac{\pi}{4} - \frac{x}{2} \right) + 2 \ln \sin \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

PE (330)

$$5. \int \frac{x dx}{1 + \cos x} = x \operatorname{tg} \frac{x}{2} + 2 \ln \cos \frac{x}{2}.$$

PE (331)

$$6. \int \frac{x dx}{1 - \cos x} = -x \operatorname{ctg} \frac{x}{2} + 2 \ln \cos \frac{x}{2}.$$

PE (332)

$$7. \int \frac{x \cos x}{(1 + \sin x)^2} dx = -\frac{x}{1 + \sin x} + \operatorname{tg} \left(\frac{x}{2} - \frac{\pi}{4} \right).$$

$$8. \int \frac{x \cos x}{(1 - \sin x)^2} dx = \frac{x}{1 - \sin x} + \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right).$$

$$9. \int \frac{x \sin x}{(1 + \cos x)^2} dx = \frac{x}{1 + \cos x} - \operatorname{tg} \frac{x}{2}.$$

$$10. \int \frac{x \sin x}{(1 - \cos x)^2} dx = -\frac{x}{1 - \cos x} - \operatorname{ctg} \frac{x}{2}.$$

MZ 247a

2.648

$$1. \int \frac{x + \sin x}{1 + \cos x} dx = x \operatorname{tg} \frac{x}{2}.$$

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$$2. \int \frac{x - \sin x}{1 - \cos x} dx = -x \operatorname{ctg} \frac{x}{2}.$$

$$\int \frac{x^2 dx}{[(ax - b) \sin x + (a + bx) \cos x]^2} = \frac{x \sin x + \cos x}{b[(ax - b) \sin x + (a + bx) \cos x]}.$$

GU ((333))(17)

2.651

$$\int \frac{dx}{[a + (ax + b) \operatorname{tg} x]^2} = \frac{\operatorname{tg} x}{a[a + (ax + b) \operatorname{tg} x]}.$$

GU ((333))(18)

2.652

$$\int \frac{x dx}{\cos(x+t) \cos(x-t)} = \operatorname{cosec} 2t \left\{ x \ln \frac{\cos(x-t)}{\cos(x+t)} - L(x+t) + L(x-t) \right\}$$

$$\left[t \neq n\pi; \quad |x| < \left| \frac{\pi}{2} - |t_0| \right| \right],$$

where t_0 is the value of the argument t , which is reduced by multiples of the argument π to lie in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

2.653

$$1. \int \frac{\sin x}{\sqrt{x}} dx = \sqrt{2\pi} S(\sqrt{x}) \quad (\text{cf. 8.2511.}).$$

8.251
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$$2. \int \frac{\cos x}{\sqrt{x}} dx = \sqrt{2\pi} C(\sqrt{x}) \quad (\text{cf. 8.2512.}).$$

8.251

2.654

Notation: $\Delta = \sqrt{1 - k^2 \sin^2 x}$, $k' = \sqrt{1 - k^2}$:

$$1. \int \frac{x \sin x \cos x}{\Delta} dx = -\frac{x\Delta}{k^2} + \frac{1}{k^2} E(x, k).$$

$$2. \int \frac{x \sin^3 x \cos x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{2k^2 + 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(3 - \Delta^2)x + k^2 \sin x \cos x] \Delta.$$

$$3. \int \frac{x \sin x \cos^3 x}{\Delta} dx = -\frac{k'^2}{9k^4} F(x, k) + \frac{7k^2 - 5}{9k^4} E(x, k) - \frac{1}{9k^4} [3(\Delta^2 - 3k'^2)x - k^2 \sin x \cos x] \Delta.$$

$$4. \int \frac{x \sin x dx}{\Delta^3} = -\frac{x \cos x}{k'^2 \Delta} + \frac{1}{kk'^2} \arcsin(k \sin x).$$

$$5. \int \frac{x \cos x dx}{\Delta^3} = \frac{x \sin x}{\Delta} + \frac{1}{k} \ln(k \cos x + \Delta).$$

$$6. \int \frac{x \sin x \cos x dx}{\Delta^3} = \frac{x}{k^2 \Delta} - \frac{1}{k^2} F(x, k).$$

$$7. \int \frac{x \sin^3 x \cos x dx}{\Delta^3} = x \frac{2 - k^2 \sin^2 x}{k^4 \Delta} - \frac{1}{k^4} [E(x, k) + F(x, k)].$$

$$8. \int \frac{x \sin x \cos^3 x dx}{\Delta^3} = x \frac{k^2 \sin^2 x + k^2 - 2}{k^4 \Delta} + \frac{k'^2}{k^4} F(x, k) + \frac{1}{k^4} E(x, k).$$

Integrals containing $\sin x^2$ and $\cos x^2$

In integrals containing $\sin x^2$ and $\cos x^2$, it is expedient to make the substitution $x^2 = u$.

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2.655

$$1. \int x^p \sin x^2 dx = -\frac{x^{p-1}}{2} \cos x^2 + \frac{p-1}{2} \int x^{p-2} \cos x^2 dx.$$

$$2. \int x^p \cos x^2 dx = \frac{x^{p-1}}{2} \sin x^2 - \frac{p-1}{2} \int x^{p-2} \sin x^2 dx.$$

$$3. \int x^n \sin x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^k \left[\frac{x^{n-4k+3} \cos x^2}{2^{2k-1}(n-4k+3)!!} - \frac{x^{n-4k+1} \sin x^2}{2^{2k}(n-4k+1)!!} \right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \sin x^2 dx \right\} \quad \left[r = \left[\frac{n}{4} \right] \right].$$

GU ((336))(4a)

$$4. \int x^n \cos x^2 dx = (n-1)!! \left\{ \sum_{k=1}^r (-1)^{k-1} \left[\frac{x^{n-4k+3} \sin x^2}{2^{2k-1}(n-4k+3)!!} + \frac{x^{n-4k+1} \cos x^2}{2^{2k}(n-4k+1)!!} \right] + \frac{(-1)^r}{2^{2r}(n-4r-1)!!} \int x^{n-4r} \cos x^2 dx \right\} \quad \left[r = \left[\frac{n}{4} \right] \right].$$

GU ((336))(5a)

$$5. \int x \sin x^2 dx = -\frac{\cos x^2}{2}.$$

$$6. \int x \cos x^2 dx = \frac{\sin x^2}{2}.$$

$$7. \int x^2 \sin x^2 dx = -\frac{x}{2} \cos x^2 + \frac{1}{2} \sqrt{\frac{\pi}{2}} C(x).$$

$$8. \int x^2 \cos x^2 dx = \frac{x}{2} \sin x^2 - \frac{1}{2} \sqrt{\frac{\pi}{2}} S(x).$$

$$9. \int x^3 \sin x^2 dx = -\frac{x^2}{2} \cos x^2 + \frac{1}{2} \sin x^2.$$

$$10. \int x^3 \cos x^2 dx = \frac{x^2}{2} \sin x^2 + \frac{1}{2} \cos x^2.$$

2.66 Combinations of trigonometric functions and exponentials

2.661

$$\int e^{ax} \sin^p x \cos^q x dx = \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^p x \cos^q x [a \cos x + (p+q) \sin x] - \int e^{ax} \sin^{p-1} x \cos^{q+1} x dx - \int e^{ax} \sin^{p+1} x \cos^{q-1} x dx \right\}$$

$$\begin{aligned}
&= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^q x [a \sin x - (p+q) \cos x] + \right. \\
&\quad \left. + qa \int e^{ax} \sin^{p-1} x \cos^{q-1} x dx + (p-1)(p+q) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}; \\
&= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x [a \sin x \cos x + q \sin^2 x - p \cos^2 x] + \right. \\
&\quad \left. + q(q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx + p(p-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}; \\
&= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + \right. \\
&\quad \left. + q(q-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx - \right. \\
&\quad \left. - (q-p)(p+q-1) \int e^{ax} \sin^{p-2} x \cos^q x dx \right\}; \\
&= \frac{1}{a^2 + (p+q)^2} \left\{ e^{ax} \sin^{p-1} x \cos^{q-1} x (a \sin x \cos x + q \sin^2 x - p \cos^2 x) + m \right. \\
&\quad \left. + p(p-1) \int e^{ax} \sin^{p-2} x \cos^{q-2} x dx + \right. \\
&\quad \left. + (q-p)(p+q-1) \int e^{ax} \sin^p x \cos^{q-2} x dx \right\}.
\end{aligned}$$

GU ((334))(1a)

TI (526)

TI (525)

TI (524)

For $p = m$ and $q = n$ even integers, the integral $\int e^{ax} \sin^m x \cos^n x dx$ can be reduced by means of these formulas to the integral $\int e^{ax} dx$. However, when only m or only n is even, they can be reduced to integrals of the form $\int e^{ax} \cos^n x dx$ or $\int e^{ax} \sin^m x dx$ respectively.

2.662

$$1. \int e^{ax} \sin^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[(a \sin bx - nb \cos bx) e^{ax} \sin^{n-1} bx + \right. \\
\left. + n(n-1)b^2 \int e^{ax} \sin^{n-2} bx dx \right].$$

$$2. \int e^{ax} \cos^n bx dx = \frac{1}{a^2 + n^2 b^2} \left[(a \cos bx + nb \sin bx) e^{ax} \cos^{n-1} bx + \right. \\
\left. + n(n-1)b^2 \int e^{ax} \cos^{n-2} bx dx \right].$$

$$\begin{aligned}
3. \int e^{ax} \sin^{2m} bx \, dx &= \\
&= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \sin^{2m-2k-1} bx}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \dots [a^2 + (2m-2k)^2 b^2]} \times \\
&\quad \times [a \sin bx - (2m-2k)b \cos bx] + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \dots [a^2 + 4b^2] a} = \\
&= \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m (-1)^k \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} (a \cos 2kx + 2kb \sin 2kx).
\end{aligned}$$

$$\begin{aligned}
4. \int e^{ax} \sin^{2m+1} bx \, dx &= \\
&= \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \sin^{2m-2k} bx [a \sin bx - (2m-2k+1)b \cos bx]}{(2m-2k+1)! [a^2 + (2m+1)^2 b^2] [a^2 + (2m-1)^2 b^2] \dots [a^2 + (2m-2k+1)^2 b^2]} = \\
&= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \frac{(-1)^k}{a^2 + (2k+1)^2 b^2} \binom{2m+1}{m-k} [a \sin(2k+1)bx - (2k+1)b \cos(2k+1)bx].
\end{aligned}$$

$$\begin{aligned}
5.8 \int e^{ax} \cos^{2m} bx \, dx &= \\
&= \sum_{k=0}^{m-1} \frac{(2m)! b^{2k} e^{ax} \cos^{2m-2k-1} bx [a \cos bx + (2m-2k)b \sin bx]}{(2m-2k)! [a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \dots [a^2 + (2m-2k)^2 b^2]} + \\
&\quad + \frac{(2m)! b^{2m} e^{ax}}{[a^2 + (2m)^2 b^2] [a^2 + (2m-2)^2 b^2] \dots [a^2 + 4b^2] a} = \\
&= \binom{2m}{m} \frac{e^{ax}}{2^{2m} a} + \frac{e^{ax}}{2^{2m-1}} \sum_{k=1}^m \binom{2m}{m-k} \frac{1}{a^2 + 4b^2 k^2} [a \cos 2kx + 2kb \sin 2kx].
\end{aligned}$$

$$\begin{aligned}
6. \int e^{ax} \cos^{2m+1} bx \, dx &= \\
&= \sum_{k=0}^m \frac{(2m+1)! b^{2k} e^{ax} \cos^{2m-2k} bx}{(2m-2k+1)! [a^2 + (2m-1)^2 b^2] \dots [a^2 + (2m-2k+1)^2 b^2]} = \\
&= \frac{e^{ax}}{2^{2m}} \sum_{k=0}^m \binom{2m+1}{m-k} \frac{1}{a^2 + (2k+1)^2 b^2} [a \cos(2k+1)bx + (2k+1)b \sin(2k+1)bx].
\end{aligned}$$

2.663

$$1. \int e^{ax} \sin bx \, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}.$$

2.664

$$1. \int e^{ax} \sin bx \cos cx \, dx = \frac{e^{ax}}{2} \left[\frac{a \sin(b+c)x - (b+c) \cos(b+c)x}{a^2 + (b+c)^2} + \frac{a \sin(b-c)x - (b-c) \cos(b-c)x}{a^2 + (b-c)^2} \right].$$

GU ((334))(6b)

$$2. \int e^{ax} \sin^2 bx \cos cx \, dx = \frac{e^{ax}}{4} \left[2 \frac{a \cos cx + c \sin cx}{a^2 + c^2} - \frac{a \cos(2b+c)x + (2b+c) \sin(2b+c)x}{a^2 + (2b+c)^2} - \frac{a \cos(2b-c)x + (2b-c) \sin(2b-c)x}{a^2 + (2b-c)^2} \right].$$

GU ((334))(6c)

$$3. \int e^{ax} \sin bx \cos^2 cx \, dx = \frac{e^{ax}}{4} \left[2 \frac{a \sin bx - b \cos bx}{a^2 + b^2} + \frac{a \sin(b+2c)x - (b+2c) \cos(b+2c)x}{a^2 + (b+2c)^2} + \frac{a \sin(b-2c)x - (b-2c) \cos(b-2c)x}{a^2 + (b-2c)^2} \right].$$

GU ((334))(6d)

2.665

$$1. \int \frac{e^{ax} \, dx}{\sin^p bx} = -\frac{e^{ax}[a \sin bx + (p-2)b \cos bx]}{(p-1)(p-2)b^2 \sin^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\sin^{p-2} bx}.$$

TI (530)a

$$2. \int \frac{e^{ax} \, dx}{\cos^p bx} = -\frac{e^{ax}[a \cos bx - (p-2)b \sin bx]}{(p-1)(p-2)b^2 \cos^{p-1} bx} + \frac{a^2 + (p-2)^2 b^2}{(p-1)(p-2)b^2} \int \frac{e^{ax} \, dx}{\cos^{p-2} bx}.$$

TI (529)a

By successive applications of formulas 2.665 for p a natural number, we obtain integrals of the form

$\int \frac{e^{ax} \, dx}{\sin bx}$, $\int \frac{e^{ax} \, dx}{\sin^2 bx}$, $\int \frac{e^{ax} \, dx}{\cos bx}$, $\int \frac{e^{ax} \, dx}{\cos^2 bx}$, which are not expressible in terms of a finite combination of elementary functions.

2.666

$$1. \int e^{ax} \operatorname{tg}^p x \, dx = \frac{e^{ax}}{p-1} \operatorname{tg}^{p-1} x - \frac{a}{p-1} \int e^{ax} \operatorname{tg}^{p-1} x \, dx - \int e^{ax} \operatorname{tg}^{p-2} x \, dx.$$

TI (527)

$$2. \int e^{ax} \operatorname{ctg}^p x \, dx = -\frac{e^{ax} \operatorname{ctg}^{p-1} x}{p-1} + \frac{a}{p-1} \int e^{ax} \operatorname{ctg}^{p-1} x \, dx - \int e^{ax} \operatorname{ctg}^{p-2} x \, dx.$$

TI (528)

$$3. \int e^{ax} \operatorname{tg} x \, dx = \frac{e^{ax} \operatorname{tg} x}{a} - \frac{1}{a} \int \frac{e^{ax} \, dx}{\cos^2 x} \quad (\text{see remark following 2.665}).$$

2.665

$$4. \int e^{ax} \operatorname{tg}^2 x \, dx = \frac{e^{ax}}{a} (a \operatorname{tg} x - 1) - a \int e^{ax} \operatorname{tg} x \, dx \quad (\text{see 2.6663}).$$

2.666
TI 355

$$5. \int e^{ax} \operatorname{ctg} x \, dx = \frac{e^{ax} \operatorname{ctg} x}{a} + \frac{1}{a} \int \frac{e^{ax} \, dx}{\sin^2 x} \quad (\text{see remark following 2.665}).$$

2.665

$$6. \int e^{ax} \operatorname{ctg}^2 x \, dx = -\frac{e^{ax}}{a} (a \operatorname{ctg} x + 1) + a \int e^{ax} \operatorname{ctg} x \, dx \quad (\text{see 2.6665}).$$

2.666

Integrals of type $\int R(x, e^{ax}, \sin bx, \cos cx) \, dx$

$$\text{Notation: } \sin t = -\frac{b}{\sqrt{a^2+b^2}}; \quad \cos t = \frac{a}{\sqrt{a^2+b^2}}.$$

$$\sin t = -\frac{b}{\sqrt{a^2+b^2}}; \quad \cos t = \frac{a}{\sqrt{a^2+b^2}}.$$

2.667

$$\begin{aligned} 1. \quad \int x^p e^{ax} \sin bx \, dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \sin bx - b \cos bx) \, dx; \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx+t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \sin(bx+t) \, dx. \end{aligned}$$

$$\begin{aligned} 2. \quad \int x^p e^{ax} \cos bx \, dx &= \frac{x^p e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) - \frac{p}{a^2 + b^2} \int x^{p-1} e^{ax} (a \cos bx + b \sin bx) \, dx; \\ &= \frac{x^p e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx+t) - \frac{p}{\sqrt{a^2 + b^2}} \int x^{p-1} e^{ax} \cos(bx+t) \, dx. \end{aligned}$$

$$3. \quad \int x^n e^{ax} \sin bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \sin(bx + kt).$$

$$4. \quad \int x^n e^{ax} \cos bx \, dx = e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} n! x^{n-k+1}}{(n-k+1)! (a^2 + b^2)^{k/2}} \cos(bx + kt).$$

$$5. \quad \int x e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \sin bx - \left(bx - \frac{2ab}{a^2 + b^2} \right) \cos bx \right].$$

$$6. \quad \int x e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[\left(ax - \frac{a^2 - b^2}{a^2 + b^2} \right) \cos bx + \left(bx - \frac{2ab}{a^2 + b^2} \right) \sin bx \right].$$

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$$\begin{aligned} 7. \quad \int x^2 e^{ax} \sin bx \, dx &= \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \sin bx - \right. \\ &\quad \left. - \left[bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \cos bx \right\}. \end{aligned}$$

$$\begin{aligned} 8. \quad \int x^2 e^{ax} \cos bx \, dx &= \frac{e^{ax}}{a^2 + b^2} \left\{ \left[ax^2 - \frac{2(a^2 - b^2)}{a^2 + b^2} x + \frac{2a(a^2 - 3b^2)}{(a^2 + b^2)^2} \right] \cos bx + \right. \\ &\quad \left. + \left[bx^2 - \frac{4ab}{a^2 + b^2} x + \frac{2b(3a^2 - b^2)}{(a^2 + b^2)^2} \right] \sin bx \right\}. \end{aligned}$$

2.67 Combinations of trigonometric and hyperbolic functions

2.671

$$1. \int \operatorname{sh}(ax + b) \sin(cx + d) dx = \frac{a}{a^2 + c^2} \operatorname{ch}(ax + b) \sin(cx + d) - \frac{c}{a^2 + c^2} \operatorname{sh}(ax + b) \cos(cx + d).$$

$$2. \int \operatorname{sh}(ax + b) \cos(cx + d) dx = \frac{a}{a^2 + c^2} \operatorname{ch}(ax + b) \cos(cx + d) + \frac{c}{a^2 + c^2} \operatorname{sh}(ax + b) \sin(cx + d).$$

$$3. \int \operatorname{ch}(ax + b) \sin(cx + d) dx = \frac{a}{a^2 + c^2} \operatorname{sh}(ax + b) \sin(cx + d) - \frac{c}{a^2 + c^2} \operatorname{ch}(ax + b) \cos(cx + d).$$

$$4. \int \operatorname{ch}(ax + b) \cos(cx + d) dx = \frac{a}{a^2 + c^2} \operatorname{sh}(ax + b) \cos(cx + d) + \frac{c}{a^2 + c^2} \operatorname{ch}(ax + b) \sin(cx + d).$$

GU ((354))(1)

2.672

$$1. \int \operatorname{sh} x \sin x dx = \frac{1}{2}(\operatorname{ch} x \sin x - \operatorname{sh} x \cos x).$$

$$2. \int \operatorname{sh} x \cos x dx = \frac{1}{2}(\operatorname{ch} x \cos x + \operatorname{sh} x \sin x).$$

$$3. \int \operatorname{ch} x \sin x dx = \frac{1}{2}(\operatorname{sh} x \sin x - \operatorname{ch} x \cos x).$$

$$4. \int \operatorname{ch} x \cos x dx = \frac{1}{2}(\operatorname{sh} x \cos x + \operatorname{ch} x \sin x).$$

$$\begin{aligned}
1. \quad & \int \operatorname{sh}^{2m}(ax+b) \sin^{2n}(cx+d) dx = \\
& = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{(-1)^{m+n}}{2^{2m+2n-1}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} \binom{2n}{k} \sin[(2n-2k)(cx+d)] + \\
& \quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]\}.
\end{aligned}$$

GU ((354))(3a)

$$\begin{aligned}
2. \quad & \int \operatorname{sh}^{2m}(ax+b) \sin^{2n-1}(cx+d) dx = \\
& = \frac{(-1)^{m+n}}{2^{2m+2n-2}} \binom{2m}{m} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k-1)c} \binom{2n-1}{k} \cos[(2n-2k-1)(cx+d)] + \\
& \quad + \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] - \\
& \quad - (2n-2k-1)c \operatorname{ch}[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)]\}.
\end{aligned}$$

GU ((354))(3b)

$$\begin{aligned}
3. \quad & \int \operatorname{sh}^{2m-1}(ax+b) \sin^{2n}(cx+d) dx = \\
& = \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \operatorname{ch}[(2m-2j-1)(ax+d)] + \\
& \quad + \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{(2m-2j-1)a \operatorname{ch}[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2n-2k)c \operatorname{sh}[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]\}.
\end{aligned}$$

GU ((354))(3c)

$$\begin{aligned}
4. \quad & \int \operatorname{sh}^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx = \\
& = \frac{(-1)^{n-1}}{2^{2m-2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{(2m-2j-1)a \operatorname{ch}[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] - \\
& \quad - (2n-2k-1)c \operatorname{sh}[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]\}.
\end{aligned}$$

$$\begin{aligned}
5. \quad & \int \operatorname{sh}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \\
& = \frac{(-1)^m}{2^{2m+2n}} \binom{2m}{m} \binom{2n}{n} x + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m}{j}}{(2m-2j)a} \operatorname{sh}[(2m-2j)(ax+b)] + \\
& \quad + \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] + \\
& \quad + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
\end{aligned}$$

GU ((354))(4a)

$$\begin{aligned}
6. \quad & \int \operatorname{sh}^{2m}(ax+b) \cos^{2n-1}(cx+d) dx = \\
& = \frac{(-1)^m \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-2)(cx+d)] + \\
& \quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] + \\
& \quad + (2n-2k-1)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] \}.
\end{aligned}$$

GU ((354))(4a)

$$\begin{aligned}
7. \quad & \int \operatorname{sh}^{2m-1}(ax+b) \cos^{2n}(cx+d) dx = \\
& = \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{(-1)^j \binom{2m-1}{j}}{(2m-2j-1)a} \operatorname{ch}[(2m-2j-1)(ax+d)] + \\
& \quad + \frac{1}{2^{2m-2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{ (2m-2j-1)a \operatorname{ch}[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2n-2k)c \operatorname{sh}[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
\end{aligned}$$

GU ((354))(4b)

$$8. \quad \int \operatorname{sh}^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx =$$

$$\begin{aligned}
&= \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^j \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
&\quad \times \{ (2m-2j-1)a \operatorname{ch}[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] + \\
&\quad + (2n-2k-1)c \operatorname{sh}[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \}.
\end{aligned}$$

GU ((354))(4b)

$$\begin{aligned}
9. \quad &\int \operatorname{ch}^{2m}(ax+b) \sin^{2n}(cx+d) dx = \\
&= \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{(-1)^n \binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{m-1} \frac{(-1)^k \binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] + \\
&\quad + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh}[(2m-2j)(ax+b)] + \\
&\quad + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
&\quad \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
&\quad + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
\end{aligned}$$

GU ((354))(5a)

$$\begin{aligned}
10. \quad &\int \operatorname{ch}^{2m-1}(ax+b) \sin^{2n}(cx+d) dx = \\
&= \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \operatorname{sh}[(2m-2j-1)(ax+b)] + \\
&\quad + \frac{(-1)^n}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \times \\
&\quad \times \{ (2m-2j-1)a \operatorname{sh}[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
&\quad + (2n-2k)c \operatorname{ch}[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)] \}.
\end{aligned}$$

GU ((354))(5a)

$$\begin{aligned}
11. \quad &\int \operatorname{ch}^{2m}(ax+b) \sin^{2n-1}(cx+d) dx = \\
&= \frac{(-1)^{n-1} \binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{(-1)^{k+1} \binom{2n-1}{k}}{(2n-2k-1)c} \cos[(2n-2k-1)(cx+d)] + \\
&\quad + \frac{(-1)^{n-1}}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
&\quad \times \{ (2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)] - \\
&\quad - (2n-2k-1)c \operatorname{ch}[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] \}.
\end{aligned}$$

$$\begin{aligned}
12. \quad & \int \operatorname{ch}^{2m-1}(ax+b) \sin^{2n-1}(cx+d) dx = \\
& = \frac{(-1)^{n-1}}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^k \binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{(2m-2j-1)a \operatorname{sh}[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] - \\
& \quad - (2n-2k-1)c \operatorname{ch}[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)]\}.
\end{aligned}$$

GU ((354))(5b)

$$\begin{aligned}
13. \quad & \int \operatorname{ch}^{2m}(ax+b) \cos^{2n}(cx+d) dx = \\
& = \frac{\binom{2m}{m} \binom{2n}{n}}{2^{2m+2n}} x + \frac{\binom{2m}{m}}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{k}}{(2n-2k)c} \sin[(2n-2k)(cx+d)] + \\
& \quad + \frac{\binom{2n}{n}}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{\binom{2m}{j}}{(2m-2j)a} \operatorname{sh}[(2m-2j)(ax+b)] + \\
& \quad + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n}{k}}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]\}.
\end{aligned}$$

GU ((354))(6)

$$\begin{aligned}
14. \quad & \int \operatorname{ch}^{2m-1}(ax+b) \cos^{2n}(cx+d) dx = \\
& = \frac{\binom{2n}{n}}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \frac{\binom{2m-1}{j}}{(2m-2j-1)a} \operatorname{sh}[(2m-2j-1)(ax+b)] + \\
& \quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n}{k}}{(2m-2j-1)^2 a^2 + (2n-2k)^2 c^2} \times \\
& \quad \times \{(2m-2j-1)a \operatorname{sh}[(2m-2j-1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\
& \quad + (2n-2k)c \operatorname{ch}[(2m-2j-1)(ax+b)] \sin[(2n-2k)(cx+d)]\}.
\end{aligned}$$

GU ((354))(6)

$$\begin{aligned}
15. \quad & \int \operatorname{ch}^{2m}(ax+b) \cos^{2n-1}(cx+d) dx = \\
& = \frac{\binom{2m}{m}}{2^{2m+2n-2}} \sum_{k=0}^{n-1} \frac{\binom{2n-1}{k}}{(2n-2k-1)c} \sin[(2n-2k-1)(cx+d)] + \\
& \quad + \frac{1}{2^{2m+2n-3}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m}{j} \binom{2n-1}{k}}{(2m-2j)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k-1)(cx+d)] + \\
& \quad + (2n-2k-1)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k-1)(cx+d)]\}.
\end{aligned}$$

$$\begin{aligned}
16. \quad & \int \operatorname{ch}^{2m-1}(ax+b) \cos^{2n-1}(cx+d) dx = \\
& = \frac{1}{2^{2m+2n-4}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{\binom{2m-1}{j} \binom{2n-1}{k}}{(2m-2j-1)^2 a^2 + (2n-2k-1)^2 c^2} \times \\
& \quad \times \{ (2m-2j-1)a \operatorname{sh}[(2m-2j-1)(ax+b)] \cos[(2n-2k-1)(cx+d)] + \\
& \quad + (2n-2k-1)c \operatorname{ch}[(2m-2j-1)(ax+b)] \sin[(2n-2k-1)(cx+d)] \}.
\end{aligned}$$

GU ((354))(6)

2.674

$$\begin{aligned}
1. \quad \int e^{ax} \operatorname{sh} bx \sin cx dx &= \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] - \\
& - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx].
\end{aligned}$$

$$\begin{aligned}
2. \quad \int e^{ax} \operatorname{sh} bx \cos cx dx &= \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] - \\
& - \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx].
\end{aligned}$$

$$\begin{aligned}
3. \quad \int e^{ax} \operatorname{ch} bx \sin cx dx &= \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \sin cx - c \cos cx] + \\
& + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \sin cx - c \cos cx].
\end{aligned}$$

$$\begin{aligned}
4. \quad \int e^{ax} \operatorname{ch} bx \cos cx dx &= \frac{e^{(a+b)x}}{2[(a+b)^2 + c^2]} [(a+b) \cos cx + c \sin cx] + \\
& + \frac{e^{(a-b)x}}{2[(a-b)^2 + c^2]} [(a-b) \cos cx + c \sin cx].
\end{aligned}$$

MZ 379

2.7 Logarithms and Inverse-Hyperbolic Functions

2.71 The logarithm

2.711

$$\begin{aligned}
\int \ln^m x dx &= x \ln^m x - m \int \ln^{m-1} x dx = \\
& = \frac{x}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m+1) \dots (m-k+1) \ln^{m-k} x \quad (m > 0).
\end{aligned}$$

2.72-2.73 Combinations of logarithms and algebraic functions

2.721

$$1. \int x^n \ln^m x \, dx = \frac{x^{n+1} \ln^m x}{n+1} - \frac{m}{n+1} \int x^n \ln^{m-1} x \, dx \quad (\text{see 2.722}).$$

2.722

For $n = -1$

$$2. \int \frac{\ln^m x \, dx}{x} = \frac{\ln^{m+1} x}{m+1}.$$

For $n = -1$ and $m = -1$

$$3. \int \frac{dx}{x \ln x} = \ln(\ln x).$$

2.722

$$\int x^n \ln^m x \, dx = \frac{x^{n+1}}{m+1} \sum_{k=0}^m (-1)^k (m+1)m(m-1)\dots(m-k+1) \frac{\ln^{m-k} x}{(n+1)^{k+1}}.$$

TI (604)

2.723

$$1. \int x^n \ln x \, dx = x^{n+1} \left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right].$$

TI 375

$$2. \int x^n \ln^2 x \, dx = x^{n+1} \left[\frac{\ln^2 x}{n+1} - \frac{2 \ln x}{(n+1)^2} + \frac{2}{(n+1)^3} \right].$$

TI 375

2.724

$$1. \int \frac{x^n dx}{(\ln x)^m} = -\frac{x^{n+1}}{(m-1)(\ln x)^{m-1}} + \frac{n+1}{m-1} \int \frac{x^n dx}{(\ln x)^{m-1}}.$$

For $m = 1$

$$2. \int \frac{x^n dx}{\ln x} = \text{li}(x^{n+1}).$$

2.725

$$1. \int (a+bx)^m \ln x dx = \frac{1}{(m+1)b} \left[(a+bx)^{m+1} \ln x - \int \frac{(a+bx)^{m+1} dx}{x} \right].$$

TI 374

$$2. \int (a+bx)^m \ln x dx = \frac{1}{(m+1)b} [(a+bx)^{m+1} - a^{m+1}] \ln x - \sum_{k=0}^m \frac{\binom{m}{k} a^{m-k} b^k x^{k+1}}{(k+1)^2}.$$

For $m = -1$ see 2.727 2.

249

2.726

$$1. \int (a+bx) \ln x dx = \left[\frac{(a+bx)^2}{2b} - \frac{a^2}{2b} \right] \ln x - \left(ax + \frac{1}{4}bx^2 \right).$$

$$2. \int (a+bx)^2 \ln x dx = \frac{1}{3b} [(a+bx)^3 - a^3] \ln x - \left(a^2x + \frac{abx^2}{2} + \frac{b^2x^3}{9} \right).$$

$$3. \int (a+bx)^3 \ln x dx = \frac{1}{4b} [(a+bx)^4 - a^4] \ln x - \left(a^3x + \frac{3}{4}a^2bx^2 + \frac{1}{3}ab^2x^3 + \frac{1}{16}b^3x^4 \right).$$

2.727

$$1. \int \frac{1 x dx}{(a + bx)^m} = \frac{1}{b(m-1)} \left[-\frac{\ln x}{(a + bx)^{m-1}} + \int \frac{dx}{x(a + bx)^{m-1}} \right].$$

TI 376

For $m = 1$

$$2. \int \frac{\ln x dx}{a + bx} = \frac{1}{b} \ln x \ln(a + bx) - \frac{1}{b} \int \frac{\ln(a + bx) dx}{x} \quad (\text{see 2.7282}).$$

2.728

$$3. \int \frac{\ln x dx}{(a + bx)^2} = -\frac{\ln x}{b(a + bx)} + \frac{1}{ab} \ln \frac{x}{a + bx}.$$

$$4. \int \frac{\ln x dx}{(a + bx)^3} = -\frac{\ln x}{2b(a + bx)^2} + \frac{1}{2ab(a + bx)} + \frac{1}{2a^2b} \ln \frac{x}{a + bx}.$$

$$5. \int \frac{\ln x dx}{\sqrt{a + bx}} = \frac{2}{b} \left\{ (\ln x - 2)\sqrt{a + bx} + \sqrt{a} \ln \frac{\sqrt{a + bx} + \sqrt{a}}{\sqrt{a + bx} - \sqrt{a}} \right\} \quad [a > 0];$$

$$= \frac{2}{b} \left\{ (\ln x - 2)\sqrt{a + bx} + 2\sqrt{-a} \operatorname{arctg} \sqrt{\frac{a + bx}{-a}} \right\} \quad [a < 0].$$

2.728

$$1. \int x^m \ln(a + bx) dx = \frac{1}{m+1} \left[x^{m+1} \ln(a + bx) - b \int \frac{x^{m+1} dx}{a + bx} \right].$$

$$2.^6 \int \frac{\ln(a + bx)}{x} = \ln a + \ln x + \frac{bx}{a} \Phi \left(-\frac{bx}{a}, 2, 1 \right) \quad [a > 0].$$

2.729

$$1. \int x^m \ln(a + bx) dx = \frac{1}{m+1} \left[x^{m+1} - \frac{(-a)^{m+1}}{b^{m+1}} \right] \ln(a + bx) +$$

$$+ \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} a^{k-1}}{(m-k+2)b^k}.$$

$$2. \int x \ln(a + bx) dx = \frac{1}{2} \left[x^2 - \frac{a^2}{b^2} \right] \ln(a + bx) - \frac{1}{2} \left[\frac{x^2}{2} - \frac{ax}{b} \right].$$

$$3. \int x^2 \ln(a + bx) dx = \frac{1}{3} \left[x^3 + \frac{a^3}{b^3} \right] \ln(a + bx) - \frac{1}{3} \left[\frac{x^3}{3} - \frac{ax^2}{2b} + \frac{a^2x}{b^2} \right].$$

$$4. \int x^3 \ln(a + bx) dx = \frac{1}{4} \left[x^4 - \frac{a^4}{b^4} \right] \ln(a + bx) - \frac{1}{4} \left[\frac{x^4}{4} - \frac{ax^3}{3b} + \frac{a^2x^2}{2b^2} - \frac{a^3x}{b^3} \right].$$

2.731

$$\frac{1}{2n+1} \left\{ x^{2n+1} \ln(x^2 + a^2) + (-1)^n 2a^{2n+1} \operatorname{arctg} \frac{x}{a} - 2 \sum_{k=0}^n \frac{(-1)^{n-k}}{2k+1} a^{2n-2k} x^{2k+1} \right\}.$$

2.732

$$\frac{1}{2n+1} \left\{ (x^{2n+2} + (-1)^n a^{2n+2}) \ln(x^2 + a^2) + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} a^{2n-2k+2} x^{2k} \right\}.$$

2.733

$$1. \int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) - 2x + 2a \operatorname{arctg} \frac{x}{a}.$$

DW

$$2. \int x \ln(x^2 + a^2) dx = \frac{1}{2} [(x^2 + a^2) \ln(x^2 + a^2) - x^2].$$

DW

$$3. \int x^2 \ln(x^2 + a^2) dx = \frac{1}{3} \left[x^3 \ln(x^2 + a^2) - \frac{2}{3} x^3 + 2a^2 x - 2a^3 \operatorname{arctg} \frac{x}{a} \right].$$

$$4. \int x^3 \ln(x^2 + a^2) dx = \frac{1}{4} \left[(x^4 - a^4) \ln(x^2 + a^2) - \frac{x^4}{2} + a^2 x^2 \right].$$

DW

$$5. \int x^4 \ln(x^2 + a^2) dx = \frac{1}{5} \left[x^5 \ln(x^2 + a^2) - \frac{2}{5} x^5 + \frac{2}{3} a^2 x^3 - 2a^4 x + 2a^5 \operatorname{arctg} \frac{x}{a} \right].$$

DW

2.734

$$\int x^{2n} \ln|x^2 - a^2| dx = \frac{1}{2n+1} \left\{ x^{2n+1} \ln|x^2 - a^2| + a^{2n+1} \ln \left| \frac{x+a}{x-a} \right| - 2 \sum_{k=0}^n \frac{1}{2k+1} a^{2n-2k} x^{2k+1} \right\}.$$

2.735

$$\int x^{2n+1} \ln|x^2 - a^2| dx = \frac{1}{2n+2} \left\{ (x^{2n+2} - a^{2n+2}) \ln|x^2 - a^2| - \sum_{k=1}^{n+1} \frac{1}{k} a^{2n-2k+2} x^{2k} \right\}.$$

2.736

$$1. \int \ln|x^2 - a^2| dx = x \ln|x^2 - a^2| - 2x + a \ln \left| \frac{x+a}{x-a} \right|.$$

DW

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$$2. \int x \ln|x^2 - a^2| dx = \frac{1}{2} \{ (x^2 - a^2) \ln|x^2 - a^2| - x^2 \}.$$

DW

$$3. \int x^2 \ln|x^2 - a^2| dx = \frac{1}{3} \left\{ x^3 \ln|x^2 - a^2| - \frac{2}{3} x^3 - 2a^2 x + a^3 \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

DW

$$4. \int x^3 \ln|x^2 - a^2| dx = \frac{1}{4} \left\{ (x^4 - a^4) \ln|x^2 - a^2| - \frac{x^4}{2} - a^2 x^2 \right\}.$$

$$5. \int x^4 \ln|x^2 - a^2| dx = \frac{1}{5} \left\{ x^5 \ln|x^2 - a^2| - \frac{2}{5}x^5 - \frac{2}{3}a^2x^3 - 2a^4x + a^5 \ln \left| \frac{x+a}{x-a} \right| \right\}.$$

DW

2.74 Inverse hyperbolic functions

2.741

$$1. \int \operatorname{Arsh} \frac{x}{a} dx = x \operatorname{Arsh} \frac{x}{a} - \sqrt{x^2 + a^2}.$$

DW

$$2. \int \operatorname{Arch} \frac{x}{a} dx = x \operatorname{Arch} \frac{x}{a} - \sqrt{x^2 - a^2}. \quad \left[\operatorname{Arch} \frac{x}{a} > 0 \right];$$

$$= x \operatorname{Arch} \frac{x}{a} + \sqrt{x^2 - a^2}. \quad \left[\operatorname{Arch} \frac{x}{a} < 0 \right].$$

DW
DW

$$3. \int \operatorname{Arth} \frac{x}{a} dx = x \operatorname{Arth} \frac{x}{a} + \frac{a}{2} \ln(a^2 - x^2).$$

DW

$$4. \int \operatorname{Arcth} \frac{x}{a} dx = x \operatorname{Arcth} \frac{x}{a} + \frac{a}{2} \ln(x^2 - a^2).$$

DW

2.742

$$1. \int x \operatorname{Arsh} \frac{x}{a} dx = \left(\frac{x^2}{2} + \frac{a^2}{4} \right) \operatorname{Arsh} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 + a^2}.$$

DW

$$2. \int x \operatorname{Arch} \frac{x}{a} dx = \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{Arch} \frac{x}{a} - \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} > 0 \right];$$

$$= \left(\frac{x^2}{2} - \frac{a^2}{4} \right) \operatorname{Arch} \frac{x}{a} + \frac{x}{4} \sqrt{x^2 - a^2} \quad \left[\operatorname{Arch} \frac{x}{a} < 0 \right].$$

DW

2.8 Inverse Trigonometric Functions

2.81 Arcsines and arccosines

2.811

$$\int \left(\arcsin \frac{x}{a}\right)^n dx = x \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arcsin \frac{x}{a}\right)^{n-2k} + \sqrt{a^2 - x^2} \sum_{k=1}^{[(n+1)/2]} (-1)^{k-1} \binom{n}{2k-1} \cdot (2k-1)! \left(\arcsin \frac{x}{a}\right)^{n-2k+1}.$$

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2.812

$$\int \left(\arccos \frac{x}{a}\right)^n dx = x \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot (2k)! \left(\arccos \frac{x}{a}\right)^{n-2k} + \sqrt{a^2 - x^2} \sum_{k=1}^{[(n+1)/2]} (-1)^k \binom{n}{2k-1} \cdot (2k-1)! \left(\arccos \frac{x}{a}\right)^{n-2k+1}.$$

2.813

$$1.* \int \arcsin \frac{x}{a} dx = \text{sign}(a) \left[n \arcsin \frac{x}{|a|} + \sqrt{a^2 - x^2} \right].$$

$$2.* \int \left(\arcsin \frac{x}{a}\right)^2 dx = x \left(\arcsin \frac{x}{|a|}\right)^2 + 2\sqrt{a^2 - x^2} \arcsin \frac{x}{|a|} - 2x.$$

$$3.* \int \left(\arcsin \frac{x}{a}\right)^3 dx = \text{sign}(a) \left[x \left(\arcsin \frac{x}{|a|}\right)^3 + 3\sqrt{a^2 - x^2} \left(\arcsin \frac{x}{|a|}\right)^2 - 6x \arcsin \frac{x}{|a|} - 6\sqrt{a^2 - x^2} \right].$$

2.814

$$1. \int \arccos \frac{x}{a} dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2}.$$

$$2. \int \left(\arccos \frac{x}{a}\right)^2 dx = x \left(\arccos \frac{x}{a}\right)^2 - 2\sqrt{a^2 - x^2} \arccos \frac{x}{a} - 2x.$$

$$3. \int \left(\arccos \frac{x}{a}\right)^3 dx = x \left(\arccos \frac{x}{a}\right)^3 - 3\sqrt{a^2 - x^2} \left(\arccos \frac{x}{a}\right)^2 - 6x \arccos \frac{x}{a} + 6\sqrt{a^2 - x^2}.$$

2.82 The arcsecant, the arccosecant, the arctangent, and the arccotangent

2.821

$$1. \int \operatorname{arccosec} \frac{x}{a} dx = \int \arcsin \frac{a}{x} dx = x \arcsin \frac{x}{2} + a \ln(x + \sqrt{x^2 - a^2}) \quad \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2}\right];$$

$$= x \arcsin \frac{a}{x} - a \ln(x + \sqrt{x^2 - a^2}) \quad \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0\right].$$

DW

$$2. \int \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = x \arccos \frac{a}{x} - a \ln(x + \sqrt{x^2 - a^2}) \quad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2}\right];$$

$$= x \arccos \frac{a}{x} - a \ln(x + \sqrt{x^2 - a^2}) \quad \left[-\frac{\pi}{2} < \arccos \frac{a}{x} < 0\right].$$

DW

2.822

$$1. \int \operatorname{arctg} \frac{x}{q} dx = x \operatorname{arctg} \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2).$$

DW

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$$2. \int \operatorname{arcctg} \frac{x}{a} dx = x \operatorname{arcctg} \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2).$$

DW

$$3.* \int x \operatorname{arctg} \frac{x}{a} dx = \frac{1}{2}(x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{ax}{2}$$

$$4.* \int x \operatorname{arcctg} \frac{x}{a} dx = \frac{ax}{2} + \frac{\pi x^2}{4} - \frac{1}{2}(x^2 + a^2) \operatorname{arctg} \frac{x}{a}$$

$$5.* \int x^2 \operatorname{arctg} \frac{x}{a} dx = \frac{1}{3} x^3 \operatorname{arctg} \frac{x}{a} + \frac{1}{6} a^3 \ln(x^2 + a^2) - \frac{ax^2}{6}$$

$$6.* \int x^2 \operatorname{arccotg} \frac{x}{a} dx = -\frac{1}{3} x^3 \operatorname{arctg} \frac{x}{a} - \frac{1}{6} a^3 \ln(x^2 + a^2) + \frac{\pi x^3}{6} + \frac{ax^2}{6}$$

2.83 Combinations of arcsine or arccosine and algebraic functions

2.831

$$\int x^n \arcsin \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arcsin \frac{x}{a} - \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}} \text{ (see 2.263 1., 2.264, 2.27).}$$

2.27

2.264

2.263

2.832

$$\int x^n \arccos \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \arccos \frac{x}{a} + \frac{1}{n+1} \int \frac{x^{n+1} dx}{\sqrt{a^2 - x^2}} \text{ (see 2.263 1., 2.264, 2.27).}$$

2.27

2.264

2.263

1. For $n = -1$, these integrals (that is, $\int \frac{\arcsin x}{x} dx$ and $\int \frac{\arccos x}{x} dx$) cannot be expressed as a finite combination of elementary functions.

$$2. \int \frac{\arccos x}{x} dx = -\frac{\pi}{2} \ln \frac{1}{x} - \int \frac{\arcsin x}{x} dx.$$

1. $\int x \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\left(\frac{x^2}{2} - \frac{a^2}{4} \right) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right].$
2. $\int x \arccos \frac{x}{a} dx = \frac{\pi x^2}{4} - \text{sign}(a) \left[\frac{1}{4}(2x^2 - a^2) \arcsin \frac{x}{|a|} + \frac{x}{4} \sqrt{a^2 - x^2} \right].$
3. $\int x^2 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9}(x^2 + 2a^2) \sqrt{a^2 - x^2} \right].$
4. $\int x^2 \arccos \frac{x}{a} dx = \frac{\pi x^3}{6} - \text{sign}(a) \left[\frac{x^3}{3} \arcsin \frac{x}{|a|} + \frac{1}{9}(x^2 + 2a^2) \sqrt{a^2 - x^2} \right].$
5. $\int x^3 \arcsin \frac{x}{a} dx = \text{sign}(a) \left[\left(\frac{x^4}{4} - \frac{3a^4}{32} \right) \arcsin \frac{x}{|a|} + \frac{1}{32} x(2x^2 + 3a^2) \sqrt{a^2 - x^2} \right].$
6. $\int x^3 \arccos \frac{x}{a} dx = \frac{\pi x^4}{8} - \text{sign}(a) \left[\frac{(8x^4 - 3a^4)}{32} \arcsin \frac{x}{|a|} + \frac{1}{32} x(2x^2 + 3a^2) \sqrt{a^2 - x^2} \right].$

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2.834

1. $\int \frac{1}{x^2} \arcsin \frac{x}{a} dx = -\frac{1}{x} \arcsin \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}.$
2. $\int \frac{1}{x^2} \arccos \frac{x}{a} dx = -\frac{1}{x} \arccos \frac{x}{a} - \frac{1}{a} \ln \frac{a + \sqrt{a^2 - x^2}}{x}.$

2.835

$$\int \frac{\arcsin x}{(a + bx)^2} dx = -\frac{\arcsin x}{b(a + bx)} - \frac{2}{b\sqrt{a^2 - b^2}} \arctg \sqrt{\frac{(a - b)(1 - x)}{(a + b)(1 + x)}} \quad [a^2 > b^2];$$

$$= -\frac{\arcsin x}{b(a + bx)} - \frac{1}{b\sqrt{b^2 - a^2}} \ln \frac{\sqrt{(a + b)(1 + x)} + \sqrt{(b - a)(1 - x)}}{\sqrt{(a + b)(1 + x)} - \sqrt{(b - a)(1 - x)}} \quad [a^2 < b^2].$$

$$\int \frac{x \arcsin x}{(1+cx^2)^2} dx = -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{2c\sqrt{c+1}} \operatorname{arctg} \frac{\sqrt{c+1}x}{\sqrt{1-x^2}} \quad [c > -1];$$

$$= -\frac{\arcsin x}{2c(1+cx^2)} + \frac{1}{4c\sqrt{-(c+1)}} \ln \frac{\sqrt{1-x^2} + x\sqrt{-(c+1)}}{\sqrt{1-x^2} - x\sqrt{-(c+1)}} \quad [c < -1].$$

2.837

1. $\int \frac{x \arcsin x}{\sqrt{1-x^2}} dx = x - \sqrt{1-x^2} \arcsin x.$
2. $\int \frac{x \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^2}{4} - \frac{x}{2} \sqrt{1-x^2} \arcsin x + \frac{1}{4} (\arcsin x)^2.$
3. $\int \frac{x^3 \arcsin x}{\sqrt{1-x^2}} dx = \frac{x^3}{9} + \frac{2x}{3} - \frac{1}{3}(x^2+2)\sqrt{1-x^2} \arcsin x.$

2.838

1. $\int \frac{\arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{x \arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln(1-x^2).$
2. $\int \frac{x \arcsin x}{\sqrt{(1-x^2)^3}} dx = \frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2} \ln \frac{1-x}{1+x}.$

2.84 Combinations of the arcsecant and arccosecant with powers of x

2.841

1. $\int x \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} - a\sqrt{x^2-a^2} \right\} \quad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right];$
 $= \frac{1}{2} \left\{ x^2 \arccos \frac{a}{x} + a\sqrt{x^2-a^2} \right\} \quad \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right].$

DW

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2. $\int x^2 \operatorname{arcsec} \frac{x}{a} dx = \int \arccos \frac{a}{x} dx = \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} - \frac{a}{2} x \sqrt{x^2-a^2} - \frac{a^3}{2} \ln(x + \sqrt{x^2-a^2}) \right\}$
 $\quad \left[0 < \arccos \frac{a}{x} < \frac{\pi}{2} \right];$
 $= \frac{1}{3} \left\{ x^3 \arccos \frac{a}{x} + \frac{a}{2} x \sqrt{x^2-a^2} + \frac{a^3}{2} \ln(x + \sqrt{x^2-a^2}) \right\}$
 $\quad \left[\frac{\pi}{2} < \arccos \frac{a}{x} < \pi \right].$

$$3. \int x \operatorname{arccosec} \frac{x}{a} dx = \int \arcsin \frac{a}{x} dx = \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} + a\sqrt{x^2 - a^2} \right\} \quad \left[0 < \arcsin \frac{a}{x} < \frac{\pi}{2} \right];$$

$$= \frac{1}{2} \left\{ x^2 \arcsin \frac{a}{x} - a\sqrt{x^2 - a^2} \right\} \quad \left[-\frac{\pi}{2} < \arcsin \frac{a}{x} < 0 \right].$$

DW

2.85 Combinations of the arctangent and arccotangent with algebraic functions

2.851

$$\int x^n \operatorname{arctg} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arctg} \frac{x}{a} - \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}.$$

2.852

$$1. \int x^n \operatorname{arcctg} \frac{x}{a} dx = \frac{x^{n+1}}{n+1} \operatorname{arcctg} \frac{x}{a} + \frac{a}{n+1} \int \frac{x^{n+1} dx}{a^2 + x^2}.$$

For $n = -1$

$\int \frac{\operatorname{arctg} x}{x} dx$ cannot be expressed as a finite combination of elementary functions.

$$2. \int \frac{\operatorname{arcctg} x}{x} dx = \frac{\pi}{2} \ln x - \int \frac{\operatorname{arctg} x}{x} dx.$$

2.853

$$1. \int x \operatorname{arctg} \frac{x}{a} dx = \frac{1}{2}(x^2 + a^2) \operatorname{arctg} \frac{x}{a} - \frac{ax}{2}.$$

$$2. \int x \operatorname{arcctg} \frac{x}{a} dx = \frac{1}{2}(x^2 + a^2) \operatorname{arcctg} \frac{x}{a} + \frac{ax}{2}.$$

$$3.* \int x^2 \operatorname{arctg} \frac{x}{a} dx = \frac{x^3}{3} \operatorname{arctg} \frac{x}{a} + \frac{a^3}{6} \ln(x^2 + a^2) - \frac{ax^2}{6}.$$

2.854

$$\int \frac{1}{x^2} \operatorname{arctg} \frac{x}{a} dx = -\frac{1}{x} \operatorname{arctg} \frac{x}{a} - \frac{1}{2a} \ln \frac{a^2 + x^2}{x^2}.$$

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2.855

$$\int \frac{\operatorname{arctg} x}{(\alpha + \beta x)^2} dx = \frac{1}{\alpha^2 + \beta^2} \left\{ \ln \frac{\alpha + \beta x}{\sqrt{1+x^2}} - \frac{\beta - \alpha x}{\alpha + \beta x} \operatorname{arctg} x \right\}.$$

2.856

$$1. \int \frac{x \operatorname{arctg} x}{1+x^2} dx = \frac{1}{2} \operatorname{arctg} x \ln(1+x^2) - \frac{1}{2} \int \frac{\ln(1+x^2) dx}{1+x^2}.$$

TI (689)

$$2. \int \frac{x^2 \operatorname{arctg} x}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) - \frac{1}{2} (\operatorname{arctg} x)^2.$$

TI (405)

$$3. \int \frac{x^3 \operatorname{arctg} x}{1+x^2} dx = -\frac{1}{2}x + \frac{1}{2}(1+x^2) \operatorname{arctg} x - \int \frac{x \operatorname{arctg} x}{1+x^2} dx. \quad (\text{see 2.851.})$$

2.851

$$4. \int \frac{x^4 \operatorname{arctg} x}{1+x^2} dx = -\frac{1}{6}x^2 + \frac{2}{3} \ln(1+x^2) + \left(\frac{x^3}{3} - x \right) \operatorname{arctg} x + \frac{1}{2} (\operatorname{arctg} x)^2.$$

2.857

$$\int \frac{\operatorname{arctg} x dx}{(1+x^2)^{n+1}} = \left[\sum_{k=1}^n \frac{(2n-2k)!!(2n-1)!!}{(2n)!!(2n-2k+1)!!} \frac{x}{(1+x^2)^{n-k+1}} + \frac{1}{2} \frac{(2n-1)!!}{(2)!!} \operatorname{arctg} x \right] \operatorname{arctg} x + \frac{1}{2} \sum_{k=1}^n \frac{(2n-1)!!(2n-2k)!!}{(2n)!!(2n-2k+1)!!(n-k+1)} \frac{1}{(1+x^2)^{n-k+1}}.$$

$$\int \frac{x \operatorname{arctg} x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} \operatorname{arctg} x + \sqrt{2} \operatorname{arctg} \frac{x\sqrt{2}}{\sqrt{1-x^2}} - \arcsin x.$$

2.859

$$\begin{aligned} \int \frac{\operatorname{arctg} x}{\sqrt{(a+bx^2)^3}} dx &= \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} - \frac{1}{a\sqrt{b-a}} \operatorname{arctg} \sqrt{\frac{a+bx^2}{b-a}} \quad [a < b]; \\ &= \frac{x \operatorname{arctg} x}{a\sqrt{a+bx^2}} + \frac{1}{2a\sqrt{a-b}} \ln \frac{\sqrt{a+bx^2} - \sqrt{a-b}}{\sqrt{a+bx^2} + \sqrt{a-b}} \quad [a > b]. \end{aligned}$$

3.-4. Definite Integrals of Elementary Functions

3.0 Introduction

* We omit the definition of definite and multiple integrals since they are widely known and can easily be found in any textbook on the subject. Here we give only certain theorems of a general nature which provide estimates, or which reduce the given integral to a simpler one.

3.01 Theorems of a general nature

3.011

Suppose that $f(x)$ is integrable**. A function $f(x)$ is said to be integrable over the interval (p, q) , if the integral $\int_p^q f(x) dx$ exists.

Here, we usually mean the existence of the integral in the sense of Riemann. When it is a matter of the existence of the integral in the sense of Stieltjes or Lebesgue, etc., we shall speak of integrability in the sense of Stieltjes or Lebesgue.

over the largest of the intervals (p, q) , (p, r) , (r, q) . Then (depending on the relative positions of the points p , q , and r) it is also integrable over the other two intervals and we have

$$\int_p^q f(x) dx = \int_p^r f(x) dx + \int_r^q f(x) dx.$$

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3.012

The first mean-value theorem. Suppose (1) that $f(x)$ is continuous and that $g(x)$ is integrable over the interval (p, q) , (2) that

$$m \leq f(x) \leq M$$

and (3) that $g(x)$ does not change sign anywhere in the interval (p, q) . Then, there exists at least one point

$\xi (p \leq \xi \leq q)$ such that

$$\int_p^q f(x)g(x) dx = f(\xi) \int_p^q g(x) dx.$$

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3.013

The second mean-value theorem. If $f(x)$ is monotonic and non-negative throughout the interval (p, q) , where $p < q$, and if $g(x)$ is integrable over that interval, then there exists at least one point $\xi [p \leq \xi \leq q]$ such that

$$1. \int_p^q f(x)g(x) dx = f(p) \int_p^{\xi} g(x) dx.$$

Under the conditions of Theorem 3.013 1, if $f(x)$ is nondecreasing, then

$$2. \int_p^q f(x)g(x) dx = f(q) \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q].$$

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If $f(x)$ is monotonic in the interval (p, q) , where $p < q$, and if $g(x)$ is integrable over that interval, then

$$3. \int_p^q f(x)g(x) dx = f(p) \int_p^{\xi} g(x) dx + f(q) \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q],$$

or

$$4. \int_p^q f(x)g(x) dx = A \int_p^{\xi} g(x) dx + B \int_{\xi}^q g(x) dx \quad [p \leq \xi \leq q],$$

where A and B are any two numbers satisfying the conditions

$$\begin{aligned} A &\geq f(p+0) \quad \text{and} \quad B \leq f(q-0) && [\text{if } f \text{ decreases}], \\ A &\leq f(p+0) \quad \text{and} \quad B \geq f(q-0) && [\text{if } f \text{ increases}]. \end{aligned}$$

In particular,

$$5. \int_p^q f(x)g(x) dx = f(p+0) \int_p^\xi g(x) dx + f(q-0) \int_\xi^q g(x) dx.$$

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3.02 Change of variable in a definite integral

3.020

$$\int_\alpha^\beta f(x) dx = \int_\varphi^\psi f[g(t)]g'(t) dt; \quad x = g(t).$$

This formula is valid under the following conditions:

1. $f(x)$ is continuous on some interval $A \leq x \leq B$ containing the original limits of integration α and β .
2. The equalities $\alpha = g(\varphi)$ and $\beta = g(\psi)$ hold.
3. $g(t)$ and its derivative $g'(t)$ are continuous on the interval $\varphi \leq t \leq \psi$.
4. As t varies from φ to ψ , the function $g(t)$ always varies in the same direction from $g(\varphi) = \alpha$ to $g(\psi) = \beta$.* If this last condition is not satisfied, the interval $\varphi \leq t \leq \psi$ should be partitioned into subintervals throughout each of which the condition is satisfied: $\int_\alpha^\beta f(x) dx = \int_\varphi^{\varphi_1} f[g(t)]g'(t) dt + \int_{\varphi_1}^{\varphi_2} f[g(t)]g'(t) dt + \cdots + \int_{\varphi_{n-1}}^\psi f[g(t)]g'(t) dt$.

3.021

The integral $\int_\alpha^\beta f(x) dx$ can be transformed into another integral with given limits φ and ψ by means of the linear substitution

$$x = \frac{\beta - \alpha}{\psi - \varphi}t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi}:$$

$$1. \int_\alpha^\beta f(x) dx = \frac{\beta - \alpha}{\psi - \varphi} \int_\varphi^\psi f\left(\frac{\beta - \alpha}{\psi - \varphi}t + \frac{\alpha\psi - \beta\varphi}{\psi - \varphi}\right) dt.$$

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In particular, for $\varphi = 0$ and $\psi = 1$,

$$2. \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^1 f((\beta - \alpha)t + \alpha) dt.$$

For $\varphi = 0$ and $\psi = \infty$,

$$3. \int_{\alpha}^{\beta} f(x) dx = (\beta - \alpha) \int_0^{\infty} f\left(\frac{\alpha + \beta t}{1 + t}\right) \frac{dt}{(1 + t)^2}$$

3.022

The following formulas also hold:

$$1. \int_{\alpha}^{\beta} f(x) dx = \int_{\alpha}^{\beta} f(\alpha + \beta - x) dx.$$

$$2. \int_0^{\beta} f(x) dx = \int_0^{\beta} f(\beta - x) dx.$$

$$3. \int_{-\alpha}^{\alpha} f(x) dx = \int_{-\alpha}^{\alpha} f(-x) dx.$$

3.03 General formulas

3.031

1. Suppose that a function $f(x)$ is integrable over the interval $(-p, p)$ and satisfies the relation $f(-x) = f(x)$ on that interval. (A function satisfying the latter condition is called an *even* function.) Then,

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx.$$

2. Suppose that $f(x)$ is a function that is integrable on the interval $(-p, p)$ and satisfies the relation $f(-x) = -f(x)$ on that interval. (A function satisfying the latter condition is called an *odd* function). Then,

$$\int_{-p}^p f(x) dx = 0.$$

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3.032

$$1. \int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx,$$

where $f(x)$ is a function that is integrable on the interval $(0, 1)$.

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$$2. \int_0^{2\pi} f(p \cos x + q \sin x) dx = 2 \int_0^{\pi} f(\sqrt{p^2 + q^2} \cos x) dx,$$

where $f(x)$ is integrable on the interval $(-\sqrt{p^2 + q^2}, \sqrt{p^2 + q^2})$.

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$$3. \int_0^{\frac{\pi}{2}} f(\sin 2x) \cos x dx = \int_0^{\frac{\pi}{2}} f(\cos^2 x) \cos x dx,$$

where $f(x)$ is integrable on the interval $(0, 1)$.

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3.033

1. If $f(x + \pi) = f(x)$ and $f(-x) = f(x)$, then

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) dx.$$

2. If $f(x + \pi) = -f(x)$ and $f(-x) = f(x)$, then

$$\int_0^{\infty} f(x) \frac{\sin x}{x} dx = \int_0^{\frac{\pi}{2}} f(x) \cos x dx.$$

LO V 279(4)

In formulas 3.033, it is assumed that the integrals in the left members of the formulas exist.

3.034

$$\int_0^{\infty} \frac{f(px) - f(qx)}{x} dx = [f(0) - f(+\infty)] \ln \frac{q}{p},$$

if $f(x)$ is continuous for $x \geq 0$ and if there exists a finite limit $f(+\infty) = \lim_{x \rightarrow +\infty} f(x)$.

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3.035

$$1. \int_0^{\pi} \frac{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})}{1 + 2p \cos x + p^2} dx = \frac{2\pi}{1 - p^2} f(\alpha + p) \quad [|p| < 1].$$

LA 230(16)

$$2. \int_0^{\pi} \frac{1 - p \cos x}{1 - 2p \cos x + p^2} \{f(\alpha + e^{xi}) + f(\alpha + e^{-xi})\} dx = \pi \{f(\alpha + p) + f(\alpha)\} \quad [|p| < 1].$$

BE 169

$$3. \int_0^{\pi} \frac{f(\alpha + e^{-xi}) - f(\alpha + e^{xi})}{1 - 2p \cos x + p^2} \sin x dx = \frac{\pi}{pi} \{f(\alpha + p) - f(\alpha)\} \quad [|p| < 1].$$

BE 169

In formulas 3.035, it is assumed that the function f is analytic in the closed unit circle with its center at the point α .

3.036

$$1. \int_0^{\pi} f \left(\frac{\sin^2 x}{1 + 2p \cos x + p^2} \right) dx = \int_0^{\pi} f(\sin^2 x) dx \quad [p^2 \geq 1];$$

$$= \int_0^{\pi} f \left(\frac{\sin^2 x}{p^2} \right) dx \quad [p^2 < 1].$$

$$2. \int_0^\pi F^{(n)}(\cos x) \sin^{2n} x \, dx = (2n-1)!! \int_0^\pi F(\cos x) \cos nx \, dx.$$

BE 174

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3.037

If f is analytic in the circle of radius r and if

$$f[r(\cos x + i \sin x)] = f_1(r, x) + i f_2(r, x),$$

then

$$1. \int_0^\infty \frac{f_1(r, x)}{p^2 + x^2} \, dx = \frac{\pi}{2p} f(re^{-p}).$$

LA 230(19)

$$2. \int_0^\infty f_2(r, x) \frac{x \, dx}{p^2 + x^2} = \frac{\pi}{2} [f(re^{-p}) - f(0)].$$

LA 230(20)

$$3. \int_0^\infty \frac{f_2(r, x)}{x} \, dx = \frac{\pi}{2} [f(r) - f(0)].$$

LA 230(21)

$$4. \int_0^\infty \frac{f_2(r, x)}{x(p^2 + x^2)} \, dx = \frac{\pi}{2p^2} [f(r) - f(re^{-p})].$$

LA 230(22)

3.038

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x \, dx}{\sqrt{1+x^2}} F(qx + p\sqrt{1+x^2}) &= \int_{-\infty}^{\infty} F(p \operatorname{ch} x + q \operatorname{sh} x) \operatorname{sh} x \, dx = \\ &= 2q \int_0^{\infty} F'(\operatorname{sign} p \cdot \sqrt{p^2 - q^2} \operatorname{ch} x) \operatorname{sh}^2 x \, dx \end{aligned}$$

[F is a function with a continuous derivative in the interval $(-\infty, \infty)$; all these integrals converge.]

3.04 Improper integrals

3.041

Suppose that a function $f(x)$ is defined on an interval $(p, +\infty)$ and that it is integrable over an arbitrary finite subinterval of the form (p, P) . Then, by definition

$$\int_p^{+\infty} f(x) dx = \lim_{P \rightarrow +\infty} \int_p^P f(x) dx,$$

if this limit exists. If it does exist, we say that the integral $\int_p^{+\infty} f(x) dx$ exists or that it converges. Otherwise, we say that the integral diverges.

3.042

Suppose that a function $f(x)$ is bounded and integrable in an arbitrary interval $(p, q - \eta)$ (for $0 < \eta < q - p$) but is unbounded in every interval $(q - \eta, q)$ to the left of the point q . The point q is then called a *singular point*. Then, by definition,

$$\int_p^q f(x) dx = \lim_{\eta \rightarrow 0} \int_p^{q-\eta} f(x) dx,$$

if this limit exists. In this case, we say that the integral $\int_p^q f(x) dx$ exists or that it converges.

3.043

If not only the integral of $f(x)$ but also the integral of $|f(x)|$ exists, we say that the integral of $f(x)$ converges *absolutely*.

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3.044

The integral $\int_p^{+\infty} f(x) dx$ converges absolutely if there exists a number $\alpha > 1$ such that the limit

$$\lim_{x \rightarrow +\infty} \{x^\alpha |f(x)|\}$$

exists. On the other hand, if

$$\lim_{x \rightarrow +\infty} \{x |f(x)|\} = L > 0,$$

the integral $\int_p^{+\infty} |f(x)| dx$ diverges.

3.045

Suppose that the upper limit q of the integral $\int_p^q f(x) dx$ is a singular point. Then, this integral converges absolutely if there exists a number $\alpha < 1$ such that the limit

$$\lim_{x \rightarrow q} [(q - x)^\alpha |f(x)|]$$

exists. On the other hand, if

$$\lim_{x \rightarrow q} [(q - x)|f(x)|] = L > 0,$$

the integral $\int_p^q f(x) dx$ diverges.

3.046

Suppose that the functions $f(x)$ and $g(x)$ are defined on the interval $(p, +\infty)$, that $f(x)$ is integrable over every finite interval of the form (p, P) , that the integral

$$\int_p^P f(x) dx$$

is a bounded function of P , that $g(x)$ is monotonic, and that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$. Then, the integral

$$\int_p^{+\infty} f(x)g(x) dx$$

converges.

FI II 577

3.05 The principal values of improper integrals

3.051

Suppose that a function $f(x)$ has a singular point r somewhere inside the interval (p, q) , that $f(x)$ is defined at r , and that $f(x)$ is

$$f(x) \qquad r \qquad (p, q) \qquad f(x) \qquad r \qquad f(x)$$

integrable over every portion of this interval that does not contain

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the point r . Then, by definition

$$\int_p^q f(x) dx = \lim_{\substack{\eta \rightarrow 0 \\ \eta' \rightarrow 0}} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta'}^q f(x) dx \right\},$$

Here, the limit must exist for *independent* modes of approach of η and η' to zero. If this limit does not exist but the limit

$$\lim_{\eta \rightarrow 0} \left\{ \int_p^{r-\eta} f(x) dx + \int_{r+\eta}^q f(x) dx \right\},$$

does exist, we say that this latter limit is the Cauchy *principal value* of the improper integral $\int_p^q f(x) dx$ and we say that the integral $\int_p^q f(x) dx$ exists in the sense of principal values.

FI II 603

3.052

Suppose that the function $f(x)$ is continuous over the interval (p, q) and vanishes at only one point r inside this interval. Suppose that the first derivative $f'(x)$ exists in a neighborhood of the point r . Suppose that $f'(r) \neq 0$ and that the second derivative $f''(r)$ exists at the point r itself. Then,

$$\int_p^q \frac{dx}{f(x)}$$

FI II 605

diverges, but exists in the sense of principal values.

3.053

A divergent integral of a positive function cannot exist in the sense of principal values.

3.054

Suppose that the function $f(x)$ has no singular points in the interval $(-\infty, +\infty)$. Then, by definition

Here, the limit must exist for independent approach of P and Q to $\pm\infty$. If this limit does not exist but the limit

$$\lim_{P \rightarrow +\infty} \int_{-P}^{+P} f(x) dx,$$

does exist, this last limit is called the principal value of the improper integral

$$\int_{-\infty}^{+\infty} f(x) dx.$$

FI II 607

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3.055

The principal value of an improper integral of an even function exists only when this integral converges (in the ordinary sense).

FI II 607

3.1-3.2 Power and Algebraic Functions

3.11 Rational functions

3.111

$$\int_{-\infty}^{\infty} \frac{p + qx}{r^2 + 2rx \cos \lambda + x^2} dx = \frac{\pi}{r \sin \lambda} (p - qr \cos \lambda) \quad (\text{principal value}^*)$$

(see also 3.194 8. and 3.252 1. and 2.).

3.252

3.194

* We give the values of proper and improper convergent integrals and also the principal values of divergent integrals (see 3.05) if the latter exist. Henceforth, we make no special indication of principal values.

3.112

Integrals of the form $\int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)}$,

where

$$\begin{aligned} g_n(x) &= b_0x^{2n-2} + b_1x^{2n-4} + \cdots + b_{n-1}, \\ h_n(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_n \end{aligned}$$

[All roots of $h_n(x)$ lie in the upper half-plane.]

$$1. \int_{-\infty}^{\infty} \frac{g_n(x) dx}{h_n(x)h_n(-x)} = \frac{\pi i}{a_0} \frac{M_n}{\Delta_n},$$

JE

where

$$\begin{aligned} \Delta_n &= \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}, \\ M_n &= \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix}. \end{aligned}$$

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$$2. \int_{-\infty}^{\infty} \frac{g_1(x) dx}{h_1(x)h_1(-x)} = \frac{\pi i b_0}{a_0 a_1}.$$

$$3.8 \int_{-\infty}^{\infty} \frac{g_2(x) dx}{h_2(x)h_2(-x)} = \pi i \frac{-b_0 + \frac{a_0 b_1}{a_2}}{a_0 a_1}.$$

$$4. \int_{-\infty}^{\infty} \frac{g_3(x) dx}{h_3(x)h_3(-x)} = \pi i \frac{-a_2 b_0 + a_0 b_1 - \frac{a_0 a_1 b_2}{a_3}}{a_0(a_0 a_3 - a_1 a_2)}.$$

JE

$$5. \int_{-\infty}^{\infty} \frac{g_4(x) dx}{h_4(x)h_4(-x)} = \pi i \frac{b_0(-a_1 a_4 + a_2 a_3) - a_0 a_3 b_1 + a_0 a_1 b_2 + \frac{a_0 b_3}{a_4}(a_0 a_3 - a_1 a_2)}{a_0(a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3)}.$$

JE

$$6. \int_{-\infty}^{\infty} \frac{g_5(x) dx}{h_5(x)h_5(-x)} = \pi i \frac{M_5}{a_0 \Delta_5},$$

where

$$M_5 = b_0(-a_0 a_4 a_5 + a_1 a_4^2 + a_2^2 a_5 - a_2 a_3 a_4) + a_0 b_1(-a_2 a_5 + a_3 a_4) + \\ + a_0 b_2(a_0 a_5 - a_1 a_4) + a_0 b_3(-a_0 a_3 + a_1 a_2) + \frac{a_0 b_4}{a_5}(-a_0 a_1 a_5 + a_0 a_3^2 + a_1^2 a_4 - a_1 a_2 a_3),$$

$$\Delta_5 = a_0^2 a_5^2 - 2a_0 a_1 a_4 a_5 - a_0 a_2 a_3 a_5 + a_0 a_3^2 a_4 + a_1^2 a_4^2 + a_1 a_2^2 a_5 - a_1 a_2 a_3 a_4.$$

JE

3.12 Products of rational functions and expressions that can be reduced to square roots of first- and second-degree polynomials

3.121

$$1. \int_0^1 \frac{1}{1 - 2x \cos \lambda + x^2} \frac{dx}{\sqrt{x}} = 2 \operatorname{cosec} \lambda \sum_{k=1}^{\infty} \frac{\sin k\lambda}{2k-1}.$$

BI ((10))(17)

$$2. \int_0^1 \frac{1}{q - px} \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{\sqrt{q(q-p)}} \quad [0 < p < q].$$

$$3. \int_0^1 \frac{dx}{1-2rx+r^2} \sqrt{\frac{1 \mp x}{1 \pm x}} = \pm \frac{\pi}{4r} \mp \frac{1}{r} \frac{1 \mp r}{1 \pm r} \operatorname{arctg} \frac{1+r}{1-r}.$$

LI ((14))(5, 16)

3.13-3.17 Expressions that can be reduced to square roots of third- and fourth-degree polynomials and their products with rational functions

In 3.131-3.137 we set: $\alpha = \arcsin \sqrt{\frac{a-c}{a-u}}$, $\beta = \arcsin \sqrt{\frac{c-u}{b-u}}$,

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$$\gamma = \arcsin \sqrt{\frac{u-c}{b-c}}, \quad \delta = \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}},$$

$$\{ = \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, \quad \lambda = \arcsin \sqrt{\frac{a-u}{a-b}},$$

$$\mu = \arcsin \sqrt{\frac{u-a}{u-b}}, \quad \nu = \arcsin \sqrt{\frac{a-c}{u-c}}, \quad p = \sqrt{\frac{a-b}{a-c}}, \quad q = \sqrt{\frac{b-c}{a-c}}.$$

3.131

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\alpha, p) \quad [a > b > c \geq u].$$

BY (231.00)

$$2. \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} F(\beta, p) \quad [a > b > c > u].$$

BY (232.00)

$$3. \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\gamma, q) \quad [a > b \geq u > c].$$

$$4. \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\delta, q) \quad [a > b > u \geq c].$$

BY (234.00)

$$5. \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\{\}, p) \quad [a \geq u > b > c].$$

>BY (235.00)>

$$6. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\lambda, p) \quad [a > u \geq b > c].$$

BY (236.00)

$$7. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\mu, q) \quad [u > a > b > c].$$

BY (237.00)

$$8. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} F(\nu, q) \quad [u \geq a > b > c].$$

BY (238.00)

3.132

$$1. \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{\sqrt{a-c}} [cF(\beta, p) + (a-c)E(\beta, p)] - 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u].$$

BY (232.19)

$$2. \int_c^u \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2a}{\sqrt{a-c}} F(\gamma, q) - 2\sqrt{a-c}E(\gamma, q) \quad [a > b \geq u > c].$$

BY (233.17)

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$$3. \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-a)\Pi(\delta, q^2, q) + aF(\delta, q)] \quad [a > b > u \geq c].$$

$$4. \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}} [(b-c)\Pi(\{\cdot, p^2, p\}) + cF(\{\cdot, p\})] \quad [a \geq u > b > c].$$

BY (235.16)

$$5. \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)}} = \frac{2c}{\sqrt{a-c}} F(\lambda, p) + 2\sqrt{a-c} E(\lambda, p) \quad [a > u \geq b > c].$$

BY (236.16)

$$6. \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{b\sqrt{a-c}} [a(a-b)\Pi(\mu, 1, q) + b^2 F(\mu, q)] \quad [u > a > b > c].$$

BY (237.16)

3.133

$$1.^8 \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] \quad [a > b > c \geq u].$$

BY (231.08)

$$2. \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\beta, p) - E(\beta, p)] + \frac{2}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c > u].$$

BY (232.13)

$$3. \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\gamma, q) - \frac{2}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b \geq u > c].$$

BY (233.09)

$$4. \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} E(\delta, q) \quad [a > b > u \geq c].$$

BY (234.05)

$$5. \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} [F(\{\cdot, p\}) - E(\{\cdot, p\})] + \frac{2}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}}$$

$$6. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)}} = \frac{2}{(b-a)\sqrt{a-c}} E(\nu, q) + \frac{2}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}} \\ [u > a > b > c].$$

BY (238.05)

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$$7. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\alpha, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\alpha, p) - \\ - \frac{2}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c \geq u].$$

BY (231.09)

$$8. \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\beta, p) - \frac{2}{(a-b)\sqrt{a-c}} F(\beta, p) \\ [a > b > c > u].$$

BY (232.14)

$$9. \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^3(x-c)}} = \frac{2}{(b-c)\sqrt{a-c}} F(\gamma, q) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\gamma, q) + \\ + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{b-u}} \quad [a > b > u > c].$$

BY (233.10)

$$10. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)}} = \frac{2}{(a-b)\sqrt{a-c}} F(\lambda, p) - \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\lambda, p) + \\ + \frac{2}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{u-b}} \quad [a > u > b > c].$$

BY (236.09)

$$11. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\mu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\mu, q) \\ [u > a > b > c].$$

$$12. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)}} = \frac{2\sqrt{a-c}}{(a-b)(b-c)} E(\nu, q) - \frac{2}{(b-c)\sqrt{a-c}} F(\nu, q) - \frac{2}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u \geq a > b > c].$$

BY (238.04)

$$13. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^3}} = \frac{2}{(c-b)\sqrt{a-c}} E(\alpha, p) + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(c-u)}} \quad [a > b > c > u].$$

BY (231.10)

$$14. \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\delta, q) - E(\delta, q)] + \frac{2}{b-c} \sqrt{\frac{b-u}{(a-u)(u-c)}} \quad [a > b > u > c].$$

BY (234.04)

$$15. \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\{, p) \quad [a \geq u > b > c].$$

BY (235.01)

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$$16. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} E(\lambda, p) - \frac{2}{(b-c)(a-c)} \sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a > u \geq b > c].$$

BY (236.10)

$$17. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\mu, q) - E(\mu, q)] + \frac{2}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u > a > b > c].$$

BY (237.13)

$$18. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^3}} = \frac{2}{(b-c)\sqrt{a-c}} [F(\nu, q) - E(\nu, q)] \quad [u \geq a > b > c].$$

BY (238.03)

$$\begin{aligned}
1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} &= \\
&= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c)F(\alpha, p) - 2(2a-b-c)E(\alpha, p)] + \\
&\quad + \frac{2}{3(a-c)(a-b)} \sqrt{\frac{(c-u)(b-u)}{(a-u)^3}} \quad [a > b > c \geq u].
\end{aligned}$$

BY (231.08)

$$\begin{aligned}
2. \int_u^c \frac{dx}{\sqrt{(a-x)^5(b-x)(c-x)}} &= \\
&= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c)F(\beta, p) - 2(2a-b-c)E(\beta, p)] + \\
&\quad + \frac{2[4a^2 - 3ab - 2ac + bc - u(3a-2b-c)]}{3(a-b)(a-c)^2} \sqrt{\frac{c-u}{(a-u)^3(b-u)}} \quad [a > b > c > u].
\end{aligned}$$

BY (232.13)

$$\begin{aligned}
3. \int_c^u \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} &= \\
&= \frac{2}{3(a-b)^3\sqrt{(a-c)^3}} [2(2a-b-c)E(\gamma, q) - (a-b)F(\gamma, q)] - \\
&\quad - \frac{2[5a^2 - 3ab - 3ac + bc - 2u(2a-b-c)]}{3(a-b)^2(a-c)^2} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \quad [a > b \geq u > c].
\end{aligned}$$

BY (233.09)

$$\begin{aligned}
4. \int_u^b \frac{dx}{\sqrt{(a-x)^5(b-x)(x-c)}} &= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c)E(\delta, q) - (a-b)F(\delta, q)] - \\
&\quad - \frac{2}{3(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)^3}} \quad [a > b > u \geq c].
\end{aligned}$$

BY (234.05)

$$\begin{aligned}
5. \int_b^u \frac{dx}{\sqrt{(a-x)^5(x-b)(x-c)}} &= \\
&= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [(3a-b-2c)F(\{, p) - 2(2a-b-c)E(\{, p)] + \\
&\quad + \frac{2[4a^2 - 2ab - 3ac + bc - u(3a-b-2c)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(a-u)^3(u-c)}} \quad [a > u > b > c].
\end{aligned}$$

$$\begin{aligned}
6. \int_u^\infty \frac{dx}{\sqrt{(x-a)^5(x-b)(x-c)}} &= \\
&= \frac{2}{3(a-b)^2\sqrt{(a-c)^3}} [2(2a-b-c)E(\nu, q) - (a-b)F(\nu, q)] + \\
&\quad + \frac{2[4a^2 - 2ab - 3ac + bc + u(b+2c-3a)]}{3(a-b)^2(a-c)} \sqrt{\frac{u-b}{(u-a)^3(u-c)}} \quad [u > a > b > c].
\end{aligned}$$

BY (238.05)

$$\begin{aligned}
7. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} &= \\
&= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
&\quad \times [2(a-c)(a+c-2b)E(\alpha, p) + (b-c)(3b-a-2c)F(\alpha, p)] - \\
&\quad - \frac{2[3ab - ac + 2bc - 4b^2 - u(2a-3b+c)]}{3(a-b)(b-c)^2} \sqrt{\frac{c-u}{(a-u)(b-u)^3}} \quad [a > b > c \geq u].
\end{aligned}$$

BY (231.09)

$$\begin{aligned}
8. \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)^5(c-x)}} &= \\
&= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
&\quad \times [(b-c)(3b-a-2c)F(\beta, p) + 2(a-c)(a-2b+c)E(\beta, p)] + \\
&\quad + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(a-u)(c-u)}{(b-u)^3}} \quad [a > b > c > u].
\end{aligned}$$

BY (232.14)

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$$\begin{aligned}
9. \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)^5(x-c)}} &= \\
&= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
&\quad \times [(a-b)(2a-3b+c)F(\gamma, q) + 2(a-c)(2b-a-c)E(\gamma, q)] + \\
&\quad + \frac{2[3ab + 3bc - ac - 5b^2 - 2u(a-2b+c)]}{3(a-b)^2(b-c)^2} \sqrt{\frac{(a-u)(u-c)}{(b-u)^3}} \quad [a > b > u > c].
\end{aligned}$$

BY (233.10)

$$\begin{aligned}
10. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^5(x-c)}} &= \\
&= \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
&\quad \times [(b-c)(3b-2c-a)F(\lambda, p) + 2(a-c)(a+c-2b)E(\lambda, p)] +
\end{aligned}$$

$$\begin{aligned}
11. \quad & \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \\
& = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
& \quad \times [(a-b)(2a+c-3b)F(\mu, q) + 2(a-c)(2b-a-c)E(\mu, q)] + \\
& \quad + \frac{2}{3(a-b)(b-c)} \sqrt{\frac{(u-a)(u-c)}{(u-b)^3}} \quad [u > a > b > c].
\end{aligned}$$

BY (237.12)

$$\begin{aligned}
12. \quad & \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)^5(x-c)}} = \\
& = \frac{2}{3(a-b)^2(b-c)^2\sqrt{a-c}} \times \\
& \quad \times [(a-b)(2a+c-3b)F(\nu, q) + 2(a-c)(2b-c-a)E(\nu, q)] - \\
& \quad - \frac{2[3bc+2ab-ac-4b^2+u(3b-a-2c)]}{3(a-b)^2(b-c)} \sqrt{\frac{u-a}{(u-b)^3(u-c)}} \quad [u \geq a > b > c].
\end{aligned}$$

BY (238.04)

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$$\begin{aligned}
13. \quad & \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)^5}} = \\
& = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\alpha, p) - (b-c)F(\alpha, p)] + \\
& \quad + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(a-c)(b-c)^2} \sqrt{\frac{b-u}{(a-u)(c-u)^3}} \quad [a > b > c > u].
\end{aligned}$$

BY (231.10)

$$\begin{aligned}
14. \quad & \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)^5}} = \\
& = \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\delta, q) - 2(a+b-2c)E(\delta, q)] + \\
& \quad + \frac{2[ab-3ac-2bc+4c^2+u(2a+b-3c)]}{3(b-c)^2(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)^3}} \quad [a > b > u > c].
\end{aligned}$$

BY (234.04)

$$\begin{aligned}
16. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)(x-c)^5}} &= \\
&= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [2(a+b-2c)E(\lambda, p) - (b-c)F(\lambda, p)] - \\
&\quad - \frac{2[ab-3ac-3bc+5c^2+2u(a+b-2c)]}{3(b-c)^2(a-c)^2} \sqrt{\frac{(a-u)(u-b)}{(u-c)^3}} \quad [a > u \geq b > c].
\end{aligned}$$

BY (236.10)

$$\begin{aligned}
17. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} &= \\
&= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\mu, q) - 2(a+b-2c)E(\mu, q)] + \\
&\quad + \frac{2[4c^2-ab-2ac-bc+u(3a+2b-5c)]}{3(b-c)(a-c)^2} \sqrt{\frac{u-a}{(u-b)(u-c)^3}} \quad [u > a > b > c].
\end{aligned}$$

BY (237.13)

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$$\begin{aligned}
18. \int_u^\infty \frac{dx}{\sqrt{(x-a)(x-b)(x-c)^5}} &= \\
&= \frac{2}{3(b-c)^2\sqrt{(a-c)^3}} [(2a+b-3c)F(\nu, q) - 2(a+b-2c)E(\nu, q)] + \\
&\quad + \frac{2}{3(a-c)(b-c)} \sqrt{\frac{(u-a)(u-b)}{(u-c)^3}} \quad [u \geq a > b > c].
\end{aligned}$$

BY (238.03)

3.135

$$\begin{aligned}
1.^6 \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)(b-x)^3(c-x)^3}} &= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times \\
&\quad \times [(b-c)F(\alpha, p) - (2a-b-c)E(\alpha, p)] + \\
&\quad + \frac{2(b+c-2u)}{(b-c)^2\sqrt{(a-u)(b-u)(c-u)}} \quad [a > b > c > u].
\end{aligned}$$

BY (231.13)}

$$\begin{aligned}
2. \int_u^a \frac{dx}{\sqrt{(a-x)(x-b)^3(x-c)^3}} &= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times \\
&\quad \times \frac{[(b-c)F(\lambda, p) - 2(2a-b-c)E(\lambda, p)] +}{2(a-b-c+u)} \\
&\quad + \frac{2(a-b-c+u)}{(a-b)(b-c)(a-c)} \sqrt{\frac{a-u}{(u-b)(u-c)}} \quad [a > u > b > c].
\end{aligned}$$

$$\begin{aligned}
3. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} &= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times \\
&\times [(2a-b-c)E(\mu, q) - 2(a-b)F(\mu, q)] + \\
&+ \frac{2}{(a-c)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u > a > b > c].
\end{aligned}$$

BY (236.14)

$$\begin{aligned}
4. \int_a^\infty \frac{dx}{\sqrt{(x-a)(x-b)^3(x-c)^3}} &= \frac{2}{(a-b)(b-c)^2\sqrt{a-c}} \times \\
&\times [(2a-b-c)E(\nu, q) - 2(a-b)F(\nu, q)] - \\
&- \frac{2}{(a-b)(b-c)} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u \geq a > b > c].
\end{aligned}$$

BY (238.13)

274

$$\begin{aligned}
5. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)(c-x)^3}} &= \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times \\
&\times [(2b-a-c)E(\alpha, p) - (b-c)F(\alpha, p)] + \\
&+ \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(c-u)}} \quad [a > b > c > u].
\end{aligned}$$

BY(231.12)

$$\begin{aligned}
6. \int_u^b \frac{dx}{\sqrt{(a-x)^3(b-x)(x-c)^3}} &= \frac{2}{(b-c)(a-b)\sqrt{(a-c)^3}} \times \\
&\times [(a-b)F(\delta, q) + (2b-a-c)E(\delta, q)] + \\
&+ \frac{2}{(b-c)(a-c)} \sqrt{\frac{b-u}{(a-u)(u-c)}} \quad [a > b > u > c].
\end{aligned}$$

BY (234.03)

$$\begin{aligned}
7. \int_b^u \frac{dx}{\sqrt{(a-x)^3(x-b)(x-c)^3}} &= \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times \\
&\times [(b-c)F(\{, p) - (2b-a-c)E(\{, p)] + \\
&+ \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(a-u)(u-c)}} \quad [a > u > b > c].
\end{aligned}$$

$$8. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)(x-c)^3}} = \frac{2}{(a-b)(b-c)\sqrt{(a-c)^3}} \times \\ \times [(a+c-2b)E(\nu, q) - (a-b)F(\nu, q)] + \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{u-b}{(u-a)(u-c)}} \quad [u > a > b > c].$$

BY (238.14)

$$9.^8 \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(b-c)(a-b)^2\sqrt{a-c}} \times \\ \times [(a+b-2c)E(\alpha, p) - 2(b-c)F(\alpha, p)] - \\ - \frac{2}{(a-b)(b-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c \geq u].$$

BY (231.11)

$$10. \int_u^c \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times \\ \times [(a+b-2c)E(\beta, p) - 2(b-c)F(\beta, p)] + \\ + \frac{2}{(a-b)(a-c)} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c > u].$$

BY (232.15)

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$$11. \int_c^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times \\ \times [(a-b)F(\gamma, q) - (a+b-2c)E(\gamma, q)] + \\ + \frac{2[a^2 + b^2 - ac - bc - u(a+b-2c)]}{(a-b)^2(b-c)(a-c)} \sqrt{\frac{u-c}{(a-u)(b-u)}} \\ [a > b > u > c].$$

BY (233.11)

$$12. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)}} = \frac{2}{(a-b)^2(b-c)\sqrt{a-c}} \times \\ \times [(a-b)F(\nu, q) - (a+b-2c)E(\nu, q)] + \\ + \frac{2u-a-b}{(a-b)^2\sqrt{(u-a)(u-b)(u-c)}} \quad [u > a > b > c].$$

BY (238.15)

3.136

$$1. \int_{-\infty}^u \frac{dx}{\sqrt{(a-x)^3(b-x)^3(c-x)^3}} = \\ = \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times$$

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$$\begin{aligned}
2. \int_u^\infty \frac{dx}{\sqrt{(x-a)^3(x-b)^3(x-c)^3}} &= \\
&= \frac{2}{(a-b)^2(b-c)^2\sqrt{(a-c)^3}} \times \\
&\times [(a-b)(2a-b-c)F(\nu, q) - 2(a^2+b^2+c^2-ab-ac-bc)E(\nu, q)] + \\
&+ \frac{2[u(a+b-2c) - a(a-c) - b(b-c)]}{(a-b)^2(a-c)(b-c)\sqrt{(u-a)(u-b)(u-c)}} \quad [u > a > b > c].
\end{aligned}$$

BY (238.16)

3.137

$$1. \int_{-\infty}^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2}{(a-r)\sqrt{a-c}} \left[\Pi\left(\alpha, \frac{a-r}{a-c}, p\right) - F(\alpha, p) \right]$$

$[a > b > c \geq u].$

BY (231.15)

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$$\begin{aligned}
2. \int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} &= \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \times \\
&\times \Pi\left(\beta, \frac{r-b}{r-c}, p\right) + \frac{2}{(r-b)\sqrt{a-c}} F(\beta, p) \\
&\quad [a > b > c > u, \quad r \neq 0].
\end{aligned}$$

BY (232.17)

$$3. \int_c^u \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \Pi\left(\gamma, \frac{b-c}{r-c}, q\right)$$

$[a > b \geq u > c, \quad r \neq c].$

BY (233.02)

$$\begin{aligned}
4. \int_u^b \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(x-c)}} &= \frac{2}{(r-a)(r-b)\sqrt{a-c}} \times \\
&\times \left[(b-a)\Pi\left(\delta, q^2 \frac{r-a}{r-b}, q\right) + (r-b)F(\delta, q) \right] \\
&\quad [a > b > u \geq c, \quad r \neq b].
\end{aligned}$$

BY (234.18)

BY (235.17)

$$6.8 \int_u^a \frac{dx}{(x-r)\sqrt{(a-x)(x-b)(x-c)}} = \frac{2}{(a-r)\sqrt{a-c}} \Pi\left(\lambda, \frac{a-b}{a-r}, p\right) \\ [a > u \geq b > c, \quad r \neq a].$$

BY (236.02)

$$7. \int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \times \\ \times \left[(b-a)\Pi\left(\mu, \frac{b-r}{a-b}, q\right) + (a-p)F(\mu, q) \right] \\ [u > a > b > c, \quad r \neq a].$$

BY (237.17)

$$8. \int_u^\infty \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(r-c)\sqrt{a-c}} \left[\Pi\left(\nu, \frac{r-c}{a-c}, q\right) - F(\nu, q) \right] \\ [u \geq a > b > c].$$

BY (238.06)

3.138

$$1. \int_0^u \frac{dx}{\sqrt{x(1-x)(1-k^2x)}} = 2F(\arcsin \sqrt{u}, k) \quad [0 < u < 1].$$

PE (532), JA

$$2. \int_u^1 \frac{dx}{\sqrt{x(1-x)(k'^2 + k^2x)}} = 2F(\arccos \sqrt{u}, k) \quad [0 < u < 1].$$

PE(533)

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$$3. \int_u^1 \frac{dx}{\sqrt{x(1-x)(x-k'^2)}} = 2F\left(\arcsin \frac{\sqrt{1-u}}{k}, k\right) \quad [0 < u < 1].$$

PE (534)

$$4. \int_0^u \frac{dx}{\sqrt{x(1+x)(1+k'^2x)}} = 2F(\operatorname{arctg} \sqrt{u}, k) \quad [0 < u < 1].$$

$$5. \int_0^u \frac{dx}{\sqrt{x[1+x^2+2(k'^2-k^2)x]}} = F(2 \operatorname{arctg} \sqrt{u}, k) \quad [0 < u < 1].$$

JA

$$6. \int_u^1 \frac{dx}{\sqrt{x[k'^2(1+x^2)+2(1+k^2)x]}} = F\left(\frac{\pi}{2} - 2 \operatorname{arctg} \sqrt{u}, k\right) \quad [0 < u < 1].$$

JA

$$7. \int_a^u \frac{dx}{\sqrt{(x-\alpha)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \operatorname{arctg} \sqrt{\frac{u-\alpha}{p}}, \sqrt{\frac{p+m-\alpha}{2p}}\right) \quad [\alpha < u],$$

$$8. \int_u^a \frac{dx}{\sqrt{(\alpha-x)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{p}} F\left(2 \operatorname{arctg} \sqrt{\frac{\alpha-u}{p}}, \sqrt{\frac{p-m+\alpha}{2p}}\right) \quad [u < \alpha],$$

where $p = \sqrt{(m-\alpha)^2 + n^2}$.

3.139

Notation:⁸ $\alpha = \arccos \frac{1-\sqrt{3}-u}{1+\sqrt{3}-u}, \quad \beta = \arccos \frac{\sqrt{3}-1+u}{\sqrt{3}+1-u},$
 $\gamma = \arccos \frac{\sqrt{3}+1-u}{\sqrt{3}-1+u}, \quad \delta = \arccos \frac{u-1-\sqrt{3}}{u-1+\sqrt{3}}.$

$$1.^8 \int_{-\infty}^u \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\alpha, \sin 75^\circ).$$

ZH 66 (285)

$$2. \int_u^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{3}} F(\beta, \sin 75^\circ).$$

ZH 65 (284)

$$3. \int_1^u \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\gamma, \sin 15^\circ).$$

$$4. \int_u^\infty \frac{dx}{\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{3}} F(\delta, \sin 15^\circ).$$

ZH 65 (282)

$$5. \int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt[3]{3}\sqrt[3]{2}} \left\{ \Gamma\left(\frac{1}{3}\right) \right\}^3.$$

MO 9

$$6. \int_0^1 \frac{x dx}{\sqrt{1-x^3}} = \frac{1}{\pi} \frac{\sqrt{3}}{\sqrt[3]{4}} \left\{ \Gamma\left(\frac{2}{3}\right) \right\}^3.$$

MO 9

$$7. \int_u^1 \sqrt{1-x^3} dx = \frac{1}{5} \{ \sqrt[4]{27} F(\beta, \sin 75^\circ) - 2u\sqrt{1-u^3} \}.$$

BY (244.01)

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$$8. \int_u^1 \frac{x dx}{\sqrt{1-x^3}} = (3^{-\frac{1}{4}} - 3^{\frac{1}{4}}) F(\beta, \sin 75^\circ) + 2\sqrt[4]{3} E(\beta, \sin 75^\circ) - \frac{2\sqrt{1-u^3}}{\sqrt{3}+1-u}.$$

BY (244.05)

$$9. \int_u^1 \frac{x^m dx}{\sqrt{1-x^3}} = \frac{2u^{m-2}\sqrt{1-u^3}}{2m-1} + \frac{2(m-2)}{2m-1} \int_u^1 \frac{x^{m-3} dx}{\sqrt{1-x^3}}.$$

BY (244.07)

$$10. \int_1^u \frac{x dx}{\sqrt{x^3-1}} = (3^{-\frac{1}{4}} + 3^{\frac{1}{4}}) F(\gamma, \sin 15^\circ) - 2\sqrt[4]{3} E(\gamma, \sin 15^\circ) + \frac{2\sqrt{u^3-1}}{\sqrt{3}-1+u}.$$

BY (240.05)

$$11. \int_{-\infty}^u \frac{dx}{(1-x)\sqrt{1-x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - 2E(\alpha, \sin 75^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(1+\sqrt{3}-u)\sqrt{1-u}} [u \neq 1].$$

$$12. \int_u^\infty \frac{dx}{(x-1)\sqrt{x^3-1}} = \frac{1}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - 2E(\delta, \sin 15^\circ)] + \frac{2}{\sqrt{3}} \frac{\sqrt{1+u+u^2}}{(u-1+\sqrt{3})\sqrt{u-1}} [u \neq 1].$$

BY (242.03)

$$13. \int_{-\infty}^u \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\alpha, \sin 75^\circ) - E(\alpha, \sin 75^\circ)].$$

BY (246.07)

$$14. \int_u^1 \frac{(1-x) dx}{(1+\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2-\sqrt{3}}{\sqrt[4]{27}} [F(\beta, \sin 75^\circ) - E(\beta, \sin 75^\circ)].$$

BY (244.04)

$$15. \int_1^u \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(\sqrt{3}-2)}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ).$$

BY (240.08)

$$16. \int_u^\infty \frac{(x-1) dx}{(1+\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2(2-\sqrt{3})}{\sqrt{3}} \frac{\sqrt{u^3-1}}{u^2-2u-2} - \frac{2-\sqrt{3}}{\sqrt[4]{27}} E(\delta, \sin 15^\circ).$$

BY (242.07)

$$17. \int_{-\infty}^u \frac{(1-x) dx}{(1-\sqrt{3}-x)^2 \sqrt{1-x^3}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} \left[\frac{2\sqrt[4]{3}\sqrt{1-u^3}}{u^2-2u-2} - E(\alpha, \sin 75^\circ) \right].$$

BY (246.08)

$$18. \int_1^u \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\gamma, \sin 15^\circ) - E(\gamma, \sin 15^\circ)].$$

BY (240.04)

$$19. \int_u^\infty \frac{(x-1) dx}{(1-\sqrt{3}-x)^2 \sqrt{x^3-1}} = \frac{2+\sqrt{3}}{\sqrt[4]{27}} [F(\delta, \sin 15^\circ) - E(\delta, \sin 15^\circ)].$$

BY (242.05)

BY (242.05)

$$20. \int_{-\infty}^u \frac{(x^2 + x + 1) dx}{(1 + \sqrt{3} - x)^2 \sqrt{1 - x^3}} = \frac{1}{\sqrt[4]{3}} E(\alpha, \sin 75^\circ).$$

BY (246.01)

$$21. \int_u^1 \frac{(x^2 + x + 1) dx}{(x - 1 + \sqrt{3})^2 \sqrt{1 - x^3}} = \frac{1}{\sqrt[4]{3}} E(\beta, \sin 75^\circ).$$

BY (244.02)

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$$22. \int_1^u \frac{(x^2 + x + 1) dx}{(\sqrt{3} + x - 1)^2 \sqrt{x^3 - 1}} = \frac{1}{\sqrt[4]{3}} E(\gamma, \sin 15^\circ).$$

BY (240.01)

$$23. \int_u^\infty \frac{(x^2 + x + 1) dx}{(x - 1 + \sqrt{3})^2 \sqrt{x^3 - 1}} = \frac{1}{\sqrt[4]{3}} E(\delta, \sin 15^\circ).$$

BY (242.01)

$$24. \int_1^u \frac{(x - 1) dx}{(x^2 + x + 1) \sqrt{x^3 - 1}} = \frac{4}{\sqrt[4]{27}} E(\gamma, \sin 15^\circ) - \frac{2 + \sqrt{3}}{\sqrt[4]{27}} F(\gamma, \sin 15^\circ) - \frac{2 - \sqrt{3}}{\sqrt{3}} \frac{2(u - 1)(\sqrt{3} + 1 - u)}{(\sqrt{3} - 1 + u) \sqrt{u^3 - 1}}$$

BY (240.09)

$$25. \int_{-\infty}^u \frac{(1 + \sqrt{3} - x)^2 dx}{[(1 + \sqrt{3} - x)^2 - 4\sqrt{3}p^2(1 - x)] \sqrt{1 - x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\alpha, p^2, \sin 75^\circ).$$

BY (246.02)

$$26. \int_u^1 \frac{(1 + \sqrt{3} - x)^2 dx}{[(1 + \sqrt{3} - x)^2 - 4\sqrt{3}p^2(1 - x)] \sqrt{1 - x^3}} = \frac{1}{\sqrt[4]{3}} \Pi(\beta, p^2, \sin 75^\circ).$$

BY (244.03)

BY (240.02)

$$28. \int_u^\infty \frac{(1 - \sqrt{3} - x)^2 dx}{[(1 - \sqrt{3} - x)^2 - 4\sqrt{3}p^2(x - 1)]\sqrt{x^3 - 1}} = \frac{1}{\sqrt[4]{3}} \Pi(\delta, p^2, \sin 15^\circ).$$

BY (242.02)

In 3.141 and 3.142 we set: $\alpha = \arcsin \sqrt{\frac{a-c}{a-u}}$, $\beta = \arcsin \sqrt{\frac{c-u}{b-u}}$, $\gamma = \arcsin \sqrt{\frac{u-c}{b-c}}$, $\delta = \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}$,
 $\{ = \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}$, $\lambda = \arcsin \sqrt{\frac{a-u}{a-b}}$, $\mu = \arcsin \sqrt{\frac{u-a}{u-b}}$, $\nu = \arcsin \sqrt{\frac{a-c}{u-c}}$, $p = \sqrt{\frac{a-b}{a-c}}$, $q = \sqrt{\frac{b-c}{a-c}}$.

3.141

$$1. \int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)}} dx = 2\sqrt{a-c}[F(\beta, p) - E(\beta, p)] + 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \\ [a > b > c > u].$$

BY (232.06)

$$2. \int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c}E(\gamma, q) \quad [a > b \geq u > c].$$

BY (233.01)

$$3. \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)}} dx = 2\sqrt{a-c}E(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b > u \geq c].$$

BY (234.06)

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$$4. \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c}[F(\{, p) - E(\{, p)] + 2\sqrt{\frac{(a-u)(u-b)}{u-c}} \\ [a \geq u > b > c].$$

BY (235.07)

$$5. \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)}} dx = 2\sqrt{a-c}[F(\lambda, p) - E(\lambda, p)] \quad [a > u \geq b > c]$$

BY (236.04)

BY (237.03)

$$7. \int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)}} dx = \frac{2(b-c)}{\sqrt{a-c}} F(\beta, p) - 2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}}$$

[$a > b > c > u$].

BY (232.07)

$$8. \int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\gamma, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\gamma, q) \quad [a > b \geq u > c].$$

BY (233.04)

$$9. \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\delta, q) - \frac{2(a-b)}{\sqrt{a-c}} F(\delta, q) - 2\sqrt{\frac{(b-u)(u-c)}{a-u}}$$

[$a > b > u \geq c$].

BY (234.07)

$$10. \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\{, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\{, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}}$$

[$a \geq u > b > c$].

BY (235.06)

$$11. \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)}} dx = 2\sqrt{a-c} E(\lambda, p) - \frac{2(b-c)}{\sqrt{a-c}} F(\lambda, p) \quad [a > u \geq b > c].$$

BY (236.03)

$$12. \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)}} dx = \frac{2(a-b)}{\sqrt{a-c}} F(\mu, q) - 2\sqrt{a-c} E(\mu, q) + 2\sqrt{\frac{(u-a)(u-c)}{u-b}}$$

[$u > a > b > c$].

BY (237.04)

$$13. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)}} dx = -2\sqrt{a-c} E(\beta, p) + 2\sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u].$$

$$14. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c}[F(\gamma, q) - E(\gamma, q)] \quad [a > b \geq u > c].$$

BY (233.03)

$$15. \int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)}} dx = 2\sqrt{a-c}[F(\delta, q) - E(\delta, q)] + 2\sqrt{\frac{(b-u)(u-c)}{a-u}} \\ [a > b > u \geq c].$$

BY (234.08)

$$16. \int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c}E(\lambda, p) - 2\sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a \geq u > b > c].$$

BY (235.07)

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$$17. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)}} dx = 2\sqrt{a-c}E(\lambda, p) \quad [a > u \geq b > c].$$

BY (236.01)

$$18. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)}} dx = 2\sqrt{a-c}[F(\mu, q) - E(\mu, q)] + 2\sqrt{\frac{(u-a)(u-c)}{u-b}} \\ [u > a > b > c].$$

BY (237.05)

$$19. \int_u^c \sqrt{\frac{(b-x)(c-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\beta, p) - (b-c)F(\beta, p)] + \\ + \frac{2}{3}(2b-2a+c-u)\sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u].$$

BY (232.11)

$$20. \int_c^u \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3}\sqrt{a-c}[(2a-b-c)E(\gamma, q) - 2(a-b)F(\gamma, q)] - \\ - \frac{2}{3}\sqrt{(a-u)(b-u)(u-c)} \quad [a > b \geq u > c].$$

$$21. \int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-a)F(\delta, q) + (2a-b-c)E(\delta, q)] + \\ + \frac{2}{3} (2c-b-u) \sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b > u \geq c].$$

BY (234.11)

$$22. \int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\{, p) - (b-c)F(\{, p)] + \\ + \frac{2}{3} (b+2c-2a-u) \sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a \geq u > b > c].$$

BY (235.10)

$$22. \int_b^u \sqrt{\frac{(x-b)(x-c)}{a-x}} dx = \frac{2}{3} \sqrt{a-c} [(2a-b-c)E(\{, p) - (b-c)F(\{, p)] + \\ + \frac{2}{3} (b+2c-2a-u) \sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a \geq u > b > c].$$

BY (236.07)

$$24. \int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}} dx = \frac{2}{3} \sqrt{a-c} [2(a-b)F(\mu, q) + (b+c-2a)E(\mu, q)] + \\ + \frac{2}{3} (u+2a-2b-c) \sqrt{\frac{(u-a)(u-b)}{u-c}} \quad [u > a > b > c].$$

BY (237.08)

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$$25. \int_u^c \sqrt{\frac{(a-x)(c-x)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c)E(\beta, p) - (b-c)F(\beta, p)] + \\ + \frac{2}{3} (a+c-b-u) \sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u].$$

BY (232.10)

$$26. \int_c^u \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(2b-a-c)E(\gamma, q) + (a-b)F(\gamma, q)] - \\ - \frac{2}{3} \sqrt{(a-u)(b-u)(u-c)} \quad [a > b \geq u > c].$$

BY (233.05)

$$27. \int_u^b \sqrt{\frac{(a-x)(x-c)}{b-x}} dx = \frac{2}{3} \sqrt{a-c} [(a-b)F(\delta, q) + (2b-a-c)E(\delta, q)] + \\ + \frac{2}{3} (2a+c-2b-u) \sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b > u \geq c].$$

BY (234.10)

$$28. \int_b^u \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(b-c)F(\{, p) + (a+c-2b)E(\{, p)] + \\ + \frac{2}{3} (2b-a-2c+u) \sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a \geq u > b > c].$$

BY (235.11)

$$29. \int_u^a \sqrt{\frac{(a-x)(x-c)}{x-b}} dx = \frac{2}{3} \sqrt{a-c} [(a+c-2b)E(\lambda, p) + (b-c)F(\lambda, p)] - \\ - \frac{2}{3} \sqrt{(a-u)(u-b)(u-c)} \quad [a > u \geq b > c].$$

BY (236.06)

$$30. \int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}} dx = \frac{2}{3} \frac{\sqrt{(a-c)^3}}{b-c} [(a+c-2b)E(\mu, q) - (a-b)F(\mu, q)] + \\ + \frac{2}{3} \frac{a-c}{b-c} (u+b-a-c) \sqrt{\frac{(u-a)(u-c)}{u-b}} \quad [u > a > b > c].$$

BY (237.06)

$$31. \int_u^c \sqrt{\frac{(a-x)(b-x)}{c-x}} dx = \frac{2}{3} \sqrt{a-c} [2(b-c)F(\beta, p) + (2c-a-b)E(\beta, p)] + \\ + \frac{2}{3} (a+2b-2c-u) \sqrt{\frac{(a-u)(c-u)}{b-u}} \quad [a > b > c > u].$$

BY (232.09)

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$$32. \int_c^u \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3} \sqrt{a-c} [(a+b-2c)E(\gamma, q) - (a-b)F(\gamma, q)] + \\ + \frac{2}{3} \sqrt{(a-u)(b-u)(u-c)} \quad [a > b \geq u > c].$$

$$33. \int_u^b \sqrt{\frac{(a-x)(b-x)}{x-c}} dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\delta, q) - (a-b)F(\delta, q)] + \\ + \frac{2}{3}(2c-2a-b+u)\sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b > u \geq c].$$

BY (234.09)

$$34. \int_b^u \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\{, p) - 2(b-c)F(\{, p)] + \\ + \frac{2}{3}(u+c-a-b)\sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a \geq u > b > c].$$

BY (235.09)

$$35. \int_u^a \sqrt{\frac{(a-x)(x-b)}{x-c}} dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\lambda, p) - 2(b-c)F(\lambda, p)] - \\ - \frac{2}{3}\sqrt{(a-u)(u-b)(u-c)} \quad [a > u \geq b > c].$$

BY (236.05)

$$36. \int_a^u \sqrt{\frac{(x-a)(x-b)}{x-c}} dx = \frac{2}{3}\sqrt{a-c}[(a+b-2c)E(\mu, q) - (a-b)F(\mu, q)] + \\ + \frac{2}{3}(u+2c-a-2b)\sqrt{\frac{(u-a)(u-c)}{u-b}} \quad [u > a > b > c].$$

BY (237.07)

3.142

$$1. \int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)(c-x)^3}} dx = \frac{2}{\sqrt{a-c}}F(\alpha, p) - \frac{2\sqrt{a-c}}{b-c}E(\alpha, p) + \frac{2(a-c)}{b-c}\sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a > b > c > u].$$

BY (231.05)

$$2. \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3}} dx = 2\frac{a-b}{(b-c)\sqrt{a-c}}F(\delta, q) - \frac{2\sqrt{a-c}}{b-c}E(\delta, q) + \\ + 2\frac{a-c}{b-c}\sqrt{\frac{b-u}{(a-u)(u-c)}} \quad [a > b > u > c].$$

$$3. \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\lambda, p) - \frac{2}{\sqrt{a-c}} F(\lambda, p) \quad [a \geq u > b > c].$$

BY (235.12)

$$4. \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\lambda, p) - \frac{2}{\sqrt{a-c}} F(\lambda, p) - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{u-c}} \quad [a > u \geq b > c].$$

BY (236.12)

$$5. \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\mu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\mu, q) - 2\sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u > a > b > c].$$

BY (237.10)

$$6. \int_u^\infty \sqrt{\frac{x-a}{(x-b)(x-c)^3}} dx = \frac{2\sqrt{a-c}}{b-c} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{a-c}} F(\nu, q) \quad [u \geq a > b > c].$$

BY (238.09)

$$7. \int_{-\infty}^u \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\alpha, p) - 2\frac{a-b}{b-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c \geq u].$$

BY (231.03)

$$8. \int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)}} dx = \frac{2\sqrt{a-c}}{b-c} E(\beta, p) \quad [a > b > c > u].$$

BY (232.01)

$$9. \int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{b-u}} \quad [a > b > u > c].$$

BY (233.15)

$$10. \int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{c-b} E(\lambda, p) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{u-b}} \quad [a > u > b > c].$$

$$11. \int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\mu, q) - E(\mu, q)] \quad [u > a > b > c].$$

BY (237.09)

$$12. \int_u^\infty \sqrt{\frac{x-a}{(x-b)^3(x-c)}} dx = \frac{2\sqrt{a-c}}{b-c} [F(\nu, q) - E(\nu, q)] + 2\sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u \geq a > b > c].$$

BY (238.10)

$$13. \int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\alpha, p) \quad [a > b > c \geq u].$$

BY (231.01)

$$14. \int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)}} dx = \frac{2}{\sqrt{a-c}} E(\beta, p) - \frac{2(a-b)}{a-c} \sqrt{\frac{c-u}{(a-u)(b-u)}} \quad [a > b > c > u].$$

BY (232.05)

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$$15. \int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\gamma, q) - E(\gamma, q)] + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{a-u}} \quad [a > b \geq u > c].$$

BY (233.13)

$$16. \int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)}} dx = \frac{2}{\sqrt{a-c}} [F(\delta, q) - E(\delta, q)] \quad [a > b > u \geq c].$$

BY (234.15)

$$17. \int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)}} dx = -\frac{2}{\sqrt{a-c}} E(\epsilon, p) + 2\sqrt{\frac{u-b}{(a-u)(u-c)}} \quad [a > u > b > c].$$

BY (235.08)

BY (238.07)

$$19. \int_{-\infty}^u \sqrt{\frac{b-x}{(a-x)(c-x)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{b-u}{(a-u)(c-u)}} \\ [a > b > c > u].$$

BY (231.04)

$$20. \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3}} dx = -\frac{2}{\sqrt{a-c}} E(\delta, q) + 2\sqrt{\frac{b-u}{(a-u)(u-c)}} \quad [a > b > u > c].$$

BY (234.14)

$$21. \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\nu, p) - E(\nu, p)] \quad [a \geq u > b > c].$$

BY (235.03)

$$22. \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{u-c}} \\ [a > u \geq b > c].$$

BY (236.14)

$$23. \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\mu, q) - 2\frac{b-c}{a-c} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u > a > b > c].$$

BY (237.11)

$$24. \int_u^{\infty} \sqrt{\frac{x-b}{(x-a)(x-c)^3}} dx = \frac{2}{\sqrt{a-c}} E(\nu, q) \quad [u \geq a > b > c].$$

BY (238.01)

$$25. \int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\alpha, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\alpha, p) \quad [a > b > c \geq u].$$

$$26. \int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\beta, p) - \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\beta, p) - 2\sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c > u].$$

BY (232.03)

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$$27. \int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\gamma, q) - \frac{2}{\sqrt{a-c}} F(\gamma, q) - \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{a-u}} \\ [a > b \geq u > c].$$

BY (233.14)

$$28. \int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)}} dx = \frac{2\sqrt{a-c}}{a-b} E(\delta, q) - \frac{2}{\sqrt{a-c}} F(\delta, q) \quad [a > b > u \geq c].$$

BY (234.20)

$$29. \int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)}} dx = \frac{2(b-c)}{(a-b)\sqrt{a-c}} F(\{, p) - \frac{2\sqrt{a-c}}{a-b} E(\{, p) + \\ + 2\frac{a-c}{a-b} \sqrt{\frac{u-b}{(a-u)(u-c)}} \quad [a > u > b > c].$$

BY (235.13)

$$30. \int_u^\infty \sqrt{\frac{x-c}{(x-a)^3(x-b)}} dx = \frac{2}{\sqrt{a-c}} F(\nu, q) - \frac{2\sqrt{a-c}}{a-b} E(\nu, q) + \frac{2(a-c)}{a-b} \sqrt{\frac{u-b}{(u-a)(u-c)}} \\ [u > a > b > c].$$

BY (238.08)

$$31. \int_{-\infty}^u \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\alpha, p) - E(\alpha, p)] + 2\sqrt{\frac{c-u}{(a-u)(b-u)}} \\ [a > b > c \geq u].$$

BY (231.06)

$$32. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\beta, p) - E(\beta, p)] \quad [a > b > c > u].$$

BY (232.04)

$$33. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3}} dx = -\frac{2\sqrt{a-c}}{a-b} E(\gamma, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{b-u}}$$

[$a > b > u > c$].

BY (233.16)

$$34. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} [F(\lambda, p) - E(\lambda, p)] + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{u-b}}$$

[$a > u > b > c$].

BY (236.13)

$$35. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} E(\mu, q) \quad [u > a > b > c].$$

BY (237.01)

$$36. \int_u^\infty \sqrt{\frac{x-c}{(x-a)(x-b)^3}} dx = \frac{2\sqrt{a-c}}{a-b} E(\nu, q) - 2\frac{b-c}{a-b} \sqrt{\frac{u-a}{(u-b)(u-c)}} \quad [u \geq a > b > c].$$

BY (238.11)

3.143

$$1.^6 \int_u^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F \left(\operatorname{arctg} \frac{(1+\sqrt{2})(1-u)}{(1+u)}, 2\sqrt[4]{2}(\sqrt{2}-1) \right)$$

ZH 66 (286)

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$$2. \int_u^\infty \frac{dx}{\sqrt{1+x^4}} = \frac{1}{2} F \left(\arccos \frac{u^2-1}{u^2+1}, \frac{\sqrt{2}}{2} \right).$$

ZH 66 (287)

3.144

Notation: $\alpha = \arcsin \frac{1}{\sqrt{u^2-u+1}}$.

BY (261.50)

$$2. \int_u^\infty \frac{dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = \frac{2(2u-1)}{\sqrt{u(u-1)(u^2-u+1)}} - 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1].$$

BY (261.54)

$$3. \int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x^3(x-1)^3(x^2-x+1)}} = 4 \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2u-1}{2\sqrt{u(u-1)(u^2-u-1)}} \right] \quad [u > 1].$$

BY (261.56)

$$4. \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = \frac{4}{3} \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right] \quad u \geq 1.$$

BY (261.52)

$$5. \int_u^\infty \frac{(2x-1)^2 dx}{\sqrt{x(x-1)(x^2-x+1)^3}} = 4E\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1].$$

BY (261.51)

$$6. \int_u^\infty \sqrt{\frac{x(x-1)}{(x^2-x+1)^3}} dx = \frac{4}{3}E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3}F\left(\alpha, \frac{\sqrt{3}}{2}\right) \quad [u > 1].$$

BY (261.53)

$$7. \int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x(x-1)}{x^2-x+1}} = \frac{1}{3} \left[F\left(\alpha, \frac{\sqrt{3}}{2}\right) - E\left(\alpha, \frac{\sqrt{3}}{2}\right) \right] + \frac{1}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}} \quad [u > 1].$$

BY (261.57)

$$8. \int_u^\infty \frac{dx}{(2x-1)^2} \sqrt{\frac{x^2-x+1}{x(x-1)}} = E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{3}{2(2u-1)} \sqrt{\frac{u(u-1)}{u^2-u+1}} \quad [u > 1].$$

BY (261.58)

BY (261.58)

$$9. \int_u^\infty \frac{dx}{(2x-1)^2 \sqrt{x(x-1)(x^2-x+1)}} = \frac{4}{3} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{3} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2}{2u-1} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

[$u > 1$].

BY (261.55)

$$10. \int_u^\infty \frac{dx}{\sqrt{x^5(x-1)^5(x^2-x+1)}} = \frac{40}{3} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{4}{3} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{2(2u-1)(9u^2-9u-1)}{3\sqrt{u^3(u-1)^3(u^2-u+1)}}$$

[$u > 1$].

BY (261.54)

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$$11. \int_u^\infty \frac{dx}{\sqrt{x(x-1)(x^2-x+1)^5}} = \frac{44}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{56}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) + \frac{2(2u-1)\sqrt{u(u-1)}}{9\sqrt{(u^2-u+1)^3}}$$

[$u > 1$].

BY (261.52)

$$12. \int_u^\infty \frac{dx}{(2x-1)^4 \sqrt{x(x-1)(x^2-x+1)}} = \frac{16}{27} E\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{1}{27} F\left(\alpha, \frac{\sqrt{3}}{2}\right) - \frac{8(5u^2-5u+2)}{9(2u-1)^3} \sqrt{\frac{u(u-1)}{u^2-u+1}}$$

[$u > 1$].

BY (261.55)

3.145

$$1. \int_\alpha^u \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arctg} \sqrt{\frac{q(u-\alpha)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}}\right)$$

[$\beta < \alpha < u$].

$$2. \int_\beta^u \frac{dx}{\sqrt{(\alpha-x)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F\left(2 \operatorname{arccctg} \sqrt{\frac{q(\alpha-u)}{p(u-\beta)}}, \frac{1}{2} \sqrt{\frac{-(p-q)^2 + (\alpha-\beta)^2}{pq}}\right)$$

[$\beta < u < \alpha$].

$$3. \int_u^\beta \frac{dx}{\sqrt{(x-\alpha)(x-\beta)[(x-m)^2+n^2]}} = \frac{1}{\sqrt{pq}} F \left(2 \operatorname{arctg} \sqrt{\frac{q(\beta-u)}{p(\alpha-u)}}, \frac{1}{2} \sqrt{\frac{(p+q)^2 + (\alpha-\beta)^2}{pq}} \right) \quad [u < \beta < \alpha],$$

where $(m-\alpha)^2+n^2=p^2$, $(m-\beta)^2+n^2=q^2$.* Formulas 3.145 are not valid for $\alpha+\beta=2m$. In this case, we make the substitution $x-m=z$, which leads to one of the formulas 3.152.

4. Set

$$(m_1-m)^2+(n_1+n)^2=p^2, \quad (m_1-m)^2+(n_1-n)^2=p_1^2,$$

$$\operatorname{ctg} \alpha = \sqrt{\frac{(p+p_1)^2-4n^2}{4n^2-(p-p_1)^2}};$$

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then

$$\int_{m-n \operatorname{tg} \alpha}^u \frac{dx}{\sqrt{[(x-m)^2+n^2][(x-m_1)^2+n_1^2]}} = \frac{2}{p+p_1} F \left(\alpha + \operatorname{arctg} \frac{u-m}{n}, \frac{2\sqrt{pp_1}}{p+p_1} \right) \quad [m-n \operatorname{tg} \alpha < u < m+n \operatorname{ctg} \alpha].$$

3.146

$$1. \int_0^1 \frac{1}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8} + \frac{1}{4} \sqrt{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right).$$

BI ((13))(6)

$$2. \int_0^1 \frac{x^2}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{8}.$$

$$3. \int_0^1 \frac{x^4}{1+x^4} \frac{dx}{\sqrt{1-x^4}} = -\frac{\pi}{8} + \frac{1}{4}\sqrt{2}K\left(\frac{\sqrt{2}}{2}\right).$$

BI ((13))(8)

In 3.147-3.151 we set: $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\begin{aligned} \beta &= \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, & \gamma &= \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}}, \\ \delta &= \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, & \{ &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \lambda &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \mu &= \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \end{aligned}$$

$$\nu = \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, \quad q = \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, \quad r = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}.$$

3.147

$$1. \int_u^d \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \quad [a > b > c > d > u].$$

BY (251.00)

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$$2. \int_d^u \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\beta, r) \quad [a > b > c \geq u > d].$$

BY (254.00)

$$3. \int_u^c \frac{dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\gamma, r) \quad [a > b > c > u \geq d].$$

BY (253.00)

$$4. \int_c^u \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\delta, q) \quad [a > b \geq u > c > d].$$

$$5. \int_u^b \frac{dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\{, q) \quad [a > b > u \geq c > d].$$

BY (255.00)

$$6. \int_b^u \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\lambda, r) \quad [a \geq u > b > c > d].$$

BY (256.00)

$$7. \int_u^\alpha \frac{dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\mu, r) \quad [a > u \geq b > c > d].$$

BY (257.00)

$$8. \int_a^u \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} F(\nu, q) \quad [u > a > b > c > d].$$

BY (258.00)

3.148

$$1.^8 \int_u^d \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-c) \Pi \left(\alpha, \frac{a-d}{a-c}, q \right) + c F(\alpha, q) \right\} \\ [a > b > c > d > u].$$

BY (251.03)

$$2. \int_d^u \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (d-a) \Pi \left(\beta, \frac{d-c}{a-c}, r \right) + a F(\beta, r) \right\} \\ [a > b > c \geq u > d].$$

BY (252.11)

$$3. \int_u^c \frac{x dx}{\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left(\gamma, \frac{c-d}{b-d}, r \right) + b F(\gamma, r) \right\} \\ [a > b > c > u \geq d].$$

BY (253.11)

$$5. \int_u^b \frac{x dx}{\sqrt{(a-x)(b-x)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a)\Pi \left(\left\{ \frac{b-c}{a-c}, q \right\} \right) + aF \left(\left\{ \frac{b-c}{a-c}, q \right\} \right) \right\} \\ [a > b > u \geq c > d].$$

BY (255.17)

$$6.^8 \int_b^u \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c)\Pi \left(\left\{ \lambda, \frac{a-b}{a-c}, r \right\} \right) + cF \left(\left\{ \lambda, r \right\} \right) \right\} \\ [a \geq u > b > c > d].$$

BY (256.11)

$$7. \int_u^a \frac{x dx}{\sqrt{(a-x)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-d)\Pi \left(\left\{ \mu, \frac{b-a}{b-d}, r \right\} \right) + dF \left(\left\{ \mu, r \right\} \right) \right\} \\ [a > u \geq b > c > d].$$

BY (257.11)

$$8. \int_a^u \frac{x dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b)\Pi \left(\left\{ \nu, \frac{a-d}{b-d}, q \right\} \right) + bF \left(\left\{ \nu, q \right\} \right) \right\} \\ [u > a > b > c > d].$$

BY (258.11)

3.149

$$1. \int_u^d \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(d-x)}} = \\ = \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{ (c-d)\Pi \left(\left\{ \alpha, \frac{c(a-d)}{d(a-c)}, q \right\} \right) + dF \left(\left\{ \alpha, q \right\} \right) \right\} \quad [a > b > c > d > u].$$

BY (251.04)

$$2. \int_d^u \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} = \\ = \frac{2}{ad\sqrt{(a-c)(b-d)}} \left\{ (a-d)\Pi \left(\left\{ \beta, \frac{a(d-c)}{d(a-c)}, r \right\} \right) + dF \left(\left\{ \beta, r \right\} \right) \right\} \quad [a > b > c \geq u > d].$$

$$\begin{aligned}
3. \int_u^c \frac{dx}{x\sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \\
&= \frac{2}{bc\sqrt{(a-c)(b-d)}} \left\{ (b-c)\Pi\left(\gamma, \frac{b(c-d)}{c(b-d)}, r\right) + cF(\gamma, r) \right\} \quad [a > b > c > u \geq d].
\end{aligned}$$

BY (253.12)

$$\begin{aligned}
4. \int_c^u \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \\
&= \frac{2}{cd\sqrt{(a-c)(b-d)}} \left\{ (d-c)\Pi\left(\delta, \frac{d(b-c)}{c(b-d)}, q\right) + cF(\delta, q) \right\} \quad [a > b \geq u > c > d].
\end{aligned}$$

BY (254.11)

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$$\begin{aligned}
5. \int_u^b \frac{dx}{x\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \\
&= \frac{2}{ab\sqrt{(a-c)(b-d)}} \times \left\{ (a-b)\Pi\left(\left\{\frac{a(b-c)}{b(a-c)}, q\right\}, \right) + bF(\{, q) \right\} \quad [a > b > u \geq c > d].
\end{aligned}$$

BY (255.18)

$$\begin{aligned}
6. \int_b^u \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{bc\sqrt{(a-c)(b-d)}} \times \left\{ (c-b)\Pi\left(\lambda, \frac{c(a-b)}{b(a-c)}, r\right) + bF(\lambda, r) \right\} \quad [a \geq u > b > c > d].
\end{aligned}$$

BY (256.12)

$$\begin{aligned}
7. \int_u^a \frac{dx}{x\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{ad\sqrt{(a-c)(b-d)}} \times \left\{ (d-a)\Pi\left(\mu, \frac{d(b-a)}{a(b-d)}, r\right) + aF(\mu, r) \right\} \quad [a > u \geq b > c > d].
\end{aligned}$$

BY (257.12)

$$\begin{aligned}
8. \int_a^u \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{ab\sqrt{(a-c)(b-d)}} \left\{ (b-a)\Pi\left(\nu, \frac{b(a-d)}{a(b-d)}, q\right) + aF(\nu, q) \right\} \quad [u > a > b > c > d].
\end{aligned}$$

3.151

$$\begin{aligned}
 1. \int_u^d \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(d-x)}} &= \\
 &= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \\
 &\quad \times \left[(d-c)\Pi\left(\alpha, \frac{(a-d)(p-c)}{(a-c)(p-d)}, q\right) + (p-d)F(\alpha, q) \right] \quad [a > b > c > d > u, \quad p \neq d].
 \end{aligned}$$

BY (251.39)

$$\begin{aligned}
 2. \int_d^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \\
 &= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \\
 &\quad \times \left[(d-a)\Pi\left(\beta, \frac{(d-c)(p-a)}{(a-c)(p-d)}, r\right) + (p-d)F(\beta, r) \right] \quad [a > b > c \geq u > d, \quad p \neq d].
 \end{aligned}$$

BY (252.39)

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$$\begin{aligned}
 3. \int_u^c \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(c-x)(x-d)}} &= \\
 &= \frac{2}{(p-b)(p-c)\sqrt{(a-c)(b-d)}} \times \\
 &\quad \times \left[(c-b)\Pi\left(\gamma, \frac{(c-d)(p-b)}{(b-d)(p-c)}, r\right) + (p-c)F(\gamma, r) \right] \quad [a > b > c > u \geq d, \quad p \neq c].
 \end{aligned}$$

BY (253.39)

$$\begin{aligned}
 4. \int_c^u \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \\
 &= \frac{2}{(p-c)(p-d)\sqrt{(a-c)(b-d)}} \times \\
 &\quad \times \left[(c-d)\Pi\left(\delta, \frac{(b-c)(p-d)}{(b-d)(p-c)}, q\right) + (p-c)F(\delta, q) \right] \quad [a > b \geq u > c > d, \quad p \neq c].
 \end{aligned}$$

BY (254.39)

$$\begin{aligned}
 5. \int_u^b \frac{dx}{(p-x)\sqrt{(a-x)(b-x)(x-c)(x-d)}} &= \\
 &= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \\
 &\quad \times \left[(b-c)\Pi\left(\epsilon, \frac{(b-c)(p-a)}{(b-d)(p-a)}, q\right) + (p-a)F(\epsilon, q) \right] \quad [a > b \geq u > c > d, \quad p \neq a].
 \end{aligned}$$

$$\begin{aligned}
6. \int_b^u \frac{dx}{(x-p)\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{(b-p)(p-c)\sqrt{(a-c)(b-d)}} \times \\
&\quad \times \left[(b-c)\Pi\left(\lambda, \frac{(a-b)(p-c)}{(a-c)(p-b)}, r\right) + (p-b)F(\lambda, r) \right] \quad [a \geq u > b > c > d, \quad p \neq b].
\end{aligned}$$

BY (256.39)

$$\begin{aligned}
7. \int_u^a \frac{dx}{(p-x)\sqrt{(a-x)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{(p-a)(p-d)\sqrt{(a-c)(b-d)}} \times \\
&\quad \times \left[(a-d)\Pi\left(\mu, \frac{(b-a)(p-d)}{(b-d)(p-a)}, r\right) + (p-a)F(\mu, r) \right] \quad [a > u \geq b > c > d, \quad p \neq a].
\end{aligned}$$

BY (257.39)

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$$\begin{aligned}
8. \int_a^u \frac{dx}{(p-x)\sqrt{(x-a)(x-b)(x-c)(x-d)}} &= \\
&= \frac{2}{(p-a)(p-b)\sqrt{(a-c)(b-d)}} \times \\
&\quad \times \left[(a-b)\Pi\left(\nu, \frac{(a-d)(p-b)}{(b-d)(p-a)}, q\right) + (p-a)F(\nu, q) \right] \quad [u > a > b > c > d, \quad p \neq a].
\end{aligned}$$

BY (258.39)

In 3.152–3.163 we set:

$$\begin{aligned}
\alpha &= \operatorname{arctg} \frac{u}{b}, & \beta &= \operatorname{arctg} \frac{u}{a}, \\
\gamma &= \arcsin \frac{u}{b} \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, & \delta &= \arccos \frac{u}{b}, & \varepsilon &= \arccos \frac{b}{u}, & \xi &= \arcsin \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, \\
\eta &= \arcsin \frac{u}{b}, & \zeta &= \arcsin \frac{a}{b} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}, & \{ &= \arcsin \frac{a}{u} \sqrt{\frac{u^2 - b^2}{a^2 - b^2}}, \\
\lambda &= \arcsin \sqrt{\frac{a^2 - u^2}{a^2 - b^2}}, & \mu &= \arcsin \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}, & \nu &= \arcsin \frac{a}{u}, & q &= \frac{\sqrt{a^2 - b^2}}{a}, \\
r &= \frac{b}{\sqrt{a^2 + b^2}}, & s &= \frac{a}{\sqrt{a^2 + b^2}}, & t &= \frac{b}{a}.
\end{aligned}$$

3.152

$$1. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} F(\alpha, q) \quad [a > b > 0].$$

ZH 62(258), BY (221.00)

$$2. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a} F(\beta, q) \quad [a > b > 0].$$

ZH 63 (259), BY (222.00)

$$3. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\gamma, r) \quad [b \geq u > 0].$$

ZH 63 (260)

$$4. \int_u^b \frac{dx}{\sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\delta, r) \quad [b > u \geq 0].$$

ZH 63 (261), BY (213.00)

$$5. \int_b^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\varepsilon, s) \quad [u > b > 0].$$

ZH 63 (262), BY (211.00)

$$6. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} F(\xi, s) \quad [u > b > 0].$$

ZH 63 (263), BY (212.00)

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$$7. \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\eta, t) \quad [a > b \geq u > 0].$$

$$8. \int_u^b \frac{dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{a} F(\zeta, t) \quad [a > b > u \geq 0].$$

ZH 63 (265), BY (220.00)

$$9. \int_b^u \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\{, q) \quad [a \geq u > b > 0].$$

ZH 63 (266), BY (217.00)

$$10. \int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{a} F(\lambda, q) \quad [a > u \geq b > 0].$$

ZH 63 (257), BY (218.00)

$$11. \int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\mu, t) \quad [u > a > b > 0].$$

ZH 63 (268), BY (216.00)

$$12. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{a} F(\nu, t) \quad [u \geq a > b > 0].$$

ZH 64(269), BY (215.00)

3.153

$$1. \int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} - aE(\alpha, q) \quad [u > 0, \quad a > b].$$

BY (221.09)

$$2. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\gamma, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \\ [b \geq u > 0].$$

BY (214.05)

$$3. \int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \sqrt{a^2 + b^2} E(\delta, r) - \frac{a^2}{\sqrt{a^2 + b^2}} F(\delta, r) \quad [b > u \geq 0].$$

BY (213.06)

$$4. \int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{b^2}{\sqrt{a^2 + b^2}} F(\varepsilon, s) - \sqrt{a^2 + b^2} E(\varepsilon, s) + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)}$$

$$[u > b > 0].$$

BY (211.09)

$$5. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a\{F(\eta, t) - E(\eta, t)\} \quad [a > b \geq u > 0].$$

BY (219.05)

$$6. \int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = a\{F(\zeta, t) - E(\zeta, t)\} + u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0].$$

BY (220.06)

$$7. \int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = aE(\{\lambda, q\}) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.05)

$$8. \int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = aE(\lambda, q) \quad [a > u \geq b > 0].$$

BY (218.06)

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$$9.^6 \int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = a\{F(\mu, t) - E(\mu, t)\} + u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.06)

$$10. \int_0^1 \frac{x^2 dx}{\sqrt{(1 + x^2)(1 + k^2 x^2)}} = \frac{1}{k^2} \left\{ \sqrt{\frac{1 + k^2}{2}} - E\left(\frac{\pi}{4}, \sqrt{1 - k^2}\right) \right\}.$$

BI ((14))(9)

$$1. \int_0^u \frac{x^4 dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{a}{3} \{2(a^2+b^2)E(\alpha, q) - b^2 F(\alpha, q)\} + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{a^2+u^2}{b^2+u^2}}$$

$[a > b, \quad u > 0].$

BY (221.09)

$$2. \int_0^u \frac{x^4 dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}} \{(2a^2-b^2)a^2 F(\gamma, r) - 2(a^4-b^4)E(\gamma, r)\} -$$

$$- \frac{u}{3} (2b^2 - a^2 + u^2) \sqrt{\frac{b^2-u^2}{a^2+u^2}} \quad [a \geq u > 0].$$

BY (214.05)

$$3. \int_u^b \frac{x^4 dx}{\sqrt{(a^2+x^2)(b^2-x^2)}} = \frac{1}{3\sqrt{a^2+b^2}} \{(2a^2-b^2)a^2 F(\delta, r) - 2(a^4-b^4)E(\delta, r)\} +$$

$$+ \frac{u}{3} \sqrt{(a^2+u^2)(b^2-u^2)} \quad [b > u \geq 0].$$

BY (213.06)

$$4. \int_b^u \frac{x^4 dx}{\sqrt{(a^2+x^2)(x^2-b^2)}} = \frac{1}{3\sqrt{a^2+b^2}} \{(2b^2-a^2)b^2 F(\varepsilon, s) + 2(a^4-b^4)E(\varepsilon, s)\} +$$

$$+ \frac{2b^2 - 2a^2 + u^2}{3u} \sqrt{(u^2+a^2)(u^2-b^2)} \quad [u > b > 0].$$

BY (211.09)

$$5. \int_0^u \frac{x^4 dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3} \{(2a^2+b^2)F(\eta, t) - 2(a^2+b^2)E(\eta, t)\} + \frac{u}{3} \sqrt{(a^2-u^2)(b^2-u^2)}$$

$[a > b \geq u > 0].$

BY (219.05)

$$6. \int_u^b \frac{x^4 dx}{\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{a}{3} \{(2a^2+b^2)F(\zeta, t) - 2(a^2+b^2)E(\zeta, t)\} +$$

$$+ \frac{u}{3} (u^2+a^2+2b^2) \sqrt{\frac{b^2-u^2}{a^2-u^2}} \quad [a > b > u \geq 0].$$

BY (220.06)

$$7. \int_b^u \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{2(a^2 + b^2)E(\{, q) - b^2 F(\{, q)\} - \frac{u^2 + 2a^2 + 2b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.05)

$$8. \int_u^a \frac{x^4 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{a}{3} \{2(a^2 + b^2)E(\lambda, q) - b^2 F(\lambda, q)\} + \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a > u \geq b > 0].$$

BY (218.06)

$$9. \int_a^u \frac{x^4 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{a}{3} \{(2a^2 + b^2)F(\mu, t) - 2(a^2 + b^2)E(\mu, t)\} + \frac{u}{3} (u^2 + 2a^2 + b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.06)

3.155

$$1. \int_u^a \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \{(a^2 + b^2)E(\lambda, q) - 2b^2 F(\lambda, q)\} - \frac{u}{3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a > u \geq b > 0].$$

BY (218.11)

$$2. \int_a^u \sqrt{(x^2 - a^2)(x^2 - b^2)} dx = \frac{a}{3} \{(a^2 + b^2)E(\mu, t) - (a^2 - b^2)F(\mu, t)\} + \frac{u}{3} (u^2 - a^2 - 2b^2) \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.10)

$$3. \int_0^u \sqrt{(x^2 + a^2)(x^2 + b^2)} dx = \frac{a}{3} \{2b^2 F(\alpha, q) - (a^2 + b^2)E(\alpha, q)\} + \frac{u}{3} (u^2 + a^2 + 2b^2) \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \quad [a > b, \quad u > 0].$$

$$4. \int_0^u \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{a^2 F(\gamma, r) - (a^2 - b^2)E(\gamma, r)\} + \\ + \frac{u}{3} (u^2 + 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [a \geq u > 0].$$

BY (214.12)

$$5. \int_u^b \sqrt{(a^2 + x^2)(b^2 - x^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{a^2 F(\delta, r) + 2(b^2 - a^2)E(\delta, r)\} + \\ + \frac{u}{3} \sqrt{(a^2 + u^2)(b^2 - u^2)} \quad [b > u \geq 0].$$

BY (213.13)

$$6. \int_b^u \sqrt{(a^2 + x^2)(x^2 - b^2)} dx = \frac{1}{3} \sqrt{a^2 + b^2} \{(b^2 - a^2)E(\varepsilon, s) - b^2 F(\varepsilon, s)\} + \\ + \frac{u^2 + a^2 - b^2}{3u} \sqrt{(a^2 + u^2)(u^2 - b^2)} \quad [u > b > 0].$$

BY (211.08)

298

$$7. \int_0^u \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \{(a^2 + b^2)E(\eta, t) - (a^2 - b^2)F(\eta, t)\} + \\ + \frac{u}{3} \sqrt{(a^2 - u^2)(b^2 - u^2)} \quad [a > b \geq u > 0].$$

BY (219.11)

$$8. \int_u^b \sqrt{(a^2 - x^2)(b^2 - x^2)} dx = \frac{a}{3} \{(a^2 + b^2)E(\zeta, t) - (a^2 - b^2)F(\zeta, t)\} + \\ + \frac{u}{3} (u^2 - 2a^2 - b^2) \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0].$$

BY (220.05)

$$9. \int_b^u \sqrt{(a^2 - x^2)(x^2 - b^2)} dx = \frac{a}{3} \{(a^2 + b^2)E(\{, q) - 2b^2 F(\{, q)\} + \\ + \frac{u^2 - a^2 - b^2}{3u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.09)

3.156

$$1.^6 \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{ub^2} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} - \frac{1}{ab^2} E(\beta, q) \quad [a \geq b, \quad u > 0].$$

$$2. \int_u^b \frac{dx}{x^2 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{a^2 F(\delta, r) - (a^2 + b^2) E(\delta, r)\} + \\ + \frac{1}{a^2 b^2 u} \sqrt{(a^2 + u^2)(b^2 - u^2)} \quad [b > u > 0].$$

BY (213.09)

$$3. \int_b^u \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{(a^2 + b^2) E(\varepsilon, s) - b^2 F(\varepsilon, s)\} \quad [u > b > 0].$$

BY (211.11)

$$4. \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{a^2 b^2 \sqrt{a^2 + b^2}} \{(a^2 + b^2) E(\xi, s) - b^2 F(\xi, s)\} - \\ - \frac{1}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \quad [u \geq b > 0].$$

BY (212.06)

$$5. \int_u^b \frac{dx}{x^2 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ab^2} \{F(\zeta, t) - E(\zeta, t)\} + \frac{1}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u > 0].$$

BY (220.09)

$$6. \int_b^u \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\{, q) \quad [a \geq u > b > 0].$$

BY (217.01)

299

$$7. \int_u^a \frac{dx}{x^2 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{ab^2} E(\lambda, q) - \frac{1}{a^2 b^2 u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a > u \geq b > 0].$$

BY (218.12)

$$8. \int_a^u \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{F(\mu, t) - E(\mu, t)\} + \frac{1}{a^2 u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

$$9. \int_u^\infty \frac{dx}{x^2 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{ab^2} \{F(\nu, t) - E(\nu, t)\} \quad [u \geq a > b > 0].$$

BY (215.07)

3.157

$$1. \int_0^u \frac{dx}{(p - x^2) \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{a(p + b^2)} \left\{ \frac{b^2}{p} \Pi \left(\alpha, \frac{p + b^2}{p}, q \right) + F(\alpha, q) \right\} \quad [p \neq 0].$$

BY (221.13)

$$2. \int_u^\infty \frac{dx}{(p - x^2) \sqrt{(x^2 + a^2)(x^2 + b^2)}} = -\frac{1}{a(a^2 + p)} \left\{ \Pi \left(\beta, \frac{a^2 + p}{a^2}, q \right) - F(\beta, q) \right\}.$$

BY (222.11)

$$3. \int_0^u \frac{dx}{(p - x^2) \sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{p(p + a^2) \sqrt{a^2 + b^2}} \left\{ a^2 \Pi \left(\gamma, \frac{b^2(p + a^2)}{p(a^2 + b^2)}, r \right) + pF(\gamma, r) \right\} \\ [b \geq u > 0, \quad p \neq 0].$$

BY (214.13)a

$$4. \int_u^b \frac{dx}{(p - x^2) \sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{(p - b^2) \sqrt{a^2 + b^2}} \Pi \left(\delta, \frac{b^2}{b^2 - p}, r \right) \\ [b > u \geq 0, \quad p \neq b^2].$$

BY (213.02)

$$5. \int_b^u \frac{dx}{(p - x^2) \sqrt{(a^2 + x^2)(x^2 - b^2)}} = \\ = \frac{1}{p(p - b^2) \sqrt{a^2 + b^2}} \left\{ b^2 \Pi \left(\varepsilon, \frac{p}{p - b^2}, s \right) + (p - b^2) F(\varepsilon, s) \right\} \quad [u > b > 0, \quad p \neq b^2].$$

BY (211.14)

$$6. \int_u^\infty \frac{dx}{(x^2 - p) \sqrt{(a^2 + x^2)(x^2 - b^2)}} = \frac{1}{(a^2 + p) \sqrt{a^2 + b^2}} \left\{ \Pi \left(\xi, \frac{a^2 + p}{a^2 + b^2}, s \right) - F(\xi, s) \right\} \\ [u \geq b > 0].$$

BY (212.12)

$$7. \int_0^u \frac{dx}{(p - x^2) \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{ap} \Pi \left(\eta, \frac{b^2}{p}, t \right) \quad [a > b \geq u > 0; \quad p \neq b].$$

$$8. \int_u^b \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(b^2-x^2)}} = \frac{1}{a(p-a^2)(p-b^2)} \times \\ \times \left\{ (b^2-a^2)\Pi\left(\zeta, \frac{b^2(p-a^2)}{a^2(p-b^2)}, t\right) + (p-b^2)F(\zeta, t) \right\} \\ [a > b > u \geq 0; \quad p \neq b^2].$$

BY (220.13)

$$9. \int_b^u \frac{dx}{(p-x^2)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{ap(p-b^2)} \left\{ b^2\Pi\left(\left\{, \frac{p(a^2-b^2)}{a^2(p-b^2)}, q\right\}, q\right) + (p-b^2)F(\{, q)\right\} \\ [a \geq u > b > 0; \quad p \neq b^2].$$

BY (217.12)

$$10. \int_u^a \frac{dx}{(x^2-p)\sqrt{(a^2-x^2)(x^2-b^2)}} = \frac{1}{a(a^2-p)} \Pi\left(\lambda, \frac{a^2-b^2}{a^2-p}, q\right) \quad [a > u \geq b > 0; \quad p \neq a^2].$$

BY (218.02)

$$11. \int_a^u \frac{dx}{(p-x^2)\sqrt{(x^2-a^2)(x^2-b^2)}} = \\ = \frac{1}{a(p-a^2)(p-b^2)} \left\{ (a^2-b^2)\Pi\left(\mu, \frac{p-b^2}{p-a^2}, t\right) + (p-a^2)F(\mu, t) \right\} \\ [u > a > b > 0; \quad p \neq a^2, \quad p \neq b^2].$$

BY (216.12)

$$12. \int_u^\infty \frac{dx}{(x^2-p)\sqrt{(x^2-a^2)(x^2-b^2)}} = \frac{1}{ap} \left\{ \Pi\left(\nu, \frac{p}{a^2}, t\right) - F(\nu, t) \right\} \quad [u \geq a > b > 0; \quad p \neq 0].$$

BY (215.12)

3.158

$$1. \int_0^u \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)^3}} = \frac{1}{ab^2(a^2-b^2)} \{a^2E(\alpha, q) - b^2F(\alpha, q)\} \quad [a > b; \quad u > 0].$$

BY (221.05)

BY (222.05)

$$3. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\alpha, q) - E(\alpha, q)\} + \frac{u}{a^2 \sqrt{(u^2 + a^2)(u^2 + b^2)}} \\ [a > b; \quad u > 0].$$

BY (221.06)

$$4. \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{F(\beta, q) - E(\beta, q)\} \quad [a > b, \quad u \geq 0].$$

BY (222.03)

301

$$5. \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\gamma, r) \quad [b \geq u > 0].$$

BY (214.01)a

$$6. \int_u^b \frac{dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} E(\delta, r) - \frac{u}{a^2(a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b > u \geq 0].$$

BY (213.08)

$$7. \int_b^u \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{(a^2 + b^2)u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}} \\ [u > b > 0].$$

BY (211.05)

$$8. \int_u^\infty \frac{dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{a^2 \sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\} \quad [u \geq b > 0].$$

BY (212.03)

$$9. \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^3}} = \frac{1}{b^2 \sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\} + \frac{u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}} \\ [b > u > 0].$$

$$10. \int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^3}} = \frac{u}{b^2\sqrt{(a^2+u^2)(u^2-b^2)}} - \frac{1}{b^2\sqrt{a^2+b^2}} E(\xi, s) \quad [u \geq b > 0].$$

BY (212.04)

$$11. \int_0^u \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a^2(a^2-b^2)} \left\{ aE(\eta, t) - u\sqrt{\frac{b^2-u^2}{a^2-u^2}} \right\} \quad [a > b \geq u > 0].$$

BY (219.07)

$$12. \int_u^b \frac{dx}{\sqrt{(a^2-x^2)^3(b^2-x^2)}} = \frac{1}{a(a^2-b^2)} E(\zeta, t) \quad [a > b > u \geq 0].$$

BY (220.10)

$$13. \int_b^u \frac{dx}{\sqrt{(a^2-x^2)^3(x^2-b^2)}} = \frac{1}{a(a^2-b^2)} \left\{ F(\{, q) - E(\{, q) + \frac{a}{u}\sqrt{\frac{u^2-b^2}{a^2-u^2}} \right\} \quad [a > u > b > 0].$$

BY (217.10)

$$14. \int_u^\infty \frac{dx}{\sqrt{(x^2-a^2)^3(x^2-b^2)}} = \frac{1}{a(b^2-a^2)} \left\{ E(\nu, t) - \frac{a}{u}\sqrt{\frac{u^2-b^2}{u^2-a^2}} \right\} \quad [u > a > b > 0].$$

BY (215.04)

$$15. \int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)^3}} = \frac{1}{ab^2} F(\eta, t) - \frac{1}{b^2(a^2-b^2)} \left\{ aE(\eta, t) - u\sqrt{\frac{a^2-u^2}{b^2-u^2}} \right\} \quad [a > b > u > 0].$$

BY (219.06)

302

$$16. \int_u^a \frac{dx}{\sqrt{(a^2-x^2)(x^2-b^2)^3}} = \frac{1}{ab^2(a^2-b^2)} \left\{ b^2 F(\lambda, q) - a^2 E(\lambda, q) + au\sqrt{\frac{a^2-u^2}{u^2-b^2}} \right\} \quad [a > u > b > 0].$$

BY (218.04)

$$17. \int_a^u \frac{dx}{\sqrt{(x^2-a^2)(x^2-b^2)^3}} = \frac{a}{b^2(a^2-b^2)} E(\mu, t) - \frac{1}{ab^2} F(\mu, t) \quad [u > a > b > 0].$$

$$18. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{1}{b^2(a^2 - b^2)} \left\{ aE(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} - \frac{1}{ab^2} F(\nu, t)$$

$[u \geq a > b > 0].$

BY (215.06)

3.159

$$1. \int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\alpha, q) - E(\alpha, q)\} \quad [a > b, \quad u > 0].$$

BY (221.12)

$$2. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^3}} = \frac{a}{a^2 - b^2} \{F(\beta, q) - E(\beta, q)\} + \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}}$$

$[a > b, \quad u \geq 0].$

BY (222.10)

$$3. \int_0^u \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\alpha, q) - b^2 F(\alpha, q)\} - \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}}$$

$[a > b, \quad u > 0].$

BY (221.11)

$$4. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)}} = \frac{1}{a(a^2 - b^2)} \{a^2 E(\beta, q) - b^2 F(\beta, q)\} \quad [a > b, \quad u \geq 0].$$

BY (222.07)

$$5. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\gamma, r) - E(\gamma, r)\} \quad [b \geq u > 0].$$

BY (214.04)

$$6. \int_u^b \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3(b^2 - x^2)}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\delta, r) - E(\delta, r)\} + \frac{u}{a^2 + b^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}}$$

$[b > u \geq 0].$

BY (213.07)

$$7. \int_b^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\varepsilon, s) - \frac{a^2}{u(a^2 + b^2)} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}} \quad [u > b > 0].$$

BY (211.13)

$$8. \int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)^3(x^2 - b^2)}} = \frac{1}{\sqrt{a^2 + b^2}} E(\xi, s) \quad [u \geq b > 0].$$

BY (212.01)

303

$$9. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)^3}} = \frac{u}{\sqrt{(a^2 + u^2)(b^2 - u^2)}} - \frac{1}{\sqrt{a^2 + b^2}} E(\gamma, r) \quad [b > u > 0].$$

BY (214.07)

$$10. \int_u^\infty \frac{x^2 dx}{\sqrt{(a^2 + x^2)(x^2 - b^2)^3}} = \frac{1}{\sqrt{a^2 + b^2}} \{F(\xi, s) - E(\xi, s)\} + \frac{u}{\sqrt{(a^2 + u^2)(u^2 - b^2)}} \quad [u > b > 0].$$

BY (212.10)

$$11. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{1}{a^2 - b^2} \left\{ aE(\eta, t) - u\sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} - \frac{1}{a} F(\eta, t) \quad [a > b \geq u > 0].$$

BY (219.04)

$$12. \int_u^b \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)}} = \frac{a}{a^2 - b^2} E(\zeta, t) - \frac{1}{a} F(\zeta, t) \quad [a > b > u \geq 0].$$

BY (220.08)

$$13. \int_b^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)^3(x^2 - b^2)}} = \frac{1}{a(a^2 - b^2)} \left\{ b^2 F(\{, q) - a^2 E(\{, q) + \frac{a^3}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} \right\} \quad [a > u > b > 0].$$

BY (217.06)

$$14. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)^3(x^2 - b^2)}} = \frac{a}{a^2 - b^2} \left\{ \frac{a}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} - E(\nu, t) \right\} + \frac{1}{a} F(\nu, t)$$

$$[u > a > b > 0].$$

BY (215.09)

$$15. \int_0^u \frac{x^2 dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)^3}} = \frac{1}{a^2 - b^2} \left\{ u \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} - aE(\eta, t) \right\} \quad [a > b > u > 0].$$

BY (219.12)

$$16. \int_u^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ aF(\lambda, q) - aE(\lambda, q) + u \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \right\}$$

$$[a > u > b > 0].$$

BY (218.07)

$$17. \int_a^u \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{a}{a^2 - b^2} E(\mu, t) \quad [u > a > b > 0].$$

BY (216.01)

$$18. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^3}} = \frac{1}{a^2 - b^2} \left\{ aE(\nu, t) - \frac{b^2}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} \quad [u \geq a > b > 0].$$

BY (215.11)

3.161

$$1. \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{3a^3b^4} \{2(a^2 + b^2)E(\beta, q) - b^2F(\beta, q)\} + \frac{a^2b^2 - u^2(2a^2 + b^2)}{3a^2b^4u^3}$$

$$[a > b, \quad u > 0].$$

BY (222.04)

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$$2. \int_u^b \frac{dx}{x^4 \sqrt{(x^2 + a^2)(b^2 - x^2)}} = \frac{1}{3a^4b^4\sqrt{a^2 + b^2}} \{a^2(2a^2 - b^2)F(\delta, r) - 2(a^4 - b^4)E(\delta, r)\} +$$

$$+ \frac{a^2b^2 + 2u^2(a^2 - b^2)}{3a^4b^4u^3} \sqrt{(b^2 - u^2)(a^2 + u^2)} \quad [b > u > 0].$$

$$3. \int_b^u \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{2b^2 - a^2}{3a^4 b^2 \sqrt{a^2 + b^2}} F(\varepsilon, s) + \frac{2}{3} \frac{(a^2 - b^2) \sqrt{a^2 + b^2}}{a^4 b^4} E(\varepsilon, s) + \frac{1}{3a^2 b^2 u^3} \sqrt{(u^2 + a^2)(u^2 - b^2)} \quad [u > b > 0].$$

BY (211.11)

$$4. \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{3a^4 b^4 \sqrt{a^2 + b^2}} \{2(a^4 - b^4)E(\xi, s) + b^2(2b^2 - a^2)F(\xi, s)\} - \frac{a^2 b^2 + u^2(2a^2 - b^2)}{3a^2 b^4 u^3} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}} \quad [u \geq b > 0].$$

BY (212.06)

$$5. \int_u^b \frac{dx}{x^4 \sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{3a^3 b^4} \left\{ (2a^2 + b^2)F(\zeta, t) - 2(a^2 + b^2)E(\zeta, t) + \frac{[(2a^2 + b^2)u^2 + a^2 b^2]a}{u^3} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b > u > 0].$$

BY (220.09)

$$6. \int_b^u \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \{2(a^2 + b^2)E(\{, q) - b^2 F(\{, q)\} + \frac{1}{3a^2 b^2 u^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.14)

$$7. \int_u^a \frac{dx}{x^4 \sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ 2(a^2 + b^2)E(\lambda, q) - b^2 F(\lambda, q) - \frac{2(a^2 + b^2)u^2 + a^2 b^2}{au^3} \sqrt{(a^2 - u^2)(u^2 - b^2)} \right\} \quad [a > u \geq b > 0].$$

BY (218.12)

$$8. \int_a^u \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ (2a^2 + b^2)F(\mu, t) - 2(a^2 + b^2)E(\mu, t) + \frac{[(a^2 + 2b^2)u^2 + a^2 b^2]b^2}{au^3} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \right\} \quad [u > a > b > 0].$$

$$9. \int_u^\infty \frac{dx}{x^4 \sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{3a^3 b^4} \left\{ (2a^2 + b^2)F(\nu, t) - 2(a^2 + b^2)E(\nu, t) + \right. \\ \left. + \frac{ab^2}{u^3} \sqrt{(u^2 - a^2)(u^2 - b^2)} \right\} \quad [u \geq a > b > 0].$$

BY (215.07)

3.162

$$1. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^5(x^2 + b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \{ (3a^2 - b^2)F(\alpha, q) - 2(2a^2 - b^2)E(\alpha, q) \} + \\ + \frac{u[a^2(4a^2 - 3b^2) + u^2(3a^2 - 2b^2)]}{3a^4(a^2 - b^2)\sqrt{(u^2 + a^2)^3(u^2 + b^2)}} \quad [a > b, \quad u > 0].$$

BY (221.06)

$$2. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^5(x^2 + b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \{ (3a^2 - b^2)F(\beta, q) - 2(2a^2 - b^2)E(\beta, q) \} + \\ + \frac{u}{3a^2(a^2 - b^2)} \sqrt{\frac{u^2 + b^2}{(a^2 + u^2)^3}} \quad [a > b, \quad u \geq 0].$$

BY (222.03)

$$3. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^5}} = \frac{3b^2 - a^2}{3ab^2(a^2 - b^2)^2} F(\alpha, q) + \frac{a(2a^2 - 4b^2)}{3b^4(a^2 - b^2)^2} E(\alpha, q) + \\ + \frac{u}{3b^2(a^2 - b^2)} \sqrt{\frac{u^2 + a^2}{(u^2 + b^2)^3}} \quad [a > b, \quad u > 0].$$

BY (221.05)

$$4. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)^5}} = \frac{1}{3ab^4(a^2 - b^2)^2} \{ 2a^2(a^2 - 2b^2)E(\beta, q) + b^2(3b^2 - a^2)F(\beta, q) \} - \\ - \frac{u[b^2(3a^2 - 4b^2) + u^2(2a^2 - 3b^2)]}{3b^4(a^2 - b^2)\sqrt{(u^2 + a^2)(u^2 + b^2)^3}} \quad [a > b, \quad u \geq 0].$$

BY (222.05)

$$5. \int_0^u \frac{dx}{\sqrt{(a^2 + x^2)^5(b^2 - x^2)}} = \frac{1}{3a^4\sqrt{(a^2 + b^2)^3}} \{ 2(b^2 + 2a^2)E(\gamma, r) - a^2F(\gamma, r) \} + \\ + \frac{u}{3a^2(a^2 + b^2)} \sqrt{\frac{b^2 - u^2}{(a^2 + u^2)^3}} \quad [b \geq u > 0].$$

$$6. \int_u^b \frac{dx}{\sqrt{(a^2+x^2)^5(b^2-x^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}} \{(4a^2+2b^2)E(\delta, r) - a^2F(\delta, r)\} - \frac{u[a^2(5a^2+3b^2)+u^2(4a^2+2b^2)]}{3a^4(a^2+b^2)^2} \sqrt{\frac{b^2-u^2}{(a^2+u^2)^3}} \quad [b > u > 0].$$

BY (213.08)

306

$$7. \int_b^u \frac{dx}{\sqrt{(a^2+x^2)^5(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}} \{(3a^2+2b^2)F(\varepsilon, s) - (4a^2+2b^2)E(\varepsilon, s)\} + \frac{(3a^2+b^2)u^2+2(2a^2+b^2)a^2}{3a^2(a^2+b^2)^2u} \sqrt{\frac{u^2-b^2}{(u^2+a^2)^3}} \quad [u > b > 0].$$

BY (211.05)

$$8. \int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)^5(x^2-b^2)}} = \frac{1}{3a^4\sqrt{(a^2+b^2)^3}} \{(3a^2+2b^2)F(\xi, s) - (4a^2+2b^2)E(\xi, s)\} + \frac{u}{3a^2(a^2+b^2)} \sqrt{\frac{u^2-b^2}{(a^2+u^2)^3}} \quad [u > b > 0].cr$$

BY (212.03)

$$9. \int_0^u \frac{dx}{\sqrt{(a^2+x^2)(b^2-x^2)^5}} = \frac{1}{3b^4\sqrt{(a^2+b^2)^3}} \{(2a^2+3b^2)F(\gamma, r) - (2a^2+4b^2)E(\gamma, r)\} + \frac{u[(3a^3+4b^2)b^2 - (2a^2+3b^2)u^2]}{3b^4(a^2+b^2)\sqrt{(a^2+u^2)(b^2-u^2)^3}} \quad [b > u > 0].$$

BY (214.10)

$$10. \int_u^\infty \frac{dx}{\sqrt{(a^2+x^2)(x^2-b^2)^5}} = \frac{1}{3b^4\sqrt{(a^2+b^2)^3}} \{(2a^2+4b^2)E(\xi, s) - b^2F(\xi, s)\} + \frac{u[(3a^2+4b^2)b^2 - (2a^2+3b^2)u^2]}{3b^4(a^2+b^2)\sqrt{(a^2+u^2)(u^2-b^2)^3}} \quad [u > b > 0].$$

BY (212.04)

$$11. \int_0^u \frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)^5}} = \frac{2a^2-3b^2}{3ab^4(a^2-b^2)}F(\eta, t) + \frac{2a(2b^2-a^2)}{3b^4(a^2-b^2)^2}E(\eta, t) + \frac{u[(3a^2-5b^2)b^2 - 2(a^2-2b^2)u^2]}{3b^4(a^2-b^2)^2(b^2-u^2)} \sqrt{\frac{a^2-u^2}{b^2-u^2}} \quad [a > b > a > 0].$$

$$12. \int_u^a \frac{dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)^5}} = \frac{3b^2 - a^2}{3ab^2(a^2 - b^2)^2} F(\lambda, q) + \frac{2a(a^2 - 2b^2)}{3b^4(a^2 - b^2)^2} E(\lambda, q) + \\ + \frac{u[2(2b^2 - a^2)u^2 + (3a^2 - 5b^2)b^2]}{3b^4(a^2 - b^2)^2(u^2 - b^2)} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} \\ [a > u > b > 0].$$

BY (218.04)

$$13. \int_a^u \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} = \frac{2a^2 - 3b^2}{3ab^4(a^2 - b^2)} F(\mu, t) + \frac{2a(2b^2 - a^2)}{3b^4(a^2 - b^2)^2} E(\mu, t) + \\ + \frac{u}{3b^2(a^2 - b^2)(u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.11)

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$$14. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)^5}} = \frac{(4b^2 - 2a^2)a}{3b^4(a^2 - b^2)^2} E(\nu, t) + \frac{2a^2 - 3b^2}{3ab^4(a^2 - b^2)} F(\nu, t) - \\ - \frac{(3b^2 - a^2)u^2 - (4b^2 - 2a^2)b^2}{3b^2u(a^2 - b^2)^2(u^2 - b^2)} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u \geq a > b > 0].$$

BY (215.06)

$$15. \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^5(b^2 - x^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \left\{ (4a^2 - 2b^2)E(\eta, t) - (a^2 - b^2)F(\eta, t) - \right. \\ \left. - \frac{u[(5a^2 - 3b^2)a^2 - (4a^2 - 2b^2)u^2]}{a(a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b \geq u > 0].$$

BY (219.07)

$$16. \int_u^b \frac{dx}{\sqrt{(a^2 - x^2)^5(b^2 - x^2)}} = \frac{2(2a^2 - b^2)}{3a^3(a^2 - b^2)^2} E(\zeta, r) - \frac{1}{3a^3(a^2 - b^2)} F(\zeta, t) + \\ + \frac{u}{3a^2(a^2 - b^2)(a^2 - u^2)} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0].$$

BY (220.10)

$$17. \int_b^u \frac{dx}{\sqrt{(a^2 - x^2)^5(x^2 - b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \left\{ (3a^2 - b^2)F(\{\}, q) - (4a^2 - 2b^2)E(\{\}, q) \right\} + \\ + \frac{2(2a^2 - b^2)a^2 + (b^2 - 3a^2)u^2}{3a^2u(a^2 - b^2)^2(a^2 - u^2)} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}}, \quad [a > u > b > 0].$$

$$18. \int_u^\infty \frac{dx}{\sqrt{(x^2 - a^2)^5(x^2 - b^2)}} = \frac{1}{3a^3(a^2 - b^2)^2} \{ (4a^2 - 2b^2)E(\nu, t) - (a^2 - b^2)F(\nu, t) \} + \\ + \frac{(4a^2 - 2b^2)a^2 + (b^2 - 3a^2)u^2}{3a^2u(a^2 - b^2)^2(u^2 - a^2)} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \quad [u > a > b > 0].$$

BY (215.04)

3.163

$$1. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)^2} \{ (a^2 + b^2)E(\alpha, q) - 2b^2F(\alpha, q) \} - \\ - \frac{u}{a^2(a^2 - b^2)\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u > 0].$$

BY (221.07)

308

$$2. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 + b^2)^3}} = \frac{1}{ab^2(a^2 - b^2)^2} \{ (a^2 + b^2)E(\beta, q) - 2b^2F(\beta, q) \} - \\ - \frac{u}{b^2(a^2 - b^2)\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u \geq 0].$$

BY (222.12)

$$3. \int_0^u \frac{dx}{\sqrt{(x^2 + a^2)^3(b^2 - x^2)^3}} = \frac{1}{a^2b^2\sqrt{(a^2 + b^2)^3}} \{ a^2F(\gamma, r) - (a^2 - b^2)E(\gamma, r) \} + \\ + \frac{u}{b^2(a^2 + b^2)\sqrt{(a^2 + u^2)(b^2 - u^2)}} \quad [b > u > 0].$$

BY (214.15)

$$4. \int_u^\infty \frac{dx}{\sqrt{(x^2 + a^2)^3(x^2 - b^2)^3}} = \frac{b^2 - a^2}{a^2b^2\sqrt{(a^2 + b^2)^3}} E(\xi, s) - \frac{1}{a^2\sqrt{(a^2 + b^2)^3}} F(\xi, s) + \\ + \frac{u}{b^2(a^2 + b^2)\sqrt{(u^2 + a^2)(u^2 - b^2)}} \quad [u > b > 0].$$

BY (212.05)

$$5. \int_0^u \frac{dx}{\sqrt{(a^2 - x^2)^3(b^2 - x^2)^3}} = \frac{1}{ab^2(a^2 - b^2)} F(\eta, t) - \frac{a^2 + b^2}{ab^2(a^2 - b^2)^2} E(\eta, t) + \\ + \frac{[a^4 + b^4 - (a^2 + b^2)u^2]u}{a^2b^2(a^2 - b^2)^2\sqrt{(a^2 - u^2)(b^2 - u^2)}} \quad [a > b > u > 0].$$

BY (279.08)

3.164

Notations:

$$\alpha = \arccos \frac{u^2 - \rho\bar{\rho}}{u^2 + \rho\bar{\rho}}, r = \frac{1}{2} \sqrt{-\frac{(\rho - \bar{\rho})^2}{\rho\bar{\rho}}}.$$

$$1. \int_u^\infty \frac{dx}{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{1}{\sqrt{\rho\bar{\rho}}} F(\alpha, r).$$

BY (225.00)

$$2. \int_u^\infty \frac{x^2 dx}{(x^2 - \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = \frac{2u \sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}}{(\rho + \bar{\rho})^2 (u^4 - \rho^2 \bar{\rho}^2)} - \frac{1}{(\rho + \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} E(\alpha, r).$$

BY (225.03)

$$3. \int_u^\infty \frac{x^2 dx}{(x^2 + \rho\bar{\rho})^2 \sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}} = -\frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} [F(\alpha, r) - E(\alpha, r)].$$

BY (225.07)

309

$$4. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho^2 - \bar{\rho}^2)^2} E(\alpha, r) + \frac{1}{(\rho - \bar{\rho})^2 \sqrt{\rho\bar{\rho}}} F(\alpha, r) - \frac{2u(u^2 - \rho\bar{\rho})}{(\rho + \bar{\rho})^2 (u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}}.$$

BY (225.05)

$$5. \int_u^\infty \frac{(x^2 - \rho\bar{\rho})^2 dx}{\sqrt{(x^2 + \rho^2)^3 (x^2 + \bar{\rho}^2)^3}} = -\frac{4\sqrt{\rho\bar{\rho}}}{(\rho - \bar{\rho})^2} [F(\alpha, r) - E(\alpha, r)] + \frac{2u(u^2 - \rho\bar{\rho})}{(u^2 + \rho\bar{\rho}) \sqrt{(u^2 + \rho^2)(u^2 + \bar{\rho}^2)}}.$$

BY (225.06)

$$6. \int_u^\infty \frac{\sqrt{(x^2 + \rho^2)(x^2 + \bar{\rho}^2)}}{(x^2 + \rho\bar{\rho})^2} dx = \frac{1}{\sqrt{\rho\bar{\rho}}} E(\alpha, r).$$

$$7. \int_u^\infty \frac{(x^2 - \varrho\bar{\varrho})^2 dx}{(x^2 + \varrho\bar{\varrho})^2 \sqrt{(x^2 + \varrho^2)(x^2 + \bar{\varrho}^2)}} = -\frac{4\sqrt{\varrho\bar{\varrho}}}{(\varrho - \bar{\varrho})^2} E(\alpha, r) + \frac{(\varrho + \bar{\varrho})^2}{(\varrho - \bar{\varrho})^2 \sqrt{\varrho\bar{\varrho}}} F(\alpha, r).$$

BY (225.08)

$$8. \int_u^\infty \frac{(x^2 + \varrho\bar{\varrho})^2 dx}{[(x^2 + \varrho\bar{\varrho})^2 - 4p^2\varrho\bar{\varrho}x^2] \sqrt{(x^2 + \varrho^2)(x^2 + \bar{\varrho}^2)}} = \frac{1}{\sqrt{\varrho\bar{\varrho}}} \Pi(\alpha, p^2, r).$$

BY (225.02)

3.165

Notations:

$$\alpha = \arccos \frac{u^2 - a^2}{u^2 + a^2}, \quad r = \frac{\sqrt{a^2 - b^2}}{a\sqrt{2}}.$$

$$1. \int_u^a \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{\sqrt{2}}{a\sqrt{2} + \sqrt{a^2 + b^2}} \times \\ \times F \left[\operatorname{arctg} \left(\frac{a\sqrt{2} + \sqrt{a^2 - b^2}}{\sqrt{a^2 + b^2}} \frac{a - u}{a + u} \right), \frac{2\sqrt{a\sqrt{2}(a^2 - b^2)}}{a\sqrt{2} + \sqrt{a^2 - b^2}} \right] \\ [a > b, \quad a > u \geq 0].$$

BY (264.00)

$$2. \int_u^\infty \frac{dx}{\sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a} F(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

BY (263.00, 266.00)

$$3. \int_u^\infty \frac{dx}{x^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{2a^3} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 2b^2u^2 + a^4}}{a^2u(u^2 + a^2)} \\ [a > b > 0, \quad u > 0].$$

BY (263.06)

$$4. \int_u^\infty \frac{x^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{1}{4a(a^2 - b^2)} [F(\alpha, r) - E(\alpha, r)] \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

$$5. \int_u^\infty \frac{x^2 dx}{(x^2 - a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{u\sqrt{u^4 + 2b^2u^2 + a^4}}{2(a^2 + b^2)(u^4 - a^4)} - \frac{1}{4a(a^2 + b^2)} E(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad u^2 > a^2 > 0].$$

BY (263.05, 266.02)

$$6. \int_u^\infty \frac{x^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{2(a^4 - b^4)} E(\alpha, r) - \frac{1}{4a(a^2 - b^2)} F(\alpha, r) - \\ - \frac{u(u^2 - a^2)}{2(a^2 + b^2)(u^2 + a^2)\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

BY (263.08, 266.03)

$$7. \int_u^\infty \frac{(x^2 - a^2)^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 - b^2} [F(\alpha, r) - E(\alpha, r)] + \frac{u^2 - a^2}{u^2 + a^2} \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0].$$

BY (266.08)

$$8. \int_u^\infty \frac{(x^2 + a^2)^2 dx}{\sqrt{(x^4 + 2b^2x^2 + a^4)^3}} = \frac{a}{a^2 + b^2} E(\alpha, r) - \frac{a^2 - b^2}{a^2 + b^2} \cdot \frac{u^2 - a^2}{u^2 + a^2} \cdot \frac{u}{\sqrt{u^4 + 2b^2u^2 + a^4}} \\ [|b^2| < a^2, \quad u \geq 0].$$

BY (266.06)a

$$9. \int_u^\infty \frac{(x^2 - a^2)^2 dx}{(x^2 + a^2)^2 \sqrt{x^4 + 2b^2x^2 + a^4}} = \frac{a}{a^2 - b^2} E(\alpha, r) - \frac{a^2 + b^2}{2a(a^2 - b^2)} F(\alpha, r) \\ [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0]$$

BY (263.04, 266.07)

$$10. \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 + a^2)^2} dx = \frac{1}{2a} E(\alpha, r) \quad [a^2 > b^2 > -\infty, \quad a^2 > 0, \quad u \geq 0].$$

BY (263.01, 266.01)

$$11. \int_u^\infty \frac{\sqrt{x^4 + 2b^2x^2 + a^4}}{(x^2 - a^2)^2} dx = \frac{1}{2a} [F(\alpha, r) - E(\alpha, r)] + \frac{u}{u^4 - a^4} \sqrt{u^4 + 2b^2u^2 + a^4} \\ [a > b > 0, \quad u > a].$$

$$12. \int_u^\infty \frac{(x^2 + a^2)^2 dx}{[(x^2 + a^2)^2 - 4a^2 p^2 x^2] \sqrt{x^4 + 2b^2 x^2 + a^4}} = \frac{1}{2a} \Pi(\alpha, p^2, r) \quad [a > b > 0, \quad u \geq 0].$$

BY (263.02)

3.166

Notations:

$$\alpha = \arccos \frac{u^2 - 1}{u^2 + 1}, \beta = \operatorname{arctg} \left\{ \frac{(1 + \sqrt{2})\{1 - u\}}{1 + u} \right\},$$

$$\begin{aligned} \gamma &= \arccos u, \quad \delta = \arccos \frac{1}{u}, \quad \varepsilon = \arccos \frac{1 - u^2}{1 + u^2}, \\ r &= \frac{\sqrt{2}}{2}, \quad q = 2\sqrt{3\sqrt{2} - 4} = 2^{\sqrt[4]{2}}(\sqrt{2} - 1) \approx 0,985171. \end{aligned}$$

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$$1. \int_u^\infty \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2} F(\alpha, r) \quad [u \geq 0].$$

ZH (287), BY (263.50)

$$2. \int_u^\infty \frac{dx}{x^2 \sqrt{x^4 + 1}} = \frac{1}{2} [F(\alpha, r) - 2E(\alpha, r)] + \frac{\sqrt{u^4 + 1}}{u(u^2 + 1)} \quad [u > 0].$$

BY (263.57)

$$3. \int_u^\infty \frac{x^2 dx}{(x^4 + 1)\sqrt{x^4 + 1}} = \frac{1}{2} E(\alpha, r) - \frac{1}{4} F(\alpha, r) - \frac{u(u^2 - 1)}{2(u^2 + 1)\sqrt{u^4 + 1}} \quad [u \geq 0].$$

BY (263.59)

$$4. \int_u^\infty \frac{x^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = \frac{1}{4} [F(\alpha, r) - E(\alpha, r)] \quad [u \geq 0].$$

BY (263.53)

$$5. \int_u^\infty \frac{x^2 dx}{(x^2 - 1)^2 \sqrt{x^4 + 1}} = \frac{u\sqrt{u^4 + 1}}{2(u^4 - 1)} - \frac{1}{4} E(\alpha, r) \quad [u > 1].$$

BY (263.55)

BY (263.58)

$$7. \int_u^\infty \frac{(x^2 - 1)^2 dx}{(x^2 + 1)^2 \sqrt{x^4 + 1}} = E(\alpha, r) - \frac{1}{2}F(\alpha, r) \quad [u \geq 0].$$

BY (263.54)

$$8. \int_u^\infty \frac{\sqrt{x^4 + 1} dx}{(x^2 + 1)^2} = \frac{1}{2}E(\alpha, r) \quad [u \geq 0].$$

BY (263.51)

$$9. \int_u^\infty \frac{(x^2 + 1)^2 dx}{[(x^2 + 1)^2 - 4p^2x^2]\sqrt{x^4 + 1}} = \frac{1}{2}\Pi(\alpha, p^2, r) \quad [u \geq 0].$$

BY (263.52)

$$10. \int_0^u \frac{dx}{\sqrt{x^4 + 1}} = \frac{1}{2}F(\varepsilon, r).$$

ZH 66(288)

$$11. \int_u^1 \frac{dx}{\sqrt{x^4 + 1}} = (2 - \sqrt{2})F(\beta, q) \quad [0 \leq u < 1].$$

BY (264.50)

$$12. \int_u^1 \frac{(x^2 + x\sqrt{2} + 1) dx}{(x^2 - x\sqrt{2} + 1)\sqrt{x^4 + 1}} = (2 + \sqrt{2})E(\beta, q) \quad [0 \leq u < 1].$$

BY (264.51)

$$13. \int_u^1 \frac{(1 - x)^2 dx}{(x^2 - x\sqrt{2} + 1)\sqrt{x^4 + 1}} = \frac{1}{\sqrt{2}}[F(\beta, q) - E(\beta, q)] \quad [0 \leq u < 1].$$

BY (264.55)

$$14. \int_u^1 \frac{(1 + x)^2 dx}{(x^2 - x\sqrt{2} + 1)\sqrt{x^4 + 1}} = \frac{3\sqrt{2} + 4}{2}E(\beta, q) - \frac{3\sqrt{2} - 4}{2}F(\beta, q) \quad [0 \leq u < 1].$$

$$15. \int_u^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1].$$

ZH 66 (290), BY (259.75)

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$$16. \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}} \left\{ \Gamma\left(\frac{1}{4}\right) \right\}^2.$$

$$17. \int_1^u \frac{dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}} F(\delta, r) \quad [u > 1].$$

ZH 66 (289), BY (260.75)

$$\begin{aligned} 18.^8 \int_u^1 \frac{x^2 dx}{\sqrt{1-x^4}} &= \sqrt{2} E(\gamma, r) - \frac{1}{\sqrt{2}} F(\gamma, r) \quad [u < 1]; \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \Gamma\left(\frac{3}{4}\right) \right\}^2 \quad [u = 0]. \end{aligned}$$

BY (259.76)

$$19. \int_1^u \frac{x^2 dx}{\sqrt{x^4-1}} = \frac{1}{\sqrt{2}} F(\delta, r) - \sqrt{2} E(\delta, r) + \frac{1}{u} \sqrt{u^4-1} \quad [u > 1].$$

BY (260.77)

$$20. \int_u^1 \frac{x^4 dx}{\sqrt{1-x^4}} = \frac{1}{3\sqrt{2}} F(\gamma, r) + \frac{u}{3} \sqrt{1-u^4} \quad [u < 1].$$

BY (259.76)

$$21.^3 \int_1^u \frac{x^4 dx}{\sqrt{x^4-1}} = \frac{1}{3\sqrt{2}} F(\delta, r) + \frac{1}{3} u \sqrt{u^4-1} \quad [u > 1].$$

BY (260.77)

$$22. \int_0^u \frac{dx}{\sqrt{x(1+x^3)}} = \frac{1}{\sqrt[4]{3}} F\left(\arccos \frac{1+(1-\sqrt{3})u}{1+(1+\sqrt{3})u}, \frac{\sqrt{2+\sqrt{3}}}{2}\right) \quad [u > 0].$$

$$23. \int_0^u \frac{dx}{\sqrt{x(1-x^3)}} = \frac{1}{\sqrt[4]{3}} F \left(\arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \frac{\sqrt{2 - \sqrt{3}}}{2} \right) \quad [1 \geq u > 0].$$

BY (259.50)

In 3.167 and 3.168 we set: $\alpha = \arcsin \sqrt{\frac{(a-c)(d-u)}{(a-d)(c-u)}}$,

$$\begin{aligned} \beta &= \arcsin \sqrt{\frac{(a-c)(u-d)}{(c-d)(a-u)}}, & \gamma &= \arcsin \sqrt{\frac{(b-d)(c-u)}{(c-d)(b-u)}}, \\ \delta &= \arcsin \sqrt{\frac{(b-d)(u-c)}{(b-c)(u-d)}}, & \{ &= \arcsin \sqrt{\frac{(a-c)(b-u)}{(b-c)(a-u)}}, \\ \lambda &= \arcsin \sqrt{\frac{(a-c)(u-b)}{(a-b)(u-c)}}, & \mu &= \arcsin \sqrt{\frac{(b-d)(a-u)}{(a-b)(u-d)}}, \\ \nu &= \arcsin \sqrt{\frac{(b-d)(u-a)}{(a-d)(u-b)}}, & q &= \sqrt{\frac{(b-c)(a-d)}{(a-c)(b-d)}}, & r &= \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}. \end{aligned}$$

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3.167

$$1. \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left(\alpha, \frac{a-d}{a-c}, q \right) - F(\alpha, q) \right\} \\ [a > b > c > d > u].$$

BY (251.05)

$$2. \int_u^d \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left\{ \Pi \left(\beta, \frac{d-c}{a-c}, r \right) - F(\beta, r) \right\} \\ [a > b > c \geq u > d].$$

BY (252.14)

$$3. \int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (c-b) \Pi \left(\gamma, \frac{c-d}{b-d}, r \right) + (b-d) F(\gamma, r) \right\} \\ [a > b > c > u \geq d].$$

BY (253.14)

$$4. \int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi \left(\delta, \frac{b-c}{b-d}, q \right)$$

$$5. \int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-a)\Pi\left(\lambda, \frac{b-c}{a-c}, q\right) + (a-d)F(\lambda, q) \right\}$$

$[a > b > u \geq c > d].$

BY (255.20)

$$6. \int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (b-c)\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (c-d)F(\lambda, r) \right\}$$

$[a \geq u > b > c > d].$

BY (256.13)

$$7. \int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\mu, \frac{b-a}{b-d}, r\right) \quad [a > u \geq b > c > d].$$

BY (257.02)

$$8. \int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left\{ (a-b)\Pi\left(\nu, \frac{a-d}{b-d}, q\right) + (b-d)F(\nu, q) \right\}$$

$[u > a > b > c > d].$

BY (258.14)

$$9. \int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)(d-x)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\alpha, \frac{a-d}{a-c}, q\right)$$

$[a > b > c > d > u].$

BY (251.02)

$$10. \int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d)\Pi\left(\beta, \frac{d-c}{a-c}, r\right) - (a-c)F(\beta, r) \right]$$

$[a > b > c \geq u > d].$

BY (252.13)

$$11. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\gamma, \frac{c-d}{b-d}, r\right) - F(\gamma, r) \right]$$

$[a > b > c > u \geq d].$

$$12. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2(c-d)}{\sqrt{(a-c)(b-d)}} \left[\Pi \left(\delta, \frac{b-c}{b-d}, q \right) - F(\delta, q) \right] \\ [a > b \geq u > c > d].$$

BY (254.12)

$$13. \int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-a)\Pi \left(\left\{, \frac{b-c}{a-c}, q \right\} \right) + (a-c)F(\{, q) \right] \\ [a > b > u \geq c > d].$$

BY (259.19)

$$14. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi \left(\lambda, \frac{a-b}{a-c}, r \right) \quad [a \geq u > b > c > d].$$

BY (256.02)

$$15. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d)\Pi \left(\mu, \frac{b-a}{b-d}, r \right) + (d-c)F(\mu, r) \right] \\ [a > u \geq b > c > d].$$

BY (257.13)

$$16. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-b)\Pi \left(\nu, \frac{a-d}{b-d}, q \right) + (b-c)F(\nu, q) \right] \\ [u > a > b > c > d].$$

BY (258.13)

$$17. \int_u^d \sqrt{\frac{b-x}{(a-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d)\Pi \left(\alpha, \frac{a-d}{a-c}, q \right) + (b-c)F(\alpha, q) \right] \\ [a > b > c > d > u].$$

BY (251.07)

$$18. \int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d)\Pi \left(\beta, \frac{d-c}{a-c}, r \right) - (a-b)F(\beta, r) \right] \\ [a > b > c \geq u > d].$$

BY (252.15)

$$19. \int_u^c \sqrt{\frac{b-x}{(a-x)(c-x)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \Pi\left(\gamma, \frac{c-d}{b-d}, r\right) \quad [a > b > c > u \geq d].$$

BY (253.02)

$$20. \int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (b-d) F(\delta, q) \right] \\ [a > b \geq u > c > d].$$

BY (254.14)

$$21. \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\left\{, \frac{b-c}{a-c}, q\right\} - F(\{, q)\right] \\ [a > b > u \geq c > d].$$

BY (255.21)

$$22. \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\lambda, \frac{a-b}{a-c}, r\right) - F(\lambda, r) \right] \\ [a \geq u > b > c > d].$$

BY (256.15)

$$23.^8 \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(a-d) \Pi\left(\mu, \frac{b-a}{b-d}, r\right) - (b-d) F(\mu, r) \right] \\ [a > u \geq b > c > d].$$

BY (257.15)

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$$24. \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\nu, \frac{a-d}{b-d}, q\right) \quad [u > a > b > c > d].$$

BY (258.02)

$$25. \int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)(d-x)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-d) \Pi\left(\alpha, \frac{a-d}{a-c}, q\right) + (a-c) F(\alpha, q) \right] \\ [a > b > c > d > u].$$

$$26. \int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2(a-d)}{\sqrt{(a-c)(b-d)}} \Pi\left(\beta, \frac{d-c}{a-c}, r\right) \quad [a > b > c \geq u > d].$$

BY (252.02)

$$27. \int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(b-c) \Pi\left(\gamma, \frac{c-d}{b-d}, r\right) + (a-b) F(\gamma, r) \right] \\ [a > b > c > u \geq d].$$

BY (253.15)

$$28. \int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(d-c) \Pi\left(\delta, \frac{b-c}{b-d}, q\right) + (a-d) F(\delta, q) \right] \\ [a > b \geq u > c > d].$$

BY (254.13)

$$29. \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \Pi\left(\zeta, \frac{b-c}{a-c}, q\right) \quad [a > b > u \geq c > d].$$

BY (255.02)

$$30. \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2}{\sqrt{(a-c)(b-d)}} \left[(c-b) \Pi\left(\lambda, \frac{a-b}{a-c}, r\right) + (a-c) F(\lambda, r) \right] \\ [a \geq u > b > c > d].$$

BY (256.14)

$$31. \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)}} dx = \frac{2(d-a)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\mu, \frac{b-a}{b-d}, r\right) - F(\mu, r) \right] \\ [a > u \geq b > c > d].$$

BY (257.14)

$$32. \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)}} dx = \frac{2(a-b)}{\sqrt{(a-c)(b-d)}} \left[\Pi\left(\nu, \frac{a-d}{b-d}, q\right) - F(\nu, q) \right] \\ [u > a > b > c > d].$$

BY (258.15)

3.168

$$1. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{d-a} \left[\sqrt{\frac{a-c}{b-d}} E(\gamma, r) - \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \right] \\ [a > b > c > u > d].$$

BY (253.06)

$$2. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)] \quad [a > b \geq u > c > d].$$

BY (254.04)

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$$3. \int_u^b \sqrt{\frac{x-c}{(a-x)(b-x)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\{, q) - E(\{, q)] + \\ + \frac{2}{b-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \quad [a > b > u \geq c > d].$$

BY (255.09)

$$4. \int_b^u \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \left[\sqrt{\frac{a-c}{b-d}} E(\lambda, r) - \frac{c-d}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \right] \\ [a \geq u > b > c > d].$$

BY (256.06)

$$5. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} E(\mu, r) \quad [a > u \geq b > c > d].$$

BY (257.01)

$$6. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)(x-d)^3}} dx = \frac{2}{a-d} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)] + \\ + \frac{2}{a-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} \quad [u > a > b > c > d].$$

BY (258.10)

BY (253.03)

$$8. \int_c^u \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d)E(\delta, q) - (a-b)(c-d)F(\delta, q)] \\ [a > b \geq u > c > d].$$

BY (254.15)

$$9. \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d)E(\delta, q) - (a-b)(c-d)F(\delta, q)] - \\ - \frac{2}{c-d} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \quad [a > b > u \geq c > d].$$

BY (255.06)

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$$10. \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = \frac{2}{(a-d)(c-d)\sqrt{(a-c)(b-d)}} \times \\ \times [(a-c)(b-d)E(\lambda, r) - (a-d)(b-c)F(\lambda, r)] - \\ - \frac{2}{a-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \quad [a \geq u > b > c > d].$$

BY (256.03)

$$11. \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)(x-d)^3}} dx = 2 \frac{\sqrt{(a-c)(b-d)}}{(a-d)(c-d)} E(\mu, r) - \\ - \frac{2(b-c)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r) \\ [a > u \geq b > c > d].$$

BY (257.09)

$$12. \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)(x-d)^3}} dx = \frac{2(b-d)}{(a-d)(c-d)} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} + \\ + \frac{2(a-b)}{(a-d)\sqrt{(a-c)(b-d)}} F(\nu, q) + \\ + 2 \frac{\sqrt{(a-c)(b-d)}}{(c-d)\sqrt{(a-c)(b-d)}} E(\nu, q) \quad [u > a > b > c > d].$$

$$13. \int_u^c \sqrt{\frac{a-x}{(b-x)(c-x)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] + \\ + \frac{2}{c-d} \sqrt{\frac{(a-u)(c-u)}{(b-u)(u-d)}} \quad [a > b > c > u > d].$$

BY (253.04)

$$14. \int_c^u \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\delta, q) \quad [a > b \geq u > c > d].$$

BY (254.01)

$$15. \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\{, q) - \frac{2(a-d)}{(b-d)(c-d)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \\ [a > b > u \geq c > d].$$

BY (255.08)

$$16. \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\lambda, r) - E(\lambda, r)] + \\ + \frac{2}{b-d} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \quad [a \geq u > b > c > d].$$

BY (256.05)

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$$17. \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)(x-d)^3}} dx = \frac{2}{c-d} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)] \quad [a > u \geq b > c > d].$$

BY (257.06)

$$18. \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)(x-d)^3}} dx = \frac{-2}{c-d} \sqrt{\frac{a-c}{b-d}} E(\nu, q) + \frac{2}{c-d} \sqrt{\frac{(u-a)(u-c)}{(u-b)(u-d)}} \\ [u > a > b > c > d].$$

BY (258.05)

$$19. \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)(c-x)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)] \quad [a > b > c > d > u].$$

$$20. \int_d^u \sqrt{\frac{x-d}{(a-x)(b-x)(c-x)^3}} dx = \frac{-2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\beta, r) + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \\ [a > b > c \geq u > d].$$

BY (252.06)

$$21. \int_u^b \sqrt{\frac{x-d}{(a-x)(b-x)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\{, q) - E(\{, q)] + \\ + \frac{2}{b-c} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \quad [a > b > u > c > d].$$

BY (255.05)

$$22. \int_b^u \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\lambda, r) \quad [a \geq u > b > c > d].$$

BY (256.01)

$$23. \int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} E(\mu, r) - \frac{2(c-d)}{(a-c)(b-c)} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \\ [a > u \geq b > c > d].$$

BY (257.06)

$$24. \int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)(x-c)^3}} dx = \frac{2}{b-c} \sqrt{\frac{b-d}{a-c}} [F(\nu, q) - E(\nu, q)] + \\ + \frac{2}{a-c} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \quad [u > a > b > c > d].$$

BY (258.06)

$$25. \int_u^a \sqrt{\frac{b-x}{(a-x)(c-x)^3(d-x)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\alpha, q) \quad [a > b > c > d > u].$$

BY (251.01)

$$26. \int_d^u \sqrt{\frac{b-x}{(a-x)(c-x)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\beta, r) - E(\beta, r)] + \\ + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \quad [a > b > c > u > d].$$

$$27. \int_u^b \sqrt{\frac{b-x}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{d-c} \sqrt{\frac{b-d}{a-c}} E(\lambda, q) + \frac{2}{c-d} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \\ [a > b > u > c > d].$$

BY (255.03)

$$28. \int_b^u \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] \quad [a \geq u > b > c > d].$$

BY (256.08)

$$29. \int_u^a \sqrt{\frac{x-b}{(a-x)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} [F(\mu, r) - E(\mu, r)] + \\ + \frac{2}{a-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \quad [a > u \geq b > c > d].$$

BY (257.03)

$$30. \int_a^u \sqrt{\frac{x-b}{(x-a)(x-c)^3(x-d)}} dx = \frac{2}{c-d} \sqrt{\frac{b-d}{a-c}} E(\nu, q) - \frac{2(b-c)}{(a-c)(c-d)} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \\ [u > a > b > c > d].$$

BY (258.03)

$$31. \int_u^d \sqrt{\frac{a-x}{(b-x)(c-x)^3(d-x)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\alpha, q) - \\ - \frac{a-b}{b-c} \frac{2}{\sqrt{(a-c)(b-d)}} F(\alpha, q) \quad [a > b > c > d > u].$$

BY (251.08)

$$32. \int_d^u \sqrt{\frac{a-x}{(b-x)(c-x)^3(x-d)}} dx = \\ = \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\beta, r) - 2 \frac{\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\beta, r) + \\ + 2 \frac{a-c}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(c-u)}} \quad [a > b > c > u > d].$$

$$\begin{aligned}
33. \quad \int_u^b \sqrt{\frac{a-x}{(b-x)(x-c)^3(x-d)}} dx &= \\
&= \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-c)}} F(\{, q) - 2\sqrt{\frac{(a-c)(b-d)}{(b-c)(c-d)}} E(\{, q) + \\
&\quad + \frac{2(a-c)}{(b-c)(c-d)} \sqrt{\frac{(b-u)(u-d)}{(a-u)(u-c)}} \quad [a > b > u > c > d].
\end{aligned}$$

BY (255.04)

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$$\begin{aligned}
34. \quad \int_b^u \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx &= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\lambda, r) - \\
&\quad - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\lambda, r) \quad [a \geq u > b > c > d].
\end{aligned}$$

BY (256.09)

$$\begin{aligned}
35. \quad \int_u^a \sqrt{\frac{a-x}{(x-b)(x-c)^3(x-d)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\mu, r) - \frac{2(a-d)}{(c-d)\sqrt{(a-c)(b-d)}} F(\mu, r) - \\
&\quad - \frac{2}{b-c} \sqrt{\frac{(a-u)(u-b)}{(u-c)(u-d)}} \quad [a > u \geq b > c > d].
\end{aligned}$$

BY (257.04)

$$\begin{aligned}
36. \quad \int_a^u \sqrt{\frac{x-a}{(x-b)(x-c)^3(x-d)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(b-c)(c-d)} E(\nu, q) - \frac{2(a-b)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q) - \\
&\quad - \frac{2}{c-d} \sqrt{\frac{(u-a)(u-d)}{(u-b)(u-c)}} \quad [u > a > b > c > d].
\end{aligned}$$

BY (258.04)

$$\begin{aligned}
37. \quad \int_u^d \sqrt{\frac{d-x}{(a-x)(b-x)^3(c-x)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\alpha, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\alpha, q) - \\
&\quad - \frac{2}{a-b} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \quad [a > b > c > d > u].
\end{aligned}$$

BY (251.11)

$$\begin{aligned}
 39. \int_u^c \sqrt{\frac{x-d}{(a-x)(b-x)^3(c-x)}} dx &= \\
 &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\gamma, r) - \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r) \\
 &\quad [a > b > c > u \geq d].
 \end{aligned}$$

BY (253.07)

$$\begin{aligned}
 40. \int_c^u \sqrt{\frac{x-d}{(a-x)(b-x)^3(x-c)}} dx &= \\
 &= \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\delta, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\delta, q) + \\
 &\quad + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \quad [a > b > u > c > d].
 \end{aligned}$$

BY (254.05)

$$\begin{aligned}
 41. \int_u^a \sqrt{\frac{x-d}{(a-x)(x-b)^3(x-c)}} dx &= \\
 &= \frac{2(a-d)}{(a-b)\sqrt{(a-c)(b-d)}} F(\mu, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\mu, r) + \\
 &\quad + \frac{2(b-d)}{(a-b)(b-c)} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \quad [a > u > b > c > d].
 \end{aligned}$$

BY (257.07)

$$\begin{aligned}
 42. \int_a^u \sqrt{\frac{x-d}{(x-a)(x-b)^3(x-c)}} dx &= \\
 &= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-c)} E(\nu, q) - \frac{2(c-d)}{(b-c)\sqrt{(a-c)(b-d)}} F(\nu, q) \\
 &\quad [u > a > b > c > d].
 \end{aligned}$$

BY (258.07)

$$\begin{aligned}
 43. \int_u^d \sqrt{\frac{c-x}{(a-x)(b-x)^3(d-x)}} dx &= \\
 &= \frac{2}{a-b} \sqrt{\frac{c-d}{b-d}} E(\alpha, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(b-d)} \sqrt{\frac{(a-u)(u-c)}{(b-u)(c-u)}}
 \end{aligned}$$

$$44. \int_d^u \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\beta, r) - E(\beta, r)] + \\ + \frac{2}{b-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \quad [a > b > c \geq u > d].$$

BY (252.10)

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$$45. \int_u^c \sqrt{\frac{c-x}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\gamma, r) - E(\gamma, r)] \quad [a > b > c > u \geq d].$$

BY (254.08)

$$46. \int_c^u \sqrt{\frac{x-c}{(a-x)(b-x)^3(x-d)}} dx = \frac{2}{b-a} \sqrt{\frac{a-c}{b-d}} E(\delta, q) + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \\ [a > b \geq u > c > d].$$

BY (254.08)

$$47. \int_u^a \sqrt{\frac{x-c}{(a-x)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} [F(\mu, r) - E(\mu, r)] + \\ + \frac{2}{a-b} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \quad [a > u \geq b > c > d].$$

BY (257.10)

$$48. \int_a^u \sqrt{\frac{x-c}{(x-a)(x-b)^3(x-d)}} dx = \frac{2}{a-b} \sqrt{\frac{a-c}{b-d}} E(\nu, q) \quad [u > a > b > c > d].$$

BY (258.01)

$$49. \int_u^d \sqrt{\frac{a-x}{(b-x)^3(c-x)(d-x)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\alpha, q) - E(\alpha, q)] + \\ + \frac{2}{b-d} \sqrt{\frac{(a-u)(d-u)}{(b-u)(c-u)}} \quad [a > b > c > d > u].$$

BY (251.12)

$$50. \int_d^u \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\beta, r) - \frac{2(a-b)}{(b-c)(b-d)} \sqrt{\frac{(u-d)(c-u)}{(a-u)(b-u)}} \\ [a > b > c \geq u > d].$$

BY (252.09)

$$51. \int_u^c \sqrt{\frac{a-x}{(b-x)^3(c-x)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} E(\gamma, r) \quad [a > b > c > u \geq d].$$

BY (253.01)

$$52. \int_c^u \sqrt{\frac{a-x}{(b-x)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\delta, q) - E(\delta, q)] + \\ + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(b-u)(u-d)}} \quad [a > b > u > c > d].$$

BY (254.06)

$$53. \int_u^a \sqrt{\frac{a-x}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{c-b} \sqrt{\frac{a-c}{b-d}} E(\mu, r) + \frac{2}{b-c} \sqrt{\frac{(a-u)(u-c)}{(u-b)(u-d)}} \\ [a > u > b > c > d].$$

BY (257.08)

$$54. \int_a^u \sqrt{\frac{x-a}{(x-b)^3(x-c)(x-d)}} dx = \frac{2}{b-c} \sqrt{\frac{a-c}{b-d}} [F(\nu, q) - E(\nu, q)] \quad [u > a > b > c > d].$$

BY (258.08)

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$$55. \int_u^d \sqrt{\frac{d-x}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{b-a} \sqrt{\frac{b-d}{a-c}} E(\alpha, q) + \frac{2}{a-b} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \\ [a > b > c > d > u].$$

BY (251.09)

$$56. \int_d^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\beta, q) - E(\beta, q)] \quad [a > b > c \geq u > d].$$

$$57. \int_u^c \sqrt{\frac{x-d}{(a-x)^3(b-x)(c-x)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\gamma, r) - E(\gamma, r)] + \\ + \frac{2}{a-c} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \quad [a > b > c > u \geq d].$$

BY (253.05)

$$58. \int_c^u \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \\ = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\delta, q) - \frac{2(a-d)}{(a-b)(a-c)} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \quad [a > b \geq u > c > d].$$

BY (254.03)

$$59. \int_u^b \sqrt{\frac{x-d}{(a-x)^3(b-x)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} E(\{, q) \quad [a > b > u \geq c > d].$$

BY (255.01)

$$60. \int_b^u \sqrt{\frac{x-d}{(a-x)^3(x-b)(x-c)}} dx = \frac{2}{a-b} \sqrt{\frac{b-d}{a-c}} [F(\lambda, r) - E(\lambda, r)] + \\ + \frac{2}{a-b} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \quad [a > u > b > c > d].$$

BY (256.10)

$$61. \int_u^d \sqrt{\frac{c-x}{(a-x)^3(b-x)(d-x)}} dx = \\ = \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\alpha, q) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\alpha, q) + \\ + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \quad [a > b > c > d > u].$$

BY (251.15)

$$62. \int_d^u \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx = \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\beta, r) - \\ - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\beta, r) \\ [a > b > c \geq u > d].$$

$$\begin{aligned}
63. \int_u^c \sqrt{\frac{c-x}{(a-x)^3(b-x)(x-d)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\gamma, r) - \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\gamma, r) - \\
&\quad - \frac{2}{a-d} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \quad [a > b > c > u \geq d].
\end{aligned}$$

BY (253.10)

$$\begin{aligned}
64. \int_c^u \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\delta, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\delta, q) - \\
&\quad - \frac{2}{a-b} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \quad [a > b \geq u > c > d].
\end{aligned}$$

BY (254.09)

$$\begin{aligned}
65. \int_u^b \sqrt{\frac{x-c}{(a-x)^3(b-x)(x-d)}} dx &= \\
&= \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\{, q) - \frac{2(c-d)}{(a-d)\sqrt{(a-c)(b-d)}} F(\{, q) \\
&\quad [a > b > u \geq c > d].
\end{aligned}$$

BY (255.10)

$$\begin{aligned}
66. \int_b^u \sqrt{\frac{x-c}{(a-x)^3(x-b)(x-d)}} dx &= \\
&= \frac{2(b-c)}{(a-b)\sqrt{(a-c)(b-d)}} F(\lambda, r) - \frac{2\sqrt{(a-c)(b-d)}}{(a-b)(a-d)} E(\lambda, r) + \\
&\quad + \frac{2(a-c)}{(a-b)(a-d)} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \quad [a > u > b > c > d].
\end{aligned}$$

BY (256.07)

$$\begin{aligned}
67. \int_u^d \sqrt{\frac{b-x}{(a-x)^3(c-x)(d-x)}} dx &= \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\alpha, q) - E(\alpha, q)] + \\
&\quad + \frac{2}{a-d} \sqrt{\frac{(b-u)(d-u)}{(a-u)(c-u)}} \quad [a > b > c > d > u].
\end{aligned}$$

$$68. \int_d^u \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\beta, r) \quad [a > b > c \geq u > d].$$

BY (252.01)

$$69. \int_u^c \sqrt{\frac{b-x}{(a-x)^3(c-x)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\gamma, r) - \frac{2(a-b)}{(a-c)(a-d)} \sqrt{\frac{(c-u)(u-d)}{(a-u)(b-u)}} \\ [a > b > c > u \geq d].$$

BY (253.08)

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$$70. \int_c^u \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\delta, q) - E(\delta, q)] + \\ + \frac{2}{a-c} \sqrt{\frac{(b-u)(u-c)}{(a-u)(u-d)}} \quad [a > b \geq u > c > d].$$

BY (254.07)

$$71. \int_u^b \sqrt{\frac{b-x}{(a-x)^3(x-c)(x-d)}} dx = \frac{2}{a-d} \sqrt{\frac{b-d}{a-c}} [F(\{, q) - E(\{, q)] \\ [a > b > u \geq c > d].$$

BY (255.07)

$$72. \int_b^u \sqrt{\frac{x-b}{(a-x)^3(x-c)(x-d)}} dx = \frac{-2}{a-d} \sqrt{\frac{b-d}{a-c}} E(\lambda, r) + \frac{2}{a-d} \sqrt{\frac{(u-b)(u-d)}{(a-u)(u-c)}} \\ [a \geq u > b > c > d].$$

BY (256.04)

In 3.169-3.172, we set: $\alpha = \operatorname{arctg} \frac{u}{b}$, $\beta = \operatorname{arctg} \frac{a}{u}$,

$$\gamma = \arcsin \frac{u}{b} \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, \quad \delta = \arccos \frac{u}{b}, \quad \varepsilon = \arccos \frac{b}{u}, \quad \xi = \arcsin \sqrt{\frac{a^2 + b^2}{a^2 + u^2}}, \\ \eta = \arcsin \frac{u}{b}, \quad \zeta = \arcsin \frac{a}{b} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}}, \quad \{ = \arcsin \frac{a}{u} \sqrt{\frac{u^2 - b^2}{a^2 - b^2}}, \\ \lambda = \arcsin \sqrt{\frac{a^2 - u^2}{a^2 - b^2}}, \quad u = \arcsin \sqrt{\frac{u^2 - a^2}{u^2 - b^2}}, \quad \nu = \arcsin \frac{a}{u}, \quad q = \frac{\sqrt{a^2 - b^2}}{a}, \\ r = \frac{b}{\sqrt{a^2 + b^2}}, \quad s = \frac{a}{\sqrt{a^2 + b^2}}, \quad t = \frac{b}{a}.$$

$$1. \int_0^u \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} dx = a \{F(\alpha, q) - E(\alpha, q)\} + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \quad [a > b, \quad u > 0].$$

$$2.^6 \int_0^u \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} dx = \frac{b^2}{a} F(\alpha, q) - a E(\alpha, q) + u \sqrt{\frac{a^2 + u^2}{b^2 + u^2}} \quad [a > b, \quad u > 0].$$

BY (221.04)

$$3. \int_0^u \sqrt{\frac{x^2 + a^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\gamma, r) - u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b \geq u > 0].$$

BY (214.11)

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$$4. \int_u^b \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} dx = \sqrt{a^2 + b^2} E(\delta, r) \quad [b > u \geq 0].$$

BY (213.01), ZH 64 (273)

$$5. \int_b^u \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} dx = \sqrt{a^2 + b^2} \{F(\varepsilon, s) - E(\varepsilon, s)\} + \frac{1}{u} \sqrt{(u^2 + a^2)(u^2 - b^2)} \quad [u > b > 0].$$

BY (211.03)

$$6. \int_0^u \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\gamma, r) - E(\gamma, r)\} + u \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b \geq u > 0].$$

BY (214.03)

$$7. \int_u^b \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} dx = \sqrt{a^2 + b^2} \{F(\delta, r) - E(\delta, r)\} \quad [b > u \geq 0].$$

BY (213.03)

$$8. \int_b^u \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} dx = \frac{1}{u} \sqrt{(a^2 + u^2)(u^2 - b^2)} - \sqrt{a^2 + b^2} E(\varepsilon, s) \quad [u > b > 0].$$

BY (211.04)

$$9. \int_0^u \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = a E(\eta, t) - \frac{a^2 - b^2}{a} F(\eta, t) \quad [a > b \geq u > 0].$$

$$10. \int_u^b \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} dx = aE(\zeta, t) - \frac{a^2 - b^2}{a} F(\zeta, t) - u \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u \geq 0].$$

BY (220.04)

$$11. \int_b^u \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = aE(\{\zeta, q\}) - \frac{b^2}{a} F(\{\zeta, q\}) - \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.04)

$$12. \int_u^a \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} dx = aE(\lambda, q) - \frac{b^2}{a} F(\lambda, q) \quad [a > u \geq b > 0].$$

BY (218.03)

$$13. \int_a^u \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} dx = \frac{a^2 - b^2}{a} F(\mu, t) - aE(\mu, t) + \mu \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.03)

$$14. \int_0^u \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = aE(\eta, t) \quad [a > b \geq u > 0].$$

ZH 64 (276), BY (219.01)

$$15. \int_u^b \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} dx = a \left\{ E(\zeta, t) - \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b > u \geq 0].$$

BY (220.03)

$$16. \int_b^u \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\{\zeta, q\}) - E(\{\zeta, q\}) \} + \frac{1}{u} \sqrt{(a^2 - u^2)(u^2 - b^2)} \quad [a \geq u > b > 0].$$

BY (217.03)

$$17. \int_u^a \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} dx = a \{ F(\lambda, q) - E(\lambda, q) \} \quad [a > u \geq b > 0].$$

$$18. \int_a^u \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} dx = u \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} - aE(\mu, t) \quad [u > a > b > 0].$$

BY (216.04)

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3.171

$$1. \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\varepsilon, s) \quad [u > b > 0].$$

BY (211.01), ZH 64 (274)

$$2. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{x^2 - b^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) - \frac{a^2}{b^2 u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \quad [u \geq b > 0].$$

BY (212.09)

$$3. \int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{b^2 - x^2}} = \frac{a^2 - b^2}{ab^2} F(\zeta, t) - \frac{a}{b^2} E(\zeta, t) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \quad [a > b > u > 0].$$

BY (220.12)

$$4. \int_b^u \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\{, q) - \frac{1}{a} F(\{, q) \quad [a \geq u > b > 0].$$

BY (217.11)

$$5. \int_u^a \frac{dx}{x^2} \sqrt{\frac{a^2 - x^2}{x^2 - b^2}} = \frac{a}{b^2} E(\lambda, q) - \frac{1}{a} f(\lambda, q) - \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{b^2 u} \quad [a > u \geq b > 0].$$

BY (218.10)

$$6. \int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\mu, t) - \frac{a^2 - b^2}{ab^2} F(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.08)

$$7. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + a^2}{x^2 + b^2}} = \frac{1}{a} F(\beta, q) - \frac{a}{b^2} E(\beta, q) + \frac{a^2}{b^2 u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \quad [a > b, \quad u > 0].$$

$$8. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 + b^2}{x^2 + a^2}} = \frac{1}{a} \{F(\beta, q) - E(\beta, q)\} + \frac{1}{u} \sqrt{\frac{b^2 + u^2}{a^2 + u^2}} \quad [a > b, \quad u > 0].$$

BY (222.09)

$$9. \int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 + x^2}} = \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{a^2 u} - \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) \quad [b > u > 0].$$

BY (213.10)

$$10. \int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\varepsilon, s) - E(\varepsilon, s)\} \quad [a > b > 0].$$

BY (211.07)

$$11. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 + x^2}} = \frac{\sqrt{a^2 + b^2}}{a^2} \{F(\xi, s) - E(\xi, s)\} + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 + u^2}} \quad [u \geq b > 0].$$

BY (212.11)

$$12. \int_u^b \frac{dx}{x^2} \sqrt{\frac{a^2 + x^2}{b^2 - x^2}} = \frac{\sqrt{a^2 + b^2}}{b^2} \{F(\delta, r) - E(\delta, r)\} + \frac{\sqrt{(b^2 - u^2)(a^2 + u^2)}}{b^2 u} \quad [b > u > 0].$$

BY (213.05)

$$13. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - a^2}{x^2 - b^2}} = \frac{a}{b^2} E(\nu, t) - \frac{a^2 - b^2}{ab^2} F(\nu, t) \quad [u \geq a > b > 0].$$

BY (215.08)

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$$14. \int_u^b \frac{dx}{x^2} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} = \frac{1}{u} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} - \frac{1}{a} E(\zeta, t) \quad [a > b > u > 0].$$

BY (220.11)

$$15. \int_b^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{a^2 - x^2}} = \frac{1}{a} \{F(\{, q) - E(\{, q)\} \quad [a \geq u > b > 0].$$

BY (217.08)

BY (217.08)

$$16. \int_u^a \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{u^2 - x^2}} = \frac{1}{a} \{F(\lambda, q) - E(\lambda, q)\} + \frac{\sqrt{(a^2 - u^2)(u^2 - b^2)}}{a^2 u} \quad [a > u \geq b > 0].$$

BY (218.08)

$$17. \int_a^u \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\mu, t) - \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u > a > b > 0].$$

BY (216.07)

$$18. \int_u^\infty \frac{dx}{x^2} \sqrt{\frac{x^2 - b^2}{x^2 - a^2}} = \frac{1}{a} E(\nu, t) \quad [u \geq a > b > 0].$$

BY (215.01), ZH 65 (281)

3.172

$$1. \int_0^u \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\alpha, q) - \frac{a^2 - b^2}{a^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u > 0].$$

BY (221.10)

$$2. \int_u^\infty \sqrt{\frac{x^2 + b^2}{(x^2 + a^2)^3}} dx = \frac{1}{a} E(\beta, q) \quad [a > b, \quad u \geq 0].$$

ZH 64 (271)

$$3. \int_0^u \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\alpha, q) \quad [a > b, \quad u > 0].$$

ZH 64 (270)

$$4. \int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 + b^2)^3}} dx = \frac{a}{b^2} E(\beta, q) - \frac{a^2 - b^2}{b^2} \frac{u}{\sqrt{(a^2 + u^2)(b^2 + u^2)}} \quad [a > b, \quad u \geq 0].$$

BY (222.06)

$$5. \int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\gamma, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\gamma, r) \quad [b \geq u > 0].$$

BY (214.08)

$$6. \int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\delta, r) - \frac{1}{\sqrt{a^2 + b^2}} F(\delta, r) - \frac{u}{a^2} \sqrt{\frac{b^2 - u^2}{a^2 + u^2}} \quad [b > u \geq 0].$$

BY (213.04)

$$7. \int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\varepsilon, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\varepsilon, s) - \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 + a^2}} \quad [u > b > 0].$$

BY (211.06)

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$$8. \int_u^\infty \sqrt{\frac{x^2 - b^2}{(a^2 + x^2)^3}} dx = \frac{\sqrt{a^2 + b^2}}{a^2} E(\xi, s) - \frac{b^2}{a^2 \sqrt{a^2 + b^2}} F(\xi, s) \quad [u \geq b > 0].$$

BY (212.08)

$$9. \int_0^u \sqrt{\frac{x^2 + a^2}{(b^2 - x^2)^3}} dx = \frac{a^2}{b^2 \sqrt{a^2 + b^2}} F(\gamma, r) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\gamma, r) + \frac{(a^2 + b^2)u}{b^2 \sqrt{(a^2 + u^2)(b^2 - u^2)}} \quad [b > u > 0].$$

BY (214.09)

$$10. \int_u^\infty \sqrt{\frac{x^2 + a^2}{(x^2 - b^2)^3}} dx = \frac{1}{\sqrt{a^2 + b^2}} F(\xi, s) - \frac{\sqrt{a^2 + b^2}}{b^2} E(\xi, s) + \frac{(a^2 + b^2)u}{b^2 \sqrt{(a^2 + u^2)(u^2 - b^2)}} \quad [u > b > 0].$$

BY (212.07)

$$11. \int_0^u \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \left\{ F(\eta, t) - E(\eta, t) + \frac{u}{a} \sqrt{\frac{b^2 - u^2}{a^2 - u^2}} \right\} \quad [a > b \geq u > 0].$$

BY (219.09)

$$12. \int_u^b \sqrt{\frac{b^2 - x^2}{(a^2 - x^2)^3}} dx = \frac{1}{a} \{ F(\zeta, t) - E(\zeta, t) \} \quad [a > b > u \geq 0].$$

$$13. \int_b^u \sqrt{\frac{x^2 - b^2}{(a^2 - x^2)^3}} dx = \frac{1}{u} \sqrt{\frac{u^2 - b^2}{a^2 - u^2}} - \frac{1}{a} E(\lambda, q) \quad [a > u > b > 0].$$

BY (217.07)

$$14. \int_u^\infty \sqrt{\frac{x^2 - b^2}{(x^2 - a^2)^3}} dx = \frac{1}{a} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - b^2}{u^2 - a^2}} \quad [u > a > b > 0].$$

BY (215.05)

$$15. \int_0^u \sqrt{\frac{a^2 - x^2}{(b^2 - x^2)^3}} dx = \frac{a}{b^2} [F(\eta, t) - E(\eta, t)] + \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{b^2 - u^2}} \quad [a > b > u > 0].$$

BY (219.10)

$$16. \int_u^a \sqrt{\frac{a^2 - x^2}{(x^2 - b^2)^3}} dx = \frac{u}{b^2} \sqrt{\frac{a^2 - u^2}{u^2 - b^2}} - \frac{a}{b^2} E(\lambda, q) \quad [a > u > b > 0].$$

BY (218.05)

$$17. \int_a^u \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\mu, t) - E(\mu, t)] \quad [u > a > b > 0].$$

BY (216.05)

$$18. \int_u^\infty \sqrt{\frac{x^2 - a^2}{(x^2 - b^2)^3}} dx = \frac{a}{b^2} [F(\nu, t) - E(\nu, t)] + \frac{1}{u} \sqrt{\frac{u^2 - a^2}{u^2 - b^2}} \quad [u \geq a > b > 0].$$

BY (215.03)

3.173

$$1. \int_u^1 \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{1 - x^2}} = \sqrt{2} \left[F \left(\arccos u, \frac{\sqrt{2}}{2} \right) - E \left(\arccos u, \frac{\sqrt{2}}{2} \right) \right] + \frac{\sqrt{1 - u^4}}{u} \quad [u < 1].$$

BY (259.77)

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$$2. \int_1^u \frac{dx}{x^2} \sqrt{\frac{x^2 + 1}{x^2 - 1}} = \sqrt{2} E \left(\arccos \frac{1}{u}, \frac{\sqrt{2}}{2} \right) \quad [u > 1].$$

In 3.174 and 3.175, we take: $\alpha = \arccos \frac{1+(1-\sqrt{3})u}{1+(1+\sqrt{3})u}$,

$$\beta = \arccos \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u}, \quad p = \frac{\sqrt{2 + \sqrt{3}}}{2}, \quad q = \frac{\sqrt{2 - \sqrt{3}}}{2}.$$

3.174

$$1. \int_0^u \frac{dx}{[1 + (1 + \sqrt{3})x]^2} \sqrt{\frac{1 - x + x^2}{x(1 + x)}} = \frac{1}{\sqrt[4]{3}} E(\alpha, p) \quad [u > 0].$$

BY (260.51)

$$2. \int_0^u \frac{dx}{[1 + (\sqrt{3} - 1)x]^2} \sqrt{\frac{1 + x + x^2}{x(1 - x)}} = \frac{1}{\sqrt[4]{3}} E(\beta, q) \quad [1 \geq u > 0].$$

BY (259.51)

$$3. \int_0^u \frac{dx}{1 - x + x^2} \sqrt{\frac{x(1 + x)}{1 - x + x^2}} = \frac{1}{\sqrt[4]{27}} E(\alpha, p) + \frac{2 - \sqrt{3}}{\sqrt[4]{27}} F(\alpha, p) - \frac{2(2 + \sqrt{3})}{\sqrt{3}} \frac{1 + (1 - \sqrt{3})u}{1 + (1 + \sqrt{3})u} \times \\ \times \sqrt{\frac{u(1 + u)}{1 - u + u^2}} \quad [u > 0].$$

BY (260.54)

$$4. \int_0^u \frac{dx}{1 + x + x^2} \sqrt{\frac{x(1 - x)}{1 + x + x^2}} = \frac{4}{\sqrt[4]{27}} E(\beta, q) - \frac{2 + \sqrt{3}}{\sqrt[4]{27}} F(\beta, q) - \frac{2(2 - \sqrt{3})}{\sqrt{3}} \frac{1 - (1 + \sqrt{3})u}{1 + (\sqrt{3} - 1)u} \times \\ \times \sqrt{\frac{u(1 - u)}{1 + u + u^2}} \quad [1 \geq u > 0].$$

BY (259.55)

3.175

$$1. \int_0^u \frac{dx}{1 + x} \sqrt{\frac{x}{1 + x^3}} = \frac{1}{\sqrt[4]{27}} [F(\alpha, p) - 2E(\alpha, p)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 - u + u^2)}}{\sqrt{1 + u}[1 + (1 + \sqrt{3})u]} \quad [u > 0].$$

BY (260.55)

$$2. \int_0^u \frac{dx}{1 - x} \sqrt{\frac{x}{1 - x^3}} = \frac{1}{\sqrt[4]{27}} [F(\beta, q) - 2E(\beta, q)] + \frac{2}{\sqrt{3}} \frac{\sqrt{u(1 + u + u^2)}}{\sqrt{1 - u}[1 + (\sqrt{3} - 1)u]} \quad [0 < u < 1].$$

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**3.18 Expressions that can be reduced to fourth roots of second-degree polynomials
and their products with rational functions**

3.181

$$1. \int_b^u \frac{dx}{\sqrt[4]{(a-x)(x-b)}} = \sqrt{a-b} \left\{ 2 \left[\mathbf{E} \left(\frac{1}{\sqrt{2}} \right) + E \left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] - \left[\mathbf{K} \left(\frac{1}{\sqrt{2}} \right) + F \left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] \right\} \quad [a \geq u > b].$$

BY (271.05)

$$2. \int_a^u \frac{dx}{\sqrt[4]{(x-a)(x-b)}} = \sqrt{\frac{a-b}{2}} F \left[\left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) - 2E \left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \right] + \frac{2(2u-a-b)\sqrt[4]{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}} \quad [u > a > b].$$

BY (272.05)

3.182

$$1. \int_b^u \frac{dx}{\sqrt[4]{[(a-x)(x-b)]^3}} = \frac{2}{\sqrt{a-b}} \left[\mathbf{K} \left(\frac{1}{\sqrt{2}} \right) + F \left(\arccos \sqrt[4]{\frac{4(a-u)(u-b)}{(a-b)^2}}, \frac{1}{\sqrt{2}} \right) \right] \quad [a \geq u > b].$$

BY (271.01)

$$2. \int_a^u \frac{dx}{\sqrt[4]{[(x-a)(x-b)]^3}} = \frac{\sqrt{2}}{\sqrt{a-b}} F \left(\arccos \frac{a-b-2\sqrt{(u-a)(u-b)}}{a-b+2\sqrt{(u-a)(u-b)}}, \frac{1}{\sqrt{2}} \right) \quad [u > a > b].$$

BY (272.00)

In 3.183- 3.186 we set: $\alpha = \arccos \frac{1}{\sqrt[4]{u^2+1}}$,

$$\beta = \arccos \sqrt[4]{1-u^2}, \quad \gamma = \arccos \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}}.$$

3.183

$$1. \int_0^u \frac{dx}{\sqrt[4]{x^2+1}} = \sqrt{2} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{2u}{\sqrt[4]{u^2+1}} \quad [u > 0].$$

BY (273.55)

332

$$2. \int_0^u \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{2} \left[2E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right] \quad [0 < u \leq 1].$$

BY (271.55)

$$3. \int_1^u \frac{dx}{\sqrt[4]{x^2-1}} = F\left(\gamma, \frac{1}{\sqrt{2}}\right) - 2E\left(\gamma, \frac{1}{\sqrt{2}}\right) + \frac{2u\sqrt[4]{u^2-1}}{1+\sqrt{u^2-1}} \quad [u > 1].$$

BY (272.55)

3.184

$$1. \int_0^u \frac{x^2 dx}{\sqrt[4]{1-x^2}} = \frac{2\sqrt{2}}{5} \left[2E\left(\beta, \frac{1}{\sqrt{2}}\right) - F\left(\beta, \frac{1}{\sqrt{2}}\right) \right] - \frac{2u}{5} \sqrt[4]{(1-u^2)^3} \quad [0 < u \leq 1].$$

BY (271.59)

$$2. \int_1^u \frac{dx}{x^2 \sqrt[4]{x^2-1}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1}{2} F\left(\gamma, \frac{1}{\sqrt{2}}\right) - \frac{1-\sqrt{u^2-1}}{1+\sqrt{u^2-1}} \cdot \frac{\sqrt{u^2-1}}{u} \quad [u > 1].$$

BY (272.54)

3.185

$$1. \int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^3}} = \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0].$$

BY (273.50)

$$2. \int_0^u \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \sqrt{2} F\left(\beta, \frac{1}{\sqrt{2}}\right) \quad [0 < u \leq 1].$$

BY (271.51)

BY (272.50)

$$4. \int_0^u \frac{x^2 dx}{\sqrt[4]{(1-x^2)^3}} = \frac{2\sqrt{2}}{3} F\left(\beta, \frac{1}{\sqrt{2}}\right) - \frac{2}{3} u \sqrt[4]{1-u^2} \quad [0 < u \leq 1].$$

BY (271.54)

$$5. \int_0^u \frac{dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) - \sqrt{2} F\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0].$$

BY (273.54)

$$6. \int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^5}} = 2\sqrt{2} \left[F\left(\alpha, \frac{1}{\sqrt{2}}\right) - 2E\left(\alpha, \frac{1}{\sqrt{2}}\right) \right] + \frac{2u}{\sqrt[4]{u^2+1}} \quad [u > 0].$$

BY (273.56)

$$7. \int_0^u \frac{x^2 dx}{\sqrt[4]{(x^2+1)^7}} = \frac{1}{3\sqrt{2}} F\left(\alpha, \frac{1}{\sqrt{2}}\right) - \frac{u}{6\sqrt[4]{(u^2+1)^3}} \quad [u > 0].$$

BY (273.53)

3.186

$$1. \int_0^u \frac{1 + \sqrt{x^2+1}}{(x^2+1)\sqrt[4]{x^2+1}} dx = 2\sqrt{2} E\left(\alpha, \frac{1}{\sqrt{2}}\right) \quad [u > 0].$$

BY (273.51)

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$$2. \int_0^u \frac{dx}{(1 + \sqrt{1-x^2})\sqrt[4]{1-x^2}} = \sqrt{2} \left[F\left(\beta, \frac{1}{\sqrt{2}}\right) - E\left(\beta, \frac{1}{\sqrt{2}}\right) \right] + \frac{u\sqrt[4]{1-u^2}}{1 + \sqrt{1-u^2}} \quad [0 < u \leq 1].$$

BY (271.58)

$$3. \int_1^u \frac{dx}{(x^2 + 2\sqrt{x^2-1})\sqrt[4]{x^2-1}} = \frac{1}{2} \left[F\left(\gamma, \frac{1}{\sqrt{2}}\right) - E\left(\gamma, \frac{1}{\sqrt{2}}\right) \right] \quad [u > 1].$$

BY (272.53)}

BY (271.57)}

$$5. \int_1^u \frac{x^2 dx}{(x^2 + 2\sqrt{x^2 - 1})\sqrt[4]{(x^2 - 1)^3}} = E\left(\gamma, \frac{1}{\sqrt{2}}\right) \quad [u > 1].$$

BY (272.51)}

3.19- 3.23 Combinations of powers of x and powers of binomials of the form $(\alpha + \beta x)$

3.191

$$1. \int_0^u x^{\nu-1}(u-x)^{\mu-1} dx = u^{\mu+\nu-1} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET II 185(7)

$$2. \int_u^\infty x^{-\nu}(x-u)^{\mu-1} dx = u^{\mu-\nu} B(\nu-\mu, \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0].$$

ET II 201(6)

$$3. \int_0^1 x^{\nu-1}(1-x)^{\mu-1} dx = \int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 774(1)

3.192

$$1. \int_0^1 \frac{x^p dx}{(1-x)^p} = p\pi \operatorname{cosec} p\pi \quad [p^2 < 1].$$

BI ((3))(4)

$$2. \int_0^1 \frac{x^p dx}{(1-x)^{p+1}} = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0].$$

BI ((3))(5)

$$3. \int_0^1 \frac{(1-x)^p}{x^{p+1}} dx = -\pi \operatorname{cosec} p\pi \quad [-1 < p < 0].$$

BI ((4))(6)

$$4. \int_1^{\infty} (x-1)^{p-\frac{1}{2}} \frac{dx}{x} = \pi \sec p\pi \quad \left[-\frac{1}{2} < p < \frac{1}{2} \right].$$

BI ((23))(7)

3.193

$$\int_0^n x^{\nu-1} (n-x)^n dx = \frac{n! n^{\nu+n}}{\nu(\nu+1)(\nu+2)\dots(\nu+n)} \quad [\operatorname{Re} \nu > 0].$$

EH I 2

3.194

$$1. \int_0^u \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^\mu}{\mu} {}_2F_1(\nu, \mu; 1+\mu; -\beta u) \quad [|\arg(1+\beta u)| < \pi, \operatorname{Re} \mu > 0].$$

ET I 310(20)

334

$$2.^6 \int_u^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \frac{u^{\mu-\nu}}{\beta^\nu(\nu-\mu)} {}_2F_1\left(\nu, \nu-\mu; \nu-\mu+1; -\frac{1}{\beta u}\right) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu].$$

ET I 310(21)

$$3. \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^\nu} = \beta^{-\mu} B(\mu, \nu-\mu) \quad [|\arg \beta| < \pi, \operatorname{Re} \nu > \operatorname{Re} \mu > 0].$$

FI II 775a, ET I 310(19)

$$4.^7 \int_0^\infty \frac{x^{\mu-1} dx}{(1+\beta x)^{n+1}} = (-1)^n \frac{\pi}{\beta^\mu} \binom{\mu-1}{n} \operatorname{cosec}(\mu\pi) \quad [|\arg \beta| < \pi, 0 < \operatorname{Re} \mu < n+1].$$

ET I 308(6)

$$5. \int_0^u \frac{x^{\mu-1} dx}{1+\beta x} = \frac{u^\mu}{\mu} {}_2F_1(1, \mu; 1+\mu; -u\beta) \quad [|\arg(1+u\beta)| < \pi, \operatorname{Re} \mu > 0].$$

ET I 308(5)

BI ((16))(4)

$$7. \int_0^{\infty} \frac{x^m dx}{(a+bx)^{n+\frac{1}{2}}} = 2^{m+1} m! \frac{(2n-2m-3)!!}{(2n-1)!!} \frac{a^{m-n+\frac{1}{2}}}{b^{m+1}} \left[m < n - \frac{1}{2}, \quad a > 0, \quad b > 0 \right].$$

BI ((21))(2)

$$8. \int_0^1 \frac{x^{n-1} dx}{(1+x)^m} = 2^{-n} \sum_{k=0}^{\infty} \binom{m-n-1}{k} \frac{(-2)^{-k}}{n+k}.$$

BI ((3))(1)

3.195

$$\int_0^{\infty} \frac{(1+x)^{p-1}}{(x+a)^{p+1}} dx = \frac{1-a^{-p}}{p(a-1)} \quad [a > 0].$$

LI ((19))(6)

3.196

$$1. \int_0^u (x+\beta)^{\nu} (u-x)^{\mu-1} dx = \frac{\beta^{\nu} u^{\mu}}{\mu} {}_2F_1 \left(1, -\nu; 1+\mu; -\frac{u}{\beta} \right) \left[\left| \arg \frac{u}{\beta} \right| < \pi \right].$$

ET II 185(8)

$$2. \int_u^{\infty} (x+\beta)^{-\nu} (x-u)^{\mu-1} dx = (u+\beta)^{\mu-\nu} B(\nu-\mu, \mu) \left[\left| \arg \frac{u}{\beta} \right| < \pi, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > 0 \right].$$

ET II 201(7)

$$3. \int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} dx = (b-a)^{\mu+\nu-1} B(\mu, \nu) \quad [b > a, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

EH I 10(13)

$$4. \int_1^{\infty} \frac{dx}{(a-bx)(x-1)^{\nu}} = -\frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{b-a} \right)^{\nu} \quad [a < b, \quad b > 0, \quad 0 < \nu < 1].$$

LI ((23))(5)

$$5. \int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^{\nu}} = \frac{\pi}{b} \operatorname{cosec} \nu\pi \left(\frac{b}{a-b} \right)^{\nu} \quad [a > b > 0, \quad 0 < \nu < 1].$$

$$1. \int_0^{\infty} x^{\nu-1}(\beta+x)^{-\mu}(x+\gamma)^{-\varrho} dx = \beta^{-\mu}\gamma^{\nu-\varrho}B(\nu, \mu-\nu+\varrho) {}_2F_1\left(\mu, \nu; \mu+\varrho; 1-\frac{\gamma}{\beta}\right) \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > \operatorname{Re}(\nu - \varrho)].$$

ET II 233(9)

$$2. \int_u^{\infty} x^{-\lambda}(x+\beta)^{\nu}(x-u)^{\mu-1} dx = u^{\mu+\nu-\lambda}B(\lambda-\mu-\nu, \mu) {}_2F_1\left(-\nu, \lambda-\mu-\nu; \lambda-\nu; -\frac{\beta}{u}\right) \\ \left[\left| \arg \frac{u}{\beta} \right| < \pi \quad \text{or} \quad \left| \frac{\beta}{u} \right| < 1, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda - \nu) \right].$$

ET II 201(8)

$$3. \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu} dx = B(\lambda, \mu) {}_2F_1(\nu, \lambda; \lambda+\mu; \beta) \\ [\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0, \quad |\beta| < 1].$$

WH

$$4. \int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1+ax)^{-\mu-\nu} dx = (1+a)^{-\mu}B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad a > -1].$$

BI((5)4, EH I 10(11)

$$5. \int_0^{\infty} x^{\lambda-1}(1+x)^{\nu}(1+\alpha x)^{\mu} dx = B(\lambda, -\mu-\nu-\lambda) {}_2F_1(-\mu, \lambda; -\mu-\nu; 1-\alpha) \\ [|\arg \alpha| < \pi, \quad -\operatorname{Re}(\mu + \nu) > \operatorname{Re} \lambda > 0].$$

EH I 60(12), ET I 310(23)

$$6. \int_1^{\infty} x^{\lambda-\nu}(x-1)^{\nu-\mu-1}(\alpha x-1)^{-\lambda} dx = \alpha^{-\lambda}B(\mu, \nu-\mu) {}_2F_1(\nu, \mu; \lambda; \alpha^{-1}) \\ [1 + \operatorname{Re} \nu > \operatorname{Re} \lambda > \operatorname{Re} \mu, \quad |\arg(\alpha - 1)| < \pi].$$

EH I 115(6)

$$7. \int_0^{\infty} x^{\mu-\frac{1}{2}}(x+a)^{-\mu}(x+b)^{-\mu} dx = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2\mu} \frac{\Gamma(\mu-\frac{1}{2})}{\Gamma(\mu)} \quad [\operatorname{Re} \mu > 0].$$

$$8. \int_0^u x^{\nu-1} (x+\alpha)^\lambda (u-x)^{\mu-1} dx = \alpha^\lambda u^{\mu+\nu-1} B(\mu, \nu) {}_2F_1\left(-\lambda, \nu; \mu + \nu; -\frac{u}{\alpha}\right) \\ \left[\left| \arg\left(\frac{u}{\alpha}\right) \right| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0 \right].$$

ET II 186(9)

$$9. \int_0^\infty x^{\lambda-1} (1+x)^{-\mu+\nu} (x+\beta)^{-\nu} dx = B(\mu-\lambda, \lambda) {}_2F_1(\nu, \mu-\lambda; \mu; 1-\beta) \quad [\operatorname{Re} \mu > \operatorname{Re} \lambda > 0].$$

EH I 205

$$10. \int_0^1 \frac{x^{q-1} dx}{(1-x)^q (1+px)} = \frac{\pi}{(1+p)^q} \operatorname{cosec} q\pi \quad [0 < q < 1, \quad p > -1].$$

BI ((5))(1)

$$11. \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p (1+qx)^p} = \frac{2\Gamma\left(p + \frac{1}{2}\right) \Gamma(1-p)}{\sqrt{\pi}} \cos^{2p}(\operatorname{arctg} \sqrt{q}) \frac{\sin[(2p-1) \operatorname{arctg}(\sqrt{q})]}{(2p-1) \sin[\operatorname{arctg}(\sqrt{q})]} \\ \left[-\frac{1}{2} < p < 1, \quad q > 0 \right].$$

BI ((11))(1)

336

$$12. \int_0^1 \frac{x^{p-\frac{1}{2}} dx}{(1-x)^p (1+qx)^p} = \frac{\Gamma\left(p + \frac{1}{2}\right) \Gamma(1-p)}{\sqrt{\pi}} \frac{(1-\sqrt{q})^{1-2p} - (1+\sqrt{q})^{1-2p}}{(2p-1)\sqrt{q}} \\ \left[-\frac{1}{2} < p < 1, \quad 0 < q < 1 \right].$$

BI ((11))(2)

3.198

$$\int_0^1 x^{\mu-1} (1-x)^{\nu-1} [ax + b(1-x) + c]^{-(\mu+\nu)} dx = (a+c)^{-\mu} (b+c)^{-\nu} B(\mu, \nu) \\ [a \geq 0, \quad b \geq 0, \quad c > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

FI II 787

3.199

$$\int_a^b (x-a)^{\mu-1} (b-x)^{\nu-1} (x-c)^{-\mu-\nu} dx = (b-a)^{\mu+\nu-1} (b-c)^{-\mu} (a-c)^{-\nu} B(\mu, \nu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad c < a < b].$$

EH I 10(14)

3.211

$$\int_0^1 x^{\lambda-1} (1-x)^{\mu-1} (1-ux)^{-\rho} (1-vx)^{-\sigma} dx = B(\mu, \lambda) F_1(\lambda, \rho, \sigma, \lambda + \mu; u, v)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0].$$

EH I 231(5)

3.212

$$\int_0^\infty [(1+ax)^{-p} + (1+bx)^{-p}] x^{q-1} dx = 2(ab)^{-\frac{q}{2}} B(q, p-q) \cos \left\{ q \arccos \left[\frac{a+b}{2\sqrt{ab}} \right] \right\}$$

$$[p > q > 0].$$

BI ((19))(9)

3.213

$$\int_0^\infty [(1+ax)^{-p} - (1+bx)^{-p}] x^{q-1} dx = -2i(ab)^{-\frac{q}{2}} B(q, p-q) \sin \left\{ q \arccos \left[\frac{a+b}{2\sqrt{ab}} \right] \right\}$$

$$[p > q > 0].$$

BI ((19))(10)

3.214

$$\int_0^1 [(1+x)^{\mu-1} (1-x)^{\nu-1} + (1+x)^{\nu-1} (1-x)^{\mu-1}] dx = 2^{\mu+\nu-1} B(\mu, \nu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

LI((1))(15), EH I 10(10)

3.215

$$\int_0^1 \{ a^\mu x^{\mu-1} (1-ax)^{\nu-1} + (1-a)^\nu x^{\nu-1} [1 - (1-a)x]^{\mu-1} \} dx = B(\mu, \nu)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |a| < 1].$$

BI ((1))(16)

3.216

$$1. \int_0^1 \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 775

$$2. \int_1^\infty \frac{x^{\mu-1} + x^{\nu-1}}{(1+x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 775

3.217

$$\int_0^\infty \left\{ \frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1} x^p} \right\} dx = \pi \operatorname{ctg} p\pi \quad [0 < p < 1, b > 0].$$

BI((18))(13)

3.218

$$\int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p x^p} dx = \pi \operatorname{ctg} p\pi \quad [p < 1] \quad (\text{cf. 3.217}).$$

3.217
BI ((18))(7)

337

3.219

$$\int_0^\infty \left\{ \frac{x^\nu}{(x+1)^{\nu+1}} - \frac{x^\mu}{(x+1)^{\mu+1}} \right\} dx = \psi(\mu+1) - \psi(\nu+1) \quad [\operatorname{Re} \mu > -1, \operatorname{Re} \nu > -1].$$

BI ((19))(13)

3.221

$$1. \int_a^\infty \frac{(x-a)^{p-1}}{x-b} dx = \pi(a-b)^{p-1} \operatorname{cosec} p\pi \quad [a > b, 0 < p < 1].$$

LI ((24))(8)

$$2. \int_{-\infty}^a \frac{(a-x)^{p-1}}{x-b} dx = -\pi(b-a)^{p-1} \operatorname{cosec} p\pi \quad [a < b, 0 < p < 1].$$

3.222

$$1. \int_0^1 \frac{x^{\mu-1} dx}{1+x} = \beta(\mu) \quad [\operatorname{Re} \mu > 0].$$

WH

$$2. \int_0^\infty \frac{x^{\mu-1} dx}{x+a} = \begin{cases} \pi \operatorname{cosec}(\mu\pi) a^{\mu-1} & \text{for } a > 0, \\ -\pi \operatorname{ctg}(\mu\pi) (-a)^{\mu-1} & \text{for } a < 0, \end{cases} \\ [0 < \operatorname{Re} \mu < 1].$$

FI II 718, FI II 737
BI((18))(2), ET II 249(28)

3.223

$$1. \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\gamma+x)} = \frac{\pi}{\gamma-\beta} (\beta^{\mu-1} - \gamma^{\mu-1}) \operatorname{cosec}(\mu\pi) \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 2].$$

ET I 309(7)

$$2. \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x)(\alpha-x)} = \frac{\pi}{\alpha+\beta} [\beta^{\mu-1} \operatorname{cosec}(\mu\pi) + \alpha^{\mu-1} \operatorname{ctg}(\mu\pi)] \\ [|\arg \beta| < \pi, \quad \alpha > 0, \quad 0 < \operatorname{Re} \mu < 2].$$

ET I 309(8)

$$3. \int_0^\infty \frac{x^{\mu-1} dx}{(a-x)(b-x)} = \pi \operatorname{ctg}(\mu\pi) \frac{a^{\mu-1} - b^{\mu-1}}{b-a} \quad [a > b > 0, \quad 0 < \operatorname{Re} \mu < 2].$$

ET I 309(9)

3.224

$$\int_0^\infty \frac{(x+\beta)x^{\mu-1} dx}{(x+\gamma)(x+\delta)} = \pi \operatorname{cosec}(\mu\pi) \left\{ \frac{\gamma-\beta}{\gamma-\delta} \gamma^{\mu-1} + \frac{\delta-\beta}{\delta-\gamma} \delta^{\mu-1} \right\} \\ [|\arg \gamma| < \pi, \quad |\arg \delta| < \pi, \quad 0 < \operatorname{Re} \mu < 1].$$

ET I 309(10)

3.225

$$1. \int_1^\infty \frac{(x-1)^{p-1}}{x^2} dx = (1-p)\pi \operatorname{cosec} p\pi \quad [-1 < p < 1].$$

$$2. \int_1^{\infty} \frac{(x-1)^{1-p}}{x^3} dx = \frac{1}{2} p(1-p)\pi \operatorname{cosec} p\pi \quad [0 < p < 1].$$

BI ((23))(1)

$$3. \int_0^{\infty} \frac{x^p dx}{(1+x)^3} = \frac{\pi}{2} p(1-p) \operatorname{cosec} p\pi \quad [-1 < p < 2].$$

BI ((16))(5)

3.226

$$1. \int_0^1 \frac{x^n dx}{\sqrt{1-x}} = 2 \frac{(2n)!!}{(2n+1)!!}$$

BI ((8))(1)

$$2. \int_0^1 \frac{x^{n-\frac{1}{2}} dx}{\sqrt{1-x}} = \frac{(2n-1)!!}{(2n)!!} \pi.$$

BI ((8))(2)

3.227

$$1. \int_0^{\infty} \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x} dx = \beta^{1-\mu} \gamma^{\nu-1} B(\nu, \mu-\nu) {}_2F_1\left(\mu-1, \nu; \mu; 1-\frac{\gamma}{\beta}\right) \\ \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu].$$

ET II 217(9)

$$2. \int_0^{\infty} \frac{x^{-\varrho}(\beta-x)^{-\sigma}}{\gamma+x} dx = \pi \gamma^{-\varrho} (\beta-\gamma)^{-\sigma} \operatorname{cosec}(\varrho\pi) I_{1-\gamma/\beta}(\sigma, \varrho) \\ \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad -\operatorname{Re} \sigma < \operatorname{Re} \varrho < 1].$$

ET II 217(10)

3.228

$$1. \int_a^b \frac{(x-a)^{\nu}(b-x)^{-\nu}}{x-c} dx = \pi \operatorname{cosec}(\nu\pi) \left[1 - \left(\frac{a-c}{b-c} \right)^{\nu} \right] \quad \text{for } c < a; \\ = \pi \operatorname{cosec}(\nu\pi) \left[1 - \cos(\nu\pi) \left(\frac{c-a}{b-c} \right)^{\nu} \right] \quad \text{for } a < c < b; \\ = \pi \operatorname{cosec}(\nu\pi) \left[1 - \left(\frac{c-a}{c-b} \right)^{\nu} \right] \quad \text{for } c > b \quad [|\operatorname{Re} \nu| < 1].$$

$$2. \int_a^b \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c} dx = \frac{\pi \operatorname{cosec}(\nu\pi)}{b-c} \left| \frac{a-c}{b-c} \right|^{\nu-1} \quad \text{for } c < a \text{ or } c > b;$$

$$= -\frac{\pi(c-a)^{\nu-1}}{(b-c)^\nu} \operatorname{ctg}(\nu\pi) \quad \text{for } a < c < b \quad [0 < \operatorname{Re} \nu < 1].$$

ET II 250(32)

339

$$3. \int_a^b \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x-c} dx = \frac{(b-a)^{\mu+\nu-1}}{b-c} \operatorname{B}(\mu, \nu) {}_2F_1 \left(1, \mu; \mu + \nu; \frac{b-a}{b-c} \right)$$

for $c < a$ or $c > b$;

$$= \pi(c-a)^{\nu-1}(b-c)^{\mu-1} \operatorname{ctg} \mu\pi - (b-a)^{\mu+\nu-2} \operatorname{B}(\mu-1, \nu) \times$$

$$\times {}_2F_1 \left(2-\mu-\nu, 1; 2-\mu; \frac{b-c}{b-a} \right) \quad \text{for } a < c < b$$

$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, \mu + \nu \neq 1, \mu \neq 1, 2, \dots].$

ET II 250(33)

$$4. \int_0^1 \frac{(1-x)^{\nu-1}x^{-\nu}}{a-bx} dx = \frac{\pi(a-b)^{\nu-1}}{a^\nu} \operatorname{cosec}(\nu\pi) \quad [0 < \operatorname{Re} \nu < 1, 0 < b < a].$$

BI ((5))(8)

$$5. \int_0^\infty \frac{x^{\nu-1}(x+a)^{1-\mu}}{x-c} dx = a^{1-\mu}(-c)^{\nu-1} \operatorname{B}(\mu-\nu, \nu) {}_2F_1 \left(\mu-1, \nu; \mu; 1 + \frac{c}{a} \right) \quad \text{for } c < 0;$$

$$= \pi c^{\nu-1}(a+c)^{1-\mu} \operatorname{ctg}[(\mu-\nu)\pi] - \frac{a^{1-\mu-\nu}}{a+c} \operatorname{B}(\mu-\nu-1, \nu) \times$$

$$\times {}_2F_1 \left(2-\mu, 1; 2-\mu+\nu; \frac{a}{a+c} \right) \quad \text{for } c > 0$$

$[a > 0, 0 < \operatorname{Re} \nu < \operatorname{Re} \mu].$

ET II 251(34)

3.229

$$\int_0^1 \frac{x^{\mu-1} dx}{(1-x)^\mu(1+ax)(1+bx)} = \frac{\pi \operatorname{cosec} \mu\pi}{a-b} \left[\frac{a}{(1+a)^\mu} - \frac{b}{(1+b)^\mu} \right] \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((5))(7)

3.231

$$1. \int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx = \pi \operatorname{ctg} p\pi \quad [p^2 < 1].$$

$$2. \int_0^1 \frac{x^{p-1} - x^{-p}}{1+x} dx = \pi \operatorname{cosec} p\pi \quad [p^2 < 1].$$

BI ((4))(1)

$$3. \int_0^1 \frac{x^p - x^{-p}}{x-1} dx = \frac{1}{p} - \pi \operatorname{ctg} p\pi \quad [p^2 < 1].$$

BI ((4))(3)

$$4. \int_0^1 \frac{x^p - x^{-p}}{1+x} dx = \frac{1}{p} - \pi \operatorname{cosec} p\pi \quad [p^2 < 1].$$

BI ((4))(2)

$$5. \int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x} dx = \psi(\nu) - \psi(\mu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 815, BI((4))(5)

$$6. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x} dx = \pi(\operatorname{ctg} p\pi - \operatorname{ctg} q\pi) \quad [p > 0, q > 0].$$

FI II 718

3.232

$$\int_0^\infty \frac{(c+ax)^{-\mu} - (c+bx)^{-\mu}}{x} dx = c^{-\mu} \ln \frac{b}{a} \quad [\operatorname{Re} \mu > -1; a > 0; b > 0; c > 0].$$

BI ((18))(14)

340

3.233

$$\int_0^\infty \left\{ \frac{1}{1+x} - (1+x)^{-\nu} \right\} \frac{dx}{x} = \psi(\nu) + C \quad [\operatorname{Re} \nu > 0].$$

EH I 17, WH

3.234

$$1. \int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = \pi a^{-q} \operatorname{ctg} q\pi \quad [0 < q < 1, a > 0].$$

$$2. \int_0^1 \left(\frac{x^{q-1}}{1+ax} + \frac{x^{-q}}{a+x} \right) dx = \pi a^{-q} \operatorname{cosec} q\pi \quad [0 < q < 1, \quad a > 0].$$

BI ((5))(10)

3.235

$$\int_0^\infty \frac{(1+x)^\mu - 1}{(1+x)^\nu} \frac{dx}{x} = \psi(\nu) - \psi(\nu - \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0].$$

BI ((18))(5)

3.236

$$\int_0^1 \frac{\mu \frac{x}{2} dx}{[(1-x)(1-a^2x)]^{\frac{\mu+1}{2}}} = \frac{(1-a)^{-\mu} - (1+a)^{-\mu}}{2a\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right) \quad [-2 < \mu < 1, \quad |a| < 1].$$

BI ((12))(32)

3.237

$$\sum_{n=0}^{\infty} (-1)^{n+1} \int_n^{n+1} \frac{dx}{x+u} = \ln \frac{u \left[\Gamma\left(\frac{u}{2}\right) \right]^2}{2 \left[\Gamma\left(\frac{u+1}{2}\right) \right]^2} \quad [|\arg u| < \pi].$$

ET II 216(1)

3.238

$$1. \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} dx = -\pi \operatorname{ctg} \frac{\nu\pi}{2} |u|^{\nu-1} \operatorname{sign} u \quad [0 < \operatorname{Re} \nu < 1] \quad [u \text{ real}, \quad u \neq 0].$$

ET II 249(29)

$$2. \int_{-\infty}^{\infty} \frac{|x|^{\nu-1}}{x-u} \operatorname{sign} x dx = \pi \operatorname{tg} \frac{\nu\pi}{2} |u|^{\nu-1} \quad [0 < \operatorname{Re} \nu < 1] \quad [u \text{ real}, \quad u \neq 0].$$

ET II 249(30)

$$3. \int_a^b \frac{(b-x)^{\mu-1} (x-a)^{\nu-1}}{|x-u|^{\mu+\nu}} dx = \frac{(b-a)^{\mu+\nu-1}}{|a-u|^\mu |b-u|^\nu} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad 0 < u < a < b \quad \text{or} \quad 0 < a < b < u].$$

3.241

$$1. \int_0^1 \frac{x^{\mu-1} dx}{1+x^p} = \frac{1}{p} \beta\left(\frac{\mu}{p}\right) \quad [\operatorname{Re} \mu > 0, \quad p > 0].$$

WH, BI ((2))(13)

$$2. \int_0^\infty \frac{x^{\mu-1} dx}{1+x^\nu} = \frac{\pi}{\nu} \operatorname{cosec} \frac{\mu\pi}{\nu} = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0]$$

ET I 309(15)A, BI ((17))(10)

341

$$3. \int_0^\infty \frac{x^{p-1} dx}{1-x^q} = \frac{\pi}{q} \operatorname{ctg} \frac{p\pi}{q} \quad [p < q].$$

BI ((17))(11)

$$4. \int_0^\infty \frac{x^{\mu-1} dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu p^{n+1}} \left(\frac{p}{q}\right) \frac{\mu}{\nu} \frac{\Gamma\left(\frac{\mu}{\nu}\right) \Gamma\left(1+n-\frac{\mu}{\nu}\right)}{\Gamma(1+n)} \quad \left[0 < \frac{\mu}{\nu} < n+1, \quad p \neq 0, \quad q \neq 0\right].$$

BI ((17))(22)a

$$5. \int_0^\infty \frac{x^{p-1} dx}{(1+x^q)^2} = \frac{(p-q)\pi}{q^2} \operatorname{cosec} \frac{(p-q)\pi}{q} \quad [p < 2q].$$

BI ((17))(18)

3.242

$$\int_{-\infty}^\infty \frac{x^{2m} dx}{x^{4n} + 2x^{2n} \cos t + 1} = \frac{\pi}{n} \sin \left[\frac{(2n-2m-1)t}{2n} \right] \operatorname{cosec} t \operatorname{cosec} \frac{(2m+1)\pi}{2n} \quad [m < n, \quad t^2 < \pi^2].$$

FI II 642

3.243

$$\int_0^\infty \frac{x^{\mu-1} dx}{(1+x^{2\nu})(1+x^{3\nu})} = -\frac{\pi}{8\nu} \frac{\operatorname{cosec}\left(\frac{\mu\pi}{3\nu}\right)}{1-4\cos^2\left(\frac{\mu\pi}{3\nu}\right)} \quad [0 < \operatorname{Re} \mu < 5 \operatorname{Re} \nu].$$

$$1. \int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1+x^q} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0].$$

BI ((2))(14)

$$2. \int_0^1 \frac{x^{p-1} - x^{q-p-1}}{1-x^q} dx = \frac{\pi}{q} \operatorname{ctg} \frac{p\pi}{q} \quad [q > p > 0].$$

BI ((2))(16)

$$3. \int_0^1 \frac{x^{\nu-1} - x^{\mu-1}}{1-x^\nu} dx = \frac{1}{\nu} \left[\mathbf{C} + \psi \left(\frac{\mu}{\nu} \right) \right] \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0].$$

BI ((2))(17)

$$4. \int_{-\infty}^{\infty} \frac{x^{2m} - x^{2n}}{1-x^{2l}} dx = \frac{\pi}{l} \left[\operatorname{ctg} \left(\frac{2m+1}{2l} \pi \right) - \operatorname{ctg} \left(\frac{2n+1}{2l} \pi \right) \right] \quad [m < l, \quad n < l].$$

FI II 640

3.245

$$\int_0^{\infty} [x^{\nu-\mu} - x^\nu(1+x)^{-\mu}] dx = \frac{\nu}{\nu-\mu+1} \mathbf{B}(\nu, \mu-\nu) \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0].$$

BI ((16))(13)

3.246

$$\int_0^{\infty} \frac{1-x^q}{1-x^r} x^{p-1} dx = \frac{\pi}{r} \sin \frac{q\pi}{r} \operatorname{cosec} \frac{p\pi}{r} \operatorname{cosec} \frac{(p+q)\pi}{r} \quad [p+q < r, \quad p > 0].$$

ET I 331(33), BI ((17))(12)

Integrals of the form $\int f(x^p \pm x^{-p}, x^q \pm x^{-q}, \dots) \frac{dx}{x}$ can be transformed by the substitution $x = e^t$ or $x = e^{-t}$. For example, instead of $\int_0^1 (x^{1+p} + x^{1-p})^{-1} dx$, we should seek to evaluate $\int_0^{\infty} \operatorname{sech} px dx$ and, instead of $\int_0^1 \frac{x^{n-m-1} + x^{n+m-1}}{1+2x^n \cos a + x^{2n}} dx$, we should seek to evaluate $\int_0^{\infty} \operatorname{ch} mx (\operatorname{ch} nx - \cos a)^{-1} dx$ (see 3.514 2.).

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3.247

$$1. \int_0^1 \frac{x^{\alpha-1}(1-x)^{n-1}}{1-\xi x^b} dx = (n-1)! \sum_{k=0}^{\infty} \frac{\xi^k}{(\alpha+kb), (\alpha+kb+1) \dots (\alpha+kb+k-1)} \quad [b > 0, \quad |\xi| < 1].$$

$$2. \int_0^\infty \frac{(1-x^p)x^{\nu-1}}{1-x^{np}} dx = \frac{\pi}{np} \sin\left(\frac{\pi}{n}\right) \operatorname{cosec} \frac{(p+\nu)\pi}{np} \operatorname{cosec} \frac{\pi\nu}{np} \quad [0 < \operatorname{Re} \nu < (n-1)p].$$

ET I 311(33)

3.248

$$1.^0 \int_0^\infty \frac{x^{\mu-1} dx}{\sqrt{1+x^\nu}} = \frac{1}{\nu} B\left(\frac{\mu}{\nu}, \frac{1}{2} - \frac{\mu}{\nu}\right) \quad [\operatorname{Re} \nu > \operatorname{Re} 2\mu > 0].$$

BI ((21))(9)

$$2. \int_0^1 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!}.$$

BI ((8))(14)

$$3. \int_0^1 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}.$$

BI ((8))(13)

$$4.* \int_{-\infty}^\infty \frac{dx}{(1+x^2)\sqrt{4+3x^2}} = \frac{\pi}{3}.$$

$$5.* \int_0^\infty \frac{dx}{(1+x^2)^{3/2} \left[1 + \frac{4x^2}{3(1+x^2)^2} + \left(1 + \frac{4x^2}{3(1+x^2)^2}\right)^{1/2}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}.$$

3.249

$$1.^0 \int_0^\infty \frac{dx}{(x^2+a^2)^n} = \frac{(2n-3)!!}{2 \cdot (2n-2)!!} \frac{\pi}{a^{2n-1}}.$$

FI II 743

$$2. \int_0^a (a^2-x^2)^{n-\frac{1}{2}} dx = a^{2n} \frac{(2n-1)!!}{2(2n)!!} \pi.$$

$$3. \int_{-1}^1 \frac{(1-x^2)^n dx}{(a-x)^{n+1}} = 2^{n+1} Q_n(a).$$

EH II 181(31)

$$4. \int_0^1 \frac{x^\mu dx}{1+x^2} = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1].$$

BI ((2))(7)

$$5. \int_0^1 (1-x^2)^{\mu-1} dx = 2^{2\mu-2} B(\mu, \mu) = \frac{1}{2} B\left(\frac{1}{2}, \mu\right) \quad [\operatorname{Re} \mu > 0].$$

FI II 784

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$$6. \int_0^1 (1-\sqrt{x})^{p-1} dx = \frac{2}{p(p+1)} \quad [p > 0].$$

BI ((7))(7)

$$7. \int_0^1 (1-x^\mu)^{-\frac{1}{\nu}} dx = \frac{1}{\mu} B\left(\frac{1}{\mu}, 1-\frac{1}{\nu}\right) \quad [\operatorname{Re} \mu > 0, \quad |\nu| > 1].$$

$$8.* \int_{-\infty}^{\infty} \left(1 + \frac{x^2}{n-1}\right)^{-n/2} dx = \sqrt{\pi(n-1)} \Gamma\left(\frac{n-1}{2}\right) / \Gamma\left(\frac{n}{2}\right) \quad [\text{integer } n > 1].$$

3.251

$$1. \int_0^1 x^{\mu-1} (1-x^\lambda)^{\nu-1} dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \lambda > 0].$$

FI II 787

$$2. \int_0^{\infty} x^{\mu-1} (1+x^2)^{\nu-1} dx = \frac{1}{2} B\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right) \quad \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \left(\nu + \frac{1}{2}\mu\right) < 1 \right].$$

$$3. \int_1^{\infty} x^{\mu-1} (x^p - 1)^{\nu-1} dx = \frac{1}{p} B\left(1 - \nu - \frac{\mu}{p}, \nu\right) \quad [p > 0, \operatorname{Re} \nu > 0, \operatorname{Re} \mu < p - p \operatorname{Re} \nu].$$

ET I 311(32)

$$4. \int_0^{\infty} \frac{x^{2m} dx}{(ax^2 + c)^n} = \frac{(2m-1)!!(2n-2m-3)!!\pi}{2 \cdot (2n-2)!! a^m c^{n-m-1} \sqrt{ac}} \quad [a > 0, c > 0, n > m+1].$$

GU ((141))(8a)

$$5. \int_0^{\infty} \frac{x^{2m+1} dx}{(ax^2 + c)^n} = \frac{m!(n-m-2)!}{2(n-1)! a^{m+1} c^{n-m-1}} \quad [ac > 0, n > m+1 \geq 1].$$

GU ((141))(8b)

$$6. \int_0^{\infty} \frac{x^{\mu+1}}{(1+x^2)^2} dx = \frac{\mu\pi}{4 \sin \frac{\mu\pi}{2}} \quad [-2 < \operatorname{Re} \mu < 2].$$

WH

$$7. \int_0^1 \frac{x^{\mu} dx}{(1+x^2)^2} = -\frac{1}{4} + \frac{\mu-1}{4} \beta\left(\frac{\mu-1}{2}\right) \quad [\operatorname{Re} \mu > 1].$$

LI ((3))(11)

$$8. \int_0^1 x^{q+p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{p\pi}{q^2} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p].$$

BI ((9))(22)

$$9. \int_0^1 x^{\frac{q}{p}-1} (1-x^q)^{-\frac{1}{p}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{\pi}{p} \quad [p > 1, q > 0].$$

BI ((9))(23a)

$$10. \int_0^1 x^{p-1} (1-x^q)^{-\frac{p}{q}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0].$$

BI ((9))(20)

$$11. \int_0^{\infty} x^{\mu-1} (1 + \beta x^p)^{-\nu} dx = \frac{1}{p} \beta^{-\frac{\mu}{p}} B\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right) \\ [|\arg \beta| < \pi, p > 0, 0 < \operatorname{Re} \mu < p \operatorname{Re} \nu].$$

$$1. \int_0^{\infty} \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial c^{n-1}} \left[\frac{1}{\sqrt{ac-b^2}} \operatorname{arcctg} \frac{b}{\sqrt{ac-b^2}} \right] \quad [a > 0, \quad ac > b^2].$$

GW ((131))(4)

$$2. \int_{-\infty}^{\infty} \frac{dx}{(ax^2 + 2bx + c)^n} = \frac{(2n-3)!!\pi a^{n-1}}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}} \quad [a > 0, \quad ac > b^2].$$

GW ((131))(5)

$$3. \int_0^{\infty} \frac{dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{(-2)^n}{(2n+1)!!} \frac{\partial^n}{\partial c^n} \left\{ \frac{1}{\sqrt{c}(\sqrt{ac}+b)} \right\} \quad [a \geq 0, \quad c > 0, \quad b > -\sqrt{ac}].$$

GW ((213))(4)

$$\begin{aligned} 4. \int_0^{\infty} \frac{x dx}{(ax^2 + 2bx + c)^n} &= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac-b^2)} - \frac{b}{2(ac-b^2)^{\frac{3}{2}}} \operatorname{arcctg} \frac{b}{\sqrt{ac-b^2}} \right\} \\ &\quad \text{for } ac > b^2; \\ &= \frac{(-1)^n}{(n-1)!} \frac{\partial^{n-2}}{\partial c^{n-2}} \left\{ \frac{1}{2(ac-b^2)} + \frac{b}{4(b^2-ac)^{\frac{3}{2}}} \ln \frac{b+\sqrt{b^2-ac}}{b-\sqrt{b^2-ac}} \right\} \\ &\quad \text{for } b^2 > ac > 0; \\ &= \frac{a^{n-2}}{2(n-1)(2n-1)b^{2n-2}} \quad \text{for } ac = b^2 \quad [a > 0, \quad b > 0, \quad n \geq 2]. \end{aligned}$$

GW ((141))(5)

$$5. \int_{-\infty}^{\infty} \frac{x dx}{(ax^2 + 2bx + c)^n} = -\frac{(2n-3)!!\pi b a^{n-2}}{(2n-2)!!(ac-b^2)^{\frac{(2n-1)}{2}}} \quad [ac > b^2, \quad a > 0, \quad n \geq 2].$$

GW ((141))(6)

$$\begin{aligned} 6. \int_{-\infty}^{\infty} \frac{x^m dx}{(ax^2 + 2bx + c)^n} &= \frac{(-1)^m \pi a^{n-m-1} b^m}{(2n-2)!!(ac-b^2)^{n-\frac{1}{2}}} \times \\ &\quad \times \sum_{k=0}^{[m/2]} \binom{m}{2k} (2k-1)!!(2n-2k-3)!! \left(\frac{ac-b^2}{b^2} \right)^k \\ &\quad [ac > b^2, \quad 0 \leq m \leq 2n-2]. \end{aligned}$$

$$7. \int_0^{\infty} \frac{x^n dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{c}(\sqrt{ac}+b)^{n+1}} \quad [a \geq 0, \quad c > 0, \quad b > -\sqrt{ac}].$$

GW ((213))(5a)

$$8. \int_0^{\infty} \frac{x^{n+1} dx}{(ax^2 + 2bx + c)^{n+\frac{3}{2}}} = \frac{n!}{(2n+1)!!\sqrt{a}(\sqrt{ac}+b)^{n+1}} \quad [a > 0, \quad c \geq 0, \quad b > -\sqrt{ac}].$$

GW ((213))(5b)

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$$9. \int_0^{\infty} \frac{x^{n+\frac{1}{2}} dx}{(ax^2 + 2bx + c)^{n+1}} = \frac{(2n-1)!!\pi}{2^{2n+\frac{1}{2}}(b+\sqrt{ac})^{n+\frac{1}{2}}n!\sqrt{a}} \quad [a > 0, \quad c > 0, \quad b+\sqrt{ac} > 0].$$

LI ((21))(19)

$$10.^6 \int_0^{\infty} \frac{x^{\mu-1} dx}{(1+2x \cos t + x^2)^{\nu}} = 2^{\nu-\frac{1}{2}}(\sin t)^{\frac{1}{2}-\nu} t \Gamma\left(\nu + \frac{1}{2}\right) B(\mu, 2\nu-\mu) P_{\mu-\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t) \\ [0 < t < \pi, \quad 0 < \operatorname{Re} \mu < \operatorname{Re} 2\nu].$$

ET I 310(22)

$$11. \int_0^{\infty} (1+2\beta x+x^2)^{\mu-\frac{1}{2}} x^{-\nu-1} dx = 2^{-\mu}(\beta^2-1)^{\frac{\mu}{2}} \Gamma(1-\mu) B(\nu-2\mu+1, -\nu) P_{\nu-\mu}^{\mu}(\beta) \\ [\operatorname{Re} \nu < 0, \quad \operatorname{Re}(2\mu-\nu) < 1, \quad |\arg(\beta \pm 1)| < \pi]. \\ = -\pi \operatorname{cosec} \nu \pi C_{\nu}^{\frac{1}{2}-\mu}(\beta) \\ \left[-2 < \operatorname{Re}\left(\frac{1}{2}-\mu\right) < \operatorname{Re} \nu < 0, \quad |\arg(\beta \pm 1)| < \pi\right].$$

EH I 178(24)

EH I 160(33)

$$12. \int_0^{\infty} \frac{x^{\mu-1} dx}{x^2 + 2ax \cos t + a^2} = -\pi a^{\mu-2} \operatorname{cosec} t \operatorname{cosec}(\mu\pi) \sin[(\mu-1)t] \\ [a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 2].$$

FI II 738, BI((20))(3)

$$13. \int_0^{\infty} \frac{x^{\mu-1} dx}{(x^2 + 2ax \cos t + a^2)^2} = \frac{\pi a^{\mu-4}}{2} \operatorname{cosec} \mu\pi \operatorname{cosec}^3 t \times \\ \times \{(\mu-1) \sin t \cos[(\mu-2)t] - \sin[(\mu-1)t]\} \\ [a > 0, \quad 0 < |t| < \pi, \quad 0 < \operatorname{Re} \mu < 4].$$

$$14. \int_0^{\infty} \frac{x^{\mu-1} dx}{\sqrt{1+2x \cos t + x^2}} = \pi \operatorname{cosec}(\mu\pi) P_{\mu-1}(\cos t) \quad [-\pi < t < \pi, \quad 0 < \operatorname{Re} \mu < 1].$$

ET I 310(17)

3.253

$$\int_{-1}^1 \frac{(1+x)^{2\mu-1}(1-x)^{2\nu-1}}{(1+x^2)^{\mu+\nu}} dx = 2^{\mu+\nu-2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

FI II 787

3.254

$$\begin{aligned} 1. \int_0^u x^{\lambda-1}(u-x)^{\mu-1}(x^2+\beta^2)^{\nu} dx &= \\ &= \beta^{2\nu} u^{\lambda+\mu-1} B(\lambda, \mu) {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; \frac{-u^2}{\beta^2}\right) \\ &\quad \left[\operatorname{Re}\left(\frac{u}{\beta}\right) > 0, \quad \lambda > 0, \quad \operatorname{Re} \mu > 0\right]. \end{aligned}$$

ET II 186(10)

346

$$\begin{aligned} 2.^6 \int_u^{\infty} (x^{-\lambda}(x-u)^{\mu-1}(x^2+\beta^2)^{\nu}) dx &= \\ &= u^{\mu-\lambda+2\nu} \frac{\Gamma(\mu)\Gamma(\lambda-\mu-2\nu)}{\Gamma(\lambda-2\nu)} \times \\ &\quad \times {}_3F_2\left(-\nu, \frac{\lambda-\mu}{2}-\nu, \frac{1+\lambda-\mu}{2}-\nu; \frac{\lambda}{2}-\nu, \frac{1+\lambda}{2}-\nu; -\frac{\beta^2}{u^2}\right) \\ &\quad \left[|u| > |\beta| \quad \text{or} \quad \operatorname{Re}\left(\frac{\beta}{u}\right) > 0, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda-2\nu)\right]. \end{aligned}$$

ET II 202(9)

3.255

$$\begin{aligned} \int_0^1 \frac{x^{\mu+\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}}{(c+2bx-ax^2)^{\mu+1}} dx &= \\ &= \frac{\sqrt{\pi}}{\{a+(\sqrt{c+2b-a}+\sqrt{c})^2\}^{\mu+\frac{1}{2}} \sqrt{c+2b-a}} \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu+1)} \\ &\quad \left[a+(\sqrt{c+2b-a}+\sqrt{c})^2 > 0, \quad c+2b-a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right]. \end{aligned}$$

$$1. \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \cos\left(\frac{q-p}{4}\pi\right) \sec\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right) \quad [p > 0, q > 0, p+q < 2].$$

BI ((8))(25)

$$2. \int_0^1 \frac{x^{p-1} - x^{q-1}}{(1-x^2)^{\frac{p+q}{2}}} dx = \frac{1}{2} \sin\left(\frac{q-p}{4}\pi\right) \operatorname{cosec}\left(\frac{q+p}{4}\pi\right) B\left(\frac{p}{2}, \frac{q}{2}\right) \quad [p > 0, q > 0, p+q < 2].$$

BI ((8))(26)

3.257

$$\int_0^\infty \left[\left(ax + \frac{b}{x}\right)^2 + c \right]^{-p-1} dx = \frac{\sqrt{\pi}\Gamma\left(p + \frac{1}{2}\right)}{2ac^{p+\frac{1}{2}}\Gamma(p+1)}.$$

BI ((20))(4)

3.258

$$1. \int_b^\infty (x - \sqrt{x^2 - a^2})^n dx = \frac{a^2}{2(n-1)} (b - \sqrt{b^2 - a^2})^{n-1} - \frac{1}{2(n+1)} (b - \sqrt{b^2 - a^2})^{n+1} \quad [0 < a \leq b, n \geq 2].$$

GW ((215))(5)

$$2. \int_b^\infty (\sqrt{x^2 + 1} - x)^n dx = \frac{(\sqrt{b^2 + 1} - b)^{n-1}}{2(n-1)} + \frac{(\sqrt{b^2 + 1} - b)^{n+1}}{2(n+1)} \quad [n \geq 2].$$

GW ((214))(7)

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$$3. \int_0^\infty (\sqrt{x^2 + a^2} - x)^n dx = \frac{na^{n+1}}{n^2 - 1} \quad [n \geq 2].$$

GW ((214))(6a)

$$4. \int_0^\infty \frac{dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n}{a^{n-1}(n^2 - 1)} \quad [n \geq 2].$$

GW ((214))(5a)

$$5. \int_0^\infty x^m (\sqrt{x^2 + a^2} - x)^n dx = \frac{n \cdot m! a^{m+n+1}}{(n-m-1)(n-m+1) \dots (m+n+1)} \quad [a > 0, 0 \leq m \leq n-2].$$

$$6. \int_0^\infty \frac{x^m dx}{(x + \sqrt{x^2 + a^2})^n} = \frac{n \cdot m!}{(n - m - 1)(n - m + 1) \dots (m + n + 1)a^{n-m-1}} \quad [a > 0, \quad 0 \leq m \leq n - 2].$$

GW ((214))(5)

$$7. \int_a^\infty (x-a)^m (x-\sqrt{x^2 - a^2})^n dx = \frac{n \cdot (n - m - 2)!(2m + 1)!a^{m+n+1}}{2^m(n + m + 1)!} \quad [a > 0, \quad n \geq m + 2].$$

GW ((215))(6)

3.259

$$1.^6 \int_0^1 x^{p-1}(1-x)^{n-1}(1+bx^m)^l dx = (n-1)! \sum_{k=0}^\infty \binom{l}{k} \frac{b^k \Gamma(p+km)}{\Gamma(p+n+km)} \quad [|b| < 1 \text{ unless } l = 0, 1, 2, \dots; p, n, p+ml > 0].$$

BI ((1))(14)

$$2. \int_0^u x^{\nu-1}(u-x)^{\mu-1}(x^m + \beta^m)^\lambda dx = \beta^{m\lambda} u^{\mu+\nu+1} B(\mu, \nu) \times \\ \times {}_{m+1}F_m \left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \dots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \dots, \frac{\mu+\nu+m-1}{m}; \frac{-u^m}{\beta^m} \right) \\ \left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \left| \arg \left(\frac{u}{\beta} \right) \right| < \frac{\pi}{m} \right].$$

ET II 186(11)

$$3. \int_0^\infty x^{\lambda-1}(1+\alpha x^p)^{-\mu}(1+\beta x^p)^{-\nu} dx = \frac{1}{p} \alpha^{-\frac{\lambda}{p}} B \left(\frac{\lambda}{p}, \mu + \nu - \frac{\lambda}{p} \right) {}_2F_1 \left(\nu, \frac{\lambda}{p}; \mu + \nu; 1 - \frac{\beta}{\alpha} \right) \\ [|\arg \alpha| < \pi, \quad |\arg \beta| < \pi, \quad p > 0, \quad 0 < \operatorname{Re} \lambda < 2 \operatorname{Re}(\mu + \nu)].$$

ET I 312(35)

3.261

$$1. \int_0^1 \frac{(1-x \cos t)x^{\mu-1} dx}{1-2x \cos t + x^2} = \sum_{h=0}^\infty \frac{\cos kt}{\mu + k} \quad [\operatorname{Re} \mu > 0, \quad t \neq 2n\pi].$$

BI ((6))(9)

$$2. \int_0^1 \frac{(x^\nu + x^{-\nu}) dx}{1+2x \cos t + x^2} = \frac{\pi \sin \nu t}{\sin t \sin \nu \pi} \quad [\nu^2 < 1, \quad t \neq (2n+1)\pi].$$

$$3. \int_0^1 \frac{(x^{1+p} + x^{1-p}) dx}{(1 + 2x \cos t + x^2)^2} = \frac{\pi(p \sin t \cos pt - \cos t \sin pt)}{2 \sin^3 t \sin p\pi} \quad [p^2 < 1, \quad t \neq (2n+1)\pi].$$

BI ((6))(18)

$$4. \int_0^1 \frac{x^{\mu-1}}{1 + 2ax \cos t + a^2 x^2} \cdot \frac{dx}{(1-x)^\mu} = \frac{\pi \operatorname{cosec} t \operatorname{cosec} \mu\pi}{(1 + 2a \cos t + a^2)^{\frac{\mu}{2}}} \sin \left(t - \mu \operatorname{arctg} \frac{a \sin t}{1 + a \cos t} \right) \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

BI ((6))(21)

3.262

$$\int_0^\infty \frac{x^{-p} dx}{1+x^3} = \frac{\pi}{3} \operatorname{cosec} \frac{(1-p)\pi}{3} \quad [-2 < p < 1].$$

LI ((18))(3)

3.263

$$\int_0^\infty \frac{x^\nu dx}{(x+\gamma)(x^2+\beta^2)} = \frac{\pi}{2(\beta^2+\gamma^2)} \left[\gamma \beta^{\nu-1} \sec \frac{\nu\pi}{2} + \beta^\nu \operatorname{cosec} \frac{\nu\pi}{2} - 2\gamma^\nu \operatorname{cosec}(\nu\pi) \right] \\ [\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \pi, \quad -1 < \operatorname{Re} \nu < 2, \quad \nu \neq 0].$$

ET II 216(7)

3.264

$$1. \int_0^\infty \frac{x^{p-1} dx}{(a^2+x^2)(b^2-x^2)} = \frac{\pi}{2} \frac{a^{p-2} + b^{p-2} \cos \frac{p\pi}{2}}{a^2 + b^2} \operatorname{cosec} \frac{p\pi}{2} \quad [0 < p < 4, \quad a > 0, \quad b > 0].$$

BI ((19))(14)

$$2. \int_0^\infty \frac{x^{\mu-1} dx}{(\beta+x^2)(\gamma+x^2)} = \frac{\pi}{2} \frac{\gamma^{\frac{\mu}{2}-1} - \beta^{\frac{\mu}{2}-1}}{\beta - \gamma} \operatorname{cosec} \frac{\mu\pi}{2} \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < 4].$$

ET I 309(4)

3.265

$$\int_0^1 \frac{1-x^{\mu-1}}{1-x} dx = \psi(\mu) + \mathbf{C} \quad [\operatorname{Re} \mu > 0]; \\ = \psi(1-\mu) + \mathbf{C} - \pi \operatorname{ctg}(\mu\pi) \quad [\operatorname{Re} \mu > 0].$$

3.266

$$\int_0^{\infty} \frac{(x^\nu - a^\nu) dx}{(x-a)(\beta+x)} = \frac{\pi}{a+\beta} \left\{ \beta^\nu \operatorname{cosec}(\nu\pi) - a^\nu \operatorname{ctg}(\nu\pi) - \frac{a^\nu}{\pi} \ln \frac{\beta}{a} \right\}$$

[$|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1, \quad \nu \neq 0$].

ET II 216(8)

3.267

$$1. \int_0^1 \frac{x^{3n} dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}} \frac{\Gamma\left(n + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right) \Gamma(n+1)}.$$

BI ((9))(6)

349

$$2. \int_0^1 \frac{x^{3n-1} dx}{\sqrt[3]{1-x^3}} = \frac{(n-1)! \Gamma\left(\frac{2}{3}\right)}{3\Gamma\left(n + \frac{2}{3}\right)}.$$

BI ((9))(7)

3.268

$$1. \int_0^1 \left(\frac{1}{1-x} - \frac{px^{p-1}}{1-x^p} \right) dx = \ln p.$$

BI ((5))(14)

$$2. \int_0^1 \frac{1-x^\mu}{1-x} x^{\nu-1} dx = \psi(\mu+\nu) - \psi(\nu) \quad [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((2))(3)

$$3. \int_0^1 \left[\frac{n}{1-x} - \frac{x^{\mu-1}}{1-\sqrt[n]{x}} \right] dx = n\mathbf{C} + \sum_{k=1}^n \psi\left(\mu + \frac{n-k}{n}\right) \quad [\operatorname{Re} \mu > 0].$$

BI ((13))(10)

3.269

BI ((4))(12)

$$2. \int_0^1 \frac{x^p - x^{-p}}{1+x^2} x dx = \frac{1}{p} - \frac{\pi}{2} \operatorname{cosec} \frac{p\pi}{2} \quad [p^2 < 1].$$

BI ((4))(8)

$$3. \int_0^1 \frac{x^\mu - x^\nu}{1-x^2} dx = \frac{1}{2} \psi \left(\frac{\nu+1}{2} \right) - \frac{1}{2} \psi \left(\frac{\mu+1}{2} \right) \quad [\operatorname{Re} \mu > -1, \operatorname{Re} \nu > -1].$$

BI ((2))(9)

3.271

$$1. \int_0^\infty \frac{x^p - x^q}{x-1} \frac{dx}{x+a} = \frac{\pi}{1+a} \left(\frac{a^p - \cos p\pi}{\sin p\pi} - \frac{a^q - \cos q\pi}{\sin q\pi} \right) \quad [p^2 < 1, q^2 < 1, a > 0].$$

BI ((19))(2)

$$2. \int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^p - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ \frac{a^{2p} - 1}{\sin(2p\pi)} - \frac{1}{\pi} a^p \ln a \right\} \quad \left[p^2 < \frac{1}{4} \right].$$

BI ((19))(3)

$$3. \int_0^\infty \frac{x^p - a^p}{x-a} \frac{x^{-p} - 1}{x-1} dx = \frac{\pi}{a-1} \left\{ 2(a^p - 1) \operatorname{ctg} p\pi - \frac{1}{\pi} (a^p + 1) \ln a \right\} \quad [p^2 < 1].$$

BI ((18))(9)

$$4. \int_0^\infty \frac{x^p - a^p}{x-a} \frac{1-x^{-p}}{1-x} x^q dx = \frac{\pi}{a-1} \left\{ \frac{a^{p+q} - 1}{\sin[(p+q)\pi]} + \frac{a^p - a^q}{\sin[(q-p)\pi]} \right\} \frac{\sin p\pi}{\sin q\pi} \\ [(p+q)^2 < 1, (p-q)^2 < 1].$$

BI ((19))(4)

$$5. \int_0^\infty \left(\frac{x^p - x^{-p}}{1-x} \right)^2 dx = 2(1 - 2p\pi \operatorname{ctg} 2p\pi) \quad \left[0 < p^2 < \frac{1}{4} \right].$$

BI ((16))(3)

3.272

$$1. \int_0^1 \frac{x^{n-1} + x^{n-\frac{1}{2}} - 2x^{2n-1}}{1-x} dx = 2 \ln 2.$$

BI ((8))(8)

$$2. \int_0^1 \frac{x^{n-1} + x^{n-\frac{2}{3}} + x^{n-\frac{1}{3}} - 3x^{3n-1}}{1-x} dx = 3 \ln 3.$$

BI ((8))(9)

3.273

$$1. \int_0^1 \frac{\sin t - a^n x^n \sin[(n+1)t] + a^{n+1} x^{n+1} \sin nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \sin kt}{\Gamma(p+k)} \quad [p > 0].$$

BI ((6))(13)

$$2. \int_0^1 \frac{\cos t - ax - a^n x^n \cos[(n+1)t] + a^{n+1} x^{n+1} \cos nt}{1 - 2ax \cos t + a^2 x^2} (1-x)^{p-1} dx = \Gamma(p) \sum_{k=1}^n \frac{(k-1)! a^{k-1} \cos kt}{\Gamma(p+k)} \quad [p > 0].$$

BI ((6))(14)

$$3. \int_0^1 x \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} dx = \sum_{k=1}^n \frac{\sin kt}{k+1}.$$

BI ((6))(12)

$$4. \int_0^1 \frac{1 - x \cos t - x^{n+1} \cos[(n+1)t] + x^{n+2} \cos nt}{1 - 2x \cos t + x^2} dx = \sum_{k=0}^n \frac{\cos kt}{k+1}.$$

BI ((6))(11)

3.274

$$1. \int_0^\infty \frac{x^{\mu-1}(1-x)}{1-x^n} dx = \frac{\pi}{n} \sin \frac{\pi}{n} \operatorname{cosec} \frac{\mu\pi}{n} \operatorname{cosec} \frac{(\mu+1)\pi}{n} \quad [0 < \operatorname{Re} \mu < n-1].$$

$$2. \int_0^1 \frac{1-x^n}{(1+x)^{n+1}} \frac{dx}{1-x} = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k}.$$

BI ((5))(3)

$$3. \int_0^\infty \frac{x^q - 1}{x^p - x^{-p}} \frac{dx}{x} = \frac{\pi}{2p} \operatorname{tg} \frac{q\pi}{2p} \quad [p > q].$$

BI ((18))(6)

3.275

$$1. \int_0^1 \left\{ \frac{x^{n-1}}{1-x^{\frac{1}{p}}} - \frac{px^{np-1}}{1-x} \right\} dx = p \ln p \quad [p > 0].$$

BI ((13))(9)

$$2. \int_0^1 \left\{ \frac{nx^{n-1}}{1-x^n} - \frac{x^{mn-1}}{1-x} \right\} dx = \mathbf{C} + \frac{1}{n} \sum_{k=1}^n \psi \left(m + \frac{n-k}{n} \right).$$

BI ((5))(13)

$$3. \int_0^1 \left(\frac{x^{p-1}}{1-x} - \frac{qx^{pq-1}}{1-x^q} \right) dx = \ln q \quad [q > 0].$$

BI ((5))(12)

$$4. \int_0^\infty \left\{ \frac{1}{1+x^{2n}} - \frac{1}{1+x^{2m}} \right\} \frac{dx}{x} = 0.$$

BI ((18))(17)

351

3.276

$$1. \int_0^\infty \frac{\left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx}{x^2} = \frac{\sqrt{\pi}}{2bc^{p+\frac{1}{2}}} \frac{\Gamma \left(p + \frac{1}{2} \right)}{\Gamma(p+1)} \quad \left[p > -\frac{1}{2} \right].$$

BI ((20))(19)

$$2. \int_0^\infty \left(a + \frac{b}{x^2} \right) \left[\left(ax + \frac{b}{x} \right)^2 + c \right]^{-p-1} dx = \frac{\sqrt{\pi}}{c^{p+\frac{1}{2}}} \frac{\Gamma \left(p + \frac{1}{2} \right)}{\Gamma(p+1)} \quad \left[p > -\frac{1}{2} \right].$$

$$1. \int_0^\infty \frac{x^{\mu-1} [\sqrt{1+x^2} + \beta]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu}{2}-1} (\beta^2-1)^{\frac{\nu}{2}+\frac{\mu}{4}} \Gamma\left(\frac{\mu}{2}\right) \Gamma(1-\mu-\nu) P_{\frac{\nu+\mu}{2}-1}^{\frac{\nu+\mu}{2}}(\beta)$$

$$[\operatorname{Re} \beta > -1, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu].$$

ET I 310(25)

$$2. \int_0^\infty \frac{x^{\mu-1} [\sqrt{\beta^2+x^2} + x]^\nu}{\sqrt{\beta^2+x^2}} dx = \frac{\beta^{\mu+\nu-1}}{2^\mu} B\left(\mu, \frac{1-\mu-\nu}{2}\right) \quad [\operatorname{Re} \beta > 0, \quad 0 < \operatorname{Re} \mu < 1 - \operatorname{Re} \nu].$$

ET I 311(28)

$$3. \int_0^\infty \frac{x^{\mu-1} [\cos t \pm i \sin t \sqrt{1+x^2}]^\nu}{\sqrt{1+x^2}} dx = 2^{\frac{\mu-1}{2}} \sin^{\frac{1-\mu}{2}} t \frac{\Gamma\left(\frac{\mu}{2}\right) \Gamma(1-\mu-\nu)}{\Gamma(-\nu)} \times$$

$$\times \left[\pi^{-\frac{1}{2}} Q_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu-1}{2}}(\cos t) \mp \frac{i}{2} \pi \frac{1}{2} P_{-\frac{\mu+1}{2}-\nu}^{\frac{\mu-1}{2}}(\cos t) \right]$$

$$[\operatorname{Re} \mu > 0].$$

ET I 311 (27)

$$4. \int_0^\infty \frac{x^{\mu-1} [\sqrt{(\beta^2-1)(x^2+1)} + \beta]^\nu}{\sqrt{x^2+1}} dx$$

$$= \frac{2^{\frac{\mu-1}{2}}}{\sqrt{\pi}} e^{-\frac{1}{2}i\pi(\mu-1)} \frac{\Gamma\left(\frac{\mu}{2}\right) \Gamma(1-\mu-\nu)}{\Gamma(-\nu)} (\beta^2-1)^{\frac{1-\mu}{4}} Q_{-\nu-\frac{\mu+1}{2}}^{\frac{\mu-1}{2}}(\beta)$$

$$[\operatorname{Re} \beta > 1, \quad \operatorname{Re} \nu < 0, \quad \operatorname{Re} \mu < 1 - \operatorname{Re} \nu].$$

ET I 311(26)

$$5. \int_u^\infty \frac{(x-u)^{\mu-1} (\sqrt{x+1} - \sqrt{x-1})^{2\nu}}{\sqrt{x^2-1}} dx = \frac{2^{\nu+\frac{1}{2}}}{\sqrt{\pi}} e^{(\mu-\frac{1}{2})\pi i} (u^2-1)^{\frac{2\mu-1}{4}} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u)$$

$$[|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu].$$

ET II 202(10)

$$6. \int_1^\infty \frac{x^{\mu-1} [(x-\sqrt{x^2-1})^\nu + (x-\sqrt{x^2-1})^{-\nu}]}{\sqrt{x^2-1}} dx = 2^{-\mu} B\left(\frac{1-\mu+\nu}{2}, \frac{1-\mu-\nu}{2}\right)$$

$$[\operatorname{Re} \mu < 1 + \operatorname{Re} \nu].$$

ET I 311(29)

$$7. \int_0^u \frac{(u-x)^{\mu-1} [(\sqrt{x+2} + \sqrt{x})^{2\nu} + (\sqrt{x+2} - \sqrt{x})^{2\nu}]}{\sqrt{x(x+2)}} dx = 2 \frac{2\mu+1}{2} \sqrt{\pi[u(u+2)]^{\mu-\frac{1}{2}}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(u+1)$$

[|arg u| < π, Re μ > 0].

ET II 186(12)

3.278

$$\int_0^\infty \left(\frac{x^p}{1+x^{2p}} \right)^q \frac{dx}{1-x^2} = 0.$$

3.3- 3.4 Exponential Functions

3.31 Exponential functions

3.310

$$\int_0^\infty e^{-px} dx = \frac{1}{p} \quad [\text{Re } p > 0].$$

3.311

$$1. \int_0^\infty \frac{dx}{1+e^{px}} = \frac{\ln 2}{p}.$$

LO III 284a

$$2. \int_0^\infty \frac{e^{-\mu x}}{1+e^{-x}} dx = \beta(\mu) \quad [\text{Re } \mu > 0].$$

EH I 20(3), ET I 144(7)

$$3. \int_{-\infty}^\infty \frac{e^{-px}}{1+e^{-qx}} dx = \frac{\pi}{q} \operatorname{cosec} \frac{p\pi}{q} \quad [q > p > 0 \quad \text{or} \quad 0 > p > q] \quad (\text{cf. 3.241 2.}).$$

3.241

BI ((28))(7)

$$4. \int_0^{\infty} \frac{e^{-qx} dx}{1 - ae^{-px}} = \sum_{k=0}^{\infty} \frac{a^k}{q + kp} \quad [0 < a < 1].$$

BI ((27))(7)

$$5. \int_0^{\infty} \frac{1 - e^{\nu x}}{e^x - 1} dx = \psi(\nu) + C + \pi \operatorname{ctg}(\pi\nu) \quad [\operatorname{Re} \nu < 1] \quad (\text{cf. 3.265}).$$

3.265
EH I 16(16)

$$6. \int_0^{\infty} \frac{e^{-x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) + C \quad [\operatorname{Re} \nu > 0].$$

WH, EH I 16(14)

$$7. \int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}} dx = \psi(\nu) - \psi(\mu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad (\text{cf. 3.231 5}).$$

3.231
BI ((27))(8)

$$8. \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{b - e^{-x}} = \pi b^{\mu-1} \operatorname{ctg}(\mu\pi) \quad [b > 0, 0 < \operatorname{Re} \mu < 1].$$

ET I 120(14)a

$$9. \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{b + e^{-x}} = \pi b^{\mu-1} \operatorname{cosec}(\mu\pi) \quad [|\arg b| < \pi, 0 < \operatorname{Re} \mu < 1].$$

ET I 120(15)a

$$10. \int_0^{\infty} \frac{e^{-px} - e^{-qx}}{1 + e^{-(p+q)x}} dx = \frac{\pi}{p+q} \operatorname{ctg} \frac{p\pi}{p+q} \quad [p > 0, q > 0].$$

GW ((311))(16c)

$$11. \int_0^{\infty} \frac{e^{px} - e^{qx}}{e^{rx} - e^{sx}} dx = \frac{1}{r-s} \left[\psi \left(\frac{r-q}{r-s} \right) - \psi \left(\frac{r-p}{r-s} \right) \right] \quad [r > s, r > p, r > q].$$

GW ((311))(16)

$$12. \int_0^{\infty} \frac{a^x - b^x}{c^x - d^x} dx = \frac{1}{\ln \frac{c}{d}} \left\{ \psi \left(\frac{\ln \frac{c}{b}}{\ln \frac{c}{d}} \right) - \psi \left(\frac{\ln \frac{c}{a}}{\ln \frac{c}{d}} \right) \right\} \quad [c > a > 0, b > 0, d > 0].$$

GW ((311))(16a)

3.312

$$1. \int_0^{\infty} \left(1 - e^{-\frac{x}{\beta}}\right)^{\nu-1} e^{-\mu x} dx = \beta B(\beta\mu, \nu) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0, \operatorname{Re} \mu > 0].$$

LI((25))(13), EH I 11(24)

$$2. \int_0^{\infty} (1 - e^{-x})^{-1} (1 - e^{-\alpha x}) (1 - e^{-\beta x}) e^{-px} dx = \psi(p+\alpha) + \psi(p+\beta) - \psi(p+\alpha+\beta) - \psi(p) \\ [\operatorname{Re} p > 0, \operatorname{Re} p > -\operatorname{Re} \alpha, \operatorname{Re} p > -\operatorname{Re} \beta, \operatorname{Re} p > -\operatorname{Re}(\alpha + \beta)].$$

ET I 145(15)

$$3. \int_0^{\infty} (1 - e^{-x})^{\nu-1} (1 - \beta e^{-x})^{-\varrho} e^{-\mu x} dx = B(\mu, \nu) {}_2F_1(\varrho, u; \mu + \nu; \beta) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, |\arg(1 - \beta)| < \pi].$$

EH I 116(15)

3.313

$$1.7 \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{1 - e^{-x}} = \pi \cot \pi \mu \quad [0 < \operatorname{Re} \mu < 1].$$

$$2.7 \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{(1 + e^{-x})^{\nu}} = B(\mu, \nu - \mu) \quad [0 < \operatorname{Re} \mu < \operatorname{Re} \nu].$$

3.314

ET I 120(21)

3.315

$$1. \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{(e^{\beta} + e^{-x})^{\nu} (e^{\gamma} + e^{-x})^{\varrho}} = \exp[\gamma(\mu - \varrho) - \beta\nu] B(\mu, \nu + \varrho - \mu) {}_2F_1(\nu, \mu; \nu + \varrho; 1 - e^{\nu - \beta})$$

$$[|\operatorname{Im} \beta| < \pi, \quad |\operatorname{Im} \gamma| < \pi, \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\nu + \varrho)].$$

ET I 121(22)

$$2. \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^{-x})} = \frac{\pi(\beta^{\mu-1} - \gamma^{\mu-1})}{\gamma - \beta} \operatorname{cosec}(\mu\pi)$$

$$[|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \beta \neq \gamma, \quad 0 < \operatorname{Re} \mu < 2].$$

ET I 120(18)

354

3.316

$$\int_{-\infty}^{\infty} \frac{(1 + e^{-x})^{\nu} - 1}{(1 + e^{-x})^{\mu}} dx = \psi(\mu) - \psi(\mu - \nu) \quad [\operatorname{Re} \mu > \operatorname{Re} \nu > 0] \quad (\text{cf. 3.235}).$$

3.235
BI ((28))(8)

3.317

$$1. \int_{-\infty}^{\infty} \left\{ \frac{1}{1 + e^{-x}} - \frac{1}{(1 + e^{-x})^{\mu}} \right\} dx = \mathbf{C} + \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad (\text{cf. 3.233}).$$

3.233
BI ((28))(10)

$$2. \int_{-\infty}^{\infty} \left\{ \frac{1}{(1 + e^{-x})^{\nu}} - \frac{1}{(1 + e^{-x})^{\mu}} \right\} dx = \psi(\mu) - \psi(\nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0] \quad (\text{cf. 3.219}).$$

3.219
BI ((28))(11)

3.318

3.318

$$1. \int_0^\infty \frac{[\beta + \sqrt{1 - e^{-x}}]^{-\nu} + [\beta - \sqrt{1 - e^{-x}}]^{-\nu}}{\sqrt{1 - e^{-x}}} e^{-\mu x} dx =$$

$$= \frac{2^{\mu+1} e^{(\mu-\nu)\pi i} (\beta^2 - 1)^{(\mu-\nu)/2} \Gamma(\mu) Q_{\mu-1}^{\nu-\mu}(\beta)}{\Gamma(\nu)} \quad [\operatorname{Re} \mu > 0].$$

ET I 145(18)

$$2.7 \int_u^\infty \frac{1}{\sqrt{1 - e^{-2x}}} \left\{ e^{-u} \sqrt{1 - e^{-2x}} - e^{-x} \sqrt{1 - e^{-2u}} \right\}^\nu e^{-\mu x} dx =$$

$$= \frac{2^{-\frac{1}{2}(\mu+\nu)} \sqrt{\pi} e^{-\frac{\pi}{2}(\mu+\nu)} \Gamma(\mu) \Gamma(\nu+1) P_{-\frac{1}{2}(\mu+\nu)}^{-\frac{1}{2}(\mu+\nu)}(\sqrt{1 - e^{-2u}})}{\Gamma[(\mu + \nu + 1)/2]}$$

$$[u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 145(19)

3.32- 3.34 Exponentials of more complicated arguments

3.321

$$1.7 \quad \Phi(u) = \frac{\sqrt{\pi}}{2} \operatorname{erf}(u) = \int_0^u e^{-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{k!(2k+1)};$$

$$= e^{-u^2} \sum_{k=0}^{\infty} \frac{2^k u^{2k+1}}{(2k+1)!}. \quad [cf. 8.25]$$

8.25
AD 6.700

$$2. \int_0^u e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \Phi(qu) \quad [q > 0].$$

$$3. \int_0^\infty e^{-q^2 x^2} dx = \frac{\sqrt{\pi}}{2q} \quad [q > 0].$$

FI II 624

3.322

$$1.7 \int_u^\infty \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[1 - \Phi\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}}\right) \right] \quad [\operatorname{Re} \beta > 0, \quad u \geq 0].$$

$$2. \int_0^{\infty} \exp\left(-\frac{x^2}{4\beta} - \gamma x\right) dx = \sqrt{\pi\beta} \exp(\beta\gamma^2) \left[1 - \Phi(\gamma\sqrt{\beta})\right] \quad [\operatorname{Re} \beta > 0].$$

NT 27(1)a

3.323

$$1.7 \int_1^{\infty} \exp(-qx - x^2) dx = \frac{\pi^{1/2}}{2} e^{q^2/4} \left[1 - \Phi\left(1 + \frac{1}{2}q\right)\right].$$

BI ((29))(4)

$$2.7 \int_{-\infty}^{\infty} \exp(-p^2 x^2 \pm qx) dx = \exp\left(\frac{q^2}{4p^2}\right) \frac{\sqrt{\pi}}{|p|}.$$

BI ((28))(1)

$$3.6 \int_0^{\infty} \exp(-\beta^2 x^4 - 2\gamma^2 x^2) dx = 2^{-\frac{3}{2}} \frac{\gamma}{\beta} e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}}\left(\frac{\gamma^4}{2\beta^2}\right) \quad \left[|\arg \beta| < \frac{\pi}{4}\right].$$

ET I 147(34)a

3.324

$$1. \int_0^{\infty} \exp\left(-\frac{\beta}{4x} - \gamma x\right) dx = \sqrt{\frac{\beta}{\gamma}} K_1(\sqrt{\beta\gamma}) \quad [\operatorname{Re} \beta \geq 0, \operatorname{Re} \gamma > 0].$$

ET I 146(25)

$$2. \int_{-\infty}^{\infty} \exp\left[-\left(x - \frac{b}{x}\right)^{2n}\right] dx = \frac{1}{n} \Gamma\left(\frac{1}{2n}\right).$$

3.325

$$\int_0^{\infty} \exp\left(-ax^2 - \frac{b}{x^2}\right) dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{ab}) \quad [a > 0, b > 0].$$

FI II 644

3.326⁸

Exponentials of exponentials

3.327

$$\int_0^{\infty} \exp(-ae^{nx}) dx = -\frac{1}{n} \text{Ei}(-a) \quad [n \geq 1, \quad \text{Re } a \geq 0, \quad a \neq 0].$$

LI ((26))(5)

3.328

$$\int_{-\infty}^{\infty} \exp(-e^x) e^{\mu x} dx = \Gamma(\mu) \quad [\text{Re } \mu > 0].$$

NH 145(14)

3.329

$$\int_0^{\infty} \left[\frac{a \exp(-ce^{\alpha x})}{1 - e^{-\alpha x}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right] dx = e^{-c} \ln \frac{b}{a} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((27))(12)

3.331

$$1. \int_0^{\infty} \exp(-\beta e^{-x} - \mu x) dx = \beta^{-\mu} \gamma(\mu, \beta) \quad [\text{Re } \mu > 0].$$

ET I 147(36)

$$2. \int_0^{\infty} \exp(-\beta e^x - \mu x) dx = \beta^{\mu} \Gamma(-\mu, \beta) \quad [\text{Re } \beta > 0].$$

ET I 147(37)

356

$$3. \int_0^{\infty} (1 - e^{-x})^{\nu-1} \exp(\beta e^{-x} - \mu x) dx = \text{B}(\mu, \nu) \beta^{-\frac{\mu-\nu}{2}} e^{\frac{\beta}{2}} M_{\frac{\nu-\mu}{2}, \frac{\nu+\mu-1}{2}}(\beta) \\ [\text{Re } \mu > 0, \quad \text{Re } \nu > 0].$$

ET I 147(38)

$$4. \int_0^{\infty} (1 - e^{-x})^{\nu-1} \exp(-\beta e^x - \mu x) dx = \Gamma(\nu) \beta^{\frac{\mu-1}{2}} e^{-\frac{\beta}{2}} W_{\frac{1-\mu-2\nu}{2}, \frac{-\mu}{2}}(\beta)$$

3.332

$$\int_0^{\infty} (1 - e^{-x})^{\nu-1} (1 - \lambda e^{-x})^{-\varrho} \exp(\beta e^{-x} - \mu x) dx = B(\mu, \nu) \Phi_1(\mu, \varrho, \nu, \lambda, \beta)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad |\arg(1 - \lambda)| < \pi].$$

ET I 147(40)

3.333

$$1. \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{\exp(e^{-x}) - 1} = \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 1].$$

ET I 121(24)

$$2.^3 \int_{-\infty}^{\infty} \frac{e^{-\mu x} dx}{\exp(e^{-x}) + 1} = (1 - 2^{1-\mu}) \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 0, \quad \mu \neq 1];$$

$$= \ln 2 \quad [\mu = 1].$$

ET I 121(25)

3.334

$$\int_0^{\infty} (e^x - 1)^{\nu-1} \exp\left[-\frac{\beta}{e^x - 1} - \mu x\right] dx = \Gamma(\mu - \nu + 1) e^{\frac{\beta}{2}} \beta^{\frac{\nu-1}{2}} W_{\frac{\nu-2\mu-1}{2}, \frac{\nu}{2}}(\beta)$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu - 1].$$

ET I 137(41)

Exponentials of hyperbolic functions

3.335

$$\int_0^{\infty} (e^{\nu x} + e^{-\nu x} \cos \nu \pi) \exp(-\beta \operatorname{sh} x) dx = -\pi [\mathbf{E}_{\nu}(\beta) + N_{\nu}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

EH II 35(34)

3.336

$$1. \int_0^{\infty} \exp(-\nu x - \beta \operatorname{sh} x) dx = \pi \operatorname{cosec} \nu \pi [\mathbf{J}_{\nu}(\beta) - J_{\nu}(\beta)]$$

$$\left[|\arg \beta| < \frac{\pi}{2} \quad \text{and} \quad |\arg \beta| = \frac{\pi}{2} \quad \text{for} \quad \operatorname{Re} \nu > 0; \quad \nu - \text{not an integer} \right].$$

$$2. \int_0^{\infty} \exp(nx - \beta \operatorname{sh} x) dx = \frac{1}{2} [S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi N_n(\beta)] \quad [\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots].$$

WA 342(6)

$$3. \int_0^{\infty} \exp(-nx - \beta \operatorname{sh} x) dx = \frac{1}{2} (-1)^{n+1} [S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi N_n(\beta)] \\ [\operatorname{Re} \beta > 0; \quad n = 0, 1, 2, \dots].$$

EH II 84(47)

357

3.337

$$1. \int_{-\infty}^{\infty} \exp(-\alpha x - \beta \operatorname{ch} x) dx = 2K_{\alpha}(\beta) \quad \left[\left| \arg \beta \right| < \frac{\pi}{2} \right].$$

WA 201(7)

$$2. \int_{-\infty}^{\infty} \exp(-\nu x + i\beta \operatorname{ch} x) dx = i\pi e^{\frac{i\nu\pi}{2}} H_{\nu}^{(1)}(\beta) \quad [0 < \arg z < \pi].$$

EH II 21(27)

$$3. \int_{-\infty}^{\infty} \exp(-\nu x - i\beta \operatorname{ch} x) dx = -i\pi e^{-\frac{i\nu\pi}{2}} H_{\nu}^{(2)}(\beta) \quad [-\pi < \arg z < 0].$$

EH II 21(30)

Exponentials of trigonometric functions and logarithms

3.338

$$1. \int_0^{\pi} \{ \exp[i(\nu - 1)x - \beta \sin x] - \exp[i(\nu + 1)x - \beta \sin x] \} dx = 2\pi [\mathbf{J}'_{\nu}(\beta) + i\mathbf{E}'_{\nu}(\beta)] \\ [\operatorname{Re} \beta > 0].$$

EH II 36

$$2. \int_0^{\pi} \exp[\pm i(\nu x - \beta \sin x)] dx = \pi [\mathbf{J}_{\nu}(\beta) \pm i\mathbf{E}_{\nu}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

EH II 35(32)

$$3. \int_0^{\infty} \exp[-\gamma(x - \beta \sin x)] dx = \frac{1}{\gamma} + 2 \sum_{k=1}^{\infty} \frac{\gamma_k^{J(k\beta)}}{\gamma^2 + k^2} \quad [\operatorname{Re} \gamma > 0].$$

$$4.6 \int_{-\pi}^{\pi} \frac{\exp \left[\frac{a + b \sin x + c \cos x}{1 + p \sin x + q \cos x} \right]}{1 + p \sin x + q \cos x} dx = \frac{2\pi}{\sqrt{1 - p^2 - q^2}} e^{-\alpha} I_0(\beta),$$

$$\text{with } \alpha = \frac{bp + cq - a}{1 - p^2 - q^2}; \quad \beta = \sqrt{\alpha^2 - \frac{a^2 - b^2 - c^2}{1 - p^2 - q^2}}; \quad [p^2 + q^2 < 1].$$

3.339⁶

$$\int_0^{\pi} \exp(z \cos x) dx = \pi I_0(z).$$

BI ((277))(2)a

3.341

$$\int_0^{\frac{\pi}{2}} \exp(-p \operatorname{tg} x) dx = \operatorname{ci}(p) \sin p - \operatorname{si}(p) \cos(p) \quad [p > 0].$$

BI ((271))(2)a

3.342

$$\int_0^1 \exp(-px \ln x) dx = \int_0^1 x^{-px} dx = \sum_{k=1}^{\infty} \frac{p^k - 1}{k^k}.$$

BI ((29))(1)

3.35 Combinations of exponentials and rational functions

3.351

$$1. \int_0^u x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}} - e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} \quad [u > 0].$$

ET I 134(5)

$$2. \int_u^{\infty} x^n e^{-\mu x} dx = e^{-u\mu} \sum_{k=0}^n \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}} \quad [u > 0, \operatorname{Re} \mu > 0].$$

ET I 33(4)

$$3. \int_0^{\infty} x^n e^{-\mu x} dx = n! \mu^{-n-1} \quad [\operatorname{Re} \mu > 0].$$

$$4. \int_u^\infty \frac{e^{-px} dx}{x^{n+1}} = (-1)^{n+1} \frac{p^n \operatorname{Ei}(-pu)}{n!} + \frac{e^{-pu}}{u^n} \sum_{k=0}^{n-1} \frac{(-1)^k p^k u^k}{n(n-1)\dots(n-k)} \quad [p > 0].$$

NT 21(3)

$$5. \int_1^\infty \frac{e^{-\mu x} dx}{x} = -\operatorname{Ei}(-\mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((104))(10)

$$6. \int_{-\infty}^u \frac{e^x}{x} dx = \operatorname{li}(e^u) = \operatorname{Ei}(u) \quad [u < 0].$$

$$7.* \int_0^u x e^{-\mu x} dx = \frac{1}{\mu^2} - \frac{1}{\mu^2} e^{-\mu u} (1 + \mu u) \quad [u > 0]$$

$$8.* \int_0^u x^2 e^{-\mu x} dx = \frac{2}{\mu^3} - \frac{1}{\mu^3} e^{-\mu u} (2 + 2\mu u - \mu^2 u^2) \quad [u > 0]$$

$$9.* \int_0^u x^3 e^{-\mu x} dx = \frac{6}{\mu^4} - \frac{1}{\mu^4} e^{-\mu u} (6 + 6\mu u + 3\mu^2 u^2 + \mu^3 u^3) \quad [u > 0]$$

3.352

$$1. \int_0^u \frac{e^{-\mu x} dx}{x + \beta} = e^{\mu\beta} [\operatorname{Ei}(-\mu u - \mu\beta) - \operatorname{Ei}(-\mu\beta)] \quad [|\arg \beta| < \pi].$$

ET II 217(12)

$$2. \int_u^\infty \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \operatorname{Ei}(-\mu u - \mu\beta) \quad [u \geq 0, \quad |\arg(u + \beta)| < \pi, \quad \operatorname{Re} \mu > 0].$$

ET I 134(6), JA

$$3. \int_u^v \frac{e^{-\mu x} dx}{x + \alpha} = e^{\alpha\mu} \{ \operatorname{Ei}[-(\alpha + v)\mu] - \operatorname{Ei}[-(\alpha + u)\mu] \} \quad [-\alpha < n, \quad \text{or} \quad -\alpha > v, \quad \operatorname{Re} \mu > 0].$$

$$4. \int_0^{\infty} \frac{e^{-\mu x} dx}{x + \beta} = -e^{\beta\mu} \operatorname{Ei}(-\mu\beta) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET II 217(11)

$$5.7 \int_u^{\infty} \frac{e^{-px} dx}{a - x} = e^{-pa} \operatorname{Ei}(pa - pu) \quad [p > 0, a < u]$$

ET II 251(37)

$$6.8 \int_0^{\infty} \frac{e^{-\mu x} dx}{a - x} = e^{-\mu a} \operatorname{Ei}(a\mu) \quad [a < 0, \operatorname{Re} \mu > 0].$$

BI ((91))(4)

$$7. \int_{-\infty}^{\infty} \frac{e^{ipx} dx}{x - a} = i\pi e^{iap} \quad [p > 0].$$

ET II 251(38)

3.353

$$1. \int_u^{\infty} \frac{e^{-\mu x} dx}{(x + \beta)^n} = e^{-u\mu} \sum_{k=1}^{n-1} \frac{(k-1)!(-\mu)^{n-k-1}}{(n-1)!(u + \beta)^k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \operatorname{Ei}[-(u + \beta)\mu]$$

$$[n \geq 2, |\arg(u + \beta)| < \pi, \operatorname{Re} \mu > 0].$$

ET I 134(10)

359

$$2.7 \int_0^{\infty} \frac{e^{-\mu x} dx}{(x + \beta)^n} = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (k-1)!(-\mu)^{n-k-1} \beta^{-k} - \frac{(-\mu)^{n-1}}{(n-1)!} e^{\beta\mu} \operatorname{Ei}(-\beta\mu)$$

$$[n \geq 2, |\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET I 134(9), BI ((92))(2)

$$3. \int_0^{\infty} \frac{e^{-px} dx}{(a + x)^2} = pe^{\alpha p} \operatorname{Ei}(-ap) + \frac{1}{a} \quad [p > 0, a > 0].$$

LI ((281))(28), LI ((281))(29)

BI ((80))(6)

$$5.7 \int_0^{\infty} \frac{x^n e^{-\mu x}}{x + \beta} dx = (-1)^{n-1} \beta^n e^{\beta\mu} \operatorname{Ei}(-\beta\mu) + \sum_{k=1}^n (k-1)! (-\beta)^{n-k} \mu^{-k} \\ [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0; n \geq 0].$$

BI ((91))(3)A, LET I 135(11)

3.354

$$1. \int_0^{\infty} \frac{e^{-\mu x} dx}{\beta^2 + x^2} = \frac{1}{\beta} [\operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((91))(7)

$$2. \int_0^{\infty} \frac{x e^{-\mu x} dx}{\beta^2 + x^2} = -\operatorname{ci}(\beta\mu) \cos \beta\mu - \operatorname{si}(\beta\mu) \sin \beta\mu \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((91))(8)

$$3.7 \int_0^{\infty} \frac{e^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2\beta} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu) - e^{\beta\mu} \operatorname{Ei}(-\beta\mu)] \quad [|\arg(\pm\beta)| < \pi, \quad \operatorname{Re} \mu > 0]$$

BI ((91))(14)

$$4. \int_0^{\infty} \frac{x e^{-\mu x} dx}{\beta^2 - x^2} = \frac{1}{2} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu) + e^{\beta\mu} \operatorname{Ei}(-\beta\mu)] \quad [|\arg(\pm\beta)| < \pi, \quad \operatorname{Re} \mu > 0; \\ \text{for } \beta > 0 \text{ one should replace } \operatorname{Ei}(\beta\mu) \text{ in this formula with } \overline{\operatorname{Ei}}(\beta\mu)].$$

BI ((91))(15)

$$5.8 \int_{-\infty}^{\infty} \frac{e^{-ipx} dx}{a^2 + x^2} = \frac{\pi}{a} e^{-|ap|} \quad [a \neq 0], \quad p \text{ real.}$$

ET I 118(1)a

3.355

$$1. \int_0^{\infty} \frac{e^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^3} \{ \operatorname{ci}(\beta\mu) \sin \beta\mu - \operatorname{si}(\beta\mu) \cos \beta\mu - \\ - \beta\mu [\operatorname{ci}(\beta\mu) \cos \beta\mu + \operatorname{si}(\beta\mu) \sin \beta\mu] \}.$$

LI ((92))(6)

$$2. \int_0^{\infty} \frac{x e^{-\mu x} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^2} \{-\beta\mu[\text{ci}(\beta\mu) \sin \beta\mu - \text{si}(\beta\mu) \cos \beta\mu]\} \quad [\text{Re } \beta > 0, \quad \text{Re } \mu > 0].$$

BI ((92))(7)

360

$$3.7 \int_0^{\infty} \frac{e^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^3} [(ap-1)e^{ap} \text{Ei}(-ap) + (1+ap)e^{-ap} \text{Ei}(ap)] \quad [\text{Im}(a^2) \neq 0, \quad p > 0].$$

BI ((92))(8)

$$4.7 \int_0^{\infty} \frac{x e^{-px} dx}{(a^2 - x^2)^2} = \frac{1}{4a^2} \{-2 + ap[e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap)]\} \quad [\text{Im}(a^2) \neq 0, \quad p > 0].$$

LI ((92))(9)

3.356

$$1. \int_0^{\infty} \frac{x^{2n+1} e^{-px}}{a^2 + x^2} dx = (-1)^{n-1} a^{2n} [\text{ci}(ap) \cos ap + \text{si}(ap) \sin ap] + \frac{1}{p^{2n}} \sum_{k=1}^n (2n-2k+1)! (-a^2 p^2)^{k-1} \quad [p > 0].$$

BI ((91))(12)

$$2. \int_0^{\infty} \frac{x^{2n} e^{-px}}{a^2 + x^2} dx = (-1)^n a^{2n-1} [\text{ci}(ap) \sin ap - \text{si}(ap) \cos ap] + \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n-2k)! (-a^2 p^2)^{k-1} \quad [p > 0].$$

BI ((91))(11)

$$3. \int_0^{\infty} \frac{x^{2n+1} e^{-px}}{a^2 - x^2} dx = \frac{1}{2} a^{2n} [e^{ap} \text{Ei}(-ap) + e^{-ap} \text{Ei}(ap)] - \frac{1}{p^{2n}} \sum_{k=1}^n (2n-2k+1)! (a^2 p^2)^{k-1} \quad [p > 0].$$

BI ((91))(17)

$$4. \int_0^{\infty} \frac{x^{2n} e^{-px}}{a^2 - x^2} dx = \frac{1}{2} a^{2n-1} [e^{-ap} \text{Ei}(ap) - e^{ap} \text{Ei}(-ap)] - \frac{1}{p^{2n-1}} \sum_{k=1}^n (2n-2k)! (a^2 p^2)^{k-1} \quad [p > 0].$$

$$1. \int_0^{\infty} \frac{e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu)(\sin a\mu + \cos a\mu) + \\ + \text{si}(a\mu)(\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

BI ((92))(18)

$$2. \int_0^{\infty} \frac{x e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2a} \{ \text{ci}(a\mu)(\sin a\mu - \cos a\mu) - \\ - \text{si}(a\mu)(\sin a\mu + \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

BI ((92))(19)

$$3. \int_0^{\infty} \frac{x^2 e^{-\mu x} dx}{a^3 + a^2 x + ax^2 + x^3} = \frac{1}{2} \{ -\text{ci}(a\mu)(\sin a\mu + \cos a\mu) - \\ - \text{si}(a\mu)(\sin a\mu - \cos a\mu) - e^{a\mu} \text{Ei}(-a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

BI ((92))(20)

$$4. \int_0^{\infty} \frac{e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2a^2} \{ \text{ci}(a\mu)(\sin a\mu - \cos a\mu) - \\ - \text{si}(a\mu)(\sin a\mu + \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

BI ((92))(21)

$$5. \int_0^{\infty} \frac{x e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2a} \{ -\text{ci}(a\mu)(\sin a\mu + \cos a\mu) - \\ - \text{si}(a\mu)(\sin a\mu - \cos a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

BI ((92))(22)

$$6. \int_0^{\infty} \frac{x^2 e^{-\mu x} dx}{a^3 - a^2 x + ax^2 - x^3} = \frac{1}{2} \{ \text{ci}(a\mu)(\cos a\mu - \sin a\mu) + \\ + \text{si}(a\mu)(\cos a\mu + \sin a\mu) + e^{-a\mu} \text{Ei}(a\mu) \} \quad [\text{Re } \mu > 0, \quad a > 0].$$

$$1. \int_0^{\infty} \frac{e^{-px}}{a^4 - x^4} dx = \frac{1}{4a^3} \{e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) + 2 \operatorname{ci}(ap) \sin ap - 2 \operatorname{si}(ap) \cos ap\}$$

$$[p > 0, \quad a > 0].$$

BI ((91))(18)

$$2. \int_0^{\infty} \frac{x e^{-px}}{a^4 - x^4} dx = \frac{1}{4a^2} \{e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) - 2 \operatorname{ci}(ap) \cos ap - 2 \operatorname{si}(ap) \sin ap\}$$

$$[p > 0, \quad a > 0].$$

BI ((91))(19)

$$3. \int_0^{\infty} \frac{x^2 e^{-px}}{a^4 - x^4} dx = \frac{1}{4a} \{e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) - 2 \operatorname{ci}(ap) \sin ap + 2 \operatorname{si}(ap) \cos ap\}$$

$$[p > 0, \quad a > 0].$$

BI ((91))(20)

$$4. \int_0^{\infty} \frac{x^3 e^{-px}}{a^4 - x^4} dx = \frac{1}{4} \{e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) + 2 \operatorname{ci}(ap) \cos ap + 2 \operatorname{si}(ap) \sin ap\}$$

$$[p > 0, \quad a > 0].$$

BI ((91))(21)

$$5. \int_0^{\infty} \frac{x^{4n} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-3} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) + 2 \operatorname{ci}(ap) \sin ap - 2 \operatorname{si}(ap) \cos ap] -$$

$$- \frac{1}{p^{4n-3}} \sum_{k=1}^n (4n - 4k)! (a^4 p^4)^{k-1} \quad [p > 0, \quad a > 0].$$

BI ((91))(22)

$$6. \int_0^{\infty} \frac{x^{4n+1} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-2} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) - 2 \operatorname{ci}(ap) \cos ap - 2 \operatorname{si}(ap) \sin ap] -$$

$$- \frac{1}{p^{4n-2}} \sum_{k=1}^n (4n - 4k + 1)! (a^4 p^4)^{k-1} \quad [p > 0, \quad a > 0].$$

BI ((91))(23)

$$7. \int_0^{\infty} \frac{x^{4n+2} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n-1} [e^{-ap} \operatorname{Ei}(ap) - e^{ap} \operatorname{Ei}(-ap) - 2 \operatorname{ci}(ap) \sin ap + 2 \operatorname{si}(ap) \cos ap] -$$

$$- \frac{1}{p^{4n-1}} \sum_{k=1}^n (4n - 4k + 2)! (a^4 p^4)^{k-1} \quad [p > 0, \quad a > 0].$$

BI ((91))(24)

$$8. \int_0^{\infty} \frac{x^{4n+3} e^{-px}}{a^4 - x^4} dx = \frac{1}{4} a^{4n} [e^{ap} \operatorname{Ei}(-ap) + e^{-ap} \operatorname{Ei}(ap) + 2 \operatorname{ci}(ap) \cos ap + 2 \operatorname{si}(ap) \sin ap] - \frac{1}{p^{4n}} \sum_{k=1}^n (4n - 4k + 3)! (a^4 p^4)^{k-1} \quad [p > 0, \quad a > 0].$$

BI ((91))(25)

3.359

$$\int_{-\infty}^{\infty} \frac{(i-x)^n e^{-ipx}}{(i+x)^n i+x^2} dx = (-1)^{n-1} 2\pi p e^{-p} L_{n-1}(2p) \quad \text{for } p > 0; \\ = 0 \quad \text{for } p < 0.$$

ET I 118(2)

3.36- 3.37 Combinations of exponentials and algebraic functions

3.361⁸

$$1. \int_0^u \frac{e^{-qx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{q}} \Phi(\sqrt{qu}).$$

$$2. \int_0^{\infty} \frac{e^{-qx}}{\sqrt{x}} dx = \sqrt{\frac{\pi}{q}} \quad [q > 0].$$

BI((98))(10)

$$3. \int_{-1}^{\infty} \frac{e^{-qx}}{\sqrt{1+x}} dx = e^q \sqrt{\frac{\pi}{q}} \quad [q > 0].$$

BI ((104))(16)

3.362

$$1. \int_1^{\infty} \frac{e^{-\mu x}}{\sqrt{x-1}} dx = \sqrt{\frac{\pi}{\mu}} e^{-\mu} \quad [\operatorname{Re} \mu > 0].$$

BI ((104))(11)a

$$2. \int_0^{\infty} \frac{e^{-\mu x}}{\sqrt{x+\beta}} dx = \sqrt{\frac{\pi}{\mu}} e^{\beta\mu} [1 - \Phi(\sqrt{\beta\mu})] \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi].$$

3.363

$$1. \int_u^\infty \frac{\sqrt{x-u}}{x} e^{-\mu x} dx = \sqrt{\frac{\pi}{\mu}} e^{-u\mu} - \pi\sqrt{u} [1 - \Phi(\sqrt{u\mu})] \quad [u > 0, \operatorname{Re} \mu > 0].$$

ET I 136(23)

$$2. \int_u^\infty \frac{e^{-\mu x} dx}{x\sqrt{x-u}} = \frac{\pi}{\sqrt{u}} [1 - \Phi(\sqrt{u\mu})] \quad [u > 0, \operatorname{Re} \mu \geq 0].$$

ET I 136(26)

3.364

$$1. \int_0^2 \frac{e^{-px} dx}{\sqrt{x(2-x)}} = \pi e^{-p} I_0(p) \quad [p > 0].$$

GW ((312))(7a)

$$2. \int_{-1}^1 \frac{e^{2x} dx}{\sqrt{1-x^2}} = \pi I_0(2).$$

BI ((277))(2a)

$$3. \int_0^\infty \frac{e^{-px} dx}{\sqrt{x(x+a)}} = e^{\frac{ap}{2}} K_0\left(\frac{ap}{2}\right) \quad [a > 0, p > 0].$$

GW ((312))(8a)

363

3.365

$$1. \int_0^u \frac{x e^{-\mu x} dx}{\sqrt{u^2 - x^2}} = \frac{\pi u}{2} [\mathbf{L}_1(\mu u) - I_1(\mu u)] + u \quad [u > 0, \operatorname{Re} \mu > 0].$$

ET I 136(28)

$$2. \int_u^\infty \frac{x e^{-\mu x} dx}{\sqrt{x^2 - u^2}} = u K_1(u\mu) \quad [u > 0, \operatorname{Re} \mu > 0].$$

ET I 136(29)

3.366

$$1. \int_0^{2u} \frac{(u-x)e^{-\mu x} dx}{\sqrt{2ux-x^2}} = \pi u e^{-u\mu} I_1(u\mu) \quad [\operatorname{Re} \mu > 0].$$

ET I 136(31)

$$2. \int_0^\infty \frac{(x+\beta)e^{-\mu x} dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu) \quad [\operatorname{Re} \mu > 0, \quad |\arg \beta| < \pi].$$

ET I 136(30)

$$3. \int_0^\infty \frac{x e^{-\mu x} dx}{\sqrt{x^2+\beta^2}} = \frac{\beta\pi}{2} [\mathbf{H}_1(\beta\mu) - N_1(\beta\mu)] - \beta \quad \left[|\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0 \right].$$

ET I 136(27)

3.367

$$\int_0^\infty \frac{e^{-\mu x} dx}{(1+\cos t+x)\sqrt{x^2+2x}} = \frac{\exp\left(2\mu \cos^2 \frac{t}{2}\right)}{\sin t} \left(t - \sin t \int_0^u K_0(v) e^{-v \cos t} dv \right) \quad [\operatorname{Re} \mu > 0].$$

ET I 136(33)

3.368

$$\int_0^\infty \frac{e^{-\mu x} dx}{x + \sqrt{x^2 + \beta^2}} = \frac{\pi}{2\beta\mu} [\mathbf{H}_1(\beta\mu) - N_1(\beta\mu)] - \frac{1}{\beta^2\mu^2} \quad \left[|\arg \beta| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0 \right].$$

ET I 136(32)

3.369

$$\int_0^\infty \frac{e^{-\mu x} dx}{\sqrt{(x+a)^3}} = \frac{2}{\sqrt{a}} - 2\sqrt{\pi\mu} e^{\alpha\mu} (1 - \Phi(\sqrt{a\mu})) \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0].$$

ET I 135(20)

3.371

$$\begin{aligned} \int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} dx &= \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n-1}{2} \mu^{-n-\frac{1}{2}} \\ &= \sqrt{\pi} 2^{-n} \mu^{-n-1/2} (2n-1)!! \quad [n \geq 0] \quad [\operatorname{Re} \mu > 0]. \end{aligned}$$

$$\int_0^{\infty} x^{n-\frac{1}{2}}(2+x)^{n-\frac{1}{2}}e^{-px} dx = \frac{(2n-1)!!}{p^n} e^p K_n(p) \quad [p > 0, \quad n = 0, 1, 2, \dots].$$

GW ((312))(8)

3.373

$$\int_0^{\infty} \left[(x + \sqrt{x^2 + \beta^2})^n + (x - \sqrt{x^2 + \beta^2})^n \right] e^{-\mu x} dx = 2\beta^{n+1} O_n(\beta\mu) \quad [\operatorname{Re} \mu > 0].$$

WA 05(1)

3.374

$$1. \int_0^{\infty} \frac{(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = \frac{1}{2} [S_n(\mu) - \pi \mathbf{E}_n(\mu) - \pi N_n(\mu)] \quad [\operatorname{Re} \mu > 0].$$

ET I 37(35)

364

$$2. \int_0^{\infty} \frac{(x - \sqrt{1+x^2})^n}{\sqrt{1+x^2}} e^{-\mu x} dx = -\frac{1}{2} [S_n(\mu) + \pi \mathbf{E}_n(\mu) + \pi N_n(\mu)] \quad [\operatorname{Re} \mu > 0].$$

ET I 137(36)

3.38- 3.39 Combinations of exponentials and arbitrary powers

3.381

$$1. \int_0^u x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \gamma(\nu, \mu u) \quad [\operatorname{Re} \nu > 0]$$

EH I 266(22), EH II 133(1)

$$\begin{aligned} 2. \int_0^u x^{p-1} e^{-x} dx &= \sum_{k=0}^{\infty} (-1)^k \frac{u^{p+k}}{k!(p+k)}; \\ &= e^{-u} \sum_{k=0}^{\infty} \frac{u^{p+k}}{p(p+1)\dots(p+k)}. \end{aligned}$$

AD 6.705

$$3. \int_u^{\infty} x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \Gamma(\nu, \mu u) \quad [u > 0, \quad \operatorname{Re} \mu > 0].$$

$$4. \int_0^{\infty} x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^{\nu}} \Gamma(\nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 779

$$5. \int_0^{\infty} x^{\nu-1} e^{-(p+iq)x} dx = \Gamma(\nu)(p^2 + q^2)^{-\frac{\nu}{2}} \exp\left(-i\nu \operatorname{arctg} \frac{q}{p}\right) \\ [p > 0, \operatorname{Re} \nu > 0 \text{ or } p = 0, 0 < \operatorname{Re} \nu < 1].$$

EH I 12(32)

$$6. \int_u^{\infty} \frac{e^{-x}}{x^{\nu}} dx = u^{-\frac{\nu}{2}} e^{-\frac{u}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(u) \quad [u > 0].$$

WH

3.382

$$1.^6 \int_0^u (u-x)^{\nu} e^{-\mu x} dx = (-\mu)^{-\nu-1} e^{-u\mu} \gamma(\nu+1, -u\mu) \quad [\operatorname{Re} \nu > -1, u > 0].$$

ET I 137(6)

$$2. \int_u^{\infty} (x-u)^{\nu} e^{-\mu x} dx = \mu^{-\nu-1} e^{-u\mu} \Gamma(\nu+1) \quad [u > 0, \operatorname{Re} \nu > -1, \operatorname{Re} \mu > 0].$$

ET I 137(5), ET II 202(11)

$$3. \int_0^{\infty} (1+x)^{-\nu} e^{-\mu x} dx = \mu^{\frac{\nu}{2}-1} e^{\frac{\mu}{2}} W_{-\frac{\nu}{2}, \frac{(1-\nu)}{2}}(\mu) \quad [\operatorname{Re} \mu > 0].$$

WH

$$4. \int_0^{\infty} (x+\beta)^{\nu} e^{-\mu x} dx = \mu^{-\nu-1} e^{\beta\mu} \Gamma(\nu+1, \beta\mu) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET I 137(4), ET II 233(10)

$$5. \int_0^u (a+x)^{\mu-1} e^{-x} dx = e^a [\gamma(\mu, a+u) - \gamma(\mu, a)] \quad [\operatorname{Re} \mu > 0].$$

EH II 139

$$6. \int_{-\infty}^{\infty} (\beta + ix)^{-\nu} e^{-ipx} dx = 0 \quad \text{for } p > 0;$$

$$= \frac{2\pi(-p)^{\nu-1} e^{\beta p}}{\Gamma(\nu)} \quad \text{for } p < 0 \quad [\operatorname{Re} \nu > 0, \operatorname{Re} \beta > 0].$$

ET I 118(4)

$$7. \int_{-\infty}^{\infty} (\beta - ix)^{-\nu} e^{-ipx} dx = \frac{2\pi p^{\nu-1} e^{-\beta p}}{\Gamma(\nu)} \quad \text{for } p > 0;$$

$$= 0 \quad \text{for } p < 0 \quad [\operatorname{Re} \nu > 0, \operatorname{Re} \beta > 0].$$

ET I 118(3)

3.383

$$1. \int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{\beta x} dx = B(\mu, \nu) u^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; \beta u) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET II 187(14)

$$2. \int_0^u x^{\mu-1} (u-x)^{\mu-1} e^{\beta x} dx = \sqrt{\pi} \left(\frac{u}{\beta}\right)^{u-\frac{1}{2}} \exp\left(\frac{\beta u}{2}\right) \Gamma(\mu) I_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right) \quad [\operatorname{Re} \mu > 0].$$

ET II 187(13)

$$3. \int_u^{\infty} x^{\mu-1} (x-u)^{\mu-1} e^{-\beta x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{u}{\beta}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) K_{\mu-\frac{1}{2}}\left(\frac{\beta u}{2}\right)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \beta u > 0].$$

ET II 202(12)

$$4. \int_u^{\infty} x^{\nu-1} (x-u)^{\mu-1} e^{-\beta x} dx = \beta^{-\frac{\mu+\nu}{2}} u^{\frac{\mu+\nu-2}{2}} \Gamma(\mu) \exp\left(-\frac{\beta u}{2}\right) W_{\frac{\nu-\mu}{2}, \frac{1-\mu-\nu}{2}}(\beta u)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \beta u > 0].$$

ET II 202(13)

$$5.7 \int_0^{\infty} e^{-px} x^{q-1} (1+ax)^{-\nu} dx = a^{-q} \Gamma(q) \Psi(q, q+1-\nu, p/a)$$

$$= p^{-q} \Gamma(q) \left[\sum_{k=0}^n (q)_k (\nu)_k (a/p)^k / k! + O(a/p)^{N+1} \right]$$

$\operatorname{Re} q > 0, \operatorname{Re} p > 0, \operatorname{Re} a > 0, \nu$ complex, $N = 0, 1, \dots$. For $\nu = 0$ the integral equals $p^{-q}\Gamma(q)$.

$$6. \int_0^\infty x^{\nu-1}(x+\beta)^{-\nu+\frac{1}{2}}e^{-\mu x} dx = 2^{\nu-\frac{1}{2}}\Gamma(\nu)\mu^{-\frac{1}{2}}e^{\frac{\beta\mu}{2}}D_{1-2\nu}(\sqrt{2\beta\mu})$$

$$[|\arg \beta| < \pi, \operatorname{Re} \nu > 0, \operatorname{Re} \mu \geq 0, \mu \neq 0].$$

ET I 39(20), EH II 119(2)a

$$7. \int_0^\infty x^{\nu-1}(x+\beta)^{-\nu-\frac{1}{2}}e^{-\mu x} dx = 2^\nu\Gamma(\nu)\beta^{-\frac{1}{2}}e^{\frac{\beta\mu}{2}}D_{-2\nu}(\sqrt{2\beta\mu})$$

$$[|\arg \beta| < \pi, \operatorname{Re} \nu > 0, \operatorname{Re} \mu \geq 0].$$

ET I 139(21), EH II 119(1)a

$$8. \int_0^\infty x^{\nu-1}(x+\beta)^{\nu-1}e^{-\mu x} dx = \frac{1}{\sqrt{\pi}}\left(\frac{\beta}{\mu}\right)^{\nu-\frac{1}{2}}e^{\frac{\beta\mu}{2}}\Gamma(\nu)K_{\frac{1}{2}-\nu}\left(\frac{\beta\mu}{2}\right)$$

$$[|\arg \beta| < \pi, \operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET II 233(11), EH II 19(16)A, EH II 82(22)a

366

$$9. \int_u^\infty \frac{(x-u)^\nu e^{-\mu x}}{x} dx = u^\nu\Gamma(\nu+1)\Gamma(-\nu, u\mu) \quad [u > 0, \operatorname{Re} \nu > -1, \operatorname{Re} \mu > 0].$$

ET I 138(8)

$$10. \int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{x+\beta} dx = \beta^{\nu-1}e^{\beta\mu}\Gamma(\nu)\Gamma(1-\nu, \beta\mu) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

EH II 137(3)

3.384

$$1. \int_{-1}^1 (1-x)^{\nu-1}(1+x)^{\mu-1}e^{-ipx} dx = 2^{\mu+\nu-1}\mathbf{B}(\mu, \nu)e^{ip}{}_1F_1(\mu; \nu + \mu; -2ip)$$

$$[\operatorname{Re} \nu > 0, \operatorname{Re} \mu > 0].$$

ET I 119(13)

$$2. \int_u^v (x-u)^{2\mu-1}(v-x)^{2\nu-1}e^{-px} dx =$$

$$= \mathbf{B}(2\mu, 2\nu)(v-u)^{\mu+\nu-1}p^{-\mu-\nu} \exp\left(-p\frac{u+v}{2}\right)M_{\mu-\nu, \mu+\nu-\frac{1}{2}}(vp-up)$$

$$\begin{aligned}
3. \int_u^\infty (x + \beta)^{2\nu-1} (x - u)^{2\rho-1} e^{-\mu x} dx &= \\
&= \frac{(u + \beta)^{\nu+\rho-1}}{\mu^{\nu+\rho}} \exp\left[\frac{(\beta - u)\mu}{2}\right] \Gamma(2\rho) W_{\nu-\rho, \nu+\rho-\frac{1}{2}}(u\mu + \beta\mu) \\
&\quad [u > 0, \quad |\arg(\beta + u)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \rho > 0].
\end{aligned}$$

ET I 139(22)

$$\begin{aligned}
4. \int_u^\infty (x + \beta)^\nu (x - u)^{-\nu} e^{-\mu x} dx &= \frac{1}{\mu} \nu \pi \operatorname{cosec}(\nu\pi) e^{-\frac{(\beta+u)\mu}{2}} k_{2\nu} \left[\frac{(\beta + u)\mu}{2}\right] \\
&\quad [\nu \neq 0, \quad u > 0, \quad |\arg(u + \beta)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu < 1].
\end{aligned}$$

ET I 139(17)

$$\begin{aligned}
5. \int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu+\frac{1}{2}} e^{-\mu x} dx &= \frac{1}{\sqrt{\mu}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{1-2\nu}(2\sqrt{u\mu}) \\
&\quad [u > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET I 139(18)

$$\begin{aligned}
6. \int_u^\infty (x - u)^{\nu-1} (x + u)^{-\nu-\frac{1}{2}} e^{-\mu x} dx &= \frac{1}{\sqrt{u}} 2^{\nu-\frac{1}{2}} \Gamma(\nu) D_{-2\nu}(2\sqrt{u\mu}) \\
&\quad [u > 0, \quad \operatorname{Re} \mu \geq 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET I 139(19)

$$\begin{aligned}
7.^6 \int_{-\infty}^\infty (\beta - ix)^{-\mu} (\gamma - ix)^{-\nu} e^{-ipx} dx &= \frac{2\pi e^{-\beta p} p^{\mu+\nu-1}}{\Gamma(\mu + \nu)} {}_1F_1(\nu; \mu + \nu; (\beta - \gamma)p) \quad \text{for } p > 0; \\
&= 0 \quad \text{for } p < 0 \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu + \nu) > 1].
\end{aligned}$$

ET I 119(10)

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$$\begin{aligned}
8.^6 \int_{-\infty}^\infty (\beta + ix)^{-\mu} (\gamma + ix)^{-\nu} e^{-ipx} dx &= 0 \quad \text{for } p > 0; \\
&= \frac{2\pi e^{\gamma p} (-p)^{\mu+\nu-1}}{\Gamma(\mu + \nu)} {}_1F_1[\mu; \mu + \nu; (\beta - \gamma)p] \quad \text{for } p < 0 \\
&\quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad \operatorname{Re}(\mu + \nu) > 1].
\end{aligned}$$

ET I 19(11)

3.385

$$\int_0^1 x^{\nu-1} (1-x)^{\lambda-1} (1-\beta x)^{-\varrho} e^{-\mu x} dx = B(\nu, \lambda) \Phi_1(\nu, \varrho, \lambda + \nu, \beta, -\mu)$$

$$[\operatorname{Re} \lambda > 0, \operatorname{Re} \nu > 0, |\arg(1-\beta)| < \pi].$$

ET I 39(24)

3.386

$$1. \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 - ix} = 2\pi e^{-\beta_0 p} \beta_0^{\nu_0} \prod_{k=1}^n (\beta_0 + \beta_k)^{\nu_k}$$

$$\left[\operatorname{Re} \nu_0 > -1, \operatorname{Re} \beta_k > 0, \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \arg ix = \frac{\pi}{2} \operatorname{sign} x, p > 0 \right].$$

ET I 118(8)

$$2. \int_{-\infty}^{\infty} \frac{(ix)^{\nu_0} \prod_{k=1}^n (\beta_k + ix)^{\nu_k} e^{-ipx} dx}{\beta_0 + ix} = 0$$

$$\left[\operatorname{Re} \nu_0 > -1, \operatorname{Re} \beta_k > 0, \sum_{k=0}^n \operatorname{Re} \nu_k < 1, \arg ix = \frac{\pi}{2} \operatorname{sign} x, p > 0 \right].$$

ET I 119(9)

3.387

$$1.^6 \int_{-1}^1 (1-x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) I_{\nu-\frac{1}{2}}(\mu) \quad \left[\operatorname{Re} \nu > 0, |\arg \mu| < \frac{\pi}{2} \right].$$

WA 172(2)a

$$2.^6 \int_{-1}^1 (1-x^2)^{\nu-1} e^{i\mu x} dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) J_{\nu-\frac{1}{2}}(\mu) \quad [\operatorname{Re} \nu > 0].$$

WA 25(3), WA 48(4)a

368

$$3. \int_1^{\infty} (x^2-1)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\mu) \quad \left[|\arg \mu| < \frac{\pi}{2}, \operatorname{Re} \nu > 0 \right].$$

$$4. \int_1^{\infty} (x^2-1)^{\nu-1} e^{i\mu x} dx = i \frac{\sqrt{\pi}}{2} \left(\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(1)}(\mu) \quad [\text{Im } \mu > 0, \quad \text{Re } \nu > 0];$$

$$= -i \frac{\sqrt{\pi}}{2} \left(-\frac{2}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) H_{\frac{1}{2}-\nu}^{(2)}(-\mu) \quad [\text{Im } \mu < 0, \quad \text{Re } \nu > 0].$$

EH II 83(29)a
EH II 83(28)a

$$5. \int_0^u (u^2-x^2)^{\nu-1} e^{\mu x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) [I_{\nu-\frac{1}{2}}(u\mu) + \mathbf{L}_{\nu-\frac{1}{2}}(u\mu)]. \quad [u > 0, \quad \text{Re } \nu > 0].$$

ET II 188(20)a

$$6. \int_u^{\infty} (x^2-u^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) K_{\nu-\frac{1}{2}}(u\mu) \quad [u > 0, \quad \text{Re } \mu > 0, \quad \text{Re } \nu > 0].$$

ET II 203(17)a

$$7. \int_0^{\infty} (x^2+u^2)^{\nu-1} e^{-\mu x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} \Gamma(\nu) [\mathbf{H}_{\nu-\frac{1}{2}}(u\mu) - N_{\nu-\frac{1}{2}}(u, \mu)]$$

$$[\arg u < \pi, \quad \text{Re } \mu > 0].$$

ET I 138(10)

3.388

$$1. \int_0^{2u} (2ux-x^2)^{\nu-1} e^{-\mu x} dx = \sqrt{\pi} \left(\frac{2u}{\mu}\right)^{\nu-\frac{1}{2}} e^{-u\mu} \Gamma(\nu) I_{\nu-\frac{1}{2}}(u\mu) \quad [u > 0, \quad \text{Re } \nu > 0].$$

ET I 138(14)

$$2. \int_0^{\infty} (2\beta x+x^2)^{\nu-1} e^{-\mu x} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\beta}{\mu}\right)^{\nu-\frac{1}{2}} e^{\beta\mu} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\beta\mu)$$

$$[|\arg \beta| < \pi; \quad \text{Re } \nu > 0, \quad \text{Re } \mu > 0].$$

ET I 138(13)

$$3. \int_0^{\infty} (x^2+ix)^{\nu-1} e^{-\mu x} dx = -\frac{i\sqrt{\pi}e^{\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}} \Gamma(\nu) H_{\nu-\frac{1}{2}}^{(2)}\left(\frac{\mu}{2}\right) \quad [\text{Re } \mu > 0, \quad \text{Re } \nu > 0].$$

$$4. \int_0^{\infty} (x^2 - ix)^{\nu-1} e^{-\mu x} dx = \frac{i\sqrt{\pi}e^{-\frac{i\mu}{2}}}{2\mu^{\nu-\frac{1}{2}}}\Gamma(\nu)H_{\nu-\frac{1}{2}}^{(1)}\left(\frac{\mu}{2}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET I 138(16)

3.389

$$1. \int_0^u x^{2\nu-1}(u^2-x^2)^{\varrho-1}e^{\mu x} dx = \frac{1}{2}\mathbf{B}(\nu, \varrho)u^{2\nu+2\varrho-2} {}_1F_2\left(\nu; \frac{1}{2}, \nu + \varrho; \frac{\mu^2 u^2}{4}\right) + \frac{\mu}{2}\mathbf{B}\left(\nu + \frac{1}{2}, \varrho\right)u^{2\nu+2\varrho-1} {}_1F_2\left(\nu + \frac{1}{2}; \frac{3}{2}, \nu + \varrho + \frac{1}{2}; \frac{\mu^2 u^2}{4}\right) \quad [\operatorname{Re} \varrho > 0, \operatorname{Re} \nu > 0].$$

ET II 188(21)

369

$$2.7 \int_0^{\infty} x^{2\nu-1}(u^2+x^2)^{\varrho-1}e^{-\mu x} dx = \frac{u^{2\nu+2\varrho-2}}{2\sqrt{\pi}\Gamma(1-\varrho)}G_{13}^{31}\left(\frac{\mu^2 u^2}{4} \left| \begin{matrix} 1-\nu \\ 1-\varrho-\nu, 0, \frac{1}{2} \end{matrix} \right.\right) \quad \left[|\arg u| < \frac{\pi}{2}, \operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0\right].$$

ET II 234(15)a

$$3.7 \int_0^u x(u^2-x^2)^{\nu-1}e^{\mu x} dx = \frac{u^{2\nu}}{2\nu} + \frac{\sqrt{\pi}}{2}\left(\frac{\mu}{2}\right)^{\frac{1}{2}-\nu}u^{\nu+\frac{1}{2}}\Gamma(\nu)[I_{\nu+\frac{1}{2}}(\mu u) + L_{\nu+\frac{1}{2}}(\mu u)] \quad [\operatorname{Re} \nu > 0].$$

ET II 188(19)a

$$4. \int_u^{\infty} x(x^2-u^2)^{\nu-1}e^{-\mu x} dx = 2^{\nu-\frac{1}{2}}(\sqrt{\pi})^{-1}\mu^{\frac{1}{2}-\nu}u^{\nu+\frac{1}{2}}\Gamma(\nu)K_{\nu+\frac{1}{2}}(u\mu) \quad [\operatorname{Re}(u\mu) > 0].$$

ET II 203(16)a

$$5. \int_{-\infty}^{\infty} \frac{(ix)^{-\nu}e^{-ipx} dx}{\beta^2+x^2} = \pi\beta^{-\nu-1}e^{-|p|\beta} \quad \left[|\nu| < 1, \operatorname{Re} \beta > 0, \arg ix = \frac{\pi}{2} \operatorname{sign} x\right].$$

ET I 118(5)

$$6. \int_0^{\infty} \frac{x^{\nu}e^{-\mu x}}{\beta^2+x^2} dx = \frac{1}{2}\Gamma(\nu)\beta^{\nu-1} \left[\exp\left(i\mu\beta + i\frac{(\nu-1)\pi}{2}\right) \times \Gamma(1-\nu, i\beta\mu) + \exp\left(-i\beta\mu - i\frac{(\nu-1)\pi}{2}\right) \Gamma(1-\nu, -i\beta\mu) \right] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0, \operatorname{Re} \nu > -1].$$

$$7. \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x} dx}{1+x^2} = \pi \operatorname{cosec}(\nu\pi) V_{\nu}(2\mu, 0) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET I 138(9)

$$8. \int_{-\infty}^{\infty} \frac{(\beta + ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta + \gamma)^{-\nu} e^{-p\gamma} \quad [\operatorname{Re} \nu > -1, p > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

ET I 118(6)

$$9.^6 \int_{-\infty}^{\infty} \frac{(\beta - ix)^{-\nu} e^{-ipx}}{\gamma^2 + x^2} dx = \frac{\pi}{\gamma} (\beta + \gamma)^{-\nu} e^{\gamma p} \quad [p < 0, \operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, \operatorname{Re} \nu > -1].$$

ET I 118(7)

3.391

$$\int_0^{\infty} \left[(\sqrt{x+2\beta} + \sqrt{x})^{2\nu} - (\sqrt{x+2\beta} - \sqrt{x})^{2\nu} \right] e^{-\mu x} dx = 2^{\nu+1} \frac{\nu}{\mu} \beta^{\nu} e^{\beta\mu} K_{\nu}(\beta\mu) \\ [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET I 140(30)

3.392

$$1. \int_0^{\infty} (x + \sqrt{1+x^2})^{\nu} e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) + \frac{\nu}{\mu} S_{0,\nu}(\mu) [\operatorname{Re} \mu > 0].$$

ET I 140(25)

$$2. \int_0^{\infty} (\sqrt{1+x^2} - x)^{\nu} e^{-\mu x} dx = \frac{1}{\mu} S_{1,\nu}(\mu) - \frac{\nu}{\mu} S_{0,\nu}(\mu) \quad [\operatorname{Re} \mu > 0].$$

ET I 140(26)

370

$$3. \int_0^{\infty} \frac{(x + \sqrt{1+x^2})^{\nu}}{\sqrt{1+x^2}} e^{-\mu x} dx = \pi \operatorname{cosec} \nu\pi [J_{-\nu}(\mu) - J_{\nu}(\mu)] \quad [\operatorname{Re} \mu > 0].$$

ET I 140(27), EH II 35(33)

$$4. \int_0^{\infty} \frac{(\sqrt{1+x^2} - x)^{\nu}}{\sqrt{1+x^2}} e^{-\mu x} dx = S_{0,\nu}(\mu) - \nu S_{-1,\nu}(\mu) \quad [\operatorname{Re} \mu > 0].$$

3.393

$$\int_0^{\infty} \frac{(x + \sqrt{x^2 + 4\beta^2})^{2\nu}}{\sqrt{x^3 + 4\beta^2 x}} e^{-\mu x} dx = \frac{\sqrt{\mu\pi^3}}{2^{2\nu+3/2}\beta^{2\nu}} [J_{\nu+1/4}(\beta\mu)N_{\nu-1/4}(\beta\mu) - J_{\nu-1/4}(\beta\mu)N_{\nu+1/4}(\beta\mu)]$$

[Re $\beta > 0$, Re $\mu > 0$].

ET I 140(33)

3.394

$$\int_0^{\infty} \frac{(1 + \sqrt{1 + x^2})^{\nu+1/2}}{x^{\nu+1}\sqrt{1 + x^2}} e^{-\mu x} dx = \sqrt{2}\Gamma(-\nu)D_{\nu}(\sqrt{2i\mu})D_{\nu}(\sqrt{-2i\mu}) \quad [\text{Re } \mu \geq 0, \text{ Re } \nu < 0].$$

ET I 140(32)

3.395

$$1. \int_1^{\infty} \frac{(\sqrt{x^2 - 1} + x)^{\nu} + (\sqrt{x^2 - 1} - x)^{-\nu}}{\sqrt{x^2 - 1}} e^{-\mu x} dx = 2K_{\nu}(\mu) \quad [\text{Re } \mu > 0].$$

ET I 140(29)

$$2. \int_1^{\infty} \frac{(x + \sqrt{x^2 - 1})^{2\nu} + (x - \sqrt{x^2 - 1})^{2\nu}}{\sqrt{x(x^2 - 1)}} e^{-\mu x} dx = \sqrt{\frac{2\mu}{\pi}} K_{\nu+1/4}\left(\frac{\mu}{2}\right) K_{\nu-1/4}\left(\frac{\mu}{2}\right)$$

[Re $\mu > 0$].

ET I 140(34)

$$3. \int_0^{\infty} \frac{(x + \sqrt{x^2 + 1})^{\nu} + \cos \nu\pi(x + \sqrt{x^2 + 1})^{-\nu}}{\sqrt{x^2 + 1}} e^{-\mu x} dx = -\pi[E_{\nu}(\mu) + N_{\nu}(\mu)] \quad [\text{Re } \mu > 0].$$

EH II 35(34)

3.41- 3.44 Combinations of rational functions of powers and exponentials

3.411

$$1. \int_0^{\infty} \frac{x^{\nu-1} dx}{e^{\mu x} - 1} = \frac{1}{\mu^{\nu}} \Gamma(\nu)\zeta(\nu) \quad [\text{Re } \mu > 0, \text{ Re } \nu > 1].$$

FI II 792a

$$2. \int_0^{\infty} \frac{x^{2n-1} dx}{e^{px} - 1} = (-1)^{n-1} \left(\frac{2\pi}{p}\right)^{2n} \frac{B_{2n}}{4n} \quad [n = 1, 2, \dots].$$

$$3. \int_0^{\infty} \frac{x^{\nu-1} dx}{e^{\mu x} + 1} = \frac{1}{\mu^{\nu}} (1 - 2^{1-\nu}) \Gamma(\nu) \zeta(\nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 792a, WH

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$$4. \int_0^{\infty} \frac{x^{2n-1} dx}{e^{px} + 1} = (1 - 2^{1-2n}) \left(\frac{2\pi}{p} \right)^{2n} \frac{|B_{2n}|}{4n} \quad [n = 1, 2, \dots].$$

BI((83))(2), EH I 39(25)

$$5. \int_0^{\ln 2} \frac{x dx}{1 - e^{-x}} = \frac{\pi^2}{12}.$$

BI ((104))(5)

$$6.8 \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{1 - \beta e^{-x}} dx = \Gamma(\nu) \sum_{n=0}^{\infty} (\mu + n)^{-\nu} \beta^n = \Gamma(\nu) \Phi(\beta, \nu, \mu)$$

$[\operatorname{Re} \mu > 0 \text{ and either } |\beta| \leq 1, \beta \neq 1, \operatorname{Re} \nu > 0; \text{ or } \beta = 1, \operatorname{Re} \nu > 1].$

EH I 27(3)

$$7. \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{1 - e^{-\beta x}} dx = \frac{1}{\beta^{\nu}} \Gamma(\nu) \zeta \left(\nu, \frac{\mu}{\beta} \right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 1].$$

ET I 144(10)

$$8.7 \int_0^{\infty} \frac{x^{n-1} e^{-px}}{1 + e^x} dx = (n-1)! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(p+k)^n} = (n-1)! \Phi(-1, n, p+1)$$

$[p > -1; n = 1, 2, \dots].$

BI ((83))(9)

$$9. \int_0^{\infty} \frac{x e^{-x} dx}{e^x - 1} = \frac{\pi^2}{6} - 1 \quad (\text{cf. 4.231 3.}).$$

4.231
BI ((82))(1)

4.251
BI ((82))(2)

$$11. \int_0^{\infty} \frac{x e^{-3x}}{e^{-x} + 1} dx = \frac{\pi^2}{12} - \frac{3}{4} \quad (\text{cf. 4.251 5.}).$$

4.251
BI ((82))(3)

$$12. \int_0^{\infty} \frac{x e^{-2nx}}{1 + e^x} dx = -\frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}}{k^2} \quad (\text{cf. 4.251 6.}).$$

4.251
BI ((82))(5)

$$13. \int_0^{\infty} \frac{x e^{-(2n-1)x}}{1 + e^x} dx = \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} \quad (\text{cf. 4.251 5.}).$$

4.251
BI ((82))(4)

$$14.^7 \int_0^{\infty} \frac{x^2 e^{-nx}}{1 - e^{-x}} dx = 2 \sum_{k=n}^{\infty} \frac{1}{k^3} = 2 \left(\zeta(3) - \sum_{k=1}^{n-1} \frac{1}{k^3} \right) \quad [n = 1, 2, \dots]. \quad (\text{cf. 4.261 12.}).$$

4.261
BI ((82))(9)

$$15.^7 \int_0^{\infty} \frac{x^2 e^{-nx}}{1 + e^{-x}} dx = 2 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^3} = (-1)^{n+1} \left(\frac{3}{2} \zeta(3) + 2 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^3} \right) \quad [n = 1, 2, \dots].$$

(cf. 4.261 11.).

4.261
LI ((82))(10)

$$16. \int_{-\infty}^{\infty} \frac{x^2 e^{-\mu x}}{1 + e^{-x}} dx = \pi^3 \csc^3 \mu\pi (2 - \sin^2 \mu\pi) \quad [0 < \operatorname{Re} \mu < 1].$$

ET I 120(17)a

$$17. \int_0^{\infty} \frac{x^3 e^{-nx}}{1 - e^{-x}} dx = \frac{\pi^4}{15} - 6 \sum_{k=1}^{n-1} \frac{1}{k^4} \quad (\text{cf. 4.262 5.}).$$

4.262
BI ((82))(12)

$$18.^7 \int_0^{\infty} \frac{x^3 e^{-nx}}{1 + e^{-x}} dx = 6 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^4} = (-1)^{n+1} \left(\frac{7}{120} \pi^4 - 6 \sum_{k=1}^{n-1} \frac{(-1)^k}{k^4} \right) \quad (\text{cf. 4.262 4.}).$$

4.262
LI ((82))(13)

$$19.^* \int_0^{\infty} e^{-px} (e^{-x} - 1)^n \frac{dx}{x} = - \sum_{k=0}^n (-1)^k \binom{n}{k} \ln(p + n - k)$$

LI ((89))(10)

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$$20.^* \int_0^{\infty} e^{-px} (e^{-x} - 1)^n \frac{dx}{x^2} = \sum_{k=0}^n (-1)^k \binom{n}{k} (p + n - k) \ln(p + n - k).$$

LI ((89))(15)

$$21. \int_0^{\infty} x^{n-1} \frac{1 - e^{-mx}}{1 - e^{-x}} dx = (n-1)! \sum_{k=1}^m \frac{1}{k^n} \quad (\text{cf. 4.272 11.}).$$

4.272
LI ((83))(8)

$$22.7 \int_0^{\infty} \frac{x^{p-1}}{e^{rx} - q} dx = \frac{1}{qr^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{q^k}{k^p} = \Gamma(p) r^{-p} \Phi(q, p, 1) \quad [p > 0, \quad r > 0, \quad -1 < q < 1].$$

BI ((83))(5)

$$23. \int_{-\infty}^{\infty} \frac{x e^{\mu x}}{\beta + e^x} dx = \pi \beta^{\mu-1} \operatorname{cosec}(\mu\pi) [\ln \beta - \pi \operatorname{ctg}(\mu\pi)] \quad [|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 1].$$

BI ((101))(5), ET I 120(16)a

$$24. \int_{-\infty}^{\infty} \frac{x e^{\mu x}}{e^{\nu x} - 1} dx = \left(\frac{\pi}{\nu} \operatorname{cosec} \frac{\mu\pi}{\nu} \right)^2 \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0]. \quad (\text{cf. 4.254 2}).$$

4.254
LI ((101))(3)

$$25. \int_0^{\infty} x \frac{1 + e^{-x}}{e^x - 1} dx = \frac{\pi^2}{3} - 1 \quad (\text{cf. 4.231 4}).$$

4.231
BI ((82))(6)

$$26. \int_0^{\infty} x \frac{1 - e^{-x}}{1 + e^{-3x}} e^{-x} dx = \frac{2\pi^2}{27}.$$

LI ((82))(7)a

$$27. \int_0^{\infty} \frac{1 - e^{-\mu x}}{1 + e^x} \frac{dx}{x} = \ln \left[\frac{\Gamma\left(\frac{\mu}{2} + 1\right)}{\Gamma\left(\frac{\mu + 1}{2}\right)} \sqrt{\pi} \right] \quad [\operatorname{Re} \mu > -1].$$

BI ((93))(4)

$$28. \int_0^{\infty} \frac{e^{-\nu x} - e^{-\mu x}}{e^{-x} + 1} \frac{dx}{x} = \ln \frac{\Gamma\left(\frac{\nu}{2}\right) \Gamma\left(\frac{\mu + 1}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\nu + 1}{2}\right)} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

$$29. \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 + e^{rx}} \frac{dx}{x} = \ln \left[\operatorname{tg} \frac{p\pi}{2r} \operatorname{ctg} \frac{q\pi}{2r} \right] \quad [|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0].$$

(cf. 4.267 18.).

4.267
BI ((103))(3)

$$30. \int_{-\infty}^{\infty} \frac{e^{px} - e^{qx}}{1 - e^{rx}} \frac{dx}{x} = \ln \left[\sin \frac{p\pi}{r} \operatorname{cosec} \frac{q\pi}{r} \right] \quad [|r| > |p|, \quad |r| > |q|, \quad rp > 0, \quad rq > 0]$$

(cf. 4.267 19.).

4.267
BI ((103))(4)

$$31. \int_0^{\infty} \frac{e^{-qx} + e^{(q-p)x}}{1 - e^{-px}} x dx = \left(\frac{\pi}{p} \operatorname{cosec} \frac{q\pi}{p} \right)^2 \quad [0 < q < p].$$

BI ((82))(8)

$$32. \int_0^{\infty} \frac{e^{-px} - e^{(p-q)x}}{e^{-qx} + 1} \frac{dx}{x} = \ln \operatorname{ctg} \frac{p\pi}{2q} \quad [0 < p < q].$$

BI ((93))(7)

3.412

$$\int_0^{\infty} \left\{ \frac{a + be^{-px}}{ce^{px} + g + he^{-px}} - \frac{a + be^{-qx}}{ce^{qx} + g + he^{-qx}} \right\} \frac{dx}{x} = \frac{a + b}{c + g + h} \ln \frac{p}{q} \quad [p > 0, \quad q > 0].$$

BI ((96))(7)

373

3.413

$$1. \int_0^{\infty} \frac{(1 - 3^{-\beta x})(1 - e^{-\gamma x})e^{\mu x}}{1 - e^{-x}} \frac{dx}{x} = \ln \frac{\Gamma(\mu)\Gamma(\beta + \gamma + \mu)}{\Gamma(\mu + \beta)\Gamma(\mu + \gamma)}$$

[$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \mu > -\operatorname{Re} \beta, \quad \operatorname{Re} \mu > -\operatorname{Re} \gamma, \quad \operatorname{Re} \mu > -\operatorname{Re}(\beta + \gamma)$]
(cf. 4.267 25.)

4.267
BI ((93))(13)

$$3. \int_0^{\infty} \frac{e^{-px} - e^{-qx}}{1 + e^{-x}} \frac{1 + e^{-(2n+1)x}}{x} dx =$$

$$= \ln \left\{ \frac{q(q+2)(q+4) \dots (q+2n)(p+1)(p+3) \dots (p+2n-1)}{p(p+2)(p+4) \dots (p+2n)(q+1)(q+3) \dots (q+2n-1)} \right\}$$

[Re $p > -2n$, Re $q \geq 2n$] (cf. 4.267 14.).

4.267
BI ((93))(11)

3.414

$$\int_0^{\infty} \frac{(1 - e^{-\beta x})(1 - e^{-\gamma x})(1 - e^{-\delta x})e^{-\mu x}}{1 - e^{-x}} \frac{dx}{x} =$$

$$= \ln \frac{\Gamma(\mu)\Gamma(\mu + \beta + \gamma)\Gamma(\mu + \beta + \delta)\Gamma(\mu + \gamma + \delta)}{\Gamma(\mu + \beta)\Gamma(\mu + \gamma)\Gamma(\mu + \delta)\Gamma(\mu + \beta + \gamma + \delta)}$$

[2 Re $\mu > |\text{Re } \beta| + |\text{Re } \gamma| + |\text{Re } \delta|]$ (cf. 4.267 31.).

4.267
BI ((93))(14)
ET I 145(17)

3.415

$$1. \int_0^{\infty} \frac{x dx}{(x^2 + \beta^2)(e^{\mu x} - 1)} = \frac{1}{2} \left[\ln \left(\frac{\beta\mu}{2\pi} \right) - \frac{\pi}{\beta\mu} - \psi \left(\frac{\beta\mu}{2\pi} \right) \right] \quad [\text{Re } \beta > 0, \text{ Re } \mu > 0].$$

BI ((97))(20), EH I 18(27)

$$2.^7 \int_0^{\infty} \frac{x dx}{(x^2 + \beta^2)^2(e^{2\pi x} - 1)} = -\frac{1}{8\beta^3} - \frac{1}{4\beta^2} + \frac{1}{4\beta} \psi'(\beta);$$

$$= \frac{1}{4\beta^4} \sum_{k=0}^{\infty} \frac{|B_{2k+2}|}{\beta^{2k}} \quad [\text{Re } \beta > 0].$$

BI((97))(22), EH I 22(12)

$$3.^8 \int_0^{\infty} \frac{x dx}{(x^2 + \beta^2)(e^{\mu x} - 1)} = \frac{1}{2} \left[\Psi \left(\frac{\beta\mu}{2\pi} + \frac{1}{2} \right) - \ln \left(\frac{\beta\mu}{2\pi} \right) \right] \quad [\text{Re } \beta > 0, \text{ Re } \mu > 0]$$

$$4.^8 \int_0^{\infty} \frac{x dx}{(x^2 + \beta^2)^2(e^{2\pi x} + 1)} = \frac{1}{4\beta^2} - \frac{1}{4\beta} \Psi' \left(\beta + \frac{1}{2} \right) \quad [\text{Re } \beta > 0, \text{ Re } \mu > 0]$$

$$1. \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \frac{2n-1}{2n+1} \quad [n = 1, 2, \dots].$$

BI ((88))(4)

$$2. \int_0^\infty \frac{(1+ix)^{2n} - (1-ix)^{2n}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n+1} \quad [n = 1, 2, \dots].$$

BI ((87))(1)

$$3.7 \int_0^\infty \frac{(1+ix)^{2n-1} - (1-ix)^{2n-1}}{i} \frac{dx}{e^{\pi x} + 1} = \frac{1}{2n} [1 - 2^{2n} B_{2n}] \quad [n = 1, 2, \dots].$$

BI ((87))(2)

374

3.417

$$1. \int_{-\infty}^\infty \frac{x dx}{a^2 e^x + b^2 e^{-x}} = \frac{\pi}{2ab} \ln \frac{b}{a} \quad [ab > 0] \quad (\text{cf. 4.231 8.}).$$

4.231
BI ((101))(1)

$$2. \int_{-\infty}^\infty \frac{x dx}{a^2 e^x - b^2 e^{-x}} = \frac{\pi^2}{4ab} \quad (\text{cf. 4.231 10.}).$$

4.231
LI ((101))(2)

3.418

$$1.6 \int_0^\infty \frac{x dx}{e^x + e^{-x} - 1} = 1.171\,953\,6193\dots = \frac{1}{3} \left[\psi' \left(\frac{1}{3} \right) - \frac{2}{3} \pi^2 \right]$$

LI ((88))(1)

$$2.6 \int_0^\infty \frac{x e^{-x} dx}{e^x + e^{-x} - 1} = 0.311\,821\,1319\dots = \frac{1}{6} \left[\psi' \left(\frac{1}{3} \right) - \frac{5}{6} \pi^2 \right]$$

$$3. \int_0^{\ln 2} \frac{x dx}{e^x + 2e^{-x} - 2} = \frac{\pi}{8} \ln 2.$$

BI ((104))(7)

3.419

$$1. \int_{-\infty}^{\infty} \frac{x dx}{(\beta + e^x)(1 + e^{-x})} = \frac{(\ln \beta)^2}{2(\beta - 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.232 2.}).$$

4.232
BI ((101))(16)

$$2. \int_{-\infty}^{\infty} \frac{x dx}{(\beta + e^x)(1 - e^{-x})} = \frac{\pi^2 + (\ln \beta)^2}{2(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.232 3.}).$$

4.232
BI ((101))(17)

$$3. \int_{-\infty}^{\infty} \frac{x^2 dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2] \ln \beta}{3(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.261 4.}).$$

4.261
BI ((102))(6)

$$4. \int_{-\infty}^{\infty} \frac{x^3 dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{4(\beta + 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.262 3.}).$$

4.262
BI ((102))(9)

$$5. \int_{-\infty}^{\infty} \frac{x^4 dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{15(\beta + 1)} [7\pi^2 + 3(\ln \beta)^2] \ln \beta \quad (\text{cf. 4.263 1.}).$$

4.263

$$6. \int_{-\infty}^{\infty} \frac{x^5 dx}{(\beta + e^x)(1 - e^{-x})} = \frac{[\pi^2 + (\ln \beta)^2]^2}{6(\beta + 1)} [3\pi^2 + (\ln \beta)^2] \quad (\text{cf. 4.264 3}).$$

4.264
BI ((102))(11)

$$7. \int_{-\infty}^{\infty} \frac{(x - \ln \beta)x dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-[4\pi^2 + (\ln \beta)^2] \ln \beta}{6(\beta - 1)} \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.257 4}).$$

4.257
BI ((102))(7)375
3.421

$$1. \int_0^{\infty} (e^{-\nu x} - 1)^n (e^{-\rho x} - 1)^m e^{-\mu x} \frac{dx}{x^2} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{l=0}^m (-1)^l \binom{m}{l} \times \\ \times \{(m-l)\rho + (n-k)\nu + \mu\} \ln[(m-l)\rho + (n-k)\nu + \mu] \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \rho > 0].$$

BI ((89))(17)

$$2. \int_0^{\infty} (1 - e^{-\nu x})^n (1 - e^{-\rho x}) e^{-x} \frac{dx}{x^3} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (\rho + k\nu + 1)^2 \times \\ \times \ln(\rho + k\nu + 1) + \frac{1}{2} \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (k\nu + 1)^2 \ln(k\nu + 1) \\ [n \geq 2, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \rho > 0].$$

BI ((89))(31)

$$3. \int_{-\infty}^{\infty} \frac{x e^{-\mu x} dx}{(\beta + e^{-x})(\gamma + e^x)} = \frac{\pi(\beta^{\mu-1} \ln \beta - \gamma^{\mu-1} \ln \gamma)}{(\beta - \gamma) \sin \mu\pi} + \frac{\pi^2(\beta^{\mu-1} - \gamma^{\mu-1}) \cos \mu\pi}{(\gamma - \beta) \sin^2 \mu\pi} \\ [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad \beta \neq \gamma, \quad 0 < \operatorname{Re} \mu < 2].$$

ET I 120(19)

4.267
BI ((89))(11)

$$\begin{aligned}
 5. \int_0^{\infty} (1 - e^{-px})(1 - e^{-qx})(1 - e^{-rx})e^{-x} \frac{dx}{x} = \\
 = (p + q + 1) \ln(p + q + 1) + \\
 + (p + r + 1) \ln(p + r + 1) + (q + r + 1) \ln(q + r + 1) - \\
 - (p + 1) \ln(p + 1) - (q + 1) \ln(q + 1) - (r + 1) \ln(r + 1) - \\
 - (p + q + r) \ln(p + q + r) \quad [p > 0, \quad q > 0, \quad r > 0] \quad (\text{cf. 4.268 3.}).
 \end{aligned}$$

4.268
BI ((89))(14)

3.422

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{x(x - a)e^{\mu x} dx}{(\beta - e^x)(1 - e^{-x})} = \frac{-\pi^2}{e^a - 1} \operatorname{cosec}^2 \mu\pi [(e^{\alpha\mu} + 1) \ln \mu - 2\pi \operatorname{ctg} \mu\pi (e^{\alpha\mu} - 1)] \\
 [a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \mu| < 1] \quad (\text{cf. 4.257 5.}).
 \end{aligned}$$

4.257
BI ((102))(8)a

3.423

$$1. \int_0^{\infty} \frac{x^{\nu-1}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1) - \zeta(\nu)] \quad [\operatorname{Re} \nu > 2].$$

ET I 313(10)

$$2.^6 \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(e^x - 1)^2} dx = \Gamma(\nu) [\zeta(\nu - 1, \mu + 2) - (\mu + 1) \zeta(\nu, \mu + 2)] \quad [\operatorname{Re} \mu > -2, \quad \operatorname{Re} \nu > 2].$$

ET I 313(11)

$$3.^7 \int_0^{\infty} \frac{x^q e^{-px} dx}{(1 - ae^{-px})^2} = \Gamma(p + 1) p^{-q-1} \Phi(a, q, 1) \quad [-1 \leq a < 1, \quad q > -1, \quad p > 0]$$

BI ((85))(13)

$$4.7 \int_0^{\infty} \frac{x^{\nu-1} e^{-\mu x}}{(1 - \beta e^{-x})^2} dx = \Gamma(\nu) [\Phi(\beta, \nu - 1, \mu) - (\mu - 1)\Phi(\beta, \nu, \mu)]$$

[Re $\nu > 0$, Re $\mu > 0$, $|\arg(1 - \beta)| < \pi$] (cf. 9.550)

9.550
ET I 313(12)

$$5. \int_{-\infty}^{\infty} \frac{x e^x dx}{(\beta + e^x)^2} = \frac{1}{\beta} \ln \beta \quad [|\arg \beta| < \pi] \quad (\text{cf. 4.231 5.}).$$

4.231
BI ((101))(10)

3.424

$$1.7 \int_0^{\infty} \frac{(1+a)e^x - a}{(1-e^x)^2} e^{-ax} x^n dx = n! \zeta(n, a) \quad [a > -1, n = 1, 2, \dots]$$

BI ((85))(15)

$$2. \int_0^{\infty} \frac{(1+a)e^x + a}{(1+e^x)^2} e^{-ax} x^n dx = n! \Phi(-1, n, a+1) \quad [a > -1, n = 1, 2, \dots]$$

BI ((85))(14)

$$3. \int_{-\infty}^{\infty} \frac{a^2 e^x + b^2 e^{-x}}{(a^2 e^x - b^2 e^{-x})^2} x^2 dx = \frac{\pi^2}{2ab} \quad [ab > 0].$$

BI ((102))(3)a

$$4. \int_{-\infty}^{\infty} \frac{a^2 e^x - b^2 e^{-x}}{(a^2 e^x + b^2 e^{-x})^2} x^2 dx = \frac{\pi}{ab} \ln \frac{b}{a} \quad [ab > 0].$$

BI ((102))(1)

$$5. \int_0^{\infty} \frac{e^x - e^{-x} + 2}{(e^x - 1)^2} x^2 dx = \frac{2}{3} \pi^2 - 2.$$

$$1.7 \int_{-\infty}^{\infty} \frac{x e^x dx}{(a^2 + b^2 e^{2x})^n} = \frac{\sqrt{\pi} \Gamma\left(n - \frac{1}{2}\right)}{4a^{2n-1} b \Gamma(n)} \left[2 \ln \frac{a}{2b} - \mathcal{C} - \psi\left(n - \frac{1}{2}\right) \right] \quad [ab > 0, \quad n > 0]$$

(cf. 4.231 7.).

4.231

BI((101))(13), LI((101))(13)

$$2.7 \int_{-\infty}^{\infty} \frac{(a^2 e^x - e^{-x}) x^2 dx}{(a^2 e^x + e^{-x})^{p+1}} = -\frac{1}{a^{p+1}} B\left(\frac{p}{2}, \frac{p}{2}\right) \ln a \quad [a > 0, \quad p > 0].$$

BI ((102))(5)

3.426

$$1. \int_{-\infty}^{\infty} \frac{(e^x - a e^{-x}) x^2 dx}{(a + e^x)^2 (1 + e^{-x})^2} = \frac{(\ln a)^2}{a - 1}.$$

BI ((102))(12)

$$2. \int_{-\infty}^{\infty} \frac{(e^x - a e^{-x}) x^2 dx}{(a + e^x)^2 (1 - e^{-x})^2} = \frac{\pi^2 + (\ln a)^2}{a + 1}.$$

BI ((102))(13)

3.427

$$1. \int_0^{\infty} \left(\frac{e^{-x}}{x} + \frac{e^{-\mu x}}{e^{-x} - 1} \right) dx = \psi(\mu) \quad [\operatorname{Re} \mu > 0] \quad (\text{cf. 4.281 4.}).$$

4.281

WH

377

$$2.7 \int_0^{\infty} \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x} dx = \mathcal{C} \quad (\text{cf. 4.281 1.}).$$

4.281

BI ((94))(1)

$$3. \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{1+e^{-x}} \right) \frac{e^{-2x}}{x} dx = \frac{1}{2} \ln \frac{\pi}{4}.$$

BI ((94))(5)

$$4. \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x - 1} \right) \frac{e^{-\mu x}}{x} dx = \ln \Gamma(\mu) - \left(\mu - \frac{1}{2} \right) \ln \mu + \mu - \frac{1}{2} \ln(2\pi) \quad [\operatorname{Re} \mu > 0].$$

WH

$$5. \int_0^{\infty} \left(\frac{1}{2} e^{-2x} - \frac{1}{e^x + 1} \right) \frac{dx}{x} = -\frac{1}{2} \ln \pi.$$

BI ((94))(6)

$$6. \int_0^{\infty} \left(\frac{e^{\mu x} - 1}{1 - e^{-x}} - \mu \right) \frac{e^{-x}}{x} dx = -\ln \Gamma(\mu) - \ln \sin(\pi\mu) + \ln \pi \quad [\operatorname{Re} \mu < 1].$$

EH I 21(6)

$$7. \int_0^{\infty} \left(\frac{e^{-\nu x}}{1 - e^{-x}} - \frac{e^{-\mu x}}{x} \right) dx = \ln \mu - \psi(\nu) \quad (\text{cf. 4.281 5}).$$

4.281
BI ((94))(3)

$$8. \int_0^{\infty} \left(\frac{n}{x} - \frac{e^{-\mu x}}{1 - e^{-x/n}} \right) e^{-x} dx = n\psi(n\mu+n) - n \ln n \quad [\operatorname{Re} \mu > 0, \quad n = 1, 2, \dots].$$

BI ((94))(4)

$$9. \int_0^{\infty} \left(\mu - \frac{1 - e^{-\mu x}}{1 - e^{-x}} \right) \frac{e^{-x}}{x} dx = \ln \Gamma(\mu + 1) \quad [\operatorname{Re} \mu > -1].$$

WH

$$10. \int_0^{\infty} \left(\nu e^{-x} - \frac{e^{-\mu x} - e^{-(\mu+\nu)x}}{e^x - 1} \right) \frac{dx}{x} = \ln \frac{\Gamma(\mu + \nu + 1)}{\Gamma(\mu + 1)} \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > 0]$$

(cf. 4.267 33.).

$$11. \int_0^{\infty} [(1 - e^x)^{-1} + x^{-1} - 1]e^{-xz} dx = \psi(z) - \ln z \quad [\operatorname{Re} z > 0].$$

EH I 18(24)

3.428

$$1. \int_0^{\infty} \left(\nu e^{-\mu x} - \frac{1}{\mu} e^{-x} - \frac{1}{\mu} \frac{e^{-1} - e^{-\mu \nu x}}{1 - e^{-x}} \right) \frac{dx}{x} = \frac{1}{\mu} \ln \Gamma(\mu \nu) - \nu \ln \mu \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((94))(18)

$$2. \int_0^{\infty} \left(\frac{n-1}{2} + \frac{n-1}{1-e^{-x}} + \frac{e^{(1-\mu)x}}{1-e^{x/n}} + \frac{e^{-n\mu x}}{1-e^{-x}} \right) e^{-x} \frac{dx}{x} = \frac{n-1}{2} \ln 2\pi - \left(n\mu + \frac{1}{2} \right) \ln n$$

[$\operatorname{Re} \mu > 0, n = 1, 2, \dots$].

BI ((94))(14)

$$3. \int_0^{\infty} \left(n\mu - \frac{n-1}{2} - \frac{n}{1-e^{-x}} - \frac{e^{(1-\mu)x}}{1-e^{x/n}} \right) \frac{e^{-x}}{x} dx = \sum_{k=0}^{n-1} \ln \Gamma \left(\mu - \frac{k}{n} + 1 \right)$$

[$\operatorname{Re} \mu > 0, n = 1, 2, \dots$].

BI ((94))(13)

$$4. \int_0^{\infty} \left(\frac{e^{-\nu x}}{1-e^x} - \frac{e^{-\mu \nu x}}{1-e^{\mu x}} - \frac{e^x}{1-e^x} + \frac{e^{\mu x}}{1-e^{\mu x}} \right) \frac{dx}{x} = \nu \ln \mu \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

LI ((94))(15)

378

$$5. \int_0^{\infty} \left[\frac{1}{e^x - 1} - \frac{\mu e^{-\mu x}}{1 - e^{-\mu x}} + \left(a\mu - \frac{\mu + 1}{2} \right) e^{-\mu x} + (1 - a\mu) e^{-x} \right] \frac{dx}{x} =$$

$$= \frac{\mu - 1}{2} \ln(2\pi) + \left(\frac{1}{2} - a\mu \right) \ln \mu \quad [\operatorname{Re} \mu > 0].$$

BI ((94))(16)

$$6. \int_0^{\infty} \left[\frac{e^{-\nu x}}{1 - e^{-x}} - \frac{e^{-\mu \nu x}}{1 - e^{-\mu x}} - \frac{(\mu - 1)e^{-\mu x}}{1 - e^{-\mu x}} - \frac{\mu - 1}{2} e^{-\mu x} \right] \frac{dx}{x} = \frac{\mu - 1}{2} \ln(2\pi) + \left(\frac{1}{2} - \mu \nu \right) \ln \mu$$

[$\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0$]. (cf. 4.267-37.)

4.267
BI ((94))(17)

$$7. \int_0^{\infty} \left[1 - e^{-x} - \frac{(1 - e^{-\nu x})(1 - e^{-\mu x})}{1 - e^{-x}} \right] \frac{dx}{x} = \ln B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0]$$

(cf. 4.267 35.).

4.267
BI ((94))(12)

3.429

$$\int_0^{\infty} [e^{-x} - (1+x)^{-\mu}] \frac{dx}{x} = \psi(\mu) \quad [\operatorname{Re} \mu > 0].$$

NH 184(7)

3.431

$$1. \int_0^{\infty} \left(e^{-\mu x} - 1 + \mu x - \frac{1}{2} \mu^2 x^2 \right) x^{\nu-1} dx = \frac{-1}{\nu(\nu+1)(\nu+2)\mu^\nu} \Gamma(\nu+3)$$

[$\operatorname{Re} \mu > 0, -2 > \operatorname{Re} \nu > -3$].

LI ((90))(5)

$$2. \int_0^{\infty} \left[x^{-1} - \frac{1}{2} x^{-2} (x+2)(1 - e^{-x}) \right] e^{-px} dx = -1 + \left(p + \frac{1}{2} \right) \ln \left(1 + \frac{1}{p} \right) \quad [\operatorname{Re} p > 0].$$

ET I 144(6)

3.432

$$1. \int_0^{\infty} x^{\nu-1} e^{-mx} (e^{-x} - 1)^n dx = \Gamma(\nu) \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(n+m-k)^\nu} \quad [n = 0, 1, \dots, \operatorname{Re} \nu > 0].$$

LI ((90))(10)

$$2. \int_0^{\infty} [x^{\nu-1} e^{-x} - e^{-\mu x} (1 - e^{-x})^{\nu-1}] dx = \Gamma(\nu) - \frac{\Gamma(\mu)}{\Gamma(\mu + \nu)} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

LI ((81))(14)

3.433

$$\int_0^{\infty} x^{p-1} \left[e^{-x} + \sum_{k=1}^n (-1)^k \frac{x^{k-1}}{(k-1)!} \right] dx = \Gamma(p) \quad [-n < p < -n+1, \quad n = 0, 1, \dots].$$

FI II 805

3.434

$$1. \int_0^{\infty} \frac{e^{-\nu x} - e^{-\mu x}}{x^{\rho+1}} dx = \frac{\mu^{\rho} - \nu^{\rho}}{\rho} \Gamma(1-\rho) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad \operatorname{Re} \rho < 1].$$

BI ((90))(6)

$$2. \int_0^{\infty} \frac{e^{-\mu x} - e^{-\nu x}}{x} dx = \ln \frac{\nu}{\mu} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

FI II 634

379

3.435

$$1. \int_0^{\infty} \left\{ (x+1)e^{-x} - e^{-\frac{x}{2}} \right\} \frac{dx}{x} = 1 - \ln 2.$$

LI ((89))(19)

$$2.7 \int_0^{\infty} \frac{1 - e^{-\mu x}}{x(x+\beta)} dx = \frac{1}{\beta} [\ln(\beta\mu\mathbf{C}) - e^{\beta\mu} \operatorname{Ei}(-\beta\mu)] \quad [|\arg \beta| < \pi, \quad \operatorname{Re} \mu > 0].$$

ET II 217 (18)

$$3. \int_0^{\infty} \left(\frac{1}{1+x} - e^{-x} \right) \frac{dx}{x} = \mathbf{C}.$$

FI II 7 95, 802

$$4. \int_0^{\infty} \left(e^{-\mu x} - \frac{1}{1+ax} \right) \frac{dx}{x} = \ln \frac{a}{\mu} - \mathbf{C} \quad [a > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((92))(10)

3.436

$$\int_0^{\infty} \left\{ \frac{e^{-npx} - e^{-nqx}}{n} - \frac{e^{-mpx} - e^{-mqx}}{m} \right\} \frac{dx}{x^2} = (q-p) \ln \frac{m}{n} \quad [p > 0, \quad q > 0].$$

BI ((89))(28)

3.437

$$\int_0^{\infty} \left\{ pe^{-x} - \frac{1 - e^{-px}}{x} \right\} \frac{dx}{x} = p \ln p - p \quad [p > 0].$$

BI ((89))(24)

3.438

$$1. \int_0^{\infty} \left\{ \left(\frac{1}{2} + \frac{1}{x} \right) e^{-x} - \frac{1}{x} e^{-\frac{x}{2}} \right\} \frac{dx}{x} = \frac{\ln 2 - 1}{2}.$$

BI ((89))(19)

$$2.7 \int_0^{\infty} \left\{ \frac{p^3}{6} e^{-x} - \frac{p^2}{2x} + \frac{p}{x^2} - \frac{1 - e^{-px}}{x^3} \right\} \frac{dx}{x} = \frac{p^3}{6} \ln p - \frac{11}{36} p^3 \quad [p > 0].$$

BI ((89))(33)

$$3. \int_0^{\infty} \left(e^{-x} - e^{-2x} - \frac{1}{x} e^{-2x} \right) \frac{dx}{x} = 1 - \ln 2.$$

BI ((89))(25)

$$4. \int_0^{\infty} \left\{ \left(p - \frac{1}{2} \right) e^{-x} + \frac{x+2}{2x} (e^{-px} - e^{-\frac{x}{2}}) \right\} \frac{dx}{x} = \left(p - \frac{1}{2} \right) (\ln p - 1) \quad [p > 0].$$

BI ((89))(22)

3.439

$$\int_0^{\infty} \left\{ (p-q)e^{-rx} + \frac{1}{mx} (e^{-mpx} - e^{-mqx}) \right\} \frac{dx}{x} = p \ln p - q \ln q - (p-q) \left(1 + \ln \frac{r}{m} \right) \quad [p > 0, \quad q > 0, \quad r > 0].$$

LI((89))(26), LI((89))(27)

3.441

3.442

$$1. \int_0^{\infty} \left\{ 1 - \frac{x+2}{2x}(1 - e^{-x}) \right\} e^{-qx} \frac{dx}{x} = -1 + \left(q + \frac{1}{2} \right) \ln \frac{q+1}{q} \quad [q > 0].$$

BI ((89))(23)

$$2. \int_0^{\infty} \left(\frac{e^{-x} - 1}{x} + \frac{1}{1+x} \right) \frac{dx}{x} = C - 1.$$

BI ((92))(16)

$$3. \int_0^{\infty} \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = -C + \ln \frac{a}{p} \quad [p > 0].$$

BI ((92))(11)

380

3.443

$$1. \int_0^{\infty} \left\{ \frac{e^{-x}p^2}{2} - \frac{p}{x} + \frac{1 - e^{-px}}{x^2} \right\} \frac{dx}{x} = \frac{p^2}{2} \ln p - \frac{3}{4}p^2 \quad [p > 0].$$

BI ((89))(32)

$$2. \int_0^{\infty} \frac{(1 - e^{-px})^n e^{-qx}}{x^3} dx = \frac{1}{2} \sum_{k=2}^n (-1)^{k-1} \binom{n}{k} (q+kp)^2 \ln(q+kp) \\ [n > 2, \quad q > 0, \quad pn + q > 0] \quad (\text{cf. 4.268 4}).$$

4.268
BI ((89))(30)

$$3. \int_0^{\infty} (1 - e^{-px})^2 e^{-qx} \frac{dx}{x^2} = (2p+q) \ln(2p+q) - 2(p+q) \ln(p+q) + q \ln q \\ [q > 0, \quad 2p > -q] \quad (\text{cf. 4.268 2}).$$

3.45 Combinations of powers and algebraic functions of exponentials

3.451

$$1. \int_0^{\infty} x e^{-x} \sqrt{1 - e^{-x}} dx = \frac{4}{3} \left(\frac{4}{3} - \ln 2 \right).$$

BI ((99))(1)

$$2. \int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} dx = \frac{\pi}{4} \left(\frac{1}{2} + \ln 2 \right) \quad (\text{cf. 4.241 9.}).$$

4.241
BI ((99))(2)

3.452

$$1. \int_0^{\infty} \frac{x dx}{\sqrt{e^x - 1}} = 2\pi \ln 2.$$

FI II 643a, BI ((99))(4)

$$2. \int_0^{\infty} \frac{x^2 dx}{\sqrt{e^x - 1}} = 4\pi \left\{ (\ln 2)^2 + \frac{\pi^2}{12} \right\}.$$

BI ((99))(5)

$$3. \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^x - 1}} = \frac{\pi}{2} [2 \ln 2 - 1].$$

BI ((99))(6)

$$4. \int_0^{\infty} \frac{x e^{-x} dx}{\sqrt{e^{2x} - 1}} = 1 - \ln 2.$$

BI ((99))(8)

$$5. \int_0^{\infty} \frac{x e^{-2x} dx}{\sqrt{e^x - 1}} = \frac{3}{4} \pi \left(\ln 2 - \frac{7}{12} \right).$$

BI ((99))(7)

3.453

$$1. \int_0^{\infty} \frac{x e^x}{a^2 e^x - (a^2 - b^2)} \frac{dx}{\sqrt{e^x - 1}} = \frac{2\pi}{ab} \ln \left(1 + \frac{b}{a} \right) \quad [ab > 0] \quad (\text{cf. 4.298 17.}).$$

4.298
BI ((99))(16)

$$2. \int_0^{\infty} \frac{x e^x dx}{[a^2 e^x - (a^2 + b^2)] \sqrt{e^x - 1}} = \frac{2\pi}{ab} \operatorname{arctg} \frac{b}{a} \quad [ab > 0] \quad (\text{cf. 4.298 18.}).$$

4.298
BI ((99))(17)

381

3.454

$$1. \int_0^{\infty} \frac{x e^{-2nx} dx}{\sqrt{e^{2x} + 1}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right\}.$$

LI ((99))(10)

$$2. \int_0^{\infty} \frac{x e^{-(2n-1)x} dx}{\sqrt{e^{2x} - 1}} = -\frac{(2n-2)!!}{(2n-1)!!} \left\{ \ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right\}.$$

LI ((99))(9)

3.455

$$1. \int_0^{\infty} \frac{x^2 e^x dx}{\sqrt{(e^x - 1)^3}} = 8\pi \ln 2.$$

BI ((99))(11)

$$2. \int_0^{\infty} \frac{x^3 e^x dx}{\sqrt{(e^x - 1)^3}} = 24\pi \left[(\ln 2)^2 + \frac{\pi^2}{12} \right].$$

BI ((99))(12)

3.456

$$1. \int_0^{\infty} \frac{x dx}{\sqrt[3]{e^{3x} - 1}} = \frac{\pi}{3\sqrt{3}} \left[\ln 3 + \frac{\pi}{3\sqrt{3}} \right].$$

BI ((99))(13)

$$2. \int_0^{\infty} \frac{x dx}{\sqrt[3]{(e^{3x} - 1)^2}} = \frac{\pi}{3\sqrt{3}} \left[\ln 3 - \frac{\pi}{3\sqrt{3}} \right] \quad (\text{cf. 4.244 3.}).$$

4.244
BI ((99))(14)

3.457

$$1. \int_0^{\infty} x e^{-x} (1 - e^{-2x})^{n-1/2} dx = \frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [\mathbf{C} + \psi(n+1) + 2 \ln 2] \quad (\text{cf. 4.241 5.}).$$

4.241
BI ((99))(3)

$$2. \int_{-\infty}^{\infty} \frac{x e^x dx}{(a + e^x)^{n+3/2}} = \frac{2}{(2n+1)a^{n+1/2}} [\ln(4a) - 3\mathbf{C} - 2\psi(2n) - \psi(n)].$$

BI ((101))(12)

$$3. \int_{-\infty}^{\infty} \frac{x dx}{(a^2 e^x + e^{-x})^\mu} = \frac{-1}{2a^\mu} \mathbf{B} \left(\frac{\mu}{2}, \frac{\mu}{2} \right) \ln a \quad [a > 0, \quad \text{Re } \mu > 0].$$

BI ((101))(14)

3.458

$$1.7 \int_0^{\ln 2} x e^x (e^x - 1)^{p-1} dx = \frac{1}{p} \left[\ln 2 + \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{p+k+1} \right] \quad [p > -1]$$

BI ((104))(4)

$$2. \int_{-\infty}^{\infty} \frac{x e^x dx}{(a + e^x)^{\nu+1}} = \frac{1}{\nu a^\nu} [\ln a - \mathbf{C} - \psi(\nu)] \quad [a > 0];$$

$$= \frac{1}{\nu a^\nu} \left[\ln a - \sum_{k=1}^{\nu-1} \frac{1}{k} \right] \quad [\nu = 1, 2, \dots].$$

3.46- 3.48 Combinations of exponentials of more complicated arguments and powers

3.461

$$1. \int_u^\infty \frac{e^{-p^2 x^2}}{x^{2n}} dx = \frac{(-1)^n 2^{n-1} p^{2n-1} \sqrt{\pi}}{(2n-1)!!} [1 - \Phi(pu)] + \\ + \frac{e^{-p^2 u^2}}{2u^{2n-1}} \sum_{k=0}^{n-1} \frac{(-1)^k 2^{k+1} (pu)^{2k}}{(2n-1)(2n-3)\dots(2n-2k-1)} \quad [p > 0].$$

NT 21(4)

$$2. \int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}} \quad [p > 0, \quad n = 0, 1, \dots].$$

FI II 743

$$3. \int_0^\infty x^{2n+1} e^{-px^2} dx = \frac{n!}{2p^{n+1}} \quad [p > 0].$$

BI ((81))(7)

$$4. \int_{-\infty}^\infty (x+ai)^{2n} e^{-x^2} dx = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \sum_{k=0}^n (-1)^k \frac{(2a)^{2k} n!}{(2k)!(n-k)!}.$$

BI ((100))(12)

$$5.3 \int_u^\infty e^{-\mu x^2} \frac{dx}{x^2} = \frac{1}{u} e^{-\mu u^2} - \sqrt{\mu\pi} [1 - \Phi(\sqrt{\mu u})] \quad \left[\left| \arg \mu \right| < \frac{\pi}{2}, \quad u > 0 \right].$$

ET I 135(19a)

3.462

$$1. \int_0^\infty x^{\nu-1} e^{-\beta x^2 - \gamma x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0].$$

EH II 119(3)A, ET I 313(13)

$$2. \int_{-\infty}^\infty x^n e^{-px^2+2qx} dx = \frac{1}{2^{n-1} p} \sqrt{\frac{\pi}{p}} \frac{d^{n-1}}{dq^{n-1}} (qe^{q^2/p}) \quad [p > 0];$$

$$= n! e^{q^2/p} \sqrt{\frac{\pi}{p}} \left(\frac{q}{p}\right)^n \sum_{k=0}^{E(n/2)} \frac{1}{(n-2k)!} \left(\frac{p}{4q}\right)^k \quad [p > 0].$$

$$3. \int_{-\infty}^{\infty} (ix)^{\nu} e^{-\beta^2 x^2 - iqx} dx = 2^{-\frac{\nu}{2}} \sqrt{\pi} \beta^{-\nu-1} \exp\left(-\frac{q^2}{8\beta^2}\right) D_{\nu}\left(\frac{q}{\beta\sqrt{2}}\right) \\ \left[\operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1, \arg ix = \frac{\pi}{2} \operatorname{sign} x\right].$$

ET I 121(23)

$$4. \int_{-\infty}^{\infty} x^n \exp[-(x - \beta)^2] dx = (2i)^{-n} \sqrt{\pi} H_n(i\beta).$$

EH II 195(31)

$$5. \int_0^{\infty} x e^{-\mu x^2 - 2\nu x} dx = \frac{1}{2\mu} - \frac{\nu}{2\mu} \sqrt{\frac{\pi}{\mu}} e^{\frac{\nu^2}{\mu}} \left[1 - \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \quad \left[|\arg \nu| < \frac{\pi}{2}, \operatorname{Re} \mu > 0\right].$$

ET I 146(31)a

$$6. \int_{-\infty}^{\infty} x e^{-px^2 + 2qx} dx = \frac{q}{p} \sqrt{\frac{\pi}{p}} \exp\left(\frac{q^2}{p}\right) \quad [\operatorname{Re} p > 0].$$

BI ((100))(7)

383

$$7. \int_0^{\infty} x^2 e^{-\mu x^2 - 2\nu x} dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}} \frac{2\nu^2 + \mu}{4} e^{\frac{\nu^2}{\mu}} \left[1 - \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \quad \left[|\arg \nu| < \frac{\pi}{2}, \operatorname{Re} \mu > 0\right].$$

ET I 146(32)

$$8. \int_{-\infty}^{\infty} x^2 e^{-\mu x^2 + 2\nu x} dx = \frac{1}{2\mu} \sqrt{\frac{\pi}{\mu}} \left(1 + 2\frac{\nu^2}{\mu}\right) e^{\frac{\nu^2}{\mu}} \quad [|\arg \nu| < \pi, \operatorname{Re} \mu > 0].$$

BI ((100))(8)a

3.463

$$\int_0^{\infty} (e^{-x^2} - e^{-x}) \frac{dx}{x} = \frac{1}{2} \mathbf{C}.$$

BI ((89))(5)

3.464

$$\int_0^{\infty} (e^{-\mu x^2} - e^{-\nu x^2}) \frac{dx}{x^2} = \sqrt{\pi}(\sqrt{\nu} - \sqrt{\mu}) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

FI II 645

3.465

$$\int_0^{\infty} (1 + 2\beta x^2) e^{-\mu x^2} dx = \frac{\mu + \beta}{2} \sqrt{\frac{\pi}{\mu^3}} \quad [\operatorname{Re} \mu > 0].$$

ET I 136(24)a

3.466

$$1. \int_0^{\infty} \frac{e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = [1 - \Phi(\beta\mu)] \frac{\pi}{2\beta} e^{\beta^2 \mu^2} \quad \left[\operatorname{Re} \beta > 0, |\arg \mu| < \frac{\pi}{4} \right].$$

NT 19(13)

$$2. \int_0^{\infty} \frac{x^2 e^{-\mu^2 x^2}}{x^2 + \beta^2} dx = \frac{\sqrt{\pi}}{2\mu} - \frac{\pi\beta}{2} e^{\mu^2 \beta^2} [1 - \Phi(\beta\mu)] \quad \left[\operatorname{Re} \beta > 0, |\arg \mu| < \frac{\pi}{4} \right].$$

ET II 217(16)

$$3. \int_0^1 \frac{e^{x^2} - 1}{x^2} dx = \sum_{k=1}^{\infty} \frac{1}{k!(2k-1)}.$$

FI II 683

3.467

$$\int_0^{\infty} \left(e^{-x^2} - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{1}{2} \mathbf{C}.$$

BI ((92))(12)

3.468

$$1. \int_{u\sqrt{2}}^{\infty} \frac{e^{-x^2}}{\sqrt{x^2 - u^2}} \frac{dx}{x} = \frac{\pi}{4u} [1 - \Phi(u)]^2 \quad [u > 0].$$

NT 33(17)

3.469

$$1. \int_0^{\infty} e^{-\mu x^4 - 2\nu x^2} dx = \frac{1}{4} \sqrt{\frac{2\nu}{\mu}} \exp\left(\frac{\nu^2}{2\mu}\right) K_{\frac{1}{4}}\left(\frac{\nu^2}{2\mu}\right) \quad [\operatorname{Re} \mu \geq 0].$$

ET I 146(23)

$$2. \int_0^{\infty} (e^{-x^4} - e^{-x}) \frac{dx}{x} = \frac{3}{4} \mathbf{C}.$$

BI ((89))(7)

$$3. \int_0^{\infty} (e^{-x^4} - e^{-x^2}) \frac{dx}{x} = \frac{1}{4} \mathbf{C}.$$

BI ((89))(6)

384

3.471

$$1. \int_0^u \exp\left(-\frac{\beta}{x}\right) \frac{dx}{x^2} = \frac{1}{\beta} \exp\left(-\frac{\beta}{u}\right).$$

ET II 188(22)

$$2. \int_0^u x^{\nu-1} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} u^{\frac{2\mu+\nu-1}{2}} \exp\left(-\frac{\beta}{2u}\right) \Gamma(\mu) W_{\frac{1-2\mu-\nu}{2}, \frac{\nu}{2}}\left(\frac{\beta}{u}\right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0, \quad u > 0].$$

ET II 187(18)

$$3. \int_0^u x^{-\mu-1} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{-\mu} u^{\mu-1} \Gamma(\mu) \exp\left(-\frac{\beta}{u}\right) \quad [\operatorname{Re} \mu > 0, \quad u > 0].$$

ET II 187(16)

$$4. \int_0^u x^{-2\mu} (u-x)^{\mu-1} e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi u}} \beta^{\frac{1}{2}-\mu} e^{-\frac{\beta}{2u}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right) \\ [u > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0].$$

$$5. \int_u^\infty x^{\nu-1}(x-u)^{\mu-1} e^{-\frac{\beta}{x}} dx = B(1-\mu-\nu, \mu) u^{\mu+\nu-1} {}_1F_1\left(1-\mu-\nu; 1-\nu; \frac{\beta}{u}\right) \\ [0 < \operatorname{Re} \mu < \operatorname{Re}(1-\nu), \quad u > 0].$$

ET II 203(15)

$$6. \int_u^\infty x^{-2\mu}(x-u)^{\mu-1} e^{-\frac{\beta}{x}} dx = \sqrt{\frac{\pi}{u}} \beta^{\frac{1}{2}-\mu} \Gamma(\mu) \exp\left(\frac{\beta}{2u}\right) I_{\mu-\frac{1}{2}}\left(\frac{\beta}{2u}\right) \quad [\operatorname{Re} \mu > 0, \quad u > 0].$$

ET II 202(14)

$$7. \int_0^\infty x^{\nu-1}(x+\gamma)^{\mu-1} e^{-\frac{\beta}{x}} dx = \beta^{\frac{\nu-1}{2}} \gamma^{\frac{\nu-1}{2}+\mu} \Gamma(1-\mu-\nu) e^{\frac{\beta}{2\gamma}} W_{\frac{\nu-1}{2}+\mu, -\frac{\nu}{2}}\left(\frac{\beta}{\gamma}\right) \\ [|\arg \gamma| < \pi, \quad \operatorname{Re}(1-\mu) > \operatorname{Re} \nu > 0].$$

ET II 234(13)a

$$8. \int_0^u x^{-2\mu}(u^2-x^2)^{\mu-1} e^{-\frac{\beta}{x}} dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) K_{\mu-\frac{1}{2}}\left(\frac{\beta}{u}\right) \\ [\operatorname{Re} \beta > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 188(23)a

$$9. \int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} dx = 2 \left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

ET II 82(23)A, LET I 146(29)

$$10. \int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2}\left(x-\frac{\beta^2}{x}\right)\right] dx = 2\beta^\nu e^{\frac{i\nu\pi}{2}} K_{-\nu}(\beta\mu) \\ [\operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) < 0; \quad \text{note } K_{-\nu} \equiv K_\nu].$$

EH II 82(24)

$$11. \int_0^\infty x^{\nu-1} \exp\left[\frac{i\mu}{2}\left(x+\frac{\beta^2}{x}\right)\right] dx = i\pi\beta^\nu e^{-\frac{i\nu\pi}{2}} H_{-\nu}^{(1)}(\beta\mu) \quad [\operatorname{Im} \mu > 0, \quad \operatorname{Im}(\beta^2\mu) > 0].$$

EH II 21(33)

385

$$12. \int_0^\infty x^{\nu-1} \exp\left(-x-\frac{\mu^2}{4x}\right) dx = 2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu) \quad \left[|\arg \mu| < \frac{\pi}{2}, \quad \operatorname{Re} \mu^2 > 0; \quad \text{note } K_{-\nu} \equiv K_\nu\right].$$

$$13. \int_0^{\infty} \frac{x^{\nu-1} e^{-\frac{\beta}{x}}}{x+\gamma} dx = \gamma^{\nu-1} e^{\frac{\beta}{\gamma}} \Gamma(1-\nu) \Gamma\left(\nu, \frac{\beta}{\gamma}\right) \quad [|\arg \gamma| < \pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 218(19)

$$14. \int_0^1 \frac{\exp\left(1 - \frac{1}{x}\right) - x^{\nu}}{x(1-x)} dx = \psi(\nu) \quad [\operatorname{Re} \nu > 0].$$

BI ((80))(7)

3.472

$$1. \int_0^{\infty} \left(\exp\left(-\frac{a}{x^2}\right) - 1\right) e^{-\mu x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} [\exp(-2\sqrt{a\mu}) - 1] \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0].$$

ET I 146(30)

$$2. \int_0^{\infty} x^2 \exp\left(-\frac{a}{x^2} - \mu x^2\right) dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu^3}} (1 + 2\sqrt{a\mu}) \exp(-2\sqrt{a\mu}) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} a > 0].$$

ET I 146(26)

$$3. \int_0^{\infty} \exp\left(-\frac{a}{x^2} - \mu x^2\right) \frac{dx}{x^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{a\mu}) \quad [\operatorname{Re} \mu > 0, \quad a > 0].$$

ET I 146(28)a

$$4. \int_0^{\infty} \exp\left[-\frac{1}{2a} \left(x^2 + \frac{1}{x^2}\right)\right] \frac{dx}{x^4} = \sqrt{\frac{a\pi}{2}} (1+a) e^{-1/a} \quad [a > 0].$$

BI ((98))(14)

3.473

$$\int_0^{\infty} \exp(-x^n) x^{(m+1/2)n-1} dx = \frac{(2m-1)!!}{2^m n} \sqrt{\pi}.$$

BI ((98))(6)

3.474

$$1. \int_0^1 \left\{ \frac{n \exp(1-x^{-n})}{1-x^n} - \frac{x^{np}}{1-x} \right\} \frac{dx}{x} = \frac{1}{n} \sum_{k=1}^n \psi\left(p + \frac{k-1}{n}\right) \quad [p > 0].$$

$$2. \int_0^1 \left\{ \frac{n \exp(1 - x^{-n})}{1 - x^n} - \frac{\exp(1 - \frac{1}{x})}{1 - x} \right\} \frac{dx}{x} = -\ln n.$$

BI ((80))(9)

3.475

$$1.7 \int_0^\infty \left\{ \exp(-x^2) - \frac{1}{1 + x^{2n}} \right\} \frac{dx}{x} = -\frac{1}{2} \mathbf{C}. \quad [n \in \mathbb{N}]$$

BI ((92))(14)

386

$$2. \int_0^\infty \left\{ \exp(-x^{2n}) - \frac{1}{1 + x^2} \right\} \frac{dx}{x} = -2^{-n} \mathbf{C}.$$

BI ((92))(13)

$$3. \int_0^\infty \left\{ \exp(-x^{2n}) - e^{-x} \right\} \frac{dx}{x} = (1 - 2^{-n}) \mathbf{C}.$$

BI ((89))(8)

3.476

$$1. \int_0^\infty [\exp(-\nu x^p) - \exp(-\mu x^p)] \frac{dx}{x} = \frac{1}{p} \ln \frac{\mu}{\nu} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((89))(3)

$$2. \int_0^\infty [\exp(-x^p) - \exp(-x^q)] \frac{dx}{x} = \frac{p - q}{pq} \mathbf{C} \quad [p > 0, q > 0].$$

BI ((89))(9)

3.477

$$1.7 \int_{-\infty}^\infty \frac{\exp(-a|x|)}{x - u} dx = \frac{\operatorname{sign} u}{\pi} [\exp(a|u|) \operatorname{Ei}(-a|u|) - \exp(-a|u|) \operatorname{Ei}(a|u|)] \quad [a > 0].$$

ET II 251(35)

3.478

$$1. \int_0^{\infty} x^{\nu-1} \exp(-\mu x^p) dx = \frac{1}{p} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, p > 0].$$

BI((81))(8)A, ET I 313(15, 16)

$$2. \int_0^{\infty} x^{\nu-1} [1 - \exp(-\mu x^p)] dx = -\frac{1}{|p|} \mu^{-\frac{\nu}{p}} \Gamma\left(\frac{\nu}{p}\right) \\ [\operatorname{Re} \mu > 0 \quad \text{and} \quad -p < \operatorname{Re} \nu < 0 \quad \text{for} \quad p > 0, \quad 0 < \operatorname{Re} \nu < -p \quad \text{for} \quad p < 0].$$

ET I 313(18, 19)

$$3. \int_0^u x^{\nu-1} (u-x)^{\mu-1} \exp(\beta x^n) dx = B(\mu, \nu) u^{\mu+\nu-1} {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n}; \right. \\ \left. \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, \dots, \frac{\mu+\nu+n-1}{n}; \beta u^n\right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, n = 2, 3, \dots].$$

ET II 187(15)

$$4. \int_0^{\infty} x^{\nu-1} \exp(-\beta x^p - \gamma x^{-p}) dx = \frac{2}{p} \left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}(2\sqrt{\beta\gamma}) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

ET I 313(17)

3.479

$$1. \int_0^{\infty} \frac{x^{\nu-1} \exp(-\beta\sqrt{1+x})}{\sqrt{1+x}} dx = \frac{2}{\sqrt{\pi}} \left(\frac{\beta}{2}\right)^{\frac{1}{2}-\nu} \Gamma(\nu) K_{\frac{1}{2}-\nu}(\beta) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0].$$

ET I 313(14)

387

$$2.^8 \int_0^{\infty} \frac{x^{\nu-1} \exp(i\mu\sqrt{1+x^2})}{\sqrt{1+x^2}} dx = i \frac{\sqrt{\pi}}{2} \left(\frac{\mu}{2}\right)^{\frac{1-\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) H_{\frac{1-\nu}{2}}^{(1)}(\mu) \quad [\operatorname{Im} \mu > 0, \operatorname{Re} \nu > 2].$$

EH II 83(30)

3.481

$$1. \int_{-\infty}^{\infty} x e^x \exp(-\mu e^x) dx = -\frac{1}{\mu}(\mathbf{C} + \ln \mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((100))(13)

$$2. \int_{-\infty}^{\infty} x e^x \exp(-\mu e^{2x}) dx = -\frac{1}{4}[\mathbf{C} + \ln(4\mu)]\sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0].$$

BI ((100))(14)

3.482

$$1.^3 \int_0^{\infty} \exp(nx - \beta \operatorname{sh} x) dx = \frac{1}{2}[S_n(\beta) - \pi \mathbf{E}_n(\beta) - \pi N_n(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 168(11)

$$2. \int_0^{\infty} \exp(-nx - \beta \operatorname{sh} x) dx = (-1)^{n+1} \frac{1}{2}[S_n(\beta) + \pi \mathbf{E}_n(\beta) + \pi N_n(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 168(12)

$$3. \int_0^{\infty} \exp(-\nu x - \beta \operatorname{sh} x) dx = \frac{\pi}{\sin \nu\pi} [\mathbf{J}_\nu(\beta) - J_\nu(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 168(13)

3.483³

$$\int_{-\infty}^{\infty} \frac{\exp(\nu \operatorname{Arsh} x - iax)}{\sqrt{1+x^2}} dx = \begin{cases} 2 \exp\left(-\frac{i\nu\pi}{2}\right) K_\nu(a) & \text{for } a > 0, \\ 2 \exp\left(\frac{i\nu\pi}{2}\right) K_\nu(-a) & \text{for } a < 0 \end{cases} \quad [|\operatorname{Re} \nu| < 1].$$

ET I 122(32)

3.484

$$\int_0^{\infty} \left[\left(1 + \frac{a}{qx}\right)^{qx} - \left(1 + \frac{a}{px}\right)^{px} \right] \frac{dx}{x} = (e^a - 1) \ln \frac{q}{p} \quad [p > 0, \quad q > 0].$$

BI ((89))(34)

3.485

$$\int_0^{\frac{\pi}{2}} \exp(-\operatorname{tg}^2 x) dx = \frac{\pi e}{2} [1 - \Phi(1)].$$

3.486⁶

$$\int_0^1 x^{-x} dx = \int_0^1 e^{-x \ln x} dx = \sum_{k=1}^{\infty} k^{-k} = 1.2912859970627 \dots$$

FI II 483

3.5 Hyperbolic Functions

3.51 Hyperbolic functions

3.511

$$1. \int_0^{\infty} \frac{dx}{\operatorname{ch} ax} = \frac{\pi}{2a} \quad [a > 0].$$

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$$2. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} bx} dx = \frac{\pi}{2b} \operatorname{tg} \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((27))(10)a

$$3. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{ch} bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} - \frac{1}{b} \beta \left(\frac{a+b}{2b} \right) \quad [b > |a|].$$

GW ((351))(3b)

$$4. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{ch} bx} dx = \frac{\pi}{2b} \sec \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((4))(14)a

$$5. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{ch} bx}{\operatorname{sh} cx} dx = \frac{\pi}{2c} \frac{\sin \frac{a\pi}{c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|].$$

$$6. \int_0^{\infty} \frac{\operatorname{ch} ax \operatorname{ch} bx}{\operatorname{ch} cx} dx = \frac{\pi}{c} \frac{\cos \frac{a\pi}{2c} \cos \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|].$$

BI ((27))(5)a

$$7. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch} cx} dx = \frac{\pi}{c} \frac{\sin \frac{a\pi}{2c} \sin \frac{b\pi}{2c}}{\cos \frac{a\pi}{c} + \cos \frac{b\pi}{c}} \quad [c > |a| + |b|].$$

BI ((27))(6)a

$$8. \int_0^{\infty} \frac{dx}{\operatorname{ch} x^2} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}$$

BI ((98))(25)

$$9. \int_{-\infty}^{\infty} \frac{\operatorname{sh}^2 ax}{\operatorname{sh}^2 x} dx = 1 - a\pi \operatorname{ctg} a\pi \quad [a^2 < 1].$$

BI ((16))(3)a

$$10. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{sh} bx}{\operatorname{ch}^2 bx} dx = \frac{a\pi}{2b^2} \sec \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((27))(16)a

3.512

$$1. \int_0^{\infty} \frac{\operatorname{ch} 2\beta x}{\operatorname{ch}^{2\nu} ax} dx = \frac{4^{\nu-1}}{a} \operatorname{B} \left(\nu + \frac{\beta}{a}, \nu - \frac{\beta}{a} \right) \quad [\operatorname{Re}(\nu \pm \beta) > 0, \quad a > 0, \quad \beta > 0].$$

LI((27))(17)A, EH I 11(26)

$$2. \int_0^{\infty} \frac{\operatorname{sh}^{\mu} x}{\operatorname{ch}^{\nu} x} dx = \frac{1}{2} \operatorname{B} \left(\frac{\mu+1}{2}, \frac{\nu-\mu}{2} \right) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re}(\mu - \nu) < 0].$$

EH I 11(23)

3.513

$$1. \int_0^{\infty} \frac{dx}{a + b \operatorname{sh} x} = \frac{1}{\sqrt{a^2 + b^2}} \ln \frac{a + b + \sqrt{a^2 + b^2}}{a + b - \sqrt{a^2 + b^2}} \quad [ab \neq 0].$$

$$2. \int_0^\infty \frac{dx}{a + b \operatorname{ch} x} = \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctg} \frac{\sqrt{b^2 - a^2}}{a + b} \quad [b^2 > a^2];$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \quad [b^2 < a^2].$$

GW ((351))(7)

$$3. \int_0^\infty \frac{dx}{a \operatorname{sh} x + b \operatorname{ch} x} = \frac{2}{\sqrt{b^2 - a^2}} \operatorname{arctg} \frac{\sqrt{b^2 - a^2}}{a + b} \quad [b^2 > a^2];$$

$$= \frac{1}{\sqrt{a^2 - b^2}} \ln \frac{a + b + \sqrt{a^2 - b^2}}{a + b - \sqrt{a^2 - b^2}} \quad [a^2 > b^2].$$

GW ((351))(9)

$$4. \int_0^\infty \frac{dx}{a + b \operatorname{ch} x + c \operatorname{sh} x} = \frac{2}{\sqrt{b^2 - a^2 - c^2}} \left[\operatorname{arctg} \frac{\sqrt{b^2 - a^2 - c^2}}{a + b + c} + \epsilon \pi \right]$$

$$[b^2 > a^2 + c^2; \quad \epsilon = 0 \quad \text{for } (b - a)(a + b + c) > 0,$$

$$|\epsilon| = 1 \quad \text{for } (b - a)(a + b + c) < 0,$$

$$\text{also } \epsilon = 1 \quad \text{for } a < b + c \quad \text{and } \epsilon = -1 \quad \text{for } a > b + c];$$

$$= \frac{1}{\sqrt{a^2 - b^2 + c^2}} \ln \frac{a + b + c + \sqrt{a^2 - b^2 + c^2}}{a + b + c - \sqrt{a^2 - b^2 + c^2}} [b^2 < a^2 + c^2, \quad a^2 \neq b^2];$$

$$= \frac{1}{c} \ln \frac{a + c}{a} \quad [a = b \neq 0, \quad c \neq 0];$$

$$= \frac{2(a - b)}{c(a - b - c)} \quad [b^2 = a^2 + c^2, \quad c(a - b - c) < 0].$$

GW ((351))(6)

3.514

$$1. \int_0^\infty \frac{dx}{\operatorname{ch} ax + \cos t} = \frac{t}{a} \operatorname{cosec} t \quad [0 < t < \pi, \quad a > 0].$$

BI ((27))(22)a

$$2. \int_0^\infty \frac{\operatorname{ch} ax - \cos t_1}{\operatorname{ch} bx - \cos t_2} dx = \frac{\pi}{b} \frac{\sin \frac{a(\pi - t_2)}{b}}{\sin t_2 \sin \frac{a}{b} \pi} - \frac{\pi - t_2}{b \sin t_2} \cos t_1 \quad [0 < |a| < b, \quad 0 < t_2 < \pi].$$

BI ((6))(20)a

$$3. \int_0^\infty \frac{\operatorname{ch} ax dx}{(\operatorname{ch} x + \cos t)^2} = \frac{\pi(-\cos t \sin at + a \sin t \cos at)}{\sin^3 t \sin a\pi} \quad [0 < a^2 < 1, \quad 0 < t < \pi].$$

$$4. \int_0^\infty \frac{\operatorname{sh} ax \operatorname{sh} bx}{(\operatorname{ch} ax + \cos t)^2} dx = \frac{b\pi}{a^2} \operatorname{cosec} t \operatorname{cosec} \frac{b\pi}{a} \sin \frac{bt}{a} \quad [0 < |b| < a, \quad 0 < t < \pi].$$

BI ((27))(27)a

3.515

$$\int_{-\infty}^{\infty} \left(1 - \frac{\sqrt{2} \operatorname{ch} x}{\sqrt{\operatorname{ch} 2x}} \right) dx = -\ln 2.$$

BI ((21))(12)a

390

3.516

$$1. \int_0^\infty \frac{dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^\mu} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(z + \sqrt{z^2 - 1} \operatorname{ch} x)^\mu} = Q_{\mu-1}(z) \quad [\operatorname{Re} \mu > -1].$$

For a suitable choice of a single-valued branch of the integrand, this formula is valid for arbitrary values of z in the z -plane cut from -1 to $+1$ provided $\mu < 0$. If $\mu > 0$, this formula ceases to be valid for points at which the denominator vanishes.

CO, WH

$$2. \int_0^\infty \frac{dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{n+1}} = Q_n(\beta).$$

EH II 181(32)

$$3. \int_0^\infty \frac{\operatorname{ch} \gamma x dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{\nu+1}} = \frac{e^{-i\gamma\pi} \Gamma(\nu - \gamma + 1) Q_\nu^\gamma(\beta)}{\Gamma(\nu + 1)} \\ [\operatorname{Re}(\nu \pm \gamma) > -1, \quad \nu \neq -1, -2, -3, \dots].$$

EH I 157(12)

$$4. \int_0^\infty \frac{\operatorname{sh}^{2\mu} x dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{\nu+1}} = \frac{2^\mu e^{-i\mu\pi} \Gamma(\nu - 2\mu + 1) \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(\nu + 1)} Q_{\nu-\mu}^\mu(\beta) \\ [\operatorname{Re}(\nu - 2\mu + 1) > 0, \quad \operatorname{Re}(\nu + 1) > 0].$$

EH I 155(2)

3.517

$$1. \int_0^\infty \frac{\operatorname{ch}\left(\gamma + \frac{1}{2}\right) x dx}{(\beta + \operatorname{ch} x)^{\nu + \frac{1}{2}}} = \sqrt{\frac{\pi}{2}} (\beta^2 - 1)^{-\frac{\nu}{2}} \frac{\Gamma(\nu + \gamma + 1) \Gamma(\nu - \gamma) P_\gamma^{-\nu}(\beta)}{\Gamma\left(\nu + \frac{1}{2}\right)}$$

[$\operatorname{Re}(\nu - \gamma) > 0$, $\operatorname{Re}(\nu + \gamma + 1) > 0$].

EH I 156(11)

$$2. \int_0^a \frac{\operatorname{ch}\left(\gamma + \frac{1}{2}\right) x dx}{(\operatorname{ch} a - \operatorname{ch} x)^{\nu + \frac{1}{2}}} = \sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\frac{1}{2} - \nu\right)}{\operatorname{sh}^\nu a} P_\gamma^\nu(\operatorname{ch} a) \quad \left[\operatorname{Re} \nu < \frac{1}{2}, \quad a > 0 \right].$$

EH I 156(8)

3.518

$$1. \int_0^\infty \frac{\operatorname{sh}^{2\mu} x dx}{(\operatorname{ch} a + \operatorname{sh} a \operatorname{ch} x)^{\nu + 1}} = \frac{2^\mu e^{-i\mu\pi}}{\sqrt{\pi} \operatorname{sh}^\mu a} \frac{\Gamma(\nu - 2\mu + 1) \Gamma\left(\mu + \frac{1}{2}\right)}{\Gamma(\nu + 1)} Q_{\nu - \mu}^\mu(\operatorname{ch} a)$$

[$\operatorname{Re}(\nu + 1) > 0$, $\operatorname{Re}(\nu - 2\mu + 1) > 0$, $a > 0$].

EH I 155(3a)

$$2.* \int_0^\infty \frac{\operatorname{sh}^{2\mu+1} x dx}{(\beta + \operatorname{ch} x)^{\nu+1}} = 2^\mu (\beta^2 - 1)^{\frac{\mu-\nu}{2}} \Gamma(\nu - 2\mu) \Gamma(\mu + 1) P_\mu^{\mu-\nu}(\beta)$$

[$\operatorname{Re}(\nu - \mu) > \operatorname{Re} \mu > -1$, β does not lie on the ray $(-\infty, +1)$ of the real axis].

EH I 155(1)

391

$$3. \int_0^\infty \frac{\operatorname{sh}^{2\mu-1} x \operatorname{ch} x dx}{(1 + a \operatorname{sh}^2 x)^\nu} = \frac{1}{2} a^{-\mu} B(\mu, \nu - \mu) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0, \quad a > 0].$$

EH I 11(22)

$$4.7 \int_0^\infty \frac{\operatorname{sh}^{\mu-1} x (\operatorname{ch} x + 1)^{\nu-1} dx}{(\beta + \operatorname{ch} x)^\varrho} = 2^{\mu+\nu-\rho-2} B\left(\frac{1}{2}\mu, \varrho + 2 - \mu - \nu\right) \times$$

$$\times {}_2F_1\left(\varrho, \varrho + 2 - \mu - \nu; \rho + 2 - \frac{1}{2}\mu - 2; \frac{1}{2} - \frac{1}{2}\beta\right)$$

[$\operatorname{Re} \mu > 0$, $\operatorname{Re}(\varrho - \mu - \nu) > -2$, $|\arg(1 + \beta)| < \pi$].

$$5.6 \int_0^\infty \frac{\text{sh}^{\mu-1} x (\text{ch} x - 1)^{\nu-1} dx}{(\beta + \text{ch} x)^{\varrho}} = 2^{-(2-\mu-\nu+\varrho)} {}_2F_1 \left(\varrho, 2 - \mu - \nu + \varrho; 1 + \varrho - \frac{\mu}{2}; \frac{1 - \beta}{2} \right) \\ \times \text{B} \left(2 - \mu - \nu + \varrho, -1 + \nu + \frac{\mu}{2} \right) \\ [\beta \notin (-\infty, -1), \quad \text{Re}(2 + \varrho) > \text{Re}(\mu + \nu), \quad \text{Re}(2\nu + \mu) > 2].$$

EHI 115(10)

$$6.7 \int_0^\infty \frac{\text{sh}^{\mu-1} x \text{ch}^{\nu-1} x}{(\text{ch}^2 x - \beta)^{\varrho}} dx = \frac{1}{2^2} F_1 \left(\varrho, 1 + \varrho - \frac{\mu + \nu}{2}; 1 + \varrho - \frac{\nu}{2}; \beta \right) \text{B} \left(\frac{\mu}{2}, 1 + \varrho - \frac{\mu + \nu}{2} \right) \\ [\beta \notin (1, \infty), \quad \text{Re} \mu > 0, \quad 2 \text{Re}(1 + \varrho) > \text{Re}(\mu + \nu)].$$

EHI 115(9)

3.519

$$\int_0^{\frac{\pi}{2}} \frac{\text{sh}[(r-p) \text{tg} x]}{\text{sh}(r \text{tg} x)} dx = \pi \sum_{k=1}^{\infty} \frac{1}{k\pi + r} \sin \frac{pk\pi}{r} \quad [p^2 < r^2].$$

BI ((274))(13)

3.52- 3.53 Combinations of hyperbolic functions and algebraic functions

3.521

$$1. \int_0^\infty \frac{x dx}{\text{sh} ax} = \frac{\pi^2}{4a^2} \quad [a > 0].$$

GW ((352))(2b)

$$2. \int_0^\infty \frac{x dx}{\text{ch} x} = 2\mathbf{G} = \pi \ln 2 - 4L \left(\frac{\pi}{4} \right) = 1.831931188\dots$$

LI III 225(103a), BI((84))(1)a

$$3. \int_1^\infty \frac{dx}{x \text{sh} ax} = -2 \sum_{k=0}^{\infty} \text{Ei}[-(2k+1)a] \quad [a > 0].$$

LI ((104))(14)

$$4. \int_1^\infty \frac{dx}{x \text{ch} ax} = 2 \sum_{k=0}^{\infty} (-1)^{k+1} \text{Ei}[-(2k+1)a] \quad [a > 0].$$

LI ((104))(13)

$$1. \int_0^{\infty} \frac{x dx}{(b^2 + x^2) \operatorname{sh} ax} = \frac{\pi}{2ab} + \pi \sum_{k=1}^{\infty} \frac{(-1)^k}{ab + k\pi} \quad [a > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \frac{x dx}{(b^2 + x^2) \operatorname{sh} \pi x} = \frac{1}{2b} - \beta(b+1) \quad [b > 0].$$

BI((97))(16), GW((352))(8)

$$3. \int_0^{\infty} \frac{dx}{(b^2 + x^2) \operatorname{ch} ax} = \frac{2\pi}{b} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2ab + (2k-1)\pi} \quad [a > 0, \quad b > 0].$$

BI ((97))(5)

$$4. \int_0^{\infty} \frac{dx}{(b^2 + x^2) \operatorname{ch} \pi x} = \frac{1}{b} \beta\left(b + \frac{1}{2}\right) \quad [b > 0].$$

BI ((97))(4)

$$5. \int_0^{\infty} \frac{x dx}{(1 + x^2) \operatorname{sh} \pi x} = \ln 2 - \frac{1}{2}.$$

BI ((97))(7)

$$6. \int_0^{\infty} \frac{dx}{(1 + x^2) \operatorname{ch} \pi x} = 2 - \frac{\pi}{2}.$$

BI ((97))(1)

$$7. \int_0^{\infty} \frac{x dx}{(1 + x^2) \operatorname{sh} \frac{\pi x}{2}} = \frac{\pi}{2} - 1.$$

BI ((97))(8)

$$8. \int_0^{\infty} \frac{dx}{(1 + x^2) \operatorname{ch} \frac{\pi x}{2}} = \ln 2.$$

$$9. \int_0^{\infty} \frac{x dx}{(1+x^2) \operatorname{sh} \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} [\pi + 2 \ln(\sqrt{2} + 1)] - 2.$$

BI ((97))(9)

$$10. \int_0^{\infty} \frac{dx}{(1+x^2) \operatorname{ch} \frac{\pi x}{4}} = \frac{1}{\sqrt{2}} [\pi - 2 \ln(\sqrt{2} + 1)].$$

BI ((97))(3)

3.523

$$1. \int_0^{\infty} \frac{x^{\beta-1}}{\operatorname{sh} ax} dx = \frac{2^{\beta}-1}{2^{\beta-1}a^{\beta}} \Gamma(\beta) \zeta(\beta) \quad [\operatorname{Re} \beta > 1, \quad a > 0].$$

WH

$$2. \int_0^{\infty} \frac{x^{2n-1}}{\operatorname{sh} ax} dx = \frac{2^{2n}-1}{2n} \left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \quad [a > 0, \quad n = 1, 2, \dots].$$

WH, GW((352))(2a)

$$3. \int_0^{\infty} \frac{x^{\beta-1}}{\operatorname{ch} ax} dx = \frac{2}{(2a)^{\beta}} \Gamma(\beta) \Phi\left(-1, \beta, \frac{1}{2}\right) = \frac{2}{(2a)^{\beta}} \Gamma(\beta) \sum_{k=0}^{\infty} (-1)^k \left(\frac{2}{2k+1}\right)^{\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

EH I 35, ET I 322(1)

$$4. \int_0^{\infty} \frac{x^{2n}}{\operatorname{ch} ax} dx = \left(\frac{\pi}{2a}\right)^{2n+1} |E_{2n}| \quad [a > 0].$$

BI((84))(12)A, GW((352))(1a)

$$5. \int_0^{\infty} \frac{x^2 dx}{\operatorname{ch} x} = \frac{\pi^3}{8} \quad (\text{cf. 4.261 6}).$$

4.261
BI ((84))(3)

$$6. \int_0^{\infty} \frac{x^3 dx}{\operatorname{sh} x} = \frac{\pi^4}{8} \quad (\text{cf. 4.262 1. and 2}).$$

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$$7. \int_0^{\infty} \frac{x^4 dx}{\operatorname{ch} x} = \frac{5}{32} \pi^5.$$

BI ((84))(7)

$$8. \int_0^{\infty} \frac{x^5 dx}{\operatorname{sh} x} = \frac{\pi^6}{4}.$$

BI ((84))(8)

$$9. \int_0^{\infty} \frac{x^6 dx}{\operatorname{ch} x} = \frac{61}{128} \pi^7.$$

BI ((84))(9)

$$10. \int_0^{\infty} \frac{x^7 dx}{\operatorname{sh} x} = \frac{17}{16} \pi^8.$$

BI ((84))(10)

$$11. \int_0^{\infty} \frac{\sqrt{x} dx}{\operatorname{ch} x} = \sqrt{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{(2k+1)^3}}.$$

BI ((98))(7)a

$$12. \int_0^{\infty} \frac{dx}{\sqrt{x} \operatorname{ch} x} = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((98))(25)a

3.524

$$1. \int_0^{\infty} x^{\mu-1} \frac{\operatorname{sh} \beta x}{\operatorname{sh} \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[\mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] - \zeta \left[\mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] \right\}$$

[Re $\gamma > |\operatorname{Re} \beta|$, Re $\mu > -1$].

$$2. \int_0^{\infty} x^{2m} \frac{\operatorname{sh} ax}{\operatorname{sh} bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \operatorname{tg} \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((112))(20)a

$$3. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} bx} \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}} \right\} \quad [b > |a|, \quad p < 1].$$

BI ((131))(2)a

$$4. \int_0^{\infty} x^{2m+1} \frac{\operatorname{sh} ax}{\operatorname{ch} bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \operatorname{sec} \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((112))(18)a

$$5. \int_0^{\infty} x^{\mu-1} \frac{\operatorname{ch} \beta x}{\operatorname{sh} \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \zeta \left[\mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] + \zeta \left[\mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] \right\} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \quad \operatorname{Re} \mu > 1].$$

ET I 323(12)

$$6. \int_0^{\infty} x^{2m} \frac{\operatorname{ch} ax}{\operatorname{ch} bx} dx = \frac{\pi}{2b} \frac{d^{2m}}{da^{2m}} \operatorname{sec} \frac{a\pi}{2b} \quad [b > |a|].$$

BI((112))(17)

$$7. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{ch} bx} \frac{dx}{x^p} = \Gamma(1-p) \sum_{k=0}^{\infty} (-1)^k \left\{ \frac{1}{[b(2k+1)-a]^{1-p}} + \frac{1}{[b(2k+1)+a]^{1-p}} \right\} \\ [b > |a|, \quad p < 1].$$

BI((131))(1)a

$$8. \int_0^{\infty} x^{2m+1} \frac{\operatorname{ch} ax}{\operatorname{sh} bx} dx = \frac{\pi}{2b} \frac{d^{2m+1}}{da^{2m+1}} \operatorname{tg} \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((112))(19)a

394

$$9.^8 \int_0^{\infty} x^2 \frac{\operatorname{sh} ax}{\operatorname{sh} bx} dx = \frac{\pi^3}{4b^3} \sin \frac{a\pi}{2b} \operatorname{sec}^3 \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((84))(18)

BI ((84))(18)

$$10. \int_0^{\infty} x^4 \frac{\text{sh } ax}{\text{sh } bx} dx = 8 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \cdot \sin \frac{a\pi}{2b} \cdot \left(2 + \sin^2 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(17)a

$$11. \int_0^{\infty} x^6 \frac{\text{sh } ax}{\text{sh } bx} dx = 16 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \sin \frac{a\pi}{2b} \left(45 - 30 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(21)a

$$12. \int_0^{\infty} x \frac{\text{sh } ax}{\text{ch } bx} dx = \frac{\pi^2}{4b^2} \sin \frac{a\pi}{2b} \sec^2 \frac{a\pi}{2b} \quad [b > |a|].$$

BI ((84))(15)a

$$13. \int_0^{\infty} x^3 \frac{\text{sh } ax}{\text{ch } bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \sin \frac{a\pi}{2b} \cdot \left(6 - \cos^2 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(14)a

$$14. \int_0^{\infty} x^5 \frac{\text{sh } ax}{\text{ch } bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \sin \frac{a\pi}{2b} \left(120 - 60 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(18)a

$$15. \int_0^{\infty} x^7 \frac{\text{sh } ax}{\text{ch } bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \sin \frac{a\pi}{2b} \left(5040 - 4200 \cos^2 \frac{a\pi}{2b} + 546 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(22)a

$$16. \int_0^{\infty} x \frac{\text{ch } ax}{\text{sh } bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^2 \quad [b > |a|].$$

BI ((84))(16)a

$$17. \int_0^{\infty} x^3 \frac{\text{ch } ax}{\text{sh } bx} dx = 2 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^4 \left(1 + 2 \sin^2 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

$$18. \int_0^{\infty} x^5 \frac{\operatorname{ch} ax}{\operatorname{sh} bx} dx = 8 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^6 \left(15 - 15 \cos^2 \frac{a\pi}{2b} + 2 \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(19)a

$$19. \int_0^{\infty} x^7 \frac{\operatorname{ch} ax}{\operatorname{sh} bx} dx = 16 \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^8 \left(315 - 420 \cos^2 \frac{a\pi}{2b} + 126 \cos^4 \frac{a\pi}{2b} - 4 \cos^6 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI((82))(23)a

$$20. \int_0^{\infty} x^2 \frac{\operatorname{ch} ax}{\operatorname{ch} bx} dx = \frac{\pi^3}{8b^3} \left(2 \sec^3 \frac{a\pi}{2b} - \sec \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((84))(17)a

$$21. \int_0^{\infty} x^4 \frac{\operatorname{ch} ax}{\operatorname{ch} bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^5 \left(24 - 20 \cos^2 \frac{a\pi}{2b} + \cos^4 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(16)a

$$22. \int_0^{\infty} x^6 \frac{\operatorname{ch} ax}{\operatorname{ch} bx} dx = \left(\frac{\pi}{2b} \sec \frac{a\pi}{2b} \right)^7 \left(720 - 840 \cos^2 \frac{a\pi}{2b} + 182 \cos^4 \frac{a\pi}{2b} - \cos^6 \frac{a\pi}{2b} \right) \quad [b > |a|].$$

BI ((82))(20)a

$$23. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{ch} bx} \cdot \frac{dx}{x} = \ln \operatorname{tg} \left(\frac{a\pi}{4b} + \frac{\pi}{4} \right) \quad [b > |a|].$$

BI ((95))(3)a

3.525

$$1. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} \pi x} \cdot \frac{dx}{1+x^2} = -\frac{a}{2} \cos a + \frac{1}{2} \sin a \ln[2(1 + \cos a)] \quad [\pi \geq |a|].$$

BI ((97))(10)a

395

$$2. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \sin a + \frac{1}{2} \cos a \ln \frac{1 - \sin a}{1 + \sin a} \quad [\pi \geq 2|a|].$$

BI ((97))(11)a

BI ((97))(12)a

$$4. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{sh} \frac{\pi}{2} x} \cdot \frac{x dx}{1+x^2} = \frac{\pi}{2} \cos a - 1 + \frac{1}{2} \sin a \ln \frac{1+\sin a}{1-\sin a} \quad \left[\frac{\pi}{2} > |a| \right].$$

BI ((97))(13)a

$$5. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{ch} \pi x} \cdot \frac{x dx}{1+x^2} = -2 \sin \frac{a}{2} + \frac{\pi}{2} \sin a - \cos a \ln \operatorname{tg} \frac{a+\pi}{4} \quad [\pi > |a|].$$

GW ((352))(12)

$$6. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{ch} \pi x} \cdot \frac{dx}{1+x^2} = 2 \cos \frac{a}{2} - \frac{\pi}{2} \cos a - \sin a \ln \operatorname{tg} \frac{a+\pi}{4} \quad [\pi > |a|].$$

GW ((352))(11)

$$7. \int_0^{\infty} \frac{\operatorname{sh} ax}{\operatorname{sh} bx} \cdot \frac{dx}{c^2+x^2} = \frac{\pi}{c} \sum_{k=1}^{\infty} \frac{\sin \frac{k(b-a)\pi}{b}}{bc+k\pi} \quad [b \geq |a|].$$

BI ((97))(18)

$$8. \int_0^{\infty} \frac{\operatorname{ch} ax}{\operatorname{sh} bx} \cdot \frac{x dx}{c^2+x^2} = \frac{\pi}{2bc} + \pi \sum_{k=1}^{\infty} \frac{\cos \frac{k(b-a)\pi}{b}}{bc+k\pi} \quad [b > |a|].$$

BI ((97))(19)

3.526

$$1. \int_0^{\infty} \frac{\operatorname{sh} ax \operatorname{ch} bx}{\operatorname{ch} cx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \left\{ \operatorname{tg} \frac{(a+b+c)\pi}{4c} \operatorname{ctg} \frac{(b+c-a)\pi}{4c} \right\} \quad [c > |a|+|b|].$$

BI ((93))(10)a

$$2. \int_0^{\infty} \frac{\operatorname{sh}^2 ax}{\operatorname{sh} bx} \cdot \frac{dx}{x} = \frac{1}{2} \ln \sec \frac{a}{b} \pi \quad [b > |2a|].$$

BI ((95))(5)a

$$3. \int_0^{\infty} \frac{x^{\mu-1}}{\operatorname{sh} \beta x \operatorname{ch} \gamma x} dx = \frac{\Gamma(\mu)}{(2\gamma)^\mu} \left\{ \Phi \left[-1, \mu, \frac{1}{2} \left(1 + \frac{\beta}{\gamma} \right) \right] + \Phi \left[-1, \mu, \frac{1}{2} \left(1 - \frac{\beta}{\gamma} \right) \right] \right\} \\ [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, \operatorname{Re} \mu > 0].$$

$$1. \int_0^{\infty} \frac{x^{\mu-1}}{\operatorname{sh}^2 ax} dx = \frac{4}{(2a)^\mu} \Gamma(\mu) \zeta(\mu-1) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 2].$$

BI ((86))(7)a

$$2. \int_0^{\infty} \frac{x^{2m}}{\operatorname{sh}^2 ax} dx = \frac{\pi^{2m}}{a^{2m+1}} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots].$$

BI((86))(5)a

$$3.^6 \int_0^{\infty} \frac{x^{\mu-1}}{\operatorname{ch}^2 ax} dx = \frac{4}{(2a)^\mu} (1 - 2^{2-\mu}) \Gamma(\mu) \zeta(\mu-1) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0, \quad \mu \neq 2; \quad \text{integral equals } (1/a^2) \ln 2 \text{ if } \mu = 2].$$

BI ((86))(6)a

$$4. \int_0^{\infty} \frac{x dx}{\operatorname{ch}^2 ax} = \frac{\ln 2}{a^2} \quad [a \neq 0].$$

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$$5. \int_0^{\infty} \frac{x^{2m}}{\operatorname{ch}^2 ax} dx = \frac{(2^{2m} - 2)\pi^{2m}}{(2a)^{2m} a} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots].$$

BI((86))(2)a

$$6. \int_0^{\infty} x^{\mu-1} \frac{\operatorname{sh} ax}{\operatorname{ch}^2 ax} dx = \frac{2\Gamma(\mu)}{a^\mu} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{\mu-1}} \quad [\operatorname{Re} \mu > 1, \quad a > 0].$$

BI ((86))(15)a

$$7. \int_0^{\infty} \frac{x \operatorname{sh} ax}{\operatorname{ch}^2 ax} dx = \frac{\pi}{2a^2} \quad [a > 0].$$

BI ((86))(8)a

$$8. \int_0^{\infty} x^{2m+1} \frac{\operatorname{sh} ax}{\operatorname{ch}^2 ax} dx = \frac{2m+1}{a} \left(\frac{\pi}{2a}\right)^{2m+1} |E_{2m}| \quad [a > 0, \quad m = 0, 1, \dots].$$

$$9. \int_0^{\infty} x^{2m+1} \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax} dx = \frac{2^{2m+1} - 1}{a^2 (2a)^{2m}} (2m+1)! \zeta(2m+1) \quad [a \neq 0, \quad m = 1, 2, \dots].$$

BI ((86))(13)a

$$10. \int_0^{\infty} x^{2m} \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax} dx = \frac{2^{2m} - 1}{a} \left(\frac{\pi}{a}\right)^{2m} |B_{2m}| \quad [a > 0, \quad m = 1, 2, \dots].$$

BI ((86))(14)a

$$11. \int_0^{\infty} \frac{x \operatorname{sh} ax}{\operatorname{ch}^{2\mu+1} ax} dx = \frac{\sqrt{\pi}}{4\mu a^2} \frac{\Gamma(\mu)}{\left(\mu + \frac{1}{2}\right)} \quad [\mu > 0, \quad a > 0].$$

LI ((86))(9)

$$12. \int_{-\infty}^{\infty} \frac{x^2 dx}{\operatorname{sh}^2 x} = \frac{\pi^2}{3}.$$

BI ((102))(2)a

$$13. \int_0^{\infty} x^2 \frac{\operatorname{ch} ax}{\operatorname{sh}^2 ax} dx = \frac{\pi^2}{2a^3} \quad [a > 0].$$

BI ((86))(11)a

$$14. \int_0^{\infty} x^2 \frac{\operatorname{sh} ax}{\operatorname{ch}^2 ax} dx = \frac{\ln 2}{2a^3} \quad [a \neq 0].$$

BI ((86))(10)a

$$15.* \int_0^{\infty} \frac{\theta \frac{x}{2} dx}{\operatorname{ch} x} = \ln 2.$$

BI ((93))(17)a

3.528

$$1. \int_0^{\infty} \frac{(1 + xi)^{2n-1} - (1 - xi)^{2n-1}}{i \operatorname{sh} \frac{\pi x}{2}} dx = 2.$$

$$2. \int_0^\infty \frac{(1+xi)^{2n} - (1-xi)^{2n}}{i \operatorname{sh} \frac{\pi x}{2}} dx = (-1)^{n+1} 2|E_{2n}| + 2 \quad [n = 0, 1, \dots].$$

BI ((87))(7)

397

3.529

$$1. \int_0^\infty \left(\frac{1}{\operatorname{sh} x} - \frac{1}{x} \right) \frac{dx}{x} = -\ln 2.$$

BI ((94))(10)a

$$2. \int_0^\infty \frac{\operatorname{ch} ax - 1}{\operatorname{sh} bx} \cdot \frac{dx}{x} = -\ln \cos \frac{a\pi}{2b} \quad [b > |a|].$$

GW ((352))(66)

$$3. \int_0^\infty \left(\frac{a}{\operatorname{sh} ax} - \frac{b}{\operatorname{sh} bx} \right) \frac{dx}{x} = (b-a) \ln 2.$$

BI ((94))(11)a

3.531

$$1.^7 \int_0^\infty \frac{x dx}{2 \operatorname{ch} x - 1} = \frac{4}{\sqrt{3}} \left[\frac{\pi}{3} \ln 2 - L(\pi/3) \right] \quad 1.1719536193 \dots$$

LI ((88))(1)

$$2.^* \int_0^\infty \frac{x dx}{\operatorname{ch} 2x + \cos 2t} = \frac{t \ln 2 - L(t)}{\sin 2t}.$$

LO III 402

$$3. \int_0^\infty \frac{x^2 dx}{\operatorname{ch} x + \cos t} = \frac{t}{3} \cdot \frac{\pi^2 - t^2}{\sin t} \quad [0 < t < \pi].$$

BI ((88))(3)a

$$4. \int_0^\infty \frac{x^4 dx}{\operatorname{ch} x + \cos t} = \frac{t}{15} \frac{(\pi^2 - t^2)(7\pi^2 - 3t^2)}{\sin t} \quad [0 < t < \pi].$$

$$5.3 \int_0^\infty \frac{x^{2m} dx}{\operatorname{ch} x - \cos 2a\pi} = 2(2m)! \operatorname{cosec} 2a\pi \sum_{k=1}^\infty \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \frac{1}{2}]$$

$$= 2(2^{2m-1} - 1)\pi^{2m} |B_{2m}| \quad [a = \frac{1}{2}].$$

BI ((88))(5)a

$$6.3 \int_0^\infty \frac{x^{\mu-1} dx}{\operatorname{ch} x - \cos t} = \frac{i\Gamma(\mu)}{\sin t} [e^{-it}\Phi(e^{-it}, \mu, 1) - e^{it}\Phi(e^{it}, \mu, 1)]$$

$$[\operatorname{Re} \mu > 0, \quad 0 < t < 2\pi, \quad t \neq \pi].$$

$$= (2 - 2^{3-\mu})\Gamma(\mu)\zeta(\mu - 1) \quad [\mu \neq 2, \quad t = \pi];$$

$$= 2 \ln 2 \quad [\mu = 2, \quad t = \pi].$$

ET I 323(5)

$$7. \int_0^\infty \frac{x^\mu dx}{\operatorname{ch} x + \cos t} = \frac{2\Gamma(\mu + 1)}{\sin t} \sum_{k=1}^\infty (-1)^{k-1} \frac{\sin kt}{k^{\mu+1}} \quad [\mu > -1, \quad 0 < t < \pi].$$

BII ((96))(14)a

$$8. \int_0^u \frac{x dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2} \operatorname{cosec} 2t [L(\theta+t) - L(\theta-t) - 2L(t)] \quad [\theta = \operatorname{arctg}(\theta u \operatorname{ctg} t), \quad t \neq n\pi].$$

LO III 402

3.532

$$1. \int_0^\infty \frac{x^n dx}{a \operatorname{ch} x + b \operatorname{sh} x} = \frac{(2n)!}{a+b} \sum_{k=0}^\infty \frac{1}{(2k+1)^{n+1}} \left(\frac{b-a}{b+a}\right)^k \quad [a > 0, \quad b > 0, \quad n > -1].$$

GW ((352))(5)

398

$$2. \int_0^u \frac{x \operatorname{ch} x dx}{\operatorname{ch} 2x - \cos 2t} = \frac{1}{2} \operatorname{cosec} t \left\{ L\left(\frac{\theta+t}{2}\right) - L\left(\frac{\theta-t}{2}\right) + L\left(\pi - \frac{\psi+t}{2}\right) \right.$$

$$\left. + L\left(\frac{\psi-t}{2}\right) - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi-t}{2}\right) \right\}$$

$$\left[\operatorname{tg} \frac{\theta}{2} = \theta \frac{u}{2} \operatorname{ctg} \frac{t}{2}, \quad \operatorname{tg} \frac{\psi}{2} = \operatorname{cth} \frac{u}{2} \operatorname{ctg} \frac{t}{2}; \quad t \neq n\pi \right].$$

LO III 288a

3.533

$$1. \int_0^{\infty} \frac{x \operatorname{ch} x \, dx}{\operatorname{ch} 2x - \cos 2t} = \operatorname{cosec} t \left[\frac{\pi}{2} \ln 2 - L \left(\frac{t}{2} \right) - L \left(\frac{(\pi - t)}{2} \right) \right] \quad [t \neq m\pi].$$

LO III 403

$$2.^6 \int_0^{\infty} x \frac{\operatorname{sh} ax \, dx}{(\operatorname{ch} ax - \cos t)^2} = \frac{\pi - t}{a^2} \operatorname{cosec} t \quad [a > 0, \quad 0 < t < \pi] \quad (\text{cf. 3.5141}).$$

3.514
BI ((88))(11)a

$$3. \int_0^{\infty} x^3 \frac{\operatorname{sh} x \, dx}{(\operatorname{ch} x + \cos t)^2} = \frac{t(\pi^2 - t^2)}{\sin t} \quad [0 < t < \pi] \quad (\text{cf. 3.531 3}).$$

3.531
BI ((88))(13)

$$4.^* \int_0^{\infty} x^{2m+1} \frac{\operatorname{sh} x \, dx}{(\operatorname{ch} x - \cos 2a\pi)^2} = 2(2m+1)! \operatorname{cosec} 2a\pi \sum_{k=1}^{\infty} \frac{\sin 2ka\pi}{k^{2m+1}} \quad [0 < a < 1, \quad a \neq \frac{1}{2}].$$

$$= 2(2m+1)(2^{2m-1} - 1)\pi^{2m} |B_{2m}|, \quad [a = \frac{1}{2}].$$

BI ((88))(14)

3.534

$$1. \int_0^1 \sqrt{1-x^2} \operatorname{ch} ax \, dx = \frac{\pi}{2a} I_1(a).$$

WA 94(9)

$$2. \int_0^1 \frac{\operatorname{ch} ax}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2} I_0(a).$$

WA 94(9)

3.535

$$\int_0^1 \frac{x}{\sqrt{\operatorname{ch} 2a - \operatorname{ch} 2ax}} \cdot \frac{dx}{\operatorname{sh} ax} = \frac{\pi}{2\sqrt{2}a^2} \cdot \frac{\operatorname{arcsin}(\operatorname{th} a)}{\operatorname{sh} a} \quad [a > 0].$$

$$1. \int_0^{\infty} \frac{x^2}{\operatorname{ch} x^2} dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

BI ((98))(7)

$$2. \int_0^{\infty} \frac{x^2 \theta x^2}{\operatorname{ch} x^2} dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((98))(8)

$$3. \int_0^{\infty} \operatorname{sh}(\nu \operatorname{Arsh} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} dx = \frac{\sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{2^{\mu\pi}} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \\ [-1 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|].$$

ET I 324(14)

$$4. \int_0^{\infty} \operatorname{ch}(\nu \operatorname{Arch} x) \frac{x^{\mu-1}}{\sqrt{1+x^2}} dx = \frac{\cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{2^{\mu\pi}} \Gamma(\mu) \Gamma\left(\frac{1-\mu-\nu}{2}\right) \times \Gamma\left(\frac{1-\mu+\nu}{2}\right) \\ [0 < \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|].$$

ET I 324(15)

3.54 Combinations of hyperbolic functions and exponentials

$$1. \int_0^{\infty} e^{-\mu x} \operatorname{sh}^{\nu} \beta x dx = \frac{1}{2^{\nu+1} \beta} \operatorname{B}\left(\frac{\mu}{2\beta} - \frac{\nu}{2}, \nu + 1\right) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1, \operatorname{Re} \mu > \operatorname{Re} \beta \nu].$$

EH I 11(25), ET I 163(5)

$$2. \int_0^{\infty} e^{-\mu x} \frac{\operatorname{sh} \beta x}{\operatorname{sh} b x} dx = \frac{1}{2b} \left[\psi\left(\frac{1}{2} + \frac{\mu + \beta}{2b}\right) - \psi\left(\frac{1}{2} + \frac{\mu - \beta}{2b}\right) \right] \quad [\operatorname{Re}(\mu + b \pm \beta) > 0].$$

EH I 16(14)a

$$3. \int_{-\infty}^{\infty} e^{-\mu x} \frac{\operatorname{sh} \mu x}{\operatorname{sh} \beta x} dx = \frac{\pi}{2\beta} \operatorname{tg} \frac{\mu\pi}{\beta} \quad [\operatorname{Re} \beta > 2|\operatorname{Re} \mu|].$$

BI ((18))(6)

$$4. \int_0^{\infty} e^{-x} \frac{\operatorname{sh} ax}{\operatorname{sh} x} dx = \frac{1}{a} - \frac{\pi}{2} \operatorname{ctg} \frac{a\pi}{2} \quad [0 < a < 2].$$

BI ((4))(3)

$$5. \int_0^{\infty} \frac{e^{-px} dx}{(\operatorname{ch} px)^{2q+1}} = \frac{2^{2q-2}}{p} B(q, q) - \frac{1}{2qp} \quad [p > 0, \quad q > 0].$$

LI ((27))(19)

$$6. \int_0^{\infty} e^{-\mu x} \frac{dx}{\operatorname{ch} x} = \beta \left(\frac{\mu+1}{2} \right) \quad [\operatorname{Re} \mu > -1].$$

ET I 163(7)

$$7. \int_0^{\infty} e^{-\mu x} \theta x dx = \beta \left(\frac{\mu}{2} \right) - \frac{1}{\mu} \quad [\operatorname{Re} \mu > 0].$$

ET I 163(9)

$$8. \int_0^{\infty} \frac{e^{-\mu x}}{\operatorname{ch}^2 x} dx = \mu \beta \left(\frac{\mu}{2} \right) - 1 \quad [\operatorname{Re} \mu > 0].$$

ET I 163(8)

$$9. \int_0^{\infty} e^{-\mu x} \frac{\operatorname{sh} \mu x}{\operatorname{ch}^2 \mu x} dx = \frac{1}{\mu} (1 - \ln 2) \quad [\operatorname{Re} \mu > 0].$$

LI ((27))(15)

$$10. \int_0^{\infty} e^{-qx} \frac{\operatorname{sh} px}{\operatorname{sh} qx} dx = \frac{1}{p} - \frac{\pi}{2q} \operatorname{ctg} \frac{p\pi}{2q} \quad [0 < p < 2q].$$

BI ((27))(9)a

3.542

$$1. \int_0^{\infty} e^{-\mu x} (\operatorname{ch} \beta x - 1)^{\nu} dx = \frac{1}{2^{\nu} \beta} B \left(\frac{\mu}{\beta} - \nu, 2\nu + 1 \right) \left[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > \operatorname{Re} \beta \nu \right].$$

$$2. \int_0^{\infty} e^{-\mu x} (\operatorname{ch} x - \operatorname{ch} u)^{\nu-1} dx = -i \sqrt{\frac{2}{\pi}} e^{i\pi\nu} \Gamma(\nu) \operatorname{sh}^{\nu-\frac{1}{2}} u Q_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\operatorname{ch} u) \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu - 1].$$

EH I 155(4), ET I 164(23)

3.543

$$1. \int_{-\infty}^{\infty} \frac{e^{-ibx} dx}{\operatorname{sh} x + \operatorname{sh} t} = -\frac{i\pi e^{itb}}{\operatorname{sh} \pi b \operatorname{ch} t} (\operatorname{ch} \pi b - e^{-2itb}) \quad [t > 0].$$

ET I 121(30)

$$2. \int_0^{\infty} \frac{e^{-\mu x}}{\operatorname{ch} x - \cos t} dx = 2 \operatorname{cosec} t \sum_{k=1}^{\infty} \frac{\sin kt}{\mu + k} \quad [\operatorname{Re} \mu > -1, \quad t \neq 2n\pi].$$

BI ((6))(10)a

$$3. \int_0^{\infty} \frac{1 - e^{-x} \cos t}{\operatorname{ch} x - \cos t} e^{-(\mu-1)x} dx = 2 \sum_{k=0}^{\infty} \frac{\cos kt}{\mu + k} \quad [\operatorname{Re} \mu > 0, \quad t \neq 2n\pi].$$

BI ((6))(9)a

$$4. \int_0^{\infty} \frac{e^{px} - \cos t}{(\operatorname{ch} px + \cos t)^2} dx = \frac{1}{p} \left(t \operatorname{cosec} t + \frac{1}{1 + \cos t} \right) \quad [p > 0].$$

BI ((27))(26)a

3.544

$$\int_u^{\infty} \frac{\exp \left[- \left(n + \frac{1}{2} \right) x \right]}{\sqrt{2}(\operatorname{ch} x - \operatorname{ch} u)} dx = Q_n(\operatorname{ch} u), \quad [u > 0].$$

EH II 181(33)

3.545

$$1. \int_0^{\infty} \frac{\operatorname{sh} ax}{e^{px} + 1} dx = \frac{\pi}{2p} \operatorname{cosec} \frac{a\pi}{p} - \frac{1}{2a} \quad [p > a, \quad p > 0].$$

BI ((27))(3)

3.546

$$1. \int_0^{\infty} e^{-\beta x^2} \operatorname{sh} ax \, dx = \frac{1}{2} \frac{\sqrt{\pi}}{\sqrt{\beta}} \exp \frac{a^2}{4\beta} \Phi \left(\frac{a}{2\sqrt{\beta}} \right) \quad [\operatorname{Re} \beta > 0].$$

ET I 166(38)a

$$2. \int_0^{\infty} e^{-\beta x^2} \operatorname{ch} ax \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \frac{a^2}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

FI II 720a

$$3. \int_0^{\infty} e^{-\beta x^2} \operatorname{sh}^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} - 1 \right) \quad [\operatorname{Re} \beta > 0].$$

ET I 166(40)

$$4. \int_0^{\infty} e^{-\beta x^2} \operatorname{ch}^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left(\exp \frac{a^2}{\beta} + 1 \right) \quad [\operatorname{Re} \beta > 0].$$

ET I 166(41)

401

3.547

$$1. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{sh} \gamma x \, dx = \frac{\pi}{2} \operatorname{ctg} \frac{\gamma\pi}{2} [J_{\gamma}(\beta) - \mathbf{J}_{\gamma}(\beta)] - \frac{\pi}{2} [\mathbf{E}_{\gamma}(\beta) + N_{\gamma}(\beta)] = \gamma S_{-1, \gamma}(\beta) \quad [\operatorname{Re} \beta > 0].$$

WA 341(5), ET I 168(14)a

$$2. \int_0^{\infty} \exp(-\beta \operatorname{ch} x) \operatorname{sh} \gamma x \operatorname{sh} x \, dx = \frac{\gamma}{\beta} K_{\gamma}(\beta).$$

$$3. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{ch} \gamma x \, dx = \frac{\pi}{2} \operatorname{tg} \frac{\pi\gamma}{2} [\mathbf{J}_{\gamma}(\beta) - J_{\gamma}(\beta)] - \frac{\pi}{2} [\mathbf{E}_{\gamma}(\beta) + N_{\gamma}(\beta)] = S_{0, \gamma}(\beta) \quad [\operatorname{Re} \beta > 0, \quad \gamma \text{ not an integer}].$$

ET I 168(16)A, WA 341(4), EH II 84(50)

$$5. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{sh} \gamma x \operatorname{ch} x \, dx = \frac{\gamma}{\beta} S_{0, \gamma}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 168(7), EH II 85(51)

$$6. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{sh}[(2n+1)x] \operatorname{ch} x \, dx = O_{2n+1}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 167(5)

$$7. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{ch} \gamma x \operatorname{ch} x \, dx = \frac{1}{\beta} S_{1, \gamma}(\beta) \quad [\operatorname{Re} \beta > 0].$$

$$8. \int_0^{\infty} \exp(-\beta \operatorname{sh} x) \operatorname{ch} 2nx \operatorname{ch} x \, dx = O_{2n}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 168(6)

$$9. \int_0^{\infty} \exp(-\beta \operatorname{ch} x) \operatorname{sh}^{2\nu} x \, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) K_{\nu}(\beta) \quad \left[\operatorname{Re} \beta > 0, \operatorname{Re} \nu > -\frac{1}{2}\right].$$

EH II 82(20)

$$10. \int_0^{\infty} \exp[-2(\beta \operatorname{cth} x + \mu x)] \operatorname{sh}^{2\nu} x \, dx = \frac{1}{4} \beta^{\frac{\nu-1}{2}} \Gamma(\mu - \nu) \times \\ \times [W_{-\mu+\frac{1}{2}, \nu}(4\beta) - (\mu - \nu)W_{-\mu-\frac{1}{2}, \nu}(4\beta)] \\ [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > \operatorname{Re} \nu].$$

ET I 165(31)

$$11. \int_0^{\infty} \exp\left(-\frac{\beta^2}{2} \operatorname{sh} x\right) \operatorname{sh}^{\nu-1} x \operatorname{ch}^{\nu} x \, dx = -\pi D_{\nu}\left(\beta e^{\frac{i\pi}{4}}\right) D_{\nu}\left(\beta e^{-\frac{i\pi}{4}}\right) \quad \left[\operatorname{Re} \nu > 0, |\arg \beta| \leq \frac{\pi}{4}\right].$$

EH II 120(10)

$$12. \int_0^{\infty} \frac{\exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}} \, dx = \frac{1}{2} \sqrt{\pi^3 \beta} [J_{\nu+\frac{1}{4}}(\beta) J_{\nu-\frac{1}{4}}(\beta) + N_{\nu+\frac{1}{4}}(\beta) N_{\nu-\frac{1}{4}}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

$$13. \int_0^{\infty} \frac{\exp(-2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} [J_{\nu+\frac{1}{4}}(\beta) N_{\nu-\frac{1}{4}}(\beta) - J_{\nu-\frac{1}{4}}(\beta) N_{\nu+\frac{1}{4}}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 169(21)

402

$$14. \int_0^{\infty} \frac{\exp(-2\beta \operatorname{sh} x) \operatorname{sh} 2\nu x}{\sqrt{\operatorname{sh} x}} dx = \frac{1}{4i} \sqrt{\frac{\pi^3 \beta}{2}} [e^{\nu\pi i} H_{\frac{1}{2}+\nu}^{(1)}(\beta) H_{\frac{1}{2}-\nu}^{(2)}(\beta) - e^{-\nu\pi i} H_{\frac{1}{2}-\nu}^{(1)}(\beta) H_{\frac{1}{2}+\nu}^{(2)}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 170(24)

$$15. \int_0^{\infty} \frac{\exp(-2\beta \operatorname{sh} x) \operatorname{ch} 2\nu x}{\sqrt{\operatorname{sh} x}} dx = \frac{1}{4} \sqrt{\frac{\pi^3 \beta}{2}} [e^{\nu\pi i} H_{\frac{1}{2}+\nu}^{(1)}(\beta) H_{\frac{1}{2}-\nu}^{(2)}(\beta) + e^{-\nu\pi i} H_{\frac{1}{2}-\nu}^{(1)}(\beta) H_{\frac{1}{2}+\nu}^{(2)}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 170(25)

$$16. \int_0^{\infty} \frac{\exp(-2\beta \operatorname{ch} x) \operatorname{ch} 2\nu x}{\sqrt{\operatorname{ch} x}} dx = \sqrt{\frac{\beta}{\pi}} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 170(26)

$$17.^8 \int_0^{\infty} \frac{\exp[-2\beta(\operatorname{ch} x - 1)] \operatorname{ch} 2\nu x}{\sqrt{\operatorname{ch} x}} dx = \sqrt{\frac{\beta}{\pi}} \cdot e^{2\beta} K_{\nu+\frac{1}{4}}(\beta) K_{\nu-\frac{1}{4}}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 170(27)

$$18. \int_0^{\infty} \frac{\cos \left[\left(\nu + \frac{1}{4} \right) \pi \right] \exp(-2\nu x - 2\beta \operatorname{sh} x) + \sin \left[\left(\nu + \frac{1}{4} \right) \pi \right] \exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} [J_{\frac{1}{4}+\nu}(\beta) J_{\frac{1}{4}-\nu}(\beta) + N_{\frac{1}{4}+\nu}(\beta) N_{\frac{1}{4}-\nu}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

ET I 169(22)

$$19. \int_0^{\infty} \frac{\sin \left[\left(\nu + \frac{1}{4} \right) \pi \right] \exp(-2\nu x - 2\beta \operatorname{sh} x) - \cos \left[\left(\nu + \frac{1}{4} \right) \pi \right] \exp(2\nu x - 2\beta \operatorname{sh} x)}{\sqrt{\operatorname{sh} x}} dx = \frac{1}{2} \sqrt{\pi^3 \beta} [J_{\frac{1}{4}+\nu}(\beta) N_{\frac{1}{4}-\nu}(\beta) - J_{\frac{1}{4}-\nu}(\beta) N_{\frac{1}{4}+\nu}(\beta)] \quad [\operatorname{Re} \beta > 0].$$

$$20. \int_0^{\infty} \frac{\exp[-\beta(\operatorname{ch} x - 1)] \operatorname{ch} \nu x \operatorname{sh} x}{\sqrt{\operatorname{ch} x(\operatorname{ch} x - 1)}} dx = e^{\beta} K_{\nu}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 169(19)

3.548

$$1. \int_0^{\infty} e^{-\mu x^4} \operatorname{sh} ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \quad [\operatorname{Re} \mu > 0, \quad a \geq 0].$$

ET I 166(42)

$$2. \int_0^{\infty} e^{-\mu x^4} \operatorname{ch} ax^2 dx = \frac{\pi}{4} \sqrt{\frac{a}{2\mu}} \exp\left(\frac{a^2}{8\mu}\right) I_{-\frac{1}{4}}\left(\frac{a^2}{8\mu}\right) \quad [\operatorname{Re} \mu > 0, \quad a > 0],$$

ET I 166(43)

3.549

$$1. \int_0^{\infty} e^{-\beta x} \operatorname{sh}[(2n+1) \operatorname{Arsh} x] dx = O_{2n+1}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.547 6.}).$$

3.547
ET I 167(5)

$$2. \int_0^{\infty} e^{-\beta x} \operatorname{ch}(2n \operatorname{Arsh} x) dx = O_{2n}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.547 8.}).$$

3.547
ET I 168(6)

403

$$3. \int_0^{\infty} e^{-\beta x} \operatorname{sh}(\nu \operatorname{Arsh} x) dx = \frac{\nu}{\beta} S_{0, \nu}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.547 5.}).$$

3.547
ET I 168(7)

$$4. \int_0^{\infty} e^{-\beta x} \operatorname{ch}(\nu \operatorname{Arsh} x) dx = \frac{1}{\beta} S_{1, \nu}(\beta) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.547 7.}).$$

A number of other integrals containing hyperbolic functions and exponentials, depending on $\text{Arsh } x$ or $\text{Arch } x$ can be found by first making the substitution $x = \text{sh } t$ or $x = \text{ch } t$.

3.55- 3.56 Combinations of hyperbolic functions, exponentials, and powers

3.551

$$1. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \text{sh } \gamma x \, dx = \frac{1}{2} \Gamma(\mu) [(\beta-\gamma)^{-\mu} - (\beta+\gamma)^{-\mu}] \quad [\text{Re } \beta > -1, \quad \text{Re } \beta > |\text{Re } \gamma|].$$

ET I 164(18)

$$2. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \text{ch } \gamma x \, dx = \frac{1}{2} \Gamma(\mu) [(\beta-\gamma)^{-\mu} + (\beta+\gamma)^{-\mu}] \quad [\text{Re } \mu > 0, \quad \text{Re } \beta > |\text{Re } \gamma|].$$

ET I 164(19)

$$3. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \text{cth } x \, dx = \Gamma(\mu) \left[2^{1-\mu} \zeta \left(\mu, \frac{\beta}{2} \right) - \beta^{-\mu} \right] \quad [\text{Re } \mu > 1, \quad \text{Re } \beta > 0].$$

ET I 164(21)

$$4. \int_0^{\infty} x^n e^{-(p+mq)x} \text{sh}^m qx \, dx = 2^{-m} n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+2kq)^{n+1}} \quad [p > 0, \quad q > 0, \quad m < p+qm].$$

LI ((81))(4)

$$5. \int_0^1 \frac{e^{-\beta x}}{x} \text{sh } \gamma x \, dx = \frac{1}{2} \left[\ln \frac{\beta+\gamma}{\beta-\gamma} \text{Ei}(\gamma-\beta) - \text{Ei}(-\gamma-\beta) \right] \quad [\beta > \gamma].$$

BI ((80))(4)

$$6. \int_0^{\infty} \frac{e^{-\beta x}}{x} \text{sh } \gamma x \, dx = \frac{1}{2} \ln \frac{\beta+\gamma}{\beta-\gamma} \quad [\text{Re } \beta > |\text{Re } \gamma|].$$

ET I 163(12)

$$7. \int_1^{\infty} \frac{e^{-\beta x}}{x} \text{ch } \gamma x \, dx = \frac{1}{2} [-\text{Ei}(\gamma-\beta) - \text{Ei}(-\gamma-\beta)] \quad [\text{Re } \beta > |\text{Re } \gamma|].$$

$$8.^6 \int_0^\infty x e^{-x} \operatorname{cth} x \, dx = \frac{\pi^2}{4} - 1.$$

BI ((82))(6)

$$9. \int_0^\infty e^{-\beta x} \theta x \frac{dx}{x} = \ln \frac{\beta}{4} + 2 \ln \frac{\Gamma\left(\frac{\beta}{4}\right)}{\Gamma\left(\frac{\beta}{4} + \frac{1}{2}\right)} \quad [\operatorname{Re} \beta > 0].$$

ET I 164(16)

$$10.^6 \int_0^\infty x e^{-x} \operatorname{cth}(x/2) \, dx = \frac{\pi^2}{3} - 1.$$

404
3.552

$$1. \int_0^\infty \frac{x^{\mu-1} e^{-\beta x}}{\operatorname{sh} x} \, dx = 2^{1-\mu} \Gamma(\mu) \zeta\left[\mu, \frac{1}{2}(\beta+1)\right] \quad [\operatorname{Re} \mu > 1, \operatorname{Re} \beta > -1].$$

ET I 164(20)

$$2. \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\operatorname{sh} ax} \, dx = \frac{1}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, m = 1, 2, \dots].$$

EH I 38(24)a

$$3. \int_0^\infty \frac{x^{\mu-1} e^{-x}}{\operatorname{ch} x} \, dx = 2^{1-\mu} (1-2^{1-2\mu}) \Gamma(\mu) \zeta(\mu) \quad [\operatorname{Re} \mu > 0, \mu \neq 1] \quad (= \ln 2 \text{ if } \mu = 1).$$

EH I 32(5)

$$4. \int_0^\infty \frac{x^{2m-1} e^{-ax}}{\operatorname{ch} ax} \, dx = \frac{1-2^{1-2m}}{2m} |B_{2m}| \left(\frac{\pi}{a}\right)^{2m} \quad [a > 0, m = 1, 2, \dots].$$

EH I 39(25)a

4.261
BI((84))(4)

$$6. \int_0^{\infty} \frac{x^3 e^{-2nx}}{\operatorname{sh} x} dx = \frac{\pi^4}{8} - 12 \sum_{k=1}^n \frac{1}{(2k+1)^4} \quad [n = 0, 1, \dots]. \quad (\text{cf. 4.262 6.}).$$

4.262
BI ((84))(6)

3.553

$$1. \int_0^{\infty} \frac{\operatorname{sh}^2 ax}{\operatorname{sh} x} \frac{e^{-x} dx}{x} = \frac{1}{2} \ln(a\pi \operatorname{cosec} a\pi) \quad [a < 1].$$

BI ((95))(7)

$$2. \int_0^{\infty} \frac{\operatorname{sh}^2 \frac{\pi}{2}}{\operatorname{ch} x} \cdot \frac{e^{-x} dx}{x} = \frac{1}{2} \ln \frac{4}{\pi} \quad (\text{cf. 4.267 2.}).$$

4.267
BI ((95))(4)

3.554

$$1. \int_0^{\infty} e^{-\beta x} (1 - \operatorname{sech} x) \frac{dx}{x} = 2 \ln \frac{\Gamma\left(\frac{\beta+3}{4}\right)}{\Gamma\left(\frac{\beta+1}{4}\right)} - \ln \frac{\beta}{4} \quad [\operatorname{Re} \beta > 0].$$

ET I 164(17)

$$2. \int_0^{\infty} e^{-\beta x} \left(\frac{1}{x} - \operatorname{cosech} x \right) dx = \psi\left(\frac{\beta+1}{2}\right) - \ln \frac{\beta}{2} \quad [\operatorname{Re} \beta > 0].$$

ET I 163(10)

$$3. \int_0^{\infty} \left[\frac{\operatorname{sh}\left(\frac{1}{2} - \beta\right)x}{\operatorname{sh} \frac{x}{2}} - (1 - 2\beta)e^{-x} \right] \frac{dx}{x} = 2 \ln \Gamma(\beta) - \ln \pi + \ln(\sin \pi\beta) \quad [0 < \operatorname{Re} \beta < 1].$$

$$4. \int_0^{\infty} e^{-\beta x} \left(\frac{1}{x} - \operatorname{cth} x \right) dx = \psi \left(\frac{\beta}{2} \right) - \ln \frac{\beta}{2} + \frac{1}{\beta} \quad [\operatorname{Re} \beta > 0].$$

ET I 163(11)

405

$$5. \int_0^{\infty} \left\{ -\frac{\operatorname{sh} qx}{\operatorname{sh} \frac{x}{2}} + 2qe^{-x} \right\} \frac{dx}{x} = 2 \ln \Gamma \left(q + \frac{1}{2} \right) + \ln \cos \pi q - \ln \pi \quad \left[q^2 < \frac{1}{2} \right].$$

WH

$$6. \int_0^{\infty} x^{\mu-1} e^{-\beta x} (\operatorname{cth} x - 1) dx = 2^{1-\mu} \Gamma(\mu) \zeta \left(\mu, \frac{\beta}{2} + 1 \right) \quad [\operatorname{Re} \beta > 0; \operatorname{Re} \mu > 1].$$

ET I 164(22)

3.555

$$1. \int_0^{\infty} \frac{\operatorname{sh}^2 ax}{1 - e^{px}} \cdot \frac{dx}{x} = \frac{1}{4} \ln \left(\frac{p}{2a\pi} \sin \frac{2a\pi}{p} \right) \quad [0 < 2|a| < p]. \quad (\text{cf. 3.545 2.}).$$

3.545
BI ((93))(15)

$$2. \int_0^{\infty} \frac{\operatorname{sh}^2 ax}{e^x + 1} \cdot \frac{dx}{x} = -\frac{1}{4} \ln(a\pi \operatorname{ctg} a\pi) \quad \left[a < \frac{1}{2} \right] \quad (\text{cf. 3.545 1.}).$$

3.545
BI ((93))(9)

3.556

$$1. \int_{-\infty}^{\infty} x \frac{1 - e^{px}}{\operatorname{sh} x} dx = -\frac{\pi^2}{2} \operatorname{tg}^2 \frac{p\pi}{2} \quad [p < 1] \quad (\text{cf. 4.255 3.}).$$

4.255
BI ((101))(4)

$$2. \int_0^{\infty} \frac{1 - e^{-px}}{\operatorname{sh} x} \cdot \frac{1 - e^{-(p+1)x}}{x} dx = 2p \ln 2 \quad [p > -1].$$

$$\begin{aligned}
1. \int_0^\infty \frac{e^{-px} - e^{-qx}}{\operatorname{ch} x - \cos \frac{m}{n}\pi} \cdot \frac{dx}{x} &= 2 \operatorname{cosec} \frac{m}{n}\pi \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{n+q+k}{2n} \right) \Gamma \left(\frac{p+k}{2n} \right)}{\Gamma \left(\frac{n+p+k}{2n} \right) \Gamma \left(\frac{q+k}{2n} \right)} \\
&\quad [m+n \text{ odd}]; \\
&= 2 \operatorname{cosec} \frac{m}{n}\pi \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{n+q-k}{n} \right) \Gamma \left(\frac{p+k}{n} \right)}{\Gamma \left(\frac{n+p-k}{n} \right) \Gamma \left(\frac{q+k}{n} \right)} \\
&\quad [m+n \text{ even}]; \quad [p > -1, \quad q > -1].
\end{aligned}$$

BI ((96))(1)

$$\begin{aligned}
2. \int_0^\infty \frac{(1 - e^{-x})^2}{\operatorname{ch} x + \cos \frac{m}{n}\pi} \cdot \frac{dx}{x} &= \\
&= 2 \operatorname{cosec} \frac{m}{n}\pi \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \times \ln \frac{\left[\Gamma \left(\frac{n+k+1}{2n} \right) \right]^2 \Gamma \left(\frac{k+2}{2n} \right) \Gamma \left(\frac{k}{2n} \right)}{\left[\Gamma \left(\frac{k+1}{2n} \right) \right]^2 \Gamma \left(\frac{n+k}{2n} \right) \Gamma \left(\frac{n+k+2}{2n} \right)} \\
&\quad [m+n \text{ odd}]; \\
&= 2 \operatorname{cosec} \frac{m}{n}\pi \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \times \ln \frac{\left[\Gamma \left(\frac{n-k+1}{n} \right) \right]^2 \Gamma \left(\frac{k+2}{n} \right) \Gamma \left(\frac{k}{n} \right)}{\left[\Gamma \left(\frac{k+1}{n} \right) \right]^2 \Gamma \left(\frac{n-k}{n} \right) \Gamma \left(\frac{n-k+2}{n} \right)} \\
&\quad [m+n \text{ even}].
\end{aligned}$$

BI ((96))(2)

$$\begin{aligned}
3. \int_0^\infty \left[e^{-x} \operatorname{tg} \frac{m}{2n}\pi - \frac{e^{-px} \sin \frac{m}{n}\pi}{\operatorname{ch} x + \cos \frac{m}{n}\pi} \right] \cdot \frac{dx}{x} &= \\
&= \operatorname{tg} \left(\frac{m}{2n}\pi \right) \ln(2n) + 2 \sum_{k=1}^{n-1} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{p+n+k}{2n} \right)}{\Gamma \left(\frac{p+k}{2n} \right)} \quad [m+n \text{ odd}]; \\
&= \operatorname{tg} \left(\frac{m}{2n}\pi \right) \ln n + 2 \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \left(\frac{km}{n}\pi \right) \ln \frac{\Gamma \left(\frac{p+n-k}{n} \right)}{\Gamma \left(\frac{p+k}{n} \right)} \quad [m+n \text{ even}].
\end{aligned}$$

BI ((96))(3)

$$4. \int_0^{\infty} \frac{1 + e^{-x}}{\operatorname{ch} x + \cos a} \cdot \frac{dx}{x^{1-p}} = 2 \operatorname{sec} \frac{a}{2} \Gamma(p) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos \left(k - \frac{1}{2}\right) a}{k^p} \quad [p > 0].$$

LI ((96))(5)

$$5. \int_0^{\infty} \frac{x^q e^{-\frac{x}{2}} \operatorname{ch} \frac{x}{2}}{\operatorname{ch} x + \cos \lambda} dx = \frac{\Gamma(q+1)}{\cos \frac{\lambda}{2}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos \left(k - \frac{1}{2}\right) \lambda}{k^{q+1}} \quad [q > -1].$$

LI ((96))(5a)

$$6. \int_0^{\infty} x \frac{e^{-x} - \cos a}{\operatorname{ch} x - \cos a} dx = |a|\pi - \frac{a^2}{2} - \frac{\pi^2}{3}.$$

BI ((88))(8)

$$7. \int_0^{\infty} x^{2m+1} \frac{e^{-x} - \cos a\pi}{\operatorname{ch} x - \cos a\pi} dx = 2 \cdot (2m+1)! \sum_{k=1}^{\infty} \frac{\cos ka\pi}{k^{2m+2}}.$$

BI ((88))(6)

3.558

$$1. \int_0^{\infty} x \frac{1 - e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}} dx = \frac{2n\pi^2}{3} - 4 \sum_{k=1}^{n-1} \frac{n-k}{k^2}.$$

BI ((85))(3)

$$2. \int_0^{\infty} x \frac{1 - (-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}} dx = \frac{n\pi^2}{3} + 4 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^2}.$$

LI ((85))(1)

407

$$3. \int_0^{\infty} x^2 \frac{1 - e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}} dx = 8n\zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}.$$

$$4. \int_0^{\infty} x^2 e^x \frac{1 - e^{-2nx}}{\operatorname{sh}^2 x} dx = 8n \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} - 8 \sum_{k=1}^{n-1} \frac{n-k}{(2k-1)^3}.$$

LI ((85))(6)

$$5. \int_0^{\infty} x^2 \frac{1 + (-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}} dx = 6n\zeta(3) - 8 \sum_{k=1}^{n-1} \frac{n-k}{k^3}.$$

LI ((85))(4)

$$6. \int_0^{\infty} x^3 \frac{1 - e^{-nx}}{\operatorname{sh}^2 \frac{x}{2}} dx = \frac{4}{15} n\pi^4 - 24 \sum_{k=1}^{n-1} \frac{n-k}{k^4}.$$

BI ((85))(9)

$$7. \int_0^{\infty} x^3 \frac{1 + (-1)^n e^{-nx}}{\operatorname{ch}^2 \frac{x}{2}} dx = \frac{7}{30} n\pi^4 + 24 \sum_{k=1}^{n-1} (-1)^k \frac{n-k}{k^4}.$$

BI ((85))(8)

3.559

$$\int_0^{\infty} e^{-x} \left[a - \frac{1}{2} + \frac{(1 - e^{-x})(1 - ax) - xe^{-x}}{4 \operatorname{sh}^2 \frac{x}{2}} e^{(2-a)x} \right] \frac{dx}{x} = a - \frac{1}{2} + \ln \Gamma(a) - \frac{1}{2} \ln(2\pi) \quad [a > 0].$$

BI ((96))(6)

3.561

$$\int_0^{\infty} \frac{e^{-2x} \theta \frac{x}{2}}{x \operatorname{ch} x} dx = 2 \ln \frac{\pi}{2\sqrt{2}}.$$

BI ((93))(18)

3.562

$$1. \int_0^{\infty} x^{2\mu-1} e^{-\beta x^2} \operatorname{sh} \gamma x dx = \frac{1}{2} \Gamma(2\mu) (2\beta)^{-\mu} \exp\left(\frac{\gamma^2}{8\beta}\right) \left[D_{-2\mu} \left(-\frac{\gamma}{\sqrt{2\beta}}\right) - D_{-2\mu} \left(\frac{\gamma}{\sqrt{2\beta}}\right) \right] \left[\operatorname{Re} \mu > -\frac{1}{2}, \operatorname{Re} \beta > 0 \right].$$

ET I 166(44)

ET I 166(45)

$$3. \int_0^{\infty} x e^{-\beta x^2} \operatorname{sh} \gamma x \, dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp \frac{\gamma^2}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

BI((81))(12)A, ET I 165(34)

$$4. \int_0^{\infty} x e^{-\beta x^2} \operatorname{ch} \gamma x \, dx = \frac{\gamma}{4\beta} \sqrt{\frac{\pi}{\beta}} \exp \frac{\gamma^2}{4\beta} \Phi \left(\frac{\gamma}{2\sqrt{\beta}} \right) + \frac{1}{2\beta} \quad [\operatorname{Re} \beta > 0].$$

ET I 166(35)

408

$$5. \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{sh} \gamma x \, dx = \frac{\sqrt{\pi}(2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp \left(\frac{\gamma^2}{4\beta} \right) \Phi \left(\frac{\gamma}{2\sqrt{\beta}} \right) + \frac{\gamma}{4\beta^2} \quad [\operatorname{Re} \beta > 0].$$

ET I 166(36)

$$6. \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{ch} \gamma x \, dx = \frac{\sqrt{\pi}(2\beta + \gamma^2)}{8\beta^2 \sqrt{\beta}} \exp \left(\frac{\gamma^2}{4\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

ET I 166(37)

3.6- 4.1 Trigonometric Functions

3.61 Rational functions of sines and cosines and trigonometric functions of multiple angles

3.611

$$1. \int_0^{2\pi} (1 - \cos x)^n \sin nx \, dx = 0.$$

BI ((68))(10)

$$2. \int_0^{2\pi} (1 - \cos x)^n \cos nx \, dx = (-1)^n \frac{\pi}{2^{n-1}}.$$

$$3. \int_0^\pi (\cos t + i \sin t \cos x)^n dx = \int_0^\pi (\cos t + i \sin t \cos x)^{-n-1} dx = \pi P_n(\cos t).$$

EH I 158(23)a

3.612

$$1.^6 \int_0^\pi \frac{\sin nx \cos mx}{\sin x} dx = 0 \quad \text{for } n \leq m;$$

$$= \pi \quad \text{for } n > m \quad \text{if } m+n \text{ is odd and positive}$$

$$= 0 \quad \text{for } n > m, \quad \text{if } m+n \text{ is even.}$$

LI ((64))(3)

$$2. \int_0^\pi \frac{\sin nx}{\sin x} dx = 0 \quad \text{for } n \text{ even;}$$

$$= \pi \quad \text{for } n \text{ odd.}$$

BI ((64))(1, 2)

$$3. \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} dx = \frac{\pi}{2}.$$

FI II 145

$$4. \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x} dx = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{k-1}}{2n-1} \right).$$

GW ((332))(21b)

$$5. \int_0^\pi \frac{\sin 2nx}{\cos x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\cos x} dx = (-1)^{n-1} 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{2n-1} \right).$$

GW ((332))(22a)

$$6. \int_0^\pi \frac{\cos(2n+1)x}{\cos x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\cos(2n+1)x}{\cos x} dx = (-1)^n \pi.$$

GW ((332))(22b)

$$7.^7 \int_0^{\frac{\pi}{2}} \frac{\sin 2nx \cos^{2m+1} x}{\sin x} dx = \frac{\pi}{2} \quad [n > m \geq 0]$$

$$1.^6 \int_0^\pi \frac{\cos nx \, dx}{1 + a \cos x} = \frac{\pi}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2}-1}{a} \right)^n \quad [a^2 < 1, \quad n \geq 0].$$

BI ((64))(12)

$$2.^6 \int_0^\pi \frac{\cos nx \, dx}{1 - 2a \cos x + a^2} = \frac{\pi a^n}{1 - a^2} \quad [a^2 < 1, \quad n \geq 0].$$

$$= \frac{\pi}{(a^2 - 1)a^n} \quad [a^2 > 1, \quad n \geq 0].$$

BI ((65))(3)

$$3. \int_0^\pi \frac{\sin nx \sin x \, dx}{1 - 2a \cos x + a^2} = \frac{\pi}{2} a^{n-1} \quad [a^2 < 1, \quad n \geq 1];$$

$$= \frac{\pi}{2a^{n+1}} \quad [a^2 > 1, \quad n \geq 1].$$

BI((65))(4), GW((332))(34a)

$$4.^* \int_0^\pi \frac{\cos nx \cos x \, dx}{1 - 2a \cos x + a^2} = \frac{\pi}{2} \cdot \frac{1+a^2}{1-a^2} a^{n-1} \quad [a^2 < 1, \quad n \geq 1];$$

$$= \frac{\pi}{2a^{n+1}} \cdot \frac{a^2+1}{a^2-1} \quad [a^2 > 1, \quad n \geq 1];$$

$$= \frac{\pi a}{1-a^2} \quad [n=0, \quad a^2 < 1].$$

$$= \frac{\pi}{a(a^2-1)} \quad [n=0, \quad a^2 > 1].$$

BI((65))(5), GW((332))(34b)

$$5. \int_0^\pi \frac{\cos(2n-1)x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\cos 2nx \cos x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1].$$

BI ((65))(9, 10)

$$6. \int_0^\pi \frac{\cos(2n-1)x \cos 2x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1].$$

BI ((65))(12)

$$7. \int_0^\pi \frac{\sin 2nx \sin x \, dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \frac{\sin(2n-1)x \sin 2x \, dx}{1 - 2a \cos 2x + a^2} = 0 \quad [a^2 \neq 1].$$

$$8. \int_0^\pi \frac{\sin(2n-1)x \sin x dx}{1-2a \cos 2x + a^2} = \frac{\pi}{2} \cdot \frac{a^{n-1}}{1+a} \quad [a^2 < 1];$$

$$= \frac{\pi}{2} \cdot \frac{1}{(1+a)a^n} \quad [a^2 > 1].$$

BI ((65))(8)

$$9. \int_0^\pi \frac{\cos(2n-1)x \cos x dx}{1-2a \cos 2x + a^2} = \frac{\pi}{2} \cdot \frac{a^{n-1}}{1-a} \quad [a^2 < 1];$$

$$= \frac{\pi}{2} \cdot \frac{1}{(a-1)a^n} \quad [a^2 > 1].$$

BI ((65))(11)

$$10. \int_0^\pi \frac{\sin nx - a \sin(n-1)x}{1-2a \cos x + a^2} \sin mx dx = 0 \quad \text{for } m < n;$$

$$= \frac{\pi}{2} a^{m-n} \quad \text{for } m \geq n; \quad [a^2 < 1].$$

LI ((65))(13)

410

$$11.^6 \int_0^\pi \frac{\cos nx - a \cos(n-1)x}{1-2a \cos x + a^2} \cos mx dx = \frac{\pi}{2} (a^{|m|-n} - 1) \quad [a^2 < 1].$$

BI ((65))(14)

$$12. \int_0^\pi \frac{\sin nx - a \sin[(n+1)x]}{1-2a \cos x + a^2} dx = 0 \quad [a^2 < 1].$$

BI ((68))(13)

$$13. \int_0^\pi \frac{\cos nx - a \cos[(n+1)x]}{1-2a \cos x + a^2} dx = \pi a^n \quad [a^2 < 1].$$

BI ((68))(14)

3.614⁷

$$\int_0^\pi \frac{\sin x}{a^2 - 2ab \cos x + b^2} \cdot \frac{\sin px \cdot dx}{1 - 2a^p \cos px + a^{2p}} =$$

$$= \frac{\pi b^{p-1}}{2a^{p+1}(1-b^p)} \quad [0 < b \leq a \leq 1, \quad p = 1, 2, 3, \dots];$$

$$= \frac{\pi a^{p-1}}{2b(b^p - a^{2p})} \quad [0 < a \leq 1, \quad a < b, \quad \text{copyright } \textcircled{c} 1996 \text{ by Academic Press, Inc. All rights reserved.}]$$

$$1. \int_0^{\pi/2} \frac{\cos 2nx \, dx}{1 - a^2 \sin^2 x} = \frac{(-1)^n \pi}{2\sqrt{1 - a^2}} \left(\frac{1 - \sqrt{1 - a^2}}{a} \right)^{2n} \quad [a^2 < 1].$$

BI ((47))(27)

$$2. \int_0^{\pi} \frac{\cos x \sin 2nx \, dx}{1 + (a + b \sin x)^2} = -\frac{\pi}{b} \sin \left\{ 2n \operatorname{arctg} \sqrt{\frac{s}{2}} \right\} \operatorname{tg}^{2n} \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right).$$

$$3. \int_0^{\pi} \frac{\cos x \cos(2n + 1)x \, dx}{1 + (a + b \sin x)^2} = \frac{\pi}{b} \cos \left\{ (2n + 1) \operatorname{arctg} \sqrt{\frac{s}{2}} \right\} \operatorname{tg}^{2n+1} \left(\frac{1}{2} \arccos \sqrt{\frac{s}{2a^2}} \right),$$

where $s = -(1 + b^2 - a^2) + \sqrt{(1 + b^2 - a^2)^2 + 4a^2}$.

BI ((65))(21, 22)

3.616

$$1. \int_0^{\pi} (1 - 2a \cos x + a^2)^n \, dx = \pi \sum_{k=0}^n \binom{n}{k}^2 a^{2k}.$$

BI ((63))(1)

$$2.^* \int_0^{\pi} \frac{dx}{(1 - 2a \cos x + a^2)^n} = \frac{1}{2} \int_0^{2\pi} \frac{dx}{(1 - 2a \cos x + a^2)^n} =$$

$$= \frac{\pi}{(1 - a^2)^n} \sum_{k=0}^{n-1} \frac{(n + k - 1)!}{(k!)^2 (n - k - 1)!} \left(\frac{a^2}{1 - a^2} \right)^k \quad [a^2 < 1]$$

$$= \frac{\pi}{(a^2 - 1)^n} \sum_{k=0}^{n-1} \frac{(n + k - 1)!}{(k!)^2 (n - k - 1)!} \frac{1}{(a^2 - 1)^k} \quad [a^2 > 1].$$

BI ((331))(63)

411

$$3. \int_0^{\pi} (1 - 2a \cos x + a^2)^n \cos nx \, dx = (-1)^n \pi a^n.$$

BI ((63))(2)

$$4. \int_0^{\pi} (1 - 2a \cos x + a^2)^n \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} (1 - 2a \cos x + a^2)^n \cos mx \, dx =$$

$$= 0 [n < m];$$

$$5. \int_0^{2\pi} \frac{\sin nx \, dx}{(1 - 2a \cos 2x + a^2)^m} = 0.$$

GW ((332))(32a)

$$6. \int_0^\pi \frac{\sin x \, dx}{(1 - 2a \cos 2x + a^2)^m} = \frac{1}{2(m-1)a} \left[\frac{1}{(1-a)^{2m-2}} - \frac{1}{(1+a)^{2m-2}} \right] \quad [a \neq 0, \pm 1],$$

GW ((332))(32c)

$$\begin{aligned} 7. \int_0^\pi \frac{\cos nx \, dx}{(1 - 2a \cos x + a^2)^m} &= \\ &= \frac{1}{2} \int_0^{2\pi} \frac{\cos nx \, dx}{(1 - 2a \cos x + a^2)^m} = \\ &= \frac{a^{2m+n-2} \pi}{(1-a^2)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} \left(\frac{1-a^2}{a^2} \right)^k \quad [a^2 < 1]; \\ &= \frac{\pi}{a^n (a^2-1)^{2m-1}} \sum_{k=0}^{m-1} \binom{m+n-1}{k} \binom{2m-k-2}{m-1} (a^2-1)^k \quad [a^2 > 1]. \end{aligned}$$

GW ((332))(31)

$$8. \int_0^{\frac{\pi}{2}} \frac{\cos 2nx \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \binom{2n}{n} \frac{(b^2 - a^2)^n}{(2ab)^{2n+1}} \pi \quad [a > 0, \quad b > 0].$$

GW ((332))(30b)

3.617

$$\int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)^{n+1/2}} = \frac{2}{|1+a|^{2n+1}} F_n \left(\frac{2\sqrt{|a|}}{|1+a|} \right), \quad |a| \neq 1$$

with

$$F_n(k) = \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{n+1/2}},$$

where the $F_n(k)$ satisfy the recurrence relation

$$F_{n+1}(k) = F_n(k) + \frac{k}{2n+1} \frac{dF_n(k)}{dk}, \quad n = 0, 1, 2, \dots$$

and

$$F_0(k) = K(k) \equiv \int_0^{\pi/2} \frac{dx}{(1 - k^2 \sin^2 x)^{1/2}}$$

is the complete elliptic integral of the first kind.

Introducing the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{1/2} dx,$$

the derivatives

$$\frac{dK}{dk} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}, \quad \frac{dE}{dk} = \frac{E(k) - K(k)}{k}$$

combined with the recurrence relation lead to

$$\begin{aligned} F_1(k) &= F_0(k) + k \frac{dF_0(k)}{dk} \\ &= K(k) + \frac{E(k)}{1 - k^2} - K(k) = \frac{E(k)}{1 - k^2}, \\ F_2(k) &= \frac{E(k)}{1 - k^2} + \frac{k}{3} \frac{d}{dk} \left[\frac{E(k)}{1 - k^2} \right] \\ &= \frac{1}{3(1 - k^2)} \left[\left(\frac{4 - 2k^2}{1 - k^2} \right) E(k) - K(k) \right]. \end{aligned}$$

3.62 Powers of trigonometric functions

3.621

$$1. \int_0^{\pi/2} \sin^{\mu-1} x dx = \int_0^{\pi/2} \cos^{\mu-1} x dx = 2^{\mu-2} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right).$$

$$2. \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x dx = \int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}} x dx = \frac{1}{6\sqrt{2\pi}} \left[\Gamma\left(\frac{1}{4}\right) \right]^2.$$

$$3. \int_0^{\frac{\pi}{2}} \sin^{2m} x dx = \int_0^{\frac{\pi}{2}} \cos^{2m} x dx = \frac{(2m-1)!!}{(2m)!!} \frac{\pi}{2}.$$

FI II 151

$$4. \int_0^{\frac{\pi}{2}} \sin^{2m+1} x dx = \int_0^{\frac{\pi}{2}} \cos^{2m+1} x dx = \frac{(2m)!!}{(2m+1)!!}.$$

FI II 151

413

$$5. \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

LO V 113(50), LO V 122, FI II 788

3.622

$$1. \int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1].$$

BI ((42))(1)

$$2. \int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu} x dx = \frac{1}{2} \beta\left(\frac{\mu+1}{2}\right) \quad [\operatorname{Re} \mu > -1].$$

BI ((34))(1)

$$3. \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n} x dx = (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1}.$$

BI ((34))(2)

$$4. \int_0^{\frac{\pi}{4}} \operatorname{tg}^{2n+1} x dx = (-1)^{n+1} \frac{\ln 2}{2} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k}.$$

3.623

$$1. \int_0^{\frac{\pi}{2}} \operatorname{tg}^{\mu-1} x \cos^{2\nu-2} x \, dx = \int_0^{\frac{\pi}{2}} \operatorname{ctg}^{\mu-1} x \sin^{2\nu-2} x \, dx = \\ = \frac{1}{2} B\left(\frac{\mu}{2}, \nu - \frac{\mu}{2}\right) \quad [0 < \operatorname{Re} \mu < 2 \operatorname{Re} \nu].$$

BI((42))(6), BI((45))(22)

$$2.^6 \int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu} x \sin^2 x \, dx = \frac{1+\mu}{4} \beta\left(\frac{\mu+1}{2}\right) - \frac{1}{4} \quad [\operatorname{Re} \mu > -1].$$

BI ((34))(4)

$$3.^6 \int_0^{\frac{\pi}{4}} \operatorname{tg}^{\mu} x \cos^2 x \, dx = \frac{1-\mu}{4} \beta\left(\frac{\mu+1}{2}\right) + \frac{1}{4} \quad [\operatorname{Re} \mu > -1].$$

BI ((34))(5)

3.624

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^p x}{\cos^{p+2} x} \, dx = \frac{1}{p+1} \quad [p > -1].$$

GW ((331))(34b)

$$2.^3 \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-\frac{1}{2}} x}{\cos^{2\mu-1} x} \, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-\frac{1}{2}} x}{\sin^{2\mu-1} x} \, dx = \frac{1}{2} \left\{ \frac{\Gamma\left(\frac{\mu}{2} + \frac{1}{4}\right) \Gamma(1-\mu)}{\Gamma\left(\frac{5}{4} - \frac{\mu}{2}\right)} \right\} \quad \left[-\frac{1}{2} < \operatorname{Re} \mu < 1\right].$$

LI ((55))(12)

414

$$3. \int_0^{\frac{\pi}{4}} \frac{\cos^{n-\frac{1}{2}} 2x}{\cos^{2n+1} x} \, dx = \frac{(2n-1)!!}{2 \cdot (2n)!!} \pi.$$

BI ((38))(3)

$$4. \int_0^{\frac{\pi}{4}} \frac{\cos^{\mu} 2x}{\cos^{2(\mu+1)} x} \, dx = 2^{2\mu} B(\mu+1, \mu+1) \quad [\operatorname{Re} \mu > -1].$$

$$5. \int_0^{\frac{\pi}{4}} \frac{\sin^{2\mu-2} x}{\cos^\mu 2x} dx = 2^{1-2\mu} B(2\mu-1, 1-\mu) = \frac{\Gamma\left(\mu - \frac{1}{2}\right) \Gamma(1-\mu)}{2\sqrt{\pi}} \quad \left[\frac{1}{2} < \operatorname{Re} \mu < 1\right].$$

BI ((35))(4)

$$6. \int_0^{\frac{\pi}{2}} \left(\frac{\sin ax}{\sin x}\right)^2 dx = \frac{a\pi}{2} - \frac{1}{2} \sin \pi a [2a\beta(a) - 1], \quad [a > 0].$$

3.625

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x \cos^p 2x}{\cos^{2p+2n+1} x} dx = \frac{(n-1)!}{2} \cdot \frac{\Gamma(p+1)}{\Gamma(p+n+1)} = \frac{(n-1)!}{2(p+n)(p+n-1)\dots(p+1)} = \frac{1}{2} B(n, p+1) \quad [p > -1],$$

(cf. 3.251 1.).

3.251

BI ((35))(2)

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x \cos^p 2x}{\cos^{2p+2n+2} x} dx = \frac{1}{2} B\left(n + \frac{1}{2}, p+1\right) \quad [p > -1] \quad (\text{cf. 3.251 1.}).$$

3.251

BI ((35))(3)

$$3. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m} x} dx = \frac{(2n-2)!!(2m-1)!!}{(2n+2m-1)!!}.$$

BI ((38))(6)

$$4. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x \cos^{m-\frac{1}{2}} 2x}{\cos^{2n+2m+1} x} + dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \cdot \frac{\pi}{2}.$$

BI ((38))(7)

3.626

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n-1} x}{\cos^{2n+2} x} \sqrt{\cos 2x} dx = \frac{(2n-2)!!}{(2n+1)!!} \quad (\text{cf. 3.251 1.}).$$

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^{2n} x}{\cos^{2n+3} x} \sqrt{\cos 2x} dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \quad (\text{cf. 3.251 1.}).$$

415
3.627

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^\mu x}{\cos^\mu x} dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^\mu x}{\sin^\mu x} dx = \frac{\Gamma(\mu)\Gamma\left(\frac{1}{2}-\mu\right)}{2^\mu\sqrt{\pi}} \sin \frac{\mu\pi}{2} \quad \left[-1 < \operatorname{Re} \mu < \frac{1}{2}\right].$$

BI ((55))(12)a

3.628

$$\int_0^{\frac{\pi}{2}} \sec^{2p+1} x \frac{d \sin^{2p} x}{dx} dx = \frac{1}{\sqrt{\pi}} \Gamma(p+1)\Gamma\left(\frac{1}{2}-p\right) \quad \left[\frac{1}{2} > p > 0\right].$$

WA 691

3.63 Powers of trigonometric functions and trigonometric functions of linear functions

3.631

$$1. \int_0^\pi \sin^{\nu-1} x \sin ax dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \quad [\operatorname{Re} \nu > 0].$$

LO V 121(67a), WA 337a

$$2.^7 \int_0^{\pi/2} \sin^{\nu-2} x \sin \nu x dx = \frac{1}{1-\nu} \cos \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu > 1].$$

GW((332))(16d), FI I 152

$$3.^6 \int_0^\pi \sin^\nu x \sin \nu x dx = 2^{-\nu} \pi \sin \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu > -1].$$

$$4. \int_0^{\pi} \sin^n x \sin 2mx \, dx = 0.$$

GW ((332))(11a)

$$\begin{aligned} 5. \int_0^{\pi} \sin^{2n} x \sin(2m+1)x \, dx &= \int_0^{\pi/2} \sin^{2n} x \sin(2m+1)x \, dx = \\ &= \frac{(-1)^m 2^{n+1} n! (2n-1)!!}{(2n-2m-1)!! (2m+2n+1)!!} \quad [m \leq n]*; \\ &= \frac{(-1)^n 2^{n+1} n! (2m-2n-1)!! (2n-1)!!}{(2m+2n+1)!!} \quad [m \geq n].* \end{aligned}$$

* For $m = n$ we should set $(2n - 2m - 1)!! = 1$.

GW ((332))(11b)

$$\begin{aligned} 6. \int_0^{\pi} \sin^{2n+1} x \sin(2m+1)x \, dx &= 2 \int_0^{\pi/2} \sin^{2n+1} x \sin(2m+1)x \, dx = \\ &= \frac{(-1)^m \pi}{2^{2n+1}} \binom{2n+1}{n-m} \quad [n \geq m]; \\ &= 0 \quad [n < m]. \end{aligned}$$

BI((40))(12), GW((332))(11c)

416

$$7. \int_0^{\pi} \sin^n x \cos(2m+1)x \, dx = 0.$$

GW ((332))(12a)

$$8. \int_0^{\pi} \sin^{\nu-1} x \cos ax \, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \quad [\operatorname{Re} \nu > 0].$$

LO V 121(68)A, WA 337a

$$9. \int_0^{\pi/2} \cos^{\nu-1} x \cos ax \, dx = \frac{\pi}{2^{\nu} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)} \quad [\operatorname{Re} \nu > 0].$$

$$10. \int_0^{\pi/2} \sin^{\nu-2} x \cos \nu x dx = \frac{1}{\nu-1} \sin \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu > 1].$$

GW((332))(16b), FI II 15 2

$$11. \int_0^{\pi} \sin^{\nu} x \cos \nu x dx = \frac{\pi}{2^{\nu}} \cos \frac{\nu\pi}{2} \quad [\operatorname{Re} \nu > -1].$$

LO V 121(70)a

$$12. \int_0^{\pi} \sin^{2n} x \cos 2mx dx = 2 \int_0^{\pi/2} \sin^{2n} x \cos 2mx dx = \frac{(-1)^m}{2^{2n}} \binom{2n}{n-m} \pi \quad [n \geq m];$$

$$= 0 \quad [n < m].$$

BI((40))(16), GW((332))(12b)

$$13.7 \int_0^{\pi} \sin^{2n+1} x \cos 2mx dx =$$

$$= 2 \int_0^{\pi/2} \sin^{2n+1} x \cos 2mx dx = \frac{(-1)^m 2^{n+1} n! (2n+1)!!}{(2n-2m+1)!! (2m+2n+1)!!} \quad [n \geq m-1];$$

$$= \frac{(-1)^{n+1} 2^{n+1} n! (2m-2n-3)!! (2n+1)!!}{(2m+2n+1)!!} \quad [n < m-1].$$

GW ((332))(12c)

$$14. \int_0^{\pi/2} \cos^{\nu-2} x \sin \nu x dx = \frac{1}{\nu-1} \quad [\operatorname{Re} \nu > 1].$$

GW((332))(16c), FI II 152

$$15. \int_0^{\pi} \cos^m x \sin nx dx = [1 - (-1)^{m+n}] \int_0^{\pi/2} \cos^m x \sin nx dx =$$

$$= [1 - (-1)^{m+n}] \left\{ \sum_{k=0}^{r-1} \frac{m!}{(m-k)!} \frac{(m+n-2k-2)!!}{(m+n)!!} + s \frac{m!(n-m-2)!!}{(m+n)!!} \right\}$$

$$\left[r = \begin{cases} m & [m \leq n], \\ n & [m \geq n], \end{cases} \quad s = \begin{cases} 2 & [n-m = 4l+2 > 0], \\ 1 & [n-m = 2l+1 > 0], \\ 0 & [n-m = 4l \text{ or } n-m < 0] \end{cases} \right].$$

GW ((332))(13a)

$$16. \int_0^{\pi/2} \cos^n x \sin nx dx = \frac{1}{2^{n+1}} \sum_{k=1}^n \frac{2^k}{k}.$$

$$\begin{aligned}
17. \quad & \int_0^\pi \cos^n x \cos mx \, dx = \\
& = [1 + (-1)^{m+n}] \int_0^{\frac{\pi}{2}} \cos^n x \cos mx \, dx = \\
& = [1 + (-1)^{m+n}] \begin{cases} \frac{n!}{(m-n)(m-n+2)\dots(m+n)} & [n < m]; \\ \frac{\pi}{2^{n+1}} \binom{n}{k} & [m \leq n \text{ and } n-m = 2k]; \\ \frac{n!}{(2k+1)!!(2m+2k+1)!!} & [m < n \text{ and } n-m = 2k+1]; \end{cases} \\
& \text{where } s = \begin{cases} 0 & [m-n = 2k], \\ 1 & [m-n = 4k+1], \\ -1 & [m-n = 4k-1]. \end{cases}
\end{aligned}$$

GW ((332))(15a)

417

$$18.^6 \quad \int_0^\pi \cos^m x \cos ax \, dx = \frac{(-1)^m \sin a\pi}{2^m(m+a)} {}_2F_1 \left(-m, -\frac{a+m}{2}; 1 - \frac{a+m}{2}; -1 \right)$$

$$[a \neq 0, \pm 1, \pm 2, \dots].$$

WA 313

$$19. \quad \int_0^{\frac{\pi}{2}} \cos^{\nu-2} x \cos \nu x \, dx = 0 \quad [\operatorname{Re} \nu > 1].$$

GW((332))(16a), FI II 152

$$20.* \quad \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}} \quad [\operatorname{Re} n > -1].$$

LO V 122(78), FI II 153

3.632

$$1. \quad \int_0^\pi \sin^{p-1} x \cos \left[a \left(\frac{\pi}{2} - x \right) \right] dx = 2^{p-1} \frac{\Gamma \left(\frac{p-a}{2} \right) \Gamma \left(\frac{p+a}{2} \right)}{\Gamma(p-a)\Gamma(p+a)} \Gamma(p) \quad [p^2 < a^2].$$

BI ((62))(11)

$$2. \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\nu-1} x \sin \left[a \left(x + \frac{\pi}{2} \right) \right] dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu B \left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2} \right)} \quad [\operatorname{Re} \nu > 0].$$

$$3.^* \int_0^{\frac{\pi}{2}} \cos^p x \sin[(p+2n)x] dx = (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^k 2^k}{p+k+1} \binom{n-1}{k} \quad [n > 0].$$

LI ((41))(12)

$$4. \int_{-\pi}^{\pi} \cos^{n-1} x \cos[m(x-a)] dx = [1 - (-1)^{n+m}] \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1} x \cos[m(x-a)] dx = \\ = \frac{[1 - (-1)^{n+m}] \pi \cos ma}{2^{n-1} n B\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}\right)} \quad [n \geq m].$$

LO V 123(80), LO V 139(94a)

418

$$5. \int_0^{\frac{\pi}{2}} \cos^{p+q-2} x \cos[(p-q)x] dx = \frac{\pi}{2^{p+q-1} (p+q-1) B(p, q)} \quad [p+q > 1].$$

WH

3.633

$$1. \int_0^{\frac{\pi}{2}} \cos^{p-1} x \sin ax \sin x dx = \frac{a\pi}{2^{p+1} p(p+1) B\left(\frac{p+a}{2} + 1, \frac{p-a}{2} + 1\right)}.$$

LO V 150(110)

$$2. \int_0^{\frac{\pi}{2}} \cos^n x \sin nx \sin 2mx dx = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \cos 2mx dx = \frac{\pi}{2^{n+2}} \binom{n}{m}.$$

BI ((42))(19, 20)

$$3. \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos[(n+1)x] \cos 2mx dx = \frac{\pi}{2^{n+1}} \binom{n-1}{m-1} \quad [n > m-1].$$

BI ((42))(21)

$$4. \int_0^{\frac{\pi}{2}} \cos^{p+q} x \cos px \cos qx dx = \frac{\pi}{2^{p+q+2}} \left[1 + \frac{1}{(p+q+1) B(p+1, q+1)} \right] \quad [p+q > -1].$$

$$5.6 \int_0^{\frac{\pi}{2}} \cos^{p+q} x \sin px \sin qx dx = \frac{\pi}{2^{p+q+2}} \sum_{k=1}^{\infty} \binom{p}{k} \binom{q}{k} = \frac{\pi}{2^{p+q+2}} \left[\frac{\Gamma(p+q+1)}{\Gamma(p+1)\Gamma(q+1)} - 1 \right] \\ [p+q > -1].$$

BI ((42))(16)

3.634

$$1. \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{\nu-1} x \sin(\mu+\nu)x dx = \sin \frac{\mu\pi}{2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI((42))(23), FI II 814a

$$2. \int_0^{\frac{\pi}{2}} \sin^{\mu-1} x \cos^{\nu-1} x \cos(\mu+\nu)x dx = \cos \frac{\mu\pi}{2} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI((42))(24), FI II 814a

$$3. \int_0^{\frac{\pi}{2}} \cos^{p+n-1} x \sin px \cos[(n+1)x] \sin x dx = \frac{\pi}{2^{p+n+1}} \frac{\Gamma(p+n)}{n!\Gamma(p)} \quad [p > -n].$$

BI ((42))(15)

3.635

$$1. \int_0^{\frac{\pi}{4}} \cos^{\mu-1} 2x \operatorname{tg} x dx = \frac{1}{4} \left[\psi \left(\frac{\mu+1}{2} \right) - \psi \left(\frac{\mu}{2} \right) \right] \quad [\operatorname{Re} \mu > 0].$$

BI ((34))(7)

419

$$2.7 \int_0^{\frac{\pi}{2}} \cos^{p+2n} x \sin px \operatorname{tg} x dx = \frac{\pi}{2^{p+2n+1}\Gamma(p)} \sum_{k=0}^{\infty} \binom{n}{k} \frac{\Gamma(p+n-k)}{(n-k)!} = \\ = \frac{p\pi}{2^{p+2n+1}} \frac{\Gamma(p+2n)}{\Gamma(n+1)\Gamma(p+n+1)} \quad [p > -2n].$$

BI ((42))(22)

$$3. \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin[(n+1)x] \operatorname{ctg} x dx = \frac{\pi}{2}.$$

BI ((45))(18)

3.636

$$1. \int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x \sin 2x \, dx = \frac{\mu\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \quad [0 < \operatorname{Re} \mu < 2].$$

BI ((45))(20)a

$$2. \int_0^{\frac{\pi}{2}} \operatorname{tg}^{\pm\mu} x \cos 2x \, dx = \mp \frac{\mu\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1].$$

BI ((45))(21)

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{2\mu} x}{\cos x} \, dx = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{2\mu} x}{\sin x} \, dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma(-\mu)}{2\sqrt{\mu}} \quad \left[-\frac{1}{2} < \operatorname{Re} \mu < 1\right], \quad (\text{cf. 3.251 1}).$$

3.251

BI ((45))(13, 14)

3.637

$$1. \int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \sin^{q-2} x \sin qx \, dx = -\cos \frac{(p+q)\pi}{2} \operatorname{B}(p+q-1, 1-p) \quad [p+q > 1 > p].$$

GW ((332))(15d)

$$2. \int_0^{\frac{\pi}{2}} \operatorname{tg}^p x \sin^{q-2} x \cos qx \, dx = \sin \frac{(p+q)\pi}{2} \operatorname{B}(p+q-1, 1-p) \quad [p+q > 1 > p].$$

GW ((332))(15b)

$$3. \int_0^{\frac{\pi}{2}} \operatorname{ctg}^p x \cos^{q-2} x \sin qx \, dx = \cos \frac{p\pi}{2} \operatorname{B}(p+q-1, 1-p) \quad [p+q > 1 > p].$$

GW ((332))(15c)

$$4. \int_0^{\frac{\pi}{2}} \operatorname{ctg}^p x \cos^{q-2} x \cos qx \, dx = \sin \frac{p\pi}{2} \operatorname{B}(p+q-1, 1-p) \quad [p+q > 1 > p].$$

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^{2\mu} x dx}{\cos^{\mu+\frac{1}{2}} 2x \cos x} = \frac{\pi}{2} \sec \mu\pi \quad \left[\left| \operatorname{Re} \mu \right| < \frac{1}{2} \right], \quad (\text{cf. 3.192 2}).$$

3.192
BI ((38))(8)

420

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^{\mu-\frac{1}{2}} 2x dx}{\cos^{\mu} 2x \cos x} = \frac{2}{2\mu-1} \cdot \frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(1-\mu)}{\sqrt{\pi}} \sin\left(\frac{2\mu-1}{4}\pi\right) \quad \left[-\frac{1}{2} < \operatorname{Re} \mu < 1\right].$$

BI ((38))(17)

$$3. \int_0^{\frac{\pi}{2}} \frac{\cos^{p-1} x \sin px}{\sin x} dx = \frac{\pi}{2} \quad [p > 0].$$

GW((332))(17), BI((45))(5)

3.64- 3.65 Powers and rational functions of trigonometric functions

3.641

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^{p-1} x \cos^{-p} x}{a \cos x + b \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^{-p} x \cos^{p-1} x}{a \sin x + b \cos x} dx = \frac{\pi \operatorname{cosec} p\pi}{a^{1-p} b^p} \quad [ab > 0, \quad 0 < p < 1].$$

GW ((331))(62)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^{1-p} x \cos^p x}{(\sin x + \cos x)^3} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^p x \cos^{1-p} x}{(\sin x + \cos x)^3} dx = \frac{(1-p)p}{2} \pi \operatorname{cosec} p\pi \quad [-1 < p < 2].$$

BI((48))(5)

3.642

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} = \frac{1}{2a^{2\mu} b^{2\nu}} B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

BI ((48))(28)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^{n-1} x \cos^{n-1} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^n} = \frac{B\left(\frac{n}{2}, \frac{n}{2}\right)}{2(ab)^n} \quad [ab > 0].$$

$$\begin{aligned}
3. \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} &= \\
&= \frac{1}{2} \int_0^{\pi} \frac{\sin^{2n} x dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} = \\
&= \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{1}{2} \int_0^{\pi} \frac{\cos^{2n} x dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^{n+1}} = \frac{(2n-1)!!\pi}{2^{n+1}n!ab^{2n+1}} \\
&\quad [ab > 0].
\end{aligned}$$

GW ((331))(58)

$$\begin{aligned}
4. \int_0^{\frac{\pi}{2}} \frac{\cos^{p+2n} x \cos px dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^{n+1}} &= \pi \sum_{k=0}^n \binom{2n-k}{n} \binom{p+k-1}{k} \frac{b^{p-1}}{(2a)^{2n-k+1}(a+b)^{p+k}} \\
&\quad [a > 0, \quad b > 0, \quad p > -2n-1].
\end{aligned}$$

GW ((332))(30)

3.643

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos^p x \cos px dx}{1-2a \cos 2x + a^2} = \frac{\pi}{2^{p+1}} \cdot \frac{(1+a)^{p-1}}{1-a} \quad [a^2 < 1, \quad p > -1].$$

GW ((332))(33c)

421

$$\begin{aligned}
2. \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x \cos^{\mu} x \cos \beta x}{(1-2a \cos 2x + a^2)^m} dx &= \frac{(-1)^n \pi (1-a)^{2n-2m+1}}{2^{2m-\beta-1} (1+a)^{2m+\beta+1}} \sum_{k=0}^{m-1} \sum_{l=0}^{m-k-1} \binom{\beta}{k} \binom{2n}{l} \times \\
&\quad \times \binom{2m-k-l-2}{m-1(-2)^l} (a-1)^k \\
&\quad [a^2 < 1, \quad \beta = 2m - 2n - \mu - 2, \quad \mu > -1].
\end{aligned}$$

GW ((332))(33)

3.644

$$\begin{aligned}
1. \int_0^{\pi} \frac{\sin^m x}{p+q \cos x} dx &= 2^{m-2} \frac{p}{q^2} \sum_{\nu=1}^k \left(\frac{p^2 - q^2}{-4q^2} \right)^{\nu-1} B \left(\frac{m+1-2\nu}{2}, \frac{m+1-2\nu}{2} \right) + \left(\frac{p^2 - q^2}{-q^2} \right)^k A; \\
A &= \frac{\pi p}{q^2} \left(1 - \sqrt{1 - \frac{q^2}{p^2}} \right) \quad [m = 2k + 2]; \\
A &= \frac{1}{q} \ln \frac{p+q}{p-q} \quad [m = 2k + 1] \quad [k \geq 1, \quad q \neq 0, \quad p^2 - q^2 \geq 0].
\end{aligned}$$

$$2. \int_0^\pi \frac{\sin^m x}{1 + \cos x} dx = 2^{m-1} B\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \quad [m \geq 2].$$

$$3. \int_0^\pi \frac{\sin^m x}{1 - \cos x} dx = 2^{m-1} B\left(\frac{m-1}{2}, \frac{m+1}{2}\right) \quad [m \geq 2].$$

$$4. \int_0^\pi \frac{\sin^2 x}{p + q \cos x} dx = \frac{p\pi}{q^2} \left(1 - \sqrt{1 - \frac{q^2}{p^2}}\right).$$

$$5. \int_0^\pi \frac{\sin^3 x}{p + q \cos x} dx = 2\frac{p}{q^2} + \frac{1}{q} \left(1 - \frac{p^2}{q^2}\right) \ln \frac{p+q}{p-q}.$$

3.645

$$\int_0^\pi \frac{\cos^n x dx}{(a + b \cos x)^{n+1}} = \frac{\pi}{2^n(a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^n (-1)^k \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \left(\frac{a+b}{a-b}\right)^k$$

$[a^2 > b^2].$

LI ((64))(16)

3.646

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos^n x \sin nx \sin 2x}{1 - 2a \cos 2x + a^2} dx = \frac{\pi}{4a} \left[\left(\frac{1+a}{2}\right)^n - \frac{1}{2^n} \right] \quad [a^2 < 1].$$

BI ((50))(6)

422

$$2. \int_0^{\frac{\pi}{2}} \frac{1 - a \cos 2nx}{1 - 2a \cos 2nx + a^2} \cos^m x \cos mx dx = \frac{\pi}{2^{m+2}} \sum_{k=1}^{\infty} \binom{m}{kn} a^k + \frac{\pi}{2^{m+1}} \quad [a^2 < 1].$$

LI ((50))(7)

3.647

$$\int_0^{\frac{\pi}{2}} \frac{\cos^p x \cos px dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2b} \cdot \frac{a^{p-1}}{(a+b)^p} \quad [p > -1, \quad a > 0, \quad b > 0].$$

BI ((47))(20)

3.648

$$1. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^l x dx}{1 + \cos \frac{m}{n} \pi \sin 2x} = \frac{1}{2n} \operatorname{cosec} \frac{m}{n} \pi \sum_{k=0}^{n-1} (-1)^{k-1} \sin \frac{km}{n} \pi \times$$

$\times \left[\psi \left(\frac{n+l+k}{2} \right) - \psi \left(\frac{l+k}{2} \right) \right] \quad [m+n \text{ is odd};$

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$$1 \quad m \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \cdot km$$

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \, dx}{1 + \cos x \sin 2x} = \pi \operatorname{cosec} t \sin \mu t \operatorname{cosec}(\mu\pi) \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

BI ((47))(4)

3.649

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \sin 2x \, dx}{1 \mp 2a \cos 2x + a^2} = \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 - \left(\frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1];$$

$$= \frac{\pi}{4a} \operatorname{cosec} \frac{\mu\pi}{2} \left[1 + \left(\frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \quad [-2 < \operatorname{Re} \mu < 1].$$

BI ((50))(3)

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x (1 \mp a \cos 2x)}{1 \mp 2a \cos 2x + a^2} \, dx = \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 + \left(\frac{1-a}{1+a} \right)^\mu \right] \quad [a^2 < 1];$$

$$= \frac{\pi}{4} \sec \frac{\mu\pi}{2} \left[1 - \left(\frac{a-1}{a+1} \right)^\mu \right] \quad [a^2 > 1] \quad [|\operatorname{Re} \mu| < 1].$$

BI ((50))(4)

3.651

$$1. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x \, dx}{1 + \sin x \cos x} = \frac{1}{3} \left[\psi \left(\frac{\mu+2}{3} \right) - \psi \left(\frac{\mu+1}{3} \right) \right] \quad [\operatorname{Re} \mu > -1].$$

BI ((36))(3)

423

$$2. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x \, dx}{1 - \sin x \cos x} = \frac{1}{3} \left[\beta \left(\frac{\mu+2}{3} \right) + \beta \left(\frac{\mu+1}{3} \right) \right] \quad [\operatorname{Re} \mu > -1].$$

BI ((36))(4)a

3.652

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^\mu x \, dx}{(\sin x + \cos x) \sin x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^\mu x \, dx}{(\sin x + \cos x) \cos x} = \pi \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((49))(1)

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^\mu x \, dx}{(\sin x - \cos x) \sin x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^\mu x \, dx}{(\cos x - \sin x) \cos x} = -\pi \operatorname{ctg} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu+\frac{1}{2}} x \, dx}{(\sin x + \cos x) \cos x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu-\frac{1}{2}} x \, dx}{(\sin x + \cos x) \cos x} = \pi \sec \mu\pi \quad \left[\left| \operatorname{Re} \mu \right| < \frac{1}{2} \right].$$

BI ((61))(1, 2)

3.653

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{1-2\mu} x \, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{1-2\mu} x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2a^{2\mu} b^{2-2\mu} \sin \mu\pi} \quad [0 < \operatorname{Re} \mu < 1].$$

GW ((331))(59b)

$$2.7 \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x \, dx}{1 - a \sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x \, dx}{1 - a \cos^2 x} = \frac{\pi \sec \frac{\mu\pi}{2}}{2\sqrt{(1-a)^{\mu+1}}} \quad [|\operatorname{Re} \mu| < 1, \quad a < 1].$$

BI ((49))(6)

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} t \sec \frac{\mu\pi}{2} \cos \left[\left(\frac{\pi}{2} - t \right) \mu \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

BI((49))(7), BI((47))(21)

$$4. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm\mu} x \sin 2x}{1 - \cos^2 t \sin^2 2x} dx = \pi \operatorname{cosec} 2t \operatorname{cosec} \frac{\mu\pi}{2} \sin \left[\left(\frac{\pi}{2} - t \right) \mu \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

BI ((47))(22)a

$$5. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x \sin^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x \cos^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} 2t \sec \frac{\mu\pi}{2} \cos \left[\frac{\mu\pi}{2} - (\mu+1)t \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

BI((47))(23)A, BI((49))(10)

424

$$6. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu} x \cos^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu} x \sin^2 x \, dx}{1 - \cos^2 t \sin^2 2x} = \frac{\pi}{2} \operatorname{cosec} 2t \sec \frac{\mu\pi}{2} \cos \left[\frac{\mu\pi}{2} - (\mu-1)t \right] \quad [|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

$$1. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu+1} x \cos^2 x dx}{(1 + \cos t \sin 2x)^2} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu+1} x \sin^2 x dx}{(1 + \cos t \sin 2x)^2} = \frac{\pi(\mu \sin t \cos \mu t - \cos t \sin \mu t)}{2 \sin \mu \pi \sin^3 t}$$

$$[|\operatorname{Re} \mu| < 1, \quad t^2 < \pi^2].$$

BI((48))(3), BI((49))(22)

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm \mu} x dx}{(\sin x + \cos x)^2} = \frac{\mu \pi}{\sin \mu \pi} \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((56))(9a)

$$3. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\pm(\mu-1)} x dx}{\cos^2 x - \sin^2 x} = \pm \frac{\pi}{2} \operatorname{ctg} \frac{\mu \pi}{2} \quad [0 < \operatorname{Re} \mu < 2].$$

BI ((45))(27, 29)

3.655

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{2\mu-1} x dx}{1 - 2a(\cos t_1 \sin^2 x + \cos t_2 \cos^2 x) + a^2} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{2\mu-1} x dx}{1 - 2a(\cos t_1 \cos^2 x + \cos t_2 \sin^2 x) + a^2} =$$

$$= \frac{\pi \operatorname{cosec} \mu \pi}{(1 - 2a \cos t_2 + a^2)^\mu (1 - 2a \cos t_1 + a^2)^{1-\mu}}$$

$$[0 < \operatorname{Re} \mu < 1, \quad t_1^2 < \pi^2, \quad t_2^2 < \pi^2].$$

BI ((50))(18)

3.656

$$1. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x dx}{1 - \sin^2 x \cos^2 x} = \frac{1}{12} \left\{ -\psi \left(\frac{\mu+1}{6} \right) - \psi \left(\frac{\mu+2}{6} \right) + \right.$$

$$\left. + \psi \left(\frac{\mu+4}{6} \right) + \psi \left(\frac{\mu+5}{6} \right) + 2\psi \left(\frac{\mu+2}{3} \right) - 2\psi \left(\frac{\mu+1}{3} \right) \right\}$$

$$[\operatorname{Re} \mu > -1], \quad (\text{cf. 3.651 1. and 2.}).$$

3.651

LI ((36))(10)

$$2. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^{\mu-1} x \cos^2 x dx}{1 - \sin^2 x \cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg}^{\mu-1} x \sin^2 x dx}{1 - \sin^2 x \cos^2 x} = \frac{\pi}{4\sqrt{3}} \operatorname{cosec} \frac{\mu \pi}{6} \operatorname{cosec} \left(\frac{2 + \mu}{6} \pi \right)$$

$$[0 < \operatorname{Re} \mu < 4].$$

LI ((47))(26)

3.66 Forms containing powers of linear functions of trigonometric functions

3.661

$$1. \int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} dx = 0.$$

BI ((68))(9)

425

$$2. \int_0^{2\pi} (a \sin x + b \cos x)^{2n} dx = \frac{(2n-1)!!}{(2n)!!} \cdot 2\pi(a^2 + b^2)^n.$$

BI ((68))(8)

$$\begin{aligned} 3. \int_0^\pi (a+b \cos x)^n dx &= \frac{1}{2} \int_0^{2\pi} (a+b \cos x)^n dx = \pi(a^2 - b^2)^{\frac{n}{2}} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right) = \\ &= \frac{\pi}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} a^{n-2k} (a^2 - b^2)^k \quad [a^2 > b^2]. \end{aligned}$$

GW ((332))(37a)

$$\begin{aligned} 4. \int_0^\pi \frac{dx}{(a+b \cos x)^{n+1}} &= \frac{1}{2} \int_0^{2\pi} \frac{dx}{(a+b \cos x)^{n+1}} = \frac{\pi}{(a^2 - b^2)^{\frac{n+1}{2}}} P_n \left(\frac{a}{\sqrt{a^2 - b^2}} \right) = \\ &= \frac{\pi}{2^n (a+b)^n \sqrt{a^2 - b^2}} \sum_{k=0}^n \frac{(2n-2k-1)!!(2k-1)!!}{(n-k)!k!} \cdot \left(\frac{a+b}{a-b} \right)^k \\ &\quad [a > |b|]. \end{aligned}$$

GW((332))(38), LI((64))(14)

3.662

$$1. \int_0^{\frac{\pi}{2}} (\sec x - 1)^\mu \sin x dx = \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu \cos x dx = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1].$$

BI ((55))(13)

$$2. \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu \sin 2x dx = (1 - \mu)\mu\pi \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 2].$$

$$3. \int_0^{\frac{\pi}{2}} (\sec x - 1)^\mu \operatorname{tg} x \, dx = \int_0^{\frac{\pi}{2}} (\operatorname{cosec} x - 1)^\mu \operatorname{ctg} x \, dx = -\pi \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0],$$

(cf. 3.192 2.).

3.192
BI ((46))(4,6)

$$4. \int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^\mu \frac{dx}{\sin 2x} = -\frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0].$$

BI ((38))(22)a

$$5. \int_0^{\frac{\pi}{4}} (\operatorname{ctg} x - 1)^\mu \frac{dx}{\cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1].$$

BI ((38))(11)a

3.663

$$1. \int_0^u (\cos x - \cos u)^{\nu - \frac{1}{2}} \cos ax \, dx = \sqrt{\frac{\pi}{2}} \sin^\nu u \Gamma\left(\nu + \frac{1}{2}\right) P_{a - \frac{1}{2}}^{-\nu}(\cos u)$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}; \quad a > 0, \quad 0 < u < \pi \right].$$

EH I 159(27), ET I 22(28)

426

$$2. \int_0^u (\cos x - \cos u)^{\nu - 1} \cos[(\nu + \beta)x] \, dx = \frac{\sqrt{\pi} \Gamma(\beta + 1) \Gamma(\nu) \Gamma(2\nu) \sin^{2\nu - 1} u}{2^\nu \Gamma(\beta + 2\nu) \Gamma\left(\nu + \frac{1}{2}\right)} C_\beta^\nu(\cos u)$$

$$[\operatorname{Re} \nu > 0, \quad \operatorname{Re} \beta > -1, \quad 0 < u < \pi].$$

EH I 178(23)

3.664

$$1. \int_0^\pi (z + \sqrt{z^2 - 1} \cos x)^q \, dx = \pi P_q(z)$$

$$\left[\operatorname{Re} z > 0, \quad \arg(z + \sqrt{z^2 - 1} \cos x) = \arg z \quad \text{for } x = \frac{\pi}{2} \right].$$

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$$3. \int_0^\pi (z + \sqrt{z^2 - 1} \cos x)^q \cos nx \, dx = \frac{\pi}{(q+1)(q+2)\dots(q+n)} P_q^n(z).$$

$$\left[\begin{array}{l} \text{Re } z > 0, \quad \arg(z + \sqrt{z^2 - 1} \cos x) = \arg z \quad \text{for } x = \frac{\pi}{2}, \\ z \text{ lies outside the interval } (-1, 1) \text{ of the real axis} \end{array} \right].$$

WH, SM 483(15)

$$4. \int_0^\pi (z + \sqrt{z^2 - 1} \cos x)^\mu \sin^{2\nu-1} x \, dx =$$

$$= \frac{2^{2\nu-1} \Gamma(\mu+1) [\Gamma(\nu)]^2}{\Gamma(2\nu+\mu)} C_\mu^\nu(z) =$$

$$= \frac{\sqrt{\pi} \Gamma(\nu) \Gamma(2\nu) \Gamma(\mu+1)}{\Gamma(2\nu+\mu) \Gamma(\nu+\frac{1}{2})} C_\mu^\nu(z) = 2^\nu \sqrt{\frac{\pi}{2}} (z^2 - 1)^{\frac{1}{4} - \frac{\nu}{2}} \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(z)$$

[Re $\nu > 0$].

EH I 155(6)A, EH I 178(22)

$$5. \int_0^{2\pi} [\beta + \sqrt{\beta^2 - 1} \cos(a-x)]^\nu (\gamma + \sqrt{\gamma^2 - 1} \cos x)^{\nu-1} \, dx =$$

$$= 2\pi P_\nu[\beta\gamma - \sqrt{\beta^2 - 1} \sqrt{\gamma^2 - 1} \cos a] \quad [\text{Re } \beta > 0, \quad \text{Re } \gamma > 0].$$

EH I 157(18)

3.665

$$1. \int_0^\pi \frac{\sin^{\mu-1} x \, dx}{(a + b \cos x)^\mu} = \frac{2^{\mu-1}}{\sqrt{(a^2 - b^2)^\mu}} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \quad [\text{Re } \mu > 0, \quad 0 < b < a].$$

FI II 790a

427

$$2. \int_0^\pi \frac{\sin^{2\mu-1} x \, dx}{(1 + 2a \cos x + a^2)^\nu} = B\left(\mu, \frac{1}{2}\right) F\left(\nu, \nu - \mu + \frac{1}{2}; \mu + \frac{1}{2}; a^2\right) \quad [\text{Re } \mu > 0, \quad |a| < 1].$$

EH I 81(9)

3.666

$$1. \int_0^\pi (\beta + \cos x)^{\mu-\nu-\frac{1}{2}} \sin^{2\nu} x \, dx = \frac{2^{\nu+\frac{1}{2}} e^{-i\mu\pi} (\beta^2 - 1)^{\frac{\mu}{2}} \Gamma(\nu + \frac{1}{2}) Q_{\nu-\frac{1}{2}}^\mu(\beta)}{\Gamma\left(\nu + \mu + \frac{1}{2}\right)}$$

$$\left[\text{Re} \left(\nu + \mu + \frac{1}{2} \right) > 0, \quad \text{Re } \nu > -\frac{1}{2} \right].$$

$$2.6 \int_0^\pi (\operatorname{ch} \beta + \operatorname{sh} \beta \cos x)^{\mu+\nu} \sin^{-2\nu} x \, dx = \frac{\sqrt{\pi}}{2^\nu} \operatorname{sh}^\nu(\beta) \Gamma\left(\frac{1}{2} - \nu\right) P_\mu^\nu(\operatorname{ch} \beta) \quad \left[\operatorname{Re} \nu < \frac{1}{2}\right].$$

EH I 156(7)

$$3. \int_0^\pi (\cos t + i \sin t \cos x)^\mu \sin^{2\nu-1} x \, dx = 2^{\nu-\frac{1}{2}} \sqrt{\pi} \sin^{\frac{1}{2}-\nu} t \Gamma(\nu) P_{\mu+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos t) \\ [\operatorname{Re} \nu > 0, \quad t^2 < \pi^2].$$

EH I 158(23)

$$4. \int_0^{2\pi} [\cos t + i \sin t \cos(a-x)]^\nu \cos mx \, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \cos ma P_\nu^m(\cos t) \quad \left[0 < t < \frac{\pi}{2}\right].$$

EH I 159(25)

$$5. \int_0^{2\pi} [\cos t + i \sin t \cos(a-x)]^\nu \sin mx \, dx = \frac{i^{3m} 2\pi \Gamma(\nu+1)}{\Gamma(\nu+m+1)} \sin ma P_\nu^m(\cos t) \quad \left[0 < t < \frac{\pi}{2}\right].$$

EH I 159(26)

3.667

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^{\mu-1} 2x \, dx}{(\cos x + \sin x)^{2\mu}} = \frac{\sqrt{\pi}}{2^{\mu+1}} \frac{\Gamma(\mu)}{\Gamma\left(\mu + \frac{1}{2}\right)} \quad [\operatorname{Re} \mu > 0].$$

BI ((37))(1)

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^\mu x \, dx}{(\cos x - \sin x)^{\mu+1} \cos x} = -\pi \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0], \quad (\text{cf. 3.192 2}).$$

3.192
BI ((37))(16)

$$3. \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^\mu}{\sin^\mu x \sin 2x} \, dx = -\frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [-1 < \operatorname{Re} \mu < 0].$$

$$4. \int_0^{\frac{\pi}{4}} \frac{\sin^\mu x dx}{(\cos x - \sin x)^\mu \sin 2x} = \frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

LI ((37))(20)a

$$5. \int_0^{\frac{\pi}{4}} \frac{\sin^\mu x dx}{(\cos x - \sin x)^\mu \cos^2 x} = \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1].$$

BI ((37))(17)

$$6. \int_0^{\frac{\pi}{4}} \frac{\sin^\mu x dx}{(\cos x - \sin x)^{\mu-1} \cos^3 x} = \frac{1-\mu}{2} \mu\pi \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1].$$

BI((35))(24), BI((37))(18)

$$7. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x \cos^{\nu-1} x}{(\sin x + \cos x)^{\mu+\nu}} dx = B(\mu, \nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((48))(8)

3.668

$$1. \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)^{\cos 2t} dx = \frac{\pi}{2 \sin(\pi \cos^2 t)}.$$

FI II 788

$$2. \int_u^v \frac{(\cos u - \cos x)^{\mu-1}}{(\cos x - \cos v)^\mu} \cdot \frac{\sin x dx}{1 - 2a \cos x + a^2} = \frac{(1 - 2a \cos u + a^2)^{\mu-1}}{(1 - 2a \cos v + a^2)^\mu} \cdot \frac{\pi}{\sin \mu\pi} \\ [0 < \operatorname{Re} \mu < 1, \quad a^2 < 1].$$

BI ((73))(2)

3.669

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{p-1} x \cos^{q-p-1} x dx}{(a \cos x + b \sin x)^q} = \int_0^{\frac{\pi}{2}} \frac{\sin^{q-p-1} x \cos^{p-1} x}{(a \sin x + b \cos x)^q} dx = \frac{B(p, q-p)}{a^{q-p} b^p} \\ [q > p > 0, \quad ab > 0].$$

BI ((331))(9)

3.67 Square roots of expressions containing trigonometric functions

3.671

$$1. \int_0^{\frac{\pi}{2}} \sin^\alpha x \cos^\beta x \sqrt{1 - k^2 \sin^2 x} dx = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right) \\ [\alpha > -1, \quad \beta > -1, \quad |k| < 1].$$

GW ((331))(93)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^\alpha x \cos^\beta x}{\sqrt{1 - k^2 \sin^2 x}} dx = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) F\left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right) \\ [\alpha > -1, \quad \beta > -1, \quad |k| < 1].$$

GW ((331))(92)

$$3. \int_0^\pi \frac{\sin^{2n} x dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2^n} \sum_{j=0}^{\infty} \frac{(2j-1)!!(2n+2j-1)!!}{2^{2j} j!(n+j)!} k^{2j} \quad [k^2 < 1]; \\ = \frac{(2n-1)!!\pi}{2^n \sqrt{1 - k^2}} \sum_{j=0}^{\infty} \frac{[(2j-1)!!]^2}{2^{2j} j!(n+j)!} \left(\frac{k^2}{k^2-1}\right)^j \quad \left[k^2 < \frac{1}{2}\right].$$

LI ((67))(2)

429

3.672

$$1. \int_0^{\frac{\pi}{4}} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\cos x(\cos x - \sin x)}} = 2 \cdot \frac{(2n)!!}{(2n+1)!!}.$$

BI ((39))(5)

$$2. \int_0^{\frac{\pi}{4}} \frac{\sin^n x}{\cos^{n+1} x} \cdot \frac{dx}{\sqrt{\sin x(\cos x - \sin x)}} = \frac{(2n-1)!!}{(2n)!!} \pi.$$

BI ((39))(6)

3.673

$$\int_u^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x - \sin u}} = \sqrt{2} K\left(\sin \frac{\pi - 2u}{4}\right).$$

BI ((74))(11)

3.674

$$1. \int_0^\pi \frac{dx}{\sqrt{1 \pm 2p \cos x + p^2}} = 2\mathbf{K}(p) \quad [p^2 < 1].$$

BI ((67))(5)

$$2. \int_0^\pi \frac{\sin x \, dx}{\sqrt{1 - 2p \cos x + p^2}} = 2 \quad [p^2 \leq 1];$$

$$= \frac{2}{p} \quad [p^2 \geq 1].$$

BI ((67))(6)

$$3. \int_0^\pi \frac{\cos x \, dx}{\sqrt{1 - 2p \cos x + p^2}} = \frac{2}{p} [\mathbf{K}(p) - \mathbf{E}(p)] \quad [p^2 < 1].$$

BI ((67))(7)

3.675

$$1. \int_u^\pi \frac{\sin \left(n + \frac{1}{2} \right) x \, dx}{\sqrt{2(\cos u - \cos x)}} = \frac{\pi}{2} P_n(\cos u).$$

WH

$$2. \int_0^u \frac{\cos \left(n + \frac{1}{2} \right) x \, dx}{\sqrt{2(\cos x - \cos u)}} = \frac{\pi}{2} P_n(\cos u).$$

FI II 684, WH

3.676

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin x \, dx}{\sqrt{1 + p^2 \sin^2 x}} = \frac{1}{p} \operatorname{arctg} p.$$

BI ((60))(5)

$$2. \int_0^{\frac{\pi}{2}} \operatorname{tg}^2 x \sqrt{1 - p^2 \sin^2 x} \, dx = \infty.$$

$$3. \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{p^2 \cos^2 x + q^2 \sin^2 x}} = \frac{1}{p} \mathbf{K} \left(\frac{\sqrt{p^2 - q^2}}{p} \right) \quad [0 < q < p].$$

FI II 165

3.677

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x \, dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right).$$

BI ((60))(2)

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos^2 x \, dx}{\sqrt{1 + \sin^2 x}} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((60))(3)

3.678

$$1. \int_0^{\frac{\pi}{4}} (\sec^{\frac{1}{2}} 2x - 1) \frac{dx}{\operatorname{tg} x} = \ln 2.$$

BI ((38))(23)

$$2. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^2 x \, dx}{\sqrt{1 - k^2 \sin^2 2x}} = \sqrt{1 - k^2} - \mathbf{E}(k) + \frac{1}{2} \mathbf{K}(k).$$

BI ((39))(2)

$$3. \int_0^u \sqrt{\frac{\cos 2x - \cos 2u}{\cos 2x + 1}} \, dx = \frac{\pi}{2} (1 - \cos u) \quad \left[u^2 < \frac{\pi^2}{4} \right].$$

LI ((74))(6)

$$4. \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \sqrt{\operatorname{cosec} x} \, dx = \frac{(2n-1)!!}{(2n)!!} \pi.$$

BI ((38))(24)

$$5. \int_0^{\frac{\pi}{4}} \frac{(\cos x - \sin x)^{n-\frac{1}{2}}}{\cos^{n+1} x} \operatorname{tg}^m x \sqrt{\operatorname{cosec} x} \, dx = \frac{(2n-1)!!(2m-1)!!}{(2n+2m)!!} \pi.$$

3.679

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1 - \cos^2 \beta \cos^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= \frac{1}{\sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - \mathbf{KE}(\beta, k') - \mathbf{EF}(\beta, k') + \mathbf{KF}(\beta, k') \right\} .^*$$

*

$$k' = \sqrt{1 - k^2}$$

MO 138

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1 - (1 - k'^2 \sin^2 \beta) \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= \frac{1}{k'^2 \sin \beta \cos \beta \sqrt{1 - k'^2 \sin^2 \beta}} \left\{ \frac{\pi}{2} - \mathbf{KE}(\beta, k') - \mathbf{EF}(\beta, k') + \mathbf{KF}(\beta, k') \right\} .^*$$

MO 138

431

$$3. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1 - k^2 \sin^2 \beta \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\mathbf{KE}(\beta, k) - \mathbf{EF}(\beta, k)}{k^2 \sin \beta \cos \beta \sqrt{1 - k^2 \sin^2 \beta}} .$$

MO 138

3.68 Various forms of powers of trigonometric functions

3.681

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^\rho} = \frac{1}{2} \mathbf{B}(\mu, \nu) \mathbf{F}(\rho, \mu; \mu + \nu; k^2) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

EH I 115(7)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^{2\mu-1} x \cos^{2\nu-1} x dx}{(1 - k^2 \sin^2 x)^{\mu+\nu}} = \frac{\mathbf{B}(\mu, \nu)}{2(1 - k^2)^\mu} \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

EH I 10(20)

BI ((54))(10)

$$4.8 \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu+1} x dx}{\cos^{\mu} x (1 - k^2 \sin^2 x)^{\frac{\mu+1}{2}}} = \frac{(1+k)^{-\mu} - (1-k)^{-\mu}}{2k\mu\sqrt{\pi}} \Gamma\left(1 + \frac{\mu}{2}\right) \Gamma\left(\frac{1-\mu}{2}\right) \\ [-2 < \operatorname{Re} \mu < 1].$$

BI ((61))(5)

3.682

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x \cos^{\nu} x}{(a - b \cos^2 x)^{\rho}} dx = \frac{1}{2a^{\rho}} B\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) F\left(\frac{\nu+1}{2}, \rho; \frac{\mu+\nu}{2} + 1; \frac{b}{a}\right) \\ [\operatorname{Re} \mu > -1, \operatorname{Re} \nu > -1, a > |b| \geq 0].$$

GW ((331))(64)

3.683

$$1. \int_0^{\frac{\pi}{4}} (\sin^n 2x - 1) \operatorname{tg}\left(\frac{\pi}{4} + x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^n 2x - 1) \operatorname{ctg} x dx = -\frac{1}{2} \sum_{k=1}^n \frac{1}{k} = \\ = -\frac{1}{2} [\mathbf{C} + \psi(n+1)] \quad [n \geq 0].$$

BI((34))(8), BI((35))(11)

$$2. \int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x - 1) \operatorname{cosec}^{\mu} 2x \operatorname{tg}\left(\frac{\pi}{4} + x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x - 1) \sec^{\mu} 2x \operatorname{ctg} x dx = \\ = \frac{1}{2} [\mathbf{C} + \psi(1-\mu)]; \quad [\operatorname{Re} \mu < 1].$$

BI ((35))(20)

432

$$3. \int_0^{\frac{\pi}{4}} (\sin^{2\mu} 2x - 1) \operatorname{cosec}^{\mu} 2x \operatorname{tg}\left(\frac{\pi}{4} + x\right) dx = \int_0^{\frac{\pi}{4}} (\cos^{2\mu} 2x - 1) \sec^{\mu} 2x \operatorname{ctg} x dx = \\ = -\frac{1}{2\mu} + \frac{\pi}{2} \operatorname{ctg} \mu\pi.$$

BI ((35))(21)

$$4. \int_0^{\frac{\pi}{4}} (1 - \sec^{\mu} 2x) \operatorname{ctg} x dx = \int_0^{\frac{\pi}{4}} (1 - \operatorname{cosec}^{\mu} 2x) \operatorname{tg}\left(\frac{\pi}{4} + x\right) dx = \frac{1}{2} [\mathbf{C} + \psi(1-\mu)] \\ [\operatorname{Re} \mu < 1].$$

$$\int_0^{\frac{\pi}{4}} \frac{(\operatorname{ctg}^{\mu} x - 1) dx}{(\cos x - \sin x) \sin x} = \int_0^{\frac{\pi}{2}} \frac{(\operatorname{tg}^{\mu} x - 1) dx}{(\sin x - \cos x) \cos x} = -\mathbf{C}\psi(1-\mu) \quad [\operatorname{Re} \mu < 1].$$

BI ((37))(9)

3.685

$$\begin{aligned} 1. \int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x - \sin^{\nu-1} 2x) \operatorname{tg} \left(\frac{\pi}{4} + x \right) dx &= \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x - \cos^{\nu-1} 2x) \operatorname{ctg} x dx = \\ &= \frac{1}{2} [\psi(\nu) - \psi(\mu)] \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]. \end{aligned}$$

BI((34))(9), BI((35))(12)

$$\begin{aligned} 2. \int_0^{\frac{\pi}{2}} (\sin^{\mu-1} x - \sin^{\nu-1} x) \frac{dx}{\cos x} &= \int_0^{\frac{\pi}{2}} (\cos^{\mu-1} x - \cos^{\nu-1} x) \frac{dx}{\sin x} = \frac{1}{2} \left[\psi \left(\frac{\nu}{2} \right) - \psi \left(\frac{\mu}{2} \right) \right] \\ &[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0]. \end{aligned}$$

BI ((46))(2)

$$3. \int_0^{\frac{\pi}{2}} (\sin^{\mu} x - \operatorname{cosec}^{\mu} x) \frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} (\cos^{\mu} x - \operatorname{sec}^{\mu} x) \frac{dx}{\sin x} = -\frac{\pi}{2} \operatorname{tg} \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1].$$

BI ((46))(1, 3)

$$\begin{aligned} 4. \int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x - \operatorname{cosec}^{\mu} 2x) \operatorname{ctg} \left(\frac{\pi}{4} + x \right) dx &= \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x - \operatorname{sec}^{\mu} 2x) \operatorname{tg} x dx = \\ &= \frac{1}{2\mu} - \frac{\pi}{2} \operatorname{cosec} \mu\pi \quad [|\operatorname{Re} \mu| < 1]. \end{aligned}$$

BI ((35))(19, 22)

$$\begin{aligned} 5. \int_0^{\frac{\pi}{4}} (\sin^{\mu} 2x - \operatorname{cosec}^{\mu} 2x) \operatorname{tg} \left(\frac{\pi}{4} + x \right) dx &= \int_0^{\frac{\pi}{4}} (\cos^{\mu} 2x - \operatorname{sec}^{\mu} 2x) \operatorname{ctg} x dx = \\ &= -\frac{1}{2\mu} + \frac{\pi}{2} \operatorname{ctg} \mu\pi \quad [|\operatorname{Re} \mu| < 1]. \end{aligned}$$

BI ((35))(14)

$$\begin{aligned} 6. \int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x + \operatorname{cosec}^{\mu} 2x) \operatorname{ctg} \left(\frac{\pi}{4} + x \right) dx &= \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x + \operatorname{sec}^{\mu} 2x) \operatorname{tg} x dx = \frac{\pi}{4} \operatorname{cosec} \mu\pi \\ &[0 < \operatorname{Re} \mu < 1]. \end{aligned}$$

$$7. \int_0^{\frac{\pi}{4}} (\sin^{\mu-1} 2x - \operatorname{cosec}^{\mu} 2x) \operatorname{tg} \left(\frac{\pi}{4} + x \right) dx = \int_0^{\frac{\pi}{4}} (\cos^{\mu-1} 2x - \sec^{\mu} 2x) \operatorname{ctg} x dx = \frac{\pi}{2} \operatorname{ctg} \mu\pi$$

$$[0 < \operatorname{Re} \mu < 1].$$

BI((35))(7), LI((34))(10)

3.686

$$\int_0^{\frac{\pi}{2}} \frac{\operatorname{tg} x dx}{\cos^{\mu} x + \sec^{\mu} x} = \int_0^{\frac{\pi}{2}} \frac{\operatorname{ctg} x dx}{\sin^{\mu} x + \operatorname{cosec}^{\mu} x} = \frac{\pi}{4\mu}.$$

BI((47))(28), BI((49))(14)

3.687

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x + \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-1} x + \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\cos \left(\frac{\nu - \mu}{4} \pi \right)}{2 \cos \left(\frac{\nu + \mu}{4} \pi \right)} B \left(\frac{\mu}{2}, \frac{\nu}{2} \right)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, \operatorname{Re}(\mu + \nu) < 2].$$

BI ((46))(7)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu-1} x - \sin^{\nu-1} x}{\cos^{\mu+\nu-1} x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu-1} x - \cos^{\nu-1} x}{\sin^{\mu+\nu-1} x} dx = \frac{\sin \left(\frac{\nu - \mu}{4} \pi \right)}{2 \sin \left(\frac{\nu + \mu}{4} \pi \right)} B \left(\frac{\mu}{2}, \frac{\nu}{2} \right)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, \operatorname{Re}(\mu + \nu) < 4].$$

BI((46))(8)

$$3. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x + \sin^{\nu} x}{\sin^{\mu+\nu} x + 1} \operatorname{ctg} x dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x + \cos^{\nu} x}{\cos^{\mu+\nu} x + 1} \operatorname{tg} x dx = \frac{\pi}{\mu + \nu} \sec \left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2} \right)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((49))(15)A, BI ((47))(29)

$$4. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x - \sin^{\nu} x}{\sin^{\mu+\nu} x - 1} \operatorname{ctg} x dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{\mu} x - \cos^{\nu} x}{\cos^{\mu+\nu} x - 1} \operatorname{tg} x dx = \frac{\pi}{\mu + \nu} \operatorname{tg} \left(\frac{\mu - \nu}{\mu + \nu} \cdot \frac{\pi}{2} \right)$$

$$[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI((149))(16)A, BI((47))(30)

BI ((49))(12)

$$6. \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x - \sec^\mu x}{\cos^\nu x - \sec^\nu x} \operatorname{tg} x \, dx = \frac{\pi}{2\nu} \operatorname{tg} \left(\frac{\mu}{\nu} \cdot \frac{\pi}{2} \right) \quad [|\operatorname{Re} \nu| > |\operatorname{Re} \mu|].$$

BI ((49))(13)

3.688

$$1. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\nu x - \operatorname{tg}^\mu x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \psi(\mu) - \psi(\nu) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((37))(10)

$$2. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x - \operatorname{tg}^{1-\mu} x}{\cos x - \sin x} \cdot \frac{dx}{\sin x} = \pi \operatorname{ctg} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((37))(11)

434

$$3. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x) \, dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 1].$$

BI ((35))(9)

$$4. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x) \operatorname{tg} x \, dx = \frac{1}{\mu} - \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \quad [0 < \operatorname{Re} \mu < 2].$$

BI ((35))(15)

$$5. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1} x - \operatorname{ctg}^{\mu-1} x}{\cos 2x} \, dx = \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \quad [|\operatorname{Re} \mu| < 2].$$

BI ((35))(10)

$$6. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x - \operatorname{ctg}^\mu x}{\cos 2x} \operatorname{tg} x \, dx = -\frac{1}{\mu} + \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \quad [-2 < \operatorname{Re} \mu < 0].$$

BI ((35))(23)

$$7. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^\mu x + \operatorname{ctg}^\mu x}{1 + \cos t \sin 2x} \, dx = \pi \operatorname{cosec} t \operatorname{cosec} \mu\pi \sin \mu t \quad [t \neq n\pi, |\operatorname{Re} \mu| < 1].$$

$$8. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1} x + \operatorname{ctg}^{\mu} x}{(\sin x + \cos x) \cos x} dx = \pi \operatorname{cosec} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((37))(3)

$$9. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu} x - \operatorname{ctg}^{\mu} x}{(\sin x + \cos x) \cos x} dx = -\pi \operatorname{cosec} \mu\pi + \frac{1}{\mu} \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((37))(4)

$$10. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\nu} x - \operatorname{ctg}^{\mu} x}{(\cos x - \sin x) \cos x} dx = \psi(1-\mu) - \psi(1+\nu) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} \nu > -1].$$

BI ((37))(5)

$$11. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu-1} x - \operatorname{ctg}^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \operatorname{ctg} \mu\pi \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((37))(7)

$$12. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\mu} x - \operatorname{ctg}^{\mu} x}{(\cos x - \sin x) \cos x} dx = \pi \operatorname{ctg} \mu\pi - \frac{1}{\mu} \quad [0 < \operatorname{Re} \mu < 1].$$

BI ((37))(8)

$$13. \int_0^{\frac{\pi}{4}} \frac{1}{\operatorname{tg}^{\mu} x + \operatorname{ctg}^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{8\mu} \quad [\operatorname{Re} \mu \neq 0].$$

BI ((37))(12)

$$14. \int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^{\mu} x + \operatorname{ctg}^{\mu} x)^{\nu}} \cdot \frac{dx}{\operatorname{tg} x} = \int_0^{\frac{\pi}{2}} \frac{1}{(\operatorname{tg}^{\mu} x + \operatorname{ctg}^{\mu} x)^{\nu}} \cdot \frac{dx}{\sin 2x} = \frac{\sqrt{\pi}}{2^{2\nu+1}\mu} \frac{\Gamma(\nu)}{\Gamma\left(\nu + \frac{1}{2}\right)} \quad [\nu > 0].$$

BI((49))(25), BI((49))(26)

$$15. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^{\mu} x - \operatorname{ctg}^{\mu} x)(\operatorname{tg}^{\nu} x - \operatorname{ctg}^{\nu} x) dx = \frac{2\pi \sin \frac{\mu\pi}{2} \sin \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, |\operatorname{Re} \nu| < 1].$$

$$16. \int_0^{\frac{\pi}{4}} (\operatorname{tg}^{\mu} x + \operatorname{ctg}^{\mu} x)(\operatorname{tg}^{\nu} x + \operatorname{ctg}^{\nu} x) dx = \frac{2\pi \cos \frac{\mu\pi}{2} \cos \frac{\nu\pi}{2}}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, \quad |\operatorname{Re} \nu| < 1].$$

BI ((35))(16)

435

$$17. \int_0^{\frac{\pi}{4}} \frac{(\operatorname{tg}^{\mu} x - \operatorname{ctg}^{\mu} x)(\operatorname{tg}^{\nu} x + \operatorname{ctg}^{\nu} x)}{\cos 2x} dx = -\pi \frac{\sin \mu\pi}{\cos \mu\pi + \cos \nu\pi} \quad [|\operatorname{Re} \mu| < 1, \quad |\operatorname{Re} \nu| < 1].$$

BI ((35))(25)

$$18. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\nu} x - \operatorname{ctg}^{\nu} x}{\operatorname{tg}^{\mu} x - \operatorname{ctg}^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \operatorname{tg} \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1].$$

BI ((37))(14)

$$19. \int_0^{\frac{\pi}{4}} \frac{\operatorname{tg}^{\nu} x + \operatorname{ctg}^{\nu} x}{\operatorname{tg}^{\mu} x + \operatorname{ctg}^{\mu} x} \cdot \frac{dx}{\sin 2x} = \frac{\pi}{4\mu} \operatorname{sec} \frac{\nu\pi}{2\mu} \quad [0 < \operatorname{Re} \nu < 1].$$

BI ((37))(13)

$$20. \int_0^{\frac{\pi}{2}} \frac{(1 + \operatorname{tg} x)^{\nu} - 1}{(1 + \operatorname{tg} x)^{\mu + \nu}} \cdot \frac{dx}{\sin x \cos x} = \psi(\mu + \nu) - \psi(\mu) \quad [\mu > 0, \quad \nu > 0].$$

BI ((49))(29)

3.689

$$1. \int_0^{\frac{\pi}{2}} \frac{(\sin^{\mu} x + \operatorname{cosec}^{\mu} x) \operatorname{ctg} x dx}{\sin^{\nu} x - 2 \cos t + \operatorname{cosec}^{\nu} x} = \frac{\pi}{\nu} \operatorname{cosec} t \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t}{\nu} \quad [\mu < \nu].$$

LI ((50))(14)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^{\mu} x - 2 \cos t_1 + \operatorname{cosec}^{\mu} x}{\sin^{\nu} x + 2 \cos t_2 + \operatorname{cosec}^{\nu} x} \cdot \operatorname{ctg} x dx = \frac{\pi}{\nu} \operatorname{cosec} t_2 \operatorname{cosec} \frac{\mu\pi}{\nu} \sin \frac{\mu t_2}{\nu} - \frac{t_2}{\nu} \operatorname{cosec} t_2 \cos t_1$$

[$\nu > \mu > 0$ or $\nu < \mu < 0$ or $\mu > 0, \nu < 0$ and $\mu + \nu < 0$
or $\mu < 0, \nu > 0$ and $\mu + \nu > 0$].

BI ((50))(15)

3.69- 3.71 Trigonometric functions of more complicated arguments

3.69- 3.71 Trigonometric functions of more complicated arguments

3.691

$$1. \int_0^{\infty} \sin(ax^2) dx = \int_0^{\infty} \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0].$$

FI II 743a, ET I 64(7)a

$$2. \int_0^1 \sin(ax^2) dx = \sqrt{\frac{\pi}{2a}} S(\sqrt{a}) \quad [a > 0].$$

$$3. \int_0^1 \cos(ax^2) dx = \sqrt{\frac{\pi}{2a}} C(\sqrt{a}) \quad [a > 0].$$

ET I 8(5)a

$$4. \int_0^{\infty} \sin(ax^2) \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) + \sin \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, \quad b > 0].$$

ET I 82(1)a

$$5. \int_0^{\infty} \sin(ax^2) \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} - \sin \frac{b^2}{a} \right\} = \frac{1}{2} \sqrt{\frac{\pi}{a}} \cos\left(\frac{b^2}{a} + \frac{\pi}{4}\right) \\ [a > 0, \quad b > 0].$$

ET I 82(18), BI((70))(13) GW((334))(5a)

436

$$6. \int_0^{\infty} \cos ax^2 \sin 2bx dx = \sqrt{\frac{\pi}{2a}} \left\{ \sin \frac{b^2}{a} C\left(\frac{b}{\sqrt{a}}\right) - \cos \frac{b^2}{a} S\left(\frac{b}{\sqrt{a}}\right) \right\} \quad [a > 0, \quad b > 0].$$

ET I 83(3)a

$$7. \int_0^{\infty} \cos ax^2 \cos 2bx dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} + \sin \frac{b^2}{a} \right\} \quad [a > 0, \quad b > 0].$$

GW((334))(5a), BI((70))(14), ET I 24(7)

$$8. \int_0^{\infty} (\cos ax + \sin ax) \sin(b^2 x^2) dx = \\ = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2C\left(\frac{b^2}{2b}\right)\right) \cos\left(\frac{b^2}{4b^2}\right) - \left(1 - 2S\left(\frac{b^2}{2b}\right)\right) \sin\left(\frac{b^2}{4b^2}\right) \right\} \quad [a > 0, \quad b > 0].$$

$$9. \int_0^{\infty} (\cos ax + \sin ax) \cos(b^2 x^2) dx = \\ = \frac{1}{2b} \sqrt{\frac{\pi}{2}} \left\{ \left(1 + 2C\left(\frac{a}{2b}\right)\right) \sin\left(\frac{a^2}{4b^2}\right) + \left(1 - 2S\left(\frac{a}{2b}\right)\right) \cos\left(\frac{a^2}{4b^2}\right) \right\} \quad [a > 0, \quad b > 0].$$

ET I 25(21)

$$10. \int_0^{\infty} \sin(a^2 x^2) \sin 2bx \sin 2cx dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0, \quad c > 0].$$

ET I 84(15)

$$11. \int_0^{\infty} \sin(a^2 x^2) \cos 2bx \cos 2cx dx = \frac{\sqrt{\pi}}{2a} \cos \frac{2bc}{a^2} \cos\left(\frac{b^2 + c^2}{a^2} + \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0, \quad c > 0].$$

ET I 84(21)

$$12. \int_0^{\infty} \cos(a^2 x^2) \sin 2bx \sin 2cx dx = \frac{\sqrt{\pi}}{2a} \sin \frac{2bc}{a^2} \sin\left(\frac{b^2 + c^2}{a^2} - \frac{\pi}{4}\right) \quad [a > 0, \quad b > 0, \quad c > 0].$$

ET I 25(19)

$$13. \int_0^{\infty} \sin(ax^2) \cos(bx^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{a-b}} \right) \quad [a > b > 0]; \\ = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left(\frac{1}{\sqrt{b+a}} - \frac{1}{\sqrt{b-a}} \right) \quad [b > a > 0].$$

BI ((177))(21)

$$14. \int_0^{\infty} (\sin^2 ax^2 - \sin^2 bx^2) dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0].$$

BI ((178))(1)

$$15. \int_0^{\infty} (\cos^2 ax^2 - \sin^2 bx^2) dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{b}} + \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0].$$

BI ((178))(3)

BI ((178))(5)

437

$$17. \int_0^{\infty} (\sin^4 ax^2 - \sin^4 bx^2) dx = \frac{1}{64}(8 - \sqrt{2}) \left(\sqrt{\frac{\pi}{b}} - \sqrt{\frac{\pi}{a}} \right) \quad [a > 0, \quad b > 0].$$

BI ((178))(2)

$$18. \int_0^{\infty} (\cos^4 ax^2 - \sin^4 bx^2) dx = \frac{1}{8} \left(\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{b}} \right) + \frac{1}{32} \left(\sqrt{\frac{\pi}{2a}} - \sqrt{\frac{\pi}{2b}} \right) \quad [a > 0, \quad b > 0].$$

BI ((178))(4)

$$19. \int_0^{\infty} (\cos^4 ax^2 - \cos^4 bx^2) dx = \frac{1}{64}(8 + \sqrt{2}) \left(\sqrt{\frac{\pi}{a}} - \sqrt{\frac{\pi}{b}} \right) \quad [a > 0, \quad b > 0].$$

BI ((178))(6)

$$20. \int_0^{\infty} \sin^{2n} ax^2 dx = \int_0^{\infty} \cos^{2n} ax^2 dx = \infty.$$

BI ((177))(5, 6)

$$21. \int_0^{\infty} \sin^{2n+1}(ax^2) dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n (-1)^{n+k} \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \quad [a > 0].$$

BI ((70))(9)

$$22. \int_0^{\infty} \cos^{2n+1}(ax^2) dx = \frac{1}{2^{2n+1}} \sum_{k=0}^n \binom{2n+1}{k} \sqrt{\frac{\pi}{2(2n-2k+1)a}} \quad [a > 0].$$

BI((177))(7A), BI((70))(10)

3.692

$$1. \int_0^{\infty} [\sin(a - x^2) + \cos(a - x^2)] dx = \sqrt{\frac{\pi}{a}} \sin a.$$

GW((333))(30c), BI((178))(7a)

$$2. \int_0^{\infty} \cos\left(\frac{x^2}{2} - \frac{\pi}{8}\right) \cos ax \, dx = \sqrt{\frac{\pi}{2}} \cos\left(\frac{a^2}{2} - \frac{\pi}{8}\right) \quad [a > 0].$$

ET I 24(8)

$$3. \int_0^{\infty} \sin[a(1-x^2)] \cos bx \, dx = -\frac{1}{2} \sqrt{\frac{\pi}{a}} \cos\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \quad [a > 0].$$

ET I 23(2)

$$4. \int_0^{\infty} \cos[a(1-x^2)] \cos bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} \sin\left(a + \frac{b^2}{4a} + \frac{\pi}{4}\right) \quad [a > 0].$$

ET I 24(10)

$$5. \int_0^{\infty} \sin\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \int_0^{\infty} \cos\left(ax^2 + \frac{b^2}{a}\right) \cos 2bx \, dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0].$$

BI ((70))(19, 20)

3.693

$$1. \int_0^{\infty} \sin(ax^2 + 2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) - \sin \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \right\} \quad [a > 0]$$

[see Use of Tables]

BI ((70))(3)

$$2. \int_0^{\infty} \cos(ax^2 + 2bx) \, dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \frac{b^2}{a} \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) + \sin \frac{b^2}{a} \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \right\} \quad [a > 0]$$

[see Use of Tables]

BI ((70))(4)

438

3.694

$$1. \int_0^{\infty} \sin(ax^2 + 2bx + c) \, dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \sin c + \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \cos c \right\} +$$

$$+ \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \sin c - \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \cos c \right\}$$

[a > 0] [see Use of Tables]

$$2. \int_0^{\infty} \cos(ax^2+2bx+c) dx = \sqrt{\frac{\pi}{2a}} \cos \frac{b^2}{a} \left\{ \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \cos c - \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \sin c \right\} + \\ + \sqrt{\frac{\pi}{2a}} \sin \frac{b^2}{a} \left\{ \left(\frac{1}{2} - S_2 \left(\frac{b^2}{a} \right) \right) \cos c + \left(\frac{1}{2} - C_2 \left(\frac{b^2}{a} \right) \right) \sin c \right\} \\ [a > 0] \quad [\text{see Use of Tables}]$$

GW ((334))(4b)

3.695

$$1. \int_0^{\infty} \sin(a^3 x^3) \sin(bx) dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) - \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) \right\} \\ [a > 0, \quad b > 0].$$

ET I 83(5)

$$2. \int_0^{\infty} \cos(a^3 x^3) \cos(bx) dx = \frac{\pi}{6a} \sqrt{\frac{b}{3a}} \left\{ J_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + J_{-\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) + \frac{\sqrt{3}}{\pi} K_{\frac{1}{3}} \left(\frac{2b}{3a} \sqrt{\frac{b}{3a}} \right) \right\} \\ [a > 0, \quad b > 0].$$

ET I 24(11)

3.696

$$1. \int_0^{\infty} \sin(ax^4) \sin(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin \left(\frac{b^2}{8a} - \frac{3}{8}\pi \right) J_{\frac{1}{4}} \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET I 83(2)

$$2. \int_0^{\infty} \sin(ax^4) \cos(bx^2) dx = -\frac{\pi}{4} \sqrt{\frac{b}{2a}} \sin \left(\frac{b^2}{8a} - \frac{\pi}{8} \right) J_{-\frac{1}{4}} \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET I 84(19)

$$3. \int_0^{\infty} \cos(ax^4) \sin(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos \left(\frac{b^2}{8a} - \frac{3}{8}\pi \right) J_{\frac{1}{4}} \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET I 83(4), ET I 25(24)

$$4. \int_0^{\infty} \cos(ax^4) \cos(bx^2) dx = \frac{\pi}{4} \sqrt{\frac{b}{2a}} \cos \left(\frac{b^2}{8a} - \frac{\pi}{8} \right) J_{-\frac{1}{4}} \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET I 25(25)

3.697

$$\int_0^{\infty} \sin\left(\frac{a^2}{x}\right) \sin(bx) dx = \frac{a\pi}{2\sqrt{b}} J_1(2a\sqrt{b}) \quad [a > 0, \quad b > 0].$$

ET I 83(6)

439

3.698

$$1. \int_0^{\infty} \sin\left(\frac{a^2}{x^2}\right) \sin(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab - \cos 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0].$$

ET I 83(9)

$$2.^8 \int_0^{\infty} \sin\left(\frac{a^2}{x^2}\right) \cos(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab - e^{-2ab}]$$

ET I 24(13)

$$3. \int_0^{\infty} \cos\left(\frac{a^2}{x^2}\right) \sin(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\sin 2ab + \cos 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0].$$

ET I 84(12)

$$4. \int_0^{\infty} \cos\left(\frac{a^2}{x^2}\right) \cos(b^2 x^2) dx = \frac{1}{4b} \sqrt{\frac{\pi}{2}} [\cos 2ab - \sin 2ab + e^{-2ab}] \quad [a > 0, \quad b > 0].$$

ET I 24(14)

3.699

$$1. \int_0^{\infty} \sin\left(a^2 x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab + \sin 2ab) \quad [a > 0, \quad b > 0].$$

BI ((70))(27)

$$2. \int_0^{\infty} \cos\left(a^2 x^2 + \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} (\cos 2ab - \sin 2ab) \quad [a > 0, \quad b > 0].$$

BI ((70))(28)

$$4. \int_0^{\infty} \sin\left(a^2 x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0].$$

GW ((334))(9b)a

$$5. \int_0^{\infty} \cos\left(a^2 x^2 - \frac{b^2}{x^2}\right) dx = \frac{\sqrt{2\pi}}{4a} e^{-2ab} \quad [a > 0, \quad b > 0].$$

GW ((334))(9b)a

3.711

$$\int_0^u \sin(a\sqrt{u^2 - x^2}) \cos bx \, dx = \frac{\pi au}{2\sqrt{a^2 + b^2}} J_1(u\sqrt{a^2 + b^2}) \quad [a > 0, \quad b > 0, \quad u > 0].$$

ET I 27(37)

3.712

$$1. \int_0^{\infty} \sin(ax^p) \, dx = \frac{\Gamma\left(\frac{1}{p}\right) \sin \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1].$$

EH I 13(40)

440

$$2. \int_0^{\infty} \cos(ax^p) \, dx = \frac{\Gamma\left(\frac{1}{p}\right) \cos \frac{\pi}{2p}}{pa^{\frac{1}{p}}} \quad [a > 0, \quad p > 1].$$

EH I 13(39)

3.713

$$1. \int_0^{\infty} \sin(ax^p + bx^q) \, dx = \frac{1}{p} \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-\frac{kq+1}{p}} \Gamma\left(\frac{kq+1}{p}\right) \sin\left[\frac{k(q-p)+1}{2p}\pi\right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0].$$

BI ((70))(7)

$$2. \int_0^{\infty} \cos(ax^p + bx^q) \, dx = \frac{1}{p} \sum_{k=0}^{\infty} \frac{(-b)^k}{k!} a^{-(kq+1)/p} \Gamma\left(\frac{kq+1}{p}\right) \cos\left[\frac{k(q-p)+1}{2p}\pi\right] \\ [a > 0, \quad b > 0, \quad p > 0, \quad q > 0].$$

3.714

$$1. \int_0^{\infty} \cos(z \operatorname{sh} x) dx = K_0(z) \quad [\operatorname{Re} z > 0].$$

WA 202(14)

$$2. \int_0^{\infty} \sin(z \operatorname{ch} x) dx = \frac{\pi}{2} J_0(z) \quad [\operatorname{Re} z > 0].$$

MO 36

$$3. \int_0^{\infty} \cos(z \operatorname{ch} x) dx = -\frac{\pi}{2} N_0(z) \quad [\operatorname{Re} z > 0].$$

MO 37

$$4. \int_0^{\infty} \cos(z \operatorname{sh} x) \operatorname{ch} \mu x dx = \cos \frac{\mu\pi}{2} K_{\mu}(z) \quad [\operatorname{Re} z > 0, \quad |\operatorname{Re} \mu| < 1].$$

WA 202(13)

$$5. \int_0^{\pi} \cos(z \operatorname{ch} x) \sin^{2\mu} x dx = \sqrt{\pi} \left(\frac{2}{z}\right)^{\mu} \Gamma\left(\mu + \frac{1}{2}\right) I_{\mu}(z) \quad \left[\operatorname{Re} z > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right].$$

WH

3.715

$$1. \int_0^{\pi} \sin(z \sin x) \sin ax dx = \sin a\pi s_{0,a}(z) = \sin a\pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \quad [a > 0].$$

WA 338(13)

$$\begin{aligned} 2. \int_0^{\pi} \sin(z \sin x) \sin nx dx &= \frac{1}{2} \int_{-\pi}^{\pi} \sin(z \sin x) \sin nx dx = \\ &= [1 - (-1)^n] \int_0^{\frac{\pi}{2}} \sin(z \sin x) \sin nx dx = [1 - (-1)^n] \frac{\pi}{2} J_n(z) \\ &\quad [n = 0, \pm 1, \pm 2, \dots]. \end{aligned}$$

$$3. \int_0^{\frac{\pi}{2}} \sin(z \sin x) \sin 2x \, dx = \frac{2}{z^2} (\sin z - z \cos z).$$

LI ((43))(14)

$$4. \int_0^{\pi} \sin(z \sin x) \cos ax \, dx = (1 + \cos a\pi) s_{0,a}(z) = \\ = (1 + \cos a\pi) \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \quad [a > 0].$$

WA 338(14)

$$5. \int_0^{\pi} \sin(z \sin x) \cos[(2n+1)x] \, dx = 0.$$

GW ((334))(53b)

$$6. \int_0^{\pi} \cos(z \sin x) \sin ax \, dx = -a(1 - \cos a\pi) s_{-1,a}(z) = \\ = -a(1 - \cos a\pi) \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\ [a > 0].$$

WA 338(12)

$$7. \int_0^{\pi} \cos(z \sin x) \sin 2nx \, dx = 0.$$

GW ((334))(54a)

$$8. \int_0^{\pi} \cos(z \sin x) \cos ax \, dx = -a \sin a\pi s_{-1,a}(z) = \\ = -a \sin a\pi \left\{ -\frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{a^2(2^2 - a^2)(4^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\ [a > 0].$$

WA 338(11)

$$9. \int_0^{\pi} \cos(z \sin x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(z \sin x) \cos nx \, dx = \\ = [1 + (-1)^n] \int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos nx \, dx = [1 + (-1)^n] \frac{\pi}{2} J_n(z).$$

$$10. \int_0^{\frac{\pi}{2}} \cos(z \sin x) \cos^{2n} x \, dx = \frac{\pi}{2} \frac{(2n-1)!!}{z^n} J_n(z) \quad \left[\operatorname{Re} n > -\frac{1}{2} \right].$$

FI II 486, WA 35a

$$11. \int_0^{\frac{\pi}{2}} \sin(z \cos x) \sin 2x \, dx = \frac{2}{z^2} (\sin z - z \cos z).$$

LI ((43))(15)

$$\begin{aligned} 12. \int_0^{\frac{\pi}{2}} \sin(z \cos x) \cos ax \, dx &= \cos \frac{a\pi}{2} s_{0,a}(z) = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{J}_\nu(z) - \mathbf{J}_{-\nu}(z)] = \\ &= -\frac{\pi}{4} \sec \frac{a\pi}{4} [\mathbf{E}_\nu(z) + \mathbf{E}_{-\nu}(z)] = \\ &= \cos \frac{a\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k-1)^2 - a^2]} \quad [a > 0]. \end{aligned}$$

WA 339

442

$$13. \int_0^{\pi} \sin(z \cos x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(z \cos x) \cos nx \, dx = \pi \sin \frac{n\pi}{2} J_n(z).$$

GW ((334))(55b)

$$14. \int_0^{\frac{\pi}{2}} \sin(z \cos x) \cos[(2n+1)x] \, dx = (-1)^n \frac{\pi}{2} J_{2n+1}(z).$$

WA 30(8)

$$15. \int_0^{\frac{\pi}{2}} \sin(a \cos x) \operatorname{tg} x \, dx = \operatorname{si}(a) + \frac{\pi}{2} \quad [a > 0].$$

BI ((43))(17)

$$16. \int_0^{\frac{\pi}{2}} \sin(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_\nu(z) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 358(1)

$$\begin{aligned} 17. \int_0^{\frac{\pi}{2}} \cos(z \cos x) \cos ax \, dx &= -a \sin \frac{a\pi}{2} s_{-1,a}(z) = \\ &= \frac{\pi}{4} \sec \frac{a\pi}{2} [\mathbf{J}_\nu(z) + \mathbf{J}_{-\nu}(z)] = \frac{\pi}{4} \operatorname{cosec} \frac{a\pi}{2} [\mathbf{E}_\nu(z) - \mathbf{E}_{-\nu}(z)] = \\ &= -a \sin \frac{a\pi}{2} \left\{ -\frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k}}{(1^2 - a^2)(3^2 - a^2) \dots [(2k)^2 - a^2]} \right\} \\ &\quad [a > 0]. \end{aligned}$$

$$18. \int_0^\pi \cos(z \cos x) \cos nx \, dx = \frac{1}{2} \int_{-\pi}^\pi \cos(z \cos x) \cos nx \, dx = \pi \cos \frac{n\pi}{2} J_n(z).$$

GW ((334))(56b)

$$19. \int_0^{\frac{\pi}{2}} \cos(z \cos x) \cos 2nx \, dx = (-1)^n \cdot \frac{\pi}{2} J_{2n}(z).$$

WA 30(9)

$$20. \int_0^{\frac{\pi}{2}} \cos(z \cos x) \sin^{2\nu} x \, dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{z}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(z) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

WA 35, WH

$$21. \int_0^\pi \cos(z \cos x) \sin^{2\mu} x \, dx = \sqrt{\pi} \left(\frac{2}{z}\right)^\mu \Gamma\left(\mu + \frac{1}{2}\right) J_\mu(z) \quad \left[\operatorname{Re} \mu > -\frac{1}{2}\right].$$

WH

3.716

$$1. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \, dx = \frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) - e^a \operatorname{Ei}(-a)] \quad [a > 0] \quad (\text{cf. 3.723 1.}).$$

3.723
BI ((43))(1)

$$2. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \, dx = \frac{\pi}{2} e^{-a} \quad [a \geq 0].$$

BI ((43))(2)

$$3. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \quad [a \geq 0].$$

BI ((43))(7)

$$4. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \sin^2 x \, dx = \frac{1-a}{4} \pi e^{-a} \quad [a \geq 0].$$

$$5. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \cos^2 x \, dx = \frac{1+a}{4} \pi e^{-a} \quad [a \geq 0].$$

BI ((43))(9)

$$6. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \operatorname{tg} x \, dx = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

BI ((43))(5)

$$7. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \operatorname{tg} x \, dx = -\frac{1}{2} [e^{-a} \overline{\operatorname{Ei}}(a) + e^a \operatorname{Ei}(-a)] \quad [a > 0] \quad (\text{cf. 3.723 5.}).$$

3.723
BI ((43))(6)

$$8. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x) \sin^2 x \operatorname{tg} x \, dx = \frac{2-a}{4} \pi e^{-a} \quad [a > 0].$$

BI ((43))(11)

$$9. \int_0^{\frac{\pi}{2}} \sin^2(a \operatorname{tg} x) \, dx = \frac{\pi}{4} (1 - e^{-2a}) \quad [a \geq 0] \quad (\text{cf. 3.742 1.}).$$

3.742
BI ((43))(3)

$$10. \int_0^{\frac{\pi}{2}} \cos^2(a \operatorname{tg} x) \, dx = \frac{\pi}{4} (1 + e^{-2a}) \quad [a \geq 0] \quad (\text{cf. 3.742 3.}).$$

3.742
BI ((43))(4)

$$11. \int_0^{\frac{\pi}{2}} \sin^2(a \operatorname{tg} x) \operatorname{ctg}^2 x \, dx = \frac{\pi}{4} (e^{-2a} + 2a - 1) \quad [a \geq 0].$$

$$12. \int_0^{\frac{\pi}{2}} [1 - \sec^2 x \cos(\operatorname{tg} x)] \frac{dx}{\operatorname{tg} x} = C.$$

BI ((51))(14)

$$13. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{ctg} x) \sin 2x \, dx = \frac{a\pi}{2} e^{-a} \quad [a \geq 0] \quad (\text{cf. 3.716 3.}),$$

3.716

and in general, formulas 3.716 remain valid if we replace $\operatorname{tg} x$ in the argument of the sine or cosine with $\operatorname{ctg} x$ if we also replace $\sin x$ with $\cos x$, $\cos x$ with $\sin x$, hence $\operatorname{tg} x$ with $\operatorname{ctg} x$, $\operatorname{ctg} x$ with $\operatorname{tg} x$, $\sec x$ with $\operatorname{cosec} x$, and $\operatorname{cosec} x$ with $\sec x$ in the factors.

Analogously,

3.717

$$\int_0^{\frac{\pi}{2}} \sin(a \operatorname{cosec} x) \sin(a \operatorname{ctg} x) \frac{dx}{\cos x} = \int_0^{\frac{\pi}{2}} \sin(a \sec x) \sin(a \operatorname{tg} x) \frac{dx}{\sin x} = \frac{\pi}{2} \sin a \quad [a \geq 0].$$

BI ((52))(11, 12)

3.718

$$1. \int_0^{\frac{\pi}{2}} \sin\left(\frac{\pi}{2}p - a \operatorname{tg} x\right) \operatorname{tg}^{p-1} x \, dx = \int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}p - a \operatorname{tg} x\right) \operatorname{tg}^p x \, dx = \frac{\pi}{2} e^{-a} \\ [p^2 < 1, \quad p \neq 0, \quad a \geq 0].$$

BI ((44))(5, 6)

$$2. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{tg} x - \nu x) \sin^{\nu-2} x \, dx = 0 \quad [\operatorname{Re} \nu > 0, \quad a > 0].$$

NH 157(15)

444

$$3. \int_0^{\frac{\pi}{2}} \sin(n \operatorname{tg} x + \nu x) \frac{\cos^{\nu-1} x}{\sin x} \, dx = \frac{\pi}{2} \quad [\operatorname{Re} \nu > 0].$$

BI ((51))(15)

$$5. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x + \nu x) \cos^\nu x \, dx = 2^{-\nu-1} \pi e^{-a} \quad [\operatorname{Re} \nu > -1, \quad a \geq 0].$$

BI ((44))(4)

$$6. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x - \gamma x) \cos^\nu x \, dx = \frac{\pi a^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}+1}} \cdot \frac{W_{\frac{\gamma}{2}, -\frac{\nu+1}{2}}(2a)}{\Gamma\left(1 + \frac{\gamma + \nu}{2}\right)}$$

$$\left[a > 0, \quad \operatorname{Re} \nu > -1, \quad \frac{\nu + \gamma}{2} \neq -1, -2, \dots \right].$$

EH I 274(13)a

$$7. \int_0^{\frac{\pi}{2}} \frac{\sin nx - \sin(nx - a \operatorname{tg} x)}{\sin x} \cos^{n-1} x \, dx = \begin{cases} \pi/2 & [n = 0, \quad a > 0], \\ \pi(1 - e^{-a}) & [n = 1, \quad a \geq 0]. \end{cases}$$

LO V 153(114)

3.719

$$1.^6 \int_0^\pi \sin(\nu x - z \sin x) \, dx = \pi \mathbf{E}_\nu(z).$$

WA 336(2)

$$2. \int_0^\pi \cos(nx - z \sin x) \, dx = \pi J_n(z).$$

WH

$$3. \int_0^\pi \cos(\nu x - z \sin x) \, dx = \pi \mathbf{J}_\nu(z).$$

WA 336(1)

3.72- 3.74 Combinations of trigonometric and rational functions

3.721

$$1. \int_0^\infty \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2} \operatorname{sign} a.$$

$$2. \int_1^{\infty} \frac{\sin(ax)}{x} dx = -\text{si}(a).$$

BI 203(1)

$$3. \int_1^{\infty} \frac{\cos(ax)}{x} dx = -\text{ci}(a).$$

BI 203(5)

3.722

$$1. \int_0^{\infty} \frac{\sin(ax)}{x + \beta} dx = \text{ci}(a\beta) \sin(a\beta) - \cos(a\beta) \text{si}(a\beta) \quad [|\arg \beta| < \pi, \quad a > 0].$$

BI((16))(1), FI II 646a

445

$$2.^7 \int_{-\infty}^{\infty} \frac{\sin(ax)}{x + \beta} dx = \pi e^{ia\beta} \quad [a > 0, \quad \text{Im } \beta > 0].$$

$$3. \int_0^{\infty} \frac{\cos(ax)}{x + \beta} dx = -\sin(a\beta) \text{si}(a\beta) - \cos(a\beta) \text{ci}(a\beta) \quad [|\arg \beta| < \pi, \quad a > 0].$$

ET I 8(7), BI((160))(2)

$$4.^* \int_{-\infty}^{\infty} \frac{\cos(ax)}{x + \beta} dx = -i\pi e^{iab} \quad [a > 0, \quad \text{Im } \beta > 0].$$

$$5.^* \int_0^{\infty} \frac{\sin(ax)}{\beta - x} dx = \sin(\beta a) \text{ci}(\beta a) - \cos(\beta a) [\text{si}(\beta a) + \pi] \\ [a > 0, \quad \beta \text{ not real and positive}].$$

FI II 646, BI((161))(1)

$$6.^8 \int_{-\infty}^{\infty} \frac{\sin(ax)}{\beta - x} dx = -\pi e^{ia\beta} \quad [a > 0, \quad \text{Im } \beta > 0].$$

$$7.* \int_0^{\infty} \frac{\cos(ax)}{\beta - x} dx = -\cos(a\beta) \operatorname{ci}(a\beta) + \sin(a\beta)[\operatorname{si}(a\beta) + \pi]$$

[$a > 0$, β not real and positive].

ET I 8(8), BI((161))(2)a

$$8.7 \int_{-\infty}^{\infty} \frac{\cos(ax)}{\beta - x} dx = -\pi e^{ia\beta} \quad [a > 0, \operatorname{Im} \beta > 0].$$

3.723

$$1.7 \int_0^{\infty} \frac{\sin(ax)}{\beta^2 + x^2} dx = \frac{1}{2\beta} [e^{-a\beta} \operatorname{Ei}(a\beta) - e^{a\beta} \operatorname{Ei}(-a\beta)] \quad [a > 0, \beta > 0].$$

ET I 65(14), BI((160))(3)

$$2. \int_0^{\infty} \frac{\cos(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2\beta} e^{-a\beta} \quad [a \geq 0, \operatorname{Re} \beta > 0].$$

FI II 741, 750, ET I 8(11), WH

$$3. \int_0^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \frac{\pi}{2} e^{-a\beta} \quad [a > 0, \operatorname{Re} \beta > 0].$$

FI II 741, 750, ET I 65(15), WH

$$4. \int_{-\infty}^{\infty} \frac{x \sin(ax)}{\beta^2 + x^2} dx = \pi e^{-a\beta} \quad [a > 0, \operatorname{Re} \beta > 0].$$

BI ((202))(10)

$$5.7 \int_0^{\infty} \frac{x \cos(ax)}{\beta^2 + x^2} dx = -\frac{1}{2} [e^{-a\beta} \operatorname{Ei}(a\beta) + e^{a\beta} \operatorname{Ei}(-a\beta)] \quad [a > 0, \beta > 0].$$

BI ((160))(6)

$$6. \int_{-\infty}^{\infty} \frac{\sin[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \sin(ab) \quad [a > 0, b > 0, c > 0].$$

LI ((202))(9)

$$7. \int_{-\infty}^{\infty} \frac{\cos[a(b-x)]}{c^2 + x^2} dx = \frac{\pi}{c} e^{-ac} \cos(ab) \quad [a > 0, b > 0, c > 0].$$

$$8. \int_0^{\infty} \frac{\sin(ax)}{\beta^2 - x^2} dx = \frac{1}{\beta} \left[\sin(a\beta) \operatorname{ci}(a\beta) - \cos(a\beta) \left(\operatorname{si}(a\beta) + \frac{\pi}{2} \right) \right] \quad [|\arg \beta| < \pi, \quad a > 0].$$

BI ((161))(3)

$$9. \int_0^{\infty} \frac{\cos(ax)}{b^2 - x^2} dx = \frac{\pi}{2b} \sin(ab) \quad [a > 0, \quad b > 0].$$

BI((161))(5), ET I 9(15)

$$10. \int_0^{\infty} \frac{x \sin(ax)}{b^2 - x^2} dx = -\frac{\pi}{2} \cos(ab) \quad [a > 0].$$

FI II 647, ET II 252(45)

$$11. \int_0^{\infty} \frac{x \cos(ax)}{\beta^2 + x^2} dx = \cos(a\beta) \operatorname{ci}(a\beta) + \sin(a\beta) \left[\operatorname{si}(a\beta) + \frac{\pi}{2} \right] \quad [|\arg \beta| < \pi, \quad a > 0].$$

BI ((161))(6)

$$12. \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x-b)} dx = \pi \frac{\cos(ab) - 1}{b} \quad [a > 0, \quad b > 0].$$

ET II 252(44)

3.724

$$1. \int_{-\infty}^{\infty} \frac{b + cx}{p + 2qx + x^2} \sin(ax) dx = \left(\frac{cq - b}{\sqrt{p - q^2}} \sin(aq) + c \cos(aq) \right) \pi e^{-a\sqrt{p - q^2}} \\ [a > 0, \quad p > q^2].$$

BI ((202))(12)

$$2. \int_{-\infty}^{\infty} \frac{b + cx}{p + 2qx + x^2} \cos(ax) dx = \left(\frac{b - cq}{\sqrt{p - q^2}} \cos(aq) + c \sin(aq) \right) \pi e^{-a\sqrt{p - q^2}} \\ [a > 0, \quad p > q^2].$$

BI ((202))(13)

3.725

$$1. \int_0^{\infty} \frac{\sin(ax) dx}{x(\beta^2 + x^2)} = \frac{\pi}{2\beta^2} (1 - e^{-a\beta}) \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((172))(1)

$$2. \int_0^{\infty} \frac{\sin(ax) dx}{x(b^2 - x^2)} = \frac{\pi}{2b^2} (1 - \cos(ab)) \quad [a > 0].$$

BI ((172))(4)

$$3. \int_0^{\infty} \frac{\sin(ax) \cos(bx)}{x(x^2 + \beta^2)} dx = \frac{\pi}{2\beta^2} e^{-\beta b} \operatorname{sh}(a\beta) \quad [0 < a < b]:$$

$$= -\frac{\pi}{2\beta^2} e^{-a\beta} \operatorname{ch}(b\beta) + \frac{\pi}{2\beta^2} \quad [a > b > 0].$$

ET I 19(4)

447

3.726

$$1.7 \int_0^{\infty} \frac{x \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} =$$

$$= \pm \frac{1}{4b} \left[e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab) - 2 \operatorname{ci}(ab) \sin(ab) + 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right] +$$

$$+ \frac{\pi e^{-ab} - \pi \cos(ab)}{4b} \quad [a > 0, \quad b > 0]$$

ET I 65(21)A, BI((176))(10, 13)

$$2.7 \int_0^{\infty} \frac{x^2 \sin(ax) dx}{b^3 \pm b^2 x + bx^2 \pm x^3} =$$

$$= \frac{1}{4} \left[e^{ab} \operatorname{Ei}(-ab) - e^{-ab} \operatorname{Ei}(ab) + 2 \operatorname{ci}(ab) \sin(ab) - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) \right] \pm$$

$$\pm \pi (e^{-ab} + \cos(ab)) \quad [a > 0, \quad b > 0]$$

ET I 66(22), BI((176))(11, 14)

3.727

$$1. \int_0^{\infty} \frac{\cos(ax) dx}{b^4 + x^4} = \frac{\pi\sqrt{2}}{4b^3} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} + \sin \frac{ab}{\sqrt{2}} \right) \quad [a > 0, \quad b > 0].$$

$$2.^8 \int_0^\infty \frac{\sin(ax) dx}{b^4 - x^4} = \frac{1}{4b^3} \left[2 \sin(ab) \operatorname{ci}(ab) - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) + e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab) \right] \quad [a > 0, \quad b > 0],$$

BI ((161))(12)

$$3. \int_0^\infty \frac{\cos(ax) dx}{b^4 - x^4} = \frac{\pi}{4b^3} [e^{-ab} + \sin(ab)] \quad [a > 0, \quad b > 0] \quad (\text{cf. 3.723 2. and 3.723 9.}).$$

3.723
BI ((161))(16)

$$4. \int_0^\infty \frac{x \sin(ax) dx}{b^4 + x^4} = \frac{\pi}{2b^2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \sin \frac{ab}{\sqrt{2}} \quad [a > 0, \quad b > 0].$$

BI ((160))(23)a

$$5. \int_0^\infty \frac{x \sin(ax) dx}{b^4 - x^4} = \frac{\pi}{4b^2} [e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0],$$

(cf. 3.723 3. and 3.723 10.).

3.723
BI ((161))(13)

$$6.^7 \int_0^\infty \frac{x \cos(ax) dx}{b^4 - x^4} = \frac{1}{4b^2} \left[2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) - e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab) \right] \quad [a > 0, \quad b > 0],$$

(cf. 3.723 5. and 3.723 11.).

3.723
BI ((161))(17)

$$7. \int_0^\infty \frac{x^2 \cos(ax) dx}{b^4 + x^4} = \frac{\pi\sqrt{2}}{4b} \exp\left(-\frac{ab}{\sqrt{2}}\right) \left(\cos \frac{ab}{\sqrt{2}} - \sin \frac{ab}{\sqrt{2}} \right) \quad [a > 0, \quad b > 0].$$

$$8.7 \int_0^{\infty} \frac{x^2 \sin(ax) dx}{b^4 - x^4} = \frac{1}{4b} [2 \sin(ab) \operatorname{ci}(ab) - 2 \cos(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) - e^{-ab} \operatorname{Ei}(ab) + e^{ab} \operatorname{Ei}(-ab)]$$

[a > 0, b > 0],

3.723
BI ((161))(14)

$$9. \int_0^{\infty} \frac{x^2 \cos(ax) dx}{b^4 - x^4} = \frac{\pi}{4b} (\sin(ab) - e^{-ab}) \quad [a > 0, b > 0],$$

BI ((161))(18)

$$10. \int_0^{\infty} \frac{x^3 \sin(ax) dx}{b^4 + x^4} = \frac{\pi}{2} \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}} \quad [a > 0, b > 0].$$

BI ((160))(24)

$$11. \int_0^{\infty} \frac{x^3 \sin(ax) dx}{b^4 - x^4} = \frac{-\pi}{4} [e^{-ab} - \cos(ab)] \quad [a > 0, b > 0],$$

BI ((161))(15)

$$12.7 \int_0^{\infty} \frac{x^3 \cos(ax) dx}{b^4 - x^4} = \frac{1}{4} \left[2 \cos(ab) \operatorname{ci}(ab) + 2 \sin(ab) \left(\operatorname{si}(ab) + \frac{\pi}{2} \right) + e^{-ab} \operatorname{Ei}(ab) + e^{ab} \operatorname{Ei}(-ab) \right] \quad [a > 0, b > 0],$$

3.723
BI((161))(19)

3.728

$$1. \int_0^{\infty} \frac{\cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(\beta e^{-a\gamma} - \gamma e^{-a\beta})}{2\beta\gamma(\beta^2 - \gamma^2)} \quad [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

BI ((175))(1)

$$2. \int_0^{\infty} \frac{x \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(e^{-a\beta} - e^{-a\gamma})}{2(\gamma^2 - \beta^2)} \quad [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

$$3. \int_0^{\infty} \frac{x^2 \cos(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(\beta e^{-a\beta} - \gamma e^{-a\gamma})}{2(\beta^2 - \gamma^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

BI ((175))(2)

$$4. \int_0^{\infty} \frac{x^3 \sin(ax) dx}{(\beta^2 + x^2)(\gamma^2 + x^2)} = \frac{\pi(\beta^2 e^{-a\beta} - \gamma^2 e^{-a\gamma})}{2(\beta^2 - \gamma^2)} \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

BI ((174))(2)

$$5. \int_0^{\infty} \frac{\cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(b \sin(ac) - c \sin(ab))}{2bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((175))(3)

$$6. \int_0^{\infty} \frac{x \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(\cos(ab) - \cos(ac))}{2(b^2 - c^2)} \quad [a > 0].$$

BI ((174))(3)

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$$7. \int_0^{\infty} \frac{x^2 \cos(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(c \sin(ac) - b \sin(ab))}{2(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((175))(4)

$$8. \int_0^{\infty} \frac{x^3 \sin(ax) dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(b^2 \cos(ab) - c^2 \cos(ac))}{2(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((174))(4)

3.729

$$1. \int_0^{\infty} \frac{\cos(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b^3} (1 + ab) e^{-ab} \quad [a > 0, \quad b > 0].$$

BI ((170))(7)

$$2. \int_0^{\infty} \frac{x \sin(ax) dx}{(b^2 + x^2)^2} = \frac{\pi}{4b} a e^{-ab} \quad [a > 0, \quad b > 0].$$

$$3. \int_0^{\infty} \cos(px) \frac{1-x^2}{(1+x^2)^2} dx = \frac{\pi p}{2} e^{-p}.$$

BI ((43))(10)a

$$4. \int_0^{\infty} \frac{x^3 \sin(ax) dx}{(b^2+x^2)^2} = \frac{\pi}{4} (2-ab)e^{-ab} \quad [a > 0, \quad b > 0].$$

BI ((170))(4)

3.731

Notations: $2A^2 = \sqrt{b^4 + c^2} + b^2$, $2B^2 = \sqrt{b^4 + c^2} - b^2$,

$$1. \int_0^{\infty} \frac{\cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} \frac{e^{-aA}(B \cos(aB) + A \sin(aB))}{\sqrt{b^4 + c^2}} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((176))(3)

$$2. \int_0^{\infty} \frac{x \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2c} e^{-aA} \sin(aB) \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((176))(1)

$$3. \int_0^{\infty} \frac{(x^2 + b^2) \cos(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} \frac{e^{-aA}(A \cos(aB) - B \sin(aB))}{\sqrt{b^4 + c^2}} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((176))(4)

$$4. \int_0^{\infty} \frac{x(x^2 + b^2) \sin(ax) dx}{(x^2 + b^2)^2 + c^2} = \frac{\pi}{2} e^{-aA} \cos(aB) \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((176))(2)

3.732

$$1. \int_0^{\infty} \left[\frac{1}{\beta^2 + (\gamma - x)^2} - \frac{1}{\beta^2 + (\gamma + x)^2} \right] \sin(ax) dx = \frac{\pi}{\beta} e^{-a\beta} \sin(a\gamma) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}].$$

ET I 65(16)

$$3. \int_0^\infty \left[\frac{\gamma + x}{\beta^2 + (\gamma + x)^2} - \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \sin(ax) dx = \pi e^{-a\beta} \cos(a\gamma)$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma + i\beta \text{ is not real}].$

LI ((175))(17)

$$4. \int_0^\infty \left[\frac{\gamma + x}{\beta^2 + (\gamma + x)^2} + \frac{\gamma - x}{\beta^2 + (\gamma - x)^2} \right] \cos(ax) dx = \pi e^{-a\beta} \sin(a\gamma) \quad [a > 0, \quad |\operatorname{Im} a| < \operatorname{Re} \beta].$$

LI ((176))(21)

3.733

$$1. \int_0^\infty \frac{\cos(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b^3} \exp(-ab \cos t) \frac{\sin(t + ab \sin t)}{\sin 2t}$$

$[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}].$

BI ((176))(7)

$$2. \int_0^\infty \frac{x \sin(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b^2} \exp(-ab \cos t) \frac{\sin(ab \sin t)}{\sin 2t} \quad [a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}].$$

BI((176))(5), ET I 66(23)

$$3. \int_0^\infty \frac{x^2 \cos(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2b} \exp(-ab \cos t) \frac{\sin(t - ab \sin t)}{\sin 2t}$$

$[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}].$

BI ((176))(8)

$$4. \int_0^\infty \frac{x^3 \sin(ax) dx}{x^4 + 2b^2x^2 \cos 2t + b^4} = \frac{\pi}{2} \exp(-ab \cos t) \frac{\sin(2t - ab \sin t)}{\sin 2t}$$

$[a > 0, \quad b > 0, \quad |t| < \frac{\pi}{2}].$

BI ((176))(6)

$$5. \int_0^\infty \frac{\sin(ax) dx}{x(x^4 + 2b^2x^2 \cos 2t + b^4)} = \frac{\pi}{2b^4} \left[1 - \exp(-ab \cos t) \frac{\sin(2t + ab \sin t)}{\sin 2t} \right]$$

3.734

$$1. \int_0^{\infty} \frac{\sin(ax) dx}{x(b^4 + x^4)} = \frac{\pi}{2b^4} \left[1 - \exp\left(-\frac{ab}{\sqrt{2}}\right) \cos \frac{ab}{\sqrt{2}} \right] \quad [a > 0, \quad b > 0].$$

BI ((172))(7)

$$2. \int_0^{\infty} \frac{\sin(ax) dx}{x(b^4 - x^4)} = \frac{\pi}{4b^4} [2 - e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0].$$

BI ((172))(10)

3.735

$$\int_0^{\infty} \frac{\sin(ax) dx}{x(b^2 + x^2)^2} = \frac{\pi}{2b^4} \left[1 - \frac{1}{2} e^{-ab}(2 + ab) \right] \quad [a > 0, \quad b > 0].$$

WH, BI ((172))(22)

451

3.736

$$1. \int_0^{\infty} \frac{\cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^5} [\sin(ab) + (2 + ab)e^{-ab}] \quad [a > 0, \quad b > 0]$$

(cf. 3.723 3. and 3.729 1.).

3.729

3.723
BI ((176))(5)

$$2. \int_0^{\infty} \frac{x \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^4} [(1 + ab)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0]$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723
BI ((174))(5)

3.729

3.723
BI ((175))(6)

$$4. \int_0^\infty \frac{x^3 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b^2} [(1 - ab)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0]$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723
BI ((174))(6)}

$$5. \int_0^\infty \frac{x^4 \cos(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8b} [\sin(ab) + (ab - 2)e^{-ab}] \quad [a > 0, \quad b > 0]$$

(cf. 3.723 2. and 9. and 3.729 1.).

3.729

3.723
BI ((175))(7)

$$6. \int_0^\infty \frac{x^5 \sin(ax) dx}{(b^2 + x^2)(b^4 - x^4)} = \frac{\pi}{8} [(ab - 3)e^{-ab} - \cos(ab)] \quad [a > 0, \quad b > 0],$$

(cf. 3.723 3. and 10. and 3.729 2.).

3.729

3.723
BI ((174))(7)

3.737

$$1. \int_0^\infty \frac{\cos(ax) dx}{(b^2 + x^2)^n} = \frac{\pi e^{-ab}}{(2b)^{2n-1} (n-1)!} \sum_{k=0}^{n-1} \frac{(2n-k-2)! (2ab)^k}{k! (n-k-1)!};$$

$$2. \int_0^\infty \frac{x \sin(ax) dx}{(x^2 + \beta^2)^{n+1}} = \frac{\pi a e^{-a\beta}}{2^{2n} n! \beta^{2n-1}} \sum_{k=0}^{n-1} \frac{(2n-k-2)!(2a\beta)^k}{k!(n-k-1)!} \quad [a > 0, \quad \operatorname{Re} \beta > 0].$$

GW ((333))(66c)

$$3. \int_0^\infty \frac{\sin(ax) dx}{x(\beta^2 + x^2)^{n+1}} = \frac{\pi}{2\beta^{2n+2}} \left[1 - \frac{e^{-a\beta}}{2^n n!} F_n(a\beta) \right]$$

$[a > 0, \quad \operatorname{Re} \beta > 0, \quad F_0(z) = 1, \quad F_1(z) = z + 2, \dots, F_n(z) = (z + 2n)F_{n-1}(z) - zF'_{n-1}(z)]$.

GW ((333))(66e)

452

$$4. \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^3} = \frac{\pi a}{16b^3} (1 + ab)e^{-ab} \quad [a > 0, \quad b > 0].$$

BI((170))(5), ET I 67(35)a

$$5. \int_0^\infty \frac{x \sin(ax) dx}{(b^2 + x^2)^4} = \frac{\pi a}{96b^5} (3 + 3ab + a^2b^2)e^{-ab} \quad [a > 0, \quad b > 0].$$

BI((170))(6), ET I 67(35)a

3.738

$$1. \int_0^\infty \frac{x^{m-1} \sin(ax)}{x^{2n} + \beta^{2n}} dx = -\frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[-a\beta \sin \frac{(2k-1)\pi}{2n} \right] \times$$

$$\times \cos \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \quad [m \text{ is even};$$

$$\left[a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m \leq 2n \right].$$

ET I 67(38)

$$2. \int_0^\infty \frac{x^{m-1} \cos(ax)}{x^{2n} + \beta^{2n}} dx = \frac{\pi \beta^{m-2n}}{2n} \sum_{k=1}^n \exp \left[-a\beta \sin \frac{(2k-1)\pi}{2n} \right] \times$$

$$\times \sin \left\{ \frac{(2k-1)m\pi}{2n} + a\beta \cos \frac{(2k-1)\pi}{2n} \right\} \quad [m \text{ is odd};$$

$$\left[a > 0, \quad |\arg \beta| < \frac{\pi}{2n}, \quad 0 < m < 2n + 1 \right].$$

$$1. \int_0^{\infty} \frac{\sin(ax) dx}{x(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} = \frac{\pi(-1)^n}{(2n)!2^{2n+1}} \left[2 \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} e^{2(k-n)a} + (-1)^n \binom{2n}{n} \right] \quad [a > 0, \quad n \geq 0].$$

LI ((174))(8)

$$2. \int_0^{\infty} \frac{\cos(ax) dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} = \frac{(-1)^n}{(2n+1)!} \frac{\pi}{2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} e^{(2k-2n-1)a} \quad [a \geq 0, \quad n \geq 0]$$

$$= \frac{\pi 2^{-2n-1}}{(2n+1)(n!)^2} \quad [a = 0, \quad n \geq 0].$$

BI((175))(8)

$$3. \int_0^{\infty} \frac{x \sin(ax) dx}{(x^2+1^2)(x^2+3^2)\dots[x^2+(2n+1)^2]} = \frac{\pi(-1)^n}{(2n+1)!2^{2n+1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n-2k+1) e^{(2k-2n-1)a} \quad [a > 0, \quad n \geq 0].$$

LI ((174))(9)

453

$$4. \int_0^{\infty} \frac{\cos ax dx}{(x^2+2^2)(x^2+4^2)\dots(x^2+4n^2)} = \frac{\pi 2^{1-2n}}{(2n)!} \sum_{k=1}^n (-1)^k k \binom{2n}{n-k} e^{-2ak} \quad [n \geq 1, \quad a \geq 0].$$

3.741

$$1. \int_0^{\infty} \frac{\sin(ax) \sin(bx)}{x} dx = \frac{1}{4} \ln \left(\frac{a+b}{a-b} \right)^2 \quad [a > 0, \quad b > 0, \quad a \neq b].$$

FI II 647

$$2. \int_0^{\infty} \frac{\sin(ax) \cos(bx)}{x} dx = \frac{\pi}{2} \quad [a > b \geq 0];$$

$$= \frac{\pi}{4} \quad [a = b > 0];$$

$$= 0 \quad [b > a \geq 0].$$

FI II 645

$$3. \int_0^{\infty} \frac{\sin(ax) \sin(bx)}{x^2} dx = \frac{a\pi}{2} \quad [0 < a \leq b];$$

$$= \frac{b\pi}{2} \quad [0 < b \leq a].$$

$$1. \int_0^{\infty} \frac{\sin(ax) \sin(bx)}{\beta^2 + x^2} dx = \frac{\pi}{4\beta} (e^{-|a-b|\beta} - e^{-(a+b)\beta}) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0].$$

BI((162))(1)A, GW((333))(71a)

$$2. \int_0^{\infty} \frac{\sin(ax) \cos(bx)}{\beta^2 + x^2} dx = \frac{1}{4\beta} e^{-a\beta} \{e^{b\beta} \operatorname{Ei}[\beta(a-b)] + e^{-b\beta} \operatorname{Ei}[\beta(a+b)]\} - \\ - \frac{1}{4\beta} e^{a\beta} \{e^{b\beta} \operatorname{Ei}[-\beta(a+b)] + e^{-b\beta} \operatorname{Ei}[\beta(b-a)]\}.$$

BI ((162))(3)

$$3. \int_0^{\infty} \frac{\cos(ax) \cos(bx)}{\beta^2 + x^2} dx = \frac{\pi}{4\beta} [e^{-|a-b|\beta} + e^{-(a+b)\beta}] \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0].$$

BI((163))(1)A, GW((333))(71c)

$$4. \int_0^{\infty} \frac{x \cos(ax) \cos(bx)}{\beta^2 + x^2} dx = -\frac{1}{4} e^{a\beta} \{e^{b\beta} \operatorname{Ei}[-\beta(a+b)] + e^{-b\beta} \operatorname{Ei}[\beta(b-a)]\} - \\ - \frac{1}{4} e^{-a\beta} \{e^{b\beta} \operatorname{Ei}[\beta(a-b)] + e^{-b\beta} \operatorname{Ei}[\beta(a+b)]\} \quad [a \neq b]; \\ = \infty \quad [a = b].$$

BI ((163))(2)

$$5. \int_0^{\infty} \frac{x \sin(ax) \cos(bx)}{x^2 + \beta^2} dx = \frac{\pi}{2} e^{-a\beta} \operatorname{ch}(b\beta) \quad [0 < b < a]; \\ = \frac{\pi}{4} e^{-2a\beta} \quad [0 < b = a]; \\ = -\frac{\pi}{2} e^{-b\beta} \operatorname{sh}(a\beta) \quad [0 < a < b].$$

BI ((162))(4)

454

$$6. \int_0^{\infty} \frac{\sin(ax) \sin(bx)}{p^2 - x^2} dx = -\frac{\pi}{2p} \cos(ap) \sin(bp) \quad [a > b > 0]; \\ = -\frac{\pi}{4p} \sin(2ap) \quad [a = b > 0]; \\ = -\frac{\pi}{2p} \sin(ap) \cos(bp) \quad [b > a > 0].$$

BI ((166))(1)

BI ((166))(1)

$$\begin{aligned}
7. \int_0^\infty \frac{\sin(ax) \cos(bx)}{p^2 - x^2} x dx &= -\frac{\pi}{2} \cos(ap) \cos(bp) \quad [a > b > 0]; \\
&= -\frac{\pi}{4} \cos(2ap) \quad [a = b > 0]; \\
&= \frac{\pi}{2} \sin(ap) \sin(bp) \quad [b > a > 0].
\end{aligned}$$

BI ((166))(2)

$$\begin{aligned}
8. \int_0^\infty \frac{\cos(ax) \cos(bx)}{p^2 - x^2} dx &= \frac{\pi}{2p} \sin(ap) \cos(bp) \quad [a > b > 0]; \\
&= \frac{\pi}{4p} \sin(2ap) \quad [a = b > 0]; \\
&= \frac{\pi}{2p} \cos(ap) \sin(bp) \quad [b > a > 0].
\end{aligned}$$

BI ((166))(3)

3.743

$$1. \int_0^\infty \frac{\sin(ax)}{\sin(bx)} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi}{2\beta} \cdot \frac{\text{sh}(a\beta)}{\text{sh}(b\beta)} \quad [0 < a < b, \quad \text{Re } \beta > 0].$$

ET I 80(21)

$$2. \int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{x dx}{x^2 + \beta^2} = -\frac{\pi}{2} \cdot \frac{\text{sh}(a\beta)}{\text{ch}(b\beta)} \quad [0 < a < b, \quad \text{Re } \beta > 0].$$

ET I 81(30)

$$3. \int_0^\infty \frac{\cos(ax)}{\sin(bx)} \cdot \frac{x dx}{x^2 + \beta^2} = \frac{\pi}{2} \cdot \frac{\text{ch}(a\beta)}{\text{sh}(b\beta)} \quad [0 < a < b, \quad \text{Re } \beta > 0].$$

ET I 23(37)

$$4. \int_0^\infty \frac{\cos(ax)}{\cos(bx)} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi}{2\beta} \cdot \frac{\text{ch}(a\beta)}{\text{ch}(b\beta)} \quad [0 < a < b, \quad \text{Re } \beta > 0].$$

ET I 23(36)

$$\begin{aligned}
5.^6 \text{ P.V. } \int_0^\infty \frac{\sin(ax)}{\sin x} \cdot \frac{dx}{b^2 - x^2} &= 0 \quad \text{if } 0 \leq a \leq 1 \\
&= \frac{\pi}{b} \sin(a-1)b \quad \text{if } 1 \leq a \leq 2 \quad [b \text{ real, } b/\pi \notin \mathbb{Z}].
\end{aligned}$$

$$\int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(x^2 + \beta^2)} = \frac{\pi}{2\beta^2} \cdot \frac{\text{sh}(a\beta)}{\text{ch}(b\beta)} \quad [0 < a < b, \quad \text{Re } \beta > 0].$$

ET I 82(32)

3.745

$$\int_0^\infty \frac{\sin(ax)}{\cos(bx)} \cdot \frac{dx}{x(c^2 - x^2)} = 0 \quad [0 < a < b, \quad c > 0].$$

ET I 82(31)

455

3.746

$$1. \int_0^\infty \frac{dx}{x^{n+1}} \prod_{k=0}^n \sin(a_k x) = \frac{\pi}{2} \prod_{k=1}^n a_k \quad \left[a_0 > \sum_{k=1}^n a_k, \quad a_k > 0 \right].$$

FI II 646

$$2. \int_0^\infty \frac{\sin(ax)}{x^{n+1}} dx \prod_{k=1}^n \sin(a_k x) \prod_{j=1}^m \cos(b_j x) = \frac{\pi}{2} \prod_{k=1}^n a_k \quad \left[a > \sum_{k=1}^n |a_k| + \sum_{j=1}^m |b_j| \right].$$

WH

3.747

$$1.^7 \int_0^{\frac{\pi}{2}} \frac{x^m}{\sin x} dx = \left(\frac{\pi}{2}\right)^m \left[\frac{1}{m} + \sum_{k=1}^{\infty} \frac{2^{2k-1} - 1}{4^{2k-1}(m+2k)} \zeta(2k) \right] = 2\pi \mathbf{G} - \frac{7}{2} \zeta(3). \quad [m = 2].$$

LI ((206))(2)

$$2. \int_0^{\frac{\pi}{2}} \frac{x dx}{\sin x} = \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2} - x) dx}{\cos x} = 2\mathbf{G}.$$

BI((204))(18)

$$3. \int_0^\infty \frac{x dx}{(x^2 + b^2) \sin(ax)} = \frac{\pi}{2 \text{sh}(ab)} \quad [b > 0].$$

GW ((333))(79c)

BI ((218))(4)

$$5. \int_0^{\frac{\pi}{2}} x \operatorname{tg} x \, dx = \infty.$$

BI ((205))(2)

$$6. \int_0^{\frac{\pi}{4}} x \operatorname{tg} x \, dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} = 0.1857845358 \dots$$

BI ((204))(1)

$$7. \int_0^{\frac{\pi}{2}} x \operatorname{ctg} x \, dx = \frac{\pi}{2} \ln 2.$$

FI II 623

$$8. \int_0^{\frac{\pi}{4}} x \operatorname{ctg} x \, dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G} = 0.7301810584 \dots$$

BI ((204))(2)

$$9. \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \operatorname{tg} x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \operatorname{tg} x \, dx = \frac{\pi}{2} \ln 2.$$

GW((333))(33b), BI((218))(12)

$$10. \int_0^{\infty} \operatorname{tg} ax \frac{dx}{x} = \frac{\pi}{2} \quad [a > 0].$$

LO V 279(5)

$$11. \int_0^{\frac{\pi}{2}} \frac{x \operatorname{ctg} x}{\cos 2x} \, dx = \frac{\pi}{4} \ln 2.$$

BI ((206))(12)

3.748

$$1. \int_0^{\frac{\pi}{4}} x^m \operatorname{tg} x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \sum_{k=1}^{\infty} \frac{(4^k - 1)\zeta(2k)}{4^{2k-1}(m+2k)}.$$

$$2. \int_0^{\frac{\pi}{2}} x^p \operatorname{ctg} x \, dx = \left(\frac{\pi}{2}\right)^p \left\{ \frac{1}{p} - 2 \sum_{k=1}^{\infty} \frac{1}{4^k (p+2k)} \zeta(2k) \right\}.$$

LI ((205))(7)

$$3. \int_0^{\frac{\pi}{4}} x^m \operatorname{ctg} x \, dx = \frac{1}{2} \left(\frac{\pi}{4}\right)^m \left[\frac{2}{m} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1} (m+2k)} \right].$$

LI ((204))(6)

3.749

$$1. \int_0^{\infty} \frac{x \operatorname{tg}(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} + 1} \quad [a > 0, \quad b > 0].$$

GW ((333))(79a)

$$2. \int_0^{\infty} \frac{x \operatorname{ctg}(ax) \, dx}{x^2 + b^2} = \frac{\pi}{e^{2ab} - 1} \quad [a > 0, \quad b > 0].$$

GW ((333))(79b)

$$3. \int_0^{\infty} \frac{x \operatorname{tg}(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \operatorname{ctg}(ax) \, dx}{b^2 - x^2} = \int_0^{\infty} \frac{x \operatorname{cosec}(ax) \, dx}{b^2 - x^2} = \infty.$$

BI ((161))(7, 8, 9)

3.75 Combinations of trigonometric and algebraic functions

3.751

$$1. \int_0^{\infty} \frac{\sin(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} [\cos(a\beta) - \sin(a\beta) + 2C(\sqrt{a\beta}) \sin(a\beta) - 2S(\sqrt{a\beta}) \cos(a\beta)]$$

$$[a > 0, \quad |\arg \beta| < \pi].$$

ET I 65(12)a

$$2.* \int_0^{\infty} \frac{\cos(ax) \, dx}{\sqrt{x+\beta}} = \sqrt{\frac{\pi}{2a}} [\cos(a\beta) + \sin(a\beta) - 2C(\sqrt{a\beta}) \cos(a\beta) - 2S(\sqrt{a\beta}) \sin(a\beta)]$$

$$[a > 0, \quad |\arg \beta| < \pi].$$

$$3. \int_u^\infty \frac{\sin(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\sin(au) + \cos(au)] \quad [a > 0, \quad u > 0].$$

ET I 65(13)

$$4. \int_u^\infty \frac{\cos(ax)}{\sqrt{x-u}} dx = \sqrt{\frac{\pi}{2a}} [\cos(au) - \sin(au)] \quad [a > 0, \quad u > 0].$$

ET I 8(10)

3.752

$$1.^8 \int_0^1 \sin(ax) \sqrt{1-x^2} dx = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{(2k-1)!!(2k+3)!!} = \frac{\pi}{2a} \mathbf{H}_i, (a) \quad [a > 0].$$

BI ((149))(6)

$$2. \int_0^1 \cos(ax) \sqrt{1-x^2} dx = \frac{\pi}{2a} J_1(a).$$

KU 65(6)a

3.753

$$1.^8 \int_0^1 \frac{\sin(ax) dx}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{(-1)^k a^{2k+1}}{[(2k+1)!!]^2} = \frac{\pi}{2} \mathbf{H}_0, (a) [a > 0].$$

BI ((149))(9)

457

$$2. \int_0^1 \frac{\cos(ax) dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_0(a).$$

WA 30(7)a

$$3. \int_1^\infty \frac{\sin(ax) dx}{\sqrt{x^2-1}} = \frac{\pi}{2} J_0(a). \quad [a > 0].$$

WA 200(14)

$$4. \int_1^\infty \frac{\cos(ax) dx}{\sqrt{x^2-1}} = -\frac{\pi}{2} N_0(a).$$

$$5. \int_0^1 \frac{x \sin(ax)}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_1(a) \quad [a > 0].$$

WA 30(6)

3.754

$$1. \int_0^\infty \frac{\sin(ax) dx}{\sqrt{\beta^2 + x^2}} = \frac{\pi}{2} [I_0(a\beta) - \mathbf{L}_0(a\beta)] \quad [a > 0, \operatorname{Re} \beta > 0].$$

ET I 66(26)

$$2. \int_0^\infty \frac{\cos(ax) dx}{\sqrt{\beta^2 + x^2}} = K_0(a\beta) \quad [a > 0, \operatorname{Re} \beta > 0].$$

WA 191(1), GW((333))(78a)

$$3. \int_0^\infty \frac{x \sin(ax)}{\sqrt{(\beta^2 + x^2)^3}} dx = aK_0(a\beta) \quad [a > 0, \operatorname{Re} \beta > 0].$$

ET I 66(27)

3.755

$$1. \int_0^\infty \frac{\sqrt{\sqrt{x^2 + \beta^2} - \beta} \sin(ax) dx}{\sqrt{x^2 + \beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0].$$

ET I 66(31)

$$2. \int_0^\infty \frac{\sqrt{\sqrt{x^2 + \beta^2} + \beta} \cos(ax) dx}{\sqrt{x^2 + \beta^2}} = \sqrt{\frac{\pi}{2a}} e^{-a\beta} \quad [a > 0, \operatorname{Re} \beta > 0].$$

ET I 10(25)

3.756

$$1. \int_0^\infty \frac{\sin(ax)}{x^{\frac{n}{2}-1}} \prod_{k=2}^n \sin(a_k x) dx = 0 \quad \left[a_k > 0, \quad a > \sum_{k=2}^n a_k \right]$$

ET I 80(22)

$$2. \int_0^\infty x^{\frac{n}{2}-1} \cos(ax) \prod_{k=1}^n \cos(a_k x) dx = 0 \quad \left[a_k > 0, \quad a > \sum_{k=1}^n a_k \right].$$

3.757

$$1. \int_0^\infty \frac{\sin(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}}.$$

BI ((177))(1)

$$2. \int_0^\infty \frac{\cos(ax)}{\sqrt{x}} dx = \sqrt{\frac{\pi}{2a}}.$$

BI ((177))(2)

458

3.76- 3.77 Combinations of trigonometric functions and powers

3.761

$$1. \int_0^1 x^{\mu-1} \sin(ax) dx = \frac{-i}{2\mu} [{}_1F_1(\mu; \mu+1; ia) - {}_1F_1(\mu; \mu+1; -ia)]$$

$$[a > 0, \quad \operatorname{Re} \mu > -1, \quad \mu \neq 0].$$

ET I 68(2)a

$$2. \int_u^\infty x^{\mu-1} \sin x dx = \frac{i}{2} [e^{-\frac{\pi}{2}iu} \Gamma(\mu, iu) - e^{\frac{\pi}{2}iu} \Gamma(\mu, -iu)] \quad [\operatorname{Re} \mu > -1].$$

EH II 149(2)

$$3. \int_1^\infty \frac{\sin(ax)}{x^{2n}} dx = \frac{a^{2n-1}}{(2n-1)!} \left[\sum_{k=1}^{2n-1} \frac{(2n-k-1)!}{a^{2n-k}} \sin\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^n \operatorname{ci}(a) \right]$$

$$[a > 0].$$

LI ((203))(15)

$$4. \int_0^\infty x^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin \frac{\mu\pi}{2} = \frac{\pi \sec \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)} \quad [a > 0; \quad 0 < |\operatorname{Re} \mu| < 1].$$

FI II 809a, BI((150))(1)

$$5. \int_0^\pi x^m \sin(nx) dx = \frac{(-1)^{n+1}}{n^{m+1}} \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k)!} (n\pi)^{m-2k} -$$

$$- (-1)^{[m/2]} \frac{m! [m - 2E[m/2] - 1]}{n^{m+1}}$$

$$6.7 \int_0^1 x^{\mu-1} \cos(ax) dx = \frac{1}{2\mu} [{}_1F_1(\mu; \mu+1; ia) + {}_1F_1(\mu, \mu+1; -ia)] \quad [a > 0, \operatorname{Re} \mu > 0].$$

ET I 11(2)

$$7. \int_u^\infty x^{\mu-1} \cos x dx = \frac{1}{2} [e^{-\frac{\pi}{2}i\mu} \Gamma(\mu, iu) + e^{\frac{\pi}{2}i\mu} \Gamma(\mu, -iu)] \quad [\operatorname{Re} \mu < 1].$$

EH II 149(1)

$$8. \int_1^\infty \frac{\cos(ax)}{x^{2n+1}} dx = \frac{a^{2n}}{(2n)!} \left[\sum_{k=1}^{2n} \frac{(2n-k)!}{a^{2n-k+1}} \cos\left(a + (k-1)\frac{\pi}{2}\right) + (-1)^{n+1} \operatorname{ci}(a) \right] \quad [a > 0].$$

LI ((203))(16)

$$9. \int_0^\infty x^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos \frac{\mu\pi}{a} = \frac{\pi \operatorname{cosec} \frac{\mu\pi}{2}}{2a^\mu \Gamma(1-\mu)} \quad [a > 0, 0 < \operatorname{Re} \mu < 1].$$

FI II 809a, BI((150))(2)

$$10. \int_0^\pi x^m \cos(nx) dx = \frac{(-1)^n}{n^{m+1}} \sum_{k=0}^{[(m-1)/2]} (-1)^k \frac{m!}{(m-2k-1)!} (n\pi)^{m-2k-1} + (-1)^{[(m+1)/2]} \frac{2[(m+1)/2] - m}{n^{m+1}} \cdot m!$$

GW ((333))(7)

459

$$11. \int_0^{\frac{\pi}{2}} x^m \cos x dx = \sum_{k=0}^{[m/2]} (-1)^k \frac{m!}{(m-2k)!} \left(\frac{\pi}{2}\right)^{m-2k} + (-1)^{[m/2]} (2[m/2] - m) m!.$$

GW ((333))(9c)

$$12. \int_0^{2n\pi} x^m \cos kx dx = - \sum_{j=0}^{m-1} \frac{j!}{k^{j+1}} \binom{m}{j} (2n\pi)^{m-j} \cos \frac{j+1}{2} \pi.$$

BI ((226))(2)

3.762

$$1. \int_0^{\infty} x^{\mu-1} \sin(ax) \sin(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) [|b-a|^{-\mu} - (b+a)^{-\mu}]$$

$$[a > 0, \quad b > 0, \quad a \neq b, \quad -2 < \operatorname{Re} \mu < 1]$$

(for $\mu = 0$, see 3.741 1., for $\mu = -1$, see 3.741 3.).

3.741

BI((149))(7), ET I 321(40)

$$2. \int_0^{\infty} x^{\mu-1} \sin(ax) \cos(bx) dx = \frac{1}{2} \sin \frac{\mu\pi}{2} \Gamma(\mu) [(a+b)^{-\mu} + |a-b|^{-\mu} \operatorname{sign}(a-b)]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1] \quad (\text{for } \mu = 0 \text{ see 3.741 2.}).$$

3.741

BI((159))(8)A, ET I 321(41)

$$3. \int_0^{\infty} x^{\mu-1} \cos(ax) \cos(bx) dx = \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) [(a+b)^{-\mu} + |a-b|^{-\mu}]$$

$$[a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

ET I 20(17)

3.763

$$1. \int_0^{\infty} \frac{\sin(ax) \sin(bx) \sin(cx)}{x^{\nu}} dx =$$

$$= \frac{1}{4} \cos \frac{\nu\pi}{2} \Gamma(1-\nu) [(c+a-b)^{\nu-1} - (c+a+b)^{\nu-1} -$$

$$- |c-a+b|^{\nu-1} \operatorname{sign}(a-b-c) + |c-a-b|^{\nu-1} \operatorname{sign}(a+b-c)]$$

$$[c > 0, \quad 0 < \operatorname{Re} \nu < 4, \quad \nu \neq 1, 2, 3, \quad a \geq b > 0].$$

GW((333))(26a)A, ET I 79(13)

$$2. \int_0^{\infty} \frac{\sin(ax) \sin(bx) \sin(cx)}{x} dx = 0 \quad [c < a-b \quad \text{and} \quad c > a+b];$$

$$= \frac{\pi}{8} \quad [c = a-b \quad \text{and} \quad c = a+b]; \quad [a \geq b > 0, \quad c > 0].$$

$$= \frac{\pi}{4} [a-b < c < a+b].$$

FI II 645

460

$$4. \int_0^{\infty} \frac{\sin(ax) \sin(bx) \sin(cx)}{x^3} dx = \frac{\pi bc}{2} \quad [0 < c < a - b \quad \text{and} \quad c > a + b];$$

$$= \frac{\pi bc}{2} - \frac{\pi(a-b-c)^2}{8} \quad [a - b < c < a + b];$$

$$[a \geq b > 0, \quad c > 0].$$

BI((157))(20), ET I 79(12)

3.764

$$1. \int_0^{\infty} x^p \sin(ax+b) dx = \frac{1}{a^{p+1}} \Gamma(1+p) \cos\left(b + \frac{p\pi}{2}\right) \quad [a > 0, \quad -1 < p < 0].$$

GW ((333))(30a)

$$2. \int_0^{\infty} x^p \cos(ax+b) dx = -\frac{1}{a^{p+1}} \Gamma(1+p) \sin\left(b + \frac{p\pi}{2}\right) \quad [a > 0, \quad -1 < p < 0].$$

GW ((333))(30b)

3.765

$$1. \int_0^{\infty} \frac{\sin(ax) dx}{x^\nu(x+\beta)} = \frac{i}{2\beta^\nu} \Gamma(1-\nu) [e^{-ia\beta} \Gamma(\nu, -ia\beta) - e^{ia\beta} \Gamma(\nu, ia\beta)]$$

$$[a > 0, \quad -1 < \operatorname{Re} \nu < 2, \quad |\arg \beta| < \pi].$$

ET I 219(34)

$$2. \int_0^{\infty} \frac{\cos(ax) dx}{x^\nu(x+\beta)} = \frac{\Gamma(1-\nu)}{2\beta^\nu} [e^{ia\beta} \Gamma(\nu, ia\beta) + e^{-ia\beta} \Gamma(\nu, -ia\beta)]$$

$$[a > 0, \quad |\operatorname{Re} \nu| < 1, \quad |\arg \beta| < \pi].$$

ET II 221(52)

3.766

$$1. \int_0^{\infty} \frac{x^{\mu-1} \sin(ax)}{1+x^2} dx = \frac{\pi}{2} \sec \frac{\mu\pi}{2} \operatorname{sh} a +$$

$$+ \frac{1}{2} \sin \frac{\mu\pi}{2} \Gamma(\mu) \{ \exp[-a + i\pi(1-\mu)] \gamma(1-\mu, -a) - e^a \gamma(1-\mu, a) \}$$

$$[a > 0, \quad -1 < \operatorname{Re} \mu < 3].$$

$$2. \int_0^{\infty} \frac{x^{\mu-1} \cos(ax)}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec} \frac{\mu\pi}{2} \operatorname{ch} a + \frac{1}{2} \cos \frac{\mu\pi}{2} \Gamma(\mu) \{ \exp[-a + i\pi(1-\mu)] \gamma(1-\mu, -a) - e^a \gamma(1-\mu, a) \} \\ [a > 0, \quad 0 < \operatorname{Re} \mu < 3].$$

ET I 319(24)

$$3. \int_0^{\infty} \frac{x^{2\mu+1} \sin(ax)}{x^2 + b^2} dx = -\frac{\pi}{2} b^{2\mu} \sec(\mu\pi) \operatorname{sh}(ab) + \frac{\sin(\mu\pi)}{2a^{2\mu}} \Gamma(2\mu) [{}_1F_1(1; 1-2\mu; ab) + {}_1F_1(1; 1-2\mu; -ab)] \\ \left[a > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2} \right].$$

ET II 220(39)

461

$$4. \int_0^{\infty} \frac{x^{2\mu+1} \cos(ax)}{x^2 + b^2} dx = -\frac{\pi}{2} b^{2(\mu+\frac{1}{2})} \operatorname{cosec}[(\mu + \frac{1}{2})\pi] \operatorname{ch}(ab) + \frac{\cos[(\mu + \frac{1}{2})\pi]}{2a^{2(\mu+\frac{1}{2})}} \Gamma[2(\mu + \frac{1}{2})] [{}_1F_1(1; 1-2(\mu + \frac{1}{2}); ab) + {}_1F_1(1; 1-2(\mu + \frac{1}{2}); -ab)] \\ \left[a > 0, \quad -1 < \operatorname{Re} \mu < \frac{1}{2} \right].$$

ET II 221(56)

3.767

$$1. \int_0^{\infty} \frac{x^{\beta-1} \sin\left(ax - \frac{\beta\pi}{2}\right)}{\gamma^2 + x^2} dx = -\frac{\pi}{2} \gamma^{\beta-2} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad 0 < \operatorname{Re} \beta < 2].$$

BI ((160))(20)

$$2. \int_0^{\infty} \frac{x^{\beta} \cos\left(ax - \frac{\beta\pi}{2}\right)}{\gamma^2 + x^2} dx = \frac{\pi}{2} \gamma^{\beta-1} e^{-a\gamma} \quad [a > 0, \quad \operatorname{Re} \gamma > 0, \quad |\operatorname{Re} \beta| < 1].$$

BI ((160))(21)

$$3. \int_0^{\infty} \frac{x^{\beta-1} \sin\left(ax - \frac{\beta\pi}{2}\right)}{x^2 - b^2} dx = \frac{\pi}{2} b^{\beta-2} \cos\left(ab - \frac{\pi\beta}{2}\right) \quad [a > 0, \quad b > 0, \quad 0 < \operatorname{Re} \beta < 2].$$

$$4. \int_0^\infty \frac{x^\beta \cos\left(ax - \frac{\beta\pi}{2}\right)}{x^2 - b^2} dx = -\frac{\pi}{2} b^{\beta-1} \sin\left(ab - \frac{\pi\beta}{2}\right) \quad [a > 0, \quad b > 0, \quad |\beta| < 1].$$

GW ((333))(82)

3.768

$$1. \int_u^\infty (x-u)^{\mu-1} \sin(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \sin\left(au + \frac{\mu\pi}{2}\right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

ET II 203(19)

$$2. \int_u^\infty (x-u)^{\mu-1} \cos(ax) dx = \frac{\Gamma(\mu)}{a^\mu} \cos\left(au + \frac{\mu\pi}{2}\right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

ET II 204(24)

$$3.^* \int_0^1 (1-x)^\nu \sin(ax) dx = \frac{1}{a} - \frac{\Gamma(\nu+1)}{a^{\nu+1}} C_\nu(a) = a^{-\nu-1/2} s_{\nu+1/2, 1/2}(a)$$

with $C_\nu(a)$ the Young's function

$$C_\nu(a) = \frac{\frac{1}{2}a^\nu}{\Gamma(\nu+1)} [{}_1F_1(1; \nu+1; ia) + {}_1F_1(1; \nu+1; -ia)]$$

$$= \sum_{n=0}^{\infty} (-1)^n a^{\nu+2n} / \Gamma(\nu+2n+1) \quad [a > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 11(3)a

462

$$4.^3 \int_0^1 (1-x)^\nu \cos(ax) dx = \frac{i}{2} a^{-\nu-1} \left\{ \exp\left[\frac{i}{2}(\nu\pi - 2a)\right] \gamma(\nu+1, -ia) - \exp\left[-\frac{i}{2}(\nu\pi - 2a)\right] \gamma(\nu+1, ia) \right\} =$$

$$= \Gamma(\nu+1) \sum_{n=0}^{\infty} (-a^2)^n / \Gamma(\nu+2+2n) \quad [a > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 11(3)a

$$5. \int_0^u x^{\nu-1} (u-x)^{\mu-1} \sin(ax) dx = \frac{u^{\mu+\nu-1}}{2i} B(\mu, \nu) [{}_1F_1(\nu; \mu+\nu; iau) - {}_1F_1(\nu; \mu+\nu; -iau)]$$

$$[a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0].$$

ET II 189(26)

ET II 189(32)

$$7. \int_0^u x^{\mu-1}(u-x)^{\mu-1} \sin(ax) dx = \sqrt{\pi} \left(\frac{u}{a}\right)^{\mu-1/2} \sin \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left(\frac{au}{2}\right) \quad [\operatorname{Re} \mu > 0].$$

ET II 189(25)

$$8. \int_u^\infty x^{\mu-1}(x-u)^{\mu-1} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{u}{a}\right)^{\mu-1/2} \Gamma(\mu) \left[\cos \frac{au}{2} J_{1/2-\mu} \left(\frac{au}{2}\right) - \sin \frac{au}{2} N_{1/2-\mu} \left(\frac{au}{2}\right) \right] \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \right].$$

ET II 203(20)

$$9. \int_0^u x^{\mu-1}(u-x)^{\mu-1} \cos(ax) dx = \sqrt{\pi} \left(\frac{u}{a}\right)^{\mu-1/2} \cos \frac{au}{2} \Gamma(\mu) J_{\mu-1/2} \left(\frac{au}{2}\right) \quad [\operatorname{Re} \mu > 0].$$

ET II 189(31)

$$10. \int_u^\infty x^{\mu-1}(x-u)^{\mu-1} \cos(ax) dx = -\frac{\sqrt{\pi}}{2} \left(\frac{u}{a}\right)^{\mu-1/2} \Gamma(\mu) \left[\sin \frac{au}{2} J_{1/2-\mu} \left(\frac{au}{2}\right) - \cos \frac{au}{2} N_{1/2-\mu} \left(\frac{au}{2}\right) \right] \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \right].$$

ET II 204(25)

$$11.^3 \int_0^1 x^{\nu-1}(1-x)^{\mu-1} \sin(ax) dx = -\frac{i}{2} B(\mu, \nu) [{}_1F_1(\nu; \nu+\mu; ia) - {}_1F_1(\nu; \nu+\mu; -ia)] \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1, \quad \nu \neq 0]. \quad ETI68(5)a,$$

ET I 317(5)

$$12.^3 \int_0^1 x^{\nu-1}(1-x)^{\mu-1} \cos(ax) dx = \frac{1}{2} B(\mu, \nu) [{}_1F_1(\nu; \nu+\mu; ia) + {}_1F_1(\nu; \nu+\mu; -ia)] \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

ET I 11(5)

$$13. \int_0^1 x^\mu (1-x)^\mu \sin(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \sin a \quad [a > 0, \operatorname{Re} \mu > -1].$$

ET I 68(4)

$$14. \int_0^1 x^\mu (1-x)^\mu \cos(2ax) dx = \frac{\sqrt{\pi}}{(2a)^{\mu+\frac{1}{2}}} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(a) \cos a \quad [a > 0, \operatorname{Re} \mu > -1].$$

ET I 11(4)

3.769

$$1. \int_0^\infty [(\beta+ix)^{-\nu} - (\beta-ix)^{-\nu}] \sin(ax) dx = -\frac{\pi a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)} \quad [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0].$$

ET I 70(15)

$$2. \int_0^\infty [(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}] \cos(ax) dx = \frac{\pi a^{\nu-1} e^{-a\beta}}{\Gamma(\nu)} \quad [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0].$$

ET I 13(19)

$$3. \int_0^\infty x [(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}] \sin(ax) dx = -\frac{\pi a^{\nu-2} (\nu-1-a\beta)}{\Gamma(\nu)} e^{-a\beta} \\ [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0].$$

ET I 70(16)

$$4. \int_0^\infty x^{2n} [(\beta-ix)^{-\nu} - (\beta+ix)^{-\nu}] \sin(ax) dx = \frac{(-1)^n i}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta) \\ [a > 0, \operatorname{Re} \beta > 0, 0 \leq 2n < \operatorname{Re} \nu].$$

ET I 70(17)

$$5. \int_0^\infty x^{2n} [(\beta+ix)^{-\nu} + (\beta-ix)^{-\nu}] \cos(ax) dx = \frac{(-1)^n}{\Gamma(\nu)} (2n)! \pi a^{\nu-2n-1} e^{-a\beta} L_{2n}^{\nu-2n-1}(a\beta) \\ [a > 0, \operatorname{Re} \beta > 0, 0 \leq 2n < \operatorname{Re} \nu].$$

ET I 13(20)

$$7. \int_0^{\infty} x^{2n+1} [(\beta + ix)^{-\nu} - (\beta - ix)^{-\nu}] \cos(ax) dx =$$

$$= \frac{(-1)^{n+1}}{\Gamma(\nu)} (2n+1)! \pi a^{\nu-2n-2} e^{-a\beta} L_{2n+1}^{\nu-2n-2}(a\beta)$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq 2n < \operatorname{Re} \nu - 1].$$

ET I 13(21)

464
3.771

$$1. \int_0^{\infty} (\beta^2 + x^2)^{\nu-\frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) [I_{-\nu}(a\beta) - \mathbf{L}_{\nu}(a\beta)]$$

$$\left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2}, \quad \nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots \right].$$

EH II 38a, ET I 68(6)

$$2. \int_0^{\infty} (\beta^2 + x^2)^{\nu-\frac{1}{2}} \cos(ax) dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\beta}{a}\right)^{\nu} \cos(\pi\nu) \Gamma\left(\nu + \frac{1}{2}\right) K_{-\nu}(a\beta)$$

$$\left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].$$

WA 191(1)A, GW((333))(78)a

$$3. \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \sin(ax) dx =$$

$$= \frac{a}{2} u^{2\mu+2\nu-1} \mathbf{B}\left(\mu, \nu + \frac{1}{2}\right) {}_1F_2\left(\nu + \frac{1}{2}; \frac{3}{2}, \mu + \nu + \frac{1}{2}; -\frac{a^2 u^2}{4}\right)$$

$$\left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 189(29)

$$4. \int_0^u x^{2\nu-1} (u^2 - x^2)^{\mu-1} \cos(ax) dx = \frac{1}{2} u^{2\mu+2\nu-2} \mathbf{B}(\mu, \nu) {}_1F_2\left(\nu; \frac{1}{2}, \mu + \nu; -\frac{a^2 u^2}{4}\right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 190(35)

$$6. \int_0^u (u^2 - x^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_\nu(au) \quad \left[a > 0, u > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 69(7), WA 358(1)a

$$7. \int_u^\infty (x^2 - u^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_{-\nu}(au) \quad \left[a > 0, u > 0, |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

EH II 81(12)A, ET I 69(8), WA 187(3)a

$$8. \int_0^u (u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = \frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(au) \quad \left[a > 0, u > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 11(8)

465

$$9. \int_u^\infty (x^2 - u^2)^{\nu - \frac{1}{2}} \cos(ax) dx = -\frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) N_{-\nu}(au) \quad \left[a > 0, u > 0, |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

WA 187(4)A, EH II 82(13)A, ET I 11(9)

$$10. \int_0^u x(u^2 - x^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu+1}(au) \quad \left[a > 0, u > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 69(9)

$$11. \int_u^\infty x(x^2 - u^2)^{\nu - \frac{1}{2}} \sin(ax) dx = \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) N_{-\nu-1}(au) \quad \left[a > 0, u > 0, -\frac{1}{2} < \operatorname{Re} \nu < 0 \right].$$

ET I 69(10)

$$12.7 \int_0^u x(u^2 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = -\frac{u^{\nu+1}}{a^\nu} s_{\nu-1, \nu+1}(au) =$$

$$\text{Printed from Table of Integrals, Series, and Products (CD-ROM). Copyright © 1996 by Academic Press, Inc. All rights reserved.} \\ = \frac{1}{2} \left(\frac{2u}{a}\right)^\nu u^{2\nu+1} - \frac{\sqrt{\pi}}{2} u \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \mathbf{H}_{\nu+1}(au)$$

$$13. \int_u^\infty x(x^2 - u^2)^{\nu-1/2} \cos(ax) dx = \frac{\sqrt{\pi}u}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_{-\nu-1}(au) \\ \left[a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET I 12(11)

3.772

$$1. \int_0^\infty (x^2 + 2\beta x)^{\nu-1/2} \sin(ax) dx = \\ = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) [J_{-\nu}(a\beta) \cos(a\beta) + N_{-\nu}(a\beta) \sin(a\beta)] \\ \left[a > 0, \quad |\arg \beta| < \pi, \quad \frac{1}{2} > \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET I 69(12)

$$2. \int_0^\infty (x^2 + 2\beta x)^{\nu-1/2} \cos(ax) dx = \\ = -\frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) [N_{-\nu}(a\beta) \cos(a\beta) - J_{-\nu}(a\beta) \sin(a\beta)] \\ \left[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 12(13)

466

$$3. \int_0^{2u} (2ux - x^2)^{\nu-1/2} \sin(ax) dx = \sqrt{\pi} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \sin(au) J_\nu(au) \\ \left[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 69(13)a

$$4. \int_{2u}^\infty (x^2 - 2ux)^{\nu-1/2} \sin(ax) dx = \\ = \frac{\sqrt{\pi}}{2} \left(\frac{2\beta}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) [J_{-\nu}(au) \cos(au) - N_{-\nu}(au) \sin(au)] \\ \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

$$5. \int_0^{2u} (2ux - x^2)^{\nu-1/2} \cos(ax) dx = \sqrt{\pi} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(au) \cos(au) \\ \left[a > 0, \quad u > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 12(4)

$$6. \int_{2u}^\infty (x^2 - 2ux)^{\nu-1/2} \cos(ax) dx = \\ = -\frac{\sqrt{\pi}}{2} \left(\frac{2u}{a}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) [J_{-\nu}(au) \sin(au) + N_{-\nu}(au) \cos(au)] \\ \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 12(12)

3.773

$$1.^6 \int_0^\infty \frac{x^{2\nu}}{(x^2 + \beta^2)^{\mu+1}} \sin(ax) dx = \\ = \frac{1}{2} \beta^{2\nu-2\mu} a \operatorname{B}\left(1 + \nu, \mu - \nu\right) {}_1F_2\left(\nu + 1; \nu + 1 - \mu; \frac{3}{2}; \frac{\beta^2 a^2}{4}\right) + \\ + \frac{\sqrt{\pi} a^{2\nu-2\mu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu - \mu)}{\Gamma\left(\mu - \nu + \frac{3}{2}\right)} {}_1F_2\left(\mu + 1; \mu - \nu + \frac{3}{2}, \mu - \nu + 1; \frac{\beta^2 a^2}{4}\right) = \\ = \frac{\sqrt{\pi}}{2\Gamma(\mu + 1)} \beta^{2\nu-2\mu-1} G_{13}^{21} \left(\frac{a^2 \beta^2}{4} \left| \begin{matrix} -\nu + \frac{1}{2} \\ \mu - \nu + \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \right. \right) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu + 1].$$

ET I 71(28)A, ET II 234(17)

$$2. \int_0^\infty \frac{x^{2m+1} \sin(ax)}{(z + x^2)^{n+1}} dx = \frac{(-1)^{n+m}}{n!} \cdot \frac{\pi}{2} \frac{d^n}{dz^n} (z^m e^{-a\sqrt{z}}) \quad [a > 0, \quad 0 \leq m \leq n, \quad |\arg z| < \pi].$$

ET I 68(39)

467

$$3. \int_0^\infty \frac{x^{2m+1} \sin(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} = \frac{(-1)^{m+1} \sqrt{\pi}}{2^n \beta^n \Gamma\left(n + \frac{1}{2}\right)} \frac{d^{2m+1}}{da^{2m+1}} [a^n K_n(a\beta)] \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 \leq m \leq n].$$

ET I 67(37)

$$4. \int_0^\infty \frac{x^{2\nu} \cos(ax) dx}{(x^2 + \beta^2)^{\mu+1}} = \\ = \frac{1}{2} \beta^{2\nu-2\mu-1} \operatorname{B}\left(\nu + \frac{1}{2}, \mu - \nu + \frac{1}{2}\right) {}_1F_2\left(\nu + \frac{1}{2}; \nu - \mu + \frac{1}{2}, \frac{1}{2}; \frac{\beta^2 a^2}{4}\right) + \\ + \frac{\sqrt{\pi} a^{2\nu-2\mu+1}}{4^{\mu-\nu+1}} \frac{\Gamma(\nu - \mu - \frac{1}{2})}{\Gamma\left(\mu - \nu + \frac{3}{2}\right)} {}_1F_2\left(\mu + 1; \mu - \nu + 1, \mu - \nu + \frac{3}{2}; \frac{\beta^2 a^2}{4}\right) = \\ = \frac{\sqrt{\pi}}{2\Gamma(\mu + 1)} \beta^{2\nu-2\mu-1} G_{13}^{21} \left(\frac{a^2 \beta^2}{4} \left| \begin{matrix} -\nu + \frac{1}{2} \\ \mu - \nu + \frac{1}{2}, \frac{1}{2}, 0 \end{matrix} \right. \right)$$

$$5. \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(z+x^2)^{n+1}} = (-1)^{m+n} \frac{\pi}{2 \cdot n!} \cdot \frac{d^n}{dz^n} (z^{m-\frac{1}{2}} e^{-a\sqrt{z}})$$

$$[a > 0, \quad n+1 > m \geq 0, \quad |\arg z| < \pi].$$

ET I 10(28)

$$6.7 \int_0^\infty \frac{x^{2m} \cos(ax) dx}{(\beta^2 + x^2)^{n+\frac{1}{2}}} = \frac{(-1)^m \sqrt{\pi}}{2^n \beta^n \Gamma\left(n + \frac{1}{2}\right)} \cdot \frac{d^{2m}}{da^{2m}} \{a^n K_n(a\beta)\}$$

$$[a > 0, \quad \operatorname{Re} \beta > 0, \quad 0 \leq m \leq n].$$

ET I 14(28)

3.774

$$1. \int_0^\infty \frac{\sin(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)} \left[\sin \frac{\nu\pi}{2} I_\nu(ab) + \frac{i}{2} \mathbf{J}_\nu(iab) - \frac{i}{2} \mathbf{J}_\nu(-iab) \right]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 70(19)

$$2. \int_0^\infty \frac{\cos(ax) dx}{\sqrt{x^2 + b^2} (x + \sqrt{x^2 + b^2})^\nu} = \frac{\pi}{b^\nu \sin(\nu\pi)} \left[\frac{1}{2} \mathbf{J}_\nu(iab) + \frac{1}{2} \mathbf{J}_\nu(-iab) - \cos \frac{\nu\pi}{2} I_\nu(ab) \right]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 12(15)

$$3. \int_0^\infty \frac{(x + \sqrt{x^2 + \beta^2})^\nu}{\sqrt{x(x^2 + \beta^2)}} \sin(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a\beta}{2} \right)$$

$$\left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{3}{2} \right].$$

ET I 71(23)

468

$$4. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x(x^2 + \beta^2)}} \cos(ax) dx = \sqrt{\frac{a\pi}{2}} \beta^\nu I_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a\beta}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a\beta}{2} \right)$$

$$\left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET I 12(17)

$$5. \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu+\frac{1}{2}} \sqrt{x^2 + \beta^2}} \sin(ax) dx = \frac{1}{\beta} \sqrt{\frac{2}{a}} \Gamma\left(\frac{3}{4} - \frac{\nu}{2}\right) W_{\frac{\nu}{2}, \frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, \frac{1}{4}}(a\beta)$$

$$6. \int_0^\infty \frac{(\beta + \sqrt{x^2 + \beta^2})^\nu}{x^{\nu + \frac{1}{2}} \sqrt{\beta^2 + x^2}} \cos(ax) dx = \frac{1}{\beta \sqrt{2a}} \Gamma\left(\frac{1}{4} - \frac{\nu}{2}\right) W_{\frac{\nu}{2}, -\frac{1}{4}}(a\beta) M_{-\frac{\nu}{2}, -\frac{1}{4}}(a\beta) \\ \left[a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET I 12(18)

3.775

$$1. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu - (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \sin(ax) dx = 2\beta^\nu \sin \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET I 70(20)

$$2. \int_0^\infty \frac{(\sqrt{x^2 + \beta^2} + x)^\nu + (\sqrt{x^2 + \beta^2} - x)^\nu}{\sqrt{x^2 + \beta^2}} \cos(ax) dx = 2\beta^\nu \cos \frac{\nu\pi}{2} K_\nu(a\beta) \\ [a > 0, \quad \operatorname{Re} \beta > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET I 13(22)

$$3. \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \sin(ax) dx = \pi u^\nu \left[J_\nu(au) \cos \frac{\nu\pi}{2} - N_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET I 70(22)

$$4. \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x^2 - u^2}} \cos(ax) dx = -\pi u^\nu \left[N_\nu(au) \cos \frac{\nu\pi}{2} + J_\nu(au) \sin \frac{\nu\pi}{2} \right] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET I 13(25)

$$5. \int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \sin(ax) dx = \frac{\pi}{2} u^\nu \operatorname{cosec} \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) - \mathbf{J}_{-\nu}(au)] \\ [a > 0, \quad u > 0].$$

ET I 70(21)

$$6. \int_0^u \frac{(x + i\sqrt{u^2 - x^2})^\nu + (x - i\sqrt{u^2 - x^2})^\nu}{\sqrt{u^2 - x^2}} \cos(ax) dx = \frac{\pi}{2} u^\nu \sec \frac{\nu\pi}{2} [\mathbf{J}_\nu(au) + \mathbf{J}_{-\nu}(au)] \\ [a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < 1].$$

$$\begin{aligned}
7.6 \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \sin(ax) dx &= \\
&= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[J_{1/4+\nu/2} \left(\frac{au}{2}\right) N_{1/4-\nu/2} \left(\frac{au}{2}\right) + J_{1/4-\nu/2} \left(\frac{au}{2}\right) N_{1/4+\nu/2} \left(\frac{au}{2}\right) \right] \\
&\quad \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2} \right].
\end{aligned}$$

ET I 71(25)

$$\begin{aligned}
8.6 \int_u^\infty \frac{(x + \sqrt{x^2 - u^2})^\nu + (x - \sqrt{x^2 - u^2})^\nu}{\sqrt{x(x^2 - u^2)}} \cos(ax) dx &= \\
&= -\sqrt{\left(\frac{\pi}{2}\right)^3} au^\nu \left[J_{-1/4+\nu/2} \left(\frac{au}{2}\right) N_{-1/4-\nu/2} \left(\frac{au}{2}\right) + J_{-1/4-\nu/2} \left(\frac{au}{2}\right) N_{-1/4+\nu/2} \left(\frac{au}{2}\right) \right] \\
&\quad \left[a > 0, \quad u > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2} \right].
\end{aligned}$$

ET I 13(26)

$$\begin{aligned}
9. \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \sin(ax) dx &= \\
&= \pi\beta^\nu \left[N_\nu(\beta a) \sin\left(\beta a - \frac{\nu\pi}{2}\right) + J_\nu(\beta a) \cos\left(\beta a - \frac{\nu\pi}{2}\right) \right] \\
&\quad [a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET I 71(26)

$$\begin{aligned}
10. \int_0^\infty \frac{(x + \beta + \sqrt{x^2 + 2\beta x})^\nu + (x + \beta - \sqrt{x^2 + 2\beta x})^\nu}{\sqrt{x^2 + 2\beta x}} \cos(ax) dx &= \\
&= \pi\beta^\nu \left[J_\nu(\beta a) \sin\left(\beta a - \frac{\nu\pi}{2}\right) - N_\nu(\beta a) \cos\left(\beta a - \frac{\nu\pi}{2}\right) \right] \\
&\quad [a > 0, \quad |\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET I 13(23)

$$\begin{aligned}
11. \int_0^{2u} \frac{(\sqrt{2u+x} + i\sqrt{2u-x})^{4\nu} + (\sqrt{2u+x} - i\sqrt{2u-x})^{4\nu}}{\sqrt{4u^2x - x^3}} \cos(ax) dx &= \\
&= (4u)^{2\nu} \pi^{3/2} \sqrt{\frac{a}{2}} J_{\nu-1/4}(au) J_{-\nu-1/4}(au) \quad [a > 0, \quad u > 0].
\end{aligned}$$

ET I 14(27)

3.776

$$1. \int_0^{\infty} \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \sin(ax) dx = \frac{a}{b^p} \quad [a > 0, \quad b > 0, \quad p > 0].$$

BI ((170))(1)

$$2. \int_0^{\infty} \frac{a^2(b+x)^2 + p(p+1)}{(b+x)^{p+2}} \cos(ax) dx = \frac{p}{b^{p+1}} \quad [a > 0, \quad b > 0, \quad p > 0].$$

BI ((170))(2)

470

3.78- 3.81 Rational functions of x and of trigonometric functions

3.781

$$1. \int_0^{\infty} \left(\frac{\sin x}{x} - \frac{1}{1+x} \right) \frac{dx}{x} = 1 - C \quad (\text{cf. 3.784 4. and 3.781 2.}).$$

3.781

3.784
BI ((173))(7)

$$2. \int_0^{\infty} \left(\cos x - \frac{1}{1+x} \right) \frac{dx}{x} = -C.$$

BI ((173))(8)

3.782

$$1. \int_0^u \frac{1 - \cos x}{x} dx - \int_u^{\infty} \frac{\cos x}{x} dx = C + \ln u \quad [u > 0].$$

GW ((333))(31)

$$2. \int_0^{\infty} \frac{1 - \cos ax}{x^2} dx = \frac{a\pi}{2} \quad [a \geq 0].$$

BI ((158))(1)

3.783

$$1. \int_0^{\infty} \left[\frac{\cos x - 1}{x^2} + \frac{1}{2(1+x)} \right] \frac{dx}{x} = \frac{1}{2} \mathbf{C} - \frac{3}{4}.$$

BI ((173))(19)

$$2. \int_0^{\infty} \left(\cos x - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\mathbf{C}.$$

EH I 17, BI((273))(21)

3.784

$$1. \int_0^{\infty} \frac{\cos ax - \cos bx}{x} dx = \ln \frac{b}{a} \quad [a > 0, \quad b > 0].$$

FI II 635, GW((333))(20)

$$2. \int_0^{\infty} \frac{a \sin bx - b \sin ax}{x^2} dx = ab \ln \frac{a}{b} \quad [a > 0, \quad b > 0].$$

FI II 647

$$3. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{(b-a)\pi}{2} \quad [a \geq 0, \quad b \geq 0].$$

BI((158))(12), FI II 645

$$4. \int_0^{\infty} \frac{\sin x - x \cos x}{x^2} dx = 1.$$

BI ((158))(3)

$$5. \int_0^{\infty} \frac{\cos ax - \cos bx}{x(x+\beta)} dx = \frac{1}{\beta} \left[\text{ci}(a\beta) \cos a\beta + \text{si}(a\beta) \sin a\beta - \right. \\ \left. - \text{ci}(b\beta) \cos b\beta - \text{si}(b\beta) \sin b\beta + \ln \frac{b}{a} \right] \\ [a > 0, \quad b > 0, \quad |\arg \beta| < \pi].$$

$$6. \int_0^{\infty} \frac{\cos ax + x \sin ax}{1+x^2} dx = \pi e^{-a} \quad [a > 0].$$

GW ((333))(73)

$$7. \int_0^{\infty} \frac{\sin ax - ax \cos ax}{x^3} dx = \frac{\pi}{4} a^2 \operatorname{sign} a.$$

LI ((158))(5)

$$8. \int_0^{\infty} \frac{\cos ax - \cos bx}{x^2(x^2 + \beta^2)} dx = \frac{\pi[(b-a)\beta + e^{-b\beta} - e^{-a\beta}]}{2\beta^3} \quad [a > 0, \quad b > 0, \quad |\arg \beta| < \pi].$$

BI((173))(20)A, ET II 222(59)

3.785

$$\int_0^{\infty} \frac{1}{x} \sum_{k=1}^n a_k \cos b_k x dx = -\sum_{k=1}^n a_k \ln b_k \quad \left[b_k > 0, \quad \sum_{k=1}^n a_k = 0 \right].$$

FI II 649

3.786

$$1. \int_0^{\infty} \frac{(1 - \cos ax) \sin bx}{x^2} dx = \frac{b}{2} \ln \frac{b^2 - a^2}{b^2} + \frac{a}{2} \ln \frac{a+b}{a-b} \quad [a > 0, \quad b > 0].$$

ET I 81(29)

$$2. \int_0^{\infty} \frac{(1 - \cos ax) \cos bx}{x} dx = \ln \frac{\sqrt{|a^2 - b^2|}}{b} \quad [0 > 0, \quad b > 0, \quad a \neq b].$$

FI II 647

$$3. \int_0^{\infty} \frac{(1 - \cos ax) \cos bx}{x^2} dx = \frac{\pi}{2}(a-b) \quad [a < b \leq a]; \\ = 0 \quad [0 < a \leq b].$$

ET I 20(16)

3.787

$$1. \int_0^{\infty} \frac{(\cos a - \cos na) \sin mx}{x} dx = \frac{\pi}{2}(\cos a - 1) \quad [m > na > 0]; \\ = \frac{\pi}{2} \cos a \quad [na > m].$$

$$2. \int_0^{\infty} \frac{\sin^2 ax - \sin^2 bx}{x} dx = \frac{1}{2} \ln \frac{a}{b} \quad [a > 0, \quad b > 0].$$

GW ((333))(20b)

$$3. \int_0^{\infty} \frac{x^3 - \sin^3 x}{x^5} dx = \frac{13}{32} \pi.$$

BI ((158))(6)

$$4. \int_0^{\infty} \frac{(3 - 4 \sin^2 ax) \sin^2 ax}{x} dx = \frac{1}{2} \ln 2 \quad [a \text{ real}, \quad a \neq 0].$$

BI ((155))(6)

3.788

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{x} - \operatorname{ctg} x \right) dx = \ln \frac{\pi}{2}.$$

GW ((333))(61a)

3.789

$$\int_0^{\frac{\pi}{2}} \frac{4x^2 \cos x + (\pi - x)x}{\sin x} dx = \pi^2 \ln 2.$$

LI ((206))(10)

472

3.791

$$1. \int_0^{\frac{\pi}{2}} \frac{x dx}{1 + \sin x} = \ln 2.$$

GW ((333))(55a)

$$2. \int_0^{\pi} \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 4\mathbf{G}.$$

GW ((333))(55c)

$$3. \int_0^{\frac{\pi}{2}} \frac{x \cos x}{1 + \sin x} dx = \pi \ln 2 - 2\mathbf{G}.$$

$$4. \int_0^{\pi} \frac{\left(\frac{\pi}{2} - x\right) \cos x}{1 - \sin x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right) \cos x}{1 - \sin x} dx = \pi \ln 2 + 4\mathbf{G} = 5.8414484669 \dots$$

BI((207))(3), GW((333))(56c)

$$5. \int_0^{\frac{\pi}{2}} \frac{x^2 dx}{1 - \cos x} = -\frac{\pi^2}{4} + \pi \ln 2 + 4\mathbf{G} = 3.3740473667 \dots$$

BI ((207))(3)

$$6. \int_0^{\pi} \frac{x^2 dx}{1 - \cos x} = 4\pi \ln 2.$$

BI ((219))(1)

$$7. \int_0^{\frac{\pi}{2}} \frac{x^{p+1} dx}{1 - \cos x} = -\left(\frac{\pi}{2}\right)^{p+1} + \left(\frac{\pi}{2}\right)^p (p+1) \left\{ \frac{2}{p} - \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\} \quad [p > 0].$$

LI ((207))(4)

$$8. \int_0^{\frac{\pi}{2}} \frac{x dx}{1 + \cos x} = \frac{\pi}{2} - \ln 2.$$

GW ((333))(55a)

$$9. \int_0^{\frac{\pi}{2}} \frac{x \sin x dx}{1 - \cos x} = \frac{\pi}{2} \ln 2 + 2\mathbf{G}.$$

GW ((333))(56a)

$$10. \int_0^{\pi} \frac{x \sin x dx}{1 - \cos x} = 2\pi \ln 2.$$

GW ((333))(56b)

$$11. \int_0^{\pi} \frac{x - \sin x}{1 - \cos x} dx = \frac{\pi}{2} + \int_0^{\frac{\pi}{2}} \frac{x - \sin x}{1 - \cos x} dx = 2.$$

GW ((333))(57a)

3.792

$$1. \int_{-\pi}^{\pi} \frac{dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2} \quad [a^2 < 1].$$

FI II 485

473

$$2. \int_0^{\frac{\pi}{2}} \frac{x \cos x dx}{1 + 2a \sin x + a^2} = \frac{\pi}{2a} \ln(1 + a) - \sum_{k=0}^{\infty} (-1)^k \frac{a^{2k}}{(2k + 1)^2} \quad [a^2 < 1].$$

LI ((241))(2)

$$3. \int_0^{\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{\pi}{a} \ln(1 + a) \quad [a^2 < 1, \quad a \neq 0];$$

$$= \frac{\pi}{a} \ln \left(1 + \frac{1}{a} \right) \quad [a^2 < 1];$$

BI ((221))(2)

$$4. \int_0^{2\pi} \frac{x \sin x dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{a} \ln(1 - a) \quad [a^2 < 1, \quad a \neq 0];$$

$$= \frac{2\pi}{a} \ln \left(1 - \frac{1}{a} \right) \quad [a^2 > 1].$$

BI ((223))(4)

$$5. \int_0^{2\pi} \frac{x \sin nx dx}{1 - 2a \cos x + a^2} = \frac{2\pi}{1 - a^2} \left[(a^{-n} - a^n) \ln(1 - a) + \sum_{k=1}^{n-1} \frac{a^{-k} - a^k}{n - k} \right] \quad [a^2 < 1, \quad a \neq 0].$$

BI ((223))(5)

$$6. \int_0^{\infty} \frac{\sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{4a} \left[\left| \frac{1+a}{1-a} \right| - 1 \right] \quad [a \text{ real}, \quad a \neq 0, \quad a \neq 1].$$

GW ((333))(62b)

$$7. \int_0^{\infty} \frac{\sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{1 + a - 2a^{[b]+1}}{(1 - a)^2(1 - a)} \quad [b \neq 0, 1, 2, \dots];$$

$$= \frac{\pi}{2} \frac{1 + a - a^b - a^{b+1}}{(1 - a)^2(1 - a)} \quad [b = 0, 1, 2, \dots]; \quad [0 < a < 1].$$

$$8. \int_0^\infty \frac{\sin x \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2(1-a)} a^{[b]} \quad [b \neq 0, 1, 2, \dots];$$

$$= \frac{\pi}{2(1-a)} a^b + \frac{\pi}{4} a^{b-1} \quad [b = 1, 2, 3, \dots];$$

$[0 < a < 1, \quad b > 0; \quad \text{for } b = 0, \text{ see 3.7926}].$

3.792
ET I 19(5)

$$9. \int_0^\infty \frac{(1 - a \cos x) \sin bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{1 - a^{[b]+1}}{1 - a} \quad [b \neq 1, 2, 3, \dots];$$

$$= \frac{\pi}{2} \cdot \frac{1 - a^b}{1 - a} + \frac{\pi a^b}{4} \quad [b = 1, 2, 3, \dots]. \quad [0 < a < 1, \quad b > 0].$$

ET I 82(33)

$$10.^3 \int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta(1-a^2)} \frac{1 + ae^{-b\beta}}{1 - ae^{-b\beta}} \quad [a^2 < 1, \quad b \geq 0].$$

BI ((192))(1)

$$11. \int_0^\infty \frac{1}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{a\pi}{\beta(1-a^2)} \frac{\sin b\beta}{1 - 2a \cos b\beta + a^2} \quad [a^2 < 1, \quad b > 0].$$

BI ((193))(1)

474

$$12. \int_0^\infty \frac{\sin bcx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{e^{-\beta bc} - a^c}{(1 - ae^{-b\beta})(1 - ae^{b\beta})} \quad [a^2 < 1, \quad b > 0, \quad c > 0].$$

BI ((192))(8)

$$13. \int_0^\infty \frac{\sin bx}{1 - 2a \cos bx + a^2} \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \frac{1}{e^{b\beta} - a} \quad [a^2 < 1, \quad b > 0];$$

$$= \frac{\pi}{2a} \frac{1}{ae^{b\beta} - 1} \quad [a^2 > 1, \quad b > 0].$$

BI ((192))(2)

BI ((193))(5)

$$15. \int_0^\infty \frac{\cos bcx}{1 - 2a \cos bx + a^2} \frac{dx}{\beta^2 - x^2} = \frac{\pi}{2\beta(1 - a^2)} \frac{(1 - a^2) \sin \beta bc + 2a^{c+1} \sin \beta b}{1 - 2a \cos \beta b + a^2}$$

$[a^2 < 1, \quad b > 0, \quad c > 0].$

BI ((193))(9)

$$16. \int_0^\infty \frac{1 - a \cos bx}{1 - 2a \cos bx + a^2} \frac{dx}{1 + x^2} = \frac{\pi}{2} \frac{e^b}{e^b - a} \quad [a^2 < 1, \quad b > 0].$$

FI II 719

$$17. \int_0^\infty \frac{\cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi(e^{\beta - \beta b} + ae^{\beta b})}{2\beta(1 - a^2)(e^\beta - a)} \quad [0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0].$$

ET I 21(21)

$$18. \int_0^\infty \frac{\sin bx \sin x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi}{2\beta} \frac{\operatorname{sh} b\beta}{e^\beta - a} \quad [0 \leq b < 1];$$

$$= \frac{\pi}{4\beta(ae^\beta - 1)} [a^m e^{\beta(m+1-b)} - e^{(1-b)\beta}] -$$

$$- \frac{\pi}{4\beta(ae^{-\beta} - 1)} [a^m e^{-(m+1-b)\beta} - e^{-(1-b)\beta}]$$

$[m \leq b \leq m + 1] \quad [0 < a < 1, \quad \operatorname{Re} \beta > 0].$

ET I 81(27)

$$19. \int_0^\infty \frac{(\cos x - a) \cos bx}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x^2 + \beta^2} = \frac{\pi \operatorname{ch} \beta b}{2\beta(e^\beta - a)} \quad [0 \leq b < 1, \quad |a| < 1, \quad \operatorname{Re} \beta > 0].$$

ET I 21(23)

$$20. \int_0^\infty \frac{\sin x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x} = \int_0^\infty \frac{\operatorname{tg} x}{(1 - 2a \cos 2x + a^2)^{n+1}} \frac{dx}{x} =$$

$$= \int_0^\infty \frac{\operatorname{tg} x}{(1 - 2a \cos 4x + a^2)^{n+1}} \frac{dx}{x} = \frac{\pi}{2(1 - a^2)^{2n+1}} \sum_{k=0}^n \binom{n}{k}^2 a^{2k}.$$

BI ((187))(14)

BI ((223))(9)

$$2. \int_0^{2\pi} \frac{\cos nx - a \cos[(n+1)x]}{1 - 2a \cos x + a^2} x dx = 2\pi a^n \quad [a^2 < 1].$$

BI ((223))(13)

475

3.794

$$1.^3 \int_0^\pi \frac{x dx}{1 + a^2 + 2a \cos x} = \frac{\pi^2}{2(1-a^2)} + \frac{4}{(1-a^2)} \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)^2} \quad [a^2 < 1].$$

$$2. \int_0^{2\pi} \frac{x \sin nx}{1 \pm a \cos x} dx = \frac{2\pi}{\sqrt{1-a^2}} \left[(\mp 1)^n \frac{(1 + \sqrt{1-a^2})^n - (1 - \sqrt{1-a^2})^n}{a^n} \times \right. \\ \left. \times \ln \frac{2\sqrt{1 \pm a}}{\sqrt{1+a} + \sqrt{1-a}} + \sum_{k=0}^{n-1} \frac{(\mp 1)^k}{n-k} \frac{(1 + \sqrt{1-a^2})^k - (1 - \sqrt{1-a^2})^k}{a^k} \right] \\ [a^2 < 1].$$

BI ((223))(2)

$$3.^3 \int_0^{2\pi} \frac{x \cos nx}{1 \pm a \cos x} dx = \frac{2\pi^2}{\sqrt{1-a^2}} \left(\frac{1 - \sqrt{1-a^2}}{\mp a} \right)^n \quad [a^2 < 1].$$

BI ((223))(3)

$$4. \int_0^\pi \frac{x \sin x dx}{a + b \cos x} = \frac{\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a-b)} \quad [a > |b| > 0].$$

GW ((333))(53a)

$$5. \int_0^{2\pi} \frac{x \sin x dx}{a + b \cos x} = \frac{2\pi}{b} \ln \frac{a + \sqrt{a^2 - b^2}}{2(a+b)} \quad [a > |b| > 0].$$

GW ((333))(53b)

$$6. \int_0^\infty \frac{\sin x}{a \pm b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [a^2 > b^2]; \\ = 0 \quad [a^2 < b^2].$$

$$\int_{-\infty}^{\infty} \frac{(b^2 + c^2 + x^2)x \sin ax - (b^2 - c^2 - x^2)c \operatorname{sh} ac}{[x^2 + (b - c)^2][x^2 + (b + c)^2](\cos ax + \operatorname{ch} ac)} dx = \pi \quad [c > b > 0];$$

$$= \frac{2\pi}{e^{ab} + 1} \quad [b > c > 0] \quad [a > 0].$$

BI ((202))(18)

3.796

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos x \pm \sin x}{\cos x \mp \sin x} x dx = \mp \frac{\pi}{4} \ln 2 - \mathbf{G}.$$

BI ((207))(8, 9)

$$2. \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} x dx = \frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G}.$$

BI ((204))(23)

3.797

$$1. \int_0^{\pi/4} \left(\frac{\pi}{4} - x \operatorname{tg} x \right) \operatorname{tg} x dx = \frac{1}{2} \ln 2 + \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{\pi}{8} \ln 2.$$

BI ((204))(8)

476

$$2. \int_0^{\pi/4} \frac{\left(\frac{\pi}{4} - x \right) \operatorname{tg} x dx}{\cos 2x} = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

BI ((204))(19)

$$3. \int_0^{\pi/4} \frac{\frac{\pi}{4} - x \operatorname{tg} x}{\cos 2x} dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

BI ((204))(20)

3.798

$$1. \int_0^{\infty} \frac{\operatorname{tg} x}{a + b \cos 2x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [a^2 > b^2];$$

$$= 0 \quad [a^2 < b^2] \quad [a > 0].$$

$$2. \int_0^{\infty} \frac{\operatorname{tg} x}{a + b \cos 4x} \cdot \frac{dx}{x} = \frac{\pi}{2\sqrt{a^2 - b^2}} \quad [a^2 > b^2];$$

$$= 0 \quad [a^2 < b^2] \quad [a > 0].$$

BI ((181))(3)

3.799

$$1. \int_0^{\frac{\pi}{2}} \frac{x dx}{(\sin x + a \cos x)^2} = \frac{a}{1+a^2} \frac{\pi}{2} - \frac{\ln a}{1+a^2} \quad [a > 0].$$

BI ((208))(5)

$$2. \int_0^{\frac{\pi}{4}} \frac{x dx}{(\cos x + a \sin x)^2} = \frac{1}{1+a^2} \ln \frac{1+a}{\sqrt{2}} + \frac{\pi}{4} \cdot \frac{1-a}{(1+a)(1+a^2)} \quad [a > 0].$$

BI ((204))(24)

$$3. \int_0^{\pi} \frac{a \cos x + b}{(a + b \cos x)^2} x^2 dx = \frac{2\pi}{b} \ln \frac{2(a-b)}{a + \sqrt{a^2 - b^2}} \quad [a > |b| > 0].$$

GW ((333))(58a)

3.811

$$1. \int_0^{\pi} \frac{\sin x}{1 - \cos t_1 \cos x} \cdot \frac{x dx}{1 - \cos t_2 \cos x} = \pi \operatorname{cosec} \frac{t_1 + t_2}{2} \operatorname{cosec} \frac{t_1 - t_2}{2} \ln \frac{1 + \operatorname{tg} \frac{t_1}{2}}{1 + \operatorname{tg} \frac{t_2}{2}}$$

(cf. 3.794 4.).

3.794

BI ((222))(5)

$$2. \int_0^{\frac{\pi}{2}} \frac{x dx}{(\cos x \pm \sin x) \sin x} = \frac{\pi}{4} \ln 2 + \mathbf{G}.$$

BI ((208))(16, 17)

$$3. \int_0^{\frac{\pi}{4}} \frac{x dx}{(\cos x + \sin x) \sin x} = -\frac{\pi}{8} \ln 2 + \mathbf{G}.$$

BI ((204))(29)

BI ((204))(28)

$$5. \int_0^{\frac{\pi}{4}} \frac{\sin x}{\sin x + \cos x} \frac{x dx}{\cos^2 x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

BI ((204))(30)

477

3.812

$$1. \int_0^{\pi} \frac{x \sin x dx}{a + b \cos^2 x} = \frac{\pi}{\sqrt{ab}} \operatorname{arctg} \sqrt{\frac{b}{a}} \quad [a > 0, \quad b > 0];$$

$$= \frac{\pi}{2\sqrt{-ab}} \ln \frac{\sqrt{a} + \sqrt{-b}}{\sqrt{a} - \sqrt{-b}} \quad [a > -b > 0].$$

GW ((333))(60a)

$$2. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x dx}{1 + a \cos^2 x} = \frac{\pi}{a} \ln \frac{1 + \sqrt{1+a}}{2} \quad [a > -1, \quad a \neq 0].$$

BI ((207))(10)

$$3. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x dx}{1 + a \sin^2 x} = \frac{\pi}{a} \ln \frac{2(1 + a - \sqrt{1+a})}{2} \quad [a > -1, \quad a \neq 0].$$

BI ((207))(2)

$$4.^7 \int_0^{\pi} \frac{x dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}} \quad [a^2 > 1];$$

$$= 0 \quad [0 < a^2 < 1]$$

BI ((219))(10)

$$5.^7 \int_0^{\pi} \frac{x \sin x dx}{a^2 - \cos^2 x} = \frac{\pi}{2a} \ln \left| \frac{1+a}{1-a} \right| \quad [0 < a^2 < 1]$$

BI ((219))(13)

$$6.^6 \int_0^{\pi} \frac{x \sin 2x dx}{a^2 - \cos^2 x} = \pi \ln\{4(1-a^2)\} \quad [0 \leq a^2 < 1];$$

$$= 2\pi \ln[2(1-a^2 + a\sqrt{a^2-1})] \quad [a^2 > 1].$$

$$7. \int_0^{\frac{\pi}{2}} \frac{x \sin x dx}{\cos^2 t - \sin^2 x} = -2 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}.$$

BI ((207))(1)

$$8. \int_0^{\pi} \frac{x \sin x dx}{1 - \cos^2 t \sin^2 x} = \pi(\pi - 2t) \operatorname{cosec} 2t.$$

BI ((219))(12)

$$9. \int_0^{\pi} \frac{x \cos x dx}{\cos^2 t - \cos^2 x} = 4 \operatorname{cosec} t \sum_{k=0}^{\infty} \frac{\sin(2k+1)t}{(2k+1)^2}.$$

BI ((219))(17)

$$10. \int_0^{\pi} \frac{x \sin x dx}{\operatorname{tg}^2 t + \cos^2 x} = \frac{\pi}{2}(\pi - 2t) \operatorname{ctg} t.$$

BI ((219))(14)

$$11. \int_0^{\infty} \frac{x(a \cos x + b) \sin x dx}{\operatorname{ctg}^2 t + \cos^2 x} = 2a\pi \ln \cos \frac{t}{2} + \pi b t \operatorname{tg} t.$$

BI ((219))(18)

3.813

$$1. \int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{4} \int_0^{2\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab} \quad [a > 0, \quad b > 0].$$

GW ((333))(36)

478

$$2. \int_0^{\infty} \frac{1}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{dx}{x^2 + \delta^2} = \frac{\pi \operatorname{sh}(2a\delta)}{4\delta(\beta^2 \operatorname{sh}^2(a\delta) - \gamma^2 \operatorname{ch}^2(a\delta))} \left[\frac{\beta}{\gamma} - \frac{\gamma}{\beta} - \frac{2}{\operatorname{sh}(2a\delta)} \right] \\ \left[\left| \arg \frac{\beta}{\gamma} \right| < \pi, \quad \operatorname{Re} \delta > 0, \quad a > 0 \right].$$

GW((333))(81), ET II 222(63)

$$3. \int_0^{\infty} \frac{\sin x dx}{x(a^2 \sin^2 x + b^2 \cos^2 x)} = \frac{\pi}{2ab} \quad [ab > 0].$$

$$4. \int_0^{\infty} \frac{\sin^2 x dx}{x(a^2 \cos^2 x + b^2 \sin^2 x)} = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0].$$

BI ((181))(11)

$$5. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{a^2 - b^2} \ln \frac{a+b}{2b} \quad [a > 0, \quad b > 0, \quad a \neq b].$$

GW ((333))(52a)

$$6. \int_0^{\pi} \frac{x \sin 2x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{2\pi}{a^2 - b^2} \ln \frac{a+b}{2a} \quad [a > 0, \quad b > 0, \quad a \neq b].$$

GW ((333))(52b)

$$7. \int_0^{\infty} \frac{\sin 2x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{a(a+b)} \quad [a > 0, \quad b > 0].$$

BI ((182))(3)

$$8. \int_0^{\infty} \frac{\sin 2ax}{\beta^2 \sin^2 ax + \gamma^2 \cos^2 ax} \cdot \frac{x dx}{x^2 + \delta^2} = \frac{\pi}{2(\beta^2 \operatorname{sh}^2(a\delta) - \gamma^2 \operatorname{ch}^2(a\delta))} \left[\frac{\beta - \gamma}{\beta + \gamma} - e^{-2a\delta} \right] \\ \left[a > 0, \quad \left| \arg \frac{\beta}{\gamma} \right| < \pi, \quad \operatorname{Re} \delta > 0 \right].$$

ET II 222(64), GW((333))(80)

$$9. \int_0^{\infty} \frac{(1 - \cos x) \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0].$$

BI ((182))(7a)

$$10. \int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2a(a+b)} \quad [a > 0, \quad b > 0].$$

BI ((182))(4)

$$11. \int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{2}{a+b} \quad [a > 0, \quad b > 0].$$

$$1. \int_0^{\frac{\pi}{2}} \frac{(1 - x \operatorname{ctg} x) dx}{\sin^2 x} = \frac{\pi}{4}.$$

BI ((206))(9)

$$2. \int_0^{\frac{\pi}{4}} \frac{x \operatorname{tg} x dx}{(\sin x + \cos x) \cos x} = -\frac{\pi}{8} \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

BI ((204))(30)

$$3. \int_0^{\infty} \frac{\operatorname{tg} x}{a^2 \cos^2 x + b^2 \sin^2 x} \frac{dx}{x} = \frac{\pi}{2ab} \quad [a > 0, \quad b > 0].$$

BI ((181))(9)

$$4. \int_0^{\frac{\pi}{2}} \frac{x \operatorname{ctg} x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2a^2} \ln \frac{a+b}{b} \quad [a > 0, \quad b > 0].$$

LI ((208))(20)

479

$$5. \int_0^{\frac{\pi}{2}} \frac{(\frac{\pi}{2} - x) \operatorname{tg} x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{2} \int_0^{\pi} \frac{(\frac{\pi}{2} - x) \operatorname{tg} x dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \\ = \frac{\pi}{2b^2} \ln \frac{a+b}{a} \quad [a > 0, \quad b > 0].$$

GW ((333))(59)

$$6. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0].$$

BI ((182))(6)

$$7. \int_0^{\infty} \frac{\operatorname{tg} x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2ab} \quad [a > 0, \quad b > 0].$$

BI ((181))(10)a

$$8. \int_0^{\infty} \frac{\sin^2 2x \operatorname{tg} x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2b} \cdot \frac{1}{a+b} \quad [a > 0, \quad b > 0].$$

$$9. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x} = \frac{\pi}{2a} \cdot \frac{1}{a+b} \quad [a > 0, \quad b > 0].$$

BI ((182))(5)a

$$10. \int_0^{\infty} \frac{\sin^2 x \cos x}{a^2 \cos^2 2x + b^2 \sin^2 2x} \cdot \frac{dx}{x \cos 4x} = -\frac{\pi}{8b} \frac{a}{a^2 + b^2} \quad [a > 0, \quad b > 0].$$

BI ((186))(12)a

$$11. \int_0^{\infty} \frac{\sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2 - a^2}{b^2 + a^2} \quad [a > 0, \quad b > 0].$$

BI ((186))(4)a

$$12. \int_0^{\infty} \frac{\sin x \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2a} \cdot \frac{b}{a^2 + b^2} \quad [a > 0, \quad b > 0].$$

BI ((186))(7)a

$$13. \int_0^{\infty} \frac{\sin x \cos^2 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = \frac{\pi}{2ab} \cdot \frac{b^2}{a^2 + b^2} \quad [a > 0, \quad b > 0].$$

BI ((186))(8)a

$$14. \int_0^{\infty} \frac{\sin^3 x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \cos 2x} = -\frac{\pi}{2b} \cdot \frac{a}{a^2 + b^2} \quad [a > 0, \quad b > 0].$$

BI ((186))(10)

$$15. \int_0^{\infty} \frac{1 - \cos x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x \sin x} = \frac{\pi}{2ab} \quad [a > 0, \quad b > 0].$$

BI ((186))(3)a

3.815

$$1. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1 + a \sin^2 x)(1 + b \sin^2 x)} = \frac{\pi}{a-b} \ln \left\{ \frac{1 + \sqrt{1+b}}{1 + \sqrt{1+a}} \cdot \frac{\sqrt{1+a}}{\sqrt{1+b}} \right\} \quad [a > 0, \quad b > 0],$$

(cf. 3.812 3.).

3.812
BI ((208))(22)

$$2. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1+a \sin^2 x)(1+b \cos^2 x)} = \frac{\pi}{a+ab+b} \ln \frac{(1+\sqrt{1+n})\sqrt{1+a}}{1+\sqrt{1+a}} \quad [a > 0, \quad b > 0],$$

(cf. 3.812 2. and 3.).

3.812
BI ((208))(24)

480

$$3. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1+a \cos^2 x)(1+b \cos^2 x)} = \frac{\pi}{a-b} \ln \frac{1+\sqrt{1+a}}{1+\sqrt{1+b}} \quad [a > 0, \quad b > 0], \quad (\text{cf. 3.812 2.}).$$

3.812
BI ((208))(23)

$$4. \int_0^{\frac{\pi}{2}} \frac{x \sin 2x \, dx}{(1-\sin^2 t_1 \cos^2 x)(1-\sin^2 t_2 \cos^2 x)} = \frac{2\pi}{\cos^2 t_1 - \cos^2 t_2} \ln \frac{\cos \frac{t_1}{2}}{\cos \frac{t_2}{2}} \quad [-\pi < t_1 < \pi, \quad -\pi < t_2 < \pi].$$

BI ((208))(21)

3.816

$$1. \int_0^{\pi} \frac{x^2 \sin 2x}{(a^2 - \cos^2 x)^2} \, dx = \pi^2 \frac{\sqrt{a^2-1}-a}{a(a^2-1)} \quad [a > 1].$$

LI ((220))(9)

$$2.7 \int_0^{\pi} \frac{(a^2-1-\sin^2 x) \cos x}{(a^2-\cos^2 x)^2} x^2 \, dx = \frac{\pi}{a} \ln \left| \frac{1-a}{1+a} \right| \quad [a^2 > 1] \quad (\text{cf. 3.812 5.}).$$

3.812
BI ((220))(12)

$$4. \int_0^\pi \frac{a \cos 2x + \sin^2 x}{(a - \sin^2 x)^2} x^2 dx = \pi \ln(4a) \quad [a > 1], \quad (\text{cf. 3.812 6.}).$$

3.817

$$1. \int_0^\infty \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0].$$

$$2. \int_0^\infty \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0].$$

$$3. \int_0^\infty \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0].$$

$$4. \int_0^\infty \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3 b} \quad [ab > 0].$$

$$5. \int_0^\infty \frac{\operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0].$$

$$6. \int_0^\infty \frac{\operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4} \cdot \frac{a^2 + b^2}{a^3 b^3} \quad [ab > 0].$$

$$7. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4ab^3} \quad [ab > 0].$$

BI ((182))(11)

$$8. \int_0^{\infty} \frac{\operatorname{tg} x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^2} \cdot \frac{dx}{x} = \frac{\pi}{4a^3b} \quad [ab > 0].$$

BI ((182))(10)

3.818

$$1. \int_0^{\infty} \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5} \quad [ab > 0].$$

BI ((181))(16)

$$2. \int_0^{\infty} \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5b^3} \quad [ab > 0].$$

BI ((182))(13)

$$3. \int_0^{\infty} \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5b^3} \quad [ab > 0].$$

BI ((182))(14)

$$4. \int_0^{\infty} \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3b^5} \quad [ab > 0].$$

LI ((181))(19)

$$5. \int_0^{\infty} \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{64} \cdot \frac{3a^2 + b^2}{a^3b^5} \quad [ab > 0].$$

BI ((182))(17)

$$6. \int_0^{\infty} \frac{\operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2b^2 + 3b^4}{a^5b^5} \quad [ab > 0].$$

$$7. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^2 + b^2}{a^3 b^5} \quad [ab > 0].$$

BI ((182))(16)

$$8. \int_0^{\infty} \frac{\operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{3a^4 + 2a^2 b^2 + 3b^4}{a^5 b^5} \quad [ab > 0].$$

BI ((181))(18)

$$9. \int_0^{\infty} \frac{\operatorname{tg} x \cos^2 2x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^3} \cdot \frac{dx}{x} = \frac{\pi}{16} \cdot \frac{a^2 + 3b^2}{a^5 b^3} \quad [ab > 0].$$

BI ((182))(15)

3.819

$$1. \int_0^{\infty} \frac{\sin x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^6 + 3a^4 b^2 + 3a^2 b^4 + 5b^6}{a^7 b^7} \quad [ab > 0].$$

BI ((181))(20)

$$2. \int_0^{\infty} \frac{\sin x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5} \quad [ab > 0].$$

BI ((182))(18)

$$3. \int_0^{\infty} \frac{\sin x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5} \quad [ab > 0].$$

BI ((182))(19)

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$$4. \int_0^{\infty} \frac{\sin^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + a^2 b^2 + b^4}{a^5 b^7} \quad [ab > 0].$$

BI ((181))(23)

$$5. \int_0^{\infty} \frac{\sin^3 x \cos x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5 b^5} \quad [ab > 0].$$

$$6. \int_0^{\infty} \frac{\sin x \cos^3 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3} \quad [ab > 0].$$

BI ((182))(23)

$$7. \int_0^{\infty} \frac{\sin^3 x \cos^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + b^2}{a^5 b^5} \quad [ab > 0].$$

BI ((182))(27)

$$8. \int_0^{\infty} \frac{\sin x \cos^4 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3} \quad [ab > 0].$$

BI ((182))(24)

$$9. \int_0^{\infty} \frac{\sin^5 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3 b^7} \quad [ab > 0].$$

BI ((181))(24)

$$10. \int_0^{\infty} \frac{\sin^3 x \cos x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{128} \cdot \frac{5a^4 + 2a^2 b^2 + b^4}{a^5 b^7} \quad [ab > 0].$$

BI ((182))(22)

$$11. \int_0^{\infty} \frac{\sin^5 x \cos^3 x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{512} \cdot \frac{5a^2 + b^2}{a^3 b^7} \quad [ab > 0].$$

BI ((182))(30)

$$12. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^4 + 2a^2 b^2 + b^4}{a^5 b^7} \quad [ab > 0].$$

BI ((182))(21)

$$13. \int_0^{\infty} \frac{\sin^4 x \operatorname{tg} x}{(a^2 \cos^2 x + b^2 \sin^2 x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{5a^2 + b^2}{a^3 b^7} \quad [ab > 0].$$

BI ((182))(29)

$$14. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^4 + 2a^2 b^2 + 5b^4}{a^7 b^5} \quad [ab > 0].$$

$$15. \int_0^{\infty} \frac{\sin^3 4x \operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{8} \cdot \frac{a^2 + b^2}{a^5 b^5} \quad [ab > 0].$$

BI ((182))(28)

$$16. \int_0^{\infty} \frac{\cos^4 2x \operatorname{tg} x}{(a^2 \cos^2 2x + b^2 \sin^2 2x)^4} \cdot \frac{dx}{x} = \frac{\pi}{32} \cdot \frac{a^2 + 5b^2}{a^7 b^3} \quad [ab > 0].$$

BI ((182))(25)

3.82- 3.83 Powers of trigonometric functions combined with other powers

3.821

$$1. \int_0^{\pi} x \sin^p x \, dx = \frac{\pi^2}{2^{p+1}} \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2} + 1\right)\right]^2} \quad [p > -1].$$

BI((218))(7), LO V 121(71)

483

$$2. \int_0^{r\pi} x \sin^n x \, dx = \frac{\pi^2}{2} \cdot \frac{(2m-1)!!}{(2m)!!} r^2 \quad [n = 2m];$$

$$= (-1)^{r+1} \pi \frac{(2m)!!}{(2m+1)!!} r \quad [n = 2m+1], \quad [r \text{ is a natural number}].$$

GW ((333))(8c)

$$3. \int_0^{\frac{\pi}{2}} x \cos^n x \, dx = -\sum_{k=0}^{m-1} \frac{(n-2k+1)(n-2k+3)\dots(n-1)}{(n-2k)(n-2k+2)\dots n} \frac{1}{n-2k} +$$

$$+ \begin{cases} \frac{\pi}{2} \cdot \frac{(2m-2)!!}{(2m-1)!!} & [n = 2m-1]; \\ \frac{\pi^2}{8} \cdot \frac{(2m-1)!!}{(2m)!!} & [n = 2m]. \end{cases}$$

GW ((333))(9b)

$$4. \int_0^{\pi} x \cos^{2m} x \, dx = \frac{\pi^2}{2} \frac{(2m-1)!!}{(2m)!!}.$$

$$5. \int_{r\pi}^{s\pi} x \cos^{2m} x \, dx = \frac{\pi^2}{2} (s^2 - r^2) \frac{(2m-1)!!}{(2m)!!}.$$

BI ((226))(3)

$$6. \int_0^\infty \frac{\sin^p x}{x} \, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} = 2^{p-2} B\left(\frac{p}{2}, \frac{p}{2}\right);$$

[p is a fraction with odd numerator and denominator].

LO V 278, FI II 808

$$7. \int_0^\infty \frac{\sin^{2n+1} x}{x} \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}.$$

BI ((151))(4)

$$8. \int_0^\infty \frac{\sin^{2n} x}{x} \, dx = \infty.$$

BI ((151))(3)

$$9. \int_0^\infty \frac{\sin^2 ax}{x^2} \, dx = \frac{a\pi}{2} \quad [a > 0].$$

LO V 307, 312, FI II 632

$$10. \int_0^\infty \frac{\sin^{2m} ax}{x^2} \, dx = \frac{(2m-3)!!}{(2m-2)!!} \cdot \frac{a\pi}{2} \quad [a > 0].$$

GW ((333))(14b)

$$11. \int_0^\infty \frac{\sin^{2m+1} ax}{x^3} \, dx = \frac{(2m-3)!!}{(2m)!!} (2m+1) \frac{a^2\pi}{4} \quad [a > 0].$$

GW ((333))(14d)

$$12. \int_0^\infty \frac{\sin^p x}{x^m} \, dx = \frac{p}{m-1} \int_0^\infty \frac{\sin^{p-1} x}{x^{m-1}} \cos x \, dx \quad [p > m-1 > 0];$$

$$= \frac{p(p-1)}{(m-1)(m-2)} \int_0^\infty \frac{\sin^{p-2} x}{x^{m-2}} \, dx - \frac{p^2}{(m-1)(m-2)} \int_0^\infty \frac{\sin^p x}{x^{m-2}} \, dx$$

$$[p > m-1 > 1].$$

$$13. \int_0^{\infty} \frac{\sin^{2n} px}{\sqrt{x}} dx = \infty.$$

BI ((177))(5)

$$14. \int_0^{\infty} \sin^{2n+1} px \frac{dx}{\sqrt{x}} = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}$$

BI ((177))(7)

3.822

$$1. \int_0^{\frac{\pi}{2}} x^p \cos^m x dx = -\frac{p(p-1)}{m^2} \int_0^{\frac{\pi}{2}} x^{p-2} \cos^m x dx + \frac{m-1}{m} \int_0^{\frac{\pi}{2}} x^p \cos^{m-2} x dx$$

[$m > 1, p > 1$].

GW ((333))(9a)

$$2. \int_0^{\infty} x^{-\frac{1}{2}} \cos^{2n+1}(px) dx = \frac{1}{2^{2n}} \sqrt{\frac{\pi}{2p}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \frac{1}{\sqrt{2k+1}}$$

BI ((177))(8)

3.823

$$\int_0^{\infty} x^{\mu-1} \sin^2 ax dx = -\frac{\Gamma(\mu) \cos \frac{\mu\pi}{2}}{2^{\mu+1} a^{\mu}} \quad [a > 0, -2 < \operatorname{Re} \mu < 0].$$

ET I 319(15), GW((333))(19c)a

3.824

$$1. \int_0^{\infty} \frac{\sin^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 - e^{-2a\beta}) \quad [a > 0, \operatorname{Re} \beta > 0].$$

BI ((160))(10)

$$2. \int_0^{\infty} \frac{\cos^2 ax}{x^2 + \beta^2} dx = \frac{\pi}{4\beta} (1 + e^{-2a\beta}) \quad [a > 0, \operatorname{Re} \beta > 0].$$

BI ((160))(11)

$$3.7 \int_0^{\infty} \sin^{2m} x \frac{dx}{a^2 + x^2} = \frac{(-1)^m}{2^{2m+1}} \cdot \frac{\pi}{a} \left\{ 2^{2m} \operatorname{sh}^{2m} a - 2 \sum_{k=0}^m (-1)^k \binom{2m}{k} \operatorname{sh} [2(m-k)a] \right\} \quad [a > 0].$$

$$4.7 \int_0^\infty \sin^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1}}{2^{2m+2}a} \left\{ e^{(2m+1)a} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a] + \right. \\ \left. + e^{-(2m+1)a} \sum_{k=0}^{2m+1} (-1)^{k-1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m+1-2k)a] \right\} \\ [a > 0].$$

BI ((160))(14)

$$5.7 \int_0^\infty \sin^{2m+1} x \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1} e^{-(2m+1)a} \sum_{k=0}^m (-1)^{m+k} \binom{2m+1}{k} e^{2ka}} \\ [|\arg a| < \pi/2], m = 0, 1, 2, \dots$$

BI ((160))(15)

$$6.7 \int_0^\infty \cos^{2m} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}a} \binom{2m}{m} + \frac{\pi}{2^{2m}a} \sum_{k=1}^m \binom{2m}{m+k} e^{-2ka} \quad [a > 0].$$

BI ((160))(16)

485

$$7. \int_0^\infty \cos^{2m+1} x \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{2m+1}a} \sum_{k=1}^m \binom{2m+1}{m+k+1} e^{-(2k+1)a} \quad [a > 0].$$

BI ((160))(17)

$$8. \int_0^\infty \cos^{2m+1} x \frac{x dx}{a^2 + x^2} = -\frac{e^{-(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{2ka} \operatorname{Ei}[(2m-2k+1)a] - \\ -\frac{e^{(2m+1)a}}{2^{2m+2}} \sum_{k=0}^{2m+1} \binom{2m+1}{k} e^{-2ka} \operatorname{Ei}[(2k-2m-1)a].$$

BI ((160))(18)

$$9. \int_0^\infty \frac{\cos^2 ax}{b^2 - x^2} dx = \frac{\pi}{4b} \sin 2ab \quad [a > 0, \quad b > 0].$$

BI ((161))(10)

$$10. \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{\beta^2 + x^2} dx = \frac{\pi}{8\beta} \left[1 - \frac{1}{2} e^{-2(a+b)\beta} + e^{-2b\beta} - \frac{1}{2} e^{2(b-a)\beta} - e^{-2a\beta} \right] \quad [a > b];$$

$$\begin{aligned}
 11. \int_0^\infty \frac{x \sin 2ax \cos^2 bx}{\beta^2 + x^2} dx &= \frac{\pi}{8} [2e^{-2a\beta} + e^{-2(a+b)\beta} + e^{2(b-a)\beta}] \quad [a > 0]; \\
 &= \frac{\pi}{8} [e^{-4a\beta} + 2e^{-2a\beta}] \quad [a = b]; \\
 &= \frac{\pi}{8} [2e^{-2a\beta} + e^{-2(a+b)\beta} - e^{2(a-b)\beta}] \quad [a < b].
 \end{aligned}$$

LI ((162))(5)

3.825

$$1. \int_0^\infty \frac{\sin^2 ax dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi(b - c + ce^{-2ab} - be^{-2ac})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((174))(15)

$$2. \int_0^\infty \frac{\cos^2 ax dx}{(b^2 + x^2)(c^2 + x^2)} = \frac{\pi(b - c + be^{-2ac} - ce^{-2ab})}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI ((175))(14)

$$3.^3 \int_0^\infty \frac{\sin^2 ax dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(c \sin 2ab - b \sin 2ac)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c].$$

LI ((174))(16)

$$4.^3 \int_0^\infty \frac{\cos^2 ax dx}{(b^2 - x^2)(c^2 - x^2)} = \frac{\pi(b \sin 2ac - c \sin 2ab)}{4bc(b^2 - c^2)} \quad [a > 0, \quad b > 0, \quad c > 0, \quad b \neq c].$$

LI ((175))(15)

486

3.826

$$1. \int_0^\infty \frac{\sin^2 ax dx}{x^2(b^2 + x^2)} = \frac{\pi}{4b^2} \left[2a - \frac{1}{b}(1 - e^{-2ab}) \right] \quad [a > 0, \quad b > 0].$$

BI ((172))(13)

3.827

$$1.^* \int_0^{\infty} \frac{\sin^3 ax}{x^{\nu}} dx = \frac{3 - 3^{\nu-1}}{4} a^{\nu-1} \cos \frac{\nu\pi}{2} \Gamma(1-\nu) \quad [a > 0, \quad 0 < \operatorname{Re} \nu < 4].$$

GW ((333))(19f)

$$2. \int_0^{\infty} \frac{\sin^3 ax}{x} dx = \frac{\pi}{4} \operatorname{sign} a.$$

LO V 277

$$3. \int_0^{\infty} \frac{\sin^3 ax}{x^2} dx = \frac{3}{4} a \ln 3.$$

BI ((156))(2)

$$4. \int_0^{\infty} \frac{\sin^3 ax}{x^3} dx = \frac{3}{8} a^2 \pi \operatorname{sign} a.$$

BI((156))(7)A, LO V 312

$$5. \int_0^{\infty} \frac{\sin^4 ax}{x^2} dx = \frac{a\pi}{4} \quad [a > 0].$$

BI ((156))(3)

$$6. \int_0^{\infty} \frac{\sin^4 ax}{x^3} dx = a^2 \ln 2.$$

BI ((156))(8)

$$7. \int_0^{\infty} \frac{\sin^4 ax}{x^4} dx = \frac{a^3 \pi}{3} \quad [a > 0].$$

BI((156))(11), LO V 312

$$8. \int_0^{\infty} \frac{\sin^5 ax}{x^2} dx = \frac{5}{16} a(3 \ln 3 - \ln 5).$$

$$9. \int_0^{\infty} \frac{\sin^5 ax}{x^3} dx = \frac{5}{32} a^2 \pi \quad [a > 0].$$

BI ((156))(9)

$$10. \int_0^{\infty} \frac{\sin^5 ax}{x^4} dx = \frac{5}{96} a^3 (25 \ln 5 - 27 \ln 3).$$

BI ((156))(12)

$$11. \int_0^{\infty} \frac{\sin^5 ax}{x^5} dx = \frac{115}{384} a^4 \pi \quad [a > 0].$$

BI((156))(13), LO V 312

$$12. \int_0^{\infty} \frac{\sin^6 ax}{x^2} dx = \frac{3}{16} a \pi \quad [a > 0].$$

BI ((156))(5)

$$13. \int_0^{\infty} \frac{\sin^6 ax}{x^3} dx = \frac{3}{16} a^2 (8 \ln 2 - 3 \ln 3).$$

BI ((156))(10)

487

$$14. \int_0^{\infty} \frac{\sin^6 ax}{x^5} dx = \frac{1}{16} a^4 (27 \ln 3 - 32 \ln 2).$$

BI ((156))(14)

$$15. \int_0^{\infty} \frac{\sin^6 ax}{x^6} dx = \frac{11}{40} a^5 \pi \quad [a > 0].$$

LO V 312

3.828

$$1. \int_0^{\infty} \frac{\sin px \sin qx}{x} dx = \ln \sqrt{\frac{p+q}{|p-q|}} \quad [p \neq q].$$

$$2. \int_0^{\infty} \sin qx \sin px \frac{dx}{x^2} = \frac{1}{2}p\pi \quad [p \leq q];$$

$$= \frac{1}{2}q\pi \quad [p \geq q].$$

BI ((157))(1)

$$3. \int_0^{\infty} \frac{\sin^2 ax \sin bx}{x} dx = \frac{\pi}{4} \quad [0 < b < 2a];$$

$$= \frac{\pi}{8} \quad [b = 2a];$$

$$= 0 \quad [b > 2a].$$

BI ((151))(10)

$$4. \int_0^{\infty} \frac{\sin^2 ax \cos bx}{x} dx = \frac{1}{4} \ln \frac{4a^2 - b^2}{b^2}$$

BI ((151))(12)

$$5. \int_0^{\infty} \frac{\sin^2 ax \cos 2bx}{x^2} dx = \frac{\pi}{2}(a - b) \quad [b < a];$$

$$= 0 \quad [b \geq a].$$

FI III 648a, BI((157))(5)A}\Cr

$$6. \int_0^{\infty} \frac{\sin 2ax \cos^2 bx}{x} dx = \frac{\pi}{2} \quad [a > b];$$

$$= \frac{3}{8}\pi \quad [a = b];$$

$$= \frac{\pi}{4} \quad [a < b].$$

BI ((151))(9)

$$7. \int_0^{\infty} \frac{\sin^2 ax \sin bx \sin cx}{x^2} dx = \frac{\pi}{16} (|b-2a-c| - |2a-b-c| + 2c) \quad [a > 0, \quad b > 0, \quad c > 0].$$

BI((157))(9)A, ET I 79(15)

$$8. \int_0^{\infty} \frac{\sin^2 ax \sin bx \sin cx}{x} dx = \frac{1}{4} \ln \frac{b+c}{b-c} + \frac{1}{8} \ln \frac{(2a-b+c)(2a+b-c)}{(2a+b+c)(2a-b-c)}$$

$$[a > 0, \quad b > 0, \quad c > 0, \quad b \neq c].$$

$$9. \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^2} dx = \frac{\pi}{4}a \quad [0 \leq a \leq b];$$

$$= \frac{\pi}{4}b \quad [0 \leq b \leq a].$$

BI ((157))(3)

488

$$10. \int_0^\infty \frac{\sin^2 ax \sin^2 bx}{x^4} dx = \frac{1}{6}a^2\pi(3b - a) \quad [0 \leq a \leq b];$$

$$= \frac{1}{6}b^2\pi(3a - b) \quad [0 \leq b \leq a].$$

BI ((157))(27)

$$11. \int_0^\infty \frac{\sin^2 ax \cos^2 bx}{x^2} dx = \frac{2a - b}{4}\pi \quad [a \geq b > 0];$$

$$= \frac{a\pi}{4} \quad [0 < a \leq b].$$

BI ((157))(6)

$$12. \int_0^\infty \frac{\sin^3 ax \sin 3bx}{x^4} dx = \frac{a^3\pi}{2} \quad [b > a];$$

$$= \frac{\pi}{16}[8a^3 - 9(a - b)^3] \quad [a \leq 3b \leq 3a];$$

$$= \frac{9b\pi}{8}(a^2 - b^2) \quad [3b \leq a].$$

LI ((157))(28)

BI ((157))(28)

$$13. \int_0^\infty \frac{\sin^3 ax \cos bx}{x} dx = 0 \quad [b > 3a];$$

$$= -\frac{\pi}{16} \quad [b = 3a];$$

$$= -\frac{\pi}{8} \quad [3a > b > a];$$

$$= \frac{\pi}{16} \quad [b = a];$$

$$= \frac{\pi}{4} \quad [a > b] \quad [a > 0, \quad b > 0].$$

BI ((151))(15)

$$14. \int_0^\infty \frac{\sin^3 ax \cos 3bx}{x^2} dx = \frac{3}{8} \left\{ (a + b) \ln[3(a + b)] + (b - a) \ln[3(b - a)] - \right.$$

$$\left. - \frac{1}{2}(a + 3b) \ln(a + 3b) - \frac{1}{2}(3b - a) \ln(3b - a) \right\} \quad [a > 0, \quad b > 0].$$

$$\begin{aligned}
15. \int_0^\infty \frac{\sin^3 ax \cos bx}{x^3} dx &= \frac{\pi}{8}(3a^2 - b^2) & [b < a]; \\
&= \frac{\pi b^2}{4} & [a = b]; \\
&= \frac{\pi}{16}(3a - b)^2 & [a < b < 3a]; \\
&= 0 & [3a < b]; \quad [a > 0, \quad b > 0].
\end{aligned}$$

BI((157))(19), ET I 19(10)

489

$$\begin{aligned}
16. \int_0^\infty \frac{\sin^3 ax \sin bx}{x^4} dx &= \frac{b\pi}{24}(9a^2 - b^2) & [0 < b \leq a]; \\
&= \frac{\pi}{48}[24a^3 - (3a - b)^3] & [0 < a \leq b \leq 3a]; \\
&= \frac{\pi a^3}{2} & [0 < 3a \leq b].
\end{aligned}$$

ET I 79(16)

$$\begin{aligned}
17. \int_0^\infty \frac{\sin^3 ax \sin^2 bx}{x} dx &= \frac{\pi}{8} & [2b > 3a]; \\
&= \frac{5\pi}{32} & [2b = 3a]; \\
&= \frac{3\pi}{16} & [3a > 2b > a]; \\
&= \frac{3\pi}{32} & [2b = a]; \\
&= 0 & [a > 2b]; \quad [a > 0, \quad b > 0].
\end{aligned}$$

BI ((151))(14)

$$\begin{aligned}
18. \int_0^\infty \frac{\sin^2 ax \cos^3 bx}{x} dx &= \frac{1}{16} \ln \frac{(2a + b)^3 (b - 2a)^3 (2a + 3b)(3b - 2a)}{9b^8} \\
& & [b > 2a > 0 \quad \text{or} \quad 2a > 3b > 0]; \\
&= \frac{1}{16} \ln \frac{(2a + b)^3 (2a - b)^3 (2a + 3b)(3b - 2a)}{9b^8} & [3b > 2a > b].
\end{aligned}$$

BI ((151))(13)

$$\begin{aligned}
19. \int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx}{x} dx &= \\
&= \frac{\pi}{16} [1 + \text{sign}(c - a + b) + \text{sign}(c + a - b) - 2 \text{sign}(c - a) - 2 \text{sign}(c - b)] \\
& & [a > 0, \quad b > 0, \quad c > 0].
\end{aligned}$$

$$\begin{aligned}
20. \int_0^\infty \frac{\sin^2 ax \sin^2 bx \sin 2cx \, dx}{x^2} &= \frac{a-b-c}{16} \ln 4(a-b-c)^2 - \\
&- \frac{a+b+c}{16} \ln 4(a+b+c)^2 + \frac{a+b-c}{16} \ln 4(a+b-c)^2 - \\
&- \frac{a-b+c}{16} \ln 4(a-b+c)^2 + \frac{a+c}{8} \ln 4(a+c)^2 - \\
&- \frac{a-c}{8} \ln 4(a-c)^2 + \frac{b+c}{8} \ln 4(b+c)^2 - \frac{b-c}{8} \ln 4(b-c)^2 - \\
&- \frac{1}{2} c \ln 2c \quad [a > 0, \quad b > 0, \quad c > 0].
\end{aligned}$$

BI ((157))(10)

490

$$\begin{aligned}
21. \int_0^\infty \frac{\sin^2 ax \sin^3 bx}{x^3} \, dx &= \frac{3b^2\pi}{16} \quad [2a > 3b]; \\
&= \frac{a^2\pi}{12} \quad [2a = 3b]; \\
&= \frac{6b^2 - (3b - 2a)^2}{32} \pi \quad [3b > 2a > b]; \\
&= \frac{a^2\pi}{4} \quad [b > 2a]; \quad [a > 0, \quad b > 0].
\end{aligned}$$

BI ((157))(18)

3.829

$$1. \int_0^\infty \frac{x^n - \sin^n x}{x^{n+2}} \, dx = \frac{\pi}{2^n(n+1)!} \sum_{k=0}^{[(n-1)/2]} (-1)^k \binom{n}{k} (n-2k)^{n+1}$$

GW ((333))(63)

$$2. \int_0^\infty (1 - \cos^{2m-1} x) \frac{dx}{x^2} = \int_0^\infty (1 - \cos^{2m} x) \frac{dx}{x^2} = \frac{m\pi}{2^{2m}} \binom{2m}{m}.$$

BI ((158))(7, 8)

3.831

$$1. \int_0^\infty \frac{\sin^{2n} ax - \sin^{2n} bx}{x} \, dx = \frac{(2n-1)!!}{(2n)!!} \ln \frac{b}{a} \quad [ab > 0, \quad n = 1, 2, \dots].$$

FI II 651

$$2. \int_0^\infty \frac{\cos^{2n} ax - \cos^{2n} bx}{x} \, dx = \left[1 - \frac{(2n-1)!!}{(2n)!!} \right] \ln \frac{b}{a} \quad [ab > 0, \quad n = 0, 1, \dots].$$

$$3. \int_0^{\infty} \frac{\cos^{2m+1} ax - \cos^{2m+1} bx}{x} dx = \ln \frac{b}{a} \quad [ab > 0, \quad m = 0, 1, \dots].$$

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$$4. \int_0^{\infty} \frac{\cos^m ax \cos max - \cos^m bx \cos mbx}{x} dx = \left(1 - \frac{1}{2^m}\right) \ln \frac{b}{a} \quad [ab > 0, \quad m = 0, 1, \dots].$$

LI ((155))(8)

3.832

$$1. \int_0^{\frac{\pi}{2}} x \cos^{p-1} x \sin ax dx = \frac{\pi}{2^{p+1}} \Gamma(p) \frac{\psi\left(\frac{p+a+1}{2}\right) - \psi\left(\frac{p-a+1}{2}\right)}{\Gamma\left(\frac{p+a+1}{2}\right) \Gamma\left(\frac{p-a+1}{2}\right)} \\ [p > 0, \quad -(p+1) < a < p+1].$$

BI ((205))(6)

$$2.^3 \int_0^{\infty} \sin^{2m+1} x \sin 2mx \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} [(1 - e^{-2a})^{2m} - 1] \operatorname{sh} a \quad [a > 0, \quad m = 0, 1, \dots] \\ [\text{for } m = -1 \text{ see 3.723.2}].$$

3.723
BI ((162))(17)

491

$$3. \int_0^{\infty} \sin^{2m-1} x \sin[(2m-1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m+1} \pi}{2^{2m} a} (1 - e^{-2a})^{2m-1} \quad [a > 0, \quad m = 1, 2, \dots].$$

BI ((162))(11)

$$4. \int_0^{\infty} \sin^{2m-1} x \sin[(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2^{2m} a} e^{-2a} (1 - e^{-2a})^{2m-1} \\ [a > 0, \quad m = 1, 2, \dots].$$

BI ((162))(12)

$$5. \int_0^{\infty} \sin^{2m+1} x \sin[3(2m+1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2a} e^{-3(2m+1)a} \operatorname{sh}^{2m+1} a \quad [a > 0].$$

$$6.^3 \int_0^\infty \sin^{2m} x \sin[(2m-1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^a [(1-e^{-2a})^{2m} - (1+e^{-2a})] \\ [a \geq 0, \quad m = 0, 1, \dots].$$

BI ((162))(13)

$$7. \int_0^\infty \sin^{2m} x \sin(2mx) \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} [(1-e^{-2a})^{2m} - 1] \quad [a > 0, \quad m = 0, 1, \dots].$$

BI ((162))(14)

$$8. \int_0^\infty \sin^{2m} x \sin[(2m+2)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1}} e^{-2a} (1-e^{-2a})^{2m} \quad [a > 0, \quad m = 0, 1, \dots].$$

BI ((162))(15)

$$9. \int_0^\infty \sin^{2m} x \sin 4mx \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2} e^{-4ma} \operatorname{sh}^{2m} a \quad [a > 0, \quad m = 1, 2, \dots].$$

BI ((162))(16)

$$10. \int_0^\infty \sin^{2m} x \cos x \frac{dx}{x^2} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2} \quad [m = 1, 2, \dots].$$

GW ((333))(15a)

$$11. \int_0^\infty \sin^{2m} x \cos[(2m-1)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m} a} [(1-e^{-2a})^{2m-1} - 1] \operatorname{sh} a \\ [a > 0, \quad m = 1, 2, \dots].$$

BI ((162))(25)

$$12. \int_0^\infty \sin^{2m} x \cos(2mx) \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} (1-e^{-2a})^{2m} \quad [a > 0, \quad m = 0, 1, \dots].$$

BI ((162))(26)

$$13. \int_0^\infty \sin^{2m} x \cos[(2m+2)x] \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m+1} a} e^{-2a} (1-e^{-2a})^{2m} \quad [a > 0, \quad m = 0, 1, \dots].$$

$$14. \int_0^{\infty} \sin^{2m} x \cos 4mx \frac{dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2a} e^{-4ma} \operatorname{sh}^{2m} a \quad [a > 0, \quad m = 0, 1, \dots].$$

BI ((162))(28)

$$15. \int_0^{\infty} \sin^{2m+1} x \cos x \frac{dx}{x} = \frac{(2m-1)!!}{(2m+2)!!} \cdot \frac{\pi}{2} \quad [m = 0, 1, \dots].$$

GW ((333))(15)

492

$$16.^3 \int_0^{\infty} \sin^{2m+1} x \cos x \frac{dx}{x^3} = \frac{(2m-3)!!}{(2m)!!} \cdot \frac{\pi}{2} \quad [m = 1, 2, \dots].$$

GW ((333))(15b)

$$17. \int_0^{\infty} \sin^{2m-1} x \cos[(2m-1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m}} [(1 - e^{-2a})^{2m-1} - 1] \\ [m = 1, 2, \dots, \quad a > 0].$$

BI ((162))(23)

$$18.^3 \int_0^{\infty} \sin^{2m+1} x \cos 2mx \frac{x dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2^{2m+2}} \{e^a [(1 - e^{-2a})^{2m+1} - 1] - e^{-a}\} \\ [m = 0, 1, \dots, \quad a \geq 0].$$

BI ((162))(29)

$$19. \int_0^{\infty} \sin^{2m-1} x \cos[(2m+1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^m \pi}{2^{2m}} e^{-2a} (1 - e^{-2a})^{2m-1} \\ [m = 1, 2, \dots, \quad a > 0].$$

BI ((162))(24)

$$20. \int_0^{\infty} \sin^{2m+1} x \cos[2(2m+1)x] \frac{x dx}{a^2 + x^2} = \frac{(-1)^{m-1} \pi}{2} e^{-2(2m+1)a} \operatorname{sh}^{2m+1} a \\ [m = 0, 1, \dots, \quad a > 0].$$

BI ((162))(30)

$$21. \int_0^{\infty} \cos^m x \sin mx \frac{x dx}{a^2 + x^2} = \frac{1}{2^{m+1} a} \sum_{k=1}^m \binom{m}{k} [e^{-2ka} \operatorname{Ei}(2ka) - e^{2ka} \operatorname{Ei}(-2ka)] \quad [a > 0].$$

$$22. \int_0^{\infty} \cos^n sx \sin n sx \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{n+1}} [(1 + e^{-2as})^n - 1] \quad [s > 0, \operatorname{Re} a > 0, n \geq 0].$$

BI ((163))(9)

$$23. \int_0^{\infty} \cos^n sx \sin n sx \frac{x dx}{a^2 - x^2} = \frac{\pi}{2} (2^{-n} - \cos^n as \cos nas) \quad [n = 0, 1, \dots].$$

BI ((166))(10)

$$24. \int_0^{\infty} \cos^{m-1} x \sin[(m+1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^m} e^{-2a} (1 + e^{-2a})^{m-1} \quad [a > 0, m = 1, 2, \dots].$$

BI ((163))(6)

$$25. \int_0^{\infty} \cos^m x \sin[(m+1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^{m+1}} e^{-a} (1 + e^{-2a})^m \quad [m = 0, 1, \dots, a > 0].$$

BI ((163))(10)

$$26.^3 \int_0^{\infty} \cos^m x \sin[(m-1)x] \frac{x dx}{a^2 + x^2} = \frac{\pi}{2^m} \operatorname{ch} a [(1 + e^{-2a})^{m-1} - 1] \quad [m = 0, 1, \dots, a \geq 0].$$

BI ((163))(7)

$$27. \int_0^{\infty} \cos^m x \sin(3mx) \frac{x dx}{a^2 + x^2} = \frac{\pi}{2} e^{-3a} \operatorname{ch}^m a \quad [a > 0, m = 1, 2, \dots].$$

BI ((163))(11)

$$28. \int_0^{\infty} \cos^n sx \cos n sx \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{n+1} a} (1 + e^{-2as})^n \quad [n = 0, 1, \dots].$$

BI ((163))(16)

$$29. \int_0^{\infty} \cos^n sx \cos n sx \frac{dx}{a^2 - x^2} = \frac{\pi}{2a} \cos^n as \sin nas \quad [n = 0, 1, \dots].$$

$$30. \int_0^{\infty} \cos^{m-1} x \cos[(m+1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^m a} e^{-2a} (1 + e^{-2a})^{m-1} \quad [m = 1, 2, \dots, \quad a > 0].$$

BI ((163))(14)

$$31. \int_0^{\infty} \cos^m x \cos[(m-1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{m+1} a} e^a [(1 + e^{-2a})^m - (1 - e^{-2a})] \\ [m = 0, 1, \dots, \quad a > 0].$$

BI ((163))(15)

$$32. \int_0^{\infty} \cos^m x \cos[(m+1)x] \frac{dx}{a^2 + x^2} = \frac{\pi}{2^{m+1} a} e^{-a} (1 + e^{-2a})^m \quad [m = 0, 1, \dots, \quad a > 0].$$

BI ((163))(17)

$$33. \int_0^{\infty} \sin^p x \cos x \frac{dx}{x^q} = \frac{p}{q-1} \int_0^{\infty} \frac{\sin^{p-1} x}{x^{q-1}} dx - \frac{p+1}{q-1} \int_0^{\infty} \frac{\sin^{p+1} x}{x^{q-1}} dx \quad [p > q-1 > 0]; \\ = \frac{p(p-1)}{(q-1)(q-2)} \int_0^{\infty} \sin^{p-2} x \cos x \frac{dx}{x^{q-2}} - \\ - \frac{(p+1)^2}{(q-1)(q-2)} \int_0^{\infty} \sin^p x \cos x \frac{dx}{x^{q-2}} \quad [p > q-1 > 1].$$

GW ((333))(18)

$$34. \int_0^{\infty} \cos^{2m} x \cos 2nx \sin x \frac{dx}{x} = \int_0^{\infty} \cos^{2m-1} x \cos 2nx \sin x \frac{dx}{x} = \frac{\pi}{2^{2m+1}} \binom{2m}{m+n}.$$

BI ((152))(5, 6)

$$35. \int_0^{\infty} \cos^p ax \sin bx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [b > ap, \quad p > -1].$$

BI ((153))(12)

$$36. \int_0^{\infty} \cos^p ax \sin pax \cos x \frac{dx}{x} = \frac{\pi}{2^{p+1}} (2^p - 1) \quad [p > -1].$$

BI ((153))(2)

BI ((157))(15)

3.833

$$1. \int_0^\infty \sin^{2m+1} x \cos^{2n} x \frac{dx}{x} = \int_0^\infty \sin^{2m+1} x \cos^{2n-1} x \frac{dx}{x} = \frac{(2m-1)!!(2n-1)!!}{2^{m+n+1}(m+n)!!} \pi.$$

$$= \frac{1}{2} B\left(m + \frac{1}{2}, n + \frac{1}{2}\right).$$

GW ((333))(24)

BI ((151))(24, 25)

$$2. \int_0^\infty \sin^{2m+1} 2x \cos^{2n-1} 2x \cos^2 x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2m-1)!!(2n-1)!!}{(2m+2n)!!}.$$

LI ((152))(4)

494

3.834

$$1. \int_0^\infty \frac{\sin^{2m+1} x}{1 - 2a \cos x + a^2} \cdot \frac{dx}{x} = \frac{(-1)^m \pi (1+a)^{4m}}{2^{2m+2} a^{2m+1}} \left\{ \left| \frac{1-a}{1+a} \right|^{2m-1} - \sum_{k=0}^{2m} (-1)^k \binom{m - \frac{1}{2}}{k} \left(\frac{4a}{(1+a)^2} \right)^k \right\} \quad [|a| \neq 1].$$

GW ((333))(62a)

$$2. \int_0^\infty \frac{\sin^{2m+1} x \cos^n x}{(1 - 2a \cos x + a^2)^p} \cdot \frac{dx}{x} =$$

$$= \frac{n! \pi}{2^{n+1} (2m+n+1)! (1+a)^{2p}} \sum_{k=0}^n \frac{(-1)^k (2m+2n-2k+1)!! (2m+2k-1)!!}{k! (n-k)!} \times$$

$$\times F\left(m+n-k + \frac{3}{2}, p; 2m+n+2; \frac{4a}{(1+a)^2}\right) \quad [a \neq \pm 1].$$

GW ((333))(62)

3.835

$$1. \int_0^\infty \frac{\cos^{2m} x \cos 2mx \sin x}{a^2 \cos^2 x + b^2 \sin^2 x} \cdot \frac{dx}{x} = \frac{\pi}{2} \frac{b^{2m-1}}{a(a+b)^{2m}} \quad [ab > 0].$$

BI ((182))(31)a

3.836

$$1. \int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin mx}{x} dx = \frac{\pi}{2} \quad [m \geq n].$$

LI ((159))(12)

$$2.^3 \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos mx dx = \frac{n\pi}{2^n} \sum_{0 \leq k < \frac{m+n}{2}} \frac{(-1)^k (n+m-2k)^{n-1}}{k!(n-k)!} \quad [0 \leq m < n];$$

$$= 0 \quad [m \geq n] \quad [n \geq 2];$$

$$= \frac{\pi}{4} \quad [m = n = 1].$$

GI((159))(14), ET I 20(11)

$$3. \int_0^\infty \left(\frac{\sin x}{x}\right)^{n-1} \sin nx \cos x \frac{dx}{x} = \frac{\pi}{2} \quad [n \geq 1].$$

BI ((159))(20)

$$4.^* \int_0^\infty \left(\frac{\sin x}{x}\right)^n \frac{\sin(ax)}{x} dx = \frac{\pi}{2} \left[1 - \frac{1}{2^{n-1}n!} \sum_{0 \leq k < \frac{n}{2}(1-a)} (-1)^k \binom{n}{k} (n+an-2k)^n \right]$$

[all real a , $n \geq 1$].

ET I 20(11)

495

$$5.^7 \frac{2}{\pi} \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos bx dx = \frac{\pi}{2^{n-2}(n-1)!} \sum_{k=0}^{[r]} (-1)^k \binom{n}{k} (n-b-2k)^{n-1},$$

where $0 \leq b < n$, $n \geq 1$, $r = (n-b)/2$, and $[r]$ is the largest integer contained in r .

LO V 340(14)

$$6. \int_0^\infty \left(\frac{\sin x}{x}\right)^n \cos anx dx = 0 \quad [a \leq -1 \text{ or } a \geq 1, n \geq 2; \text{ for } n = 1 \text{ see 3.741.2}].$$

$$1. \int_0^{\frac{\pi}{2}} \frac{x^2 dx}{\sin^2 x} = \pi \ln 2.$$

BI ((206))(9)

$$2. \int_0^{\frac{\pi}{4}} \frac{x^2 dx}{\sin^2 x} = -\frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 + \mathbf{G} = 0.8435118417\dots$$

BI ((204))(10)

$$3. \int_0^{\frac{\pi}{4}} \frac{x^2 dx}{\cos^2 x} = \frac{\pi^2}{16} + \frac{\pi}{4} \ln 2 - \mathbf{G}.$$

GW ((333))(35a)

$$4. \int_0^{\frac{\pi}{4}} \frac{x^{p+1} dx}{\sin^2 x} = -\left(\frac{\pi}{4}\right)^{p+1} + (p+1) \left(\frac{\pi}{4}\right)^p \left\{ \frac{1}{p} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}(p+2k)} \zeta(2k) \right\} \quad [p > 0].$$

LI ((204))(14)

$$5. \int_0^{\frac{\pi}{2}} \frac{x^2 \cos x}{\sin^2 x} dx = -\frac{\pi^2}{4} + 4\mathbf{G} = 1.1964612764\dots$$

BI ((206))(7)

$$6. \int_0^{\frac{\pi}{2}} \frac{x^3 \cos x}{\sin^3 x} dx = -\frac{\pi^3}{16} + \frac{3}{2}\pi \ln 2.$$

BI ((206))(8)

$$7. \int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n} x \frac{dx}{x^m} = 0 \quad \left[n > \frac{m-1}{2}, \quad m > 0 \right].$$

BI ((180))(16)

$$8. \int_0^{\infty} \frac{\cos 2nx}{\cos x} \sin^{2n+1} x \frac{dx}{x^m} = 0 \quad \left[n > \frac{m-2}{2}, \quad m > 0 \right].$$

BI ((180))(17)

$$10.^3 \int_0^\pi \frac{x \sin(2n+1)x}{\sin x} dx = \frac{1}{2}\pi^2 \quad [n = 0, 1, 2, \dots].$$

$$11.^3 \int_0^\pi \frac{x \sin 2nx}{\sin x} dx = -4 \sum_{k=1}^n (2k-1)^{-2} \quad [n = 1, 2, 3, \dots].$$

3.838

$$1. \int_0^{\frac{\pi}{2}} \frac{x \cos^{p-1} x}{\sin^{p+1} x} dx = \frac{\pi}{2p} \sec \frac{\pi p}{2} \quad [p < 1].$$

BI ((206))(13)a

496

$$2. \int_0^{\frac{\pi}{4}} \frac{x \sin^{p-1} x}{\cos^{p+1} x} dx = \frac{\pi}{4p} - \frac{1}{2p} \beta \left(\frac{p+1}{2} \right) \quad [p > -1].$$

LI ((204))(15)

$$3. \int_0^{\frac{\pi}{4}} \frac{x \sin^{2m-1} x}{\cos^{2m+1} x} dx = \frac{\pi}{8m} (1 - \cos m\pi) + \frac{1}{2m} \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{2m-2k-1}.$$

BI ((204))(17)

$$4. \int_0^{\frac{\pi}{4}} \frac{x \sin^{2m} x}{\cos^{2m+2} x} dx = \frac{1}{2(2m+1)} \left[\frac{\pi}{2} + (-1)^{m-1} \ln 2 + \sum_{k=0}^{m-1} \frac{(-1)^{k-1}}{m-k} \right].$$

BI ((204))(16)

3.839

$$1. \int_0^{\frac{\pi}{4}} x \operatorname{tg}^2 x dx = \frac{\pi}{4} - \frac{\pi^3}{32} - \frac{1}{2} \ln 2.$$

BI ((204))(3)

BI ((204))(7)

$$3. \int_0^{\frac{\pi}{4}} \frac{x^2 \operatorname{tg} x}{\cos^2 x} dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{\pi^2}{16} \quad (\text{cf. 3.839 1.}).$$

3.839
BI ((204))(13)

$$4. \int_0^{\frac{\pi}{4}} \frac{x^2 \operatorname{tg}^2 x}{\cos^2 x} dx = \frac{1}{3} \left(1 - \frac{\pi}{4} \ln 2 - \frac{\pi}{2} + \frac{\pi^2}{16} + \mathbf{G} \right) \quad (\text{cf. 3.839 2.}).$$

3.839
BI ((204))(12)

$$5. \int_0^{\frac{\pi}{2}} x \cos^p x \operatorname{tg} x dx = \frac{\pi}{2^{p+1} p} \cdot \frac{\Gamma(p+1)}{\left[\Gamma\left(\frac{p}{2}+1\right) \right]^2} \quad [p > -1].$$

BI ((205))(3)

$$6. \int_0^{\frac{\pi}{2}} x \sin^p x \operatorname{ctg} x dx = \frac{\pi}{2p} - \frac{2^{p-1}}{p} \mathbf{B}\left(\frac{p+1}{2}, \frac{p+1}{2}\right) \quad [p > -1].$$

BI ((206))(11)

$$7. \int_0^{\infty} \sin^{2n} x \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} \cdot \frac{(2n-1)!!}{(2n)!!}.$$

GW ((333))(16)

$$8. \int_0^{\infty} \cos^s rx \operatorname{tg} qx \frac{dx}{x} = \frac{\pi}{2} \quad [s > -1].$$

BI ((151))(26)

$$9. \int_0^{\infty} \frac{\cos[(2n-1)x]}{\cos x} \cdot \left(\frac{\sin x}{x}\right)^{2n} dx = (-1)^{n-1} \frac{2^{2n}-1}{(2n)!} \cdot 2^{2n-1} \pi |B_{2n}|.$$

$$10. \int_0^{\infty} \operatorname{tg}^r px \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \sec \frac{r\pi}{2} \theta^r pq \quad [r^2 < 1].$$

BI ((160))(19)

3.84 Integrals containing the expressions $\sqrt{1 - k^2 \sin^2 x}$, $\sqrt{1 - k^2 \cos^2 x}$, and similar expressions

Notation: $k' = 1 - k^2$

3.841

$$1. \int_0^{\infty} \sin x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = \mathbf{E}(k).$$

BI ((154))(8)

497

$$2. \int_0^{\infty} \sin x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = \mathbf{E}(k).$$

BI ((154))(20)

$$3. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \sin^2 x} \frac{dx}{x} = \mathbf{E}(k).$$

BI ((154))(9)

$$4. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \cos^2 x} \frac{dx}{x} = \mathbf{E}(k).$$

BI ((154))(21)

3.842

$$\begin{aligned} 1. \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \sin^2 x}} \frac{dx}{x} &= \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \sin^2 x}} \cdot \frac{dx}{x} = \\ &= \int_0^{\infty} \frac{\sin x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \cos^2 x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{1}{\sqrt{2}} \right). \end{aligned}$$

BI ((183))(4, 5, 9, 10)

$$3. \int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \\ = \int_0^{\infty} \frac{\sin x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} = \frac{dx}{x} = \mathbf{K}(k).$$

BI ((183))(12, 13, 21, 22)

$$4. \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sqrt{1-k^2 \sin^2 x}} dx = \frac{1}{2k^2} [-\pi k' + 2\mathbf{E}(k)].$$

BI ((211))(1)

$$5. \int_0^{\frac{\pi}{2}} \frac{x \sin x \cos x}{\sqrt{1-k^2 \cos^2 x}} dx = \frac{1}{2k^2} [\pi - 2\mathbf{E}(k)].$$

BI ((214))(1)

$$6. \int_0^{\alpha} \frac{x \sin x dx}{\cos^2 x \sqrt{\sin^2 \alpha - \sin^2 x}} = \frac{\pi \sin^2 \frac{\alpha}{2}}{\cos^2 \alpha}.$$

LO III 284

$$7. \int_0^{\beta} \frac{x \sin x dx}{(1 - \sin^2 \alpha \sin^2 x) \sqrt{\sin^2 \beta - \sin^2 x}} = \frac{\pi \ln \frac{\cos \alpha + \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}{2 \cos \beta \cos^2 \frac{\alpha}{2}}}{2 \cos \alpha \sqrt{1 - \sin^2 \alpha \sin^2 \beta}}.$$

LO III 284

3.843

$$1. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \sin^2 2x} \frac{dx}{x} = \mathbf{E}(k).$$

BI ((154))(10)

498

$$2. \int_0^{\infty} \operatorname{tg} x \sqrt{1 - k^2 \cos^2 2x} \frac{dx}{x} = \mathbf{E}(k).$$

$$3. \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \sin^2 2x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 + \cos^2 2x}} \frac{dx}{x} = \frac{1}{\sqrt{2}} \mathbf{K} \left(\frac{1}{\sqrt{2}} \right).$$

BI ((183))(6, 11)

$$4. \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 - k^2 \sin^2 2x}} \frac{dx}{x} = \int_0^{\infty} \frac{\operatorname{tg} x}{\sqrt{1 - k^2 \cos^2 2x}} \frac{dx}{x} = \mathbf{K}(k).$$

BI ((183))(14, 23)

3.844

$$1. \int_0^{\infty} \frac{\sin x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)].$$

BI ((185))(20)

$$2. \int_0^{\infty} \frac{\sin x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)].$$

BI ((185))(21)

$$3. \int_0^{\infty} \frac{\sin x \cos^3 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2)\mathbf{K}(k) - 2(1 + k^2)\mathbf{E}(k)].$$

BI ((185))(22)

$$4. \int_0^{\infty} \frac{\sin x \cos^4 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2 + k^2)\mathbf{K}(k) - 2(1 + k^2)\mathbf{E}(k)].$$

BI ((185))(23)

$$5. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2)\mathbf{E}(k) - 2k'^2 \mathbf{K}(k)].$$

BI ((185))(24)

$$6. \int_0^{\infty} \frac{\sin^3 x \cos^2 x}{\sqrt{1 - k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1 + k'^2)\mathbf{E}(k) - 2k'^2 \mathbf{K}(k)].$$

BI ((185))(25)

BI ((184))(16)

$$8. \int_0^{\infty} \frac{\sin^4 x \operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+3k^2)k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)].$$

BI ((184))(18)

3.845

$$1. \int_0^{\infty} \frac{\sin x \cos x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((185))(6)

$$2. \int_0^{\infty} \frac{\sin x \cos^2 x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \frac{1}{2} \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((185))(7)

$$3. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{\sqrt{1+\cos^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{E} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BU ((184))(8)

499

3.846

$$1. \int_0^{\infty} \frac{\sin x \cos x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)].$$

BI ((185))(9)

$$2. \int_0^{\infty} \frac{\sin x \cos^2 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)].$$

BI ((185))(10)

$$3. \int_0^{\infty} \frac{\sin x \cos^3 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2)k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)].$$

$$4. \int_0^{\infty} \frac{\sin x \cos^4 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2)k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)].$$

BI ((185))(12)

$$5. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)].$$

BI ((185))(13)

$$6. \int_0^{\infty} \frac{\sin^3 x \cos^2 x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)].$$

BI ((185))(14)

$$7. \int_0^{\infty} \frac{\sin^2 x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)].$$

BI ((184))(9)

$$8. \int_0^{\infty} \frac{\sin^4 x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+k^2) \mathbf{K}(k) - 2(1+k^2) \mathbf{E}(k)].$$

BI ((184))(11)

3.847

$$\int_0^{\infty} \frac{\sin x \cos x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \int_0^{\infty} \frac{\sin x \cos^2 x}{\sqrt{1+\sin^2 x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((185))(3, 4)

3.848

$$1. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{K}(k) - \mathbf{E}(k)].$$

BI ((185))(15)

$$2. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)].$$

$$3. \int_0^{\infty} \frac{\cos^4 2x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2-3k^2)k'^2 \mathbf{K}(k) - 2(k'^2 - k^2) \mathbf{E}(k)].$$

BI ((184))(13)

$$4. \int_0^{\infty} \frac{\sin^2 4x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \cdot \frac{dx}{x} = \frac{4}{3k^4} [(1+k'^2) \mathbf{E}(k) - 2k'^2 \mathbf{K}(k)].$$

BI ((184))(17)

$$5. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{4k^2} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)].$$

BI ((185))(26)

500

$$6. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{k^2} [\mathbf{K}(k) - \mathbf{E}(k)].$$

BI ((184))(19)

$$7. \int_0^{\infty} \frac{\cos^4 2x \operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{3k^4} [(2+k^2) \mathbf{K}(k) - 2(1+k^2) \mathbf{E}(k)].$$

BI ((184))(20)

3.849

$$1. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1+\cos^2 2x}} \cdot \frac{dx}{x} = \frac{1}{2\sqrt{2}} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((185))(8)

$$2. \int_0^{\infty} \frac{\sin^3 x \cos x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \frac{\sqrt{2}}{8} \left[2\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{K} \left(\frac{\sqrt{2}}{2} \right) \right].$$

BI ((185))(5)

$$3. \int_0^{\infty} \frac{\cos^2 2x \operatorname{tg} x}{\sqrt{1+\sin^2 2x}} \cdot \frac{dx}{x} = \sqrt{2} \left[\mathbf{K} \left(\frac{\sqrt{2}}{2} \right) - \mathbf{E} \left(\frac{\sqrt{2}}{2} \right) \right].$$

powers

3.851

$$1. \int_0^{\infty} x \sin(ax^2) \sin(2bx) dx = \frac{b}{2a} \sqrt{\frac{\pi}{2a}} \left(\cos \frac{b^2}{a} + \sin \frac{b^2}{a} \right) \quad [a \geq 0, \quad b > 0].$$

BI ((150))(4)

$$2. \int_0^{\infty} x \sin(ax^2) \cos(2bx) dx = \frac{1}{2a} - \frac{b}{a} \sqrt{\frac{\pi}{2a}} \left[\sin \frac{b^2}{a} C \left(\frac{b}{\sqrt{a}} \right) - \cos \frac{b^2}{a} S \left(\frac{b}{\sqrt{a}} \right) \right].$$

BI ((150))(5)a

$$3. \int_0^{\infty} x \cos(ax^2) \sin(2bx) dx = \frac{b}{2a} \sqrt{\frac{\pi}{2a}} \left(\sin \frac{b^2}{a} - \cos \frac{b^2}{a} \right) \quad [a > 0, \quad b > 0], \quad (\text{cf. 3.691 7.}).$$

3.691
BI ((150))(7)

$$4. \int_0^{\infty} x \cos(ax^2) \cos(2bx) dx = \frac{b}{a} \sqrt{\frac{\pi}{2a}} \left[\cos \frac{b^2}{a} C \left(\frac{b}{\sqrt{a}} \right) + \sin \frac{b^2}{a} S \left(\frac{b}{\sqrt{a}} \right) \right].$$

BI ((150))(6)a

$$5. \int_0^{\infty} \sin(ax^2) \cos(bx) \frac{dx}{x^2} = \frac{b\pi}{2} \left\{ S \left(\frac{b}{2\sqrt{a}} \right) - C \left(\frac{b}{2\sqrt{a}} \right) + \sqrt{a\pi} \sin \left(\frac{b^2}{4a} + \frac{\pi}{4} \right) \right\} \\ [a > 0, \quad b > 0], \quad (\text{cf. 3.691 7.}).$$

3.691
ET I 23(3)a

3.852

$$1. \int_0^{\infty} \frac{\sin(ax^2)}{x^2} dx = \sqrt{\frac{a\pi}{2}} \quad [a \geq 0].$$

BI ((177))(10)a

$$\begin{aligned}
2. \int_0^\infty \sin(ax^2) \cos(bx^2) \frac{dx}{x^2} &= \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} + \sqrt{a-b}) \quad [a > b > 0]; \\
&= \frac{1}{2} \sqrt{\pi a} \quad [b = a \geq 0]; \\
&= \frac{1}{2} \sqrt{\frac{\pi}{2}} (\sqrt{a+b} - \sqrt{b-a}) \quad [b > a > 0], \quad (\text{cf. 3.852 1.}).
\end{aligned}$$

3.852
BI ((177))(23)

$$3. \int_0^\infty \frac{\sin^2(a^2 x^2)}{x^4} dx = \frac{2\sqrt{\pi}}{3} a^3 \quad [a \geq 0].$$

GW ((333))(19e)

$$4. \int_0^\infty \frac{\sin^3(a^2 x^2)}{x^2} dx = \frac{3 - \sqrt{3}}{8} \sqrt{\pi} a \quad [a \geq 0].$$

GW ((333))(19g)

$$5. \int_0^\infty (\sin x^2 - x^2 \cos x^2) \frac{dx}{x^4} = \frac{1}{3} \sqrt{\frac{\pi}{2}}.$$

BI ((178))(8)

$$6. \int_0^\infty \left\{ \cos x^2 - \frac{1}{1+x^2} \right\} \frac{dx}{x} = -\frac{1}{2} C.$$

BI ((173))(22)

3.853

$$\begin{aligned}
1. \int_0^\infty \frac{\sin(ax^2)}{\beta^2 + x^2} dx &= \frac{\pi}{2\beta} \left[\sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) - \sin(a\beta^2) \right] \\
& \quad [a > 0, \quad \text{Re } \beta > 0].
\end{aligned}$$

ET II 219(33)a

$$\begin{aligned}
2. \int_0^\infty \frac{\cos(ax^2)}{\beta^2 + x^2} dx &= \frac{\pi}{2\beta} \left[\cos(a\beta^2) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right] \\
& \quad [a > 0, \quad \text{Re } \beta > 0].
\end{aligned}$$

$$3. \int_0^{\infty} \frac{x^2 \sin(ax^2)}{\beta^2 + x^2} dx = \frac{\beta\pi}{2} \left[\sin(a\beta^2) - \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) + \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right] - \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad [a > 0, \operatorname{Re} \beta > 0].$$

ET II 219(32)a

$$4. \int_0^{\infty} \frac{x^2 \cos(ax^2)}{\beta^2 + x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{2a}} - \frac{\beta\pi}{2} \left[\cos(a\beta^2) - \sqrt{2} \cos\left(a\beta^2 + \frac{\pi}{4}\right) C(\sqrt{a}\beta) - \sqrt{2} \sin\left(a\beta^2 + \frac{\pi}{4}\right) S(\sqrt{a}\beta) \right] \quad [a > 0, \operatorname{Re} \beta > 0].$$

ET II 221(50)a

3.854

$$1. \int_0^{\infty} (\cos(ax^2) - \sin(ax^2)) \frac{dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b^3 \sqrt{2}} \quad [a > 0, b > 0].$$

LI ((178))(11)A, BI ((168))(25)

502

$$2. \int_0^{\infty} (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{x^4 + b^4} = \frac{\pi e^{-ab^2}}{2b\sqrt{2}} \quad [a > 0, b > 0].$$

LI ((178))(12)

$$3. \int_0^{\infty} (\cos(ax^2) + \sin(ax^2)) \frac{x^2 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b^3} \left(a + \frac{1}{2b^2} \right) \quad [a > 0, b > 0].$$

LI ((178))(14)

$$4. \int_0^{\infty} (\cos(ax^2) - \sin(ax^2)) \frac{x^4 dx}{(x^4 + b^4)^2} = \frac{\pi e^{-ab^2}}{4\sqrt{2}b} \left(\frac{1}{2b^2} - a \right) \quad [a > 0, b > 0].$$

BI ((178))(15)

3.855

$$1. \int_0^{\infty} \frac{\sin(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I \frac{1}{4} \left(\frac{a\beta}{2} \right) K \frac{1}{4} \left(\frac{a\beta}{2} \right) \quad [a > 0, \operatorname{Re} \beta > 0].$$

$$2. \int_0^\infty \frac{\cos(ax^2)}{\sqrt{\beta^2 + x^4}} dx = \frac{1}{2} \sqrt{\frac{a\pi}{2}} I_{-\frac{1}{4}} \left(\frac{a\beta}{2} \right) K_{\frac{1}{4}} \left(\frac{a\beta}{2} \right) \quad [a > 0, \quad \operatorname{Re} \beta > 0].$$

ET I 9(22)

$$3. \int_0^u \frac{\sin(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[J_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \right]^2 \quad [a > 0].$$

ET I 66(29)

$$4. \int_u^\infty \frac{\sin(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) N_{\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \quad [a > 0].$$

ET I 66(30)

$$5. \int_0^u \frac{\cos(a^2 x^2)}{\sqrt{u^4 - x^4}} dx = \frac{a}{4} \sqrt{\frac{\pi^3}{2}} \left[J_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) \right]^2.$$

ET I 9(23)

$$6. \int_u^\infty \frac{\cos(a^2 x^2)}{\sqrt{x^4 - u^4}} dx = -\frac{a}{4} \sqrt{\frac{\pi^3}{2}} J_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right) N_{-\frac{1}{4}} \left(\frac{a^2 u^2}{2} \right).$$

ET I 10(24)

3.856

$$1. \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} + x^2)^\nu}{\sqrt{\beta^4 + x^4}} \sin(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \left[\operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right].$$

ET I 71(23)

$$2. \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} + x^2)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \left[\operatorname{Re} \nu < \frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right].$$

ET I 12(16)

$$3. \int_0^\infty \frac{(\sqrt{\beta^4 + x^4} - x^2)^\nu}{\sqrt{\beta^4 + x^4}} \cos(a^2 x^2) dx = \frac{a}{2} \sqrt{\frac{\pi}{2}} \beta^{2\nu} I_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) K_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 \beta^2}{2} \right) \\ \left[\operatorname{Re} \nu > -\frac{3}{2}, \quad |\arg \beta| < \frac{\pi}{4} \right].$$

ET I 12(17)

$$4. \int_0^\infty \frac{\sin(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{x^2 + \sqrt{\beta^4 + x^4}}} = \frac{\operatorname{sh} \frac{a^2 \beta^2}{2}}{\sqrt{2} \beta^2} K_0 \left(\frac{a^2 \beta^2}{2} \right) \quad \left[|\arg \beta| < \frac{\pi}{4} \right].$$

ET I 66(32)

$$5. \int_0^\infty \frac{\cos(a^2 x^2) dx}{\sqrt{\beta^4 + x^4} \sqrt{(x^2 + \sqrt{\beta^4 + x^4})^3}} = \frac{\operatorname{sh} \frac{a^2 \beta^2}{2}}{2\sqrt{2} \beta^4} K_1 \left(\frac{a^2 \beta^2}{2} \right) \quad \left[|\arg \beta| < \frac{\pi}{4} \right].$$

ET I 10(27)

$$6. \int_0^\infty \frac{\sqrt{\sqrt{\beta^4 + x^4} + x^2}}{\sqrt{\beta^4 + x^4}} \sin(a^2 x^2) dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a^2 \beta^2}{2}} I_0 \left(\frac{a^2 \beta^2}{2} \right) \quad \left[|\arg \beta| < \frac{\pi}{4} \right],$$

ET I 67(33)

3.857

$$1. \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 - R_1}{R_2 + R_1}} \sin(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \sin ab \\ \left[R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right].$$

ET I 67(34)

$$2. \int_0^\infty \frac{x^2}{R_1 R_2} \sqrt{\frac{R_2 + R_1}{R_2 - R_1}} \cos(ax^2) dx = \frac{1}{2\sqrt{b}} K_0(ac) \cos ab \\ \left[R_1 = \sqrt{c^2 + (b - x^2)^2}, \quad R_2 = \sqrt{c^2 + (b + x^2)^2}, \quad a > 0, \quad c > 0 \right].$$

ET I 10(26)

3.858

$$1. \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \sin(a^2 x^2) dx = \\ = -\frac{a}{4} \sqrt{\frac{\pi}{a}} u^{2\nu} \left[J_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) N_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) + J_{\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) N_{\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) \right]$$

$$\begin{aligned}
2. \int_u^\infty \frac{(x^2 + \sqrt{x^4 - u^4})^\nu + (x^2 - \sqrt{x^4 - u^4})^\nu}{\sqrt{x^4 - u^4}} \cos(a^2 x^2) dx = \\
= -\frac{a}{4} \sqrt{\frac{\pi^3}{a}} u^{2\nu} \left[J_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) N_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) + J_{-\frac{1}{4} - \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) N_{-\frac{1}{4} + \frac{\nu}{2}} \left(\frac{a^2 u^2}{2} \right) \right] \\
\left[\operatorname{Re} \nu < \frac{3}{2} \right].
\end{aligned}$$

ET I 13(26)

3.859

$$\int_0^\infty \left[\cos(x^{2^n}) - \frac{1}{1 + x^{2^{n+1}}} \right] \frac{dx}{x} = -\frac{1}{2^n} \mathbf{C}.$$

BI ((173))(24)

3.861

$$\begin{aligned}
1. \int_0^\infty \sin^{2n+1}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-m+\frac{1}{2}} (2m-1)!!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{2n+1}{n+k} (2k-1)^{m-\frac{1}{2}} \\
\text{[the sign + is taken when } m \equiv 0(\bmod 4) \text{ or } m \equiv 1(\bmod 4), \\
\text{the sign - is taken when } m \equiv 2(\bmod 4) \text{ or } m \equiv 3(\bmod 4)].
\end{aligned}$$

BI ((177))(19)a

$$\begin{aligned}
2. \int_0^\infty \sin^{2n}(ax^2) \frac{dx}{x^{2m}} = \pm \frac{\sqrt{\pi} a^{m-\frac{1}{2}}}{2^{2n-2m+1} (2m-1)!!} \sum_{k=1}^n (-1)^k \binom{2n}{n+k} k^{m-\frac{1}{2}} \\
\text{[the sign + is taken when } m \equiv 0(\bmod 4) \text{ or } m \equiv 3(\bmod 4), \\
\text{the sign - is taken when } m \equiv 2(\bmod 4) \text{ or } m \equiv 1(\bmod 4)].
\end{aligned}$$

BI ((177))(18)A, LI ((177))(18)

3.862

$$\begin{aligned}
\int_0^\infty [\cos(ax^2 \sqrt{n}) + \sin(ax^2 \sqrt{n})] \left(\frac{\sin x^2}{x^2} \right)^n dx = \\
= \frac{\sqrt{\pi}}{(2n-1)!! \sqrt{2}} \sum_{k=0}^n (-1)^k \binom{n}{k} (n-2k + a\sqrt{n})^{n-\frac{1}{2}} \quad [a > \sqrt{n} > 0].
\end{aligned}$$

BI ((178))(9)

3.863

$$1. \int_0^{\infty} x^2 \cos(ax^4) \sin(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[\sin\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} - \frac{\pi}{8}\right) J_{\frac{3}{4}}\left(\frac{b^2}{2a}\right) \right]$$

$[a > 0, \quad b > 0].$

ET I 25(22)

$$2. \int_0^{\infty} x^2 \cos(ax^4) \cos(2bx^2) dx = -\frac{\pi}{8} \sqrt{\frac{b^3}{a^3}} \left[\sin\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{3}{4}}\left(\frac{b^2}{2a}\right) + \cos\left(\frac{b^2}{2a} + \frac{\pi}{8}\right) J_{-\frac{1}{4}}\left(\frac{b^2}{2a}\right) \right]$$

$[a > 0, \quad b > 0].$

ET I 25(23)

3.864

$$1. \int_0^{\infty} \sin \frac{b}{x} \sin ax \frac{dx}{x} = \frac{\pi}{2} N_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0].$$

WA 204(3)a

505

$$2. \int_0^{\infty} \cos \frac{b}{x} \cos ax \frac{dx}{x} = -\frac{\pi}{2} N_0(2\sqrt{ab}) + K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0].$$

WA 204(4)A, ET I 24 (12)

3.865

$$1. \int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} u^{\mu-\frac{3}{2}} \Gamma(\mu) J_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$

$[a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \mu < 1].$

ET II 189(30)

$$2. \int_u^{\infty} \frac{(x-u)^{\mu-1}}{x^{2\mu}} \sin \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \sin \frac{a}{2u} J_{\mu-\frac{1}{2}}\left(\frac{a}{2u}\right) \quad [a > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 203(21)

$$3. \int_0^u \frac{(u^2 - x^2)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = -\frac{\sqrt{\pi}}{2} \left(\frac{2}{a}\right)^{\mu-\frac{1}{2}} \Gamma(\mu) u^{\mu-\frac{3}{2}} N_{\frac{1}{2}-\mu}\left(\frac{a}{u}\right)$$

$[a > 0, \quad u > 0, \quad 0 < \operatorname{Re} \mu < 1].$

$$4. \int_u^\infty \frac{(x-u)^{\mu-1}}{x^{2\mu}} \cos \frac{a}{x} dx = \sqrt{\frac{\pi}{u}} a^{\frac{1}{2}-\mu} \Gamma(\mu) \cos \frac{a}{2u} J_{\mu-\frac{1}{2}} \left(\frac{a}{2u} \right) \quad [a > 0, \quad u > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 204(26)

3.866

$$1. \int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \sin(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a} \right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_\mu(2ab) - J_{-\mu}(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

ET I 322(42)

$$2. \int_0^\infty x^{\mu-1} \sin \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a} \right)^\mu \sec \frac{\mu\pi}{2} [J_\mu(2ab) + J_{-\mu}(2ab) + I_\mu(2ab) - I_{-\mu}(2ab)]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

ET I 322(43)

$$3. \int_0^\infty x^{\mu-1} \cos \frac{b^2}{x} \cos(a^2 x) dx = \frac{\pi}{4} \left(\frac{b}{a} \right)^\mu \operatorname{cosec} \frac{\mu\pi}{2} [J_{-\mu}(2ab) - J_\mu(2ab) + I_{-\mu}(2ab) - I_\mu(2ab)]$$

$$[a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

ET I 322(44)

3.867

$$1. \int_0^1 \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax - \cos \frac{a}{x}}{1-x^2} dx = \frac{\pi}{2} \sin a \quad [a > 0].$$

GW ((334))(7a)

$$2. \int_0^1 \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{1}{2} \int_0^\infty \frac{\cos ax + \cos \frac{a}{x}}{1+x^2} dx = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

GW ((334))(7b)

506

3.868

$$1. \int_0^\infty \sin \left(a^2 x + \frac{b^2}{x} \right) \frac{dx}{x} = \pi J_0(2ab) \quad [a > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \cos\left(a^2x + \frac{b^2}{x}\right) \frac{dx}{x} = -\pi N_0(2ab). \quad [a > 0, \quad b > 0].$$

GW ((334))(11a)

$$3. \int_0^{\infty} \sin\left(a^2x - \frac{b^2}{x}\right) \frac{dx}{x} = 0 \quad [a > 0, \quad b > 0].$$

GW ((334))(11b)

$$4. \int_0^{\infty} \cos\left(a^2x - \frac{b^2}{x}\right) \frac{dx}{x} = 2K_0(2ab) \quad [a > 0, \quad b > 0].$$

GW ((334))(11b)

3.869

$$1. \int_0^{\infty} \sin\left(ax - \frac{b}{x}\right) \frac{x dx}{\beta^2 + x^2} = \frac{\pi}{2} \exp\left(-\alpha\beta - \frac{b}{\beta}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0].$$

ET II 220(42)

$$2. \int_0^{\infty} \cos\left(ax - \frac{b}{x}\right) \frac{dx}{\beta^2 + x^2} = \frac{\pi}{2\beta} \exp\left(-a\beta - \frac{b}{\beta}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \beta > 0].$$

ET II 222(58)

3.871

$$1. \int_0^{\infty} x^{\mu-1} \sin\left[a\left(x + \frac{b^2}{x}\right)\right] dx = \pi b^{\mu} \left[J_{\mu}(2ab) \cos \frac{\mu\pi}{2} - N_{\mu}(2ab) \sin \frac{\mu\pi}{2} \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \mu < 1].$$

ET I 319(17)

$$2. \int_0^{\infty} x^{\mu-1} \cos\left[a\left(x + \frac{b^2}{x}\right)\right] dx = -\pi b^{\mu} \left[J_{\mu}(2ab) \sin \frac{\mu\pi}{2} + N_{\mu}(2ab) \cos \frac{\mu\pi}{2} \right] \\ [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

ET I 321(35)

$$3. \int_0^{\infty} x^{\mu-1} \sin\left[a\left(x - \frac{b^2}{x}\right)\right] dx = 2b^{\mu} K_{\mu}(2ab) \sin \frac{\mu\pi}{2} \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

$$4. \int_0^{\infty} x^{\mu-1} \cos \left[a \left(x - \frac{b^2}{x} \right) \right] dx = 2b^{\mu} K_{\mu}(2ab) \cos \frac{\mu\pi}{2} \quad [a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < 1].$$

ET I 321(36)

3.872

$$1. \int_0^1 \sin \left[a \left(x + \frac{1}{x} \right) \right] \sin \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} = \\ = \frac{1}{2} \int_0^{\infty} \sin \left[a \left(x + \frac{1}{x} \right) \right] \sin \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1-x^2} = -\frac{\pi}{4} \sin 2a \quad [a \geq 0].$$

BI ((149))(15), GW ((334))(8a)

507

$$2. \int_0^1 \cos \left[a \left(x + \frac{1}{x} \right) \right] \cos \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1+x^2} = \\ = \frac{1}{2} \int_0^{\infty} \cos \left[a \left(x + \frac{1}{x} \right) \right] \cos \left[a \left(x - \frac{1}{x} \right) \right] \frac{dx}{1+x^2} = \frac{\pi}{4} e^{-2a} \quad [a \geq 0].$$

GW ((334))(8b)

3.873

$$1. \int_0^{\infty} \sin \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2a}} [\sin(2ab) + \cos(2ab) + e^{-2ab}] \quad [a > 0, \quad b > 0].$$

ET I 24(15)

$$2. \int_0^{\infty} \cos \frac{a^2}{x^2} \cos b^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{4\sqrt{2a}} [\cos(2ab) - \sin(2ab) + e^{-2ab}] \quad [a > 0, \quad b > 0].$$

ET I 24(16)

3.874

$$1. \int_0^{\infty} \sin \left(a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \sin \left(2ab + \frac{\pi}{4} \right) \quad [a > 0, \quad b > 0].$$

BI ((179))(6)A, GW((334))(10a)

$$2. \int_0^{\infty} \cos \left(a^2 x^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2b} \cos \left(2ab + \frac{\pi}{4} \right) \quad [a > 0, \quad b > 0].$$

$$3. \int_0^{\infty} \sin\left(a^2 x^2 - \frac{b^2}{x^2}\right) \frac{dx}{x^2} = -\frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab} \quad [a \geq 0, \quad b > 0].$$

GW ((335))(10b)

$$4. \int_0^{\infty} \cos\left(a^2 x^2 - \frac{b^2}{x^2}\right) \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\sqrt{2}b} e^{-2ab} \quad [a \geq 0, \quad b > 0].$$

GW ((334))(10b)

$$5. \int_0^{\infty} \sin\left(ax - \frac{b}{x}\right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad [a > 0, \quad b > 0].$$

BI ((179))(13)a

$$6. \int_0^{\infty} \cos\left(ax - \frac{b}{x}\right)^2 \frac{dx}{x^2} = \frac{\sqrt{2\pi}}{4b} \quad [a > 0, \quad b > 0].$$

BI ((179))(14)a

3.875

$$1. \int_u^{\infty} \frac{x \sin(p\sqrt{x^2 - u^2})}{x^2 + a^2} \cos bx \, dx = \frac{\pi}{2} \exp(-p\sqrt{a^2 + u^2}) \operatorname{ch} ab \quad [0 < b < p].$$

ET I 27(39)

$$2. \int_u^{\infty} \frac{x \sin(p\sqrt{x^2 - u^2})}{a^2 + x^2 - u^2} \cos bx \, dx = \frac{\pi}{2} e^{-ap} \cos(b\sqrt{u^2 - a^2}) \quad [0 < b < p, \quad a > 0].$$

ET I 27(38)

$$3.^6 \int_0^{\infty} \frac{\sin(p\sqrt{a^2 + x^2})}{(a^2 + x^2)^{3/2}} \cos bx \, dx = \frac{\pi p}{2a} e^{-ab} \quad [0 < p < b, \quad a > 0].$$

ET I 26(29)

3.876

$$1. \int_0^{\infty} \frac{\sin(p\sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} \cos bx \, dx = \frac{\pi}{2} J_0(a\sqrt{p^2 - b^2}) \quad [0 < b < p];$$

$$= 0 \quad [b > p > 0]; \quad [a > 0].$$

$$2. \int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}} \cos bx \, dx = -\frac{\pi}{2} N_0(a\sqrt{p^2-b^2}) \quad [0 < b < p];$$

$$= K_0(a\sqrt{b^2-p^2}) \quad [b > p > 0]; \quad [a > 0].$$

ET I 26(34)

$$3. \int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{x^2+c^2} \cos bx \, dx = \frac{\pi}{2c} e^{-bc} \cos(p\sqrt{a^2-c^2}) \quad [c > 0, \quad b > p].$$

ET I 26(33)

$$4. \int_0^\infty \frac{\sin(p\sqrt{x^2+a^2})}{(x^2+c^2)\sqrt{x^2+a^2}} \cos bx \, dx = \frac{\pi}{2c} \frac{e^{-bc} \sin(p\sqrt{a^2-c^2})}{\sqrt{a^2-c^2}} \quad [c \neq a];$$

$$= \frac{\pi}{2} e^{-ba} \frac{p}{a} \quad [c = a]; \quad [b > p, \quad c > 0].$$

ET I 26(31)a

$$5.6 \int_0^\infty \frac{\cos(p\sqrt{x^2+a^2})}{x^2+a^2} \cos bx \, dx = \frac{\pi}{2a} e^{-ab} \quad [b > p > 0; \quad a > 0].$$

ET I 27(35)a

$$6.6 \int_0^\infty \frac{x \cos(p\sqrt{x^2+a^2})}{x^2+a^2} \sin bx \, dx = \frac{\pi}{2} e^{-ab} \quad [a > 0, \quad b > p > 0].$$

ET I 85(29)a

$$7. \int_0^u \frac{\cos(p\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}} \cos bx \, dx = \frac{\pi}{2} J_0(u\sqrt{b^2+p^2}).$$

ET I 28(42)

$$8. \int_u^\infty \frac{\cos(p\sqrt{x^2-u^2})}{\sqrt{x^2-u^2}} \cos bx \, dx = K_0(u\sqrt{p^2-b^2}) \quad [0 < b < |p|].$$

$$= -\frac{\pi}{2} N_0(u\sqrt{b^2-p^2}) \quad [b > |p|].$$

ET I 28(43)

3.877

$$1. \int_0^u \frac{\sin(p\sqrt{u^2-x^2})}{\sqrt{(u^2-x^2)^3}} \cos bx \, dx = \sqrt{\frac{\pi^3 p}{8}} J \frac{1}{4} \left[\frac{u}{2} (\sqrt{b^2+p^2} - b) \right] J \frac{1}{4} \left[\frac{u}{2} (\sqrt{b^2+p^2} + b) \right]$$

$$2. \int_u^\infty \frac{\sin(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx \, dx = -\sqrt{\frac{\pi^3 p}{8}} J_{\frac{1}{4}} \left[\frac{u}{2} (b - \sqrt{b^2 - p^2}) \right] N_{\frac{1}{4}} \left[\frac{u}{2} (b + \sqrt{b^2 - p^2}) \right]$$

$$[b > p > 0].$$

ET I 27(41)

$$3. \int_0^u \frac{\cos(p\sqrt{u^2 - x^2})}{\sqrt[4]{(u^2 - x^2)^3}} \cos bx \, dx = \sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[\frac{u}{2} (\sqrt{p^2 + b^2} - b) \right] J_{-\frac{1}{4}} \left[\frac{u}{2} (\sqrt{p^2 + b^2} + b) \right]$$

$$[u > 0, \quad p > 0].$$

ET I 28(44)

509

$$4. \int_u^\infty \frac{\cos(p\sqrt{x^2 - u^2})}{\sqrt[4]{(x^2 - u^2)^3}} \cos bx \, dx = -\sqrt{\frac{\pi^3 p}{8}} J_{-\frac{1}{4}} \left[\frac{u}{2} (b - \sqrt{b^2 - p^2}) \right] N_{\frac{1}{4}} \left[\frac{u}{2} (b + \sqrt{b^2 - p^2}) \right]$$

$$[b > p > 0].$$

ET I 28(45)

3.878

$$1. \int_0^\infty \frac{\sin(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 \, dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{a^2}{2} (p - \sqrt{p^2 - b^2}) \right] J_{\frac{1}{4}} \left[\frac{a^2}{2} (p + \sqrt{p^2 - b^2}) \right]$$

$$[p > b > 0].$$

ET I 26(32)

$$2. \int_0^\infty \frac{\cos(p\sqrt{x^4 + a^4})}{\sqrt{x^4 + a^4}} \cos bx^2 \, dx =$$

$$= -\frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{a^2}{2} (p - \sqrt{p^2 - b^2}) \right] N_{\frac{1}{4}} \left[\frac{a^2}{2} (p + \sqrt{p^2 - b^2}) \right] \quad [a > 0, \quad p > b > 0].$$

ET I 27(36)

$$3. \int_0^u \frac{\cos(p\sqrt{u^4 - x^4})}{\sqrt{u^4 - x^4}} \cos bx^2 \, dx = \frac{1}{2} \sqrt{\left(\frac{\pi}{2}\right)^3} b J_{-\frac{1}{4}} \left[\frac{u^2}{2} (\sqrt{p^2 + b^2} - p) \right] J_{-\frac{1}{4}} \left[\frac{u^2}{2} (\sqrt{p^2 + b^2} + p) \right]$$

$$[p > 0, \quad b > 0].$$

ET I 28(46)

3.879

$$\int_0^{\infty} \sin ax^p \frac{dx}{x} = \frac{\pi}{2p} \quad [a > 0, \quad p > 0].$$

GW ((334))(6)

3.881

$$1. \int_0^{\frac{\pi}{2}} x \sin(a \operatorname{tg} x) dx = \frac{\pi}{4} e^{-a} [\mathbf{C} + \ln 2a - e^{2a} \operatorname{Ei}(-2a)] \quad [a > 0].$$

BI ((205))(9)

$$2. \int_0^{\infty} \sin(a \operatorname{tg} x) \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0].$$

BI ((151))(6)

$$3. \int_0^{\infty} \sin(a \operatorname{tg} x) \cos x \frac{dx}{x} = \frac{\pi}{2} (1 - e^{-a}) \quad [a > 0].$$

BI ((151))(19)

$$4. \int_0^{\infty} \cos(a \operatorname{tg} x) \sin x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

BI ((151))(20)

$$5. \int_0^{\infty} \sin(a \operatorname{tg} x) \sin 2x \frac{dx}{x} = \frac{1+a}{2} \pi e^{-a} \quad [a > 0].$$

BI ((152))(11)

$$6. \int_0^{\infty} \cos(a \operatorname{tg} x) \sin^3 x \frac{dx}{x} = \frac{1-a}{4} \pi e^{-a} \quad [a > 0].$$

BI ((151))(23)

$$7. \int_0^{\infty} \sin(a \operatorname{tg} x) \operatorname{tg} \frac{x}{2} \cos^2 x \frac{dx}{x} = \frac{1+a}{4} \pi e^{-a} \quad [a > 0].$$

$$8. \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} x) \frac{x dx}{\sin 2x} = -\frac{\pi}{4} \operatorname{Ei}(-a) \quad [a > 0].$$

BI ((206))(15)

$$9. \int_0^{\frac{\pi}{2}} \sin(a \operatorname{ctg} x) \frac{x dx}{\sin^2 x} = \frac{1 - e^{-a}}{2a} \pi \quad [a > 0].$$

LI ((206))(14)

$$10. \int_0^{\frac{\pi}{2}} x \cos(a \operatorname{tg} x) \operatorname{tg} x dx = -\frac{\pi}{4} e^{-a} [C + \ln 2a + e^{2a} \operatorname{Ei}(-2a)] \quad [a > 0].$$

BI ((205))(10)

$$11. \int_0^{\infty} \cos(a \operatorname{tg} x) \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

BI ((151))(21)

$$12. \int_0^{\infty} \cos(a \operatorname{tg} x) \sin^2 x \operatorname{tg} x \frac{dx}{x} = \frac{1 - a}{16} \pi e^{-a} \quad [a > 0].$$

BI ((152))(15)

$$13. \int_0^{\infty} \sin(a \operatorname{tg} x) \operatorname{tg}^2 x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

BI ((152))(9)

$$14. \int_0^{\infty} \cos(a \operatorname{tg} 2x) \operatorname{tg} x \frac{dx}{x} = \frac{\pi}{2} e^{-a} \quad [a > 0].$$

BI ((151))(22)

$$15. \int_0^{\infty} \sin(a \operatorname{tg} 2x) \cos^2 2x \operatorname{tg} x \frac{dx}{x} = \frac{1 + a}{4} \pi e^{-a} \quad [a > 0].$$

BI ((152))(13)

BI ((152))(10)

$$17. \int_0^{\infty} \sin(a \operatorname{tg} 2x) \operatorname{tg} x \operatorname{ctg} 2x \frac{dx}{x} = \frac{\pi}{2}(1 - e^{-a}) \quad [a > 0].$$

BI ((180))(6)

3.882

$$1. \int_0^{\infty} \sin(a \operatorname{tg}^2 x) \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\exp(-a \theta b) - e^{-a}] \quad [a > 0, \quad b > 0].$$

BI ((160))(22)

$$2. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \cos x \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} [\operatorname{ch} b \exp(-a \theta b) - e^{-a} \operatorname{sh} b] \quad [a > 0, \quad b > 0].$$

BI ((163))(3)

$$3. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \operatorname{cosec} 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \operatorname{sh} 2b} \exp(-a \theta b) \quad [a > 0, \quad b > 0].$$

BI ((191))(10)

$$4. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \operatorname{tg} x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2 \operatorname{ch} b} [e^{-a} \operatorname{ch} b - \exp(-a \theta b) \operatorname{sh} b] \quad [a > 0, \quad b > 0].$$

BI ((163))(4)

$$5. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \operatorname{ctg} x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\operatorname{cth} b \exp(-a \theta b) - e^{-a}] \quad [a > 0, \quad b > 0].$$

BI ((163))(5)

$$6. \int_0^{\infty} \cos(a \operatorname{tg}^2 x) \operatorname{ctg} 2x \frac{x dx}{b^2 + x^2} = \frac{\pi}{2} [\operatorname{cth} 2b \exp(-a \theta b) - e^{-a}] \quad [a > 0, \quad b > 0].$$

BI ((191))(11)

511

3.883

$$2. \int_0^1 x^{\mu-1} \sin(\beta \ln x) dx = -\frac{\beta}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

ET I 319(19)

$$3. \int_0^1 x^{\mu-1} \cos(\beta \ln x) dx = \frac{\mu}{\beta^2 + \mu^2} \quad [\operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

ET I 321(38)

3.884

$$\int_{-\infty}^{\infty} \frac{\sin a\sqrt{|x|}}{x-b} \operatorname{sign} x dx = \cos a\sqrt{|b|} + \exp\{-a\sqrt{|b|}\} \quad [a > 0].$$

ET II 253(46)

3.89- 3.91 Trigonometric functions and exponentials

3.891

$$1. \int_0^{2\pi} e^{imx} \sin nx dx = 0 \quad [m \neq n; \quad m = n = 0];$$

$$= \pi i \quad [m = n \neq 0].$$

$$2. \int_0^{2\pi} e^{imx} \cos nx dx = 0 \quad [m \neq n];$$

$$= \pi \quad [m = n \neq 0];$$

$$= 2\pi \quad [m = n = 0].$$

3.892

$$1. \int_0^{\pi} e^{i\beta x} \sin^{\nu-1} x dx = \frac{\pi e^{i\beta \frac{\pi}{2}}}{2^{\nu-1} \nu B\left(\frac{\nu + \beta + 1}{2}, \frac{\nu - \beta + 1}{2}\right)} \quad [\operatorname{Re} \nu > -1].$$

NH 158, EH I 12(29)

$$2. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^{\nu-1} x dx = \frac{\pi}{2^{\nu-1} \nu B\left(\frac{\nu + \beta + 1}{2}, \frac{\nu - \beta + 1}{2}\right)} \quad [\operatorname{Re} \nu > -1].$$

$$\begin{aligned}
3.6 \int_0^{\frac{\pi}{2}} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x dx &= \frac{1}{2^{2\mu+2\nu+1}} \left\{ \exp \left[i\pi \left(\beta - \nu - \frac{1}{2} \right) \right] B(\beta - \mu - \nu, 2\nu + 1) \times \right. \\
&\quad \times F(-2\mu, \beta - \mu - \nu; 1 + \beta - \mu + \nu; -1) + \exp \left[i\pi \left(\mu + \frac{1}{2} \right) \right] \times \\
&\quad \times B(\beta - \mu - \nu, 2\mu + 1) F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1) \left. \right\} \\
&\quad \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
\end{aligned}$$

EH I 80(6)

512

$$4. \int_0^{\pi} e^{i2\beta x} \sin^{2\mu} x \cos^{2\nu} x dx = \frac{\pi \exp[i\pi(\beta - \nu)] F(-2\nu, \beta - \mu - \nu; 1 + \beta + \mu - \nu; -1)}{4^{\mu+\nu} (2\mu + 1) B(1 - \beta + \mu + \nu, 1 + \beta + \mu - \nu)}.$$

EH I 80(8)

$$\begin{aligned}
5. \int_0^{\frac{\pi}{2}} e^{i(\mu+\nu)x} \sin^{\mu-1} x \cos^{\nu-1} x dx &= e^{i\mu\frac{\pi}{2}} B(\mu, \nu) = \\
&= \frac{1}{2^{\mu+\nu-1}} e^{i\mu\frac{\pi}{2}} \left\{ \frac{1}{\mu} F(1 - \nu, 1; \mu + 1; -1) + \frac{1}{\nu} F(1 - \mu, 1; \nu + 1; -1) \right\} \\
&\quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

EH I 80(7)

3.893

$$1. \int_0^{\infty} e^{-px} \sin(qx + \lambda) dx = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda) \quad [p > 0].$$

BI ((261))(3)

$$2. \int_0^{\infty} e^{-px} \cos(qx + \lambda) dx = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda) \quad [p > 0].$$

BI ((261))(4)

$$3. \int_0^{\infty} e^{-x \cos t} \cos(t - x \sin t) dx = 1.$$

BI ((261))(7)

$$5. \int_0^\infty \frac{e^{-2px} \sin[(2n+1)x]}{\sin x} dx = \frac{1}{2p} + \sum_{k=1}^n \frac{p}{p^2 + k^2} \quad [p > 0].$$

BI ((267))(15)

$$6. \int_0^\infty \frac{e^{-px} \sin 2nx}{\sin x} dx = 2p \sum_{k=0}^{n-1} \frac{1}{p^2 + (2k+1)^2} \quad [p > 0].$$

GW ((335))(15c)

$$7. \int_0^\infty e^{-px} \cos[(2n+1)x] \operatorname{tg} x dx = \frac{2n+1}{p^2 + (2n+1)^2} + (-1)^n 2 \sum_{k=0}^{n-1} \frac{(-1)^k (2k+1)}{p^2 + (2k+1)^2} \quad [p > 0].$$

LI ((267))(16)

3.894

$$\int_{-\pi}^{\pi} [\beta + \sqrt{\beta^2 - 1} \cos x]^\nu e^{inx} dx = \frac{2\pi \Gamma(\nu+1) P_\nu^m(\beta)}{\Gamma(\nu+m+1)} \quad [\operatorname{Re} \beta > 0].$$

ET I 157(15)

3.895

$$1. \int_0^\infty e^{-\beta x} \sin^{2m} x dx = \frac{(2m)!}{\beta(\beta^2 + 2^2)(\beta^2 + 4^2) \dots [\beta^2 + (2m)^2]}; \quad [\operatorname{Re} \beta > 0].$$

FI II 615, WA 620a

$$2.* \int_0^\pi e^{-px} \sin^{2m} x dx = \frac{(2m)!(1 - e^{-p\pi})}{p(p^2 + 2^2)(p^2 + 4^2) \dots [p^2 + (2m)^2]}.$$

GW ((335))(4a)

513

$$3.* \int_0^{\frac{\pi}{2}} e^{-px} \sin^{2m} x dx = \frac{(2m)!}{p(p^2 + 2^2)(p^2 + 4^2) \dots [p^2 + (2m)^2]} \times$$

$$\times \left\{ 1 - e^{-\frac{p\pi}{2}} \left[1 + \frac{p^2}{2!} + \frac{p^4}{4!} + \dots + \frac{p^{2m}}{(2m)!} \right] \right\}$$

$$4. \int_0^{\infty} e^{-\beta x} \sin^{2m+1} x \, dx = \frac{(2m+1)!}{(\beta^2+1^2)(\beta^2+3^2)\dots[\beta^2+(2m+1)^2]} \quad [\operatorname{Re} \beta > 0].$$

FI II 615, WA 620a

$$5.* \int_0^{\pi} e^{-px} \sin^{2m+1} x \, dx = \frac{(2m+1)!(1+e^{-p\pi})}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]}.$$

GW ((335))(4b)

$$6.* \int_0^{\frac{\pi}{2}} e^{-px} \sin^{2m+1} x \, dx = \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]} \times \\ \times \left\{ 1 - pe^{\frac{p\pi}{2}} \left[1 + \frac{p^2+1^2}{3!} + \dots + \frac{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!} \right] \right\}$$

BI ((270))(5)

$$7. \int_0^{\infty} e^{-px} \cos^{2m} x \, dx = \frac{(2m)!}{p(p^2+2^2)\dots[p^2+(2m)^2]} \times \\ \times \left\{ 1 + \frac{p^2}{2!} + \frac{p^2(p^2+2^2)}{4!} + \dots + \frac{p^2(p^2+2^2)\dots[p^2+(2m-2)^2]}{(2m)!} \right\} \\ [p > 0].$$

BI ((262))(3)

$$8.* \int_0^{\frac{\pi}{2}} e^{-px} \cos^{2m} x \, dx = \frac{(2m)!}{p(p^2+2^2)\dots[p^2+(2m)^2]} \times \\ = \times \left\{ -e^{-p\frac{\pi}{2}} + 1 + \frac{p^2}{2!} + \frac{p^2(p^2+2^2)}{4!} + \dots + \frac{p^2(p^2+2^2)\dots[p^2+(2m-2)^2]}{(2m)!} \right\}$$

BI ((270))(6)

$$9.7 \int_0^{\infty} e^{-px} \cos^{2m+1} x \, dx = \frac{(2m+1)!p}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]} \times \\ = \times \left\{ 1 + \frac{p^2+1^2}{3!} + \frac{(p^2+1^2)(p^2+3^2)}{5!} + \dots + \frac{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!} \right\} \\ [p > 0].$$

BI ((262))(4)

$$10.^7 \int_0^{\frac{\pi}{2}} e^{-px} \cos^{2m+1} x dx = \frac{(2m+1)!}{(p^2+1^2)(p^2+3^2)\dots[p^2+(2m+1)^2]} \times \\ = \times \left\{ e^{-p\frac{\pi}{2}} + p \left[1 + \frac{p^2+1^2}{3!} + \dots + \frac{(p^2+1)(p^2+3^2)\dots[p^2+(2m-1)^2]}{(2m+1)!} \right] \right\}$$

BI ((270))(7)

$$11. \int_0^{\infty} e^{-\beta x} \sin^{2n} x \sin ax dx = -\frac{1}{(-4)^{n+1}(2n+1)} \left\{ \frac{1}{\binom{\frac{a}{2} + i\frac{\beta}{2} + n}{2n+1}} + \frac{1}{\binom{\frac{a}{2} - i\frac{\beta}{2} + n}{2n+1}} \right\} \\ [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 80(19)

$$12.^7 \int_0^{\infty} e^{-\beta x} \sin^{2n-1} x \sin ax dx = \frac{-i}{(-4)^{n+1}n} \left\{ \frac{1}{\binom{\frac{a}{2} - i\frac{\beta}{2} + n - \frac{1}{2}}{2n}} - \frac{1}{\binom{\frac{a}{2} + i\frac{\beta}{2} + n - \frac{1}{2}}{2n}} \right\} \\ [\operatorname{Re} \beta > 0, \quad a > 0, n > 0].$$

ET I 80(20)a

$$13.^* \int_0^{\infty} e^{-\beta x} \sin^{2n} x \cos ax dx = \frac{(-1)^n i}{(2n+1)2^{2n+2}} \left\{ \frac{1}{\binom{\frac{a}{2} + i\frac{\beta}{2} + n}{2n+1}} - \frac{1}{\binom{\frac{a}{2} - i\frac{\beta}{2} + n}{2n+1}} \right\} \\ [\operatorname{Re} \beta > 0, \quad a \geq 0].$$

ET I 20(12)a

$$14.^* \int_0^{\infty} e^{-\beta x} \sin^{2n-1} x \cos ax dx = \frac{(-1)^n}{2^{2n+2}n} \left\{ \frac{1}{\binom{\frac{a}{2} - i\frac{\beta}{2} + n - \frac{1}{2}}{2n}} + \frac{1}{\binom{\frac{a}{2} + i\frac{\beta}{2} + n - \frac{1}{2}}{2n}} \right\} \\ [\operatorname{Re} \beta > 0, \quad a \geq 0, n > 0].$$

$$1. \int_{-\infty}^{\infty} e^{-q^2 x^2} \sin[p(x + \lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \sin p\lambda.$$

BI ((269))(2)

$$2. \int_{-\infty}^{\infty} e^{-q^2 x^2} \cos[p(x + \lambda)] dx = \frac{\sqrt{\pi}}{q} e^{-\frac{p^2}{4q^2}} \cos p\lambda.$$

BI ((269))(3)

515

$$\begin{aligned} 3. \int_0^{\infty} e^{-ax^2} \sin bx dx &= \frac{b}{2a} \exp\left(-\frac{b^2}{4a}\right) {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) = \\ &= \frac{b}{2a} {}_1F_1\left(1; \frac{3}{2}; -\frac{b^2}{4a}\right); \\ &= \frac{b}{2a} \sum_{k=1}^{\infty} \frac{1}{(2k-1)!!} \left(-\frac{b^2}{2a}\right)^{k-1} \quad [a > 0]. \end{aligned}$$

FI II 720
ET I 73(18)

$$4. \int_0^{\infty} e^{-\beta x^2} \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp\left(-\frac{b^2}{4\beta}\right) \quad [\operatorname{Re} \beta > 0].$$

BI ((263))(2)

3.897

$$\begin{aligned} 1. \int_0^{\infty} e^{-\beta x^2 - \nu x} \sin bx dx &= -\frac{i}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp\frac{(\gamma - ib)^2}{4\beta} \left[1 - \Phi\left(\frac{\gamma - ib}{2\sqrt{\beta}}\right)\right] - \right. \\ &\quad \left. - \exp\frac{(\gamma + ib)^2}{4\beta} \left[1 - \Phi\left(\frac{\gamma + ib}{2\sqrt{\beta}}\right)\right] \right\} \quad [\operatorname{Re} \beta > 0]. \end{aligned}$$

ET I 74(27)

$$\begin{aligned} 2. \int_0^{\infty} e^{-\beta x^2 - \gamma x} \cos bx dx &= \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ \exp\frac{(\gamma - ib)^2}{4\beta} \left[1 - \Phi\left(\frac{\gamma - ib}{2\sqrt{\beta}}\right)\right] + \right. \\ &\quad \left. + \exp\frac{(\gamma + ib)^2}{4\beta} \left[1 - \Phi\left(\frac{\gamma + ib}{2\sqrt{\beta}}\right)\right] \right\} \quad [\operatorname{Re} \beta > 0]. \end{aligned}$$

ET I 15(16)

3.898

$$1. \int_0^{\infty} e^{-\beta x^2} \sin ax \sin bx dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} - e^{-\frac{(a+b)^2}{4\beta}} \right\} \quad [\operatorname{Re} \beta > 0].$$

$$2. \int_0^{\infty} e^{-\beta x^2} \cos ax \cos bx \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta}} \left\{ e^{-\frac{(a-b)^2}{4\beta}} + e^{-\frac{(a+b)^2}{4\beta}} \right\} \quad [\operatorname{Re} \beta > 0].$$

BI ((263))(5)

$$3. \int_0^{\infty} e^{-px^2} \sin^2 ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \left(1 - e^{-\frac{a^2}{p}} \right) \quad [p > 0].$$

BI ((263))(6)

3.899

$$1.7 \int_0^{\infty} \frac{e^{-p^2 x^2} \sin[(2n+1)x]}{\sin x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=1}^n e^{-\left(\frac{k}{p}\right)^2} \right] \quad [p > 0].$$

BI ((267))(17)

$$2.* \int_0^{\infty} \frac{e^{-p^2 x^2} \cos[(4n+1)x]}{\cos x} \, dx = \frac{\sqrt{\pi}}{p} \left[\frac{1}{2} + \sum_{k=1}^{2n} (-1)^{k-n} e^{-\left(\frac{k}{p}\right)^2} \right] \quad [p > 0].$$

BI ((267))(18)

516

$$3. \int_0^{\infty} \frac{e^{-px^2} \, dx}{1 - 2a \cos x + a^2} = \frac{\sqrt{\frac{\pi}{p}}}{1 - a^2} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^k \exp\left(-\frac{k^2}{4p}\right) \right\} \quad [a^2 < 1, \quad p > 0];$$

$$= \frac{\sqrt{\frac{\pi}{p}}}{a^2 - 1} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} a^{-k} \exp\left(-\frac{k^2}{4p}\right) \right\} \quad [a^2 > 1, \quad p > 0].$$

LI ((266))(1)

3.911

$$1. \int_0^{\infty} \frac{\sin ax}{e^{\beta x} + 1} \, dx = \frac{1}{2a} - \frac{\pi}{2\beta \operatorname{sh} \frac{a\pi}{\beta}} \quad [a > 0, \quad \operatorname{Re} \beta > 0].$$

BI ((264))(1)

$$2. \int_0^{\infty} \frac{\sin ax}{e^{\beta x} - 1} \, dx = \frac{\pi}{2\beta} \operatorname{cth} \left(\frac{\pi a}{\beta} \right) - \frac{1}{2a} \quad [a > 0, \quad \operatorname{Re} \beta > 0].$$

$$3.6 \int_0^{\infty} \frac{\sin ax}{e^x - 1} e^{\frac{x}{2}} dx = \frac{1}{2} \pi \theta(a\pi) \quad [a > 0].$$

ET I 73(13)

$$4. \int_0^{\infty} \frac{\sin ax}{1 - e^{-x}} e^{-nx} dx = \frac{\pi}{2} - \frac{1}{2a} + \frac{\pi}{e^{2\pi a} - 1} - \sum_{k=1}^{n-1} \frac{a}{a^2 + k^2} \quad [a > 0].$$

BI ((264))(8)

$$5. \int_0^{\infty} \frac{\sin ax}{e^{\beta x} - e^{\gamma x}} dx = \frac{1}{2i(\beta - \gamma)} \left[\psi \left(\frac{\beta + ia}{\beta - \gamma} \right) - \psi \left(\frac{\beta - ia}{\beta - \gamma} \right) \right] \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

GW ((335))(8)

$$6. \int_0^{\infty} \frac{\sin ax dx}{e^{\beta x}(e^{-x} - 1)} = \frac{i}{2} [\psi(\beta + ia) - \psi(\beta - ia)] \quad [\operatorname{Re} \beta > -1].$$

ET 73(15)

3.912

$$1. \int_0^{\infty} e^{-\beta x} (1 - e^{-\gamma x})^{\nu-1} \sin ax dx = -\frac{i}{2\gamma} \left[\operatorname{B} \left(\nu, \frac{\beta - ia}{\gamma} \right) - \operatorname{B} \left(\nu, \frac{\beta + ia}{\gamma} \right) \right] \\ [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, \operatorname{Re} \nu > 0, a > 0].$$

ET I 73(17)

$$2. \int_0^{\infty} e^{-\beta x} (1 - e^{-\gamma x})^{\nu-1} \cos ax dx = \frac{1}{2\gamma} \left[\operatorname{B} \left(\nu, \frac{\beta - ia}{\gamma} \right) + \operatorname{B} \left(\nu, \frac{\beta + ia}{\gamma} \right) \right] \\ [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, \operatorname{Re} \nu > 0, a > 0].$$

ET I 15(10)

3.913

$$1. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta x} \cos^{\nu} x (\beta^2 e^{ix} + \nu^2 e^{-ix})^{\mu} dx = \frac{\pi {}_2F_1 \left(-\mu, \frac{\beta}{2} - \frac{\nu}{2} - \frac{\mu}{2}; 1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}; \frac{\beta^2}{\nu^2} \right)}{2^{\nu}(\nu + 1) \operatorname{B} \left(1 + \frac{\beta}{2} + \frac{\nu}{2} - \frac{\mu}{2}, 1 - \frac{\beta}{2} + \frac{\nu}{2} + \frac{\mu}{2} \right)} \\ [\operatorname{Re} \nu > -1, |\nu| > |\beta|].$$

EH I 81(11)a

$$\begin{aligned}
2. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{iux} \cos^\mu x (a^2 e^{ix} + b^2 e^{-ix})^\nu dx &= \\
&= \frac{\pi b^{2\nu} {}_2F_1\left(-\nu, \frac{u+\mu+\nu}{2}; 1 + \frac{\mu-\nu-u}{2}; \frac{a^2}{b^2}\right)}{2^\mu(\mu+1)\text{B}\left(1 - \frac{u+\nu-\mu}{2}, 1 + \frac{u+\mu+\nu}{2}\right)} && \text{for } a^2 < b^2; \\
&= \frac{\pi a^{2\nu} {}_2F_1\left(-\nu, \frac{u+\mu-\nu}{2}; 1 + \frac{\mu-\nu+u}{2}; \frac{b^2}{a^2}\right)}{2^\mu(\mu+1)\text{B}\left(1 + \frac{u+\mu-\nu}{2}, 1 + \frac{\mu+\nu-u}{2}\right)} && \text{for } b^2 < a^2 \quad [\text{Re } \mu > -1].
\end{aligned}$$

ET I 122(31)a

3.914

$$\int_0^\infty e^{-\beta\sqrt{\gamma^2+x^2}} \cos bx \, dx = \frac{\beta\gamma}{\sqrt{\beta^2+b^2}} K_1(\gamma\sqrt{\beta^2+b^2}) \quad [\text{Re } \beta > 0, \text{ Re } \gamma > 0].$$

ET I 16(26)

3.915

$$1. \int_0^\pi e^{a \cos x} \sin x \, dx = \frac{2}{a} \text{sh } a.$$

GW ((337))(15c)

$$2. \int_0^\pi e^{i\beta \cos x} \cos nx \, dx = i^n \pi J_n(\beta).$$

EH II 81(2)

$$3. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i\beta \sin x} \cos^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) J_\nu(\beta) \quad \left[\text{Re } \nu > -\frac{1}{2}\right].$$

EH II 81(6)

$$4. \int_0^\pi e^{\pm\beta \cos x} \sin^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^\nu \Gamma\left(\nu + \frac{1}{2}\right) I_\nu(\beta) \quad \left[\text{Re } \nu > -\frac{1}{2}\right].$$

GW ((337))(15b)

$$5. \int_0^{\pi} e^{i\beta \cos x} \sin^{2\nu} x \, dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(\beta) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

WA 34(2), WA 60(6)

3.916

$$1. \int_0^{\frac{\pi}{2}} e^{-p^2 \operatorname{tg} x} \frac{\sin \frac{x}{2} \sqrt{\cos x}}{\sin 2x} \, dx = \left[C(p) - \frac{1}{2}\right]^2 + \left[S(p) - \frac{1}{2}\right]^2 .m$$

NT 33(18)a

$$2. \int_0^{\frac{\pi}{2}} \frac{\exp(-p \operatorname{tg} x) \, dx}{\sin 2x + a \cos 2x + a} = -\frac{1}{2} e^{ap} \operatorname{Ei}(-ap) \quad [p > 0], \quad (\text{cf. 3552 4. and 6.})$$

BI ((273))(11)

$$3. \int_0^{\frac{\pi}{2}} \frac{\exp(-p \operatorname{ctg} x) \, dx}{\sin 2x + a \cos 2x - a} = -\frac{1}{2} e^{-ap} \operatorname{Ei}(ap) \quad [p > 0], \quad (\text{cf. 3.552 4. and 6.}).$$

3.552
BI ((273))(12)

518

$$4. \int_0^{\frac{\pi}{2}} \frac{\exp(-p \operatorname{tg} x) \sin 2x \, dx}{(1-a^2) - 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)] \quad [p > 0].$$

BI ((273))(13)

$$5. \int_0^{\frac{\pi}{2}} \frac{\exp(-p \operatorname{ctg} x) \sin 2x \, dx}{(1-a^2) + 2a^2 \cos 2x - (1+a^2) \cos^2 2x} = -\frac{1}{4} [e^{-ap} \operatorname{Ei}(ap) + e^{ap} \operatorname{Ei}(-ap)] \quad [p > 0].$$

BI ((273))(14)

3.917

$$1. \int_0^{\frac{\pi}{2}} e^{-2\beta \operatorname{ctg} x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \sin \left[\beta - \left(\nu - \frac{1}{2} \right) x \right] \, dx = \frac{\sqrt{\pi}}{2(2\beta)^{\nu}} \Gamma\left(\nu + \frac{1}{2}\right) J_{\nu}(\beta) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

$$2. \int_0^{\frac{\pi}{2}} e^{-2\beta \operatorname{ctg} x} \cos^{\nu-1/2} x \sin^{-(\nu+1)} x \cos \left[\beta - \left(\nu - \frac{1}{2} \right) x \right] dx = \frac{\sqrt{\pi}}{2(2\beta)^\nu} \Gamma \left(\nu + \frac{1}{2} \right) N_\nu(\beta) \\ \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 186(8)

3.918

$$1. \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x}{\sin^{2\mu+2} x} e^{i\gamma(\beta-\mu x)-2\beta \operatorname{ctg} x} dx = \frac{i\gamma}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) H_{\mu+\frac{1}{2}}^{(\varepsilon)}(\beta) \\ \left[\varepsilon = 1, 2; \quad \gamma = (-1)^{\varepsilon+1}; \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1 \right].$$

GW ((337))(16)

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x \sin(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \operatorname{ctg} x} dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) J_{\mu+\frac{1}{2}}(\beta) \\ \left[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1 \right].$$

WH

$$3. \int_0^{\frac{\pi}{2}} \frac{\cos^\mu x \cos(\beta - \mu x)}{\sin^{2\mu+2} x} e^{-2\beta \operatorname{ctg} x} dx = -\frac{1}{2} \sqrt{\frac{\pi}{2\beta}} (2\beta)^{-\mu} \Gamma(\mu+1) N_{\mu+\frac{1}{2}}(\beta) \\ \left[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > -1 \right].$$

GW ((337))(17b)

3.919

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin^{2n+2} x} \cdot \frac{dx}{\exp(2\pi \operatorname{ctg} x) - 1} = (-1)^{n-1} \frac{2n-1}{4(2n+1)}.$$

BI ((275))(6), LI ((275))(6)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin^{2n+2} x} \frac{dx}{\exp(\pi \operatorname{ctg} x) - 1} = (-1)^{n-1} \frac{n}{2n+1}.$$

BI ((275))(7), LI ((275))(7)

519

3.92 Trigonometric functions of more complicated arguments combined with exponentials

exponentials

3.921

$$\int_0^{\infty} e^{-\gamma x} \cos ax^2 (\cos \gamma x - \sin \gamma x) dx = \sqrt{\frac{\pi}{8a}} \exp\left(-\frac{\gamma^2}{2a}\right) \quad [a > 0, \quad \operatorname{Re} \gamma \geq |\operatorname{Im} \gamma|].$$

ET I 26(28)

3.922

$$\begin{aligned} 1. \int_0^{\infty} e^{-\beta x^2} \sin ax^2 dx &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-\beta x^2} \sin ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} - \beta}{\beta^2 + a^2}} = \\ &= \frac{\sqrt{\pi}}{2 \sqrt[4]{\beta^2 + a^2}} \sin\left(\frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad a > 0]. \end{aligned}$$

FI II 750, BI ((263))(8)

$$\begin{aligned} 2. \int_0^{\infty} e^{-\beta x^2} \cos ax^2 dx &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-\beta x^2} \cos ax^2 dx = \sqrt{\frac{\pi}{8}} \sqrt{\frac{\sqrt{\beta^2 + a^2} + \beta}{\beta^2 + a^2}} = \\ &= \frac{\sqrt{\pi}}{2 \sqrt[4]{\beta^2 + a^2}} \cos\left(\frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad a > 0]. \end{aligned}$$

FI II 750, BI ((263))(9)

$$\begin{aligned} 3. \int_0^{\infty} e^{-\beta x^2} \sin ax^2 \cos bx dx &= -\frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \sin Aa - C \cos Aa) = \\ &= \frac{\sqrt{\pi}}{2 \sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \sin\left\{\frac{1}{2} \operatorname{arctg} \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\}. \end{aligned}$$

LI ((263))(10), GW ((337))(5)

$$\begin{aligned} 4. \int_0^{\infty} e^{-\beta x^2} \cos ax^2 \cos bx dx &= \frac{1}{2} \sqrt{\frac{\pi}{\beta^2 + a^2}} e^{-A\beta} (B \cos Aa + C \sin Aa) = \\ &= \frac{\sqrt{\pi}}{2 \sqrt[4]{\beta^2 + a^2}} \exp\left(-\frac{\beta b^2}{4(\beta^2 + a^2)}\right) \cos\left\{\frac{1}{2} \operatorname{arctg} \frac{a}{\beta} - \frac{ab^2}{4(\beta^2 + a^2)}\right\}. \end{aligned}$$

LI ((263))(11), GW ((337))(5)

[In formulas 3.922 3 and 4. $a > 0, b > 0, \operatorname{Re} \beta > 0,$

$$A = \frac{b^2}{4(a^2 + \beta^2)}, \quad B = \sqrt{\frac{1}{2}(\sqrt{\beta^2 + a^2} + \beta)}, \quad C = \sqrt{\frac{1}{2}(\sqrt{\beta^2 + a^2} - \beta)}.$$

If a is complex, $\operatorname{Re} \beta > |\operatorname{Im} a|$.

520

3.923

$$\begin{aligned}
 1. \int_{-\infty}^{\infty} \exp[-(ax^2 + 2bx + c)] \sin(px^2 + 2qx + r) dx &= \\
 &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \times \\
 &\quad \times \sin \left\{ \frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \quad [a > 0].
 \end{aligned}$$

GW ((337))(3), BI ((296))(6)

$$\begin{aligned}
 2. \int_{-\infty}^{\infty} \exp[-(ax^2 + 2bx + c)] \cos(px^2 + 2qx + r) dx &= \\
 &= \frac{\sqrt{\pi}}{\sqrt[4]{a^2 + p^2}} \exp \frac{a(b^2 - ac) - (aq^2 - 2bpq + cp^2)}{a^2 + p^2} \times \\
 &\quad \times \cos \left\{ \frac{1}{2} \operatorname{arctg} \frac{p}{a} - \frac{p(q^2 - pr) - (b^2p - 2abq + a^2r)}{a^2 + p^2} \right\} \quad [a > 0].
 \end{aligned}$$

GW ((337))(3), BI ((269))(7)

3.924

$$1. \int_0^{\infty} e^{-\beta x^4} \sin bx^2 dx = \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{\frac{1}{4}}\left(\frac{b^2}{8\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

ET 73(22)

$$2. \int_0^{\infty} e^{-\beta x^4} \cos bx^2 dx = \frac{\pi}{4} \sqrt{\frac{b}{2\beta}} \exp\left(-\frac{b^2}{8\beta}\right) I_{-\frac{1}{4}}\left(\frac{b^2}{8\beta}\right) \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

ET I 15(12)

3.925

$$1. \int_0^{\infty} e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\frac{p^2}{x^2}} \sin 2a^2 x^2 dx = \frac{\sqrt{\pi}}{4a} e^{-2ap} (\cos 2ap + \sin 2ap)$$

$$[a > 0, \quad b > 0].$$

BI ((268))(12)

BI ((268))(13)

3.926

$$1. \int_0^{\infty} e^{-(\beta x^2 + \frac{\gamma}{x^2})} \sin ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [v \cos(2v\sqrt{\gamma}) + u \sin(2v\sqrt{\gamma})]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

BI ((268))(14)

521

$$2. \int_0^{\infty} e^{-(\beta x^2 + \frac{\gamma}{x^2})} \cos ax^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{a^2 + \beta^2}} e^{-2u\sqrt{\gamma}} [u \cos(2v\sqrt{\gamma}) - v \sin(2v\sqrt{\gamma})]$$

$$[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

BI ((268))(15)

[In formulas 3.926 1., 3.926 2.

$$u = \sqrt{\frac{\sqrt{a^2 + \beta^2} + \beta}{2}}, \quad v = \sqrt{\frac{\sqrt{a^2 + \beta^2} - \beta}{2}}.]$$

3.927

$$\int_0^{\infty} e^{-\frac{p}{x}} \sin^2 \frac{a}{x} dx = a \operatorname{arctg} \frac{2a}{p} + \frac{p}{4} \ln \frac{p^2}{p^2 + 4a^2} \quad [a > 0, \quad p > 0].$$

LI ((268))(4)

3.928

$$1. \int_0^{\infty} \exp \left[- \left(p^2 x^2 + \frac{q^2}{x^2} \right) \right] \sin \left(a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \sin \{ A + 2rs \sin(A+B) \} .$$

BI ((268))(22)

$$2. \int_0^{\infty} \exp \left[- \left(p^2 x^2 + \frac{q^2}{x^2} \right) \right] \cos \left(a^2 x^2 + \frac{b^2}{x^2} \right) dx = \frac{\sqrt{\pi}}{2r} e^{-2rs \cos(A+B)} \cos \{ A + 2rs \sin(A+B) \} .$$

BI ((268))(23)

[In formulas 3.928 1., 3.928 2. $a^2 + p^2 > 0$ and

$$r = \sqrt[4]{a^4 + p^4}, \quad s = \sqrt[4]{b^4 + q^4}, \quad A = \frac{1}{2} \operatorname{arctg} \frac{a^2}{p^2}, \quad B = \frac{1}{2} \operatorname{arctg} \frac{b^2}{q^2}.]$$

3.929

$$\int_0^{\infty} [e^{-x} \cos(p\sqrt{x}) + pe^{-x^2} \sin px] dx = 1.$$

LI ((268))(3)

3.93 Trigonometric and exponential functions of trigonometric functions

3.931

$$1. \int_0^{\frac{\pi}{2}} e^{-p \cos x} \sin(p \sin x) dx = \operatorname{Ei}(-p) - \operatorname{ci}(p).$$

NT 13(27)

$$2. \int_0^{\pi} e^{-p \cos x} \sin(p \sin x) dx = - \int_{-\pi}^0 e^{-p \cos x} \sin(p \sin x) dx = -2 \operatorname{si}(p).$$

GW ((337))(11b)

$$3. \int_0^{\frac{\pi}{2}} e^{-p \cos x} \cos(p \sin x) dx = -\operatorname{si}(p).$$

NT 13(26)

$$4. \int_0^{\frac{\pi}{2}} e^{-p \cos x} \cos(p \sin x) dx = \frac{1}{2} \int_0^{2\pi} e^{-p \cos x} \cos(p \sin x) dx = \pi.$$

GW ((337))(11a)

522

3.932

$$1. \int_0^{\pi} e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \sin(p \sin x) \sin mx dx = \frac{\pi}{2} \cdot \frac{p^m}{m!}.$$

$$2. \int_0^\pi e^{p \cos x} \cos(p \sin x) \cos mx \, dx = \frac{1}{2} \int_0^{2\pi} e^{p \cos x} \cos(p \sin x) \cos mx \, dx = \frac{\pi}{2} \cdot \frac{p^m}{m!}.$$

BI ((277))(8), GW ((337))(13b)

3.933

$$\int_0^\pi e^{p \cos x} \sin(p \sin x) \operatorname{cosec} x \, dx = \pi \operatorname{sh} p.$$

BI ((278))(1)

3.934

$$1. \int_0^\pi e^{p \cos x} \sin(p \sin x) \operatorname{tg} \frac{x}{2} \, dx = \pi(1 - e^p).$$

BI ((271))(8)

$$2. \int_0^\pi e^{p \cos x} \sin(p \sin x) \operatorname{ctg} \frac{x}{2} \, dx = \pi(e^p - 1).$$

BI ((272))(5)

3.935

$$\int_0^\pi e^{p \cos x} \cos(p \sin x) \frac{\sin 2nx}{\sin x} \, dx = \pi \sum_{k=0}^{n-1} \frac{p^{2k+1}}{(2k+1)!} \quad [p > 0].$$

LI ((278))(3)

3.936

$$1. \int_0^{2\pi} e^{p \cos x} \cos(p \sin x - mx) \, dx = 2 \int_0^\pi e^{p \cos x} \cos(p \sin x - mx) \, dx = \frac{2\pi p^m}{m!}.$$

BI ((277))(9), GW ((337))(14a)

$$2. \int_0^{2\pi} e^{p \sin x} \sin(p \cos x + mx) \, dx = \frac{2\pi p^m}{m!} \sin \frac{m\pi}{2} \quad [p > 0].$$

GW ((337))(14b)

$$4. \int_0^{2\pi} e^{\cos x} \sin(mx - \sin x) dx = 0.$$

WH

$$5. \int_0^\pi e^{\beta \cos x} \cos(ax + \beta \sin x) dx = \beta^{-a} \sin(a\pi) \gamma(a, \beta).$$

EH II 137(2)

3.937

$$1. \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(a \cos x + b \sin x - mx) dx = \\ = i\pi[(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A+iB) \frac{m}{2} I_m(\sqrt{C-iD}) - (A-iB) \frac{m}{2} I_m(\sqrt{C+iD}) \right\}.$$

GW ((337))(9b)

523

$$2. \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(a \cos x + b \sin x - mx) dx = \\ = \pi[(b-p)^2 + (a+q)^2]^{-\frac{m}{2}} \left\{ (A+iB)^{\frac{m}{2}} I_m(\sqrt{C-iD}) + (A-iB)^{\frac{m}{2}} I_m(\sqrt{C+iD}) \right\}.$$

[In formulas 3.937 1. and 3.937 2. $(b-p)^2 + (a+q)^2 > 0$, $m = 0, 1, 2, \dots$, $A = p^2 - q^2 + a^2 - b^2$,

$B = 2(pq + ab)$, $C = p^2 + q^2 - a^2 - b^2$, $D = 2(ap + bq)$.]

$$3. \int_0^{2\pi} \exp(p \cos x + q \sin x) \sin(q \cos x - p \sin x + mx) dx = \frac{2\pi}{m!} (p^2 + q^2)^{\frac{m}{2}} \sin \left(m \operatorname{arctg} \frac{q}{p} \right).$$

GW ((337))(12)

GW ((337))(9a)

$$4. \int_0^{2\pi} \exp(p \cos x + q \sin x) \cos(q \cos x - p \sin x + mx) dx = \frac{2\pi}{m!} (p^2 + q^2)^{\frac{m}{2}} \cos \left(m \operatorname{arctg} \frac{q}{p} \right).$$

$$1. \int_0^\pi e^{r(\cos px + \cos qx)} \sin(r \sin px) \sin(r \sin qx) dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(pk+1)\Gamma(qk+1)} r^{(p+q)k}.$$

BI ((277))(14)

$$2. \int_0^\pi e^{r(\cos px + \cos qx)} \cos(r \sin px) \cos(r \sin qx) dx = \frac{\pi}{2} \left(2 + \sum_{k=1}^{\infty} \frac{r^{(p+q)k}}{\Gamma(pk+1)\Gamma(qk+1)} \right).$$

BI ((277))(15)

3.939

$$1. \int_0^\pi e^{q \cos x} \frac{\sin rx}{1 - 2p^r \cos rx + p^{2r}} \sin(q \sin x) dx = \frac{\pi}{2pr} \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \quad [r > 0, \quad 0 < p < 1].$$

BI ((278))(15)

$$2.3 \int_0^\pi e^{q \cos x} \frac{1 - p^r \cos rx}{1 - 2p^r \cos rx + p^{2r}} \cos(q \sin x) dx = \frac{\pi}{2} \left[2 + \sum_{k=1}^{\infty} \frac{(pq)^{kr}}{\Gamma(kr+1)} \right] \quad [r > 0, \quad 0 < p < 1].$$

BI ((278))(16)

$$3. \int_0^{\frac{\pi}{2}} \frac{e^{p \cos 2x} \cos(p \sin 2x) dx}{\cos^2 x + q^2 \sin^2 x} = \frac{\pi}{2q} \exp\left(p \frac{q-1}{q+1}\right).$$

BI ((273))(8)

3.94- 3.97 Combinations involving trigonometric functions, exponentials, and powers

3.941

$$1. \int_0^\infty e^{-px} \sin qx \frac{dx}{x} = \operatorname{arctg} \frac{q}{p} \quad [p > 0].$$

BI ((365))(1)

524

$$2. \int_0^\infty e^{-px} \cos qx \frac{dx}{x} = \infty.$$

$$1. \int_0^{\infty} e^{-px} \cos px \frac{x dx}{b^4 + x^4} = \frac{\pi}{4b^2} \exp(-bp\sqrt{2}) \quad [p > 0, \quad b > 0].$$

BI ((386))(6)a

$$2. \int_0^{\infty} e^{-px} \cos px \frac{x dx}{b^4 - x^4} = \frac{\pi}{4b^2} e^{-bp} \sin bp \quad [p > 0, \quad b > 0].$$

BI ((386))(7)a

3.943

$$\int_0^{\infty} e^{-\beta x} (1 - \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \beta^2}{\beta^2} \quad [\operatorname{Re} \beta > 0].$$

BI ((367))(6)

3.944

$$1. \int_0^u x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \gamma[\mu, (\beta + i\delta)u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \gamma[\mu, (\beta - i\delta)u] \\ [\operatorname{Re} \mu > -1].$$

ET I 318(8)

$$2. \int_u^{\infty} x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{i}{2} (\beta + i\delta)^{-\mu} \Gamma[\mu, (\beta + i\delta)u] - \frac{i}{2} (\beta - i\delta)^{-\mu} \Gamma[\mu, (\beta - i\delta)u] \\ [\operatorname{Re} \beta > |\operatorname{Im} \delta|].$$

ET I 318(9)

$$3. \int_0^u x^{\mu-1} e^{-\beta x} \cos \delta x dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \gamma[\mu, (\beta + i\delta)u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \gamma[\mu, (\beta - i\delta)u] \\ [\operatorname{Re} \mu > 0].$$

ET I 320(28)

$$4. \int_u^{\infty} x^{\mu-1} e^{-\beta x} \cos \delta x dx = \frac{1}{2} (\beta + i\delta)^{-\mu} \Gamma[\mu, (\beta + i\delta)u] + \frac{1}{2} (\beta - i\delta)^{-\mu} \Gamma[\mu, (\beta - i\delta)u] \\ [\operatorname{Re} \beta > |\operatorname{Im} \delta|].$$

ET I 320(29)

$$5. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \sin \delta x dx = \frac{\Gamma(\mu)}{(\beta^2 + \delta^2)^{\frac{\mu}{2}}} \sin \left(\mu \operatorname{arctg} \frac{\delta}{\beta} \right) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta|].$$

$$6. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \cos \delta x \, dx = \frac{\Gamma(\mu)}{(\delta^2 + \beta^2)^{\frac{\mu}{2}}} \cos \left(\mu \operatorname{arctg} \frac{\delta}{\beta} \right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \delta|].$$

FI II 812, BI ((361))(10)

$$7. \int_0^{\infty} x^{\mu-1} \exp(-ax \cos t) \sin(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \sin(\mu t) \\ \left[\operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2} \right].$$

EH I 13(36)

525

$$8. \int_0^{\infty} x^{\mu-1} \exp(-ax \cos t) \cos(ax \sin t) \, dx = \Gamma(\mu) a^{-\mu} \cos(\mu t) \\ \left[\operatorname{Re} \mu > -1, \quad a > 0, \quad |t| < \frac{\pi}{2} \right].$$

EH I 13(35)

$$9. \int_0^{\infty} x^{p-1} e^{-qx} \sin(qx \operatorname{tg} t) \, dx = \frac{1}{q^p} \Gamma(p) \cos^p t \sin pt \quad \left[|t| < \frac{\pi}{2}, \quad q > 0 \right].$$

LO V 288(16)

$$10. \int_0^{\infty} x^{p-1} e^{-qx} \cos(qx \operatorname{tg} t) \, dx = \frac{1}{q^p} \Gamma(p) \cos^p(t) \cos pt \quad \left[|t| < \frac{\pi}{2}, \quad q > 0 \right].$$

LO V 288(15)

$$11. \int_0^{\infty} x^n e^{-\beta x} \sin bx \, dx = n! \left(\frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n} (-1)^k \binom{n+1}{2k+1} \left(\frac{b}{\beta} \right)^{2k+1} = \\ = (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{b}{b^2 + \beta^2} \right) \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

GW ((336))(3), ET I 72(3)

$$12. \int_0^{\infty} x^n e^{-\beta x} \cos bx \, dx = n! \left(\frac{\beta}{\beta^2 + b^2} \right)^{n+1} \sum_{0 \leq 2k \leq n+1} (-1)^k \binom{n+1}{2k} \left(\frac{b}{\beta} \right)^{2k} = \\ = (-1)^n \frac{\partial^n}{\partial \beta^n} \left(\frac{\beta}{b^2 + \beta^2} \right) \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

$$13. \int_0^{\infty} x^{n-1/2} e^{-\beta x} \sin bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \frac{\sqrt{\sqrt{\beta^2 + b^2} - \beta}}{\sqrt{\beta^2 + b^2}} \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

ET I 72(6)

$$14. \int_0^{\infty} x^{n-1/2} e^{-\beta x} \cos bx \, dx = (-1)^n \sqrt{\frac{\pi}{2}} \frac{d^n}{d\beta^n} \frac{\sqrt{\sqrt{\beta^2 + b^2} + \beta}}{\sqrt{\beta^2 + b^2}} \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

ET I 15(6)

3.945

$$1. \int_0^{\infty} (e^{-\beta x} \sin ax - e^{-\gamma x} \sin bx) \frac{dx}{x^r} = \\ = \Gamma(1-r) \left\{ (b^2 + \gamma^2) \frac{r-1}{2} \sin \left[(r-1) \operatorname{arctg} \frac{b}{\gamma} \right] - (a^2 + \beta^2) \frac{r-1}{2} \sin \left[(r-1) \operatorname{arctg} \frac{a}{\beta} \right] \right\} \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1].$$

BI((371))(6)

$$2. \int_0^{\infty} (e^{-\beta x} \cos ax - e^{-\gamma x} \cos bx) \frac{dx}{x^r} = \\ = \Gamma(1-r) \left\{ (a^2 + \beta^2) \frac{r-1}{2} \cos \left[(r-1) \operatorname{arctg} \frac{a}{\beta} \right] - (b^2 + \gamma^2) \frac{r-1}{2} \cos \left[(r-1) \operatorname{arctg} \frac{b}{\gamma} \right] \right\} \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0, \quad r < 2, \quad r \neq 1].$$

BI ((371))(7)

526

$$3. \int_0^{\infty} (ae^{-\beta x} \sin bx - be^{-\gamma x} \sin ax) \frac{dx}{x^2} = ab \left[\frac{1}{2} \ln \frac{a^2 + \gamma^2}{b^2 + \beta^2} + \frac{\gamma}{a} \operatorname{arctg} \frac{\gamma}{a} - \frac{\beta}{b} \operatorname{arctg} \frac{\beta}{b} \right] \\ [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

BI ((368))(22)

3.946

$$1. \int_0^{\infty} e^{-px} \sin^{2m+1} ax \frac{dx}{x} = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \operatorname{arctg} \frac{(2m-2k+1)a}{p} \\ [m = 0, 1, \dots, \quad p > 0].$$

GW ((336))(9a)

3.947

$$1. \int_0^{\infty} e^{-\beta x} \sin \gamma x \sin ax \frac{dx}{x} = \frac{1}{4} \ln \frac{\beta^2 + (a + \gamma)^2}{\beta^2 + (a - \gamma)^2} \quad [\operatorname{Re} \beta > |\operatorname{Im} \gamma|, \quad a > 0].$$

BI ((365))(5)

$$2. \int_0^{\infty} e^{-px} \sin ax \sin bx \frac{dx}{x^2} = \frac{a}{2} \operatorname{arctg} \frac{2pb}{p^2 + a^2 - b^2} + \frac{b}{2} \operatorname{arctg} \frac{2pa}{p^2 + b^2 - a^2} + \frac{p}{4} \ln \frac{p^2 + (a - b)^2}{p^2 + (a + b)^2} \\ [p > 0].$$

BI ((368))(1), FI II 744

$$3. \int_0^{\infty} e^{-px} \sin ax \cos bx \frac{dx}{x} = \frac{1}{2} \operatorname{arctg} \frac{2pa}{p^2 - a^2 + b^2} + s \frac{\pi}{2} \\ [a \geq 0, \quad p > 0, \quad s = 0 \quad \text{for} \quad p^2 - a^2 + b^2 \geq 0 \quad \text{and} \quad s = 1 \quad \text{for} \quad p^2 - a^2 + b^2 < 0].$$

GW ((336))(10b)

3.948

$$1. \int_0^{\infty} e^{-\beta x} (\sin ax - \sin bx) \frac{dx}{x} = \operatorname{arctg} \frac{(a - b)\beta}{ab + \beta^2} \quad [\operatorname{Re} \beta > 0], \quad (\text{cf. 3.951 2.}).$$

3.951
BI ((367))(7)

$$2. \int_0^{\infty} e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x} = \frac{1}{2} \ln \frac{b^2 + \beta^2}{a^2 + \beta^2} \quad [\operatorname{Re} \beta > 0], \quad (\text{cf. 3.951 3.}).$$

3.951
BI ((367))(8), FI II 748a

$$3. \int_0^{\infty} e^{-\beta x} (\cos ax - \cos bx) \frac{dx}{x^2} = \frac{\beta}{2} \ln \frac{a^2 + \beta^2}{b^2 + \beta^2} + b \operatorname{arctg} \frac{b}{\beta} - a \operatorname{arctg} \frac{a}{\beta} \quad [\operatorname{Re} p > 0].$$

$$4. \int_0^{\infty} e^{-\beta x} (\sin^2 ax - \sin^2 bx) \frac{dx}{x^2} = a \operatorname{arctg} \frac{2a}{p} - b \operatorname{arctg} \frac{2b}{p} - \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2} \quad [p > 0].$$

BI ((368))(25)

$$5. \int_0^{\infty} e^{-\beta x} (\cos^2 ax - \cos^2 bx) \frac{dx}{x^2} = -a \operatorname{arctg} \frac{2a}{p} + b \operatorname{arctg} \frac{2b}{p} + \frac{p}{4} \ln \frac{p^2 + 4a^2}{p^2 + 4b^2} \quad [p > 0].$$

BI ((368))(26)

527

3.949

$$1. \int_0^{\infty} e^{-px} \sin ax \sin bx \sin cx \frac{dx}{x} = -\frac{1}{4} \operatorname{arctg} \frac{a+b+c}{p} + \frac{1}{4} \operatorname{arctg} \frac{a+b-c}{p} + \frac{1}{4} \operatorname{arctg} \frac{a-b+c}{p} + \frac{1}{4} \operatorname{arctg} \frac{-a+b+c}{p} \quad [p > 0].$$

BI ((365))(11)

$$2.^8 \int_0^{\infty} e^{-\beta x} \sin^2 ax \sin bx \frac{dx}{x} = \frac{1}{2} \operatorname{arctg} \frac{b}{p} - \frac{1}{2} \left[\frac{1}{2} \operatorname{arctg} \frac{2pb}{p^2 + 4a^2 - b^2} + s \frac{\pi}{2} \right] \left[s = \begin{cases} 1, & p^2 + 4a^2 - b^2 < 0 \\ 0, & p^2 + 4a^2 - b^2 \geq 0 \end{cases} \right]$$

BI ((365))(8)

$$3. \int_0^{\infty} e^{-\beta x} \sin^2 ax \cos bx \frac{dx}{x} = \frac{1}{8} \ln \frac{[p^2 + (2a+b)^2][p^2 + (2a-b)^2]}{(p^2 + b^2)^2} \quad [p > 0].$$

BI ((365))(9)

$$4. \int_0^{\infty} e^{-\beta x} \sin ax \cos^2 bx \frac{dx}{x} = \frac{1}{2} \operatorname{arctg} \frac{a}{p} \left[\operatorname{arctg} \frac{2pb}{p^2 + 4a^2 - b^2} + s \frac{\pi}{2} \right] \left[s = \begin{cases} 1, & p^2 + 4a^2 - b^2 < 0 \\ 0, & p^2 + 4a^2 - b^2 \geq 0 \end{cases} \right]$$

BI ((365))(10)

$$5. \int_0^{\infty} e^{-px} \sin^2 ax \sin bx \sin cx \frac{dx}{x} = \frac{1}{8} \ln \frac{p^2 + (b+c)^2}{p^2 + (b-c)^2} + \frac{1}{16} \ln \frac{[p^2 + (2a-b+c)^2][p^2 + (2a+b-c)^2]}{[p^2 + (2a+b+c)^2][p^2 + (2a-b-c)^2]} \quad [p > 0].$$

BI ((365))(15)

3.951

$$2. \int_0^{\infty} \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \operatorname{arctg} \frac{(\beta - \gamma)b}{b^2 + \beta\gamma} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma \geq 0].$$

BI ((367))(3)

$$3. \int_0^{\infty} \frac{e^{-\gamma x} - e^{-\beta x}}{x} \cos bx \, dx = \frac{1}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma \geq 0].$$

BI ((367))(4)

$$4. \int_0^{\infty} \frac{e^{-\gamma x} - e^{-\beta x}}{x} \sin bx \, dx = \frac{b}{2} \ln \frac{b^2 + \beta^2}{b^2 + \gamma^2} + \beta \operatorname{arctg} \frac{b}{\beta} - \gamma \operatorname{arctg} \frac{b}{\gamma} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

BI ((368))(21)a

$$5. \int_0^{\infty} \frac{x}{e^{\beta x} - 1} \cos bx \, dx = \frac{1}{2b^2} - \frac{\pi^2}{2\beta^2} \operatorname{cosech}^2 \frac{b\pi}{\beta} \quad [\operatorname{Re} \beta > 0].$$

ET I 15(18)

$$6. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) \cos bx \, dx = \ln b - \frac{1}{2} [\psi(ib) + \psi(-ib)] \quad [b > 0].$$

ET I 15(9)

$$7. \int_0^{\infty} \frac{1 - \cos ax}{e^{2\pi x} - 1} \cdot \frac{dx}{x} = \frac{a}{4} + \frac{1}{2} \ln \frac{1 - e^{-a}}{a} \quad [a > 0].$$

BI ((387))(10)

528

$$8. \int_0^{\infty} (e^{-\beta x} - e^{-\gamma x} \cos ax) \frac{dx}{x} = \frac{1}{2} \ln \frac{a^2 + \gamma^2}{\beta^2} \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0].$$

BI ((367))(10)

$$9. \int_0^{\infty} \frac{\cos px - e^{-px}}{b^4 + x^4} \frac{dx}{x} = \frac{\pi}{2b^4} \exp\left(-\frac{1}{2}bp\sqrt{2}\right) \sin\left(\frac{1}{2}bp\sqrt{2}\right) \quad [p > 0].$$

$$10. \int_0^{\infty} \left(\frac{1}{e^x - 1} - \frac{\cos x}{x} \right) dx = C.$$

NT 65(8)

$$11. \int_0^{\infty} \left(ae^{-px} - \frac{e^{-qx}}{x} \sin ax \right) \frac{dx}{x} = \frac{a}{2} \ln \frac{a^2 + q^2}{p^2} + q \operatorname{arctg} \frac{a}{q} - a \quad [p > 0, \quad q > 0].$$

BI ((368))(24)

$$12. \int_0^{\infty} \frac{x^{2m} \sin bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{2} \operatorname{cth} b\pi - \frac{1}{2b} \right] \quad [b > 0].$$

GW ((336))(15a)

$$13. \int_0^{\infty} \frac{x^{2m+1} \cos bx}{e^x - 1} dx = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{2} \operatorname{cth} b\pi - \frac{1}{2b} \right] \quad [b > 0].$$

GW ((336))(15b)

$$14. \int_0^{\infty} \frac{x^{2m} \sin bx dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \theta \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0].$$

GW ((336))(14a)

$$15. \int_0^{\infty} \frac{x^{2m+1} \cos bx dx}{e^{(2n+1)cx} - e^{(2n-1)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \theta \frac{b\pi}{2c} - \sum_{k=1}^n \frac{b}{b^2 + (2k-1)^2 c^2} \right] \quad [b > 0].$$

GW ((336))(14b)

$$16. \int_0^{\infty} \frac{x^{2m} \sin bx dx}{e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m}}{\partial b^{2m}} \left[\frac{\pi}{4c} \operatorname{cth} \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0].$$

GW ((336))(14c)

$$17. \int_0^{\infty} \frac{x^{2m+1} \cos bx dx}{e^{2n cx} - e^{(2n-2)cx}} = (-1)^m \frac{\partial^{2m+1}}{\partial b^{2m+1}} \left[\frac{\pi}{4c} \operatorname{cth} \frac{b\pi}{2c} - \frac{1}{2b} - \sum_{k=1}^{n-1} \frac{b}{b^2 + (2k)^2 c^2} \right] \quad [b > 0, \quad c > 0].$$

$$18. \int_0^{\infty} \frac{\cos ax - \cos bx}{e^{(2m+1)px} - e^{(2m-1)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{\operatorname{ch} \frac{b\pi}{2p}}{\operatorname{ch} \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^m \ln \frac{b^2 + (2k-1)^2 p^2}{a^2 + (2k-1)^2 p^2} \quad [p > 0].$$

GW ((336))(16a)

$$19. \int_0^{\infty} \frac{\cos ax - \cos bx}{e^{2mpx} - e^{(2m-2)px}} \frac{dx}{x} = \frac{1}{2} \ln \frac{a \operatorname{sh} \frac{b\pi}{2p}}{b \operatorname{sh} \frac{a\pi}{2p}} - \frac{1}{2} \sum_{k=1}^{m-1} \ln \frac{b^2 + 4k^2 p^2}{a^2 + 4k^2 p^2} \quad [p > 0].$$

GW ((336))(16b)

$$20. \int_0^{\infty} \frac{\sin x \sin bx}{1 - e^{-x}} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{(b+1) \operatorname{sh}[(b-1)\pi]}{(b-1) \operatorname{sh}[(b+1)\pi]} \quad [b^2 \neq 1].$$

LO V 305

529

$$21. \int_0^{\infty} \frac{\sin^2 ax}{1 - e^{-x}} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{2a\pi}{\operatorname{sh} 2a\pi}.$$

LO V 306, BI ((387))(5)

3.952

$$1. \int_0^{\infty} x e^{-p^2 x^2} \sin ax \, dx = \frac{a\sqrt{\pi}}{4p^3} \exp\left(-\frac{a^2}{4p^2}\right).$$

BI ((362))(1)

$$2. \int_0^{\infty} x e^{-p^2 x^2} \cos ax \, dx = \frac{1}{2p^2} - \frac{a}{4p^3} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \quad [a > 0].$$

BI ((362))(2)

$$3. \int_0^{\infty} x^2 e^{-p^2 x^2} \sin ax \, dx = \frac{a}{4p^4} + \frac{2p^2 - a^2}{8p^5} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!} \left(\frac{a}{p}\right)^{2k+1} \quad [a > 0].$$

BI ((362))(4)

BI ((362))(5)

$$5. \int_0^{\infty} x^3 e^{-p^2 x^2} \sin ax \, dx = \sqrt{\pi} \frac{6ap^2 - a^3}{16p^7} \exp\left(-\frac{a^2}{4p^2}\right).$$

BI ((362))(6)

$$6.^3 \int_0^{\infty} e^{-p^2 x^2} \sin ax \frac{dx}{x} = \frac{a\sqrt{\pi}}{2p} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} \left(\frac{a}{2p}\right)^{2k} = \frac{\pi}{2} \Phi\left(\frac{a}{2p}\right).$$

BI ((365))(21)

$$7. \int_0^{\infty} x^{\mu-1} e^{-\beta x^2} \sin \gamma x \, dx = \frac{\gamma e^{-\frac{\gamma^2}{4\beta}}}{2\beta^{\frac{\mu+1}{2}}} \Gamma\left(\frac{1+\mu}{2}\right) {}_1F_1\left(1-\frac{\mu}{2}; \frac{3}{2}; \frac{\gamma^2}{4\beta}\right) \\ [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > -1].$$

ET I 318(10)

$$8.* \int_0^{\infty} x^{\mu-1} e^{-\beta x^2} \cos ax \, dx = \frac{1}{2} \beta^{-\mu/2} \Gamma(\mu/2) e^{-a^2/4\beta} {}_1F_1(-\mu/2+1/2; 1/2; a^2/4\beta) \\ [\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0, a > 0].$$

ET I 320(30)

$$9. \int_0^{\infty} x^{2n} e^{-\beta^2 x^2} \cos ax \, dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+1} \beta^{2n+1}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n}\left(\frac{a}{\beta\sqrt{2}}\right) = \\ = (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+1}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n}\left(\frac{a}{2\beta}\right) \quad \left[|\arg \beta| < \frac{\pi}{4}, a > 0\right].$$

WH, ET I 15(13)

$$10. \int_0^{\infty} x^{2n+1} e^{-\beta^2 x^2} \sin ax \, dx = (-1)^n \frac{\sqrt{\pi}}{2^{n+\frac{3}{2}} \beta^{2n+2}} \exp\left(-\frac{a^2}{8\beta^2}\right) D_{2n+1}\left(\frac{a}{\beta\sqrt{2}}\right) = \\ = (-1)^n \frac{\sqrt{\pi}}{(2\beta)^{2n+2}} \exp\left(-\frac{a^2}{4\beta^2}\right) H_{2n+1}\left(\frac{a}{2\beta}\right) \quad \left[|\arg \beta| < \frac{\pi}{4}, a > 0\right].$$

WH, ET I 74(23)

$$\begin{aligned}
2. \int_0^{\infty} x^{\mu-1} e^{-\gamma x - \beta x^2} \cos ax \, dx &= \\
&= \frac{1}{2(2\beta)^{\frac{\mu}{2}}} \exp\left(\frac{\gamma^2 - a^2}{8\beta}\right) \Gamma(\mu) \left\{ \exp\left(-\frac{ia\gamma}{4\beta}\right) D_{-\mu}\left(\frac{\gamma - ia}{\sqrt{2\beta}}\right) + \{exp\} \frac{ia\gamma}{4\beta} D_{-\mu}\left(\frac{\gamma + ia}{\sqrt{2\beta}}\right) \right\} \\
&\quad [\operatorname{Re} \mu > 0, \operatorname{Re} \beta > 0, a > 0].
\end{aligned}$$

ET I 16(18)

$$\begin{aligned}
3. \int_0^{\infty} x e^{-\gamma x - \beta x^2} \sin ax \, dx &= \frac{i\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp\left[-\frac{(\gamma - ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma - ia}{2\sqrt{\beta}}\right)\right] - \right. \\
&\quad \left. - (\gamma + ia) \exp\left[-\frac{(\gamma + ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma + ia}{2\sqrt{\beta}}\right)\right] \right\} \quad [\operatorname{Re} \beta > 0, a > 0].
\end{aligned}$$

ET I 74(28)

$$\begin{aligned}
4. \int_0^{\infty} x e^{-\gamma x - \beta x^2} \cos ax \, dx &= -\frac{\sqrt{\pi}}{8\sqrt{\beta^3}} \left\{ (\gamma - ia) \exp\left[\frac{(\gamma - ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma - ia}{2\sqrt{\beta}}\right)\right] + \right. \\
&\quad \left. + (\gamma + ia) \exp\left[\frac{(\gamma + ia)^2}{4\beta}\right] \left[1 - \Phi\left(\frac{\gamma + ia}{2\sqrt{\beta}}\right)\right] \right\} + \frac{1}{2\beta} \\
&\quad [\operatorname{Re} \beta > 0, a > 0].
\end{aligned}$$

ET I 16(17)

3.954

$$\begin{aligned}
1. \int_0^{\infty} e^{-\beta x^2} \sin ax \frac{x \, dx}{\gamma^2 + x^2} &= -\frac{\pi}{4} e^{\beta\gamma^2} \left[2 \operatorname{sh} a\gamma + e^{-\gamma a} \Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a} \Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right) \right] \\
&\quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0].
\end{aligned}$$

ET I 74(26)a

$$\begin{aligned}
2. \int_0^{\infty} e^{-\beta x^2} \cos ax \frac{dx}{\gamma^2 + x^2} &= \frac{\pi}{4\gamma} e^{\beta\gamma^2} \left[2 \operatorname{ch} a\gamma - e^{-\gamma a} \Phi\left(\gamma\sqrt{\beta} - \frac{a}{2\sqrt{\beta}}\right) - e^{\gamma a} \Phi\left(\gamma\sqrt{\beta} + \frac{a}{2\sqrt{\beta}}\right) \right] \\
&\quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0].
\end{aligned}$$

ET I 15(15)

3.955

$$\int_0^{\infty} x^{\nu} e^{-\frac{x^2}{2}} \cos\left(\beta x - \nu \frac{\pi}{2}\right) dx = \sqrt{\frac{\pi}{2}} e^{-\frac{\beta^2}{4}} D_{\nu}(\beta) \quad [\operatorname{Re} \nu > -1].$$

3.956

$$\int_0^{\infty} e^{-x^2} (2x \cos x - \sin x) \sin x \frac{dx}{x^2} = \sqrt{\pi} \frac{e-1}{2e}.$$

BI ((369))(19)

3.957

$$\begin{aligned} 1. \int_0^{\infty} x^{\mu-1} \exp\left(\frac{-\beta^2}{4x}\right) \sin ax \, dx &= \frac{i}{2\mu} \beta^\mu a^{-\frac{\mu}{2}} \times \\ &\times \left[\exp\left(-\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{\frac{\pi i}{4}} \sqrt{a}\right) - \{exp\} \left(\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{-\frac{\pi i}{4}} \sqrt{a}\right) \right] \\ &[\operatorname{Re} \beta > 0, \operatorname{Re} \mu < 1, a > 0]. \end{aligned}$$

ET I 318(12)

$$\begin{aligned} 2. \int_0^{\infty} x^{\mu-1} \exp\left(\frac{-\beta^2}{4x}\right) \cos ax \, dx &= \\ &= \frac{1}{2\mu} \beta^\mu a^{-\frac{\mu}{2}} \left[\exp\left(-\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{\frac{\pi i}{4}} \sqrt{a}\right) + \{exp\} \left(\frac{i}{4}\mu\pi\right) K_\mu\left(\beta e^{-\frac{\pi i}{4}} \sqrt{a}\right) \right] \\ &[\operatorname{Re} \beta > 0, \operatorname{Re} \mu < 1, a > 0]. \end{aligned}$$

ET I 320(32)a

3.958

$$\begin{aligned} 1. \int_{-\infty}^{\infty} x^n e^{-(ax^2+bx+c)} \sin(px+q) \, dx &= -\left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!} a^k \times \\ &\times \sum_{j=0}^{n-2k} \binom{n-2k}{j} b^{n-2k-j} p^j \sin\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right) \quad [a > 0]. \end{aligned}$$

GW ((37))(1b)

$$\begin{aligned} 2. \int_{-\infty}^{\infty} x^n e^{-(ax^2+bx+c)} \cos(px+q) \, dx &= \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!k!} a^k \times \\ &\times \sum_{j=0}^{n-2k} \binom{n-2k}{j} p^j \cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right) \\ &[a > 0]. \end{aligned}$$

GW ((337))(1a)

3.959

$$\int_0^{\infty} x e^{-p^2 x^2} \operatorname{tg} ax \, dx = \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2 k^2}{p^2}\right) \quad [p > 0].$$

BI ((362))(15)

3.961

$$1. \int_0^{\infty} \exp(-\beta\sqrt{\gamma^2 + x^2}) \sin ax \frac{x \, dx}{\sqrt{\gamma^2 + x^2}} = \frac{a\gamma}{\sqrt{a^2 + \beta^2}} K_1(\gamma\sqrt{a^2 + \beta^2})$$

[Re $\beta > 0$, Re $\gamma > 0$, $a > 0$].

ET I 75(36)

532

$$2. \int_0^{\infty} \exp[-\beta\sqrt{\gamma^2 + x^2}] \cos ax \frac{dx}{\sqrt{\gamma^2 + x^2}} = K_0(\gamma\sqrt{a^2 + \beta^2}) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \gamma > 0, a > 0].$$

ET I 17(27)

3.962

$$1. \int_0^{\infty} \frac{\sqrt{\sqrt{\gamma^2 + x^2} - \gamma} \exp(-\beta\sqrt{\gamma^2 + x^2})}{\sqrt{\gamma^2 + x^2}} \sin ax \, dx = \sqrt{\frac{\pi}{2}} \frac{a \exp(-\gamma\sqrt{a^2 + \beta^2})}{\sqrt{\beta^2 + a^2} \sqrt{\beta + \sqrt{a^2 + \beta^2}}}$$

[Re $\beta > 0$, Re $\gamma > 0$, $a > 0$].

ET I 75(38)

$$2. \int_0^{\infty} \frac{x \exp(-\beta\sqrt{\gamma^2 + x^2})}{\sqrt{\gamma^2 + x^2} \sqrt{\sqrt{\gamma^2 + x^2} - \gamma}} \cos ax \, dx = \sqrt{\frac{\pi}{2}} \frac{\sqrt{\beta + \sqrt{a^2 + \beta^2}}}{\sqrt{a^2 + \beta^2}} \exp[-\gamma\sqrt{a^2 + \beta^2}]$$

[Re $\beta > 0$, Re $\gamma > 0$, $a > 0$].

ET I 17(29)

3.963

$$1. \int_0^{\infty} e^{-\operatorname{tg}^2 x} \frac{\sin x}{\cos^2 x} \frac{dx}{x} = \frac{\sqrt{\pi}}{2}.$$

BI ((391))(1)

3.339
BI ((396))(3)

$$3. \int_0^{\frac{\pi}{2}} x e^{-\operatorname{tg}^2 x} \sin 4x \frac{dx}{\cos^2 x} = -\frac{3}{2} \sqrt{\pi}.$$

BI ((396))(5)

$$4. \int_0^{\frac{\pi}{2}} x e^{-\operatorname{tg}^2 x} \sin^2 2x \frac{dx}{\cos^2 x} = 2\sqrt{\pi}.$$

BI ((396))(6)

3.964

$$1. \int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg}^2 x} \frac{p \sin x - \cos x}{\cos^3 x} dx = -\sin p \operatorname{si}(p) - \operatorname{ci}(p) \cos p \quad [p > 0].$$

LI ((396))(4)

$$2. \int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg}^2 x} \frac{p - \cos^2 x}{\cos^4 x \operatorname{ctg} x} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \quad [p > 0].$$

BI ((396))(7)

$$3.8 \int_0^{\frac{\pi}{2}} x e^{-p \operatorname{tg}^2 x} \frac{p - 2 \cos^2 x}{\cos^6 x \operatorname{ctg} x} dx = \frac{1 + 2p}{8p} \sqrt{\frac{\pi}{p}} \quad [p > 0].$$

BI ((396))(8)

3.965

$$1. \int_0^{\infty} x e^{-\beta x} \sin ax^2 \sin \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \quad \left[|\arg \beta| < \frac{\pi}{4}, \quad a > 0 \right].$$

ET I 84(17)

533

$$2. \int_0^{\infty} x e^{-\beta x} \cos ax^2 \cos \beta x dx = \frac{\beta}{4} \sqrt{\frac{\pi}{2a^3}} e^{-\frac{\beta^2}{2a}} \quad [a > 0, \quad \operatorname{Re} \beta > |\operatorname{Im} \beta|].$$

$$1. \int_0^{\infty} x e^{-px} \cos(2x^2 + px) dx = 0 \quad [p > 0].$$

BI ((361))(16)

$$2. \int_0^{\infty} x e^{-px} \cos(2x^2 - px) dx = \frac{p\sqrt{\pi}}{8} \exp\left(-\frac{1}{4}p^2\right) \quad [p > 0].$$

BI ((361))(17)

$$3. \int_0^{\infty} x^2 e^{-px} [\sin(2x^2 + px) + \cos(2x^2 + px)] dx = 0 \quad [p > 0].$$

BI ((361))(18)

$$4. \int_0^{\infty} x^2 e^{-px} [\sin(2x^2 - px) - \cos(2x^2 - px)] dx = \frac{\sqrt{\pi}}{16} (2 - p^2) \exp\left(-\frac{1}{4}p^2\right).$$

BI ((361))(19)

$$5.^3 \int_0^{\infty} x^{\mu-1} e^{-x} \cos(x+ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \cos \frac{\mu\pi}{4} D_{-\mu} \left(\frac{1}{\sqrt{a}} \right) \quad [\operatorname{Re} \mu > 0, \quad a > 0].$$

ET I 321(37)

$$6.^6 \int_0^{\infty} x^{\mu-1} e^{-x} \sin(x+ax^2) dx = \frac{e^{\frac{1}{4a}} \Gamma(\mu)}{(2a)^{\frac{\mu}{2}}} \sin \frac{\mu\pi}{4} D_{-\mu} \left(\frac{1}{\sqrt{a}} \right) \quad [\operatorname{Re} \mu > -1, \quad a > 0].$$

ET I 319(18)

3.967

$$1. \int_0^{\infty} e^{-\frac{\beta^2}{x^2}} \sin a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \sin(\sqrt{2}a\beta) \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 75(30)A, BI((369))(3)a

$$2. \int_0^{\infty} e^{-\frac{\beta^2}{x^2}} \cos a^2 x^2 \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2\beta} e^{-\sqrt{2}a\beta} \cos(\sqrt{2}a\beta) \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

$$3. \int_0^{\infty} x^2 e^{-\beta x^2} \cos ax^2 dx = \frac{\sqrt{\pi}}{4 \sqrt[4]{(a^2 + \beta^2)^3}} \cos \left(\frac{3}{2} \operatorname{arctg} \frac{a}{\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

ET I 14(3)a

3.968

$$1. \int_0^{\infty} e^{-\beta x^2} \sin ax^4 dx = -\frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}} \left(\frac{\beta^2}{8a} \right) \{ \cos \} \left(\frac{\beta^2}{8a} + \frac{\pi}{8} \right) + N_{\frac{1}{4}} \left(\frac{\beta^2}{8a} \right) \{ \sin \} \left(\frac{\beta^2}{8a} + \frac{\pi}{8} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 75(34)

534

$$2. \int_0^{\infty} e^{-\beta x^2} \cos ax^4 dx = \frac{\pi}{8} \sqrt{\frac{\beta}{a}} \left[J_{\frac{1}{4}} \left(\frac{\beta^2}{8a} \right) \sin \left(\frac{\beta^2}{8a} + \frac{\pi}{8} \right) - N_{\frac{1}{4}} \left(\frac{\beta^2}{8a} \right) \{ \cos \} \left(\frac{\beta^2}{8a} + \frac{\pi}{8} \right) \right] \\ [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 16(24)

3.969

$$1. \int_0^{\infty} e^{-p^2 x^4 + q^2 x^2} [2px \cos(2pqx^3) + q \sin(2pqx^3)] dx = \frac{\sqrt{\pi}}{2}.$$

BI ((363))(7)

$$2. \int_0^{\infty} e^{-p^2 x^4 + q^2 x^2} [2px \sin(2pqx^3) - q \cos(2pqx^3)] dx = 0.$$

BI ((363))(8)

3.971

$$1. \int_0^{\infty} \exp \left(-px^2 - \frac{q}{x^2} \right) \sin \left(ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2} = \\ = \frac{1}{2} \int_{-\infty}^{\infty} \exp \left(-px^2 - \frac{q}{x^2} \right) \sin \left(ax^2 + \frac{b}{x^2} \right) \frac{dx}{x^2} = \\ = \frac{\sqrt{\pi}}{2s} \exp[-2rs \cos(A+B)] \sin[A + 2rs \sin(A+B)].$$

BI ((369))(16, 17)

3.972

$$\begin{aligned}
 1. \int_0^\infty \exp[-\beta\sqrt{\gamma^4+x^4}] \sin ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}} &= \\
 &= \sqrt{\frac{a\pi}{8}} I_{1/4} \left[\frac{\gamma^2}{2}(\sqrt{\beta^2+a^2}-\beta) \right] K_{1/4} \left[\frac{\gamma^2}{4}(\sqrt{\beta^2+a^2}+\beta) \right] \\
 &\quad \left[\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0 \right].
 \end{aligned}$$

ET I 75(37)

535

$$\begin{aligned}
 2. \int_0^\infty \exp[-\beta\sqrt{\gamma^4+x^4}] \cos ax^2 \frac{dx}{\sqrt{\gamma^4+x^4}} &= \\
 &= \sqrt{\frac{a\pi}{8}} I_{-1/4} \left[\frac{\gamma^2}{2}(\sqrt{\beta^2+a^2}-\beta) \right] K_{1/4} \left[\frac{\gamma^2}{4}(\sqrt{\beta^2+a^2}+\beta) \right] \\
 &\quad \left[\operatorname{Re} \beta > 0, \quad |\arg \gamma| < \frac{\pi}{4}, \quad a > 0 \right].
 \end{aligned}$$

ET I 17(28)

3.973

$$1. \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \frac{dx}{x} = \frac{\pi}{2} (e^p - 1) \quad [p > 0, \quad a > 0].$$

WH, FI II 725

$$\begin{aligned}
 2. \int_0^\infty \exp(p \cos ax) \sin(p \sin ax + bx) \frac{x dx}{c^2+x^2} &= \frac{\pi}{2} \exp(-cb + pe^{-ac}) \\
 &\quad [a > 0, \quad b > 0, \quad c > 0, \quad p > 0].
 \end{aligned}$$

BI ((372))(3)

$$\begin{aligned}
 3. \int_0^\infty \exp(p \cos ax) \cos(p \sin ax + bx) \frac{dx}{c^2+x^2} &= \frac{\pi}{2c} \exp(-cb + pe^{-ac}) \\
 &\quad [a > 0, \quad b > 0, \quad c > 0, \quad p > 0].
 \end{aligned}$$

BI ((372))(4)

BI ((366))(2)

$$5. \int_0^\infty \exp(p \cos x) \sin(p \sin x) \cos nx \frac{dx}{x} = \frac{p^n}{n!} \cdot \frac{\pi}{4} + \frac{\pi}{2} \sum_{k=n+1}^\infty \frac{p^k}{k!} \quad [p > 0].$$

LI ((366))(3)

$$6. \int_0^\infty \exp(p \cos x) \cos(p \sin x) \sin nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=0}^{n-1} \frac{p^k}{k!} + \frac{p^n \pi}{n!} \frac{\pi}{4} \quad [p > 0].$$

LI ((366))(4)

3.974

$$1. \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi[e^p - \exp(pe^{-ab})]}{2b \operatorname{sh} ab} \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(4)

$$2. \int_0^\infty [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi[e^p - \exp(pe^{-ab})]}{2 \operatorname{sh} ab} \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(5)

$$3. \int_0^\infty \exp(p \cos ax) \sin(p \sin ax + ax) \operatorname{cosec} ax \frac{dx}{b^2 + x^2} = \frac{\pi[e^p - \exp(pe^{-ab} - ab)]}{2b \operatorname{sh} ab} \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(6)

$$4. \int_0^\infty \exp(p \cos ax) \cos(p \sin ax + ax) \operatorname{cosec} ax \frac{x dx}{b^2 + x^2} = \frac{\pi[e^p - \exp(pe^{-ab} - ab)]}{2 \operatorname{sh} ab} \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(7)

$$5. \int_0^\infty \exp(p \cos ax) \sin(p \sin ax) \frac{x dx}{b^2 - x^2} = \frac{\pi}{2} [1 - \exp(p \cos ab) \cos(p \sin ab)] \\ [p > 0, \quad a > 0].$$

$$6. \int_0^{\infty} \exp(p \cos ax) \cos(p \sin ax) \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \exp(p \cos ab) \sin(p \sin ab) \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((378))(2)

$$7. \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \operatorname{tg} ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \cdot \theta ab [\exp(pe^{-ab}) - e^p] \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((372))(14)

$$8. \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \operatorname{ctg} ax \frac{dx}{b^2 + x^2} = \frac{\pi}{2b} \operatorname{cth} ab [e^p - \exp(pe^{-ab})] \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((372))(15)

$$9. \int_0^{\infty} \exp(p \cos ax) \sin(p \sin ax) \operatorname{cosec} ax \frac{dx}{b^2 - x^2} = \frac{\pi}{2b} \operatorname{cosec} ab [e^p - \exp(p \cos ab) \cos(p \sin ab)] \\ [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(12)

$$10. \int_0^{\infty} [1 - \exp(p \cos ax) \cos(p \sin ax)] \operatorname{cosec} ax \frac{x dx}{b^2 - x^2} = \\ = -\frac{\pi}{2} \exp(p \cos ab) \sin(p \sin ab) \operatorname{cosec} ab \quad [a > 0, \quad b > 0, \quad p > 0].$$

BI ((391))(13)

3.975

$$1. \int_0^{\infty} \frac{\sin\left(\beta \operatorname{arctg} \frac{x}{\gamma}\right)}{(\gamma^2 + x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} - 1} = \frac{1}{2} \zeta(\beta, \gamma) - \frac{1}{4\gamma^\beta} - \frac{\gamma^{1-\beta}}{2(\beta-1)} \quad [\operatorname{Re} \beta > 1, \quad \operatorname{Re} \gamma > 0].$$

WH, ET I 26(7)

$$2. \int_0^{\infty} \frac{\sin(\beta \operatorname{arctg} x)}{(1+x^2)^{\frac{\beta}{2}}} \cdot \frac{dx}{e^{2\pi x} + 1} = \frac{1}{2(\beta-1)} - \frac{\zeta(\beta)}{2^\beta} \quad [\operatorname{Re} \beta > 1].$$

$$\int_0^{\infty} (1+x^2)^{\beta-\frac{1}{2}} e^{-px^2} \cos[2px+(2\beta-1) \operatorname{arctg} x] dx = \frac{e^{-p}}{2p^{\beta}} \sin \pi\beta\Gamma(\beta) \quad [\operatorname{Re} \beta > 0, \quad p > 0].$$

WH

3.98- 3.99 Combinations of trigonometric and hyperbolic functions

3.981

$$1. \int_0^{\infty} \frac{\sin ax}{\operatorname{sh} \beta x} dx = \frac{\pi}{2\beta} \theta \frac{a\pi}{2\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((264))(16)

$$2. \int_0^{\infty} \frac{\sin ax}{\operatorname{ch} \beta x} dx = -\frac{\pi}{2\beta} \theta \frac{a\pi}{2\beta} - \frac{i}{2\beta} \left[\psi \left(\frac{\beta + ai}{4\beta} \right) - \psi \left(\frac{\beta - ai}{4\beta} \right) \right] \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

GW ((335))(12), ET I 88(1)

537

$$3. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \beta x} dx = \frac{\pi}{2\beta} \operatorname{sech} \frac{a\pi}{2\beta} \quad [\operatorname{Re} \beta > 0, \quad \text{all real } a].$$

BI ((264))(14)

$$4. \int_0^{\infty} \sin ax \frac{\operatorname{sh} \beta x}{\operatorname{sh} \gamma x} dx = \frac{\pi}{2\gamma} \frac{\operatorname{sh} \frac{a\pi}{\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} + \frac{i}{2\gamma} \left[\psi \left(\frac{\beta + \gamma + ia}{2\gamma} \right) - \psi \left(\frac{\beta + \gamma - ia}{2\gamma} \right) \right] \\ \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0].$$

ET I 88(5)

$$5. \int_0^{\infty} \cos ax \frac{\operatorname{sh} \beta x}{\operatorname{sh} \gamma x} dx = \frac{\pi}{2\gamma} \frac{\sin \frac{\pi\beta}{\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma].$$

BI ((265))(7)

$$6. \int_0^{\infty} \sin ax \frac{\operatorname{sh} \beta x}{\operatorname{ch} \gamma x} dx = \frac{\pi}{\gamma} \frac{\sin \frac{\beta\pi}{2\gamma} \operatorname{sh} \frac{a\pi}{2\gamma}}{\operatorname{ch} \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma; \quad a > 0].$$

$$7. \int_0^{\infty} \cos ax \frac{\text{sh } \beta x}{\text{ch } \gamma x} dx = \frac{1}{4\gamma} \left\{ \psi \left(\frac{3\gamma - \beta + ia}{4\gamma} \right) + \psi \left(\frac{3\gamma - \beta - ia}{4\gamma} \right) - \psi \left(\frac{3\gamma + \beta - ia}{4\gamma} \right) - \right. \\ \left. - \psi \left(\frac{3\gamma + \beta + ia}{4\gamma} \right) + \frac{2\pi \sin \frac{\pi\beta}{\gamma}}{\cos \frac{\pi\beta}{\gamma} + \text{ch } \frac{\pi a}{\gamma}} \right\} \quad [|\text{Re } \beta| < \text{Re } \gamma, \quad a > 0].$$

ET I 31(13)

$$8. \int_0^{\infty} \sin ax \frac{\text{ch } \beta x}{\text{sh } \gamma x} dx = \frac{\pi}{2\gamma} \cdot \frac{\text{sh } \frac{\pi a}{\gamma}}{\text{ch } \frac{\pi a}{\gamma} + \cos \frac{\pi\beta}{\gamma}} \quad [|\text{Re } \beta| < \text{Re } \gamma, \quad a > 0].$$

BI ((265))(4)

$$9. \int_0^{\infty} \sin ax \frac{\text{ch } \beta x}{\text{ch } \gamma x} dx = \frac{i}{4\gamma} \left[\psi \left(\frac{3\gamma + \beta + ia}{4\gamma} \right) - \psi \left(\frac{3\gamma + \beta - ia}{4\gamma} \right) + \psi \left(\frac{3\gamma - \beta + ia}{4\gamma} \right) - \right. \\ \left. - \psi \left(\frac{3\gamma - \beta - ia}{4\gamma} \right) - \frac{2\pi i \text{sh } \frac{\pi a}{\gamma}}{\text{ch } \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \right] \quad [|\text{Re } \beta| < \text{Re } \gamma, \quad a > 0].$$

ET I 88(6)

538

$$10. \int_0^{\infty} \cos ax \frac{\text{ch } \beta x}{\text{ch } \gamma x} dx = \frac{\pi}{\gamma} \frac{\cos \frac{\beta\pi}{2\gamma} \text{ch } \frac{a\pi}{2\gamma}}{\text{ch } \frac{a\pi}{\gamma} + \cos \frac{\beta\pi}{\gamma}} \quad [|\text{Re } \beta| < \text{Re } \gamma, \quad \text{all real } a].$$

BI ((265))(6)

$$11. \int_0^{\frac{\pi}{2}} \cos^{2m} x \text{ch } \beta x dx = \frac{(2m)! \text{sh } \frac{\pi\beta}{2}}{\beta(\beta^2 + 2^2) \dots [\beta^2 + (2m)^2]} \quad [\text{Re } \beta > 0].$$

WA 620a

3.982

$$1. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch}^2 \beta x} dx = \frac{a\pi}{2\beta^2 \operatorname{sh} \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((264))(16)

$$2. \int_0^{\infty} \sin ax \frac{\operatorname{sh} \beta x}{\operatorname{ch}^2 \gamma x} dx = \frac{\pi \left(a \sin \frac{\beta\pi}{2\gamma} \operatorname{ch} \frac{a\pi}{2\gamma} - \beta \cos \frac{\beta\pi}{2\gamma} \operatorname{sh} \frac{a\pi}{2\gamma} \right)}{\gamma^2 \left(\operatorname{ch} \frac{a\pi}{\gamma} - \cos \frac{\beta\pi}{\gamma} \right)} \quad [|\operatorname{Re} \beta| < 2 \operatorname{Re} \gamma, \quad a > 0].$$

ET I 88(9)

$$3.8 \int_0^{\infty} \frac{\sin^2 x \cos ax}{\sin h^2 x} dx = \frac{\pi}{4} \left\{ \frac{a+2}{1-e^{-\pi(a+2)}} - \frac{2a}{1-e^{-\pi a}} + \frac{a-2}{1-e^{-\pi(a-2)}} \right\} = I(a)$$

$$\left[I(0) = \frac{1}{2}(\pi \coth \pi - 1), \quad I(\pm 2) = \frac{1}{4} + \frac{\pi}{2}(\coth 2\pi - \coth \pi) \right]$$

3.983

$$1.6 \int_0^{\infty} \frac{\cos ax dx}{b \operatorname{ch} \beta x + c} = \frac{\pi \sin \left(\frac{a}{\beta} \operatorname{Arch} \frac{c}{b} \right)}{\beta \sqrt{c^2 - b^2} \operatorname{sh} \frac{a\pi}{\beta}} \quad [c > b > 0]$$

$$= \frac{\pi \operatorname{sh} \left(\frac{a}{\beta} \arccos \frac{c}{b} \right)}{\beta \sqrt{b^2 - c^2} \operatorname{sh} \frac{a\pi}{\beta}} \quad [b > |c| > 0]; \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

GW ((335))(13a)

$$2. \int_0^{\infty} \frac{\cos ax dx}{\operatorname{ch} \beta x + \cos \gamma} = \frac{\pi}{\beta} \frac{\operatorname{sh} \frac{a\gamma}{\beta}}{\sin \gamma \operatorname{sh} \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta < \operatorname{Im} \bar{\beta} \gamma, \quad a > 0].$$

BI ((267))(3)

$$3.3 \int_0^{\infty} \frac{\cos ax dx}{\operatorname{ch} x - \operatorname{ch} b} = -\pi \operatorname{cth} a\pi \frac{\sin ab}{\operatorname{sh} b} \quad [a > 0, \quad b > 0].$$

$$4. \int_0^{\infty} \frac{\cos ax \, dx}{1 + 2 \operatorname{ch} \left(\sqrt{\frac{2}{3}} \pi x \right)} = \frac{\sqrt{\frac{\pi}{2}}}{1 + 2 \operatorname{ch} \left(\sqrt{\frac{2}{3}} \pi a \right)} \quad [a > 0].$$

ET I 30(9)

$$5. \int_0^{\infty} \frac{\sin ax \operatorname{sh} \beta x}{\operatorname{ch} \gamma x + \cos \delta} \, dx = \frac{\pi \left\{ \sin \left[\frac{\beta}{\gamma} (\pi - \delta) \right] \operatorname{sh} \left[\frac{a}{\gamma} (\pi + \delta) \right] - \sin \left[\frac{\beta}{\gamma} (\pi + \delta) \right] \operatorname{sh} \left[\frac{a}{\gamma} (\pi - \delta) \right] \right\}}{\gamma \sin \delta \left(\operatorname{ch} \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)}$$

[$\operatorname{Re} \gamma > |\operatorname{Re} \bar{\gamma} \delta|$, $|\operatorname{Re} \beta| < \operatorname{Re} \gamma$, $a > 0$].

BI ((267))(2)

$$6. \int_0^{\infty} \frac{\cos ax \operatorname{ch} \beta x}{\operatorname{ch} \gamma x + \cos b} \, dx = \frac{\pi \left\{ \cos \left[\frac{\beta}{\gamma} (\pi - b) \right] \operatorname{ch} \left[\frac{a}{\gamma} (\pi + b) \right] - \{ \cos \} \left[\frac{\beta}{\gamma} (\pi + b) \right] \operatorname{ch} \left[\frac{a}{\gamma} (\pi - b) \right] \right\}}{\gamma \sin b \left(\operatorname{ch} \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma} \right)}$$

[$|\operatorname{Re} \beta| < \operatorname{Re} \gamma$, $0 < b < \pi$, $a < 0$].

BI ((267))(6)

$$7. \int_0^{\infty} \frac{\cos ax \, dx}{(\beta + \sqrt{\beta^2 - 1} \operatorname{ch} x)^{\nu+1}} = \Gamma(\nu + 1 - ai) e^{a\pi} \frac{Q_{\nu}^{ai}(\beta)}{\Gamma(\nu + 1)}$$

[$\operatorname{Re} \nu > -1$, $|\arg(\beta + 1)| < \pi$, $a > 0$].

ET I 30(10)

3.984

$$1.^6 \lim_{c \uparrow 1} \int_0^{\infty} \frac{\sin ax \operatorname{sh} cx}{\operatorname{ch} x + \cos b} \, dx = \pi \frac{\operatorname{ch} ab}{\operatorname{sh} a\pi} \quad [|b| \leq \pi, \quad a \text{ real}].$$

BI ((267))(1)

$$2.^6 \lim_{c \uparrow 1} \int_0^{\infty} \frac{\cos ax \operatorname{ch} cx}{\operatorname{ch} x + \cos b} \, dx = -\pi \operatorname{ctg} b \frac{\operatorname{sh} ab}{\operatorname{sh} a\pi} \quad [0 < |b| < \pi, \quad a \text{ real}].$$

BI ((267))(5)

ET I 80(10)

$$4. \int_0^{\infty} \frac{\cos ax \operatorname{ch} \frac{\beta}{2} x}{\operatorname{ch} \beta x + \operatorname{ch} \gamma} dx = \frac{\pi \cos \frac{a\gamma}{\beta}}{2\beta \operatorname{ch} \frac{\gamma}{2} \operatorname{ch} \frac{a\pi}{\beta}} \quad [\pi \operatorname{Re} \beta > |\operatorname{Im}(\bar{\beta}\gamma)|].$$

ET I 31(16)

$$5. \int_0^{\infty} \frac{\sin ax \operatorname{sh} \beta x}{\operatorname{ch} 2\beta x + \cos 2ax} dx = \frac{a\pi}{4(a^2 + \beta^2)} \quad [a > 0, \operatorname{Re} \beta > 0].$$

BI ((267))(7)

$$6. \int_0^{\infty} \frac{\cos ax \operatorname{ch} \beta x}{\operatorname{ch} 2\beta x + \cos 2ax} dx = \frac{\beta\pi}{4(a^2 + \beta^2)} \quad [\operatorname{Re} \beta > 0, a > 0].$$

BI ((267))(8)

540

$$7.8 \int_0^{\infty} \frac{\operatorname{sh}^{2\mu-1} x \operatorname{ch}^{2\nu-2\mu+1} x}{(\operatorname{ch}^2 x - \beta \operatorname{sh}^2 x)^{\nu}} dx = \frac{1}{2} \operatorname{B}(\mu, \nu - \mu) {}_2F_1(\nu, \mu; \nu; \beta) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > 0].$$

EH I 115(12)

3.985

$$1. \int_0^{\infty} \frac{\cos ax dx}{\operatorname{ch}^{\nu} \beta x} = \frac{2^{\nu-2}}{\beta \Gamma(\nu)} \Gamma\left(\frac{\nu}{2} + \frac{ai}{2\beta}\right) \Gamma\left(\frac{\nu}{2} - \frac{ai}{2\beta}\right) \quad [\operatorname{Re} \beta > 0, \operatorname{Re} \nu > 0, a > 0].$$

ET I 30(5)

$$2. \int_0^{\infty} \frac{\cos ax dx}{\operatorname{ch}^{2n} \beta x} = \frac{4^{n-1} \pi a}{2(2n-1)! \beta^2 \operatorname{sh} \frac{a\pi}{2\beta}} \prod_{k=1}^{n-1} \left(\frac{a^2}{4\beta^2} + k^2 \right);$$

$$= \frac{\pi a (a^2 + 2^2 \beta^2)(a^2 + 4^2 \beta^2) \dots [a^2 + (2n-2)^2 \beta^2]}{2(2n-1)! \beta^{2n} \operatorname{sh} \frac{a\pi}{2\beta}} \quad [n \geq 2, a > 0].$$

ET I 30(3)

3.986

$$1. \int_0^{\infty} \frac{\sin \beta x \sin \gamma x}{\operatorname{ch} \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\operatorname{sh} \frac{\beta \pi}{2\delta} \operatorname{sh} \frac{\gamma \pi}{2\delta}}{\operatorname{ch} \frac{\beta}{\delta} \pi + \operatorname{ch} \frac{\gamma}{\delta} \pi} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta].$$

BI ((264))(19)

$$2. \int_0^{\infty} \frac{\sin \alpha x \cos \beta x}{\operatorname{sh} \gamma x} dx = \frac{\pi \operatorname{sh} \frac{\pi \alpha}{\gamma}}{2\gamma \left(\operatorname{ch} \frac{\alpha \pi}{\gamma} + \operatorname{ch} \frac{\beta \pi}{\gamma} \right)} \quad [|\operatorname{Im}(\alpha + \beta)| < \operatorname{Re} \gamma].$$

LI ((264))(20)

$$3. \int_0^{\infty} \frac{\cos \beta x \cos \gamma x}{\operatorname{ch} \delta x} dx = \frac{\pi}{\delta} \cdot \frac{\operatorname{ch} \frac{\beta \pi}{2\delta} \operatorname{ch} \frac{\gamma \pi}{2\delta}}{\operatorname{ch} \frac{\beta \pi}{\delta} + \operatorname{ch} \frac{\gamma \pi}{\delta}} \quad [|\operatorname{Im}(\beta + \gamma)| < \operatorname{Re} \delta].$$

BI ((264))(21)

$$4^3 \int_0^{\infty} \frac{\sin^2 \beta x}{\operatorname{sh}^2 \pi x} dx = \frac{\beta}{\pi(e^{2\beta} - 1)} + \frac{\beta - 1}{2\pi} = \frac{\beta \operatorname{cth} \beta - 1}{2\pi} \quad [|\operatorname{Im} \beta| < \pi].$$

EH I 44(3)

541

3.987

$$1. \int_0^{\infty} \sin ax(1 - \theta \beta x) dx = \frac{1}{a} - \frac{\pi}{2\beta \operatorname{sh} \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0].$$

ET I 88(4)a

$$2. \int_0^{\infty} \sin ax(\operatorname{cth} \beta x - 1) dx = \frac{\pi}{2\beta} \operatorname{cth} \frac{a\pi}{2\beta} - \frac{1}{a} \quad [\operatorname{Re} \beta > 0].$$

ET I 88(3)

3.988

ET I 37(66)

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos ax \operatorname{ch}(2b \cos x)}{\sqrt{\cos x}} dx = \frac{\pi}{2} \sqrt{\pi b} I_{\frac{a}{2}-\frac{1}{4}}(b) I_{-\frac{a}{2}-\frac{1}{4}}(b) \quad [a > 0].$$

ET I 37(67)

$$3. \int_0^{\infty} \frac{\cos ax dx}{\sqrt{\operatorname{ch} x \cos b}} = \frac{\pi P_{-\frac{1}{2}+ia}(\cos b)}{\sqrt{2} \operatorname{ch} a\pi} \quad [a > 0, \quad b > 0].$$

ET I 30(7)

3.989

$$1. \int_0^{\infty} \frac{\sin \frac{a^2 x^2}{\pi} \sin bx}{\operatorname{sh} ax} dx = \frac{\pi}{2a} \sin \frac{\pi b^2}{4a^2} \operatorname{cosech} \frac{\pi b}{2a} \quad [a > 0, \quad b > 0].$$

ET I 93(44)

$$2. \int_0^{\infty} \frac{\cos \frac{a^2 x^2}{\pi} \sin bx}{\operatorname{sh} ax} dx = \frac{\pi}{2a} \frac{\operatorname{ch} \frac{\pi b}{a} - \cos \frac{\pi b^2}{4a^2}}{\operatorname{sh} \frac{\pi b}{2a}} \quad [a > 0, \quad b > 0].$$

ET I 93(45)

$$3. \int_0^{\infty} \frac{\sin \frac{x^2}{\pi} \cos ax}{\operatorname{ch} x} dx = \frac{\pi}{2} \frac{\cos \frac{a^2 \pi}{4} - \frac{1}{\sqrt{2}}}{\operatorname{ch} \frac{a\pi}{2}}.$$

ET I 36(54)

$$4. \int_0^{\infty} \frac{\cos \frac{x^2}{\pi} \cos ax}{\operatorname{ch} x} dx = \frac{\pi}{2} \cdot \frac{\sin \frac{a^2 \pi}{4} + \frac{1}{\sqrt{2}}}{\operatorname{ch} \frac{a\pi}{2}}.$$

ET I 36(55)

542

$$5. \int_0^{\infty} \frac{\sin(\pi a x^2) \cos bx}{\operatorname{ch} \pi x} dx = - \sum_{k=0}^{\infty} \exp \left[- \left(k + \frac{1}{2} \right) b \right] \sin \left[\left(k + \frac{1}{2} \right)^2 \pi a \right] +$$

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$$+ \frac{1}{\sqrt{a}} \sum_{k=0}^{\infty} \exp \left[- \frac{b \left(k + \frac{1}{2} \right)}{a} \right] \sin \left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k + \frac{1}{2} \right)^2 \pi}{a} \right]$$

$$6. \int_0^{\infty} \frac{\cos(\pi a x^2) \cos b x}{\operatorname{ch} \pi x} dx = \sum_{k=0}^{\infty} (-1)^k \exp \left[- \left(k + \frac{1}{2} \right) b \right] \cos \left[\left(k + \frac{1}{2} \right)^2 \pi a \right] + \\ + \frac{1}{\sqrt{a}} \sum_{k=0}^{\infty} \exp \left[- \frac{b \left(k + \frac{1}{2} \right)}{a} \right] \cos \left[\frac{\pi}{4} - \frac{b^2}{4\pi a} + \frac{\left(k + \frac{1}{2} \right)^2 \pi}{a} \right] \\ [a > 0, \quad b > 0].$$

ET I 36(57)

3.991

$$1. \int_0^{\infty} \sin \pi x^2 \sin a x \operatorname{cth} \pi x dx = \frac{1}{2} \theta \frac{a}{2} \sin \left(\frac{\pi}{4} + \frac{a^2}{4\pi} \right).$$

ET I 93(42)

$$2. \int_0^{\infty} \cos \pi x^2 \sin a x \operatorname{cth} \pi x dx = \frac{1}{2} \theta \frac{a}{2} \left[1 \cos \left(\frac{\pi}{4} + \frac{a^2}{4\pi} \right) \right].$$

ET I 93(43)

3.992

$$1. \int_0^{\infty} \frac{\sin \pi x^2 \cos a x}{1 + 2 \operatorname{ch} \left(\frac{2}{\sqrt{3}} \pi x \right)} dx = -\sqrt{3} + \frac{\cos \left(\frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \operatorname{ch} \frac{a}{\sqrt{3}} - 2}.$$

ET I 37(60)

$$2. \int_0^{\infty} \frac{\cos \pi x^2 \cos a x}{1 + 2 \operatorname{ch} \left(\frac{2}{\sqrt{3}} \pi x \right)} dx = 1 - \frac{\sin \left(\frac{\pi}{12} - \frac{a^2}{4\pi} \right)}{4 \operatorname{ch} \frac{a}{\sqrt{3}} - 2}.$$

ET I 37(61)

3.993

$$\int_0^{\infty} \frac{\sin x^2 + \cos x^2}{\operatorname{ch}(\sqrt{\pi} x)} \cos a x dx = \frac{\sqrt{\pi}}{2} \cdot \frac{\sin a^2 + \cos a^2}{\operatorname{ch}(\sqrt{\pi} a)}.$$

$$1. \int_0^{\infty} \frac{\sin(2a \operatorname{ch} x) \cos bx}{\sqrt{\operatorname{ch} x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[J_{\frac{1}{4}+\frac{ib}{2}}(a) N_{\frac{1}{4}-\frac{ib}{2}}(a) + J_{\frac{1}{4}-\frac{ib}{2}}(a) N_{\frac{1}{4}+\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 37(62)

$$2. \int_0^{\infty} \frac{\cos(2a \operatorname{ch} x) \cos bx}{\sqrt{\operatorname{ch} x}} dx = -\frac{\pi}{4} \sqrt{a\pi} \left[J_{-\frac{1}{4}+\frac{ib}{2}}(a) N_{-\frac{1}{4}-\frac{ib}{2}}(a) + J_{-\frac{1}{4}-\frac{ib}{2}}(a) N_{-\frac{1}{4}+\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 37(63)

$$3. \int_0^{\infty} \frac{\sin(2a \operatorname{sh} x) \sin bx}{\sqrt{\operatorname{sh} x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[I_{\frac{1}{4}-\frac{ib}{2}}(a) K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{\frac{1}{4}+\frac{ib}{2}}(a) K_{\frac{1}{4}-\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 93(47)

$$4. \int_0^{\infty} \frac{\cos(2a \operatorname{sh} x) \sin bx}{\sqrt{\operatorname{sh} x}} dx = -\frac{i}{2} \sqrt{\pi a} \left[I_{-\frac{1}{4}-\frac{ib}{2}}(a) K_{-\frac{1}{4}+\frac{ib}{2}}(a) - I_{-\frac{1}{4}+\frac{ib}{2}}(a) K_{-\frac{1}{4}-\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 93(48)

$$5. \int_0^{\infty} \frac{\sin(2a \operatorname{sh} x) \cos bx}{\sqrt{\operatorname{sh} x}} dx = \frac{\sqrt{\pi a}}{2} \left[I_{\frac{1}{4}-\frac{ib}{2}}(a) K_{\frac{1}{4}+\frac{ib}{2}}(a) + I_{\frac{1}{4}+\frac{ib}{2}}(a) K_{\frac{1}{4}-\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 37(64)

$$6. \int_0^{\infty} \frac{\cos(2a \operatorname{sh} x) \cos bx}{\sqrt{\operatorname{sh} x}} dx = \frac{\sqrt{\pi a}}{2} \left[I_{-\frac{1}{4}-\frac{ib}{2}}(a) K_{-\frac{1}{4}+\frac{ib}{2}}(a) + I_{-\frac{1}{4}+\frac{ib}{2}}(a) K_{-\frac{1}{4}-\frac{ib}{2}}(a) \right] \\ [a > 0, \quad b > 0].$$

ET I 37(65)

$$7. \int_0^{\infty} \sin(a \operatorname{ch} x) \sin(a \operatorname{sh} x) \frac{dx}{\operatorname{sh} x} = \frac{\pi}{2} \sin a \quad [a > 0].$$

BI ((264))(22)

3.995

$$1. \int_0^{\frac{\pi}{2}} \frac{\sin(2a \cos^2 x) \operatorname{ch}(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \sin \frac{2ac}{b+c} \quad [b > 0, \quad c > 0].$$

BI ((273))(9)

$$2. \int_0^{\frac{\pi}{2}} \frac{\cos(2a \cos^2 x) \operatorname{ch}(a \sin 2x)}{b^2 \cos^2 x + c^2 \sin^2 x} dx = \frac{\pi}{2bc} \cos \frac{2ac}{b+c} \quad [b > 0, \quad c > 0].$$

BI ((273))(10)

3.996

$$1. \int_0^{\infty} \sin(a \operatorname{sh} x) \operatorname{sh} \beta x dx = \sin \frac{\beta\pi}{2} K_{\beta}(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0].$$

EH II 82(26)

$$2. \int_0^{\infty} \cos(a \operatorname{sh} x) \operatorname{ch} \beta x dx = \cos \frac{\beta\pi}{2} K_{\beta}(a) \quad [|\beta| < 1, \quad a > 0].$$

WA 202(13)

544

$$3. \int_0^{\frac{\pi}{2}} \cos(a \sin x) \operatorname{ch}(\beta \cos x) dx = \frac{\pi}{2} J_0(\sqrt{a^2 - \beta^2}).$$

MO 40

$$4. \int_0^{\infty} \sin \left(a \operatorname{ch} x - \frac{1}{2} \beta \pi \right) \operatorname{ch} \beta x dx = \frac{\pi}{2} J_{\beta}(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0].$$

WA 199(12)

$$5. \int_0^{\infty} \cos \left(a \operatorname{ch} x - \frac{1}{2} \beta \pi \right) \operatorname{ch} \beta x dx = -\frac{\pi}{2} N_{\beta}(a) \quad [|\operatorname{Re} \beta| < 1, \quad a > 0].$$

WA 199(13)

3.997

$$2. \int_0^\pi \sin^\nu x \operatorname{ch}(\beta \cos x) dx = \sqrt{\pi} \left(\frac{2}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) I_{\frac{\nu}{2}}(\beta) \quad [\operatorname{Re} \nu > -1].$$

WH

$$3. \int_0^{\frac{\pi}{2}} \frac{dx}{\operatorname{ch}(\operatorname{tg} x) \cos x \sqrt{\sin 2x}} = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((276))(13)

$$4. \int_0^{\frac{\pi}{2}} \frac{\operatorname{tg}^q x}{\operatorname{ch}(\operatorname{tg} x) + \cos \lambda} \frac{dx}{\sin 2x} = \frac{\Gamma(q)}{\sin \lambda} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\lambda}{k^q} \quad [q > 0].$$

BI ((275))(20)

4.11- 4.12 Combinations involving trigonometric and hyperbolic functions and powers

4.111

$$1. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\theta \frac{a\pi}{2\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.981 1.}).$$

3.981
GW ((336))(17a)

$$2. \int_0^\infty \frac{\cos ax}{\operatorname{sh} \beta x} \cdot x^{2m+1} dx = (-1)^m \frac{\pi}{2\beta} \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\theta \frac{a\pi}{2\beta} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.981 1.}).$$

3.981
GW ((336))(17b)

$$3. \int_0^\infty \frac{\sin ax}{\operatorname{ch} \beta x} \cdot x^{2m+1} dx = (-1)^{m+1} \frac{\pi}{2\beta} \cdot \frac{\partial^{2m+1}}{\partial a^{2m+1}} \left(\frac{1}{\operatorname{ch} \frac{a\pi}{2\beta}} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.981 3.}).$$

$$4. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \beta x} \cdot x^{2m} dx = (-1)^m \frac{\pi}{2\beta} \cdot \frac{\partial^{2m}}{\partial a^{2m}} \left(\frac{1}{\operatorname{ch} \frac{a\pi}{2\beta}} \right) \quad [\operatorname{Re} \beta > 0] \quad (\text{cf. 3.981 3}).$$

3.981
GW ((336))(18a)

545

$$5. \int_0^{\infty} x \frac{\sin 2ax}{\operatorname{ch} \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{\operatorname{sh} \frac{a\pi}{\beta}}{\operatorname{ch}^2 \frac{a\pi}{\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((364))(6)a

$$6. \int_0^{\infty} x \frac{\cos 2ax}{\operatorname{sh} \beta x} dx = \frac{\pi^2}{4\beta^2} \cdot \frac{1}{\operatorname{ch}^2 \frac{a\pi}{\beta}} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((364))(1)a

$$7. \int_0^{\infty} \frac{\sin ax}{\operatorname{ch} \beta x} \frac{dx}{x} = 2 \operatorname{arctg} \left(\exp \frac{\pi a}{2\beta} \right) - \frac{\pi}{2} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((387))(1), ET I 89(13), LI ((298))(17)

4.112

$$1. \int_0^{\infty} (x^2 + \beta^2) \frac{\cos ax}{\operatorname{ch} \frac{\pi x}{2\beta}} dx = \frac{2\beta^3}{\operatorname{ch}^3 a\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 32(19)

$$2. \int_0^{\infty} x(x^2 + 4\beta^2) \frac{\cos ax}{\operatorname{sh} \frac{\pi x}{2\beta}} dx = \frac{6\beta^4}{\operatorname{ch}^4 a\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 32(20)

$$\begin{aligned}
1. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \pi x} \cdot \frac{dx}{x^2 + \beta^2} &= -\frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{\beta \sin \pi\beta} + \\
&+ \frac{1}{2\beta^2} [{}_2F_1(1, -\beta; 1 - \beta; -e^{-a}) + {}_2F_1(1, \beta; 1 + \beta; -e^{-a})] = \\
&= \frac{1}{2\beta^2} - \frac{\pi e^{-a\beta}}{2\beta \sin \pi\beta} - \sum_{k=1}^{\infty} \frac{(-1)^k e^{-ak}}{k^2 - \beta^2} \\
&\quad [\operatorname{Re} \beta > 0, \quad \beta \neq 0, 1, 2, \dots, a > 0].
\end{aligned}$$

ET I 90(18)

$$\begin{aligned}
2. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \pi x} \cdot \frac{dx}{x^2 + m^2} &= \frac{(-1)^m a e^{-ma}}{2m} + \frac{1}{2m} \sum_{k=1}^{m-1} \frac{(-1)^k e^{-ka}}{m-k} + \frac{(-1)^m e^{-ma}}{2m} \ln(1 + e^{-a}) + \\
&+ \frac{1}{2m!} \frac{d^{m-1}}{dz^{m-1}} \left[\frac{(1+z)^{m-1}}{z} \ln(1+z) \right]_{z=e^{-a}} \quad [a > 0].
\end{aligned}$$

ET I 89(17)

$$3. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \pi x} \cdot \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin ax}{\operatorname{sh} \pi x} \frac{dx}{1+x^2} = -\frac{a}{2} \operatorname{ch} a + \operatorname{sh} a \ln \left(2 \operatorname{ch} \frac{a}{2} \right).$$

GW ((336))(21b)

$$4. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin ax}{\operatorname{sh} \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{2} \operatorname{sh} a - \operatorname{ch} a \operatorname{arctg}(\operatorname{sh} a).$$

GW ((336))(21a)

$$5. \int_0^\infty \frac{\sin ax}{\operatorname{sh} \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} = -\frac{\pi}{\sqrt{2}} e^{-a} + \frac{\operatorname{sh} a}{\sqrt{2}} \ln \frac{2 \operatorname{ch} a + \sqrt{2}}{2 \operatorname{ch} a - \sqrt{2}} + \sqrt{2} \operatorname{ch} a \operatorname{arctg} \frac{\sqrt{2}}{2 \operatorname{sh} a} \quad [a > 0].$$

LI ((389))(1)

$$6. \int_0^\infty \frac{\sin ax}{\operatorname{ch} \frac{\pi}{4} x} \cdot \frac{x dx}{1+x^2} = \frac{\pi}{\sqrt{2}} e^{-a} + \frac{\operatorname{sh} a}{\sqrt{2}} \ln \frac{2 \operatorname{ch} a + \sqrt{2}}{2 \operatorname{ch} a - \sqrt{2}} - \sqrt{2} \operatorname{ch} a \operatorname{arctg} \left(\frac{1}{\sqrt{2} \operatorname{sh} a} \right) \quad [a > 0].$$

BI ((388))(1)

$$8. \int_0^{\infty} \frac{\cos ax}{\operatorname{sh} \frac{\pi}{2} x} \cdot \frac{x dx}{1+x^2} = 2 \operatorname{sh} a \operatorname{arctg}(e^{-a}) + \frac{\pi}{2} e^{-a} - 1 \quad [a > 0].$$

BI ((389))(11)

$$9. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \cdot \frac{x dx}{x^2 + \beta^2} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(k + \frac{1}{2}\right)^2 e^{-a\beta} - \beta e^{-(k+\frac{1}{2})a}}{\beta \left[\left(k + \frac{1}{2}\right)^2 - \beta^2\right]} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

ET I 32(26)

$$10. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \cdot \frac{dx}{\left(m + \frac{1}{2}\right)^2 + x^2} = \frac{(-1)^m e^{-(m+\frac{1}{2})a}}{2m+1} [a + \ln(1 + e^{-a})] + \\ + \frac{e^{-\frac{a}{2}}}{2m+1} \sum_{k=0}^{m-1} \frac{(-1)^k e^{-ak}}{k-m} + \frac{e^{-\frac{a}{2}}}{(2m+1)(m+1)} \times \\ \times {}_2F_1(1, m+1; m+2; -e^{-a}) \quad [a > 0].$$

ET I 32(25)

$$11. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x} \cdot \frac{dx}{1+x^2} = 2 \operatorname{ch} \frac{a}{2} - [e^a \operatorname{arctg}(e^{-\frac{a}{2}}) + e^{-a} \operatorname{arctg}(e^{\frac{a}{2}})] \quad [a > 0].$$

ET I 32(21)

$$12. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \frac{\pi}{2} x} \cdot \frac{dx}{1+x^2} = ae^{-a} + \operatorname{ch} a \ln(1 + e^{-2a}) \quad [a > 0].$$

BI ((388))(6)

$$13. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \frac{\pi}{4} x} \cdot \frac{dx}{1+x^2} = \frac{\pi}{\sqrt{2}} e^{-a} + \frac{2 \operatorname{sh} a}{\sqrt{2}} \operatorname{arctg} \left(\frac{1}{\sqrt{2} \operatorname{sh} a} \right) - \frac{\operatorname{ch} a}{\sqrt{2}} \ln \frac{2 \operatorname{ch} a + \sqrt{2}}{2 \operatorname{ch} a - \sqrt{2}} \quad [a > 0].$$

BI ((388))(5)

$$1. \int_0^{\infty} \frac{\sin ax \operatorname{sh} \beta x}{x \operatorname{sh} \gamma x} dx = \operatorname{arctg} \left(\operatorname{tg} \frac{\beta\pi}{2\gamma} \theta \frac{a\pi}{2\gamma} \right) \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma, \quad a > 0].$$

BI ((387))(6)a

$$2. \int_0^{\infty} \frac{\cos ax \operatorname{sh} \beta x}{x \operatorname{ch} \gamma x} dx = \frac{1}{2} \ln \frac{\operatorname{ch} \frac{a\pi}{2\gamma} + \sin \frac{\beta\pi}{2\gamma}}{\operatorname{ch} \frac{a\pi}{2\gamma} - \sin \frac{\beta\pi}{2\gamma}} \quad [|\operatorname{Re} \beta| < \operatorname{Re} \gamma].$$

ET I 33(34)

4.115

$$1. \int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{sh} \pi x} dx = \frac{\pi e^{-ab} \sin b\beta}{2 \sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{k e^{-ak} \sin k\beta}{k^2 - b^2}$$

$$[0 < \operatorname{Re} \beta < \pi, \quad a > 0, \quad b > 0].$$

BI ((389))(23)

$$2. \int_0^{\infty} \frac{x \sin ax}{x^2 + 1} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{sh} \pi x} dx = \frac{1}{2} e^{-a} (a \sin \beta - \beta \cos \beta) - \frac{1}{2} \operatorname{sh} a \sin \beta \ln[1 + 2e^{-a} \cos \beta + e^{-2a}] +$$

$$+ \operatorname{ch} a \cos \beta \operatorname{arctg} \frac{\sin \beta}{e^a + \cos \beta} \quad [|\operatorname{Re} \beta| < \pi, \quad a > 0].$$

LI ((389))(10)

$$3. \int_0^{\infty} \frac{x \sin ax}{x^2 + 1} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{sh} \frac{\pi}{2} x} dx = \frac{\pi}{2} e^{-a} \sin \beta + \frac{1}{2} \cos \beta \operatorname{sh} a \ln \frac{\operatorname{ch} a + \sin \beta}{\operatorname{ch} a - \sin \beta} - \sin \beta \operatorname{ch} a \operatorname{arctg} \left(\frac{\cos \beta}{\operatorname{sh} a} \right)$$

$$\left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right].$$

BI ((389))(8)

$$4. \int_0^{\infty} \frac{\cos ax}{x^2 + b^2} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{sh} \pi x} dx = \frac{\pi}{2b} \cdot \frac{e^{-ab} \sin b\beta}{\sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{e^{-ak} \sin k\beta}{k^2 - b^2}$$

$$[0 < \operatorname{Re} \beta < \pi, \quad a > 0, \quad b > 0].$$

BI ((389))(22)

$$6. \int_0^{\infty} \frac{\cos ax}{x^2 + 1} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{sh} \frac{\pi}{2} x} dx = \frac{\pi}{2} e^{-a} \sin \beta - \frac{1}{2} \operatorname{ch} a \cos \beta \ln \frac{\operatorname{ch} a + \sin \beta}{\operatorname{ch} a - \sin \beta} + \operatorname{sh} a \sin \beta \operatorname{arctg} \frac{\cos \beta}{\operatorname{sh} a}$$

$$\left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0, \quad b > 0 \right].$$

BI ((389))(18)

$$7. \int_0^{\infty} \frac{\sin ax}{x^2 + \frac{1}{4}} \cdot \frac{\operatorname{sh} \beta x}{\operatorname{ch} \pi x} dx = e^{-\frac{a}{2}} \left(a \sin \frac{\beta}{2} - \beta \cos \frac{\beta}{2} \right) - \operatorname{sh} \frac{a}{2} \sin \frac{\beta}{2} \ln(1 + 2e^{-a} \cos \beta + e^{-2a}) +$$

$$+ \operatorname{ch} \frac{a}{2} \cos \frac{\beta}{2} \operatorname{arctg} \frac{\sin \beta}{1 + e^{-a} \cos \beta} \quad [|\operatorname{Re} \beta| < \pi, \quad a > 0].$$

ET I 91(26)

548

$$8. \int_0^{\infty} \frac{\sin ax}{x^2 + \beta^2} \cdot \frac{\operatorname{ch} \gamma x}{\operatorname{sh} \pi x} dx = \frac{1}{2\beta^2} - \frac{\pi}{2\beta} \cdot \frac{e^{-a\beta} \cos \beta \gamma}{\sin \beta \pi} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{-ak} \cos k\gamma}{k^2 - \beta^2}$$

$$[0 \leq \operatorname{Re} \beta, \quad |\operatorname{Re} \gamma| < \pi, \quad a > 0].$$

BI ((389))(21)

$$9. \int_0^{\infty} \frac{\sin ax}{x^2 + 1} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{sh} \pi x} dx = -\frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta) + \frac{1}{2} \operatorname{sh} a \cos \beta \ln(1 + 2e^{-a} \cos \beta + e^{-2a}) +$$

$$+ \operatorname{ch} a \sin \beta \operatorname{arctg} \frac{\sin \beta}{e^a + \cos \beta} \quad [|\operatorname{Re} \beta| < \pi, \quad a > 0].$$

ET I 91(25), LI ((389))(9)

$$10. \int_0^{\infty} \frac{\sin ax}{x^2 + 1} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{sh} \frac{\pi}{2} x} dx = -\frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \operatorname{sh} a \sin \beta \ln \frac{\operatorname{ch} a + \sin \beta}{\operatorname{ch} a - \sin \beta} + \operatorname{ch} a \cos \beta \operatorname{arctg} \frac{\cos \beta}{\operatorname{sh} a}$$

$$\left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right],$$

BI ((389))(7)

$$11. \int_0^{\infty} \frac{x \cos ax}{x^2 + b^2} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{sh} \pi x} dx = \frac{\pi}{2} \cdot \frac{e^{-ab} \cos b\beta}{\sin b\pi} + \sum_{k=1}^{\infty} (-1)^k \frac{k e^{-ak} \cos k\beta}{k^2 - b^2} \quad [|\operatorname{Re} \beta| < \pi, \quad a > 0].$$

$$12. \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{sh} \pi x} dx = \frac{1}{2} e^{-a} (a \cos \beta + \beta \sin \beta) - \frac{1}{2} + \frac{1}{2} \operatorname{ch} a \cos \beta \ln[1 + 2e^{-a} \cos \beta + e^{-2a}] + \operatorname{sh} a \sin \beta \operatorname{arctg} \frac{\sin \beta}{e^a + \cos \beta} \quad [|\operatorname{Re} \beta| < \pi, \quad a > 0].$$

BI ((389))(19)

$$13. \int_0^\infty \frac{x \cos ax}{x^2 + 1} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{sh} \frac{\pi}{2} x} dx = -1 + \frac{\pi}{2} e^{-a} \cos \beta + \frac{1}{2} \operatorname{ch} a \sin \beta \ln \frac{\operatorname{ch} a + \sin \beta}{\operatorname{ch} a - \sin \beta} + \operatorname{sh} a \cos \beta \operatorname{arctg} \frac{\cos \beta}{\operatorname{sh} a} \quad \left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right].$$

BI ((389))(17)

$$14. \int_0^\infty \frac{\cos ax}{x^2 + 1} \cdot \frac{\operatorname{ch} \beta x}{\operatorname{ch} \frac{\pi}{2} x} dx = a e^{-a} \cos \beta + \beta e^{-a} \sin \beta + \operatorname{sh} a \sin \beta \operatorname{arctg} \frac{e^{-2a} \sin 2\beta}{1 + e^{-2a} \cos 2\beta} + \frac{1}{2} \operatorname{ch} a \cos \beta \ln(1 + 2e^{-2a} \cos 2\beta + e^{-4a}) \quad \left[|\operatorname{Re} \beta| < \frac{\pi}{2}, \quad a > 0 \right].$$

ET I 34(37)

4.116

$$1.^6 \int_0^\infty x \cos 2ax \theta x dx \text{ the integral is divergent.}$$

BI ((364))(2)

549

$$2. \int_0^\infty \cos ax \theta \beta x \frac{dx}{x} = \ln \operatorname{cth} \frac{a\pi}{4\beta} \quad [\operatorname{Re} \beta > 0, \quad a > 0].$$

BI ((387))(8)

4.117

$$1. \int_0^\infty \frac{\sin ax}{1 + x^2} \theta \frac{\pi x}{2} dx = a \operatorname{ch} a - \operatorname{sh} a \ln(2 \operatorname{sh} a) \quad [a > 0].$$

BI ((388))(3)

$$2. \int_0^\infty \frac{\sin ax}{1 + x^2} \theta \frac{\pi x}{4} dx = -\frac{\pi}{2} e^a + \operatorname{sh} a \ln \operatorname{cth} \frac{a}{2} + 2 \operatorname{ch} a \operatorname{arctg}(e^a).$$

$$3. \int_0^{\infty} \frac{\sin ax}{1+x^2} \operatorname{cth} \pi x dx = \frac{a}{2} e^{-a} - \operatorname{sh} a \ln(1 - e^{-a}) \quad [a > 0].$$

BI ((389))(5)

$$4. \int_0^{\infty} \frac{\sin ax}{1+x^2} \operatorname{cth} \frac{\pi}{2} x dx = \operatorname{sh} a \ln \operatorname{cth} \frac{a}{2} \quad [a > 0].$$

BI ((389))(6)

$$5. \int_0^{\infty} \frac{x \cos ax}{1+x^2} \theta \frac{\pi}{2} x dx = -ae^{-a} - \operatorname{ch} a \ln(1 - e^{-2a}) \quad [a > 0].$$

BI ((388))(7)

$$6. \int_0^{\infty} \frac{x \cos ax}{1+x^2} \theta \frac{\pi}{4} x dx - \frac{\pi}{2} e^a + \operatorname{ch} a \ln \operatorname{cth} \frac{a}{2} + 2 \operatorname{sh} a \operatorname{arctg}(e^a) \quad [a > 0].$$

BI ((388))(8)

$$7. \int_0^{\infty} \frac{x \cos ax}{1+x^2} \operatorname{cth} \pi x dx = -\frac{a}{2} e^{-a} - \frac{1}{2} - \operatorname{ch} a \ln(1 - e^{-a}).$$

BI ((389))(15)A, ET I 33(31)a

$$8. \int_0^{\infty} \frac{x \cos ax}{1+x^2} \operatorname{cth} \frac{\pi}{2} x dx = -1 + \operatorname{ch} a \ln \operatorname{cth} \frac{a}{2} \quad [a > 0].$$

BI ((389))(12)

$$9. \int_0^{\infty} \frac{x \cos ax}{1+x^2} \operatorname{cth} \frac{\pi}{4} x dx = -2 + \frac{\pi}{2} e^{-a} + \operatorname{ch} a \ln \operatorname{cth} \frac{a}{2} + 2 \operatorname{sh} a \operatorname{arctg}(e^{-a}) \quad [a > 0].$$

BI ((389))(13)

4.118

$$\int_0^{\infty} \frac{x \sin ax}{\operatorname{ch}^2 x} dx = -\frac{d}{da} \left(\frac{\pi a}{2 \operatorname{sh} \frac{\pi a}{2}} \right) \quad [a > 0].$$

$$\int_0^{\infty} \frac{1 - \cos px}{\operatorname{sh} qx} \cdot \frac{dx}{x} = \ln \left(\operatorname{ch} \frac{p\pi}{2q} \right).$$

BI ((387))(2)a

4.121

$$1. \int_0^{\infty} \frac{\sin ax - \sin bx}{\operatorname{ch} \beta x} \cdot \frac{dx}{x} = 2 \operatorname{arctg} \frac{\exp \frac{a\pi}{2\beta} - \exp \frac{b\pi}{2\beta}}{1 + \exp \frac{(a+b)\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0].$$

GW ((336))(19b)

$$2. \int_0^{\infty} \frac{\cos ax - \cos bx}{\operatorname{sh} \beta x} \cdot \frac{dx}{x} = \ln \frac{\operatorname{ch} \frac{b\pi}{2\beta}}{\operatorname{ch} \frac{a\pi}{2\beta}} \quad [\operatorname{Re} \beta > 0].$$

GW ((336))(19a)

550

4.122

$$1.^6 \int_0^{\infty} \frac{\cos \beta x \sin \gamma x}{\operatorname{ch} \delta x} \cdot \frac{dx}{x} = \operatorname{arctg} \frac{\operatorname{sh} \frac{\gamma\pi}{2\delta}}{\operatorname{ch} \frac{\beta\pi}{2\delta}} \quad [\operatorname{Re} \delta > |\operatorname{Im} \beta| + |\operatorname{Im} \gamma|].$$

ET I 93(46)a

$$2. \int_0^{\infty} \sin^2 ax \frac{\operatorname{ch} \beta x}{\operatorname{sh} x} \cdot \frac{dx}{x} = \frac{1}{4} \ln \frac{\operatorname{ch} 2a\pi + \cos \beta\pi}{1 + \cos \beta\pi} \quad [|\operatorname{Re} \beta| < 1].$$

BI ((387))(7)

4.123

$$1. \int_0^{\infty} \frac{\sin x}{\operatorname{ch} ax + \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \operatorname{arctg} \frac{1}{a} - \frac{1}{a}.$$

BI ((390))(1)

$$2. \int_0^{\infty} \frac{\sin x}{\operatorname{ch} ax - \cos x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{a}{1 + a^2} - \operatorname{arctg} \frac{1}{a}.$$

$$3. \int_0^{\infty} \frac{\sin 2x}{\operatorname{ch} 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{1}{2a} \cdot \frac{1 + 2a^2}{1 + a^2} - \operatorname{arctg} \frac{1}{a}.$$

BI ((390))(4)

$$4. \int_0^{\infty} \frac{\operatorname{ch} ax \sin x}{\operatorname{ch} 2ax - \cos 2x} \cdot \frac{x dx}{x^2 - \pi^2} = \frac{-1}{2a(1 + a^2)}.$$

LI ((390))(3)

$$5. \int_0^{\infty} \frac{\cos ax}{\operatorname{ch} \pi x + \cos \pi \beta} \cdot \frac{dx}{x^2 + \gamma^2} = \frac{\pi e^{-a\gamma}}{2\gamma(\cos \gamma\pi + \cos \beta\pi)} + \\ + \frac{1}{\operatorname{sh} \beta\pi} \sum_{k=0}^{\infty} \left\{ \frac{\exp[-(2k+1-\beta)a]}{\gamma^2 - (2k+1-\beta)^2} - \frac{\exp[-(2k+1+\beta)a]}{\gamma^2 - (2k+1+\beta)^2} \right\} \\ [0 < \operatorname{Re} \beta < 1, \quad \operatorname{Re} \gamma > 0, \quad a > 0].$$

ET I 33(27)

$$6. \int_0^{\infty} \frac{\sin ax \operatorname{sh} bx}{\cos 2ax + \operatorname{ch} 2bx} x^{p-1} dx = \frac{\Gamma(p)}{(a^2 + b^2)^{\frac{p}{2}}} \sin \left(p \operatorname{arctg} \frac{a}{b} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^p} \quad [p > 0].$$

BI ((364))(8)

$$7. \int_0^{\infty} \sin ax^2 \frac{\sin \frac{\pi x}{2} \operatorname{sh} \frac{\pi x}{2}}{\cos \pi x + \operatorname{ch} \pi x} \cdot x dx = \frac{1}{4} \left[\frac{\partial \theta_1(z, q)}{\partial z} \right]_{z=0, q=e^{-2a}} \quad [a > 0].$$

ET I 93(49)

4.124

$$1. \int_0^1 \frac{\cos px \operatorname{ch}(q\sqrt{1-x^2})}{\sqrt{1-x^2}} dx = \frac{\pi}{2} J_0(\sqrt{p^2 - q^2}).$$

MO (40)

$$2. \int_u^{\infty} \cos ax \operatorname{ch} \sqrt{\beta(u^2 - x^2)} \cdot \frac{dx}{\sqrt{u^2 - x^2}} = \frac{\pi}{2} J_0 \left(\frac{u}{\sqrt{a^2 - \beta^2}} \right).$$

ET I 34(38)

$$1. \int_0^\infty \text{sh}(a \sin x) \cos(a \cos x) \sin x \sin 2nx \frac{dx}{x} = \frac{(-1)^{n-1} a^{2n-1} \pi}{(2n-1)!} \frac{\pi}{8} \left[1 + \frac{a^2}{2n(2n+1)} \right].$$

LI ((367))(14)

$$2. \int_0^\infty \text{ch}(a \sin x) \cos(a \cos x) \sin x \cos(2n-1)x \frac{dx}{x} = \frac{(-1)^{n-1} a^{2(n-1)} \pi}{[2(n-1)]!} \frac{\pi}{8} \left[1 - \frac{a^2}{2n(2n-1)} \right].$$

LI ((367))(15)

$$3. \int_0^\infty \text{sh}(a \sin x) \cos(a \cos x) \cos x \cos 2nx \frac{dx}{x} = \frac{\pi}{2} \sum_{k=n+1}^\infty \frac{(-1)^k a^{2k+1}}{(2k+1)!} + \frac{(-1)^n a^{2n+1} 3\pi}{(2n+1)!} \frac{\pi}{8} + \frac{(-1)^{n-1} a^{2n-1} \pi}{(2n-1)!} \frac{\pi}{8}.$$

LI ((367))(21)

4.126

$$1. \int_0^\infty \sin(a \cos bx) \text{sh}(a \sin bx) \frac{x dx}{c^2 - x^2} = \frac{\pi}{2} [\cos(a \cos bc) \text{ch}(a \sin bc) - 1] \quad [b > 0].$$

BI ((381))(2)

$$2. \int_0^\infty \sin(a \cos bx) \text{ch}(a \sin bx) \frac{dx}{c^2 - x^2} = \frac{\pi}{2c} \cos(a \cos bc) \text{sh}(a \sin bc) \quad [b > 0, \quad c > 0].$$

BI ((381))(1)

$$3. \int_0^\infty \cos(a \cos bx) \text{sh}(a \sin bx) \frac{x dx}{c^2 - x^2} = \frac{\pi}{2} [a \cos bc - \sin(a \cos bc) \text{ch}(a \sin bc)] \quad [b > 0].$$

BI ((381))(4)

$$4. \int_0^\infty \cos(a \cos bx) \text{ch}(a \sin bx) \frac{dx}{c^2 - x^2} = -\frac{\pi}{2c} \sin(a \cos bc) \text{sh}(a \sin bc) \quad [b > 0].$$

BI ((381))(3)

4.13 Combinations of trigonometric and hyperbolic functions and exponentials

4.131

$$1. \int_0^{\infty} \sin ax \operatorname{sh}^{\nu} \gamma x e^{-\beta x} dx = -\frac{i\Gamma(\nu+1)}{2^{\nu+2}\gamma} \left\{ \frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)} \right\}$$

[Re $\nu > -2$, Re $\gamma > 0$, |Re($\gamma\nu$)| < Re β].

ET I 91(30)a

$$2. \int_0^{\infty} \cos ax \operatorname{sh}^{\nu} \gamma x e^{-\beta x} dx = \frac{\Gamma(\nu+1)}{2^{\nu+2}\gamma} \left\{ \frac{\Gamma\left(\frac{\beta-\nu\gamma-ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma-ai}{2\gamma}+1\right)} - \frac{\Gamma\left(\frac{\beta-\nu\gamma+ai}{2\gamma}\right)}{\Gamma\left(\frac{\beta+\nu\gamma+ai}{2\gamma}+1\right)} \right\}$$

[Re $\nu > -1$, Re $\gamma > 0$, |Re($\gamma\nu$)| < Re β].

ET I 34(40)a

552

$$3. \int_0^{\infty} e^{-\beta x} \frac{\sin ax}{\operatorname{sh} \gamma x} dx = \sum_{k=1}^{\infty} \frac{2a}{a^2 + [\beta + (2k-1)\gamma]^2};$$

$$= \frac{1}{2\gamma i} \left[\psi\left(\frac{\beta + \gamma + ia}{2\gamma}\right) - \psi\left(\frac{\beta + \gamma - ia}{2\gamma}\right) \right] \quad [\operatorname{Re} \beta > |\operatorname{Re} \gamma|].$$

ET I 91(28)
BI ((264))(9)a

$$4. \int_0^{\infty} e^{-x} \frac{\sin ax}{\operatorname{sh} x} dx = \frac{\pi}{2} \operatorname{cth} \frac{a\pi}{2} - \frac{1}{a}.$$

ET I 91(29)

4.132

$$1. \int_0^{\infty} \frac{\sin ax \operatorname{sh} \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\operatorname{sh} \frac{2\pi a}{\gamma}}{\operatorname{ch} \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}} +$$

$$+ \frac{i}{2\gamma} \left[\psi\left(\frac{\beta}{\gamma} + i\frac{a}{\gamma} + 1\right) - \psi\left(\frac{\beta}{\gamma} - i\frac{a}{\gamma} + 1\right) \right] \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|, a > 0].$$

$$2. \int_0^{\infty} \frac{\sin ax \operatorname{ch} \beta x}{e^{\gamma x} - 1} dx = -\frac{a}{2(a^2 + \beta^2)} + \frac{\pi}{2\gamma} \cdot \frac{\operatorname{sh} \frac{2\pi a}{\gamma}}{\operatorname{ch} \frac{2\pi a}{\gamma} - \cos \frac{2\pi \beta}{\gamma}} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|].$$

BI ((265))(5)a

$$3. \int_0^{\infty} \frac{\sin ax \operatorname{ch} \beta x}{e^{\gamma x} + 1} dx = \frac{a}{2(a^2 + \beta^2)} - \frac{\pi}{\gamma} \cdot \frac{\operatorname{sh} \frac{a\pi}{\gamma} \cos \frac{\beta\pi}{\gamma}}{\operatorname{ch} \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|].$$

ET I 92(35)

$$4. \int_0^{\infty} \frac{\cos ax \operatorname{sh} \beta x}{e^{\gamma x} - 1} dx = \frac{\beta}{2(a^2 + \beta^2)} - \frac{\pi}{2\gamma} \cdot \frac{\sin \frac{2\pi\beta}{\gamma}}{\operatorname{ch} \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|].$$

LI ((265))(8)

$$5. \int_0^{\infty} \frac{\cos ax \operatorname{sh} \beta x}{e^{\gamma x} + 1} dx = -\frac{\beta}{2(a^2 + \beta^2)} + \frac{\pi}{\gamma} \cdot \frac{\sin \frac{\pi\beta}{\gamma} \operatorname{ch} \frac{\pi a}{\gamma}}{\operatorname{ch} \frac{2a\pi}{\gamma} - \cos \frac{2\beta\pi}{\gamma}} \quad [\operatorname{Re} \gamma > |\operatorname{Re} \beta|].$$

ET I 34(39)

4.133

$$1. \int_0^{\infty} \sin ax \operatorname{sh} \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp \gamma(\beta^2 - a^2) \sin(2a\beta\gamma) \quad [\operatorname{Re} \gamma > 0].$$

ET I 92(37)

553

$$2. \int_0^{\infty} \cos ax \operatorname{ch} \beta x \exp\left(-\frac{x^2}{4\gamma}\right) dx = \sqrt{\pi\gamma} \exp \gamma(\beta^2 - a^2) \cos(2a\beta\gamma) \quad [\operatorname{Re} \gamma > 0].$$

ET I 35(41)

4.134

$$1. \int_0^{\infty} e^{-\beta x^2} (\operatorname{ch} x - \cos x) dx = \sqrt{\frac{\pi}{\beta}} \operatorname{ch} \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$2. \int_0^{\infty} e^{-\beta x^2} (\operatorname{ch} x - \cos x) dx = \sqrt{\frac{\pi}{\beta}} \operatorname{sh} \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

4.135

$$1. \int_0^{\infty} \sin ax^2 \operatorname{ch} 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \sin\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \\ [\operatorname{Re} \beta > 0].$$

LI ((268))(7)

$$2. \int_0^{\infty} \cos ax^2 \operatorname{ch} 2\gamma x e^{-\beta x^2} dx = \frac{1}{2} \sqrt[4]{\frac{\pi^2}{a^2 + \beta^2}} \exp\left(-\frac{\beta\gamma^2}{a^2 + \beta^2}\right) \cos\left(\frac{a\gamma^2}{a^2 + \beta^2} + \frac{1}{2} \operatorname{arctg} \frac{a}{\beta}\right) \\ [\operatorname{Re} \beta > 0].$$

LI ((268))(8)

4.136

$$1. \int_0^{\infty} (\operatorname{sh} x^2 + \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{ch} \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$2. \int_0^{\infty} (\operatorname{sh} x^2 - \sin x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{sh} \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$3. \int_0^{\infty} (\operatorname{ch} x^2 + \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{ch} \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$4. \int_0^{\infty} (\operatorname{ch} x^2 - \cos x^2) e^{-\beta x^4} dx = \frac{\sqrt{2}\pi}{4\sqrt{\beta}} I_{-\frac{1}{4}}\left(\frac{1}{8\beta}\right) \operatorname{sh} \frac{1}{8\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

4.137

$$1. \int_0^{\infty} \sin 2x^2 \operatorname{sh} 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} + \frac{\pi}{4}\right) \quad [\operatorname{Re} \beta > 0].$$

$$2. \int_0^{\infty} \sin 2x^2 \operatorname{ch} 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}} \left(\frac{1}{\beta} \right) \cos \left(\frac{1}{\beta} - \frac{\pi}{4} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

554

$$3. \int_0^{\infty} \cos 2x^2 \operatorname{sh} 2x^2 e^{-\beta x^4} dx = \frac{-\pi}{\sqrt[4]{128\beta^2}} J_{\frac{1}{4}} \left(\frac{1}{\beta} \right) \sin \left(\frac{1}{\beta} - \frac{\pi}{4} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$4. \int_0^{\infty} \cos 2x^2 \operatorname{ch} 2x^2 e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{128\beta^2}} J_{-\frac{1}{4}} \left(\frac{1}{\beta} \right) \sin \left(\frac{1}{\beta} + \frac{\pi}{4} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

4.138

$$1. \int_0^{\infty} (\sin 2x^2 \operatorname{ch} 2x^2 + \cos 2x^2 \operatorname{sh} 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}} \left(\frac{1}{\beta} \right) \cos \left(\frac{1}{\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$2. \int_0^{\infty} (\sin 2x^2 \operatorname{ch} 2x^2 - \cos 2x^2 \operatorname{sh} 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{\frac{1}{4}} \left(\frac{1}{\beta} \right) \sin \left(\frac{1}{\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$3. \int_0^{\infty} (\cos 2x^2 \operatorname{ch} 2x^2 + \sin 2x^2 \operatorname{sh} 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}} \left(\frac{1}{\beta} \right) \cos \left(\frac{1}{\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$4. \int_0^{\infty} (\cos 2x^2 \operatorname{ch} 2x^2 - \sin 2x^2 \operatorname{sh} 2x^2) e^{-\beta x^4} dx = \frac{\pi}{\sqrt[4]{32\beta^2}} J_{-\frac{1}{4}} \left(\frac{1}{\beta} \right) \sin \left(\frac{1}{\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

4.14 Combinations of trigonometric and hyperbolic functions, exponentials, and powers

4.141

$$1. \int_0^{\infty} x e^{-\beta x^2} \operatorname{ch} x \sin x \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} + \sin \frac{1}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$2. \int_0^{\infty} x e^{-\beta x^2} \operatorname{sh} x \cos x \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} - \sin \frac{1}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$3. \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{ch} x \cos x \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{1}{2\beta} - \frac{1}{\beta} \sin \frac{1}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$4. \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{sh} x \sin x \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\sin \frac{1}{2\beta} + \frac{1}{\beta} \cos \frac{1}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

MI 32

4.142

$$1. \int_0^{\infty} x e^{-\beta x^2} (\operatorname{sh} x + \sin x) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \operatorname{ch} \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$2. \int_0^{\infty} x e^{-\beta x^2} (\operatorname{sh} x - \sin x) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \operatorname{sh} \frac{1}{4\beta} \quad [\operatorname{Re} \beta > 0].$$

ME 24

555

$$3. \int_0^{\infty} x^2 e^{-\beta x^2} (\operatorname{ch} x + \cos x) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left(\operatorname{ch} \frac{1}{4\beta} + \frac{1}{2\beta} \operatorname{sh} \frac{1}{4\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

ME 24

$$4. \int_0^{\infty} x^2 e^{-\beta x^2} (\operatorname{ch} x - \cos x) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta^3}} \left(\operatorname{sh} \frac{1}{4\beta} + \frac{1}{2\beta} \operatorname{ch} \frac{1}{4\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

4.143

$$1. \int_0^{\infty} x e^{-\beta x^2} (\operatorname{ch} x \sin x + \operatorname{sh} x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \cos \frac{1}{2\beta} \quad [\operatorname{Re} \beta > 0].$$

MI 32

$$2. \int_0^{\infty} x e^{-\beta x^2} (\operatorname{ch} x \sin x - \operatorname{sh} x \cos x) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \sin \frac{1}{2\beta} \quad [\operatorname{Re} \beta > 0].$$

MI 32

4.144

$$\int_0^{\infty} e^{-x^2} \operatorname{sh} x^2 \cos ax \frac{dx}{x^2} = \sqrt{\frac{\pi}{2}} e^{-\frac{a^2}{8}} - \frac{\pi a}{4} \left[1 - \Phi \left(\frac{a}{\sqrt{8}} \right) \right] \quad [a > 0].$$

ET I 35(44)

4.145

$$1. \int_0^{\infty} x e^{-\beta x^2} \operatorname{ch}(2ax \sin t) \sin(2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left(-\frac{a^2}{\beta} \cos 2t \right) \cos \left(t - \frac{a^2}{\beta} \sin 2t \right) \quad [\operatorname{Re} \beta > 0].$$

BI ((363))(5)

$$2. \int_0^{\infty} x e^{-\beta x^2} \operatorname{sh}(2ax \sin t) \cos(2ax \cos t) dx = \frac{a}{2} \sqrt{\frac{\pi}{\beta^3}} \exp \left(-\frac{a^2}{\beta} \cos 2t \right) \sin \left(t - \frac{a^2}{\beta} \sin 2t \right) \quad [\operatorname{Re} \beta > 0].$$

BI ((363))(6)

4.146

$$1.^8 \int_0^{\infty} e^{-\beta x^2} \operatorname{sh} ax \sin bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left(\frac{a^2 - b^2}{4\beta} \right) \sin \frac{ab}{2\beta} \quad [\operatorname{Re} \beta > 0].$$

$$2.^8 \int_0^{\infty} e^{-\beta x^2} \operatorname{ch} ax \cos bx dx = \frac{1}{2} \sqrt{\frac{\pi}{\beta}} \exp \left(\frac{a^2 - b^2}{4\beta} \right) \cos \frac{ab}{2\beta} \quad [\operatorname{Re} \beta > 0].$$

$$3. \int_0^{\infty} x e^{-\beta x^2} \operatorname{ch} ax \sin ax dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left(\cos \frac{a^2}{2\beta} + \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

$$4. \int_0^{\infty} x e^{-\beta x^2} \operatorname{sh} ax \cos ax \, dx = \frac{a}{4\beta} \sqrt{\frac{\pi}{\beta}} \left(\cos \frac{a^2}{2\beta} - \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

$$5.8 \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{ch} ax \sin ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\sin \frac{a^2}{2\beta} + \frac{a^2}{\beta} \cos \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

$$6.8 \int_0^{\infty} x^2 e^{-\beta x^2} \operatorname{ch} ax \cos ax \, dx = \frac{1}{4} \sqrt{\frac{\pi}{\beta^3}} \left(\cos \frac{a^2}{2\beta} - \frac{a^2}{\beta} \sin \frac{a^2}{2\beta} \right) \quad [\operatorname{Re} \beta > 0].$$

556

4.2- 4.4 Logarithmic Functions

4.21 Logarithmic Functions

4.211

$$1. \int_e^{\infty} \frac{dx}{\ln \frac{1}{x}} = -\infty$$

BI((33))(9)

$$2. \int_0^u \frac{dx}{\ln x} = \operatorname{li} u$$

FI III 653, FI II 606

4.212

$$1.7 \int_0^1 \frac{dx}{a + \ln x} = e^{-a} \operatorname{Ei}(a) [a > 0]. \quad (\text{cf. note after 4.212.9})$$

4.212
BI((31))(4)

$$2. \int_0^1 \frac{dx}{a - \ln x} = -e^a \operatorname{Ei}(-a) [a > 0].$$

$$3.7 \int_0^1 \frac{dx}{(a \ln x)^2} = -\frac{1}{a} + e^{-a} \text{Ei}(a) \quad [a > 0].$$

BI((31))(14)

$$4. \int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^a \text{Ei}(-a) \quad [a > 0].$$

BI((31))(16)

$$5.8 \int_0^1 \frac{\ln x \, dx}{(a + \ln x)^2} = 1 + (1 - a)e^{-a} \text{Ei}(a) \quad [a > 0].$$

BI((31))(15)

$$6. \int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1 + a)e^a \text{Ei}(-a) \quad [a > 0].$$

BI((31))(17)

$$7. \int_1^e \frac{\ln x \, dx}{(1 + \ln x)^2} = \frac{e}{2} - 1.$$

BI((33))(10)

$$8.7 \int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{1}{(n-1)!} e^{-a} \text{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! a^{k-n} \quad [a > 0, n \text{ odd}].$$

BI((31))(22)

$$9. \int_0^1 \frac{dx}{(a - \ln x)^n} = \frac{(-1)^n}{(n-1)!} e^a \text{Ei}(-a) + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! (-a)^{k-n} \quad [a > 0, n \text{ odd}].$$

BI((31))(23)

In integrals of the form $\int \frac{(\ln x)^m}{[a^n + (\ln x)^n]^l} dx$ it is convenient to make the substitution $x = e^{-t}$.

Results 4.212.3, 4.212.5, and 4.212.8 [for $n > 1$] and 4.213.6., 4.213.8 below are divergent but may be considered to be valid if defined as follows:

$$\int_0^a \frac{f(z) dz}{(z - z_0)^n} = \frac{1}{(n-1)!} \left(\frac{d}{dz_0} \right)^{n-1} P \int_0^a \frac{f(z)}{z - z_0} dz$$

where $a > z_0 > 0$, $n = 1, 2, 3, \dots$ and P indicates the Cauchy principal value.

4.213

$$1. \int_0^1 \frac{dx}{a^2 + (\ln x)^2} = \frac{1}{a} [\text{ci}(a) \sin a - \text{si}(a) \cos a] \quad [a > 0].$$

BI ((31))(6)

$$2.7 \int_0^1 \frac{dx}{a^2 - (\ln x)^2} = \frac{1}{2a} [e^{-a} \text{Ei}(a) - e^a \text{Ei}(-a)] \quad [a > 0], \quad (\text{cf. 4.212 1. and 2.})$$

4.212
BI ((31))(8)

$$3. \int_0^1 \frac{\ln x dx}{a^2 + (\ln x)^2} = \text{ci}(a) \cos a + \text{si}(a) \sin a \quad [a > 0].$$

BI ((31))(7)

$$4.7 \int_0^1 \frac{\ln x dx}{a^2 - (\ln x)^2} = -\frac{1}{2} [e^{-a} \text{Ei}(a) + e^a \text{Ei}(-a)] \quad [a > 0], \quad (\text{cf. 4.212 1. and 2.})$$

4.212
BI ((31))(9)

$$5. \int_0^1 \frac{dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a^3} [\text{ci}(a) \sin a - \text{si}(a) \cos a] - \frac{1}{2a^2} [\text{ci}(a) \cos a + \text{si}(a) \sin a] \quad [a > 0].$$

$$6.7 \int_0^1 \frac{dx}{[a^2 - (\ln x)^2]^2} = \frac{1}{4a^3} [(a-1)e^a \operatorname{Ei}(-a) + (1+a)e^{-a} \operatorname{Ei}(a)] \quad [a > 0].$$

BI ((31))(20)

$$7. \int_0^1 \frac{\ln x \, dx}{[a^2 + (\ln x)^2]^2} = \frac{1}{2a} [\operatorname{ci}(a) \sin a - \operatorname{si}(a) \cos a] - \frac{1}{2a^2} \quad [a > 0].$$

BI ((31))(19)

$$8.7 \int_0^1 \frac{\ln x \, dx}{[a^2 - (\ln x)^2]^2} = \frac{1}{4a^2} \{2 + a[e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a)]\} \quad [a > 0].$$

LI ((31))(21)

4.214

$$1. \int_0^1 \frac{dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^3} [e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a) - 2 \operatorname{ci}(a) \sin a + 2 \operatorname{si}(a) \cos a] \quad [a > 0].$$

BI ((31))(10)

$$2. \int_0^1 \frac{\ln x \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a^2} [e^a \operatorname{Ei}(-a) + e^{-a} \operatorname{Ei}(a) - 2 \operatorname{ci}(a) \cos a - 2 \operatorname{si}(a) \sin a] \quad [a > 0].$$

BI ((31))(11)

$$3. \int_0^1 \frac{(\ln x)^2 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4a} [e^a \operatorname{Ei}(-a) - e^{-a} \operatorname{Ei}(a) + 2 \operatorname{ci}(a) \sin a - 2 \operatorname{si}(a) \cos a] \quad [a > 0].$$

BI ((31))(12)

558

$$4.7 \int_0^1 \frac{(\ln x)^3 \, dx}{a^4 - (\ln x)^4} = -\frac{1}{4} [e^a \operatorname{Ei}(-a) + e^{-a} \operatorname{Ei}(a) + 2 \operatorname{ci}(a) \cos a + 2 \operatorname{si}(a) \sin a] \quad [a > 0].$$

BI ((31))(13)

4.215

$$1. \int_0^1 \left(\ln \frac{1}{x} \right)^{\mu-1} dx = \Gamma(\mu) \quad [\operatorname{Re} \mu > 0].$$

$$2. \int_0^1 \frac{dx}{\left(\ln \frac{1}{x}\right)^\mu} = \frac{\pi}{\Gamma(\mu)} \operatorname{cosec} \mu\pi \quad [\operatorname{Re} \mu < 1].$$

BI ((31))(1)

$$3. \int_0^1 \sqrt{\ln \frac{1}{x}} dx = \frac{\sqrt{\pi}}{2}.$$

BI ((32))(1)

$$4. \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi}.$$

BI ((32))(3)

4.216

$$\int_0^{\frac{1}{e}} \frac{dx}{\sqrt{(\ln x)^2 - 1}} = K_0(1).$$

GW ((32))(2)

4.22 Logarithms of more complicated arguments

4.221

$$1. \int_0^1 \ln x \ln(1-x) dx = 2 - \frac{\pi^2}{6}.$$

BI ((30))(7)

$$2. \int_0^1 \ln x \ln(1+x) dx = 2 - \frac{\pi^2}{12} - 2 \ln 2.$$

BI ((30))(8)

$$3. \int_0^1 \ln \frac{1-ax}{1-a} \frac{dx}{\ln x} = -\sum_{k=1}^{\infty} a^k \frac{\ln(1+k)}{k} \quad [a < 1].$$

$$1. \int_0^{\infty} \ln \frac{a^2 + x^2}{b^2 + x^2} dx = (a - b)\pi \quad [a > 0, \quad b > 0].$$

GW ((322))(20)

$$2. \int_0^{\infty} \ln x \ln \frac{a^2 + x^2}{b^2 + x^2} dx = \pi(b - a) + \pi \ln \frac{a^a}{b^b} \quad [a > 0, \quad b > 0].$$

BI ((33))(1)

$$3. \int_0^{\infty} \ln x \ln \left(1 + \frac{b^2}{x^2}\right) dx = \pi b(\ln b - 1) \quad [b > 0].$$

BI ((33))(2)

$$4. \int_0^{\infty} \ln(1 + a^2 x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[\frac{1 + ab}{a} \ln(1 + ab) - b \right] \quad [a > 0, \quad b > 0].$$

BI ((33))(3)

559

$$5. \int_0^{\infty} \ln(a^2 + x^2) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi[(a+b) \ln(a+b) - a \ln a - b] \quad [a > 0, \quad b > 0].$$

BI ((33))(4)

$$6. \int_0^{\infty} \ln \left(1 + \frac{a^2}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi[(a+b) \ln(a+b) - a \ln a - b \ln b] \quad [a > 0, \quad b > 0].$$

BI ((33))(5)

$$7. \int_0^{\infty} \ln \left(a^2 + \frac{1}{x^2}\right) \ln \left(1 + \frac{b^2}{x^2}\right) dx = 2\pi \left[\frac{1 + ab}{a} \ln(1 + ab) - b \ln b \right] \quad [a > 0, \quad b > 0].$$

BI ((33))(7)

4.223

$$1. \int_0^{\infty} \ln(1 + e^{-x}) dx = \frac{\pi^2}{12}.$$

$$2. \int_0^{\infty} \ln(1 - e^{-x}) dx = -\frac{\pi^2}{6}.$$

BI ((256))(11)

$$3. \int_0^{\infty} \ln(1 + 2e^{-x} \cos t + e^{-2x}) dx = \frac{\pi^2}{6} - \frac{t^2}{2} \quad [|t| < \pi].$$

BI ((256))(18)

4.224

$$1. \int_0^u \ln \sin x dx = L\left(\frac{\pi}{2} - u\right) - L\left(\frac{\pi}{2}\right).$$

LO III 186(15)

$$2. \int_0^{\frac{\pi}{4}} \ln \sin x dx = -\frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G}.$$

BI ((285))(1)

$$3. \int_0^{\frac{\pi}{2}} \ln \sin x dx = \frac{1}{2} \int_0^{\pi} \ln \sin x dx = -\frac{\pi}{2} \ln 2.$$

FI II 629,643

$$4. \int_0^u \ln \cos x dx = -L(u).$$

LO III 184(10)

$$5. \int_0^{\frac{\pi}{4}} \ln \cos x dx = -\frac{\pi}{4} \ln 2 + \frac{1}{2} \mathbf{G}.$$

BI ((286))(1)

$$6. \int_0^{\frac{\pi}{2}} \ln \cos x dx = -\frac{\pi}{2} \ln 2.$$

BI 306(1)

$$7. \int_0^{\frac{\pi}{2}} (\ln \sin x)^2 dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right].$$

$$8. \int_0^{\frac{\pi}{2}} (\ln \cos x)^2 dx = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right].$$

560

$$9. \int_0^{\pi} \ln(a + b \cos x) dx = \pi \ln \frac{a + \sqrt{a^2 - b^2}}{2} \quad [a \geq |b| > 0].$$

$$10. \int_0^{\pi} \ln(1 \pm \sin x) dx = -\pi \ln 2 \pm 4\mathbf{G}.$$

$$11.^7 \int_0^{\frac{\pi}{2}} \ln(1 + a \sin x)^2 dx = \pi \ln(a/2) + 4\mathbf{G} + 4 \sum_{k=1}^{\infty} \frac{bk}{k} \sum_{n=1}^k \frac{(-1)^{n+1}}{2n-1} \quad b = \frac{(1-a)}{(1+a)} \quad [a > 0]$$

$$= -\pi \ln 2 - 4\mathbf{G} \quad [a = -1].$$

$$12. \int_0^{\pi} \ln(1 + a \cos x)^2 dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2}}{2} \quad [a^2 \leq 1].$$

$$13. \int_0^{\frac{\pi}{2}} \ln(1 + 2a \sin x + a^2) dx = \sum_{k=0}^{\infty} \frac{2^{2k} (k!)^2}{(2k+1) \cdot (2k+1)!!} \left(\frac{2a}{1+a^2} \right)^{2k+1} \quad [a^2 \leq 1].$$

$$14.^8 \int_0^{n\pi} \ln(b^2 - 2ab \cos x + a^2) dx = n\pi \ln a^2 \quad [a^2 \geq b^2 > 0];$$

$$= n\pi \ln b^2 \quad [b^2 > a^2 > 0].$$

4.225

$$1. \int_0^{\frac{\pi}{4}} \ln(\cos x - \sin x) dx = -\frac{\pi}{8} \ln 2 - \frac{1}{2} \mathbf{G}.$$

GW ((322))(9b)

$$2. \int_0^{\frac{\pi}{4}} \ln(\cos x + \sin x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\cos x + \sin x) dx = -\frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

GW ((322))(9a)

$$3. \int_0^{2\pi} \ln(1 + a \sin x + b \cos x) dx = 2\pi \ln \frac{1 + \sqrt{1 - a^2 - b^2}}{2} \quad [a^2 + b^2 < 1].$$

BI ((332))(2)

$$4. \int_0^{2\pi} \ln(1 + a^2 + b^2 + 2a \sin x + 2b \cos x) dx = 0 \quad [a^2 + b^2 \leq 1];$$

$$= 2\pi \ln(a^2 + b^2) \quad [a^2 + b^2 \geq 1].$$

BI ((322))(3)

4.226

$$1. \int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x)^2 dx = -2\pi \ln 2 \quad [a^2 \leq 1];$$

$$= 2\pi \ln \frac{a + \sqrt{a^2 - 1}}{2} = 2\pi(\text{Arch } a - \ln 2) \quad [a > 1].$$

FI II 644, 687

561

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + a \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(1 + a \sin^2 x) dx = \int_0^{\frac{\pi}{2}} \ln(1 + a \cos^2 x) dx =$$

$$= \frac{1}{2} \int_0^{\pi} \ln(1 + a \cos^2 x) dx = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \quad [a \geq -1].$$

BI ((308))(15), GW((322))(12)

$$3. \int_0^u \ln(1 - \sin^2 \alpha \sin^2 x) dx = (\pi - 2\theta) \ln \text{ctg } \frac{\alpha}{2} + 2u \ln \left(\frac{1}{2} \sin \alpha \right) - \frac{\pi}{2} \ln 2 +$$

$$4. \int_0^{\frac{\pi}{2}} \ln[1 - \cos^2 x (\sin^2 \alpha - \sin^2 \beta \sin^2 x)] dx = \pi \ln \left[\frac{1}{2} \left(\cos^2 \frac{\alpha}{2} + \sqrt{\cos^4 \frac{\alpha}{2} + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2}} \right) \right] \\ [\alpha > \beta > 0].$$

$$5. \int_0^u \ln \left(1 - \frac{\sin^2 x}{\sin^2 a} \right) dx = -u \ln \sin^2 a - L \left(\frac{\pi}{2} - \alpha + u \right) + L \left(\frac{\pi}{2} - \alpha - u \right) \\ \left[-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad |\sin u| \leq |\sin \alpha| \right].$$

$$6. \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \frac{a+b}{2} \\ [a > 0, \quad b > 0].$$

$$7. \int_0^{\frac{\pi}{2}} \ln \frac{1 + \sin t \cos^2 x}{1 - \sin t \cos^2 x} dx = \pi \ln \frac{1 + \sin \frac{t}{2}}{\cos \frac{t}{2}} = \pi \ln \operatorname{ctg} \frac{\pi - t}{4} \quad \left[|t| < \frac{\pi}{2} \right].$$

4.227

$$1. \int_0^u \ln \operatorname{tg} x dx = L(u) + L \left(\frac{\pi}{2} - u \right) - L \left(\frac{\pi}{2} \right).$$

$$2. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x dx = - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \operatorname{tg} x dx = -\mathbf{G}.$$

$$3. \int_0^{\frac{\pi}{2}} \ln(a \operatorname{tg} x) dx = \frac{\pi}{2} \ln a \quad [a > 0].$$

$$4.7 \int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^n dx = n!(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}} = \frac{1}{2} \left(\frac{\pi}{2}\right)^{n+1} |E_n| \quad [n \text{ even}].$$

BI ((286))(21)

$$5.7 \int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2n} dx \left(\frac{\pi^{2n+1}}{2} |E_{2n}| \right).$$

BI ((307))(15)

$$6. \int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2n+1} dx = 0.$$

BI ((307))(14)

$$7. \int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^2 dx = \frac{\pi^3}{16}.$$

BI ((286))(16)

$$8. \int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^4 dx = \frac{5}{64} \pi^5.$$

BI ((286))(19)

$$9. \int_0^{\frac{\pi}{4}} \ln(1 + \operatorname{tg} x) dx = \frac{\pi}{8} \ln 2.$$

BI ((287))(1)

$$10. \int_0^{\frac{\pi}{2}} \ln(1 + \operatorname{tg} x) dx = \frac{\pi}{4} \ln 2 + \mathbf{G}.$$

BI ((308))(9)

$$11. \int_0^{\frac{\pi}{4}} \ln(1 - \operatorname{tg} x) dx = \frac{\pi}{8} \ln 2 - \mathbf{G}.$$

BI ((287))(2)

BI ((308))(10)

$$13. \int_0^{\frac{\pi}{4}} \ln(1 + \operatorname{ctg} x) dx = \frac{\pi}{8} \ln 2 + \mathbf{G}.$$

BI ((287))(3)

$$14. \int_0^{\frac{\pi}{4}} \ln(\operatorname{ctg} x - 1) dx = \frac{\pi}{8} \ln 2.$$

BI ((287))(4)

$$15. \int_0^{\frac{\pi}{4}} \ln(\operatorname{tg} x + \operatorname{ctg} x) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\operatorname{tg} x + \operatorname{ctg} x) dx = \frac{\pi}{2} \ln 2.$$

BI ((287))(5), BI ((308))(11)

$$16. \int_0^{\frac{\pi}{4}} \ln(\operatorname{ctg} x - \operatorname{tg} x)^2 dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\operatorname{ctg} x - \operatorname{tg} x)^2 dx = \frac{\pi}{2} \ln 2.$$

BI ((287))(6), BI ((308))(12)

$$17. \int_0^{\frac{\pi}{2}} \ln(a^2 + b^2 \operatorname{tg}^2 x) dx = \frac{1}{2} \int_0^{\pi} \ln(a^2 + b^2 \operatorname{tg}^2 x) dx = \pi \ln(a + b)$$

$$[a > 0, \quad b > 0].$$

GW ((322))(17)

563
4.228

$$1. \int_0^{\frac{\pi}{2}} \ln(\sin t \sin x + \sqrt{1 - \cos^2 t \sin^2 x}) dx = \frac{\pi}{2} \ln 2 - 2L\left(\frac{t}{2}\right) - 2L\left(\frac{\pi - t}{2}\right).$$

LO III 290

$$2. \int_0^u \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) dx = -\left(\frac{\pi}{2} - t - \varphi\right) \ln \cos t + \frac{1}{2}L(u + \varphi) - \frac{1}{2}L(u - \varphi) - L(\varphi)$$

$$\left[\cos \varphi = \frac{\sin u}{\sin t}; \quad 0 \leq u \leq t \leq \frac{\pi}{2} \right].$$

$$3. \int_0^t \ln(\cos x + \sqrt{\cos^2 x - \cos^2 t}) dx = -\left(\frac{\pi}{2} - t\right) \ln \cos t.$$

LO III 285

$$4. \int_0^u \ln \frac{\sin u + \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}}{\sin u - \sin t \cos x \sqrt{\sin^2 u - \sin^2 x}} dx = \pi \ln \left[\operatorname{tg} \frac{t}{2} \sin u + \sqrt{\operatorname{tg}^2 \frac{t}{2} \sin^2 u + 1} \right]$$

$$[t > 0, \quad u > 0].$$

LO III 283

$$5. \int_0^{\frac{\pi}{4}} \sqrt{\ln \operatorname{ctg} x} dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

BI ((297))(9)

$$6. \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\ln \operatorname{ctg} x}} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((304))(24)

$$7. \int_0^{\frac{\pi}{4}} \ln(\sqrt{\operatorname{tg} x} + \sqrt{\operatorname{ctg} x}) dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sqrt{\operatorname{tg} x} + \sqrt{\operatorname{ctg} x}) dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

BI ((287))(7), BI ((308))(22)

$$8. \int_0^{\frac{\pi}{4}} \ln(\sqrt{\operatorname{ctg} x} - \sqrt{\operatorname{tg} x})^2 dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sqrt{\operatorname{ctg} x} - \sqrt{\operatorname{tg} x})^2 dx = \frac{\pi}{4} \ln 2 - \mathbf{G}.$$

BI ((287))(8), BI ((308))(23)

4.229

$$1. \int_0^1 \ln \left(\ln \frac{1}{x} \right) dx = -\mathbf{C}.$$

FI II 807

$$2. \int_0^1 \frac{dx}{\ln \left(\ln \frac{1}{x} \right)} = 0.$$

$$3. \int_0^1 \ln \left(\ln \frac{1}{x} \right) \frac{dx}{\sqrt{\ln \frac{1}{x}}} = -(C + 2 \ln 2) \sqrt{\pi}.$$

BI ((32))(4)

564

$$4. \int_0^1 \ln \left(\ln \frac{1}{x} \right) \left(\ln \frac{1}{x} \right)^{\mu-1} dx = \psi(\mu) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((30))(10)

If the integrand contains $\ln \left(\ln \frac{1}{x} \right)$, it is convenient to make the substitution $\ln \frac{1}{x} = u$, i.e., $x = e^{-u}$.

$$5.7 \int_0^1 \ln(a + \ln x) dx = \ln a - e^{-a} \operatorname{Ei}(a) \quad [a > 0].$$

BI ((30))(5)

$$6. \int_0^1 \ln(a - \ln x) dx = \ln a - e^a \operatorname{Ei}(-a) \quad [a > 0].$$

BI ((30))(6)

$$7. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln \ln \operatorname{tg} x dx = \frac{\pi}{2} \ln \left\{ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \sqrt{2\pi} \right\}.$$

BI ((308))(28)

4.23 Combinations of logarithms and rational functions

4.231

$$1. \int_0^1 \frac{\ln x}{1+x} dx = -\frac{\pi^2}{12}.$$

FI II 483a

$$2. \int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6}.$$

FI II 714

BI ((108))(7)

$$4. \int_0^1 \frac{1+x}{1-x} \ln x \, dx = 1 - \frac{\pi^2}{3}.$$

BI ((108))(9)

$$5.7 \int_0^\infty \frac{\ln x \, dx}{(x+a)^2} = \frac{\ln a}{a} \quad [a > 0].$$

BI ((139))(1)

$$6. \int_0^1 \frac{\ln x}{(1+x)^2} \, dx = -\ln 2.$$

BI ((111))(1)

$$7.7 \int_0^\infty \ln x \frac{dx}{(a^2 + b^2 x^2)^n} = \frac{\Gamma(n - \frac{1}{2})\sqrt{\pi}}{4(n-1)!a^{2n-1}b} \left[2 \ln \frac{a}{2b} - \mathbf{C} - \psi \left(n - \frac{1}{2} \right) \right] \quad [a > 0, \quad b > 0].$$

LI ((139))(3)

$$8. \int_0^\infty \frac{\ln x \, dx}{a^2 + b^2 x^2} = \frac{\pi}{2ab} \ln \frac{a}{b} \quad [ab > 0].$$

BI ((135))(6)

$$9. \int_0^\infty \frac{\ln px}{q^2 + x^2} \, dx = \frac{\pi}{2q} \ln pq \quad [p > 0, \quad q > 0].$$

BI ((135))(4)

$$10. \int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = -\frac{\pi^2}{4ab} \quad [ab > 0].$$

LI ((324))(7b)

565

$$11. \int_0^a \frac{\ln x \, dx}{x^2 + a^2} = \frac{\pi \ln a}{4a} - \frac{\mathbf{G}}{a} \quad [a > 0].$$

$$12. \int_0^1 \frac{\ln x}{1+x^2} dx = -\int_1^\infty \frac{\ln x}{1+x^2} dx = -\mathbf{G}.$$

FI II 482, 614

$$13. \int_0^1 \frac{\ln x dx}{1-x^2} = -\frac{\pi^2}{8}.$$

BI ((108))(11)

$$14. \int_0^1 \frac{x \ln x}{1+x^2} dx = -\frac{\pi^2}{48}.$$

GW ((324))(7b)

$$15. \int_0^1 \frac{x \ln x}{1-x^2} dx = -\frac{\pi^2}{24}.$$

$$16. \int_0^1 \ln x \frac{1-x^{2n+2}}{(1-x^2)^2} dx = -\frac{(n+1)\pi^2}{8} + \sum_{k=1}^n \frac{n-k+1}{(2k-1)^2}.$$

BI ((111))(5)

$$17. \int_0^1 \ln x \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = -\frac{(n+1)\pi^2}{12} - \sum_{k=1}^n (-1)^k \frac{n-k+1}{k^2}$$

BI ((111))(2)

$$18. \int_0^1 \ln x \frac{1-x^{n+1}}{(1-x)^2} dx = -\frac{(n+1)\pi^2}{6} + \sum_{k=1}^n \frac{n-k+1}{k^2}.$$

BI ((111))(3)

4.232

$$1. \int_u^v \frac{\ln x dx}{(x+u)(x+v)} = \frac{\ln uv}{2(v-u)} \ln \frac{(u+v)^2}{4uv}.$$

$$2. \int_0^{\infty} \frac{\ln x \, dx}{(x + \beta)(x + \gamma)} = \frac{(\ln \beta)^2 - (\ln \gamma)^2}{2(\beta - \gamma)} \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi].$$

ET II 218(24)

$$3. \int_0^{\infty} \frac{\ln x}{x + a} \frac{dx}{x - 1} = \frac{\pi^2 + (\ln a)^2}{2(a + 1)} \quad [a > 0].$$

BI ((140))(10)

4.233

$$1.^3 \int_0^1 \frac{\ln x \, dx}{1 + x + x^2} = \frac{2}{9} \left[\frac{2\pi^2}{3} - \psi' \left(\frac{1}{3} \right) \right] = -0.781\,302\,4129\dots$$

LI ((113))(1)

$$2.^3 \int_0^1 \frac{\ln x \, dx}{1 - x + x^2} = \frac{1}{3} \left[\frac{2\pi^2}{3} - \psi' \left(\frac{1}{3} \right) \right] = -1.171\,953\,619\,34\dots$$

LI ((113))(2)

$$3.^7 \int_0^1 \frac{x \ln x \, dx}{1 + x + x^2} = -\frac{1}{9} \left[\frac{7\pi^2}{6} - \psi' \left(\frac{1}{3} \right) \right] = -0.157\,660\,149\,17\dots$$

LI ((113))(2)

$$4.^3 \int_0^1 \frac{x \ln x \, dx}{1 - x + x^2} = \frac{1}{6} \left[\frac{5\pi^2}{6} - \psi' \left(\frac{1}{3} \right) \right] = -0.311\,821\,131\,9\dots$$

LI ((113))(4)

$$5. \int_0^{\infty} \frac{\ln x \, dx}{x^2 + 2xa \cos t + a^2} = \frac{t \ln a}{a \sin t} \quad [a > 0, \quad 0 < t < \pi].$$

GW ((324))(13c)

566

4.234

$$1. \int_1^{\infty} \frac{\ln x \, dx}{(1 + x^2)^2} = \ln 2.$$

$$2. \int_0^1 \frac{x \ln x dx}{(1+x^2)^2} = -\frac{1}{4} \ln 2.$$

BI ((111))(4)

$$3. \int_0^\infty \frac{1+x^2}{(1-x^2)^2} \ln x dx = 0.$$

BI ((142))(2)a

$$4. \int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x dx = -\frac{\pi}{2}.$$

BI ((142))(1)a

$$5. \int_0^1 \frac{x^2 \ln x dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})}.$$

BI ((112))(21)

$$6. \int_0^\infty \frac{\ln x dx}{(a^2+b^2x^2)(1+x^2)} = \frac{b\pi}{2a(b^2-a^2)} \ln \frac{a}{b} \quad [ab > 0].$$

BI ((317))(16)a

$$7. \int_0^\infty \frac{\ln x}{x^2+a^2} \cdot \frac{dx}{1+b^2x^2} = \frac{\pi}{2(1-a^2b^2)} \left(\frac{1}{a} \ln a + b \ln b \right) \quad [a > 0, \quad b > 0].$$

LI ((140))(12)

$$8. \int_0^\infty \frac{x^2 \ln x dx}{(a^2+b^2x^2)(1+x^2)} = \frac{a\pi}{2b(b^2-a^2)} \ln \frac{b}{a} \quad [ab > 0].$$

LI ((140))(12), BI ((317))(15)a

4.235

$$1. \int_0^\infty \ln x \frac{(1-x)x^{n-2}}{1-x^{2n}} dx = -\frac{\pi^2}{4n^2} \operatorname{tg}^2 \frac{\pi}{2n} \quad [n > 1].$$

BI ((135))(10)

$$2. \int_0^\infty \ln x \frac{(1-x^2)x^{m-1}}{1-x^{2n}} dx = -\frac{\pi^2 \sin \frac{m+1}{n} \pi \sin \frac{\pi}{n}}{4n \sin \frac{m\pi}{2n} \sin \left(\frac{m+2}{2n} \pi \right)}$$

LI ((135))(12)

$$3. \int_0^{\infty} \ln x \frac{(1-x^2)x^{n-2}}{1-x^{2n}} dx = -\frac{\pi^2}{4n^2} \operatorname{tg}^2 \frac{\pi}{n} \quad [n > 2].$$

BI ((135))(11)

$$4. \int_0^1 \ln x \frac{x^{m-1} + x^{n-m-1}}{1-x^n} dx = -\frac{\pi^2}{n^2 \sin^2 \left(\frac{m}{n} \pi \right)} \quad [n > m].$$

BI ((108))(15)

4.236

$$1. \int_0^1 \left\{ \frac{1 + (p-1) \ln x}{1-x} + \frac{x \ln x}{(1-x)^2} \right\} x^{p-1} dx = -1 + \psi'(p) \quad [p > 0].$$

BI ((111))(6)A, GW ((326))(13)

$$2. \int_0^1 \left[\frac{1}{1-x} + \frac{x \ln x}{(1-x)^2} \right] dx = \frac{\pi^2}{6} - 1.$$

GW ((326))(13a)

567

4.24 Combinations of logarithms and algebraic functions

4.241

$$1. \int_0^1 \frac{x^{2n} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right).$$

BI ((118))(5)a

$$2. \int_0^1 \frac{x^{2n+1} \ln x}{\sqrt{1-x^2}} dx = \frac{(2n)!!}{(2n+1)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right).$$

BI ((118))(5)a

$$3. \int_0^1 x^{2n} \sqrt{1-x^2} \ln x dx = \frac{(2n-1)!!}{(2n+2)!!} \cdot \frac{\pi}{2} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2 \right).$$

$$4. \int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x \, dx = \frac{(2n)!!}{(2n+3)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} - \frac{1}{2n+3} \right).$$

BI ((117))(5), GW ((324))(53b)

$$5. \int_0^1 \ln x \cdot \sqrt{(1-x^2)^{2n-1}} \, dx = -\frac{(2n-1)!!}{4 \cdot (2n)!!} \pi [\psi(n+1) + \mathbf{C} + \ln 4].$$

BI ((117))(3)

$$6. \int_0^{\sqrt{\frac{1}{2}}} \frac{\ln x \, dx}{\sqrt{1-x^2}} = -\frac{\pi}{4} \ln 2 - \frac{1}{2} \mathbf{G}.$$

BI ((145))(1)

$$7. \int_0^1 \frac{\ln x \, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2} \ln 2.$$

FI II 614, 643

$$8. \int_1^{\infty} \frac{\ln x \, dx}{x^2 \sqrt{x^2-1}} = 1 - \ln 2.$$

BI ((144))(17)

$$9. \int_0^1 \sqrt{1-x^2} \ln x \, dx = -\frac{\pi}{8} - \frac{\pi}{4} \ln 2.$$

BI ((117))(1), GW ((324))(53c)

$$10. \int_0^1 x \sqrt{1-x^2} \ln x \, dx = \frac{1}{3} \ln 2 - \frac{4}{9}.$$

BI ((117))(2)

$$11. \int_0^1 \frac{\ln x \, dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8} \left[\Gamma\left(\frac{1}{4}\right) \right]^2.$$

4.242

$$1. \int_0^{\infty} \frac{\ln x \, dx}{\sqrt{(a^2 + x^2)(x^2 + b^2)}} = \frac{1}{2a} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \ln ab \quad [a > b > 0].$$

BY (800.04)

$$2. \int_0^b \frac{\ln x \, dx}{\sqrt{(a^2 + x^2)(b^2 - x^2)}} = \frac{1}{2\sqrt{a^2 + b^2}} \left[\mathbf{K} \left(\frac{b}{\sqrt{a^2 + b^2}} \right) \ln ab - \frac{\pi}{2} \mathbf{K} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \right] \\ [a > 0, \quad b > 0].$$

BY (800.02)

568

$$3. \int_b^{\infty} \frac{\ln x \, dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{1}{2\sqrt{a^2 + b^2}} \left[\mathbf{K} \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \ln ab + \frac{\pi}{2} \mathbf{K} \left(\frac{b}{\sqrt{a^2 + b^2}} \right) \right] \\ [a > 0, \quad b > 0].$$

BY (800.06)

$$4. \int_0^b \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(b^2 - x^2)}} = \frac{1}{2a} \left[\mathbf{K} \left(\frac{b}{a} \right) \ln ab - \frac{\pi}{2} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \right] \quad [a > b > 0].$$

BY (800.01)

$$5. \int_b^a \frac{\ln x \, dx}{\sqrt{(a^2 - x^2)(x^2 - b^2)}} = \frac{1}{2a} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \ln ab.$$

BY (800.03)

$$6. \int_a^{\infty} \frac{\ln x \, dx}{\sqrt{(x^2 - a^2)(x^2 - b^2)}} = \frac{1}{2a} \left[\mathbf{K} \left(\frac{b}{a} \right) \ln ab + \frac{\pi}{2} \mathbf{K} \left(\frac{\sqrt{a^2 - b^2}}{a} \right) \right] \quad [a > b > 0].$$

BY (800.05)

4.243

$$\int_0^1 \frac{x \ln x}{\sqrt{1 - x^4}} \, dx = -\frac{\pi}{8} \ln 2.$$

4.244

$$1. \int_0^1 \frac{\ln x \, dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8} \left[\Gamma\left(\frac{1}{3}\right) \right]^3.$$

GW ((324))(54b)

$$2. \int_0^1 \frac{\ln x \, dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}} \left(\ln 3 + \frac{\pi}{3\sqrt{3}} \right).$$

BI ((118))(7)

$$3. \int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}} \left(\frac{\pi}{3\sqrt{3}} - \ln 3 \right).$$

BI ((118))(8)

4.245

$$1. \int_0^1 \frac{x^{4n+1} \ln x}{\sqrt{1-x^4}} \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{8} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$$

GW ((324))(56a)

$$2. \int_0^1 \frac{x^{4n+3} \ln x}{\sqrt{1-x^4}} \, dx = \frac{(2n)!!}{4 \cdot (2n+1)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right).$$

GW ((324))(56c)

4.246

$$\int_0^1 (1-x^2)^{n-\frac{1}{2}} \ln x \, dx = -\frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{4} \left[2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right].$$

GW ((324))(55)

4.247

$$1.^6 \int_0^1 \frac{\ln x}{\sqrt[2n]{1-x^{2n}}} \, dx = -\frac{\pi B\left(\frac{1}{2n}, \frac{1}{2n}\right)}{8n^2 \sin \frac{\pi}{2n}} \quad [n > 1].$$

$$2.^6 \int_0^1 \frac{\ln x \, dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi B\left(\frac{1}{2n}, \frac{1}{2n}\right)}{8 \sin \frac{\pi}{2n}}.$$

GW ((324))(54)

4.25 Combinations of logarithms and powers

4.251

$$1. \int_0^\infty \frac{x^{\mu-1} \ln x}{\beta+x} dx = \frac{\pi \beta^{\mu-1}}{\sin \mu \pi} (\ln \beta - \pi \operatorname{ctg} \mu \pi) \quad [|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 1].$$

BI ((135))(1)

$$2. \int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} dx = \pi a^{\mu-1} \left(\operatorname{ctg} \mu \pi \ln a - \frac{\pi}{\sin^2 \mu \pi} \right) \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

ET I 314(5)

$$3.^* \int_0^1 \frac{x^{\mu-1} \ln x}{x+1} dx = \beta'(\mu) \quad [\operatorname{Re} \mu > 0].$$

GW ((324))(6), ET I 314(3)

$$4. \int_0^1 \frac{x^{\mu-1} \ln x}{1-x} dx = -\psi'(\mu) = -\zeta(2, \mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((108))(8)

$$5. \int_0^1 \ln x \frac{x^{2n} dx}{1+x} = -\frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k^2}.$$

BI ((108))(4)

$$6. \int_0^1 \ln x \frac{x^{2n-1} dx}{1+x} = \frac{\pi^2}{12} + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k^2}.$$

BI ((108))(5)

4.252

$$2. \int_0^{\infty} \frac{x^{\mu-1} \ln x \, dx}{(x+\beta)(x-1)} = \frac{\pi}{(\beta+1) \sin^2 \mu\pi} [\pi - \beta^{\mu-1} (\sin \mu\pi \ln \beta - \pi \cos \mu\pi)]$$

$$[|\arg \beta| < \pi, \quad 0 < \operatorname{Re} \mu < 2, \quad \mu \neq 1].$$

BI ((140))(11)

$$3. \int_0^{\infty} \frac{x^{p-1} \ln x}{1-x^2} \, dx = -\frac{\pi^2}{4} \operatorname{cosec}^2 \frac{p\pi}{2} \quad [0 < p < 2] \quad (\text{see also 4.254 2}).$$

4.254

$$4.6 \int_0^{\infty} \frac{x^{\mu-1} \ln x}{(x+a)^2} \, dx = \frac{(1-\mu)a^{\mu-2}\pi}{\sin \mu\pi} \left(\ln a - \pi \operatorname{ctg} \mu\pi + \frac{1}{\mu-1} \right)$$

$$[|\arg a| < \pi \quad 0 < \operatorname{Re} \mu < 2 \quad (\mu \neq 1)].$$

GW ((324))(13b)

570

4.253

$$1. \int_0^1 x^{\mu-1} (1-x^r)^{\nu-1} \ln x \, dx = \frac{1}{r^2} B\left(\frac{\mu}{r}, \nu\right) \left[\psi\left(\frac{\mu}{r}\right) - \psi\left(\frac{\mu}{r} + \nu\right) \right]$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad r > 0].$$

GW ((324))(3b)A, BI ((107))(5)a

$$2. \int_0^1 \frac{x^{p-1}}{(1-x)^{p+1}} \ln x \, dx = -\frac{\pi}{p} \operatorname{cosec} p\pi \quad [0 < p < 1].$$

Bi ((319))(10)a

$$3. \int_u^{\infty} \frac{(x-u)^{\mu-1} \ln x \, dx}{x^{\lambda}} = u^{\mu-\lambda} B(\lambda-\mu, \mu) [\ln u + \psi(\lambda) - \psi(\lambda-\mu)]$$

$$[0 < \operatorname{Re} \mu < \operatorname{Re} \lambda].$$

ET II 203(18)

$$4.8 \int_0^1 \ln x \left(\frac{x}{a^2+x^2} \right)^p \frac{dx}{x} = \frac{\ln a}{2a^p} B\left(\frac{p}{2}, \frac{p}{2}\right) \quad [a > 0, \quad p > 0].$$

$$5. \int_1^{\infty} (x-1)^{p-1} \ln x \, dx = \frac{\pi}{p} \operatorname{cosec} \pi p \quad [-1 < p < 0].$$

BI ((289))(12)a

$$6.7 \int_0^{\infty} \ln x \frac{dx}{(a+x)^{\mu+1}} = \frac{1}{\mu a^{\mu}} (\ln a - \mathbf{C} - \psi(\mu))$$

$$[\operatorname{Re} \mu > 0, \quad a \neq 0, \quad |\arg a| < \pi].$$

NT 68(7)

$$7.7 \int_0^{\infty} \ln x \frac{dx}{(a+x)^{n+\frac{1}{2}}} = \frac{2}{(2n-1)a^{n-\frac{1}{2}}} \left(\ln a + 2 \ln 2 - 2 \sum_{k=1}^{n-1} \frac{1}{2k-1} \right)$$

$$[|\arg a| < \pi, \quad n = 1, 2, \dots].$$

BI ((142))(5)

4.254

$$1. \int_0^1 \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{1}{q^2} \psi' \left(\frac{p}{q} \right) \quad [p > 0, \quad q > 0].$$

GW ((324))(5)

$$2. \int_0^{\infty} \frac{x^{p-1} \ln x}{1-x^q} \, dx = -\frac{\pi^2}{q^2 \sin^2 \frac{p\pi}{q}} \quad [0 < p < q].$$

BI ((135))(8)

$$3. \int_0^{\infty} \frac{\ln x}{x^q - 1} \frac{dx}{x^p} = \frac{\pi^2}{q^2 \sin^2 \frac{p\pi}{q}} \quad [p < 1, \quad p+q > 1].$$

BI ((140))(2)

$$4.3 \int_0^1 \frac{x^{p-1} \ln x}{1+x^q} \, dx = \frac{1}{q^2} \beta' \left(\frac{p}{q} \right) \quad [p > 0, \quad q > 0].$$

GW ((324))(7)

$$5. \int_0^{\infty} \frac{x^{p-1} \ln x}{1+x^q} \, dx = -\frac{\pi^2}{q^2} \frac{\cos \frac{p\pi}{q}}{\sin^2 \frac{p\pi}{q}} \quad [0 < p < q].$$

BI ((135))(7)

$$6. \int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} dx = -\frac{\pi^2}{8q^2} \quad [q > 0].$$

BI ((108))(12)

571

4.255

$$1. \int_0^1 \ln x \frac{(1-x^2)x^{p-2}}{1+x^{2p}} dx = -\left(\frac{\pi}{2p}\right)^2 \frac{\sin \frac{\pi}{2p}}{\cos^2 \frac{\pi}{2p}} \quad [p > 1].$$

BI ((108))(13)

$$2. \int_0^1 \ln x \frac{(1+x^2)x^{p-2}}{1-x^{2p}} dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2 \frac{\pi}{2p} \quad [p > 1].$$

BI ((108))(14)

$$3. \int_0^\infty \ln x \frac{1-x^p}{1-x^2} dx = \frac{\pi^2}{4} \operatorname{tg}^2 \frac{p\pi}{2} \quad [p < 1].$$

BI ((140))(3)

4.256

$$\int_0^1 \ln \frac{1}{x} \frac{x^{\mu-1} dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2} \operatorname{B} \left(\frac{\mu}{n}, \frac{m}{n} \right) \left[\psi \left(\frac{\mu+m}{n} \right) - \psi \left(\frac{\mu}{n} \right) \right] \quad [\operatorname{Re} \mu > 0].$$

LI ((118))(12)

4.257

$$1. \int_0^\infty \frac{x^\nu \ln \frac{x}{\beta} dx}{(x+\beta)(x+\gamma)} = \frac{\pi \left[\gamma^\nu \ln \frac{\gamma}{\beta} + \pi(\beta^\nu - \gamma^\nu) \operatorname{ctg} \nu\pi \right]}{\sin \nu\pi(\gamma - \beta)} \quad [|\arg \beta| < \pi, \quad |\arg \gamma| < \pi, \quad |\operatorname{Re} \nu| < 1].$$

ET II 219(30)

$$2. \int_0^\infty \ln \frac{x}{q} \left(\frac{x^p}{q^{2p} + x^{2p}} \right) \frac{dx}{x} = 0 \quad [q > 0].$$

$$3. \int_0^{\infty} \ln \frac{x}{q} \left(\frac{x^p}{q^{2p} + x^{2p}} \right)^r \frac{dx}{q^2 + x^2} = 0 \quad [q > 0].$$

BI ((140))(4)a

$$4. \int_0^{\infty} \ln x \ln \frac{x}{a} \frac{dx}{(x-1)(x-a)} = \frac{[4\pi^2 + (\ln a)^2] \ln a}{6(a-1)} \quad [a > 0], \quad [a = 1 \text{ see 4.261 5.}].$$

4.261
BI ((141))(5)

$$5. \int_0^{\infty} \ln x \ln \frac{x}{a} \frac{x^p dx}{(x-1)(x-a)} = \frac{\pi^2 [(a^p + 1) \ln a - 2\pi(a^p - 1) \operatorname{ctg} p\pi]}{(a-1) \sin^2 p\pi} \quad [p^2 < 1, \quad a > 0].$$

BI ((141))(6)

4.26-4.27 Combinations involving powers of the logarithm and other powers

4.261

$$1. \int_0^1 (\ln x)^2 \frac{dx}{1 + 2x \cos t + x^2} = \frac{t(\pi^2 - t^2)}{6 \sin t} \quad [0 < t < \pi].$$

BI ((113))(7)

$$2. \int_0^1 \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{1}{2} \int_0^{\infty} \frac{(\ln x)^2 dx}{x^2 - x + 1} = \frac{10\pi^3}{81\sqrt{3}}.$$

GW ((324))(16c)

572

$$3. \int_0^1 \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{1}{2} \int_0^{\infty} \frac{(\ln x)^2 dx}{x^2 + x + 1} = \frac{8\pi^3}{81\sqrt{3}}.$$

GW ((324))(16b)

$$4. \int_0^{\infty} (\ln x)^2 \frac{dx}{(x-1)(x+a)} = \frac{[\pi^2 + (\ln a)^2] \ln a}{3(1+a)} \quad [a > 0].$$

$$5. \int_0^{\infty} (\ln x)^2 \frac{dx}{(1-x)^2} = \frac{2}{3}\pi^2.$$

BI ((139))(4)

$$6. \int_0^1 (\ln x)^2 \frac{dx}{1+x^2} = \frac{\pi^3}{16}.$$

BI ((109))(3)

$$7. \int_0^1 (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{1}{2} \int_0^{\infty} (\ln x)^2 \frac{1+x^2}{1+x^4} dx = \frac{3\sqrt{2}}{64}\pi^3.$$

BI ((109))(5), BI ((135))(13)

$$8.^3 \int_0^1 (\ln x)^2 \frac{1-x}{1-x^6} dx = \frac{1}{36} \left(\frac{4\sqrt{3}\pi^3}{27} - \psi'' \left(\frac{1}{3} \right) \right).$$

$$9. \int_0^1 (\ln x)^2 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \left[(\ln 2)^2 + \frac{\pi^2}{12} \right].$$

BI ((118))(13)

$$10. \int_0^{\infty} (\ln x)^2 \frac{x^{\mu-1}}{1+x} dx = \frac{\pi^3(2 - \sin^2 \mu\pi)}{\sin^3 \mu\pi} \quad [0 < \operatorname{Re} \mu < 1].$$

ET I 315(10)

$$11.^7 \int_0^1 (\ln x)^{2x^n} \frac{dx}{1+x} = (-1)^n \left(\frac{3}{2}\zeta(3) + 2 \sum_{k=1}^n \frac{(-1)^k}{k^3} \right) \quad [n = 0, 1, \dots]$$

BI ((109))(1)

$$12.^7 \int_0^1 (\ln x)^2 \frac{x^n dx}{1-x} = 2 \left(\zeta(3) - \sum_{k=1}^n \frac{1}{k^3} \right) \quad [n = 0, 1, \dots]$$

BI ((109))(2)

$$13.^7 \int_0^1 (\ln x)^2 \frac{x^{2n} dx}{1-x^2} = \frac{7}{4}\zeta(3) - 2 \sum_{k=1}^n \frac{1}{(2k-1)^3} \quad [n = 0, 1, \dots].$$

$$14. \int_0^{\infty} (\ln x)^2 \frac{x^{p-1} dx}{x^2 + 2x \cos t + 1} = \frac{\pi \sin(1-p)t}{\sin t \sin p\pi} \{ \pi^2 - t^2 + 2\pi \operatorname{ctg} p\pi [\pi \operatorname{ctg} p\pi + t \operatorname{ctg}(1-p)t] \}$$

$$[0 < t < \pi, \quad 0 < p < 2 \quad (p \neq 1)].$$

GW ((324))(17)

$$15. \int_0^1 (\ln x)^2 \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{2 \cdot (2n)!!} \pi \left\{ \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}.$$

GW ((324))(60a)

$$16. \int_0^1 (\ln x)^2 \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \left\{ -\frac{\pi^2}{12} - \sum_{k=1}^{2n+1} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}.$$

GW ((324))(60b)

573

$$17.7 \int_0^1 (\ln x)^2 x^{\mu-1} (1-x)^{\nu-1} dx = B(\mu, \nu) \{ [\psi(\mu) - \psi(\nu + \mu)]^2 + \psi^1(\mu) - \psi^1(\mu + \nu) \}$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

ET I 315(11)

$$18. \int_0^1 (\ln x)^2 \frac{1-x^{n+1}}{(1-x)^2} dx = 2(n+1)\zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{k^3}.$$

LI ((111))(8)

$$19. \int_0^1 (\ln x)^2 \frac{1+(-1)^n x^{n+1}}{(1+x)^2} dx = \frac{3}{2}(n+1)\zeta(3) - 2 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^3}.$$

LI ((111))(7)

$$20.7 \int_0^1 (\ln x)^2 \frac{1-x^{2n+2}}{(1-x^2)^2} dx = \frac{7}{4}(n+1)\zeta(3) - 2 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^3}.$$

$$21. \int_0^1 (\ln x)^2 x^{p-1} (1-x^r)^{q-1} dx = \frac{1}{r^3} B\left(\frac{p}{r}, q\right) \left\{ \psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r} + q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r} + q\right) \right]^2 \right\}$$

$$[p > 0, \quad q > 0, \quad r > 0].$$

GW ((324))(8a)

4.262

$$1. \int_0^1 (\ln x)^3 \frac{dx}{1+x} = -\frac{7}{120} \pi^4.$$

BI ((109))(9)

$$2. \int_0^1 (\ln x)^3 \frac{dx}{1-x} = -\frac{\pi^4}{15}.$$

BI ((109))(11)

$$3. \int_0^\infty (\ln x)^3 \frac{dx}{(x+a)(x-1)} = \frac{[\pi^2 + (\ln a)^2]^2}{4(a+1)} \quad [a > 0].$$

BI ((141))(2)

$$4. \int_0^1 (\ln x)^3 \frac{x^n dx}{1+x} = (-1)^{n+1} \left[\frac{7\pi^4}{120} - 6 \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \right] \quad [n = 1, 2, \dots].$$

BI ((109))(10)

$$5. \int_0^1 (\ln x)^3 \frac{x^n dx}{1-x} = -\frac{\pi^4}{15} + 6 \sum_{k=0}^{n-1} \frac{1}{(k+1)^4} \quad [n = 1, 2, \dots].$$

BI ((109))(12)

$$6. \int_0^1 (\ln x)^3 \frac{x^{2n} dx}{1-x^2} = -\frac{\pi^4}{16} + 6 \sum_{k=0}^{n-1} \frac{1}{(2k+1)^4} \quad [n = 1, 2, \dots].$$

BI ((109))(14)

$$7. \int_0^1 (\ln x)^3 \frac{1-x^{n+1}}{(1-x)^2} dx = -\frac{(n+1)\pi^4}{15} + 6 \sum_{k=1}^n \frac{n-k+1}{k^4}.$$

$$8. \int_0^1 (\ln x)^3 \frac{1 + (-1)^n x^{n+1}}{(1+x)^2} dx = -\frac{7(n+1)\pi^4}{120} + 6 \sum_{k=1}^n (-1)^{k-1} \frac{n-k+1}{k^4}.$$

BI ((111))(10)

$$9. \int_0^1 (\ln x)^3 \frac{1 - x^{2n+2}}{(1-x^2)^2} dx = -\frac{(n+1)\pi^4}{16} + 6 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^4}.$$

BI ((111))(12)

574

4.263

$$1.^8 \int_0^\infty (\ln x)^4 \frac{dx}{(x-1)(x+a)} = \frac{\ln a [\pi^2 + (\ln a)^2] [7\pi^2 + 3(\ln a)^2]}{15(1+a)} \quad [a > 0].$$

BI ((141))(3)

$$2. \int_0^1 (\ln x)^4 \frac{dx}{1+x^2} = \frac{5\pi^5}{64}.$$

BI ((109))(17)

$$3. \int_0^1 (\ln x)^4 \frac{dx}{1+2x \cos t + x^2} = \frac{t(\pi^2 - t^2)(7\pi^2 - 3t^2)}{30 \sin t} \quad [|t| < \pi].$$

BI ((113))(8)

4.264

$$1. \int_0^1 (\ln x)^5 \frac{dx}{1+x} = -\frac{31\pi^6}{252}.$$

BI ((109))(20)

$$2. \int_0^1 (\ln x)^5 \frac{dx}{1-x} = -\frac{8\pi^6}{63}.$$

BI ((109))(21)

$$3. \int_0^\infty (\ln x)^5 \frac{dx}{(x-1)(x+a)} = \frac{[\pi^2 + (\ln a)^2]^2 [3\pi^2 + (\ln a)^2]}{6(1+a)} \quad [a > 0].$$

$$\int_0^1 (\ln x)^6 \frac{dx}{1+x^2} = \frac{61\pi^7}{256}.$$

BI ((109))(25)

4.266

$$1. \int_0^1 (\ln x)^7 \frac{dx}{1+x} = -\frac{127\pi^8}{240}.$$

BI ((109))(28)

$$2. \int_0^1 (\ln x)^7 \frac{dx}{1-x} = -\frac{8\pi^8}{15}.$$

BI ((109))(29)

4.267

$$1. \int_0^1 \frac{1-x}{1+x} \frac{dx}{\ln x} = \ln \frac{2}{\pi}.$$

BI ((127))(3)

$$2. \int_0^1 \frac{(1-x)^2}{1+x^2} \frac{dx}{\ln x} = \ln \frac{\pi}{4}.$$

BI ((128))(2)

$$3.^8 \int_0^1 \frac{(1-x)^2}{1+2x \cos \frac{mx}{n} + x^2} \cdot \frac{dx}{\ln x} =$$

$$= \frac{1}{\sin \frac{m\pi}{n}} \sum_{k=1}^{n-1} (-1)^k \sin \frac{km\pi}{n} \ln \frac{\left\{ \Gamma \left(\frac{n+k+1}{2n} \right) \right\}^2 \Gamma \left(\frac{k+2}{2n} \right) \Gamma \left(\frac{k}{2n} \right)}{\left\{ \Gamma \left(\frac{k+1}{2n} \right) \right\}^2 \Gamma \left(\frac{n+k}{2n} \right) \Gamma \left(\frac{n+k+2}{2n} \right)}$$

[m + n is odd];

$$= \frac{1}{\sin \frac{m\pi}{n}} \sum_{k=1}^{[\frac{1}{2}(n-1)]} (-1)^k \sin \frac{km\pi}{n} \ln \frac{\left\{ \Gamma \left(\frac{n-k+1}{n} \right) \right\}^2 \Gamma \left(\frac{k+2}{n} \right) \Gamma \left(\frac{k}{n} \right)}{\left\{ \Gamma \left(\frac{k+1}{n} \right) \right\}^2 \Gamma \left(\frac{n-k}{n} \right) \Gamma \left(\frac{n-k+2}{n} \right)}$$

[m + n is even]; [m < n].

$$5. \int_0^1 \frac{1-x}{1+x} \cdot \frac{x^2}{1+x^2} \cdot \frac{dx}{\ln x} = \ln \frac{2\sqrt{2}}{\pi}.$$

$$6. \int_0^1 (4-x)^p \frac{dx}{\ln x} = \sum_{k=1}^{\infty} (-1)^k \frac{p}{k} \ln(1+k) \quad [p \geq 1].$$

BI ((123))(2)

$$7. \int_0^1 \left(\frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} = \ln \Gamma(p+1).$$

GW ((326))(10)

$$8. \int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \ln \frac{p}{q} \quad [p > 0, \quad q > 0].$$

FI II 647

$$9. \int_0^1 \frac{x^{p-1} - x^{q-1}}{\ln x} \cdot \frac{dx}{1+x} = \ln \frac{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q+1}{2}\right)} \quad [p > 0, \quad q > 0].$$

FI II 186

$$10. \int_0^1 \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{p-1} - x^{-p}}{(1+x) \ln x} dx = \ln \left(\operatorname{tg} \frac{p\pi}{2} \right) \quad [0 < p < 1].$$

FI II 816

$$11. \int_0^1 (x^p - x^q) x^{r-1} \frac{dx}{\ln x} = \ln \frac{p+r}{r+q} \quad [r > 0, \quad p > 0, \quad q > 0].$$

LI ((123))(5)

$$12. \int_0^1 \frac{x^p - x^q}{(1-ax)^n} \frac{dx}{x \ln x} = \ln \frac{p}{q} + \sum_{k=1}^{\infty} \binom{n+k-1}{k} a^k \ln \frac{p+k}{q+k} \quad [p > 0, \quad q > 0, \quad a^2 < 1].$$

BI ((130))(15)

$$13. \int_0^1 (x^p - 1)(x^q - 1) \frac{dx}{\ln x} = \ln \frac{p+q+1}{(p+1)(q+1)} \quad [p > -1, \quad q > -1, \quad p+q > -1]$$

$$14. \int_0^1 \frac{x^p - x^q}{1+x} \cdot \frac{1+x^{2n+1}}{x \ln x} dx = \ln \frac{\Gamma\left(\frac{p}{2} + n + 1\right) \Gamma\left(\frac{q+1}{2} + n\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{q}{2} + n + 1\right) \Gamma\left(\frac{p+1}{2} + n\right) \Gamma\left(\frac{q+1}{2}\right) \Gamma\left(\frac{p}{2}\right)} [p > 0, q > 0].$$

BI ((127))(7)

$$15. \int_0^1 \frac{x^p - x^q}{1-x} \cdot \frac{1-x^r}{\ln x} dx = \ln \frac{\Gamma(q+1)\Gamma(p+r+1)}{\Gamma(p+1)\Gamma(q+r+1)} [p > -1, q > -1, p+r > -1, q+r > -1].$$

GW ((324))(23)

576

$$16. \int_0^1 \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \frac{\Gamma\left(\frac{p+r}{2r}\right) \Gamma\left(\frac{q}{2r}\right)}{\Gamma\left(\frac{q+r}{2r}\right) \Gamma\left(\frac{p}{2r}\right)} [p > 0, q > 0, r > 0].$$

GW ((324))(21)

$$17. \int_0^1 \frac{1-x^{2p-2q}}{1+x^{2p}} \frac{x^{q-1} dx}{\ln x} = \ln \operatorname{tg} \frac{q\pi}{4p} [0 < q < p] \quad (\text{see also 3.524 27}).$$

3.524
BI ((128))(6)

$$18. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1+x^r) \ln x} dx = \ln \left(\operatorname{tg} \frac{p\pi}{2r} \operatorname{ctg} \frac{q\pi}{2r} \right) [0 < p < r, 0 < q < r].$$

GW ((324))(22), BI ((143))(2)

$$19. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{(1-x^r) \ln x} dx = \ln \left(\frac{\sin \frac{p\pi}{r}}{\sin \frac{q\pi}{r}} \right) [0 < p < r, 0 < q < r].$$

BI ((143))(4)

$$20. \int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x^{2n}} \cdot \frac{1-x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+2}{2n}\right) \Gamma\left(\frac{q}{2n}\right)}{\Gamma\left(\frac{q+2}{2n}\right) \Gamma\left(\frac{p}{2n}\right)} [p > 0, q > 0].$$

$$21. \int_0^1 \frac{x^{p-1} - x^{q-1}}{1 + x^{2(2n+1)}} \frac{1 + x^2}{\ln x} dx = \ln \frac{\Gamma\left(\frac{p+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{q+2}{4(2n+1)}\right) \Gamma\left(\frac{p+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{q}{4(2n+1)}\right)}{\Gamma\left(\frac{q+4n+4}{4(2n+1)}\right) \Gamma\left(\frac{p+2}{4(2n+1)}\right) \Gamma\left(\frac{q+4n+2}{4(2n+1)}\right) \Gamma\left(\frac{p}{4(2n+1)}\right)}$$

[$p > 0, \quad q > 0$].

BI ((128))(7)

$$22. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{1 + x^{2(2n+1)}} \frac{1 + x^2}{\ln x} dx = \ln \left\{ \operatorname{tg} \frac{p\pi}{4(2n+1)} \cdot \operatorname{tg} \frac{(p+2)\pi}{4(2n+1)} \cdot \operatorname{ctg} \frac{q\pi}{4(2n+1)} \cdot \operatorname{ctg} \frac{(q+2)\pi}{4(2n+1)} \right\}$$

[$0 < p < 4n, \quad 0 < q < 4n$].

BI ((143))(5)

$$23. \int_0^\infty \frac{x^{p-1} - x^{q-1}}{1 - x^{2n}} \frac{1 - x^2}{\ln x} dx = \ln \frac{\sin \frac{p\pi}{2n} \cdot \sin \frac{(q+2)\pi}{2n}}{\sin \frac{q\pi}{2n} \cdot \sin \frac{(p+2)\pi}{2n}} \quad [0 < p < 2n, \quad 0 < q < 2n].$$

BI ((143))(6)

$$24. \int_0^1 (1-x^p)(1-x^q) \frac{x^{r-1} dx}{\ln x} = \ln \frac{(p+q+r)r}{(p+r)(q+r)} \quad [p > 0, \quad q > 0, \quad r > 0].$$

BI ((123))(8)

$$25. \int_0^1 (1-x^p)(1-x^q) \frac{x^{r-1} dx}{(1-x) \ln x} = \ln \frac{\Gamma(p+r)\Gamma(q+r)}{\Gamma(p+q+r)\Gamma(r)}$$

[$r > 0, \quad r+p > 0, \quad r+q > 0, \quad r+p+q > 0$].

FI II 815a

$$26. \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{\ln x} = \ln \frac{(p+q+1)(q+r+1)(r+p+1)}{(p+q+r+1)(p+1)(q+1)(r+1)}$$

[$p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > -1$].

GW ((324))(19c)

$$28. \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{x^{s-1} dx}{\ln x} = \ln \frac{(p+q+s)(p+r+s)(q+r+s)s}{(p+s)(q+s)(r+s)(p+q+r+s)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0].$

BI ((123))(10)

$$29. \int_0^1 (1-x^p)(1-x^q) \frac{x^{s-1} dx}{(1-x^r) \ln x} = \ln \frac{\Gamma\left(\frac{p+s}{r}\right) \Gamma\left(\frac{q+s}{r}\right)}{\Gamma\left(\frac{s}{r}\right) \Gamma\left(\frac{p+q+s}{r}\right)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0].$

GW ((324))(23a)

$$30. \int_0^\infty (1-x^p)(1-x^q) \frac{x^{s-1} dx}{(1-x^{p+q+2s}) \ln x} =$$

$$= 2 \int_0^1 (1-x^p)(1-x^q) \frac{x^{s-1} dx}{(1-x^{p+q+2s}) \ln x} = 2 \ln \left\{ \sin \frac{s\pi}{p+q+2s} \operatorname{cosec} \frac{(p+s)\pi}{p+q+2s} \right\}$$

$[s > 0, \quad s+p > 0, \quad s+p+q > 0].$

GW ((324))(23b)a

$$31. \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{x^{s-1} dx}{(1-x) \ln x} = \ln \frac{\Gamma(p+s)\Gamma(q+s)\Gamma(r+s)\Gamma(p+q+r+s)}{\Gamma(p+q+s)\Gamma(p+r+s)\Gamma(q+r+s)\Gamma(s)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0]^* . < ftn1 >$

* These restrictions can be somewhat weakened by writing, for example, $s > 0, p+s > 0, q+s > 0,$

$r+s > 0, p+q+s > 0, p+r+s > 0, q+r+s > 0, p+q+r+s > 0,$ in 4.267 31. and 32.

BI ((127))(11)

$$32. \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{x^{s-1} dx}{(1-x^t) \ln x} = \ln \frac{\Gamma\left(\frac{p+s}{t}\right) \Gamma\left(\frac{q+s}{t}\right) \Gamma\left(\frac{r+s}{t}\right) \Gamma\left(\frac{p+q+r+s}{t}\right)}{\Gamma\left(\frac{p+q+s}{t}\right) \Gamma\left(\frac{q+r+s}{t}\right) \Gamma\left(\frac{p+r+s}{t}\right) \Gamma\left(\frac{s}{t}\right)}$$

$[p > 0, \quad q > 0, \quad r > 0, \quad s > 0, \quad t > 0]^* .$

GW ((324))(23b)

$$34. \int_0^1 \left\{ \frac{x^\mu - x}{x-1} - x(\mu-1) \right\} \frac{dx}{x \ln x} = \ln \Gamma(\mu) \quad [\operatorname{Re} \mu > 0].$$

WH, BI ((127))(18)

$$35. \int_0^1 \left\{ 1 - x - \frac{(1-x^p)(1-x^q)}{1-x} \right\} \frac{dx}{x \ln x} = \ln \{B(p, q)\} \quad [p > 0, \quad q > 0].$$

BI ((130))(18)

578

$$36. \int_0^1 \left\{ \frac{x^{p-1}}{1-x} - \frac{x^{pq-1}}{1-x^q} - \frac{1}{x(1-x)} + \frac{1}{x(1-x^q)} \right\} \frac{dx}{\ln x} = q \ln p \quad [p > 0].$$

BI ((130))(20)

$$37. \int_0^1 \left\{ \frac{x^{q-1}}{1-x} - \frac{x^{pq-1}}{1-x^p} - \frac{p-1}{1-x^p} x^{p-1} - \frac{p-1}{2} x^{p-1} \right\} \frac{dx}{\ln x} = \frac{1-p}{2} \ln(2\pi) + \left(pq - \frac{1}{2} \right) \ln p$$

$$[p > 0, \quad q > 0].$$

BI ((130))(22)

$$38. \int_0^1 \frac{(1-x^p)(1-x^q) - (1-x)^2}{x(1-x) \ln x} dx = \ln B(p, q) \quad [p > 0, \quad q > 0].$$

GW ((324))(24)

$$39.^6 \int_0^1 (x^p - 1)^n \frac{dx}{\ln x} = \sum_{k=0}^n \binom{n}{n-k} (-1)^{n-k} \ln(pk+1) \quad [n > 0, \quad pn > -1]$$

GW ((324))(19d), BI ((123))(12)a

$$40.^6 \int_0^1 \frac{(1-x^p)^n}{1-x} \frac{dx}{\ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+1] \quad [n > 1, \quad pn > -1]$$

BI ((127))(12)

BI ((123))(12)

$$42.^6 \int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(1-x) \ln x} = \sum_{k=0}^n (-1)^{k-1} \ln \Gamma[(n-k)p+q]$$

$$[n > 1, \quad q > 0, \quad pn > -q]$$

BI ((127))(13)

$$43.^6 \int_0^1 (x^p-1)^n (x^q-1)^m \frac{x^{r-1} dx}{\ln x} = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^m (-1)^k \binom{m}{k} [r+(m-k)q+(n-j)p]$$

$$[n \geq 0, \quad m \geq 0, \quad n+m > 0, \quad r > 0, \quad pn+qm+r > 0].$$

BI ((123))(16)

4.268

$$1. \int_0^1 \frac{(x^p-x^q)(1-x^r)}{(\ln x)^2} dx = (p+1) \ln(p+1) - (q+1) \ln(q+1) -$$

$$-(p+r+1) \ln(p+r+1) + (q+r+1) \ln(q+r+1)$$

$$[p > -1, \quad q > -1, \quad p+r > -1, \quad q+r > -1].$$

GW ((324))(26)

$$2. \int_0^1 (x^p-x^q)^2 \frac{dx}{(\ln x)^2} = (2p+1) \ln(2p+1) + (2q+1) \ln(2q+1) - 2(p+q+1) \ln(p+q+1)$$

$$\left[p > -\frac{1}{2}, \quad q > -\frac{1}{2} \right]$$

GW ((324))(26a)

$$3. \int_0^1 (1-x^p)(1-x^q)(1-x^r) \frac{dx}{(\ln x)^2} =$$

$$= (p+q+1) \ln(p+q+1) + (q+r+1) \ln(q+r+1) + (p+r+1) \ln(p+r+1) -$$

$$-(p+1) \ln(p+1) - (q+1) \ln(q+1) - (r+1) \ln(r+1) - (p+q+r) \ln(p+q+r)$$

$$[p > -1, \quad q > -1, \quad r > -1, \quad p+q > -1, \quad p+r > -1, \quad q+r > -1, \quad p+q+r > 0].$$

BI ((124))(4)

579

$$4. \int_0^1 (1-x^p)^n x^{q-1} \frac{dx}{(\ln x)^2} = \frac{1}{2} \sum_{k=0}^n (-1)^k \binom{n}{k} (pk+q)^2 \ln(pk+q) \quad \left[q > 0, \quad p > -\frac{q}{n} \right].$$

$$5. \int_0^1 (1-x^p)^n (1-x^q)^m x^{r-1} \frac{dx}{(\ln x)^2} = \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{k=0}^m (-1)^k \binom{m}{k} \times \\ \times [(m-k)q + (n-j)p + r] \ln[(m-k)q + (n-j)p + r] \\ [r > 0, \quad mq + r > 0, \quad np + r > 0, \quad mq + np + r > 0].$$

BI ((124))(8)

$$6. \int_0^1 [(q-r)x^{p-1} + (r-p)x^{q-1} + (p-q)x^{r-1}] \frac{dx}{(\ln x)^2} = \\ = (q-r)p \ln p + (r-p)q \ln q + (p-q)r \ln r \quad [p > 0, \quad q > 0, \quad r > 0].$$

BI ((124))(9)

$$7. \int_0^1 \left[\frac{x^{p-1}}{(p-q)(p-r)(p-s)} + \frac{x^{q-1}}{(q-p)(q-r)(q-s)} + \frac{x^{r-1}}{(r-p)(r-q)(r-s)} + \right. \\ \left. + \frac{x^{s-1}}{(s-p)(s-q)(s-r)} \right] \frac{dx}{(\ln x)^2} = \frac{1}{2} \left[\frac{p^2 \ln p}{(p-q)(p-r)(p-s)} + \frac{q^2 \ln q}{(q-p)(q-r)(q-s)} + \right. \\ \left. + \frac{r^2 \ln r}{(r-p)(r-q)(r-s)} + \frac{s^2 \ln s}{(s-p)(s-q)(s-r)} \right] \\ [p > 0, \quad q > 0, \quad r > 0, \quad s > 0].$$

BI ((124))(16)

4.269

$$1. \int_0^1 \sqrt{\ln \frac{1}{x}} \cdot \frac{dx}{1+x^2} = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{(2k+1)^3}}.$$

BI ((115))(33)

$$2. \int_0^1 \frac{dx}{\sqrt{\ln \frac{1}{x}} \cdot (1+x)^2} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{2k+1}}.$$

BI ((133))(2)

$$3. \int_0^1 \sqrt{\ln \frac{1}{x}} \cdot x^{p-1} dx = \frac{1}{2} \sqrt{\frac{\pi}{p^3}} \quad [p > 0].$$

GW ((324))(1c)

BI ((133))(1)

$$5. \int_0^1 \frac{\sin t - x^n \sin[(n+1)t] + x^{n+1} \sin nt}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^n \frac{\sin kt}{\sqrt{k}} \quad [|t| < \pi].$$

BI ((133))(5)

580

$$6. \int_0^1 \frac{\cos t - x - x^{n-1} \cos nt + x^n \cos[(n-1)t]}{1 - 2x \cos t + x^2} \cdot \frac{dx}{\sqrt{\ln \frac{1}{x}}} = \sqrt{\pi} \sum_{k=1}^{n-1} \frac{\cos kt}{\sqrt{k}} \quad [|t| < \pi].$$

BI ((133))(6)

$$7. \int_u^v \frac{dx}{x \cdot \sqrt{\ln \frac{x}{u} \ln \frac{v}{x}}} = \pi \quad [uv > 0].$$

BI ((145))(37)

4.271

$$1. \int_0^1 (\ln x)^{2n} \frac{dx}{1+x} = \frac{2^{2n} - 1}{2^{2n}} \cdot (2n)! \zeta(2n+1).$$

BI ((110))(1)

$$2. \int_0^1 (\ln x)^{2n-1} \frac{dx}{1+x} = \frac{1 - 2^{2n-1}}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((110))(2)

$$3. \int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x} = -\frac{1}{n} 2^{2n-2} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI (110)(5), GW((324))(9a)

$$4. \int_0^1 (\ln x)^{p-1} \frac{dx}{1-x} = e^{i(p-1)\pi} \Gamma(p) \zeta(p) \quad [p > 1].$$

$$5. \int_0^1 (\ln x)^n \frac{dx}{1+x^2} = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+1}}.$$

BI ((110))(11)

$$6. \int_0^1 (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{1}{2} \int_0^{\infty} (\ln x)^{2n} \frac{dx}{1+x^2} = \frac{\pi^{2n+1}}{2^{2n+2}} |E_{2n}|.$$

GW ((324))(10)a

$$7. \int_0^{\infty} \frac{(\ln x)^{2n+1}}{1+bx+x^2} dx = 0 \quad [|b| < 2].$$

BI ((135))(2)

$$8. \int_0^1 (\ln x)^{2n} \frac{dx}{1-x^2} = \frac{2^{2n+1}-1}{2^{2n+1}} \cdot (2n)! \zeta(2n+1) \quad [n = 1, 2, \dots].$$

BI ((110))(12)

$$9. \int_0^{\infty} (\ln x)^{2n} \frac{dx}{1-x^2} = 0.$$

BI ((312))(7)a

$$10. \int_0^1 (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1}{2} \int_0^{\infty} (\ln x)^{2n-1} \frac{dx}{1-x^2} = \frac{1-2^{2n}}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((290))(17)A, BI((312))(6)a

$$11. \int_0^1 (\ln x)^{2n-1} \frac{x dx}{1-x^2} = -\frac{1}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((290))(19)a

$$12. \int_0^1 (\ln x)^{2n} \frac{1+x^2}{(1-x^2)^2} dx = \frac{2^{2n}-1}{2} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((296))(17)a

$$13. \int_0^1 (\ln x)^{2n+1} \frac{(\cos 2a\pi - x) dx}{1 - 2x \cos 2a\pi + x^2} = -(2n+1)! \sum_{k=1}^{\infty} \frac{\cos 2ak\pi}{k^{2n+2}} \quad [a \text{ is not an integer}].$$

LI ((113))(10)

$$14.^6 \int_0^{\infty} (\ln x)^n \frac{x^{\nu-1} dx}{a^2 + 2ax \cos t + x^2} = -\pi \operatorname{cosec} t \frac{d^n}{d\nu^n} \left[a^{\nu-2} \frac{\sin(\nu-1)t}{\sin \nu\pi} \right] \\ [a > 0, \quad 0 < \operatorname{Re} \nu < 2, \quad 0 < |t| < \pi].$$

ET I 315(12)

$$15. \int_0^1 (\ln x)^n \frac{x^{p-1} dx}{1-x^q} = -\frac{1}{q^{n+1}} \psi^{(n)} \left(\frac{p}{q} \right) \quad [p > 0, \quad q > 0].$$

GW ((324))(9)

$$16.^3 \int_0^1 (\ln x)^n \frac{x^{p-1} dx}{1+x^q} = \frac{1}{q^{n+1}} \beta^{(n)} \left(\frac{p}{q} \right) \quad [p > 0, \quad q > 0].$$

GW ((324))(10)

4.272

$$1. \int_0^1 \frac{\left[\ln \left(\frac{1}{x} \right) \right]^{q-1} dx}{1 + 2x \cos t + x^2} = \operatorname{cosec} t \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kt}{k^q} \quad [|t| < \pi, \quad q < 1].$$

LI ((130))(1)

$$2. \int_0^1 \left(\ln \frac{1}{x} \right)^{q-1} \frac{(1+x) dx}{1 + 2x \cos t + x^2} = \sec \frac{t}{2} \Gamma(q) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos \left[\left(k - \frac{1}{2} \right) t \right]}{k^q} \\ \left[|t| < \pi, \quad q < \frac{1}{2} \right].$$

LI ((130))(5)

$$3.^* \int_0^1 \left[\ln \left(\frac{1}{x} \right) \right]^{\mu} \frac{x^{\nu-1} dx}{1 - 2ax \cos t + x^2 a^2} = \frac{\Gamma(\mu+1)}{a \sin t} \sum_{k=1}^{\infty} \frac{a^k \sin kt}{(\nu+k-1)^{\mu+1}} \\ [a > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0, \quad -\pi < t < \pi].$$

$$4. \int_0^1 \left(\ln \frac{1}{x}\right)^{r-1} \frac{\cos \lambda - px}{1 + p^2 x^2 - 2px \cos \lambda} x^{q-1} dx = \Gamma(r) \sum_{k=1}^{\infty} \frac{p^{k-1} \cos k\lambda}{(q+k-1)^r} \quad [r > 0, \quad q > 0].$$

BI ((113))(11)

$$5. \int_1^{\infty} (\ln x)^p \frac{dx}{x^2} = \Gamma(1+p) \quad [p > -1].$$

BI ((149))(1)

$$6. \int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^{\mu}} \Gamma(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

BI ((107))(3)

$$7. \int_0^1 \left(\ln \frac{1}{x}\right)^{n-\frac{1}{2}} x^{\nu-1} dx = \frac{(2n-1)!!}{(2\nu)^n} \sqrt{\frac{\pi}{\nu}} \quad [\operatorname{Re} \nu > 0].$$

BI ((107))(2)

$$8. \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1+x} dx = (n-1)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(\nu+k)^n} \quad [\operatorname{Re} \nu > 0].$$

BI ((110))(4)

582

$$9. \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{x^{\nu-1}}{1-x} dx = (n-1)! \zeta(n, \nu) \quad [\operatorname{Re} \nu > 0].$$

BI ((110))(7)

$$10. \int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} (x-1)^n \left(a + \frac{nx}{x-1}\right) x^{a-1} dx = \Gamma(\mu) \sum_{k=0}^n \frac{(-1)^k n(n-1) \dots (n-k+1)}{(a+n-k)^{\mu-1} k!} \quad [\operatorname{Re} \mu > 0].$$

LI ((110))(10)

$$11. \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} \frac{1-x^m}{1-x} dx = (n-1)! \sum_{k=1}^m \frac{1}{k^n}.$$

$$12. \int_0^1 \left(\ln \frac{1}{x}\right)^{\mu-1} \frac{x^{\nu-1} dx}{1-x^2} = \Gamma(\mu) \sum_{k=0}^{\infty} \frac{1}{(\nu+2k)^\mu} = \frac{1}{2^\mu} \Gamma(\mu) \zeta\left(\mu, \frac{\nu}{2}\right)$$

[Re $\mu > 0$, Re $\nu > 0$].

BI ((110))(13)

$$13. \int_0^1 \frac{x^q - x^{-q}}{1-x^2} \left(\ln \frac{1}{x}\right)^p dx = \Gamma(p+1) \sum_{k=1}^{\infty} \left\{ \frac{1}{(2k+q-1)^{p+1}} - \frac{1}{(2k-q-1)^{p+1}} \right\}$$

[$p > -1$, $q^2 < 1$].

LI ((326))(12)a

$$14. \int_0^1 \left(\ln \frac{1}{x}\right)^{r-1} \frac{x^{p-1} dx}{(1+x^q)^s} = \Gamma(r) \sum_{k=0}^{\infty} \frac{-s}{k} \frac{1}{(p+kq)^r}$$

[$p > 0$, $q > 0$, $r > 0$, $0 < s < r+2$].

GW ((324))(11)

$$15. \int_0^1 \left(\ln \frac{1}{x}\right)^n (1+x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{1}{(p+kq)^{n+1}} \quad [p > 0, q > 0].$$

BI ((107))(6)

$$16. \int_0^1 \left(\ln \frac{1}{x}\right)^n (1-x^q)^m x^{p-1} dx = n! \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(p+kq)^{n+1}} \quad [p > 0, q > 0].$$

BI ((107))(7)

$$17. \int_0^1 \left(\ln \frac{1}{x}\right)^{p-1} \frac{x^{q-1} dx}{1-ax^q} = \frac{1}{aq^p} \Gamma(p) \sum_{k=1}^{\infty} \frac{a^k}{k^p} \quad [p > 0, q > 0, a < 1].$$

LI ((110))(8)

$$18. \int_0^1 \left(\ln \frac{1}{x}\right)^{2-\frac{1}{n}} (x^{p-1} - x^{q-1}) dx = \frac{n}{n-1} \Gamma\left(\frac{1}{n}\right) \left(q^{1-\frac{1}{n}} - p^{1-\frac{1}{n}}\right) \quad [q > p > 0].$$

BI ((133))(4)

LI ((110))(16)

4.273

$$\int_u^v \left(\ln \frac{x}{u}\right)^{p-1} \left(\ln \frac{v}{x}\right)^{q-1} \frac{dx}{x} = B(p, q) \left(\ln \frac{v}{u}\right)^{p+q-1} \quad [p > 0, \quad q > 0, \quad uv > 0].$$

BI ((145))(36)

4.274

$$\int_0^{\frac{1}{e}} \frac{\sqrt[q]{x} dx}{x \sqrt{-(1 + \ln x)}} = \frac{\sqrt[q]{q\pi}}{\sqrt[q]{e}} \quad [q > 0].$$

BI ((145))(4)

583

4.275

$$1. \int_0^1 \left[\left(\ln \frac{1}{x}\right)^{q-1} - x^{p-1} (1-x)^{q-1} \right] dx = \frac{\Gamma(q)}{\Gamma(p+q)} [\Gamma(p+q) - \Gamma(p)] \quad [p > 0, \quad q > 0].$$

BI ((107))(8)

$$2. \int_0^1 \left[x - \left(\frac{1}{1 - \ln x}\right)^q \right] \frac{dx}{x \ln x} = -\psi(q) \quad [q > 0].$$

BI ((126))(5)

4.28 Combinations of rational functions of $\ln x$ and powers

4.281

$$1. \int_0^1 \left[\frac{1}{\ln x} + \frac{1}{1-x} \right] dx = \mathbf{C}.$$

BI ((127))(15)

$$2. \int_1^\infty \frac{dx}{x^2(\ln p - \ln x)} = \frac{1}{p} \operatorname{li}(p).$$

LA 281(30)

$$4. \int_0^1 \left[\frac{1}{\ln x} + \frac{x^{\mu-1}}{1-x} \right] dx = -\psi(\mu) \quad [\operatorname{Re} \mu > 0].$$

WH

$$5. \int_0^1 \left[\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right] dx = \ln p - \psi(q) \quad [p > 0, \quad q > 0].$$

BI ((127))(17)

$$6. \int_0^1 \left[\frac{1}{1-x^2} + \frac{1}{2x \ln x} \right] \frac{dx}{\ln x} = \frac{\ln 2}{2}.$$

LI ((130))(19)

$$7. \int_0^1 \left[q - \frac{1}{2} + \frac{(1-x)(1+q \ln x) + x \ln x}{(1-x)^2} x^{q-1} \right] \frac{dx}{\ln x} = \frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2} \quad [q > 0].$$

BI ((128))(15)

4.282

$$1. \int_0^1 \frac{\ln x}{4\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x} = \frac{1}{4} - \frac{1}{2} \mathbf{C}.$$

BI ((129))(1)

$$2. \int_0^1 \frac{1}{a^2 + (\ln x)^2} \cdot \frac{dx}{1+x^2} = \frac{1}{2a} \beta \left(\frac{2a + \pi}{4\pi} \right) \quad \left[a > -\frac{\pi}{2} \right].$$

BI ((129))(9)

$$3. \int_0^1 \frac{1}{\pi^2 + (\ln x)^2} \frac{dx}{1+x^2} = \frac{4 - \pi}{4\pi}.$$

BI ((129))(6)

$$4. \int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{dx}{1-x^2} = \frac{1}{2} \left(\frac{1}{2} - \ln 2 \right).$$

$$5. \int_0^1 \frac{\ln x}{a^2 + (\ln x)^2} \cdot \frac{x dx}{1 - x^2} = \frac{1}{2} \left[\frac{\pi}{2a} + \ln \frac{\pi}{a} + \psi \left(\frac{a}{\pi} \right) \right] \quad [a > 0].$$

BI ((129))(14)

$$6. \int_0^1 \frac{\ln x}{\pi^2 + (\ln x)^2} \cdot \frac{x dx}{1 - x^2} = \frac{1}{2} \left(\frac{1}{2} - \mathbf{C} \right).$$

BI ((129))(13)

$$7. \int_0^1 \frac{1}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1 + x^2} = \frac{\ln 2}{4\pi}.$$

BI ((129))(7)

$$8. \int_0^1 \frac{\ln x}{\pi^2 + 4(\ln x)^2} \cdot \frac{dx}{1 - x^2} = \frac{2 - \pi}{16}.$$

BI ((129))(11)

$$9.^* \int_0^1 \frac{1}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1 + x^2} = \frac{1}{8\pi\sqrt{2}} \quad [\pi + 2\ln(\sqrt{2} - 1)].$$

BI ((129))(8)

$$10. \int_0^1 \frac{\ln x}{\pi^2 + 16(\ln x)^2} \cdot \frac{dx}{1 - x^2} = -\frac{\pi}{32\sqrt{2}} + \frac{1}{16} + \frac{1}{16\sqrt{2}} \ln(\sqrt{2} - 1).$$

BI ((129))(12)

$$11. \int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{dx}{1 - x} = -\frac{\pi^2}{a^4} \sum_{k=1}^{\infty} |B_{2k}| \left(\frac{2\pi}{a} \right)^{2k-2}$$

BI ((129))(4)

$$12. \int_0^1 \frac{\ln x}{[a^2 + (\ln x)^2]^2} \frac{x dx}{1 - x^2} = -\frac{\pi^2}{4a^4} \sum_{k=1}^{\infty} |B_{2k}| \left(\frac{\pi}{a} \right)^{2k-2}$$

$$13. \int_0^1 \frac{x^p - x^{-p}}{x^2 - 1} \frac{dx}{q^2 + (\ln x)^2} = \frac{2\pi}{q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin kp\pi}{2q + k\pi} \quad [p^2 < 1].$$

BI ((132))(13)a

4.283

$$1. \int_0^1 \left(\frac{x-1}{\ln x} - x \right) \frac{dx}{\ln x} = \ln 2 - 1.$$

BI ((132))(17)a

$$2. \int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2} - 1.$$

BI ((127))(20)

$$3. \int_0^1 \left(\frac{1}{\ln x} + \frac{x}{1-x} + \frac{x}{2} \right) \frac{dx}{x \ln x} = \frac{\ln 2\pi}{2}.$$

BI ((127))(23)

$$4. \int_0^1 \left[\frac{1}{(\ln x)^2} - \frac{x}{(1-x)^2} \right] dx = \mathbf{C} - \frac{1}{2}.$$

GW ((326))(8a)

$$5. \int_0^1 \left(\frac{1}{1-x^2} + \frac{1}{2 \ln x} - \frac{1}{2} \right) \frac{dx}{\ln x} = \frac{\ln 2 - 1}{2}.$$

BI ((128))(14)

$$6. \int_0^1 \left(\frac{1}{\ln x} + \frac{1}{2} \cdot \frac{1+x}{1-x} - \ln x \right) \frac{dx}{\ln x} = \frac{\ln 2\pi}{2}.$$

BI ((127))(22)

$$7. \int_0^1 \left[\frac{1}{1 - \ln x} - x \right] \frac{dx}{x \ln x} = -\mathbf{C}.$$

GW ((326))(11a)

$$8. \int_0^1 \left[\frac{x^q - 1}{x(\ln x)^2} - \frac{q}{\ln x} \right] dx = q \ln q - q \quad [q > 0].$$

BI ((126))(2)

$$9. \int_0^1 \left[x + \frac{1}{a \ln x - 1} \right] \frac{dx}{x \ln x} = \ln \frac{a}{q} + C \quad [a > 0, \quad q > 0].$$

BI ((126))(8)

$$10. \int_0^1 \left[\frac{1}{\ln x} + \frac{1+x}{2(1-x)} \right] \frac{x^{p-1}}{\ln x} dx = -\ln \Gamma(p) + \left(p - \frac{1}{2} \right) \ln p - p + \frac{\ln 2\pi}{2} \quad [p > 0].$$

GW ((326))(9)

$$11. \int_0^1 \left[p - 1 - \frac{1}{1-x} + \left(\frac{1}{2} - \frac{1}{\ln x} \right) x^{p-1} \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) \ln p + p - \frac{\ln 2\pi}{2} \quad [p > 0].$$

BI ((127))(25)

$$12. \int_0^1 \left[-\frac{1}{(\ln x)^2} + \frac{(p-2)x^p - (p-1)x^{p-1}}{(1-x)^2} \right] dx = -\psi(p) + p - \frac{3}{2} \quad [p > 0]$$

GW ((326))(8)

$$13. \int_0^1 \left[\left(p - \frac{1}{2} \right) x^3 + \frac{1}{2} \left(1 - \frac{1}{\ln x} \right) (x^{2p-1} - 1) \right] \frac{dx}{\ln x} = \left(\frac{1}{2} - p \right) (\ln p - 1) \quad [p > 0].$$

BI ((132))(23)a

$$14. \int_0^1 \left[\left(q - \frac{1}{2} \right) \frac{x^{p-1} - x^{r-1}}{\ln x} + \frac{px^{pq-1}}{1-x^p} - \frac{rx^{rq-1}}{1-x^r} \right] \frac{dx}{\ln x} = (p-r) \left[\frac{1}{2} - q - \ln \Gamma(q) + \frac{\ln 2\pi}{2} \right] \quad [q > 0].$$

BI ((132))(13)

4.284

$$1. \int_0^1 \left[\frac{x^q - 1}{x(\ln x)^3} - \frac{q}{x(\ln x)^2} - \frac{q^2}{2 \ln x} \right] dx = \frac{q^2}{2} \ln q - \frac{3}{4} q^2 \quad [q > 0].$$

$$2. \int_0^1 \left[\frac{x^q - 1}{x(\ln x)^4} - \frac{q}{x(\ln x)^3} - \frac{q^2}{2x(\ln x)^2} - \frac{q^3}{6 \ln x} \right] dx = \frac{q^3}{6} \ln q - \frac{11}{36} q^3 \quad [q > 0].$$

BI ((126))(4)

4.285

$$\int_0^1 \frac{x^{p-1} dx}{(q + \ln x)^n} = \frac{p^{n-1}}{(n-1)!} e^{-pq} \text{Ei}(pq) - \frac{1}{(n-1)! q^{n-1}} \sum_{k=1}^{n-1} (n-k-1)! (pq)^{k-1} \quad [p > 0, \quad q < 0].$$

BI ((125))(21)

In integrals of the form $\int \frac{x^a (\ln x)^n dx}{[b \pm (\ln x)^m]^l}$, we should make the substitution $x = e^t$ or $x = e^{-t}$ and then seek the resulting integrals in 3.351-3.356.

4.29- 4.32 Combinations of logarithmic functions of more complicated arguments and powers

4.291

$$1. \int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}.$$

FI II 483

586

$$2. \int_0^1 \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}.$$

FI II 714

$$3. \int_0^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}.$$

BI ((145))(2)

$$4. \int_0^1 \ln \left(1 - \frac{x}{2} \right) \frac{dx}{x} = \frac{1}{2} (\ln 2)^2 - \frac{\pi^2}{12}.$$

$$5. \int_0^1 \frac{\ln \frac{1+x}{2}}{1-x} dx = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}.$$

BI ((115))(1)

$$6. \int_0^1 \frac{\ln(1+x)}{1+x} dx = \frac{1}{2}(\ln 2)^2.$$

BI ((114))(14)a

$$7. \int_0^\infty \frac{\ln(1+ax)}{1+x} dx = \frac{\pi}{4} \ln(1+a^2) - \int_0^a \frac{\ln u du}{1+u^2} \quad [a > 0].$$

GI II (2209)

$$8. \int_0^1 \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{8} \ln 2.$$

FI II 157

$$9. \int_0^\infty \frac{\ln(1+x)}{1+x^2} dx = \frac{\pi}{4} \ln 2 + \mathbf{G}.$$

BI ((136))(1)

$$10. \int_0^1 \frac{\ln(1-x)}{1+x^2} dx = \frac{\pi}{8} \ln 2 - \mathbf{G}.$$

BI ((114))(17)

$$11. \int_1^\infty \frac{\ln(x-1)}{1+x^2} dx = \frac{\pi}{8} \ln 2.$$

BI ((144))(4)

$$12. \int_0^1 \frac{\ln(1+x)}{x(1+x)} dx = \frac{\pi^2}{12} - \frac{1}{2}(\ln 2)^2.$$

BI ((144))(4)

$$14. \int_0^1 \frac{\ln(1+x)}{(ax+b)^2} dx = \frac{1}{a(a-b)} \ln \frac{a+b}{b} + \frac{2 \ln 2}{b^2 - a^2} \quad [a \neq b, \quad ab > 0];$$

$$= \frac{1}{2a^2} (1 - \ln 2) \quad [a = b].$$

LI ((114))(5)a

$$15. \int_0^\infty \frac{\ln(1+x)}{(ax+b)^2} dx = \frac{\ln \frac{a}{b}}{a(a-b)} \quad [ab > 0].$$

BI ((139))(5)

$$16. \int_0^1 \ln(a+x) \frac{dx}{a+x^2} = \frac{1}{2\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln[(1+a)a] \quad [a > 0].$$

BI ((114))(20)

$$17. \int_0^\infty \ln(a+x) \frac{dx}{(b+x)^2} = \frac{a \ln a - b \ln b}{b(a-b)} \quad [a > 0, \quad b > 0, \quad a \neq b].$$

LI ((139))(6)

587

$$18. \int_0^a \frac{\ln(1+ax)}{1+x^2} dx = \frac{1}{2} \operatorname{arctg} a \ln(1+a^2).$$

GI II (2195)

$$19. \int_0^1 \frac{\ln(1+ax)}{1+ax^2} dx = \frac{1}{2\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln(1+a) \quad [a > 0].$$

BI ((114))(21)

$$20. \int_0^1 \frac{\ln(ax+b)}{(1+x)^2} dx = \frac{1}{a-b} \left[\frac{1}{2}(a+b) \ln(a+b) - b \ln b - a \ln 2 \right] \quad [a > 0, \quad b > 0, \quad a \neq b].$$

BI ((114))(22)

BI ((139))(8)

$$22. \int_0^{\infty} \ln(a+x) \frac{x dx}{(b^2+x^2)^2} = \frac{1}{2(a^2+b^2)} \left(\ln b + \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right) \quad [a > 0, \quad b > 0].$$

BI ((139))(9)

$$23. \int_0^1 \ln(1+x) \frac{1+x^2}{(1+x)^4} dx = -\frac{1}{3} \ln 2 + \frac{23}{72}.$$

LI ((114))(12)

$$24. \int_0^1 \ln(1+x) \frac{1+x^2}{a^2+x^2} \cdot \frac{dx}{1+a^2x^2} = \frac{1}{2a(1+a^2)} \left[\frac{\pi}{2} \ln(1+a^2) - 2 \arctg a \cdot \ln a \right] \quad [a > 0].$$

LI ((114))(11)

$$25. \int_0^1 \ln(1+x) \frac{1-x^2}{(ax+b)^2} \cdot \frac{dx}{(bx+a)^2} = \frac{1}{a^2-b^2} \left\{ \frac{1}{a-b} \left[\frac{a+b}{ab} \ln(a+b) - \frac{1}{a} \ln b - \frac{1}{b} \ln a \right] + \frac{4 \ln 2}{b^2-a^2} \right\} \quad [a > 0, \quad b > 0, \quad a^2 \neq b^2].$$

LI ((114))(13)

$$26. \int_0^{\infty} \ln(1+x) \frac{1-x^2}{(ax+b)^2} \cdot \frac{dx}{(bx+a)^2} = \frac{1}{ab(a^2-b^2)} \ln \frac{b}{a} \quad [a > 0, \quad b > 0].$$

LI ((139))(14)

$$27. \int_0^1 \ln(1+ax) \frac{1-x^2}{(1+x^2)^2} dx = \frac{1}{2} \frac{(1+a)^2}{1+a^2} \ln(1+a) - \frac{1}{2} \cdot \frac{a}{1+a^2} \ln 2 - \frac{\pi}{4} \cdot \frac{a^2}{1+a^2} \quad [a > -1].$$

BI ((114))(23)}

$$28. \int_0^{\infty} \ln(a+x) \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{1}{a^2+b^2} \left(a \ln \frac{b}{a} - \frac{b\pi}{2} \right) \quad [a > 0, \quad b > 0].$$

BI ((139))

$$29. \int_0^{\infty} \ln(a-x)^2 \frac{b^2-x^2}{(b^2+x^2)^2} dx = \frac{2}{a^2+b^2} \left(a \ln \frac{a}{b} - \frac{b\pi}{2} \right) \quad [a > 0, \quad b > 0].$$

$$30. \int_0^{\infty} \ln(a-x)^2 \frac{x dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2} \left(\ln b - \frac{a\pi}{2b} + \frac{a^2}{b^2} \ln a \right) \quad [a > 0, \quad b > 0].$$

BI ((139))(10)

588

4.292

$$1. \int_0^1 \frac{\ln(1 \pm x)}{\sqrt{1-x^2}} dx = -\frac{\pi}{2} \ln 2 \pm 2\mathbf{G}.$$

GW ((325))(20)

$$2. \int_0^1 \frac{x \ln(1 \pm x)}{\sqrt{1-x^2}} dx = -1 \pm \frac{\pi}{2}.$$

GW ((325))(22c)

$$3. \int_{-a}^a \frac{\ln(1+bx)}{\sqrt{a^2-x^2}} dx = \pi \ln \frac{1+\sqrt{1-a^2b^2}}{2} \quad \left[0 \leq |b| \leq \frac{1}{a} \right].$$

BI ((145))(16, 17)A, GW ((325))(21e)

$$4. \int_0^1 \frac{x \ln(1+ax)}{\sqrt{1-x^2}} dx = -1 + \frac{\pi}{2} \cdot \frac{1-\sqrt{1-a^2}}{a} + \frac{\sqrt{1-a^2}}{a} \arcsin a \quad [|a| \leq 1];$$

$$= -1 + \frac{\pi}{2a} + \frac{\sqrt{a^2-1}}{a} \ln(a + \sqrt{a^2-1}) \quad [a \geq 1].$$

GW ((325))(22)

$$5. \int_0^1 \frac{\ln(1+ax)}{x\sqrt{1-x^2}} dx = \frac{1}{2} \arcsin a (\pi - \arcsin a) = \frac{\pi^2}{8} - \frac{1}{2} (\arccos a)^2 \quad [|a| \leq 1].$$

BI ((120))(4), GW ((325))(21a)

4.293

$$1. \int_0^1 x^{\mu-1} \ln(1+x) dx = \frac{1}{\mu} [\ln 2 - \beta(\mu+1)] \quad [\operatorname{Re} \mu > -1].$$

BI ((106))(4)a

$$3. \int_0^{\infty} x^{\mu-1} \ln(1+x) dx = \frac{\pi}{\mu \sin \mu\pi} \quad [-1 < \operatorname{Re} \mu < 0].$$

GW ((325))(3a)

$$4. \int_0^1 x^{2n-1} \ln(1+x) dx = \frac{1}{2n} \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}$$

GW ((325))(2b)

$$5. \int_0^1 x^{2n} \ln(1+x) dx = \frac{1}{2n+1} \left[\ln 4 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right].$$

GW ((325))(2c)

$$6. \int_0^1 x^{n-\frac{1}{2}} \ln(1+x) dx = \frac{2 \ln 2}{2n+1} + \frac{(-1)^n \cdot 4}{2n+1} \left[\pi - \sum_{k=0}^n \frac{(-1)^k}{2k+1} \right].$$

GW ((325))(2f)

$$7. \int_0^{\infty} x^{\mu-1} \ln|1-x| dx = \frac{\pi}{\mu} \operatorname{ctg}(\mu\pi) \quad [-1 < \operatorname{Re} \mu < 0].$$

BI ((134))(4), ET I 315(18)

$$8. \int_0^1 x^{\mu-1} \ln(1-x) dx = -\frac{1}{\mu} [\psi(\mu+1) - \psi(1)] \quad [\operatorname{Re} \mu > -1].$$

ET I 316(19)

589

$$9.7 \int_1^{\infty} x^{\mu-1} \ln(x-1) dx = \frac{1}{\mu} [\pi \operatorname{ctg}(\mu\pi) + \psi(\mu+1) + \mathcal{C}] \quad [\operatorname{Re} \mu < 0].$$

ET I 316(20)

BI ((134))(3)

$$11. \int_0^{\infty} \frac{x^{\mu-1} \ln(1+x)}{1+x} dx = \frac{\pi}{\sin \mu\pi} [\mathbf{C} + \psi(1-\mu)] \quad [-1 < \operatorname{Re} \mu < 1].$$

ET I 316(21)

$$12. \int_0^1 \frac{\ln(1+x)}{(1+x)^{\mu+1}} dx = \frac{-\ln 2}{2^{\mu}\mu} + \frac{2^{\mu}-1}{2^{\mu}\mu^2}.$$

BI ((114))(6)

$$13. \int_0^1 \frac{x^{\mu-1} \ln(1-x)}{(1-x)^{1-\nu}} dx = \mathbf{B}(\mu, \nu) [\psi(\nu) - \psi(\mu+\nu)] \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET I 316(122)

$$14. \int_0^{\infty} \frac{x^{\mu-1} \ln(\gamma+x)}{(\gamma+x)^{\nu}} dx = \gamma^{\mu-\nu} \mathbf{B}(\mu, \nu-\mu) [\psi(\nu) - \psi(\nu-\mu) + \ln \gamma] \quad [0 < \operatorname{Re} \mu < \operatorname{Re} \nu].$$

ET I 316(23)

4.294

$$1. \int_0^1 \ln(1+x) \frac{(p-1)x^{p-1} - px^{-p}}{x} dx = 2 \ln 2 - \frac{\pi}{\sin p\pi} \quad [0 < p < 1].$$

BI ((114))(2)

$$2. \int_0^1 \ln(1+x) \frac{1+x^{2n+1}}{1+x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} - \sum_{j=1}^{2n+1} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}.$$

BI ((114))(7)

$$3. \int_0^1 \ln(1+x) \frac{1-x^{2n}}{1+x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{j=1}^{2n} \frac{1}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}.$$

BI ((114))(8)

$$4. \int_0^1 \ln(1+x) \frac{1-x^{2n}}{1-x} dx = 2 \ln 2 \cdot \sum_{k=0}^{n-1} \frac{1}{2k+1} + \sum_{i=1}^{2n} \frac{(-1)^i}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}.$$

$$5. \int_0^1 \ln(1+x) \frac{1-x^{2n+1}}{1-x} dx = 2 \ln 2 \sum_{k=0}^n \frac{1}{2k+1} + \sum_{j=1}^{2n+1} \frac{(-1)^j}{j} \sum_{k=1}^j \frac{(-1)^{k-1}}{k}.$$

BI ((114))(10)

$$6. \int_0^1 \ln(1-x) \frac{1-(-1)^n x^n}{1-x} dx = \sum_{j=1}^n \frac{(-1)^j}{j} \sum_{k=1}^j \frac{1}{k}.$$

BI ((114))(15)

$$7. \int_0^1 \ln(1-x) \frac{1-x^n}{1-x} dx = -\sum_{j=1}^n \frac{1}{j} \sum_{k=1}^j \frac{1}{k}.$$

BI ((114))(16)

$$8. \int_0^\infty \ln(1-x)^2 x^p dx = \frac{2\pi}{p+1} \operatorname{ctg} p\pi \quad [-2 < p < -1].$$

BI ((134))(13)a

590

$$9. \int_0^1 [\ln(1+x)]^n (1+x)^r dx = (-1)^{n-1} \frac{n!}{(r+1)^{n+1}} + 2^{r+1} \sum_{k=0}^n \frac{(-1)^k n! (\ln 2)^{n-k}}{(n-k)!(r+1)^{k+1}}.$$

LI ((106))(34)a

$$10. \int_0^1 [\ln(1-x)]^n (1-x)^r dx = (-1)^n \frac{n!}{(r+1)^{n+1}} \quad [r > -1].$$

BI ((106))(35)a

$$11. \int_0^1 \left(\ln \frac{1}{1-x^2} \right)^n x^{2q-1} dx = \frac{n!}{2} \zeta(n+1, q+1) \quad [-1 < q < 0].$$

BI ((311))(15)a

$$12. \int_0^1 (\ln x)^{2n} \ln(1-x^2) \frac{dx}{x} = -\frac{\pi^{2n+2}}{2(n+1)(2n+1)} |B_{2n+2}|.$$

$$13.^6 \int_0^1 [\ln(1/x)]^m \ln(1-x^2) dx = -\sum_{n=1}^{\infty} \frac{\Gamma(m+1)}{n(2n+1)^{m+1}} [m+1 > 0, \quad n+1 > 0].$$

4.295

$$1. \int_0^{\infty} \ln(\mu x^2 + \beta) \frac{dx}{\gamma + x^2} = \frac{\pi}{\sqrt{\gamma}} \ln(\sqrt{\mu\gamma} + \sqrt{\beta}) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg \gamma| < \pi].$$

ET II 218(27)

$$2. \int_0^1 \ln(1+x^2) \frac{dx}{x^2} = \frac{\pi}{2} - \ln 2.$$

GW ((325))(2g)

$$3. \int_0^{\infty} \ln(1+x^2) \frac{dx}{x^2} = \pi.$$

GW ((325))(4c)

$$4. \int_0^{\infty} \ln(1+x^2) \frac{dx}{(a+x)^2} = \frac{2a}{1+a^2} \left(\frac{\pi}{2a} + \ln a \right) \quad [a > 0].$$

BI ((319))(6)a

$$5. \int_0^1 \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 - \mathbf{G}.$$

BI ((114))(24)

$$6. \int_1^{\infty} \ln(1+x^2) \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2 + \mathbf{G}.$$

BI ((114))(5)

$$7. \int_0^{\infty} \ln(a^2 + b^2 x^2) \frac{dx}{c^2 + g^2 x^2} = \frac{\pi}{cg} \ln \frac{ag + bc}{g} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

$$8. \int_0^{\infty} \ln(a^2 + b^2 x^2) \frac{dx}{c^2 - g^2 x^2} = -\frac{\pi}{cg} \operatorname{arctg} \frac{bc}{ag} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

BI ((136))(15)a

$$9. \int_0^{\infty} \frac{\ln(1 + p^2 x^2) - \ln(1 + q^2 x^2)}{x^2} dx = \pi(p - q) \quad [p > 0, \quad q > 0].$$

FI II 645

$$10. \int_0^1 \ln \frac{1 + a^2 x^2}{1 + a^2} \frac{dx}{1 - x^2} = -(\operatorname{arctg} a)^2.$$

BI ((115))(2)

591

$$11. \int_0^1 \ln(1 - x^2) \frac{dx}{x} = -\frac{\pi^2}{12}.$$

$$12. \int_0^{\infty} \ln(1 - x^2)^2 \frac{dx}{x^2} = 0.$$

BI ((142))(9)a

$$13. \int_0^1 \ln(1 - x^2) \frac{dx}{1 + x^2} = \frac{\pi}{4} \ln 2 - \mathbf{G}.$$

GW ((325))(17)

$$14. \int_1^{\infty} \ln(x^2 - 1) \frac{dx}{1 + x^2} = \frac{\pi}{4} \ln 2 + \mathbf{G}.$$

BI ((144))(6)

$$15. \int_0^{\infty} \ln(a^2 - x^2)^2 \frac{dx}{b^2 + x^2} = \frac{\pi}{b} \ln(a^2 + b^2) \quad [b > 0].$$

BI ((136))(16)

$$16. \int_0^{\infty} \ln(a^2 - x^2)^2 \frac{b^2 - x^2}{(b^2 + x^2)^2} dx = -\frac{2b\pi}{a^2 + b^2} \quad [b > 0].$$

$$17. \int_0^1 \ln(1+x^2) \frac{dx}{x(1+x^2)} = \frac{1}{2} \left[\frac{\pi^2}{12} - \frac{1}{2} (\ln 2)^2 \right].$$

BI ((114))(25)

$$18. \int_0^\infty \ln(1+x^2) \frac{dx}{x(1+x^2)} = \frac{\pi^2}{12}.$$

BI ((141))(9)

$$19. \int_0^1 \ln(\cos^2 t + x^2 \sin^2 t) \frac{dx}{1-x^2} = -t^2.$$

BI ((114))(27)a

$$20. \int_0^\infty \ln(a^2+b^2x^2) \frac{dx}{(c+gx)^2} = \frac{2 \ln b}{cg} + \frac{b^2}{a^2g^2+b^2c^2} \left(\frac{a}{b} \pi + 2 \frac{c}{g} \ln \frac{c}{g} + 2 \frac{a^2g}{b^2c} \ln \frac{a}{b} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

$$21. \int_0^1 \ln(a^2+b^2x^2) \frac{dx}{(c+gx)^2} = \\ = \frac{2}{c(c+g)} \ln a + \frac{b^2}{a^2g^2+b^2c^2} \left[\frac{2a}{b} \operatorname{arcctg} \frac{a}{b} + \frac{cb^2-ga^2}{b^2(c+g)} \ln \frac{a^2+b^2}{a^2} - 2 \frac{c}{g} \ln \frac{c+g}{c} \right] \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

BI ((114))(28)a

$$22. \int_0^\infty \frac{\ln(1+p^2x^2)}{r^2+q^2x^2} dx = \int_0^\infty \frac{\ln(p^2+x^2)}{q^2+r^2x^2} dx = \frac{\pi}{qr} \ln \frac{q+pr}{q} \quad [qr > 0, \quad p > 0].$$

FI II 745a, BI ((318))(1)A, BI ((318))(4)a

$$23. \int_0^\infty \frac{\ln(1+a^2x^2)}{b^2+c^2x^2} \frac{dx}{d^2+g^2x^2} = \frac{\pi}{b^2g^2-c^2d^2} \left[\frac{g}{d} \ln \left(1 + \frac{ad}{g} \right) - \frac{c}{b} \ln \left(1 + \frac{ab}{c} \right) \right] \\ [a > 0, \quad b > 0, \quad c > 0, \quad d > 0, \quad g > 0, \quad b^2g^2 \neq c^2d^2].$$

BI ((141))(10)

$$24. \int_0^\infty \frac{\ln(1+a^2x^2)}{b^2+c^2x^2} \frac{x^2 dx}{d^2+g^2x^2} = \frac{\pi}{b^2g^2-c^2d^2} \left[\frac{b}{c} \ln \left(1 + \frac{ab}{c} \right) - \frac{d}{g} \ln \left(1 + \frac{ad}{g} \right) \right]$$

$$25. \int_0^{\infty} \ln(a^2 + b^2x^2) \frac{dx}{(c^2 + g^2x^2)^2} = \frac{\pi}{2c^3g} \left(\ln \frac{ag + bc}{g} - \frac{bc}{ag + bc} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

GW ((325))(18a)

$$26. \int_0^{\infty} \ln(a^2 + b^2x^2) \frac{x^2 dx}{(c^2 + g^2x^2)^2} = \frac{\pi}{2cg^3} \left(\ln \frac{ag + bc}{g} + \frac{bc}{ag + bc} \right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

GW ((325))(18b)

$$27. \int_0^1 \ln(1+ax^2)\sqrt{1-x^2} dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} + \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\} \quad [a > 0].$$

BI ((117))(6)

$$28. \int_0^1 \ln(1+a-ax^2)\sqrt{1-x^2} dx = \frac{\pi}{2} \left\{ \ln \frac{1+\sqrt{1+a}}{2} - \frac{1}{2} \frac{1-\sqrt{1+a}}{1+\sqrt{1+a}} \right\} \quad [a > 0].$$

BI ((117))(7)

$$29. \int_0^1 \ln(1-a^2x^2) \frac{dx}{\sqrt{1-x^2}} = \pi \ln \frac{1+\sqrt{1-a^2}}{2} \quad [a^2 < 1].$$

BI ((119))(1)

$$30.^6 \int_0^1 \ln(1-a^2x^2) \frac{dx}{x\sqrt{1-x^2}} = -\left(\arccos |a| - \frac{\pi}{2} \right)^2$$

LI ((120))(11)

$$31. \int_0^1 \ln(1-x^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \ln \frac{k'}{k} \mathbf{K}(k) - \frac{\pi}{2} \mathbf{K}(k').$$

BI ((120))(12)

$$32. \int_0^1 \ln(1 \pm kx^2) \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln \frac{2 \pm 2k}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k').$$

$$33. \int_0^1 \frac{\ln(1 - k^2 x^2)}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx = \ln k' \mathbf{K}(k).$$

BI ((119))(27)

$$34. \int_0^1 \ln(1 - k^2 x^2) \sqrt{\frac{1 - k^2 x^2}{1 - x^2}} dx = (2 - k^2) \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k).$$

BI ((119))(3)

$$35. \int_0^1 \sqrt{\frac{1 - x^2}{1 - k^2 x^2}} \ln(1 - k^2 x^2) dx = \frac{1}{k^2} (1 + k'^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k).$$

BI ((119))(7)

$$36. \int_{-1}^1 \ln(1 - x^2) \frac{dx}{(a + bx)\sqrt{1 - x^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}} \ln \frac{\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}} \quad [a > 0, \quad b > 0, \quad a \neq b].$$

BI ((145))(15)

$$37. \int_0^1 \ln(1 - x^2) (px^{p-1} - qx^{q-1}) dx = \psi\left(\frac{q}{2} + 1\right) - \psi\left(\frac{p}{2} + 1\right) \quad [p > -2, \quad q > -2].$$

BI ((106))(15)

$$38. \int_0^1 \ln(1 + ax^2) \frac{dx}{\sqrt{1 - x^2}} = \pi \ln \frac{1 + \sqrt{1 + a}}{2} \quad [a \geq -1].$$

GW ((325))(21b)

593

$$39. \int_0^1 \ln(1 + x^2) x^{\mu-1} dx = \frac{1}{\mu} \left[\ln 2 - \beta\left(\frac{\mu}{2} + 1\right) \right] \quad [\operatorname{Re} \mu > -2].$$

BI ((106))(12)

$$40. \int_0^\infty \ln(1 + x^2) x^{\mu-1} dx = \frac{\pi}{\mu \sin \frac{\mu\pi}{2}} \quad [-2 < \operatorname{Re} \mu < 0].$$

$$41. \int_0^{\infty} \ln(1+x^2) \frac{x^{\mu-1} dx}{1+x} = \frac{\pi}{\sin \mu\pi} \left\{ \ln 2 - (1-\mu) \sin \frac{\mu\pi}{2} \beta \left(\frac{1-\mu}{2} \right) - (2-\mu) \cos \frac{\mu\pi}{2} \beta \left(\frac{2-\mu}{2} \right) \right\} \\ [-2 < \operatorname{Re} \mu < 1].$$

ET I 316(25)

4.296

$$1. \int_0^1 \ln(1+2x \cos t + x^2) \frac{dx}{x} = \frac{\pi^2}{6} - \frac{t^2}{2}.$$

BI ((114))(34)

$$2. \int_{-\infty}^{\infty} \ln(a^2 - 2ax \cos t + x^2) \frac{dx}{1+x^2} = \pi \ln(1 + 2a |\sin t| + a^2).$$

BI ((145))(28)

$$3. \int_0^{\infty} \ln(1+2x \cos t + x^2) x^{\mu-1} dx = \frac{2\pi}{\mu} \frac{\cos \mu t}{\sin \mu\pi} \quad [|t| < \pi, \quad -1 < \operatorname{Re} \mu < 0].$$

ET I 316(27)

$$4. \int_0^{\infty} \ln \left(\frac{x^2 + 2ax \cos t + a^2}{x^2 - 2ax \cos t + a^2} \right) \frac{x dx}{x^2 + b^2} = \frac{1}{2} \pi^2 - \pi t + \pi \operatorname{arctg} \frac{(a^2 - b^2) \cos t}{(a^2 + b^2) \sin t + 2ab} \\ [a > 0, \quad b > 0, \quad 0 < t < \pi]$$

4.297

$$1. \int_0^1 \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} = \frac{1}{a-b} \left[(a+b) \ln \frac{a+b}{2} - a \ln a - b \ln b \right] \quad [a > 0, \quad b > 0].$$

BI ((115))(16)

$$2. \int_0^{\infty} \ln \frac{ax+b}{bx+a} \frac{dx}{(1+x)^2} = 0 \quad [ab > 0].$$

BI ((139))(23)

$$3. \int_0^1 \ln \frac{1-x}{x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2.$$

$$4. \int_0^1 \ln \frac{1+x}{1-x} \frac{dx}{1+x^2} = \mathbf{G}.$$

BI ((115))(17)

$$5. \int_0^\infty \ln \left(\frac{1+x}{1-x} \right)^2 \frac{dx}{x(1+x^2)} = \frac{\pi^2}{2}.$$

BI ((141))(13)

$$6. \int_u^v \ln \frac{v+x}{u+x} \frac{dx}{x} = \frac{1}{2} \left(\ln \frac{v}{u} \right)^2 \quad [uv > 0].$$

BI ((145))(33)

$$7. \int_0^\infty \frac{b \ln(1+ax) - a \ln(1+bx)}{x^2} dx = ab \ln \frac{b}{a} \quad [a > 0, \quad b > 0].$$

FI II 647

594

$$8. \int_0^1 \ln \frac{1+ax}{1-ax} \frac{dx}{x\sqrt{1-x^2}} = \pi \arcsin a \quad [|a| \leq 1]$$

GW ((325))(21c), BI ((122))(2)

$$9. \int_u^v \ln \left(\frac{1+ax}{1-ax} \right) \frac{dx}{\sqrt{(x^2-u^2)(v^2-x^2)}} = \frac{\pi}{v} F \left(\arcsin av, \frac{u}{v} \right) \quad [|av| < 1].$$

BI ((145))(35)

4.298

$$1. \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1).$$

BI ((137))(1)

$$2. \int_0^\infty \ln \frac{1+x^2}{x} \frac{x^{2n}}{1+x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1).$$

$$3. \int_0^{\infty} \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1-x} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1).$$

BI ((137))(2)

$$4. \int_0^{\infty} \ln \frac{1+x^2}{x} \frac{x^{2n}}{1-x} dx = -\frac{\ln 2}{2n} - \frac{1}{4n^2} + \frac{1}{2n} \beta(2n+1).$$

BI ((137))(4)

$$5. \int_0^{\infty} \ln \frac{1+x^2}{x} \frac{x^{2n-1}}{1+x^2} dx = \frac{\ln 2}{2n} + \frac{1}{4n^2} - \frac{1}{2n} \beta(2n+1).$$

BI ((137))(10)

$$6. \int_0^1 \ln \frac{1+x^2}{x} x^{2n} dx = \frac{1}{2n+1} \left\{ (-1)^n \frac{\pi}{2} + \ln 2 - \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right\}.$$

BI ((294))(8)

$$7. \int_0^1 \ln \frac{1+x^2}{x} x^{2n-1} dx = \frac{1}{2n} \left\{ (-1)^{n+1} \ln 2 + \ln 2 - \frac{1}{2n} + (-1)^{n+1} \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right\}.$$

BI ((294))(9a)

$$8. \int_0^1 \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{2} \ln 2.$$

BI ((115))(7)

$$9. \int_0^{\infty} \ln \frac{1+x^2}{x} \frac{dx}{1+x^2} = \pi \ln 2.$$

BI ((137))(8)

$$10. \int_0^{\infty} \ln \frac{1+x^2}{x} \frac{dx}{1-x^2} = 0.$$

BI ((137))(9)

$$11. \int_0^1 \ln \frac{1-x^2}{x} \frac{dx}{1+x^2} = \frac{\pi}{4} \ln 2.$$

BI ((115))(9)

$$12. \int_1^{\infty} \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2.$$

BI ((144))(8)

$$13. \int_0^1 \ln \frac{1+x^2}{x+1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 - \mathbf{G}.$$

BI ((115))(18)

$$14. \int_1^{\infty} \ln \frac{1+x^2}{x-1} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2 + \mathbf{G}.$$

BI ((144))(9)

595

$$15. \int_0^1 \ln \frac{1+x^2}{1-x} \frac{dx}{1+x^2} = \frac{3\pi}{8} \ln 2.$$

BI ((115))(19)

$$16. \int_0^{\infty} \ln \frac{1+x^2}{x^2} \frac{x dx}{1+x^2} = \frac{\pi^2}{12}.$$

BI ((138))(3)

$$17. \int_0^{\infty} \ln \frac{a^2 + b^2 x^2}{x^2} \frac{dx}{c^2 + g^2 x^2} = \frac{\pi}{cg} \ln \frac{ag + bc}{c} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

BI ((138))(6, 7, 9, 10)a

$$18. \int_0^{\infty} \ln \frac{a^2 + b^2 x^2}{x^2} \frac{dx}{c^2 - g^2 x^2} = \frac{1}{cg} \operatorname{arctg} \frac{ag}{bc} \quad [a > 0, \quad b > 0, \quad c > 0, \quad g > 0].$$

BI ((138))(8, 11)a

$$19. \int_0^{\infty} \ln \frac{1+x^2}{x^2} \frac{x^2 dx}{(1+x^2)^2} = \frac{\pi}{4} (\ln 4 - 1).$$

$$20. \int_0^1 \ln \left(\frac{1-x^2}{x^2} \right)^2 \sqrt{1-x^2} dx = \pi.$$

FI II 643a

$$21. \int_0^1 \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = \frac{1}{2} \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} \frac{dx}{x} = -\frac{t^2}{2} \quad [|t| < \pi].$$

BI ((115))(23), BI ((134))(15)

$$22. \int_0^\infty \ln \frac{1+2x \cos t + x^2}{(1+x)^2} x^{p-1} dx = -\frac{2\pi(1-\cos pt)}{p \sin p\pi} \quad [0 < |p| < 1, \quad |t| < \pi].$$

BI ((134))(17)

$$23. \int_0^1 \ln \frac{1+x^2 \sin t}{1-x^2 \sin t} \frac{dx}{\sqrt{1-x^2}} = \pi \ln \operatorname{ctg} \left(\frac{\pi-t}{4} \right) \quad [|t| < \pi].$$

GW ((325))(21d)

4.299

$$1. \int_0^\infty \ln \frac{(x+1)(x+a^2)}{(x+a)^2} \frac{dx}{x} = (\ln a)^2 \quad [a > 0].$$

BI ((134))(14)

$$2. \int_0^1 \ln \frac{(1-ax)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{2\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln(1+a) \quad [a > 0].$$

BI ((115))(25)

$$3. \int_0^1 \ln \frac{(1-a^2x^2)(1+ax^2)}{(1-ax^2)^2} \frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}} \operatorname{arctg} \sqrt{a} \ln(1+a) \quad [a > 0].$$

BI ((115))(26)

$$4. \int_0^1 \ln \frac{(x+1)(x+a^2)}{(x+a)^2} x^{\mu-1} dx = \frac{\pi(a^\mu - 1)^2}{\mu \sin \mu\pi} \quad [a > 0, \quad \operatorname{Re} \mu > 0].$$

BI ((134))(16)

4.311

$$1.^3 \int_0^{\infty} \ln(a^3 - x^3) \frac{dx}{x^3} \quad \text{integral divergent.}$$

BI ((134))(7)

596

$$2. \int_0^{\infty} \ln(1 + x^3) \frac{dx}{1 - x + x^2} = \frac{2\pi}{\sqrt{3}} \ln 3.$$

LI ((136))(8)

$$3. \int_0^{\infty} \ln(1 + x^3) \frac{dx}{1 + x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9}.$$

LI ((136))(6)

$$4. \int_0^{\infty} \ln(1 + x^3) \frac{x dx}{1 + x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9}.$$

LI ((136))(7)

$$5. \int_0^{\infty} \ln(1 + x^3) \frac{1 - x}{1 + x^3} dx = -\frac{2}{9} \pi^2.$$

BI ((136))(9)

$$6.^8 \int_0^{\infty} \ln \left| 1 - \frac{n^3}{a^3} \right| \frac{dx}{x^3} = -\frac{\pi\sqrt{3}}{6a^2}$$

4.312

$$1. \int_0^{\infty} \ln \frac{1 + x^3}{x^3} \frac{dx}{1 + x^3} = \frac{\pi}{\sqrt{3}} \ln 3 + \frac{\pi^2}{9}.$$

BI ((138))(12)

$$2. \int_0^{\infty} \ln \frac{1 + x^3}{x^3} \frac{x dx}{1 + x^3} = \frac{\pi}{\sqrt{3}} \ln 3 - \frac{\pi^2}{9}.$$

4.313

$$1. \int_0^{\infty} \ln x \ln(1 + a^2 x^2) \frac{dx}{x^2} = \pi a(1 - \ln a) \quad [a > 0].$$

BI ((134))(18)

$$2. \int_0^{\infty} \ln(1 + c^2 x^2) \ln(a^2 + b^2 x^2) \frac{dx}{x^2} = 2\pi \left[\left(c + \frac{b}{a} \right) \ln(b + ac) - \frac{b}{a} \ln b - c \ln c \right] \\ [a > 0, \quad b > 0, \quad c > 0].$$

BI ((134))(20, 21)a

$$3. \int_0^{\infty} \ln(1 + c^2 x^2) \ln \left(a^2 + \frac{b^2}{x^2} \right) \frac{dx}{x^2} = 2\pi \left[\frac{a + bc}{b} \ln(a + bc) - \frac{a}{b} \ln a - c \right] \\ [a > 0, \quad a + bc > 0].$$

BI ((134))(22, 23)a

$$4. \int_0^{\infty} \ln x \ln \frac{1 + a^2 x^2}{1 + b^2 x^2} \frac{dx}{x^2} = \pi(a - b) + \pi \ln \frac{b^b}{a^a} \quad [a > 0, \quad b > 0].$$

BI ((134))(24)

$$5. \int_0^{\infty} \ln x \ln \frac{a^2 + 2bx + x^2}{a^2 - 2bx + x^2} \frac{dx}{x} = 2\pi \ln a \arcsin \frac{b}{a} \quad [a \geq |b|].$$

BI ((134))(25)

$$6. \int_0^{\infty} \ln(1 + x) \frac{x \ln x - x - a}{(x + a)^2} \frac{dx}{x} = \frac{(\ln a)^2}{2(a - 1)} \quad [a > 0].$$

BI ((141))(7)

$$7. \int_0^{\infty} \ln(1 - x)^2 \frac{x \ln x - x - a}{(x + a)^2} \frac{dx}{x} = \frac{\pi^2 + (\ln a)^2}{1 + a} \quad [a > 0].$$

LI ((141))(8)

4.314

$$1. \int_0^1 \ln(1 + ax) \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=1}^{\infty} \frac{a^k}{k} \ln \frac{p+k}{q+k} + \ln \frac{p}{q} \quad [a > 0, \quad p > 0, \quad q > 0].$$

$$2. \int_0^\infty \left[\frac{(q-1)x}{(1+x)^2} - \frac{1}{x+1} + \frac{1}{(1+x)^q} \right] \frac{dx}{x \ln(1+x)} = \ln \Gamma(q) \quad [q > 0].$$

BI ((143))(7)

$$3. \int_0^1 \frac{x \ln x + 1 - x}{x(\ln x)^2} \ln(1+x) dx = \ln \frac{4}{\pi}.$$

BI ((126))(12)

$$4. \int_0^1 \frac{\ln(1-x^2) dx}{x(q^2 + (\ln x)^2)} = -\frac{\pi}{q} \ln \Gamma\left(\frac{q+\pi}{\pi}\right) + \frac{\pi}{2q} \ln 2q + \ln \frac{q}{\pi} - 1 \quad [q > 0].$$

LI ((327))(12)a

4.315

$$1. \int_0^1 \ln(1+x)(\ln x)^{n-1} \frac{dx}{x} = (-1)^{n-1} (n-1)! \left(1 - \frac{1}{2^n}\right) \zeta(n+1).$$

BI ((116))(3)

$$2. \int_0^1 \ln(1+x)(\ln x)^{2n} \frac{dx}{x} = \frac{2^{2n+1} - 1}{(2n+1)(2n+2)} \pi^{2n+2} |B_{2n+2}|.$$

BI ((116))(1)

$$3. \int_0^1 \ln(1-x)(\ln x)^{n-1} \frac{dx}{x} = (-1)^n (n-1)! \zeta(n+1).$$

BI ((116))(4)

$$4. \int_0^1 \ln(1-x)(\ln x)^{2n} \frac{dx}{x} = -\frac{2^{2n}}{(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|.$$

BI ((116))(2)

4.316

$$1. \int_0^1 \ln(1-ax^r) \left(\ln \frac{1}{x}\right)^p \frac{dx}{x} = -\frac{1}{r^{p+1}} \Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k}{k^{p+2}} \quad [p > -1, \quad a < 1, \quad r > 0].$$

$$2. \int_0^1 \ln(1 - 2ax \cos t + a^2 x^2) \left(\ln \frac{1}{x} \right)^p \frac{dx}{x} = -2\Gamma(p+1) \sum_{k=1}^{\infty} \frac{a^k \cos kt}{k^{p+2}}.$$

LI ((116))(8)

4.317

$$1. \int_0^{\infty} \ln \frac{\sqrt{1+x^2} + a}{\sqrt{1+x^2} - a} \frac{dx}{\sqrt{1+x^2}} = \pi \arcsin a \quad [|a| < 1].$$

BI ((142))(11)

$$2. \int_0^1 \ln \frac{\sqrt{1-a^2x^2} - x\sqrt{1-a^2}}{1-x} \frac{dx}{x} = \frac{1}{2}(\arcsin a)^2.$$

BI ((115))(32)

$$3. \int_0^1 \ln \frac{1 + \cos t \sqrt{1-x^2}}{1 - \cos t \sqrt{1-x^2}} \frac{dx}{x^2 + \operatorname{tg}^2 v} = \pi \operatorname{ctg} t \frac{\cos \frac{v-t}{2}}{\sin \frac{v+t}{2}}.$$

BI ((115))(30)

$$4. \int_0^1 \ln \left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \frac{x dx}{1-x^2} = \frac{\pi^2}{2}.$$

BI ((115))(31)

598

$$5. \int_0^1 \ln\{\sqrt{1+kx} + \sqrt{1-kx}\} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{\pi}{8} \mathbf{K}(k').$$

BI ((121))(8)

$$6. \int_0^1 \ln\{\sqrt{1+kx} - \sqrt{1-kx}\} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{4} \ln(4k) \mathbf{K}(k) + \frac{3}{8} \pi \mathbf{K}(k').$$

BI ((121))(9)

$$7. \int_0^1 \ln\{1 + \sqrt{1-k^2x^2}\} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) + \frac{\pi}{4} \mathbf{K}(k').$$

$$8. \int_0^1 \ln\{1 - \sqrt{1 - k^2 x^2}\} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = \frac{1}{2} \ln k \mathbf{K}(k) - \frac{3}{4} \pi \mathbf{K}(k').$$

BI ((121))(7)

$$9. \int_0^1 \ln \frac{1 + p\sqrt{1 - x^2}}{1 - p\sqrt{1 - x^2}} \frac{dx}{1 - x} = \pi \arcsin p \quad [p^2 < 1].$$

BI ((115))(29)

$$10. \int_0^1 \ln \frac{1 + q\sqrt{1 - k^2 x^2}}{1 - q\sqrt{1 - k^2 x^2}} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = \pi F(\arcsin q, k') \quad [q^2 < 1].$$

BI ((122))(15)

$$11.* \int_{-\infty}^{\infty} \ln \left| \frac{1 + 2\sqrt{1 + x^2}}{1 - 2\sqrt{1 + x^2}} \right| \frac{dx}{\sqrt{1 + x^2}} = \pi^2/3$$

4.318

$$1. \int_0^1 \frac{\ln(1 - x^q)}{1 + (\ln x)^2} \frac{dx}{x} = \pi \left[\ln \Gamma \left(\frac{q}{2\pi} + 1 \right) - \frac{\ln q}{2} + \frac{q}{2\pi} \left(\ln \frac{q}{2\pi} - 1 \right) \right] \quad [q > 0].$$

BI ((126))(11)

$$2. \int_0^{\infty} \ln(1 + x^r) \left[\frac{(p - r)x^p - (q - r)x^q}{\ln x} + \frac{x^q - x^p}{(\ln x)^2} \right] \frac{dx}{x^{r+1}} = r \ln \left(\operatorname{tg} \frac{q\pi}{2r} \operatorname{ctg} \frac{p\pi}{2r} \right) \quad [p < r, q < r].$$

BI ((143))(9)

In integrals containing $\ln(a + bx^r)$, it is useful to make the substitution $x^r = t$ and then to seek the resulting integral in the tables.

For example,

$$\int_0^{\infty} x^{p-1} \ln(1 + x^r) dx = \frac{1}{r} \int_0^{\infty} t^{\frac{p}{r}-1} \ln(1 + t) dt = \frac{\pi}{p \sin \frac{p\pi}{r}} \quad (\text{see 4.293 3}).$$

4.319

$$1. \int_0^{\infty} \ln(1-e^{-2a\pi x}) \frac{dx}{1+x^2} = -\pi \left[\frac{1}{2} \ln 2a\pi + a(\ln a - 1) - \ln \Gamma(a+1) \right] \quad [a > 0].$$

BI ((354))(6)

$$2. \int_0^{\infty} \ln(1+e^{-2a\pi x}) \frac{dx}{1+x^2} = \pi \left[\ln \Gamma(2a) - \ln \Gamma(a) + a(1 - \ln a) - \left(2a - \frac{1}{2}\right) \ln 2 \right] \quad [a > 0].$$

BI ((354))(7)

599

$$3. \int_0^{\infty} \ln \frac{a+be^{-px}}{a+be^{-qx}} \frac{dx}{x} = \ln \frac{a}{a+b} \ln \frac{p}{q} \quad \left[\frac{b}{a} > -1, \quad pq > 0 \right].$$

FI II 635, BI ((354))(1)

4.321

$$1. \int_{-\infty}^{\infty} x \ln \operatorname{ch} x \, dx = 0.$$

BI ((358))(2)a

$$2. \int_0^{\infty} \ln \operatorname{ch} x \frac{dx}{1-x^2} = 0.$$

BI ((138))(20)a

4.322

$$1.^7 \int_0^{\pi} x \ln \sin x \, dx = \frac{1}{2} \int_0^{\pi} x \ln \cos^2 x \, dx = -\frac{\pi^2}{2} \ln 2.$$

BI ((432))(1, 2) FI II 643

$$2. \int_0^{\infty} \frac{\ln \sin^2 ax}{b^2+x^2} dx = \frac{\pi}{b} \ln \frac{1-e^{-2ab}}{2} \quad [a > 0, \quad b > 0].$$

GW ((338))(28b)

$$4. \int_0^{\infty} \frac{\ln \sin^2 ax}{b^2 - x^2} dx = -\frac{\pi^2}{2b} + a\pi \quad [a > 0, \quad b > 0].$$

BI ((418))(1)

$$5. \int_0^{\infty} \frac{\ln \cos^2 ax}{b^2 - x^2} dx = a\pi \quad [a > 0].$$

BI ((418))(2)

$$6. \int_0^{\infty} \frac{\ln \cos^2 x}{x^2} dx = -\pi.$$

FI II 686

$$7.7 \int_0^{\frac{\pi}{4}} x^{\mu-1} \ln \sin x dx = -\frac{1}{2\mu} \left(\frac{\pi}{4}\right)^{\mu} \left[\ln 2 + \frac{2}{\mu} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)} \right] \quad [\operatorname{Re} \mu > 0].$$

LI ((425))(1)

$$8.7 \int_0^{\frac{\pi}{2}} x^{\mu-1} \ln \sin x dx = -\frac{1}{\mu} \left(\frac{\pi}{2}\right)^{\mu} \left[\frac{1}{\mu} - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(\mu + 2k)} \right] \quad [\operatorname{Re} \mu > 0].$$

LI ((430))(1)

$$9. \int_0^{\frac{\pi}{2}} \ln(1 - \cos x) x^{\mu-1} dx = \frac{-1}{\mu} \left(\frac{\pi}{2}\right)^{\mu} \left[\frac{2}{\mu} - \sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^{2k-1}(\mu + 2k)} \right] \quad [\operatorname{Re} \mu > 0].$$

LI ((430))(2)

$$10. \int_0^{\infty} \ln(1 \pm 2p \cos \beta x + p^2) \frac{dx}{q^2 + x^2} = \frac{\pi}{q} \ln(1 \pm pe^{-\beta q}) \quad [p^2 < 1];$$

$$= \frac{\pi}{q} \ln(p \pm e^{-\beta q}) \quad [p^2 > 1]$$

FI II 718a

4.323

$$1.^7 \int_0^\pi x \ln \operatorname{tg}^2 x \, dx = 0.$$

BI ((432))(3)

600

$$2. \int_0^\infty \frac{\ln \operatorname{tg}^2 ax}{b^2 + x^2} \, dx = \frac{\pi}{b} \ln \theta ab \quad [a > 0, \quad b > 0].$$

GW ((338))(28c)

$$3. \int_0^\infty \ln \left(\frac{1 + \operatorname{tg} x}{1 - \operatorname{tg} x} \right)^2 \frac{dx}{x} = \frac{\pi^2}{2}.$$

GW ((338))(26)

4.324

$$1. \int_0^\infty \ln \left(\frac{1 + \sin x}{1 - \sin x} \right)^2 \frac{dx}{x} = \pi^2.$$

GW ((338))(25)

$$2. \int_0^\infty \ln \frac{1 + 2a \cos px + a^2}{1 + 2a \cos qx + a^2} \frac{dx}{x} = \ln(1+a) \ln \frac{q^2}{p^2} \quad [-1 < a \leq 1];$$

$$= \ln \left(1 + \frac{1}{a} \right) \ln \frac{q^2}{p^2} \quad [a < -1 \quad \text{or} \quad a \geq 1].$$

GW ((338))(27)

$$3. \int_0^\infty \ln(a^2 \sin^2 px + b^2 \cos^2 px) \frac{dx}{c^2 + x^2} = \frac{\pi}{c} [\ln(a \operatorname{sh} cp + b \operatorname{ch} cp) - cp]$$

$$[a > 0, \quad b > 0, \quad c > 0, \quad p > 0].$$

GW ((338))(29)

4.325

$$1.^3 \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x} = -C \ln 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\ln k}{k} = -C \ln 2 + 0.159\,868\,905 \dots = -\frac{1}{2} (\ln 2)^2.$$

$$2. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{x + e^{i\lambda}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-ik\lambda} (\mathbf{C} + \ln k).$$

GW ((325))(26)

$$3. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{(1+x)^2} = \int_1^{\infty} \ln \ln x \frac{dx}{(1+x)^2} = \frac{1}{2} \left[\psi \left(\frac{1}{2} \right) + \ln 2\pi \right] = \frac{1}{2} \left(\ln \frac{\pi}{2} - \mathbf{C} \right).$$

BI ((147))(7)

$$4. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x^2} = \int_1^{\infty} \ln \ln x \frac{dx}{1+x^2} = \frac{\pi}{2} \ln \frac{\sqrt{2\pi} \Gamma \left(\frac{3}{4} \right)}{\Gamma \left(\frac{1}{4} \right)}.$$

BI ((148))(1)

$$5. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+x+x^2} = \int_1^{\infty} \ln \ln x \frac{dx}{1+x+x^2} = \frac{\pi}{\sqrt{3}} \ln \frac{\sqrt[3]{2\pi} \Gamma \left(\frac{2}{3} \right)}{\Gamma \left(\frac{1}{3} \right)}.$$

BI ((148))(2)

$$6. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1-x+x^2} = \int_1^{\infty} \ln \ln x \frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}} \left[\frac{5}{6} \ln 2\pi - \ln \Gamma \left(\frac{1}{6} \right) \right].$$

BI ((148))(5)

601

$$7. \int_0^1 \ln \ln \left(\frac{1}{x} \right) \frac{dx}{1+2x \cos t + x^2} = \int_1^{\infty} \ln \ln x \frac{dx}{1+2x \cos t + x^2} = \frac{\pi}{2 \sin t} \ln \frac{(2\pi)^{t/\pi} \Gamma \left(\frac{1}{2} + \frac{t}{2\pi} \right)}{\Gamma \left(\frac{1}{2} - \frac{t}{2\pi} \right)}.$$

BI ((147))(9)

$$8. \int_0^1 \ln \ln \frac{1}{x} x^{\mu-1} dx = -\frac{1}{\mu} (\mathbf{C} + \ln \mu) \quad [\operatorname{Re} \mu > 0].$$

$$\begin{aligned}
9. \int_1^\infty \ln \ln x \frac{x^{n-2} dx}{1+x^2+x^4+\dots+x^{2n-2}} &= \\
&= \frac{\pi}{2n} \operatorname{tg} \frac{\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \quad [n \text{ is even}]. \\
&= \frac{\pi}{2n} \operatorname{tg} \frac{\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{k\pi}{n} \ln \frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \quad [n \text{ is odd}].
\end{aligned}$$

BI ((148))(4)

$$\begin{aligned}
10. \int_0^1 \ln \ln \left(\frac{1}{x}\right) \frac{dx}{(1+x^2)\sqrt{\ln \frac{1}{x}}} &= \int_1^\infty \ln \ln x \frac{dx}{(1+x^2)\sqrt{\ln x}} = \\
&= \sqrt{\pi} \sum_{k=0}^\infty \frac{(-1)^{k+1}}{\sqrt{2k+1}} [\ln(2k+1) + 2 \ln 2 + \mathbf{C}].
\end{aligned}$$

BI ((147))(4)

$$11. \int_0^1 \ln \ln \left(\frac{1}{x}\right) \frac{x^{\mu-1} dx}{\sqrt{\ln \frac{1}{x}}} = -(\mathbf{C} + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0].$$

BI ((147))(3)

$$12. \int_0^1 \ln \ln \left(\frac{1}{x}\right) \left(\ln \frac{1}{x}\right)^{\mu-1} x^{\nu-1} dx = \frac{1}{\nu^\mu} \Gamma(\mu) [\psi(\mu) - \ln(\nu)] \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

BI ((147))(2)

4.326

$$1. \int_0^1 \ln(a - \ln x) x^{\mu-1} dx = \frac{1}{\mu} [\ln a - e^{a\mu} \operatorname{Ei}(-a\mu)] \quad [\operatorname{Re} \mu > 0, a > 0].$$

BI ((107))(23)

$$2. \int_0^{\frac{1}{e}} \ln \left(2 \ln \frac{1}{x} - 1\right) \frac{x^{2\mu-1}}{\ln x} dx = -\frac{1}{2} [\operatorname{Ei}(-\mu)]^2 \quad [\operatorname{Re} \mu > 0].$$

602
4.327

$$1. \int_0^1 \ln[a^2 + (\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma\left(\frac{2a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2a+\pi}{4\pi}\right)} + \frac{\pi}{2} \ln \frac{\pi}{2} \quad \left[a > -\frac{\pi}{2}\right].$$

BI ((147))(10)

$$2. \int_0^1 \ln[a^2 + 4(\ln x)^2] \frac{dx}{1+x^2} = \pi \ln \frac{2\Gamma\left(\frac{a+3\pi}{4\pi}\right)}{\Gamma\left(\frac{a+\pi}{4\pi}\right)} + \frac{\pi}{2} \ln \pi \quad [a > -\pi].$$

BI ((147))(16)a

$$3. \int_0^\infty \ln[a^2 + (\ln x)^2] x^{\mu-1} dx = \frac{2}{\mu} [-\cos a\mu \operatorname{ci}(a\mu) - \sin a\mu \operatorname{si}(a\mu) + \ln a] \\ [a > 0, \quad \operatorname{Re} \mu > 0].$$

GW ((325))(28)

If the integrand contains a logarithm whose argument also contains a logarithm, for example, if the integrand contains $\ln \ln \frac{1}{x}$, it is useful to make the substitution $\ln x = t$ and then seek the transformed integral in the tables.

4.33- 4.34 Combinations of logarithms and exponentials

4.331

$$1. \int_0^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} (\mathbf{C} + \ln \mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((256))(2)

$$2. \int_1^\infty e^{-\mu x} \ln x dx = -\frac{1}{\mu} \operatorname{Ei}(-\mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((260))(5)

4.332

$$1. \int_0^{\infty} \frac{\ln x \, dx}{e^x + e^{-x} - 1} = \frac{2\pi}{\sqrt{3}} \left[\frac{5}{6} \ln 2\pi - \ln \Gamma \left(\frac{1}{6} \right) \right] \quad (\text{cf. 4.325 6.}).$$

4.325
BI ((257))(6)

$$2. \int_0^{\infty} \frac{\ln x \, dx}{e^x + e^{-x} + 1} = \frac{\pi}{\sqrt{3}} \ln \left[\frac{\Gamma \left(\frac{2}{3} \right)}{\Gamma \left(\frac{1}{3} \right)} \sqrt{2\pi} \right] \quad (\text{cf. 4.325 5.}).$$

4.325
BI ((257))(7)A, LI ((260))(3)

4.333

$$\int_0^{\infty} e^{-\mu x^2} \ln x \, dx = -\frac{1}{4} (\mathbf{C} + \ln 4\mu) \sqrt{\frac{\pi}{\mu}} \quad [\text{Re } \mu > 0].$$

BI ((256))(8), FI II 807a

4.334

$$\int_0^{\infty} \frac{\ln x \, dx}{e^{x^2} + 1 + e^{-x^2}} = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{k=1}^{\infty} (-1)^k \frac{\mathbf{C} + \ln 4k}{\sqrt{k}} \sin \frac{k\pi}{3}.$$

BI ((357))(13)

603

4.335

$$1. \int_0^{\infty} e^{-\mu x} (\ln x)^2 \, dx = \frac{1}{\mu} \left[\frac{\pi^2}{6} + (\mathbf{C} + \ln \mu)^2 \right] \quad [\text{Re } \mu > 0].$$

ET I 149(13)

$$2. \int_0^{\infty} e^{-x^2} (\ln x)^2 \, dx = \frac{\sqrt{\pi}}{8} \left[(\mathbf{C} + 2 \ln 2)^2 + \frac{\pi^2}{2} \right].$$

$$3.7 \int_0^{\infty} e^{-\mu x} (\ln x)^3 dx = -\frac{1}{\mu} \left[(\mathbf{C} + \ln \mu)^3 + \frac{\pi^2}{2} (\mathbf{C} + \ln \mu) + 2\zeta(3) \right].$$

MI 26

4.336

$$1.6 \int_0^{\infty} \frac{e^{-x}}{\ln x} dx = -0.154479567.$$

BI ((260))(9)

$$2. \int_0^{\infty} \frac{e^{-\mu x} dx}{\pi^2 + (\ln x)^2} = \nu'(\mu) - e^{\mu} \quad [\operatorname{Re} \mu > 0].$$

MI 26

4.337

$$1. \int_0^{\infty} e^{-\mu x} \ln(\beta + x) dx = \frac{1}{\mu} [\ln \beta - e^{\mu\beta} \operatorname{Ei}(-\beta\mu)] \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

BI ((256))(3)

$$2. \int_0^{\infty} e^{-\mu x} \ln(1 + \beta x) dx = -\frac{1}{\mu} e^{\frac{\mu}{\beta}} \operatorname{Ei}\left(-\frac{\mu}{\beta}\right) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET I 148(4)

$$3. \int_0^{\infty} e^{-\mu x} \ln|a - x| dx = \frac{1}{\mu} [\ln a - e^{-a\mu} \operatorname{Ei}(a\mu)] \quad [a > 0, \operatorname{Re} \mu > 0].$$

BI ((256))(4)

$$4.7 \int_0^{\infty} e^{-\mu x} \ln \left| \frac{\beta}{\beta - x} \right| dx = \frac{1}{\mu} [e^{-\beta\mu} \operatorname{Ei}(\beta\mu)] \quad [\operatorname{Re} \mu > 0].$$

MI 26

4.338

$$1. \int_0^{\infty} e^{-\mu x} \ln(\beta^2 + x^2) dx = \frac{2}{\mu} [\ln \beta - \operatorname{ci}(\beta\mu) \cos(\beta\mu) - \operatorname{si}(\beta\mu) \sin(\beta\mu)]$$

$$[\operatorname{Re} \beta > 0, \operatorname{Re} \mu > 0].$$

$$2. \int_0^{\infty} e^{-\mu x} \ln(x^2 - \beta^2)^2 dx = \frac{2}{\mu} [\ln \beta^2 - e^{\beta\mu} \text{Ei}(-\beta\mu) - e^{\beta\mu} \text{Ei}(\beta\mu)]$$

[Im $\beta > 0$, Re $\mu > 0$].

BI ((256))(5)

4.339

$$\int_0^{\infty} e^{-\mu x} \ln \left| \frac{x+1}{x-1} \right| dx = \frac{1}{\mu} [e^{-\mu} (\ln 2\mu + \gamma) - e^{\mu} \text{Ei}(-2\mu)] \quad [\text{Re } \mu > 0].$$

MI 27

604

4.341

$$\int_0^{\infty} e^{-\mu x} \ln \frac{\sqrt{x+ai} + \sqrt{x-ai}}{\sqrt{2a}} dx = \frac{\pi}{4\mu} [\mathbf{H}_0(a\mu) - N_0(a\mu)] \quad [a > 0, \text{Re } \mu > 0].$$

ET I 149(20)

4.342

$$1. \int_0^{\infty} e^{-2nx} \ln(\text{sh } x) dx = \frac{1}{2n} \left[\frac{1}{n} + \ln 2 - 2\beta(2n+1) \right].$$

BI ((256))(17)

$$2. \int_0^{\infty} e^{-\mu x} \ln(\text{ch } x) dx = \frac{1}{\mu} \left[\beta \left(\frac{\mu}{2} \right) - \frac{1}{\mu} \right] \quad [\text{Re } \mu > 0].$$

ET I 165(32)

$$3. \int_0^{\infty} e^{-\mu x} [\ln(\text{sh } x) - \ln x] dx = \frac{1}{\mu} \left[\ln \frac{\mu}{2} - \frac{1}{2\mu} - \psi \left(\frac{\mu}{2} \right) \right] \quad [\text{Re } \mu > 0].$$

ET I 165(33)

4.343

$$\int_0^{\pi} e^{\mu \cos x} [\ln(2\mu \sin^2 x) + \mathbf{C}] dx = -\pi K_0(\mu).$$

WA 95(16)

4.35- 4.36 Combinations of logarithms, exponentials, and powers

4.351

$$1. \int_0^1 (1-x)e^{-x} \ln x \, dx = \frac{1-e}{e}.$$

BI ((352))(1)

$$2. \int_0^1 e^{\mu x} (\mu x^2 + 2x) \ln x \, dx = \frac{1}{\mu^2} [(1-\mu)e^\mu - 1].$$

BI ((352))(2)

$$3. \int_1^\infty \frac{e^{-\mu x} \ln x}{1+x} \, dx = \frac{1}{2} e^\mu [\text{Ei}(-\mu)]^2 \quad [\text{Re } \mu > 0].$$

NT 32(10)

4.352

$$1. \int_0^\infty x^{\nu-1} e^{-\mu x} \ln x \, dx = \frac{1}{\mu^\nu} \Gamma(\nu) [\psi(\nu) - \ln \mu] \quad [\text{Re } \mu > 0, \text{ Re } \nu > 0].$$

BI ((353))(3), ET I 315(10)a

$$2. \int_0^\infty x^n e^{-\mu x} \ln x \, dx = \frac{n!}{\mu^{n+1}} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \mathbf{C} - \ln \mu \right] \quad [\text{Re } \mu > 0].$$

ET I 148(7)

$$3. \int_0^\infty x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \sqrt{\pi} \frac{(2n-1)!!}{2^n \mu^{n+\frac{1}{2}}} \left[2 \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right) - \mathbf{C} - \ln 4\mu \right] \\ [\text{Re } \mu > 0].$$

ET I 148(10)

$$4. \int_0^\infty x^{\mu-1} e^{-x} \ln x \, dx = \Gamma'(\mu) \quad [\text{Re } \mu > 0].$$

GW ((324))(83a)

4.353

$$1. \int_0^{\infty} (x - \nu)x^{\nu-1} e^{-x} \ln x \, dx = \Gamma(\nu) \quad [\operatorname{Re} \nu > 0].$$

GW ((324))(84)

605

$$2. \int_0^{\infty} \left(\mu x - n - \frac{1}{2} \right) x^{n-\frac{1}{2}} e^{-\mu x} \ln x \, dx = \frac{(2n-1)!!}{(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0].$$

BI ((357))(2)

$$3. \int_0^1 (\mu x + n + 1)x^n e^{\mu x} \ln x \, dx = e^{\mu} \sum_{k=0}^n (-1)^{k-1} \frac{n!}{(n-k)! \mu^{k+1}} + (-1)^n \frac{n!}{\mu^{n+1}} \quad [\mu \neq 0].$$

GW ((324))(82)

4.354

$$1.^6 \int_0^{\infty} \frac{x^{\nu-1} \ln x}{e^x + 1} \, dx = \Gamma(\nu) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\nu}} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0].$$

$$= -\frac{1}{2} (\ln 2)^2 \quad \text{for } \nu = 1.$$

GW ((324))(86a)

$$2. \int_0^{\infty} \frac{x^{\nu-1} \ln x}{(e^x + 1)^2} \, dx = \Gamma(\nu) \sum_{k=2}^{\infty} \frac{(-1)^k (k-1)}{k^{\nu}} [\psi(\nu) - \ln k] \quad [\operatorname{Re} \nu > 0].$$

GW ((324))(86b)

$$3. \int_0^{\infty} \frac{(x - \nu)e^x - \nu}{(e^x + 1)^2} x^{\nu-1} \ln x \, dx = \Gamma(\nu) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{\nu}} \quad [\operatorname{Re} \nu > 0].$$

GW ((324))(87a)

$$4. \int_0^{\infty} \frac{(x - 2n)e^x - 2n}{(e^x + 1)^2} x^{2n-1} \ln x \, dx = \frac{2^{2n-1} - 1}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

GW ((324))(87b)

4.355

$$1. \int_0^{\infty} x^2 e^{-\mu x^2} \ln x \, dx = \frac{1}{8\mu} (2 - \ln 4\mu - C) \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0].$$

BI ((357))(1a)

$$2. \int_0^{\infty} x(\mu x^2 - \nu x - 1) e^{-\mu x^2 + 2\nu x} \ln x \, dx = \frac{1}{4\mu} + \frac{\nu}{4\mu} \sqrt{\frac{\pi}{\mu}} \exp\left(\frac{\nu^2}{\mu}\right) \left[1 + \Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right] \\ [\operatorname{Re} \mu > 0].$$

BI ((358))(1)

$$3. \int_0^{\infty} (\mu x^2 - n) x^{2n-1} e^{-\mu x^2} \ln x \, dx = \frac{(n-1)!}{4\mu^n} \quad [\operatorname{Re} \mu > 0].$$

BI ((353))(4)

$$4. \int_0^{\infty} (2\mu x^2 - 2n - 1) x^{2n} e^{-\mu x^2} \ln x \, dx = \frac{(2n-1)!!}{2(2\mu)^n} \sqrt{\frac{\pi}{\mu}} \quad [\operatorname{Re} \mu > 0].$$

BI ((353))(5)

4.356

$$1. \int_0^{\infty} \exp\left[-\mu\left(\frac{x}{a} + \frac{a}{x}\right)\right] \ln x \frac{dx}{x} = 2 \ln a K_0(2\mu) \quad [a > 0, \operatorname{Re} \mu > 0].$$

GW ((324))(91)

606

$$2. \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n+1)x - 2b] x^{n-\frac{1}{2}} \, dx = \\ = 2 \left(\frac{b}{a}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k)!}{(n-k)!(2k)!(2\sqrt{ab})^k} \quad [a > 0, b > 0].$$

BI ((357))(4)

$$3. \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 + (2n-1)x - 2b] \frac{dx}{x^{n+\frac{3}{2}}} = \\ = 2 \left(\frac{a}{b}\right)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-k-1)!(2k)!(2\sqrt{ab})^k} \quad [a > 0, b > 0].$$

For $n = \frac{1}{2}$:

$$4. \int_0^{\infty} \exp\left(-ax - \frac{a}{x}\right) \ln x \frac{ax^2 - b}{x^2} dx = 2K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0].$$

GW ((324))(92c)

For $n = 0$:

$$5. \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - x - 2b}{x\sqrt{x}} dx = 2\sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}. \quad [a > 0, \quad b > 0].$$

BI ((357))(7), GW((324))(92a)

For $n = -1$:

$$6. \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x \frac{2ax^2 - 3x - 2b}{\sqrt{x}} dx = \frac{1 + 2\sqrt{ab}}{a} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \\ [a > 0, \quad b > 0].$$

LI ((357))(6), GW ((324))(92b)

$$7.* \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x \left(a - \frac{b}{x^2}\right) dx = K_0(2\sqrt{ab}) \quad [a > 0, \quad b > 0].$$

$$8.* \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln x [2ax^2 - (2n + 1)x - 2b] x^{n-\frac{3}{2}} dx = \\ = 4(b/a)^{(2n+1)/4} K_{n+\frac{1}{2}}(2\sqrt{ab}) = \\ = 2(b/a)^{\frac{n}{2}} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(2k)!(2\sqrt{ab})^k} \quad [n = 0, 1, \dots, a > 0, \quad b > 0].$$

$$9.* \int_0^{\infty} \exp\left(-ax - \frac{b}{x}\right) \ln[(ax^2 - b) \cos(\alpha \ln x) + \alpha x \sin(\alpha \ln x)] \frac{dx}{x^2} = \\ = 2 \cos(\alpha \ln \sqrt{b/a}) K_{i\alpha}(2\sqrt{ab}) \quad [a > 0, \quad b > 0, \quad -\infty < \alpha < \infty].$$

$$11.* \int_0^{\infty} x^{\alpha} \ln x \left[a - \frac{\alpha}{x} - \frac{b}{x^2} \right] \exp \left(-ax - \frac{b}{x} \right) dx = 2 \left(\frac{b}{a} \right)^{\alpha/2} K_{\alpha}(2\sqrt{ab})$$

$$[a > 0, \quad b > 0, \quad -\infty < \alpha < \infty].$$

4.357

$$1. \int_0^{\infty} \exp \left(-\frac{1+x^4}{2ax^2} \right) \ln x \frac{1+ax^2-x^4}{x^2} dx = -\frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0].$$

BI ((357))(8)

$$2. \int_0^{\infty} \exp \left(-\frac{1+x^4}{2ax^2} \right) \ln x \frac{x^4+ax^2-1}{x^4} dx = \frac{\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0].$$

BI ((357))(9)

$$3. \int_0^{\infty} \exp \left(-\frac{1+x^4}{2ax^2} \right) \ln x \frac{x^4+3ax-1}{x^6} dx = \frac{(1+a)\sqrt{2a^3\pi}}{2\sqrt[4]{e}} \quad [a > 0].$$

BI ((357))(10)

4.358

$$1.^6 \int_1^{\infty} x^{\nu-1} e^{-\mu x} (\ln x)^m dx = \frac{\partial^m}{\partial \nu^m} \{ \mu^{-\nu} \Gamma(\nu, \mu) \} \quad [m = 0, 1, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

MI 26

$$2. \int_0^{\infty} x^{\nu-1} e^{-\mu x} (\ln x)^2 dx = \frac{\Gamma(\nu)}{\mu^{\nu}} \{ [\psi(\nu) - \ln \mu]^2 + \zeta(2, \nu) \} \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

MI 26

$$3.* \int_0^{\infty} x^{\nu-1} e^{-\mu x} (\ln x)^3 dx = \frac{\Gamma(\nu)}{\mu^{\nu}} \{ [\psi(\nu) - \ln \mu]^3 + 3\zeta(2, \nu)[\psi(\nu) - \ln \mu] - 2\zeta(3, \nu) \}$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

MI 26

4.359

$$1. \int_0^{\infty} e^{-\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \frac{1}{\mu} [\lambda(\mu, p-1) - \lambda(\mu, q-1)] \quad [\operatorname{Re} \mu > 0, \quad p > 0, \quad q > 0].$$

MI 27

$$2. \int_0^1 e^{\mu x} \frac{x^{p-1} - x^{q-1}}{\ln x} dx = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \ln \frac{p+k}{q+k} \quad [\operatorname{Re} \mu > 0, \quad p > 0, \quad q > 0].$$

BI ((352))(9)

608

4.361

$$1. \int_0^{\infty} \frac{(x+1)e^{-\mu x}}{\pi^2 + (\ln x)^2} dx = \nu'(\mu) - \nu''(\mu) \quad [\operatorname{Re} \mu > 0].$$

MI 27

$$2. \int_0^{\infty} \frac{e^{-\mu x} dx}{x[\pi^2 + (\ln x)^2]} = e^{\mu} - \nu(\mu) \quad [\operatorname{Re} \mu > 0].$$

MI 27

4.362

$$1. \int_0^1 x e^x \ln(1-x) dx = 1 - e.$$

BI ((352))(5)a

$$2. \int_1^{\infty} e^{-\mu x} \ln(2x-1) \frac{dx}{x} = \frac{1}{2} \left[\operatorname{Ei} \left(-\frac{\mu}{2} \right) \right]^2 \quad [\operatorname{Re} \mu > 0].$$

ET I 148(8)

4.363

$$1. \int_0^{\infty} e^{-\mu x} \ln(a+x) \frac{\mu(x+a) \ln(x+a) - 2}{x+a} dx = \frac{1}{4} \int_0^{\infty} e^{-\mu x} \ln(a-x)^2 \frac{\mu(x-a) \ln(x-a)^2 - 4}{x-a} dx = (\ln a)^2 \quad [\operatorname{Re} \mu > 0, \quad a > 0].$$

$$2. \int_0^1 x(1-x)(2-x)e^{-(1-x)^2} \ln(1-x) dx = \frac{1-e}{4e}.$$

BI ((352))(4)

4.364

$$1. \int_0^\infty e^{-\mu x} \ln[(x+a)(x+b)] \frac{dx}{x+a+b} = e^{(a+b)\mu} \{ \text{Ei}(-a\mu) \text{Ei}(-b\mu) - \ln(ab) \text{Ei}[-(a+b)\mu] \} \quad [a > 0, \quad b > 0, \quad \text{Re } \mu > 0].$$

BI ((354))(11)

$$2. \int_0^\infty e^{-\mu x} \ln(x+a+b) \left(\frac{1}{x+a} + \frac{1}{x+b} \right) dx = (1 + \ln a \ln b) \ln(a+b) + e^{-(a+b)\mu} \{ \text{Ei}(-a\mu) \text{Ei}(-b\mu) + (1 - \ln(ab)) \text{Ei}[-(a+b)\mu] \} \quad [a > 0, \quad b > 0, \quad \text{Re } \mu > 0].$$

BI ((354))(12)

4.365

$$\int_0^\infty \left[e^{-x} - \frac{x}{(1+x)^{p+1} \ln(1+x)} \right] \frac{dx}{x} = \ln p \quad [p > 0].$$

BI ((354))(15)

609

4.366

$$1. \int_0^\infty e^{-\mu x} \ln \left(1 + \frac{x^2}{a^2} \right) \frac{dx}{x} = [\text{ci}(a\mu)]^2 + [\text{si}(a\mu)]^2 \quad [\text{Re } \mu > 0].$$

NT 32(11)a

$$2. \int_0^\infty e^{-\mu x} \ln \left| 1 - \frac{x^2}{a^2} \right| \frac{dx}{x} = \text{Ei}(a\mu) \text{Ei}(-a\mu) \quad [\text{Re } \mu > 0].$$

ME 18

$$3. \int_0^\infty x e^{-\mu x^2} \ln \left| \frac{1+x^2}{1-x^2} \right| dx = \frac{1}{\mu} [\text{ch } \mu \text{ sh } i(\mu) - \text{sh } \mu \text{ ch } i(\mu)] \quad [\text{Re } \mu > 0]; \quad (\text{cf. 4.339}).$$

4.367

$$\int_0^{\infty} x e^{-\mu x^2} \ln \frac{x + \sqrt{x^2 + 2\beta}}{\sqrt{2\beta}} dx = \frac{e^{\beta\mu}}{4\mu} K_0(\beta\mu) \quad [|\arg \beta| < \pi, \operatorname{Re} \mu > 0].$$

ET I 149(19)

4.368

$$\int_0^{2u} e^{-\mu x^2} \ln \frac{x^2(4u^2 - x^2)}{u^4} \frac{dx}{\sqrt{4u^2 - x^2}} = \frac{\pi}{2} e^{-2u^2\mu} \left[\frac{\pi}{2} N_0(2iu^2\mu) - (C - \ln 2) J_0(2iu^2\mu) \right] \\ [\operatorname{Re} \mu > 0].$$

ET I 149(21)a

4.369

$$1. \int_0^{\infty} x^{\nu-1} e^{-\mu x} [\psi(\nu) - \ln x] dx = \frac{\Gamma(\nu) \ln \mu}{\mu^{\nu}} \quad [\operatorname{Re} \nu > 0].$$

ET I 149(12)

$$2. \int_0^{\infty} x^n e^{-\mu x} \left\{ \left[\ln x - \frac{1}{2} \psi(n+1) \right]^2 - \frac{1}{2} \psi'(n+1) \right\} dx = \\ = \frac{n!}{\mu^{n+1}} \left\{ \left[\ln \mu - \frac{1}{2} \psi(n+1) \right]^2 + \frac{1}{2} \psi'(n+1) \right\} \quad [\operatorname{Re} \mu > 0].$$

MI 26

4.37 Combinations of logarithms and hyperbolic functions

4.371

$$1. \int_0^{\infty} \frac{\ln x}{\operatorname{ch} x} dx = \pi \ln \left[\frac{\sqrt{2\pi} \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right].$$

LI ((260))(1)a

$$2. \int_0^{\infty} \frac{\ln x dx}{\operatorname{ch} x + \cos t} = \frac{\pi}{\sin t} \ln \frac{(2\pi)^{\frac{t}{\pi}} \Gamma\left(\frac{\pi+t}{2\pi}\right)}{\Gamma\left(\frac{\pi-t}{2\pi}\right)} \quad [t^2 < \pi^2].$$

$$3. \int_0^{\infty} \frac{\ln x \, dx}{\operatorname{ch}^2 x} = \psi\left(\frac{1}{2}\right) + \ln \pi = \ln \pi - 2 \ln 2 - \mathcal{C}.$$

BI ((257))(4)a

610

4.372

$$\begin{aligned} 1. \int_1^{\infty} \ln x \frac{\operatorname{sh} mx}{\operatorname{sh} nx} dx &= \\ &= \frac{\pi}{2n} \operatorname{tg} \frac{m\pi}{2n} \ln 2\pi + \frac{\pi}{n} \sum_{k=1}^{n-1} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n+k}{2n}\right)}{\Gamma\left(\frac{k}{2n}\right)} \quad [m+n \text{ is odd}]; \\ &= \frac{\pi}{2n} \operatorname{tg} \frac{m\pi}{2n} \ln \pi + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \sin \frac{km\pi}{n} \ln \frac{\Gamma\left(\frac{n-k}{n}\right)}{\Gamma\left(\frac{k}{n}\right)} \quad [m+n \text{ is even}]. \end{aligned}$$

BI ((148))(3)a

$$\begin{aligned} 2. \int_1^{\infty} \ln x \frac{\operatorname{ch} mx}{\operatorname{ch} nx} dx &= \\ &= \frac{\pi}{2n} \frac{\ln 2\pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^n (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n+2k-1}{4n}\right)}{\Gamma\left(\frac{2k-1}{4n}\right)} \quad [m+n \text{ is odd}]; \\ &= \frac{\pi}{2n} \frac{\ln \pi}{\cos \frac{m\pi}{2n}} + \frac{\pi}{n} \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k-1} \cos \frac{(2k-1)m\pi}{2n} \ln \frac{\Gamma\left(\frac{2n-2k+1}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \quad [m+n \text{ is even}]. \end{aligned}$$

BI ((148))(6)a

4.373

$$1. \int_0^{\infty} \frac{\ln(a^2 + x^2)}{\operatorname{ch} bx} dx = \frac{\pi}{b} \left[2 \ln \frac{2\Gamma\left(\frac{2ab+3\pi}{4\pi}\right)}{\Gamma\left(\frac{2ab+\pi}{4\pi}\right)} - \ln \frac{2b}{\pi} \right] \quad \left[b > 0, \quad a > -\frac{\pi}{2b} \right].$$

BI ((258))(11)a

$$2. \int_0^{\infty} \ln(1+x^2) \frac{dx}{\operatorname{ch} \frac{\pi x}{2}} = 2 \ln \frac{4}{\pi}.$$

$$3. \int_0^{\infty} \ln(a^2+x^2) \frac{\operatorname{sh}\left(\frac{2}{3}\pi x\right)}{\operatorname{sh}\pi x} dx = 2 \sin \frac{\pi}{3} \ln \frac{6\Gamma\left(\frac{a+4}{6}\right)\Gamma\left(\frac{a+5}{6}\right)}{\Gamma\left(\frac{a+1}{6}\right)\Gamma\left(\frac{a+2}{6}\right)} \quad [a > -1].$$

BI ((258))(12)

611

$$4. \int_0^{\infty} \ln(1+x^2) \frac{dx}{\operatorname{sh}^2 ax} = \frac{2}{a} \left[\ln \frac{a}{\pi} + \frac{\pi}{2a} - \psi\left(\frac{\pi+a}{\pi}\right) \right] \quad [a > 0].$$

BI ((258))(5)

$$5. \int_0^{\infty} \ln(1+x^2) \frac{\operatorname{ch}\frac{\pi}{2}x}{\operatorname{sh}^2\frac{\pi}{2}x} dx = \frac{2\pi-4}{\pi}.$$

BI ((258))(3)

$$6. \int_0^{\infty} \ln(1+x^2) \frac{\operatorname{ch}\frac{\pi}{4}x}{\operatorname{sh}^2\frac{\pi}{4}x} dx = 4\sqrt{2} - \frac{16}{\pi} + \frac{8\sqrt{2}}{\pi} \ln(\sqrt{2}+1).$$

BI ((258))(2)

4.374

$$1. \int_0^{\infty} \ln(\cos^2 t + e^{-2x} \sin^2 t) \frac{dx}{\operatorname{sh} x} = -2t^2.$$

BI ((259))(10)a

$$2. \int_0^{\infty} \ln(a+be^{-2x}) \frac{dx}{\operatorname{ch}^2 x} = \frac{2}{(b-a)} \left[\frac{a+b}{2} \ln(a+b) - a \ln a - b \ln b \right] \quad [a > 0, \quad a+b > 0].$$

LI ((259))(14)

4.375

$$1. \int_0^{\infty} \ln \operatorname{ch} \frac{x}{2} \frac{dx}{\operatorname{ch} x} = \mathbf{G} + \frac{\pi}{4} \ln 2.$$

$$2. \int_0^{\infty} \ln \operatorname{cth} x \frac{dx}{\operatorname{ch} x} = \frac{\pi}{2} \ln 2.$$

BI ((259))(16)

4.376

$$1. \int_0^{\infty} \frac{\ln x}{\sqrt{x} \operatorname{ch} x} dx = 2\sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k+1}} \{\ln(2k+1) + 2 \ln 2 + \mathbf{C}\}.$$

BI ((147))(4)

$$2. \int_0^{\infty} \ln x \frac{(\mu+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} x^{\mu} dx = 2\Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^{\mu+1}} \quad [\operatorname{Re} \mu > -1].$$

BI ((356))(10)

$$3. \int_0^{\infty} \ln x \frac{(n+1) \operatorname{ch} x - x \operatorname{sh} x}{\operatorname{ch}^2 x} x^n dx = \frac{(-1)^n}{2^n} \beta^{(n)} \left(\frac{1}{2} \right).$$

$$4. \int_0^{\infty} \ln 2x \frac{n \operatorname{sh} 2ax - ax}{\operatorname{sh}^2 ax} x^{2n-1} dx = -\frac{1}{n} \left(\frac{\pi}{a} \right)^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((356))(9a)

$$5. \int_0^{\infty} \ln x \frac{ax \operatorname{ch} ax - (2n+1) \operatorname{sh} ax}{\operatorname{sh}^2 ax} x^{2n} dx = 2 \frac{2^{2n+1} - 1}{(2a)^{2n+1}} (2n)! \zeta(2n+1).$$

BI ((356))(14)

$$6. \int_0^{\infty} \ln x \frac{ax \operatorname{ch} ax - 2n \operatorname{sh} ax}{\operatorname{sh}^2 ax} x^{2n-1} dx = \frac{2^{2n-1} - 1}{2n} |B_{2n}| \left(\frac{\pi}{a} \right)^{2n} \quad [n = 1, 2, \dots, a > 0].$$

BI ((356))(15)

612

$$7. \int_0^{\infty} \ln \frac{(2n+1) \operatorname{ch} ax - ax \operatorname{sh} ax}{\operatorname{ch}^2 ax} x^{2n} dx = -\left(\frac{\pi}{2a} \right)^{2n+1} |E_{2n}| \quad [a > 0].$$

BI ((356))(11)

BI ((356))(2)

$$9.6 \int_0^\infty \ln x \frac{2ax \operatorname{ch} ax - (2n+1) \operatorname{sh} ax}{\operatorname{sh}^3 ax} x^{2n} dx = \frac{1}{a} \left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \quad [a > 0, \quad n = 1, 2, \dots].$$

BI ((356))(6)a

$$10. \int_0^\infty \ln x \frac{x \operatorname{sh} x - 6 \operatorname{sh}^2\left(\frac{x}{2}\right) - 6 \cos^2 \frac{t}{2}}{(\operatorname{ch} x + \cos t)^2} x^2 dx = \frac{(\pi - t^2)t}{3 \sin t} \quad [0 < t < \pi].$$

BI ((356))(16)a

$$11. \int_0^\infty \ln(1+x^2) \frac{\operatorname{ch} \pi x + \pi x \operatorname{sh} \pi x}{\operatorname{ch}^2 \pi x} \frac{dx}{x^2} = 4 - \pi.$$

BI ((356))(12)

$$12. \int_0^\infty \ln(1+4x^2) \frac{\operatorname{ch} \pi x + \pi x \operatorname{sh} \pi x}{\operatorname{ch}^2 \pi x} \frac{dx}{x^2} = 4 \ln 2.$$

BI ((356))(13)

4.377

$$\int_0^\infty \ln 2x \frac{ax - n(1 - e^{-2ax})}{\operatorname{sh}^2 ax} x^{2n-1} dx = \frac{1}{2n} \left(\frac{\pi}{a}\right)^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

LI ((356))(8)a

4.38- 4.41 Logarithms and trigonometric functions

4.381

$$1. \int_0^1 \ln x \sin ax dx = -\frac{1}{a} [\mathbf{C} + \ln a - \operatorname{ci}(a)] \quad [a > 0].$$

GW ((338))(2a)

$$2. \int_0^1 \ln x \cos ax dx = -\frac{1}{a} \left[\operatorname{si}(a) + \frac{\pi}{2} \right] \quad [a > 0].$$

$$3. \int_0^{2\pi} \ln x \sin nx \, dx = -\frac{1}{n} [\mathbf{C} + \ln(2n\pi) - \text{ci}(2n\pi)].$$

GW ((338))(1a)

$$4. \int_0^{2\pi} \ln x \cos nx \, dx = -\frac{1}{n} \left[\text{si}(2n\pi) + \frac{\pi}{2} \right].$$

GW ((338))(1b)

4.382

$$1. \int_0^{\infty} \ln \left| \frac{x+a}{x-a} \right| \sin bx \, dx = \frac{\pi}{b} \sin ab \quad [a < 0, \quad b > 0].$$

ET I 77(11)

$$2. \int_0^{\infty} \ln \left| \frac{x+a}{x-a} \right| \cos bx \, dx = \frac{2}{b} [\cos ab \, \text{si}(ab) - \sin ab \, \text{ci}(ab)] \quad [a > 0, \quad b > 0].$$

ET I 18(9)

613

$$3. \int_0^{\infty} \ln \frac{a^2 + x^2}{b^2 + x^2} \cos cx \, dx = \frac{\pi}{c} (e^{-bc} - e^{-ac}) \quad [a > 0, \quad b > 0, \quad c > 0].$$

FI III 648a, BI ((337))(5)

$$4. \int_0^{\infty} \ln \frac{x^2 + x + a^2}{x^2 - x + a^2} \sin bx \, dx = \frac{2\pi}{b} \exp \left(-b\sqrt{a^2 - \frac{1}{4}} \right) \sin \frac{b}{2} \quad [b > 0].$$

ET I 77(12)

$$5. \int_0^{\infty} \ln \frac{(x+\beta)^2 + \gamma^2}{(x-\beta)^2 + \gamma^2} \sin bx \, dx = \frac{2\pi}{b} e^{-\gamma b} \sin \beta b \quad [\text{Re } \gamma > 0, \quad |\text{Im } \beta| \leq \text{Re } \gamma, \quad b > 0].$$

ET I 77(13)

4.383

$$1. \int_0^{\infty} \ln(1 + e^{-\beta x}) \cos bx \, dx = \frac{\beta}{2b^2} - \frac{\pi}{2b \, \text{sh} \left(\frac{\pi b}{\beta} \right)} \quad [\text{Re } \beta > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \ln(1 - e^{-\beta x}) \cos bx \, dx = \frac{\beta}{2b^2} - \frac{\pi}{2b} \operatorname{cth} \left(\frac{\pi b}{\beta} \right) \quad [\operatorname{Re} \beta > 0, \quad b > 0].$$

ET I 18(14)

4.384

$$1. \int_0^1 \ln(\sin \pi x) \sin 2n\pi x \, dx = 0.$$

GW ((338))(3a)

$$\begin{aligned} 2.^7 \int_0^1 \ln(\sin \pi x) \sin(2n+1)\pi x \, dx &= 2 \int_0^{\frac{1}{2}} \ln(\sin \pi x) \sin(2n+1)\pi x \, dx = \\ &= \frac{2}{(2n+1)\pi} \left[\ln 2 - \frac{1}{2n+1} - 2 \sum_{k=1}^n \frac{1}{2k-1} \right]. \end{aligned}$$

GW ((338))(3b)

$$\begin{aligned} 3.^6 \int_0^1 \ln(\sin \pi x) \cos 2n\pi x \, dx &= 2 \int_0^{\frac{1}{2}} \ln(\sin \pi x) \cos 2n\pi x \, dx = \\ &= -\ln 2 \quad [n = 0]; \\ &= -\frac{1}{2n} \quad [n > 0]. \end{aligned}$$

GW ((338))(3c)

$$4. \int_0^1 \ln(\sin \pi x) \cos(2n+1)\pi x \, dx = 0.$$

GW ((338))(3d)

$$5. \int_0^{\frac{\pi}{2}} \ln \sin x \sin x \, dx = \ln 2 - 1.$$

BI ((305))(4)

$$6. \int_0^{\frac{\pi}{2}} \ln \sin x \cos x \, dx = -1.$$

BI ((305))(5)

$$7. \int_0^{\frac{\pi}{2}} \ln \sin x \cos 2nx \, dx = \begin{cases} -\frac{\pi}{4n}, & n > 0 \\ -\frac{\pi}{2} \ln 2, & n = 0. \end{cases}$$

LI ((305))(6)

$$8. \int_0^{\pi} \ln \sin x \cos[2m(x-n)] \, dx = -\frac{\pi \cos 2mn}{2m}.$$

LI ((330))(8)

$$9. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^2 x \, dx = \frac{\pi}{8}(1 - \ln 4).$$

BI ((305))(7)

$$10. \int_0^{\frac{\pi}{2}} \ln \sin x \cos^2 x \, dx = -\frac{\pi}{8}(1 + \ln 4).$$

BI ((305))(8)

$$11. \int_0^{\frac{\pi}{2}} \ln \sin x \sin x \cos^2 x \, dx = \frac{1}{9}(\ln 8 - 4).$$

BI ((305))(9)

$$12. \int_0^{\frac{\pi}{2}} \ln \sin x \operatorname{tg} x \, dx = -\frac{\pi^2}{24}.$$

BI ((305))(11)

$$13. \int_0^{\frac{\pi}{2}} \ln \sin 2x \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \sin 2x \cos x \, dx = 2(\ln 2 - 1).$$

BI ((305))(16, 17)

$$14. \int_0^{\pi} \frac{\ln(1 + p \cos x)}{\cos x} \, dx = \pi \arcsin p \quad [p^2 < 1].$$

$$15. \int_0^\pi \ln \sin x \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^2}{2} \quad [a^2 < 1];$$

$$= \frac{\pi}{a^2 - 1} \ln \frac{a^2 - 1}{2a^2} \quad [a^2 > 1].$$

BI ((331))(8)

$$16. \int_0^\pi \ln \sin bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^{2b}}{2} \quad [a^2 < 1].$$

BI ((331))(10)

$$17. \int_0^\pi \ln \cos bx \frac{dx}{1 - 2a \cos x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^{2b}}{2} \quad [a^2 < 1].$$

BI ((331))(11)

$$18. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{1}{2} \int_0^\pi \ln \sin x \frac{dx}{1 - 2a \cos 2x + a^2} =$$

$$= \frac{\pi}{2(1 - a^2)} \ln \frac{1 - a}{2} \quad [a^2 < 1];$$

$$= \frac{\pi}{2(a^2 - 1)} \ln \frac{a - 1}{2a} \quad [a^2 > 1].$$

BI ((321))(1), BI ((331))(13)

$$19. \int_0^\pi \ln \sin bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 - a^b}{2} \quad [a^2 < 1].$$

BI ((331))(18)

$$20. \int_0^\pi \ln \cos bx \frac{dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1 - a^2} \ln \frac{1 + a^b}{2} \quad [a^2 < 1].$$

BI ((331))(21)

615

$$21. \int_0^{\frac{\pi}{2}} \frac{\ln \cos x dx}{1 - 2p \cos 2x + p^2} = \frac{\pi}{2(1 - p^2)} \ln \frac{1 + p}{2} \quad [p^2 < 1];$$

$$= \frac{\pi}{2(p^2 - 1)} \ln \frac{p + 1}{2p} \quad [p^2 > 1].$$

$$\begin{aligned}
22. \int_0^\pi \ln \sin x \frac{\cos x dx}{1 - 2a \cos x + a^2} &= \frac{\pi}{2a} \frac{1+a^2}{1-a^2} \ln(1-a^2) - \frac{a\pi \ln 2}{1-a^2} \quad [a^2 < 1] \\
&= \frac{\pi}{2a} \frac{a^2+1}{a^2-1} \ln \frac{a^2-1}{a^2} - \frac{\pi \ln 2}{a(a^2-1)} \quad [a^2 > 1].
\end{aligned}$$

LI ((331))(9)

$$23. \int_0^\pi \ln \sin bx \frac{\cos x dx}{1 - 2a \cos 2x + a^2} = \int_0^\pi \ln \cos bx \frac{\cos x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [0 < a < 1].$$

BI ((331))(19, 22)

$$\begin{aligned}
24. \int_0^\pi \ln \sin x \frac{\cos^2 x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{4a} \frac{1+a}{1-a} \ln(1-a) - \frac{\pi \ln 2}{2(1-a)} \quad [0 < a < 1]; \\
&= \frac{\pi}{4a} \frac{a+1}{a-1} \ln \frac{a-1}{a} - \frac{\pi \ln 2}{2a(a-1)} \quad [a > 1].
\end{aligned}$$

BI ((331))(16)

$$\begin{aligned}
25. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} &= \frac{1}{2} \int_0^\pi \ln \sin x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} = \\
&= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1-a) - a^2 \ln 2 \right\} \quad [a^2 < 1]; \\
&= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{a-1}{a} - \ln 2 \right\} \quad [a^2 > 1].
\end{aligned}$$

BI ((321))(2), BI ((331))(15), LI ((321))(2)

$$\begin{aligned}
26. \int_0^{\frac{\pi}{2}} \ln \cos x \frac{\cos 2x dx}{1 - 2a \cos 2x + a^2} &= \frac{\pi}{2a(1-a^2)} \left\{ \frac{1+a^2}{2} \ln(1+a) - a^2 \ln 2 \right\} \quad [a^2 < 1] \\
&= \frac{\pi}{2a(a^2-1)} \left\{ \frac{1+a^2}{2} \ln \frac{1+a}{a} - \ln 2 \right\} \quad [a^2 > 1].
\end{aligned}$$

BI ((321))(9)

4.385

$$1. \int_0^\pi \ln \sin x \frac{dx}{a + b \cos x} = \frac{\pi}{\sqrt{a^2 - b^2}} \ln \frac{\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}} \quad [a > 0, \quad a > b].$$

BI ((331))(6)

$$\begin{aligned}
2. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{(a \sin x \pm b \cos x)^2} &= \int_0^{\frac{\pi}{2}} \ln \cos x \frac{dx}{(a \cos x \pm b \sin x)^2} = \\
&= \frac{1}{b(a^2 + b^2)} \left(\mp a \ln \frac{a}{b} - \frac{b\pi}{2} \right) \quad [a > 0, \quad b > 0].
\end{aligned}$$

$$3. \int_0^{\frac{\pi}{2}} \frac{\ln \sin x \, dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{\ln \cos x \, dx}{b^2 \sin^2 x + a^2 \cos^2 x} = \frac{\pi}{2ab} \ln \frac{b}{a+b} \quad [a > 0, \quad b > 0].$$

BI ((317))(4, 10)

$$4. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{\sin 2x \, dx}{(a \sin^2 x + b \cos^2 x)^2} = \\ = \int_0^{\frac{\pi}{2}} \ln \cos x \frac{\sin 2x \, dx}{(b \sin^2 x + a \cos^2 x)^2} = \frac{1}{2b(b-a)} \ln \frac{a}{b} \quad [a > 0, \quad b > 0].$$

BI ((319))(3, 7), LI ((319))(3)

$$5. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{a^2 \sin^2 x - b^2 \cos^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} \, dx = \\ = \int_0^{\frac{\pi}{2}} \ln \cos x \frac{a^2 \cos^2 x - b^2 \sin^2 x}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} \, dx = \frac{\pi}{2b(a+b)} \quad [a > 0, \quad b > 0].$$

LI ((319))(2, 8)

4.386

$$1. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{\sin x}{\sqrt{1 + \sin^2 x}} \, dx = \int_0^{\frac{\pi}{2}} \frac{\cos x \ln \cos x}{\sqrt{1 + \cos^2 x}} \, dx = -\frac{\pi}{8} \ln 2.$$

BI ((322))(1, 6)

$$2. \int_0^{\frac{\pi}{2}} \frac{\sin^3 x \ln \sin x}{\sqrt{1 + \sin^2 x}} \, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x \ln \cos x}{\sqrt{1 + \cos^2 x}} \, dx = \frac{\ln 2 - 1}{4}.$$

BI ((322))(2, 7)

$$3. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{1}{2} \mathbf{K}(k) \ln k - \frac{\pi}{4} \mathbf{K}(k').$$

BI ((322))(3)

$$4. \int_0^{\frac{\pi}{2}} \frac{\ln \cos x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \mathbf{K}(k) \ln \frac{k'}{k} - \frac{\pi}{4} \mathbf{K}(k').$$

$$\begin{aligned}
1. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^\mu x \cos^\nu x dx &= \int_0^{\frac{\pi}{2}} \ln \cos x \cos^\mu x \sin^\nu x dx = \\
&= \frac{1}{4} B\left(\frac{\mu+1}{2}, \frac{\nu+1}{2}\right) \left[\psi\left(\frac{\mu+1}{2}\right) - \psi\left(\frac{\mu+\nu+2}{2}\right) \right] \\
&= \qquad \qquad \qquad [\operatorname{Re} \mu > -1, \operatorname{Re} \nu > -1].
\end{aligned}$$

GW ((338))(6c)

$$2. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{\mu-1} x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\mu}{2}\right)}{4 \Gamma\left(\frac{\mu+1}{2}\right)} \left[\psi\left(\frac{\mu}{2}\right) - \psi\left(\frac{\mu+1}{2}\right) \right] \quad [\operatorname{Re} \mu > 0].$$

GW ((338))(6a)

617

$$3. \int_0^{\frac{\pi}{2}} \ln \sin x \cos^{\nu-1} x dx = \frac{\sqrt{\pi} \Gamma\left(\frac{\nu}{2}\right)}{4 \Gamma\left(\frac{\nu+1}{2}\right)} \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right) \right] \quad [\operatorname{Re} \nu > 0].$$

GW ((338))(6b)

$$4. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left\{ \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} - \ln 2 \right\}.$$

FI II 811

$$5. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{2n+1} x dx = \frac{(2n)!!}{(2n+1)!!} \left\{ \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right\}.$$

BI ((305))(13)

$$\begin{aligned}
6. \int_0^{\frac{\pi}{2}} \ln \sin x \cos^{2n} x dx &= -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} \left[\sum_{k=1}^n \frac{1}{k} + \ln 4 \right] = \\
&= -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} [C + \psi(n+1) + \ln 4].
\end{aligned}$$

BI ((305))(14)

$$7. \int_0^{\frac{\pi}{2}} \ln \sin x \cos^{2n+1} x dx = -\frac{(2n)!!}{(2n+1)!!} \sum_{k=0}^n \frac{1}{2k+1} =$$

$$8. \int_0^{\frac{\pi}{2}} \ln \cos x \sin^{2n} x dx = -\frac{(2n-1)!!}{2^{n+1} \cdot n!} \frac{\pi}{2} \{C + 2 \ln 2 + \psi(n+1)\}.$$

BI ((306))(8)

$$9. \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{2n} x dx = -\frac{(2n-1)!!}{2^{n+1} \cdot n!} \frac{\pi}{2} \left\{ \ln 2 + \sum_{k=1}^{2n} \frac{(-1)^k}{k} \right\}.$$

BI ((306))(10)

$$10. \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{2n-1} x dx = \frac{2^{n-1}(n-1)!}{(2n-1)!!} \left[\ln 2 + \sum_{k=1}^{2n-1} \frac{(-1)^k}{k} \right].$$

BI ((306))(9)

4.388

$$1. \int_0^{\frac{\pi}{4}} \ln \sin x \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[\frac{1}{2} \ln 2 + (-1)^n \frac{\pi}{4} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k-1} \right].$$

BI ((288))(1)

$$2. \int_0^{\frac{\pi}{4}} \ln \sin x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{4n} \left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^k}{n-k} \right].$$

LI ((288))(2)

$$3. \int_0^{\frac{\pi}{4}} \ln \cos x \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[-\frac{1}{2} \ln 2 + (-1)^{n+1} \frac{\pi}{4} + \sum_{k=0}^n \frac{(-1)^{k-1}}{2n-2k+1} \right].$$

BI ((288))(10)

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$$4. \int_0^{\frac{\pi}{4}} \ln \cos x \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{4n} \left[-\ln 2 + (-1)^n \ln 2 + \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k} \right].$$

BI ((288))(11)

BI ((310))(4)

$$6. \int_0^{\frac{\pi}{2}} \ln \sin x \frac{dx}{\operatorname{tg}^{p-1} x \sin 2x} = \frac{1}{4} \frac{\pi}{p-1} \sec \frac{p\pi}{2} \quad [p^2 < 1].$$

BI ((310))(3)

4.389

$$1. \int_0^{\pi} \ln \sin x \sin^{2n} 2x \cos 2x dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4n+2}.$$

BI ((330))(9)

$$2. \int_0^{\frac{\pi}{4}} \ln \sin x \cos^n 2x \sin 2x dx = -\frac{1}{4(n+1)} \{C + \psi(n+2) + \ln 2\}.$$

BI ((285))(2)

$$3. \int_0^{\frac{\pi}{4}} \ln \cos x \cos^{\mu-1} 2x \operatorname{tg} 2x dx = \frac{1}{4(1-\mu)} \beta(\mu) \quad [\operatorname{Re} \mu > 0].$$

BI ((286))(2)

$$4. \int_0^{\frac{\pi}{2}} \ln \sin x \sin^{\mu-1} x \cos x dx = \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{\mu-1} x \sin x dx = -\frac{1}{\mu^2} \quad [\operatorname{Re} \mu > 0].$$

BI ((306))(11)

$$5.3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \cos x \cos^p x \cos px dx = \frac{\pi}{2^{p+1}} [C + \psi(p+1) - 2 \ln 2] \quad [p > -1].$$

$$6. \int_0^{\frac{\pi}{2}} \ln \cos x \cos^{p-1} x \sin px \sin x dx = \frac{\pi}{2^{p+2}} \left[C + \psi(p) - \frac{1}{p} - 2 \ln 2 \right] \quad [p > 0].$$

BI ((306))(12)

4.391

$$1. \int_0^{\frac{\pi}{4}} (\ln \cos 2x)^n \cos^{p-1} 2x \operatorname{tg} x dx = \int_0^{\frac{\pi}{4}} (\ln \sin 2x)^n \sin^{p-1} 2x \operatorname{tg} \left(\frac{\pi}{4} - x \right) dx = \frac{1}{2} \beta^{(n)}(p) \quad [p > 0].$$

$$2. \int_0^{\frac{\pi}{4}} (\ln \sin 2x)^n \sin^{p-1} 2x \operatorname{tg} \left(\frac{\pi}{4} + x \right) dx = \frac{(-1)^n n!}{2} \zeta(n+1, p).$$

BI ((285))(17)

619

$$3. \int_0^{\frac{\pi}{4}} (\ln \cos 2x)^{2n-1} \operatorname{tg} x dx = \frac{1 - 2^{2n-1}}{4n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((286))(7)

$$4. \int_0^{\frac{\pi}{4}} (\ln \cos 2x)^{2n} \operatorname{tg} x dx = \frac{2^{2n} - 1}{2^{2n+1}} (2n)! \zeta(2n+1).$$

BI ((286))(8)

4.392

$$1. \int_0^{\frac{\pi}{4}} \ln(\sin x \cos x) \frac{\sin^{2n} x}{\cos^{2n+2} x} dx = \frac{1}{2n+1} \left[(-1)^{n+1} \frac{\pi}{2} - \ln 2 + \frac{1}{2n+1} + 2 \sum_{k=0}^{n-1} \frac{(-1)^{k-1}}{2n-2k-1} \right].$$

BI ((294))(8)

$$2. \int_0^{\frac{\pi}{4}} \ln(\sin x \cos x) \frac{\sin^{2n-1} x}{\cos^{2n+1} x} dx = \frac{1}{2n} \left[(-1)^n \ln 2 - \ln 2 + \frac{1}{2n} + (-1)^n \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right].$$

BI ((294))(9)

4.393

$$1. \int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \sin x dx = \ln 2.$$

BI ((307))(3)

$$2. \int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \cos x dx = -\ln 2.$$

BI ((307))(4)

$$4. \int_0^{\frac{\pi}{4}} \frac{\ln \operatorname{tg} x}{\cos 2x} dx = -\frac{\pi^2}{8}.$$

GW ((338))(10b)a

$$5. \int_0^{\frac{\pi}{2}} \sin x \ln \operatorname{ctg} \frac{x}{2} dx = \ln 2.$$

LO III 290

4.394

$$1. \int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg} x dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{2(1-a^2)} \ln \frac{1-a}{1+a} \quad [a^2 < 1];$$

$$= \frac{\pi}{2(a^2-1)} \ln \frac{a-1}{a+1} \quad [a^2 > 1].$$

BI ((321))(15)

$$2. \int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg} x \cos 2x dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{4a} \frac{1+a^2}{1-a^2} \ln \frac{1-a}{1+a} \quad [a^2 < 1];$$

$$= \frac{\pi}{4a} \frac{a^2+1}{a^2-1} \ln \frac{a-1}{a+1} \quad [a^2 > 1].$$

BI ((321))(16)

620

$$3. \int_0^{\pi} \frac{\ln \operatorname{tg} bx dx}{1 - 2a \cos 2x + a^2} = \frac{\pi}{1-a^2} \ln \frac{1-a^b}{1+a^b} \quad [0 < a < 1, \quad b > 0].$$

BI ((331))(24)

$$4. \int_0^{\pi} \frac{\ln \operatorname{tg} bx \cos x dx}{1 - 2a \cos 2x + a^2} = 0 \quad [0 < a < 1].$$

BI ((331))(25)

$$5. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x dx}{1 - a \sin 2x} = -\frac{\arcsin a}{4a} (\pi + \arcsin a) \quad [a^2 \leq 1].$$

$$6. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x dx}{1 - a^2 \sin^2 2x} = -\frac{\pi}{4a} \arcsin a \quad [a^2 < 1].$$

BI ((291))(9)

$$7. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \frac{\cos 2x dx}{1 + a^2 \sin^2 x} = -\frac{\pi}{4a} \operatorname{Arsh} a = -\frac{\pi}{4a} \ln(a + \sqrt{1 + a^2}) \quad [a^2 < 1].$$

BI ((291))(10)

$$8. \int_0^u \frac{\sin x \ln \operatorname{ctg} \frac{x}{2}}{1 - \cos^2 \alpha \sin^2 x} dx = \operatorname{cosec} 2\alpha \left\{ \frac{\pi}{2} \ln 2 + L(\varphi - \alpha) - L(\varphi + \alpha) - L\left(\frac{\pi}{2} - 2\alpha\right) \right\}$$

$$[\operatorname{tg} \varphi = \operatorname{ctg} \alpha \cos u; \quad 0 < u < \pi].$$

LO III 290

$$9. \int_0^{\frac{\pi}{4}} \frac{\ln \operatorname{tg} x \sin 2x dx}{1 - \cos^2 t \sin^2 2x} = \operatorname{cosec} 2t \left[L\left(\frac{\pi}{2} - t\right) - \left(\frac{\pi}{2} - t\right) \ln 2 \right].$$

LO III 290a

4.395

$$1. \int_0^{\frac{\pi}{2}} \frac{\ln \operatorname{tg} x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\ln k' K(k).$$

BI ((322))(11)

$$2. \int_u^{\frac{\pi}{4}} \frac{\ln \operatorname{tg} x \sin 4x dx}{(\sin^2 u + \operatorname{tg}^2 v \sin^2 2x) \sqrt{\sin^2 2x - \sin^2 u}} = -\frac{\pi}{2} \frac{\cos^2 v}{\sin u \sin v} \ln \frac{\sin v + \sqrt{1 - \cos^2 u \cos^2 v}}{\sin u (1 + \sin v)}$$

$$\left[0 < u < \frac{\pi}{2}, \quad 0 < v < \frac{\pi}{2} \right].$$

LO III 285a

4.396

$$1. \int_0^{\frac{\pi}{2}} \ln(a \operatorname{tg} x) \sin^{\mu-1} 2x dx = 2^{\mu-2} \ln a \frac{\left\{ \Gamma\left(\frac{a}{2}\right) \right\}^2}{\Gamma(a)} \quad [a > 0, \quad \operatorname{Re} \mu > 0]$$

LI ((307))(8)

BI ((307))(9)

$$3. \int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \cos^{q-1} x \operatorname{ctg} x \sin[(q+1)x] dx = -\frac{\pi}{2} [C + \psi(q+1)] \quad [q > -1].$$

BI ((307))(11)

621

$$4. \int_0^{\frac{\pi}{2}} \ln \operatorname{tg} x \cos^{q-1} x \cos[(q+1)x] dx = -\frac{\pi}{2q} \quad [q > 0].$$

BI ((307))(10)

$$5. \int_0^{\frac{\pi}{4}} (\ln \operatorname{tg} x)^n \operatorname{tg}^p x dx = \frac{1}{2^{n+1}} B^{(n)} \left(\frac{p+1}{2} \right) \quad [p > -1].$$

LI ((286))(22)

$$6. \int_0^{\frac{\pi}{2}} (\ln \operatorname{tg} x)^{2n-1} \frac{dx}{\cos 2x} = \frac{1-2^{2n}}{2n} \pi^{2n} |B_{2n}| \quad [n = 1, 2, \dots].$$

BI ((312))(6)

$$7. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x \operatorname{tg}^{2n+1} x dx = \frac{(-1)^{n+1}}{4} \left[\frac{\pi^2}{12} + \sum_{k=1}^n \frac{(-1)^k}{k^2} \right].$$

GW ((338))(8a)

4.397

$$1. \int_0^{\frac{\pi}{2}} \ln(1 + p \sin x) \frac{dx}{\sin x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1].$$

BI ((313))(1)

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + p \cos x) \frac{dx}{\cos x} = \frac{\pi^2}{8} - \frac{1}{2} (\arccos p)^2 \quad [p^2 < 1].$$

BI ((313))(8)

$$3. \int_0^{\pi} \ln(1 + p \cos x) \frac{dx}{\cos x} = \pi \arcsin p \quad [p^2 < 1].$$

$$4. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1 + \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) - \alpha \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right].$$

LO III 291

$$5. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln(1 - \cos \alpha \cos x)}{1 - \cos^2 \alpha \cos^2 x} dx = \frac{L\left(\frac{\pi}{2} - \alpha\right) + (\pi - \alpha) \ln \sin \alpha}{\sin \alpha \cos \alpha} \quad \left[0 < \alpha < \frac{\pi}{2}\right].$$

LO III 291

$$6. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \cos nx dx = \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx dx =$$

$$= -\frac{\pi}{n} a^n \quad [a^2 < 1];$$

$$= -\frac{\pi}{na^n} \quad [a^2 > 1].$$

GW ((338))(13a)

BI ((330))(11), BI ((332))(5)

$$7. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx =$$

$$= \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx = \frac{\pi}{2} \left(\frac{a^{n+1}}{n+1} - \frac{a^{n-1}}{n-1} \right)$$

$$[a^2 > 1].$$

BI ((330))(10), BI ((332))(4)

622

$$8. \int_0^{\pi} \ln(1 - 2a \cos x + a^2) \sin nx \sin x dx =$$

$$= \frac{1}{2} \int_0^{2\pi} \ln(1 - 2a \cos x + a^2) \cos nx \cos x dx = -\frac{\pi}{2} \left(\frac{a^{n+1}}{n+1} + \frac{a^{n-1}}{n-1} \right) \quad [a^2 < 1].$$

BI ((330))(12), BI ((332))(6)

$$9. \int_0^{\pi} \ln(1 - 2a \cos 2x + a^2) \cos(2n - 1)x dx = 0 \quad [a^2 < 1].$$

BI ((330))(15)

BI ((330))(13)

$$11. \int_0^\pi \ln(1-2a \cos 2x+a^2) \sin(2n-1)x \sin x \, dx = \frac{\pi}{2} \left(\frac{a^n}{n} - \frac{a^{n-1}}{n-1} \right) \quad [a^2 < 1].$$

BI ((330))(14)

$$12. \int_0^\pi \ln(1-2a \cos 2x+a^2) \cos 2nx \cos x \, dx = 0 \quad [a^2 < 1].$$

BI ((330))(16)

$$13. \int_0^\pi \ln(1-2a \cos 2x+a^2) \cos(2n-1)x \cos x \, dx = -\frac{\pi}{2} \left(\frac{a^n}{n} + \frac{a^{n-1}}{n-1} \right) \quad [a^2 < 1].$$

BI ((330))(17)

$$14. \int_0^{\frac{\pi}{2}} \ln(1+2a \cos 2x+a^2) \sin^2 x \, dx = -\frac{a\pi}{4} \quad [a^2 < 1];$$

$$= \frac{\pi \ln a^2}{4} - \frac{\pi}{4a} \quad [a^2 > 1].$$

BI ((309))(22), LI ((309))(22)

$$15. \int_0^{\frac{\pi}{2}} \ln(1+2a \cos 2x+a^2) \cos^2 x \, dx = \frac{a\pi}{4} \quad [a^2 < 1];$$

$$= \frac{\pi \ln a^2}{4} + \frac{\pi}{4a} \quad [a^2 > 1].$$

BI ((309))(23), LI ((309))(23)

$$16. \int_0^\pi \frac{\ln(1-2a \cos x+a^2)}{1-2b \cos x+b^2} \, dx = \frac{2\pi \ln(1-ab)}{1-b^2} \quad [a^2 \leq 1, \quad b^2 < 1].$$

BI ((331))(26)

4.398

$$1. \int_0^\pi \ln \frac{1+2a \cos x+a^2}{1-2a \cos x+a^2} \sin(2n+1)x \, dx = (-1)^n \frac{2\pi a^{2n+1}}{2n+1} \quad [a^2 < 1].$$

$$2. \int_0^{2\pi} \ln \frac{1 - 2a \cos x + a^2}{1 - 2a \cos nx + a^2} \cos mx \, dx = 2\pi \left(\frac{n}{m} a^{\frac{m}{n}} - \frac{a^m}{m} \right) \quad [a^2 \leq 1];$$

$$= 2\pi \left(\frac{n}{m} a^{-\frac{m}{n}} - \frac{a^{-m}}{m} \right) \quad [a^2 \geq 1].$$

BI ((332))(9)

$$3. \int_0^\pi \ln \frac{1 + 2a \cos 2x + a^2}{1 + 2a \cos 2nx + a^2} \operatorname{ctg} x \, dx = 0.$$

BI ((331))(5), LI((331))(5)

623

4.399

$$1. \int_0^{\frac{\pi}{2}} \ln(1 + a \sin^2 x) \sin^2 x \, dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+a}}{2} - \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) \quad [a > -1].$$

BI ((309))(14)

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + a \sin^2 x) \cos^2 x \, dx = \frac{\pi}{2} \left(\ln \frac{1 + \sqrt{1+a}}{2} + \frac{1}{2} \frac{1 - \sqrt{1+a}}{1 + \sqrt{1+a}} \right) \quad [a > -1].$$

BI ((309))(15)

$$3. \int_0^{\frac{\pi}{2}} \frac{\ln(1 - \cos^2 \beta \cos^2 x)}{1 - \cos^2 \alpha \cos^2 x} \, dx = -\frac{\pi}{\sin \alpha} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sin \beta} \quad \left[0 < \beta < \frac{\pi}{2}, \quad 0 < \alpha < \frac{\pi}{2} \right].$$

LO III 285

4.411

$$1. \int_0^\pi \ln \frac{1 + \sin x}{1 + \cos \lambda \sin x} \frac{dx}{\sin x} = \lambda^2 \quad [\lambda^2 < \pi^2].$$

BI ((331))(2)

$$2. \int_0^{\frac{\pi}{2}} \ln \frac{p + q \sin ax}{p - q \sin ax} \frac{dx}{\sin ax} = \int_0^{\frac{\pi}{2}} \ln \frac{p + q \cos ax}{p - q \cos ax} \frac{dx}{\cos ax} = \int_0^{\frac{\pi}{2}} \ln \frac{p + q \operatorname{tg} ax}{p - q \operatorname{tg} ax} \frac{dx}{\operatorname{tg} ax} = \pi \operatorname{arcsin} \frac{q}{p}$$

$$[p > q > 0].$$

FI II 695a, BI ((315))(5, 13,17)a

4.412

$$1. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\sin 2x} = \pm \frac{\pi^2}{8}.$$

BI ((293))(1)

$$2. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) \frac{dx}{\operatorname{tg} 2x} = \pm \frac{\pi^2}{16}.$$

BI ((293))(2)

$$3. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2n} \frac{dx}{\sin 2x} = \pm \frac{2^{2n+2} - 1}{4(n+1)(2n+1)} \pi^{2n+2} |B_{2n+2}|.$$

BI ((294))(24)

$$4. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \operatorname{tg} x)^{2n-1} \frac{dx}{\sin 2x} = \pm \frac{1 - 2^{2n+1}}{2^{2n+2} n} (2n)! \zeta(2n+1).$$

BI ((294))(25)

$$5. \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} \left(\frac{\pi}{4} \pm x \right) (\ln \sin 2x)^{n-1} \frac{dx}{\operatorname{tg} 2x} = \frac{(-1)^{n-1}}{2} (n-1)! \zeta(n+1).$$

LI ((294))(20)

624

4.413

$$1. \int_0^{\frac{\pi}{2}} \ln(p^2 + q^2 \operatorname{tg}^2 x) \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{ab} \ln \frac{ap + bq}{a}$$

$$[a > 0, \quad b > 0, \quad p > 0, \quad q > 0].$$

BI ((318))(1-4)a

$$2. \int_0^{\frac{\pi}{2}} \ln(1 + q^2 \operatorname{tg}^2 x) \frac{1}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} =$$

$$= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p^2 - r^2}{pr} \ln \left(1 + \frac{qr}{p} \right) + \frac{t^2 - s^2}{st} \ln \left(1 + \frac{qt}{s} \right) \right\}$$

$$[q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0].$$

$$\begin{aligned}
3. \int_0^{\frac{\pi}{2}} \ln(1 + q^2 \operatorname{tg}^2 x) \frac{\sin^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} &= \\
&= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{t}{s} \ln \left(1 + \frac{qt}{s} \right) - \frac{r}{p} \ln \left(1 + \frac{qr}{p} \right) \right\} \\
&\quad [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0].
\end{aligned}$$

BI ((320))(20)

$$\begin{aligned}
4. \int_0^{\frac{\pi}{2}} \ln(1 + q^2 \operatorname{tg}^2 x) \frac{\cos^2 x}{p^2 \sin^2 x + r^2 \cos^2 x} \frac{dx}{s^2 \sin^2 x + t^2 \cos^2 x} &= \\
&= \frac{\pi}{p^2 t^2 - s^2 r^2} \left\{ \frac{p}{r} \ln \left(1 + \frac{qr}{p} \right) - \frac{s}{t} \ln \left(1 + \frac{qt}{s} \right) \right\} \\
&\quad [q > 0, \quad p > 0, \quad r > 0, \quad s > 0, \quad t > 0].
\end{aligned}$$

BI ((320))(21)

$$5. \int_0^{\pi} \frac{\ln \operatorname{tg} r x \, dx}{1 - 2p \cos x + p^2} = \frac{\pi}{1 - p^2} \ln \frac{1 - p^{2r}}{1 + p^{2r}} \quad [p^2 < 1].$$

BI ((331))(12)

4.414

$$1. \int_0^{\frac{\pi}{2}} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \ln k' \mathbf{K}(k).$$

BI ((323))(1)

$$2. \int_0^{\frac{\pi}{2}} \ln(1 - k^2 \sin^2 x) \frac{\sin^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2} \{ (k^2 - 2 + \ln k') \mathbf{K}(k) + (2 - \ln k') \mathbf{E}(k) \}.$$

BI ((323))(3)

$$3. \int_0^{\frac{\pi}{2}} \ln(1 - k^2 \sin^2 x) \frac{\cos^2 x \, dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{k^2} [(1 + k'^2 - k'^2 \ln k') \mathbf{K}(k) - (2 - \ln k') \mathbf{E}(k)].$$

BI ((323))(6)

625

$$4. \int_0^{\frac{\pi}{2}} \ln(1 - k^2 \sin^2 x) \frac{dx}{\sqrt{(1 - k^2 \sin^2 x)^3}} = \frac{1}{k'^2} [(k^2 - 2) \mathbf{K}(k) + (2 + \ln k') \mathbf{E}(k)].$$

$$5. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin^2 x dx}{\sqrt{(1-k^2 \sin^2 x)^3}} = \frac{1}{k^2 k'^2} [(2+\ln k')\mathbf{E}(k) - (1+k'^2+k'^2 \ln k')\mathbf{K}(k)].$$

BI ((323))(10)

$$6. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\cos^2 x dx}{\sqrt{(1-k^2 \sin^2 x)^3}} = \frac{1}{k^2} [(1+k'^2+\ln k')\mathbf{K}(k) - (2+\ln k')\mathbf{E}(k)].$$

BI ((323))(16)

$$7. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \sqrt{1-k^2 \sin^2 x} dx = (1+k'^2)\mathbf{K}(k) - (2-\ln k')\mathbf{E}(k).$$

BI ((324))(18)

$$8. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \sin^2 x \sqrt{1-k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ (-2+11k^2-6k^4+3k'^2 \ln k')\mathbf{K}(k) + [2-10k^2-3(1-2k^2) \ln k']\mathbf{E}(k) \right\}.$$

BI ((324))(20)

$$9. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \cos^2 x \sqrt{1-k^2 \sin^2 x} dx = \frac{1}{9k^2} \left\{ (2+7k^2-3k^4-3k'^2 \ln k')\mathbf{K}(k) - [2+8k^2-3(1+k^2) \ln k']\mathbf{E}(k) \right\}.$$

BI ((324))(21), LI ((324))(21)

$$10. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x dx}{\sqrt{(1-k^2 \sin^2 x)^{2n+1}}} = \frac{2}{(2n-1)^2 k^2} \{ [1+(2n-1) \ln k'] k'^{1-2n} - 1 \}.$$

BI ((324))(17)

4.415

$$1. \int_0^{\infty} \ln x \sin ax^2 dx = -\frac{1}{4} \sqrt{\frac{\pi}{2a}} \left(\ln 4a + \mathbf{C} - \frac{\pi}{2} \right) \quad [a > 0].$$

GW ((338))(19)

4.416

$$\begin{aligned}
 1. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln \left(1 + \sqrt{\sin^2 \beta - \cos^2 \beta \operatorname{tg}^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx = \\
 = \operatorname{cosec} 2\alpha \{ (2\alpha + 2\gamma - \pi) \ln \cos \beta + 2L(\alpha) - 2L(\gamma) + L(\alpha + \gamma) - L(\alpha - \gamma) \} \\
 \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right].
 \end{aligned}$$

LO III 291

626

$$\begin{aligned}
 2. \int_0^{\frac{\pi}{2}} \frac{\cos x \ln \left(1 - \sqrt{\sin^2 \beta - \cos^2 \beta \operatorname{tg}^2 \alpha \sin^2 x} \right)}{1 - \sin^2 \alpha \cos^2 x} dx = \\
 = \operatorname{cosec} 2\alpha \{ (\pi + 2\alpha - 2\gamma) \ln \cos \beta + 2L(\alpha) + 2L(\gamma) - L(\alpha + \gamma) + L(\alpha - \gamma) \} \\
 \left[\cos \gamma = \frac{\sin \alpha}{\sin \beta}; \quad 0 < \alpha < \beta < \frac{\pi}{2} \right].
 \end{aligned}$$

LO III 291

$$\begin{aligned}
 3. \int_{\beta}^{\frac{\pi}{2}} \frac{\ln(\sin x + \sqrt{\sin^2 x - \sin^2 \beta})}{1 - \cos^2 \alpha \cos^2 x} dx = \\
 = -\operatorname{cosec} \alpha \left\{ \operatorname{arctg} \left(\frac{\operatorname{tg} \beta}{\sin \alpha} \right) \ln \sin \beta + \frac{\pi}{2} \ln \frac{1 + \sin \alpha}{\sin \alpha + \sqrt{1 - \cos^2 \alpha \cos^2 \beta}} \right\} \\
 \left[0 < \alpha < \pi, \quad 0 < \beta < \frac{\pi}{2} \right].
 \end{aligned}$$

LO III 285

$$4.7 \int_0^{\frac{\pi}{4}} \ln \operatorname{tg} x (\ln \cos 2x)^{n-1} \operatorname{tg} 2x dx = \frac{1}{2} (-1)^n (n-1)! (1 - 2^{-(n+1)}) \zeta(n+1)$$

4.42- 4.43 Combinations of logarithms, trigonometric functions, and powers

4.421

$$1. \int_0^{\infty} \ln x \sin ax \frac{dx}{x} = -\frac{\pi}{2} (C + \ln a) \quad [a > 0].$$

FI II 810a

$$3. \int_0^{\infty} \ln ax \cos bx \frac{\beta' dx}{\beta^2 + x^2} = \frac{\pi}{2} e^{-b\beta'} \ln(a\beta') + \frac{\pi}{4} [e^{b\beta'} \operatorname{Ei}(-b\beta') - e^{-b\beta'} \operatorname{Ei}(b\beta')] \\ [\beta' = \beta \operatorname{sign} \beta; \quad a > 0, \quad b > 0].$$

$$4. \int_0^{\infty} \ln ax \sin bx \frac{x dx}{x^2 - c^2} = \frac{\pi}{2} \{-\operatorname{si}(bc) \sin bc + \cos bc [\ln ac - \operatorname{ci}(bc)]\} \\ [a > 0, \quad b > 0, \quad c > 0].$$

$$5. \int_0^{\infty} \ln ax \cos bx \frac{dx}{x^2 - c^2} = \frac{\pi}{2c} \{\sin bc [\operatorname{ci}(bc) - \ln ac] - \cos bc \operatorname{si}(bc)\} \\ [a > 0, \quad b > 0, \quad c > 0].$$

627
4.422

$$1. \int_0^{\infty} \ln x \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^{\mu}} \sin \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a + \frac{\pi}{2} \operatorname{ctg} \frac{\mu\pi}{2} \right] \quad [a > 0, \quad |\operatorname{Re} \mu| < 1].$$

$$2. \int_0^{\infty} \ln x \cos ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^{\mu}} \cos \frac{\mu\pi}{2} \left[\psi(\mu) - \ln a - \frac{\pi}{2} \operatorname{tg} \frac{\mu\pi}{2} \right] \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

4.423

$$1. \int_0^{\infty} \ln x \frac{\cos ax - \cos bx}{x} dx = \ln \frac{a}{b} \left(\mathbf{C} + \frac{1}{2} \ln ab \right) \quad [a > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \ln x \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2} [(a-b)(\mathbf{C}-1) + a \ln a - b \ln b] \quad [a > 0, \quad b > 0].$$

$$3. \int_0^{\infty} \ln x \frac{\sin^2 ax}{x^2} dx = -\frac{a\pi}{2} (\mathbf{C} + \ln 2a - 1) \quad [a > 0].$$

GW ((338))(20b)

4.424

$$1. \int_0^{\infty} (\ln x)^2 \sin ax \frac{dx}{x} = \frac{\pi}{2} \mathbf{C}^2 + \frac{\pi^3}{24} + \pi \mathbf{C} \ln a + \frac{\pi}{2} (\ln a)^2 \quad [a > 0].$$

ET I 77(9), FI II 810a

$$2.^6 \int_0^{\infty} (\ln x)^2 \sin ax x^{\mu-1} dx = \frac{\Gamma(\mu)}{a^{\mu}} \sin \frac{\mu\pi}{2} \left[\psi'(\mu) + \psi^2(\mu) + \pi\psi(\mu) \operatorname{ctg} \frac{\mu\pi}{2} - 2\psi(\mu) \ln a - \pi \ln a \operatorname{ctg} \frac{\mu\pi}{2} + (\ln a)^2 - \frac{1}{4}\pi^2 \right] \quad [a > 0, \quad 0 < \operatorname{Re} \mu < 1].$$

ET I 77(10)

4.425

$$1. \int_0^{\infty} \ln(1+x) \cos ax \frac{dx}{x} = \frac{1}{2} \{[\operatorname{si}(a)]^2 + [\operatorname{ci}(a)]^2\} \quad [a > 0].$$

ET I 18(8)

$$2. \int_0^{\infty} \ln \left(\frac{b+x}{b-x} \right)^2 \cos ax \frac{dx}{x} = -2\pi \operatorname{si}(ab) \quad [a \geq 0, \quad b > 0].$$

ET I 18(11)

$$3. \int_0^{\infty} \ln(1+b^2x^2) \sin ax \frac{dx}{x} = -\pi \operatorname{Ei} \left(-\frac{a}{b} \right) \quad [a > 0, \quad b > 0].$$

GW ((338))(24), ET I 77(14)

$$4. \int_0^1 \ln(1-x^2) \cos(p \ln x) \frac{dx}{x} = \frac{1}{2p^2} + \frac{\pi}{2p} \operatorname{cth} \frac{p\pi}{2}.$$

LI ((309))(1)a

4.426

$$1. \int_0^{\infty} \ln \frac{b^2 + x^2}{c^2 + x^2} \sin ax \, x \, dx = \frac{\pi}{a^2} [(1+ac)e^{-ac} - (1+ab)e^{-ab}] \quad [b \geq 0, \quad c \geq 0, \quad a > 0].$$

GW ((338))(23)

$$2. \int_0^{\infty} \ln \frac{b^2 x^2 + p^2}{c^2 x^2 + p^2} \sin ax \frac{dx}{x} = \pi \left[\text{Ei} \left(-\frac{ap}{c} \right) - \text{Ei} \left(-\frac{ap}{b} \right) \right] \quad [b > 0, \quad c > 0, \quad p > 0, \quad a > 0].$$

ET I 77(15)

4.427

$$\int_0^{\infty} \ln(x + \sqrt{\beta^2 + x^2}) \frac{\sin ax}{\sqrt{\beta^2 + x^2}} dx = \frac{\pi}{2} K_0(a\beta) + \frac{\pi}{2} \ln(\beta) [I_0(a\beta) - \mathbf{L}(a\beta)]$$

$$[\text{Re } \beta > 0, \quad a > 0].$$

ET I 77(16)

4.428

$$1. \int_0^{\infty} \ln \cos^2 ax \frac{\cos bx}{x^2} dx = \pi b \ln 2 - a\pi \quad [a > 0, \quad b > 0].$$

ET I 22(29)

$$2. \int_0^{\infty} \ln(4 \cos^2 ax) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \text{ch}(bc) \ln(1 + e^{-2ac}) \quad \left[a < b < 2a < \frac{\pi}{c} \right].$$

ET I 22(30)

$$3. \int_0^{\infty} \ln \cos^2 ax \frac{\sin bx}{x(1+x^2)} dx = \pi \ln(1 + e^{-2a}) \text{sh } b - \pi \ln 2(1 - e^{-b}) \quad [a > 0, \quad b > 0].$$

ET I 82(36)

$$4. \int_0^{\infty} \ln \cos^2 ax \frac{\cos bx}{x^2(1+x^2)} dx = -\pi \ln(1 + e^{-2a}) \text{ch } b + (b + e^{-b})\pi \ln 2 - a\pi \quad [a > 0, \quad b > 0].$$

ET I 22(31)

4.429

$$\int_0^1 \frac{(1+x)x}{\ln x} \sin(\ln x) dx = \frac{\pi}{4}.$$

BI ((326))(2)a

4.431

$$1. \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\sin bx}{x^2 + c^2} x dx = -\pi \operatorname{sh}(bc) \ln(1 \pm e^{-c}) \quad [b > 0, \quad c > 0].$$

ET I 22(32)

$$2. \int_0^\infty \ln(2 \pm 2 \cos x) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \operatorname{ch}(bc) \ln(1 \pm e^{-c}) \quad [b > 0, \quad c > 0].$$

ET I 22(32)

$$3. \int_0^\infty \ln(1 + 2a \cos x + a^2) \frac{\sin bx}{x} dx = -\frac{\pi}{2} \sum_{k=1}^{[b]} \frac{(-a)^k}{k} [1 + \operatorname{sign}(b-k)] \quad [0 < a < 1, \quad b > 0].$$

ET I 82(25)

$$4. \int_0^\infty \ln(1 - 2a \cos x + a^2) \frac{\cos bx}{x^2 + c^2} dx = \frac{\pi}{c} \ln(1 - ae^{-c}) \operatorname{ch}(bc) + \frac{\pi}{c} \sum_{k=1}^{[b]} \frac{a^k}{k} \operatorname{sh}[c(b-k)]$$

$$[|a| < 1, \quad b > 0, \quad c > 0].$$

ET I 22(33)

629

4.432

$$1. \int_0^\infty \ln(1 - k^2 \sin^2 x) \frac{\sin x}{\sqrt{1 - k^2 \sin^2 x}} \frac{dx}{x} = \int_0^\infty \ln(1 - k^2 \cos^2 x) \frac{\sin x}{\sqrt{1 - k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k).$$

BI ((412, 414))(4)

$$2. \int_0^{\frac{\pi}{2}} \ln(1 - k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1 - k^2 \sin^2 x}} x dx =$$

$$= \frac{1}{k^2} \{ \pi k' (1 - \ln k') + (2 - k^2) \mathbf{K}(k) - (4 - \ln k') \mathbf{E}(k) \}.$$

$$3. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2 \cos^2 x}} x dx = \frac{1}{k^2} \{-\pi - (2-k^2) \mathbf{K}(k) + (4-\ln k') \mathbf{E}(k)\}.$$

BI ((426))(6)

$$4. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(2-k^2 - k'^2 \ln k') \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k)\}.$$

BI ((412))(5)

$$5. \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(k^2 - 2 + \ln k') \mathbf{K}(k) + (2-\ln k') \mathbf{E}(k)\}.$$

BI ((414))(5)

$$\begin{aligned} 6. \int_0^{\infty} \ln(1 \pm k \sin^2 x) \frac{\sin x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} &= \int_0^{\infty} \ln(1 \pm k \cos^2 x) \frac{\sin x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1 \pm k \sin^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1 \pm k \cos^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1 \pm k \sin^2 2x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 2x}} \frac{dx}{x} = \\ &= \int_0^{\infty} \ln(1 \pm k^2 \cos^2 2x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \frac{dx}{x} = \\ &= \frac{1}{2} \ln \frac{2(1 \pm k)}{\sqrt{k}} \mathbf{K}(k) - \frac{\pi}{8} \mathbf{K}(k'). \end{aligned}$$

BI ((413))(1--6), BI ((415))(1--6)

$$7. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(k^2 - 2 + \ln k') \mathbf{K}(k) + (2-\ln k') \mathbf{E}(k)\}.$$

BI ((412))(6)

630

$$8. \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(2-k^2 - k'^2 \ln k') \mathbf{K}(k) - (2-\ln k') \mathbf{E}(k)\}.$$

BI ((414))(6a)

BI ((412))(7)

$$10. \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(k^2-2+\ln k')\mathbf{K}(k)+(2-\ln k')\mathbf{E}(k)\}.$$

BI ((414))(7)

$$11. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \ln k' \mathbf{K}(k).$$

BI ((412, 414))(9)

$$12. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \frac{\sin^2 x \operatorname{tg} x}{\sqrt{1-k^2 \sin^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(k^2-2+\ln k')\mathbf{K}(k)+(2-\ln k')\mathbf{E}(k)\}.$$

BI ((412))(8)

$$13. \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\sin^2 x \operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \frac{1}{k^2} \{(2-k^2-k'^2 \ln k')\mathbf{K}(k)-(2-\ln k')\mathbf{E}(k)\}.$$

BI ((414))(8)

$$14. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \frac{\sin^2 x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} = \int_0^{\infty} \ln(1-k^2 \cos^2 x) \frac{\sin x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} = \frac{1}{k'^2} \{(k^2-2)\mathbf{K}(k) + (2+\ln k')\mathbf{E}(k)\}.$$

BI ((412, 414))(13)

$$15. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1-k^2 \sin^2 x)^3}} x dx = \frac{1}{k^2} \left\{ (1+\ln k') \frac{\pi}{k'} - (2+\ln k')\mathbf{K}(k) \right\}.$$

BI ((426))(9)

$$16. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1-k^2 \cos^2 x)^3}} x dx = \frac{1}{k^2} \{-\pi + (2+\ln k')\mathbf{K}(k)\}.$$

BI ((426))(15)

$$\begin{aligned}
17. \int_0^\infty \ln(1-k^2 \sin^2 x) \frac{\sin x \cos x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1-k^2 \cos^2 x) \frac{\sin^3 x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} = \\
&= \frac{1}{k^2} \{(2-k^2 + \ln k')\mathbf{K}(k) - (2 + \ln k')\mathbf{E}(k)\}.
\end{aligned}$$

BI ((412))(14), BI((414))(15)

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$$\begin{aligned}
18. \int_0^\infty \ln(1-k^2 \sin^2 x) \frac{\sin^3 x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} &= \\
= \int_0^\infty \ln(1-k^2 \cos^2 x) \frac{\sin x \cos x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} &= \\
= \frac{1}{k^2 k'^2} \{(2 + \ln k')\mathbf{E}(k) - (2 - k^2 + k'^2 \ln k')\mathbf{K}(k)\}.
\end{aligned}$$

BI ((412))(15), BI((414))(14)

$$\begin{aligned}
19. \int_0^\infty \ln(1-k^2 \sin^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1-k^2 \cos^2 x) \frac{\sin^2 x \operatorname{tg} x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} = \\
&= \frac{1}{k^2} \{(2-k^2 + \ln k')\mathbf{K}(k) - (2 + \ln k')\mathbf{E}(k)\}.
\end{aligned}$$

BI ((412))(16), BI((414))(17)

$$\begin{aligned}
20. \int_0^\infty \ln(1-k^2 \sin^2 x) \frac{\sin^2 x \operatorname{tg} x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} &= \\
= \int_0^\infty \ln(1-k^2 \cos^2 x) \frac{\sin x \cos^2 x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} &= \\
= \frac{1}{k^2 k'^2} \{(2 + \ln k')\mathbf{E}(k) - (2 - k^2 + k'^2 \ln k')\mathbf{K}(k)\}.
\end{aligned}$$

BI ((412))(17), BI((414))(16)

$$\begin{aligned}
21. \int_0^\infty \ln(1-k^2 \sin^2 x) \frac{\operatorname{tg} x}{\sqrt{(1-k^2 \sin^2 x)^3}} \frac{dx}{x} &= \int_0^\infty \ln(1-k^2 \cos^2 x) \frac{\operatorname{tg} x}{\sqrt{(1-k^2 \cos^2 x)^3}} \frac{dx}{x} = \\
&= \frac{1}{k'^2} \{(k^2 - 2)\mathbf{K}(k) + (2 + \ln k')\mathbf{E}(k)\}.
\end{aligned}$$

$$22. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \sqrt{1-k^2 \sin^2 x} \sin x \frac{dx}{x} = \int_0^{\infty} \ln(1-k^2 \cos^2 x) \sqrt{1-k^2 \cos^2 x} \sin x \frac{dx}{x} = \\ = (2-k^2)\mathbf{K}(k) - (2-\ln k')\mathbf{E}(k)$$

BI ((412, 414))(1)

$$23. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \sin^2 x) \sqrt{1-k^2 \sin^2 x} \sin x \cos x \cdot x dx = \\ = \frac{1}{27k^2} \{3\pi k'^3(1-3\ln k') + (22k'^2 + 6k^4 - 3k'^2 \ln k')\mathbf{K}(k) - \\ - (2-k^2)(14-6\ln k')\mathbf{E}(k)\}$$

BI ((426))(1)

$$24. \int_0^{\frac{\pi}{2}} \ln(1-k^2 \cos^2 x) \sqrt{1-k^2 \cos^2 x} \sin x \cos x \cdot x dx = \\ = \frac{1}{27k^2} \{-3\pi - (22k'^2 + 6k^4 - 3k'^2 \ln k')\mathbf{K}(k) + (2-k^2)(14-6\ln k')\mathbf{E}(k)\}.$$

BI ((426))(2)

632

$$25. \int_0^{\infty} \ln(1-k^2 \sin^2 x) \sqrt{1-k^2 \sin^2 x} \operatorname{tg} x \frac{dx}{x} = \int_0^{\infty} \ln(1-k^2 \cos^2 x) \sqrt{1-k^2 \cos^2 x} \operatorname{tg} x \frac{dx}{x} = \\ = (2-k^2)\mathbf{K}(k) - (2-\ln k')\mathbf{E}(k).$$

BI ((412, 414))(2)

$$26. \int_0^{\infty} \ln(\sin^2 x + k' \cos^2 x) \frac{\sin x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \int_0^{\infty} \ln(\sin^2 x + k' \cos^2 x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 x}} \frac{dx}{x} = \\ = \int_0^{\infty} \ln(\sin^2 2x + k' \cos^2 2x) \frac{\operatorname{tg} x}{\sqrt{1-k^2 \cos^2 2x}} \frac{dx}{x} = \\ = \frac{1}{2} \ln \left[\frac{2(\sqrt{k'})^3}{1+k'} \right] \mathbf{K}(k).$$

BI ((415))(19--21)

4.44 Combinations of logarithms, trigonometric functions, and exponentials

4.441

$$1.7 \int_0^{\infty} e^{-qx} \sin px \ln x dx = \frac{1}{p^2+q^2} \left[q \operatorname{arctg} \frac{p}{q} - pC - \frac{p}{2} \ln(p^2-q^2) \right] \quad [q > 0, \quad p > 0].$$

$$2. \int_0^{\infty} e^{-qx} \cos px \ln x \, dx = -\frac{1}{p^2 + q^2} \left[\frac{q}{2} \ln(p^2 + q^2) + p \operatorname{arctg} \frac{p}{q} + qC \right] \quad [q > 0].$$

BI ((467))(2)

4.442

$$\int_0^{\frac{\pi}{2}} \frac{e^{-p \operatorname{tg} x} \ln \cos x \, dx}{\sin x \cos x} = -\frac{1}{2} [\operatorname{ci}(p)]^2 + \frac{1}{2} [\operatorname{si}(p)]^2 \quad [\operatorname{Re} p > 0].$$

NT 32(11)

4.5 Inverse Trigonometric Functions

4.51 Inverse trigonometric functions

4.511

$$\int_0^{\infty} \operatorname{arctg} px \operatorname{arctg} qx \, dx = \frac{\pi}{2} \left\{ \frac{1}{p} \ln \left(1 + \frac{p}{q} \right) + \frac{1}{q} \ln \left(1 + \frac{q}{p} \right) \right\} \quad [p > 0, \quad q > 0].$$

BI ((77))(8)

4.512

$$\int_0^{\pi} \operatorname{arctg}(\cos x) \, dx = 0.$$

BI ((345))(1)

4.52 Combinations of arcsines, arccosines, and powers

4.521

$$1. \int_0^1 \frac{\arcsin x}{x} \, dx = \frac{\pi}{2} \ln 2.$$

FI II 614, 623

$$2. \int_0^1 \frac{\arccos x}{1 \pm x} \, dx = \mp \frac{\pi}{2} \ln 2 + 2G.$$

BI ((231))(7, 8)

$$3. \int_0^1 \arcsin x \frac{x}{1+qx^2} dx = \frac{\pi}{2q} \ln \frac{2\sqrt{1+q}}{1+\sqrt{1+q}} \quad [q > -1].$$

BI ((231))(1)

$$4. \int_0^1 \arcsin x \frac{x}{1-p^2x^2} dx = \frac{\pi}{2p^2} \ln \frac{1+\sqrt{1-p^2}}{2\sqrt{1-p^2}} \quad [p^2 < 1].$$

LI ((231))(3)

$$5. \int_0^1 \arccos x \frac{dx}{\sin^2 \lambda - x^2} = 2 \operatorname{cosec} \lambda \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}.$$

BI ((231))(10)

$$6. \int_0^1 \arcsin x \frac{dx}{x(1+qx^2)} = \frac{\pi}{2} \ln \frac{1+\sqrt{1+q}}{\sqrt{1+q}} \quad [q > -1].$$

BI ((235))(10)

$$7. \int_0^1 \arcsin x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1].$$

BI ((234))(2)

$$8. \int_0^1 \arccos x \frac{x}{(1+qx^2)^2} dx = \frac{\pi}{4q} \frac{\sqrt{1+q}-1}{1+q} \quad [q > -1].$$

BI ((234))(4)

4.522

$$1. \int_0^1 x \sqrt{1-k^2x^2} \arccos x dx = \frac{1}{9k^2} \left[\frac{3}{2}\pi + k'^2 \mathbf{K}(k) - 2(1+k'^2) \mathbf{E}(k) \right].$$

BI ((236))(9)

$$2. \int_0^1 x \sqrt{1-k^2x^2} \arcsin x dx = \frac{1}{9k^2} \left[-\frac{3}{2}\pi k'^3 - k'^2 \mathbf{K}(k) + 2(1+k'^2) \mathbf{E}(k) \right].$$

$$3. \int_0^1 x \sqrt{k'^2 + k^2 x^2} \arcsin x \, dx = \frac{1}{9k^2} \left[\frac{3}{2} \pi + k'^2 \mathbf{K}(k) - 2(1 + k'^2) \mathbf{E}(k) \right].$$

BI((236))(5)

$$4. \int_0^1 \frac{x \arcsin x}{\sqrt{1 - k^2 x^2}} \, dx = \frac{1}{k^2} \left[-\frac{\pi}{2} k' + \mathbf{E}(k) \right].$$

BI ((237))(1)

$$5. \int_0^1 \frac{x \arccos x}{\sqrt{1 - k^2 x^2}} \, dx = \frac{1}{k^2} \left[\frac{\pi}{2} - \mathbf{E}(k) \right].$$

BI ((240))(1)

$$6. \int_0^1 \frac{x \arcsin x}{\sqrt{k'^2 + k^2 x^2}} \, dx = \frac{1}{k^2} \left[\frac{\pi}{2} - \mathbf{E}(k) \right].$$

BI ((238))(1)

$$7. \int_0^1 \frac{x \arccos x}{\sqrt{k'^2 + k^2 x^2}} \, dx = \frac{1}{k^2} \left[-\frac{\pi}{2} k' + \mathbf{E}(k) \right].$$

BI ((241))(1)

$$8. \int_0^1 \frac{x \arcsin x \, dx}{(x^2 - \cos^2 \lambda) \sqrt{1 - x^2}} = \frac{2}{\sin \lambda} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\lambda]}{(2k+1)^2}.$$

BI ((243))(11)

$$9. \int_0^1 \frac{x \arcsin kx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \, dx = -\frac{\pi}{2k} \ln k'.$$

BI ((239))(1)

$$10. \int_0^1 \frac{x \arccos kx}{\sqrt{(1-x^2)(1-k^2 x^2)}} \, dx = \frac{\pi}{2k} \ln(1+k).$$

BI ((242))(1)

$$1. \int_0^1 x^{2n} \arcsin x \, dx = \frac{1}{2n+1} \left[\frac{\pi}{2} - \frac{2^n n!}{(2n+1)!!} \right].$$

BI ((229))(1)

$$2. \int_0^1 x^{2n-1} \arcsin x \, dx = \frac{\pi}{4n} \left[1 - \frac{(2n-1)!!}{2^n n!} \right].$$

BI ((229))(2)

$$3. \int_0^1 x^{2n} \arccos x \, dx = \frac{2^n n!}{(2n+1)(2n+1)!!}.$$

BI ((229))(4)

$$4. \int_0^1 x^{2n-1} \arccos x \, dx = \frac{\pi}{4n} \frac{(2n-1)!!}{2^n n!}.$$

BI ((229))(5)

$$5. \int_{-1}^1 (1-x^2)^n \arccos x \, dx = \pi \frac{2^n n!}{(2n+1)!!}.$$

BI ((254))(2)

$$6. \int_{-1}^1 (1-x^2)^{n-\frac{1}{2}} \arccos x \, dx = \frac{\pi^2}{2} \frac{(2n-1)!!}{2^n n!}.$$

BI ((254))(3)

4.524

$$1. \int_0^1 (\arcsin x)^2 \frac{dx}{x^2 \sqrt{1-x^2}} = \pi \ln 2.$$

BI ((243))(13)

$$2. \int_0^1 (\arccos x)^2 \frac{dx}{(\sqrt{1-x^2})^3} = \pi \ln 2.$$

4.53- 4.54 Combinations of arctangents, arccotangents, and powers

4.531

$$1. \int_0^1 \frac{\operatorname{arctg} x}{x} dx = \int_1^\infty \frac{\operatorname{arcctg} x}{x} dx = \mathbf{G}.$$

FI II 482, BI ((253))(8)

$$2. \int_0^\infty \frac{\operatorname{arcctg} x}{1 \pm x} dx = \pm \frac{\pi}{4} \ln 2 + \mathbf{G}.$$

BI ((248))(6, 7)

$$3. \int_0^1 \frac{\operatorname{arcctg} x}{x(1+x)} dx = -\frac{\pi}{8} \ln 2 + \mathbf{G}.$$

BI ((235))(11)

$$4. \int_0^\infty \frac{\operatorname{arctg} x}{1-x^2} dx = -\mathbf{G}.$$

BI ((248))(2)

$$5. \int_0^1 \operatorname{arctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{(1+p)^2}{1+q^2} + \frac{q^2-p}{(1+p)(p^2+q^2)} \operatorname{arctg} q \quad [p > -1].$$

BI ((243))(7)

$$6. \int_0^1 \operatorname{arcctg} qx \frac{dx}{(1+px)^2} = \frac{1}{2} \frac{q}{p^2+q^2} \ln \frac{1+q^2}{(1+p)^2} + \frac{p}{p^2+q^2} \operatorname{arctg} q + \frac{1}{1+p} \operatorname{arcctg} q \quad [p > -1].$$

BI ((234))(10)

635

$$7. \int_0^1 \frac{\operatorname{arctg} x}{x(1+x^2)} dx = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

$$8. \int_0^{\infty} \frac{x \operatorname{arctg} x}{1+x^4} dx = \frac{\pi^2}{16}.$$

BI ((248))(3)

$$9. \int_0^{\infty} \frac{x \operatorname{arctg} x}{1-x^4} dx = -\frac{\pi}{8} \ln 2.$$

BI ((248))(4)

$$10. \int_0^{\infty} \frac{x \operatorname{arctg} x}{1-x^4} dx = \frac{\pi}{8} \ln 2.$$

BI ((248))(12)

$$11. \int_0^{\infty} \frac{\operatorname{arcctg} x}{x\sqrt{1+x^2}} dx = \int_0^{\infty} \frac{\operatorname{arcctg} x}{\sqrt{1+x^2}} dx = 2\mathbf{G}.$$

BI ((251))(3, 10)

$$12. \int_0^1 \frac{\operatorname{arctg} x}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(1+\sqrt{2}).$$

FI II 694

$$13. \int_0^1 \frac{x \operatorname{arctg} x dx}{\sqrt{(1+x^2)(1+k'^2x^2)}} = \frac{1}{k^2} \left[F\left(\frac{\pi}{4}, k\right) - \frac{\pi}{2\sqrt{2(1+k'^2)}} \right].$$

BI ((294))(14)

4.532

$$1. \int_0^1 x^p \operatorname{arctg} x dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} - \beta\left(\frac{p}{2} + 1\right) \right] \quad [p > -2].$$

BI ((229))(7)

$$2. \int_0^{\infty} x^p \operatorname{arctg} x dx = \frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2} \quad [-1 > p > -2].$$

BI ((246))(1)

$$3. \int_0^1 x^p \operatorname{arcctg} x dx = \frac{1}{2(p+1)} \left[\frac{\pi}{2} + \beta\left(\frac{p}{2} + 1\right) \right] \quad [p > -1].$$

$$4. \int_0^{\infty} x^p \operatorname{arctg} x \, dx = -\frac{\pi}{2(p+1)} \operatorname{cosec} \frac{p\pi}{2} \quad [-1 < p < 0].$$

BI ((246))(2)

$$5. \int_0^{\infty} \left(\frac{x^p}{1+x^{2p}} \right)^{2q} \operatorname{arctg} x \frac{dx}{x} = \frac{\sqrt{\pi^3}}{2^{2q+2} p} \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)} \quad [q > 0].$$

BI ((250))(10)

4.533

$$1. \int_0^{\infty} (1 - x \operatorname{arctg} x) \, dx = \frac{\pi}{4}.$$

BI ((246))(3)

$$2. \int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x \right) \frac{dx}{1-x} = -\frac{\pi}{8} \ln 2 + \mathbf{G}.$$

BI ((232))(2)

$$3. \int_0^1 \left(\frac{\pi}{4} - \operatorname{arctg} x \right) \frac{1+x}{1-x} \frac{dx}{1+x^2} = \frac{\pi}{8} \ln 2 + \frac{1}{2} \mathbf{G}.$$

BI ((235))(25)

636

$$4. \int_0^1 \left(x \operatorname{arctg} x - \frac{1}{x} \operatorname{arctg} x \right) \frac{dx}{1-x^2} = -\frac{\pi}{4} \ln 2.$$

BI ((232))(1)

4.534

$$\int_0^{\infty} (\operatorname{arctg} x)^2 \frac{dx}{x^2 \sqrt{1+x^2}} = \int_0^{\infty} (\operatorname{arctg} x)^2 \frac{x \, dx}{\sqrt{1+x^2}} = -\frac{\pi^2}{4} + 4 \mathbf{G}.$$

BI ((251))(9, 17)

$$1. \int_0^1 \frac{\operatorname{arctg} px}{1+p^2x} dx = \frac{1}{2p^2} \operatorname{arctg} p \ln(1+p^2).$$

BI ((231))(19)

$$2. \int_0^1 \frac{\operatorname{arcctg} px}{1+p^2x} dx = \frac{1}{p^2} \left\{ \frac{\pi}{4} + \frac{1}{2} \operatorname{arcctg} p \right\} \ln(1+p^2) \quad [p > 0].$$

BI ((231))(24)

$$3. \int_0^\infty \frac{\operatorname{arctg} qx}{(p+x)^2} dx = -\frac{q}{1+p^2q^2} \left(\ln pq - \frac{\pi}{2} pq \right) \quad [p > 0, \quad q > 0].$$

BI ((249))(1)

$$4. \int_0^\infty \frac{\operatorname{arcctg} qx}{(p+x)^2} dx = \frac{q}{1+p^2q^2} \left(\ln pq + \frac{\pi}{2} pq \right) \quad [p > 0, \quad q > 0].$$

BI ((249))(8)

$$5. \int_0^\infty \frac{x \operatorname{arcctg} px}{q^2+x^2} dx = \frac{\pi}{2} \ln \frac{1+pq}{pq} \quad [p > 0, \quad q > 0].$$

BI ((248))(9)

$$6. \int_0^\infty \frac{x \operatorname{arcctg} px}{x^2-q^2} dx = \frac{\pi}{4} \ln \frac{1+p^2q^2}{p^2q^2} \quad [p > 0, \quad q > 0].$$

BI ((248))(10)

$$7. \int_0^\infty \frac{\operatorname{arctg} px}{x(1+x^2)} dx = \frac{\pi}{2} \ln(1+p) \quad [p \geq 0].$$

FI II 745

$$8. \int_0^\infty \frac{\operatorname{arctg} px}{x(1-x^2)} dx = \frac{\pi}{4} \ln(1+p^2) \quad [p \geq 0].$$

BI ((250))(6)

BI ((250))(3)

$$10. \int_0^{\infty} \operatorname{arctg} qx \frac{dx}{x(1-p^2x^2)} = \frac{\pi}{4} \ln \frac{p^2+q^2}{p^2} \quad [p \geq 0].$$

BI ((250))(6)

$$11. \int_0^{\infty} \frac{x \operatorname{arctg} qx}{(p^2+x^2)^2} dx = \frac{\pi q}{4p(1+pq)} \quad [p > 0, \quad q \geq 0].$$

BI ((252))(12)a

$$12. \int_0^{\infty} \frac{x \operatorname{arctg} qx}{(p^2+x^2)^2} dx = \frac{\pi}{4p^2(1+pq)} \quad [p > 0, \quad q \geq 0].$$

BI ((252))(20)a

$$13. \int_0^1 \frac{\operatorname{arctg} qx}{x\sqrt{1-x^2}} dx = \frac{\pi}{2} \ln(q + \sqrt{1+q^2}).$$

BI ((244))(11)

$$14.* \int_{-\infty}^{\infty} \frac{x \arctan(\alpha x) dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \frac{\pi}{\beta^2-\gamma^2} \log\left(\frac{1+|\alpha\beta|}{1+|\alpha\gamma|}\right) \operatorname{sign}(\alpha) & (\alpha, \beta, \gamma \text{ real}; \beta \neq \gamma) \\ \frac{\pi\alpha}{2|\beta|(1+|\alpha\beta|)} & (\beta = \gamma) \end{cases}$$

637

$$15.* \int_{-\infty}^{\infty} \frac{x \arctan(\alpha/x) dx}{(x^2+\beta^2)(x^2+\gamma^2)} = \begin{cases} \frac{\pi}{\beta^2-\gamma^2} \log\left(\frac{1+|\alpha/\gamma|}{1+|\alpha/\beta|}\right) \operatorname{sign}(\alpha) & (\alpha, \beta, \gamma \text{ real}; \beta \neq \gamma) \\ \frac{\pi\alpha}{2\beta^2(|\beta|+|\alpha|)} & (\beta = \gamma) \end{cases}$$

4.536

$$1. \int_0^{\infty} \operatorname{arctg} qx \arcsin x \frac{dx}{x^2} = \frac{1}{2} q\pi \ln \frac{1+\sqrt{1+q^2}}{\sqrt{1+q^2}} + \frac{\pi}{2} \ln(q + \sqrt{1+q^2}) - \frac{\pi}{2} \operatorname{arctg} q.$$

$$2. \int_0^{\infty} \frac{\operatorname{arctg} px - \operatorname{arctg} qx}{x} dx = \frac{\pi}{2} \ln \frac{p}{q} \quad [p > 0, \quad q > 0].$$

FI II 635

$$3. \int_0^{\infty} \frac{\operatorname{arctg} px \operatorname{arctg} qx}{x^2} dx = \frac{\pi}{2} \ln \frac{(p+q)^{p+q}}{p^p q^q} \quad [p > 0, \quad q > 0].$$

FI II 745

4.537

$$1.^8 \int_0^1 \operatorname{arctg}(\sqrt{1-x^2}) \frac{dx}{1-x^2 \cos^2 \lambda} = \frac{\pi}{2 \cos \lambda} \ln \left[\cos \left(\frac{\pi - 4\lambda}{8} \right) \operatorname{cosec} \left(\frac{\pi + 4\lambda}{8} \right) \right].$$

BI ((245))(9)

$$2. \int_0^1 \operatorname{arctg} \left(p\sqrt{1-x^2} \right) \frac{dx}{1-x^2} = \frac{1}{2} \pi \ln \left(p + \sqrt{1+p^2} \right) \quad [p > 0].$$

BI ((245))(10)

$$3. \int_0^1 \operatorname{arctg} \left(\operatorname{tg} \lambda \sqrt{1-k^2 x^2} \right) \sqrt{\frac{1-x^2}{1-k^2 x^2}} dx = \frac{\pi}{2k^2} [E(\lambda, k) - k'^2 F(\gamma, k)] - \frac{\pi}{2k^2} \operatorname{ctg} \gamma \left(1 - \sqrt{1-k^2 \sin^2 \gamma} \right).$$

BI ((245))(12)

$$4. \int_0^1 \operatorname{arctg} \left(\operatorname{tg} \lambda \sqrt{1-k^2 x^2} \right) \sqrt{\frac{1-k^2 x^2}{1-x^2}} dx = \frac{\pi}{2} E(\lambda, k) - \frac{\pi}{2} \operatorname{ctg} \lambda \left(1 - \sqrt{1-k^2 \sin^2 \lambda} \right).$$

BI ((245))(11)

$$5. \int_0^1 \frac{\operatorname{arctg} \left(\operatorname{tg} \lambda \sqrt{1-k^2 x^2} \right)}{\sqrt{(1-x^2)(1-k^2 x^2)}} dx = \frac{\pi}{2} F(\lambda, k).$$

BI ((245))(13)

4.538

$$1. \int_0^{\infty} \operatorname{arctg} x^2 \frac{dx}{1+x^2} = \int_0^{\infty} \operatorname{arctg} x^3 \frac{dx}{1+x^2};$$

$$= \int_0^{\infty} \operatorname{arcctg} x^2 \frac{dx}{1+x^2} = \int_0^{\infty} \operatorname{arcctg} x^3 \frac{dx}{1+x^2} = \frac{\pi^2}{8}.$$

$$2. \int_0^{\infty} \frac{1-x^2}{x^2} \operatorname{arctg} x^2 dx = \frac{\pi}{2}(\sqrt{2}-1).$$

BI ((244))(10)a

4.539

$$\int_0^{\infty} x^{s-1} \operatorname{arctg}(ae^{-x}) dx = 2^{-s-1} \Gamma(s) a \Phi\left(-a^2, s+1, \frac{1}{2}\right).$$

ET I 222(47)

4.541

$$\int_0^{\infty} \operatorname{arctg}\left(\frac{p \sin qx}{1+p \cos qx}\right) \frac{x dx}{1+x^2} = \frac{\pi}{2} \ln(1+pe^{-q}) \quad [p > -e^q].$$

BI ((341))(14)a

4.55 Combinations of inverse trigonometric functions and exponentials

4.551

$$1. \int_0^1 (\arcsin x) e^{-bx} dx = \frac{\pi}{2b} [I_0(b) - L_0(b)].$$

ET I 160(1)

$$2. \int_0^1 x(\arcsin x) e^{-bx} dx = \frac{\pi}{2b^2} [L_0(b) - I_0(b) + bL_1(b) - bI_1(b)] + \frac{1}{b}.$$

ET I 161(2)

$$3.* \int_0^{\infty} \left(\operatorname{arctg} \frac{x}{a}\right) e^{-bx} dx = \frac{1}{b} [\operatorname{ci}(ab) \sin(ab) - \operatorname{si}(ab) \cos(ab)] \quad [\operatorname{Re} b > 0].$$

ET I 161(3)

$$4.* \int_0^{\infty} \left(\operatorname{arcctg} \frac{x}{a}\right) e^{-bx} dx = \frac{1}{b} \left[\frac{\pi}{2} - \operatorname{ci}(ab) \sin(ab) + \operatorname{si}(ab) \cos(ab)\right] \quad [\operatorname{Re} b > 0].$$

ET I 161(4)

4.552

$$\int_0^{\infty} \frac{\operatorname{arctg} \frac{x}{q}}{e^{2\pi x} - 1} dx = \frac{1}{2} \left[\ln \Gamma(q) - \left(q - \frac{1}{2} \right) \ln q + q - \frac{1}{2} \ln 2\pi \right] \quad [q > 0].$$

WH

4.553

$$\int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arccctg} x - e^{-px} \right) \frac{dx}{x} = C + \ln p \quad [p > 0].$$

NT 66(12)

4.56 A combination of the arctangent and a hyperbolic function

4.561

$$\int_{-\infty}^{\infty} \frac{\operatorname{arctg} e^{-x}}{\operatorname{ch}^{2q} px} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\Pi(x)}{\operatorname{ch}^{2q} px} dx = \frac{\sqrt{\pi^3}}{4p} \frac{\Gamma(q)}{\Gamma\left(q + \frac{1}{2}\right)} \quad [q > 0].$$

LI ((282))(10)

4.57 Combinations of inverse and direct trigonometric functions

4.571

$$\int_0^{\frac{\pi}{2}} \arcsin(k \sin x) \frac{\sin x dx}{\sqrt{1 - k^2 \sin^2 x}} = -\frac{\pi}{2k} \ln k'.$$

BI ((344))(2)

4.572

$$\int_0^{\infty} \left(\frac{2}{\pi} \operatorname{arccctg} x - \cos px \right) dx = C + \ln p \quad [p > 0].$$

NT 66(12)

4.573

$$1. \int_0^{\infty} \operatorname{arccctg} qx \sin px dx = \frac{\pi}{2p} \left(1 - e^{-\frac{p}{q}} \right) \quad [p > 0, \quad q > 0]$$

$$2. \int_0^{\infty} \operatorname{arctg} qx \cos px \, dx = \frac{1}{2p} \left[e^{-\frac{p}{q}} \operatorname{Ei} \left(\frac{p}{q} \right) - e^{\frac{p}{q}} \operatorname{Ei} \left(-\frac{p}{q} \right) \right] \quad [p > 0, \quad q > 0]$$

BI ((347))(2)a

$$3. \int_0^{\infty} \operatorname{arctg} rx \frac{\sin px \, dx}{1 \pm 2q \cos px + q^2} = \pm \frac{\pi}{2pq} \ln \frac{1 \pm q}{1 \pm qe^{-\frac{p}{r}}} \quad [p^2 < 1, \quad r > 0, \quad p > 0];$$

$$= \pm \frac{\pi}{2pq} \ln \frac{q \pm 1}{q \pm e^{-\frac{p}{r}}} \quad [q^2 > 1, \quad r > 0, \quad p > 0].$$

BI ((347))(10)

$$4. \int_0^{\infty} \operatorname{arctg} px \frac{\operatorname{tg} x \, dx}{q^2 \cos^2 x + r^2 \sin^2 x} = \frac{\pi}{2r^2} \ln \left(1 + \frac{r}{q} \theta \frac{1}{p} \right) \quad [p > 0, \quad q > 0, \quad r > 0].$$

BI ((347))(9)

4.574

$$1. \int_0^{\infty} \operatorname{arctg} \left(\frac{2a}{x} \right) \sin(bx) \, dx = \frac{\pi}{b} e^{-ab} \operatorname{sh}(ab) \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 87(8)

$$2.7 \int_0^{\infty} \operatorname{arctg} \frac{a}{x} \cos(bx) \, dx = \frac{1}{2b} [e^{-ab} \operatorname{Ei}(ab) - e^{ab} \operatorname{Ei}(-ab)] \quad [a > 0, \quad b > 0].$$

ET I 29(7)

$$3. \int_0^{\infty} \operatorname{arctg} \left[\frac{2ax}{x^2 + c^2} \right] \sin(bx) \, dx = \frac{\pi}{b} e^{-b\sqrt{a^2+c^2}} \operatorname{sh}(ab) \quad [b > 0].$$

ET I 87(9)

$$4. \int_0^{\infty} \operatorname{arctg} \left(\frac{2}{x^2} \right) \cos(bx) \, dx = \frac{\pi}{b} e^{-b} \sin b \quad [b > 0].$$

ET I 29(8)

4.575

$$1. \int_0^{\pi} \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \sin nx \, dx = \frac{\pi}{2n} p^n \quad [p^2 < 1].$$

$$2. \int_0^\pi \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \sin nx \cos x dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} + \frac{p^{n-1}}{n-1} \right) \quad [p^2 < 1].$$

BI ((345))(5)

$$3. \int_0^\pi \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \cos nx \sin x dx = \frac{\pi}{4} \left(\frac{p^{n+1}}{n+1} - \frac{p^{n-1}}{n-1} \right) \quad [p^2 < 1].$$

BI ((345))(6)

4.576

$$1. \int_0^\pi \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \frac{dx}{\sin x} = \frac{\pi}{2} \ln \frac{1+p}{1-p} \quad [p^2 < 1].$$

BI((346))(1)

$$2. \int_0^\pi \operatorname{arctg} \frac{p \sin x}{1 - p \cos x} \frac{dx}{\operatorname{tg} x} = -\frac{\pi}{2} \ln(1 - p^2) \quad [p^2 < 1].$$

BI((346))(3)

640

4.577

$$1. \int_0^{\frac{\pi}{2}} \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1 - k^2 \sin^2 x}) \frac{\sin^2 x dx}{\sqrt{1 - k^2 \sin^2 x}} = \\ = \frac{\pi}{2k^2} [F(\lambda, k) - E(\lambda, k) + \operatorname{ctg} \lambda (1 - \sqrt{1 - k^2 \sin^2 \lambda})].$$

BI ((344))(4)

$$2. \int_0^{\frac{\pi}{2}} \operatorname{arctg}(\operatorname{tg} \lambda \sqrt{1 - k^2 \sin^2 x}) \frac{\cos^2 x dx}{\sqrt{1 - k^2 \sin^2 x}} = \\ = \frac{\pi}{2k^2} [E(\lambda, k) - k'^2 F(\lambda, k) + \operatorname{ctg} \lambda (\sqrt{1 - k^2 \sin^2 \lambda} - 1)].$$

BI ((344))(5)

4.58 A combination involving an inverse and a direct trigonometric function and a power

4.581.

$$\int_0^{\infty} \operatorname{arctg} x \cos px \frac{dx}{x} = \int_0^{\infty} \operatorname{arctg} \frac{x}{p} \cos x \frac{dx}{x} = -\frac{\pi}{2} \operatorname{Ei}(-p) \quad [\operatorname{Re}(p) > 0].$$

ET I 29(3), NT 25(13)

4.59 Combinations of inverse trigonometric functions and logarithms

4.591

$$1. \int_0^1 \arcsin x \ln x \, dx = 2 - \ln 2 - \frac{1}{2}\pi.$$

BI ((339))(1)

$$2. \int_0^1 \arccos x \ln x \, dx = \ln 2 - 2.$$

BI ((339))(2)

4.592

$$\int_0^1 \arccos x \frac{dx}{\ln x} = -\sum_{k=0}^{\infty} \frac{(2k-1)!! \ln(2k+2)}{2^k k! (2k+1)}.$$

BI ((339))(8)

4.593

$$1. \int_0^1 \operatorname{arctg} x \ln x \, dx = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{1}{48} \pi^2.$$

BI ((339))(3)

$$2. \int_0^1 \operatorname{arctg} x \ln x \, dx = -\frac{1}{48} \pi^2 - \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

BI ((339))(4)

4.594

$$\int_0^1 \operatorname{arctg} x (\ln x)^{n-1} (\ln x + n) \, dx = \frac{n!}{(-2)^{n+1}} (2^{-n} - 1) \zeta(n+1).$$

4.6 Multiple Integrals

4.60 Change of variables in multiple integrals

4.601

$$1. \iint_{(\sigma)} f(x, y) dx dy = \iint_{(\sigma')} f[\varphi(u, v), \psi(u, v)] |\Delta| du dv,$$

641

where $x = \varphi(u, v)$, $y = \psi(u, v)$, and $\Delta = \frac{\partial \varphi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \psi}{\partial u} \frac{\partial \varphi}{\partial v} \equiv \frac{\Delta(\varphi, \psi)}{\Delta(u, v)}$ is the Jacobian determinant of the functions φ and ψ .

$$2. \iiint_{(V)} f(x, y, z) dx dy dz = \iiint_{(V')} f[\varphi(u, v, w), \psi(u, v, w), \text{chi}(u, v, w)] |\Delta| du dv dw,$$

where $x = \varphi(u, v, w)$, $y = \psi(u, v, w)$, and $z = \text{chi}(u, v, w)$ and where

$$\Delta = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & \frac{\partial \varphi}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \text{chi}}{\partial u} & \frac{\partial \text{chi}}{\partial v} & \frac{\partial \text{chi}}{\partial w} \end{vmatrix} \equiv \frac{D(\varphi, \psi, \text{chi})}{D(u, v, w)}$$

is the Jacobian determinant of the functions φ , ψ , and chi .

Here, we assume, both in (4.601 2.) and in (4.601 1.) that

(a) the functions φ , ψ , and chi and also their first partial derivatives are continuous in the region of integration;

(b) the Jacobian does not change sign in this region;

(c) there exists a one-to-one correspondence between the old variables x , y , z and the new ones u , v , w in the region of integration;

(d) when we change from the variables x, y, z to the variables u, v, w , the region V (resp. σ) is mapped into the region V' (resp. σ').

4.602

Transformation to polar coordinates:

$$x = r \cos \varphi, \quad y = r \sin \varphi; \quad \frac{D(x, y)}{D(r, \varphi)} = r.$$

4.603

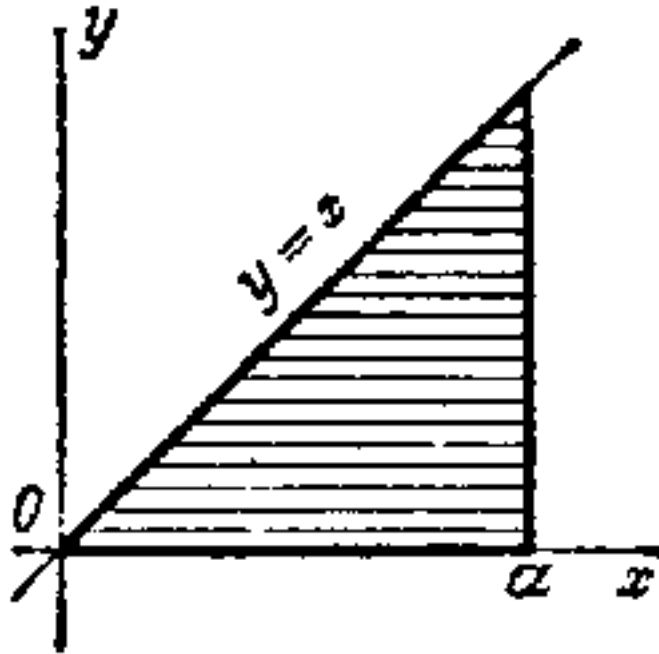
Transformation to spherical coordinates:

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \quad \frac{D(x, y, z)}{D(r, \theta, \varphi)} = r^2 \sin \theta.$$

4.61 Change of the order of integration and change of variables

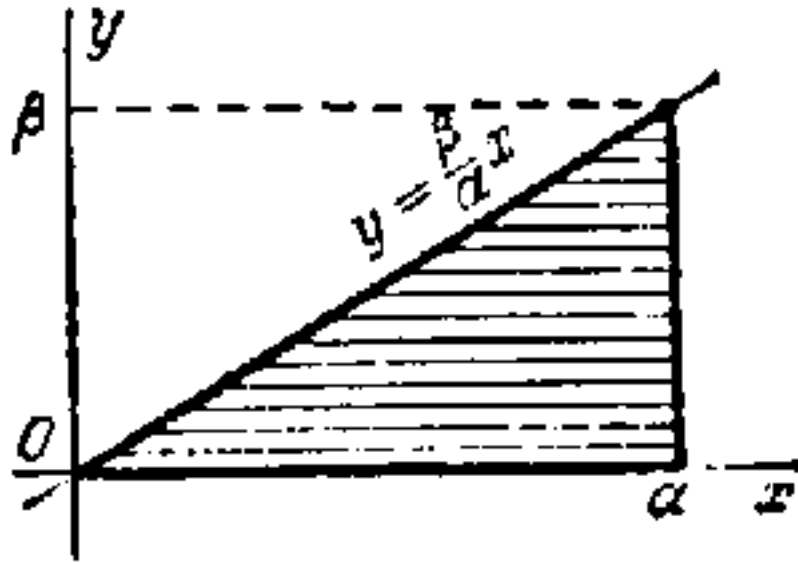
4.611

- $$\int_0^\alpha dx \int_0^x f(x, y) dy = \int_0^\alpha dy \int_y^\alpha f(x, y) dx.$$



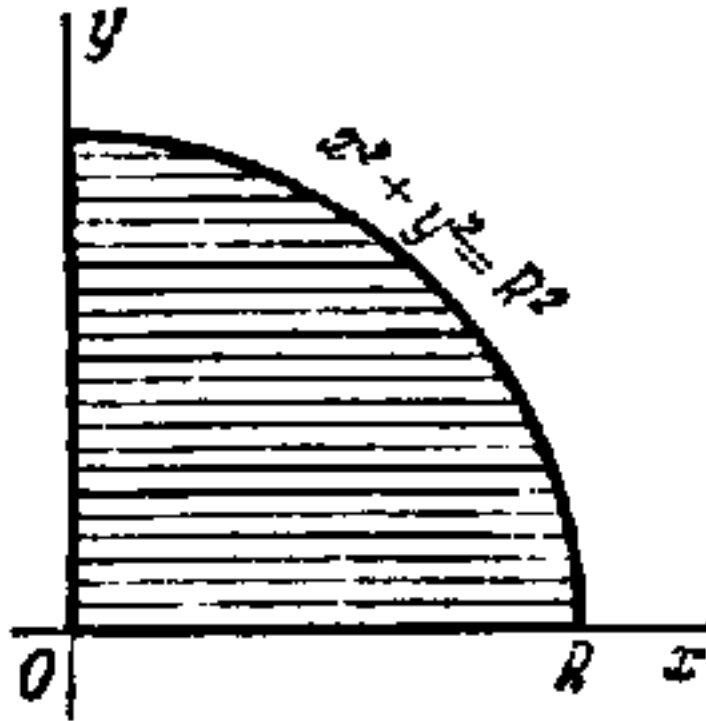
642

$$2. \int_0^{\alpha} dx \int_0^{\frac{\beta}{\alpha}x} f(x, y) dy = \int_0^{\beta} dy \int_{\frac{\alpha}{\beta}y}^{\alpha} f(x, y) dx.$$

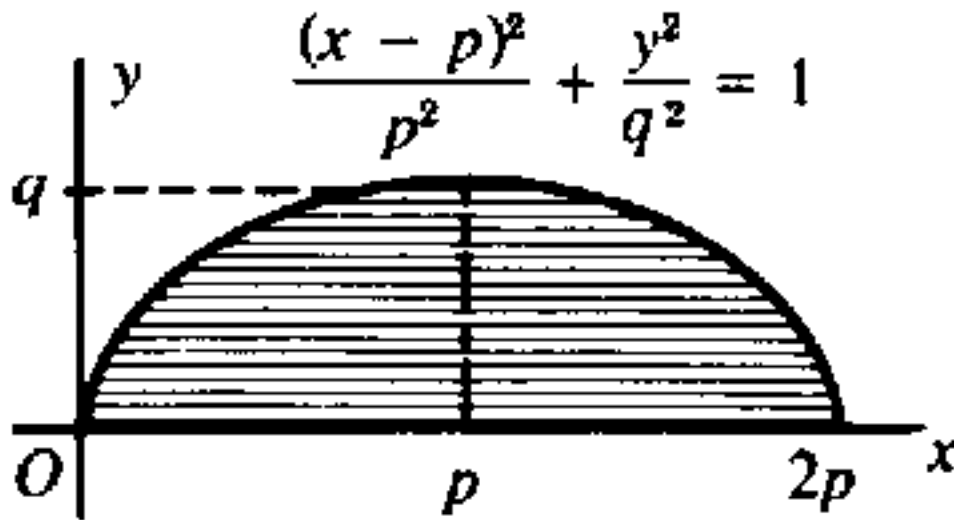


4.612

$$\begin{aligned}
 1. \quad & \int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x, y) dy = \\
 & = \int_0^R dy \int_0^{\sqrt{R^2-y^2}} f(x, y) dx.
 \end{aligned}$$

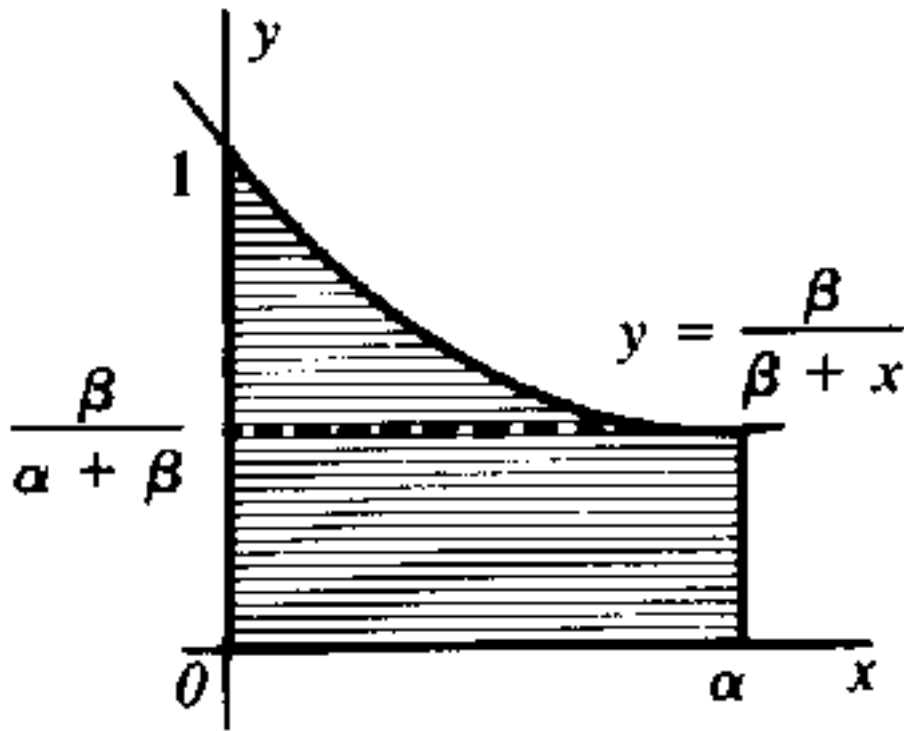


$$\begin{aligned}
 2. \quad & \int_0^{2p} dx \int_0^{q/p\sqrt{2px-x^2}} f(x, y) dy = \\
 & = \int_0^q dy \int_{p[1-\sqrt{1-(y/q)^2}]}^{p[1+\sqrt{1-(y/q)^2}]} f(x, y) dx.
 \end{aligned}$$

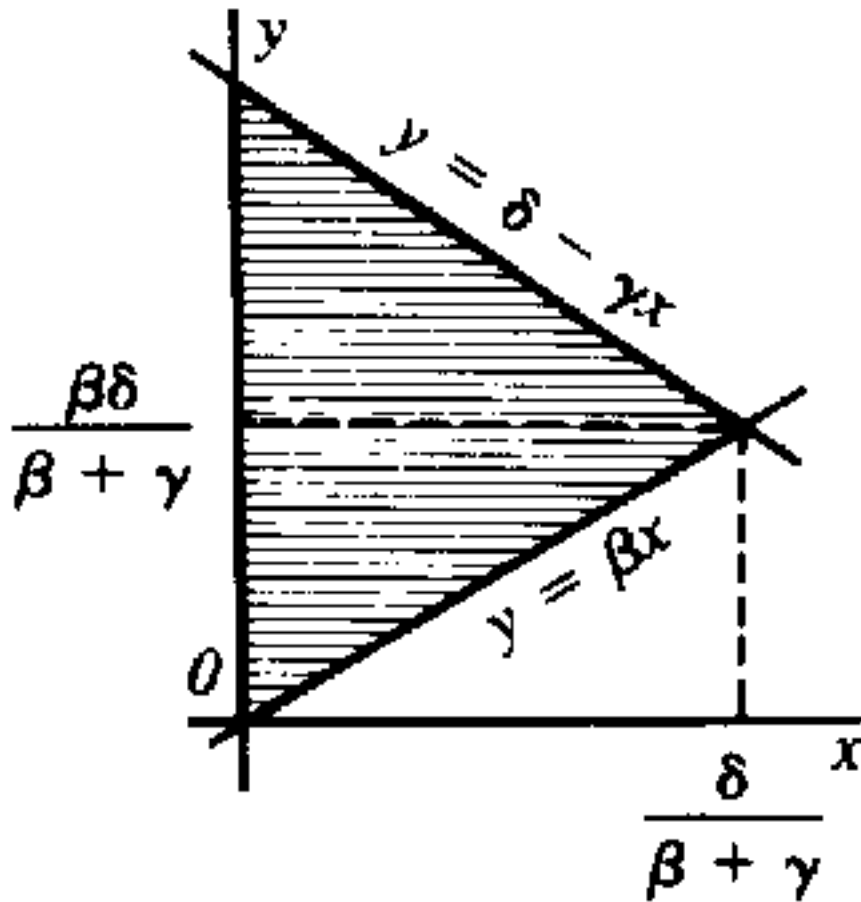


4.613

$$\begin{aligned}
 1. \quad & \int_0^\alpha dx \int_0^{\beta/(\beta+x)} f(x, y) dy = \\
 & = \int_0^{\beta/(\beta+\alpha)} dy \int_0^\alpha f(x, y) dx + \\
 & \quad + \int_{\beta/(\beta+\alpha)}^1 dy \int_0^{\beta(1-y)/y} f(x, y) dx.
 \end{aligned}$$

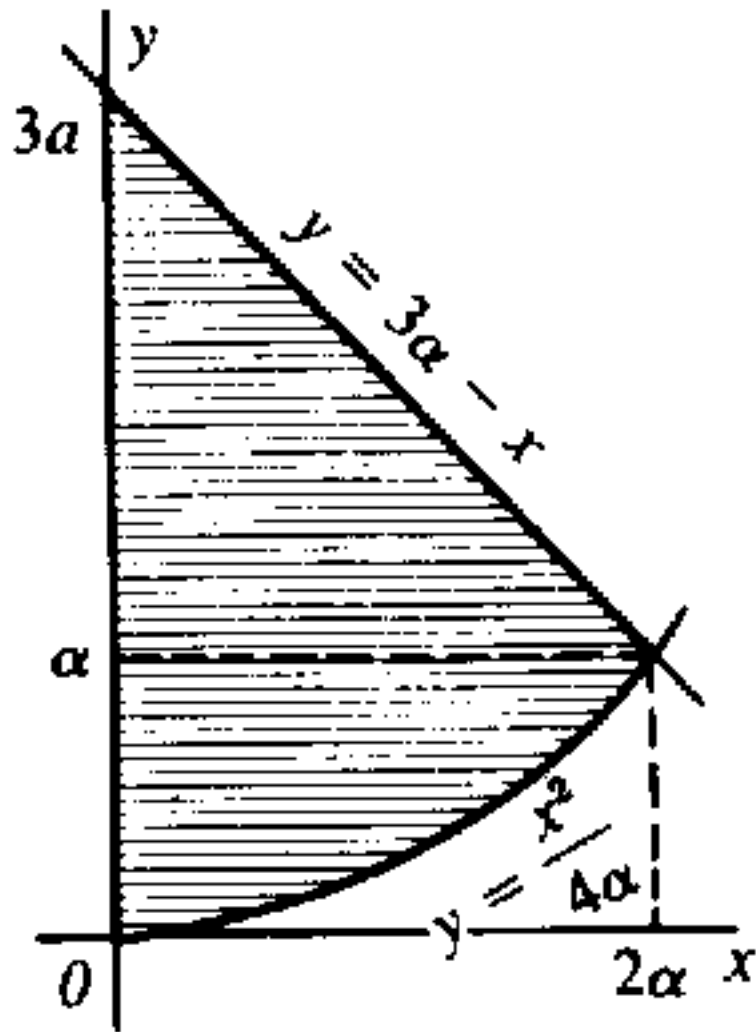


$$\begin{aligned}
 2. \quad & \int_0^\alpha dx \int_{\beta x}^{\delta - \nu x} f(x, y) dy = \\
 & = \int_0^{\alpha\beta} dy \int_0^{y/\beta} f(x, y) dx + \\
 & \quad + \int_{\alpha\beta}^\delta dy \int_0^{(\delta-y)/\gamma} f(x, y) dx. \\
 & \left[\alpha = \frac{\delta}{\beta + \gamma}, \quad \alpha > 0, \quad \beta > 0, \quad \gamma > 0 \right].
 \end{aligned}$$

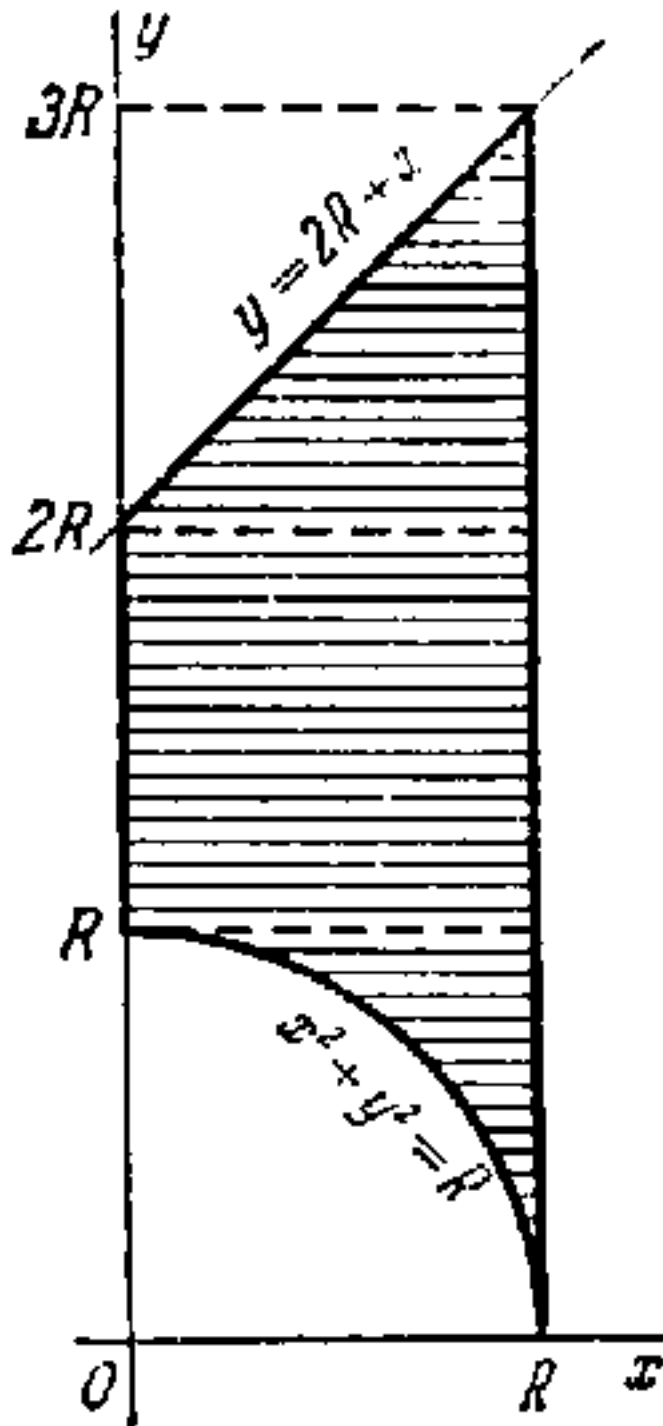


643

$$\begin{aligned}
 3. \int_0^{2\alpha} dx \int_{x^2/4\alpha}^{3\alpha-x} f(x, y) dy &= \\
 &= \int_0^\alpha dy \int_0^{2\sqrt{\alpha y}} f(x, y) dx + \\
 &\quad + \int_\alpha^{3\alpha} dy \int_0^{3\alpha-y} f(x, y) dx.
 \end{aligned}$$



$$\begin{aligned}
 4. \int_0^R dx \int_{\sqrt{R^2-x^2}}^{x+2R} f(x, y) dy &= \int_0^R dy \int_{\sqrt{R^2-y^2}}^R f(x, y) dx + \\
 &+ \int_R^{2R} dy \int_0^R f(x, y) dx + \\
 &+ \int_{2R}^{3R} dy \int_{y-2R}^R f(x, y) dx.
 \end{aligned}$$

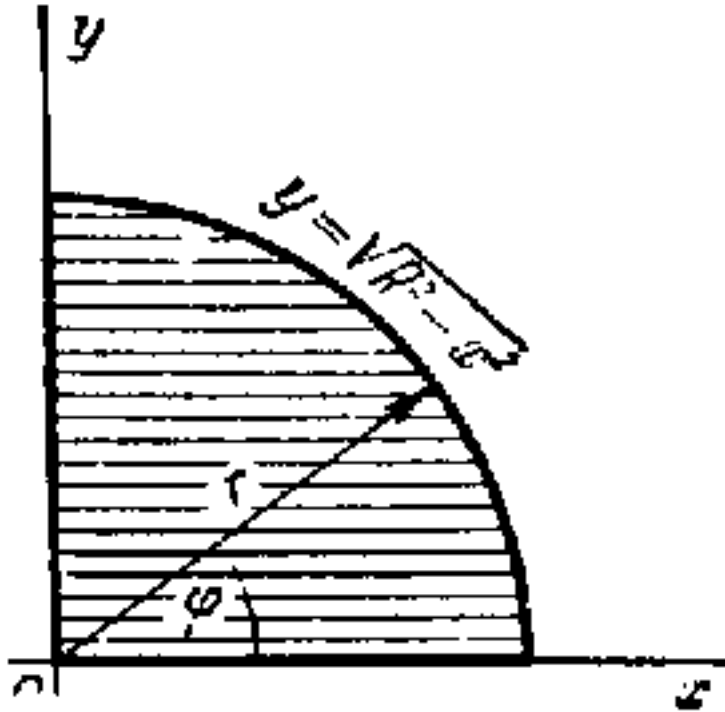


4.614

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{2R \cos \varphi} f(r, \varphi) dr = \int_0^{2R} dr \int_0^{\arccos \frac{r}{2R}} f(r, \varphi) d\varphi.$$

4.615

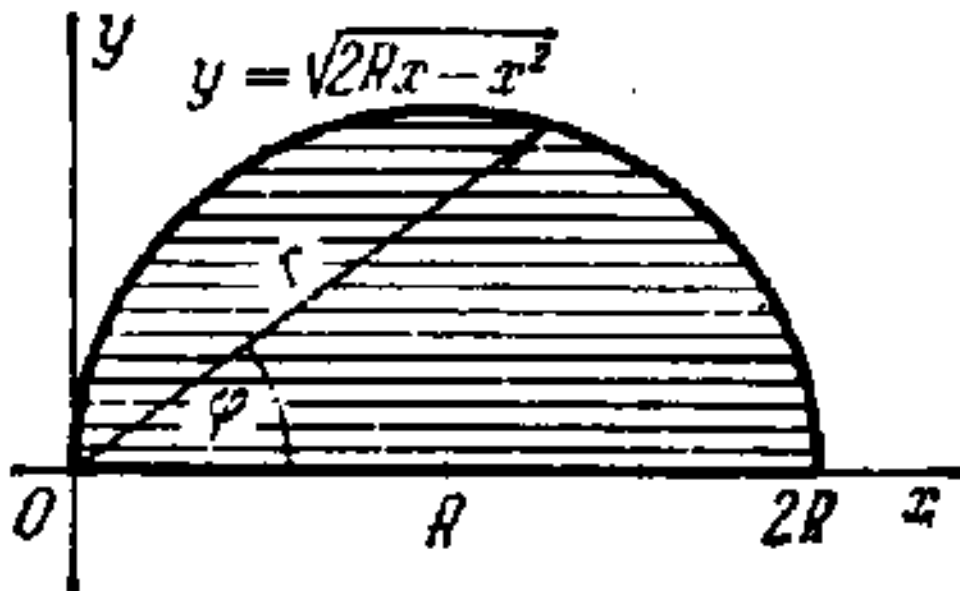
$$\int_0^R dx \int_0^{\sqrt{R^2-x^2}} f(x, y) dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^R f(r \cos \varphi, \sin \varphi) r dr.$$



644

4.616

$$\begin{aligned} \int_0^{2R} dx \int_0^{\sqrt{2R-x^2}} f(x, y) dy &= \\ &= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2R \cos \varphi} f(r \cos \varphi, \sin \varphi) r dr. \end{aligned}$$



4.617

$$\int_{\alpha}^{\beta} dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy = \int_0^{\beta} dx \int_0^{\varphi_2(x)} f(x, y) dy - \int_0^{\beta} dx \int_0^{\varphi_1(x)} f(x, y) dy - \int_0^{\alpha} dx \int_0^{\varphi_2(x)} f(x, y) dy + \int_0^{\alpha} dx \int_0^{\varphi_1(x)} f(x, y) dy [\varphi_1(x) \leq \varphi_2(x) \text{ for } \alpha \leq x \leq \beta].$$

4.618

$$\begin{aligned} \int_0^{\gamma} dx \int_0^{\varphi(x)} f(x, y) dy &= \int_0^{\gamma} dx \int_0^1 f[x, z\varphi(x)]\varphi(x) dz & [y = z\varphi(x)]; \\ &= \gamma \int_0^1 dz \int_0^{\varphi(\gamma z)} f(\gamma z, y) dy & [x = \gamma z]. \end{aligned}$$

4.619

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} f(x, y) dy = \int_{x_0}^{x_1} dx \int_0^1 (y_1 - y_0) f[x, y_0 + (y_1 - y_0)t] dt \quad [y = y_0 + (y_1 - y_0)t].$$

4.62 Double and triple integrals with constant limits

General formulas

$$1. \int_0^\pi d\omega \int_0^\infty f'(p \operatorname{ch} x + q \cos \omega \operatorname{sh} x) \operatorname{sh} x dx = -\frac{\pi \operatorname{sign} p}{\sqrt{p^2 - q^2}} f\left(\operatorname{sign} p \sqrt{p^2 - q^2}\right) \\ \left[p^2 > q^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right]$$

LO III 389

$$2. \int_0^{2\pi} d\omega \int_0^\infty f'[p \operatorname{ch} x + (q \cos \omega + r \sin \omega) \operatorname{sh} x] \operatorname{sh} x dx = \\ = -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2}) \quad \left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

LO III 390

$$3. \int_0^\pi \int_0^\pi \frac{dx dy}{\sin x \sin^2 y} f' \left[\frac{p - q \cos x}{\sin x \sin y} + r \operatorname{ctg} y \right] = -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f\left(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2}\right) \\ \left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

LO III 280

645

$$4. \int_{-\infty}^\infty dx \int_{-\infty}^\infty f'(p \operatorname{ch} x \operatorname{ch} y + q \operatorname{sh} x \operatorname{ch} y + r \operatorname{sh} y) \operatorname{ch} y dy = \\ = -\frac{2\pi \operatorname{sign} p}{\sqrt{p^2 - q^2 - r^2}} f(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2}) \quad \left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

LO III 390

$$5. \int_0^\infty dx \int_0^\pi f(p \operatorname{ch} x + q \cos \omega \operatorname{sh} x) \operatorname{sh}^2 x \sin \omega d\omega = 2 \int_0^\infty f(\operatorname{sign} p \sqrt{p^2 - q^2} \operatorname{ch} x) \operatorname{sh}^2 x dx \\ \left[\lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

LO III 391

$$6. \int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f[p \operatorname{ch} x + (q \cos \omega + r \sin \omega) \sin \theta \operatorname{sh} x] \operatorname{sh}^2 x \sin \theta d\theta = \\ = 4 \int_0^\infty f(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2} \operatorname{ch} x) \operatorname{sh}^2 x dx \quad \left[p^2 > q^2 + r^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

$$7. \int_0^\infty dx \int_0^{2\pi} d\omega \int_0^\pi f\{p \operatorname{ch} x + [(q \cos \omega + r \sin \omega) \sin \theta + s \operatorname{ch} \theta] \operatorname{sh} x\} \operatorname{sh}^2 x \sin \theta d\theta = \\ = 4\pi \int_0^\infty f(\operatorname{sign} p \sqrt{p^2 - q^2 - r^2 - s^2} \operatorname{ch} x) \operatorname{sh}^2 x dx \quad \left[p^2 > q^2 + r^2 + s^2, \quad \lim_{x \rightarrow +\infty} f(x) = 0 \right].$$

LO III 391

4.621

$$1. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = \frac{\pi}{2\sqrt{1 - k^2}}.$$

LO I 252(90)

$$2. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\cos y \sqrt{1 - k^2 \sin^2 x \sin^2 y}}{1 - k^2 \sin^2 y} dx dy = K(k).$$

LO I 252(91)

$$3. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\sin \alpha \sin y dx dy}{\sqrt{1 - \sin^2 \alpha \sin^2 x \sin^2 y}} = \frac{\pi \alpha}{2}.$$

LO I 253

4.622

$$1. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{1 - \cos x \cos y \cos z} = 4\pi K^2 \left(\frac{\sqrt{2}}{2} \right).$$

MO 137

$$2. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos y \cos z - \cos x \cos z - \cos x \cos y} = \sqrt{3}\pi K^2 \left(\sin \frac{\pi}{12} \right).$$

MO 137

$$3. \int_0^\pi \int_0^\pi \int_0^\pi \frac{dx dy dz}{3 - \cos x - \cos y - \cos z} = 4\pi [18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}] K^2 [(2 - \sqrt{3})(\sqrt{3} - \sqrt{2})].$$

MO 137

$$\int_0^\infty \int_0^\infty \varphi(a^2 x^2 + b^2 y^2) dx dy = \frac{\pi}{2ab} \int_0^\infty \varphi(x^2) x dx.$$

$$\begin{aligned} \int_0^\pi \int_0^{2\pi} f(\alpha \cos \theta + \beta \sin \theta \cos \psi + \gamma \sin \theta \sin \psi) \sin \theta d\theta d\psi = \\ = 2\pi \int_0^\pi f(R \cos p) \sin p dp = 2\pi \int_{-1}^1 f(Rt) dt \quad \left[R = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \right]. \end{aligned}$$

$$\begin{aligned} P_l(a, b) &= \int_0^a dx \int_0^b dy (x^2 + y^2 + 1)^{-3/2} P_l \left(1/\sqrt{x^2 + y^2 + 1} \right) \\ P_{2l}(a, b) &= \frac{1}{l(2l+1)2^{2l}} \frac{ab}{\sqrt{a^2 + b^2 + 1}} \sum_{k=0}^{l-1} \frac{(-1)^{l-k-1} 2^{2k} \binom{2l+2k}{l+k} \binom{l+k}{l-k-1}}{\binom{2k}{k} (2k+1)} \times \\ &\quad \times (2l+2k+1) \sum_{j=0}^k \frac{\binom{2j}{j}}{2^{2j}} \frac{1}{(a^2 + b^2 + 1)^j} \left(\frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right). \\ p_{2l+1}(a, b) &= \frac{1}{2^{2l+1}(2l+1)} \sum_{k=0}^l \frac{(-1)^{l+k}}{2^{2k}} \binom{l}{k} \binom{l+k+1}{k} \binom{2l+2k+1}{l+k} \times \\ &\quad \times \left\{ \frac{1}{(b^2 + 1)^k} \frac{b}{\sqrt{b^2 + 1}} \operatorname{arctg}^{-1} \frac{a}{\sqrt{b^2 + 1}} + \frac{1}{(a^2 + 1)^k} \frac{a}{\sqrt{a^2 + 1}} \operatorname{arctg}^{-1} \frac{b}{\sqrt{a^2 + 1}} + \right. \\ &\quad \left. + ab \sum_{j=1}^k \frac{2^{2j-1}}{j \binom{2j}{j}} \cdot \frac{1}{(a^2 + b^2 + 1)^j} \left(\frac{1}{(a^2 + 1)^{k-j+1}} + \frac{1}{(b^2 + 1)^{k-j+1}} \right) \right\}. \end{aligned}$$

4.63-4.64 Multiple integrals

$$\int_p^x dt_{n-1} \int_p^{t_{n-1}} dt_{n-2} \dots \int_p^{t_1} f(t) dt = \frac{1}{(n-1)!} \int_p^x (x-t)^{n-1} f(t) dt,$$

$$\int_p^x dt_{n-1} \int_p^{t_{n-1}} dt_{n-2} \dots \int_p^{t_1} f(t) dt = \frac{1}{(n-1)!} \int_p^x (x-t)^{n-1} f(t) dt,$$

where $f(t)$ is continuous on the interval $[p, q]$ and $p \leq x \leq q$.

4.632

$$1. \quad \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq h}} dx_1 dx_2 \dots dx_n = \frac{h^n}{n!} \quad [\text{the volume of an } n\text{-dimensional simplex}]$$

FI III 472

FI II 692

$$2. \quad \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq R^2} dx_1 dx_2 \dots dx_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n \quad [\text{the volume of an } n\text{-dimensional sphere}]$$

FI III 473

647

4.633

$$\int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1} dx_1 dx_2 \dots dx_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \quad [n > 1]$$

[Half-area of the surface of an $(n+1)$ -dimensional sphere $x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1$].

FI III {474}

4.634⁸

$$\begin{aligned} & \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1} dx_1 dx_2 \dots dx_n = \\ & = \frac{q_1^{p_1} q_2^{p_2} \dots q_n^{p_n}}{\alpha_1 \alpha_2 \dots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} + 1\right)} \quad [\alpha_i > 0, \quad p_i > 0, \quad q_i > 0, \quad i = 1, 2, \dots, n]. \end{aligned}$$

FI III 477

4.635

$$\begin{aligned} 1.^8 \quad & \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \geq 1}} f\left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \dots + \left(\frac{x_n}{q_n}\right)^{\alpha_n}\right] \times \\ & \times x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1} dx_1 dx_2 \dots dx_n = \\ & = \frac{q_1^{p_1} q_2^{p_2} \dots q_n^{p_n}}{\alpha_1 \alpha_2 \dots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \dots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}\right)} \int_1^\infty f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} - 1} dx \end{aligned}$$

$$\begin{aligned}
2.8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ \left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \leq 1}} f \left[\left(\frac{x_1}{q_1}\right)^{\alpha_1} + \left(\frac{x_2}{q_2}\right)^{\alpha_2} + \cdots + \left(\frac{x_n}{q_n}\right)^{\alpha_n} \right] \times \\
& \times x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n = \\
& = \frac{q_1^{p_1} q_2^{p_2} \cdots q_n^{p_n}}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \int_0^1 f(x) x^{\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n} - 1} dx
\end{aligned}$$

FI III 487

under the assumptions that the one-dimensional integral on the right covers absolutely and that the numbers q_i , α_i , and p_i are positive.

648

In particular,

$$\begin{aligned}
3. \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-q(x_1+x_2+\cdots+x_n)} dx_1 dx_2 \cdots dx_n = \\
& = \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)} \int_0^1 x^{p_1+p_2+\cdots+p_n-1} e^{-qx} dx \quad [n > 0, \quad p_1 > 0, \quad p_2 > 0, \dots, p_n > 0].
\end{aligned}$$

FI III 479

$$\begin{aligned}
4.8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \cdots - x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n = \\
& = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(1 - \mu + \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)} \\
& \quad [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad \mu > 1].
\end{aligned}$$

FI III 480

4.636

$$\begin{aligned}
1.8 \quad & \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n} \geq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n = \\
& = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\mu - \frac{p_1}{\alpha_1} - \frac{p_2}{\alpha_2} - \cdots - \frac{p_n}{\alpha_n}\right) \Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \cdots + \frac{p_n}{\alpha_n}\right)}
\end{aligned}$$

$$2.8 \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n})^\mu} dx_1 dx_2 \cdots dx_n =$$

$$= \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_n \left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} - \mu \right)} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\Gamma\left(\frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}\right)}$$

$$\left[\mu < \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n} \right].$$

FI III 480

649

$$3.8 \int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n} \leq 1}} x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} \sqrt{\frac{1 - x_1^{\alpha_1} - x_2^{\alpha_2} - \dots - x_n^{\alpha_n}}{1 + x_1^{\alpha_1} + x_2^{\alpha_2} + \dots + x_n^{\alpha_n}}} dx_1 dx_2 \cdots dx_n =$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p_1}{\alpha_1}\right) \Gamma\left(\frac{p_2}{\alpha_2}\right) \cdots \Gamma\left(\frac{p_n}{\alpha_n}\right)}{\alpha_1 \alpha_2 \cdots \alpha_n} \frac{1}{\Gamma(m)} \left\{ \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \right\},$$

where $m = \frac{p_1}{\alpha_1} + \frac{p_2}{\alpha_2} + \dots + \frac{p_n}{\alpha_n}$.

4.637

$$\int \int \cdots \int_{\substack{x_1 \leq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \dots + x_n \leq 1}} f(x_1 + x_2 + \dots + x_n) \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} dx_1 dx_2 \cdots dx_n}{(q_1 x_1 + q_2 x_2 + \dots + q_n x_n + r)^{p_1 + p_2 + \dots + p_n}} =$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \dots + p_n)} \int_0^1 f(x) \frac{x^{p_1 + p_2 + \dots + p_n - 1}}{(q_1 x + r)^{p_1} (q_2 x + r)^{p_2} \cdots (q_n x + r)^{p_n}} dx,$$

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where $f(x)$ is continuous on the interval $(0, 1)$

$$[q_1 \geq 0, \quad q_2 \geq 0, \dots, q_n \geq 0; \quad r > 0].$$

4.638

$$1. \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1} e^{-(q_1 x_1 + q_2 x_2 + \dots + q_n x_n)}}{(r_0 + r_1 x_1 + r_2 x_2 + \dots + r_n x_n)^s} dx_1 dx_2 \cdots dx_n =$$

$$= \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(s)} \int_0^\infty \frac{e^{-r_0 x} x^{s-1} dx}{(q_1 + r_1 x)^{p_1} (q_2 + r_2 x)^{p_2} \cdots (q_n + r_n x)^{p_n}}$$

$$2. \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1}}{(r_0 + r_1 x_1 + r_2 x_2 + \dots + r_n x_n)^s} dx_1 dx_2 \dots dx_n =$$

$$= \frac{\Gamma(p_1)\Gamma(p_2)\dots\Gamma(p_n)\Gamma(s - p_1 - p_2 - \dots - p_n)}{r_1^{p_1} r_2^{p_2} \dots r_n^{p_n} r_0^{s-p_1-p_2-\dots-p_n} \Gamma(s)} \quad [p_i > 0, \quad r_i > 0, \quad s > 0].$$

$$3. \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{x_1^{p_1-1} x_2^{p_2-1} \dots x_n^{p_n-1}}{[1 + (r_1 x_1)^{q_1} + (r_2 x_2)^{q_2} + \dots + (r_n x_n)^{q_n}]^s} dx_1 dx_2 \dots dx_n =$$

$$= \frac{\Gamma\left(\frac{p_1}{q_1}\right) \Gamma\left(\frac{p_2}{q_2}\right) \dots \Gamma\left(\frac{p_n}{q_n}\right) \Gamma\left(s - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \dots - \frac{p_n}{q_n}\right)}{q_1 q_2 \dots q_n r_1^{p_1 q_1} r_2^{p_2 q_2} \dots r_n^{p_n q_n} \Gamma(s)} \quad [p_i > 0, \quad q_i > 0, \quad r_i > 0, \quad s > 0].$$

650

4.639

$$1. \int \int \dots \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \dots + p_n x_n)^{2m} dx_1 dx_2 \dots dx_n =$$

$$= \frac{(2m - 1)!!}{2^m} \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2} + m + 1\right)} (p_1^2 + p_2^2 + \dots + p_n^2)^m.$$

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$$2. \int \int \dots \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1} (p_1 x_1 + p_2 x_2 + \dots + p_n x_n)^{2m+1} dx_1 dx_2 \dots dx_n = 0.$$

FI III 483

4.641

$$1. \int \int \dots \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \dots + p_n x_n} dx_1 dx_2 \dots dx_n =$$

$$= \sqrt{\pi^n} \sum_{k=0}^\infty \frac{1}{k! \Gamma\left(\frac{n}{2} + k + 1\right)} \left(\frac{p_1^2 + p_2^2 + \dots + p_n^2}{4}\right)^k.$$

FI III 483

$$2. \int \int \dots \int_{x_1^2 + x_2^2 + \dots + x_n^2 \leq 1} e^{p_1 x_1 + p_2 x_2 + \dots + p_n x_n} dx_1 dx_2 \dots dx_n = \frac{(2\pi)^n I_n(\sqrt{p_1^2 + p_2^2 + \dots + p_n^2})}{(p_1^2 + p_2^2 + \dots + p_n^2)^{n/2}}.$$

4.642

$$\int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2} f(\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}) dx_1 dx_2 \cdots dx_n = \frac{2\sqrt{\pi^n}}{\Gamma\left(\frac{n}{2}\right)} \int_0^R x^{n-1} f(x) dx,$$

where $f(x)$ is a function that is continuous on the interval $(0, R)$.

4.643

$$\begin{aligned} \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 x_2 \cdots x_n) (1-x_1)^{p_1-1} (1-x_2)^{p_2-1} \cdots (1-x_n)^{p_n-1} \times \\ x_2^{p_1} x_3^{p_1+p_2} \cdots x_n^{p_1+p_2+\cdots+p_{n-1}} dx_1 dx_2 \cdots dx_n = \\ = \frac{\Gamma(p_1)\Gamma(p_2)\cdots\Gamma(p_n)}{\Gamma(p_1+p_2+\cdots+p_n)} \int_0^1 f(x)(1-x)^{p_1+p_2+\cdots+p_n-1} dx \end{aligned}$$

FI III 485

under the assumption that the integral on the right converges absolutely.

4.644

$$\begin{aligned} \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = 1}^{n-1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{|x_n|} = \\ = 2 \int \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1} f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n) \frac{dx_1 dx_2 \cdots dx_{n-1}}{\sqrt{1-x_1^2-x_2^2-\cdots-x_{n-1}^2}} = \\ = \frac{2\sqrt{\pi^{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi f(\sqrt{p_1^2+p_2^2+\cdots+p_n^2} \cos x) \sin^{n-2} x dx \quad [n \geq 3], \end{aligned}$$

FI III 488

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where $f(x)$ is continuous on the interval $\left[-\sqrt{p_1^2+p_2^2+\cdots+p_n^2}, \sqrt{p_1^2+p_2^2+\cdots+p_n^2}\right]$.

4.645

Suppose that two functions $f(x_1, x_2, \dots, x_n)$ and $g(x_1, x_2, \dots, x_n)$ are continuous in a closed bounded region D and that the smallest and greatest values of the function g in D are m and M respectively. Let $\varphi(u)$ denote a function that is continuous for $m \leq u \leq M$

. We denote by $\psi(u)$ the integral

$$1. \quad \psi(u) = \int \int \cdots \int_{m \leq g(x_1, x_2, \dots, x_n) \leq u} f(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n,$$

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over that portion of the region D on which the inequality $m \leq g(x_1, x_2, \dots, x_n) \leq u$ is satisfied. Then

$$2. \quad \int \int \cdots \int_{m \leq g(x_1, x_2, \dots, x_n) \leq M} f(x_1, x_2, \dots, x_n) \varphi[g(x_1, x_2, \dots, x_n)] dx_1 dx_2 \cdots dx_n = \\ = (S) \int_m^M \varphi(u) d\psi(u) = (R) \int_m^M \varphi(u) \frac{d\psi(u)}{du} du,$$

where the middle integral must be understood in the sense of Stieltjes. If the derivative $\frac{d\psi}{du}$ exists and is continuous, the Riemann integral on the right exists.

M may be $+\infty$ in formulas 4.645 2., in which case $\int_m^{+\infty}$ should be understood to mean $\lim_{M \rightarrow +\infty} \int_m^M$.

4.646³

$$\int \int \cdots \int_{\substack{x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \\ x_1 + x_2 + \cdots + x_n \leq 1}} \frac{x_1^{p_1-1} x_2^{p_2-1} \cdots x_n^{p_n-1}}{(q_1 x_1 + q_2 x_2 + \cdots + q_n x_n)^r} dx_1 dx_2 \cdots dx_n = \\ = \frac{\Gamma(p_1) \Gamma(p_2) \cdots \Gamma(p_n)}{\Gamma(p_1 + p_2 + \cdots + p_n - r + 1) \Gamma(r)} \int_0^\infty \frac{x^{r-1} dx}{(1 + q_1 x)^{p_1} (1 + q_2 x)^{p_2} \cdots (1 + q_n x)^{p_n}} \\ [p_1 > 0, \quad p_2 > 0, \dots, p_n > 0, \quad q_2 > 0, \quad q_2 > 0, \dots, q_n > 0, \quad p_1 + p_2 + \cdots + p_n > r > 0].$$

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4.647

$$\int \int \cdots \int_{0 \leq x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1} \exp \left\{ \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \right\} dx_1 dx_2 \cdots dx_n = \\ = \frac{2\sqrt{\pi^n}}{n(p_1^2 + p_2^2 + \cdots + p_n^2)^{\frac{n}{4} - \frac{1}{2}}} I_{\frac{n}{2} - 1}(\sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}).$$

FI III 495

4.648

5. Indefinite Integrals of Special Functions

5.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1 - k^2}$ (cf. 8.1).

5.11 Complete elliptic integrals

5.111

$$1. \int \mathbf{K}(k)k^{2p+3} dk = \frac{1}{(2p+3)^2} \left\{ 4(p+1)^2 \int \mathbf{K}(k)k^{2p+1} dk + k^{2p+2}[\mathbf{E}(k) - (2p+3)\mathbf{K}(k)k'^2] \right\}.$$

BY (610.04)

$$2. \int \mathbf{E}(k)k^{2p+3} dk = \frac{1}{4p^2 + 16p + 15} \left\{ 4(p+1)^2 \int \mathbf{E}(k)k^{2p+1} dk - \right. \\ \left. - \mathbf{E}(k)k^{2p+2}[(2p+3)k'^2 - 2] - k^{2p+2}k'^2 \mathbf{K}(k) \right\}$$

BY (611.04)

5.112

$$1. \int \mathbf{K}(k) dk = \frac{\pi k}{2} \left[1 + \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(2j+1)2^{4j}(j!)^4} \right].$$

BY (610.00)

$$2.^6 \int \mathbf{E}(k) dk = \frac{\pi k}{2} \left[1 - \sum_{j=1}^{\infty} \frac{[(2j)!]^2 k^{2j}}{(4j^2 - 1)2^{4j}(j!)^4} \right].$$

BY (611.00)

$$3. \int \mathbf{K}(k)k dk = \mathbf{E}(k) - k'^2 \mathbf{K}(k).$$

$$4. \int \mathbf{E}(k)k \, dk = \frac{1}{3}[(1 + k^2)\mathbf{E}(k) - k'^2\mathbf{K}(k)].$$

BY (611.01)

$$5. \int \mathbf{K}(k)k^3 \, dk = \frac{1}{9}[(4 + k^2)\mathbf{E}(k) - k'^2(4 + 3k^2)\mathbf{K}(k)].$$

BY (610.02)

$$6. \int \mathbf{E}(k)k^3 \, dk = \frac{1}{45}[(4 + k^2 + 9k^4)\mathbf{E}(k) - k'^2(4 + 3k^2)\mathbf{K}(k)].$$

BY 611.02

$$7. \int \mathbf{K}(k)k^5 \, dk = \frac{1}{225}[(64 + 16k^2 + 9k^4)\mathbf{E}(k) - k'^2(64 + 48k^2 + 45k^4)\mathbf{K}(k)].$$

BY (610.03)

653

$$8. \int \mathbf{E}(k)k^5 \, dk = \frac{1}{1575}[(64 + 16k^2 + 9k^4 + 225k^6)\mathbf{E}(k) - k'^2(64 + 48k^2 + 45k^4)\mathbf{K}(k)].$$

BY (611.03)

$$9. \int \frac{\mathbf{K}(k)}{k^2} \, dk = -\frac{\mathbf{E}(k)}{k}$$

BY (612.05)

$$10. \int \frac{\mathbf{E}(k)}{k^2} \, dk = \frac{1}{k}[k'^2\mathbf{K}(k) - 2\mathbf{E}(k)].$$

BY (612.02)

$$11. \int \frac{\mathbf{E}(k)}{k'^2} \, dk = k\mathbf{K}(k).$$

BY (612.01)

BY (612.03)

$$13. \int \frac{k\mathbf{E}(k)}{k'^2} dk = \mathbf{K}(k) - \mathbf{E}(k).$$

BY (612.04)

5.113

$$1. \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -\mathbf{E}(k).$$

BY (612.06)

$$2. \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k} = 2\mathbf{E}(k) - k'^2 \mathbf{K}(k).$$

BY (612.09)

$$3. \int [(1 + k^2)\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k} = -k'^2 \mathbf{K}(k).$$

BY (612.12)

$$4. \int [\mathbf{K}(k) - \mathbf{E}(k)] \frac{dk}{k^2} = \frac{1}{k} [\mathbf{E}(k) - k'^2 \mathbf{K}(k)].$$

BY (612.07)

$$5. \int [\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k^2 k'^2} = \frac{1}{k} [\mathbf{K}(k) - \mathbf{E}(k)].$$

$$6. \int [(1 + k^2)\mathbf{E}(k) - k'^2 \mathbf{K}(k)] \frac{dk}{k k'^4} = \frac{\mathbf{E}(k)}{k'^2}.$$

BY (612.13)

5.114

$$\int \frac{k\mathbf{K}(k) dk}{[\mathbf{E}(k) - k'^2 \mathbf{K}(k)]^2} = \frac{1}{k'^2 \mathbf{K}(k) - \mathbf{E}(k)}.$$

BY (612.11)

5.115

$$1. \int \Pi\left(\frac{\pi}{2}, r^2, k\right) k dk = (k^2 - r^2)\Pi\left(\frac{\pi}{2}, r^2, k\right) - \mathbf{K}(k) + \mathbf{E}(k).$$

BY (612.14)

$$2. \int \left[\mathbf{K}(k) - \Pi\left(\frac{\pi}{2}, r^2, k\right) \right] k dk = k^2 \mathbf{K}(k) - (k^2 - r^2)\Pi\left(\frac{\pi}{2}, r^2, k\right).$$

BY (612.15)

$$3. \int \left[\frac{\mathbf{E}(k)}{k'^2} + \Pi\left(\frac{\pi}{2}, r^2, k\right) \right] k dk = (k^2 - r^2)\Pi\left(\frac{\pi}{2}, r^2, k\right).$$

BY (612.16)

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5.12 Elliptic integrals

5.121

$$\int_0^x \frac{F(x, k) dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{[F(x, k)]^2}{2} \quad \left[0 < x \leq \frac{\pi}{2} \right].$$

BY (630.01)

5.122

$$\int_0^x E(x, k) \sqrt{1 - k^2 \sin^2 x} dx = \frac{[\mathbf{E}(x, k)]^2}{2}.$$

BY (630.32)

5.123

$$1. \int_0^x F(x, k) \sin x dx = -\cos x F(x, k) + \frac{1}{k} \arcsin(k \sin x).$$

BY (630.11)

$$2. \int_0^x F(x, k) \cos x dx = \sin x F(x, k) + \frac{1}{k} \operatorname{Arch} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - \frac{1}{k} \operatorname{Arch} \left(\frac{1}{k'} \right).$$

5.124

$$1. \int_0^x E(x, k) \sin x \, dx = -\cos x E(x, k) + \frac{1}{2k} \left[k \sin x \sqrt{1 - k^2 \sin^2 x} + \arcsin(k \sin x) \right].$$

BY (630.12)

$$2. \int_0^x E(x, k) \cos x \, dx = \sin x E(x, k) + \frac{1}{2k} \left[k \cos x \sqrt{1 - k^2 \sin^2 x} - k'^2 \operatorname{Arch} \sqrt{\frac{1 - k^2 \sin^2 x}{k'^2}} - k + k'^2 \operatorname{Arch} \left(\frac{1}{k'} \right) \right].$$

BY (630.22)

5.125

$$1. \int_0^x \Pi(x, \alpha^2, k) \sin x \, dx = -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{k^2 - \alpha^2}} \operatorname{arctg} \left[\sqrt{\frac{k^2 - \alpha^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 < k^2];$$

$$= -\cos x \Pi(x, \alpha^2, k) + \frac{1}{\sqrt{\alpha^2 - k^2}} \operatorname{Arth} \left[\sqrt{\frac{\alpha^2 - k^2}{1 - k^2 \sin^2 x}} \sin x \right] \quad [\alpha^2 > k^2].$$

BY (630.13)

$$2. \int_0^x \Pi(x, \alpha^2, k) \cos x \, dx = \sin x \Pi(x, \alpha^2, k) - f + f_0,$$

where

$$f = \frac{1}{2\sqrt{(1 - \alpha^2)(\alpha^2 - k^2)}} \operatorname{arctg} \left[\frac{2(1 - \alpha^2)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(2k^2 - \alpha^2 - \alpha^2 k^2)}{2\alpha^2 \sqrt{(1 - \alpha^2)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}} \right]$$

for $(1 - \alpha^2)(\alpha^2 - k^2) > 0$;

$$= \frac{1}{2\sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)}} \ln \left[\frac{2(\alpha^2 - 1)(\alpha^2 - k^2) + (1 - \alpha^2 \sin^2 x)(\alpha^2 + \alpha^2 k^2 - 2k^2)}{1 - \alpha^2 \sin^2 x} \right] +$$

$$+ \frac{2\alpha^2 \sqrt{(\alpha^2 - 1)(\alpha^2 - k^2)} \cos x \sqrt{1 - k^2 \sin^2 x}}{1 - \alpha^2 \sin^2 x} \quad \text{for } (1 - \alpha^2)(\alpha^2 - k^2) < 0,$$

f_0 is the value of f at $x = 0$.

BY (630.23)

Integration with respect to the modulus

5.126

$$\int F(x, k)k dk = E(x, k) - k'^2 F(x, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x.$$

BY (613.01)

5.127

$$\int E(x, k)k dk = \frac{1}{3} \left[(1 + k^2)E(x, k) - k'^2 F(x, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x \right].$$

BY (613.02)

5.128

$$\int \Pi(x, r^2, k)k dk = (k^2 - r^2)\Pi(x, r^2, k) - F(x, k) + E(x, k) + (\sqrt{1 - k^2 \sin^2 x} - 1) \operatorname{ctg} x.$$

BY (613.03)

5.13 Jacobian elliptic functions

5.131

$$1. \int \operatorname{sn}^m u du = \frac{1}{m+1} \left[\operatorname{sn}^{m+1} u \operatorname{cn} u \operatorname{dn} u + (m+2)(1+k^2) \int \operatorname{sn}^{m+2} u du - (m+3)k^2 \int \operatorname{sn}^{m+4} u du \right].$$

SI 259, PE(567)

$$2. \int \operatorname{cn}^m u du = \frac{1}{(m+1)k'^2} \left[-\operatorname{cn}^{m+1} u \operatorname{sn} u \operatorname{dn} u + (m+2)(1-2k^2) \int \operatorname{cn}^{m+2} u du + (m+3)k^2 \int \operatorname{cn}^{m+4} u du \right].$$

$$3. \int \operatorname{dn}^m u \, du = \frac{1}{(m+1)k'^2} \left[k^2 \operatorname{dn}^{m+1} u \operatorname{sn} u \operatorname{cn} u + (m+2)(2-k^2) \int \operatorname{dn}^{m+2} u \, du - (m+3) \int \operatorname{dn}^{m+4} u \, du \right].$$

PE (569)

By using formulas 5.131, we can reduce the integrals $\int \operatorname{sn}^m u \, du$, $\int \operatorname{cn}^m u \, du$, $\int \operatorname{dn}^m u \, du$ to the integrals 5.132, 5.133 and 5.134.

5.132

$$1. \int \frac{du}{\operatorname{sn} u} = \ln \frac{\operatorname{sn} u}{\operatorname{cn} u + \operatorname{dn} u};$$

$$= \ln \frac{\operatorname{dn} u - \operatorname{cn} u}{\operatorname{sn} u}.$$

SI 266(4)
ZH 87(164)

656

$$2. \int \frac{du}{\operatorname{cn} u} = \frac{1}{k'} \ln \frac{k' \operatorname{sn} u + \operatorname{dn} u}{\operatorname{cn} u}.$$

SI 266(5)

$$3. \int \frac{du}{\operatorname{dn} u} = \frac{1}{k'} \operatorname{arctg} \frac{k' \operatorname{sn} u - \operatorname{cn} u}{k' \operatorname{sn} u + \operatorname{cn} u};$$

$$= \frac{1}{k'} \operatorname{arccos} \frac{\operatorname{cn} u}{\operatorname{dn} u};$$

$$= \frac{1}{ik'} \ln \frac{\operatorname{cn} u + ik' \operatorname{sn} u}{\operatorname{dn} u};$$

$$= \frac{1}{k'} \operatorname{arcsin} \frac{k' \operatorname{sn} u}{\operatorname{dn} u}.$$

JA
SI 266(6)
JA
ZH 88(166)

5.133

$$1. \int \operatorname{sn} u \, du = \frac{1}{k} \ln(\operatorname{dn} u - k \operatorname{cn} u);$$

$$= \frac{1}{k} \operatorname{Arch} \frac{\operatorname{dn} u - k^2 \operatorname{cn} u}{1 - k^2};$$

$$= \frac{1}{k} \operatorname{Arsh} \left(k \frac{\operatorname{dn} u - \operatorname{cn} u}{1 - k^2} \right);$$

$$\begin{aligned}
 2. \quad \int \operatorname{cn} u \, du &= \frac{1}{k} \arccos(\operatorname{dn} u); \\
 &= \frac{i}{k} \ln(\operatorname{dn} u - ik \operatorname{sn} u); \\
 &= \frac{1}{k} \arcsin(k \operatorname{sn} u).
 \end{aligned}$$

JA
SI 265(2)A, ZH 87(162)
ZH 87(162)

$$\begin{aligned}
 3. \quad \int \operatorname{dn} u \, du &= \arcsin(\operatorname{sn} u); \\
 &= \operatorname{am} u = i \ln(\operatorname{cn} u - i \operatorname{sn} u).
 \end{aligned}$$

SI 266(3), ZH 87(163)

5.134

$$1. \quad \int \operatorname{sn}^2 u \, du = \frac{1}{k^2} [u - E(\operatorname{am} u, k)].$$

PE (564)}

$$2. \quad \int \operatorname{cn}^2 u \, du = \frac{1}{k^2} [E(\operatorname{am} u, k) - k'^2 u].$$

PE (565)

$$3. \quad \int \operatorname{dn}^2 u \, du = E(\operatorname{am} u, k).$$

PE (566)

657
5.135

$$\begin{aligned}
 1. \quad \int \frac{\operatorname{sn} u}{\operatorname{cn} u} \, du &= \frac{1}{k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{cn} u}; \\
 &= \frac{1}{2k'} \ln \frac{\operatorname{dn} u + k'}{\operatorname{dn} u - k'}.
 \end{aligned}$$

$$2. \int \frac{\operatorname{sn} u}{\operatorname{dn} u} du = \frac{i}{kk'} \ln \frac{ik' - k \operatorname{cn} u}{\operatorname{dn} u};$$

$$= \frac{1}{kk'} \operatorname{arcctg} \frac{k \operatorname{cn} u}{k'}.$$

ZH 88(169)
SI 266(8)

$$3. \int \frac{\operatorname{cn} u}{\operatorname{sn} u} du = \ln \frac{1 - \operatorname{dn} u}{\operatorname{sn} u};$$

$$= \frac{1}{2} \ln \frac{1 - \operatorname{dn} u}{1 + \operatorname{dn} u}.$$

ZH 88(168)
SI 266(10)

$$4. \int \frac{\operatorname{cn} u}{\operatorname{dn} u} du = -\frac{1}{k} \ln \frac{1 - k \operatorname{sn} u}{\operatorname{dn} u};$$

$$= \frac{1}{2k} \ln \frac{1 + k \operatorname{sn} u}{1 - k \operatorname{sn} u}.$$

ZH 88(171)
SI 266(9)

$$5. \int \frac{\operatorname{dn} u}{\operatorname{cn} u} du = \frac{1}{2} \ln \frac{1 + \operatorname{sn} u}{1 - \operatorname{sn} u};$$

$$= \ln \frac{1 + \operatorname{sn} u}{\operatorname{cn} u}.$$

JA
ZH 88(172)

$$6. \int \frac{\operatorname{dn} u}{\operatorname{sn} u} du = \frac{1}{2} \ln \frac{1 - \operatorname{cn} u}{1 + \operatorname{cn} u}.$$

ZH 87(170)

5.136

$$1. \int \operatorname{sn} u \operatorname{cn} u du = -\frac{1}{k^2} \operatorname{dn} u.$$

$$2. \int \operatorname{sn} u \operatorname{dn} u du = -\operatorname{cn} u.$$

5.137

$$1. \int \frac{\operatorname{sn} u}{\operatorname{cn}^2 u} du = \frac{1}{k'^2} \frac{dn u}{\operatorname{cn} u}.$$

ZH 88(173)

$$2. \int \frac{\operatorname{sn} u}{\operatorname{dn}^2 u} du = -\frac{1}{k'^2} \frac{\operatorname{cn} u}{\operatorname{dn} u}.$$

ZH 88(175)

658

$$3. \int \frac{\operatorname{cn} u}{\operatorname{sn}^2 u} du = -\frac{dn u}{\operatorname{sn} u}.$$

ZH 88(174)

$$4. \int \frac{\operatorname{cn} u}{\operatorname{dn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{dn} u}.$$

ZH 88(177)

$$5. \int \frac{dn u}{\operatorname{sn}^2 u} du = -\frac{\operatorname{cn} u}{\operatorname{sn} u}.$$

ZH 88(176)

$$6. \int \frac{dn u}{\operatorname{cn}^2 u} du = \frac{\operatorname{sn} u}{\operatorname{cn} u}.$$

ZH 88(178)

5.138

$$1. \int \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \ln \frac{\operatorname{sn} u}{\operatorname{dn} u}.$$

ZH 88(183)

$$2. \int \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \ln \frac{dn u}{\operatorname{cn} u}.$$

$$3. \int \frac{dn u}{sn u cn u} du = \ln \frac{sn u}{cn u}.$$

ZH 88(184)

5.139

$$1. \int \frac{cn u dn u}{sn u} du = \ln sn u.$$

ZH 88(179)

$$2. \int \frac{sn u dn u}{cn u} du = \ln \frac{1}{cn u}.$$

ZH 88(180)

$$3. \int \frac{sn u cn u}{dn u} du = -\frac{1}{k^2} \ln dn u.$$

ZH 88(181)

5.14 Weierstrass elliptic functions

5.141

$$1. \int \wp(u) du = -\zeta(u).$$

$$2. \int \wp^2(u) du = \frac{1}{6} \wp'(u) + \frac{1}{12} g_2 u.$$

ZH 120(192)

$$3. \int \wp^3(u) du = \frac{1}{120} \wp'''(u) - \frac{3}{20} g_2 \zeta(u) + \frac{1}{10} g_3 u.$$

ZH 120(193)

$$4. \int \frac{du}{\mathcal{P}(u) - \mathcal{P}(v)} = \frac{1}{\mathcal{P}'(v)} \left[2u\zeta(v) + \ln \frac{\sigma(u-v)}{\sigma(u+v)} \right].$$

$$5. \int \frac{\alpha \mathcal{P}(u) + \beta}{\gamma \mathcal{P}(u) + \delta} du = \frac{au}{\gamma} - \frac{\alpha\delta - \beta\gamma}{\gamma^2 \mathcal{P}'(v)} \left[\ln \frac{\sigma(u+v)}{\sigma(u-v)} - 2u\zeta(v) \right], \text{ where } \mathcal{P}'(v) = -\frac{\delta}{\gamma}.$$

ZH 120(195)

5.2 The Exponential-Integral Function

5.21 The exponential-integral function

5.211

$$\int_x^\infty \text{Ei}(-\beta x) \text{Ei}(-\gamma x) dx = \left(\frac{1}{\beta} + \frac{1}{\gamma} \right) \text{Ei}[-(\beta + \gamma)x] - x \text{Ei}(-\beta x) \text{Ei}(-\gamma x) - \frac{e^{-\beta x}}{\beta} \text{Ei}(-\gamma x) - \frac{e^{-\gamma x}}{\gamma} \text{Ei}(-\beta x) \quad [\text{Re}(\beta + \gamma) > 0].$$

NT 53(2)

5.22 Combinations of the exponential-integral function and powers

5.221

$$1. \int_x^\infty \frac{\text{Ei}[-a(x+b)]}{x^{n+1}} dx = \left[\frac{1}{x^n} - \frac{(-1)^n}{b^n} \right] \frac{\text{Ei}[-a(x+b)]}{n} + \frac{e^{-ab}}{n} \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{b^{n-k}} \int_x^\infty \frac{e^{-ax}}{x^{k+1}} dx [a > 0, \quad b > 0].$$

NT 52(3)

$$2. \int_x^\infty \frac{\text{Ei}[-a(x+b)]}{x^2} dx = \left(\frac{1}{x} + \frac{1}{b} \right) \text{Ei}[-a(x+b)] - \frac{e^{-ab} \text{Ei}(-ax)}{b} \quad [a > 0, \quad b > 0].$$

NT 52(4)

5.23 Combinations of the exponential-integral and the exponential

5.231

$$1. \int_0^x e^x \text{Ei}(-x) dx = -\ln x - \mathbf{C} + e^x \text{Ei}(-x).$$

$$2. \int_0^x e^{-\beta x} \operatorname{Ei}(-\alpha x) dx = -\frac{1}{\beta} \left\{ e^{-\beta x} \operatorname{Ei}(-\alpha x) + \ln \left(1 + \frac{\beta}{\alpha} \right) - \operatorname{Ei}[-(\alpha + \beta)x] \right\}.$$

ET II 308(12)

5.3 The Sine-Integral and the Cosine-Integral

5.31

$$1. \int \cos \alpha x \operatorname{ci}(\beta x) dx = \frac{\sin \alpha x \operatorname{ci}(\beta x)}{\alpha} - \frac{\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)}{2\alpha}.$$

NT 49(1)

660

$$2. \int \sin \alpha x \operatorname{ci}(\beta x) dx = -\frac{\cos \alpha x \operatorname{ci}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) + \operatorname{ci}(\alpha x - \beta x)}{2\alpha}.$$

NT 49(2)

5.32

$$1. \int \cos \alpha x \operatorname{si}(\beta x) dx = \frac{\sin \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{ci}(\alpha x + \beta x) - \operatorname{ci}(\alpha x - \beta x)}{2\alpha}.$$

NT 49(3)

$$2. \int \sin \alpha x \operatorname{si}(\beta x) dx = -\frac{\cos \alpha x \operatorname{si}(\beta x)}{\alpha} + \frac{\operatorname{si}(\alpha x + \beta x) - \operatorname{si}(\alpha x - \beta x)}{2\alpha}.$$

NT 49(4)

5.33

$$1. \int \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{ci}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)) + \\ + \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x - \alpha x)) - \frac{1}{\alpha} \sin \alpha x \operatorname{ci}(\beta x) - \frac{1}{\beta} \sin \beta x \operatorname{ci}(\alpha x).$$

$$2. \int \operatorname{si}(\alpha x) \operatorname{si}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{si}(\beta x) - \frac{1}{2\beta} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\alpha x - \beta x)) - \\ - \frac{1}{2\alpha} (\operatorname{si}(\alpha x + \beta x) + \operatorname{si}(\beta x + \alpha x)) + \frac{1}{\alpha} \cos \alpha x \operatorname{si}(\beta x) + \frac{1}{\beta} \cos \beta x \operatorname{si}(\alpha x).$$

NT 54(6)

$$3. \int \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) dx = x \operatorname{si}(\alpha x) \operatorname{ci}(\beta x) + \frac{1}{\alpha} \cos \alpha x \operatorname{ci}(\beta x) - \\ - \frac{1}{\beta} \sin \beta x \operatorname{si}(\alpha x) - \left(\frac{1}{2\alpha} + \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x + \beta x) - \left(\frac{1}{2\alpha} - \frac{1}{2\beta} \right) \operatorname{ci}(\alpha x - \beta x).$$

NT 54(10)

5.34

$$1. \int_x^\infty \operatorname{si}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b} \right) \operatorname{si}[a(x+b)] - \frac{\cos ab \operatorname{si}(ax) + \sin ab \operatorname{ci}(ax)}{b} [a > 0, \quad b > 0].$$

NT 52(6)

$$2. \int_x^\infty \operatorname{ci}[a(x+b)] \frac{dx}{x^2} = \left(\frac{1}{x} + \frac{1}{b} \right) \operatorname{ci}[a(x+b)] + \frac{\sin ab \operatorname{si}(ax) - \cos ab \operatorname{ci}(ax)}{b} [a > 0, \quad b > 0].$$

NT 52(5)

5.4 The Probability Integral and Fresnel Integrals

5.41

$$\int \Phi(\alpha x) dx = x\Phi(\alpha x) + \frac{e^{-\alpha^2 x^2}}{\alpha\sqrt{\pi}}.$$

NT 12(20)a

661

5.42

$$\int S(\alpha x) dx = xS(\alpha x) + \frac{\cos \alpha^2 x^2}{\alpha\sqrt{2\pi}}.$$

5.43

$$\int C(\alpha x) dx = xC(\alpha x) - \frac{\sin \alpha^2 x^2}{\alpha \sqrt{2\pi}}.$$

NT 12(21)a

5.5 Bessel Functions

5.51

$$\int J_p(x) dx = 2 \sum_{k=0}^{\infty} J_{p+2k+1}(x).$$

JA, MO 30

5.52

$$1. \int x^{p+1} Z_p(x) dx = x^{p+1} Z_{p+1}(x)^*$$

WA 146(1)

$$2. \int x^{-p+1} Z_p(x) dx = -x^{-p+1} Z_{p-1}(x)^*.$$

WA 146(2)

5.53

$$\int \left[(\alpha^2 - \beta^2)x - \frac{p^2 - q^2}{x} \right] Z_p(\alpha x) Z_q(\beta x) dx = \beta x Z_p(\alpha x) Z_{q-1}(\beta x) - \alpha x Z_{p-1}(\alpha x) Z_q(\beta x) + (p - q) Z_p(\alpha x) Z_q(\beta x)^*$$

JA, MO 30, WA 148(7)a

5.54

$$1.^6 \int x Z_p(\alpha x) Z_p(\beta x) dx = \frac{\beta x Z_p(\alpha x) Z_{p-1}(\beta x) + \alpha x Z_{p-1}(\alpha x) Z_p(\beta x)^*}{\alpha^2 - \beta^2}.$$

WA 148(8)a

$$2. \int x [Z_p(\alpha x)]^2 dx = \frac{x^2}{2} \{ [Z_p(\alpha x)]^2 - Z_{p-1}(\alpha x) Z_{p+1}(\alpha x) \}^*.$$

WA 149(11)

5.55

$$\int \frac{1}{x} Z_p(\alpha x) Z_q(\alpha x) dx = \alpha x \frac{Z_{p-1}(\alpha x) Z_q(\alpha x) - Z_p(\alpha x) Z_{q-1}(\alpha x)}{p^2 - q^2} - \frac{Z_p(\alpha x) Z_q(\alpha x)^*}{p + q}$$

* In formulas 5.52- 5.56, $Z_p(x)$ and Z are arbitrary Bessel

functions.

WA 149(13)

5.56

$$1. \int Z_1(x) dx = -Z_0(x)^*.$$

JA

$$2. \int x Z_0(x) dx = x Z_1(x)^*.$$

JA

6.-7. Definite Integrals of Special Functions

6.1 Elliptic Integrals and Functions

6.1 Elliptic Integrals and Functions

Notation: $k' = \sqrt{1 - k^2}$ (cf. 8.1).

6.11 Forms containing $F(x, k)$

6.111

$$\int_0^{\frac{\pi}{2}} F(x, k) \operatorname{ctg} x \, dx = \frac{\pi}{4} \mathbf{K}(k') + \frac{1}{2} \ln k \mathbf{K}(k).$$

BI ((350))(1)

6.112

$$1. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1 + k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k').$$

BI ((350))(6)

$$2. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1 - k \sin^2 x} \, dx = \frac{1}{4k} \mathbf{K}(k) \ln \frac{2}{(1-k)\sqrt{k}} - \frac{\pi}{16k} \mathbf{K}(k').$$

BI ((350))(7)

$$3. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} \, dx = -\frac{1}{2k^2} \ln k' \mathbf{K}(k).$$

BI ((350))(2)A, BY(802.12)a

6.113

$$1. \int_0^{\frac{\pi}{2}} F(x, k') \frac{\sin x \cos x \, dx}{\cos^2 x + k \sin^2 x} = \frac{1}{4(1-k)} \ln \frac{2}{(1+k)\sqrt{k}} \mathbf{K}(k').$$

BI ((350))(5)

$$2. \int_0^{\frac{\pi}{2}} F(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \\ = -\frac{1}{k^2 \sin t \cos t} \left[\mathbf{K}(k) \operatorname{arctg}(k' \operatorname{tg} t) - \frac{\pi}{2} F(t, k) \right].$$

BI ((350))(12)

6.114

$$\int_u^v F(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} \mathbf{K}(k) \mathbf{K}(\sqrt{1 - tg^2 u \operatorname{ctg}^2 v})$$

$$[k^2 = 1 - \operatorname{ctg}^2 u \cdot \operatorname{ctg}^2 v].$$

BI ((351))(9)

663

6.115

$$\int_0^1 F(\arcsin x, k) \frac{x dx}{1 + kx^2} = \frac{1}{4k} \mathbf{K}(k) \ln \frac{(1+k)\sqrt{k}}{2} + \frac{\pi}{16k} \mathbf{K}(k') \quad (\text{cf. 6.112 2.}).$$

6.112

BI ((466))(1)

This and similar formulas can be obtained from formulas 6.111 6.113 by means of the substitution $x = \arcsin t$.

6.12 Forms containing $E(x, k)$

6.121

$$\int_0^{\frac{\pi}{2}} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 x} dx = \frac{1}{2k^2} \{(1 + k'^2) \mathbf{K}(k) - (2 + \ln k') \mathbf{E}(k)\}.$$

BI ((350))(4)

6.122

$$\int_0^{\frac{\pi}{2}} E(x, k) \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{1}{2} \{\mathbf{E}(k) \mathbf{K}(k) - \ln k'\}.$$

BI ((350))(10), BY (630.02)

6.123

$$\int_0^{\frac{\pi}{2}} E(x, k) \frac{\sin x \cos x}{1 - k^2 \sin^2 t \sin^2 x} \cdot \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} =$$

$$= -\frac{1}{k^2 \sin t \cos t} \left[\mathbf{E}(k) \operatorname{arctg}(k' \operatorname{tg} t) - \frac{\pi}{2} E(t, k) + \frac{\pi}{2} \operatorname{ctg} t (1 - \sqrt{1 - k^2 \sin^2 t}) \right].$$

BI ((350))(13)

6.124

$$\int_u^n E(x, k) \frac{dx}{\sqrt{(\sin^2 x - \sin^2 u)(\sin^2 v - \sin^2 x)}} = \frac{1}{2 \cos u \sin v} E(k) \mathbf{K} \left(\sqrt{1 - \frac{\operatorname{tg}^2 u}{\operatorname{tg}^2 v}} \right) + \frac{k^2 \sin v}{2 \cos u} \mathbf{K} \left(\sqrt{1 - \frac{\sin^2 2u}{\sin^2 2v}} \right) [k^2 = 1 - \operatorname{ctg}^2 u \operatorname{ctg}^2 v].$$

BI ((351))(10)

6.13 Integration of elliptic integrals with respect to the modulus

6.131

$$\int_0^1 F(x, k) k dk = \frac{1 - \cos x}{\sin x} = \operatorname{tg} \frac{x}{2}.$$

BY (616.03)

6.132

$$\int_0^1 E(x, k) k dk = \frac{\sin^2 x + 1 - \cos x}{3 \sin x}.$$

BY (616.04)

6.133

$$\int_0^1 \Pi(x, r^2, k) k dk = \operatorname{tg} \frac{x}{2} - r \ln \sqrt{\frac{1 + r \sin x}{1 - r \sin x}} - r^2 \mathbf{\Pi}(x, r^2, 0).$$

BY (616.05)

6.14- 6.15 Complete elliptic integrals

6.141

$$1. \int_0^1 \mathbf{K}(k) dk = 2\mathbf{G}.$$

FI II 755

$$2. \int_0^1 \mathbf{K}(k') dk = \frac{\pi^2}{4}.$$

664

6.142

$$\int_0^1 \left(\mathbf{K}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2\mathbf{G}.$$

BY (615.05)

6.143

$$\int_0^1 \mathbf{K}(k) \frac{dk}{k'} = \mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right).$$

BY (615.08)

6.144

$$\int_0^1 \mathbf{K}(k) \frac{dk}{1+k} = \frac{\pi^2}{8}.$$

BY (615.09)

6.145

$$\int_0^1 \left(\mathbf{K}(k') - \ln \frac{4}{k} \right) \frac{dk}{k} = \frac{1}{12} [24(\ln 2)^2 - \pi^2].$$

BY (615.13)

6.146

$$n^2 \int_0^1 k^n \mathbf{K}(k) dk = (n-1)^2 \int_0^1 k^{n-2} \mathbf{K}(k) dk + 1.$$

BY (615.12)

6.147

$$n \int_0^1 k^n \mathbf{K}(k') dk = (n-1) \int_0^1 k^{n-2} \mathbf{E}(k) dk \quad [n > 1] \quad (\text{see } \mathbf{6.152}).$$

6.152
BY (615.11)

6.148

$$1. \int_0^1 \mathbf{E}(k) dk = \frac{1}{2} + \mathbf{G}.$$

BY (615.02)

$$2. \int_0^1 \mathbf{E}(k') dk = \frac{\pi^2}{8}.$$

BY (615.04)

6.149

$$1. \int_0^1 \left(\mathbf{E}(k) - \frac{\pi}{2} \right) \frac{dk}{k} = \pi \ln 2 - 2\mathbf{G} + 1 - \frac{\pi}{2}.$$

BY (615.06)

$$2. \int_0^1 (\mathbf{E}(k') - 1) \frac{dk}{k} = 2 \ln 2 - 1.$$

BY (615.07)

6.151

$$\int_0^1 \mathbf{E}(k) \frac{dk}{k'} = \frac{1}{8} \left[4\mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right) + \frac{\pi^2}{\mathbf{K}^2 \left(\frac{\sqrt{2}}{2} \right)} \right].$$

BY (615.10)

6.152

$$(n+2) \int_0^1 k^n \mathbf{E}(k') dk = (n+1) \int_0^1 k^n \mathbf{K}(k') dk \quad [n > 1] \quad (\text{see } \mathbf{6.147}).$$

6.147
BY (615.14)

6.153⁶

6.154

$$\int_0^{\frac{\pi}{2}} \frac{E(p \sin x)}{1 - p^2 \sin^2 x} \sin x \, dx = \frac{\pi}{2\sqrt{1-p^2}} \quad [p^2 > 1].$$

FI II 489

6.16 The theta function

665

6.161

$$1. \int_0^{\infty} x^{s-1} \theta_2(0|ix^2) \, dx = 2^s (1 - 2^{-s}) \pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \quad [\operatorname{Re} s > 2].$$

ET I 339(20)

$$2. \int_0^{\infty} x^{s-1} [\theta_3(0|ix^2) - 1] \, dx = \pi^{-\frac{s}{2}} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \quad [\operatorname{Re} s > 2].$$

ET I 339(21)

$$3. \int_0^{\infty} x^{s-1} [1 - \theta_4(0|ix^2)] \, dx = (1 - 2^{1-s}) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s) \quad [\operatorname{Re} s > 2].$$

ET I 339(22)

$$4. \int_0^{\infty} x^{s-1} [\theta_4(0|ix^2) + \theta_2(0|ix^2) - \theta_3(0|ix^2)] \, dx = -(2^s - 1)(2^{1-s} - 1) \pi^{-\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \zeta(s).$$

ET I 339(24)

6.162

$$1. \int_0^{\infty} e^{-ax} \theta_4\left(\frac{b\pi}{2l} \middle| \frac{i\pi x}{l^2}\right) \, dx = \frac{l}{\sqrt{a}} \operatorname{ch}(b\sqrt{a}) \operatorname{cosec}(l\sqrt{a}) \quad [\operatorname{Re} a > 0, \quad |b| \leq l].$$

ET I 224(1)a

$$2. \int_0^{\infty} e^{-ax} \theta_1\left(\frac{b\pi}{2l} \middle| \frac{i\pi x}{l^2}\right) \, dx = -\frac{l}{\sqrt{a}} \operatorname{sh}(b\sqrt{a}) \operatorname{sech}(l\sqrt{a}) \quad [\operatorname{Re} a > 0, \quad |b| \leq l].$$

$$3. \int_0^{\infty} e^{-ax} \theta_2 \left(\frac{(1+b)\pi}{2l} \middle| \frac{i\pi x}{l^2} \right) dx = -\frac{l}{\sqrt{a}} \operatorname{sh}(b\sqrt{a}) \operatorname{sech}(l\sqrt{a}) \quad [\operatorname{Re} a > 0, \quad |b| \leq l].$$

ET I 224(3)a

$$4. \int_0^{\infty} e^{-ax} \theta_3 \left(\frac{(1+b)\pi}{2l} \middle| \frac{i\pi x}{l^2} \right) dx = \frac{l}{\sqrt{a}} \operatorname{ch}(b\sqrt{a}) \operatorname{cosech}(l\sqrt{a}) \quad [\operatorname{Re} a > 0, \quad |b| \leq l].$$

ET I 224(4)a

6.163

$$1. \int_0^{\infty} e^{-(a-\mu)x} \theta_3(\pi \sqrt{\mu} x | i\pi x) dx = \frac{1}{2\sqrt{a}} [\operatorname{coth}(\sqrt{a} + \sqrt{\mu}) + \operatorname{coth}(\sqrt{a} - \sqrt{\mu})] \quad [\operatorname{Re} a > 0].$$

ET I 224(7)a

$$2.* \int_0^{\infty} \theta_3(i\pi k x | i\pi x) e^{-(k^2+l^2)x} dx = \frac{\sinh 2l}{l(\cosh 2l - \cos 2k)}.$$

6.164

$$\begin{aligned} \int_0^{\infty} [\theta_4(0|ie^{2x}) + \theta_2(0|ie^{2x}) - \theta_3(0|ie^{2x})] e^{\frac{1}{2}x} \cos(ax) dx = \\ = \frac{1}{2} (2^{\frac{1}{2}+ia} - 1)(1 - 2^{\frac{1}{2}-ia}) \pi^{-\frac{1}{4}-\frac{1}{2}ia} \Gamma\left(\frac{1}{4} + \frac{1}{2}ia\right) \zeta\left(\frac{1}{2} + ia\right) \quad [a > 0]. \end{aligned}$$

ET I 61(11)

6.165

$$\begin{aligned} \int_0^{\infty} e^{\frac{1}{2}x} [\theta_3(0|ie^{2x}) - 1] \cos(ax) dx = \\ = 2(1+4a^2)^{-1} \left\{ 1 + \left[\left(a^2 + \frac{1}{4} \right) \pi^{-\frac{1}{2}ia - \frac{1}{4}} \Gamma\left(\frac{1}{2}ia + \frac{1}{4}\right) \zeta\left(ia + \frac{1}{2}\right) \right] \right\} \quad [a > 0]. \end{aligned}$$

ET I 61(12)

666

6.17 Generalized elliptic integrals

1. Set

$$\Omega_j(k) \equiv \int_0^\pi [1 - k^2 \cos \phi]^{-(j+\frac{1}{2})} d\phi,$$

$$\alpha_m(j) = \frac{\pi}{(64)^m} \frac{j!}{(2j)!} \frac{(4m+2j)!}{(2m+j)!} \left(\frac{1}{m}\right)^2, \quad \lambda = (\pi/2)\sqrt{(2j+1)k^2/(1-k^2)},$$

then

$$\begin{aligned} \Omega_j(k) &= \sum_{m=0}^{\infty} \alpha_m(j) k^{4m} \\ &= \sqrt{\pi/(2j+1)k^2} (1-k^2)^{-j} \left[\operatorname{erf} \lambda + (1/2)(2j+1)^{-1} \left(1 + \frac{1}{2k^2}\right) \times \right. \\ &\quad \times \left\{ \operatorname{erf} \lambda - (2/\sqrt{\pi})(\lambda e^{-\lambda^2}) \left(1 + \frac{2}{3}\lambda^2\right) \right\} - (1/12)(2j+1)^{-2} \left(16 + \frac{13}{k^2} + \frac{1}{k^4}\right) \times \\ &\quad \times \left. \left\{ \operatorname{erf} \lambda - (2/\sqrt{\pi})(\lambda e^{-\lambda^2}) \left(1 + \frac{2}{3}\lambda^2 + \frac{4}{15}\lambda^4\right) \right\} + \dots \right], \end{aligned}$$

while for large λ

$$\begin{aligned} \lim_{j \rightarrow \infty} \Omega_j(k) &= \sqrt{\pi/(2j+1)k^2} (1-k^2)^{-j} \times \\ &\times \left[1 + (1/2)(2j+1)^{-1} \left\{ 1 + \frac{1}{2k^2} \right\} - (4/3)(2j+1)^{-2} \left\{ 1 + \frac{13}{16k^2} + \frac{1}{16k^4} \right\} + \dots \right]. \end{aligned}$$

2. Set

$$R_\mu(k, \alpha, \delta) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\delta-2\alpha-1}(\theta/2) d\theta}{[1 - k^2 \cos \theta]^{\mu+\frac{1}{2}}},$$

$$0 < k < 1, \quad \operatorname{Re} \delta > \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -1/2,$$

$$M_\nu(\mu, \alpha, \delta) = \left[(-1)^\nu 2^\nu \left(\mu + \frac{1}{2}\right)_\nu / \nu! \right] \Gamma(\alpha) \Gamma(\delta - \alpha + \nu) / \Gamma(\delta + \nu),$$

with $(\lambda)_\nu = \Gamma(\lambda + \nu) / \Gamma(\lambda)$, and

$$W_\nu(\mu, \alpha, \delta) = \left[2^\nu \left(\mu + \frac{1}{2}\right)_\nu / \nu! \right] \Gamma(\alpha + \nu) \Gamma(\delta - \alpha) / \Gamma(\delta + \nu),$$

then:

for small k ;

$$\begin{aligned}
 R_\mu(k, \alpha, \delta) &= (1 - k^2)^{-(\mu + \frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1 - k^2)]^\nu M_\nu(\mu, \alpha, \delta) \\
 &= (1 + k^2)^{-(\mu + \frac{1}{2})} \sum_{\nu=0}^{\infty} [k^2 / (1 + k^2)]^\nu W_\nu(\mu, \alpha, \delta),
 \end{aligned}$$

for k^2 close to 1;

$$\begin{aligned}
 R_\mu(k, \alpha, \delta) &= \left[\Gamma(\delta - \alpha) \Gamma\left(\mu + \alpha - \delta + \frac{1}{2}\right) \Gamma\left(\mu + \frac{1}{2}\right) \right] (2k^2)^{\alpha - \delta} (1 - k^2)^{\delta - \alpha - \mu - \frac{1}{2}} \times \\
 &\quad \times \left\{ \Gamma\left(\delta - \alpha - \mu - \frac{1}{2}\right) \Gamma(\alpha) \left[\Gamma\left(\delta - \mu - \frac{1}{2}\right) (2k^2)^{\mu + \frac{1}{2}} \right] \right\}, \\
 &\quad \left[\operatorname{Re}\left(\mu + \alpha - \delta + \frac{1}{2}\right) \quad \text{not an integer} \right] \\
 &= \left[2^{\mu + \frac{1}{2}} k^{2\mu + 1} \Gamma\left(\mu + \frac{1}{2}\right) \Gamma(1 - \alpha) \right] \times \\
 &\quad \times \sum_{n=0}^{\infty} \left[\Gamma(\delta - \alpha + n) \Gamma(1 - \alpha + n) \Gamma\left(\alpha - \delta + \mu - n + \frac{1}{2}\right) n! \right] [2k^2 / (1 - k^2)]^{\alpha - \delta + \mu - n + \frac{1}{2}} \\
 &\quad \left[\alpha - \delta + \mu + \frac{1}{2} = m, \quad \text{with } m \text{ a non-negative integer} \right]
 \end{aligned}$$

6.2- 6.3 The Exponential-Integral Function and Functions Generated by It

6.21 The logarithm-integral

6.211

$$\int_0^1 \operatorname{li}(x) dx = -\ln 2.$$

BI ((79))(5)

6.212

$$1. \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) x dx = 0.$$

$$2. \int_0^1 \operatorname{li}(x) x^{p-1} dx = -\frac{1}{p} \ln(p+1) \quad [p > -1].$$

BI ((255))(2)

$$3. \int_0^1 \operatorname{li}(x) \frac{dx}{x^{q+1}} = \frac{1}{q} \ln(1-q) \quad [q < 1].$$

BI ((255))(3)

$$4. \int_1^\infty \operatorname{li}(x) \frac{dx}{x^{q+1}} = -\frac{1}{q} \ln(q-1) \quad [q > 1].$$

BI ((255))(4)

6.213

$$1. \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = \frac{1}{1+a^2} \left(a \ln a - \frac{\pi}{2}\right) \quad [a > 0].$$

BI ((475))(1)

$$2. \int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \sin(a \ln x) dx = -\frac{1}{1+a^2} \left(\frac{\pi}{2} + a \ln a\right) \quad [a > 0].$$

BI ((475))(9)

$$3. \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = -\frac{1}{1+a^2} \left(\ln a + \frac{\pi}{2} a\right) \quad [a > 0].$$

BI ((475))(2)

$$4. \int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \cos(a \ln x) dx = \frac{1}{1+a^2} \left(\ln a - \frac{\pi}{2} a\right) \quad [a > 0].$$

BI ((475))(10)

$$5. \int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x} = \frac{\ln(1+a^2)}{2a} \quad [a > 0].$$

$$6. \int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x} = -\frac{\operatorname{arctg} a}{a}.$$

BI ((479))(2)

$$7. \int_0^1 \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(a \ln a + \frac{\pi}{2} \right) \quad [a > 0].$$

BI ((479))(3)

$$8. \int_1^\infty \operatorname{li}(x) \sin(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(\frac{\pi}{2} - a \ln a \right) \quad [a > 0].$$

BI ((479))(13)

$$9. \int_0^1 \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = \frac{1}{1+a^2} \left(\ln a - \frac{\pi}{2} a \right) \quad [a > 0].$$

BI ((479))(4)

$$10. \int_1^\infty \operatorname{li}(x) \cos(a \ln x) \frac{dx}{x^2} = -\frac{1}{1+a^2} \left(\ln a + \frac{\pi}{2} a \right) \quad [a > 0].$$

BI ((479))(14)

$$11. \int_0^1 \operatorname{li}(x) \sin(a \ln x) x^{p-1} dx = \frac{1}{a^2 + p^2} \left\{ \frac{a}{2} \ln[(1+p)^2 + a^2] - p \operatorname{arctg} \frac{a}{1+p} \right\} \quad [p > 0].$$

BI ((477))(1)

$$12. \int_0^1 \operatorname{li}(x) \cos(a \ln x) x^{p-1} dx = -\frac{1}{a^2 + p^2} \left\{ a \operatorname{arctg} \frac{a}{1+p} + \frac{p}{2} \ln[(1+p)^2 + a^2] \right\} \quad [p > 0].$$

BI ((477))(2)

6.214

$$1. \int_0^1 \operatorname{li}\left(\frac{1}{x}\right) \left(\ln \frac{1}{x}\right)^{p-1} dx = -\pi \operatorname{ctg} p\pi \cdot \Gamma(p) \quad [0 < p < 1].$$

BI ((340))(1)

$$2. \int_1^{\infty} \operatorname{li}\left(\frac{1}{x}\right) (\ln x)^{p-1} dx = -\frac{\pi}{\sin p\pi} \Gamma(p) \quad [0 < p < 1].$$

BI ((340))(9)

6.215

$$1. \int_0^1 \operatorname{li}(x) \frac{x^{p-1}}{\sqrt{\ln\left(\frac{1}{x}\right)}} dx = -2\sqrt{\frac{\pi}{p}} \operatorname{Arsh} \sqrt{p} = -2\sqrt{\frac{\pi}{p}} \ln(\sqrt{p} + \sqrt{p+1}) \quad [p > 0].$$

BI ((444))(3)

$$2. \int_0^1 \operatorname{li}(x) \frac{dx}{x^{p+1} \sqrt{\ln\left(\frac{1}{x}\right)}} = -2\sqrt{\frac{\pi}{p}} \arcsin \sqrt{p} \quad [1 > p > 0].$$

BI ((444))(4)

6.216

$$1. \int_0^1 \operatorname{li}(x) \left[\ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{dx}{x} = -\frac{1}{p} \Gamma(p) \quad [0 < p \leq 1].$$

BI ((444))(1)

$$2. \int_0^1 \operatorname{li}(x) \left[\ln\left(\frac{1}{x}\right) \right]^{p-1} \frac{dx}{x^2} = -\frac{\pi \Gamma(p)}{\sin p\pi} \quad [0 < p < 1].$$

BI ((444))(2)

6.22- 6.23 The exponential-integral function

6.221

$$\int_0^p \operatorname{Ei}(\alpha x) dx = p \operatorname{Ei}(\alpha p) + \frac{1 - e^{\alpha p}}{\alpha}.$$

NT 11(7)

6.222

$$\int_0^{\infty} \operatorname{Ei}(-px) \operatorname{Ei}(-qx) dx = \left(\frac{1}{p} + \frac{1}{q}\right) \ln(p+q) - \frac{\ln q}{p} - \frac{\ln p}{q} \quad [p > 0, \quad q > 0].$$

6.223

$$\int_0^{\infty} \text{Ei}(-\beta x) x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu\beta^{\mu}} \quad [\text{Re } \beta \geq 0, \quad \text{Re } \mu > 0].$$

NT 55(7), ET I 325(10)

6.224

$$1. \int_0^{\infty} \text{Ei}(-\beta x) e^{-\mu x} dx = -\frac{1}{\mu} \ln\left(1 + \frac{\mu}{\beta}\right) \quad [\text{Re}(\beta + \mu) \geq 0, \quad \mu > 0];$$

$$= -1/\beta \quad [\mu = 0].$$

FI II 652, NT 48(8)

$$2. \int_0^{\infty} \text{Ei}(ax) e^{-\mu x} dx = -\frac{1}{\mu} \ln\left(\frac{\mu}{a} - 1\right) \quad [a > 0, \quad \text{Re } \mu > 0, \quad \mu > a].$$

ET I 178(23)A, BI ((283))(3)

670

6.225

$$1. \int_0^{\infty} \text{Ei}(-x^2) e^{-\mu x^2} dx = -\sqrt{\frac{\pi}{\mu}} \text{Arsh } \sqrt{\mu} = -\sqrt{\frac{\pi}{\mu}} \ln(\sqrt{\mu} + \sqrt{1 + \mu}) \quad [\text{Re } \mu > 0].$$

BI ((283))(5), ET I 178(25)a

$$2. \int_0^{\infty} \text{Ei}(-x^2) e^{px^2} dx = -\sqrt{\frac{\pi}{p}} \arcsin \sqrt{p} \quad [1 > p > 0].$$

NT 59(9)a

6.226

$$1. \int_0^{\infty} \text{Ei}\left(-\frac{1}{4x}\right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(\sqrt{\mu}) \quad [\text{Re } \mu > 0].$$

MI 34

$$2. \int_0^{\infty} \text{Ei}\left(\frac{a^2}{4x}\right) e^{-\mu x} dx = -\frac{2}{\mu} K_0(a\sqrt{\mu}) \quad [a > 0, \quad \text{Re } \mu > 0].$$

$$3. \int_0^{\infty} \text{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2} dx = \sqrt{\frac{\pi}{\mu}} \text{Ei}(-\sqrt{\mu}) \quad [\text{Re } \mu > 0].$$

MI 34

$$4. \int_0^{\infty} \text{Ei}\left(-\frac{1}{4x^2}\right) e^{-\mu x^2 + \frac{1}{4x^2}} dx = \sqrt{\frac{\pi}{\mu}} [\cos \sqrt{\mu} \text{ci } \sqrt{\mu} - \sin \sqrt{\mu} \text{si } \sqrt{\mu}] \quad [\text{Re } \mu > 0].$$

MI 34

6.227

$$1. \int_0^{\infty} \text{Ei}(-x) e^{-\mu x} x dx = \frac{1}{\mu(\mu+1)} - \frac{1}{\mu^2} \ln(1+\mu) \quad [\text{Re } \mu > 0].$$

MI 34

$$2. \int_0^{\infty} \left[\frac{e^{-ax} \text{Ei}(ax)}{x-b} - \frac{e^{ax} \text{Ei}(-ax)}{x+b} \right] dx = 0 \quad [a > 0, \quad b < 0];$$

$$= \pi^2 e^{-ab} \quad [a > 0, \quad b > 0].$$

ET II 253(1)a

6.228

$$1. \int_0^{\infty} \text{Ei}(-x) e^x x^{\nu-1} dx = -\frac{\pi \Gamma(\nu)}{\sin \nu \pi} \quad [0 < \text{Re } \nu < 1].$$

ET II 308(13)

$$2. \int_0^{\infty} \text{Ei}(-\beta x) e^{-\mu x} x^{\nu-1} dx = -\frac{\Gamma(\nu)}{\nu(\beta+\mu)^{\nu}} {}_2F_1\left(1, \nu; \nu+1; \frac{\mu}{\beta+\mu}\right)$$

$$[|\arg \beta| < \pi, \quad \text{Re}(\beta+\mu) > 0, \quad \text{Re } \nu > 0].$$

ET II 308(14)

6.229

$$\int_0^{\infty} \text{Ei}\left(-\frac{1}{4x^2}\right) \exp\left(-\mu x^2 + \frac{1}{4x^2}\right) \frac{dx}{x^2} = 2\sqrt{\pi} (\cos \sqrt{\mu} \text{si } \sqrt{\mu} - \sin \sqrt{\mu} \text{ci } \sqrt{\mu}) \quad [\text{Re } \mu > 0].$$

MI 34

6.231

$$\int_{-\ln a}^{\infty} [\text{Ei}(-a) - \text{Ei}(-e^{-x})] e^{-\mu x} dx = \frac{1}{\mu} \gamma(\mu, a) \quad [a < 1, \quad \text{Re } \mu > 0].$$

671
6.232

$$1. \int_0^{\infty} \text{Ei}(-ax) \sin bx \, dx = -\frac{\ln\left(1 + \frac{b^2}{a^2}\right)}{2b} \quad [a > 0, \quad b > 0].$$

BI ((473))(1)a

$$2. \int_0^{\infty} \text{Ei}(-ax) \cos bx \, dx = -\frac{1}{b} \operatorname{arctg} \frac{b}{a} \quad [a > 0, \quad b > 0].$$

BI ((473))(2)a

6.233

$$1. \int_0^{\infty} \text{Ei}(-x)e^{-\mu x} \sin \beta x \, dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\beta}{2} \ln[(1 + \mu)^2 + \beta^2] - \mu \operatorname{arctg} \frac{\beta}{1 + \mu} \right\} \\ [\operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

BI ((473))(7)a

$$2. \int_0^{\infty} \text{Ei}(-x)e^{-\mu x} \cos \beta x \, dx = -\frac{1}{\beta^2 + \mu^2} \left\{ \frac{\mu}{2} \ln[(1 + \mu)^2 + \beta^2] + \beta \operatorname{arctg} \frac{\beta}{1 + \mu} \right\} \\ [\operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

BI ((473))(8)a

6.234

$$\int_0^{\infty} \text{Ei}(-x) \ln x \, dx = \mathbf{C} + 1.$$

NT 56(10)

6.24- 6.26 The sine- and cosine-integral functions

6.241

$$1. \int_0^{\infty} \operatorname{si}(px) \operatorname{si}(qx) \, dx = \frac{\pi}{2p} \quad [p \geq q].$$

BI II 653, NT 54(8)

$$3. \int_0^{\infty} \text{si}(px) \text{ci}(qx) dx = \frac{1}{4q} \ln \left(\frac{p+q}{p-q} \right)^2 + \frac{1}{4p} \ln \frac{(p^2 - q^2)^2}{q^4} \quad [p \neq q];$$

$$= \frac{1}{q} \ln 2 \quad [p = q].$$

FI II 653, NT 54(10, 12)

6.242

$$\int_0^{\infty} \frac{\text{ci}(ax)}{\beta + x} dx = -\frac{1}{2} \{ [\text{si}(a\beta)]^2 + [\text{ci}(a\beta)]^2 \} \quad [a > 0, \quad |\arg \beta| < \pi].$$

ET II 224(1)

6.243

$$1. \int_{-\infty}^{\infty} \frac{\text{si}(a|x|)}{x-b} \text{sign } x dx = \pi \text{ci}(a|b|) \quad [a > 0, \quad b > 0].$$

ET II 253(3)

$$2. \int_{-\infty}^{\infty} \frac{\text{ci}(a|x|)}{x-b} dx = -\pi \text{sign } b \cdot \text{si}(a|b|) \quad [a > 0].$$

ET II 253(2)

672

6.244

$$1. \int_0^{\infty} [\text{si}(px)] \frac{x dx}{q^2 + x^2} = \frac{\pi}{2} \text{Ei}(-pq) \quad [p > 0, \quad q > 0].$$

BI ((255))(6)

$$2. \int_0^{\infty} [\text{si}(px)] \frac{x dx}{q^2 - x^2} = -\frac{\pi}{2} \text{ci}(pq) \quad [p > 0, \quad q > 0].$$

BI ((255))(6)

6.245

$$1. \int_0^{\infty} \text{ci}(px) \frac{dx}{q^2 + x^2} = \frac{\pi}{2q} \text{Ei}(-pq) \quad [p > 0, \quad q > 0].$$

$$2. \int_0^{\infty} \text{ci}(px) \frac{dx}{q^2 - x^2} = \frac{\pi}{2q} \text{si}(pq) \quad [p > 0, \quad q > 0].$$

BI ((255))(8)

6.246

$$1. \int_0^{\infty} \text{si}(ax)x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^{\mu}} \sin \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \text{Re } \mu < 1].$$

NT 56(9), ET I 325(12)a

$$2. \int_0^{\infty} \text{ci}(ax)x^{\mu-1} dx = -\frac{\Gamma(\mu)}{\mu a^{\mu}} \cos \frac{\mu\pi}{2} \quad [a > 0, \quad 0 < \text{Re } \mu < 1].$$

NT 56(8), ET I 325(13)a

6.247

$$1. \int_0^{\infty} \text{si}(\beta x)e^{-\mu x} dx = -\frac{1}{\mu} \text{arctg} \frac{\mu}{\beta} \quad [\text{Re } \mu > 0].$$

NT 49(12), ET I 177(18)

$$2. \int_0^{\infty} \text{ci}(\beta x)e^{-\mu x} dx = -\frac{1}{\mu} \ln \sqrt{1 + \frac{\mu^2}{\beta^2}} \quad [\text{Re } \mu > 0].$$

NT 49(11), ET I 178(19)a

6.248

$$1.^8 \int_0^{\infty} \text{si}(x)e^{-\mu x^2} x dx = \frac{\pi}{4\mu} \left[\Phi \left(\frac{1}{2\sqrt{\mu}} \right) - 1 \right] \quad [\text{Re } \mu > 0].$$

MI 34

$$2. \int_0^{\infty} \text{ci}(x)e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} \text{Ei} \left(-\frac{1}{4\mu} \right) \quad [\text{Re } \mu > 0].$$

MI 34

6.249

$$\int_0^{\infty} \left[\text{si}(x^2) + \frac{\pi}{2} \right] e^{-\mu x} dx = \frac{\pi}{\mu} \left\{ \left[S \left(\frac{\mu^2}{4} \right) - \frac{1}{2} \right]^2 + \left[C \left(\frac{\mu^2}{4} \right) - \frac{1}{2} \right]^2 \right\} \quad [\text{Re } \mu > 0].$$

6.251

$$1. \int_0^{\infty} \operatorname{si}\left(\frac{1}{x}\right) e^{-\mu x} dx = \frac{2}{\mu} \operatorname{kei}(2\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0].$$

MI 34

673

$$2. \int_0^{\infty} \operatorname{ci}\left(\frac{1}{x}\right) e^{-\mu x} dx = -\frac{2}{\mu} \operatorname{ker}(2\sqrt{\mu}) \quad [\operatorname{Re} \mu > 0].$$

MI 34

6.252

$$\begin{aligned} 1. \int_0^{\infty} \sin px \operatorname{si}(qx) dx &= \begin{cases} -\frac{\pi}{2p} & [p^2 > q^2] \\ -\frac{\pi}{4p} & [p^2 = q^2]; \\ 0 & [p^2 < q^2]. \end{cases} \end{aligned}$$

FI II 652, NT 50(8)

$$\begin{aligned} 2.^6 \int_0^{\infty} \cos px \operatorname{si}(qx) dx &= \begin{cases} -\frac{1}{4p} \ln\left(\frac{p+q}{p-q}\right)^2 & [p \neq 0, \quad p^2 \neq q^2]; \\ \frac{1}{q} & [p = 0]. \end{cases} \end{aligned}$$

FI II 652, NT 50(10)

$$\begin{aligned} 3. \int_0^{\infty} \sin px \operatorname{ci}(qx) dx &= \begin{cases} -\frac{1}{4p} \ln\left(\frac{p^2}{q^2} - 1\right)^2 & [p \neq 0, \quad p^2 \neq q^2]; \\ 0 & [p = 0]. \end{cases} \end{aligned}$$

FI II 652, NT 50(9)

$$\begin{aligned} 4. \int_0^{\infty} \cos px \operatorname{ci}(qx) dx &= \begin{cases} -\frac{\pi}{2p} & [p^2 > q^2]; \\ -\frac{\pi}{4p} & [p^2 = q^2]; \\ 0 & [p^2 < q^2]. \end{cases} \end{aligned}$$

6.253

$$\begin{aligned}
\int_0^\infty \frac{\operatorname{si}(ax) \sin bx}{1 - 2r \cos x + r^2} dx &= -\frac{\pi(r^m + r^{m+1})}{4b(1-r)(1-r^2)} \quad [b = a - m]; \\
&= -\frac{\pi(2 + 2r - r^m - r^{m+1})}{4b(1-r)(1-r^2)} \quad [b = a + m]; \\
&= -\frac{\pi r^{m+1}}{2b(1-r)(1-r^2)} \quad [a - m - 1 < b < a - m]; \\
&= -\frac{\pi(1 + r - r^{m+1})}{2b(1-r)(1-r^2)} \quad [a + m < b < a + m + 1].
\end{aligned}$$

ET I 97(10)

6.254

$$1. \int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2} \right] \sin bx \frac{dx}{x} = \frac{1}{2} \left[L_2 \left(\frac{a}{b} \right) - L_2 \left(-\frac{a}{b} \right) \right] \quad [a > 0, \quad b > 0].$$

ET I 97(12)

$$2. \int_0^\infty \left[\operatorname{si}(ax) + \frac{\pi}{2} \right] \cos bx \cdot \frac{dx}{x} = \frac{\pi}{2} \ln \frac{a}{b} \quad [a > 0, \quad b > 0].$$

ET I 41(11)

674

6.255

$$1. \int_{-\infty}^\infty [\cos ax \operatorname{ci}(a|x|) + \sin(a|x|) \operatorname{si}(a|x|)] \frac{dx}{x-b} = -\pi [\operatorname{sign} b \cos ab \operatorname{si}(a|b|) - \sin ab \operatorname{ci}(a|b|)]$$

$$[a > 0].$$

ET II 253(4)

$$2. \int_{-\infty}^\infty [\sin ax \operatorname{ci}(a|x|) - \operatorname{sign} x \cos ax \operatorname{si}(a|x|)] \frac{dx}{x-b} = -\pi [\sin(a|b|) \operatorname{si}(a|b|) + \cos ab \operatorname{ci}(a|b|)]$$

$$[a > 0].$$

ET II 253(5)

6.256

$$\int_0^\infty [\operatorname{si}^2(x) + \operatorname{ci}^2(x)] \cos ax dx = \frac{\pi}{a} \ln(1+a) \quad [a > 0].$$

6.257

$$\int_0^{\infty} \operatorname{si}\left(\frac{a}{x}\right) \sin bx \, dx = -\frac{\pi}{2b} J_0(2\sqrt{ab}) \quad [b > 0].$$

ET I 96(9)

6.258

$$\begin{aligned} 1. \int_0^{\infty} \left[\operatorname{si}(ax) + \frac{\pi}{2} \right] \sin bx \frac{dx}{x^2 + c^2} &= \\ &= \frac{\pi}{4c} \{ e^{-bc} [\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)] + e^{bc} [\operatorname{Ei}(-ac) - \operatorname{Ei}(-bc)] \} \quad [0 < b \leq a, \quad c > 0]; \\ &= \frac{\pi}{4c} e^{-bc} [\operatorname{Ei}(ac) - \operatorname{Ei}(-ac)] \quad [0 < a \leq b, \quad c > 0]. \end{aligned}$$

BI ((460))(1)

$$\begin{aligned} 2. \int_0^{\infty} \left[\operatorname{si}(ax) + \frac{\pi}{2} \right] \cos bx \frac{x \, dx}{x^2 + c^2} &= -\frac{\pi}{4} \{ e^{-bc} [\operatorname{Ei}(bc) - \operatorname{Ei}(-ac)] + e^{bc} [\operatorname{Ei}(-bc) - \operatorname{Ei}(-ac)] \} \\ &\quad [0 < b \leq a, \quad c > 0]; \\ &= \frac{\pi}{4} e^{-bc} [\operatorname{Ei}(-ac) - \operatorname{Ei}(ac)] \quad [0 < a \leq b, \quad c > 0]. \end{aligned}$$

BI ((460))(2, 5)

6.259

$$\begin{aligned} 1. \int_0^{\infty} \operatorname{si}(ax) \sin bx \frac{dx}{x^2 + c^2} &= \frac{\pi}{2c} \operatorname{Ei}(-ac) \operatorname{sh}(bc) \quad [0 < b \leq a, \quad c > 0]; \\ &= \frac{\pi}{4c} e^{-cb} [\operatorname{Ei}(-bc) + \operatorname{Ei}(bc) - \operatorname{Ei}(-ac) - \operatorname{Ei}(ac)] + \\ &\quad + \frac{\pi}{2c} \operatorname{Ei}(-bc) \operatorname{sh}(bc) \quad [0 < a \leq b, \quad c > 0]. \end{aligned}$$

ET I 96(8)

$$\begin{aligned} 2. \int_0^{\infty} \operatorname{ci}(ax) \sin bx \frac{x \, dx}{x^2 + c^2} &= -\frac{\pi}{2} \operatorname{sh}(bc) \operatorname{Ei}(-ac) \quad [0 < b \leq a, \quad c > 0]; \\ &= -\frac{\pi}{2} \operatorname{sh}(bc) \operatorname{Ei}(-bc) + \frac{\pi}{4} e^{-bc} [\operatorname{Ei}(-bc) + \operatorname{Ei}(bc) - \\ &\quad - \operatorname{Ei}(-ac) - \operatorname{Ei}(ac)] \quad [0 < a \leq b, \quad c > 0]. \end{aligned}$$

BI ((460))(3)A, ET I 97(15)a

675

6.261

$$1. \int_0^{\infty} \text{si}(bx) \cos ax e^{-px} dx = -\frac{1}{2(a^2 + p^2)} \left[\frac{a}{2} \ln \frac{p^2 + (a+b)^2}{p^2 + (a-b)^2} + p \operatorname{arctg} \frac{2bp}{b^2 - a^2 - p^2} \right]$$

$$[a > 0, \quad b > 0, \quad p > 0].$$

ET I 40(8)

$$2. \int_0^{\infty} \text{si}(\beta x) \cos ax e^{-\mu x} dx = -\frac{\operatorname{arctg} \frac{\mu+ai}{\beta}}{2(\mu+ai)} - \frac{\operatorname{arctg} \frac{\mu-ai}{\beta}}{2(\mu-ai)} \quad [a > 0, \quad \operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

ET I 40(9)

6.262

$$1. \int_0^{\infty} \text{ci}(bx) \sin ax e^{-\mu x} dx = \frac{1}{2(a^2 + \mu^2)} \left\{ \mu \operatorname{arctg} \frac{2a\mu}{\mu^2 + b^2 - a^2} - \frac{a}{2} \ln \frac{(\mu^2 + b^2 - a^2)^2 + 4a^2\mu^2}{b^4} \right\}$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0].$$

ET I 98(16)a

$$2. \int_0^{\infty} \text{ci}(bx) \cos ax e^{-px} dx = \frac{-1}{2(a^2 + p^2)} \left\{ \frac{p}{2} \ln \frac{[(b^2 + p^2 - a^2)^2 + 4a^2p^2]}{b^4} + a \operatorname{arctg} \frac{2ap}{b^2 + p^2 - a^2} \right\}$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} p > 0].$$

ET I 41(16)

$$3. \int_0^{\infty} \text{ci}(\beta x) \cos ax e^{-\mu x} dx = \frac{-\ln \left[1 + \frac{(\mu+ai)^2}{\beta^2} \right]}{4(\mu+ai)} - \frac{\ln \left[1 + \frac{(\mu-ai)^2}{\beta^2} \right]}{4(\mu-ai)}$$

$$[a > 0, \quad \operatorname{Re} \mu > |\operatorname{Im} \beta|].$$

ET I 41(17)

6.263

$$1. \int_0^{\infty} [\text{ci}(x) \cos x + \text{si}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2} - \mu \ln \mu}{1 + \mu^2} \quad [\operatorname{Re} \mu > 0].$$

ME 26a, ET I 178(21)a

$$2. \int_0^{\infty} [\text{si}(x) \cos x - \text{ci}(x) \sin x] e^{-\mu x} dx = \frac{-\frac{\pi}{2}\mu + \ln \mu}{1 + \mu^2} \quad [\operatorname{Re} \mu > 0].$$

$$3. \int_0^{\infty} [\sin x - x \operatorname{ci}(x)] e^{-\mu x} dx = \frac{\ln(1 + \mu^2)}{2\mu^2} \quad [\operatorname{Re} \mu > 0].$$

ME 26

6.264

$$1. \int_0^{\infty} \operatorname{si}(x) \ln x dx = C + 1.$$

NT 46(10)

$$2. \int_0^{\infty} \operatorname{ci}(x) \ln x dx = \frac{\pi}{2}.$$

NT 56(11)

6.27 The hyperbolic-sine- and cosine-integral functions

6.271

$$1. \int_0^{\infty} \operatorname{shi}(x) e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu + 1}{\mu - 1} = \frac{1}{\mu} \operatorname{Arcth} \mu \quad [\operatorname{Re} \mu > 1].$$

MI 34

$$2. \int_0^{\infty} \operatorname{chi}(x) e^{-\mu x} dx = -\frac{1}{2\mu} \ln(\mu^2 - 1) \quad [\operatorname{Re} \mu > 1].$$

MI 34

6.272

$$\int_0^{\infty} \operatorname{chi}(x) e^{-px^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{p}} \operatorname{Ei} \left(\frac{1}{4p} \right) \quad [p > 0].$$

MI 35

6.273

$$1. \int_0^{\infty} [\operatorname{ch} x \operatorname{shi}(x) - \operatorname{sh} x \operatorname{chi}(x)] e^{-\mu x} dx = \frac{\ln \mu}{\mu^2 - 1} \quad [\operatorname{Re} \mu > 0].$$

$$2. \int_0^{\infty} [\operatorname{ch} x \operatorname{chi}(x) + \operatorname{sh} x \operatorname{shi}(x)] e^{-\mu x} dx = \frac{\mu \ln \mu}{1 - \mu^2} \quad [\operatorname{Re} \mu > 2].$$

MI 35

6.274

$$\int_0^{\infty} [\operatorname{ch} x \operatorname{shi}(x) - \operatorname{sh} x \operatorname{chi}(x)] e^{-\mu x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{\mu}} e^{\frac{1}{4\mu}} \operatorname{Ei} \left(-\frac{1}{4\mu} \right) \quad [\operatorname{Re} \mu > 0].$$

MI 35

6.275

$$\int_0^{\infty} [x \operatorname{chi}(x) - \operatorname{sh} x] e^{-\mu x} dx = -\frac{\ln(\mu^2 - 1)}{2\mu^2} \quad [\operatorname{Re} \mu > 1].$$

MI 35

6.276

$$\int_0^{\infty} [\operatorname{ch} x \operatorname{chi}(x) + \operatorname{sh} x \operatorname{shi}(x)] e^{-\mu x^2} x dx = \frac{1}{8} \sqrt{\frac{\pi}{\mu^3}} \exp \left(\frac{1}{4\mu} \right) \operatorname{Ei} \left(-\frac{1}{4\mu} \right) \quad [\operatorname{Re} \mu > 0].$$

MI 35

6.277

$$1. \int_0^{\infty} [\operatorname{chi}(x) + \operatorname{ci}(x)] e^{-\mu x} dx = -\frac{\ln(\mu^4 - 1)}{2\mu} \quad [\operatorname{Re} \mu > 1].$$

MI 34

677

$$2. \int_0^{\infty} [\operatorname{chi}(x) - \operatorname{ci}(x)] e^{-\mu x} dx = \frac{1}{2\mu} \ln \frac{\mu^2 + 1}{\mu^2 - 1} \quad [\operatorname{Re} \mu > 1].$$

MI 35

6.28- 6.31 The probability integral

6.281

$$1.^6 \int_0^{\infty} [1 - \Phi(px)] x^{2q-1} dx = \frac{\Gamma(q + \frac{1}{2})}{2\sqrt{\pi} q p^{2q}} \quad [\operatorname{Re} q > 0, \operatorname{Re} p > 0].$$

$$2.6 \int_0^\infty \left[1 - \Phi \left(at^\alpha \pm \frac{b}{t^\alpha} \right) \right] dt = \frac{2b}{\sqrt{\pi}} \left(\frac{b}{a} \right)^{\frac{1-\alpha}{2\alpha}} \left[K_{\frac{1+\alpha}{2\alpha}}(2ab) \pm K_{\frac{1-\alpha}{2\alpha}}(2ab) \right] e^{\pm 2ab}$$

$[a > 0, \quad b > 0, \quad \alpha \neq 0].$

6.282

$$1. \int_0^\infty \Phi(qt) e^{-pt} dt = \frac{1}{p} \left[1 - \Phi \left(\frac{p}{2q} \right) \right] \exp \left(\frac{p^2}{4q^2} \right) \quad [\operatorname{Re} p > 0, \quad |\arg q| < \pi/4].$$

MO 175, EH II 148(11)

$$2. \int_0^\infty \left[\Phi \left(x + \frac{1}{2} \right) - \Phi \left(\frac{1}{2} \right) \right] e^{-\mu x + \frac{1}{4}} dx = \frac{1}{(\mu+1)(\mu+2)} \exp \frac{(\mu+1)^2}{4} \left[1 - \Phi \left(\frac{\mu+1}{2} \right) \right].$$

ME 27

6.283

$$1. \int_0^\infty e^{\beta x} [1 - \Phi(\sqrt{\alpha x})] dx = \frac{1}{\beta} \left[\frac{\sqrt{\alpha}}{\sqrt{\alpha - \beta}} - 1 \right] \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta < \operatorname{Re} \alpha].$$

ET II 307(5)

$$2. \int_0^\infty \Phi(\sqrt{qt}) e^{-pt} dt = \frac{\sqrt{q}}{p} \frac{1}{\sqrt{p+q}} \quad [\operatorname{Re} p > 0, \quad \operatorname{Re}(q+p) > 0].$$

EH II 148(12)

6.284

$$\int_0^\infty \left[1 - \Phi \left(\frac{q}{2\sqrt{x}} \right) \right] e^{-px} dx = \frac{1}{p} e^{-q\sqrt{p}} \quad \left[\operatorname{Re} p > 0, \quad |\arg q| < \frac{\pi}{4} \right].$$

EH II 148(13)

EF 147(235)

6.285

$$1. \int_0^\infty [1 - \Phi(x)] e^{-\mu^2 x^2} dx = \frac{\operatorname{arctg} \mu}{\sqrt{\pi} \mu} \quad [\operatorname{Re} \mu > 0].$$

MI 37

$$2. \int_0^\infty \Phi(iat) e^{-a^2 t^2 - st} dt = \frac{-1}{2ai\sqrt{\pi}} \exp \left(\frac{s^2}{4a^2} \right) \operatorname{Ei} \left(-\frac{s^2}{4a^2} \right) \quad \left[\operatorname{Re} s > 0, \quad |\arg a| < \frac{\pi}{4} \right].$$

678
6.286

$$1. \int_0^{\infty} [1 - \Phi(\beta x)] e^{\mu^2 x^2} x^{\nu-1} dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\nu\beta^{\nu}} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2} + 1; \frac{\mu^2}{\beta^2}\right) \\ [\operatorname{Re} \beta^2 > \operatorname{Re} \mu^2, \operatorname{Re} \nu > 0].$$

ET II 306(2)

$$2. \int_0^{\infty} \left[1 - \Phi\left(\frac{\sqrt{2}x}{2}\right)\right] e^{\frac{x^2}{2}} x^{\nu-1} dx = 2^{\frac{\nu}{2}-1} \sec \frac{\nu\pi}{2} \Gamma\left(\frac{\nu}{2}\right) \quad [0 < \operatorname{Re} \nu < 1].$$

ET I 325(9)

6.287

$$1. \int_0^{\infty} \Phi(\beta x) e^{-\mu x^2} x dx = \frac{\beta}{2\mu\sqrt{\mu + \beta^2}} \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \operatorname{Re} \mu > 0].$$

ME 27a, ET I 176(4)

$$2. \int_0^{\infty} [1 - \Phi(\beta x)] e^{-\mu x^2} x dx = \frac{1}{2\mu} \left(1 - \frac{\beta}{\sqrt{\mu + \beta^2}}\right) \quad [\operatorname{Re} \mu > -\operatorname{Re} \beta^2, \operatorname{Re} \mu > 0].$$

NT 49(14), ET I 177(9)

6.288

$$\int_0^{\infty} \Phi(iax) e^{-\mu x^2} x dx = \frac{ai}{2\mu\sqrt{\mu - a^2}} \quad [a > 0, \operatorname{Re} \mu > \operatorname{Re} a^2].$$

MI 37a

6.289

$$1. \int_0^{\infty} \Phi(\beta x) e^{(\beta^2 - \mu^2)x^2} x dx = \frac{\beta}{2\mu(\mu^2 - \beta^2)} \quad \left[\operatorname{Re} \mu^2 > \operatorname{Re} \beta^2, |\arg \mu| < \frac{\pi}{4}\right].$$

ET I 176(5)

$$2. \int_0^{\infty} [1 - \Phi(\beta x)] e^{(\beta^2 - \mu^2)x^2} x dx = \frac{1}{2\mu(\mu + \beta)} \quad \left[\operatorname{Re} \mu^2 > \operatorname{Re} \beta^2, \arg \mu < \frac{\pi}{4}\right]$$

$$3. \int_0^{\infty} \Phi(\sqrt{b-ax}) e^{-(a+\mu)x^2} x dx = \frac{\sqrt{b-a}}{2(\mu+a)\sqrt{\mu+b}} \quad [\operatorname{Re} \mu > -a > 0, \quad b > a].$$

ME 27

6.291

$$\int_0^{\infty} \Phi(ix) e^{-(\mu x + x^2)} x dx = \frac{i}{\sqrt{\pi}} \left[\frac{1}{\mu} + \frac{\mu}{4} \operatorname{Ei} \left(-\frac{\mu^2}{4} \right) \right] \quad [\operatorname{Re} \mu > 0].$$

MI 37

6.292

$$\int_0^{\infty} [1 - \Phi(x)] e^{-\mu^2 x^2} x^2 dx = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\operatorname{arctg} \mu}{\mu^3} - \frac{1}{\mu^2(\mu^2 + 1)} \right\} \quad \left[\left| \arg \mu \right| < \frac{\pi}{4} \right].$$

MI 37

6.293

$$\int_0^{\infty} \Phi(x) e^{-\mu x^2} \frac{dx}{x} = \frac{1}{2} \ln \frac{\sqrt{\mu+1} + 1}{\sqrt{\mu+1} - 1} = \operatorname{Arcth} \sqrt{\mu+1} \quad [\operatorname{Re} \mu > 0].$$

MI 37a

679

6.294

$$1. \int_0^{\infty} \left[1 - \Phi \left(\frac{\beta}{x} \right) \right] e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^2} \exp(-2\beta\mu) \quad \left[\left| \arg \beta \right| < \frac{\pi}{4}, \quad \left| \arg \mu \right| < \frac{\pi}{4} \right].$$

ET I 177(11)

$$2. \int_0^{\infty} \left[1 - \Phi \left(\frac{1}{x} \right) \right] e^{-\mu^2 x^2} \frac{dx}{x} = -\operatorname{Ei}(-2\mu) \quad \left[\left| \arg \mu \right| < \frac{\pi}{4} \right].$$

MI 37

6.295

$$1. \int_0^{\infty} \left[1 - \Phi \left(\frac{1}{x} \right) \right] \exp \left(-\mu^2 x^2 + \frac{1}{x^2} \right) dx = \frac{1}{\sqrt{\pi}\mu} [\sin 2\mu \operatorname{ci}(2\mu) - \cos 2\mu \operatorname{si}(2\mu)] \quad \left[\left| \arg \mu \right| < \frac{\pi}{4} \right].$$

$$2. \int_0^{\infty} \left[1 - \Phi\left(\frac{1}{x}\right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) x dx = \frac{\pi}{2\mu} [H_1(2\mu) - N_1(2\mu)] - \frac{1}{\mu^2} \quad \left[|\arg \mu| < \frac{\pi}{4} \right].$$

MI 37

$$3. \int_0^{\infty} \left[1 - \Phi\left(\frac{1}{x}\right) \right] \exp\left(-\mu^2 x^2 + \frac{1}{x^2}\right) \frac{dx}{x} = \frac{\pi}{2} [H_0(2\mu) - N_0(2\mu)] \quad \left[|\arg \mu| < \frac{\pi}{4} \right].$$

MI 37

6.296

$$\int_0^{\infty} \left\{ (x^2 + a^2) \left[1 - \Phi\left(\frac{a}{\sqrt{2}x}\right) \right] - \sqrt{\frac{2}{\pi}} ax \cdot e^{-\frac{a^2}{2x^2}} \right\} e^{-\mu^2 x^2} x dx = \frac{1}{2\mu^4} e^{-a\mu\sqrt{2}} \quad \left[|\arg \mu| < \frac{\pi}{4}, \quad a > 0 \right].$$

MI 38a

6.297

$$1. \int_0^{\infty} \left[1 - \Phi\left(\gamma x + \frac{\beta}{x}\right) \right] e^{(\gamma^2 - \mu)x^2} x dx = \frac{1}{2\sqrt{\mu}(\sqrt{\mu} + \gamma)} \exp[-2(\beta\gamma + \beta\sqrt{\mu})] \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > 0].$$

ET I 177(12)a

$$2. \int_0^{\infty} \left[1 - \Phi\left(\frac{b + 2ax^2}{2x}\right) \right] \exp[-(\mu^2 - a^2)x^2 + ab]x dx = \frac{e^{-b\mu}}{2\mu(\mu + a)} \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0].$$

MI 38

$$3. \int_0^{\infty} \left\{ \left[1 - \Phi\left(\frac{b - 2ax^2}{2x}\right) \right] e^{-ab} + \left[1 - \Phi\left(\frac{b + 2ax^2}{2x}\right) \right] e^{ab} \right\} e^{-\mu x^2} x dx = \frac{1}{\mu} \exp(-b\sqrt{a^2 + \mu}) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0].$$

MI 38

6.298

$$\int_0^{\infty} \left\{ 2 \operatorname{ch} ab - e^{-ab} \Phi\left(\frac{b - 2ax^2}{2x}\right) - e^{ab} \Phi\left(\frac{b + 2ax^2}{2x}\right) \right\} e^{-(\mu - a^2)x^2} x dx = \frac{1}{\mu - a^2} \exp(-b\sqrt{\mu}) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0].$$

MI 38

$$\int_0^{\infty} \operatorname{ch}(2\nu t) \exp[(a \operatorname{ch} t)^2][1 - \Phi(a \operatorname{ch} t)] dt = \frac{1}{2 \cos(\nu\pi)} \exp\left(\frac{1}{2}a^2\right) K_{\nu}(a^2) \\ \left[\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 308(10)

6.311

$$\int_0^{\infty} [1 - \Phi(ax)] \sin bx \, dx = \frac{1}{b} \left(1 - e^{-\frac{b^2}{4a^2}}\right) \quad [a > 0, \quad b > 0].$$

ET I 96(4)

6.312

$$\int_0^{\infty} \Phi(ax) \sin bx^2 \, dx = \frac{1}{4\sqrt{2\pi b}} \left(\ln \frac{b + a^2 + a\sqrt{2b}}{b + a^2 - a\sqrt{2b}} + 2 \operatorname{arctg} \frac{a\sqrt{2b}}{b - a^2} \right) \quad [a > 0, \quad b > 0].$$

ET I 96(3)

6.313

$$1. \int_0^{\infty} \sin(\beta x)[1 - \Phi(\sqrt{\alpha x})] \, dx = \frac{1}{\beta} - \left(\frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} [(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha]^{-\frac{1}{2}} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

ET II 307(6)

$$2. \int_0^{\infty} \cos(\beta x)[1 - \Phi(\sqrt{\alpha x})] \, dx = \left(\frac{\frac{\alpha}{2}}{\alpha^2 + \beta^2} \right)^{\frac{1}{2}} [(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha]^{-\frac{1}{2}} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

ET II 307(7)

6.314

$$1. \int_0^{\infty} \sin(bx) \left[1 - \Phi\left(\sqrt{\frac{a}{x}}\right) \right] \, dx = b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \cos[(2ab)^{\frac{1}{2}}] \quad [\operatorname{Re} a > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \cos(bx) \left[1 - \Phi \left(\sqrt{\frac{a}{x}} \right) \right] dx = -b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \sin[(2ab)^{\frac{1}{2}}] \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET II 307(9)

6.315

$$1. \int_0^{\infty} x^{\nu-1} \sin(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma \left(1 + \frac{1}{2} \nu \right) \beta}{\sqrt{\pi} (\nu + 1) \alpha^{\nu+1}} {}_2F_2 \left(\frac{\nu + 1}{2}, \frac{\nu}{2} + 1; \frac{3}{2}, \frac{\nu + 3}{2}; -\frac{\beta^2}{4\alpha^2} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 307(3)

$$2. \int_0^{\infty} x^{\nu-1} \cos(\beta x) [1 - \Phi(\alpha x)] dx = \frac{\Gamma \left(\frac{1}{2} + \frac{1}{2} \nu \right)}{\sqrt{\pi} \nu \alpha^{\nu}} {}_2F_2 \left(\frac{\nu}{2}, \frac{\nu + 1}{2}; \frac{1}{2}, \frac{\nu}{2} + 1; -\frac{\beta^2}{4\alpha^2} \right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 307(4)

681

$$3. \int_0^{\infty} [1 - \Phi(ax)] \cos bx \cdot x dx = \frac{1}{2a^2} \exp \left(-\frac{b^2}{4a^2} \right) - \frac{1}{b^2} \left[1 - \exp \left(-\frac{b^2}{4a^2} \right) \right] \quad [a > 0, \quad b > 0].$$

ET I 40(5)

$$4. \int_0^{\infty} [\Phi(ax) - \Phi(bx)] \cos px \frac{dx}{x} = \frac{1}{2} \left[\operatorname{Ei} \left(-\frac{p^2}{4b^2} \right) - \operatorname{Ei} \left(-\frac{p^2}{4a^2} \right) \right] \quad [a > 0, \quad b > 0, \quad p > 0].$$

ET I 40(6)

$$5. \int_0^{\infty} x^{-\frac{1}{2}} \Phi(a\sqrt{x}) \sin bx dx = \frac{1}{2\sqrt{2\pi}b} \left\{ \ln \left[\frac{b + a\sqrt{2b} + a^2}{b - a\sqrt{2b} + a^2} \right] + 2 \operatorname{arctg} \left[\frac{a\sqrt{2b}}{b - a^2} \right] \right\} \quad [a > 0, \quad b > 0].$$

ET I 96(3)

6.316

$$\int_0^{\infty} e^{\frac{1}{2}x^2} \left[1 - \Phi \left(\frac{x}{\sqrt{2}} \right) \right] \sin bx dx = \sqrt{\frac{\pi}{2}} e^{\frac{b^2}{2}} \left[1 - \Phi \left(\frac{b}{\sqrt{2}} \right) \right] \quad [b > 0].$$

$$\int_0^{\infty} e^{-a^2 x^2} \Phi(iax) \sin bx \, dx = \frac{i}{a} \frac{\sqrt{\pi}}{2} e^{-\frac{b^2}{4a^2}} \quad [b > 0].$$

ET I 96(2)

6.318

$$\int_0^{\infty} [1 - \Phi(x)] \operatorname{si}(2px) \, dx = \frac{2}{\pi p} (1 - e^{-p^2}) - \frac{2}{\sqrt{\pi}} (1 - \Phi(p)) \quad [p > 0].$$

NT 61(13)a

6.32 Fresnel integrals

6.321

$$1. \int_0^{\infty} \left[\frac{1}{2} - S(px) \right] x^{2q-1} \, dx = \frac{\sqrt{2}\Gamma\left(q + \frac{1}{2}\right) \sin \frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}} \quad \left[0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0 \right].$$

NT 56(14)a

$$2. \int_0^{\infty} \left[\frac{1}{2} - C(px) \right] x^{2q-1} \, dx = \frac{\sqrt{2}\Gamma\left(q + \frac{1}{2}\right) \cos \frac{2q+1}{4}\pi}{4\sqrt{\pi}qp^{2q}} \quad \left[0 < \operatorname{Re} q < \frac{3}{2}, \quad p > 0 \right].$$

NT 56(13)a

6.322

$$1. \int_0^{\infty} S(t)e^{-pt} \, dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right) \right] + \sin \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right) \right] \right\}.$$

MO 173a

$$2. \int_0^{\infty} C(t)e^{-pt} \, dt = \frac{1}{p} \left\{ \cos \frac{p^2}{4} \left[\frac{1}{2} - S\left(\frac{p}{2}\right) \right] - \sin \frac{p^2}{4} \left[\frac{1}{2} - C\left(\frac{p}{2}\right) \right] \right\}.$$

MO 172a

6.323

$$1. \int_0^{\infty} S(\sqrt{t})e^{-pt} \, dx = \frac{(\sqrt{p^2+1}-p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}.$$

$$2. \int_0^{\infty} C(\sqrt{t})e^{-pt} dt = \frac{(\sqrt{p^2+1}+p)^{\frac{1}{2}}}{2p\sqrt{p^2+1}}.$$

EF 122(58)a

6.324

$$1. \int_0^{\infty} \left[\frac{1}{2} - S(x) \right] \sin 2px dx = (1 + \sin p^2 - \cos p^2)/4p \quad [p > 0].$$

NT 61(12)a

$$2. \int_0^{\infty} \left[\frac{1}{2} - C(x) \right] \sin 2px dx = (1 - \sin p^2 - \cos p^2)/4p \quad [p > 0].$$

NT 61(11)a

6.325

$$1. \int_0^{\infty} S(x) \sin b^2 x^2 dx = \frac{1}{b} \sqrt{\pi} 2^{-\frac{5}{2}} \quad [0 < b^2 < 1];$$

$$= 0 \quad [b^2 > 1].$$

ET I 98(21)a

$$2. \int_0^{\infty} C(x) \cos b^2 x^2 dx = \frac{\sqrt{\pi}}{b} 2^{-\frac{5}{2}} \quad [0 < b^2 < 1];$$

$$= 0 \quad [b^2 > 1].$$

ET I 42(22)

6.326

$$1. \int_0^{\infty} \left[\frac{1}{2} - S(x) \right] \text{si}(2px) dx = (\pi/8)^{1/2} (S(p)+C(p)-1) - (1+\sin p^2 - \cos p^2)/4p$$

$$[p > 0].$$

NT 61(15)a

$$2. \int_0^{\infty} \left[\frac{1}{2} - C(x) \right] \text{si}(2px) dx = (\pi/8)^{1/2} (S(p)-C(p)) - (1-\sin p^2 - \cos p^2)/4p \quad [p > 0].$$

NT 61(14)a

6.4 The Gamma Function and Functions Generated by It

6.41 The gamma function

6.411

$$\begin{aligned} \int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x) dx &= -i\pi 2^{1-\alpha-\beta} \Gamma(\alpha+\beta) \quad [\operatorname{Re}(\alpha+\beta) < 1, \operatorname{Im} \alpha, \operatorname{Im} \beta > 0]; \\ &= i\pi 2^{1-\alpha-\beta} \Gamma(\alpha+\beta) \quad [\operatorname{Re}(\alpha+\beta) < 1, \operatorname{Im} \alpha, \operatorname{Im} \beta < 0]; \\ &= 0 \quad [\operatorname{Re}(\alpha+\beta) < 1, \operatorname{Im} \alpha, \operatorname{Im} \beta < 0]. \end{aligned}$$

ET II 297(1)

ET II 297(2)

ET II 297(3)

683

6.412

$$\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s) ds = 2\pi i \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \\ [\operatorname{Re} \alpha, \operatorname{Re} \beta, \operatorname{Re} \gamma, \operatorname{Re} \delta > 0].$$

ET II 302(32)

6.413

$$1. \int_0^{\infty} |\Gamma(a+ix)\Gamma(b+ix)|^2 dx = \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\Gamma(a+b)}{2\Gamma\left(a+b+\frac{1}{2}\right)} \quad [a > 0, b > 0].$$

ET II 302(27)

$$2. \int_0^{\infty} \left| \frac{\Gamma(a+ix)}{\Gamma(b+ix)} \right|^2 dx = \frac{\sqrt{\pi}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b-a-\frac{1}{2}\right)}{2\Gamma(b)\Gamma\left(b-\frac{1}{2}\right)\Gamma(b-a)} \quad \left[0 < a < b - \frac{1}{2} \right].$$

ET II 302(28)

6.414

ET II 297(4)

$$2. \int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \quad [\operatorname{Re}(\alpha+\beta) > 1].$$

ET II 297(5)

$$3. \int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)} dx = 0 \quad [\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \quad \operatorname{Im} \gamma, \quad \operatorname{Im} \delta > 0].$$

ET II 299(18)

$$4. \int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x)\Gamma(\delta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)} dx = \frac{\pm 2\pi^2 i \Gamma(\alpha+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)]\Gamma(\alpha-\gamma)\Gamma(\alpha-\delta)\Gamma(\beta-\gamma)\Gamma(\beta-\delta)}$$

$[\operatorname{Re}(\alpha+\beta-\gamma-\delta) > 1, \operatorname{Im} \gamma, \operatorname{Im} \delta < 0$. In the numerator, we take the plus sign of $\operatorname{Im} \gamma > \operatorname{Im} \delta$ and the minus sign if $\operatorname{Im} \gamma < \operatorname{Im} \delta$.]

$$5. \int_{-\infty}^{\infty} \frac{\Gamma(\alpha-\beta-\gamma+x+1) dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)} = \frac{\pi \exp \left[\pm \frac{1}{2} \pi (\delta - \gamma) i \right]}{\Gamma(\beta+\gamma-1) \Gamma \left[\frac{1}{2} (\alpha+\beta) \right] \Gamma \left[\frac{1}{2} (\gamma-\delta+1) \right]}$$

ET II 300(19)

$[\operatorname{Re}(\beta+\gamma) > 1, \delta = \alpha - \beta - \gamma + 1, \operatorname{Im} \delta \neq 0$. The sign is plus in the argument if the exponential for $\operatorname{Im} \delta > 0$ and minus for $\operatorname{Im} \delta < 0$.]

$$6. \int_{-\infty}^{\infty} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\Gamma(\alpha+\beta+\gamma+\delta-3)}{\Gamma(\alpha+\beta-1)\Gamma(\beta+\gamma-1)\Gamma(\gamma+\delta-1)\Gamma(\delta+\alpha-1)} \quad [\operatorname{Re}(\alpha+\beta+\gamma+\delta) > 3].$$

ET II 300(21)

ET II 300(20)

$$2. \int_{-\infty}^{\infty} \frac{R(x) dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\int_0^1 R(t) \cos \left[\frac{1}{2} \pi (2t + \alpha - \beta) \right] dt}{\Gamma\left(\frac{\alpha+\beta}{2}\right) \Gamma\left(\frac{\gamma+\delta}{2}\right) \Gamma(\alpha+\delta-1)}$$

$[\alpha + \delta = \beta + \gamma, \quad \operatorname{Re}(\alpha + \beta + \gamma + \delta) > 2, \quad R(x+1) = -R(x)].$

6.42 Combinations of the gamma function, the exponential, and powers

6.421

$$1. \int_{-\infty}^{\infty} \Gamma(\alpha+x)\Gamma(\beta-x) \exp[2(\pi n + \theta)xi] dx = 2\pi i \Gamma(\alpha + \beta) (2 \cos \theta)^{-\alpha-\beta} \exp[(\beta - \alpha)i\theta] \times$$

$$\times [\eta_n(\beta) \exp(2n\pi\beta i) - \eta_n(-\alpha) \exp(-2n\pi\alpha i)]$$

$$\left[\operatorname{Re}(\alpha + \beta) < 1; \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}; \quad n - \text{an integer}; \quad \eta_n(\zeta) = 0, \right.$$

$$\left. \text{if } \left(\frac{1}{2} - n\right) \operatorname{Im} \zeta > 0, \quad \eta_n(\zeta) = \operatorname{sign}\left(\frac{1}{2} - n\right), \quad \text{if } \left(\frac{1}{2} - n\right) \operatorname{Im} \zeta < 0 \right].$$

$$2. \int_{-\infty}^{\infty} \frac{e^{\pi icx} dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+kx)\Gamma(\delta-kx)} = 0$$

$[\operatorname{Re}(\alpha + \beta + \gamma + \delta) > 2, \quad c, \quad k - \text{are real: } |c| > |k| + 1].$

$$3. \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx =$$

$$= 2\pi i \operatorname{sign}\left(n + \frac{1}{2}\right) \frac{(2 \cos \theta)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} \exp[-(2\pi n + \pi - \theta)\alpha i + \theta i(\beta - 1)]$$

$$\left[\operatorname{Re}(\beta - \alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n - \text{an integer}, \quad \left(n + \frac{1}{2}\right) \operatorname{Im} \alpha < 0 \right].$$

$$4. \int_{-\infty}^{\infty} \frac{\Gamma(\alpha+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx = 0$$

$$\left[\operatorname{Re}(\beta - \alpha) > 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad n - \text{an integer}, \quad \left(n + \frac{1}{2}\right) \operatorname{Im} \alpha > 0 \right].$$

$$\begin{aligned}
1. \quad & \int_{-i\infty}^{i\infty} \Gamma(s-k-\lambda) \Gamma\left(\lambda+\mu-s+\frac{1}{2}\right) \Gamma\left(\lambda-\mu-s+\frac{1}{2}\right) z^s ds = \\
& = 2\pi i \Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-k+\mu\right) z^\lambda e^{\frac{z}{2}} W_{k,\mu}(z) \\
& \left[\operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re} \lambda > |\operatorname{Re} \mu| - \frac{1}{2}, \quad |\arg z| < \frac{3\pi}{2} \right].
\end{aligned}$$

ET II 302(29)

$$\begin{aligned}
2. \quad & \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+s) \Gamma(-s) \Gamma(1-c-s) x^s ds = 2\pi i \Gamma(\alpha) \Gamma(\alpha-c+1) \Psi(\alpha, c; x) \\
& \left[-\operatorname{Re} \alpha < \gamma < \min(0, 1 - \operatorname{Re} c), \quad -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2} \right].
\end{aligned}$$

EH I 256(5)

$$3. \quad \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(-s) \Gamma(\beta+s) t^s ds = 2\pi i \Gamma(\beta) (1+t)^{-\beta} \quad [0 > \gamma > \operatorname{Re}(1-\beta), \quad |\arg t| < \pi].$$

EH I 256, BU 75

$$\begin{aligned}
4. \quad & \int_{-\infty i}^{\infty i} \Gamma\left(\frac{t-p}{2}\right) \Gamma(-t) (\sqrt{2})^{t-p-2} z^t dt = 2\pi i e^{\frac{1}{4}z^2} \Gamma(-p) D_p(z) \\
& \left[|\arg z| < \frac{3}{4}\pi; \quad p - \text{not a positive integer} \right].
\end{aligned}$$

WH

$$\begin{aligned}
5. \quad & \int_{-i\infty}^{i\infty} \Gamma(s) \Gamma\left(\frac{1}{2}\nu + \frac{1}{4} - s\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4} - s\right) \left(\frac{z^2}{2}\right)^s ds = \\
& = 2\pi i \cdot 2^{\frac{1}{4}-\frac{1}{2}\nu} z^{-\frac{1}{2}} e^{\frac{3}{4}z^2} \Gamma\left(\frac{1}{2}\nu + \frac{1}{4}\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{4}\right) D_\nu(z) \\
& \left[|\arg z| < \frac{3}{4}\pi, \quad \nu \neq \frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2}, \dots \right].
\end{aligned}$$

EH II 120

$$6.^3 \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{2}x\right)^{-s} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}s\right) \left[\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}s\right)\right]^{-1} ds = 4\pi i J_\nu(x) \quad [x > 0, \quad -\operatorname{Re} \nu < c < 1].$$

EH II 21(34)

$$7. \int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\Gamma(-s) \left(-\frac{1}{2}iz\right)^{\nu+2s} ds = -2\pi^2 e^{\frac{1}{2}i\nu\pi} H_\nu^{(1)}(z) \\ \left[|\arg(-iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re} \nu < c \right].$$

EH II 83(34)

$$8. \int_{-c-i\infty}^{-c+i\infty} \Gamma(-\nu-s)\Gamma(-s) \left(\frac{1}{2}iz\right)^{\nu+2s} ds = 2\pi^2 e^{-\frac{1}{2}i\nu\pi} H_\nu^{(2)}(z) \\ \left[|\arg(iz)| < \frac{\pi}{2}, \quad 0 < \operatorname{Re} \nu < c \right].$$

EH II 83(35)

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$$9. \int_{-i\infty}^{i\infty} \Gamma(-s) \frac{\left(\frac{1}{2}x\right)^{\nu+2s}}{\Gamma(\nu+s+1)} ds = 2\pi i J_\nu(x) \quad [x > 0, \quad \operatorname{Re} \nu > 0].$$

EH II 83(36)

$$10. \int_{-i\infty}^{i\infty} \Gamma(-s)\Gamma(-2\nu-s)\Gamma\left(\nu+s+\frac{1}{2}\right) (-2iz)^s ds = -\pi^{\frac{5}{2}} e^{-i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(1)}(z) \\ \left[|\arg(-iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3 \dots \right].$$

EH II 83(37)

$$11. \int_{-i\infty}^{i\infty} \Gamma(-s)\Gamma(-2\nu-s)\Gamma\left(\nu+s+\frac{1}{2}\right) (2iz)^s ds = \pi^{\frac{5}{2}} e^{i(z-\nu\pi)} \sec(\nu\pi) (2z)^{-\nu} H_\nu^{(2)}(z) \\ \left[|\arg(iz)| < \frac{3}{2}\pi, \quad 2\nu \neq \pm 1, \quad \pm 3 \dots \right].$$

EH II 84(38)

$$12. \int_{-i\infty}^{i\infty} \Gamma(s)\Gamma\left(\frac{1}{2}-s-\nu\right)\Gamma\left(\frac{1}{2}-s+\nu\right) (2z)^s ds = 2^{\frac{3}{2}} \pi^{\frac{3}{2}} iz^{\frac{1}{2}} e^z \sec(\nu\pi) K_\nu(z) \\ \left[|\arg z| < \frac{3\pi}{2}, \quad 2\nu \neq \pm 1, \quad \pm 3, \dots \right].$$

$$13. \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-s)}{s\Gamma(1+s)} x^{2s} ds = 4\pi \int_{2x}^{\infty} \frac{J_0(t)}{t} dt \quad [x > 0].$$

$$14. \int_{-i\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s ds = 2\pi i \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha, \beta; \gamma; z)$$

[For $\arg(-z) < \pi$, the path of integration must separate the poles of the integrand at the points $s = 0, 1, 2, 3, \dots$ from the poles $s = -\alpha - n$ and $s = -\beta - n$ (for $n = 0, 1, 2, \dots$)].

$$15. \int_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s ds = \frac{2\pi i\Gamma(\alpha)}{\Gamma(\gamma)} {}_1F_1(\alpha; \gamma; z) \\ \left[-\frac{\pi}{2} < \arg(-z) < \frac{\pi}{2}, \quad 0 > \delta > -\operatorname{Re} \alpha, \quad \gamma \neq 0, 1, 2, \dots \right].$$

EH I 256(4)

EH I 62(15)

$$16. \int_{-i\infty}^{i\infty} \left[\frac{\Gamma\left(\frac{1}{2}-s\right)}{\Gamma(s)} \right]^2 z^s ds = 2\pi i z^{\frac{1}{2}} [2\pi^{-1} K_0(4z^{\frac{1}{4}}) - N_0(4z^{\frac{1}{4}})] \quad [z > 0].$$

ET II 303(33)

$$17. \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)\Gamma\left(\lambda-\mu-s+\frac{1}{2}\right)}{\Gamma(\lambda-k-s+1)} z^s ds = 2\pi i z^\lambda e^{-\frac{z}{2}} W_{k,\mu}(z) \\ \left[\operatorname{Re} \lambda > |\operatorname{Re} \mu| - \frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right].$$

ET II 302(30)

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$$18. \int_{-i\infty}^{i\infty} \frac{\Gamma(k-\lambda+s)\Gamma\left(\lambda+\mu-s+\frac{1}{2}\right)}{\Gamma\left(\mu-\lambda+s+\frac{1}{2}\right)} z^s ds = 2\pi i \frac{\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma(2\mu+1)} z^\lambda e^{-\frac{z}{2}} M_{k,\mu}(z) \\ \left[\operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(\lambda+\mu) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right].$$

$$19. \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds = 2\pi i G_{pq}^{mn} \left(z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

$$\left[p + q < 2(m + n); \quad |\arg z| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi; \right.$$

$$\left. \operatorname{Re} a_k < 1, \quad k = 1, \dots, n; \quad \operatorname{Re} b_j > 0, \quad j = 1, \dots, m \right].$$

ET II 303(34)

6.423

$$1. \int_0^{\infty} e^{-\alpha x} \frac{dx}{\Gamma(1+x)} = \nu(e^{-\alpha}).$$

MI 39, EH III 222(16)

$$2. \int_0^{\infty} e^{-\alpha x} \frac{dx}{\Gamma(x + \beta + 1)} = e^{\beta\alpha} \nu(e^{-\alpha}, \beta).$$

MI 39, EH III 222(16)

$$3. \int_0^{\infty} e^{-\alpha x} \frac{x^m}{\Gamma(x+1)} dx = \mu(e^{-\alpha}, m) \Gamma(m+1) \quad [\operatorname{Re} m > -1].$$

MI 39, EH III 222(17)

$$4. \int_0^{\infty} e^{-\alpha x} \frac{x^m}{\Gamma(x+n+1)} dx = e^{n\alpha} \mu(e^{-\alpha}, m, n) \Gamma(m+1).$$

MI 39, EH III 222(17)

6.424

$$\int_{-\infty}^{\infty} \frac{R(x) \exp[(2\pi n + \theta)xi] dx}{\Gamma(\alpha + x)\Gamma(\beta - x)} = \frac{\left[2 \cos\left(\frac{\theta}{2}\right) \right]^{\alpha + \beta - 2}}{\Gamma(\alpha + \beta - 1)} \exp\left[\frac{1}{2}\theta(\beta - \alpha)i\right] \int_0^1 R(t) \exp(2\pi nti) dt$$

$$[\operatorname{Re}(\alpha + \beta) > 1, \quad -\pi < \theta < \pi, \quad n - \text{an integer}, \quad R(x+1) = R(x)].$$

ET II 299(16)

6.43 Combinations of the gamma function and trigonometric functions

6.431

$$1. \int_{-\infty}^{\infty} \frac{\sin rx dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \sin \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad [|r| < \pi];$$

$$= 0 \quad [|r| > \pi]; \quad [r - \text{real}; \quad \operatorname{Re}(p+q) > 1].$$

$$2. \int_{-\infty}^{\infty} \frac{\cos rx \, dx}{\Gamma(p+x)\Gamma(q-x)} = \frac{\left(2 \cos \frac{r}{2}\right)^{p+q-2} \cos \frac{r(q-p)}{2}}{\Gamma(p+q-1)} \quad \begin{array}{l} [|r| < \pi]; \\ = 0 \quad [|r| > \pi]; \quad [r - \text{real}; \quad \text{Re}(p+q) > 1]. \end{array}$$

MO 10a, ET II 299(13, 14)

6.432

$$\int_{-\infty}^{\infty} \frac{\sin(m\pi x)}{\sin(\pi x)} \frac{dx}{\Gamma(\alpha+x)\Gamma(\beta-x)} = 0 \quad [m - \text{an even integer}];$$

$$= \frac{2^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)} \quad \begin{array}{l} [m - \text{an odd integer}] \\ [\text{Re}(\alpha+\beta) > 1]. \end{array}$$

ET II 298(11, 12)

6.433

$$1. \int_{-\infty}^{\infty} \frac{\sin \pi x \, dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\sin \left[\frac{\pi}{2}(\beta-\alpha) \right]}{2\Gamma \left(\frac{\alpha+\beta}{2} \right) \Gamma \left(\frac{\gamma+\delta}{2} \right) \Gamma(\alpha+\delta-1)}$$

$$[\alpha+\delta = \beta+\gamma, \quad \text{Re}(\alpha+\beta+\gamma+\delta) > 2].$$

ET II 300(22)

$$2. \int_{-\infty}^{\infty} \frac{\cos \pi x \, dx}{\Gamma(\alpha+x)\Gamma(\beta-x)\Gamma(\gamma+x)\Gamma(\delta-x)} = \frac{\cos \left[\frac{\pi}{2}(\beta-\alpha) \right]}{2\Gamma \left(\frac{\alpha+\beta}{2} \right) \Gamma \left(\frac{\gamma+\delta}{2} \right) \Gamma(\alpha+\delta-1)}$$

$$[\alpha+\delta = \beta+\gamma, \quad \text{Re}(\alpha+\beta+\gamma+\delta) > 2].$$

ET II 301(23)

6.44 *The logarithm of the gamma function* Here, we are violating our usual order of presentation of the formulas in order to make it easier to examine the integrals involving the gamma function.*

6.441

$$1. \int_p^{p+1} \ln \Gamma(x) \, dx = \frac{1}{2} \ln 2\pi + p \ln p - p.$$

$$2. \int_0^1 \ln \Gamma(x) dx = \int_0^1 \ln \Gamma(1-x) dx = \frac{1}{2} \ln 2\pi.$$

FI II 783

$$3. \int_0^1 \ln \Gamma(x+q) dx = \frac{1}{2} \ln 2\pi + q \ln q - q \quad [q \geq 0].$$

NH 89(17), ET II 304(40)

$$4. \int_0^z \ln \Gamma(x+1) dx = \frac{z}{2} \ln 2\pi - \frac{z(z+1)}{2} + z \ln \Gamma(z+1) - \ln G(z+1),$$

where $G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z(z+1)}{2} - \frac{Cz^2}{2}\right) \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}$

WH

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$$5. \int_0^n \ln \Gamma(\alpha+x) dx = \sum_{k=0}^{n-1} (a+k) \ln(a+k) - na + \frac{1}{2}n \ln(2\pi) - \frac{1}{2}n(n-1) \\ [a \geq 0; \quad n = 1, 2, \dots].$$

ET II 304(41)

6.442

$$\int_0^1 \exp(2\pi n x i) \ln \Gamma(a+x) dx = (2\pi n i)^{-1} [\ln a - \exp(-2\pi n a i) \text{Ei}(2\pi n a i)] \\ [a > 0; \quad n = \pm 1, \pm 2, \dots].$$

ET II 304(38)

6.443

$$1. \int_0^1 \ln \Gamma(x) \sin 2\pi n x dx = \frac{1}{2\pi n} [\ln(2\pi n) + C].$$

NH 203(5), ET II 304(42)

$$2. \int_0^1 \ln \Gamma(x) \sin(2n+1)\pi x dx = \frac{1}{(2n+1)\pi} \left[\ln\left(\frac{\pi}{2}\right) + 2 \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) + \frac{1}{2n+1} \right].$$

$$3. \int_0^1 \ln \Gamma(x) \cos 2\pi n x \, dx = \frac{1}{4n}.$$

NH 203(6), ET II 305(44)

$$4.7 \int_0^1 \ln \Gamma(x) \cos(2n+1)\pi x \, dx = \frac{2}{\pi^2} \left[\frac{1}{(2n+1)^2} (\mathbf{C} + \ln 2\pi) + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - (2n+1)^2} \right].$$

$$5. \int_0^1 \sin(2\pi n x) \ln \Gamma(a+x) \, dx = -(2\pi n)^{-1} [\ln a + \cos(2\pi n a) \operatorname{ci}(2\pi n a) - \sin(2\pi n a) \operatorname{si}(2\pi n a)]$$

$$[a > 0; \quad n = 1, 2, \dots].$$

ET II 304(36)

$$6. \int_0^1 \cos(2\pi n x) \ln \Gamma(a+x) \, dx = -(2\pi n)^{-1} [\sin(2\pi n a) \operatorname{ci}(2\pi n a) + \cos(2\pi n a) \operatorname{si}(2\pi n a)]$$

$$[a > 0; \quad n = 1, 2, \dots].$$

ET II 304(37)

6.45 The incomplete gamma function

6.451

$$1. \int_0^{\infty} e^{-\alpha x} \gamma(\beta, x) \, dx = \frac{1}{\alpha} \Gamma(\beta) (1 + \alpha)^{-\beta} \quad [\beta > 0].$$

MI 39

$$2. \int_0^{\infty} e^{-\alpha x} \Gamma(\beta, x) \, dx = \frac{1}{\alpha} \Gamma(\beta) \left[1 - \frac{1}{(\alpha + 1)^\beta} \right] \quad [\beta > 0].$$

MI 39

6.452

$$1. \int_0^{\infty} e^{-\mu x} \gamma\left(\nu, \frac{x^2}{8a^2}\right) \, dx = \frac{1}{\mu} 2^{-\nu-1} \Gamma(2\nu) e^{(a\mu)^2} D_{-2\nu}(2a\mu)$$

$$\left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > 0 \right].$$

$$2. \int_0^{\infty} e^{-\mu x} \gamma\left(\frac{1}{4}, \frac{x^2}{8a^2}\right) dx = \frac{2^{\frac{3}{4}} \sqrt{a}}{\sqrt{\mu}} e^{(a\mu)^2} K_{\frac{1}{4}}(a^2 \mu^2) \quad \left[|\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \mu > 0\right].$$

ET I 179(35)

6.453

$$\int_0^{\infty} e^{-\mu x} \Gamma\left(\nu, \frac{a}{x}\right) dx = 2a^{\frac{1}{2}\nu} \mu^{\frac{1}{2}\nu-1} K_{\nu}(2\sqrt{\mu a}) \quad \left[|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re} \mu > 0\right].$$

ET I 179(32)

6.454

$$\int_0^{\infty} e^{-\beta x} \gamma(\nu, \alpha x^{\frac{1}{2}}) dx = 2^{-\frac{1}{2}\nu} \alpha^{\nu} \beta^{-\frac{1}{2}\nu-1} \Gamma(\nu) \exp\left(\frac{\alpha^2}{8\beta}\right) D_{-\nu}\left(\frac{\alpha}{\sqrt{2\beta}}\right) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 309(19), MI 39a

6.455

$$1. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \Gamma(\nu, \alpha x) dx = \frac{\alpha^{\nu} \Gamma(\mu + \nu)}{\mu(\alpha + \beta)^{\mu+\nu}} {}_2F_1\left(1, \mu + \nu; \mu + 1; \frac{\beta}{\alpha + \beta}\right) \\ [\operatorname{Re}(\alpha + \beta) > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\mu + \nu) > 0].$$

ET II 309(16)

$$2. \int_0^{\infty} x^{\mu-1} e^{-\beta x} \gamma(\nu, \alpha x) dx = \frac{\alpha^{\nu} \Gamma(\mu + \nu)}{\nu(\alpha + \beta)^{\mu+\nu}} {}_2F_1\left(1, \mu + \nu; \nu + 1; \frac{\alpha}{\alpha + \beta}\right) \\ [\operatorname{Re}(\alpha + \beta) > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\mu + \nu) > 0].$$

(15)

6.456

$$1. \int_0^{\infty} e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \gamma\left(\nu, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39a

$$2. \int_0^{\infty} e^{-\alpha x} (4x)^{\nu-\frac{1}{2}} \Gamma\left(\nu, \frac{1}{4x}\right) dx = \frac{\sqrt{\pi} \Gamma(2\nu, \sqrt{\alpha})}{\alpha^{\nu+\frac{1}{2}}}.$$

MI 39a

6.457

$$1. \int_0^{\infty} e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \gamma\left(\nu + 1, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu + \frac{1}{2}}}.$$

MI 39

$$2. \int_0^{\infty} e^{-\alpha x} \frac{(4x)^\nu}{\sqrt{x}} \Gamma\left(\nu + 1, \frac{1}{4x}\right) dx = \sqrt{\pi} \frac{\Gamma(2\nu + 1, \sqrt{\alpha})}{\alpha^{\nu + \frac{1}{2}}}.$$

MI 39

6.458

$$\int_0^{\infty} x^{1-2\nu} \exp(\alpha x^2) \sin(bx) \Gamma(\nu, \alpha x^2) dx = \pi^{\frac{1}{2}} 2^{-\nu} \alpha^{\nu-1} \Gamma\left(\frac{3}{2} - \nu\right) \exp\left(\frac{b^2}{8\alpha}\right) D_{2\nu-2} \left[\frac{b}{(2\alpha)^{\frac{1}{2}}}\right] \\ \left[|\arg \alpha| < \frac{3\pi}{2}, \quad 0 < \operatorname{Re} \nu < 1 \right].$$

ET II 309(18)

6.46-6.47 The function $\psi(x)$

6.461

$$\int_1^x \psi(x) dx = \ln \Gamma(x).$$

691

6.462

$$\int_0^1 \psi(\alpha + x) dx = \ln \alpha \quad [\alpha > 0].$$

ET II 305(1)

6.463

$$\int_0^{\infty} x^{-\alpha} [\mathbf{C} + \psi(1+x)] = -\pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \quad [1 < \operatorname{Re} \alpha < 2].$$

ET II 305(6)

6.464

6.465

$$1.7 \int_0^1 \psi(x) \sin \pi x \, dx = -\frac{2}{\pi} \left[C + \ln 2\pi + 2 \sum_{k=2}^{\infty} \frac{\ln k}{4k^2 - 1} \right].$$

NH 204

$$2. \int_0^1 \psi(x) \sin(2\pi n x) \, dx = -\frac{1}{2}\pi \quad [n = 1, 2, \dots].$$

ET II 305(3)

6.466

$$\int_0^{\infty} [\psi(\alpha + ix) - \psi(\alpha - ix)] \sin xy \, dx = i\pi e^{-\alpha y} (1 - e^{-y})^{-1} \quad [\alpha > 0, \quad y > 0].$$

ET I 96(1)

6.467

$$1. \int_0^1 \sin(2\pi n x) \psi(\alpha + x) \, dx = \sin(2\pi n \alpha) \operatorname{ci}(2\pi n \alpha) + \cos(2\pi n \alpha) \operatorname{si}(2\pi n \alpha) \\ [\alpha \geq 0; \quad n = 1, 2, \dots].$$

ET II 305(4)

$$2. \int_0^1 \cos(2\pi n x) \psi(\alpha + x) \, dx = \sin(2\pi n \alpha) \operatorname{si}(2\pi n \alpha) - \cos(2\pi n \alpha) \operatorname{ci}(2\pi n \alpha) \\ [\alpha > 0; \quad n = 1, 2, \dots].$$

ET II 305(5)

6.468

$$\int_0^1 \psi(x) \sin^2 \pi x \, dx = -\frac{1}{2} [C + \ln(2\pi)].$$

NH 204

6.469

$$1. \int_0^1 \psi(x) \sin \pi x \cos \pi x \, dx = -\frac{\pi}{4}.$$

$$2.7 \int_0^1 \psi(x) \sin \pi x \sin(n\pi x) dx = \frac{n}{1-n^2} \quad [n - \text{even}];$$

$$= \frac{1}{2} \ln \frac{n-1}{n+1} \quad [n - \text{odd}].$$

NH 204(8)a

6.471

$$1. \int_0^\infty x^{-\alpha} [\ln x - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) \zeta(\alpha) \quad [0 < \operatorname{Re} \alpha < 1].$$

ET II 306(7)

692

$$2. \int_0^\infty x^{-\alpha} [\ln(1+x) - \psi(1+x)] dx = \pi \operatorname{cosec}(\pi\alpha) [\zeta(\alpha) - (\alpha-1)^{-1}] \quad [0 < \operatorname{Re} \alpha < 1].$$

ET II 306(8)

$$3. \int_0^\infty [\psi(x+1) - \ln x] \cos(2\pi xy) dx = \frac{1}{2} [\psi(y+1) - \ln y].$$

ET II 306(12)

6.472

$$1. \int_0^\infty x^{-\alpha} [(1+x)^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) [\zeta(1+\alpha) - \alpha^{-1}] \quad [|\operatorname{Re} \alpha| < 1].$$

ET II 306(9)

$$2. \int_0^\infty x^{-\alpha} [x^{-1} - \psi'(1+x)] dx = -\pi\alpha \operatorname{cosec}(\pi\alpha) \zeta(1+\alpha) \quad [-2 < \operatorname{Re} \alpha < 0].$$

ET II 306(10)

6.473

$$\int_0^\infty x^{-\alpha} \psi^{(n)}(1+x) dx = (-1)^{n-1} \frac{\pi \Gamma(\alpha+n)}{\Gamma(\alpha) \sin \pi\alpha} \zeta(\alpha+n) \quad [n = 1, 2, \dots; \quad 0 < \operatorname{Re} \alpha < 1].$$

6.5- 6.7 Bessel Functions

6.51 Bessel functions

6.511

$$1. \int_0^{\infty} J_{\nu}(bx) dx = \frac{1}{b} \quad [\operatorname{Re} \nu > -1, \quad b > 0].$$

ET II 22(3)

$$2. \int_0^{\infty} N_{\nu}(bx) dx = -\frac{1}{b} \operatorname{tg} \left(\frac{\nu\pi}{2} \right) \quad [|\operatorname{Re} \nu| < 1, \quad b > 0].$$

WA 432(7), ET II 96(1)

$$3. \int_0^a J_{\nu}(x) dx = 2 \sum_{k=0}^{\infty} J_{\nu+2k+1}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 333(1)

$$4. \int_0^a J_{\frac{1}{2}}(t) dt = 2S(\sqrt{a}).$$

WA 599(4)

$$5. \int_0^a J_{-\frac{1}{2}}(t) dt = 2C(\sqrt{a}).$$

WA 599(3)

$$6. \int_0^a J_0(x) dx = aJ_0(a) + \frac{\pi a}{2} [J_1(a)H_0(a) - J_0(a)H_1(a)] \quad [a > 0].$$

ET II 7(2)

$$7. \int_0^a J_1(x) dx = 1 - J_0(a) \quad [a > 0].$$

$$8. \int_a^\infty J_0(x) dx = 1 - aJ_0(a) + \frac{\pi a}{2} [J_0(a)H_1(a) - J_1(a)H_0(a)] \quad [a > 0].$$

ET II 7(3)

693

$$9. \int_a^\infty J_1(x) dx = J_0(a) \quad [a > 0].$$

ET II 18(2)

$$10. \int_a^b N_\nu(x) dx = 2 \sum_{n=0}^{\infty} [N_{\nu+2n+1}(b) - N_{\nu+2n+1}(a)].$$

ET II 339(46)

$$11. \int_0^a I_\nu(x) dx = 2 \sum_{n=0}^{\infty} (-1)^n I_{\nu+2n+1}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 364(1)

6.512

$$1. \int_0^\infty J_\mu(ax) J_\nu(bx) dx = b^\nu a^{-\nu-1} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+1}{2}\right)} F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

$[a > 0, \quad b > 0, \quad \operatorname{Re}(\mu+\nu) > -1, \quad b > a.]$

For $a < b$, the positions of μ and ν should be reversed].

ET II 48(6)

$$2.7 \int_0^\infty J_{\nu+n}(\alpha t) J_{\nu-n-1}(\beta t) dt = \frac{\beta^{\nu-n-1} \Gamma(\nu)}{\alpha^{\nu-n} n! \Gamma(\nu-n)} F\left(\nu, -n; \nu-n; \frac{\beta^2}{\alpha^2}\right) \quad [0 < \beta < \alpha];$$

$$= (-1)^n \frac{1}{2\alpha} \quad [0 < \beta = \alpha];$$

$$= 0 \quad [0 < \alpha < \beta] \quad [\operatorname{Re}(\nu) > 0, n \geq 0].$$

MO 50

$$3. \left. \begin{aligned} \int_0^\infty J_\nu(ax) J_{\nu-1}(\beta x) dx &= \frac{\beta^{\nu-1}}{\alpha^\nu} \quad [\beta < \alpha]; \\ &= \frac{1}{2\beta} \quad [\beta = \alpha]; \\ &= 0 \quad [\beta > \alpha]; \end{aligned} \right\} \quad [\operatorname{Re} \nu > 0].$$

$$4. \int_0^\infty J_{\nu+2n+1}(ax)J_\nu(bx) dx = b^\nu a^{-\nu-1} P_n^{(\nu,0)} \left(1 - \frac{2b^2}{a^2}\right) \quad [\operatorname{Re} \nu > -1 - n, \quad 0 < b < a];$$

$$= 0 \quad [\operatorname{Re} \nu > -1 - n, \quad 0 < a < b].$$

ET II 47(5)

$$5. \int_0^\infty J_{\nu+n}(ax)N_{\nu-n}(ax) dx = (-1)^{n+1} \frac{1}{2a} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}; \quad a > 0; \quad n = 0, 1, 2, \dots \right].$$

ET II 347(57)

$$6. \int_0^\infty J_1(bx)N_0(ax) dx = -\frac{b^{-1}}{\pi} \ln \left(1 - \frac{b^2}{a^2}\right) \quad [0 < b < a].$$

ET II 21(31)

$$7. \int_0^a J_\nu(x)J_{\nu+1}(x) dx = \sum_{n=0}^\infty [J_{\nu+n+1}(a)]^2 \quad [\operatorname{Re} \nu > -1].$$

ET II 338(37)

694

6.513

$$1. \int_0^\infty [J_\mu(ax)]^2 J_\nu(bx) dx = a^{2\mu} b^{-2\mu-1} \frac{\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{[\Gamma(\mu+1)]^2 \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \times$$

$$\times \left[F \left(\frac{1-\nu+2\mu}{2}, \frac{1+\nu+2\mu}{2}; \mu+1; \frac{1-\sqrt{1-\frac{4a^2}{b^2}}}{2} \right) \right]^2$$

$$[\operatorname{Re} \nu + \operatorname{Re} 2\mu > -1, \quad 0 < 2a < b].$$

ET II 52(33)

$$2. \int_0^\infty [J_\mu(ax)]^2 K_\nu(bx) dx = \frac{b^{-1}}{2} \Gamma\left(\frac{2\mu+\nu+1}{2}\right) \Gamma\left(\frac{2\mu-\nu+1}{2}\right) \left[P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} \left(\sqrt{1 + \frac{4a^2}{b^2}} \right) \right]^2$$

$$[2 \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \operatorname{Re} b > 2|\operatorname{Im} a|].$$

ET II 138(18)

ET II 65(20)

$$4. \int_0^{\infty} J_{\mu}(ax)J_{-\mu}(ax)K_{\nu}(bx) dx = \frac{\pi}{2b} \sec\left(\frac{\nu\pi}{2}\right) P_{\frac{1}{2}\nu-\frac{1}{2}}^{\mu}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) \\ [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re} b > 2|\operatorname{Im} a|].$$

ET II 138(21)

$$5. \int_0^{\infty} [K_{\mu}(ax)]^2 J_{\nu}(bx) dx = \frac{e^{2\mu\pi i} \Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b \Gamma\left(\frac{1+\nu-2\mu}{2}\right)} \left[Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) \right]^2 \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) > -\frac{1}{2} \right].$$

ET II 66(28)

$$6. \int_0^z J_{\mu}(x)J_{\nu}(z-x) dx = 2 \sum_{k=0}^{\infty} (-1)^k J_{\mu+\nu+2k+1}(z) \quad [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1] \\ [\text{see also } \mathbf{6.683} \text{ } 3].$$

6.683
WA 414(2)

$$7. \int_0^z J_{\mu}(x)J_{-\mu}(z-x) dx = \sin z \quad [-1 < \operatorname{Re} \mu < 1].$$

WA 415(4)

$$8. \int_0^z J_{\mu}(x)J_{1-\mu}(z-x) dx = J_0(z) - \cos(z) \quad [-1 < \operatorname{Re} \mu < 2].$$

WA 415(4)

695
6.514

$$1. \int_0^{\infty} J_{\nu}\left(\frac{a}{x}\right) J_{\nu}(bx) dx = b^{-1} J_{2\nu}(2\sqrt{ab}) \quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$2. \int_0^{\infty} J_{\nu} \left(\frac{a}{x} \right) N_{\nu}(bx) dx = b^{-1} \left[N_{2\nu}(2\sqrt{ab}) + \frac{2}{\pi} K_{2\nu}(\sqrt{2ab}) \right] \\ \left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{3}{2} \right].$$

ET II 110(12)

$$3. \int_0^{\infty} J_{\nu} \left(\frac{a}{x} \right) K_{\nu}(bx) dx = b^{-1} e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}[2e^{\frac{1}{4}i\pi}\sqrt{ab}] + b^{-1} e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}[2e^{-\frac{1}{4}i\pi}\sqrt{ab}] \\ \left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2} \right].$$

ET II 141(31)

$$4. \int_0^{\infty} N_{\nu} \left(\frac{a}{x} \right) J_{\nu}(bx) dx = -\frac{2b^{-1}}{\pi} \left[K_{2\nu}(2\sqrt{ab}) - \frac{\pi}{2} N_{2\nu}(2\sqrt{ab}) \right] \quad \left[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 62(37)a

$$5. \int_0^{\infty} N_{\nu} \left(\frac{a}{x} \right) N_{\nu}(bx) dx = -b^{-1} J_{2\nu}(2\sqrt{ab}) \quad \left[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 110(14)

$$6. \int_0^{\infty} N_{\nu} \left(\frac{a}{x} \right) K_{\nu}(bx) dx = -b^{-1} e^{\frac{1}{2}\nu\pi i} K_{2\nu}(2e^{\frac{1}{4}\pi i}\sqrt{ab}) - b^{-1} e^{-\frac{1}{2}\nu\pi i} K_{2\nu}(2e^{-\frac{1}{4}\pi i}\sqrt{ab}) \\ \left[a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2} \right].$$

ET II 143(37)

$$7. \int_0^{\infty} K_{\nu} \left(\frac{a}{x} \right) N_{\nu}(bx) dx = -2b^{-1} \left[\sin \left(\frac{3\nu\pi}{2} \right) \ker_{2\nu} (2\sqrt{ab}) + \cos \left(\frac{3\nu\pi}{2} \right) \operatorname{kei}_{2\nu} (2\sqrt{ab}) \right] \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 113(28)

$$8. \int_0^{\infty} K_{\nu} \left(\frac{a}{x} \right) K_{\nu}(bx) dx = \pi b^{-1} K_{2\nu}(2\sqrt{ab}) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

6.515

$$1. \int_0^{\infty} J_{\mu}\left(\frac{a}{x}\right) N_{\mu}\left(\frac{a}{x}\right) K_0(bx) dx = -2b^{-1} J_{2\mu}(2\sqrt{ab}) K_{2\mu}(2\sqrt{ab}) \quad [a > 0, \operatorname{Re} b > 0].$$

ET II 143(42)

$$2. \int_0^{\infty} \left[K_{\mu}\left(\frac{a}{x}\right) \right]^2 K_0(bx) dx = 2\pi b^{-1} K_{2\mu}(2e^{\frac{1}{4}\pi i} \sqrt{ab}) K_{2\mu}(2e^{-\frac{1}{4}\pi i} \sqrt{ab}) \quad [\operatorname{Re} a > 0, \operatorname{Re} b > 0].$$

ET II 147(59)

$$3. \int_0^{\infty} H_{\mu}^{(1)}\left(\frac{a^2}{x}\right) H_{\mu}^{(2)}\left(\frac{a^2}{x}\right) J_0(bx) dx = 16\pi^{-2} b^{-1} \cos \mu\pi K_{2\mu}(2e^{\pi i/4} a\sqrt{b}) K_{2\mu}(2e^{-\pi i/4} a\sqrt{b}) \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4} \right].$$

ET II 17(36)

696

6.516

$$1. \int_0^{\infty} J_{2\nu}(a\sqrt{x}) J_{\nu}(bx) dx = b^{-1} J_{\nu}\left(\frac{a^2}{4b}\right) \quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 58(16)

$$2. \int_0^{\infty} J_{2\nu}(a\sqrt{x}) N_{\nu}(bx) dx = -b^{-1} H_{\nu}\left(\frac{a^2}{4b}\right) \quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 111(18)

$$3. \int_0^{\infty} J_{2\nu}(a\sqrt{x}) K_{\nu}(bx) dx = \frac{\pi}{2} b^{-1} \left[I_{\nu}\left(\frac{a^2}{4b}\right) - L_{\nu}\left(\frac{a^2}{4b}\right) \right] \quad \left[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 144(45)

$$4. \int_0^{\infty} N_{2\nu}(a\sqrt{x}) J_{\nu}(bx) dx = 2 \sec(\nu\pi) b^{-1} \left[\frac{1}{2} \cos(\nu\pi) N_{\nu}\left(\frac{a^2}{4b}\right) - N_{-\nu}\left(\frac{a^2}{4b}\right) + H_{-\nu}\left(\frac{a^2}{4b}\right) \right] \\ \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$\begin{aligned}
5. \quad \int_0^\infty N_{2\nu}(a\sqrt{x})N_\nu(bx) dx &= \\
&= \frac{b^{-1}}{2} \left[\sec(\nu\pi)J_{-\nu}\left(\frac{a^2}{4b}\right) + \operatorname{cosec}(\nu\pi)H_{-\nu}\left(\frac{a^2}{4b}\right) - 2\operatorname{ctg}(2\nu\pi)H_\nu\left(\frac{a^2}{4b}\right) \right] \\
&\quad \left[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
\end{aligned}$$

ET II 111(19)

$$\begin{aligned}
6. \quad \int_0^\infty N_{2\nu}(a\sqrt{x})K_\nu(bx) dx &= \frac{\pi b^{-1}}{2} \left[\operatorname{cosec}(2\nu\pi)L_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{ctg}(2\nu\pi)L_\nu\left(\frac{a^2}{4b}\right) - \right. \\
&\quad \left. - \operatorname{tg}(\nu\pi)I_\nu\left(\frac{a^2}{4b}\right) - \frac{\sec(\nu\pi)}{\pi}K_\nu\left(\frac{a^2}{4b}\right) \right] \\
&\quad \left[\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
\end{aligned}$$

ET II 144(46)

$$\begin{aligned}
7. \quad \int_0^\infty K_{2\nu}(a\sqrt{x})J_\nu(bx) dx &= \frac{1}{4}\pi b^{-1} \sec(\nu\pi) \left[H_{-\nu}\left(\frac{a^2}{4b}\right) - N_{-\nu}\left(\frac{a^2}{4b}\right) \right] \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
\end{aligned}$$

ET II 70(22)

$$\begin{aligned}
8. \quad \int_0^\infty K_{2\nu}(a\sqrt{x})N_\nu(bx) dx &= \\
&= -\frac{1}{4}\pi b^{-1} \left[\sec(\nu\pi)J_{-\nu}\left(\frac{a^2}{4b}\right) - \operatorname{cosec}(\nu\pi)H_{-\nu}\left(\frac{a^2}{4b}\right) + 2\operatorname{cosec}(2\nu\pi)H_\nu\left(\frac{a^2}{4b}\right) \right] \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
\end{aligned}$$

ET II 114(34)

697

$$\begin{aligned}
9. \quad \int_0^\infty K_{2\nu}(a\sqrt{x})K_\nu(bx) dx &= \frac{\pi b^{-1}}{4\cos(\nu\pi)} \left\{ K_\nu\left(\frac{a^2}{4b}\right) + \frac{\pi}{2\sin(\nu\pi)} \left[L_{-\nu}\left(\frac{a^2}{4b}\right) - L_\nu\left(\frac{a^2}{4b}\right) \right] \right\} \\
&\quad \left[\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
\end{aligned}$$

ET II 147(63)

$$10. \quad \int_0^\infty I_{2\nu}(a\sqrt{x})K_\nu(bx) dx = \frac{\pi b^{-1}}{2} \left[I_\nu\left(\frac{a^2}{4b}\right) + L_\nu\left(\frac{a^2}{4b}\right) \right] \quad \left[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

6.517

$$\int_0^z J_0(\sqrt{z^2 - x^2}) dx = \sin z.$$

MO 48

6.518

$$\int_0^\infty K_{2\nu}(2z \operatorname{sh} x) dx = \frac{\pi^2}{8 \cos \nu\pi} (J_\nu^2(z) + N_\nu^2(z)) \quad \left[\operatorname{Re} z > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right].$$

MO 45

6.519

$$1. \int_0^{\pi/2} J_{2\nu}(2z \cos x) dx = \frac{\pi}{2} J_\nu^2(z) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WH

$$2. \int_0^{\pi/2} J_{2\nu}(2z \sin x) dx = \frac{\pi}{2} J_\nu^2(z) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 42(1)a

6.52 Bessel functions combined with x and x^2

6.521

$$1. \int_0^1 x J_\nu(\alpha x) J_\nu(\beta x) dx = 0 \quad [\alpha \neq \beta];$$

$$= \frac{1}{2} \{J_{\nu+1}(\alpha)\}^2 \quad [\alpha = \beta] \quad [J_\nu(\alpha) = J_\nu(\beta) = 0, \quad \nu > -1].$$

WH

$$2. \int_0^\infty x K_\nu(\alpha x) J_\nu(\beta x) dx = \frac{b^\nu}{a^\nu (b^2 + a^2)} \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 63(2)

$$3. \int_0^\infty x K_\nu(\alpha x) K_\nu(\beta x) dx = \frac{\pi (ab)^{-\nu} (a^{2\nu} - b^{2\nu})}{2 \sin(\nu\pi) (a^2 - b^2)} \quad [|\operatorname{Re} \nu| < 1, \quad \operatorname{Re}(a+b) > 0].$$

$$4. \int_0^a x J_\nu(\lambda x) K_\nu(\mu x) dx = (\mu^2 + \lambda^2)^{-1} \left[\left(\frac{\lambda}{\mu} \right)^\nu + \lambda a J_{\nu+1}(\lambda a) K_\nu(\mu a) - \mu a J_\nu(\lambda a) K_{\nu+1}(\mu a) \right] \\ [\operatorname{Re} \nu > -1].$$

ET II 367(26)

6.522

$$1. \int_0^\infty x [J_\mu(ax)]^2 K_\nu(bx) dx = \Gamma\left(\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\mu - \frac{1}{2}\nu + 1\right) b^{-2} \times \\ \times (1 + 4a^2 b^{-2})^{-\frac{1}{2}} P_{\frac{1}{2}\nu}^{-\mu}[(1 + 4a^2 b^{-2})^{\frac{1}{2}}] P_{\frac{1}{2}\nu-1}^{-\mu}[(1 + 4a^2 b^{-2})^{\frac{1}{2}}] \\ [\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2\operatorname{Re} \mu > |\operatorname{Re} \nu| - 2].$$

ET II 138(19)

698

$$2. \int_0^\infty x [K_\mu(ax)]^2 J_\nu(bx) dx = \frac{2e^{2\mu\pi i} \Gamma\left(1 + \frac{1}{2}\nu + \mu\right)}{b(4a^2 + b^2)^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\nu - \mu\right)} \times \\ \times Q_{\frac{1}{2}\nu}^{-\mu}[(1 + 4a^2 b^{-2})^{\frac{1}{2}}] Q_{\frac{1}{2}\nu-1}^{-\mu}[(1 + 4a^2 b^{-2})^{\frac{1}{2}}] \\ \left[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu \pm \mu\right) > -1 \right].$$

ET II 66(27)a

$$3. \int_0^\infty x K_0(ax) J_\nu(bx) J_\nu(cx) dx = r_1^{-1} r_2^{-1} (r_2 - r_1)^\nu (r_2 + r_1)^{-\nu}, \\ r_1 = [a^2 + (b - c)^2]^{\frac{1}{2}}, \quad r_2 = [a^2 + (b + c)^2]^{\frac{1}{2}} \\ [c > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} a > |\operatorname{Im} b|].$$

ET II 63(6)

$$4. \int_0^\infty x I_0(ax) K_0(bx) J_0(cx) dx = (a^4 + b^4 + c^4 - 2a^2 b^2 + 2a^2 c^2 + 2b^2 c^2)^{-\frac{1}{2}} \\ [\operatorname{Re} b > \operatorname{Re} a, \quad c > 0].$$

ET II 16(27)

$$5. \int_0^\infty x J_0(ax) K_0(bx) J_0(cx) dx = (a^4 + b^4 + c^4 - 2a^2 c^2 + 2a^2 b^2 + 2b^2 c^2)^{-\frac{1}{2}} \\ [\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0].$$

$$6. \int_0^\infty x J_0(ax) N_0(ax) J_0(bx) dx = 0 \quad [0 < b < 2a];$$

$$= -2\pi^{-1} b^{-1} [b^2 - 4a^2]^{-\frac{1}{2}} \quad [0 < 2a < b < \infty].$$

ET II 15(21)

$$7. \int_0^\infty x J_\mu(ax) J_{\mu+1}(ax) K_\nu(bx) dx = \Gamma\left(\mu + \frac{3+\nu}{2}\right) \Gamma\left(\mu + \frac{3-\nu}{2}\right) b^{-2} (1 + 4a^2 b^{-2})^{-\frac{1}{2}} \times$$

$$\times P_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu} [(1 + 4a^2 b^{-2})^{\frac{1}{2}}] P_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu-1} [(1 + 4a^2 b^{-2})^{\frac{1}{2}}]$$

$$[\operatorname{Re} b > 2|\operatorname{Im} a|, \quad 2 \operatorname{Re} \mu > |\operatorname{Re} \nu| - 3].$$

ET II 138(20)

$$8. \int_0^\infty x K_{\mu-\frac{1}{2}}(ax) K_{\mu+\frac{1}{2}}(ax) J_\nu(bx) dx =$$

$$= -\frac{2e^{2\mu\pi i} \Gamma\left(\frac{1}{2}\nu + \mu + 1\right)}{b \Gamma\left(\frac{1}{2}\nu - \mu\right) (b^2 + 4a^2)^{\frac{1}{2}}} Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu + \frac{1}{2}} [(1 + 4a^2 b^{-2})^{\frac{1}{2}}] Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu - \frac{1}{2}} [(1 + 4a^2 b^{-2})^{\frac{1}{2}}]$$

$$\left[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} \mu| < 1 + \frac{1}{2} \operatorname{Re} \nu \right].$$

ET II 67(29)a

$$9. \int_0^\infty x I_{\frac{1}{2}\nu}(ax) K_{\frac{1}{2}}(ax) J_\nu(bx) dx = b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} \quad [b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 65(16)

699

$$10. \int_0^\infty x J_{\frac{1}{2}\nu}(ax) N_{\frac{1}{2}\nu}(ax) J_\nu(bx) dx =$$

$$= 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1; \quad 0 < b < 2a];$$

$$= -2\pi^{-1} b^{-1} (b^2 - 4a^2)^{-\frac{1}{2}} \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b < \infty].$$

ET II 55(48)

$$11. \int_0^\infty x J_{\frac{1}{2}(\nu+n)}(ax) J_{\frac{1}{2}(\nu-n)}(ax) J_\nu(bx) dx =$$

$$= 2\pi^{-1} b^{-1} (4a^2 - b^2)^{-\frac{1}{2}} T_n\left(\frac{b}{2a}\right) \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 0 < b < 2a];$$

$$= 0 \quad [a > 0, \quad \operatorname{Re} \nu > -1, \quad 2a < b < \infty].$$

$$12. \int_0^{\infty} x I_{\frac{1}{2}(\nu-\mu)}(ax) K_{\frac{1}{2}(\nu+\mu)}(ax) J_{\nu}(bx) dx = 2^{-\mu} a^{-\mu} b^{-1} (b^2 + 4a^2)^{-\frac{1}{2}} [b + (b^2 + 4a^2)^{\frac{1}{2}}]^{\mu}$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - \mu) > -2].$$

ET II 66(23)

$$13. \int_0^{\infty} x J_{\mu}(xa \sin \varphi) K_{\nu-\mu}(ax \cos \varphi \cos \psi) J_{\nu}(xa \sin \psi) dx = \frac{(\sin \varphi)^{\mu} (\sin \psi)^{\nu} (\cos \varphi)^{\nu-\mu} (\cos \psi)^{\mu-\nu}}{a^2 (1 - \sin^2 \varphi \sin^2 \psi)}$$

$$\left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1 \right].$$

ET II 64(10)

$$14.^6 \int_0^{\infty} 7x J_{\mu}(xa \sin \varphi \cos \psi) J_{\nu-\mu}(ax) J_{\nu}(xa \cos \varphi \sin \psi) dx =$$

$$= -2\pi^{-1} a^{-2} \sin(\mu\pi) (\sin \varphi)^{\mu} (\sin \psi)^{\nu} (\cos \varphi)^{-\nu} (\cos \psi)^{-\mu} [\cos(\varphi + \psi) \cos(\varphi - \psi)]^{-1}$$

$$\left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -1 \right].$$

ET II 54(39)

6.523

$$\int_0^{\infty} x [2\pi^{-1} K_0(ax) - N_0(ax)] K_0(bx) dx = 2\pi^{-1} [(a^2 + b^2)^{-1} + (b^2 - a^2)^{-1}] \ln \frac{b}{a}$$

$$[\operatorname{Re} b > |\operatorname{Im} a|, \quad \operatorname{Re}(a + b) > 0].$$

ET II 145(50)

6.524

$$1. \int_0^{\infty} x J_{\nu}^2(ax) J_{\nu}(bx) N_{\nu}(bx) dx = 0 \quad \left[0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2} \right];$$

$$= -(2\pi ab)^{-1} \quad \left[0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 352(14)

$$2. \int_0^{\infty} x [J_0(ax) K_0(bx)]^2 dx = \frac{\pi}{8ab} - \frac{1}{4ab} \arcsin \left(\frac{b^2 - a^2}{b^2 + a^2} \right) \quad [a > 0, \quad b > 0].$$

ET II 373(9)

6.525

$$1. \int_0^{\infty} x^2 J_1(ax) K_0(bx) J_0(cx) dx = 2a(a^2 + b^2 - c^2) [(a^2 + b^2 + c^2)^2 - 4a^2 c^2]^{-\frac{3}{2}}$$

$$[c > 0, \quad \operatorname{Re} b \geq |\operatorname{Im} a|, \quad \operatorname{Re} a > 0].$$

$$2. \int_0^{\infty} x^2 I_0(ax) K_1(bx) J_0(cx) dx = 2b(b^2 + c^2 - a^2)[(a^2 + b^2 + c^2)^2 - 4a^2 b^2]^{-\frac{3}{2}}.$$

ET II 16(28)

6.526

$$1. \int_0^{\infty} x J_{\frac{1}{2}\nu}(ax^2) J_{\nu}(bx) dx = (2a)^{-1} J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 56(1)

$$2. \int_0^{\infty} x J_{\frac{1}{2}\nu}(ax^2) N_{\nu}(bx) dx = (4a)^{-1} \left[N_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \operatorname{tg}\left(\frac{\nu\pi}{2}\right) J_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \operatorname{sec}\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 109(9)

$$3. \int_0^{\infty} x J_{\frac{1}{2}\nu}(ax^2) K_{\nu}(bx) dx = \frac{\pi}{8a \cos\left(\frac{\nu\pi}{2}\right)} \left[\mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - N_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 140(27)

$$4. \int_0^{\infty} x N_{\frac{1}{2}\nu}(ax^2) J_{\nu}(bx) dx = -(2a)^{-1} \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 61(35)

$$5. \int_0^{\infty} x N_{\frac{1}{2}\nu}(ax^2) K_{\nu}(bx) dx = \\ = \frac{\pi}{4a \sin(\nu\pi)} \left[\cos\left(\frac{\nu\pi}{2}\right) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \sin\left(\frac{\nu\pi}{2}\right) J_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET II 141(28)

$$6. \int_0^{\infty} x K_{\frac{1}{2}\nu}(ax^2) J_{\nu}(bx) dx = \frac{\pi}{4a} \left[I_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 68(9)

$$8. \int_0^{\infty} x K_{\frac{1}{2}\nu}(ax^2) K_{\nu}(bx) dx = \frac{\pi}{8a} \left\{ \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) + \pi \operatorname{cosec}(\nu\pi) \left[L_{-\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) - L_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \right] \right\}$$

[Re $a > 0$, $|\operatorname{Re} \nu| < 1$].

ET II 146(52)

6.527

$$1. \int_0^{\infty} x^2 J_{2\nu}(2ax) J_{\nu-\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu+\frac{1}{2}}(a^2) \quad \left[a > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 355(33)

701

$$2. \int_0^{\infty} x^2 J_{2\nu}(2ax) J_{\nu+\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \operatorname{Re} \nu > -2].$$

ET II 355(35)

$$3. \int_0^{\infty} x^2 J_{2\nu}(2ax) N_{\nu+\frac{1}{2}}(x^2) dx = -\frac{1}{2} a H_{\nu-\frac{1}{2}}(a^2) \quad [a > 0, \operatorname{Re} \nu > -2].$$

ET II 355(36)

6.528

$$\int_0^{\infty} x K_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) I_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) J_{\nu}(bx) dx = K_{\frac{1}{4}\nu}\left(\frac{x^2}{4}\right) I_{\frac{1}{4}\nu}\left(\frac{b^2}{4}\right) \quad [b > 0, \nu > -1].$$

MO 183a

6.529

$$1. \int_0^{\infty} x J_{\nu}(2\sqrt{ax}) K_{\nu}(2\sqrt{ax}) J_{\nu}(bx) dx = \frac{1}{2} b^{-2} e^{-\frac{2a}{b}} \quad [\operatorname{Re} a > 0, b > 0, \operatorname{Re} \nu > -1].$$

ET II 70(23)

$$2. \int_0^a x J_{\lambda}(2a) I_{\lambda}(2x) J_{\mu}(2\sqrt{a^2-x^2}) I_{\mu}(2\sqrt{a^2-x^2}) dx =$$

$$= \frac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)} \times$$

$$\times {}_1F_4\left(\frac{\lambda+\mu+1}{2}, \lambda+1, \mu+1, \frac{\lambda+\mu+3}{2}, \frac{\lambda+\mu+1}{2}\right)$$

[Re $\lambda > -1$, $\operatorname{Re} \mu > -1$].

6.53- 6.54 Combinations of Bessel functions and rational functions

6.531

$$1. \int_0^{\infty} \frac{N_{\nu}(bx)}{x+a} dx = \frac{\pi}{\sin(\pi\nu)} [E_{\nu}(ab) + N_{\nu}(ab)] + 2 \operatorname{ctg}(\pi\nu) [J_{\nu}(ab) - J_{\nu}(ab)]$$

$$\left[b > 0, \quad |\arg a| < \pi, \quad |\operatorname{Re} \nu| < 1, \quad \nu \neq 0, \quad \pm \frac{1}{2} \right].$$

ET II 97(5)

$$2. \int_0^{\infty} \frac{N_{\nu}(bx)}{x-a} dx = \pi \{ \operatorname{ctg}(\nu\pi) [N_{\nu}(ab) + E_{\nu}(ab)] + J_{\nu}(ab) + 2 [\operatorname{ctg}(\nu\pi)]^2 [J_{\nu}(ab) - J_{\nu}(ab)] \}$$

$$[b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET II 98(9)

$$3. \int_0^{\infty} \frac{K_{\nu}(bx)}{x+a} dx = \frac{\pi^2}{2} [\operatorname{cosec}(\nu\pi)]^2 [I_{\nu}(ab) + I_{-\nu}(ab) - e^{-\frac{1}{2}i\nu\pi} J_{\nu}(iab) - e^{\frac{1}{2}i\nu\pi} J_{-\nu}(iab)]$$

$$[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad |\operatorname{Re} \nu| < 1].$$

ET II 128(5)

6.532

$$1. \int_0^{\infty} \frac{J_{\nu}(x)}{x^2 + a^2} dx = \frac{\pi [J_{\nu}(a) - J_{\nu}(a)]}{a \sin(\nu\pi)} \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 340(2)

702

$$2. \int_0^{\infty} \frac{N_{\nu}(x)}{x^2 + a^2} dx = \frac{1}{\cos \frac{\nu\pi}{2}} \left[-\frac{\pi}{2a} \operatorname{tg} \left(\frac{\nu\pi}{2} \right) I_{\nu}(ab) - \frac{1}{a} K_{\nu}(ab) + \right.$$

$$\left. + \frac{b \sin \left(\frac{\nu\pi}{2} \right)}{1 - \nu^2} {}_1F_2 \left(1; \frac{3 - \nu}{2}, \frac{3 + \nu}{2}; \frac{a^2 b^2}{4} \right) \right]$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1].$$

ET II 99(13)

$$3. \int_0^{\infty} \frac{N_{\nu}(bx)}{x^2 - a^2} dx = \frac{\pi}{2a} \left\{ J_{\nu}(ab) + \operatorname{tg} \left(\frac{\nu\pi}{2} \right) \left\{ \operatorname{tg} \left(\frac{\nu\pi}{2} \right) [J_{\nu}(ab) - J_{\nu}(ab)] - E_{\nu}(ab) - N_{\nu}(ab) \right\} \right\}$$

$$[b > 0, \quad a > 0, \quad |\operatorname{Re} \nu| < 1].$$

$$4. \int_0^{\infty} \frac{x J_0(ax)}{x^2 + k^2} dx = K_0(ak) \quad [a > 0, \operatorname{Re} k > 0].$$

WA 466(5)

$$5. \int_0^{\infty} \frac{N_0(ax)}{x^2 + k^2} dx = -\frac{K_0(ak)}{k} \quad [a > 0, \operatorname{Re} k > 0].$$

WA 466(6)

$$6. \int_0^{\infty} \frac{J_0(ax)}{x^2 + k^2} dx = \frac{\pi}{2k} [I_0(ak) - L_0(ak)] \quad [a > 0, \operatorname{Re} k > 0].$$

WA 467(7)

6.533

$$1. \int_0^z J_p(x) J_q(z-x) \frac{dx}{x} = \frac{J_{p+q}(z)}{p} \quad [\operatorname{Re} p > 0, \operatorname{Re} q > -1].$$

WA 415(3)

$$2. \int_0^z \frac{J_p(x)}{x} \frac{J_q(z-x)}{z-x} dx = \left(\frac{1}{p} + \frac{1}{q} \right) \frac{J_{p+q}(z)}{z} \quad [\operatorname{Re} p > 0, \operatorname{Re} q > 0].$$

WA 415(5)

$$3. \int_0^{\infty} [J_0(ax) - 1] J_1(bx) \frac{dx}{x} = \frac{-b}{4} \left[1 + 2 \ln \frac{a}{b} \right] \quad [0 < b < a];$$

$$= -\frac{a^2}{4b} \quad [0 < a < b].$$

ET II 21(28)a

$$4. \int_0^{\infty} [1 - J_0(ax)] J_0(bx) \frac{dx}{x} = 0 \quad [0 < a < b];$$

$$= \ln \frac{a}{b} \quad [0 < b < a].$$

ET II 14(16)

6.534

$$\int_0^{\infty} \frac{x^3 J_0(x)}{x^4 - a^4} dx = \frac{1}{2} K_0(a) - \frac{1}{4} \pi N_0(a) \quad [a > 0].$$

6.535

$$\int_0^{\infty} \frac{x}{x^2 + a^2} [J_{\nu}(x)]^2 dx = I_{\nu}(a)K_{\nu}(a) \quad [\operatorname{Re} a > 0, \operatorname{Re} \nu > -1].$$

ET II 342(26)

703

6.536

$$\int_0^{\infty} \frac{x^3 J_0(bx)}{x^4 + a^4} dx = \operatorname{ker}(ab) \quad \left[b > 0, \quad |\arg a| < \frac{1}{4}\pi \right].$$

ET II 8(9), MO 46a

6.537

$$\int_0^{\infty} \frac{x^2 J_0(bx)}{x^4 + a^4} dx = -\frac{1}{a^2} \operatorname{kei}(ab) \quad \left[b > 0, \quad |\arg a| < \frac{\pi}{4} \right].$$

MO 46a

6.538

$$1. \int_0^{\infty} J_1(ax)J_1(bx) \frac{dx}{x^2} = \frac{a+b}{\pi} \left[E \left(\frac{2i\sqrt{ab}}{|b-a|} \right) - K \left(\frac{2i\sqrt{ab}}{|b-a|} \right) \right] \quad [a > 0, \quad b > 0].$$

ET II 21(30)

$$2.^8 \int_0^{\infty} x^{-1} J_{\nu+2n+1}(x)J_{\nu+2m+1}(x) dx = 0 \quad [m \neq n \text{ with } m, n \text{ integers, } \nu > -1];$$

$$= (4n + 2\nu + 2)^{-1} \quad [m = n, \quad \nu > -1].$$

EH II 64

6.539

$$1. \int_a^b \frac{dx}{x[J_{\nu}(x)]^2} = \frac{\pi}{2} \left[\frac{N_{\nu}(b)}{J_{\nu}(b)} - \frac{N_{\nu}(a)}{J_{\nu}(a)} \right] \quad [J_{\nu}(x) \neq 0 \text{ for } x \in [a, b]].$$

ET II 338(41)

$$2. \int_a^b \frac{dx}{x[N_{\nu}(x)]^2} = \frac{\pi}{2} \left[\frac{J_{\nu}(a)}{N_{\nu}(a)} - \frac{J_{\nu}(b)}{N_{\nu}(b)} \right] \quad [N_{\nu}(x) \neq 0 \text{ for } x \in [a, b]].$$

$$3. \int_a^b \frac{dx}{x J_\nu(x) N_\nu(x)} = \frac{\pi}{2} \ln \left[\frac{J_\nu(a) N_\nu(b)}{J_\nu(b) N_\nu(a)} \right].$$

ET II 339(50)

6.541

$$1. \int_0^\infty x J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} = I_\nu(bc) K_\nu(ac) \quad [0 < b < a, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1];$$

$$= I_\nu(ac) K_\nu(bc) \quad [0 < a < b, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 49(10)

$$2.8 \int_0^\infty x^{1-2n} J_\nu(ax) J_\nu(bx) \frac{dx}{x^2 + c^2} =$$

$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc) K_\nu(ac) - \frac{1}{2} \left(\frac{b}{a}\right)^\nu \frac{\pi}{\sin(\pi\nu)} \sum_{p=0}^{n-1} \frac{(a^2 c^2/4)^p}{p! \Gamma(1-\nu+p)} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2/4)^k}{k! \Gamma(1+\nu+k)} \right],$$

$$[0 < b < a]$$

$$= \left(-\frac{1}{c^2}\right)^n \left[I_\nu(bc) K_\nu(ac) - \frac{1}{2\nu} \left(\frac{b}{a}\right)^\nu \sum_{p=0}^{n-1} \frac{(a^2 c^2/4)^p}{p! (1-\nu)_p} \sum_{k=0}^{n-1-p} \frac{(b^2 c^2/4)^k}{k! (1+\nu)_k} \right],$$

$$[n = 1, 2, \dots, \quad \operatorname{Re} \nu > n-1, \quad \operatorname{Re} c > 0, \quad 0 < b < a].$$

704

$$3.7 \int_0^\infty \frac{x^{\alpha-1}}{(x^2 + z^2)^\rho} J_\mu(cx) J_\nu(cx) dx = \frac{1}{2} \left(\frac{c}{2}\right)^{2\rho-\alpha} \times$$

$$\times \Gamma \left[\begin{matrix} (\mu + \nu + \alpha)/2 - \rho, 1 + 2\rho - \alpha \\ (\mu - \nu - \alpha)/2 + \rho + 1, (\mu + \nu - \alpha)/2 + \rho + 1, (\nu - \mu - \alpha)/2 + \rho + 1 \end{matrix} \right] \times$$

$$\times {}_3F_4 \left(\begin{matrix} \frac{1-\alpha}{2} + \rho, 1 - \frac{\alpha}{2} + \rho, \rho; \rho + 1 - \frac{\mu + \nu + \alpha}{2}, \rho + 1 + \frac{\mu - \nu - \alpha}{2}, \\ \rho + 1 + \frac{\mu + \nu - \alpha}{2}, \rho + 1 + \frac{\nu - \mu - \alpha}{2}; c^2 z^2 \end{matrix} \right) + \frac{z^{\alpha-2\rho}}{2} \left(\frac{cz}{2}\right)^{\mu+\nu} \times$$

$$\times \Gamma \left[\begin{matrix} \rho - (\alpha + \mu + \nu)/2, (\alpha + \mu + \nu)/2 \\ \rho, \mu + 1, \nu + 1 \end{matrix} \right] {}_3F_4 \left(\begin{matrix} 1 + \frac{\mu + \nu}{2}, 1 + \frac{\mu + \nu}{2}, \\ \frac{\alpha + \mu + \nu}{2}; 1 - \rho + \frac{\alpha + \mu + \nu}{2}, \mu + 1, \nu + 1, \mu + \nu + 1; c^2 z^2 \end{matrix} \right)$$

$$\left[\Gamma \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} = \frac{\Gamma(a_1) \dots \Gamma(a_p)}{\Gamma(b_1) \dots \Gamma(b_q)} \right] c, \quad \operatorname{Re} z, \quad \operatorname{Re}(\alpha + \mu + \nu) > 0; \quad \operatorname{Re}(\alpha - 2\rho) < 1 \right].$$

6.542

$$\int_0^\infty \frac{J_\nu(ax) N_\nu(bx) - J_\nu(bx) N_\nu(ax)}{x \{ [J_\nu(bx)]^2 + [N_\nu(bx)]^2 \}} dx = -\frac{\pi}{2} \left(\frac{b}{a}\right)^\nu \quad [0 < b < a]$$

$$\int_0^\infty J_\mu(bx) \left\{ \cos \left[\frac{1}{2}(\nu - \mu)\pi \right] J_\nu(ax) - \sin \left[\frac{1}{2}(\nu - \mu)\pi \right] N_\nu(ax) \right\} \frac{x dx}{x^2 + r^2} = I_\mu(br)K_\nu(ar)$$

[Re $\nu > 0$, $a \geq b > 0$, Re $\mu > |\text{Re } \nu| - 2$].

6.544

$$1. \int_0^\infty J_\nu\left(\frac{a}{x}\right) N_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = -\frac{1}{a} \left[\frac{2}{\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - N_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right] \quad \left[a > 0, b > 0, |\text{Re } \nu| < \frac{1}{2} \right].$$

ET II 357(47)

$$2. \int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{1}{a} J_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \quad \left[a > 0, b > 0, \text{Re } \nu > -\frac{1}{2} \right].$$

ET II 57(10)

$$3. \int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{1}{a} e^{\frac{1}{2}i\nu\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi}\right) + \frac{1}{a} e^{-\frac{1}{2}i\nu\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi}\right)$$

[Re $b > 0$, $a > 0$, $|\text{Re } \nu| < \frac{1}{2}$].

ET II 142(32)

$$4. \int_0^\infty N_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{2}{a\pi} \left[K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) + \frac{\pi}{2} N_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right] \quad \left[a > 0, b > 0, |\text{Re } \nu| < \frac{1}{2} \right].$$

ET II 62(38)

$$5. \int_0^\infty N_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{4}{a} \left[e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}} e^{\frac{1}{4}i\pi}\right) + e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}} e^{-\frac{1}{4}i\pi}\right) \right]$$

[Re $b > 0$, $a > 0$, $|\text{Re } \nu| < \frac{1}{2}$].

ET II 143(38)

705

$$6. \int_0^\infty K_\nu\left(\frac{a}{x}\right) J_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{i}{a} \left[e^{\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) - e^{-\frac{1}{2}\nu\pi i} K_{2\nu}\left(e^{-\frac{1}{4}\pi i} \frac{2\sqrt{a}}{\sqrt{b}}\right) \right]$$

[Re $a > 0$, $b > 0$, $|\text{Re } \nu| < \frac{5}{2}$].

$$7. \int_0^\infty K_\nu\left(\frac{a}{x}\right) N_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{2}{a} \left[\sin\left(\frac{3}{2}\pi\nu\right) \operatorname{kei}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) - \cos\left(\frac{3}{2}\pi\nu\right) \operatorname{ker}_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \right] \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{5}{2} \right].$$

ET II 113(29)

$$8. \int_0^\infty K_\nu\left(\frac{a}{x}\right) K_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = \frac{\pi}{a} K_{2\nu}\left(\frac{2\sqrt{a}}{\sqrt{b}}\right) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 146(55)

6.55 Combinations of Bessel functions and algebraic functions

6.551

$$1. \int_0^1 x^{1/2} J_\nu(xy) dx = \sqrt{2} y^{-3/2} \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2}\nu\right)} + \\ + y^{-1/2} \left[\left(\nu - \frac{1}{2}\right) J_\nu(y) S_{-1/2, \nu-1}(y) - J_{\nu-1}(y) S_{1/2, \nu}(y) \right] \\ \left[y > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 21(1)

$$2. \int_1^\infty x^{1/2} J_\nu(xy) dx = y^{-1/2} \left[J_{\nu-1}(y) S_{1/2, \nu}(y) + \left(\frac{1}{2} - \nu\right) J_\nu(y) S_{-1/2, \nu-1}(y) \right] \quad [y > 0].$$

ET II 22(2)

6.552

$$1. \int_0^\infty J_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \quad [\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 23(11)

WA 477(3)

MO 44

$$2. \int_0^\infty N_\nu(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = -\frac{1}{\pi} \sec\left(\frac{1}{2}\nu\pi\right) K_{\nu/2}\left(\frac{1}{2}ay\right) \left[K_{\nu/2}\left(\frac{1}{2}ay\right) + \pi \sin\left(\frac{1}{2}\nu\pi\right) I_{\nu/2}\left(\frac{1}{2}ay\right) \right] \\ [y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| < 1].$$

$$3. \int_0^{\infty} K_{\nu}(xy) \frac{dx}{(x^2 + a^2)^{1/2}} = \frac{\pi^2}{8} \sec\left(\frac{1}{2}\nu\pi\right) \left\{ \left[J_{\nu/2}\left(\frac{1}{2}ay\right) \right]^2 + \left[N_{\nu/2}\left(\frac{1}{2}ay\right) \right]^2 \right\}$$

[Re $a > 0$, Re $y > 0$, |Re ν | < 1].

ET II 128(6)

$$4. \int_0^1 J_{\nu}(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} \left[J_{\nu/2}\left(\frac{1}{2}y\right) \right]^2 \quad [y > 0, \text{ Re } \nu > -1].$$

ET II 24(22)a

706

$$5. \int_0^1 N_0(xy) \frac{dx}{(1-x^2)^{1/2}} = \frac{\pi}{2} J_0\left(\frac{1}{2}y\right) N_0\left(\frac{1}{2}y\right) \quad [y > 0].$$

ET II 102(26)a

$$6. \int_1^{\infty} J_{\nu}(xy) \frac{dx}{(x^2-1)^{1/2}} = -\frac{\pi}{2} J_{\nu/2}\left(\frac{1}{2}y\right) N_{\nu/2}\left(\frac{1}{2}y\right) \quad [y > 0].$$

ET II 24(23)a

$$7. \int_1^{\infty} N_{\nu}(xy) \frac{dx}{(x^2-1)^{1/2}} = \frac{\pi}{4} \left\{ \left[J_{\nu/2}\left(\frac{1}{2}y\right) \right]^2 - \left[N_{\nu/2}\left(\frac{1}{2}y\right) \right]^2 \right\} \quad [y > 0].$$

ET II 102(27)

6.553

$$\int_0^{\infty} x^{-1/2} I_{\nu}(x) K_{\nu}(x) K_{\mu}(2x) dx = \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{4} - \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{4} + \nu + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{4} + \nu - \frac{1}{2}\mu\right)}{4\Gamma\left(\frac{3}{4} + \nu + \frac{1}{2}\mu\right) \Gamma\left(\frac{3}{4} + \nu - \frac{1}{2}\mu\right)}$$

[|Re μ | $< \frac{1}{2}$, 2 Re $\nu > |\text{Re } \mu| - \frac{1}{2}$].

ET II 372(2)

6.554

$$1. \int_0^{\infty} x J_0(xy) \frac{dx}{(a^2 + x^2)^{1/2}} = y^{-1} e^{-ay} \quad [y > 0, \text{ Re } a > 0].$$

$$2. \int_0^1 x J_0(xy) \frac{dx}{(1-x^2)^{1/2}} = y^{-1} \sin y \quad [y > 0].$$

ET II 7(5)a

$$3. \int_1^\infty x J_0(xy) \frac{dx}{(x^2-1)^{1/2}} = y^{-1} \cos y \quad [y > 0].$$

ET II 7(6)a

$$4. \int_0^\infty x J_0(xy) \frac{dx}{(x^2+a^2)^{3/2}} = a^{-1} e^{-ay} \quad [y > 0, \operatorname{Re} a > 0].$$

ET II 7(7)a

$$5. \int_0^\infty \frac{x J_0(ax)}{\sqrt{x^4+4k^4}} dx = K_0(ak) J_0(ak) \quad [a > 0, k > 0].$$

WA 473(1)

6.555

$$\int_0^\infty x^{1/2} J_{2\nu-1}(ax^{1/2}) N_\nu(xy) dx = -\frac{a}{2y^2} \mathbf{H}_{\nu-1} \left(\frac{a^2}{4y} \right) \quad \left[a > 0, y > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 111(17)

6.556

$$\int_0^\infty J_\nu [a(x^2+1)^{1/2}] \frac{dx}{\sqrt{x^2+1}} = -\frac{\pi}{2} J_{\nu/2} \left(\frac{a}{2} \right) N_{\nu/2} \left(\frac{a}{2} \right) \quad [\operatorname{Re} \nu > -1, a > 0].$$

MO 46

6.56- 6.58 Combinations of Bessel functions and powers

6.561

$$1. \int_0^1 x^\nu J_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma \left(\nu + \frac{1}{2} \right) [J_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) J_{\nu-1}(a)] \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 333(2)a

$$2. \int_0^1 x^\nu N_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma \left(\nu + \frac{1}{2} \right) [N_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) N_{\nu-1}(a)] \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$3. \int_0^1 x^\nu I_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) \\ [I_\nu(a)\mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a)I_{\nu-1}(a)] \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 364(2)a

$$4. \int_0^1 x^\nu K_\nu(ax) dx = 2^{\nu-1} a^{-\nu} \pi^{\frac{1}{2}} \Gamma\left(\nu + \frac{1}{2}\right) \\ [K_\nu(a)\mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a)K_{\nu-1}(a)] \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 367(21)a

$$5. \int_0^1 x^{\nu+1} J_\nu(ax) dx = a^{-1} J_{\nu+1}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 333(3)a

$$6. \int_0^1 x^{\nu+1} N_\nu(ax) dx = a^{-1} N_{\nu+1}(a) + 2^{\nu+1} a^{-\nu-2} \pi^{-1} \Gamma(\nu+1) \quad [\operatorname{Re} \nu > -1].$$

ET II 339(44)a

$$7. \int_0^1 x^{\nu+1} I_\nu(ax) dx = a^{-1} I_{\nu+1}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 365(3)a

$$8. \int_0^1 x^{\nu+1} K_\nu(ax) dx = 2^\nu a^{-\nu-2} \Gamma(\nu+1) - a^{-1} K_{\nu+1}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 367(22)a

$$9. \int_0^1 x^{1-\nu} J_\nu(ax) dx = \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)} - a^{-1} J_{\nu-1}(a).$$

ET II 333(4)a

ET II 339(45)a

$$11. \int_0^1 x^{1-\nu} I_\nu(ax) dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1} \Gamma(\nu)}.$$

ET II 365(4)a

$$12. \int_0^1 x^{1-\nu} K_\nu(ax) dx = 2^{-\nu} a^{\nu-2} \Gamma(1-\nu) - a^{-1} K_{\nu-1}(a) \quad [\operatorname{Re} \nu < 1].$$

ET II 367(23)a

$$13.^7 \int_0^1 x^\mu J_\nu(ax) dx = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{a^{\mu+1} \Gamma\left(\frac{\nu-\mu+1}{2}\right)} + a^{-\mu} \{(\mu+\nu-1) J_\nu(a) S_{\mu-1, \nu-1}(a) - J_{\nu-1}(a) S_{\mu, \nu}(a)\}$$

$$[a > 0, \quad \operatorname{Re}(\mu+\nu) > -1].$$

ET II 22(8)a

$$14. \int_0^\infty x^\mu J_\nu(ax) dx = 2^\mu a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu\right)} \quad \left[-\operatorname{Re} \nu - 1 < \operatorname{Re} \mu < \frac{1}{2}, \quad a > 0\right].$$

EH II 49(19)

$$15. \int_0^\infty x^\mu N_\nu(ax) dx = 2^\mu \operatorname{ctg} \left[\frac{1}{2}(\nu+1-\mu)\pi \right] a^{-\mu-1} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \frac{1}{2}\mu\right)}$$

$$\left[|\operatorname{Re} \nu| - 1 < \mu < \frac{1}{2}, \quad a > 0 \right].$$

ET II 97(3)a

708

$$16. \int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)$$

$$[\operatorname{Re}(\mu+1 \pm \nu) > 0, \quad \operatorname{Re} a > 0].$$

$$17. \int_0^\infty \frac{J_\nu(ax)}{x^{\nu-q}} dx = \frac{\Gamma\left(\frac{1}{2}q + \frac{1}{2}\right)}{2^{\nu-q} a^{q-\nu+1} \Gamma\left(\nu - \frac{1}{2}q + \frac{1}{2}\right)} \left[-1 < \operatorname{Re} q < \operatorname{Re} \nu - \frac{1}{2}\right].$$

WA 428(1), KU 144(5)

$$18. \int_0^\infty \frac{N_\nu(x)}{x^{\nu-\mu}} dx = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \nu\right) \sin\left(\frac{1}{2}\mu - \nu\right) \pi}{2^{\nu-\mu} \pi} \left[|\operatorname{Re} \nu| < \operatorname{Re}(1 + \mu - \nu) < \frac{3}{2}\right].$$

WA 430(5)

6.562

$$1. \int_0^\infty x^\mu N_\nu(bx) \frac{dx}{x+a} = (2a)^\mu \pi^{-1} \left\{ \sin\left[\frac{1}{2}\pi(\mu - \nu)\right] \Gamma\left[\frac{1}{2}(\mu + \nu + 1)\right] \Gamma\left[\frac{1}{2}(1 + \mu - \nu)\right] S_{-\mu, \nu}(ab) - 2 \cos\left[\frac{1}{2}\pi(\mu - \nu)\right] \Gamma\left(1 + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\mu - \frac{1}{2}\nu\right) S_{-\mu-1, \nu}(ab) \right\} \left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\mu \pm \nu) > -1, \quad \operatorname{Re} \mu < \frac{3}{2}\right].$$

ET II 98(8)

$$2. \int_0^\infty \frac{x^\nu J_\nu(ax)}{x+k} dx = \frac{\pi k^\nu}{2 \cos \nu \pi} [H_{-\nu}(ak) - N_{-\nu}(ak)] \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{3}{2}, \quad a > 0, \quad |\arg k| < \pi\right].$$

WA 479(7)

$$3. \int_0^\infty x^\mu K_\nu(bx) \frac{dx}{x+a} = 2^{\mu-2} \Gamma\left[\frac{1}{2}(\mu + \nu)\right] \Gamma\left[\frac{1}{2}(\mu - \nu)\right] b^{-\mu} {}_1F_2\left(1; 1 - \frac{\mu + \nu}{2}, 1 - \frac{\mu - \nu}{2}; \frac{a^2 b^2}{4}\right) - 2^{\mu-3} \Gamma\left[\frac{1}{2}(\mu - \nu - 1)\right] \Gamma\left[\frac{1}{2}(\mu + \nu - 1)\right] ab^{1-\mu} {}_1F_2\left(1; \frac{3 - \mu - \nu}{2}, \frac{3 - \mu + \nu}{2}; \frac{a^2 b^2}{4}\right) - \pi a^\mu \operatorname{cosec}[\pi(\mu - \nu)] \{K_\nu(ab) + \pi \cos(\mu\pi) \operatorname{cosec}[\pi(\nu + \mu)] I_\nu(ab)\} \left[\operatorname{Re} b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| - 1\right].$$

ET II 127(4)

709
6.563

$$\int_0^\infty x^{\varrho-1} J_\nu(bx) \frac{dx}{(x+a)^{1+\mu}} = \frac{\pi a^{\varrho-\mu-1}}{\sin[(\varrho+\nu-\mu)\pi]\Gamma(\mu+1)} \times$$

$$\times \left\{ \sum_{m=0}^\infty \frac{(-1)^m \left(\frac{1}{2}ab\right)^{\nu+2m} \Gamma(\varrho+\nu+2m)}{m!\Gamma(\nu+m+1)\Gamma(\varrho+\nu-\mu+2m)} - \right.$$

$$\left. - \sum_{m=0}^\infty \frac{\left(\frac{1}{2}ab\right)^{\mu+1-\varrho+m} \Gamma(\mu+m+1) \sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{m!\Gamma\left[\frac{1}{2}(\mu+\nu-\varrho+m+3)\right]} \frac{\sin\left[\frac{1}{2}(\varrho+\nu-\mu-m)\pi\right]}{\Gamma\left[\frac{1}{2}(\mu-\nu-\varrho+m+3)\right]} \right\}$$

$$\left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\varrho+\nu) > 0, \quad \operatorname{Re}(\varrho-\mu) < \frac{5}{2} \right].$$

ET II 23(10), WA 479

6.564

$$1. \int_0^\infty x^{\nu+1} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{2}{\pi b}} a^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(ab) \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 23(15)

$$2. \int_0^\infty x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{x^2+a^2}} = \sqrt{\frac{\pi}{2b}} a^{\frac{1}{2}-\nu} [I_{\nu-\frac{1}{2}}(ab) - L_{\nu-\frac{1}{2}}(ab)]$$

$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 23(16)

6.565

$$1. \int_0^\infty x^{-\nu} (x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^\nu a^{-2\nu} b^\nu \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu\left(\frac{ab}{2}\right) K_\nu\left(\frac{ab}{2}\right)$$

$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 477(4), ET II 23(17)

$$2. \int_0^\infty x^{\nu+1} (x^2+a^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = \frac{\sqrt{\pi} b^{\nu-1}}{2^\nu e^{ab} \Gamma\left(\nu+\frac{1}{2}\right)} \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$3. \int_0^\infty x^{\nu+1} (x^2+a^2)^{-\nu-\frac{3}{2}} J_\nu(bx) dx = \frac{b^\nu \sqrt{\pi}}{2^{\nu+1} a e^{ab} \Gamma\left(\nu + \frac{3}{2}\right)} \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 24(19)

710

$$4. \int_0^\infty \frac{J_\nu(bx) x^{\nu+1}}{(x^2+a^2)^{\mu+1}} dx = \frac{a^{\nu-\mu} b^\mu}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(ab) \quad \left[-1 < \operatorname{Re} \nu < \operatorname{Re}\left(2\mu + \frac{3}{2}\right), \quad a > 0, \quad b > 0\right].$$

MO 43

$$5. \int_0^\infty x^{\nu+1} (x^2+a^2)^\mu N_\nu(bx) dx = 2^{\nu-1} \pi^{-1} a^{2\mu+2} (1+\mu)^{-1} \Gamma(\nu) b^{-\nu} \times \\ \times {}_1F_2\left(1; 1-\nu, 2+\mu; \frac{a^2 b^2}{4}\right) - 2^\mu a^{\mu+\nu+1} [\sin(\nu\pi)]^{-1} \times \\ \times \Gamma(\mu+1) b^{-1-\mu} [I_{\mu+\nu+1}(ab) - 2 \cos(\mu\pi) K_{\mu+\nu+1}(ab)] \\ [b > 0, \quad \operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < -2 \operatorname{Re} \mu].$$

ET II 100(19)

$$6. \int_0^\infty x^{1-\nu} (x^2+a^2)^\mu N_\nu(bx) dx = 2^\mu a^{\mu-\nu+1} b^{-1-\mu} \left\{ \frac{\cos(\nu\pi)}{\pi} \Gamma(\mu+1) \times \right. \\ \left. \times \Gamma(\nu) I_{\nu-\mu-1}(ab) - 2 \operatorname{cosec}(\nu\pi) [\Gamma(-\mu)]^{-1} K_{\nu-\mu-1}(ab) \right\} - \\ - \frac{a^{2\mu+2} \operatorname{ctg}(\nu\pi) b^\nu}{2^{\nu+1} (\mu+1) \Gamma(\nu+1)} {}_1F_2\left(1; \nu+1, \mu+2; \frac{a^2 b^2}{4}\right) \\ \left[b > 0, \quad \operatorname{Re} a > 0, \quad \frac{1}{2} + 2 \operatorname{Re} \mu < \operatorname{Re} \nu < 1 \right].$$

ET II 100(20)

$$7. \int_0^\infty x^{1+\nu} (x^2+a^2)^\mu K_\nu(bx) dx = 2^\nu \Gamma(\nu+1) a^{\nu+\mu+1} b^{-1-\mu} S_{\mu-\nu, \mu+\nu+1}(ab) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 128(8)

$$8. \int_0^\infty \frac{x^{\varrho-1} J_\nu(ax)}{(x^2+k^2)^{\mu+1}} dx = \frac{a^\nu k^{\varrho+\nu-2\mu-2} \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu\right) \Gamma\left(\mu+1 - \frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{2^{\nu+1} \Gamma(\mu+1) \Gamma(\nu+1)} \times \\ \times {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2} - \mu, \nu+1; \frac{a^2 k^2}{4}\right) + \\ + \frac{a^{2\mu+2-\varrho} \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\varrho - \mu - 1\right)}{\Gamma\left(\mu + \frac{1}{2} + \frac{1}{2}\varrho\right) \Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\varrho\right)} \times \\ \times {}_1F_2\left(\frac{\nu-\varrho}{2}; \frac{\nu-\varrho}{2} - \mu, \nu+\varrho; \frac{a^2 k^2}{4}\right)$$

$$\begin{aligned}
1. \int_0^\infty x^\mu N_\nu(bx) \frac{dx}{x^2 + a^2} &= 2^{\mu-2} \pi^{-1} b^{1-\mu} \times \\
&\times \cos \left[\frac{\pi}{2} (\mu - \nu + 1) \right] \Gamma \left(\frac{1}{2} \mu + \frac{1}{2} \nu - \frac{1}{2} \right) \Gamma \left(\frac{1}{2} \mu - \frac{1}{2} \nu - \frac{1}{2} \right) \times \\
&\times {}_1F_2 \left(1; 2 - \frac{\mu + 1 + \nu}{2}, 2 - \frac{\mu + 1 - \nu}{2}; \frac{a^2 b^2}{4} \right) - \\
&- \frac{1}{2} \pi a^{\mu-1} \operatorname{cosec} \left[\frac{\pi}{2} (\mu + \nu + 1) \right] \operatorname{ctg} \left[\frac{\pi}{2} (\mu - \nu + 1) \right] I_\nu(ab) - \\
&- a^{\mu-1} \operatorname{cosec} \left[\frac{\pi}{2} (\mu - \nu + 1) \right] K_\nu(ab) \\
&\left[b > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2} \right].
\end{aligned}$$

ET II 100(17)

$$2. \int_0^\infty x^{\nu+1} J_\nu(ax) \frac{dx}{x^2 + b^2} = b^\nu K_\nu(ab) \quad \left[a > 0, \quad \operatorname{Re} b > 0, \quad -1 < \operatorname{Re} \nu < \frac{3}{2} \right].$$

EH II 96(58)

$$3. \int_0^\infty x^\nu K_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi^2 b^{\nu-1}}{4 \cos \nu \pi} [H_{-\nu}(ab) - N_{-\nu}(ab)] \quad \left[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 468(9)

$$4. \int_0^\infty x^{-\nu} K_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi^2}{4b^{\nu+1} \cos \nu \pi} [H_\nu(ab) - N_\nu(ab)] \quad \left[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].$$

WA 468(10)

$$5. \int_0^\infty x^{-\nu} J_\nu(ax) \frac{dx}{x^2 + b^2} = \frac{\pi}{2b^{\nu+1}} [I_\nu(ab) - L_\nu(ab)] \quad \left[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{5}{2} \right].$$

WA 468(11)

6.567

$$\begin{aligned}
1. \int_0^1 x^{\nu+1} (1-x^2)^\mu J_\nu(bx) dx &= 2^\mu \Gamma(\mu+1) b^{-(\mu+1)} J_{\nu+\mu+1}(b) \\
&[b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1].
\end{aligned}$$

ET II 26(33)a

$$\begin{aligned}
2. \int_0^1 x^{\nu+1} (1-x^2)^\mu N_\nu(bx) dx &= \\
&= b^{-(\mu+1)} \Gamma(\mu+1) \Gamma(\nu+\mu+1) \Gamma(\nu-\mu) \Gamma(\nu+\mu+1) \Gamma(\nu-\mu) \Gamma(\nu+\mu+1) \\
&[b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1].
\end{aligned}$$

$$3. \int_0^1 x^{1-\nu}(1-x^2)^\mu J_\nu(bx) dx = \frac{2^{1-\nu} s_{\nu+\mu, \mu-\nu+1}(b)}{b^{\mu+1} \Gamma(\nu)} \quad [b > 0, \quad \operatorname{Re} \mu > -1].$$

ET II 25(31)a

$$4. \int_0^1 x^{1-\nu}(1-x^2)^\mu N_\nu(bx) dx = b^{-(\mu+1)} [2^{1-\nu} \pi^{-1} \cos(\nu\pi) \Gamma(1-\nu) \times \\ \times s_{\mu+\nu, \mu-\nu+1}(b) - 2^\mu \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) J_{\mu-\nu+1}(b)] \\ [b > 0, \quad \operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1]. \quad cr$$

ET II 104(37)a

$$5. \int_0^1 x^{1-\nu}(1-x^2)^\mu K_\nu(bx) dx = 2^{-\nu-2} b^\nu (\mu+1)^{-1} \Gamma(-\nu) {}_1F_2 \left(1; \nu+1, \mu+2; \frac{b^2}{4} \right) + \\ + \pi 2^{\mu-1} b^{-(\mu+1)} \operatorname{cosec}(\nu\pi) \Gamma(\mu+1) I_{\mu-\nu+1}(b) \\ [\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1].$$

ET II 129(12)a

$$6. \int_0^1 x^{1-\nu} J_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} H_{\nu-\frac{1}{2}}(b) \quad [b > 0].$$

ET II 24(24)a

$$7. \int_0^1 x^{1+\nu} N_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \operatorname{cosec}(\nu\pi) [\cos(\nu\pi) J_{\nu+\frac{1}{2}}(b) - H_{-\nu-\frac{1}{2}}(b)] \\ [b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 102(28)a

$$8. \int_0^1 x^{1-\nu} N_\nu(bx) \frac{dx}{\sqrt{1-x^2}} = \sqrt{\frac{\pi}{2b}} \left\{ \operatorname{ctg}(\nu\pi) [H_{\nu-\frac{1}{2}}(b) - N_{\nu-\frac{1}{2}}(b)] - J_{\nu-\frac{1}{2}}(b) \right\} \\ [b > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 102(30)a

$$9. \int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma \left(\nu + \frac{1}{2} \right) \left[J_\nu \left(\frac{b}{2} \right) \right]^2 \quad \left[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$10. \int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} N_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) J_\nu\left(\frac{b}{2}\right) N_\nu\left(\frac{b}{2}\right) \quad \left[b > 0, \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 102(31)a

$$11. \int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} K_\nu(bx) dx = 2^{\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) I_\nu\left(\frac{b}{2}\right) K_\nu\left(\frac{b}{2}\right) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 129(10)a

$$12. \int_0^1 x^\nu (1-x^2)^{\nu-\frac{1}{2}} I_\nu(bx) dx = 2^{-\nu-1} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[I_\nu\left(\frac{b}{2}\right)\right]^2$$

ET II 365(5)a

$$13. \int_0^1 x^{\nu+1} (1-x^2)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = 2^{-\nu} \frac{b^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} - \nu\right) \sin b \quad \left[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right].$$

ET II 25(27)a

713

$$14. \int_1^\infty x^\nu (x^2-1)^{\nu-\frac{1}{2}} N_\nu(bx) dx = 2^{\nu-2} \sqrt{\pi} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[J_\nu\left(\frac{b}{2}\right) J_{-\nu}\left(\frac{b}{2}\right) - N_\nu\left(\frac{b}{2}\right) N_{-\nu}\left(\frac{b}{2}\right)\right] \quad \left[|\operatorname{Re} \nu| < \frac{1}{2}, \quad b > 0\right].$$

ET II 103(32)a

$$15. \int_1^\infty x^\nu (x^2-1)^{\nu-\frac{1}{2}} K_\nu(bx) dx = \frac{2^{\nu-1}}{\sqrt{\pi}} b^{-\nu} \Gamma\left(\nu + \frac{1}{2}\right) \left[K_\nu\left(\frac{b}{2}\right)\right]^2 \quad \left[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 129(11)a

$$16. \int_1^\infty x^{-\nu} (x^2-1)^{-\nu-\frac{1}{2}} J_\nu(bx) dx = -2^{-\nu-1} \sqrt{\pi} b^\nu \Gamma\left(\frac{1}{2} - \nu\right) J_\nu\left(\frac{b}{2}\right) N_\nu\left(\frac{b}{2}\right) \quad \left[b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right].$$

ET II 25(26)a

6.568

$$1. \int_0^{\infty} x^{\nu} N_{\nu}(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\nu-1} J_{\nu}(ab) \quad \left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2} \right].$$

ET II 101(22)

$$2. \int_0^{\infty} x^{\mu} N_{\nu}(bx) \frac{dx}{x^2 - a^2} = \frac{\pi}{2} a^{\mu-1} J_{\nu}(ab) + 2^{\mu} \pi^{-1} a^{\mu-1} \cos \left[\frac{\pi}{2} (\mu - \nu + 1) \right] \times \\ \times \Gamma \left(\frac{\mu - \nu + 1}{2} \right) \Gamma \left(\frac{\mu + \nu + 1}{2} \right) S_{-\mu, \nu}(ab) \\ \left[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| - 1 < \operatorname{Re} \mu < \frac{5}{2} \right].$$

ET II (101)(25)

6.569

$$\int_0^1 x^{\lambda} (1-x)^{\mu-1} J_{\nu}(ax) dx = \\ = \frac{\Gamma(\mu) \Gamma(1 + \lambda + \nu) 2^{-\nu} a^{\nu}}{\Gamma(\nu + 1) \Gamma(1 + \lambda + \mu + \nu)} \times \\ \times {}_2F_3 \left(\frac{\lambda + 1 + \nu}{2}, \frac{\lambda + 2 + \nu}{2}; \nu + 1, \frac{\lambda + 1 + \mu + \nu}{2}, \frac{\lambda + 2 + \mu + \nu}{2}; -\frac{a^2}{4} \right) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + \nu) > -1].$$

ET II 193(56)a

6.571

$$1. \int_0^{\infty} \left[(x^2 + a^2)^{\frac{1}{2}} \pm x \right]^{\mu} J_{\nu}(bx) \frac{dx}{\sqrt{x^2 + a^2}} = a^{\mu} I_{\frac{1}{2}(\nu \mp \mu)} \left(\frac{ab}{2} \right) K_{\frac{1}{2}(\nu \pm \mu)} \left(\frac{ab}{2} \right) \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{3}{2} \right].$$

ET II 26(38)

714

$$2. \int_0^{\infty} [(x^2 + a^2)^{\frac{1}{2}} - x]^{\mu} N_{\nu}(bx) \frac{dx}{\sqrt{x^2 + a^2}} = \\ = a^{\mu} \left[\operatorname{ctg}(\nu\pi) I_{\frac{1}{2}(\mu+\nu)} \left(\frac{ab}{2} \right) K_{\frac{1}{2}(\mu-\nu)} \left(\frac{ab}{2} \right) - \operatorname{cosec}(\nu\pi) I_{\frac{1}{2}(\mu-\nu)} \left(\frac{ab}{2} \right) K_{\frac{1}{2}(\mu+\nu)} \left(\frac{ab}{2} \right) \right] \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{3}{2}, \quad |\operatorname{Re} \nu| < 1 \right].$$

$$\begin{aligned}
3. \int_0^\infty [(x^2 + a^2)^{\frac{1}{2}} + x]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} &= \\
&= \frac{\pi^2}{4} a^\mu \operatorname{cosec}(\nu\pi) \left[J_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right) N_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right) - N_{\frac{1}{2}(\nu-\mu)}\left(\frac{ab}{2}\right) J_{-\frac{1}{2}(\nu+\mu)}\left(\frac{ab}{2}\right) \right] \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].
\end{aligned}$$

ET II 130(15)

6.572

$$\begin{aligned}
1. \int_0^\infty x^{-\mu} [(x^2 + a^2)^{\frac{1}{2}} + a]^\mu J_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} &= \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}{ab\Gamma(\nu+1)} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\nu - \mu) > -1].
\end{aligned}$$

ET II 26(40)

$$\begin{aligned}
2. \int_0^\infty x^{-\mu} [(x^2 + a^2)^{\frac{1}{2}} + a]^\mu K_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} &= \\
&= \frac{\Gamma\left(\frac{1+\nu-\mu}{2}\right) \Gamma\left(\frac{1-\nu-\mu}{2}\right)}{2ab} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(iab) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(-iab) \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET II 130(18), BU 87(6a)

$$\begin{aligned}
3. \int_0^\infty x^{-\mu} [(x^2 + a^2)^{\frac{1}{2}} - a]^\mu N_\nu(bx) \frac{dx}{\sqrt{x^2 + a^2}} &= \\
&= -\frac{1}{ab} W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \left\{ \frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma(\nu+1)} \operatorname{tg}\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) + \right. \\
&\quad \left. + \operatorname{sec}\left(\frac{\nu-\mu}{2}\pi\right) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ab) \right\} \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} + \frac{1}{2} \operatorname{Re} \mu \right].
\end{aligned}$$

ET II 105(42)

715

6.573

$$\begin{aligned}
1. \int_0^\infty x^{\nu-M+1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx &= 0, \quad M = \sum_{i=1}^k \mu_i \\
&\quad \left[a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k - \frac{1}{2} \right].
\end{aligned}$$

$$2. \int_0^\infty x^{\nu-M-1} J_\nu(bx) \prod_{i=1}^k J_{\mu_i}(a_i x) dx = 2^{\nu-M-1} b^{-\nu} \Gamma(\nu) \prod_{i=1}^k \frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}, \quad M = \sum_{i=1}^k \mu_i$$

$$\left[a_i > 0, \quad \sum_{i=1}^k a_i < b < \infty, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k + \frac{3}{2} \right].$$

WA 460(16)A, ET II 54(43)

6.574

$$1. \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\alpha^\nu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \beta^{\nu-\lambda+1} \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma(\nu+1)} \times$$

$$\times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{\alpha^2}{\beta^2}\right)$$

$$[\operatorname{Re}(\nu+\mu-\lambda+1) > 0, \quad \operatorname{Re} \lambda > -1, \quad 0 < \alpha < \beta].$$

WA 439(2)A, MO 49

If we reverse the positions of ν and μ and at the same time reverse the positions of α and β , the function on the right hand side of this equation will change. Thus, the right hand side represents a function of $\frac{\alpha}{\beta}$ that is not analytic at $\frac{\alpha}{\beta} = 1$.

For $\alpha = \beta$, we have the following equation

$$2. \int_0^\infty J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu+\mu+\lambda+1}{2}\right) \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)}$$

$$[\operatorname{Re}(\nu+\mu+1) > \operatorname{Re} \lambda > 0, \quad \alpha > 0].$$

MO 49, WA 441(2)a

$$3. \int_0^\infty J_\nu(\alpha t) J_\mu(\beta t) t^{-\lambda} dt = \frac{\beta^\mu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda \alpha^{\mu-\lambda+1} \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right) \Gamma(\mu+1)} \times$$

$$\times F\left(\frac{\nu+\mu-\lambda+1}{2}, \frac{-\nu+\mu-\lambda+1}{2}; \mu+1; \frac{\beta^2}{\alpha^2}\right)$$

$$[\operatorname{Re}(\nu+\mu-\lambda+1) > 0, \quad \operatorname{Re} \lambda > -1, \quad 0 < \beta < \alpha].$$

MO 50, WA 440(3)a

716

If $\mu - \nu + \lambda + 1$ (or $\nu - \mu + \lambda + 1$) is a negative integer, the right hand side of equation 6.574 1. (or 6.574 3.) vanishes. The

$$\mu - \nu + \lambda + 1 \quad \nu - \mu + \lambda + 1$$

cases in which the hypergeometric function F in 6.574 3. (or 6.574 1.) can be reduced to an elementary function are then especially important.

6.575

$$1. \int_0^\infty J_{\nu+1}(\alpha t) J_\mu(\beta t) t^{\mu-\nu} dt = 0 \quad [\alpha < \beta];$$

$$= \frac{(\alpha^2 - \beta^2)^{\nu-\mu} \beta^\mu}{2^{\nu-\mu} \alpha^{\nu+1} \Gamma(\nu - \mu + 1)} \quad [\alpha \geq \beta] \quad [\operatorname{Re} \mu > \operatorname{Re}(\nu + 1) > 0].$$

MO 51

$$2. \int_0^\infty \frac{J_\nu(x) J_\mu(x)}{x^{\nu+\mu}} dx = \frac{\sqrt{\pi} \Gamma(\nu + \mu)}{2^{\nu+\mu} \Gamma\left(\nu + \mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \frac{1}{2}\right)} \quad [\operatorname{Re}(\nu + \mu) > 0].$$

KU 147(17), WA 434(1)

6.576

$$1. \int_0^\infty x^{\mu-\nu+1} J_\mu(x) K_\nu(x) dx = \frac{1}{2} \Gamma(\mu - \nu + 1) \quad [\operatorname{Re} \mu > -1, \operatorname{Re}(\mu - \nu) > -1].$$

ET II 370(47)

$$2. \int_0^\infty x^{-\lambda} J_\nu(ax) J_\nu(bx) dx = \frac{2^\nu b^\nu \Gamma\left(\nu + \frac{1-\lambda}{2}\right)}{2^\lambda (a+b)^{2\nu-\lambda+1} \Gamma(\nu+1) \Gamma\left(\frac{1+\lambda}{2}\right)} \times$$

$$\times F\left[\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu + 1; \frac{4ab}{(a+b)^2}\right]$$

$$[a > 0, b > 0, 2 \operatorname{Re} \nu + 1 > \operatorname{Re} \lambda > -1].$$

ET II 47(4)

$$3. \int_0^\infty x^{-\lambda} K_\mu(ax) J_\nu(bx) dx = \frac{b^\nu \Gamma\left(\frac{\nu - \lambda + \mu + 1}{2}\right) \Gamma\left(\frac{\nu - \lambda - \mu + 1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(1 + \nu)} \times$$

$$\times F\left(\frac{\nu - \lambda + \mu + 1}{2}, \frac{\nu - \lambda - \mu + 1}{2}; \nu + 1; -\frac{b^2}{a^2}\right)$$

$$[\operatorname{Re}(a \pm ib) > 0, \operatorname{Re}(\nu - \lambda + 1) > |\operatorname{Re} \mu|].$$

EH II 52(31), ET II 63(4), WA 449(1)

$$\begin{aligned}
5. \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx &= \\
&= \frac{b^\nu \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{2^{\lambda+1} \Gamma(\nu+1) a^{-\lambda+\nu+1}} \times \\
&\quad \times F\left(\frac{1}{2} - \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu; \nu+1; \frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(\nu+1-\lambda \pm \mu) > 0, \quad a > b].
\end{aligned}$$

EH II 93(35)

$$\begin{aligned}
6. \int_0^\infty x^{-\lambda} N_\mu(ax) J_\nu(bx) dx &= \frac{2}{\pi} \sin \frac{\pi(\nu-\mu-\lambda)}{2} \int_0^\infty x^{-\lambda} K_\mu(ax) I_\nu(bx) dx \\
&[a > b, \quad \operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\nu-\lambda+1 \pm \mu) > 0]; \quad (\text{see } \mathbf{6.576} \text{ 5.}).
\end{aligned}$$

6.576
EH II 93(37)

$$7. \int_0^\infty x^{\mu+\nu+1} J_\mu(ax) K_\nu(bx) dx = 2^{\mu+\nu} a^\mu b^\nu \frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}} \quad [\operatorname{Re} \mu > |\operatorname{Re} \nu| - 1, \quad \operatorname{Re} b > |\operatorname{Im} a|].$$

ET 137(16), EH II 93(36), B 449(2)

6.577

$$\begin{aligned}
1.^8 \int_0^\infty x^{\nu-\mu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} &= (-1)^n c^{\nu-\mu+2n} I_\mu(ac) K_\nu(bc) \\
[a > 0, \quad b > a, \quad \operatorname{Re} c > 0, \quad 2+\operatorname{Re} \mu-2n > \operatorname{Re} \nu > -1-n, \quad n \geq 0 \text{ an integer}].
\end{aligned}$$

ET II 49(13)

$$\begin{aligned}
2.^8 \int_0^\infty x^{\mu-\nu+1+2n} J_\mu(ax) J_\nu(bx) \frac{dx}{x^2+c^2} &= (-1)^n c^{\mu-\nu+2n} I_\nu(bc) K_\mu(ac) \\
[b > 0, \quad a > b, \quad \operatorname{Re} \nu-2n+2 > \operatorname{Re} \mu > -n-1, \quad n \geq 0 \text{ an integer}].
\end{aligned}$$

ET II 49(15)

6.578

$$1. \int_0^{\infty} x^{\varrho-1} J_{\lambda}(ax) J_{\mu}(bx) J_{\nu}(cx) dx = \frac{2^{\varrho-1} a^{\lambda} b^{\mu} c^{-\lambda-\mu-\varrho} \Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)}{\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma\left(1-\frac{\lambda+\mu-\nu+\varrho}{2}\right)} \times \\ \times F_4\left(\frac{\lambda+\mu-\nu+\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \lambda+1, \mu+1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right) \\ \left[\operatorname{Re}(\lambda+\mu+\nu+\varrho) > 0, \quad \operatorname{Re} \varrho < \frac{5}{2}, \quad a > 0, \quad b > 0, \quad c > 0, \quad c > a+b \right].$$

ET II 351(9)

$$2. \int_0^{\infty} x^{\varrho-1} J_{\lambda}(ax) J_{\mu}(bx) K_{\nu}(cx) dx = \\ = \frac{2^{\varrho-2} a^{\lambda} b^{\mu} c^{-\varrho-\lambda-\mu}}{\Gamma(\lambda+1)\Gamma(\mu+1)} \Gamma\left(\frac{\varrho+\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{\varrho+\lambda+\mu+\nu}{2}\right) \times \\ \times F_4\left(\frac{\varrho+\lambda+\mu-\nu}{2}, \frac{\varrho+\lambda+\mu+\nu}{2}; \lambda+1, \mu+1; -\frac{a^2}{c^2}, -\frac{b^2}{c^2}\right) \\ [\operatorname{Re}(\varrho+\lambda+\mu) > |\operatorname{Re} \nu|, \quad \operatorname{Re} c > |\operatorname{Im} a| + |\operatorname{Im} b|].$$

ET II 373(8)

$$3. \int_0^{\infty} x^{\lambda-\mu-\nu+1} J_{\nu}(ax) J_{\mu}(bx) J_{\lambda}(cx) dx = 0 \\ \left[\operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{1}{2}, \quad c > b > 0, \quad 0 < a < c-b \right].$$

ET II 53(36)

$$4. \int_0^{\infty} x^{\lambda-\mu-\nu-1} J_{\nu}(ax) J_{\mu}(bx) J_{\lambda}(cx) dx = \frac{2^{\lambda-\mu-\nu-1} a^{\nu} b^{\mu} \Gamma(\lambda)}{c^{\lambda} \Gamma(\mu+1) \Gamma(\nu+1)} \\ \left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda-\mu-\nu) < \frac{5}{2}, \quad c > b > 0, \quad 0 < a < c-b \right].$$

ET II 53(37)

$$5. \int_0^{\infty} x^{1+\mu} N_{\mu}(ax) J_{\nu}(bx) J_{\nu}(cx) dx = 0 \quad [0 < b < c, \quad 0 < a < c-b].$$

ET II 352(13)

$$7. \int_0^\infty x^{\mu+1} I_\nu(ax) K_\mu(bx) J_\nu(cx) dx = \frac{1}{\sqrt{2\pi}} a^{-\mu-1} b^\mu c^{-\mu-1} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{4})\pi i} (v^2+1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q^\mu + \frac{1}{2\nu} - \frac{1}{2}(iv),$$

$$2acv = b^2 - a^2 + c^2 \quad [\operatorname{Re} b > |\operatorname{Re} a|, \quad c > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1].$$

ET II 66(22)

$$8. \int_0^\infty x^{1-\mu} J_\mu(ax) J_\nu(bx) J_\nu(cx) dx = \frac{c^{\mu-1} (\operatorname{sh} u)^{\mu-\frac{1}{2}}}{\sqrt{\frac{1}{2}\pi^3 a^\mu b^{1-\mu}}} e^{(\mu-\frac{1}{2})\pi i} \sin[(\mu-\nu)\pi] Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\operatorname{ch} u),$$

$$2bc \operatorname{ch} u = a^2 - b^2 - c^2 \quad \left[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad 0 < c < a - b, \quad b > 0 \right];$$

$$= \frac{b^{\mu-1} c^{\mu-1}}{\sqrt{2\pi} a^\mu} (\sin v)^{\mu-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cos v), \quad 2bc \cos v = b^2 + c^2 - a^2$$

$$\left[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad |a-b| < c < a+b, \quad a > 0, \quad b > 0 \right];$$

$$= 0$$

$$\left[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad 0 < c < b-a \quad \text{or} \quad a+b < c < \infty, \quad a > 0, \quad b > 0 \right].$$

ET II 52(34)

$$9. \int_0^\infty J_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1-\nu} dx = \frac{2^{\nu-1} \Delta^{2\nu-1}}{(abc)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)},$$

where Δ is the area of a triangle whose sides are a , b , and c . In the case in which the segments whose lengths are a , b , and c cannot form a triangle, the value of the integral is zero $[\operatorname{Re} \nu > -\frac{1}{2}]$.

719

$$10. \int_0^\infty x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx) dx =$$

$$= \frac{\sqrt{\pi} c^\nu \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{2^{\frac{3}{2}} (ab)^{\nu+1} (u^2 - 1)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(u), \quad 2abu = a^2 + b^2 + c^2$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad \operatorname{Re}(\nu \pm \mu) > -1, \quad \operatorname{Re} \nu > -1].$$

ET II 67(30)

MO 52, WA 451(3)

$$\begin{aligned}
 12. \quad \int_0^\infty x^{\nu+1} [J_\nu(ax)]^2 N_\nu(bx) dx &= 0 \quad \left[a > 0, \quad 0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right]; \\
 &= \frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \\
 &\quad \left[a > 0, \quad 2a < b < \infty, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
 \end{aligned}$$

ET II 109(3)

$$\begin{aligned}
 13. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) N_\nu(ax) J_\nu(bx) dx &= 0 \quad \left[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2}, \quad 0 < b < 2a \right]; \\
 &= -\frac{2^{3\nu+1} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \\
 &\quad \left[a > 0, \quad 2a < b < \infty, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
 \end{aligned}$$

ET II 55(49)

$$\begin{aligned}
 14. \quad \int_0^\infty x^{\nu+1} J_\mu(xa \sin \psi) J_\nu(xa \sin \varphi) K_\mu(xa \cos \varphi \cos \psi) dx &= \\
 &= \frac{2^\nu \Gamma(\mu + \nu + 1) (\sin \varphi)^\nu \left(\cos \frac{\alpha}{2}\right)^{2\nu+1}}{a^{\nu+2} (\cos \psi)^{2\nu+2}} P_\nu^{-\mu}(\cos \alpha), \quad \operatorname{tg} \frac{1}{2} \alpha = \operatorname{tg} \psi \cos \varphi \\
 &\quad \left[a > 0, \quad \frac{\pi}{2} > \varphi > 0, \quad 0 < \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1 \right].
 \end{aligned}$$

ET II 64(11)

$$\begin{aligned}
 15. \quad \int_0^\infty x^{\nu+1} J_\nu(ax) K_\nu(bx) J_\nu(cx) dx &= \frac{2^{3\nu} (abc)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} [(a^2 + b^2 + c^2)^2 - 4a^2 c^2]^{\nu+\frac{1}{2}}} \\
 &\quad \left[\operatorname{Re} b > |\operatorname{Im} a|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
 \end{aligned}$$

ET II 63(8)

$$16. \quad \int_0^\infty x^{\nu+1} I_\nu(ax) K_\nu(bx) J_\nu(cx) dx = \frac{2^{3\nu} (abc)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} [(b^2 - a^2 + c^2)^2 + 4a^2 c^2]^{\nu+\frac{1}{2}}}$$

$$1. \int_0^\infty x^{2\nu+1} J_\nu(ax) N_\nu(ax) J_\nu(bx) N_\nu(bx) dx = \frac{a^{2\nu} \Gamma(3\nu+1)}{2\pi b^{4\nu+2} \Gamma\left(\frac{1}{2}-\nu\right) \Gamma\left(2\nu+\frac{3}{2}\right)} \times \\ \times F\left(\nu+\frac{1}{2}, 3\nu+1; 2\nu+\frac{3}{2}; \frac{a^2}{b^2}\right) \\ \left[0 < a < b, \quad -\frac{1}{3} < \operatorname{Re} \nu < \frac{1}{2}\right].$$

EH II 94(45), ET II 352(15)

$$2. \int_0^\infty x^{2\nu+1} J_\nu(ax) K_\nu(ax) J_\nu(bx) K_\nu(bx) dx = \frac{2^{\nu-3} a^{2\nu} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right) \Gamma\left(\frac{3\nu+1}{2}\right)}{\sqrt{\pi} b^{4\nu+2} \Gamma(\nu+1)} \times \\ \times F\left(\nu+\frac{1}{2}, \frac{3\nu+1}{2}; 2\nu+1; 1-\frac{a^4}{b^4}\right) \\ \left[0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{3}\right].$$

ET II 373(10)

$$3. \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^4 dx = \frac{\Gamma(\nu)\Gamma(2\nu)}{2\pi \left[\Gamma\left(\nu+\frac{1}{2}\right)\right]^2 \Gamma(3\nu)} \quad [\operatorname{Re} \nu > 0].$$

ET II 342(25)

$$4. \int_0^\infty x^{1-2\nu} [J_\nu(ax)]^2 [J_\nu(bx)]^2 dx = \frac{a^{2\nu-1} \Gamma(\nu)}{2\pi b \Gamma\left(\nu+\frac{1}{2}\right) \Gamma\left(2\nu+\frac{1}{2}\right)} F\left(\nu, \frac{1}{2}-\nu; 2\nu+\frac{1}{2}; \frac{a^2}{b^2}\right).$$

ET II 351(10)

$$1. \int_0^a x^{\lambda-1} J_\mu(x) J_\nu(a-x) dx = 2^\lambda \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\lambda+\mu+m) \Gamma(\lambda+m)}{m! \Gamma(\lambda) \Gamma(\mu+m+1)} J_{\lambda+\mu+\nu+2m}(a) \\ [\operatorname{Re}(\lambda+\mu) > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 354(25)

ET II 354(27)

$$3. \int_0^a x^\mu (a-x)^\nu J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{2\pi} \Gamma(\mu + \nu + 1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a) \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 354(28), EH II 46(6)

$$4. \int_0^a x^\mu (a-x)^{\nu+1} J_\mu(x) J_\nu(a-x) dx = \frac{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{2\pi} \Gamma(\mu + \nu + 2)} a^{\mu+\nu+\frac{3}{2}} J_{\mu+\nu+\frac{1}{2}}(a) \\ \left[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2} \right].$$

ET II 354(29)

$$5. \int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x) dx = \frac{2^\mu \Gamma\left(\mu + \frac{1}{2}\right) \Gamma(\nu - \mu)}{\sqrt{\pi} \Gamma(\mu + \nu + 1)} a^\mu J_\nu(a) \\ \left[\operatorname{Re} \nu > \operatorname{Re} \mu > -\frac{1}{2} \right].$$

ET II 355(30)

6.582

$$\int_0^\infty x^{\mu-1} |x-b|^{-\mu} K_\mu(|x-b|) K_\nu(x) dx = \frac{1}{\sqrt{\pi}} (2b)^{-\mu} \Gamma\left(\frac{1}{2} - \mu\right) \Gamma(\mu+\nu) \Gamma(\mu-\nu) K_\nu(b) \\ \left[b > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu| \right].$$

ET II 374(14)

6.583

$$\int_0^\infty x^{\mu-1} (x+b)^{-\mu} K_\mu(x+b) K_\nu(x) dx = \frac{\sqrt{\pi} \Gamma(\mu+\nu) \Gamma(\mu-\nu)}{2^\mu b^\mu \Gamma\left(\mu + \frac{1}{2}\right)} K_\nu(b) \\ [|\arg b| < \pi, \quad \operatorname{Re} \mu > |\operatorname{Re} \nu|].$$

ET II 374(15)

6.584

$$1. \int_0^\infty \frac{x^{\varrho-1} [H_\nu^{(1)}(ax) - e^{\varrho\pi i} H_\nu^{(1)}(axe^{\pi i})]}{(x^2 - r^2)^{m+1}} dx = \frac{\pi i}{m!} \left(\frac{d}{dr}\right)^m [r^{\varrho-2} H_\nu^{(1)}(ar)]$$

$$\left[m = 0, 1, 2, \dots, \quad \text{Im } r > 0, \quad a > 0, \quad |\text{Re } \nu| < \text{Re } \varrho < 2m + \frac{7}{2} \right].$$

WA 465

722

$$2. \int_0^\infty \left[\cos \frac{1}{2}(\varrho - \nu)\pi J_\nu(ax) + \sin \frac{1}{2}(\varrho - \nu)\pi N_\nu(ax) \right] \frac{x^{\varrho-1}}{(x^2 - k^2)^{m+1}} dx =$$

$$= \frac{(-1)^{m+1}}{2^m \cdot m!} \left(\frac{d}{k dk}\right)^m [k^{\varrho-2} K_\nu(ak)]$$

$$\left[m = 0, 1, 2, \dots, \quad \text{Re } k > 0, \quad a > 0, \quad |\text{Re } \nu| < \text{Re } \varrho < 2m + \frac{7}{2} \right].$$

WA 466(2)

$$3. \int_0^\infty \{ \cos \nu\pi J_\nu(ax) - \sin \nu\pi N_\nu(ax) \} \frac{x^{1-\nu} dx}{(x^2 + k^2)^{m+1}} = \frac{a^m K_{\nu+m}(ak)}{2^m \cdot m! k^{\nu+m}}$$

$$\left[m = 0, 1, 2, \dots, \quad \text{Re } k > 0, \quad a > 0, \quad -2m - \frac{3}{2} < \text{Re } \nu < 1 \right].$$

WA 466(3)

$$4. \int_0^\infty \left\{ \cos \left[\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu \right) \pi \right] J_\nu(ax) + \sin \left[\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu \right) \pi \right] N_\nu(ax) \right\} \frac{x^{\varrho-1}}{(x^2 + k^2)^{\mu+1}} dx =$$

$$= \frac{\pi k^{\varrho-2\mu-2}}{2 \sin \nu\pi \cdot \Gamma(\mu+1)} \left[\frac{\left(\frac{1}{2}ak\right)^\nu \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu\right)}{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\varrho + \frac{1}{2}\nu - \mu\right)} {}_1F_2\left(\frac{\varrho+\nu}{2}; \frac{\varrho+\nu}{2} - \mu, \nu+1; \frac{a^2k^2}{4}\right) - \right.$$

$$\left. - \frac{\left(\frac{1}{2}ak\right)^{-\nu} \Gamma\left(\frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{\Gamma(1-\nu)\Gamma\left(\frac{1}{2}\varrho - \frac{1}{2}\nu - \mu\right)} {}_1F_2\left(\frac{\varrho-\nu}{2}; \frac{\varrho-\nu}{2} - \mu, 1-\nu; \frac{a^2k^2}{4}\right) \right]$$

$$\left[a > 0, \quad \text{Re } k > 0, \quad |\text{Re } \nu| < \text{Re } \varrho < 2 \text{Re } \mu + \frac{7}{2} \right].$$

WA 407(1)

$$5.^8 \int_0^\infty \left[\prod_{j,n} J_{\mu_j}(b_n x) \right] \left\{ \cos \left[\frac{1}{2} \left(\varrho + \sum_j \mu_j - \nu \right) \pi \right] J_\nu(ax) + \right.$$

$$\left. + \sin \left[\frac{1}{2} \left(\varrho + \sum_j \mu_j - \nu \right) \pi \right] N_\nu(ax) \right\} \frac{x^{\varrho-1}}{x^2 + k^2} dx$$

6.59 Combinations of powers and Bessel functions of more complicated arguments

6.591

$$1. \int_0^{\infty} x^{2\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_{\nu}(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} J_{1+2\nu}(\sqrt{2ab}) K_{1+2\nu}(\sqrt{2ab})$$

$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 142(35)

723

$$2. \int_0^{\infty} x^{2\nu+\frac{1}{2}} N_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_{\nu}(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} N_{2\nu+1}(\sqrt{2ab}) K_{2\nu+1}(\sqrt{2ab})$$

$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 143(41)

$$3. \int_0^{\infty} x^{2\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) K_{\nu}(bx) dx = \sqrt{2\pi} b^{-\nu-1} a^{\nu+\frac{1}{2}} K_{2\nu+1}(e^{\frac{1}{4}i\pi}\sqrt{2ab}) K_{2\nu+1}(e^{-\frac{1}{4}i\pi}\sqrt{2ab})$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

ET II 146(56)

$$4. \int_0^{\infty} x^{-2\nu+\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_{\nu}(bx) dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} K_{2\nu-1}(\sqrt{2ab}) \times$$

$$\times [\sin(\nu\pi) J_{2\nu-1}(\sqrt{2ab}) + \cos(\nu\pi) N_{2\nu-1}(\sqrt{2ab})]$$

$$[a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 142(34)

$$5. \int_0^{\infty} x^{-2\nu+\frac{1}{2}} N_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) K_{\nu}(bx) dx = -\sqrt{\frac{\pi}{2}} b^{\nu-1} a^{\frac{1}{2}-\nu} \sec(\nu\pi) K_{2\nu-1}(\sqrt{2ab}) \times$$

$$\times [J_{2\nu-1}(\sqrt{2ab}) - J_{1-2\nu}(\sqrt{2ab})] \quad [a > 0, \quad \operatorname{Re} \nu < 1].$$

ET II 143(40)

$$6. \int_0^{\infty} x^{-2\nu+\frac{1}{2}} J_{\frac{1}{2}-\nu}\left(\frac{a}{x}\right) J_{\nu}(bx) dx =$$

$$= -\frac{1}{2} i \operatorname{cosec}(2\nu\pi) b^{\nu-1} a^{\frac{1}{2}-\nu} [e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v) - e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)],$$

$$u = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}; \quad v = \left(\frac{1}{2}ab\right)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i} \quad \left[a > 0, \quad b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 3 \right].$$

$$7. \int_0^\infty x^{-2\nu+\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(\frac{a}{x}\right) N_\nu(bx) dx = \sqrt{2\pi} b^{\nu-1} a^{\frac{1}{2}-\nu} N_{2\nu-1}(\sqrt{2ab}) K_{2\nu-1}(\sqrt{2ab})$$

$$\left[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > \frac{1}{6} \right].$$

ET II 113(30)

$$8. \int_0^\infty x^{\varrho-1} J_\mu(ax) J_\nu\left(\frac{b}{x}\right) dx = \frac{a^{\nu-\varrho} b^\nu \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\varrho - \frac{1}{2}\nu\right)}{2^{2\nu-\varrho+1} \Gamma(\nu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\varrho + 1\right)} \times$$

$$\times {}_0F_3\left(\nu+1, \frac{\nu-\mu-\varrho}{2}+1, \frac{\nu+\mu-\varrho}{2}+1; \frac{a^2 b^2}{16}\right) +$$

$$+ \frac{a^\mu b^{\mu+\varrho} \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\mu - \frac{1}{2}\varrho\right)}{2^{2\mu+\varrho+1} \Gamma(\mu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\varrho + 1\right)} \times$$

$$\times {}_0F_3\left(\mu+1, \frac{\mu-\nu+\varrho}{2}+1, \frac{\nu+\mu+\varrho}{2}+1; \frac{a^2 b^2}{16}\right)$$

$$\left[a > 0, \quad b > 0, \quad -\operatorname{Re}\left(\mu + \frac{3}{2}\right) < \operatorname{Re} \varrho < \operatorname{Re}\left(\nu + \frac{3}{2}\right) \right].$$

WA 480(1)

724

6.592

$$1. \int_0^\infty x^\lambda (1-x)^{\mu-1} N_\nu(a\sqrt{x}) dx = 2^{-\nu} a^\nu \operatorname{ctg}(\nu\pi) \frac{\Gamma(\mu)\Gamma\left(\lambda+1+\frac{1}{2}\nu\right)}{\Gamma(1+\nu)\Gamma\left(\lambda+1+\mu+\frac{1}{2}\nu\right)} \times$$

$$\times {}_1F_2\left(\lambda+1+\frac{1}{2}\nu; 1+\nu, \lambda+1+\mu+\frac{1}{2}\nu; -\frac{a^2}{4}\right) -$$

$$- 2^\nu a^{-\nu} \operatorname{cosec}(\nu\pi) \frac{\Gamma(\mu)\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma(1-\nu)\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)} \times$$

$$\times {}_1F_2\left(\lambda-\frac{1}{2}\nu+1; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; -\frac{a^2}{4}\right)$$

$$\left[\operatorname{Re} \lambda > -1 + \frac{1}{2}|\operatorname{Re} \nu|, \quad \operatorname{Re} \mu > 0 \right].$$

ET II 197(76)a

$$2.* \int_0^1 x^\lambda (1-x)^{\mu-1} K_\nu(a\sqrt{x}) dx =$$

$$= 2^{-\nu-1} a^{-\nu} \frac{\Gamma(\nu)\Gamma(\mu)\Gamma\left(\lambda+1-\frac{1}{2}\nu\right)}{\Gamma\left(\lambda+1+\mu-\frac{1}{2}\nu\right)} {}_1F_2\left(\lambda+1-\frac{1}{2}\nu; 1-\nu, \lambda+1+\mu-\frac{1}{2}\nu; \frac{a^2}{4}\right) +$$

$$3. \int_1^{\infty} x^{\lambda}(x-1)^{\mu-1} J_{\nu}(a\sqrt{x}) dx = 2^{2\lambda} a^{-2\lambda} G_{13}^{20} \left(\frac{a^2}{4} \left| \begin{matrix} 0 \\ -\mu, \lambda + \frac{1}{2}\nu, \lambda - \frac{1}{2}\nu \end{matrix} \right. \right) \Gamma(\mu) \\ \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{4} - \operatorname{Re} \lambda \right].$$

ET II 205(36)a

$$4. \int_1^{\infty} x^{\lambda}(x-1)^{\mu-1} K_{\nu}(a\sqrt{x}) dx = \Gamma(\mu) 2^{2\lambda-1} a^{-2\lambda} G_{13}^{30} \left(\frac{a^2}{4} \left| \begin{matrix} 0 \\ -\mu, \frac{1}{2}\nu + \lambda, -\frac{1}{2}\nu + \lambda \end{matrix} \right. \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 209(60)a

725

$$5. \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} J_{\nu}(a\sqrt{x}) dx = \pi \left[J_{\frac{1}{2}\nu} \left(\frac{1}{2}a \right) \right]^2 \quad [\operatorname{Re} \nu > -1].$$

ET II 194(59)a

$$6. \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} I_{\nu}(a\sqrt{x}) dx = \pi \left[I_{\frac{1}{2}\nu} \left(\frac{1}{2}a \right) \right]^2 \quad [\operatorname{Re} \nu > -1].$$

ET II 197(79)

$$7. \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} K_{\nu}(a\sqrt{x}) dx = \frac{1}{2} \pi \sec \left(\frac{1}{2} \nu \pi \right) \left[I_{\frac{\nu}{2}} \left(\frac{a}{2} \right) + I_{-\frac{\nu}{2}} \left(\frac{a}{2} \right) \right] K_{\frac{\nu}{2}} \left(\frac{a}{2} \right) \\ [|\operatorname{Re} \nu| < 1].$$

ET II 198(85)a

$$8. \int_1^{\infty} x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}} K_{\nu}(a\sqrt{x}) dx = \left[K_{\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2 \quad [\operatorname{Re} a > 0].$$

ET II 208(56)a

$$9. \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}} N_{\nu}(a\sqrt{x}) dx = \pi \left\{ \operatorname{ctg}(\nu\pi) \left[J_{\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2 - \operatorname{cosec}(\nu\pi) \left[J_{-\frac{\nu}{2}} \left(\frac{a}{2} \right) \right]^2 \right\} \\ [|\operatorname{Re} \nu| < 1].$$

$$10. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_{\nu} (a\sqrt{x}) dx = \Gamma(\mu) 2^{\mu} a^{-\mu} J_{\nu-\mu}(a) \quad \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4} \right].$$

ET II 205(34)a

$$11. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} J_{-\nu} (a\sqrt{x}) dx = \Gamma(\mu) 2^{\mu} a^{-\mu} [\cos(\nu\pi) J_{\nu-\mu}(a) - \sin(\nu\pi) N_{\nu-\mu}(a)] \\ \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4} \right].$$

ET II 205(35)a

$$12. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} K_{\nu} (a\sqrt{x}) dx = \Gamma(\mu) 2^{\mu} a^{-\mu} K_{\nu-\mu}(a) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 209(59)a

$$13. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} N_{\nu} (a\sqrt{x}) dx = 2^{\mu} a^{-\mu} N_{\nu-\mu}(a) \Gamma(\mu) \quad \left[a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4} \right].$$

ET II 206(40)a

$$14. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_{\nu}^{(1)} (a\sqrt{x}) dx = 2^{\mu} a^{-\mu} H_{\nu-\mu}^{(1)}(a) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Im} a > 0].$$

ET II 206(45)a

$$15. \int_1^{\infty} x^{-\frac{1}{2}\nu} (x-1)^{\mu-1} H_{\nu}^{(2)} (a\sqrt{x}) dx = 2^{\mu} a^{-\mu} H_{\nu-\mu}^{(2)}(a) \Gamma(\mu) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Im} a < 0].$$

ET II 207(48)a

$$16. \int_0^1 x^{-\frac{1}{2}\nu} (1-x)^{\mu-1} J_{\nu} (a\sqrt{x}) dx = \frac{2^{2-\nu} a^{-\mu}}{\Gamma(\nu)} s_{\mu+\nu-1, \mu-\nu}(a) \quad [\operatorname{Re} \mu > 0].$$

ET II 194(64)a

$$17. \int_0^1 x^{-\frac{1}{2}\nu} (1-x)^{\mu-1} N_{\nu} (a\sqrt{x}) dx = \frac{2^{2-\nu} a^{-\mu} \operatorname{ctg}(\nu\pi)}{\Gamma(\nu)} s_{\mu+\nu-1, \mu-\nu}(a) \\ - 2^{\mu} a^{-\mu} \operatorname{cosec}(\nu\pi) J_{\mu-\nu}(a) \Gamma(\mu) \\ [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu < 1].$$

$$1. \int_0^{\infty} \sqrt{x} J_{2\nu-1}(a\sqrt{x}) J_{\nu}(bx) dx = \frac{1}{2} ab^{-2} J_{\nu-1}\left(\frac{a^2}{4b}\right) \quad \left[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 58(15)

$$2. \int_0^{\infty} \sqrt{x} J_{2\nu-1}(a\sqrt{x}) K_{\nu}(bx) dx = \frac{\pi a}{4b^2} \left[I_{\nu-1}\left(\frac{a^2}{4b}\right) - L_{\nu-1}\left(\frac{a^2}{4b}\right) \right] \quad \left[\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 144(44)

6.594

$$1. \int_0^{\infty} x^{\nu} I_{2\nu-1}(a\sqrt{x}) J_{2\nu-1}(a\sqrt{x}) K_{\nu}(bx) dx = \sqrt{\pi} 2^{-\nu} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) \quad [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 148(65)

$$2. \int_0^{\infty} x^{\nu} I_{2\nu-1}(a\sqrt{x}) N_{2\nu-1}(a\sqrt{x}) K_{\nu}(bx) dx = \sqrt{\pi} 2^{-\nu-1} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \left[H_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) + \sin(\nu\pi) N_{\nu-\frac{1}{2}}\left(\frac{a^2}{2b}\right) \right] \quad [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 148(66)

$$3. \int_0^{\infty} x^{\nu} J_{2\nu-1}(a\sqrt{x}) K_{2\nu-1}(a\sqrt{x}) K_{\nu}(bx) dx = \pi^2 2^{-\nu-2} a^{2\nu-1} b^{-2\nu-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \left[H_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) - N_{\frac{1}{2}-\nu}\left(\frac{a^2}{2b}\right) \right] \quad [\operatorname{Re} b > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 148(67)

6.595

$$1. \int_0^{\infty} x^{\nu+1} J_{\nu}(cx) \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i) dx = 0, \quad z_i = \sqrt{x^2 + b_i^2} \quad \left[a_i > 0, \quad \operatorname{Re} b_i > 0, \quad \sum_{i=1}^n a_i < c; \quad \operatorname{Re} \left(\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2} \right) > \operatorname{Re} \nu > -1 \right].$$

EH II 52(33), ET II 60(26)

$$1. \int_0^\infty J_\nu(\alpha\sqrt{x^2+z^2}) \frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu+1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} J_{\nu-\mu-1}(\alpha z) \\ \left[\alpha > 0, \quad \operatorname{Re}\left(\frac{1}{2}\nu - \frac{1}{4}\right) > \operatorname{Re}\mu > -1 \right].$$

WA 457(5)

$$2. \int_0^\infty \frac{J_\nu(\alpha\sqrt{t^2+1})}{\sqrt{t^2+1}} dt = -\frac{\pi}{2} J_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right) N_{\frac{\nu}{2}}\left(\frac{\alpha}{2}\right) \quad [\operatorname{Re}\nu > -1, \quad \alpha > 0].$$

MO 46

$$3. \int_0^\infty K_\nu(\alpha\sqrt{x^2+z^2}) \frac{x^{2\mu+1}}{\sqrt{(x^2+z^2)^\nu}} dx = \frac{2^\mu \Gamma(\mu+1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\alpha z) \quad [\alpha > 0, \quad \operatorname{Re}\mu > -1].$$

WA 457(6)

$$4.^6 \int_0^\infty J_\nu(\beta x) \frac{J_{\mu-1}\{\alpha\sqrt{x^2+z^2}\}}{(x^2+z^2)^{\frac{1}{2}\mu+\frac{1}{2}}} x^{\nu+1} dx = \frac{\alpha^{\mu-1} z^\nu}{2^{\mu-1} \Gamma(\mu)} K_\nu(\beta x) \\ [\alpha < \beta, \quad \operatorname{Re}(\mu+2) > \operatorname{Re}\nu > -1].$$

WA 459(11)A, ET II 59(19)

$$5. \int_0^\infty J_\nu(\beta x) \frac{J_\mu\{\alpha\sqrt{x^2+z^2}\}}{\sqrt{(x^2+z^2)^\mu}} x^{\nu-1} dx = \frac{2^{\nu-1} \Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu} \\ [\operatorname{Re}(\mu+2) > \operatorname{Re}\nu > 0, \quad \beta > \alpha > 0].$$

WA 459(12)

$$6.^6 \int_0^\infty J_\nu(\beta x) \frac{J_\mu(\alpha\sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^\mu}} x^{\nu+1} dx \\ = 0 \quad [0 < \alpha < \beta]; \\ = \frac{\beta^\nu}{\alpha^\mu} \left(\frac{\sqrt{\alpha^2 - \beta^2}}{z} \right)^{\mu-\nu-1} J_{\mu-\nu-1}\left\{ z\sqrt{\alpha^2 - \beta^2} \right\} \quad [\alpha > \beta > 0]; \quad [\operatorname{Re}\mu > \operatorname{Re}\nu > -1].$$

$$7. \int_0^\infty J_\nu(\beta x) \frac{K_\mu(\alpha\sqrt{x^2+z^2})}{\sqrt{(x^2+z^2)^\mu}} x^{\nu+1} dx = \frac{\beta^\nu}{\alpha^\mu} \left(\frac{\sqrt{\alpha^2+\beta^2}}{z} \right)^{\mu-\nu-1} K_{\mu-\nu-1}(z\sqrt{z\alpha^2+\beta^2})$$

$$\left[\alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1, \quad |\arg z| < \frac{\pi}{2} \right].$$

KU 151(31), WA 416(2)

$$8.^6 \int_0^\infty J_\nu(\beta t) \frac{K_\mu(\alpha\sqrt{t^2-y^2})}{\sqrt{(t^2-y^2)^\mu}} t^{\nu+1} dt = \frac{\pi}{2} \frac{\beta^\nu}{\alpha^\mu} \left\{ \frac{\sqrt{\alpha^2+\beta^2}}{y} \right\}^{\mu-\nu-1} \exp\left[-\frac{\pi i}{2} \left(\mu-\nu-\frac{1}{2}\right)\right] \times$$

$$\times \left\{ J_{\mu-\nu-1}\left[y\sqrt{\alpha^2+\beta^2}\right] - i N_{\mu-\nu-1}\left[y\sqrt{\alpha^2+\beta^2}\right] \right\}$$

[Re $\mu < 1$. Here, it is assumed that the integration contour does not contain the singularity $t = y$, which can be excluded by going upwards around it, and that the sign of $\sqrt{t^2 - y^2}$ is chosen in such a way that the expression in question is positive for $t > y; \alpha > 0, \beta > 0, y > 0$].

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$$9. \int_0^\infty J_\nu(ux) K_\mu(v\sqrt{x^2-y^2}) (x^2-y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx = \frac{\pi}{2} \exp\left[-i\pi\left(\mu-\nu-\frac{1}{2}\right)\right] \cdot \frac{u^\nu}{v^\mu} \cdot$$

$$\cdot \left[\frac{\sqrt{u^2+v^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)}\left(y\sqrt{u^2+v^2}\right)$$

$$\left[\operatorname{Re} \mu < 1, \quad \operatorname{Re} \nu > -1, \quad u > 0, \quad v > 0; \quad \arg \sqrt{x^2-y^2} = 0 \quad \text{for } x > y;$$

$$\text{if } x < y, \text{ then } \arg(x^2-y^2)^\sigma = \pi\sigma, \quad \text{where } \sigma = \frac{1}{2} \text{ or } \sigma = -\frac{\mu}{2} \right].$$

MO 43
WA 416(3)

$$10.^6 \int_0^\infty J_\nu(ux) H_\mu^{(2)}(v\sqrt{x^2+y^2}) (x^2+y^2)^{-\frac{\mu}{2}} x^{\nu+1} dx = \frac{u^\nu}{v^\mu} \left[\frac{\sqrt{v^2-u^2}}{y} \right]^{\mu-\nu-1} H_{\mu-\nu-1}^{(2)}\left(y\sqrt{v^2-u^2}\right)$$

$$[u < v] \quad [\operatorname{Re} \mu < \operatorname{Re} \nu, \quad \operatorname{Re} \nu > -1, \quad u > 0, \quad v > 0, \quad y > 0; \quad \arg \sqrt{v^2-u^2} = 0$$

$$\text{for } v > u, \quad \arg(v^2-u^2)^\sigma = -\pi\sigma \quad \text{for } v < u, \quad \text{where } \sigma = \frac{1}{2} \quad \text{or} \quad \sigma = \frac{\mu-\nu-1}{2}].$$

MO 43

$$11. \int_0^\infty J_\nu(\beta x) J_\mu(\alpha\sqrt{x^2+z^2}) J_\mu(\gamma\sqrt{x^2+z^2}) \frac{x^{\nu-1}}{(x^2+z^2)^\mu} dx = \frac{2^{\nu-1}\Gamma(\nu)}{\beta^\nu} \frac{J_\mu(\alpha z)}{z^\mu} \frac{J_\mu(\gamma z)}{z^\mu}$$

$$\left[\alpha > 0; \quad \beta > \alpha + \gamma; \quad \gamma > 0, \quad \operatorname{Re}\left(2\mu + \frac{5}{2}\right) > \operatorname{Re} \nu > 0 \right].$$

$$12.^6 \int_0^\infty J_\nu(\beta t) t^{\nu-1} \prod_{k=1}^n J_\mu(\alpha_k \sqrt{t^2 + x^2}) \sqrt{(t^2 + x^2)^{-n\mu}} dt = 2^{\nu-1} \beta^{-\nu} \Gamma(\nu) \prod_{k=1}^n [x^{-\mu} J_\mu(\alpha_k x)]$$

$$\left[x > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \dots, \alpha_n > 0, \quad \beta > \prod_{k=1}^n \alpha_k; \quad \operatorname{Re} \left(n\mu + \frac{1}{2}n + \frac{1}{2} \right) > \operatorname{Re} \nu > 0 \right].$$

MO 43

$$13. \int_0^\infty \frac{J_\nu^2(\sqrt{a^2 + x^2})}{(a^2 + x^2)^\nu} x^{2\nu-2} dx = \frac{\Gamma\left(\nu - \frac{1}{2}\right)}{2a^{\nu+1}\sqrt{\pi}} \mathbf{H}_\nu(2a) \quad \left[\operatorname{Re} \nu > \frac{1}{2} \right].$$

WA 457(8)

6.597

$$\int_0^\infty t^{\nu+1} J_\mu \left[b(t^2 + y^2)^{\frac{1}{2}} \right] (t^2 + y^2)^{-\frac{1}{2}\mu} (t^2 + \beta^2)^{-1} J_\nu(at) dt =$$

$$= \beta^\nu J_\mu \left[b(y^2 - \beta^2)^{\frac{1}{2}} \right] (y^2 - \beta^2)^{-\frac{1}{2}\mu} K_\nu(a\beta) \quad [a \geq b, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < 2 + \operatorname{Re} \mu].$$

EH II 95(56)

6.598

$$\int_0^1 x^{\frac{\mu}{2}} (1-x)^{\frac{\nu}{2}} J_\mu(a\sqrt{x}) J_\nu(b\sqrt{1-x}) dx = 2a^\mu b^\nu (a^2 + b^2)^{-\frac{1}{2}(\nu + \mu + 1)} J_{\nu + \mu + 1}(\sqrt{a^2 + b^2})$$

$$[\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1].$$

EH II 46a

729

6.61 Combinations of Bessel functions and exponentials

6.611

$$1. \int_0^\infty e^{-\alpha x} J_\nu(\beta x) dx = \frac{\beta^{-\nu} \left[\sqrt{\alpha^2 + \beta^2} - \alpha \right]^\nu}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \nu > -1, \quad \operatorname{Re}(\alpha \pm i\beta) > 0].$$

EH II 49(18), WA 422(8)

$$2. \int_0^\infty e^{-\alpha x} N_\nu(\beta x) dx = (\alpha^2 + \beta^2)^{-\frac{1}{2}} \operatorname{cosec}(\nu\pi) \times$$

$$\times \left\{ \beta^\nu \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^{-\nu} \cos(\nu\pi) - \beta^{-\nu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha \right]^\nu \right\}$$

$$[\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad |\operatorname{Re} \nu| < 1].$$

$$3. \int_0^\infty e^{-\alpha x} K_\nu(\beta x) dx = \frac{\pi}{\beta \sin(\nu\pi)} \frac{\sin(\nu\theta)}{\sin\theta} \left[\cos\theta = \frac{\alpha}{\beta}; \quad \theta \rightarrow \frac{\pi}{2} \text{ for } \beta \rightarrow \infty \right];$$

$$= \frac{\pi \operatorname{cosec}(\nu\pi)}{2\sqrt{\alpha^2 - \beta^2}} \left[\beta^{-\nu} \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right)^\nu - \beta^\nu \left(\sqrt{\alpha^2 - \beta^2} + \alpha \right)^{-\nu} \right]$$

[|Re ν| < 1, Re(α + β) > 0].

ET I 197(24), MO 180
ET II 131(22)

$$4. \int_0^\infty e^{-\alpha x} I_\nu(\beta x) dx = \frac{\beta^\nu}{\sqrt{\alpha^2 - \beta^2} \left(\alpha + \sqrt{\alpha^2 - \beta^2} \right)^\nu} \quad [\operatorname{Re} \nu > -1, \operatorname{Re} \alpha > |\operatorname{Re} \beta|].$$

MO 180, ET I 195(1)

$$5. \int_0^\infty e^{-\alpha x} H_\nu^{(1,2)}(\beta x) dx = \frac{\left(\sqrt{\alpha^2 + \beta^2} - \alpha \right)^\nu}{\beta^\nu \sqrt{\alpha^2 + \beta^2}} \left\{ 1 \pm \frac{i}{\sin(\nu\pi)} \left[\cos(\nu\pi) - \frac{\left(\alpha + \sqrt{\alpha^2 + \beta^2} \right)^{2\nu}}{b^{2\nu}} \right] \right\}$$

[−1 < Re ν < 1; a plus sign corresponds to the function $H_\nu^{(1)}$, a minus sign to the function $H_\nu^{(2)}$].

$$6. \int_0^\infty e^{-\alpha x} H_0^{(1)}(\beta x) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 - \frac{2i}{\pi} \ln \left[\frac{\alpha}{\beta} + \sqrt{1 + \left(\frac{\alpha}{\beta} \right)^2} \right] \right\} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

MO 180, ET I 188(53)
MO 180, ET I 188(54, 55)

$$7. \int_0^\infty e^{-\alpha x} H_0^{(2)}(\beta x) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ 1 + \frac{2i}{\pi} \ln \left[\frac{\alpha}{\beta} + \sqrt{1 + \left(\frac{\alpha}{\beta} \right)^2} \right] \right\} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

MO 180, ET I 188(53)}

$$8. \int_0^\infty e^{-\alpha x} N_0(\beta x) dx = \frac{-2}{\pi \sqrt{\alpha^2 + \beta^2}} \ln \frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{\beta} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

MO 47, ET I 187(44)

730

$$9. \int_0^\infty e^{-\alpha x} K_0(\beta x) dx = \frac{\arccos \frac{\alpha}{\beta}}{\sqrt{\beta^2 - \alpha^2}} \quad [0 < \alpha < \beta, \operatorname{Re}(\alpha + \beta) > 0];$$

$$\frac{1}{\sqrt{\alpha^2 - \beta^2}} \ln \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - 1} \right) \quad [0 < \beta < \alpha, \operatorname{Re}(\alpha + \beta) > 0].$$

6.612

$$1. \int_0^{\infty} e^{-2\alpha x} J_0(x) N_0(x) dx = \frac{\mathbf{K} \left[\alpha(\alpha^2 + 1)^{-\frac{1}{2}} \right]}{\pi(\alpha^2 + 1)^{\frac{1}{2}}} \quad [\operatorname{Re} \alpha > 0].$$

ET II 347(58)

$$2. \int_0^{\infty} e^{-2\alpha x} I_0(x) K_0(x) dx = \frac{1}{2} \mathbf{K} \left[(1 - \alpha^2)^{\frac{1}{2}} \right] \quad [0 < \alpha < 1];$$

$$= \frac{1}{2\alpha} \mathbf{K} \left[\left(1 - \frac{1}{\alpha^2} \right)^{\frac{1}{2}} \right] \quad [1 < \alpha < \infty].$$

ET II 370(48)

$$3. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta x) J_{\nu}(\gamma x) dx = \frac{1}{\pi \sqrt{\gamma \beta}} Q_{\nu - \frac{1}{2}} \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2\beta\gamma} \right)$$

$$\left[\operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 426(2), ET II 50(17)

$$4. \int_0^{\infty} e^{-\alpha x} [J_0(\beta x)]^2 dx = \frac{2}{\pi \sqrt{\alpha^2 + 4\beta^2}} \mathbf{K} \left(\frac{2\beta}{\sqrt{\alpha^2 + 4\beta^2}} \right).$$

MO 178

$$5. \int_0^{\infty} e^{-2\alpha x} J_1^2(\beta x) dx = \frac{(2\alpha^2 + \beta^2) \mathbf{K} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) - 2(\alpha^2 + \beta^2) \mathbf{E} \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)}{\pi \beta^2 \sqrt{\alpha^2 + \beta^2}}.$$

WA 428(3)

6.613⁷

$$\int_0^{\infty} e^{-x^2} J_{\nu + \frac{1}{2}} \left(\frac{x^2}{2} \right) dx = \frac{\Gamma(\nu + 1)}{\sqrt{\pi}} D_{-\nu - 1} \left(z e^{\frac{\pi}{4} i} \right) D_{-\nu - 1} \left(z e^{-\frac{\pi}{4} i} \right) \quad [\operatorname{Re} \nu > -1].$$

MO 122

6.614

$$1. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta \sqrt{x}) dx = \frac{\beta}{4} \sqrt{\frac{\pi}{\alpha^3}} \exp \left(-\frac{\beta^2}{8\alpha} \right) \left[I_{\frac{1}{2}(\nu - 1)} \left(\frac{\beta^2}{8\alpha} \right) - I_{\frac{1}{2}(\nu + 1)} \left(\frac{\beta^2}{8\alpha} \right) \right].$$

$$2. \int_0^{\infty} e^{-\alpha x} N_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{-\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \left\{ \operatorname{ctg}(\nu\pi) \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{\frac{1}{2}, \nu} \left(\frac{\beta}{\alpha} \right) - \operatorname{cosec}(\nu\pi) W_{\frac{1}{2}, \nu} \left(\frac{\beta}{\alpha} \right) \right\}$$

[Re $\alpha > 0$, |Re ν | < 1].

ET I 188(50)a

731

$$3. \int_0^{\infty} e^{-\alpha x} I_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{\sqrt{\alpha\beta}} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)} M_{-\frac{1}{2}, \nu} \left(\frac{\beta}{\alpha} \right) \quad [\operatorname{Re} \alpha > 0, \operatorname{Re} \nu > -1].$$

ET I 197(20)a

$$4. \int_0^{\infty} e^{-\alpha x} K_{2\nu} (2\sqrt{\beta x}) dx = \frac{e^{\frac{1}{2}\frac{\beta}{\alpha}}}{2\sqrt{\alpha\beta}} \Gamma(\nu+1)\Gamma(1-\nu) W_{-\frac{1}{2}, \nu} \left(\frac{\beta}{\alpha} \right) \quad [\operatorname{Re} \alpha > 0, |\operatorname{Re} \nu| < 1].$$

ET I 199(37)a

$$5. \int_0^{\infty} e^{-\alpha x} K_1 (\beta\sqrt{x}) dx = \frac{\beta}{8} \sqrt{\frac{\pi}{\alpha^3}} \exp \left(\frac{\beta^2}{8\alpha} \right) \left[K_1 \left(\frac{\beta^2}{8\alpha} \right) - K_0 \left(\frac{\beta^2}{8\alpha} \right) \right].$$

MO 181

6.615

$$\int_0^{\infty} e^{-\alpha x} J_{\nu} (2\beta\sqrt{x}) J_{\nu} (2\gamma\sqrt{x}) dx = \frac{1}{\alpha} I_{\nu} \left(\frac{2\beta\gamma}{\alpha} \right) \exp \left(-\frac{\beta^2 + \gamma^2}{\alpha} \right) \quad [\operatorname{Re} \nu > -1].$$

MO 178

6.616

$$1. \int_0^{\infty} e^{-\alpha x} J_0 (\beta\sqrt{x^2 + 2\gamma x}) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left[\gamma \left(\alpha - \sqrt{\alpha^2 + \beta^2} \right) \right].$$

MO 179

$$2. \int_1^{\infty} e^{-\alpha x} J_0 (\beta\sqrt{x^2 - 1}) dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \exp \left(-\sqrt{\alpha^2 + \beta^2} \right).$$

MO 179

$$4. \int_{-\infty}^{\infty} e^{-itx} H_0^{(2)}(r\sqrt{\alpha^2 - t^2}) dt = 2i \frac{e^{-i\alpha\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}}$$

[$-\pi < \arg \sqrt{\alpha^2 - t^2} \leq 0$, $-\pi < \arg \alpha \leq 0$, r and x are real].

$$5.^3 \int_{-1}^1 e^{-ax} I_0(b\sqrt{1-x^2}) dx = 2(a^2+b^2)^{-1/2} \operatorname{sh} \sqrt{a^2+b^2} \quad [a > 0, \quad b > 0].$$

6.617

$$1. \int_0^{\infty} K_{q-p}(2z \operatorname{sh} x) e^{(p+q)x} dx = \frac{\pi^2}{4 \sin[(p-q)\pi]} [J_p(z)N_q(z) - J_q(z)N_p(z)]$$

[$\operatorname{Re} z > 0$, $-1 < \operatorname{Re}(p-q) < 1$].

$$2. \int_0^{\infty} K_0(2z \operatorname{sh} x) e^{-2px} dx = -\frac{\pi}{4} \left\{ J_p(z) \frac{\partial N_p(z)}{\partial p} - N_p(z) \frac{\partial J_p(z)}{\partial p} \right\} \quad [\operatorname{Re} z > 0].$$

732

6.618

$$1. \int_0^{\infty} e^{-\alpha x^2} J_{\nu}(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1].$$

WA 432(5), ET II 29(8)

$$2. \int_0^{\infty} e^{-\alpha x^2} N_{\nu}(\beta x) dx = -\frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[\operatorname{tg} \frac{\nu\pi}{2} I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \right]$$

[$\operatorname{Re} \alpha > 0$, $\beta > 0$, $|\operatorname{Re} \nu| < 1$].

WA 432(6), ET II 106(3)

$$3. \int_0^{\infty} e^{-\alpha x^2} K_{\nu}(\beta x) dx = \frac{1}{4} \sec\left(\frac{\nu\pi}{2}\right) \frac{\sqrt{\pi}}{\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) K_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \nu| < 1].$$

$$4. \int_0^{\infty} e^{-\alpha x^2} I_{\nu}(\beta x) dx = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} \exp\left(\frac{\beta^2}{8\alpha}\right) I_{\frac{1}{2}\nu}\left(\frac{\beta^2}{8\alpha}\right) \quad [\operatorname{Re} \nu > -1, \operatorname{Re} \alpha > 0].$$

EH II 92(27)

$$5. \int_0^{\infty} e^{-\alpha x^2} J_{\mu}(\beta x) J_{\nu}(\beta x) dx =$$

$$= 2^{-\nu-\mu-1} \alpha^{-\frac{\nu+\mu+1}{2}} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)} \times$$

$$\times {}_3F_3\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \mu+1, \nu+1, \nu+\mu+1; -\frac{\beta^2}{\alpha}\right)$$

$$[\operatorname{Re}(\nu+\mu) > -1, \operatorname{Re} \alpha > 0].$$

EH II 50(21)a

6.62- 6.63 Combinations of Bessel functions, exponentials, and powers

6.621

$$1. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta x) x^{\mu-1} dx =$$

$$= \frac{\left(\frac{\beta}{2\alpha}\right)^{\nu} \Gamma(\nu+\mu)}{\alpha^{\mu} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; -\frac{\beta^2}{\alpha^2}\right);$$

$$= \frac{\left(\frac{\beta}{2\alpha}\right)^{\nu} \Gamma(\nu+\mu)}{\alpha^{\mu} \Gamma(\nu+1)} \left(1 + \frac{\beta^2}{\alpha^2}\right)^{\frac{1}{2}-\mu} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2} + 1; \nu+1; -\frac{\beta^2}{\alpha^2}\right);$$

$$= \frac{\left(\frac{\beta}{2}\right)^{\nu} \Gamma(\nu+\mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$[\operatorname{Re}(\nu+\mu) > 0, \operatorname{Re}(\alpha+i\beta) > 0, \operatorname{Re}(\alpha-i\beta) > 0];$$

$$= (\alpha^2 + \beta^2)^{-\frac{1}{2}\mu} \Gamma(\nu+\mu) P_{\mu-1}^{-\nu} \left[\alpha(\alpha^2 + \beta^2)^{-\frac{1}{2}}\right] \quad [\alpha > 0, \beta > 0, \operatorname{Re}(\nu+\mu) > 0].$$

ET II 29(6)

WA 421(3)

WA 421(3)

WA 421(2)

733

$$2. \int_0^{\infty} e^{-\alpha x} N_{\nu}(\beta x) x^{\mu-1} dx =$$

$$= \operatorname{ctg} \nu \pi \frac{\left(\frac{\beta}{2}\right)^{\nu} \Gamma(\nu+\mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right) -$$

$$\frac{\left(\frac{\beta}{2}\right)^{-\nu} \Gamma(\nu+\mu)}{\sqrt{(\alpha^2 + \beta^2)^{\nu+\mu}} \Gamma(\nu+1)} F\left(\frac{\nu+\mu}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{\beta^2}{\alpha^2 + \beta^2}\right)$$

$$3. \int_0^{\infty} x^{\mu-1} e^{-\alpha x} K_{\nu}(\beta x) dx = \frac{\sqrt{\pi}(2\beta)^{\nu}}{(\alpha + \beta)^{\mu+\nu}} \frac{\Gamma(\mu + \nu)\Gamma(\mu - \nu)}{\Gamma\left(\mu + \frac{1}{2}\right)} F\left(\mu + \nu, \nu + \frac{1}{2}; \mu + \frac{1}{2}; \frac{\alpha - \beta}{\alpha + \beta}\right)$$

[Re $\mu > |\text{Re } \nu|$, Re $(\alpha + \beta) > 0$].

ET II 131(23)A, EH II 50(26)

$$4. \int_0^{\infty} x^{m+1} e^{-\alpha x} J_{\nu}(\beta x) dx = (-1)^{m+1} \beta^{-\nu} \frac{d^{m+1}}{d\alpha^{m+1}} \left[\frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^{\nu}}{\sqrt{\alpha^2 + \beta^2}} \right]$$

[$\beta > 0$, Re $\nu > -m - 2$].

ET II 28(3)

6.622

$$1. \int_0^{\infty} (J_0(x) - e^{-\alpha x}) \frac{dx}{x} = \ln 2\alpha \quad [\alpha > 0].$$

NT 66(13)

$$2. \int_0^{\infty} \frac{e^{i(u+x)}}{u+x} J_0(x) dx = \frac{\pi}{2} i H_0^{(1)}(u).$$

MO 44

$$3. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} I_{\nu}(x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} Q_{\nu-\frac{1}{2}}(\operatorname{ch} \alpha).$$

WA 424(5)

6.623

$$1. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta x) x^{\nu} dx = \frac{(2\beta)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}(\alpha^2 + \beta^2)^{\nu+\frac{1}{2}}} \left[\operatorname{Re} \nu > -\frac{1}{2}, \operatorname{Re} \alpha > |\operatorname{Im} \beta| \right].$$

WA 422(5)

734

$$2. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta x) x^{\nu+1} dx = \frac{2\alpha(2\beta)^{\nu} \Gamma\left(\nu + \frac{3}{2}\right)}{\sqrt{\pi}(\alpha^2 + \beta^2)^{\nu+\frac{3}{2}}} \quad [\operatorname{Re} \nu > -1, \operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

$$3. \int_0^{\infty} e^{-\alpha x} J_{\nu}(\beta x) \frac{dx}{x} = \frac{(\sqrt{\alpha^2 + \beta^2} - \alpha)^{\nu}}{\nu \beta^{\nu}} \quad [\operatorname{Re} \nu > 0; \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|] \quad (\text{cf. } \mathbf{6.611} \text{ 1}).$$

6.611
WA 422(7)

6.624

$$1. \int_0^{\infty} x e^{-\alpha x} K_0(\beta x) dx = \frac{1}{\alpha^2 - \beta^2} \left\{ \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \ln \left[\frac{\alpha}{\beta} + \sqrt{\left(\frac{\alpha}{\beta}\right)^2 - 1} \right] - 1 \right\}.$$

MO 181

$$2. \int_0^{\infty} \sqrt{x} e^{-\alpha x} K_{\pm \frac{1}{2}}(\beta x) dx = \sqrt{\frac{\pi}{2\beta}} \frac{1}{\alpha + \beta}.$$

MO 181

$$3. \int_0^{\infty} e^{-tz(z^2-1)^{-\frac{1}{2}}} K_{\mu}(t) t^{\nu} dt = \frac{\Gamma(\nu - \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} e^{i\mu\pi} Q_{\nu}^{\mu}(z) \quad [\operatorname{Re}(\nu \pm \mu) > -1].$$

EH II 57(7)

$$4. \int_0^{\infty} e^{-tz(z^2-1)^{-\frac{1}{2}}} I_{-\mu}(t) t^{\nu} dt = \frac{\Gamma(-\nu - \mu)}{(z^2 - 1)^{\frac{1}{2}\nu}} P_{\nu}^{\mu}(z) \quad [\operatorname{Re}(\nu + \mu) < 0].$$

EH II 57(8)

$$5. \int_0^{\infty} e^{-tz(z^2-1)^{-\frac{1}{2}}} I_{\mu}(t) t^{\nu} dt = \frac{\Gamma(\nu + \mu + 1)}{(z^2 - 1)^{-\frac{1}{2}(\nu+1)}} P_{\nu}^{-\mu}(z) \quad [\operatorname{Re}(\nu + \mu) > -1].$$

EH II 57(9)

$$6. \int_0^{\infty} e^{-t \cos \theta} J_{\mu}(t \sin \theta) t^{\nu} dt = \Gamma(\nu + \mu + 1) P_{\nu}^{-\mu}(\cos \theta) \quad \left[\operatorname{Re}(\nu + \mu) > -1, \quad 0 \leq \theta < \frac{1}{2}\pi \right].$$

EH II 57(10)

$$7. \int_0^{\infty} \frac{J_{\nu}(bx) x^{\nu}}{e^{\pi x} - 1} dx = \frac{(2b)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{2\nu+1}} \quad [\operatorname{Re} \nu > 0, \quad |\operatorname{Im} b| < \pi].$$

$$1. \int_0^1 x^{\lambda-\nu-1} (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{2^{-\nu} \alpha^\nu \Gamma(\lambda) \Gamma(\mu)}{\Gamma(\lambda+\mu) \Gamma(\nu+1)} {}_2F_2 \left(\lambda, \nu + \frac{1}{2}; \lambda + \mu, 2\nu + 1; \pm 2i\alpha \right) \\ [\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0].$$

ET II 194(58)a

$$2. \int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm i\alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma(\mu) \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1 \left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2i\alpha \right) \\ \left[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 194(57)a

$$3. \int_0^1 x^\nu (1-x)^{\mu-1} e^{\pm \alpha x} J_\nu(\alpha x) dx = \frac{(2\alpha)^\nu \Gamma\left(\nu + \frac{1}{2}\right) \Gamma(\mu)}{\sqrt{\pi} \Gamma(\mu + 2\nu + 1)} {}_1F_1 \left(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2\alpha \right) \\ \left[\operatorname{Re} \mu > 0, \operatorname{Re} \nu > -\frac{1}{2} \right].$$

BU 9(16a), ET II 197(77)a

$$4. \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} e^{\pm \alpha x} I_\nu(\alpha x) dx = \frac{\left(\frac{1}{2}\alpha\right)^\nu \Gamma(\lambda + \nu) \Gamma(\mu)}{\Gamma(\nu + 1) \Gamma(\lambda + \mu + \nu)} \times \\ \times {}_2F_2 \left(\nu + \frac{1}{2}, \lambda + \nu; 2\nu + 1, \mu + \lambda + \nu; \pm 2\alpha \right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re}(\lambda + \nu) > 0].$$

ET II 197(78)a

$$5. \int_0^1 x^{\mu-\{ (1-x)^{2\{-1} I_{\mu-\{ \left(\frac{1}{2}xz\right)} e^{-\frac{1}{2}xz} dx = \frac{\Gamma(2\{)}{\sqrt{\pi} \Gamma(1+2\mu)} e^{\frac{x}{2}} z^{-\{-\frac{1}{2}} M_{\{,u}(z) \\ \left[\operatorname{Re} \left(\left\{ -\frac{1}{2} - \mu \right\} < 0, \operatorname{Re} \{ > 0 \right] \right.$$

BU 129(14a)

$$7. \int_1^{\infty} x^{-\lambda}(x-1)^{\mu-1} e^{-\alpha x} K_{\nu}(\alpha x) dx = \Gamma(\mu) \sqrt{\pi} (2\alpha)^{\lambda} G_{23}^{30} \left(2\alpha \left| \begin{array}{c} 0, \frac{1}{2} - \lambda \\ -\mu, \nu - \lambda, -\nu - \lambda \end{array} \right. \right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \alpha > 0].$$

$$8. \int_1^{\infty} x^{-\nu}(x-1)^{\mu-1} e^{-\alpha x} I_{\nu}(\alpha x) dx = \frac{(2\alpha)^{\nu-\mu} \Gamma\left(\frac{1}{2} - \mu + \nu\right) \Gamma(\mu)}{\sqrt{\pi} \Gamma(1 - \mu + 2\nu)} \times \\ \times {}_1F_1\left(\frac{1}{2} - \mu + \nu; 1 - \mu + 2\nu; -2\alpha\right) \\ \left[0 < \operatorname{Re} \mu < \frac{1}{2} + \operatorname{Re} \nu, \operatorname{Re} \alpha > 0\right].$$

$$9. \int_1^{\infty} x^{-\nu}(x-1)^{\mu-1} e^{-\alpha x} K_{\nu}(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}\mu - \frac{1}{2}} e^{-\alpha} W_{-\frac{1}{2}\mu, \nu - \frac{1}{2}\mu}(2\alpha) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \alpha > 0].$$

736

$$10. \int_1^{\infty} x^{-\mu - \frac{1}{2}(x-1)^{\mu-1}} e^{-\alpha x} K_{\nu}(\alpha x) dx = \sqrt{\pi} \Gamma(\mu) (2\alpha)^{-\frac{1}{2}} e^{-\alpha} W_{-\mu, \nu}(2\alpha) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} \alpha > 0].$$

$$11.^3 \int_{-1}^1 (1-x^2)^{-1/2} x e^{-\alpha x} I_1(b\sqrt{1-x^2}) dx = \frac{2}{b} \left\{ \operatorname{sh} a - a(a^2 + b^2)^{-1/2} \operatorname{sh} \sqrt{a^2 + b^2} \right\} \\ [a > 0, b > 0].$$

6.626

$$1. \int_0^{\infty} x^{\lambda-1} e^{-\alpha x} J_{\mu}(\beta x) J_{\nu}(\gamma x) dx = \frac{\beta^{\mu} \gamma^{\nu}}{\Gamma(\nu+1)} 2^{-\nu-\mu} \alpha^{-\lambda-\mu-\nu} \sum_{m=0}^{\infty} \frac{\Gamma(\lambda + \mu + \nu + 2m)}{m! \Gamma(\mu + m + 1)} \times \\ \times F\left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2}\right) \left(-\frac{\beta^2}{4\alpha^2}\right)^m \\ [\operatorname{Re}(\lambda + \mu + \nu) > 0, \operatorname{Re}(\alpha \pm i\beta \pm i\gamma) > 0].$$

$$2. \int_0^{\infty} e^{-2\alpha x} J_{\nu}(\beta x) J_{\mu}(\beta x) x^{\nu+\mu} dx = \frac{\Gamma\left(\nu + \mu + \frac{1}{2}\right) \beta^{\nu+\mu}}{\sqrt{\pi^3}} \times \\ \times \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu+\mu} \varphi \cos(\nu - \mu) \varphi}{(\alpha^2 + \beta^2 \cos^2 \varphi)^{\nu+\mu} \sqrt{\alpha^2 + \beta^2 \cos^2 \varphi}} d\varphi \\ \left[\operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re}(\nu + \mu) > -\frac{1}{2} \right].$$

WA 427(1)

$$3. \int_0^{\infty} e^{-2\alpha x} J_0(\beta x) J_1(\beta x) x dx = \frac{\mathbf{K}\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right) - \mathbf{E}\left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right)}{2\pi\beta\sqrt{\alpha^2 + \beta^2}}.6$$

WA 427(2)

$$4. \int_0^{\infty} e^{-2\alpha x} I_0(\beta x) I_1(\beta x) x dx = \frac{1}{2\pi\beta} \left\{ \frac{\alpha}{\alpha^2 - \beta^2} \mathbf{E}\left(\frac{\beta}{\alpha}\right) - \frac{1}{\alpha} \mathbf{K}\left(\frac{\beta}{\alpha}\right) \right\} \quad [\operatorname{Re} \alpha > \operatorname{Re} \beta].$$

WA 428(5)

6.627

$$\int_0^{\infty} \frac{x^{-1/2}}{x+a} e^{-x} K_{\nu}(x) dx = \frac{\pi e^a K_{\nu}(a)}{\sqrt{a} \cos(\nu\pi)} \quad \left[|\arg a| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 368(29)

6.628

$$1. \int_0^{\infty} e^{-x \cos \beta} J_{-\nu}(x \sin \beta) x^{\mu} dx = \Gamma(\mu - \nu + 1) \mathbf{P}_{\mu}^{\nu}(\cos \beta) \quad \left[0 < \beta < \frac{\pi}{2}, \quad \operatorname{Re}(\mu - \nu) > -1 \right].$$

WA 424(3), WH

737

$$2. \int_0^{\infty} e^{-x \cos \beta} N_{\nu}(x \sin \beta) x^{\mu} dx = -\frac{\sin \mu \pi}{\sin(\mu + \nu) \pi} \frac{\Gamma(\mu - \nu + 1)}{\pi} \times \\ \times \left[Q_{\mu}^{\nu}(\cos \beta + 0 \cdot i) e^{\frac{1}{2} \nu \pi i} + Q_{\mu}^{\nu}(\cos \beta - 0 \cdot i) e^{-\frac{1}{2} \nu \pi i} \right] \\ \left[\operatorname{Re}(\mu + \nu) > -1, \quad 0 < \beta < \frac{\pi}{2} \right].$$

WA 424(4)

$$3. \int_0^1 e^{\frac{x\mu}{2}} (1-x)^{2\nu-1} x^{\mu-\nu} J_{\mu-\nu} \left(\frac{ixu}{2} \right) dx = 2^{2(\nu-\mu)} e^{\frac{\pi}{2}(\mu-\nu)i} \frac{\mathbf{B}(2\nu, 2\mu - 2\nu + 1)}{\Gamma(\mu - \nu + 1)} \frac{e^{\frac{\pi}{2}}}{u^{\nu+\frac{1}{2}}} M_{\nu, \mu}(u).$$

$$4. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} I_{\nu}(x \operatorname{sh} \alpha) x^{\mu} dx = \Gamma(\nu + \mu + 1) P_{\mu}^{-\nu}(\operatorname{ch} \alpha) \quad [\operatorname{Re} \mu > -2].$$

WA 423(1)

$$5. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} K_{\nu}(x \operatorname{sh} \alpha) x^{\mu} dx = \frac{\sin \mu \pi}{\sin(\nu + \mu) \pi} \Gamma(\mu - \nu + 1) Q_{\mu}^{\nu}(\operatorname{ch} \alpha) \quad [\operatorname{Re}(\mu + 1) > |\operatorname{Re} \nu|].$$

WA 423(2)

$$6. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} I_{\nu}(x) x^{\mu-1} dx = \frac{\frac{\cos \nu \pi}{\sin(\mu + \nu) \pi} Q^{\mu} - \frac{1}{2} \nu - \frac{1}{2}(\operatorname{ch} \alpha)}{\sqrt{\frac{\pi}{2}} (\operatorname{sh} \alpha)^{\mu - \frac{1}{2}}} \quad [\operatorname{Re}(\mu + \nu) > 0, \quad \operatorname{Re}(\operatorname{ch} \alpha) > 1].$$

WA 424(6)

$$7. \int_0^{\infty} e^{-x \operatorname{ch} \alpha} K_{\nu}(x) x^{\mu-1} dx = \sqrt{\frac{\pi}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu) \frac{P_{\nu - \frac{1}{2}}^{\frac{1}{2} - \mu}(\operatorname{ch} \alpha)}{(\operatorname{sh} \alpha)^{\mu - \frac{1}{2}}} \\ [\operatorname{Re} \mu > |\operatorname{Re} \nu|, \quad \operatorname{Re}(\operatorname{ch} \alpha) > -1].$$

WA 424(7)

6.629

$$\int_0^{\infty} x^{-1/2} e^{-x \alpha \cos \varphi \cos \psi} J_{\mu}(\alpha x \sin \varphi) J_{\nu}(\alpha x \sin \psi) dx = \Gamma\left(\mu + \nu + \frac{1}{2}\right) \alpha^{-\frac{1}{2}} P_{\nu - \frac{1}{2}}^{-\mu}(\cos \varphi) P_{\mu - \frac{1}{2}}^{-\nu}(\cos \psi) \\ \left[\alpha > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{1}{2} \right].$$

ET II 50(19)

6.631

$$1. \int_0^{\infty} x^{\mu} e^{-\alpha x^2} J_{\nu}(\beta x) dx = \frac{\beta^{\nu} \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + \frac{1}{2}\right)}{2^{\nu+1} \alpha^{\frac{1}{2}(\mu + \nu + 1)} \Gamma(\nu + 1)} {}_1F_1\left(\frac{\nu + \mu + 1}{2}; \nu + 1; -\frac{\beta^2}{4\alpha}\right); \\ = \frac{\Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + \frac{1}{2}\right)}{\beta \alpha^{\frac{1}{2} \mu} \Gamma(\nu + 1)} \exp\left(-\frac{\beta^2}{8\alpha}\right) M_{\frac{1}{2} \mu, \frac{1}{2} \nu}\left(\frac{\beta^2}{4\alpha}\right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re}(\mu + \nu) > -1].$$

$$2. \int_0^\infty x^\mu e^{-\alpha x^2} N_\nu(\beta x) dx = -\alpha^{-\frac{1}{2}\mu} \beta^{-1} \sec\left(\frac{\nu-\mu}{2}\pi\right) \exp\left(-\frac{\beta^2}{8\alpha}\right) \times$$

$$\times \left\{ \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(1+\nu)} \sin\left(\frac{\nu-\mu}{2}\pi\right) M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) + W_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right) \right\}$$

[Re $\alpha > 0$, Re $\mu > |\text{Re } \nu| - 1$, $\beta > 0$].

ET II 106(4)

$$3. \int_0^\infty x^\mu e^{-\alpha x^2} K_\nu(\beta x) dx = \frac{1}{2} \alpha^{-\frac{1}{2}\mu} \beta^{-1} \Gamma\left(\frac{1+\nu+\mu}{2}\right) \Gamma\left(\frac{1-\nu+\mu}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{-\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{\beta^2}{4\alpha}\right)$$

[Re $\mu > |\text{Re } \nu| - 1$].

ET II 132(25)

$$4. \int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left(-\frac{\beta^2}{4\alpha}\right) \quad [\text{Re } \alpha > 0, \text{ Re } \nu > -1].$$

WA 43(4), ET II 29(10)

$$5. \int_0^\infty x^{\nu-1} e^{-\alpha x^2} J_\nu(\beta x) dx = 2^{\nu-1} \beta^{-\nu} \gamma\left(\nu, \frac{\beta^2}{4\alpha}\right) \quad [\text{Re } \alpha > 0, \text{ Re } \nu > 0].$$

ET II 30(11)

$$6. \int_0^\infty x^{\nu+1} e^{\pm i\alpha x^2} J_\nu(\beta x) dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} \exp\left[\pm i\left(\frac{\nu+1}{2}\pi - \frac{\beta^2}{4\alpha}\right)\right]$$

$$\left[\alpha > 0, \quad -1 < \text{Re } \nu < \frac{1}{2}, \quad \beta > 0 \right].$$

ET II 30(12)

$$7. \int_0^\infty x e^{-\alpha x^2} J_\nu(\beta x) dx = \frac{\sqrt{\pi}\beta}{8\alpha^{\frac{3}{2}}} \exp\left(-\frac{\beta^2}{8\alpha}\right) \left[I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) - I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{\beta^2}{8\alpha}\right) \right]$$

[Re $\alpha > 0$, Re $\nu > -2$].

$$8. \int_0^1 x^{n+1} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[e^\alpha - e^{-\alpha} \sum_{r=-n}^n I_r(2\alpha) \right] \quad [n = 0, 1, \dots].$$

ET II 365(8)a

$$9. \int_1^\infty x^{1-n} e^{-\alpha x^2} I_n(2\alpha x) dx = \frac{1}{4\alpha} \left[e^\alpha - e^{-\alpha} \sum_{r=1-n}^{n-1} I_r(2\alpha) \right] \quad [n = 1, 2, \dots].$$

ET II 367(20)a

$$10. \int_0^\infty e^{-x^2} x^{2n+\mu+1} J_\mu(2x\sqrt{z}) dx = \frac{n!}{2} e^{-z} z^{\frac{1}{2}\mu} L_n^\mu(z) \quad [n = 0, 1, \dots; \quad n + \operatorname{Re} \mu > -1].$$

BU 135(5)

6.632

$$\int_0^\infty x^{-\frac{1}{2}} \exp[-(x^2 + a^2 - 2ax \cos \varphi)^{\frac{1}{2}}] [x^2 + a^2 - 2ax \cos \varphi]^{-\frac{1}{2}} K_\nu(x) dx = \pi a^{-\frac{1}{2}} \sec(\nu\pi) P_{\nu-\frac{1}{2}}(-\cos \varphi) K_\nu(a) \quad \left[|\arg a| + |\operatorname{Re} \varphi| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 368(32)

739

6.633

$$1. \int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu \alpha^{-\frac{\mu+\nu+\lambda+2}{2}}}{2^{\nu+\mu+1} \Gamma(\nu+1)} \sum_{m=0}^\infty \frac{\Gamma\left(m + \frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2}\lambda + 1\right)}{m! \Gamma(m + \mu + 1)} \left(-\frac{\beta^2}{4\alpha}\right)^m \times F\left(-m, -\mu - m; \nu + 1; \frac{\gamma^2}{\beta^2}\right) \quad [\operatorname{Re} \alpha > 0, \operatorname{Re}(\mu + \nu + \lambda) > -2, \beta > 0, \gamma > 0].$$

EH II 49(20)A, ET II 51(24)a

$$2. \int_0^\infty e^{-\varrho^2 x^2} J_p(\alpha x) J_p(\beta x) x dx = \frac{1}{2\varrho^2} \exp\left(-\frac{\alpha^2 + \beta^2}{4\varrho^2}\right) I_p\left(\frac{\alpha\beta}{2\varrho^2}\right) \quad \left[\operatorname{Re} p > -1, \quad |\arg \varrho| < \frac{\pi}{4}, \quad \alpha > 0, \quad \beta > 0\right].$$

KU 146(16)A, WA 433(1)

$$4. \int_0^{\infty} x e^{-\alpha x^2} I_{\nu}(\beta x) J_{\nu}(\gamma x) dx = \frac{1}{2\alpha} \exp\left(\frac{\beta^2 - \gamma^2}{4\alpha}\right) J_{\nu}\left(\frac{\beta\gamma}{2\alpha}\right) \quad [\operatorname{Re} \alpha > 0, \operatorname{Re} \nu > -1].$$

ET II 63(1)

$$5. \int_0^{\infty} x^{\lambda-1} e^{-\alpha x^2} J_{\mu}(\beta x) J_{\nu}(\beta x) dx = \\ = 2^{-\nu-\mu-1} \alpha^{-\frac{1}{2}(\nu+\lambda+\mu)} \beta^{\nu+\mu} \frac{\Gamma\left(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu\right)}{\Gamma(\mu+1)\Gamma(\nu+1)} \times \\ \times {}_3F_3\left[\frac{\nu}{2} + \frac{\mu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{\mu}{2} + 1, \frac{\nu+\mu+\lambda}{2}; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right] \\ [\operatorname{Re}(\nu+\lambda+\mu) > 0, \operatorname{Re} \alpha > 0].$$

WA 434, EH II 50(21)

6.634

$$\int_0^{\infty} x e^{-\frac{x^2}{2a}} [I_{\nu}(x) + I_{-\nu}(x)] K_{\nu}(x) dx = a e^a K_{\nu}(a) \quad [\operatorname{Re} a > 0, -1 < \operatorname{Re} \nu < 1].$$

ET II 371(49)

6.635

$$1. \int_0^{\infty} x^{-1} e^{-\frac{\alpha}{x}} J_{\nu}(\beta x) dx = 2J_{\nu}\left(\sqrt{2\alpha\beta}\right) K_{\nu}\left(\sqrt{2\alpha\beta}\right) \quad [\operatorname{Re} \alpha > 0, \beta > 0].$$

ET II 30(15)

$$2. \int_0^{\infty} x^{-1} e^{-\frac{\alpha}{x}} N_{\nu}(\beta x) dx = 2N_{\nu}\left(\sqrt{2\alpha\beta}\right) K_{\nu}\left(\sqrt{2\alpha\beta}\right) \quad [\operatorname{Re} \alpha > 0, \beta > 0].$$

ET II 106(5)

740

$$3. \int_0^{\infty} x^{-1} e^{-\frac{\alpha}{x} - \beta x} J_{\nu}(\gamma x) dx = 2J_{\nu}\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2 + \gamma^2} - \beta\right]^{\frac{1}{2}}\right\} K_{\nu}\left\{\sqrt{2\alpha}\left[\sqrt{\beta^2 + \gamma^2} + \beta\right]^{\frac{1}{2}}\right\} \\ [\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0].$$

ET II 30(16)

6.636

$$\int_0^{\infty} x^{-\frac{1}{2}} e^{-\alpha\sqrt{x}} J_{\nu}(\beta x) dx = \frac{\sqrt{2}}{\sqrt{\pi}\beta} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}}\alpha e^{\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}) D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}}\alpha e^{-\frac{1}{4}\pi i}\beta^{-\frac{1}{2}}) \\ \left[\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 30(17)

6.637

$$1. \int_0^{\infty} (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] J_{\nu}(\gamma x) dx = I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} \times \\ \times K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} \\ \left[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -1 \right].$$

ET II 31(20)

$$2. \int_0^{\infty} (\beta^2 + x^2)^{-\frac{1}{2}} \exp\left[-\alpha(\beta^2 + x^2)^{\frac{1}{2}}\right] N_{\nu}(\gamma x) dx = \\ = -\sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} \times \\ \times \left(\frac{1}{\pi} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} + \alpha \right] \right\} + \sin\left(\frac{\nu\pi}{2}\right) I_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[(\alpha^2 + \gamma^2)^{\frac{1}{2}} - \alpha \right] \right\} \right) \\ \left[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \gamma > 0, \quad |\operatorname{Re} \nu| < 1 \right].$$

ET II 106(6)

$$3. \int_0^{\infty} (x^2 + \beta^2)^{-\frac{1}{2}} \exp\left[-\alpha(x^2 + \beta^2)^{\frac{1}{2}}\right] K_{\nu}(\gamma x) dx = \\ = \frac{1}{2} \sec\left(\frac{\nu\pi}{2}\right) K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[\alpha + (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right\} K_{\frac{1}{2}\nu} \left\{ \frac{1}{2}\beta \left[\alpha - (\alpha^2 - \gamma^2)^{\frac{1}{2}} \right] \right\} \\ \left[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\gamma + \beta) > 0, \quad |\operatorname{Re} \nu| < 1 \right].$$

ET II 132(26)

6.64 Combinations of Bessel functions of more complicated arguments, exponentials, and titles

6.641

$$\int_0^{\infty} \sqrt{x} e^{-\alpha x} J_{\pm\frac{1}{4}}(x^2) dx = \frac{\sqrt{\pi\alpha}}{4} \left[\mathbf{H}_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) - N_{\mp\frac{1}{4}}\left(\frac{\alpha^2}{4}\right) \right].$$

6.642

$$1. \int_0^{\infty} x^{-1} e^{-\alpha x} N_{\nu} \left(\frac{2}{x} \right) dx = N_{\nu}(\sqrt{\alpha}) K_{\nu}(\sqrt{\alpha}).$$

MI 44

$$2. \int_0^{\infty} x^{-1} e^{-\alpha x} H_{\nu}^{(1,2)} \left(\frac{2}{x} \right) dx = H_{\nu}^{(1,2)}(\sqrt{\alpha}) K_{\nu}(\sqrt{\alpha}).$$

MI 44, EH II 91(26)

741

6.643

$$1. \int_0^{\infty} x^{\mu-\frac{1}{2}} e^{-\alpha x} J_{2\nu}(2\beta\sqrt{x}) dx = \frac{\Gamma\left(\mu + \nu + \frac{1}{2}\right)}{\beta\Gamma(2\nu + 1)} e^{-\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \left[\operatorname{Re}\left(\mu + \nu + \frac{1}{2}\right) > 0 \right],$$

(cf. **6.631** 1.).

6.631
BU 14(13a), MI 42a

$$2. \int_0^{\infty} x^{\mu-\frac{1}{2}} e^{-\alpha x} I_{2\nu}(2\beta\sqrt{x}) dx = \frac{\Gamma\left(\mu + \nu + \frac{1}{2}\right)}{\Gamma(2\nu + 1)} \beta^{-1} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} M_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \left[\operatorname{Re}\left(\mu + \nu + \frac{1}{2}\right) > 0 \right].$$

MI 45

$$3. \int_0^{\infty} x^{\mu-\frac{1}{2}} e^{-\alpha x} K_{2\nu}(2\beta\sqrt{x}) dx = \frac{\Gamma\left(\mu + \nu + \frac{1}{2}\right) \Gamma\left(\mu - \nu + \frac{1}{2}\right)}{2\beta} e^{\frac{\beta^2}{2\alpha}} \alpha^{-\mu} W_{-\mu,\nu}\left(\frac{\beta^2}{\alpha}\right) \left[\operatorname{Re}\left(\mu + \nu + \frac{1}{2}\right) > 0 \right], \quad (\text{cf. } \mathbf{6.631} \text{ 3.}).$$

6.631
MI 47a

$$5. \int_0^{\infty} x^{-\frac{1}{2}} e^{-\alpha x} N_{2\nu}(\beta\sqrt{x}) dx = -\sqrt{\frac{\pi}{\alpha}} \frac{\exp\left(-\frac{\beta^2}{8\alpha}\right)}{\cos(\nu\pi)} \left[\sin(\nu\pi) I_{\nu}\left(\frac{\beta^2}{8\alpha}\right) + \frac{1}{\pi} K_{\nu}\left(\frac{\beta^2}{8\alpha}\right) \right] \\ \left[|\operatorname{Re} \nu| < \frac{1}{2} \right].$$

MI 44

$$6. \int_0^{\infty} x^{\frac{1}{2}m} e^{-\alpha x} K_m(2\sqrt{x}) dx = \frac{\Gamma(m+1)}{2\alpha} \left(\frac{1}{\alpha}\right)^{\frac{1}{2}m-\frac{1}{2}} e^{\frac{1}{2\alpha}} W_{-\frac{1}{2}(m+1), -\frac{1}{2}m}\left(\frac{1}{\alpha}\right).$$

MI 48a

6.644

$$\int_0^{\infty} e^{-\beta x} J_{2\nu}(2a\sqrt{x}) J_{\nu}(bx) dx = \exp\left(-\frac{a^2\beta}{\beta^2+b^2}\right) J_{\nu}\left(\frac{a^2b}{\beta^2+b^2}\right) \frac{1}{\sqrt{\beta^2+b^2}} \\ \left[\operatorname{Re} \beta > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 58(17)

6.645

$$1. \int_1^{\infty} (x^2-1)^{-\frac{1}{2}\nu} e^{-\alpha x} J_{\nu}(\beta\sqrt{x^2-1}) dx = I_{\frac{1}{2}\nu} \left[\frac{1}{2} (\sqrt{\alpha^2+\beta^2} - \alpha) \right] K_{\frac{1}{2}\nu} \left[\frac{1}{2} (\sqrt{\alpha^2+\beta^2} + \alpha) \right].$$

MO 179a

742

$$2. \int_1^{\infty} (x^2-1)^{\frac{1}{2}\nu} e^{-\alpha x} J_{\nu}(\beta\sqrt{x^2-1}) dx = \sqrt{\frac{2}{\pi}} \beta^{\nu} (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}} K_{\nu+\frac{1}{2}}(\sqrt{\alpha^2 + \beta^2}).$$

MO 179a

$$3. \int_{-1}^1 (1-x^2)^{-1/2} e^{-ax} I_1(b\sqrt{1-x^2}) dx = \frac{2}{b} \left\{ \operatorname{ch} \sqrt{a^2+b^2} - \operatorname{ch} a \right\} \quad [a > 0, \quad b > 0].$$

6.646

$$2. \int_1^{\infty} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}\nu} e^{-\alpha x} I_{\nu}(\beta\sqrt{x^2-1}) dx = \frac{\exp(-\sqrt{\alpha^2-\beta^2})}{\sqrt{\alpha^2-\beta^2}} \left(\frac{\beta}{\alpha+\sqrt{\alpha^2-\beta^2}}\right)^{\nu}$$

[Re $\nu > -1$, $\alpha > \beta$].

MO 180

$$3.7 \int_b^{\infty} e^{-pt} \left(\frac{t-b}{t+b}\right)^{\nu/2} K_{\nu}[a(t^2-b^2)^{1/2}] dt = \frac{\Gamma(\nu+1)}{2sa^{\nu}} [x^{\nu} e^{-bs} \Gamma(-\nu, bx) - y^{\nu} e^{bs} \Gamma(-\nu, by)]$$

where $x = p-s$, $y = p+s$, $s = (p^2 - a^2)^{1/2}$ [Re $(p+a) > 0$, |Re (ν) | < 1].

ME 39a

6.647

$$1. \int_0^{\infty} x^{-\lambda-\frac{1}{2}} (\beta+x)^{\lambda-\frac{1}{2}} e^{-\alpha x} K_{2\mu}[\sqrt{x(\beta+x)}] dx = \frac{1}{\beta} e^{\frac{1}{2}\alpha\beta} \Gamma\left(\frac{1}{2}-\lambda+\mu\right) \Gamma\left(\frac{1}{2}-\lambda-\mu\right) W_{\lambda,\mu}(z_1) \times$$

$\times W_{\lambda,\mu}(z_2)$,

$$z_1 = \frac{1}{2}\beta(\alpha + \sqrt{\alpha^2 - 1}),$$

$$z_2 = \frac{1}{2}\beta(\alpha - \sqrt{\alpha^2 - 1}),$$

$$\left[|\arg \beta| < \pi, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \lambda + |\operatorname{Re} \mu| < \frac{1}{2}\right].$$

ET II 377(37)

$$2. \int_0^{\infty} (\alpha+x)^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-x \operatorname{ch} t} K_{\nu}[\sqrt{x(\alpha+x)}] dx = \frac{1}{2} \sec\left(\frac{\nu\pi}{2}\right) e^{\frac{1}{2}\alpha \operatorname{ch} t} K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^t\right) K_{\frac{1}{2}\nu}\left(\frac{1}{4}\alpha e^{-t}\right)$$

[$-1 < \operatorname{Re} \nu < 1$].

ET II 377(36)

743

$$3.7 \int_0^{\alpha} x^{\lambda-\frac{1}{2}} (\alpha-x)^{-\lambda-\frac{1}{2}} e^{-x \operatorname{sh} t} I_{2\mu}[\sqrt{x(\alpha-x)}] dx =$$

$$= e^{-(\alpha/2) \operatorname{sh} t} \frac{2\Gamma\left(\frac{1}{2}+\lambda+\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\mu\right)}{\alpha[\Gamma(2\mu+1)]^2} M_{\lambda,\mu}\left(\frac{1}{2}\alpha e^t\right) M_{-\lambda,\mu}\left(\frac{1}{2}\alpha e^{-t}\right)$$

[Re $\mu > |\operatorname{Re} \lambda| - \frac{1}{2}$].

$$\int_{-\infty}^{\infty} e^{\alpha x} \left(\frac{\alpha + \beta e^x}{\alpha e^x + \beta} \right)^{\nu} K_{2\nu} \left[(\alpha^2 + \beta^2 + 2\alpha\beta \operatorname{ch} x)^{\frac{1}{2}} \right] dx = 2K_{\nu+\varrho}(\alpha)K_{\nu-\varrho}(\beta)$$

[Re $\alpha > 0$, Re $\beta > 0$].

ET II 379(45)

6.649

$$1. \int_0^{\infty} K_{\mu-\nu}(2z \operatorname{sh} x) e^{(\nu+\mu)x} dx = \frac{\pi^2}{4 \sin[(\nu-\mu)\pi]} [J_{\nu}(z)N_{\mu}(z) - J_{\mu}(z)N_{\nu}(z)]$$

[Re $z > 0$, $-1 < \operatorname{Re}(\nu-\mu) < 1$].

MO 44

$$2. \int_0^{\infty} J_{\nu+\mu}(2x \operatorname{sh} t) e^{(\nu-\mu)t} dt = K_{\nu}(x)I_{\mu}(x) \quad \left[\operatorname{Re}(\nu-\mu) < \frac{3}{2}, \operatorname{Re}(\nu+\mu) > -1, x > 0 \right].$$

EH II 97(68)

$$3. \int_0^{\infty} N_{\nu-\mu}(2x \operatorname{sh} t) e^{-(\nu+\mu)t} dt = \frac{1}{\sin[\pi(\mu-\nu)]} \{ I_{\mu}(x)K_{\nu}(x) - \cos[(\nu-\mu)\pi]I_{\nu}(x)K_{\mu}(x) \}$$

[$|\operatorname{Re}(\nu-\mu)| < 1$, $\operatorname{Re}(\nu+\mu) > -\frac{1}{2}$, $x > 0$].

EH II 97(73)

$$4. \int_0^{\infty} K_0(2z \operatorname{sh} x) e^{-2\nu x} dx = -\frac{\pi}{4} \left\{ J_{\nu}(z) \frac{\partial N_{\nu}(z)}{\partial \nu} - N_{\nu}(z) \frac{\partial J_{\nu}(z)}{\partial \nu} \right\}.$$

6.65 Combinations of Bessel and exponential functions of more complicated arguments and powers

6.651

$$1. \int_0^{\infty} x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} I_{\mu} \left(\frac{1}{4}\alpha^2 x^2 \right) J_{\nu}(\beta x) dx = \frac{1}{\sqrt{2\pi}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{21} \left(\frac{\beta^2}{2\alpha^2} \left| \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right. \right),$$

$$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu,$$

$$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu,$$

[$|\arg \alpha| < \frac{\pi}{4}$, $\beta > 0$, $-\frac{3}{2} - \operatorname{Re}(2\mu + \nu) < \operatorname{Re} \lambda < 0$].

ET II 68(8)

$$2. \int_0^{\infty} x^{\lambda+\frac{1}{2}} e^{-\frac{1}{4}\alpha^2 x^2} K_{\mu}\left(\frac{1}{4}\alpha^2 x^2\right) J_{\nu}(\beta x) dx = \sqrt{\frac{\pi}{2}} 2^{\lambda+1} \beta^{-\lambda-\frac{3}{2}} G_{23}^{12} \left(\frac{\beta^2}{2\alpha^2} \left| \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right. \right),$$

$$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu,$$

$$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu,$$

$$\left[|\arg \alpha| < \frac{\pi}{4}, \quad \operatorname{Re}(\lambda + \nu \pm 2\mu) > -\frac{3}{2} \right].$$

ET II 69(15)

$$3. \int_0^{\infty} x^{2\mu-\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\mu}\left(\frac{1}{4}\alpha^2 x^2\right) J_{\nu}(\beta x) dx =$$

$$= 2^{\mu-\nu+\frac{1}{2}} (\pi\alpha)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} + \mu\right) \frac{\beta^{\nu-2\mu-1}}{\Gamma\left(\frac{1}{2} - \mu + \nu\right)} {}_1F_1\left(\frac{1}{2} + \mu; \frac{1}{2} - \mu + \nu; -\frac{\beta^2}{2\alpha}\right)$$

$$\left[\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > 2 \operatorname{Re} \mu + \frac{1}{2} > -\frac{1}{2} \right].$$

ET II 68(6)

$$4. \int_0^{\infty} x^{2\mu+\nu+1} e^{-\frac{1}{4}\alpha^2 x^2} K_{\mu}\left(\frac{1}{4}\alpha^2 x^2\right) J_{\nu}(\beta x) dx =$$

$$= \sqrt{\pi} 2^{\mu} \alpha^{-2\mu-2\nu-2} \beta^{\nu} \frac{\Gamma(1+2\mu+\nu)}{\Gamma\left(\mu+\nu+\frac{3}{2}\right)} {}_1F_1\left(1+2\mu+\nu; \mu+\nu+\frac{3}{2}; -\frac{\beta^2}{2\alpha^2}\right)$$

$$\left[|\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(2\mu+\nu) > -1, \quad \beta > 0 \right].$$

ET II 69(13)

$$5. \int_0^{\infty} x^{2\mu+\nu+1} e^{-\frac{1}{2}\alpha^2 x^2} I_{\mu}\left(\frac{1}{2}\alpha^2 x^2\right) K_{\nu}(\beta x) dx =$$

$$= \frac{2^{\mu-\frac{1}{2}}}{\sqrt{\pi}} \beta^{-\mu-\frac{3}{2}} \alpha^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} \Gamma(2\mu+\nu+1) \Gamma\left(\mu+\frac{1}{2}\right) \exp\left(\frac{\beta^2}{8\alpha}\right) W_{k,m}\left(\frac{\beta^2}{4\alpha}\right),$$

$$2k = -3\mu - \nu - \frac{1}{2},$$

$$2m = \mu + \nu + \frac{1}{2}$$

$$\left[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -1 \right].$$

745

$$6. \int_0^{\infty} x e^{-\frac{1}{4}\alpha x^2} J_{\frac{1}{2}\nu} \left(\frac{1}{4}\beta x^2 \right) J_{\nu}(\gamma x) dx = 2(\alpha^2 + \beta^2)^{-\frac{1}{2}} \exp \left(-\frac{\alpha\gamma^2}{\alpha^2 + \beta^2} \right) J_{\frac{1}{2}\nu} \left(\frac{\beta\gamma^2}{\alpha^2 + \beta^2} \right) \\ [\gamma > 0, \quad \operatorname{Re} \alpha > |\operatorname{Im} \beta|, \quad \operatorname{Re} \nu > -1].$$

ET II 56(2)

$$7. \int_0^{\infty} x e^{-\frac{1}{4}\alpha x^2} I_{\frac{1}{2}\nu} \left(\frac{1}{4}\alpha x^2 \right) J_{\nu}(\beta x) dx = \left(\frac{1}{2}\pi\alpha \right)^{-\frac{1}{2}} \beta^{-1} \exp \left(-\frac{\beta^2}{2\alpha} \right) \\ [\operatorname{Re} \alpha > 0, \quad \beta > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 67(3)

$$8. \int_0^{\infty} x^{1-\nu} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu} \left(\frac{1}{4}\alpha^2 x^2 \right) J_{\nu}(\beta x) dx = \sqrt{\frac{2}{\pi}} \frac{\beta^{\nu-1}}{\alpha} \exp \left(-\frac{\beta^2}{4\alpha^2} \right) D_{-2\nu} \left(\frac{\beta}{\alpha} \right) \\ \left[|\arg \alpha| < \frac{1}{4}\pi, \quad \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 67(1)

$$9. \int_0^{\infty} x^{-\nu-1} e^{-\frac{1}{4}\alpha^2 x^2} I_{\nu+1} \left(\frac{1}{4}\alpha^2 x^2 \right) J_{\nu}(\beta x) dx = \sqrt{\frac{2}{\pi}} \beta^{\nu} \exp \left(-\frac{\beta^2}{4\alpha^2} \right) D_{-2\nu-3} \left(\frac{\beta}{\alpha} \right) \\ \left[|\arg \alpha| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1, \quad \beta > 0 \right].$$

ET II 67(2)

6.652

$$\int_0^{\infty} x^{2\nu} e^{-\left(\frac{x^2}{8} + \alpha x\right)} I_{\nu} \left(\frac{x^2}{8} \right) dx = \frac{\Gamma(4\nu + 1)}{2^{4\nu}\Gamma(\nu + 1)} \frac{e^{\frac{\alpha^2}{2}}}{\alpha^{\nu+1}} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(\alpha^2) \quad \left[\operatorname{Re} \left(\nu + \frac{1}{4} \right) > 0 \right].$$

MI 45

6.653

$$1. \int_0^{\infty} \exp \left[-\frac{1}{2}x - \frac{1}{2x}(a^2 + b^2) \right] I_{\nu} \left(\frac{ab}{x} \right) \frac{dx}{x} = 2I_{\nu}(a)K_{\nu}(b) \quad [0 < a < b]; \\ = 2K_{\nu}(a)I_{\nu}(b) \quad [0 < b < a] \quad [\operatorname{Re} \nu > -1].$$

$$2. \int_0^{\infty} \exp \left[-\frac{1}{2}x - \frac{1}{2x}(z^2 + w^2) \right] K_{\nu} \left(\frac{zw}{x} \right) \frac{dx}{x} = 2K_{\nu}(z)K_{\nu}(w) \\ \left[|\arg z| < \pi, \quad |\arg w| < \pi, \quad |\arg(z+w)| < \frac{1}{4}\pi \right].$$

WA 483(1), EH II 53(36)

6.654

$$\int_0^{\infty} x^{-\frac{1}{2}} e^{-\frac{\beta^2}{8x} - \alpha x} K_{\nu} \left(\frac{\beta^2}{8x} \right) dx = \sqrt{4\pi} \alpha^{-\frac{1}{2}} K_{2\nu}(\beta\sqrt{\alpha}).$$

ME 39

6.655

$$\int_0^{\infty} x(\beta^2 + x^2)^{-\frac{1}{2}} \exp \left(-\frac{\alpha^2 \beta}{\beta^2 + x^2} \right) J_{\nu} \left(\frac{\alpha^2 x}{\beta^2 + x^2} \right) J_{\nu}(\gamma x) dx = \gamma^{-1} e^{-\beta\gamma} J_{2\nu}(2\alpha\sqrt{\gamma}) \\ \left[\operatorname{Re} \beta > 0, \quad \gamma > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 58(14)

6.656

$$1. \int_0^{\infty} e^{-(\xi-z) \operatorname{ch} t} J_{2\nu}[2(z\xi)^{\frac{1}{2}} \operatorname{sh} t] dt = I_{\nu}(z)K_{\nu}(\xi) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\xi - z) > 0 \right].$$

EH II 98(78)

746

$$2. \int_0^{\infty} e^{-(\xi+z) \operatorname{ch} t} K_{2\nu}[2(z\xi)^{\frac{1}{2}} \operatorname{sh} t] dt = \frac{1}{2} K_{\nu}(z)K_{\nu}(\xi) \sec(\nu\pi) \quad \left[|\operatorname{Re} \nu| < \frac{1}{2}, \quad \operatorname{Re}(z^{\frac{1}{2}} + \xi^{\frac{1}{2}})^2 \geq 0 \right].$$

EH II 98(79)

6.66 Combinations of Bessel, hyperbolic, and exponential functions

Bessel and hyperbolic functions

6.661

$$1. \int_0^{\infty} \operatorname{sh}(ax)K_{\nu}(bx) dx = \frac{\pi}{2} \frac{\operatorname{cosec} \left(\frac{\nu\pi}{2} \right) \sin \left[\nu \arcsin \left(\frac{a}{b} \right) \right]}{\sqrt{b^2 - a^2}} \quad [\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 2].$$

ET II 133(32)

$$2. \int_0^{\infty} \operatorname{ch}(ax)K_{\nu}(bx) dx = \frac{\pi \cos \left[\nu \arcsin \left(\frac{a}{b} \right) \right]}{2\sqrt{b^2 - a^2} \cos \left(\frac{\nu\pi}{2} \right)} \quad [\operatorname{Re} b > |\operatorname{Re} a|, \quad |\operatorname{Re} \nu| < 1].$$

6.662

$$1. \int_0^{\infty} \operatorname{ch}(\beta x)K_0(\alpha x)J_0(\gamma x) dx = \frac{\mathbf{K}(k)}{\sqrt{u+v}},$$

$$u = \frac{1}{2} \left\{ [(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2]^{\frac{1}{2}} + \alpha^2 - \beta^2 - \gamma^2 \right\},$$

$$v = \frac{1}{2} \left\{ [(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2 \right\},$$

$$k^2 = v(u+v)^{-1} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0].$$

$$2. \int_0^{\infty} \operatorname{sh}(\beta x)K_1(\alpha x)J_0(\gamma x) dx = a^{-1} \left[u\mathbf{E}(k) - \mathbf{K}(k)E(u) + \frac{\mathbf{K}(k) \operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} \right],$$

$$\operatorname{cn}^2 u = 2\gamma^2 \left\{ [(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2]^{\frac{1}{2}} - \alpha^2 + \beta^2 + \gamma^2 \right\}^{-1},$$

$$k^2 = \frac{1}{2} \left\{ 1 - (\alpha^2 - \beta^2 - \gamma^2)[(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2]^{-\frac{1}{2}} \right\} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0].$$

747

6.663

$$1. \int_0^{\infty} K_{\nu \pm \mu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu \mp \nu)t] dt = \frac{1}{2} K_{\mu}(z)K_{\nu}(z) \quad [\operatorname{Re} z > 0].$$

$$2. \int_0^{\infty} N_{\mu+\nu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu-\nu)t] dt = \frac{\pi}{4} [J_{\mu}(z)J_{\nu}(z) - N_{\mu}(z)N_{\nu}(z)] \quad [z > 0].$$

$$3. \int_0^{\infty} J_{\mu+\nu}(2z \operatorname{ch} t) \operatorname{ch}[(\mu-\nu)t] dt = -\frac{\pi}{4} [J_{\mu}(z)N_{\nu}(z) + J_{\nu}(z)N_{\mu}(z)] \quad [z > 0].$$

$$4. \int_0^{\infty} J_{\mu+\nu}(2z \operatorname{sh} t) \operatorname{ch}[(\mu - \nu)t] dt = \frac{1}{2}[I_{\nu}(z)K_{\mu}(z) + I_{\mu}(z)K_{\nu}(z)] \\ \left[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0 \right].$$

EH II 97(71)

$$5. \int_0^{\infty} J_{\mu+\nu}(2z \operatorname{sh} t) \operatorname{sh}[(\mu - \nu)t] dt = \frac{1}{2}[I_{\nu}(z)K_{\mu}(z) - I_{\mu}(z)K_{\nu}(z)] \\ \left[\operatorname{Re}(\nu + \mu) > -1, \quad |\operatorname{Re}(\mu - \nu)| < \frac{3}{2}, \quad z > 0 \right].$$

EH II 97(72)

6.664

$$1. \int_0^{\infty} J_0(2z \operatorname{sh} t) \operatorname{sh}(2\nu t) dt = \frac{\sin(\nu\pi)}{\pi} [K_{\nu}(z)]^2 \quad \left[|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0 \right].$$

EH II 97(69)

$$2. \int_0^{\infty} N_0(2z \operatorname{sh} t) \operatorname{ch}(2\nu t) dt = -\frac{\cos(\nu\pi)}{\pi} [K_{\nu}(z)]^2 \quad \left[|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0 \right].$$

EH II 97(70)

$$3. \int_0^{\infty} N_0(2z \operatorname{sh} t) \operatorname{sh}(2\nu t) dt = \frac{1}{\pi} \left[I_{\nu}(z) \frac{\partial K_{\nu}(z)}{\partial \nu} - K_{\nu}(z) \frac{\partial I_{\nu}(z)}{\partial \nu} \right] - \frac{1}{\pi} \cos(\nu\pi) [K_{\nu}(z)]^2 \\ \left[|\operatorname{Re} \nu| < \frac{3}{4}, \quad z > 0 \right].$$

EH II 97(75)

$$4. \int_0^{\infty} K_0(2z \operatorname{sh} t) \operatorname{ch} 2\nu t dt = \frac{\pi^2}{8} \{ J_{\nu}^2(z) + N_{\nu}^2(z) \} \quad [\operatorname{Re} z > 0].$$

MO 44

$$5. \int_0^{\infty} K_{2\mu}(z \operatorname{sh} 2t) \operatorname{cth}^{2\nu} t dt = \frac{1}{4z} \Gamma\left(\frac{1}{2} + \mu - \nu\right) \Gamma\left(\frac{1}{2} - \mu - \nu\right) W_{\nu, \mu}(iz) W_{\nu, \mu}(-iz) \\ \left[|\arg z| \leq \frac{\pi}{2}, \quad |\operatorname{Re} \mu| + \operatorname{Re} \nu < \frac{1}{2} \right].$$

MO 119

$$6. \int_0^{\infty} \operatorname{ch}(2\mu x) K_{2\nu}(2a \operatorname{ch} x) dx = \frac{1}{2} K_{\mu+\nu}(a) K_{\mu-\nu}(a) \quad [\operatorname{Re} a > 0].$$

ET II 378(42)

748

6.665

$$\int_0^{\infty} \operatorname{sech} x \operatorname{ch}(2\lambda x) I_{2\mu}(a \operatorname{sech} x) dx = \frac{\Gamma\left(\frac{1}{2} + \lambda + \mu\right) \Gamma\left(\frac{1}{2} - \lambda + \mu\right)}{2a[\Gamma(2\mu + 1)]^2} M_{\lambda, \mu}(a) M_{-\lambda, \mu}(a) \left[|\operatorname{Re} \lambda| - \operatorname{Re} \mu < \frac{1}{2} \right].$$

ET II 378(43)

Bessel, hyperbolic, and algebraic functions

6.666

$$\int_0^{\infty} x^{\nu+1} \operatorname{sh}(\alpha x) \operatorname{cosech} \pi x J_{\nu}(\beta x) dx = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} n^{\nu+1} \sin(n\alpha) K_{\nu}(n\beta) \quad [|\operatorname{Re} \alpha| < \pi, \operatorname{Re} \nu > -1].$$

ET II 41(3), WA 469(12)

6.667

$$1.^3 \int_0^a y^{-1} \operatorname{ch}(y \operatorname{sh} t) I_{2\nu}(x) dx = \frac{\pi}{2} I_{\nu}\left(\frac{1}{2} a e^t\right) I_{\nu}\left(\frac{1}{2} a e^{-t}\right), \quad y = (a^2 - x^2)^{\frac{1}{2}} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 365(10)

$$2. \int_0^a y^{-1} \operatorname{ch}(y \operatorname{sh} t) K_{2\nu}(x) dx = \frac{\pi^2}{4} \operatorname{cosec}(\nu\pi) [I_{-\nu}(a e^t) I_{-\nu}(a e^{-t}) - I_{\nu}(a e^t) I_{\nu}(a e^{-t})], \quad y = (a^2 - x^2)^{\frac{1}{2}} \quad \left[|\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 367(25)

6.668

$$1. \int_0^\infty e^{-\alpha x} \operatorname{sh}(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}},$$

$$r_1 = [\gamma^2 + (\beta - \alpha)^2]^{\frac{1}{2}}, \quad r_2 = [\gamma^2 + (\beta + \alpha)^2]^{\frac{1}{2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0].$$

ET II 12(52)

$$2. \int_0^\infty e^{-\alpha x} \operatorname{ch}(\beta x) J_0(\gamma x) dx = (\alpha\beta)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 + r_1)^{\frac{1}{2}} (r_2 - r_1)^{-\frac{1}{2}},$$

$$r_1 = [\gamma^2 + (\beta - \alpha)^2]^{\frac{1}{2}}, \quad r_2 = [\gamma^2 + (\beta + \alpha)^2]^{\frac{1}{2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \gamma > 0].$$

ET II 12(54)

749

6.669

$$1. \int_0^\infty \left[\operatorname{cth} \left(\frac{1}{2} x \right) \right]^{2\lambda} e^{-\beta \operatorname{ch} x} J_{2\mu}(\alpha \operatorname{sh} x) dx = \frac{\Gamma \left(\frac{1}{2} - \lambda + \mu \right)}{\alpha \Gamma(2\mu + 1)} M_{-\lambda, \mu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} - \beta \right] \times$$

$$\times W_{\lambda, \mu} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} + \beta \right]$$

$$\left[\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re} (\mu - \lambda) > -\frac{1}{2} \right].$$

BU 86(5b)A, ET II 363(34)

$$2. \int_0^\infty \left[\operatorname{cth} \left(\frac{1}{2} x \right) \right]^{2\lambda} e^{-\beta \operatorname{ch} x} N_{2\mu}(\alpha \operatorname{sh} x) dx =$$

$$= -\frac{\sec[(\mu + \lambda)\pi]}{\alpha} W_{\lambda, \mu} \left(\sqrt{\alpha^2 + \beta^2} + \beta \right) W_{-\lambda, \mu} \left(\sqrt{\alpha^2 + \beta^2} - \beta \right) -$$

$$-\frac{\operatorname{tg}[(\mu + \lambda)\pi] \Gamma \left(\frac{1}{2} - \lambda + \mu \right)}{\alpha \Gamma(2\mu + 1)} W_{\lambda, \mu} \left(\sqrt{\alpha^2 + \beta^2} + \beta \right) M_{-\lambda, \mu} \left(\sqrt{\alpha^2 + \beta^2} - \beta \right)$$

$$\left[\operatorname{Re} \beta > |\operatorname{Re} \alpha|, \quad \operatorname{Re} \lambda < \frac{1}{2} - |\operatorname{Re} \mu| \right].$$

ET II 363(35)

$$3. \int_0^\infty e^{-\frac{1}{2}(a_1 + a_2)t \operatorname{ch} x} \left[\operatorname{cth} \left(\frac{1}{2} x \right) \right]^{2\nu} K_{2\mu} (t\sqrt{a_1 a_2} \operatorname{sh} x) dx = \frac{\Gamma \left(\frac{1}{2} + \mu - \nu \right) \Gamma \left(\frac{1}{2} - \mu - \nu \right)}{2t\sqrt{a_1 a_2}} W_{\nu, \mu}(a_1 t) \times$$

$$\times W_{\nu, \mu}(a_2 t)$$

$$\left[\operatorname{Re} \nu < \operatorname{Re} \frac{1 \pm 2\mu}{2}, \quad \operatorname{Re} [t(\sqrt{a_1} + \sqrt{a_2})^2] > 0 \right].$$

$$4. \int_0^\infty e^{-\frac{1}{2}(a_1+a_2)t \operatorname{ch} x} \left[\operatorname{cth} \left(\frac{x}{2} \right) \right]^{2\nu} I_{2\mu} (t\sqrt{a_1 a_2} \operatorname{sh} x) dx = \frac{\Gamma \left(\frac{1}{2} + \mu - \nu \right)}{t\sqrt{a_1 a_2} \Gamma(1+2\mu)} W_{\nu, \mu}(a_1 t) M_{\nu, \mu}(a_2 t)$$

$$\left[\operatorname{Re} \left(\frac{1}{2} + \mu - \nu \right) > 0, \quad \operatorname{Re} \mu > 0, \quad a_1 > a_2 \right].$$

BU 86(5c)

$$5. \int_{-\infty}^\infty e^{2\nu s - \frac{x+y}{2} \theta s} I_{2\mu} \left(\frac{\sqrt{xy}}{\operatorname{ch} s} \right) \frac{ds}{\operatorname{ch} s} = \frac{\Gamma \left(\frac{1}{2} + \mu + \nu \right) \Gamma \left(\frac{1}{2} + \mu - \nu \right)}{\sqrt{xy} [\Gamma(1+2\mu)]^2} M_{\nu, \mu}(x) M_{-\nu, \mu}(y)$$

$$\left[\operatorname{Re} \left(\pm \nu + \frac{1}{2} + \mu \right) > 0 \right].$$

BU 83(3a)a

750

$$6. \int_{-\infty}^\infty e^{2\nu s - \frac{x+y}{2} \theta s} J_{2\mu} \left(\frac{\sqrt{xy}}{\operatorname{ch} s} \right) \frac{ds}{\operatorname{ch} s} = \frac{\Gamma \left(\frac{1}{2} + \mu + \nu \right) \Gamma \left(\frac{1}{2} + \mu - \nu \right)}{\sqrt{xy} [\Gamma(1+2\mu)]^2} M_{\nu, \mu}(x) M_{\nu, \mu}(y)$$

$$\left[\operatorname{Re} \left(\mp \nu + \frac{1}{2} + \mu \right) > 0 \right]$$

BU 84(3b)a

6.67- 6.68 Combinations of Bessel and trigonometric functions

6.671

$$1. \int_0^\infty J_\nu(\alpha x) \sin \beta x dx = \begin{cases} \frac{\sin \left(\nu \arcsin \frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 - \beta^2}} & [\beta < \alpha]; \\ \infty \quad \text{or} \quad 0 & [\beta = \alpha]; \\ \frac{\alpha^\nu \cos \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} \left(\beta + \sqrt{\beta^2 - \alpha^2} \right)^\nu} & [\beta > \alpha]. \end{cases} \quad [\operatorname{Re} \nu > -2].$$

WA 444(4)

$$2. \int_0^\infty J_\nu(\alpha x) \cos \beta x dx = \begin{cases} \frac{\cos \left(\nu \arcsin \frac{\beta}{\alpha} \right)}{\sqrt{\alpha^2 - \beta^2}} & [\beta < \alpha]; \\ \infty \quad \text{or} \quad 0 & [\beta = \alpha]; \\ \frac{-\alpha^\nu \sin \frac{\nu\pi}{2}}{\sqrt{\beta^2 - \alpha^2} \left(\beta + \sqrt{\beta^2 - \alpha^2} \right)^\nu} & [\beta > \alpha]. \end{cases} \quad [\operatorname{Re} \nu > -1].$$

$$\begin{aligned}
3. \int_0^\infty N_\nu(ax) \sin(bx) dx &= \operatorname{ctg} \left(\frac{\nu\pi}{2} \right) (a^2 - b^2)^{-\frac{1}{2}} \sin \left[\nu \arcsin \left(\frac{b}{a} \right) \right] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 2]; \\
&= \frac{1}{2} \operatorname{cosec} \left(\frac{\nu\pi}{2} \right) (b^2 - a^2)^{-\frac{1}{2}} \times \\
&\quad \times \left\{ a^{-\nu} \cos(\nu\pi) \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu - a^\nu \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \\
&\quad [0 < a < b, \quad |\operatorname{Re} \nu| < 2].
\end{aligned}$$

ET I 103(33)

$$\begin{aligned}
4. \int_0^\infty N_\nu(ax) \cos(bx) dx &= \frac{\operatorname{tg} \left(\frac{\nu\pi}{2} \right)}{(a^2 - b^2)^{\frac{1}{2}}} \cos \left[\nu \arcsin \left(\frac{b}{a} \right) \right] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1]; \\
&= -\sin \left(\frac{\nu\pi}{2} \right) (b^2 - a^2)^{-\frac{1}{2}} \left\{ a^{-\nu} \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu + \operatorname{ctg}(\nu\pi) + \right. \\
&\quad \left. + a^\nu \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \operatorname{cosec}(\nu\pi) \right\} \\
&\quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET I 47(29)

751

$$\begin{aligned}
5. \int_0^\infty K_\nu(ax) \sin(bx) dx &= \frac{1}{4} \pi a^{-\nu} \operatorname{cosec} \left(\frac{\nu\pi}{2} \right) (a^2 + b^2)^{-\frac{1}{2}} \left\{ \left[(b^2 + a^2)^{\frac{1}{2}} + b \right]^\nu - \left[(b^2 + a^2)^{\frac{1}{2}} - b \right]^\nu \right\} \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 2, \quad \nu \neq 0].
\end{aligned}$$

ET I 105(48)

$$\begin{aligned}
6. \int_0^\infty K_\nu(ax) \cos(bx) dx &= \frac{\pi}{4} (b^2 + a^2)^{-\frac{1}{2}} \sec \left(\frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \left[b + (b^2 + a^2)^{\frac{1}{2}} \right]^\nu + a^\nu \left[b + (b^2 + a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \\
&\quad [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET I 49(40)

$$\begin{aligned}
7. \int_0^\infty J_0(ax) \sin(bx) dx &= 0 \quad [0 < b < a]; \\
&= \frac{1}{\sqrt{b^2 - a^2}} \quad [0 < a < b].
\end{aligned}$$

ET I 99(1)

$$\begin{aligned}
8. \int_0^\infty J_0(ax) \cos(bx) dx &= \frac{1}{\sqrt{a^2 - b^2}} \quad [0 < b < a]; \\
&= \infty \quad [a = b]; \\
&= 0 \quad [0 < a < b].
\end{aligned}$$

$$9. \int_0^{\infty} J_{2n+1}(ax) \sin(bx) dx = (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n+1} \left(\frac{b}{a} \right) \quad [0 < b < a];$$

$$= 0 \quad [0 < a < b].$$

ET I 99(2)

$$10. \int_0^{\infty} J_{2n}(ax) \cos(bx) dx = (-1)^n \frac{1}{\sqrt{a^2 - b^2}} T_{2n} \left(\frac{b}{a} \right) \quad [0 < b < a];$$

$$= 0 \quad [0 < a < b].$$

ET I 43(2)

$$11. \int_0^{\infty} N_0(ax) \sin(bx) dx = \frac{2 \arcsin \left(\frac{b}{a} \right)}{\pi \sqrt{a^2 - b^2}} \quad [0 < b < a];$$

$$= \frac{2}{\pi} \frac{1}{\sqrt{b^2 - a^2}} \ln \left[\frac{b}{a} - \sqrt{\frac{b^2}{a^2} - 1} \right] \quad [0 < a < b].$$

ET I 103(31)

$$12. \int_0^{\infty} N_0(ax) \cos(bx) dx = 0 \quad [0 < b < a];$$

$$= -\frac{1}{\sqrt{b^2 - a^2}} \quad [0 < a < b].$$

ET I 47(28)

$$13. \int_0^{\infty} K_0(\beta x) \sin \alpha x dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \ln \left(\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} + 1} \right) \quad [\alpha > 0, \quad \beta > 0].$$

WA 425(11)A, MO 48

$$14. \int_0^{\infty} K_0(\beta x) \cos \alpha x dx = \frac{\pi}{2\sqrt{\alpha^2 + \beta^2}} \quad [\alpha \text{ and } \beta \text{ are real; } \beta > 0].$$

WA 425(10)A, MO 48

752

6.672

$$1. \int_0^{\infty} J_{\nu}(ax) J_{\nu}(bx) \sin(cx) dx =$$

$$= 0 \quad [\operatorname{Re} \nu > -1, \quad 0 < c < b - a, \quad 0 < a < b];$$

$$= \frac{1}{2\sqrt{ab}} P_{\nu - \frac{1}{2}} \left(\frac{b^2 + a^2 - c^2}{2ab} \right) \quad [\operatorname{Re} \nu > -1, \quad b - a < c < b + a, \quad 0 < a < b];$$

$$= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}} Q_{\nu - \frac{1}{2}} \left(\frac{b^2 + a^2 - c^2}{2ab} \right) \quad [\operatorname{Re} \nu > -1, \quad b + a < c, \quad 0 < a < b]$$

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$$2. \int_0^\infty J_\nu(x)J_{-\nu}(x) \cos(bx) dx = \frac{1}{2}P_{\nu-\frac{1}{2}}\left(\frac{1}{2}b^2 - 1\right) \quad [0 < b < 2];$$

$$= 0 \quad [2 < b].$$

ET I 46(21)

$$3. \int_0^\infty K_\nu(ax)K_\nu(bx) \cos(cx) dx = \frac{\pi^2}{4\sqrt{ab}} \sec(\nu\pi)P_{\nu-\frac{1}{2}}[(a^2 + b^2 + c^2)(2ab)^{-1}]$$

$$\left[\operatorname{Re}(a + b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 50(51)

$$4. \int_0^\infty K_\nu(ax)I_\nu(bx) \cos(cx) dx = \frac{1}{2\sqrt{ab}}Q_{\nu-\frac{1}{2}}\left(\frac{a^2 + b^2 + c^2}{2ab}\right)$$

$$\left[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 49(47)

$$5. \int_0^\infty \sin(2ax)[J_\nu(x)]^2 dx = \frac{1}{2}P_{\nu-\frac{1}{2}}(1 - 2a^2) \quad [0 < a < 1, \quad \operatorname{Re} \nu > -1];$$

$$= \frac{1}{\pi} \cos(\nu\pi)Q_{\nu-\frac{1}{2}}(2a^2 - 1) \quad [a > 1, \quad \operatorname{Re} \nu > -1];$$

ET II 343(30)

$$6. \int_0^\infty \cos(2ax)[J_\nu(x)]^2 dx = \frac{1}{\pi}Q_{\nu-\frac{1}{2}}(1 - 2a^2) \quad \left[0 < a < 1, \quad \operatorname{Re} \nu > -\frac{1}{2}\right];$$

$$= -\frac{1}{\pi} \sin(\nu\pi)Q_{\nu-\frac{1}{2}}(2a^2 - 1) \quad \left[a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 344(32)

$$7. \int_0^\infty \sin(2ax)J_0(x)N_0(x) dx = 0 \quad [0 < a < 1];$$

$$= -\frac{\mathbf{K}\left[(1 - a^{-2})^{\frac{1}{2}}\right]}{\pi a} \quad [a > 1].$$

ET II 348(60)

$$8. \int_0^\infty K_0(ax)I_0(bx) \cos(cx) dx = \frac{1}{\sqrt{c^2 + (a + b)^2}} \mathbf{K}\left\{\frac{2\sqrt{ab}}{\sqrt{c^2 + (a + b)^2}}\right\} \quad [\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0].$$

$$9. \int_0^{\infty} \cos(2ax) J_0(x) N_0(x) dx = -\frac{1}{\pi} K(a) \quad [0 < a < 1];$$

$$= -\frac{1}{\pi a} K\left(\frac{1}{a}\right) \quad [a > 1].$$

ET II 348(61)

753

$$10. \int_0^{\infty} \cos(2ax) [N_0(x)]^2 dx = \frac{1}{\pi} K(\sqrt{1-a^2}) \quad [0 < a < 1];$$

$$= \frac{2}{\pi a} K\left(\sqrt{1-\frac{1}{a^2}}\right) \quad [a > 1].$$

ET II 348(62)

6.673

$$1. \int_0^{\infty} \left[J_{\nu}(ax) \cos\left(\frac{\nu\pi}{2}\right) - N_{\nu}(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \sin(bx) dx =$$

$$= 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 2];$$

$$= \frac{1}{2a^{\nu} \sqrt{b^2 - a^2}} \left\{ \left[b + (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} + \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} \right\} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 2].$$

ET I 104(39)

$$2. \int_0^{\infty} \left[N_{\nu}(ax) \cos\left(\frac{\nu\pi}{2}\right) + J_{\nu}(ax) \sin\left(\frac{\nu\pi}{2}\right) \right] \cos(bx) dx =$$

$$= 0 \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1];$$

$$= -\frac{1}{2a^{\nu} \sqrt{b^2 - a^2}} \left\{ \left[b + (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} + \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{\nu} \right\} \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1].$$

ET I 48(32)

6.674

$$1. \int_0^a \sin(a-x) J_{\nu}(x) dx = a J_{\nu+1}(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 334(12)

$$2. \int_0^a \cos(a-x) J_{\nu}(x) dx = a J_{\nu}(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a) \quad [\operatorname{Re} \nu > -1].$$

$$3. \int_0^a \sin(a-x) J_{2n}(x) dx = a J_{2n+1}(a) + (-1)^n 2n \left[\cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right]$$

$$[n = 0, 1, 2, \dots].$$

ET II 334(10)

$$4. \int_0^a \cos(a-x) J_{2n}(x) dx = a J_{2n}(a) - (-1)^n 2n \left[\sin a - 2 \sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a) \right]$$

$$[n = 0, 1, 2, \dots].$$

ET II 335(21)

$$5. \int_0^a \sin(a-x) J_{2n+1}(x) dx = a J_{2n+2}(a) + (-1)^n (2n+1) \left[\sin a - 2 \sum_{m=0}^n (-1)^m J_{2m+1}(a) \right]$$

$$[n = 0, 1, 2, \dots].$$

ET II 334(11)

$$6. \int_0^a \cos(a-x) J_{2n+1}(x) dx = a J_{2n+1}(a) + (-1)^n (2n+1) \left[\cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a) \right]$$

$$[n = 0, 1, 2, \dots].$$

ET II 336(22)

$$7. \int_0^z \sin(z-x) J_0(x) dx = z J_1(z).$$

WA 415(2)

$$8. \int_0^z \cos(z-x) J_0(x) dx = z J_0(z).$$

WA 415(1)

754

6.675

$$1. \int_0^\infty J_\nu(a\sqrt{x}) \sin(bx) dx = \frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{a^2}{8b}\right) - \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{a^2}{8b}\right) \right]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4].$$

$$\begin{aligned}
2. \int_0^{\infty} J_{\nu}(a\sqrt{x}) \cos(bx) dx &= \\
&= -\frac{a\sqrt{\pi}}{4b^{\frac{3}{2}}} \left[\sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu - \frac{1}{2}}\left(\frac{a^2}{8b}\right) + \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu + \frac{1}{2}}\left(\frac{a^2}{8b}\right) \right] \\
&\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2].
\end{aligned}$$

ET I 53(22)a

$$3. \int_0^{\infty} J_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{b} \cos\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0].$$

ET I 110(22)

$$4. \int_0^{\infty} J_0(a\sqrt{x}) \cos(bx) dx = \frac{1}{b} \sin\left(\frac{a^2}{4b}\right) \quad [a > 0, \quad b > 0].$$

ET I 53(21)

6.676

$$\begin{aligned}
1. \int_0^{\infty} J_{\nu}(a\sqrt{x}) J_{\nu}(b\sqrt{x}) \sin(cx) dx &= \frac{1}{c} J_{\nu}\left(\frac{ab}{2c}\right) \cos\left(\frac{a^2 + b^2}{4c} - \frac{\nu\pi}{2}\right) \\
&\quad [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2].
\end{aligned}$$

ET I 111(29)a

$$\begin{aligned}
2. \int_0^{\infty} J_{\nu}(a\sqrt{x}) J_{\nu}(b\sqrt{x}) \cos(cx) dx &= \frac{1}{c} J_{\nu}\left(\frac{ab}{2c}\right) \sin\left(\frac{a^2 + b^2}{4c} - \frac{\nu\pi}{2}\right) \\
&\quad [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1].
\end{aligned}$$

ET I 54(27)

$$3. \int_0^{\infty} J_0(a\sqrt{x}) K_0(a\sqrt{x}) \sin(bx) dx = \frac{1}{2b} K_0\left(\frac{a^2}{2b}\right) \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 111(31)

$$4. \int_0^{\infty} J_0(\sqrt{ax}) K_0(\sqrt{ax}) \cos(bx) dx = \frac{\pi}{4b} \left[I_0\left(\frac{a}{2b}\right) - L_0\left(\frac{a}{2b}\right) \right] \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 54(29)

$$5. \int_0^{\infty} K_0(\sqrt{ax}) N_0(\sqrt{ax}) \cos(bx) dx = -\frac{1}{2b} K_0\left(\frac{a}{2b}\right) \quad [\operatorname{Re} \sqrt{a} > 0, \quad b > 0].$$

$$6. \int_0^{\infty} K_0\left(\sqrt{ax}e^{\frac{1}{4}\pi i}\right) K_0\left(\sqrt{ax}e^{-\frac{1}{4}\pi i}\right) \cos(bx) dx = \frac{\pi^2}{8b} \left[H_0\left(\frac{a}{2b}\right) - N_0\left(\frac{a}{2b}\right) \right] \quad [\operatorname{Re} a > 0, b > 0].$$

ET I 54(31)

6.677

$$1. \int_a^{\infty} J_0\left(b\sqrt{x^2 - a^2}\right) \sin(cx) dx = 0 \quad [0 < c < b];$$

$$= \frac{\cos\left(a\sqrt{c^2 - b^2}\right)}{\sqrt{c^2 - b^2}} \quad [0 < b < c].$$

ET I 113(47)

755

$$2. \int_a^{\infty} J_0\left(b\sqrt{x^2 - a^2}\right) \cos(cx) dx = \frac{\exp\left(-a\sqrt{b^2 - c^2}\right)}{\sqrt{b^2 - c^2}} \quad [0 < c < b];$$

$$= \frac{-\sin\left(a\sqrt{c^2 - b^2}\right)}{\sqrt{c^2 - b^2}} \quad [0 < b < c].$$

ET I 57(48)a

$$3.^6 \int_0^{\infty} J_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos \beta x dx = \frac{\cos z\sqrt{\alpha^2 - \beta^2}}{\sqrt{\alpha^2 - \beta^2}} \quad [0 < \beta < \alpha, z > 0];$$

$$= 0 \quad [0 < \alpha < \beta, z > 0].$$

MO 47a

$$4. \int_0^{\infty} N_0\left(\alpha\sqrt{x^2 + z^2}\right) \cos \beta x dx = \frac{1}{\sqrt{\alpha^2 - \beta^2}} \sin\left(z\sqrt{\alpha^2 - \beta^2}\right) \quad [0 < \beta < \alpha, z > 0];$$

$$= -\frac{1}{\sqrt{\beta^2 - \alpha^2}} \exp\left(-z\sqrt{\beta^2 - \alpha^2}\right) \quad [0 < \alpha < \beta, z > 0].$$

MO 47a

$$5. \int_0^{\infty} K_0\left[\alpha\sqrt{x^2 + \beta^2}\right] \cos(\gamma x) dx = \frac{\pi}{2\sqrt{\alpha^2 + \gamma^2}} \exp\left(-\beta\sqrt{\alpha^2 + \gamma^2}\right)$$

$$[\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0, \gamma > 0].$$

ET I 56(43)

$$6. \int_0^a J_0\left(b\sqrt{a^2 - x^2}\right) \cos(cx) dx = \frac{\sin\left(a\sqrt{b^2 + c^2}\right)}{\sqrt{b^2 + c^2}} \quad [b > 0].$$

$$7. \int_0^{\infty} J_0(b\sqrt{x^2 - a^2}) \cos(cx) dx = \frac{\text{ch}(a\sqrt{b^2 - c^2})}{\sqrt{b^2 - c^2}} \quad [0 < c < b, \quad a > 0];$$

$$= 0 \quad [0 < b < c, \quad a > 0].$$

ET I 57(49)

$$8. \int_0^{\infty} H_0^{(1)}(\alpha\sqrt{\beta^2 - x^2}) \cos(\gamma x) dx = -i \frac{\exp(i\beta\sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2}}$$

$$\left[\pi > \arg \sqrt{\beta^2 - x^2} \geq 0, \quad \alpha > 0, \quad \gamma > 0 \right].$$

ET I 59(59)

$$9. \int_0^{\infty} H_0^{(2)}(\alpha\sqrt{\beta^2 - x^2}) \cos(\gamma x) dx = \frac{i \exp(-i\beta\sqrt{\alpha^2 + \gamma^2})}{\sqrt{\alpha^2 + \gamma^2}}$$

$$\left[-\pi < \arg \sqrt{\beta^2 - x^2} \leq 0, \quad \alpha > 0, \quad \gamma > 0 \right].$$

ET I 58(58)

6.678

$$\int_0^{\infty} \left[K_0(2\sqrt{x}) + \frac{\pi}{2} N_0(2\sqrt{x}) \right] \sin(bx) dx = \frac{\pi}{2b} \sin\left(\frac{1}{b}\right) \quad [b > 0].$$

ET I 111(34)

6.679

$$1. \int_0^{\infty} J_{2\nu} \left[2b \operatorname{sh} \left(\frac{x}{2} \right) \right] \sin(bx) dx = -i [I_{\nu-ib}(a) K_{\nu+ib}(a) - I_{\nu+ib}(a) K_{\nu-ib}(a)]$$

$$[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 115(59)

756

$$2. \int_0^{\infty} J_{2\nu} \left[2a \operatorname{sh} \left(\frac{x}{2} \right) \right] \cos(bx) dx = I_{\nu-ib}(a) K_{\nu+ib}(a) + I_{\nu+ib}(a) K_{\nu-ib}(a)$$

$$\left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$3. \int_0^{\infty} J_{2\nu} \left[2a \operatorname{ch} \left(\frac{x}{2} \right) \right] \cos(bx) dx = -\frac{\pi}{2} [J_{\nu+ib}(a)N_{\nu-ib}(a) + J_{\nu-ib}(a)N_{\nu+ib}(a)].$$

ET I 59(63)

$$4. \int_0^{\infty} J_0 \left[2a \operatorname{sh} \left(\frac{x}{2} \right) \right] \sin(bx) dx = \frac{2}{\pi} \operatorname{sh}(\pi b) [K_{ib}(a)]^2 \quad [a > 0, \quad b > 0].$$

ET I 115(58)

$$5. \int_0^{\infty} J_0 \left[2a \operatorname{sh} \left(\frac{x}{2} \right) \right] \cos(bx) dx = [I_{ib}(a) + I_{-ib}(a)] K_{ib}(a) \quad [a > 0, \quad b > 0].$$

ET I 59(62)

$$6. \int_0^{\infty} N_0 \left[2a \operatorname{sh} \left(\frac{x}{2} \right) \right] \cos(bx) dx = -\frac{2}{\pi} \operatorname{ch}(\pi b) [K_{ib}(a)]^2 \quad [a > 0, \quad b > 0].$$

ET I 59(65)

$$7. \int_0^{\infty} K_0 \left[2a \operatorname{sh} \left(\frac{x}{2} \right) \right] \cos(bx) dx = \frac{\pi^2}{4} \{ [J_{ib}(a)]^2 + [N_{ib}(a)]^2 \} \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 59(66)

6.681

$$1. \int_0^{\frac{\pi}{2}} \cos(2\mu x) J_{2\nu}(2a \cos x) dx = \frac{\pi}{2} J_{\nu+\mu}(a) J_{\nu-\mu}(a) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 361(23)

$$2. \int_0^{\frac{\pi}{2}} \cos(2\mu x) N_{2\nu}(2a \cos x) dx = \frac{\pi}{2} [\operatorname{ctg}(2\nu\pi) J_{\nu+\mu}(a) J_{\nu-\mu}(a) - \operatorname{cosec}(2\nu\pi) J_{\mu-\nu}(a) J_{-\mu-\nu}(a)] \quad \left[|\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 361(24)

$$3. \int_0^{\frac{\pi}{2}} \cos(2\mu x) I_{2\nu}(2a \cos x) dx = \frac{\pi}{2} I_{\nu-\mu}(a) I_{\nu+\mu}(a) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 59(61)

$$4. \int_0^{\frac{\pi}{2}} \cos(\nu x) K_{\nu}(2a \cos x) dx = \frac{\pi}{2} I_0(a) K_{\nu}(a) \quad [\operatorname{Re} \nu < 1].$$

$$5. \int_0^\pi J_0(2z \cos x) \cos 2nx \, dx = (-1)^n \pi J_n^2(z).$$

MO 45

$$6. \int_0^\pi J_0(2z \sin x) \cos 2nx \, dx = \pi J_n^2(z).$$

WA 43(3), MO 45

$$7. \int_0^{\frac{\pi}{2}} \cos(2n\pi) N_0(2a \sin x) \, dx = \frac{\pi}{2} J_n(a) N_n(a) \quad [n = 0, 1, 2, \dots].$$

ET II 360(16)

$$8. \int_0^\pi \sin(2\mu x) J_{2\nu}(2a \sin x) \, dx = \pi \sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a) \quad [\operatorname{Re} \nu > -1].$$

ET II 360(13)

757

$$9. \int_0^\pi \cos(2\mu x) J_{2\nu}(2a \sin x) \, dx = \pi \cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 360(14)

$$10. \int_0^{\frac{\pi}{2}} J_{\nu+\mu}(2z \cos x) \cos[(\nu - \mu)x] \, dx = \frac{\pi}{2} J_\nu(z) J_\mu(z) \quad [\operatorname{Re}(\nu + \mu) > -1].$$

MO 42

$$11. \int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] I_{\mu+\nu}(2a \cos x) \, dx = \frac{\pi}{2} I_\mu(a) I_\nu(a) \quad [\operatorname{Re}(\mu + \nu) > -1].$$

WA 484(2), ET II 378(39)

$$12. \int_0^{\frac{\pi}{2}} \cos[(\mu - \nu)x] K_{\mu+\nu}(2a \cos x) \, dx = \frac{\pi^2}{4} \operatorname{cosec}[(\mu + \nu)\pi] [I_{-\mu}(a) I_{-\nu}(a) - I_\mu(a) I_\nu(a)] \\ [|\operatorname{Re}(\mu + \nu)| < 1].$$

$$13. \int_0^{\frac{\pi}{2}} K_{\nu-m}(2a \cos x) \cos[(m+\nu)x] dx = (-1)^m \frac{\pi^2}{4} I_m(a) K_\nu(a) \quad [|\operatorname{Re}(\nu-m)| < 1].$$

WA 485(4)

6.682

$$1.7 \int_0^{\frac{\pi}{2}} J_{\nu-\frac{1}{2}}(x \sin t) \sin^{\nu+\frac{1}{2}} t dt = \sqrt{\frac{\pi}{2x}} J_\nu(x) \\ [x > 0; \nu \geq 0 \text{ integer or half-integer}]$$

MO 42a

$$2. \int_0^{\frac{\pi}{2}} J_\nu(z \sin x) \sin^\nu x \cos^{2\nu} x dx = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right) z^{-\nu} J_\nu^2\left(\frac{z}{2}\right) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

MO 42a

6.683

$$1. \int_0^{\frac{\pi}{2}} J_\nu(z \sin x) I_\mu(z \cos x) \operatorname{tg}^{\nu+1} x dx = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{\mu-\nu}{2}\right)}{\Gamma\left(\frac{\mu+\nu}{2} + 1\right)} J_\mu(z) \quad [\operatorname{Re} \nu > \operatorname{Re} \mu > -1].$$

WA 407(4)

$$2. \int_0^{\frac{\pi}{2}} J_\nu(z_1 \sin x) J_\mu(z_2 \cos x) \sin^{\nu+1} x \cos^{\mu+1} x dx = \frac{z_1^\nu z_2^\mu J_{\nu+\mu+1}\left(\sqrt{z_1^2 + z_2^2}\right)}{\sqrt{(z_1^2 + z_2^2)^{\nu+\mu+1}}} \\ [\operatorname{Re} \nu > -1, \operatorname{Re} \mu > -1].$$

WA 410(1)

$$3. \int_0^{\frac{\pi}{2}} J_\nu(z \cos^2 x) J_\mu(z \sin^2 x) \sin x \cos x dx = \frac{1}{z} \sum_{k=0}^{\infty} (-1)^k J_{\nu+\mu+2k+1}(z) \\ [\operatorname{Re} \nu > -1, \operatorname{Re} \mu > -1] \quad (\text{see also } \mathbf{6.513} \text{ 6}).$$

6.513
WA 414(1)

758

$$4. \int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} (\cos \theta)^{2\nu+1} d\theta = \frac{s_{\mu+\nu, \nu-\mu+1}(z)}{2^{\mu-1} z^{\nu+1} \Gamma(\mu)} \quad [\operatorname{Re} \nu > -1].$$

WA 407(2)

$$5. \int_0^{\frac{\pi}{2}} J_\mu(z \sin \theta) (\sin \theta)^{1-\mu} d\theta = \frac{\mathbf{H}_{\mu-\frac{1}{2}}(z)}{\sqrt{\frac{2z}{\pi}}}.$$

WA 407(3)

$$6. \int_0^{\frac{\pi}{2}} J_\mu(a \sin \theta) (\sin \theta)^{\mu+1} (\cos \theta)^{2\varrho+1} d\theta = 2^\varrho \Gamma(\varrho+1) a^{-\varrho-1} J_{\varrho+\mu+1}(a) \\ [\operatorname{Re} \varrho > -1, \quad \operatorname{Re} \mu > -1].$$

WA 406(1), EH II 46(5)

$$7. \int_0^{\frac{\pi}{2}} J_\nu(2z \sin \theta) (\sin \theta)^\nu (\cos \theta)^{2\nu} d\theta = \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m z^{\nu+2m} \Gamma\left(\nu+m+\frac{1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right)}{m! \Gamma(\nu+m+1) \Gamma(2\nu+m+1)}; \\ = \frac{1}{2} z^{-\nu} \sqrt{\pi} \Gamma\left(\nu+\frac{1}{2}\right) [J_\nu(z)]^2 \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

EH II 47(10)

$$8. \int_0^{\frac{\pi}{2}} J_\nu(z \sin \theta) (\sin \theta)^{\nu+1} (\cos \theta)^{-2\nu} d\theta = 2^{-\nu} \frac{z^{\nu-1}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}-\nu\right) \sin z \quad \left[-1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

EH II 68(39)

$$9. \int_0^{\frac{\pi}{2}} J_\nu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) (\sin \theta)^{2\nu+1} (\cos \theta)^{2\nu+1} d\theta = \frac{\Gamma\left(\frac{1}{2}+\nu\right) J_{2\nu+\frac{1}{2}}(z)}{2^{2\nu+\frac{3}{2}} \Gamma(\nu+1) \sqrt{z}} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 409(1)

$$10. \int_0^{\frac{\pi}{2}} J_\mu(z \sin^2 \theta) J_\nu(z \cos^2 \theta) \sin^{2\mu+1} \theta \cos^{2\nu+1} \theta d\theta = \frac{\Gamma\left(\mu+\frac{1}{2}\right) \Gamma\left(\nu+\frac{1}{2}\right) J_{\mu+\nu+\frac{1}{2}}(z)}{2\sqrt{\pi} \Gamma(\mu+\nu+1) \sqrt{2z}} \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$1. \int_0^\pi (\sin x)^{2\nu} \frac{J_\nu \left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x} \right)}{\left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x} \right)^\nu} dx = 2^\nu \sqrt{\pi} \Gamma \left(\nu + \frac{1}{2} \right) \frac{J_\nu(\alpha)}{\alpha^\nu} \frac{J_\nu(\beta)}{\beta^\nu} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 362(27)

$$2. \int_0^\pi (\sin x)^{2\nu} \frac{N_\nu \left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x} \right)}{\left(\sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos x} \right)^\nu} dx = 2^\nu \sqrt{\pi} \Gamma \left(\nu + \frac{1}{2} \right) \frac{J_\nu(\alpha)}{\alpha^\nu} \frac{N_\nu(\beta)}{\beta^\nu} \\ \left[|\alpha| < |\beta|, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 362(28)

759

6.685

$$\int_0^{\frac{\pi}{2}} \sec x \cos(2\lambda x) K_{2\mu}(a \sec x) dx = \frac{\pi}{2a} W_{\lambda, \mu}(a) W_{-\lambda, \mu}(a) \quad [\operatorname{Re} a > 0].$$

ET II 378(41)

6.686

$$1. \int_0^\infty \sin(ax^2) J_\nu(bx) dx = -\frac{\sqrt{\pi}}{2\sqrt{a}} \sin \left(\frac{b^2}{8a} - \frac{\nu+1}{4} \pi \right) J_{\frac{1}{2}\nu} \left(\frac{b^2}{8a} \right) \quad [a > 0, b > 0, \operatorname{Re} \nu > -3].$$

ET II 34(13)

$$2. \int_0^\infty \cos(ax^2) J_\nu(bx) dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \cos \left(\frac{b^2}{8a} - \frac{\nu+1}{4} \pi \right) J_{\frac{1}{2}\nu} \left(\frac{b^2}{8a} \right) \quad [a > 0, b > 0, \operatorname{Re} \nu > -1].$$

ET II 38(38)

$$3. \int_0^\infty \sin(ax^2) N_\nu(bx) dx = -\frac{\sqrt{\pi}}{4\sqrt{a}} \sec \left(\frac{\nu\pi}{2} \right) \times \\ \times \left[\cos \left(\frac{b^2}{8a} - \frac{3\nu+1}{4} \pi \right) J_{\frac{1}{2}\nu} \left(\frac{b^2}{8a} \right) - \sin \left(\frac{b^2}{8a} + \frac{\nu-1}{4} \pi \right) N_{\frac{1}{2}\nu} \left(\frac{b^2}{8a} \right) \right] \\ [a > 0, b > 0, -3 < \operatorname{Re} \nu < 3].$$

ET II 107(7)

ET II 107(8)

$$5. \int_0^{\infty} \sin(ax^2) J_1(bx) dx = \frac{1}{b} \sin \frac{b^2}{4a} \quad [a > 0, \quad b > 0].$$

ET II 19(16)

$$6. \int_0^{\infty} \cos(ax^2) J_1(bx) dx = \frac{2}{b} \sin^2 \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET II 20(20)

$$7. \int_0^{\infty} \sin^2(ax^2) J_1(bx) dx = \frac{1}{2b} \cos \left(\frac{b^2}{8a} \right) \quad [a > 0, \quad b > 0].$$

ET II 19(17)

6.687

$$\int_0^{\infty} \cos \left(\frac{x^2}{2a} \right) K_{2\nu} \left(x e^{i\frac{\pi}{4}} \right) K_{2\nu} \left(x e^{-i\frac{\pi}{4}} \right) dx = \frac{\Gamma \left(\frac{1}{4} + \nu \right) \Gamma \left(\frac{1}{4} - \nu \right) \sqrt{\pi}}{8\sqrt{a}} W_{\frac{1}{4}, \nu} \left(a e^{i\frac{\pi}{2}} \right) W_{\frac{1}{4}, \nu} \left(a e^{-i\frac{\pi}{2}} \right) \\ \left[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{4} \right].$$

ET II 372(1)

760

6.688

$$1. \int_0^{\frac{\pi}{2}} J_{\nu}(\mu z \sin t) \cos(\mu x \cos t) dt = \frac{\pi}{2} J_{\frac{\nu}{2}} \left(\mu \frac{\sqrt{x^2 + z^2} + x}{2} \right) J_{\frac{\nu}{2}} \left(\mu \frac{\sqrt{x^2 + z^2} - x}{2} \right) \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re} z > 0].$$

MO 46

$$2. \int_0^{\frac{\pi}{2}} (\sin x)^{\nu+1} \cos(\beta \cos x) J_{\nu}(\alpha \sin x) dx = 2^{-\frac{1}{2}} \sqrt{\pi} \alpha^{\nu} (\alpha^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} J_{\nu + \frac{1}{2}} \left[(\alpha^2 + \beta^2)^{\frac{1}{2}} \right] \\ [\operatorname{Re} \nu > -1].$$

ET II 361(19)

6.69- 6.74 Combinations of Bessel and trigonometric functions and powers

6.691

$$\int_0^{\infty} x \sin(bx) K_0(ax) dx = \frac{\pi b}{2} (a^2 + b^2)^{-\frac{3}{2}} \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 105(47)

6.692

$$1. \int_0^{\infty} x K_{\nu}(ax) I_{\nu}(bx) \sin(cx) dx = -\frac{1}{2} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} Q_{\nu - \frac{1}{2}}^1(u), \quad u = (2ab)^{-1} (a^2 + b^2 + c^2)$$

$$\left[\operatorname{Re} a > |\operatorname{Re} b|, \quad c > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET I 106(54)

$$2. \int_0^{\infty} x K_{\nu}(ax) K_{\nu}(bx) \sin(cx) dx = \frac{\pi}{4} (ab)^{-\frac{3}{2}} c (u^2 - 1)^{-\frac{1}{2}} \Gamma\left(\frac{3}{2} + \nu\right) \Gamma\left(\frac{3}{2} - \nu\right) P_{\nu - \frac{1}{2}}^{-1}(u),$$

$$u = (2ab)^{-1} (a^2 + b^2 + c^2) \quad \left[\operatorname{Re}(a + b) > 0, \quad c > 0, \quad |\operatorname{Re} \nu| < \frac{3}{2} \right].$$

ET I 107(61)

6.693

$$1. \int_0^{\infty} J_{\nu}(\alpha x) \sin \beta x \frac{dx}{x} = \frac{1}{\nu} \sin\left(\nu \arcsin \frac{\beta}{\alpha}\right) \quad \left. \begin{array}{l} [\beta \leq \alpha] \\ [\beta \geq \alpha] \end{array} \right\} \quad [\operatorname{Re} \nu > -1].$$

$$= \frac{\alpha^{\nu} \sin \frac{\nu\pi}{2}}{\nu (\beta + \sqrt{\beta^2 - \alpha^2})^{\nu}}$$

WA 443(2)

$$2. \int_0^{\infty} J_{\nu}(\alpha x) \cos \beta x \frac{dx}{x} = \frac{1}{\nu} \cos\left(\nu \arcsin \frac{\beta}{\alpha}\right) \quad \left. \begin{array}{l} [\beta \leq \alpha] \\ [\beta \geq \alpha] \end{array} \right\} \quad [\operatorname{Re} \nu > 0].$$

$$= \frac{\alpha^{\nu} \cos \frac{\nu\pi}{2}}{\nu (\beta + \sqrt{\beta^2 - \alpha^2})^{\nu}}$$

WA 443(3)

$$\begin{aligned}
3. \int_0^\infty N_\nu(ax) \sin(bx) \frac{dx}{x} &= -\frac{1}{\nu} \operatorname{tg} \left(\frac{\nu\pi}{2} \right) \sin \left[\nu \arcsin \left(\frac{b}{a} \right) \right] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1]; \\
&= \frac{1}{2\nu} \sec \left(\frac{\nu\pi}{2} \right) \left\{ a^{-\nu} \cos(\nu\pi) \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^\nu - a^\nu \left[b - (b^2 - a^2)^{\frac{1}{2}} \right]^{-\nu} \right\} \\
&\quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET I 103(35)

$$\begin{aligned}
4. \int_0^\infty J_\nu(ax) \sin(bx) \frac{dx}{x^2} &= \frac{\sqrt{a^2 - b^2} \sin \left[\nu \arcsin \left(\frac{b}{a} \right) \right]}{\nu^2 - 1} - \frac{b \cos \left[\nu \arcsin \left(\frac{b}{a} \right) \right]}{\nu(\nu^2 - 1)} \\
&\quad [0 < b < a, \quad \operatorname{Re} \nu > 0]; \\
&= \frac{-a^\nu \cos \left(\frac{\nu\pi}{2} \right) \left[b + \nu \sqrt{b^2 - a^2} \right]}{\nu(\nu^2 - 1) \left[b + \sqrt{b^2 - a^2} \right]^\nu} \quad [0 < a < b, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET I 99(6)

$$\begin{aligned}
5. \int_0^\infty J_\nu(ax) \cos(bx) \frac{dx}{x^2} &= \frac{a \cos \left[(\nu - 1) \arcsin \left(\frac{b}{a} \right) \right]}{2\nu(\nu - 1)} + \frac{a \cos \left[(\nu + 1) \arcsin \left(\frac{b}{a} \right) \right]}{2\nu(\nu + 1)} \\
&\quad [0 < b < a, \quad \operatorname{Re} \nu > 1]; \\
&= \frac{a^\nu \sin \left(\frac{\nu\pi}{2} \right)}{2\nu(\nu - 1) \left[b + \sqrt{b^2 - a^2} \right]^{\nu-1}} - \frac{a^{\nu+2} \sin \left(\frac{\nu\pi}{2} \right)}{2\nu(\nu + 1) \left[b + \sqrt{b^2 - a^2} \right]^{\nu+1}} \\
&\quad [0 < a < b, \quad \operatorname{Re} \nu > 1].
\end{aligned}$$

ET I 44(6)

$$\begin{aligned}
6. \int_0^\infty J_0(\alpha x) \sin x \frac{dx}{x} &= \frac{\pi}{2} \quad [0 < \alpha < 1]; \\
&= \operatorname{arccosec} \alpha \quad [\alpha > 1].
\end{aligned}$$

WH

$$\begin{aligned}
7. \int_0^\infty J_0(x) \sin \beta x \frac{dx}{x} &= \frac{\pi}{2} \quad [\beta > 1]; \\
&= \arcsin \beta \quad [\beta^2 < 1]; \\
&= -\frac{\pi}{2} \quad [\beta < -1].
\end{aligned}$$

NT 66(13)

$$9. \int_0^z J_\nu(x) \sin(z-x) \frac{dx}{x} = \frac{2}{\nu} \sum_{k=0}^{\infty} (-1)^k J_{\nu+2k+1}(z) \quad [\operatorname{Re} \nu > 0].$$

WA 416(4)

$$10. \int_0^z J_\nu(x) \cos(z-x) \frac{dx}{x} = \frac{1}{\nu} J_\nu(z) + \frac{2}{\nu} \sum_{k=1}^{\infty} (-1)^k J_{\nu+2k}(z) \quad [\operatorname{Re} \nu > 0].$$

WA 416(5)

762

6.694*

$$\begin{aligned} \int_0^\infty \left[\frac{J_1(ax)}{x} \right]^2 \sin(bx) dx &= \\ &= \frac{1}{2}b - \left(\frac{4a}{3\pi} \right) \left[\left(1 + \frac{b^2}{4a^2} \right) \mathbf{E} \left(\frac{b}{2a} \right) + \left(1 - \frac{b^2}{4a^2} \right) \mathbf{K} \left(\frac{b}{2a} \right) \right] \quad [0 \leq b \leq 2a]. \\ &= \frac{1}{2}b - \frac{2b}{3\pi} \left[\left(1 + \frac{b^2}{4a^2} \right) \mathbf{E} \left(\frac{2a}{b} \right) - \left(1 - \left(\frac{4a^2}{b^2} \right)^{-1} \right) \mathbf{K} \left(\frac{2a}{b} \right) \right] \quad [0 \leq 2a \leq b]. \end{aligned}$$

ET I 102(22)

6.695

$$1. \int_0^\infty \frac{\sin \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\operatorname{sh} \alpha \beta}{\beta} K_0(\beta u) \quad [\alpha > 0, \operatorname{Re} \beta > 0, u > \alpha].$$

MO 46

$$2. \int_0^\infty \frac{\cos \alpha x}{\beta^2 + x^2} J_0(ux) dx = \frac{\pi e^{-\alpha \beta}}{2\beta} I_0(\beta u) \quad [\alpha > 0, \operatorname{Re} \beta > 0, -\alpha < u < \alpha].$$

MO 46

$$3. \int_0^\infty \frac{x}{x^2 + \beta^2} \sin(\alpha x) J_0(\gamma x) dx = \frac{\pi}{2} e^{-\alpha \beta} I_0(\gamma \beta) \quad [\alpha > 0, \operatorname{Re} \beta > 0, 0 < \gamma < \alpha].$$

ET II 10(36)

6.696

$$\int_0^{\infty} [1 - \cos(\alpha x)] J_0(\beta x) \frac{dx}{x} = \text{Arch} \left(\frac{\alpha}{\beta} \right) \quad [0 < \beta < \alpha];$$

$$= 0 \quad [0 < \alpha < \beta].$$

ET II 11(43)

6.697

$$1. \int_{-\infty}^{\infty} \frac{\sin[\alpha(x + \beta)]}{x + \beta} J_0(x) dx = 2 \int_0^{\alpha} \frac{\cos \beta u}{\sqrt{1 - u^2}} du \quad [0 \leq \alpha \leq 1];$$

$$= \pi J_0(\beta) \quad [1 \leq \alpha < \infty].$$

WA 463(1), ET II 345(42)

WA 463(2)

$$2. \int_0^{\infty} \frac{\sin(x + t)}{x + t} J_0(t) dt = \frac{\pi}{2} J_0(x) \quad [x > 0].$$

WA 475(4)

$$3. \int_0^{\infty} \frac{\cos(x + t)}{x + t} J_0(t) dt = -\frac{\pi}{2} N_0(x) \quad [x > 0].$$

WA 475(5)

$$4. \int_{-\infty}^{\infty} \frac{|x|}{x + \beta} \sin[\alpha(x + \beta)] J_0(bx) dx = 0 \quad [0 \leq \alpha < b].$$

WA 464(5), ET II 345(43)a

$$5. \int_{-\infty}^{\infty} \frac{\sin[\alpha(x + \beta)]}{x + \beta} [J_{n+\frac{1}{2}}(x)]^2 dx = \pi [J_{n+\frac{1}{2}}(\beta)]^2 \quad [2 \leq \alpha < \infty, \quad n = 0, 1, \dots].$$

ET II 346(45)

$$6. \int_{-\infty}^{\infty} \frac{\sin[\alpha(x + \beta)]}{x + \beta} J_{n+\frac{1}{2}}(x) J_{-n-\frac{1}{2}}(x) dx = \pi J_{n+\frac{1}{2}}(\beta) J_{-n-\frac{1}{2}}(\beta) \quad [2 \leq \alpha < \infty, \quad n = 0, 1, \dots].$$

ET II 346(46)

$$7. \int_{-\infty}^{\infty} \frac{J_{\mu}[a(z+x)]}{(z+x)^{\mu}} \frac{J_{\nu}[a(\zeta+x)]}{(\zeta+x)^{\nu}} dx = \frac{\Gamma(\mu+\nu)\sqrt{\pi}\sqrt{\frac{2}{a}}}{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \frac{J_{\mu+\nu-\frac{1}{2}}[a(z-\zeta)]}{(z-\zeta)^{\mu+\nu-\frac{1}{2}}} \\ [\operatorname{Re}(\mu+\nu) > 0].$$

WA 463(3)

6.698

$$1. \int_0^{\infty} \sqrt{x} J_{\nu+\frac{1}{4}}(ax) J_{-\nu+\frac{1}{4}}(ax) \sin(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \quad [0 < b < 2a]; \\ = 0 \quad [0 < 2a < b].$$

ET I 102(26)

$$2. \int_0^{\infty} \sqrt{x} J_{\nu-\frac{1}{4}}(ax) J_{-\nu-\frac{1}{4}}(ax) \cos(bx) dx = \sqrt{\frac{2}{\pi b}} \frac{\cos\left[2\nu \arccos\left(\frac{b}{2a}\right)\right]}{\sqrt{4a^2-b^2}} \quad [0 < b < 2a]; \\ = 0 \quad [0 < 2a < b].$$

ET I 46(24)

$$3. \int_0^{\infty} \sqrt{x} I_{\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \sin(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b+\sqrt{a^2+b^2})^{2\nu}}{\sqrt{a^2+b^2}} \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{5}{4} \right].$$

ET I 106(56)

$$4. \int_0^{\infty} \sqrt{x} I_{-\frac{1}{4}-\nu}\left(\frac{1}{2}ax\right) K_{-\frac{1}{4}+\nu}\left(\frac{1}{2}ax\right) \cos(bx) dx = \sqrt{\frac{\pi}{2b}} a^{-2\nu} \frac{(b+\sqrt{a^2+b^2})^{2\nu}}{\sqrt{a^2+b^2}} \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu < \frac{3}{4} \right].$$

ET I 50(49)

6.699

$$1. \int_0^{\infty} x^{\lambda} J_{\nu}(ax) \sin(bx) dx = 2^{1+\lambda} a^{-(2+\lambda)} b \frac{\Gamma\left(\frac{2+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda}{2}\right)} F\left(\frac{2+\lambda+\nu}{2}, \frac{2+\lambda-\nu}{2}; \frac{3}{2}; \frac{b^2}{a^2}\right)$$

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$$\left[0 < b < a, \quad -\operatorname{Re} \nu - 1 < 1 + \operatorname{Re} \lambda < \frac{3}{2} \right].$$

$$\begin{aligned}
2. \int_0^\infty x^\lambda J_\nu(ax) \cos(bx) dx &= \\
&= \frac{2^\lambda a^{-(1+\lambda)} \Gamma\left(\frac{1+\lambda+\nu}{2}\right)}{\Gamma\left(\frac{\nu-\lambda+1}{2}\right)} F\left(\frac{1+\lambda+\nu}{2}, \frac{1+\lambda-\nu}{2}; \frac{1}{2}; \frac{b^2}{a^2}\right) \\
&\quad \left[0 < b < a, \quad -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}\right]; \\
&= \frac{\left(\frac{a}{2}\right)^\nu b^{-(\nu+1+\lambda)} \Gamma(1+\lambda+\nu) \cos\left[\frac{\pi}{2}(1+\lambda+\nu)\right]}{\Gamma(\nu+1)} F\left(\frac{1+\lambda+\nu}{2}, \frac{2+\lambda+\nu}{2}; \nu+1; \frac{a^2}{b^2}\right) \\
&\quad \left[0 < a < b, \quad -\operatorname{Re} \nu < 1 + \operatorname{Re} \lambda < \frac{3}{2}\right].
\end{aligned}$$

ET I 45(13)

$$\begin{aligned}
3. \int_0^\infty x^\lambda K_\mu(ax) \sin(bx) dx &= \frac{2^\lambda b \Gamma\left(\frac{2+\mu+\lambda}{2}\right) \Gamma\left(\frac{2+\lambda-\mu}{2}\right)}{a^{2+\lambda}} F\left(\frac{2+\mu+\lambda}{2}, \frac{2+\lambda-\mu}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 2, \quad \operatorname{Re} a > 0, \quad b > 0].
\end{aligned}$$

ET I 106(50)

$$\begin{aligned}
4. \int_0^\infty x^\lambda K_\mu(ax) \cos(bx) dx &= 2^{\lambda-1} a^{-\lambda-1} \Gamma\left(\frac{\mu+\lambda+1}{2}\right) \Gamma\left(\frac{1+\lambda-\mu}{2}\right) \times \\
&\quad \times F\left(\frac{\mu+\lambda+1}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; -\frac{b^2}{a^2}\right) \\
&\quad [\operatorname{Re}(-\lambda \pm \mu) < 1, \quad \operatorname{Re} a > 0, \quad b > 0].
\end{aligned}$$

ET I 49(42)

$$\begin{aligned}
5. \int_0^\infty x^\nu \sin(ax) J_\nu(bx) dx &= \frac{\sqrt{\pi} 2^\nu b^\nu (a^2 - b^2)^{-\nu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \nu\right)} \left[0 < b < a, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}\right]; \\
&= 0 \quad \left[0 < a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}\right].
\end{aligned}$$

ET II 32(4)

$$\begin{aligned}
6. \int_0^\infty x^\nu \cos(ax) J_\nu(bx) dx &= \\
&= -2^\nu \frac{\sin(\nu\pi)}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) b^\nu (a^2 - b^2)^{-\nu-\frac{1}{2}} \quad \left[0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right]; \\
&= 2^\nu \frac{b^\nu}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \nu\right) (b^2 - a^2)^{-\nu-\frac{1}{2}} \quad \left[0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right].
\end{aligned}$$

$$\begin{aligned}
7. \int_0^\infty x^{\nu+1} \sin(ax) J_\nu(bx) dx &= \\
&= -2^{1+\nu} a \frac{\sin(\nu\pi)}{\sqrt{\pi}} b^\nu \Gamma\left(\nu + \frac{3}{2}\right) (a^2 - b^2)^{-\nu - \frac{3}{2}} \quad \left[0 < b < a, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}\right]; \\
&= -\frac{2^{1+\nu}}{\sqrt{\pi}} ab^\nu \Gamma\left(\nu + \frac{3}{2}\right) (b^2 - a^2)^{-\nu - \frac{3}{2}} \quad \left[0 < a < b, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2}\right].
\end{aligned}$$

ET II 32(3)

765

$$\begin{aligned}
8. \int_0^\infty x^{\nu+1} \cos(ax) J_\nu(bx) dx &= 2^{1+\nu} \sqrt{\pi} ab^\nu \frac{(a^2 - b^2)^{-\nu - \frac{3}{2}}}{\Gamma\left(-\frac{1}{2} - \nu\right)} \quad \left[0 < b < a, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}\right]; \\
&= 0 \quad \left[0 < a < b, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}\right].
\end{aligned}$$

ET II 36(28)

$$9. \int_0^1 x^\nu \sin(ax) J_\nu(ax) dx = \frac{1}{2\nu + 1} [\sin a J_\nu(a) - \cos a J_{\nu+1}(a)] \quad [\operatorname{Re} \nu > -1].$$

ET II 334(9)a

$$10. \int_0^1 x^\nu \cos(ax) J_\nu(ax) dx = \frac{1}{2\nu + 1} [\cos a J_\nu(a) + \sin a J_{\nu+1}(a)] \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 335(20)

$$\begin{aligned}
11. \int_0^\infty x^{1+\nu} K_\nu(ax) \sin(bx) dx &= \sqrt{\pi} (2a)^\nu \Gamma\left(\frac{3}{2} + \nu\right) b (b^2 + a^2)^{-\frac{3}{2} - \nu} \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}\right].
\end{aligned}$$

ET I 105(49)

$$\begin{aligned}
12. \int_0^\infty x^\mu K_\mu(ax) \cos(bx) dx &= \frac{1}{2} \sqrt{\pi} (2a)^\mu \Gamma\left(\mu + \frac{1}{2}\right) (b^2 + a^2)^{-\mu - \frac{1}{2}} \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}\right].
\end{aligned}$$

ET I 49(41)

ET I 104(36)

$$14. \int_0^{\infty} x^{\nu} N_{\nu}(ax) \cos(bx) dx = 0 \quad \left[0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right];$$

$$= -2^{\nu} \sqrt{\pi} a^{\nu} \frac{(b^2 - a^2)^{-\nu - \frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \nu\right)} \quad \left[0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 47(30)

6.711

$$1. \int_0^{\infty} x^{\nu - \mu} J_{\mu}(ax) J_{\nu}(bx) \sin(cx) dx = 0 \quad [0 < c < b - a, \quad -1 < \operatorname{Re} \nu < 1 + \operatorname{Re} \mu].$$

ET I 103(28)

$$2. \int_0^{\infty} x^{\nu - \mu + 1} J_{\mu}(ax) J_{\nu}(bx) \cos(cx) dx = 0 \quad [0 < c < b - a, \quad a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \operatorname{Re} \mu].$$

ET I 47(25)

766

$$3. \int_0^{\infty} x^{\nu - \mu - 2} J_{\mu}(ax) J_{\nu}(bx) \sin(cx) dx = 2^{\nu - \mu - 1} a^{\mu} b^{-\nu} \frac{c \Gamma(\nu)}{\Gamma(\mu + 1)}$$

$$[0 < a, \quad 0 < b, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu + 3].$$

ET I 103(29)

$$4. \int_0^{\infty} x^{\varrho - \mu - 1} J_{\mu}(ax) J_{\varrho}(bx) \cos(cx) dx = 2^{\varrho - \mu - 1} b^{-\varrho} a^{\mu} \frac{\Gamma(\varrho)}{\Gamma(\mu + 1)}$$

$$[b > 0, \quad a > 0, \quad 0 < c < b - a, \quad 0 < \operatorname{Re} \varrho < \operatorname{Re} \mu + 2].$$

ET I 47(26)

$$5. \int_0^{\infty} x^{1 - 2\nu} \sin(2ax) J_{\nu}(x) N_{\nu}(x) dx = -\frac{\Gamma\left(\frac{3}{2} - \nu\right) a}{2\Gamma\left(2\nu - \frac{1}{2}\right) \Gamma(2 - \nu)} F\left(\frac{3}{2} - \nu, \frac{3}{2} - 2\nu; 2 - \nu; a^2\right)$$

$$\left[0 < \operatorname{Re} \nu < \frac{3}{2}, \quad 0 < a < 1 \right].$$

$$1. \int_0^{\infty} x^{\nu} [J_{\nu}(ax) \cos(ax) + N_{\nu}(ax) \sin(ax)] \sin(bx) dx = \frac{\sqrt{\pi}(2a)^{\nu}}{\Gamma\left(\frac{1}{2} - \nu\right)} (b^2 + 2ab)^{-\nu - \frac{1}{2}}$$

$$\left[b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET I 104(40)

$$2. \int_0^{\infty} x^{\nu} [N_{\nu}(ax) \cos(ax) - J_{\nu}(ax) \sin(ax)] \cos(bx) dx = -\frac{\sqrt{\pi}(2a)^{\nu}}{\Gamma\left(\frac{1}{2} - \nu\right)} (b^2 + 2ab)^{-\nu - \frac{1}{2}}.$$

ET I 48(35)

$$3. \int_0^{\infty} x^{\nu} [J_{\nu}(ax) \cos(ax) - N_{\nu}(ax) \sin(ax)] \sin(bx) dx =$$

$$= 0 \quad \left[0 < b < 2a, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right];$$

$$= \frac{2^{\nu} \sqrt{\pi} b^{\nu}}{\Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 2ab)^{-\nu - \frac{1}{2}} \quad \left[2a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET I 104(41)

$$4. \int_0^{\infty} x^{\nu} [J_{\nu}(ax) \sin(ax) + N_{\nu}(ax) \cos(ax)] \cos(bx) dx$$

$$= 0 \quad \left[0 < b < 2a, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right];$$

$$= -\frac{\sqrt{\pi}(2a)^{\nu}}{\Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 2ab)^{-\nu - \frac{1}{2}} \quad \left[0 < 2a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 48(33)

767

6.713

$$1. \int_0^{\infty} x^{1-2\nu} \sin(2ax) \left\{ [J_{\nu}(x)]^2 - [N_{\nu}(x)]^2 \right\} dx =$$

$$= \frac{\sin(2\nu\pi) \Gamma\left(\frac{3}{2} - \nu\right) \Gamma\left(\frac{3}{2} - 2\nu\right) a}{\pi \Gamma(2 - \nu)} F\left(\frac{3}{2} - \nu, \frac{3}{2} - 2\nu; 2 - \nu; a^2\right)$$

$$\left[0 < \operatorname{Re} \nu < \frac{3}{4}, \quad 0 < a < 1 \right].$$

ET II 348(64)

$$\begin{aligned}
3. \int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x)N_{\nu-1}(x) + N_\nu(x)J_{\nu-1}(x)] dx &= \\
&= -\frac{\Gamma\left(\frac{3}{2}-\nu\right)a}{\Gamma\left(2\nu-\frac{3}{2}\right)\Gamma(2-\nu)} F\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right) \quad \left[\frac{1}{2} < \operatorname{Re} \nu < \frac{5}{2}, \quad 0 < a < 1\right].
\end{aligned}$$

6.714

$$\begin{aligned}
1. \int_0^\infty \sin(2ax) [x^\nu J_\nu(x)]^2 dx &= \\
&= \frac{a^{-2\nu}\Gamma\left(\frac{1}{2}+\nu\right)}{2\sqrt{\pi}\Gamma(1-\nu)} F\left(\frac{1}{2}+\nu, \frac{1}{2}; 1-\nu; a^2\right) \quad \left[0 < a < 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right], \\
&= \frac{a^{-4\nu-1}\Gamma\left(\frac{1}{2}+\nu\right)}{2\Gamma(1+\nu)\Gamma\left(\frac{1}{2}-2\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \quad \left[a > 1, \quad |\operatorname{Re} \nu| < \frac{1}{2}\right].
\end{aligned}$$

768

$$\begin{aligned}
2. \int_0^\infty \cos(2ax) [x^\nu J_\nu(x)]^2 dx &= \frac{a^{-2\nu}\Gamma(\nu)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\nu+\frac{1}{2}, \frac{1}{2}; 1-\nu; a^2\right) + \\
&\quad + \frac{\Gamma(-\nu)\Gamma\left(\frac{1}{2}+2\nu\right)}{2\pi\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^2\right) \\
&\quad \left[0 < a < 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}\right]; \\
&= -\frac{\sin(\nu\pi)a^{-4\nu-1}\Gamma\left(\frac{1}{2}+2\nu\right)}{\Gamma(1+\nu)\Gamma\left(\frac{1}{2}-\nu\right)} F\left(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; \frac{1}{a^2}\right) \\
&\quad \left[a > 1, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}\right].
\end{aligned}$$

6.715

$$1. \int_0^\infty \frac{x^\nu}{x+\beta} \sin(x+\beta) J_\nu(x) dx = \frac{\pi}{2} \sec(\nu\pi) \beta^\nu J_{-\nu}(\beta) \quad \left[|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 340(8)

$$2. \int_0^\infty \frac{x^\nu}{x+\beta} \cos(x+\beta) J_\nu(x) dx = -\frac{\pi}{2} \sec(\nu\pi) \beta^\nu N_{-\nu}(\beta) \quad \left[|\arg \beta| < \pi, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 340(9)

6.716

$$1. \int_0^a x^\lambda \sin(a-x) J_\nu(x) dx = 2a^{\lambda+1} \sum_{n=0}^\infty \frac{(-1)^n \Gamma(\nu - \lambda + 2n) \Gamma(\nu + \lambda + 1)}{\Gamma(\nu - \lambda) \Gamma(\nu + \lambda + 3 + 2n)} (\nu + 2n + 1) J_{\nu+2n+1}(a) \\ [\operatorname{Re}(\lambda + \nu) > -1].$$

ET II 335(16)

$$2. \int_0^a x^\lambda \cos(a-x) J_\nu(x) dx = \frac{a^{\lambda+1} J_\nu(a)}{\lambda + \nu + 1} + 2a^{\lambda+1} \times \\ \times \sum_{n=1}^\infty \frac{(-1)^n \Gamma(\nu - \lambda + 2n - 1) \Gamma(\nu + \lambda + 1)}{\Gamma(\nu - \lambda) \Gamma(\nu + \lambda + 2n + 2)} (\nu + 2n) J_{\nu+2n}(a) \\ [\operatorname{Re}(\lambda + \nu) > -1].$$

ET II 335(26)

6.717

$$\int_{-\infty}^\infty \frac{\sin[a(x+\beta)]}{x^\nu(x+\beta)} J_{\nu+2n}(x) dx = \pi \beta^{-\nu} J_{\nu+2n}(\beta) \quad \left[1 \leq a < \infty, \quad n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 345(44)

769

6.718

$$1. \int_0^\infty \frac{x^\nu}{x^2 + \beta^2} \sin(\alpha x) J_\nu(\gamma x) dx = \beta^{\nu-1} \operatorname{sh}(\alpha\beta) K_\nu(\beta\gamma) \\ \left[0 < \alpha \leq \gamma, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{3}{2} \right].$$

$$2. \int_0^{\infty} \frac{x^{\nu+1}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \beta^{\nu} \operatorname{ch}(\alpha\beta) K_{\nu}(\beta\gamma) \left[0 < \alpha \leq \gamma, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 37(33)

$$3. \int_0^{\infty} \frac{x^{1-\nu}}{x^2 + \beta^2} \sin(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu} e^{-\alpha\beta} I_{\nu}(\beta\gamma) \left[0 < \gamma \leq \alpha, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 33(9)

$$4. \int_0^{\infty} \frac{x^{-\nu}}{x^2 + \beta^2} \cos(\alpha x) J_{\nu}(\gamma x) dx = \frac{\pi}{2} \beta^{-\nu-1} e^{-\alpha\beta} I_{\nu}(\beta\gamma) \left[0 < \gamma \leq \alpha, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 37(34)

6.719

$$1.^6 \int_0^{\alpha} \frac{\sin(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\alpha\beta) J_{\frac{1}{2}\nu+n+\frac{1}{2}} \left(\frac{1}{2}\alpha \right) J_{\frac{1}{2}\nu-n-\frac{1}{2}} \left(\frac{1}{2}\alpha \right) \quad [\operatorname{Re} \nu > -2].$$

ET II 335(17)

$$2. \int_0^{\alpha} \frac{\cos(\beta x)}{\sqrt{\alpha^2 - x^2}} J_{\nu}(x) dx = \frac{\pi}{2} J_0(\alpha\beta) \left[J_{\frac{1}{2}\nu} \left(\frac{1}{2}\alpha \right) \right]^2 + \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(\alpha\beta) J_{\frac{1}{2}\nu+n} \left(\frac{1}{2}\alpha \right) J_{\frac{1}{2}\nu-n} \left(\frac{1}{2}\alpha \right). \quad [\operatorname{Re} \nu > -1].$$

ET II 336(27)

6.721

$$1. \int_0^{\infty} \sqrt{x} J_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = 2^{-\frac{3}{2}} a^{-2} \sqrt{\pi b} J_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \quad [b > 0].$$

ET I 108(1)

$$2. \int_0^{\infty} \sqrt{x} J_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = 2^{-\frac{3}{2}} a^{-2} \sqrt{\pi b} J_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \quad [b > 0].$$

ET I 51(1)

$$3. \int_0^{\infty} \sqrt{x} N_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = -2^{-\frac{3}{2}} \sqrt{\pi b} a^{-2} H_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right).$$

$$4. \int_0^{\infty} \sqrt{x} N_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = -2^{-\frac{3}{2}} \sqrt{\pi} b a^{-2} H_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right).$$

ET I 52(7)

770

$$5. \int_0^{\infty} \sqrt{x} K_{\frac{1}{4}}(a^2 x^2) \sin(bx) dx = 2^{-\frac{5}{2}} \sqrt{\pi^3} b a^{-2} \left[I_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) - L_{\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \right] \quad \left[|\arg a| < \frac{\pi}{4}, b > 0 \right].$$

ET I 109(11)

$$6. \int_0^{\infty} \sqrt{x} K_{-\frac{1}{4}}(a^2 x^2) \cos(bx) dx = 2^{-\frac{5}{2}} \sqrt{\pi^3} b a^{-2} \left[I_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) - L_{-\frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \right] \quad [b > 0].$$

ET I 52(10)

6.722

$$1. \int_0^{\infty} \sqrt{x} K_{\frac{1}{8}+\nu}(a^2 x^2) I_{\frac{1}{8}-\nu}(a^2 x^2) \sin(bx) dx = \sqrt{2\pi} b^{-\frac{3}{2}} \frac{\Gamma\left(\frac{5}{8}-\nu\right)}{\Gamma\left(\frac{5}{4}\right)} W_{\nu, \frac{1}{8}} \left(\frac{b^2}{8a^2} \right) M_{-\nu, \frac{1}{8}} \left(\frac{b^2}{8a^2} \right) \left[\operatorname{Re} \nu < \frac{5}{8}, |\arg a| < \frac{\pi}{4}, b > 0 \right].$$

ET I 109(13)

$$2. \int_0^{\infty} \sqrt{x} J_{-\frac{1}{8}-\nu}(a^2 x^2) J_{-\frac{1}{8}+\nu}(a^2 x^2) \cos(bx) dx = \sqrt{\frac{2}{\pi}} b^{-\frac{3}{2}} \left[e^{-\frac{i\pi}{8}} W_{\nu, -\frac{1}{8}} \left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2} \right) W_{-\nu, -\frac{1}{8}} \left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2} \right) + e^{\frac{i\pi}{8}} W_{\nu, -\frac{1}{8}} \left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2} \right) W_{-\nu, -\frac{1}{8}} \left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2} \right) \right] \quad [b > 0].$$

ET I 52(6)

$$3. \int_0^{\infty} \sqrt{x} J_{\frac{1}{8}-\nu}(a^2 x^2) J_{\frac{1}{8}+\nu}(a^2 x^2) \sin(bx) dx = \sqrt{\frac{2}{\pi}} b^{-\frac{3}{2}} \left[e^{\frac{\pi i}{8}} W_{\nu, \frac{1}{8}} \left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2} \right) W_{-\nu, \frac{1}{8}} \left(\frac{b^2 e^{\frac{\pi i}{2}}}{8a^2} \right) + e^{-\frac{i\pi}{8}} W_{\nu, \frac{1}{8}} \left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2} \right) W_{-\nu, \frac{1}{8}} \left(\frac{b^2 e^{-\frac{\pi i}{2}}}{8a^2} \right) \right] \quad [b > 0].$$

$$4. \int_0^{\infty} \sqrt{x} K_{\frac{1}{8}-\nu}(a^2 x^2) I_{-\frac{1}{8}-\nu}(a^2 x^2) \cos(bx) dx = \sqrt{2\pi} b^{-\frac{3}{2}} \frac{\Gamma\left(\frac{3}{8}-\nu\right)}{\Gamma\left(\frac{3}{4}\right)} W_{\nu, -\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) M_{-\nu, -\frac{1}{8}}\left(\frac{b^2}{8a^2}\right) \\ \left[\operatorname{Re} \nu < \frac{3}{8}, \quad b > 0 \right].$$

ET I 52(12)

771

6.723

$$\int_0^{\infty} x J_{\nu}(x^2) [\sin(\nu\pi) J_{\nu}(x^2) - \cos(\nu\pi) N_{\nu}(x^2)] J_{4\nu}(4ax) dx = \frac{1}{4} J_{\nu}(a^2) J_{-\nu}(a^2) \\ [a > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 375(20)

6.724

$$1. \int_0^{\infty} x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \sin(bx) dx = \\ = \frac{\sqrt{\pi} a^{2\nu} \Gamma(\lambda - \nu + 1) b^{2\nu-2\lambda-1}}{4^{2\nu-\lambda} \Gamma(2\nu+1) \Gamma\left(\nu - \lambda + \frac{1}{2}\right)} {}_0F_3\left(2\nu+1, \nu-\lambda, \nu-\lambda + \frac{1}{2}; \frac{a^2 b^2}{16}\right) + \\ + \frac{a^{2\lambda+2} \Gamma(\nu - \lambda - 1) b}{2^{2\lambda+3} \Gamma(\nu + \lambda + 2)} {}_0F_3\left(\frac{3}{2}, \lambda - \nu + 2, \lambda + \nu + 2; \frac{a^2 b^2}{16}\right) \\ \left[-\frac{5}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu, \quad a > 0, \quad b > 0 \right].$$

ET I 109(15)

$$2. \int_0^{\infty} x^{2\lambda} J_{2\nu}\left(\frac{a}{x}\right) \cos(bx) dx = \\ = 4^{\lambda-2\nu} \sqrt{\pi} a^{2\nu} b^{2\nu-2\lambda-1} \frac{\Gamma\left(\lambda - \nu + \frac{1}{2}\right)}{\Gamma(2\nu+1) \Gamma(\nu - \lambda)} {}_0F_3\left(2\nu+1, \nu - \lambda + \frac{1}{2}, \nu - \lambda; \frac{a^2 b^2}{16}\right) + \\ + 4^{-\lambda-1} a^{2\lambda+1} \frac{\Gamma\left(\nu - \lambda - \frac{1}{2}\right)}{\Gamma\left(\nu + \lambda + \frac{3}{2}\right)} {}_0F_3\left(\frac{1}{2}, \lambda - \nu + \frac{3}{2}, \nu + \lambda + \frac{3}{2}; \frac{a^2 b^2}{16}\right) \\ \left[-\frac{3}{4} < \operatorname{Re} \lambda < \operatorname{Re} \nu - \frac{1}{2}, \quad a > 0, \quad b > 0 \right].$$

ET I 53(14)

6.725

$$1. \int_0^{\infty} \frac{\sin(bx)}{\sqrt{x}} J_{\nu}(a\sqrt{x}) dx = -\sqrt{\frac{\pi}{b}} \sin\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{\nu}{2}}\left(\frac{a^2}{8b}\right) \\ [\operatorname{Re} \nu > -3, \quad a > 0, \quad b > 0].$$

$$2. \int_0^{\infty} \frac{\cos(bx)}{\sqrt{x}} J_{\nu}(a\sqrt{x}) dx = \sqrt{\frac{\pi}{b}} \cos\left(\frac{a^2}{8b} - \frac{\nu\pi}{4} - \frac{\pi}{4}\right) J_{\frac{1}{2}\nu}\left(\frac{a^2}{8b}\right) \\ [\operatorname{Re} \nu > -1, \quad a > 0, \quad b > 0].$$

ET I 54(25)

$$3. \int_0^{\infty} x^{\frac{1}{2}\nu} J_{\nu}(a\sqrt{x}) \sin(bx) dx = 2^{-\nu} a^{\nu} b^{-\nu-1} \cos\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \left[-2 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0\right].$$

ET I 110(28)

772

$$4. \int_0^{\infty} x^{\frac{1}{2}\nu} J_{\nu}(a\sqrt{x}) \cos(bx) dx = 2^{-\nu} b^{-\nu-1} a^{\nu} \sin\left(\frac{a^2}{4b} - \frac{\nu\pi}{2}\right) \left[-1 < \operatorname{Re} \nu < \frac{1}{2}, \quad a > 0, \quad b > 0\right].$$

ET I 54(26)

6.726

$$1. \int_0^{\infty} x(x^2 + b^2)^{-\frac{1}{2}\nu} J_{\nu}(a\sqrt{x^2 + b^2}) \sin(cx) dx = \\ = \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{3}{2}} c(a^2 - c^2)^{\frac{1}{2}\nu-\frac{3}{4}} J_{\nu-\frac{3}{2}}(b\sqrt{a^2 - c^2}) \left[0 < c < a, \quad \operatorname{Re} \nu > \frac{1}{2}\right]; \\ = 0 \quad \left[0 < a < c, \quad \operatorname{Re} \nu > \frac{1}{2}\right].$$

ET I 111(37)

$$2. \int_0^{\infty} (x^2 + b^2)^{-\frac{1}{2}\nu} J_{\nu}(a\sqrt{x^2 + b^2}) \cos(cx) dx = \\ = \sqrt{\frac{\pi}{2}} a^{-\nu} b^{-\nu+\frac{1}{2}} (a^2 - c^2)^{\frac{1}{2}\nu-\frac{1}{4}} J_{\nu-\frac{1}{2}}(b\sqrt{a^2 - c^2}) \left[0 < c < a, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right]; \\ = 0 \quad \left[0 < a < c, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET I 55(37)

$$3. \int_0^{\infty} x(x^2 + b^2)^{\frac{1}{2}\nu} K_{\pm\nu}(a\sqrt{x^2 + b^2}) \sin(cx) dx = \sqrt{\frac{\pi}{2}} a^{\nu} b^{\nu+\frac{3}{2}} c(a^2 + c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} K_{-\nu-\frac{3}{2}}(b\sqrt{a^2 + c^2}) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0].$$

$$4. \int_0^{\infty} (x^2 + b^2)^{\mp \frac{1}{2}\nu} K_{\nu} \left(a\sqrt{x^2 + b^2} \right) \cos(cx) dx = \sqrt{\frac{\pi}{2}} a^{\mp \nu} b^{\frac{1}{2}\mp \nu} (a^2 + c^2)^{\pm \frac{1}{2}\nu - \frac{1}{4}} K_{\pm \nu - \frac{1}{2}} \left(b\sqrt{a^2 + c^2} \right) \\ [\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0, \quad c > 0].$$

ET I 56(45)

$$5. \int_0^{\infty} (x^2 + a^2)^{-\frac{1}{2}\nu} N_{\nu} \left(b\sqrt{x^2 + a^2} \right) \cos(cx) dx = \\ = \sqrt{\frac{a\pi}{2}} (ab)^{-\nu} (b^2 - c^2)^{\frac{1}{2}\nu - \frac{1}{4}} N_{\nu - \frac{1}{2}} \left(a\sqrt{b^2 - c^2} \right) \quad \left[0 < c < b, \quad a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right]; \\ = -\sqrt{\frac{2a}{\pi}} (ab)^{-\nu} (c^2 - b^2)^{\frac{1}{2}\nu - \frac{1}{4}} K_{\nu - \frac{1}{2}} \left(a\sqrt{c^2 - b^2} \right) \quad \left[0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 56(41)

6.727

$$1. \int_0^a \frac{\sin(cx)}{\sqrt{a^2 - x^2}} J_{\nu} \left(b\sqrt{a^2 - x^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - c) \right] J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + c) \right] \\ [\operatorname{Re} \nu > -1, \quad c > 0, \quad a > 0].$$

ET I 113(48)

773

$$2. \int_a^{\infty} \frac{\sin(cx)}{\sqrt{x^2 - a^2}} J_{\nu} \left(b\sqrt{x^2 - a^2} \right) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} (c - \sqrt{c^2 + b^2}) \right] J_{-\frac{1}{2}\nu} \left[\frac{a}{2} (c + \sqrt{c^2 + b^2}) \right] \\ [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 113(49)

$$3. \int_a^{\infty} \frac{\cos(cx)}{\sqrt{x^2 - a^2}} J_{\nu} \left(b\sqrt{x^2 - a^2} \right) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} (c - \sqrt{c^2 - b^2}) \right] N_{-\frac{1}{2}\nu} \left[\frac{a}{2} (c + \sqrt{c^2 - b^2}) \right] \\ [0 < b < c, \quad a > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 58(54)

$$4. \int_0^a (a^2 - x^2)^{\frac{1}{2}\nu} \cos I_{\nu} \left(\sqrt{a^2 - x^2} \right) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma \left(\nu + \frac{3}{2} \right)} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 409(2)

6.728

$$1. \int_0^{\infty} x \sin(ax^2) J_{\nu}(bx) dx = \\ = \frac{\sqrt{\pi} b}{8a^{\frac{3}{2}}} \left[\cos \left(\frac{b^2}{8a} - \frac{\nu\pi}{4} \right) J_{\frac{1}{2}\nu - \frac{1}{2}} \left(\frac{b^2}{8a} \right) - \sin \left(\frac{b^2}{8a} - \frac{\nu\pi}{4} \right) J_{\frac{1}{2}\nu + \frac{1}{2}} \left(\frac{b^2}{8a} \right) \right] \\ \text{Printed From Table of Integrals, Series, and Products CD-ROM. Copyright } \text{\textcircled{c}} \text{ 1996 by Academic Press, Inc. All Rights reserved.} \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -4].$$

$$2. \int_0^{\infty} x \cos(ax^2) J_{\nu}(bx) dx = \frac{\sqrt{\pi}b}{8a^{\frac{3}{2}}} \left[\cos\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{b^2}{8a}\right) + \sin\left(\frac{b^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{b^2}{8a}\right) \right] \\ [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2].$$

ET II 38(39)

$$3. \int_0^{\infty} J_0(\beta x) \sin(\alpha x^2) x dx = \frac{1}{2\alpha} \cos \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0].$$

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$$4. \int_0^{\infty} J_0(\beta x) \cos(\alpha x^2) x dx = \frac{1}{2\alpha} \sin \frac{\beta^2}{4\alpha} \quad [\alpha > 0, \quad \beta > 0].$$

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$$5. \int_0^{\infty} x^{\nu+1} \sin(ax^2) J_{\nu}(bx) dx = \frac{b^{\nu}}{2^{\nu+1} a^{\nu+1}} \cos\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \quad \left[a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 34(15)

$$6. \int_0^{\infty} x^{\nu+1} \cos(ax^2) J_{\nu}(bx) dx = \frac{b^{\nu}}{2^{\nu+1} a^{\nu+1}} \sin\left(\frac{b^2}{4a} - \frac{\nu\pi}{2}\right) \quad \left[a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 38(40)

6.729

$$1. \int_0^{\infty} x \sin(ax^2) J_{\nu}(bx) J_{\nu}(cx) dx = \frac{1}{2a} \cos\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_{\nu}\left(\frac{bc}{2a}\right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -2].$$

ET II 51(26)

774

$$2. \int_0^{\infty} x \cos(ax^2) J_{\nu}(bx) J_{\nu}(cx) dx = \frac{1}{2a} \sin\left(\frac{b^2 + c^2}{4a} - \frac{\nu\pi}{2}\right) J_{\nu}\left(\frac{bc}{2a}\right) \\ [a > 0, \quad b > 0, \quad c > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 51(27)

6.731

$$\begin{aligned}
1. \int_0^{\infty} x \sin(ax^2) J_{\nu}(bx^2) J_{2\nu}(2cx) dx &= \\
&= \frac{1}{2\sqrt{b^2 - a^2}} \sin\left(\frac{ac^2}{b^2 - a^2}\right) J_{\nu}\left(\frac{bc^2}{b^2 - a^2}\right) \quad [0 < a < b, \operatorname{Re} \nu > -1]; \\
&= \frac{1}{2\sqrt{a^2 - b^2}} \cos\left(\frac{ac^2}{a^2 - b^2}\right) J_{\nu}\left(\frac{bc^2}{a^2 - b^2}\right) \quad [0 < b < a, \operatorname{Re} \nu > -1].
\end{aligned}$$

ET II 356(41)a

$$\begin{aligned}
2.* \int_0^{\infty} x \cos(ax^2) J_{\nu}(bx^2) J_{2\nu}(2cx) dx &= \\
&= \frac{1}{2\sqrt{b^2 - a^2}} \cos\left(\frac{ac^2}{b^2 - a^2}\right) J_{\nu}\left(\frac{bc^2}{b^2 - a^2}\right) \quad \left[0 < a < b, \operatorname{Re} \nu > -\frac{1}{2}\right]; \\
&= \frac{1}{2\sqrt{a^2 - b^2}} \sin\left(\frac{ac^2}{a^2 - b^2}\right) J_{\nu}\left(\frac{bc^2}{a^2 - b^2}\right) \quad \left[0 < b < a, \operatorname{Re} \nu > -\frac{1}{2}\right].
\end{aligned}$$

ET II 356(42)a

6.732

$$\int_0^{\infty} x^2 \cos\left(\frac{x^2}{2a}\right) N_1(x) K_1(x) dx = -a^3 K_0(a) \quad [a > 0].$$

ET II 371(52)

6.733

$$1. \int_0^{\infty} \sin\left(\frac{a}{2x}\right) [\sin x J_0(x) + \cos x N_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) N_0(\sqrt{a}) \quad [a > 0].$$

ET II 346(51)

$$2. \int_0^{\infty} \cos\left(\frac{a}{2x}\right) [\sin x N_0(x) - \cos x J_0(x)] \frac{dx}{x} = \pi J_0(\sqrt{a}) N_0(\sqrt{a}) \quad [a > 0].$$

ET II 347(52)

$$3. \int_0^{\infty} x \sin\left(\frac{a}{2x}\right) K_0(x) dx = \frac{\pi a}{2} J_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0].$$

ET II 368(34)

$$4. \int_0^{\infty} x \cos\left(\frac{a}{2x}\right) K_0(x) dx = -\frac{\pi a}{2} N_1(\sqrt{a}) K_1(\sqrt{a}) \quad [a > 0].$$

6.734

$$\int_0^{\infty} \cos(a\sqrt{x}) K_{\nu}(bx) \frac{dx}{\sqrt{x}} = \frac{\pi}{2\sqrt{b}} \sec(\nu\pi) \left[D_{\nu-\frac{1}{2}} \left(\frac{a}{\sqrt{2b}} \right) D_{-\nu-\frac{1}{2}} \left(-\frac{a}{\sqrt{2b}} \right) + D_{\nu-\frac{1}{2}} \left(-\frac{a}{\sqrt{2b}} \right) D_{-\nu-\frac{1}{2}} \left(\frac{a}{\sqrt{2b}} \right) \right] \left[\operatorname{Re} b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 132(27)

775

6.735

$$1. \int_0^{\infty} x^{\frac{1}{4}} \sin(2a\sqrt{x}) J_{-\frac{1}{4}}(x) dx = \sqrt{\pi} a^{\frac{3}{2}} J_{\frac{3}{4}}(a^2) \quad [a > 0].$$

ET II 341(10)

$$2. \int_0^{\infty} x^{\frac{1}{4}} \cos(2a\sqrt{x}) J_{\frac{1}{4}}(x) dx = \sqrt{\pi} a^{\frac{3}{2}} J_{-\frac{3}{4}}(a^2) \quad [a > 0].$$

ET II 341(12)

$$3. \int_0^{\infty} x^{\frac{1}{4}} \sin(2a\sqrt{x}) J_{\frac{3}{4}}(x) dx = \sqrt{\pi} a^{\frac{3}{2}} J_{-\frac{1}{4}}(a^2) \quad [a > 0].$$

ET II 341(11)

$$4. \int_0^{\infty} x^{\frac{1}{4}} \cos(2a\sqrt{x}) J_{-\frac{3}{4}}(x) dx = \sqrt{\pi} a^{\frac{3}{2}} J_{\frac{1}{4}}(a^2) \quad [a > 0].$$

ET II 341(13)

6.736

$$1. \int_0^{\infty} x^{-\frac{1}{2}} \sin \cos(4a\sqrt{x}) J_0(x) dx = -2^{-\frac{3}{2}} \sqrt{\pi} \left[\cos \left(a^2 - \frac{\pi}{4} \right) J_0(a^2) - \sin \left(a^2 - \frac{\pi}{4} \right) N_0(a^2) \right] \quad [a > 0].$$

ET II 341(18)

$$2. \int_0^{\infty} x^{-\frac{1}{2}} \cos x \cos(4a\sqrt{x}) J_0(x) dx = -2^{-\frac{3}{2}} \sqrt{\pi} \left[\sin \left(a^2 - \frac{\pi}{4} \right) J_0(a^2) + \cos \left(a^2 - \frac{\pi}{4} \right) N_0(a^2) \right]$$

$$3. \int_0^{\infty} x^{-\frac{1}{2}} \sin x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 + \frac{\pi}{4}\right) J_0(a^2) \quad [a > 0].$$

ET II 341(16)

$$4. \int_0^{\infty} x^{-\frac{1}{2}} \cos x \sin(4a\sqrt{x}) J_0(x) dx = \sqrt{\frac{\pi}{2}} \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) \quad [a > 0].$$

ET II 342(20)

$$5. \int_0^{\infty} x^{-\frac{1}{2}} \sin x \cos(4a\sqrt{x}) N_0(x) dx = 2^{-\frac{3}{2}} \sqrt{\pi} \left[3 \sin\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) - \cos\left(a^2 - \frac{\pi}{4}\right) N_0(a^2) \right] \\ [a > 0].$$

ET II 347(55)

$$6. \int_0^{\infty} x^{-\frac{1}{2}} \cos x \cos(4a\sqrt{x}) N_0(x) dx = -2^{-\frac{3}{2}} \sqrt{\pi} \left[3 \cos\left(a^2 - \frac{\pi}{4}\right) J_0(a^2) + \sin\left(a^2 - \frac{\pi}{4}\right) N_0(a^2) \right] \\ [a > 0].$$

ET II 347(56)

6.737

$$1. \int_0^{\infty} \frac{\sin(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_{\nu}(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{b}{2} (a - \sqrt{a^2 - c^2}) \right] J_{-\frac{1}{2}\nu} \left[\frac{b}{2} (a + \sqrt{a^2 - c^2}) \right] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re} \nu > -1].$$

ET II 35(19)

$$2. \int_0^{\infty} \frac{\cos(a\sqrt{x^2 + b^2})}{\sqrt{x^2 + b^2}} J_{\nu}(cx) dx = -\frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{b}{2} (a - \sqrt{a^2 - c^2}) \right] N_{-\frac{1}{2}\nu} \left[\frac{b}{2} (a + \sqrt{a^2 - c^2}) \right] \\ [a > 0, \quad \operatorname{Re} b > 0, \quad c > 0, \quad a > c, \quad \operatorname{Re} \nu > -1].$$

ET II 39(44)

776

$$3. \int_0^a \frac{\cos(b\sqrt{a^2 - x^2})}{\sqrt{a^2 - x^2}} J_{\nu}(cx) dx = \frac{\pi}{2} J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} - b) \right] J_{\frac{1}{2}\nu} \left[\frac{a}{2} (\sqrt{b^2 + c^2} + b) \right] \\ [c > 0, \quad \operatorname{Re} \nu > -1].$$

$$4. \int_0^a x^{\nu+1} \frac{\cos(\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} I_\nu(x) dx = \frac{\sqrt{\pi} a^{2\nu+1}}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)} \quad [\operatorname{Re} \nu > -1].$$

ET II 365(9)

$$5. \int_0^\infty x^{\nu+1} \frac{\sin(a\sqrt{b^2+x^2})}{\sqrt{b^2+x^2}} J_\nu(cx) dx = \sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2-c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} J_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \\ \left[0 < c < a, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}\right]; \\ = 0 \quad \left[0 < a < c, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}\right].$$

ET II 35(20)

$$6. \int_0^\infty x^{\nu+1} \frac{\cos(a\sqrt{x^2+b^2})}{\sqrt{x^2+b^2}} J_\nu(cx) dx = -\sqrt{\frac{\pi}{2}} b^{\frac{1}{2}+\nu} c^\nu (a^2-c^2)^{-\frac{1}{4}-\frac{1}{2}\nu} N_{-\nu-\frac{1}{2}}\left(b\sqrt{a^2-c^2}\right) \\ \left[0 < c < a, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}\right]; \\ = \sqrt{\frac{2}{\pi}} b^{\frac{1}{2}+\nu} c^\nu (c^2-a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} K_{\nu+\frac{1}{2}}\left(b\sqrt{c^2-a^2}\right) \\ \left[0 < a < c, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}\right].$$

ET II 39(45)

6.738

$$1. \int_0^a x^{\nu+1} \sin(b\sqrt{a^2-x^2}) J_\nu(x) dx = \sqrt{\frac{\pi}{2}} a^{\nu+\frac{3}{2}} b(1+b^2)^{-\frac{1}{2}\nu-\frac{3}{4}} J_{\nu+\frac{3}{2}}\left(a\sqrt{1+b^2}\right) \quad [\operatorname{Re} \nu > -1].$$

ET II 335(19)

$$2. \int_0^\infty x^{\nu+1} \cos(a\sqrt{x^2+b^2}) J_\nu(cx) dx = \\ = \sqrt{\frac{\pi}{2}} a b^{\nu+\frac{3}{2}} c^\nu (a^2-c^2)^{-\frac{1}{2}\nu-\frac{3}{4}} \left[\cos(\pi\nu) J_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right) - \sin(\pi\nu) N_{\nu+\frac{3}{2}}\left(b\sqrt{a^2-c^2}\right) \right] \\ \left[0 < c < a, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < -\frac{1}{2}\right]; \\ = 0 \quad \left[0 < a < c, \operatorname{Re} b > 0, -1 < \operatorname{Re} \nu < -\frac{1}{2}\right].$$

ET II 39(43)

$$\int_0^t x^{-\frac{1}{2}} \frac{\cos(b\sqrt{t-x})}{\sqrt{t-x}} J_{2\nu}(a\sqrt{x}) dx = \pi J_\nu \left[\frac{\sqrt{t}}{2} (\sqrt{a^2 + b^2} + b) \right] J_\nu \left[\frac{\sqrt{t}}{2} (\sqrt{a^2 + b^2} - b) \right] \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

EH II 47(7)

$$1. \int_0^1 \frac{\cos(\mu \arccos x)}{\sqrt{1-x^2}} J_\nu(ax) dx = \frac{\pi}{2} J_{\frac{1}{2}(\mu+\nu)} \left(\frac{a}{2} \right) J_{\frac{1}{2}(\nu-\mu)} \left(\frac{a}{2} \right) \quad [\operatorname{Re}(\mu + \nu) > -1, \quad a > 0].$$

ET II 41(54)

$$2. \int_0^1 \frac{\cos[(\nu+1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx = \sqrt{\frac{\pi}{a}} \cos \left(\frac{a}{2} \right) J_{\nu+\frac{1}{2}} \left(\frac{a}{2} \right) \quad [\operatorname{Re} \nu > -1, \quad a > 0].$$

ET II 40(53)

$$3. \int_0^1 \frac{\cos[(\nu-1) \arccos x]}{\sqrt{1-x^2}} J_\nu(ax) dx = \sqrt{\frac{\pi}{a}} \sin \left(\frac{a}{2} \right) J_{\nu-\frac{1}{2}} \left(\frac{a}{2} \right) \quad [\operatorname{Re} \nu > 0, \quad a > 0].$$

ET II 40(52)a

6.75 Combinations of Bessel, trigonometric, and exponential functions and powers

$$1. \int_0^\infty e^{-\frac{1}{2}ax} \sin(bx) I_0 \left(\frac{1}{2}ax \right) dx = \frac{1}{\sqrt{2b}} \frac{1}{\sqrt{b^2 + a^2}} \sqrt{b + \sqrt{b^2 + a^2}} \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 105(44)

$$2. \int_0^\infty e^{-\frac{1}{2}ax} \cos(bx) I_0 \left(\frac{1}{2}ax \right) dx = \frac{a}{\sqrt{2b}} \frac{1}{\sqrt{a^2 + b^2} \sqrt{b + \sqrt{a^2 + b^2}}} \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET I 48(38)

$$3. \int_0^\infty e^{-bx} \cos(ax) J_0(cx) dx = \frac{\left[\sqrt{(b^2 + c^2 - a^2)^2 + 4a^2 b^2} + b^2 + c^2 - a^2 \right]^{\frac{1}{2}}}{\sqrt{2} \sqrt{(b^2 - c^2 - a^2)^2 + 4a^2 b^2}} \quad [c > 0].$$

6.752

$$1. \int_0^{\infty} e^{-ax} J_0(bx) \sin(cx) \frac{dx}{x} = \arcsin \left(\frac{2c}{\sqrt{a^2 + (c+b)^2} + \sqrt{a^2 + (c-b)^2}} \right) \\ [\operatorname{Re} a > |\operatorname{Im} b|, \quad c > 0].$$

ET I 101(17)

$$2. \int_0^{\infty} e^{-ax} J_1(cx) \sin(bx) \frac{dx}{x} = \frac{b}{c}(1-r), \quad \left[b^2 = \frac{c^2}{1-r^2} - \frac{a^2}{r^2}, \quad c > 0 \right].$$

ET II 19(15)

778

6.753

$$1. \int_0^{\infty} \frac{\sin(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_{\nu}(xa \sin \varphi) dx = \nu^{-1} \left(\operatorname{tg} \frac{\varphi}{2} \right)^{\nu} \sin(\nu\psi) \\ \left[\operatorname{Re} \nu > -1, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2} \right].$$

ET II 33(10)

$$2. \int_0^{\infty} \frac{\cos(xa \sin \psi)}{x} e^{-xa \cos \varphi \cos \psi} J_{\nu}(xa \sin \varphi) dx = \nu^{-1} \left(\operatorname{tg} \frac{\varphi}{2} \right)^{\nu} \cos(\nu\psi) \\ \left[\operatorname{Re} \nu > 0, \quad a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2} \right].$$

ET II 38(35)

$$3.* \int_0^{\infty} x^{\nu+1} e^{-sx} \sin(bx) I_{\nu}(ax) dx = \\ = -\frac{2(2a)^{\nu}}{\sqrt{\pi}} \Gamma(\nu + 3/2) R^{-2\nu-3} [b \cos(\nu + 3/2)\varphi + s \sin(\nu + 3/2)\varphi], \\ [\operatorname{Re} \nu > -3/2, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|, \quad R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad \varphi = \arg(s^2 + a^2 - b^2 - 2ibs)] .$$

$$4.* \int_0^{\infty} x^{\nu+1} e^{-sx} \cos(bx) J_{\nu}(ax) dx = \\ = \frac{2(2a)^{\nu}}{\sqrt{\pi}} \Gamma(\nu + 3/2) R^{-2\nu-3} [s \cos(\nu + 3/2)Q - b \sin(\nu + 3/2)\varphi], \\ [\operatorname{Re} \nu > -1, \quad \operatorname{Re} s > |\operatorname{Im} a| + |\operatorname{Im} b|, \quad R^4 = (s^2 + a^2 - b^2)^2 + 4b^2 s^2, \quad Q = \arg(s^2 + a^2 - b^2 - 2ibs)] .$$

$$\begin{aligned}
5. \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \sin(ax \sin \psi) J_\nu(ax \sin \varphi) dx &= \\
&= 2^\nu \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \sin \left[\left(\nu + \frac{3}{2}\right) \beta \right], \\
\operatorname{tg} \frac{\beta}{2} = \operatorname{tg} \psi \cos \varphi &\quad \left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -1 \right].
\end{aligned}$$

ET II 34(12)

$$\begin{aligned}
6. \int_0^\infty x^\nu e^{-ax \cos \varphi \cos \psi} \cos(ax \sin \psi) J_\nu(ax \sin \varphi) dx &= \\
&= 2^\nu \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} a^{-\nu-1} (\sin \varphi)^\nu (\cos^2 \psi + \sin^2 \psi \cos^2 \varphi)^{-\nu-\frac{1}{2}} \cos \left[\left(\nu + \frac{1}{2}\right) \beta \right], \\
\operatorname{tg} \frac{\beta}{2} = \operatorname{tg} \psi \cos \varphi &\quad \left[a > 0, \quad 0 < \varphi, \quad \psi < \frac{\pi}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
\end{aligned}$$

ET II 38(37)

779

6.754

$$1. \int_0^\infty e^{-x^2} \sin(bx) I_0(x^2) dx = \frac{\sqrt{\pi}}{2^{\frac{3}{2}}} e^{-\frac{b^2}{8}} I_0\left(\frac{b^2}{8}\right) \quad [b > 0].$$

ET I 108(9)

$$2. \int_0^\infty e^{-ax} \cos(x^2) J_0(x^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - N_0\left(\frac{a^2}{16}\right) \cos\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\ [a > 0].$$

MI 42

$$3. \int_0^\infty e^{-ax} \sin(x^2) J_0(x^2) dx = \frac{1}{4} \sqrt{\frac{\pi}{2}} \left[J_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} - \frac{\pi}{4}\right) - N_0\left(\frac{a^2}{16}\right) \sin\left(\frac{a^2}{16} + \frac{\pi}{4}\right) \right] \\ [a > 0].$$

MI 42

6.755

$$1. \int_0^\infty x^{-\nu} e^{-x} \sin(4a\sqrt{x}) I_\nu(x) dx = (2^{\frac{3}{2}} a)^{\nu-1} e^{-a^2} W_{\frac{1}{2}-\frac{3}{2}\nu, \frac{1}{2}-\frac{1}{2}\nu}(2a^2) \quad [a > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 366(14)

$$2. \int_0^\infty x^{-\nu-\frac{1}{2}} e^{-x} \cos(4a\sqrt{x}) I_\nu(x) dx = 2^{\frac{3}{2}\nu-1} a^{\nu-1} e^{-a^2} W_{-\frac{3}{2}\nu, \frac{1}{2}\nu}(2a^2) \quad \left[a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 366(16)

$$3. \int_0^{\infty} x^{-\nu} e^x \sin(4a\sqrt{x}) K_{\nu}(x) dx = (2^{\frac{3}{2}}a)^{\nu-1} \pi \frac{\Gamma\left(\frac{3}{2} - 2\nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)} e^{a^2} W_{\frac{3}{2}\nu - \frac{1}{2}, \frac{1}{2} - \frac{1}{2}\nu}(2a^2) \\ \left[a > 0, \quad 0 < \operatorname{Re} \nu < \frac{3}{4} \right].$$

ET II 369(38)

$$4. \int_0^{\infty} x^{-\nu - \frac{1}{2}} e^x \cos(4a\sqrt{x}) K_{\nu}(x) dx = 2^{\frac{3}{2}\nu-1} \pi a^{\nu-1} \frac{\Gamma\left(\frac{1}{2} - 2\nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)} e^{a^2} W_{\frac{3}{2}\nu, -\frac{1}{2}\nu}(2a^2) \\ \left[a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{4} \right].$$

ET II 369(42)

$$5. \int_0^{\infty} x^{\varrho - \frac{3}{2}} e^{-x} \sin(4a\sqrt{x}) K_{\nu}(x) dx = \\ = \frac{\sqrt{\pi} a \Gamma(\varrho + \nu) \Gamma(\varrho - \nu)}{2^{\varrho-2} \Gamma\left(\varrho + \frac{1}{2}\right)} {}_2F_2\left(\varrho + \nu, \varrho - \nu; \frac{3}{2}, \varrho + \frac{1}{2}; -2a^2\right) \quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|].$$

ET II 369(39)

780

$$6. \int_0^{\infty} x^{\varrho-1} e^{-x} \cos(4a\sqrt{x}) K_{\nu}(x) dx = \\ = \frac{\sqrt{\pi} \Gamma(\varrho + \nu) \Gamma(\varrho - \nu)}{2^{\varrho} \Gamma\left(\varrho + \frac{1}{2}\right)} {}_2F_2\left(\varrho + \nu, \varrho - \nu; \frac{1}{2}, \varrho + \frac{1}{2}; -2a^2\right) \quad [\operatorname{Re} \varrho > |\operatorname{Re} \nu|].$$

ET II 370(43)

$$7. \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} \cos(4a\sqrt{x}) I_0(x) dx = \frac{1}{\sqrt{2\pi}} e^{-a^2} K_0(a^2) \quad [a > 0].$$

ET II 366(15)

ET II 369(40)

$$9. \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} \cos(4a\sqrt{x}) K_0(x) dx = \frac{1}{\sqrt{2}} \pi^{\frac{3}{2}} e^{-a^2} I_0(a^2).$$

ET II 369(41)

6.756

$$\begin{aligned} 1. \int_0^{\infty} x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_{\nu}(bx) dx &= \\ &= \frac{i}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) - D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ &\quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1]. \end{aligned}$$

ET II 34(17)

$$\begin{aligned} 2. \int_0^{\infty} x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_{\nu}(bx) dx &= \\ &= \frac{1}{\sqrt{2\pi b}} \Gamma\left(\nu + \frac{1}{2}\right) D_{-\nu-\frac{1}{2}}\left(\frac{a}{\sqrt{b}}\right) \left[D_{-\nu-\frac{1}{2}}\left(\frac{ia}{\sqrt{b}}\right) + D_{-\nu-\frac{1}{2}}\left(-\frac{ia}{\sqrt{b}}\right) \right] \\ &\quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right]. \end{aligned}$$

ET II 39(42)

$$3. \int_0^{\infty} x^{-\frac{1}{2}} e^{-a\sqrt{x}} \sin(a\sqrt{x}) J_0(bx) dx = \frac{1}{2b} a I_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \quad \left[|\arg a| < \frac{\pi}{4}, \quad b > 0 \right].$$

ET II 11(40)

$$4. \int_0^{\infty} x^{-\frac{1}{2}} e^{-a\sqrt{x}} \cos(a\sqrt{x}) J_0(bx) dx = \frac{a}{2b} I_{-\frac{1}{4}}\left(\frac{a^2}{4b}\right) K_{\frac{1}{4}}\left(\frac{a^2}{4b}\right) \quad \left[|\arg a| < \frac{\pi}{4}, \quad b > 0 \right].$$

ET II 12(49)

6.757

$$\begin{aligned} 1. \int_0^{\infty} e^{-bx} \sin[a(1-e^{-x})] J_{\nu}(ae^{-x}) dx &= \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\nu - b + 2n + 1) \Gamma(\nu + b)}{\Gamma(\nu - b + 1) \Gamma(\nu + b + 2n + 2)} (\nu + 2n - 1) J_{\nu+2n+1}(a) \quad [\operatorname{Re} b > -\operatorname{Re} \nu]. \end{aligned}$$

$$\begin{aligned}
2. \int_0^\infty e^{-bx} \cos [a(1 - e^{-x})] J_\nu(ae^{-x}) dx &= \\
&= \frac{J_\nu(a)}{\nu + b} + \sum_{n=0}^\infty 2(-1)^n \frac{\Gamma(\nu - b + 2n)\Gamma(\nu + b)}{\Gamma(\nu - b + 1)\Gamma(\nu + b + 2n + 1)} (\nu + 2n) J_{\nu+2n}(a) \\
&\quad [\operatorname{Re} b > -\operatorname{Re} \nu].
\end{aligned}$$

ET I 193(27)

6.758

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(\mu-\nu)\theta} (\cos \theta)^{\nu+\mu} (\lambda z)^{-\nu-\mu} J_{\nu+\mu}(\lambda z) d\theta &= \pi(2az)^{-\mu} (2bz)^{-\nu} J_\mu(az) J_\nu(bz); \\
\lambda &= \sqrt{2 \cos \theta (a^2 e^{i\theta} + b^2 e^{-i\theta})} \quad [\operatorname{Re}(\nu + \mu) > -1].
\end{aligned}$$

EH II 48(12)

6.76 Combinations of Bessel, trigonometric, and hyperbolic functions

6.761

$$\begin{aligned}
\int_0^\infty \operatorname{ch} x \cos(2a \operatorname{sh} x) J_\nu(be^x) J_\nu(be^{-x}) dx &= \frac{J_{2\nu}(2\sqrt{b^2 - a^2})}{2\sqrt{b^2 - a^2}} \quad [0 < a < b, \operatorname{Re} \nu > -1]; \\
&= 0 \quad [0 < b < a, \operatorname{Re} \nu > -1].
\end{aligned}$$

ET II 359(10)

6.762

$$\begin{aligned}
\int_0^\infty \operatorname{ch} x \sin(2a \operatorname{sh} x) [J_\nu(be^x) N_\nu(be^{-x}) - N_\nu(be^x) J_\nu(be^{-x})] dx &= \\
&= 0 \quad \left[0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right]; \\
&= -\frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-\frac{1}{2}} K_{2\nu} \left[2(a^2 - b^2)^{\frac{1}{2}} \right] \quad \left[0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].
\end{aligned}$$

ET II 360(12)

6.763

$$\begin{aligned}
\int_0^\infty \operatorname{ch} x \cos(2a \operatorname{sh} x) N_\nu(be^x) N_\nu(be^{-x}) dx &= \\
&= -\frac{1}{2} (b^2 - a^2)^{-\frac{1}{2}} J_{2\nu} \left[2(b^2 - a^2)^{\frac{1}{2}} \right] \quad [0 < a < b, \quad |\operatorname{Re} \nu| < 1]; \\
&= \frac{2}{\pi} \cos(\nu\pi) (a^2 - b^2)^{-\frac{1}{2}} K_{2\nu} \left[2(a^2 - b^2)^{\frac{1}{2}} \right] \quad [0 < b < a, \quad |\operatorname{Re} \nu| < 1].
\end{aligned}$$

ET II 360(11)

6.77 Combinations of Bessel functions and the logarithm, or arctangent

6.771

$$\int_0^{\infty} x^{\mu+\frac{1}{2}} \ln x J_{\nu}(ax) dx = \frac{2^{\mu-\frac{1}{2}} \Gamma\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) a^{\mu+\frac{3}{2}}} \left[\psi\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right) + \psi\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right) - \ln \frac{a^2}{4} \right]$$

$$\left[a > 0, \quad -\operatorname{Re} \nu - \frac{3}{2} < \operatorname{Re} \mu < 0 \right].$$

ET II 32(25)

6.772

$$1. \int_0^{\infty} \ln x J_0(ax) dx = -\frac{1}{a} [\ln(2a) + \mathbf{C}].$$

WA 430(4)A, ET II 10(27)

782

$$2. \int_0^{\infty} \ln x J_1(ax) dx = -\frac{1}{a} \left[\ln\left(\frac{a}{2}\right) + \mathbf{C} \right].$$

ET II 19(11)

$$3. \int_0^{\infty} \ln(a^2 + x^2) J_1(bx) dx = \frac{2}{b} [K_0(ab) + \ln a].$$

ET II 19(12)

$$4. \int_0^{\infty} J_1(tx) \ln \sqrt{1+t^4} dt = \frac{2}{x} \operatorname{ker} x.$$

MO 46

6.773

$$\int_0^{\infty} \frac{\ln(x + \sqrt{x^2 + a^2})}{\sqrt{x^2 + a^2}} J_0(bx) dx = \left[\frac{1}{2} K_0^2\left(\frac{ab}{2}\right) + \ln a I_0\left(\frac{ab}{2}\right) K_0\left(\frac{ab}{2}\right) \right] \quad [a > 0, \quad b > 0].$$

ET II 10(28)

6.774

$$\int_0^{\infty} \ln \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x} J_0(bx) \frac{dx}{\sqrt{x^2 + a^2}} = K_0^2 \left(\frac{ab}{2} \right) \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET II 10(29)

6.775

$$\int_0^{\infty} x \left[\ln \left(1 + \sqrt{a^2 + x^2} \right) - \ln x \right] J_0(bx) dx = \frac{1}{b^2} (1 - e^{-ab}) \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET II 12(55)

6.776

$$\int_0^{\infty} x \ln \left(1 + \frac{a^2}{x^2} \right) J_0(bx) dx = \frac{2}{b} \left[\frac{1}{b} - aK_1(ab) \right] \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET II 10(30)

6.777

$$\int_0^{\infty} J_1(tx) \operatorname{arctg} t^2 dt = -\frac{2}{x} \operatorname{kei} x.$$

MO 46

6.78 Combinations of Bessel and other special functions

6.781

$$\int_0^{\infty} \operatorname{si}(ax) J_0(bx) dx = -\frac{1}{b} \arcsin \left(\frac{b}{a} \right) \quad [0 < b < a];$$

$$= 0 \quad [0 < a < b].$$

ET II 13(6)

6.782

$$1. \int_0^{\infty} \operatorname{Ei}(-x) J_0(2\sqrt{zx}) dx = \frac{e^{-z} - 1}{z}.$$

NT 60(4)

$$2. \int_0^{\infty} \operatorname{si}(x) J_0(2\sqrt{zx}) dx = -\frac{\sin z}{z}.$$

$$3. \int_0^{\infty} \text{ci}(x) J_0(2\sqrt{zx}) dx = \frac{\cos z - 1}{z}.$$

NT 60(5)

$$4. \int_0^{\infty} \text{Ei}(-x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\text{Ei}(-z) - \mathbf{C} - \ln z}{\sqrt{z}}.$$

NT 60(7)

$$5. \int_0^{\infty} \text{si}(x) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = -\frac{\frac{\pi}{2} - \text{si}(z)}{\sqrt{z}}.$$

NT 60(9)

783

$$6. \int_0^{\infty} \text{ci}(z) J_1(2\sqrt{zx}) \frac{dx}{\sqrt{x}} = \frac{\text{ci}(z) - \mathbf{C} - \ln z}{\sqrt{z}}.$$

NT 60(8)

$$7. \int_0^{\infty} \text{Ei}(-x) N_0(2\sqrt{zx}) dx = \frac{\mathbf{C} + \ln z - e^2 \text{Ei}(-z)}{\pi z}.$$

NT 63(5)

6.783

$$1. \int_0^{\infty} x \text{si}(a^2 x^2) J_0(bx) dx = -\frac{2}{b^2} \sin\left(\frac{b^2}{4a^2}\right) \quad [a > 0].$$

ET II 13(7)a

$$2. \int_0^{\infty} x \text{ci}(a^2 x^2) J_0(bx) dx = \frac{2}{b^2} \left[1 - \cos\left(\frac{b^2}{4a^2}\right) \right] \quad [a > 0].$$

ET II 13(8)a

$$3. \int_0^{\infty} \text{ci}(a^2 x^2) J_0(bx) dx = \frac{1}{b} \left[\text{ci}\left(\frac{b^2}{4a^2}\right) + \ln\left(\frac{b^2}{4a^2}\right) + 2\mathbf{C} \right] \quad [a > 0].$$

$$4. \int_0^{\infty} \text{si}(a^2 x^2) J_1(bx) dx = \frac{1}{b} \left[-\text{si} \left(\frac{b^2}{4a^2} \right) - \frac{\pi}{2} \right] \quad [a > 0].$$

ET II 20(25)a

6.784

$$1. \int_0^{\infty} x^{\nu+1} [1 - \Phi(ax)] J_{\nu}(bx) dx = a^{-\nu} \frac{\Gamma \left(\nu + \frac{3}{2} \right)}{b^2 \Gamma(\nu + 2)} \exp \left(-\frac{b^2}{8a^2} \right) M_{\frac{1}{2}\nu + \frac{1}{2}, \frac{1}{2}\nu + \frac{1}{2}} \left(\frac{b^2}{4a^2} \right) \\ \left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \text{Re } \nu > -1 \right].$$

ET II 92(22)

$$2. \int_0^{\infty} x^{\nu} [1 - \Phi(ax)] J_{\nu}(bx) dx = \sqrt{\frac{2}{\pi}} \frac{a^{\frac{1}{2}-\nu} \Gamma \left(\nu + \frac{1}{2} \right)}{b^{\frac{3}{2}} \Gamma \left(\nu + \frac{3}{2} \right)} \exp \left(-\frac{b^2}{8a^2} \right) M_{\frac{1}{2}\nu - \frac{1}{4}, \frac{1}{2}\nu + \frac{1}{4}} \left(\frac{b^2}{4a^2} \right) \\ \left[|\arg a| < \frac{\pi}{4}, \quad \text{Re } \nu > -\frac{1}{2}, \quad b > 0 \right].$$

ET II 92(23)

6.785

$$\int_0^{\infty} \frac{\exp \left(\frac{a^2}{2x} - x \right)}{x} \left[1 - \Phi \left(\frac{a}{\sqrt{2x}} \right) \right] K_{\nu}(x) dx = \frac{\pi^{\frac{5}{2}}}{4} \sec(\nu\pi) \left\{ [J_{\nu}(a)]^2 + [N_{\nu}(a)]^2 \right\} \\ \left[\text{Re } a > 0, \quad |\text{Re } \nu| < \frac{1}{2} \right].$$

ET II 370(46)

784

6.786

$$\int_0^{\infty} x^{\nu-2\mu+2n+2} e^{x^2} \Gamma(\mu, x^2) N_{\nu}(bx) dx = \\ = (-1)^n \frac{\Gamma \left(\frac{3}{2} - \mu + \nu + n \right) \Gamma \left(\frac{3}{2} - \mu + n \right)}{b \Gamma(1 - \mu)} \exp \left(\frac{b^2}{8} \right) W_{\mu - \frac{1}{2}\nu - n - 1, \frac{1}{2}\nu} \left(\frac{b^2}{4} \right) \\ \left[n - \text{an integer}, \quad b > 0, \quad \text{Re}(\nu - \mu + n) > -\frac{3}{2}, \quad \text{Re}(-\mu + n) > -\frac{3}{2}, \quad \text{Re } \nu < \frac{1}{2} - 2n \right].$$

ET II 108(2)

6.787

$$\int_0^\infty \frac{x^{\nu+2n-\frac{1}{2}}}{B(a+x, a-x)} J_\nu(bx) dx = 0 \quad \left[\pi \leq b < \infty, \quad -1 < \operatorname{Re} \nu < 2a - 2n - \frac{7}{2} \right].$$

ET II 92(21)

6.79 Integration of Bessel functions with respect to the order

6.791

$$1. \int_{-\infty}^\infty K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi K_{iy-iz}(a+b) \quad [|\arg a| + |\arg b| < \pi].$$

ET II 382(21)

$$2. \int_{-\infty}^\infty J_{\nu-x}(a) J_{\mu+x}(a) dx = J_{\mu+\nu}(2a) \quad [\operatorname{Re}(\mu + \nu) > 1].$$

ET II 379(1)

$$3. \int_{-\infty}^\infty J_{\{+x}(a) J_{\lambda-x}(a) J_{\mu+x}(a) J_{\nu-x}(a) dx =$$

$$= \frac{\Gamma(\{ + \lambda + \mu + \nu + 1)}{\Gamma(\{ + \lambda + 1) \Gamma(\lambda + \mu + 1) \Gamma(\mu + \nu + 1) \Gamma(\nu + \{ + 1)} \times$$

$$\times {}_4F_5 \left(\frac{\{ + \lambda + \mu + \nu + 1}{2}, \frac{\{ + \lambda + \mu + \nu + 1}{2}, \frac{\{ + \lambda + \mu + \nu}{2} + 1, \frac{\{ + \lambda + \mu + \nu}{2} + 1; \right.$$

$$\left. \{ + \lambda + \mu + \nu + 1, \{ + \lambda + 1, \lambda + \mu + 1, \mu + \nu + 1, \nu + \{ + 1; -4a^2 \right)$$

$$[\operatorname{Re}(\{ + \lambda + \mu + \nu) > -1].$$

ET II 379(3)

6.792

$$1. \int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\pi z} K_{i(y-z)}(a-b) \quad [a > b > 0].$$

ET II 382(22)

$$2. \int_{-\infty}^\infty e^{i\varrho x} K_{\nu+ix}(\alpha) K_{\nu-ix}(\beta) dx = \pi \left(\frac{\alpha e^\rho + \beta}{\alpha + \beta e^\rho} \right)^\nu K_{2\nu} \left(\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \operatorname{ch} \varrho} \right)$$

$$[|\arg \alpha| + |\arg \beta| + |\operatorname{Im} \varrho| < \pi].$$

$$3. \int_{-\infty}^{\infty} e^{(\pi-\gamma)x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\beta y - \alpha z} K_{iy-iz}(c)$$

$[0 < \gamma < \pi, \quad a > 0, \quad b > 0, \quad c > 0, \quad \alpha, \beta, \gamma \text{ --- the angles of the triangle with sides } a, b, c] .$

ET II 382(24), EH II 55(44)a

785

$$4. \int_{-\infty}^{\infty} e^{-cxi} H_{\nu-ix}^{(2)}(a) H_{\nu+ix}^{(2)}(b) dx = 2i \left(\frac{h}{k}\right)^{2\nu} H_{2\nu}^{(2)}(hk),$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad \text{Im } c = 0] .$$

ET II 380(11)

$$5. \int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{cxi} J_{\mu+x}(a) J_{\nu-x}(b) dx =$$

$$= \left[\frac{2 \cos\left(\frac{c}{2}\right)}{a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}} \right]^{\frac{1}{2}\mu + \frac{1}{2}\nu} \exp\left[\frac{c}{2}(\nu - \mu)i\right] J_{\mu+\nu} \left\{ \left[2 \cos\left(\frac{c}{2}\right) \left(a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci} \right) \right]^{\frac{1}{2}} \right\}$$

$$= 0 \quad [a > 0, \quad b > 0, \quad |c| \geq \pi, \quad \text{Re}(\mu + \nu) > 1] .$$

$[b > 0, \quad a > 0, \quad |c| < \pi, \quad \text{Re}(\mu + \nu) > 1] ;$

EH II 54(41), ET II 379(2)

6.793

$$1. \int_{-\infty}^{\infty} e^{-cxi} [J_{\nu-ix}(a) N_{\nu+ix}(b) + N_{\nu-ix}(a) J_{\nu+ix}(b)] dx = -2 \left(\frac{h}{k}\right)^{2\nu} J_{2\nu}(hk),$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad \text{Im } c = 0] .$$

ET II 380(9)

$$2. \int_{-\infty}^{\infty} e^{-cxi} [J_{\nu-ix}(a) J_{\nu+ix}(b) - N_{\nu-ix}(a) N_{\nu+ix}(b)] dx = 2 \left(\frac{h}{k}\right)^{2\nu} N_{2\nu}(hk),$$

$$h = \sqrt{ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}}, \quad k = \sqrt{ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c}} \quad [a, b > 0, \quad \text{Im } c = 0] .$$

ET II 380(10)

$$3.* \int_{-\infty}^{\infty} e^{i\gamma x} \text{sech}(\pi x) [J_{-ix}(\alpha) J_{ix}(\beta) - J_{ix}(\alpha) J_{-ix}(\beta)] dx = 2i H(\sigma) \text{sgn}(\beta - \alpha) J_0(\sigma^{\frac{1}{2}})$$

$[\alpha, \beta, \gamma \in \mathbb{R}, \quad \alpha, \beta > 0, \quad \sigma = \alpha^2 + \beta^2 - 2\alpha\beta \cosh \gamma, \quad H(\sigma) \text{ the Heaviside step function}] .$

$$1. \int_0^{\infty} K_{ix}(a)K_{ix}(b) \operatorname{ch}[(\pi - \varphi)x] dx = \frac{\pi}{2} K_0 \left(\sqrt{a^2 + b^2 - 2ab \cos \varphi} \right).$$

EH II 55(42)

$$2. \int_0^{\infty} \operatorname{ch}\left(\frac{\pi}{2}x\right) K_{ix}(a) dx = \frac{\pi}{2} \quad [a > 0].$$

ET II 382(19)

$$3. \int_0^{\infty} \operatorname{ch}(\varrho x) K_{ix+\nu}(a) K_{-ix+\nu}(a) dx = \frac{\pi}{2} K_{2\nu} \left[2a \cos\left(\frac{\varrho}{2}\right) \right] \quad [2|\arg a| + |\operatorname{Re} \varrho| < \pi].$$

ET II 383(28)

$$4. \int_{-\infty}^{\infty} \operatorname{sech}\left(\frac{\pi}{2}x\right) J_{ix}(a) dx = 2 \sin a \quad [a > 0].$$

ET II 380(6)

$$5. \int_{-\infty}^{\infty} \operatorname{cosech}\left(\frac{\pi}{2}x\right) J_{ix}(a) dx = -2i \cos a \quad [a > 0].$$

ET II 380(7)

786

$$6. \int_0^{\infty} \operatorname{sech}(\pi x) \left\{ [J_{ix}(a)]^2 + [N_{ix}(a)]^2 \right\} dx = -N_0(2a) - E_0(2a) \quad [a > 0].$$

ET II 380(12)

$$7. \int_0^{\infty} x \operatorname{sh}\left(\frac{\pi}{2}x\right) K_{ix}(a) dx = \frac{\pi a}{2} \quad [a > 0].$$

ET II 382(20)

$$8. \int_0^{\infty} x \theta(\pi x) K_{ix}(\beta) K_{ix}(\alpha) dx = \frac{\pi}{2} \sqrt{\alpha\beta} \frac{\exp(-\beta - \alpha)}{\alpha + \beta} \quad [|\arg \beta| < \pi, \quad |\arg \alpha| < \pi].$$

ET II 175(4)

$$9. \int_0^{\infty} x \operatorname{sh}(\pi x) K_{2ix}(\alpha) K_{ix}(\beta) dx = \frac{\pi^{\frac{3}{2}} \alpha}{2^{\frac{5}{2}} \sqrt{\beta}} \exp\left(-\beta - \frac{\alpha^2}{8\beta}\right) \quad \left[\beta > 0, \quad |\arg \alpha| < \frac{\pi}{4}\right].$$

$$10. \int_0^\infty \frac{x \operatorname{sh}(\pi x)}{x^2 + n^2} K_{ix}(\alpha) K_{ix}(\beta) dx = \frac{\pi^2}{2} I_n(\beta) K_n(\alpha) \quad [0 < \beta < \alpha; \quad n = 0, 1, 2, \dots];$$

$$= \frac{\pi^2}{2} I_n(\alpha) K_n(\beta) \quad [0 < \alpha < \beta; \quad n = 0, 1, 2, \dots].$$

ET II 176(8)

$$11. \int_0^\infty x \operatorname{sh}(\pi x) K_{ix}(\alpha) K_{ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2}{4} \exp \left[-\frac{\gamma}{2} \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha\beta}{\gamma^2} \right) \right]$$

$$\left[|\arg \alpha| + |\arg \beta| < \frac{\pi}{2}, \quad \gamma > 0 \right]$$

ET II 176(9)

$$12. \int_0^\infty x \operatorname{sh} \left(\frac{\pi}{2} x \right) K_{\frac{1}{2}ix}(\alpha) K_{\frac{1}{2}ix}(\beta) K_{ix}(\gamma) dx = \frac{\pi^2 \gamma}{2\sqrt{\gamma^2 + 4\alpha\beta}} \exp \left[-\frac{(\alpha + \beta)\sqrt{\gamma^2 + 4\alpha\beta}}{2\sqrt{\alpha\beta}} \right]$$

$$[|\arg \alpha| + |\arg \beta| < \pi, \quad \gamma > 0].$$

ET II 176(10)

$$13. \int_0^\infty x \operatorname{sh}(\pi x) K_{\frac{1}{2}ix+\lambda}(\alpha) K_{\frac{1}{2}ix-\lambda}(\alpha) K_{ix}(\gamma) dx = 0 \quad [0 < \gamma < 2\alpha];$$

$$= \frac{\pi^2 \gamma}{2^{2\lambda+1} \alpha^{2\lambda} z} \quad [(\gamma + z)^{2\lambda} + (\gamma - z)^{2\lambda}],$$

$$z = \sqrt{\gamma^2 - 4\alpha^2} \quad [0 < 2\alpha < \gamma].$$

ET II 176(11)

6.795

$$1. \int_0^\infty \cos(bx) K_{ix}(a) dx = \frac{\pi}{2} e^{-a \operatorname{ch} b} \quad \left[|\operatorname{Im} b| < \frac{\pi}{2}, \quad a > 0 \right].$$

EH II 55(46), ET II 175(2)

$$2. \int_0^\infty J_x(ax) J_{-x}(ax) \cos(\pi x) dx = \frac{1}{4} (1 - a^2)^{-\frac{1}{2}} \quad [|a| < 1].$$

ET II 380(4)

$$3. \int_0^\infty x \sin(ax) K_{ix}(b) dx = \frac{\pi b}{2} \operatorname{sh} a \exp(-b \operatorname{ch} a) \quad \left[|\operatorname{Im} a| < \frac{\pi}{2}, \quad b > 0 \right].$$

$$4. \int_{-\infty}^{\infty} \frac{\sin[(\nu + ix)\pi]}{n + \nu + ix} K_{\nu+ix}(a) K_{\nu-ix}(b) dx = \pi^2 I_n(a) K_{n+2\nu}(b) \quad [0 < a < b; \quad n = 0, 1, \dots];$$

$$= \pi^2 K_{n+2\nu}(a) I_n(b) \quad [0 < b < a; \quad n = 0, 1, \dots].$$

ET II 382(25)

$$5. \int_0^{\infty} x \sin\left(\frac{1}{2}\pi x\right) K_{\frac{1}{2}ix}(a) K_{ix}(b) dx = \frac{\pi^{\frac{3}{2}} b}{\sqrt{2a}} \exp\left(-a - \frac{b^2}{8a}\right) \quad \left[|\arg a| < \frac{\pi}{2}, \quad b > 0\right].$$

ET II 175(6)

6.796

$$1. \int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\operatorname{sh}(\pi x)} J_{ix}(a) dx = -i \exp(ia \operatorname{ch} b) \quad [a > 0, \quad b > 0].$$

ET II 380(8)

$$2. \int_0^{\infty} \cos(bx) \operatorname{ch}\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \cos(a \operatorname{sh} b).$$

EH II 55(47)

$$3. \int_0^{\infty} \sin(bx) \operatorname{sh}\left(\frac{1}{2}\pi x\right) K_{ix}(a) dx = \frac{\pi}{2} \sin(a \operatorname{sh} b).$$

EH II 55(48)

$$4. \int_0^{\infty} \cos(bx) \operatorname{ch}(\pi x) [K_{ix}(a)]^2 dx = -\frac{\pi^2}{4} N_0 \left[2a \operatorname{sh}\left(\frac{b}{2}\right)\right] \quad [a > 0, \quad b > 0].$$

ET II 383(27)

$$5. \int_0^{\infty} \sin(bx) \operatorname{sh}(\pi x) [K_{ix}(a)]^2 dx = \frac{\pi^2}{4} J_0 \left[2a \operatorname{sh}\left(\frac{b}{2}\right)\right] \quad [a > 0, \quad b > 0].$$

ET II 382(26)

6.797

$$\begin{aligned}
1. \int_0^{\infty} x e^{\pi x} \operatorname{sh}(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = \\
= i 2^{\nu} \sqrt{\pi} \Gamma\left(\frac{1}{2} + \nu\right) (ab)^{\nu} (a+b)^{-\nu} K_{\nu}(a+b) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET II 381(14)

$$\begin{aligned}
2. \int_0^{\infty} x e^{\pi x} \operatorname{sh}(\pi x) \operatorname{ch}(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = \frac{i \pi^{\frac{3}{2}} 2^{\nu}}{\Gamma\left(\frac{1}{2} - \nu\right)} (b-a)^{-\nu} H_{\nu}^{(2)}(b-a) \\
\left[0 < a < b, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}\right].
\end{aligned}$$

ET II 381(15)

$$\begin{aligned}
3. \int_0^{\infty} x e^{\pi x} \operatorname{sh}(\pi x) \Gamma\left(\frac{\nu + ix}{2}\right) \Gamma\left(\frac{\nu - ix}{2}\right) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = \\
= i \pi 2^{2-\nu} (ab)^{\nu} (a^2 + b^2)^{-\frac{1}{2}\nu} H_{\nu}^{(2)}\left(\sqrt{a^2 + b^2}\right) \quad [a > 0, \quad b > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET II 381(16)

$$\begin{aligned}
4. \int_0^{\infty} x \operatorname{sh}(\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) dx = 2^{\nu-1} \pi^{\frac{3}{2}} (ab)^{\lambda} (a+b)^{-\lambda} \Gamma\left(\lambda + \frac{1}{2}\right) K_{\lambda}(a+b) \\
[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0, \quad b > 0].
\end{aligned}$$

ET II 176(12)

$$\begin{aligned}
5. \int_0^{\infty} x \operatorname{sh}(2\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) dx = \frac{2^{\lambda} \pi^{\frac{5}{2}}}{\Gamma\left(\frac{1}{2} - \lambda\right)} \left(\frac{ab}{|b-a|}\right)^{\lambda} K_{\lambda}(|b-a|) \\
\left[a > 0, \quad 0 < \operatorname{Re} \lambda < \frac{1}{2}, \quad b > 0\right].
\end{aligned}$$

ET II 176(13)

$$\begin{aligned}
6. \int_0^{\infty} x \operatorname{sh}(\pi x) \Gamma\left(\lambda + \frac{1}{2} ix\right) \Gamma\left(\lambda - \frac{1}{2} ix\right) K_{ix}(a) K_{ix}(b) dx = 2\pi^2 \left(\frac{ab}{2\sqrt{a^2 + b^2}}\right) K_{2\lambda}\left(\sqrt{a^2 + b^2}\right) \\
\left[|\arg a| < \frac{\pi}{2}, \quad \operatorname{Re} \lambda > 0, \quad b > 0\right].
\end{aligned}$$

$$7. \int_0^{\infty} \frac{x \theta(\pi x) K_{ix}(a) K_{ix}(b)}{\Gamma\left(\frac{3}{4} + \frac{1}{2}ix\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}ix\right)} dx = \frac{1}{2} \sqrt{\frac{\pi ab}{a^2 + b^2}} \exp\left(-\sqrt{a^2 + b^2}\right) \quad \left[|\arg a| < \frac{\pi}{2}, \quad b > 0\right],$$

(see also **7.335**).

7.335
ET II 177(15)

6.8 Functions Generated by Bessel Functions

6.81 Struve functions

6.811

$$1. \int_0^{\infty} \mathbf{H}_{\nu}(bx) dx = -\frac{\operatorname{ctg}\left(\frac{\nu\pi}{2}\right)}{b} \quad [-2 < \operatorname{Re} \nu < 0, \quad b > 0].$$

ET II 158(1)

$$2. \int_0^{\infty} \mathbf{H}_{\nu}\left(\frac{a^2}{x}\right) \mathbf{H}_{\nu}(bx) dx = -\frac{J_{2\nu}(2a\sqrt{b})}{b} \quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}\right].$$

ET II 170(37)

$$3. \int_0^{\infty} \mathbf{H}_{\nu-1}\left(\frac{a^2}{x}\right) \mathbf{H}_{\nu}(bx) \frac{dx}{x} = -\frac{1}{a\sqrt{b}} J_{2\nu-1}(2a\sqrt{b}) \quad \left[a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}\right].$$

ET II 170(38)

6.812

$$1. \int_0^{\infty} \frac{\mathbf{H}_{\nu}(bx) dx}{x^2 + a^2} = \frac{\pi}{2a} [I_1(ab) - \mathbf{L}_1(ab)] \quad [\operatorname{Re} a > 0, \quad b > 0].$$

ET II 158(6)

$$2. \int_0^{\infty} \frac{\mathbf{H}_{\nu}(bx)}{x^2 + a^2} dx = -\frac{\pi}{2a \sin\left(\frac{\nu\pi}{2}\right)} L_{\nu}(ab) + \frac{b \operatorname{ctg}\left(\frac{\nu\pi}{2}\right)}{1 - \nu^2} {}_1F_2\left(1; \frac{3 - \nu}{2}; \frac{3 + \nu}{2}; \frac{a^2 b^2}{2}\right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < 2].$$

$$1. \int_0^{\infty} x^{s-1} \mathbf{H}_{\nu}(ax) dx = \frac{2^{s-1} \Gamma\left(\frac{s+\nu}{2}\right)}{a^s \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}s + 1\right)} \operatorname{tg}\left(\frac{s+\nu}{2}\pi\right) \left[a > 0, \quad -1 - \operatorname{Re} \nu < \operatorname{Re} s < \min\left(\frac{3}{2}, 1 - \operatorname{Re} \nu\right) \right].$$

WA 429(2), ET I 335(52)

$$2. \int_0^{\infty} x^{-\nu-1} \mathbf{H}_{\nu}(x) dx = \frac{2^{-\nu-1} \pi}{\Gamma(\nu+1)} \left[\operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 383(2)

$$3. \int_0^{\infty} x^{-\mu-\nu} \mathbf{H}_{\mu}(x) \mathbf{H}_{\nu}(x) dx = \frac{2^{-\mu-\nu} \sqrt{\pi} \Gamma(\mu+\nu)}{\Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\mu + \nu + \frac{1}{2}\right)} \quad [\operatorname{Re}(\mu + \nu) > 0].$$

WA 435(2), ET II 384(8)

$$4. \int_0^1 x^{\nu+1} \mathbf{H}_{\nu}(ax) dx = \frac{1}{a} \mathbf{H}_{\nu+1}(a) \quad \left[a > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 158(2)a

$$5. \int_0^1 x^{1-\nu} \mathbf{H}_{\nu}(ax) dx = \frac{a^{\nu-1}}{2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} - \frac{1}{a} \mathbf{H}_{\nu-1}(a) \quad [a > 0].$$

ET II 158(3)a

6.814

$$1. \int_0^{\infty} \frac{x^{\lambda} \mathbf{H}_{\nu}(bx)}{(x^2 + a^2)^{1-\mu}} dx = \frac{1}{\sqrt{2b}} \frac{a^{\lambda+2\mu-\frac{3}{2}}}{\Gamma(1-\mu)} G_{24}^{22} \left(\frac{a^2 b^2}{4} \middle| \begin{matrix} l, m \\ l, m - \mu, h, k \end{matrix} \right),$$

$$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}, \quad m = \frac{3}{4} - \frac{\lambda}{2}$$

$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda + \nu) > -2, \quad \operatorname{Re}(\lambda + 2\mu) < \frac{5}{2}, \quad \operatorname{Re}(\lambda + 2\mu + \nu) < 2 \right].$$

ET II 159(10)

$$2. \int_0^{\infty} \frac{x^{\nu+1} \mathbf{H}_{\nu}(bx)}{(x^2 + a^2)^{1-\mu}} dx = \frac{2^{\mu-1} \pi a^{\mu+\nu} b^{-\mu}}{\Gamma(1-\mu) \cos[(\mu+\nu)\pi]} [I_{-\mu-\nu}(ab) - \mathbf{L}_{\mu+\nu}(ab)]$$

$$\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > \frac{3}{2}, \quad \operatorname{Re}(\mu + \nu) < \frac{1}{2}, \quad \operatorname{Re}(\mu + \nu + \nu) < \frac{3}{2} \right].$$

6.815

$$1. \int_0^1 x^{\frac{1}{2}\nu} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = 2^\mu a^{-\mu} \Gamma(\mu) \mathbf{H}_{\mu+\nu}(a) \quad \left[\operatorname{Re} \nu > -\frac{3}{2}, \operatorname{Re} \mu > 0 \right].$$

ET II 199(88)a

790

$$2. \int_0^1 x^{\lambda-\frac{1}{2}\nu-\frac{3}{2}} (1-x)^{\mu-1} \mathbf{H}_\nu(a\sqrt{x}) dx = \frac{\mathbf{B}(\lambda, \mu) a^{\nu+1}}{2^\nu \sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)} {}_2F_3\left(1, \lambda; \frac{3}{2}, \nu + \frac{3}{2}, \lambda + \mu; -\frac{a^2}{4}\right) \\ [\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0].$$

ET II 199(89)a

6.82 Combinations of Struve functions, exponentials, and powers

6.821

$$1.^6 \int_0^\infty e^{-\alpha x} \mathbf{H}_{-n-\frac{1}{2}}(\beta x) dx = (-1)^n \beta^{n+\frac{1}{2}} (\alpha + \sqrt{\alpha^2 + \beta^2})^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

ET I 206(6)

$$2.^6 \int_0^\infty e^{-\alpha x} \mathbf{L}_{-n-\frac{1}{2}}(\beta x) dx = \beta^{n+\frac{1}{2}} (\alpha + \sqrt{\alpha^2 - \beta^2})^{-n-\frac{1}{2}} \frac{1}{\sqrt{\alpha^2 - \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|].$$

ET I 208(26)

$$3. \int_0^\infty e^{-\alpha x} \mathbf{H}_0(\beta x) dx = \frac{2}{\pi} \frac{\ln\left(\frac{\sqrt{\alpha^2 + \beta^2} + \beta}{\alpha}\right)}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Im} \beta|].$$

ET II 205(1)

$$4. \int_0^\infty e^{-\alpha x} \mathbf{L}_0(\beta x) dx = \frac{2}{\pi} \frac{\arcsin\left(\frac{\beta}{\alpha}\right)}{\sqrt{\alpha^2 + \beta^2}} \quad [\operatorname{Re} \alpha > |\operatorname{Re} \beta|].$$

ET II 207(18)

6.822

$$\int_0^{\infty} e^{(\nu+1)x} \mathbf{H}_{\nu}(a \operatorname{sh} x) dx = \sqrt{\frac{\pi}{a}} \operatorname{cosec}(\nu\pi) \left[\operatorname{sh} \left(\frac{a}{2} \right) I_{\nu+\frac{1}{2}} \left(\frac{a}{2} \right) - \operatorname{ch} \left(\frac{a}{2} \right) I_{-\nu-\frac{1}{2}} \left(\frac{a}{2} \right) \right] \\ [\operatorname{Re} a > 0, \quad -2 < \operatorname{Re} \nu < 0].$$

ET II 385(11)

6.823

$$1. \int_0^{\infty} x^{\lambda} e^{-\alpha x} \mathbf{H}_{\nu}(bx) dx = \frac{b^{\nu+1} \Gamma(\lambda + \nu + 2)}{2^{\nu} a^{\lambda+\nu+2} \sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)} {}_3F_2 \left(1, \frac{\lambda + \nu}{2} + 1, \frac{\lambda + \nu + 3}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{b^2}{a^2} \right) \\ [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\lambda + \nu) > -2].$$

ET II 161(19)

$$2. \int_0^{\infty} x^{\nu} e^{-\alpha x} \mathbf{L}_{\nu}(\beta x) dx = \frac{(2\beta)^{\nu} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} (\sqrt{\alpha^2 - \beta^2})^{2\nu+1}} - \frac{\Gamma(2\nu + 1) \left(\frac{\beta}{\alpha}\right)^{\nu}}{\sqrt{\frac{\pi}{2}} \alpha (\beta^2 - \alpha^2)^{\frac{1}{2}\nu + \frac{1}{4}}} P_{\nu - \frac{1}{2}}^{\nu - \frac{1}{2}} \left(\frac{\beta}{\alpha}\right) \\ \left[\operatorname{Re} \alpha > |\operatorname{Re} \beta|, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET I 209(35)a

791

6.824

$$1. \int_0^{\infty} t^{\nu} e^{-at} \mathbf{L}_{2\nu}(2\sqrt{t}) dt = \frac{1}{a^{2\nu+1}} e^{\frac{1}{a}} \Phi\left(\frac{1}{\sqrt{a}}\right).$$

MI 51

$$2. \int_0^{\infty} t^{\nu} e^{-at} \mathbf{L}_{-2\nu}(\sqrt{t}) dt = \frac{1}{\Gamma\left(\frac{1}{2} - 2\nu\right) a^{2\nu+1}} e^{\frac{1}{a}} \gamma\left(\frac{1}{2} - 2\nu, \frac{1}{a}\right).$$

MI 51

6.825

$$\int_0^{\infty} x^{s-1} e^{-\alpha^2 x^2} \mathbf{H}_{\nu}(\beta x) dx = \frac{\beta^{\nu+1} \Gamma\left(\frac{1}{2} + \frac{s}{2} + \frac{\nu}{2}\right)}{2^{\nu+1} \sqrt{\pi} \alpha^{\nu+s+1} \Gamma\left(\nu + \frac{3}{2}\right)} {}_2F_2 \left(1, \frac{\nu + s + 1}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{\beta^2}{4\alpha^2} \right)$$

6.83 Combinations of Struve and trigonometric functions

6.831

$$\int_0^{\infty} x^{-\nu} \sin(ax) \mathbf{H}_{\nu}(bx) dx = 0 \quad \left[0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2} \right];$$

$$= \sqrt{\pi} 2^{-\nu} b^{-\nu} \frac{(b^2 - a^2)^{\nu - \frac{1}{2}}}{\Gamma\left(\nu + \frac{1}{2}\right)} \quad \left[0 < a < b, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 162(21)

6.832

$$\int_0^{\infty} \sqrt{x} \sin(ax) \mathbf{H}_{\frac{1}{4}}(b^2 x^2) dx = -2^{-\frac{3}{2}} \sqrt{\pi} \frac{\sqrt{a}}{b^2} N_{\frac{1}{4}}\left(\frac{a^2}{4b^2}\right) \quad [a > 0].$$

ET I 109(14)

6.84- 6.85 Combinations of Struve and Bessel functions

6.841

$$\int_0^{\infty} \mathbf{H}_{\nu-1}(ax) N_{\nu}(bx) dx = -a^{\nu-1} b^{-\nu} \quad \left[0 < b < a, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right];$$

$$= 0 \quad \left[0 < a < b, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 114(36)

6.842

$$\int_0^{\infty} [\mathbf{H}_0(ax) - N_0(ax)] J_0(bx) dx = \frac{4}{\pi(a+b)} \mathbf{K} \left[\frac{|a-b|}{a+b} \right] \quad [a > 0, \quad b > 0].$$

ET II 15(22)

6.843

$$1. \int_0^{\infty} J_{2\nu}(a\sqrt{x}) \mathbf{H}_{\nu}(bx) dx = -\frac{1}{b} N_{\nu}\left(\frac{a^2}{4b}\right) \quad \left[a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{5}{4} \right].$$

ET II 164(10)

$$2. \int_0^{\infty} K_{2\nu}(2a\sqrt{x}) \mathbf{H}_{\nu}(bx) dx = \frac{2^{\nu}}{\pi b} \Gamma(\nu+1) S_{-\nu-1, \nu}\left(\frac{a^2}{b}\right) \quad [\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1].$$

792
6.844

$$\int_0^\infty \left[\cos\left(\frac{\mu - \nu}{2}\pi\right) J_\mu(a\sqrt{x}) - \sin\left(\frac{\mu - \nu}{2}\pi\right) N_\mu(a\sqrt{x}) \right] K_\mu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx = \\ = \frac{1}{a^2} W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2b}\right) \quad \left[|\arg a| < \frac{\pi}{4}, \quad b > 0, \quad \operatorname{Re} \nu > |\operatorname{Re} \mu| - 2 \right].$$

ET II 169(35)

6.845

$$1. \int_0^\infty \left[\mathbf{H}_{-\nu}\left(\frac{a}{x}\right) - N_{-\nu}\left(\frac{a}{x}\right) \right] J_\nu(bx) dx = \frac{4}{\pi b} \cos(\nu\pi) K_{2\nu}(2\sqrt{ab}) \\ \left[|\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 73(7)

$$2. \int_0^\infty \left[J_{-\nu}\left(\frac{a^2}{x}\right) + \sin(\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{x}\right) \right] \mathbf{H}_\nu(bx) dx = \frac{1}{b} \left[\frac{2}{\pi} K_{2\nu}(2a\sqrt{b}) - N_{2\nu}(2a\sqrt{b}) \right] \\ \left[a > 0, \quad b > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right].$$

ET II 170(39)

6.846

$$\int_0^\infty \left[\frac{2}{\pi} K_{2\nu}(2a\sqrt{x}) + N_{2\nu}(2a\sqrt{x}) \right] \mathbf{H}_\nu(bx) dx = \frac{1}{b} J_\nu\left(\frac{a^2}{b}\right) \quad \left[a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 169(30)

6.847

$$\int_0^\infty \left[\cos\frac{\nu\pi}{2} J_\nu(ax) + \sin\frac{\nu\pi}{2} \mathbf{H}_\nu(ax) \right] \frac{dx}{x^2 + k^2} = \frac{\pi}{2k} [I_\nu(ak) - \mathbf{L}_\nu(ak)] \\ \left[a > 0, \quad \operatorname{Re} k > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 2 \right].$$

ET II 384(5)A, WA 467(8)

6.848

$$1. \int_0^\infty x [I_\nu(ax) - \mathbf{L}_{-\nu}(ax)] J_\nu(bx) dx = \frac{2}{\pi} \left(\frac{a}{b}\right)^{\nu-1} \cos(\nu\pi) \frac{1}{a^2 + b^2} \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} \right].$$

$$2. \int_0^{\infty} x [\mathbf{H}_{-\nu}(ax) - N_{-\nu}(ax)] J_{\nu}(bx) dx = 2 \frac{\cos(\nu\pi)}{a^{\nu}\pi} b^{\nu-1} \frac{1}{a+b} \left[|\arg a| < \pi, \quad -\frac{1}{2} < \operatorname{Re} \nu, \quad b > 0 \right].$$

ET II 73(5)

6.849

$$1. \int_0^{\infty} x K_{\nu}(ax) \mathbf{H}_{\nu}(bx) dx = a^{-\nu-1} b^{\nu+1} \frac{1}{a^2 + b^2} \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 164(12)

793

$$2. \int_0^{\infty} x [K_{\mu}(ax)]^2 \mathbf{H}_0(bx) dx = -2^{-\mu-1} \pi a^{-2\mu} \frac{[(z+b)^{2\mu} + (z-b)^{2\mu}]}{bz} \sec(\mu\pi), \quad z = \sqrt{4a^2 + b^2} \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \mu| < \frac{3}{2} \right].$$

ET II 166(18)

6.851

$$1. \int_0^{\infty} x \left\{ [J_{\frac{1}{2}\nu}(ax)]^2 - [N_{\frac{1}{2}\nu}(ax)]^2 \right\} \mathbf{H}_{\nu}(bx) dx = 0 \quad \left[0 < b < 2a, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right]; \\ = \frac{4}{\pi b} \frac{1}{\sqrt{b^2 - 4a^2}} \quad \left[0 < 2a < b, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right].$$

ET II 164(7)

$$2. \int_0^{\infty} x^{\nu+1} \left\{ [J_{\nu}(ax)]^2 - [N_{\nu}(ax)]^2 \right\} \mathbf{H}_{\nu}(bx) dx = \\ = 0 \quad \left[0 < b < 2a, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0 \right]; \\ = \frac{2^{3\nu+2} a^{2\nu} b^{-\nu-1}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} (b^2 - 4a^2)^{-\nu-\frac{1}{2}} \quad \left[0 < 2a < b, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0 \right].$$

ET II 163(6)

6.852

$$1. \int_0^{\infty} x^{1-\mu-\nu} J_{\nu}(x) \mathbf{H}_{\mu}(x) dx = \frac{(2\nu-1)2^{-\mu-\nu}}{(\mu+\nu-1)\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)} \left[\operatorname{Re} \nu > \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > 1 \right].$$

$$\begin{aligned}
2. \int_0^\infty x^{\mu-\nu+1} N_\mu(ax) \mathbf{H}_\nu(bx) dx &= \\
&= 0 \quad \left[0 < b < a, \quad \operatorname{Re}(\nu - \mu) > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2} \right]; \\
&= \frac{2^{1+\mu-\nu} a^\mu b^{-\nu}}{\Gamma(\nu - \mu)} (b^2 - a^2)^{\nu-\mu-1} \quad \left[0 < a < b, \quad \operatorname{Re}(\nu - \mu) > 0, \quad -\frac{3}{2} < \operatorname{Re} \mu < \frac{1}{2} \right].
\end{aligned}$$

ET II 163(3)

$$\begin{aligned}
3. \int_0^\infty x^{\mu+\nu+1} K_\mu(ax) \mathbf{H}_\nu(bx) dx &= \frac{2^{\mu+\nu+1} b^{\nu+1}}{\sqrt{\pi} a^{\mu+2\nu+3}} \Gamma\left(\mu + \nu + \frac{3}{2}\right) F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\frac{b^2}{a^2}\right) \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2} \right].
\end{aligned}$$

ET II 165(13)

794
6.853

$$\begin{aligned}
1. \int_0^\infty x^{1-\mu} [\sin(\mu\pi) J_{\mu+\nu}(ax) + \cos(\mu\pi) N_{\mu+\nu}(ax)] \mathbf{H}_\nu(bx) dx &= \\
&= 0 \quad \left[0 < b < a, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2} \right]; \\
&= \frac{b^\nu (b^2 - a^2)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)} \quad \left[0 < a < b, \quad 1 < \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re} \nu > -\frac{3}{2}, \quad \operatorname{Re}(\nu - \mu) < \frac{1}{2} \right].
\end{aligned}$$

ET II 163(4)

$$\begin{aligned}
2. \int_0^\infty x^{\lambda+\frac{1}{2}} [I_\mu(ax) - \mathbf{L}_{-\mu}(ax)] J_\nu(bx) dx &= \\
&= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi} b^{-\lambda-\frac{3}{2}} G_{33}^{22} \left(\frac{b^2}{a^2} \left| \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right) \\
&\quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re}(\mu + \nu + \lambda) > -\frac{3}{2}, \quad -\operatorname{Re} \nu - \frac{5}{2} < \operatorname{Re}(\lambda - \mu) < 1 \right].
\end{aligned}$$

ET II 76(21)

$$\begin{aligned}
3. \int_0^\infty x^{\lambda+\frac{1}{2}} [\mathbf{H}_\mu(ax) - N_\mu(ax)] J_\nu(bx) dx &= \\
&= 2^{\lambda+\frac{1}{2}} \frac{\cos(\mu\pi)}{\pi^2} b^{-\lambda-\frac{3}{2}} G_{33}^{23} \left(\frac{b^2}{a^2} \left| \begin{matrix} \frac{1-\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right) \\
&\quad \left[b > 0, \quad |\arg a| < \pi, \quad \operatorname{Re}(\lambda + \mu) < 1, \quad \operatorname{Re}(\lambda + \nu) + \frac{3}{2} > |\operatorname{Re} \mu| \right].
\end{aligned}$$

$$4. \int_0^{\infty} \sqrt{x} \left[I_{\nu-\frac{1}{2}}(ax) - \mathbf{L}_{\nu-\frac{1}{2}}(ax) \right] J_{\nu}(bx) dx = \sqrt{\frac{2}{\pi}} a^{\nu-\frac{1}{2}} b^{-\nu} \frac{1}{\sqrt{a^2+b^2}} \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 74(11)

$$5. \int_0^{\infty} x^{\mu-\nu+1} \left[I_{\mu}(ax) - \mathbf{L}_{\mu}(ax) \right] J_{\nu}(bx) dx = \frac{2^{\mu-\nu+1} a^{\mu-1} b^{\nu-2\mu-1}}{\sqrt{\pi} \Gamma\left(\nu-\mu+\frac{1}{2}\right)} F\left(1, \frac{1}{2}; \nu-\mu+\frac{1}{2}; -\frac{b^2}{a^2}\right) \left[-1 < 2 \operatorname{Re} \mu + 1 < \operatorname{Re} \nu + \frac{1}{2}, \quad \operatorname{Re} a > 0, \quad b > 0 \right].$$

ET II 74(13)

795

$$6. \int_0^{\infty} x^{\mu-\nu+1} \left[I_{\mu}(ax) - \mathbf{L}_{-\mu}(ax) \right] J_{\nu}(bx) dx = \frac{2^{\mu-\nu+1} a^{-\mu-1} b^{\nu-1}}{\Gamma\left(\frac{1}{2}-\mu\right) \Gamma\left(\frac{1}{2}+\nu\right)} F\left(1, \frac{1}{2}+\mu; \frac{1}{2}+\nu; -\frac{b^2}{a^2}\right) \left[\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > -1, \quad b > 0 \right].$$

ET II 75(18)

6.854

$$1. \int_0^{\infty} x \mathbf{H}_{\frac{1}{2}\nu}(ax^2) K_{\nu}(bx) dx = \frac{\Gamma\left(\frac{1}{2}\nu+1\right)}{2^{1-\frac{1}{2}\nu} a \pi} S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad [a > 0, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \nu > -2].$$

ET II 150(75)

$$2. \int_0^{\infty} x \mathbf{H}_{\frac{1}{2}\nu}(ax^2) J_{\nu}(bx) dx = -\frac{1}{2a} N_{\frac{1}{2}\nu}\left(\frac{b^2}{4a}\right) \quad \left[a > 0, \quad b > 0, \quad -2 < \operatorname{Re} \nu < \frac{3}{2} \right].$$

ET II 73(3)

6.855

$$1. \int_0^{\infty} x^{2\nu+\frac{1}{2}} \left[I_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) - \mathbf{L}_{\nu+\frac{1}{2}}\left(\frac{a}{x}\right) \right] J_{\nu}(bx) dx = 2^{\frac{3}{2}} \frac{a^{\nu+\frac{1}{2}}}{\sqrt{\pi} b^{\nu+1}} J_{2\nu+1}(\sqrt{2ab}) K_{2\nu+1}(\sqrt{2ab}) \left[\operatorname{Re} a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2} \right].$$

$$2. \int_0^\infty \left[\mathbf{H}_{-\nu-1} \left(\frac{a}{x} \right) - N_{-\nu-1} \left(\frac{a}{x} \right) \right] J_\nu(bx) \frac{dx}{x} = -\frac{4}{\pi\sqrt{ab}} \cos(\nu\pi) K_{-2\nu-1} (2\sqrt{ab}) \\ \left[|\arg a| < \pi, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 74(8)

$$3. \int_0^\infty x^{2\nu+\frac{1}{2}} \left[\mathbf{H}_{\nu+\frac{1}{2}} \left(\frac{a}{x} \right) - N_{\nu+\frac{1}{2}} \left(\frac{a}{x} \right) \right] J_\nu(bx) dx = \\ = -2^{\frac{5}{2}} \pi^{-\frac{3}{2}} a^{\nu+\frac{1}{2}} b^{-\nu-1} \sin(\nu\pi) K_{2\nu+1} \left(\sqrt{2abe}^{\frac{1}{4}\pi i} \right) K_{2\nu+1} \left(\sqrt{2abe}^{-\frac{1}{4}\pi i} \right) \\ \left[|\arg a| < \pi, \quad b > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{6} \right].$$

ET II 74(9)

6.856

$$\int_0^\infty x N_\nu(a\sqrt{x}) K_\nu(a\sqrt{x}) \mathbf{H}_\nu(bx) dx = \frac{1}{2b^2} \exp\left(-\frac{a^2}{2b}\right) \left[b > 0, \quad |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].$$

ET II 169(32)

796

6.857

$$1. \int_0^\infty x \exp\left(\frac{a^2 x^2}{8}\right) K_{\frac{1}{2}\nu} \left(\frac{a^2 x^2}{8} \right) \mathbf{H}_\nu(bx) dx = \\ = \frac{2}{\sqrt{\pi}} a^{-\frac{\nu}{2}-1} b^{\frac{\nu}{2}-1} \cos\left(\frac{\nu\pi}{2}\right) \Gamma\left(-\frac{1}{2}\nu\right) \exp\left(\frac{b^2}{2a^2}\right) W_{k,m} \left(\frac{b^2}{a^2} \right), \\ k = \frac{1}{4}\nu, \quad m = \frac{1}{2} + \frac{1}{4}\nu \quad \left[|\arg a| < \frac{3}{4}\pi, \quad b > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < 0 \right].$$

ET II 167(24)

$$2. \int_0^\infty x^{\sigma-2} \exp\left(-\frac{1}{2}a^2 x^2\right) K_\mu \left(\frac{1}{2}a^2 x^2 \right) \mathbf{H}_\nu(bx) dx = \\ = \frac{\sqrt{\pi}}{2^{\nu+2}} a^{-\nu-\sigma} b^{\nu+1} \frac{\Gamma\left(\frac{\nu+\sigma}{2} + \mu\right) \Gamma\left(\frac{\nu+\sigma}{2} - \mu\right)}{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{\nu+\sigma}{2}\right)} \times \\ \times {}_3F_3 \left(1, \frac{\nu+\sigma}{2} + \mu, \frac{\nu+\sigma}{2} - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{\nu+\sigma}{2}; -\frac{b^2}{4a^2} \right) \\ \left[b > 0, \quad |\arg a| < \frac{\pi}{4}, \quad \operatorname{Re}(\sigma + \nu) > 2|\operatorname{Re} \mu| \right].$$

ET II 167(23)

6.86 Lommel functions

6.861

$$\int_0^\infty x^{\lambda-1} s_{\mu, \nu}(x) dx = \frac{\Gamma\left[\frac{1}{2}(1+\lambda+\mu)\right] \Gamma\left[\frac{1}{2}(1-\lambda-\mu)\right] \Gamma\left[\frac{1}{2}(1+\mu+\nu)\right] \Gamma\left[\frac{1}{2}(1+\mu-\nu)\right]}{2^{2-\lambda-\mu} \Gamma\left[\frac{1}{2}(\nu-\lambda)+1\right] \Gamma\left[1-\frac{1}{2}(\lambda+\nu)\right]} \left[-\operatorname{Re} \mu < \operatorname{Re} \lambda + 1 < \frac{5}{2}\right].$$

ET II 385(17)

6.862

$$\begin{aligned} 1. \int_0^u x^{\lambda-\frac{1}{2}\mu-\frac{1}{2}}(u-x)^{\sigma-1} s_{\mu, \nu}(a\sqrt{x}) dx &= \\ &= \Gamma(\sigma) \frac{a^{\mu+1} u^{\lambda+\sigma} \Gamma(\lambda+1)}{(\mu-\nu+1)(\mu+\nu+1)\Gamma(\lambda+\sigma+1)} \times \\ &\quad \times {}_2F_3\left(1, 1+\lambda; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}, \lambda+\sigma+1; -\frac{a^2 u}{4}\right) \quad [\operatorname{Re} \lambda > -1, \operatorname{Re} \sigma > 0]. \end{aligned}$$

ET II 199(92)

797

$$2. \int_u^\infty x^{\frac{1}{2}\nu} (x-u)^{\mu-1} S_{\lambda, \nu}(a\sqrt{x}) dx = \frac{B\left[\mu, \frac{1}{2}(1-\lambda-\nu)-\mu\right] u^{\frac{1}{2}\mu+\frac{1}{2}\nu}}{a^\mu} S_{\lambda+\mu, \mu+\nu}(a\sqrt{u}) \left[|\arg(a\sqrt{u})| < \pi, \quad 0 < 2 \operatorname{Re} \mu < 1 - \operatorname{Re}(\lambda + \nu)\right].$$

ET II 211(71)

6.863

$$\int_0^\infty \sqrt{x} e^{-\alpha x} s_{\mu, \frac{1}{4}}\left(\frac{x^2}{2}\right) dx = 2^{-2\mu-1} \sqrt{\alpha} \Gamma\left(2\mu + \frac{3}{2}\right) S_{-\mu-1, \frac{1}{4}}\left(\frac{\alpha^2}{2}\right) \left[\operatorname{Re} \alpha > 0, \operatorname{Re} \mu > -\frac{3}{4}\right].$$

ET I 209(38)

6.864

$$\begin{aligned} \int_0^\infty \exp[(\mu+1)x] s_{\mu, \nu}(a \operatorname{sh} x) dx &= 2^{\mu-2} \pi \operatorname{cosec}(\mu\pi) \Gamma(\varrho) \Gamma(\sigma) \times \\ &\quad \times \left[I_\varrho\left(\frac{a}{2}\right) I_\sigma\left(\frac{a}{2}\right) - I_{-\varrho}\left(\frac{a}{2}\right) I_{-\sigma}\left(\frac{a}{2}\right) \right], \\ 2\varrho &= \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [a > 0, \quad -2 < \operatorname{Re} \mu < 0]. \end{aligned}$$

$$\int_0^{\infty} \sqrt{\operatorname{sh} x} \operatorname{ch}(\nu x) S_{\mu, \frac{1}{2}}(a \operatorname{ch} x) dx = \frac{B\left(\frac{1}{4} - \frac{\mu + \nu}{2}, \frac{1}{4} - \frac{\mu - \nu}{2}\right)}{\sqrt{a} 2^{\mu + \frac{3}{2}}} S_{\mu + \frac{1}{2}, \nu}(a) \\ \left[|\arg a| < \pi, \quad \operatorname{Re} \mu + |\operatorname{Re} \nu| < \frac{1}{2} \right]$$

ET II 388(31)

6.866

$$1. \int_0^{\infty} x^{-\mu-1} \cos(ax) s_{\mu, \nu}(x) dx = \\ = 0 \quad [a > 1]; \\ = 2^{\mu - \frac{1}{2}} \sqrt{\pi} \Gamma\left(\frac{\mu + \nu + 1}{2}\right) \Gamma\left(\frac{\mu - \nu + 1}{2}\right) (1 - a^2)^{\frac{1}{2}\mu + \frac{1}{4}} P_{\nu - \frac{1}{2}}^{-\mu - \frac{1}{2}}(a) \quad [0 < a < 1].$$

ET II 386(18)

$$2. \int_0^{\infty} x^{-\mu} \sin(ax) S_{\mu, \nu}(x) dx = 2^{-\mu - \frac{1}{2}} \sqrt{\pi} \Gamma\left(1 - \frac{\mu + \nu}{2}\right) \Gamma\left(1 - \frac{\mu - \nu}{2}\right) (a^2 - 1)^{\frac{1}{2}\mu - \frac{1}{4}} P_{\nu - \frac{1}{2}}^{\mu - \frac{1}{2}}(a) \\ [a > 1, \quad \operatorname{Re} \mu < 1 - |\operatorname{Re} \nu|].$$

ET II 387(23)

6.867

$$1. \int_0^{\frac{\pi}{2}} \cos(2\mu x) S_{2\mu-1, 2\nu}(a \cos x) dx = \\ = \frac{\pi 2^{2\mu-3} a^{2\mu} \operatorname{cosec}(2\nu\pi)}{\Gamma(1 - \mu - \nu) \Gamma(1 - \mu + \nu)} \left[J_{\mu+\nu}\left(\frac{a}{2}\right) N_{\mu-\nu}\left(\frac{a}{2}\right) - J_{\mu-\nu}\left(\frac{a}{2}\right) N_{\mu+\nu}\left(\frac{a}{2}\right) \right] \\ [\operatorname{Re} \mu > -2, \quad |\operatorname{Re} \nu| < 1].$$

ET II 388(29)

798

$$2. \int_0^{\frac{\pi}{2}} \cos[(\mu + 1)x] s_{\mu, \nu}(a \cos x) dx = 2^{\mu-2} \pi \Gamma(\varrho) \Gamma(\sigma) J_{\varrho}\left(\frac{a}{2}\right) J_{\sigma}\left(\frac{a}{2}\right), \\ 2\varrho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1 \quad [\operatorname{Re} \mu > -2].$$

ET II 386(21)

6.868

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2\mu x)}{\cos x} S_{2\mu, 2\nu}(a \sec x) dx = \frac{\pi 2^{2\mu-1}}{a} W_{\mu, \nu}(ae^{i\frac{\pi}{2}}) W_{\mu, \nu}(ae^{-i\frac{\pi}{2}}) \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu < 1].$$

6.869

$$1. \int_0^\infty x^{1-\mu-\nu} J_\nu(ax) S_{\mu, -\mu-2\nu}(x) dx = \frac{\sqrt{\pi} a^{\nu-1} \Gamma(1-\mu-\nu)}{2^{\mu+2\nu} \Gamma\left(\nu + \frac{1}{2}\right)} (a^2-1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$$

$$\left[a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu + \nu) < 1 \right].$$

ET II 388(28)

$$2. \int_0^\infty x^{-\mu} J_\nu(ax) s_{\nu+\mu, -\nu+\mu+1}(x) dx =$$

$$= 2^{\nu-1} \Gamma(\nu) a^{-\nu} (1-a^2)^\mu \left[0 < a < 1, \quad \operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < \frac{3}{2} \right];$$

$$= 0 \quad \left[1 < a, \quad \operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < \frac{3}{2} \right].$$

ET II 92(24)

$$3. \int_0^\infty x K_\nu(bx) s_{\mu, \frac{1}{2}\nu}(ax^2) dx = \frac{1}{4a} \Gamma\left(\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\mu - \frac{1}{2}\nu + 1\right) S_{-\mu-1, \frac{1}{2}\nu}\left(\frac{b^2}{4a}\right)$$

$$\left[\operatorname{Re} \mu > \frac{1}{2} |\operatorname{Re} \nu| - 2, \quad a > 0, \quad \operatorname{Re} b > 0 \right].$$

ET II 151(78)

6.87 Thomson functions

6.871

$$1. \int_0^\infty e^{-\beta x} \operatorname{ber} x dx = \frac{\left(\sqrt{\beta^4 + 1} + \beta^2\right)^{\frac{1}{2}}}{\sqrt{2(\beta^4 + 1)}}.$$

ME 40

$$2. \int_0^\infty e^{-\beta x} \operatorname{bei} x dx = \frac{\left(\sqrt{\beta^4 + 1} - \beta^2\right)^{\frac{1}{2}}}{\sqrt{2(\beta^4 + 1)}}.$$

ME 40

799

6.872

$$1. \int_0^{\infty} e^{-\beta x} \operatorname{ber}_{\nu}(2\sqrt{x}) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right) \cos\left(\frac{1}{2\beta} + \frac{3\nu\pi}{4}\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right) \cos\left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi\right) \right].$$

MI 49

$$2. \int_0^{\infty} e^{-\beta x} \operatorname{bei}_{\nu}(2\sqrt{x}) dx = \frac{1}{2\beta} \sqrt{\frac{\pi}{\beta}} \left[J_{\frac{1}{2}(\nu-1)}\left(\frac{1}{2\beta}\right) \sin\left(\frac{1}{2\beta} + \frac{3\nu\pi}{4}\right) - J_{\frac{1}{2}(\nu+1)}\left(\frac{1}{2\beta}\right) \sin\left(\frac{1}{2\beta} + \frac{3\nu+6}{4}\pi\right) \right].$$

MI 49

$$3. \int_0^{\infty} e^{-\beta x} \operatorname{ber}(2\sqrt{x}) dx = \frac{1}{\beta} \cos \frac{1}{\beta}.$$

ME 40

$$4. \int_0^{\infty} e^{-\beta x} \operatorname{bei}(2\sqrt{x}) dx = \frac{1}{\beta} \sin \frac{1}{\beta}.$$

ME 40

$$5. \int_0^{\infty} e^{-\beta x} \operatorname{ker}(2\sqrt{x}) dx = -\frac{1}{2\beta} \left[\cos \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} + \sin \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right].$$

MI 50

$$6. \int_0^{\infty} e^{-\beta x} \operatorname{kei}(2\sqrt{x}) dx = -\frac{1}{2\beta} \left[\sin \frac{1}{\beta} \operatorname{ci} \frac{1}{\beta} - \cos \frac{1}{\beta} \operatorname{si} \frac{1}{\beta} \right].$$

MI 50

$$7. \int_0^{\infty} e^{-\beta x} \operatorname{ber}_{\nu}(2\sqrt{x}) \operatorname{bei}_{\nu}(2\sqrt{x}) dx = \frac{1}{2\beta} J_{\nu}\left(\frac{2}{\beta}\right) \sin\left(\frac{2}{\beta} + \frac{3\nu\pi}{2}\right) \quad [\operatorname{Re} \nu > -1].$$

MI 49

6.873

6.874

$$1. \int_0^{\infty} \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{ber}_{2\nu}(2\sqrt{2x}) dx = \sqrt{\frac{\pi}{\beta}} J_{\nu}\left(\frac{1}{\beta}\right) \cos\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

MI 49

$$2. \int_0^{\infty} \frac{e^{-\beta x}}{\sqrt{x}} \operatorname{bei}_{2\nu}(2\sqrt{2x}) dx = \sqrt{\frac{\pi}{\beta}} J_{\nu}\left(\frac{1}{\beta}\right) \sin\left(\frac{1}{\beta} - \frac{3\pi}{4} + \frac{3\nu\pi}{2}\right) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

MI 49

$$3. \int_0^{\infty} x^{\frac{\nu}{2}} \operatorname{ber}_{\nu}(\sqrt{x}) e^{-\beta x} dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \cos\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \quad [\operatorname{Re} \nu > -1].$$

ME 40

$$4. \int_0^{\infty} x^{\frac{\nu}{2}} \operatorname{bei}_{\nu}(\sqrt{x}) e^{-\beta x} dx = \frac{2^{-\nu}}{\beta^{1+\nu}} \sin\left(\frac{1}{4\beta} + \frac{3\nu\pi}{4}\right) \quad [\operatorname{Re} \nu > -1].$$

ME 40

800

6.875

$$1. \int_0^{\infty} e^{-\beta x} \left[\operatorname{ker}(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{ber}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[\ln \beta \cos \frac{1}{\beta} + \frac{\pi}{4} \sin \frac{1}{\beta} \right].$$

MI 50

$$2. \int_0^{\infty} e^{-\beta x} \left[\operatorname{kei}(2\sqrt{x}) - \frac{1}{2} \ln x \operatorname{bei}(2\sqrt{x}) \right] dx = \frac{1}{\beta} \left[\ln \beta \sin \frac{1}{\beta} - \frac{\pi}{4} \cos \frac{1}{\beta} \right].$$

MI 50

6.876

$$1. \int_0^{\infty} x \operatorname{kei} x J_1(ax) dx = -\frac{1}{2a} \operatorname{arctg} a^2 \quad [a > 0].$$

ET II 21(32)

$$2. \int_0^{\infty} x \operatorname{ker} x J_1(ax) dx = \frac{1}{2a} \ln(1 + a^4)^{\frac{1}{2}} \quad [a > 0].$$

6.9 Mathieu Functions

Notation: $k^2 = q$. For definition of the coefficients $A_p^{(m)}$ and $B_p^{(m)}$ see 8.6

6.91 Mathieu functions

6.911

$$1. \int_0^{2\pi} \operatorname{ce}_m(z, q) \operatorname{ce}_p(z, q) dz = 0 \quad [m \neq p].$$

MA

$$2. \int_0^{2\pi} [\operatorname{ce}_{2n}(z, q)]^2 dz = 2\pi [A_0^{(2n)}]^2 + \pi \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^2 = \pi.$$

MA

$$3. \int_0^{2\pi} [\operatorname{ce}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^{\infty} [A_{2r+1}^{(2n+1)}]^2 = \pi.$$

MA

$$4. \int_0^{2\pi} \operatorname{se}_m(z, q) \operatorname{se}_p(z, q) dz = 0 \quad [m \neq p].$$

MA

$$5. \int_0^{2\pi} [\operatorname{se}_{2n+1}(z, q)]^2 dz = \pi \sum_{r=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2 = \pi.$$

MA

MA

$$7. \int_0^{2\pi} \text{se}_m(z, q) \text{ce}_p(z, q) dz = 0 \quad [m = 1, 2, \dots; \quad p = 1, 2, \dots].$$

MA

6.92 Combinations of Mathieu, hyperbolic, and trigonometric functions

6.921

$$1. \int_0^\pi \text{ch}(2k \cos u \text{ sh } z) \text{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} (-1)^n \text{Ce}_{2n}(z, -q) \quad [q > 0].$$

MA

801

$$2. \int_0^\pi \text{ch}(2k \sin u \text{ ch } z) \text{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\text{ce}_{2n}(0, q)} (-1)^n \text{Ce}_{2n}(z, -q) \quad [q > 0].$$

MA

$$3. \int_0^\pi \text{sh}(2k \sin u \text{ ch } z) \text{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\text{se}'_{2n+1}(0, q)} (-1)^n \text{Ce}_{2n+1}(z, -q) \quad [q > 0].$$

MA

$$4. \int_0^\pi \text{sh}(2k \cos u \text{ sh } z) \text{ce}_{2n+1}(u, q) du = \frac{\pi k A_1^{(2n+1)}}{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} (-1)^{n+1} \text{Se}_{2n+1}(z, -q) \quad [q > 0].$$

MA

$$5. \int_0^\pi \text{sh}(2k \sin u \text{ sin } z) \text{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\text{se}'_{2n+1}(0, q)} \text{se}_{2n+1}(z, q) \quad [q > 0].$$

MA

6.922

MA

$$2. \int_0^\pi \sin u \operatorname{sh} z \cos(2k \cos u \operatorname{ch} z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+1}(z, q) \quad [q > 0].$$

MA

$$3. \int_0^\pi \sin u \operatorname{sh} z \sin(2k \cos u \operatorname{ch} z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+2}(z, q) \quad [q > 0].$$

MA

$$4. \int_0^\pi \cos u \operatorname{ch} z \sin(2k \sin u \operatorname{sh} z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} \operatorname{Se}_{2n+2}(z, q) \quad [q > 0].$$

MA

$$5. \int_0^\pi \sin u \operatorname{ch} z \operatorname{ch}(2k \cos u \operatorname{sh} z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi B_1^{(2n+1)}}{2 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} (-1)^n \operatorname{Ce}_{2n+1}(z, -q) \quad [q > 0].$$

MA

$$6. \int_0^\pi \cos u \operatorname{sh} z \operatorname{ch}(2k \sin u \operatorname{ch} z) \operatorname{ce}_{2n+1}(u, q) du = \frac{\pi A_1^{(2n+1)}}{2 \operatorname{ce}_{2n+1}(0, q)} (-1)^n \operatorname{Se}_{2n+1}(z, -q) \quad [q > 0].$$

MA

$$7. \int_0^\pi \sin u \operatorname{ch} z \operatorname{sh}(2k \cos u \operatorname{sh} z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} (-1)^{n+1} \operatorname{Se}_{2n+2}(z, -q) \quad [q > 0].$$

MA

802

$$8. \int_0^\pi \cos u \operatorname{sh} z \operatorname{sh}(2k \sin u \operatorname{ch} z) \operatorname{se}_{2n+2}(u, q) du = \frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}(0, q)} (-1)^n \operatorname{Se}_{2n+2}(z, -q)$$

$$1. \int_0^{\infty} \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+1}(z, q) \quad [q > 0].$$

MA

$$2. \int_0^{\infty} \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+1}(u, q) du = -\frac{\pi B_1^{(2n+1)}}{4 \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Gey}_{2n+1}(z, q) \quad [q > 0].$$

MA

$$3. \int_0^{\infty} \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Gey}_{2n+2}(z, q) \quad [q > 0].$$

MA

$$4. \int_0^{\infty} \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{sh} z \operatorname{sh} u \operatorname{Se}_{2n+2}(u, q) du = -\frac{k\pi B_2^{(2n+2)}}{4 \operatorname{se}_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+2}(z, q) \quad [q > 0].$$

MA

$$5. \int_0^{\infty} \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}\left(\frac{1}{2}\pi, q\right)} \operatorname{Ce}_{2n}(z, q) \quad [q > 0].$$

MA

$$6. \int_0^{\infty} \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n}(u, q) du = -\frac{\pi A_0^{(2n)}}{2 \operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{Fey}_{2n}(z, q) \quad [q > 0].$$

MA

$$7. \int_0^{\infty} \sin(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k\pi A_1^{(2n+1)}}{2 \operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Fey}_{2n+1}(z, q) \quad [q > 0].$$

$$8. \int_0^{\infty} \cos(2k \operatorname{ch} z \operatorname{ch} u) \operatorname{Ce}_{2n+1}(u, q) du = \frac{k\pi A_1^{(2n+1)}}{2 \operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Ce}_{2n+1}(z, q) \quad [q > 0].$$

MA

6.924

$$1. \int_0^{\pi} \cos(2k \cos u \cos z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{ce}_{2n}(z, q) \quad [q > 0].$$

MA

803

$$2. \int_0^{\pi} \sin(2k \cos u \cos z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{ce}_{2n+1}(z, q) \quad [q > 0].$$

MA

$$3. \int_0^{\pi} \cos(2k \cos u \operatorname{ch} z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{Ce}_{2n}(z, q) \quad [q > 0].$$

MA

$$4. \int_0^{\pi} \cos(2k \sin u \operatorname{sh} z) \operatorname{ce}_{2n}(u, q) du = \frac{\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q)} \operatorname{Ce}_{2n}(z, q) \quad [q > 0].$$

MA

$$5. \int_0^{\pi} \sin(2k \cos u \operatorname{ch} z) \operatorname{ce}_{2n+1}(u, q) du = -\frac{\pi k A_1^{(2n+1)}}{\operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Ce}_{2n+1}(z, q) \quad [q > 0].$$

MA

$$6. \int_0^{\pi} \sin(2k \sin u \operatorname{sh} z) \operatorname{se}_{2n+1}(u, q) du = \frac{\pi k B_1^{(2n+1)}}{\operatorname{se}'_{2n+1}(0, q)} \operatorname{Se}_{2n+1}(z, q) \quad [q > 0].$$

MA

6.925

Notation: $z_1 = 2k\sqrt{\operatorname{ch}^2 \xi - \sin^2 \eta}$, $\operatorname{tg} \alpha = \theta \xi \operatorname{tg} \eta$

$$1. \int_0^{2\pi} \sin [z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = 0.$$

MA

$$2. \int_0^{2\pi} \cos [z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n}(\theta, q) d\theta = \frac{2\pi A_0^{(2n)}}{\operatorname{ce}_{2n}(0, q) \operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{Ce}_{2n}(\xi, q) \operatorname{ce}_{2n}(\eta, q).$$

MA

$$3. \int_0^{2\pi} \sin [z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = -\frac{2\pi k A_1^{(2n+1)}}{\operatorname{ce}_{2n+1}(0, q) \operatorname{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Ce}_{2n+1}(\xi, q) \operatorname{ce}_{2n+1}(\eta, q).$$

MA

$$4. \int_0^{2\pi} \cos [z_1 \cos(\theta - \alpha)] \operatorname{ce}_{2n+1}(\theta, q) d\theta = 0.$$

MA

$$5. \int_0^{2\pi} \sin [z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = \frac{2\pi k B_1^{(2n+1)}}{\operatorname{se}_{2n+1}(0, q) \operatorname{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \operatorname{Se}_{2n+1}(\xi, q) \operatorname{se}_{2n+1}(\eta, q).$$

MA

$$6. \int_0^{2\pi} \cos [z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+1}(\theta, q) d\theta = 0.$$

MA

$$7. \int_0^{2\pi} \sin [z_1 \cos(\theta - \alpha)] \operatorname{se}_{2n+2}(\theta, q) d\theta = 0.$$

MA

804
6.926

$$\int_0^\pi \sin u \sin z \sin(2k \cos u \cos z) \operatorname{se}_{2n+2}(u, q) du = -\frac{\pi k B_2^{(2n+2)}}{2 \operatorname{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)} \operatorname{se}_{2n+2}(z, q) \quad [q > 0].$$

MA

6.93 Combinations of Mathieu and Bessel functions

6.931

$$1. \int_0^\pi J_0\{k[2(\cos 2u + \cos 2z)]^{\frac{1}{2}}\} \operatorname{ce}_{2n}(u, q) du = \frac{\pi [A_0^{(2n)}]^2}{\operatorname{ce}_{2n}(0, q) \operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{ce}_{2n}(z, q).$$

MA

$$2. \int_0^{2\pi} N_0\{k[2(\cos 2u + \operatorname{ch} 2z)]^{\frac{1}{2}}\} \operatorname{ce}_{2n}(u, q) du = \frac{2\pi [A_0^{(2n)}]^2}{\operatorname{ce}_{2n}(0, q) \operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \operatorname{Fey}_{2n}(z, q).$$

MA

7.1- 7.2 Associated Legendre Functions

7.11 Associated Legendre functions

7.111

$$\int_{\cos \varphi}^1 P_\nu(x) dx = \sin \varphi P_\nu^{-1}(\cos \varphi).$$

MO 90

7.112

$$1. \int_{-1}^1 P_n^m(x) P_k^m(x) dx = 0 \quad [n \neq k];$$

$$= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \quad [n = k].$$

$$2. \int_{-1}^1 Q_n^m(x) P_k^m(x) dx = (-1)^m \frac{1 - (-1)^{n+k}(n+m)!}{(k-n)(k+n+1)(n-m)!}.$$

EH I 171(18)

$$3. \int_{-1}^1 P_\nu(x) P_\sigma(x) dx = \frac{2\pi \sin \pi(\sigma - \nu) + 4 \sin(\pi\nu) \sin(\pi\sigma) [\psi(\nu + 1) - \psi(\sigma + 1)]}{\pi^2(\sigma - \nu)(\sigma + \nu + 1)} \\ [\sigma + \nu + 1 \neq 0]; \\ = \frac{\pi^2 - 2(\sin \pi\nu)^2 \psi'(\nu + 1)}{\pi^2 \left(\nu + \frac{1}{2}\right)} \quad [\sigma = \nu].$$

EH I 170(9)a
EH I 170(7)

805

$$4. \int_{-1}^1 Q_\nu(x) Q_\sigma(x) dx = \\ = \frac{[\psi(\nu + 1) - \psi(\sigma + 1)][1 + \cos(\pi\sigma) \cos(\nu\pi)] - \frac{\pi}{2} \sin \pi(\nu - \sigma)}{(\sigma - \nu)(\sigma + \nu + 1)} \\ [\sigma + \nu + 1 \neq 0; \quad \nu, \quad \sigma \neq -1, -2, -3, \dots]; \\ = \frac{\frac{1}{2}\pi^2 - \psi'(\nu + 1)[1 + (\cos \nu\pi)^2]}{2\nu + 1} \quad [\nu = \sigma, \quad \nu \neq -1, -2, -3, \dots].$$

EH I 170(12)
EH I 170(11)

$$5. \int_{-1}^1 P_\nu(x) Q_\sigma(x) dx = \\ = \frac{1 - \cos \pi(\sigma - \nu) - 2\pi^{-1} \sin(\pi\nu) \cos(\pi\sigma) [\psi(\nu + 1) - \psi(\sigma + 1)]}{(\nu - \sigma)(\nu + \sigma + 1)} \\ [\operatorname{Re} \nu > 0, \quad \operatorname{Re} \sigma > 0, \quad \sigma \neq \nu]; \\ = - \frac{\sin(2\nu\pi) \psi'(\nu + 1)}{\pi(2\nu + 1)} \quad [\operatorname{Re} \nu > 0, \quad \sigma = \nu].$$

EH I 171(14)
EH I 170(13)

7.113

$$\text{Notation: } A = \frac{\Gamma(\frac{1}{2} + \frac{\nu}{2}) \Gamma(1 + \frac{\sigma}{2})}{\Gamma(\frac{1}{2} + \frac{\sigma}{2}) \Gamma(1 + \frac{\nu}{2})}$$

$$1. \int_0^1 P_\nu(x) P_\sigma(x) dx = \frac{A \sin \frac{\pi\sigma}{2} \cos \frac{\pi\nu}{2} - A^{-1} \sin \frac{\pi\nu}{2} \cos \frac{\pi\sigma}{2}}{\frac{1}{2} \sin(\frac{\sigma - \nu}{2}) (\sigma + \nu + 1)}.$$

EH I 171(15)

$$2. \int_0^1 Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\nu+1) - \psi(\sigma+1) - \frac{\pi}{2} \left[(A - A^{-1}) \sin \frac{\pi(\sigma+\nu)}{2} (A + A^{-1}) \sin \frac{\pi(\sigma-\nu)}{2} \right]}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \operatorname{Re} \sigma > 0].$$

EH I 171(16)

$$3. \int_0^1 P_\nu(x) Q_\sigma(x) dx = \frac{A^{-1} \cos \frac{\pi(\nu-\sigma)}{2} - 1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re} \nu > 0, \operatorname{Re} \sigma > 0].$$

EH I 171(17)

7.114

$$1. \int_1^\infty P_\nu(x) Q_\sigma(x) dx = \frac{1}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\sigma-\nu) > 0, \operatorname{Re}(\sigma+\nu) > -1].$$

ET II 324(19)

806

$$2. \int_1^\infty Q_\nu(x) Q_\sigma(x) dx = \frac{\psi(\sigma+1) - \psi(\nu+1)}{(\sigma-\nu)(\sigma+\nu+1)} \quad [\operatorname{Re}(\nu+\sigma) > -1; \sigma, \nu \neq -1, -2, -3, \dots].$$

EH I 170(5)

$$3. \int_1^\infty [Q_\nu(x)]^2 dx = \frac{\psi'(\nu+1)}{2\nu+1} \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

EH I 170(6)

7.115

$$\int_1^\infty Q_\nu(x) dx = \frac{1}{\nu(\nu+1)} \quad [\operatorname{Re} \nu > 0].$$

ET II 324(18)

7.12- 7.13 Combinations of associated Legendre functions and powers

7.121

$$\int_{\cos \varphi}^1 x P_\nu(x) dx = \frac{-\sin \varphi}{(\nu-1)(\nu+2)} [\sin \varphi P_\nu(\cos \varphi) + \cos \varphi P_\nu^1(\cos \varphi)].$$

MO 90

7.122

$$1. \int_0^1 \frac{[P_n^m(x)]^2}{1-x^2} dx = \frac{1}{2m} \frac{(n+m)!}{(n-m)!} \quad [0 < m \leq n].$$

MO 74

$$2. \int_0^1 [P_\nu^\mu(x)]^2 \frac{dx}{1-x^2} = -\frac{\Gamma(1+\mu+\nu)}{2\mu\Gamma(1-\mu+\nu)} \quad [\operatorname{Re} \mu < 0, \quad \nu+\mu-\text{a positive integer}].$$

EH I 172(26)

$$3. \int_0^1 [P_\nu^{n-\nu}(x)]^2 \frac{dx}{1-x^2} = -\frac{n!}{2(n-\nu)\Gamma(1-n+2\nu)} \quad [n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > n].$$

ET II 315(9)

7.123

$$\int_{-1}^1 P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} = 0 \quad [0 \leq m \leq n, \quad 0 \leq k \leq n; \quad m \neq k].$$

MO 74

7.124

$$\int_{-1}^1 x^k (z-x)^{-1} (1-x^2)^{\frac{1}{2}m} P_n^m(x) dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z) \cdot z^k$$

$[m \leq n; k = 0, 1, \dots, n-m; z$ in the complex plane with a cut along the interval $(-1, 1)$ on the real axis].

7.125

$$\begin{aligned} \int_{-1}^1 (1-x^2)^{\frac{1}{2}m} P_k^m(x) P_l^m(x) P_n^m(x) dx &= \\ &= (-1)^m \pi^{-\frac{3}{2}} \frac{(k+m)!(l+m)!(n+m)!(s-m)!}{(k-m)!(l-m)!(n-m)!(s-k)!} \times \\ &\quad \times \frac{\Gamma\left(\frac{1}{m+\frac{1}{2}}\right) \Gamma\left(\frac{1}{l-k+\frac{1}{2}}\right) \Gamma\left(\frac{1}{l-l+\frac{1}{2}}\right) \Gamma\left(\frac{1}{l-n+\frac{1}{2}}\right)}{\Gamma\left(\frac{3}{2}\right)} \end{aligned}$$

$$1. \int_0^1 P_\nu(x) x^\sigma dx = \frac{\sqrt{\pi} 2^{-\sigma-1} \Gamma(1+\sigma)}{\Gamma\left(1+\frac{1}{2}\sigma-\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\sigma+\frac{1}{2}\nu+\frac{3}{2}\right)} \quad [\operatorname{Re} \sigma > -1].$$

EH I 171(23)

$$2. \int_0^1 x^\sigma P_\nu^m(x) dx = \frac{(-1)^m \pi^{\frac{1}{2}} 2^{-2m-1} \Gamma\left(\frac{1+\sigma}{2}\right) \Gamma(1+m+\nu)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}m\right) \Gamma\left(\frac{3}{2}+\frac{\sigma}{2}+\frac{m}{2}\right) \Gamma(1-m+\nu)} \times \\ \times {}_3F_2\left(\frac{m+\nu+1}{2}, \frac{m-\nu}{2}, \frac{m}{2}+1; m+1, \frac{3+\sigma+m}{2}; 1\right) \\ [\operatorname{Re} \sigma > -1; \quad m = 0, 1, 2, \dots].$$

ET II 313(2)

$$3. \int_0^1 x^\sigma P_\nu^\mu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{2\mu-1} \Gamma\left(\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right) \Gamma\left(\frac{3+\sigma-\mu}{2}\right)} \times \\ \times {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1-\frac{\mu}{2}; 1-\mu, \frac{3+\sigma-\mu}{2}; 1\right) \\ [\operatorname{Re} \sigma > -1, \quad \operatorname{Re} \mu < 2].$$

ET II 313(3)

$$4. \int_1^\infty x^{\mu-1} Q_\nu(ax) dx = e^{\mu\pi i} \Gamma(\mu) a^{-\mu} (a^2-1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a) \\ [|\arg(a-1)| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\nu-\mu) > -1].$$

ET II 325(26)

7.127

$$\int_{-1}^1 (1+x)^\sigma P_\nu(x) dx = \frac{2^{\sigma+1} [\Gamma(\sigma+1)]^2}{\Gamma(\sigma+\nu+2) \Gamma(1+\sigma-\nu)} \quad [\operatorname{Re} \sigma > -1].$$

ET II 316(15)

7.128

$$1. \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_\nu^\mu(x) dx = \\ = -\frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\pi^{\frac{1}{2}} e^{2\mu\pi i} \Gamma(\mu+\nu) \Gamma(\mu-\nu-1)} \times$$

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$$\begin{aligned}
2. \int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx &= \\
&= \frac{2e^{-2\mu\pi i} \Gamma\left(\frac{1}{2} + \mu\right)}{\pi^{\frac{1}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu + 1)} (z-1)^\mu Q_\nu^\mu \left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}} \right] Q_{-\nu-1}^\mu \left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}} \right] \\
&\left[-\frac{1}{2} < \operatorname{Re} \mu < 1, \quad z - \text{in the complex plane with a cut along the interval} \right. \\
&\qquad\qquad\qquad \left. (-1, 1) \text{ of the real axis} \right].
\end{aligned}$$

ET II 316(18)

7.129

$$\int_{-1}^1 P_\nu(x) P_\lambda(x) (1+x)^{\lambda+\nu} dx = \frac{2^{\lambda+\nu+1} [\Gamma(\lambda + \nu + 1)]^4}{[\Gamma(\lambda + 1) \Gamma(\nu + 1)]^2 \Gamma(2\lambda + 2\nu + 2)} \quad [\operatorname{Re}(\nu + \lambda + 1) > 0].$$

EH I 172(30)

7.131

$$\begin{aligned}
1. \int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx &= \\
&= \pi^{\frac{1}{2}} \frac{\Gamma(-\mu - \nu) \Gamma(1 - \mu + \nu)}{\Gamma\left(\frac{1}{2} - \mu\right)} (z-1)^\mu \left\{ P_\nu^\mu \left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}} \right] \right\}^2 \\
&\quad [\operatorname{Re}(\mu + \nu) < 0, \quad \operatorname{Re}(\mu - \nu) < 1, \quad |\arg(z+1)| < \pi].
\end{aligned}$$

ET II 321(6)

$$\begin{aligned}
2. \int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{3}{2}} P_\nu^\mu(x) dx &= \\
&= \frac{\pi^{\frac{1}{2}} \Gamma(1 - \mu - \nu) \Gamma(2 - \mu + \nu) (z-1)^{\mu-\frac{1}{2}} (z+1)^{-\frac{1}{2}}}{\Gamma\left(\frac{3}{2} - \mu\right)} P_\nu^\mu \left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}} \right] P_{\nu-1}^{\mu-1} \left[\left(\frac{1+z}{2}\right)^{\frac{1}{2}} \right] \\
&\quad [\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + \nu) < 1, \quad \operatorname{Re}(\mu - \nu) < 2, \quad \arg(1+z) < \pi].
\end{aligned}$$

ET II 321(7)

7.132

$$\begin{aligned}
1. \int_{-1}^1 (1-x^2)^{\lambda-1} P_\nu^\mu(x) dx &= \\
&= \frac{\pi 2^\mu \Gamma\left(\lambda + \frac{1}{2}\mu\right) \Gamma\left(\lambda - \frac{1}{2}\mu\right)}{\Gamma\left(\lambda + \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\lambda - \frac{1}{2}\nu\right) \Gamma\left(-\frac{1}{2}\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(-\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)}
\end{aligned}$$

$$2. \int_1^{\infty} (x^2-1)^{\lambda-1} P_n^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma\left(\lambda - \frac{1}{2}\mu\right) \Gamma\left(1 - \lambda + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \lambda - \frac{1}{2}\nu\right)}{\Gamma\left(1 - \frac{1}{2}\mu + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 - \lambda - \frac{1}{2}\mu\right)}$$

[Re $\lambda > \text{Re } \mu$, Re($1 - 2\lambda - \nu$) > 0 , Re($2 - 2\lambda + \nu$) > 0].

ET II 320(2)

$$3. \int_1^{\infty} (x^2-1)^{\lambda-1} Q_{\nu}^{\mu}(x) dx = e^{\mu\pi i} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right) \Gamma\left(1 - \lambda + \frac{1}{2}\nu\right) \Gamma\left(\lambda + \frac{1}{2}\mu\right) \Gamma\left(\lambda - \frac{1}{2}\mu\right)}{2^{2\lambda-\mu} \Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \lambda + \frac{1}{2}\nu\right)}$$

[|Re μ | $< 2 \text{ Re } \lambda < \text{Re } \nu + 2$].

ET II 324(23)

$$4. \int_0^1 x^{\sigma} (1-x^2)^{-\frac{1}{2}\mu} P_{\nu}^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma\left(\frac{1}{2} + \frac{1}{2}\sigma\right) \Gamma\left(1 + \frac{1}{2}\sigma\right)}{\Gamma\left(1 + \frac{1}{2}\sigma - \frac{1}{2}\nu - \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2}\sigma + \frac{1}{2}\nu - \frac{1}{2}\mu + \frac{3}{2}\right)}$$

[Re $\mu < 1$, Re $\sigma > -1$].

EH I 172(24)

$$5. \int_0^1 x^{\sigma} (1-x^2)^{\frac{1}{2}m} P_{\nu}^m(x) dx = \frac{(-1)^m 2^{-m-1} \Gamma\left(\frac{1}{2} + \frac{1}{2}\sigma\right) \Gamma\left(1 + \frac{1}{2}\sigma\right) \Gamma(1 + m + \nu)}{\Gamma(1 - m + \nu) \Gamma\left(1 + \frac{1}{2}\sigma + \frac{1}{2}m - \frac{1}{2}\nu\right) \Gamma\left(\frac{3}{2} + \frac{1}{2}\sigma + \frac{1}{2}m + \frac{1}{2}\nu\right)}$$

[Re $\sigma > -1$, m is a positive integer].

EH I 172(25), ET II 313(4)

$$6. \int_0^1 (1-x^2)^{\eta} P_{\nu}^{\mu}(x) dx = \frac{2^{\mu-1} \Gamma\left(1 + \eta - \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\sigma\right)}{\Gamma(1 - \mu) \Gamma\left(\frac{3}{2} + \eta + \frac{1}{2}\sigma - \frac{1}{2}\mu\right)} \times$$

$$\times {}_3F_2\left(\frac{\nu - \mu + 1}{2}, -\frac{\mu + \nu}{2}, 1 + \eta - \frac{\mu}{2}; 1 - \mu, \frac{3 + \sigma - \mu}{2} + \eta; 1\right)$$

[Re $\left(\eta - \frac{1}{2}\mu\right) > -1$, Re $\sigma > -1$].

$$7. \int_1^\infty x^{-\varrho} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\lambda+\mu} \Gamma(\lambda) \Gamma(-\lambda - \mu - \nu) \Gamma(1 - \lambda - \mu + \nu)}{\sqrt{\pi} \Gamma(\varrho)}$$

[Re $\mu < 1$, Re($\varrho + \mu + \nu$) > 0 , Re($\varrho + \mu - \nu$) > 1].

7.133

$$1. \int_u^\infty Q_\nu(x) (x - u)^{\mu-1} dx = \Gamma(\mu) e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(u)$$

[|arg($u - 1$)| $< \pi$, $0 < \text{Re } \mu < 1 + \text{Re } \nu$].

MO 90a

810

$$2. \int_u^\infty (x^2 - 1)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x) (x - u)^{\mu-1} dx = \Gamma(\mu) e^{\mu\pi i} (u^2 - 1)^{\frac{1}{2}\lambda + \frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(u)$$

[|arg($u - 1$)| $< \pi$, $0 < \text{Re } \mu < 1 + \text{Re}(\nu - \lambda)$].

ET II 204(30)

7.134

$$1. \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\lambda+\mu} \Gamma(\lambda) \Gamma(-\lambda - \mu - \nu) \Gamma(1 - \lambda - \mu + \nu)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 - \lambda - \mu)}$$

[Re $\lambda > 0$, Re($\lambda + \mu + \nu$) < 0 , Re($\lambda + \mu - \nu$) < 1].

ET II 321(4)

$$2. \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = -\frac{2^{\lambda-\mu} \sin \pi\nu \Gamma(\lambda - \mu) \Gamma(-\lambda + \mu - \nu) \Gamma(1 - \lambda + \mu + \nu)}{\pi \Gamma(1 - \lambda)}$$

[Re($\lambda - \mu$) > 0 , Re($\mu - \lambda - \nu$) > 0 , Re($\mu - \lambda + \nu$) > -1].

ET II 321(5)

7.135

$$1. \int_{-1}^1 (1 - x^2)^{-\frac{1}{2}\mu} (z - x)^{-1} P_{\mu+n}^\mu(x) dx = 2e^{-i\mu\pi} (z^2 - 1)^{-\frac{1}{2}\mu} Q_{\mu+n}^\mu(z)$$

[$n = 0, 1, 2, \dots$, Re $\mu + n > -1$, z ---in the complex plane with a cut along the interval $(-1, 1)$ of the real axis].

$$2. \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx =$$

$$= \frac{2^{\lambda+\mu-\rho} \Gamma(\lambda - \rho) \Gamma(\rho - \lambda - \mu - \nu) \Gamma(\rho - \lambda - \mu + \nu + 1)}{\Gamma(1 - \mu + \nu) \Gamma(-\mu - \nu) \Gamma(1 + \rho - \lambda - \mu)} \times$$

$$\times {}_3F_2 \left(\rho, \rho - \lambda - \mu - \nu, \rho - \lambda - \mu + \nu + 1; \rho - \lambda + 1, \rho - \lambda - \mu + 1; \frac{1+z}{2} \right) +$$

$$+ \frac{\Gamma(\rho - \lambda) \Gamma(\lambda)}{\Gamma(\rho) \Gamma(1 - \mu)} 2^\mu (z+1)^{\lambda-\rho} {}_3F_2 \left(\lambda, -\mu - \nu, 1 - \mu + \nu; 1 - \mu, 1 - \rho + \lambda; \frac{1+z}{2} \right)$$

[Re $\lambda > 0$, Re($\rho - \lambda - \mu - \nu$) > 0 , Re($\rho - \lambda - \mu + \nu + 1$) > 0 ,
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|arg($z + 1$)| $< \pi$].

$$\begin{aligned}
3. \int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\mu/2} (x+z)^{-\rho} P_\nu^\mu(x) dx &= \\
&= -\frac{\sin(\nu\pi)\Gamma(\lambda-\mu-\rho)\Gamma(\rho-\lambda+\mu-\nu)\Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu}\pi\Gamma(1+\rho-\lambda)} \times \\
&\quad \times {}_3F_2\left(\rho, \rho-\lambda+\mu-\nu, \rho-\lambda+\mu+\nu+1; 1+\rho-\lambda, 1+\rho-\lambda+\mu; \frac{1+z}{2}\right) + \\
&\quad + \frac{\Gamma(\lambda-\mu)\Gamma(\rho-\lambda+\mu)}{\Gamma(\rho)\Gamma(1-\mu)} (z+1)^{\lambda-\rho-\mu} \times \\
&\quad \times {}_3F_2\left(\lambda-\mu, -\nu, \nu+1; 1+\lambda-\mu-\rho, 1-\mu; \frac{1+z}{2}\right) \\
&[\operatorname{Re}(\lambda-\mu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1) > 0, \\
&\quad |\arg(z+1)| < \pi].
\end{aligned}$$

ET II 322(10)

811
7.136

$$\begin{aligned}
1. \int_{-1}^1 (1-x^2)^{\lambda-1} (1-a^2x^2)^{\mu/2} P_\nu(ax) dx &= \frac{\pi 2^\mu \Gamma(\lambda)}{\Gamma\left(\frac{1}{2}+\lambda\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)} \times \\
&\quad \times {}_2F_1\left(-\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \frac{1}{2}+\lambda; a^2\right) \\
&[\operatorname{Re} \lambda > 0, \quad -1 < a < 1].
\end{aligned}$$

ET II 318(31)

$$\begin{aligned}
2. \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{\mu/2} P_\nu^\mu(ax) dx &= \\
&= \frac{\Gamma(\lambda)\Gamma\left(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu\right)}{\Gamma\left(1-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma(1-\lambda-\mu)} \times \\
&\quad \times 2^{\mu-1} a^{\mu-\nu-1} {}_2F_1\left(\frac{1-\mu+\nu}{2}, 1-\lambda-\frac{\mu-\nu}{2}; 1-\lambda-\mu; 1-\frac{1}{a^2}\right) \\
&[\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\nu-\mu-2\lambda) > -2, \quad \operatorname{Re}(2\lambda+\mu+\nu) < 1].
\end{aligned}$$

ET II 325(25)

$$\begin{aligned}
3. \int_1^\infty (x^2-1)^{\lambda-1} (a^2x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(ax) dx &= \\
&= \frac{\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma(\lambda)\Gamma\left(1-\lambda+\frac{\mu+\nu}{2}\right) 2^{\mu-2} e^{\mu\pi i} a^{-\mu-\nu-1}}{\Gamma\left(\nu+\frac{3}{2}\right)} \times \\
&\quad \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, 1-\lambda+\frac{\mu+\nu}{2}; \nu+\frac{3}{2}; a^{-2}\right) \\
&[|\arg(a-1)| < \pi, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\lambda-\mu-\nu) < 2].
\end{aligned}$$

$$1. \int_1^{\infty} x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}Q_{\nu}^{\mu}(1+2ax) dx = \pi^{-\frac{1}{2}}e^{-\mu\pi i}\Gamma\left(\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\{Q_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}]\}^2$$

$$\left[|\arg a| < \pi, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\mu + \nu) > -1\right].$$

ET II 325(28)

812

$$2. \int_1^{\infty} x^{-\frac{1}{2}\mu-\frac{1}{2}}(x-1)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}Q_{\nu}^{\mu}(1+2ax) dx =$$

$$= -\pi^{-\frac{1}{2}}e^{-\mu\pi i}\Gamma\left(-\mu-\frac{1}{2}\right)a^{\frac{1}{2}\mu+\frac{1}{2}}(1+a^2)^{-\frac{1}{2}}Q_{\nu}^{\mu+1}[(1+a)^{\frac{1}{2}}]Q_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}]$$

$$\left[|\arg a| < \pi, \quad \operatorname{Re} \mu < -\frac{1}{2}, \quad \operatorname{Re}(\mu + \nu + 2) > 0\right].$$

ET II 326(29)

$$3. \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{1}{2}}(1+ax)^{\frac{1}{2}\mu}P_{\nu}^{\mu}(1+2ax) dx = \pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu}\{P_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}]\}^2$$

$$\left[\operatorname{Re} \mu < \frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(32)

$$4. \int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}P_{\nu}^{\mu}(1+2ax) dx =$$

$$= \pi^{\frac{1}{2}}\Gamma\left(-\frac{1}{2}-\mu\right)a^{\frac{1}{2}\mu+\frac{1}{2}}P_{\nu}^{\mu+1}[(1+a)^{\frac{1}{2}}]P_{\nu}^{\mu}[(1+a)^2] \quad \left[\operatorname{Re} \mu < -\frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(33)

$$5. \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{1}{2}}(1+ax)^{-\frac{1}{2}\mu}P_{\nu}^{\mu}(1+2ax) dx = \pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+\mu\right)a^{-\frac{1}{2}\mu}P_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}]P_{\nu}^{-\mu}[(1+a)^{\frac{1}{2}}]$$

$$\left[\operatorname{Re} \mu > -\frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(34)

$$6. \int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}}(1-x)^{\mu-\frac{3}{2}}(1+ax)^{-\frac{1}{2}\mu}P_{\nu}^{\mu}(1+2ax) dx =$$

$$= \frac{1}{2}\pi^{\frac{1}{2}}\Gamma\left(\mu-\frac{1}{2}\right)a^{\frac{1}{2}-\frac{1}{2}\mu}(1+a)^{-\frac{1}{2}}\{P_{\nu}^{1-\mu}[(1+a)^{\frac{1}{2}}]P_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}] +$$

$$+(\mu+\nu)(1-\mu+\nu)P_{\nu}^{-\mu}[(1+a)^{\frac{1}{2}}]P_{\nu}^{\mu}[(1+a)^{\frac{1}{2}}]\} \quad \left[\operatorname{Re} \mu > \frac{1}{2}, \quad |\arg a| < \pi\right].$$

ET II 319(35)

$$\begin{aligned}
8. \quad & \int_0^1 x^{-\frac{\mu}{2}-\frac{1}{2}}(1-x)^{-\mu-\frac{3}{2}}(1+ax)^{\frac{1}{2}\mu}Q_\nu^\mu(1+2ax)dx = \\
& = \frac{1}{2}\pi^{\frac{1}{2}}\Gamma\left(-\mu-\frac{1}{2}\right)(1+a)^{-\frac{1}{2}}a^{\frac{1}{2}\mu+\frac{1}{2}} \times \\
& \quad \times \{P_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]Q_\nu^\mu[(1+a)^{\frac{1}{2}}] + P_\nu^\mu[(1+a)^{\frac{1}{2}}]Q_\nu^{\mu+1}[(1+a)^{\frac{1}{2}}]\} \\
& \quad \left[\operatorname{Re} \mu < -\frac{1}{2}, \quad |\arg a| < \pi \right].
\end{aligned}$$

ET II 320(39)

813

$$\begin{aligned}
9. \quad & \int_0^y (y-x)^{\mu-1} \left[x \left(1 + \frac{1}{2}\gamma x \right) \right]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx = \\
& = \Gamma(\mu) \left(\frac{2}{\gamma} \right)^{\frac{1}{2}\mu} \left[y \left(1 + \frac{1}{2}\gamma y \right) \right]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1+\gamma y) \quad [\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0, \quad \arg \gamma y < \pi].
\end{aligned}$$

ET II 193(52)

$$\begin{aligned}
10. \quad & \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} \left(1 + \frac{1}{2}\gamma x \right)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x) dx = \\
& = \frac{\left(\frac{\gamma}{2} \right)^{-\frac{1}{2}\lambda} \Gamma(\sigma)\Gamma(\mu)y^{\sigma+\mu-1}}{\Gamma(1-\lambda)\Gamma(\sigma+\mu)} {}_3F_2 \left(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; -\frac{1}{2}\gamma y \right) \\
& \quad [\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad |\gamma y| < 1].
\end{aligned}$$

ET II 193(53)

$$\begin{aligned}
11. \quad & \int_0^y (y-x)^{\mu-1} [x(1-x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx = \Gamma(\mu)[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1-2y) \\
& \quad [\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0, \quad 0 < y < 1].
\end{aligned}$$

ET II 193(54)

$$\begin{aligned}
12. \quad & \int_0^y (y-x)^{\mu-1} x^{\sigma+\frac{1}{2}\lambda-1} (1-x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x) dx = \\
& = \frac{\Gamma(\mu)\Gamma(\sigma)y^{\sigma+\mu-1}}{\Gamma(\sigma+\mu)\Gamma(1-\lambda)} {}_3F_2(-\nu, 1+\nu, \sigma; 1-\lambda, \sigma+\mu; y) \\
& \quad [\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad 0 < y < 1].
\end{aligned}$$

$$\int_0^{\infty} (a+x)^{-\mu-\nu-2} P_{\mu} \left(\frac{a-x}{a+x} \right) P_{\nu} \left(\frac{a-x}{a+x} \right) dx = \frac{a^{-\mu-\nu-1} [\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1)\Gamma(\nu+1)]^2 \Gamma(2\mu+2\nu+2)}$$

[|arg a| < π, Re(μ + ν) > -1].

ET II 326(3)

7.14 Combinations of associated Legendre functions, exponentials, and powers

7.141

$$1. \int_1^{\infty} e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_{\nu}^{\mu}(x) dx = \frac{a^{-\lambda-\mu} e^{-a}}{\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G_{23}^{31} \left(2a \left| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, -\nu, 1+\nu \end{matrix} \right. \right)$$

[Re a > 0, Re λ > 0].

ET II 323(13)

$$2. \int_1^{\infty} e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} Q_{\nu}^{\mu}(x) dx = \frac{\Gamma(\nu+\mu+1)e^{\mu\pi i}}{2\Gamma(\nu-\mu+1)} a^{-\lambda-\mu} e^{-a} G_{23}^{22} \left(2a \left| \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, \nu+1, -\nu \end{matrix} \right. \right)$$

[Re a > 0, Re λ > 0, Re(λ + μ) > 0].

ET II 325(24)

$$3. \int_1^{\infty} e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu}^{\mu}(x) dx = -\pi^{-1} \sin(\nu\pi) a^{\mu-\lambda} e^{-a} G_{23}^{31} \left(2a \left| \begin{matrix} 1, 1-\mu \\ \lambda-\mu, 1+\nu, -\nu \end{matrix} \right. \right)$$

[Re a > 0, Re(λ - μ) > 0].

ET II 323(15)

814

$$4. \int_1^{\infty} e^{-ax} (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} Q_{\nu}^{\mu}(x) dx = \frac{1}{2} e^{\mu\pi i} a^{\mu-\lambda} e^{-a} G_{23}^{22} \left(\left| 2a \begin{matrix} 1-\mu, 1 \\ \lambda-\mu, \nu+1, -\nu \end{matrix} \right. \right)$$

[Re a > 0, Re λ > 0, Re(λ - μ) > 0].

ET II 323(14)

$$5. \int_1^{\infty} e^{-ax} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu}^{\mu}(x) dx = 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(a) \quad [\text{Re } a > 0, \text{Re } \mu < 1].$$

ET II 323(11), MO 90

7.142

$$\int_1^{\infty} e^{-\frac{1}{2}ax} \left(\frac{x+1}{x-1} \right)^{\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^{\mu}(x) dx = \frac{2}{a} W_{\mu, \nu}(a) \quad \left[\text{Re } \mu < 1, \nu - \frac{1}{2} \neq 0, \pm 1, \pm 2, \dots \right].$$

$$1. \int_0^{\infty} [x(1+x)]^{-\frac{1}{2}\mu} e^{-\beta x} P_{\nu}^{\mu}(1+2x) dx = \frac{\beta^{\mu-\frac{1}{2}}}{\sqrt{\pi}} e^{\frac{1}{2}\beta} K_{\nu+\frac{1}{2}}\left(\frac{\beta}{2}\right) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} \beta > 0].$$

ET I 179(1)

$$2. \int_0^{\infty} \left(1 + \frac{1}{x}\right)^{\frac{1}{2}\mu} e^{-\beta x} P_{\nu}^{\mu}(1+2x) dx = \frac{e^{\frac{1}{2}\beta}}{\beta} W_{\mu, \nu+\frac{1}{2}}(\beta) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} \beta > 0].$$

ET I 179(2)

7.144

$$1. \int_0^{\infty} e^{-\beta x} x^{\lambda+\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_{\nu}^{\mu}(1+x) dx =$$

$$= \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} \left\{ \frac{\sin(\nu\pi)}{2\beta^{\lambda+\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda+\mu; \mu+1: 2\beta) - \right.$$

$$\left. - \frac{\sin[(\mu+\nu)\pi]}{2^{1-\mu}\beta^{\lambda} \sin(\mu\pi)} E(\nu-\mu+1, -\nu-\mu, \lambda: 1-\mu: 2\beta) \right\}$$

$$[\operatorname{Re} \beta > 0, \operatorname{Re} \lambda > 0, \operatorname{Re}(\lambda+\mu) > 0].$$

ET I 181(16)

$$2. \int_0^{\infty} e^{-\beta x} x^{\lambda-\frac{1}{2}\mu-1} (x+2)^{\frac{1}{2}\mu} Q_{\nu}^{\mu}(1+x) dx =$$

$$= -\frac{\sin(\nu\pi)}{2\beta^{\lambda-\mu} \sin(\mu\pi)} E(-\nu, \nu+1, \lambda-\mu: 1-\mu: 2\beta) -$$

$$- \frac{\sin[(\mu-\nu)\pi]}{2^{1+\mu}\beta^{\lambda} \sin(\mu\pi)} E(\mu+\nu+1, \mu-\nu, \lambda: 1+\mu: 2\beta)$$

$$[\operatorname{Re} \beta > 0, \operatorname{Re} \lambda > 0, \operatorname{Re}(\lambda-\mu) > 0].$$

ET I 181(17)

7.145

$$1. \int_0^{\infty} \frac{e^{-\beta x}}{1+x} P_{\nu} \left[\frac{1}{(1+x)^2} - 1 \right] dx = \frac{e^{\beta}}{\beta} W_{\nu+\frac{1}{2}, 0}(\beta) W_{-\nu-\frac{1}{2}, 0}(\beta) \quad [\operatorname{Re} \beta > 0].$$

ET I 180(6)

815

$$2. \int_0^{\infty} x^{-1} e^{-\beta x} Q_{-\frac{1}{2}}(1+2x^{-2}) dx = \frac{\pi^2}{8} \left\{ \left[J_0\left(\frac{1}{2}\beta\right) \right]^2 + \left[N_0\left(\frac{1}{2}\beta\right) \right]^2 \right\} \quad [\operatorname{Re} \beta > 0].$$

$$3. \int_0^{\infty} x^{-1} e^{-ax} Q_{\nu}(1+2x^{-2}) dx = \frac{1}{2} [\Gamma(\nu+1)]^2 a^{-1} W_{-\nu-\frac{1}{2}, 0}(ai) W_{-\nu-\frac{1}{2}, 0}(-ai) \\ [\operatorname{Re} a > 0, \operatorname{Re} \nu > -1].$$

ET II 327(6)

7.146

$$1. \int_0^{\infty} x^{-\frac{1}{2}\mu} e^{-\beta x} P_{\nu}^{\mu}(\sqrt{1+x}) dx = 2^{\mu} \beta^{\frac{1}{2}\mu - \frac{5}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu + \frac{1}{4}, \frac{1}{2}\nu + \frac{1}{4}}(\beta) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} \beta > 0].$$

ET I 180(7)

$$2. \int_0^{\infty} x^{-\frac{1}{2}\mu} \frac{e^{-\beta x}}{\sqrt{1+x}} P_{\nu}^{\mu}(\sqrt{1+x}) dx = 2^{\mu} \beta^{\frac{1}{2}\mu - \frac{3}{4}} e^{\frac{\beta}{2}} W_{\frac{1}{2}\mu + \frac{1}{4}, \frac{1}{2}\nu + \frac{1}{4}}(\beta) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} \beta > 0].$$

ET I 180(8)a

$$3. \int_0^{\infty} \sqrt{x} e^{-\beta x} P_{\nu}^{\frac{1}{4}}(\sqrt{1+x^2}) P_{\nu}^{-\frac{1}{4}}(\sqrt{1+x^2}) dx = \frac{1}{2} \sqrt{\frac{\pi}{2\beta}} H_{\nu+\frac{1}{2}}^{(1)}\left(\frac{1}{2}\beta\right) H_{\nu+\frac{1}{2}}^{(2)}\left(\frac{1}{2}\beta\right) \quad [\operatorname{Re} \beta > 0].$$

ET I 180(9)

7.147

$$\int_0^{\infty} x^{\lambda-1} (x^2 + a^2)^{\frac{1}{2}\nu} e^{-\beta x} P_{\nu}^{\mu} \left[\frac{x}{(x^2 + a^2)^{\frac{1}{2}}} \right] dx = \\ = \frac{2^{-\nu-2} a^{\lambda+\nu}}{\pi \Gamma(-\mu-\nu)} G_{24}^{32} \left(\frac{a^2 \beta^2}{4} \left| 1 - \frac{\lambda}{2}, \frac{1-\lambda}{2}, 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2} \right. \right) \\ [a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \lambda > 0].$$

ET II 327(7)

7.148

$$\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu+\nu-1} \exp\left(-\frac{1-x}{1+x}y\right) P_{\nu}^{\mu}(x) dx = 2^{\nu} y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2}, \frac{1}{2}\mu}(y) \\ [\operatorname{Re} y > 0].$$

ET II 317(21)

7.149

$$\int_1^{\infty} (\alpha^2 + \beta^2 + 2\alpha\beta x)^{-\frac{1}{2}} \exp[-(\alpha^2 + \beta^2 + 2\alpha\beta x)^{\frac{1}{2}}] P_{\nu}(x) dx = 2\pi^{-1} (\alpha\beta)^{-\frac{1}{2}} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta) \\ [\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0].$$

7.15 Combinations of associated Legendre and hyperbolic functions

7.151

$$1. \int_0^{\infty} (\operatorname{sh} x)^{\alpha-1} P_{\nu}^{-\mu}(\operatorname{ch} x) dx = \frac{2^{-1-\mu} \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2}\nu - \frac{1}{2}\alpha + 1\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}\alpha - \frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 + \frac{1}{2}\mu - \frac{1}{2}\alpha\right)}$$

[$\operatorname{Re}(\alpha + \mu) > 0$, $\operatorname{Re}(\nu - \alpha + 2) > 0$, $\operatorname{Re}(1 - \alpha - \nu) > 0$].

EH I 172(28)

816

$$2. \int_0^{\infty} (\operatorname{sh} x)^{\alpha-1} Q_{\nu}^{\mu}(\operatorname{ch} x) dx = \frac{e^{i\mu\pi} 2^{\mu-\alpha} \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu\right) \Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha\right)}{\Gamma\left(1 + \frac{1}{2}\nu - \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\alpha\right)} \times$$

$$\times \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\mu\right)$$

[$\operatorname{Re}(\alpha \pm \mu) > 0$, $\operatorname{Re}(\nu - \alpha + 2) > 0$].

EH I 172(29)

7.152

$$\int_0^{\infty} e^{-\alpha x} \operatorname{sh}^{2\mu} \left(\frac{1}{2}x\right) P_{2n}^{-2\mu} \left[\operatorname{ch} \left(\frac{1}{2}x\right)\right] dx = \frac{\Gamma\left(2\mu + \frac{1}{2}\right) \Gamma(\alpha - n - \mu) \Gamma\left(\alpha + n - \mu + \frac{1}{2}\right)}{4^{\mu} \sqrt{\pi} \Gamma(\alpha + n + \mu + 1) \Gamma\left(\alpha - n + \mu + \frac{1}{2}\right)}$$

[$\operatorname{Re} \alpha > n + \operatorname{Re} \mu$, $\operatorname{Re} \mu > -\frac{1}{4}$].

ET I 181(15)

7.16 Combinations of associated Legendre functions, powers, and trigonometric functions

7.161

$$1. \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \sin(ax) P_{\nu}^{\mu}(x) dx =$$

$$= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda-1} \Gamma(\lambda+1) a}{\Gamma\left(1 + \frac{\lambda - \mu - \nu}{2}\right) \Gamma\left(\frac{3 + \lambda - \mu + \nu}{2}\right)} \times$$

$$\times {}_2F_3\left(\frac{1+\lambda}{2}, 1 + \frac{\lambda}{2}; \frac{3}{2}, 1 + \frac{\lambda - \mu - \nu}{2}, \frac{3 + \lambda - \mu + \nu}{2}; -\frac{a^2}{4}\right)$$

[$\operatorname{Re} \lambda > -1$, $\operatorname{Re} \mu < 1$].

$$\begin{aligned}
2. \int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \cos(ax) P_\nu^\mu(x) dx &= \\
&= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma\left(1 + \frac{\lambda-\mu+\nu}{2}\right) \Gamma\left(\frac{1+\lambda-\mu-\nu}{2}\right)} \times \\
&\quad \times {}_2F_3\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{2}, \frac{1+\lambda-\mu-\nu}{2}, 1 + \frac{\lambda-\mu+\nu}{2}; -\frac{a^2}{4}\right) \quad [\operatorname{Re} \lambda > 0, \operatorname{Re} \mu < 1].
\end{aligned}$$

ET II 314(8)

$$\begin{aligned}
3. \int_0^\infty (x^2-1)^{\frac{1}{2}\mu} \sin(ax) P_\nu^\mu(x) dx &= \frac{2^\mu \pi^{\frac{1}{2}} a^{-\mu-\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu\right) \Gamma\left(1 - \frac{1}{2}\mu + \frac{1}{2}\nu\right)} S_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) \\
&\quad \left[a > 0, \quad \operatorname{Re} \mu < \frac{3}{2}, \quad \operatorname{Re}(\mu + \nu) < 1 \right].
\end{aligned}$$

ET II 320(1)

817
7.162

$$\begin{aligned}
1. \int_a^\infty P_\nu(2x^2a^{-2}-1) \sin(bx) dx &= -\frac{\pi a}{4 \cos(\nu\pi)} \left\{ \left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 - \left[J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right]^2 \right\} \\
&\quad [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 0].
\end{aligned}$$

ET II 326(1)

$$\begin{aligned}
2. \int_a^\infty P_\nu(2x^2a^{-2}-1) \cos(bx) dx &= -\frac{\pi}{4} a \left[J_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) J_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) - N_{\nu+\frac{1}{2}}\left(\frac{ab}{2}\right) N_{-\nu-\frac{1}{2}}\left(\frac{ab}{2}\right) \right] \\
&\quad [a > 0, \quad b > 0, \quad -1 < \operatorname{Re} \nu < 0].
\end{aligned}$$

ET II 326(2)

$$\begin{aligned}
3. \int_0^\infty (x^2+2)^{-\frac{1}{2}} \sin(ax) P_\nu^{-1}(x^2+1) dx &= 2^{-\frac{1}{2}} \pi^{-1} a \sin(\nu\pi) [K_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a)]^2 \\
&\quad [a > 0, \quad -2 < \operatorname{Re} \nu < 1].
\end{aligned}$$

ET I 98(22)

$$\begin{aligned}
4. \int_0^\infty (x^2+2)^{-\frac{1}{2}} \sin(ax) Q_\nu^1(x^2+1) dx &= -2^{-\frac{3}{2}} \pi a K_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a) I_{\nu+\frac{1}{2}}(2^{-\frac{1}{2}}a) \\
&\quad \left[a > 0, \quad \operatorname{Re} \nu > -\frac{3}{2} \right].
\end{aligned}$$

$$5. \int_0^{\infty} \cos(ax) P_{\nu}(1+x^2) dx = -\frac{\sqrt{2}}{\pi} \sin(\nu\pi) \left[K_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right) \right]^2 \quad [a > 0, \quad -1 < \operatorname{Re} \nu < 0].$$

ET I 42(23)

$$6. \int_0^{\infty} \cos(ax) Q_{\nu}(1+x^2) dx = \frac{\pi}{\sqrt{2}} K_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right) I_{\nu+\frac{1}{2}} \left(\frac{a}{\sqrt{2}} \right) \quad [a > 0, \quad \operatorname{Re} \nu > -1].$$

ET I 42(24)

$$7. \int_0^1 \cos(ax) P_{\nu}(2x^2-1) dx = \frac{\pi}{2} J_{\nu+\frac{1}{2}} \left(\frac{a}{2} \right) J_{-\nu-\frac{1}{2}} \left(\frac{a}{2} \right) \quad [a > 0].$$

ET I 42(25)

7.163

$$1. \int_a^{\infty} (x^2-a^2)^{\frac{1}{2}\nu-\frac{1}{4}} \sin(bx) P_0^{\frac{1}{2}-\nu}(ax^{-1}) dx = b^{-\nu-\frac{1}{2}} \cos\left(ab - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) \quad \left[a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET I 98(24)

$$2. \int_0^1 x^{-1} \cos(ax) P_{\nu}(2x^{-2}-1) dx = -\frac{1}{2}\pi \operatorname{cosec}(\nu\pi) {}_1F_1(\nu+1; 1; ai) {}_1F_1(\nu+1; 1; -ai) \\ [a > 0, \quad -1 < \operatorname{Re} \nu < 0].$$

ET II 327(4)

7.164

$$1. \int_0^{\infty} x^{\frac{1}{2}} \sin(bx) [P_{\nu}^{-\frac{1}{4}}(\sqrt{1+a^2x^2})]^2 dx = \frac{\sqrt{\frac{2}{\pi}} a^{-1} b^{-\frac{1}{2}}}{\Gamma\left(\frac{5}{4}+\nu\right) \Gamma\left(\frac{1}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \right]^2 \\ \left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{1}{4} \right].$$

ET II 327(8)

818

$$2. \int_0^{\infty} x^{\frac{1}{2}} \sin(bx) P_{\nu}^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) Q_{\nu}^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) dx = \\ = \frac{\sqrt{\frac{\pi}{2}} e^{-\frac{1}{4}\pi i} \Gamma\left(\nu + \frac{5}{4}\right)}{ab^{\frac{1}{2}} \Gamma\left(\nu + \frac{3}{4}\right)} I_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) K_{\nu+\frac{1}{2}} \left(\frac{b}{2a} \right) \quad \left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{5}{4} \right].$$

$$\begin{aligned}
3. \int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) P_{\nu-1}^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} = \\
= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{5}{4}-\nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \\
\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{5}{4} \right].
\end{aligned}$$

ET II 328(10)

$$\begin{aligned}
4. \int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) P_{\nu-\frac{3}{4}}^{-\frac{3}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} = \\
= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{7}{4}+\nu\right)\Gamma\left(\frac{3}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2 \left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{7}{4} < \operatorname{Re} \nu < \frac{3}{4} \right].
\end{aligned}$$

ET II 328(11)

$$\begin{aligned}
5. \int_0^\infty x^{\frac{1}{2}} \cos(bx) [P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2})]^2 dx = \frac{a^{-1}\left(\frac{\pi b}{2}\right)^{-\frac{1}{2}}}{\Gamma\left(\frac{3}{4}+\nu\right)\Gamma\left(-\frac{1}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2 \\
\left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{3}{4} < \operatorname{Re} \nu < -\frac{1}{4} \right].
\end{aligned}$$

ET II 328(12)

$$\begin{aligned}
6. \int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) Q_\nu^{\frac{1}{4}}(\sqrt{1+a^2x^2}) dx = \frac{\sqrt{\frac{\pi}{2}} e^{\frac{1}{4}\pi i} \Gamma\left(\nu+\frac{3}{4}\right)}{ab^{\frac{1}{2}}\Gamma\left(\nu+\frac{5}{4}\right)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \\
\left[\operatorname{Re} a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -\frac{3}{4} \right].
\end{aligned}$$

ET II 328(13)

$$\begin{aligned}
7. \int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{-\frac{1}{4}}(\sqrt{1+a^2x^2}) P_\nu^{\frac{3}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} = \\
= \frac{a^{-2}b^{\frac{1}{2}}}{\sqrt{2\pi}\Gamma\left(\frac{5}{4}+\nu\right)\Gamma\left(\frac{1}{4}-\nu\right)} \left[K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2 \left[\operatorname{Re} a > 0, \quad b > 0, \quad -\frac{5}{4} < \operatorname{Re} \nu < \frac{1}{4} \right].
\end{aligned}$$

ET II 328(14)

$$8. \int_0^{\infty} x^{\frac{1}{2}} \cos(bx) P_{\nu}^{\frac{1}{4}}(\sqrt{1+a^2x^2}) P_{\nu-1}^{\frac{1}{4}}(\sqrt{1+a^2x^2}) \frac{dx}{\sqrt{1+a^2x^2}} =$$

$$= \frac{a^{-2} b^{\frac{1}{2}}}{\sqrt{2\pi} \Gamma\left(\frac{3}{4} + \nu\right) \Gamma\left(\frac{3}{4} - \nu\right)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \left[\operatorname{Re} a > 0, \quad b > 0, \quad |\operatorname{Re} \nu| < \frac{3}{4} \right].$$

ET II 329(15)

7.165

$$\int_0^{\infty} \cos(ax) P_{\nu}(\operatorname{ch} x) dx = -\frac{\sin(\nu\pi)}{4\pi^2} \Gamma\left(\frac{1+\nu+i\alpha}{2}\right) \Gamma\left(\frac{1+\nu-i\alpha}{2}\right) \Gamma\left(-\frac{\nu+i\alpha}{2}\right) \Gamma\left(-\frac{\nu-i\alpha}{2}\right)$$

$$[a > 0, \quad -1 < \operatorname{Re} \nu < 0].$$

ET II 329(18)

7.166

$$\int_0^{\pi} P_{\nu}^{-\mu}(\cos \varphi) \sin^{\alpha-1} \varphi d\varphi =$$

$$= \frac{2^{-\mu} \pi \Gamma\left(\frac{1}{2}\alpha + \frac{1}{2}\mu\right) \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\alpha - \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + 1\right) \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right)}$$

$$[\operatorname{Re}(\alpha \pm \mu) > 0].$$

MO 90, EH I 172(27)

7.167

$$\int_0^a P_{\nu}^{-\mu}(\cos x) P_{\nu}^{-\eta}[\cos(a-x)] \left[\frac{\sin(a-x)}{\sin x} \right]^{\eta} \frac{dx}{\sin x} = \frac{2^{\eta} \Gamma(\mu - \eta) \Gamma\left(\eta + \frac{1}{2}\right) (\sin a)^{\eta}}{\sqrt{\pi} \Gamma(\eta + \mu + 1)} P_{\nu}^{-\mu}(\cos a)$$

$$\left[\operatorname{Re} \mu > \operatorname{Re} \eta > -\frac{1}{2} \right].$$

ET II 329(16)

7.17 A combination of an associated Legendre function and the probability integral

7.171

$$\int_1^{\infty} (x^2 - 1)^{-\frac{1}{2}\mu} \exp(a^2 x^2) [1 - \Phi(ax)] P_{\nu}^{\mu}(x) dx =$$

$$= \pi^{-1} 2^{\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right) a^{\mu-\frac{3}{2}} e^{\frac{a^2}{2}} W_{\frac{1}{4}-\frac{1}{2}\mu, \frac{1}{4}+\frac{1}{2}\nu}(a^2)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + \nu) > -1, \quad \operatorname{Re}(\mu - \nu) > 0].$$

7.181

$$1. \int_1^{\infty} P_{\nu-\frac{1}{2}}(x)x^{\frac{1}{2}}N_{\nu}(ax) dx = 2^{-\frac{1}{2}}a^{-1} \left[\cos\left(\frac{1}{2}a\right) J_{\nu}\left(\frac{1}{2}a\right) - \sin\left(\frac{1}{2}a\right) N_{\nu}\left(\frac{1}{2}a\right) \right] \\ \left[a > 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 108(3)a

820

$$2. \int_1^{\infty} P_{\nu-\frac{1}{2}}(x)x^{\frac{1}{2}}J_{\nu}(ax) dx = -\frac{1}{\sqrt{2}a} \left[\cos\left(\frac{1}{2}a\right) N_{\nu}\left(\frac{1}{2}a\right) + \sin\left(\frac{1}{2}a\right) J_{\nu}\left(\frac{1}{2}a\right) \right] \quad \left[|\operatorname{Re} \nu| < \frac{1}{2} \right].$$

ET II 344(36)a

7.182

$$1. \int_1^{\infty} x^{\nu}(x^2-1)^{\frac{1}{2}\lambda-\frac{1}{2}}P_{\lambda}^{\lambda-1}(x)J_{\nu}(ax) dx = \frac{2^{\lambda+\nu}a^{-\lambda}\Gamma\left(\frac{1}{2}+\nu\right)}{\pi^{\frac{1}{2}}\Gamma(1-\lambda)}S_{\lambda-\nu, \lambda+\nu}(a) \\ \left[a > 0, \quad \operatorname{Re} \nu < \frac{5}{2}, \quad \operatorname{Re}(2\lambda+\nu) < \frac{3}{2} \right].$$

ET II 345(38)a

$$2. \int_1^{\infty} x^{\frac{1}{2}-\mu}(x^2-1)^{-\frac{1}{2}\mu}P_{\nu-\frac{1}{2}}^{\mu}(x)J_{\nu}(ax) dx = -2^{-\frac{3}{2}}\pi^{\frac{1}{2}}a^{\mu-\frac{1}{2}} \left[J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)N_{\nu}\left(\frac{a}{2}\right) + N_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right)J_{\nu}\left(\frac{a}{2}\right) \right] \\ \left[-\frac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad |\operatorname{Re} \nu| < \frac{1}{2} + 2 \operatorname{Re} \mu \right].$$

ET II 344(37)a

$$3. \int_1^{\infty} x^{\frac{1}{2}-\mu}(x^2-1)^{-\frac{1}{2}\mu}P_{\nu-\frac{1}{2}}^{\mu}(x)N_{\nu}(ax) dx = 2^{-\frac{3}{2}}\pi^{\frac{1}{2}}a^{\mu-\frac{1}{2}} \left[J_{\nu}\left(\frac{a}{2}\right)J_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - N_{\nu}\left(\frac{a}{2}\right)N_{\mu-\frac{1}{2}}\left(\frac{a}{2}\right) \right] \\ \left[-\frac{1}{4} < \operatorname{Re} \mu < 1, \quad a > 0, \quad \operatorname{Re}(2\mu-\nu) > -\frac{1}{2} \right].$$

ET II 349(67)a

$$4. \int_0^1 x^{\frac{1}{2}-\mu}(1-x^2)^{-\frac{1}{2}\mu}P_{\nu}^{\mu}(x)J_{\nu+\frac{1}{2}}(ax) dx = \sqrt{\frac{\pi}{2}}a^{\mu-\frac{1}{2}}J_{\frac{1}{2}-\mu}\left(\frac{1}{2}a\right)J_{\nu+\frac{1}{2}}\left(\frac{1}{2}a\right) \\ \left[\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu-\nu) < 2 \right].$$

$$5. \int_1^{\infty} x^{\frac{1}{2}-\mu} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^{\mu}(x) K_{\nu}(ax) dx = (2\pi)^{-\frac{1}{2}} a^{\mu-\frac{1}{2}} K_{\nu}\left(\frac{1}{2}a\right) K_{\mu-\frac{1}{2}}\left(\frac{1}{2}a\right) \\ [\operatorname{Re} \mu < 1, \operatorname{Re} a > 0].$$

ET II 135(5)a

$$6. \int_1^{\infty} x^{\mu+\frac{1}{2}} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^{\mu}(x) K_{\nu}(ax) dx = \sqrt{\frac{\pi}{2}} a^{-\frac{3}{2}} e^{-\frac{1}{2}a} W_{\mu,\nu}(a) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} a > 0].$$

ET II 135(3)a

$$7. \int_1^{\infty} x^{\mu-\frac{3}{2}} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^{\mu}(x) K_{\nu}(ax) dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}} e^{-\frac{1}{2}a} W_{\mu-1,\nu}(a) \quad [\operatorname{Re} \mu < 1, \operatorname{Re} a > 0].$$

ET II 135(4)a

$$8. \int_1^{\infty} x^{\mu-\frac{1}{2}} (x^2-1)^{-\frac{1}{2}\mu} P_{\nu-\frac{3}{2}}^{\mu}(x) K_{\nu}(ax) dx = \sqrt{\frac{\pi}{2}} a^{-1} e^{-\frac{1}{2}a} W_{\mu-\frac{1}{2},\nu-\frac{1}{2}}(a) \quad [\operatorname{Re} \mu < 1].$$

ET II 135(6)a

$$9. \int_1^{\infty} x^{\frac{1}{2}} (x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} P_{\mu}^{\frac{1}{2}-\nu}(2x^2-1) K_{\nu}(ax) dx = \pi^{-\frac{1}{2}} a^{-\nu} 2^{\nu-1} \left[K_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) \right]^2 \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, \operatorname{Re} a > 0 \right].$$

ET II 136(11)a

821

$$10. \int_1^{\infty} x^{\frac{1}{2}} (x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} P_{\mu}^{\frac{1}{2}-\nu}(2x^2-1) N_{\nu}(ax) dx = \\ = \pi^{\frac{1}{2}} 2^{\nu-2} a^{-\nu} \left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) - N_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) N_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) \right] \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, a > 0, \operatorname{Re} \nu + |2 \operatorname{Re} \mu + 1| < \frac{3}{2} \right].$$

ET II 108(5)a

$$11. \int_1^{\infty} x^{\frac{1}{2}} (x^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} P_{\mu}^{\frac{1}{2}-\nu}(2x^2-1) J_{\nu}(ax) dx = \\ = -2^{\nu-2} a^{-\nu} \pi^{\frac{1}{2}} \sec(\mu\pi) \left\{ \left[J_{\mu+\frac{1}{2}}\left(\frac{a}{2}\right) \right]^2 - \left[J_{-\mu-\frac{1}{2}}\left(\frac{a}{2}\right) \right]^2 \right\} \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, a > 0, \operatorname{Re} \nu - \frac{3}{2} < 2 \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re} \nu \right].$$

$$12. \int_1^{\infty} x(x^2-1)^{-\frac{1}{2}\nu} P_{\mu}^{\nu}(2x^2-1)K_{\nu}(ax) dx = 2^{-\nu} a^{\nu-1} K_{\mu+1}(a) \quad [\operatorname{Re} a > 0, \operatorname{Re} \nu < 1].$$

ET II 136(10)a

$$13. \int_0^{\infty} x(x^2+a^2)^{\frac{1}{2}\nu} P_{\mu}^{\nu}(1+2x^2a^{-2})K_{\nu}(xy) dx = 2^{-\nu} ay^{-\nu-1} S_{2\nu, 2\mu+1}(ay) \\ [\operatorname{Re} a > 0, \operatorname{Re} y > 0, \operatorname{Re} \nu < 1].$$

ET II 135(7)

$$14. \int_0^{\infty} x(x^2+a^2)^{\frac{1}{2}\nu} [(\mu - \nu) P_{\mu}^{\nu}(1+2x^2a^{-2}) + (\mu + \nu) P_{-\mu}^{\nu}(1+2x^2a^{-2})] K_{\nu}(xy) dx = \\ = 2^{1-\nu} \mu y^{-\nu-2} S_{2\nu+1, 2\mu}(ay) \quad [\operatorname{Re} a > 0, \operatorname{Re} y > 0, \operatorname{Re} \nu < 1].$$

ET II 136(8)

$$15. \int_0^{\infty} x(x^2+a^2)^{\frac{1}{2}\nu-1} [P_{\mu}^{\nu}(1+2x^2a^{-2}) + P_{-\mu}^{\nu}(1+2x^2a^{-2})] K_{\nu}(xy) dx = 2^{1-\nu} y^{-\nu} S_{2\nu-1, 2\mu}(ay) \\ [\operatorname{Re} a > 0, \operatorname{Re} y > 0, \operatorname{Re} \nu < 1].$$

ET II 136(9)

$$16. \int_0^{\infty} x^{\frac{1}{2}} (x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} P_{\mu}^{-\nu-\frac{1}{2}}(x^2+1) J_{\nu}(xy) dx = \frac{y^{-\frac{1}{2}} 2^{\frac{1}{2}-\nu} \pi^{-\frac{1}{2}} [K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y)]^2}{\Gamma\left(\nu+\mu+\frac{3}{2}\right) \Gamma\left(\nu-\mu+\frac{1}{2}\right)} \\ \left[-\frac{3}{2} - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + \frac{1}{2}, \quad y > 0 \right].$$

ET II 44(1)

$$17. \int_0^{\infty} x^{\frac{1}{2}} (x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} Q_{\mu}^{\nu+\frac{1}{2}}(x^2+1) J_{\nu}(xy) dx = 2^{-\nu-\frac{1}{2}} \pi^{\frac{1}{2}} e^{(\nu+\frac{1}{2})\pi i} y^{\nu} K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y) I_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y) \\ \left[\operatorname{Re} \nu > -1, \operatorname{Re}(2\mu+\nu) > -\frac{5}{2}, \quad y > 0 \right].$$

ET II 46(12)

7.183

$$\int_0^{\infty} x^{1-\mu} (1+a^2x^2)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax) J_{\nu}(xy) dx = \\ = i(2\pi)^{\frac{1}{2}} e^{i\pi(\mu \mp \frac{1}{2} \nu \mp \frac{1}{4})} a^{-1} y^{\mu-1} I_{\nu}\left(\frac{1}{2}a^{-1}y\right) K_{\mu}\left(\frac{1}{2}a^{-1}y\right) \\ \left[-\frac{3}{4} - \frac{1}{2} \operatorname{Re} \nu < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu, \quad y > 0, \quad \operatorname{Re} a > 0 \right].$$

$$1. \int_1^{\infty} x^{\frac{1}{2}} (x^2-1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}(x^{-1}) J_{\nu}(xa) dx = 2^{\frac{1}{2}} a^{-1-\mu} \pi^{-\frac{1}{2}} \cos \left[a + \frac{1}{2}(\nu - \mu)\pi \right] \\ \left[|\operatorname{Re} \mu| < \frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad a > 0 \right].$$

ET II 44(2)a

$$2. \int_1^{\infty} x^{-\nu} (x^2-1)^{\frac{1}{4}-\frac{1}{2}\nu} P_{\mu}^{\nu-\frac{1}{2}}(2x^{-2}-1) K_{\nu}(ax) dx = \pi^{\frac{1}{2}} 2^{-\nu} a^{-2+\nu} W_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\ \left[\operatorname{Re} \nu < \frac{3}{2}, \quad a > 0 \right].$$

ET II 370(45)a

$$3. \int_0^{\infty} x^{\nu} (1+x^2)^{\frac{1}{4}+\frac{\nu}{2}} Q_{\mu}^{\nu+\frac{1}{2}} \left(1 + \frac{2}{x^2} \right) J_{\nu}(ax) dx = \\ = -ie^{i\pi\nu} \pi^{-\frac{1}{2}} 2^{\nu} a^{-\nu-2} \left[\Gamma \left(\frac{3}{2} + \mu + \nu \right) \right]^2 \Gamma \left(\frac{1}{2} + \nu - \mu \right) \times \\ \times W_{-\mu-\frac{1}{2}, \nu+\frac{1}{2}}(a) \left[\frac{\cos(\mu\pi)}{\Gamma(2+2\nu)} M_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) + \frac{\sin(\nu\pi)}{\Gamma \left(\nu + \mu + \frac{3}{2} \right)} W_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(a) \right] \\ \left[a > 0, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}, \quad \operatorname{Re}(\mu - \nu) < \frac{1}{2} \right].$$

ET II 46(14)

$$4. \int_0^1 x^{\nu} (1-x^2)^{\frac{1}{2}\nu+\frac{1}{4}} P_{\mu}^{-\nu-\frac{1}{2}}(2x^{-2}-1) J_{\nu}(xy) dx = \\ = 2^{\nu+\frac{1}{2}} y^{\nu} \frac{\Gamma \left(\frac{3}{2} + \mu + \nu \right) \Gamma \left(\frac{1}{2} + \nu - \mu \right)}{(2\pi)^{\frac{1}{2}} \left[\Gamma \left(\frac{3}{2} + \nu \right) \right]^2} \times \\ \times {}_1F_1 \left(\nu + \mu + \frac{3}{2}; 2\nu + 2; iy \right) {}_1F_1 \left(\nu + \mu + \frac{3}{2}; 2\nu + 2; -iy \right) \\ \left[y > 0, \quad -\frac{3}{2} - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + \frac{1}{2} \right].$$

ET II 45(3)

$$5. \int_0^{\infty} x^{-\nu} (x^2+a^2)^{\frac{1}{4}-\frac{1}{2}\nu} Q_{\mu}^{\frac{1}{2}-\nu}(1+2a^2x^{-2}) K_{\nu}(xy) dx = \\ = ie^{-i\pi\nu} \pi^{\frac{1}{2}} 2^{-\nu-1} a^{-\nu-\frac{1}{2}} y^{\nu-2} \left[\Gamma \left(\frac{3}{2} + \mu - \nu \right) \right]^2 W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(iay) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(-iay)$$

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 $\left[\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \mu > -\frac{3}{2}, \quad \operatorname{Re}(\mu - \nu) > -\frac{3}{2} \right].$

$$\begin{aligned}
6. \int_0^\infty x^{-\nu} (x^2 + 1)^{\frac{1}{4} - \frac{1}{2}\nu} Q_{\mu}^{\frac{1}{2} - \nu} (1 + 2x^{-2}) J_{\nu}(ax) dx &= \\
&= 2^{-\nu} a^{-\nu-2} \frac{i e^{-i\nu\pi} \pi^{\frac{1}{2}} \Gamma\left(\frac{3}{2} + \mu - \nu\right)}{\Gamma(2\nu)} M_{\mu+\frac{1}{2}, \nu-\frac{1}{2}}(a) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(a) \\
&\quad \left[a > 0, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu + \frac{3}{2} \right].
\end{aligned}$$

ET II 47(15)a

$$\begin{aligned}
7. \int_0^\infty x^{-\nu} (x^2 + a^2)^{\frac{1}{4} - \frac{1}{2}\nu} Q_{-\frac{1}{2}}^{\frac{1}{2} - \nu} (1 + 2a^2 x^{-2}) K_{\nu}(xy) dx &= \\
&= i e^{-i\pi\nu} \pi^{\frac{3}{2}} 2^{-\nu-3} a^{\frac{1}{2}-\nu} y^{\nu-1} [\Gamma(1-\nu)]^2 \times \left\{ \left[J_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) \right]^2 + \left[N_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) \right]^2 \right\} \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \nu < 1].
\end{aligned}$$

ET II 136(12)

7.185

$$\begin{aligned}
\int_0^\infty x^{\frac{1}{2}} Q_{\nu-\frac{1}{2}} [(a^2+x^2)x^{-1}] J_{\nu}(xy) dx &= 2^{-\frac{1}{2}} \pi y^{-1} \exp \left[-\left(a^2 - \frac{1}{4}\right)^{\frac{1}{2}} y \right] J_{\nu}\left(\frac{1}{2}y\right) \\
&\quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad y > 0 \right].
\end{aligned}$$

ET II 46(10)

7.186

$$\int_0^\infty x(1+x^2)^{-\nu-1} P_{\nu}\left(\frac{1-x^2}{1+x^2}\right) J_0(xy) dx = y^{2\nu} [2^{\nu} \Gamma(\nu+1)]^{-2} K_0(y) \quad [\operatorname{Re} \nu > 0].$$

ET II 13(10)

7.187

$$1. \int_0^\infty x P_{\mu}^{\nu}(\sqrt{1+x^2}) K_{\nu}(xy) dx = y^{-\frac{3}{2}} S_{\nu+\frac{1}{2}, \mu+\frac{1}{2}}(y) \quad [\operatorname{Re} \nu < 1, \quad \operatorname{Re} y > 0].$$

ET II 137(14)

$$\begin{aligned}
2. \int_0^\infty x [P_{\lambda-\frac{1}{2}}(\sqrt{1+a^2x^2})]^2 J_0(xy) dx &= 2\pi^{-2} y^{-1} a^{-1} \cos(\lambda\pi) \left[K_{\lambda}\left(\frac{y}{2a}\right) \right]^2 \\
&\quad \left[\operatorname{Re} a > 0, \quad |\operatorname{Re} \lambda| < \frac{1}{4}, \quad y > 0 \right].
\end{aligned}$$

$$3. \int_0^{\infty} x(1+x^2)^{-\frac{1}{2}} P_{\mu}^{\nu}(\sqrt{1+x^2}) K_{\nu}(xy) dx = y^{-\frac{1}{2}} S_{\nu-\frac{1}{2}, \mu+\frac{1}{2}}(y) \quad [\operatorname{Re} \nu < 1, \operatorname{Re} y > 0].$$

ET II 137(15)

$$4. \int_0^{\infty} x P_{\mu}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) Q_{\mu}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_{\nu}(xy) dx = \\ = \frac{y^{-1} e^{-\frac{1}{2}\nu\pi i} \Gamma\left(1 + \mu + \frac{1}{2}\nu\right)}{a \Gamma\left(1 + \mu - \frac{1}{2}\nu\right)} I_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \mu > -\frac{3}{4}, \quad \operatorname{Re} \nu > -1 \right].$$

ET II 47(16)

824

$$5. \int_0^{\infty} x P_{\sigma-\frac{1}{2}}^{\mu}(\sqrt{1+a^2x^2}) Q_{\sigma-\frac{1}{2}}^{\mu}(\sqrt{1+a^2x^2}) J_0(xy) dx = y^{-2} e^{\mu\pi i} \frac{\Gamma\left(\frac{1}{2} + \sigma - \mu\right)}{\Gamma(1 + 2\sigma)} W_{\mu, \sigma}\left(\frac{y}{a}\right) M_{-\mu, \sigma}\left(\frac{y}{a}\right) \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \sigma > -\frac{1}{4}, \quad \operatorname{Re} \mu < 1 \right].$$

ET II 14(15)

$$6. \int_0^{\infty} x P_{\sigma-\frac{1}{2}}^{\mu}(\sqrt{1+a^2x^2}) P_{\sigma-\frac{1}{2}}^{-\mu}(\sqrt{1+a^2x^2}) J_0(xy) dx = 2\pi^{-1} y^{-2} \cos(\sigma\pi) W_{\mu, \sigma}\left(\frac{y}{a}\right) W_{-\mu, \sigma}\left(\frac{y}{a}\right) \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \frac{1}{4} \right].$$

ET II 14(14)

$$7. \int_0^{\infty} x \{P_{\sigma-\frac{1}{2}}^{\mu}(\sqrt{1+a^2x^2})\}^2 J_0(xy) dx = -i\pi^{-1} y^{-2} W_{\mu, \sigma}\left(\frac{y}{a}\right) \left[W_{\mu, \sigma}\left(e^{\pi i} \frac{y}{a}\right) - W_{\mu, \sigma}\left(e^{-\pi i} \frac{y}{a}\right) \right] \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad |\operatorname{Re} \sigma| < \frac{1}{4}, \quad \operatorname{Re} \mu < 1 \right].$$

ET II 14(13)

$$8. \int_0^{\infty} x(1+a^2x^2)^{-\frac{1}{2}} P_{\mu}^{-\frac{1}{2}-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) P_{\mu}^{\frac{1}{2}-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_{\nu}(xy) dx = \\ = \frac{\left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) \right]^2}{\pi a^2 \Gamma\left(\frac{\nu}{2} + \mu + \frac{3}{2}\right) \Gamma\left(\frac{\nu}{2} - \mu + \frac{1}{2}\right)} \left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{5}{4} < \operatorname{Re} \mu < \frac{1}{4} \right].$$

$$9. \int_0^{\infty} x \{P_{\mu}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2})\}^2 J_{\nu}(xy) dx = \frac{2 \left[K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) \right]^2 y^{-1}}{\pi a \Gamma\left(1+\mu+\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2}\nu-\mu\right)} \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{3}{4} < \operatorname{Re} \mu < -\frac{1}{4}, \quad \operatorname{Re} \nu > -1 \right].$$

ET II 45(7)

$$10. \int_0^{\infty} x(1+a^2x^2)^{-\frac{1}{2}} P_{\mu}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) P_{\mu+\frac{1}{2}}^{-\frac{1}{2}\nu}(\sqrt{1+a^2x^2}) J_{\nu}(xy) dx = \\ = \frac{K_{\mu+\frac{1}{2}}\left(\frac{y}{2a}\right) K_{\mu+\frac{3}{2}}\left(\frac{y}{2a}\right)}{\pi a^2 \Gamma\left(2+\frac{1}{2}\nu+\mu\right) \Gamma\left(\frac{1}{2}\nu-\mu\right)} \left[\operatorname{Re} a > 0, \quad y > 0, \quad -\frac{7}{4} < \operatorname{Re} \mu < -\frac{1}{4} \right].$$

ET II 45(8)

7.188

$$1. \int_0^{\infty} x(a^2+x^2)^{-\frac{1}{2}\mu} P_{\mu-1}^{-\nu} \left[\frac{a}{\sqrt{a^2+x^2}} \right] J_{\nu}(xy) dx = \frac{y^{\mu-2} e^{-ay}}{\Gamma(\mu+\nu)} \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > \frac{1}{2} \right].$$

ET II 45(4)

825

$$2. \int_0^{\infty} x^{\nu+1} (x^2+a^2)^{\frac{1}{2}\nu} P_{\nu} \left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}} \right) J_{\nu}(xy) dx = \frac{(2a)^{\nu+1} y^{-\nu-1}}{\pi \Gamma(-\nu)} \left[K_{\nu+\frac{1}{2}}\left(\frac{ya}{2}\right) \right]^2 \\ \left[\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0, \quad y > 0 \right].$$

ET II 45(5)

$$3. \int_0^{\infty} x^{1-\nu} (x^2+a^2)^{-\frac{1}{2}\nu} P_{\nu-1} \left(\frac{x^2+2a^2}{2a\sqrt{x^2+a^2}} \right) J_{\nu}(xy) dx = \frac{(2a)^{1-\nu} y^{\nu-1}}{\Gamma(\nu)} I_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) K_{\nu-\frac{1}{2}}\left(\frac{ay}{2}\right) \\ \left[\operatorname{Re} a > 0, \quad y > 0, \quad 0 < \operatorname{Re} \nu < 1 \right].$$

ET II 45(6)

7.189

$$1. \int_0^{\infty} (a+x)^{\mu} e^{-x} P_{\nu}^{-2\mu} \left(1 + \frac{2x}{a} \right) I_{\mu}(x) dx = 0 \\ \left[-\frac{1}{2} < \operatorname{Re} \mu < 0, \quad -\frac{1}{2} + \operatorname{Re} \mu < \operatorname{Re} \nu < -\frac{1}{2} - \operatorname{Re} \mu \right].$$

$$\begin{aligned}
2. \int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu} \left(1 + \frac{2x}{a}\right) I_\mu(x) dx &= \\
&= \frac{2^{\mu-1} \Gamma\left(\mu + \nu + \frac{1}{2}\right) \Gamma\left(\mu - \nu - \frac{1}{2}\right) e^a}{\pi^{\frac{1}{2}} \Gamma(2\mu + \nu + 1) \Gamma(2\mu - \nu)} W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a) \\
&\quad \left[|\arg a| < \pi, \quad \operatorname{Re} \mu > \left|\operatorname{Re} \nu + \frac{1}{2}\right|\right].
\end{aligned}$$

ET II 367(19)

$$\begin{aligned}
3. \int_0^\infty x^{-\mu} e^x P_\nu^{2\mu} \left(1 + \frac{2x}{a}\right) K_\mu(x+a) dx &= \\
&= \pi^{-\frac{1}{2}} 2^{\mu-1} \cos(\mu\pi) \Gamma\left(\mu + \nu + \frac{1}{2}\right) \Gamma\left(\mu - \nu + \frac{1}{2}\right) W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a) \\
&\quad \left[|\arg a| < \pi, \quad \operatorname{Re} \mu > \left|\operatorname{Re} \nu + \frac{1}{2}\right|\right].
\end{aligned}$$

ET II 373(11)

$$4. \int_0^\infty x^{-\frac{1}{2}\mu} (x+a)^{-\frac{1}{2}} e^{-x} P_{\nu-\frac{1}{2}}^\mu \left(\frac{a-x}{a+x}\right) K_\nu(a+x) dx = \sqrt{\frac{\pi}{2}} a^{-\frac{1}{2}\mu} \Gamma(\mu, 2a) \quad [a > 0, \quad \operatorname{Re} \mu < 1].$$

ET II 374(12)

$$\begin{aligned}
5. \int_0^\infty (\operatorname{sh} x)^{\mu+1} (\operatorname{ch} x)^{-2\mu-\frac{3}{2}} P_\nu^{-\mu} [\operatorname{ch}(2x)] I_{\mu-\frac{1}{2}}(a \operatorname{sech} x) dx &= \\
&= \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu + 1)}{\pi^{\frac{1}{2}} a^{\mu+\frac{3}{2}} [\Gamma(\mu + 1)]^2} M_{\nu+\frac{1}{2}, \mu}(a) M_{-\nu-\frac{1}{2}, \mu}(a) \\
&\quad [\operatorname{Re} \mu > \operatorname{Re} \nu, \quad \operatorname{Re} \mu > -\operatorname{Re} \nu - 1].
\end{aligned}$$

ET II 378(44)

826

7.19 Combinations of associated Legendre functions and functions generated by Bessel functions

7.191

$$\begin{aligned}
1. \int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{4} - \frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}}(2x^2 a^{-2} - 1) [\mathbf{H}_\nu(x) - N_\nu(x)] dx &= \\
&= 2^{-\nu-2} \pi^{\frac{1}{2}} a \operatorname{cosec}(\mu\pi) \cos(\nu\pi) \left\{ \left[N_\nu \left(\frac{1}{2} a \right) \right]^2 - \left[J_\nu \left(\frac{1}{2} a \right) \right]^2 \right\} \\
&\quad \left[-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].
\end{aligned}$$

$$\begin{aligned}
2. \int_0^\infty x^{1/2}(x^2 - a^2)^{-1/4-\nu/2} P_\mu^{\nu+1/2}(2x^2 a^{-2} - 1)[I_{-\nu}(x) - \mathbf{L}_\nu(x)] dx = \\
= 2^{-\nu-1} \pi^{1/2} a \operatorname{cosec}(2\mu\pi) \cos(\nu\pi) \left\{ \left[I_\nu \left(\frac{1}{2}a \right) \right]^2 - \left[I_{-\nu} \left(\frac{1}{2}a \right) \right]^2 \right\} \\
\left[-1 < \operatorname{Re} \mu < 0, \quad \operatorname{Re} \nu < \frac{1}{2} \right].
\end{aligned}$$

ET II 385(15)

7.192

$$\begin{aligned}
1. \int_0^1 x^{(\nu-\mu-1)/2} (1-x^2)^{(\nu-\mu-2)/4} P_{\nu-1/2}^{(\mu-\nu+2)/2}(x) S_{\mu,\nu}(ax) dx = \\
= 2^{\mu-3/2} \pi^{1/2} a^{-(\nu-\mu-1)/2} \Gamma \left(\frac{\mu+\nu+3}{4} \right) \Gamma \left(\frac{\mu-3\nu+3}{4} \right) \cos \left(\frac{\mu-\nu}{2} \pi \right) \times \\
\times \left[J_\nu \left(\frac{1}{2}a \right) N_{-(\mu-\nu+1)/2} \left(\frac{1}{2}a \right) - N_\nu \left(\frac{1}{2}a \right) J_{-(\mu-\nu+1)/2} \left(\frac{1}{2}a \right) \right] \\
\left[\operatorname{Re}(\mu-\nu) < 0, \quad a > 0, \quad |\operatorname{Re}(\mu+\nu)| < 1, \quad \operatorname{Re}(\mu-3\nu) < 1 \right].
\end{aligned}$$

ET II 387(24)a

$$\begin{aligned}
2. \int_1^\infty x^{1/2}(x^2 - 1)^{-\beta/2} P_\nu^\beta(x) S_{\mu,1/2}(ax) dx = \\
= \frac{2^{-3/2+\beta-\mu} a^{\beta-1} \Gamma \left(\frac{\beta-\mu+\nu}{2} + \frac{1}{4} \right) \Gamma \left(\frac{\beta-\mu-\nu}{2} - \frac{1}{4} \right)}{\pi^{1/2} \Gamma \left(\frac{1}{2} - \mu \right)} S_{\mu-\beta+1, \nu+1/2}(a) \\
\left[\operatorname{Re} \beta < 1, \quad a > 0, \quad \operatorname{Re}(\mu+\nu-\beta) < -\frac{1}{2}, \quad \operatorname{Re}(\mu-\nu-\beta) < \frac{1}{2} \right].
\end{aligned}$$

ET II 387(25)a

827

7.193

$$\begin{aligned}
1. \int_1^\infty x^{-\nu} (x^2 - 1)^{1/4-\nu/2} P_{\mu/2-\nu/2}^{\nu-1/2}(2x^{-2} - 1) S_{\mu,\nu}(ax) dx = \\
= \frac{2^{\mu-\nu} a^{\nu-2} \pi^{1/2} \Gamma \left(\frac{3\nu-\mu-1}{2} \right)}{\Gamma \left(\frac{1+\nu-\mu}{2} \right)} W_{\rho,\sigma}(ae^{i\pi/2}) W_{\rho,\sigma}(ae^{-i\pi/2}); \\
\rho = \frac{1}{2}(\mu+1-\nu), \quad \sigma = \nu - \frac{1}{2} \\
\left[\operatorname{Re}(\mu-\nu) < 0, \quad a > 0, \quad \operatorname{Re} \nu < \frac{3}{2}, \quad \operatorname{Re}(3\nu-\mu) > 1 \right].
\end{aligned}$$

ET II 387(27)a

7.21 Integration of associated Legendre functions with respect to the order

7.211

$$1. \int_0^{\infty} P_{-x-\frac{1}{2}}(\cos \theta) dx = \frac{1}{2} \operatorname{cosec} \left(\frac{1}{2} \theta \right) \quad [0 < \theta < \pi].$$

ET II 329(19)

$$2. \int_{-\infty}^{\infty} P_x(\cos \theta) dx = \operatorname{cosec} \left(\frac{1}{2} \theta \right) \quad [0 < \theta < \pi].$$

ET II 329(20)

7.212

$$\int_0^{\infty} x^{-1} \theta(\pi x) P_{-\frac{1}{2}+ix}(\operatorname{ch} a) dx = 2e^{-\frac{1}{2}a} \mathbf{K}(e^{-a}) \quad [a > 0].$$

ET II 330(22)

7.213

$$\int_0^{\infty} \frac{x \theta(\pi x)}{a^2 + x^2} P_{-\frac{1}{2}+ix}(\operatorname{ch} b) dx = Q_{a-\frac{1}{2}}(\operatorname{ch} b) \quad [\operatorname{Re} a > 0].$$

ET II 387(23)

7.214

$$\int_0^{\infty} \operatorname{sh}(\pi x) \cos(ax) P_{-\frac{1}{2}+ix}(b) dx = \frac{1}{\sqrt{2(b + \operatorname{ch} a)}} \quad [a > 0, \quad |b| < 1].$$

ET I 42(27)

7.215

$$\begin{aligned} \int_0^{\infty} \cos(bx) P_{-\frac{1}{2}+ix}^{\mu}(\operatorname{ch} a) dx &= 0 \quad [0 < a < b]; \\ &= \frac{\sqrt{\frac{\pi}{2}} (\operatorname{sh} a)^{\mu}}{\Gamma\left(\frac{1}{2} - \mu\right) (\operatorname{ch} a - \operatorname{ch} b)^{\mu+\frac{1}{2}}} \quad [0 < b < a]. \end{aligned}$$

$$\int_0^{\infty} \cos(bx) \Gamma(\mu + ix) \Gamma(\mu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\mu}(\operatorname{ch} a) dx = \frac{\sqrt{\frac{\pi}{2}} \Gamma(\mu) (\operatorname{sh} a)^{\mu-\frac{1}{2}}}{(\operatorname{ch} a + \operatorname{ch} b)^{\mu}}$$

$[a > 0, \quad b > 0, \quad \operatorname{Re} \mu > 0].$

ET II 330(24)

7.217

$$1. \int_{-\infty}^{\infty} \left(\nu - \frac{1}{2} + ix\right) \Gamma\left(\frac{1}{2} - ix\right) \Gamma\left(2\nu - \frac{1}{2} + ix\right) P_{\nu+ix-1}^{\frac{1}{2}-\nu}(\cos \theta) I_{\nu-\frac{1}{2}+ix}(a) K_{\nu-\frac{1}{2}+ix}(b) dx =$$

$$= \sqrt{2\pi} (\sin \theta)^{\nu-\frac{1}{2}} \left(\frac{ab}{\omega}\right)^{\nu} K_{\nu}(\omega); \quad \omega = (a^2 + b^2 + 2ab \cos \theta)^{\frac{1}{2}}.$$

ET II 383(29)

$$2. \int_0^{\infty} x e^{\pi x} \theta(\pi x) P_{-\frac{1}{2}+ix}(-\cos \theta) H_{ix}^{(2)}(ka) H_{ix}^{(2)}(kb) dx = -\frac{2(ab)^{\frac{1}{2}}}{\pi R} e^{-ikR};$$

$$R = (a^2 + b^2 - 2ab \cos \theta)^{\frac{1}{2}} \quad [a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Im} k \leq 0].$$

ET II 381(17)

$$3. \int_0^{\infty} x e^{\pi x} \operatorname{sh}(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\nu}(-\cos \theta) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx =$$

$$= i(2\pi)^{\frac{1}{2}} (\sin \theta)^{\nu-\frac{1}{2}} \left(\frac{ab}{R}\right)^{\nu} H_{\nu}^{(2)}(R); \quad R = (a^2 + b^2 - 2ab \cos \theta)^{\frac{1}{2}}$$

$[a > 0, \quad b > 0, \quad 0 < \theta < \pi, \quad \operatorname{Re} \nu > 0].$

ET II 381(18)

$$4. \int_0^{\infty} x \operatorname{sh}(\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix) K_{ix}(a) K_{ix}(b) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\lambda}(\beta) dx = \frac{\pi^{\frac{1}{2}}}{\sqrt{2}} \left(\frac{ab}{z}\right)^{\lambda} (\beta^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{4}} K_{\lambda}(z);$$

$$z = \sqrt{a^2 + b^2 + 2ab\beta} \quad \left[|\arg a| < \frac{\pi}{2}, \quad |\arg(\beta - 1)| < \pi, \quad \operatorname{Re} \lambda > 0\right].$$

ET II 177(16)

7.22 Combinations of Legendre polynomials, rational functions, and algebraic functions

7.221

$$1. \int_{-1}^1 P_n(x) P_m(x) dx = 0 \quad [m \neq n]$$

$$= \frac{2}{2n+1} \quad [m = n].$$

$$\begin{aligned}
2.6 \quad \int_0^1 P_n(x)P_m(x) dx &= \frac{1}{2n+1} \quad [m = n]; \\
&= 0 \quad [n - m \text{ is even, } m \neq n]; \\
&= \frac{(-1)^{\frac{1}{2}(m+n-1)} m!n!}{2^{m+n-1}(m-n)(n+m+1) \left[\left(\frac{n}{2}\right)! \left(\frac{m-1}{2}\right)! \right]^2} \quad [n\text{-even, } m\text{-odd}].
\end{aligned}$$

WH

$$3. \int_0^{2\pi} P_{2n}(\cos \varphi) d\varphi = 2\pi \left[\binom{2n}{n} 2^{-2n} \right]^2.$$

MO 70, EH II 183(50)

829

7.222

$$1. \int_{-1}^1 x^m P_n(x) dx = 0 \quad [m < n].$$

$$2. \int_{-1}^1 (1+x)^{m+n} P_m(x)P_n(x) dx = \frac{2^{m+n+1} [(m+n)!]^4}{(m!n!)^2 (2m+2n+1)!}.$$

ET II 277(15)

$$3. \int_{-1}^1 (1+x)^{m-n-1} P_m(x)P_n(x) dx = 0 \quad [m > n].$$

ET II 278(16)

$$4. \int_{-1}^1 (1-x^2)^n P_{2m}(x) dx = \frac{2n^2}{(n-m)(2m+2n+1)} \int_{-1}^1 (1-x^2)^{n-1} P_{2m}(x) dx \quad [m < n].$$

WH

$$5. \int_0^1 x^2 P_{n+1}(x)P_{n-1}(x) dx = \frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

WH

7.223

7.224

[z belongs to the complex plane with a discontinuity along the interval from -1 to $+1$].

$$1. \int_{-1}^1 (z-x)^{-1} P_n(x) dx = 2Q_n(z).$$

ET II 277(7)

$$2. \int_{-1}^1 x(z-x)^{-1} P_0(x) dx = 2Q_1(z).$$

ET II 277(8)

$$3. \int_{-1}^1 x^{n+1}(z-x)^{-1} P_n(x) dx = 2z^{n+1}Q_n(z) - \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

ET II 277(9)

$$4. \int_{-1}^1 x^m(z-x)^{-1} P_n(x) dx = 2z^m Q_n(z) \quad [m \leq n].$$

ET II 277(10)a

$$5. \int_{-1}^1 (z-x)^{-1} P_m(x) P_n(x) dx = 2P_m(z) Q_n(z) \quad [m \leq n].$$

ET II 278(18)a

$$6. \int_{-1}^1 (z-x)^{-1} P_n(x) P_{n+1}(x) dx = 2P_{n+1}(z) Q_n(z) - \frac{2}{n+1}.$$

ET II 278(19)

$$7. \int_{-1}^1 x(z-x)^{-1} P_m(x) P_n(x) dx = 2zP_m(z) Q_n(z) \quad [m < n].$$

ET II 278(21)

7.225

$$1. \int_{-1}^x (x-t)^{-\frac{1}{2}} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1+x)^{-\frac{1}{2}} [T_n(x) + T_{n+1}(x)].$$

EH II 187(43)

$$2. \int_x^1 (t-x)^{-\frac{1}{2}} P^{-\frac{1}{2}} P_n(t) dt = \left(n + \frac{1}{2}\right)^{-1} (1-x)^{-\frac{1}{2}} [T_n(x) - T_{n+1}(x)].$$

EH II 187(44)

830

$$3. \int_{-1}^1 (1-x)^{-1/2} P_n(x) dx = \frac{2^{3/2}}{2n+1}.$$

EH II 183(49)

$$4. \int_{-1}^1 (\cosh 2p - x)^{-1/2} P_n(x) dx = \frac{2\sqrt{2}}{2n+1} \exp[-(2n+1)p] \quad [p > 0].$$

WH

7.226

$$1. \int_{-1}^1 (1-x^2)^{-1/2} P_{2m}(x) dx = \left[\frac{\Gamma\left(\frac{1}{2} + m\right)}{m!} \right]^2.$$

ET II 276(4)

$$2. \int_{-1}^1 x(1-x^2)^{-1/2} P_{2m+1}(x) dx = \frac{\Gamma\left(\frac{1}{2} + m\right) \Gamma\left(\frac{3}{2} + m\right)}{m!(m+1)!}$$

ET II 276(5)

$$3. \int_{-1}^1 (1+px^2)^{-m-3/2} P_{2m}(x) dx = \frac{2}{2m+1} (-p)^m (1+p)^{-m-1/2} \quad [|p| < 1].$$

7.227

$$\int_0^1 x(a^2 + x^2)^{-1/2} P_n(1 - 2x^2) dx = \frac{[a + (a^2 + 1)^{1/2}]^{-2n-1}}{2n + 1} \quad [\operatorname{Re} a > 0].$$

ET II 278(23)

7.228⁶

$$\frac{1}{2}\Gamma(1 + \mu) \int_{-1}^1 P_l(x)(z - x)^{-\mu-1} dx = (z^2 - 1)^{-\mu/2} e^{-i\pi\mu} Q_l^\mu(z),$$

$$[l = 0, 1, 2, \dots, \quad |\arg(z - 1)| < \pi]$$

7.23 Combinations of Legendre polynomials and powers

7.231

$$1. \int_0^1 x^\lambda P_{2m}(x) dx = \frac{(-1)^m \Gamma\left(m - \frac{1}{2}\lambda\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\lambda\right)}{2\Gamma\left(-\frac{1}{2}\lambda\right) \Gamma\left(m + \frac{3}{2} + \frac{1}{2}\lambda\right)} \quad [\operatorname{Re} \lambda > -1].$$

EH II 183(51)

$$2.^6 \int_0^1 x^\lambda P_{2m+1}(x) dx = \frac{(-1)^m \Gamma\left(m + \frac{1}{2} - \frac{1}{2}\lambda\right) \Gamma\left(1 + \frac{1}{2}\lambda\right)}{2\Gamma\left(\frac{1}{2} - \frac{1}{2}\lambda\right) \Gamma\left(m + 2 + \frac{1}{2}\lambda\right)} \quad [\operatorname{Re} \lambda > -2].$$

EH II 183(52)

831

7.232

$$1. \int_{-1}^1 (1-x)^{a-1} P_m(x) P_n(x) dx =$$

$$= \frac{2^a \Gamma(a) \Gamma(n-a+1)}{\Gamma(1-a) \Gamma(n+a+1)} {}_4F_3(-m, m+1, a, a; 1, a+n+1, a-n; 1) \quad [\operatorname{Re} a > 0].$$

ET II 278(17)

$$2. \int_{-1}^1 (1-x)^{a-1} (1+x)^{b-1} P_n(x) dx = \frac{2^{a+b-1} \Gamma(a) \Gamma(b)}{\Gamma(a+b)} {}_3F_2(-n, 1+n, a; 1, a+b; 1)$$

$$[\operatorname{Re} a > 0, \quad \operatorname{Re} b > 0].$$

$$3. \int_0^1 (1-x)^{\mu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu)n!}{\Gamma(\mu+n+1)} P_n^{(\mu, -\mu)}(1-\gamma) \quad [\operatorname{Re} \mu > 0].$$

ET II 190(37)a

$$4. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} P_n(1-\gamma x) dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_3F_2 \left(-n, n+1, \nu; 1, \mu+\nu; \frac{1}{2}\gamma \right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0].$$

ET II 190(38)

7.233

$$\int_0^1 x^{2\mu-1} P_n(1-2x^2) dx = \frac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n+1)\Gamma(\mu-n)} \quad [\operatorname{Re} \mu > 0].$$

ET II 278(22)

7.24 Combinations of Legendre polynomials and other elementary functions

7.241

$$\int_0^\infty P_n(1-x)e^{-ax} dx = e^{-a} a^n \left(\frac{1}{a} \frac{d}{da} \right)^n \left(\frac{e^a}{a} \right); \\ = a^n \left(1 + \frac{1}{2} \frac{d}{da} \right)^n \left(\frac{1}{a^{n+1}} \right) \quad [\operatorname{Re} a > 0].$$

ET I 171(2)

7.242

$$\int_0^\infty P_n(e^{-x})e^{-ax} dx = \frac{(a-1)(a-2)\dots(a-n+1)}{(a+n)(a+n-2)\dots(a-n+2)} \quad [n \geq 2, \operatorname{Re} a > 0].$$

ET I 171(3)

7.243

$$1. \int_0^\infty P_{2n}(\operatorname{ch} x)e^{-ax} dx = \frac{(a^2-1^2)(a^2-3^2)\dots[a^2-(2n-1)^2]}{a(a^2-2^2)(a^2-4^2)\dots[a^2-(2n)^2]} \quad [\operatorname{Re} a > 2n].$$

ET I 171(6)

$$2. \int_0^\infty P_{2n+1}(\operatorname{ch} x)e^{-ax} dx = \frac{a(a^2-2^2)(a^2-4^2)\dots[a^2-(2n)^2]}{(a^2-1)(a^2-3^2)\dots[a^2-(2n+1)^2]} \quad [\operatorname{Re} a > 2n+1].$$

$$3. \int_0^\infty P_{2n}(\cos x) e^{-ax} dx = \frac{(a^2 + 1^2)(a^2 + 3^2) \dots [a^2 + (2n - 1)^2]}{a(a^2 + 2^2)(a^2 + 4^2) \dots [a^2 + (2n)^2]} \quad [\operatorname{Re} a > 0].$$

ET I 171(4)

$$4. \int_0^\infty P_{2n+1}(\cos x) e^{-ax} dx = \frac{a(a^2 + 2^2)(a^2 + 4^2) \dots [a^2 + (2n)^2]}{(a^2 + 1^2)(a^2 + 3^2) \dots [a^2 + (2n + 1)^2]} \quad [\operatorname{Re} a > 0].$$

ET I 171(5)

832

7.244

$$1. \int_0^1 P_n(1 - 2x^2) \sin ax dx = \frac{\pi}{2} \left[J_{n+\frac{1}{2}} \left(\frac{a}{2} \right) \right]^2 \quad [a > 0].$$

ET I 94(2)

$$2. \int_0^1 P_n(1 - 2x^2) \cos ax dx = \frac{\pi}{2} (-1)^n J_{n+\frac{1}{2}} \left(\frac{a}{2} \right) J_{-n-\frac{1}{2}} \left(\frac{a}{2} \right) \quad [a > 0].$$

ET I 38(1)

7.245

$$1. \int_0^{2\pi} P_{2m+1}(\cos \theta) \cos \theta d\theta = \frac{\pi}{2^{4m+1}} \binom{2m}{m} \binom{2m+2}{m+1}.$$

MO 70, EH II 183(5)

$$2. \int_0^\pi P_m(\cos \theta) \sin n\theta d\theta = \frac{2(n-m+1)(n-m+3) \dots (n+m-1)}{(n-m)(n-m+2) \dots (n+m)} \quad [n > m, \quad n+m \text{ is odd}];$$

$$= 0 \quad [n \leq m \quad \text{or} \quad n+m \text{ is even}].$$

MO 71

7.246

$$\int_0^\pi P_n(1 - 2 \sin^2 x \sin^2 \theta) \sin x dx = \frac{2 \sin(2n+1)\theta}{(2n+1) \sin \theta}.$$

$$\int_0^1 P_{2n+1}(x) \sin ax \frac{dx}{\sqrt{x}} = (-1)^{n+1} \sqrt{\frac{\pi}{2a}} J_{2n+\frac{3}{2}}(a) \quad [a > 0].$$

ET I 94(1)

7.248

$$1. \int_{-1}^1 (a^2 + b^2 - 2abx)^{-\frac{1}{2}} \sin[\lambda(a^2 + b^2 - 2abx)^{\frac{1}{2}}] P_n(x) dx = \pi(ab)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(a\lambda) J_{n+\frac{1}{2}}(b\lambda) \\ [a > 0, \quad b > 0].$$

ET II 277(11)

$$2. \int_{-1}^1 (a^2 + b^2 - 2abx)^{-\frac{1}{2}} \cos[\lambda(a^2 + b^2 - 2abx)^{\frac{1}{2}}] P_n(x) dx = -\pi(ab)^{-\frac{1}{2}} J_{n+\frac{1}{2}}(a\lambda) N_{n+\frac{1}{2}}(b\lambda) \\ [0 \leq a \leq b].$$

ET II 277(12)

7.249

$$1. \int_{-1}^1 P_n(x) \arcsin x dx = 0 \quad [n - \text{even}]; \\ = \pi \left\{ \frac{(n-2)!!}{2^{\frac{1}{2}(n+1)} \left(\frac{n+1}{2}\right)!} \right\}^2 \quad [n - \text{odd}].$$

WH

* I. J. Good, Proc. Camb. Philos. Soc. 51(1955), 385-388.

$$2. P_n(x) = \frac{1}{t} \sum_{r=0}^{t-1} \left(x + \sqrt{x^2 - 1} \cos \frac{2\pi r}{t} \right)^n \quad [t > n].*$$

833

7.25 Combinations of Legendre polynomials and Bessel functions

7.251

$$1. \int_0^1 x P_n(1 - 2x^2) N_\nu(xy) dx = \pi^{-1} y^{-1} [S_{2n+1}(y) + \pi N_{2n+1}(y)]$$

ET II 108(1)

$$2. \int_0^1 x P_n(1-2x^2) K_0(xy) dx = y^{-1} \left[(-1)^{n+1} K_{2n+1}(y) + \frac{i}{2} S_{2n+1}(iy) \right] \quad [y > 0].$$

ET II 134(1)

$$3. \int_0^1 x P_n(1-2x^2) J_0(xy) dx = y^{-1} J_{2n+1}(y) \quad [y > 0].$$

ET II 13(1)

$$4. \int_0^1 x P_n(1-2x^2) [J_0(ax)]^2 dx = \frac{1}{2(2n+1)} \{ [J_n(a)]^2 + [J_{n+1}(a)]^2 \}.$$

ET II 338(39)a

$$5. \int_0^1 x P_n(1-2x^2) J_0(ax) N_0(ax) dx = \frac{1}{2(2n+1)} [J_n(a) N_n(a) + J_{n+1}(a) N_{n+1}(a)].$$

ET II 339(48)a

$$6. \int_0^1 x^2 P_n(1-2x^2) J_1(xy) dx = y^{-1} (2n+1)^{-1} [(n+1) J_{2n+2}(y) - n J_{2n}(y)] \quad [y > 0].$$

ET II 20(23)

$$7. \int_0^1 x^{\mu-1} P_n(2x^2-1) J_\nu(ax) dx = \frac{2^{-\nu-1} a^\nu \left[\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu\right) \right]^2}{\Gamma(\nu+1) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + n + 1\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - n\right)} \times \\ \times {}_2F_3\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu}{2}; \nu+1, \frac{\mu+\nu}{2} + n + 1, \frac{\mu+\nu}{2} - n; -\frac{a^2}{4}\right) \\ [a > 0, \quad \text{Re}(\mu + \nu) > 0].$$

ET II 337(32)a

7.252

$$\int_0^1 e^{-ax} P_n(1-2x) I_0(ax) dx = \frac{e^{-a}}{2n+1} [I_n(a) + I_{n+1}(a)] \quad [a > 0].$$

$$\int_0^{\frac{\pi}{2}} \sin(2x) P_n(\cos 2x) J_0(a \sin x) dx = a^{-1} J_{2n+1}(a).$$

ET II 361(20)

7.254

$$\int_0^1 x P_n(1-2x^2) [I_0(ax) - L_0(ax)] dx = (-1)^n [I_{2n+1}(a) - \mathbf{L}_{2n+1}(a)] \quad [a > 0].$$

ET II 385(14)a

7.3- 7.4 Orthogonal Polynomials

7.31 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and powers

7.311

$$1. \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = 0 \quad \left[n > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 280(1)

834

$$2. \int_0^1 x^{n+2\varrho} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\Gamma(2\nu+n)\Gamma(2\varrho+n+1)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\varrho+\frac{1}{2}\right)}{2^{n+1}\Gamma(2\nu)\Gamma(2\varrho+1)n!\Gamma(n+\nu+\varrho+1)} \left[\operatorname{Re} \varrho > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 280(2)

$$3. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\beta+\nu+\frac{1}{2}}\Gamma(\beta+1)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2\nu+n)\Gamma\left(\beta-\nu+\frac{3}{2}\right)}{n!\Gamma(2\nu)\Gamma\left(\beta-\nu-n+\frac{3}{2}\right)\Gamma\left(\beta+\nu+n+\frac{3}{2}\right)} \left[\operatorname{Re} \beta > -1, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 280(3)

$$4. \int_{-1}^1 (1-x)^\alpha (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(n+2\nu)}{n!\Gamma(2\nu)\Gamma(\alpha+\beta+2)} \times$$

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 $\times {}_3F_2\left(-n, n+2\nu, \alpha+1; \nu+\frac{1}{2}, \alpha+\beta+2; 1\right)$

$[\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]$

7.312

In the following integrals, z belongs to the complex plane with a cut along the interval of the real axis from -1 to 1 .

$$1. \int_{-1}^1 x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \left[m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 281(5)

$$2. \int_{-1}^1 x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) - \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu)\Gamma(\nu+n+1)} \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 281(6)

$$3.^6 \int_{-1}^1 (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{\frac{3}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z) Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z) \left[m \leq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 283(17)

7.313

$$1. \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = 0 \quad \left[m \neq n, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 282(12), MO 98a, EH I 177(16)

$$2. \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi 2^{1-2\nu} \Gamma(2\nu+n)}{n!(n+\nu)[\Gamma(\nu)]^2} \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 281(8), MO 98a, EH I 177(17)

835

7.314

$$1. \int_{-1}^1 (1-x)^{\nu-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi^{\frac{1}{2}} \Gamma\left(\nu - \frac{1}{2}\right) \Gamma(2\nu+n)}{n! \Gamma(\nu) \Gamma(2\nu)} \left[\operatorname{Re} \nu > \frac{1}{2} \right].$$

$$2. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{2\nu-1} [C_n^\nu(x)]^2 dx = \frac{2^{3\nu-\frac{1}{2}} [\Gamma(2\nu+n)]^2 \Gamma\left(2n+\nu+\frac{1}{2}\right)}{(n!)^2 \Gamma(2\nu) \Gamma\left(3\nu+2n+\frac{1}{2}\right)} \quad [\operatorname{Re} \nu > 0].$$

ET II 282(10)

$$3. \int_{-1}^1 (1-x)^{3\nu+2n-\frac{3}{2}} (1+x)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi^{\frac{1}{2}} \left[\Gamma\left(\nu+\frac{1}{2}\right) \right]^2 \Gamma\left(\nu+2n+\frac{1}{2}\right) \Gamma(2\nu+2n) \Gamma\left(3\nu+2n-\frac{1}{2}\right)}{2^{2\nu+2n} \left[n! \Gamma\left(\nu+n+\frac{1}{2}\right) \Gamma(2\nu) \right]^2 \Gamma\left(2\nu+2n+\frac{1}{2}\right)} \quad \left[\operatorname{Re} \nu > \frac{1}{6} \right].$$

ET II 282(11)

$$4. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{\nu+m-n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx = (-1)^m \frac{2^{2-2\nu-m+n} \pi^{\frac{3}{2}} \Gamma(2\nu+n)}{m!(n-m)! [\Gamma(\nu)]^2 \Gamma\left(\frac{1}{2}+\nu+m\right)} \frac{\Gamma\left(\nu-\frac{1}{2}+m-n\right) \Gamma\left(\frac{1}{2}-\nu+m-n\right)}{\Gamma\left(\frac{1}{2}-\nu-n\right) \Gamma\left(\frac{1}{2}+m-n\right)} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}; \quad n \geq m \right].$$

ET II 282(13a)

$$5. \int_{-1}^1 (1-x)^{2\nu-1} (1+x)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{2^{3\nu-\frac{1}{2}} \Gamma\left(\nu+\frac{1}{2}\right) \Gamma(2\nu+m) \Gamma(2\nu+n)}{m! n! \Gamma(2\nu) \Gamma\left(\frac{1}{2}-\nu\right)} \frac{\Gamma\left(\nu+\frac{1}{2}+m+n\right) \Gamma\left(\frac{1}{2}-\nu+n-m\right)}{\Gamma\left(\nu+\frac{1}{2}+n-m\right) \Gamma\left(3\nu+\frac{1}{2}+m+n\right)} \quad [\operatorname{Re} \nu > 0].$$

ET II 282(14)

$$6. \int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{3\nu+m+n-\frac{3}{2}} C_m^\nu(x) C_n^\nu(x) dx = \frac{2^{4\nu+m+n-1} \left[\Gamma\left(\nu+\frac{1}{2}\right) \Gamma(2\nu+m+n) \right]^2 \Gamma\left(\nu+m+n+\frac{1}{2}\right) \Gamma\left(3\nu+m+n-\frac{1}{2}\right)}{\Gamma\left(\nu+m+\frac{1}{2}\right) \Gamma\left(\nu+n+\frac{1}{2}\right) \Gamma(2\nu+m) \Gamma(2\nu+n) \Gamma(4\nu+2m+2n)} \quad \left[\operatorname{Re} \nu > \frac{1}{6} \right].$$

$$\begin{aligned}
7. \int_{-1}^1 (1-x)^\alpha (1+x)^{\nu-\frac{1}{2}} C_m^\mu(x) C_n^\nu(x) dx &= \\
&= \frac{2^{\alpha+\nu+\frac{1}{2}} \Gamma(\alpha+1) \Gamma\left(\nu+\frac{1}{2}\right) \Gamma\left(\nu-\alpha+n-\frac{1}{2}\right)}{m!n! \Gamma\left(\nu-\alpha-\frac{1}{2}\right) \Gamma\left(\nu-\alpha+n+\frac{3}{2}\right)} \frac{\Gamma(2\mu+m)\Gamma(2\nu+n)}{\Gamma(2\mu)\Gamma(2\nu)} \times \\
&\quad \times {}_4F_3\left(-m, m+2\mu, \alpha+1, \alpha-\nu+\frac{3}{2}; \mu+\frac{1}{2}, \nu+\alpha+n+\frac{3}{2}, \alpha-\nu-n+\frac{3}{2}; 1\right) \\
&\quad \left[\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
\end{aligned}$$

ET II 283(16)

7.315

$$\int_{-1}^1 (1-x^2)^{\frac{1}{2}\nu-1} C_{2n}^\nu(ax) dx = \frac{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\right)} C_n^{\frac{1}{2}\nu}(2a^2-1) \quad [\operatorname{Re} \nu > 0].$$

ET II 283(19)

7.316

$$\int_{-1}^1 (1-x^2)^{\nu-1} C_n^\nu(\cos \alpha \cos \beta + x \sin \alpha \sin \beta) dx = \frac{2^{2\nu-1} n! [\Gamma(\nu)]^2}{\Gamma(2\nu+n)} C_n^\nu(\cos \alpha) C_n^\nu(\cos \beta) \quad [\operatorname{Re} \nu > 0].$$

ET II 283(20)

7.317

$$\begin{aligned}
1. \int_0^1 (1-x)^{\mu-1} x^{\lambda-\frac{1}{2}} C_n^\lambda(1-\gamma x) dx &= \frac{\Gamma(2\lambda+n) \Gamma\left(\lambda+\frac{1}{2}\right) \Gamma(\mu)}{\Gamma(2\lambda) \Gamma\left(\lambda+\mu+n+\frac{1}{2}\right)} P_n^{(\alpha, \beta)}(1-\gamma) \\
\alpha = \lambda + \mu - \frac{1}{2} \quad \beta = \lambda - \mu - \frac{1}{2} \quad &\left[\operatorname{Re} \lambda > -1, \quad \lambda \neq 0, \quad -\frac{1}{2}, \quad \operatorname{Re} \mu > 0 \right].
\end{aligned}$$

ET II 190(39)a

$$\begin{aligned}
2. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_n^\lambda(1-\gamma x) dx &= \frac{\Gamma(2\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(2\lambda) \Gamma(\mu+\nu)} \times \\
&\quad \times {}_3F_2\left(-n, n+2\lambda, \nu; \lambda+\frac{1}{2}, \mu+\nu; \frac{\gamma}{2}\right) \\
&\quad [2\lambda \neq 0, -1, -2, \dots, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

$$\int_0^1 x^{2\nu}(1-x^2)^{\sigma-1} C_n^\nu(1-x^2y) dx = \frac{\Gamma(2\nu+n)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(\sigma)}{2\Gamma(2\nu)\Gamma\left(n+\nu+\sigma+\frac{1}{2}\right)} P_n^{(\alpha,\beta)}(1-y),$$

$$\alpha = \nu + \sigma - \frac{1}{2}, \quad \beta = \nu - \sigma - \frac{1}{2} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \sigma > 0 \right].$$

ET II 283(21)

837
7.319

$$1. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{\frac{1}{2}}) dx = (-1)^n \frac{\Gamma(\lambda+n)\Gamma(\mu)\Gamma(\nu)}{n!\Gamma(\lambda)\Gamma(\mu+\nu)} {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 191(41)a

$$2. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{\frac{1}{2}}) dx = \frac{(-1)^n 2\Gamma(\mu)\Gamma(\lambda+n+1)\Gamma\left(\nu+\frac{1}{2}\right)}{n!\Gamma(\lambda)\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times$$

$$\times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$

$$\left[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and some elementary functions

7.321

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n!\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 281(7), MO 99a

7.322

$$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab)$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$1. \int_0^\pi C_n^\nu(\cos \varphi)(\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots];$$

$$= 2^{-2\nu} \pi \Gamma(2\nu + 1) [\Gamma(1 + \nu)]^{-2} \quad [n = 0].$$

EH I 177(18)

$$2. \int_0^\pi C_n^\nu(\cos \psi \cos \psi' + \sin \psi \sin \psi' \cos \varphi)(\sin \varphi)^{2\nu-1} d\varphi =$$

$$= 2^{2\nu-1} n! [\Gamma(\nu)]^2 C_n^\nu(\cos \psi) C_n^\nu(\cos \psi') [\Gamma(2\nu + n)]^{-1} \quad [\operatorname{Re} \nu > 0].$$

EH I 177(20)

7.324

$$1. \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n+1}^\nu(x) \sin ax \, dx = (-1)^n \pi \frac{\Gamma(2n+2\nu+1) J_{2n+\nu+1}(a)}{(2n+1)! \Gamma(\nu) (2a)^\nu}$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0 \right].$$

ET I 94(4)

838

$$2. \int_0^1 (1-x^2)^{\nu-\frac{1}{2}} C_{2n}^\nu(x) \cos ax \, dx = \frac{(-1)^n \pi \Gamma(2n+2\nu) J_{\nu+2n}(a)}{(2n)! \Gamma(\nu) (2a)^\nu} \quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad a > 0 \right].$$

ET I 38(3)a

7.33 Combinations of the polynomials $C_n^\nu(x)$ and Bessel functions. Integration of Gegenbauer functions with respect to the index

7.331

$$1. \int_1^\infty x^{2n+1-\nu} (x^2-1)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n} \left(\frac{1}{x} \right) J_\nu(xy) \, dx =$$

$$= (-1)^n 2^{2n-\nu+1} y^{-\nu+2n-1} [(2n)!]^{-1} \Gamma(2\nu-2n) [\Gamma(\nu-2n)]^{-1} \cos y$$

$$\left[y > 0, \quad 2n - \frac{1}{2} < \operatorname{Re} \nu < 2n + \frac{1}{2} \right].$$

ET II 44(10)a

$$2. \int_1^\infty x^{2n-\nu+2} (x^2-1)^{\nu-2n-\frac{3}{2}} C_{2n+1}^{\nu-2n-1} \left(\frac{1}{x} \right) J_\nu(xy) \, dx =$$

$$= (-1)^n 2^{2n-\nu+2} y^{-\nu+2n} \Gamma(2\nu-2n-1) [(2n+1)! \Gamma(\nu-2n-1)]^{-1} \sin y$$

$$\left[y > 0, \quad 2n + \frac{1}{2} < \operatorname{Re} \nu < 2n + \frac{3}{2} \right].$$

$$\begin{aligned}
1. \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n+1}^{\nu+\frac{1}{2}} [(x^2 + \beta^2)^{-\frac{1}{2}} \beta] J_{\nu+\frac{3}{2}+2n} [(x^2 + \beta^2)^{\frac{1}{2}} a] J_\nu(xy) dx = \\
= (-1)^n 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-\frac{1}{2}} \sin[\beta(a^2 - y^2)^{\frac{1}{2}}] C_{2n+1}^{\nu+\frac{1}{2}} \left[\left(1 - \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \quad [0 < y < a]; \\
= 0 \quad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1].
\end{aligned}$$

ET II 59(23)

$$\begin{aligned}
2. \int_0^\infty x^{\nu+1} (x^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{3}{4}} C_{2n}^{\nu+\frac{1}{2}} [\beta(x^2 + \beta^2)^{-\frac{1}{2}}] J_{\nu+\frac{1}{2}+2n} [(x^2 + \beta^2)^{\frac{1}{2}} a] J_\nu(xy) dx = \\
= (-1)^n 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^\nu (a^2 - y^2)^{-\frac{1}{2}} \cos[\beta(a^2 - y^2)^{\frac{1}{2}}] C_{2n}^{\nu+\frac{1}{2}} \left[\left(1 - \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \quad [0 < y < a]; \\
= 0 \quad [a < y < \infty] \quad [a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1].
\end{aligned}$$

ET II 59(24)

7.333

$$\begin{aligned}
1. \int_0^\pi (\sin x)^{\nu+1} \cos(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx = \\
= (-1)^{\frac{n}{2}} \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 0, 2, 4, \dots]; \\
= 0 \quad [n = 1, 3, 5, \dots] \quad [\operatorname{Re} \nu > -1].
\end{aligned}$$

WA 414(2)a

839

$$\begin{aligned}
2. \int_0^\pi (\sin x)^{\nu+1} \sin(a \cos \theta \cos x) C_n^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx = \\
= 0 \quad [n = 0, 2, 4, \dots]; \\
= (-1)^{\frac{n-1}{2}} \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} (\sin \theta)^\nu C_n^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+\frac{1}{2}+n}(a) \quad [n = 1, 3, 5, \dots] \quad [\operatorname{Re} \nu > -1].
\end{aligned}$$

WA 414(3)a

7.334

$$\begin{aligned}
1. \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{J_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{J_{\nu+n}(\beta)}{\beta^\nu}, \\
\omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{\frac{1}{2}} \quad \left[n = 0, 1, 2, \dots; \quad \operatorname{Re} \nu > -\frac{1}{2} \right].
\end{aligned}$$

ET II 362(29)

$$2. \int_0^\pi (\sin x)^{2\nu} C_n^\nu(\cos x) \frac{N_\nu(\omega)}{\omega^\nu} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(\alpha)}{\alpha^\nu} \frac{N_{\nu+n}(\beta)}{\beta^\nu},$$

$$\omega = (\alpha^2 + \beta^2 - 2\alpha\beta \cos x)^{\frac{1}{2}} \quad \left[|\alpha| < |\beta|, \quad \operatorname{Re} \nu - \frac{1}{2} \right].$$

Integration of gegenbauer functions with respect to the index

7.335

$$\int_{c-i\infty}^{c+i\infty} [\sin(\alpha\pi)]^{-1} t^\alpha C_\alpha^\nu(z) d\alpha = -2i(1 + 2tz + t^2)^{-\nu}$$

$$[-2 < \operatorname{Re} \nu < c < 0, \quad |\arg(z \pm 1)| < \pi].$$

7.336

$$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x) \left(\nu - \frac{1}{2} + ix \right) K_{\nu-\frac{1}{2}+ix}(a) I_{\nu-\frac{1}{2}+ix}(b) C_{-\frac{1}{2}+ix}^\nu(-\cos \varphi) dx = \frac{2^{-\nu+1} (ab)^\nu}{\Gamma(\nu)} \omega^{-\nu} K_\nu(\omega),$$

$$\omega = \sqrt{a^2 + b^2 - 2ab \cos \varphi}.$$

7.34 Combinations of Chebyshev polynomials and powers

7.341

$$\int_{-1}^1 [T_n(x)]^2 dx = 1 - (4n^2 - 1)^{-1}.$$

7.342

$$\int_{-1}^1 U_n[x(1-y^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}} + yz] dx = \frac{2}{n+1} U_n(y) U_n(z) \quad [|y| < 1, \quad |z| < 1].$$

7.343

$$1. \int_{-1}^1 T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = 0 \quad [m \neq n];$$

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$$= \pi \quad [m = n = 0].$$

$$2. \int_{-1}^1 \sqrt{1-x^2} U_n(x) U_m(x) dx = 0 \quad [m \neq n];$$

$$= \frac{\pi}{2} \quad [m = n].$$

ET II 274(28)
ET II 274(27), MO 105a

7.344

$$1. \int_{-1}^1 (y-x)^{-1} (1-y^2)^{-\frac{1}{2}} T_n(y) dy = \pi U_{n-1}(x) \quad [n = 1, 2, \dots].$$

EH II 187(47)

$$2. \int_{-1}^1 (y-x)^{-1} (1-y^2)^{\frac{1}{2}} U_{n-1}(y) dy = -\pi T_n(x) \quad [n = 1, 2, \dots].$$

EH II 187(48)

7.345

$$1. \int_{-1}^1 (1-x)^{-\frac{1}{2}} (1+x)^{m-n-\frac{3}{2}} T_m(x) T_n(x) dx = 0 \quad [m > n].$$

ET II 272(10)

$$2. \int_{-1}^1 (1-x)^{-\frac{1}{2}} (1+x)^{m+n-\frac{3}{2}} T_m(x) T_n(x) dx = \frac{\pi(2m+2n-2)!}{2^{m+n}(2m-1)!(2n-1)!} \quad [m+n \neq 0].$$

ET II 272(11)

$$3. \int_{-1}^1 (1-x)^{\frac{1}{2}} (1+x)^{m+n+\frac{3}{2}} U_m(x) U_n(x) dx = \frac{\pi(2m+2n+2)!}{2^{m+n+2}(2m+1)!(2n+1)!}.$$

ET II 274(31)

$$4. \int_{-1}^1 (1-x)^{\frac{1}{2}} (1+x)^{m-n-\frac{1}{2}} U_m(x) U_n(x) dx = 0 \quad [m > n].$$

$$5. \int_{-1}^1 (1-x)(1+x)^{\frac{1}{2}} U_m(x) U_n(x) dx = \frac{2^{\frac{5}{2}}(m+1)(n+1)}{\left(m+n+\frac{3}{2}\right)\left(m+n+\frac{5}{2}\right)[1-4(m-n)^2]}.$$

ET II 274(29)

$$6. \int_{-1}^1 (1+x)^{-\frac{1}{2}}(1-x)^{\alpha-1} T_m(x) T_n(x) dx = \frac{\pi^{\frac{1}{2}} 2^{\alpha-\frac{1}{2}} \Gamma(\alpha) \Gamma\left(n-\alpha+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\alpha\right) \Gamma\left(\alpha+n+\frac{1}{2}\right)} {}_4F_3\left(-m, m, \alpha, \alpha+\frac{1}{2}; \frac{1}{2}, \alpha+n+\frac{1}{2}, \alpha-n+\frac{1}{2}; 1\right) \quad [\operatorname{Re} \alpha > 0].$$

ET II 272(12)

$$7. \int_{-1}^1 (1+x)^{\frac{1}{2}}(1-x)^{\alpha-1} U_m(x) U_n(x) dx = \frac{\pi^{\frac{1}{2}} 2^{\alpha-\frac{1}{2}}(m+1)(n+1) \Gamma(\alpha) \Gamma\left(n-\alpha+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-\alpha\right) \Gamma\left(\frac{3}{2}+\alpha+n\right)} \times {}_4F_3\left(-m, m+2, \alpha, \alpha-\frac{1}{2}; \frac{3}{2}, \alpha+n+\frac{3}{2}, \alpha-n-\frac{1}{2}; 1\right) \quad [\operatorname{Re} \alpha > 0].$$

ET II 275(32)

841
7.346

$$\int_0^1 x^{s-1} T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s 2^s \operatorname{B}\left(\frac{1}{2}+\frac{1}{2}s+\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}s-\frac{1}{2}n\right)} \quad [\operatorname{Re} s > 0].$$

ET II 324(2)

7.347

$$1. \int_{-1}^1 (1-x)^\alpha (1+x)^\beta T_n(x) dx = \frac{2^{\alpha+\beta+2n+1} (n!)^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n)! \Gamma(\alpha+\beta+2)} \times {}_3F_2\left(-n, n, \alpha+1; \frac{1}{2}, \alpha+\beta+2; 1\right) \quad [\operatorname{Re} \alpha > -1, \operatorname{Re} \beta > -1].$$

ET II 271(2)

7.348

$$\int_{-1}^1 (1-x^2)^{-\frac{1}{2}} U_{2n}(xz) dx = \pi P_n(2z^2 - 1) \quad [|z| < 1].$$

ET II 275(33)

7.349

$$\int_{-1}^1 (1-x^2)^{-\frac{1}{2}} T_n(1-x^2y) dx = \frac{1}{2}\pi [P_n(1-y) + P_{n-1}(1-y)].$$

ET II 222(14)

7.35 Combinations of Chebyshev polynomials and some elementary functions

7.351

$$\int_0^1 x^{-\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} e^{-\frac{2a}{x}} T_n(x) dx = \pi^{\frac{1}{2}} D_{n-\frac{1}{2}}(2a^{\frac{1}{2}}) D_{-n-\frac{1}{2}}(2a^{\frac{1}{2}}) \quad [\operatorname{Re} a > 0].$$

ET II 272(13)

7.352

$$1. \int_0^\infty \frac{x U_n[a(a^2+x^2)^{-\frac{1}{2}}]}{(a^2+x^2)^{\frac{1}{2}n+1} (e^{\pi x} + 1)} dx = \frac{a^{-n}}{2n} - 2^{-n-1} \zeta\left(n+1, \frac{a+1}{2}\right) \quad [\operatorname{Re} a > 0].$$

ET II 275(39)

$$2. \int_0^\infty \frac{x U_n[a(a^2+x^2)^{-\frac{1}{2}}]}{(a^2+x^2)^{\frac{1}{2}n+1} (e^{2\pi x} - 1)} dx = \frac{1}{2} \zeta(n+1, a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n} \quad [\operatorname{Re} a > 0].$$

ET II 276(40)

7.353

$$\begin{aligned} 1. \int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} \operatorname{sech}\left(\frac{1}{2}\pi x\right) T_n[a(a^2+x^2)^{-\frac{1}{2}}] dx \\ = 2^{1-2n} \left[\zeta\left(n, \frac{a+1}{4}\right) - \zeta\left(n, \frac{a+3}{4}\right) \right] = \\ = 2^{1-n} \Phi\left(-1, n, \frac{a+1}{2}\right) \quad [\operatorname{Re} a > 0]. \end{aligned}$$

$$2. \int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} \left[\operatorname{ch} \left(\frac{1}{2}\pi x \right) \right]^{-2} T_n[a(a^2+x^2)^{-\frac{1}{2}}] dx = \pi^{-1} n 2^{1-n} \zeta \left(n+1, \frac{a+1}{2} \right) \quad [\operatorname{Re} a > 0].$$

ET II 273(20)

7.354

$$1. \int_{-1}^1 \sin(xyz) \cos[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z] T_{2n+1}(x) dx = (-1)^n \pi T_{2n+1}(y) J_{2n+1}(x).$$

ET II 271(4)

$$2. \int_{-1}^1 \sin(xyz) \sin[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z] U_{2n+1}(x) dx = (-1)^n \pi (1-y^2)^{\frac{1}{2}} U_{2n+1}(y) J_{2n+2}(z).$$

ET II 274(25)

$$3. \int_{-1}^1 \cos(xyz) \cos[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z] T_{2n}(x) dx = (-1)^n \pi T_{2n}(y) J_{2n}(z).$$

ET II 271(5)

$$4. \int_{-1}^1 \cos(xyz) \sin[(1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}z] U_{2n}(x) dx = (-1)^n \pi (1-y^2)^{\frac{1}{2}} U_{2n}(y) J_{2n+1}(z).$$

ET II 274(24)

7.355

$$1. \int_0^1 T_{2n+1}(x) \sin ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(a) \quad [a > 0].$$

ET I 94(3)a

$$2. \int_0^1 T_{2n}(x) \cos ax \frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n}(a) \quad [a > 0].$$

ET I 38(2)a

7.36 Combinations of Chebyshev polynomials and Bessel functions

7.361

$$\int_0^1 (1-x^2)^{-\frac{1}{2}} T_n(x) J_\nu(xy) dx = \frac{1}{2} \pi J_{\frac{1}{2}(\nu+n)} \left(\frac{1}{2}y \right) J_{\frac{1}{2}(\nu-n)} \left(\frac{1}{2}y \right) \quad [y > 0, \operatorname{Re} \nu > -n-1].$$

7.362

$$\int_1^{\infty} (x^2 - 1)^{-\frac{1}{2}} T_n \left(\frac{1}{x} \right) K_{2\mu}(ax) dx = \frac{\pi}{2a} W_{\frac{1}{2}n, \mu}(a) W_{-\frac{1}{2}n, \mu}(a) \quad [\operatorname{Re} a > 0].$$

ET II 366(17)a

7.37- 7.38 Hermite polynomials

7.371

$$\int_0^x H_n(y) dy = [2(n+1)]^{-1} [H_{n+1}(x) - H_{n+1}(0)].$$

EH II 194(27)

7.372

$$\int_{-1}^1 (1-t^2)^{\alpha-\frac{1}{2}} H_{2n}(\sqrt{x}t) dx = \frac{(-1)^n \pi^{\frac{1}{2}} (2n)! \Gamma\left(\alpha + \frac{1}{2}\right) L_n^\alpha(x)}{\Gamma(n + \alpha + 1)} \quad \left[\operatorname{Re} a > -\frac{1}{2} \right].$$

EH II 195(34)

843

7.373

$$1. \int_0^x e^{-y^2} H_n(y) dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x). \quad (\text{see } 8.956).$$

8.956
EH II 194(26)

$$2. \int_{-\infty}^{\infty} e^{-x^2} H_{2m}(xy) dx = \sqrt{\pi} \frac{(2m)!}{m!} (y^2 - 1)^m.$$

EH II 195(28)

7.374

$$1. \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \quad [m \neq n];$$

$$= 2^n \cdot n! \sqrt{\pi} \quad [m = n].$$

$$2. \int_{-\infty}^{\infty} e^{-2x^2} H_m(x) H_n(x) dx = (-1)^{\frac{1}{2}(m+n)} 2^{\frac{m+n-1}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \quad [m+n \text{ is even}].$$

ET II 289(10)a

$$3. \int_{-\infty}^{\infty} e^{-x^2} H_m(ax) H_n(x) dx = 0 \quad [m < n].$$

ET II 290(20)a

$$4. \int_{-\infty}^{\infty} e^{-x^2} H_{2m+n}(ax) H_n(x) dx = \sqrt{\pi} 2^n \frac{(2m+n)!}{m!} (a^2 - 1)^m a^n.$$

ET II 291(21)a

$$5. \int_{-\infty}^{\infty} e^{-2\alpha^2 x^2} H_m(x) H_n(x) dx = 2^{\frac{m+n-1}{2}} \alpha^{-m-n-1} (1-2\alpha^2)^{\frac{m+n}{2}} \Gamma\left(\frac{m+n+1}{2}\right) \times \\ \times {}_2F_1\left(-m, -n; \frac{1-m-n}{2}; \frac{\alpha^2}{2\alpha^2-1}\right) \\ [\operatorname{Re} \alpha^2 > 0, \quad \alpha^2 \neq 1/2, \quad m+n \text{ is even}].$$

ET II 289(12)a

$$6. \int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(x) dx = \pi^{\frac{1}{2}} y^n 2^n.$$

ET II 288(2)A, EH II 195(31)

$$7. \int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(x) H_n(x) dx = 2^n \pi^{\frac{1}{2}} m! y^{n-m} L_m^{n-m}(-2y^2) \quad [m \leq n].$$

BU 148(15), ET II 289(13)a

$$8. \int_{-\infty}^{\infty} e^{-(x-y)^2} H_n(\alpha x) dx = \pi^{\frac{1}{2}} (1-\alpha^2)^{\frac{n}{2}} H_n\left[\frac{\alpha y}{(1-\alpha^2)^{\frac{1}{2}}}\right].$$

ET II 290(17)a

$$9. \int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(\alpha x) H_n(\alpha x) dx = \\ = \pi^{\frac{1}{2}} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k} \binom{n}{k} (1-\alpha^2)^{\frac{m+n}{2}-k} H_{m+n-2k}\left[\frac{\alpha y}{(1-\alpha^2)^{\frac{1}{2}}}\right].$$

$$10. \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2u}} H_n(x) dx = (2\pi u)^{\frac{1}{2}} (1-2u)^{\frac{n}{2}} H_n[y(1-2u)^{-\frac{1}{2}}] \quad \left[0 \leq u < \frac{1}{2}\right].$$

EH II 195(30)

844

7.375

$$1. \int_{-\infty}^{\infty} e^{-2x^2} H_k(x) H_m(x) H_n(x) dx = \pi^{-1} 2^{\frac{1}{2}(m+n+k-1)} \Gamma(s-k) \Gamma(s-m) \Gamma(s-n) \\ 2s = k + m + n + 1 \quad [k + m + n \text{ is even}].$$

ET II 290(14)a

$$2. \int_{-\infty}^{\infty} e^{-x^2} H_k(x) H_m(x) H_n(x) dx = \frac{2^{\frac{m+n+k}{2}} \pi^{\frac{1}{2}} k! m! n!}{(s-k)! (s-m)! (s-n)!}, \\ 2s = m + n + k \quad [k + m + n \text{ is even}].$$

ET II 290(15)a

7.376

$$1. \int_{-\infty}^{\infty} e^{ixy} e^{-\frac{x^2}{2}} H_n(x) dx = (2\pi)^{\frac{1}{2}} e^{-\frac{y^2}{2}} H_n(y) i^n.$$

MO 165a

$$2. \int_0^{\infty} e^{-2\alpha x^2} x^\nu H_{2n}(x) dx = (-1)^n 2^{2n-\frac{3}{2}-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi} \alpha^{\frac{1}{2}(\nu+1)}} F\left(-n, \frac{\nu+1}{2}; \frac{1}{2}; \frac{1}{2\alpha}\right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1].$$

BU 150(18a)

$$3. \int_0^{\infty} e^{-2\alpha x^2} x^\nu H_{2n+1}(x) dx = (-1)^n 2^{2n-\frac{1}{2}\nu} \frac{\Gamma\left(\frac{\nu}{2}+1\right) \Gamma\left(n+\frac{3}{2}\right)}{\sqrt{\pi} \alpha^{\frac{1}{2}\nu+1}} F\left(-n, \frac{\nu}{2}+1; \frac{3}{2}; \frac{1}{2\alpha}\right) \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -2].$$

BU 150(18b)

7.377

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x+y) H_n(x+z) dx = 2^n \pi^{\frac{1}{2}} m! z^{n-m} L_m^{n-m}(-2yx) \quad [m \leq n].$$

$$\int_0^{\infty} x^{\alpha-1} e^{-\beta x} H_n(x) dx = 2^n \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! \Gamma(\alpha + n - 2m)}{m!(n-2m)!} (-1)^m 2^{-2m} \beta^{2m-\alpha-n}$$

[Re $\alpha > 0$, if n is even; Re $\alpha > -1$, if n is odd; Re $\beta > 0$].

ET I 172(11)a

7.379

$$1. \int_{-\infty}^{\infty} x e^{-x^2} H_{2m+1}(xy) dx = \pi^{\frac{1}{2}} \frac{(2m+1)!}{m!} y(y^2-1)^m.$$

EH II 195(28)

$$2. \int_{-\infty}^{\infty} x^n e^{-x^2} H_n(xy) dx = \pi^{\frac{1}{2}} n! P_n(y).$$

EH II 195(29)

7.381

$$\int_{-\infty}^{\infty} (x \pm ic)^{\nu} e^{-x^2} H_n(x) dx = 2^{n-1-\nu} \pi^{\frac{1}{2}} \frac{\Gamma\left(\frac{n-\nu}{2}\right)}{\Gamma(-\nu)} \exp\left[\pm \frac{1}{2} \pi(\nu+n)i\right] \quad [c > 0].$$

ET II 288(3)a

845

7.382

$$\int_0^{\infty} x^{-1} (x^2 + a^2)^{-1} e^{-x^2} H_{2n+1}(x) dx = (-2)^n (\pi)^{\frac{1}{2}} a^{-2} [2^{\nu} n! - (2n+1)! e^{\frac{1}{2} a^2} D_{-2n-2}(a\sqrt{2})].$$

ET II 288(4)a

7.383

$$1. \int_0^{\infty} e^{-xp} H_{2n+1}(\sqrt{x}) dx = (-1)^n 2^n (2n+1)!! \pi^{\frac{1}{2}} (p-1)^n p^{-n-\frac{3}{2}} \quad [\text{Re } p > 0].$$

EF 151(261)A, ET I 172(12)a

$$2. \int_0^{\infty} e^{-(b-\beta x)} H_{2n+1}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \sqrt{\alpha-\beta} \frac{(2n+1)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{3}{2}}}$$

[Re $(b-\beta) > 0$].

$$3. \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-(b-\beta)x} H_{2n}(\sqrt{(\alpha-\beta)x}) dx = (-1)^n \sqrt{\pi} \frac{(2n)!}{n!} \frac{(b-\alpha)^n}{(b-\beta)^{n+\frac{1}{2}}} \quad [\operatorname{Re}(b-\beta) > 0].$$

ET I 172(16)a

$$4. \int_0^{\infty} x^{a-\frac{1}{2}n-1} e^{-bx} H_n(\sqrt{x}) dx = 2^n \Gamma(a) b^{-a} {}_2F_1 \left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1-a; b \right)$$

$\left[\operatorname{Re} a > \frac{1}{2}n, \text{ if } n \text{ is even; } \operatorname{Re} a > \frac{1}{2}n - \frac{1}{2}, \text{ if } n \text{ is odd; } \operatorname{Re} b > 0. \right.$
 If a is even, only the first $1 + \binom{n}{2}$ terms are kept in the series for ${}_2F_1$.

ET I 172(14)a

$$5. \int_0^{\infty} x^{-\frac{1}{2}} e^{-px} H_{2n}(\sqrt{x}) dx = (-1)^n 2^n (2n-1)!! \pi^{\frac{1}{2}} (p-1)^n p^{-n-\frac{1}{2}}.$$

MO 177a

7.384

$$\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-bx} \left[H_n \left(\frac{\alpha + \sqrt{x}}{\lambda} \right) + H_n \left(\frac{a - \sqrt{x}}{\lambda} \right) \right] dx = \sqrt{\frac{2\pi}{b}} (1 - \lambda^{-2} b^{-1})^{\frac{n}{2}} H_n \left(\frac{\alpha}{\sqrt{\lambda^2 - \frac{1}{b}}} \right)$$

[$\operatorname{Re} b > 0$].

ET I 173(17)a

7.385

$$1. \int_0^{\infty} \frac{e^{-bx}}{\sqrt{e^x - 1}} H_{2n}[\sqrt{s(1-e^{-x})}] dx = (-1)^n 2^{2n} \sqrt{\pi} \frac{(2n)! \Gamma \left(b + \frac{1}{2} \right)}{\Gamma(n+b+1)} L_n^n(s) \quad \left[\operatorname{Re} b > -\frac{1}{2} \right].$$

ET I 174(23)a

$$2. \int_0^{\infty} e^{-bx} H_{2n+1}[\sqrt{s}\sqrt{1-e^{-x}}] dx = (-1)^n 2^{2n} \sqrt{\pi s} \frac{(2n+1)! \Gamma(b)}{\Gamma \left(n+b+\frac{3}{2} \right)} L_n^b(s) \quad [\operatorname{Re} b > 0].$$

ET I 174(24)a

$$\int_0^{\infty} x^{-\frac{n+1}{2}} e^{-\frac{q^2}{4x}} H_n \left(\frac{q}{2\sqrt{x}} \right) e^{-px} dx = 2^n \pi^{\frac{1}{2}} p^{\frac{n-1}{2}} e^{-q\sqrt{p}}.$$

EF 129(117)

7.387

$$1. \int_0^{\infty} e^{-x^2} \operatorname{sh}(\sqrt{2}\beta x) H_{2n+1}(x) dx = 2^{n-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{2n+1} e^{\frac{1}{2}\beta^2}.$$

ET II 289(7)a

$$2. \int_0^{\infty} e^{-x^2} \operatorname{ch}(\sqrt{2}\beta x) H_{2n}(x) dx = 2^{n-1} \pi^{\frac{1}{2}} \beta^{2n} e^{\frac{1}{2}\beta^2}.$$

ET II 289(8)a

7.388

$$1. \int_0^{\infty} e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(x) dx = (-1)^n 2^{n-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{2n+1} e^{-\frac{1}{2}\beta^2}.$$

ET II 288(5)a

$$2. \int_0^{\infty} e^{-x^2} \sin(\sqrt{2}\beta x) H_{2n+1}(ax) dx = (-1)^n 2^{-1} \pi^{\frac{1}{2}} (a^2-1)^{n+\frac{1}{2}} e^{-\frac{1}{2}\beta^2} H_{2n+1} \left(\frac{a\beta}{\sqrt{2}(a^2-1)^{\frac{1}{2}}} \right).$$

ET II 290(18)a

$$3. \int_0^{\infty} e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(x) dx = (-1)^n 2^{n-1} \pi^{\frac{1}{2}} \beta^{2n} e^{-\frac{1}{2}\beta^2}.$$

ET II 289(6)a

$$4. \int_0^{\infty} e^{-x^2} \cos(\sqrt{2}\beta x) H_{2n}(ax) dx = 2^{-1} \pi^{\frac{1}{2}} (1-a^2)^n e^{-\frac{1}{2}\beta^2} H_{2n} \left[\frac{a\beta}{\sqrt{2}(a^2-1)^{\frac{1}{2}}} \right].$$

ET II 290(19)a

$$5. \int_0^{\infty} e^{-y^2} [H_n(y)]^2 \cos(\sqrt{2}\beta y) dy = \pi^{\frac{1}{2}} 2^{n-1} n! e^{-\frac{\beta^2}{2}} L_n(\beta^2).$$

$$6. \int_0^{\infty} e^{-x^2} \sin(bx) H_n(x) H_{n+2m+1}(x) dx = 2^n (-1)^m \sqrt{\frac{\pi}{2}} n! b^{2m+1} e^{-\frac{b^2}{4}} L_n^{2m+1} \left(\frac{b^2}{2} \right) \quad [b > 0].$$

ET I 39(11)a

$$7. \int_0^{\infty} e^{-x^2} \cos(bx) H_n(x) H_{n+2m}(x) dx = 2^{n-\frac{1}{2}} \sqrt{\frac{\pi}{2}} n! (-1)^m b^{2m} e^{-\frac{b^2}{4}} L_n^{2m} \left(\frac{b^2}{2} \right) \quad [b > 0].$$

ET I 39(11)a

7.389

$$\int_0^{\pi} (\cos x)^n H_{2n}[a(1 - \sec x)^{\frac{1}{2}}] dx = 2^{-n} (-1)^n \pi \frac{(2n)!}{(n!)^2} [H_n(a)]^2.$$

ET II 292(31)

7.39 Jacobi polynomials

7.391

$$\begin{aligned} 1. \int_{-1}^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx &= \\ &= 0 \quad [m \neq n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]; \\ &= \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (\alpha+\beta+1+2n) \Gamma(\alpha+\beta+n+1)} \quad [m = n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1]. \end{aligned}$$

ET II 285(5, 9)

847

$$\begin{aligned} 2. \int_{-1}^1 (1-x)^\varrho (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx &= \frac{2^{\varrho+\sigma+1} \Gamma(\varrho+1) \Gamma(\sigma+1) \Gamma(n+1+\alpha)}{n! \Gamma(\varrho+\sigma+2) \Gamma(1+\alpha)} \times \\ &\quad \times {}_3F_2(-n, \alpha+\beta+n+1, \varrho+1; \alpha+1, \varrho+\sigma+2; 1) \\ &\quad [\operatorname{Re} \varrho > -1, \quad \operatorname{Re} \sigma > -1]. \end{aligned}$$

ET II 284(3)

$$3.^6 \int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+1) \Gamma(\sigma-\beta+1)}{n! \Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2)} \quad [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \sigma > -1].$$

$$4. \int_{-1}^1 (1-x)^\varrho (1+x)^\beta P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\beta+\varrho+1} \Gamma(\varrho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\varrho+n)}{n! \Gamma(\alpha-\varrho) \Gamma(\beta+\varrho+n+2)} \\ [\operatorname{Re} \varrho > -1, \quad \operatorname{Re} \beta > -1].$$

ET II 284(2)

$$5. \int_{-1}^1 (1-x)^{\alpha-1} (1+x)^\beta [P_n^{(\alpha, \beta)}(x)]^2 dx = \frac{2^{\alpha+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! \alpha \Gamma(\alpha+\beta+n+1)} \\ [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > -1].$$

ET II 285(6)

$$6. \int_{-1}^1 (1-x)^{2\alpha} (1+x)^\beta [P_n^{(\alpha, \beta)}(x)]^2 dx = \frac{2^{4\alpha+\beta+1} \Gamma\left(\alpha + \frac{1}{2}\right) [\Gamma(\alpha+n+1)]^2 \Gamma(\beta+2n+1)}{\sqrt{\pi} (n!)^2 \Gamma(\alpha+1) \Gamma(2\alpha+\beta+2n+2)} \\ \left[\operatorname{Re} \alpha > -\frac{1}{2}, \quad \operatorname{Re} \beta > -1 \right].$$

ET II 285(7)

$$7. \int_{-1}^1 (1-x)^\varrho (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_n^{(\varrho, \beta)}(x) dx = \\ = \frac{2^{\varrho+\beta+1} \Gamma(\varrho+n+1) \Gamma(\beta+n+1) \Gamma(\alpha+\beta+2n+1)}{n! \Gamma(\beta+\varrho+2n+2) \Gamma(\alpha+\beta+n+1)} \quad [\operatorname{Re} \varrho > -1, \quad \operatorname{Re} \beta > -1].$$

ET II 285(10)

$$8. \int_{-1}^1 (1-x)^{\varrho-1} (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_n^{(\varrho, \beta)}(x) dx = \frac{2^{\varrho+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1) \Gamma(\varrho)}{n! \Gamma(\alpha+1) \Gamma(\varrho+\beta+n+1)} \\ [\operatorname{Re} \beta > -1, \quad \operatorname{Re} \varrho > 0].$$

ET II 286(11)

$$9.7 \int_{-1}^1 (1-x)^\alpha (1+x)^\sigma P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \sigma)}(x) dx = \\ = \frac{2^{\alpha+\sigma+1} \Gamma(\alpha+n+1) \Gamma(\alpha+\beta+m+n+1) \Gamma(\sigma+m+1) \Gamma(\sigma-\beta+1)}{m! (n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\alpha+\sigma+m+n+2) \Gamma(\sigma-\beta+m-n+1)} \\ [\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \sigma > -1].$$

ET II 286(12)

$$10.6 \int_{-1}^1 (1-x)^\varrho (1+x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\varrho, \beta)}(x) dx = \\ = \frac{2^{\beta+\varrho+1} \Gamma(\alpha+\beta+m+n+1) \Gamma(\beta+n+1) \Gamma(\varrho+m+1)}{m! (n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\beta+\varrho+m+n+2)} \frac{\Gamma(\alpha-\varrho-m+n)}{\Gamma(\alpha-\varrho)} \\ [\operatorname{Re} \varrho > -1, \quad \operatorname{Re} \beta > -1].$$

$$11. \int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha, \beta)}(y) dy = \frac{1}{2n} [P_{n-1}^{(\alpha+1, \beta+1)}(0) - (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x)].$$

EH II 173(38)

7.392

$$1. \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)} {}_3F_2 \left(-n, n+\alpha+\beta+1, \lambda; \alpha+1, \lambda+\mu; \frac{1}{2}\gamma \right) \\ [\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0].$$

ET II 192(46)a

$$2. \int_0^1 x^{\lambda-1} (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(\gamma x - 1) dx = (-1)^n \frac{\Gamma(\beta+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\beta+1)\Gamma(\lambda+\mu)} {}_3F_2 \left(-n, n+\alpha+\beta+1, \lambda; \beta+1, \lambda+\mu; \frac{1}{2}\gamma \right) \\ [\operatorname{Re} \lambda > 0, \operatorname{Re} \mu > 0].$$

ET II 192(47)a

$$3. \int_0^1 x^\alpha (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(1-\gamma x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} P_n^{(\alpha+\mu, \beta-\mu)}(1-\gamma) \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} \mu > 0].$$

ET II 191(43)a

$$4. \int_0^1 x^\beta (1-x)^{\mu-1} P_n^{(\alpha, \beta)}(\gamma x - 1) dx = \frac{\Gamma(\beta+n+1)\Gamma(\mu)}{\Gamma(\beta+\mu+n+1)} P_n^{(\alpha-\mu, \beta+\mu)}(\gamma-1) \\ [\operatorname{Re} \beta > -1, \operatorname{Re} \mu > 0].$$

ET II 191(44)a

7.393

$$1. \int_0^1 (1-x^2)^\nu \sin bx P_{2n+1}^{(\nu, \nu)}(x) dx = \frac{(-1)^n \sqrt{\pi} \Gamma(2n+\nu+2) J_{2n+\nu+\frac{3}{2}}(b)}{2^{\frac{1}{2}-\nu} (2n+1)! b^{\nu+\frac{1}{2}}} \\ [b > 0, \operatorname{Re} \nu > -1].$$

ET I 94(5)

7.41- 7.42 Laguerre polynomials

7.411

$$1. \int_0^t L_n(x) dx = L_n(t) - L_{n+1}(t)/(n+1).$$

MO 110

$$2. \int_0^t L_n^\alpha(x) dx = L_n^\alpha(t) - L_{n+1}^\alpha(t) - \binom{n+\alpha}{n} + \binom{n+1+\alpha}{n+1}.$$

EH II 189(16)a

849

$$3. \int_0^t L_{n-1}^{\alpha+1}(x) dx = -L_n^\alpha(t) + \binom{n+\alpha}{n}.$$

EH II 189(15)a

$$4. \int_0^t L_m(x)L_n(t-x) dx = L_{m+n}(t) - L_{m+n+1}(t).$$

EH II 191(31)

$$5. \sum_{k=0}^{\infty} \left[\int_0^t (L_k(x)/k!) dx \right]^2 = e^t - 1 \quad [t \geq 0].$$

MO 110

7.412

$$1. \int_0^1 (1-x)^{\mu-1} x^\alpha L_n^\alpha(ax) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\mu)}{\Gamma(\alpha+\mu+n+1)} L_n^{\alpha+\mu}(a) \quad [\operatorname{Re} \alpha > -1, \operatorname{Re} \mu > 0].$$

EH II 191(30)A, BU 129(14c)

$$2. \int_0^1 (1-x)^{\mu-1} x^{\lambda-1} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha+n+1)\Gamma(\lambda)\Gamma(\mu)}{n!\Gamma(\alpha+1)\Gamma(\lambda+\mu)} {}_2F_2(-n, \lambda; \alpha+1, \lambda+\mu; \beta)$$

[Re $\lambda > 0$, Re $\mu > 0$].

7.413

$$\int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta[(1-x)y] dx = \frac{(m+n)! \Gamma(\alpha+m+1) \Gamma(\beta+n+1)}{m! n! \Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y) \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} \beta > -1].$$

ET II 293(7)

7.414

$$1. \int_0^\infty e^{-x} L_n^\alpha(x) dx = e^{-y} [L_n^\alpha(y) - L_{n-1}^\alpha(y)].$$

EH II 191(29)

$$2. \int_0^\infty e^{-bx} L_n(\lambda x) L_n(\mu x) dx = \frac{(b-\lambda-\mu)^n}{b^{n+1}} P_n \left[\frac{b^2 - (\lambda+\mu)b + 2\lambda\mu}{b(b-\lambda-\mu)} \right] \quad [\operatorname{Re} b > 0].$$

ET I 175(34)

$$3. \int_0^\infty e^{-x\alpha} L_n^\alpha(x) L_m^\alpha(x) dx = 0 \quad [m \neq n, \operatorname{Re} \alpha > -1]; \\ = \frac{\Gamma(\alpha+n+1)}{n!} \quad [m = n, \operatorname{Re} \alpha > 0].$$

BU 115(8), ET II 292(2)

BU 115(8), ET II 293(3)

850

$$4. \int_0^\infty e^{-bx} x^\alpha L_n^\alpha(\lambda x) L_m^\alpha(\mu x) dx = \frac{\Gamma(m+n+\alpha+1)}{m! n!} \frac{(b-\lambda)^n (b-\mu)^m}{b^{m+n+\alpha+1}} \times \\ \times F \left[-m, -n; -m-n-\alpha, \frac{b(b-\lambda-\mu)}{(b-\lambda)(b-\mu)} \right] \\ [\operatorname{Re} \alpha > -1, \operatorname{Re} b > 0].$$

ET I 175(35)

$$4^* (1) \int_0^\infty e^{-x} x^{\alpha+1/2} L_n^\alpha(x) L_m^\alpha(x) dx = \frac{\Gamma(\alpha+n+1)^2 \Gamma(\alpha+m+1) \Gamma\left(\alpha + \frac{3}{2}\right) \Gamma\left(m - \frac{1}{2}\right)}{n! m! \Gamma(\alpha+1) \Gamma\left(-\frac{1}{2}\right)} \times \\ \times {}_3F_2 \left(-n, \alpha + \frac{3}{2}, \frac{3}{2}; \alpha + 1, \frac{3}{2} - m; 1 \right).$$

$$5. \int_0^\infty e^{-bx} L_n^a(x) dx = \sum_{m=0}^n \binom{a+m-1}{m} \frac{(b-1)^{n-m}}{b^{n-m+1}} \quad [\operatorname{Re} b > 0].$$

$$6. \int_0^\infty e^{-bx} L_n(x) dx = (b-1)^n b^{-n-1} \quad [\operatorname{Re} b > 0].$$

ET I 174(25)

$$7. \int_0^\infty e^{-st} t^\beta L_n^\alpha(t) dt = \frac{\Gamma(\beta+1)\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} s^{-\beta-1} F\left(-n, \beta+1; \alpha+1; \frac{1}{s}\right) \\ [\operatorname{Re} \beta > -1, \operatorname{Re} s > 0].$$

BU 119(4b), EH II 191(133)

$$8. \int_0^\infty e^{-st} t^\alpha L_n^\alpha(t) dt = \frac{\Gamma(\alpha+n+1)(s-1)^n}{n!s^{\alpha+n+1}} \quad [\operatorname{Re} \alpha > -1, \operatorname{Re} s > 0].$$

EH II 191(32), MO 176a

$$9. \int_0^\infty e^{-x} x^{\alpha+\beta} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n} (\alpha+\beta)! \binom{\alpha+m}{n} \binom{\beta+n}{m} \quad [\operatorname{Re}(\alpha+\beta) > -1].$$

ET II 293(4)

$$10.6 \int_0^\infty e^{-bx} x^{2a} [L_n^\alpha(x)]^2 dx = \frac{2^{2a} \Gamma\left(a + \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{\pi(n!)^2 b^{2a+1}} \times \\ \times F\left(-n, a + \frac{1}{2}; \frac{1}{2} - n; \left(1 - \frac{2}{b}\right)^2\right) \Gamma(a+n+1) \\ \left[\operatorname{Re} a > -\frac{1}{2}, \operatorname{Re} b > 0 \right].$$

ET I 174(30)

$$11. \int_0^\infty e^{-x} x^{\gamma-1} L_n^\mu(x) dx = \frac{\Gamma(\gamma)\Gamma(1+\mu+n-\gamma)}{n!\Gamma(1+\mu-\gamma)} \quad [\operatorname{Re} \gamma > 0].$$

BU 120(4b)

$$12. \int_0^\infty e^{-x(s+\frac{a_1+a_2}{2})} x^{\mu+\beta} L_k^\mu(a_1x) L_k^\mu(a_2x) dx = \\ = \frac{\Gamma(1+\mu+\beta)\Gamma(1+\mu+k)}{k!k!\Gamma(1+\mu)} \left\{ \frac{d^k}{dh^k} \left[\frac{F\left(\frac{1+\mu+\beta}{2}, 1 + \frac{\mu+\beta}{2}; 1+\mu; \frac{A^2}{B^2}\right)}{(1-h)^{1+\mu} B^{1+\mu+\beta}} \right] \right\}_{h=0}, \\ A^2 = \frac{4a_1a_2h}{(1-h)^2}; \quad B = s + \frac{a_1+a_2}{2} \frac{1+h}{1-h}$$

$$13. \int_0^\infty e^{-x(s+\frac{a_1+a_2}{2})} x^\mu L_k^\mu(a_1x) L_k^\mu(a_2x) dx = \frac{\Gamma(1+\mu+k)}{b_0^{1+\mu+k}} \cdot \frac{b_2^k}{k!} \cdot P_k^{(\mu,0)}\left(\frac{b_1^2}{b_0b_2}\right),$$

$$b_0 = s + \frac{a_1+a_2}{2}, \quad b_1^2 = b_0b_2 + 2a_1a_2, \quad b_2 = s - \frac{a_1+a_2}{2}$$

$$\left[\operatorname{Re} \mu > -1, \quad \operatorname{Re}\left(s + \frac{a_1+a_2}{2}\right) > 0 \right].$$

BU 144(22)

7.415

$$\int_0^1 (1-x)^{\mu-1} x^{\lambda-1} e^{-\beta x} L_n^\alpha(\beta x) dx = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} B(\lambda, \mu) {}_2F_2(\alpha+n+1, \lambda; \alpha+1, \lambda+\mu; -\beta)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0].$$

ET II 193(51)a

7.416

$$\int_{-\infty}^\infty x^{m-n} \exp\left[-\frac{1}{2}(x-y)^2\right] L_n^{m-n}(x^2) dx = \frac{(2\pi)^{\frac{1}{2}}}{n!} i^{n-m} 2^{-\frac{n+m}{2}} H_n\left(\frac{iy}{\sqrt{2}}\right) H_m\left(\frac{iy}{\sqrt{2}}\right).$$

BU 149(15b), ET II 293(8)a

7.417

$$1. \int_0^\infty x^{\nu-2n-1} e^{-ax} \sin(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n i \Gamma(\nu) \frac{b^{2n} [(a-ib)^{-\nu} - (a+ib)^{-\nu}]}{2(2n)!}$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n].$$

ET I 95(12)

$$2. \int_0^\infty x^{\nu-2n-2} e^{-ax} \sin(bx) L_{2n+1}^{\nu-2n-2}(ax) dx = (-1)^{n+1} \Gamma(\nu) \frac{b^{2n+1} [(a+ib)^{-\nu} + (a-ib)^{-\nu}]}{2(2n+1)!}$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n+1].$$

ET I 95(13)

$$3. \int_0^\infty x^{\nu-2n} e^{-ax} \cos(bx) L_{2n-1}^{\nu-2n}(ax) dx = i(-1)^{n+1} \Gamma(\nu) \frac{b^{2n-1} [(a-ib)^{-\nu} - (a+ib)^{-\nu}]}{2(2n-1)!}$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n-1].$$

ET I 39(12)

$$4. \int_0^\infty x^{\nu-2n-1} e^{-ax} \cos(bx) L_{2n}^{\nu-2n-1}(ax) dx = (-1)^n \Gamma(\nu) \frac{b^{2n} [(a+ib)^{-\nu} + (a-ib)^{-\nu}]}{2(2n)!}$$

$$[b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 2n].$$

7.418

$$1. \int_0^{\infty} e^{-\frac{1}{2}x^2} \sin(bx) L_n(x^2) dx = (-1)^n \frac{i}{2} n! \frac{1}{\sqrt{2\pi}} \{ [D_{-n-1}(ib)]^2 - [D_{-n-1}(-ib)]^2 \} \quad [b > 0].$$

ET I 95(14)

$$2. \int_0^{\infty} e^{-\frac{1}{2}x^2} \cos(bx) L_n(x^2) dx = \sqrt{\frac{\pi}{2}} (n!)^{-1} e^{-\frac{1}{2}b^2} 2^{-n} \left[H_n \left(\frac{b}{\sqrt{2}} \right) \right]^2 \quad [b > 0].$$

ET I 39(14)

$$3. \int_0^{\infty} x^{2n+1} e^{-\frac{1}{2}x^2} \sin(bx) L_n^{n+\frac{1}{2}} \left(\frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left(\frac{b^2}{2} \right) \quad [b > 0].$$

ET I 95(15)

852

$$4. \int_0^{\infty} x^{2n} e^{-\frac{1}{2}x^2} \cos(bx) L_n^{n-\frac{1}{2}} \left(\frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} L_n^{n+\frac{1}{2}} \left(\frac{1}{2}b^2 \right) \quad [b > 0].$$

ET I 39(16)

$$5. \int_0^{\infty} x e^{-\frac{1}{2}x^2} L_n^{\alpha} \left(\frac{1}{2}x^2 \right) L_n^{\frac{1}{2}-\alpha} \left(\frac{1}{2}x^2 \right) \sin(xy) dx = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} y e^{-\frac{1}{2}y^2} L_n^{\alpha} \left(\frac{1}{2}y^2 \right) L_n^{\frac{1}{2}-\alpha} \left(\frac{1}{2}y^2 \right).$$

ET II 294(11)

$$6. \int_0^{\infty} e^{-\frac{1}{2}x^2} L_n^{\alpha} \left(\frac{1}{2}x^2 \right) L_n^{-\frac{1}{2}-\alpha} \left(\frac{1}{2}x^2 \right) \cos(xy) dx = \left(\frac{\pi}{2} \right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{\alpha} \left(\frac{1}{2}y^2 \right) L_n^{-\alpha-\frac{1}{2}} \left(\frac{1}{2}y^2 \right).$$

ET II 294(12)

7.419

$$\begin{aligned} \int_0^{\infty} x^{n+2\nu-\frac{1}{2}} \exp[-(1+a)x] L_n^{2\nu}(ax) K_{\nu}(x) dx &= \\ &= \frac{\pi^{\frac{1}{2}} \Gamma \left(n + \nu + \frac{1}{2} \right) \Gamma \left(n + 3\nu + \frac{1}{2} \right)}{2^{n+2\nu+\frac{1}{2}} n! \Gamma(2\nu+1)} F \left(n + \nu + \frac{1}{2}, n + 3\nu + \frac{1}{2}; 2\nu + 1; -\frac{1}{2}a \right) \\ &\quad \left[\operatorname{Re} a > -2, \operatorname{Re}(n + \nu) > -\frac{1}{2}, \operatorname{Re}(n + 3\nu) > -\frac{1}{2} \right]. \end{aligned}$$

$$1. \int_0^{\infty} x e^{-\frac{1}{2}\alpha x^2} L_n \left(\frac{1}{2}\beta x^2 \right) J_0(xy) dx = \frac{(\alpha - \beta)^n}{\alpha^{n+1}} e^{-\frac{1}{2\alpha}y^2} L_n \left[\frac{\beta y^2}{2\alpha(\beta - \alpha)} \right] \quad [y > 0, \quad \operatorname{Re} \alpha > 0].$$

ET II 13(4)a

$$2. \int_0^{\infty} x e^{-x^2} L_n(x^2) J_0(xy) dx = \frac{2^{-2n-1}}{n!} y^{2n} e^{-\frac{1}{4}y^2}.$$

ET II 13(5)

$$3. \int_0^{\infty} x^{2n+\nu+1} e^{-\frac{1}{2}x^2} L_n^{\nu+n} \left(\frac{1}{2}x^2 \right) J_{\nu}(xy) dx = y^{2n+\nu} e^{-\frac{1}{2}y^2} L_n^{\nu+n} \left(\frac{1}{2}y^2 \right) \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

MO 183

$$4. \int_0^{\infty} x^{\nu+1} e^{-\beta x^2} L_n^{\nu}(\alpha x^2) J_{\nu}(xy) dx = 2^{-\nu-1} \beta^{-\nu-n-1} (\beta - \alpha)^n y^{\nu} e^{-\frac{y^2}{4\beta}} L_n^{\nu} \left[\frac{\alpha y^2}{4\beta(\alpha - \beta)} \right].$$

ET II 43(5)

$$5. \int_0^{\infty} e^{-\frac{1}{2q}x^2} x^{\nu+1} L_n^{\nu} \left[\frac{x^2}{2q(1-q)} \right] J_{\nu}(xy) dx = \frac{q^{n+\nu+1}}{(q-1)^n} e^{-\frac{qy^2}{2}} y^{\nu} L_n^{\nu} \left(\frac{y^2}{2} \right) \quad [\nu > 0].$$

MO 183

7.422

$$\begin{aligned} 1. \int_0^{\infty} x^{\nu+1} e^{-\beta x^2} [L_n^{\frac{1}{2}\nu}(\alpha x^2)]^2 J_{\nu}(xy) dx &= \\ &= \frac{y^{\nu}}{\pi n!} \Gamma \left(n + 1 + \frac{1}{2}\nu \right) (2\beta)^{-\nu-1} e^{-\frac{y^2}{4\beta}} \times \\ &\times \sum_{l=0}^n \frac{(-1)^l \Gamma \left(n - l + \frac{1}{2} \right) \Gamma \left(l + \frac{1}{2} \right)}{\Gamma \left(l + 1 + \frac{1}{2}\nu \right) (n-l)!} \left(\frac{2\alpha - \beta}{\beta} \right)^{2l} L_{2l}^{\nu} \left[\frac{\alpha y^2}{2\beta(2\alpha - \beta)} \right] \\ & \quad [y > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1]. \end{aligned}$$

ET II 43(7)

853

$$\begin{aligned} 2.7 \int_0^{\infty} x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}(\alpha x^2) L_n^{\sigma}(\alpha x^2) J_{\nu}(xy) dx &= \\ &= (-1)^{m+n} (2\alpha)^{-\nu-1} y^{\nu} e^{-\frac{y^2}{4\alpha}} L_n^{\sigma+m-n} \left(\frac{y^2}{4\alpha} \right) L_m^{\nu-\sigma+m-n} \left(\frac{y^2}{4\alpha} \right) \\ & \quad [y > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1] \end{aligned}$$

$$1. \int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n+1}\left(\frac{x}{2\sqrt{2}}\right) \sin(xy) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n+1}\left(\frac{y}{2\sqrt{2}}\right).$$

ET II 294(13)a

$$2. \int_0^\infty e^{-\frac{1}{2}x^2} L_n\left(\frac{1}{2}x^2\right) H_{2n}\left(\frac{x}{2\sqrt{2}}\right) \cos(xy) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n\left(\frac{1}{2}y^2\right) H_{2n}\left(\frac{y}{2\sqrt{2}}\right).$$

ET II 294(14)a

7.5 Hypergeometric Functions

7.51 Combinations of hypergeometric functions and powers

7.511

$$\int_0^\infty F(a, b; c; -z) z^{-s-1} dx = \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(c)\Gamma(-s)}{\Gamma(a)\Gamma(b)\Gamma(c+s)}$$

[$c \neq 0, -1, -2, \dots$, $\operatorname{Re} s < 0$, $\operatorname{Re}(a+s) > 0$, $\operatorname{Re}(b+s) > 0$].

EH I 79(4)

7.512

$$1. \int_0^1 x^{\alpha-\gamma} (1-x)^{\gamma-\beta-1} F(\alpha, \beta; \gamma; x) dx =$$

$$= \frac{\Gamma\left(1 + \frac{\alpha}{2}\right) \Gamma(\gamma) \Gamma(\alpha - \gamma + 1) \Gamma\left(\gamma - \frac{\alpha}{2} - \beta\right)}{\Gamma(1 + \alpha) \Gamma\left(1 + \frac{\alpha}{2} - \beta\right) \Gamma\left(\gamma - \frac{\alpha}{2}\right)}$$

[$\operatorname{Re} \alpha + 1 > \operatorname{Re} \gamma > \operatorname{Re} \beta$, $\operatorname{Re}\left(\gamma - \frac{\alpha}{2} - \beta\right) > 0$].

ET II 398(1)

$$2. \int_0^1 x^{\varrho-1} (1-x)^{\beta-\gamma-n} F(-n, \beta; \gamma; x) dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\beta-\gamma+1)\Gamma(\gamma-\varrho+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\varrho)\Gamma(\beta-\gamma+\varrho+1)}$$

[$n = 0, 1, 2, \dots$; $\operatorname{Re} \varrho > 0$, $\operatorname{Re}(\beta-\gamma) > n-1$].

ET II 398(2)

$$3. \int_0^1 x^{\varrho-1} (1-x)^{\beta-\varrho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\beta-\varrho)\Gamma(\gamma-\alpha-\varrho)}{\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\varrho)}$$

[$\operatorname{Re} \varrho > 0$, $\operatorname{Re}(\beta-\varrho) > 0$, $\operatorname{Re}(\gamma-\alpha-\varrho) > 0$].

$$4. \int_0^1 x^{\gamma-1} (1-x)^{\varrho-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma + \varrho - \alpha - \beta)}{\Gamma(\gamma + \varrho - \alpha)\Gamma(\gamma + \varrho - \beta)}$$

[Re $\gamma > 0$, Re $\varrho > 0$, Re($\gamma + \varrho - \alpha - \beta$) > 0].

ET II 399(4)

854

$$5. \int_0^1 x^{\varrho-1} (1-x)^{\sigma-1} F(\alpha, \beta; \gamma; x) dx = \frac{\Gamma(\varrho)\Gamma(\sigma)}{\Gamma(\varrho + \sigma)} {}_3F_2(\alpha, \beta, \varrho; \gamma, \varrho + \sigma; 1)$$

[Re $\varrho > 0$, Re $\sigma > 0$, Re($\gamma + \sigma - \alpha - \beta$) > 0].

ET II 399(5)

$$6.* \int_0^1 x^{\lambda-1} (1-x)^{\beta-\lambda-1} F\left(\alpha, \beta; \lambda; \frac{zx}{b}\right) dx = B(\lambda, \beta - \lambda)(1 - z/b)^{-\alpha}.$$

BU 9

$$7. \int_0^1 x^{\gamma-1} (1-x)^{\delta-\gamma-1} F(\alpha, \beta; \gamma; xz) F(\delta - \alpha, \delta - \beta; \delta - \gamma; (1-x)\zeta) dx =$$

$$= \frac{\Gamma(\gamma)\Gamma(\delta - \gamma)}{\Gamma(\delta)} (1 - \zeta)^{2\alpha - \delta} F(\alpha, \beta; \delta; z + \zeta - z\zeta)$$

[$0 < \text{Re } \gamma < \text{Re } \delta$, $|\arg(1 - z)| < \pi$, $|\arg(1 - \zeta)| < \pi$].

ET II 400(11)

$$8. \int_0^1 x^{\gamma-1} (1-x)^{\epsilon-1} (1-xz)^{-\delta} F(\alpha, \beta; \gamma; xz) F\left[\delta, \beta - \gamma; \epsilon; \frac{(1-x)z}{(1-xz)}\right] dx =$$

$$= \frac{\Gamma(\gamma)\Gamma(\epsilon)}{\Gamma(\gamma + \epsilon)} F(\alpha + \delta, \beta; \gamma + \epsilon; z)$$

[Re $\gamma > 0$, Re $\epsilon > 0$, $|\arg(z - 1)| < \pi$].

ET II 400(12), Eh I 78(3)

$$9. \int_0^1 x^{\gamma-1} (1-x)^{\varrho-1} (1-zx)^{-\sigma} F(\alpha, \beta; \gamma; x) dx =$$

$$= \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma + \varrho - \alpha - \beta)}{\Gamma(\gamma + \varrho - \alpha)\Gamma(\gamma + \varrho - \beta)} (1 - z)^{-\sigma} \times$$

$$\times {}_3F_2\left(\varrho, \sigma, \gamma + \varrho - \alpha - \beta; \gamma + \varrho - \alpha, \gamma + \varrho - \beta; \frac{z}{z - 1}\right)$$

[Re $\gamma > 0$, Re $\varrho > 0$, Re($\gamma + \varrho - \alpha - \beta$) > 0 , $|\arg(1 - z)| < \pi$].

$$10. \int_0^\infty x^{\gamma-1}(x+z)^{-\sigma} F(\alpha, \beta; \gamma; -x) dx = \frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\Gamma(\alpha+\beta-\gamma+\sigma)} \times \\ \times F(\alpha-\gamma+\sigma, \beta-\gamma+\sigma; \alpha+\beta-\gamma+\sigma; 1-z) \\ [\operatorname{Re} \gamma > 0, \operatorname{Re}(\alpha-\gamma+\sigma) > 0, \operatorname{Re}(\beta-\gamma+\sigma) > 0, |\arg z| < \pi].$$

$$11. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; \nu, b_2, \dots, b_q; ax) dx = \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_pF_q(a_1, \dots, a_p; \mu+\nu, b_2, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, p \leq q+1; \text{ if } p = q+1, \text{ then } |a| < 1].$$

ET II 200(94)

$$12. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; ax) dx = \\ = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} {}_{p+1}F_{q+1}(\nu, a_1, \dots, a_p; \mu+\nu, b_1, \dots, b_q; a) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0, p \leq q+1, \text{ if } p = q+1, \text{ then } |a| < 1].$$

ET II 200(95)

855
7.513

$$\int_0^1 x^{s-1}(1-x^2)^\nu F(-n, a; b; x^2) dx = \frac{1}{2} B\left(\nu+1, \frac{s}{2}\right) {}_3F_2\left(-n, a, \frac{s}{2}; b, \nu+1+\frac{s}{2}; 1\right) \\ [\operatorname{Re} s > 0, \operatorname{Re} \nu > -1].$$

ET I 336(4)

7.52 Combinations of hypergeometric functions and exponentials

7.521

$$\int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q, t) dt = \frac{1}{s} {}_{p+1}F_q(1, a_1, \dots, a_p; b_1, \dots, b_q, s^{-1}) \quad [p \leq q].$$

EH I 192

7.522

$$1. \int_0^\infty e^{-\lambda x} x^{\gamma-1} {}_2F_1(\alpha, \beta; \delta; -x) dx = \frac{\Gamma(\delta)\lambda^{-\gamma}}{\Gamma(\alpha)\Gamma(\beta)} E(\alpha, \beta, \gamma; \delta; \lambda) \\ [\operatorname{Re} \lambda > 0, \operatorname{Re} \gamma > 0].$$

EH I 205(10)

$$2. \int_0^\infty e^{-bx} x^{a-1} F\left(\frac{1}{2} + \nu, \frac{1}{2} - \nu; a; -\frac{x}{2}\right) dx = 2^a e^b \frac{1}{\sqrt{\pi}} \Gamma(a) (2b)^{\frac{1}{2}-a} K_\nu(b) \\ [\operatorname{Re} a > 0, \operatorname{Re} b > 0].$$

$$3. \int_0^\infty e^{-bx} x^{\gamma-1} F(2\alpha, 2\beta; \gamma; -\lambda x) dx = \Gamma(\gamma) b^{-\gamma} \left(\frac{b}{\lambda}\right)^{\alpha+\beta-\frac{1}{2}} e^{\frac{b}{2\lambda}} W_{\frac{1}{2}-\alpha-\beta, \alpha-\beta} \left(\frac{b}{\lambda}\right)$$

$$[\operatorname{Re} b > 0, \quad \operatorname{Re} \gamma > 0, \quad |\arg \lambda| < \pi].$$

BU 78(30), ET I 212(4)

$$4.^6 \int_0^\infty e^{-xt} t^{b-1} F(a, a-c+1; b; -t) dt = x^{a-b} \Gamma(b) \Psi(a, c; x) \quad [\operatorname{Re} b > 0, \quad \operatorname{Re} x > 0].$$

EH I 273(11)

$$5. \int_0^\infty e^{-x} x^{s-1} {}_pF_q(a_1, \dots, a_p, b_1, \dots, b_q; ax) dx = \Gamma(s) {}_{p+1}F_q(s, a_1, \dots, a_p; b_1, \dots, b_q; a)$$

$$[p < q, \quad \operatorname{Re} s > 0].$$

ET I 337(11)

$$6. \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2(-n, n+1; 1, \beta; x) dx = \Gamma(\beta) \mu^{-\beta} P_n \left(1 - \frac{2}{\mu}\right) \quad [\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0].$$

ET I 218(6)

$$7. \int_0^\infty x^{\beta-1} e^{-\mu x} {}_2F_2\left(-n, n; \beta, \frac{1}{2}; x\right) dx = \Gamma(\beta) \mu^{-\beta} \cos \left[2n \arcsin \left(\frac{1}{\sqrt{\mu}}\right)\right]$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \beta > 0].$$

ET I 218(7)

$$8. \int_0^\infty x^{\varrho_n-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \varrho_1, \dots, \varrho_n; \lambda x) dx = \Gamma(\varrho_n) \mu^{-\varrho_n} {}_mF_{n-1}\left(a_1, \dots, a_m; \varrho_1, \dots, \varrho_{n-1}; \frac{\lambda}{\mu}\right)$$

$$[m \leq n; \quad \operatorname{Re} \varrho_n > 0, \quad \operatorname{Re} \mu > 0, \quad \text{if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \quad \text{if } m = n].$$

ET I 219(16)a

$$9. \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n(a_1, \dots, a_m; \varrho_1, \dots, \varrho_n; \lambda x) dx = \Gamma(\sigma) \mu^{-\sigma} {}_{m+1}F_n\left(a_1, \dots, a_m, \sigma; \varrho_1, \dots, \varrho_n; \frac{\lambda}{\mu}\right)$$

$$[m \leq n, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re} \mu > 0, \quad \text{if } m < n; \operatorname{Re} \mu > \operatorname{Re} \lambda, \quad \text{if } m = n].$$

ET I 219(17)

856
7.523

$$\int_0^1 x^{\gamma-1}(1-x)^{\varrho-1}e^{-xz}F(\alpha, \beta; \gamma; x) dx =$$

$$= \frac{\Gamma(\gamma)\Gamma(\varrho)\Gamma(\gamma + \varrho - \alpha - \beta)}{\Gamma(\gamma + \varrho - \alpha)\Gamma(\gamma + \varrho - \beta)}e^{-z} {}_2F_2(\varrho, \gamma + \varrho - \alpha - \beta; \gamma + \varrho - \alpha, \gamma + \varrho - \beta; z)$$

$$[\operatorname{Re} \gamma > 0, \operatorname{Re} \varrho > 0, \operatorname{Re}(\gamma + \varrho - \alpha - \beta) > 0].$$

ET II 400(8)

7.524

$$1. \int_0^\infty e^{-\lambda x} F\left(\alpha, \beta; \frac{1}{2}; -x^2\right) dx = \lambda^{\alpha+\beta-1} S_{1-\alpha-\beta, \alpha-\beta}(\lambda) \quad [\operatorname{Re} \lambda > 0].$$

ET II 401(13)

$$2. \int_0^\infty e^{-st} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; t^2) dx = s^{-1} {}_{p+2}F_q\left(a_1, \dots, a_p, 1, \frac{1}{2}; b_1, \dots, b_q; \frac{4}{s^2}\right) \quad [p < q].$$

MO 176

$$3. \int_0^\infty e^{-st} {}_0F_q\left(\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}, 1; \frac{t^q}{q^q}\right) dt = s^{-1} \exp(s^{-q}).$$

MO 176

7.525

$$1. \int_0^\infty x^{\sigma-1} e^{-\mu x} {}_mF_n[a_1, \dots, a_m; \varrho_1, \dots, \varrho_n; (\lambda x)^k] dx =$$

$$= \Gamma(\sigma)\mu^{-\sigma} {}_{m+k}F_n\left[a_1, \dots, a_m, \frac{\sigma}{k}, \frac{\sigma+1}{k}, \dots, \frac{\sigma+k-1}{k}; \varrho_1, \dots, \varrho_n; \left(\frac{k\lambda}{\mu}\right)^k\right]$$

$$\left[\begin{array}{l} m+k \leq n+1, \quad \operatorname{Re} \sigma > 0; \quad \operatorname{Re} \mu > 0, \quad \text{if } m+k \leq n; \\ \operatorname{Re}\left(\mu + k\lambda e^{\frac{2\pi i}{k}}\right) > 0; \quad r = 0, 1, \dots, k-1 \quad \text{for } m+k = n+1 \end{array} \right].$$

$$1. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} s^{-b} F\left(a, b; a+b-c+1; 1-\frac{1}{s}\right) dx = 2\pi i \frac{\Gamma(a+b-c+1)}{\Gamma(b)\Gamma(b-c+1)} t^{b-1} \Psi(a; c; t)$$

$$\left[\operatorname{Re} b > 0, \quad \operatorname{Re}(b-c) > -1, \quad \gamma > \frac{1}{2} \right].$$

EH I 273(12)

$$2. \int_0^\infty e^{-t} t^{\gamma-1} (x+t)^{-\alpha} (y+t)^{-a'} F\left[a, a'; \gamma; \frac{t(x+y+t)}{(x+t)(y+t)}\right] dt = \Gamma(\gamma) \Psi(a, c; x) \Psi(a', c; y),$$

$$\gamma = a + a' - c + 1 \quad [\operatorname{Re} \gamma > 0, \quad xy \neq 0].$$

EH I 287(21)

857

$$3. \int_0^\infty x^{\gamma-1} (x+y)^{-\alpha} (x+z)^{-\beta} e^{-x} F\left[\alpha, \beta; \gamma; \frac{x(x+y+z)}{(x+y)(x+z)}\right] dx =$$

$$= \Gamma(\gamma) (zy)^{-\frac{1}{2}-\mu} e^{\frac{y+z}{2}} W_{\nu, \mu}(y) W_{\lambda, \mu}(z),$$

$$2\nu = 1 - \alpha + \beta - \gamma; \quad 2\lambda = 1 + \alpha - \beta - \gamma; \quad 2\mu = \alpha + \beta - \gamma$$

$$[\operatorname{Re} \gamma > 0, \quad |\arg y| < \pi, \quad |\arg z| < \pi].$$

ET II 401(15)

7.527

$$1. \int_0^\infty (1-e^{-x})^{\lambda-1} e^{-\mu x} F(\alpha, \beta; \gamma; \delta e^{-x}) dx = B(\mu, \lambda) {}_3F_2(\alpha, \beta, \mu; \gamma, \mu+\lambda; \delta)$$

$$[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0, \quad |\arg(1-\delta)| < \pi].$$

ET I 213(9)

$$2. \int_0^\infty (1-e^{-x})^\mu e^{-\alpha x} F(-n, \mu+\beta+n; \beta; e^{-x}) dx = \frac{B(\alpha, \mu+n+1)B(\alpha, \beta+n-\alpha)}{B(\alpha, \beta-\alpha)}$$

$$[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu > -1].$$

ET I 213(10)

$$3. \int_0^\infty (1-e^{-x})^{\gamma-1} e^{-\mu x} F(\alpha, \beta; \gamma; 1-e^{-x}) dx = \frac{\Gamma(\mu)\Gamma(\gamma-\alpha-\beta+\mu)\Gamma(\gamma)}{\Gamma(\gamma-\alpha+\mu)\Gamma(\gamma-\beta+\mu)}$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \mu > \operatorname{Re}(\alpha+\beta-\gamma), \quad \operatorname{Re} \gamma > 0].$$

ET I 213(11)

$$4. \int_0^\infty (1-e^{-x})^{\gamma-1} e^{-\mu x} F[\alpha, \beta; \gamma; \delta(1-e^{-x})] dx = B(\mu, \gamma) F(\alpha, \beta; \mu+\gamma; \delta)$$

$$[\operatorname{Re} \mu > 0, \quad \operatorname{Re} \gamma > 0, \quad |\arg(1-\delta)| < \pi].$$

ET I 213(12)

7.53 Hypergeometric and trigonometric functions

7.531

$$1. \int_0^\infty x \sin \mu x F\left(\alpha, \beta; \frac{3}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-2} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ \left[\mu > 0, \quad \operatorname{Re} \alpha > \frac{1}{2}, \quad \operatorname{Re} \beta > \frac{1}{2}\right].$$

ET I 115(6)

$$2. \int_0^\infty \cos \mu x F\left(\alpha, \beta; \frac{1}{2}; -c^2 x^2\right) dx = 2^{-\alpha-\beta+1} \pi c^{-\alpha-\beta} \mu^{\alpha+\beta-1} \frac{K_{\alpha-\beta}\left(\frac{\mu}{c}\right)}{\Gamma(\alpha)\Gamma(\beta)} \\ [\mu > 0, \quad \operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad c > 0].$$

ET I 61(9)

7.54 Combinations of hypergeometric and Bessel functions

7.541

$$\int_0^\infty x^{\alpha+\beta-2\nu-1} (x+1)^{-\nu} e^{xz} K_\nu[(x+1)z] F(\alpha, \beta; \alpha+\beta-2\nu; -x) dx = \\ = \pi^{-\frac{1}{2}} \cos(\nu\pi) \Gamma\left(\frac{1}{2}-\alpha+\nu\right) \Gamma\left(\frac{1}{2}-\beta+\nu\right) \Gamma(\gamma) \times \\ \times (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} W_{\frac{1}{2}\gamma, \frac{1}{2}(\beta-\alpha)}(2z), \quad \gamma = \alpha + \beta - 2\nu \\ \left[\operatorname{Re}(\alpha + \beta - 2\nu) > 0, \operatorname{Re}\left(\frac{1}{2} - \alpha + \nu\right) > 0, \operatorname{Re}\left(\frac{1}{2} - \beta + \nu\right) > 0, |\arg z| < \frac{3\pi}{2}\right].$$

ET II 401(16)

858

7.542

$$1. \int_0^\infty x^{\sigma-1} {}_pF_{p-1}(a_1, \dots, a_p; b_1, \dots, b_{p-1}; -\lambda x^2) N_\nu(xy) dx = \\ = \frac{\Gamma(b_1) \dots \Gamma(b_{p-1})}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2, p+3}^{p+2, 1} \left(\frac{y^2}{4\lambda} \middle| \begin{matrix} b_0^*, \dots, b_{p+1}^*, l \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \right), \\ a_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \dots, p; \quad b_0^* = 1 - \frac{\sigma}{2}; \quad b_j^* = b_j - \frac{\sigma}{2}, \\ j = 1, \dots, p-1; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\ \left[|\arg \lambda| < \pi, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0 \right].$$

$$\begin{aligned}
2. \quad & \int_0^\infty x^{\sigma-1} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) N_\nu(xy) dx = \\
& = \frac{\Gamma(b_1) \dots \Gamma(b_p)}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+2, p+3}^{p+2, 1} \left(\frac{y^2}{4\lambda} \middle| \begin{matrix} b_0^*, \dots, b_p^*, l \\ h, k, a_1^*, \dots, a_p^* \end{matrix} \right), \\
b_0^* &= 1 - \frac{\sigma}{2}; \quad a_j^* = a_j - \frac{\sigma}{2}, \quad b_j^* = b_j - \frac{\sigma}{2}; \quad j = 1, \dots, p; \quad h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2} \\
& \left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|, \quad \operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}, \quad y > 0 \right].
\end{aligned}$$

ET II 119(54)

$$\begin{aligned}
3. \quad & \int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) N_\nu(xy) dx = \\
& = -\pi^{-1} 2^{\sigma-1} y^{-\sigma} \cos \left[\frac{\pi}{2} (\sigma - \nu) \right] \Gamma \left(\frac{\sigma + \nu}{2} \right) \Gamma \left(\frac{\sigma - \nu}{2} \right) \times \\
& \quad \times {}_{p+2}F_q \left(a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; -\frac{4\lambda}{y^2} \right) \\
& \quad [y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|].
\end{aligned}$$

ET II 119(55)

$$\begin{aligned}
4. \quad & \int_0^\infty x^{\sigma-1} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\lambda x^2) K_\nu(xy) dx = \\
& = 2^{\sigma-2} y^{-\sigma} \Gamma \left(\frac{\sigma + \nu}{2} \right) \Gamma \left(\frac{\sigma - \nu}{2} \right) {}_{p+2}F_q \left(a_1, \dots, a_p, \frac{\sigma + \nu}{2}, \frac{\sigma - \nu}{2}; b_1, \dots, b_q; \frac{4\lambda}{y^2} \right) \\
& \quad [\operatorname{Re} y > 0, \quad p \leq q - 1, \quad \operatorname{Re} \sigma > |\operatorname{Re} \nu|].
\end{aligned}$$

ET II 153(88)

$$\begin{aligned}
5. \quad & \int_0^\infty x^{2\varrho} {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; -\lambda x^2) J_\nu(xy) dx = \\
& = \frac{2^{2\varrho} \Gamma(b_1) \dots \Gamma(b_p)}{y^{2\varrho+1} \Gamma(a_1) \dots \Gamma(a_p)} G_{p+1, p+2}^{p+1, 1} \left(\frac{y^2}{4\lambda} \middle| \begin{matrix} 1, b_1, \dots, b_p \\ h, a_1, \dots, a_p, k \end{matrix} \right), \\
& \quad h = \frac{1}{2} + \varrho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \varrho - \frac{1}{2}\nu, \\
& \quad \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \varrho < \frac{1}{2} + 2 \operatorname{Re} a_r, \quad r = 1, \dots, p \right].
\end{aligned}$$

ET II 91(18)

859

$$\begin{aligned}
6. \quad & \int_0^\infty x^{2\varrho} {}_{m+1}F_m(a_1, \dots, a_{m+1}; b_1, \dots, b_m; -\lambda^2 x^2) J_\nu(xy) dx = \\
& = \frac{2^{2\varrho} \Gamma(b_1) \dots \Gamma(b_m) y^{-2\varrho-1}}{\Gamma(a_1) \dots \Gamma(a_{m+1})} G_{m+1, m+3}^{m+2, 1} \left(\frac{y^2}{4\lambda^2} \middle| \begin{matrix} 1, b_1, \dots, b_m \\ h, a_1, \dots, a_{m+1}, k \end{matrix} \right), \\
& \quad h = \frac{1}{2} + \varrho + \frac{1}{2}\nu, \quad k = \frac{1}{2} + \varrho - \frac{1}{2}\nu, \\
& \quad \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\varrho + \nu) > -1, \quad \operatorname{Re}(\varrho - a_r) < \frac{1}{4}; \quad r = 1, \dots, m+1 \right].
\end{aligned}$$

$$7. \int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^\delta \Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} y^{-\delta-1} G_{24}^{22} \left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1-\alpha, & 1-\beta \\ \frac{1+\delta+\nu}{2}, & 0, & 1-\gamma, & \frac{1+\delta-\nu}{2} \end{matrix} \right. \right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 - \operatorname{Re} \nu - 2 \min(\operatorname{Re} \alpha, \operatorname{Re} \beta) < \operatorname{Re} \delta < -\frac{1}{2} \right].$$

ET II 82(9)

$$8. \int_0^\infty x^\delta F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^\delta y^{-\delta-1} \Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} G_{24}^{31} \left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1, & \gamma \\ \frac{1+\delta+\nu}{2}, & \alpha, & \beta, & \frac{1+\delta-\nu}{2} \end{matrix} \right. \right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\operatorname{Re} \nu - 1 < \operatorname{Re} \delta < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{1}{2} \right].$$

ET II 81(6)

$$9. \int_0^\infty x^{\nu+1} F(\alpha, \beta; \gamma; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu+1} \Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} y^{-\nu-2} G_{13}^{30} \left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} \gamma \\ \nu+1, & \alpha, & \beta \end{matrix} \right. \right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2} \right].$$

ET II 81(5)

$$10. \int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) J_\nu(xy) dx = \frac{2^{\nu-\alpha-\beta+2} \Gamma(\nu+1)}{\lambda^{\alpha+\beta} \Gamma(\alpha)\Gamma(\beta)} y^{\alpha+\beta-\nu-2} K_{\alpha-\beta} \left(\frac{y}{\lambda} \right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2} \right].$$

ET II 81(3)

$$11. \int_0^\infty x^{\nu+1} F(\alpha, \beta; \nu+1; -\lambda^2 x^2) K_\nu(xy) dx = 2^{\nu+1} \lambda^{-\alpha-\beta} y^{\alpha+\beta-\nu-2} \Gamma(\nu+1) S_{1-\alpha-\beta, \alpha-\beta} \left(\frac{y}{\lambda} \right) \\ [\operatorname{Re} y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 152(86)

$$12. \int_0^\infty x^{\nu+1} F\left(\alpha, \beta; \frac{\beta+\nu}{2}+1; -\lambda^2 x^2\right) J_\nu(xy) dx = \frac{\Gamma\left(\frac{\beta+\nu+2}{2}\right) y^{\beta-1} \lambda^{-\nu-\beta-1}}{\pi^{\frac{1}{2}} \Gamma(\alpha)\Gamma(\beta) 2^{\beta-1}} \left[K_{\frac{1}{2}(\nu-\beta+1)} \left(\frac{y}{2\lambda} \right) \right]^2 \\ \left[y > 0, \quad -1 < \operatorname{Re} \nu < 2 \max(\operatorname{Re} \alpha, \operatorname{Re} \beta) - \frac{3}{2} \right].$$

ET II 81(4)

$$13. \int_0^\infty x^{\sigma+\frac{1}{2}} F(\alpha, \beta; \gamma; -\lambda^2 x^2) N_\nu(xy) dx = \frac{\lambda^{-\sigma-1} y^{-\frac{1}{2}} \Gamma(\gamma)}{\sqrt{2} \Gamma(\alpha)\Gamma(\beta)} G_{35}^{41} \left(\frac{y^2}{4\lambda^2} \left| \begin{matrix} 1-p, & \gamma-p, & l \\ h, & k, & \alpha-p, & \beta-p, & l \end{matrix} \right. \right),$$

$$14. \int_0^{\infty} x^{\nu+2} F\left(\frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -\lambda^2 x^2\right) N_{\nu}(xy) dx = \frac{2^{\nu} y^{-\nu-1}}{\pi^{\frac{1}{2}} \lambda^2 \Gamma\left(\frac{1}{2} - \nu\right)} K_{\nu}\left(\frac{y}{2\lambda}\right) K_{\nu+1}\left(\frac{y}{2\lambda}\right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < -\frac{1}{2} \right].$$

ET II 117(49)

$$15. \int_0^{\infty} x^{\nu+2} F\left(1, 2\nu + \frac{3}{2}; \nu + 2; -\lambda^2 x^2\right) N_{\nu}(xy) dx = \pi^{-\frac{1}{2}} 2^{-\nu} \lambda^{-2\nu-3} \frac{\Gamma(\nu+2)}{\Gamma\left(2\nu + \frac{3}{2}\right)} \left[K_{\nu}\left(\frac{y}{2\lambda}\right) \right]^2 \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 117(50)

$$16. \int_0^{\infty} x^{\nu+2} F\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\lambda^2 x^2\right) N_{\nu}(xy) dx = \frac{\pi^{\frac{1}{2}} 2^{-\mu-\nu-1} \lambda^{-\mu-2\nu-3} y^{\mu+\nu}}{\Gamma\left(\mu + \nu + \frac{3}{2}\right)} K_{\mu}\left(\frac{y}{\lambda}\right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad -\frac{3}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -\frac{3}{2} \right].$$

ET II 118(51)

$$17. \int_0^{\infty} x^{2\alpha+\nu} F\left(\alpha - \nu - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_{\nu}(xy) dx = \\ = \frac{i\Gamma\left(\frac{1}{2} + \alpha\right) \Gamma\left(\frac{1}{2} + \alpha + \nu\right)}{\pi 2^{1-\nu-2\alpha} \lambda^{2\alpha-1} y^{\nu+2}} W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(\frac{y}{\lambda}\right) \left[W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{-i\pi} \frac{y}{\lambda}\right) - W_{\frac{1}{2}-\alpha, -\frac{1}{2}-\nu}\left(e^{i\pi} \frac{y}{\lambda}\right) \right] \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu < -\frac{1}{2}, \quad \operatorname{Re}(\alpha + \nu) > -\frac{1}{2} \right].$$

ET II 80(1)

$$18. \int_0^{\infty} x^{2\alpha-\nu} F\left(\nu + \alpha - \frac{1}{2}, \alpha; 2\alpha; -\lambda^2 x^2\right) J_{\nu}(xy) dx = \\ = \frac{2^{2\alpha-\nu} \Gamma\left(\frac{1}{2} + \alpha\right) y^{\nu-2}}{\lambda^{2\alpha-1} \Gamma(2\nu)} M_{\alpha-\frac{1}{2}, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right) W_{\frac{1}{2}-\alpha, \nu-\frac{1}{2}}\left(\frac{y}{\lambda}\right).$$

ET II 80(2)

$$1. \int_0^\infty x^{-2\alpha-1} F\left(\frac{1}{2} + \alpha, 1 + \alpha; 1 + 2\alpha; -\frac{4\lambda^2}{x^2}\right) J_\nu(xy) dx = \lambda^{-2\alpha} I_{\frac{1}{2}\nu+\alpha}(\lambda y) K_{\frac{1}{2}\nu-\alpha}(\lambda y) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \alpha > -\frac{1}{2} \right].$$

ET II 81(7)

$$2. \int_0^\infty x^{\nu+1-4\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \nu + 1; -\frac{\lambda^2}{x^2}\right) J_\nu(xy) dx = \\ = \frac{\Gamma(\nu)}{\Gamma(2\alpha)} 2^\nu \lambda^{1-2\alpha} y^{2\alpha-\nu-1} I_\nu\left(\frac{1}{2}\lambda y\right) K_{2\alpha-\nu-1}\left(\frac{1}{2}\lambda y\right) \\ \left[y > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re} \alpha - 1 < \operatorname{Re} \nu < 4 \operatorname{Re} \alpha - \frac{3}{2} \right].$$

ET II 81(8)

7.544

$$\int_0^\infty x^{\nu+1} (1+x)^{-2\alpha} F\left[\alpha, \nu + \frac{1}{2}; 2\nu + 1; \frac{4x}{(1+x)^2}\right] J_\nu(xy) dx = \\ = \frac{\Gamma(\nu+1)\Gamma(\nu-\alpha+1)}{\Gamma(\alpha)} 2^{2\nu-2\alpha+1} y^{2(\alpha-\nu-1)} J_\nu(y) \quad \left[y > 0, \quad -1 < \operatorname{Re} \nu < 2 \operatorname{Re} \alpha - \frac{3}{2} \right].$$

ET II 82(10)

7.6 Confluent Hypergeometric Functions

7.61 Combinations of confluent hypergeometric functions and powers

7.611

$$1. \int_0^\infty x^{-1} W_{k,\mu}(x) dx = \frac{\pi^{\frac{3}{2}} 2^k \sec(\mu\pi)}{\Gamma\left(\frac{3}{4} - \frac{1}{2}k + \frac{1}{2}\mu\right) \Gamma\left(\frac{3}{4} - \frac{1}{2}k - \frac{1}{2}\mu\right)} \quad \left[|\operatorname{Re} \mu| < \frac{1}{2} \right].$$

ET II 406(22)

$$2. \int_0^\infty x^{-1} M_{k,\mu}(x) W_{\lambda,\mu}(x) dx = \frac{\Gamma(2\mu+1)}{(k-\lambda)\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \quad \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\lambda) > 0 \right].$$

$$\begin{aligned}
3. \int_0^\infty x^{-1} W_{k,\mu}(x) W_{\lambda,\mu}(x) dx &= \\
&= \frac{1}{(k-\lambda) \sin(2\mu\pi)} \left[\frac{1}{\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-\lambda-\mu\right)} - \frac{1}{\Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\mu\right)} \right] \\
&\quad \left[|\operatorname{Re} \mu| < \frac{1}{2} \right].
\end{aligned}$$

BU 116(12), ET II 409(40)

862

$$4. \int_0^\infty \{W_{\{\cdot,\mu\}}(z)\}^2 \frac{dz}{z} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi\left(\frac{1}{2}+\mu-\{\cdot\}\right) - \psi\left(\frac{1}{2}-\mu-\{\cdot\}\right)}{\Gamma\left(\frac{1}{2}+\mu-\{\cdot\}\right) \Gamma\left(\frac{1}{2}-\mu-\{\cdot\}\right)} \left[|\operatorname{Re} \mu| < \frac{1}{2} \right].$$

BU 117(12a)

$$5. \int_0^\infty \frac{1}{z} [W_{\{\cdot,0\}}(z)]^2 dx = \frac{\psi'\left(\frac{1}{2}-\{\cdot\}\right)}{\left[\Gamma\left(\frac{1}{2}-\{\cdot\}\right)\right]^2}.$$

BU 117(12b)

$$\begin{aligned}
6. \int_0^\infty x^{\varrho-1} W_{k,\mu}(x) W_{-k,\mu}(x) dx &= \frac{\Gamma(\varrho+1) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2} + \mu\right) \Gamma\left(\frac{1}{2}\varrho + \frac{1}{2} - \mu\right)}{2\Gamma\left(1 + \frac{1}{2}\varrho + k\right) \Gamma\left(1 + \frac{1}{2}\varrho - k\right)} \\
&\quad [\operatorname{Re} \varrho > 2|\operatorname{Re} \mu| - 1].
\end{aligned}$$

ET II 409(41)

$$\begin{aligned}
7. \int_0^\infty x^{\varrho-1} W_{k,\mu}(x) W_{\lambda,\nu}(x) dx &= \\
&= \frac{\Gamma(1+\mu+\nu+\varrho) \Gamma(1-\mu+\nu+\varrho) \Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\lambda-\nu\right) \Gamma\left(\frac{3}{2}-k+\nu+\varrho\right)} \times \\
&\quad \times {}_3F_2\left(1+\mu+\nu+\varrho, 1-\mu+\nu+\varrho, \frac{1}{2}-\lambda-\nu; 1+2\nu, \frac{3}{2}-k+\nu+\varrho; 1\right) + \\
&\quad + \frac{\Gamma(1+\mu-\nu+\varrho) \Gamma(1-\mu-\nu+\varrho) \Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{3}{2}-k-\nu+\varrho\right)} \times \\
&\quad \times {}_3F_2\left(1+\mu-\nu+\varrho, 1-\mu-\nu+\varrho, \frac{1}{2}-\lambda-\nu; 1-2\nu, \frac{3}{2}-k-\nu+\varrho; 1\right)
\end{aligned}$$

$$1. \int_0^{\infty} t^{b-1} {}_1F_1(a; c; -t) dt = \frac{\Gamma(b)\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a].$$

$$2. \int_0^{\infty} t^{b-1} \Psi(a, c; t) dt = \frac{\Gamma(b)\Gamma(a-b)\Gamma(b-c+1)}{\Gamma(a)\Gamma(a-c+1)} \quad [0 < \operatorname{Re} b < \operatorname{Re} a \quad \operatorname{Re} c < \operatorname{Re} b+1].$$

EH I 285(11)

863

7.613

$$1. \int_0^t x^{\gamma-1} (t-x)^{c-\gamma-1} {}_1F_1(a; \gamma; x) dx = t^{c-1} \frac{\Gamma(\gamma)\Gamma(c-\gamma)}{\Gamma(c)} {}_1F_1(a; c; t) \quad [\operatorname{Re} c > \operatorname{Re} \gamma > 0].$$

BU 9(16)A, EH I 271(16)

$$2. \int_0^t x^{\beta-1} (t-x)^{\gamma-1} {}_1F_1(t; \beta; x) dx = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} t^{\beta+\gamma-1} {}_1F_1(t; \beta+\gamma; t) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0].$$

ET II 401(1)

$$3. \int_0^1 x^{\lambda-1} (1-x)^{2\mu-\lambda} {}_1F_1\left(\frac{1}{2} + \mu - \nu; \lambda; xz\right) dx = B(\lambda, 1+2\mu-\lambda) e^{\frac{1}{2}z} z^{-\frac{1}{2}-\mu} M_{\nu, \mu}(z) \quad [\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\mu - \lambda) > -1].$$

BU 14(14)

$$4. \int_0^t x^{\beta-1} (t-x)^{\delta-1} {}_1F_1(t; \beta; x) {}_1F_1(\gamma; \delta; t-x) dx = \frac{\Gamma(\beta)\Gamma(\delta)}{\Gamma(\beta+\delta)} t^{\beta+\delta-1} {}_1F_1(t+\gamma; \beta+\delta; t) \quad [\operatorname{Re} \beta > 0, \quad \operatorname{Re} \delta > 0].$$

ET II 402(2), EH I 271(15)

$$5. \int_0^t x^{\mu-\frac{1}{2}} (t-x)^{\nu-\frac{1}{2}} M_{k, \mu}(x) M_{\lambda, \nu}(t-x) dx = \frac{\Gamma(2\mu+1)\Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} t^{\mu+\nu} M_{k+\lambda, \mu+\nu+\frac{1}{2}}(t) \quad \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

BU 128(14), ET II 402(7)

$$6. \int_0^1 x^{\beta-1} (1-x)^{\sigma-\beta-1} {}_1F_1(\alpha; \beta; \lambda x) {}_1F_1[\sigma-\alpha; \sigma-\beta; \mu(1-x)] dx = \frac{\Gamma(\beta)\Gamma(\sigma-\beta)}{\Gamma(\sigma)} e^{\lambda} {}_1F_1(\alpha; \sigma; \mu-\lambda) \quad [0 < \operatorname{Re} \beta < \operatorname{Re} \sigma].$$

7.62- 7.63 Combinations of confluent hypergeometric functions and exponentials

7.621

$$1. \int_0^{\infty} e^{-st} t^{\alpha} M_{\mu, \nu}(t) dt = \frac{\Gamma\left(\alpha + \nu + \frac{3}{2}\right)}{\left(\frac{1}{2} + s\right)^{\alpha + \nu + \frac{3}{2}}} F\left(\alpha + \nu + \frac{3}{2}, -\mu + \nu + \frac{1}{2}; 2\nu + 1; \frac{2}{2s + 1}\right) \\ \left[\operatorname{Re}\left(\alpha + \mu + \frac{3}{2}\right) > 0, \quad \operatorname{Re} s > \frac{1}{2} \right].$$

BU 118(1), MO 176a, EH I 270(12)a

$$2. \int_0^{\infty} e^{-st} t^{\mu - \frac{1}{2}} M_{\lambda, \mu}(qt) dt = q^{\mu + \frac{1}{2}} \Gamma(2\mu + 1) \left(s - \frac{1}{2}q\right)^{\lambda - \mu - \frac{1}{2}} \left(s + \frac{1}{2}q\right)^{-\lambda - \mu - \frac{1}{2}} \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \frac{|\operatorname{Re} q|}{2} \right].$$

BU 119(4c), MO 176a, EH I 271(13)a

864

$$3. \int_0^{\infty} e^{-st} t^{\alpha} W_{\lambda, \mu}(qt) dt = \\ = \frac{\Gamma\left(\alpha + \mu + \frac{3}{2}\right) \Gamma\left(\alpha - \mu + \frac{3}{2}\right) q^{\mu + \frac{1}{2}}}{\Gamma(\alpha - \lambda + 2)} \left(s + \frac{1}{2}q\right)^{-\alpha - \mu - \frac{3}{2}} \times \\ \times F\left(\alpha + \mu + \frac{3}{2}, \mu - \lambda + \frac{1}{2}; \alpha - \lambda + 2; \frac{2s - q}{2s + q}\right) \\ \left[\operatorname{Re}\left(\alpha \pm \mu + \frac{3}{2}\right) > 0, \quad \operatorname{Re} s > -\frac{q}{2}, \quad q > 0 \right].$$

EH I 271(14)A, BU 121(6), MO 176

$$4. \int_0^{\infty} e^{-st} t^{b-1} {}_1F_1(a; c; kt) dt = \Gamma(b) s^{-b} F(a, b; c; ks^{-1}) \quad [|s| > |k|]; \\ = \Gamma(b) (s - k)^{-b} F\left(c - a, b; c; \frac{k}{k - s}\right) \quad [|s - k| > |k|]; \\ [\operatorname{Re} b > 0, \quad \operatorname{Re} s > \max(0, \operatorname{Re} k)].$$

EH I 269(5)

$$5. \int_0^{\infty} t^{c-1} {}_1F_1(a; c; t) e^{-st} dt = \Gamma(c) s^{-c} (1 - s^{-1})^{-a} \quad [\operatorname{Re} c > 0, \quad \operatorname{Re} s > 1].$$

$$\begin{aligned}
6. \quad \int_0^\infty t^{b-1} \Psi(a, c; t) e^{-st} dt &= \\
&= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)} F(b, b-c+1; a+b-c+1; 1-s) \\
&\quad [\operatorname{Re} b > 0, \quad \operatorname{Re} c < \operatorname{Re} b + 1, \quad |1-s| < 1]; \\
&= \frac{\Gamma(b)\Gamma(b-c+1)}{\Gamma(a+b-c+1)} s^{-b} F(a, b; a+b-c+1; 1-s^{-1}) \quad \left[\operatorname{Re} s > \frac{1}{2} \right].
\end{aligned}$$

EH I 270(7)

$$\begin{aligned}
7. \quad \int_0^\infty e^{-\frac{b}{2}x} x^{\nu-1} M_{\{\cdot, \mu\}}(bx) dx &= \frac{\Gamma(1+2\mu)\Gamma(\{-\nu\})\Gamma\left(\frac{1}{2} + \mu + \nu\right)}{\Gamma\left(\frac{1}{2} + \mu + \{\cdot\}\right)\Gamma\left(\frac{1}{2} + \mu - \nu\right)} b^\nu \\
&\quad \left[\operatorname{Re}\left(\nu + \frac{1}{2} + \mu\right) > 0, \quad \operatorname{Re}(\{-\nu\}) > 0 \right].
\end{aligned}$$

BU 119(3)A, ET I 215(11)a

$$8. \quad \int_0^\infty e^{-sx} M_{\{\cdot, \mu\}}(x) \frac{dx}{x} = \frac{2\Gamma(1+2\mu)e^{-i\pi\{\cdot\}}}{\Gamma\left(\frac{1}{2} + \mu + \{\cdot\}\right)} \left(\frac{s - \frac{1}{2}}{s + \frac{1}{2}} \right)^{\frac{1}{2}} Q_{\mu-\frac{1}{2}}^{\{\cdot\}}(2s) \quad \left[\operatorname{Re}\left(\frac{1}{2} + \mu\right) > 0, \quad \operatorname{Re} s > \frac{1}{2} \right].$$

BU 119(4a)

865

$$\begin{aligned}
9. \quad \int_0^\infty e^{-sx} W_{\{\cdot, \mu\}}(x) \frac{dx}{x} &= \frac{\pi}{\cos\left(\frac{\pi\mu}{2}\right)} \left(\frac{s - \frac{1}{2}}{s + \frac{1}{2}} \right)^{\frac{1}{2}} P_{\mu-\frac{1}{2}}^{\{\cdot\}}(2s) \\
&\quad \left[\operatorname{Re}\left(\frac{1}{2} \pm \mu\right) > 0, \quad \operatorname{Re} s > -\frac{1}{2} \right].
\end{aligned}$$

BU 121(7)

$$\begin{aligned}
10. \quad \int_0^\infty x^{k+2\mu-1} e^{-\frac{3}{2}x} W_{k, \mu}(x) dx &= \frac{\Gamma\left(k + \mu + \frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2k + 6\mu + 5)\right]}{\left(k + 3\mu + \frac{1}{2}\right)\Gamma\left[\frac{1}{4}(2\mu - 2k + 3)\right]} \\
&\quad \left[\operatorname{Re}(k + \mu) > -\frac{1}{2}, \quad \operatorname{Re}(k + 3\mu) > -\frac{1}{2} \right].
\end{aligned}$$

$$11. \int_0^{\infty} e^{-\frac{1}{2}x} x^{\nu-1} W_{\{\mu\}}(x) dx = \frac{\Gamma\left(\nu + \frac{1}{2} - \mu\right) \Gamma\left(\nu + \frac{1}{2} + \mu\right)}{\Gamma(\nu - \{+1\})} \left[\operatorname{Re}\left(\nu + \frac{1}{2} \pm \mu\right) > 0 \right].$$

BU 122(8b)

$$12. \int_0^{\infty} e^{\frac{1}{2}x} x^{\nu-1} W_{\{\mu\}}(x) dx = \Gamma(-\{-\mu\}) \frac{\Gamma\left(\frac{1}{2} + \mu + \nu\right) \Gamma\left(\frac{1}{2} - \mu + \nu\right)}{\Gamma\left(\frac{1}{2} - \mu - \{\}\right) \Gamma\left(\frac{1}{2} + \mu - \{\}\right)} \left[\operatorname{Re}\left(\nu + \frac{1}{2} \pm \mu\right) > 0, \operatorname{Re}(\{\} + \nu) < 0 \right].$$

BU 122(8c)a

7.622

$$1. \int_0^{\infty} e^{-st} t^{c-1} {}_1F_1(a; c; t) {}_1F_1(\alpha; c; \lambda t) dt = \Gamma(c)(s-1)^{-a}(s-\lambda)^{-\alpha} s^{a+\alpha-c} F[a, \alpha; c; \lambda(s-1)^{-1}(s-\lambda)^{-1}]$$

$$[\operatorname{Re} c > 0, \operatorname{Re} s > \operatorname{Re} \lambda + 1].$$

EH I 287(22)

$$2. \int_0^{\infty} e^{-t} t^{\varrho} {}_1F_1(a; c; t) \Psi(a'; c'; \lambda t) dt = C \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(\gamma)} \lambda^{\sigma} F(c-a, \beta; \gamma; 1-\lambda^{-1}),$$

$$\varrho = c-1, \quad \sigma = -c, \quad \beta = c-c'+1, \quad \gamma = c-a+a'-c'+1, \quad C = \frac{\Gamma(a'-a)}{\Gamma(a')},$$

or

$$\varrho = c+c'-2, \quad \sigma = 1-c-c', \quad \beta = c+c'-1, \quad \gamma = a'-a+c, \quad C = \frac{\Gamma(a'-a-c'+1)}{\Gamma(a'-c'+1)}.$$

EH I 287(24)

866

$$3. \int_0^{\infty} x^{\nu-1} e^{-bx} M_{\lambda_1, \mu_1 - \frac{1}{2}}(a_1 x) \dots M_{\lambda_n, \mu_n - \frac{1}{2}}(a_n x) dx = a_1^{\mu_1} \dots a_n^{\mu_n} (b+A)^{-\nu-M} \Gamma(\nu+M) \times F_A\left(\nu+M; \mu_1 - \lambda_1, \dots, \mu_n - \lambda_n; 2\mu_1, \dots, 2\mu_n; \frac{a_1}{b+A}, \dots, \frac{a_n}{b+A}\right),$$

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 $M = \mu_1 + \dots + \mu_n, \quad A = \frac{1}{2}(a_1 + \dots + a_n)$

$$\left[\dots \right]$$

$$1. \int_0^{\infty} e^{-x} x^{c+n-1} (x+y)^{-1} {}_1F_1(a; c; x) dx = (-1)^n \Gamma(c) \Gamma(1-a) y^{c+n-1} \Psi(c-a, c; y) \\ [-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \quad n = 0, 1, 2, \dots, \quad |\arg y| < \pi].$$

EH I 285(16)

$$2. \int_0^t x^{-1} (t-x)^{k-1} e^{\frac{1}{2}(t-x)} M_{k, \mu}(x) dx = \frac{\Gamma(k) \Gamma(2\mu+1)}{\Gamma\left(k + \mu + \frac{1}{2}\right)} \pi^{\frac{1}{2}} t^{k-\frac{1}{2}} l_{\mu}\left(\frac{1}{2}t\right) \\ \left[\operatorname{Re} k > 0, \quad \operatorname{Re} \mu > -\frac{1}{2} \right].$$

ET II 402(5)

$$3. \int_0^t x^{k-1} (t-x)^{\lambda-1} e^{\frac{1}{2}(t-x)} M_{k+\lambda, \mu}(x) dx = \frac{\Gamma(\lambda) \Gamma\left(k + \mu + \frac{1}{2}\right) t^{k+\lambda-1}}{\Gamma\left(k + \lambda + \mu + \frac{1}{2}\right)} M_{k, \mu}(t) \\ \left[\operatorname{Re}(k + \mu) > -\frac{1}{2}, \quad \operatorname{Re} \lambda > 0 \right].$$

ET II 402(6)

$$4. \int_0^t x^{-k-\lambda-1} (t-x)^{\lambda-1} e^{\frac{1}{2}x} W_{k, \mu}(x) dx = \frac{\Gamma(\lambda) \Gamma\left(\frac{1}{2} - k - \lambda + \mu\right) \Gamma\left(\frac{1}{2} - k - \lambda - \mu\right)}{t^{k+1} \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right)} W_{k+\lambda, \mu}(t) \\ \left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k + \lambda) < \frac{1}{2} - |\operatorname{Re} \mu| \right].$$

ET II 405(21)

$$5. \int_1^{\infty} (x-1)^{\mu-1} x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{k, \lambda}(ax) dx = \frac{\Gamma(\mu) \Gamma\left(\frac{1}{2} - k - \lambda - \mu\right)}{\Gamma\left(\frac{1}{2} - k - \lambda\right)} a^{-\frac{1}{2}\mu} e^{\frac{1}{2}a} W_{k+\frac{1}{2}\mu, \lambda+\frac{1}{2}\mu}(a) \\ \left[|\arg(a)| < \frac{3}{2}\pi, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re}(k + \lambda) \right].$$

ET II 211(72)a

$$7. \int_1^{\infty} (x-1)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} W_{k-\mu,\lambda}(a) \quad [\operatorname{Re} \mu > 0, \operatorname{Re} a > 0].$$

$$8. \int_0^1 (1-x)^{\mu-1} x^{k-\mu-1} e^{-\frac{1}{2}ax} W_{k,\lambda}(ax) dx = \Gamma(\mu) e^{-\frac{1}{2}a} \sec[(k-\mu-\lambda)\pi] \times \left\{ \begin{aligned} & \frac{\Gamma\left(k-\mu+\lambda+\frac{1}{2}\right)}{\Gamma(2\lambda+1)} M_{k-\mu,\lambda}(a) + \cos[(k-\lambda)\pi] W_{k-\mu,\lambda}(a) \end{aligned} \right\} \left[0 < \operatorname{Re} \mu < \operatorname{Re} k - |\operatorname{Re} \lambda| + \frac{1}{2} \right].$$

7.624

$$1. \int_0^{\infty} x^{\varrho-1} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx = \frac{-\sigma \Gamma(2\mu+1) a^{\sigma}}{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2} + k + \mu\right)} G_{34}^{23} \left(a \left| \begin{array}{l} \frac{1}{2}, 1, 1-k+\varrho \\ \frac{1}{2} + \mu + \varrho, -\sigma, \sigma, \frac{1}{2} - \mu + \varrho \end{array} \right. \right) \left[|\arg a| < \pi, \operatorname{Re}(\mu + \varrho) > -\frac{1}{2}, \operatorname{Re}(k - \varrho - \sigma) > 0 \right].$$

$$2. \int_0^{\infty} x^{\varrho-1} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx = -\pi^{-\frac{1}{2}} \sigma a^{\sigma} G_{34}^{32} \left(a \left| \begin{array}{l} \frac{1}{2}, 1, 1-k+\varrho \\ \frac{1}{2} + \mu + \varrho, \frac{1}{2} - \mu + \varrho, -\sigma, \sigma \end{array} \right. \right) \left[|\arg a| < \pi, \operatorname{Re} \varrho > |\operatorname{Re} \mu| - \frac{1}{2} \right].$$

$$\begin{aligned}
4. \int_0^\infty x^{\varrho-1} (a+x)^{-\frac{1}{2}} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} M_{k,\mu}(x) dx &= \\
&= \frac{\Gamma(2\mu+1)a^\sigma}{\pi^{\frac{1}{2}}\Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{23} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k-\varrho \\ -\sigma, \varrho+\mu, \varrho-\mu, \sigma \end{matrix} \right. \right) \\
&\quad \left[|\arg a| < \pi, \quad \operatorname{Re}(\varrho+\mu) > -\frac{1}{2}, \quad \operatorname{Re}(k-\varrho-\sigma) > -\frac{1}{2} \right].
\end{aligned}$$

ET II 403(9)

$$\begin{aligned}
5. \int_0^\infty x^{\varrho-1} (a+x)^{-\frac{1}{2}} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx &= \\
&= \frac{\pi^{-\frac{1}{2}} a^\sigma}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{34}^{33} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}+k+\varrho \\ -\sigma, \varrho+\mu, \varrho-\mu, \sigma \end{matrix} \right. \right) \\
&\quad \left[|\arg a| < \pi, \quad \operatorname{Re} \varrho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k+\varrho+\sigma) < \frac{1}{2} \right].
\end{aligned}$$

ET II 406(26)

$$\begin{aligned}
6. \int_0^\infty x^{\varrho-1} (a+x)^{-\frac{1}{2}} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(x) dx &= \\
&= \pi^{-\frac{1}{2}} a^\sigma G_{34}^{32} \left(a \left| \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}-k+\varrho \\ -\sigma, \varrho+\mu, \varrho-\mu, \sigma \end{matrix} \right. \right) \quad \left[|\arg a| < \pi, \quad \operatorname{Re} \varrho > |\operatorname{Re} \mu| - \frac{1}{2} \right].
\end{aligned}$$

ET II 406(27)

7.625

$$\begin{aligned}
1. \int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] M_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx &= \\
&= \frac{\Gamma(1+\mu+\nu+\varrho)\Gamma(1+\mu-\nu+\varrho)}{\Gamma\left(\frac{3}{2}-\lambda+\mu+\varrho\right)} \alpha^{\mu+\frac{1}{2}} \beta^{-\mu-\varrho-\frac{1}{2}} \times \\
&\quad \times {}_3F_2\left(\frac{1}{2}+k+\mu, 1+\mu+\nu+\varrho, 1+\mu-\nu+\varrho; 2\mu+1, \frac{3}{2}-\lambda+\mu+\varrho; -\frac{\alpha}{\beta}\right) \\
&\quad \left[\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\varrho+\mu) > |\operatorname{Re} \nu| - 1 \right].
\end{aligned}$$

ET II 410(43)

$$\begin{aligned}
2. \int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx &= \\
&= \beta^{-\varrho} \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{1}{2}-\lambda-\nu\right) \right]^{-1} \times \\
&\quad \times G_{33}^{33} \left(\frac{\beta}{\alpha} \left| \begin{array}{c} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\varrho \\ \frac{1}{2}+\nu+\varrho, \frac{1}{2}-\nu+\varrho, -k \end{array} \right. \right) \\
&\quad [|\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \varrho + 1, \quad \operatorname{Re}(k + \lambda + \varrho) < 0].
\end{aligned}$$

ET II 410(44)a

$$\begin{aligned}
3. \int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha+\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx &= \beta^{-\varrho} G_{33}^{22} \left(\frac{\beta}{\alpha} \left| \begin{array}{c} \frac{1}{2}+\mu, \frac{1}{2}-\nu, 1-\lambda+\varrho \\ \frac{1}{2}+\nu+\varrho, \frac{1}{2}-\nu+\varrho, k \end{array} \right. \right) \\
&\quad [\operatorname{Re}(\alpha + \beta) > 0, \quad |\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \varrho + 1].
\end{aligned}$$

ET II 411(46)

$$\begin{aligned}
4. \int_0^\infty x^{\varrho-1} \exp\left[-\frac{1}{2}(\alpha-\beta)x\right] W_{k,\mu}(\alpha x) W_{\lambda,\nu}(\beta x) dx &= \\
&= \beta^{-\varrho} \left[\Gamma\left(\frac{1}{2}-\lambda+\nu\right) \Gamma\left(\frac{1}{2}-\lambda-\nu\right) \right]^{-1} G_{33}^{23} \left(\frac{\beta}{\alpha} \left| \begin{array}{c} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\varrho \\ \frac{1}{2}+\nu+\varrho, \frac{1}{2}-\nu+\varrho, k \end{array} \right. \right) \\
&\quad [\operatorname{Re} \alpha > 0, \quad |\operatorname{Re} \mu| + |\operatorname{Re} \nu| < \operatorname{Re} \varrho + 1].
\end{aligned}$$

ET II 411(45)

7.626

$$\begin{aligned}
1. \int_0^1 \left[\frac{k}{x} - \frac{1}{4}(\xi + \eta) \right] \exp\left[-\frac{1}{2}(\xi + \eta)x\right] x^c \times \\
&= \times {}_1F_1(a; c; \xi x) {}_1F_1(a; c; \eta x) dx = 0 \quad [\xi \neq \eta, \quad \operatorname{Re} c > 0]; \\
&\quad = \frac{a}{\xi} e^{-\xi} [{}_1F_1(a+1; c; \xi)]^2 \quad [\xi = \eta, \quad \operatorname{Re} c > 0] \\
&\quad \text{[where } \xi \text{ and } \eta \text{ are any two zeros of the function } {}_1F_1(a; c; x) \text{].}
\end{aligned}$$

EH I 285

$$\begin{aligned}
2. \int_1^\infty \left[\frac{k}{x} - \frac{1}{4}(\xi + \eta) \right] e^{-\frac{1}{2}(\xi+\eta)x} x^c \Psi(a, c; \xi x) \Psi(a, c; \eta x) dx &= \\
&= 0 \quad [\xi \neq \eta]; \\
&= -\xi^{-1} e^{-\xi} [\Psi(a-1, c; \xi)]^2 \quad [\xi = \eta] \\
&\quad \text{[where } \xi \text{ and } \eta \text{ are any two zeros of the function } \Psi(a, c; x) \text{].}
\end{aligned}$$

$$1. \int_0^{\infty} x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx = \frac{\Gamma(2\lambda)\Gamma\left(\frac{1}{2}-k+\mu-2\lambda\right)}{\Gamma\left(\frac{1}{2}-k+\mu\right)} a^{\lambda-\mu-\frac{1}{2}} W_{k+\lambda,\mu-\lambda}(a)$$

$$\left[|\arg a| < \pi, \quad 0 < 2 \operatorname{Re} \lambda < \frac{1}{2} - \operatorname{Re}(k+\mu) \right].$$

ET II 411(50)

$$2. \int_0^{\infty} x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} M_{k,\mu}^{-\frac{1}{2}x}(a+x) dx =$$

$$= \frac{\Gamma(2\lambda)\Gamma(2\mu+1)\Gamma\left(k+\mu-2\lambda+\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma(1-2\lambda+2\mu)} a^{\lambda-\mu-\frac{1}{2}} M_{k-\lambda,\mu-\lambda}(a)$$

$$\left[\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(k+\mu-2\lambda) > -\frac{1}{2} \right].$$

ET II 405(20)

$$3. \int_0^{\infty} x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(2\lambda) a^{\lambda-\mu-\frac{1}{2}} W_{k-\lambda,\mu-\lambda}(a)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0].$$

ET II 411(47)

$$4. \int_0^{\infty} x^{\lambda-1} (a+x)^{k-\lambda-1} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\lambda) a^{k-1} W_{k-\lambda,\mu}(a) \quad [|\arg a| < \pi, \quad \operatorname{Re} \lambda > 0].$$

ET II 411(48)

$$5. \int_0^{\infty} x^{\varrho-1} (a+x)^{-\sigma} e^{-\frac{1}{2}x} W_{k,\mu}(a+x) dx = \Gamma(\varrho) a^{\varrho} e^{\frac{1}{2}a} G_{23}^{30} \left(a \left| \begin{array}{l} 0, 1-k-\sigma \\ -\varrho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{array} \right. \right)$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \varrho > 0].$$

ET II 411(49)

$$6. \int_0^{\infty} x^{\varrho-1} (a+x)^{-\sigma} e^{\frac{1}{2}x} W_{k,\mu}(a+x) dx =$$

$$= \frac{\Gamma(\varrho) a^{\varrho} e^{-\frac{1}{2}a}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{23}^{31} \left(a \left| \begin{array}{l} k-\sigma+1, 0 \\ -\varrho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{array} \right. \right)$$

$$7. \int_0^\infty e^{-\frac{1}{2}(a+x)} \frac{(a+x)^{2\{ -1}}}{(ax)^\{ } W_{\{, \mu}(x) \frac{dx}{x} = \frac{\Gamma\left(\frac{1}{2} - \mu - \{ \right) \Gamma\left(\frac{1}{2} + \mu - \{ \right)}{a\Gamma(1 - 2\{)} W_{\{, \mu}(a) \left[\operatorname{Re}\left(\frac{1}{2} \pm \mu - \{ \right) > 0 \right].$$

BU 126(7a)

871

$$8. \int_0^\infty e^{-\frac{1}{2}x} x^{\gamma+\alpha-1} M_{\{, \mu}(x) \frac{dx}{(x+a)^\alpha} = \frac{\Gamma(1+2\mu)\Gamma\left(\frac{1}{2} + \mu + \gamma\right) \Gamma(\{ - \gamma)}{\Gamma\left(\frac{1}{2} + \mu - \gamma\right) \Gamma\left(\frac{1}{2} + \mu + \{ \right)} {}_2F_2\left(\alpha, \{ - \gamma; \frac{1}{2} + \mu - \gamma, \frac{1}{2} - \mu - \gamma; a\right) + \frac{\Gamma\left(\alpha + \gamma + \frac{1}{2} + \mu\right) \Gamma\left(-\gamma - \frac{1}{2} - \mu\right)}{\Gamma(\alpha)} a^{\gamma+\frac{1}{2}+\mu} \times {}_2F_2\left(\alpha + \gamma + \mu + \frac{1}{2}, \{ + \mu + \frac{1}{2}; 1 + 2\mu, \frac{3}{2} + \mu + \gamma; a\right) \left[\operatorname{Re}\left(\gamma + \alpha + \frac{1}{2} + \mu\right) > 0, \operatorname{Re}(\gamma - \{) < 0 \right].$$

BU 126(8a)

$$9. \int_0^\infty e^{-\frac{1}{2}x} x^{n+\mu+\frac{1}{2}} M_{\{, \mu}(x) \frac{dx}{x+a} = (-1)^{n+1} a^{n+\mu+\frac{1}{2}} e^{\frac{1}{2}a} \Gamma(1+2\mu)\Gamma\left(\frac{1}{2} - \mu + \{ \right) W_{-\{, \mu}(a) \left[n = 0, 1, 2, \dots, \operatorname{Re}\left(\mu + 1 + \frac{n}{2}\right) > 0, \operatorname{Re}\left(\{ - \mu - \frac{1}{2}\right) < n, |\arg a| < \pi \right].$$

BU 127(10a)a

7.628

$$1. \int_0^\infty e^{-st} e^{-t^2} t^{2c-2} {}_1F_1(a; c; t^2) dt = 2^{1-2c} \Gamma(2c-1) \Psi\left(c - \frac{1}{2}, a + \frac{1}{2}; \frac{1}{4}s^2\right) \left[\operatorname{Re} c > \frac{1}{2}, \operatorname{Re} s > 0 \right].$$

EH I 270(11)

$$2. \int_0^\infty t^{2\nu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{-3\nu, \nu}\left(\frac{t^2}{a}\right) dt = \frac{1}{2\sqrt{\pi}} \Gamma(4\nu+1) a^{-\nu} s^{-4\nu} e^{as^2/8} K_{2\nu}\left(\frac{as^2}{8}\right) \left[\operatorname{Re} a > 0, \operatorname{Re} \nu > -\frac{1}{4}, \operatorname{Re} s > 0 \right].$$

$$3. \int_0^{\infty} t^{2\mu-1} e^{-\frac{1}{2a}t^2} e^{-st} M_{\lambda, \mu} \left(\frac{t^2}{a} \right) dt = 2^{-3\mu-\lambda} \Gamma(4\mu+1) a^{\frac{1}{2}(\lambda+\mu-1)} s^{\lambda-\mu-1} e^{\frac{as^2}{8}} W_{-\frac{1}{2}(\lambda+3\mu), \frac{1}{2}(\lambda-\mu)} \left(\frac{as^2}{4} \right) \\ \left[\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{4}, \quad \operatorname{Re} s > 0 \right].$$

ET I 215(13)

7.629

$$1.^8 \int_0^{\infty} t^k \exp \left(\frac{a}{2t} \right) e^{-st} W_{k, \mu} \left(\frac{a}{t} \right) dt = 2^{1-2k} \sqrt{a} s^{-k-\frac{1}{2}} S_{2k, 2\mu} (2\sqrt{a} s) \\ \left[|\arg a| < \pi, \quad \operatorname{Re}(k \pm \mu) > -\frac{1}{2}, \quad \operatorname{Re} s > 0 \right].$$

ET I 217(21)

872

$$2. \int_0^{\infty} t^{-k} \exp \left(-\frac{a}{2t} \right) e^{-st} W_{k, \mu} \left(\frac{a}{t} \right) dt = 2\sqrt{a} s^{k-\frac{1}{2}} K_{2\mu} (2\sqrt{a} s) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0].$$

ET I 217(22)

7.631

$$1. \int_0^{\infty} x^{\varrho-1} \exp \left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1}) \right] W_{k, \mu}(\alpha^{-1}x) W_{\lambda, \nu}(\beta x^{-1}) dx = \\ = \beta^{\varrho} \left[\Gamma \left(\frac{1}{2} - k + \mu \right) \Gamma \left(\frac{1}{2} - k - \mu \right) \right]^{-1} \times \\ \times G_{24}^{41} \left(\frac{\beta}{\alpha} \left| \begin{matrix} 1+k, & 1-\lambda-\varrho \\ \frac{1}{2} + \mu, & \frac{1}{2} - \mu, & \frac{1}{2} + \nu - \varrho, & \frac{1}{2} - \nu - \varrho \end{matrix} \right. \right) \\ \left[|\arg \alpha| < \frac{3}{2}\pi, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(k + \varrho) < -|\operatorname{Re} \nu| - \frac{1}{2} \right].$$

ET II 412(55)

$$2. \int_0^{\infty} x^{\varrho-1} \exp \left[\frac{1}{2}(\alpha^{-1}x - \beta x^{-1}) \right] W_{k, \mu}(\alpha^{-1}x) W_{\lambda, \nu}(\beta x^{-1}) dx = \\ = \beta^{\varrho} \left[\Gamma \left(\frac{1}{2} - k + \mu \right) \Gamma \left(\frac{1}{2} - k - \mu \right) \Gamma \left(\frac{1}{2} - \lambda + \nu \right) \Gamma \left(\frac{1}{2} - \lambda - \nu \right) \right]^{-1} \times \\ \times G_{24}^{42} \left(\frac{\beta}{\alpha} \left| \begin{matrix} 1+k, & 1+\lambda-\varrho \\ \frac{1}{2} + \mu, & \frac{1}{2} - \mu, & \frac{1}{2} + \nu - \varrho, & \frac{1}{2} - \nu - \varrho \end{matrix} \right. \right) \\ \left[|\arg \alpha| < \frac{3}{2}\pi, \quad |\arg \beta| < \frac{3}{2}\pi, \quad \operatorname{Re}(\lambda - \varrho) < \frac{1}{2} - |\operatorname{Re} \mu|, \quad \operatorname{Re}(k + \varrho) < \frac{1}{2} - |\operatorname{Re} \nu| \right].$$

$$= \beta^{\varrho} G_{24}^{40} \left(\begin{matrix} \beta \\ \alpha \end{matrix} \middle| \begin{matrix} 1-k, & 1-\lambda-\varrho \\ \frac{1}{2}+\mu, & \frac{1}{2}-\mu, & \frac{1}{2}+\nu-\varrho, & \frac{1}{2}-\nu-\varrho \end{matrix} \right) \quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0].$$

7.632

$$\int_0^{\infty} e^{-st} (e^t - 1)^{\mu - \frac{1}{2}} \exp\left(-\frac{1}{2}\lambda e^t\right) M_{k, \mu}(\lambda e^t - \lambda) dt = \frac{\Gamma(2\mu + 1)\Gamma\left(\frac{1}{2} + k - \mu + s\right)}{\Gamma(s + 1)} W_{-k - \frac{1}{2}s, \mu - \frac{1}{2}s}(\lambda) \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} s > \operatorname{Re}(\mu - k) - \frac{1}{2} \right].$$

ET I 216(15)

873

7.64 Combinations of confluent hypergeometric and trigonometric functions

7.641

$$\int_0^{\infty} \cos(ax) {}_1F_1(\nu+1; 1; ix) {}_1F_1(\nu+1; 1; -ix) dx = -a^{-1} \sin(\nu\pi) P_{\nu}(2a^{-2} - 1) \quad [0 < a < 1]; \\ = 0 \quad [1 < a < \infty] \quad [-1 < \operatorname{Re} \nu < 0].$$

ET II 402(4)

7.642

$$\int_0^{\infty} \cos(2xy) {}_1F_1(a; c; -x^2) dx = \frac{1}{2} \pi^{\frac{1}{2}} \frac{\Gamma(c)}{\Gamma(a)} y^{2a-1} e^{-y^2} \Psi\left(c - \frac{1}{2}, a + \frac{1}{2}; y^2\right).$$

EH I 285(12)

7.643

$$1. \int_0^{\infty} x^{4\nu} e^{-\frac{1}{2}x^2} \sin(bx) {}_1F_1\left(\frac{1}{2} - 2\nu; 2\nu + 1; \frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{4\nu} c^{-\frac{1}{2}b^2} {}_1F_1\left(\frac{1}{2} - 2\nu; 1 + 2\nu; \frac{1}{2}b^2\right) \\ \left[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4} \right].$$

ET I 115(5)

$$2. \int_0^{\infty} x^{2\nu-1} e^{-\frac{1}{4}x^2} \sin(bx) M_{3\nu, \nu}\left(\frac{1}{2}x^2\right) dx = \sqrt{\frac{\pi}{2}} b^{2\nu-1} e^{-\frac{1}{4}b^2} M_{3\nu, \nu}\left(\frac{1}{2}b^2\right) \\ \left[b > 0, \quad \operatorname{Re} \nu > -\frac{1}{4} \right].$$

ET I 116(10)

$$4. \int_0^{\infty} x^{-2\nu} e^{\frac{1}{4}x^2} \sin(bx) W_{3\nu-1, \nu} \left(\frac{1}{2}x^2 \right) dx = \sqrt{\frac{\pi}{2}} b^{-2\nu} e^{\frac{1}{4}b^2} W_{3\nu-1, \nu} \left(\frac{1}{2}b^2 \right) \quad \left[\operatorname{Re} \nu < \frac{1}{2}, b > 0 \right].$$

ET I 116(9)

7.644

$$1. \int_0^{\infty} x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} \sin(2ax^{\frac{1}{2}}) M_{k, \mu}(x) dx = \pi^{\frac{1}{2}} a^{k+\mu-1} \frac{\Gamma(3-2\mu)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} \exp\left(-\frac{a^2}{2}\right) W_{\varrho, \sigma}(a^2),$$

$$2\varrho = k - 3\mu + 1, \quad 2\sigma = k + \mu - 1 \quad [a > 0, \operatorname{Re}(k + \mu) > 0].$$

ET II 403(10)

$$2. \int_0^{\infty} x^{\varrho-1} \sin(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{k, \mu}(x) dx = \frac{c\Gamma(1+\mu+\varrho)\Gamma(1-\mu+\varrho)}{\Gamma\left(\frac{3}{2}-k+\varrho\right)} \times$$

$$\times {}_2F_2\left(1+\mu+\varrho, 1-\mu+\varrho; \frac{3}{2}, \frac{3}{2}-k+\varrho; -\frac{c^2}{4}\right)$$

$$[\operatorname{Re} \varrho > |\operatorname{Re} \mu| - 1].$$

ET II 407(28)

874

$$3. \int_0^{\infty} x^{\varrho-1} \sin(cx^{\frac{1}{2}}) e^{\frac{1}{2}x} W_{k, \mu}(x) dx =$$

$$= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-k+\mu\right)\Gamma\left(\frac{1}{2}-k-\mu\right)} G_{23}^{22} \left(\frac{c^2}{4} \left| \begin{array}{l} \frac{1}{2} + \mu - \varrho, \frac{1}{2} - \mu - \varrho \\ \frac{1}{2}, -k - \varrho, 0 \end{array} \right. \right)$$

$$\left[c > 0, \operatorname{Re} \varrho > |\operatorname{Re} \mu| - 1, \operatorname{Re}(k + \varrho) < \frac{1}{2} \right].$$

ET II 407(29)

$$4. \int_0^{\infty} x^{\varrho-1} \cos(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{k, \mu}(x) dx = \frac{\Gamma\left(\frac{1}{2}+\mu+\varrho\right)\Gamma\left(\frac{1}{2}-\mu+\varrho\right)}{\Gamma(1-k+\varrho)} \times$$

$$\times {}_2F_2\left(\frac{1}{2}+\mu+\varrho, \frac{1}{2}-\mu+\varrho; \frac{1}{2}, 1-k+\varrho; -\frac{c^2}{4}\right)$$

$$\left[\operatorname{Re} \varrho > |\operatorname{Re} \mu| - \frac{1}{2} \right].$$

$$\begin{aligned}
5. \int_0^\infty x^{\varrho-1} \cos(cx^{\frac{1}{2}}) e^{\frac{1}{2}x} W_{k, \mu}(x) dx &= \\
&= \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right)} G_{23}^{22} \left(\frac{c^2}{4} \left| \begin{array}{c} \frac{1}{2} + \mu - \varrho, \frac{1}{2} - \mu - \varrho \\ 0, -k - \varrho, \frac{1}{2} \end{array} \right. \right) \\
&\quad \left[c > 0, \quad \operatorname{Re} \varrho > |\operatorname{Re} \mu| - \frac{1}{2}, \quad \operatorname{Re}(k + \varrho) < \frac{1}{2} \right].
\end{aligned}$$

ET II 407(31)

7.65 Combinations of confluent hypergeometric functions and Bessel functions

7.651

$$\begin{aligned}
1. \int_0^\infty J_\nu(xy) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) dx &= \\
&= ay^{-\mu-1} \frac{\Gamma(\nu+1)}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\nu\right)} [a + (a^2 + y^2)^{\frac{1}{2}}]^\mu (a^2 + y^2)^{-\frac{1}{2}} \\
&\quad \left[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} a > 0 \right].
\end{aligned}$$

ET II 85(19)

875

$$\begin{aligned}
2. \int_0^\infty M_{k, \frac{1}{2}\nu}(-iax) M_{-k, \frac{1}{2}\nu}(-iax) J_\nu(xy) dx &= \\
&= \frac{ae^{-\frac{1}{2}(\nu+1)\pi i} [\Gamma(1+\nu)]^2}{\Gamma\left(\frac{1}{2} + k + \frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} - k + \frac{1}{2}\nu\right)} y^{-1-2k} \times \\
&\quad \times (a^2 - y^2)^{-\frac{1}{2}} \{ [a + (a^2 - y^2)^{\frac{1}{2}}]^{2k} + [a - (a^2 - y^2)^{\frac{1}{2}}]^{2k} \} \quad [0 < y < a]; \\
&= 0 \quad [a < y < \infty] \quad \left[a > 0, \quad \operatorname{Re} \nu > -1, \quad |\operatorname{Re} k| < \frac{1}{4} \right].
\end{aligned}$$

ET II 85(18)

7.652

$$\begin{aligned}
\int_0^\infty M_{-\mu, \frac{1}{2}\nu} \{ a[(b^2 + x^2)^{\frac{1}{2}} - b] \} W_{\mu, \frac{1}{2}\nu} \{ a[(b^2 + x^2)^{\frac{1}{2}} + b] \} J_\nu(xy) dx &= \\
&= \frac{ay^{-2\mu-1} \Gamma(1+\nu) [(a^2 + y^2)^{\frac{1}{2}} + a]^{2\mu}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu - \mu\right) (A^2 + Y^2)^{\frac{1}{2}}} \exp[-b(a^2 + y^2)^{\frac{1}{2}}] \\
&\quad \left[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{4}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} b > 0 \right].
\end{aligned}$$

ET II 87(29)

7.66 Combinations of confluent hypergeometric functions, Bessel functions, and powers

7.661

$$\begin{aligned}
 1. \int_0^\infty x^{-1} W_{k, \mu}(ax) M_{-k, \mu}(ax) J_0(xy) dx &= \\
 &= e^{-ik\pi} \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2} + \mu + k\right)} P_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] Q_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \\
 &\quad \left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} k < \frac{3}{4} \right].
 \end{aligned}$$

ET II 18(44)

$$\begin{aligned}
 2. \int_0^\infty x^{-1} W_{k, \mu}(ax) W_{-k, \mu}(ax) J_0(xy) dx &= \frac{1}{2} \pi \cos(\mu\pi) P_{\mu-\frac{1}{2}}^k \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] P_{\mu-\frac{1}{2}}^{-k} \left[\left(1 + \frac{y^2}{a^2}\right)^{\frac{1}{2}} \right] \\
 &\quad \left[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2} \right].
 \end{aligned}$$

ET II 18(45)

$$\begin{aligned}
 3. \int_0^\infty x^{2\mu-\nu} W_{k, \mu}(ax) M_{-k, \mu}(ax) J_\nu(xy) dx &= \\
 &= 2^{2\mu-\nu+2k} a^{2k} y^{\nu-2\mu-2k-1} \frac{\Gamma(2\mu+1)}{\Gamma\left(\nu-k-\mu+\frac{1}{2}\right)} \times \\
 &\quad \times {}_3F_2\left(\frac{1}{2}-k, 1-k, \frac{1}{2}-k+\mu; 1-2k, \frac{1}{2}-k-\mu+\nu; -\frac{y^2}{a^2}\right) \\
 &\quad \left[y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(2\mu+2k-\nu) < \frac{1}{2} \right].
 \end{aligned}$$

ET II 85(20)

876

$$\begin{aligned}
 4. \int_0^\infty x^{2\varrho-\nu} W_{k, \mu}(iax) W_{k, \mu}(-iax) J_\nu(xy) dx &= \\
 &= 2^{2\varrho-\nu} y^{\nu-2\varrho-1} \pi^{-\frac{1}{2}} \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \right]^{-1} G_{44}^{24} \left(\frac{y^2}{a^2} \left| \begin{array}{l} \frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \varrho+\frac{1}{2}, -k, k, \varrho-\nu+\frac{1}{2} \end{array} \right. \right) \\
 &\quad \left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \varrho > |\operatorname{Re} \mu| - 1, \quad \operatorname{Re}(2\varrho+2k-\nu) < \frac{1}{2} \right].
 \end{aligned}$$

$$\begin{aligned}
5. \int_0^\infty x^{2\varrho-\nu} W_{k,\mu}(ax) M_{-k,\mu}(ax) J_\nu(xy) dx &= \\
&= \frac{2^{2\varrho-\nu} \Gamma(2\mu+1)}{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}-k+\mu\right)} y^{\nu-2\varrho-1} G_{44}^{23} \left(\frac{y^2}{a^2} \left| \begin{array}{c} \frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \varrho+\frac{1}{2}, -k, k, \varrho-\nu+\frac{1}{2} \end{array} \right. \right) \\
\left[y > 0, \operatorname{Re} a > 0, \operatorname{Re} \varrho > -1, \operatorname{Re}(\varrho+\mu) > -1, \operatorname{Re}(2e+2k+\nu) < \frac{1}{2} \right].
\end{aligned}$$

ET II 86(21)a

$$\begin{aligned}
6. \int_0^\infty x^{2\varrho-\nu} W_{k,\mu}(ax) W_{-k,\mu}(ax) J_\nu(xy) dx &= \\
&= \frac{\Gamma(\varrho+1+\mu)\Gamma(\varrho+1-\mu)\Gamma(2\varrho+2)}{\Gamma\left(\frac{3}{2}+k+\varrho\right)\Gamma\left(\frac{3}{2}-k+\varrho\right)\Gamma(1+\nu)} y^\nu 2^{-\nu-1} a^{-2\varrho-1} \times \\
&\quad \times {}_4F_3 \left(\varrho+1, \varrho+\frac{3}{2}, \varrho+1+\mu, \varrho+1-\mu; \frac{3}{2}+k+\varrho, \frac{3}{2}-k+\varrho, 1+\nu; -\frac{y^2}{a^2} \right) \\
&\quad [y > 0, \operatorname{Re} \varrho > |\operatorname{Re} \mu| - 1, \operatorname{Re} a > 0].
\end{aligned}$$

ET II 86(22)a

7.662

$$\begin{aligned}
1. \int_0^\infty x^{-1} M_{-\mu, \frac{1}{4}\nu} \left(\frac{1}{2}x^2 \right) W_{\mu, \frac{1}{4}\nu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \frac{\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{4}\nu-\mu\right)} I_{\frac{1}{4}\nu-\mu} \left(\frac{1}{4}y^2 \right) K_{\frac{1}{4}\nu+\mu} \left(\frac{1}{4}y^2 \right) \\
&\quad [y > 0, \operatorname{Re} \nu > -1].
\end{aligned}$$

ET II 86(24)

$$\begin{aligned}
2. \int_0^\infty x^{-1} M_{\alpha-\beta, \frac{1}{4}\nu-\gamma} \left(\frac{1}{2}x^2 \right) W_{\alpha+\beta, \frac{1}{4}\nu+\gamma} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \\
&= \frac{\Gamma\left(1+\frac{1}{2}\nu-2\gamma\right)}{\Gamma\left(1+\frac{1}{2}\nu-2\beta\right)} y^{-2} M_{\alpha-\gamma, \frac{1}{4}\nu-\beta} \left(\frac{1}{2}y^2 \right) W_{\alpha+\gamma, \frac{1}{4}\nu+\beta} \left(\frac{1}{2}y^2 \right) \\
&\quad \left[y > 0, \operatorname{Re} \beta < \frac{1}{8}, \operatorname{Re} \nu > -1, \operatorname{Re}(\nu-4\gamma) > -2 \right].
\end{aligned}$$

ET II 86(25)

877

$$3. \int_0^\infty x^{-1} M_{k,0}(iax^2) M_{k,0}(-iax^2) K_0(xy) dx = \frac{\pi}{16} \left\{ \left[J_k \left(\frac{y^2}{8a} \right) \right]^2 + \left[N_k \left(\frac{y^2}{8a} \right) \right]^2 \right\} \quad [a > 0].$$

$$4. \int_0^{\infty} x^{-1} M_{k, \mu}(iax^2) M_{k, \mu}(-iax^2) K_0(xy) dx = ay^{-2} [\Gamma(2\mu+1)]^2 W_{-\mu, k} \left(\frac{iy^2}{4a} \right) W_{-\mu, k} \left(-\frac{iy^2}{4a} \right) \\ \left[a > 0, \quad \operatorname{Re} y > 0, \quad \operatorname{Re} \mu > -\frac{1}{2} \right].$$

ET II 152(84)

7.663

$$1. \int_0^{\infty} x^{2\varrho} {}_1F_1(a; b; -\lambda x^2) J_{\nu}(xy) dx = \frac{2^{2\varrho} \Gamma(b)}{\Gamma(a) y^{2\varrho+1}} G_{23}^{21} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} 1, b \\ \frac{1}{2} + \varrho + \frac{1}{2}\nu, a, \frac{1}{2} + \varrho - \frac{1}{2}\nu \end{matrix} \right. \right) \\ \left[y > 0, \quad -1 - \operatorname{Re} \nu < 2 \operatorname{Re} \varrho < \frac{1}{2} + 2 \operatorname{Re} a, \quad \operatorname{Re} \lambda > 0 \right].$$

ET II 88(6)

$$2. \int_0^{\infty} x^{\nu+1} {}_1F_1 \left(2a - \nu; a + 1; -\frac{1}{2} x^2 \right) J_{\nu}(xy) dx = \frac{2^{\nu-a+\frac{1}{2}} \Gamma(a+1)}{\pi^{\frac{1}{2}} \Gamma(2a-\nu)} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} K_{a-\nu-\frac{1}{2}} \left(\frac{1}{4} y^2 \right) \\ \left[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(4a-3\nu) > \frac{1}{2} \right].$$

ET II 87(1)

$$3. \int_0^{\infty} x^a {}_1F_1 \left(a; \frac{1+a+\nu}{2}; -\frac{1}{2} x^2 \right) J_{\nu}(xy) dx = y^{a-1} {}_1F_1 \left(a; \frac{1+a+\nu}{2}; -\frac{y^2}{2} \right) \\ \left[y > 0, \quad \operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a+\nu) > -1 \right].$$

ET II 87(2)

$$4. \int_0^{\infty} x^{\nu+1-2a} {}_1F_1 \left(a; 1+\nu-a; -\frac{1}{2} x^2 \right) J_{\nu}(xy) dx = \\ = \frac{\pi^{\frac{1}{2}} \Gamma(1+\nu-a)}{\Gamma(a)} 2^{-2a+\nu+\frac{1}{2}} y^{2a-\nu-1} e^{-\frac{1}{4}y^2} I_{a-\frac{1}{2}} \left(\frac{1}{4} y^2 \right) \\ \left[y > 0, \quad \operatorname{Re} a - 1 < \operatorname{Re} \nu < 4 \operatorname{Re} a - \frac{1}{2} \right].$$

ET II 87(3)

$$5. \int_0^{\infty} x {}_1F_1(\lambda; 1; -x^2) J_0(xy) dx = [2^{2\lambda-1} \Gamma(\lambda)]^{-1} y^{2\lambda-2} e^{-\frac{1}{4}y^2} \quad [y > 0, \quad \operatorname{Re} \lambda > 0].$$

ET II 18(46)

878

$$\begin{aligned}
7. \int_0^\infty x^{2b-\nu-1} {}_1F_1(a; b; -\lambda x^2) J_\nu(xy) dx \\
= \frac{2^{2b-2a-\nu-1} \Gamma(b)}{\Gamma(a-b+\nu+1)} \lambda^{-a} y^{2a-2b+\nu} \times \\
\times {}_1F_1\left(a; 1+a-b+\nu; -\frac{y^2}{4\lambda}\right) \\
\left[y > 0, \quad 0 < \operatorname{Re} b < \frac{3}{4} + \operatorname{Re}\left(a + \frac{1}{2}\nu\right), \quad \operatorname{Re} \lambda > 0 \right].
\end{aligned}$$

ET II 88(5)

7.664

$$\begin{aligned}
1. \int_0^\infty x W_{\frac{1}{2}\nu, \mu}\left(\frac{a}{x}\right) W_{-\frac{1}{2}\nu, \mu}\left(\frac{a}{x}\right) K_\nu(xy) dx = 2ay^{-1} K_{2\mu}[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}] K_{2\mu}[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \\
[\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0].
\end{aligned}$$

ET II 152(85)

$$\begin{aligned}
2. \int_0^\infty x W_{\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) J_\nu(xy) dx = \\
= -4y^{-1} \left\{ \sin\left[\left(\mu - \frac{1}{2}\nu\right)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right) + \cos\left[\left(\mu - \frac{1}{2}\nu\right)\pi\right] N_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\
[y > 0, \quad \operatorname{Re}(\nu \pm 2\mu) > -1].
\end{aligned}$$

ET II 87(27)

$$\begin{aligned}
3. \int_0^\infty x W_{\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) W_{-\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) N_\nu(xy) dx = \\
= 4y^{-1} \left\{ \cos\left[\left(\mu - \frac{1}{2}\nu\right)\pi\right] J_{2\mu}\left(2y^{\frac{1}{2}}\right) - \sin\left[\left(\mu - \frac{1}{2}\nu\right)\pi\right] N_{2\mu}\left(2y^{\frac{1}{2}}\right) \right\} K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\
\left[y > 0, \quad |\operatorname{Re} \mu| < \frac{1}{4} \right].
\end{aligned}$$

ET II 117(48)

$$\begin{aligned}
4. \int_0^\infty x W_{-\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) M_{\frac{1}{2}\nu, \mu}\left(\frac{2}{x}\right) J_\nu(xy) dx = \frac{4\Gamma(1+2\mu)y^{-1}}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu + \mu\right)} J_{2\mu}\left(2y^{\frac{1}{2}}\right) K_{2\mu}\left(2y^{\frac{1}{2}}\right) \\
\left[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{4} \right].
\end{aligned}$$

$$\begin{aligned}
5. \int_0^\infty x W_{-\frac{1}{2}\nu, \mu} \left(\frac{ia}{x} \right) W_{-\frac{1}{2}\nu, \mu} \left(-\frac{ia}{x} \right) J_\nu(xy) dx &= \\
&= 4ay^{-1} \left[\Gamma \left(\frac{1}{2} + \mu + \frac{1}{2}\nu \right) \Gamma \left(\frac{1}{2} - \mu + \frac{1}{2}\nu \right) \right]^{-1} K_\mu[(2ia y)^{\frac{1}{2}}] K_\mu[(-2ia y)^{\frac{1}{2}}] \\
&\quad \left[y > 0, \quad \operatorname{Re} a > 0, \quad |\operatorname{Re} \mu| < \frac{1}{2}, \quad \operatorname{Re} \nu > -1 \right].
\end{aligned}$$

ET II 87(28)

879

7.665

$$\begin{aligned}
1. \int_0^\infty x^{-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}}) K_{\frac{1}{2}\nu-\mu} \left(\frac{1}{2}x \right) M_{k, \mu}(x) dx &= \\
&= \frac{\Gamma(2\mu+1)}{a\Gamma \left(k + \frac{1}{2}\nu + 1 \right)} W_{\frac{1}{2}(k-\mu), \frac{1}{2}k-\frac{1}{4}\nu} \left(\frac{a^2}{2} \right) M_{\frac{1}{2}(k+\mu), \frac{1}{2}k+\frac{1}{4}\nu} \left(\frac{a^2}{2} \right) \\
&\quad \left[a > 0, \quad \operatorname{Re} k > -\frac{1}{4}, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1 \right].
\end{aligned}$$

ET II 405(18)

$$\begin{aligned}
2. \int_0^\infty x^{\frac{1}{2}c+\frac{1}{2}c'-1} \Psi(a, c; x) {}_1F_1(a'; c'; -x) J_{c+c'-2}[2(xy)^{\frac{1}{2}}] dx &= \\
&= \frac{\Gamma(c')}{\Gamma(a+a')} y^{\frac{1}{2}c+\frac{1}{2}c'-1} \Psi(c'-a', c+c'-a-a'; y) {}_1F_1(a'; a+a'; -y) \\
&\quad \left[\operatorname{Re} c' > 0, \quad 1 < \operatorname{Re}(c+c') < 2 \operatorname{Re}(a+a') + \frac{1}{2} \right].
\end{aligned}$$

EH I 287(23)

7.666

$$\begin{aligned}
\int_0^\infty x^{\frac{1}{2}c-\frac{1}{2}} {}_1F_1(a; c; -2x^{\frac{1}{2}}) \Psi(a, c; 2x^{\frac{1}{2}}) J_{c-1}[2(xy)^{\frac{1}{2}}] dx &= \\
&= 2^{-c} \frac{\Gamma(c)}{\Gamma(a)} y^{a-\frac{1}{2}c-\frac{1}{2}} [1+(1+y)^{\frac{1}{2}}]^{c-2a} (1+y)^{-\frac{1}{2}} \quad \left[\operatorname{Re} c > 2, \quad \operatorname{Re}(c-2a) < \frac{1}{2} \right].
\end{aligned}$$

EH I 285(13)

7.67 Combinations of confluent hypergeometric functions, Bessel functions, exponentials, and powers

7.671

$$\begin{aligned}
1. \int_0^\infty x^{k-\frac{3}{2}} \exp \left[-\frac{1}{2}(a+1)x \right] K_\nu \left(\frac{1}{2}ax \right) M_{k, \nu}(x) dx &= \\
&= \frac{\pi^{\frac{1}{2}} \Gamma(k) \Gamma(k+2\nu)}{a^{k+\nu} \Gamma \left(k + \nu + \frac{1}{2} \right)} {}_2F_1 \left(k, k+2\nu; 2\nu+1; -\frac{1}{a} \right)
\end{aligned}$$

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$$\begin{aligned}
2. \int_0^\infty x^{-k-\frac{3}{2}} \exp\left[-\frac{1}{2}(a-1)x\right] K_\mu\left(\frac{1}{2}ax\right) W_{k,\mu}(x) dx &= \\
&= \frac{\pi\Gamma(-k)\Gamma(2\mu-k)\Gamma(-2\mu-k)}{\Gamma\left(\frac{1}{2}-k\right)\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} 2^{2k+1} a^{k-\nu} {}_2F_1(-k, 2\mu-k; -2k; 1-a^{-1}) \\
&\quad [\operatorname{Re} a > 0, \quad \operatorname{Re} k < 2 \operatorname{Re} \mu < -\operatorname{Re} k].
\end{aligned}$$

ET II 408(36)

880
7.672

$$\begin{aligned}
1. \int_0^\infty x^{2\varrho} e^{-\frac{1}{2}ax^2} M_{k,\mu}(ax^2) J_\nu(xy) dx &= \\
&= \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu+k+\frac{1}{2}\right)} 2^{2\varrho} y^{-2\varrho-1} G_{23}^{21} \left(\frac{y^2}{4a} \left| \begin{array}{c} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\varrho+\frac{1}{2}\nu, k, \frac{1}{2}+\varrho-\frac{1}{2}\nu \end{array} \right. \right) \\
&\quad \left[y > 0, \quad -1 - \operatorname{Re} \left(\frac{1}{2}\nu + \mu \right) < \operatorname{Re} \varrho < \operatorname{Re} k - \frac{1}{4}, \quad \operatorname{Re} a > 0 \right].
\end{aligned}$$

ET II 83(10)

$$\begin{aligned}
2. \int_0^\infty x^{2\varrho} e^{-\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx &= \\
&= \frac{\Gamma\left(1+\mu+\frac{1}{2}\nu+\varrho\right)\Gamma\left(1-\mu+\frac{1}{2}\nu+\varrho\right)2^{-\nu-1}}{\Gamma(\nu+1)\Gamma\left(\frac{3}{2}-k+\frac{1}{2}\nu+\varrho\right)} a^{-\frac{1}{2}\nu-\varrho-\frac{1}{2}} y^\nu \times \\
&\quad \times {}_2F_2\left(\lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-k+\lambda; -\frac{y^2}{4a}\right), \quad \lambda = 1 + \frac{1}{2}\nu + \varrho \\
&\quad \left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \left(\varrho \pm \mu + \frac{1}{2}\nu \right) > -1 \right].
\end{aligned}$$

ET II 85(16)

$$\begin{aligned}
3. \int_0^\infty x^{2\varrho} e^{\frac{1}{2}ax^2} W_{k,\mu}(ax^2) J_\nu(xy) dx &= \frac{2^{2\varrho} y^{-2\varrho-1}}{\Gamma\left(\frac{1}{2}+\mu-k\right)\Gamma\left(\frac{1}{2}-\mu-k\right)} \times \\
&\quad \times G_{23}^{22} \left(\frac{y^2}{4a} \left| \begin{array}{c} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\varrho+\frac{1}{2}\nu, -k, \frac{1}{2}+\varrho-\frac{1}{2}\nu \end{array} \right. \right) \\
&\quad \left[y > 0, \quad |\arg a| < \pi, \quad -1 - \operatorname{Re} \left(\frac{1}{2}\nu \pm \mu \right) < \operatorname{Re} \varrho < -\frac{1}{4} - \operatorname{Re} k \right].
\end{aligned}$$

$$\begin{aligned}
4. \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu} \left(\frac{1}{2}x^2 \right) N_\nu(xy) dx &= \\
&= \frac{2^\lambda y^{-\frac{1}{2}} \Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2}+k+\mu\right)} G_{34}^{31} \left(\frac{y^2}{2} \left| \begin{matrix} -\mu-\lambda, & \mu-\lambda, & l \\ h, & \{, & k-\lambda-\frac{1}{2}, & l \end{matrix} \right. \right), \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \{ = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
& \quad \left[y > 0, \quad \operatorname{Re}(k-\lambda) > 0, \quad \operatorname{Re}(2\lambda+2\mu\pm\nu) > -\frac{5}{2} \right].
\end{aligned}$$

ET II 116(45)

881

$$\begin{aligned}
5. \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu} \left(\frac{1}{2}x^2 \right) N_\nu(xy) dx &= \\
&= 2^\lambda \left[\Gamma\left(\frac{1}{2}-k+\mu\right) \Gamma\left(\frac{1}{2}-k-\mu\right) \right]^{-1} \times \\
& \quad \times G_{34}^{32} \left(\frac{y^2}{2} \left| \begin{matrix} -\mu-\lambda, & \mu-\lambda, & l \\ h, & \{, & -\frac{1}{2}-k-\lambda, & l \end{matrix} \right. \right) y^{-\frac{1}{2}}, \\
& \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad \{ = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
& \quad \left[y > 0, \quad \operatorname{Re}(k+\lambda) < 0, \quad \operatorname{Re}(2\lambda\pm 2\mu\pm\nu) > -\frac{5}{2} \right].
\end{aligned}$$

ET II 117(47)

$$\begin{aligned}
6. \int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu-\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(x^2) J_\nu(xy) dx &= (2\nu+1)2^{-\nu}y^{\nu-1} \left[1 - \Phi\left(\frac{1}{2}y\right) \right] \\
& \quad \left[y > 0, \quad \operatorname{Re}\nu > -\frac{1}{2} \right].
\end{aligned}$$

ET II 82(1)

$$7. \int_0^\infty x^{-1} e^{-\frac{1}{2}x^2} M_{\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}(x^2) J_\nu(xy) dx = \frac{\Gamma(\nu+2)y^\nu}{\Gamma\left(\nu+\frac{3}{2}\right)2^\nu} \left[1 - \Phi\left(\frac{1}{2}y\right) \right] \quad [y > 0, \operatorname{Re}\nu > -1].$$

ET II 82(2)

$$\begin{aligned}
8. \int_0^\infty e^{-\frac{1}{4}x^2} M_{k, \frac{1}{2}\nu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \frac{2^{-k}\Gamma(\nu+1)}{\Gamma\left(k+\frac{1}{2}\nu+\frac{1}{2}\right)} y^{2k-1} e^{-\frac{1}{2}y^2} \\
& \quad \left[y > 0, \quad \operatorname{Re}\nu > -1, \quad \operatorname{Re}k < \frac{1}{2} \right].
\end{aligned}$$

ET II 83(7)

$$\begin{aligned}
10. \quad \int_0^\infty x^{\nu-2\mu} e^{\frac{1}{4}x^2} W_{k, \pm\mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \\
&= \frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{k+\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right), \\
2\alpha &= \frac{1}{2} + k + \nu - 3\mu, \quad 2\beta = \frac{1}{2} - k + \nu - \mu \\
&[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1].
\end{aligned}$$

ET II 84(14)

882

$$\begin{aligned}
11. \quad \int_0^\infty x^{\nu-2\mu} e^{-\frac{1}{4}x^2} W_{k, \pm\mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \frac{\Gamma(1+\nu-2\mu)}{\Gamma\left(\frac{1}{2} + \mu - k\right)} 2^{\frac{1}{2}(\frac{1}{2}+k-3\mu+\nu)} y^{\mu-k-\frac{3}{2}} e^{\frac{1}{4}y^2} W_{\alpha, \beta} \left(\frac{1}{2}y^2 \right), \\
2\alpha &= k + 3\mu - \nu - \frac{1}{2}, \quad 2\beta = k - \mu + \nu + \frac{1}{2} \\
&[y > 0, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\nu - 2\mu) > -1, \quad \operatorname{Re}\left(k - \mu + \frac{1}{2}\nu\right) < -\frac{1}{4}].
\end{aligned}$$

ET II 84(15)

$$\begin{aligned}
12. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k, \mu} \left(\frac{1}{2}x^2 \right) J_\nu(xy) dx &= \\
&= \frac{\Gamma(2\mu+1)}{\Gamma\left(\frac{1}{2} + k - \mu + \nu\right)} 2^{\frac{1}{2}(\frac{1}{2}-k+3\mu-\nu)} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right), \\
2\alpha &= \frac{1}{2} + k + 3\mu - \nu, \quad 2\beta = -\frac{1}{2} + k - \mu + \nu \\
&[y > 0, \quad -\frac{1}{2} < \operatorname{Re} \mu < \operatorname{Re}\left(k + \frac{1}{2}\nu\right) - \frac{1}{4}].
\end{aligned}$$

ET II 83(8)

$$\begin{aligned}
13. \quad \int_0^\infty x^{2\mu-\nu} e^{-\frac{1}{4}x^2} M_{k, \mu} \left(\frac{1}{2}x^2 \right) N_\nu(xy) dx &= \\
&= \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} e^{-\frac{1}{4}y^2} \Gamma(2\mu+1) \times \\
&\quad \times \Gamma\left(\frac{1}{2} - k - \mu\right) \left\{ \cos[(\nu-2\mu)\pi] \frac{\Gamma(2\mu-\nu-1)}{\Gamma(2\beta+1)} M_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) - \right. \\
&\quad \left. - \sin[(\nu+k-\mu)\pi] W_{\alpha, \beta} \left(\frac{1}{2}y^2 \right) \right\}, \quad 2\alpha = 3\mu - \nu + k + \frac{1}{2}, \quad 2\beta = \mu - \nu - k + \frac{1}{2} \\
&[y > 0, \quad -1 < 2 \operatorname{Re} \mu < \operatorname{Re}(2k + \nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu - \nu) > -1].
\end{aligned}$$

$$\begin{aligned}
14. \quad & \int_0^{\infty} x^{2\mu+\nu} e^{-\frac{1}{4}x^2} M_{k,\mu} \left(\frac{1}{2}x^2 \right) N_{\nu}(xy) dx = \\
& = \pi^{-1} 2^{\mu+\beta} y^{k-\mu-\frac{3}{2}} \Gamma(2\mu+1) \times \\
& \quad \times \Gamma \left(\frac{1}{2} - \mu - k \right) e^{-\frac{1}{4}y^2} \left\{ \cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma \left(\mu + \nu - k + \frac{3}{2} \right)} M_{\alpha,\beta} \left(\frac{1}{2}y^2 \right) + \right. \\
& \quad \left. + \sin[(\mu-k)\pi] W_{\alpha,\beta} \left(\frac{1}{2}y^2 \right) \right\}, \quad 2\alpha = 3\mu + \nu + k + \frac{1}{2}, \quad 2\beta = \mu + \nu - k + \frac{1}{2} \\
& \quad \left[y > 0, \quad -1 < 2 \operatorname{Re} \mu < \operatorname{Re}(2k - \nu) + \frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -1 \right].
\end{aligned}$$

ET II 116(43)

883

$$\begin{aligned}
15. \quad & \int_0^{\infty} x^{2\mu+\nu} e^{-\frac{1}{2}ax^2} M_{k,\mu}(ax^2) K_{\nu}(xy) dx = 2^{\mu-k-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+k)} y^{k-\mu-\frac{3}{2}} \times \\
& \quad \times \Gamma(2\mu+1) \Gamma(2\mu+\nu+1) \exp \left(\frac{y^2}{8a} \right) W_{\{,m} \left(\frac{y^2}{4a} \right), \\
& \quad 2\{ = -3\mu - \nu - k - \frac{1}{2}, \quad 2m = \mu + \nu - k + \frac{1}{2} \\
& \quad \left[\operatorname{Re} y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu + \nu) > -1 \right].
\end{aligned}$$

ET II 152(82)

7.673

$$\begin{aligned}
1.* \quad & \int_0^{\infty} e^{-\frac{1}{2}ax} x^{\frac{1}{2}(\mu-\nu-1)} M_{\{\frac{1}{2}\mu} (ax) J_{\nu}(2\sqrt{bx}) dx = \\
& = \left(\frac{b}{a} \right)^{\frac{\frac{1}{2}-1}{2} - \frac{1+\mu}{4}} a^{-\frac{1}{2}(\mu+1-\nu)} \Gamma(1+\mu) e^{-\frac{b}{2a}} \frac{1}{\Gamma \left(1 + \frac{\{+\nu}{2} - \frac{1+\mu}{4} \right)} \times \\
& \quad \times M_{\frac{1}{2}(\{-\nu-1\} + \frac{3}{4}(1+\mu), \frac{\frac{1}{2}+\nu}{2} - \frac{1+\mu}{4})} \left(\frac{b}{a} \right) \\
& \quad \left[\operatorname{Re}(1+\mu) > 0, \quad \operatorname{Re} \left(\{ + \frac{\nu-\mu}{2} \right) > -\frac{3}{4}, \quad \operatorname{Im} b = 0 \right].
\end{aligned}$$

BU 128(12)a

$$\begin{aligned}
2. \quad & \int_0^{\infty} e^{\frac{1}{2}ax} x^{\frac{1}{2}(\nu-1\mp\mu)} W_{\{\frac{1}{2}\mu} (ax) J_{\nu}(2\sqrt{bx}) dx = a^{-\frac{1}{2}(\nu+1\mp\mu)} \frac{\Gamma(\nu+1\mp\mu) e^{\frac{b}{2a}}}{\Gamma \left(\frac{1\pm\mu}{2} - \{ \right)} \left(\frac{a}{b} \right)^{\frac{1}{2}(\{+1\} + \frac{1}{4}(1\mp\nu))} \times \\
& \quad \times W_{\frac{1}{2}(\{+1-\nu\} - \frac{3}{4}(1\mp\mu), \frac{1}{2}(\{+\nu\} + \frac{1}{4}(1\mp\mu))} \left(\frac{b}{a} \right) \\
& \quad \left[\operatorname{Re} \left(\frac{\nu\mp\mu}{2} + \{ \right) < \frac{3}{4}, \quad \operatorname{Re} \nu > -1 \right].
\end{aligned}$$

BU 128(13)

7.674

$$2. \int_0^\infty x^{\varrho-1} e^{-\frac{1}{2}x} \{ I_{\lambda+\nu}(ax^{\frac{1}{2}}) K_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{k,\mu}(x) \} dx = \frac{\pi^{-\frac{1}{2}}}{2} G_{45}^{24} \left(a^2 \left| \begin{array}{c} 0, \frac{1}{2}, \frac{1}{2} + \mu - \varrho, \frac{1}{2} - \mu - \varrho \\ \lambda, \nu, -\lambda, -\nu, k - \varrho \end{array} \right. \right) \\ \left[|\operatorname{Re} \mu| < \operatorname{Re}(\lambda + \varrho) + \frac{1}{2}, \quad |\operatorname{Re} \mu| < \operatorname{Re}(\nu + \varrho) + \frac{1}{2} \right].$$

ET II 409(38)

Combinations of Struve functions and confluent hypergeometric functions

7.675

$$1. \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} M_{k,\mu} \left(\frac{1}{2}x^2 \right) H_\nu(xy) dx = \frac{2^{-\lambda} \Gamma(2\mu+1)}{y^{\frac{1}{2}} \Gamma\left(\frac{1}{2} + k + \mu\right)} G_{34}^{22} \left(\frac{y^2}{2} \left| \begin{array}{c} l, -\mu - \lambda, \mu - \lambda \\ l, k - \lambda - \frac{1}{2}, h, \{ \end{array} \right. \right), \\ h = \frac{1}{4} + \frac{1}{2}\nu, \quad \{ = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu \\ \left[\operatorname{Re}(2\lambda + 2\mu + \nu) > -\frac{7}{2}, \operatorname{Re}(k - \lambda) > 0, y > 0, \operatorname{Re}(2\lambda - 2k + \nu) < -\frac{1}{2} \right].$$

ET II 171(42)

$$2. \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,\mu} \left(\frac{1}{2}x^2 \right) H_\nu(xy) dx = \\ = 2^{\frac{1}{4}-\lambda-\frac{1}{2}\nu} \pi^{-\frac{1}{2}} y^{\nu+1} \frac{\Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda + \mu\right) \Gamma\left(\frac{7}{4} + \frac{1}{2}\nu + \lambda - \mu\right)}{\Gamma\left(\nu + \frac{3}{2}\right) \Gamma\left(\frac{9}{4} + \lambda - k - \frac{1}{2}\nu\right)} \times \\ \times {}_3F_3 \left(1, \frac{7}{4} + \frac{\nu}{2} + \lambda + \mu, \frac{7}{4} + \frac{\nu}{2} + \lambda - \mu; \frac{3}{2}, \nu + \frac{3}{2}, \frac{9}{4} + \lambda - k + \frac{\nu}{2}; -\frac{y^2}{2} \right) \\ \left[\operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{4}, \quad y > 0 \right].$$

ET II 171(43)

$$3. \int_0^\infty x^{2\lambda+\frac{1}{2}} e^{\frac{1}{4}x^2} W_{k,\mu} \left(\frac{1}{2}x^2 \right) H_\nu(xy) dx = \\ = \left[2^\lambda \Gamma\left(\frac{1}{2} - k + \mu\right) \Gamma\left(\frac{1}{2} - k - \mu\right) \right]^{-1} y^{-\frac{1}{2}} G_{34}^{23} \left(\frac{y^2}{2} \left| \begin{array}{c} l, -\mu - \lambda, \mu - \lambda \\ l, -k - \lambda - \frac{1}{2}, h, \{ \end{array} \right. \right), \\ h = \frac{1}{4} + \frac{1}{2}\nu, \quad \{ = \frac{1}{4} - \frac{1}{2}\nu, \quad l = \frac{3}{4} + \frac{1}{2}\nu$$

$$\left[y > 0, \operatorname{Re}(2\lambda + \nu) > 2|\operatorname{Re} \mu| - \frac{7}{2}, \operatorname{Re}(2k + 2\lambda + \nu) < -\frac{1}{2}, \operatorname{Re}(k + \lambda) < 0 \right].$$

$$4. \int_0^{\infty} e^{\frac{1}{2}x^2} W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu}(x^2) H_{\nu}(xy) dx = 2^{-\nu-1} y^{\nu} \pi e^{\frac{1}{4}y^2} \left[1 - \Phi\left(\frac{y}{2}\right) \right] \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 171(44)

7.68 Combinations of confluent hypergeometric functions and other special functions

Combinations of confluent hypergeometric functions and associated Legendre functions

7.681

$$1. \int_0^{\infty} x^{-\frac{1}{2}} (a+x)^{\mu} e^{-\frac{1}{2}x} P_{\nu}^{-2\mu} \left(1 + 2\frac{x}{a} \right) M_{k, \mu}(x) dx =$$

$$= -\frac{\sin(\nu\pi)}{\pi\Gamma(k)} \Gamma(2\mu+1) \Gamma\left(k-\mu+\nu+\frac{1}{2}\right) \Gamma\left(k-\mu-\nu-\frac{1}{2}\right) e^{\frac{1}{2}a} W_{\varrho, \sigma}(a),$$

$$\varrho = \frac{1}{2} - k + \mu, \quad \sigma = \frac{1}{2} + \nu$$

$$\left[|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k-\mu) > \left| \operatorname{Re} \nu + \frac{1}{2} \right| \right].$$

ET II 403(11)

$$2. \int_0^{\infty} x^{-\frac{1}{2}} (a+x)^{-\mu} e^{-\frac{1}{2}x} P_{\nu}^{-2\mu} \left(1 + 2\frac{x}{a} \right) M_{k, \mu}(x) dx =$$

$$= \frac{\Gamma(2\mu+1) \Gamma\left(k+\mu+\nu+\frac{1}{2}\right) \Gamma\left(k+\mu-\nu-\frac{1}{2}\right) e^{\frac{1}{2}a}}{\Gamma\left(k+\mu+\frac{1}{2}\right) \Gamma(2\mu+\nu+1) \Gamma(2\mu-\nu)} W_{\frac{1}{2}-k-\mu, \frac{1}{2}+\nu}(a)$$

$$\left[|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu) > \left| \operatorname{Re} \nu + \frac{1}{2} \right| \right].$$

ET II 403(12)

$$3. \int_0^{\infty} x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\nu-\frac{3}{2}}^{\mu} \left(1 + 2\frac{x}{a} \right) W_{k, \nu}(x) dx =$$

$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho, \sigma}(a), \quad 2\varrho = \frac{1}{2} + 2\mu + \nu - k, \quad 2\sigma = k + 3\nu - \frac{3}{2}$$

$$[|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + 2\nu) < 1].$$

ET II 407(32)

$$4. \int_0^{\infty} x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{k+\mu+\nu-\frac{3}{2}}^{\mu} \left(1 + 2\frac{x}{a} \right) W_{k, \nu}(x) dx =$$

$$= \frac{\Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho, \sigma}(a), \quad 2\varrho = \frac{1}{2} - k + \nu, \quad 2\sigma = k + 2\mu + 3\nu - \frac{3}{2}$$

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$$[|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + 2\nu) < 1].$$

$$\begin{aligned}
5. \int_0^\infty x^{\mu-\frac{1}{4}k-\frac{1}{2}\nu-\frac{1}{2}}(a+x)^{\frac{1}{2}\nu}e^{-\frac{1}{2}x}Q_{\mu-k+\frac{3}{2}}^\nu\left(1+2\frac{x}{a}\right)M_{k,\nu}(x)dx &= \\
&= \frac{e^{\nu\pi i}\Gamma(1+2\mu-\nu)\Gamma(1+2\mu)\Gamma\left(\frac{5}{2}-k+\mu+\nu\right)}{2\Gamma\left(\frac{1}{2}+k+\mu\right)}a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)}e^{\frac{1}{2}a}W_{\varrho,\sigma}(a), \\
2\varrho &= \frac{1}{2}-k-\mu+2\nu, \quad 2\sigma = k-3\mu-\frac{3}{2} \\
&\left[|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu-\nu) > -1\right].
\end{aligned}$$

ET II 404(14)

7.682

$$\begin{aligned}
1. \int_0^\infty x^{-\frac{1}{2}}e^{-\frac{1}{2}x}P_\nu^{-2\mu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\mu}(x)dx &= \\
&= \frac{\Gamma(2\mu+1)\Gamma\left(k+\frac{1}{2}\nu\right)\Gamma\left(k-\frac{1}{2}\nu-\frac{1}{2}\right)e^{\frac{1}{2}a}}{2^{2\mu}a^{\frac{1}{4}}\Gamma\left(k+\mu+\frac{1}{2}\right)\Gamma\left(\mu+\frac{1}{2}\nu+\frac{1}{2}\right)\Gamma\left(\mu-\frac{1}{2}\nu\right)}W_{\frac{3}{4}-k,\frac{1}{4}+\frac{1}{2}\nu}(a) \\
&\left[|\arg a| < \pi, \quad \operatorname{Re} k > \frac{1}{2}\operatorname{Re} \nu - \frac{1}{2}, \quad \operatorname{Re} k > -\frac{1}{2}\operatorname{Re} \nu\right].
\end{aligned}$$

ET II 404(13)

$$\begin{aligned}
2. \int_0^\infty x^{\frac{1}{2}(k+\mu+\nu)-1}(a+x)^{-\frac{1}{2}}e^{-\frac{1}{2}x}Q_{k-\mu-\nu-1}^{1-k+\mu-\nu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\mu}(x)dx &= \\
&= e^{(1-k+\mu-\nu)\pi i}2^{\mu-k-\nu}a^{\frac{1}{2}(k+\mu-1)}\frac{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma(1+2\mu)\Gamma(k+\mu+\nu)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}e^{\frac{1}{2}a}W_{\varrho,\sigma}(a), \\
2\varrho &= \frac{1}{2}-k-\frac{1}{2}\nu, \quad \sigma = \mu + \frac{1}{2}\nu \\
&\left[|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(k+\mu+\nu) > 0\right].
\end{aligned}$$

ET II 404(15)

$$\begin{aligned}
3. \int_0^\infty x^{\nu-\frac{1}{2}}e^{-\frac{1}{2}x}Q_{2k-2\nu-3}^{2\mu-2\nu}\left[\left(1+\frac{x}{a}\right)^{\frac{1}{2}}\right]M_{k,\mu}(x)dx &= \\
&= e^{2(\mu-\nu)\pi i}2^{2\mu-2\nu-1}a^{\frac{1}{2}(k+\mu-1)}e^{\frac{1}{2}a}\frac{\Gamma(2\mu+1)\Gamma(\nu+1)\Gamma\left(k+\mu-2\nu-\frac{1}{2}\right)}{\Gamma\left(k+\mu+\frac{1}{2}\right)}W_{\varrho,\sigma}(a), \\
2\varrho &= 1-k+\mu-2\nu, \quad 2\sigma = k-\mu-2\nu-2 \\
&\left[|\arg a| < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(k+\mu-2\nu) > \frac{1}{2}\right].
\end{aligned}$$

$$\begin{aligned}
4. \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-3}^\mu \left[\left(1 + \frac{x}{a}\right)^{\frac{1}{2}} \right] W_{k,\nu}(x) dx &= \\
&= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\sigma}(a), \\
2\varrho &= 1-k+\mu+\nu, \quad 2\sigma = k+\mu+3\nu-2 \\
[|\arg a| < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1].
\end{aligned}$$

ET II 408(34)

$$\begin{aligned}
5.* \int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} P_{2k+\mu+2\nu-2}^\mu \left[\left(1 + \frac{x}{a}\right)^{\frac{1}{2}} \right] W_{k,\nu}(x) dx &= \\
&= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma\left(\frac{3}{2}-k-\mu-\nu\right)} a^{-\frac{1}{2}+\frac{1}{2}k-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\varrho,\sigma}(a), \\
2\varrho &= \mu+\nu-k, \quad 2\sigma = k+\mu+3\nu-1 \quad [|\arg a| < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0].
\end{aligned}$$

ET II 408(35)

A combination of confluent hypergeometric functions and orthogonal polynomials

7.683⁷

$$\begin{aligned}
\int_0^1 e^{-\frac{1}{2}ax} x^\alpha (1-x)^{\frac{\mu-\alpha}{2}-1} L_n^\alpha(ax) M_{\{-\frac{1+\alpha}{2}, \frac{\mu-\alpha-1}{2}\}} [a(1-x)] dx &= \\
&= \frac{\Gamma(\mu-\alpha)}{\Gamma(1+\mu)} \frac{\Gamma(1+n+\alpha)}{n!} a^{-\frac{1+\alpha}{2}} M_{\{+n, \frac{\mu}{2}\}}(a) \\
&\quad [\operatorname{Re} a > -1, \quad \operatorname{Re}(\mu-\alpha) > 0, \quad n = 0, 1, 2, \dots].
\end{aligned}$$

BU 129(14b)

A combination of hypergeometric and confluent hypergeometric functions

7.684

$$\begin{aligned}
\int_0^\infty x^{\varrho-1} e^{-\frac{1}{2}x} M_{\gamma+\varrho, \beta+\varrho+\frac{1}{2}}(x) {}_2F_1\left(\alpha, \beta; \gamma; -\frac{\lambda}{x}\right) dx &= \\
&= \frac{\Gamma(\alpha+\beta+2\varrho)\Gamma(2\beta+2\varrho)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta+\gamma+2\varrho)} \lambda^{\frac{1}{2}\beta+\varrho-\frac{1}{2}} e^{\frac{1}{2}\lambda} W_{k,\mu}(\lambda); \\
k &= \frac{1}{2} - \alpha - \frac{1}{2}\beta - \varrho, \quad \mu = \frac{1}{2}\beta + \varrho \\
[|\arg \lambda| < \pi, \quad \operatorname{Re}(\beta+\varrho) > 0, \quad \operatorname{Re}(\alpha+\beta+2\varrho) > 0, \quad \operatorname{Re} \gamma > 0].
\end{aligned}$$

ET II 405(19)

7.69 Integration of confluent hypergeometric functions with respect to the index

7.691

$$\int_{-\infty}^{\infty} \operatorname{sech}(\pi x) W_{ix,0}(\alpha) W_{-ix,0}(\beta) dx = 2 \frac{(a\beta)^{\frac{1}{2}}}{\alpha + \beta} \exp\left[-\frac{1}{2}(\alpha + \beta)\right].$$

ET II 414(61)

7.692

$$\int_{-i\infty}^{i\infty} \Gamma(-a)\Gamma(c-a)\Psi(a, c; x)\Psi(c-a, c; y) da = 2\pi i\Gamma(c)\Psi(c, 2c; x+y).$$

EH I 285(15)

888

7.693

$$\begin{aligned} 1. \int_{-\infty}^{\infty} \Gamma(ix)\Gamma(2k+ix)W_{k+ix, k-\frac{1}{2}}(\alpha)W_{-k-ix, k-\frac{1}{2}}(\beta) dx = \\ = 2\pi^{\frac{1}{2}}\Gamma(2k)(a\beta)^k(\alpha + \beta)^{\frac{1}{2}-2k}K_{2k-\frac{1}{2}}\left(\frac{\alpha + \beta}{2}\right). \end{aligned}$$

ET II 414(62)

$$\begin{aligned} 2. \int_{-i\infty}^{i\infty} \Gamma\left(\frac{1}{2} + \nu + \mu + x\right)\Gamma\left(\frac{1}{2} + \nu + \mu - x\right) \times \\ = \times \Gamma\left(\frac{1}{2} + \nu - \mu + x\right)\Gamma\left(\frac{1}{2} + \nu - \mu - x\right) M_{\mu+ix, \nu}(\alpha)M_{\mu-ix, \nu}(\beta) dx = \\ = \frac{2\pi(a\beta)^{\nu+\frac{1}{2}}[\Gamma(2\nu+1)]^2\Gamma(2\nu+2\mu+1)\Gamma(2\nu-2\mu+1)}{(\alpha + \beta)^{2\nu+1}\Gamma(4\nu+2)} M_{2\mu, 2\nu+\frac{1}{2}}(\alpha + \beta) \\ \left[\operatorname{Re} \nu > |\operatorname{Re} \mu| - \frac{1}{2} \right]. \end{aligned}$$

ET II 413(59)

7.694

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2\varrho x i} \Gamma\left(\frac{1}{2} + \nu + ix\right)\Gamma\left(\frac{1}{2} + \nu - ix\right) M_{ix, \nu}(\alpha)M_{ix, \nu}(\beta) dx = \\ = \frac{2\pi(\alpha\beta)^{\frac{1}{2}}}{\operatorname{ch} \varrho} \exp[-(\alpha + \beta)\theta\varrho] J_{2\nu}\left(\frac{2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\operatorname{ch} \varrho}\right) \left[|\operatorname{Im} \varrho| < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2} \right]. \end{aligned}$$

ET II 414(60)

7.7 Parabolic-Cylinder Functions* See Whitaker, E. T., & Watson, G. N.,
Modern Analysis, Cambridge University Press 1952, page 437 for
 definition.

7.71 Parabolic-cylinder functions

7.711

$$1. \int_{-\infty}^{\infty} D_n(x)D_m(x) dx = 0 \quad [m \neq n];$$

$$= n!(2\pi)^{\frac{1}{2}} \quad [m = n].$$

WH

$$2. \int_0^{\infty} D_{\mu}(\pm t)D_{\nu}(t) dt = \frac{\pi 2^{\frac{1}{2}(\mu+\nu+1)}}{\mu - \nu} \left[\frac{1}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\mu\right)\Gamma\left(-\frac{1}{2}\nu\right)} \mp \frac{1}{\Gamma\left(\frac{1}{2} - \frac{1}{2}\nu\right)\Gamma\left(-\frac{1}{2}\mu\right)} \right]$$

[when the lower sign is taken, $\text{Re } \mu > \text{Re } \nu$].

BU 11 117(13a), EH II 122(21)

$$3. \int_0^{\infty} [D_{\nu}(t)]^2 dt = \pi^{\frac{1}{2}} 2^{-\frac{3}{2}} \frac{\psi\left(\frac{1}{2} - \frac{1}{2}\nu\right) - \psi\left(-\frac{1}{2}\nu\right)}{\Gamma(-\nu)}.$$

BU 117(13b)A, EH II 122(22)a

889

7.72 Combinations of parabolic-cylinder functions, powers, and exponentials

7.721

$$1. \int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2} (x - z)^{-1} D_n(x) dx = \pm i e^{\mp n\pi i} (2\pi)^{\frac{1}{2}} n! e^{-\frac{1}{4}z^2} D_{-n-1}(\mp iz)$$

$$2. \int_1^{\infty} x^{\nu} (x-1)^{\frac{1}{2}\mu - \frac{1}{2}\nu - 1} \exp\left[-\frac{(x-1)^2 a^2}{4}\right] D_{\mu}(ax) dx = 2^{\mu-\nu-2} a^{\frac{\mu}{2} - \frac{\nu}{2} - 1} \Gamma\left(\frac{\mu-\nu}{2}\right) D_{\nu}(a) \quad [\operatorname{Re}(\mu - \nu) > 0].$$

ET II 395(4)a
WH

7.722

$$1. \int_0^{\infty} e^{-\frac{3}{4}x^2} x^{\nu} D_{\nu+1}(x) dx = 2^{-\frac{1}{2} - \frac{1}{2}\nu} \Gamma(\nu+1) \sin \frac{1}{4}(1-\nu)\pi \quad [\operatorname{Re} \nu > -1].$$

WH

$$2. \int_0^{\infty} e^{-\frac{1}{4}x^2} x^{\mu-1} D_{-\nu}(x) dx = \frac{\pi^{\frac{1}{2}} 2^{-\frac{1}{2}\mu - \frac{1}{2}\nu} \Gamma(\mu)}{\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)} \quad [\operatorname{Re} \mu > 0].$$

EH II 122(20)

$$3. \int_0^{\infty} e^{-\frac{3}{4}x^2} x^{\nu} D_{\nu-1}(x) dx = 2^{-\frac{1}{2}\nu-1} \Gamma(\nu) \sin \frac{1}{4}\pi\nu \quad [\operatorname{Re} \nu > -1].$$

ET II 395(2)

7.723

$$1. \int_0^{\infty} e^{-\frac{1}{4}x^2} x^{\nu} (x^2+y^2)^{-1} D_{\nu}(x) dx = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \Gamma(\nu+1) y^{\nu-1} e^{\frac{1}{4}y^2} D_{-\nu-1}(y) \quad [\operatorname{Re} y > 0, \operatorname{Re} \nu > -1].$$

EH II 121(18)A, ET II 396(6)a

$$2. \int_0^{\infty} e^{-\frac{1}{4}x^2} x^{\nu-1} (x^2+y^2)^{-\frac{1}{2}} D_{\nu}(x) dx = y^{\nu-1} \Gamma(\nu) e^{\frac{1}{4}y^2} D_{-\nu}(y) \quad [\operatorname{Re} y > 0, \operatorname{Re} \nu > 0].$$

ET II 396(7)

$$3. \int_0^1 x^{2\nu-1} (1-x^2)^{\lambda-1} e^{\frac{a^2 x^2}{4}} D_{-2\lambda-2\nu}(ax) dx = \frac{\Gamma(\lambda)\Gamma(2\nu)}{\Gamma(2\lambda+2\nu)} 2^{\lambda-1} e^{\frac{a^2}{4}} D_{-2\nu}(a) \quad [\operatorname{Re} \lambda > 0, \operatorname{Re} \nu > 0].$$

ET II 395(3)a

7.724

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\mu}} e^{\frac{1}{4}x^2} D_{\nu}(x) dx = (2\pi\mu)^{\frac{1}{2}} (1-\mu)^{\frac{1}{2}\nu} e^{\frac{y^2}{4-4\mu}} D_{\nu}[y(1-\mu)^{-\frac{1}{2}}] \quad [0 < \operatorname{Re} \mu < 1].$$

EH II 121(15)

7.725

$$1. \int_0^{\infty} e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu-2}(\sqrt{2t}) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(\sqrt{p+1}-1)^{\nu+1}}{(\nu+1)p^{\nu+1}} \quad [\operatorname{Re} \nu > -1].$$

MO 175

890

$$2. \int_0^{\infty} e^{-pt} (2t)^{\frac{\nu-1}{2}} e^{-\frac{t}{2}} D_{-\nu}(\sqrt{2t}) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{(\sqrt{p+1}-1)^{\nu}}{p^{\nu}\sqrt{p+1}} \quad [\operatorname{Re} \nu > -1].$$

MO 175

$$3. \int_0^{\infty} e^{-bx} D_{2n+1}(\sqrt{2x}) dx = (-2)^n \Gamma\left(n + \frac{3}{2}\right) \left(b - \frac{1}{2}\right)^n \left(b + \frac{1}{2}\right)^{-n-\frac{3}{2}} \quad \left[\operatorname{Re} b > -\frac{1}{2}\right].$$

ET I 210(3)

$$4. \int_0^{\infty} (\sqrt{x})^{-1} e^{-bx} D_{2n}(\sqrt{2x}) dx = (-2)^n \Gamma\left(n + \frac{1}{2}\right) \left(b - \frac{1}{2}\right)^n \left(b + \frac{1}{2}\right)^{-n-\frac{1}{2}} \quad \left[\operatorname{Re} b > -\frac{1}{2}\right].$$

ET I 210(5)

$$5. \int_0^{\infty} x^{-\frac{1}{2}(\nu+1)} e^{-sx} D_{\nu}(\sqrt{x}) dx = \sqrt{\pi} \left(1 + \sqrt{\frac{1}{2} + 2s}\right)^{\nu} \frac{1}{\sqrt{\frac{1}{4} + s}} \quad \left[\operatorname{Re} s > -\frac{1}{4}, \operatorname{Re} \nu < 1\right].$$

ET I 210(7)

$$6. \int_0^{\infty} e^{-zt} t^{-1+\frac{\beta}{2}} D_{-\nu}[2(kt)^{\frac{1}{2}}] dt = \frac{2^{1-\beta-\frac{\nu}{2}} \pi^{\frac{1}{2}} \Gamma(\beta)}{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\beta + \frac{1}{2}\right)} (z+k)^{-\frac{\beta}{2}} F\left(\frac{\nu}{2}, \frac{\beta}{2}; \frac{\nu+\beta+1}{2}; \frac{z-k}{z+k}\right) \\ \left[\operatorname{Re}(z+k) > 0, \operatorname{Re} \frac{z}{k} > 0\right].$$

$$\int_{-\infty}^{\infty} e^{ixy - \frac{(1+\lambda)x^2}{4}} D_{\nu}[x(1-\lambda)^{\frac{1}{2}}] dx = (2\pi)^{\frac{1}{2}} \lambda^{\frac{1}{2}\nu} e^{-\frac{(1+\lambda)y^2}{4\lambda}} D_{\nu}[i(\lambda^{-1}-1)^{\frac{1}{2}}y] \quad [\operatorname{Re} \lambda > 0].$$

EH II 121(16)

7.727

$$\int_0^{\infty} \frac{e^{\frac{1}{2}x} e^{-bx}}{(e^x - 1)^{\mu + \frac{1}{2}}} \exp\left(-\frac{a}{1 - e^{-x}}\right) D_{2\mu}\left(\frac{2\sqrt{a}}{\sqrt{1 - e^{-x}}}\right) dx = e^{-a} 2^{b+\mu} \Gamma(b+\mu) D_{-2b}(2\sqrt{a})$$

[$\operatorname{Re} a > 0, \operatorname{Re} b > -\operatorname{Re} \mu$].

ET I 211(13)

7.728

$$\int_0^{\infty} (2t)^{-\frac{k}{2}} e^{-pt} e^{-\frac{q^2}{8t}} D_{\nu-1}\left(\frac{q}{\sqrt{2t}}\right) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} p^{\frac{1}{2}\nu-1} e^{-q\sqrt{p}}.$$

MO 175

7.73 Combinations of parabolic-cylinder and hyperbolic functions

7.731

$$1. \int_0^{\infty} \operatorname{ch}(2\mu x) \exp[-(a \operatorname{sh} x)^2] D_{2k}(2a \operatorname{ch} x) dx = 2^{k-\frac{3}{2}} \pi^{\frac{1}{2}} a^{-1} W_{k, \mu}(2a^2) \quad [\operatorname{Re} a^2 > 0].$$

ET II 398(20)

$$2. \int_0^{\infty} \operatorname{ch}(2\mu x) \exp[(a \operatorname{sh} x)^2] D_{2k}(2a \operatorname{ch} x) dx = \frac{\Gamma(\mu - k) \Gamma(-\mu - k)}{2^{k+\frac{5}{2}} a \Gamma(-2k)} W_{k+\frac{1}{2}, \mu}(2a^2)$$

[$|\arg a| < \frac{3\pi}{4}, \operatorname{Re} k + |\operatorname{Re} \mu| < 0$].

ET II 398(21)

891

7.74 Combinations of parabolic-cylinder and trigonometric functions

7.741

$$1. \int_0^{\infty} \sin(bx) \{[D_{-n-1}(ix)]^2 - [D_{-n-1}(-ix)]^2\} dx = (-1)^{n+1} \frac{i}{n!} \pi \sqrt{2\pi} e^{-\frac{1}{2}b^2} L_n(b^2) \quad [b > 0].$$

$$2. \int_0^{\infty} e^{-\frac{1}{4}x^2} \sin(bx) D_{2n+1}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n+1} e^{-\frac{1}{2}b^2} \quad [b > 0].$$

ET I 115(1)

$$3. \int_0^{\infty} e^{-\frac{1}{4}x^2} \cos(bx) D_{2n}(x) dx = (-1)^n \sqrt{\frac{\pi}{2}} b^{2n} e^{-\frac{1}{2}b^2} \quad [b > 0].$$

ET I 60(2)

$$4. \int_0^{\infty} e^{-\frac{1}{4}x^2} \sin(bx) [D_{2\nu-\frac{1}{2}}(x) - D_{2\nu-\frac{1}{2}}(-x)] dx = \sqrt{2\pi} \sin \left[\left(\nu - \frac{1}{4} \right) \pi \right] b^{2\nu-\frac{1}{2}} e^{-\frac{1}{2}b^2} \\ \left[\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0 \right].$$

ET I 115(2)

$$5. \int_0^{\infty} e^{-\frac{1}{2}x^2} \cos(bx) [D_{2\nu-\frac{1}{2}}(x) + D_{2\nu-\frac{1}{2}}(-x)] dx = \frac{2^{\frac{1}{4}-2\nu} \sqrt{\pi} b^{2\nu-\frac{1}{2}} e^{-\frac{1}{4}b^2}}{\operatorname{cosec} \left[\left(\nu + \frac{1}{4} \right) \pi \right]} \left[\operatorname{Re} \nu > \frac{1}{4}, \quad b > 0 \right].$$

ET I 61(4)

7.742

$$1. \int_0^{\infty} x^{2\varrho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = 2^{\nu-\varrho-\frac{1}{2}} \pi^{\frac{1}{2}} a \frac{\Gamma(2\varrho+1)}{\Gamma(\varrho-\nu+1)} \times \\ \times {}_2F_2 \left(\varrho + \frac{1}{2}, \varrho + 1; \frac{3}{2}, \varrho - \nu + 1; -\frac{a^2}{2} \right) \left[\operatorname{Re} \varrho > -\frac{1}{2} \right].$$

ET II 396(8)

$$2. \int_0^{\infty} x^{2\varrho-1} \sin(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\varrho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left(\frac{a^2}{2} \left| \begin{array}{l} \frac{1}{2} - \varrho, 1 - \varrho \\ -\varrho - \nu, \frac{1}{2}, 0 \end{array} \right. \right) \\ \left[a > 0, \quad \operatorname{Re} \varrho > -\frac{1}{2}, \quad \operatorname{Re}(\varrho + \nu) < \frac{1}{2} \right].$$

ET II 396(9)

$$3. \int_0^{\infty} x^{2\varrho-1} \cos(ax) e^{-\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\nu-\varrho} \Gamma(2\varrho) \pi^{\frac{1}{2}}}{\Gamma \left(\varrho - \nu + \frac{1}{2} \right)} {}_2F_2 \left(\varrho, \varrho + \frac{1}{2}; \frac{1}{2}, \varrho - \nu + \frac{1}{2}; -\frac{a^2}{2} \right) \\ [\operatorname{Re} \varrho > 0].$$

$$4. \int_0^{\infty} x^{2\varrho-1} \cos(ax) e^{\frac{x^2}{4}} D_{2\nu}(x) dx = \frac{2^{\varrho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22} \left(\frac{a^2}{2} \left| \begin{matrix} \frac{1}{2} - \varrho, 1 - \varrho \\ -\varrho - \nu, 0, \frac{1}{2} \end{matrix} \right. \right) \\ \left[a > 0, \quad \operatorname{Re} \varrho > 0, \quad \operatorname{Re}(\varrho + \nu) < \frac{1}{2} \right].$$

ET II 396(11)

7.743

$$\int_0^{\frac{\pi}{2}} (\cos x)^{-\mu-2} (\sin x)^{-\nu} D_{\nu}(a \sin x) D_{\mu}(a \cos x) dx = - \left(\frac{1}{2} \pi \right)^{\frac{1}{2}} (1+\mu)^{-1} D_{\mu+\nu+1}(a) \\ [\operatorname{Re} \nu < 1, \quad \operatorname{Re} \mu < -1].$$

ET II 397(19)

7.744

$$1. \int_0^{\infty} \sin(bx) [D_{-\nu-\frac{1}{2}}(\sqrt{2x}) - D_{-\nu-\frac{1}{2}}(-\sqrt{2x})] D_{\nu-\frac{1}{2}}(\sqrt{2x}) dx = \\ = -\sqrt{2\pi} \sin \left[\left(\frac{1}{4} + \frac{1}{2} \nu \right) \pi \right] b^{-\nu-\frac{1}{2}} \frac{(1 + \sqrt{1+b^2})^{\nu}}{\sqrt{1+b^2}} \quad [b > 0].$$

ET I 115(4)

$$2. \int_0^{\infty} \cos(bx) [D_{-2\nu-\frac{1}{2}}(\sqrt{2x}) + D_{-2\nu-\frac{1}{2}}(-\sqrt{2x})] D_{2\nu-\frac{1}{2}}(\sqrt{2x}) dx = \\ = -\frac{\sqrt{\pi} \sin \left[\left(\nu - \frac{1}{4} \right) \pi \right] (1 + \sqrt{1+b^2})^{2\nu}}{\sqrt{1+b^2} b^{2\nu+\frac{1}{2}}} \quad [b > 0].$$

ET I 60(3)

7.75 Combinations of parabolic-cylinder and Bessel functions

7.751

$$1. \int_0^{\infty} [D_n(ax)]^2 J_1(xy) dx = (-1)^{n-1} y^{-1} \left[D_n \left(\frac{y}{a} \right) \right]^2 \quad [y > 0].$$

ET II 20(24)

$$2. \int_0^{\infty} J_0(xy) D_n(ax) D_{n+1}(ax) dx = (-1)^n y^{-1} D_n \left(\frac{y}{a} \right) D_{n+1} \left(\frac{y}{a} \right) \quad \left[y > 0, \quad |\arg a| < \frac{1}{4} \pi \right]$$

$$3. \int_0^{\infty} J_0(xy) D_{\nu}(x) D_{\nu+1}(x) dx = 2^{-1} y^{-1} [D_{\nu}(-y) D_{\nu+1}(y) - D_{\nu+1}(-y) D_{\nu}(y)].$$

ET II 397(17)a

7.752

$$1. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}x^2} D_{2\nu-1}(x) J_{\nu}(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu-1}(y) - D_{2\nu-1}(-y)]$$

$$\left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 76(1), MO 183

893

$$2. \int_0^{\infty} x^{\nu} e^{\frac{1}{4}x^2} D_{2\nu-1}(x) J_{\nu}(xy) dx = 2^{\frac{1}{2}-\nu} \pi \sin(\nu\pi) y^{-\nu} \Gamma(2\nu) e^{\frac{1}{4}y^2} K_{\nu} \left(\frac{1}{4}y^2 \right)$$

$$\left[y > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right].$$

ET II 77(4)

$$3. \int_0^{\infty} x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu}(x) J_{\nu}(xy) dx = \frac{1}{2} \sec(\nu\pi) y^{\nu-1} e^{-\frac{1}{4}y^2} [D_{2\nu+1}(y) - D_{2\nu+1}(-y)]$$

$$[y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 78(13)

$$4. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}x^2} D_{2\nu+1}(x) J_{\nu}(xy) dx = \frac{1}{2} \sec(\nu\pi) e^{-\frac{1}{4}y^2} y^{\nu} [D_{2\nu}(y) + D_{2\nu}(-y)]$$

$$\left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 77(5)

$$5. \int_0^{\infty} x^{\nu+1} e^{-\frac{1}{4}x^2} D_{2\nu+2}(x) J_{\nu}(xy) dx = -\frac{1}{2} \sec(\nu\pi) y^{\nu} e^{-\frac{1}{4}y^2} [D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$$

$$[\operatorname{Re} \nu > -1, \quad y > 0].$$

ET II 78(16)

$$6. \int_0^{\infty} x^{\nu+1} e^{\frac{1}{4}x^2} D_{2\nu+2}(x) J_{\nu}(xy) dx = \pi^{-1} \sin(\nu\pi) \Gamma(2\nu+3) y^{-\nu-2} e^{\frac{1}{4}y^2} K_{\nu+1} \left(\frac{1}{4}y^2 \right)$$

$$7. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}x^2} D_{-2\nu}(x) J_{\nu}(xy) dx = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{-\nu} e^{-\frac{1}{4}y^2} I_{\nu} \left(\frac{1}{4}y^2 \right) \quad \left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 77(8)

$$8. \int_0^{\infty} x^{\nu} e^{\frac{1}{4}x^2} D_{-2\nu}(x) J_{\nu}(xy) dx = y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu}(y) \quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad y > 0 \right].$$

ET II 77(9), EH II 121(17)

$$9. \int_0^{\infty} x^{\nu} e^{\frac{1}{4}x^2} D_{-2\nu-2}(x) J_{\nu}(xy) dx = (2\nu+1)^{-1} y^{\nu} e^{\frac{1}{4}y^2} D_{-2\nu-1}(y) \quad \left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 77(10)

$$10. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_{\nu}(xy) dx = \frac{2^{\mu-\frac{1}{2}} \Gamma \left(\nu + \frac{1}{2} \right) y^{\nu}}{\Gamma(\nu - \mu + 1) a^{1+2\nu}} {}_1F_1 \left(\nu + \frac{1}{2}; \nu - \mu + 1; -\frac{y^2}{2a^2} \right) \\ \left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 77(11)

$$11. \int_0^{\infty} x^{\nu} e^{\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_{\nu}(xy) dx = \frac{\Gamma \left(\frac{1}{2} + \nu \right) a^{2k} 2^{m+\mu}}{\Gamma \left(\frac{1}{2} - \mu \right) y^{\mu+\frac{3}{2}}} e^{\frac{y^2}{4a^2}} W_{k,m} \left(\frac{y^2}{4a^2} \right), \\ 2k = \frac{1}{2} + \mu - \nu, \quad 2m = \frac{1}{2} + \mu + \nu \\ \left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \operatorname{Re} \left(\frac{1}{2} - 2\mu \right) \right].$$

ET II 78(12)

894

$$12. \int_0^{\infty} x^{\nu+1} e^{-\frac{1}{4}a^2x^2} D_{2\mu}(ax) J_{\nu}(xy) dx = \frac{2^{\mu} \Gamma \left(\nu + \frac{3}{2} \right) y^{\nu}}{\Gamma \left(\nu - \mu + \frac{3}{2} \right) a^{2\nu+2}} {}_1F_1 \left(\nu + \frac{3}{2}; \nu - \mu + \frac{3}{2}; -\frac{y^2}{2a^2} \right) \\ \left[y > 0, \quad |\arg a| < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1 \right].$$

$$13. \int_0^\infty x^{\nu+1} e^{\frac{1}{4}a^2 x^2} D_{2\mu}(ax) J_\nu(xy) dx = \frac{\Gamma\left(\frac{3}{2} + \nu\right) 2^{\frac{1}{2}+m+\mu} a^{2k+1}}{\Gamma(-\mu)y^{\mu+2}} e^{\frac{y^2}{4a^2}} W_{k,m}\left(\frac{y^2}{2a^2}\right),$$

$$2k = \mu - \nu - 1, \quad 2m = \mu + \nu + 1$$

$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2} - 2 \operatorname{Re} \mu \right].$$

ET II 79(24)

$$14. \int_0^\infty x^{\lambda+\frac{1}{2}} e^{\frac{1}{4}a^2 x^2} D_\mu(ax) J_\nu(xy) dx = \frac{2^{\lambda-\frac{1}{2}} \pi^{-\frac{1}{2}}}{\Gamma(-\mu)y^{\lambda+\frac{3}{2}}} G_{23}^{22} \left(\frac{y^2}{2a^2} \left| \begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right)$$

$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad \operatorname{Re} \mu < -\operatorname{Re} \lambda < \operatorname{Re} \nu + \frac{3}{2} \right].$$

ET II 80(26)

$$15. \int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-1}(x) J_\nu(xy) dx = (2\nu+1)y^{\nu-1} e^{\frac{1}{4}y^2} D_{-2\nu-2}(y) \quad \left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 79(20)

$$16. \int_0^\infty x^{\nu+1} e^{-\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{-\nu-2} e^{-\frac{1}{4}y^2} I_{\nu+1}\left(\frac{1}{4}y^2\right) \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 79(21)

$$17. \int_0^\infty x^{\nu+1} e^{\frac{1}{4}x^2} D_{-2\nu-3}(x) J_\nu(xy) dx = y^\nu e^{\frac{1}{4}y^2} D_{-2\nu-3}(y) \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 79(22)

$$18. \int_0^\infty x^\nu e^{\frac{1}{4}a^2 x^2} D_{\frac{1}{2}\nu-\frac{1}{2}}(ax) N_\nu(xy) dx = -\pi^{-1} 2^{\frac{3}{4}\nu+\frac{3}{4}} a^{-\nu} y^{-1} \Gamma(\nu+1) e^{\frac{y^2}{4a^2}} W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu}\left(\frac{y^2}{2a^2}\right)$$

$$\left[y > 0, \quad |\arg a| < \frac{3}{4}\pi, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{2}{3} \right].$$

ET II 115(39)

7.753

$$1. \int_0^\infty x^{\nu-\frac{1}{2}} e^{-(x+a)^2} I_{\nu-\frac{1}{2}}(2ax) D_\nu(2x) dx = \frac{1}{2} \pi^{-\frac{1}{2}} \Gamma(\nu) a^{\nu-\frac{1}{2}} D_{-\nu}(2a) \quad [\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 0].$$

ET II 397(12)

895
7.754

$$1. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}x^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu-1}(x) - D_{2\nu-1}(-x)\} J_{\nu}(xy) dx = \\ = \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu-1}(y) - D_{2\nu-1}(-y)\} \quad \left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 76(2, 3)

$$2. \int_0^{\infty} x^{\nu} e^{-\frac{1}{4}x^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu+1}(x) - D_{2\nu+1}(-x)\} J_{\nu}(xy) dx = \\ = \mp y^{\nu} e^{-\frac{1}{4}y^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu}(y) + D_{2\nu}(-y)\} \quad \left[y > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 77(6, 7)

$$3. \int_0^{\infty} x^{\nu+1} e^{-\frac{1}{4}x^2} \{[1 \pm 2 \cos(\nu\pi)]D_{2\nu}(x) + D_{2\nu}(-x)\} J_{\nu}(xy) dx = \\ = \pm y^{\nu-1} e^{-\frac{1}{4}y^2} \{[1 \pm 2 \cos(\nu\pi)]D_{2\nu+1}(y) - D_{2\nu+1}(-y)\} \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 78(14, 15)

$$4. \int_0^{\infty} x^{\nu+1} e^{-\frac{1}{4}x^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu+2}(x) + D_{2\nu+2}(-x)\} J_{\nu}(xy) dx = \\ = \pm y^{\nu} e^{-\frac{1}{4}y^2} \{[1 \mp 2 \cos(\nu\pi)]D_{2\nu+2}(y) + D_{2\nu+2}(-y)\} \quad [y > 0, \quad \operatorname{Re} \nu > -1].$$

ET II 78(17, 18)

7.755

$$1. \int_0^{\infty} x^{-\frac{1}{2}} D_{\nu}(a^{\frac{1}{2}} x^{\frac{1}{2}}) D_{-\nu-1}(a^{\frac{1}{2}} x^{\frac{1}{2}}) J_0(xy) dx = \\ = 2^{-\frac{3}{2}} \pi a^{-\frac{1}{2}} P_{-\frac{1}{4}}^{\frac{1}{2}\nu+\frac{1}{4}} \left[\left(1 + \frac{4y^2}{a^2} \right)^{\frac{1}{2}} \right] P_{\frac{1}{4}}^{\frac{1}{2}\nu-\frac{1}{4}} \left[\left(1 + \frac{4y^2}{a^2} \right)^{\frac{1}{2}} \right] \quad [y > 0, \quad \operatorname{Re} a > 0].$$

ET II 17(43)

$$3. \int_0^{\infty} D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i}x^{-\frac{1}{2}})D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i}x^{-\frac{1}{2}})J_{\nu}(xy) dx = 2^{\frac{1}{2}}\pi^{\frac{1}{2}}y^{-1} \left[\Gamma\left(\nu + \frac{1}{2}\right) \right]^{-1} \exp[-a(2y)^{\frac{1}{2}}] \\ \left[y > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2} \right].$$

ET II 80(28)a

896

$$4. \int_0^{\infty} x^{\frac{1}{2}}D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})N_{\nu}(xy) dx = y^{-\frac{3}{2}} \exp(-ay^{\frac{1}{2}}) \sin \left[ay^{\frac{1}{2}} - \frac{1}{2} \left(\nu - \frac{1}{2} \right) \pi \right]. \\ \left[y > 0, \quad |\arg a| < \frac{1}{4}\pi \right].$$

ET II 115(40)

$$5. \int_0^{\infty} x^{\frac{1}{2}}D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})K_{\nu}(xy) dx = 2^{-1}y^{-\frac{3}{2}}\pi \exp[-a(2y)^{\frac{1}{2}}] \\ [\operatorname{Re} y > 0, \quad |\arg a| < \frac{1}{4}\pi].$$

ET II 151(81)

Combinations of parabolic-cylinder and Struve functions

7.756

$$\int_0^{\infty} x^{-\nu}e^{-\frac{1}{4}x^2}[D_{\mu}(x) - D_{\mu}(-x)]\mathbf{H}_{\nu}(xy) dx = \\ = \frac{2^{\frac{3}{2}}\Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu + \nu + 1\right)}y^{\mu+\nu} \sin\left(\frac{1}{2}\mu\pi\right) {}_1F_1\left(\frac{1}{2}\mu + \frac{1}{2}; \frac{1}{2}\mu + \nu + 1; -\frac{1}{2}y^2\right) \\ \left[y > 0, \quad \operatorname{Re}(\mu + \nu) > -\frac{3}{2}, \quad \operatorname{Re} \mu > -1 \right].$$

ET II 171(41)

7.76 Combinations of parabolic-cylinder functions and degenerate hypergeometric functions

7.761

$$\begin{aligned}
1. \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-1} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt &= \\
&= \frac{\pi^{\frac{1}{2}}}{2^{c+\frac{1}{2}\nu}} \frac{\Gamma(2c)\Gamma\left(\frac{1}{2}\nu - c + a\right)}{\Gamma\left(\frac{1}{2}\nu\right)\Gamma\left(a + \frac{1}{2} + \frac{1}{2}\nu\right)} F\left(a, c + \frac{1}{2}; a + \frac{1}{2} + \frac{1}{2}\nu; 1-p\right) \\
&\quad [|1-p| < 1, \quad \operatorname{Re} c > 0, \quad \operatorname{Re} \nu > 2 \operatorname{Re}(c-a)].
\end{aligned}$$

EH II 121(12)

$$\begin{aligned}
2. \int_0^\infty e^{\frac{1}{4}t^2} t^{2c-2} D_{-\nu}(t) {}_1F_1\left(a; c; -\frac{1}{2}pt^2\right) dt &= \\
&= \frac{\pi^{\frac{1}{2}}}{2^{c+\frac{1}{2}\nu-\frac{1}{2}}} \frac{\Gamma(2c-1)\Gamma\left(\frac{1}{2}\nu + \frac{1}{2} - c + a\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right)\Gamma\left(a + \frac{1}{2}\nu\right)} F\left(a, c - \frac{1}{2}; a + \frac{1}{2}\nu; 1-p\right) \\
&\quad \left[|1-p| < 1, \quad \operatorname{Re} c > \frac{1}{2}, \quad \operatorname{Re} \nu > 2 \operatorname{Re}(c-a) - 1 \right].
\end{aligned}$$

EH II 121(13)

897

7.77 Integration of a parabolic-cylinder function with respect to the index

7.771

$$\begin{aligned}
\int_0^\infty \cos(ax) D_{x-\frac{1}{2}}(\beta) D_{-x-\frac{1}{2}}(\beta) dx &= \frac{1}{2} \left(\frac{\pi}{\cos a} \right)^{\frac{1}{2}} \exp\left(-\frac{\beta^2 \cos a}{2}\right) \quad \left[|a| < \frac{1}{2}\pi \right]; \\
&= 0 \quad \left[|a| > \frac{1}{2}\pi \right].
\end{aligned}$$

ET II 298(22)

7.772

$$\begin{aligned}
1. \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} &\left[\frac{\left(\operatorname{tg} \frac{1}{2}\varphi\right)^\nu}{\cos \frac{1}{2}\varphi} D_\nu(-e^{\frac{1}{4}i\pi}\xi) D_{-\nu-1}(e^{\frac{1}{4}i\pi}\eta) + \right. \\
&\left. + \frac{\left(\operatorname{ctg} \frac{1}{2}\varphi\right)^\nu}{\sin \frac{1}{2}\varphi} D_{-\nu-1}(e^{\frac{1}{4}i\pi}\xi) D_\nu(-e^{\frac{1}{4}i\pi}\eta) \right] \frac{d\nu}{\sin \nu\pi} \\
&= -2i(2\pi)^{\frac{1}{2}} \exp\left[-\frac{1}{4}i(\xi^2 - \eta^2) \cos \varphi - \frac{1}{2}i\xi\eta \sin \varphi\right].
\end{aligned}$$

$$2. \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\left(\operatorname{tg} \frac{1}{2}\varphi\right)^\nu}{\cos \frac{1}{2}\varphi} D_\nu(-e^{\frac{1}{4}i\pi}\zeta) D_{-\nu-1}(e^{\frac{1}{4}i\pi}\eta) \frac{d\nu}{\sin \nu\pi} =$$

$$= -2iD_0 \left[e^{\frac{1}{4}i\pi} \left(\zeta \cos \frac{1}{2}\varphi + \eta \sin \frac{1}{2}\varphi \right) \right] D_{-1} \left[e^{\frac{1}{4}i\pi} \left(\eta \cos \frac{1}{2}\varphi - \zeta \sin \frac{1}{2}\varphi \right) \right].$$

EH II 125(8)

7.773

$$1. \int_{c-i\infty}^{c+i\infty} D_\nu(z) t^\nu \Gamma(-\nu) d\nu = 2\pi i e^{-\frac{1}{4}z^2 - zt - \frac{1}{2}t^2} \quad \left[c < 0, \quad |\arg t| < \frac{\pi}{4} \right].$$

EH II 126(10)

$$2. \int_{c-i\infty}^{c+i\infty} [D_\nu(x) D_{-\nu-1}(iy) + D_\nu(-x) D_{-\nu-1}(iy)] \frac{t^{-\nu-1} d\nu}{\sin(-\nu\pi)} =$$

$$= \frac{2\pi i}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}} (1+t^2)^{-\frac{1}{2}} \exp \left[\frac{1}{4} \frac{1-t^2}{1+t^2} (x^2 + y^2) + i \frac{txy}{1+t^2} \right] \quad \left[-1 < c < 0, \quad |\arg t| < \frac{1}{2}\pi \right].$$

EH II 126(11)

7.774

$$\int_{c-i\infty}^{c+i\infty} D_\nu \left[k^{\frac{1}{2}}(1+i)\xi \right] D_{-\nu-1} \left[k^{\frac{1}{2}}(1+i)\eta \right] \Gamma\left(-\frac{1}{2}\nu\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}\nu\right) d\nu =$$

$$= 2^{\frac{1}{2}} \pi^2 H_0^{(2)} \left[\frac{1}{2} k(\xi^2 + \eta^2) \right] \quad [-1 < c < 0, \quad \operatorname{Re} ik \geq 0].$$

EH II 125(9)

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7.8 Meijer's and MacRobert's Functions(G and E)

7.81 Combinations of the functions G and E and the elementary functions

7.811

$$1. \int_0^\infty G_{p,q}^{m,n} \left(\eta x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) G_{\sigma,\tau}^{\mu,\nu} \left(\omega x \left| \begin{matrix} c_1, \dots, c_\sigma \\ d_1, \dots, d_\tau \end{matrix} \right. \right) dx =$$

$$= \frac{1}{\eta} G_{q+\sigma, p+\tau}^{n+\mu, m+\nu} \left(\frac{\omega}{\eta} \left| \begin{matrix} -b_1, \dots, -b_m, c_1, \dots, c_\sigma, -b_{m+1}, \dots, -b_q \\ -a_1, \dots, -a_n, d_1, \dots, d_\tau, -a_{n+1}, \dots, -a_p \end{matrix} \right. \right)$$

$[m, n, p, q, \mu, \nu, \sigma, \tau]$ —are integers; $1 \leq n \leq p < q < p + \tau - \sigma,$
 $\frac{1}{2}p + \frac{1}{2}q - n < m \leq q, \quad 0 \leq \nu \leq \sigma, \quad \frac{1}{2}\sigma + \frac{1}{2}\tau - \nu < \mu \leq \tau;$
 $\operatorname{Re}(b_j + d_k) > -1 \quad (j = 1, \dots, m; k = 1, \dots, \mu),$
 $\operatorname{Re}(a_j + c_k) < 1 \quad (j = 1, \dots, n; k = 1, \dots, \tau);$

$$\begin{aligned}
b_j - b_k & \quad (j = 1, \dots, m; k = 1, \dots, m; j \neq k), \\
a_j - a_k & \quad (j = 1, \dots, n; k = 1, \dots, n; j \neq k), \\
d_j - d_k & \quad (j = 1, \dots, \mu; k = 1, \dots, \mu; j \neq k), \\
a_j + d_k & \quad (j = 1, \dots, n; k = 1, \dots, n);
\end{aligned}$$

must not be positive integers:

$$\begin{aligned}
a_j - b_k & \quad (j = 1, \dots, n; k = 1, \dots, m), \\
c_j - d_k & \quad (j = 1, \dots, \nu; k = 1, \dots, \mu); \\
\omega \neq 0, \quad \eta \neq 0, \quad |\arg \eta| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \quad |\arg \omega| < \left(\mu + \nu - \frac{1}{2}\sigma - \frac{1}{2}\tau\right) \pi \Big].
\end{aligned}$$

Formula 7.811 1 also holds for four sets of restrictions. See C. S. Meijer, *Neue Integraldarstellungen für Whittakersche Funktionen*, *Nederl. Akad. Wetensch. Proc.* **44** (1941), 82-92.

899

$$\begin{aligned}
2. \int_0^1 x^{\varrho-1} (1-x)^{\sigma-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \Gamma(\sigma) G_{p+1, q+1}^{m, n+1} \left(\alpha \left| \begin{matrix} 1-\varrho, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\varrho-\sigma \end{matrix} \right. \right) \\
\left[(p+q) < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \right. \\
\left. \operatorname{Re}(\varrho + b_j) > 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0, \right.
\end{aligned}$$

ET II 422(14)

either

$$\begin{aligned}
p+q \leq 2(m+n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}\varrho - \frac{1}{2}q\right) \pi, \\
\operatorname{Re}(\varrho + b_j) > 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0, \\
\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q) \left(\varrho - \frac{1}{2}\right) \right] > -\frac{1}{2},
\end{aligned}$$

or

$$p < q \quad (\text{or } p \leq q \text{ for } |\alpha| < 1), \quad \operatorname{Re}(p + b_j) > 0; \quad j = 1, \dots, m; \quad \operatorname{Re} \sigma > 0].$$

ET II 417(1)

$$\begin{aligned}
3. \int_1^\infty x^{-\varrho} (x-1)^{\sigma-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \Gamma(\sigma) G_{p+1, q+1}^{m+1, n} \left(\alpha \left| \begin{matrix} a_1, \dots, a_p, \varrho \\ \varrho - \sigma, b_1, \dots, b_q \end{matrix} \right. \right) \\
\left[p+q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}\varrho - \frac{1}{2}q\right) \pi, \right. \\
\left. \operatorname{Re}(\varrho - \sigma - a_j) > -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0, \right.
\end{aligned}$$

either

$$\begin{aligned}
 p + q &\leq 2(m + n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \\
 \operatorname{Re}(\varrho - \sigma - a_j) &> -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0, \\
 \operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q - p) \left(\varrho - \sigma + \frac{1}{2}\right) \right] &> -\frac{1}{2},
 \end{aligned}$$

or

$$q < p \quad (\text{or } q \leq p \text{ for } |\alpha| > 1), \quad \operatorname{Re}(\varrho - \sigma - a_j) > -1; \quad j = 1, \dots, n; \quad \operatorname{Re} \sigma > 0].$$

ET II 417(2)

900

$$\begin{aligned}
 4. \int_0^\infty x^{\varrho-1} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \frac{\prod_{j=1}^m \Gamma(b_j + \varrho) \prod_{j=1}^n \Gamma(1 - a_j - \varrho)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \varrho) \prod_{j=n+1}^p \Gamma(a_j + \varrho)} \alpha^{-\varrho} \\
 &\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \right. \\
 &\quad \left. - \min_{1 \leq j \leq m} \operatorname{Re} b_j < \operatorname{Re} \varrho < 1 - \max_{1 \leq j \leq n} \operatorname{Re} a_j \right].
 \end{aligned}$$

ET II 418(3)A, ET I 337(14)

$$\begin{aligned}
 5. \int_0^\infty x^{\varrho-1} (x + \beta)^{-\sigma} G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \frac{\beta^{\varrho-\sigma}}{\Gamma(\sigma)} G_{p+1, q+1}^{m+1, n+1} \left(\alpha \beta \left| \begin{matrix} 1 - \varrho, a_1, \dots, a_p \\ \sigma - \varrho, b_1, \dots, b_q \end{matrix} \right. \right) \\
 &\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \quad |\arg \beta| < \pi \right. \\
 &\quad \left. \operatorname{Re}(\varrho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\varrho - \sigma + a_j) < 1, \quad j = 1, \dots, n, \right.
 \end{aligned}$$

either

$$\begin{aligned}
 p \leq q, \quad p + q &\leq 2(m + n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \quad |\arg \beta| < \pi \\
 \operatorname{Re}(\varrho = b_j) &> 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\varrho - \sigma + a_j) < 1, \quad j = 1, \dots, n, \\
 \operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j - (q - p) \left(\varrho - \sigma - \frac{1}{2}\right) \right] &> 1,
 \end{aligned}$$

$$p \geq q, \quad p + q \leq 2(m + n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \quad |\arg \beta| < \pi,$$

$$\operatorname{Re}(\varrho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\varrho - \sigma + a_j) < 1, \quad j = 1, \dots, n,$$

$$\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p - q) \left(\varrho - \frac{1}{2}\right) \right] > 1 \Bigg].$$

ET II 418(4)

7.812

$$1. \int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} E \left(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; \frac{z}{x^m} \right) dx =$$

$$= \Gamma(\gamma - \beta) m^{\beta-\gamma} E(a_1, \dots, a_{p+m}; \varrho_1, \dots, \varrho_{q+m}; z),$$

$$a_{p+k} = \frac{\beta + k - 1}{m}, \quad \varrho_{q+k} = \frac{\gamma + k - 1}{m}, \quad k = 1, \dots, m$$

$$[\operatorname{Re} \gamma > \operatorname{Re} \beta > 0, \quad m = 1, 2, \dots].$$

ET II 414(2)

901

$$2. \int_0^\infty x^{\varrho-1} (1+x)^{-\sigma} E[a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; (1+x)z] dx =$$

$$= \Gamma(\varrho) E(a_1, \dots, a_p, \sigma - \varrho; \varrho_1, \dots, \varrho_q, \sigma; z) \quad [\operatorname{Re} \sigma > \operatorname{Re} \varrho > 0].$$

ET II 415(3)

$$3. \int_0^\infty (1+x)^{-\beta} x^{s-1} G_{p,q}^{m,n} \left(\frac{ax}{1+x} \middle| a_1, \dots, a_p \right. \\ \left. b_1, \dots, b_q \right) dx = \Gamma(\beta - s) G_{p+1,q+1}^{m,n+1} \left(a \middle| 1-s, a_1, \dots, a_p \right. \\ \left. b_1, \dots, b_q, 1-\beta \right)$$

$$\left[-\min \operatorname{Re} b_k < \operatorname{Re} s < \operatorname{Re} \beta, \quad 1 \leq k \leq m; (p+q) < 2(m+n), \right.$$

$$\left. |\arg a| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi \right].$$

ET I 338(19)

7.813

$$1. \int_0^\infty x^{-\varrho} e^{-\beta x} G_{pq}^{mn} \left(\alpha x \middle| a_1, \dots, a_p \right. \\ \left. b_1, \dots, b_q \right) dx = \beta^{\varrho-1} G_{p+1,q}^{m,n+1} \left(\frac{\alpha}{\beta} \middle| \varrho, a_1, \dots, a_p, b_1, \dots, b_q \right)$$

$$\left[p+q < 2(m+n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi, \right.$$

$$\left. |\arg \beta| < \frac{1}{2}\pi, \quad \operatorname{Re}(b_j - \varrho) > -1, \quad j = 1, \dots, m \right].$$

ET II 419(5)

7.814

$$\begin{aligned}
1. \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; xz) dx &= \\
&= \pi \operatorname{cosec}(\beta\pi) \left[E(a_1, \dots, a_p; 1-\beta, \varrho_1, \dots, \varrho_q; e^{\pm i\pi} z) - \right. \\
&\quad \left. - z^{-\beta} E(a_1 + \beta, \dots, a_p + \beta; 1 + \beta, \varrho_1 + \beta, \dots, \varrho_l + \beta; e^{\pm i\pi} z) \right]
\end{aligned}$$

$[p \geq q + 1, \operatorname{Re}(a_r + \beta) > 0, r = 1, \dots, p, |\arg z| < \pi$. The formula holds also for $p < q + 1$, provided the integral converges].

902

$$\begin{aligned}
2. \int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; x^{-m} z) dx &= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}} E(a_1, \dots, a_{p+m}; \varrho_1, \dots, \varrho_q; m^{-m} z) \\
&\left[\operatorname{Re} \beta > 0, \quad a_{p+k} = \frac{\beta + k - 1}{m}, \quad k = 1, \dots, m; m = 1, 2, \dots \right].
\end{aligned}$$

ET II 415(5)

ET II 415(4)

7.815

$$\begin{aligned}
1. \int_0^\infty \sin(cx) G_{pq}^{mn} \left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \pi^{\frac{1}{2}} c^{-1} G_{p+2, q}^{m, n+1} \left(\frac{4\alpha}{c^2} \left| \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2} \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\
&\quad \left. c > 0, \quad \operatorname{Re} b_j > -1, \quad j = 1, 2, \dots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \dots, n \right].
\end{aligned}$$

ET II 420(7)

$$\begin{aligned}
2. \int_0^\infty \cos(cx) G_{pq}^{mn} \left(\alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \pi^{\frac{1}{2}} c^{-1} G_{p+2, q}^{m, n+1} \left(\frac{4\alpha}{c^2} \left| \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right. \right) \\
&\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\
&\quad \left. c > 0, \quad \operatorname{Re} b_j > -\frac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j < \frac{1}{2}, \quad j = 1, \dots, n \right].
\end{aligned}$$

ET II 420(8)

7.82 Combinations of the functions G and E and Bessel functions

7.821

$$1. \int_0^\infty x^{-\varrho} J_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = G_{p+2, q}^{m, n+1} \left(\alpha \left| \begin{matrix} \varrho - \frac{1}{2}\nu, a_1, \dots, a_p, \varrho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ \left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\ \left. -\frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \varrho < 1 + \frac{1}{2} \operatorname{Re} \nu + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right].$$

ET II 420(9)

903

$$2. \int_0^\infty x^{-\varrho} N_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = G_{p+3, q+1}^{m, n+2} \left(\alpha \left| \begin{matrix} \varrho - \frac{1}{2}\nu, \varrho + \frac{1}{2}\nu, a_1, \dots, a_p, \varrho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \varrho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix} \right. \right) \\ \left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\ \left. -\frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \varrho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2} |\operatorname{Re} \nu| + 1 \right]$$

ET II 420(10)

$$3. \int_0^\infty x^{-\varrho} K_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \frac{1}{2} G_{p+2, q}^{m, n+2} \left(\alpha \left| \begin{matrix} \varrho - \frac{1}{2}\nu, \varrho + \frac{1}{2}\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ \left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\ \left. \operatorname{Re} \varrho < 1 - \frac{1}{2} |\operatorname{Re} \nu| + \min_{1 \leq j \leq m} \operatorname{Re} b_j \right].$$

ET II 421(11)

7.822

$$1. \int_0^\infty x^{2\varrho} J_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = \\ = \frac{2^{2\varrho}}{y^{2\varrho+1}} G_{p+2, q}^{m, n+1} \left(\frac{4\lambda}{y^2} \left| \begin{matrix} h, a_1, \dots, a_p, k \\ b_1, \dots, b_q \end{matrix} \right. \right), \quad h = \frac{1}{2} - \varrho - \frac{1}{2}\nu, \quad k = \frac{1}{2} - \varrho + \frac{1}{2}\nu \\ \left[p + q < 2(m + n), \quad |\arg \lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \quad \operatorname{Re} \left(b_j + \varrho + \frac{1}{2}\nu \right) > -\frac{1}{2}, \right. \\ \left. j = 1, 2, \dots, m, \quad \operatorname{Re}(a_j + \varrho) < \frac{3}{4}, \quad j = 1, \dots, n, \quad y > 0 \right].$$

$$\begin{aligned}
2. \int_0^\infty x^{\frac{1}{2}} N_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \\
&= (2\lambda)^{-\frac{1}{2}} y^{-\frac{1}{2}} G_{q+1, p+3}^{n+2, m} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q, l \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, l \end{matrix} \right. \right) \\
&\quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu, \quad l = -\frac{1}{4} - \frac{1}{2}\nu \\
&\quad \left[p + q < 2(m + n), \quad |\arg \lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \quad y > 0, \right. \\
&\quad \left. \operatorname{Re} a_j < 1, \quad j = 1, \dots, n, \quad \operatorname{Re} \left(b_j \pm \frac{1}{2}\nu \right) > -\frac{3}{4}, \quad j = 1, \dots, m \right].
\end{aligned}$$

ET II 119(56)

904

$$\begin{aligned}
3. \int_0^\infty x^{\frac{1}{2}} K_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx &= \\
&= 2^{-\frac{3}{2}} \lambda^{-\frac{1}{2}} y^{-\frac{1}{2}} G_{q, p+2}^{n+2, m} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ h, k, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p \end{matrix} \right. \right), \quad h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu \\
&\quad \left[\operatorname{Re} y > 0, \quad p + q < 2(m + n), \quad |\arg \lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right. \\
&\quad \left. \operatorname{Re} b_j > \frac{1}{2} |\operatorname{Re} \nu| - \frac{3}{4}, \quad j = 1, \dots, m \right].
\end{aligned}$$

ET II 153(90)

7.823

$$\begin{aligned}
1. \int_0^\infty x^{\beta-1} J_\nu(x) E(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; x^{-2m} z) dx &= \\
&= (2\pi)^{-m} (2m)^{\beta-1} \left\{ \exp \left[\frac{1}{2} \pi (\beta - \nu - 1) i \right] E [a_1, \dots, a_{p+2m}; \varrho_1, \dots, \varrho_q; (2m)^{-2m} z e^{-m\pi i}] \right. \\
&\quad \left. + \exp \left[-\frac{1}{2} \pi (\beta - \nu - 1) i \right] E [a_1, \dots, a_{p+2m}; \varrho_1, \dots, \varrho_q; (2m)^{-2m} z e^{m\pi i}] \right\}, \\
a_{p+k} &= \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad m = 1, 2, \dots, ; \quad k = 1, \dots, m \\
&\quad \left[\operatorname{Re}(\beta + \nu) > 0, \quad \operatorname{Re}(2a_r m - \beta) > -\frac{3}{2}, \quad r = 1, \dots, p \right]
\end{aligned}$$

ET II 415(7)

$$\begin{aligned}
2. \int_0^\infty x^{\beta-1} K_\nu(x) E(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; x^{-2m} z) dx &= (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1} \times \\
&\quad \times E [a_1, \dots, a_{p+2m}; \varrho_1, \dots, \varrho_q; (2m)^{-2m} z], \\
a_{p+k} &= \frac{\beta + \nu + 2k - 2}{2m}, \quad a_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}, \quad k = 1, 2, \dots, m \\
&\quad [\operatorname{Re} \beta > |\operatorname{Re} \nu|, \quad m = 1, 2, \dots].
\end{aligned}$$

$$1. \int_0^\infty x^{\frac{1}{2}} \mathbf{H}_\nu(xy) G_{pq}^{mn} \left(\lambda x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = (2\lambda y)^{-\frac{1}{2}} G_{q+1, p+3}^{n+1, m+1} \left(\frac{y^2}{4\lambda} \left| \begin{matrix} l, \frac{1}{2} - b_1, \dots, \frac{1}{2} - b_q \\ l, \frac{1}{2} - a_1, \dots, \frac{1}{2} - a_p, h, k \end{matrix} \right. \right)$$

$$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$$

$$\left[p + q < 2(m + n), \quad |\arg \lambda| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \quad y > 0, \right.$$

$$\left. \operatorname{Re} a_j < \min \left(1, \frac{3}{4} - \frac{1}{2}\nu \right), \quad j = 1, \dots, n, \quad \operatorname{Re} (2b_j + \nu) > -\frac{5}{2}, \quad j = 1, \dots, m \right].$$

ET II 172(47)

$$2. \int_0^\infty x^{-\varrho} \mathbf{H}_\nu(2\sqrt{x}) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx = G_{p+3, q+1}^{m+1, n+1} \left(\alpha \left| \begin{matrix} \varrho - \frac{1}{2} - \frac{1}{2}\nu, a_1, \dots, a_p, \varrho + \frac{1}{2}\nu, \varrho - \frac{1}{2}\nu \\ \varrho - \frac{1}{2} - \frac{1}{2}\nu, b_1, \dots, b_q \end{matrix} \right. \right)$$

$$\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right.$$

$$\left. \max \left(-\frac{3}{4}, \operatorname{Re} \frac{\nu - 1}{2} \right) + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \varrho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2} \operatorname{Re} \nu + \frac{3}{2} \right]$$

ET II 421(12)

7.83 Combinations of the functions G and E and other special functions

7.831

$$\int_1^\infty x^{-\varrho} (x-1)^{\sigma-1} F(k + \sigma - \varrho, \lambda + \sigma - \varrho; \sigma; 1-x) G_{pq}^{mn} \left(\alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx =$$

$$= \Gamma(\sigma) G_{p+2, q+2}^{m+2, n} \left(\alpha \left| \begin{matrix} a_1, \dots, a_p, k + \lambda + \sigma - \varrho, \varrho \\ k, \lambda, b_1, \dots, b_q \end{matrix} \right. \right)$$

$$\left[p + q < 2(m + n), \quad |\arg \alpha| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi, \right.$$

$$\left. \operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n, \right.$$

or

$$p + q \leq 2(m + n), \quad |\arg \alpha| \leq \left(m + n - \frac{1}{2}p - \frac{1}{2}q \right) \pi,$$

$$\operatorname{Re} \sigma > 0, \quad \operatorname{Re} k \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n,$$

$$\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left(k + \frac{1}{2} \right) \right] > -\frac{1}{2},$$

$$\operatorname{Re} \left[\sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p) \left(\lambda + \frac{1}{2} \right) \right] > -\frac{1}{2}.$$

$$\int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\{\mu\}}(x) E(a_1, \dots, a_p; \varrho_1, \dots, \varrho_q; x^{-m} z) dx =$$

$$= (2\pi)^{\frac{1}{2} - \frac{1}{2}m} m^{\beta + \{\frac{1}{2}\}} E(a_1, \dots, a_{p+2m}; \varrho_1, \dots, \varrho_{q+m}; m^{-m} z),$$

$$a_{p+k} = \frac{\beta + k + \mu - \frac{1}{2}}{m}, \quad a_{p+m+k} = \frac{\beta - \mu + k - \frac{1}{2}}{m}, \quad \varrho_{q+k} = \frac{\beta - \{+k\}}{m}, \quad k = 1, \dots, m$$

$$\left[\operatorname{Re} \beta > |\operatorname{Re} \mu| - \frac{1}{2}, \quad m = 1, 2, \dots \right].$$

ET II 416(10)

8.-9. Special Functions

8.1 Elliptic Integrals and Functions

8.11 Elliptic integrals

8.110

1. Every integral of the form $\int R(x, \sqrt{P(x)}) dx$, where $P(x)$ is a third- or fourth-degree polynomial, can be reduced to a linear combination of integrals leading to elementary functions and the following three integrals:

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \int \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx, \quad \int \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

which are called respectively *elliptic integrals of the first, second, and third kind in the Legendre normal form*. The results of this reduction for the more frequently encountered integrals are given in formulas 3.13-3.17. The number k is called the *modulus** of these integrals, the number $k' = \sqrt{1-k^2}$ is called the complementary modulus, and the number n is called the parameter of the integral of the third kind.

BY (110.04)}

* The quantity k is sometimes called the *module* of the functions.

2. By means of the substitution $x = \sin \varphi$, elliptic integrals can be reduced to the normal trigonometric form

$$\int \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}, \quad \int \sqrt{1-k^2 \sin^2 \varphi} d\varphi, \quad \int \frac{d\varphi}{(1-n \sin^2 \varphi)\sqrt{1-k^2 \sin^2 \varphi}}.$$

BY (110.04)

The results of reducing integrals of trigonometric functions to normal form are given in 2.58-2.62.

3. Elliptic integrals from 0 to $\frac{\pi}{2}$ are called *complete elliptic integrals*.

8.111

Notations:

$$1. \quad \Delta\varphi = \sqrt{1 - k^2 \sin^2 \varphi}; \quad k' = \sqrt{1 - k^2}; \quad k^2 < 1.$$

2. The elliptic integral of the first kind:

$$F(\varphi, k) = \int_0^\varphi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

908

3. The elliptic integral of the second kind:

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \alpha} \, d\alpha = \int_0^{\sin \varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}} \, dx.$$

FI II 135

4. The elliptic integral of the third kind:

$$\Pi(\varphi, n, k) = \int_0^\varphi \frac{d\alpha}{(1 - n \sin^2 \alpha) \sqrt{1 - k^2 \sin^2 \alpha}} = \frac{\int_0^{\sin \varphi} dx}{(1 - nx^2) \sqrt{(1 - x^2)(1 - k^2 x^2)}}$$

BY (110.04)

$$5. \quad D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2} = \int_0^\varphi \frac{\sin^2 \alpha \, d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin \varphi} \frac{x^2 \, dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

8.112

Complete elliptic integrals

In writing complete elliptic integrals, the modulus k , which acts as an independent variable, is often omitted and we write

$$K(\equiv K(k)), \quad K'(\equiv K'(k)), \quad E(\equiv E(k)), \quad E'(\equiv E'(k)).$$

Series representations

8.113

$$1. \quad K = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n} + \dots \right\} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

FI II 487, WH

$$2. \quad K = \frac{\pi}{1+k'} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\frac{1-k'}{1+k'}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\frac{1-k'}{1+k'}\right)^4 + \dots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \left(\frac{1-k'}{1+k'}\right)^{2n} + \dots \right\}.$$

DW

909

$$3. \quad K = \ln \frac{4}{k'} + \left(\frac{1}{2}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2}\right) k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4}\right) k'^4 + \\ + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{2}{5 \cdot 6}\right) k'^6 + \dots$$

DW

See also 8.197 1., 8.197 2.

8.114

$$1.^6 \quad E = \frac{\pi}{2} \left\{ 1 - \frac{1}{2^2} k^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} k^4 - \dots - \left[\frac{(2n-1)!!}{2^n n!}\right]^2 \frac{k^{2n}}{2n-1} - \dots \right\} = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

$$2. \quad E = \frac{(1+k')\pi}{4} \left\{ 1 + \frac{1}{2^2} \left(\frac{1-k'}{1+k'} \right)^2 + \frac{1^2}{2^2 \cdot 4^2} \left(\frac{1-k'}{1+k'} \right)^4 + \dots + \left[\frac{(2n-3)!!}{2^n n!} \right]^2 \left(\frac{1-k'}{1+k'} \right)^{2n} + \dots \right\}$$

DW

$$3. \quad E = 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{1 \cdot 2} \right) k'^2 + \frac{1^2 \cdot 3}{2^2 \cdot 4} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{1}{3 \cdot 4} \right) k'^4 + \\ + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \left(\ln \frac{4}{k'} - \frac{2}{1 \cdot 2} - \frac{2}{3 \cdot 4} - \frac{1}{5 \cdot 6} \right) k'^6 + \dots$$

DW

8.115

$$D = \pi \left\{ \frac{1}{1} \left(\frac{1}{2} \right)^2 + \frac{2}{3} \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^2 + \dots + \frac{n}{2n-1} \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k^{2(n-1)} + \dots \right\}.$$

ZH 43(158)

8.116

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{1-k^2 \sin^2 \varphi}}{1-n^2 \sin^2 \varphi} d\varphi = \sqrt{n'^2 - k'^2} \left(\frac{\arccos \frac{1}{n'}}{n' \sqrt{n'^2 - 1}} + \mathbf{R} \right),$$

where

$$\mathbf{R} = \frac{k'^2}{2} \left(p + \frac{1}{2} \right) \frac{1}{n'^3} + \frac{k'^4}{16} \left[-1 + \left(p + \frac{1}{4} \right) \frac{1}{n'^3} \left(1 + \frac{6}{n'^2} \right) \right] + \\ + \frac{k'^6}{16} \left[-\frac{7}{16} - \frac{1}{n'^2} + \left(p + \frac{1}{6} \right) \frac{1}{n'^3} \left(\frac{3}{8} + \frac{1}{n'^2} + \frac{5}{n'^4} \right) \right] + \\ + \frac{15k'^8}{256} \left[-\frac{37}{144} - \frac{21}{40n'^2} - \frac{1}{n'^4} + \left(p + \frac{1}{8} \right) \frac{1}{n'^3} \left(\frac{5}{24} + \frac{9}{20n'^2} + \frac{1}{n'^4} + \frac{14}{3n'^6} \right) \right] + \dots, \\ p = \ln \frac{4}{k'}, \quad k' = 4e^{-p}, \quad k'^2 = 1 - k^2, \quad n'^2 = 1 - n^2.$$

ZH 44(163)

Trigonometric series

8.117

For *small* values of k and φ , we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K} \varphi - \sin \varphi \cos \varphi \left(a_0 + \frac{2}{3} a_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a_2 \sin^4 \varphi + \dots \right),$$

where

$$a_0 = \frac{2}{\pi} \mathbf{K} - 1; \quad a_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k^{2n}.$$

ZH 10(19)

$$2. \quad E(\varphi, k) = \frac{2}{\pi} \mathbf{E} \varphi + \sin \varphi \cos \varphi \left(b_0 + \frac{2}{3} b_1 \sin^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b_2 \sin^4 \varphi + \dots \right),$$

where

$$b_0 = 1 - \frac{2}{\pi} \mathbf{E}, \quad b_n = b_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 \frac{k^{2n}}{2n-1}.$$

ZH 27(86)

8.118

For k close to 1, we may use the series

$$1. \quad F(\varphi, k) = \frac{2}{\pi} \mathbf{K}' \ln \operatorname{tg} \left(\frac{\varphi}{2} + \frac{\pi}{4} \right) - \frac{\operatorname{tg} \varphi}{\cos \varphi} \left(a'_0 - \frac{2}{3} a'_1 \operatorname{tg}^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} a'_2 \operatorname{tg}^4 \varphi - \dots \right),$$

where

$$a'_0 = \frac{2}{\pi} \mathbf{K}' - 1; \quad a'_n = a_{n-1} - \left[\frac{(2n-1)!!}{2^n n!} \right]^2 k'^{2n}.$$

ZH 10(23)

$$2. \quad E(\varphi, k) = \frac{2}{\pi} (\mathbf{K}' - \mathbf{E}') \ln \operatorname{tg} \left(\frac{\varphi}{2} + \frac{\pi}{2} \right) + \frac{\operatorname{tg} \varphi}{\cos \varphi} \left(b'_1 - \frac{2}{3} b'_2 \operatorname{tg}^2 \varphi + \frac{2 \cdot 4}{3 \cdot 5} b'_3 \operatorname{tg}^4 \varphi - \dots \right) + \frac{1}{\sin \varphi} [1 - \cos \varphi \sqrt{1 - k^2 \sin^2 \varphi}],$$

$$b'_0 = \frac{2}{\pi}(\mathbf{K}' - \mathbf{E}'), \quad b'_n = b'_{n-1} - \left[\frac{(2n-3)!!}{2^{n-1}(n-1)!} \right]^2 \left(\frac{2n-1}{2n} \right) k'^{2n}$$

ZH 27(90)

For the expansion of complete elliptic integrals in Legendre polynomials, see 8.928.

8.119

Representation in the form of an infinite product:

$$1. \quad \mathbf{K}(k) = \frac{\pi}{2} \prod_{n=1}^{\infty} (1 + k_n),$$

where

$$k_n = \frac{1 - \sqrt{1 - k_{n-1}^2}}{1 + \sqrt{1 - k_{n-1}^2}}; \quad k_0 = k.$$

FI II 166

See also 8.197.

911

8.12 Functional relations between elliptic integrals

8.121

$$1. \quad F(-\varphi, k) = -F(\varphi, k).$$

JA

$$2. \quad E(-\varphi, k) = -E(\varphi, k).$$

JA

$$3. \quad F(n\pi \pm \varphi, k) = 2n\mathbf{K}(k) \pm F(\varphi, k).$$

$$4. \quad E(n\pi \pm \varphi, k) = 2n\mathbf{E}(k) \pm E(\varphi, k).$$

8.122

$$\mathbf{E}(k)\mathbf{K}'(k) + \mathbf{E}'(k)\mathbf{K}(k) - \mathbf{K}(k)\mathbf{K}'(k) = \frac{\pi}{2}.$$

FI II 691, 791

8.123

$$1. \quad \frac{\partial F}{\partial k} = \frac{1}{k'^2} \left(\frac{E - k'^2 F}{k} - \frac{k \sin \varphi \cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \right).$$

MO 138, BY (710.07)

$$2. \quad \frac{d\mathbf{K}(k)}{dk} = \frac{\mathbf{E}(k)}{kk'^2} - \frac{\mathbf{K}(k)}{k}.$$

FI II 691

$$3. \quad \frac{\partial E}{\partial k} = \frac{E - F}{k}.$$

MO 138

$$4. \quad \frac{d\mathbf{E}(k)}{dk} = \frac{\mathbf{E}(k) - \mathbf{K}(k)}{k}.$$

FI II 690

8.124

1. The functions \mathbf{K} and \mathbf{K}' satisfy the equation

$$\frac{d}{dk} \left\{ kk'^2 \frac{du}{dk} \right\} - ku = 0.$$

2. The functions E and $E' - K'$ satisfy the equation

$$k'^2 \frac{d}{dk} \left(k \frac{du}{dk} \right) + ku = 0.$$

WH

8.125

$$\left. \begin{aligned} 1. \quad F \left(\psi, \frac{1-k'}{1+k'} \right) &= (1+k') F(\varphi, k) \\ 2. \quad E \left(\psi, \frac{1-k'}{1+k'} \right) &= \frac{2}{1+k'} [E(\varphi, k) + k' F(\varphi, k)] - \frac{1-k'}{1+k'} \sin \psi \end{aligned} \right\} [\operatorname{tg}(\psi-\varphi) = k' \operatorname{tg} \varphi]$$

MO 131
MO 130

912

$$\left. \begin{aligned} 3. \quad F \left(\psi, \frac{2\sqrt{k}}{1+k} \right) &= (1+k) F(\varphi, k). \\ 4. \quad E \left(\psi, \frac{2\sqrt{k}}{1+k} \right) &= \frac{1}{1+k} [2E(\varphi, k) - k'^2 F(\varphi, k) + \\ &\quad + 2k \frac{\sin \varphi \cos \varphi}{1+k \sin^2 \varphi} \sqrt{1-k^2 \sin^2 \varphi}] \end{aligned} \right\} \left[\sin \psi = \frac{(1+k) \sin \varphi}{1+k \sin^2 \varphi} \right].$$

MO 131

8.126

In particular,

$$1. \quad \mathbf{K} \left(\frac{1-k'}{1+k'} \right) = \frac{1+k'}{2} \mathbf{K}(k).$$

MO 130

$$2. \quad \mathbf{E} \left(\frac{1-k'}{1+k'} \right) = \frac{1}{1+k'} [\mathbf{E}(k) + k' \mathbf{K}(k)].$$

MO 130

$$3. \quad \mathbf{K} \left(\frac{2\sqrt{k}}{1+k} \right) = (1+k) \mathbf{K}(k).$$

$$4. \quad E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}[2E(k) - k'^2 K(k)].$$

MO 130

8.127

k_1	$\sin \varphi_1$	$\cos \varphi_1$	$F(\varphi_1, k_1)$	$E(\varphi_1, k_1)$
$i \frac{k}{k'}$	$k' \frac{\sin \varphi}{\Delta \varphi}$	$\frac{\cos \varphi}{\Delta \varphi}$	$k' F(\varphi, k)$	$\frac{1}{k'} \left[E(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$
k'	$-i \operatorname{tg} \varphi$	$\sec \varphi$	$-i F(\varphi, k)$	$i[E(\varphi, k) - F(\varphi, k) - \Delta \varphi \operatorname{tg} \varphi]$
$\frac{1}{k}$	$k \sin \varphi$	$\Delta \varphi$	$k F(\varphi, k)$	$\frac{1}{k}[E(\varphi, k) - k'^2 F(\varphi, k)]$
$\frac{1}{k'}$	$-k' \operatorname{tg} \varphi$	$\frac{\Delta \varphi}{\cos \varphi}$	$-ik' F(\varphi, k)$	$\frac{i}{k'}[E(\varphi, k) - k'^2 F(\varphi, k) - \Delta \varphi \operatorname{tg} \varphi]$
$\frac{k'}{ik}$	$\frac{-ik \sin \varphi}{\Delta \varphi}$	$\frac{1}{\Delta \varphi}$	$-ik F(\varphi, k)$	$\frac{i}{k} \left[E(\varphi, k) - F(\varphi, k) - \frac{k^2 \sin \varphi \cos \varphi}{\Delta \varphi} \right]$

8.111
MO 131

8.128

In particular,

$$1. \quad K\left(i \frac{k}{k'}\right) = k' K(k) [\operatorname{Im}(k) < 0].$$

MO 130

913

$$2. \quad K'\left(i \frac{k}{k'}\right) = k' [K(k') - i K(k)] [\operatorname{Im}(k) < 0].$$

MO 130

$$3. \quad K\left(\frac{1}{k}\right) = k [K(k) + i K'(k)] [\operatorname{Im}(k) < 0].$$

MO 130

For integrals of elliptic integrals, see 6.14-6.15. For indefinite integrals of complete elliptic integrals, see 5.11.

8.129

Special values:

$$1. \quad K\left(\sin \frac{\pi}{4}\right) = K\left(\frac{\sqrt{2}}{2}\right) = K'\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{4\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

MO 130

$$2. \quad K'(\sqrt{2}-1) = \sqrt{2}K(\sqrt{2}-1).$$

MO 130

$$3. \quad K'\left(\sin \frac{\pi}{12}\right) = \sqrt{3}K\left(\sin \frac{\pi}{12}\right).$$

MO 130

$$4. \quad K'\left(\operatorname{tg}^2 \frac{\pi}{8}\right) = K'\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = 2K\left(\operatorname{tg}^2 \frac{\pi}{8}\right).$$

MO 130

8.13 Elliptic functions

8.130

Definition and general properties.

1. A single-valued function $f(z)$ of a complex variable, which is not a constant, is said to be elliptic if it has two periods $2\omega_1$ and $2\omega_2$, that is

$$f(z + 2m\omega_1 + 2n\omega_2) = f(z) \quad [m, n \text{ integers}].$$

The ratio of the periods of an analytic function cannot be a real number. For an elliptic function $f(z)$, the z -plane can be partitioned

$$f(z) = z$$

into parallelograms--the period parallelograms--the vertices of which are the points $z_0 + 2m\omega_1 + 2n\omega_2$. At corresponding points of these parallelograms, the function $f(z)$ has the same value.

ZH 117, SI 299

2. Suppose that α is the angle between the sides a and b of one of the period parallelograms. Then,

$$\tau = \frac{\omega_1}{\omega_2} = \frac{a}{b}e^{i\alpha}, \quad q = e^{i\pi\tau} = e^{-\frac{a}{b}\pi \sin \alpha} \left[\cos \left(\frac{a}{b}\pi \cos \alpha \right) + i \sin \left(\frac{a}{b}\pi \cos \alpha \right) \right].$$

3. The *derivative* of an elliptic function is also an elliptic function with the same periods.

SM III 598

4. A nonconstant elliptic function has a finite number of poles in a period parallelogram: it can have no more than two simple and one second-order pole in such a parallelogram. Suppose that these poles lie at the points a_1, a_2, \dots, a_n and that their orders are $\alpha_1, \alpha_2, \dots, \alpha_n$. Suppose that the zeros of an analytic function that occur in a single parallelogram are b_1, b_2, \dots, b_m and that the orders of the zeros are $\beta_1, \beta_2, \dots, \beta_m$, respectively. Then,

$$\gamma = \alpha_1 + \alpha_2 + \dots + \alpha_n = \beta_1 + \beta_2 + \dots + \beta_m.$$

ZH 118

The number γ representing this sum is called the *order* of the elliptic function.

914

5. The sum of the residues of an elliptic function with respect to all the poles belonging to a period parallelogram is equal to zero.

6. The difference between the sum of all the zeros and the sum of all the poles of an elliptic function that are located in a period parallelogram is equal to one of its periods.

7. Every two elliptic functions with the same periods are related by an algebraic relationship.

GO II 151

8.⁷ A nonconstant single-valued function cannot have more than two periods.

GO II 147

9. An elliptic function of order γ assumes *an arbitrary value* γ times in a period parallelogram.

SM 601, SI 301

8.14 Jacobian elliptic functions

8.141

Consider the upper limit φ of the integral

$$u = \int_0^{\varphi} \frac{da}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of u . Using the notation

$$\varphi = \operatorname{am} u$$

we call this upper limit the *amplitude*. The quantity u is called the *argument*, and its dependence on φ is written

$$u = \operatorname{arg} \varphi.$$

8.142

The amplitude is an *infinitely-many-valued* function of u and has a period of $4\mathbf{K}i$. The *branch points* of the amplitude correspond to the values of the argument

$$u = 2m\mathbf{K} + (2n + 1)\mathbf{K}'i,$$

ZH 67- 69

where m and n are arbitrary integers (see also 8.151).

8.143

The first two of the following functions

$$\begin{aligned} \operatorname{sn} u &= \sin \varphi = \sin \operatorname{am} u, & \operatorname{cn} u &= \cos \varphi = \cos \operatorname{am} u, \\ \operatorname{dn} u &= \Delta \varphi = \sqrt{1 - k^2 \sin^2 \varphi} = \frac{d\varphi}{du} \end{aligned}$$

are called, respectively, the *sine-amplitude* and the *cosine-amplitude* while the third may be called the

delta amplitude. All these elliptic functions were exhibited by Jacobi and they bear his name.

SI 16

The Jacobian elliptic functions are *doubly-periodic* functions and have *two simple poles* in a period parallelogram.

ZH 69

915

8.144

$$\left. \begin{aligned} 1. \quad u &= \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \\ 2. \quad u &= \int_1^{\operatorname{cn} u} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}} \\ 3. \quad u &= \int_1^{\operatorname{dn} u} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}} \end{aligned} \right\}$$

SI 21(23)

8.145

Power series representations:

$$\begin{aligned} 1. \quad \operatorname{sn} u &= u - \frac{1+k^2}{3!}u^3 + \frac{1+14k^2+k^4}{5!}u^5 - \frac{1+135k^2+135k^4+k^6}{7!}u^7 + \\ &+ \frac{1+1228k^2+5478k^4+1228k^6+k^8}{9!}u^9 - \dots \quad [|u| < |\mathbf{K}'|]. \end{aligned}$$

ZH 81(97)

$$\begin{aligned} 2. \quad \operatorname{cn} u &= 1 - \frac{1}{2!}u^2 + \frac{1+4k^2}{4!}u^4 - \frac{1+44k^2+16k^4}{6!}u^6 + \\ &+ \frac{1+408k^2+912k^4+64k^6}{8!}u^8 - \dots \quad [|u| < |\mathbf{K}'|]. \end{aligned}$$

ZH 81(98)

$$\begin{aligned} 3. \quad \operatorname{dn} u &= 1 - \frac{k^2}{2!}u^2 + \frac{k^2(4+k^2)}{4!}u^4 - \frac{k^2(16+44k^2+k^4)}{6!}u^6 + \\ &+ \frac{k^2(64+912k^2+408k^4+k^6)}{8!}u^8 - \dots \quad [|u| < |\mathbf{K}'|]. \end{aligned}$$

$$4. \quad \operatorname{am} u = u - \frac{k^2}{3!}u^3 + \frac{k^2(4+k^2)}{5!}u^5 - \frac{k^2(16+44k^2+k^4)}{7!}u^7 + \\ + \frac{k^2(64+912k^2+408k^4+k^6)}{9!}u^9 - \dots \quad [|u| < |K'|].$$

LA 380(4)

8.146

Representation as a trigonometric series or a product ($q = e^{-\frac{\pi K'}{K}}$)* * The expansions 1-22 are valid in every strip of the form

$\left| \operatorname{Im} \frac{\pi u}{2K} \right| < \frac{1}{2} \pi \operatorname{Im} \tau$. The expansions 23-25 are valid in an arbitrary bounded portion of u .

:

$$1. \quad \operatorname{sn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}.$$

WH, ZH 84(108)

916

$$2. \quad \operatorname{cn} u = \frac{2\pi}{kK} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}.$$

WH, ZH 84(109)

$$3. \quad \operatorname{dn} u = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K}.$$

WH, ZH 84(110)

$$4. \quad \operatorname{am} u = \frac{\pi u}{2K} + 2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1+q^{2n}} \sin \frac{n\pi u}{K}.$$

WH

$$5. \quad \frac{1}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right].$$

LA 369(3)

LA 369(3)

$$7. \frac{1}{\operatorname{dn} u} = \frac{\pi}{2k'K} \left[1 + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos \frac{n\pi u}{K} \right].$$

LA 369(3)

$$8. \frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\pi}{2k'K} \left[\operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin \frac{n\pi u}{K} \right].$$

LA 369(4)

$$9. \frac{\operatorname{sn} u}{\operatorname{dn} u} = -\frac{2\pi}{kk'K} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K}.$$

LA 369(4)

$$10. \frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin \frac{\pi n u}{K} \right].$$

LA 369(5)

$$11. \frac{\operatorname{cn} u}{\operatorname{dn} u} = -\frac{2\pi}{kK} \sum_{n=1}^{\infty} (-1)^n \frac{q^{n-\frac{1}{2}}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K}.$$

LA 369(5)

$$12. \frac{\operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\frac{1}{\sin \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{2n-1}} \sin(2n-1) \frac{\pi u}{2K} \right].$$

LA 369(6)

$$13. \frac{\operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left[\frac{1}{\cos \frac{\pi u}{2K}} - 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n-1}}{1-q^{2n-1}} \cos(2n-1) \frac{\pi u}{2K} \right].$$

LA 369(6)

$$14. \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{q^n}{1+q^n} \sin \frac{n\pi u}{K} \right].$$

$$15. \quad \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} = \frac{\pi}{2K} \left\{ \operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + (-1)^n q^n} \sin \frac{n\pi u}{K} \right\}.$$

LA 369(7)

917

$$16. \quad \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{4\pi^2}{k^2 K} \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K}.$$

LA 369(7)

$$17. \quad \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} = \frac{\pi}{2(1-k^2)K} \left[\operatorname{tg} \frac{\pi u}{2K} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1 - q^n} \sin \frac{n\pi u}{K} \right].$$

LA 369(8)

$$18. \quad \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} = \frac{\pi}{2K} \left[\operatorname{ctg} \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + (-1)^n q^n} \sin \frac{n\pi u}{K} \right].$$

LA 369(8)

$$19. \quad \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} = \frac{\pi}{K} \left[\frac{1}{\sin \frac{\pi u}{K}} + 4 \sum_{n=1}^{\infty} \frac{q^{2(2n-1)}}{1 - q^{2(2n-1)}} \sin(2n-1) \frac{\pi u}{K} \right].$$

LA 369(8)

$$20. \quad \ln \operatorname{sn} u = \ln \frac{2K}{\pi} + \ln \sin \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 + q^n} \sin^2 \frac{n\pi u}{2K}.$$

LA 369(2)

$$21. \quad \ln \operatorname{cn} u = \ln \cos \frac{\pi u}{2K} - 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 + (-1)^n q^n} \sin^2 \frac{n\pi u}{2K}.$$

LA 369(2)

$$23. \quad \operatorname{sn} u = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1 - 2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}.$$

ZH 86(145)

$$24. \quad \operatorname{cn} u = \frac{2\sqrt{k'}\sqrt[4]{q}}{\sqrt{k}} \cos \frac{\pi u}{2K} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n} \cos \frac{\pi u}{K} + q^{4n}}{1 - 2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}.$$

ZH 86(146)

$$25. \quad \operatorname{dn} u = \sqrt{k'} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}{1 - 2q^{2n-1} \cos \frac{\pi u}{K} + q^{4n-2}}.$$

ZH 86(147)

$$26. \quad \operatorname{sn}^3 u = \sum_{n=0}^{\infty} \left[\frac{1+k^2}{2k^3} - \frac{(2n+1)^2}{2k^3} \frac{\pi^2}{4K^2} \right] \frac{2\pi q^{n+\frac{1}{2}} \sin(2n+1) \frac{\pi u}{2K}}{K(1-q^{2n+1})} \quad \left[\left| \operatorname{Im} \frac{u}{2K} \right| < \operatorname{Im} \tau \right].$$

MO 147

$$27. \quad \frac{1}{\operatorname{sn}^2 u} = \frac{\pi^2}{4K^2} \operatorname{cosec}^2 \frac{\pi u}{2K} + \frac{K-E}{K} - \frac{2\pi^2}{K^2} \sum_{n=1}^{\infty} \frac{nq^{2n} \cos \frac{n\pi u}{K}}{1-q^{2n}} \quad \left[\left| \operatorname{Im} \frac{u}{2K} \right| < \frac{1}{2} \operatorname{Im} \tau \right].$$

MO 148

8.147

$$1. \quad \operatorname{sn} u = \frac{\pi}{2kK} \sum_{n=-\infty}^{\infty} \frac{1}{\sin \frac{\pi}{2K} [u - (2n-1)iK']}.$$

MO 149

918

$$2. \quad \operatorname{cn} u = \frac{\pi i}{2kK} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sin \frac{\pi}{2K} [u - (2n-1)iK']}.$$

$$3. \quad \operatorname{dn} u = \frac{\pi i}{2K} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\operatorname{tg} \frac{\pi}{2K} [u - (2n-1)iK']}.$$

8.148

The Weierstrass expansions of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$:

$$\operatorname{sn} u = \frac{B}{A}, \quad \operatorname{cn} u = \frac{C}{A}, \quad \operatorname{dn} u = \frac{D}{A},$$

where

$$A = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} a_{n+1} \frac{u^{2n+2}}{(2n+2)!}$$

$$[a_2 = 2k^2, \quad a_3 = 8(k^2 + k^4), \quad a_4 = 32(k^2 + k^6) + 68k^4, \quad a_5 = 128(k^2 + k^8) + 480(k^4 + k^6), \quad a_6 = 512(k^2 + k^{10}) + 3008(k^4 + k^8) + 5400k^6, \dots]$$

$$B = \sum_{n=0}^{\infty} (-1)^n b_n \frac{u^{2n+1}}{(2n+1)!}$$

$$[b_0 = 1, \quad b_1 = 1 + k^2, \quad b_2 = 1 + k^4 + 4k^2, \quad b_3 = 1 + k^6 + 9(k^2 + k^4), \\ b_4 = 1 + k^8 + 16(k^2 + k^6) - 6k^4, \quad b_5 = 1 + k^{10} + 25(k^2 + k^8) - 494(k^4 + k^6), \\ b_6 = 1 + k^{12} + 36(k^2 + k^{10}) - 5781(k^4 + k^8) - 12184k^6, \dots].$$

$$C = \sum_{n=0}^{\infty} (-1)^n c_n \frac{u^{2n}}{(2n)!}$$

$$[c_0 = 1, \quad c_1 = 1, \quad c_2 = 1 + 2k^2, \quad c_3 = 1 + 6k^2 + 8k^4, \quad c_4 = 1 + 12k^2 + 60k^4 + 32k^6, \\ c_5 = 1 + 20k^2 + 348k^4 + 448k^6 + 128k^8, \quad c_6 = 1 + 30k^2 + 2372k^4 + 4600k^6 + 2880k^8 + 512k^{10}, \dots].$$

$$D = \sum_{n=0}^{\infty} (-1)^n d_n \frac{u^{2n}}{(2n)!}$$

$$[d_0 = 1, \quad d_1 = k^2, \quad d_2 = 2k^2 + k^4, \quad d_3 = 8k^2 + 6k^4 + k^6, \quad d_4 = 32k^2 + 60k^4 + 12k^4 + k^8, \\ d_5 = 128k^2 + 448k^4 + 348k^6 + 20k^8 + k^{10}, \\ d_6 = 512k^2 + 2880k^4 + 4600k^6 + 2372k^8 + 30k^{10} + k^{12}, \dots].$$

ZH 82-83(105,106,107)

919

8.15 Properties of Jacobian elliptic functions and functional relationships between

them

8.151

The periods, zeros, poles, and residues of Jacobian elliptic functions:

1.

	Periods	Zeros	Poles	Residues
sn u	$4mK + 2nK'i$	$2mK + 2nK'i$	$2mk + (2n + 1)K'i$	$(-1)^m \frac{1}{k}$
cn u	$4mK + 2n(K + K'i)$	$(2m + 1)K + 2nK'i$	$2mK + (2n + 1)K'i$	$(-1)^{m-1} \frac{i}{k}$
dn u	$2mK + 4nK'i$	$(2m + 1)K + (2n + 1)K'i$	$2mK + (2n + 1)K'i$	$(-1)^{n-1} \frac{i}{k}$

SM 630, ZH 69-72}

2.

$u^* = u + K$	$u + iK$	$u + K + iK'$	$u + 2K$	$u + 2iK'$	$u + 2K + 2iK'$
sn $u^* = \frac{\text{cn } u}{\text{dn } u}$	$\frac{1}{k \text{sn } u}$	$\frac{1}{k} \frac{\text{dn } u}{\text{cn } u}$	$-\text{sn } u$	sn u	$-\text{sn } u$
cn $u^* = -k' \frac{\text{sn } u}{\text{dn } u}$	$-\frac{i}{k} \frac{\text{dn } u}{\text{sn } u}$	$-\frac{ik'}{k \text{cn } u}$	$-\text{cn } u$	$-\text{cn } u$	cn u
dn $u^* = k' \frac{1}{\text{dn } u}$	$-\frac{i \text{cn } u}{\text{sn } u}$	$ik' \frac{\text{sn } u}{\text{cn } u}$	dn u	$-\text{dn } u$	$-\text{dn } u$

SM 630

3.

$u^* = 0$	$-u$	$\frac{1}{2}K$	$\frac{1}{2}(K + iK')$	$\frac{1}{2}iK'$	$u + 2mK + 2nK'i$
sn $u^* = 0$	$-\text{sn } u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{\sqrt{1+k+i\sqrt{1-k}}}{\sqrt{2k}}$	$\frac{i}{\sqrt{k}}$	$(-1)^m \text{sn } u$
cn $u^* = 1$	cn u	$\frac{\sqrt{k'}}{\sqrt{1+k'}}$	$\frac{(1-i)\sqrt{k'}}{\sqrt{2k}}$	$\frac{\sqrt{1+k}}{\sqrt{k}}$	$(-1)^{m+n} \text{cn } u$
dn $u^* = 1$	dn u	$\sqrt{k'}$	$\frac{\sqrt{k'}(\sqrt{1+k'} - i\sqrt{1-k'})}{\sqrt{2}}$	$\sqrt{1+k}$	$(-1)^n \text{dn } u$

SI 19, SI 18(13), WH

920

8.152

Transformation formulas

u_1	k_1	$n(u_1, k_1)$	$\text{cn}(u_1, k)$	$\text{ln}(u_1, k)$
ku	$\frac{1}{k}$	$k \text{sn}(u, k)$	dn (u, k)	cn (u, k)
iu	k'	$i \frac{\text{sn}(u, k)}{\text{cn}(u, k)}$	$\frac{1}{\text{cn}(u, k)}$	$\frac{\text{dn}(u, k)}{\text{cn}(u, k)}$
$k'u$	$i \frac{k}{k'}$	$k' \frac{\text{sn}(u, k)}{\text{dn}(u, k)}$	$\frac{\text{cn}(u, k)}{\text{dn}(u, k)}$	$\frac{1}{\text{dn}(u, k)}$
iku	$i \frac{k'}{k}$	$ik \frac{\text{sn}(u, k)}{\text{dn}(u, k)}$	$\frac{1}{\text{dn}(u, k)}$	$\frac{\text{cn}(u, k)}{\text{dn}(u, k)}$
$ik'u$	$\frac{1}{k'}$	$ik' \frac{\text{sn}(u, k)}{\text{cn}(u, k)}$	$\frac{\text{dn}(u, k)}{\text{cn}(u, k)}$	$\frac{1}{\text{cn}(u, k)}$
$(1+k)u$	$\frac{2\sqrt{k}}{1+k}$	$(1+k) \frac{\text{sn}(u, k)}{1+k \text{sn}^2(u, k)}$	$\frac{\text{cn}(u, k) \text{dn}(u, k)}{1+k \text{sn}^2(u, k)}$	$\frac{1-k \text{sn}^2(u, k)}{1+k \text{sn}^2(u, k)}$
	$1-k'$	$\text{sn}(u, k) \text{cn}(u, k)$	$1 - (1+k') \text{sn}^2(u, k)$	$1 - (1-k')$

$$1. \quad \operatorname{sn}(iu, k) = i \frac{\operatorname{sn}(u, k')}{\operatorname{cn}(u, k')}.$$

$$2. \quad \operatorname{cn}(iu, k) = \frac{1}{\operatorname{cn}(u, k')}.$$

SI 50(65)

$$3. \quad \operatorname{dn}(iu, k) = \frac{\operatorname{dn}(u, k')}{\operatorname{cn}(u, k')}.$$

SI 50(65)

$$4. \quad \operatorname{sn}(u, k) = k^{-1} \operatorname{sn}(ku, k^{-1}).$$

$$5. \quad \operatorname{cn}(u, k) = \operatorname{dn}(ku, k^{-1}).$$

$$6. \quad \operatorname{dn}(u, k) = \operatorname{cn}(ku, k^{-1}).$$

$$7. \quad \operatorname{sn}(u, ik) = \frac{1}{\sqrt{1+k^2}} \frac{\operatorname{sn}(u\sqrt{1+k^2}, k(1+k^2)^{-\frac{1}{2}})}{\operatorname{dn}(u\sqrt{1+k^2}, k(1+k^2)^{-\frac{1}{2}})}.$$

$$8. \quad \operatorname{cn}(u, ik) = \frac{\operatorname{sn}(u(1+k^2)^{\frac{1}{2}}, k(1+k^2)^{-\frac{1}{2}})}{\operatorname{dn}(u(1+k^2)^{\frac{1}{2}}, k(1+k^2)^{-\frac{1}{2}})}.$$

$$9. \quad \operatorname{dn}(u, ik) = \frac{1}{\operatorname{dn}(u(1+k^2)^{\frac{1}{2}}, k(1+k^2)^{-\frac{1}{2}})}.$$

Functional relations

8.154

$$1. \quad \operatorname{sn}^2 u = \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{dn} 2u}.$$

MO 146

$$3. \quad \operatorname{dn}^2 u = \frac{\operatorname{dn} 2u + k^2 \operatorname{cn} 2u + k'^2}{1 + \operatorname{dn} 2u}.$$

$$4. \quad \operatorname{sn}^2 u + \operatorname{cn}^2 u = 1.$$

$$5. \quad \operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1.$$

8.155

$$1. \quad \frac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} = k^2 \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn}^2 u}.$$

$$2. \quad \frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u} = \frac{\operatorname{sn}^2 u \operatorname{dn}^2 u}{\operatorname{cn}^2 u}.$$

8.156

$$1. \quad \operatorname{sn}(u \pm v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v \pm \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

922

$$2. \quad \operatorname{cn}(u \pm v) = \frac{\operatorname{cn} u \operatorname{cn} v \mp \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

$$3. \quad \operatorname{dn}(u \pm v) = \frac{\operatorname{dn} u \operatorname{dn} v \mp k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

$$1. \quad \operatorname{sn} \frac{u}{2} = \pm \frac{1}{k} \sqrt{\frac{1 - \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm \sqrt{\frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}}.$$

SI 47(61), SU 67(15)

$$2. \quad \operatorname{cn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{dn} u}} = \pm \frac{k'}{k} \sqrt{\frac{1 - \operatorname{dn} u}{\operatorname{dn} u - \operatorname{cn} u}}.$$

SI 48(62), SI 67(16)

$$3. \quad \operatorname{dn} \frac{u}{2} = \pm \sqrt{\frac{\operatorname{cn} u + \operatorname{dn} u}{1 + \operatorname{cn} u}} = \pm k' \sqrt{\frac{1 - \operatorname{cn} u}{\operatorname{dn} u + \operatorname{cn} u}}.$$

SI 48(63), SI 67(17)

8.158

$$\left. \begin{array}{l} 1. \quad \frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u. \\ 2. \quad \frac{d}{du} \operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u \\ 3. \quad \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{dn} u \operatorname{cn} u. \end{array} \right\}$$

SI 21(21)

8.159

Jacobian elliptic functions are solutions of the following differential equations:

$$\left. \begin{array}{l} 1. \quad \frac{d}{du} \operatorname{sn} u = \sqrt{(1 - \operatorname{sn}^2 u)(1 - k^2 \operatorname{sn}^2 u)}. \\ 2. \quad \frac{d}{du} \operatorname{cn} u = -\sqrt{(1 - \operatorname{cn}^2 u)(k'^2 + k^2 \operatorname{cn}^2 u)}, \\ 3. \quad \frac{d}{du} \operatorname{dn} u = -\sqrt{(1 - \operatorname{dn}^2 u)(\operatorname{dn}^2 u - k'^2)}. \end{array} \right\}$$

SI 21(22)

For the indefinite integrals of Jacobi's elliptic functions, see 5.13.

8.16 The Weierstrass function $\wp(u)$

8.160

The Weierstrass elliptic function $\wp(u)$ is defined by

$$1. \quad \wp(u) = \frac{1}{u^2} + \sum'_{m,n} \left\{ \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right\},$$

where the symbol \sum' means that the summation is made over all combinations of integers m and n except for the combination $m = n = 0$; $2\omega_1$ and $2\omega_2$ are the periods of the function $\wp(u)$. Obviously,

$$2. \quad \wp(u + 2m\omega_1 + 2n\omega_2) = \wp(u) \quad \text{and} \quad \text{Im} \left(\frac{\omega_1}{\omega_2} \right) \neq 0,$$

923

$$3. \quad \frac{d}{du} \wp(u) = -2 \sum_{m,n} \frac{1}{(u - 2m\omega_1 - 2n\omega_2)^3},$$

where the summation is made over all integral values of m and n .

The series 8.160 1. and 8.160 3. converge everywhere except at the poles, that is, at the points $2m\omega_1 + 2n\omega_2$ (where m and n are integers).

4. The function $\wp(u)$ is a *second-order periodic function* and has *one second-order pole* in a period parallelogram.

SI 306

8.161

The function $\wp(u)$ satisfies the differential equation

$$1. \quad \left[\frac{d}{du} \wp(u) \right]^2 = 4\wp^3(u) - g_2\wp(u) - g_3,$$

SI 142, 310, WH

where

$$2. \quad g_2 = 60 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-4}; \quad g_3 = 140 \sum'_{m,n} (m\omega_1 + n\omega_2)^{-6}.$$

WH, SI 310

The functions g_2 and g_3 are called the *invariants* of the function $\wp(u)$.

8.162

$$u = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} = \int_{\wp(u)}^{\infty} \frac{dz}{\sqrt{4(z - e_1)(z - e_2)(z - e_3)}},$$

where $e_1, e_2,$ and e_3 are the roots of the equation $4z^3 - g_2z - g_3 = 0$; that is,

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}.$$

SI 142, 143, 144

8.163

$\wp(\omega_1) = e_1, \wp(\omega_1 + \omega_2) = e_2, \wp(\omega_2) = e_3$. Here, it is assumed that if $e_1, e_2,$ and e_3 lie on a straight line in the complex plane, e_2 lies between e_1 and e_3 .

8.164

The number $\Delta = g_2^3 - 27g_3^2$ is called the *discriminant* of the function $\wp(u)$. If $\Delta > 0$, all roots $e_1, e_2,$ and e_3 of the equation $4z^3 - g_2z - g_3 = 0$ (where g_2 and g_3 are real numbers) are *real*. In this case, the roots $e_1, e_2,$ and e_3 are numbered in such a way that $e_1 > e_2 > e_3$.

1. If $\Delta > 0$, then

$$\omega_1 = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}, \quad \omega_2 = i \int_{-\infty}^{e_3} \frac{dz}{\sqrt{g_3 + g_2z - 4z^3}},$$

where ω_1 is real and ω_2 is a purely imaginary number. Here, the values of the radical in the integrand are chosen in such a way that ω_1 and $\frac{\omega_2}{i}$ will be positive.

2. If $\Delta < 0$, the root e_2 of the equation $4z^3 - g_2z - g_3 = 0$ is *real* and the remaining two roots (e_1 and e_3) are *complex conjugates*.

Suppose that $e_1 = \alpha = i\beta,$ and $e_3 = \alpha - i\beta$. In this case, it is convenient to take

$$\omega' = \int_{e_1}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}} \quad \text{and} \quad \omega'' = \int_{e_3}^{\infty} \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}.$$

924

In the first integral, the integration is taken over a path lying entirely in the upper half-plane and in the second over a path lying entirely in the lower-half plane.

SI 151(21, 22)

8.165

Series representation:

$$1. \quad \wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{4 \cdot 5} + \frac{g_3 u^4}{4 \cdot 7} + \frac{g_2^2 u^6}{2^4 \cdot 3 \cdot 5^2} + \frac{3g_2 g_3 u^8}{2^4 \cdot 5 \cdot 7 \cdot 11} + \dots$$

WH

8.166

Functional relations

$$1. \quad \wp(u) = \wp(-u), \quad \wp'(u) = -\wp'(-u).$$

$$2. \quad \wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4} \left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right]^2.$$

SI 163(32)

8.167

$$\wp(u; g_2, g_3) = \mu^2 \wp \left(\mu u; \frac{g_2}{\mu^4}, \frac{g_3}{\mu^6} \right) \text{ (the formula for homogeneity).}$$

SI 149(13)

The special case: $\mu = i$.

$$1. \quad \wp(u; g_2, g_3) = -\wp(iu; g_2, -g_3).$$

8.168

An arbitrary elliptic function can be expressed in terms of the elliptic function $\wp(u)$ having the same periods as the original function and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

$$\wp(u)$$

and its derivative $\wp'(u)$. This expression is rational with respect to $\wp(u)$ and linear with respect to $\wp'(u)$.

8.169

A connection with the Jacobian elliptic functions. For $\Delta > 0$ (see 8.164 1.).

$$\begin{aligned} 1. \quad \wp\left(\frac{u}{\sqrt{e_1 - e_2}}\right) &= e_1 + (e_1 - e_3) \frac{\operatorname{cn}^2(u; k)}{\operatorname{sn}^2(u; k)}; \\ &= e_2 + (e_1 - e_3) \frac{\operatorname{dn}^2(u; k)}{\operatorname{sn}^2(u; k)}; \\ &= e_3 + (e_1 - e_3) \frac{1}{\operatorname{sn}^2(u, k)}; \end{aligned}$$

SI 145(5), ZH 120(197-199)a

$$2. \quad \omega_1 = \frac{K}{\sqrt{e_1 - e_3}}, \quad \omega_2 = \frac{iK'}{\sqrt{e_1 - e_3}},$$

SI 154(29)

where

$$3. \quad k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \quad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}.$$

SI 145(7)

For $\Delta < 0$ (see 8.164 2.)

$$4. \quad \wp\left(\frac{u}{\sqrt[4]{9\alpha^2 + \beta^2}}\right) = e_2 + \sqrt{9\alpha^2 + \beta^2} \frac{1 + \operatorname{cn}(2u; k)}{1 - \operatorname{cn}(2u; k)};$$

SI 147(12)

$$5. \quad \omega' = \frac{K - iK'}{2\sqrt{9\alpha^2 + \beta^2}}, \quad \omega'' = \frac{K + iK'}{\sqrt[4]{9\alpha^2 + \beta^2}},$$

SI 153(28)

where

$$6. \quad k = \sqrt{\frac{1}{2} - \frac{3e_2}{\sqrt{9\alpha^2 + \beta^2}}}; \quad k' = \sqrt{\frac{1}{2} + \frac{3e_2}{\sqrt{9\alpha^2 + \beta^2}}}.$$

SI 147

For $\Delta = 0$, all the roots e_1 , e_2 , and e_3 are real and if $g_2g_3 \neq 0$, two of them are equal to each other. If $e_1 = e_2 \neq e_3$, then

$$7. \quad \wp(u) = \frac{3g_3}{g_2} - \frac{9g_3}{2g_2} \operatorname{cth}^2 \left(u \sqrt{-\frac{9g_3}{2g_2}} \right).$$

SI 148

If $e_1 \neq e_2 = e_3$, then

$$8. \quad \wp(u) = -\frac{3g_3}{2g_2} + \frac{9g_3}{2g_2} \frac{1}{\sin^2 \left(u \sqrt{\frac{9g_3}{2g_2}} \right)}.$$

SI 149

If $g_2 = g_3 = 0$, then $e_1 = e_2 = e_3 = 0$, and

$$9. \quad \wp(u) = \frac{1}{u^2}.$$

SI 149

8.17 The functions $\zeta(u)$ and $\sigma(u)$

8.171

Definitions:

$$1. \quad \zeta(u) = \frac{1}{u} - \int_0^u \left(\wp(z) - \frac{1}{z^2} \right) dz.$$

SI 181(45)

$$2. \quad \sigma(u) = u \exp \left\{ \int_0^u \left(\wp(z) - \frac{1}{z^2} \right) dz \right\}.$$

8.172

Series and infinite-product representation

$$1. \quad \zeta(u) = \frac{1}{u} + \sum' \left(\frac{1}{u - 2m\omega_1 - 2n\omega_2} + \frac{1}{2m\omega_1 + 2n\omega_2} + \frac{u}{(2m\omega_1 - 2n\omega_2)^2} \right).$$

SI 307(8)

$$2. \quad \sigma(u) = u \prod' \left(1 - \frac{u}{2m\omega_1 + 2n\omega_2} \right) \exp \left\{ \frac{u}{2m\omega_1 + 2n\omega_2} + \frac{u^2}{2(2m\omega_1 + 2n\omega_2)^2} \right\}.$$

SI 308(9)

8.173

$$1. \quad \zeta(u) = u - \frac{g_2 u^3}{2^2 \cdot 3 \cdot 5} - \frac{g_3 u^5}{2^2 \cdot 5 \cdot 7} - \frac{g_2^2 u^7}{2^4 \cdot 3 \cdot 5^2 \cdot 7} - \frac{3g_2 g_3 u^9}{2^4 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \dots$$

SI 181(49)

$$2. \quad \zeta(u) = u - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} - \frac{g_2^2 u^9}{2^9 \cdot 3^2 \cdot 5 \cdot 7} - \frac{3g_2 g_3 u^{11}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11} - \dots$$

SI 181(49)

926

8.174

$$\begin{aligned} \zeta(u) &= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{\pi}{2\omega_1} \sum_{n=1}^{\infty} \left\{ \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} + n\pi \frac{\omega_2}{\omega_1} \right) + \right. \\ &\quad \left. + \operatorname{ctg} \left(\frac{\pi u}{2\omega_1} - n\pi \frac{\omega_2}{\omega_1} \right) \right\}; \\ &= \frac{\zeta(\omega_1)}{\omega_1} u + \frac{\pi}{2\omega_1} \operatorname{ctg} \frac{\pi u}{2\omega_1} + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin \frac{\pi n u}{\omega_1} \end{aligned}$$

MO 155

MO 154

Functional relations and properties

8.175

8.176

$$1. \quad \zeta(u + 2\omega_1) = \zeta(u) + 2\zeta(\omega_1).$$

SI 184(57)

$$2. \quad \zeta(u + 2\omega_2) = \zeta(u) + 2\zeta(\omega_2).$$

SI 184(57)

$$3. \quad \sigma(u + 2\omega_1) = -\sigma(u) \exp\{2(u + \omega_1)\zeta(\omega_1)\}.$$

SI 185(60)

$$4. \quad \sigma(u + 2\omega_2) = -\sigma(u) \exp\{2(u + \omega_2)\zeta(\omega_2)\}.$$

SI 185(60)

$$5. \quad \omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{\pi}{2}i.$$

SI 186(62)

8.177

$$1. \quad \zeta(u + v) - \zeta(u) - \zeta(v) = \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}.$$

SI 182(53)

$$2. \quad \wp(u) - \wp(v) = -\frac{\sigma(u - v)\sigma(u + v)}{\sigma^2(u)\sigma^2(v)}.$$

SI 183(54)

$$3. \quad \zeta(u - v) + \zeta(u + v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}.$$

SI 182(51)

8.178

$$1. \quad \zeta(u; \omega_1, \omega_2) = t\zeta(tu; t\omega_1, t\omega_2).$$

MO 154

$$2.^7 \quad \sigma(u; \omega_1, \omega_2) = t\sigma(tu; t\omega_1, t\omega_2).$$

MO 156

For the (indefinite) integrals of Weierstrass elliptic functions, see 5.14.

8.18- 8.19 Theta functions

8.180

Theta functions are defined as the sums (for $|q| < 1$) of the following series:

$$1. \quad \vartheta_4(u) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu.$$

WH

927

$$2. \quad \vartheta_1(u) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} q^{(n-\frac{1}{2})^2} \sin(2n-1)u.$$

WH

$$3. \quad \vartheta_2(u) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{(2n+1)ui} = 2 \sum_{n=1}^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n-1)u.$$

WH

$$4. \quad \vartheta_3(u) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu.$$

WH

The notations $\vartheta(u, q)$, $\vartheta(u|\tau)$, where τ and q are related by $q = e^{i\pi\tau}$, are also used. q is called the *nome* of the theta function and τ its *parameter*.

8.181

Representation of theta functions in terms of infinite products

$$1. \quad \vartheta_4(u) = \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2u + q^{2(2n-1)})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$2. \quad \vartheta_3(u) = \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2u + q^{2(2n-1)})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$3. \quad \vartheta_1(u) = 2\sqrt[4]{q} \sin u \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}).$$

SI 200(9), ZH 90(9)

$$4. \quad \vartheta_2(u) = 2\sqrt{2} \cos u \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2u + q^{4n})(1 - q^{2n}).$$

SI 200(0), ZH 90(9)

Functional relations and properties

8.182

Quasiperiodicity.

Suppose that $q = e^{\pi\tau i}$ ($\text{Im } \tau > 0$). Then, theta functions that are periodic functions of u are called *quasiperiodic functions* of τ and u . This property follows from the equations

$$1. \quad \vartheta_4(u + \pi) = \vartheta_4(u).$$

$$2. \vartheta_4(u + \tau\pi) = -\frac{1}{q}e^{-2iu}\vartheta_4(u).$$

SI 200(10)

$$3. \vartheta_1(u + \pi) = -\vartheta_1(u).$$

SI 200(10)

$$4. \vartheta_1(u + \tau\pi) = -\frac{1}{q}e^{-2iu}\vartheta_1(u).$$

SI 200(10)

$$5. \vartheta_2(u + \pi) = -\vartheta_2(u).$$

SI 200(10)

$$6. \vartheta_2(u + \tau\pi) = \frac{1}{q}e^{-2iu}\vartheta_2(u).$$

SI 200(10)

$$7. \vartheta_3(u + \pi) = \vartheta_3(u).$$

SI 200(10)

$$8. \vartheta_3(u + \tau\pi) = \frac{1}{q}e^{-2iu}\vartheta_3(u).$$

SI 200(10)

$$9^*. \vartheta_4(u + \pi) = \vartheta_4(u).$$

LW 6(1.3.5)

928

8.183

$$1. \vartheta_4\left(u + \frac{1}{2}\pi\right) = \vartheta_3(u).WH$$

$$2. \vartheta_1\left(u + \frac{1}{2}\pi\right) = \vartheta_2(u).$$

WH

$$3. \vartheta_2\left(u + \frac{1}{2}\pi\right) = -\vartheta_1(u).$$

WH

$$4. \vartheta_3\left(u + \frac{1}{2}\pi\right) = \vartheta_4(u).$$

WH

$$5. \vartheta_4\left(u + \frac{1}{2}\pi\tau\right) = iq^{-\frac{1}{4}}e^{-iu}\vartheta_1(u).$$

WH

$$6. \vartheta_1\left(u + \frac{1}{2}\pi\tau\right) = iq^{-\frac{1}{4}}e^{-iu}\vartheta_4(u).$$

WH

$$7. \vartheta_2\left(u + \frac{1}{2}\pi\tau\right) = q^{-\frac{1}{4}}e^{-iu}\vartheta_3(u).$$

WH

$$8. \vartheta_3\left(u + \frac{1}{2}\pi\tau\right) = q^{-\frac{1}{4}}e^{-iu}\vartheta_2(u).$$

WH

8.184

Even and odd theta functions

$$1. \vartheta_1(-u) = -\vartheta_1(u)$$

WH

WH

$$3. \vartheta_3(-u) = \vartheta_3(u)$$

WH

$$4. \vartheta_4(-u) = \vartheta_4(u).$$

WH

8.185

$$\vartheta_4^4(u) + \vartheta_2^4(u) = \vartheta_1^4(u) + \vartheta_3^4(u).$$

WH

8.186⁷

Considering the theta functions as functions of two independent variables u and τ , we have

$$\pi i \frac{\partial^2 \vartheta_k(u|\tau)}{\partial u^2} + 4 \frac{\partial \vartheta_k(u|\tau)}{\partial \tau} = 0 \quad [k = 1, 2, 3, 4].$$

WH

8.187

We denote the partial derivatives of the theta functions with respect to u by a prime and consider them as functions of the single argument u . Then,

$$1. \vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$$

WH

$$2. \frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}.$$

WH

8.188

$$\vartheta_1(u)\vartheta_2(u)\vartheta_3(u)\vartheta_4(0) = \frac{1}{2}\vartheta_1(2u)\vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$$

The zeros of the theta functions:

$$1.^{\text{S}} \quad \vartheta_4(u) = 0 \quad \text{for} \quad u = 2m\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2}.$$

SI 201

$$2.^* \quad \vartheta_1(u) = 0 \quad \text{for} \quad u = 2m\frac{\pi}{2} + 2n\frac{\pi\tau}{2}.$$

SI 201

$$3. \quad \vartheta_2(u) = 0 \quad \text{for} \quad u = (2m-1)\frac{\pi}{2} + 2n\frac{\pi\tau}{2}.$$

SI 201

$$4. \quad \vartheta_3(u) = 0 \quad \text{for} \quad u = (2m-1)\frac{\pi}{2} + (2n-1)\frac{\pi\tau}{2} \quad [m \text{ and } n\text{-integers or zero}].$$

SI 201

For integrals of theta functions, see 6.16.

Connections with the Jacobian elliptic functions:

$$\text{For } \tau = i\frac{K'}{K}, \quad \text{i.e. for } q = \exp\left(-\pi\frac{K'}{K}\right),$$

$$1. \quad \text{sn } u = \frac{1}{\sqrt{k}} \frac{\vartheta_1\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \frac{1}{\sqrt{k}} \frac{H(u)}{\Theta(u)}.$$

SI 206(22), SI 209(35)

$$2. \quad \text{cn } u = \sqrt{\frac{k'}{k}} \frac{\vartheta_2\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{\frac{k'}{k}} \frac{H_1(u)}{\Theta(u)}.$$

$$3. \quad \operatorname{dn} u = \sqrt{k'} \frac{\vartheta_3\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)} = \sqrt{k'} \frac{\Theta_1(u)}{\Theta(u)}.$$

SI 207(24), SI 209(35)

8.192

Series representation of the functions H , H_1 , Θ , Θ_1 .

$$1. \quad \Theta(u) = \vartheta_4\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos \frac{n\pi u}{K}.$$

SI 207(25), SI 212(42)

$$2. \quad H(u) = \vartheta_1\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \sqrt[4]{q^{(2n+1)^2}} \sin(2n-1) \frac{\pi u}{2K}.$$

SI 207(25), SI 212(43)

$$3. \quad \Theta_1(u) = \vartheta_3\left(\frac{\pi u}{2K}\right) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos \frac{n\pi u}{K}.$$

SI 207(25), SI 212(45)

$$4. \quad H_1(u) = \vartheta_2\left(\frac{\pi u}{2K}\right) = 2 \sum_{n=1}^{\infty} \sqrt[4]{q^{(2n-1)^2}} \cos(2n-1) \frac{\pi u}{2K}.$$

SI 207(25), SI 212(44)

In formulas 8.192 $q = \exp\left(-\pi \frac{K'}{K}\right)$.

930

8.193

Connections with the Weierstrass elliptic functions

$$1. \quad \wp(u) = e_1 + \left[\frac{H_1(u\sqrt{\lambda})H'(0)}{H_1(0)H(u\sqrt{\lambda})} \right]^2 \lambda = e_2 + \left[\frac{\Theta_1(u\sqrt{\lambda})H'(0)}{\Theta_1(0)H(u\sqrt{\lambda})} \right]^2 \lambda = e_3 + \left[\frac{\Theta(u\sqrt{\lambda})H'(0)}{\Theta(0)H(u\sqrt{\lambda})} \right]^2 \lambda.$$

$$2. \quad \zeta(u) = \frac{\eta_1 u}{\omega_1} + \sqrt{\lambda} \frac{H'(u\sqrt{\lambda})}{H(u\sqrt{\lambda})}.$$

SI 234(73)

$$3. \quad \sigma(u) = \frac{1}{\sqrt{\lambda}} \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right) \frac{H(u\sqrt{\lambda})}{H'(0)},$$

SI 234(72)

where

$$\lambda = e_1 - e_3; \quad \eta_1 = \zeta(\omega_1) = -\frac{\omega_1 \lambda H'''(0)}{3 H'(0)}.$$

SI 236

8.194

The connection with elliptic integrals:

$$1. \quad E(u, k) = u - u \frac{\Theta''(0)}{\Theta(0)} + \frac{\Theta'(u)}{\Theta(u)}.$$

SI 228(65)

$$2. \quad \Pi(u, -k^2 \sin^2 a, k) = \int_0^u \frac{d\varphi}{1 - k^2 \sin^2 a \operatorname{sn}^2 \varphi} = u + \frac{\operatorname{sn} a}{\operatorname{cn} a \operatorname{dn} a} \left[\frac{\Theta'(a)}{\Theta(a)} u + \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} \right].$$

SI 232(69)

$$q\text{-series and products} \quad \left[q = \exp\left(-\pi \frac{K'}{K}\right) \right]$$

8.195

$$\frac{\pi}{2} \left[1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right]^2 = K = \frac{\pi}{2} \Theta^2(K) \quad (\text{cf. 8.1971.}).$$

8.196

$$E = K - K \frac{\Theta''(0)}{\Theta(0)} = K - \frac{2\pi^2}{K} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{n^2}}{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}}.$$

SI 230(67)

8.197

$$1. \quad 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \sqrt{\frac{2K}{\pi}} = \vartheta_3(0) \quad (\text{c.f. 8.195}).$$

8.195
WH

$$2. \quad \sum_{n=1}^{\infty} q^{\left(\frac{2n-1}{2}\right)^2} = \sqrt{\frac{kK}{2\pi}} = \frac{1}{2} \vartheta_2(0).$$

WH

931

$$3. \quad 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^4 = k.$$

SI 206(17, 18)

$$4. \quad \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^4 = k'$$

SI 206(19, 20)

$$5. \quad 2^4 \sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 - q^{2n-1}} \right)^2 = 2\sqrt{k} \frac{K}{\pi}.$$

WH

8.198

$$1. \quad \lambda = \frac{1}{2} \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} = \frac{\sum_{n=0}^{\infty} q^{(2n+1)^2}}{1 + 2 \sum_{n=1}^{\infty} q^{4n^2}} \quad \left[\text{for } 0 < k < 1, \text{ we have } 0 < \lambda < \frac{1}{2} \right]$$

WH

The series

$$2. \quad q = \lambda + 2\lambda^5 + 15\lambda^9 + 150\lambda^{13} + 1707\lambda^{17} + \dots \text{ is used to determine } q \text{ from the given modulus } k.$$

WH

8.199

*Identities involving products of theta functions

$$1. \quad \theta_1(x, q)\theta_1(y, q) = \theta_3(x + y, q^2)\theta_2(x - y, q^2) - \theta_2(x + y, q^2)\theta_3(x - y, q^2).$$

LW 7(1.4.7)

$$2. \quad \theta_1(x, q)\theta_2(y, q) = \theta_1(x + y, q^2)\theta_4(x - y, q^2) + \theta_4(x + y, q^2)\theta_1(x - y, q^2).$$

LW 8(1.4.8)

$$3. \quad \theta_2(x, q)\theta_2(y, q) = \theta_2(x + y, q^2)\theta_3(x - y, q^2) + \theta_3(x + y, q^2)\theta_2(x - y, q^2).$$

LW 8(1.4.9)

$$4. \quad \theta_3(x, q)\theta_3(y, q) = \theta_3(x + y, q^2)\theta_3(x - y, q^2) + \theta_2(x + y, q^2)\theta_2(x - y, q^2).$$

LW 8(1.4.10)

$$5. \quad \theta_3(x, q)\theta_4(y, q) = \theta_4(x + y, q^2)\theta_4(x - y, q^2) - \theta_1(x + y, q^2)\theta_1(x - y, q^2).$$

$$6. \quad \theta_4(x, q)\theta_4(y, q) = \theta_3(x + y, q^2)\theta_3(x - y, q^2) - \theta_2(x + y, q^2)\theta_2(x - y, q^2).$$

LW 8(1.4.12)

$$7. \quad \theta_1(x+y)\theta_1(x-y)\theta_4^2(0) = \theta_3^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_3^2(y) = \theta_1^2(x)\theta_4^2(y) - \theta_4^2(x)\theta_1^2(y).$$

LW 8(1.4.16)

$$8. \quad \theta_2(x+y)\theta_2(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_2^2(y) - \theta_1^2(x)\theta_3^2(y) = \theta_2^2(x)\theta_4^2(y) - \theta_3^2(x)\theta_1^2(y).$$

LW 8(1.4.17)

$$9. \quad \theta_3(x+y)\theta_3(x-y)\theta_4^2(0) = \theta_4^2(x)\theta_3^2(y) - \theta_1^2(x)\theta_2^2(y) = \theta_3^2(x)\theta_4^2(y) - \theta_2^2(x)\theta_1^2(y).$$

LW 8(1.4.18)

$$10. \quad \theta_4(x + y)\theta_4(x - y)\theta_4^2(0) = \theta_4^2(x)\theta_4^2(y) - \theta_1^2(x)\theta_1^2(y).$$

LW 8(1.4.15)

$$11. \quad \theta_4(x+y)\theta_4(x-y)\theta_4^2(0) = \theta_3^2(x)\theta_3^2(y) - \theta_2^2(x)\theta_2^2(y) = \theta_4^2(x)\theta_4^2(y) - \theta_1^2(x)\theta_1^2(y).$$

LW 9(1.4.19)

$$12. \quad \theta_1(x+y)\theta_1(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_1^2(y) = \theta_4^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_4^2(y).$$

LW 9(1.4.23)

932

$$13. \quad \theta_2(x+y)\theta_2(x-y)\theta_3^2(0) = \theta_2^2(x)\theta_3^2(y) - \theta_4^2(x)\theta_1^2(y) = \theta_3^2(x)\theta_2^2(y) - \theta_1^2(x)\theta_4^2(y).$$

LW 9(1.4.24)

$$14. \quad \theta_3(x+y)\theta_3(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_1^2(y) + \theta_3^2(x)\theta_3^2(y) = \theta_2^2(x)\theta_2^2(y) + \theta_4^2(x)\theta_4^2(y).$$

$$15. \quad \theta_4(x+y)\theta_4(x-y)\theta_3^2(0) = \theta_1^2(x)\theta_2^2(y) + \theta_3^2(x)\theta_4^2(y) = \theta_2^2(x)\theta_1^2(y) + \theta_4^2(x)\theta_3^2(y).$$

LW 9(1.4.26)

$$16. \quad \theta_1(x+y)\theta_1(x-y)\theta_2^2(0) = \theta_1^2(x)\theta_2^2(y) - \theta_2^2(x)\theta_1^2(y) = \theta_4^2(x)\theta_3^2(y) - \theta_3^2(x)\theta_4^2(y).$$

LW 9(1.4.30)

$$17. \quad \theta_2(x+y)\theta_2(x-y)\theta_2^2(0) = \theta_2^2(x)\theta_2^2(y) - \theta_1^2(x)\theta_1^2(y) = \theta_3^2(x)\theta_3^2(y) - \theta_4^2(x)\theta_4^2(y).$$

LW 10(1.4.31)

$$18. \quad \theta_3(x+y)\theta_3(x-y)\theta_2^2(0) = \theta_3^2(x)\theta_2^2(y) + \theta_4^2(x)\theta_1^2(y) = \theta_2^2(x)\theta_3^2(y) + \theta_1^2(x)\theta_4^2(y).$$

LW 10(1.4.32)

$$19. \quad \theta_4(x+y)\theta_4(x-y)\theta_2^2(0) = \theta_4^2(x)\theta_2^2(y) + \theta_3^2(x)\theta_1^2(y) = \theta_1^2(x)\theta_3^2(y) + \theta_2^2(x)\theta_4^2(y).$$

LW 10(1.4.33)

$$20. \quad \theta_3^2(x)\theta_3^2(0) = \theta_4^2(x)\theta_4^2(0) + \theta_2^2(x)\theta_2^2(0).$$

LW 11(1.4.49)

$$21. \quad \theta_4^2(x)\theta_3^2(0) = \theta_1^2(x)\theta_2^2(0) + \theta_3^2(x)\theta_4^2(0).$$

LW 11(1.4.50)

$$22. \quad \theta_4^2(x)\theta_2^2(0) = \theta_1^2(x)\theta_3^2(0) + \theta_2^2(x)\theta_4^2(0).$$

LW 11(1.4.51)

$$23. \quad \theta_3^2(x)\theta_2^2(0) = \theta_1^2(x)\theta_4^2(0) + \theta_2^2(x)\theta_3^2(0).$$

$$24. \theta_3^4(x)\theta_2^4(0) + \theta_4^4(0).$$

LW 11(1.4.53)

8.199(1)

Theta functions as infinite products

$$1. \theta_1(z) = 2q^{\frac{1}{4}} \sin z \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2z + q^{4n}).$$

LW 15(1.6.23)

$$2. \theta_2(z) = 2q^{\frac{1}{4}} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2z + q^{4n}).$$

LW 15(1.5.24)

$$3. \theta_3(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2z + q^{4n-2}).$$

LW 15(1.6.25)

$$4. \theta_4(z) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos 2z + q^{4n-2}).$$

LW 15(1.6.26)

8.199(2)

Derivatives of ratios of theta functions

$$1. \frac{d}{dx}(\theta_1/\theta_4) = \theta_4^2(0)\theta_2(x)\theta_3(x)/\theta_4^2(x).$$

LW 19(1.9.3)

$$2. \frac{d}{dx}(\theta_2/\theta_4) = -\theta_3^2(0)\theta_1(x)\theta_3(x)/\theta_4^2(x).$$

$$3. \quad \frac{d}{dx}(\theta_3/\theta_4) = -\theta_2^2(0)\theta_1(x)\theta_2(x)/\theta_4^2(x).$$

LW 19(1.9.7)

$$4. \quad \frac{d}{dx}(\theta_1/\theta_3) = \theta_3^2(0)\theta_2(x)\theta_4(x)/\theta_3^2(x).$$

LW 19(1.9.8)

$$5. \quad \frac{d}{dx}(\theta_2/\theta_3) = -\theta_4^2(0)\theta_1(x)\theta_4(x)/\theta_3^2(x).$$

LW 19(1.9.9)

933

$$6. \quad \frac{d}{dx}(\theta_1/\theta_2) = \theta_2^2(0)\theta_3(x)\theta_4(x)/\theta_2^2(x).$$

LW 19(1.9.10)

$$7. \quad \frac{d}{dx}(\theta_4/\theta_1) = -\theta_4^2(0)\theta_2(x)\theta_3(x)/\theta_1^2(x).$$

LW 19(1.9.11)

$$8. \quad \frac{d}{dx}(\theta_4/\theta_2) = \theta_3^2(0)\theta_1(x)\theta_3(x)/\theta_2^2(x).$$

LW 20(1.9.12)

$$9. \quad \frac{d}{dx}(\theta_4/\theta_3) = \theta_2^2(0)\theta_1(x)\theta_2(x)/\theta_3^2(x).$$

LW 20(1.9.13)

$$10. \quad \frac{d}{dx}(\theta_3/\theta_1) = -\theta_3^2(0)\theta_2(x)\theta_4(x)/\theta_1^2(x).$$

LW 20(1.9.14)

$$11. \quad \frac{d}{dx}(\theta_3/\theta_2) = \theta_4^2(0)\theta_1(x)\theta_4(x)/\theta_2^2(x).$$

$$12. \quad \frac{d}{dx}(\theta_2/\theta_1) = -\theta_2^2(0)\theta_3(x)\theta_4(x)/\theta_1^2(x).$$

LW 20(1.9.16)

8.199(3)

Derivatives of theta functions

$$1. \quad \frac{d}{du} \ln \theta_1(u) = \cot u + 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - 2q^{2n} \cos 2u + q^{4n}}.$$

$$2. \quad \frac{d}{du} \ln \theta_2(u) = -\tan u - 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n}}{1 + 2q^{2n} \cos 2u + q^{4n}}.$$

$$3. \quad \frac{d}{du} \ln \theta_3(u) = -4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 + 2q^{2n} \cos 2u + q^{4n-2}}.$$

$$4. \quad \frac{d}{du} \ln \theta_4(u) = 4 \sin 2u \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - 2q^{2n} \cos 2u + q^{4n-2}}.$$

$$5. \quad \frac{d^2}{du^2} \ln \theta_2(u) = - \sum_{n=-\infty}^{\infty} \operatorname{sech}^2 \{i(u + n\pi\tau)\}.$$

8.2 The Exponential-Integral Function and Functions Generated by It

8.21 The exponential-integral function $Ei(x)$

8.211

$$1. \quad Ei(x) = - \int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt = \operatorname{li}(e^x) \quad [x < 0].$$

$$2.^8 \operatorname{Ei}(x) = - \lim_{\varepsilon \rightarrow +0} \left[\int_{-x}^{-\varepsilon} \frac{e^{-t}}{t} dt + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt \right] pv \int_{-\infty}^x \frac{e^t}{t} dt \quad [x > 0].$$

$$3.^7 \operatorname{Ei}(x) = \frac{1}{2} \{ \operatorname{Ei}(x + i0) + \operatorname{Ei}(x - i0) \} \quad [x > 0].$$

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934

8.212

$$\begin{aligned} 1. \quad \operatorname{Ei}(-x) &= C = \ln x + \int_0^x \frac{e^{-t} - 1}{t} dt \quad [x > 0]; \\ &= C + e^{-x} \ln x + \int_0^x e^{-t} \ln t dt \quad [x > 0]. \end{aligned}$$

NT 11(10)

NT 11(1)

$$2.^7 \operatorname{Ei}(x) = e^x \left[\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x-t)^2} \right] \quad [\text{integral is divergent}].$$

8.211

$$3. \quad \operatorname{Ei}(-x) = e^{-x} \left[-\frac{1}{x} + \int_0^{\infty} \frac{e^{-t} dt}{(x+t)^2} \right] \quad [x > 0] \quad (\text{cf. } \mathbf{8.211} \text{ 1.}).$$

8.211

LA 281(28)

$$4. \quad \operatorname{Ei}(\pm x) = \pm e^{\pm x} \int_0^1 \frac{dt}{x \pm \ln t} \quad [x > 0] \quad (\text{cf. } \mathbf{8.211} \text{ 1.}).$$

8.211

$$5. \quad \operatorname{Ei}(\pm xy) = \pm e^{\pm xy} \int_0^{\infty} \frac{e^{-xt}}{y \mp t} dt \quad [\operatorname{Re} y > 0, \quad x > 0].$$

NT 19(11)

$$6. \quad \text{Ei}(\pm x) = -e^{\pm x} \int_0^{\infty} \frac{e^{-it}}{t \pm ix} dt \quad [x > 0].$$

NT 23(2, 3)

$$7.8 \quad \text{Ei}(xy) = e^{xy} \int_0^1 \frac{t^{y-1}}{x + \ln t} dt;$$

LA 283(46)a

LA 282(44)a

$$8. \quad \text{Ei}(-xy) = -e^{-xy} \int_0^1 \frac{t^{y-1}}{x - \ln t} dt;$$

$$= x^{-1} e^{-xy} \left[\int_0^1 \frac{t^{x-1}}{(y - \ln t)^2} dt - y^{-1} \right] \quad [x > 0, \quad y > 0].$$

LA 283(47)a

LA 282(45)a

$$9. \quad \text{Ei}(x) = e^x \int_1^{\infty} \frac{1}{x - \ln t} \frac{dt}{t^2} \quad [x > 0].$$

LA 283(48)

$$10. \quad \text{Ei}(-x) = -e^{-x} \int_1^{\infty} \frac{1}{x + \ln t} \frac{dt}{t^2} \quad [x > 0].$$

LA 283(48)

$$11. \quad \text{Ei}(-x) = -e^{-x} \int_0^{\infty} \frac{t \cos t + x \sin t}{t^2 + x^2} dt \quad [x > 0].$$

NT 23(6)

$$12. \quad \text{Ei}(-x) = -e^{-x} \int_0^{\infty} \frac{t \cos t - x \sin t}{t^2 + x^2} dt \quad [x < 0].$$

NT 23(6)

$$13. \quad \text{Ei}(-x) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos t}{t} \arctg \frac{t}{x} dt \quad [\text{Re } x > 0].$$

$$14. \quad \text{Ei}(-x) = \frac{2e^{-x}}{\pi} \int_0^{\infty} \frac{x \cos t - t \sin t}{t^2 + x^2} \ln t \, dt \quad [x > 0].$$

NT 26(7)

935

$$15. \quad \text{Ei}(x) = 2 \ln x - \frac{2e^x}{\pi} \int_0^{\infty} \frac{x \cos t + t \sin t}{t^2 + x^2} \ln t \, dt \quad [x > 0].$$

NT 27(8)

$$16. \quad \text{Ei}(-x) = -x \int_1^{\infty} e^{-tx} \ln t \, dt \quad [x > 0].$$

NT 32(12)

See also 3.327, 3.881 8., 3.916 2. and 3., 4.326 1., 4.326 2., 4.331 2., 4.351 3., 4.425 3., 4.581. For integrals of the exponential-integral function, see 6.22-6.23, 6.78.

Series and asymptotic representations

8.213

$$1. \quad \text{li}(x) = \mathbf{C} + \ln(-\ln x) + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \quad [0 < x < 1].$$

NT 3(9)

$$2. \quad \text{li}(x) = \mathbf{C} + \ln \ln x + \sum_{k=1}^{\infty} \frac{(\ln x)^k}{k \cdot k!} \quad [x > 1].$$

NT 3(10)

8.214

$$1. \quad \text{Ei}(x) = \mathbf{C} + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!} \quad [x < 0].$$

8.215

$$\operatorname{Ei}(z) = \frac{e^z}{z} \left[\sum_{k=0}^n \frac{k!}{z^k} + R_n(z) \right], \quad |R_n(z)| = O(|z|^{-n-1}),$$

$[z \rightarrow \infty, |\arg(-z)| \leq \pi - \delta; \quad \delta > 0 \text{ small}]. \quad |R_n(z)| \leq (n+1)!|z|^{-n-1} [\operatorname{Re} z \leq 0].$

where

$$|R_n| < \frac{n!}{|x|^{n+1} \cos \frac{\varphi}{2}}, \quad x = |x|e^{i\varphi}, \quad \varphi^2 < \pi^2.$$

NT 37(9)

8.216⁷

$$\operatorname{Ei}(nx) - \operatorname{Ei}(-nx) = e^{nx'} \left(\frac{1}{nx} + \frac{1}{n^2x^2} + \frac{k_n}{n^3x^3} \right),$$

where

$$x' = x \operatorname{sign} \operatorname{Re}(x), \quad k_n = O(1), \quad \text{and } n \rightarrow \infty.$$

NT 39(15)

8.217

Functional relations:

$$\begin{aligned} 1. \quad e^{x'} \operatorname{Ei}(-x') - e^{-x'} \operatorname{Ei}(x') &= -2 \int_0^\infty \frac{x' \sin t}{t^2 + x'^2} dt = \\ &= \frac{4}{\pi} \int_0^\infty \frac{x' \cos t}{t^2 + x'^2} \ln t dt - 2e^{-x'} \ln x' \quad [x' = x \operatorname{sign} \operatorname{Re} x]. \end{aligned}$$

NT 27(9)

NT 24(11)

936

$$3. \quad \text{Ei}(-x) - \text{Ei}\left(-\frac{1}{x}\right) = \frac{2}{\pi} \int_0^\infty \frac{\cos t}{t} \operatorname{arctg} \frac{t\left(x - \frac{1}{x}\right)}{1+t^2} dt \quad [\operatorname{Re} x > 0].$$

NT 25(14)

$$4. \quad \text{Ei}(-\alpha x) \text{Ei}(-\beta x) - \ln(\alpha\beta) \text{Ei}[-(\alpha+\beta)x] = e^{-(\alpha+\beta)x} \int_0^\infty \frac{e^{-tx} \ln[(\alpha+t)(\beta+t)]}{t+\alpha+\beta} dt.$$

NT 32(9)

See also 3.723 1. and 5., 3.742 2. and 4., 3.824 4., 4.573 2.

For a connection with a degenerate hypergeometric function, see 9.237.

For integrals of the exponential-integral function, see 5.21, 5.22, 5.23, and 6.22-6.23.

8.218

Two numerical values:

$$1. \quad \text{Ei}(-1) = -0.219 \quad 383 \quad 934 \quad 395 \quad 520 \quad 273 \quad 665 \dots$$

NT 89

$$2. \quad \text{Ei}(1) = 1.895 \quad 117 \quad 816 \quad 355 \quad 936 \quad 755 \quad 478 \dots$$

NT 89

8.22 The hyperbolic-sine-integral $\operatorname{shi} x$ and the hyperbolic-cosine-integral $\operatorname{chi} x$

8.221

$$1. \quad \operatorname{shi} x = \int_0^x \frac{\operatorname{sh} t}{t} dt = -i \left[\frac{\pi}{2} + \operatorname{si}(ix) \right] \quad (\text{see } \mathbf{8.2301}).$$

8.230

$$2. \quad \text{chi } x = C + \ln x + \int_0^x \frac{\text{ch } t - 1}{t} dt.$$

8.23 The sine integral and the cosine integral: $\text{si}(x)$ and $\text{ci}(x)$

8.230

$$1.* \quad \text{si}(x) = -\int_x^\infty \frac{\sin t}{t} dt = -\frac{\pi}{2} + \text{Si}(x), \text{ where } \text{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$$

NT 11(3)

$$2.* \quad \text{ci}(x) = -\int_x^\infty \frac{\cos t}{t} dt = C + \ln x + \int_0^x \frac{\cos t - 1}{t} dt \quad [\text{ci}(x) \text{ is also written } \text{Ci}(x)].$$

NT 11(2)

8.231

$$1. \quad \text{si}(xy) = -\int_x^\infty \frac{\sin ty}{t} dt.$$

NT 18(7)

$$2. \quad \text{ci}(xy) = -\int_x^\infty \frac{\cos ty}{t} dt.$$

NT 18(6)

937

$$3. \quad \text{si}(x) = -\int_0^{\frac{\pi}{2}} e^{-x \cos t} \cos(x \sin t) dt.$$

NT 13(26)

8.232

$$1. \quad \text{si}(x) = -\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(2k-1)!}.$$

$$2.7 \quad \text{ci}(x) = C + \ln x + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{2k(2k)!}.$$

NT 7(3)

8.233

$$1. \quad \text{ci}(x) \pm i \text{si}(x) = \text{Ei}(\pm ix).$$

NT 6a

$$2. \quad \text{ci}(x) - \text{ci}(xe^{\pm\pi i}) = \mp\pi i.$$

NT 7(5)

$$3. \quad \text{si}(x) + \text{si}(-x) = -\pi.$$

NT 7(7)

8.234

$$1.7 \quad \text{Ei}(-x) - \text{ci}(x) = \int_0^{\frac{\pi}{2}} e^{-x \cos \varphi} \sin(s \sin \varphi) d\varphi.$$

NT 13(27)

$$2. \quad [\text{ci}(x)]^2 + [\text{si}(x)]^2 = -2 \int_0^{\frac{\pi}{2}} \frac{\exp(-x \operatorname{tg} \varphi) \ln \cos \varphi}{\sin \varphi \cos \varphi} d\varphi \quad [\operatorname{Re} x > 0] \quad (\text{see also 4.366}).$$

4.366
NT 32(11)

See also 3.341, 3.351 1. and 2., 3.354 1 and 2., 3.721 2. and 3., 3.722 1., 3., 5. and 7., 3.723 8. and 11., 4.338 1., 4.366 1.

8.235

$$1. \quad \lim_{x \rightarrow +\infty} (x^\rho \text{si}(x)) = 0, \quad \lim_{x \rightarrow +\infty} (x^\rho \text{ci}(x)) = 0 \quad [\rho < 1].$$

NT 38(5)

For integrals of the sine integral and cosine integral, see 6.24-6.26, 6.781, 6.782, and 6.783. For indefinite integrals of the sine-integral and cosine-integral, see 5.3.

8.24 The logarithm-integral $\text{li}(x)$

8.240

$$1. \quad \text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Ei}(\ln x) \quad [x < 1].$$

JA

$$2. \quad \text{li}(x) = \lim_{\varepsilon \rightarrow 0} \left[\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right] = \text{Ei}(\ln x) \quad [x > 1].$$

JA

$$3. \quad \text{li}\{\exp(-xe^{\pm i\pi})\} = \text{Ei}(-xe^{\pm i\pi}) = \text{Ei}(x \mp i0) = \text{Ei}(x) \pm i\pi = \text{li}(e^x) \pm i\pi \quad [x > 0].$$

JA, NT 2(6)

938

Integral representations

8.241

$$1. \quad \text{li}(x) = \int_{-\infty}^{\ln x} \frac{e^t}{t} dt = x \ln \ln \frac{1}{x} - \int_{-\ln x}^{\infty} e^{-t} \ln t dt \quad [x < 1].$$

LA 281(33)

$$\begin{aligned} 2. \quad \text{li}(x) &= x \int_0^1 \frac{dt}{\ln x + \ln t}; \\ &= \frac{x}{\ln x} + x \int_0^1 \frac{dt}{(\ln x + \ln t)^2}; \\ &= x \int_1^{\infty} \frac{1}{\ln x - \ln t} \frac{dt}{t^2} \quad [x < 1]. \end{aligned}$$

$$3. \quad \text{li}(a^x) = \frac{1}{\ln a} \int_{-\infty}^x \frac{a^t}{t} dt \quad [x > 0].$$

For integrals of the logarithm integral, see 6.21

8.25 The probability integral $\Phi(x)$, the Fresnel integrals $S(x)$, $C(x)$, the error function $\text{erf}(x)$ and the complementary error function $\text{erfc}(x)$

8.250

Definition:

$$1. \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad (\text{also called the error function and denoted by } \text{erf}(x)).$$

$$2. \quad S(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \sin t^2 dt.$$

$$3. \quad C(x) = \frac{2}{\sqrt{2\pi}} \int_0^x \cos t^2 dt.$$

$$4^* \quad \text{erfc}(x) = 1 - \text{erf}(x) = 1 - \Phi(x) \quad (\text{called the complementary error function}).$$

Integral representations

8.251

$$1. \quad \Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt \quad (\text{see also } \mathbf{3.3611}).$$

$$2. \quad S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt.$$

$$3. \quad C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt.$$

3.361

8.252

$$1. \quad \Phi(xy) = \frac{2y}{\sqrt{\pi}} \int_0^x e^{-t^2 y^2} dt.$$

939

$$2. \quad S(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \sin(t^2 y^2) dt.$$

$$3. \quad C(xy) = \frac{2y}{\sqrt{2\pi}} \int_0^x \cos(t^2 y^2) dt.$$

$$4. \quad \left. \begin{aligned} \Phi(xy) &= 1 - \frac{2}{\sqrt{\pi}} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} ty dt}{\sqrt{t^2 + x^2}} \\ &= 1 - \frac{2x}{\pi} e^{-x^2 y^2} \int_0^\infty \frac{e^{-t^2 y^2} dt}{t^2 + x^2} \end{aligned} \right\} \quad [\operatorname{Re} y^2 > 0]$$

NT 19(13)a
NT 19(11)a

$$5.7 \quad \Phi\left(\frac{-y}{2xi}\right) - \Phi\left(\frac{y}{2xi}\right) = \frac{4xie^{\frac{y^2}{4x^2}}}{\sqrt{\pi}} \int_0^\infty e^{-t^2 y^2} \sin(ty) dt \quad [\operatorname{Re} x^2 > 0].$$

NT 28(3)a

$$6. \quad \Phi\left(\frac{y}{2x}\right) = 1 - \frac{2}{\sqrt{\pi}} x e^{-\frac{y^2}{4x^2}} \int_0^\infty e^{-t^2 x^2 - ty} dt \quad [\operatorname{Re} x^2 > 0].$$

NT 27(1)a

See also 3.322, 3.362 2., 3.363, 3.468, 3.897, 6.511 4. and 5.

8.253

Series representations:

$$1. \quad \Phi(x) = \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)(k-1)!};$$

$$\frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{(2k+1)!!}.$$

NT 7(9)a

$$2. \quad S(x) = \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(2k+1)!(4k+3)};$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} - \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} \right\}.$$

NT 8(14)a
NT 10(13)a

$$3. \quad C(x) \frac{2}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(2k)!(4k+1)};$$

$$= \frac{2}{\sqrt{2\pi}} \left\{ \sin x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1} x^{4k+3}}{(4k+3)!!} + \cos x^2 \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{4k+1}}{(4k+1)!!} \right\}$$

NT 10(12)a
NT 8(13)a

For the expansions in Bessel functions, see 8.515 2., 8.515 3.

Asymptotic representations

8.254⁷

$$\Phi(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi z}} \left[\sum_{k=0}^n (-1)^k \frac{(2k-1)!!}{(2z^z)^k} + O(|z|^{-2n-2}) \right], [z \rightarrow \infty, |\arg -z| \leq \pi - \delta; \delta > 0 \text{ small}]$$

940

where

$$|R_n| < \frac{\Gamma(n + \frac{1}{2})}{|x|^{n+\frac{1}{2}} \cos \frac{\varphi}{2}}, \quad x = |x|e^{i\varphi} \quad \text{and} \quad \varphi^2 < \pi^2.$$

NT 37(10)a

8.255

$$1. \quad S(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi x}} \cos x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty].$$

$$2. \quad C(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi x}} \sin x^2 + O\left(\frac{1}{x^2}\right) \quad [x \rightarrow \infty].$$

MO 127a

8.256

Functional relations:

$$1. \quad C(z) + iS(z) = \sqrt{\frac{i}{2}} \Phi\left(\frac{z}{\sqrt{i}}\right) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{it^2} dt.$$

$$2. \quad C(z) - iS(z) = \frac{1}{\sqrt{2i}} \Phi(z\sqrt{i}) = \frac{2}{\sqrt{2\pi}} \int_0^z e^{-it^2} dt.$$

$$3. \quad [\cos u^2 C(u) + \sin u^2 S(u)] = \frac{1}{2} [\cos u^2 + \sin u^2] + \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \sin t^2 dt \quad [\operatorname{Re} u \geq 0].$$

NT 28(6)a

$$4. \quad [\cos u^2 S(u) - \sin u^2 C(u)] = \frac{1}{2} [\cos u^2 - \sin u^2] - \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2ut} \cos t^2 dt \quad [\operatorname{Re} u \geq 0].$$

NT 28(5)a

$$5. \quad \left[C(x) - \frac{1}{2}\right]^2 + \left[S(x) - \frac{1}{2}\right]^2 + \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\exp(-x^2 \operatorname{tg} \varphi) \sin \frac{\varphi}{2} \sqrt{\cos \varphi}}{\sin 2\varphi} d\varphi. \quad \text{see also 6.322.}$$

6.322
NT 33(18)a

For a connection with a degenerate hypergeometric function, see 9.236.

For a connection with a parabolic-cylinder function, see 9.254.

8.257

NT 38(11)

$$2. \lim_{x \rightarrow +\infty} \left(x^\varrho \left[C(x) - \frac{1}{2} \right] \right) = 0 \quad [\varrho < 1].$$

NT 38(11)

$$3. \lim_{x \rightarrow +\infty} S(x) = \frac{1}{2}.$$

NT 38(12)a

941

$$4. \lim_{x \rightarrow +\infty} C(x) = \frac{1}{2}.$$

NT 38(12)a

For integrals of the probability integral, see 6.28-6.31.

For integrals of Fresnel's sine-integral and cosine-integral, see 6.32.

8.258

Integrals involving the complementary error function

$$1. \int_0^\infty \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{\sqrt{\beta\pi}} \left(-\arccos\left(\frac{1}{1+\beta}\right) + 2 \operatorname{arctg}(\sqrt{\beta}) \right) \quad [\beta > 0].$$

$$2. \int_0^\infty x \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{2\beta} \left(1 - \frac{4 \operatorname{arctg}(\sqrt{1+\beta})}{\pi \sqrt{1+\beta}} \right) \quad [\beta > 0].$$

$$3. \int_0^\infty x^3 \operatorname{erfc}^2(x) e^{-\beta x^2} dx = \frac{1}{2\beta^2} \left(1 - \frac{4 \operatorname{arctg}(\sqrt{1+\beta})}{\pi \sqrt{1+\beta}} \right) + \frac{1}{\beta\pi} \left(\frac{1}{(1+\beta)(\beta^2+2\beta+2)} - \frac{\operatorname{arctg}(\sqrt{1+\beta})}{(1+\beta)^{\frac{3}{2}}} \right) \quad [\beta > 0].$$

8.26 Lobachevskiy's function $L(x)$

8.260

Definition:

$$L(x) = -\int_0^x \ln \cos t \, dt.$$

LO III 184(10)

For integral representations of the function $L(x)$, see also 3.531 8., 3.532 2., 3.533, and 4.224.

8.261

Representation in the form of a series:

$$L(x) = x \ln 2 - \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin 2kx}{k^2}.$$

LO III 185(11)

942

8.262

Functional relationships:

1. $L(-x) = -L(x) \left[-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right].$

LO III 185(13)

2. $L(\pi - x) = \pi \ln 2 - L(x).$

LO III 286

3. $L(\pi + x) = \pi \ln 2 + L(x).$

LO III 286

8.3 Euler's Integrals of the First and Second Kinds and Functions Generated by Them

8.31 The gamma function (Euler's integral of the second kind): $\Gamma(z)$

8.310⁷

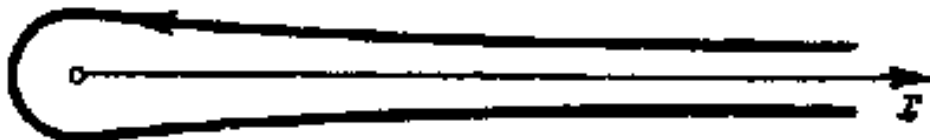
Definition:

$$1. \quad \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad [\operatorname{Re} z > 0]. \quad (\text{Euler}).$$

FI II 777(6)

Generalization:

$$2. \quad \Gamma(z) = -\frac{1}{2i \sin \pi z} \int_C (-t)^{z-1} e^{-t} dt$$

for z not an integer.The contour C is shown in the drawing.

$\Gamma(z)$ is an analytic function z with simple poles at the points $z = -l$ (for $l = 0, 1, 2, \dots$) which correspond to residues $\frac{(-1)^l}{l!}$.

WH, MO 1

Integral representations

8.311

$$\Gamma(z) = \frac{1}{e^{2\pi iz} - 1} \int_{-\infty}^{(0+)} e^{-t} t^{z-1} dt.$$

MO 2
WH

8.312

$$1. \quad \Gamma(z) = \int_0^1 \left(\ln \frac{1}{t} \right)^{z-1} dt \quad [\operatorname{Re} z > 0].$$

FI II 778

$$2. \quad \Gamma(z) = x^z \int_0^{\infty} e^{-xt} t^{z-1} dt \quad [\operatorname{Re} z > 0, \operatorname{Re} x > 0].$$

FI II 779(8)

$$3. \quad \Gamma(z) = \frac{2a^z e^a}{\sin \pi z} \int_0^{\infty} e^{-at^2} (1+t^2)^{z-\frac{1}{2}} \cos[2at + (2z-1) \operatorname{arctg} t] dt \quad [a > 0].$$

WH

$$4. \quad \Gamma(z) = \frac{1}{2 \sin \pi z} \int_0^{\infty} e^{-t^2} t^{z-1} (1+t^2)^{\frac{z}{2}} \{3 \sin[t+z \operatorname{arcctg}(-t)] + \sin[t+(z-2) \operatorname{arcctg}(-t)]\} dt$$

[arcctg denotes an obtuse angle].

WH

943

$$5. \quad \Gamma(y) = x^y e^{-i\beta y} \int_0^{\infty} t^{y-1} \exp(-xte^{-i\beta}) dt \quad \left[x, y, \beta \text{ real, } x > 0, y > 0, |\beta| < \frac{\pi}{2} \right].$$

MO 8

$$6. \quad \Gamma(z) = \frac{b^z}{2 \sin \pi z} \int_{-\infty}^{\infty} e^{bti} (it)^{z-1} dt \quad [b > 0, 0 < \operatorname{Re} z < 1].$$

$$7. \quad \left. \begin{aligned} \Gamma(z) &= \frac{(\sqrt{a^2 + b^2})^z}{\cos\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^\infty e^{-at} \cos(bt) t^{z-1} dt; \\ &= \frac{(\sqrt{a^2 + b^2})^z}{\sin\left(z \operatorname{arctg} \frac{b}{a}\right)} \int_0^\infty e^{-at} \sin(bt) t^{z-1} dt \end{aligned} \right\} [a > 0, \quad b \geq 0, \quad \operatorname{Re} z > 0].$$

NH 152(1)a
NH 152(2)

$$8. \quad \left. \begin{aligned} \Gamma(z) &= \frac{b^z}{\cos \frac{\pi z}{2}} \int_0^\infty \cos(bt) t^{z-1} dt; \\ &= \frac{b^z}{\sin \frac{\pi z}{2}} \int_0^\infty \sin(bt) t^{z-1} dt \end{aligned} \right\} [b > 0, \quad 0 < \operatorname{Re} z < 1].$$

NH 152(5)

$$\left. \begin{aligned} 9. \quad \Gamma(z) &= \int_0^\infty e^{-t} (t-z) t^{z-1} \ln t dt; \\ 10. \quad \Gamma(z) &= \int_{-\infty}^\infty \exp(zt - e^t) dt \end{aligned} \right\} [\operatorname{Re} z > 0].$$

NH 145(14)
NH 173(7)

$$\left. \begin{aligned} 11. \quad \Gamma(z) \cos \alpha x &= \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \cos(\lambda t \sin \alpha) dt; \\ 12. \quad \Gamma(x) \sin \alpha x &= \lambda^x \int_0^\infty t^{x-1} e^{-\lambda t \cos \alpha} \sin(\lambda t \sin \alpha) dt \end{aligned} \right\} \left[\lambda > 0, \quad x > 0, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2} \right].$$

WH

$$13. \quad \Gamma(-z) = \int_0^\infty \left[\frac{e^{-t} - \sum_{k=0}^n (-1)^k \frac{t^k}{k!}}{t^{z+1}} \right] dt \quad [n = [\operatorname{Re} z]].$$

MO 2

8.315

$$1.^7 \frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} dt \quad (\text{For } C \text{ see 8.310.2})$$

8.310

944

$$2.^8 \int_{-\infty}^{\infty} \frac{e^{bti}}{(a+it)^z} dt = \frac{2\pi e^{-ab} b^{z-1}}{\Gamma(z)}$$

$$\int_{-\infty}^{\infty} \frac{e^{-bti}}{(a+it)^z} dt = 0 \quad \left[\text{Re } a > 0, b > 0, \text{Re } z > 0, |\arg(a+it)| < \frac{1}{2}\pi \right].$$

NH 155(8), MO 7
NH 155(8), MO 7

$$3. \frac{1}{\Gamma(z)} = a^{1-z} \frac{e^a}{\pi} \int_0^{\frac{\pi}{2}} \cos(a \operatorname{tg} \theta - z\theta) \cos^{z-2} \theta d\theta \quad [\text{Re } z > 1].$$

NH 157(14)

See also 3.324 2., 3.326, 3.328, 3.381 4., 3.382 2., 3.389 2., 3.433, 3.434, 3.478 1., 3.551 1., 2., 3.827 1., 4.267 7., 4.272, 4.353 1., 4.369 1., 6.214, 6.223, 6.246, 6.281.

8.32 Representation of the gamma function as series and products

8.321

Representation in the form of a series:

$$1.^6 \Gamma(z+1) = \sum_{k=0}^{\infty} c_k z^k$$

$$\left[c_0 = 1, c_{n+1} = \frac{\sum_{k=0}^n (-1)^{k+1} s_{k+1} c_{n-k}}{n+1}; s_1 = \mathbf{C}, s_n = \zeta(n) \quad \text{for } n \geq 2, |z| < 1 \right].$$

$$2. \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$

$$\left[d_0 = 1, d_{n+1} = \frac{\sum_{k=0}^n (-1)^k s_{k+1} d_{n-k}}{n+1}; s_1 = \mathbf{C}, s_n = \zeta(n) \text{ for } n \geq 2 \right].$$

NH 41(4, 6)

Infinite-product representation

8.322

$$\Gamma(z) = e^{-\mathbf{C}z} \frac{1}{z} \prod_{k=1}^{\infty} \frac{e^{\frac{z}{k}}}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0];$$

$$= \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}} \quad [\operatorname{Re} z > 0];$$

$$= \lim_{n \rightarrow \infty} \frac{n^z}{z} \prod_{k=1}^n \frac{k}{z+k} \quad [\operatorname{Re} z > 0].$$

SM 267(130)
WH
SM 269

945

8.323⁷

$$\Gamma(z) = 2z^z e^{-z} \prod_{k=1}^{\infty} {}_2\sqrt{\operatorname{B}\left(2^{k-1}z, \frac{1}{2}\right)}.$$

NH 98(12)

8.324⁷

$$\Gamma(1+z) = 4^z \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right)}{\sqrt{\pi}}.$$

MO 3

8.325

$$1. \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right) \left(1 - \frac{\gamma}{\beta+k}\right) \right].$$

NH 62(2)

$$2. \frac{e^{\mathbf{C}x}\Gamma(z+1)}{\Gamma(z-x+1)} = \prod_{k=1}^{\infty} \left[\left(1 - \frac{x}{z+k}\right) e^{\frac{x}{k}} \right] \quad [z \neq 0, -1, -2, \dots; \operatorname{Re} z > 0, \operatorname{Re}(z-x) > 0].$$

$$3.7 \frac{\sqrt{\pi}}{\Gamma\left(1 + \frac{z}{2}\right)\Gamma\left(\frac{1}{2} - \frac{z}{2}\right)} = \prod_{k=1}^{\infty} \left(1 - \frac{z}{2k-1}\right) \left(1 + \frac{z}{2k}\right).$$

MO 2

8.326

$$1. \frac{\frac{[\Gamma(x)]^2}{\Gamma(2x)}}{B(x+iy, x-iy)} = \left| \frac{\Gamma(x)}{\Gamma(x-iy)} \right|^2 = \prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2}\right) \quad [x, y \text{ real}, \quad x \neq 0, \quad -1, -2, \dots].$$

LO V, NH 63(4)

$$2. \frac{\Gamma(x+iy)}{\Gamma(x)} = \frac{x e^{-iCy}}{x+iy} \prod_{n=1}^{\infty} \frac{\exp\left(\frac{iy}{n}\right)}{1 + \frac{iy}{x+n}} \quad [x, r \text{ real}, \quad x \neq 0, \quad -1, -2, \dots].$$

MO 2

8.327

Asymptotic representation for large values of $|z|$:

$$\Gamma(z) = z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + O(z^{-5}) \right\} \quad [|\arg z| < \pi].$$

WH

For z real and positive, the remainder of the series is less than the last term that is retained.

8.328

$$1. \lim_{|y| \rightarrow \infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi} \quad [x \text{ and } y \text{ are real}].$$

MO 6

$$2. \lim_{|z| \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1.$$

MO 6

8.33 Functional relations involving the gamma function

8.331

$$\Gamma(x+1) = x\Gamma(x).$$

8.332

- $$\left. \begin{array}{l} 1. \quad |\Gamma(iy)|^2 = \frac{\pi}{y \operatorname{sh} \pi y} \\ 2. \quad \left| \Gamma\left(\frac{1}{2} + iy\right) \right|^2 = \frac{\pi}{\operatorname{ch} \pi y} \end{array} \right\} \quad [y \text{ is real}].$$
- $$3. \quad \Gamma(1+ix)\Gamma(1-ix) = \frac{\pi x}{\operatorname{sh} x\pi} \quad [x \text{ is real}].$$
- $$4. \quad \Gamma(1+x+iy)\Gamma(1-x+iy)\Gamma(1+x-iy)\Gamma(1-x-iy) = \frac{2\pi^2(x^2+y^2)}{\operatorname{ch} 2y\pi - \cos 2x\pi} \quad [x \text{ and } y \text{ are real}].$$

LO V
LO V
MO 3

8.333

$$[\Gamma(n+1)]^n = G(n+1) \prod_{k=1}^n k^k,$$

where n is a natural number and

$$G(z+1) = (2\pi)^{\frac{z}{2}} \exp\left[-\frac{z(z+1)}{2} - \frac{C}{2}z^2\right] \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n \exp\left(-z + \frac{z^2}{2n}\right) \right\}.$$

WH

8.334

$$1. \quad \prod_{k=1}^n \frac{1}{\Gamma\left(-z \exp \frac{2\pi ki}{n}\right)} = -z^n \prod_{k=1}^{\infty} \left[1 - \left(\frac{z}{k}\right)^n\right] \quad [n = 2, 3, 4, \dots].$$

MO 2

Special cases

8.335⁷

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \quad [\text{product theorem}].$$

FI II 782a, WH

$$1. \quad \Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right) \quad [\text{doubling formula}].$$

$$2. \quad \Gamma(3x) = \frac{3^{3x-\frac{1}{2}}}{2\pi} \Gamma(x) \Gamma\left(x + \frac{1}{3}\right) \Gamma\left(x + \frac{2}{3}\right).$$

$$3. \quad \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) \Gamma\left(1 - \frac{k}{n}\right) = \frac{(2\pi)^{n-1}}{n}.$$

WH

947

$$4^* \quad \sum_{n=0}^{\infty} \frac{\Gamma\left(n - \frac{1}{2}\right)^2}{4(n!)^2 \Gamma\left(-\frac{1}{2}\right)^2} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{1024} + \frac{25}{65536} + \dots = \frac{1}{\pi}.$$

8.336

$$\Gamma\left(-\frac{yz + xi}{2y}\right) \Gamma(1 - z) = (2i)^{z+1} y \Gamma\left(1 + \frac{yz - xi}{2y}\right) \int_0^{\infty} e^{-tx} \sin^z(ty) dt$$

[Re(yi) > 0, Re(x - yzi) > 0].

NH 133(10)

For a connection with the psi function, 8.334 1.

For a connection with the beta function, see 8.384 1.

For integrals of the gamma function, see 8.412 4., 8.414, 9.223, 9.242 3., 9.242 4.

8.337

1. $[\Gamma'(x)]^2 < \Gamma(x)\Gamma''(x) \quad [x > 0].$

2. For $x > 0$, $\min \Gamma(1 + x) = 0.88560 \dots$ is attained when $x = 0.46163$.

JA

Particular values

8.338

1. $\Gamma(1) = \Gamma(2) = 1.$

MO 1

2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

3. $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}.$

4. $\left[\Gamma\left(\frac{1}{4}\right)\right]^4 = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2[(4k+1)^2-1]}{[(4k-1)^2-1](4k+1)^2}.$

MO 1a

5. $\prod_{k=1}^8 \Gamma\left(\frac{k}{3}\right) = \frac{640}{3^6} \left(\frac{\pi}{\sqrt{3}}\right)^3.$

WH

8.339

For n a natural number

$$1. \quad \Gamma(n) = (n-1)!$$

$$2. \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!$$

$$3. \quad \Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{2^n \sqrt{\pi}}{(2n-1)!!}.$$

$$4. \quad \frac{\Gamma\left(p + n + \frac{1}{2}\right)}{\Gamma\left(p - n + \frac{1}{2}\right)} = \frac{(4p^2 - 1^2)(4p^2 - 3^2) \dots [4p^2 - (2n-1)^2]}{2^{2n}}.$$

WA 221

948

8.34 The logarithm of the gamma function

8.341

Integral representation:

$$1. \quad \ln \Gamma(z) = \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right) \frac{e^{-tz}}{t} dt \quad [\operatorname{Re} z > 0].$$

WH

$$2.^7 \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + 2 \int_0^\infty \frac{\operatorname{arctg} \frac{t}{z}}{e^{2\pi t} - 1} dt$$

$\left[\operatorname{Re} z > 0 \text{ and } \operatorname{arctg} w = \int_0^w \frac{du}{1+u^2} \text{ is taken over a rectangular path in the } w\text{-plane} \right]$

WH

$$3. \quad \ln \Gamma(z) = \int_0^\infty \left\{ \frac{e^{-zt} - e^{-t}}{1 - e^{-t}} + (z-1)e^{-t} \right\} \frac{dt}{t} \quad [\operatorname{Re} z > 0].$$

$$4. \quad \ln \Gamma(z) = \int_0^\infty \left\{ (z-1)e^{-t} + \frac{(1+t)^{-z} - (1+t)^{-1}}{\ln(1+t)} \right\} \frac{dt}{t} \quad [\operatorname{Re} z > 0].$$

WH

$$5. \quad \ln \Gamma(x) = \frac{\ln \pi - \ln \sin \pi x}{2} + \frac{1}{2} \int_0^\infty \left\{ \frac{\operatorname{sh} \left(\frac{1}{2} - x \right) t}{\operatorname{sh} \frac{t}{2}} - (1-2x)e^{-t} \right\} \frac{dt}{t} \quad [0 < x < 1].$$

WH

$$6. \quad \ln \Gamma(z) = \int_0^1 \left\{ \frac{t^z - t}{t-1} - t(z-1) \right\} \frac{dt}{t \ln t} \quad [\operatorname{Re} z > 0].$$

WH

$$7. \quad \ln \Gamma(z) = \int_0^\infty \left[(z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1 - e^{-t}} \right] \frac{dt}{t} \quad [\operatorname{Re} z > 0].$$

NH 187(7)

See also 3.427 9., 3.554 5.

8.342

Series representations:

$$\begin{aligned} 1. \quad \ln \Gamma(z+1) &= \frac{1}{2} \left[\ln \left(\frac{\pi 2}{\sin \pi z} \right) - \ln \frac{1+z}{1-z} \right] + (1-\mathbf{C})z + \\ &+ \sum_{k=1}^{\infty} \frac{1 - \zeta(2k+1)}{2k+1} z^{2k+1} = \\ &= -\mathbf{C}z + \sum_{k=2}^{\infty} (-1)^k \frac{z^k}{k} \zeta(k) \quad [|z| < 1]. \end{aligned}$$

NH 38(16, 12)

$$2. \quad \ln \Gamma(1+x) = \frac{1}{2} \ln \frac{\pi x}{\sin \pi x} - \mathbf{C}x - \sum_{n=1}^{\infty} \frac{x^{2n+1}}{2n+1} \{\zeta(2n+1)\} \quad [|x| < 1].$$

$$1. \quad \ln \Gamma(x) = \ln \sqrt{2\pi} + \sum_{n=1}^{\infty} \left\{ \frac{1}{2n} \cos 2n\pi x + \frac{1}{n\pi} (\mathbf{C} + \ln 2n\pi) \sin 2n\pi x \right\} \quad [0 < x < 1].$$

FI III 558

949

$$2. \quad \ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{m}{(m+1)(m+2)} \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}} \quad [|\arg z| < \pi].$$

MO 9

8.344⁷

Asymptotic expansion for large values of $|z|$:

$$\ln \Gamma(z) = z \ln z - z - \frac{1}{2} \ln z + \ln \sqrt{2\pi} + \sum_{k=1}^{n-1} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + R_n(z),$$

where

$$|R_n(z)| < \frac{|B_{2n}|}{2n(2n-1)|z|^{2n-1} \cos^{2n-1} \left(\frac{1}{2} \arg z \right)}.$$

MO5

For integrals of $\ln \Gamma(x)$, see 6.44.

8.35 The incomplete gamma functions

8.350

Definition:

$$1. \quad \gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt \quad [\operatorname{Re} \alpha > 0].$$

EH II 133(1), NH 1(1)

$$2.^7 \quad \Gamma(\alpha, x) = \int_x^{\infty} e^{-t} t^{\alpha-1} dt.$$

8.351

1. $\gamma^*(\alpha, x) = \frac{x^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha, x)$ is an analytic function with respect to α and x .

EH II 133(5)

2. Another definition of $\gamma(\alpha, x)$, that is also suitable for the case $\operatorname{Re} \alpha \leq 0$:

$$\gamma(\alpha, x) = \frac{x^\alpha}{\alpha} e^{-x} \Phi(1, 1 + \alpha; x) = \frac{x^\alpha}{\alpha} \Phi(a, 1 + a; -x).$$

EH II 133(3)

3. For fixed x , $\Gamma(\alpha, x)$ is an entire function of α . For nonintegral α , $\Gamma(\alpha, x)$ is a multiple-valued function of x with a branch point at $x = 0$.

4. A second definition of $\Gamma(\alpha, x)$:

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \Psi(1, 1 + \alpha; x) = e^{-x} \Psi(1 - \alpha, 1 - \alpha; x).$$

EH II 133(4)

8.352

Special cases:

$$1. \quad \gamma(1 + n, x) = n! \left[1 - e^{-x} \left(\sum_{m=0}^n \frac{x^m}{m!} \right) \right] \quad [n = 0, 1, \dots].$$

EH II 136(17, 16), NH 6(11)

$$2. \quad \Gamma(1 + n, x) = n! e^{-x} \sum_{m=0}^n \frac{x^m}{m!} \quad [n = 0, 1, \dots].$$

EH II 136(16, 18)

950

$$3.^7 \quad \Gamma(-n, x) = \frac{(-1)^n}{n!} \left[-\operatorname{Ei}(-x) - e^{-x} \sum_{m=0}^{n-1} (-1)^m \frac{m!}{x^{m+1}} \right] \quad [n = 1, 2, \dots].$$

Integral representations:

$$1. \quad \gamma(\alpha, x) = x^\alpha \operatorname{cosec} \pi\alpha \int_0^\pi e^x \cos \theta \cos(\alpha\theta + x \sin \theta) d\theta \quad [x \neq 0, \operatorname{Re} \alpha > 0, \alpha \neq 1, 2, \dots].$$

EH II 137(2)

$$2. \quad \gamma(\alpha, x) = x^{\frac{1}{2}\alpha} \int_0^\infty e^{-t} t^{\frac{1}{2}\alpha-1} J_\alpha(2\sqrt{xt}) dt \quad [\operatorname{Re} \alpha > 0].$$

EH II 138(4)

$$3. \quad \Gamma(\alpha, x) = \frac{\rho^{-x} x^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{e^{-t} t^{-\alpha}}{x+t} dt \quad [\operatorname{Re} \alpha < 1, x > 0].$$

EH II 137(3), NH 19(12)

$$4. \quad \Gamma(\alpha, x) = \frac{2x^{\frac{1}{2}\alpha} e^{-x}}{\Gamma(1-\alpha)} \int_0^\infty e^{-t} t^{-\frac{1}{2}\alpha} K_\alpha[2\sqrt{xt}] dt \quad [\operatorname{Re} \alpha < 1].$$

EH II 138(5)

$$5. \quad \Gamma(\alpha, xy) = y^\alpha e^{-xy} \int_0^\infty e^{-ty} (t+x)^{\alpha-1} dt \quad [\operatorname{Re} y > 0, x > 0, \operatorname{Re} \alpha > 1].$$

(See also 3.936 5.- 3.944 1.-4.)

3.944

3.936
NH 19(10)

For integrals of the gamma function, see 6.45.

8.354

Series representations:

$$1. \quad \gamma(\alpha, x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)}.$$

$$2. \quad \Gamma(\alpha, x) = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \quad [\alpha \neq 0, -1, -2, \dots].$$

EH II 135(5), LE 340(2)

$$3. \quad \Gamma(\alpha, x) - \Gamma(\alpha, x+y)\gamma(\alpha, x+y) - \gamma(\alpha, x) = \\ = e^{-x} x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k [1 - e^{-y} e_k(y)] \Gamma(1-\alpha+k)}{x^k \Gamma(1-\alpha)}, \\ e_k(x) = \sum_{m=0}^k \frac{x^m}{m!} \quad [|y| < |x|].$$

EH II 139(2)

$$4. \quad \gamma(\alpha, x) = \Gamma(\alpha) e^{-x} x^{\frac{1}{2}\alpha} \sum_{n=0}^{\infty} x^{\frac{1}{2}n} I_{n+\alpha}(2\sqrt{x}) \sum_{m=0}^n \frac{(-1)^m}{m!} \quad [x \neq 0, \alpha \neq 0, -1, -2, \dots].$$

EH II 139(3)

$$5. \quad \Gamma(\alpha, x) = e^{-x} x^{\alpha} \sum_{n=0}^{\infty} \frac{L_n^{\alpha}(x)}{n+1} \quad [x > 0].$$

EH II 140(5)

8.355

$$\Gamma(\alpha, x)\gamma(\alpha, y) = e^{-x-y} (xy)^{\alpha} \sum_{n=0}^{\infty} \frac{n! \Gamma(\alpha)}{(n+1) \Gamma(\alpha+n+1)} L_n^{\alpha}(x) L_n^{\alpha}(y) \\ [y > 0, x \geq y, \alpha \neq 0, -1, \dots].$$

EH II 139(4)

951

8.356

Functional relations:

$$1. \quad \nu(\alpha+1, x) = \alpha \gamma(\alpha, x) - x^{\alpha} e^{-x}.$$

EH II 134(2)

EH II 134(3)

$$3. \quad \Gamma(\alpha, x) + \gamma(\alpha, x) = \Gamma(\alpha).$$

EH II 134(1)

$$4. \quad \frac{d\gamma(\alpha, x)}{dx} = -\frac{d\Gamma(\alpha, x)}{dx} = x^{\alpha-1}e^{-x}.$$

EH II 135(8)

$$5. \quad \frac{\Gamma(\alpha + n, x)}{\Gamma(\alpha + n)} = \frac{\Gamma(\alpha, x)}{\Gamma(\alpha)} + e^{-x} \sum_{s=0}^{n-1} \frac{x^{\alpha+s}}{\Gamma(\alpha + s + 1)}.$$

NH 4(3)

$$6. \quad \Gamma(\alpha)\Gamma(\alpha + n, x) - \Gamma(\alpha + n)\Gamma(\alpha, x) = \Gamma(\alpha + n)\gamma(\alpha, x) - \Gamma(\alpha)\Gamma(\alpha + n, x).$$

NH 5

8.357

Asymptotic representation for large values of $|x|$:

$$\Gamma(\alpha, x) = x^{\alpha-1}e^{-x} \left[\sum_{m=0}^{M-1} \frac{(-1)^m \Gamma(1 - \alpha + m)}{x^m \Gamma(1 - \alpha)} + O(|x|^{-M}) \right] \\ \left[|x| \rightarrow \infty, -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}, M = 1, 2, \dots \right].$$

EH II 135(6), NH 37(7), LE 340(3)

8.358

Representation as a continued fraction:

$$\Gamma(\alpha, x) = \frac{e^{-x} x^\alpha}{x + \frac{1-\alpha}{1 + \frac{1}{x + \frac{2-\alpha}{1 + \frac{2}{x + \frac{3-\alpha}{1+\dots}}}}}}$$

EH II 136(13), NH 42(9)

8.359

Relationships with other functions:

1. $\Gamma(0, x) = -\text{Ei}(-x).$

EH II 143(1)

2. $\Gamma\left(0, \ln \frac{1}{x}\right) = -\text{li}(x).$

EH II 143(2)

3. $\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\Phi(x).$

EH II 147(2)

4. $\gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi}\Phi(x).$

EH II 147(1)

952

8.36 The psi function $\psi(x)$

8.360

Definition:

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

8.361

Integral representations:

1.⁸ $\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right) dt \quad [\text{Re } z > 0].$

NH 183(1), WH

$$3. \quad \psi(z) = \ln z - \frac{1}{2z} - 2 \int_0^{\infty} \frac{t \, dt}{(t^2 + z^2)(e^{2\pi t} - 1)} \quad [\operatorname{Re} z > 0].$$

WH

$$4. \quad \psi(z) = \int_0^1 \left(\frac{1}{-\ln t} - \frac{t^{z-1}}{1-t} \right) dt \quad [\operatorname{Re} z > 0].$$

WH

$$\left. \begin{aligned} 5. \quad \psi(z) &= \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt - \mathbf{C}, \\ 6. \quad \psi(z) &= \int_0^{\infty} \left\{ (1+t)^{-1} - (1+t)^{-z} \right\} \frac{dt}{t} - \mathbf{C}, \\ 7. \quad \psi(z) &= \int_0^1 \frac{t^{z-1} - 1}{t-1} dt - \mathbf{C} \end{aligned} \right\} \quad [\operatorname{Re} z > 0].$$

WH
FI II 796, WH

$$8. \quad \psi(z) = \ln z + \int_0^{\infty} e^{-tz} \left[\frac{1}{t} - \frac{1}{1 - e^{-t}} \right] dt \quad [\operatorname{Re} z > 0].$$

MO 4

See also 3.244 3., 3.311 6., 3.317 1., 3.457, 3.458 2., 3.471 14., 4.253 1. and 6., 4.275 2., 4.281 4., 4.282 5.

For integrals of the psi function, see 6.46-6.47.

Series representation

8.362

$$\begin{aligned} 1. \quad \psi(x) &= -\mathbf{C} - \sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k+1} \right); \\ &= -\mathbf{C} - \frac{1}{x} + x \sum_{k=1}^{\infty} \frac{1}{k(x+k)}. \end{aligned}$$

$$2. \quad \psi(x) = \ln x - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \ln \left(1 + \frac{1}{x+k} \right) \right].$$

MO 4

953

$$3. \quad \psi(x) = -C + \frac{\pi^2}{6}(x-1) - (x-1) \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \sum_{n=0}^{k-1} \frac{1}{x+n}.$$

NH 54(12)

8.363

$$1. \quad \psi(x+1) = -C + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1}.$$

NH 37(5)

$$2. \quad \psi(x+1) = \frac{1}{2x} - \frac{\pi}{2} \operatorname{ctg} \pi x - \frac{x^2}{1-x^2} - C + \sum_{k=1}^{\infty} [1 - \zeta(2k+1)] x^{2k}.$$

NH 38(10)

$$3. \quad \psi(x) - \psi(y) = \sum_{k=0}^{\infty} \left(\frac{1}{y+k} - \frac{1}{x+k} \right)$$

(see also 3.219, 3.231 5., 3.311 7., 3.688 20., 4.253 1., 4.295 37.) .

4.295

4.253

3.688

3.311

3.231

$$4. \quad \psi(x + iy) - \psi(x - iy) = \sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x + k)^2}.$$

$$5. \quad \psi\left(\frac{p}{q}\right) = -\mathbf{C} + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{q}{p+kq} \right) \quad (\text{see also 3.244 3.}).$$

$$6. \quad \psi\left(\frac{p}{q}\right) = -\mathbf{C} - \ln(2q) - \frac{\pi}{2} \operatorname{ctg} \frac{p\pi}{q} + 2 \sum_{k=1}^{\left[\frac{q+1}{2}\right]-1} \left[\cos \frac{2kp\pi}{q} \ln \sin \frac{k\pi}{q} \right]$$

$[q = 2, 3, \dots, p, = 1, 2, \dots, q-1].$

$$7. \quad \psi\left(\frac{p}{q}\right) - \psi\left(\frac{p-1}{q}\right) = q \sum_{n=2k=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(p+kq)^n - 1}.$$

$$8.7 \quad \psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, x).$$

Infinite-product representation

8.364

$$1. \quad e^{\psi(x)} = x \prod_{k=0}^{\infty} \left(1 + \frac{1}{x+k} \right) e^{-\frac{1}{x+k}}.$$

See also 8.37.

For a connection with Riemann's zeta function, see 9.533 2.

For a connection with the gamma function, see 4.325 12. and 4.352 1.

For a connection with the beta function, see 4.253 1.

For series of psi functions, see 8.403 2., 8.446, and 8.447 3. (Bessel functions) 8.761 (derivatives of associated Legendre functions with respect to the degree), 9.153, 9.154 (hypergeometric function), 9.237 (confluent hypergeometric function).

For integrals containing psi functions, see 6.46-6.47.

954

8.365

Functional relations:

$$1. \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

JA

$$2. \quad \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) = 2\beta(x) \quad (\text{cf. 8.37 0}).$$

8.37

$$3. \quad \psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k}.$$

GA 154(64)a

$$4. \quad \psi(n+1) = -C + \sum_{k=1}^n \frac{1}{k}.$$

$$5. \quad \lim_{n \rightarrow \infty} [\psi(z+n) - \ln n] = 0.$$

MO 3

$$6. \quad \psi(nz) = \frac{1}{n} \sum_{k=0}^{n-1} \psi\left(z + \frac{k}{n}\right) + \ln n \quad [n = 2, 3, 4, \dots].$$

MO 3

$$7. \quad \psi(x-n) = \psi(x) - \sum_{k=1}^n \frac{1}{x-k}.$$

$$8. \quad \psi(1-z) = \psi(z) + \pi \operatorname{ctg} \pi z.$$

GA 155(68)a

$$9. \quad \psi\left(\frac{1}{2} + z\right) = \psi\left(\frac{1}{2} - z\right) + \pi \operatorname{tg} \pi z.$$

JA

$$10. \quad \psi\left(\frac{3}{4} - n\right) = \psi\left(\frac{1}{4} + n\right) + \pi \quad [n = 0, \pm 1, \pm 2, \dots].$$

8.366

Particular values

$$1. \quad \psi(1) = -\mathbf{C} \quad (\text{cf. 8.367 1.}).$$

8.367

$$2. \quad \psi\left(\frac{1}{2}\right) = -\mathbf{C} - 2 \ln 2 = -1.963\,510\,026\dots$$

$$3. \quad \psi\left(\frac{1}{2} \pm n\right) = -\mathbf{C} + 2 \left[\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right].$$

JA

$$4. \quad \psi\left(\frac{1}{4}\right) = -\mathbf{C} - \frac{\pi}{2} - 3 \ln 2.$$

GA 157a

$$5. \quad \psi\left(\frac{3}{4}\right) = -\mathbf{C} + \frac{\pi}{2} - 3 \ln 2.$$

GA 157a

$$6. \quad \psi\left(\frac{1}{3}\right) = -\mathbf{C} - \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$$

GA 157a

$$7. \quad \psi\left(\frac{2}{3}\right) = -\mathbf{C} + \frac{\pi}{2} \sqrt{\frac{1}{3}} - \frac{3}{2} \ln 3.$$

GA 157a

955

$$8. \quad \psi'(1) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\dots$$

JA

$$9. \quad \psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2} = 4.934\,802\,200\,5\dots$$

JA

$$10. \quad \psi'(-n) = \infty$$

$$11. \quad \psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}$$

$$12. \quad \psi'\left(\frac{1}{2} + n\right) = \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2} \quad \left. \vphantom{\sum_{k=1}^n} \right\} [n - \text{a natural number}].$$

$$13. \quad \psi'\left(\frac{1}{2} - n\right) = \frac{\pi^2}{2} + 4 \sum_{k=1}^n \frac{1}{(2k-1)^2}$$

1. $C = -\psi(1) = 0.577\,215\,664\,90\dots$

FI II 319, 795

2. $C = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^{n-1} \frac{1}{k} - \ln n \right].$

FI II 801a

3. $C = \lim_{x \rightarrow 1+0} \left[\zeta(x) - \frac{1}{x-1} \right].$

FI II 804

Integral representations:

4. $C = -\int_0^{\infty} e^{-t} \ln t \, dt.$

FI II 807

5. $C = -\int_0^1 \ln \left(\ln \frac{1}{t} \right) dt.$

FI II 807

6. $C = \int_0^1 \left[\frac{1}{\ln t} + \frac{1}{1-t} \right] dt.$

DW

7. $C = -\int_0^{\infty} \left[\cos t - \frac{1}{1+t} \right] \frac{dt}{t}.$

MO 10

8. $C = 1 - \int_0^{\infty} \left[\frac{\sin t}{t} - \frac{1}{1+t} \right] \frac{dt}{t}.$

$$9. \quad C = -\int_0^{\infty} \left[e^{-t} - \frac{1}{1+t} \right] \frac{dt}{t}.$$

FI II 795, 802

$$10. \quad C = -\int_0^{\infty} \left[e^{-t} - \frac{1}{1+t^2} \right] \frac{dt}{t}.$$

DW, MO 10

$$11. \quad C = \int_0^{\infty} \left[\frac{1}{e^t - 1} - \frac{1}{te^t} \right] dt.$$

DW

956

$$12. \quad C = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^{\infty} \frac{e^{-t}}{t} dt.$$

FI II 802

See also 8.361 5.-8.361 7., 3.311 6., 3.435 3. and 4., 3.476 2., 3.481 1. and 2., 3.951 10., 4.283 9., 4.331 1., 4.421 1., 4.424 1., 4.553 , 4.572, 6.234, 6.264 1., 6.468.

13. Asymptotic expansions

$$C = \sum_{k=1}^{n-1} \frac{1}{k} - \ln n + \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} - \frac{1}{240n^8} + \dots$$

$$\dots + \frac{B_{2r}}{2r} \frac{1}{n^{2r}} + \frac{B_{2r+2}}{2(r+1)} \frac{\theta}{n^{2r+2}} \quad [0 < \theta < 1].$$

FI II 827

8.37 The function $\beta(x)$

Definition:

8.370

$$\beta(x) = \frac{1}{2} \left[\psi \left(\frac{x+1}{2} \right) - \psi \left(\frac{x}{2} \right) \right].$$

8.371

Integral representations:

$$1.^3 \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt \quad [\operatorname{Re} x > 0].$$

WH

$$2. \beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt \quad [\operatorname{Re} x > 0].$$

MO 4

$$3. \beta\left(\frac{x+1}{2}\right) = \int_0^\infty \frac{e^{-xt}}{\operatorname{ch} t} dt \quad [\operatorname{Re} x > -1] \quad (\text{cf 8.371 1.}).$$

8.371

See also 3.241 1., 3.251 7., 3.522 2. and 4., 3.623 2. and 3., 4.282 2., 4.389 3., 4.532 1. and 3.

Series representation

8.372

$$1.^7 \beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} \quad [-x \notin \mathbb{N}].$$

NH 37, 101(1)

$$2.^7 \beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)} \quad [-x \notin \mathbb{N}].$$

NH 101(2)

$$3.^7 \beta(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^k} \quad [-x \notin \mathbb{N}].$$

NH 246(7)

8.373

$$1.^6 \beta(x+1) = \ln 2 + \sum_{k=1}^{\infty} (-1)^k (1 - 2^{-k}) \zeta(k+1) x^k \quad [|x| < 1].$$

NH 37(5)

957

$$2.^6 \beta(x+1) = \ln 2 - 1 + \frac{1}{2x} - \frac{\pi}{2 \sin \pi x} + \frac{1}{1-x^2} - \sum_{k=1}^{\infty} [1 - (1 - 2^{-2k}) \zeta(2k+1)] x^{2k} \quad [0 < |x| < 2; x \neq \pm 1].$$

NH 38(11)

8.374⁷

$$\beta^{(n)}(x) = (-1)^n n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(x+k)^{n+1}} \quad [-x \notin \mathbb{N}].$$

NH 37(2)

8.375

Representation in the form of a finite sum:

$$1.^6 \beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin \frac{p\pi}{q}} - \sum_{k=0}^{\left[\frac{q-1}{2}\right]} \cos \frac{p(2k+1)\pi}{q} \ln \sin \frac{(2k+1)\pi}{2q} \\ [q = 2, 3, \dots, p = 1, 2, 3, \dots, q-1] \quad (\text{see also 8.362 5.-7.}).$$

8.362
NH 23(9)

$$2. \beta(n) = (-1)^{n+1} \ln 2 + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}}{k}.$$

Functional relations

8.376

$$\sum_{k=0}^{2n} (-1)^k \beta \left(\frac{x+k}{2n+1} \right) = (2n+1)\beta(x).$$

NH 19

8.377

$$\sum_{k=1}^n \beta(2^k x) = \psi(2^n x) - \psi(x) - n \ln 2.$$

NH 20(10)

8.38 The beta function (Euler's integral of the first kind): $B(x,y)$

Integral representation

8.380

$$\begin{aligned} 1. \quad B(x, y) &= \int_0^1 t^{x-1} (1-t)^{y-1} dt^*; \\ &= 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0]. \end{aligned}$$

* This equation is used as the definition of the function B

 (x, y) .

FI II 774(1)

$$2. \quad B(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} \varphi \cos^{2y-1} \varphi d\varphi \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0].$$

KU 10

958

$$3. \quad B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = 2 \int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \operatorname{Re} y > 0].$$

$$4. \quad B(x, y) = 2^{2-y-x} \int_{-1}^1 \frac{(1+t)^{2x-1}(1-t)^{2y-1}}{(1+t^2)^{x+y}} dt \quad [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0].$$

MO 7

$$\left. \begin{aligned} 5. \quad B(x, y) &= \int_0^1 \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt = \int_1^\infty \frac{t^{x-1} + t^{y-1}}{(1+t)^{x+y}} dt \\ 6. \quad B(x, y) &= \frac{1}{2^{x+y-1}} \int_0^1 [(1+t)^{x-1}(1-t)^{y-1} + (1+t)^{y-1}(1-t)^{x-1}] dt \end{aligned} \right\} \\ [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0].$$

BI ((1))(15)

$$\left. \begin{aligned} 7. \quad B(x, y) &= z^y(1+z)^x \int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+z)^{x+y}} dt \\ 8. \quad B(x, y) &= z^y(1+z)^x \int_0^{\frac{\pi}{2}} \frac{\cos^{2x-1} \varphi \sin^{2y-1} \varphi}{(z + \cos^2 \varphi)^{x+y}} d\varphi \end{aligned} \right\}, \\ [\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0, \quad 0 > z > -1, \quad \operatorname{Re}(x+y) < 1].$$

NH 163(8)

See also 3.196 3., 3.198, 3.199, 3.215, 3.238 3., 3.251 1.-;3., 11., 3.253, 3.312 1., 3.512 1. and 2., 3.541 1., 3.542 1., 3.621 5., 3.623 1., 3.631 1., 8., 9., 3.632 2., 3.633 1., 4., 3.634 1., 2., 3.637, 3.642 1., 3.667 8., 3.681 2.

$$9. \quad B(x, x) = \frac{1}{2^{2x-2}} \int_0^1 (1-t^2)^{x-1} dt = \frac{1}{2^{2x-1}} \int_0^1 \frac{(1-t)^{x-1}}{\sqrt{t}} dt.$$

See 8.384 4., 8.382 3., and also 3.621 1., 3.642 2., 3.665 1., 3.821 6., 3.839 6.

$$10. \quad B(x+y, x-y) = 4^{1-x} \int_0^\infty \frac{\operatorname{ch} 2yt}{\operatorname{ch}^{2x} t} dt \quad [\operatorname{Re} x > |\operatorname{Re} y|, \quad \operatorname{Re} x > 0].$$

MO 9

$$11. \quad B\left(x, \frac{y}{z}\right) = z \int_0^1 (1-t^z)^{x-1} t^{y-1} dt \quad \left[\operatorname{Re} z > 0, \quad \operatorname{Re} \frac{y}{z} > 0, \quad \operatorname{Re} x > 0 \right].$$

FI II 787a

8.381

$$3. \quad B(x+iy, x-iy) = 2^{1-2x} \alpha e^{-2i\gamma y} \int_{-\infty}^{\infty} \frac{e^{2i\alpha y t} dt}{\operatorname{ch}^{2x}(\alpha t - \gamma)} \quad [y, \alpha, \gamma \text{ are real, } \alpha > 0; \operatorname{Re} x > 0].$$

MI 8a

For an integral representation of $\ln B(x, y)$, see 3.428 7.

959

$$\begin{aligned} 4. \quad \frac{1}{B(x, y)} &= \frac{2^{x+y-1}(x+y-1)}{\pi} \int_0^{\frac{\pi}{2}} \cos[(x-y)t] \cos^{x+y-2} t dt; \\ &= \frac{2^{x+y-2}(x+y-1)}{\pi \cos\left[(x-y)\frac{\pi}{2}\right]} \int_0^{\pi} \cos[(x-y)t] \sin^{x+y-2} t dt; \\ &= \frac{2^{x+y-2}(x+y-1)}{\pi \sin\left[(x-y)\frac{\pi}{2}\right]} \int_0^{\pi} \sin[(x-y)t] \sin^{x+y-2} t dt. \end{aligned}$$

NH 159(9)a

NH 159(8)a

NH 158(5)a

Series representation

8.382

$$1. \quad B(x, y) = \frac{1}{y} \sum_{n=0}^{\infty} (-1)^n y \frac{(y-1) \dots (y-n)}{n!(x+n)} \quad [y > 0].$$

WH

$$\begin{aligned} 2. \quad \ln B\left(\frac{1+x}{2}, \frac{1}{2}\right) &= \ln \sqrt{2\pi} + \frac{1}{2} \left[\ln \left(\frac{\operatorname{tg} \frac{\pi x}{2}}{x} \right) - \ln \left(\frac{1+x}{1-x} \right) \right] + \\ &+ \sum_{k=0}^{\infty} \frac{1 - (1 - 2^{-2k}) \zeta(2k+1)}{2k+1} x^{2k+1} \quad [|x| < 2]. \end{aligned}$$

NH 39(17)

$$3. \quad B\left(z, \frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2^k k!} \frac{1}{z+k} + \frac{1}{z} \quad (\text{see also 8.384 and 8.380 9.}).$$

8.380

8.384

WH

8.383

Infinite-product representation:

$$(x + y + 1) B(x + 1, y + 1) = \prod_{k=1}^{\infty} \frac{k(x + y + k)}{(x + k)(y + k)} \quad [x, \quad y \neq -1, \quad -2, \dots].$$

MO 2

8.384

Functional relations involving the beta function:

$$1. \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = B(y, x)$$

FI II 779

$$2. \quad B(x, y) B(x + y, z) = B(y, z) B(y + z, x).$$

MO 6

$$3. \quad \sum_{k=0}^{\infty} B(x, y + k) = B(x - 1, y).$$

WH

$$4. \quad B(x, x) = 2^{1-2x} B\left(\frac{1}{2}, x\right) \quad (\text{see also 8.380 9. and 8.382 3.})$$

8.382

8.380

FI II 784

960

$$6. \quad \frac{1}{B(n, m)} = m \binom{n+m-1}{n-1} = n \binom{n+m-1}{m-1} \quad [m \text{ and } n - \text{natural numbers}].$$

For a connection with the psi function, see 4.253 1.

8.39 The incomplete beta function $B_x(p, q)$

8.391⁷

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt = \frac{x^p}{p} {}_2F_1(p, 1-q; p+1; x).$$

ET I 373

8.392

$$I_x(p, q) = \frac{B_x(p, q)}{B(p, q)}.$$

ET II 429

8.4- 8.5 Bessel Functions and Functions Associated with Them

8.40 Definitions

8.401

Bessel functions $Z_\nu(z)$ are solutions of the differential equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z} \frac{dZ_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0.$$

KU 37(1)

Special types of Bessel functions are what are called Bessel functions of the first kind $J_\nu(z)$, Bessel functions of the second kind $N_\nu(z)$ (also called Neumann functions and often written $Y_\nu(z)$), and Bessel functions of the third kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ (also

$$J_\nu(z)$$

$$N_\nu(z)$$

$$Y_\nu(z)$$

$$H_\nu^{(1)}(z) \quad H_\nu^{(2)}(z)$$

called Hankel's functions).

8.402

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} \quad [|\arg z| < \pi].$$

KU 55(1)

8.403

$$1. \quad N_\nu(z) = \frac{1}{\sin \nu\pi} [\cos \nu\pi J_\nu(z) - J_{-\nu}(z)] \quad [\text{for nonintegral } \nu, \quad |\arg z| < \pi].$$

KU 41(3)

$$\begin{aligned} 2. \quad \pi N_n(z) &= 2J_n(z) \ln \frac{z}{2} - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - \\ &\quad - \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+n)!} \left(\frac{z}{2}\right)^{n+2k} [\psi(k+1) + \psi(k+n+1)]; \\ &= 2J_n(z) \left(\ln \frac{z}{2} + \mathcal{C}\right) - \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} - \\ &\quad - \left(\frac{z}{2}\right)^n \frac{1}{n!} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k!(k+n)!} \left[\sum_{m=1}^{n+k} \frac{1}{m} + \sum_{m=1}^k \frac{1}{m} \right] \\ &\quad [n+1 - \text{a natural number}, \quad |\arg z| < \pi]. \end{aligned}$$

KU 43(10)

KU 44, WA 75(3)a

961

8.404

$$\left. \begin{aligned} 1. \quad N_{-n}(z) &= (-1)^n N_n(z) \\ 2. \quad J_{-n}(z) &= (-1)^n J_n(z) \end{aligned} \right\} \quad [n - \text{a natural number}].$$

KU 41(2)

8.405⁷

$$\left. \begin{aligned} 1. \quad H_\nu^{(1)}(z) &= J_\nu(z) + iN_\nu(z). \\ 2. \quad H_\nu^{(2)}(z) &= J_\nu(z) - iN_\nu(z). \end{aligned} \right\}$$

$$J_\nu(z) \quad N_\nu(z)$$

$$H_\nu^{(1)}(z) \quad H_\nu^{(2)}(z)$$

we shall write simply the letter Z instead of the letters J , N , $H^{(1)}$, and $H^{(2)}$.

Modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$

8.406

$$1. \quad I_\nu(z) = e^{-\frac{\pi}{2}\nu i} J_\nu\left(e^{\frac{\pi}{2}i} z\right) \quad \left[-\pi < \arg z \leq \frac{\pi}{2}\right].$$

WA 92

$$2. \quad I_\nu(z) = e^{\frac{3}{2}\pi\nu i} J_\nu\left(e^{-\frac{3}{2}\pi i} z\right) \quad \left[\frac{\pi}{2} < \arg z \leq \pi\right].$$

WA 92

For integral ν ,

$$3. \quad I_n(z) = i^{-n} J_n(iz).$$

KU 46(1)

8.407

$$1. \quad K_\nu(z) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}(iz).$$

$$2. \quad K_\nu(z) = \frac{\pi i}{2} e^{-\frac{\pi}{2}\nu i} H_{-\nu}^{(1)}(iz).$$

WA 92(8)

For the differential equation defining these functions, see 8.494.

8.41 Integral representations of the functions $J_\nu(z)$ and $N_\nu(z)$

8.411

$$1.7 \quad J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\theta + iz \sin \theta} d\theta; \\ = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta \quad [n = 0, 1, 2, \dots].$$

$$2. \quad J_{2n}(z) = \frac{1}{\pi} \int_0^\pi \cos 2n\theta \cos(z \sin \theta) d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos 2n\theta \cos(z \sin \theta) d\theta \quad [n - \text{an integer}].$$

WA 30(7)

$$3. \quad J_{2n+1}(z) = \frac{1}{\pi} \int_0^\pi \sin(2n+1)\theta \sin(z \sin \theta) d\theta = \\ = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin(2n+1)\theta \sin(z \sin \theta) d\theta \quad [n - \text{an integer}].$$

WA 30(6)

$$4. \quad J_\nu(z) = 2 \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin^{2\nu} \theta \cos(z \cos \theta) d\theta \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WH

$$5. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi \sin^{2\nu} \theta \cos(z \cos \theta) d\theta \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$6. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(z \sin \theta) \cos^{2\nu} \theta d\theta \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

KU 65(5), WA 35(4)a

$$7. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^\pi e^{\pm iz \cos \varphi} \sin^{2\nu} \varphi d\varphi \quad \left[\operatorname{Re} \left(\nu + \frac{1}{2} \right) > 0 \right].$$

WH

$$8. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \cos zt dt \quad \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

$$9. \quad J_\nu(x) = 2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2} - \nu\right) \Gamma\left(\frac{1}{2}\right)} \int_1^\infty \frac{\sin xt}{(t^2 - 1)^{\nu + \frac{1}{2}}} dt \quad \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, x > 0\right].$$

MO 37

$$10. \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 e^{izt} (1 - t^2)^{\nu - \frac{1}{2}} dt \quad \left[\operatorname{Re} \nu > -\frac{1}{2}\right].$$

WA 34(3)

963

$$11. \quad J_\nu(x) = \frac{2}{\pi} \int_0^\infty \sin\left(x \operatorname{ch} t - \frac{\nu\pi}{2}\right) \operatorname{ch} \nu t dt.$$

WA 199(12)

$$12. \quad J_\nu(z) = \frac{2^{\nu+1} z^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu - \frac{1}{2}} \theta \sin\left(z - \nu\theta + \frac{1}{2}\theta\right)}{\sin^{2\nu+1} \theta} e^{-2z \operatorname{ctg} \theta} d\theta$$

$$\left[|\arg z| < \frac{\pi}{2}, \operatorname{Re}\left(\nu + \frac{1}{2}\right) > 0\right].$$

WH

$$13.^8 \quad J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta - \frac{\sin \nu\pi}{\pi} \int_0^\infty e^{-\nu\theta - z \operatorname{sh} \theta} d\theta$$

$$[\operatorname{Re} z > 0].$$

WA 195(4)

$$14. \quad J_\nu(z) = \frac{e^{\pm \nu\pi i}}{\pi} \left[\int_0^\pi \cos(\nu\theta + z \sin \theta) d\theta - \sin \nu\pi \int_0^\infty e^{-\nu\theta + z \operatorname{sh} \theta} d\theta \right]$$

$$\left[\text{for } \frac{\pi}{2} < |\arg z| < \pi, \text{ with the upper sign taken for } \arg z > \frac{\pi}{2} \right.$$

$$\left. \text{and the lower sign taken for } \arg z < -\frac{\pi}{2} \right].$$

WH

8.412

$$1. \quad J_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] dt \quad \left[|\arg z| < \frac{\pi}{2}\right].$$

$$2. \quad J_\nu(z) = \frac{z^\nu}{2^{\nu+1}\pi i} \int_{-\infty}^{(0+)} t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) dt.$$

WA 195(1)

$$3. \quad J_\nu(z) = \frac{z}{2^{\nu+1}\pi i} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k!} \int_{-\infty}^{(0+)} e^t t^{-\nu-k-1} dt.$$

WA 195(1)

$$4. \quad J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)} \left(\frac{x}{2}\right)^{\nu+2t} dt \quad [\operatorname{Re} \nu \geq 0, \quad x > 0].$$

WA 214(7)

$$5.7 \quad J_\nu(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \left(\frac{z}{2}\right)^\nu}{2\pi i \Gamma\left(\frac{1}{2}\right)} \int_A^{(1+, -1-)} (t^2 - 1)^{\nu-\frac{1}{2}} \cos(zt) dt$$

$\left[\nu \neq \frac{1}{2}, \frac{3}{2}, \dots \text{ The point } A \text{ falls to the right of the point } t = 1, \right.$
 $\left. \text{and } \arg(t-1) = \arg(t+1) = 0 \text{ at the point } A \right].$

WH

964

$$6.7 \quad J_\nu(z) = \frac{1}{2\pi} \int_{-\pi+\infty i}^{\pi+\infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0].$$

The path of integration is taken around the semi-infinite strip $y \geq 0, -\pi \leq x \leq \pi$.

8.413

$$\frac{J_\nu(\sqrt{z^2 + \zeta^2})}{(z^2 - \zeta^2)^{\frac{\nu}{2}}} = \frac{1}{\pi(z + \zeta)^\nu} \left\{ \int_0^\infty e^{\zeta \cos t} \cos(z \sin t - \nu t) dt - \sin \nu \pi \int_0^\infty \exp(-z \operatorname{sh} t - \zeta \operatorname{ch} t - \nu t) dt \right\} \quad [\operatorname{Re}(z + \zeta) > 0].$$

$$\int_{2x}^{\infty} \frac{J_0(t)}{t} dt = \frac{1}{4\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\Gamma(-t)}{t\Gamma(1+t)} x^{2t} dt \quad [x > 0]$$

MO 41

See 3.715 2., 9., 10., 13., 14., 19.- 21., 3.865 1., 2., 4., 3.996 4. For an integral representation of $J_0(z)$ see 3.714 2., 3.753 2., 3., and 4.124. For an integral representation of $J_1(z)$ see 3.697, 3.711, 3.752 2., and 3.753 5.

8.415

$$1. \quad N_0(x) = \frac{4}{\pi^2} \int_0^1 \frac{\arcsin t}{\sqrt{1-t^2}} \sin(xt) dt - \frac{4}{\pi^2} \int_1^{\infty} \frac{\ln(t + \sqrt{t^2-1})}{\sqrt{t^2-1}} \sin(xt) dt \quad [x > 0].$$

MO 37

$$2. \quad N_\nu(x) = -2 \frac{\left(\frac{x}{2}\right)^{-\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}\right)} \int_1^{\infty} \frac{\cos xt}{(t^2-1)^{\nu+\frac{1}{2}}} dt \quad \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0\right].$$

KU 89(28)A, MO 38

$$3. \quad N_\nu(x) = -\frac{2}{\pi} \int_0^{\infty} \cos\left(x \operatorname{ch} t - \frac{\nu\pi}{2}\right) \operatorname{ch} \nu t dt \quad [-1 < \operatorname{Re} \nu < 1, \quad x > 0].$$

WA 199(13)

$$4. \quad N_\nu(z) = \frac{1}{\pi} \int_0^{\infty} \sin(z \sin \theta - \nu\theta) d\theta - \frac{1}{\pi} \int_0^{\infty} (e^{\nu t} + e^{-\nu t} \cos \nu\pi) e^{-z \operatorname{sh} t} dt \quad [\operatorname{Re} z > 0].$$

WA 197(1)

$$5. \quad N_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left[\int_0^{\frac{\pi}{2}} \sin(z \sin \theta) \cos^{2\nu} \theta d\theta - \int_0^{\infty} e^{-z \operatorname{sh} \theta} \operatorname{ch}^{2\nu} \theta d\theta \right] \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0 \right].$$

WA 181(5)a

For an integral representation of $N_0(z)$, see 3.714 3., 3.753 4., 3.864. See also 3.865 3.

8.42 Integral representations of the functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

8.421

$$\left. \begin{aligned} 1. \quad H_\nu^{(1)}(x) &= \frac{e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^{\infty} e^{ix \operatorname{ch} t - \nu t} dt = \\ &= \frac{2e^{-\frac{\nu\pi i}{2}}}{\pi i} \int_0^{\infty} e^{ix \operatorname{ch} t} \operatorname{ch} \nu t dt \\ 2. \quad H_\nu^{(2)}(x) &= -\frac{e^{\frac{\nu\pi i}{2}}}{\pi i} \int_{-\infty}^{\infty} e^{-ix \operatorname{ch} t - \nu t} dt = \\ &= -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi i} \int_0^{\infty} e^{-ix \operatorname{ch} t} \operatorname{ch} \nu t dt \end{aligned} \right\} \quad [-1 < \operatorname{Re} \nu < 1, \quad x > 0].$$

WA 199(11)

WA 199(10)

$$3. \quad H_\nu^{(1)}(z) = -\frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} t e^{i(z-\nu t+\frac{t}{2})}}{\sin^{2\nu+1} t} \exp(-2z \operatorname{ctg} t) dt$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0 \right].$$

WA 186(5)

$$4. \quad H_\nu^{(2)}(z) = \frac{2^{\nu+1} i z^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{\nu-\frac{1}{2}} t e^{-i(z-\nu t+\frac{t}{2})}}{\sin^{2\nu+1} t} \exp(-2z \operatorname{ctg} t) dt$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0 \right].$$

WA 186(6)

966

$$5. \quad H_\nu^{(1)}(x) = -\frac{2i \left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \nu\right)} \int_1^{\infty} \frac{e^{ixt}}{(t^2 - 1)^{\nu+\frac{1}{2}}} dt \quad \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0 \right].$$

$$6. H_\nu^{(2)}(x) = \frac{2i \left(\frac{x}{2}\right)^{-\nu}}{\sqrt{\pi}\Gamma\left(\frac{1}{2} - \nu\right)} \int_1^\infty \frac{e^{-ixt}}{(t^2 - 1)^{\nu + \frac{1}{2}}} dt \quad \left[-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}, \quad x > 0\right].$$

WA 187(2)

$$7. H_\nu^{(1)}(z) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} \int_0^\infty \exp\left[\frac{1}{2}iz\left(t + \frac{1}{t}\right)\right] t^{-\nu-1} dt$$

$[0 < \arg z < \pi; \text{ or } \arg z = 0 \text{ and } -1 < \operatorname{Re} \nu < 1].$

MO 38

$$8. H_\nu^{(1)}(xz) = -\frac{i}{\pi} e^{-\frac{1}{2}i\nu\pi} z^\nu \int_0^\infty \exp\left[\frac{1}{2}ix\left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1} dt$$

$[0 < \arg z < \frac{\pi}{2}, \quad x > 0, \operatorname{Re} \nu > -1; \text{ or } \arg z = \frac{\pi}{2}, \quad x > 0 \text{ and } -1 < \operatorname{Re} \nu < 1].$

MO 38

$$9. H_\nu^{(1)}(xz) = \frac{\sqrt{\frac{2}{\pi}} x^\nu \exp\left[i\left(xz - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)\right]}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty \left(1 + \frac{it}{2z}\right)^{\nu - \frac{1}{2}} t^{\nu - \frac{1}{2}} e^{-xt} dt$$

$\left[\operatorname{Re} \nu > -\frac{1}{2}, \quad -\frac{\pi}{2} < \arg z < \frac{3}{2}\pi, \quad x > 0\right].$

MO 39

$$10. H_\nu^{(1)}(z) = \frac{-2ie^{-i\nu\pi} \left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{iz \operatorname{ch} t} \operatorname{sh}^{2\nu} t dt$$

$\left[0 < \arg z < \pi, \operatorname{Re} \nu > -\frac{1}{2} \text{ or } \arg z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}\right].$

MO 38

$$11. H_0^{(1)}(x) = -\frac{i}{\pi} \int_{-\infty}^\infty \frac{\exp(i\sqrt{x^2 + t^2})}{\sqrt{x^2 + t^2}} dt \quad [x > 0].$$

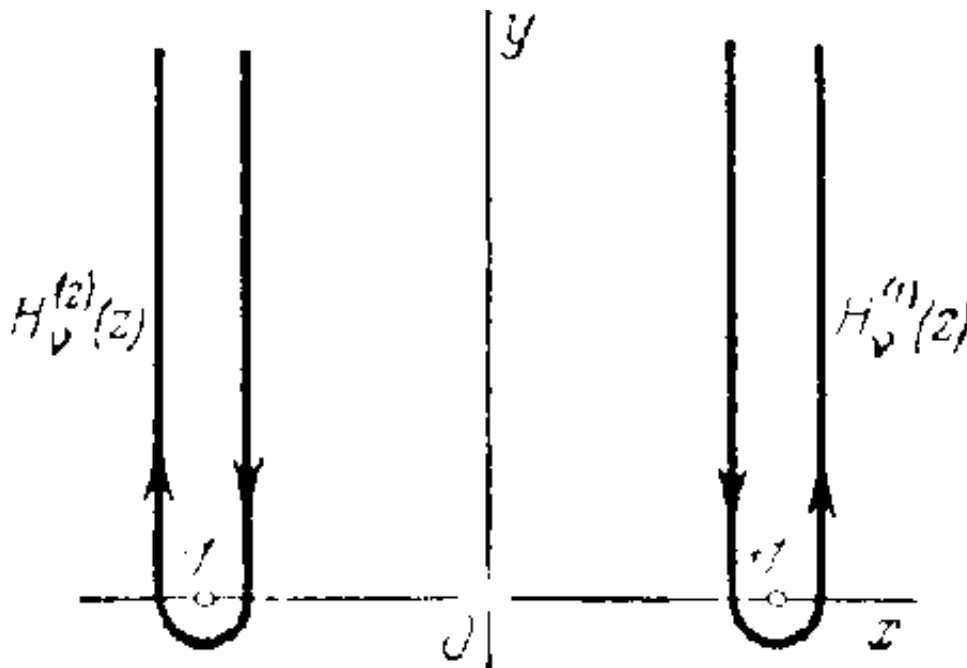
MO 38

8.422

$$1. H_\nu^{(1)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma\left(\frac{1}{2}\right)} \int_{1+i\infty}^{(1+)} e^{izt} (t^2 - 1)^{\nu - \frac{1}{2}} dt \quad [-\pi < \arg z < 2\pi].$$

$$2. H_\nu^{(2)}(z) = \frac{\Gamma\left(\frac{1}{2} - \nu\right) \left(\frac{z}{2}\right)^\nu}{\pi i \Gamma\left(\frac{1}{2}\right)} \times \\ \times \int_{-1+\infty i}^{(-1-)} e^{izt} (t^2 - 1)^{\nu - \frac{1}{2}} dt \\ [-2\pi < \arg z < \pi].$$

The paths of integration are shown in the drawing.



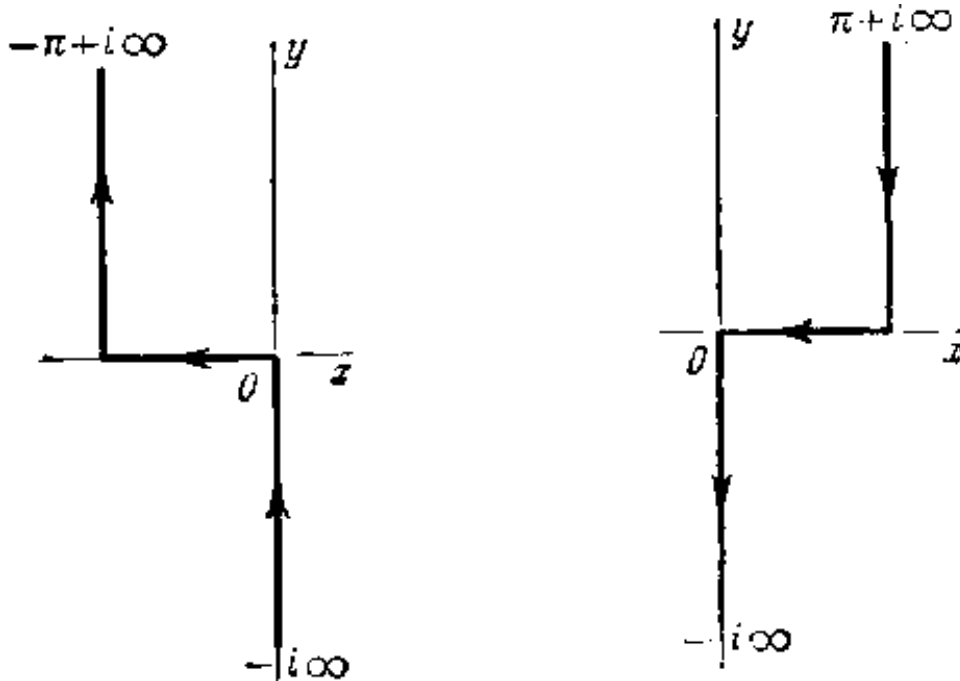
8.423

$$1. H_\nu^{(1)}(z) = -\frac{1}{\pi} \int_{-\infty i}^{-\pi + \infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0].$$

$$2. H_\nu^{(2)}(z) = -\frac{1}{\pi} \int_{\pi + \infty i}^{-\infty i} e^{-iz \sin \theta + i\nu \theta} d\theta \quad [\operatorname{Re} z > 0].$$

WA 197(2)a

The path of integration for formula 1 is shown in the left hand drawing and for formula 2 in the right hand drawing.



8.424

$$\begin{aligned}
 1. \quad H_\nu^{(1)}(z)J_\nu(\zeta) &= \frac{1}{\pi i} \int_0^{\gamma+i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2 + \zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right) \frac{dt}{t} \\
 2. \quad H_\nu^{(2)}(z)J_\nu(\zeta) &= \frac{i}{\pi} \int_0^{\gamma-i\infty} \exp\left[\frac{1}{2}\left(t - \frac{z^2 + \zeta^2}{t}\right)\right] I_\nu\left(\frac{z\zeta}{t}\right) \frac{dt}{t}
 \end{aligned}$$

$[\gamma > 0, \quad \text{Re } \nu > -1, \quad |\zeta| < |z|].$

MO 45

968

8.43 Integral representations of the function $I_\nu(z)$ and $K_\nu(z)$

The function $I_\nu(Z)$

8.431

$$\begin{aligned}
 1. \quad I_\nu(z) &= \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{\pm zt} dt \\
 2. \quad I_\nu(z) &= \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} \text{ch } zt dt
 \end{aligned}$$

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$$5. \quad I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos \nu \theta \, d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \operatorname{ch} t - \nu t} \, dt \quad \left[|\arg z| \leq \frac{\pi}{2}, \operatorname{Re} \nu > 0 \right].$$

WA 201(4)

See also 3.383 2., 3.387 1., 3.471 6., 3.714 5.

For an integral representation of $I_0(z)$ and $I_1(z)$, see 3.366 1., 3.534 3.856 6.

The function $K_\nu(Z)$

8.432

$$1. \quad K_\nu(z) = \int_0^\infty e^{-z \operatorname{ch} t} \operatorname{ch} \nu t \, dt \quad \left[|\arg z| < \frac{\pi}{2} \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right].$$

MO 39

$$2. \quad K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z \operatorname{ch} t} \operatorname{sh}^{2\nu} t \, dt$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2}, \operatorname{Re} z > 0; \text{ or } \operatorname{Re} z = 0 \text{ and } -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2} \right].$$

WA 190(5), WH

969

$$3. \quad K_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} \, dt$$

$$\left[\operatorname{Re}\left(\nu + \frac{1}{2}\right) > 0, \quad |\arg z| < \frac{\pi}{2}; \text{ or } \operatorname{Re} z = 0 \text{ and } \nu = 0 \right].$$

WA 190(4)

$$4. \quad K_\nu(x) = \frac{1}{\cos \frac{\nu \pi}{2}} \int_0^\infty \cos(x \operatorname{sh} t) \operatorname{ch} \nu t \, dt \quad [x > 0, \quad -1 < \operatorname{Re} \nu < 1].$$

WA 202(13)

$$6.7 \quad K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t - \frac{z^2}{4t}} dt}{t^{\nu+1}} \quad \left[|\arg z| < \frac{\pi}{2}, \quad \operatorname{Re} z^2 > 0 \right].$$

WA 203(15)

$$7.7 \quad K_\nu(xz) = \frac{z^\nu}{2} \int_0^\infty \exp\left[-\frac{x}{2}\left(t + \frac{z^2}{t}\right)\right] t^{-\nu-1} dt \quad \left[|\arg z| < \frac{\pi}{4} \text{ or } |\arg z| = \frac{\pi}{4} \text{ and } \operatorname{Re} \nu < 1 \right].$$

MO 39

$$8. \quad K_\nu(xz) = \sqrt{\frac{\pi}{2z}} \frac{x^\nu e^{-xz}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-xt} t^{\nu-\frac{1}{2}} \left(1 + \frac{t}{2z}\right)^{\nu-\frac{1}{2}} dt \quad \left[|\arg z| < \pi, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad x > 0 \right].$$

MO 39

$$9. \quad K_\nu(xz) = \frac{\sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{x}{2z}\right)^\nu \int_0^\infty \frac{\exp(-x\sqrt{t^2 + z^2})}{\sqrt{t^2 + z^2}} t^{2\nu} dt \quad \left[\operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} z > 0, \quad \operatorname{Re} \sqrt{t^2 + z^2} > 0, \quad x > 0 \right].$$

MO 39

See also 3.337 4., 3.383 3., 3.387 3., 6., 3.388 2., 3.389 4., 3.391, 3.395 1., 3.471 9., 3.483, 3.547 2., 3.856, 3.871 3., 4., 7.141 5.

8.433

$$K_{\frac{1}{3}}\left(\frac{2x\sqrt{x}}{3\sqrt{3}}\right) = \frac{3}{\sqrt{x}} \int_0^\infty \cos(t^3 + xt) dt.$$

KU 98(31), WA 211(2)

For an integral representation of $K_0(z)$, see 3.754 2., 3.864, 4.343, 4.356, 4.367.

970

8.44 Series representation

The function $J_\nu(Z)$

8.440

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k} \quad [|\arg z| < \pi].$$

8.441

Special cases:

$$1. \quad J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} (k!)^2}.$$

$$2. \quad J_1(z) = -J_0'(z) = \frac{z}{2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k! (k+1)!}.$$

$$3. \quad J_{\frac{1}{3}}(z) = \frac{\sqrt[3]{\frac{z}{2}}}{\Gamma\left(\frac{4}{3}\right)} \sum_{k=0}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 1 \cdot 4 \cdot 7 \cdot \dots \cdot (3k+1)}.$$

$$4. \quad J_{-\frac{1}{3}}(z) = \frac{1}{\Gamma\left(\frac{2}{3}\right)} \sqrt[3]{\frac{2}{z}} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(z\sqrt{3})^{2k}}{2^{2k} k! \cdot 2 \cdot 5 \cdot 8 \cdot \dots \cdot (3k-1)} \right\}.$$

For the expansion of $J_\nu(z)$ in Laguerre polynomials, see 8.975 3.

8.442

$$1.^7 \quad J_\nu(z) J_\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}z\right)^{\mu+\nu+2m} (\mu+\nu+m+1)_m}{m! \Gamma(\mu+m+1) \Gamma(\nu+m+1)},$$

$$2.^8 \quad J_\nu(az) J_\mu(bz) = \frac{\left(\frac{az}{2}\right)^\nu \left(\frac{bz}{2}\right)^\mu}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{az}{2}\right)^{2k} F\left(-k, -\nu-k; \mu+1; \frac{b^2}{a^2}\right)}{k! \Gamma(\nu+k+1)}.$$

The function $N_\nu(Z)$

8.443

$$N_\nu(z) = \frac{1}{\sin \nu\pi} \left\{ \cos \nu\pi \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} - \left(\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(k - \nu + 1)} \right\} \quad [\nu \neq \text{an integer}] \quad (\text{cf. 8.4031}).$$

8.403

For $\nu + 1$ a natural number, see 8.403 2.; for ν a negative integer see 8.404 1

971

8.444

Special cases:

$$1. \quad \pi N_0(z) = 2J_0(z) \left(\ln \frac{z}{2} + C \right) - 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^k \frac{1}{m}.$$

KU 44

$$2. \quad \pi N_1(z) = 2J_1(z) \left(\ln \frac{z}{2} + C \right) - \frac{2}{z} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \left(\frac{z}{2}\right)^{2k-1}}{k!(k-1)!} \left\{ \frac{1}{k} + 2 \sum_{m=1}^{k-1} \frac{1}{m} \right\}.$$

The functions $I_\nu(z)$ and $K_n(z)$

8.445

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$

WH

8.446⁷

$$K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{k!} \left(\frac{z}{2}\right)^{n-2k} +$$

$$+ (-1)^{n+1} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{n+2k} \left[\frac{1}{k!} \left(\frac{z}{2}\right)^{k/(k+1)} - \frac{1}{k!(n+k+1)} \right]$$

8.447

Special cases:

$$1. \quad I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2}.$$

$$2. \quad I_1(z) = I_0'(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!}.$$

$$3. \quad K_0(z) = -\ln \frac{z}{2} I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2} \psi(k+1).$$

WA 95(14)

972

8.45 Asymptotic expansions of Bessel functions

8.451

For large values of $|z|$ * An estimate of the remainders in formulas 8.451 is given in 8.451 7. and 8.451 8.

$$1. \quad J_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma\left(\nu + 2k + \frac{1}{2}\right)}{(2k)! \Gamma\left(\nu - 2k + \frac{1}{2}\right)} + R_1 \right] - \right. \\ \left. - \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma\left(\nu + 2k + \frac{3}{2}\right)}{(2k+1)! \Gamma\left(\nu - 2k - \frac{1}{2}\right)} + R_2 \right] \right\} \\ [|\arg z| < \pi] (\text{see 8.339 4.}).$$

8.339
WA 222(1, 3)

$$2. \quad N_{\pm\nu}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k}} \frac{\Gamma\left(\nu + 2k + \frac{1}{2}\right)}{(2k)! \Gamma\left(\nu - 2k + \frac{1}{2}\right)} + R_1 \right] + \right. \\ \left. + \cos\left(z \mp \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2z)^{2k+1}} \frac{\Gamma\left(\nu + 2k + \frac{3}{2}\right)}{(2k+1)! \Gamma\left(\nu - 2k - \frac{1}{2}\right)} + R_2 \right] \right\} \\ [|\arg z| < \pi] \text{ (see 8.339 4.)}$$

8.339
WA 222(2, 4, 5)

$$3. \quad H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \left[\sum_{k=0}^{n-1} \frac{(-1)^k}{(2iz)^k} \frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k! \Gamma\left(\nu - k + \frac{1}{2}\right)} + \theta_1 \frac{(-1)^n}{(2iz)^n} \frac{\Gamma\left(\nu + n + \frac{1}{2}\right)}{k! \Gamma\left(\nu - n + \frac{1}{2}\right)} \right] \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, |\arg z| < \pi \right] \text{ (see 8.339 4.)}$$

8.339
WA 221(5)

$$4. \quad H_{\nu}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \left[\sum_{k=0}^{n-1} \frac{1}{(2iz)^k} \frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k! \Gamma\left(\nu - k + \frac{1}{2}\right)} + \theta_2 \frac{1}{(2iz)^n} \frac{\Gamma\left(\nu + n + \frac{1}{2}\right)}{k! \Gamma\left(\nu - n + \frac{1}{2}\right)} \right] \\ \left[\operatorname{Re} \nu > -\frac{1}{2}, |\arg z| < \pi \right] \text{ (see 8.339 4.)}$$

8.339
WA 221(6)

973

For indices of the form $\nu = \frac{2n-1}{2}$ (where n is a natural number), the series 8.451 terminate. In this case, the closed formulas 8.46 are valid for all values.

[The sign + is taken for $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, the sign - for $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$ *)] (see 8.339 4.)* The contradiction that this condition contains at first glance is explained by the so-called Stokes phenomenon (see Watson, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge Univ. Press, 1944, page 201).

$$6. \quad K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[\sum_{k=0}^{n-1} \frac{1}{(2z)^k} \frac{\Gamma\left(\nu + k + \frac{1}{2}\right)}{k! \Gamma\left(\nu - k + \frac{1}{2}\right)} + \theta_3 \frac{\Gamma\left(\nu + n + \frac{1}{2}\right)}{(2z)^n n! \Gamma\left(\nu - n + \frac{1}{2}\right)} \right] \quad (\text{see 8.339 4.}).$$

8.339
WA 231, 245(9)
WA 226(2,3)

An estimate of the remainders of the asymptotic series in formulas 8.451:

$$7. \quad |R_1| < \left| \frac{\Gamma\left(\nu + 2n + \frac{1}{2}\right)}{(2z)^{2n} (2n)! \Gamma\left(\nu - 2n + \frac{1}{2}\right)} \right| \left[n > \frac{\nu}{2} - \frac{1}{4} \right].$$

WA 231

$$8. \quad |R_2| < \left| \frac{\Gamma\left(\nu + 2n + \frac{3}{2}\right)}{(2z)^{2n+1} (2n+1)! \Gamma\left(\nu - 2n - \frac{1}{2}\right)} \right| \left[n \geq \frac{\nu}{2} - \frac{3}{4} \right].$$

WA 231

For $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, ν real, and $n + \frac{1}{2} > |\nu|$

$$|\theta_1| < 1, \text{ if } \operatorname{Im} z \geq 0; \quad |\theta_1| < |\sec(\arg z)|, \text{ if } \operatorname{Im} z \leq 0.$$

WA 245

For $-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}$, ν real, and $n + \frac{1}{2} > |\nu|$

$$|\theta_2| < 1, \text{ if } \operatorname{Im} z \leq 0; \quad |\theta_2| < |\sec(\arg z)|, \text{ if } \operatorname{Im} z \geq 0.$$

WA 246

974

For ν real,

$$\begin{aligned} |\theta_3| < 1 \text{ and } \operatorname{Re} \theta_3 \geq 0, \text{ if } \operatorname{Re} z \geq 0; \\ |\theta_3| < |\operatorname{cosec}(\arg z)|, \text{ if } \operatorname{Re} z < 0. \end{aligned}$$

WA 245

For ν and z real and $n \geq \nu - \frac{1}{2}$,

$$0 \leq |\theta_3| \leq 1.$$

WA 231

In particular, it follows from 8.451 7. and 8.451 8. that for real positive values of z and ν , the errors $|R_1|$ and $|R_2|$ are less than the absolute value of the first discarded term. For values of $|\arg z|$ close to π , the series 8.451 1. and 8.451 2. may not be suitable for calculations. In particular, the error for $|\arg z| > \pi$ can be greater in absolute value than the first discarded term.

"Approximation by tangents"

8.452

For large values of the index (where the argument is less than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \operatorname{ch} \alpha$. Then, for large values of ν , the following expansions are valid:

$$\begin{aligned} 1. \quad J_\nu \left(\frac{\nu}{\operatorname{ch} \alpha} \right) \sim \frac{\exp(\nu\theta\alpha - \nu\alpha)}{\sqrt{2\nu\pi\theta\alpha}} \left\{ 1 + \frac{1}{\nu} \left(\frac{1}{8} \operatorname{cth} \alpha - \frac{5}{24} \operatorname{cth}^3 \alpha \right) + \right. \\ \left. + \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{cth}^2 \alpha - \frac{231}{576} \operatorname{cth}^4 \alpha + \frac{1155}{3456} \operatorname{cth}^6 \alpha \right) + \dots \right\}. \end{aligned}$$

$$2. \quad N_\nu \left(\frac{\nu}{\operatorname{ch} \alpha} \right) \sim - \frac{\exp(\nu\alpha - \nu\theta\alpha)}{\sqrt{\frac{\pi}{2}\nu\theta\alpha}} \left\{ 1 - \frac{1}{\nu} \left(\frac{1}{8} \operatorname{cth} \alpha - \frac{5}{24} \operatorname{cth}^3 \alpha \right) + \right. \\ \left. + \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{cth}^2 \alpha - \frac{231}{576} \operatorname{cth}^4 \alpha + \frac{1155}{3456} \operatorname{cth}^6 \alpha \right) + \dots \right\}.$$

WA 270(5)

8.453

For large values of the index (where the argument is greater than the index).

Suppose that $x > 0$ and $\nu > 0$. Let us set $\nu/x = \cos \beta$. Then, for large values of ν , the following expansions are valid:

$$1. \quad J_\nu(\nu \sec \beta) \sim \sqrt{\frac{2}{\nu\pi \operatorname{tg} \beta}} \left\{ \left[1 - \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{ctg}^2 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right] \cos \left(\nu \operatorname{tg} \beta - \nu\beta - \frac{\pi}{4} \right) + \right. \\ \left. + \left[\frac{1}{\nu} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \dots \right] \sin \left(\nu \operatorname{tg} \beta - \nu\beta - \frac{\pi}{4} \right) \right\}.$$

WA 271(4)

975

$$2. \quad N_\nu(\nu \sec \beta) \sim \sqrt{\frac{2}{\nu\pi \operatorname{tg} \beta}} \left\{ \left[1 - \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{ctg}^2 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right] \sin \left(\nu \operatorname{tg} \beta - \nu\beta - \frac{\pi}{4} \right) - \right. \\ \left. - \left[\frac{1}{\nu} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \dots \right] \cos \left(\nu \operatorname{tg} \beta - \nu\beta - \frac{\pi}{4} \right) \right\}.$$

WA 271(5)

$$3. \quad H_\nu^{(1)}(\nu \sec \beta) \sim \frac{\exp \left[\nu i (\operatorname{tg} \beta - \beta) - \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2}\nu \operatorname{tg} \beta}} \left\{ 1 - \frac{i}{\nu} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \right. \\ \left. - \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{ctg}^2 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right\}.$$

WA 271(1)

$$4. \quad H_\nu^{(2)}(\nu \sec \beta) \sim \frac{\exp \left[-\nu i (\operatorname{tg} \beta - \beta) + \frac{\pi}{4} i \right]}{\sqrt{\frac{\pi}{2}\nu \operatorname{tg} \beta}} \left\{ 1 + \frac{i}{\nu} \left(\frac{1}{8} \operatorname{ctg} \beta + \frac{5}{24} \operatorname{ctg}^3 \beta \right) - \right. \\ \left. - \frac{1}{\nu^2} \left(\frac{9}{128} \operatorname{ctg}^2 \beta + \frac{231}{576} \operatorname{ctg}^4 \beta + \frac{1155}{3456} \operatorname{ctg}^6 \beta \right) + \dots \right\}.$$

$|x - \nu|$

$x^{\frac{1}{3}}$

$|x - \nu|$

may use the following formulas:

8.454

Suppose that $x > 0$ and $\nu > 0$, we set

$$w = \sqrt{\frac{x^2}{\nu^2} - 1};$$

Then,

$$1. \quad H_{\nu}^{(1)}(x) = \frac{w}{\sqrt{3}} \exp \left\{ \left[\frac{\pi}{6} + \nu \left(w - \frac{w^3}{3} - \operatorname{arctg} w \right) \right] i \right\} H_{\frac{1}{3}}^{(1)} \left(\frac{\nu}{3} w^3 \right) + O \left(\frac{1}{|\nu|} \right).$$

$$2. \quad H_{\nu}^{(2)}(x) = \frac{w}{\sqrt{3}} \exp \left\{ \left[-\frac{\pi}{6} - \nu \left(w - \frac{w^3}{3} - \operatorname{arctg} w \right) \right] i \right\} H_{\frac{1}{3}}^{(2)} \left(\frac{\nu}{3} w^3 \right) + O \left(\frac{1}{|\nu|} \right).$$

MO 34

The absolute value of the error $O \left(\frac{1}{|\nu|} \right)$ is then less than $24\sqrt{2} \left| \frac{1}{\nu} \right|$.

976

8.455

For x real and ν a natural number ($\nu = n$), if $n \gg 1$, the following approximations are valid:

$$\begin{aligned} 1.7 \quad J_n(x) &\sim \frac{1}{\pi} \sqrt{\frac{2(n-x)}{3x}} K_{\frac{1}{3}} \left\{ \frac{[2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\} [n > x] \quad (\text{see also 8.433}); \\ &\quad \frac{1}{2} e^{\frac{2}{3}\pi i} \sqrt{\frac{2(n-x)}{3x}} H_{\frac{1}{3}}^{(1)} \left\{ \frac{i [2(n-x)]^{\frac{3}{2}}}{3\sqrt{x}} \right\} [n > x]; \\ &\quad \frac{1}{\sqrt{3}} \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] + J_{-\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} [x > n]. \end{aligned}$$

(see also 8.441 3., 8.441 4.).

8.441

8.441

$$2. \quad N_n(x) \text{ t } \sqrt{\frac{2(x-n)}{3x}} \left\{ J_{-\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] - J_{\frac{1}{3}} \left[\frac{\{2(x-n)\}^{\frac{3}{2}}}{3\sqrt{x}} \right] \right\} [x > n].$$

WA 276(3)

An estimate of the error in formulas 8.455 has not yet been achieved.

8.456

$$J_\nu^2(z) + N_\nu^2(z) \text{ t } \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma\left(\nu+k+\frac{1}{2}\right)}{k! \Gamma\left(\nu-k+\frac{1}{2}\right)} [|\arg z| < \pi] \quad (\text{see also 8.479 1.}).$$

8.479
 WA 250(5)

8.457

$$J_\nu^2(x) + J_{\nu+1}^2(x) \text{ t } \frac{2}{\pi x} [x \gg |\nu|].$$

WA 223

8.46 Bessel functions of order equal to an integer plus one-half

The function $J_\nu(Z)$

8.461

$$1. \quad J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \sin\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!} (2z)^{2k} + \right. \\ \left. + \cos\left(z - \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!} (2z)^{2k+1} \right\} \\ [n+1 - \text{is a natural number}] \quad (\text{cf. 8.451 1.}).$$

$$2. \quad J_{-n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \cos\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (n+2k)!}{(2k)!(n-2k)!(2z)^{2k}} - \right. \\ \left. - \sin\left(z + \frac{\pi}{2}n\right) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^k (n+2k+1)!}{(2k+1)!(n-2k-1)!(2z)^{2k+1}} \right\} \\ [n+1 - \text{is a natural number}] \quad (\text{cf. 8.451 1.}).$$

8.451
KU 58(7), WA 67(5)

8.462

$$1. \quad J_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{-n+k-1} (n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{-n+k-1} (n+k)!}{k!(n-k)!(2z)^k} \right\} \\ [n+1 - \text{is a natural number}].$$

KU 59(6), WA 66(1)

$$2. \quad J_{-n-\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}} \left\{ e^{iz} \sum_{k=0}^n \frac{i^{n+k} (n+k)!}{k!(n-k)!(2z)^k} + e^{-iz} \sum_{k=0}^n \frac{(-i)^{n+k} (n+k)!}{k!(n-k)!(2z)^k} \right\} \\ [n+1 - \text{is a natural number}].$$

KU 59(7), WA 67(4)

8.463

$$1. \quad J_{n+\frac{1}{2}}(z) = (-1)^n z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(zdz)^n} \left(\frac{\sin z}{z} \right).$$

KU 58(4)

$$2. \quad J_{-n-\frac{1}{2}}(z) = z^{n+\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{d^n}{(zdz)^n} \left(\frac{\cos z}{z} \right).$$

KU 58(5)

8.464

Special cases:

$$1. \quad J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$$

$$2. \quad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z.$$

DW

$$3. \quad J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$$

DW

$$4. \quad J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\sin z - \frac{\cos z}{z} \right).$$

DW

$$5.8 \quad J_{\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \left(\frac{3}{z^2} - 1 \right) \sin z - \frac{3}{z} \cos z \right\}.$$

DW

$$6. \quad J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left\{ \frac{3}{z} \sin z + \left(\frac{3}{z^2} - 1 \right) \cos z \right\}.$$

DW

978

The function $N_{N+12}(Z)$

8.465

$$1. \quad N_{n+\frac{1}{2}}(z) = (-1)^{n-1} J_{-n-\frac{1}{2}}(z).$$

JA

$$2. \quad N_{-n-\frac{1}{2}}(z) = (-1)^n J_{n+\frac{1}{2}}(z).$$

JA

The functions $H_{N+12(z)}^{(1,2)}$, $I_{n+\frac{1}{2}}(z)$, $K_{n+\frac{1}{2}}(z)$

8.466

$$1. \quad H_{n-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} i^{-n} e^{iz} \sum_{k=0}^{n-1} (-1)^k \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k} \quad (\text{cf. 8.451 3.}).$$

8.451

$$2. \quad H_{n-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} i^n e^{-iz} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{1}{(2iz)^k} \quad (\text{cf. 8.451 4.}).$$

8.451

8.467

$$I_{\pm(n+\frac{1}{2})}(z) = \frac{1}{\sqrt{2\pi z}} \left[e^z \sum_{k=0}^n \frac{(-1)^k (n+k)!}{k!(n-k)!(2z)^k} \pm (-1)^{n+1} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k} \right] \quad (\text{cf. 8.451 5.}).$$

8.451
KU 60a

8.468

$$K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k} \quad (\text{cf. 8.451 6.}).$$

8.451
KU 60

8.469

Special cases:

$$1. \quad N_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos z.$$

$$2. N_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z.$$

$$3. K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

WA 95(13)

$$4. H_{\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz}}{i}.$$

MO 27

$$5. H_{\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{-iz}}{-i}.$$

MO 27

$$6. H_{-\frac{1}{2}}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{iz}.$$

MO 27

$$7. H_{-\frac{1}{2}}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-iz}.$$

MO 27

979

8.47- 8.48 Functional relations

8.471⁸

Recursion formulas:

$$1. zZ_{\nu-1}(z) + zZ_{\nu+1}(z) = 2\nu Z_{\nu}(z).$$

KU 56(13), WA 56(1), WA 79(1), WA 88(3)

$$2. Z_{\nu-1}(z) - Z_{\nu+1}(z) = 2 \frac{d}{dz} Z_{\nu}(z).$$

Sonin and Nielsen, in their construction of the theory of Bessel functions, defined Bessel functions as analytic functions of z that satisfy the recursion relations 8.471. Z denotes $J, N, H^{(1)}, H^{(2)}$ or any linear combination of these functions, the coefficients of which are independent of z and ν .

8.472

Consequences of the recursion formulas for Z defined as above:

$$1. \quad z \frac{d}{dz} Z_\nu(z) + \nu Z_\nu(z) = z Z_{\nu-1}(z).$$

KU 56(11), WA 56(3), WA 79(3), WA 88(5)

$$2. \quad z \frac{d}{dz} Z_\nu(z) - \nu Z_\nu(z) = -z Z_{\nu+1}(z).$$

KU 56(10), WA 56(4), WA 79(4), WA 88(6)

$$3. \quad \left(\frac{d}{z dz} \right)^m (z^\nu Z_\nu(z)) = z^{\nu-m} Z_{\nu-m}(z).$$

KU 56(8), WA 57(5), WA 89(9)

$$4. \quad \left(\frac{d}{z dz} \right)^m (z^{-\nu} Z_\nu(z)) = (-1)^m z^{-\nu-m} Z_{\nu+m}(z).$$

WA 89(10), Ku 55(5), WA 57(6)

$$5. \quad Z_{-n}(z) = (-1)^n Z_n(z) [n \text{ is a natural number}]. \quad (\text{cf. 8.404}).$$

8.404

8.473

Special cases:

$$1. \quad J_2(z) = \frac{2}{z} J_1(z) - J_0(z).$$

$$2. \quad N_2(z) = \frac{2}{z}N_1(z) - N_0(z).$$

$$3. \quad H_2^{(1,2)}(z) = \frac{2}{z}H_1^{(1,2)}(z) - H_0^{(1,2)}(z).$$

$$4. \quad \frac{d}{dz}J_0(z) = -J_1(z).$$

$$5. \quad \frac{d}{dz}N_0(z) = -N_1(z).$$

$$6. \quad \frac{d}{dz}H_0^{(1,2)}(z) = -H_1^{(1,2)}(z).$$

8.474⁸

Each of the pairs of functions $J_\nu(z)$ and $J_{-\nu}(z)$ (for $\nu \neq 0, \pm 1, \pm 2, \dots$), $J_\nu(z)$ and $N_\nu(z)$, and $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, which are solutions of equation 8.401, and also the pair $I_\nu(z)$ and $K_\nu(z)$ is a pair of linearly independent functions. The Wronskians of these pairs are, respectively,

$$-\frac{2}{\pi z} \sin \nu\pi, \frac{2}{\pi z}, -\frac{4i}{\pi z}, -\frac{1}{z}.$$

KU 52(10, 11, 12), WA 90(1, 4)

980

8.475⁶

The functions $J_\nu(z)$, and $N_\nu(z)$, $H_\nu^{(1,2)}(z)$, $I_\nu(z)$, $K_\nu(z)$ with the exception of $J_n(z)$ and $I_n(z)$ for n an integer are *non-single-valued*: $z = 0$ is a branch point for these functions. The branches of these functions that lie on opposite sides of the cut ($-\infty, 0$) are connected by the relations

8.476

$$1. \quad J_\nu(e^{m\pi i} z) = e^{m\nu\pi i} J_\nu(z).$$

WA 90(1)

WA 90(3)

$$3. \quad N_{-\nu}(e^{m\pi i} z) = e^{-m\nu\pi i} N_{-\nu}(z) + 2i \sin m\nu\pi \operatorname{cosec} \nu\pi J_{\nu}(z).$$

WA 90(4)

$$4. \quad I_{\nu}(e^{m\pi i} z) = e^{m\nu\pi i} I_{\nu}(z).$$

WA 95(17)

$$5. \quad K_{\nu}(e^{m\pi i} z) = e^{-m\nu\pi i} K_{\nu}(z) - i\pi \frac{\sin m\nu\pi}{\sin \nu\pi} I_{\nu}(z) [\nu \text{ not an integer}].$$

WA 95(18)

$$6. \quad H_{\nu}^{(1)}(e^{m\pi i} z) = e^{-m\nu\pi i} H_{\nu}^{(1)}(z) - 2e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_{\nu}(z) = \\ = \frac{\sin(1-m)\nu\pi}{\sin \nu\pi} H_{\nu}^{(1)}(z) - e^{-\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_{\nu}^{(2)}(z).$$

WA 95(5)

$$7. \quad H_{\nu}^{(2)}(e^{m\pi i} z) = e^{-m\nu\pi i} H_{\nu}^{(2)}(z) + 2e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} J_{\nu}(z) = \\ = \frac{\sin(1+m)\nu\pi}{\sin \nu\pi} H_{\nu}^{(2)}(z) + e^{\nu\pi i} \frac{\sin m\nu\pi}{\sin \nu\pi} H_{\nu}^{(1)}(z) \quad [m - \text{an integer}].$$

WA 90(6)

$$8. \quad H_{\nu}^{(1)}(e^{i\pi} z) = -H_{-\nu}^{(2)}(z) = -e^{-i\pi\nu} H_{\nu}^{(2)}(z).$$

MO 26

$$9. \quad H_{\nu}^{(2)}(e^{-i\pi} z) = -H_{-\nu}^{(1)}(z) = -e^{i\pi\nu} H_{\nu}^{(1)}(z).$$

MO 26

$$10. \quad \overline{H_{\nu}^{(2)}}(z) = H_{\nu}^{(1)}(\bar{z}).$$

$$1. \quad J_\nu(z)N_{\nu+1}(z) - J_{\nu+1}(z)N_\nu(z) = -\frac{2}{\pi z}.$$

WA 91(12)

$$2. \quad I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = \frac{1}{z}.$$

WA 95(20)

See also 3.863.

For a connection with Legendre functions, see 8.722.

For a connection with the polynomials $C_n^\lambda(t)$, see 8.936 4.

For a connection with a confluent hypergeometric function, see 9.235.

8.478

For $\nu > 0$ and $x > 0$, the product

$$x[J_\nu^2(x) + N_\nu^2(x)],$$

considered as a function of x , decreases monotonically, if $\nu > \frac{1}{2}$ and increases monotonically if $0 < \nu < \frac{1}{2}$.

981

8.479

$$1. \quad \frac{1}{\sqrt{x^2 - \nu^2}} > \frac{\pi}{2} [J_\nu^2(x) + N_\nu^2(x)] \geq \frac{1}{x} \left[x \geq \nu \geq \frac{1}{2} \right]$$

(see also 6.518, 6.664 4., 8.456).

8.456

6.664

6.518

MO 35

Relations between bessel functions of the first, second, and third kinds

8.481

$$\begin{aligned} J_\nu(z) &= \frac{N_{-\nu}(z) - N_\nu(z) \cos \nu\pi}{\sin \nu\pi} = H_\nu^{(1)}(z) - iN_\nu(z) = \\ &= H_\nu^{(2)}(z) + iN_\nu(z) = \frac{1}{2}(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)) \quad (\text{cf. 8.403 1., 8.405}). \end{aligned}$$

8.405

8.403
WA 89(1), JA

8.482

$$\begin{aligned} N_\nu(z) &= \frac{J_\nu(z) \cos \nu\pi - J_{-\nu}(z)}{\sin \nu\pi} = iJ_\nu(z) - iH_\nu^{(1)}(z) = \\ &= iH_\nu^{(2)}(z) - iJ_\nu(z) = \frac{i}{2}(H_\nu^{(2)}(z) - H_\nu^{(1)}(z)) \quad (\text{cf. 8.403 1., 8.405}). \end{aligned}$$

8.405

8.403
WA 89(3), JA

8.483

$$1. \quad H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\nu\pi i} J_\nu(z)}{i \sin \nu\pi} = \frac{N_{-\nu}(z) - e^{-\nu\pi i} N_\nu(z)}{\sin \nu\pi} = J_\nu(z) + iN_\nu(z).$$

WA 89(5)

$$2. \quad H_\nu^{(2)}(z) = \frac{e^{\nu\pi i} J_\nu(z) - J_{-\nu}(z)}{i \sin \nu\pi} = \frac{N_{-\nu}(z) - e^{\nu\pi i} N_\nu(z)}{\sin \nu\pi} = J_\nu(z) - iN_\nu(z) \quad (\text{cf. 8.405}).$$

8.405
WA 89(6)

8.484

1. $H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_{\nu}^{(1)}(z).$

WA 89(7)

2. $H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_{\nu}^{(2)}(z).$

WA 89(7)

8.485⁷

$$K_{\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \nu\pi} \quad [\nu \text{ not an integer}] \quad (\text{see also 8.407})$$

8.407
WA 92(6)

8.486

Recursion formulas for the functions $I_{\nu}(z)$ and $K_{\nu}(z)$ and their consequences:

1. $zI_{\nu-1}(z) - zI_{\nu+1}(z) = 2\nu I_{\nu}(z).$

WA 93(1)

2. $I_{\nu-1}(z) + I_{\nu+1}(z) = 2 \frac{d}{dz} I_{\nu}(z).$

WA 93(2)

3. $z \frac{d}{dz} I_{\nu}(z) + \nu I_{\nu}(z) = zI_{\nu-1}(z).$

WA 93(3)

4. $z \frac{d}{dz} I_{\nu}(z) - \nu I_{\nu}(z) = zI_{\nu+1}(z).$

WA 93(4)

$$5. \left(\frac{d}{zdz} \right)^m \{z^\nu I_\nu(z)\} = z^{\nu-m} I_{\nu-m}(z).$$

WA 93(5)

$$6. \left(\frac{d}{zdz} \right)^m \{z^{-\nu} I_\nu(z)\} = z^{-\nu-m} I_{\nu+m}(z).$$

WA 93(6)

$$7. I_{-n}(z) = I_n(z) \quad [n - \text{a natural number}].$$

WA 93(8)

$$8. I_2(z) = -\frac{2}{z} I_1(z) + I_0(z).$$

$$9. \frac{d}{dz} I_0(z) = I_1(z).$$

WA 93(7)

$$10. zK_{\nu-1}(z) - zK_{\nu+1}(z) = -2\nu K_\nu(z).$$

WA 93(1)

$$11. K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{d}{dz} K_\nu(z).$$

WA 93(2)

$$12. z\frac{d}{dz} K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z).$$

WA 93(3)

$$13. z\frac{d}{dz} K_\nu(z) - \nu K_\nu(z) = -zK_{\nu+1}(z).$$

$$14. \left(\frac{d}{zdz}\right)^m \{z^\nu K_\nu(z)\} = (-1)^m z^{\nu-m} K_{\nu-m}(z).$$

WA 93(5)

$$15. \left(\frac{d}{zdz}\right)^m \{z^{-\nu} K_\nu(z)\} = (-1)^m z^{-\nu-m} K_{\nu+m}(z).$$

WA 93(6)

$$16. K_{-\nu}(z) = K_\nu(z).$$

WA 93(8)

$$17. K_2(z) = \frac{2}{z} K_1(z) + K_0(z).$$

$$18. \frac{d}{dz} K_0(z) = -K_1(z).$$

WA 93(7)

8.486(1)⁷

Differentiation with respect to order

$$1. \frac{\partial J_\nu(z)}{\partial \nu} = J_\nu(z) \ln\left(\frac{1}{2}z\right) - \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k! \Gamma(\nu+k+1)}.$$

MS 3.1.3

$$2. \frac{\partial J_{-\nu}(z)}{\partial \nu} = -J_{-\nu}(z) \ln\left(\frac{1}{2}z\right) + \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2}z\right)^{-\nu+2k} \frac{\psi(-\nu+k+1)}{k! \Gamma(-\nu+k+1)}$$

MS 3.1.3

$$3. \frac{\partial N_\nu(z)}{\partial \nu} = \operatorname{ctg} \pi \nu \frac{\partial J_\nu(z)}{\partial \nu} - \operatorname{cosec} \pi \nu \frac{\partial J_{-\nu}(z)}{\partial \nu} - \pi \operatorname{cosec} \pi \nu N(z).$$

$$4. \quad \frac{\partial I_\nu(z)}{\partial \nu} = I_\nu(z) \ln \left(\frac{1}{2} z \right) - \sum_{k=0}^{\infty} \left(\frac{1}{2} z \right)^{\nu+2k} \frac{\psi(\nu+k+1)}{k! \Gamma(\nu+k+1)}$$

MS 3.1.3

$$5. \quad \frac{\partial K_\nu(z)}{\partial \nu} = -\pi \operatorname{ctg} \pi \nu K_\nu(z) + \frac{1}{2} \pi \operatorname{cosec} \pi \nu \left[\frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu} \right].$$

MS 3.1.3

$$6. \quad \left[\frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \frac{1}{2} \pi (\pm 1)^n N_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2} z \right)^{k-n} J_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots]$$

MS 3.2.3

$$7. \quad \left[\frac{\partial N_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = -\frac{1}{2} \pi (\pm 1)^n J_n(z) \pm (\pm 1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2} z \right)^{k-n} N_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots]$$

MS 3.2.3

$$8. \quad \left[\frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = (-1)^{n+1} K_n(z) \pm (-1)^n \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2} z \right)^{k-n} I_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots]$$

MS 3.2.3

$$9. \quad \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=\pm n} = \pm \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2} z \right)^{k-n} K_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots]$$

MS 3.2.3

$$10. \quad (-1)^n \left[\frac{\partial}{\partial \nu} I_\nu(z) \right]_{\nu=n} = -K_n(z) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k \left(\frac{1}{2} z \right)^{k-n} I_k(z)}{k!(n-k)} \quad [n = 0, 1, \dots]$$

$$11. \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=n} = \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}z\right)^{k-n} I_k(z)}{k!(n-k)}. \quad [n = 0, 1, \dots]$$

Special cases

$$12. \left[\frac{\partial J_\nu(z)}{\partial \nu} \right]_{\nu=0} = \frac{1}{2} \pi N_0(z).$$

MS 3.2.3

$$13. \left[\frac{\partial N_\nu(z)}{\partial \nu} \right]_{\nu=0} = -\frac{1}{2} \pi J_0(z).$$

MS 3.2.3

$$14. \left[\frac{\partial I_\nu(z)}{\partial \nu} \right]_{\nu=0} = -K_0(z).$$

MS 3.2.3

984

$$15. \left[\frac{\partial K_\nu(z)}{\partial \nu} \right]_{\nu=0} = 0.$$

MS 3.2.3

$$16. \left[\frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} [\sin x \operatorname{Ci}(3x) - \cos x \operatorname{Si}(2x)].$$

MS 3.3.3

$$17. \left[\frac{\partial J_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = \left(\frac{1}{2}\pi x\right)^{-\frac{1}{2}} [\cos x \operatorname{Ci}(2x) + \sin x \operatorname{Si}(2x)].$$

$$18. \left[\frac{\partial N_\nu(x)}{\partial \nu} \right]_{\nu=\frac{1}{2}} = \left(\frac{1}{2} \pi x \right)^{-\frac{1}{2}} \{ \cos x \operatorname{Ci}(2x) + \sin x [\operatorname{Si}(2x) - \pi] \}.$$

MS 3.3.3

$$19. \left[\frac{\partial N_\nu(x)}{\partial \nu} \right]_{\nu=-\frac{1}{2}} = - \left(\frac{1}{2} \pi x \right)^{-\frac{1}{2}} \{ \sin x \operatorname{Ci}(2x) - \cos x [\operatorname{Si}(2x) - \pi] \}.$$

MS 3.3.3

$$20. \left[\frac{\partial I_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = (2\pi x)^{-\frac{1}{2}} [e^x \operatorname{Ei}(-2x) \mp e^{-x} \operatorname{Ei}(2x)].$$

MS 3.3.3

$$21. \left[\frac{\partial K_\nu(x)}{\partial \nu} \right]_{\nu=\pm\frac{1}{2}} = \mp \left(\frac{\pi}{2x} \right)^{\frac{1}{2}} e^x \operatorname{Ei}(-2x).$$

MS 3.3.3

8.487

Continuity with respect to the order* The continuity of the functions $J_\nu(z)$ and $I_\nu(z)$ follows directly from the series representations of these functions.

:

$$\left. \begin{array}{l} 1. \lim_{\nu \rightarrow n} N_\nu(z) = N_n(z) \\ 2. \lim_{\nu \rightarrow n} H_\nu^{(1,2)}(z) = H_n^{(1,2)}(z) \\ 3. \lim_{\nu \rightarrow n} K_\nu(z) = K_n(z) \end{array} \right\} [n - \text{an integer}].$$

WA 183
WA 76

8.49 Differential equations leading to Bessel functions

See also 8.401

8.491

$$1. \frac{1}{z} \frac{d}{dz} (zu') + \left(\beta^2 - \frac{\nu^2}{z^2} \right) u = 0, \quad u = Z_\nu(\beta z).$$

$$2. \quad \frac{1}{z} \frac{d}{dz} (zu') + \left[(\beta\gamma z^{\gamma-1})^2 - \left(\frac{\nu\gamma}{z} \right)^2 \right] u = 0, \quad u = Z_\nu(\beta z^\gamma).$$

JA

$$3. \quad u'' + \frac{1-2\alpha}{z} u' + \left[(\beta\gamma z^{\gamma-1})^2 + \frac{\alpha^2 - \nu^2 \gamma^2}{z^2} \right] u = 0, \quad u = z^\alpha Z_\nu(\beta z^\gamma).$$

JA

$$4. \quad u'' + \left[(\beta\gamma z^{\gamma-1})^2 - \frac{4\nu^2 \gamma^2 - 1}{4z^2} \right] u = 0, \quad u = \sqrt{z} Z_\nu(\beta z^\gamma).$$

JA

$$5. \quad u'' + \left(\beta^2 - \frac{4\nu^2 - 1}{4z^2} \right) u = 0, \quad u = \sqrt{z} Z_\nu(\beta z).$$

JA

$$6. \quad u'' + \frac{1-2\alpha}{z} u' + \left(\beta^2 + \frac{\alpha^2 - \nu^2}{z^2} \right) u = 0, \quad u = z^\alpha Z_\nu(\beta z).$$

JA

$$7. \quad u'' + bz^m u = 0, \quad u = \sqrt{z} Z_{\frac{m+2}{2}} \left(\frac{2\sqrt{b}}{m+2} z^{\frac{m+2}{2}} \right).$$

JA 111(5)

$$8. \quad u'' + \frac{1}{z} u' + 4 \left(z^2 - \frac{\nu^2}{z^2} \right) u = 0, \quad u = Z_\nu(z^2).$$

WA 111(6)

$$9. \quad u'' + \frac{1}{z} u' + \frac{1}{4z} \left(1 - \frac{\nu^2}{z} \right) u = 0, \quad u = Z_\nu(\sqrt{z}).$$

$$10. \quad u'' + \frac{1-\nu}{z}u' + \frac{1}{4z}u = 0, \quad u = z^{\frac{\nu}{2}}Z_{\nu}(\sqrt{z}).$$

WA 111(9)a

$$11. \quad u'' + \beta^2\gamma^2z^{2\beta-2}u = 0, \quad u = z^{\frac{1}{2}}Z_{\frac{1}{2\beta}}(\gamma z^{\beta}).$$

WA 110(3)

$$12. \quad z^2u'' + (2\alpha - 2\beta\nu + 1)zu' + [\beta^2\gamma^2z^{2\beta} + \alpha(\alpha - 2\beta\nu)]u = 0, \quad u = z^{\beta\nu - \alpha}Z_{\nu}(\gamma z^{\beta}).$$

WA 112(21)

8.492

$$1. \quad u'' + (e^{2z} - \nu^2)u = 0, \quad u = Z_{\nu}(e^z).$$

WA 112(22)

$$2. \quad u'' + \frac{e^{\frac{2}{z}} - \nu^2}{z^4}u = 0, \quad u = zZ_{\nu}\left(e^{\frac{1}{z}}\right).$$

WA 112(22)

8.493

$$1. \quad u'' + \left(\frac{1}{z} - 2\operatorname{tg} z\right)u' - \left(\frac{\nu^2}{z^2} + \frac{\operatorname{tg} z}{z}\right)u = 0, \quad u = \sec zZ_{\nu}(z).$$

JA

$$2. \quad u'' + \left(\frac{1}{z} + 2\operatorname{ctg} z\right)u' - \left(\frac{\nu^2}{z^2} - \frac{\operatorname{ctg} z}{z}\right)u = 0, \quad u = \operatorname{cosec} zZ_{\nu}(z).$$

JA

986

8.494

$$1. \quad u'' + \frac{1}{z}u' - \left(1 + \frac{\nu^2}{z^2}\right)u = 0, \quad u = Z_{\nu}(iz) = C_1I_{\nu}(z) + C_2K_{\nu}(z).$$

$$2. \quad u'' + \frac{1}{z}u' - \left[\frac{1}{z} + \left(\frac{\nu}{2z} \right)^2 \right] u = 0, \quad u = Z_\nu(2i\sqrt{z}).$$

JA

$$3. \quad u'' + u' + \frac{1}{z^2} \left(\frac{1}{4} - \nu^2 \right) u = 0, \quad u = \sqrt{z} e^{-\frac{z}{2}} Z_\nu \left(\frac{iz}{2} \right).$$

JA

$$4. \quad u'' + \left(\frac{2\nu + 1}{z} - k \right) u' - \frac{2\nu + 1}{2z} k u = 0, \quad u = z^{-\nu} e^{\frac{1}{2}kz} Z_\nu \left(\frac{ikz}{2} \right).$$

JA

$$5. \quad u'' + \frac{1-\nu}{z}u' - \frac{1}{4} \frac{u}{z} = 0, \quad u = z^{\frac{\nu}{2}} Z_\nu(i\sqrt{z}).$$

WA 111(8)

$$6. \quad u'' \pm \frac{u}{\sqrt{z}} = 0, \quad u = \sqrt{z} Z_{\frac{2}{3}} \left(\frac{4}{3} z^{\frac{3}{4}} \right), \quad \sqrt{z} Z_{\frac{2}{3}} \left(\frac{4}{3} iz^{\frac{3}{4}} \right).$$

WA 111(10)

$$7. \quad u'' \pm zu = 0, \quad u = \sqrt{z} Z_{\frac{1}{3}} \left(\frac{2}{3} z^{\frac{3}{2}} \right), \quad \sqrt{z} Z_{\frac{1}{3}} \left(\frac{2}{3} iz^{\frac{3}{2}} \right).$$

WA 111(10)

$$8. \quad u'' - \left(c^2 + \frac{\nu(\nu+1)}{z^2} \right) u = 0, \quad u = \sqrt{z} Z_{\nu+\frac{1}{2}}(icz).$$

WA 108(1)

$$9. \quad u'' - \frac{2\nu}{z}u' - c^2u = 0, \quad u = z^{\nu+\frac{1}{2}} Z_{\nu+\frac{1}{2}}(icz).$$

WA 109(3, 4)

$$10. \quad u'' - c^2 z^{2\nu-2} u = 0, \quad u = \sqrt{z} Z_{\frac{1}{2\nu}} \left(i \frac{c}{\nu} z^\nu \right).$$

$$1. \quad u'' + \frac{1}{z}u' + \left(i - \frac{\nu^2}{z^2}\right)u = 0, \quad u = Z_\nu(z\sqrt{i}).$$

JA

$$2. \quad u'' + \left(\frac{1}{z} \mp 2i\right)u' - \left(\frac{\nu^2}{z^2} \pm \frac{i}{z}\right)u = 0, \quad u = e^{\pm iz} Z_\nu(z).$$

JA

$$3. \quad u'' + \frac{1}{z}u' + se^{i\alpha}u = 0, \quad u = Z_0\left(\sqrt{s}ze^{\frac{i}{2}\alpha}\right).$$

JA

$$4. \quad u'' + \left(se^{i\alpha} + \frac{1}{4z^2}\right)u = 0, \quad u = \sqrt{z}Z_0\left(\sqrt{s}ze^{\frac{i}{2}\alpha}\right).$$

JA

8.496

$$1. \quad \frac{d^2}{dz^2}\left(z^4 \frac{d^2u}{dz^2}\right) - z^2u = 0, \quad u = \frac{1}{z}\{Z_2(2\sqrt{z}) + \overline{Z_2(2i\sqrt{z})}\}.$$

WA 122(7)

987

$$2. \quad \frac{d^2}{dz^2}\left(z^{\frac{16}{5}} \frac{d^2u}{dz^2}\right) - z^{\frac{8}{5}}u = 0, \quad u = z^{-\frac{7}{10}}\left\{Z_{\frac{5}{6}}\left(\frac{5}{3}z^{\frac{3}{5}}\right) + \overline{Z_{\frac{5}{6}}\left(\frac{5}{3}iz^{\frac{3}{5}}\right)}\right\}.$$

WA 122(8)

$$3. \quad \frac{d^2}{dz^2}\left(z^{12} \frac{d^2u}{dz^2}\right) - z^6u = 0, \quad u = z^{-4}\{Z_{10}(2z^{-\frac{1}{2}}) + \overline{Z_{10}(2iz^{-\frac{1}{2}})}\}.$$

WA 122(9)

$$4. \quad \frac{d^4u}{dz^4} + \frac{2}{z} \frac{d^3u}{dz^3} - \frac{2\nu^2 + 1}{z^2} \frac{d^2u}{dz^2} + \frac{2\nu^2 + 1}{z^3} \frac{du}{dz} + \left(\frac{\nu^4 - 4\nu^2}{z^4} - 1\right)u = 0,$$

$u = A_1J_\nu(z) + A_2N_\nu(z) + A_3I_\nu(z) + A_4K_\nu(z)$, where A_1, A_2, A_3, A_4 are constants

8.51- 8.52 Series of Bessel functions

8.511

Generating function for Bessel functions:

$$1. \quad \exp \frac{1}{2} \left(t - \frac{1}{t} \right) z = J_0(z) + \sum_{k=1}^{\infty} [t^k + (-t)^{-k}] J_k(z) = \sum_{k=-\infty}^{\infty} J_k(z) t^k \quad [|z| < |t|].$$

KU 119(12)

$$2. \quad \exp \left(t - \frac{1}{t} \right) z = \left\{ \sum_{k=-\infty}^{\infty} t^k J_k(z) \right\} \left\{ \sum_{m=-\infty}^{\infty} t^m J_m(z) \right\}.$$

WA 40

$$3. \quad \exp(\pm iz \sin \varphi) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\varphi \pm 2i \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\varphi.$$

KU 120(13)

$$\begin{aligned} 4. \quad \exp(iz \cos \varphi) &= \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} (2k+1) i^k J_{k+\frac{1}{2}}(z) P_k(\cos \varphi); \\ &= \sum_{k=-\infty}^{\infty} i^k J_k(z) e^{ik\varphi}; \\ &= J_0(z) + 2 \sum_{k=1}^{\infty} i^k J_k(z) \cos k\varphi. \end{aligned}$$

MO 27
MO 27
WA 401(1)

$$5. \quad \sqrt{\frac{i}{\pi}} e^{iz \cos 2\varphi} \int_{-\infty}^{\sqrt{2z} \cos \varphi} e^{-it^2} dt = \frac{1}{2} J_0(z) + \sum_{k=1}^{\infty} e^{\frac{1}{4}k\pi i} J_{\frac{k}{2}}(z) \cos k\varphi.$$

MO 28

The series $\sum J_k(z)$

8.512

$$1. \quad J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) = 1.$$

WA 44

$$2. \quad \sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!}{k!} J_{n+2k}(z) = \left(\frac{z}{2}\right)^n \quad [n = 1, 2, \dots].$$

WA 45

$$3. \quad \sum_{k=0}^{\infty} \frac{(4k+1)(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) = \sqrt{2z/\pi}.$$

8.513

$$1. \quad \sum_{k=1}^{\infty} (2k)^{2p} J_{2k}(z) = \sum_{k=0}^p Q_{2k}^{(2p)} z^{2k} \quad [p = 1, 2, 3, \dots].$$

WA 46(1)

988

$$2. \quad \sum_{k=0}^{\infty} (2k+1)^{2p+1} J_{2k+1}(z) = \sum_{k=0}^p Q_{2k+1}^{(2p+1)} z^{2k+1} \quad [p = 0, 1, 2, 3, \dots].$$

WA 46(2)

$$\left[\text{In formulas 8.513} \quad Q_k^{(p)} = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^m \binom{k}{m} (k-2m)^p}{2^k k!} \right].$$

8.513

In particular:

$$3. \quad \sum_{k=0}^{\infty} (2k+1)^3 J_{2k+1}(z) = \frac{1}{2}(z+z^3).$$

$$4. \sum_{k=1}^{\infty} (2k)^2 J_{2k}(z) = \frac{1}{2} z^2.$$

WA 47(4)

$$5. \sum_{k=1}^{\infty} 2k(2k+1)(2k+2) J_{2k+1}(z) = \frac{1}{2} z^3.$$

WA 47(4)

8.514

$$1. \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) = \frac{\sin z}{2}.$$

WH

$$2. J_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(z) = \cos z.$$

WH

$$3. \sum_{k=1}^{\infty} (-1)^{k+1} (2k)^2 J_{2k}(z) = \frac{z \sin z}{2}.$$

WA 32(9)

$$4. \sum_{k=0}^{\infty} (-1)^k (2k+1)^2 J_{2k+1}(z) = \frac{z \cos z}{2}.$$

WA 32(10)

$$5. J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos 2k\theta = \cos(z \sin \theta).$$

KU 120(14), WA 32

$$6. \sum_{k=0}^{\infty} J_{2k+1}(z) \sin(2k+1)\theta = \frac{\sin(z \sin \theta)}{2}.$$

$$7. \sum_{k=0}^{\infty} J_{2k+1}(x) = \frac{1}{2} \int_0^x J_0(t) dt \quad [x \text{ real}].$$

WA 638

8.515

$$1. \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \left(\frac{2z+t}{2z} \right)^k J_{\nu+k}(z) = \left(\frac{z}{z+t} \right)^{\nu} J_{\nu}(z+t).$$

AD (9140)

$$2. \sum_{k=1}^{\infty} J_{2k-\frac{1}{2}}(x^2) = S(x).$$

MO 127a

$$3. \sum_{k=0}^{\infty} J_{2k+\frac{1}{2}}(x^2) = C(x).$$

MO 127a

989

8.516

$$\sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{2n+2k}(2z \sin \theta) = (z \sin \theta)^{2n}$$

WA 47

The series $\sum a_k J_k(kx)$ and $\sum a_k J'_k(kx)$

8.517

$$\left. \begin{array}{l} 1. \sum_{k=1}^{\infty} J_k(kz) = \frac{z}{2(1-z)} \\ 2. \sum_{k=1}^{\infty} (-1)^k J_k(kz) = -\frac{z}{2(1+z)} \\ 3. \sum_{k=1}^{\infty} J_{2k}(2kz) = \frac{z^2}{2(1-z^2)} \end{array} \right\} \left[\left| \frac{z \exp \sqrt{1-z^2}}{1 + \sqrt{1-z^2}} \right| < 1 \right].$$

$$1. \sum_{k=1}^{\infty} \frac{J'_k(kx)}{k} = \frac{1}{2} + \frac{x}{4} \quad [0 \leq x < 1].$$

MO 58

$$2. \sum_{k=1}^{\infty} (-1)^{k-1} \frac{J'_k(kx)}{k} = \frac{1}{2} - \frac{x}{4} \quad [0 \leq x < 1].$$

MO 58

$$3. \sum_{k=1}^{\infty} k J'_k(kx) = \frac{1}{2(1-x)^2} \quad [0 \leq x < 1].$$

MO 58

$$4. \sum_{k=1}^{\infty} (-1)^{k-1} J'_k(kx) k = \frac{1}{2(1+x)^2} \quad [0 \leq x < 1].$$

MO 58

The series $\sum A_K J_0(Kx)$

8.519

If, on the interval $[0 \leq x \leq \pi]$, a function $f(x)$ possesses a continuous derivative with respect to x that is of bounded variation, then

$$1. f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k J_0(kx) \quad [0 < x < \pi],$$

where

$$2. a_0 = 2f(0) + \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin \varphi) d\varphi.$$

$$3. a_n = \frac{2}{\pi} \int_0^{\pi} du \int_0^{\frac{\pi}{2}} u f'(u \sin \varphi) \cos nud\varphi.$$

WH

8.521

Examples:

$$1. \sum_{k=1}^{\infty} J_0(kx) = -\frac{1}{2} + \frac{1}{x} + 2 \sum_{m=1}^n \frac{1}{\sqrt{x^2 - 4m^2\pi^2}} \quad [2n\pi < x < 2(n+1)\pi].$$

MO 59

$$2. \sum_{k=1}^{\infty} (-1)^{k+1} J_0(kx) = \frac{1}{2} \quad [0 < x < \pi].$$

KU 124(12)

990

$$3. \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} J_0\left\{(2k-1)x\right\} \frac{\pi^2}{8} - \frac{|x|}{2} \quad [-\pi < x < \pi];$$

$$= \frac{\pi^2}{8} + \sqrt{x^2 - \pi^2} - \frac{x}{2} - \pi \arccos \frac{\pi}{x} \quad [\pi < x < 2\pi].$$

MO 59
KU 124

$$4. \sum_{k=1}^{\infty} e^{-kz} J_0(k\sqrt{x^2 + y^2}) = \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{1}{\sqrt{(2ki\pi + z)^2 + x^2 + y^2}} - \frac{1}{\sqrt{(2ki\pi - z)^2 + x^2 + y^2}} \right\};$$

$$= \frac{1}{r} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} B_{2k} r^{2k-1} P_{2k-1} \left(\frac{z}{r} \right) \quad [0 < r < 2\pi],$$

MO 59

where $r = \sqrt{x^2 + y^2 + z^2}$ and where the radical indicates the square root with a positive real part. In formula 8.521 4., the first equation holds when x and y are real and $\operatorname{Re} z > 0$; the second equation holds when x , y , and z are all real.

The series $\sum a_k Z_0(kx) \sin kx$ and $\sum a_k Z_0(kx) \cos kx$

8.522

$$1. \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - (2\pi l + tx)^2}} + \frac{1}{x\sqrt{1-t^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - (2\pi l - tx)^2}}.$$

$$2. \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}.$$

MO 59

$$3. \sum_{k=1}^{\infty} N_0(kx) \cos kxt = -\frac{1}{\pi} \left(\mathbf{C} + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l + tx)^2 - x^2}} - \frac{1}{2\pi l} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} - \frac{1}{2\pi l} \right\}.$$

MO 60

In formulas 8.522, $x > 0$, $0 \leq t < 1$, $2\pi m < x(1-t) < 2(m+1)\pi$, $2n\pi < x(1+t) < 2(n+1)\pi$, $m+1$ and $n+1$ are natural numbers.

8.523

$$1. \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=1}^m \frac{1}{\sqrt{x^2 - [(2l-1)\pi + tx]^2}} + \sum_{l=1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}.$$

MO 60

991

$$2. \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \frac{1}{2\pi} \left\{ \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^m \frac{1}{l} \right\} + \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}.$$

MO 60

$$3. \sum_{k=1}^{\infty} (-1)^k N_0(kx) \cos kxt = -\frac{1}{\pi} \left(\mathbf{C} + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \left\{ \sum_{l=1}^m \frac{1}{l} + \sum_{l=1}^n \frac{1}{l} \right\} - \sum_{l=m+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}.$$

$$x > 0, 0 \leq t < 1 \quad (2m-1)\pi < x(1-t) < (2m+1)\pi \quad (2n-1)\pi < x(1+t) < (2n+1)\pi \quad m \quad n$$

8.524⁶

$$1. \quad \sum_{k=1}^{\infty} J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - (2l\pi - tx)^2}}.$$

MO 60

$$2. \quad \sum_{k=1}^{\infty} J_0(kx) \sin kxt = \sum_{l=0}^m \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l}.$$

MO 60

$$3. \quad \sum_{k=1}^{\infty} N_0(kx) \cos kxt = -\frac{1}{\pi} \left(\mathbf{C} + \ln \frac{x}{4\pi} \right) - \sum_{l=0}^m \frac{1}{\sqrt{(2\pi l - tx)^2 - x^2}} + \\ + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi + tx)^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{(2l\pi - tx)^2 - x^2}} - \frac{1}{2l\pi} \right\}.$$

MO 61

In formulas 8.524, $x > 0, t > 1, 2m\pi < x(t-1) < 2(m+1)\pi, 2n\pi < x(t+1) < 2(n+1)\pi, m+1$ and $n+1$ are natural numbers.

8.525

$$1. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \cos kxt = -\frac{1}{2} + \sum_{l=m+1}^n \frac{1}{\sqrt{x^2 - [(2l-1)\pi - tx]^2}}.$$

MO 61

992

$$2. \quad \sum_{k=1}^{\infty} (-1)^k J_0(kx) \sin kxt = \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} + \\ + \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\ - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}$$

$$\begin{aligned}
3. \quad \sum_{k=1}^{\infty} (-1)^k N_0(kx) \cos kxt &= -\frac{1}{\pi} \left(C + \ln \frac{x}{4\pi} \right) + \frac{1}{2\pi} \sum_{l=1}^n \frac{1}{l} - \\
&\quad - \sum_{l=1}^m \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \\
&\quad - \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi + tx]^2 - x^2}} - \frac{1}{2l\pi} \right\} - \\
&\quad - \sum_{l=n+1}^{\infty} \left\{ \frac{1}{\sqrt{[(2l-1)\pi - tx]^2 - x^2}} - \frac{1}{2l\pi} \right\}.cr
\end{aligned}$$

MO 61

In formulas 8.525, $x > 0, t > 1, (2m-1)\pi < x(t-1) < (2m+1)\pi, (2n-1)\pi < x(t+1) < (2n+1)\pi$, m and n are natural numbers.

8.526

$$\begin{aligned}
1. \quad \sum_{k=1}^{\infty} K_0(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x\sqrt{1+t^2}} + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi - tx)^2}} - \frac{1}{2l\pi} \right\} + \\
&\quad + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2l\pi + tx)^2}} - \frac{1}{2l\pi} \right\}.
\end{aligned}$$

MO 61

$$\begin{aligned}
2. \quad \sum_{k=1}^{\infty} (-1)^k K_0(kx) \cos kxt &= \frac{1}{2} \left(C + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi - xt]^2}} - \frac{1}{2l\pi} \right\} + \\
&\quad + \frac{\pi}{2} \sum_{l=1}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + [(2l-1)\pi + xt]^2}} - \frac{1}{2l\pi} \right\} \quad [x > 0, \quad t \text{ real}], \\
&\quad \text{(see also 8.66).}
\end{aligned}$$

8.66

MO 62

8.53 Expansion in products of Bessel functions

8.530

Suppose that $r > 0, \varrho > 0, \varphi > 0$, and $R = \sqrt{r^2 + \varrho^2 - 2r\varrho \cos \varphi}$; that is, suppose that r, ϱ , and R are the sides of a triangle such that the angle between the sides r and ϱ is equal to φ . Suppose also that

$\varrho < r$ and that ψ is the angle opposite the side ϱ , so that

$$1. \quad 0 < \psi < \frac{\pi}{2}, \quad e^{2i\psi} = \frac{r - \varrho e^{-i\varphi}}{r - \varrho e^{i\varphi}}.$$

When these conditions are satisfied, we have the "summation theorem" for Bessel functions:

$$2. \quad e^{i\nu\psi} Z_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\varrho) Z_{\nu+k}(mr) e^{ik\varphi} \quad [m\text{—an arbitrary complex number }].$$

WA 394(6)

For $Z_\nu = J_\nu$ and ν an integer, the restriction $\varrho < r$ is superfluous.

8.531

Special cases:

$$1. \quad J_0(mR) = J_0(m\varrho)J_0(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)J_k(mr) \cos k\varphi.$$

WA 391(1)

MO 31

$$2. \quad H_0^{(1,2)}(mR) = J_0(m\varrho)H_0^{(1,2)}(mr) + 2\sum_{k=1}^{\infty} J_k(m\varrho)H_k^{(1,2)}(mr) \cos k\varphi.$$

MO 31

$$\begin{aligned} 3. \quad J_0(z \sin \alpha) &= J_0^2\left(\frac{z}{2}\right) + 2\sum_{k=1}^{\infty} J_k^2\left(\frac{z}{2}\right) \cos 2k\alpha; \\ &= \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \left(2k + \frac{1}{2}\right) \frac{(2k-1)!!}{2^k k!} J_{2k+\frac{1}{2}}(z) P_{2k}(\cos \alpha). \end{aligned}$$

MO 31

8.532

The term "summation theorem" is also applied to the formula

$$1. \quad \frac{Z_\nu(mR)}{R^\nu} = 2^\nu m^{-\nu} \Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) \frac{J_{\nu+k}(m\varrho)}{\varrho^\nu} \frac{Z_{\nu+k}(mr)}{r^\nu} C_k^\nu(\cos \varphi).$$

$[\nu \neq -1, -2, -3, \dots]$; the conditions on $r, \varrho, R, \varphi,$ and m are the same as in formula 8.530; for $Z_\nu = J_\nu$ and ν an integer, formula 8.532 1 is valid for arbitrary $r, \varrho,$ and φ].

8.533

Special cases:

$$1. \quad \frac{e^{imR}}{R} = \frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(1)}(mr) P_k(\cos \varphi).$$

MO 31
WA 398(4)

$$2. \quad \frac{e^{-imR}}{R} = -\frac{\pi i}{2\sqrt{r\varrho}} \sum_{k=0}^{\infty} (2k+1) J_{k+\frac{1}{2}}(m\varrho) H_{k+\frac{1}{2}}^{(2)}(mr) P_k(\cos \varphi).$$

MO 31

8.534

A degenerate addition theorem ($r \rightarrow \infty$):

$$\begin{aligned} e^{im\varrho \cos \varphi} &= \sqrt{\frac{\pi}{2m\varrho}} \sum_{k=0}^{\infty} i^k (2k+1) J_{k+\frac{1}{2}}(m\varrho) P_k(\cos \varphi); \\ &= 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) i^k (m\varrho)^{-\nu} J_{\nu+k}(m\varrho) C_k^\nu(\cos \varphi) \quad [\nu \neq 0, -1, -2, \dots]. \end{aligned}$$

WA 401(2)
WA 401(1)

994

8.535

The term "product theorem" is also applied to the formula

For $Z_\nu = J_\nu$, it is valid for all values of λ and z .

8.536

$$1. \sum_{k=0}^{\infty} \frac{(2n+2k)(2n+k-1)!}{k!} J_{n+k}^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \quad [n > 0].$$

WA 47(1)
MO 32

$$2. 2 \sum_{k=n}^{\infty} \frac{k\Gamma(n+k)}{\Gamma(k-n+1)} J_k^2(z) = \frac{(2n)!}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \quad [n > 0].$$

WA 47(2)

$$3. J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z) = 1.$$

WA 41(3)

8.537

$$\sum_{k=-\infty}^{\infty} Z_{\nu-k}(t) J_k(z) = Z_\nu(z+t) \quad [|z| < |t|].$$

WA 158(2)

In particular:

$$\sum_{k=-\infty}^{\infty} J_k(z) J_{n-k}(z) = J_n(2z).$$

WA 41

8.538

$$1. \sum_{k=-\infty}^{\infty} (-1)^k J_{-\nu+k}(t) J_k(z) = J_{-\nu}(z+t) \quad [|z| < |t|].$$

$$2. \sum_{k=-\infty}^{\infty} Z_{\nu+k}(t) J_k(z) = Z_{\nu}(t-z) \quad [|z| < |t|].$$

WA 159(5)

8.54 The zeros of Bessel functions

8.541

For arbitrary real ν , the function $J_{\nu}(z)$ has infinitely many real zeros. For $\nu > -1$, all its zeros are real.

A Bessel function $Z_{\nu}(z)$ has no multiple zeros except possibly the coordinate origin.

WA 528

8.542

All zeros of the function $N_0(z)$ with positive real parts are real.

WA 531

8.543

If $-(2s+2) < \nu < -(2s+1)$, where s is a natural number or 0, then $J_{\nu}(z)$ has exactly $4s+2$ complex roots, two of which are purely imaginary. If $-(2s+1) < \nu < -2s$, where s is a natural number, then the function $J_{\nu}(z)$ has exactly $4s$ complex zeros none of which are purely imaginary.

WA 532

995

8.544

If x_{ν} and x'_{ν} are, respectively, the smallest positive zeros of the functions $J_{\nu}(z)$ and $J'_{\nu}(z)$ for $\nu > 0$, then $x_{\nu} > \nu$ and $x'_{\nu} > \nu$.

Suppose also that y_{ν} is the smallest positive zero of the function $N_{\nu}(z)$. Then, $x_{\nu} < y_{\nu} < x'_{\nu}$.

WA 534, 536

Suppose that $z_{\nu,m}$ (for $m = 1, 2, 3, \dots$) are the zeros of the function $z^{-\nu} J_{\nu}(z)$, numbered in order of the absolute value of their real parts. Here, we assume that $\nu \neq -1, -2, -3, \dots$. Then, for arbitrary z

$$J_{\nu}(z) = \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(\nu+1)} \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{z_{\nu,m}^2}\right).$$

WA 550
WA 526, 530

8.545

The number of zeros of the function $z^{-\nu} J_\nu(z)$ that occur between the imaginary axis and the line on which

$$\operatorname{Re} z = \left(m + O\left(\frac{1}{m}\right) \right) \pi + \left[\frac{1}{2} \operatorname{Re} \nu + \frac{1}{4} \right] \pi,$$

WA 548

is $m + O\left(\frac{1}{m}\right)$, for large values of m .

8.546

For $\nu \geq 0$, the number of zeros of the function $K_\nu(z)$ that occur in the region $\operatorname{Re} z < 0$, $|\arg z| < \pi$ is equal to the even number closest to $\nu - \frac{1}{2}$.

8.547

Large zeros of the functions $J_\nu(z) \cos \alpha - N_\nu(z) \sin \alpha$, where ν and α are real numbers, are given by the asymptotic expansion

$$\begin{aligned} x_{\nu,m} \sim & \left(m + \frac{1}{2}\nu - \frac{1}{4} \right) \pi - \alpha - \frac{4\nu^2 - 1}{8 \left[\left(m + \frac{1}{2}\nu - \frac{1}{4} \right) \pi - \alpha \right]} - \\ & - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384 \left[\left(m + \frac{1}{2}\nu - \frac{1}{4} \right) \pi - \alpha \right]^3} - \dots \end{aligned}$$

KU 109(24), WA 558
WA 562

8.548

In particular, large zeros of the function $J_0(z)$ are given by the expansion

$$x_{0,m} \sim \frac{\pi}{4}(4m - 1) + \frac{1}{2\pi(4m - 1)} - \frac{31}{6\pi^3(4m - 1)^3} + \frac{3779}{15\pi^5(4m - 1)^5} - \dots$$

KU 109(25), WA 556

This series is suitable for calculating all (except the smallest x_{01}) zeros of the function $J_0(z)$ correctly to at least five digits.

996

8.549

To calculate the roots $x_{\nu,m}$ of the function $J_{\nu}(z)$ of smallest absolute value, we may use the identity

$$\sum_{m=1}^{\infty} \frac{1}{x_{\nu,m}^{16}} = \frac{429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738}{2^{16}(\nu+1)^8(\nu+2)^4(\nu+3)^2(\nu+4)^2(\nu+5)(\nu+6)(\nu+7)(\nu+8)}.$$

KU 112(27)a
WA 554

8.55 Struve functions

8.550

Definitions:

$$1. \quad \mathbf{H}_{\nu}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)}.$$

WA 358(2)

$$2. \quad \mathbf{L}_{\nu}(z) = -ie^{-i\nu\frac{\pi}{2}} \mathbf{H}_{\nu}(ze^{i\frac{\pi}{2}}) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu+1}}{\Gamma\left(m + \frac{3}{2}\right) \Gamma\left(\nu + m + \frac{3}{2}\right)}.$$

WA 360(11)

8.551

Integral representations:

$$1. \quad \mathbf{H}_{\nu}(z) = \frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin zt dt = \frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi$$

$$\left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

WA 358(1)

$$2. \quad \mathbf{L}_{\nu}(z) = \frac{2\left(\frac{z}{2}\right)^{\nu}}{\sqrt{\pi}\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \operatorname{sh}(z \cos \varphi) (\sin \varphi)^{2\nu} d\varphi \left[\operatorname{Re} \nu > -\frac{1}{2} \right].$$

8.552

Special cases:

$$1.^6 \mathbf{H}_n(z) = \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{z}{2}\right)^{n-2m-1}}{\Gamma\left(n + \frac{1}{2} - m\right)} - \mathbf{E}_n(z) \quad [n = 1, 2, \dots].$$

EH II 40(66), WA 337(1)

997

$$2.^6 \mathbf{H}_{-n}(z) = (-1)^{n+1} \frac{1}{\pi} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma\left(n - m - \frac{1}{2}\right) \left(\frac{z}{2}\right)^{-n+2m+1}}{\Gamma\left(m + \frac{3}{2}\right)} - \mathbf{E}_{-n}(z) [n = 1, 2, \dots].$$

EH II 40(67), WA 337(2)

$$3. \mathbf{H}_{n+\frac{1}{2}}(z) = N_{n+\frac{1}{2}}(z) + \frac{1}{\pi} \sum_{m=0}^n \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{z}{2}\right)^{-2m+n-\frac{1}{2}}}{\Gamma(n+1-m)} [n = 0, 1, \dots].$$

EH II 39(64)

$$4. \mathbf{H}_{-(n+\frac{1}{2})}(z) = (-1)^n J_{n+\frac{1}{2}}(z) [n = 0, 1, \dots].$$

EH II 39(65)

$$5. \mathbf{L}_{-(n+\frac{1}{2})}(z) = I_{n+\frac{1}{2}}(z) [n = 0, 1, \dots].$$

EH II 39(65)

$$6. \mathbf{H}_{\frac{1}{2}}(z) = \frac{\sqrt{2}}{\sqrt{\pi z}} (1 - \cos z).$$

EH II 39, WA 364(3)

$$7. \mathbf{H}_{\frac{3}{2}}(z) = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} \left(1 + \frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\sin z + \frac{\cos z}{z}\right).$$

8.553

Functional relations:

$$1. \quad \mathbf{H}_\nu(z e^{im\pi}) = e^{i\pi(\nu+1)m} \mathbf{H}_\nu(z) [m = 1, 2, 3, \dots].$$

WA 362(5)

$$2. \quad \frac{d}{dz} [z^\nu \mathbf{H}_\nu(z)] = z^\nu \mathbf{H}_{\nu-1}(z).$$

WA 358

$$3. \quad \frac{d}{dz} [z^{-\nu} \mathbf{H}_\nu(z)] = 2^{-\nu} \pi^{-\frac{1}{2}} \left[\Gamma\left(\nu + \frac{3}{2}\right) \right]^{-1} - z^{-\nu} \mathbf{H}_{\nu+1}(z).$$

WA 359

$$4. \quad \mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z) = 2\nu z^{-1} \mathbf{H}_\nu(z) + \pi^{-\frac{1}{2}} \left(\frac{z}{2}\right)^\nu \left[\Gamma\left(\nu + \frac{3}{2}\right) \right]^{-1}.$$

WA 359(5)

$$5. \quad \mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu+1}(z) = 2\mathbf{H}'_\nu(z) - \pi^{-\frac{1}{2}} \left(\frac{z}{2}\right)^\nu \left[\Gamma\left(\nu + \frac{3}{2}\right) \right]^{-1}.$$

WA 359(6)

8.554

Asymptotic representations:

$$\mathbf{H}_\nu(\xi) = N_\nu(\xi) + \frac{1}{\pi} \sum_{m=0}^{p-1} \frac{\Gamma\left(m + \frac{1}{2}\right) \left(\frac{\xi}{2}\right)^{-2m+\nu-1}}{\Gamma\left(\nu + \frac{1}{2} - m\right)} + O(|\xi|^{\nu-2p-1}) [|\arg \xi| < \pi].$$

EH II 39(63), WA 363(2)

For the asymptotic representation of $N_\nu(\xi)$, see 8.451 2.

998
8.555

The differential equation for Struve functions:

$$z^2 y'' + zy' + (z^2 - \nu^2)y = \frac{1}{\sqrt{\pi}} \frac{4 \left(\frac{z}{2}\right)^{\nu+1}}{\Gamma\left(\nu + \frac{1}{2}\right)}.$$

WA 359(10)

**8.56 Thomson functions and their generalizations: $ber_\nu(z)$, $bei_\nu(z)$, $her_\nu(z)$, $hei_\nu(z)$,
 $ker(z)$, $kei(z)$**

8.561

$$\left. \begin{array}{l} 1. \quad ber_\nu(z) + i bei_\nu(z) = J_\nu(z e^{\frac{3}{4}\pi i}). \\ 2. \quad ber_\nu(z) - i bei_\nu(z) = J_\nu(z e^{-\frac{3}{4}\pi i}). \end{array} \right\}$$

WA 96(6)

8.562

$$\left. \begin{array}{l} 1. \quad her_\nu(z) + i hei_\nu(z) = H_\nu^{(1)}(z e^{\frac{3}{4}\pi i}) \\ 2. \quad her_\nu(z) - i hei_\nu(z) = H_\nu^{(1)}(z e^{-\frac{3}{4}\pi i}) \end{array} \right\} \quad (\text{see also 8.567}).$$

8.567
WA 96(7)

8.563

$$\left. \begin{array}{l} 1. \quad ber_0(z) \equiv ber(z); \quad bei_0(z) \equiv bei(z). \\ 2. \quad ker(z) \equiv -\frac{\pi}{2} hei_0(z); \quad kei(z) \equiv \frac{\pi}{2} her_0(z). \end{array} \right\}$$

WA 96(8)

For integral representations, see 6.251, 6.536, 6.537, 6.772 4., 6.777.

Series representation

8.564

$$1. \quad \text{ber}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{2^{4k} [(2k)!]^2}.$$

WA 96(3)

$$2. \quad \text{bei}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{4k+2}}{2^{4k+2} [(2k+1)!]^2}.$$

WA 96(4)

$$3. \quad \text{ker}(z) = \left(\ln \frac{2}{z} - \mathbf{C} \right) \text{ber}(z) + \frac{\pi}{4} \text{bei}(z) + \sum_{k=1}^{\infty} (-1)^k \frac{z^{4k}}{2^{4k} [(2k)!]^2} \sum_{m=1}^{2k} \frac{1}{m}.$$

WA 96(9)A, DW

$$4. \quad \text{kei}(z) = \left(\ln \frac{2}{z} - \mathbf{C} \right) \text{bei}(z) - \frac{\pi}{4} \text{ber}(z) + \sum_{k=0}^{\infty} (-1)^k \frac{z^{4k+2}}{2^{4k+2} [(2k+1)!]^2} \sum_{m=1}^{2k+1} \frac{1}{m}.$$

WA 96(10)A, DW

999

8.565

$$\text{ber}_{\nu}^2(z) + \text{bei}_{\nu}^2(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2\nu+4k}}{k! \Gamma(\nu+k+1) \Gamma(\nu+2k+1)}.$$

WA 163(6)

Asymptotic representation

8.566

$$1. \quad \text{ber}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \cos \beta(z) \quad \left[|\arg z| < \frac{\pi}{4} \right].$$

WA 227(1)

WA 227(1)

$$2. \operatorname{bei}(z) = \frac{e^{\alpha(z)}}{\sqrt{2\pi z}} \sin \beta(z) \quad \left[|\arg z| < \frac{\pi}{4} \right].$$

WA 227(1)

$$3. \operatorname{ker}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \cos \beta(-z) \quad \left[|\arg z| < \frac{5}{4}\pi \right].$$

WA 227(2)

$$4. \operatorname{kei}(z) = \sqrt{\frac{\pi}{2z}} e^{\alpha(-z)} \sin \beta(-z) \quad \left[|\arg z| < \frac{5}{4}\pi \right],$$

WA 227(2)

where

$$\alpha(z) \sim \frac{z}{\sqrt{2}} + \frac{1}{8z\sqrt{2}} - \frac{25}{384z^3\sqrt{2}} - \frac{13}{128z^4} - \dots,$$

$$\beta(z) \sim \frac{z}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8z\sqrt{2}} - \frac{1}{16z^2} - \frac{25}{384z^3\sqrt{2}} + \dots$$

8.567

Functional relations

$$\left. \begin{array}{l} 1. \operatorname{ker}(z) + i\operatorname{kei}(z) = K_0(z\sqrt{i}) \\ 2. \operatorname{ker}(z) - i\operatorname{kei}(z) = K_0(z\sqrt{-i}) \end{array} \right\} \quad (\text{see } 8.562).$$

8.562
WA 96(5), DW

For integrals of Thomson's functions, see 6.87.

8.57 Lommel functions

8.570

Definitions of the Lommel functions $s_{\mu,\nu}(z)$ and $S_{\mu,\nu}(z)$:

$$\begin{aligned}
 1. \quad s_{\mu,\nu}(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m z^{\mu+1+2m}}{[(\mu+1)^2 - \nu^2][(\mu+3)^2 - \nu^2] \dots [(\mu+2m+1)^2 - \nu^2]}; \\
 &= z^{\mu-1} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+2} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{3}{2}\right)} \\
 &\quad [\mu \pm \nu \text{ is not a negative odd integer}].
 \end{aligned}$$

EH II 40(69), WA 377(2)

1000

$$\begin{aligned}
 2. \quad S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + \left[2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right) \right] \times \\
 &\quad \times \frac{\cos\left[\frac{1}{2}(\mu - \nu)\pi\right] J_{-\nu}(z) - \cos\left[\frac{1}{2}(\mu + \nu)\pi\right] J_{\nu}(z)}{\sin \nu \pi} \\
 &\quad [\mu \pm \nu \text{ is a positive odd integer, } \nu \text{ is an odd integer}]; \\
 &= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\right) \times \\
 &\quad \times \left\{ \sin\left[\frac{1}{2}(\mu - \nu)\pi\right] J_{\nu}(z) - \cos\left[\frac{1}{2}(\mu - \nu)\pi\right] N_{\nu}(z) \right\} \\
 &\quad [\mu \pm \nu \text{ is a positive odd integer, } \nu \text{ is an integer}].
 \end{aligned}$$

EH II 41(71), WA 379(3)

EH II 40(71), WA 379(2)

Integral representations

8.571

$$s_{\mu,\nu}(z) = \frac{\pi}{2} \left[N_{\nu}(z) \int_0^z z^{\mu} J_{\nu}(z) dz - J_{\nu}(z) \int_0^z z^{\mu} N_{\nu}(z) dz \right].$$

WA 378(9)

8.572

$$\begin{aligned}
 s_{\mu,\nu}(z) &= 2^{\mu} \left(\frac{z}{2}\right)^{\frac{1}{2}(1+\nu+\mu)} \Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu\right) \times \int_0^{\frac{\pi}{2}} J_{\frac{1}{2}(1+\mu-\nu)}(z \sin \theta) (\sin \theta)^{\frac{1}{2}(1+\nu-\mu)} (\cos \theta)^{\nu+\mu} d\theta \\
 &\quad [\operatorname{Re}(\nu + \mu + 1) > 0].
 \end{aligned}$$

$$1. S_{1,2n}(z) = zO_{2n}(z).$$

WA 382(1)

$$2. S_{0,2n+1}(z) = \frac{z}{2n+1}O_{2n+1}(z).$$

WA 382(1)

$$3. S_{-1,2n}(z) = \frac{1}{4n}S_{2n}(z).$$

WA 382(2)

$$4. S_{0,2n+1}(z) = \frac{1}{2}S_{2n+1}(z).$$

WA 382(2)

$$5. s_{\nu,\nu}(z) = \Gamma\left(\nu + \frac{1}{2}\right) \sqrt{\pi} 2^{\nu-1} \mathbf{H}_{\nu}(z).$$

EH II 42(84)

$$6. S_{\nu,\nu}(z) = [\mathbf{H}_{\nu}(z) - N_{\nu}(z)] 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right).$$

EH II 42(84)

8.574

Connections with other special functions:

$$1. \mathbf{J}_{\nu}(z) = \frac{1}{\pi} \sin(\nu\pi) [s_{0,\nu}(z) - \nu s_{-1,\nu}(z)].$$

EH II 41(82)

$$2. \mathbf{E}_{\nu}(z) = -\frac{1}{\pi} [(1 + \cos \nu\pi) s_{0,\nu}(z) + \nu(1 - \cos \nu\pi) s_{-1,\nu}(z)].$$

A connection with a hypergeometric function

$$3. \quad s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2 \left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4} \right).$$

EH II 40(69), WA 378(10)

8.575

Functional relations:

$$1. \quad s_{\mu+2,\nu}(z) = z^{\mu+1} - [(\mu+1)^2 - \nu^2] s_{\mu,\nu}(z).$$

EH II 41(73), WA 380(1)

$$2. \quad s_{\mu,\nu}(z) + \left(\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z).$$

EH II 41(74), WA 380(2)

$$3. \quad s'_{\mu,\nu}(z) - \left(\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu - \nu - 1) s_{\mu-1,\nu+1}(z).$$

EH II 41(75), WA 380(3)

$$4. \quad \left(2\frac{\nu}{z}\right) s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) - (\mu - \nu - 1) s_{\mu-1,\nu+1}(z).$$

EH II 41(76), WA 380(4)

$$5. \quad 2s_{\mu,\nu}(z) = (\mu + \nu - 1) s_{\mu-1,\nu-1}(z) + (\mu - \nu - 1) s_{\mu-1,\nu+1}(z).$$

EH II 41(77), WA 380(5)

In formulas 8.575 1.- 5., $s_{\mu,\nu}(z)$ can be replaced with $S_{\mu,\nu}(z)$.

8.576

Asymptotic expansion of $S_{\mu,\nu}(z)$. In the case in which $\mu \pm \nu$ is not a positive odd integer, the following asymptotic expansion is valid for $S_{\mu,\nu}(z)$:

8.577

Lommel functions satisfy the following differential equation:

$$z^2 w'' + zw' + (z^2 - \nu^2)w = z^{\mu+1}.$$

WA 377(1), EH II 40(68)

8.578

Lommel functions of two variables $U_\nu(w, z), V_\nu(w, z)$:

Definition

$$1. \quad U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z).$$

EH II 42(87), WA 591(5)

$$2. \quad V_\nu(w, z) = \cos \left[\frac{1}{2} \left(w + \frac{z^2}{w} + \nu\pi \right) \right] + U_{-\nu+2}(w, z).$$

EH II 42(88), WA 591(6)

Particular values:

$$3. \quad U_0(z, z) = V_0(z, z) = \frac{1}{2} \{ J_0(z) + \cos z \}.$$

WA 591(9)

1002

$$4. \quad U_1(z, z) = -V_1(z, z) = \frac{1}{2} \sin z.$$

WA 591(10)

$$5. \quad U_{2n}(z, z) = V_{2n}(z, z) = \frac{(-1)^n}{2} \left\{ \cos z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m} J_{2m}(z) \right\} \quad [n \geq 1], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

$$6. \quad U_{2n+1}(z, z) - V_{2n+1}(z, z) = \frac{(-1)^n}{2} \left\{ \sin z - \sum_{m=0}^{n-1} (-1)^m \varepsilon_{2m+1} J_{2m+1}(z) \right\}, \quad [n \geq 0], \quad \varepsilon_m = \begin{cases} 2, & m > 0, \\ 1, & m = 0. \end{cases}$$

WA 591(12)

$$7. \quad V_n(w, z) = (-1)^n U_n \left(\frac{z^2}{w}, z \right).$$

$$8. \quad U_\nu(w, 0) = \frac{\left(\frac{w}{2}\right)^{\frac{1}{2}}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left(\frac{w}{2}\right).$$

WA 593(9)

$$9. \quad V_{-\nu+2}(w, 0) = \frac{\left(\frac{w}{2}\right)^{\frac{1}{2}}}{\Gamma(\nu-1)} S_{\nu-\frac{3}{2}, \frac{1}{2}} \left(\frac{w}{2}\right).$$

WA 593(10)

8.579

Functional relations:

$$1. \quad 2 \frac{\partial}{\partial w} U_\nu(w, z) = U_{\nu-1}(w, z) + \left(\frac{z}{w}\right)^2 U_{\nu+1}(w, z).$$

WA 593(2)

$$2. \quad 2 \frac{\partial}{\partial w} V_\nu(w, z) = V_{\nu+1}(w, z) + \left(\frac{z}{w}\right)^2 V_{\nu-1}(w, z).$$

WA 593(4)

3. The function $U_\nu(w, z)$ is a particular solution of the differential equation

$$\frac{\partial^2 U}{\partial z^2} - \frac{1}{z} \frac{\partial U}{\partial z} + \frac{z^2 U}{w^2} = \left(\frac{w}{z}\right)^{\nu-2} J_\nu(z).$$

$V_\nu(w, z)$

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{z} \frac{\partial V}{\partial z} + \frac{z^2 V}{w^2} = \left(\frac{w}{z}\right)^{-\nu} J_{-\nu+2}(z).$$

WA 592(3)

8.58 Anger and Weber functions $\mathbf{J}_\nu(z)$ and $\mathbf{E}_\nu(z)$

8.580

Definitions:

1. The Anger function $\mathbf{J}_\nu(z)$:

$$\mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta.$$

WA 336(1), EH II 35(32)

1003

2. The Weber function $\mathbf{E}_\nu(z)$:

$$\mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu\theta - z \sin \theta) d\theta.$$

WA 336(2), EH II 35(32)

8.581

Series representations:

$$1. \quad \mathbf{J}_\nu(z) = \cos \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{\Gamma\left(n+1+\frac{1}{2}\nu\right) \Gamma\left(n+1-\frac{1}{2}\nu\right)} + \\ + \sin \frac{\nu\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+1}}{\Gamma\left(n+\frac{3}{2}+\frac{1}{2}\nu\right) \Gamma\left(n+\frac{3}{2}-\frac{1}{2}\nu\right)}.$$

EH II 36(36), WA 337(3)

For the asymptotic expansion of $J_\nu(z)$ and $N_\nu(z)$, see 8.451.

8.584

The Anger and Weber functions satisfy the differential equation

$$y'' + z^{-1}y' + \left(1 - \frac{\nu^2}{z^2}\right)y = f(\nu, z),$$

where $f(\nu, z) = \frac{z-\nu}{\pi z^2} \sin \nu\pi$ for $\mathbf{J}_\nu(z)$

WA 341(9), EH II 37(44)

and $f(\nu, z) = -\frac{1}{\pi z^2} [z + \nu + (z - \nu) \cos \nu\pi]$ for $\mathbf{E}_\nu(z)$

EH II 37(45), WA 341(10)

8.59 Neumann's and Schlöfli's polynomials: $O_n(z)$ and $S_n(z)$

8.590

Definition of Neumann's polynomials

$$1. \quad O_n(z) = \frac{1}{4} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n-1} [n \geq 1].$$

WA 299(2), EH II 33(6)

$$2. \quad O_{-n}(z) = (-1)^n O_n(z) \quad [n \geq 1].$$

WA 303(8)

$$3. \quad O_0(z) = \frac{1}{z}.$$

WA 299(3), EH II 33(7)

$$4. \quad O_1(z) = \frac{1}{z^2}.$$

EH II 33(7)

In general, $O_n(z)$ is a polynomial in z^{-1} of degree $n + 1$.

8.591

Functional relations:

$$1. \quad O'_0(z) = -O_1(z).$$

EH II 33(9), WA 301(3)

$$2. \quad 2O'_n(z) = O_{n-1}(z) - O_{n+1}(z) \quad [n \geq 1].$$

EH II 33(10), WA 301(2)

1005

$$3. \quad (n-1)O_{n+1}(z) + (n+1)O_{n-1}(z) - 2z^{-1}(n^2-1)O_n(z) = 2nz^{-1} \left(\sin n \frac{\pi}{2} \right)^2 \quad [n \geq 1].$$

EH II 33(11), WA 301(1)

$$4. \quad nzO_{n-1}(z) - (n^2 - 1)O_n(z) = (n - 1)zO'_n(z) + n \left(\sin n \frac{\pi}{2} \right)^2.$$

EH II 33(12), WA 303(4)

$$5. \quad nzO_{n+1}(z) - (n^2 - 1)O_n(z) = -(n + 1)zO'_n(z) + n \left(\sin n \frac{\pi}{2} \right)^2.$$

EH II 33(13), WA 303(5)a

8.592

The generating function:

$$\frac{1}{z - \xi} = J_0(\xi)z^{-1} + 2 \sum_{n=1}^{\infty} J_n(\xi)O_n(z) \quad [|\xi| < |z|].$$

EH II 32(1), WA 298(1)

8.593

The integral representation:

$$O_n(z) = \int_0^\infty \frac{[u + \sqrt{u^2 + z^2}]^n + [u - \sqrt{u^2 + z^2}]^n}{2z^{n+1}} e^{-u} du.$$

See also 3.547 6., 8., 3.549 1., 2.

EH II 32(3), WA 305(1)

8.594

The inequality

$$|O_n(z)| \leq 2^{n-1} n! |z|^{-n-1} e^{\frac{1}{4}|z|^2} \quad [n > 1].$$

EH II 33(8), WA 300(8)

8.595

Neumann's polynomial $O_n(z)$ satisfies the differential equation

$$z^2 \frac{d^2 y}{dz^2} + 3z \frac{dy}{dz} + (z^2 + 1 - n^2)y = z \left(\cos n \frac{\pi}{2} \right)^2 + n \left(\sin n \frac{\pi}{2} \right)^2.$$

EH II 33(14), WA 303(1)

8.596

Schl\"afli's polynomials $S_n(z)$. These are the functions that satisfy the formulas

$$1. \quad S_0(z) = 0.$$

EH II 34(18), WA 312(2)

$$\begin{aligned} 2. \quad S_n(z) &= \frac{1}{n} \left[2zO_n(z) - 2 \left(\cos n \frac{\pi}{2} \right)^2 \right] [n \geq 1]; \\ &= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-m-1)!}{m!} \left(\frac{z}{2} \right)^{2m-n} [n \geq 1]. \end{aligned}$$

EH II 34(18)

$$3. \quad S_{-n}(z) = (-1)^{n+1} S_n(z).$$

WA 313(6)

1006
8.597

Functional relations:

$$1. \quad S_{n-1}(z) + S_{n+1}(z) = 4O_n(z).$$

WA 313(7)

Other functional relations may be obtained from 8.591 by replacing $O_n(z)$ with the expression for $S_n(z)$ given by 8.596 2.

8.6 Mathieu Functions

8.60 Mathieu's equation

$$\frac{d^2 y}{dz^2} + (a - 2k^2 \cos 2z)y = 0, \quad k^2 = q.$$

MA

8.61 Periodic Mathieu functions

8.610

In general, Mathieu's equation 8.60 does not have periodic solutions. If k is a real number, there exist infinitely many *eigenvalues* a , not identically equal to zero, corresponding to the periodic solutions

$$y(z) = y(2\pi + z),$$

If k is nonzero, there are no other linearly independent periodic solutions. Periodic solutions of Mathieu's equations are called *Mathieu's periodic functions* or *Mathieu functions of the first kind*, or, more simply, *Mathieu functions*.

8.611

Mathieu's equation has four series of distinct periodic solutions:

$$1. \quad ce_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz.$$

MA

$$2. \quad ce_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z.$$

MA

$$3. \quad se_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z.$$

MA

$$4. \quad se_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z.$$

MA

5. The coefficients A and B depend on q . The eigenvalues a of the functions ce_{2n} , ce_{2n+1} , se_{2n} , se_{2n+1} are denoted by a_{2n} , a_{2n+1} , b_{2n} , b_{2n+1} .

8.612

The solutions of Mathieu's equation are normalized so that

$$\int_0^{2\pi} y^2 dx = \pi.$$

MO 65

8.613

$$1. \quad \lim_{q \rightarrow 0} ce_0(x) = \frac{1}{\sqrt{2}}.$$

2. $\lim_{q \rightarrow 0} ce_n(x) = \cos nx \quad [n \neq 0]$

3. $\lim_{q \rightarrow 0} se_n(x) = \sin nx.$

MO 65

8.62 Recursion relations for the coefficients $A_{2r}^{(2n)}, A_{2r+1}^{(2n+1)},$

$B_{2r+1}^{(2n+1)}, B_{2r+2}^{(2n+2)}$

8.621

1. $aA_0^{(2n)} - qA_2^{(2n)} = 0.$

MA

2. $(a - 4)A_2^{(2n)} - q(A_4^{(2n)} + 2A_0^{(2n)}) = 0.$

MA

3. $(a - 4r^2)A_{2r}^{(2n)} - q(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}) = 0 \quad [r \geq 2].$

MA

8.622

1. $(a - 1 - q)A_1^{(2n+1)} - qA_3^{(2n+1)} = 0.$

MA

2. $[a - (2r + 1)^2]A_{2r+1}^{(2n+1)} - q(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)}) = 0 \quad [r \geq 1].$

MA

8.623

1. $(a - 1 + q)B_1^{(2n+1)} - qB_3^{(2n+1)} = 0.$

MA

MA

8.624

$$1. (a - 4)B_2^{(2n+2)} - qB_4^{(2n+2)} = 0.$$

MA

$$2. (a - 4r^2)B_{2r}^{(2n+2)} - q(B_{2r+2}^{(2n+2)} - B_{2r-2}^{(2n+2)}) = 0 \quad [r \geq 2].$$

MA

8.625

We can determine the coefficients A and B from equations 8.612, 8.613 and 8.621-8.624 provided a is known. Suppose, for example, that we need to determine the coefficients $A_{2r}^{(2n)}$ for the function $ce_{2n}(z, q)$. From the recursion formulas, we have

$$1. \begin{vmatrix} a & -q & 0 & 0 & 0 & \dots \\ -2q & a-4 & -q & 0 & 0 & \dots \\ 0 & -q & a-16 & -q & 0 & \dots \\ 0 & 0 & -q & a-36 & -q & \dots \\ 0 & 0 & 0 & -q & a-64 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

ST

For given q in equation 8.625 1., we may determine the eigenvalues

$$2. a = a_0, a_2, a_4, \dots [|a_0| \leq |a_2| \leq |a_4| \leq \dots].$$

If we now set $a = a_{2n}$, we can determine the coefficients $A_{2r}^{(2n)}$ from the recursion formulas 8.621 up to a proportionality coefficient. This coefficient is determined from the formula

$$3. 2[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty} [A_{2r}^{(2n)}]^2 = 1,$$

MA

which follows from the conditions of normalization.

1008

8.63 Mathieu functions with a purely imaginary argument

8.630

If, in equation 8.60, we replace z with iz , we arrive at the differential equation

$$1. \quad \frac{d^2 y}{dz^2} + (-a + 2q \operatorname{ch} 2x)y = 0.$$

We can find the solutions of this equation if we replace the argument z with iz in the functions $\operatorname{ce}_n(z, q)$ and $\operatorname{se}_n(z, q)$. The functions obtained in this way are called *associated Mathieu functions of the first kind* and are denoted as follows:

$$2. \quad \operatorname{Ce}_{2n}(z, q), \quad \operatorname{Ce}_{2n+1}(z, q), \quad \operatorname{Se}_{2n+1}(z, q), \quad \operatorname{Se}_{2n+2}(z, q).$$

8.631

$$1. \quad \operatorname{Ce}_{2n}(z, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \operatorname{ch} 2rz.$$

MA

$$2. \quad \operatorname{Ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \operatorname{ch}(2r+1)z.$$

MA

$$3. \quad \operatorname{Se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \operatorname{sh}(2r+1)z.$$

MA

$$4. \quad \operatorname{Se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \operatorname{sh}(2r+2)z.$$

Along with each periodic solution of equation 8.60, there exists a second nonperiodic solution that is linearly independent. The nonperiodic solutions are denoted as follows:

$$fe_{2n}(z, q), \quad fe_{2n+1}(z, q), \quad ge_{2n+1}(z, q), \quad ge_{2n+2}(z, q).$$

Analogously, the second solutions of equation 8.630 1. are denoted by

$$Fe_{2n}(z, q), \quad Fe_{2n+1}(z, q), \quad Ge_{2n+1}(z, q), \quad Ge_{2n+2}(z, q).$$

8.65 Mathieu functions for negative q

8.651

If we replace the argument z in equation 8.60 with $\pm\left(\frac{\pi}{2} \pm z\right)$, we get the equation

$$\frac{d^2y}{dz^2} + (a + 2q \cos 2z)y = 0.$$

MA

This equation has the following solutions:

1009

8.652

$$1. \quad ce_{2n}(z, -q) = (-1)^n ce_{2n}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$2. \quad ce_{2n+1}(z, -q) = (-1)^n se_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$3. \quad se_{2n+1}(z, -q) = (-1)^n ce_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

$$4. \operatorname{se}_{2n+2}(z, -q) = (-1)^n \operatorname{se}_{2n+2}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$5. \operatorname{fe}_{2n}(z, -q) = (-1)^{n+1} \operatorname{fe}_{2n}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$6. \operatorname{ge}_{2n+1}(z, -q) = (-1)^n \operatorname{ge}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$7. \operatorname{ge}_{2n+1}(z, -q) = (-1)^n \operatorname{fe}_{2n+1}\left(\frac{1}{2}\pi - z, q\right).$$

MA

$$8. \operatorname{ge}_{2n+2}(z, -q) = (-1)^n \operatorname{ge}_{2n+2}\left(\frac{1}{2}\pi - z, q\right).$$

MA

8.653

Analogously, if we replace z with $\frac{\pi}{2}i + z$ in equation 8.630 1. we get the equation

$$\frac{d^2y}{dz^2} - (a + 2q \operatorname{ch} z)y = 0.$$

It has the following solutions:

8.654

$$1. \operatorname{Ce}_{2n}(z, -q) = (-1)^n \operatorname{Ce}_{2n}\left(\frac{\pi}{2}i + z, q\right).$$

MA

$$2. \operatorname{Ce}_{2n+1}(z, -q) = (-1)^{n+1}i \operatorname{Se}_{2n+1}\left(\frac{1}{2}\pi i + z, q\right).$$

MA

$$3. \operatorname{Se}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ce}_{2n+1}\left(\frac{\pi}{2}i + z, q\right).$$

MA

$$4. \operatorname{Se}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Se}_{2n+2}\left(\frac{\pi}{2}i + z, q\right).$$

MA

$$5. \operatorname{Fe}_{2n}(z, -q) = (-1)^n \operatorname{Fe}_{2n}\left(\frac{1}{2}\pi i + z, q\right).$$

MA

$$6. \operatorname{Fe}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Ge}_{2n+1}\left(\frac{\pi}{2}i + z, q\right).$$

MA

1010

$$7. \operatorname{Ge}_{2n+1}(z, -q) = (-1)^{n+1} i \operatorname{Fe}_{2n+1}\left(\frac{\pi}{2}i + z, q\right).$$

MA

$$8. \operatorname{Ge}_{2n+2}(z, -q) = (-1)^{n+1} \operatorname{Ge}_{2n+2}\left(\frac{\pi}{2}i + z, q\right).$$

MA

8.66 Representation of Mathieu functions as series of Bessel functions

8.661

$$\begin{aligned} 1. \operatorname{ce}_{2n}(z, q) &= \frac{\operatorname{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_{2r}(2k \cos z); \\ &= \frac{\operatorname{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(2k \sin z). \end{aligned}$$

$$\begin{aligned}
2. \quad \text{ce}_{2n+1}(z, q) &= -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{kA_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z); \\
&= \frac{\text{ce}_{2n+1}(0, q)}{kA_1^{(2n+1)}} \text{ctg } z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z).
\end{aligned}$$

MA
MA

$$\begin{aligned}
3. \quad \text{se}_{2n+1}(z, q) &= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \text{tg } z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} J_{2r+1}(2k \cos z); \\
&= \frac{\text{se}'_{2n+1}(0, q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(2k \sin z).
\end{aligned}$$

MA
MA

$$\begin{aligned}
4. \quad \text{se}_{2n+2}(z, q) &= \frac{-\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \text{tg } z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} J_{2r+2}(2k \cos z); \\
&= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \text{ctg } z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} I_{2r+2}(2k \sin z).
\end{aligned}$$

MA
MA

8.662

$$1. \quad \text{fe}_{2n}(z, q) = -\frac{\pi \text{fe}'_{2n}(0, q)}{2 \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \text{Im}[J_r(ke^{iz})N_r(ke^{-iz})].$$

MA

$$\begin{aligned}
2. \quad \text{fe}_{2n+1}(z, q) &= \frac{\pi k \text{fe}'_{2n+1}(0, q)}{2 \text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)} \times \\
&\quad \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} \text{Im}[J_r(ke^{iz})N_{r+1}(ke^{-iz}) + J_{r+1}(ke^{iz})N_r(ke^{-iz})].
\end{aligned}$$

MA

1011

$$\begin{aligned}
3. \quad \text{ge}_{2n+1}(z, q) &= -\frac{\pi k \text{ge}_{2n+1}(0, q)}{2 \text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)} \times \\
&\quad \times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} \text{Re}[J_r(ke^{iz})N_{r+1}(ke^{-iz}) - J_{r+1}(ke^{iz})N_r(ke^{-iz})].
\end{aligned}$$

$$4. \quad \text{ge}_{2n+2}(z, q) = -\frac{\pi k^2 \text{ge}_{2n+2}(0, q)}{2 \text{se}'_{2n+2}\left(\frac{1}{2}\pi, q\right)} \times \\ \times \sum_{r=0}^{\infty} (-1)^r \text{Re}[J_k(ke^{iz})N_{r+2}(ke^{-iz}) - J_{r+2}(ke^{iz})N_r(ke^{-iz})].$$

MA

The expansions of the functions Fe_n and Ge_n as series of the functions N_ν are denoted, respectively, by Fey_n and Gey_n and the expansions of these functions as series of the functions K_ν are denoted, respectively, by Fek_n and Gek_n .

8.663

$$1. \quad \text{Fey}_{2n}(z, q) = \frac{\text{ce}_{2n}(0, q)}{A_0^{(2n)}} \sum_{r=0}^{\infty} A_{2r}^{(2n)} N_{2r}(2k \text{sh } z), \quad k^2 = q[|\text{sh } z| > 1, \quad \text{Re } z > 0]; \\ = \frac{\text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} N_{2r}(2k \text{ch } z) \quad [|\text{ch } z| > 1]; \\ = \frac{\text{ce}_{2n}(0, q) \text{ce}_{2n}\left(\frac{\pi}{2}, q\right)}{[A_0^{(2n)}]^2} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(ke^{-z}) N_r(ke^z).$$

MA
MA
MA

$$2. \quad \text{Fey}_{2n+1}(z, q) = \frac{\text{ce}_{2n+1}(0, q) \text{cth } z}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(2n+1)} N_{2r+1}(2k \text{sh } z), \quad k^2 = q \\ [|\text{sh } z| > 1, \quad \text{Re } z > 0]; \\ = -\frac{\text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k A_1^{(2n+1)}} \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} N_{2r+1}(2k \text{ch } z) \quad [|\text{ch } z| > 1]; \\ = -\frac{\text{ce}_{2n+1}(0, q) \text{ce}'_{2n+1}\left(\frac{\pi}{2}, q\right)}{k [A_1^{(2n+1)}]^2} \times \\ \times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} [J_r(ke^{-z}) N_{r+1}(ke^z) + J_{r+1}(ke^{-z}) N_r(ke^z)].$$

MA
MA
MA

$$\begin{aligned}
3. \quad \text{Gey}_{2n+1}(z, q) &= \frac{\text{se}'_{2n+1}(0, q)}{kB_1^{(2n+1)}} \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} N_{2r+1}(2k \text{sh } z) \quad [|\text{sh } z| > 1, \quad \text{Re } z > 0]; \\
&= \frac{\text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{kB_1^{(2n+1)}} \theta z \sum_{r=0}^{\infty} (-1)^r (2r+1) B_{2r+1}^{(2n+1)} N_{2r+1}(2k \text{ch } z) \quad [|\text{ch } z| > 1]; \\
&= \frac{\text{se}_{2n+1}(0, q) \text{se}_{2n+1}\left(\frac{\pi}{2}, q\right)}{k[B_1^{(2n+1)}]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} [J_r(ke^{-z})N_{r+1}(ke^z) - \\
&\quad - J_{r+1}(ke^{-z})N_r(ke^z)].
\end{aligned}$$

MA
MA
MA

$$\begin{aligned}
4. \quad \text{Gey}_{2n+2}(z, q) &= \frac{\text{se}'_{2n+2}(0, q)}{k^2 B_2^{(2n+2)}} \text{cth } z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} N_{2r+2}(2k \text{sh } z) \\
&\quad [|\text{sh } z| > 1, \quad \text{Re } z > 0]; \\
&= -\frac{\text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 B_2^{(2n+2)}} \theta z \sum_{r=0}^{\infty} (-1)^r (2r+2) B_{2r+2}^{(2n+2)} N_{2r+2}(2k \text{ch } z) \\
&\quad [|\text{ch } z| > 1]; \\
&= \frac{\text{se}'_{2n+2}(0, q) \text{se}'_{2n+2}\left(\frac{\pi}{2}, q\right)}{k^2 [B_2^{(2n+2)}]^2} \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} [J_r(ke^{-z})N_{r+2}(ke^z) - \\
&\quad - J_{r+2}(ke^{-z})N_r(ke^z)].
\end{aligned}$$

MA
MA
MA

8.664

$$1. \quad \text{Fek}_{2n}(z, q) = \frac{\text{ce}_{2n}(0, q)}{\pi A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} K_{2r}(-2ik \text{sh } z), \quad k^2 = q \quad [|\text{sh } z| > 1, \quad \text{Re } z > 0].$$

MA

$$\begin{aligned}
2. \quad \text{Fek}_{2n+1}(z, q) &= \frac{\text{ce}_{2n+1}(0, q)}{\pi k A_1^{(2n+1)}} \text{cth } z \sum_{r=0}^{\infty} (-1)^r (2r+1) A_{2r+1}^{(2n+1)} K_{2r+1}(-2ik \text{sh } z), \quad k^2 = q \\
&\quad [|\text{sh } z| > 1, \quad \text{Re } z > 0]
\end{aligned}$$

MA

MA

$$4. \text{Gek}_{2n+2}(z, q) = \frac{se'_{2n+2}\left(\frac{\pi}{2}, q\right)}{\pi k^2 B_2^{(2n+2)}} \theta z \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)} K_{2r+2}(-2ik \operatorname{ch} z).$$

MA

8.67 The general theory

If $i\mu$ is not an integer, the general solution of equation 8.60 can be found in the form

1013
8.671

$$1. \quad y = Ae^{\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{2rzi} + Be^{-\mu z} \sum_{r=-\infty}^{\infty} c_{2r} e^{-2rzi}.$$

MA

The coefficients c_{2r} can be determined from the homogeneous system of linear algebraic equations

$$2. \quad c_{2r} + \xi_{2r}(c_{2r+2} + c_{2r-2}) = 0, r = \dots, -2, -1, 0, 1, 2, 2 \dots,$$

MA

where

$$\xi_{2r} = \frac{q}{(2r - i\mu)^2 - a}.$$

The condition that this system be compatible yields an equation that μ must satisfy:

$$3.7 \quad \Delta(i\mu) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

4. $\operatorname{ch} \mu\pi = 1 - 2\Delta(0) \sin^2 \left(\frac{\pi\sqrt{a}}{2} \right)$, where $\Delta(0)$ is the value that is assumed by the determinant of the preceding article if we set $\mu = 0$ in the expressions for ξ_{2r} .

5. If the pair (a, q) is such that $|\operatorname{ch} \mu\pi| < 1$, then $\mu = i\beta$, $\operatorname{Im}\beta = 0$, and the solution 8.671 1. is bounded on the real axis.

6. If $|\operatorname{ch} \mu\pi| > 1$, μ may be real or complex and the solution 8.671 1. will not be bounded on the real axis.

7. If $\operatorname{ch} \mu\pi = \pm 1$, $i\mu$ will be an integer. In this case, one of the solutions will be of period π or 2π (depending on whether n is even or odd). The second solution is nonperiodic (see 8.61 and 8.64).

8.7- 8.8 Associated Legendre Functions

8.70 Introduction

8.700

An *associated Legendre function* is a solution of the differential equation

$$1. \quad (1 - z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - z^2} \right] u = 0,$$

in which ν and μ are arbitrary complex constants.

This equation is a special case of (Riemann's) hypergeometric equation (see 9.151). The points

$$+1, -1, \infty$$

are, in general, its *singular points*, specifically, its ordinary branch points.

1014

We are interested, on the one hand, in solutions of the equation that correspond to real values of the independent variable z that lie in the interval $[-1, 1]$ and, on the other hand, in solutions corresponding to an arbitrary complex number z such that $\operatorname{Re} z > 1$. These are multiple-valued in the z -plane. To separate these functions into single-valued branches, we make a cut along the real axis from

$-\infty$ to $+1$. We are also interested in those solutions of equation 8.700 1. for which ν or μ or both are integers. Of especial significance is the case in which $\mu = 0$.

8.701

In connection with this, we shall use the following notations:

The letter z will denote *an arbitrary complex variable*; the letter x will denote a *real* variable that varies over the interval $[-1, +1]$.

We shall sometimes set $x = \cos \varphi$, where φ is a real number.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation 8.700 1., that are single-valued and regular for $|z| < 1$ and, in particular, uniquely determined for $z = x$.

We shall use the symbols $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ to denote those solutions of equation 8.700 1. that are single-valued and regular for $\operatorname{Re} z > 1$. When these functions cannot be unrestrictedly extended without violating their single-valuedness we make a cut along the real axis to the left of the point $z = 1$. The values of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ on the upper and lower boundaries of that portion of the cuts lying between the points -1 and $+1$ are denoted respectively by

$$P_\nu^\mu(x \pm i0), \quad Q_\nu^\mu(x \pm i0).$$

The letters n and m denote natural numbers or zero. The letters ν and μ denote arbitrary complex numbers unless the contrary is stated.

The upper index will be omitted when it is equal to zero. That is, we set

$$P_\nu^0(z) = P_\nu(z), \quad Q_\nu^0(z) = Q_\nu(z).$$

The *linearly independent* functions

8.702

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\frac{\mu}{2}} F \left(-\nu, \nu+1; \quad 1-\mu; \quad \frac{1-z}{2} \right) \\ \left[\arg \frac{z+1}{z-1} = 0, \text{ if } z \text{ is real and greater than } 1 \right] \text{ and}$$

$$Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i}\Gamma(\nu + \mu + 1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu + \frac{3}{2}\right)}(z^2-1)^{\frac{\mu}{2}}z^{-\nu-\mu-1}F\left(\frac{\nu + \mu + 2}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2}\right)$$

$[\arg(z^2 - 1) = 0$ when z is real and greater than 1; $\arg z = 0$ when z is real and greater than zero] which are solutions of the differential equation 8.700 1., are called *associated Legendre functions* (or *spherical functions*) of the *first* and *second kinds* respectively. They are uniquely defined, respectively, in the intervals $|1 - z| < 2$ and $|z| > 1$ with the portion of the real axis that lies between $-\infty$ and $+1$ excluded. They can be extended by means of hypergeometric series to the entire z -plane where the above-mentioned cut was made. These expressions for $P_{\nu}^{\mu}(z)$ and $Q_{\nu}^{\mu}(z)$ lose their meaning when $1 - \mu$ and $\nu + \frac{3}{2}$ are nonpositive integers respectively.

MO 80

1015

When z is a real number lying on the interval $[-1, +1]$, so that ($z = x = \cos \varphi$), we take the following functions as linearly independent Solutions of the equation

8.704

$$P_{\nu}^{\mu}(x) = \frac{1}{2} \left[e^{\frac{1}{2}\mu\pi i} P_{\nu}^{\mu}(\cos \varphi + i0) + e^{-\frac{1}{2}\mu\pi i} P_{\nu}^{\mu}(\cos \varphi - i0) \right] ;$$

$$= \frac{1}{\Gamma(1 - \mu)} \left(\frac{1+x}{1-x} \right)^{\frac{\mu}{2}} F\left(-\nu, \nu + 1; 1 - \mu; \frac{1-x}{2}\right) .$$

EH I 143(6)

EH I 143(1)

8.705

$$Q_{\nu}^{\mu}(x) = \frac{1}{2} e^{-\mu\pi i} \left[e^{-\frac{1}{2}\mu\pi i} Q_{\nu}^{\mu}(x + i0) + e^{\frac{1}{2}\mu\pi i} Q_{\nu}^{\mu}(x - i0) \right] ;$$

$$= \frac{\pi}{2 \sin \mu\pi} \left[P_{\nu}^{\mu}(x) \cos \mu\pi - \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} P_{\nu}^{-\mu}(x) \right] \quad (\text{cf. 8.732 5.})$$

8.732

EH I 143(2)

If $\mu = \pm m$ is an integer, the last equation loses its meaning. In this case, we get the following formulas by passing to the limit:

8.706

$$1. \quad Q_{\nu}^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} Q_{\nu}(x) \quad (\text{cf. 8.752 1}).$$

8.752
EH I 149(7)

$$2. \quad Q_{\nu}^{-m}(x) = (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} Q_{\nu}^m(x).$$

EH I 144(18)

The functions $Q_{\nu}^{\mu}(z)$ are not defined when $\nu + \mu$ is equal to a negative integer. Therefore, we must exclude the cases when $\nu + \mu = -1, -2, -3, \dots$ for these formulas.

The functions

$$P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z).$$

are *linearly independent solutions* of the differential equation for $\nu + \mu \neq 0, \pm 1, \pm 2, \dots$

8.707

Nonetheless, two linearly independent solutions can always be found. Specifically, for $\nu \pm \mu$ not an integer, the differential equation

8.700 1. has the following solutions:

$$1. \quad P_{\nu}^{\pm\mu}(\pm z), \quad Q_{\nu}^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

respectively, for $z = x = \cos \varphi$,

$$2. \quad P_{\nu}^{\pm\mu}(\pm x), \quad Q_{\nu}^{\pm\mu}(\pm x), \quad P_{-\nu-1}^{\pm\mu}(\pm x), \quad Q_{-\nu-1}^{\pm\mu}(\pm x).$$

If $\nu \pm \mu$ is not an integer, the solutions

$\nu \pm \mu$

3. $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$, respectively, and $P_\nu^\mu(x)$, $Q_\nu^\mu(x)$

are linearly independent. If $\nu \pm \mu$ is an integer but μ itself is not an integer, the following functions are linearly independent solutions of equation 8.700 1.:

4. $P_\nu^\mu(z)$, $P_\nu^{-\mu}(z)$, respectively, and $P_\nu^\mu(x)$, $P_\nu^{-\mu}(x)$.

If $\mu = \pm m, \nu = n$, or $\nu = -n - 1$, the following functions are linearly independent solutions of equation 8.700 1. for $n \geq m$:

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5. $P_n^m(z)$, $Q_n^m(z)$, respectively, and $P_n^m(x)$, $Q_n^m(x)$,

and for $n < m$, the following functions will be linearly independent solutions

6. $P_n^{-m}(z)$, $Q_n^m(z)$, respectively, and $P_n^{-m}(x)$, $Q_n^m(x)$.

8.71 Integral representations

8.711

1.
$$P_\nu^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^\mu \sqrt{\pi} \Gamma\left(\mu + \frac{1}{2}\right)} \int_{-1}^1 \frac{(1 - t^2)^{\mu - \frac{1}{2}}}{(z + t\sqrt{z^2 - 1})^{\mu - \nu}} dt \quad \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad |\arg(z \pm 1)| < \pi \right].$$

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2.
$$P_\nu^m(z) = \frac{(\nu + 1)(\nu + 2) \dots (\nu + m)}{\pi} \int_0^\pi [z + \sqrt{z^2 - 1} \cos \varphi]^\nu \cos m\varphi d\varphi;$$
$$= (-1)^m \frac{\nu(\nu - 1) \dots (\nu - m + 1)}{\pi} \int_0^\pi \frac{\cos m\varphi d\varphi}{[z + \sqrt{z^2 - 1} \cos \varphi]^{\nu + 1}}$$
$$\left[|\arg z| < \frac{\pi}{2}, \quad \arg(z + \sqrt{z^2 - 1} \cos \varphi) = \arg z \quad \text{for} \quad \varphi = \frac{\pi}{2} \right] \quad (\text{cf. 8.822 1}).$$

$$3. \quad Q_{\nu}^{\mu}(z) = \sqrt{\pi} \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\mu} \Gamma\left(\mu + \frac{1}{2}\right) \Gamma(\nu - \mu + 1)} (z^2 - 1)^{\frac{\mu}{2}} \int_0^{\infty} \frac{\operatorname{sh}^{2\mu} t dt}{(z + \sqrt{z^2 - 1} \operatorname{ch} t)^{\nu + \mu + 1}}$$

[Re($\nu \pm \mu$) > -1, |arg($z \pm 1$)| < π] (cf. 8.822 2.).

$$4. \quad Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i} \Gamma(\nu + 1)}{\Gamma(\nu - \mu + 1)} \int_0^{\infty} \frac{\operatorname{ch} \mu t dt}{(z + \sqrt{z^2 - 1} \operatorname{ch} t)^{\nu + 1}}$$

[Re($\nu + \mu$) > -1, $\nu \neq -1, -2, -3, \dots$, |arg($z \pm 1$)| < π].

$$5. \quad \int_{-1}^1 P_l^2(x) P_l^0(x) dx = -\frac{l!}{(l-2)!} \frac{1}{2l+1} = -\frac{l(l-1)}{2l+1}.$$

8.712

$$Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i} \Gamma(\nu + \mu + 1)}{2^{\nu+1} \Gamma(\nu + 1)} (z^2 - 1)^{-\frac{\mu}{2}} \int_{-1}^1 (1 - t^2)^{\nu} (z - t)^{-\nu - \mu - 1} dt$$

[Re($\nu + \mu$) > -1, Re μ > -1, |arg($z \pm 1$)| < π] (cf. 8.821 2.).

1017
8.713

$$1. \quad Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{2\pi}} (z^2 - 1)^{\frac{\mu}{2}} \left\{ \int_0^{\pi} \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(z - \cos t)^{\mu + \frac{1}{2}}} - \cos \nu \pi \int_0^{\infty} \frac{e^{-(\nu + \frac{1}{2})t} dt}{(z + \operatorname{ch} t)^{\mu + \frac{1}{2}}} \right\}$$

[Re μ > $-\frac{1}{2}$, Re($\nu + \mu$) > -1, |arg($z \pm 1$)| < π].

$$2. P_{\nu}^{-\mu}(z) = \frac{(z^2 - 1)^{\frac{\mu}{2}}}{2^{\nu} \Gamma(\mu - \nu) \Gamma(\nu + 1)} \int_0^{\infty} \frac{\text{sh}^{2\nu+1} t}{(z + \text{ch } t)^{\nu+\mu+1}} dt$$

[Re $z > -1$, $|\arg(z \pm 1)| < \pi$, Re($\nu + 1$) > 0 , Re($\mu - \nu$) > 0] .

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$$3. P_{\nu}^{-\mu}(z) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\left(\mu + \frac{1}{2}\right) (z^2 - 1)^{\frac{\mu}{2}}}{\Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} \int_0^{\infty} \frac{\text{ch}\left(\nu + \frac{1}{2}\right) t dt}{(z + \text{ch } t)^{\mu+\frac{1}{2}}}$$

[Re $z > -1$, $|\arg(z \pm 1)| < \pi$, Re($\nu + \mu$) > -1 , Re($\mu - \nu$) > 0] .

MO 89

8.714

$$1. P_{\nu}^{\mu}(\cos \varphi) = \sqrt{\frac{2}{\pi}} \frac{\sin^{\mu} \varphi}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_0^{\varphi} \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(\cos t - \cos \varphi)^{\mu+\frac{1}{2}}} \quad \left[0 < \varphi < \pi, \text{Re } \mu < \frac{1}{2}\right]; \quad (\text{cf. 8.823})$$

8.823
MO 87

$$2. P_{\nu}^{-\mu}(\cos \varphi) = \frac{\Gamma(2\mu + 1) \sin^{\mu} \varphi}{2^{\mu} \Gamma(\mu + 1) \Gamma(\nu + \mu + 1) \Gamma(\mu - \nu)} \int_0^{\infty} \frac{t^{\nu+\mu} dt}{(1 + 2t \cos \varphi + t^2)^{\mu+\frac{1}{2}}}$$

[Re($\nu + \mu$) > -1 , Re($\mu - \nu$) > 0] .

MO 89

$$3. Q_{\nu}^{\mu}(\cos \varphi) = \frac{1}{2^{\mu+1}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \times$$

$$\times \int_0^{\infty} \left[\frac{\text{sh}^{2\mu} t}{(\cos \varphi + i \sin \varphi \text{ch } t)^{\nu+\mu+1}} + \frac{\text{sh}^{2\mu} t}{(\cos \varphi - i \sin \varphi \text{ch } t)^{\nu+\mu+1}} \right] dt$$

[Re($\nu + \mu + 1$) > 0 , Re($\nu - \mu + 1$) > 0 , Re $\mu > -\frac{1}{2}$] .

MO 89

1018

$$4. P_{\nu}^{\mu}(\cos \varphi) = \frac{i}{2^{\mu}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \times$$

$$\times \int_0^{\infty} \left[\frac{\text{sh}^{2\mu} t}{(\cos \varphi + i \sin \varphi \text{ch } t)^{\nu+\mu+1}} - \frac{\text{sh}^{2\mu} t}{(\cos \varphi - i \sin \varphi \text{ch } t)^{\nu+\mu+1}} \right] dt$$

[Re($\nu \pm \mu + 1$) > 0 , Re $\mu > -\frac{1}{2}$] .

$$1. \quad P_{\nu}^{\mu}(\operatorname{ch} \alpha) = \frac{\sqrt{2} \operatorname{sh}^{\mu} \alpha}{\sqrt{\pi} \Gamma\left(\frac{1}{2} - \mu\right)} \int_0^{\alpha} \frac{\operatorname{ch}\left(\nu + \frac{1}{2}\right) t dt}{(\operatorname{ch} \alpha - \operatorname{ch} t)^{\mu + \frac{1}{2}}} \quad \left[\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2} \right].$$

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$$2. \quad Q_{\nu}^{\mu}(\operatorname{ch} \alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{\mu\pi i} \operatorname{sh}^{\mu} \alpha}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_{\alpha}^{\infty} \frac{e^{-(\nu + \frac{1}{2})t} dt}{(\operatorname{ch} t - \operatorname{ch} \alpha)^{\mu + \frac{1}{2}}} \quad \left[\alpha > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\nu + \mu) > -1 \right].$$

MO 87

See also 3.277 1., 4., 5., 7., 3.318, 3.516 3., 3.518 1., 2., 3.542 2., 3.663 1., 3.894, 3.988 3., 6.622 3., 6.628 1., 4.-7., and also 8.742.

8.72 Asymptotic series for large values of $|\nu|$

8.721

⁶ For real values of μ , $|\nu| \gg 1$, $|\nu| \gg |\mu|$, $|\arg \nu| < \pi$, we have:

$$1. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \Gamma(\nu + \mu + 1) \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k + \frac{1}{2}\right) \cos\left[\left(\nu + k + \frac{1}{2}\right)\varphi + \frac{\pi}{4}(2k - 1) + \frac{\mu\pi}{2}\right]}{\Gamma\left(\mu - k + \frac{1}{2}\right) k! \Gamma\left(\nu + k + \frac{3}{2}\right) (2 \sin \varphi)^{k + \frac{1}{2}}}$$

$$\left[\nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \frac{7}{2}, \dots; \quad \text{for } \frac{\pi}{6} < \varphi < \frac{5\pi}{6} \right]$$

this series also converges for complex values of ν and μ . In the remaining cases, it is an asymptotic expansion

for $|\nu| \gg |\mu|$, $|\nu| \gg 1$, if $\nu > 0, \mu > 0$ and $0 < \varepsilon \leq \varphi \leq \pi - \varepsilon$.

MO 92

1019

$$2. \quad Q_{\nu}^{\mu}(\cos \varphi) = \sqrt{\pi} \Gamma(\nu + \mu + 1) \times$$

$$\times \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\mu + k + \frac{1}{2}\right) \cos\left[\left(\nu + k + \frac{1}{2}\right)\varphi - \frac{\pi}{4}(2k - 1) + \frac{\mu\pi}{2}\right]}{\Gamma\left(\mu - k + \frac{1}{2}\right) k! \Gamma\left(\nu + k + \frac{3}{2}\right) (2 \sin \varphi)^{k + \frac{1}{2}}}$$

$$\left[\nu + \mu \neq -1, -2, -3, \dots; \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \dots; \quad \text{for } \frac{\pi}{6} < \varphi < \frac{5\pi}{6} \right]$$

for $|\nu| \gg |\mu|$, $|\nu| \gg 1$, if $\nu > 0$, $\mu > 0$, $0 < \varepsilon \leq \varphi \leq \pi - \varphi$.

EH I 147(6), MO 92

$$3. \quad P_{\nu}^{\mu}(\cos \varphi) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} \frac{\cos \left[\left(\nu + \frac{1}{2}\right) \varphi - \frac{\pi}{4} + \frac{\mu\pi}{2} \right]}{\sqrt{2 \sin \varphi}} \left[1 + O\left(\frac{1}{\nu}\right) \right]$$

$$\left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon, |\nu| \gg \frac{1}{\varepsilon} \right].$$

MO 92

For $\nu > 0$, $\mu > 0$ and $\nu > \mu$, it follows from formulas 8.721 1. and 8.721 2. that

$$4. \quad \nu^{-\mu} P_{\nu}^{\mu}(\cos \varphi) = \sqrt{\frac{2}{\nu\pi \sin \varphi}} \cos \left[\left(\nu + \frac{1}{2}\right) \varphi - \frac{\pi}{4} + \frac{\mu\pi}{2} \right] + O\left(\frac{1}{\sqrt{\nu^3}}\right).$$

$$5. \quad \nu^{-\mu} Q_{\nu}^{\mu}(\cos \varphi) = \sqrt{\frac{\pi}{2\nu \sin \varphi}} \cos \left[\left(\nu + \frac{1}{2}\right) \varphi + \frac{\pi}{4} + \frac{\mu\pi}{2} \right] O\left(\frac{1}{\sqrt{\nu^3}}\right)$$

$$\left[0 < \varepsilon \leq \varphi \leq \pi - \varepsilon; \quad \nu \gg \frac{1}{\varepsilon} \right].$$

MO 92

8.722

If φ is sufficiently close to 0 or π that $\nu\varphi$ or $\nu(\pi - \varphi)$ is small in comparison with 1, the asymptotic formulas 8.721 become unsuitable. In this case, the following asymptotic representation is applicable for $\mu \leq 0$, $\nu \gg 1$, and *small* values of φ :

$$1. \quad \left[\left(\nu + \frac{1}{2}\right) \cos \frac{\varphi}{2} \right]^{\mu} P_{\nu}^{-\mu}(\cos \varphi) = J_{\mu}(\eta) + \sin^2 \frac{\varphi}{2} \left[\frac{J_{\mu+1}(\eta)}{2\eta} - J_{\mu+2}(\eta) + \frac{\eta}{6} J_{\mu+3}(\eta) \right] + O\left(\sin^4 \frac{\varphi}{2}\right),$$

where $\eta = (2\nu + 1) \sin \frac{\varphi}{2}$. In particular, it follows that

$$2. \quad \lim_{\nu \rightarrow \infty} \nu^{\mu} P_{\nu}^{-\mu} \left(\cos \frac{x}{\nu} \right) = J_{\mu}(x) [x \geq 0, \mu \geq 0].$$

MO 93

We can see how the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ behave for large $|\nu|$ and real values of $z > \frac{3}{2\sqrt{2}}$:

$$1. \quad P_\nu^\mu(\operatorname{ch} \alpha) = \frac{2^\mu}{\sqrt{\pi}} \left\{ \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{\Gamma(-\nu - \mu)} \frac{e^{(\mu-\nu)\alpha} \operatorname{sh}^\mu \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}}\right) + \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu - \mu + 1)} \frac{e^{(\nu+\mu+1)\alpha} \operatorname{sh}^\mu \alpha}{(e^{2\alpha} - 1)^{\mu+\frac{1}{2}}} F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; -\nu + \frac{1}{2}; \frac{1}{1 - e^{2\alpha}}\right) \right\} \\ \left[\nu \neq \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots; \quad a > \frac{1}{2} \ln 2 \right]$$

MO 94

$$2. \quad Q_\nu^\mu(\operatorname{ch} \alpha) = e^{\mu\pi i} 2^\mu \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} \frac{e^{-(\nu+\mu+1)\alpha}}{(1 - e^{-2\alpha})^{\mu+\frac{1}{2}}} \operatorname{sh}^\mu \alpha \times \\ \times F\left(\mu + \frac{1}{2}, -\mu + \frac{1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - e^{2\alpha}}\right) \\ \left[\mu + \nu + 1 \neq 0, \quad -1, -2, \dots; \quad \alpha > \frac{1}{2} \ln 2 \right].$$

MO 94

See also 8.776

8.724

The inequalities

$$\left. \begin{aligned} 1. \quad |P_\nu^{\pm\mu}(\cos \varphi)| &< \sqrt{\frac{8}{\nu\pi}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi}, \\ 2. \quad |Q_\nu^{\pm\mu}(\cos \varphi)| &< \sqrt{\frac{2\pi}{\nu}} \frac{\Gamma(\nu \pm \mu + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{\mu+\frac{1}{2}} \varphi}, \\ 3. \quad |P_\nu^{\pm m}(\cos \varphi)| &< \frac{2}{\sqrt{\nu\pi}} \frac{\Gamma(\nu \pm m + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{m+\frac{1}{2}} \varphi}, \\ 4. \quad |Q_\nu^{\pm m}(\cos \varphi)| &< \sqrt{\frac{\pi}{\nu}} \frac{\Gamma(\nu \pm m + 1)}{\Gamma(\nu + 1)} \frac{1}{\sin^{m+\frac{1}{2}} \varphi} \end{aligned} \right\} \quad \left[\nu \text{ and } \mu \text{ are arbitrary real numbers satisfying the inequalities } \nu \geq 1, \nu - \mu + 1 > 0, \mu \geq 0 \right].$$

$$5.^8 \quad \left| \sqrt{\sin \theta} P_n^m(\cos \theta) \right| < \frac{\Gamma(n + 1/2)}{\Gamma(n - m + 1)} 2^{(m+n)^2/n} x \operatorname{Sup}_{0 < t < \infty} \left| \sqrt{t} J_m(t) \right| \quad \left[\text{uniformly } 0 \leq m \leq n \right]$$

$$1. P_\nu^\mu(z) = \frac{\Gamma(\nu + \mu + 1)\Gamma(\mu - \nu)}{\pi\Gamma(\mu + 1)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} \sin \mu\pi \left[F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z\right) - \frac{\sin \nu\pi}{\sin \mu\pi} e^{\mp i\mu\pi} \left(\frac{z-1}{z+1}\right)^\mu F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z\right) \right]$$

AS 8.10.1

$$2. Q_\nu^\mu(z) = \frac{1}{2} e^{i\mu\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\mu + 1)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} \Gamma(\mu - \nu) \left[F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z\right) - e^{\mp i\nu\pi} \left(\frac{z-1}{z+1}\right)^\mu F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z\right) \right]$$

AS 8.10.2

$$3. Q_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi} \csc[\pi(\nu - \mu)]}{2\pi\Gamma(1 + \mu)} \left[e^{\mp i\nu\pi} \left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu} F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} - \frac{1}{2}z\right) - \left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}\mu} F\left(-\nu, \nu + 1; 1 + \mu; \frac{1}{2} + \frac{1}{2}z\right) \right]$$

AS 8.10.3

8.73- 8.74 Functional relations

8.731

$$1. (z^2 - 1) \frac{dP_\nu^\mu(z)}{dz} = (\nu - \mu + 1)P_{\nu+1}^\mu(z) - (\nu + 1)zP_\nu^\mu(z) \quad (\text{cf. 8.832 1., 8.914 2.}).$$

8.914

8.832

EH I 161(10), MO 81

$$1(1)^* (z^2 - 1) \frac{dP_\nu^\mu(z)}{dz} = \nu z P_\nu^\mu(z) - (\nu + \mu) P_{\nu-1}^\mu(z).$$

AS 8.5.4

$$1(2)^* (z^2 - 1) \frac{dP_\nu^\mu(z)}{dz} = (\nu + \mu)(\nu - \mu + 1)(z^2 - 1)P_\nu^{\mu-1}(z) - \mu z P_\nu^\mu(z).$$

2. $(2\nu+1)zP_\nu^\mu(z) = (\nu-\mu+1)P_{\nu+1}^\mu(z) + (\nu+\mu)P_{\nu-1}^\mu(z)$ (cf. 8.832 2., 8.914 1.).

8.914

8.832

EH I 160(2), MO 81

3. $P_\nu^{\mu+2}(z) + 2(\mu+1)\frac{z}{\sqrt{z^2-1}}P_\nu^{\mu+1}(z) = (\nu-\mu)(\nu+\mu+1)P_\nu^\mu(z)$.

MO 82, EH I 160(1)

3(1)* $P_\nu^{\mu+1}(z) = (z^2-1)^{-\frac{1}{2}} [(\nu-\mu)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)]$.

AS 8.5.1

4. $P_{\nu+1}^\mu(z) - P_{\nu-1}^\mu(z) = (2\nu+1)\sqrt{z^2-1}P_\nu^{\mu-1}(z)$.

EH I 160(3), MO 82

4(1)* $(\nu-\mu+1)P_{\nu+1}^\mu(z) = (2\nu+1)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)$.

AS 334(8.5.3)

4(2)* $P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{\frac{1}{2}}P_\nu^{\mu-1}(z)$.

AS 334(8.5.5)

5. $P_{-\nu-1}^\mu(z) = P_\nu^\mu(z)$ (cf. 8.820, 8.832 4.).

8.832

8.820

EH I 140(1), MO 82

8.732

1022

$$2.^* \quad (2\nu + 1)zQ_\nu^\mu(z) = (\nu - \mu + 1)Q_{\nu+1}^\mu(z) + (\nu + \mu)Q_{\nu-1}^\mu(z) \quad (\text{cf. 8.832 4.}).$$

$$3. \quad Q_\nu^{\mu+2}(z) + 2(\mu + 1) \frac{z}{\sqrt{z^2 - 1}} Q_\nu^{\mu+1}(z) = (\nu - \mu)(\nu + \mu + 1)Q_\nu^\mu(z).$$

$$4. \quad Q_{\nu-1}^\mu(z) - Q_{\nu+1}^\mu(z) = -(2\nu + 1)\sqrt{z^2 - 1}Q_\nu^{\mu-1}(z).$$

$$5. \quad e^{-\mu\pi i} Q_\nu^\mu(x \pm i0) = e^{\pm \frac{1}{2}\mu\pi i} \left[Q_\nu^\mu(x) \mp i \frac{\pi}{2} P_\nu^\mu(x) \right].$$

8.733

$$\begin{aligned} 1. \quad (1 - x^2) \frac{dP_\nu^\mu(x)}{dx} &= (\nu + 1)xP_\nu^\mu(x) - (\nu - \mu + 1)P_{\nu+1}^\mu(x) \quad (\text{cf. 8.731 1.}); \\ &= -\nu x P_\nu^\mu(x) + (\nu + \mu)P_{\nu-1}^\mu(x); \\ &= -\sqrt{1 - x^2} P_\nu^{\mu+1}(x) - \mu x P_\nu^\mu(x); \\ &= (\nu - \mu + 1)(\nu + \mu)\sqrt{1 - x^2} P_\nu^{\mu-1}(x) + \mu x P_\nu^\mu(x). \end{aligned}$$

$$2. \quad (2\nu + 1)xP_\nu^\mu(x) = (\nu - \mu + 1)P_{\nu+1}^\mu(x) + (\nu + \mu)P_{\nu-1}^\mu(x) \quad (\text{cf. 8.731 2.}).$$

$$3. \quad P_{\nu}^{\mu+2}(x) + 2(\mu+1) \frac{x}{\sqrt{1-x^2}} P_{\nu}^{\mu+1}(x) + (\nu-\mu)(\nu+\mu+1) P_{\nu}^{\mu}(x) = 0 \quad (\text{cf. 8.731 3.}).$$

8.731
MO 82

$$4. \quad P_{\nu-1}^{\mu}(x) - P_{\nu+1}^{\mu}(x) = (2\nu+1) \sqrt{1-x^2} P_{\nu}^{\mu-1}(x) \quad (\text{cf. 8.731 4.}).$$

8.731
MO 82

$$5. \quad P_{-\nu-1}^{\mu}(x) = P_{\nu}^{\mu}(x) \quad (\text{cf. 8.731 5.}).$$

8.731

8.734

$$1. \quad (\nu+\mu+1)zQ_{\nu}^{\mu}(z) + \sqrt{z^2-1}Q_{\nu}^{\mu+1}(z) = (\nu-\mu+1)Q_{\nu+1}^{\mu}(z).$$

MO 82

$$2. \quad (\nu+\mu)Q_{\nu-1}^{\mu}(z) + \sqrt{z^2-1}Q_{\nu}^{\mu+1}(z) = (\nu-\mu)zQ_{\nu}^{\mu}(z).$$

MO 82

$$3. \quad Q_{\nu-1}^{\mu}(z) - zQ_{\nu}^{\mu}(z) = -(\nu-\mu+1)\sqrt{z^2-1}Q_{\nu}^{\mu-1}(z).$$

MO 82

$$4. \quad zQ_{\nu}^{\mu}(z) - Q_{\nu+1}^{\mu}(z) = -(\nu+\mu)\sqrt{z^2-1}Q_{\nu}^{\mu-1}(z).$$

MO 82

$$5. \quad (\nu+\mu)(\nu+\mu+1)Q_{\nu-1}^{\mu}(z) + (2\nu+1)\sqrt{z^2-1}Q_{\nu}^{\mu+1}(z) = (\nu-\mu)(\nu-\mu+1)Q_{\nu+1}^{\mu}(z).$$

8.735

$$1. (\nu + \mu + 1)xP_\nu^\mu(x) + \sqrt{1 - x^2}P_\nu^{\mu+1}(x) = (\nu - \mu + 1)P_{\nu+1}^\mu(x).$$

MO 83

$$2. (\nu - \mu)xP_\nu^\mu(x) - (\nu + \mu)P_{\nu-1}^\mu(x) = \sqrt{1 - x^2}P_\nu^{\mu+1}(x).$$

MO 83

$$3. P_{\nu-1}^\mu(x) - xP_\nu^\mu(x) = (\nu - \mu + 1)\sqrt{1 - x^2}P_\nu^{\mu-1}(x).$$

MO 83

$$4. xP_\nu^\mu(x) - P_{\nu+1}^\mu(x) = (\nu + \mu)\sqrt{1 - x^2}P_\nu^{\mu-1}(x).$$

MO 83

$$5. (\nu - \mu)(\nu - \mu + 1)P_{\nu+1}^\mu(x) = (\nu + \mu)(\nu + \mu + 1)P_{\nu-1}^\mu(x) + (2\nu + 1)\sqrt{1 - x^2}P_\nu^{\mu+1}(x).$$

MO 83

8.736

$$1. P_\nu^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[P_\nu^\mu(z) - \frac{2}{\pi} e^{-\mu\pi i} \sin \mu\pi Q_\nu^\mu(z) \right].$$

MO 83

1023

$$2. P_\nu^\mu(-z) = e^{\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin [(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z < 0] \quad (\text{cf. 8.833 1.}).$$

8.833
MO 83

$$3. P_\nu^\mu(-z) = e^{-\nu\pi i} P_\nu^\mu(z) - \frac{2}{\pi} \sin [(\nu + \mu)\pi] e^{-\mu\pi i} Q_\nu^\mu(z) \quad [\text{Im } z > 0] \quad (\text{cf. 8.833 2.}).$$

$$4. \quad Q_{\nu}^{-\mu}(z) = e^{-2\mu\pi i} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_{\nu}^{\mu}(z).$$

$$5. \quad Q_{\nu}^{\mu}(-z) = -e^{-\nu\pi i} Q_{\nu}^{\mu}(z) \quad [\text{Im } z < 0] \quad (\text{cf. 8.833 3.}).$$

$$6. \quad Q_{\nu}^{\mu}(-z) = -e^{\nu\pi i} Q_{\nu}^{\mu}(z) \quad [\text{Im } z > 0] \quad (\text{cf. 8.833 4.}).$$

$$7.^6 \quad Q_{\nu}^{\mu}(z) \sin[(\nu + \mu)\pi] - Q_{-\nu-1}^{\mu}(z) \sin[(\nu - \mu)\pi] = \pi e^{\mu\pi i} \cos \nu\pi P_{\nu}^{\mu}(z).$$

8.737

$$1. \quad P_{\nu}^{-\mu}(x) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[\cos \mu\pi P_{\nu}^{\mu}(x) - \frac{2}{\pi} \sin(\mu\pi) Q_{\nu}^{\mu}(x) \right].$$

$$2. \quad P_{\nu}^{\mu}(-x) = \cos[(\nu + \mu)\pi] P_{\nu}^{\mu}(x) - \frac{2}{\pi} \sin[(\nu + \mu)\pi] Q_{\nu}^{\mu}(x).$$

$$3. \quad Q_{\nu}^{\mu}(-x) = -\cos[(\nu + \mu)\pi] Q_{\nu}^{\mu}(x) - \frac{\pi}{2} \sin[(\nu + \mu)\pi] P_{\nu}^{\mu}(x).$$

$$4. \quad Q_{-\nu-1}^{\mu}(x) = \frac{\sin[(\nu + \mu)\pi]}{\sin[(\nu - \mu)\pi]} Q_{\nu}^{\mu}(x) - \frac{\pi \cos \nu\pi \cos \mu\pi}{\sin[(\nu - \mu)\pi]} P_{\nu}^{\mu}(x).$$

8.738

$$1.^6 \quad Q_{\nu}^{\mu}(i \operatorname{ctg} \varphi) = \exp \left[i\pi \left(\mu - \frac{1}{4} \right) \right] \sqrt{\pi} \Gamma(\nu + \mu + 1) \sqrt{\frac{1}{2} \sin \varphi} P_{-\mu - \frac{1}{2}}^{-\nu - \frac{1}{2}}(\cos \varphi) \quad \left[0 < \varphi < \frac{\pi}{2} \right].$$

MO 83

$$2.^6 \quad P_{\nu}^{\mu}(i \operatorname{ctg} \varphi) = \sqrt{\frac{2}{\pi}} \exp \left[i\pi \left(\nu + \frac{1}{4} \right) \right] \frac{\sqrt{\sin \varphi}}{\Gamma(-\nu - \mu)} Q_{-\mu - \frac{1}{2}}^{-\nu - \frac{1}{2}}(\cos \varphi - i0) \quad \left[0 < \varphi < \frac{\pi}{2} \right].$$

MO 83

8.739

$$e^{-\mu\pi i} Q_{\nu}^{\mu}(\operatorname{ch} \alpha) = \frac{\sqrt{\pi} \Gamma(\nu + \mu + 1)}{\sqrt{2 \operatorname{sh} \alpha}} P_{-\mu - \frac{1}{2}}^{-\nu - \frac{1}{2}}(\operatorname{cth} \alpha) \quad [\operatorname{Re}(\operatorname{ch} \alpha) > 0].$$

MO 83

8.741

$$1. \quad P_{\nu}^{-\mu}(x) \frac{dP_{\nu}^{\mu}(x)}{dx} - P_{\nu}^{\mu}(x) \frac{dP_{\nu}^{-\mu}(x)}{dx} = \frac{2 \sin \mu\pi}{\pi(1-x^2)}.$$

MO 83

$$2. \quad P_{\nu}^{\mu}(x) \frac{dQ_{\nu}^{\mu}(x)}{dx} - Q_{\nu}^{\mu}(x) \frac{dP_{\nu}^{\mu}(x)}{dx} = \frac{2^{2\mu}}{1-x^2} \frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right) \Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right) \Gamma\left(\frac{\nu-\mu}{2}+1\right)}.$$

MO 83

1024

8.742

$$1. \quad \frac{\Gamma(\nu - \mu - 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \mu\pi P_{\nu}^{\mu}(\cos \varphi) - \frac{2}{\pi} \sin \mu\pi Q_{\nu}^{\mu}(\cos \varphi) \right\} = \\ = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^{\mu} \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_0^{\varphi} \frac{\cos\left(\nu + \frac{1}{2}\right) t dt}{(\cos t - \cos \varphi)^{\frac{1}{2} - \mu}} \quad \left[\operatorname{Re} \mu > -\frac{1}{2} \right].$$

$$2. \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left\{ \cos \nu \pi P_\nu^\mu(\cos \varphi) - \frac{2}{\pi} \sin \nu \pi Q_\nu^\mu(\cos \varphi) \right\} = \\ = \sqrt{\frac{2}{\pi}} \frac{\operatorname{cosec}^\mu \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_\varphi^\pi \frac{\cos \left[\left(\nu + \frac{1}{2}\right) (t - \pi) \right] dt}{(\cos \varphi - \cos t)^{\frac{1}{2} - \mu}} \left[\operatorname{Re} \mu > -\frac{1}{2} \right].$$

MO 88

$$3. P_\nu^\mu(\cos \varphi) \cos(\nu + \mu)\pi - \frac{2}{\pi} Q_\nu^\mu(\cos \varphi) \sin(\nu + \mu)\pi = \\ = \sqrt{\frac{2}{\pi}} \frac{\sin^\mu \varphi}{\Gamma\left(\frac{1}{2} - \mu\right)} \int_\varphi^\pi \frac{\cos \left[\left(\nu + \frac{1}{2}\right) (t - \pi) \right] dt}{(\cos \varphi - \cos t)^{\mu + \frac{1}{2}}} \left[\operatorname{Re} \mu < \frac{1}{2} \right].$$

MO 88

$$4. \cos \mu \pi P_\nu^\mu(\cos \varphi) - \frac{2}{\pi} \sin \mu \pi Q_\nu^\mu(\cos \varphi) = \\ = \frac{1}{2^\mu \sqrt{\pi}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \frac{\sin^\mu \varphi}{\Gamma\left(\mu + \frac{1}{2}\right)} \int_0^\pi \frac{\sin^{2\mu} t dt}{(\cos \varphi \pm i \sin \varphi \cos t)^{\nu - \mu}} \\ \left[\operatorname{Re} \mu > -\frac{1}{2}, \quad 0 < \varphi < \pi \right].$$

MO 38

For integrals of Legendre functions, see 7.11-7.21.

8.75 Special cases and particular values

8.751

$$1. P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu + m + 1)(1 - x^2)^{\frac{m}{2}}}{2^m \Gamma(\nu - m + 1)m!} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1 - x}{2}\right).$$

MO 84

$$2. P_\nu^m(z) = \frac{\Gamma(\nu + m + 1)(z^2 - 1)^{\frac{m}{2}}}{2^m m! \Gamma(\nu - m + 1)} F\left(m - \nu, m + \nu + 1; m + 1; \frac{1 - z}{2}\right).$$

MO 84

1025

$$3.^* Q_{n - \frac{3}{2}}^\mu(z) = \frac{e^{\mu\pi i} \Gamma\left(\mu + n + \frac{3}{2}\right)}{2^{n + \frac{3}{2}} (n + 1)!} (z^2 - 1)^{\frac{\mu}{2}} \pi^{\frac{1}{2}} z^{-n - \mu - 3/2} F\left(\frac{\mu + n + \frac{5}{2}}{2}, \frac{\mu + n + \frac{3}{2}}{2}; n + 2; \frac{1}{z^2}\right).$$

$$1. P_\nu^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\nu(x).$$

WH, MO 84, EH I 148(6)

$$2. P_\nu^{-m}(x) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(x) = (1-x^2)^{-\frac{m}{2}} \int_x^1 \dots \int_x^1 P_\nu(x) (dx)^m \quad [m \geq 1].$$

HO 99a, MO 85, EH I 149(10)a

$$3. P_\nu^{-m}(z) = (z^2-1)^{-\frac{m}{2}} \int_1^z \dots \int_1^z P_\nu(z) (dz)^m \quad [m \geq 1].$$

MO 85, EH I 149(8)

$$4. Q_\nu^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m}{dz^m} Q_\nu(z).$$

WH, MO 85, EH I 148(5)

$$5. Q_\nu^{-m}(z) = (-1)^m (z^2-1)^{-\frac{m}{2}} \int_z^\infty \dots \int_z^\infty Q_\nu(z) (dz)^m \quad [m \geq 1].$$

MO 85, EH I 149(9)

Special values of the indices

8.753

$$1. P_0^\mu(\cos \varphi) = \frac{1}{\Gamma(1-\mu)} \operatorname{ctg}^\mu \frac{\varphi}{2}.$$

MO 84

$$2. P_\nu^{-1}(\cos \varphi) = -\frac{1}{\nu(\nu+1)} \frac{dP_\nu(\cos \varphi)}{d\varphi}.$$

MO 84

$$3. P_n^m(z) \equiv 0, \quad P_n^m(x) \equiv 0 \quad \text{for } m > n.$$

8.754

$$1. P_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\operatorname{ch} \alpha) = \sqrt{\frac{2}{\pi \operatorname{sh} \alpha}} \operatorname{ch} \nu \alpha.$$

MO 85

$$2. P_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \cos \nu \varphi.$$

MO 85

$$3. P_{\nu-\frac{1}{2}}^{-\frac{1}{2}}(\cos \varphi) = \sqrt{\frac{2}{\pi \sin \varphi}} \frac{\sin \nu \varphi}{\nu}.$$

MO 85

$$4. Q_{\nu-\frac{1}{2}}^{\frac{1}{2}}(\operatorname{ch} \alpha) = i \sqrt{\frac{\pi}{2 \operatorname{sh} \alpha}} e^{-\nu \alpha}.$$

MO 85

1026

8.755

$$1. P_{\nu}^{-\nu}(\cos \varphi) = \frac{1}{\Gamma(1+\nu)} \left(\frac{\sin \varphi}{2} \right)^{\nu}.$$

MO 85

$$2. P_{\nu}^{-\nu}(\operatorname{ch} \alpha) = \frac{1}{\Gamma(1+\nu)} \left(\frac{\operatorname{sh} \alpha}{2} \right)^{\nu}.$$

MO 85

Special values of legendre functions

8.756

$$1. P_{\nu}^{\mu}(0) = \frac{2^{\mu} \sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu}{2}+1\right) \Gamma\left(\frac{-\nu-\mu+1}{2}\right)}.$$

$$2. \quad \frac{dP_{\nu}^{\mu}(0)}{dx} = \frac{2^{\mu+1} \sin \frac{1}{2}(\nu + \mu)\pi \Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu - \mu + 1}{2}\right)}.$$

MO 84

$$3. \quad Q_{\nu}^{\mu}(0) = -2^{\mu-1} \sqrt{\pi} \sin \frac{1}{2}(\nu + \mu)\pi \frac{\Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\Gamma\left(\frac{\nu - \mu}{2} + 1\right)}.$$

MO84

$$4. \quad \frac{dQ_{\nu}^{\mu}(0)}{dx} = 2^{\mu} \sqrt{\pi} \cos \frac{1}{2}(\nu + \mu)\pi \frac{\Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\Gamma\left(\frac{\nu - \mu + 1}{2}\right)}.$$

MO 84

8.76 Derivatives with respect to the order

8.761

$$q \frac{\partial P_{\nu}^{-\mu}(x)}{\partial \nu} = \frac{1}{\Gamma(\mu + 1)} \left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} \sum_{n=1}^{\infty} \frac{(-\nu)(1-\nu) \dots (n-1-\nu)(\nu+1)(\nu+2) \dots (\nu+n)}{(\mu+1)(\mu+2) \dots (\mu+n) 1 \cdot 2 \dots n} \times \\ \times [\psi(\nu+n+1) - \psi(\nu-n+1)] \left(\frac{1-x}{2}\right)^n \quad [\nu \neq 0, \pm 1, \pm 2, \dots; \operatorname{Re} \mu > -1].$$

MO 94

8.762

$$1. \quad \left[\frac{\partial P_{\nu}(\cos \varphi)}{\partial \nu} \right]_{\nu=0} = 2 \ln \cos \frac{\varphi}{2}.$$

MO 94

$$2. \quad \left[\frac{\partial P_{\nu}^{-1}(\cos \varphi)}{\partial \nu} \right]_{\nu=0} = -\operatorname{tg} \frac{\varphi}{2} - 2 \operatorname{ctg} \frac{\varphi}{2} \ln \cos \frac{\varphi}{2}.$$

MO 94

$$3. \left[\frac{\partial P_\nu^{-1}(\cos \varphi)}{\partial \nu} \right]_{\nu=1} = -\frac{1}{2} \operatorname{tg} \frac{\varphi}{2} \sin^2 \frac{\varphi}{2} + \sin \varphi \ln \cos \frac{\varphi}{2}.$$

MO 94

For a connection with the polynomials $C_n^\lambda(x)$, see 8.936.

For a connection with a hypergeometric function, see 8.77.

8.77 Series representation

For a representation in the form of a series, see 8.721. It is also possible to represent associated Legendre functions in the form of a series by expressing them in terms of a hypergeometric function.

8.771

$$1. P_\nu^\mu(z) = \left(\frac{z+1}{z-1} \right)^{\frac{\mu}{2}} \frac{1}{\Gamma(1-\mu)} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right).$$

MO 15

$$2. Q_\nu^\mu(z) = \frac{e^{\mu\pi i} \Gamma(\nu+\mu+1)}{2^{\nu+1} \Gamma\left(\nu+\frac{3}{2}\right)} \frac{\Gamma\left(\frac{1}{2}\right) (z^2-1)^{\frac{\mu}{2}}}{2^{\nu+\mu+1}} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right).$$

MO 15

See also 8.702, 8.703, 8.704, 8.723, 8.751, 8.772.

The analytic continuation for $|z| > 1$

The formulas are consequences of theorems on the analytic continuation of hypergeometric series (see 9.154 and 9.155):

8.772

$$1. P_\nu^\mu(z) = \frac{\sin(\nu+\mu)\pi\Gamma(\nu+\mu+1)}{2^{\nu+1}\sqrt{\pi}\cos\nu\pi\Gamma\left(\nu+\frac{3}{2}\right)} (z^2-1)^{\frac{\mu}{2}} z^{-\nu-\mu-1} F\left(\frac{\nu+\mu}{2}+1, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) + \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}, \frac{1}{z^2}\right) + \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\nu-\mu+1)} (z^2-1)^{\frac{\mu}{2}} z^{\nu-\mu} F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}, \frac{1}{z^2}\right).$$

[$2\nu \neq \pm 1, \pm 3, \pm 5, \dots$; $|z| > 1$; $|\arg(z \pm 1)| < \pi$].

$$\begin{aligned}
2. \quad P_\nu^\mu(z) &= \frac{\Gamma\left(-\nu - \frac{1}{2}\right) (z^2 - 1)^{-\frac{\nu+1}{2}}}{2^{\nu+1} \sqrt{\pi} \Gamma(-\nu - \mu)} F\left(\frac{\nu - \mu + 1}{2}, \frac{\nu + \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - z^2}\right) + \\
&\quad + \frac{2^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} (z^2 - 1)^{\frac{\nu}{2}} F\left(\frac{\mu - \nu}{2}, -\frac{\mu + \nu}{2}; \frac{1}{2} - \nu; \frac{1}{1 - z^2}\right) \\
&\quad [2\nu \neq \pm 1, \pm 3, \pm 5; \dots; \quad |1 - z^2| > 1; \quad |\arg(z \pm 1)| < \pi] .
\end{aligned}$$

MO 85

1028

$$3. \quad P_\nu^\mu(z) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^\nu F\left(-\nu, -\nu - \mu; 1 - \mu; \frac{z-1}{z+1}\right) \left[\left| \frac{z-1}{z+1} \right| < 1 \right]$$

MO 86

8.773

$$\begin{aligned}
1. \quad Q_\nu^\mu(z) &= e^{\mu\pi i} \frac{\sqrt{\pi} \Gamma(\nu + \mu + 1)}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)} (z^2 - 1)^{-\frac{\nu+1}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\nu - \mu + 1}{2}; \nu + \frac{3}{2}; \frac{1}{1 - z^2}\right) \\
&\quad [\nu + \mu \neq -1, -2, -3, \dots; \quad |\arg(z \pm 1)| < \pi; \quad |1 - z^2| > 1] .
\end{aligned}$$

MO 86

$$\begin{aligned}
2. \quad Q_\nu^\mu(z) &\frac{1}{2} e^{\mu\pi i} \left\{ \Gamma(\mu) \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu + 1; 1 - \mu; \frac{1-z}{2}\right) + \right. \\
&\quad \left. + \frac{\Gamma(-\mu) \Gamma(\nu + \mu + 1)}{\Gamma(\nu - \mu + 1)} \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} F\left(-\nu, \nu + 1; 1 + \mu; \frac{1-z}{2}\right) \right\} \\
&\quad [|\arg(z \pm 1)| < \pi, \quad |1 - z| < 2] .
\end{aligned}$$

MO 86

8.774

$$\begin{aligned}
P_\nu^\mu(i \operatorname{ctg} \varphi) &= \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{\Gamma(-\nu - \mu)} e^{-i(\nu+1)\frac{\pi}{2}} \left(\operatorname{tg} \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \nu + \frac{3}{2}; \sin^2 \frac{\varphi}{2}\right) + \\
&\quad + \sqrt{\frac{\sin \varphi}{2\pi}} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu - \mu + 1)} e^{i\nu\frac{\pi}{2}} \left(\operatorname{ctg} \frac{\varphi}{2}\right)^{\nu+\frac{1}{2}} F\left(\frac{1}{2} + \mu, \frac{1}{2} - \mu; \frac{1}{2} - \nu; \sin^2 \frac{\varphi}{2}\right) \\
&\quad \left[2\nu \neq \pm 1, \pm 3, \pm 5, \dots, 0 < \varphi < \frac{\pi}{2}\right] .
\end{aligned}$$

$$1.6 \quad P_{\nu}^{\mu}(x) = \frac{2^{\mu} \cos\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu - \mu}{2} + 1\right)} (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) +$$

$$+ \frac{2^{\mu+1} \sin\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu - \mu + 1}{2}\right)} \times$$

$$\times x(1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{-\nu + \mu + 1}{2}; \frac{3}{2}; x^2\right).$$

MO 87

1029

$$2.6 \quad Q_{\nu}^{\mu}(x) = -\frac{\sqrt{\pi} \sin\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{2^{1-\mu} \Gamma\left(\frac{\nu - \mu}{2} + 1\right)} (1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu + 1}{2}, \frac{\mu - \nu}{2}; \frac{1}{2}; x^2\right) +$$

$$+ 2^{\mu} \sqrt{\pi} \frac{\cos\left(\frac{1}{2}(\nu + \mu)\pi\right) \Gamma\left(\frac{\nu + \mu}{2} + 1\right)}{\Gamma\left(\frac{\nu - \mu + 1}{2}\right)} \times$$

$$\times x(1 - x^2)^{\frac{\mu}{2}} F\left(\frac{\nu + \mu}{2} + 1, \frac{\mu - \nu + 1}{2}; \frac{3}{2}; x^2\right)$$

MO 87

8.776

For $|z| \gg 1$

$$1. \quad P_{\nu}^{\mu}(z) = \left\{ \frac{2^{\nu} \Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(\nu - \mu + 1)} z^{\nu} + \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{2^{\nu+1} \sqrt{\pi} \Gamma(-\nu - \mu)} z^{-\nu-1} \right\} \left(1 + O\left(\frac{1}{z^2}\right)\right)$$

$$[2\nu \neq \pm 1, \pm 3, \pm 5, \dots, |\arg z| < \pi].$$

MO 87

$$2. \quad Q_{\nu}^{\mu}(z) = \sqrt{\pi} \frac{e^{\mu\pi i} \Gamma(\mu + \nu + 1)}{2^{\nu+1} \Gamma\left(\nu + \frac{3}{2}\right)} z^{-\nu-1} \left(1 + O\left(\frac{1}{z^2}\right)\right) [2\nu \neq -3, -5, -7, \dots; |\arg z| < \pi].$$

MO 87

8.777

Set $\zeta = z + \sqrt{z^2 - 1}$. The variable ζ is uniquely defined by this equation on the entire z -plane in which a cut is made from $-\infty$ to

+1. Here, we are considering that branch of the variable ζ for which values of ζ exceeding 1 correspond to real values of z exceeding 1. In this case,

$$1. \quad P_{\nu}^{\mu}(z) = \frac{2^{\mu} \Gamma\left(-\nu - \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(-\nu - \mu)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu + \mu + 1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) + \\ + \frac{2^{\mu}}{\sqrt{\pi}} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu - \mu + 1)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\mu - \nu}} F\left(\frac{1}{2} + \mu, \mu - \nu; \frac{1}{2} - \nu; \frac{1}{\zeta^2}\right) \\ [2\nu \neq \pm 1, \pm 3, \pm 5, \dots; \quad |\arg(z - 1)| < \pi].$$

MO 86

$$2. \quad Q_{\nu}^{\mu}(z) = 2^{\mu} e^{\mu\pi i} \sqrt{\pi} \frac{\Gamma(\nu + \mu + 1)}{\Gamma\left(\nu + \frac{3}{2}\right)} \frac{(z^2 - 1)^{\frac{\mu}{2}}}{\zeta^{\nu + \mu + 1}} F\left(\frac{1}{2} + \mu, \nu + \mu + 1; \nu + \frac{3}{2}; \frac{1}{\zeta^2}\right) \\ [|\arg(z - 1)| < \pi].$$

MO 86

1030

8.78 The zeros of associated Legendre functions

8.781

The function $P_{\nu}^{-\mu}(\cos \varphi)$, considered as a function of ν has infinitely many zeros for $\mu \geq 0$. These are all simple and real. If a number ν_0 is a zero of the function $P_{\nu}^{-\mu}(\cos \varphi)$, the number $-\nu_0 - 1$ is also a zero of this function.

8.782

If ν and μ are both real and $\mu \leq 0$, or if ν and μ are integers, the function $P_{\nu}^{\mu}(t)$ has no *real* zeros exceeding 1. If ν and μ are both real with $\nu < \mu < 0$, the function $P_{\nu}^{\mu}(t)$ has no real zeros exceeding 1 when $\sin \mu\pi \sin(\mu - \nu)\pi > 0$, but does have one such zero when $\sin \mu\pi \sin(\mu - \nu)\pi < 0$. Finally, if $\mu \leq \nu$, the function $P_{\nu}^{\mu}(t)$ has no zeros exceeding 1 for $[\mu]$ even but does have one zero for $[\mu]$ odd.

8.783

If $\nu > -\frac{3}{2}$ and $\nu + \mu + 1 > 0$, the function $Q_{\nu}^{\mu}(t)$ has no real zeros exceeding 1.

MO 91

8.784

The function $P_{-\frac{1}{2}+i\lambda}(z)$ has infinitely many zeros for real λ . All these zeros are *real* and *greater than unity*.

8.785

For n a natural number, the function $P_n(x)$ has exactly n real zeros which lie in the closed interval $-1, +1$.

8.786

The function $Q_n(z)$ has no zeros for which $|\arg(z-1)| < \pi$ if n is a natural number. The function $Q_n(\cos \varphi)$ has exactly $n+1$ zeros in the interval $0 \leq \varphi \leq \pi$.

MO 91

8.787

The following approximate formula can be used to calculate the values of ν for which the equation $P_\nu^{-\mu}(\cos \varphi) = 0$ holds for given small values of φ :

$$\nu + \frac{1}{2} = -\frac{j_\mu}{2 \sin \frac{\varphi}{2}} \left\{ 1 - \frac{\sin^2 \frac{\varphi}{2}}{6} \left(1 - \frac{4\mu^2 - 1}{j_\mu^2} \right) + O\left(\sin^4 \frac{\varphi}{2}\right) \right\};$$

MO 93

MO 91

Here, j_μ denotes an arbitrary nonzero root of the equation $J_\mu(z) = 0$ (for $\mu \geq 0$). If φ is close to π then, instead of this formula, we can use the following formulas:

$$1. \quad \nu \approx \mu + k + \frac{\Gamma(2\mu + k + 1)}{\Gamma(\mu)\Gamma(\mu + 1)\Gamma(k + 1)} \left(\frac{\pi - \varphi}{3} \right)^{2\mu} \quad [\mu > 0; \quad k = 0, 1, 2, \dots].$$

MO 93

$$2. \quad \nu \approx k + \frac{1}{2 \ln \left(\frac{2}{\pi - \varphi} \right)} \quad [\mu = 0, \quad k = 0, 1, 2, \dots].$$

MO 93

8.79 Series of associated Legendre functions

8.791

$$1. \frac{1}{z-t} = \sum_{k=0}^{\infty} (2k+1)P_k(t)Q_k(z) \quad \left[|t + \sqrt{t^2-1}| < |z + \sqrt{z^2-1}| \right];$$

1031

Here, t must lie inside an ellipse passing through the point z with foci at the points ± 1 .

$$2. \frac{1}{\sqrt{1-2tz+t^2}} \ln \frac{z-t+\sqrt{1-2tz+t^2}}{\sqrt{z^2-1}} = \sum_{k=0}^{\infty} t^k Q_k(z) \quad [\operatorname{Re} z > 1, \quad |t| < 1].$$

MO 78

8.792

$$P_{\nu}^{-\alpha}(\cos \varphi) P_{\nu}^{-\beta}(\cos \psi) = \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right] P_k^{-\alpha}(\cos \varphi) P_k^{-\beta}(\cos \psi) \\ [a \geq 0, \quad \beta \geq 0, \quad \nu \text{ real}, \quad -\pi < \varphi \pm \psi < \pi].$$

MO 94

8.793

$$P_{\nu}^{-\mu}(\cos \varphi) = \frac{\sin \nu \pi}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k^{-\mu}(\cos \varphi) \quad [\mu \geq 0, \quad 0 < \varphi < \pi].$$

MO 94

Addition theorems

8.794

$$1. P_{\nu}(\cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi) = \\ = P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_{\nu}^{-k}(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi; \\ = P_{\nu}(\cos \psi_1) P_{\nu}(\cos \psi_2) + 2 \sum_{k=1}^{\infty} \frac{\Gamma(\nu-k+1)}{\Gamma(\nu+k+1)} P_{\nu}^k(\cos \psi_1) P_{\nu}^k(\cos \psi_2) \cos k\varphi \\ [0 \leq \psi_1 < \pi, \quad 0 \leq \psi_2 < \pi, \quad \psi_1 + \psi_2 < \pi; \quad \varphi \text{ real}] \quad (\text{cf. 8.814, 8.844 1}).$$

$$\begin{aligned}
2. \quad Q_\nu(\cos \psi_1 \cos \psi_2 + \sin \psi_1 \sin \psi_2 \cos \varphi) &= \\
&= P_\nu(\cos \psi_1)Q_\nu(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_\nu^{-k}(\cos \psi_1)Q_\nu^k(\cos \psi_2) \cos k\varphi \\
\left[0 < \psi_1 < \frac{\pi}{2}, \quad 0 < \psi_2 < \pi, \quad 0 < \psi_1 + \psi_2 < \pi; \quad \varphi \text{ real} \right] & \quad (\text{cf. 8.844 3.}).
\end{aligned}$$

8.795

$$\begin{aligned}
1. \quad P_\nu \left(z_1 z_2 - \sqrt{z_1^2 - 1} \sqrt{z_2^2 - 1} \cos \varphi \right) &= P_\nu(z_1)P_\nu(z_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_\nu^k(z_1)P_\nu^{-k}(z_2) \cos k\varphi \\
\left[\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0, \quad |\arg(z_1 - 1)| < \pi, \quad |\arg(z_2 - 1)| < \pi \right] & .
\end{aligned}$$

MO 91

$$\begin{aligned}
2. \quad Q_\nu \left(x_1 x_2 - \sqrt{x_1^2 - 1} \sqrt{x_2^2 - 1} \cos \varphi \right) &= P_\nu(x_1)Q_\nu(x_2) + 2 \sum_{k=1}^{\infty} (-1)^k P_\nu^{-k}(x_1)Q_\nu^k(x_2) \cos k\varphi \\
\left[1 < x_1 < x_2, \quad \nu \neq -1, -2, -3, \dots; \quad \varphi \text{ real} \right] &
\end{aligned}$$

MO 91

$$\begin{aligned}
3. \quad Q_n \left(x_1 x_2 + \sqrt{x_1^2 + 1} \sqrt{x_2^2 + 1} \operatorname{ch} \alpha \right) &= \sum_{k=n+1}^{\infty} \frac{1}{(k-n-1)!(k+n)!} Q_n^k(ix_1)Q_n^k(ix_2)e^{-k\alpha} \\
\left[x_1 > 0, \quad x_2 > 0; \quad \alpha > 0 \right] & .
\end{aligned}$$

MO 91

1032

8.796

$$\begin{aligned}
P_\nu(-\cos \psi_1 \cos \psi_2 - \sin \psi_1 \sin \psi_2 \cos \varphi) &= P_\nu(-\cos \psi_1)P_\nu(\cos \psi_2) + 2 \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(\nu + k + 1)}{\Gamma(\nu - k + 1)} \times \\
&\quad \times P_\nu^{-k}(-\cos \psi_1)P_\nu^k(\cos \psi_2) \cos k\varphi \\
\left[0 < \psi_2 < \psi_1 < \pi; \quad \varphi \text{ real} \right] & \quad (\text{cf. 8.844 2.}).
\end{aligned}$$

See also 8.934 3.

8.81 Associated Legendre functions with integral indices

8.810

For *integral* values of ν and μ , the differential equation 8.700 1. (with $|\nu| > |\mu|$) has a simple solution in the real domain, namely:

$$u = P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

The functions $P_n^m(x)$ are called *associated Legendre functions* (or *spherical functions*) of the first kind. The number n is called the *degree* and the number m is called the *order* of the function $P_n^m(x)$. The functions

$$\cos m\vartheta P_n^m(\cos \varphi), \quad \sin m\vartheta P_n^m(\cos \varphi),$$

which depend on the angles φ and ϑ , are also called Legendre functions of the first kind, or, more specifically, *tesseral harmonics* for $m < n$ and *sectoral harmonics* for $m = n$. These last functions are periodic with respect to the angles φ and ϑ . Their periods are respectively π and 2π . They are single-valued and continuous everywhere on the surface of the unit sphere $x_1^2 + x_2^2 + x_3^2 = 1$ (where $x_1 = \sin \varphi \cos \vartheta$, $x_2 = \sin \varphi \sin \vartheta$, $x_3 = \cos \varphi$) and they are solutions of the differential equation

$$\frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial Y}{\partial \varphi} \right) + \frac{1}{\sin^2 \varphi} \frac{\partial^2 Y}{\partial \vartheta^2} + n(n+1)Y = 0.$$

8.811⁷

The integral representation

$$P_n^m(\cos \varphi) = \frac{(-1)^m (n+m)!}{\Gamma\left(m + \frac{1}{2}\right) (n-m)!} \sqrt{\frac{2}{\pi}} \sin^{-m} \varphi \int_0^\varphi (\cos t - \cos \varphi)^{m-\frac{1}{2}} \cos\left(n + \frac{1}{2}\right) t dt.$$

MO 75

1033

8.812

The series representation:

$$\begin{aligned}
 P_n^m(x) &= \frac{(-1)^m (n+m)!}{2^m m! (n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ 1 - \frac{(n-m)(m+n+1)}{1!(m+1)} \frac{1-x}{2} + \right. \\
 &\quad \left. + \frac{(n-m)(n-m+1)(m+n+1)(m+n+2)}{2!(m+1)(m+2)} \left(\frac{1-x}{2}\right)^2 - \dots \right\}; \\
 &= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} \left\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)} x^{n-m-2} + \right. \\
 &\quad \left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-m-4} - \dots \right\}; \\
 &= \frac{(-1)^m (2n-1)!!}{(n-m)!} (1-x^2)^{\frac{m}{2}} x^{n-m} F\left(\frac{m-n}{2}, \frac{m-n+1}{2}; \frac{1}{2} - n; \frac{1}{x^2}\right).
 \end{aligned}$$

MO 73

8.813

Special cases:

$$1. \quad P_1^1(x) = -(1-x^2)^{\frac{1}{2}} = -\sin \varphi.$$

MO 73

$$2. \quad P_2^1(x) = -3(1-x^2)^{\frac{1}{2}} x = -\frac{3}{2} \sin 2\varphi.$$

MO 73

$$3. \quad P_2^2(x) = 3(1-x^2) = \frac{3}{2}(1 - \cos 2\varphi).$$

MO 73

$$4. \quad P_3^1(x) = -\frac{3}{2}(1-x^2)^{\frac{1}{2}}(5x^2-1) = -\frac{3}{8}(\sin \varphi + 5 \sin 3\varphi).$$

MO 73

$$5. \quad P_3^2(x) = 15(1-x^2)x = \frac{15}{4}(\cos \varphi - \cos 3\varphi).$$

$$6. \quad P_3^3(x) = -15(1-x^2)^{\frac{3}{2}} = -\frac{15}{4}(3 \sin \varphi - \sin 3\varphi).$$

MO 73

Functional relations

For recursion formulas, see 8.731.

8.814

$$\begin{aligned} P_n(\cos \varphi_1 \cos \varphi_2 + \sin \varphi_1 \sin \varphi_2 \cos \Theta) &= \\ &= P_n(\cos \varphi_1)P_n(\cos \varphi_2) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_n^m(\cos \varphi_1)P_n^m(\cos \varphi_2) \cos m\Theta \\ & \quad [0 \leq \varphi_1 \leq \pi, \quad 0 \leq \varphi_2 \leq \pi] \quad (\text{"addition theorem"}). \end{aligned}$$

MO 74

1034

8.815

If

$$Y_{n_1}(\varphi, \vartheta) = a_0 P_{n_1}(\cos \varphi) + \sum_{m=1}^{n_1} (a_m \cos m\vartheta + b_m \sin m\vartheta) P_{n_1}^m(\cos \varphi),$$

$$Z_{n_2}(\varphi, \vartheta) = \alpha_0 P_{n_2}(\cos \varphi) + \sum_{m=1}^{n_2} (\alpha_m \cos m\vartheta + \beta_m \sin m\vartheta) P_{n_2}^m(\cos \varphi),$$

then

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_{n_1}(\varphi, \vartheta) Z_{n_2}(\varphi, \vartheta) = 0,$$

$$\int_0^{2\pi} d\vartheta \int_0^\pi \sin \varphi d\varphi Y_n(\varphi, \vartheta) P_n[\cos \varphi \cos \psi + \sin \varphi \sin \psi \cos(\vartheta - \theta)] = \frac{4\pi}{2n+1} Y_n(\psi, \theta).$$

$$(\cos \varphi + i \sin \varphi \cos \vartheta)^n = P_n(\cos \varphi) + 2 \sum_{m=1}^n (-1)^m \frac{n!}{(n+m)!} \cos m\vartheta P_n^m(\cos \varphi).$$

MO 75

For integrals of the functions, $P_n^m(x)$, see 7.112 1., 7.122 1.

8.82- 8.83 Legendre functions

8.820

The differential equation

$$\frac{d}{dz} \left[(1-z^2) \frac{du}{dz} \right] + \nu(\nu+1)u = 0 \quad (\text{cf. 8.700 1.}),$$

8.700

where the parameter ν can be an arbitrary number, has the following two linearly independent solutions:

$$1. \quad P_\nu(z) = F \left(-\nu, \nu+1; 1; \frac{1-z}{2} \right).$$

$$2. \quad Q_\nu(z) = \frac{\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)} z^{-\nu-1} F \left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{z^2} \right).$$

SM 518(137)

The functions $P_\nu(z)$ and $Q_\nu(z)$ are called *Legendre functions of the first and second kind* respectively. If ν is not an integer, the function $P_\nu(z)$ has *singularities* at $z = -1$ and $z = \infty$. However, if $\nu = n = 0, 1, 2, \dots$, the function $P_\nu(z)$ becomes the *Legendre polynomial* $P_n(z)$ (see 8.91). For $\nu = -n = -1, -2, \dots$, we have

$$P_{-n-1}(z) = P_n(z).$$

3. If $\nu \neq 0, 1, 2, \dots$, the function $Q_\nu(z)$ has singularities at the points $z = \pm 1$ and $z = \infty$. These points are branch points of the function. On the other hand, if $\nu = n = 0, 1, 2, \dots$, the function $Q_n(z)$ is single-valued for $|z| > 1$ and regular for $z = \infty$.

1035

4. In the right half-plane,

$$P_\nu(z) = \left(\frac{1+z}{2} \right)^\nu F \left(-\nu, -\nu; 1; \frac{z-1}{z+1} \right) \quad [\operatorname{Re} z > 0].$$

5. The function $P_\nu(z)$ is uniquely determined by equations 8.820 1. and 8.820 4. within a circle of radius 2 with its center at the point $z = 1$ in the right half-plane.

For $z = x = \cos \varphi$, a solution of equation 8.820 is the function

$$6. \quad P_\nu(x) = P_\nu(\cos \varphi) = F \left(-\nu, \nu + 1; 1; \sin^2 \frac{\varphi}{2} \right);$$

In general,

$$7. \quad P_\nu(z) = P_{-\nu-1}(z) = P_\nu(x) = P_{-\nu-1}(x), \quad \text{for } z = x.$$

8. The function $Q_\nu(z)$ for $|z| > 1$ is uniquely determined by equation 8.820 2. everywhere in the z -plane in which a cut is made from the point $z = -\infty$ to the point $z = 1$. By means of a hypergeometric series, the function can be continued analytically inside the unit circle. On the cut ($-1 \leq x \leq +1$) of the real axis, the function $Q_\nu(x)$ is determined by the equation

$$9. \quad Q_\nu(x) = \frac{1}{2} [Q_\nu(x + i0) + Q_\nu(x - i0)].$$

HO 52(53), WH

Integral representations

8.821

Here, A is a point on the real axis to the right of the point $t = 1$ and to the right of z if z is real. At the point A , we set

$$\arg(t - 1) = \arg(t + 1) = 0 \quad \text{and} \quad [|\arg(t - z)| < \pi].$$

WH

$$2. \quad Q_\nu(z) = \frac{1}{4i \sin \nu\pi} \int_A^{(1-, 1+)} \frac{(t^2 - 1)^\nu}{2^\nu (z - t)^{\nu+1}} dt.$$

[ν is not an integer; the point A is at the end of the major axis of an ellipse to the right of $t = 1$ drawn in the t -plane with foci at the points ± 1 and with a minor axis sufficiently small that the point z lies outside it. The contour begins at the point A , follows the path $(1-, -1+)$ and returns to A ; $|\arg z| \leq \pi$ and $|\arg(z - t)| \rightarrow \arg z$ as $t \rightarrow 0$ on the contour; $\arg(t + 1) = \arg(t - 1) = 0$ at the point A ; z does not lie on the real axis between -1 and 1 .]

For $\nu = n$ an integer,

$$3. \quad Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^1 (1 - t^2)^n (z - t)^{-n-1} dt.$$

SM 517(134), WH

1036

8.822

$$1. \quad P_\nu(z) = \frac{1}{\pi} \int_0^\pi \frac{d\varphi}{(z + \sqrt{z^2 - 1} \cos \varphi)^{\nu+1}} = \frac{1}{\pi} \int_0^\pi (z + \sqrt{z^2 - 1} \cos \varphi)^\nu d\varphi$$

$$\left[\operatorname{Re} z > 0 \quad \text{and} \quad \arg \left\{ z + \sqrt{z^2 - 1} \cos \varphi \right\} = \arg z \quad \text{for} \quad \varphi = \frac{\pi}{2} \right].$$

WH

$$2. \quad Q_\nu(z) = \int_0^\infty \frac{d\varphi}{(z + \sqrt{z^2 - 1} \operatorname{ch} \varphi)^{\nu+1}},$$

$$\left[\operatorname{Re} \nu > -1; \quad \text{if } \nu \text{ is not an integer, } \left\{ (z + \sqrt{z^2 - 1}) \operatorname{ch} \varphi \right\} \text{ for } \varphi = 0 \text{ has its principal value} \right].$$

WH

8.823

$$P_\nu(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos\left(\nu + \frac{1}{2}\right) \varphi}{\sqrt{2(\cos \varphi - \cos \theta)}} d\varphi.$$

WH

8.824

$$\begin{aligned} Q_n(z) &= 2^n n! \int_z^\infty \dots \int_z^\infty \frac{(dz)^{n+1}}{(z^2 - 1)^{n+1}} = 2^n \int_z^\infty \frac{(t - z)^n}{(t^2 - 1)^{n+1}} dt; \\ &= \frac{(-1)^n}{(2n - 1)!!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \int_z^\infty \frac{dt}{(t^2 - 1)^{n+1}} \right] \quad [\operatorname{Re} z > 1]. \end{aligned}$$

WH, MO 78

8.825

$$Q_n(z) = \frac{1}{2} \int_{-1}^1 \frac{P_n(t)}{z - t} dt \quad [|\arg(z - 1)| < \pi].$$

WH, MO 78

See also 6.622 3., 8.842.

8.826

Fourier series:

$$\begin{aligned} 1. \quad P_n(\cos \varphi) &= \frac{2^{n+2}}{\pi} \frac{n!}{(2n + 1)!!} \left[\sin(n + 1)\varphi + \frac{1}{1} \frac{n + 1}{2n + 3} \sin(n + 3)\varphi + \right. \\ &\quad \left. + \frac{1 \cdot 3(n + 1)(n + 2)}{1 \cdot 2(2n + 3)(2n + 5)} \sin(n + 5)\varphi + \dots \right] \quad [0 < \varphi < \pi]. \end{aligned}$$

MO 79

$$\begin{aligned} 2. \quad Q_n(\cos \varphi) &= 2^{n+1} \frac{n!}{(2n + 1)!!} \left[\cos(n + 1)\varphi + \frac{1}{1} \frac{n + 1}{2n + 3} \cos(n + 3)\varphi + \right. \\ &\quad \left. + \frac{1 \cdot 3}{1 \cdot 2} \frac{(n + 1)(n + 2)}{(2n + 3)(2n + 5)} \cos(n + 5)\varphi + \dots \right] \quad [0 < \varphi < \pi]. \end{aligned}$$

MO 79

The expressions for Legendre functions in terms of a hypergeometric function (see 8.820) provide other series representations of these functions.

Special cases and particular values

8.827

$$1. \quad Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x} = \operatorname{Arth} x.$$

JA

1037

$$2. \quad Q_1(x) = \frac{x}{2} \ln \frac{1+x}{1-x} - 1.$$

JA

$$3. \quad Q_2(x) = \frac{1}{4}(3x^2 - 1) \ln \frac{1+x}{1-x} - \frac{3}{2}x.$$

JA

$$4. \quad Q_3(x) = \frac{1}{4}(5x^3 - 3x) \ln \frac{1+x}{1-x} - \frac{5}{2}x^2 + \frac{2}{3}.$$

JA

$$5. \quad Q_4(x) = \frac{1}{16}(35x^4 - 30x^2 + 3) \ln \frac{1+x}{1-x} - \frac{35}{8}x^3 + \frac{55}{24}x.$$

JA

$$6. \quad Q_5(x) = \frac{1}{16}(63x^5 - 70x^3 + 15x) \ln \frac{1+x}{1-x} - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}.$$

JA

8.828

$$1. \quad P_\nu(1) = 1.$$

$$2. P_\nu(0) = -\frac{1}{2} \frac{\sin \nu\pi}{\sqrt{\pi^3}} \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right).$$

MO 79

8.829

$$Q_\nu(0) = \frac{1}{4\sqrt{\pi}} (1 - \cos \nu\pi) \Gamma\left(\frac{\nu+1}{2}\right) \Gamma\left(-\frac{\nu}{2}\right).$$

MO 79

Functional relationships

8.831

$$1. Q_\nu(x) = \frac{\pi}{2 \sin \nu\pi} [\cos \nu\pi P_\nu(x) - P_\nu(-x)] \quad [\nu \neq 0, \pm 1, \pm 2, \dots].$$

MO 76

$$2. Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x) \quad [n = 0, 1, 2, \dots],$$

where

$$3. W_{n-1}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2(n-2k)-1}{(2k+1)(n-k)} P_{n-2k-1}(x) = \sum_{k=1}^n \frac{1}{k} P_{k-1}(x) P_{n-k}(x)$$

and

$$W_{-1}(x) \equiv 0 \quad (\text{see also 8.839}).$$

8.839

SM 516(131), MO 76

$$4. \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{\nu+k+1} \right) P_k(\cos \varphi) = \frac{\pi}{\sin \nu\pi} P_\nu(\cos \varphi)$$

$[\nu \text{ not an integer; } 0 \leq \varphi < \pi].$

$$5. \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu - k} - \frac{1}{\nu + k + 1} \right) P_k(\cos \varphi) P_k(\cos \psi) = \frac{\pi}{\sin \nu \pi} P_\nu(\cos \varphi) P_\nu(\cos \psi)$$

[ν not an integer, $-\pi < \varphi + \psi < \pi$, $-\pi < \varphi - \psi < \pi$].

MO 77

See also 8.521 4.

1038

8.832

$$1. (z^2 - 1) \frac{d}{dz} P_\nu(z) = (\nu + 1) [P_{\nu+1}(z) - zP_\nu(z)].$$

WH

$$2. (2\nu + 1)zP_\nu(z) = (\nu + 1)P_{\nu+1}(z) + \nu P_{\nu-1}(z).$$

WH

$$3. (z^2 - 1) \frac{d}{dz} Q_\nu(z) = (\nu + 1) [Q_{\nu+1}(z) - zQ_\nu(z)].$$

WH

$$4. (2\nu + 1)zQ_\nu(z) = (\nu + 1)Q_{\nu+1}(z) + \nu Q_{\nu-1}(z).$$

WH

8.833

$$1. P_\nu(-z) = e^{\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z) \quad [\text{Im } z < 0].$$

MO 77

$$2. P_\nu(-z) = e^{-\nu\pi i} P_\nu(z) - \frac{2}{\pi} \sin \nu\pi Q_\nu(z) \quad [\text{Im } z > 0].$$

MO 77

$$3. Q_\nu(-z) = -e^{-\nu\pi i} Q_\nu(z) \quad [\text{Im } z < 0].$$

$$4. Q_\nu(-z) = -e^{\nu\pi i} Q_\nu(z) \quad [\operatorname{Im} z > 0].$$

MO 77

8.834

$$1. Q_\nu(x \pm i0) = Q_\nu(x) \mp \frac{\pi i}{2} P_\nu(x).$$

MO 77

$$2. Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z) \quad (\text{see 8.831 3.}).$$

8.831
MO 77

8.835

$$1. Q_\nu(z) - Q_{-\nu-1}(z) = \pi \operatorname{ctg} \nu\pi P_\nu(z) \quad [\sin \nu\pi \neq 0].$$

MO 77

$$2. Q_{-\nu-1}(\cos \varphi) = Q_\nu(\cos \varphi) - \pi \operatorname{ctg} \nu\pi P_\nu(\cos \varphi) \quad [\sin \nu\pi \neq 0].$$

MO 77

$$3. Q_\nu(-\cos \varphi) = -\cos \nu\pi Q_\nu(\cos \varphi) - \frac{\pi}{2} \sin \nu\pi P_\nu(\cos \varphi).$$

MO 77

8.836

$$1. Q_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} \left[(z^2 - 1)^n \ln \frac{z+1}{z-1} \right] - \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1}.$$

MO 79

$$2. Q_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n \ln \frac{1+x}{1-x} \right] - \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x}.$$

$$1. \quad P_\nu(x) = P_\nu(\cos \varphi) = F\left(-\nu, \nu + 1; 1; \sin^2 \frac{\varphi}{2}\right) \quad (\text{cf. 8.820 6.}).$$

8.820
MO 76

1039

$$2. \quad P_\nu(z) = \frac{\operatorname{tg} \nu\pi}{2^{\nu+1}\sqrt{\pi}} \frac{\Gamma(\nu+1)}{\Gamma\left(\nu + \frac{3}{2}\right)} z^{-\nu-1} F\left(\frac{\nu}{2} + 1, \frac{\nu+1}{2}; \nu + \frac{3}{2}; \frac{1}{z^2}\right) + \\ + \frac{2^\nu}{\sqrt{\pi}} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu+1)} z^\nu F\left(\frac{1-\nu}{2}, -\frac{\nu}{2}; \frac{1}{2} - \nu; \frac{1}{z^2}\right).$$

MO 78

See also 8.820.

For integrals of Legendre functions, see 7.1-7.2.

8.838

Inequalities:

$$1. \quad |P_\nu(\cos \varphi) - P_{\nu+2}(\cos \varphi)| \leq 2C_0 \sqrt{\frac{1}{\nu\pi}}.$$

MO 78

$$2. \quad |Q_\nu(\cos \varphi) - Q_{\nu+2}(\cos \varphi)| < C_0 \sqrt{\frac{\pi}{\nu}}.$$

MO 78

$[0 \leq \varphi \leq \pi, \nu > 1, C_0$ is a number that does not depend on the values of ν or φ].

With regard to the zeros of Legendre functions of the second kind, see 8.784, 8.785, and 8.786. For the expansion of Legendre functions in series of associated Legendre functions, see 8.794, 8.795, and 8.796.

8.839

A differential equation leading to the functions $W_{n-1}(x)$ (see 8.831 3.):

$$(1-x^2)\frac{d^2W_{n-1}}{dx^2} - 2x\frac{dW_{n-1}}{dx} + (n+1)nW_{n-1} = 2\frac{dP_n(x)}{dx}.$$

MO 76

8.84 Conical functions

8.840

Let us set

$$\nu = -\frac{1}{2} + i\lambda,$$

where λ is a real parameter, in the defining differential equation 8.700 1. for associated Legendre functions. We then obtain the differential equation of the so-called conical functions. A conical function is a special case of the associated Legendre function. However, the Legendre functions

$$P_{-\frac{1}{2}+i\lambda}(x), \quad Q_{-\frac{1}{2}+i\lambda}(x)$$

have certain peculiarities that make us distinguish them as a special class--the class of conical functions. The most important of these peculiarities is the following

8.841

The functions

$$P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\varphi}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 4^2} \sin^4 \frac{\varphi}{2} + \dots$$

1040

are real for real values of φ . Also,

$$P_{-\frac{1}{2}+i\lambda}(x) \equiv P_{-\frac{1}{2}-i\lambda}(x).$$

8.842

Integral representations:

$$1. \quad P_{-\frac{1}{2}+i\lambda}(\cos \varphi) = \frac{2}{\pi} \int_0^\varphi \frac{\operatorname{ch} \lambda u du}{\sqrt{2(\cos u - \cos \varphi)}} = \frac{2}{\pi} \operatorname{ch} \lambda \pi \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\cos \varphi + \operatorname{ch} u)}}.$$

MO 95

$$2.^6 \quad Q_{-\frac{1}{2}\mp i\lambda}(\cos \varphi) = \pm i \operatorname{sh} \lambda \pi \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\operatorname{ch} u + \cos \varphi)}} + \int_0^\infty \frac{\cos \lambda u du}{\sqrt{2(\operatorname{ch} u - \cos \varphi)}}.$$

MO 95

Functional relations (see also 8.73)

8.843

$$P_{-\frac{1}{2}+i\lambda}(-\cos \varphi) = \frac{\operatorname{ch} \lambda \pi}{\pi} \left[Q_{-\frac{1}{2}+i\lambda}(\cos \varphi) + Q_{-\frac{1}{2}-i\lambda}(\cos \varphi) \right].$$

MO 95

8.844

$$\begin{aligned} 1. \quad & P_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) = \\ & = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2) \dots [4\lambda^2 + (2k-1)^2]} \\ & \quad \left[0 < \vartheta < \frac{\pi}{2}, \quad 0 < \psi < \pi, \quad 0 < \psi + \vartheta < \pi \right] \quad (\text{cf. 8.794 1.}). \end{aligned}$$

8.794

MO 95

$$\begin{aligned} 2. \quad & P_{-\frac{1}{2}+i\lambda}(-\cos \psi \cos \vartheta - \sin \psi \sin \vartheta \cos \varphi) = \\ & = P_{-\frac{1}{2}+i\lambda}(\cos \psi) P_{-\frac{1}{2}+i\lambda}(-\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) P_{-\frac{1}{2}+i\lambda}^k(-\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1)(4\lambda^2 + 3^2) \dots [4\lambda^2 + (2k-1)^2]} \\ & \quad \left[0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. 8.796}). \end{aligned}$$

$$\begin{aligned}
3. \quad Q_{-\frac{1}{2}+i\lambda}(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) &= \\
&= P_{-\frac{1}{2}+i\lambda}(\cos \psi) Q_{-\frac{1}{2}+i\lambda}(\cos \vartheta) + 2 \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} P_{-\frac{1}{2}+i\lambda}^k(\cos \psi) Q_{-\frac{1}{2}+i\lambda}^k(\cos \vartheta) \cos k\varphi}{(4\lambda^2 + 1)(4\lambda^2 + 3^2) \dots [4\lambda^2 + (2k - 1)^2]} \\
&\quad \left[0 < \psi < \frac{\pi}{2} < \vartheta, \quad \psi + \vartheta < \pi \right] \quad (\text{cf. 8.794 2.}).
\end{aligned}$$

8.794
MO 96

Regarding the zeros of conical functions, see 8.784.

8.85 Toroidal functions* Sometimes called torus functions.

1041
8.850

Solutions of the differential equation

$$1. \quad \frac{d^2 u}{d\eta^2} + \frac{\text{ch } \eta}{\text{sh } \eta} \frac{du}{d\eta} - \left(n^2 - \frac{1}{4} + \frac{m^2}{\text{sh}^2 \eta} \right) u = 0,$$

are called toroidal functions. They are equivalent (under a coordinate transformation) to associated Legendre functions. In particular, the functions

$$P_{n-\frac{1}{2}}^m(\text{ch } \eta), \quad Q_{n-\frac{1}{2}}^m(\text{sh } \eta).$$

MO 96

are solutions of equation 8.850 1.

The following formulas, obtained from the formulas obtained earlier for associated Legendre functions, are valid for toroidal functions:

8.851

Integral representations:

$$1. \quad P_{n-\frac{1}{2}}^m(\text{ch } \eta) = \frac{\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n-m+\frac{1}{2}\right)} \frac{(\text{sh } \eta)^m}{2^m \sqrt{\pi} \Gamma\left(m+\frac{1}{2}\right)} \int_0^\pi \frac{\sin^{2m} \varphi d\varphi}{(\text{ch } \eta + \text{sh } \eta \cos \varphi)^{n+m+\frac{1}{2}}} =$$

$$\begin{aligned}
2. \quad Q_{n-\frac{1}{2}}^m(\operatorname{ch} \eta) &= (-1)^m \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(n - m + \frac{1}{2}\right)} \int_0^\infty \frac{\operatorname{ch} mtdt}{(\operatorname{ch} \eta + \operatorname{sh} \eta \operatorname{ch} t)^{n+\frac{1}{2}}}; \quad [n \geq m] \\
&= (-1)^m \frac{\Gamma\left(n + m + \frac{1}{2}\right)}{\Gamma\left(n + \frac{1}{2}\right)} \int_0^{\ln \operatorname{cth} \frac{\eta}{2}} (\operatorname{ch} \eta - \operatorname{sh} \eta \operatorname{ch} t)^{n-\frac{1}{2}} \operatorname{ch} mtdt.
\end{aligned}$$

MO 96

8.852

Functional relations:

$$\begin{aligned}
1. \quad Q_{n-\frac{1}{2}}^m(\operatorname{ch} \eta) &= (-1)^m \frac{2^m \Gamma\left(n + m + \frac{1}{2}\right) \sqrt{\pi}}{\Gamma(n+1)} \operatorname{sh}^m \eta e^{-(n+m+\frac{1}{2})\eta} \times \\
&\quad \times F\left(m + \frac{1}{2}, n + m + \frac{1}{2}; n + 1; e^{-2\eta}\right).
\end{aligned}$$

MO 96

1042

$$2. \quad P_{n-\frac{1}{2}}^{-m}(\operatorname{ch} \eta) = \frac{2^{-2m}}{\Gamma(m+1)} (1 - e^{-2\eta})^m e^{-(n+\frac{1}{2})\eta} F\left(m + \frac{1}{2}, n + m + \frac{1}{2}; 2m + 1; 1 - e^{-2\eta}\right).$$

MO 96

8.853

An asymptotic representation $P_{n-\frac{1}{2}}(\operatorname{ch} \eta)$ for large values of n :

$$\begin{aligned}
P_{n-\frac{1}{2}}(\operatorname{ch} \eta) &= \frac{\Gamma(n) e^{(n-\frac{1}{2})\eta}}{\sqrt{\pi} \Gamma\left(n + \frac{1}{2}\right)} \times \\
&\quad \times \left[\frac{2\Gamma^2\left(n + \frac{1}{2}\right)}{\pi n! \Gamma(n)} \ln(4e^\eta) e^{-2n\eta} F\left(\frac{1}{2}, n + \frac{1}{2}; n + 1; e^{-2\eta}\right) + A + B \right],
\end{aligned}$$

where

Here,

$$u_r = \sum_{s=1}^r \frac{1}{s}, \quad v_{r-\frac{1}{2}} = \sum_{s=1}^r \frac{2}{2s-1} \quad [r - \text{a natural number}].$$

MO 97

8.9 Orthogonal Polynomials

8.90 Introduction

8.901

Suppose that $w(x)$ is a nonnegative real function of a real variable x . Let (a, b) be a fixed interval on the x -axis. Let us suppose further that, for $n = 0, 1, 2, \dots$, the integral

$$\int_a^b x^n w(x) dx$$

exists and that the integral

$$\int_a^b w(x) dx$$

is positive. In this case, there exists a sequence of polynomials $p_0(x), p_1(x), \dots, p_n(x), \dots$, that is uniquely

1043

determined by the following conditions:

1. $p_n(x)$ is a polynomial of degree n and the coefficient of x^n in this polynomial is positive.
2. The polynomials $p_0(x), p_1(x), \dots$ are orthonormal; that is,

We say that the polynomials $p_n(x)$ constitute a system of orthogonal polynomials on the interval (a, b) with the weight function $w(x)$.

8.902

If q_n is the coefficient of x^n in the polynomial $p_n(x)$, then

$$1. \sum_{k=0}^n p_k(x)p_k(y) = \frac{q_n}{q_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y} \quad (\text{Darboux-Christoffel formula})$$

EH II 159(10)

$$2. \sum_{k=0}^n [p_k(x)]^2 = \frac{q_n}{q_{n+1}} [p_n(x)p'_{n+1}(x) - p'_n(x)p_{n+1}(x)] .$$

EH II 159(11)

8.903

Between any three consecutive orthogonal polynomials, there is a dependence

$$p_n(x) = (A_n x + B_n)p_{n-1}(x) - C_n p_{n-2}(x) \quad [n = 2, 3, 4, \dots] .$$

In this formula, $A_n, B_n,$ and C_n are constants and

$$A_n = \frac{q_n}{q_{n-1}}, \quad C_n = \frac{q_n q_{n-2}}{q_{n-1}^2} .$$

MO 102

8.904

Examples of normalized systems of orthogonal polynomials:

Notation and name	Interval	Weight
$\left(n + \frac{1}{2}\right)^{\frac{1}{2}} P_n(x)$, see 8.91	$(-1, +1)$	1
$2^\lambda \Gamma(\lambda) \left[\frac{(n + \lambda)n!}{2\pi \Gamma(2\lambda + n)} \right]^{\frac{1}{2}} C_n^\lambda(x)$, see 8.93	$(-1, +1)$	$(1 - x^2)^{\lambda - \frac{1}{2}}$
$\sqrt{\frac{\varepsilon_n}{\pi}} T_n(x)$, $\varepsilon_0 = 1, \varepsilon_n = 2$ for $n = 1, 2, 3, \dots$, see 8.94	$(-1, +1)$	$(1 - x^2)^{-\frac{1}{2}}$
$2^{-\frac{n}{2}} \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} H_n(x)$, see 8.95	$(-\infty, \infty)$	e^{-x^2}
$\left[\frac{\Gamma(n + 1)\Gamma(\alpha + \beta + 1 + n)}{\Gamma(\alpha + 1 + n)\Gamma(\beta + 1 + n)} \right]^{\frac{1}{2}} P_n^{(\alpha, \beta)}(x)$, see 8.96	$(-1, +1)$	$(1 - x)^\alpha (1 + x)^\beta$

8.95

8.94

8.93

8.91

Cf. 7.221 1., 7.313, 7.343, 7.374 1., 7.391 1., 7.414 3.

1044

8.91 Legendre polynomials

8.910

Definition. The Legendre polynomials $P_n(z)$ are polynomials satisfying equation 8.700 1. with $\mu = 0$ and $\nu = n$: that is, they satisfy the equation

$$1. \quad (1 - z^2) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + n(n+1)u = 0.$$

This equation has a polynomial solution if, and only if, n is an integer. Thus, Legendre polynomials constitute a special type of associated Legendre function.

Legendre polynomials of degree n are of the form

$$2. \quad P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n.$$

8.911

Legendre polynomials written in expanded form:

$$1. \quad P_n(z) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} z^{n-2k} =$$

$$= \frac{(2n)!}{2^n (n!)^2} \left(z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} z^{n-4} - \dots \right);$$

$$= \frac{(2n-1)!!}{n!} z^n F \left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2} \right).$$

$$2. \quad P_{2n}(z) = (-1)^n \frac{(2n-1)!!}{2^n n!} \left(1 - \frac{2n(2n+1)}{2!} z^2 + \frac{2n(2n-2)(2n+1)(2n+3)}{4!} z^4 - \dots \right);$$

$$= (-1)^n \frac{(2n-1)!!}{2^n n!} F \left(-n, n + \frac{1}{2}; \frac{1}{2}; z^2 \right).$$

AD (9002), MO 69

$$3. \quad P_{2n+1}(z) = (-1)^n \frac{(2n+1)!!}{2^n n!} \left(z - \frac{2n(2n+3)}{3!} z^3 + \frac{2n(2n-2)(2n+3)(2n+5)}{5!} z^5 - \dots \right);$$

$$= (-1)^n \frac{(2n+1)!!}{2^n n!} z F \left(-n, n + \frac{3}{2}; \frac{3}{2}; z^2 \right).$$

AD (9002), MO 69

$$4. \quad P_n(\cos \varphi) = \frac{(2n-1)!!}{2^n n!} \left(\cos n\varphi + \frac{1}{1} \frac{n}{2n-1} \cos(n-2)\varphi + \right.$$

$$+ \frac{1 \cdot 3}{1 \cdot 2} \frac{n(n-1)}{(2n-1)(2n-3)} \cos(n-4)\varphi +$$

$$\left. + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos(n-6)\varphi - \dots \right).$$

WH

$$5. \quad P_{2n}(\cos \varphi) = (-1)^n \frac{(2n-1)!!}{2^n n!} \times$$

$$\times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{2!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n-1)!!} \cos^{2n} \varphi \right\}.$$

AD (9011)

1045

$$6. \quad P_{2n+1}(\cos \varphi) = (-1)^n \frac{(2n+1)!!}{2^n n!} \cos \varphi \times$$

$$\times \left\{ \sin^{2n} \varphi - \frac{(2n)^2}{3!} \sin^{2n-2} \varphi \cos^2 \varphi + \dots + (-1)^n \frac{2^n n!}{(2n+1)!!} \cos^{2n} \varphi \right\}.$$

AD (9012)

$$7. \quad P_n(z) = \sum_{k=0}^n \frac{(-1)^k (n+k)!}{(n-k)!(k!)^2 2^{k+1}} [(1-z)^k + (-1)^n (1+z)^k].$$

WH

8.912

Special cases:

1. $P_0(x) = 1.$

JA

2. $P_1(x) = x = \cos \varphi.$

JA

3. $P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3 \cos 2\varphi + 1).$

JA

4. $P_3(x) = \frac{1}{2}(5x^3 - 3x) = \frac{1}{8}(5 \cos 3\varphi + 3 \cos \varphi).$

JA

5. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) = \frac{1}{64}(35 \cos 4\varphi + 20 \cos 2\varphi + 9).$

JA

6. $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) = \frac{1}{128}(63 \cos 5\varphi + 35 \cos 3\varphi + 30 \cos \varphi).$

JA

7.* $P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5) = \frac{1}{512}(231 \cos 6\varphi + 126 \cos 4\varphi + 105 \cos 2\varphi + 50).$

8.* $P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x) =$
 $= \frac{1}{1024}(429 \cos 7\varphi + 231 \cos 5\varphi + 189 \cos 3\varphi + 175 \cos \varphi).$

$$\begin{aligned}
 9.* \quad P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35) \\
 &= \frac{1}{16384}(6435 \cos 8\varphi - 3432 \cos 6\varphi + 2772 \cos 4\varphi - 2520 \cos 2\varphi + 1225).
 \end{aligned}$$

8.913

Integral representations:

$$1. \quad P_n(\cos \varphi) = \frac{2}{\pi} \int_{\varphi}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right)t}{\sqrt{2(\cos \varphi - \cos t)}} dt.$$

WH

See also 3.611 3., 3.661 3., 4.

2.⁷ Schläfli's integral formula:

$$P_n(z) = \frac{1}{2\pi i} \int_C \frac{(t^2 - 1)^n}{2^n(t - z)^{n+1}} dt,$$

with C a closed contour containing z .

1046

3.* Laplace integral formula:

$$P_n(z) = \frac{1}{\pi} \int_0^{\pi} \left[x + (x^2 - 1)^{1/2} \cos \varphi \right]^n d\varphi, \quad |x| \leq 1$$

SA 180(19)
SA 175(9)

Functional relations

8.914

Recurrence formulas:

$$1. \quad (n + 1)P_{n+1}(z) - (2n + 1)zP_n(z) + nP_{n-1}(z) = 0$$

WH

8.915

$$1.^* \sum_{k=0}^n (2k+1)P_k(x)P_k(y) = (n+1) \frac{P_n(x)P_{n+1}(y) - P_n(y)P_{n+1}(x)}{y-x}.$$

(Christoffel summation formula)

MO 70

$$1(1).^* (y-x) \sum_{k=0}^n (2k+1)P_k(x)Q_k(y) = 1-(n+1) [P_{n+1}(x)Q_n(y) - P_n(x)Q_{n+1}(y)].$$

AS 335(8.9.2)

$$2.^7 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (2n-4k-1)P_{n-2k-1}(z) = P'_n(z) \quad (\text{summation theorem})$$

MO 70

$$3.^7 \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} (2n-4k-3)P_{n-2k-2}(z) = zP'_n(z) - nP_n(z)$$

SM 491(42), WH

$$4.^* \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (2n-4k+1) [k(2n-2k+1) - 2] P_{n-2k}(z) = z^2 P''_n(z) - n(n-1)P_n(z).$$

WH

$$5.^* \sum_{k=0}^m \frac{a_{m-k}a_k a_{n-k}}{a_{n+m-k}} \left(\frac{2n+2m-4k+1}{2n+2m-2k+1} \right) P_{n+m-2k}(z) = P_n(z)P_m(z)$$

$$\left[a_k = \frac{(2k-1)!!}{k!}, \quad m \leq n \right].$$

AD (9036)

8.916

$$1. \quad P_n(\cos \varphi) = \frac{(2n-1)!!}{2^n n!} e^{\mp i n \varphi} F\left(\frac{1}{2}, -n; \frac{1}{2} - n; e^{\pm 2i\varphi}\right).$$

$$2. P_n(\cos \varphi) = F\left(n+1, -n; 1; \sin^2 \frac{\varphi}{2}\right).$$

MO 69

$$3. P_n(\cos \varphi) = (-1)^n F\left(n+1, -n; 1; \cos^2 \frac{\varphi}{2}\right).$$

WH

1047

$$4. P_n(\cos \varphi) = \cos^n \varphi F\left(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; 1; -\operatorname{tg}^2 \varphi\right).$$

HO 23

$$5. P_n(\cos \varphi) = \cos^{2n} \frac{\varphi}{2} F\left(-n, -n; 1; -\operatorname{tg}^2 \frac{\varphi}{2}\right).$$

HO 23, 29, WH

See also 8.911 1., 8.911 2., 8.911 3. For a connection with other functions, see 8.936 3., 8.836, 8.962 2. For integrals of Legendre polynomials, see 7.22-7.25. For the zeros of Legendre polynomials, see 8.785.

8.917

Inequalities:

$$1. \text{ For } x > 1 P_0(x) < P_1(x) < P_2(x) < \dots < P_n(x) < \dots$$

MO 71

$$2. \text{ For } x > -1 P_0(x) + P_1(x) + \dots + P_n(x) > 0.$$

MO 71

$$3. [P_n(\cos \varphi)]^2 > \frac{\sin(2n+1)\varphi}{(2n+1)\sin \varphi} \quad [0 < \varphi < \pi].$$

$$4. \quad \sqrt{n \sin \varphi} |P_n(\cos \varphi)| \leq 1.$$

MO 71

$$5. \quad |P_n(\cos \varphi)| \leq 1.$$

WH

6.* Let $n \geq 2$. The successive relative maxima of $|P_n(x)|$, when x decreases from 1 to 0, form a decreasing sequence. More precisely, if $\mu_1, \mu_2, \dots, \mu_{[n/2]}$ denote these maxima corresponding to decreasing values of x , we have

$$1 > \mu_1 > \mu_2 > \dots > \mu_{[n/2]}.$$

SZ 162(7.3.1)

7.* Let $n \geq 2$. The successive relative maxima of $(\sin \theta)^{\frac{1}{2}} |P_n(\cos \theta)|$ when θ increases from 0 to $\pi/2$, form an increasing sequence.

8.* We have

$$(\sin \theta)^{\frac{1}{2}} |P_n(\cos \theta)| < (2/\pi)^{\frac{1}{2}} n^{-\frac{1}{2}}, \quad 0 \leq \theta \leq \pi.$$

SZ 163(7.3.8)

SZ 163(7.3.2)

Here the constant $(2/\pi)^{\frac{1}{2}}$ cannot be replaced by a smaller one.

$$9.* \max_{0 \leq \theta \leq \pi} (\sin \theta)^{\frac{1}{2}} |P_n(\cos \theta)| \cong (2/\pi)^{\frac{1}{2}} n^{-\frac{1}{2}}, \quad n \rightarrow \infty.$$

10.* Stieltjes' first theorem:

$$|P_n(\cos \theta)| \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{4}{\sqrt{n \sin \theta}}, \quad n = 1, 2, \dots, 0 < \theta < \pi.$$

SA 197(8)

SZ 164(7.3.12)

11.* Stieltjes' second theorem:

SA 199(15)

$$12.* \quad \left| \frac{dP_n(x)}{dx} \right| < \frac{2}{\sqrt{\pi}} \frac{\sqrt{n}}{1-x^2}, \quad |x| < 1, \quad n = 1, 2, \dots$$

SA 201(18)

$$13.* \quad |P_{n+1}(x) + P_n(x)| < 6 \left(\frac{2}{\pi n} \right)^{\frac{1}{2}} (1-x)^{-\frac{1}{2}}, \quad |x| < 1, \quad n = 0, 1, \dots$$

SA 201(19)

1048
8.918

Asymptotic approximations:

$$1. \quad P_n(\cos \theta) = \left(\frac{2}{\pi n \sin \varphi} \right)^{\frac{1}{2}} \cos \left[\left(n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] + O(n^{-\frac{3}{2}}), \quad \varepsilon \leq \theta \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2m$$

(Laplace's formula).

SA 208(1)

$$2. \quad P_n(\cos \theta) = \left(\frac{2}{\pi n \sin \theta} \right)^{\frac{1}{2}} \left\{ \left(1 - \frac{1}{4n} \right) \cos \left[\left(n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] + \frac{1}{8n} \cos \theta \sin \left[\left(n + \frac{1}{2} \right) \theta - \frac{\pi}{4} \right] \right\} + O(n^{-\frac{5}{2}}), \quad \varepsilon \leq \theta \leq \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2$$

(Bonnet-Heine formula).

SA 208(2)

8.92 Series of Legendre polynomials

8.921

The generating function:

$$\begin{aligned} \frac{1}{\sqrt{1-2tz+t^2}} &= \sum_{k=0}^{\infty} t^k P_k(z) && \left[|t| < \min |z \pm \sqrt{z^2-1}| \right]; \\ &= \sum_{k=0}^{\infty} \frac{1}{t^{k+1}} P_k(z) && \left[|t| > \max |z \pm \sqrt{z^2-1}| \right]. \end{aligned}$$

$$1. \quad z^{2n} = \frac{1}{2n+1} P_0(z) + \sum_{k=1}^{\infty} (4k+1) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+1)(2n+3)\dots(2n+2k+1)} P_{2k}(z).$$

MO 72

$$2. \quad z^{2n+1} = \frac{3}{2n+3} P_1(z) + \sum_{k=1}^{\infty} (4k+3) \frac{2n(2n-2)\dots(2n-2k+2)}{(2n+3)(2n+5)\dots(2n+2k+3)} P_{2k+1}(z).$$

MO 72

$$3. \quad \frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 P_{2k}(x) \quad [|x| < 1, \quad (-1)!! \equiv 1].$$

MO 72, LA 385(15)

$$4. \quad \frac{x}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+3) \frac{(2k-1)!!(2k+1)!!}{2^{2k+1} k!(k+1)!} P_{2k+1}(x) \quad [|x| < 1, \quad (-1)!! \equiv 1].$$

LA 385(17)

$$5. \quad \sqrt{1-x^2} = \frac{\pi}{2} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} (4k+1) \frac{(2k-3)!!(2k-1)!!}{2^{2k+1} k!(k+1)!} P_{2k}(x) \right\} \quad [|x| < 1, \quad (-1)!! \equiv 1].$$

LA 385(18)

$$6.* \quad \sqrt{\frac{1-x}{2}} = \frac{2}{3} P_0(x) - 2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+3)} P_n(x) \quad [-1 \leq x \leq 1].$$

$$7.* \quad \frac{1-\rho^2}{(1-2\rho x+\rho^2)^{\frac{1}{2}}} = 1 + \sum_{n=0}^{\infty} (2n+1)\rho^n P_n(x), \quad |\rho| < 1, \quad |x| \leq 1.$$

SA 170(4)

1049
8.923

$$\arcsin x = \frac{\pi}{2} \sum_{k=1}^{\infty} \left\{ \frac{(2k-1)!!}{2^k k!} \right\}^2 [P_{2k+1}(x) - P_{2k-1}(x)] + \pi x/2 \quad [|x| < 1, \quad (-1)!! \equiv 1].$$

8.924

$$\begin{aligned}
1. \quad & -\frac{1 + \cos n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{1 + \cos n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k + 5)n^2(n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k + 3)^2]} P_{2k+2}(\cos \theta) - \\
& - \frac{3(1 - \cos n\pi)}{2(n^2 - 2^2)} P_1(\cos \theta) - \\
& - \frac{1 - \cos n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k + 3)(n^2 - 1^2) \dots [n^2 - (2k - 1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k + 2)^2]} P_{2k+1}(\cos \theta) = \cos n\theta.
\end{aligned}$$

AD (9060.1)

$$\begin{aligned}
2. \quad & \frac{-\sin n\pi}{2(n^2 - 1)} P_0(\cos \theta) - \frac{\sin n\pi}{2} \sum_{k=0}^{\infty} \frac{(4k + 5)n^2(n^2 - 2^2) \dots [n^2 - (2k)^2]}{(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k + 3)^2]} P_{2k+2}(\cos \theta) + \\
& + \frac{3 \sin n\pi}{2(n^2 - 2^2)} P_1(\cos \theta) + \\
& + \frac{\sin n\pi}{2} \sum_{k=1}^{\infty} \frac{(4k + 3)(n^2 - 1^2)(n^2 - 3^2) \dots [n^2 - (2k - 1)^2]}{(n^2 - 2^2)(n^2 - 4^2) \dots [n^2 - (2k + 2)^2]} P_{2k+1}(\cos \theta) = \sin n\theta.
\end{aligned}$$

AD (9060.2)

$$\begin{aligned}
3. \quad & \frac{2^{n-1} n!}{(2n - 1)!!} P_n(\cos \theta) + \\
& - n \sum_{k=1}^{\lfloor n/2 \rfloor} (2n - 4k + 1) \frac{2^{n-2k-1} (n - k - 1)! (2k - 3)!!}{(2n - 2k + 1)!! k!} P_{n-2k}(\cos \theta) = \cos n\theta.
\end{aligned}$$

AD (9061.1)

$$\begin{aligned}
4. \quad & \frac{(2n - 1)!! P_{n-1}(\cos \theta)}{2^{n-1} (n - 1)!} - \\
& - \frac{n}{2^{n+1}} \sum_{k=0}^{\infty} \frac{(2n + 2k - 1)!! (2k - 1)!! (2n + 4k + 3)}{2^{2k} (n + k + 1)! (k + 1)!} P_{n+2k+1}(\cos \theta) = \frac{4 \sin \theta}{\pi}.
\end{aligned}$$

AD (9061.2)

8.925

$$1. \quad \sum_{k=1}^{\infty} \frac{4k - 1}{2^{2k} (2k - 1)^2} \left[\frac{(2k - 1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = 1 - \frac{2\theta}{\pi}.$$

$$2. \quad \sum_{k=1}^{\infty} \frac{4k + 1}{2^{2k+1} (2k - 1)(k + 1)} \left[\frac{(2k - 1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{1}{2} - \frac{2 \sin \theta}{\pi}.$$

$$3. \sum_{k=1}^{\infty} \frac{k(4k-1)}{2^{2k-1}(2k-1)} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k-1}(\cos \theta) = \frac{2 \operatorname{ctg} \theta}{\pi}.$$

AD (9062.3)

$$4. \sum_{k=1}^{\infty} \frac{4k+1}{2^{2k}} \left[\frac{(2k-1)!!}{k!} \right]^2 P_{2k}(\cos \theta) = \frac{2}{\pi \sin \theta} - 1.$$

AD (9062.4)

8.926

$$1. \sum_{n=1}^{\infty} \frac{1}{n} P_n(\cos \theta) = \ln \frac{2 \operatorname{tg} \frac{\pi - \theta}{4}}{\sin \theta} = -\ln \sin \frac{\theta}{2} - \ln \left(1 + \sin \frac{\theta}{2} \right).$$

AD (9063.2)

$$2. \sum_{n=1}^{\infty} \frac{1}{n+1} P_n(\cos \theta) = \ln \frac{1 + \sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} - 1.$$

AD (9063.1)

8.927

$$\sum_{k=0}^{\infty} \cos \left(k + \frac{1}{2} \right) \beta P_k(\cos \varphi) = \frac{1}{\sqrt{2(\cos \beta - \cos \varphi)}} \quad [0 \leq \beta < \varphi < \pi];$$

$$= 0 \quad [0 < \varphi < \beta < \pi].$$

MO 72

8.928

$$1. \sum_{n=1}^{\infty} \frac{(-1)^n (4k+1) [(2n-1)!!]^3}{2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4\mathbf{K}(\sin \theta)}{\pi^2} - 1.$$

AD (9064.1)

$$2. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4n+1) [(2n-1)!!]^3}{(2n-1)(2n+2)2^{3n} (n!)^3} P_{2n}(\cos \theta) = \frac{4\mathbf{E}(\sin \theta)}{\pi^2} - \frac{1}{2}.$$

For series of products of Bessel functions and Legendre polynomials, see 8.511 4., 8.531 3., 8.533 1., 8.533 2.⁶, and 8.534.

8.93 Gegenbauer polynomials $C_n^\lambda(t)$

8.930

Definition. The polynomials $C_n^\lambda(t)$ of degree n are the coefficients of α^n in the power-series expansion of the function

$$(1 - 2t\alpha + \alpha^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) \alpha^n.$$

WH

Thus, the polynomials $C_n^\lambda(t)$ are a *generalization of the Legendre polynomials*.

$$1.* \quad C_0^\lambda(t) = 1.$$

$$2.* \quad C_1^\lambda(t) = 2\lambda t.$$

$$3.* \quad C_2^\lambda(t) = 2\lambda(\lambda + 1)t^2 - \lambda.$$

1051

$$4.* \quad C_3^\lambda(t) = \frac{1}{3}\lambda(4\lambda^2 + 12\lambda + 8)t^3 - 2\lambda(\lambda + 1)t.$$

$$5.* \quad C_4^\lambda(t) = \frac{2}{3}\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^4 - 2\lambda(\lambda^3 + 3\lambda + 2)t^2 + \frac{1}{2}\lambda(\lambda + 1).$$

$$6.* \quad C_5^\lambda(t) = \frac{1}{15}\lambda(4\lambda^4 + 40\lambda^3 + 140\lambda^2 + 200\lambda + 96)t^5 - \frac{1}{3}\lambda(4\lambda^3 + 24\lambda^2 + 44\lambda + 24)t^3 + \lambda(\lambda^2 + 3\lambda + 2)t.$$

$$7.* \quad C_6^\lambda(t) = \frac{1}{45}\lambda(\lambda^5 + 60\lambda^4 + 340\lambda^3 + 900\lambda^2 + 1096\lambda + 480)t^6 - \\ - \frac{1}{3}\lambda(2\lambda^4 + 20\lambda^3 + 70\lambda^2 + 100\lambda + 48)t^4 + \\ + \lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)t^2 + \frac{1}{6}\lambda(\lambda^2 + 3\lambda + 2).$$

8.931

Integral representation:

$$C_n^\lambda(t) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)} \frac{\Gamma\left(\frac{2\lambda + 1}{2}\right)}{\Gamma(\lambda)} \int_0^\pi (t + \sqrt{t^2 - 1} \cos \varphi)^n \sin^{2\lambda-1} \varphi d\varphi.$$

MO 99

See also 3.252 11., 3.663 2., 3.664 4.

Functional relations

8.932

Expressions in terms of hypergeometric functions:

$$1. \quad C_n^\lambda(t) = \frac{\Gamma(2\lambda + n)}{\Gamma(n + 1)\Gamma(2\lambda)} F\left(2\lambda + n, -n; \lambda + \frac{1}{2}; \frac{1-t}{2}\right)^* ; \\ = \frac{2^n \Gamma(\lambda + n)}{n!\Gamma(\lambda)} t^n F\left(-\frac{n}{2}, \frac{1-n}{2}; 1 - \lambda - n; \frac{1}{t^2}\right).$$

* This equation is used for defining the generalized functions $C_n^\lambda(t)$, where the subscript n can be an arbitrary

number.

MO 99

MO 97

$$2. \quad C_{2n}^\lambda(t) = \frac{(-1)^n}{(\lambda + n)B(\lambda, n + 1)} F\left(-n, n + \lambda; \frac{1}{2}; t^2\right).$$

MO 99

$$3. \quad C_{2n+1}^\lambda(t) = \frac{(-1)^n 2t}{B(\lambda, n + 1)} F\left(-n, n + \lambda + 1; \frac{3}{2}; t^2\right).$$

Recursion formulas:

$$1. (n+2)C_{n+2}^\lambda(t) = 2(\lambda+n+1)tC_{n+1}^\lambda(t) - (2\lambda+n)C_n^\lambda(t).$$

Mo 98

1052

$$2. nC_n^\lambda(t) = 2\lambda [tC_{n-1}^{\lambda+1}(t) - C_{n-2}^{\lambda+1}(t)].$$

WH

$$3. (2\lambda+n)C_n^\lambda(t) = 2\lambda [C_n^{\lambda+1}(t) - tC_{n-1}^{\lambda+1}(t)].$$

WH

$$4. nC_n^\lambda(t) = (2\lambda+n-1)tC_{n-1}^\lambda(t) - 2\lambda(1-t^2)C_{n-2}^{\lambda+1}(t).$$

WH

8.934

$$1. C_n^\lambda(t) = \frac{(-1)^n \Gamma(2\lambda+n) \Gamma\left(\frac{2\lambda+1}{2}\right)}{2^n \Gamma(2\lambda) \Gamma\left(\frac{2\lambda+1}{2}+n\right)} \frac{(1-t^2)^{\frac{1}{2}-\lambda}}{n!} \frac{d^n}{dt^n} \left[(1-t^2)^{\lambda+n-\frac{1}{2}} \right].$$

WH

$$2. C_n^\lambda(\cos \varphi) = \sum_{\substack{k,l=0 \\ k+l=n}}^n \frac{\Gamma(\lambda+k)\Gamma(\lambda+l)}{k!l! [\Gamma(\lambda)]^2} \cos(k-l)\varphi.$$

MO 99

$$3. C_n^\lambda(\cos \psi \cos \vartheta + \sin \psi \sin \vartheta \cos \varphi) = \\ = \frac{\Gamma(2\lambda-1)}{[\Gamma(\lambda)]^2} \sum_{k=0}^n \frac{2^{2k} (n-k)! [\Gamma(\lambda+k)]^2}{\Gamma(2\lambda+n+k)} (2\lambda+2k-1) \sin^k \psi \sin^k \vartheta \times \\ \times C_{n-k}^{\lambda+k}(\cos \psi) C_{n-k}^{\lambda+k}(\cos \vartheta) C_k^{\lambda-\frac{1}{2}}(\cos \varphi) \\ \left[\psi, \vartheta, \varphi \text{ real; } \lambda \neq \frac{1}{2} \right] \quad \text{[“summation theorem”] (see also 8.794–8.796).}$$

$$4. \lim_{\lambda \rightarrow 0} \Gamma(\lambda) C_n^\lambda(\cos \varphi) = \frac{2 \cos n\varphi}{n}.$$

MO 98

For orthogonality, see 8.904, 7.313.

8.935

Derivatives:

$$1. \frac{d^k}{dt^k} C_n^\lambda(t) = 2^k \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} C_{n-k}^{\lambda+k}(t).$$

MO 99

In particular,

$$2. \frac{dC_n^\lambda(t)}{dt} = 2\lambda C_{n-1}^{\lambda+1}(t).$$

WH

For integrals of the polynomials $C_n^\lambda(x)$ see 7.31-7.33.

8.936

Connections with other functions:

$$1. C_n^\lambda(t) = \frac{\Gamma(2\lambda + n)\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(2\lambda)\Gamma(n+1)} \left\{ \frac{1}{4}(t^2 - 1) \right\}^{\frac{1}{4} - \frac{\lambda}{2}} P_{\lambda+n-\frac{1}{2}}^{\frac{1}{2}-\lambda}(t).$$

MO 98

$$2. C_{n-m}^{m+\frac{1}{2}}(t) = \frac{1}{(2m-1)!!} \frac{d^m P_n(t)}{dt^m} = (-1)^m \frac{(1-t^2)^{-\frac{m}{2}} m! 2^m}{(2m)!} P_n^m(t)$$

[$m+1$ a natural number].

$$3. C_n^{\frac{1}{2}}(t) = P_n(t).$$

$$4. J_{\lambda-\frac{1}{2}}(r \sin \vartheta \sin \alpha)(r \sin \vartheta \sin \alpha)^{-\lambda+\frac{1}{2}} e^{-ir \cos \vartheta \cos \alpha} = \\ = \sqrt{2} \frac{\Gamma(\lambda)}{\Gamma\left(\lambda + \frac{1}{2}\right)} \sum_{k=0}^{\infty} (\lambda+k) i^{-k} \frac{J_{\lambda+k}(r) C_k^\lambda(\cos \vartheta) C_k^\lambda(\cos \alpha)}{r^\lambda C_k^\lambda(1)}.$$

MO 99

$$5. \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}} \left(t \sqrt{\frac{2}{\lambda}} \right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(t).$$

MO 99a

See also 8.932.

8.937

Special cases and particular values:

$$1. C_n^1(\cos \varphi) = \frac{\sin(n+1)\varphi}{\sin \varphi}.$$

MO 99

$$2. C_0^0(\cos \varphi) = 1.$$

MO 98

$$3. C_0^\lambda(t) \equiv 1.$$

MO 98

$$4. C_n^\lambda(1) \equiv \binom{2\lambda+n-1}{n}.$$

MO 98

8.938

A differential equation leading to the polynomials $C_n^\lambda(t)$:

$$y'' + \frac{(2\lambda + 1)t}{t^2 - 1}y' - \frac{n(2\lambda + n)}{t^2 - 1}y = 0 \quad (\text{cf. 9.174}).$$

9.174
WH

For series of products of Bessel functions and the polynomials $C_n^\lambda(x)$, see 8.532, 8.534.

8.939

Differentiation and Rodrigues' formulas and orthogonality relation

$$1. \quad \frac{d}{dt}C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t).$$

MS 5.3.2

$$2. \quad \frac{d^m}{dt^m}C_n^\lambda(t) = 2^m \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + m - 1)C_{n-m}^{\lambda+m}(t).$$

MS 5.3.2

$$3. \quad \frac{d}{dt}C_{n-1}^\lambda(t) = t \frac{d}{dt}C_n^\lambda(t) - nC_n^\lambda(t).$$

MS 5.3.2

$$4. \quad \frac{d}{dt}C_{n+1}^\lambda(t) = t \frac{d}{dt}C_n^\lambda(t) + (2\lambda + n)C_n^\lambda(t).$$

MS 5.3.2

$$5. \quad (1-t^2) \frac{d}{dt}C_n^\lambda(t) = (n + 2\lambda - 1)C_{n-1}^\lambda(t) - ntC_n^\lambda(t) = (n + 2\lambda)tC_n^\lambda(t) - (n + 1)C_{n+1}^\lambda(t) = 2\lambda(1-t^2)C_{n-1}^{\lambda+1}(t).$$

MS 5.3.2

$$6. \quad \frac{d}{dt} [C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t)] = 2(n + \lambda)C_n^\lambda(t).$$

$$\begin{aligned}
7. \quad C_n^\lambda(t) &= \frac{(-1)^n 2^\lambda (2\lambda + 1)(2\lambda + 2) \dots (2\lambda + n - 1) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! \left(\lambda + \frac{1}{2}\right) \left(\lambda + \frac{3}{2}\right) \dots \left(\lambda + n - \frac{1}{2}\right)} \frac{d^n}{dt^n} \left[(1 - t^2)^{n + \lambda - \frac{1}{2}} \right] = \\
&= \frac{(-1)^n \Gamma\left(\lambda + \frac{1}{2}\right) \Gamma(n + 2\lambda) (1 - t^2)^{\frac{1}{2} - \lambda}}{2^n n! \Gamma(2\lambda) \Gamma\left(n + \lambda + \frac{1}{2}\right)} \frac{d^n}{dt^n} \left[(1 - t^2)^{n + \lambda - \frac{1}{2}} \right].
\end{aligned}$$

[Rodrigues' formula]

MS 5.3.2

$$8. \quad \int_{-1}^1 C_n^\lambda(t) C_m^\lambda(t) (1 - t^2)^{\lambda - \frac{1}{2}} dt = \begin{cases} 0, & n \neq m \\ \frac{\pi 2^{1-2\lambda} \Gamma(n + 2\lambda)}{n! (\lambda + n) [\Gamma(\lambda)]^2}, & n = m. \end{cases} \quad (\lambda \neq 0)$$

[Orthogonality relation]

MS 5.3.2

8.94 The Chebyshev polynomials $T_n(x)$ and $U_n(x)$

8.940

Definition

1. Chebyshev's polynomials of the first kind

$$\begin{aligned}
T_n(x) &= \cos(n \arccos x) = \frac{1}{2} \left[(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n \right] = \\
&= x^n - \binom{n}{2} x^{n-2} (1-x^2) + \binom{n}{4} x^{n-4} (1-x^2)^2 - \binom{n}{6} x^{n-6} (1-x^2)^3 + \dots
\end{aligned}$$

NA 66, 71

2. Chebyshev's polynomials of the second kind:

$$\begin{aligned}
U_n(x) &= \frac{\sin[(n+1) \arccos x]}{\sin[\arccos x]} = \frac{1}{2i\sqrt{1-x^2}} \left[(x + i\sqrt{1-x^2})^{n+1} - (x - i\sqrt{1-x^2})^{n+1} \right] = \\
&= \binom{n+1}{1} x^n - \binom{n+1}{3} x^{n-2} (1-x^2) + \binom{n+1}{5} x^{n-4} (1-x^2)^2 - \dots
\end{aligned}$$

Recursion formulas:

1. $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$

NA 358

2. $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0.$

3. $T_n(x) = U_n(x) - xU_{n-1}(x).$

EH II 184(3)

4. $(1 - x^2)U_{n-1}(x) = xT_n(x) - T_{n+1}(x).$

EH II 184(4)

For the orthogonality, see 7.343 and 8.904.

1055

8.942

Relations with other functions:

1. $T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right).$

MO 104

2. $T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}.$

MO 104

3. $U_n(x) = \frac{(-1)^n (n+1)}{\sqrt{1-x^2} (2n+1)!!} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}.$

EH II 185(15)

See also 8.962 3.

8.943

Special cases

1. $T_0(x) = 1.$

2. $T_1(x) = x.$

3. $T_2(x) = 2x^2 - 1.$

4. $T_3(x) = 4x^3 - 3x.$

5. $T_4(x) = 8x^4 - 8x^2 + 1.$

6. $T_5(x) = 16x^5 - 20x^3 + 5x.$

7. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$

8. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x.$

9. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1.$

10. $U_0(x) = 1.$

11. $U_1(x) = 2x.$

12. $U_2(x) = 4x^2 - 1.$

13. $U_3(x) = 8x^3 - 4x.$

14. $U_4(x) = 16x^4 - 12x^2 + 1.$

15. $U_5(x) = 32x^5 - 32x^3 + 6x.$

16. $U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.$

17. $U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x.$

18. $U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1.$

8.944

Particular values:

- | | |
|--------------------------|--------------------------|
| 1. $T_n(1) = 1.$ | 2. $T_n(-1) = (-1)^n.$ |
| 3. $T_{2n}(0) = (-1)^n.$ | 4. $T_{2n+1}(0) = 0.$ |
| 5. $U_{2n+1}(0) = 0.$ | 6. $U_{2n}(0) = (-1)^n.$ |

1056

8.945

The generating function:

1.
$$\frac{1-t^2}{1-2tx+t^2} = T_0(x) + 2\sum_{k=1}^{\infty} T_k(x)t^k.$$

$$2. \frac{1}{1 - 2tx + t^2} = \sum_{k=0}^{\infty} U_k(x)t^k.$$

MO 104a, EH II 186(31)

8.946

Zeros.

The polynomials $T_n(x)$ and $U_n(x)$ only have real simple zeros. All these zeros lie in the interval $(-1, +1)$.

8.947

The functions $T_n(x)$ and $\sqrt{1-x^2}U_{n-1}(x)$ are two linearly independent solutions of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$$

NA 69(58)

8.948

Of all polynomials of degree n with leading coefficient equal to 1, the one that deviates the least from zero on the interval $[-1, +1]$ is the polynomial $2^{-n+1}T_n(x)$.

8.949

Differentiation and Rodrigues' formulas and orthogonality relations

$$1. \frac{d}{dx}T_n(x) = nU_{n-1}(x).$$

MS 5.7.2

$$2. \frac{d^m}{dx^m}T_n(x) = 2^{m-1}\Gamma(m)nC_{n-m}^m(x).$$

MS 5.7.2

$$3. (1-x^2)\frac{d}{dx}T_n(x) = n[T_{n-1}(x) - xT_n(x)] = n[xT_n(x) - T_{n+1}(x)].$$

MS 5.7.2

$$4. \frac{d}{dx}U_n(x) = 2C_{n-1}^2(x).$$

MS 5.7.2

$$5. \quad \frac{d^m}{dx^m} U_n(x) = 2^m m! C_{n-m}^{m+1}(x).$$

MS 5.7.2

$$6. \quad (1-x^2) \frac{d}{dx} U_n(x) = (n+1)U_{n-1}(x) - nxU_n(x) = (n+2)xU_n(x) - (n+1)U_{n+1}(x).$$

MS 5.7.2

$$7. \quad T_n(x) = \frac{(-1)^n \pi^{\frac{1}{2}} (1-x^2)^{\frac{1}{2}}}{2^{n+1} \Gamma\left(n + \frac{1}{2}\right)} \frac{d^n}{dx^n} \left[(1-x^2)^{n-\frac{1}{2}} \right]. \quad [\text{Rodrigues' formula}]$$

MS 5.7.2

$$8. \quad U_n(x) = \frac{(-1)^n \pi^{\frac{1}{2}} (n+1) (1-x^2)^{-\frac{1}{2}}}{2^{n+1} \Gamma\left(n + \frac{3}{2}\right)} \frac{d^n}{dx^n} \left[(1-x^2)^{n+\frac{1}{2}} \right]. \quad [\text{Rodrigues' formula}]$$

MS 5.7.2

1057

$$9. \quad \int_{-1}^1 T_m(x) T_n(x) (1-x^2)^{-\frac{1}{2}} dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0. \end{cases} \quad [\text{Orthogonality relation}]$$

MS 5.7.2

$$10. \quad \int_{-1}^1 U_m(x) U_n(x) (1-x^2)^{-\frac{1}{2}} dx = \begin{cases} 0, & m \neq n \\ \pi/8, & m = n. \end{cases} \quad [\text{Orthogonality relation}]$$

MS 5.7.2

8.95 The Hermite polynomials $H_n(x)$

8.950

Definition

or

$$2. \quad H_n(x) = 2^n x^n - 2^{n-1} \binom{n}{2} x^{n-2} + 2^{n-2} \cdot 1 \cdot 3 \cdot \binom{n}{4} x^{n-4} - 2^{n-3} \cdot 1 \cdot 3 \cdot 5 \cdot \binom{n}{6} x^{n-6} + \dots$$

MO 105a

$$3.* \quad H_0(x) = 1.$$

$$4.* \quad H_1(x) = 2x.$$

$$5.* \quad H_2(x) = 4x^2 - 2.$$

$$6.* \quad H_3(x) = 8x^3 - 12x.$$

$$7.* \quad H_4(x) = 16x^4 - 48x^2 + 12.$$

$$8.* \quad H_5(x) = 32x^5 - 160x^3 + 120x.$$

$$9.* \quad H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$$

$$10.* \quad H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x.$$

$$11.* \quad H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680.$$

8.951

The integral representation:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + it)^n e^{-t^2} dt.$$

MO 106a

Functional relations

8.952

Recursion formulas:

$$1. \quad \frac{dH_n(x)}{dx} = 2nH_{n-1}(x).$$

SM 569(22)

$$2. \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

SM 570(23)

For the orthogonality, see 7.374 1. and 8.904.

$$3.* \quad nH_n(x) = -nH'_{n-1}(x) + xH'_n(x).$$

MS 5.6.2

$$4.* \quad H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x).$$

MS 5.6.2

1058

8.953

The connection with other functions:

$$1. \quad H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \Phi\left(-n, \frac{1}{2}; x^2\right).$$

$$2. \quad H_{2n+1}(x) = (-1)^n 2 \frac{(2n+1)!}{n!} x \Phi \left(-n, \frac{3}{2}; x^2 \right).$$

MO 106a

For a connection with the polynomials $C_n^\lambda(x)$, see 8.936 5.

For a connection with the Laguerre polynomials, see 8.972 2. and 8.972 3.

For a connection with functions of a parabolic cylinder, see 9.253.

8.954

Inequalities:

$$1.* \quad |H_n(x)| \leq 2^{\frac{n}{2} - [n/2]} \frac{n!}{[[n/2]]!} e^{2x\sqrt{[n/2]}} \quad [x > 0].$$

MO 106a

$$2.* \quad |H_n(x)| < k\sqrt{n!} 2^{n/2} e^{x^2/2}, \quad k = 1.086435.$$

SA 324

8.955

Asymptotic representation:

$$1. \quad H_{2n}(x) = (-1)^n 2^n (2n-1)!! e^{x^2/2} \left[\cos(\sqrt{4n+1}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right].$$

SM 579

$$2. \quad H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!! \sqrt{2n+1} e^{x^2/2} \left[\sin(\sqrt{4n+3}x) + O\left(\frac{1}{\sqrt[4]{n}}\right) \right].$$

SM 579

8.956

Special cases and particular values:

$$7. \quad H_{2n+1}(0) = 0.$$

Series of hermite polynomials

8.957

The generating function:

$$1. \quad \exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x).$$

SM 569(21)

$$2. \quad \frac{1}{e} \operatorname{sh} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} H_{2k+1}(x).$$

MO 106a

1059

$$3. \quad \frac{1}{e} \operatorname{ch} 2x = \sum_{k=0}^{\infty} \frac{1}{(2k)!} H_{2k}(x).$$

MO 106a

$$4. \quad e \sin 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} H_{2k+1}(x).$$

MO 106a

$$5. \quad e \cos 2x = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} H_{2k}(x).$$

"The summation theorem":

$$1. \frac{\left(\sum_{k=1}^r a_k^2\right)^{\frac{n}{2}}}{n!} H_n \left(\frac{\sum_{k=1}^r a_k x_k}{\sqrt{\sum_{k=1}^r a_k^2}} \right) = \sum_{m_1+m_2+\dots+m_r=n} \prod_{k=1}^r \left\{ \frac{a_k^{m_k}}{m_k!} H_{m_k}(x_k) \right\}.$$

MO 106a

2. A special case:

$$1. 2^{\frac{n}{2}} H_n(x+y) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2}).$$

MO 107a

8.959

Hermite polynomials satisfy the differential equation

$$1. \frac{d^2 u_n}{dx^2} - 2x \frac{du_n}{dx} + 2n u_n = 0;$$

SM 566(9)

A second solution of this differential equation is provided by the functions:

$$2. u_{2n} = (-1)^n A x \Phi \left(\frac{1}{2} - n; \frac{3}{2}; x^2 \right),$$

$$3. u_{2n+1} = (-1)^n B \Phi \left(-\frac{1}{2} - n; \frac{1}{2}; x^2 \right) \quad [A \text{ and } B \text{-arbitrary constants}].$$

MO 107

8.959(1)

Rodrigues' formula and orthogonality relation

$$2. \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0, & m \neq n \\ \pi^{1/2} 2^n n!, & m = n. \end{cases}$$

8.96 Jacobi's polynomials

8.960

Definition

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{\alpha+n} (1+x)^{\beta+n}] ; \\ &= \frac{1}{2^n} \sum_{m=0}^n \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m. \end{aligned}$$

EH II 169(2)
EH II 169(10), CO

1060

8.961

Functional relations:

$$1. P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x).$$

EH II 169(13)

$$\begin{aligned} 2. 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha, \beta)}(x) &= \\ &= (2n+\alpha+\beta+1) [(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2] P_n^{(\alpha, \beta)}(x) - \\ &\quad - 2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha, \beta)}(x). \end{aligned}$$

EH II 169(11)

$$\begin{aligned} 3. (2n+\alpha+\beta)(1-x^2) \frac{d}{dx} P_n^{(\alpha, \beta)}(x) &= n [(\alpha - \beta) - (2n + \alpha + \beta)x] P_n^{(\alpha, \beta)}(x) + \\ &\quad + 2(n + \alpha)(n + \beta) P_{n-1}^{(\alpha, \beta)}(x). \end{aligned}$$

EH II 170(15)

$$4. \quad \frac{d^m}{dx^m} \left[P_n^{(\alpha, \beta)}(x) \right] = \frac{1}{2^m} \frac{\Gamma(n+m+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m, \beta+m)}(x) \quad [m = 1, 2, \dots, n].$$

EH II 170(17)

$$5. \quad \left(n + \frac{1}{2}\alpha + \frac{1}{2}\beta + 1 \right) (1-x)P_n^{(\alpha+1, \beta)}(x) = (n+\alpha+1)P_n^{(\alpha, \beta)}(x) - (n+1)P_{n+1}^{(\alpha, \beta)}(x).$$

EH II 173(32)

$$6. \quad \left(n + \frac{1}{2}\alpha + \frac{1}{2}\beta + 1 \right) (1+x)P_n^{(\alpha, \beta+1)}(x) = (n+\beta+1)P_n^{(\alpha, \beta)}(x) + (n+1)P_{n+1}^{(\alpha, \beta)}(x).$$

EH II 173(33)

$$7. \quad (1-x)P_n^{(\alpha+1, \beta)}(x) + (1+x)P_n^{(\alpha, \beta+1)}(x) = 2P_n^{(\alpha, \beta)}(x).$$

EH II 173(34)

$$8. \quad (2n+\alpha+\beta)P_n^{(\alpha-1, \beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha, \beta)}(x) - (n+\beta)P_{n-1}^{(\alpha, \beta)}(x).$$

EH II 173(35)

$$9. \quad (2n+\alpha+\beta)P_n^{(\alpha, \beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha, \beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha, \beta)}(x).$$

EH II 173(36)

$$10. \quad P_n^{(\alpha, \beta-1)}(x) - P_n^{(\alpha-1, \beta)}(x) = P_{n-1}^{(\alpha, \beta)}(x).$$

EH II 173(37)

8.962

Connections with other functions:

$$1. \quad P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n \Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} F \left(n+\alpha+\beta+1, -n; 1+\beta; \frac{1+x}{2} \right);$$

$$= \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} F \left(n+\alpha+\beta+1, -n; 1+\alpha; \frac{1-x}{2} \right);$$

$$= \frac{\Gamma(n+1+\alpha)}{n! \Gamma(1+\alpha)} \left(\frac{1+x}{2} \right)^n F \left(-n, \beta+1, \alpha+\beta+1; \frac{x-1}{x+1} \right);$$

$$= \frac{\Gamma(n+1+\beta)}{n! \Gamma(1+\beta)} (x-1)^n F \left(-n, \alpha+1, \alpha+\beta+1; \frac{x+1}{x-1} \right).$$

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$$2. P_n(x) = P_n^{(0,0)}(x).$$

CO, EH II 179(3)

$$3. T_n(x) = \frac{2^{2n}(n!)^2}{(2n)!} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x).$$

CO, EH II 184(5)a

1061

$$4. C_n^\nu(x) = \frac{\Gamma(n+2\nu)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(2\nu)\Gamma\left(n+\nu+\frac{1}{2}\right)} P_n^{(\nu-1/2, \nu-1/2)}(x).$$

MO 108a, EH II 174(4)

8.963

The generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta},$$

$$R = \sqrt{1-2xz+z^2} \quad [|z| < 1].$$

EH II 172(29)

8.964

The Jacobi polynomials constitute the *unique* rational solution of the differential (hypergeometric) equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x] y' + n(n + \alpha + \beta + 1)y = 0.$$

EH II 169(14)

8.965

Asymptotic representation

$$P_n^{(\alpha, \beta)}(\cos \theta) = \frac{\cos \left\{ \left[n + \frac{1}{2}(\alpha + \beta + 1) \right] \theta - \left(\frac{1}{2}\alpha + \frac{1}{4} \right) \pi \right\}}{\sqrt{\pi n} \left(\frac{1}{\sin \frac{1}{2}\theta} \right)^{\alpha + \frac{1}{2}} \left(\frac{1}{\cos \frac{1}{2}\theta} \right)^{\beta + \frac{1}{2}}} + O(n^{-\frac{3}{2}})$$

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$$[\operatorname{Im} \alpha = \operatorname{Im} \beta = 0, \quad 0 < \theta < \pi].$$

8.966

A limit relationship:

$$\lim_{n \rightarrow \infty} \left[n^{-\alpha} P_n^{(\alpha, \beta)} \left(\cos \frac{z}{n} \right) \right] = \left(\frac{z}{2} \right)^{-\alpha} J_\alpha(z).$$

EH II 173(41)

8.967

If $\alpha > -1$ and $\beta > -1$, all the zeros of the polynomial $P_n^{(\alpha, \beta)}(x)$ are simple and they lie in the interval $(-1, 1)$.

8.97 The Laguerre polynomials

8.970

Definition.

$$\begin{aligned} 1. \quad L_n^\alpha(x) &= \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}); && \text{[Rodrigues' formula]} \\ &= \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!}. \end{aligned}$$

MO 109, EH II 188(7)
EH II 188(5), MO 108

$$2. \quad L_n^0(x) = L_n(x).$$

ET I 369

$$3.* \quad L_0^\alpha(x) = 1.$$

1062

$$4.* \quad L_1^\alpha(x) = -x + \alpha + 1.$$

$$5.* \quad L_2^\alpha(x) = \frac{1}{2} [x^2 - 2(\alpha + 2)x + (\alpha + 1)(\alpha + 2)].$$

$$6.* \quad L_3^\alpha(x) = -\frac{1}{6} [x^3 - 3(\alpha + 3)x^2 + 3(\alpha + 2)(\alpha + 3)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)] .$$

$$7.* \quad L_4^\alpha(x) = \frac{1}{24} [x^4 - 4(\alpha + 4)x^3 + 6(\alpha + 3)(\alpha + 4)x^2 - 4(\alpha + 2)(\alpha + 3)(\alpha + 4)x + (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)] .$$

$$8.* \quad L_5^\alpha(x) = -\frac{1}{120} [x^5 - 5(\alpha + 5)x^4 + 10(\alpha + 4)(\alpha + 5)x^3 - 10(\alpha + 3)(\alpha + 4)(\alpha + 5)x^2 + 5(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)x - (\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)] .$$

8.971

Functional relations:

$$1. \quad \frac{d}{dx} [L_n^\alpha(x) - L_{n+1}^\alpha(x)] = L_n^\alpha(x).$$

EH II 189(16)

$$2. \quad \frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = \frac{nL_n^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x)}{x}.$$

EH II 189(15), SM 575(42)a

$$3. \quad x \frac{d}{dx} L_n^\alpha(x) = nL_n^\alpha(x) - (n + \alpha)L_{n-1}^\alpha(x); \\ = (n + 1)L_{n+1}^\alpha(x) - (n + \alpha + 1 - x)L_n^\alpha(x).$$

EH II 189(12), MO 109

$$4. \quad xL_n^{\alpha+1}(x) = (n + \alpha + 1)L_n^\alpha(x) - (n + 1)L_{n+1}^\alpha(x); \\ = (n + \alpha)L_{n-1}^\alpha(x) - (n - x)L_n^\alpha(x).$$

SM 575(43)A, EH II 190(23)

$$5. \quad L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x).$$

SM 575(44)A, EH II 190(24)

$$6. \quad (n+1)L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0 \quad [n = 1, 2, \dots] .$$

$$7.* \quad (n + \alpha)L_n^{\alpha-1}(x) = (n + 1)L_{n+1}^\alpha(x) - (n + 1 - x)L_n^\alpha(x)$$

MS 5.5.2

$$8.* \quad nL_n^\alpha(x) = (2n + \alpha - 1 - x)L_{n-1}^\alpha(x) - (n + \alpha - 1)L_{n-2}^\alpha(x) \quad n = 2, 3, \dots$$

MS 5.5.2

8.972

Connections with other functions:

$$1. \quad L_n^\alpha(x) = \binom{n + \alpha}{n} \Phi(-n, \alpha + 1; x).$$

MO 109, FI II 189(14)

$$2. \quad H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-\frac{1}{2}}(x^2).$$

EH II 193(2), SM 576(47)

$$3. \quad H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2).$$

EH II 193(3), SM 577(48)

8.973

Special cases:

$$1. \quad L_0^\alpha(x) = 1$$

EH II 188(6)

$$2. \quad L_1^\alpha(x) = \alpha + 1 - x.$$

EH II 188(6)

$$4. L_n^{-n}(x) = (-1)^n \frac{x^n}{n!}.$$

MO 109

$$5. L_1(x) = 1 - x.$$

$$6. L_2(x) = 1 - 2x + \frac{x^2}{2}.$$

MO 109

8.974

Finite sums:

$$1. \sum_{m=0}^n \frac{m!}{\Gamma(m + \alpha + 1)} L_m^\alpha(x) L_m^\alpha(y) = \frac{(n+1)!}{\Gamma(n + \alpha + 1)(x-y)} [L_n^\alpha(x) L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x) L_n^\alpha(y)]$$

EH II 188(9)

$$2. \sum_{m=0}^n \frac{\Gamma(\alpha - \beta + m)}{\Gamma(\alpha - \beta)m!} L_{n-m}^\beta(x) = L_n^\alpha(x).$$

MO 110, EH II 192(39)

$$3. \sum_{m=0}^n L_m^\alpha(x) = L_n^{\alpha+1}(x).$$

EH II 192(38)

$$4. \sum_{m=0}^n L_m^\alpha(x) L_{n-m}^\beta(y) = L_n^{\alpha+\beta+1}(x+y).$$

EH II 192(41)

8.975

Arbitrary functions:

$$1. (1-z)^{-\alpha-1} \exp \frac{xz}{z-1} = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n \quad [|z| < 1].$$

EH II 189(17), MO 109

$$2. e^{-xz}(1+z)^\alpha = \sum_{n=0}^{\infty} L_n^{\alpha-n}(x) z^n \quad [|z| < 1].$$

MO 110, EH II 189(19)

$$3. J_\alpha(2\sqrt{xz})e^z(xz)^{-\frac{1}{2}\alpha} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+\alpha+1)} L_n^\alpha(x) \quad [\alpha > -1].$$

EH II 189(18), MO 109

8.976

Other series of Laguerre polynomials:

$$1. \sum_{n=0}^{\infty} n! \frac{L_n^\alpha(x)L_n^\alpha(y)z^n}{\Gamma(n+\alpha+1)} = \frac{(xyz)^{-\frac{1}{2}\alpha}}{1-z} \exp\left(-z\frac{x+y}{1-z}\right) I_\alpha\left(2\frac{\sqrt{xyz}}{1-z}\right) \quad [|z| < 1].$$

EH II 189(20)

$$2. \sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+1} = e^x x^{-\alpha} \Gamma(\alpha, x) \quad [\alpha > -1, x > 0].$$

EH II 215(19)

$$3.^6 [L_n^\alpha(x)]^2 = \frac{\Gamma(n+\alpha+1)}{2^{2n}n!} \sum_{k=0}^n \binom{2n-2k}{n-k} \frac{(2k)!}{k!} \frac{1}{\Gamma(\alpha+k+1)} L_{2k}^{2\alpha}(2x).$$

MO 110

$$4.^6 L_n^\alpha(x)L_n^\alpha(y) = \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^n \frac{L_{n-k}^{\alpha+2k}(x+y)}{\Gamma(1+\alpha+k)} \frac{(xy)^k}{k!}.$$

8.977

Summation theorems:

$$1. \quad L_n^{\alpha_1 + \alpha_2 + \dots + \alpha_k + k - 1}(x_1 + x_2 + \dots + x_k) = \sum_{(i_1 + i_2 + \dots + i_k = n)} L_{i_1}^{\alpha_1}(x_1) L_{i_2}^{\alpha_2}(x_2) \dots L_{i_k}^{\alpha_k}(x_k).$$

MO 110

$$2. \quad L_n^\alpha(x + y) = e^y \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} y^k L_n^{\alpha+k}(x).$$

MO 110

8.978

Limit relations and asymptotic behavior:

$$1. \quad L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right).$$

EH II 191(35)

$$2. \quad \lim_{n \rightarrow \infty} \left[n^{-\alpha} L_n^\alpha \left(\frac{x}{n} \right) \right] = x^{-\frac{1}{2}\alpha} J_\alpha(2\sqrt{x}).$$

EH II 191(36)

$$3. \quad L_n^\alpha(x) = \frac{1}{\sqrt{\pi}} e^{\frac{1}{2}x} x^{-\frac{1}{2}\alpha - \frac{1}{4}} n^{\frac{1}{2}\alpha - \frac{1}{4}} \cos \left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right] + O(n^{\frac{1}{2}\alpha - \frac{3}{4}}) \quad [\text{Im } \alpha = 0, \quad x > 0].$$

EH II 199(1)

8.979

Laguerre polynomials satisfy the following differential equation:

$$x \frac{d^2 u}{dx^2} + (\alpha - x + 1) \frac{du}{dx} + nu = 0.$$

EH II 188(10), SM 574(34)

8.980

Orthogonality relation

$$\int_0^{\infty} e^{-x} x^{\alpha} L_m^{\alpha}(x) L_n^{\alpha}(x) dx = \begin{cases} 0, & m \neq n \\ \Gamma(1 + \alpha) \binom{n + \alpha}{n}, & m = n \end{cases}$$

MS 5.5.2

8.981

Behavior of relative maxima of $|L_n^{\alpha}(x)|$

1. Let α be arbitrary and real. The sequence formed by the relative maxima of $|L_n^{\alpha}(x)|$ and by the value of this function at $x = 0$, is decreasing for $x < \alpha + \frac{1}{2}$, and increasing for $x > \alpha + \frac{1}{2}$. The successive relative maxima of $|H_n(x)|$ form a decreasing sequence for $x \leq 0$, and an increasing sequence for $x \geq 0$.

SZ 174(7.6.1)

2. Let α be an arbitrary real number. The successive relative maxima of

$$(7.6.3) \quad e^{-x/2} x^{(\alpha+1)/2} |L_n^{\alpha}(x)| \quad \text{and} \quad e^{-x/2} x^{\alpha/2 + \frac{1}{4}} |L_n^{\alpha}(x)|$$

form an increasing sequence provided $x > x_0$. In the first case

$$(7.6.4) \quad x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq 1, \\ \frac{\alpha^2 - 1}{2n + \alpha + 1} & \text{if } \alpha^2 > 1. \end{cases}$$

1065

In the second case

$$(7.6.5) \quad x_0 = \begin{cases} 0 & \text{if } \alpha^2 \leq \frac{1}{4}, \\ \left(\alpha^2 - \frac{1}{4}\right)^{\frac{1}{2}} & \text{if } \alpha^2 > \frac{1}{4}. \end{cases}$$

SZ 174(7.6.2)

In the first case we take n so large that $2n + \alpha + 1 > 0$.

8.982

Asymptotic and limiting behavior of $L_n^\alpha(x)$

1. Let α be arbitrary and real, c and w fixed positive constants, and let $n \rightarrow \infty$. Then

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2 - \frac{1}{4}} O(n^{\alpha/2 - \frac{1}{4}}) & \text{if } cn^{-1} \leq x \leq \omega, \\ O(n^\alpha) & \text{if } 0 \leq x \leq cn^{-1}. \end{cases}$$

These bounds are precise as regards their orders in n . For $\alpha \geq -\frac{1}{2}$, both bounds hold in both intervals, that is,

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2 - \frac{1}{4}} O(n^{\alpha/2 - \frac{1}{4}}), & 0 < x \leq \omega, \alpha \geq -\frac{1}{2}. \\ O(n^\alpha), & \end{cases}$$

SZ 175(7.6.4)

2. Let α be arbitrary and real. Then for an arbitrary complex z

$$\lim_{n \rightarrow \infty} n^{-\alpha} L_n(z/n) = z^{-\alpha/2} J_\alpha(2z^{\frac{1}{2}}),$$

SZ 191(8.1.3)

uniformly if z is bounded.

9.1 Hypergeometric Functions

9.10 Definition

9.1007

A hypergeometric series, also called a Gaussian hypergeometric function, is a series of the form

$$F(\alpha, \beta; \gamma; z) = 1 + \frac{\alpha \cdot \beta}{\gamma \cdot 1} z + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1) \cdot 1 \cdot 2} z^2 + \frac{\alpha(\alpha + 1)(\alpha + 2)\beta(\beta + 1)(\beta + 2)}{\gamma(\gamma + 1)(\gamma + 2) \cdot 1 \cdot 2 \cdot 3} z^3 + \dots$$

A hypergeometric series terminates if α or β is equal to a negative integer or to zero. For $\gamma = -n$ ($n = 0, 1, 2, \dots$), the hypergeometric series is indeterminate if neither α nor β is equal to $-m$ (where $m < n$ and m is a natural number). However,

$$1. \quad \lim_{\gamma \rightarrow -n} \frac{F(\alpha, \beta; \gamma; z)}{\Gamma(\gamma)} = \frac{\alpha(\alpha+1)\dots(\alpha+n)\beta(\beta+1)\dots(\beta+n)}{(n+1)!} \times z^{n+1} F(\alpha+n+1, \beta+n+1; n+2; z).$$

EH I 62(16)

1066
9.102

If we exclude these values of the parameters α, β, γ , a hypergeometric series converges in the unit circle $|z| < 1$. F then has a branch point at $z = 1$. Then we have the following conditions for convergence on the unit circle:

1. $1 > \operatorname{Re}(\alpha + \beta - \gamma) \geq 0$. The series converges throughout the entire unit circle except at the point $z = 1$.
2. $\operatorname{Re}(\alpha + \beta - \gamma) < 0$. The series converges (absolutely) throughout the entire unit circle.
3. $\operatorname{Re}(\alpha + \beta - \gamma) \geq 1$. The series diverges on the entire unit circle.

FI II 410, WH

9.11 Integral representations

9.111

$$F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt \quad [\operatorname{Re} \gamma > \operatorname{Re} \beta > 0].$$

WH

9.112⁸

$$F(p, n+p; n+1; z^2) = \frac{z^{-n}}{2\pi} \frac{\Gamma(p)n!}{\Gamma(p+n)} \int_0^{2\pi} \frac{\cos nt dt}{(1-2z \cos t + z^2)^p}$$

$[n = 0, 1, 2, \dots; \quad p \neq 0, -1, -2, \dots; \quad |z| < 1].$

WH, MO 16

9.113

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(-t)}{\Gamma(\gamma+t)} (-z)^t dt,$$

Here, $|\arg(-z)| < \pi$ and the path of integration is chosen in such a way that the poles of the functions $\Gamma(\alpha + t)$ and $\Gamma(\beta + t)$ lie to the left of the path of integration and the poles of the function $\Gamma(-t)$ lie to the right of it.

9.114

$$F\left(-m, -\frac{p+m}{2}; 1 - \frac{p+m}{2}; -1\right) = \frac{(-2)^m(p+m)}{\sin p\pi} \int_0^\pi \cos^m \varphi \cos p\varphi d\varphi$$

[$m+1$ a natural number; $p \neq 0, \pm 1, \dots$].

EH I 80(8), MO 16

See also 3.194 1., 2., 5., 3.196 1., 3.197 6., 9., 3.259 3, 3.312 3., 3.518 4.-6., 3.665 2., 3.671 1., 2., 3.681 1., 3.984 7.

9.12 The representation of elementary functions in terms of a hypergeometric function

9.121

1. $F(-n, \beta; \beta; -z) = (1+z)^n$ [β arbitrary]

EH I 101(4), GA 127 Ia

2. $F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n + (t-z)^n}{2t^n}$.

GA 127 II

3. $\lim_{\omega \rightarrow \infty} F\left(-n, \omega; 2\omega; -\frac{z}{t}\right) = \left(1 + \frac{z}{2t}\right)^n$.

GA 127 IIIa

4. $F\left(-\frac{n-1}{2}, -\frac{n-2}{2}; \frac{3}{2}; \frac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}$.

GA 127 IV

1067

5. $F\left(1-n, 1; 2; -\frac{z}{t}\right) = \frac{(t+z)^n - t^n}{nzt^{n-1}}$.

$$6. \quad F(1, 1; 2; -z) = \frac{\ln(1+z)}{z}.$$

GA 127 VI

$$7. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{\ln \frac{1+z}{1-z}}{2z}.$$

GA 127 VII

$$\begin{aligned} 8. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(1, k; 1; \frac{z}{k}\right) &= 1 + z \lim_{k \rightarrow \infty} F\left(1, k; 2; \frac{z}{k}\right) = \\ &= 1 + z + \frac{z^2}{2} \lim_{k \rightarrow \infty} F\left(1, k; 3; \frac{z}{k}\right) = \dots = e^z. \end{aligned}$$

GA 127 VIII

$$9. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z + e^{-z}}{2} = \operatorname{ch} z.$$

GA 127 IX

$$10. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; \frac{z^2}{4kk'}\right) = \frac{e^z - e^{-z}}{2z} = \frac{\operatorname{sh} z}{z}.$$

GA 127 X

$$11. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{3}{2}; -\frac{z^2}{4kk'}\right) = \frac{\sin z}{z}.$$

GA 127 XI

$$12. \quad \lim_{\substack{k \rightarrow \infty \\ k' \rightarrow \infty}} F\left(k, k'; \frac{1}{2}; -\frac{z^2}{4kk'}\right) = \cos z.$$

GA 127 XII

$$13. \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{z}{\sin z}.$$

GA 127 XIII

GA 127 XIV

$$15. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{z}{\operatorname{tg} z}.$$

GA 127 XV

$$16. \quad F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z}.$$

GA 127 XVI

$$17. \quad F\left(\frac{n+2}{2}, -\frac{n-2}{2}; \frac{3}{2}; \sin^2 z\right) = \frac{\sin nz}{n \sin z \cos z}.$$

GA 127 XVII

$$18. \quad F\left(-\frac{n-2}{2}, -\frac{n-1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz}{n \sin z \cos^{n-1} z}.$$

GA 127 XVIII

$$19. \quad F\left(\frac{n+2}{2}, \frac{n+1}{2}; \frac{3}{2}; -\operatorname{tg}^2 z\right) = \frac{\sin nz \cos^{n+1} z}{n \sin z}.$$

GA 127 XIX

$$20. \quad F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \sin^2 z\right) = \cos nz.$$

EH I 101(11), GA 127 XX

1068

$$21. \quad F\left(\frac{n+1}{2}, -\frac{n-1}{2}; \frac{1}{2}; \sin^2 z\right) = \frac{\cos nz}{\cos z}.$$

EH I 101(11), GA 127 XXI

$$22. \quad F\left(-\frac{n}{2}, -\frac{n-1}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \frac{\cos nz}{\cos^n z}.$$

$$23. \quad F\left(\frac{n+1}{2}, \frac{n}{2}; \frac{1}{2}; -\operatorname{tg}^2 z\right) = \cos nz \cos^n z.$$

$$24. \quad F\left(\frac{1}{2}, 1; 2; 4z(1-z)\right) = \frac{1}{1-z} \quad \left[|z| \leq \frac{1}{2}; |z(1-z)| \leq \frac{1}{4}\right].$$

$$25. \quad F\left(\frac{1}{2}, 1; 1; \sin^2 z\right) = \sec z.$$

$$26. \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = \frac{\arcsin z}{z} \quad (\text{cf. 9.121 13.}).$$

9.121

$$27. \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = \frac{\operatorname{arctg} z}{z} \quad (\text{cf. 9.121 15.}).$$

9.121

$$28. \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = \frac{\operatorname{Arsh} z}{z} \quad (\text{cf. 9.121 26.}).$$

9.121

$$29. \quad F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz} \quad (\text{cf. 9.121 16.}).$$

9.121

$$30. \quad F\left(1 + \frac{n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; z^2\right) = \frac{\sin(n \arcsin z)}{nz\sqrt{1-z^2}} \quad (\text{cf. 9.121 17.}).$$

$$31. \quad F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; z^2\right) = \cos(n \arcsin z) \quad (\text{cf. 9.121 20}).$$

$$32. \quad F\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; z^2\right) = \frac{\cos(n \arcsin z)}{\sqrt{1-z^2}}. \quad (\text{cf. 9.121 21}).$$

The representation of special functions in terms of a hypergeometric function

For complete elliptic integrals, see 8.113 1. and 8.114 1.;

for integrals of Bessel functions, see 6.574 1., 3., 6.576 2.-5., 6.621 1.-3.;

for Legendre polynomials, 8.911 and 8.916. (All these hypergeometric series terminate; that is, these series are finite sums);

for Legendre functions, see 8.820 and 8.837;

for associated Legendre functions, see 8.702, 8.703, 8.751, 8.77, 8.852, and 8.853;

for Chebyshev polynomials, see 8.942 1.;

for Jacobi's polynomials, see 8.962;

for Gegenbauer polynomials $C_n^\lambda(x)$, see 8.932;

for integrals of parabolic-cylinder functions, see 7.725 6.

9.122

Particular values:

1069

$$\begin{aligned}
 2. \quad F(\alpha, \beta; \gamma; 1) &= F(-\alpha, -\beta; \gamma - \alpha - \beta; 1), \operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta) \\
 &= \frac{1}{F(-\alpha, \beta; \gamma - \alpha; 1)}, \quad \operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta) \\
 &= \frac{1}{F(\alpha, -\beta; \gamma - \beta; 1)}, \quad \operatorname{Re} \gamma > \operatorname{Re}(\alpha + \beta)
 \end{aligned}$$

GA 148(51)

GA 148(50)

GA 148(49)

$$3. \quad F\left(1, 1; \frac{3}{2}; \frac{1}{2}\right) = \frac{\pi}{2} \text{ (cf. 9.121 14).}$$

9.121

9.13 Transformation formulas and the analytic continuation of functions defined by hypergeometric series

9.130

The series $F(\alpha, \beta; \gamma; z)$ defines an analytic function that, speaking generally, has singularities at the points $z = 0, 1$, and ∞ (In the general case, there are branch points). We make a cut in the z -plane along the real axis from $z = 1$ to $z = \infty$; that is, we require that $|\arg(-z)| < \pi$ for $|z| \geq 1$. Then, the series $f(\alpha, \beta; \gamma; z)$ will, in the cut plane, yield a single-valued analytic continuation which we can obtain by means of the formulas below (provided $\gamma + 1$ is not a natural number and $\alpha - \beta$ and $\gamma - \alpha - \beta$ are not integers). These formulas make it possible to calculate the values of F in the given region even in the case in which $|z| > 1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α, β, γ .

Transformation formulas

9.131

$$\begin{aligned}
 1. \quad F(\alpha, \beta; \gamma; z) &= (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right); \\
 &= (1-z)^{-\beta} F\left(\beta, \gamma - \alpha; \gamma; \frac{z}{z-1}\right);
 \end{aligned}$$

$$2. \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z) + \\ + (1 - z)^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - z).$$

EH I 94, MO 13

9.132

$$1. \quad F(\alpha, \beta; \gamma; z) = \frac{(1 - z)^{-\alpha}\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} F\left(\alpha, \gamma - \beta; \alpha - \beta + 1; \frac{1}{1 - z}\right) + \\ + (1 - z)^{-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} F\left(\beta, \gamma - \alpha; \beta - \alpha + 1; \frac{1}{1 - z}\right).$$

MO 13

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$$2.^8 \quad F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-z)^{-\alpha} z^{-\alpha} F\left(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; \frac{1}{z}\right) + \\ + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-z)^{-\beta} z^{-\beta} F\left(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; \frac{1}{z}\right).$$

GA 220(93)

9.133

$$F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; z\right) = F\left[\alpha, \beta; \alpha + \beta + \frac{1}{2}; 4z(1 - z)\right] \quad \left[|z| \leq \frac{1}{2}, |z(1 - z)| \leq \frac{1}{4}\right].$$

WH

9.134

$$1. \quad F(\alpha, \beta; 2\beta; z) = \left(1 - \frac{z}{2}\right)^{-\alpha} F\left[\frac{\alpha}{2}, \frac{\alpha + 1}{2}; \beta + \frac{1}{2}; \left(\frac{z}{2 - z}\right)^2\right].$$

MO 13, EH I 111(4)

$$2. \quad F(2\alpha, 2\alpha + 1 - \gamma; \gamma; z) = (1 + z)^{-2\alpha} F\left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{4z}{(1 + z)^2}\right).$$

GA 225(100)

9.135

$$F\left(\alpha, \beta; \alpha + \beta + \frac{1}{2}; \sin^2 \varphi\right) = F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \sin^2 \frac{\varphi}{2}\right) \\ \left[x = \sin^2 \frac{\varphi}{2} \text{ real}; \quad \frac{1 - \sqrt{2}}{2} < x < \frac{1}{2} \right].$$

MO 13

9.136

We set

$$A = \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\beta + \frac{1}{2}\right)}, \quad B = \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right) \Gamma\left(-\frac{1}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)};$$

then

$$1. \quad F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \frac{1 - \sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) + B\sqrt{z}F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}; \frac{3}{2}; z\right)$$

GA 227(106)

$$2. \quad F\left(2\alpha, 2\beta; \alpha + \beta + \frac{1}{2}; \frac{1 + \sqrt{z}}{2}\right) = AF\left(\alpha, \beta; \frac{1}{2}; z\right) - B\sqrt{z}F\left(\alpha + \frac{1}{2}, \beta + \frac{1}{2}; \frac{3}{2}; z\right)$$

GA 227(107)

$$3. \quad \frac{\left(\alpha - \frac{1}{2}\right)\left(\beta - \frac{1}{2}\right)}{\alpha + \beta - \frac{1}{2}} A\sqrt{z}F\left(\alpha, \beta; \frac{3}{2}; z\right) = F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1 + \sqrt{z}}{2}\right) - \\ - F\left(2\alpha - 1, 2\beta - 1; \alpha + \beta - \frac{1}{2}; \frac{1 - \sqrt{z}}{2}\right).$$

GA 229(110)

Gauss' recursion formulas:

1. $\gamma[\gamma - 1 - (2\gamma - \alpha - \beta - 1)z]F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)zF(\alpha, \beta; \gamma + 1; z) + \gamma(\gamma - 1)(z - 1)F(\alpha, \beta; \gamma - 1; z) = 0.$
2. $(2\alpha - \gamma - \alpha z + \beta z)F(\alpha, \beta; \gamma; z) + (\gamma - \alpha)F(\alpha - 1, \beta; \gamma; z) + \alpha(z - 1)F(\alpha + 1, \beta; \gamma; z) = 0.$
3. $(2\beta - \gamma - \beta z + \alpha z)F(\alpha, \beta; \gamma; z) + (\gamma - \beta)F(\alpha, \beta - 1; \gamma; z) + \beta(z - 1)F(\alpha, \beta + 1; \gamma; z) = 0.$
4. $\gamma F(\alpha, \beta - 1; \gamma; z) - \gamma F(\alpha - 1, \beta; \gamma; z) + (\alpha - \beta)zF(\alpha, \beta; \gamma + 1; z) = 0.$
5. $\gamma(\alpha - \beta)F(\alpha, \beta; \gamma; z) - \alpha(\gamma - \beta)F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)F(\alpha, \beta + 1; \gamma + 1; z) = 0.$
6. $\gamma(\gamma + 1)F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)F(\alpha, \beta; \gamma + 1; z) - \alpha\beta zF(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
7. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha)F(\alpha, \beta + 1; \gamma + 1; z) - \alpha(1 - z)F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
8. $\gamma F(\alpha, \beta; \gamma; z) + (\beta - \gamma)F(\alpha + 1, \beta; \gamma + 1; z) - \beta(1 - z)F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
9. $\gamma(\gamma - \beta z - \alpha)F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \alpha)F(\alpha - 1, \beta; \gamma; z) + \alpha\beta z(1 - z)F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
10. $\gamma(\gamma - \alpha z - \beta)F(\alpha, \beta; \gamma; z) - \gamma(\gamma - \beta)F(\alpha, \beta - 1; \gamma; z) + \alpha\beta z(1 - z)F(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
11. $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha, \beta + 1; \gamma; z) + \alpha zF(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
12. $\gamma F(\alpha, \beta; \gamma; z) - \gamma F(\alpha + 1, \beta; \gamma; z) + \beta zF(\alpha + 1, \beta + 1; \gamma + 1; z) = 0.$
13. $\gamma[\alpha - (\gamma - \beta)z]F(\alpha, \beta; \gamma; z) - \alpha\gamma(1 - z)F(\alpha + 1, \beta; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)zF(\alpha, \beta; \gamma + 1; z) = 0.$
14. $\gamma[\beta - (\gamma - \alpha)z]F(\alpha, \beta; \gamma; z) - \beta\gamma(1 - z)F(\alpha, \beta + 1; \gamma; z) + (\gamma - \alpha)(\gamma - \beta)zF(\alpha, \beta; \gamma + 1; z) = 0.$
15. $\gamma(\gamma + 1)F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)F(\alpha, \beta + 1; \gamma + 1; z) + \alpha(\gamma - \beta)zF(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
16. $\gamma(\gamma + 1)F(\alpha, \beta; \gamma; z) - \gamma(\gamma + 1)F(\alpha + 1, \beta; \gamma + 1; z) + \beta(\gamma - \alpha)zF(\alpha + 1, \beta + 1; \gamma + 2; z) = 0.$
17. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \beta)F(\alpha, \beta; \gamma + 1; z) - \beta F(\alpha, \beta + 1; \gamma + 1; z) = 0.$
18. $\gamma F(\alpha, \beta; \gamma; z) - (\gamma - \alpha)F(\alpha, \beta; \gamma + 1; z) - \alpha F(\alpha + 1, \beta; \gamma + 1; z) = 0.$

MO 13-14

9.14 A generalized hypergeometric series

The series

$$1. {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \dots (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k \dots (\beta_q)_k} \frac{z^k}{k!}$$

is called a *generalized hypergeometric series* (see also 9.210).

MO 14

For integral representations, see 3.254 2., 3.259 2., and 3.478 3.

1072

9.15 The hypergeometric differential equation

9.151

A hypergeometric series is one of the solutions of the differential equation

$$z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{du}{dz} - \alpha\beta u = 0,$$

WH

which is called the *hypergeometric equation*.

The solution of the hypergeometric differential equation

9.152

The hypergeometric differential equation 9.151 possesses *two linearly independent solutions*. These solutions have analytic continuations to the entire z -plane except possibly for the three points 0, 1, and ∞ . Generally speaking, the points $z = 0, 1, \infty$ are branch points of at least one of the branches of each solution of the hypergeometric differential equation. The ratio $w(z)$ of two linearly independent solutions satisfies the differential equation

$$2\frac{w'''}{w'} - 3\left(\frac{w''}{w'}\right)^2 = \frac{1-a_1^2}{z^2} + \frac{1-a_2^2}{(z-1)^2} + \frac{a_1^2+a_2^2-a_3^2-1}{z(z-1)},$$

where

$$a_1^2 = (1-\gamma)^2, \quad a_2^2 = (\gamma-\alpha-\beta)^2, \quad a_3^2 = (\alpha-\beta)^2.$$

If α, β, γ are real, the function $w(z)$ maps the upper ($\text{Im } z > 0$) or the lower ($\text{Im } z < 0$) half-plane onto a curvilinear triangle whose

$$\alpha, \beta, \gamma \qquad w(z) \qquad z > 0 \qquad z < 0$$

angles are $\pi a_1, \pi a_2, \pi a_3$. The vertices of this triangle are the images of the points $z = 0, z = 1$, and $z = \infty$.

9.153

Within the unit circle $|z| < 1$, the linearly independent solutions $u_1(z)$ and $u_2(z)$ of the hypergeometric differential equation are given by the following formulas:

1. If γ is not an integer,

$$u_1 = F(\alpha, \beta; \gamma; z),$$

$$u_2 = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z).$$

2. If $\gamma = 1$, then

$$u_1 = F(\alpha, \beta; 1; z),$$

$$u_2 = F(\alpha, \beta; 1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(k!)^2} \times$$

$$\times \{ \psi(\alpha + k) - \psi(\alpha) + \psi(\beta + k) - \psi(\beta) - 2\psi(k + 1) + 2\psi(1) \}$$

(see 9.14 2.).

9.14

1073

3. If $\gamma = m + 1$ (where m is a natural number), and if neither α nor β is a positive number not exceeding m , then

$$u_1 = F(\alpha, \beta; m + 1; z),$$

$$u_2 = F(\alpha, \beta; m + 1; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha)_k (\beta)_k}{(1 + m)_k} \{ h(k) - h(0) \} - \sum_{k=1}^m \frac{(k - 1)! (-m)_k}{(1 - \alpha)_k (1 - \beta)_k} z^{-k}$$

(see 9.14 2.),

9.14

where

$$h(n) = \psi(\alpha + n) + \psi(\beta + n) - \psi(m + 1 + n) - \psi(n + 1) \quad [n + 1 \text{ a natural number}].$$

4.⁷ Suppose that $\gamma = m + 1$ (where m is a natural number) and that α or β is equal to $m' + 1$, where $0 \leq m' < m$. Then, for example, for $\alpha = m' + 1$, we obtain

$$u_1 = F(1 + m', \beta; 1 + m; z),$$

$$u_2 = z^{-m} F(1 + m' - m, \beta - m; 1 - m; z).$$

In this case, u_2 is a polynomial in z^{-1} .

5. If $\gamma = 1 - m$ (where m is a natural number) and if α and β are both different from the numbers $0, -1, -2, \dots, 1 - m$, then

$$\begin{aligned} u_1 &= z^m F(\alpha + m, \beta + m; 1 + m; z), \\ u_2 &= z^m F(\alpha + m, \beta + m; 1 + m; z) \ln z + \sum_{k=1}^{\infty} z^k \frac{(\alpha + m)_k (\beta + m)_k}{(1 + m)_k k!} \{h^*(k) - h^*(0)\} - \\ &\quad - \sum_{k=1}^{\infty} \frac{(k-1)! (-m)_k}{(1 - \alpha - m)_k (1 - \beta - m)_k} z^{m-n} \quad (\text{see 9.14 2.}), \end{aligned}$$

9.14

where

$$h^*(n) = \psi(\alpha + m + n) + \psi(\beta + m + n) - \psi(1 + m + n) - \psi(1 + n).$$

We note that

$$\psi(\alpha + n) - \psi(\alpha) = \frac{1}{\alpha} + \frac{1}{\alpha + 1} + \dots + \frac{1}{\alpha + n - 1} \quad (\text{cf. 8.365 3.})$$

and that, for $\alpha = -\lambda$, where λ is a natural number or zero and $n = \lambda + 1, \lambda + 2, \dots$ the expression

$$(\alpha)_k [\psi(\alpha + n) - \psi(\alpha)]$$

in formulas 9.153 2.-5. should be replaced with the expression

$$(-1)^\lambda \lambda! (n - \lambda - 1)!.$$

1074

6. Suppose that $\gamma = 1 - m$ (where m is a natural number) and that α or β is an integer $(-m')$, where m' is one of the following numbers: $0, 1, \dots, m - 1$. Suppose, for example, that $\alpha = -m'$. Then,

$$u_1 = F(-m', \beta; 1 - m; z),$$

$$u_2 = F(-m' + m, \beta + m; 1 + m; z).$$

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7. For $\gamma = \frac{1}{2}(\alpha + \beta + 1)$

$$u_1 = F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); z\right),$$

$$u_2 = F\left(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); 1 - z\right)$$

are two linearly independent solutions of the hypergeometric differential equation provided α, β and γ are not zero or negative integers.

The analytic continuation of a solution that is regular at the point $z = 0$

9.154

Formulas 9.153 make possible the analytic continuation, by means of the hypergeometric series, of the function $F(\alpha, \beta; \gamma; z)$ defined inside the circle $|z| < 1$ to the region $|z| > 1, |\arg(-z)| < \pi$. Here, it is assumed that $\alpha - \beta$ is not an integer. In the event that $\alpha - \beta$ is an integer (for example, if $\beta = \alpha + m$, where m is a natural number), then, for $|z| > 1, |\arg(-z)| < \pi$ we have:

$$1. \quad \frac{\Gamma(\alpha)\Gamma(\alpha+m)}{\Gamma(\gamma)} F(\alpha, \alpha+m; \gamma; z) = \\ = \frac{\sin \pi(\gamma - \alpha)}{\pi} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha+k)\Gamma(1-\gamma+\alpha+k)\Gamma(m-k)}{k!} (-z)^{-\alpha-k} + \right. \\ \left. + (-z)^{-\alpha-m} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+m+k)\Gamma(1-\gamma+\alpha+m+k)}{k!(k+m)!} g(k) z^{-k} \right\},$$

where

$$2. \quad g(n) = \ln(-z) + \pi \operatorname{ctg} \pi(\gamma - \alpha) + \psi(n+1) + \psi(n+m+1) - \\ - \psi(\alpha+m+n) - \psi(1-\gamma+\alpha+m+n).$$

For $m = 0$, we should set $\sum_{k=0}^{m-1} = 0$.

9.155

This formula loses its meaning when α, γ , or $\alpha - \gamma + 1$ is equal to one of the numbers $0, -1, -2, \dots$. In this last case, we have

1. If α is a nonpositive integer and γ is not an integer, $F(\alpha, \alpha+m; \gamma; z)$ is a polynomial in z .

1075

2. Suppose that γ is a nonpositive integer and that α is not an integer. We then set $\gamma = -\lambda$, where $\lambda = 0, 1, 2, \dots$. Then,

$$\frac{\Gamma(\alpha+\lambda+1)\Gamma(\alpha+\lambda+m+1)}{\Gamma(\lambda+2)} z^{\lambda+1} F(\alpha+\lambda+1, \alpha+\lambda+m+1; \lambda+2; z)$$

$$\frac{\Gamma(\alpha + \lambda + 1)\Gamma(\alpha + \lambda + m + 1)}{\Gamma(\lambda + 2)} z^{\lambda+1} F(\alpha + \lambda + 1, \alpha + \lambda + m + 1; \lambda + 2; z)$$

is a solution of the hypergeometric equation that is regular at the point $z = 0$. This solution is equal to the right hand member of formula 9.154 1 if we replace γ with λ in this equation and in formula 9.154 2.

3. If $\alpha - \gamma + 1$ is a nonpositive integer and if α and γ are not themselves integers, we may use the formula

$$F(\alpha, \alpha + m; \gamma; z) = (1 - z)^{\gamma - 2\alpha - m} F(\gamma - \alpha - m, \gamma - \alpha; \gamma; z)$$

and apply formula 9.154 1. to its right hand member provided $\gamma - \alpha - m > 0$. However, if $\alpha - \gamma - m \leq 0$, the right member of this expression is a polynomial taken to the $(1 - z)$ th power.

4.⁷ If α, β , and γ are integers, the hypergeometric differential equation always has a solution that is regular for $z = 0$ and that is of the form

$$R_1(z) + \ln(1 - z)R_2(z),$$

where $R_1(z)$ and $R_2(z)$ are rational functions of z . To get a solution of this form, we need to apply formulas 9.137 1.-9.137 3. to the function $F(\alpha, \beta; \gamma; z)$. However, if $\gamma = -\lambda$, where $\lambda + 1$ is a natural number, formulas 9.137 1. and 9.137 2. should be applied not to $F(\alpha, \beta; \gamma; z)$ but to the function $z^{\lambda+1} F(\alpha + \lambda + 1, \beta + \lambda + 1; \lambda + 2, z)$.

By successive applications of these formulas, we can reduce the positive values of the parameters to one or zero. Furthermore, we can obtain the desired form of the solution from the formulas

$$F(1, 1; 2; z) = -z^{-1} \ln(1 - z),$$

$$F(0, \beta; \gamma; z) = F(\alpha, 0; \gamma; z) = 1.$$

MO 19-20

9.16 Riemann's differential equation

9.160

The hypergeometric differential equation is a particular case of Riemann's differential equation

The coefficients of this equation have poles at the points $a, b,$ and $c,$ and the numbers $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are called the indices corresponding to these poles. The indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are related by the following equation:

$$\alpha + \alpha' + \beta - \beta' + \gamma + \gamma' - 1 = 0.$$

1076

2. The differential equations 9.160 1. are written diagrammatically as follows:

$$3. \quad u = P \left\{ \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ z \end{array} \right\}.$$

The singular points of the equation appear in the first row in this scheme, the indices corresponding to them appear beneath them, and the independent variable appears in the fourth column.

9.161

The two following transformation formulas are valid for Riemann's P -equation:

$$1. \quad \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l P \left\{ \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ z \end{array} \right\} = P \left\{ \begin{array}{ccc} a & b & c \\ \alpha+k & \beta-k-1 & \gamma+l \\ \alpha'+k & \beta'-k-l & \gamma'+l \\ z \end{array} \right\}.$$

$$2.^7 \quad P \left\{ \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ z \end{array} \right\} = P \left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ z_1 \end{array} \right\}.$$

The first of these formulas means that if

$$u = P \left\{ \begin{array}{cccc} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\},$$

the function

$$u_1 = \left(\frac{z-a}{z-b} \right)^k \left(\frac{z-c}{z-b} \right)^l u$$

satisfies a second-order differential equation having the same singular points as equation 9.161 2. and indices equal to $\alpha + k, \alpha' + k; \beta - k - l, \beta' - k - l; \gamma + l, \gamma' + l$. The second transformation formula converts a differential equation with singularities at the points $a, b,$ and $c,$ indices $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$, and an independent variable z into a differential equation with the same indices, singular points $a_1, b_1,$ and $c_1,$ and independent variable z_1 . The variable z_1 is connected with the variable z by the bilinear transformation

$$z = \frac{Az_1 + B}{Cz_1 + D} \quad [AD - BC \neq 0].$$

The same transformation connects the points $a_1, b_1,$ and c_1 with the points $a, b,$ and $c.$

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9.162

By the successive application of the two transformation formulas 9.161 1. and 9.161 2., we can convert Riemann's differential equation into the hypergeometric differential equation. Thus, the solution of Riemann's differential equation can be expressed in terms of a hypergeometric function.

1077

For $k = -\alpha, l = -\gamma,$ and $z_1 = \frac{(z-a)(c-b)}{(z-b)(c-a)},$ we have

$$\begin{aligned} 1. \quad u &= P \left\{ \begin{array}{cccc} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{array} \right\} = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{array}{cccc} a & b & c & \\ 0 & \beta + \alpha + \gamma & 0 & z \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma & \end{array} \right\} = \\ &= \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma P \left\{ \begin{array}{cccc} 0 & \infty & 1 & \\ 0 & \beta + \alpha + \gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma & \end{array} \right\}. \end{aligned}$$

$$2. \quad u = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma F \left(\alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)} \right).$$

If the constants $a, b, c; \alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are permuted in a suitable manner, Riemann's equation remains unchanged. Thus, we obtain a set of 24 solutions of differential equations having the following form (provided none of the differences $\alpha - \alpha', \beta - \beta', \gamma - \gamma'$ are integers):

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9.163

$$1. \quad u_1 = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha + \beta + \gamma, \alpha + \beta' + \gamma; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}.$$

$$2. \quad u_2 = \left(\frac{z-a}{z-b} \right)^{\alpha'} \left(\frac{z-c}{z-b} \right)^\gamma F \left\{ \alpha' + \beta + \gamma, \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}.$$

$$3. \quad u_3 = \left(\frac{z-a}{z-b} \right)^\alpha \left(\frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha + \beta + \gamma', \alpha + \beta' + \gamma'; 1 + \alpha - \alpha'; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}.$$

$$4. \quad u_4 = \left(\frac{z-a}{z-b} \right)^{\alpha'} \left(\frac{z-c}{z-b} \right)^{\gamma'} F \left\{ \alpha' + \beta + \gamma', \alpha' + \beta' + \gamma; 1 + \alpha' - \alpha; \frac{(c-b)(z-a)}{(c-a)(z-b)} \right\}.$$

9.164

$$1.* \quad u_5 = \left(\frac{z-b}{z-c} \right)^\beta \left(\frac{z-a}{z-c} \right)^\alpha F \left\{ \beta + \gamma + \alpha, \beta + \gamma' + \alpha; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}.$$

$$2. \quad u_6 = \left(\frac{z-b}{z-c} \right)^{\beta'} \left(\frac{z-a}{z-c} \right)^\alpha F \left\{ \beta' + \gamma + \alpha, \beta' + \gamma' + \alpha; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}.$$

$$3. \quad u_7 = \left(\frac{z-b}{z-c} \right)^\beta \left(\frac{z-a}{z-c} \right)^{\alpha'} F \left\{ \beta + \gamma + \alpha', \beta + \gamma' + \alpha'; 1 + \beta - \beta'; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}.$$

$$4. \quad u_8 = \left(\frac{z-b}{z-c} \right)^{\beta'} \left(\frac{z-a}{z-c} \right)^{\alpha'} F \left\{ \beta' + \gamma + \alpha', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(a-c)(z-b)}{(a-b)(z-c)} \right\}.$$

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9.165

$$1. \quad u_9 = \left(\frac{z-c}{z-a} \right)^{\gamma} \left(\frac{z-b}{z-a} \right)^{\beta} F \left\{ \gamma + \alpha + \beta, \gamma + \alpha' + \beta; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}.$$

$$2. \quad u_{10} = \left(\frac{z-c}{z-a} \right)^{\gamma'} \left(\frac{z-b}{z-a} \right)^{\beta} F \left\{ \gamma' + \alpha + \beta, \gamma' + \alpha' + \beta; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}.$$

$$3. \quad u_{11} = \left(\frac{z-c}{z-a} \right)^{\gamma} \left(\frac{z-b}{z-a} \right)^{\beta'} F \left\{ \gamma + \alpha + \beta', \gamma + \alpha' + \beta'; 1 + \gamma - \gamma'; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}.$$

$$4. \quad u_{12} = \left(\frac{z-c}{z-a} \right)^{\gamma'} \left(\frac{z-b}{z-a} \right)^{\beta'} F \left\{ \gamma' + \alpha + \beta', \gamma' + \alpha' + \beta'; 1 + \gamma' - \gamma; \frac{(b-a)(z-c)}{(b-c)(z-a)} \right\}.$$

9.166

$$1. \quad u_{13} = \left(\frac{z-a}{z-c} \right)^{\alpha} \left(\frac{z-b}{z-c} \right)^{\beta} F \left\{ \alpha + \gamma + \beta, \alpha + \gamma' + \beta; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}.$$

$$2. \quad u_{14} = \left(\frac{z-a}{z-c} \right)^{\alpha'} \left(\frac{z-b}{z-c} \right)^{\beta} F \left\{ \alpha' + \gamma + \beta, \alpha' + \gamma' + \beta; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}.$$

$$3. \quad u_{15} = \left(\frac{z-a}{z-c} \right)^{\alpha} \left(\frac{z-b}{z-c} \right)^{\beta'} F \left\{ \alpha + \gamma + \beta', \alpha + \gamma' + \beta'; 1 + \alpha - \alpha'; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}.$$

$$4. \quad u_{16} = \left(\frac{z-a}{z-c} \right)^{\alpha'} \left(\frac{z-b}{z-c} \right)^{\beta'} F \left\{ \alpha' + \gamma + \beta', \alpha' + \gamma' + \beta'; 1 + \alpha' - \alpha; \frac{(b-c)(z-a)}{(b-a)(z-c)} \right\}.$$

$$1. \quad u_{17} = \left(\frac{z-c}{z-b} \right)^\gamma \left(\frac{z-a}{z-b} \right)^\alpha F \left\{ \gamma + \beta + \alpha, \gamma + \beta' + \alpha; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}.$$

$$2. \quad u_{18} = \left(\frac{z-c}{z-b} \right)^{\gamma'} \left(\frac{z-a}{z-b} \right)^\alpha F \left\{ \gamma' + \beta + \alpha, \gamma' + \beta' + \alpha; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}.$$

$$3. \quad u_{19} = \left(\frac{z-c}{z-b} \right)^\gamma \left(\frac{z-a}{z-b} \right)^{\alpha'} F \left\{ \gamma + \beta + \alpha', \gamma + \beta' + \alpha'; 1 + \gamma - \gamma'; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}.$$

$$4. \quad u_{20} = \left(\frac{z-c}{z-b} \right)^{\gamma'} \left(\frac{z-a}{z-b} \right)^{\alpha'} F \left\{ \gamma' + \beta + \alpha', \gamma' + \beta' + \alpha'; 1 + \gamma' - \gamma; \frac{(a-b)(z-c)}{(a-c)(z-b)} \right\}.$$

9.168

$$1. \quad u_{21} = \left(\frac{z-b}{z-a} \right)^\beta \left(\frac{z-c}{z-a} \right)^\gamma F \left\{ \beta + \alpha + \gamma, \beta + \alpha' + \gamma; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}.$$

$$2. \quad u_{22} = \left(\frac{z-b}{z-a} \right)^{\beta'} \left(\frac{z-c}{z-a} \right)^\gamma F \left\{ \beta' + \alpha + \gamma, \beta' + \alpha' + \gamma; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}.$$

1079

$$3. \quad u_{23} = \left(\frac{z-b}{z-a} \right)^\beta \left(\frac{z-c}{z-a} \right)^{\gamma'} F \left\{ \beta + \alpha + \gamma', \beta + \alpha' + \gamma'; 1 + \beta - \beta'; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}.$$

$$4. \quad u_{24} = \left(\frac{z-b}{z-a} \right)^{\beta'} \left(\frac{z-c}{z-a} \right)^{\gamma'} F \left\{ \beta' + \alpha + \gamma', \beta' + \alpha' + \gamma'; 1 + \beta' - \beta; \frac{(c-a)(z-b)}{(c-b)(z-a)} \right\}.$$

WH

9.17 Representation of certain second-order differential equations by means of a Reimann scheme

9.171

The hypergeometric equation (see 9.151):

$$u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ 0 & \alpha & 0 \\ 1 - \gamma & \beta & \gamma - \alpha - \beta \end{array} \right. z \left. \right\}.$$

WH

9.172

The associated Legendre's equation defining the functions $P_n^m(z)$ for n and m integers (see 8.700 1.):

$$1. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ \frac{1}{2}m & n + 1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{array} \right. \frac{1-z}{2} \left. \right\}.$$

WH

$$2. \quad u = P \left\{ \begin{array}{ccc} 0 & \infty & 1 \\ -\frac{1}{2}n & \frac{1}{2}m & 0 \\ \frac{n+1}{2} & -\frac{1}{2}m & \frac{1}{2} \end{array} \right. \frac{1-z^2}{1-z} \left. \right\}.$$

WH

9.173

The function $P_n^m \left(1 - \frac{z^2}{2n^2} \right)$ satisfies the equation

$$u = P \left\{ \begin{array}{ccc} 4n^2 & \infty & 0 \\ \frac{1}{2}m & n + 1 & \frac{1}{2}m \\ -\frac{1}{2}m & -n & -\frac{1}{2}m \end{array} \right. z^2 \left. \right\}.$$

WH

The function $J_m(z)$ satisfies the limiting form of this equation obtained as $n \rightarrow \infty$.

9.174

The equation defining the polynomials $C_n^\lambda(z)$ (see 8.938):

$$u = P \left\{ \begin{matrix} -1 & \infty & 1 \\ \frac{1}{2} - \lambda & n + 2\lambda & \frac{1}{2} - \lambda \\ 0 & -n & 0 \end{matrix} \quad z \right\}.$$

WH

1080

9.175

Bessel's equation (see 8.401) is the limiting form of the equations:

$$1. \quad u = P \left\{ \begin{matrix} 0 & \infty & c \\ n & ic & \frac{1}{2} + ic \\ -n & -ic & \frac{1}{2} - ic \end{matrix} \quad z \right\},$$

WH

$$2. \quad u = e^{iz} P \left\{ \begin{matrix} 0 & \infty & c \\ n & \frac{1}{2} & 0 \\ -n & \frac{3}{2} - 2ic & 2ic - 1 \end{matrix} \quad z \right\},$$

WH

$$3. \quad u = P \left\{ \begin{matrix} 0 & \infty & c^2 \\ \frac{1}{2}n & \frac{1}{2}(c-n) & 0 \\ -\frac{1}{2}n & -\frac{1}{2}(c+n) & n+1 \end{matrix} \quad z^2 \right\},$$

WH

as $c \rightarrow \infty$.

9.18 Hypergeometric functions of two variables

9.180

$$1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n.$$

Region of convergence

$$|x| < 1, \quad |y| < 1.$$

AK 16

$$2. \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m! n!} x^m y^n.$$

EH I 224(7), AK 14(12)

Region of convergence

$$|x| + |y| < 1.$$

AK 17

$$3. \quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n.$$

EH I 224(8), AK 14(13)

Region of convergence

$$|x| < 1, \quad |y| < 1.$$

AK 17

$$4. \quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m! n!} x^m y^n.$$

EH I 224(9), AK 14(14)

Region of convergence

$$|\sqrt{x}| + |\sqrt{y}| < 1.$$

1081
9.181

The functions $F_1, F_2, F_3,$ and F_4 satisfy the following systems of partial differential equations for z :

1. System of equations for $z = F_1$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y(1-x)\frac{\partial^2 z}{\partial x\partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - \beta y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 233(9)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x(1-y)\frac{\partial^2 z}{\partial x\partial y} + [\gamma - (\alpha + \beta' + 1)y]\frac{\partial z}{\partial x} - \beta' x\frac{\partial z}{\partial y} - \alpha\beta' z = 0.$$

2. System of equations for $z = F_2$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} - xy\frac{\partial^2 z}{\partial x\partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - \beta y\frac{\partial z}{\partial y} - \alpha\beta z = 0,$$

EH I 234(10)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} - xy\frac{\partial^2 z}{\partial x\partial y} + [\gamma' - (\alpha + \beta' + 1)y]\frac{\partial z}{\partial x} - \beta' x\frac{\partial z}{\partial y} - \alpha\beta' z = 0.$$

3. System of equations for $z = F_3$:

$$x(1-x)\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} + [\gamma - (\alpha + \beta + 1)x]\frac{\partial z}{\partial x} - \alpha\beta z = 0,$$

EH I 234(11)

$$y(1-y)\frac{\partial^2 z}{\partial y^2} + x\frac{\partial^2 z}{\partial x\partial y} + [\gamma - (\alpha' + \beta' + 1)y]\frac{\partial z}{\partial y} - \alpha'\beta' z = 0.$$

4. System of equations for $z = F_4$:

$$(1-y)\frac{\partial^2 z}{\partial y^2} - x^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\partial y} + [\gamma' - (\alpha + \beta + 1)y]\frac{\partial z}{\partial y} - (\alpha + \beta + 1)x\frac{\partial z}{\partial x} - \alpha\beta z = 0.$$

AK 44

9.182

For certain relationships between the parameters and the argument, hypergeometric functions of two variables can be expressed in terms of hypergeometric functions of a single variable or in terms of elementary functions:

$$1. \quad F_1(\alpha, \beta, \beta', \beta + \beta'; x, y) = (1-y)^{-\alpha} F\left(\alpha, \beta; \beta + \beta'; \frac{x-y}{1-y}\right).$$

EH I 238(1), AK 24(28)

$$2. \quad F_2(\alpha, \beta, \beta', \beta, \gamma'; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right).$$

EH I 238(2), AK 23

$$3. \quad F_2(\alpha, \beta, \beta', \alpha, \alpha; x, y) = (1-x)^{-\beta} (1-y)^{-\beta'} F\left[\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right].$$

EH I 238(3)

1082

$$4. \quad F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma; x, y) = (1-y)^{\alpha + \beta - \gamma} F(\alpha, \beta; \gamma; x + y - xy).$$

EH I 238(4), AK 25(35)

$$5. \quad F_4[\alpha, \gamma + \gamma' - \alpha - 1, \gamma, \gamma'; x(1-y), y(1-x)] = \\ = F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma; x) F(\alpha, \gamma + \gamma' - \alpha - 1; \gamma'; y).$$

EH I 238(5)

$$6. \quad F_4\left[\alpha, \beta, \alpha, \beta; -\frac{x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right] = \frac{(1-x)^\beta (1-y)^\alpha}{(1-xy)}.$$

$$7. \quad F_4 \left[\alpha, \beta, \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)} \right] = \\ = (1-x)^\alpha (1-y)^\alpha F(\alpha, 1+\alpha-\beta; \beta; xy).$$

EH I 238(7)

$$8. \quad F_4 \left[\alpha, \beta, 1+\alpha-\beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)} \right] = \\ = (1-y)^\alpha F \left[\alpha, \beta; 1+\alpha-\beta; -\frac{x(1-y)}{1-x} \right].$$

EH I 238(8)

$$9. \quad F_4 \left(\alpha, \alpha + \frac{1}{2}, \gamma, \frac{1}{2}; x, y \right) = \frac{1}{2} (1 + \sqrt{y})^{-2\alpha} F \left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1 + \sqrt{y})^2} \right) + \\ + \frac{1}{2} (1 - \sqrt{y})^{-2\alpha} F \left(\alpha, \alpha + \frac{1}{2}; \gamma; \frac{x}{(1 - \sqrt{y})^2} \right).$$

AK 23

$$10. \quad F_1(\alpha, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta')} F(\alpha, \beta; \gamma-\beta'; x).$$

EH I 239(10), AK 22(23)

$$11. \quad F_1(\alpha, \beta, \beta', \gamma; x, x) = F(\alpha, \beta + \beta'; \gamma; x).$$

EH I 239(11), AK 23(25)

9.183

Functional relations between hypergeometric functions of two variables:

$$1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) = (1-x)^{-\beta} (1-y)^{-\beta} F_1 \left(\gamma - \alpha, \beta, \beta', \gamma; \frac{x}{x-1}, \frac{y}{y-1} \right); \\ = (1-x)^{-\alpha} F_1 \left(\alpha, \gamma - \beta - \beta', \beta', \gamma; \frac{x}{x-1}, \frac{y-x}{1-x} \right); \\ = (1-y)^{-\alpha} F_1 \left(\alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1} \right); \\ = (1-x)^{\gamma-\alpha-\beta} (1-y)^{-\beta'} F_1 \left(\gamma - \alpha, \gamma - \beta - \beta', \beta', \gamma; x, \frac{x-y}{1-y} \right); \\ = (1-x)^{-\beta} (1-y)^{\gamma-\alpha-\beta'} F_1 \left(\gamma - \alpha, \beta, \gamma - \beta - \beta', \gamma; \frac{x-y}{x-1}, y \right).$$

$$\begin{aligned}
2. \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= (1-x)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \beta', \gamma, \gamma'; \frac{x}{x-1}, \frac{y}{1-x}\right); \\
&= (1-y)^{-\alpha} F_2\left(\alpha, \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{1-y}, \frac{y}{y-1}\right); \\
&= (1-x-y)^{-\alpha} F_2\left(\alpha, \gamma - \beta, \gamma' - \beta', \gamma, \gamma'; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right).
\end{aligned}$$

EH I 240(8), AK 32(6)

EH I 240(7)

EH I 240(6)

$$\begin{aligned}
3.7 \quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) &= \frac{\Gamma(\gamma')\Gamma(\beta - \alpha)}{\Gamma(\gamma' - \alpha)\Gamma(\beta)} (-y)^{-\alpha} F_4\left(\alpha, \alpha + 1 - \gamma', \gamma, \alpha + 1 - \beta; \frac{x}{y}, \frac{1}{y}\right) + \\
&+ \frac{\Gamma(\gamma')\Gamma(\alpha - \beta)}{\Gamma(\gamma' - \beta)\Gamma(\alpha)} (-y)^{-\beta} F_4\left(\beta + 1 - \gamma', \beta, \gamma, \beta + 1 - \alpha; \frac{x}{y}, \frac{1}{y}\right).
\end{aligned}$$

EH I 240(9), AK 26(37)

9.184

Integral representations:

Double integrals of the Euler type

$$\begin{aligned}
1. \quad F_1(\alpha, \beta, \beta', \gamma; x, y) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')} \times \\
&\times \int\int_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\
&[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re}(\gamma - \beta - \beta') > 0].
\end{aligned}$$

EH I 230(1), AK 28(1)

$$\begin{aligned}
2. \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta)\Gamma(\gamma' - \beta')} \times \\
&\times \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} du dv \\
&[\operatorname{Re} \beta > 0, \quad \operatorname{Re} \beta' > 0, \quad \operatorname{Re}(\gamma - \beta) > 0, \quad \operatorname{Re}(\gamma' - \beta') > 0].
\end{aligned}$$

EH I 230(2), AK 28(2)

$$\begin{aligned}
3. \quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) &= \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')} \times \\
&\times \int\int_{\substack{u \geq 0, v \geq 0 \\ u+v \leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{-\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} (1-vy)^{-\alpha'} du dv
\end{aligned}$$

$$\begin{aligned}
4. \quad F_4[\alpha, \beta, \gamma, \gamma'; x(1-y), y(1-x)] &= \\
&= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \int_0^1 \int_0^1 u^{\alpha-1} v^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-v)^{\gamma'-\beta-1} \times \\
&\quad \times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1} du dv \\
&\quad [\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\gamma-\alpha) > 0, \quad \operatorname{Re}(\gamma'-\beta) > 0] .
\end{aligned}$$

EH I 230(4)

1084

Integrals of the Mellin-Barnes type

9.185

The functions F_1, F_2, F_3 and F_4 can be represented by means of double integrals of the following form:

$$F(x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s, t) \Gamma(-s) \Gamma(-t) (-x)^s (-y)^t ds dt.$$

$\Psi(s, t)$	$F(x, y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\beta')\Gamma(\gamma+s+t)}$	$F_1(\alpha, \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s)\Gamma(\beta'+t)\Gamma(\gamma')}{\Gamma(\beta')\Gamma(\gamma+s)\Gamma(\gamma'+t)}$	$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$
$\frac{\Gamma(\alpha+s)\Gamma(\alpha'+t)\Gamma(\beta+s)\Gamma(\beta'+t)}{\Gamma(\alpha')\Gamma(\beta')\Gamma(\gamma+s+t)}$	$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$
$\frac{\Gamma(\alpha+s+t)\Gamma(\beta+s+t)\Gamma(\gamma')}{\Gamma(\gamma+s)\Gamma(\gamma'+t)}$	$F_4(\alpha, \beta, \gamma, \gamma'; x, y)$

$[\alpha, \alpha', \beta, \beta'$ may not be negative integers].

EH I 232(9-13), AK 41(33)

9.19 A hypergeometric function of several variables

$$\begin{aligned}
F_A(\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n) &= \\
&= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n} m_1! \dots m_n!} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.
\end{aligned}$$

ET I 385

9.2 Confluent Hypergeometric Functions

9.20 Introduction

9.201.

A confluent hypergeometric function is obtained by taking the limit as $c \rightarrow \infty$ in the solution of Riemann's differential equation

$$P \left\{ \begin{array}{ccc} 0 & \infty & c \\ \frac{1}{2} + \mu & -c & c - \lambda \\ \frac{1}{2} - \mu & 0 & \lambda \end{array} \right. z \Bigg\}.$$

WH

1085

9.202

The equation obtained by means of this limiting process is of the form

$$1. \quad \frac{d^2 u}{dz^2} + \frac{du}{dz} + \left(\frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0.$$

WH

Equation 9.202 1. has the following two linearly independent solutions:

$$2. \quad z^{\frac{1}{2} + \mu} e^{-z} \Phi \left(\frac{1}{2} + \mu - \lambda, 2\mu + 1; z \right),$$

$$3. \quad z^{\frac{1}{2} - \mu} e^{-z} \Phi \left(\frac{1}{2} - \mu - \lambda, -2\mu + 1; z \right),$$

which are defined for all values of $\mu \neq \pm \frac{1}{2}, \pm \frac{2}{2}, \pm \frac{3}{2}, \dots$

MO 111

9.21 The functions $\Phi(\alpha, \gamma; z)$ and $\Phi(\alpha, \gamma; z)$

9.210

The series

$$1. \quad \Phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{z^3}{3!} + \dots$$

is also called a *confluent hypergeometric function*.

A second notation: $\Phi(\alpha, \gamma; z) = {}_1F_1(\alpha; \gamma; z)$.

$$2. \quad \Psi(\alpha, \gamma; z) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)} \Phi(\alpha, \gamma; z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(\alpha-\gamma+1, 2-\gamma; z).$$

EH I 257(7)

3. Bateman's function $k_\nu(x)$ is defined by

$$k_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \tan \theta - \nu \theta) d\theta \quad [x, \nu \text{ real}].$$

EH I 267

9.211

Integral representation:

$$1. \quad \Phi(\alpha, \gamma; z) = \frac{2^{1-\gamma} e^{\frac{1}{2}z}}{B(\alpha, \gamma-\alpha)} \int_{-1}^1 (1-t)^{\gamma-\alpha-1} (1+t)^{\alpha-1} e^{\frac{1}{2}zt} dt \quad [0 < \text{Re } \alpha < \text{Re } \gamma].$$

MO 114

$$2. \quad \Phi(\alpha, \gamma; z) = \frac{1}{B(\alpha, \gamma-\alpha)} z^{1-\gamma} \int_0^z e^t t^{\alpha-1} (z-t)^{\gamma-\alpha-1} dt \quad [0 < \text{Re } \alpha < \text{Re } \gamma].$$

MO 114

$$3. \quad \Phi(-\nu, \alpha+1; z) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} e^z z^{-\frac{\alpha}{2}} \int_0^\infty e^{-t} t^{\nu+\frac{\alpha}{2}} J_\alpha(2\sqrt{zt}) dt$$

$$\left[\text{Re } (\alpha + \nu + 1) > 0, \quad |\arg z| < \frac{\pi}{2} \right].$$

$$4.^8 \quad \Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} dt \quad [\operatorname{Re} \alpha > 0, \operatorname{Re} z > 0].$$

EH I 255(2)

1086

Functional relations

9.212

$$1. \quad \Phi(\alpha, \gamma; z) = e^z \Phi(\gamma - \alpha, \gamma; -z).$$

MO 112

$$2. \quad \frac{\tilde{z}}{\gamma} \Phi(\alpha + 1, \gamma + 1; z) = \Phi(\alpha + 1, \gamma; z) - \Phi(\alpha, \gamma; z).$$

MO 112

$$3. \quad \alpha \Phi(\alpha + 1, \gamma + 1; z) = (\alpha - \gamma) \Phi(\alpha, \gamma + 1; z) + \gamma \Phi(\alpha, \gamma; z).$$

MO 112

$$4. \quad \alpha \Phi(\alpha + 1, \gamma; z) = (z + 2\alpha - \gamma) \Phi(\alpha, \gamma; z) + (\gamma - \alpha) \Phi(\alpha - 1, \gamma; z).$$

MO 112

9.213

$$\frac{d\Phi}{dz} = \frac{\alpha}{\gamma} \Phi(\alpha + 1, \gamma + 1; z).$$

MO 112

9.214

$$\lim_{\gamma \rightarrow -n} \frac{1}{\Gamma(\gamma)} \Phi(\alpha, \gamma; z) = z^{n+1} \binom{\alpha + n}{n + 1} \Phi(\alpha + n + 1, n + 2; z) \quad [n = 0, 1, 2, \dots].$$

MO 112

9.215

$$1. \quad \Phi(\alpha, \alpha; z) = e^z.$$

$$2. \quad \Phi(\alpha, 2\alpha; 2z) = 2^{\alpha-\frac{1}{2}} \exp \left[\frac{1}{4}(1-2\alpha)\pi i \right] \Gamma \left(\alpha + \frac{1}{2} \right) e^z z^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}} \left(ze^{\frac{\pi}{2}i} \right).$$

MO 112

$$3. \quad \Phi \left(p + \frac{1}{2}, 2p + 1; 2iz \right) = \Gamma(p + 1) \left(\frac{z}{2} \right)^{-p} e^{iz} J_p(z).$$

MO 15

For a representation of special functions in terms of a confluent hypergeometric function $\Phi(\alpha, \gamma; z)$, see:

for the probability integral, 9.236;

for integrals of Bessel functions, 6.631 1.;

for Hermite polynomials, 8.953 and 8.959;

for Laguerre polynomials, 8.972 1.;

for parabolic-cylinder functions, 9.240;

for the functions $M_{\lambda, \mu}(z)$, 9.220 2. and 9.220 3.

9.216

The function $\Phi(\alpha, \gamma; z)$ is a solution of the differential equation

$$1. \quad z \frac{d^2 F}{dz^2} + (\gamma - z) \frac{dF}{dz} - \alpha F = 0.$$

MO 111

This equation has two linearly independent solutions:

$$2. \quad \Phi(\alpha, \gamma; z)$$

$$3. \quad z^{1-\gamma} \Phi(\alpha - \gamma + 1, 2 - \gamma; z)$$

9.22- 9.23 The Whittaker functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$

9.220

If we make the change of variable $u = e^{-z/2} W$ in equation 9.202 1., we obtain the equation

$$1. \quad \frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\lambda}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) W = 0.$$

MO 115

1087

Equation 9.220 1. has the following two linearly independent solutions:

$$2. \quad M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}} e^{-z/2} \Phi \left(\mu - \lambda + \frac{1}{2}, 2\mu + 1; z \right).$$

$$3. \quad M_{\lambda,-\mu}(z) = z^{-\mu+\frac{1}{2}} e^{-z/2} \Phi \left(-\mu - \lambda + \frac{1}{2}, 2\mu + 1; z \right).$$

MO 115

To obtain solutions that are also suitable for $2\mu = \pm 1, \pm 2, \dots$, we introduce Whittaker's function

$$4. \quad W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right)} M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)} M_{\lambda,-\mu}(z),$$

WH

which, for 2μ approaching an integer, is also a solution of equation 9.220 1.

For the functions $M_{\lambda,\mu}(z)$ and $W_{\lambda,\mu}(z)$, $z = 0$ is a branch point and $z = \infty$ is an essential singular point. Therefore, we shall examine these functions only for $|\arg z| < \pi$.

These functions $W_{\lambda,\mu}(z)$ and $W_{-\lambda,\mu}(-z)$ are linearly independent solutions of equation 9.220 1.

Integral representations

9.221

$$M_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{2^{2\mu} B\left(\mu + \lambda + \frac{1}{2}, \mu - \lambda + \frac{1}{2}\right)} \int_{-1}^1 (1+t)^{\mu-\lambda-\frac{1}{2}} (1-t)^{\mu+\lambda-\frac{1}{2}} e^{\frac{1}{2}zt} dt,$$

WH

if the integral converges. See also 6.631 1. and 7.623 3.

9.222

$$1. \quad W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-z/2}}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} \int_0^\infty e^{-zt} t^{\mu-\lambda-\frac{1}{2}} (1+t)^{\mu-\lambda-\frac{1}{2}} dt.$$

$$\left[\operatorname{Re}(\mu - \lambda) > -\frac{1}{2}, \quad |\arg z| < \frac{\pi}{2} \right].$$

MO 118

$$2. \quad W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-z/2}}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt.$$

$$\left[\operatorname{Re}(\mu - \lambda) > -\frac{1}{2}, \quad |\arg z| < \pi \right].$$

WH

9.223

$$W_{\lambda,\mu}(z) = \frac{e^{-\frac{z}{2}}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(u - \lambda) \Gamma\left(-u - \mu + \frac{1}{2}\right) \Gamma\left(-u + \mu + \frac{1}{2}\right)}{\Gamma\left(-\lambda + \mu + \frac{1}{2}\right) \Gamma\left(-\lambda - \mu + \frac{1}{2}\right)} z^u du$$

1088

[the path of integration is chosen in such a way that the poles of the function $\Gamma(u - \lambda)$ are separated from the poles of the functions $\Gamma(-u - \mu + \frac{1}{2})$ and $\Gamma(-u + \mu + \frac{1}{2})$]. See also 7.142.

9.224

$$W_{\mu, \frac{1}{2} + \mu}(z) = z^{\mu+1} e^{-\frac{1}{2}z} \int_0^\infty (1+t)^{2\mu} e^{-zt} dt = z^{-\mu} e^{\frac{1}{2}z} \int_z^\infty t^{2\mu} e^{-t} dt \quad [\operatorname{Re} z > 0].$$

WH

9.225

$$1. \quad W_{\lambda, \mu}(x) W_{-\lambda, \mu}(x) = -x \int_0^\infty \operatorname{th}^{2\lambda} \frac{t}{2} \{ J_{2\mu}(x \operatorname{sh} t) \sin(\mu - \lambda)\pi + N_{2\mu}(x \operatorname{sh} t) \cos(\mu - \lambda)\pi \} dt \quad \left[|\operatorname{Re} \mu| - \operatorname{Re} \lambda < \frac{1}{2}; \quad x > 0 \right].$$

MO 119

$$2. \quad W_{\{\mu\}}(z_1) W_{\lambda, \mu}(z_2) = \frac{(z_1 z_2)^{\mu + \frac{1}{2}} \exp \left[-\frac{1}{2}(z_1 + z_2) \right]}{\Gamma(1 - \{-\lambda\})} \times \\ \times \int_0^\infty e^{-t} t^{-\{-\lambda\}} (z_1 + t)^{-\frac{1}{2} + \{-\mu\}} (z_2 + t)^{-\frac{1}{2} + \lambda - \mu} \times \\ \times F \left(\frac{1}{2} - \{+\mu\}, \frac{1}{2} - \lambda + \mu; 1 - \{-\lambda\}; \Theta \right) dt, \\ \Theta = \frac{t(z_1 + z_2 + t)}{(z_1 + t)(z_2 + t)} \\ [z_1 \neq 0, \quad z_2 \neq 0, \quad |\arg z_1| < \pi, \quad |\arg z_2| < \pi, \quad \operatorname{Re}(\{+\lambda\}) < 1]$$

MO 119

See also 3.334, 3.381 6., 3.382 3., 3.383 4., 8., 3.384 3., 3.471 2.

9.226

Series representations

$$M_{0, \mu}(z) = z^{\frac{1}{2} + \mu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{2^{4k} k! (\mu + 1)(\mu + 2) \dots (\mu + k)} \right\}.$$

WH

Asymptotic representations

9.227

For large values of $|z|$

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^\lambda \left(1 + \frac{\sum_{k=1}^{\infty} \left[\mu^2 - \left(\lambda - \frac{1}{2} \right)^2 \right] \left[\mu^2 - \left(\lambda - \frac{3}{2} \right)^2 \right] \dots \left[\mu^2 - \left(\lambda - k + \frac{1}{2} \right)^2 \right]}{k! z^k} \right)$$

$[|\arg z| \leq \pi - \alpha < \pi]$.

WH

1089

9.228

For large values of $|\lambda|$

$$M_{\lambda,\mu}(z) \sim \frac{1}{\sqrt{\pi}} \Gamma(2\mu + 1) \lambda^{-\mu - \frac{1}{4}} z^{\frac{1}{4}} \cos \left(2\sqrt{\lambda z} - \mu\pi - \frac{1}{4}\pi \right).$$

MO 118

9.229

$$1. \quad W_{\lambda,\mu} \sim - \left(\frac{4z}{\lambda} \right)^{\frac{1}{4}} e^{-\lambda + \lambda \ln \lambda} \sin \left(2\sqrt{\lambda z} - \lambda\pi - \frac{\pi}{4} \right).$$

MO 118

$$2. \quad W_{-\lambda,\mu} \sim \left(\frac{z}{4\lambda} \right)^{\frac{1}{4}} e^{\lambda - \lambda \ln \lambda - 2\sqrt{\lambda z}}$$

MO 118

Formulas 9.228 and 9.229 are applicable for

$$|\lambda| \gg 1, |\lambda| \gg |z|, |\lambda| \gg |\mu|, z \neq 0, |\arg \sqrt{z}| < \frac{3\pi}{4} \text{ and } |\arg \lambda| < \frac{\pi}{2}.$$

MO 118

Functional relations

9.231

$$2. \quad z^{-\frac{1}{2}-\mu} M_{\lambda,\mu}(z) = (-z)^{-\frac{1}{2}-\mu} M_{-\lambda,\mu}(-z) \quad [2\mu \neq -1, -2, -3, \dots].$$

WH

9.232

$$1. \quad W_{\lambda,\mu}(z) = W_{\lambda,-\mu}(z).$$

MO 116

$$2. \quad W_{-\lambda,\mu}(-z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu + \lambda\right)} M_{-\lambda,\mu}(-z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu + \lambda\right)} M_{-\lambda,-\mu}(-z) \\ \left[|\arg(-z)| < \frac{3}{2}\pi \right].$$

WH

9.233

$$1. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} e^{i\pi\lambda} W_{-\lambda,\mu}(e^{i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu + \lambda + \frac{1}{2}\right)} \exp\left[i\pi\left(\lambda - \mu - \frac{1}{2}\right)\right] W_{\lambda,\mu}(z) \\ \left[-\frac{3}{2}\pi < \arg z < \frac{\pi}{2}; \quad 2\mu \neq -1, -2, \dots \right].$$

MO 117

1090

$$2. \quad M_{\lambda,\mu}(z) = \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu - \lambda + \frac{1}{2}\right)} e^{-i\pi\lambda} W_{-\lambda,\mu}(e^{-i\pi}z) + \frac{\Gamma(2\mu+1)}{\Gamma\left(\mu + \lambda + \frac{1}{2}\right)} \exp\left[-i\pi\left(\lambda - \mu - \frac{1}{2}\right)\right] W_{\lambda,\mu}(z) \\ \left[-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi; \quad 2\mu \neq -1, -2, \dots \right].$$

MO 117

9.234

Recursion formulas

WH

$$2. \quad W_{\mu,\lambda}(z) = \sqrt{z}W_{\mu-\frac{1}{2},\lambda-\frac{1}{2}}(z) + \left(\frac{1}{2} - \lambda - \mu\right) W_{\mu-1,\lambda}(z).$$

WH

$$3. \quad z\frac{d}{dz}W_{\lambda,\mu}(z) = \left(\lambda - \frac{1}{2}z\right) W_{\lambda,\mu}(z) - \left[\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right] W_{\lambda-1,\mu}(z).$$

WH

$$4. \quad \left[\left(\mu + \frac{1-z}{2}\right) W_{\lambda,\mu}(z) - z\frac{d}{dz}W_{\lambda,\mu}(z)\right] \left(\mu + \frac{1}{2} + \lambda\right) = \\ = \left[\left(\mu + \frac{1+z}{2}\right) W_{\lambda,\mu+1}(z) + z\frac{d}{dz}W_{\lambda,\mu+1}(z)\right] \left(\mu + \frac{1}{2} - \lambda\right)$$

MO 117

$$5. \quad \left(\frac{3}{2} + \lambda + \mu\right) \left(\frac{1}{2} + \lambda + \mu\right) zW_{\lambda,\mu}(z) = z(z + 2\mu + 1)\frac{d}{dz}W_{\lambda+1,\mu+1}(z) + \\ + \left[\frac{1}{2}z^2 + \left(\mu - \lambda - \frac{1}{2}\right)z + 2\mu^2 + 2\mu + \frac{1}{2}\right] W_{\lambda+1,\mu+1}(z).$$

MO 117

Connections with other functions

9.235

$$1. \quad M_{0,\mu}(z) = 2^{2\mu}\Gamma(\mu + 1)\sqrt{z}I_{\mu}\left(\frac{z}{2}\right).$$

MO 125a

$$2. \quad W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}K_{\mu}\left(\frac{z}{2}\right).$$

MO 125

9.236

$$1. \quad \Phi(x) = 1 - \frac{e^{\frac{x^2}{2}}}{\sqrt{\pi x}}W_{-\frac{1}{4},\frac{1}{4}}(x^2) = \frac{2x}{\sqrt{\pi}}\Phi\left(\frac{1}{2}, \frac{3}{2}; -x^2\right).$$

$$2. \operatorname{li}(z) = -\frac{\sqrt{z}}{\sqrt{\ln \frac{1}{2}}} W_{-\frac{1}{2}, 0}(-\ln z).$$

1091

$$3. \Gamma(\alpha, x) = e^{-x} \Psi(1 - \alpha, 1 - \alpha; x).$$

EH I 266(21)

$$4. \gamma(\alpha, x) = \frac{x^\alpha}{\alpha} \Phi(\alpha, \alpha + 1; -x).$$

EH I 266(22)

9.237

$$W_{\lambda, \mu}(z) = \frac{(-1)^{2\mu} z^{\mu + \frac{1}{2}} e^{-\frac{1}{2}z}}{\Gamma\left(\frac{1}{2} - \mu - \lambda\right) \Gamma\left(\frac{1}{2} + \mu - \lambda\right)} \times$$

$$\times \left\{ \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu + k - \lambda + \frac{1}{2}\right)}{k! (2\mu + k)!} z^k \left[\psi(k + 1) + \psi(2\mu + k + 1) - \psi\left(\mu + k - \lambda + \frac{1}{2}\right) - \ln z \right] + \right.$$

$$\left. + (-z)^{-2\mu} \sum_{k=0}^{2\mu-1} \frac{\Gamma(2\mu - k) \Gamma\left(k - \mu - \lambda + \frac{1}{2}\right)}{k!} (-z)^k \right\}^*$$

$$\left[|\arg z| < \frac{3\pi}{2}; \quad 2\mu + 1 \text{ is a natural number} \right].$$

MO 116

2. Set $\lambda - \mu - \frac{1}{2} = l$, where $l + 1$ is a natural number. Then

$$W_{l + \mu + \frac{1}{2}, \mu}(z) = (-1)^l z^{\mu + \frac{1}{2}} e^{-\frac{1}{2}z} (2\mu + 1)(2\mu + 2) \dots (2\mu + l) \Phi(-l, 2\mu + 1; z) =$$

$$= (-1)^l z^{\mu + \frac{1}{2}} e^{-\frac{1}{2}z} L_l^{2\mu}(z).$$

MO 116

9.238

$$1. J_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-ix} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2ix\right).$$

EH I 265(9)

$$2. I_\nu(x) = \frac{2^{-\nu}}{\Gamma(\nu+1)} x^\nu e^{-x} \Phi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right).$$

EH I 265(10)

$$3. K_\nu(x) = \sqrt{\pi} e^{-x} (2x)^\nu \Psi\left(\frac{1}{2} + \nu, 1 + 2\nu; 2x\right).$$

EH I 265(13)

1092

9.24- 9.25 Parabolic cylinder functions $D_p(z)$

9.240

$$\begin{aligned} D_p(z) &= 2^{\frac{1}{4} + \frac{p}{2}} W_{\frac{1}{4} + \frac{p}{2}, -\frac{1}{4}}\left(\frac{z^2}{2}\right) z^{-\frac{1}{2}} = \\ &= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \Phi\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) - \frac{\sqrt{2\pi}z}{\Gamma\left(-\frac{p}{2}\right)} \Phi\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\} \end{aligned}$$

MO 120a

are called *parabolic cylinder functions*.

Integral representations

9.241

$$1. D_p(z) = \frac{1}{\sqrt{\pi}} 2^{p+\frac{1}{2}} e^{-\frac{\pi}{2}pi} e^{\frac{z^2}{4}} \int_{-\infty}^{\infty} x^p e^{-2x^2+2ixz} dx \quad [\operatorname{Re} p > -1; \text{ for } x < 0 \arg x^p = p\pi i]$$

MO 122

9.242

$$1.^* D_p(z) = -\frac{\Gamma(p+1)}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0+)} e^{-zt - \frac{1}{2}t^2} (-t)^{-p-1} dt \quad [|\arg(-t)| \leq \pi].$$

WH

$$2. D_p(z) = 2^{\frac{1}{2}(p-1)} \frac{\Gamma\left(\frac{p}{2} + 1\right)}{i\pi} \int_{-\infty}^{(-1+)} e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}p-1} (1-t)^{\frac{1}{2}(p-1)} dt$$

$$\left[|\arg z| < \frac{\pi}{4}; \quad |\arg(1+t)| \leq \pi \right].$$

WH

$$3. D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{-\infty i}^{\infty i} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

$$\left[|\arg z| < \frac{3}{4}\pi; \quad p \text{ is not a positive integer} \right].$$

WH

$$4. D_p(z) = \frac{1}{2\pi i} e^{-\frac{1}{4}z^2} \int_{\infty}^{(0-)} \frac{\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right) \Gamma(-t)}{\Gamma(-p)} (\sqrt{2})^{t-p-2} z^t dt$$

[for all values of $\arg z$; also, the contours encircle the poles of the function $\Gamma(-t)$ but they do not encircle the poles of the function $\Gamma\left(\frac{1}{2}t - \frac{1}{2}p\right)$].

WH

1093
9.243

$$1. D_n(z) = (-1)^\mu \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} (\sqrt{n})^{n+1} e^{\frac{1}{4}z^2 - \frac{1}{2}n} \left\{ \int_{-\infty}^{\infty} e^{-n(t-1)^2} \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt + \int_0^{\infty} [e^{\frac{1}{2}n(1-t^2)} t^n - e^{-n(t-1)^2}] \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt - \int_{-\infty}^0 e^{-n(t-1)^2} \frac{\cos(zt\sqrt{n})}{\sin(zt\sqrt{n})} dt \right\}$$

$$2. D_n(z) = (-1)^\mu 2^{n+2} (2\pi)^{-\frac{1}{2}} e^{\frac{1}{4}z^2} \int_0^\infty t^n e^{-2t^2} \frac{\cos(2zt)}{\sin} dt$$

[n is a natural number, $\mu = \lfloor \frac{n}{2} \rfloor$, and the cosine or sine is chosen according as n is even or odd].

WH

9.244

$$1. D_{-p-1}[(1+i)z] = \frac{e^{-\frac{iz^2}{2}}}{2^{\frac{p-1}{2}} \Gamma\left(\frac{p+1}{2}\right)} \int_0^\infty \frac{e^{-ix^2z^2} x^p}{(1+x^2)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p > -1, \operatorname{Re} iz^2 \geq 0].$$

MO 122

$$2. D_p[(1+i)z] = \frac{2^{\frac{p+1}{2}}}{\Gamma\left(-\frac{p}{2}\right)} \int_1^\infty e^{-\frac{i}{2}z^2x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}} dx \quad [\operatorname{Re} p < 0; \operatorname{Re} iz^2 \geq 0].$$

MO 122

See also 3.383 6., 7., 3.384 2., 6., 3.966 5., 6.

9.245

$$1.* D_p(x) D_{-p-1}(x) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{cth}^{p+\frac{1}{2}} \frac{t}{2} \frac{1}{\sqrt{\operatorname{sh} t}} \sin \frac{x^2 \operatorname{sh} t + p\pi}{2} dt \quad [x \text{ is real, } \operatorname{Re} p < 0].$$

MO 122

$$2. D_p(z e^{\frac{\pi}{4}i}) D_p(z e^{-\frac{\pi}{4}i}) = \frac{1}{\Gamma(-p)} \int_0^\infty \operatorname{cth}^p t \exp\left(-\frac{z^2}{2} \operatorname{sh} 2t\right) \frac{dt}{\operatorname{sh} t} \quad \left[|\arg z| < \frac{\pi}{4}; \operatorname{Re} p < 0\right]$$

MO 122

See also 6.613.

9.246

Asymptotic expansions. If $|z| \gg 1$, $|z| \gg |p|$, then

$$1. D_p(z) \sim e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots\right) \quad \left[|\arg z| < \frac{3}{4}\pi\right]$$

$$2. \quad D_p(z) \sim e^{-z^{2/4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} e^{z^{2/4}} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right) \quad \left[\frac{\pi}{4} < \arg z < \frac{5}{4}\pi \right].$$

MO 121

$$3. \quad D_p(z) \sim e^{-z^{2/4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \dots \right) - \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} e^{z^{2/4}} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \dots \right) \quad \left[-\frac{\pi}{4} > \arg z > \frac{5}{4}\pi \right].$$

MO 121

Functional relations

9.247

Recursion formulas:

$$1. \quad D_{p+1}(z) - zD_p(z) + pD_{p-1}(z) = 0.$$

WH

$$2. \quad \frac{d}{dz} D_p(z) + \frac{1}{2} z D_p(z) - p D_{p-1}(z) = 0.$$

WH

$$3. \quad \frac{d}{dz} D_p(z) - \frac{1}{2} z D_p(z) + D_{p+1}(z) = 0.$$

MO 121

9.248

Linear relations:

9.249

$$D_p[(1+i)x] + D_p[-(1+i)x] = \frac{2^{1+p/2}}{\Gamma(-p)} \exp\left[-\frac{i}{2}\left(x^2 + p\frac{\pi}{2}\right)\right] \int_0^\infty \frac{\cos xt}{t^{p+1}} e^{-it^2/4} dt$$

[x real; $-1 < \operatorname{Re} p < 0$].

MO 122

9.251

$$D_n(z) = (-1)^n e^{z^2/4} \frac{d^n}{dz^n} (e^{-z^2/2}) \quad [n = 0, 1, 2, \dots].$$

WH

1095

9.252

$$D_p(ax+by) = \exp\frac{(bx-ay)^2}{4} \left(\frac{a}{\sqrt{a^2+b^2}}\right)^p \sum_{k=0}^{\infty} \binom{p}{k} D_{p-k}(\sqrt{a^2+b^2}x) D_k(\sqrt{a^2+b^2}y) \left(\frac{b}{a}\right)^k$$

[$a > b > 0$, $x > 0$, $y > 0$, $\operatorname{Re} p \geq 0$] [“summation theorem”].

MO 124

Connections with other functions

9.253

$$D_n(z) = -2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right).$$

MO 123a

9.254

$$1. \quad D_{-1}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right].$$

MO 123

$$2.^3 \quad D_{-2}(z) = e^{\frac{z^2}{4}} \sqrt{\frac{\pi}{2}} \left\{ \sqrt{\frac{\pi}{2}} e^{-\frac{z^2}{2}} - z \left[1 - \Phi\left(\frac{z}{\sqrt{2}}\right)\right] \right\}.$$

Differential equations leading to parabolic cylinder functions:

$$1. \quad \frac{d^2 u}{dz^2} + \left(p + \frac{1}{2} - \frac{z^2}{4} \right) u = 0,$$

$$u = D_p(z), D_p(-z), D_{-p-1}(iz), D_{-p-1}(-iz)$$

(These four solutions are linearly dependent. See 9.248)

$$2. \quad \frac{d^2 u}{dz^2} + (z^2 + \lambda)u = 0, \quad u = D_{-\frac{1+i\lambda}{2}}[\pm(1+i)z].$$

EH II 118(12,13)A, MO 123

$$3.7 \quad \frac{d^2 u}{dz^2} + z \frac{du}{dz} + (p+1)u = 0, \quad u = e^{-\frac{z^2}{4}} D_p(z).$$

MO 123

9.26 Confluent hypergeometric series of two variables

9.261

$$1.6 \quad \Phi_1(\alpha, \beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n \quad [|x| < 1].$$

EH I 225(20)

$$2. \quad \Phi_2(\beta, \beta', \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m (\beta')_m}{(\gamma)_{m+n} m! n!} x^m y^n.$$

EH I 225(21)A, ET I 385

$$3. \quad \Phi_3(\beta, \gamma, x, y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n.$$

EH I 225(22)

The functions Φ_1, Φ_2, Φ_3 satisfy the following systems of partial differential equations:

$$1. \quad z = \Phi_1(\alpha, \beta, \gamma, x, y) \begin{cases} x(1-x) \frac{\partial^2 z}{\partial x^2} + y(1-x) \frac{\partial^2 z}{\partial x \partial y} + [\gamma - (\alpha + \beta + 1)x] \frac{\partial z}{\partial x} - \beta y \frac{\partial z}{\partial y} - \alpha \beta z = 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + (\gamma - y) \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial x} - \alpha z = 0. \end{cases}$$

EH I 235(23)

$$2. \quad z = \Phi_2(\beta, \beta', \gamma, x, y) \begin{cases} x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z = 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + (\gamma - y) \frac{\partial z}{\partial y} - \beta' z = 0. \end{cases}$$

EH I 235(24)

$$3. \quad z = \Phi_3(\beta, \gamma, x, y) \begin{cases} x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + (\gamma - x) \frac{\partial z}{\partial x} - \beta z = 0, \\ y \frac{\partial^2 z}{\partial y^2} + x \frac{\partial^2 z}{\partial x \partial y} + \gamma \frac{\partial z}{\partial y} - z = 0. \end{cases}$$

EH I 235(25)

9.3 Meijer's G-Function

9.30 Definition

$$\mathbf{G}_{p,q}^{m,n} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{1}{2\pi i} \int \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds$$

$[0 \leq m \leq q, 0 \leq n \leq p]$, and the poles of $\Gamma(b_j - s)$ must not coincide with the poles of $\Gamma(1 - a_k + s)$ for any j and k (where $j = 1, \dots, m; k = 1, \dots, n$). Besides 9.301, the following notations are also used

$$\mathbf{G}_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right), \quad \mathbf{G}_{pq}^{mn}(x), \quad \mathbf{G}(x).$$

9.302

Three types of integration paths L in the right member of 9.301. can be exhibited:

1) The path L runs from $-\infty$ to $+\infty$ in such a way that the poles of the functions $\Gamma(1 - a_k + s)$ lie to the left, and the poles of the functions $\Gamma(b_j - s)$ lie to the right of L (for $j = 1, 2, \dots, m$ and $k = 1, 2, \dots, n$). In this case, the conditions under which the integral 9.301 converges are of the form

$$p + q < 2(m + n), \quad |\arg x| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right) \pi.$$

EH I 207(2)

2) L is a loop, beginning and ending at $+\infty$, that encircles the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \dots, m$) once in the negative direction. All the poles of the functions $\Gamma(1 - a_k + s)$ must remain outside this loop. Then, the conditions under which the integral 9.301 converges are:

$$q \geq 1 \text{ and either } p < q \text{ or } p = q \text{ and } |x| < 1.$$

EH I 207(3)

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3) L is a loop, beginning and ending at $-\infty$, that encircles the poles of the functions $\Gamma(1 - a_k + s)$ (for $k = 1, 2, \dots, n$) once in the positive direction. All the poles of the functions $\Gamma(b_j - s)$ (for $j = 1, 2, \dots, m$) must remain outside this loop. The conditions under which the integral in 9.301 converges are

$$p \geq 1 \text{ and either } p > q \text{ or } p = q \text{ and } |x| > 1.$$

EH I 207(4)

The function $G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$ is analytic with respect to x ; it is symmetric with respect to the parameters a_1, \dots, a_n and also with respect to $a_{n+1}, \dots, a_p; b_1, \dots, b_m; b_{m+1}, \dots, b_q$.

EH I 208

9.303⁷

If no two b_j (for $j = 1, 2, \dots, n$) differ by an integer, then, under the conditions that either $p < q$ or $p = q$ and $|x| < 1$,

$$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{n=1}^m \frac{\prod_{j=1}^m \Gamma(b_j - b_h) \prod_{j=1}^n \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^q \Gamma(1 + b_h - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_h)} x^{b_h} \times \\ \times {}_pF_{q-1} [1 + b_h - a_1, \dots, 1 + b_h - a_p; 1 + b_h - b_1, \dots, \\ \dots, \dots, 1 + b_h - b_q; (-1)^{p-m-n} x]^*).$$

EH I 208(5)

The prime by the product symbol denotes the omission of the product when $j = h$. The asterisk under the symbol for the function ${}_pF_{q-1}$ denotes the omission of the h th parameter.

9.304⁷

If no two a_k (for $k = 1, 2, \dots, n$) differ by an integer, then, under the conditions that $q < p$ or $q = p$ and $|x| > 1$,

$$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = \sum_{h=1}^n \frac{\prod_{j=1}^n \Gamma(a_h - a_j) \prod_{j=1}^m \Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^p \Gamma(a_j - a_h + 1) \prod_{j=m+1}^q \Gamma(a_h - b_j)} x^{a_h - 1} \times \\ \times {}_qF_{p-1} [1 + b_1 - a_h, \dots, 1 + b_q - a_h; 1 + a_1 - a_h, \dots, \\ \dots, \dots, 1 + a_p - a_h; (-1)^{q-m-n} x^{-1}]^*).$$

EH I 208(6)

9.31 Functional relations

If one of the parameters a_j (for $j = 1, 2, \dots, n$) coincides with one of the parameters b_j (for $j = m + 1, m + 2, \dots, q$), the order of the G -function decreases. For example,

$$1. \quad G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, a_1 \end{matrix} \right. \right) = G_{p-1, q-1}^{m, n-1} \left(x \left| \begin{matrix} a_2, \dots, a_p \\ b_1, \dots, b_{q-1} \end{matrix} \right. \right) \quad [n, p, q \geq 1].$$

1098

An analogous relationship occurs when one of the parameters b_j (for $j = 1, 2, \dots, m$) coincides with one of the a_j (for $j = n + 1, \dots, p$). In this case, it is m and not n that decreases by one unit.

The G -function with $p > q$ can be transformed into the G -function with $p < q$ by means of the relationships:

$$2. \quad G_{pq}^{mn} \left(x^{-1} \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{qp}^{nm} \left(x \left| \begin{matrix} 1 - b_s \\ 1 - a_r \end{matrix} \right. \right).$$

EH I 209(9)

$$3. \quad x \frac{d}{dx} G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) = G_{pq}^{mn} \left(x \left| \begin{matrix} a_1 - 1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) + (a_1 - 1) G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right) \quad [n \geq 1].$$

EH I 210(13)

9.32 A differential equation for the G -function

$G_{pq}^{mn} \left(x \left| \begin{matrix} a_r \\ b_s \end{matrix} \right. \right)$ satisfies the following linear q th-order differential equation

$$\left[(-1)^{p-m-n} x \prod_{j=1}^p \left(x \frac{d}{dx} - a_j + 1 \right) - \prod_{j=1}^q \left(x \frac{d}{dx} - b_j \right) \right] y = 0 \quad [p \leq q].$$

EH I 210(1)

9.33 Series of G -functions

$$\begin{aligned} G_{pq}^{mn} \left(\lambda x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) &= \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{r!} (1 - \lambda)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1 + r, b_2, \dots, b_q \end{matrix} \right. \right) \\ & \quad [|\lambda - 1| < 1, \quad m \geq 1, \quad \text{if } m = 1 \text{ and } p < q, \lambda \text{ may be arbitrary}]; \\ &= \lambda^{b_q} \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda - 1)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{q-1}, b_q + r \end{matrix} \right. \right) \\ & \quad [m < q, \quad |\lambda - 1| < 1]; \\ &= \lambda^{a_1 - 1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\lambda - \frac{1}{\lambda} \right)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1 - r, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ & \quad \left[n \geq 1, \quad \text{Re } \lambda > \frac{1}{2} \text{ (if } n = 1 \text{ and } p > q, \text{ then } \lambda \text{ may be arbitrary)} \right]; \\ &= \lambda^{a_p - 1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\lambda} - 1 \right)^r G_{pq}^{mn} \left(x \left| \begin{matrix} a_1, \dots, a_{p-1}, a_p - r \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ & \quad \left[n < p, \quad \text{Re } \gamma > \frac{1}{2} \right]. \end{aligned}$$

$$1. J_\nu(x)x^\mu = 2^\mu G_{02}^{10} \left(\frac{1}{4}x^2 \left| \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu \right. \right).$$

EH I 219(44)

$$2. N_\nu(x)x^\mu = 2^\mu G_{13}^{20} \left(\frac{1}{4}x^2 \left| \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \right. \right. \\ \left. \left. \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu - \frac{1}{2} \right. \right).$$

EH I 219(46)

$$3. K_\nu(x)x^\mu = 2^{\mu-1} G_{02}^{20} \left(\frac{1}{4}x^2 \left| \frac{1}{2}\mu + \frac{1}{2}\nu, \frac{1}{2}\mu - \frac{1}{2}\nu \right. \right).$$

EH I 219(47)

$$4. K_\nu(x) = e^x \sqrt{\pi} G_{12}^{20} \left(2x \left| \frac{1}{2} \right. \right. \\ \left. \left. \nu, -\nu \right. \right).$$

EH I 219(49)

$$5. \mathbf{H}_\nu(x)x^\mu = 2^\mu G_{13}^{11} \left(\frac{1}{4}x^2 \left| \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu \right. \right. \\ \left. \left. \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu \right. \right).$$

EH I 220(51)

$$6. S_{\mu,\nu}(x) = 2^{\mu-1} \frac{1}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu}{2}\right)} G_{13}^{31} \left(\frac{1}{4}x^2 \left| \frac{1}{2} + \frac{1}{2}\mu \right. \right. \\ \left. \left. \frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \right. \right).$$

EH I 220(55)

$$7.7 \quad {}_2F_1(a, b; c; -x) = \frac{\Gamma(c)x}{\Gamma(a)\Gamma(b)} G_{22}^{12} \left(x \left| \begin{matrix} -a, -b \\ -1, -c \end{matrix} \right. \right).$$

EH I 222(74)a

$$8. {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{p,q+1}^{1,p} \left(-x \left| \begin{matrix} 1-a_1, \dots, 1-a_p \\ 0, 1-b_1, \dots, 1-b_q \end{matrix} \right. \right);$$

$$\frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G_{q+1,p}^{1,p} \left(\frac{1}{x} \left| \begin{matrix} 1, b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix} \right. \right).$$

$$9. \quad W_{k,m}(x) = \frac{2^k \sqrt{x} e^{\frac{1}{2}x}}{\sqrt{2\pi}} G_{24}^{40} \left(\frac{x^2}{4} \left| \begin{array}{c} \frac{1}{4} - \frac{1}{2}k, \frac{3}{4} - \frac{1}{2}k \\ \frac{1}{2} + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{array} \right. \right).$$

EH I 221(70)

1100

9.4 MacRobert's E -Function

9.41 Representation by means of multiple integrals

$$E(p; \alpha_r; q; \varrho_s; x) = \frac{\Gamma(\alpha_{q+1})}{\Gamma(\varrho_1 - \alpha_1)\Gamma(\varrho_2 - \alpha_2)\dots\Gamma(\varrho_q - \alpha_q)} \times \\ \times \prod_{\mu=1}^q \int_0^\infty \lambda_\mu^{\varrho_\mu - \alpha_\mu - 1} (1 - \lambda_\mu)^{-\varrho_\mu} d\lambda_\mu \prod_{\nu=2}^{p-q-1} \int_0^\infty e^{-\lambda_{q+\nu}} \lambda_{q+\nu}^{\alpha_{q+\nu} - 1} d\lambda_{q+\nu} \times \\ \times \int_0^\infty e^{-\lambda_p} \lambda_p^{\alpha_p - 1} \left[1 + \frac{\lambda_{q+2}\lambda_{q+3}\dots\lambda_p}{(1 + \lambda_1)\dots(1 + \lambda_q)x} \right]^{-\alpha_{q+1}} d\lambda_p$$

[$|\arg x| < \pi, p \geq q + 1, \alpha_r$ and ϱ_s are bounded by the condition that the integrals on the right be convergent.]

EH I 204(3)

9.42 Functional relations

$$1. \quad \alpha_1 x E(\alpha_1, \dots, \alpha_p; \varrho_1, \dots, \varrho_q; x) = x E(\alpha_1 + 1, \alpha_2, \dots, \alpha_p; \varrho_1, \dots, \varrho_q; x) + \\ + E(\alpha_1 + 1, \alpha_2 + 1, \dots, \alpha_p + 1; \varrho_1 + 1, \dots, \varrho_q + 1; x).$$

EH I 205(7)

$$2. \quad (\varrho_1 - 1)x E(\alpha_1, \dots, \alpha_p; \varrho_1, \dots, \varrho_q; x) = x E(\alpha_1, \dots, \alpha_p; \varrho_1 - 1, \varrho_2, \dots, \varrho_q; x) + \\ + E(\alpha_1 + 1, \dots, \alpha_p + 1; \varrho_1 + 1, \dots, \varrho_q + 1; x).$$

EH I 205(9)

$$3. \quad \frac{d}{dx} E(\alpha_1, \dots, \alpha_p; \varrho_1, \dots, \varrho_q; x) = x^{-2} E(\alpha_1 + 1, \dots, \alpha_p + 1; \varrho_1 + 1, \dots, \varrho_q + 1; x).$$

9.5 Riemann's Zeta Functions $\zeta(z, q)$, and $\zeta(z)$, and the Functions $\Phi(z, s, v)$ and $\xi(s)$

9.51 Definition and integral representations

9.511

$$\begin{aligned} \zeta(z, q) &= \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1} e^{-qt}}{1 - e^{-t}} dt; \\ &= \frac{1}{2} q^{-z} + \frac{q^{1-z}}{z-1} + 2 \int_0^\infty (q^2 + t^2)^{-\frac{z}{2}} \left[\sin \left(z \operatorname{arctg} \frac{t}{q} \right) \right] \frac{dt}{e^{2\pi t} - 1} \\ &\quad [0 < q < 1, \quad \operatorname{Re} z > 1]. \end{aligned}$$

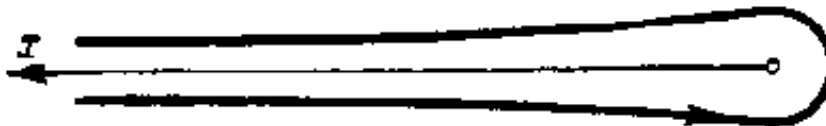
WH

9.512

$$\zeta(z, q) = -\frac{\Gamma(1-z)}{2\pi i} \int_\infty^{(0+)} \frac{(-\theta)^{z-1} e^{-q\theta}}{1 - e^{-\theta}} d\theta.$$

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This equation is valid for all values of z except for $z = 1, 2, 3, \dots$. It is assumed that the path of integration (see drawing below) does not pass through the points $2n\pi i$ (where n is a natural number).



See also 4.251 4., 4.271 1., 4., 8., 4.272 9., 12., 4.294 11.

9.513

$$1. \quad \zeta(z) = \frac{1}{(1 - 2^{1-z})\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt \quad [\operatorname{Re} z > 0].$$

WH

WH

$$3. \quad \zeta(z) = \frac{\pi^{\frac{z}{2}}}{\Gamma(\frac{z}{2})} \left[\frac{1}{z(z-1)} + \int_1^\infty \left(t^{\frac{1-z}{2}} + t^{\frac{z}{2}} \right) t^{-1} \sum_{k=1}^\infty e^{-k^2 \pi i} dt \right].$$

WH

$$4. \quad \zeta(z) = \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty (1+t^2)^{-\frac{z}{2}} \sin(z \operatorname{arctg} t) \frac{dt}{e^{\pi t} + 1}.$$

WH

$$5. \quad \zeta(z) = \frac{2^{z-1}}{2^z - 1} \frac{z}{z-1} + \frac{2}{2^z - 1} \int_0^\infty \left(\frac{1}{4} + t^2 \right)^{-z/2} \sin(z \operatorname{arctg} 2t) \frac{dt}{e^{2\pi t} - 1}.$$

WH

See also 3.411 1., 3.523 1., 3.527 1., 3., 4.271 8.

9.52 Representation as a series or as an infinite product

9.521

$$1. \quad \zeta(z, q) = \sum_{n=0}^\infty \frac{1}{(q+n)^z} \quad [\operatorname{Re} z > 1, \quad q \neq 0, \quad -1, -2, \dots].$$

WH

$$2. \quad \zeta(z, q) = \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin \frac{z\pi}{2} \sum_{n=1}^\infty \frac{\cos 2\pi qn}{n^{1-z}} + \cos \frac{z\pi}{2} \sum_{n=1}^\infty \frac{\sin 2\pi qn}{n^{1-z}} \right] \\ [\operatorname{Re} z < 0, \quad 0 < q \leq 1].$$

WH

$$3.^8 \quad \zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^\infty F_n(z),$$

where

$$3.^8 \quad \zeta(z, q) = \sum_{n=0}^N \frac{1}{(q+n)^z} - \frac{1}{(1-z)(N+q)^{z-1}} - \sum_{n=N}^{\infty} F_n(z),$$

where

$$\begin{aligned} F_n(z) &= \frac{1}{1-z} \left(\frac{1}{(n+1+q)^{z-1}} - \frac{1}{(n+q)^{z-1}} \right) - \frac{1}{(n+1+q)^z} = \\ &= z \int_n^{n+1} \frac{(t-n) dt}{(t+q)^{z+1}} \quad [\operatorname{Re} z > 1]. \end{aligned}$$

WH

1102
9.522

$$1. \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad [\operatorname{Re} z > 1].$$

WH

$$2. \quad \zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^z} \quad [\operatorname{Re} z > 0].$$

WH

9.523

$$\left. \begin{aligned} 1.^7 \quad \zeta(z) &= \prod \frac{1}{1-p^{-z}}, \\ 2. \quad \ln \zeta(z) &= \sum_p \sum_{k=1}^{\infty} \frac{1}{k p^{kz}}. \end{aligned} \right\} \begin{array}{l} \text{The product and the summation are} \\ \text{taken over all primes } p. \end{array} \quad [\operatorname{Re} z > 1].$$

WH

9.524⁷

$$\frac{\zeta'(z)}{\zeta(z)} = - \sum_{k=1}^{\infty} \frac{\Lambda(k)}{k^z}, \quad [\operatorname{Re} z > 1]$$

where $\Lambda(k) = 0$ when k is not a power of a prime and $\Lambda(k) = \ln p$ when k is a power of a prime p . [for $\operatorname{Re} z > 1$].

WH

9.53 Functional relations

9.531

$$\zeta(-n, q) = -\frac{B'_{n+2}(q)}{(n+1)(n+2)} = \frac{-B_{n+1}(q)}{n+1} \quad [n \text{ is a nonnegative integer}].$$

[see EH I 27(11).]

WH

9.532

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} z^k \zeta(k, q) = \ln \frac{e^{-\mathbf{C}z} \Gamma(q)}{\Gamma(z+q)} - \frac{z}{q} + \sum_{k=1}^{\infty} \frac{qz}{k(q+k)} \quad [|z| < q].$$

WH

9.533

$$1. \quad \lim_{z \rightarrow 1} \frac{\zeta(z, q)}{\Gamma(1-z)} = -1.$$

WH

$$2. \quad \lim_{z \rightarrow 1} \left\{ \zeta(z, q) - \frac{1}{z-1} \right\} = -\psi(q).$$

WH

$$3. \quad \left\{ \frac{d}{dz} \zeta(z, q) \right\}_{z=0} = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi.$$

WH

9.534

$$\zeta(z, 1) = \zeta(z).$$

9.535

$$1. \quad \zeta(z) = \frac{1}{2^z - 1} \zeta\left(z, \frac{1}{2}\right) \quad [\operatorname{Re} z > 1].$$

WH

$$2. \quad 2^z \Gamma(1-z) \zeta(1-z) \sin \frac{z\pi}{2} \pi^{1-z} \zeta(z).$$

WH

$$3. \quad 2^{1-z} \Gamma(z) \zeta(z) \cos \frac{z\pi}{2} = \pi^z \zeta(1-z).$$

WH

$$4. \quad \Gamma\left(\frac{z}{2}\right) \pi^{-\frac{z}{2}} \zeta(z) = \Gamma\left(\frac{1-z}{2}\right) \pi^{\frac{z-1}{2}} \zeta(1-z).$$

WH

9.536

$$\lim_{z \rightarrow 1} \left\{ \zeta(z) - \frac{1}{z-1} \right\} = C.$$

9.537

$$\text{Set } z = \frac{1}{2} + it; \quad \text{Then, } \Xi(t) = \frac{(z-1)\Gamma\left(\frac{z}{2} + 1\right)}{\sqrt{\pi^z}} \zeta(z) = \Xi(-t)$$

is an even function of t with real coefficients in its expansion in powers of t^2 .

JA

9.54 Singular points and zeros

9.541⁷

1. $z = 1$ is the only singular point of the function $\zeta(z)$.

WH

2. The function $\zeta(z)$ has simple zeros at the points $-2n$, where n is a natural number. All other zeros of the function $\zeta(z)$ lie in the strip $0 < \operatorname{Re} z < 1$.

3.⁸ Riemann's hypothesis: All zeros of the function $\zeta(z)$ lie on the straight line $\operatorname{Re} z = \frac{1}{2}$. It has been shown that a countably infinite set of zeros of the zeta function lie on this line. The first 1, 500, 000, 001 zeros lying in $0 < \operatorname{Im} z < 545, 439, 823.215$ are known to have $\operatorname{Re} z = 1/20$

WH

9.542

Particular values:

$$\left. \begin{array}{l} 1. \quad \zeta(2m) = \frac{2^{2m-1} \pi^{2m} |B_{2m}|}{(2m)!}, \\ 2. \quad \zeta(1-2m) = -\frac{B_{2m}}{2m}, \\ 3. \quad \zeta(-2m) = 0 \\ 4. \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi. \end{array} \right\} [m \text{ is a natural number}].$$

WH

9.55 The Lerch function $\Phi(z, s, v)$

9.550

Definition:

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n \quad [|z| < 1, \quad v \neq 0, \quad -1, \dots].$$

EH I 27(1)

1104

Functional relations

9.551

$$\Phi(z, s, v) = z^m \Phi(z, s, m+v) + \sum_{n=0}^{m-1} (v+n)^{-s} z^n$$

$$[m = 1, 2, 3, \dots, v \neq 0, -1, -2, \dots].$$

EH I 27(1)

9.552

9.552

$$\Phi(z, s, v) = iz^{-v}(2\pi)^{s-1}\Gamma(1-s) \left[e^{-i\pi\frac{s}{2}}\Phi\left(e^{-2\pi iv}, 1-s, \frac{\ln z}{2\pi i}\right) - e^{i\pi(\frac{s}{2}-2v)}\Phi\left(e^{2\pi iv}, 1-s, 1-\frac{\ln z}{2\pi i}\right) \right].$$

EH I 29(7)

Series representation

9.553

$$\Phi(z, s, v) = z^{-v}\Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\ln z + 2\pi ni)^{s-1} e^{2\pi nvi}$$

$$[0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\ln z + 2\pi ni)| \leq \pi].$$

EH I 28(6)

9.554

$$\Phi(z, m, v) = z^{-v} \left\{ \sum_{n=0}^{\infty} \zeta(m-n, v) \frac{(\ln z)^n}{n!} + \frac{(\ln z)^{m-1}}{(m-1)!} \left[\psi(m) - \psi(v) - \ln\left(\ln \frac{1}{z}\right) \right] \right\}^*$$

$$[m = 2, 3, 4, \dots, |\ln z| < 2\pi, \quad v \neq 0, \quad -1, -2, \dots].$$

EH I 30(9)

* The prime on the symbol \sum means that the term corresponding to

$n = m - 1$ is omitted.

9.555

$$\Phi(z, -m, v) = \frac{m!}{z^v} \left(\ln \frac{1}{z}\right)^{-m-1} - \frac{1}{z^v} \sum_{r=0}^{\infty} \frac{B_{m+r+1}(v)(\ln z)^r}{r!(m+r+1)} \quad [|\ln z| < 2\pi].$$

EH I 30(11)

Integral representation

9.556

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(v-1)t}}{e^t - z} dt$$

$$[\operatorname{Re} v > 0, \quad \text{or } |z| \leq 1, \quad z \neq 1, \quad \operatorname{Re} s > 0, \quad \text{or } z = 1, \quad \operatorname{Re} s > 1].$$

Limit relationships

9.557

$$\lim_{z \rightarrow 1} (1-z)^{1-s} \Phi(z, s, v) = \Gamma(1-s) \quad [\operatorname{Re} s < 1].$$

EH I 30(12)

9.558

$$\lim_{z \rightarrow 1} \frac{\Phi(z, 1, v)}{-\ln(1-z)} = 1.$$

EH I 30(13)

A connection with a hypergeometric function

9.559

$$\Phi(z, 1, v) = v^{-1} {}_2F_1(1, v; 1+v; z) \quad [|z| < 1].$$

EH I 30(10)

1105

9.56 The function $\xi(s)$

9.561

$$\xi(s) = \frac{1}{2} s(s-1) \frac{\Gamma\left(\frac{1}{2}s\right)}{\pi^{\frac{1}{2}s}} \zeta(s).$$

EH III 190(10)

9.562

$$\xi(1-s) = \xi(s).$$

EH III 190(11)

9.6 Bernoulli Numbers and Polynomials, Euler Numbers, the Functions

$\nu(x), \nu(x, \alpha), \mu(x, \beta), \mu(x, \beta, \alpha), \lambda(x, y)$ and Euler Polynomials

9.61 Bernoulli numbers

9.610

The numbers B_n , representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad [0 < |t| < 2\pi],$$

are called *Bernoulli* numbers. Thus, the function $\frac{t}{e^t - 1}$ is a generating function for the Bernoulli numbers.

GE 48(57), FI II 520

9.611

Integral representations

1. $B_{2n} = (-1)^{n-1} 4n \int_0^{\infty} \frac{x^{2n-1}}{e^{2\pi x} - 1} dx \quad [n = 1, 2, \dots] \quad (\text{cf. 3.411 2., 4.}).$
2. $B_{2n} = (-1)^{n-1} \pi^{-2n} \int_0^{\infty} \frac{x^{2n}}{\text{sh}^2 x} dx \quad [n = 1, 2, \dots].$
3. $B_{2n} = (-1)^{n-1} \frac{2n(1-2n)}{\pi} \int_0^{\infty} x^{2n-2} \ln(1 - e^{-2\pi x}) dx \quad [n = 1, 2, \dots].$

3.411
FI II 721a

See also 3.523 2., 4.271 3.

Properties and functional relations

9.612⁸

A symbolic notation:

$$(B + \alpha)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k \alpha^{n-k}, \quad [n \geq z];$$

$$B_n = (B + 1)^{[n]} = \sum_{k=0}^n \binom{n}{k} B_k, \quad [n \geq z];$$

hence by recursion

$$B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n+1-k)!}, \quad [n \geq z].$$

9.613

All the Bernoulli numbers are rational numbers.

1106

9.614

Every number B_n can be represented in the form

$$B_n = C_n - \sum \frac{1}{k+1},$$

where C_n is an integer and the sum is taken over all $k > 0$ such that $k + 1$ is a prime and k is a divisor of n .

GE 64

9.615⁸

All the Bernoulli numbers with odd index are equal to zero except that $B_1 = -\frac{1}{2}$; that is, $B_{2n+1} = 0$ for n a natural number.

GE 52, FI II 521

$$B_{2n} = -\frac{1}{2n+1} + \frac{1}{2} - \sum_{k=1}^{n-1} \frac{2n(2n-1)\dots(2n-k+2)}{(2k)!} B^{2k} \quad [n \geq 1].$$

9.616

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \zeta(2n) \quad [n \geq 0]. \quad (\text{cf. 9.542})$$

GE 56(79), FI II 721a

9.617

$$B_{2n}(-1)^{n-1} = \frac{2(2n)!}{(2\pi)^{2n}} \frac{1}{\prod_p \left(1 - \frac{1}{p^{2n}}\right)} \quad [n \geq 1] \quad (\text{cf. 9.523})$$

9.523

(where the product is taken over all primes p).

For a connection with Riemann's zeta function, see 9.542.

For a connection with the Euler numbers, see 9.635.

For a table of values of the Bernoulli numbers, see 9.71

9.618⁶

$$\text{Symbolic notation: } (B + \alpha)^{[n]} \equiv \sum_{k=0}^n \binom{n}{k} B_k \alpha^{n-k} \quad [n = 0, 1, \dots].$$

CE 337

9.619

An inequality $|(B - \theta)^{[n]}| \leq |B_n| \quad [0 < \theta < 1]$.

9.62 Bernoulli polynomials

9.620

The Bernoulli polynomials $B_n(x)$ are defined by

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$$

GE 51(62)

or symbolically, $B_n(x) = (B + x)^{[n]}$.

GE 52(68)

9.621

The generating function

$$\frac{e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^{n-1}}{n!} \quad [0 < |t| < 2\pi] \quad (\text{cf. 1.213})$$

GE 65(89)a

9.622

Series representation

$$1.^7 \quad B_n(x) = -2 \frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos\left(2\pi kx - \frac{1}{2}\pi n\right)}{k^n}, \quad [n > 1, 1 \geq x \geq 0, n = 1, 1 > x > 0].$$

AS 805(23.1.16)

1107

$$2.^7 \quad B_{2n-1}(x) = 2 \frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}}, \quad [n > 1, 1 \geq x \geq 0, n = 1, 1 > x > 0].$$

AS 805(23.1.17)

$$3.^* \quad B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}} \quad [0 \leq x \leq 1, \quad n = 1, 2, \dots].$$

GE 71

9.623

Functional relations and properties:

$$1. \quad B_{m+1}(n) = B_{m+1} + (m+1) \sum_{k=1}^{n-1} k^m \quad [n \text{ and } m \text{ are natural numbers}] \quad (\text{see also 0.121})$$

GE 51(65)

$$2. \quad B_n(x+1) - B_n(x) = nx^{n-1}.$$

$$3. \quad B'_n(x) = nB_{n-1}(x) \quad [n = 1, 2, \dots].$$

GE 66

$$4. \quad B_n(1-x) = (-1)^n B_n(x).$$

GE 66

$$5.* \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \quad [n = 0, 1, \dots]$$

AS 804(23.1.9)

9.624⁷

$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \quad [m = 1, 2, \dots; n = 0, 1, \dots; \text{“summation theorem”}].$$

GE 67

9.625

For n odd, the differences

$$B_n(x) - B_n$$

vanish on the interval $[0, 1]$ only at the points $0, \frac{1}{2}$, and 1 . They change sign at the point $x = \frac{1}{2}$. For n even, these differences vanish at the end points of the interval $[0, 1]$. Within this interval, they do not change sign and their greatest absolute value occurs at the point $x = \frac{1}{2}$.

9.626

The polynomials

$$B_{2n}(x) - B_{2n} \quad \text{and} \quad B_{2n+2}(x) - B_{2n+2}$$

have opposite signs in the interval $(0, 1)$.

GE 87

9.627

Special cases:

$$\left. \begin{array}{l} 1. \quad B_1(x) = x - \frac{1}{2}. \\ 2. \quad B_2(x) = x^2 - x + \frac{1}{6}. \\ 3. \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2x}. \\ 4. \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}. \\ 5. \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \end{array} \right\}$$

GE 70

1108

9.628

Particular values:

$$1. \quad B_n(0) = B_n.$$

$$2. \quad B_1(1) = -B_1 = \frac{1}{2}, \quad B_n(1) = B_n \quad [n \neq 1].$$

GE 76

9.63 Euler numbers

9.630

The numbers E_n , representing the coefficients of $\frac{t^n}{n!}$ in the expansion of the function

$$\frac{1}{\operatorname{ch} t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \left[|t| < \frac{\pi}{2} \right],$$

are known as the *Euler numbers*. Thus, the function $\frac{1}{\operatorname{ch} t}$ is a generating function for the Euler numbers.

9.631

A recursion formula

$$(E + 1)^{[n]} + (E - 1)^{[n]} = 0 \quad [n \geq 1], \quad E_0 = 1.$$

CE 329

Properties of the euler numbers

9.632

The Euler numbers are integers.

9.633

The Euler numbers of odd index are equal to zero; the signs of two adjacent numbers of even indices are opposite; that is,

$$E_{2n+1} = 0, \quad E_{4n} > 0, \quad E_{4n+2} < 0.$$

CE 329

9.634

If $\alpha, \beta, \gamma, \dots$ are the divisors of the number $n - m$, the difference $E_{2n} - E_{2m}$ is divisible by those of the numbers $2\alpha + 1, 2\beta + 1, 2\gamma + 1, \dots$, that are primes.

9.635

A connection with the Bernoulli numbers (symbolic notation):

$$1.^6 \quad E_{n-1} + 4(-1)^n(3^{n-1} - 1)B_1 = \frac{(4B - 1)^{[n]}(4B - 3)^{[n]}}{2n} + 4(-1)^{n+1}(3^{n-1} - 1)B_1.$$

CE 330

$$2. \quad B_n = \frac{n(E + 1)^{[n-1]}}{2^n(2^n - 1)} \quad [n \geq 2].$$

CE 330

$$3.^6 \quad \left(B + \frac{1}{4}\right)^{[2n+1]} = -4^{-2n-1}(2n + 1)E_{2n} \quad [n \geq 0].$$

For a table of values of the Euler numbers, see 9.72.

1109

9.64 The functions

$\nu(x)$, $\nu(x, \alpha)$, $\mu(x, \beta)$, $\mu(x, \beta, \alpha)$, $\lambda(x, y)$

9.640

$$1. \quad \nu(x) = \int_0^{\infty} \frac{x^t dt}{\Gamma(t+1)}.$$

EH III 217(1)

$$2. \quad \nu(x, \alpha) = \int_0^{\infty} \frac{x^{\alpha+t} dt}{\Gamma(\alpha+t+1)}.$$

EH III 217(1)

$$3. \quad \mu(x, \beta) = \int_0^{\infty} \frac{x^t t^{\beta} dt}{\Gamma(\beta+1)\Gamma(t+1)}.$$

EH III 217(2)

$$4. \quad \mu(x, \beta, \alpha) = \int_0^{\infty} \frac{x^{\alpha+t} t^{\beta} dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)}.$$

EH III 217(2)

$$5. \quad \lambda(x, y) = \int_0^y \frac{\Gamma(u+1) du}{x^u}.$$

MI 9

9.65 Euler polynomials

9.650

The Euler polynomials are defined by

9.651

The generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

AS 804 (23.1.1)

9.652

Series representation:

$$1. \quad E_n(x) = 4 \frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin\left((2k+1)\pi x - \frac{1}{2}\pi n\right)}{(2k+1)^{n+1}}. \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 1, \quad 1 > x > 0].$$

AS 804 (23.1.16)

$$2. \quad E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}}. \quad [n = 1, 2, \dots, \quad 1 \geq x \geq 0].$$

AS 804 (23.1.17)

$$3. \quad E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}}. \quad [n > 0, \quad 1 \geq x \geq 0, \quad n = 0, \quad 1 > x > 0].$$

AS 804 (23.1.18)

9.653

Functional relations and properties:

$$1. \quad E_m(n+1) = 2 \sum_{k=1}^n (-1)^{n-k} k^m + (-1)^{n+1} E_m(0), \quad [m, n \text{ natural numbers}].$$

AS 804 (23.1.4)

1110

As 804 (23.1.5)

$$3. E_n(x+1) + E_n(x) = 2x^n. \quad [n = 0, 1, \dots]$$

As 804 (23.1.6)

$$4. {}^8 E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x + \frac{k}{m}\right). \quad [n = 0, 1, \dots, m = 1, 3, \dots]$$

As 804 (23.1.10)

$$5. E_n(mx) = \frac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right) \quad [n = 0, 1, \dots, m = 2, 4, \dots].$$

AS 804 (23.1.10)

9.654

Special cases:

$$1. E_1(x) = x - \frac{1}{2}.$$

$$2. E_2(x) = x^2 - x.$$

$$3. E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

$$4. E_4(x) = x^4 - 2x^3 + x.$$

$$5. E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}.$$

9.655

Particular values:

$$1. \quad E_{2n+1} = 0. \quad [n = 0, 1, \dots]$$

As 805 (23.1.19)

$$2. \quad E_n = (0) = -E_n(1) = -2(n+1)^{-1}(2^{n+1} - 1)B_{n+1}. \quad [n = 1, 2, \dots]$$

As 805 (23.1.20)

$$3. \quad E_n \left(\frac{1}{2} \right) = 2^{-n} E_n. \quad [n = 0, 1, \dots]$$

As 805 (23.1.21)

$$4. \quad E_{2n-1} \left(\frac{1}{3} \right) = -E_{2n-1} \left(\frac{2}{3} \right) = -(2n)^{-1}(1-3^{1-2n})(2^{2n}-1)B_{2n}. \quad [n = 1, 2, \dots]$$

As 806 (23.1.22)

9.7 Constants

9.71 Bernoulli numbers

$$\begin{aligned} B_0 &= 1, & B_{18} &= \frac{43\,867}{798}, \\ B_1 &= -\frac{1}{2}, & B_{20} &= -\frac{174\,611}{330}, \\ B_2 &= \frac{1}{6}, & B_{22} &= \frac{854\,513}{138}, \\ B_4 &= -\frac{1}{30}, & B_{24} &= -\frac{236\,364\,091}{2730}, \\ B_6 &= \frac{1}{42}, & B_{26} &= \frac{8\,553\,103}{6}, \\ B_8 &= -\frac{1}{30}, & B_{28} &= -\frac{23\,749\,461\,029}{870}, \\ B_{10} &= \frac{5}{66}, & B_{30} &= \frac{8\,615\,841\,276\,005}{14\,322}, \\ B_{12} &= -\frac{691}{2730}, & B_{32} &= -\frac{7\,709\,321\,041\,217}{510}, \\ B_{14} &= \frac{7}{6}, & B_{34} &= \frac{2\,577\,687\,858\,367}{6}, \\ B_{16} &= -\frac{3617}{510}, & & \end{aligned}$$

1111

9.72 Euler numbers

The Bernoulli and Euler numbers of odd index (with the exception of B_1) are equal to zero.

9.73 Euler's and Catalan's constants

Euler's constant

$$C = 0.577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ \dots$$

Catalan's constant

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915\ 965\ 594\ \dots$$

9.74 Stirling numbers

9.740

The **Stirling number of the first kind** $S_n^{(m)}$ is defined by the requirement that $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles.

AS 824 (23.1.3)

9.741

Generating functions:

$$1. \quad x(x-1)\dots(x-n+1) = \sum_{m=0}^n S_n^{(m)} x^m.$$

AS 824 (24.1.3)

1112

$$2. \quad \{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \quad [|x| < 1]$$

Recurrence relations:

$$1.8 \quad S_{n+1}^{(m)} = S_n^{(m-1)} - nS_n^{(m)}; \quad S_n^{(0)} = \delta_{0n}; \quad S_n^{(1)} = (-1)^{n-1}(n-1)!; \quad S_n^{(n)} = 1. \quad [n \geq m \geq 1]$$

AS 824 (24.1.3)

$$2. \quad \binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m+r)} \quad [n \geq m \geq r]$$

AS 824 (24.1.3)

9.743

Functional relations and properties

$$1. \quad x(x-h)(x-2h)\dots(x-mh+h) = \frac{h^m \Gamma\left(\frac{x}{h} + 1\right)}{\Gamma\left(\frac{x}{h} - m + 1\right)} = h^m \sum_{k=1}^m \left(\frac{x}{h}\right)^k S_k^{(m)}.$$

1113

$$2. \quad [(x+1)(x+2)\dots(x+m)]^{-1} = \left[\binom{x+m}{m} m! \right]^{-1} = \left[\sum_{k=1}^m (x+m)^k S_k^{(m)} \right]^{-1}.$$

$$3. \quad [(x+h)(x+2h)\dots(x+mh)]^{-1} = \frac{\Gamma\left(\frac{x}{h} + 1\right)}{h^m \Gamma\left(\frac{x}{h} + m + 1\right)} = \left[h^m \sum_{k=1}^m \left(\frac{x}{h} + m\right)^k S_k^{(m)} \right]^{-1}.$$

9.744⁷

The Stirling number of the second kind $S_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

9.745

Generating functions:

$$1. \quad x^n = \sum_{m=0}^n S_n^{(m)} x(x-1)\dots(x-m+1).$$

$$2. (e^x - 1)^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}.$$

As 824 (24.1.4)

$$3. [(1-x)(1-2x)\dots(1-mx)]^{-1} = \sum_{n=m}^{\infty} S_n^{(m)} x^{n-m} \quad [|x| < m^{-1}]$$

As 824 (24.1.4)

9.746

Closed form:

$$\delta_n^{(m)} = \frac{1}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^n.$$

AS 824 (24.1.4)

9.747

Recurrence relations:

$$1. S_{n+1}^{(m)} = m S_n^{(m)} + S_n^{(m-1)}; \delta_n^{(0)} = \delta_{0n}; \delta_n^{(1)} = \delta_n^{(n)} = 1. \quad [n \geq m \geq 1]$$

AS 825(24.1.4)

$$2. \binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)}. \quad [n \geq m \geq r]$$

AS 825 (24.1.4)

$$3. S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} S_{n-m+k}^{(k)}.$$

AS 824 (24.1.3)

9.748⁸

Particular values:

Stirling numbers of the first kind $S_n^{(m)}$

m	$S_1^{(m)}$	$S_2^{(m)}$	$S_3^{(m)}$	$S_4^{(m)}$	$S_5^{(m)}$	$S_6^{(m)}$	$S_7^{(m)}$	$S_8^{(m)}$	$S_9^{(m)}$
1	1	-1	2	-6	24	-120	720	-5040	40320
2		1	-3	11	-50	274	-1764	13068	-109584
3			1	-6	35	-225	1624	-13132	118124

Stirling numbers of the second kind $S_n^{(m)}$

m	$S_1^{(m)}$	$S_2^{(m)}$	$S_3^{(m)}$	$S_4^{(m)}$	$S_5^{(m)}$	$S_6^{(m)}$	$S_7^{(m)}$	$S_8^{(m)}$	$S_9^{(m)}$
1	1	1	1	1	1	1	1	1	1
2		1	3	7	15	31	63	127	255
3			1	6	25	90	301	966	3025
4				1	10	65	350	1701	7770
5					1	15	140	1050	6951
6						1	21	266	2646
7							1	28	462
8								1	36
9									1

9.749⁸

Relationship between Stirling numbers of the first kind and derivatives of $(\ln x)^{-m}$:

$$1. \quad \frac{d^n}{dx^n} \left(\frac{1}{\ln^m x} \right) = \frac{1}{\ln^m x} \sum_{k=1}^n \frac{(-1)^k (m)_k S_n^{(x)}}{x^n \ln^k x},$$

where

$$(m)_k = \Gamma(m+k)/\Gamma(m) \quad [m, n \text{ positive integers}]$$

10. Vector Field Theory

10.1- 10.8 Vectors, Vector Operators, and Integral Theorems

10.11 Products of vectors

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$ be arbitrary vectors, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the set of orthogonal unit vectors in terms of which the components of \mathbf{a} , \mathbf{b} , and \mathbf{c} are expressed. Two different products involving pairs of vectors are defined, namely, the scalar product, written $\mathbf{a} \cdot \mathbf{b}$, and the vector product, written either $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$. Their properties are as follows:

$$1. \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (\text{scalar product}).$$

$$2. \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (\text{vector product}).$$

$$3. \quad \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (\text{triple scalar product}).$$

$$4. \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (\text{triple vector product}).$$

10.12 Properties of scalar product

$$1. \quad \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \quad (\text{commutative}).$$

$$2. \quad \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{b} \times \mathbf{c} \cdot \mathbf{a} = \mathbf{c} \times \mathbf{a} \cdot \mathbf{b} = -\mathbf{a} \times \mathbf{c} \cdot \mathbf{b} = -\mathbf{b} \times \mathbf{a} \cdot \mathbf{c} = -\mathbf{c} \times \mathbf{b} \cdot \mathbf{a}.$$

Note: $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ is also written $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$; thus (2) may also be written

$$3. \quad [\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}].$$

10.13 Properties of vector product

$$1. \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad (\text{anticommutative}).$$

$$2. \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}.$$

$$3. \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

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10.14 Differentiation of vectors

If $\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t))$, $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))$, $\mathbf{c}(t) = (c_1(t), c_2(t), c_3(t))$, $\phi(t)$ is a scalar and all functions of t are

$$\mathbf{a}(t) = (a_1(t), a_2(t), a_3(t)) \quad \mathbf{b}(t) = (b_1(t), b_2(t), b_3(t)) \quad \mathbf{c}(t) = (c_1(t), c_2(t), c_3(t)) \quad \phi(t)$$

t

differentiable, then

$$1. \quad \frac{d\mathbf{a}}{dt} = \frac{da_1}{dt} \mathbf{i} + \frac{da_2}{dt} \mathbf{j} + \frac{da_3}{dt} \mathbf{k}.$$

$$2. \quad \frac{d}{dt}(\mathbf{a} + \mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}.$$

$$3. \quad \frac{d}{dt}(\phi\mathbf{a}) = \frac{d\phi}{dt} \mathbf{a} + \phi \frac{d\mathbf{a}}{dt}.$$

$$4. \quad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}) = \frac{d\mathbf{a}}{dt} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{d\mathbf{b}}{dt}.$$

$$5. \quad \frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} + \mathbf{a} \times \frac{d\mathbf{b}}{dt}.$$

$$6. \quad \frac{d}{dt}(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}) = \frac{d\mathbf{a}}{dt} \times \mathbf{b} \cdot \mathbf{c} + \mathbf{a} \times \frac{d\mathbf{b}}{dt} \cdot \mathbf{c} + \mathbf{a} \times \mathbf{b} \cdot \frac{d\mathbf{c}}{dt}.$$

$$7. \quad \frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} \right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} \right).$$

10.21 Operators grad, div, and curl

In cartesian coordinates $O\{x_1, x_2, x_3\}$, in which system it is convenient to denote the triad of unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the vector operator ∇ , called either "del" or "nabla", has the form

$$1. \quad \nabla \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}.$$

If $\Phi(x, y, z)$ is any differentiable scalar function, the gradient of Φ , written $\text{grad } \Phi$, is

$$2. \quad \text{grad } \Phi \equiv \nabla \Phi = \frac{\partial \Phi}{\partial x_1} \mathbf{e}_1 + \frac{\partial \Phi}{\partial x_2} \mathbf{e}_2 + \frac{\partial \Phi}{\partial x_3} \mathbf{e}_3.$$

The divergence of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written $\text{div } \mathbf{f}$, is

$$3. \quad \text{div } \mathbf{f} \equiv \nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

The curl, or rotation, of the differentiable vector function $\mathbf{f} = (f_1, f_2, f_3)$, written either $\text{curl } \mathbf{f}$ or $\text{rot } \mathbf{f}$, is

$$4. \quad \text{curl } \mathbf{f} \equiv \text{rot } \mathbf{f} \equiv \nabla \times \mathbf{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \mathbf{e}_3,$$

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or equivalently,

$$\text{curl } \mathbf{f} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

10.31 Properties of the operator ∇

Let $\Phi(x_1, x_2, x_3)$, $\Psi(x_1, x_2, x_3)$ be any two differentiable scalar functions, $\mathbf{f}(x_1, x_2, x_3)$, $\mathbf{g}(x_1, x_2, x_3)$ any two differentiable vector functions, and \mathbf{a} an arbitrary vector. Define the scalar operator ∇^2 , called the Laplacian, by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Then, in terms of the operator ∇ , we have the following:

1. $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi.$
2. $\nabla(\Phi\Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi.$
3. $\nabla(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \nabla)\mathbf{g} + (\mathbf{g} \cdot \nabla)\mathbf{f} + \mathbf{f} \times (\nabla \times \mathbf{g}) + \mathbf{g} \times (\nabla \times \mathbf{f}).$
4. $\nabla \cdot (\Phi\mathbf{f}) = \Phi(\nabla \cdot \mathbf{f}) + \mathbf{f} \cdot \nabla\Phi.$
5. $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = (\nabla \times \mathbf{f}) \cdot \mathbf{g} - \mathbf{f} \cdot (\nabla \times \mathbf{g}).$

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The equivalent results in terms of grad, div, and curl are as follows:

- 1'. $\text{grad}(\Phi + \Psi) = \text{grad } \Phi + \text{grad } \Psi.$
- 2'. $\text{grad}(\Phi\Psi) = \Phi \text{grad } \Psi + \Psi \text{grad } \Phi.$
- 3'. $\text{grad}(\mathbf{f} \cdot \mathbf{g}) = (\mathbf{f} \cdot \text{grad})\mathbf{g} + (\mathbf{g} \cdot \text{grad})\mathbf{f} + \mathbf{f} \times \text{curl } \mathbf{g} + \mathbf{g} \times \text{curl } \mathbf{f}.$
- 4'. $\text{div}(\Phi\mathbf{f}) = \Phi \text{div } \mathbf{f} + \mathbf{f} \cdot \text{grad } \Phi.$
- 5'. $\text{div}(\mathbf{f} \times \mathbf{g}) = \mathbf{g} \cdot \text{curl } \mathbf{f} - \mathbf{f} \cdot \text{curl } \mathbf{g}.$
- 6'. $\text{curl}(\Phi\mathbf{f}) = \Phi \text{curl } \mathbf{f} + \text{grad } \Phi \times \mathbf{f}.$
- 7'. $\text{curl}(\mathbf{f} \times \mathbf{g}) = \mathbf{f} \text{div } \mathbf{g} - \mathbf{g} \text{div } \mathbf{f} + (\mathbf{g} \cdot \text{grad})\mathbf{f} - (\mathbf{f} \cdot \text{grad})\mathbf{g}.$
- 8'. $\text{curl}(\text{curl } \mathbf{f}) = \text{grad}(\text{div } \mathbf{f}) - \nabla^2 \mathbf{f}.$
- 9'. $\text{curl}(\text{grad } \Phi) \equiv \mathbf{0}.$
- 10'. $\text{div}(\text{curl } \mathbf{f}) \equiv 0.$
- 11'. $\nabla^2(\Phi\Psi) = \Phi \nabla^2 \Psi + 2 \text{grad } \Phi \cdot \text{grad } \Psi + \Psi \nabla^2 \Phi.$

The expression $(\mathbf{a} \cdot \nabla)$ or, equivalently $(\mathbf{a} \cdot \text{grad})$, defined by

$$(\mathbf{a} \cdot \nabla) \equiv a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3},$$

is the directional derivative operator in the direction of vector \mathbf{a} .

10.41 Solenoidal fields

A vector field \mathbf{f} is said to be solenoidal if $\text{div } \mathbf{f} \equiv 0$. We have the following representation.

10.411

Representation theorem for vector Helmholtz equation. If u is a solution of the scalar Helmholtz equation

$$\nabla^2 u + \lambda^2 u = 0,$$

and \mathbf{m} is a constant unit vector, then the vectors

$$\mathbf{X} = \text{curl}(\mathbf{m}u), \quad \mathbf{Y} = \frac{1}{\lambda} \text{curl } \mathbf{X}$$

are independent solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{H} + \lambda^2 \mathbf{H} = \mathbf{0}$$

involving a solenoidal vector \mathbf{H} . The general solution of the equation is

$$\mathbf{H} = \text{curl}(\mathbf{m}u) + \frac{1}{\lambda} \text{curl} \text{curl}(\mathbf{m}u).$$

10.51- 10.61 Orthogonal curvilinear coordinates

Consider a transformation from the cartesian coordinates $O\{x_1, x_2, x_3\}$ to the general orthogonal curvilinear coordinates $O\{u_1, u_2, u_3\}$:

$$x_1 = x_1(u_1, u_2, u_3), \quad x_2 = x_2(u_1, u_2, u_3), \quad x_3 = x_3(u_1, u_2, u_3).$$

Then,

$$1. \quad dx_i = \frac{\partial x_i}{\partial u_1} du_1 + \frac{\partial x_i}{\partial u_2} du_2 + \frac{\partial x_i}{\partial u_3} du_3 \quad (i = 1, 2, 3),$$

and the length element dl may be determined from

$$2. \quad dl^2 = g_{11} du_1^2 + g_{22} du_2^2 + g_{33} du_3^2 + 2g_{23} du_2 du_3 + 2g_{31} du_3 du_1 + 2g_{12} du_1 du_2,$$

where

$$3. \quad g_{ij} = \frac{\partial x_1}{\partial u_i} \frac{\partial x_1}{\partial u_j} + \frac{\partial x_2}{\partial u_i} \frac{\partial x_2}{\partial u_j} + \frac{\partial x_3}{\partial u_i} \frac{\partial x_3}{\partial u_j} = g_{ji}, \quad g_{ij} = 0, \quad i \neq j.$$

1118

provided the Jacobian of the transformation

$$4. \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} \\ \frac{\partial x_1}{\partial u_3} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_3}{\partial u_3} \end{vmatrix}$$

does not vanish (see 4.313).

Define the metrical coefficients

$$5. \quad h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}};$$

then the volume element dV in orthogonal curvilinear coordinates is

$$6. \quad dV = h_1 h_2 h_3 du_1 du_2 du_3,$$

and the surface elements of area ds_i on the surfaces $u_i = \text{const.}$, for $i = 1, 2, 3$, are

$$7. \quad ds_1 = h_2 h_3 du_2 du_3, \quad ds_2 = h_1 h_3 du_1 du_3, \quad ds_3 = h_1 h_2 du_1 du_2.$$

Denote by \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 the triad of orthogonal unit vectors that are tangent to the u_1 , u_2 , and u_3 coordinate lines through any given point P , and choose their sense so that they form a right-handed set in this order. Then in terms of this triad of vectors and the components f_{u_1} , f_{u_2} , and f_{u_3} of \mathbf{f} along the coordinate line,

$$8. \quad \mathbf{f} = f_{u_1} \mathbf{e}_1 + f_{u_2} \mathbf{e}_2 + f_{u_3} \mathbf{e}_3.$$

MF I 115

10.611

$\text{grad } \Phi$, $\text{div } \mathbf{f}$, $\text{curl } \mathbf{f}$, and ∇^2 in general orthogonal curvilinear coordinates.

$$1. \quad \text{grad } \Phi = \frac{\mathbf{e}_1}{h_1} \frac{\partial \Phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2} \frac{\partial \Phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3} \frac{\partial \Phi}{\partial u_3}.$$

$$2^3 \quad \operatorname{div} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3 f_{u_1}) + \frac{\partial}{\partial u_2} (h_3 h_1 f_{u_2}) + \frac{\partial}{\partial u_3} (h_1 h_2 f_{u_3}) \right).$$

$$3. \quad \operatorname{curl} \mathbf{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_{u_1} & h_2 f_{u_2} & h_3 f_{u_3} \end{vmatrix}.$$

$$4. \quad \nabla^2 \equiv \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial u_3} \right) \right).$$

MF I 21-31

10.612

Cylindrical polar coordinates. In terms of the coordinates $O\{r, \phi, z\}$, that is, $u_1 = r$, $u_2 = \phi$, $u_3 = z$, where

$x_1 = r \cos \phi$, $x_2 = r \sin \phi$, $x_3 = z$ for $-\pi < \phi \leq \pi$, it follows that

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

1119

and

$$2. \quad \operatorname{grad} \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi + \frac{\partial \Phi}{\partial z} \mathbf{e}_z,$$

$$3. \quad \operatorname{div} \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z},$$

$$4. \quad \operatorname{curl} \mathbf{f} = \frac{1}{r} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ f_r & r f_\phi & f_z \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

10.613

Spherical polar coordinates. In terms of the coordinates $O\{r, \theta, \phi\}$, that is, $u_1 = r, u_2 = \theta, u_3 = \phi$, where $x_1 = r \sin \theta \cos \phi, x_2 = r \sin \theta \sin \phi, x_3 = r \cos \theta$, for $0 \leq \theta \leq \pi, -\pi < \phi \leq \pi$, we have

$$1. \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin \theta,$$

and

$$2. \quad \text{grad } \Phi = \frac{\partial \Phi}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \mathbf{e}_\phi,$$

$$3. \quad \text{div } \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (f_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial f_\phi}{\partial \phi},$$

$$4. \quad \text{curl } \mathbf{f} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_r & r f_\theta & r \sin \theta f_\phi \end{vmatrix},$$

$$5. \quad \nabla^2 \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}.$$

10.614

Elliptic cylinder coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = u_1 u_2, \quad x_2 = \sqrt{(u_1^2 - c^2)(1 - u_2^2)}, \quad x_3 = u_3.$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{u_1^2 - c^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - c^2 u_2^2}{1 - u_2^2}}, \quad h_3 = 1.$$

MF I 657

1120

10.615

Parabolic cylinder coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \frac{1}{2}(u_1^2 - u_2^2), \quad x_2 = u_1 u_2, \quad x_3 = u_3.$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = 1.$$

MF I 658

10.616

Conical coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \frac{u_1}{a} \sqrt{(a^2 - u_2^2)(a^2 + u_3^2)}, \quad x_2 = \frac{u_1}{b} \sqrt{(b^2 + u_2^2)(b^2 - u_3^2)},$$

$$x_3 = \frac{u_1 u_2 u_3}{ab} \quad \text{with} \quad a^2 + b^2 = 1.$$

$$2. \quad h_1 = 1, \quad h_2 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 - u_2^2)(b^2 + u_2^2)}}, \quad h_3 = u_1 \sqrt{\frac{u_2^2 + u_3^2}{(a^2 + u_3^2)(b^2 - u_3^2)}}.$$

10.617

Rotational parabolic coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = u_1 u_2 u_3, \quad x_2 = u_1 u_2 \sqrt{1 - u_3^2}, \quad x_3 = \frac{1}{2}(u_1^2 - u_2^2).$$

$$2. \quad h_1 = \sqrt{u_1^2 + u_2^2}, \quad h_2 = \sqrt{u_1^2 + u_2^2}, \quad h_3 = \frac{u_1 u_2}{\sqrt{1 - u_3^2}}.$$

MF I 660

10.618

Rotational prolate spheroidal coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 - a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2.$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{u_1^2 - a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 - a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 - a^2)(1 - u_2^2)}{1 - u_3^2}}.$$

MF I 661

10.619

Rotational oblate spheroidal coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = u_3 \sqrt{(u_1^2 + a^2)(1 - u_2^2)}, \quad x_2 = \sqrt{(u_1^2 + a^2)(1 - u_2^2)(1 - u_3^2)}, \quad x_3 = u_1 u_2.$$

$$2. \quad h_1 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{u_1^2 + a^2}}, \quad h_2 = \sqrt{\frac{u_1^2 + a^2 u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{\frac{(u_1^2 + a^2)(1 - u_2^2)}{1 - u_3^2}}.$$

MF I 662

10.620

Ellipsoidal coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2(a^2 - b^2)}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2(b^2 - a^2)}},$$

$$x_3 = u_1 u_2 u_3 / ab.$$

1121

$$2. \quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = \sqrt{\frac{(u_2^2 - u_1^2)(u_2^2 - u_3^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}.$$

MF I 663

10.621

Paraboloidal coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = \sqrt{\frac{(u_1^2 - a^2)(u_2^2 - a^2)(u_3^2 - a^2)}{a^2 - b^2}}, \quad x_2 = \sqrt{\frac{(u_1^2 - b^2)(u_2^2 - b^2)(u_3^2 - b^2)}{b^2 - a^2}},$$

$$x_3 = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2 - a^2 - b^2).$$

$$2. \quad h_1 = \sqrt{\frac{(u_1^2 - u_2^2)(u_1^2 - u_3^2)}{(u_1^2 - a^2)(u_1^2 - b^2)}}, \quad h_2 = u_2 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_2^2 - a^2)(u_2^2 - b^2)}}, \quad h_3 = u_3 \sqrt{\frac{(u_3^2 - u_1^2)(u_3^2 - u_2^2)}{(u_3^2 - a^2)(u_3^2 - b^2)}}.$$

MF I 664

10.622

Bispherical coordinates $O\{u_1, u_2, u_3\}$.

$$1. \quad x_1 = au_3 \frac{\sqrt{1 - u_2^2}}{u_1 - u_2}, \quad x_2 = a \frac{\sqrt{(1 - u_2^2)(1 - u_3^2)}}{u_1 - u_2}, \quad x_3 = \frac{\sqrt{u_1^2 - 1}}{u_1 - u_2}.$$

$$2. \quad h_1 = \frac{a}{(u_1 - u_2)\sqrt{u_1^2 - 1}}, \quad h_2 = \frac{a}{(u_1 - u_2)\sqrt{1 - u_2^2}}, \quad h_3 = \left(\frac{a}{u_1 - u_2}\right) \sqrt{\frac{1 - u_2^2}{1 - u_3^2}}.$$

MF I 665

10.71- 10.72 Vector integral theorems

10.711

Gauss's divergence theorem. Let V be a volume bounded by a simple closed surface S and let \mathbf{f} be a continuously differentiable vector field defined in V and on S . Then, if $d\mathbf{S}$ is the outward drawn vector element of area,

$$\int_S \mathbf{f} \cdot d\mathbf{S} = \int_V \operatorname{div} \mathbf{f} dV.$$

KE 39

10.712

Green's theorems. Let Φ and Ψ be scalar fields which, together with $\nabla^2\Phi$ and $\nabla^2\Psi$, are defined both in a volume V and on its surface S , which we assume to be simple and closed. Then, if $\partial/\partial n$ denotes differentiation along the outward drawn normal to S , we have

10.713

Green's first theorem.

$$\int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_V (\Phi \nabla^2 \Psi + \operatorname{grad} \Phi \cdot \operatorname{grad} \Psi) dV.$$

KE 212

10.714

Green's second theorem.

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right) dS = \int_V (\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi) dV.$$

10.715

Special cases.

$$1. \int_S (\Phi \operatorname{grad} \Phi) \cdot d\mathbf{S} = \int_V (\Phi \nabla^2 \Phi + (\operatorname{grad} \Phi)^2) dV.$$

$$2. \int_S \frac{\partial \Phi}{\partial n} dS = \int_V \nabla^2 \Phi dV.$$

MV 81

10.716

Green's reciprocal theorem. If Φ and Ψ are harmonic, so that $\nabla^2 \Phi = \nabla^2 \Psi = 0$, then

$$3. \int_S \Phi \frac{\partial \Psi}{\partial n} dS = \int_S \Psi \frac{\partial \Phi}{\partial n} dS.$$

MM 105

10.717

Green's representation theorem. If Φ and $\nabla^2 \Phi$ are defined within a volume V bounded by a simple closed surface S , and P is an interior point of V , then in three dimensions

$$4. \Phi(P) = -\frac{1}{4\pi} \int_V \frac{1}{r} \nabla^2 \Phi dV + \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS.$$

KE 219

If Φ is harmonic within V , so that $\nabla^2 \Phi = 0$, then the previous result becomes

$$5. \Phi(P) = \frac{1}{4\pi} \int_S \frac{1}{r} \frac{\partial \Phi}{\partial n} dS - \frac{1}{4\pi} \int_S \Phi \frac{\partial}{\partial n} \left(\frac{1}{r} \right) dS.$$

In the case of two dimensions, result (4) takes the form

$$6. \quad \Phi(p) = \frac{1}{2\pi} \int_S \nabla^2 \Phi(q) \ln |p - q| dS + \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int_C \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq,$$

MM 116

where C is the boundary of the planar region S , and result (5) the form

$$7. \quad \Phi(p) = \frac{1}{2\pi} \int_C \Phi(q) \frac{\partial}{\partial n_q} \ln |p - q| dq - \frac{1}{2\pi} \int_C \ln |p - q| \frac{\partial}{\partial n_q} \Phi(q) dq.$$

VL 280

10.718

Green's representation theorem in R^n . If Φ is twice differentiable within a region Ω in R^n bounded by the surface Σ with outward drawn unit normal \mathbf{n} , then for $p \notin \Sigma$ and $n > 3$

$$\Phi(p) = \frac{-1}{(n-2)\sigma_n} \int_{\Omega} \frac{\nabla^2 \Phi(q)}{|p-q|^{n-2}} d\Omega_q + \frac{1}{(n-2)\sigma_n} \int_{\Sigma} \left(\frac{1}{|p-q|^{n-2}} \frac{\partial \Phi(q)}{\partial n_q} - \Phi(q) \frac{\partial}{\partial n_q} \frac{1}{|p-q|^{n-2}} \right) d\Sigma_q,$$

where

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

VL 279

is the area of the unit sphere in R^n .

1123

10.719

Green's theorem of the arithmetic mean. If Φ is harmonic in a sphere, then the value of Φ at the center of the sphere is the arithmetic mean of its value on the surface.

KE 223

10.720

Poisson's integral in three dimensions. If Φ is harmonic in the interior of a spherical volume V of radius R and is continuous on the surface of the sphere on which, in terms of the spherical polar coordinates (r, θ, ϕ) , it satisfies the boundary condition

$\Phi(R, \theta, \phi) = f(\theta, \phi)$, then

$$\Phi(r, \theta, \phi) = \frac{R(R^2 - r^2)}{4\pi} \int_0^\pi \int_{-\pi}^\pi \frac{f(\theta', \phi') \sin \theta' d\theta' d\phi'}{(r^2 + R^2 - 2rR \cos \gamma)^{3/2}},$$

where

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

KE 241

10.721

Poisson's integral in two dimensions. If Φ is harmonic in the interior of a circular disk S of radius R and is continuous on the boundary of the disk on which, in terms of the polar coordinates (r, θ) , it satisfies the boundary condition $\Phi(R, \theta) = f(\theta)$, then

$$\Phi(r, \theta) = \frac{(R^2 - r^2)}{2\pi} \int_{-\pi}^\pi \frac{f(\phi) d\phi}{r^2 + R^2 - 2rR \cos (\theta - \phi)}.$$

10.722

Stokes's theorem. Let a simple closed curve C be spanned by a surface S . Define the positive normal \mathbf{n} to S , and the positive sense of description of the curve C with line element $d\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if \mathbf{f} is continuously differentiable vector field defined on S and C with vector element $d\mathbf{S} = \mathbf{n}dS$,

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \int_S \text{curl } \mathbf{f} \cdot d\mathbf{S},$$

MM 143

where the line integral around C is taken in the positive sense.

10.723

Planar case of Stokes's theorem. If a region R in the (x, y) -plane is bounded by a simple closed curve C , and $f_1(x, y), f_2(x, y)$ are any two functions having continuous first derivatives in R and on C , then

$$\oint_C (f_1 dx + f_2 dy) = \int \int_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy,$$

MM 143

where the line integral is taken in the anticlockwise sense.

10.81 Integral rate of change theorems

10.811

Rate of change of volume integral bounded by a moving closed surface. Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a

1124

simple closed surface $S(t)$ moving with velocity \mathbf{v} . Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \frac{\partial f}{\partial t} dV + \int_{S(t)} f \mathbf{v} \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad}.$$

By virtue of Gauss's theorem this also takes the form

$$\frac{D}{Dt} \int_{V(t)} f dV = \int_{V(t)} \left(\frac{Df}{Dt} + f \text{div } \mathbf{v} \right) dV.$$

10.812

Rate of change of flux through a surface. Let \mathbf{q} be a vector function that may also depend on the time t , and \mathbf{n} be the unit outward drawn normal to the surface S that moves with velocity \mathbf{v} . Defining the flux of \mathbf{q} through S as

$$m = \int_S \mathbf{q} \cdot \mathbf{n} \, dS,$$

then

$$\frac{Dm}{Dt} = \int_S \left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v} \operatorname{div} \mathbf{q} + \operatorname{curl} (\mathbf{q} \times \mathbf{v}) \right) \cdot \mathbf{n} \, dS.$$

MV 90

10.813

Rate of change of the circulation around a given moving curve. Let C be a closed curve, moving with velocity \mathbf{v} , on which is defined a vector field \mathbf{q} . Defining the circulation ζ of \mathbf{q} around C by

$$\zeta = \int_C \mathbf{q} \cdot d\mathbf{r},$$

then

$$\frac{D\zeta}{Dt} = \int_C \left(\frac{\partial \mathbf{q}}{\partial t} + (\operatorname{curl} \mathbf{q}) \times \mathbf{v} \right) \cdot d\mathbf{r}.$$

MV 94

11. Algebraic Inequalities

11.1-11.3 General Algebraic Inequalities

11.11 Algebraic inequalities involving real numbers

11.111

Lagrange's identity. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two sets of real numbers; then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 = \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

BB 3

11.112

Cauchy-Schwarz-Buniakowsky inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two arbitrary sets of real numbers; then

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right).$$

The equality holds if, and only if, the sequences a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are proportional.

MT 30

11.113

Minkowski's inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two sets of nonnegative real numbers and let $p > 1$; then

$$\left(\sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^n a_k^p \right)^{1/p} + \left(\sum_{k=1}^n b_k^p \right)^{1/p}.$$

The equality holds if, and only if, the sequences a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are proportional.

MT 55

11.114

Hölder's inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two sets of nonnegative real numbers, and let $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$; then

$$\left(\sum_{k=1}^n a_k^p \right)^{1/p} \left(\sum_{k=1}^n b_k^q \right)^{1/q} \geq \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, the sequences $a_1^p, a_2^p, \dots, a_n^p$ and $b_1^q, b_2^q, \dots, b_n^q$ are proportional.

MT 50

1126

11.115

Chebyshev's inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be two arbitrary sets of real numbers such that either $a_1 \geq a_2 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq \dots \geq b_n$, or $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$; then

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n} \right) \left(\frac{b_1 + b_2 + \dots + b_n}{n} \right) \leq \frac{1}{n} \sum_{k=1}^n a_k b_k.$$

The equality holds if, and only if, either $a_1 = a_2 = \dots = a_n$ or $b_1 = b_2 = \dots = b_n$.

11.116

Arithmetic-geometric inequality. Let a_1, a_2, \dots, a_n be any set of positive numbers, with arithmetic mean

$$A_n = \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)$$

and geometric mean

$$G_n = (a_1 a_2 \dots a_n)^{1/n};$$

then $A_n \geq G_n$ or, equivalently,

$$\left(\frac{a_1 + a_2 + \dots + a_n}{n} \right) \geq (a_1 a_2 \dots a_n)^{1/n}.$$

The equality holds only in the event that all of the numbers a_i are equal.

BB 4

11.117

Carleman's inequality. If a_1, a_2, \dots, a_n is any set of positive numbers, then the geometric and arithmetic means satisfy the inequality

$$a_1, a_2, \dots, a_n$$

$$\sum_{r=1}^n G_r \leq e A_n$$

or, equivalently,

$$\sum_{r=1}^n (a_1 a_2 \dots a_r)^{1/r} \leq e \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right),$$

where e is the best possible constant in this inequality.

MT 131

11.118

An inequality involving absolute values. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be two arbitrary sets of real numbers; then

$$\sum_{i,j=1}^n \{|a_i - b_j|^p + |b_i - a_j|^p - |a_i - a_j|^p - |b_i - b_j|^p\} \geq 0, \quad 0 < p \leq 2.$$

1127

11.21 Algebraic inequalities involving complex numbers

If α, β are any two real numbers, the complex number $z = \alpha + i\beta$ with real part α and imaginary part β has for its modulus $|z|$ the nonnegative number

$$|z| = \sqrt{\alpha^2 + \beta^2},$$

and for its argument (amplitude) $\arg z$ the angle $\arg z = \theta$ such that

$$\cos \theta = \frac{\alpha}{|z|} \quad \text{and} \quad \sin \theta = \frac{\beta}{|z|},$$

where $-\pi < \theta \leq \pi$. The complex number $\bar{z} = \alpha - i\beta$ is said to be the **complex conjugate** of $z = \alpha + i\beta$.

$$\text{If } z = r e^{i\theta} = r(\cos \theta + i \sin \theta),$$

then

$$z^n = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta),$$

and setting $r = 1$ we have **de Moivre's theorem**

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

It follows directly that, if $z = e^{i\theta}$, then

$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \sin \theta = -\frac{i}{2} \left(z - \frac{1}{z} \right),$$

and

$$\cos r\theta = \frac{1}{2} \left(z^r + \frac{1}{z^r} \right), \quad \sin r\theta = -\frac{i}{2} \left(z^r - \frac{1}{z^r} \right).$$

If $w = z^{p/q}$ with p, q integral, and $z = re^{i\theta}$, then the q roots of w_0, w_1, \dots, w_{q-1} of z are

$$w_k = r^{p/q} \left[\cos \left(\frac{p\theta + 2k\pi}{q} \right) + i \sin \left(\frac{p\theta + 2k\pi}{q} \right) \right],$$

with $k = 0, 1, 2, \dots, q - 1$.

11.211

Simple properties and inequalities involving the modulus and the complex conjugate. If the real part of z is denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$, then

$$z + \bar{z} = 2 \operatorname{Re} z = 2\alpha,$$

$$z - \bar{z} = 2i \operatorname{Im} z = 2i\beta,$$

$$z = \overline{(\bar{z})},$$

$$\frac{1}{\bar{z}} = \overline{\left(\frac{1}{z} \right)},$$

$$\overline{(z^n)} = (\bar{z})^n,$$

$$\left| \frac{\bar{z}_1}{\bar{z}_2} \right| = \frac{|\bar{z}_1|}{|\bar{z}_2|},$$

$$\overline{(z_1 + z_2 + \dots + z_n)} = \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n,$$

a, b

(i) $|a + b| \leq |a| + |b|$ (triangle inequality),

(ii) $|a - b| \geq ||a| - |b||$.

11.31 Inequalities for sets of complex numbers

11.311

Complex Cauchy-Schwarz-Buniakowsky inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two arbitrary sets of complex numbers; then

$$\left| \sum_{k=1}^n a_k b_k \right|^2 \leq \left(\sum_{k=1}^n |a_k|^2 \right) \left(\sum_{k=1}^n |b_k|^2 \right).$$

The equality holds if, and only if, the sequences $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$ and b_1, b_2, \dots, b_n are proportional.

MT 42

11.312

Complex Minkowski inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two arbitrary sets of complex numbers, and let the real number p be such that $p > 1$; then

$$\left(\sum_{k=1}^n |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^n |b_k|^p \right)^{1/p}.$$

MT 56

11.313

Complex Hölder inequality. Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be any two arbitrary sets of complex numbers, and let the real numbers p, q be such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \left(\sum_{k=1}^n |b_k|^q \right)^{1/q} \geq \left| \sum_{k=1}^n a_k b_k \right|.$$

The equality holds if, and only if, the sequences

$$|a_1|^p, |a_2|^p, \dots, |a_n|^p \quad \text{and} \quad |b_1|^p, |b_2|^p, \dots, |b_n|^p$$

are proportional and $\arg a_k b_k$ is independent of k for $k = 1, 2, \dots, n$.

MT 53

12. Integral Inequalities

12.1- 12.5 Properties of Integrals and Integral Inequalities

12.11 Mean value theorems

12.111

First mean value theorem. Let $f(x)$ and $g(x)$ be two bounded functions integrable in $[a, b]$ and let $g(x)$ be of one sign in this interval. Then

$$\int_a^b f(x)g(x) dx = f(\xi) \int_a^b g(x) dx,$$

CA 105

with

$$a \leq \xi \leq b.$$

12.112

Second mean value theorem.

(i) Let $f(x)$ be a bounded, monotonic decreasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function.

Then,

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx,$$

with $a \leq \xi \leq b$.

$$\int_a^b f(x)g(x) dx = f(a) \int_a^\xi g(x) dx,$$

$$a \leq \xi \leq b$$

(ii) Let $f(x)$ be a bounded, monotonic increasing, and nonnegative function in $[a, b]$, and let $g(x)$ be a bounded integrable function.

Then,

$$\int_a^b f(x)g(x) dx = f(b) \int_\eta^b g(x) dx,$$

with $a \leq \eta \leq b$.

(iii) Let $f(x)$ be bounded and monotonic in $[a, b]$, and let $g(x)$ be a bounded integrable function which experiences only a finite number of sign changes in $[a, b]$. Then,

$$\int_a^b f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(b-0) \int_\xi^b g(x) dx,$$

CA 107

with

$$a \leq \xi \leq b.$$

1130

12.113

First mean value theorem for infinite integrals. Let $f(x)$ be bounded for $x \geq a$, and integrable in the arbitrary interval $[a, b]$, and let $g(x)$ be of one sign in $x \geq a$ and such that $\int_a^\infty g(x)dx$ is finite. Then,

$$\int_a^\infty f(x)g(x) dx = \mu \int_a^\infty g(x) dx,$$

CA 123

where $m \leq \mu \leq M$ and m, M are, respectively, the lower and upper bounds of $f(x)$ for $x \geq a$.

12.114⁶

Second mean value theorem for infinite integrals. Let $f(x)$ be bounded and monotonic when $x \geq a$, and $g(x)$ be bounded and

$$f(x) \qquad x \geq a \qquad g(x)$$

integrable in the arbitrary interval $[a, b]$ in which it experiences only a finite number of changes of sign. Then, provided $\int_a^\infty g(x)dx$ is finite,

$$\int_a^\infty f(x)g(x) dx = f(a+0) \int_a^\xi g(x) dx + f(\infty) \int_\xi^\infty g(x) dx,$$

CA 123

with $a \leq \xi \leq \infty$.

12.21 Differentiation of definite integral containing a parameter

12.211

Differentiation when limits are finite. Let $\phi(\alpha)$ and $\psi(\alpha)$ be twice differentiable functions in some interval $c \leq \alpha \leq d$, and let $f(x, \alpha)$ be both integrable with respect to x over the interval $\phi(\alpha) \leq x \leq \psi(\alpha)$ and differentiable with respect to α . Then,

$$\frac{d}{d\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha) dx = \left(\frac{d\psi}{d\alpha} \right) f(\psi(\alpha), \alpha) - \left(\frac{d\phi}{d\alpha} \right) f(\phi(\alpha), \alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial \alpha} dx.$$

FI II 680

12.212

Differentiation when a limit is infinite. Let $f(x, \alpha)$ and $\partial f / \partial \alpha$ both be integrable with respect to x over the semi-infinite region $x \geq a, b \leq \alpha < c$. Then, if the integral

$$f(\alpha) = \int_a^\infty f(x, \alpha) dx$$

exists for all $b \leq \alpha \leq c$, and if

$$\int_a^\infty \frac{\partial f}{\partial \alpha} dx$$

is uniformly convergent for α in $[b, c]$, it follows that

$$\frac{d}{d\alpha} \int_a^\infty f(x, \alpha) dx = \int_a^\infty \frac{\partial f}{\partial \alpha} dx.$$

12.31 Integral inequalities

1131
12.311

Cauchy-Schwarz-Buniakowsky inequality for integrals. Let $f(x)$ and $g(x)$ be any two real integrable functions on $[a, b]$. Then,

$$\left(\int_a^b f(x)g(x) dx \right)^2 \leq \left(\int_a^b f^2(x) dx \right) \left(\int_a^b g^2(x) dx \right),$$

and the equality will hold if, and only if, $f(x) = kg(x)$, with k real.

BB 21

12.312

Hölder's inequality for integrals. Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^q$ are integrable on $[a, b]$ with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then

$$\int_a^b f(x)g(x) dx \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} \left(\int_a^b |g(x)|^q dx \right)^{1/q}.$$

The equality holds if, and only if, $\alpha|f(x)|^p = \beta|g(x)|^q$, where α and β are positive constants.

BB 21

12.313

Minkowski's inequality for integrals. Let $f(x)$ and $g(x)$ be any two real functions for which $|f(x)|^p$ and $|g(x)|^p$ are integrable on $[a, b]$ for $p > 0$; then

$$\left(\int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)|^p dx \right)^{1/p}.$$

The equality holds if, and only if, $f(x) = kg(x)$ for some real $k \geq 0$.

BB 21

12.314

Chebyshev's inequality for integrals. Let f_1, f_2, \dots, f_n be nonnegative integrable functions on $[a, b]$ which are all either monotonic increasing or monotonic decreasing; then

$$\int_a^b f_1(x) dx \int_a^b f_2(x) dx \dots \int_a^b f_n(x) dx \leq (b-a)^{n-1} \int_a^b f_1(x) f_2(x) \dots f_n(x) dx.$$

MT 39

12.315

Young's inequality for integrals. Let $f(x)$ be a real-valued continuous strictly monotonic increasing function on the interval $[0, a]$, with $f(0) = 0$ and $b \leq f(a)$. Then

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy,$$

where $f^{-1}(y)$ denotes the function inverse to $f(x)$. The equality holds if, and only if, $b = f(a)$.

BB 15

12.316

Steffensen's inequality for integrals. Let $f(x)$ be nonnegative and monotonic decreasing in $[a, b]$ and $g(x)$ be such that $0 \leq g(x) \leq 1$ in $[a, b]$. Then

$$\int_{b-k}^b f(x) dx \leq \int_a^b f(x)g(x) dx \leq \int_a^{a+k} f(x) dx,$$

1132

where

$$k = \int_a^b g(x) dx.$$

MT 107

12.317

Gram's inequality for integrals. Let $f_1(x), f_2(x), \dots, f_n(x)$ be real square integrable functions on $[a, b]$; then

$$\begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x)f_2(x) dx & \cdots & \int_a^b f_1(x)f_n(x) dx \\ \int_a^b f_2(x)f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x)f_n(x) dx \\ \vdots & \vdots & \cdots & \vdots \\ \int_a^b f_n(x)f_1(x) dx & \int_a^b f_n(x)f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix} \geq 0.$$

MT 47

12.318

Ostrowski's inequality for integrals. Let $f(x)$ be a monotonic function integrable on $[a, b]$, and let $f(a)f(b) \geq 0, |f(a)| \geq |f(b)|$.

Then, if g is a real function integrable on $[a, b]$,

$$\left| \int_a^b f(x)g(x) dx \right| \leq |f(a)| \max_{a \leq \xi \leq b} \left| \int_a^\xi g(x) dx \right|.$$

12.41 Convexity and Jensen's inequality

A function $f(x)$ is said to be **convex** on an interval $[a, b]$ if for any two points x_1, x_2 in $[a, b]$

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

A function $f(x)$ is said to be **concave** on an interval $[a, b]$ if for any two points x_1, x_2 in $[a, b]$ the function $-f(x)$ is convex in that interval.

If the function $f(x)$ possesses a second derivative in the interval $[a, b]$, then a necessary and sufficient condition for it to be convex on that interval is that $f''(x) > 0$ for all x in $[a, b]$.

A function $f(x)$ is said to be **logarithmically convex** on the interval $[a, b]$ if $f > 0$ and $\log f(x)$ is concave on $[a, b]$.

If $f(x)$ and $g(x)$ are logarithmically convex on the interval $[a, b]$, then the functions $f(x) + g(x)$ and $f(x)g(x)$ are also logarithmically convex on $[a, b]$.

MT 17

12.411

Jensen's inequality. Let $f(x), p(x)$ be two functions defined for $a \leq x \leq b$ such that $\alpha \leq f(x) \leq \beta$

1133

and $p(x) \geq 0$, with $p(x) \not\equiv 0$. Let $\phi(u)$ be a convex function defined on the interval $\alpha \leq u \leq \beta$; then

$$\phi \left(\frac{\int_a^b f(x)p(x) dx}{\int_a^b p(x) dx} \right) \leq \frac{\int_a^b \phi(f)p(x) dx}{\int_a^b p(x) dx}.$$

HL 151

12.51 Fourier series and related inequalities

The trigonometric **Fourier series** representation of the function $f(x)$ integrable on $[-\pi, \pi]$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where the **Fourier coefficients** a_n and b_n of $f(x)$ are given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

(See 0.320-0.328 for convergence of Fourier series on $(-l, l)$.)

TF 1

12.511

Riemann-Lebesgue lemma. If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin tx dx \rightarrow 0$$

$$\lim_{t \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos tx \, dx \rightarrow 0.$$

TF 11

12.512

Dirichlet lemma.

$$\int_0^{\pi} \frac{\sin \left(n + \frac{1}{2} \right) x}{2 \sin \frac{1}{2} x} \, dx = \frac{\pi}{2},$$

in which $\frac{\sin \left(n + \frac{1}{2} \right) x}{2 \sin \frac{1}{2} x}$ is called the **Dirichlet kernel**.

ZY 21

12.513

Parseval's theorem for trigonometric Fourier series. If $f(x)$ is square integrable on $[-\pi, \pi]$, then

$$\frac{a_0^2}{2} + \sum_{r=1}^{\infty} (a_r^2 + b_r^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx.$$

ZY 10

12.514

Integral representation of n th partial

1134

sum. If $f(x)$ is integrable on $[-\pi, \pi]$, then the n th partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^n (a_r \cos rx + b_r \sin rx)$$

has the following integral representation in terms of the Dirichlet kernel,

$$s_n(x) = \frac{a_0}{2} + \sum_{r=1}^n (a_r \cos rx + b_r \sin rx)$$

$$s_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2}t} dt.$$

ZY 20

12.515

Generalized Fourier series. Let the set of functions $\{\phi_n\}_{n=0}^{\infty}$ form an **orthonormal set** over $[a, b]$, so that

$$\int_a^b \phi_m(x)\phi_n(x) dx = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$$

Then the **generalized Fourier series** representation of an integrable function $f(x)$ on $[a, b]$ is

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x),$$

where the generalized Fourier coefficients of $f(x)$ are given by

$$c_n = \int_a^b f(x)\phi_n(x) dx.$$

12.516

Bessel's inequality for generalized Fourier series. For any square integrable function defined on $[a, b]$,

$$\sum_{n=0}^{\infty} c_n^2 \leq \int_a^b f^2(x) dx,$$

where the c_n are the generalized Fourier coefficients of $f(x)$.

12.517

Parseval's theorem for generalized Fourier series. If $f(x)$ is a square integrable function defined on $[a, b]$ and $\{\phi_n(x)\}_{n=0}^{\infty}$ is a

complete orthonormal set of continuous functions defined on $[a, b]$, then

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x) dx,$$

where the c_n are generalized Fourier coefficients of $f(x)$.

13. Matrices and Related Results

13.1- 13.4 Matrices and Quadratic Forms

13.11- 13.12 Special matrices

13.111

Diagonal matrix. A square matrix \mathbf{A} of the form

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix}$$

in which all entries away from the **leading diagonal** are zero.

13.112

Identity matrix and null matrix. The **identity matrix** is a diagonal matrix \mathbf{I} in which all entries in the leading diagonal are unity. The **null matrix** is all zeros.

13.113

Reducible and irreducible matrices. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is said to be **reducible**, if the indices $1, 2, \dots, n$ can be divided into two disjoint nonempty sets $i_1, i_2, \dots, i_\mu; j_1, j_2, \dots, j_\nu$ ($\mu + \nu = n$), such that

$$a_{i_\alpha j_\beta} = 0 \quad (\alpha = 1, 2, \dots, \mu; \beta = 1, 2, \dots, \nu).$$

Otherwise \mathbf{A} will be said to be irreducible.

13.114

Equivalent matrices. An $m \times n$ matrix \mathbf{A} is **equivalent** to an $m \times n$ matrix \mathbf{B} if, and only if, $\mathbf{B} = \mathbf{P}\mathbf{A}\mathbf{Q}$ for suitable nonsingular $m \times m$ and $n \times n$ matrices \mathbf{P} and \mathbf{Q} , respectively.

13.115

Transpose of a matrix. If $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix with element a_{ij} in the i th row and the j th column, then the transpose \mathbf{A}^T of \mathbf{A} is the $n \times m$ matrix

$$\mathbf{A}^T = [b_{ij}] \quad \text{with} \quad b_{ij} = a_{ji},$$

that is, the matrix derived from \mathbf{A} by interchanging rows and columns.

1136

13.116

Adjoint matrix. If \mathbf{A} is an $n \times n$ matrix, then its **adjoint**, denoted by $\text{adj } \mathbf{A}$, is the transpose of the matrix of cofactors A_{ij} of \mathbf{A} , so that

$$\text{adj } \mathbf{A} = [A_{ij}]^T \quad (\text{see 14.13}).$$

14.13

13.117

Inverse matrix. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with a nonsingular determinant $|\mathbf{A}|$, then its **inverse** \mathbf{A}^{-1} is given by

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}.$$

13.118

Trace of a matrix. The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, written $\text{tr } \mathbf{A}$, is defined to be the sum of the terms on the leading diagonal, so that

$$\text{tr } \mathbf{A} = a_{11} + a_{22} + \dots + a_{nn}.$$

13.119

Symmetric matrix. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **symmetric** if $a_{ij} = a_{ji}$ for $i, j = 1, 2, \dots, n$.

13.120

Skew-symmetric matrix. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for $i, j = 1, 2, \dots, n$.

13.121

Triangular matrices. An $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is of **upper triangular type** if $a_{ij} = 0$ for $i > j$ and of **lower triangular type** if $a_{ij} = 0$ for $j > i$.

13.122

Orthogonal matrices. A real $n \times n$ matrix \mathbf{A} is **orthogonal** if, and only if, $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

13.123

Hermitian transpose of a matrix. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with complex elements, then its **hermitian transpose** \mathbf{A}^H is defined to be

$$\mathbf{A}^H = [\bar{a}_{ji}],$$

with the bar denoting the complex conjugate operation.

13.124

Hermitian matrix. An $n \times n$ matrix \mathbf{A} is **hermitian** if $\mathbf{A} = \mathbf{A}^H$, or equivalently, if $\mathbf{A} = \bar{\mathbf{A}}^T$, with the bar denoting the complex conjugate operation.

13.125

Unitary matrix. An $n \times n$ matrix \mathbf{A} is **unitary** if $\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A} = \mathbf{I}$.

13.126

Eigenvalues and eigenvectors. If \mathbf{A} is an $n \times n$ matrix, each eigenvector \mathbf{X} corresponding to λ satisfies the equation

$$\mathbf{A}\mathbf{X} = \lambda\mathbf{X},$$

while the **eigenvalues** λ satisfy the **characteristic equation**

1137

13.127.

Nilpotent matrix. An $n \times n$ matrix \mathbf{A} is **nilpotent** if $\mathbf{A}^k = \mathbf{0}$ for some k .

13.128.

Idempotent matrix. An $n \times n$ matrix \mathbf{A} is **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

13.129.

Positive definite. An $n \times n$ matrix \mathbf{A} is **positive definite** if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.130.

Non-negative definite. An $n \times n$ matrix \mathbf{A} is **non-negative definite** if $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$, for $\mathbf{x} \neq \mathbf{0}$ an n element column vector.

13.131.

Diagonally dominant. An $n \times n$ matrix \mathbf{A} is **diagonally dominant** if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i .

13.21 Quadratic forms

A **quadratic form** involving the n real variables x_1, x_2, \dots, x_n that are associated with the real $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar expression

$$Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j.$$

In terms of matrix notation, if \mathbf{x} is the $n \times 1$ column vector with real elements x_1, x_2, \dots, x_n , and \mathbf{x}^T is the transpose of \mathbf{x} , then

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}.$$

Employing the inner product notation, this same quadratic form may also be written

$$Q(\mathbf{x}) \equiv (\mathbf{x}, \mathbf{A} \mathbf{x}).$$

If the $n \times n$ matrix \mathbf{A} is hermitian, so that $\overline{\mathbf{A}}^T = \mathbf{A}$, where the bar denotes the complex conjugate operation, then the quadratic form associated with the hermitian matrix \mathbf{A} and the vector \mathbf{x} which may have complex elements is the real quadratic form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}).$$

It is always possible to express an arbitrary quadratic form

$$Q(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} x_i x_j$$

in the form

$$Q(\mathbf{x}) = (\mathbf{x}, \mathbf{Ax}),$$

where $\mathbf{A} = [a_{ij}]$ is a symmetric matrix, by defining

$$a_{ii} = \alpha_{ii} \quad \text{for } i = 1, 2, \dots, n$$

1138

and

$$a_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) \quad \text{for } i, j = 1, 2, \dots, n \quad \text{and } i \neq j.$$

13.211

Sylvester's law of inertia. When a quadratic form Q in n variables is reduced by a nonsingular linear transformation to the form

$$Q = y_1^2 + y_2^2 + \dots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \dots - y_r^2,$$

the number p of positive squares appearing in the reduction is an invariant of the quadratic form Q , and does not depend on the

$$Q = y_1^2 + y_2^2 + \dots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \dots - y_r^2,$$

p

Q

method of reduction itself.

ML 377

13.212

Rank. The **rank** of the quadratic form Q in the above canonical form is the total number r of squared terms (both positive and negative) appearing in its reduced form.

ML 360

13.213

Signature. The **signature** of the quadratic form Q above is the number s of positive squared terms appearing in its reduced form. It is sometimes also defined to be $2s - r$.

ML 378

13.214

Positive definite and semidefinite quadratic form. The quadratic form $Q(\mathbf{x}) = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is said to be **positive definite** when $Q(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. It is said to be **positive semidefinite** if $Q(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$.

ML 394

13.215

Basic theorems on quadratic forms.

1. Two real quadratic forms are **equivalent** under the group of linear transformations if, and only if, they have the same rank and the same signature.

2. A real quadratic form in n variables is positive definite if, and only if, its canonical form is

$$Q = z_1^2 + z_2^2 + \dots + z_n^2.$$

3. A real symmetric matrix \mathbf{A} is positive definite if, and only if, there exists a real nonsingular matrix \mathbf{M} such that $\mathbf{A} = \mathbf{M}\mathbf{M}^T$.

4. Any real quadratic form in n variables may be reduced to the diagonal form

$$Q = \lambda_1 z_1^2 + \lambda_2 z_2^2 + \dots + \lambda_n z_n^2, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

by a suitable orthogonal point-transformation.

5. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ is positive definite if, and only if, every eigenvalue of \mathbf{A} is positive; it is positive semidefinite if, and only if, all the eigenvalues of \mathbf{A} are nonnegative, and it is indefinite if the eigenvalues of \mathbf{A} are of both signs.

6. The necessary conditions for an hermitian matrix \mathbf{A} to be positive definite are

(i) $a_{ii} > 0$ for all i ,

(ii) $a_{ii}a_{jj} > |a_{ij}|^2$ for $i \neq j$,

(iii) the element of largest modulus must lie on the leading diagonal,

(iv) $|\mathbf{A}| > 0$.

1139

7. The quadratic form $Q = (\mathbf{x}, \mathbf{A}\mathbf{x})$ with \mathbf{A} hermitian will be positive definite if all the principal minors in the top left-hand corner of \mathbf{A} are positive, so that

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \dots$$

ML 353-379

13.31 Differentiation of matrices

If the $n \times m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have elements that are differentiable functions of t , so that

$$\mathbf{A}(t) = [a_{ij}(t)], \quad \mathbf{B}(t) = [b_{ij}(t)],$$

then

$$1. \quad \frac{d}{dt} \mathbf{A}(t) = \left[\frac{d}{dt} a_{ij}(t) \right];$$

13.41 The matrix exponential

If \mathbf{A} is a square matrix, and z is any complex number, then the matrix exponential $e^{\mathbf{A}z}$ is defined to be

$$e^{\mathbf{A}z} = \mathbf{I} + \mathbf{A}z + \dots + \frac{\mathbf{A}^n z^n}{n!} + \dots = \sum_{r=0}^{\infty} \frac{1}{r!} \mathbf{A}^r z^r.$$

1140

13.411

Basic properties.

1. $e^0 = \mathbf{I}$, $e^{\mathbf{I}z} = \mathbf{I}e^z$, $e^{\mathbf{A}(z_1+z_2)} = e^{\mathbf{A}z_1} \cdot e^{\mathbf{A}z_2}$, $e^{-\mathbf{A}z} = (e^{\mathbf{A}z})^{-1}$,
 $e^{\mathbf{A}z} \cdot e^{\mathbf{B}z} = e^{(\mathbf{A}+\mathbf{B})z}$ when $\mathbf{A} + \mathbf{B}$ is defined and $\mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A}$

2. $\frac{d^r}{dz^r}(e^{\mathbf{A}z}) = \mathbf{A}^r e^{\mathbf{A}z} = e^{\mathbf{A}z} \mathbf{A}^r.$

ML 340

3. If the square matrix \mathbf{A} can be expressed in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix},$$

with \mathbf{B} and \mathbf{C} square matrices, then

$$e^{\mathbf{A}z} = \begin{pmatrix} e^{\mathbf{B}z} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{C}z} \end{pmatrix}.$$

14. Determinants

14.1- 14.3 Properties of Determinants

14.11 Expansion of second-and third-order determinants

$$1. \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$2. \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

14.12 Basic properties

Let $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_{ij}]$ be $n \times n$ matrices, then the following statements are true.

1. If any two rows (or columns) of a square matrix are interchanged, then the sign of the associated determinant is changed.
2. If any two rows (or columns) of a determinant are identical, the determinant is zero.
3. A determinant is not changed in value if any multiple of a row (or column) is added to any other row (or column).
4. $|k\mathbf{A}| = k^n|\mathbf{A}|$ for any scalar k .
5. $|\mathbf{A}^T| = |\mathbf{A}|$, where \mathbf{A}^T is the transpose of \mathbf{A} .
6. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.
7. $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$.
8. If the elements a_{ij} of \mathbf{A} are functions of x , then

$$\frac{d|\mathbf{A}|}{dx} = \sum_{i,j=1}^n \frac{da_{ij}}{dx} A_{ij} \quad (\text{see 14.13}).$$

14.13 Minors and cofactors of a determinant

The **minor** M_{ij} of the element a_{ij} in the n th-order determinant $|\mathbf{A}|$ associated with the square $n \times n$ matrix \mathbf{A} is the $(n - 1)$ th-order determinant derived from \mathbf{A} by deletion of the i th row and j th column. The cofactor A_{ij} of the element a_{ij} is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

ML 20

1142

14.14 Principal minors

A **principal minor** is one whose elements are situated symmetrically with respect to the leading diagonal of \mathbf{A} .

ML 197

14.15 Laplace expansion of a determinant

The n th-order determinant denoted by $|\mathbf{A}|$, or $\det \mathbf{A}$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ may be expanded either by elements of the i th row as

$$|\mathbf{A}| = \sum_{j=1}^n a_{ij} A_{ij},$$

or by elements of the j th column as

$$|\mathbf{A}| = \sum_{i=1}^n a_{ij} A_{ij},$$

where A_{ij} is the cofactor of element a_{ij} . The cofactors A_{ij} satisfy the following n linear equations:

$$\sum_{j=1}^n a_{ij} A_{kj} = \delta_{ik} |\mathbf{A}|, \quad \sum_{i=1}^n a_{ij} A_{ik} = \delta_{jk} |\mathbf{A}|,$$

for $i, j, k = 1, 2, \dots, n$ and, $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$.

14.16 Jacobi's theorem

Let M_r be an r -rowed minor of the n th-order determinant $|\mathbf{A}|$, associated with the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, in which the rows i_1, i_2, \dots, i_r are represented together with the columns k_1, k_2, \dots, k_r .

Define the **complementary minor** to M_r to be the $(n - k)$ -rowed minor obtained from $|\mathbf{A}|$ by deleting all the rows and columns associated with M_r , and the **signed complementary minor** $M^{(r)}$ to M_r to be

$$M^{(r)} = (-1)^{i_1+i_2+\dots+i_r+k_1+k_2+\dots+k_r} \times (\text{complementary minor to } M_r).$$

Then, if Δ is the matrix of cofactors given by

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix},$$

and M_r and M'_r are corresponding r -rowed minors of $|\mathbf{A}|$ and Δ , it follows that

$$M'_r = |\mathbf{A}|^{r-1} M^{(r)}.$$

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Corollary. If $|\mathbf{A}| = 0$, then

$$A_{pk}A_{nq} = A_{nk}A_{pq}.$$

14.17 Hadamard's theorem

If $|\mathbf{A}|$ is an $n \times n$ determinant with elements a_{ij} that may be complex, then $|\mathbf{A}| \neq 0$ if

$$|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|.$$

14.18 Hadamard's inequality

Let $\mathbf{A} = [a_{ij}]$ be an arbitrary $n \times n$ nonsingular matrix with real elements and determinant $|\mathbf{A}|$. Then

$$|\mathbf{A}|^2 \leq \prod_{i=1}^n \left(\sum_{k=1}^n a_{ik}^2 \right).$$

This result is also true when \mathbf{A} is hermitian.

ML 418

Deductions.

1. If $M = \max |a_{ij}|$, then

$$|\mathbf{A}| \leq M^n n^{n/2}.$$

ML 419

2. If the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is positive definite, then

$$|\mathbf{A}| \leq a_{11} a_{22} \dots a_{nn}.$$

BL 126

3. If the real $n \times n$ matrix \mathbf{A} is diagonally dominant, so that $\sum_{j \neq i}^n |a_{ij}| < |a_{ii}|$ for $i = 1, 2, \dots, n$, then $|\mathbf{A}| \neq 0$.

14.21 Cramer's rule

If n linear equations

If n linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \cdot & \quad \cdot \quad \cdots \quad \cdot \quad \cdot \\ \cdot & \quad \cdot \quad \cdots \quad \cdot \quad \cdot \\ \cdot & \quad \cdot \quad \cdots \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

1144

have a nonsingular coefficient matrix $\mathbf{A} = [a_{ij}]$, so that $|\mathbf{A}| \neq 0$, then there is a unique solution

$$x_j = \frac{A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_n}{|\mathbf{A}|}$$

for $j = 1, 2, \dots, n$, where A_{ij} is the cofactor of element a_{ij} in the coefficient matrix \mathbf{A} .

ML 134

14.31 Some special determinants

14.311

Vandermonde's determinant (alternant).

Third order.

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

and, in general, the n th-order Vandermonde's determinant is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

$$\begin{vmatrix} \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

where the right-hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}n(n-1)$ pairs of numbers taken from x_1, x_2, \dots, x_n , with the order of the differences taken in the reverse order of the suffixes that are involved.

ML 17

14.312

Circulants.

Second order.

$$\begin{vmatrix} x_1 & x_2 \\ x_2 & x_1 \end{vmatrix} = (x_1 + x_2)(x_1 - x_2).$$

Third order.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = (x_1 + x_2 + x_3)(x_1 + \omega x_2 + \omega^2 x_3)(x_1 + \omega^2 x_2 + \omega x_3),$$

where ω and ω^2 are the complex cube roots of 1.

1145

In general, the n th-order circulant determinant is

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{vmatrix} = \prod_{j=1}^{n-1} (x_1 + x_2 \omega_j + x_3 \omega_j^2 + \cdots + x_n \omega_j^{n-1}),$$

where ω_j is an n th root of 1.

The eigenvalues λ (see 15.61) of an $n \times n$ circulant matrix are

$$\lambda_j = x_1 + x_2 \omega_j + x_3 \omega_j^2 + \cdots + x_n \omega_j^{n-1},$$

14.313

Jacobian determinant. If f_1, f_2, \dots, f_n are n real-valued functions which are differentiable with respect to x_1, x_2, \dots, x_n , then the Jacobian $J_f(x)$ of the f_i with respect to the x_j is the determinant

$$J_f(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

The notation

$$\frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x_1, x_2, \dots, x_n)}$$

is also used to denote the Jacobian $J_f(x)$.

14.314

Hessian determinants. The Jacobian of the derivatives $\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \dots, \frac{\partial \phi}{\partial x_n}$ of a function

1146

$\phi(x_1, x_2, \dots, x_n)$ with respect to x_1, x_2, \dots, x_n is called the Hessian H of ϕ , so that

$$H = \begin{vmatrix} \frac{\partial^2 \phi}{\partial x_1^2} & \frac{\partial^2 \phi}{\partial x_1 \partial x_2} & \frac{\partial^2 \phi}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 \phi}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \phi}{\partial x_2 \partial x_1} & \frac{\partial^2 \phi}{\partial x_2^2} & \frac{\partial^2 \phi}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 \phi}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \phi}{\partial x_n \partial x_1} & \frac{\partial^2 \phi}{\partial x_n \partial x_2} & \frac{\partial^2 \phi}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 \phi}{\partial x_n^2} \end{vmatrix}.$$

14.315

Wronskian determinants.

$$f_1, f_2, \dots, f_n \quad n \quad x$$

(a, b) . Then the Wronskian $W(x)$ of f_1, f_2, \dots, f_n is defined by

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_n^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where $f_i^{(r)} = \frac{d^r f_i}{dx^r}$.

14.316

Properties.

1. $\frac{dW}{dx}$ follows from $W(x)$ by replacing the last row of the determinant defining $W(x)$ by the n th derivatives $f_1^{(n)}, f_2^{(n)}, \dots, f_n^{(n)}$.

2. If constants k_1, k_2, \dots, k_n exist, not all zero, such that

$$k_1 f_1 + k_2 f_2 + \dots + k_n f_n = 0$$

for all x in (a, b) , then $W(x) = 0$ for all x in (a, b) .

3. The vanishing of the Jacobian throughout (a, b) is necessary, but not sufficient, for the linear dependence of f_1, f_2, \dots, f_n .

14.317

Gram-Kowalewski theorem on linear dependence. A necessary and sufficient condition for n functions f_1, f_2, \dots, f_n square

integrable over $a \leq x \leq b$ to be linearly dependent in this interval is the

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vanishing of the Gram determinant

$$G(f_1, f_2, \dots, f_n) = \begin{vmatrix} \int_a^b f_1^2(x) dx & \int_a^b f_1(x)f_2(x) dx & \cdots & \int_a^b f_1(x)f_n(x) dx \\ \int_a^b f_2(x)f_1(x) dx & \int_a^b f_2^2(x) dx & \cdots & \int_a^b f_2(x)f_n(x) dx \\ \vdots & \vdots & \cdots & \vdots \\ \int_a^b f_n(x)f_1(x) dx & \int_a^b f_n(x)f_2(x) dx & \cdots & \int_a^b f_n^2(x) dx \end{vmatrix}.$$

$$n \quad f_1, f_2, \dots, f_n \quad a \leq n \leq b$$

$$G(f_1, f_2, \dots, f_n) \geq 0,$$

and the equality sign holds only when the functions are linearly dependent in $a \leq n \leq b$.

SA 4 (Corollary 1)

14.319

The rank of the matrix corresponding to the Gram determinant $G(f_1, f_2, \dots, f_n)$ gives the maximum number of linearly independent functions f_1, f_2, \dots, f_n in $a \leq x \leq b$. If the rank is r , then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

SA 3 (Theorem 4)

15. Norms

15.1- 15.9 Vector Norms

15.11 General properties

The **vector norm** $\|\mathbf{x}\|$ of an $n \times 1$ column vector \mathbf{x} is a nonnegative number having the property that

(a) $\|\mathbf{x}\| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x}\| = 0$ if, and only if, $\mathbf{x} = \mathbf{0}$;

(b) $\|k\mathbf{x}\| = |k| \|\mathbf{x}\|$ for any scalar k ;

(c) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

15.21 Principal vector norms

15.211

The norm $\|\mathbf{x}\|_1$. If \mathbf{x} is a vector with complex components x_1, x_2, \dots, x_n , then

$$\|\mathbf{x}\|_1 = \sum_{r=1}^n |x_r|.$$

VA 15

15.212

The norm $\|\mathbf{x}\|_2$ (euclidean or L_2 norm). If \mathbf{x} is a vector with complex components x_1, x_2, \dots, x_n , then

$$\|\mathbf{x}\|_2 = \left(\sum_{r=1}^n |x_r|^2 \right)^{\frac{1}{2}}.$$

VA8

15.213

The norm $\|\mathbf{x}\|_\infty$. If \mathbf{x} is a vector with complex components x_1, x_2, \dots, x_n , then

$$\|\mathbf{x}\|_\infty = \max_i |x_i|.$$

VA 15

15.31 Matrix norms

15.311

General properties. The **matrix norm** $\|\mathbf{A}\|$ of a square matrix \mathbf{A} is a nonnegative number associated with \mathbf{A} having the property that

(a) $\|\mathbf{A}\| > 0$ when $\mathbf{A} \neq \mathbf{0}$ and $\|\mathbf{A}\| = 0$ if, and only if, $\mathbf{A} = \mathbf{0}$;

(b) $\|k\mathbf{A}\| = |k| \|\mathbf{A}\|$ for any scalar k ;

(c) $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$;

(d) $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \|\mathbf{B}\|$.

VA 9

The matrix norm $\|\mathbf{A}\|$ associated with $\mathbf{A} = [a_{ij}]$, and the vector norm $\|\mathbf{x}\|$ associated with the column vector \mathbf{x} for which the matrix product \mathbf{Ax} is defined, are said to be **compatible** if

$$\|\mathbf{Ax}\| \leq \|\mathbf{A}\| \|\mathbf{x}\|.$$

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15.312

Induced norms. When a vector \mathbf{z} with norm $\|\mathbf{z}\|$ exists such that the maximum is attained in the expression

$$\|\mathbf{A}\| = \max_{\|\mathbf{z}\|=1} \|\mathbf{Az}\|,$$

then $\|\mathbf{A}\|$ is a matrix norm and is said to be the **natural norm induced** by, or **subordinate** to, the vector norm $\|\mathbf{z}\|$.

NO 428

15.313

Natural norm of unit matrix. If \mathbf{I} is the unit matrix, then for any natural norm

$$\|\mathbf{I}\| = 1.$$

NO 429

15.41 Principal natural norms

The natural matrix norms induced on matrix $\mathbf{A} = [a_{ij}]$ by the 1, 2, and ∞ vector norms are as follows:

15.411

Maximum absolute column sum norm.

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|.$$

NO 429

15.412

Spectral norm. If \mathbf{A}^H denotes the hermitian transpose of the square matrix $\mathbf{A} = [a_{ij}]$, so that $\mathbf{A}^H = [\bar{a}_{ji}]$ with a bar denoting the complex conjugate operation, then

$$\|\mathbf{A}\|_2 = \{\max \text{ eigenvalue of } \mathbf{A}^H \mathbf{A}\}^{\frac{1}{2}},$$

or, equivalently,

$$\|\mathbf{A}\|_2 = \{\max \text{ eigenvalue of } \mathbf{A}^H \mathbf{A}\}^{\frac{1}{2}},$$

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2 \neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}.$$

NO 429

15.413

Maximum absolute row sum norm.

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|.$$

NO 429

15.51 Spectral radius of a square matrix

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix with elements that may be complex, and with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the **spectral radius** $\rho(\mathbf{A})$ of \mathbf{A} is the number

$$\rho(\mathbf{A}) = \max_{1 \leq i \leq n} |\lambda_i|.$$

VA 9

1150

15.511

Inequalities concerning matrix norms and the spectral radius.

- $\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_\infty.$

NO 431

- If \mathbf{A} is any arbitrary $n \times n$ matrix with elements that may be complex, and the $n \times n$ matrix \mathbf{U} is unitary, so that $\mathbf{U}^H = \mathbf{U}^{-1}$, with H denoting the hermitian transpose of \mathbf{A} (see 13.123), then

$$\|\mathbf{AU}\| = \|\mathbf{UA}\| = \|\mathbf{A}\|.$$

3. If \mathbf{A} is any nonsingular $n \times n$ matrix with elements that may be complex with eigenvalues $\lambda_1, \lambda_2, \lambda_n$, then

$$\frac{1}{\|\mathbf{A}^{-1}\|} \leq |\lambda| \leq \|\mathbf{A}\|.$$

VA 16

4. For any square matrix \mathbf{A} with spectral radius $\rho(\mathbf{A})$ and any natural norm $\|\mathbf{A}\|$,

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|.$$

NO 430

5. If the square matrix \mathbf{A} is hermitian, then

$$\rho(\mathbf{A}) = \|\mathbf{A}\|.$$

6. If the square matrix \mathbf{A} is hermitian and $P_m(x)$ is any polynomial of degree m with real coefficients, then

$$\|P_m(\mathbf{A})\| = \rho(P_m(\mathbf{A})).$$

7. If \mathbf{A} is any arbitrary $n \times n$ matrix with elements that may be complex, then the sequence of matrices $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \dots$ converges to the null matrix as $n \rightarrow \infty$ if, and only if, $\rho(\mathbf{A}) < 1$.

NO 303

15.512

Deductions from Gerschgorin's theorem (see 15.814).

1. Let \mathbf{A} be any arbitrary $n \times n$ matrix with elements that may be complex; then

$$\rho(\mathbf{A}) \leq \min \left(\max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|, \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \right).$$

VA 17

2. Let \mathbf{A} be any arbitrary $n \times n$ matrix with elements that may be complex, and x_1, x_2, \dots, x_n be any set of n positive numbers; then

$$\rho(\mathbf{A}) \leq \min \left(\max_{1 \leq i \leq n} \left(\frac{\sum_{j=1}^n |a_{ij}| x_j}{x_i} \right), \max_{1 \leq j \leq n} \left(x_j \sum_{i=1}^n \frac{|a_{ij}|}{x_i} \right) \right).$$

VA 18

15.61 Inequalities involving eigenvalues of matrices

The **eigenvalues** (**characteristic values** or **latent roots**) λ of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ are the solutions to the characteristic equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0.$$

1151

When expanded, the determinant $|\mathbf{A} - \lambda \mathbf{I}|$ is called the **characteristic polynomial** and it has the form

$$|\mathbf{A} - \lambda \mathbf{I}| = (-1)^n \lambda^n + c_{n-1} \lambda^{n-1} + c_{n-2} \lambda^{n-2} + \dots + c_1 \lambda + c_0.$$

The zeros of this polynomial satisfy the characteristic equation and so are the eigenvalues of \mathbf{A} . In the characteristic polynomial the coefficients have the form

$$c_{n-r} = (-1)^{n-r} \quad (\text{sum of all principal minors of } |\mathbf{A}| \text{ of order } r).$$

It then follows that

$$\begin{aligned} b_{n-1} &= (-1)^n (a_{11} + a_{22} + \dots + a_{nn}), \\ b_{n-2} &= (-1)^n \sum_{i < j} (a_{ii} a_{jj} - a_{ij} a_{ji}), \\ b_0 &= |\mathbf{A}|. \end{aligned}$$

1. all the eigenvalues λ lie within or on the circle $|z| \leq r$, where r is the positive root of

$$|b_n| + |b_{n-1}|z + |b_{n-2}|z^2 + \cdots + |b_1|z^{n-1} - z^n = 0;$$

MG 122

2. all the eigenvalues λ lie within the circle

$$|z| < 1 + \max_i |b_i|;$$

MG 123

3. when $b_n \neq 0$ the eigenvalue λ of smallest modulus lies in the annulus $R \leq |z| \leq \frac{R}{2^{1/n} - 1}$, where R is the positive root of

$$|b_n| - |b_{n-1}|z - |b_{n-2}|z^2 - \cdots - z^n = 0;$$

MG 126

4. all the eigenvalues λ lie on or outside the circle

$$|z| = \min_k \left[\frac{|b_n|}{(|b_n| + |b_k|)} \right];$$

MG 126

5. if the eigenvalues λ are ordered so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_p| > 1 \geq |\lambda_{p+1}| \geq \cdots \geq |\lambda_n|,$$

then

$$|z_1 z_2 \cdots z_p| \leq N, \quad |z_p| \leq N^{\frac{1}{p}},$$

where

$$N^2 = 1 + |b_1|^2 + |b_2|^2 + \cdots + |b_n|^2;$$

6. all the eigenvalues λ lie in or on the circle

$$|z| \leq \sum_{j=1}^n |b_j|^{\frac{1}{j}};$$

MG 126

7. all the eigenvalues λ lie on the disk

$$\left| z + \frac{b_1}{2} \right| \leq \left| \frac{b_1}{2} \right| + |b_2|^{\frac{1}{2}} + |b_3|^{\frac{1}{3}} + \cdots + |b_n|^{\frac{1}{n}};$$

MG 145

1153

8. all the eigenvalues λ lie in the annulus $m \leq |z| \leq M$, where

$$m^2 = \max \left\{ 0, \min_{1 \leq j \leq n-1} [1 - |b_j|, |b_n|^2] \right\}$$

and

$$M^2 = \max \left\{ 1 + |b_j|, |b_n|^2 + 2 \sum_{j=1}^{n-1} |b_j|^2 \right\}.$$

The next group of inequalities are named theorems that apply to the explicit form of the characteristic polynomial $P(\lambda)$.

MG 145

15.712

Parodi's theorem. The eigenvalues λ satisfying $P(\lambda) = 0$ lie in the union of the disks

$$|z| \leq 1, \quad |z + b_1| \leq \sum_{j=1}^n |b_j|.$$

MG 143

15.713

Corollary of Brauer's theorem. If

$$|b_1| > 1 + \sum_{j=2}^n |b_j|,$$

then one and only one eigenvalue satisfying $P(\lambda) = 0$ lies on the disk

$$|z + b_1| \leq \sum_{j=2}^n |b_j|.$$

MG 141

15.714

Ballieu's theorem. For any set $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ of positive numbers, let $\mu_0 = 0$ and

$$M_\mu = \max_{0 \leq k \leq n-1} \left[\frac{\mu_k + \mu_n |b_{n-k}|}{\mu_{k+1}} \right].$$

Then all the eigenvalues satisfying $P(\lambda) = 0$ lie on the disk $|z| \leq M_\mu$.

MG 144

15.715

Routh-Hurwitz theorem.

Consider the characteristic equation

$$|\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + b_1 \lambda^{n-1} + \dots + b_{n-1} \lambda + b_n = 0$$

determining the n eigenvalues λ of the real $n \times n$ matrix \mathbf{A} . Then the eigenvalues λ all have negative real parts if

$$\Delta_1 > 0, \quad \Delta_2 > 0, \dots, \Delta_n > 0,$$

where

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 & \dots & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ b_{2k-1} & b_{2k-2} & b_{2k-3} & b_{2k-4} & b_{2k-5} & \cdot & \dots & b_k \end{vmatrix}.$$

GM 230

15.81- 15.82 Named theorems on eigenvalues

In the following theorems involving eigenvalue inequalities the elements a_{ij} of matrix \mathbf{A} enter directly, and not in the form of the coefficients of the characteristic polynomial.

15.811

Schur's inequalities. If $\mathbf{A} = [a_{ij}]$ is an $n \times n$ matrix with elements that may be complex, and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$1. \sum_{i=1}^n |\lambda_i|^2 \leq \sum_{i,j=1}^n |a_{ij}|^2,$$

$$2. \sum_{i=1}^n |\operatorname{Re} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} + \bar{a}_{ji}}{2} \right|^2,$$

$$3. \sum_{i=1}^n |\operatorname{Im} \lambda_i|^2 \leq \sum_{i,j=1}^n \left| \frac{a_{ij} - \bar{a}_{ji}}{2} \right|^2.$$

ML 309

15.812

Sturmian separation theorem. Let $\mathbf{A}_r = [a_{ij}]$ with $i, j = 1, 2, \dots, r$ and $r = 1, 2, \dots, N$ be a sequence of N symmetric matrices of increasing order. Then if $\lambda_k(\mathbf{A}_r)$ for $k = 1, 2, \dots, r$ denotes the k th eigenvalue of A_r , where the ordering is such that

$$\lambda_1(A_r) \geq \lambda_2(A_r) \geq \dots \geq \lambda_r(A_r),$$

it follows that

$$\lambda_{k+1}(A_{i+1}) \leq \lambda_k(A_i) \leq \lambda_k(A_{i+1}).$$

BL 115

15.813

Poincare's separation theorem. Let $\{\mathbf{y}^k\}$, with $k = 1, 2, \dots, K$, be a set of orthonormal vectors so that the inner product $(\mathbf{y}^k, \mathbf{y}^k) = 1$. Set

$$\mathbf{x} = \sum_{k=1}^K u_k \mathbf{y}^k,$$

so that for any square matrix \mathbf{A} for which the product \mathbf{Ax} is defined, the quadratic form

$$(\mathbf{x}, \mathbf{Ax}) = \sum_{k,l=1}^K u_k u_l (\mathbf{y}^k, \mathbf{Ay}^l).$$

1155

Then if

$$\mathbf{B}_K = (\mathbf{y}^k, \mathbf{Ay}^l) \quad \text{for } k, l = 1, 2, \dots, K,$$

it follows that

$$\begin{aligned} \lambda_i(\mathbf{B}_K) &\leq \lambda_i(\mathbf{A}) & \text{for } i = 1, 2, \dots, K, \\ \lambda_{K-j}(\mathbf{B}_K) &\geq \lambda_{N-j}(\mathbf{A}) & \text{for } j = 0, 1, 2, \dots, K-1, \end{aligned}$$

Gerschgorin's theorem. Let $\mathbf{A} = [a_{ij}]$ be any arbitrary $n \times n$ matrix with elements that may be complex, and let

$$\Lambda_i \equiv \sum_{j=1, j \neq i}^n |a_{ij}| \quad \text{for } i = 1, 2, \dots, n.$$

Then all of the eigenvalues λ_i of \mathbf{A} lie in the union of the n disks Γ_i , where

$$\Gamma_i: |z - a_{ii}| \leq \Lambda_i \quad \text{for } i = 1, 2, \dots, n.$$

VA 16

15.815

Brauer's theorem. If in Gerschgorin's theorem for a given m

$$|a_{jj} - a_{mm}| \geq \Lambda_j + \Lambda_m$$

for all $j \neq m$, then one and only one eigenvalue of \mathbf{A} lies in the disk Γ_m .

MG 141

15.816

Perron's theorem. If $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$ is an arbitrary set of positive numbers, then all the eigenvalues λ of the $n \times n$ matrix

$\mathbf{A} = [a_{ij}]$ lie on the disk $|z| \leq \mathbf{M}_\mu$, where

$$\mathbf{M}_\mu = \max_{1 \leq i \leq n} \sum_{j=1}^n \frac{\mu_j}{\mu_i} |a_{ij}|.$$

MG 141

15.817

Frobenius theorem. If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients, so that $a_{ij} > 0$ for all $i, j = 1, 2, \dots, n$, then \mathbf{A} has a positive eigenvalue λ_0 and all its eigenvalues lie on the disk

$$|z| \leq \lambda_0.$$

MG 142

15.818

Perron-Frobenius theorem. If all elements a_{ij} of an irreducible matrix \mathbf{A} are nonnegative, then $R = \min M_\lambda$ is a simple eigenvalue of \mathbf{A} and all the eigenvalues of \mathbf{A} lie on the disk

$$|z| \leq R,$$

where, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a set of nonnegative numbers, not all zero,

$$M_\lambda = \inf \left\{ \mu: \mu\lambda_i > \sum_{j=1}^n |a_{ij}|\lambda_j, \quad 1 \leq i \leq n \right\}$$

and $R = \min M_\lambda$.

1156

Furthermore, if \mathbf{A} has exactly p eigenvalues ($p \leq n$) on the circle $|z| = R$, then the set of all its eigenvalues is invariant under rotations $2\pi/p$ about the origin.

GM 69

15.819

Wielandt's theorem. If the $n \times n$ matrix \mathbf{A} satisfies the conditions of the Perron-Frobenius theorem and if in the $n \times n$ matrix

$$\mathbf{C} = [c_{ij}]$$

$$|c_{ij}| \leq a_{ij}, \quad i, j = 1, 2, \dots, n,$$

then any eigenvalue λ_0 of \mathbf{C} satisfies the inequality $|\lambda_0| \leq R$. The equality sign holds only when there exists an $n \times n$ matrix

$\mathbf{D} = [\pm\delta_{ij}]$ such that $\delta_{ii} = 1$ for all i , $\delta_{ij} = 0$ for all $i \neq j$, and

$$\mathbf{C} = (\lambda_0/R) \mathbf{D} \mathbf{A} \mathbf{D}^{-1}.$$

GM 69

15.820

Ostrowski's theorem. If $\mathbf{A} = [a_{ij}]$ is a matrix with positive coefficients and λ_0 is the positive eigenvalue in Frobenius' theorem, then the $n - 1$ eigenvalues $\lambda_j \neq \lambda_0$ satisfy the inequality

$$|\lambda_j| \leq \lambda_0 \frac{M^2 - m^2}{M^2 + m^2},$$

where

$$M = \max a_{ij}, \quad m = \min a_{ij} \quad \text{for } i, j = 1, 2, \dots, n.$$

MG 145

15.821

First theorem due to Lyapunov. In order that all the eigenvalues of the real $n \times n$ matrix \mathbf{A} have negative real parts, it is necessary and sufficient that if \mathbf{V} is an $n \times n$ matrix, the equation

$$\mathbf{A}^T \mathbf{V} + \mathbf{V} \mathbf{A} = -\mathbf{I}$$

has as a solution the matrix of coefficients \mathbf{V} of some positive-definite quadratic form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ (see 13.21).

GM 224

15.822

Second theorem due to Lyapunov.

If all the eigenvalues of the real matrix \mathbf{A} have negative real parts, then to an arbitrary negative-definite quadratic form $(\mathbf{x}, \mathbf{W}\mathbf{x})$ with $\mathbf{x} = \mathbf{x}(t)$ there corresponds a positive-definite quadratic form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ such that if one takes

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x},$$

then $(\mathbf{x}, \mathbf{V}\mathbf{x})$ and $(\mathbf{x}, \mathbf{W}\mathbf{x})$ satisfy

$$\frac{d}{dt}(\mathbf{x}, \mathbf{V}\mathbf{x}) = (\mathbf{x}, \mathbf{W}\mathbf{x}).$$

Conversely, if for some negative-definite form $(\mathbf{x}, \mathbf{W}\mathbf{x})$ there exists a positive-definite form $(\mathbf{x}, \mathbf{V}\mathbf{x})$ connected to $(\mathbf{x}, \mathbf{W}\mathbf{x})$ by the preceding two equations, then all the eigenvalues of \mathbf{A} have negative real parts (see 13.21, 13.31).

GM 222

1157
15.823

Hermitian matrices and diophantine relations involving circular functions of rational angles due to Calogero and Perelomov.

1. The off-diagonal hermitian matrix \mathbf{A} of rank n whose elements are given by

$$a_{jk} = (1 - \delta_{jk}) \left\{ 1 + i \operatorname{ctg} \left[\frac{(j-k)\pi}{n} \right] \right\},$$

has the integer eigenvalues

$$\lambda_s^{(a)} = 2s - n - 1 \quad \text{for } s = 1, 2, \dots, n,$$

and the corresponding eigenvectors $v^{(s)}$ have the components

$$v_j^{(s)} = \exp \left(-\frac{2\pi i s j}{n} \right) \quad \text{for } j = 1, 2, \dots, n.$$

2. The two off-diagonal hermitian matrices \mathbf{B} and \mathbf{C} whose elements are defined by the formulas

$$b_{jk} = (1 - \delta_{jk}) \sin^{-2} \left[\frac{(j-k)\pi}{n} \right],$$

$$c_{jk} = (1 - \delta_{jk}) \sin^{-4} \left[\frac{(j-k)\pi}{n} \right],$$

are related to the matrix \mathbf{A} in (1) by the equations

$$\mathbf{B} = \frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}),$$

$$\mathbf{C} = -\frac{1}{6}(\mathbf{B}^2 - 2(2 + \sigma_n^{(1)})\mathbf{B} - \sigma_n^{(2)}\mathbf{I}),$$

$$\mathbf{B} = \frac{1}{2}(\mathbf{A}^2 + 2\mathbf{A} - \sigma_n^{(1)}\mathbf{I}),$$

$$\mathbf{C} = -\frac{1}{6}(\mathbf{B}^2 - 2(2 + \sigma_n^{(1)})\mathbf{B} - \sigma_n^{(2)}\mathbf{I}),$$

where \mathbf{I} is the unit matrix and

$$\sigma_n^{(1)} = \frac{1}{3}(n^2 - 1), \quad \sigma_n^{(2)} = \frac{1}{45}(n^2 - 1)(n^2 + 11).$$

The eigenvalues of \mathbf{B} and \mathbf{C} corresponding to the eigenvector $v_j^{(s)}$ in (1) have the form

$$\lambda_s^{(b)} = \sigma_n^{(1)} - 2s(n - s) \quad \text{for } s = 1, 2, \dots, n,$$

$$\lambda_s^{(c)} = \sigma_n^{(2)} - 2s(n - s) \frac{s(n - s) + 2}{3} \quad \text{for } s = 1, 2, \dots, n.$$

1158

3. Together, the above two results imply the following diophantine summation rules:

$$(a) \quad \sum_{k=1}^{n-1} \operatorname{ctg} \left(\frac{k\pi}{n} \right) \sin \left(\frac{2sk\pi}{n} \right) = n - 2s \quad \text{for } s = 1, 2, \dots, n - 1,$$

$$(b) \quad \sum_{k=1}^{n-1} \sin^{-2} \left(\frac{k\pi}{n} \right) \cos \left(\frac{2sk\pi}{n} \right) = b_s \quad \text{for } s = 1, 2, \dots, n - 1,$$

$$(c) \quad \sum_{k=1}^{n-1} \sin^{-4} \left(\frac{k\pi}{n} \right) \cos \left(\frac{2sk\pi}{n} \right) = c_s \quad \text{for } s = 1, 2, \dots, n - 1,$$

$$(d) \quad \sum_{k=1}^{n-1} \sin^{-2p} \left(\frac{k\pi}{n} \right) = \sigma_n^{(p)},$$

with b_s , c_s , $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ as defined in (2), and

$$\sigma_n^{(3)} = \sigma_n^{(1)} \frac{2n^4 + 23n^2 + 191}{315},$$

$$\sigma_n^{(4)} = \sigma_n^{(2)} \frac{3n^4 + 10n^2 + 227}{315}.$$

15.91 Variational principles

15.911

Rayleigh quotient.

If \mathbf{A} is an hermitian matrix, the Rayleigh quotient $\rho(\mathbf{x})$ is the expression

$$\rho(\mathbf{x}) = \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})}.$$

NO 407

15.912

Basic theorems.

1. If the $n \times n$ matrix A is hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\lambda_1 \leq \rho \leq \lambda_n,$$

where ρ is the Rayleigh quotient for any $\mathbf{x} \neq \mathbf{0}$, and

$$\lambda_1 = \min_{\mathbf{x} \neq \mathbf{0}} \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \quad \text{and} \quad \lambda_n = \max_{\mathbf{x} \neq \mathbf{0}} \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})}.$$

NO 407

2. If the $n \times n$ matrix \mathbf{A} is hermitian and has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ corresponding to the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, respectively, and $\mathbf{x} \neq \mathbf{0}$ is such that

$$(\mathbf{x}, \mathbf{x}_1) = (\mathbf{x}, \mathbf{x}_2) = \cdots = (\mathbf{x}, \mathbf{x}_n) = 0,$$

then

$$\lambda_j = \min_x \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})},$$

1159

and

3. If the $n \times n$ matrix \mathbf{A} is hermitian, then the eigenvalue

$$\lambda_r = \max \left(\min \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \right),$$

where first the minimum over \mathbf{x} is taken subject to $(\mathbf{b}_i, \mathbf{x}) = 0$, $i = 1, 2, \dots, r - 1$, with the \mathbf{b}_i regarded as fixed vectors, and then the maximum over all possible \mathbf{b}_i . Also, the eigenvalue

$$\lambda_r = \min \left(\max \frac{(\mathbf{x}, \mathbf{Ax})}{(\mathbf{x}, \mathbf{x})} \right),$$

where now the maximum over \mathbf{x} is taken first subject to $(\mathbf{b}_i, \mathbf{x}) = 0$, $i = r + 1, r + 2, \dots, n$ for fixed \mathbf{b}_i , and then the minimum over all possible \mathbf{b}_i .

NO 414

4. The $(n - 1)$ eigenvalues $\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}$ obtained from the $(n - 1) \times (n - 1)$ matrix derived from an hermitian matrix \mathbf{A} from which the last row and column have been omitted separate the n eigenvalues of \mathbf{A} , so that

$$\lambda_1 < \lambda'_1 < \lambda_2 < \lambda'_2 < \dots < \lambda'_{n-1} < \lambda_n \quad (\text{see 15.812}).$$

15.812

16. Ordinary Differential Equations

16.1- 16.9 Results Relating to the Solution of Ordinary Differential Equations

16.11 First-order equations

16.111

Solution of a first-order equation.

Consider the real function $f(t, x)$ that is defined and continuous in an open set $D \subset R^2$. Then a **solution** to the first-order differential equation

$$\frac{dx}{dt} = f(t, x)$$

in the open interval $I \subset R$ is a real function $u(t)$ that is defined and is both continuous and differentiable in I , with the property that

- (i) $(t, u(t)) \in D$ for $t \in I$,
- (ii) $\frac{du}{dt} = f(t, u(t))$ for $t \in I$.

16.112

Cauchy problem.

The **Cauchy problem** for the differential equation

$$\frac{dx}{dt} = f(t, x)$$

is the problem of existence and uniqueness of the solution to this equation satisfying the initial condition

$$u(t_0) = x_0,$$

where $(t_0, u(t_0)) \in D$, the open set defined above. The solution to the initial value problem may be expressed in the form of the integral equation

$$u(t) = x_0 + \int_{t_0}^t f(\tau, u(\tau)) d\tau \quad (\text{see 16.316}).$$

16.316

16.113

Approximate solution to an equation.

The real function $\phi(t)$ is said to be an **approximate solution**, to within the error ϵ , of the differential equation

$$\frac{dx}{dt} = f(t, x)$$

1161

if ϕ' is piecewise continuous, and for a given $\epsilon > 0$ and an open interval $I \subset R$,

$$|\phi'(t) - f(t, \phi(t))| \leq \epsilon,$$

except at points of discontinuity of the derivative.

HU 3

16.114

Lipschitz continuity of a function.

The real function $f(t, x)$ defined and continuous in some open set $D \subset R^2$ is said to be

Lipschitz continuous with respect to x for some constant $k > 0$ if, for all points (t, x_1) and (t, x_2) belonging to D

$$|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|.$$

HU 5

16.21 Fundamental inequalities and related results

16.211

Gronwall's lemma.

Let the three piecewise continuous, nonnegative functions u , v , and w be defined in the interval $[0, a]$ and satisfy the inequality

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau) d\tau,$$

$$w(t) \leq u(t) + \int_0^t v(\tau)w(\tau) d\tau,$$

except at points of discontinuity of the functions. Then, except at these same points,

$$w(t) \leq u(t) + \int_0^t u(\tau)v(\tau) \exp\left(\int_\tau^t v(\sigma) d\sigma\right) d\tau.$$

BB 135

16.212

Comparison of approximate solutions of a differential equation.

Let f be a real function that is defined in an open set $D \subset R^2$, in which it is both continuous and Lipschitz continuous. In addition, let u_1 and u_2 be two approximate solutions of

$$\frac{dx}{dt} = f(t, x)$$

in an open set $I \subset R$ in the sense already defined, with

$$|u_1'(t) - f(t, u_1(t))| \leq \epsilon_1, |u_2'(t) - f(t, u_2(t))| \leq \epsilon_2,$$

except where the derivatives are discontinuous.

Then, if for all $t_0 \in I$

$$|u_1(t_0) - u_2(t_0)| \leq \delta,$$

it follows that

$$|u_1(t) - u_2(t)| \leq \delta \exp\{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k}\right) [\exp\{k|t - t_0|\} - 1].$$

HU 6

1162

16.31 First-order systems

16.311

Solution of a system of equations.

The **system** of n first-order differential equations

$$\frac{dx_1}{dt} = f_1(t, x_1, x_2, \dots, x_n),$$

$$\frac{dx_2}{dt} = f_2(t, x_1, x_2, \dots, x_n),$$

\vdots

$$\frac{dx_n}{dt} = f_n(t, x_1, x_2, \dots, x_n),$$

in which the functions f_1, f_2, \dots, f_n are real and continuous in an open set $D \subset R^{n+1}$ may be written in the concise matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

where \mathbf{x} and \mathbf{f} are $n \times 1$ column vectors. Its solution in the open interval $I \subset R$ is the vector $\mathbf{u}(t)$ with elements

$u_1(t), u_2(t), \dots, u_n(t)$ with the property that

- (i) $(t, \mathbf{u}(t)) \in D$ for $t \in I$,
- (ii) $\frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t))$ for $t \in I$.

Cauchy problem for a system.

The **Cauchy problem** for the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

is the problem of existence and uniqueness of the solution to this system satisfying the **initial vector** condition

$$\mathbf{u}(t_0) = \mathbf{x}_0,$$

where $(t_0, \mathbf{u}(t_0)) \in D$, the open set defined above in connection with the system. The solution to the initial value problem may be expressed in the form of the **vector integral equation**

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_{t_0}^t \mathbf{f}(\tau, \mathbf{u}(\tau)) d\tau.$$

16.313

Approximate solution to a system.

The real vector $\Phi(t)$ is said to be an **approximate vector solution**, to within the order ϵ , of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}),$$

1163

if the elements of Φ' are piecewise continuous, and for a given $\epsilon > 0$ and open interval $I \subset R$,

$$\|\phi'(t) - \mathbf{f}(t, \phi(t))\| \leq \epsilon,$$

except at points of discontinuity of the derivative, where $\|\mathbf{w}\|$ denotes the supremum norm

$$\|\mathbf{w}\| = \sup(|w_1|, |w_2|, \dots, |w_n|).$$

16.314

Lipschitz continuity of a vector.

The real vector $\mathbf{f}(t, x)$ defined and continuous in some open set $D \subset R^n$ is said to be

Lipschitz continuous with respect to x for some constant $k > 0$ if, for all points $(t, \mathbf{x}_1), (t, \mathbf{x}_2)$ belonging to D ,

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq k\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

HU 26

16.315

Comparison of approximate solutions of a system.

Let \mathbf{f} be a real vector defined in an open set $D \subset R \times R^n$ in which it is both continuous and Lipschitz continuous. In addition, let \mathbf{u}_1 and \mathbf{u}_2 be two approximate solutions of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x})$$

in an open set $I \subset R$ in the sense already defined, with

$$\|\mathbf{u}'_1(t) - \mathbf{f}(t, \mathbf{u}_1(t))\| \leq \epsilon_1, \quad \|\mathbf{u}'_2(t) - \mathbf{f}(t, \mathbf{u}_2(t))\| \leq \epsilon_2,$$

except where the derivatives are discontinuous. Then, if for all $t_0 \in I$

$$\|\mathbf{u}_1(t_0) - \mathbf{u}_2(t_0)\| \leq \delta,$$

it follows that

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \leq \delta \exp \{k|t - t_0|\} + \left(\frac{\epsilon_1 + \epsilon_2}{k} \right) [\exp \{k|t - t_0|\} - 1].$$

HU 27

16.316

First-order linear differential equation.

The **first-order linear differential equation** when expressed in the canonical form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

has an integrating factor

$$\mu(t) = \exp\left(\int P(t) dt\right),$$

and a general solution

$$y(t) = \frac{1}{\mu(t)} \left(\mu(t_0)y_0 + \int_{t_0}^t \mu(\xi)Q(\xi) d\xi \right),$$

where $y_0 = y(t_0)$.

1164
16.317

Linear systems of differential equations.

Consider the **homogeneous system** of linear differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x},$$

where \mathbf{x} is an $n \times 1$ column vector and $\mathbf{A}(t)$ an $n \times n$ matrix. Then a **fundamental system** of solutions of this system is a set of n linearly independent solution vectors $\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t)$,

The square matrix $\mathbf{K}(t)$ whose columns comprise the vectors $\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t)$ is called the **fundamental matrix** of the differential equation, and we have the representation

$$|\mathbf{K}(t)| = |\mathbf{K}(t_0)| \exp\left(\int_{t_0}^t \text{tr } \mathbf{A}(\tau) d\tau\right).$$

Using the fundamental matrix $\mathbf{K}(t)$ defined in terms of the homogeneous system, the unique solution to the inhomogeneous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t),$$

assuming the initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, is

$$\phi(t) = \mathbf{K}(t)\mathbf{K}(t_0)^{-1}\mathbf{x}_0 + \mathbf{K}(t)\int_{t_0}^t \mathbf{K}(\tau)^{-1}\mathbf{b}(\tau) d\tau,$$

HU 43

where $\mathbf{b}(t)$ is an $n \times 1$ column vector.

CL 69

16.41 Some special types of elementary differential equations

16.411

Variables separable.

A first-order differential equation is said to be **variables separable** if it is of the form

$$\frac{dy}{dx} = M(x)N(y),$$

or

$$P(x)Q(y) dx + R(x)S(y) dy = 0.$$

It may then be written in the form

$$M(x) dx - \frac{1}{N(y)} dy = 0,$$

or

$$M(x) dx - \frac{1}{N(y)} dy = 0,$$

$$\frac{P(x)}{R(x)} dx + \frac{S(y)}{Q(y)} dy = 0,$$

provided $R(x)Q(y) \neq 0$.

1165
16.412

Exact differential equations.

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **exact** if there exists a function $h(x, y)$ such that

$$d[h(x, y)] = M(x, y) dx + N(x, y) dy.$$

IN 16

16.413

Conditions for an exact equation.

A necessary and sufficient condition that an equation of this form is exact is that the functions $M(x, y)$ and $N(x, y)$ together with their partial derivatives $\partial M/\partial y$ and $\partial N/\partial x$ exist and are continuous in a region in which

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

IN 16

16.414

Homogeneous differential equations.

A differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is said to be **algebraically homogeneous** if, for arbitrary k ,

$$\frac{M(kx, ky)}{N(kx, ky)} = \frac{M(x, y)}{N(x, y)}.$$

Setting $y = sx$, it may then be expressed in the form

$$[M(1, s) + sN(1, s)] dx + xN(1) dx = 0,$$

in which the variables s and x are separable.

IN 18

16.51 Second-order equations

16.511

Adjoint and self-adjoint equations.

The linear second-order differential equation

$$L(u) \equiv a(x) \frac{d^2 u}{dx^2} + b(x) \frac{du}{dx} + c(x)u = 0,$$

has associated with it the adjoint equation

$$M(v) \equiv \frac{d^2}{dx^2} [a(x)v] - \frac{d}{dx} [b(x)v] + c(x)v = 0.$$

The equation $L(u) = 0$ is said to be **self-adjoint** if $L(u) \equiv M(u)$.

A linear self-adjoint second-order differential equation defined on $[\alpha, \beta]$ can always be expressed in the form

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

where $p(x)$ and $q(x)$ are continuous on $[\alpha, \beta]$ and $p(x) > 0$. The general equation $L(u) = 0$ can always be made self-adjoint and written in this form by multiplication by the factor

$$\frac{1}{a(x)} \left[\exp \int \frac{b(x)}{a(x)} dx \right],$$

when

$$p(x) = \exp \int \frac{b(x)}{a(x)} dx \quad \text{and} \quad q(x) = \frac{c(x)}{a(x)} \left[\exp \int \frac{b(x)}{a(x)} dx \right].$$

In general, if

$$L(u) = p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u, i$$

then its adjoint is

$$M(v) = (-1)^n \frac{d^n}{dx^n} [p_0 v] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [p_1 v] + \dots - \frac{d}{dx} [p_{n-1} v] + p_n v.$$

HI 391

16.512

Abel's identity.

If $p(x)$ and $q(x)$ are continuous in $[\alpha, \beta]$ in which $p(x) > 0$, and $u(x)$ and $v(x)$ are suitably differentiable with

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0,$$

then the result

$$p(x) \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) \equiv \text{const.}$$

More generally, if we consider the linear n th-order equation

$$p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n = 0,$$

and Δ is the Wronskian of a (fundamental) set of linearly independent solutions u_1, u_2, \dots, u_n , the Abel identity takes the form

$$\Delta = \Delta_0 \exp \left(- \int_{x_0}^x \frac{p_1(x)}{p_0(x)} dx \right),$$

where Δ_0 is the value of Δ at $x = x_0$.

IN 119

1167

16.513

Lagrange identity.

If the linear n th-order equation $L(u) = 0$ is defined by

$$L(u) \equiv p_0 \frac{d^n u}{dx^n} + p_1 \frac{d^{n-1} u}{dx^{n-1}} + \dots + p_{n-1} \frac{du}{dx} + p_n u,$$

then the expression

$$vL(u) - uM(v) = \frac{d}{dx} \{P(u, v)\},$$

where $M(v)$ is the adjoint of $L(u)$, is called the **Lagrange identity**. The expression $P(u, v)$, which is linear and homogeneous in

$$u, \frac{du}{dx}, \dots, \frac{d^{n-1} u}{dx^{n-1}} \quad \text{and} \quad v, \frac{dv}{dx}, \dots, \frac{d^{n-1} v}{dx^{n-1}},$$

is then known as the **bilinear concomitant**.

In the case of the second-order equation

$$L(u) = a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0,$$

with adjoint $M(v)$, the Lagrange identity becomes

$$vL(u) - uM(v) = \frac{d}{dx} \left(a(x)v\frac{du}{dx} - \frac{d}{dx}(a(x)v)u + b(x)uv \right) .$$

IN 124

16.514

The Riccati equation.

The general **Riccati equation** has the form

$$\frac{dz}{dx} + a(x)z + b(x)z^2 + c(x) = 0,$$

and an equation of this form results from the substitution

$$z = \frac{\left(p(x)\frac{du}{dx} \right)}{u}$$

in the general self-adjoint equation

$$\frac{d}{dx} \left(p(x)\frac{du}{dx} \right) + q(x)u = 0.$$

1168

The further substitution $v = u \left(\exp \int_{\alpha}^x a(x) dx \right)$ in the Riccati equation then gives the more convenient form

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0,$$

with

$$r(x) = b(x) \exp\left(-\int_{\alpha}^x a(x) dx\right) \quad \text{and} \quad s(x) = c(x) \exp\left(\int_{\alpha}^x a(x) dx\right).$$

HI 273

16.515

Solutions of the Riccati equation.

If in the Riccati equation

$$\frac{dv}{dx} + r(x)v^2 + s(x) = 0$$

$r(x) \neq 0$, while $r(x)$ and $s(x)$ are continuous on the interval $[\alpha, \beta]$, then every solution $v(x)$ may be expressed in the form

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)},$$

with A, B arbitrary constants, not both zero, and the prime denoting differentiation, while u and v are linearly independent solutions of

$$\frac{d}{dx} \left(\frac{1}{r(x)} \frac{dz}{dx} \right) + s(x)z = 0.$$

Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions of this last equation and A and B are arbitrary constants, not both zero, the function

$$\frac{1}{r(x)} \frac{Au'(x) + Bv'(x)}{Au(x) + Bv(x)}$$

is a solution of the Riccati equation wherever $Au(x) + Bv(x) \neq 0$.

IN 24

16.516

Solution of a second-order linear differential equation.

A **fundamental system** of solutions of a homogeneous second-order linear differential equation in the canonical form

$$\frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = 0$$

is a system of two linearly independent solutions $\phi_1(t)$ and $\phi_2(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{vmatrix} = \phi_1(t)\phi_2'(t) - \phi_2(t)\phi_1'(t),$$

1169

when the solution to the inhomogeneous equation

$$\frac{d^2 x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = f(t),$$

may be written

$$x(t) = \int_{t_0}^t \frac{\phi_1(\xi)\phi_2(t) - \phi_2(\xi)\phi_1(t)}{W(\xi)} f(\xi) d\xi.$$

The linear combination $c_1\phi_1(t) + c_2\phi_2(t)$ is known as the **complementary function** where c_1 and c_2 are arbitrary constants.

16.61- 16.62 Oscillation and nonoscillation theorems for second-order equations

Equations whose solutions possess an infinite number of zeros in the interval $(0, \infty)$ are said to have **oscillatory** solutions. The following theorems relate to such properties.

16.611

First basic comparison theorem.

If all solutions of the equation

$$\frac{d^2u}{dx^2} + \phi(x)u = 0$$

are oscillatory, and if

$$\psi(x) \geq \phi(x),$$

then all the solutions of

$$\frac{d^2v}{dx^2} + \psi(x)v = 0$$

are oscillatory, and conversely. That is, if $\psi(x) \geq \phi(x)$ and some solutions v are nonoscillatory, then so also must some solutions u be nonoscillatory.

BS 119

16.622

Second basic comparison theorem.

If all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

are oscillatory as $x \rightarrow \infty$, and if

$$\begin{aligned} q_2(x) &\geq q_1(x), \\ p_2(x) &\geq p_1(x) > 0, \end{aligned}$$

1170

then all the solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0$$

16.623

Interlacing of zeros.

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2 y}{dx^2} + F(x)y = 0,$$

and suppose that $y_1(x)$ has at least two zeros in the interval (a, b) . Then if x_1 and x_2 are two consecutive zeros of $y_1(x)$, the function $y_2(x)$ has one, and only one, zero in the interval (x_1, x_2) .

HI 374

16.624

Sturm separation theorem.

Let $u(x)$ and $v(x)$ be two linearly independent solutions of the self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x) = 0,$$

in which $p(x) > 0$ and $p(x), q(x)$ are continuous on $[a, b]$. Then, between any two consecutive zeros of $u(x)$ there will be one, and only one, zero of $v(x)$.

IN 224

16.625

Sturm comparison theorem.

Let $p_1(x) \geq p_2(x) > 0$ and $q_1(x) \geq q_2(x)$ be continuous functions in the differential equations

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0,$$

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0.$$

Then between any two zeros of a nontrivial solution $u(x)$ of the first equation there will be at least one zero of every nontrivial solution $v(x)$ of the second equation.

IN 228

16.626

Szegő's comparison theorem.

Suppose, under the conditions of the Sturm comparison theorem, that $p_1(x) \equiv p_2(x)$, $q_1(x) \not\equiv q_2(x)$, and $u(x) > 0$, $v(x) > 0$ for $a < x < b$, together with

$$\lim_{x \rightarrow a} p_1(x) \left(\frac{du}{dx} v - \frac{dv}{dx} u \right) = 0.$$

Then, if $u(b) = 0$, there is a point ξ in (a, b) such that $v(\xi) = 0$.

HI 379

1171

16.627

Picone's identity.

Consider the equations

$$\begin{aligned} \frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u &= 0, \\ \frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v &= 0, \end{aligned}$$

with p_1 , p_2 , q_1 , and q_2 positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then with $a < \alpha < \beta < b$, Picone's identity is

$$\left(\frac{u}{v} \left(p_1 \frac{du}{dx} v - p_2 \frac{dv}{dx} u \right) \right)_{\alpha}^{\beta} = \int_{\alpha}^{\beta} (q_2 - q_1)u^2 ds + \int_{\alpha}^{\beta} (p_1 - p_2) \left(\frac{du}{ds} \right)^2 ds + \int_{\alpha}^{\beta} \frac{p_2}{v^2} \left(v \frac{du}{ds} - u \frac{dv}{ds} \right)^2 ds.$$

IN 226

16.628

Sturm-Picone theorem.

Consider the self-adjoint equations

$$\frac{d}{dx} \left(p_1(x) \frac{du}{dx} \right) + q_1(x)u = 0$$

and

$$\frac{d}{dx} \left(p_2(x) \frac{dv}{dx} \right) + q_2(x)v = 0.$$

Let p_1 , p_2 , q_1 , and q_2 be positive and continuous for $a < x < b$, where $q_2(x) > q_1(x)$ and $p_1(x) > p_2(x)$. Then, if x_1 and x_2 is a pair of consecutive zeros of $u(x)$ in (a, b) , $v(x)$ has at least one zero in the open interval (a, b) .

IN 225

16.629

Oscillation on the half line.

Consider the self-adjoint equation

$$\frac{d}{dx} \left(p(x) \frac{du}{dx} \right) + q(x)u = 0.$$

We then have the following results.

(i) Let $p(x) > 0$ and p, q be continuous on $[0, \infty)$. If the two improper integrals

$$\int_1^{\infty} \frac{dx}{p(x)} \quad \text{and} \quad \int_1^{\infty} q(x) dx$$

diverge, then every solution $u(x)$ has infinitely many zeros on the interval $[1, \infty)$. Also, if the two integrals

$$\int_0^1 \frac{dx}{p(x)} = +\infty \quad \text{and} \quad \int_0^1 q(x) dx = +\infty,$$

then every solution $u(x)$ has infinitely many zeros on the interval $(0, 1)$.

1172

(ii) (Moore's theorem). Every nontrivial solution $u(x)$ has at most a finite number of zeros on the interval $[a, \infty)$ if the improper integral

$$\int_a^{\infty} \frac{dx}{p(x)}$$

converges, and if

$$\left| \int_a^x q(s) ds \right| < M \quad \text{for } a \leq x < \infty$$

with $M > 0$ a finite constant.

16.71 Two related comparison theorems

16.711

Theorem 1.

Consider the equations in the Sturm comparison theorem with the same assumptions on $p(x)$ and $q(x)$, and let $u(x)$, $v(x)$ be solutions such that

$$u(x_1) = v(x_1) = 0, \quad u'(x) = v'(x_1) > 0.$$

Then if $u(x)$ is increasing in $[x_1, x_2]$ and reaches a maximum at x_2 , the function $v(x)$ reaches a maximum at some point x_3 such that $x_1 < x_3 < x_2$.

HI 376

16.712

Theorem 2.

Consider the equation

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

in which $F(x)$ is continuous in (a, b) and such that

$$0 < m \leq F(x) \leq M.$$

Then, if the solution $y(x)$ has two successive zeros x_1, x_2 , it follows that

$$\pi M^{-\frac{1}{2}} \leq x_2 - x_1 \leq \pi m^{-\frac{1}{2}}.$$

16.81- 16.82 Nonoscillatory solutions

The real solution $y(x)$ of

$$\frac{d^2y}{dx^2} + F(x)y = 0$$

is said to be **nonoscillatory** in the wide sense in $(0, \infty)$ if there exists a finite number c such that the solution has no zeros in $[c, \infty)$.

HI 376

1173

16.811

Kneser's nonoscillation theorem.

Consider the equation

$$\frac{d^2y}{dx^2} + F(x)y = 0,$$

and let

$$\limsup [x^2 F(x)] = \gamma^*,$$

$$\liminf [x^2 F(x)] = \gamma_*.$$

Then the solution $y(x)$ is nonoscillatory if $\gamma^* < \frac{1}{4}$, oscillatory if $\frac{1}{4} < \gamma_*$ and no conclusion can be drawn if either γ^* or γ_* equals $\frac{1}{4}$.

HI 461

16.822

Comparison theorem for nonoscillation.

Consider the differential equations

$$\frac{d^2 y}{dx^2} + F(x)y = 0, \quad f(x) = x \int_x^\infty F(s) ds,$$

$$\frac{d^2 y}{dx^2} + G(x)y = 0, \quad g(x) = x \int_x^\infty G(s) ds,$$

where $0 < g(x) < f(x)$. Then if the first equation is nonoscillatory in the wide sense, so also is the second.

HI 460

16.823

Necessary and sufficient conditions for nonoscillation.

Consider the equation

$$\frac{d^2 y}{dx^2} + F(x)y = 0.$$

Then, if

$$\lim_{x \rightarrow \infty} \sup \left(x \int_x^\infty F(s) ds \right) = F^*,$$

$$\lim_{x \rightarrow \infty} \inf \left(x \int_x^{\infty} F(s) ds \right) = F_*,$$

it follows that:

(i) a necessary condition that the solution $y(x)$ be nonoscillatory is that $F_* \leq \frac{1}{4}$ and $F^* \leq 1$;

(ii) a sufficient condition that the solution $y(x)$ be nonoscillatory is that $F^* < \frac{1}{4}$.

1174

16.91 Some growth estimates for solutions of second-order equations

16.911

Strictly increasing and decreasing solutions.

Suppose that $G(x) > 0$ be continuous in $(-\infty, \infty)$ and such that $xG(x) \notin L(0, \infty)$. Then the equation

$$\frac{d^2 y}{dx^2} - G(x)y = 0$$

has one, and only one, solution $y_+(x)$ passing through the point $(0, 1)$ which is positive and strictly monotonic decreasing for all x , and one and only one solution $y_-(x)$ through the point $(0, 1)$ which is positive and strictly increasing for all x . The solution $y_+(x)$ has the property that

$$[G(x)]^{\frac{1}{2}} y_+(x) \in L_2(0, \infty) \quad \text{and} \quad \frac{dy_+(x)}{dx} \in L_2(0, \infty).$$

If, in addition,

$$0 < \alpha^2 \leq G(x) \leq \beta^2 < \infty,$$

then

$$e^{-\beta x} \leq y_+(x) \leq e^{-\alpha x} \quad \text{for } x > 0.$$

16.912

General result on dominant and subdominant solutions.

Consider the equations

$$\frac{d^2 y}{dx^2} - g(x)y = 0, \quad \frac{d^2 Y}{dx^2} - G(x)Y = 0,$$

where g and G are continuous on $(0, \infty)$ with $0 < g(x) < G(x)$, and $xg(x) \notin L(0, \infty)$. In addition, let y_α and Y_α be the solutions of these respective equations corresponding to

$$y_\alpha(0) = Y_\alpha(0) = 1, \quad y'_\alpha(0) = Y'_\alpha(0) = \alpha \quad \text{for} \quad -\infty < \alpha < \infty.$$

Let y_ω and Y_ω be determined, respectively, by

$$y_\omega(0) = Y_\omega(0) = 0, \quad y'_\omega(0) = Y'_\omega(0) = 1,$$

and let y_+ and Y_+ be the **subdominant solutions** for which

$$y_+(0) = Y_+(0) = 1$$

while $[y'_+(x)]^2$, $g(x)[y_+(x)]^2$, $[Y'_+(x)]^2$, and $G(x)[Y_+(x)]^2$ belong to $L(0, \infty)$. Then, if β and γ are such that $y_{-\beta} = y_+$ and $Y_{-\gamma} = Y_+$, it follows that $\beta < \gamma$ and

$$\begin{aligned} y_\alpha(x) &< Y_\alpha(x), & 0 < x < \infty, & \quad -\gamma \leq \alpha, \\ y_\omega(x) &< Y_\omega(x), \\ y_+(x) &> Y_+(x). \end{aligned}$$

1175

16.913

Estimate of dominant solution.

Let $G(x)$ be positive and continuous with continuous first- and second-order derivatives satisfying

$$G(x)G'(x) < \frac{5}{4}[G'(x)]^2.$$

Then there exists a **dominant solution** $y(x)$ of the fundamental solutions $Y_0(x)$ and $Y_1(x)$ of

$$\frac{d^2 y}{dx^2} - G(x)y = 0,$$

determined by the initial conditions

$$\begin{aligned} Y_0(0) &= 0, & Y_1(0) &= 1, \\ Y_0'(0) &= 1, & Y_1'(0) &= 0, \end{aligned}$$

such that

$$y(x) < [G(x)]^{-\frac{1}{4}} \exp\left(\int_0^x [G(\xi)]^{\frac{1}{2}} d\xi\right),$$

and a positive constant C such that the normalized subdominant solution $y_+(x)$, for which $y_+(0) = 1$ and $[y_+'(x)]^2 \in L(0, \infty)$,

$G(x)[y_+(x)]^2 \in L(0, \infty)$, satisfies

$$y_+(x) > C[G(x)]^{-\frac{1}{4}} \exp\left(-\int_0^x [G(\xi)]^{\frac{1}{2}} d\xi\right).$$

HI 443

16.914

A theorem due to Lyapunov.

Let $y(x)$ be any solution of

$$\frac{d^2 y}{dx^2} - G(x)y = 0$$

with $G(x)$ positive and continuous in $(0, \infty)$ with $xG(x) \in L(0, \infty)$. Then

$$C \exp\left(-\int_0^x [G(\xi) + 1] d\xi\right) < [y(x)]^2 + [y'(x)]^2 < C \exp\left(\int_0^x [G(\xi) + 1] d\xi\right),$$

HI 446

where $C = [y(0)]^2 + [y'(0)]^2$.

16.92 Boundedness theorems

16.921⁶

All solutions of the equation

$$\frac{d^2u}{dx^2} + (1 + \phi(x) + \psi(x))u = 0$$

1176

are bounded, provided that

$$(i) \int_0^\infty |\phi(x)| dx < \infty,$$

$$(ii) \int_0^\infty |\psi(x)| dx < \infty \quad \text{and} \quad \psi(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty.$$

BS 112

16.922

If all solutions of the equation

$$\frac{d^2u}{dx^2} + a(x)u = 0$$

are bounded, then all solutions of

$$\frac{d^2u}{dx^2} + (a(x) + b(x))u = 0$$

are also bounded if

$$\int^{\infty} |b(x)| dx < \infty.$$

BS 112

16.923

If $a(x) \rightarrow \infty$ monotonically as $x \rightarrow \infty$, then all solutions of

$$\frac{d^2 u}{dx^2} + a(x)u = 0$$

are bounded as $x \rightarrow \infty$.

BS 113

16.924

Consider the equation

$$\frac{d^2 u}{dx^2} + a(x)u = 0$$

in which

$$\int^{\infty} x|a(x)| dx < \infty.$$

Then $\lim_{x \rightarrow \infty} \left(\frac{du}{dx} \right)$ exists, and the general solution is asymptotic to $d_0 + d_1 x$ as $x \rightarrow \infty$, where d_0 and d_1 may be zero, but not simultaneously.

BS 114

16.93*

Growth of maxima of $|y|$. Sonin's theorem generalized by Pólya may be stated as follows. *Let $y(x)$ satisfy the differential equation*

$$\{k(x)y'\}' + \phi(x)y = 0,$$

where $k(x) > 0$, $\phi(x) > 0$, and both functions $k(x)$, $\phi(x)$ have a continuous derivative. Then the relative maxima of $|y|$ form an increasing or decreasing sequence according as $k(x)\phi(x)$ is decreasing or increasing.

SZ 164

17. Fourier, Laplace and Mellin Transforms

17.1- 17.4 Integral Transforms

17.11 Laplace transform

The **Laplace transform** of the function $f(x)$, denoted by $F(s)$, is defined by the integral

$$F(s) = \int_0^{\infty} f(x)e^{-sx} dx, \quad \text{Re } s > 0.$$

The functions $f(x)$ and $F(s)$ are called a **Laplace transform pair**, and knowledge of either one enables the other to be recovered.

If f is summable over all finite intervals, and there is a constant c for which

$$\int_0^{\infty} |f(x)|e^{-c|x|} dx,$$

is finite, then the Laplace transform exists when $s = \sigma + i\tau$ is such that $\sigma \geq c$.

Setting

$$F(s) = \mathcal{L}[f(x); s],$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{L}^{-1}[F(s); x].$$

The inversion of the Laplace transform is accomplished for analytic functions $F(s)$ of order $O(s^{-k})$ with $k > 1$ by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{sx} ds,$$

where γ is a real constant that exceeds the real part of all the singularities of $F(s)$.

SN 30

17.12 Basic properties of the Laplace transform

1.⁸ For a and b arbitrary constants,

$$\mathcal{L}[af(x) + bg(x)] = aF(s) + bG(s) \quad (\text{linearity}).$$

1178

2. If $n > 0$ is an integer and $\lim_{x \rightarrow \infty} f(x)e^{-sx} = 0$, then for $x > 0$,

$$\mathcal{L}[f^{(n)}(x); s] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \dots - f^{(n-1)}(0) \quad (\text{transform of a derivative}).$$

SN 32

3. If $\lim_{x \rightarrow \infty} (e^{-sx} \int_0^x f(\xi) d\xi) = 0$, then

$$\mathcal{L} \left[\int_0^x f(\xi) d\xi; s \right] = \frac{1}{s} F(s) \quad (\text{transform of an integral}).$$

SN 37

4. $\mathcal{L}[e^{-ax} f(x); s] = F(s + a)$ (shift theorem).

SU 143

5. The **Laplace convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \int_0^x f(x - \xi)g(\xi) d\xi,$$

and it has the property that $f * g = g * f$ and $f * (g * h) = (f * g) * h$. In terms of the convolution operation

$$\mathcal{L}[f * g(x); s] = F(s)G(s) \text{ (convolution (Faltung) theorem).}$$

SN 30

17.13* Table of Laplace transform pairs

$f(x)$	$F(s)$
1. 1	$1/s$
2. $x^n, n = 0, 1, 2, \dots$	$n!/s^{n+1}, \text{ Re } s > 0$
3. $x^\nu, \text{Re } \nu > -1$	$\Gamma(\nu + 1)/s^{\nu+1}, \text{ Re } s > 0$
4. ⁸ $x^{n-\frac{1}{2}}$	$\left(\frac{\sqrt{\pi}}{2}\right) \left(\frac{3}{2}\right) \left(\frac{5}{2}\right) \dots \left(\frac{n-1}{2}\right) / s^{n+\frac{1}{2}} = 1^7 (n + \frac{1}{2}) / s^{n+\frac{1}{2}}, \text{ Re } s > 0$
5. $x^{-\frac{1}{2}}(x+a)^{-1}, \arg a < \pi$	$\pi a^{-\frac{1}{2}} e^{as} \operatorname{erfc}(a^{\frac{1}{2}} s^{\frac{1}{2}}), \text{ Re } s \geq 0 = \Gamma(n + \frac{1}{2}) / s^{n+\frac{1}{2}}$
6. x for $0 < x < 1$ 1 for $x > 1$	$(1 - e^{-s})/s^2, \text{ Re } s > 0$
7. e^{-ax}	$1/(s+a), \text{ Re } s > -\text{Re } a$
8. xe^{-ax}	$1/(s+a)^2, \text{ Re } s > -\text{Re } a$
9. $(e^{-ax} - e^{-bx})/(b-a)$	$(s+a)^{-1}(s+b)^{-1}, \text{ Re } s > \{-\text{Re } a, -\text{Re } b\}$
10. $(ae^{-ax} - be^{-bx})/(b-a)$	$s(s+a)^{-1}(s+b)^{-1}, \text{ Re } s > \{-\text{Re } a, -\text{Re } b\}$
11. $(e^{ax} - 1)/a$	$s^{-1}(s-a)^{-1}, \text{ Re } s > \text{Re } a$
12. $(e^{ax} - ax - 1)/a^2$	$s^{-2}(s-a)^{-1}, \text{ Re } s > \text{Re } a$
13. $(e^{ax} - \frac{1}{2}a^2x^2 - ax - 1)/a^3$	$s^{-3}(s-a)^{-1}, \text{ Re } s > \text{Re } a$
14. $(1+ax)e^{ax}$	$s(s-a)^{-2}, \text{ Re } s > \text{Re } a$
15. $[1+(ax-1)e^{ax}]/a^2$	$s^{-1}(s-a)^{-2}, \text{ Re } ps > \text{Re } a$
16. $[2+ax+(ax-2)e^{ax}]/a^3$	$s^{-2}(s-a)^{-2}, \text{ Re } s > \text{Re } a$
17. $x^n e^{ax}, n = 0, 1, 2, \dots$	$n!(s-a)^{-(n+1)}, \text{ Re } s > \text{Re } a$
18. $(x + \frac{1}{2}ax^2) e^{ax}$	$s(s-a)^{-3}, \text{ Re } s > \text{Re } a$
19. $(1 + 2ax + \frac{1}{2}a^2x^2) e^{ax}$	$s^2(s-a)^{-3}, \text{ Re } s > \text{Re } a$
20. $\frac{1}{6}x^3 e^{ax}$	$(s-a)^{-4}, \text{ Re } s > \text{Re } a$

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1179

17.13 (continued)

$f(x)$	$F(s)$
21. $(\frac{1}{2}x^2 + \frac{1}{6}ax^3) e^{ax}$	$s(s-a)^{-4}, \text{ Re } s > \text{Re } a$
22. $(x + ax^2 + \frac{1}{6}a^2x^3) e^{ax}$	$s^2(s-a)^{-4}, \text{ Re } s > \text{Re } a$
23. $(1 + 3ax + \frac{3}{2}a^2x^2) e^{ax}$	$s^3(s-a)^{-4}, \text{ Re } s > \text{Re } a$
24. $(ae^{ax} - be^{bx})/(a-b)$	$s(s-a)^{-1}(s-b)^{-1}, \text{ Re } s > \{\text{Re } a, \text{Re } b\}$

ET I 147(37)
 ET I 146(26)
 ET I 146(27)
 ET I 146(28)
 ET I 150(1)
 ET I 154(43)
 ET I 150(2)
 ET I 155(44)
 AS 1022(29.3.19)
 AS 1022(29.3.20)
 AS 1022(29.3.21)

1180

17.13 (continued)

$f(x)$	$F(s)$
39. $[x \sin p(ax)]/(2a)$	$s(s^2 + a^2)^{-2}$, $\text{Re } s > \text{Im } a $ ET I 152(14)
40. $[\sin p(ax) + ax \cos p(ax)]/(2a)$	$s^2(s^2 + a^2)^{-2}$, $\text{Re } s > \text{Im } a $ AS 1023(29.3.23)
41. $x \cos p(ax)$	$(s^2 - a^2)(s^2 + a^2)^{-2}$, $\text{Re } s > \text{Im } a $ ET I 157(57)
42. $[\cos p(ax) - \cos p(bx)]/(b^2 - a^2)$	$s(s^2 + a^2)^{-1}(s^2 + b^2)^{-1}$, $\text{Re } s > \{ \text{Im } a , \text{Im } b \}$ AS 1023(29.3.25)
43. $[\frac{1}{2} a^2 x^2 - 1 + \cos p(ax)]/a^4$	$s^{-3}(s^2 + a^2)^{-1}$, $\text{Re } s > \text{Im } a $
44. $[1 - \cos p(ax) - \frac{1}{2} ax \sin p(ax)]/a^4$	$s^{-1}(s^2 + a^2)^{-2}$, $\text{Re } s > \text{Im } a $
45. $[\frac{1}{b} \sin p(bx) - \frac{1}{a} \sin p(ax)]/(a^2 - b^2)$	$(s^2 + a^2)^{-1}(s^2 + b^2)^{-1}$, $\text{Re } s > \{ \text{Im } a , \text{Im } b \}$
46. $[1 - \cos p(ax) + \frac{1}{2} ax \sin p(ax)]/a^2$	$s^{-1}(s^2 + a^2)^{-2}(25^2 + a^2)$, $\text{Re } s > \text{Im } a $
47. $[a \sin p(ax) - b \sin p(bx)]/(a^2 - b^2)$	$s^2(s^2 + a^2)^{-1}(s^2 + b^2)^{-1}$, $\text{Re } s > \{ \text{Im } a , \text{Im } b \}$
48. $\sin p(a + bx)$	$(s \sin pa + b \cos pa)(s^2 + b^2)^{-1}$, $\text{Re } s > \text{Im } b $
49. $\cos p(a + bx)$	$(s \cos pa - b \sin pa)(s^2 + b^2)^{-1}$, $\text{Re } s > \text{Im } b $
50. $[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sin p(bx)]/(a^2 + b^2)$	$(s^2 - a^2)^{-1}(s^2 + b^2)^{-1}$, $\text{Re } s > \{ \text{Re } a , \text{Im } b \}$

ET I 152(14)
 AS 1023(29.3.23)
 ET I 157(57)
 AS 1023(29.3.25)

1181

17.13 (continued)

$f(x)$	$F(s)$
51. $[\cosh(ax) - \cos p(bx)]/(a^2 + b^2)$	$s(s^2 - a^2)^{-1}(s^2 + b^2)^{-1}$, $\text{Re } s > \{ \text{Re } a , \text{Im } b \}$
52. $[a \sinh(ax) + b \sin p(bx)]/(a^2 + b^2)$	$s^2(s^2 - a^2)^{-1}(s^2 + b^2)^{-1}$

17.13 (continued)

$f(x)$	$F(s)$
66. $x^{\nu-1} \cosh(ax), \operatorname{Re} \nu > 0$	$\frac{1}{2} \Gamma(\nu)[(s-a)^{-\nu} + (s+a)^{-\nu}],$ $\operatorname{Re} s > \operatorname{Re} a $ ET I 164(19)
67. $x \sinh(ax)$	$2as(s^2 - a^2)^{-2}, \operatorname{Re} s > \operatorname{Re} a $
68. $x \cosh(ax)$	$(s^2 + a^2)(s^2 - a^2)^{-2}, \operatorname{Re} s > \operatorname{Re} a $
69. $\sinh(ax) - \sin p(ax)$	$2a^3(s^4 - a^4)^{-1}, \operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} a \}$ AS 1023(29.3.31)
70. $\cosh(ax) - \cos p(ax)$	$2a^2 s(s^4 - a^4)^{-1}, \operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Im} a \}$ AS 1023(29.3.32)
71. $\sinh(ax) + ax \cosh(ax)$	$2as^2(a^2 - s^2)^{-2}, \operatorname{Re} s > \operatorname{Re} a $
72. $ax \cosh(ax) - \sinh(ax)$	$2a^3(a^2 - s^2)^{-2}, \operatorname{Re} s > \operatorname{Re} a $
73. $x \sinh(ax) - \cosh(ax)$	$s(a^2 + 2a - s^2)(a^2 - s^2)^{-2}, \operatorname{Re} s > \operatorname{Re} a $
74. $[\frac{1}{a} \sinh(ax) - \frac{1}{b} \sinh(bx)] / (a^2 - b^2)$	$(a^2 - s^2)^{-1}(b^2 - s^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
75. $[\cosh(ax) - \cosh(bx)] / (a^2 - b^2)$	$s(s^2 - a^2)^{-1}(s^2 - b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
76. $[a \sinh(ax) - b \sinh(bx)] / (a^2 - b^2)$	$s^2(s^2 - a^2)^{-1}(s^2 - b^2)^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
77. $\sinh(a + bx)$	$(b \cosh a + s \sinh a)(s^2 - b^2)^{-1},$ $\operatorname{Re} s > \operatorname{Re} b $
78. $\cosh(a + bx)$	$(s \cosh a + b \sinh a)(s^2 - b^2)^{-1},$ $\operatorname{Re} s > \operatorname{Re} b $
79. $\sinh(ax) \sinh(bx)$	$2abs[s^2 - (a+b)^2]^{-1}[s^2 - (a-b)^2]^{-1},$ $\operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$

AS 1022(29.3.13)
 ET I 164(19)
 AS 1023(29.3.31)
 AS 1023(29.3.32)

17.13 (continued)

$f(x)$	$F(s)$
80. ⁸ $\cosh(ax) \cosh(bx)$	$s(s^2 - a^2 - b^2)[s^2 - (a+b)^2]^{-1}$ $[s^2 - (a-b)^2]^{-1}, \operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
81. $\sinh(ax) \cosh(bx)$	$a(s^2 - a^2 + b^2)[s^2 - (a+b)^2]^{-1}$ $[s^2 - (a-b)^2]^{-1}, \operatorname{Re} s > \{ \operatorname{Re} a , \operatorname{Re} b \}$
82. $\sinh^2(ax)$	$2a^2 s^{-1}(s^2 - 4a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $
83. $\cosh^2(ax)$	$(s^2 - 2a^2)s^{-1}(s^2 - 4a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $
84. $\sinh(ax) \cosh(ax)$	$a(s^2 - 4a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $
85. $[\cosh(ax) - 1]/a^2$	$s^{-1}(s^2 - a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $
86. $[\sinh(ax) - ax]/a^3$	$s^{-2}(s^2 - a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $
87. $[\cosh(ax)$	$s^{-3}(s^2 - a^2)^{-1}, \operatorname{Re} s > \operatorname{Re} a $

17.13 (continued)

$f(x)$	$F(s)$	
97. $\text{Si}(x) \equiv \int_0^x \frac{\sin p\xi}{\xi} d\xi$ $\equiv \frac{1}{2} \pi + \text{si}(x)$	$s^{-1} \operatorname{arccot} s, \operatorname{Re} s > 0$	ET I 177(17)
98. $\text{Ci}(x) \equiv \text{ci}(x)$ $\equiv -\int_x^\infty \frac{\cos p\xi}{\xi} d\xi$	$-\frac{1}{2} s^{-1} \ln(1+s^2), \operatorname{Re} s > 0$	ET I 178(19)
99. ⁸ $\operatorname{erf}\left(\frac{x}{2a}\right)$	$s^{-1} e^{a^2 s^2} \operatorname{erfc}(as), \operatorname{Re} s > 0, \arg a < \pi/4$	ET I 176(2)
100. $\operatorname{erf}(a\sqrt{x})$	$as^{-1}(s+a^2)^{-\frac{1}{2}}, \operatorname{Re} s > \{0, -\operatorname{Re} a^2\}$	ET I 176(4)
101. $\operatorname{erfc}(a\sqrt{x})$	$s^{-1}(s+a^2)^{-\frac{1}{2}}[(s+a^2)^{\frac{1}{2}}-a], \operatorname{Re} s > 0$	ET I 177(9)
102. ⁸ $\operatorname{erfc}(a/\sqrt{x})$	$s^{-1} e^{-2a\sqrt{s}}, \operatorname{Re} s > 0, \operatorname{Re} a > 0$	ET I 177(11)
103. $J_\nu(ax)$	$a^\nu [s + (s^2 + a^2)^{\frac{1}{2}}]^{-\nu} (s^2 + a^2)^{-\frac{1}{2}},$ $\operatorname{Re} s > \operatorname{Im} a , \operatorname{Re} \nu > -1$	ET I 182(1)
104. $xJ_\nu(ax)$	$a^\nu [s + \nu(s^2 + a^2)^{\frac{1}{2}}][s + (s^2 + a^2)^{\frac{1}{2}}]^{-\nu}$ $(s^2 + a^2)^{-\frac{3}{2}}, \operatorname{Re} s > \operatorname{Im} a , \operatorname{Re} \nu > -2$	ET I 182(2)
105. $x^{-1}J_\nu(ax)$	$a^\nu \nu^{-1} [s + (s^2 + a^2)^{\frac{1}{2}}]^{-\nu}, \operatorname{Re} s \geq \operatorname{Im} a $	ET I 182(5)
106. $x^n J_n(ax)$	$1 \cdot 3 \cdot 5 \dots (2n-1) a^n (s^2 + a^2)^{-(n+\frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Im} a $	ET I 182(4)
107. $x^\nu J_\nu(ax)$	$2^\nu \pi^{-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) a^\nu (s^2 + a^2)^{-(\nu+\frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Im} a , \operatorname{Re} \nu > -\frac{1}{2}$	ET I 182(7)
108. $x^{\nu+1} J_\nu(ax)$	$2^{\nu+1} \pi^{-\frac{1}{2}} \Gamma(\nu + \frac{3}{2}) a^\nu s (s^2 + a^2)^{-(\nu+\frac{3}{2})},$ $\operatorname{Re} s > \operatorname{Im} a , \operatorname{Re} \nu > -1$	ET I 182(8)
109. $I_\nu(ax)$	$a^\nu (s^2 - a^2)^{-\frac{1}{2}} [s + (s^2 - a^2)^{\frac{1}{2}}]^{-\nu},$ $\operatorname{Re} s > \operatorname{Re} a , \operatorname{Re} \nu > -1$	ET I 195(1)

ET I 177(17)
ET I 178(19)
ET I 176(2)
ET I 176(4)
ET I 177(9)
ET I 177(11)
ET I 182(1)
ET I 182(2)
ET I 182(5)
ET I 182(4)
ET I 182(7)
ET I 182(8)
ET I 195(1)

17.13 (continued)

$f(x)$	$F(s)$	
110. $x^\nu I_\nu(ax)$	$2^\nu \pi^{-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) a^\nu (s^2 - a^2)^{-(\nu+\frac{1}{2})},$ $\operatorname{Re} s > \operatorname{Re} a , \operatorname{Re} \nu > -\frac{1}{2}$	ET I 195(6)
111. $x^{\nu+1} I_\nu(ax)$	$2^{\nu+1} \pi^{-\frac{1}{2}} \Gamma(\nu + \frac{3}{2}) a^\nu s (s^2 - a^2)^{-(\nu+\frac{3}{2})},$ $\operatorname{Re} s > \operatorname{Re} a , \operatorname{Re} \nu > -1$	ET I 196(7)
112. $x^{-1} I_\nu(ax)$	$\nu^{-1} a^\nu [s + (s^2 - a^2)^{\frac{1}{2}}]^{-\nu},$ $\operatorname{Re} s > \operatorname{Re} a , \operatorname{Re} \nu > 0$	ET I 195(4)
113. $\sin p(2a^{\frac{1}{2}} x^{\frac{1}{2}})$	$(\pi a)^{\frac{1}{2}} s^{-\frac{3}{2}} e^{-a/s}, \operatorname{Re} s > 0$	ET I 153(32)
114. $x^{-\frac{1}{2}} \cos p(2a^{\frac{1}{2}} x^{\frac{1}{2}})$	$\pi^{\frac{1}{2}} s^{-\frac{1}{2}} e^{-a/s}, \operatorname{Re} s > 0$	ET I 158(67)
115. $x^{-1} e^{-ax} I_1(ax)$	$[(s+2a)^{\frac{1}{2}} - s^{\frac{1}{2}}][(s+2a)^{\frac{1}{2}} + s^{\frac{1}{2}}]^{-1},$ $\operatorname{Re} s > \operatorname{Re} a $	AS 1024(29.3.52)
116. $x^{-1} J_k(ax)$	$k^{-1} a^{-k} [(s^2 + a^2)^{\frac{1}{2}} - s]^k,$ $\operatorname{Re} s > \operatorname{Im} a , k > -1$	AS 1025(29.3.58)

The **Fourier transform**, also called the **exponential** or **complex Fourier transform**, of the function $f(x)$, denoted by $F(\xi)$, is defined by the integral

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{i\xi x} dx.$$

The functions $f(x)$ and $F(\xi)$ are called a **Fourier transform pair**, and knowledge of either one enables the other to be recovered.

Setting

$$F(\xi) = \mathcal{F}[f(x); \xi],$$

to emphasize the nature of the transform, we have the symbolic inverse result

$$f(x) = \mathcal{F}^{-1}[F(\xi); x].$$

The inversion of the Fourier transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi)e^{-i\xi x} d\xi.$$

17.22 Basic properties of the Fourier transform

1. For a and b arbitrary constants,

$$\mathcal{F}[af(x) + bg(x)] = aF(\xi) + bG(\xi) \quad (\text{linearity}).$$

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2. If $n > 0$ is an integer, and $\lim_{|x| \rightarrow \infty} f^{(r)}(x) = 0$ for $r = 0, 1, \dots, n - 1$ with $f^{(0)}(x) \equiv f(x)$, then

3. The **Fourier convolution** $f * g$ of two functions $f(x)$ and $g(x)$ is defined by the integral

$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi) d\xi,$$

and it has the property $f * g = g * f$, and $f * (g * h) = (f * g) * h$. In terms of the convolution operation.

$$\mathcal{F}[f * g(x); \xi] = F(\xi)G(\xi) \quad (\text{convolution (Faltung) theorem}).$$

17.23* Table of Fourier transform pairs

$f(x)$	$F(\xi)$	
1. 1	$(2\pi)^{\frac{1}{2}} \delta(\xi)$	SU 496
2. ⁷ $\frac{1}{x}$	$(\pi/2)^{\frac{1}{2}} i \operatorname{sgn} \xi$	SU 50
3. $\delta(x)$	$(2\pi)^{-\frac{1}{2}}$	SU 496
4. ⁸ $\delta(ax + b), a, b \in \mathbb{R}, a \neq 0$	$(2\pi)^{-\frac{1}{2}} e^{-ib\xi/a}$	SU 517
5. 1, $ x < a$ 0, $ x > a, a > 0$	$(2/\pi)^{\frac{1}{2}} \xi^{-1} \sin(a\xi)$	
6. ⁸ $\frac{1}{ x }$	divergent	
7. $\frac{1}{ x ^a}, 0 < \operatorname{Re} a < 1$	$\frac{(2/\pi)^{\frac{1}{2}} \Gamma(1-a) \sin\left(\frac{1}{2} a\pi\right)}{ \xi ^{1-a}}$	SN 523
8. ⁸ $e^{iax}, a \in \mathbb{R}$	$(2\pi)^{\frac{1}{2}} \delta(\xi + a)$	SU 517
9. $e^{-a x }, a > 0$	$\frac{a(2/\pi)^{\frac{1}{2}}}{a^2 + \xi^2}$	SU 517
10. ⁷ $xe^{-a x }, a > 0$	$\frac{2ai\xi(2/\pi)^{\frac{1}{2}}}{(a^2 + \xi^2)^2}$	SU 517
11. $ x e^{-a x }, a > 0$	$\frac{(2/\pi)^{\frac{1}{2}} (a^2 - \xi^2)}{(a^2 + \xi^2)^2}$	SU 517
12. $\frac{e^{-a x }}{ x ^{\frac{1}{2}}}, a > 0$	$\frac{[a + (a^2 + \xi^2)^{\frac{1}{2}}]^{\frac{1}{2}}}{x(a^2 + \xi^2)^{\frac{1}{2}}}$	SN 523
13. $e^{-a^2x^2}, a > 0$	$(a\sqrt{2})^{-1} e^{-\xi^2/4a^2}$	SU 517
14. $\frac{1}{a^2 + x^2}, \operatorname{Re} a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} e^{-a \xi }}{a}$	SU 517
15. ⁷ $\frac{x}{a^2 + x^2}, \operatorname{Re} a > 0$	$i \operatorname{sgn} \xi (\pi/2)^{\frac{1}{2}} e^{-a \xi }$	SU 517
16. $\sin(ax^2)$	$\frac{1}{(2a)^{\frac{1}{2}}} \sin\left(\frac{\xi^2}{4a} + \frac{\pi}{4}\right)$	SN 523
17. $\cos(ax^2)$	$\frac{1}{(2a)^{\frac{1}{2}}} \cos\left(\frac{\xi^2}{4a} - \frac{\pi}{4}\right)$	SN 523
18. $e^{-a x } \cos(bx),$ $a > 0, b > 0$	$a(2\pi)^{-\frac{1}{2}} \left[\frac{1}{a^2 + (b+\xi)^2} + \frac{1}{a^2 + (b-\xi)^2} \right]$	
19. $e^{-\frac{1}{2}ax^2} \sin(bx),$ $a > 0$	$\frac{1}{2} ia^{-\frac{1}{2}} \left\{ \exp\left[-\frac{1}{2}(\xi - b)^2/a\right] - \exp\left[-\frac{1}{2}(\xi + b)^2/a\right] \right\}$	

28. $e^{-ax} \log(1 + e^{-x}),$	$-1 < \operatorname{Re} a < 0$	$(\frac{\pi}{2})^{\frac{1}{2}} \frac{\csc(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121 (26)
	$-1 < \operatorname{Re} a < 0$	$(\frac{\pi}{2})^{\frac{1}{2}} \frac{\csc(\pi a - i\xi\pi)}{a - i\xi}$	ET I 121 (27)

SN 523
SU 123
SU 506
ET I 121(26)
ET I 121 (27)

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In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $1/(2\pi)^{\frac{1}{2}}$ employed in our definition of F has not been used in those tables, and that there is a difference of sign between the exponents used in the definitions of the exponential Fourier transform.

17.31 Fourier sine and cosine transforms

The **Fourier sine** and **cosine transforms** of the function $f(x)$, denoted by $F_s(\xi)$ and $F_c(\xi)$, respectively, are defined by the integrals

$$F_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\xi x) dx \quad \text{and} \quad F_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\xi x) dx.$$

The functions $f(x)$ and $F_s(\xi)$ are called a **Fourier sine transform pair**, and the functions $f(x)$ and $F_c(\xi)$ a **Fourier cosine transform pair**, and knowledge of either $F_s(\xi)$ or $F_c(\xi)$ enables $f(x)$ to be recovered.

Setting

$$F_s(\xi) = \mathcal{F}_s[f(x); \xi] \quad \text{and} \quad F_c(\xi) = \mathcal{F}_c[f(x); \xi],$$

to emphasize the nature of the transforms, we have the symbolic inverses

$$f(x) = \mathcal{F}_s^{-1}[F_s(\xi); x] \quad \text{and} \quad f(x) = \mathcal{F}_c^{-1}[F_c(\xi); x].$$

The inversion of the Fourier sine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(\xi) \sin(\xi x) d\xi \quad (x \geq 0),$$

and the inversion of the Fourier cosine transform is accomplished by means of the **inversion integral**

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(\xi) \cos(\xi x) d\xi \quad (x \geq 0).$$

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17.32 Basic properties of the Fourier sine and cosine transforms

1. For a and b arbitrary constants,

$$\mathcal{F}_s[af(x) + bg(x)] = aF_s(\xi) + bG_s(\xi)$$

and

$$\mathcal{F}_c[af(x) + bg(x)] = aF_c(\xi) + bG_c(\xi) \quad (\text{linearity}).$$

2. If $\lim_{x \rightarrow \infty} f^{(r-1)}(x) = 0$ and $\lim_{x \rightarrow \infty} \sqrt{\frac{2}{\pi}} f^{(r-1)}(x) = a_{r-1}$, then denoting the Fourier sine and cosine transforms of $f^{(r)}(x)$ by $F_s^{(r)}$ and $F_c^{(r)}$, respectively,

- (i) $F_c^{(r)}(\xi) = -a_{r-1} + \xi F_s^{(r-1)}$,
- (ii) $F_s^{(r)}(\xi) = -\xi F_c^{(r-1)}$,
- (iii) $F_c^{(2r)}(\xi) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \xi^{2n} + (-1)^r \xi^{2r} F_c(\xi)$,
- (iv) $F_c^{(2r+1)}(\xi) = -\sum_{n=0}^r (-1)^n a_{2r-2n} \xi^{2n} + (-1)^r \xi^{2r+1} F_s(\xi)$,
- (v) $F_s^{(r)}(\xi) = \xi a_{r-2} - \xi^2 F_s^{(r-2)}$,
- (vi)⁶ $F_s^{(2r)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n} + (-1)^r \xi^{2r} F_s(\xi)$,
- (vii) $F_s^{(2r+1)}(\xi) = -\sum_{n=1}^r (-1)^n \xi^{2n-1} a_{2r-2n+1} + (-1)^{r+1} \xi^{2r+1} F_c(\xi)$.

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- 3. (i) $\int_0^{\infty} F_s(\xi) G_s(\xi) \cos(\xi x) d\xi = \frac{1}{2} \int_0^{\infty} g(s)[f(s+x) + f(s-x)] ds$,
- (ii) $\int_0^{\infty} F_c(\xi) G_c(\xi) \cos p(\xi x) d\xi = \frac{1}{2} \int_0^{\infty} g(s)[f(s+x) + f(|x-s|)] ds$
(convolution (Faltung) theorem).

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4. (i) If $F_s(\xi)$ is the Fourier sine transform of $f(x)$, then the Fourier sine transform of $F_s(x)$ is $f(\xi)$.
- (ii) If $F_c(\xi)$ is the Fourier cosine transform of $f(x)$, then the Fourier cosine transform of $F_c(x)$ is $f(\xi)$.
- (iii) If $f(x)$ is an odd function in $(-\infty, \infty)$, then the Fourier sine transform of $f(x)$ in $(0, \infty)$ is $-iF(\xi)$.
- (iv) If $f(x)$ is an even function in $(-\infty, \infty)$, then the Fourier cosine transform of $f(x)$ in $(0, \infty)$ is $F(\xi)$.
- (v) The Fourier sine transform of $f(x/a)$ is $aF_s(a\xi)$.
- (vi) The Fourier cosine transform of $f(x/a)$ is $aF_c(a\xi)$.
- (vii) $\mathcal{F}_s[f(x); \xi] = F_s(|\xi|) \operatorname{sgn} \xi$.

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17.33 Table of Fourier sine transforms

$f(x)$	$F_s(\xi)$	
1. x^{-1}	$(\pi/2)^{\frac{1}{2}}, \xi > 0$	ET I 64(3)
2. $x^{-\nu}, 0 < \operatorname{Re} \nu < 2$	$(2/\pi)^{\frac{1}{2}} \xi^{\nu-1} \Gamma(1-\nu) \cos(\nu\pi/2), \xi > 0$	ET I 68(1)
3. $x^{-\frac{1}{2}}$	$\xi^{-\frac{1}{2}}, \xi > 0$	ET I 64(6)
4. $x^{-\frac{3}{2}}$	$2\xi^{\frac{1}{2}}, \xi > 0$	ET I 64(9)
5. $1, 0 < x < a$ $0, x > a$	$(2/\pi)^{\frac{1}{2}} \xi^{-1} [1 - \cos(a\xi)], \xi > 0$	ET I 63(1)
6. $x^{-1}, 0 < x < a$ $0, x > a$	$(2/\pi)^{\frac{1}{2}} \operatorname{Si}(a\xi), \xi > 0$	ET I 64(4)
7. $(a-x)^{-1}, a > 0$	$(2/\pi)^{\frac{1}{2}} \{ \sin(a\xi) \operatorname{Ci}(a\xi) - \cos(a\xi) [\frac{1}{2}\pi + \operatorname{Si}(a\xi)] \}, \xi > 0$	ET I 64(11)
8. ⁷ $(x^2 + a^2)^{-1}, a > 0$	$(2\pi)^{-\frac{1}{2}} a^{-1} [e^{-a\xi} \operatorname{Ei}(a\xi) - e^{a\xi} \operatorname{Ei}(-a\xi)], \xi > 0$	ET I 65(14)
9. $x(x^2 + a^2)^{-\frac{3}{2}}, \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}} \xi K_0(a\xi), \xi > 0$	ET I 66(27)
10. $x^{-\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{2}}, \operatorname{Re} a > 0$	$\xi^{\frac{1}{2}} I_{\frac{1}{4}}(\frac{1}{2} a\xi) K_{\frac{1}{4}}(\frac{1}{2} a\xi), \xi > 0$	ET I 66(28)
11. ⁷ $x(x^2 + a^2)^{-\nu-\frac{3}{2}}, \operatorname{Re} \nu > -1, \operatorname{Re} a > 0$	$\frac{\xi^{\nu+1}}{\sqrt{2}(2a)^\nu \Gamma(\nu+\frac{3}{2})} K_\nu(a\xi)$	
12. $\frac{x}{a^2+x^2}, \operatorname{Re} a > 0$	$(\frac{\pi}{2})^{\frac{1}{2}} e^{-a\xi}, \xi > 0$	ET I 65(15)
13. $\frac{x}{(a^2+x^2)^2}$	$\sqrt{\pi}/8a^{-1} \xi e^{-a\xi}, \xi > 0$	
14. $x^{-1}(x^2 + a^2)^{-1}, \operatorname{Re} a > 0$	$(\pi/2)^{\frac{1}{2}} a^{-2} (1 - e^{-a\xi}), \xi > 0$	ET I 65(20)
15. $x^{-1} e^{-ax}, \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}} \tan^{-1}(\frac{\xi}{a}), \xi > 0$	ET I 72(2)
16. $x^{\nu-1} e^{-ax}, \operatorname{Re} \nu > -1,$	$(2/\pi)^{\frac{1}{2}} \Gamma(\nu) (a^2 + \xi^2)^{-\nu/2} \sin$	

$f(x)$	$F_s(\xi)$	
17. $e^{-ax}, \operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}} \xi / (a^2 + \xi^2), \xi > 0$	ET I 72(1)
18. $x e^{-ax}, \operatorname{Re} a > 0$	$\frac{(2/\pi)^{\frac{1}{2}} 2a\xi}{(a^2 + \xi^2)^2}, \xi > 0$	ET I 72(3)
19. $x e^{-ax^2}, \arg a < \pi/2$	$(2a)^{-\frac{3}{2}} \xi \exp(-\xi^2/4a), \xi > 0$	ET I 73(19)
20. $\frac{\sin pax}{x}, a > 0$	$\frac{1}{(2\pi)^{\frac{1}{2}}} \log \left \frac{\xi+a}{\xi-a} \right , \xi > 0$	ET I 78(1)
21. $\frac{\sin pax}{x^2}, a > 0$	$\xi \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$ for $0 < \xi < a$ $a \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$ for $a < \xi < \infty, \xi > 0$	ET I 78(2)
22. $\sin\left(\frac{a^2}{x}\right), a > 0$	$a \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \xi^{-\frac{1}{2}} J_1(2a\xi^{\frac{1}{2}}), \xi > 0$	ET I 83(6)
23. $x^{-1} \sin\left(\frac{a^2}{x}\right), a > 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} N_0(2a\xi^{\frac{1}{2}}) + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_0(2a\xi^{\frac{1}{2}})$	ET I 83(7)
24. $x^{-2} \sin\left(\frac{a^2}{x}\right), a > 0$	$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} a^{-1} \xi^{\frac{1}{2}} J_1(2a\xi^{\frac{1}{2}}), \xi > 0$	ET I 83(8)
25. $\operatorname{cosec}(ax), \operatorname{Re} a > 0$	$(\pi/2)^{\frac{1}{2}} a^{-1} \tanh\left(\frac{1}{2} \pi a^{-1} \xi\right), \xi > 0$	ET I 88(2)
26. $\operatorname{ctnh}\left(\frac{1}{2} ax\right) - 1,$ $\operatorname{Re} a > 0$	$(2\pi)^{\frac{1}{2}} a^{-1} \operatorname{ctnh}(\pi a^{-1} \xi) - \xi, \xi > 0$	ET I 88(3)
27. $(1-x^2)^{-1} \sin p(\pi x)$	$(2/\pi)^{\frac{1}{2}} \sin p\xi, 0 \leq \xi \leq \pi$ $0, \pi < \xi$	ET I 78(4)
28. $e^{-ax^2} \sin p(bx), \operatorname{Re} a > 0$	$(2a)^{-\frac{1}{2}} \exp p[-(\xi^2 + b^2)/(4a)] \sinh(b\xi/2a),$ $\xi > 0$	ET I 78(7)
29. $x^{-1} \sin^2(ax), a > 0$	$\pi^{\frac{1}{2}} 2^{-\frac{3}{2}}, 0 < \xi < 2a$ $\pi^{\frac{1}{2}} 2^{-\frac{5}{2}}, \xi = 2a$ $0, 2a < \xi$	ET I 78(8)
30. $\sin(ax^2), a > 0$	$a^{-\frac{1}{2}} \{\cos(\xi^2/4a) C[(2\pi a)^{-\frac{1}{2}} \xi] +$ $+ \sin(\xi^2/4a) S[(2\pi a)^{-\frac{1}{2}} \xi]\}, \xi > 0$	ET I 82(1)

ET I 72(1)
ET I 72(3)
ET I 73(19)
ET I 78(1)
ET I 78(2)
ET I 83(6)
ET I 83(7)
ET I 83(8)
ET I 88(2)
ET I 88(3)
ET I 78(4)
ET I 78(7)
ET I 78(8)
ET I 82(1)

17.33 (continued)

$f(x)$	$F_s(\xi)$	
31. $\cos(ax^2), a > 0$	$a^{-\frac{1}{2}} \{\sin(\xi^2/4a) C[(2\pi a)^{-\frac{1}{2}} \xi] -$ $- \cos(\xi^2/4a) S[(2\pi a)^{-\frac{1}{2}} \xi]\}, \xi > 0$	ET I 83(3)
32. $\operatorname{arctg}(x/a), a > 0$	$(\pi/2)^{\frac{1}{2}} \xi^{-1} e^{-a\xi}, \xi > 0$	ET I 87(3)
33. ⁷ $\operatorname{arctg}(2a/x), \operatorname{Re} a > 0$	$(2\pi)^{\frac{1}{2}} e^{-a\xi} \sinh(a\xi)/\xi, \xi > 0$	ET I 87(8)
34. $x^{-1} \ln px$	$-(\pi/2)^{\frac{1}{2}} (C + \ln p\xi), \xi > 0$	ET I 76(2)
35. $\ln \left \frac{x+a}{x-a} \right , a > 0$	$(2\pi)^{\frac{1}{2}} \xi^{-1} \sin(a\xi), \xi > 0$	ET I 77(11)
36. ⁷ $x^{-1} \ln(1 + a/x^2), a > 0$	$-(2\pi)^{\frac{1}{2}} \operatorname{Ei}(-\xi/a), \xi > 0$	ET I 77(14)
37. $J_0(ax), a > 0$	$0, 0 < \xi < a$	

ET I 99(3)
 ET I 99(4)
 ET I 100(12)
 ET I 100(13)

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_s has not been used in those tables.

17.34 Table of Fourier cosine transforms

$f(x)$	$F_c(\xi)$	
1. $x^{-\nu}, 0 < \text{Re } \nu < 1$	$(\pi/2)^{\frac{1}{2}} [\Gamma(\nu)]^{-1} \sec(\frac{1}{2}\nu\pi) \xi^{\nu-1}, \xi > 0$	ET I 10(1)
2. $1, 0 < x < a$ $0, x > a$	$(2/\pi)^{\frac{1}{2}} \frac{\sin(a\xi)}{\xi}, \xi > 0$	ET I 7(1)
3. $0, 0 < x < a$ $1/x, x > a$	$-(2/\pi)^{\frac{1}{2}} \text{Ci}(a\xi), \xi > 0$	ET I 8(3)
4. $x^{-\frac{1}{2}}, 0 < x < a$ $0, x > a$	$2\xi^{-\frac{1}{2}} C(a\xi), \xi > 0$	ET I 8(5)
5. $0, 0 < x < a$ $x^{-\frac{1}{2}}, x > a$	$2\xi^{-\frac{1}{2}} [\frac{1}{2} - C(a\xi)], \xi > 0$	ET I 8(6)
6. $x^{\nu-1}, 0 < \text{Re } \nu < 1$	$(2/\pi)^{\frac{1}{2}} \Gamma(\nu) \xi^{-\nu} \sin(\frac{1}{2}\nu\pi), \xi > 0$	ET I 10(1)
7. $(x^2 + a^2)^{-1}, \text{Re } a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} e^{-a\xi}}{a}, \xi > 0$	ET I 8(11)
8. $(x^2 + a^2)^{-2}, \text{Re } a > 0$	$\frac{(\pi/2)^{\frac{1}{2}} (1+a\xi)e^{-a\xi}}{2a^3}, \xi > 0$	
9. $(x^2 + a^2)^{-\nu-\frac{1}{2}}, \text{Re } a > 0,$ $\text{Re } \nu > -\frac{1}{2}$	$\sqrt{2} \left(\frac{\xi}{2a}\right)^{\nu} \frac{K_{\nu}(a\xi)}{\Gamma(\nu+\frac{1}{2})}, \xi > 0$	ET I 11(7)
10. $(a^2 - x^2)^{\nu}, 0 < x < a$ $0, x > a$ $\text{Re } \nu > -1$	$2^{\nu} \Gamma(\nu+1) (a/\xi)^{\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}(a\xi), \xi > 0$	ET I 11(8)
11. $0, 0 < x < a$ $(x^2 - a^2)^{-\nu-\frac{1}{2}}, x > a,$ $-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$	$-2^{-(\nu+\frac{1}{2})} \Gamma(\frac{1}{2}-\nu) (\xi/a)^{\nu} Y_{\nu}(a\xi), \xi > 0$	ET I 11(9)
12. $e^{-ax}, \text{Re } a > 0$	$(2/\pi)^{\frac{1}{2}} a(a^2 + \xi^2)^{-1}, \xi > 0$	ET I 14(1)

ET I 10(1)
 ET I 7(1)
 ET I 8(3)
 ET I 8(5)
 ET I 8(6)
 ET I 10(1)
 ET I 8(11)
 ET I 11(7)

17.34 (continued)

$f(x)$	$F_c(\xi)$	
13. $x e^{-ax}$, $\operatorname{Re} a > 0$	$(2/\pi)^{\frac{1}{2}}(a^2 - \xi^2)(a^2 + \xi^2)^{-2}$, $\xi > 0$	ET I 14(5)
14. ⁷ $x^{\nu-1} e^{-ax}$, $\operatorname{Re} a > 0$, $\operatorname{Re} \nu > 0$	$(2/\pi)^{\frac{1}{2}} \Gamma(\nu)(a^2 + \xi^2)^{-\nu/2} \cos \left[\nu \tan^{-1} \left(\frac{\xi}{a} \right) \right]$, $\xi > 0$	ET I 15(7)
15. $x^{-\frac{1}{2}} e^{-ax}$, $\operatorname{Re} a > 0$	$(a^2 + \xi^2)^{-\frac{1}{2}} [(a^2 + \xi^2)^{\frac{1}{2}} + a]^{\frac{1}{2}}$, $\xi > 0$	ET I 14(4)
16. ⁷ $e^{-a^2 x^2}$, $\operatorname{Re} a > 0$	$2^{-\frac{1}{2}} a^{-1} e^{-\xi^2/4a^2}$, $\xi > 0$	ET I 15(11)
17. $x^{-1} e^{-x} \sin px$	$(2\pi)^{-\frac{1}{2}} \tan^{-1} \left(\frac{2}{\xi^2} \right)$, $\xi > 0$	ET I 19(7)
18. $\sin(ax^2)$, $a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos \left(\frac{\xi^2}{4a} \right) - \sin \left(\frac{\xi^2}{4a} \right) \right]$, $\xi > 0$	ET I 23(1)
19. $\cos(ax^2)$, $a > 0$	$\frac{1}{2\sqrt{a}} \left[\cos \left(\frac{\xi^2}{4a} \right) + \sin \left(\frac{\xi^2}{4a} \right) \right]$, $\xi > 0$	ET I 24(7)
20. $x^{-1} \sin(ax)$, $a > 0$	$(\pi/2)^{\frac{1}{2}}$, $\xi < a$ $\frac{1}{2}(\pi/2)^{\frac{1}{2}}$, $\xi = a$ 0, $\xi > a$	ET I 18(1)
21. ⁷ $x^{-2} \sin^2(ax)$, $a > 0$	$(\pi/2)^{\frac{1}{2}} (a - \frac{1}{2}\xi)$, $\xi < 2a$ 0, $2a < \xi$	ET I 19(8)
22. ⁷ $e^{-bx} \sin(ax)$, $a > 0$, $\operatorname{Re} b > 0$	$(2\pi)^{-\frac{1}{2}} \left[\frac{a+\xi}{b^2+(a+\xi)^2} + \frac{a-\xi}{b^2+(a-\xi)^2} \right]$, $\xi > 0$	ET I 19(6)
23. $(x^2 + a^2)^{-2} \sin[b(x^2 + a^2)^{\frac{1}{2}}]$, $a > 0$	$(b/a)(\pi/2)^{\frac{1}{2}} e^{-a\xi}$, $\xi > 0$	ET I 26(29)
24. $(x^2 + a^2)^{-\frac{1}{2}} \sin[b(x^2 + a^2)^{\frac{1}{2}}]$, $a > 0$	$(\pi/2)^{\frac{1}{2}} J_0[a(b^2 - \xi^2)^{\frac{1}{2}}]$, $0 < \xi < b$ 0, $b < \xi$	ET I 26(30)

ET I 14(5)
 ET I 15(7)
 ET I 14(4)
 ET I 15(11)
 ET I 19(7)
 ET I 23(1)
 ET I 24(7)
 ET I 18(1)
 ET I 19(8)
 ET I 19(6)
 ET I 26(29)

17.34 (continued)

$f(x)$	$F_c(\xi)$	
25. $x^{-2} [1 - \cos(ax)]$, $a > 0$	$\pi/2)^{\frac{1}{2}} (a - \xi)$, $\xi < a$ 0, $a < \xi$	ET I 20(16)
26. $e^{-ax^2} \sin(bx^2)$, $\operatorname{Re} a > \operatorname{Im} b $	$2^{-\frac{1}{2}} (a^2 + b^2)^{-\frac{1}{4}} \exp\{-a\xi^2/[4(a^2 + b^2)]\} \times$ $\times \sin \left[\frac{1}{2} \arctan(b/a) - \frac{1}{2} b\xi^2(a^2 + b^2)^{-1} \right]$	

1193

In deriving results for the preceding table from ET I, account has been taken of the fact that the normalization factor $\sqrt{2/\pi}$ employed in our definition of F_c has not been used in those tables.

17.35 Relationships between transforms

The following relationships exist between transforms and they may be used to derive further transform pairs from among the results given in Sections 17.13-17.34. The appropriate sections of the main body of the tables may also be used to extend the list of transform pairs.

17.351

Fourier cosine transform and Laplace transform relationship.

$$\mathcal{F}_c[f(x); \xi] = \frac{1}{\sqrt{2\pi}}\mathcal{L}[f(x); i\xi] + \frac{1}{\sqrt{2\pi}}\mathcal{L}[f(x); -i\xi].$$

17.352

Fourier sine transform and Laplace transform relationship.

$$\mathcal{F}_s[f(x); \xi] = \frac{i}{\sqrt{2\pi}}\mathcal{L}[f(x); i\xi] - \frac{i}{\sqrt{2\pi}}\mathcal{L}[f(x); -i\xi].$$

17.353

Exponential Fourier transform and Laplace transform relationship.

$$\mathcal{F}[f(x); \xi] = \sqrt{2\pi}\mathcal{L}[f(x); -i\xi] + \sqrt{2\pi}\mathcal{L}[f(-x); i\xi].$$

17.41 Mellin transform

The **Mellin transform** of the function $f(x)$, denoted by $f^*(s)$, is defined by the integral

$$f^*(s) = \int_0^{\infty} f(x)x^{s-1} dx.$$

The functions $f(x)$ and $f^*(s)$ are called a **Mellin transform pair**, and knowledge of either one enables the other to be recovered.

1194

The transform exists provided the integral

$$\int_0^{\infty} |f(x)|x^{k-1} dx$$

is bounded for some $k > 0$, and then the inversion of the Mellin transform is accomplished by means of the **inversion integral**

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s} ds,$$

where $c > k$.

Setting

$$f^*(s) = M[f(x); s]$$

to denote the Mellin transform, we have the symbolic expression for the inverse result

$$f(x) = M^{-1}[f^*(s); x].$$

MS 397(6)

17.42 Basic properties of the Mellin transform

1. For a and b arbitrary constants,

$$\mathcal{M}[af(x) + bg(x)] = af^*(s) + bg^*(s) \quad (\text{linearity}).$$

2. If $\lim_{x \rightarrow 0} x^{s-r-1} f^{(r)}(x) = 0$, $r = 0, 1, \dots, n-1$,

$$(i) \quad \mathcal{M}[f^{(n)}(x); s] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n) \quad (\text{transform of a derivative}).$$

$$(ii) \quad \mathcal{M}[x^n f^{(n)}(x); s] = (-1)^n \frac{\Gamma(s+n)}{\Gamma(s)} f^*(s) \quad (\text{transform of a derivative}).$$

SU 267(4.2.5)

SU 267(4.2.3)

3. Denoting the n th repeated integral of $f(x)$ by $I_n f(x)$, where

$$I_n f(x) = \int_0^x I_{n-1} f(u) du,$$

$$(i) \quad \mathcal{M}[I_n f(x); s] = (-1)^n \frac{\Gamma(s)}{\Gamma(n+s)} f^*(s+n) \quad (\text{transform of an integral}).$$

$$(ii) \quad \mathcal{M}[I_n^\infty f(x); s] = \frac{\Gamma(s)}{\Gamma(s+n)} f^*(s+n),$$

SU 269(4.2.15)

where

$$I_n^\infty f(x) = \int_x^\infty I_{n-1}^\infty f(u) du \quad (\text{transform of an integral}).$$

SU 269(4.2.18)

$$4. \quad \mathcal{M}[f(x)g(x); s] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(u)g^*(s-u) du \quad (\text{Mellin convolution theorem}).$$

SU 275(4.4.1)

17.43 Table of Mellin transform pairs

$f(x)$	$F^*(\xi)$	
1. e^{-x}	$\Gamma(s), \text{Re } s > 0$	SU 521(M13)
2. e^{-x^2}	$\frac{1}{2}\Gamma\left(\frac{1}{2}s\right), \text{Re } s > 0$	SU 521(M14)
3. $\cos px$	$\Gamma(s) \cos\left(\frac{1}{2}\pi s\right), 0 < \text{Re } s < 1$	SU 521(M15)
4. $\sin px$	$\Gamma(s) \sin\left(\frac{1}{2}\pi s\right), 0 < \text{Re } s < 1$	SU 521(M16)
5. $(1-x)^{-1}$	$\pi \cot(\pi s), 0 < \text{Re } s < 1$	SU 521(M1)
6. $(1+x)^{-1}$	$\pi \operatorname{cosec}(\pi s), 0 < \text{Re } s < 1$	SU 521(M2)
7. $(1+x^a)^{-b}$	$\frac{\Gamma(s/a)\Gamma(b-s/a)}{a\Gamma(b)}, 0 < \text{Re } s < ab$	SU 521(M3)
8. $\frac{T_n(x)H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{-s}\pi\Gamma(s)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}s + \frac{1}{2}n\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}s - \frac{1}{2}n\right)},$ $\text{Re } s > 0$	SU 521(M4)
9. $\frac{T_n(x^{-1})H(1-x)}{\sqrt{(1-x^2)}}$	$\frac{2^{s-2}\Gamma\left(\frac{1}{2}n + \frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\Gamma(s)},$ $\text{Re } s > n$	SU 521(M5)
10. $P_n(x)H(1-x)$	$\frac{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s + \frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}s - \frac{1}{2}n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}s + \frac{1}{2}n + 1\right)},$ $\text{Re } s > 0$	SU 521(M6)
11. $P_n(x^{-1})H(1-x)$	$\frac{2^{s-1}\Gamma\left(\frac{1}{2}s + \frac{1}{2}n + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}s - \frac{1}{2}n\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(s+1)},$ $\text{Re } s > n$	SU 521(M7)

SU 521(M13)
 SU 521(M14)
 SU 521(M15)
 SU 521(M16)
 SU 521(M1)
 SU 521(M2)
 SU 521(M3)
 SU 521(M4)
 SU 521(M5)
 SU 521(M6)
 SU 521(M7)

17.43 (continued)

$f(x)$	$F^*(\xi)$	
12. $\frac{1+x \cos p\phi}{1-2x \cos p\phi+x^2}$ $-\pi < \phi < \pi$	$\frac{\pi \cos(s\phi)}{\sin(s\pi)}, 0 < \text{Re } s < 1$	SU 521(M11)
13. $\frac{x \sin p\phi}{1-2x \cos p\phi+x^2}$	$\frac{\pi \sin(s\phi)}{\sin(s\pi)}, 0 < \text{Re } s < 1$	SU 521(M12)

17.43 (continued)

$f(x)$	$F^*(\xi)$	
21. $(1 + ax^h)^{-\nu}$ $h > 0, \quad \arg a < \pi$	$h^{-1}a^{-s/h}B(s/h, \nu - (s/h)),$ $0 < \operatorname{Re} s < h \operatorname{Re} \nu$	MS 454
22. $(1 - x^h)^{\nu-1}$ for $0 < x < 1$ for $x > 1$ $h > 0, \quad \operatorname{Re} \nu > 0$	$h^{-1}B(\nu, s/h)$	MS 454
23. $\ln(1 + ax)$ $ \arg a < \pi$	$\pi s^{-1}a^{-s} \operatorname{cosec}(\pi s), \quad -1 < \operatorname{Re} s < 0$	MS 454
24. $\operatorname{arctg} px$	$-\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad -1 < \operatorname{Re} s < 0$	MS 454
25. $\operatorname{arccot} x$	$\frac{1}{2}\pi s^{-1} \sec(\pi s/2), \quad 0 < \operatorname{Re} s < 1$	MS 454
26. $\operatorname{csch}(ax)$ $\operatorname{Re} a > 0$	$a^{-s}2(1 - 2^{-s})\Gamma(s)\zeta(s), \quad \operatorname{Re} s > 1$	MS 454
27. ⁷ $\operatorname{sech}^2(ax)$ $\operatorname{Re} a > 0$	$4a^{-s}(1 - 2^{2-s})\Gamma(s)2^{-s}\zeta(s-1), \quad \operatorname{Re} s > 2$	MS 454
28. $\operatorname{csch}^2(ax)$ $\operatorname{Re} a > 0$	$4a^{-s}\Gamma(s)2^{-s}\zeta(s-1), \quad \operatorname{Re} s > 0$	MS 454
29. $(x^2 + b^2)^{-\frac{1}{2}\nu} \times$ $\times J_\nu[a(x^2 + b^2)^{\frac{1}{2}}]$	$2^{\frac{1}{2}s-1}a^{-\frac{1}{2}s}b^{\frac{1}{2}s-\nu}\Gamma(\frac{1}{2}s)J_{\nu-\frac{1}{2}}(ab)$ $0 < \operatorname{Re} s < \frac{3}{2} + \operatorname{Re} \nu$	MS 454
30. $(a^2 - x^2)^{\frac{1}{2}\nu} \times$ $\times J_\nu[a(b^2 - x^2)^{\frac{1}{2}}]$ $0 < x < a$ $0, \quad x > a$ $\operatorname{Re} \nu > -1$	$2^{\frac{1}{2}s-1}\Gamma(\frac{1}{2}s)b^{-\frac{1}{2}s}a^{\nu+\frac{1}{2}s}J_{\nu+\frac{1}{2}}(ab)$ $\operatorname{Re} s > 0$	MS 455

MS 454
 MS 455

17.43 (continued)

$f(x)$	$F^*(\xi)$	
31. $(a^2 - x^2)^{-\frac{1}{2}\nu} \times$ $\times J_\nu[b(a^2 - x^2)^{\frac{1}{2}}]$ $0 < x < a$ $0 \quad x > a$	$2^{1-\nu}[\Gamma(\nu)]^{-1}a^{\frac{1}{2}s-\nu}b^{-\frac{1}{2}\nu}S_{\nu-1+\frac{1}{2}s, \frac{1}{2}s-\nu}(ab)$ $\operatorname{Re} s > 0$	MS 455
32. $K_\nu(\alpha x)$	$\alpha^{-s}2^{-s}\Gamma(\frac{1}{2}s - \frac{1}{2}\nu)\Gamma(\frac{1}{2}s + \frac{1}{2}\nu),$ $\operatorname{Re} s > \operatorname{Re} \nu $	MS 455

Bibliographic References

The letters and numbers following equations [visible with the **refs** stylesheet selected from the View Main] refer to the sources used by Russian editors. The key to the letters will be found preceding each entry in the Bibliography below. Roman numerals indicate the volume number of a multivolume work. Numbers without parentheses indicate page numbers, numbers in single parentheses refer to equation numbers in the original sources, and numbers in double parentheses denote the number of a table in the source.

Some formulas were changed from their form in the source material. In such cases, the letter *a* appears at the end of the bibliographic references.

As an example we may use the reference to equation 3.354-5:

ET I 118 (1) *a*.

The key below indicates that the book referred to is:

Erdélyi, A. et al., *Tables of Integral Transforms*.

The Roman numerals denotes volume one of the work, 118 is the page on which the formula will be found, (1) refers to the number of the formula in this source, and the *a* indicates that the expression appearing in the source differs in some respect from the formula in this book.

In several cases the editors have used Russian editions of works published in other languages, under such circumstances, because the pagination and numbering of equations may be altered, we have referred the reader only to the original sources and dispensed with page and equation numbers.

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