Studies in Systems, Decision and Control 7

Xiao-Heng Chang

Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems

Springer

Studies in Systems, Decision and Control

Volume 7

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Springer

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ISSN 2198-4182 ISSN 2198-4190 (electronic)
ISBN 978-3-642-55106-2 ISBN 978-3-642-55107-9 (eBook)
DOI 10.1007/978-3-642-55107-9
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2014936741

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

It is well known that robust H_∞ control and filtering are important issues for systems. In recent years, the linear matrix inequality (LMI) technique has been widely used to solve the robust H_∞ control and filtering problems for uncertain linear systems with polytopic uncertain parameters and/or norm bounded uncertain parameters.

Although a large number of design methods have been developed to deal with the robust H_∞ control and filtering problems for both continuous-time and discrete-time uncertain linear systems, the design problem of output feedback H_∞ controllers cannot be formulated in the framework of LMI. In general, the problem can be represented as a bilinear matrix inequality (BMI) problem. However, the BMI problem is nonconvex and difficult to obtain solution. To obtain LMI-based conditions for designing output feedback H_∞ controllers, some studies have to impose constraints on system matrices. In summary, those results are limited and cannot be applied to general control systems.

This monograph aims to present some new results on robust output feedback H_∞ control and filtering for uncertain linear systems. It lists an LMI decoupling approach, and the main results of this monograph are expressed in a unified LMI framework, which will provide an effective foundation for the future research. It is primarily intended for graduate students in control and filtering, but can also serve as a valuable reference material for researchers wishing to explore the area of control and filtering of linear systems.

The background required of the reader is knowledge of basic control system theory, basic Lyapunov stability theory, and basic LMI theory.

Jinzhou, China, March 2014 Xiao-Heng Chang

Acknowledgments

This monograph would not be possible without the work done in the previous results of others. I thank them for their scientific dedication and especially for their influence on my research and on this monograph. It is a great pleasure to express my thanks to those who have been involved in various aspects of research leading to the work.

The author wishes to express his hearty gratitude to advisors Prof. Guang-Hong Yang and Prof. Yuanwei Jing, Northeastern University, China, for directing the research interest of the author to the general area of controls. Special thanks to Prof. Qingling Zhang, Northeastern University, China, for the helpful suggestions on this monograph. I want to thank Prof. Shengyuan Xu at the Nanjing University of Science and Technology, China, Prof. Shaocheng Tong at the Liaoning University of Technology, China, and Prof. Bing Chen at Qingdao University, China, for all the help in my academic research. I am also grateful to Prof. Huijun Gao, Prof. Zhongdang Yu, and Prof. Shen Yin, Bohai University, China, for the support and encouragement the author has had during the writing of this monograph.

Finally, the author would like to express his gratitude to the editor Na Xu at the Springer Beijing Office. Without their appreciation and help, the publication of this book would have not gone so smoothly.

The monograph was supported in part by the National Natural Science Foundation of China (Grant No. 61104071), by the Program for Liaoning Excellent Talents in University, China (Grant No. LJQ2012095), by the Open Program of the Key Laboratory of Manufacturing Industrial Integrated Automation, Shenyang University, China (Grant No. 1120211415).

Contents

Acronyms

LMI	Linear matrix inequality
LMIs	Linear matrix inequalities
BMI	Bilinear matrix inequality

Chapter 1
Introduction and Preview

Abstract This chapter is the introduction and preview to this monograph. First, the background to robust output feedback H_∞ control and filtering is described, in which some existing studies are mentioned. Second, the contribution of this monograph is summarized. Finally, some linear matrix inequality (LMI) technique and matrix property lemmas are given, which are helpful to derive our results.

Keywords Output feedback H_∞ control · H_∞ filtering · Linear matrix inequality (LMI) technique lemmas · Matrix property lemmas

1.1 Introduction

It is well known that almost all existing physical and engineering systems unavoidably include uncertainties and other disturbances due to inaccurate modeling, component aging, measurement errors, exterior conditions, or parameter variations [40]. The term uncertainty refers to the differences or errors between models and reality, and whatever mechanism is used to express these errors will be called a representation of uncertainty [41]. In general, the norm bounded uncertainty is one of the important descriptions of parametric uncertainty; another important description of uncertainty is the so-called polytopic uncertainty. In the past few years, robust control and filtering has become a hot topic in the engineering literature and constitutes an integral part of control systems and signal processing research [27].

Robust H_∞ control is an important branch of control theory. A robust H_∞ control problem for systems with parameter uncertainty can be stated as follows: given a dynamic system with exogenous input and measured output, where the goal is to design a control law such that the L_2 gain of the mapping from the exogenous input to the regulated output is minimized or no larger than some prescribed level for all admissible uncertainties. During the past two decades, the robust H_∞ control problem has attracted great attention from both the academic and industrial communities. A

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_1, © Springer-Verlag Berlin Heidelberg 2014

great number of results on H_∞ control have been reported in the open literature. The problem of robust state feedback H_∞ control design for a class of linear time-invariant systems with parameter uncertainty in the state-space model was investigated in [34], where uncertain systems with time-invariant norm-bounded parameter uncertainty in the state matrix were considered. Robust H_∞ control design for linear systems with uncertainty in both the state and input matrices was treated in [35], where a state feedback control design that stabilizes the plant and guarantees an H_∞-norm bound constraint on disturbance attenuation for all admissible uncertainties was presented. Montagner and Peres [28] addressed the synthesis of H_∞ parameter-dependent state feedback controllers for linear time-varying systems in polytopic domains by means of linear matrix inequalities (LMIs). Xu et al. [37] dealt with problems of robust stabilization and robust H_∞ control for discrete stochastic systems with time-varying delays and time-varying norm-bounded parameter uncertainties. Shaked [31] derived stability criteria and a bounded-real-lemma (BRL) representation for linear systems with real convex polytopic uncertainties and the method was extended to the problem of H_∞ control. He et al. [20] presented a simple technique for BRL representation and concerned the H_∞ control problem of linear systems with real convex polytopic uncertainties.

The aforementioned robust H_∞ control studies are given in terms of state feedback, which follow the assumption that the system states are available for controller implementations. However, the assumption is not true in many practical cases since the states are often unavailable. For the output feedback H_∞ control, it can be considered through three approaches. The first one is called static output feedback H_∞ control. This is interesting to reduce real-time computational cost when implementing practical applications. The second approach to address the problem of output feedback H_∞ control is to use a dynamic output feedback compensator. This kind of feedback control is a good way to improve the closed-loop transient response. Finally, the third way is the so-called observer-based H_∞ control. This is based on the introduction of a state observer and is interesting when the state is not entirely available from measurements. For the robust static output feedback H_∞ control problem, a great number of control synthesis results for the uncertain linear systems in both the continuous-time and discrete-time contexts have been extensively discussed in the literature. In [8], by inserting an equality constrained condition about Lyapunov matrix, LMI conditions for solving static output feedback control problem of linear continuous- and discrete-time systems were given, and the result can be extended to design H_∞ controllers for uncertain linear systems. In [10], by introducing a slack variable with sub-triangle structure, LMI-based condition for designing robust static output feedback H_∞ controllers for linear systems with time-invariant uncertainties were proposed. For dynamic output feedback H_∞ control, a standard LMI design method is the technique of change of variables [7, 16]. However, it is well known that the standard technique cannot design robust dynamic output feedback H_∞ controllers via LMI. This is due to the fact that to linearize the matrix inequality the introduced new variables will have to be vertex-dependent and involve the controller parameters to be sought, which implies that the required controller parameters cannot be computed from the introduced variables. To overcome this difficulty of the so-called standard approach for designing dynamic output feedback H_∞ controller, an LMI technique was developed in [26] which involves solving two LMIs in conjunction with a line search. In [26], a technical lemma was used to deal with the nonlinear term $(\bar{P}^{(i)})^{-1}$, which leads to significant conservativeness (see [26] for details). It should be noted that [26] had applied the sequentially linear programming method (SLPMM) to further reduce the design conservatism, which is an LMI approximation method. However, in many cases, this kind of approximation method leads to infeasibility of the optimization, even though there exists a solution [21]. In [13, 21], sufficient conditions were suggested, which are significantly less conservative, for dynamic output feedback H_∞ control of linear discrete-time systems. The structural restriction imposed on a Lyapunov variable is bypassed by employing auxiliary slack variables with structure. It should be pointed out that the design approaches given in [13, 21] are applicable to design robust dynamic output feedback H_∞ controllers. For the robust output feedback H_∞ control problem, most of the study focuses on the static and dynamic output feedback H_∞ controls, and few attempts have been made on observer-based H_∞ control design. In [24], two useful methodologies were adopted to design observer-based non-fragile H_∞ control for continuous-time systems. The control and observer gain matrices are found directly from LMI optimization formulation by imposing equality constrained conditions. These equality constraints also appear in [23] for observer-based control. Although a lot of research on output feedback H_∞ control has been reported, the design problem of output feedback H_∞ controllers cannot be formulated in the framework of LMI. In general, the problem can be represented as a bilinear matrix inequality (BMI) problem. However, the BMI problem is nonconvex and known to be NP-hard [2]. To obtain LMI-based conditions for designing output feedback H_∞ controllers, some studies have to impose constraints on system matrices. In summary, these results are limited and cannot be applied to general control systems.

On the other hand, the problem of H_∞ filtering is of both theoretical and practical importance in control and signal processing. In comparison with traditional Kalman filtering [32], the advantage of using H_∞ filtering is that no statistical assumptions on the exogenous signals are needed. Moreover, the H_∞ filtering technique provides both a guaranteed noise attenuation level and robustness against unmodeled dynamics [29]. When there exist parameter uncertainties in the system's model, robust H_∞ filtering can provide a powerful signal estimation. It designs an asymptotically stable filter, based on an uncertain signal model, which ensures that the filtering error system is asymptotically stable and that the L_2-induced gain from the noise signals to the filtering error remains bounded by a prescribed level for all allowed uncertainties [39]. In recent years, there are considerable studies on the H_∞ filtering problem for dynamic systems. In [18], the problem of H_∞ filtering for a class of linear uncertain systems was studied, where the parameter uncertainties are assumed to reside in a polytope. Xie et al. [36] was concerned with the robust H_∞ filtering problems for linear discrete-time systems with polytopic parameter uncertainty, and soon after their results were extended by Chang and Yang [5] and Duan et al. [14]. The problem of robust H_∞ filtering for uncertain Markovian jump linear systems mode was studied in [38]. In [15], robust H_∞ filtering of complex nonlinear systems which can be represented by a fuzzy dynamic model was presented.

In this monograph, the author puts forward some new results on robust output feedback H_∞ control and filtering for the both continuous-time and discrete-time uncertain linear systems. This monograph comprises three aspects.

(1) By applying an LMI decoupling approach, three types of robust output feedback H_∞ controllers are designed. Especially, the proposed design conditions for the three types of output feedback controllers are given by strict LMI representations, under which the prescribed H_∞ performances of the closed-loop systems are guaranteed. The presented approach can solve effectively the BMI problem in the existing literature for output feedback H_∞ controller design, and the constraints imposed on system matrices have been avoided. In addition, by theoretical proof, it can be shown that the proposed design conditions include some LMI results as special cases.
(2) The problem of robust H_∞ filtering is studied for discrete-time uncertain systems based on the parameter-dependent Lyapunov function approach. With the introduction of some auxiliary matrix variables, sufficient conditions for H_∞ filter design are proposed in terms of LMIs, which guarantee the filtering error systems to be asymptotically stable and have prescribed H_∞ performances. The theoretical proof shows that the proposed conditions can provide less conservatism than some existing results in the literature. In addition, this monograph also concerns the application of the LMI decoupling approach for designing robust H_∞ filters.
(3) This monograph also studies the problems of output feedback H_∞ control and filtering for linear systems with other types of uncertainties. Different from existing results for H_∞ control and filtering, the proposed ones are toward systems with feedback uncertainties and Frobenius norm-bounded uncertainties. Sufficient conditions for the output feedback H_∞ controllers and filters design are presented in terms of solutions of a set of LMIs. The resulting design is such that the closed-loop system (filtering error system) has a prescribed H_∞ performance with respect to the uncertainties.

Finally, numerical examples will be provided to illustrate the effectiveness of the proposed design methods.

1.2 Problem Formulation and Preliminaries

1.2.1 Output Feedback H_∞ Control

The objective of H_∞ control is to find an asymptotically stable output feedback controller such that two conditions are satisfied:

For continuous-time case

(1) The closed-loop system is asymptotically stable when $w(t) = 0$.
(2) The closed-loop system has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition

$$\int_0^\infty z^T(t)z(t)\mathrm{d}t \le \gamma^2 \int_0^\infty w^T(t)w(t)\mathrm{d}t,$$

is satisfied for any nonzero $w(t) \in L_2[0,\ \infty)$.

For discrete-time case

(1) The closed-loop system is asymptotically stable when $w(k) = 0$.
(2) The closed-loop system has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition

$$\sum_{k=0}^\infty z^T(k)z(k) < \gamma^2 \sum_{k=0}^\infty w^T(k)w(k),$$

is satisfied for any nonzero $w(k) \in l_2[0,\ \infty)$.

where $z(t)$ $[z(k)]$ and $w(t)$ $[w(k)]$ denote the system controlled output variable and noise signal, respectively.

1.2.2 H_∞ Filtering

The objective of H_∞ filtering is to find an asymptotically stable filter such that two conditions are satisfied:

For continuous-time case

(1) The filtering error system is asymptotically stable when $w(t) = 0$.
(2) The filtering error system has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition

$$\int_0^\infty e^T(t)e(t)\mathrm{d}t \le \gamma^2 \int_0^\infty w^T(t)w(t)\mathrm{d}t,$$

is satisfied for any nonzero $w(t) \in L_2[0,\ \infty)$.

For discrete-time case

(1) The filtering error system is asymptotically stable when $w(k) = 0$.
(2) The filtering error system has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition

$$\sum_{k=0}^{\infty} e^T(k)e(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k),$$

is satisfied for any nonzero $w(k) \in l_2[0, \infty)$.

where $e(t)$ $[e(k)]$ and $w(t)$ $[w(k)]$ denote the system filtering error and noise signal, respectively.

1.2.3 LMI and Matrix Properties

The following preliminary lemmas will be used in this sequel:

Lemma 1.1 *Schur Complement* [1]: *Matrices* $P > 0$, Y *and* A *being appropriate dimensions and with* Y *symmetrical. Then*

$$Y + A^T P^{-1} A < 0 \Leftrightarrow \begin{bmatrix} Y & * \\ A & -P \end{bmatrix} < 0.$$

Lemma 1.2 *Congruence* [11]: *Let* X *be a full row or full column rank matrix. If* $Y < 0$, *then*

$$X^T Y X < 0.$$

Lemma 1.3 [42]: *For matrices* X, Y, *and* $J > 0$ *with appropriate dimensions, the following inequality holds*

$$XY + Y^T X^T \leq XJX^T + Y^T J^{-1} Y.$$

Lemma 1.4 [6]: *For matrices* T, P, L, *and* A *with appropriate dimensions and scalar* β, *let there be the following condition*

$$\begin{bmatrix} T & * \\ LA & -\beta L - \beta L^T + \beta^2 P \end{bmatrix} < 0,$$

then, we have

$$T + A^T PA < 0.$$

Lemma 1.5 [19, 22]: *Let the matrices N_{ij} and the condition be*

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j N_{ij} = \sum_{i=1}^{r}\theta_i^2 N_{ii} + \sum_{i=1}^{r}\sum_{i<j}^{r}\theta_i\theta_j(N_{ij}+N_{ji}) < 0. \tag{1.1}$$

Equation (1.1) is true if there exist matrices Υ_{ii} and Υ_{ji} such that the following conditions are fulfilled

$$N_{ii} < \Upsilon_{ii}, \quad i = 1, 2, \ldots, r,$$

$$N_{ij} + N_{ji} < \Upsilon_{ji} + \Upsilon_{ji}^T, \quad i, j = 1, 2, \ldots, r, \quad i < j,$$

$$\begin{bmatrix} \Upsilon_{11} & * & \ldots & * \\ \Upsilon_{21} & \Upsilon_{22} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r1} & \Upsilon_{r2} & \ldots & \Upsilon_{rr} \end{bmatrix} < 0.$$

Lemma 1.6 [4]: *From (1.2), we can obtain (1.3)*

$$\begin{bmatrix} T + A^T M^T + MA & * \\ -M^T + GA & -G - G^T + P \end{bmatrix} < 0. \tag{1.2}$$

$$T + A^T PA < 0. \tag{1.3}$$

Remark 1.1 If the matrix variables M and G are free, then the two matrix inequalities are equivalent [11].

Lemma 1.7 *From (1.4), we have (1.5)*

$$\begin{bmatrix} -V - V^T & * & * & * & * \\ A^T V^T + P & -2P + X & * & * & * \\ B^T V^T & 0 & -R & * & * \\ 0 & C & D & -S & * \\ V^T & 0 & 0 & 0 & -X \end{bmatrix} < 0. \tag{1.4}$$

$$\begin{bmatrix} PA + A^T P & * & * \\ B^T P & -R & * \\ C & D & -S \end{bmatrix} < 0. \tag{1.5}$$

Proof Note that

$$-(V - P)P^{-1}(V - P)^T \le 0, \;\; P > 0, \tag{1.6}$$

implies that

$$-VP^{-1}V^T \le -V - V^T + P. \tag{1.7}$$

Then, from (1.4), we have

$$\begin{bmatrix} -V - V^T & * & * & * & * \\ A^T V^T + P & -PX^{-1}P & * & * & * \\ B^T V^T & 0 & -R & * & * \\ & C & D & -S & * \\ {}^T & 0 & 0 & 0 & -X \end{bmatrix} < 0. \tag{1.8}$$

By using Schur complement to (1.8), one gives

$$\begin{bmatrix} -V - V^T + VX^{-1}V^T & * & * & * \\ A^T V^T + P & -PX^{-1}P & * & * \\ B^T V^T & 0 & -R & * \\ 0 & C & D & -S \end{bmatrix} < 0. \tag{1.9}$$

Obviously, if there exist matrices V and P satisfying (1.4), it implies that these matrices are nonsingular. Pre- and post-multiplying (1.9) by $\begin{bmatrix} V^{-1} & 0 & 0 & 0 \\ 0 & P^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, it yields

$$\begin{bmatrix} -V^{-1} - V^{-T} + X^{-1} & * & * & * \\ P^{-1}A^T + V^{-T} & -X^{-1} & * & * \\ B^T & 0 & -R & * \\ 0 & CP^{-1} & D & -S \end{bmatrix} < 0. \tag{1.10}$$

Applying LMI congruence property in Lemma 1.2 to (1.10) with the full row rank matrix $\begin{bmatrix} I & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, it follows that

$$\begin{bmatrix} AP^{-1} + P^{-1}A^T & * & * \\ B^T & -R & * \\ CP^{-1} & D & -S \end{bmatrix} < 0. \tag{1.11}$$

Pre- and post-multiplying (1.11) by $\begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, (1.5) can be obtained. $\square$

Lemma 1.8 *From* (1.12)*, we have* (1.5)

$$\begin{bmatrix} -V-V^T & * & * & * & * \\ AV+Q & -2Q+X & * & * & * \\ 0 & B^T & -R & * & * \\ CV & 0 & D & -S & * \\ V & 0 & 0 & 0 & -X \end{bmatrix} < 0. \tag{1.12}$$

Proof Similar to the proof of Lemma 1.7, from (1.12), we have

$$\begin{bmatrix} -V^{-1}-V^{-T}+X^{-1} & * & * & * \\ Q^{-1}A+V^{-1} & -X^{-1} & * & * \\ 0 & B^TQ^{-1} & -R & * \\ C & 0 & D & -S \end{bmatrix} < 0. \tag{1.13}$$

The matrix inequality (1.13) implies that

$$\begin{bmatrix} Q^{-1}A+A^TQ^{-1} & * & * \\ B^TQ^{-1} & -R & * \\ C & D & -S \end{bmatrix} < 0. \tag{1.14}$$

By defining $Q^{-1}=P$, (1.5) is obtained. □

Lemma 1.9 [3]: *From* (1.15)*, we can obtain* (1.16)

$$\begin{bmatrix} T+A^TM^T+MA & * \\ P^T-M^T+GA & -G-G^T \end{bmatrix} < 0. \tag{1.15}$$

$$T+A^TP^T+PA<0. \tag{1.16}$$

Remark 1.2 If the matrix variables M and G are free, then the two matrix inequalities are equivalent [11].

Lemma 1.10 *For matrices T, P, S, and A with appropriate dimensions and scalar β, from* (1.17)*, we can obtain* (1.18)

$$\begin{bmatrix} T & * \\ \beta P^T+SA & -\beta S-\beta S^T \end{bmatrix} < 0. \tag{1.17}$$

$$T+A^TP^T+PA<0. \tag{1.18}$$

Proof Pre- and post-multiplying (1.17) by the full row rank matrix $[\ I \quad \frac{1}{\beta}A^T\ ]$ and its transpose, respectively, the inequality (1.18) can be obtained. □

Lemma 1.11 [30]: *Let X, Y, and Δ be real matrices with appropriate dimensions and $\Delta^T\Delta \le I$. Then, for any scalar $\varepsilon > 0$*

$$X\Delta Y + Y^T\Delta^T X^T \le \frac{1}{\varepsilon}XX^T + \varepsilon Y^T Y.$$

Lemma 1.12 [30]: *Let X, Y, and Δ be real matrices with appropriate dimensions and $\Delta^T\Delta \le I$. Then, for any scalar $\delta > 0$*

$$X\Delta Y + Y^T\Delta^T X^T \le \delta XX^T + \frac{1}{\delta}Y^T Y.$$

Lemma 1.13 *Let X be a square nonsingular matrix and partition*

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

such that X_{11} and X_{22} are nonsingular. Then

$$X^{-1} = \begin{bmatrix} \left(X_{11} - X_{12}X_{22}^{-1}X_{21}\right)^{-1} & -X_{11}^{-1}X_{12}\left(X_{22} - X_{21}X_{11}^{-1}X_{12}\right)^{-1} \\ -X_{22}^{-1}X_{21}\left(X_{11} - X_{12}X_{22}^{-1}X_{21}\right)^{-1} & \left(X_{22} - X_{21}X_{11}^{-1}X_{12}\right)^{-1} \end{bmatrix}.$$

Lemma 1.14 [9, 12]: *Given matrices $\nu \in \mathscr{R}^n$, $\Theta = \Theta^T \in \mathscr{R}^{n\times n}$, and $N \in \mathscr{R}^{m\times n}$, if rank$(N) < n$, then*

$$\nu^T\Theta\nu < 0, \quad \forall \quad N\nu = 0, \quad \nu \ne 0, \tag{1.19}$$

if and only if there exists matrix $L \in \mathscr{R}^{n\times m}$ such that

$$\Theta + LN + N^T L^T < 0. \tag{1.20}$$

Lemma 1.15 [17]: *Given a symmetric matrix Ψ and two matrices P and Q, the problem*

$$\Psi + P^T X^T Q + Q^T XP < 0, \tag{1.21}$$

is solvable with respect to decision matrix X if and only if

$$P^{\perp T}\Psi P^{\perp} < 0, \quad Q^{\perp T}\Psi Q^{\perp} < 0, \tag{1.22}$$

where $P^{\perp}$ *and* $Q^{\perp}$ *denote arbitrary bases of the nullspaces of* P *and* Q*, respectively, i.e.,*

$$PP^{\perp} = 0, \quad QQ^{\perp} = 0. \tag{1.23}$$

Lemma 1.16 *Inversion Matrix Lemma: Let* $\bar{A}$, $\bar{B}$, $\bar{C}$*, and* $\bar{D}$ *be matrices with appropriate dimensions. Then*

$$(\bar{A} + \bar{B}\bar{C}\bar{D})^{-1} = \bar{A}^{-1} - \bar{A}^{-1}\bar{B}\left(\bar{C}^{-1} + \bar{D}\bar{A}^{-1}\bar{B}\right)^{-1}\bar{D}\bar{A}^{-1}.$$

Lemma 1.17 *Let* $\mathscr{T} > 0$, H, E, N*, and* $\Delta(k)$ *be real matrices with appropriate dimensions and* $\Delta^T(k)\Delta(k) \le I$*. Then, for any scalar* $\varepsilon > 0$

$$-N\left(I - EH\Delta(k)\right)\mathscr{T}^{-1}\left(I - EH\Delta(k)\right)^T N^T \le \mathscr{T} - N - N^T + \frac{1}{\varepsilon}NEHH^T E^T N^T + \varepsilon I.$$

Proof From (1.6) and (1.7), we have

$$\begin{aligned}&-N\left(I - EH\Delta(k)\right)\mathscr{T}^{-1}\left(I - EH\Delta(k)\right)^T N^T\\ &\le \mathscr{T} - N - N^T + NEH\Delta(k) + \Delta^T(k)H^T E^T N^T.\end{aligned}$$

By Lemma 1.11, it follows that

$$\begin{aligned}&\mathscr{T} - N - N^T + \underbrace{NEH}_{X}\Delta(k)\underbrace{I}_{Y} + \underbrace{I}_{Y^T}\Delta^T(k)\underbrace{H^T E^T N^T}_{X^T}\\ &\le \mathscr{T} - N - N^T + \tfrac{1}{\varepsilon}NEHH^T E^T N^T + \varepsilon I.\end{aligned}$$

□

Lemma 1.18 *Let* $\mathscr{T} > 0$, H, E, N*, and* $\Delta(k)$ *be real matrices with appropriate dimensions and* $\Delta^T(k)\Delta(k) \le I$*. Then, for any scalar* $\varepsilon > 0$

$$-N^T\left(I - \Delta(k)EH\right)^T\mathscr{T}^{-1}\left(I - \Delta(k)EH\right)N \le \mathscr{T} - N - N^T + \frac{1}{\varepsilon}N^T H^T E^T EHN + \varepsilon I.$$

Proof The proof can be directly obtained from the proof of Lemma 1.17 and Lemma 1.12.

Lemma 1.19 (Frobenius Norm-Bounded Property) [33]: *Let* $\Delta(k)$ *be uncertain matrix formulated as*

$$\Delta(k) = \sum_{q=1}^{m}\sum_{s=1}^{n} M_q \Delta_{qs}(k) N_s,$$

M_q and N_s are constant matrices with appropriate dimensions and

$$\bar{\Delta}(k) = \begin{bmatrix} \Delta_{11}(k) & \Delta_{12}(k) & \dots & \Delta_{1n}(k) \\ \Delta_{21}(k) & \Delta_{22}(k) & \dots & \Delta_{2n}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m1}(k) & \Delta_{m2}(k) & \dots & \Delta_{mn}(k) \end{bmatrix},$$

is an unknown real time-varying matrix satisfying

$$\sum_{q=1}^{m} \sum_{s=1}^{n} \| \Delta_{qs}(k) \| \leq 1, \ k > 0. \tag{1.24}$$

Then, for all vectors ζ

$$\max_{\|\Delta_N\|_F \leq 1} \zeta^T X \left(\sum_{q=1}^{m} \sum_{s=1}^{n} M_q \Delta_{qs}(k) N_s \right) Y \zeta$$
$$= \sqrt{\zeta^T X \left(\sum_{q=1}^{m} M_q M_q^T \right) X^T \times \zeta^T Y^T \left(\sum_{s=1}^{n} N_s^T N_s \right) Y \zeta}.$$

Lemma 1.20 *Given $M_{\mu q}$, $N_{\mu s}$, $\mu = 1, 2, \dots, r$, and Υ of appropriate dimensions with Υ being symmetric. Then*

$$\Upsilon + X_1 \left(\sum_{q=1}^{m_1} \sum_{s=1}^{n_1} M_{1q} \Delta_{1qs}(k) N_{1s} \right) Y_1 + Y_1^T \left(\sum_{q=1}^{m_1} \sum_{s=1}^{n_1} M_{1q} \Delta_{1qs}(k) N_{1s} \right)^T X_1^T + \cdots$$
$$+ X_r \left(\sum_{q=1}^{m_r} \sum_{s=1}^{n_r} M_{rq} \Delta_{rqs}(k) N_{rs} \right) Y_r + Y_r^T \left(\sum_{q=1}^{m_r} \sum_{s=1}^{n_r} M_{rq} \Delta_{rqs}(k) N_{rs} \right)^T X_r^T < 0, \tag{1.25}$$

holds for all $\Delta_{1qs} \ \dots \ \Delta_{rqs}$ satisfying (1.24) if there exist constants $\varepsilon_1 \ \dots \ \varepsilon_r$ such that the following LMI holds:

$$
\begin{bmatrix}
\Upsilon & * & \cdots & & & & & \cdots & & & \cdots & * \\
M_{11}^T X_1^T & \varepsilon_1^- I & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & \vdots \\
M_{1m_1}^T X_1^T & 0 & \cdots & 0 & \varepsilon_1^- I & & \ddots & & \ddots & & & \\
\vdots & \vdots & & & & \ddots & & & & & & \\
M_{r1}^T X_r^T & 0 & & \cdots & 0 & \varepsilon_r^- I & & & & & & \\
\vdots & \vdots & & & & & \ddots & & \ddots & & \ddots & \vdots \\
M_{rm_r}^T X_r^T & 0 & & \cdots & & \cdots & 0 & \varepsilon_r^- I & & & & \\
\varepsilon_1 N_{11} Y_1 & 0 & & \cdots & & & \cdots & 0 & \varepsilon_1^- I & & & \\
\vdots & \vdots & & & & & & & \ddots & & \ddots & \\
\varepsilon_1 N_{1n_1} Y_1 & 0 & & \cdots & & \cdots & & \cdots & 0 & \varepsilon_1^- I & & \vdots \\
\vdots & \vdots & & & & & & & & \ddots & & \\
\varepsilon_r N_{r1} Y_r & 0 & & \cdots & & \cdots & & \cdots & & \cdots & 0 \;\; \varepsilon_r^- I & \\
\vdots & \vdots & & & & & & & & & \ddots & * \\
\varepsilon_r N_{rn_r} Y_r & 0 & & \cdots & & \cdots & & \cdots & & & \cdots \;\; 0 & \varepsilon_r^- I
\end{bmatrix} < 0, \tag{1.26}
$$

where $\varepsilon_\mu^- = -\varepsilon_\mu, \; \mu = 1, 2, \ldots, r.$

Proof By using Schur complement to (1.26), it leads to

$$
\begin{aligned}
\Upsilon + \epsilon_1^{-1} X_1 \left(\sum_{q=1}^{m_1} M_{1q} M_{1q}^T \right) X_1^T + \epsilon_1 Y_1^T \left(\sum_{s=1}^{n_1} N_{1s}^T N_{1s} \right) Y_1 + \cdots \\
+ \epsilon_r^{-1} X_r \left(\sum_{q=1}^{m_r} M_{rq} M_{rq}^T \right) X_r^T + \epsilon_r Y_r^T \left(\sum_{s=1}^{n_r} N_{rs}^T N_{rs} \right) Y_r < 0.
\end{aligned}
$$

For any $\zeta \neq 0$, we have

$$
\begin{aligned}
\zeta^T \Upsilon \zeta + \varepsilon_1^{-1} \zeta^T X_1 \left(\sum_{q=1}^{m_1} M_{1q} M_{1q}^T \right) X_1^T \zeta + \varepsilon_1 \zeta^T Y_1^T \left(\sum_{s=1}^{n_1} N_{1s}^T N_{1s} \right) Y_1 \zeta + \cdots \\
+ \varepsilon_r^{-1} \zeta^T X_r \left(\sum_{q=1}^{m_r} M_{rq} M_{rq}^T \right) X_r^T \zeta + \varepsilon_r \zeta^T Y_r^T \left(\sum_{s=1}^{n_r} N_{rs}^T N_{rs} \right) Y_r \zeta < 0.
\end{aligned} \tag{1.27}
$$

By applying Lemma 1.19 and considering the fact that $a^2 + b^2 \geq 2ab$ for scalars a and b, (1.27) implies that

$$\zeta^T \Upsilon \zeta + 2\sqrt{\zeta^T X_1 \left(\sum_{q=1}^{m_1} M_{1q} M_{1q}^T\right) X_1^T \times \zeta^T Y_1^T \left(\sum_{s=1}^{n_1} N_{1s}^T N_{1s}\right) Y_1 \zeta} + \cdots$$

$$+ 2\sqrt{\zeta^T X_r \left(\sum_{q=1}^{m_r} M_{rq} M_{rq}^T\right) X_r^T \zeta \times \zeta^T Y_r^T \left(\sum_{s=1}^{n_r} N_{rs}^T N_{rs}\right) Y_r \zeta} \quad (1.28)$$

$$= \zeta^T \Upsilon \zeta + 2 \max_{\|\Delta_{1N}\|_F \leq 1} \zeta^T X_1 \left(\sum_{q=1}^{m_1} \sum_{s=1}^{n_1} M_{1q} \Delta_{1qs}(k) N_{1s}\right) Y_1 \zeta + \cdots$$

$$+ 2 \max_{\|\Delta_{rN}\|_F \leq 1} \zeta^T X_r \left(\sum_{q=1}^{m_r} \sum_{s=1}^{n_r} M_{rk} \Delta_{rqs}(k) N_{rs}\right) Y_r \zeta < 0.$$

By [25], we obtain

$$\zeta^T \Upsilon \zeta + 2\zeta^T X_1 \left(\sum_{q=1}^{m_1} \sum_{s=1}^{n_1} M_{1q} \Delta_{1qs}(k) N_{1s}\right) Y_1 \zeta + \cdots$$

$$+ 2\zeta^T X_r \left(\sum_{q=1}^{m_r} \sum_{s=1}^{n_r} M_{rk} \Delta_{rqs}(k) N_{rs}\right) Y_r \zeta < 0. \quad (1.29)$$

Thus, (1.25) follows immediately. □

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Chapter 2
Robust Static Output Feedback H_∞ Control

Abstract This chapter will focus on robust static output feedback H_∞ control design for linear systems with polytopic uncertainties and norm bounded uncertainties. First, new H_∞ performance analysis criterions are proposed for the systems by an LMI decoupling approach. Then, sufficient conditions for designing static output feedback H_∞ controllers are given in terms of solutions to a set of linear matrix inequalities (LMIs). In contrast to the existing methods for designing the static output feedback H_∞ controllers, the input matrices and output matrices of the considered systems are allowed to have uncertainties. Moreover, theoretical proof is given to show that the proposed design conditions include the existing results as special cases. Simulation examples are provided to show the effectiveness of the proposed design method.

Keywords Uncertain linear systems · Static output feedback · H_∞ controllers · Linear matrix inequalities (LMIs)

2.1 With Time-Invariant Polytopic Uncertainties

2.1.1 Discrete-Time Systems

Consider a linear discrete-time system with time-invariant polytopic uncertainties described by state-space equations

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + E(\theta)w(k), \\ z(k) &= C_1(\theta)x(k) + D(\theta)u(k) + F(\theta)w(k)), \\ y(k) &= C_2(\theta)x(k) + H(\theta)w(k), \end{aligned} \tag{2.1}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0,\ \infty)$, $z(k) \in \mathscr{R}^q$

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_2,

is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. The matrices $A(\theta)$, $B(\theta)$, $E(\theta)$, $C_1(\theta)$, $D(\theta)$, $F(\theta)$, $C_2(\theta)$, and $H(\theta)$ are constant matrices of appropriate dimensions and belong to the following uncertainty polytope [4]:

$$\begin{aligned}\Omega &= \{[\ A(\theta),\ B(\theta),\ E(\theta),\ C_1(\theta),\ D(\theta),\ F(\theta),\ C_2(\theta),\ H(\theta)\ ] \\ &= \sum_{i=1}^{r} \theta_i [\ A_i,\ B_i,\ E_i,\ C_{1i},\ D_i,\ F_i,\ C_{2i},\ H_i\ ],\ \sum_{i=1}^{r} \theta_i = 1,\ \theta_i \geq 0\}.\end{aligned} \tag{2.2}$$

Our aim is to design a static output feedback controller

$$u(k) = K y(k), \tag{2.3}$$

such that the resulting following closed-loop system (2.4) is robustly stable or simultaneously meets H_∞ performance bound requirement.

$$\begin{aligned} x(k+1) &= \big(A(\theta) + B(\theta)KC_2(\theta)\big)x(k) + \big(E(\theta) + B(\theta)KH(\theta)\big)w(k), \\ z(k) &= \big(C_1(\theta) + D(\theta)KC_2(\theta)\big)x(k) + \big(F(\theta) + D(\theta)KH(\theta)\big)w(k). \end{aligned} \tag{2.4}$$

2.1.1.1 Case A: $D(\theta) = 0$

First, based on the parameter-dependent Lyapunov function approach, the H_∞ performance analysis problem of the closed-loop system (2.4) with $D(\theta) = 0$ is concerned. A new H_∞ performance analysis criterion is given, which will play a key role in static output feedback H_∞ controller design. The following preliminary lemma is needed to prove our results.

Lemma 2.1 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta)$, $G(\theta)$, and K such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.5}$$

where $\mathscr{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Proof Construct a parameter-dependent Lyapunov function as

$$V(k) = x^T(k)P(\theta)x(k),\ \ P(\theta) > 0. \tag{2.6}$$

The difference of $V(k)$ can be given by

$$V(k+1) - V(k) = x^T(k+1)P(\theta)x(k+1) - x^T(k)P(\theta)x(k). \tag{2.7}$$

From (2.7) and recalling (2.4) with $D(\theta) = 0$, it can be verified that

$$\begin{aligned}
&V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
&= x^T(k+1)P(\theta)x(k+1) - x^T(k)P(\theta)x(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
&= \Big(\big(A(\theta) + B(\theta)KC_2(\theta)\big)x(k) + \big(E(\theta) + B(\theta)KH(\theta)\big)w(k)\Big)^T P(\theta) \\
&\quad \times \Big(\big(A(\theta) + B(\theta)KC_2(\theta)\big)x(k) + \big(E(\theta) + B(\theta)KH(\theta)\big)w(k)\Big) \\
&\quad -x^T(k)P(\theta)x(k) + \big(C_1(\theta)x(k) + F(\theta)w(k)\big)^T \big(C_1(\theta)x(k) + F(\theta)w(k)\big) \\
&\quad -\gamma^2 w^T(k)w(k) \\
&= \zeta^T(k)\Big(\big[\, A(\theta) + B(\theta)KC_2(\theta) \quad E(\theta) + B(\theta)KH(\theta)\big]^T \\
&\quad \times P(\theta)\big[\, A(\theta) + B(\theta)KC_2(\theta) \quad E(\theta) + B(\theta)KH(\theta)\big] \\
&\quad + \big[\, C_1(\theta) \quad F(\theta)\big]^T \big[\, C_1(\theta) \quad F(\theta)\big] + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix}\Big)\zeta(k),
\end{aligned} \tag{2.8}$$

where $\zeta(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}$.

Thus, $V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0$ for any $\zeta(k) \neq 0$ if

$$\begin{aligned}
&[\, A(\theta) + B(\theta)KC_2(\theta) \quad E(\theta) + B(\theta)KH(\theta)]^T \\
&\times P(\theta)\big[\, A(\theta) + B(\theta)KC_2(\theta) \quad E(\theta) + B(\theta)KH(\theta)\big] \\
&+\big[\, C_1(\theta) \quad F(\theta)\big]^T \big[\, C_1(\theta) \quad F(\theta)\big] + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0.
\end{aligned} \tag{2.9}$$

By using Schur complement to (2.9), we have

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) & -P^{-1}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.10}$$

Pre- and post-multiplying (2.10) by $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & G(\theta) & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we verify that (2.10) is equivalent to the following matrix inequality:

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \Lambda & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.11}$$

where $\Lambda = -G(\theta)P^{-1}(\theta)G^T(\theta)$.

Note that $-\big(G(\theta) - P(\theta)\big)^T P^{-1}(\theta)\big(G(\theta) - P(\theta)\big) \leq 0,\ P(\theta) > 0$ implies that $-G(\theta)P^{-1}(\theta)G^T(\theta) \leq -G(\theta) - G^T(\theta) + P(\theta)$, then, the inequality (2.11) can be verified by (2.5). If the condition (2.5) is satisfied, we have $V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) < 0$ for any $\zeta(k) \neq 0$, which implies that

$$V(\infty) - V(0) + \sum_{k=0}^{\infty} z^T(k)z(k) - \sum_{k=0}^{\infty} w^T(k)w(k) < 0.$$

With zero initial condition $\psi(0) = 0$ and $V(\infty) > 0$, we obtain $\sum_{k=0}^{\infty} z^T(k) z(k) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k)$ for any nonzero $w(k) \in l_2[0, \infty)$. Thus, the proof is completed. □

In order to obtain LMI-based conditions for designing static output feedback H_∞ controllers, the existing results [3, 5] have to impose some constraints on the system matrices, which require that the input (or output) matrix $B(\theta)$ $\big($or $C_2(\theta)\big)$ is fixed $\big($is without uncertainties i.e., $B(\theta) = B$ and $C_2(\theta) = C_2\big)$ and B (C_2) is of full column (low) rank. Obviously, those results are limited and cannot be applied to general control systems. In our study, the constraints on the input and output matrices have been avoided. Thus, our results have more advantages than the ones in [3, 5].

Remark 2.1 It is noted that the results given by [6] are only applicable to that the system input matrix (or output matrix) is with time-varying polytopic uncertainties for linear systems. See Remark 2.6 for details.

In this following, a new H_∞ performance analysis criterion is presented in the following theorem.

Theorem 2.1 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P(\theta)$, $G(\theta)$, $J(\theta)$, M, N, V, *and* U, *scalar* β *such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.12}$$

where

$$\begin{aligned} \mathscr{G}(\theta) &= -G(\theta) - G^T(\theta) + P(\theta), \\ \Sigma_1 &= -\beta NU - \beta U^T N^T, \\ \Sigma_2 &= G(\theta)B(\theta) - MU. \end{aligned}$$

Proof We are about to prove the conclusion using Lemma 2.1. Obviously, the matrix inequality (2.5) can be rewritten as follows:

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T < 0. \tag{2.13}$$

By defining $UK = V$ and considering matrices M and N, where U and N are nonsingular without loss of generality, we have

$$\begin{aligned} &\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\ &= \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T \\ &= \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} 0 \\ 0 \\ M \\ 0 \end{bmatrix} V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T \begin{bmatrix} 0 \\ 0 \\ M \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1} N^{-1} N V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T < 0.
\end{aligned}
\tag{2.14}
$$

Based on Lemma 1.3, for a positive matrix $J(\theta)$ one gives

$$
\begin{aligned}
&\begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1} N^{-1} N V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} \big(G(\theta)B(\theta) - MU\big) U^{-1} N^{-1} N V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T
\end{aligned}
$$

$$
\begin{aligned}
&\leq \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T J^{-1}(\theta) \\
&\quad \times \big(G(\theta)B(\theta) - MU\big) U^{-1} N^{-1} N V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{2.15}
$$

Then, (2.14) holds if the following condition is satisfied:

$$
\begin{aligned}
&\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
&+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T J^{-1}(\theta) \\
&\times \big(G(\theta)B(\theta) - MU\big) U^{-1} N^{-1} N V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} < 0.
\end{aligned} \tag{2.16}
$$

Without loss of generality, we assume that matrix $G(\theta)B(\theta) - MU$ is of full rank. By Schur complement to (2.16), which leads to

$$
\begin{aligned}
&\begin{bmatrix} -P(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) + J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Xi_1 \end{bmatrix} \\
&= \begin{bmatrix} -P(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) + J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Xi_2 \end{bmatrix} < 0,
\end{aligned} \tag{2.17}
$$

where

$$
\Xi_1 = -\Big(N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T J^{-1} \big(G(\theta)B(\theta) - MU\big) U^{-1} N^{-1}\Big)^{-1},
$$

$$
\Xi_2 = -NU\Big(\big(G(\theta)B(\theta) - MU\big)^T J^{-1} \big(G(\theta)B(\theta) - MU\big)\Big)^{-1} U^T N^T.
$$

For a scalar β, note that $-(V - \beta Q)Q^{-1}(V - \beta Q)^T \leq 0,\ Q > 0$ implies that $-VQ^{-1}V^T \leq -\beta V - \beta V^T + \beta^2 Q$. Then, one has

$$\begin{aligned}\Xi_2 &= -NU\Big(\big(G(\theta)B(\theta)-MU\big)^T J^{-1}\big(G(\theta)B(\theta)-MU\big)\Big)^{-1} U^T N^T \\ &\leq -\beta NU - \beta U^T N^T + \beta^2\big(G(\theta)B(\theta)-MU\big)^T J^{-1}(\theta)\big(G(\theta)B(\theta)-MU\big) \\ &= \Xi_3.\end{aligned} \tag{2.18}$$

Then, (2.17) can be guaranteed by

$$\begin{bmatrix} -P(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ G(\theta)A(\theta)+MVC_2(\theta) & G(\theta)E(\theta)+MVH(\theta) & \mathscr{G}(\theta)+J(\theta) & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Xi_3 \end{bmatrix} < 0. \tag{2.19}$$

Applying Schur complement to (2.19) yields (2.12). Thus, the proof is complete. □

Remark 2.2 In the proof of Theorem 2.1, it should be noted that the procedure from (2.16) to (2.17) needs a constraint condition, which requires that the matrix $G(\theta)B(\theta) - MU$ is of full rank. When the matrix $G(\theta)B(\theta) - MU$ is of full rank, we can know that the matrix $\big(G(\theta)B(\theta)-MU\big)^T J^{-1}\big(G(\theta)B(\theta)-MU\big)$ is invertible. In fact, the constraint condition is not necessary. For the matrix inequality (2.16), by using Lemma 1.4 with

$$T = \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta)+MVC_2(\theta) & G(\theta)E(\theta)+MVH(\theta) & \mathscr{G}(\theta)+J(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix},$$

$$A = U^{-1}N^{-1}NV\big[\, C_2(\theta) \quad H(\theta) \quad 0 \quad 0 \,\big],$$

$$L = NU,$$

$$P = \big(G(\theta)B(\theta)-MU\big)^T J^{-1}(\theta)\big(G(\theta)B(\theta)-MU\big),$$

we can also obtain the following matrix condition (2.19).

By the LMI decoupling approach, the appearance of crossing terms between $G(\theta)$ and K has been avoided in (2.12), it enables us to obtain strict LMI conditions for designing static output feedback H_∞ controllers.

In this following, based on the analysis result in Theorem 2.1, we proposed sufficient conditions for designing the static output feedback H_∞ controller in the form of (2.3), that is, to compute the gains K in (2.3) such that the closed-loop system (2.4) with $D(\theta) = 0$ is asymptotically stable with the prescribed H_∞ performance γ. As can be seen from (2.12), the appearance of crossing terms has been avoided, it makes the LMI formulation of design conditions easier. Of course, in order to obtain

LMI-based design conditions, the analysis criterion in Theorem 2.1 dependent on the premise that is the matrix parameters M, N and scalar parameter β should be known, it may lead to conservative design. However, when the system input matrices are with time-invariant polytopic uncertainties, the new condition given by Theorem 2.1 is undoubtedly effective for dealing with this case.

When the input matrix is with polytopic uncertainties and not full column rank, we choose $M = B(\theta)$ and $N = I$ in (2.12). Then, the corresponding design condition is given in the following theorem.

Theorem 2.2 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and give scalars* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalar* β, *exist matrices* U, V, P_j, J_j, *and* G_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold*

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_iA_i + B_iVC_{2i} & G_iE_i + B_iVH_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2i} & VH_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_iB_i - B_iU & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i = 1, 2, \ldots, r, \tag{2.20}$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_jA_i + B_iVC_{2j} & G_jE_i + B_iVH_j & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2j} & VH_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_jB_i - B_iU & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_iA_j + B_jVC_{2i} & G_iE_j + B_jVH_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ VC_{2i} & VH_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_iB_j - B_jU & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i, j = 1, 2, \ldots, r, \; i < j, \tag{2.21}$$

where $\mathscr{G}_j = -G_j - G_j^T + P_j$.

Furthermore, the static output feedback H_∞ *controller gain matrix in* (2.3) *can be given by*

$$K = U^{-1}V. \tag{2.22}$$

Proof First, in this case, the matrix inequality (2.12) becomes

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta)+B(\theta)VC_2(\theta) & G(\theta)E(\theta)+B(\theta)VH(\theta) & \mathscr{G}(\theta)+J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ VC_2(\theta) & VH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.23}$$

where

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = G(\theta)B(\theta) - B(\theta)U.$$

From Theorem 2.1, we know that the prescribed H_∞ performance γ can be ensured if there exist matrices $P(\theta)$, $J(\theta)$, and $G(\theta)$ satisfying (2.23). Now, assume that the aforementioned matrices have the following form:

$$\begin{aligned} P(\theta) &= \sum_{j=1}^{r} \theta_j P_j,\ \ P_j > 0,\ \ j = 1,\ 2,\ \ldots,\ r, \\ G(\theta) &= \sum_{j=1}^{r} \theta_j G_j, \\ J(\theta) &= \sum_{j=1}^{r} \theta_j J_j,\ \ J_j > 0,\ \ j = 1,\ 2,\ \ldots,\ r. \end{aligned} \tag{2.24}$$

Then, inequality (2.23) is equivalent to

$$\begin{aligned} &\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_jA_i + B_iVC_{2j} & G_jE_i + B_iVH_j & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2j} & VH_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_jB_i - B_iU & -\frac{J_j}{\beta^2} \end{bmatrix} \\ &= \sum_{i=1}^{r}\theta_i{}^2 \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_iA_i + B_iVC_{2i} & G_iE_i + B_iVH_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2i} & VH_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_iB_i - B_iU & -\frac{J_i}{\beta^2} \end{bmatrix} \\ &\quad + \sum_{i=1}^{r}\sum_{i<j}^{r}\theta_i\theta_j \end{aligned}$$

$$\times\left(\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_jA_i + B_iVC_{2j} & G_jE_i + B_iVH_j & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2j} & VH_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_jB_i - B_iU & -\frac{J_j}{\beta^2} \end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_iA_j + B_jVC_{2i} & G_iE_j + B_jVH_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ VC_{2i} & VH_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_iB_j - B_jU & -\frac{J_i}{\beta^2} \end{bmatrix}\right) < 0. \tag{2.25}$$

If LMIs (2.20) and (2.21) are satisfied, the inequality (2.25) holds. □

Remark 2.3 Theorem 2.2 presents a new condition for designing static output feedback H_∞ controllers for discrete-time linear systems with time-invariant polytopic uncertainties which is of LMIs and can be effectively solved via LMI Control Toolbox [7].

In Theorem 2.2, a significant result is proposed to design static output feedback H_∞ controllers for uncertain discrete-time linear systems. The new result overcomes the deficiencies of the existing ones, it is able to handle this case that the system input matrices are nonfixed. In addition to this, the proposed result can give less conservative design than the existing LMI methods. In order to clarify this issue thoroughly, in the following, we consider the same system input matrix with [3, 5] as $B(\theta) = B$ (B is of full column rank). In this case, three H_∞ performance analysis conclusions with different values of matrices M and N are given based on Theorem 2.1. The first conclusion chooses $M = B$ and $N = B^TB$ in Theorem 2.1, the second chooses $M = \begin{bmatrix} I \\ 0 \end{bmatrix} R_1$, where R_1 is a known matrix parameter and $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $N = I$ in Theorem 2.1, while the third chooses $M = Y^T\begin{bmatrix} I \\ 0 \end{bmatrix}$, $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $N = I$ in Theorem 2.1.

Remark 2.4 Here, there is a description of this matrix Y. Because of this matrix B is full column rank, there exist a nonsingular matrix Y such that $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$. It should be noted that for each matrix B, the corresponding Y generally is not unique. A special Y can be obtained by the following formula:

$$Y = \begin{bmatrix} (B^TB)^{-1}B^T \\ B^{T\perp T} \end{bmatrix},$$

where $B^{T\perp}$ denotes an orthogonal basis for the null space of B^T.

Theorem 2.3 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(B is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P(\theta)$, $G(\theta)$, $J(\theta)$, V, *and* U, *scalar* β *such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + BVC_2(\theta) & G(\theta)E(\theta) + BVH(\theta) & \mathscr{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ B^T BVC_2(\theta) & B^T BVH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.26}$$

where

$$\mathscr{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta),$$

$$\Sigma_1 = -\beta B^T BU - \beta U^T (B^T B)^T,$$

$$\Sigma_2 = G(\theta)B - BU.$$

Theorem 2.4 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(B is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P(\theta)$, $G(\theta)$, $J(\theta)$, V, R_1, *and* U, *scalar* β *such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta) + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V C_2(\theta) & G(\theta)E(\theta) + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V H(\theta) & \mathscr{G}(\theta) + J(\theta) & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ VC_2(\theta) & VH(\theta) & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G(\theta)B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.27}$$

where $\mathscr{G}(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Theorem 2.5 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(B is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P(\theta)$, $G(\theta)$, $J(\theta)$, M, N, V, *and U, scalar β such that the following matrix inequality holds*

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G(\theta)A(\theta)+Y^T\begin{bmatrix} I \\ 0 \end{bmatrix}VC_2(\theta) & G(\theta)E(\theta)+Y^T\begin{bmatrix} I \\ 0 \end{bmatrix}VH(\theta) & \Xi & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * \\ VC_2(\theta) & VH(\theta) & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.28}$$

where

$$\Xi = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = G(\theta)B - Y^T\begin{bmatrix} I \\ 0 \end{bmatrix}U.$$

Based on the three H_∞ performance analysis conclusions, the corresponding static output feedback H_∞ controller design results are given in the following corollaries.

Corollary 2.1 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_iA_i + BVC_{2i} & G_iE_i + BVH_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ B^TBVC_{2i} & B^TBVH_i & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_iB - BU & -\frac{J_i}{\beta^2} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, r, \tag{2.29}$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_jA_i + BVC_{2i} & G_jE_i + BVH_i & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ B^TBVC_{2j} & B^TBVH_j & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_jB - BU & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$
+\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + BVC_{2j} & G_i E_j + BVH_j & \mathscr{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ B^T BVC_{2i} & B^T BVH_i & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & G_i B - BU & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,
$$

$$
i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j, \tag{2.30}
$$

where

$$
\mathscr{G}_j = -G_j - G_j^T + P_j,
$$

$$
\Sigma_1 = -\beta B^T BU - \beta U^T (B^T B)^T.
$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Corollary 2.2 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known matrix R_1 and scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1,\ 2,\ \ldots,\ r$ such that the following matrix inequalities hold*

$$
\Pi_{ii} < 0,\ i = 1,\ 2,\ \ldots,\ r, \tag{2.31}
$$

$$
\Pi_{ij} + \Pi_{ji} < 0,\ i < j,\ i,\ j = 1,\ 2,\ \ldots,\ r, \tag{2.32}
$$

with

$$
\Pi_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V C_{2i} & G_j E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V H_i & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ VC_{2i} & VH_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}
$$

and

$$
\mathscr{G}_j = -G_j - G_j^T + P_j,
$$

$$\Sigma_{2j} = G_j B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

Corollary 2.3 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2i} & G_i E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix}$$
$$< 0, \ i = 1, 2, \ldots, r, \tag{2.33}$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2i} & G_j E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_i & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}$$
$$+ \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V C_{2j} & G_i E_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} V H_j & \mathscr{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Sigma_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix}$$
$$< 0, \ i, \ j = 1, 2, \ldots, r, \ i < j, \tag{2.34}$$

where

$$\mathscr{G}_j = -G_j - G_j^T + P_j,$$
$$\Sigma_{2j} = G_j B - Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} U.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

In order to show the contrasts between the existing static output feedback H_∞ control results and the proposed ones, we also recall the following existing results, which can directly be obtained from [3] and [6].

Lemma 2.2 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(B is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $\hat{P}$ *and* N *such that the following matrix equations hold*

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + BNC_{2i} & \hat{P}E_i + BNH_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1,\ 2,\ \ldots,\ r, \tag{2.35}$$

$$\hat{P}B = BU. \tag{2.36}$$

Remark 2.5 By regulating $P(\theta) = \hat{P}$ and using matrix inequality congruence property with $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & \hat{P} & * \\ 0 & 0 & 0 & I \end{bmatrix}$, the matrix inequality (2.10) is equivalent to

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + \hat{P}BKC_{2i} & \hat{P}E_i + \hat{P}BKH_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1,\ 2,\ \ldots,\ r. \tag{2.37}$$

By considering the condition (2.36) and defining a new variable $N = UK$, it can be known that LMIs (2.35) can guarantee the negative-definiteness of (2.37).

In addition to this, it should be noted that the single quadratic Lyapunov function approach had been used to ensure the matrix condition (2.36) in Lemma 2.2.

Remark 2.6 Obviously, the condition in Lemma 2.2 can be relaxed further with introducing a slack matrix variables G. In this case, (2.35) and (2.36) become, respectively, as follows:

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ GA_i + BNC_{2i} & GE_i + BNH_i & -G - G^T + \hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1,\ 2,\ \ldots,\ r, \tag{2.38}$$

$$GB = BU. \tag{2.39}$$

Lemma 2.3 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(B is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable* *with the H_∞ performance γ if exist matrices L, $\hat{P}_j$, and R_j, $j = 1, 2, \ldots, r$ such that the following matrix equations hold*

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0, \quad i = 1, 2, \ldots, r, \tag{2.40}$$

$$\begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < 0, \quad i, j = 1, 2, \ldots, r, \ i < j, \tag{2.41}$$

$$R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \ j = 1, 2, \ldots, r, \tag{2.42}$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Remark 2.7 Define $P(\theta) = \hat{P}(\theta)$, $L = R_1 K$ and use matrix inequality congruence property with $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & \sum_{j=1}^{r} \theta_j R_j Y & * \\ 0 & 0 & 0 & I \end{bmatrix}$ for the matrix inequality (2.10), the LMIs (2.40) and (2.41) can derived easily.

Remark 2.8 This design condition given in Lemma 2.3 is a simple extension of the results given in [6] for discrete-time linear system with time-invariant polytopic uncertainties. In fact, [6] is concerned with the problem of designing robust static output feedback controllers for discrete-time linear systems with time-varying polytopic uncertainties. The proposed technique in [6] is applicable for linear systems with the time-varying polytopic uncertainties, which may simultaneously emerge on system output and input matrices (nonfixed). However, it should be noted that this results in [6] are not enough to be used directly for system output or input matrices with time-invariant polytopic uncertainties. This is because in the product terms $R_{ij}Y_i(\sum_{\nu=1}^{r}\theta_\nu P_\nu)^{-1}(R_{ij}Y_i)^T,\ Y_iB_i = \begin{bmatrix} I \\ 0 \end{bmatrix},\ i,\ j = 1,\ 2,\ \ldots,\ r$, it does not allow $\nu = i$ or $\nu = j$. Therefore, in Lemma 2.3, we assume that the system input matrix $B(\theta)$ is fixed, it leads to the matrix Y is also fixed.

In addition to these two conditions given in Lemmas 2.2 and 2.3, which can directly be obtained from [3] and [6], the other significant result should be mentioned. The mentioned result is given is based on the study in [5] and [8].

Lemma 2.4 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $L,\ Q_j$, and $R_j,\ j = 1,\ 2,\ \ldots,\ r$ such that the following matrix equations hold*

$$\begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_iYA_iY^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i}Y^{-1} & R_iYE_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_i - R_i^T + Q_i & * \\ C_{1i}Y^{-1} & F_i & 0 & -I \end{bmatrix} < 0,$$

$$i = 1,\ 2,\ \ldots,\ r, \tag{2.43}$$

$$\begin{bmatrix} -Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_jYA_iY^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i}Y^{-1} & R_jYE_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_j - R_j^T + Q_j & * \\ C_{1i}Y^{-1} & F_i & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_iYA_jY^{-1} + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j}Y^{-1} & R_iYE_j + \begin{bmatrix} L \\ 0 \end{bmatrix} H_j & -R_i - R_i^T + Q_i & * \\ C_{1j}Y^{-1} & F_j & 0 & -I \end{bmatrix} < 0,$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j, \tag{2.44}$$

$$R_j = \begin{bmatrix} R_1 & 0 \\ 0 & R_{2j} \end{bmatrix} \ or\ R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix},\ j = 1,\ 2,\ \ldots,\ r, \tag{2.45}$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Proof In fact, the design result presented in Lemma 2.4 is an extension, which puts the dynamic output feedback H_∞ controllers design method given in [8] to robust static output feedback H_∞ controller design for linear discrete-time system with time-invariant polytopic uncertainties.

Consider the matrix inequality (2.10), pre- and post-multiplying it by M and its transpose, respectively, we have:

$$\begin{bmatrix} -Y^{-T}P(\theta)Y^{-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ YA(\theta)Y^{-1}+YBKC_2(\theta)Y^{-1} & YE(\theta)+YBKH(\theta) & -YP^{-1}(\theta)Y^T & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.46}$$

where $M = \begin{bmatrix} Y^{-T} & * & * & * \\ 0 & I & * & * \\ 0 & 0 & Y & * \\ 0 & 0 & 0 & I \end{bmatrix}$.

Again, pre- and post-multiplying it by $\begin{bmatrix} I & * & * & * \\ 0 & I & * & * \\ 0 & 0 & R(\theta) & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, it follows that:

$$\begin{bmatrix} -Y^{-T}P(\theta)Y^{-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Delta_{31} & \Delta_{32} & -R(\theta)\left(YP^{-1}(\theta)Y^T\right)R^T(\theta) & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.47}$$

where

$$\Delta_{31} = R(\theta)YA(\theta)Y^{-1} + R(\theta)YBKC_2(\theta)Y^{-1},$$

$$\Delta_{32} = R(\theta)YE(\theta) + R(\theta)YBKH(\theta).$$

Based on a fact $-R(\theta)\left(YP^{-1}(\theta)Y^T\right)R^T(\theta) \le -R(\theta) - R^T(\theta) + Y^{-T}P(\theta)Y^{-1}$ and defining $Y^{-T}P(\theta)Y^{-1} = Q(\theta)$, we can obtain the matrix inequality (2.48) to verify (2.47)

$$\begin{bmatrix} -Q(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Delta_{31} & \Delta_{32} & -R(\theta) - R^T(\theta) + Q(\theta) & * \\ C_1(\theta)Y^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.48}$$

In order to solve the controller design problem, we partition matrices $Q(\theta)$ and $R(\theta)$ as

$$Q(\theta) = \sum_{j=1}^{r} \theta_j Q_j,\ Q_j > 0,\ j = 1,\ 2,\ \ldots,\ r,$$
$$R(\theta) = \sum_{j=1}^{r} \theta_j R_j,\ R_j = \begin{bmatrix} R_1 & 0 \\ 0 & R_{2j} \end{bmatrix} \text{ or } R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix},\ j = 1,\ 2,\ \ldots,\ r. \tag{2.49}$$

By substituting the corresponding partitioned parts into (2.48), the matrix inequality is equivalent to

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j\begin{bmatrix} -Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_jYA_iY^{-1}+\begin{bmatrix} L \\ 0 \end{bmatrix}C_{2i}Y^{-1} & R_jYE_i+\begin{bmatrix} L \\ 0 \end{bmatrix}H_i & -R_j-R_j^T+Q_j & * \\ C_{1i}Y^{-1} & F_i & 0 & -I \end{bmatrix}$$
$$< 0. \tag{2.50}$$

Choosing a new variable $L = R_1K$, it is observed that the LMIs (2.43) and (2.44) can guarantee the negative-definiteness of (2.50). □

Remark 2.9 It should be pointed out that the result given in Lemma 2.4 can also obtain by using a linear transformation approach. Consider the linear transformation on the system state

$$\bar{x}(k) = Yx(k), \tag{2.51}$$

where $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, then we can obtain the following transformed closed-loop uncertain system:

$$\begin{aligned} Yx(k+1) &= Y\big(A(\theta)+BKC_2(\theta)\big)Y^{-1}Yx(k)+Y\big(E(\theta)+BKH(\theta)\big)w(k), \\ z(k) &= C_1(\theta)Y^{-1}Yx(k)+F(\theta)w(k). \end{aligned} \tag{2.52}$$

i.e.,

$$\begin{aligned} \bar{x}(k+1) &= \left(YA(\theta)Y^{-1}+YBKC_2(\theta)Y^{-1}\right)\bar{x}(k)+\left(YE(\theta)+YBKH(\theta)\right)w(k), \\ z(k) &= C_1(\theta)Y^{-1}\bar{x}(k)+F(\theta)w(k). \end{aligned} \tag{2.53}$$

Choose the Lyapunov matrix as $Y^{-T}P(\theta)Y^{-1},\ P(\theta) > 0$, then we can obtain easily the H_∞ performance analysis conclusion (2.46).

In what follows, we will study the relationships among the proposed results and the LMI conditions in Lemmas 2.2–2.4. The following theorem shows that the results suggested herein include the ones given by [3, 5, 6, 8] as special cases, it is helpful to obtain a conclusion that the new results are less conservative than the existing LMI conditions for designing static output feedback H_∞ controllers.

Theorem 2.6 *If the condition given in* Lemma 2.2 *holds, the condition in* Corollary 2.1 *also holds.*

Proof If (2.35) and (2.36) in Lemma 2.2 are satisfied, which imply that $\hat{P}B = BU$ $(\hat{P}B - BU = 0)$ and $\hat{P} > 0$. Since the matrix B is of full column rank, we have $B^TBU + U^T(B^TB)^T = B^T\hat{P}B + B^T\hat{P}B > 0$. Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequalities hold

$$
\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \hat{P}A_i + BNC_{2i} & \hat{P}E_i + BNH_i & -\hat{P} & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
+\tfrac{1}{\beta}\big(B^T BN\,[\,C_{2i} \quad H_i \quad 0 \quad 0\,]\big)^T \tag{2.54}
$$

$$
\times\big(B^T BU + U^T (B^T B)^T - \beta(\hat{P}B - BU)^T \tfrac{1}{\rho} I (\hat{P}B - BU)\big)^{-1}
$$

$$
\times\big(B^T BN\,[\,C_{2i} \quad H_i \quad 0 \quad 0\,]\big) < 0, \; i = 1,\; 2,\; \ldots,\; r.
$$

By defining $G_j = P_j = \hat{P}$, $J_j = \rho I$, $j = 1, 2, \ldots, r$, $V = N$ and applying Schur complement, the inequalities (2.29) and (2.30) in Corollary 2.1 can be obtained. □

Theorem 2.7 *If the condition given in* Lemma 2.3 *holds, the condition in* Corollary 2.2 *also holds.*

Proof First, from $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, we can know that

$$
\begin{aligned}
R_j Y B = R_j \begin{bmatrix} I \\ 0 \end{bmatrix} &= \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} R_1, \; j = 1,\; 2,\; \ldots,\; r,
\end{aligned} \tag{2.55}
$$

i.e.,

$$
R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 = 0, \; j = 1,\; 2,\; \ldots,\; r. \tag{2.56}
$$

On the other hand, the matrix inequalities (2.40) and (2.41) in Lemma 2.3 can be rewritten as follows:

$$
\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & R_i Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}
$$

$$
= \begin{bmatrix} -\hat{P}_i \\ 0 \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} \\ C_{1i} \end{bmatrix}
$$

$$\left.\begin{matrix} * & * & * \\ -\gamma^2 I & * & * \\ R_iYE + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LH_i & -R_iY-(R_iY)^T+\hat{P}_i & * \\ F_i & 0 & -I \end{matrix}\right] < 0,$$

$$i = 1,\ 2,\ \ldots,\ r, \tag{2.57}$$

and

$$\begin{aligned}
&\begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_jYA_i + \begin{bmatrix} I \\ 0 \end{bmatrix} LC_{2i} & R_jYE_i + \begin{bmatrix} I \\ 0 \end{bmatrix} LH_i & -R_jY-(R_jY)^T+\hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_iYA_j + \begin{bmatrix} I \\ 0 \end{bmatrix} LC_{2j} & R_iYE_j + \begin{bmatrix} I \\ 0 \end{bmatrix} LH_j & -R_iY-(R_iY)^T+\hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} \\
&= \left[\begin{matrix} -\hat{P}_j \\ 0 \\ R_jYA_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LC_{2i} \\ C_{1i} \end{matrix}\right. \\
&\qquad \left.\begin{matrix} * & * & * \\ -\gamma^2 I & * & * \\ R_jYE_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LH_i & -R_jY-(R_jY)^T+\hat{P}_j & * \\ F_i & 0 & -I \end{matrix}\right] \\
&+ \left[\begin{matrix} -\hat{P}_i \\ 0 \\ R_iYA_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LC_{2j} \\ C_{1j} \end{matrix}\right. \\
&\qquad \left.\begin{matrix} * & * & * \\ -\gamma^2 I & * & * \\ R_iYE_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LH_j & -R_iY-(R_iY)^T+\hat{P}_i & * \\ F_j & 0 & -I \end{matrix}\right] < 0,
\end{aligned}$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.58}$$

Note that if the above condition is satisfied, which implies that $R_jY + (R_jY)^T > 0,\ j = 1,\ 2,\ \ldots,\ r$. Because of the matrix B is of full column rank, we only can know that

$$\begin{aligned}
&B^T R_j Y B + B^T Y^T R_j^T B = B^T R_j \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix}^T R_j^T B \\
&= B^T \begin{bmatrix} R_1 \\ 0 \end{bmatrix} + \begin{bmatrix} R_1 \\ 0 \end{bmatrix}^T B = B^T \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_1^T \begin{bmatrix} I \\ 0 \end{bmatrix}^T B \\
&= B^T Y B R_1 + R_1^T B^T Y^T B > 0,\ j = 1,\ 2,\ \ldots,\ r.
\end{aligned} \tag{2.59}$$

However, we are not sure that $R_1 + R_1^T > 0$.

Combining (2.55)–(2.58), then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequality holds:

$$\begin{aligned}
&\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} & R_i Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} \\
&+\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta}\left(R_1^{-1} L \left[\, C_{2i} \quad H_i \quad 0 \quad 0 \,\right]\right)^T \\
&\times \left(2I - \beta\left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)^T \frac{1}{\rho} I \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)\right)^{-1} \left(R_1^{-1} L \left[\, C_{2i} \quad H_i \quad 0 \quad 0 \,\right]\right) \\
&< 0,\ i = 1,\ 2,\ \ldots,\ r,
\end{aligned} \tag{2.60}$$

and

$$\begin{bmatrix} -\hat{P}_j \\ 0 \\ R_j Y A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2i} \\ C_{1i} \end{bmatrix}$$

$$\left.\begin{matrix} * & * & * \\ -\gamma^2 I & * & * \\ R_j Y E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ F_i & 0 & -I \end{matrix}\right]$$

$$+\left[\begin{matrix} -\hat{P}_i \\ 0 \\ R_i Y A_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L C_{2j} \\ C_{1j} \end{matrix}\right.$$

$$\left.\begin{matrix} * & * & * \\ -\gamma^2 I & * & * \\ R_i Y E_j + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 R_1^{-1} L H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ F_j & 0 & -I \end{matrix}\right]$$

$$+\,2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta}\left(R_1^{-1} L\,[\,C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0\,]\right)^T$$

$$\times \left(4I - \beta\left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)^T\right.$$

$$\left.\times \tfrac{1}{2\rho} I\left(R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)\right)^{-1} \left(R_1^{-1} L\,[\,C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0\,]\right)$$

$$< 0,\ i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.61}$$

By defining $G_j = R_j Y$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1, 2, \ldots, r$, $V = R_1^{-1} L$, $U = I$ and applying Schur complement, the inequalities (2.31) and (2.32) in Corollary 2.2 can be obtained. □

Theorem 2.8 *If the condition given in* Lemma 2.4 *holds, the condition in* Corollary 2.3 *also holds.*

Proof First, pre- and post-multiplying (2.43) and (2.44) and by $\begin{bmatrix} Y^T & * & * & * \\ 0 & I & * & * \\ 0 & 0 & Y^T & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we have

$$\begin{bmatrix} -Y^T Q_i Y & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_i Y A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & Y^T R_i Y E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & \Omega_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < 0,$$

$$i = 1, 2, \ldots, r, \tag{2.62}$$

$$\begin{bmatrix} -Y^T Q_j Y & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_j Y A_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2i} & Y^T R_j Y E_i + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_i & \Omega_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}$$

$$+ \begin{bmatrix} -Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ Y^T R_i Y A_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L C_{2j} & Y^T R_i Y E_j + Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} L H_j & \Omega_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < 0,$$

$$i, j = 1, 2, \ldots, r, i < j, \tag{2.63}$$

where $\Omega_j = -Y^T R_j Y - Y^T R_j^T Y + Y^T Q_j Y$. Obviously, these LMIs imply that $R_1 + R_1^T > 0$.

On the other hand, from (2.45) and $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, one can be given

$$Y^T R_j Y B = Y^T \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Y^T \begin{bmatrix} I \\ 0 \end{bmatrix} R_1, \; j = 1, 2, \ldots, r. \tag{2.64}$$

Similar to the proof of Theorem 2.6, by choosing $G_j = Y^T R_j Y$, $P_j = Y^T Q_j Y$, $J_j = \rho I$, $j = 1, 2, \ldots, r$, $V = L$, $U = R_1$ and applying Schur complement, the inequalities (2.62) and (2.63) generate (2.33) and (2.34) in Corollary 2.3, respectively. □

2.1.1.2 Case B: $H(\theta) = 0$

For this case, Lemma 2.1 is changed as the following lemma.

Lemma 2.5 *Consider the closed-loop system (2.4) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta)$, $G(\theta)$, and K such that the following matrix inequality holds:*

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)KC_2(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1 G(\theta) + D(\theta)KC_2(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.65}$$

Proof Choose a parameter-dependent Lyapunov function to be

$$V(k) = x^T(k)P^{-1}(\theta)x(k), \;\; P(\theta) > 0. \tag{2.66}$$

Similar to the proof Lemma 2.1, the H_∞ performance of closed-loop system (2.4) with $H(\theta) = 0$ can be guaranteed by the following matrix inequality:

$$\begin{bmatrix} -P^{-1}(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta) + B(\theta)KC_2(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta) + D(\theta)KC_2(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.67}$$

Pre- and post-multiplying (2.67) by $\begin{bmatrix} G^T(\theta) & * & * & * \\ 0 & I & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we have:

$$\begin{bmatrix} -G^T(\theta)P^{-1}(\theta)G(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)KC_2(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1 G(\theta) + D(\theta)KC_2(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{2.68}$$

Obviously, the inequality (2.68) can be ensured by (2.65). □

Based on Lemma 2.5, the following theorem proposes another form of H_∞ performance analysis criterion.

Theorem 2.9 *Consider the closed-loop system* (2.4) *with* $H(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $M, N, U, L, P(\theta), G(\theta)$, *and* $J(\theta)$, *scalar* β *such that the following matrix inequality holds*

$$\begin{bmatrix} \mathcal{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.69}$$

where

$$\begin{aligned}\mathscr{G}(\theta) &= -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),\\ \Sigma_1 &= -\beta U N - \beta N^T U^T,\\ \Sigma_2 &= (C_2(\theta)G(\theta) - UM)^T.\end{aligned}$$

Proof Rewrite matrix inequality (2.65) in the following form:

$$\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} K \begin{bmatrix} C_2(\theta)G(\theta) & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta)G(\theta) & 0 & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \tag{2.70}$$

Let us define $KU = L$ and consider two matrix M and N, where the matrices U and N are nonsingular, from (2.70), we have

$$\begin{bmatrix} \Omega(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1} \begin{bmatrix} C_2(\theta)G(\theta) & 0 & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} C_2(\theta)G(\theta) & 0 & 0 & 0 \end{bmatrix}^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \tag{2.71}$$

where $\Omega(\theta) = -G(\theta) - G^T(\theta) + P(\theta)$.

Furthermore, one gives

$$\begin{bmatrix} \Omega(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1} \begin{bmatrix} C_2(\theta)G(\theta) - UM & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} C_2(\theta)G(\theta) - UM & 0 & 0 & 0 \end{bmatrix}^T U^{-T} N^{-T} N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T$$

$$
+\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B(\theta)LM & 0 & 0 & 0 \\ D(\theta)LM & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B(\theta)LM & 0 & 0 & 0 \\ D(\theta)LM & 0 & 0 & 0 \end{bmatrix}^T
$$

$$
= \begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1}\begin{bmatrix} C_2(\theta)G(\theta) - UM & 0 & 0 & 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} C_2(\theta)G(\theta) - UM & 0 & 0 & 0 \end{bmatrix}^T U^{-T}N^{-T}N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0. \quad (2.72)
$$

By Lemma 1.3 with a positive matrix $J(\theta)$, one can be given

$$
\begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1}\big(C_2(\theta)G(\theta) - UM\big)[\, I \quad 0 \quad 0 \quad 0\,]
$$

$$
+ [\, I \quad 0 \quad 0 \quad 0\,]^T \big(C_2(\theta)G(\theta) - UM\big)^T U^{-T}N^{-T}N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T
$$

$$
\le \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} J(\theta)[\, I \quad 0 \quad 0 \quad 0\,] + \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1}\big(C_2(\theta)G(\theta) - UM\big)J^{-1}(\theta)
$$

$$
\times \big(C_2(\theta)G(\theta) - UM\big)^T U^{-T}N^{-T}N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T
$$

$$
= \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} J(\theta)[\, I \quad 0 \quad 0 \quad 0\,] + \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix} N^{-1}U^{-1}\big(C_2(\theta)G(\theta) - UM\big)J^{-1}(\theta)
$$

$$\times \left(C_2(\theta)G(\theta) - UM\right)^T U^{-T} N^{-T} \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix}^T . \tag{2.73}$$

Then, the inequality (2.72) holds if the following inequality is satisfied:

$$\begin{bmatrix} \mathscr{G}(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * \\ C_1 G(\theta) + D(\theta)LM & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix} N^{-1}U^{-1}$$
$$\times \left(C_2(\theta)G(\theta) - UM\right) J^{-1}(\theta) \left(C_2(\theta)G(\theta) - UM\right)^T U^{-T} N^{-T} \begin{bmatrix} 0 \\ 0 \\ B(\theta)LN \\ D(\theta)LN \end{bmatrix}^T < 0, \tag{2.74}$$

where $\mathscr{G}(\theta)$ is defined in (2.69).

Without loss of generality, we assume that matrix $C_2(\theta)G(\theta) - UM$ is of full rank. Applying Schur complement to (2.74) yields

$$\begin{bmatrix} \mathscr{G}(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Omega_1 \end{bmatrix} < 0, \tag{2.75}$$

where $\Omega_1 = -N^T U^T \left(\left(C_2(\theta)G(\theta) - UM\right) J^{-1}(\theta) \left(C_2(\theta)G(\theta) - UM\right)^T \right)^{-1} UN$.

At the same time, one can be known that

$$\begin{aligned} & -N^T U^T \left(\left(C_2(\theta)G(\theta) - UM\right) J^{-1}(\theta) \left(C_2(\theta)G(\theta) - UM\right)^T \right)^{-1} UN \\ & \le -\beta UN - \beta N^T U^T + \beta^2 \left(C_2(\theta)G(\theta) - UM\right) J^{-1}(\theta) \left(C_2(\theta)G(\theta) - UM\right)^T \\ & = \Omega_2. \end{aligned} \tag{2.76}$$

The above result (2.76) implies that the inequality (2.75) can be verified by

$$\begin{bmatrix} \mathscr{G}(\theta) & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta) & -P(\theta) & * & * \\ C_1(\theta)G(\theta) + D(\theta)LM & F(\theta) & 0 & -I & * \\ 0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Omega_2 \end{bmatrix} < 0. \tag{2.77}$$

By using Schur complement, the inequality (2.77) becomes (2.69). □

In the following, we consider several combinations about the known matrices M and N to design static output feedback H_∞ controllers.

$$\left\{\begin{array}{ll}\textbf{\textit{A:}}\ M=C_2(\theta) \text{ and } N=I, & \text{the output matrix is with polytopic uncertainties}\\ \left.\begin{array}{c}\textbf{\textit{B:}}\ M=C_2 \text{ and } N=C_2C_2^T\\ \text{or}\\ \textbf{\textit{C:}}\ M=S_1[\ I\quad 0\ ]\ (S_1 \text{ is known) and } N=I\\ \text{or}\\ \textbf{\textit{D:}}\ M=[\ I\quad 0\ ]T^T \text{ and } N=I\end{array}\right\} & \begin{array}{l}\text{the output matrix is}\\ \text{fixed and}\\ \text{of full row rank}\end{array}\end{array}\right. \tag{2.78}$$

where $C_2T=[\ I\quad 0\ ]$.

Remark 2.10 As Remark 2.4, there is a description of this matrix T. Because of this matrix C_2 is full row rank, there exist a nonsingular matrix T such that $C_2T=[\ I\quad 0\ ]$. It should be noted that for each matrix C_2, the corresponding T generally is not unique. A special T can be obtained by the following formula:

$$T=[\ C_2^T(C_2C_2^T)^{-1}\quad C_2^\perp\ ],$$

where $C_2^\perp$ denotes an orthogonal basis for the null space of C_2.

For the cases A–D, the H_∞ performance analysis condition (2.69) are rewritten as follows, respectively

A:

$$\begin{bmatrix}\mathscr{G}(\theta) & * & * & * & * & *\\ 0 & -\gamma^2 I & * & * & * & *\\ A(\theta)G(\theta)+B(\theta)LC_2(\theta) & E(\theta) & -P(\theta) & * & * & *\\ C_1(\theta)G(\theta)+D(\theta)LC_2(\theta) & F(\theta) & 0 & -I & * & *\\ 0 & 0 & L^TB^T(\theta) & L^TD^T(\theta) & -\beta U-\beta U^T & *\\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2}\end{bmatrix}<0, \tag{2.79}$$

where $\Sigma_2=\big(C_2(\theta)G(\theta)-UC_2(\theta)\big)^T$.

B:

$$\begin{bmatrix}\mathscr{G}(\theta) & * & * & * & * & *\\ 0 & -\gamma^2 I & * & * & * & *\\ A(\theta)G(\theta)+B(\theta)LC_2 & E(\theta) & -P(\theta) & * & * & *\\ C_1(\theta)G(\theta)+D(\theta)LC_2 & F(\theta) & 0 & -I & * & *\\ 0 & 0 & (C_2C_2^T)^TL^TB^T(\theta) & (C_2C_2^T)^TL^TD^T(\theta) & \Sigma_1 & *\\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2}\end{bmatrix}<0, \tag{2.80}$$

where

$$\Sigma_1 = -\beta U C_2 C_2^T - \beta (C_2 C_2^T)^T U^T,$$

$$\Sigma_2 = \big(C_2 G(\theta) - U C_2\big)^T.$$

C:

$$\begin{bmatrix} \mathscr{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)LS_1[\,I \quad 0\,] & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)LS_1[\,I \quad 0\,] & F(\theta) & 0 & -I & * & * \\ 0 & 0 & L^T B^T(\theta) & L^T D^T(\theta) & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \Lambda & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.81}$$

where $\Lambda = \big(C_2 G(\theta) - U S_1[\,I \quad 0\,]\big)^T$.

D:

$$\begin{bmatrix} \mathscr{G}(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A(\theta)G(\theta) + B(\theta)L[\,I \quad 0\,]T^T & E(\theta) & -P(\theta) & * & * & * \\ C_1(\theta)G(\theta) + D(\theta)L[\,I \quad 0\,]T^T & F(\theta) & 0 & -I & * & * \\ 0 & 0 & L^T B^T(\theta) & L^T D^T(\theta) & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.82}$$

where

$$\Sigma_1 = -\beta U - \beta U^T,$$

$$\Sigma_2 = \big(C_2 G(\theta) - U[\,I \quad 0\,]T^T\big)^T.$$

Based on these analysis conditions (2.79–2.82), the corresponding design results are given in the following corollaries.

Corollary 2.4 *Consider the closed-loop system* (2.4) *with* $H(\theta) = 0$ *and* $C_2(\theta) = C_2$ *(*C_2 *is of full row rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalar* β, *exist matrices* U, L, P_j, J_j, *and* G_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold*

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L C_{2i} & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L C_{2i} & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2i} G_i - U C_{2i})^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$
$$i = 1,\ 2,\ \ldots,\ r, \tag{2.83}$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_j L C_{2i} & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_j L C_{2i} & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2i} G_j - U C_{2i})^T & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_i L C_{2j} & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_i L C_{2j} & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & (C_{2j} G_i - U C_{2j})^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$
$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.84}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by

$$K = LU^{-1}. \tag{2.85}$$

Corollary 2.5 *Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β, exist matrices U, L, P_j, J_j, and G_j, $j = 1,\ 2,\ \ldots,\ r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L C_2 & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L C_2 & F_i & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_i^T & (C_2 C_2^T)^T L^T D_i^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$
$$i = 1,\ 2,\ \ldots,\ r, \tag{2.86}$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_i L C_2 & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_i L C_2 & F_i & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_i^T & (C_2 C_2^T)^T L^T D_i^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L C_2 & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L C_2 & F_j & 0 & -I & * & * \\ 0 & 0 & (C_2 C_2^T)^T L^T B_j^T & (C_2 C_2^T)^T L^T D_j^T & \Delta_1 & * \\ 0 & 0 & 0 & 0 & \Delta_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j, \tag{2.87}$$

where

$$\Delta_1 = -\beta U C_2 C_2^T - \beta (C_2 C_2^T)^T U^T,$$
$$\Delta_{2j} = (C_2 G_j - U C_2)^T.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

Corollary 2.6 *Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known matrix S_1 and scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1,\ 2,\ \ldots,\ r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L S_1 [\, I \quad 0 \,] & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L S_1 [\, I \quad 0 \,] & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,\ i = 1,\ 2,\ \ldots,\ r, \tag{2.88}$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_i L S_1 [\, I \quad 0 \,] & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_i L S_1 [\, I \quad 0 \,] & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2j} & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_j L S_1[\, I \quad 0\,] & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L S_1[\, I \quad 0\,] & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & \Psi_1 & * \\ 0 & 0 & 0 & 0 & \Psi_{2i} & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j, \tag{2.89}$$

where

$$\begin{aligned} \Psi_1 &= -\beta U - \beta U^T, \\ \Psi_{2j} &= \left(C_2 G_j - U S_1[\, I \quad 0\,]\right)^T. \end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) *is given by* (2.85).

Corollary 2.7 *Consider the closed-loop system* (2.4) *with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if, for known scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1,\ 2,\ \ldots,\ r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_i + B_i L\, [\, I \quad 0\,] T^T & E_i & -P_i & * & * & * \\ C_{1i} G_i + D_i L\, [\, I \quad 0\,] T^T & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \left(C_2 G_i - U\, [\, I \quad 0\,] T^T\right)^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i = 1,\ 2,\ \ldots,\ r, \tag{2.90}$$

$$\begin{bmatrix} -G_j - G_j^T + P_j + J_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_i G_j + B_i L\, [\, I \quad 0\,] T^T & E_i & -P_j & * & * & * \\ C_{1i} G_j + D_i L\, [\, I \quad 0\,] T^T & F_i & 0 & -I & * & * \\ 0 & 0 & L^T B_i^T & L^T D_i^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \left(C_2 G_j - U\, [\, I \quad 0\,] T^T\right)^T & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+\begin{bmatrix} -G_i - G_i^T + P_i + J_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ A_j G_i + B_i L\, [\, I \quad 0\,] T^T & E_j & -P_i & * & * & * \\ C_{1j} G_i + D_j L\, [\, I \quad 0\,] T^T & F_j & 0 & -I & * & * \\ 0 & 0 & L^T B_j^T & L^T D_j^T & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & \left(C_2 G_i - U\, [\, I \quad 0\,] T^T\right)^T & -\frac{J_i}{\beta^2} \end{bmatrix} < 0,$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.91}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.85).

For the case $H(\theta)$=0, we also make a comparison with the existing results to reflect advantage of the proposed results. With $H(\theta)$=0, several existing design conditions can be described by the following lemmas.

Lemma 2.6 *Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $\hat{P}$ and N such that the following matrix equations hold*

$$\begin{bmatrix} -\hat{P} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i\hat{P} + B_i N C_2 & E_i & -\hat{P} & * \\ C_{1i}\hat{P} + D_i N C_2 & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1, 2, \ldots, r, \tag{2.92}$$

$$C_2\hat{P} = UC_2. \tag{2.93}$$

Lemma 2.7 *Consider the closed-loop system (2.4) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices L_1, $\hat{P}_j$, and S_j, $j = 1, 2, \ldots, r$ such that the following matrix equations hold*

$$\begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_i + B_i\,[\,L_1 \quad 0\,] & E_i & -\hat{P}_i & * \\ C_{1i} TS_i + D_i\,[\,L_1 \quad 0\,] & F_i & 0 & -I \end{bmatrix} < 0, \ i = 1, 2, \ldots, r, \tag{2.94}$$

$$\begin{aligned} &\begin{bmatrix} -TS_j - (TS_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i TS_j + B_i[\,L_1 \quad 0\,] & E_i & -\hat{P}_j & * \\ C_{1i} TS_j + D_i[\,L_1 \quad 0\,] & F_i & 0 & -I \end{bmatrix} \\ &+ \begin{bmatrix} -TS_i - (TS_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j TS_i + B_j\,[\,L_1 \quad 0\,] & E_j & -\hat{P}_i & * \\ C_{1j} TS_i + D_j\,[\,L_1 \quad 0\,] & F_j & 0 & -I \end{bmatrix} < 0, \ i, \ j = 1, 2, \ldots, r, \ i < j, \end{aligned} \tag{2.95}$$

$$S_j = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}, \ j = 1, 2, \ldots, r, \tag{2.96}$$

where $C_2T = [\,I \quad 0\,]$.

Lemma 2.8 *Consider the closed-loop system* (2.4) *with* $H(\theta) = 0$ *and* $C_2(\theta) = C_2$ *(*C_2 *is of full row rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* L_1, P_j, *and* S_j, $j = 1, 2, \ldots, r$ *such that the following matrix equations hold*

$$\begin{bmatrix} -S_i - S_i^T + Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_i T S_i + T^{-1}B_i \,[\, L_1 \quad 0 \,] & T^{-1}E_i & -Q_i & * \\ C_{1i} T S_i + D_i \,[\, L_1 \quad 0 \,] & F_i & 0 & -I \end{bmatrix} < 0,\; i = 1, 2, \ldots, r, \tag{2.97}$$

$$\begin{bmatrix} -S_j - S_j^T + Q_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_i T S_j + T^{-1}B_i \,[\, L_1 \quad 0 \,] & T^{-1}E_i & -Q_j & * \\ C_{1i} T S_j + D_i \,[\, L_1 \quad 0 \,] & F_i & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} -S_i - S_i^T + Q_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T^{-1}A_j T S_i + T^{-1}B_j \,[\, L_1 \quad 0 \,] & T^{-1}E_j & -Q_i & * \\ C_{1j} T S_i + D_j \,[\, L_1 \quad 0 \,] & F_j & 0 & -I \end{bmatrix} < 0,\; i,\; j = 1, 2, \ldots, r,\; i < j, \tag{2.98}$$

$$S_j = \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix} \;\; or \;\; S_j = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix},\; j = 1, 2, \ldots, r, \tag{2.99}$$

where $C_2 T = [\, I \quad 0 \,]$.

Remark 2.11 The design conditions given in Lemmas 2.6 and 2.7 can directly be obtained from [3] and [6], respectively. And the condition in Lemma 2.8 can be derived by applying the homologous matrix inequality technique with Lemma 2.4 or consider a linear transformation on the system state as $\bar{x}(k) = T^{-1}x(k)$, with $C_2 T = [\, I \quad 0 \,]$.

Similar to the Case A $\big(D(\theta) = 0\big)$, the following theorems show also that the proposed results include the ones given by Lemmas as special cases.

Theorem 2.10 *If the condition given in* Lemma 2.6 *hold, the condition in* Corollary 2.5 *also holds.*

Proof If (2.92) and (2.93) in Lemma 2.6 are satisfied, which implies that $C_2 \hat{P} = U C_2$, $\hat{P} > 0$. Since the matrix C_2 is of full row rank, we have $U C_2 C_2^T + C_2 C_2^T U^T = C_2 \hat{P} C_2^T + C_2 \hat{P} C_2^T > 0$. Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the LMIs (2.86) and (2.87) hold. □

Theorem 2.11 *If the condition given in* Lemma 2.7 *hold, the condition in* Corollary 2.6 *also holds.*

Proof First, the LMIs (2.94) and (2.95) in Lemma 2.7 can be rewritten as follows:

$$\begin{bmatrix} -TS_i-(TS_i)^T+\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_iTS_i+B_iL_1\,[\,I\quad 0\,] & E_i & -\hat{P}_i & * \\ C_{1i}TS_i+D_iL_1\,[\,I\quad 0\,] & F_i & 0 & -I \end{bmatrix}$$

$$=\begin{bmatrix} -TS_i-(TS_i)^T+\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_iTS_i+B_iL_1S_1^{-1}S_1\,[\,I\quad 0\,] & E_i & -\hat{P}_i & * \\ C_{1i}TS_i+D_iL_1S_1^{-1}S_1\,[\,I\quad 0\,] & F_i & 0 & -I \end{bmatrix}<0,\ i=1,\ 2,\ \ldots,\ r, \tag{2.100}$$

and

$$\begin{bmatrix} -TS_j-(TS_j)^T+\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_iTS_j+B_iL_1\,[\,I\quad 0\,] & E_i & -TS_j-(TS_j)^T+\hat{P}_j & * \\ C_{1i}TS_j+D_iL_1\,[\,I\quad 0\,] & F_i & 0 & -I \end{bmatrix}$$

$$+\begin{bmatrix} -TS_i-(TS_i)^T+\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_jTS_i+B_jL_1\,[\,I\quad 0\,] & E_j & -TS_i-(TS_i)^T+\hat{P}_i & * \\ C_{1j}TS_i+D_jL_1\,[\,I\quad 0\,] & F_j & 0 & -I \end{bmatrix}$$

$$=\begin{bmatrix} -TS_j-(TS_j)^T+\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_iTS_j+B_iL_1S_1^{-1}S_1[\,I\quad 0\,] & E_i & -\hat{P}_j & * \\ C_{1i}TS_j+D_iL_1S_1^{-1}S_1[\,I\quad 0\,] & F_i & 0 & -I \end{bmatrix}$$

$$+\begin{bmatrix} -TS_i-(TS_i)^T+\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_jTS_i+B_jL_1S_1^{-1}S_1\,[\,I\quad 0\,] & E_j & -\hat{P}_i & * \\ C_{1i}TS_i+D_iL_1S_1^{-1}S_1\,[\,I\quad 0\,] & F_j & 0 & -I \end{bmatrix}<0, \tag{2.101}$$

$$i,\ j=1,\ 2,\ \ldots,\ r,\ i<j.$$

If the above condition is satisfied, which implies that $TS_j+(TS_j)^T>0$, $j=1,\ 2,\ \ldots,\ r$. Because C_2 is of full row rank, we have

$$\begin{aligned} &C_2(TS_j+T^TS_j^T)C_2^T=C_2TS_jC_2^T+C_2S_j^TT^TC_2^T \\ &=[\,I\quad 0\,]S_jC_2^T+C_2S_j^T[\,I\quad 0\,]^T \\ &=[\,S_1\quad 0\,]C_2^T+C_2[\,S_1\quad 0\,]^T \end{aligned}$$

$$
\begin{aligned}
&= S_1[\, I \quad 0 \,]C_2^T + C_2[\, I \quad 0 \,]^T S_1^T \\
&= S_1 C_2 T C_2^T + C_2 T^T C_2^T S_1^T > 0, \;\; j = 1,\; 2,\; \ldots,\; r.
\end{aligned} \tag{2.102}
$$

However, we are not sure that $S_1 + S_1^T > 0$.

On the other hand, from $C_2 T = [\, I \quad 0 \,]$, we can know that

$$
\begin{aligned}
&C_2 T S_j = [\, I \quad 0 \,] S_j \\
&= [\, I \quad 0 \,] \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix} = [\, S_1 \quad 0 \,] = S_1 [\, I \quad 0 \,], \;\; j = 1,\; 2,\; \ldots,\; r.
\end{aligned} \tag{2.103}
$$

Then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that the following matrix inequality holds:

$$
\begin{aligned}
&\begin{bmatrix} -T S_i - (T S_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i T S_i + B_i L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & E_i & -\hat{P}_i & * \\ C_{1i} T S_i + D_i L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & F_i & 0 & -I \end{bmatrix} \\
&+ \rho \begin{bmatrix} I & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{\beta} \big((L_1 S_1^{-1})^T [\, 0 \quad 0 \quad B_i^T \quad D_i^T \,] \big)^T \\
&\times \Big(2I - \beta \big(C_2 T S_i - S_1 \; [\, I \quad 0 \,] \big) \tfrac{1}{\rho} I \big(C_2 T S_i - S_1 \; [\, I \quad 0 \,] \big)^T \Big)^{-1} \\
&\times \big((L_1 S_1^{-1})^T [\, 0 \quad 0 \quad B_i^T \quad D_i^T \,] \big) < 0, \;\; i = 1,\; 2,\; \ldots,\; r,
\end{aligned} \tag{2.104}
$$

$$
\begin{aligned}
&\begin{bmatrix} -T S_j - (T S_j)^T + \hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_i T S_j + B_i L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & E_i & -\hat{P}_j & * \\ C_{1i} T S_j + D_i L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & F_i & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} -T S_i - (T S_i)^T + \hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_j T S_i + B_j L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & E_j & -\hat{P}_i & * \\ C_{1j} T S_i + D_j L_1 S_1^{-1} S_1 \; [\, I \quad 0 \,] & F_j & 0 & -I \end{bmatrix} \\
&+ \; 2\rho \begin{bmatrix} I & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta} \big((L_1 S_1^{-1})^T [\, 0 \quad 0 \quad B_i^T + B_j^T \quad D_i^T + D_j^T \,] \big)^T
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(4I - \beta\big(C_2TS_i - S_1\,[\,I \quad 0\,] + C_2TS_j - S_1\,[\,I \quad 0\,]\big)\frac{1}{2\rho}I \\
&\times \big(C_2TS_i - S_1\,[\,I \quad 0\,] + C_2TS_j - S_1\,[\,I \quad 0\,]\big)^T\Big)^{-1} \\
&\times \big((L_1S_1^{-1})^T[\,0 \quad 0 \quad B_i^T + B_j^T \quad D_i^T + D_j^T\,]\big) < 0,\ i < j,\ i,\ j = 1,\ 2,\ \ldots,\ r.
\end{aligned} \tag{2.105}
$$

By defining $L = L_1S_1^{-1}$, $U = I$, $G_j = TS_j$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1,\ 2,\ \ldots,\ r$, and applying Schur complement, the LMIs (2.88) and (2.89) in Corollary 2.6 can be obtained. □

Theorem 2.12 *If the condition given in* Lemma 2.8 *holds, the condition in* Corollary 2.7 *also holds.*

Proof Pre- and post-multiplying (2.97) and (2.98) and by $\begin{bmatrix} T & * & * & * \\ 0 & I & * & * \\ 0 & 0 & T & * \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we have

$$
\begin{bmatrix}
-TS_iT^T - TS_i^TT^T + TQ_iT^T & * & * & * \\
0 & -\gamma^2 I & * & * \\
A_iTS_iT^T + B_iL_1\,[I \quad 0\,]T^T & E_i & -TQ_iT^T & * \\
C_{1i}TS_iT^T + D_i\ L_1[I \quad 0\,]T^T & F_i & 0 & -I
\end{bmatrix} < 0,\ i = 1,\ 2,\ \ldots,\ r, \tag{2.106}
$$

$$
\begin{aligned}
&\begin{bmatrix}
-TS_jT^T - TS_j^TT^T + TQ_jT^T & * & * & * \\
0 & -\gamma^2 I & * & * \\
A_iTS_jT^T + B_iL_1\,[I \quad 0\,]T^T & E_i & -TQ_jT^T & * \\
C_{1i}TS_jT^T + D_i\ L_1[I \quad 0\,]T^T & F_i & 0 & -I
\end{bmatrix} \\
&+ \begin{bmatrix}
-TS_iT^T - TS_i^TT^T + TQ_iT^T & * & * & * \\
0 & -\gamma^2 I & * & * \\
A_jTS_iT^T + B_jL_1\,[I \quad 0\,]T^T & E_j & -TQ_iT^T & * \\
C_{1j}TS_iT^T + D_j\ L_1[I \quad 0\,]T^T & F_j & 0 & -I
\end{bmatrix} < 0, \\
&\qquad\qquad\qquad i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j.
\end{aligned} \tag{2.107}
$$

Obviously, these LMIs imply that $S_1 + S_1^T > 0$.
From (2.99) and $C_2T = [\,I \quad 0\,]$, we have

$$
C_2TS_jT^T = [\,I \quad 0\,]S_jT^T = [\,S_1 \quad 0\,]T^T = S_1[\,I \quad 0\,]T^T,\ j = 1,\ 2,\ \ldots,\ r. \tag{2.108}
$$

Similar to the proof of Theorem 2.11, by choosing $G_j = TS_jT^T$, $J_j = \rho I$, $P_j = TQ_jT^T$, $j = 1,\ 2,\ \ldots,\ r$, $L = L_1$, $U = S_1$ and applying Schur complement, the inequalities (2.107) and (2.108) can be changed into (2.90) and (2.91) in Corollary 2.7, respectively. □

2.1.1.3 Several Further Studies

A: Application of Lemma 1.5

The proposed design results in Sects. 2.1.1.1 and 2.1.1.2 can further be relaxed by using LMI technique in Lemma 1.5, which adds more slack matrix variables. To mention a few:

For Theorem 2.2:

Theorem 2.13 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give scalars $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalar β, exist matrices U, V, P_j, J_j, G_j, $j = 1, 2, \ldots, r$, $\hat{\Upsilon}_{ii}$, $i = 1, 2, \ldots, r$, and $\hat{\Upsilon}_{ji}$, $i, j = 1, 2, \ldots, r$, $i < j$ such that the following matrix inequalities hold*

$$\begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_i + B_i V C_{2i} & G_i E_i + B_i V H_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_i - B_i U & -\frac{J_i}{\beta^2} \end{bmatrix} < \hat{\Upsilon}_{ii},$$

$$i =, 2, \ldots, r, \tag{2.109}$$

$$\begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + B_i V C_{2j} & G_j E_i + B_i V H_j & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2j} & V H_j & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_j B_i - B_i U & -\frac{J_j}{\beta^2} \end{bmatrix}$$

$$+ \begin{bmatrix} -P_i & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_i A_j + B_j V C_{2i} & G_i E_j + B_j V H_i & \mathscr{G}_i + J_i & * & * & * \\ C_{1j} & F_j & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & -\beta U - \beta U^T & * \\ 0 & 0 & 0 & 0 & G_i B_j - B_j U & -\frac{J_i}{\beta^2} \end{bmatrix}$$

$$< \hat{\Upsilon}_{ji} + \hat{\Upsilon}_{ji}^T, \; i, \; j = 1, 2, \ldots, r, \; i < j, \tag{2.110}$$

$$\begin{bmatrix} \hat{\Upsilon}_{11} & * & \dots & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Upsilon}_{r1} & \hat{\Upsilon}_{r2} & \dots & \hat{\Upsilon}_{rr} \end{bmatrix} < 0, \tag{2.111}$$

where $\mathscr{G}_j = -G_j - G_j^T + P_j$.

Furthermore, the static output feedback H_∞ *controller gain matrix in* (2.3) *is given by* (2.22).

For Corollary 2.2:

Theorem 2.14 *Consider the closed-loop system* (2.4) *with* $D(\theta) = 0$ *and give scalars* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalar* β, *exist matrices* U, V, P_j, J_j, G_j, $j = 1, 2, \dots, r$, $\hat{\Upsilon}_{ii}$, $i = 1, 2, \dots, r$, *and* $\hat{\Upsilon}_{ji}$, $i, j = 1, 2, \dots, r$, $i < j$ *such that the following matrix inequalities hold*

$$\Pi_{ii} < \hat{\Upsilon}_{ii},\ i = 1,\ 2,\ \dots,\ r, \tag{2.112}$$

$$\Pi_{ij} + \Pi_{ji} < \hat{\Upsilon}_{ji} + \hat{\Upsilon}_{ji}^T,\ i < j,\ i,\ j =,\ 2,\ \dots,\ r, \tag{2.113}$$

$$\begin{bmatrix} \hat{\Upsilon}_{11} & * & \dots & * \\ \hat{\Upsilon}_{21} & \hat{\Upsilon}_{22} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\Upsilon}_{r1} & \hat{\Upsilon}_{r2} & \dots & \hat{\Upsilon}_{rr} \end{bmatrix} < 0, \tag{2.114}$$

with

$$\Pi_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G_j A_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V C_{2i} & G_j E_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 V H_i & \mathscr{G}_j + J_j & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * \\ V C_{2i} & V H_i & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\begin{aligned}\mathcal{G}_j &= -G_j - G_j^T + P_j,\\ \Sigma_1 &= -\beta U - \beta U^T,\\ \Sigma_2 &= G_j B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 U.\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) *is given by* (2.22).

Of course, the relaxed characteristic of Lemma 1.5 can also be applied to the existing design results. To Lemma 2.3 as an example, we have

Lemma 2.9 *Consider the closed-loop system* (2.4) *with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices L, $\hat{P}_j$, R_j, $j = 1, 2, \ldots, r$, Υ_{ii}, $i = 1, 2, \ldots, r$, and Υ_{ji}, $i, j = 1, 2, \ldots, r$, $i < j$ such that the following matrix equations hold*

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} < \Upsilon_{ii},$$
$$i = 1, 2, \ldots, r, \tag{2.115}$$

$$\begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} < \Upsilon_{ji} + \Upsilon_{ji}^T,$$
$$i, j = 1, 2, \ldots, r, \ i < j, \tag{2.116}$$

$$\begin{bmatrix} \Upsilon_{11} & * & \ldots & * \\ \Upsilon_{21} & \Upsilon_{22} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{r1} & \Upsilon_{r2} & \ldots & \Upsilon_{rr} \end{bmatrix} < 0, \tag{2.117}$$

$$R_j = \begin{bmatrix} R_1 & R_{2j} \\ 0 & R_{3j} \end{bmatrix}, \ j = 1,\ 2,\ \ldots,\ r. \tag{2.118}$$

Next, we will study the relationship between the proposed results and the existing ones under considering the relaxed characteristic of Lemma 1.5 (i.e., Theorem 2.14 and Lemma 2.9).

Theorem 2.15 *If the condition given in* Lemma 2.9 *holds, the condition in* Theorem 2.14 *also holds.*

Proof The LMIs (2.115) and (2.116) in Lemma 2.9 can be rewritten as follows:

$$\begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_i Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix} - \Upsilon_{ii} < 0,$$
$$i = 1,\ 2,\ \ldots,\ r, \tag{2.119}$$

$$\begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_j Y A_i + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2i} & R_j Y E_i + \begin{bmatrix} L \\ 0 \end{bmatrix} H_i & -R_j Y - (R_j Y)^T + \hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}$$
$$+ \begin{bmatrix} -\hat{P}_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_i Y A_j + \begin{bmatrix} L \\ 0 \end{bmatrix} C_{2j} & R_i Y E_j + \begin{bmatrix} L \\ 0 \end{bmatrix} H_j & -R_i Y - (R_i Y)^T + \hat{P}_i & * \\ C_{1j} & F_j & 0 & -I \end{bmatrix} - \Upsilon_{ji} - \Upsilon_{ji}^T$$
$$< 0,\ i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.120}$$

Recall (2.56), if the LMIs (2.119) and (2.120) are satisfied, then there exist large enough $\beta > 0$ and small enough $\rho > 0$ such that

$$\Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta}\left(R_1^{-1} L \,[\, C_{2i} \quad H_i \quad 0 \quad 0 \,]\right)^T$$
$$\times \left(2I - \beta\left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)^T \frac{1}{\rho} I \left(R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)\right)^{-1} \tag{2.121}$$
$$\times \left(R_1^{-1} L \,[\, C_{2i} \quad H_i \quad 0 \quad 0 \,]\right) - \Upsilon_{ii} < 0,\ i = 1,\ 2,\ \ldots,\ r,$$

and

$$\Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{\beta}\left(R_1^{-1}L\,[\,C_{2i}+C_{2j} \quad H_i+H_j \quad 0 \quad 0\,]\right)^T$$
$$\times\left(4I - \beta\left(R_jYB - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_iYB - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)^T \frac{1}{2\rho} I \left(R_jYB - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right.\right.$$
$$\left.\left. + R_iYB - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1\right)\right)^{-1} \left(R_1^{-1}L\,[\,C_{2i}+C_{2j} \quad H_i+H_j \quad 0 \quad 0\,]\right) - \Upsilon_{ji} - \Upsilon_{ji}^T < 0,$$
$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j, \tag{2.122}$$

with

$$\Omega_{ij} = \begin{bmatrix} -\hat{P}_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ R_jYA_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LC_{2i} & R_jYE_i + \begin{bmatrix} I \\ 0 \end{bmatrix} R_1R_1^{-1}LH_i & -R_jY-(R_jY)^T+\hat{P}_j & * \\ C_{1i} & F_i & 0 & -I \end{bmatrix}.$$

Using Schur complement to (2.121) and (2.122) yields, respectively, it follows as:

$$\begin{bmatrix} \Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} - \Upsilon_{ii} & * & * \\ R_1^{-1}L\,[\,C_{2i} \quad H_i \quad 0 \quad 0\,] & -\beta I - \beta I & * \\ 0 & R_iYB - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -\frac{\rho I}{\beta^2} \end{bmatrix} < 0,\ i = 1,\ 2,\ \ldots,\ r, \tag{2.123}$$

and

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} - \Upsilon_{ji} - \Upsilon_{ji}^T \\ R_1^{-1}L\,[\,C_{2i}+C_{2j} \quad H_i+H_j \quad 0 \quad 0\,] \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} & * & * \\ & -\beta I - \beta I - \beta I - \beta I & * \\ & R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -2\frac{\rho I}{\beta^2} \end{bmatrix} < 0,$$

$$i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.124}$$

If the above LMIs (2.123) and (2.124) are satisfied, then there exists a small enough $\sigma > 0$ such that

$$\begin{bmatrix} \Omega_{ii} + \rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} & * & * \\ R_1^{-1} L \,[\, C_{2i} \quad H_i \quad 0 \quad 0 \,] & -\beta I - \beta I & * \\ 0 & R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -\frac{\rho I}{\beta^2} \end{bmatrix} < \begin{bmatrix} \Upsilon_{ii} & * & * \\ 0 & -\sigma I & * \\ 0 & 0 & -\sigma I \end{bmatrix},$$

$$i = 1,\ 2,\ \ldots,\ r, \tag{2.125}$$

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} + 2\rho \begin{bmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & I & * \\ 0 & 0 & 0 & 0 \end{bmatrix} & * & * \\ R_1^{-1} L \,[\, C_{2i} + C_{2j} \quad H_i + H_j \quad 0 \quad 0 \,] & -\beta I - \beta I \ \ -\beta I - \beta I & * \\ 0 & R_j Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 + R_i Y B - \begin{bmatrix} I \\ 0 \end{bmatrix} R_1 & -2\frac{\rho I}{\beta^2} \end{bmatrix}$$

$$< \begin{bmatrix} \Upsilon_{ji} & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \Upsilon_{ji}^T & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix},\ i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j. \tag{2.126}$$

Let us consider the following matrix:

$$
\Pi = \begin{bmatrix}
\Upsilon_{11} & * & * & * & * & * & * & * & * & \cdots & * & * & * \\
0 & -\sigma I & * & * & * & * & * & * & * & \cdots & * & * & * \\
0 & 0 & -\sigma I & * & * & * & * & * & * & \cdots & * & * & * \\
\Upsilon_{21} & 0 & 0 & \Upsilon_{22} & * & * & * & * & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & -\sigma I & * & * & * & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & -\sigma I & * & * & * & \cdots & * & * & * \\
\Upsilon_{31} & 0 & 0 & \Upsilon_{32} & 0 & 0 & \Upsilon_{33} & * & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma I & * & \cdots & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma I & \cdots & * & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\Upsilon_{r1} & 0 & 0 & \Upsilon_{r2} & 0 & 0 & \Upsilon_{r3} & 0 & 0 & \cdots & \Upsilon_{rr} & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\sigma I & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I
\end{bmatrix}. \tag{2.127}
$$

By applying the technique of elementary transformation of matrix, the matrix Π can be transformed as follows:

$$
\Gamma = \begin{bmatrix}
\Upsilon_{11} & * & * & \cdots & * & * & * & * & * & \cdots & * & * & * \\
\Upsilon_{21} & \Upsilon_{22} & * & \cdots & * & * & * & * & * & \cdots & * & * & * \\
\Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} & \cdots & * & * & * & * & * & \cdots & * & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\Upsilon_{r1} & \Upsilon_{r2} & \Upsilon_{r3} & \cdots & \Upsilon_{rr} & * & * & * & * & \cdots & * & * & * \\
0 & 0 & 0 & \cdots & 0 & -\sigma I & * & * & * & \cdots & * & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I & * & * & \cdots & * & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\sigma I & * & \cdots & * & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -\sigma I & \cdots & * & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & -\sigma I & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\sigma I & * \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\sigma I
\end{bmatrix}. \tag{2.128}
$$

If the inequality (2.117) is satisfied, which leads to $\Gamma < 0$, i.e, $\Pi < 0$. Let us define

$$
\begin{aligned}
\hat{\Upsilon}_{ii} &= \begin{bmatrix} \Upsilon_{ii} & * & * \\ 0 & -\sigma I & * \\ 0 & 0 & -\sigma I \end{bmatrix}, \ i = 1,\ 2,\ \ldots,\ r, \\
\hat{\Upsilon}_{ji} &= \begin{bmatrix} \Upsilon_{ji} & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \ i,\ j = 1,\ 2,\ \ldots,\ r,\ i < j,
\end{aligned} \tag{2.129}
$$

and unite (2.125) and (2.126), the LMIs (2.112)–(2.114) in Theorem 2.14 can be obtained by defining $G_j = R_j Y$, $P_j = \hat{P}_j$, $J_j = \rho I$, $j = 1,\ 2,\ \ldots,\ r$, $V = R_1^{-1} L$, and $U = I$. $\square$

B: Application of Lemma 1.6

When $D(\theta) = 0$, let us recollect the following H_∞ performance analysis criterion

$$\begin{aligned}&\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}^T \\ &\times P(\theta)\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix} \\ &+\begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T\begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0.\end{aligned} \tag{2.130}$$

By using Lemma 1.6 with

$$\begin{aligned}T &= \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T\begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\ A &= \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}, \\ P &= P(\theta),\end{aligned}$$

and introducing two auxiliary parameter-dependent matrix variables $G(\theta)$ and $M(\theta)$, the matrix inequality (2.130) can be guaranteed by

$$\begin{bmatrix} \Omega_1 & * \\ \Omega_2 & -G(\theta) - G^T(\theta) + P(\theta) \end{bmatrix} < 0. \tag{2.131}$$

where

$$\begin{aligned}\Omega_1 = &\begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T\begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \\ &+\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}^T M^T(\theta) \\ &+M(\theta)\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}, \\ \Omega_2 = &-M^T(\theta) + G(\theta)\begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}.\end{aligned}$$

By defining $M(\theta) = \begin{bmatrix} S(\theta) \\ 0 \end{bmatrix}$ and applying Schur complement to (2.131), it yields

$$\begin{bmatrix} \Phi_{11} & * & * & * \\ \Phi_{21} & -\gamma^2 I & * & * \\ \Phi_{31} & \Phi_{32} & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.132}$$

where

$$
\begin{aligned}
\Phi_{11} &= -P(\theta) + S(\theta)A(\theta) + S(\theta)B(\theta)KC_2(\theta) \\
&\quad + A^T(\theta)S^T(\theta) + C_2^T(\theta)K^T B^T(\theta)S^T(\theta), \\
\Phi_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)K^T B^T(\theta)S^T(\theta), \\
\Phi_{31} &= -S^T(\theta) + G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta), \\
\Phi_{32} &= G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta), \\
\mathscr{G}(\theta) &= -G(\theta) - G^T(\theta) + P(\theta).
\end{aligned}
$$

Remark 2.12 Compared with the condition (2.5), the H_∞ performance analysis criterion (2.132) is more relaxed. In fact, when $M(\theta) = 0$, the condition (2.132) reduces to (2.5) of Lemma 2.1 that implies that the condition (2.5) is a special case of (2.132).

Next, we derive new static output feedback H_∞ controller design results based on the H_∞ performance analysis criterion (2.132).

Obviously, the matrix inequality (2.132) is equivalent to

$$
\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K
$$

$$
\times \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T < 0, \tag{2.133}
$$

where $\Omega = -P(\theta) + S(\theta)A(\theta) + A^T(\theta)S^T(\theta)$.

By defining $UK = V$ and considering matrices M, R, and N, where U and N are nonsingular without loss of generality, we have

$$
\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} + \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV
$$

$$
\times \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T
$$

$$
\begin{aligned}
&= \begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&\quad + \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} R \\ 0 \\ M \\ 0 \end{bmatrix} V \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T \begin{bmatrix} R \\ 0 \\ M \\ 0 \end{bmatrix}^T \\
&= \begin{bmatrix} \Pi & * & * & * \\ E^T(\theta)S^T(\theta) + H^T(\theta)V^T R^T & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0,
\end{aligned}
\tag{2.134}
$$

where $\Pi = -P(\theta) + S(\theta)A(\theta) + A^T(\theta)S^T(\theta) + RVC_2(\theta) + C_2^T(\theta)V^T R^T$.

By Lemma 1.3, for two positive matrices $J(\theta)$ and $X(\theta)$, we have

$$\begin{aligned}
&\begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) - MU \\ 0 \end{bmatrix}^T \\
&\leq \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix} J(\theta) \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T J^{-1}(\theta) \\
&\quad \times \big(G(\theta)B(\theta) - MU\big) U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix},
\end{aligned} \tag{2.135}$$

and

$$\begin{aligned}
&\begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) - RU \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
&\leq \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} X(\theta) \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(S(\theta)B(\theta) - RU\big)^T X^{-1}(\theta) \\
&\quad \times \big(S(\theta)B(\theta) - RU\big) U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{2.136}$$

Based on (2.135) and (2.136), we can use the following matrix inequality to ensure (2.134):

$$\begin{bmatrix} \Pi + X(\theta) & * & * & * \\ E^T(\theta)S^T(\theta) + H^T(\theta)V^T R^T & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) + MVC_2(\theta) & G(\theta)E(\theta) + MVH(\theta) & \mathscr{G}(\theta) + J(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix}$$

$$+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(G(\theta)B(\theta) - MU\big)^T J^{-1}(\theta)$$

$$\times \big(G(\theta)B(\theta) - MU\big)U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \big(S(\theta)B(\theta) - RU\big)^T X^{-1}(\theta)$$

$$\times \big(S(\theta)B(\theta) - RU\big)U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} < 0. \tag{2.137}$$

Follow the same procedure of Theorem 2.1, strictly, the final H_∞ performance analysis criterion is summarized by

$$\begin{bmatrix} \Pi + X(\theta) & * & * & * & * & * & * & * \\ \Theta_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \mathscr{G}(\theta) + J(\theta) & * & * & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Sigma_1 & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & 0 & \Sigma_3 & * & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0, \tag{2.138}$$

where

$$\begin{aligned} \Theta_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)V^T R^T, \\ \Theta_{31} &= -S^T(\theta) + G(\theta)A(\theta) + MVC_2(\theta), \\ \Theta_{32} &= G(\theta)E(\theta) + MVH(\theta), \\ \Sigma_1 &= -\beta NU - \beta U^T N^T, \\ \Sigma_2 &= G(\theta)B(\theta) - MU, \\ \Sigma_3 &= -\eta NU - \eta U^T N^T, \\ \Sigma_4 &= S(\theta)B(\theta) - RU. \end{aligned}$$

Remark 2.13 In fact, the H_∞ performance analysis criterion (2.138) can be relaxed further by adding another matrix variable $\tilde{N}$. Let us rewrite the matrix inequality (2.133) as follows:

$$\begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} K\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T$$

$$+\begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix} K\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \Omega & * & * & * \\ E^T(\theta)S^T(\theta) & -\gamma^2 I & * & * \\ -S^T(\theta) + G(\theta)A(\theta) & G(\theta)E(\theta) & \mathscr{G}(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix} U^{-1}N^{-1}NV\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)B(\theta) \\ 0 \end{bmatrix}^T$$

$$+\begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}\tilde{N}^{-1}\tilde{N}V\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T \tilde{N}^T \tilde{N}^{-T} U^{-T} \begin{bmatrix} S(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T.$$

Then, the H_∞ performance analysis criterion (2.138) becomes

$$\begin{bmatrix} \Pi + X(\theta) & * & * & * & * & * & * & * \\ \Theta_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \mathscr{G}(\theta) + J(\theta) & * & * & * & * & * \\ C_1(\theta) & F(\theta) & 0 & -I & * & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Sigma_1 & * & * & * \\ \tilde{N}VC_2(\theta) & \tilde{N}VH(\theta) & 0 & 0 & 0 & \tilde{\Sigma}_3 & * & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0,$$

with $\tilde{\Sigma}_3 = -\eta\tilde{N}U - \eta U^T\tilde{N}^T$.

Based on (2.138), the static output feedback H_∞ controller design result is given in the following theorem.

Theorem 2.16 *Consider the closed-loop system (2.4) with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N, R and scalar β, η, exist matrices U, V, P_j, J_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\Delta_{ii} < 0, \ i = 1, \ 2, \ \ldots, \ r, \tag{2.139}$$

$$\Delta_{ij} + \Delta_{ji} < 0, \ i < j, \ i, \ j = 1, \ 2, \ \ldots, \ r, \tag{2.140}$$

with

$$\Delta_{ij} = \begin{bmatrix} \Lambda_{11} & * & * & * & * & * & * & * \\ \Lambda_{21} & -\gamma^2 I & * & * & * & * & * & * \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & * & * & * & * & * \\ C_{1i} & F_i & 0 & -I & * & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & -\beta NU - \beta U^T N^T & * & * & * \\ 0 & 0 & 0 & 0 & G_j B_i - MU & -\frac{J_j}{\beta^2} & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & 0 & 0 & \Xi_3 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Xi_4 & -\frac{X_j}{\eta^2} \end{bmatrix},$$

and

$$\begin{aligned} \Lambda_{11} &= -P_j + S_j A_i + A_i^T S_j^T + RVC_{2i} + C_{2i}^T V^T R^T + X_j, \\ \Lambda_{21} &= E_i^T S_j^T + H_i^T V^T R^T, \\ \Lambda_{31} &= -S_j^T + G_j A_i + MVC_{2i}, \\ \Lambda_{32} &= G_j E_i + MVH_i, \\ \Lambda_{33} &= -G_j - G_j^T + P_j + J_j, \\ \Xi_3 &= -\eta NU - \eta U^T N^T, \\ \Xi_4 &= S_j B_i - RU. \end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

C: An Extended Study for the Case $D(\theta) \neq 0$ and $H(\theta) \neq 0$

Approach 1:

In this case, the condition (2.5) is replaced to

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta) & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & \Lambda_1 & * \\ S(\theta)C_1(\theta) + S(\theta)D(\theta)KC_2(\theta) & S(\theta)F(\theta) + S(\theta)D(\theta)KH(\theta) & 0 & \Lambda_2 \end{bmatrix} < 0, \tag{2.141}$$

where

$$\Lambda_1 = -G(\theta) + G^T(\theta) + P(\theta),$$

$$\Lambda_2 = -S(\theta) + S^T(\theta) + I.$$

By considering a similar process with the condition (2.138), we can obtain the following H_∞ performance analysis criterion:

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * & * & * \\ \Theta_{31} & \Theta_{32} & \Theta_{33} & * & * & * & * & * \\ \Theta_{41} & \Theta_{42} & 0 & \Theta_{44} & * & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & -\beta NU - \beta U^T N^T & * & * & * \\ NVC_2(\theta) & NVH(\theta) & 0 & 0 & 0 & \Sigma_3 & * & * \\ 0 & 0 & 0 & 0 & G(\theta)B(\theta) - MU & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_4 & 0 & -\frac{X(\theta)}{\eta^2} \end{bmatrix} < 0, \tag{2.142}$$

where

$$\Theta_{31} = G(\theta)A(\theta) + MVC_2(\theta),$$

$$\Theta_{32} = G(\theta)E(\theta) + MVH(\theta),$$

$$\Theta_{33} = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Theta_{41} = S(\theta)C_1(\theta) + RVC_2(\theta),$$

$$\Theta_{42} = S(\theta)F(\theta) + RVH(\theta),$$

$$\Theta_{44} = -S(\theta) - S^T(\theta) + I + X(\theta),$$

$$\Sigma_3 = -\eta NU - \eta U^T N^T,$$

$$\tilde{\Sigma}_4 = S(\theta)D(\theta) - RU.$$

Theorem 2.17 *Consider the closed-loop system (2.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N, R and scalars β, η, exist matrices U, V, P_j, J_j, S_j, X_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\Delta_{ii} < 0, \ i = 1, 2, \ldots, r, \tag{2.143}$$

$$\Delta_{ij} + \Delta_{ji} < 0, \ i < j, \ i, \ j = 1, 2, \ldots, r, \tag{2.144}$$

with

$$\Delta_{ij}=\begin{bmatrix} -P_j & * & * & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * & * & * \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & * & * & * & * & * \\ \Lambda_{41} & \Lambda_{42} & 0 & \Lambda_{44} & * & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & -\beta NU-\beta U^TN^T & * & * & * \\ NVC_{2i} & NVH_i & 0 & 0 & 0 & \Sigma_3 & * & * \\ 0 & 0 & 0 & 0 & G_jB_i-MU & 0 & -\frac{J_j}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & S_jD_i-RU & 0 & -\frac{X_j}{\eta^2} \end{bmatrix},$$

and

$$\begin{aligned}
\Lambda_{31} &= G_jA_i+MVC_{2i}, \\
\Lambda_{32} &= G_jE_i+MVH_i, \\
\Lambda_{33} &= -G_j-G_j^T+P_j+J_j, \\
\Lambda_{41} &= S_jC_{1i}+RVC_{2i}, \\
\Lambda_{44} &= -S_j-S_j^T+I+X_j, \\
\Lambda_{42} &= S_jF_i+RVH_i, \\
\Sigma_3 &= -\eta NU-\eta U^TN^T.
\end{aligned}$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) *is given by* (2.22).

Approach 2:

Let us consider (2.65) with $H(\theta)\neq 0$, it follows that

$$\begin{bmatrix} -G(\theta)-G^T(\theta)+P(\theta) & * & * & * \\ 0 & -S(\theta)-S^T(\theta)+\frac{1}{\gamma^2}I & * & * \\ A(\theta)G(\theta)+B(\theta)KC_2(\theta)G(\theta) & E(\theta)S(\theta)+B(\theta)KH(\theta)S(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta)+D(\theta)KC_2(\theta)G(\theta) & F(\theta)S(\theta)+D(\theta)KH(\theta)S(\theta) & 0 & -I \end{bmatrix}$$

$$<0. \tag{2.145}$$

By defining $K=LU^{-1}$ and adding a nonsingular matrix N, the matrix inequality (2.145) can be rewritten as the following form:

$$\begin{bmatrix} -G(\theta)-G^T(\theta)+P(\theta) & * & * & * \\ 0 & -S(\theta)-S^T(\theta)+\frac{1}{\gamma^2}I & * & * \\ A(\theta)G(\theta) & E(\theta)S(\theta) & -P(\theta) & * \\ C_1(\theta)G(\theta) & F(\theta)S(\theta) & 0 & -I \end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1}\begin{bmatrix} C_2(\theta)G(\theta) & H(\theta)S(\theta) & 0 & 0 \end{bmatrix} \\
&+\begin{bmatrix} C_2(\theta)G(\theta) & H(\theta)S(\theta) & 0 & 0 \end{bmatrix}^T U^{-T}N^{-T}N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0.
\end{aligned} \tag{2.146}
$$

Considering two matrices M and R, from (2.146), we have

$$
\begin{aligned}
&\begin{bmatrix} -G(\theta) - G^T(\theta) + P(\theta) & * & * & * \\ 0 & -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2}I & * & * \\ A(\theta)G(\theta) + B(\theta)LM & E(\theta)S(\theta) + B(\theta)LR & -P(\theta) & * \\ C_1 G(\theta) + D(\theta)LM & F(\theta)S(\theta) + D(\theta)LR & 0 & -I \end{bmatrix} \\
&+\begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix} LNN^{-1}U^{-1}\begin{bmatrix} C_2(\theta)G(\theta) - UM & H(\theta)S(\theta) - UR & 0 & 0 \end{bmatrix} \\
&+\begin{bmatrix} C_2(\theta)G(\theta) - UM & H(\theta)S(\theta) - UR & 0 & 0 \end{bmatrix}^T U^{-T}N^{-T}N^T L^T \begin{bmatrix} 0 \\ 0 \\ B(\theta) \\ D(\theta) \end{bmatrix}^T < 0.
\end{aligned} \tag{2.147}
$$

Then, we can give the following H_∞ performance analysis criterion:

$$
\begin{bmatrix}
\tilde{\Gamma}_{11} & * & * & * & * & * & * & * \\
0 & \tilde{\Gamma}_{22} & * & * & * & * & * & * \\
\tilde{\Gamma}_{31} & \tilde{\Gamma}_{32} & -P(\theta) & * & * & * & * & * \\
\tilde{\Gamma}_{41} & \tilde{\Gamma}_{42} & 0 & -I & * & * & * & * \\
0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & \Sigma_1 & * & * & * \\
0 & 0 & N^T L^T B^T(\theta) & N^T L^T D^T(\theta) & 0 & \Sigma_2 & * & * \\
0 & 0 & 0 & 0 & \tilde{\Sigma}_3 & 0 & -\frac{J(\theta)}{\beta^2} & * \\
0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_4 & 0 & -\frac{X(\theta)}{\eta^2}
\end{bmatrix} < 0, \tag{2.148}
$$

where

$$
\tilde{\Gamma}_{11} = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),
$$

$$
\tilde{\Gamma}_{22} = -S(\theta) - S^T(\theta) + \frac{1}{\gamma^2}I + X(\theta),
$$

$$\tilde{\Gamma}_{31} = A(\theta)G(\theta) + B(\theta)LM,$$
$$\tilde{\Gamma}_{32} = E(\theta)S(\theta) + B(\theta)LR,$$
$$\tilde{\Gamma}_{41} = C_1(\theta)G(\theta) + D(\theta)LM,$$
$$\tilde{\Gamma}_{42} = F(\theta)S(\theta) + D(\theta)LR,$$
$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$
$$\Sigma_2 = -\eta UN - \eta N^T U^T,$$
$$\tilde{\Sigma}_3 = \big(C_2(\theta)G(\theta) - UM\big)^T,$$
$$\tilde{\Sigma}_4 = \big(H(\theta)S(\theta) - UR\big)^T.$$

Theorem 2.18 *Consider the closed-loop system (2.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N, R and scalar β, η, exist matrices U, L, P_j, J_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\Delta_{ii} < 0, \ i = 1, 2, \ldots, r, \tag{2.149}$$
$$\Delta_{ij} + \Delta_{ji} < 0, \ i < j, \ i, \ j = 1, 2, \ldots, r, \tag{2.150}$$

with

$$\Delta_{ij} = \begin{bmatrix} \Gamma_{11} & * & * & * & * & * & * & * & * \\ 0 & \Gamma_{22} & * & * & * & * & * & * & * \\ \Gamma_{31} & \Gamma_{32} & -P(\theta) & * & * & * & * & * & * \\ \Gamma_{41} & \Gamma_{42} & 0 & -I & * & * & * & * & * \\ 0 & 0 & N^T L^T B_i^T & N^T L^T D_i^T & \Sigma_1 & * & * & * & * \\ 0 & 0 & N^T L^T B_i^T & N^T L^T D_i^T & 0 & \Sigma_2 & * & * & * \\ 0 & 0 & 0 & 0 & \Sigma_3 & 0 & -\frac{J_j}{\beta^2} & * & * \\ 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{X_j}{\eta^2} & * \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix},$$

and

$$\Gamma_{11} = -G_j - G_j^T + P_j + J_j,$$
$$\Gamma_{22} = -S_j - S_j^T + X_j,$$
$$\Gamma_{31} = A_i G_j + B_i LM,$$
$$\Gamma_{32} = E_i S_j + B_i LR,$$
$$\Gamma_{41} = C_{1i} G_j + D_i LM,$$
$$\Gamma_{42} = F_i S_j + D_i LR,$$

$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$
$$\Sigma_2 = -\eta UN - \eta N^T U^T,$$
$$\Sigma_3 = (C_{2i} G_j - UM)^T,$$
$$\Sigma_4 = (H_i S_j - UR)^T.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) *is given by* (2.85).

2.1.2 Continuous-Time Systems

Consider the following continuous-time linear systems system with polytopic uncertainties

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + E(\theta)w(t), \\ z(t) &= C_1(\theta)x(t) + D(\theta)u(t) + F(\theta)w(t), \\ y(t) &= C_2(\theta)x(t) + H(\theta)w(t), \end{aligned} \tag{2.151}$$

where $x(t) \in \mathscr{R}^n$ is the state variable, $u(t) \in \mathscr{R}^m$ is the control input, $w(t) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $L_2[0, \infty)$, $z(t) \in \mathscr{R}^q$ is the controlled output variable, $y(t) \in \mathscr{R}^p$ is the measurement output. The system matrices $A(\theta)$, $B(\theta)$, $E(\theta)$, $C_1(\theta)$, $D(\theta)$, $F(\theta)$, $C_2(\theta)$, and $H(\theta)$ belong to (2.2).

Here, we consider the following static output feedback H_∞ controller

$$u(t) = Ky(t). \tag{2.152}$$

By substituting (2.152) to (2.151), the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= \big(A(\theta) + B(\theta)KC_2(\theta)\big)x(t) + \big(E(\theta) + B(\theta)KH(\theta)\big)w(t), \\ z(t) &= \big(C_1(\theta) + D(\theta)KC_2(\theta)\big)x(t) + \big(F(\theta) + D(\theta)KH(\theta)\big)w(t). \end{aligned} \tag{2.153}$$

2.1.2.1 Case A: $D(\theta) = 0$

First, we will study new H_∞ performance analysis criterions for the continuous-time closed-loop system (2.153) with $D(\theta) = 0$. In this following, a basic lemma is given.

Theorem 2.19 *Consider the closed-loop system* (2.153) *with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $P(\theta)$, $G(\theta)$, $X(\theta)$, and K such that the following matrix inequality holds*

$$\begin{bmatrix} -G(\theta)-G^T(\theta) & * & * & * & * \\ \big(A(\theta)+B(\theta)KC_2(\theta)\big)^T G^T(\theta)+P(\theta) & -2P(\theta)+X(\theta) & * & * & * \\ \big(E(\theta)+B(\theta)KH(\theta)\big)^T G^T(\theta) & 0 & -\gamma^2 I & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix}$$
$$< 0. \tag{2.154}$$

Proof Construct a parameter-dependent Lyapunov function as

$$V(t)=x^T(t)P(\theta)x(t),\ \ P(\theta)>0. \tag{2.155}$$

The derivative of $V(t)$ can be given by

$$\dot{V}(t)=\dot{x}^T(t)P(\theta)x(t)+x^T(t)P(\theta)\dot{x}(t). \tag{2.156}$$

Then, we have

$$\begin{aligned} &\dot{V}(t)+z^T(t)z(t)-\gamma^2 w^T(t)w(t)\\ &=\dot{x}^T(t)P(\theta)x(t)+x^T(t)P(\theta)\dot{x}(t)+z^T(t)z(t)-\gamma^2 w^T(t)w(t)\\ &=\Big(\big(A(\theta)+B(\theta)KC_2(\theta)\big)x(t)+\big(E(\theta)+B(\theta)KH(\theta)\big)w(t)\Big)^T P(\theta)x(t)\\ &\quad +x^T(t)P(\theta)\Big(\big(A(\theta)+B(\theta)KC_2(\theta)\big)x(t)+\big(E(\theta)+B(\theta)KH(\theta)\big)w(t)\Big)\\ &\quad +\big(C_1(\theta)x(t)+F(\theta)w(t)\big)^T\big(C_1(\theta)x(t)+F(\theta)w(t)\big)-\gamma^2 w^T(t)w(t)\\ &=\begin{bmatrix} x^T(t) & w^T(t)\end{bmatrix}\begin{bmatrix} \Omega & * \\ \big(E(\theta)+B(\theta)KH(\theta)\big)^T P(\theta) & -\gamma^2 I\end{bmatrix}\begin{bmatrix} x(t)\\ w(t)\end{bmatrix}\\ &\quad +\begin{bmatrix} x^T(t) & w^T(t)\end{bmatrix}\begin{bmatrix} C_1(\theta) & F(\theta)\end{bmatrix}^T\begin{bmatrix} C_1(\theta) & F(\theta)\end{bmatrix}\begin{bmatrix} x(t)\\ w(t)\end{bmatrix}, \end{aligned} \tag{2.157}$$

where $\Omega=P(\theta)\big(A(\theta)+B(\theta)KC_2(\theta)\big)+\big(A(\theta)+B(\theta)KC_2(\theta)\big)^T P(\theta)$.

If the following inequality

$$\begin{bmatrix} \Omega & * \\ \big(E(\theta)+B(\theta)KH(\theta)\big)^T P(\theta) & -\gamma^2 I\end{bmatrix}+\begin{bmatrix} C_1(\theta) & F(\theta)\end{bmatrix}^T\begin{bmatrix} C_1(\theta) & F(\theta)\end{bmatrix}<0, \tag{2.158}$$

holds, we have $\dot{V}(t)+z^T(t)z(t)-\gamma^2 w^T(t)w(t)<0$ for any $\begin{bmatrix} x(t)\\ w(t)\end{bmatrix}\neq 0$.

By using Schur complement, (2.158) leads to

$$\begin{bmatrix} \Omega & * & * \\ \big(E(\theta)+B(\theta)KH(\theta)\big)^T P(\theta) & -\gamma^2 I & * \\ C_1(\theta) & F(\theta) & -I \end{bmatrix} < 0. \tag{2.159}$$

Then, based on Lemma 1.7 with considering a parameter-dependent matrix $X(\theta)$ and $V = G(\theta)$, the matrix inequality (2.159) can be guarantee by (2.154). If the matrix inequality (2.154) is satisfied, we have $\dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0$ for any $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \neq 0$, which implies that

$$V(\infty) - V(0) + \int_0^\infty e^T(t)e(t)dt - \gamma^2 \int_0^\infty w^T(t)w(t)dt < 0.$$

With zero initial condition $x(0) = 0$ and $V(\infty) > 0$, we obtain $\int_0^\infty e^T(t)e(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt$ for any nonzero $w(t) \in L_2[0, \ \infty)$. Thus, the proof is completed. □

Based on the discussion in Theorem 2.19, we introduce our main results on H_∞ performance analysis for the continuous-time closed-loop system (2.153) with $D(\theta) = 0$. Obviously, the matrix inequality (2.154) is equivalent to

$$\begin{bmatrix} -G(\theta)-G^T(\theta) & * & * & * & * \\ A^T(\theta)G^T(\theta)+P(\theta) & -2P(\theta)+X(\theta) & * & * & * \\ E^T(\theta)G^T(\theta) & 0 & -\gamma^2 I & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} + \begin{bmatrix} G(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\times K\begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T K^T \begin{bmatrix} G(\theta)B(\theta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0. \tag{2.160}$$

Then, by following the same derivation with Theorem 2.1, we give the following H_∞ performance analysis criterion:

Theorem 2.20 *Consider the closed-loop system* (2.153) *with* $D(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P(\theta)$, $G(\theta)$, $X(\theta)$, $J(\theta)$, M, N, V, *and* U, *scalar* β *such that the following matrix inequality holds*

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * & * \\ \Xi_{21} & -2P(\theta)+X(\theta) & * & * & * & * & * \\ \Xi_{31} & 0 & -\gamma^2 I & * & * & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Lambda_1 & * \\ 0 & 0 & 0 & 0 & 0 & \Lambda_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.161}$$

where

$$\begin{aligned} \Xi_{11} &= -G(\theta) - G^T(\theta) + J(\theta), \\ \Xi_{21} &= A^T(\theta)G^T(\theta) + P(\theta) + C_2^T(\theta)V^T M^T, \\ \Xi_{31} &= E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T, \\ \Lambda_1 &= -\beta NU - \beta U^T N^T, \\ \Lambda_2 &= G(\theta)B(\theta) - MU. \end{aligned}$$

By the analysis condition (2.161), the corresponding static output feedback H_∞ controller design result is given in the following theorem.

Theorem 2.21 *Consider the closed-loop system* (2.153) *with* $D(\theta) = 0$ *and give a scalar* $\gamma > 0$*. Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* U, L, P_j, J_j, *and* G_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold*

$$\Upsilon_{ii} < 0, \; i = 1, 2, \ldots, r, \tag{2.162}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \; i < j, \; i, \; j = 1, 2, \ldots, r, \tag{2.163}$$

with

$$\Upsilon_{ij} = \begin{bmatrix} \Upsilon_{11} & * & * & * & * & * & * \\ \Upsilon_{21} & -2P_j + X_j & * & * & * & * & * \\ \Upsilon_{31} & 0 & -\gamma^2 I & * & * & * & * \\ 0 & C_{1i} & F_i & -I & * & * & * \\ G_j^T & 0 & 0 & 0 & -X_j & * & * \\ 0 & NVC_{2i} & NVH_i & 0 & 0 & \Phi_1 & * \\ 0 & 0 & 0 & 0 & 0 & \Phi_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\begin{aligned} \Upsilon_{11} &= -G_j - G_j^T + J_j, \\ \Upsilon_{21} &= A_i^T G_j^T + P_j + C_{2i}^T V^T M^T, \\ \Upsilon_{31} &= E_i^T G_j^T + H_i^T V^T M^T, \\ \Phi_1 &= -\beta NU - \beta U^T N^T, \end{aligned}$$

$$\Phi_2 = G_j B_i - MU.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) is given by (2.22).

2.1.2.2 Case B: $H(\theta) = 0$

In this case, by choosing a parameter-dependent Lyapunov function as $V(t) = x^T(t)Q^{-1}(\theta)x(t),\ Q(\theta) > 0$ and using Lemma 1.8, the basic lemma is given as follows.

Lemma 2.10 *Consider the closed-loop system (2.153) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices $Q(\theta)$, $G(\theta)$, $X(\theta)$, and K such that the following matrix inequality holds*

$$\begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ \big(A(\theta) + B(\theta)KC_2(\theta)\big)G(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * \\ \big(C_1(\theta) + D(\theta)KC_2(\theta)\big)G(\theta) & 0 & F(\theta) & -I & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * \\ A(\theta)G(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * \\ C_1(\theta)G(\theta) & 0 & F(\theta) & -I & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ B(\theta) \\ 0 \\ D(\theta) \\ 0 \end{bmatrix}$$

$$\times K[\, C_2(\theta)G(\theta) \quad 0 \quad 0 \quad 0 \quad 0\,] + [\, C_2(\theta)G(\theta) \quad 0 \quad 0 \quad 0 \quad 0\,]^T K^T \begin{bmatrix} 0 \\ B(\theta) \\ 0 \\ D(\theta) \\ 0 \end{bmatrix}^T < 0. \tag{2.164}$$

By applying the same process with Theorem 2.9, we obtain a new H_∞ performance analysis criterion for the continuous-time closed-loop system with $H(\theta) = 0$, which is given in the following theorem.

Theorem 2.22 *Consider the closed-loop system (2.153) with $H(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices M, N, U, L, $Q(\theta)$, $G(\theta)$, $X(\theta)$, and $J(\theta)$, scalar β such that the following matrix inequality holds*

$$\begin{bmatrix} \Omega_{11} & * & * & * & * & * & * \\ \Omega_{21} & -2Q(\theta)+X(\theta) & * & * & * & * & * \\ 0 & E^T(\theta) & -\gamma^2 I & * & * & * & * \\ \Omega_{41} & 0 & F(\theta) & -I & * & * & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{2.165}$$

where

$$\Omega_{11} = -G(\theta) - G^T(\theta) + J(\theta),$$

$$\Omega_{21} = A(\theta)G(\theta) + B(\theta)LM + Q(\theta),$$

$$\Omega_{41} = C_1(\theta)G(\theta) + D(\theta)LM,$$

$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$

$$\tilde{\Sigma}_2 = \left(C_2(\theta)G(\theta) - UM\right)^T.$$

Further, we have the following design result:

Theorem 2.23 *Consider the closed-loop system* (2.153) *with* $H(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* U, L, Q_j, X_j, J_j, *and* G_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold*

$$\Lambda_{ii} < 0, \ i = 1, 2, \ldots, r, \tag{2.166}$$

$$\Lambda_{ij} + \Lambda_{ji} < 0, \ i < j, \ i, \ j = 1, 2, \ldots, r, \tag{2.167}$$

with

$$\Lambda_{ij} = \begin{bmatrix} \Psi_{11} & * & * & * & * & * & * \\ \Psi_{21} & -2Q_j + X_j & * & * & * & * & * \\ 0 & E_i^T & -\gamma^2 I & * & * & * & * \\ \Psi_{41} & 0 & F_i & -I & * & * & * \\ G_j & 0 & 0 & 0 & -X_j & * & * \\ 0 & N^T L^T B_i^T & 0 & N^T L^T D_i^T & 0 & -\beta UN - \beta N^T U^T & * \\ 0 & 0 & 0 & 0 & 0 & \left(C_{2i}G_j - UM\right)^T & -\frac{J_j}{\beta^2} \end{bmatrix},$$

where

$$\Psi_{11} = -G_j - G_j^T + J_j,$$
$$\Psi_{21} = A_i G_j + B_i LM + Q_j,$$
$$\Psi_{41} = C_{1i} G_j + D_i LM.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.3) *is given by* (2.85).

Remark 2.14 Here, we discuss briefly the application of Lemma 1.9 to design static output feedback H_∞ controllers for the continuous-time closed-loop system (2.153). Let us rewrite the H_∞ performance analysis condition (2.158) as

$$\begin{aligned}
&\begin{bmatrix} P(\theta) \\ 0 \end{bmatrix} \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix} \\
&\quad + \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}^T \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix}^T \\
&\quad + \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix} < 0.
\end{aligned} \tag{2.168}$$

By using Lemma 1.9 with

$$\begin{aligned}
T &= \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix}^T \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} + \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix}, \\
A &= \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}, \\
P &= \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix},
\end{aligned}$$

and Schur complement, we see that the matrix inequality (2.167) can be verified by

$$\begin{bmatrix} \begin{bmatrix} 0 & * \\ 0 & -\gamma^2 I \end{bmatrix} + M(\theta)\tilde{A} + \tilde{A}^T M^T(\theta) & * & * \\ \begin{bmatrix} P(\theta) & 0 \end{bmatrix} - M^T(\theta) + G(\theta)\tilde{A} & -G(\theta) - G^T(\theta) & * \\ \begin{bmatrix} C_1(\theta) & F(\theta) \end{bmatrix} & 0 & -I \end{bmatrix} < 0, \tag{2.169}$$

where $\tilde{A} = \begin{bmatrix} A(\theta) + B(\theta)KC_2(\theta) & E(\theta) + B(\theta)KH(\theta) \end{bmatrix}$.

Now, select $M(\theta) = \begin{bmatrix} S(\theta) \\ 0 \end{bmatrix}$. Thus, (2.168) is equivalent to

$$\begin{bmatrix} \Omega_{11} & * & * & * \\ \Omega_{21} & -\gamma^2 I & * & * \\ \Omega_{31} & G(\theta)E(\theta) + G(\theta)B(\theta)KH(\theta) & -G(\theta) - G^T(\theta) & * \\ C_1(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{2.170}$$

where

$$\begin{aligned}
\Omega_{11} &= S(\theta)A(\theta) + S(\theta)B(\theta)KC_2(\theta) + A^T(\theta)S^T(\theta) + C_2^T(\theta)K^T B^T(\theta)S^T(\theta), \\
\Omega_{21} &= E^T(\theta)S^T(\theta) + H^T(\theta)K^T B^T(\theta)S^T(\theta),
\end{aligned}$$

$$\Omega_{31} = P(\theta) - S^T(\theta) + G(\theta)A(\theta) + G(\theta)B(\theta)KC_2(\theta).$$

Then, the remaining discussion can reference Theorem 2.16, and it is omitted.

Remark 2.15 For the case $D(\theta) \neq 0$ and $H(\theta) \neq 0$, we can apply the following two H_∞ performance analysis criterions to design static output feedback H_∞ controllers, which are obtained easily from (2.161) and (2.165).

$$\begin{bmatrix} \Xi_{11} & * & * & * & * & * & * & * & * \\ \Xi_{21} & -2P(\theta)+X(\theta) & * & * & * & * & * & * & * \\ \Xi_{31} & 0 & -\gamma^2 I & * & * & * & * & * & * \\ 0 & \Xi_{42} & \Xi_{43} & \Xi_{44} & * & * & * & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * & * & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Lambda_1 & * & * & * \\ 0 & NVC_2(\theta) & NVH(\theta) & 0 & 0 & 0 & \Lambda_3 & * & * \\ 0 & 0 & 0 & 0 & 0 & \Lambda_2 & 0 & -\frac{J(\theta)}{\beta^2} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_4 & 0 & -\frac{W(\theta)}{\eta^2} \end{bmatrix} < 0, \tag{2.171}$$

where Ξ_{11}, Ξ_{21}, Ξ_{31}, Λ_1, and Λ_2 are defined in (2.161), and

$$\begin{aligned} \Xi_{42} &= S(\theta)C_1(\theta) + RVC_2(\theta), \\ \Xi_{43} &= S(\theta)F(\theta) + RVH(\theta), \\ \Xi_{44} &= -S(\theta) - S^T(\theta) + I + W(\theta), \\ \Lambda_3 &= -\eta NU - \eta U^T N^T, \\ \Lambda_4 &= S(\theta)D(\theta) - RU. \end{aligned}$$

and

$$\begin{bmatrix} \Omega_{11} & * & * & * & * & * & * & * & * & * \\ \Omega_{21} & -2Q(\theta)+X(\theta) & * & * & * & * & * & * & * & * \\ 0 & \Omega_{32} & \Omega_{33} & * & * & * & * & * & * & * \\ \Omega_{41} & 0 & \Omega_{43} & -I & * & * & * & * & * & * \\ G(\theta) & 0 & 0 & 0 & -X(\theta) & * & * & * & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & \Sigma_1 & * & * & * & * \\ 0 & N^T L^T B^T(\theta) & 0 & N^T L^T D^T(\theta) & 0 & 0 & \Sigma_3 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_2 & 0 & -\frac{J(\theta)}{\beta^2} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_4 & 0 & -\frac{W(\theta)}{\beta^2} & * \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$< 0, \tag{2.172}$$

where Ω_{11}, Ω_{21}, Ω_{41}, Σ_1, and Σ_2 are defined in (2.165), and

$$\Omega_{32} = S^T(\theta)E^T(\theta) + R^T L^T B^T(\theta),$$

$$\begin{aligned}
\Omega_{33} &= -S(\theta) - S^T(\theta) + W(\theta),\\
\Omega_{43} &= F(\theta)S(\theta) + D(\theta)LR,\\
\Sigma_3 &= -\eta UN - \eta N^T U^T,\\
\Sigma_4 &= \big(H(\theta)S(\theta) - UR\big)^T.
\end{aligned}$$

Remark 2.16 In the above study, either discrete-time or continuous-time, the proposed H_∞ performance analysis conditions are given are based on Lemma 1.3 with a matrix $J(\theta) > 0$ (or another $X(\theta) > 0$). In fact, we can use Lemma 1.10 to displace the application of Lemma 1.3. The displacement avoids the appearance of the auxiliary matrix variable $J(\theta) > 0$ (or another $X(\theta) > 0$), which reduces the dimension of LMIs in those design conditions. To the case of the closed-loop system (2.153) with $D(\theta) = 0$ as an example, we rewrite the matrix inequality (2.160) as follows:

$$\begin{aligned}
&\begin{bmatrix}
-G(\theta) - G^T(\theta) & * & * & * & * \\
A^T(\theta)G^T(\theta) + C_2^T(\theta)V^T M^T + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\
E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T & 0 & -\gamma^2 I & * & * \\
0 & C_1(\theta) & F(\theta) & -I & * \\
G^T(\theta) & 0 & 0 & 0 & -X(\theta)
\end{bmatrix}\\
&+\begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}NV\begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}\\
&+\begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix}^T V^T N^T N^{-T} U^{-T} \begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0.
\end{aligned} \tag{2.173}$$

Now, for (2.173), we use Lemma 1.10 with

$$\begin{aligned}
T &= \begin{bmatrix}
-G(\theta) - G^T(\theta) & * & * & * & * \\
A^T(\theta)G^T(\theta) + Q(\theta) & -2Q(\theta) + X(\theta) & * & * & * \\
E^T(\theta)G^T(\theta) & 0 & -\gamma^2 I & * & * \\
0 & C_1(\theta) & F(\theta) & -I & * \\
G^T(\theta) & 0 & 0 & 0 & -X(\theta)
\end{bmatrix},\\
A &= U^{-1}N^{-1}NV\begin{bmatrix} 0 & C_2(\theta) & H(\theta) & 0 & 0 \end{bmatrix},\\
P &= \begin{bmatrix} G(\theta)B(\theta) - MU \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\end{aligned}$$

$$S = NU.$$

Then, a new H_∞ performance analysis condition can obtained.

$$\begin{bmatrix} -G(\theta) - G^T(\theta) & * & * & * & * & * \\ \Pi_1 & -2Q(\theta) + X(\theta) & * & * & * & * \\ E^T(\theta)G^T(\theta) + H^T(\theta)V^T M^T & 0 & -\gamma^2 I & * & * & * \\ 0 & C_1(\theta) & F(\theta) & -I & * & * \\ G^T(\theta) & 0 & 0 & 0 & -X(\theta) & * \\ \beta\big(G(\theta)B(\theta) - MU\big)^T & NVC_2(\theta) & NVH(\theta) & 0 & 0 & \Pi_2 \end{bmatrix} < 0, \tag{2.174}$$

where

$$\Pi_1 = A^T(\theta)G^T(\theta) + C_2^T(\theta)V^T M^T + Q(\theta),$$

$$\Pi_2 = -\beta NU - \beta U^T N^T.$$

From the H_∞ performance analysis result (2.174), it can easily be known that the corresponding static output feedback H_∞ controllers design conditions are of LMIs.

2.2 With Norm Bounded Uncertainties

To keep things simple, we just study discrete-time systems in this section. Consider the following linear discrete-time dynamic model with time-varying norm bounded uncertainties:

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B + \Delta B)u(k) + (E + \Delta E)w(k), \\ z(k) &= (C_1 + \Delta C1)x(k) + (D + \Delta D)u(k) + (F + \Delta F)w(k), \\ y(k) &= (C_2 + \Delta C2)x(k) + (H + \Delta H)w(k), \end{aligned} \tag{2.175}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times m}$, $E \in \mathscr{R}^{n\times f}$, $C_1 \in \mathscr{R}^{q\times n}$, $D \in \mathscr{R}^{q\times m}$, $F \in \mathscr{R}^{q\times f}$, $C_2 \in \mathscr{R}^{p\times n}$, and $H \in \mathscr{R}^{p\times f}$ are system matrices. ΔA, ΔB, ΔE, ΔC_1, ΔD, ΔF, ΔC_2, and ΔH are uncertainties formulated as [10]

$$\begin{aligned} \Delta_A &= X_A\Delta(k)Y_A, & \Delta_B &= X_B\Delta(k)Y_B, \\ \Delta_E &= X_E\Delta(k)Y_E, & \Delta_{C1} &= X_{C1}\Delta(k)Y_{C1}, \\ \Delta_D &= X_D\Delta(k)Y_D, & \Delta_F &= X_F\Delta(k)Y_F, \\ \Delta_{C2} &= X_{C2}\Delta(k)Y_{C2}, & \Delta_H &= X_H\Delta(k)Y_H, \end{aligned} \tag{2.176}$$

$$\Delta^T(k)\Delta(k) \leq I.$$

The model (2.175) is inferred as follows:

$$\begin{aligned} x(k+1) &= A_\Delta x(k) + B_\Delta u(k) + E_\Delta w(k), \\ z(k) &= C_{1\Delta} x(k) + D_\Delta u(k) + F_\Delta w(k), \\ y(k) &= C_{2\Delta} x(k) + H_\Delta w(k), \end{aligned} \tag{2.177}$$

where

$$\begin{aligned} A_\Delta &= A + \Delta A, & B_\Delta &= B + \Delta B, \\ E_\Delta &= E + \Delta E, & C_{1\Delta} &= C_1 + \Delta C_1, \\ D_\Delta &= D + \Delta D, & F_\Delta &= F + \Delta F, \\ C_{2\Delta} &= C_2 + \Delta C_2, & H_\Delta &= H + \Delta H. \end{aligned}$$

In this section, the following static output feedback controller will be designed

$$u(k) = Ky(k) = K\big(C_{2\Delta} x(k) + H_\Delta w(k)\big). \tag{2.178}$$

where K is the controller gain.

Substituting (2.178) into (2.177) yields the following closed-loop system:

$$\begin{aligned} x(k+1) &= (A_\Delta + B_\Delta K C_{2\Delta}) x(k) + (E_\Delta + B_\Delta K H_\Delta) w(k), \\ z(k) &= (C_{1\Delta} + D_\Delta K C_{2\Delta}) x(k) + (F_\Delta + D_\Delta K H_\Delta) w(k). \end{aligned} \tag{2.179}$$

Remark 2.17 Similar to the research on robust static output feedback H_∞ control design for linear systems with polytopic uncertainties, most of the literature claim that the system input or output matrix should be without uncertainties. In this study, the requirement is not necessary.

In this following, we will develop new LMI conditions to design the static output feedback H_∞ controller in the form of (2.178) such that the resulting closed-loop system (2.179) meets an H_∞ performance bound requirement. For frugality, we just consider the case $D = 0$. First, a new H_∞ performance analysis criterion is presented in the following theorem.

Theorem 2.24 *Consider the closed-loop system* (2.179) *with* $D = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ

if exist matrices P, G, J, M, N, V, and U, scalar β such that the following matrix inequality holds

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA_\Delta + MVC_{2\Delta} & GE_\Delta + MVH_\Delta & -G-G^T+P+J & * & * & * \\ C_{1\Delta} & F_\Delta & 0 & -I & * & * \\ NVC_{2\Delta} & NVH_\Delta & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} \end{bmatrix} < 0, \tag{2.180}$$

where

$$\Sigma_1 = -\beta NU - \beta U^T N^T,$$

$$\Sigma_2 = GB_\Delta - MU.$$

Proof Follow (2.12) with the Lyapunov matrix P, the inequality (2.180) can be obtained easily. □

In this following, based on the analysis result in Theorem 2.24, we proposed sufficient conditions for designing the static output feedback H_∞ controller in the form of (2.178). First, separate the certain terms and uncertain terms in (2.180), we have

$$\Omega + \Delta_\Omega < 0, \tag{2.181}$$

with

$$\Omega = \begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA+MVC_2 & GE+MVH & -G-G^T+P+J & * & * & * \\ C_1 & F & 0 & -I & * & * \\ NVC_2 & NVH & 0 & 0 & -\beta NU - \beta U^T N^T & * \\ 0 & 0 & 0 & 0 & GB-MU & -\frac{J}{\beta^2} \end{bmatrix},$$

and

$$\Delta_\Omega = \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ G\Delta_A + MV\Delta_{C2} & G\Delta_E + MV\Delta_H & 0 & * & * & * \\ \Delta_{C1} & \Delta_F & 0 & 0 & * & * \\ NV\Delta_{C2} & NV\Delta_H & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & G\Delta_B & 0 \end{bmatrix}$$

$$= \tilde{A} + \tilde{A}^T + \tilde{B} + \tilde{B}^T + \tilde{E} + \tilde{E}^T + \tilde{C}_1 + \tilde{C}_1^T + \tilde{F} + \tilde{F}^T + \tilde{C}_2 + \tilde{C}_2^T + \tilde{H} + \tilde{H}^T,$$

where

$$\tilde{A}+\tilde{A}^T=\begin{bmatrix}0\\0\\GX_A\\0\\0\\0\end{bmatrix}\Delta(k)[\,Y_A\quad 0\quad 0\quad 0\quad 0\quad 0\,]+[\,Y_A\quad 0\quad 0\quad 0\quad 0\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\GX_A\\0\\0\\0\end{bmatrix}^T,$$

$$\tilde{B}+\tilde{B}^T=\begin{bmatrix}0\\0\\0\\0\\0\\GX_B\end{bmatrix}\Delta(k)[\,0\quad 0\quad 0\quad 0\quad Y_B\quad 0\,]+[\,0\quad 0\quad 0\quad 0\quad Y_B\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\0\\0\\0\\GX_B\end{bmatrix}^T,$$

$$\tilde{E}+\tilde{E}^T=\begin{bmatrix}0\\0\\GX_E\\0\\0\\0\end{bmatrix}\Delta(k)[\,0\quad Y_E\quad 0\quad 0\quad 0\quad 0\,]+[\,0\quad Y_E\quad 0\quad 0\quad 0\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\GX_E\\0\\0\\0\end{bmatrix}^T,$$

$$\tilde{C}_1+\tilde{C}_1^T=\begin{bmatrix}0\\0\\0\\X_{C1}\\0\\0\end{bmatrix}\Delta(k)[\,Y_{C1}\quad 0\quad 0\quad 0\quad 0\quad 0\,]+[\,Y_{C1}\quad 0\quad 0\quad 0\quad 0\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\0\\X_{C1}\\0\\0\end{bmatrix}^T,$$

$$\tilde{F}+\tilde{F}^T=\begin{bmatrix}0\\0\\0\\X_F\\0\\0\end{bmatrix}\Delta(k)[\,0\quad Y_F\quad 0\quad 0\quad 0\quad 0\,]+[\,0\quad Y_F\quad 0\quad 0\quad 0\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\0\\X_F\\0\\0\end{bmatrix}^T,$$

$$\tilde{C}_2+\tilde{C}_2^T$$

$$=\begin{bmatrix}0\\0\\MVX_{C2}\\0\\NVX_{C2}\\0\end{bmatrix}\Delta(k)[\,Y_{C2}\quad 0\quad 0\quad 0\quad 0\quad 0\,]+[\,Y_{C2}\quad 0\quad 0\quad 0\quad 0\quad 0\,]^T\Delta^T(k)\begin{bmatrix}0\\0\\MVX_{C2}\\0\\NVX_{C2}\\0\end{bmatrix}^T,$$

$$\tilde{H} + \tilde{H}^T$$

$$= \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix} \Delta(k)[\,0 \quad Y_H \quad 0 \quad 0 \quad 0 \quad 0\,] + [\,0 \quad Y_H \quad 0 \quad 0 \quad 0 \quad 0\,]^T \Delta^T(k) \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix}^T .$$

From Lemma 1.11 for positive scalars ε_A, ε_B, ε_E, ε_{C1}, ε_F, ε_{C2}, and ε_H, we can know that

$$\tilde{A} + \tilde{A}^T \leq \frac{1}{\varepsilon_A} \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ GX_A \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_A[\,Y_A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T[\,Y_A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,], \tag{2.182}$$

$$\tilde{B} + \tilde{B}^T \leq \frac{1}{\varepsilon_B} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ GX_B \end{bmatrix}^T + \varepsilon_B[\,0 \quad 0 \quad 0 \quad 0 \quad Y_B \quad 0\,]^T[\,0 \quad 0 \quad 0 \quad 0 \quad Y_B \quad 0\,], \tag{2.183}$$

$$\tilde{E} + \tilde{E}^T \leq \frac{1}{\varepsilon_E} \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ GX_E \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_E[\,0 \quad Y_E \quad 0 \quad 0 \quad 0 \quad 0\,]^T[\,0 \quad Y_E \quad 0 \quad 0 \quad 0 \quad 0\,], \tag{2.184}$$

$$\tilde{C}_1 + \tilde{C}_1^T \leq \frac{1}{\varepsilon_{C1}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_{C1} \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_{C1}[\,Y_{C1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T[\,Y_{C1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,], \tag{2.185}$$

$$\tilde{F} + \tilde{F}^T \leq \frac{1}{\varepsilon_F} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ X_F \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon_F [\, 0 \quad Y_F \quad 0 \quad 0 \quad 0 \quad 0 \,]^T [\, 0 \quad Y_F \quad 0 \quad 0 \quad 0 \quad 0 \,], \tag{2.186}$$

$$\tilde{C}_2 + \tilde{C}_2^T \leq \frac{1}{\varepsilon_{C2}} \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ MVX_{C2} \\ 0 \\ NVX_{C2} \\ 0 \end{bmatrix}^T + \varepsilon_{C2} [\, Y_{C2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,]^T [\, Y_{C2} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,], \tag{2.187}$$

$$\tilde{H} + \tilde{H}^T \leq \frac{1}{\varepsilon_H} \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ MVX_H \\ 0 \\ NVX_H \\ 0 \end{bmatrix}^T + \varepsilon_H [\, 0 \quad Y_H \quad 0 \quad 0 \quad 0 \quad 0 \,]^T [\, 0 \quad Y_H \quad 0 \quad 0 \quad 0 \quad 0 \,]. \tag{2.188}$$

So far, the design result can be summarized in the following theorem.

Theorem 2.25 *Consider the closed-loop system* (2.179) *with* $D = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrix* M, N *and scalar* β, *there exist matrices* U, P, G, *and* L, *scalars* ε_A, ε_B, ε_E, ε_{C1}, ε_F, ε_{C2}, *and* ε_H *such that the following LMI holds*

$$\begin{bmatrix} M_{11} & * \\ M_{21} & M_{22} \end{bmatrix} < 0, \tag{2.189}$$

where

$$M_{11} = \begin{bmatrix} \Lambda_1 & * & * & * & * & * \\ 0 & \Lambda_2 & * & * & * & * \\ GA + MVC_2 & GE + MVH & -G - G^T + P + J & * & * & * \\ C_1 & F & 0 & -I & * & * \\ NVC_2 & NVH & 0 & 0 & \Lambda_5 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix},$$

$$\Lambda_1 = -P + \varepsilon_A Y_A^T Y_A + \varepsilon_{C1} Y_{C1}^T Y_{C1} + \varepsilon_{C2} Y_{C2}^T Y_{C2},$$

$$\Lambda_2 = -\gamma^2 I + \varepsilon_E Y_E^T Y_E + \varepsilon_F Y_F^T Y_F + \varepsilon_H Y_H^T Y_H,$$

$$\Lambda_5 = -\beta N U - \beta U^T N^T + \varepsilon_B Y_B^T Y_B,$$

$$M_{21} = \begin{bmatrix} 0 & 0 & X_A^T G^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_B^T G^T \\ 0 & 0 & X_E^T G^T & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{C1}^T & 0 & 0 \\ 0 & 0 & 0 & X_F^T & 0 & 0 \\ 0 & 0 & X_{C2}^T V^T M^T & 0 & X_{C2}^T V^T N^T & 0 \\ 0 & 0 & X_H^T V^T M^T & 0 & X_H^T V^T N^T & 0 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} -\varepsilon_A I & * & * & * & * & * & * \\ 0 & -\varepsilon_B I & * & * & * & * & * \\ 0 & 0 & -\varepsilon_E I & * & * & * & * \\ 0 & 0 & 0 & -\varepsilon_{C1} I & * & * & * \\ 0 & 0 & 0 & 0 & -\varepsilon_F I & * & * \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon_{C2} I & * \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_H I \end{bmatrix}.$$

Furthermore, the static output feedback H_∞ controller gain matrix in (2.178) *can be given by* (2.22).

Remark 2.18 When the system input or output matrices are without uncertainties, we can also get several different forms of LMI-based design conditions for the closed-loop system (2.179). Similar to the polytopic uncertainties, in contrast to the existing LMI conditions for designing the robust static output feedback H_∞ controllers, the improvement of the our results over the existing ones can shown by strict theoretical proof. A related study has been made in our preliminary work for uncertain discrete-time T-S fuzzy systems [2].

Remark 2.19 The used LMI decoupling approach brings new results for robust static output feedback H_∞ controllers design of uncertain linear systems. It is necessary to mention that when β is known parameter, the proposed design conditions are of LMIs that can be easily and effectively solved via LMI control toolbox [7]. Due to β is free parameter, the problem is then how to find the optimal value of β in order to minimize the H_∞ performance bound. One way to address the search issue is to first solve the feasibility problem of the corresponding LMI conditions using LMI control toolbox and obtain a set of initial scalar parameters. Then, applying a numerical optimization algorithm, such as the program "fminsearch" in the optimization toolbox of Matlab, a locally convergent solution to the problem is obtained. In [9], the algorithm has been used and proved to be effective.

2.3 Numerical Examples

2.3.1 Example 1

To show the less conservativeness of the presented design result on robust static output feedback H_∞ controller design, a simulation example is given with different design methods. For simplicity, here we only consider Corollary 2.1 and Lemma 2.2. Let us consider the uncertain system (2.1) with

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 1.1 & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.8 & 0.1 & 0 \\ -0.3 & 0.1 & 0.6 \\ 1.2 & -1 & 1.1 \end{bmatrix}, \\
&B_1 = B_2 = B = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix}, \\
&C_{11} = [\,-1 \quad 0 \quad 2\,], \qquad C_{12} = [\,-0.6 \quad 0.2 \quad 1\,], \\
&D_1 = D_2 = 0, \\
&F_1 = 0.3, \qquad F_2 = -0.4, \\
&C_{21} = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}, \qquad C_{22} = \begin{bmatrix} -0.8 & 1 & 1 \\ 0 & -2 & 1.2 \end{bmatrix}, \\
&H_1 = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}.
\end{aligned}
\tag{2.190}
$$

For the system (2.190), since the system input matrix is fixed, the conditions in Corollary 2.1 and Lemma 2.2 are applicable for designing robust static output feedback H_∞ controllers. Now applying two conditions to design static output feedback H_∞ controllers such that γ is minimized.

By (2.35) and (2.36), the minimum H_∞ performances $\gamma_{\min} = 11.4687$. However, apply (2.29) and (2.30) with $\beta = 6.16$, we can find the minimum H_∞ performance $\gamma_{\min} = 7.0441$. From this comparison, it can be seen that the design condition in Corollary 2.1 is much less conservative than the existing result in Lemma 2.2.

2.3.2 Example 2

In this example, to show the effectiveness of the proposed design approach, a static output feedback H_∞ control problem of the following discrete-time system is considered

$$
\begin{aligned}
x(k+1) &= (A+\Delta A)x(k) + (B+\Delta B)u(k) + Ew(k),\\
z(k) &= C_1 x(k) + Du(k) + Fw(k),\\
y(k) &= (C_2 + \Delta C2)x(k),
\end{aligned}
\tag{2.191}
$$

with

$$
\begin{aligned}
A &= \begin{bmatrix} 2.3 & 0.5 \\ 1 & 0 \end{bmatrix}, & B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix},\\
E &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & C_1 &= [\ -1 \quad 1\ ],\\
D &= 0, & F &= 2,\\
C_2 &= [\ 1 \quad 0\ ], & &\\
X_A &= \begin{bmatrix} 0.3 \\ -0.1 \end{bmatrix}, & Y_A &= [\ 0.1 \quad 0.2\ ],\\
X_B &= \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, & Y_B &= -0.3,\\
X_{C2} &= -0.1, & Y_{C2} &= [\ 0.3 \quad 0.2\ ].
\end{aligned}
$$

We need to note that the system input matrix and output matrices in the system (2.191) have uncertainties. As a result, the existing approaches in [3, 5, 6, 8] fail to design the static output feedback H_∞ controller for this example. However, the proposed design condition in Theorem 2.25 is feasible to find a static output feedback controller to stabilize the system (2.191) with the H_∞ performance index. For example, by using the MATLAB toolbox to solve LMI (2.189) in Theorem 2.25 with $\beta = 5.34$, $M = B$, $N = 1$, minimum H_∞ performances $\gamma_{\min} = 4.3174$ is obtained.

On the other hand, by LMI (2.189) in Theorem 2.25 with $\gamma = 4.3174$, the following computational results are obtained

$$
U = 6.8776, \qquad V = -15.7883. \tag{2.192}
$$

Substituting U and V in to (2.22), the static output feedback H_∞ controller gain matrix can be given as follows:

$$
K = -2.2956. \tag{2.193}
$$

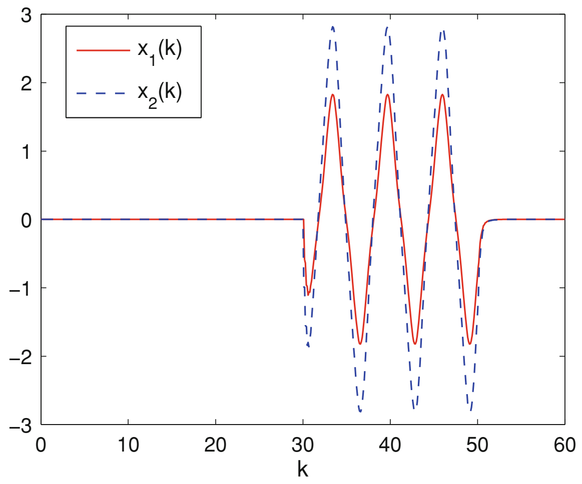

Fig. 2.1 State trajectories of the closed-loop systems

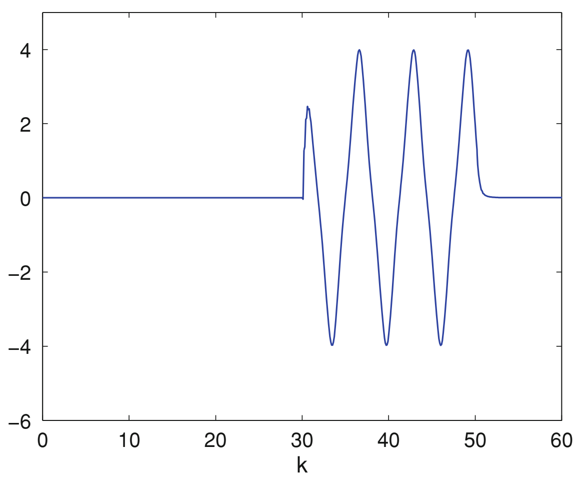

Fig. 2.2 System control signal $u(k)$

We assume the disturbance $w(k)$ as the following:

$$w(k) = \begin{cases} sin(k), & 30 \leq k \leq 50, \\ 0, & elsewhere. \end{cases} \tag{2.194}$$

Under the initial conditions $x(0) = [\ 0 \quad 0\ ]^T$, using the static output feedback controller gain (2.193) with $\Delta(k) = \sin^2(k),\ \ k = 1,\ 2,\ \ldots\ ,$ the state response of the system (2.191) is shown in Fig. 2.1. The control signal $u(k)$ is depicted in Fig. 2.2, it is able tostabilize the system with the H_∞ performance 4.3174. The ratio

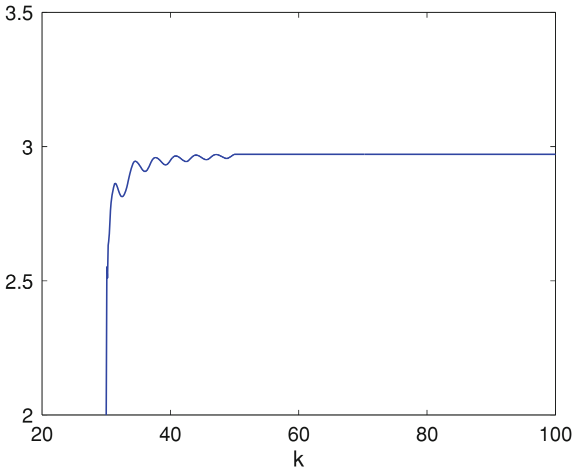

Fig. 2.3 The ratio of $\sqrt{\sum_{k=0}^{\infty} z^T(k)z(k) / \sum_{k=0}^{\infty} w^T(k)w(k)}$

of $\sqrt{\sum_{k=0}^{\infty} z^T(k)z(k) / \sum_{k=0}^{\infty} w^T(k)w(k)}$ can show the influence of the disturbance $w(k)$ on the controlled output $z(k)$, and the plot of the ratio is shown in Fig. 2.3. It can be seen that the ratio tends to a constant value 2.9716, which is less than the prescribed value, i.e., 4.3174. From Figs. 2.1, 2.2, 2.3, it can been seen the H_∞ performance is guaranteed for the system with the designed static output feedback H_∞ controller.

2.4 Conclusion

In this chapter, the robust static output feedback H_∞ control problem for the both discrete-time and continuous-time uncertain systems has been studied. Sufficient conditions for designing static output feedback H_∞ controllers have been given based on an LMI decoupling approach. The design conditions are presented in the form of LMIs. In contrast to existing LMI methods for robust static output feedback H_∞ controllers design of the uncertain linear systems, the proposed results not only allow the system input and output matrices to have uncertainties, but it can provide less conservative design. Numerical examples show the advantage of the proposed design method.

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Chapter 3
Robust Dynamic Output Feedback H_∞ Control

Abstract This chapter deals with the robust dynamic output feedback H_∞ control problem for uncertain linear systems. As a representative, just the discrete-time systems with polytopic uncertainties are considered. First, some basic results on robust dynamic output feedback H_∞ control of the systems are introduced. Then, the LMI decoupling approach is proposed to achieve the design of H_∞ control. Unlike the strategy of change of variables, the proposed results are effective for solving the robust dynamic output feedback H_∞ control problem using the LMI technique. The effectiveness of the proposed design methods is finally demonstrated through a numerical example.

Keywords Uncertain linear systems · Dynamic output feedback · H_∞ control · Polytopic uncertainties · Linear matrix inequalities (LMIs).

3.1 Problem Formulation

Consider a discrete-time linear system with time-invariant polytopic uncertainties described by state-space equations

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + E(\theta)w(k), \\ z(k) &= C_1(\theta)x(k) + D(\theta)u(k) + F(\theta)w(k)), \\ y(k) &= C_2(\theta)x(k) + H(\theta)w(k), \end{aligned} \tag{3.1}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. The matrices $A(\theta), B(\theta), E(\theta), C_1(\theta), D(\theta), F(\theta), C_2(\theta)$, and $H(\theta)$ are constant matrices of appropriate dimensions and belong to the uncertainty in (2.2).

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_3,

In this section, the following dynamic output feedback controller for discrete-time model is exploited:

$$\begin{aligned} x_F(k+1) &= A_F x_F(k) + B_F y(k), \\ u(k) &= C_F x_F(k) + D_F y(k), \end{aligned} \tag{3.2}$$

where $x_F(k) \in \mathscr{R}^{n_F}$ is the controller state, A_F, B_F, C_F, and D_F are the controller gains.

From (3.1) and (3.2), by defining the augmented state vector $\bar{x}(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}$, the closed-loop system is described as

$$\begin{aligned} &\bar{x}(k+1) \\ &= \begin{bmatrix} A(\theta) + B(\theta)D_F C_2(\theta) & B(\theta)C_F \\ B_F C_2(\theta) & A_F \end{bmatrix} \bar{x}(k) + \begin{bmatrix} B(\theta)D_F H(\theta) + E(\theta) \\ B_F H(\theta) \end{bmatrix} w(k), \\ &z(k) = [C_1(\theta) + D(\theta)D_F C_2(\theta) \;\; D(\theta)C_F]\bar{x}(k) + \big(D(\theta)D_F H(\theta) + F(\theta)\big) w(k). \end{aligned} \tag{3.3}$$

3.2 Basic Results

In this section, we present several basic results for dynamic output feedback H_∞ control for the closed-loop system (3.3) based on the LMI technique. First, we give the following lemmas, in which the H_∞ performance analysis problem for the closed-loop system (3.3) is discussed.

Lemma 3.1 *Consider the closed-loop system* (3.3) *with $D(\theta) = 0$ and $B(\theta) = B$ (B is of full column rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P_1(\theta)$, $P_2(\theta)$, Y, A_F, B_F, C_F, and D_F such that the following matrix inequality holds:*

$$\begin{bmatrix} -Y^T P_1(\theta) Y & * & * & * & * & * \\ 0 & -P_2(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta) + BD_F C_2(\theta) & BC_F & BD_F H(\theta) + E(\theta) & \Omega & * & * \\ B_F C_2(\theta) & A_F & B_F H(\theta) & 0 & -P_2^{-1}(\theta) & * \\ C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.4}$$

where $\Omega = -Y^{-1} P_1^{-1}(\theta) Y^{-T}$.

Proof Choose the parameter-dependent Lyapunov function as

$$V(k) = \bar{x}^T(k) \begin{bmatrix} Y^T P_1(\theta) Y & 0 \\ 0 & P_2(\theta) \end{bmatrix} \bar{x}(k), \;\; P_1(\theta) > 0, \;\; P_2(\theta) > 0, \tag{3.5}$$

where the matrix Y is nonsingular, then the matrix inequality (3.4) is obtained easily. □

Lemma 3.2 *Consider the closed-loop system* (3.3) *with* $H(\theta) = 0$ *and* $C_2(\theta) = C_2$ *(C_2 of full row rank). For a given scalar* $\gamma > 0$, *the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* $P_1(\theta)$, $P_2(\theta)$, T, A_F, B_F, C_F, *and* D_F *such that the following matrix inequality holds:*

$$\begin{bmatrix} -T^{-T}P_1^{-1}(\theta)T^{-1} & * & * & * & * & * \\ 0 & -P_2^{-1}(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta)+B(\theta)D_F C_2 & B(\theta)C_F & E(\theta) & -TP_1(\theta)T^T & * & * \\ B_F C_2 & A_F & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)+D(\theta)D_F C_2 & D(\theta)C_F & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.6}$$

Proof Choose the parameter-dependent Lyapunov function as

$$V(k) = \bar{x}^T(k)\begin{bmatrix} T^{-T}P_1^{-1}(\theta)T^{-1} & 0 \\ 0 & P_2^{-1}(\theta) \end{bmatrix}\bar{x}(k), \;\; P_1(\theta) > 0, \;\; P_2(\theta) > 0, \tag{3.7}$$

where the matrix T is nonsingular. □

In this following, we will study the design problem of the dynamic output feedback controller (3.2) based on the H_∞ performance analysis criteria in Lemmas 3.1 and 3.2, respectively.

For Lemma 3.1:

First, let us use the following matrix inequality to ensure (3.4)

$$\begin{bmatrix} -P_1(\theta) & * & * & * & * & * \\ 0 & -P_2(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ \Lambda & YBC_F & YBD_F H(\theta) + YE(\theta) & -P_1^{-1}(\theta) & * & * \\ B_F C_2(\theta)Y^{-1} & A_F & B_F H(\theta) & 0 & -P_2^{-1}(\theta) & * \\ C_1(\theta)Y^{-1} & 0 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.8}$$

where $\Lambda = YA(\theta)Y^{-1} + YBD_F C_2(\theta)Y^{-1}$.

Now, we assume a fact that is $YB = \begin{bmatrix} I \\ 0 \end{bmatrix}$, from (3.8), one has

$$\begin{bmatrix} -P_1(\theta) & * & * & * & * & * \\ 0 & -P_2(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^{2I} & * & * & * \\ \Pi & \begin{bmatrix} I \\ 0 \end{bmatrix} C_F & \begin{bmatrix} I \\ 0 \end{bmatrix} D_F H(\theta) + YE(\theta) & -P_1^{-1}(\theta) & * & * \\ B_F C_2(\theta) Y^{-1} & A_F & B_F H(\theta) & 0 & -P_2^{-1}(\theta) & * \\ C_1(\theta) Y^{-1} & 0 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.9}$$

where $\Pi = YA(\theta)Y^{-1} + \begin{bmatrix} I \\ 0 \end{bmatrix} D_F C_2(\theta) Y^{-1}$.

Pre- and post-multiplying (3.9) by $\begin{bmatrix} I & * & * & * & * & * \\ 0 & I & * & * & * & * \\ 0 & 0 & I & * & * & * \\ 0 & 0 & 0 & G_1(\theta) & * & * \\ 0 & 0 & 0 & 0 & G_2 & * \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, we can apply the following matrix inequality to guarantee (3.9):

$$\begin{bmatrix} -P_1(\theta) & * & * & * & * & * \\ 0 & -P_2(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ \Pi_A & G_1(\theta)\begin{bmatrix} I \\ 0 \end{bmatrix} C_F & \Pi_E & \Pi_1 & * & * \\ G_2 B_F C_2(\theta) Y^{-1} & G_2 A_F & G_2 B_F H(\theta) & 0 & \Pi_2 & * \\ C_1(\theta) Y^{-1} & 0 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.10}$$

where

$$\Pi_A = G_1(\theta) Y A(\theta) Y^{-1} + G_1(\theta) \begin{bmatrix} I \\ 0 \end{bmatrix} D_F C_2(\theta) Y^{-1},$$

$$\Pi_E = G_1(\theta) \begin{bmatrix} I \\ 0 \end{bmatrix} D_F H(\theta) + G_1(\theta) Y E(\theta),$$

$$\Pi_1 = -G_1(\theta) - G_1^T(\theta) + P_1(\theta),$$

$$\Pi_2 = -G_2 - G_2^T + P_2(\theta).$$

To facilitate the LMI presentation, we can assume that the matrices $P_1(\theta)$, $P_2(\theta)$, and $G_1(\theta)$ have the form

$$P_1(\theta) = \sum_{j=1}^{r} \theta_j P_{1j}, \;\; P_{1j} > 0, \quad j = 1, 2, \ldots, r,$$

$$P_2(\theta) = \sum_{j=1}^{r} \theta_j P_{2j}, \;\; P_{2j} > 0, \quad j = 1,\, 2, \ldots, r$$

$$G_1(\theta) = \sum_{j=1}^{r} \theta_j G_{1j} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_1 & 0 \\ 0 & G_{2j} \end{bmatrix}, \tag{3.11}$$

or

$$G_1(\theta) = \sum_{j=1}^{r} \theta_j G_{1j} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_1 & G_{2j} \\ 0 & G_{3j} \end{bmatrix}.$$

So far, substituting the above matrices $P_1(\theta)$, $P_2(\theta)$, and $G_1(\theta)$ into (3.10) and defining another four variables

$$\begin{aligned} \bar{A}_F &= G_2 A_F, \quad \bar{B}_F = G_2 B_F, \\ \bar{C}_F &= G_1 C_F, \quad \bar{D}_F = G_1 D_F. \end{aligned} \tag{3.12}$$

It is not difficult to rewrite (3.9) as

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \theta_j \Delta_{ij} < 0, \tag{3.13}$$

where

$$\Delta_{ij} = \begin{bmatrix} -P_{1j} & * & * & * & * & * \\ 0 & -P_{2j} & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_{1j} Y A_i Y^{-1} + \begin{bmatrix} \bar{D}_F \\ 0 \end{bmatrix} C_{2i} Y^{-1} & \begin{bmatrix} \bar{C}_F \\ 0 \end{bmatrix} & \begin{bmatrix} \bar{D}_F \\ 0 \end{bmatrix} H_i + G_{1j} Y E_i & \Xi_1 & * & * \\ \bar{B}_F C_{2i} Y^{-1} & \bar{A}_F & \bar{B}_F H_i & 0 & \Xi_2 & * \\ C_{1i} Y^{-1} & 0 & F_i & 0 & 0 & -I \end{bmatrix},$$

and

$$\Xi_1 = -G_{1j} - G_{1j}^T + P_{1j},$$

$$\Xi_2 = -G_2 - G_2^T + P_{2j}.$$

Then, we immediately obtain the following condition for designing the dynamic output feedback controller in (3.2):

Theorem 3.1 *Consider the closed-loop system* (3.3) *with* $D(\theta) = 0$ *and* $B(\theta) = B$ *(*B *is of full column rank). For a given scalar* $\gamma > 0$, *the system is asymptotically* *stable with the H_∞ performance γ if exist matrices $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, $\bar{D}_F$, G_2, P_{1j}, P_{2j}, and G_{1j}, $j = 1, 2, \ldots, r$ such that the following matrix equations hold:*

$$\Delta_{ii} < 0, \quad i = 1,\ 2, \ldots, r, \tag{3.14}$$

$$\Delta_{ij} + \Delta_{ji} < 0, i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.15}$$

$$G_{1j} = \begin{bmatrix} G_1 & 0 \\ 0 & G_{2j} \end{bmatrix} \text{ or } G_j = \begin{bmatrix} G_1 & G_{2j} \\ 0 & G_{3j} \end{bmatrix}, \quad j = 1, 2, \ldots, r, \tag{3.16}$$

where $\Delta_{ij}, i, j = 1, 2, \ldots, r$ are defined in (3.13).

Furthermore, the dynamic output feedback H_∞ controller gain matrix in (3.2) can be given by

$$\begin{aligned} A_F &= G_2^{-1}\bar{A}_F, \quad B_F = G_2^{-1}\bar{B}_F, \\ C_F &= G_1^{-1}\bar{C}_F, \quad D_F = G_1^{-1}\bar{D}_F. \end{aligned} \tag{3.17}$$

For Lemma 3.2:

Obviously, the matrix inequality (3.6) is equivalent to

$$\begin{bmatrix} -P_1^{-1}(\theta) & * & * & * & * & * \\ 0 & -P_2^{-1}(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ T^{-1}A(\theta)T + T^{-1}B(\theta)D_F C_2 T & T^{-1}B(\theta)C_F & T^{-1}E(\theta) & -P_1(\theta) & * & * \\ B_F C_2 T & A_F & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)T + D(\theta)D_F C_2 T & D(\theta)C_F & F(\theta) & 0 & 0 & -I \end{bmatrix}$$
$$< 0. \tag{3.18}$$

By choosing the matrix T to meet $C_2T = [I\ 0]$, (3.18) becomes

$$\begin{bmatrix} -P_1^{-1}(\theta) & * & * & * & * & * \\ 0 & -P_2^{-1}(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ T^{-1}A(\theta)T + T^{-1}B(\theta)D_F[I\ 0] & T^{-1}B(\theta)C_F & T^{-1}E(\theta) & -P_1(\theta) & * & * \\ B_F[I\ 0] & A_F & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)T + D(\theta)D_F[I\ 0] & D(\theta)C_F & F(\theta) & 0 & 0 & -I \end{bmatrix}$$
$$< 0. \tag{3.19}$$

Introduce two auxiliary matrix variables $S_1(\theta)$ and S_2 to (3.19), then the matrix inequality holds if

$$
\begin{bmatrix}
\Pi_1 & * & * & * & * & * \\
0 & \Pi_2 & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * \\
\Pi_A & T^{-1}B(\theta)C_F S_2 & T^{-1}E(\theta) & -P_1(\theta) & * & * \\
B_F[I\ \ 0]S_1(\theta) & A_F S_2 & 0 & 0 & -P_2(\theta) & * \\
\Pi_C & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I
\end{bmatrix} < 0, \tag{3.20}
$$

where

$$\Pi_1 = -S_1(\theta) - S_1^T(\theta) + P_1(\theta),$$

$$\Pi_2 = -S_2 - S_2^T + P_2(\theta),$$

$$\Pi_A = T^{-1}A(\theta)TS_1(\theta) + T^{-1}B(\theta)D_F[I\ \ 0]S_1(\theta),$$

$$\Pi_C = C_1(\theta)TS_1(\theta) + D(\theta)D_F[I\ \ 0]S_1(\theta).$$

Let us use the same form in (3.11) to define the matrices $P_1(\theta)$, $P_2(\theta)$, and consider that the matrix $S_1(\theta)$ in (3.20) has the following special form:

$$
\begin{aligned}
&S_1(\theta) = \sum_{j=1}^{r} \theta_j S_{1j} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix}, \\
&or \\
&S_1(\theta) = \sum_{j=1}^{r} \theta_j S_{1j} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}.
\end{aligned} \tag{3.21}
$$

Then, the following theorem gives a condition for designing the dynamic output feedback controller in (3.2):

Theorem 3.2 *Consider the closed-loop system (3.3) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is of full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if exist matrices $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, $\bar{D}_F$, S_2, P_{1j}, P_{2j}, and S_{1j}, $j = 1, 2, \ldots, r$ such that the following matrix equations hold:*

$$\Sigma_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.22}$$

$$\Sigma_{ij} + \Sigma_{ji} < 0, i < j, \quad i,\ j = 1, 2, \ldots, r, \tag{3.23}$$

$$S_{1j} = \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix} \ or\ \ S_{1j} = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}, \quad j = 1, 2, \ldots, r. \tag{3.24}$$

with

$$\Sigma_{ij} = \begin{bmatrix} -S_{1j} - S_{1j}^T + P_{1j} & * & * & * & * & * \\ 0 & -S_2 - S_2^T + P_{2j} & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ \Pi_A & T^{-1} B_i \bar{C}_F & T^{-1} E_i & -P_{1j} & * & * \\ [\bar{B}_F \;\; 0] & \bar{A}_F & 0 & 0 & -P_{2j} & * \\ \Pi_C & D_i \bar{C}_F & F_i & 0 & 0 & -I \end{bmatrix},$$

and

$$\begin{aligned} \Pi_A &= T^{-1} A_i T S_{1j} + T^{-1} B_i \left[\bar{D}_F \; 0\right], \\ \Pi_C &= C_{1i} T S_{1j} + D_i \left[\bar{D}_F \; 0\right]. \end{aligned}$$

Furthermore, the dynamic output feedback H_∞ controller gain matrix in (3.2) *can be given by*

$$\begin{aligned} A_F &= \bar{A}_F S_2^{-1}, \quad B_F = \bar{B}_F S_1^{-1}, \\ C_F &= \bar{C}_F S_2^{-1}, \quad D_F = \bar{D}_F S_1^{-1}. \end{aligned} \tag{3.25}$$

Remark 3.1 The design results in Theorems 3.1 and 3.2 are also the extension of the previous one of Lemmas 2.4 and 2.8, in which sufficient conditions for static output feedback H_∞ controller design of discrete-time uncertain linear systems are proposed via LMIs.

3.3 LMI Decoupling Approach

In the above section, some basic LMI design results for dynamic output feedback H_∞ control for the closed-loop system (3.3) have been derived. Though these conditions are convex, the requirements for the system input or output matrices to be fixed are strict, which might result in conservative designs. In this section, the problem of output feedback H_∞ controller design for the discrete-time closed-loop systems (3.3) is studied using the LMI decoupling approach presented in Chap. 2. In contrast to the design conditions given by Theorems 3.1 and 3.2, in the proposed results the requirement for the system input or output matrices can be avoided.

Theorem 3.3 *Consider the closed-loop system* (3.3) *with $D(\theta) = 0$ and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N and scalar β, exist matrices $\bar{A}_F$, $\bar{B}_F$, V_C, V_D, G_2, U, P_{1j}, P_{2j}, G_{1j}, and J_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold:*

$$\Omega_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.26}$$

$$\Omega_{ij} + \Omega_{ji} < 0, i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.27}$$

with

$$\Omega_{ij} = \begin{bmatrix} -P_{1j} & * & * & * & * & * & * & * \\ 0 & -P_{2j} & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * \\ G_{1j}A_i + MV_D C_{2i} & MV_C & G_{1j}E_i + MV_D H_i & \Xi_1 & * & * & * & * \\ \bar{B}_F C_{2i} & \bar{A}_F & \bar{B}_F H_i & 0 & \Xi_2 & * & * & * \\ C_{1i} & 0 & F_i & 0 & 0 & -I & * & * \\ NV_D C_{2i} & NV_C & NV_D H_i & 0 & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\Xi_1 = -G_{1j} - G_{1j}^T + P_{1j} + J_j,$$

$$\Xi_2 = -G_2 - G_2^T + P_{2j},$$

$$\Sigma_1 = -\beta NU - \beta U^T N^T,$$

$$\Sigma_2 = G_{1j}B_i - MU.$$

Furthermore, the dynamic output feedback H_∞ controller gain matrix in (3.2) can be given as

$$\begin{aligned} A_F &= G_2^{-1}\bar{A}_F, \quad B_F = G_2^{-1}\bar{B}_F, \\ C_F &= U^{-1}V_C, \quad D_F = U^{-1}V_D. \end{aligned} \tag{3.28}$$

Proof For the closed-loop system (3.3) with $D(\theta) = 0$, by choosing the parameter-dependent Lyapunov function as $V(k) = \bar{x}^T(k)\begin{bmatrix} P_1(\theta) & 0 \\ 0 & P_2(\theta) \end{bmatrix}\bar{x}(k)$, $P_1(\theta) > 0$, $P_2(\theta) > 0$, the H_∞ performance γ can be guaranteed by

$$\begin{bmatrix} -P_1(\theta) & * & * & * & * & * \\ 0 & -P_2(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ \Gamma & G_1(\theta)B(\theta)C_F & G_1(\theta)B(\theta)D_F H(\theta) + G_1(\theta)E(\theta) & \Pi_1 & * & * \\ G_2 B_F C_2(\theta) & G_2 A_F & G_2 B_F H(\theta) & 0 & \Pi_2 & * \\ C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.29}$$

where

$$\Pi_1 = -G_1(\theta) - G_1^T(\theta) + P_1(\theta),$$

$$\Pi_2 = -G_2 - G_2^T + P_2(\theta),$$

$$\Gamma = G_1(\theta)A(\theta) + G_1(\theta)B(\theta)D_F C_2(\theta).$$

By decomposing the matrix inequality (3.29), we have

$$
\begin{aligned}
&\begin{bmatrix}
-P_1(\theta) & * & * & * & * & * \\
0 & -P_2(\theta) & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * \\
G_1(\theta)A(\theta) & 0 & G_1(\theta)E(\theta) & \Pi_1 & * & * \\
G_2B_FC_2(\theta) & G_2A_F & G_2B_FH(\theta) & 0 & \Pi_2 & * \\
C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I
\end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta) \\ 0 \\ 0 \end{bmatrix} [D_FC_2(\theta)\ C_F\ \ D_FH(\theta)\ 0\ 0\ 0] \\
&+ [D_FC_2(\theta)\ C_F\ \ D_FH(\theta)\ 0\ 0\ 0]^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta) \\ 0 \\ 0 \end{bmatrix}^T < 0.
\end{aligned} \tag{3.30}
$$

Define $V_D = UD_F$ and $V_C = UC_F$, (3.30) is equivalent to

$$
\begin{aligned}
&\begin{bmatrix}
-P_1(\theta) & * & * & * & * & * \\
0 & -P_2(\theta) & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * \\
G_1(\theta)A(\theta) & 0 & G_1(\theta)E(\theta) & \Pi_1 & * & * \\
G_2B_FC_2(\theta) & G_2A_F & G_2B_FH(\theta) & 0 & \Pi_2 & * \\
C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I
\end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta) \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}N[V_DC_2(\theta)\ V_C\ \ V_DH(\theta)\ 0\ 0\ 0] \\
&+ [V_DC_2(\theta)\ \ V_C\ \ V_DH(\theta)\ \ 0\ \ 0\ \ 0]^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta) \\ 0 \\ 0 \end{bmatrix}^T < 0.
\end{aligned} \tag{3.31}
$$

Once again rewrite (3.31) as follows:

$$
\begin{bmatrix}
-P_1(\theta) & * & * & * & * & * \\
0 & -P_2(\theta) & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * \\
G_1(\theta)A(\theta)+MV_D C_2(\theta) & MV_C & G_1(\theta)E(\theta)+MV_D H(\theta) & \Pi_1 & * & * \\
G_2 B_F C_2(\theta) & G_2 A_F & G_2 B_F H(\theta) & 0 & \Pi_2 & * \\
C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I
\end{bmatrix}
$$

$$
+\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta)-MU \\ 0 \\ 0 \end{bmatrix} U^{-1}N^{-1}N[V_D C_2(\theta) \;\; V_C \;\; V_D H(\theta) \;\; 0 \;\; 0 \;\; 0]
$$

$$
+[V_D C_2(\theta) \;\; V_C \;\; V_D H(\theta) \;\; 0 \;\; 0 \;\; 0]^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1(\theta)B(\theta)-MU \\ 0 \\ 0 \end{bmatrix}^T < 0. \tag{3.32}
$$

Following the same line as in the proof of Theorem 2.1, we can establish the following matrix inequality to ensure (3.32):

$$
\begin{bmatrix}
-P_1(\theta) & * & * & * & * & * & * & * \\
0 & -P_2(\theta) & * & * & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * & * & * \\
\Sigma_A & MV_C & G_1(\theta)E(\theta)+MV_D H(\theta) & \Pi_1 + J(\theta) & * & * & * & * \\
G_2 B_F C_2(\theta) & G_2 A_F & G_2 B_F H(\theta) & 0 & \Pi_2 & * & * & * \\
C_1(\theta) & 0 & F(\theta) & 0 & 0 & -I & * & * \\
NV_D C_2(\theta) & NV_C & NV_D H(\theta) & 0 & 0 & 0 & \Sigma_1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Sigma}_2 & -\frac{J(\theta)}{\beta^2}
\end{bmatrix}
$$

$$
< 0, \tag{3.33}
$$

where

$$
\Sigma_A = G_1(\theta)A(\theta) + MV_D C_2(\theta),
$$

$$
\Sigma_1 = -\beta NU - \beta U^T N^T,
$$

$$
\tilde{\Sigma}_2 = G_1(\theta)B(\theta) - MU.
$$

Now, assume the matrices $P_1(\theta)$, $P_2(\theta)$, and $G_1(\theta)$ to be of the following form

$$P_1(\theta) = \sum_{j=1}^{r} \theta_j P_{1j}, \;\; P_{1j} > 0, \quad j = 1,\, 2, \ldots, r,$$

$$P_2(\theta) = \sum_{j=1}^{r} \theta_j P_{2j}, \;\; P_{2j} > 0, \quad j = 1,\, 2, \ldots, r, \tag{3.34}$$

$$G_1(\theta) = \sum_{j=1}^{r} \theta_j G_{1j}.$$

By defining two variables as (3.12) and combining (3.34), the LMIs (3.26) and (3.27) can be obtained. □

Theorem 3.4 *Consider the closed-loop system* (3.3) *with* $H(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* $\bar{A}_F$, L_B, $\bar{C}_F$, L_D, S_2, U, P_{1j}, P_{2j}, S_{1j}, *and* J_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold:*

$$\Upsilon_{ii} < 0, \quad i = 1,\, 2, \ldots, r, \tag{3.35}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i < j,\; i, j = 1, 2, \ldots, r, \tag{3.36}$$

with

$$\Upsilon_{ij} = \begin{bmatrix} \Xi_1 & * & * & * & * & * & * & * \\ 0 & \Xi_2 & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * \\ A_i S_{1j} + B_i L_D M & B_i \bar{C}_F & E_i & -P_{1j} & * & * & * & * \\ L_B M & \bar{A}_F & 0 & 0 & -P_{2j} & * & * & * \\ C_{1i} S_{1j} + D_i L_D M & D_i \bar{C}_F & F_i & 0 & 0 & -I & * & * \\ 0 & 0 & 0 & N^T L_D^T B_i^T & N^T L_B^T & N^T L_D^T D_i^T & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\Xi_1 = -S_{1j} - S_{1j}^T + P_{1j} + J_j,$$

$$\Xi_2 = -S_2 - S_2^T + P_{2j},$$

$$\Sigma_1 = -\beta N U - \beta U^T N^T,$$

$$\Sigma_2 = (C_{2i} S_{1j} - UM)^T.$$

Furthermore, the dynamic output feedback H_∞ *controller gain matrices in* (3.2) *are given as*

$$\begin{aligned} A_F &= \bar{A}_F S_2^{-1}, \quad B_F = L_B U^{-1}, \\ C_F &= \bar{C}_F S_2^{-1}, \quad D_F = L_D U^{-1}. \end{aligned} \tag{3.37}$$

Proof For the closed-loop system (3.3) with $H(\theta) = 0$, the parameter-dependent Lyapunov function is chosen as $V(k) = \bar{x}^T(k)\begin{bmatrix} P_1^{-1}(\theta) & 0 \\ 0 & P_2^{-1}(\theta) \end{bmatrix}\bar{x}(k),\ P_1(\theta) > 0,\ P_2(\theta) > 0$, the H_∞ performance γ is satisfied if

$$\begin{bmatrix} -P_1^{-1}(\theta) & * & * & * & * & * \\ 0 & -P_2^{-1}(\theta) & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta) + B(\theta)D_F C_2(\theta) & B(\theta)C_F & E(\theta) & -P_1(\theta) & * & * \\ B_F C_2(\theta) & A_F & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta) + D(\theta)D_F C_2(\theta) & D(\theta)C_F & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.38}$$

Adding two auxiliary matrix variables $S_1(\theta)$ and S_2, it can be seen that the matrix inequality (3.38) can be guaranteed by

$$\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ 0 & \Pi_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta)S_1(\theta) + B(\theta)D_F C_2(\theta)S_1(\theta) & B(\theta)C_F S_2 & E(\theta) & -P_1(\theta) & * & * \\ B_F C_2(\theta)S_1(\theta) & A_F S_2 & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)S_1(\theta) + D(\theta)D_F C_2(\theta)S_1(\theta) & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.39}$$

where

$$\Pi_1 = -S_1(\theta) - S_1^T(\theta) + P_1(\theta),$$

$$\Pi_2 = -S_2 - S_2^T + P_2(\theta),$$

i.e.,

$$\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ 0 & \Pi_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta)S_1(\theta) & B(\theta)C_F S_2 & E(\theta) & -P_1(\theta) & * & * \\ 0 & A_F S_2 & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)S_1(\theta) & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I \end{bmatrix}$$

$$
+\begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)D_F \\ B_F \\ D(\theta)D_F \end{bmatrix} [C_2(\theta)S_1(\theta) \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]
$$

$$
+[C_2(\theta)S_1(\theta) \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)D_F \\ B_F \\ D(\theta)D_F \end{bmatrix}^T < 0. \tag{3.40}
$$

By defining $L_B = B_F U$ and $L_D = D_F U$, (3.40) becomes

$$
\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ 0 & \Pi_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta)S_1(\theta) & B(\theta)C_F S_2 & E(\theta) & -P_1(\theta) & * & * \\ 0 & A_F S_2 & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)S_1(\theta) & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)L_D \\ L_B \\ D(\theta)L_D \end{bmatrix} N N^{-1} U^{-1} [C_2(\theta)S_1(\theta) \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]
$$

$$
+[C_2(\theta)S_1(\theta) \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T U^{-T} N^{-T} N^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)L_D \\ L_B \\ D(\theta)L_D \end{bmatrix}^T < 0. \tag{3.41}
$$

Then, for the matrix M, we have

$$
\begin{bmatrix} \Pi_1 & * & * & * & * & * \\ 0 & \Pi_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A(\theta)S_1(\theta) + B(\theta)L_D M & B(\theta)C_F S_2 & E(\theta) & -P_1(\theta) & * & * \\ L_B M & A_F S_2 & 0 & 0 & -P_2(\theta) & * \\ C_1(\theta)S_1(\theta) + D(\theta)L_D M & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I \end{bmatrix}
$$

$$
+\begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)L_D \\ L_B \\ D(\theta)L_D \end{bmatrix} NN^{-1}U^{-1}[C_2(\theta)S_1(\theta) - UM \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]
$$

$$
+ [C_2(\theta)S_1(\theta) - UM \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T U^{-T} N^{-T} N^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ B(\theta)L_D \\ L_B \\ D(\theta)L_D \end{bmatrix}^T < 0. \quad (3.42)
$$

Then, following the same line as the proof of Theorem 2.9, we can establish the following result to verify (3.42):

$$
\begin{bmatrix}
\Pi_1 + J(\theta) & * & * & * & * & * & * & * \\
0 & \Pi_2 & * & * & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * & * & * \\
\Pi_3 & B(\theta)C_F S_2 & E(\theta) & -P_1(\theta) & * & * & * & * \\
L_B M & A_F S_2 & 0 & 0 & -P_2(\theta) & * & * & * \\
\Pi_4 & D(\theta)C_F S_2 & F(\theta) & 0 & 0 & -I & * & * \\
0 & 0 & 0 & N^T L_D^T B^T(\theta) & N^T L_B^T & N^T L_D^T D^T(\theta) & \Sigma_1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J(\theta)}{\beta^2}
\end{bmatrix}
$$

$$
< 0, \quad (3.43)
$$

where

$$
\Pi_3 = A(\theta)S_1(\theta) + B(\theta)L_D M,
$$

$$
\Pi_4 = C_1(\theta)S_1(\theta) + D(\theta)L_D M,
$$

$$
\Sigma_1 = -\beta UN - \beta N^T U^T,
$$

$$
\Sigma_2 = \big(C_2(\theta)S_1(\theta) - UM\big)^T.
$$

Define $\bar{A}_F = A_F S_2$ and $\bar{C}_F = C_F S_2$, if the LMIs (3.35) and (3.36) are satisfied, the matrix inequality holds. □

Remark 3.2 In this section, the LMI decoupling approach has been applied to design the dynamic output feedback H_∞ controller in the form of (3.2). Similar to the discussion in Chap. 2, when the matrices M and N are chosen as special form and $B(\theta)$ $\big(C_2(\theta)\big)$ is fixed (and of full rank), we also can prove that the design results given by the LMI decoupling approach are less conservative than the basic LMI conditions.

3.4 A Whole Design Strategy

In the above section, we have presented meaningful LMI results to design the dynamic output feedback H_∞ controller in the form of (3.2). It is worth noting that, the design laws (i.e., solutions) of the four controller gain matrices are not unified (see (3.17), (3.25), (3.28), and (3.37)). In [1, 2], a whole trategy has been proposed to design output feedback H_∞ controllers for discrete-time linear systems. In this strategy, the designed all controller gain matrices are seen as a whole, it leads to the solutions of these gain matrices can be integrated in a unified equation expression. In this section, we will develop the design strategy to robust dynamic output feedback H_∞ control for the uncertain systems (3.1).

In the closed-loop system (3.3), by integrating the controller gain matrices, it can be rewritten in the form

$$
\begin{aligned}
\bar{x}(k+1) &= \left(\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix}\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix}\right)\bar{x}(k) \\
&\quad + \left(\begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix}\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}\begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}\right) w(k), \\
z(k) &= \left([C_1(\theta)\ \ 0] + [0\ \ D(\theta)]\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix}\right)\bar{x}(k) \\
&\quad + \left(F(\theta) + [0\ \ D(\theta)]\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}\begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}\right) w(k).
\end{aligned} \tag{3.44}
$$

In this following, two theorems are provided, in which two basic LMI design results for the closed-loop system (3.3) are given by applying the whole design strategy and considering the properties of input or output matrices. The design results presented in theorems can be seen as the extension of [2], which puts the dynamic output feedback H_∞ controllers design method to dynamic controller design for discrete-time linear system with time-invariant polytopic uncertainties.

Theorem 3.5 *Consider the closed-loop system (3.44) with $D(\theta) = 0$ and $B(\theta) = B$ (B is full column rank). For a given scalar $\gamma > 0$, then the system is asymptotically stable with the H_∞ performance γ if there exist matrices L_1, P_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix equations hold:*

$$\Psi_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.45}$$

$$\Psi_{ij} + \Psi_{ji} < 0, \quad i < j,\ i, j = 1, 2, \ldots, r, \tag{3.46}$$

$$G_j = \begin{bmatrix} G_1 & 0 \\ 0 & G_{2j} \end{bmatrix} \ \text{or}\ G_j = \begin{bmatrix} G_1 & G_{2j} \\ 0 & G_{3j} \end{bmatrix}, \quad j = 1,\ 2, \ldots, r, \tag{3.47}$$

with

$$\Psi_{ij} =$$

$$\begin{bmatrix} -P_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G_j T_u \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} T_u^{-1} + \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{2i} & 0 \end{bmatrix} T_u^{-1} & G_j T_u \begin{bmatrix} E_i \\ 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ H_i \end{bmatrix} & \Theta & * \\ [C_{1i} \;\; 0] T_u^{-1} & F_i & 0 & -I \end{bmatrix} \tag{3.48}$$

where T_u *is a nonsingular matrix satisfying* $T_u \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ *and* $\Theta = -G_j - G_j^T + P_j$.

Furthermore, the dynamic output feedback H_∞ *controller gain matrices in* (3.2) *are given as*

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = G_1^{-1} L_1. \tag{3.49}$$

Proof Choose a Lyapunov function as

$$V(k) = \bar{x}^T(k) T_u^T P(\theta) T_u \bar{x}(k), \; P(\theta) = \sum_{j=1}^{r} \theta_j P_j, \; P_j > 0, \quad j = 1, 2, \ldots, r. \tag{3.50}$$

The H_∞ performance γ of the closed-loop system (3.44) can be guaranteed by

$$\begin{bmatrix} -T_u^T P(\theta) T_u & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Delta_A & \Delta_E & -T_u^{-1} P^{-1}(\theta) T_u^{-T} & * \\ [C_1(\theta) \;\; 0] & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{3.51}$$

where

$$\Delta_A = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix},$$

$$\Delta_E = \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}.$$

Introduce an auxiliary matrix variable as

$$G(\theta) = \sum_{j=1}^{r} \theta_j G_j = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_1 & 0 \\ 0 & G_{2j} \end{bmatrix},$$

$$or \tag{3.52}$$

$$G(\theta) = \sum_{j=1}^{r} \theta_j G_j = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_1 & G_{2j} \\ 0 & G_{3j} \end{bmatrix},$$

then (3.51) is satisfied if

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)T_u\Delta_A T_u^{-1} & G(\theta)T_u\Delta_E & -G(\theta) - G^T(\theta) + P(\theta) & * \\ [C_1(\theta)\ \ 0]T_u^{-1} & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{3.53}$$

On the other hand, with the support of $L_1 = G_1 \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}$, it can be verified that

$$\begin{aligned} G(\theta)T_u\Delta_A T_u^{-1} &= G(\theta)T_u \left(\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} \right) T_u^{-1} \\ &= G(\theta)T_u \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_u^{-1} + G(\theta)T_u \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} T_u^{-1} \\ &= G(\theta)T_u \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_u^{-1} + G(\theta) \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} T_u^{-1} \\ &= G(\theta)T_u \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_u^{-1} + \begin{bmatrix} G_1 \\ 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} T_u^{-1} \\ &= G(\theta)T_u \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_u^{-1} + \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} T_u^{-1}, \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} G(\theta)T_u\Delta_E &= G(\theta)T_u \left(\begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} \right) \\ &= G(\theta)T_u \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}. \end{aligned} \tag{3.55}$$

By combining (3.52)–(3.55), the LMIs (3.45) and (3.46) can be obtained. □

Theorem 3.6 *Consider the closed-loop system (3.44) with $H(\theta) = 0$ and $C_2(\theta) = C_2$ (C_2 is full row rank). For a given scalar $\gamma > 0$, the system is asymptotically stable with the H_∞ performance γ if there exist matrices Y_1, P_j, and S_j, $j = 1, 2, \ldots, r$ such that the following matrix equations hold:*

$$\Upsilon_{ii} < 0, 1, 2, \ldots, r, \tag{3.56}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0,\ i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.57}$$

$$S_j = \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix} \ or \ S_j = \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix}, \quad j = 1, 2, \ldots, r, \tag{3.58}$$

with

$$\Upsilon_{ij} = \begin{bmatrix} -S_j - S_j^T + P_j & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T_y^{-1}\begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} T_y S_j + T_y^{-1}\begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}[Y_1 \quad 0] & T_y^{-1}\begin{bmatrix} E_i \\ 0 \end{bmatrix} & -P_j & * \\ [C_{1i} \ \ 0]T_y S_j + [0 \ \ D_i][Y_1 \ \ 0] & F_i & 0 & -I \end{bmatrix}, \tag{3.59}$$

where T_y *is a nonsingular matrix satisfying* $\begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix} T_y = [I \ \ 0]$.

Furthermore, the dynamic output feedback H_∞ *controller gain matrices in* (3.2) *are given as*

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = Y_1 S_1^{-1}. \tag{3.60}$$

Proof Choose the Lyapunov function as

$$V(k) = \bar{x}^T(k) T_y^{-T} P^{-1}(\theta) T_y^{-1} \bar{x}(k),\ P(\theta) = \sum_{j=1}^{r} \theta_j P_j,\ P_j > 0, j = 1, 2, ..., r. \tag{3.61}$$

Consider an auxiliary matrix variable as

$$S(\theta) = \sum_{j=1}^{r} \theta_j S_j = \sum_{j=1}^{r} \theta_j \begin{bmatrix} S_1 & 0 \\ 0 & S_{3j} \end{bmatrix},$$

$$or \tag{3.62}$$

$$S(\theta) = \sum_{j=1}^{r} \theta_j S_j = \sum_{j=1}^{r} \theta_j \begin{bmatrix} S_1 & 0 \\ S_{2j} & S_{3j} \end{bmatrix},$$

then the H_∞ performance γ of the closed-loop system (3.44) can be guaranteed by

$$\begin{bmatrix} -S(\theta) - S^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ T_y^{-1} \Delta_A T_y S(\theta) & T_y^{-1}\begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -P(\theta) & * \\ \Delta_C T_y S(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{3.63}$$

where

$$\Delta_A = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$\Delta_C = [C_1(\theta) \ \ 0] + [0 \ \ D(\theta)] \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}.$$

Define a variable $Y_1 = \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} S_1$, we have

$$\begin{aligned} T_y^{-1}\Delta_A T_y S(\theta) &= T_y^{-1}\left(\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}\right) T_y S(\theta) \\ &= T_y^{-1}\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_y S(\theta) + T_y^{-1}\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix} T_y S(\theta) \\ &= T_y^{-1}\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_y S(\theta) + T_y^{-1}\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} [I \ \ 0] S(\theta) \\ &= T_y^{-1}\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_y S(\theta) + T_y^{-1}\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} [S_1 \ 0] \\ &= T_y^{-1}\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} T_y S(\theta) + T_y^{-1}\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} [Y_1 \ 0], \end{aligned} \tag{3.64}$$

and

$$\begin{aligned} \Delta_C T_y S(\theta) &= \left([C_1(\theta) \ 0] + [0 \ D(\theta)] \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}\right) T_y S(\theta), \\ &= [C_1(\theta) \ 0] T_y S(\theta) + [0 \ D(\theta)][Y_1 \ 0]. \end{aligned} \tag{3.65}$$

From (3.63)–(3.65), it can be known that (3.63) holds if LMIs (3.56) and (3.57) are satisfied. □

For the whole design strategy of the dynamic output feedback controller (3.2), the LMI decoupling approach presented in Chap. 2 is also effective. The following two theorems give another design results, which are the combination of the whole strategy and the LMI decoupling approach. Especially, the new design results do not claim that the system input and output matrices must be fixed and of full rank.

Theorem 3.7 *Consider the closed-loop system* (3.44) *with* $D(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* V_A, V_B, V_C, V_D, U, P_j, G_j, *and* J_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold:*

$$\Omega_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.66}$$

$$\Omega_{ij} + \Omega_{ji} < 0, \ i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.67}$$

with

$$\Omega_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ \Phi_1 & \Phi_2 & \Phi_3 & * & * & * \\ [C_{1i} \quad 0] & F_i & 0 & -I & * & * \\ N\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 & I \\ C_{2i} & 0 \end{bmatrix} & N\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 \\ H_i \end{bmatrix} & 0 & 0 & \Omega_4 & * \\ 0 & 0 & 0 & 0 & \Omega_5 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\Omega_1 = G_j \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} + M \begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 & I \\ C_{2i} & 0 \end{bmatrix},$$

$$\Omega_2 = G_j \begin{bmatrix} E_i \\ 0 \end{bmatrix} + M \begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 \\ H_i \end{bmatrix},$$

$$\Omega_3 = -G_j - G_j^T + P_j + J_j,$$

$$\Omega_4 = -\beta NU - \beta U^T N^T,$$

$$\Omega_5 = G_j \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix} - MU.$$

Furthermore, the dynamic output feedback H_∞ controller gain matrices in (3.2) *are given as*

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = U^{-1} \begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}. \tag{3.68}$$

Proof Choose a Lyapunov function as $V(k) = \bar{x}^T(k)P(\theta)\bar{x}(k)$, $P(\theta) = \sum_{j=1}^{r} \theta_j P_j$, $P_j > 0$, $j = 1, 2, \ldots, r$.

By adding an auxiliary matrix variable $G(\theta) = \sum_{j=1}^{r} \theta_j G_j$ and defining another variable $\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix} = U \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}$ where U is a nonsingular matrix, we give immediately the following condition to ensure the H_∞ performance γ:

$$\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)\Delta_A & G(\theta)\Delta_E & -G(\theta) - G^T(\theta) + P(\theta) & * \\ [C_1(\theta) \quad 0] & F(\theta) & 0 & -I \end{bmatrix}$$

$$= \begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G(\theta)\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} & G(\theta)\begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -G(\theta) - G^T(\theta) + P(\theta) & * \\ [C_1(\theta) \quad 0] & F(\theta) & 0 & -I \end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ 0 \end{bmatrix}\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}\begin{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} & 0 & 0 \end{bmatrix} \\
&+\begin{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} & 0 & 0 \end{bmatrix}^T\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}^T\begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ 0 \end{bmatrix}^T < 0,
\end{aligned} \tag{3.69}
$$

i.e.,

$$
\begin{aligned}
&\begin{bmatrix} -P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Phi_1 & \Phi_2 & -G(\theta)-G^T(\theta)+P(\theta) & * \\ [C_1(\theta) \quad 0] & F(\theta) & 0 & -I \end{bmatrix} \\
&+\begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} - MU \\ 0 \end{bmatrix} U^{-1}N^{-1}N\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} & 0 & 0 \end{bmatrix} \\
&+\begin{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} & 0 & 0 \end{bmatrix}^T\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}^T N^T N^{-T} U^{-T}\begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} - MU \\ 0 \end{bmatrix}^T \\
&< 0,
\end{aligned} \tag{3.70}
$$

where

$$
\Phi_1 = G(\theta)\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + M\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix},
$$

$$
\Phi_2 = G(\theta)\begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + M\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}.
$$

Following the proof of Theorem 2.1, we obtain the following matrix inequality to verify (3.70):

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ \Phi_1 & \Phi_2 & \Phi_3 & * & * & * \\ [C_1(\theta) \quad 0] & F(\theta) & 0 & -I & * & * \\ N\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} & N\begin{bmatrix} V_A & V_B \\ V_C & V_D \end{bmatrix}\begin{bmatrix} 0 \\ H(\theta) \end{bmatrix} & 0 & 0 & \Upsilon_1 & * \\ 0 & 0 & 0 & 0 & \Upsilon_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{3.71}$$

where

$$\Phi_3 = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$

$$\Upsilon_1 = -\beta NU - \beta U^T N^T,$$

$$\Upsilon_2 = G(\theta)\begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} - MU.$$

□

Theorem 3.8 *Consider the closed-loop system* (3.44) *with* $H(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* $L_A, L_B, L_C, L_D, U, P_j, S_j$, *and* J_j, $j = 1, 2, \ldots, r$ *such that the following matrix inequalities hold:*

$$\Upsilon_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.72}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, i < j, \quad i, j = 1, 2, \ldots, r, \tag{3.73}$$

with

$$\Upsilon_{ij} = \begin{bmatrix} -S_j - S_j^T + P_j + J_j & * & * & * & & \\ 0 & -\gamma^2 I & * & * & & \\ \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} S_j + \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}\begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & \begin{bmatrix} E_i \\ 0 \end{bmatrix} & -P_j & * & & \\ [C_{1i} \quad 0]S_j + [0 \ D_i]\begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & F_i & 0 & -I & & \\ 0 & 0 & \Phi_1 & \Phi_2 & \Xi_1 & * \\ 0 & 0 & 0 & 0 & \Xi_2 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\Phi_1 = N^T \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T \begin{bmatrix} 0 & B_i \\ I & 0 \end{bmatrix}^T,$$

$$\Phi_2 = N^T \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T [0\ D_i]^T,$$

$$\Xi_1 = -\beta UN - \beta N^T U^T,$$

$$\Xi_2 = \left(\begin{bmatrix} 0 & I \\ C_{2i} & 0 \end{bmatrix} S_j - UM\right)^T.$$

Furthermore, the dynamic output feedback H_∞ controller gain matrices in (3.2) *are given as*

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} U^{-1}. \tag{3.74}$$

Proof Choose a Lyapunov function as $V(k) = \bar{x}^T(k)P^{-1}(\theta)\bar{x}(k)$, $P(\theta) = \sum_{j=1}^r \theta_j P_j$, $P_j > 0$, $j = 1, 2, \ldots, r$.

Introduce an auxiliary matrix variable $S(\theta) = \sum_{j=1}^r \theta_j S_j$, then the H_∞ performance γ of the closed-loop system (3.44) with $H(\theta) = 0$ can be guaranteed by

$$\begin{aligned}
&\begin{bmatrix} -S(\theta) - S^T(\theta) + P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} S(\theta) & \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -P(\theta) & * \\ [C_1(\theta) \quad 0] S(\theta) & F(\theta) & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \quad D(\theta)] \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta)\ 0\ 0\ 0 \right] \\
&+ \left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta)\ 0\ 0\ 0 \right]^T \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \quad D(\theta)] \end{bmatrix}^T < 0.
\end{aligned} \tag{3.75}$$

Assume $\begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} = \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} U$ where U is a nonsingular matrix, it is observed that the inequality (3.75) is equivalent to

$$
\begin{bmatrix} -S(\theta)-S^T(\theta)+P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} S(\theta) & \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -P(\theta) & * \\ [C_1(\theta) \quad 0]S(\theta) & F(\theta) & 0 & -I \end{bmatrix}
$$
$$
+\begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \; D(\theta)] \end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} NN^{-1}U^{-1} \left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta) \; 0 \; 0 \; 0 \right]
$$
$$
+\left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta) \; 0 \; 0 \; 0 \right]^T U^{-T}N^{-T}N^T \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \quad D(\theta)] \end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T < 0, \tag{3.76}
$$

i.e.,

$$
\begin{bmatrix} -S(\theta)-S^T(\theta)+P(\theta) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} S(\theta) + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -P(\theta) & * \\ [C_1(\theta) \quad 0]S(\theta) + [0 \quad D(\theta)] \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & F(\theta) & 0 & -I \end{bmatrix}
$$
$$
+\begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \quad D(\theta)] \end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} NN^{-1}U^{-1} \left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta) - UM \; 0 \; 0 \; 0 \right]
$$
$$
+\left[\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta) - UM \; 0 \; 0 \; 0 \right]^T U^{-T}N^{-T}N^T \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \\ [0 \quad D(\theta)] \end{bmatrix}^T . \tag{3.77}
$$

Following the proof of Theorem 2.9, we obtain the following matrix inequality to verify (3.77):

$$\begin{bmatrix} -S(\theta)-S^T(\theta)+P(\theta)+J(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} S(\theta) + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} & -P(\theta) & * & * & * \\ [C_1(\theta) \quad 0]S(\theta) + [0 \quad D(\theta)] \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix} M & F(\theta) & 0 & -I & * & * \\ 0 & 0 & \tilde{\Phi}_1 & \tilde{\Phi}_2 & \Xi_1 & * \\ 0 & 0 & 0 & 0 & \tilde{\Xi}_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix}$$

$$< 0, \tag{3.78}$$

where

$$\tilde{\Phi}_1 = N^T \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix}^T,$$

$$\tilde{\Phi}_2 = N^T \begin{bmatrix} L_A & L_B \\ L_C & L_D \end{bmatrix}^T [0 \quad D(\theta)]^T,$$

$$\Xi_1 = -\beta U N - \beta N^T U^T,$$

$$\tilde{\Xi}_2 = \left(\begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix} S(\theta) - UM \right)^T.$$

□

3.5 For the Case $D(\theta) \neq 0$ and $H(\theta) \neq 0$

It is known that the H_∞ performance γ of the closed-loop system (3.44) can be guaranteed by

$$\begin{bmatrix} \begin{bmatrix} -P(\theta) & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G(\theta) \begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix} & -G(\theta) - G^T(\theta) + \begin{bmatrix} P(\theta) & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0, \tag{3.79}$$

where

$$\mathscr{A} = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix},$$

$$\mathscr{B} = \begin{bmatrix} E(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B(\theta) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix},$$

$$\mathscr{C} = [C_1(\theta) \quad 0] + [0 \quad D(\theta)] \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2(\theta) & 0 \end{bmatrix},$$

$$\mathscr{D} = F(\theta) + [0 \quad D(\theta)] \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 \\ H(\theta) \end{bmatrix}.$$

Let us rewrite the (2,1) term in (3.79) as follows:

$$\begin{aligned} & G(\theta) \begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix} \\ &= G(\theta) \begin{bmatrix} A(\theta) & 0 & E(\theta) \\ 0 & 0 & 0 \\ C_1(\theta) & 0 & F(\theta) \end{bmatrix} + G(\theta) \begin{bmatrix} 0 & B(\theta) \\ I & 0 \\ 0 & D(\theta) \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2(\theta) & 0 & H(\theta) \end{bmatrix}. \end{aligned} \tag{3.80}$$

Similar to (3.71), the matrix inequality (3.79) can verified as

$$\begin{bmatrix} \begin{bmatrix} -P(\theta) & * \\ 0 & -\gamma^2 I \end{bmatrix} & * & * & * \\ \Xi_1 & \Xi_2 & * & * \\ NV \begin{bmatrix} 0 & I & 0 \\ C_2(\theta) & 0 & H(\theta) \end{bmatrix} & 0 & -\beta NU - \beta U^T N^T & * \\ 0 & 0 & G(\theta) \begin{bmatrix} 0 & B(\theta) \\ I & 0 \\ 0 & D(\theta) \end{bmatrix} - MU & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{3.81}$$

where

$$\Xi_1 = G(\theta) \begin{bmatrix} A(\theta) & 0 & E(\theta) \\ 0 & 0 & 0 \\ C_1(\theta) & 0 & F(\theta) \end{bmatrix} + MV \begin{bmatrix} 0 & I & 0 \\ C_2(\theta) & 0 & H(\theta) \end{bmatrix},$$

$$\Xi_2 = -G(\theta) - G^T(\theta) + \begin{bmatrix} P(\theta) & 0 \\ 0 & I \end{bmatrix} + J(\theta).$$

The matrix condition (3.81) is an H_∞ performance analysis criterion, which can be employed for dynamic output feedback H_∞ controller design. The following theorem gives the corresponding design result:

Theorem 3.9 *Consider the closed-loop system* (3.44) *and give scalars* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices*

M, N and scalar β, exist matrices U, V, P_j, J_j, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold:

$$\Upsilon_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{3.82}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad i, j = 1, 2, \ldots, r, i < j, \tag{3.83}$$

with

$$\Upsilon_{ij} = \begin{bmatrix} \begin{bmatrix} -P_j & * \\ 0 & -\gamma^2 I \end{bmatrix} & * & * & * \\ \Delta_1 & \Delta_2 & * & * \\ NV\begin{bmatrix} 0 & I & 0 \\ C_{2i} & 0 & H_i \end{bmatrix} & 0 & -\beta NU - \beta U^T N^T & * \\ 0 & 0 & G_j \begin{bmatrix} 0 & B_i \\ I & 0 \\ 0 & D_i \end{bmatrix} - MU & -\frac{J_j}{\beta^2} \end{bmatrix},$$

where

$$\Delta_1 = G_j \begin{bmatrix} A_i & 0 & E_i \\ 0 & 0 & 0 \\ C_{1i} & 0 & F_i \end{bmatrix} + MV\begin{bmatrix} 0 & I & 0 \\ C_{2i} & 0 & H_i \end{bmatrix},$$

$$\Delta_2 = -G_j - G_j^T + \begin{bmatrix} P_j & 0 \\ 0 & I \end{bmatrix} + J_j.$$

Furthermore, the dynamic output feedback H_∞ controller gain matrices in (3.2) *are given as*

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = U^{-1}V. \tag{3.84}$$

3.6 Example

The less conservativeness of the presented design results obtained by the LMI decoupling approach has been shown in Sect. 2.3. In this example, a comparative between the proposed design condition in Theorem 3.9 and the one in [3] is considered. Consider a discrete-time uncertain linear system discussed in [3], which belongs to the 2-polytopic convex polyhedron in the form of (3.1) with

$$A_1 = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & -0.2 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix},$$

$$C_{11} = [1\ 0], \quad C_{12} = [1\ 0.5],$$

$$D_1 = 1, \quad D_2 = 2,$$

$$F_1 = 0.1, \quad F_2 = 0.2,$$

$$C_{21} = [1\ 2], \quad C_{22} = [1\ 2],$$

$$H_1 = 0.2, \quad H_2 = 0.2.$$

For this example, Theorem 4.1 in [3] and Theorem 3.9 in this chapter are applicable for designing the dynamic output feedback H_∞ controller in the form of (3.2) with $n_F = 2$. The LMIs of Theorem 4.1 in [3] yield a minimum value of 3.1366 for the H_∞ performance γ. In contrast, solving LMIs (3.82) and (3.83) in this chapter with $\beta = 3.36$, $M = \begin{bmatrix} 0 & B_i \\ I & 0 \\ 0 & D_i \end{bmatrix}$ and $N = I$ yields a minimum of 1.2861, which is clearly much better. It shows that the condition proposed in Theorem 3.9 is less conservative than that proposed in [3].

3.7 Conclusion

In this chapter, the robust dynamic output feedback H_∞ control problem for discrete-time linear systems has been studied. This study is geared to systems with polytopic uncertainties based on the parameter-dependent Lyapunov approach. By applying the properties of input or output matrices, basic conditions for designing H_∞ controllers have been given. A further improvement based on the LMI decoupling approach has been also proposed. These design conditions are presented in the form of linear matrix inequalities (LMIs). A simulation example shows the less conservativeness of the proposed design methods.

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Chapter 4
Robust Observer-Based Output Feedback H_∞ Control

Abstract This chapter studies observer-based output feedback H_∞ control problem for discrete-time linear systems with polytopic uncertainties and norm bounded uncertainties. For systems with polytopic uncertainties, the descriptor systems approach is used to design observers and controllers. For systems with norm bounded uncertainties, in the so-called two-step procedure, a selective approach is introduced first, in which results of the first step are allowed to be selected in order to reduce the conservatism of previous approaches. Then, a simple LMI result for observer-based output feedback H_∞ control design for uncertain linear systems with a special case is given. Finally, the LMI decoupling approach is also considered for designing observers and controllers.

Keywords Linear discrete-time systems · Polytopic uncertainties · Norm bounded uncertainties · Observer · H_∞ control

4.1 With Time-Invariant Polytopic Uncertainties

In this section, we consider a discrete-time linear system with time-invariant polytopic uncertainties

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + E(\theta)w(k), \\ z(k) &= C_1(\theta)x(k) + D(\theta)u(k) + F(\theta)w(k), \\ y(k) &= C_2(\theta)x(k) + H(\theta)w(k), \end{aligned} \tag{4.1}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. The matrices $A(\theta)$, $B(\theta)$, $E(\theta)$, $C_1(\theta)$, $D(\theta)$, $F(\theta)$, $C_2(\theta)$, and $H(\theta)$ are constant matrices of appropriate dimensions and belong to the uncertainty in (2.2).

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_4,

Choose the following observer to estimate the state of system (4.1):

$$\begin{aligned}\hat{x}(k+1) &= A_L\hat{x}(k) + B_L u(k) + L\big(y(k) - \hat{y}(k)\big),\\ \hat{y}(k) &= C_{2L}\hat{x}(k),\end{aligned} \tag{4.2}$$

where $\hat{x}(k) \in \mathscr{R}^n$ and $\hat{y}(k) \in \mathscr{R}^f$ are the estimated state and estimated output, respectively. A_L, B_L, C_{2L}, and L are the observer gain matrices with appropriate dimensions.

Assume the following controller is employed to deal with the design of system (4.1):

$$u(k) = K\hat{x}(k), \tag{4.3}$$

where K is the controller gain.

From (4.2) and (4.3), we can see that the number of designed gain matrices is numerous, it brings great difficulties to the system design. In order to obtain LMI-based design conditions for the systems (4.1), we use the descriptor systems approach to design the observer (4.2) and controller (4.3). From the related literature [1, 4, 8], it is known that rewriting the closed-loop system by use of descriptor redundancy allows to avoid appearance of crossing terms between the system matrices and the designed ones, which makes easier the LMI formulation of synthesis conditions. To take advantage of a descriptor redundancy formulation in the case of observer-based H_∞ control, (4.1), (4.2), and (4.3) can be easily rewritten as

$$\begin{aligned}x(k+1) &= A(\theta)x(k) + B(\theta)u(k) + E(\theta)w(k),\\ \hat{x}(k+1) &= A_L\hat{x}(k) + B_L u(k) + Le_y(k),\\ 0\cdot u(k+1) &= K\hat{x}(k) - u(k),\\ 0\cdot e_y(k+1) &= C_2(\theta)x(k) + H(\theta)w(k) - C_{2L}\hat{x}(k) - e_y(k),\end{aligned} \tag{4.4}$$

where $e_y(k)$ is the output estimation error as $e_y(k) = y(k) - \hat{y}(k)$.

Let us define a new variable as $\tilde{x}(k) = \begin{bmatrix} x(k)\\ \hat{x}(k)\\ u(k)\\ e_y(k)\end{bmatrix}$, (4.4) and $z(k)$ in (4.1) can be expressed as

$$\begin{aligned}&\begin{bmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\tilde{x}(k+1) = \begin{bmatrix} A(\theta) & 0 & B(\theta) & 0\\ 0 & A_L & B_L & L\\ 0 & K & -I & 0\\ C_2(\theta) & -C_{2L} & 0 & -I\end{bmatrix}\tilde{x}(k) + \begin{bmatrix} E(\theta)\\ 0\\ 0\\ H(\theta)\end{bmatrix}w(k),\\ &z(k) = [\, C_1(\theta) \quad 0 \quad D(\theta) \quad 0\,]\tilde{x}(k) + F(\theta)w(k).\end{aligned} \tag{4.5}$$

In the following, we present sufficient design conditions guaranteeing the H_∞ performance of the system (4.1).

4.1.1 Condition A

Theorem 4.1 *Consider the system (4.1) and give a scalar $\gamma > 0$. Then the system via the observer (4.2) and the controller (4.3) is asymptotically stable with the H_∞ performance γ if exist matrices N_{AL}, N_{BL}, N_{C2L}, N_L, N_K, Q_{22}, Q_{33}, Q_{44}, P_{11j}, P_{21j}, P_{22j}, P_{31j}, P_{32j}, P_{33j}, P_{41j}, P_{42j}, P_{43j}, P_{44j}, Q_{11j}, Q_{12j}, X_{31j}, X_{32j}, X_{33j}, X_{43j}, X_{41j}, X_{42j}, X_{43j}, and X_{44j}, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{11j} & * \\ P_{21j} & P_{22j} \end{bmatrix} > 0, \ j = 1, 2, \ldots, r, \tag{4.6}$$

$$\Delta_{ii} < 0, \ i = 1, 2, \ldots, r, \tag{4.7}$$

$$\Delta_{ij} + \Delta_{ji} < 0, \ i < j, \ i, \ j = 1, 2, \ldots, r, \tag{4.8}$$

with

$$\Delta_{ij} = \begin{bmatrix} \varepsilon_{11} & * & * & * & * & * & * & * & * & * \\ \varepsilon_{21} & \varepsilon_{22} & * & * & * & * & * & * & * & * \\ Q_{33}^T B_i^T & \varepsilon_{32} & \varepsilon_{33} & * & * & * & * & * & * & * \\ C_{2i} Q_{11j} & \varepsilon_{42} & 0 & \varepsilon_{44} & * & * & * & * & * & * \\ E_i^T & 0 & 0 & H_i^T & -\gamma^2 I & * & * & * & * & * \\ \varepsilon_{61i} & A_i Q_{12j} & B_i Q_{33} & 0 & E_i & \varepsilon_{66} & * & * & * & * \\ Q_{12j}^T & \varepsilon_{72} & N_{BL} & N_L & 0 & \varepsilon_{76} & \varepsilon_{77} & * & * & * \\ X_{31j} & \varepsilon_{82} & \varepsilon_{83} & X_{34j} & 0 & \varepsilon_{86} & \varepsilon_{87} & \varepsilon_{88} & * & * \\ \varepsilon_{91} & \varepsilon_{92} & X_{43j} & \varepsilon_{94} & H_i & \varepsilon_{96} & \varepsilon_{97} & \varepsilon_{98} & \varepsilon_{99} & * \\ C_{1i} Q_{11j} & C_{1i} Q_{12j} & D_i Q_{33} & 0 & F_i & 0 & 0 & 0 & 0 & -I \end{bmatrix},$$

where

$$\varepsilon_{11} = -P_{11j} + A_i Q_{11j} + Q_{11j}^T A_i^T,$$

$$\varepsilon_{21} = -P_{21j} + Q_{12j}^T A_i^T,$$

$$\varepsilon_{22} = -P_{22j} + N_{AL} + N_{AL}^T,$$

$$\varepsilon_{32} = N_K + N_{BL}^T,$$

$$\varepsilon_{33} = -Q_{33} - Q_{33}^T,$$

$$\varepsilon_{42} = C_{2i} Q_{12j} - N_{C2L} + N_L^T,$$

$$\varepsilon_{44} = -Q_{44} - Q_{44}^T,$$

$$\varepsilon_{61} = Q_{11j}^T + A_i Q_{11j},$$

$$\varepsilon_{66} = -Q_{11j} - Q_{11j}^T + P_{11j},$$

$$\varepsilon_{72} = Q_{22}^T + N_{AL},$$

$$\varepsilon_{76} = -Q_{12j}^T + P_{21j},$$

$$\varepsilon_{77} = -Q_{22} - Q_{22}^T + P_{22j},$$

$$\varepsilon_{82} = X_{32j} + N_K,$$

$$\varepsilon_{83} = X_{33j} - Q_{33},$$

$$\varepsilon_{86} = -X_{31j} + P_{31j},$$

$$\varepsilon_{87} = -X_{32j} + P_{32j},$$

$$\varepsilon_{88} = -X_{33j} - X_{33j}^T + P_{31j},$$

$$\varepsilon_{91} = X_{41j} + C_{2i} Q_{11j},$$

$$\varepsilon_{92} = X_{42j} + C_{2i} Q_{12j} - N_{C2L},$$

$$\varepsilon_{94} = X_{44j} - Q_{44},$$

$$\varepsilon_{96} = -X_{41j} + P_{41j},$$

$$\varepsilon_{97} = -X_{42j} + P_{42j},$$

$$\varepsilon_{98} = -X_{43j} - X_{34j}^T + P_{43j},$$

$$\varepsilon_{99} = -X_{44j} - X_{44j}^T + P_{44j}.$$

Furthermore, the controller and observer gains are given by

$$K = N_K Q_{22}^{-1},$$
$$A_L = N_{AL} Q_{22}^{-1}, \quad B_L = N_{BL} Q_{33}^{-1}, \quad C_{2L} = N_{C2L} Q_{22}^{-1}, \quad L = N_L Q_{44}^{-1}.$$

respectively.

Proof For simplicity, define $E = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A(\theta) & 0 & B(\theta) & 0 \\ 0 & A_L & B_L & L \\ 0 & K & -I & 0 \\ C_2(\theta) & -C_{2L} & 0 & -I \end{bmatrix}$,
$\bar{B} = \begin{bmatrix} E(\theta) \\ 0 \\ 0 \\ H(\theta) \end{bmatrix}$, and $\bar{C} = [\ C_1(\theta) \quad 0 \quad D(\theta) \quad 0\ ]$.

Considering a parameter-dependent Lyapunov function as

$$V(k) = \tilde{x}^T(k)E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E\tilde{x}(k),\ \ E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E \geq 0, \tag{4.9}$$

and

$$E^T X^{-1}(\theta) = Q^{-T}(\theta)E. \tag{4.10}$$

From (4.5) and (4.9), we have

$$\begin{aligned}
&V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
&= \tilde{x}^T(k+1)E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E\tilde{x}(k+1) \\
&\quad - \tilde{x}^T(k)E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E\tilde{x}(k) \\
&\quad + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
&= \big(\bar{A}\tilde{x}(k) + \bar{B}w(k)\big)^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)\big(\bar{A}\tilde{x}(k) + \bar{B}w(k)\big) \\
&\quad - \tilde{x}^T(k)E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E\tilde{x}(k) + z^T(k)z(k) - \gamma^2 w^T(k)w(k) \\
&= \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix}^T [\ \bar{A} \quad \bar{B}\ ]^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)[\ \bar{A} \quad \bar{B}\ ] \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \\
&\quad + \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} -E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \\
&\quad + \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix}^T [\ \bar{C} \quad F(\theta)\ ]^T [\ \bar{C} \quad F(\theta)\ ] \begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix}.
\end{aligned}$$

Thus, $V(k+1) - V(k) + z^T(k)z(k) - \gamma^2 w^{\mathrm{T}}(k)w(k) < 0$ for any $\begin{bmatrix} \tilde{x}(k) \\ w(k) \end{bmatrix} \neq 0$ if

$$
[\ \bar{A} \quad \bar{B}\ ]^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)[\ \bar{A} \quad \bar{B}\ ] + \begin{bmatrix} -E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E & 0 \\ 0 & -\gamma^2 I \end{bmatrix}
$$
$$
+[\ \bar{C} \quad F(\theta)\ ]^T[\ \bar{C} \quad F(\theta)\ ] < 0. \tag{4.11}
$$

By using Lemma 1.6 with

$$
T = \begin{bmatrix} -E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + [\ \bar{C} \quad F(\theta)\ ]^T[\ \bar{C} \quad F(\theta)\ ],
$$

$$
A = [\ \bar{A} \quad \bar{B}\ ],
$$

$$
P = X^{-1}(\theta)P(\theta)X^{-T}(\theta),
$$

the matrix condition (4.11) can be guaranteed by

$$
\begin{bmatrix} \Pi & * \\ -M^T(\theta) + G(\theta)[\ \bar{A} \quad \bar{B}\ ] & -G(\theta) - G^T(\theta) + X^{-1}(\theta)P(\theta)X^{-T}(\theta) \end{bmatrix} < 0, \tag{4.12}
$$

where

$$
\Pi = \begin{bmatrix} -E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + [\ \bar{C} \quad F(\theta)\ ]^T[\ \bar{C} \quad F(\theta)\ ]
$$
$$
+M(\theta)[\ \bar{A} \quad \bar{B}\ ] + [\ \bar{A} \quad \bar{B}\ ]^T M^T(\theta).
$$

By defining $M(\theta) = \begin{bmatrix} Q^{-T}(\theta) \\ 0 \end{bmatrix}$, $G(\theta) = X^{-1}(\theta)$ and applying Schur complement, (4.12) leads to

$$
\begin{bmatrix} \Omega_1 & * & * & * \\ \bar{B}^T Q^{-1}(\theta) & -\gamma^2 I & * & * \\ -Q^{-1}(\theta) + X^{-1}(\theta)\bar{A} & X^{-1}(\theta)\bar{B} & \Omega_2 & * \\ \bar{C} & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{4.13}
$$

where

$$
\Omega_1 = -E^T X^{-1}(\theta)P(\theta)X^{-T}(\theta)E + Q^{-T}(\theta)\bar{A} + \bar{A}^T Q^{-1}(\theta),
$$

$$
\Omega_2 = -X^{-1}(\theta) - X^{-T}(\theta) + X^{-1}(\theta)P(\theta)X^{-T}(\theta).
$$

With the support of the equation (4.10), the matrix inequality equivalent to

$$\begin{bmatrix} \tilde{\Omega}_1 & * & * & * \\ \bar{B}^T Q^{-1}(\theta) & -\gamma^2 I & * & * \\ -Q^{-1}(\theta) + X^{-1}(\theta)\bar{A} & X^{-1}(\theta)\bar{B} & \Omega_2 & * \\ \bar{C} & F(\theta) & 0 & -I \end{bmatrix} < 0, \tag{4.14}$$

where $\tilde{\Omega}_1 = -Q^{-T}(\theta)EP(\theta)E^T Q^{-1}(\theta) + Q^{-T}(\theta)\bar{A} + \bar{A}^T Q^{-1}(\theta)$.

Pre- and post-multiplying (4.14) by $\begin{bmatrix} Q^T(\theta) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X(\theta) & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, it follows that

$$\begin{bmatrix} -EP(\theta)E^T + \bar{A}Q(\theta) + Q^T(\theta)\bar{A}^T & * & * & * \\ \bar{B}^T & -\gamma^2 I & * & * \\ X(\theta) + \bar{A}Q(\theta) & \bar{B} & -X(\theta) - X^T(\theta) + P(\theta) & * \\ \bar{C}Q(\theta) & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{4.15}$$

To cast the condition (4.15) into LMIs and the equation (4.10) to be satisfied, we partition matrices $P(\theta)$, $Q(\theta)$, and $X(\theta)$ as

$$P(\theta) = \left[\begin{array}{cc|cc} P_{11}(\theta) & * & * & * \\ P_{21}(\theta) & P_{22}(\theta) & * & * \\ \hline P_{31}(\theta) & P_{32}(\theta) & P_{33}(\theta) & * \\ P_{41}(\theta) & P_{42}(\theta) & P_{43}(\theta) & P_{44}(\theta) \end{array}\right] = \sum_{j=1}^{r} \theta_j \left[\begin{array}{cc|cc} P_{11j} & * & * & * \\ P_{21j} & P_{22j} & * & * \\ \hline P_{31j} & P_{32j} & P_{33j} & * \\ P_{41j} & P_{42j} & P_{43j} & P_{44j} \end{array}\right], \tag{4.16}$$

and

$$Q(\theta) = \left[\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_4 \end{array}\right] = \left[\begin{array}{cc|cc} Q_{11}(\theta) & Q_{12}(\theta) & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ \hline 0 & 0 & Q_{33} & 0 \\ 0 & 0 & 0 & Q_{44} \end{array}\right] = \sum_{j=1}^{r} \theta_j \left[\begin{array}{cc|cc} Q_{11j} & Q_{12j} & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ \hline 0 & 0 & Q_{33} & 0 \\ 0 & 0 & 0 & Q_{44} \end{array}\right], \tag{4.17}$$

$$X(\theta) = \left[\begin{array}{c|c} Q_1^T & 0 \\ \hline X_3 & X_4 \end{array}\right] = \left[\begin{array}{cc|cc} Q_{11}^T(\theta) & 0 & 0 & 0 \\ Q_{12}^T(\theta) & Q_{22}^T & 0 & 0 \\ \hline X_{31}(\theta) & X_{32}(\theta) & X_{33}(\theta) & X_{34}(\theta) \\ X_{41}(\theta) & X_{42}(\theta) & X_{43}(\theta) & X_{44}(\theta) \end{array}\right] = \sum_{j=1}^{r} \theta_j \left[\begin{array}{cc|cc} Q_{11j}^T & 0 & 0 & 0 \\ Q_{12j}^T & Q_{22}^T & 0 & 0 \\ \hline X_{31j} & X_{32j} & X_{33j} & X_{34j} \\ X_{41j} & X_{42j} & X_{43j} & X_{44j} \end{array}\right]. \tag{4.18}$$

Obviously, $\begin{bmatrix} P_{11j} & * \\ P_{21j} & P_{22j} \end{bmatrix} > 0, \ \ j = 1, \ 2, \ \ldots, \ r$ such that the equation $E^T X^{-1}(\theta)\ P(\theta) X^{-T}(\theta) E \geq 0$ in (4.9) holds.

From the partitions in (4.17) and (4.18), by Lemma 1.13, we have

$$Q^{-T}(\theta) = \left[\begin{array}{c|c} Q_1^T & 0 \\ \hline 0 & Q_4^T \end{array}\right]^{-1} = \left[\begin{array}{c|c} Q_1^{-T} & 0 \\ \hline 0 & Q_4^{-T} \end{array}\right], \tag{4.19}$$

and

$$X^{-1}(\theta) = \left[\begin{array}{c|c} Q_1^{-T} & 0 \\ \hline -X_4^{-1} X_3 Q_1^{-T} & X_4^{-1} \end{array}\right]. \tag{4.20}$$

By (4.19) and (4.20), we can see that $E^T X^{-1}(\theta) = Q^{-T}(\theta) E = Q_1^{-T}$, it implies that the partition in (4.17) and (4.18) satisfies the equation condition (4.10).

Define $N_{AL} = A_L Q_{22}$, $N_{BL} = B_L Q_{33}$, $N_{C2L} = C_{2L} Q_{22}$, $N_L = L Q_{44}$, and $N_K = K Q_{22}$, then with the support of (4.16–4.18), (4.15) it can be verified that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \theta_j \Delta_{ij} < 0, \tag{4.21}$$

where $\Delta_{ij}, \ i, \ j = 1, \ 2, \ \ldots, \ r$ are defined in Theorem 4.1. □

4.1.2 Condition B

Theorem 4.2 *Consider the system (4.1) and give a scalar $\gamma > 0$. Then the system via the observer (4.2) and the controller (4.3) is asymptotically stable with the H_∞ performance γ if there exist matrices Y_{AL}, Y_{BL}, Y_{C2L}, Y_L, Y_K, G_{22}, G_{33}, G_{44}, P_{11j}, P_{21j}, P_{22j}, P_{31j}, P_{32j}, P_{33j}, P_{41j}, P_{42j}, P_{43j}, P_{44j}, M_{11j}, M_{21j}, M_{31j}, M_{41j}, G_{11j}, G_{21j}, G_{31j}, and G_{41j}, $j = 1, \ 2, \ \ldots, \ r$ such that the following matrix inequalities hold:*

$$\begin{bmatrix} P_{11j} & * \\ P_{21j} & P_{22j} \end{bmatrix} > 0, \ j = 1, \ 2, \ \ldots, \ r, \tag{4.22}$$

$$\Xi_{ii} < 0, \ i = 1, \ 2, \ \ldots, \ r, \tag{4.23}$$

$$\Xi_{ij} + \Xi_{ji} < 0, \ i < j, \ i, \ j = 1, \ 2, \ \ldots, \ r, \tag{4.24}$$

with

$$\Xi_{ij} = \begin{bmatrix} \nu_{11} & * & * & * & * & * & * & * & * & * \\ \nu_{21} & \nu_{22} & * & * & * & * & * & * & * & * \\ \nu_{31} & \nu_{32} & \nu_{33} & * & * & * & * & * & * & * \\ \nu_{41} & \nu_{42} & M_{41j}B_i & \nu_{44} & * & * & * & * & * & * \\ E_i^T M_{11j}^T & E_i^T M_{21j}^T & E_i^T M_{31j}^T & \nu_{54} & -\gamma^2 I & * & * & * & * & * \\ \nu_{61} & -M_{21j}^T & \nu_{63i} & -M_{41j}^T & G_{11j}E_i & \nu_{66} & * & * & * & * \\ G_{21j}A_i & \nu_{72} & \nu_{73} & Y_L & G_{21j}E_i & \nu_{76} & \nu_{77} & * & * & * \\ G_{31j}A_i & Y_K & \nu_{83} & 0 & G_{31j}E_i & \nu_{86} & P_{32j} & \nu_{88} & * & * \\ \nu_{91} & -Y_{C2L} & G_{41j}B_i & \nu_{94} & \nu_{95} & \nu_{96} & P_{42j} & P_{43j} & \nu_{99} & * \\ C_{1i} & 0 & D_i & 0 & F_i & 0 & 0 & 0 & 0 & -I \end{bmatrix}, \tag{4.25}$$

where

$$\nu_{11} = -P_{11j} + M_{11j}A_i + A_i^T M_{11j}^T,$$

$$\nu_{21} = -P_{21j} + M_{21j}A_i,$$

$$\nu_{22} = -P_{22j} + Y_{AL} + Y_{AL}^T,$$

$$\nu_{31} = M_{31j}A_i + B_i^T M_{11j}^T,$$

$$\nu_{32} = Y_K + B_i^T M_{21j}^T + Y_{BL}^T,$$

$$\nu_{33} = B_i^T M_{31j}^T + M_{31j}B_i - G_{33} - G_{33}^T,$$

$$\nu_{41} = M_{41j}A_i + G_{44}C_{2i},$$

$$\nu_{42} = -Y_{C2L} + Y_L^T,$$

$$\nu_{44} = -G_{44} - G_{44}^T,$$

$$\nu_{54} = E_i^T M_{41j}^T + H_i^T G_{44}^T,$$

$$\nu_{61} = -M_{11j}^T + G_{11j}A_i,$$

$$\nu_{63} = -M_{31j}^T + G_{11j}B_i,$$

$$\nu_{66} = -G_{11j} - G_{11j}^T + P_{11j},$$

$$\nu_{72} = -G_{22}^T + Y_{AL},$$

$$\nu_{73} = G_{21j}B_i + Y_{BL},$$

$$\nu_{76} = -G_{21j} + P_{21j},$$

$$\nu_{77} = -G_{22} - G_{22}^T + P_{22j},$$

$$\nu_{83} = -G_{33}^T + G_{31j}B_i - G_{33},$$

$$\nu_{86} = -G_{31j} + P_{31j},$$

$$\nu_{88} = -G_{33} - G_{33}^T + P_{33j},$$

$$\nu_{91} = G_{41j}A_i + G_{44}C_{2i},$$

$$\nu_{94} = -G_{44}^T - G_{44},$$

$$\nu_{95} = G_{41j}E_i + G_{44}H_i,$$

$$\nu_{96} = -G_{41j} + P_{41j},$$

$$\nu_{99} = -G_{44} - G_{44}^T + P_{44j}.$$

Furthermore, the controller and observer gains are given by

$$K = G_{33}^{-1}Y_K,$$

$$A_L = G_{22}^{-1}Y_{AL}, \quad B_L = G_{22}^{-1}Y_{BL}, \quad C_{2L} = G_{44}^{-1}Y_{C2L}, \quad L = G_{22}^{-1}Y_L,$$

respectively.

Proof Choose a Lyapunov function candidate as

$$V(k) = \tilde{x}^T(k)E^T P(\theta)E\tilde{x}(k), \quad E^T P(\theta)E \geq 0.$$

Then, the H_∞ performance γ of the system (4.1) can be guaranteed by

$$[\ \bar{A} \quad \bar{B}\ ]^T P(\theta)[\ \bar{A} \quad \bar{B}\ ] + \begin{bmatrix} -E^T P(\theta)E & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + [\ \bar{C} \quad F(\theta)\ ]^T[\ \bar{C} \quad F(\theta)\ ] < 0. \tag{4.26}$$

As the condition A, by using Lemma 1.6, the above inequality is verified if

$$\begin{bmatrix} \Omega & * \\ -M^T(\theta) + G(\theta)[\ \bar{A} \quad \bar{B}\ ] & -G(\theta) - G^T(\theta) + P(\theta) \end{bmatrix} < 0, \tag{4.27}$$

where

$$\Omega = \begin{bmatrix} -E^T PE & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + [\ \bar{C} \quad F(\theta)\ ]^T[\ \bar{C} \quad F(\theta)\ ] + M(\theta)[\ \bar{A} \quad \bar{B}\ ] + [\ \bar{A} \quad \bar{B}\ ]^T M^T(\theta).$$

By using the Schur complement and defining $M(\theta) = \begin{bmatrix} M_1(\theta) \\ 0 \end{bmatrix}$, (4.27) gets

$$\begin{bmatrix} -E^T P(\theta)E + M_1(\theta)\bar{A} + \bar{A}^T M_1^T(\theta) & * & * & * \\ \bar{B}^T M_1^T(\theta) & -\gamma^2 I & * & * \\ -M_1^T(\theta) + G(\theta)\bar{A} & G(\theta)\bar{B} & -G(\theta) - G^T(\theta) + P(\theta) & * \\ \bar{C} & F(\theta) & 0 & -I \end{bmatrix} < 0. \tag{4.28}$$

In the case to obtain LMI-based control synthesis conditions, we define matrices $P(\theta)$ as (4.16) and

$$M_1(\theta) = \begin{bmatrix} M_{11}(\theta) & 0 & 0 & 0 \\ M_{21}(\theta) & G_{22} & 0 & 0 \\ M_{31}(\theta) & 0 & G_{33} & 0 \\ M_{41}(\theta) & 0 & 0 & G_{44} \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} M_{11j} & 0 & 0 & 0 \\ M_{21j} & G_{22} & 0 & 0 \\ M_{31j} & 0 & G_{33} & 0 \\ M_{41j} & 0 & 0 & G_{44} \end{bmatrix}. \tag{4.29}$$

$$G(\theta) = \begin{bmatrix} G_{11}(\theta) & 0 & 0 & 0 \\ G_{21}(\theta) & G_{22} & 0 & 0 \\ G_{31}(\theta) & 0 & G_{33} & 0 \\ G_{41}(\theta) & 0 & 0 & G_{44} \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_{11j} & 0 & 0 & 0 \\ G_{21j} & G_{22} & 0 & 0 \\ G_{31j} & 0 & G_{33} & 0 \\ G_{41j} & 0 & 0 & G_{44} \end{bmatrix}. \tag{4.30}$$

By combining (4.16) and (4.28)–(4.30) with $Y_{AL} = G_{22}A_L$, $Y_{BL} = G_{22}B_L$, $Y_{C2L} = G_{44}C_{2L}$, $Y_L = G_{22}L$, and $Y_K = G_{33}K$, we have $\sum_{i=1}^{r}\sum_{j=1}^{r} \theta_i\theta_j \Xi_{ij} < 0$, where $\Xi_{ij},\ i,\ j = 1,\ 2,\ \ldots,\ r$ are defined in Theorem 4.2. □

4.2 With Time-Varying Norm Bounded Uncertainties

4.2.1 The Two-Step Process with a Selection

In this section, using the two-step process approach, we study the H_∞ controller design problem for linear systems with time-varying norm bounded uncertainties.

Consider the following linear discrete-time dynamic model with time-varying norm bounded uncertainties:

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + (B + \Delta B)u(k) + (E + \Delta E)w(k), \\ z(k) &= (C_1 + \Delta C_1)x(k) + (D + \Delta D)u(k) + (F + \Delta F)w(k), \\ y(k) &= (C_2 + \Delta C_2)x(k) + (H + \Delta H)w(k), \end{aligned} \tag{4.31}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \ \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times m}$, $E \in \mathscr{R}^{n\times f}$, $C_1 \in \mathscr{R}^{q\times n}$, $D \in \mathscr{R}^{q\times m}$, $F \in \mathscr{R}^{q\times f}$, $C_2 \in \mathscr{R}^{p\times n}$, and $H \in \mathscr{R}^{p\times f}$ are system matrices. ΔA, ΔB, ΔE, ΔC_1, ΔD, ΔF, ΔC_2, and ΔH are uncertainties formulated as [9]

$$\begin{pmatrix} \Delta A & \Delta B & \Delta E \\ \Delta C_1 & \Delta D & \Delta F \\ \Delta C_2 & \Delta H & \end{pmatrix} = \begin{pmatrix} X_x \\ X_z \\ X_y \end{pmatrix} \Delta(k) \begin{pmatrix} Y_A & Y_B & Y_E \\ Y_{C1} & Y_D & Y_F \\ Y_{C2} & Y_H & \end{pmatrix}, \quad \Delta^T(k)\Delta(k) \le I. \tag{4.32}$$

The following observer is proposed to deal with the state estimation of system (4.31):

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L\big(y(k) - \hat{y}(k)\big), \\ \hat{y}(k) &= C_2\hat{x}(k), \end{aligned} \tag{4.33}$$

where $\hat{x}(k) \in \mathscr{R}^n$ and $\hat{y}(k) \in \mathscr{R}^p$ are the estimated state and estimated output, respectively, L is the observer gain.

Let us denote the estimation error as

$$e(k) = x(k) - \hat{x}(k). \tag{4.34}$$

By differentiating (4.34), we get

$$\begin{aligned} & e(k+1) \\ & = x(k+1) - \hat{x}(k+1) \\ & = (A - LC_2)e(k) + (\Delta A - L\Delta C_2)x(k) + \Delta Bu(k) \\ & \quad + \big(E + \Delta E - L(H + \Delta H)\big)w(k). \end{aligned} \tag{4.35}$$

Employ the following controller to deal with the design of system (4.31):

$$u(k) = K\hat{x}(k). \tag{4.36}$$

By defining $\bar{x}(k) = \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$ and substituting (4.36) into (4.31), the closed-loop system becomes

$$
\begin{aligned}
&\bar{x}(k+1) = \\
&\begin{bmatrix} A+\Delta A+(B+\Delta B)K & -(B+\Delta B)K \\ \Delta A+\Delta BK-L\Delta C_2 & A-\Delta BK-LC_2 \end{bmatrix}\bar{x}(k)+\begin{bmatrix} E+\Delta E \\ E+\Delta E-L(H+\Delta H) \end{bmatrix}w(k), \\
&z(k) = [\, C_1+\Delta C_1+(D+\Delta D)K \quad -(D+\Delta D)K \,]\bar{x}(k)+(F+\Delta F)w(k).
\end{aligned} \tag{4.37}
$$

In [7], a complete solution to an observer-controller structure concerning robust stabilization of uncertain T–S fuzzy systems satisfying the H_∞ performance requirement is investigated based on the two-step design approaches. Of course, the design method given in [7] is also applicable to linear systems with time-varying norm bounded uncertainties. First, based on the result in [7], we give the following H_∞ performance analysis lemma for the uncertain linear system (4.37):

Lemma 4.1 *Consider the closed-loop system* (4.37) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* P_1, P_2, K, *and* T *such that the following matrix inequality holds:*

$$
\begin{bmatrix}
-P_1 & * & * & * & * & * \\
A+\Delta A+(B+\Delta B)K & -P_1^{-1} & * & * & * & * \\
0 & (E+\Delta E)^T & -\gamma^2 I & * & * & * \\
0 & -K^T(B+\Delta B)^T & 0 & -P_2 & * & * \\
P_2\Delta A+P_2\Delta BK-T\Delta C_2 & 0 & \tilde{\Omega}_1 & \tilde{\Omega}_2 & -P_2 & * \\
C_1+\Delta C_1+(D+\Delta D)K & 0 & F+\Delta F & -(D+\Delta D)K & 0 & -I
\end{bmatrix}
< 0, \tag{4.38}
$$

where

$$
\tilde{\Omega}_1 = P_2E+P_2\Delta E-T(H+\Delta H),
$$

$$
\tilde{\Omega}_2 = P_2A-P_2\Delta BK-TC_2,
$$

$$
T = P_2L.
$$

Remark 4.1 In fact, constructing a Lyapunov function as $V(k)=\bar{x}^T(k)\begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix}\bar{x}(k)$, the H_∞ performance $\gamma > 0$ of the closed-loop system (4.37) can be ensured by

$$\begin{bmatrix} -\begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \begin{bmatrix} A+\Delta A+(B+\Delta B)K & -(B+\Delta B)K \\ \Delta A+\Delta BK-L\Delta C_2 & A-\Delta BK-LC_2 \end{bmatrix} & \begin{bmatrix} E+\Delta E \\ \Omega \end{bmatrix} & -\begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix}^{-1} & * \\ [\, C_1+\Delta C_1+(D+\Delta D)K \quad -(D+\Delta D)K \,] & F+\Delta F & 0 & -I \end{bmatrix}$$
$$< 0, \tag{4.39}$$

where $\Omega = E + \Delta E - L(H + \Delta H)$.

Then, by using the elementary transformation of matrix and the congruence of matrix inequality, (4.38) can be obtained.

Remark 4.2 For the H_∞ performance analysis condition (4.38), following the idea proposed by [7], the two-step process can be employed to design the controller and observer. In the first step, find the matrices Q_1 and Y to meet the following matrix inequality:

$$\begin{bmatrix} -Q_1 & * \\ AQ_1 + BY & -Q_1 \end{bmatrix} < 0. \tag{4.40}$$

In the second step, if the two matrices in the first step are found, denoting $P_1 = Q_1^{-1}$ and $K = YQ_1^{-1}$. Then, substituting P_1 and K into (4.38), the controller and observer can be obtained by solving the LMI (4.38) (See [7] for detailed).

This section mainly follows the problem definition of [7] and aims to improve the result in Lemma 4.1 by introducing more freedom into the design procedure. In this following, we will develop another analysis result.

Theorem 4.3 *Consider the closed-loop system* (4.37) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* P_1, P_2, G_1, G_2, K, *and* Y *such that the following matrix inequality holds:*

$$\begin{bmatrix} -G_1 - G_1^T + Q_1 & * & * & * & * & * \\ AG_1 + \Delta AG_1 + (B+\Delta B)KG_1 & -Q_1 & * & * & * & * \\ 0 & (E+\Delta E)^T & -\gamma^2 I & * & * & * \\ 0 & -K^T(B+\Delta B)^T & 0 & -P_2 & * & * \\ G_2\Delta AG_1 + G_2\Delta BKG_1 - Y\Delta C_2 G_1 & 0 & \Pi_1 & \Pi_2 & \Pi_3 & * \\ C_1G_1 + \Delta C_1 G_1 + (D+\Delta D)KG_1 & 0 & F+\Delta F & \Pi_4 & 0 & -I \end{bmatrix} < 0, \tag{4.41}$$

where

$$\Pi_1 = G_2E + G_2\Delta E - Y(H + \Delta H),$$

$$\Pi_2 = G_2A - G_2\Delta BK - YC_2,$$

$$\Pi_3 = -G_2 - G_2^T + P_2,$$

$$\Pi_4 = -(D + \Delta D)K.$$

Proof By pre- and post-multiplying (4.39) with $\begin{bmatrix} G_1^T & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$ and its transpose, respectively, and defining $Y = G_2 L$, the inequality (4.41) can obtained easily. □

Remark 4.3 In Theorem 4.3, by introducing slack matrix variables, a sufficient condition, which can guarantee observer-based H_∞ performance for uncertain linear systems, is proposed in terms of a matrix inequality. In comparison with Lemma 4.1, the proposed result in Theorem 4.3 provides more relaxation. When the matrix inequality (4.41) s chosen as a special case, that is, $G_1 = P_1^{-1}$ and $G_2 = P_2$, then it reduces to (4.38) in Lemma 4.1. Thus, we can easily know that (4.41) is less conservative than (4.38).

The H_∞ performance analysis condition has been given in Theorem 4.3. Next, we study the problem of the controller and observer design for the closed-loop system (4.37). Define

$$\Sigma = \begin{bmatrix} -G_1 - G_1^T + Q_1 & * & * & * & * & * \\ AG_1 + BN & -Q_1 & * & * & * & * \\ 0 & E^T & -\gamma^2 I & * & * & * \\ 0 & -K^T B^T & 0 & -P_2 & * & * \\ 0 & 0 & G_2 E - YH & G_2 A - YC_2 & -G_2 - G_2^T + P_2 & * \\ C_1 G_1 + DN & 0 & F & -DK & 0 & -I \end{bmatrix},$$

$$\Sigma_X = \begin{bmatrix} 0 & 0 & 0 \\ X_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ G_2 X_x & 0 & -YX_y \\ 0 & X_z & 0 \end{bmatrix},$$

and

$$\Sigma_Y = \begin{bmatrix} Y_A G_1 + Y_B N & 0 & Y_E & -Y_B K & 0 & 0 \\ Y_{C1} G_1 + Y_D N & 0 & Y_F & -Y_D K & 0 & 0 \\ Y_{C2} G_1 & 0 & Y_H & 0 & 0 & 0 \end{bmatrix},$$

with $N = KG_1$.

Then, the matrix inequality (4.41) can be rewritten as follows:

$$\Sigma + \Sigma_X \begin{bmatrix} \Delta(k) & 0 & 0 \\ 0 & \Delta(k) & 0 \\ 0 & 0 & \Delta(k) \end{bmatrix} \Sigma_Y + \Sigma_Y^T \begin{bmatrix} \Delta(k) & 0 & 0 \\ 0 & \Delta(k) & 0 \\ 0 & 0 & \Delta(k) \end{bmatrix}^T \Sigma_X^T < 0. \tag{4.42}$$

Note that $\Delta^T(k)\Delta(k) \le I$ implies that

$$\begin{bmatrix} \Delta(k) & 0 & 0 \\ 0 & \Delta(k) & 0 \\ 0 & 0 & \Delta(k) \end{bmatrix}^T \begin{bmatrix} \Delta(k) & 0 & 0 \\ 0 & \Delta(k) & 0 \\ 0 & 0 & \Delta(k) \end{bmatrix} \le I,$$

then, by Lemma 1.11 for a positive scalar ε, the matrix inequality (4.42) is satisfied if

$$\begin{bmatrix} \Sigma & * & * \\ \Sigma_X^T & -\varepsilon I & * \\ \varepsilon \Sigma_Y & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{4.43}$$

Theorem 4.4 *Consider the closed-loop system* (4.37) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* Q_1, P_2, G_1, G_2, K, N, *and* Y, *scalar* ε *such that the matrix inequality* (4.43) *holds. Furthermore, the observer and controller gain matrices can be given as* $L = G_2^{-1}Y$ *and* $K = NG_1^{-1}$.

It is noted that when $N(K)$, $Y(L)$ are unknown, the matrix inequality (4.43) in Theorem 4.4 is of BMIs (bilinear matrix inequalities). Thence, we will use the two-step process approach to solve the observer and controller design problem. However, different from the two-step procedure proposed in [7], we present a selection approach which considers an additional constraint to obtain $Q_1(P_1)$, G_1, and N by using LMIs first. And then by regulating a parameter μ, the greater region is obtained for feasibility of the second step.

In summation, the algorithm is summarized as follows:

The first step: Set μ to a big value, using the constraint as

$$Q_1 < \mu I, \ \mu > 0, \tag{4.44}$$

and we consider that (4.43) implies sub-matrix satisfying

$$\begin{bmatrix} -G_1 - G_1^{\mathrm{T}} + Q_1 & * \\ AG_1 + BN & -Q_1 \end{bmatrix} < 0. \tag{4.45}$$

Find the matrices Q_1, G_1, and Y to satisfy (4.44) and (4.45). If these matrices are found, go to Step 2. Otherwise, increase μ.

The second step: Substituting G_1, $P_1 = Q_1^{-1}$, and $K = NG_1^{-1}$ into (4.43), if the matrices G_2, P_2, N, and scalar ε are found, stop the algorithm. Otherwise, decrease μ and go to the first step till (4.44) is infeasible.

Remark 4.4 In [2], using numerical examples, it is shown that the proposed selection approach is much less conservative than the similar one in [7] for T-S fuzzy systems.

Remark 4.5 In the proposed selection approach, two-step design procedure has been improved to a certain extent. However, the two-step design method appears as a drawback. Obviously, design of the controller and optimization of the H_∞ performance are not synchronized.

4.2.2 A Simple LMI Result

In this section, we present a simple LMI result for observer-based output feedback H_∞ control design for uncertain linear systems. Let us consider the discrete-time linear dynamic model (4.31) with a special case, that is,

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + Bu(k) + Ew(k), \\ z(k) &= (C_1 + \Delta C_1)x(k) + Du(k) + Fw(k), \\ y(k) &= C_2 x(k). \end{aligned} \tag{4.46}$$

We also use the observer (4.33) to estimate the system state of (4.46), then we can give the estimation error equation as

$$e(k+1) = (A - LC_2)e(k) + \Delta A x(k) + Ew(k). \tag{4.47}$$

Combining (4.47) and the controller (4.36), the closed-loop system is given as

$$\begin{aligned} \bar{x}(k+1) &= \begin{bmatrix} A + \Delta A + BK & -BK \\ \Delta A & A - LC_2 \end{bmatrix} \bar{x}(k) + \begin{bmatrix} E \\ E \end{bmatrix} w(k), \\ z(k) &= [\, C_1 + \Delta C_1 + DK \quad -DK \,]\bar{x}(k) + Fw(k). \end{aligned} \tag{4.48}$$

By choosing the Lyapunov function as $V(k) = \bar{x}^T(k) \begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix} \bar{x}(k)$, the following matrix inequality describes the basic H_∞ performance analysis criterion for the closed-loop system (4.48).

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A+\Delta A+BK & -BK & E & -P_1^{-1} & * & * \\ \Delta A & A-LC_2 & E & 0 & -P_2^{-1} & * \\ C_1+\Delta C_1+DK & -DK & F & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.49}$$

Next, we seek methods to obtain strict LMI conditions for designing the observer and controller based on the inequality criterion (4.49). First, we extend the matrix decoupling ideas of [6] into the discrete-time case.

Lemma 4.2 *Consider the closed-loop system* (4.48) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* P_K, P_L, G_K, G_L, N_K, *and* N_L *such that the following matrix inequalities hold:*

$$\begin{bmatrix} -G_K - G_K^T + P_K & * & * & * \\ AG_K + \Delta A G_K + BN_K & -P_K & * & * \\ C_1 G_K + \Delta C_1 G_K + DN_K & 0 & -I & * \\ 0 & E^T & F^T & -\gamma^2 I \end{bmatrix} < 0, \tag{4.50}$$

$$\begin{bmatrix} -P_L & * \\ G_L A - N_L C_2 & -G_L - G_L^T + P_L \end{bmatrix} < 0. \tag{4.51}$$

Proof Define $Y = \begin{bmatrix} \tilde{G}_K^T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{G}_L & 0 \end{bmatrix}$, $N_K = KG_K$, and $N_L = G_L L$.

Multiplying the matrix of (4.49) from the left and right by Y and its transpose, respectively, it leads to

$$\begin{bmatrix} -\tilde{G}_K^T P_1 \tilde{G}_K & * & * & * & * & * \\ A\tilde{G}_K + \Delta A\tilde{G}_K + BK\tilde{G}_K & -P_1^{-1} & * & * & * & * \\ C_1\tilde{G}_K + \Delta C_1\tilde{G}_K + DK\tilde{G}_K & 0 & -I & * & * & * \\ 0 & E^T & F^T & -\gamma^2 I & * & * \\ 0 & (-BK)^T & (-DK)^T & 0 & -P_2 & * \\ \tilde{G}_L \Delta A \tilde{G}_K & 0 & 0 & \tilde{G}_L E & \tilde{G}_L A - \tilde{G}_L L C_2 & \Delta_1 \end{bmatrix} < 0, \tag{4.52}$$

where $\Delta_1 = -\tilde{G}_L P_2^{-1} \tilde{G}_L^T$.

Then, (4.52) can be guaranteed by

$$\begin{bmatrix} -\tilde{G}_K^T P_1 \tilde{G}_K & * & * & * & * & * \\ A\tilde{G}_K + \Delta A \tilde{G}_K + BK\tilde{G}_K & -P_1^{-1} & * & * & * & * \\ C_1\tilde{G}_K + \Delta C_1 \tilde{G}_K + DK\tilde{G}_K & 0 & -I & * & * & * \\ 0 & E^T & F^T & -\gamma^2 I & * & * \\ 0 & (-BK)^T & (-DK)^T & 0 & -P_2 & * \\ \tilde{G}_L \Delta A \tilde{G}_K & 0 & 0 & \tilde{G}_L E & \tilde{G}_L A - \tilde{G}_L L C_2 & \Delta_2 \end{bmatrix} < 0, \tag{4.53}$$

where $\Delta_2 = -\tilde{G}_L - \tilde{G}_L^T + P_2$.

On the other hand, since the conditions (4.50) and (4.51) are satisfied, then there exists a large enough λ such that

$$\begin{aligned} &\begin{bmatrix} -\frac{1}{\lambda^2} G_K^T P_K^{-1} G_K & * & * & * \\ A\frac{1}{\lambda}G_K + \Delta A \frac{1}{\lambda} G_K + BK\frac{1}{\lambda}G_K & -P_K & * & * \\ C_1\frac{1}{\lambda}G_K + \Delta C_1 \frac{1}{\lambda} G_K + DK\frac{1}{\lambda}G_K & 0 & -I & * \\ 0 & E^T & F^T & -\gamma^2 I \end{bmatrix} \\ &-\frac{1}{\lambda}\begin{bmatrix} 0 & (-BK)^T & (-DK)^T & 0 \\ \lambda G_L \Delta A \frac{1}{\lambda} G_K & 0 & 0 & G_L E \end{bmatrix}^T \\ &\times \begin{bmatrix} -P_L & * \\ G_L A - G_L L C_2 & -G_L - G_L^T + P_L \end{bmatrix}^{-1} \begin{bmatrix} 0 & (-BK)^T & (-DK)^T & 0 \\ \lambda G_L \Delta A \frac{1}{\lambda} G_K & 0 & 0 & G_L E \end{bmatrix} \\ &< 0. \end{aligned} \tag{4.54}$$

The satisfaction of (4.54) needs that the position of $G_L E$ is zero. The requirement can be achieved by regulating structure of the observer (4.33) as [6]. By using Schur complement to (4.54), we can obtain matrix inequality (4.53) with with $\tilde{G}_K = \frac{1}{\lambda} G_K$, $P_1 = P_K^{-1}$, $P_2 = \lambda P_L$, and $\tilde{G}_L = \lambda G_L$. □

Based on the H_∞ performance criterion given in Lemma 4.2, we will derive LMI-based conditions for designing the observer and controller for the closed-loop system (4.48). Divide the matrix (4.50) as follows:

$$\begin{aligned} &\begin{bmatrix} -G_K - G_K^T + P_K & * & * & * \\ AG_K + BKG_K & -P_K & * & * \\ C_1 G_K + DKG_K & 0 & -I & * \\ 0 & E^T & F^T & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ X_x & 0 \\ 0 & X_z \\ 0 & 0 \end{bmatrix} \Delta(k) \begin{bmatrix} Y_A G_K & 0 & 0 & 0 \\ Y_{C1} G_K & 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} Y_A G_K & 0 & 0 & 0 \\ Y_{C1} G_K & 0 & 0 & 0 \end{bmatrix}^T \Delta^T(k) \begin{bmatrix} 0 & 0 \\ X_x & 0 \\ 0 & X_z \\ 0 & 0 \end{bmatrix}^T < 0. \end{aligned} \tag{4.55}$$

Choosing $N_K = KG_K$ and $N_L = G_L L$, from (4.55) and Lemma 1.12 for a scalar $\delta > 0$, we can obtain the following control design result for the closed-loop system (4.48).

Theorem 4.5 *Consider the closed-loop system* (4.48) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* P_K, P_K, G_K, G_L, N_K, *and* N_L, *scalar* δ *such that the following matrix inequalities hold:*

$$\begin{bmatrix} -G_K - G_K^T + P_K & * & * & * & * & * \\ AG_K + BN_K & -P_K + \delta X_x X_x^T & * & * & * & * \\ C_1 G_K + DN_K & 0 & -I + \delta X_z X_z^T & * & * & * \\ 0 & E^T & F^T & -\gamma^2 I & * & * \\ Y_A G_K & 0 & 0 & 0 & -\delta I & * \\ Y_{C1} G_K & 0 & 0 & 0 & 0 & -\delta I \end{bmatrix} < 0, \tag{4.56}$$

$$\begin{bmatrix} -P_L & * \\ G_L A - N_L C_2 & -G_L - G_L^T + P_L \end{bmatrix} < 0. \tag{4.57}$$

Furthermore, the observer in (4.33) *and the controller in* (4.36) *gain matrices can be given as* $L = G_L^{-1} N_L$ *and* $K = N_K G_K^{-1}$.

Remark 4.6 It is worth noting that the descriptor systems approach used in Sect. 4.1 is also suitable for the case with time-varying norm bounded uncertainties. Especially, for the special model (4.46), we can even get LMI-based design results with less matrix dimensions.

Case A: By submitting (4.36) into (4.46) and combining $e_y(k)$ in (4.4), we have

$$\begin{aligned} x(k+1) &= (A + \Delta A)x(k) + BK\hat{x}(k) + Ew(k), \\ \hat{x}(k+1) &= A\hat{x}(k) + BK\hat{x}(k) + Le_y(k), \\ 0 \cdot e_y(k+1) &= C_2 x(k) - C_2\hat{x}(k) - e_y(k). \end{aligned} \tag{4.58}$$

Then, the corresponding descriptor system is expressed as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \\ e_y(k+1) \end{bmatrix} = \begin{bmatrix} A + \Delta A & BK & 0 \\ 0 & A + BK & L \\ C_2 & -C_2 & -I \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \\ e_y(k) \end{bmatrix} + \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} w(k). \tag{4.59}$$

Reminder: The design approach of Theorem 4.1 is applicable to the descriptor system (4.59).

Case B: By submitting $y(k) - \hat{y}(k) = C_2 x(k) - C_2\hat{x}(k)$ into (4.33) and combining $u(k)$ in (4.4), we have

$$x(k+1) = (A + \Delta A)x(k) + Bu(k) + Ew(k),$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + LC_2x(k) - LC_2\hat{x}(k), \tag{4.60}$$

$$0 \cdot u(k+1) = K\hat{x}(k) - u(k).$$

Then, the corresponding descriptor system is expressed as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \\ u(k+1) \end{bmatrix} = \begin{bmatrix} A + \Delta A & 0 & B \\ LC_2 & A - LC_2 & B \\ 0 & K & -I \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} w(k). \tag{4.61}$$

Reminder: The design approach of Theorem 4.2 is applicable to the descriptor system (4.61).

Remark 4.7 Different from the system (4.31), the system (4.46) meets that in system equation the measured output is free of disturbances. However, in most practical applications the measurements made in physical systems are not free of errors caused by the presence of disturbance. Thus, the design method given in Theorem 4.5 is parochial for the more general case.

4.2.3 LMI Decoupling Approach

For these discussions in Remarks 4.5 and 4.7, the results given in Sects. 4.2.1 and 4.2.2 are still parochial for observer-based output feedback H_∞ control design. In this section, we apply the LMI decoupling approach presented in Chapter 2 to observer-based output feedback H_∞ control design for linear systems with time-varying norm bounded uncertainties. The LMI decoupling approach reflects good superiority. For simplicity, it supposes that there is uncertainty only in the system matrix A. Then, the uncertain system (4.31) becomes as follows:

$$x(k+1) = (A + \Delta A)x(k) + Bu(k) + Ew(k),$$

$$z(k) = C_1x(k) + Du(k) + Fw(k), \tag{4.62}$$

$$y(k) = C_2x(k) + Hw(k),$$

Case A: $D = 0$

In this case, the estimation error equation (4.47) becomes

$$e(k+1) = (A - LC_2)e(k) + \Delta Ax(k) + (E - LH)w(k). \tag{4.63}$$

Considering the controller (4.36), we have the following closed-loop system:

$$\bar{x}(k+1) = \begin{bmatrix} A + \Delta A + BK & -BK \\ \Delta A & A - LC_2 \end{bmatrix} \bar{x}(k) + \begin{bmatrix} E \\ E - LH \end{bmatrix} w(k), \tag{4.64}$$

$$z(k) = [\, C_1 \quad 0 \,]\bar{x}(k) + Fw(k).$$

By choosing the Lyapunov function as $V(k) = \bar{x}^T(k) \begin{bmatrix} P_1 & * \\ 0 & P_2 \end{bmatrix} \bar{x}(k)$, the basic H_∞ performance analysis criterion can be given directly as

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A + \Delta A + BK & -BK & E & -P_1^{-1} & * & * \\ \Delta A & A - LC_2 & E - LH & 0 & -P_2^{-1} & * \\ C_1 & 0 & F & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.65}$$

Obviously, we can utilize the following matrix inequality to verify (4.65):

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A + G_1 \Delta A + G_1 BK & -G_1 BK & G_1 E & \Theta_1 & * & * \\ G_2 \Delta A & G_2 A - G_2 L C_2 & G_2 E - G_2 LH & 0 & \Theta_2 & * \\ C_1 & 0 & F & 0 & 0 & -I \end{bmatrix} < 0, \tag{4.66}$$

where

$$\Theta_1 = -G_1 - G_1 + P_1,$$

$$\Theta_2 = -G_2 - G_2^T + P_2.$$

In the matrix inequality (4.66), there is a nonlinear term $G_1 BK$, which makes it difficult to obtain LMI-based design conditions. To overcome this difficulty, we integrate the LMI decoupling approach presented in Chapter 2 to design the observer and controller. The matrix inequality (4.66) can be rewritten as

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1A + G_1\Delta A & 0 & G_1E & \Theta_1 & * & * \\ G_2\Delta A & G_2A - G_2LC_2 & G_2E - G_2LH & 0 & \Theta_2 & * \\ C_1 & 0 & F & 0 & 0 & -I \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1B \\ 0 \\ 0 \end{bmatrix} K[\, I \quad -I \quad 0 \quad 0 \quad 0 \quad 0\,] + [\, I \quad -I \quad 0 \quad 0 \quad 0 \quad 0\,]^T K^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1B \\ 0 \\ 0 \end{bmatrix}^T < 0. \tag{4.67}$$

By using the same derivation with Theorem 2.1, we can know that (4.67) is satisfied if

$$\begin{bmatrix} -P_1 & * & * & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * \\ G_1A + G_1\Delta A + MV & -MV & G_1E & \Theta_1 + J & * & * & * & * \\ G_2\Delta A & G_2A - G_2LC_2 & G_2E - G_2LH & 0 & \Theta_2 & * & * & * \\ C_1 & 0 & F & 0 & 0 & -I & * & * \\ NV & -NV & 0 & 0 & 0 & 0 & \Xi_1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Xi_2 & -\frac{J}{\beta^2} \end{bmatrix}$$

$$< 0, \tag{4.68}$$

where

$$\Xi_1 = -\beta NU - \beta U^T N^T,$$

$$\Xi_2 = G_1B - MU.$$

Finally, by using Lemma 1.11 from the matrix inequality (4.68) with $G_L = G_2L$, we can give the following design result for the observer (4.33) and the controller (4.36):

Theorem 4.6 *Consider the closed-loop system* (4.64) *and give a scalar* $\gamma > 0$*. Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* U, V, G_L, P_1, P_2, G_1, G_2, *and* J, *scalar* ε *such that the following matrix inequality holds:*

$$\begin{bmatrix} -P_1+\varepsilon Y_A^T Y_A & * & * & * & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\ G_1 A + MV & -MV & G_1 E & \Theta_1 + J & * & * & * & * & * \\ 0 & G_2 A - G_L C_2 & G_2 E - G_L H & 0 & \Theta_2 & * & * & * & * \\ C_1 & 0 & F & 0 & 0 & -I & * & * & * \\ NV & -NV & 0 & 0 & 0 & 0 & \Xi_1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Xi_2 & -\frac{J}{\beta^2} & * \\ 0 & 0 & 0 & X_x^T G_1^T & X_x^T G_2^T & 0 & 0 & 0 & -\varepsilon I \end{bmatrix}$$

$$< 0, \tag{4.69}$$

where Ξ_1 *and* Ξ_2 *are defined in* (4.68).

Furthermore, the observer (4.33) *and the controller* (4.36) *gain matrices can be given as* $L = G_2^{-1} G_L$ *and* $K = U^{-1} V$.

At the end of this section, in order to clarify the advantages of the proposed result, we recall the analysis condition given by [5] for observer-based output feedback H_∞ control for the closed-loop system (4.48).

Lemma 4.3 *Consider the closed-loop system* (4.64) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* U, N_K, N_L, P_1, *and* P_2 *such that the following matrix formulas hold:*

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ 0 & -P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ P_1 A + P_1 \Delta A + B N_K & -B N_K & P_1 E & -P_1 & * & * \\ P_2 \Delta A & P_2 A - N_L C_2 & P_2 E - N_L H & 0 & -P_2 & * \\ C_1 & 0 & F & 0 & 0 & -I \end{bmatrix} < 0, \tag{4.70}$$

$$P_1 B = BU. \tag{4.71}$$

For the matrix condition in (4.68) and the condition in Lemma 4.3, our comparison conclusion is that:

Theorem 4.7 *If the condition in Lemma 4.3 holds, the condition in* (4.68) *with* $M = B$ *and* $N = B^T B$ *also holds.*

Proof This proof follows strictly the proof of Theorem 2.6. □

Case B: $H = 0$

For this case $H = 0$, we can obtain the following closed-loop system:

$$\bar{x}(k+1) = \begin{bmatrix} A + \Delta A + BK & -BK \\ \Delta A & A - LC_2 \end{bmatrix} \bar{x}(k) + \begin{bmatrix} E \\ E \end{bmatrix} w(k), \tag{4.72}$$

$$z(k) = [\ C_1 + DK \quad -DK\ ]\bar{x}(k) + Fw(k).$$

Choosing the Lyapunov function as $V(k) = \bar{x}^T(k) \begin{bmatrix} P_1^{-1} & * \\ 0 & P_2^{-1} \end{bmatrix} \bar{x}(k)$, the H_∞ performance γ of the closed-loop system (4.72) can be guaranteed by

$$\begin{bmatrix} -P_1^{-1} & * & * & * & * & * \\ 0 & -P_2^{-1} & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A + \Delta A + BK & -BK & E & -P_1 & * & * \\ \Delta A & A - LC_2 & E & 0 & -P_2 & * \\ C_1 + DK & -DK & F & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.73}$$

From (4.73), we can know that the matrix structure in (4.72) only allows to add an auxiliary matrix variable G_1 in the next step analysis condition, i.e.,

$$\begin{bmatrix} -G_1 - G_1^T + P_1 & * & * & * & * & * \\ 0 & -G_1 - G_1^T + P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AG_1 + \Delta AG_1 + BN_K & -BN_K & E & -P_1 & * & * \\ \Delta AG_1 & AG_1 - LC_2G_1 & E & 0 & -P_2 & * \\ C_1G_1 + DN_K & -DN_K & F & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.74}$$

Note that there is a nonlinear term LC_2G_1, that lets us rewrite (4.74) as

$$\Omega + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -I \\ 0 \end{bmatrix} L\, [\, 0 \quad C_2G_1 \quad 0 \quad 0 \quad 0\,] + [\, 0 \quad C_2G_1 \quad 0 \quad 0 \quad 0\,]^T L^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -I \\ 0 \end{bmatrix}^T < 0, \tag{4.75}$$

where $\Omega = \begin{bmatrix} -G_1 - G_1^T + P_1 & * & * & * & * & * \\ 0 & -G_1 - G_1^T + P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AG_1 + \Delta AG_1 + BN_K & -BN_K & E & -P_1 & * & * \\ \Delta AG_1 & AG_1 & E & 0 & -P_2 & * \\ C_1G_1 + DN_K & -DN_K & F & 0 & 0 & -I \end{bmatrix}$.

Using the same derivation as Theorem 2.9, the following matrix condition can be introduced to ensure (4.75)

$$
:\begin{bmatrix}
-G_1-G_1^T+P_1 & * & * & * & * & * & * & * \\
0 & -G_1-G_1^T+P_2+J & * & * & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * & * & * \\
AG_1+\Delta AG_1+BN_K & -BN_K & E & -P_1 & * & * & * & * \\
\Delta AG_1 & AG_1-\tilde{L}M & E & 0 & -P_2 & * & * & * \\
C_1G_1+DN_K & -DN_K & F & 0 & 0 & -I & * & * \\
0 & 0 & 0 & 0 & -N^T\tilde{L}^T & 0 & \Sigma_1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2}
\end{bmatrix} < 0, \tag{4.76}
$$

where

$$
\Sigma_1 = -\beta UN - \beta N^T U^T,
$$

$$
\Sigma_2 = (C_2G_1 - UM)^T.
$$

Based on Lemma 1.12 with a scalar $\delta > 0$, form (4.76), we can derive the following design condition:

Theorem 4.8 *Consider the closed-loop system* (4.72) *and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N and scalar β, there exist matrices U, $\tilde{L}$, N_K, P_1, P_2, G_1, G_2, and J, scalar δ such that the following matrix inequality holds:*

$$
\begin{bmatrix}
-G_1-G_1^T+P_1 & * & * & * & * & * & * & * & * \\
0 & -G_1-G_1^T+P_2+J & * & * & * & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * & * & * & * \\
AG_1+BN_K & -BN_K & E & \Pi_1 & * & * & * & * & * \\
0 & AG_1-\tilde{L}M & E & \delta X_x X_x^T & \Pi_2 & * & * & * & * \\
C_1G_1+DN_K & -DN_K & F & 0 & 0 & -I & * & * & * \\
0 & 0 & 0 & 0 & -N^T\tilde{L}^T & 0 & \Sigma_1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} & * \\
Y_A G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta I
\end{bmatrix}
< 0, \tag{4.77}
$$

where Σ_1 and Σ_2 are defined in (4.76)*, and*

$$
\Pi_1 = -P_1 + \delta X_x X_x^T,
$$

$$
\Pi_2 = -P_2 + \delta X_x X_x^T.
$$

Furthermore, the observer (4.33) *and the controller* (4.36) *gain matrices can be given as $L = \tilde{L}U^{-1}$ and $K = N_K G_1^{-1}$.*

As previously mentioned, different from the case $D(\theta) = 0$, the situation of the system structure in (4.72) only allows to add an auxiliary matrix variable G_1 for the case $H(\theta) = 0$. It implies that the design condition given in Theorem 4.8 is conservative. To overcome this problem, we construct a new closed-loop system of the following form:

$$\tilde{x}(k+1) = \begin{bmatrix} A+BK & LC_2 \\ \Delta A & A+\Delta A-LC_2 \end{bmatrix}\tilde{x}(k) + \begin{bmatrix} 0 \\ E \end{bmatrix} w(k), \tag{4.78}$$

$$z(k) = [\ C_1+DK \quad C_1\ ]\tilde{x}(k) + Fw(k),$$

with $\tilde{x}(k) = \begin{bmatrix} \hat{x}(k) \\ e(k) \end{bmatrix}$.

For the closed-loop system (4.78), we choose the Lyapunov function as $V(k) = \tilde{x}^T(k)\begin{bmatrix} P_1^{-1} & * \\ 0 & P_2^{-1} \end{bmatrix}\tilde{x}(k)$, then the H_∞ performance analysis criterion is given as

$$\begin{bmatrix} -P_1^{-1} & * & * & * & * & * \\ 0 & -P_2^{-1} & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ A+BK & LC_2 & 0 & -P_1 & * & * \\ \Delta A & A+\Delta A-LC_2 & E & 0 & -P_2 & * \\ C_1+DK & C_1 & F & 0 & 0 & -I \end{bmatrix} < 0. \tag{4.79}$$

By introducing two auxiliary matrix variables G_1 and G_2, we can apply the following matrix inequality to verify (4.79)

$$\begin{bmatrix} -G_1-G_1^T+P_1 & * & * & * & * & * \\ 0 & -G_2-G_2^T+P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AG_1+BN_K & LC_2G_2 & 0 & -P_1 & * & * \\ \Delta AG_1 & AG_2+\Delta AG_2-LC_2G_2 & E & 0 & -P_2 & * \\ C_1G_1+DN_K & C_1G_2 & F & 0 & 0 & -I \end{bmatrix} < 0, \tag{4.80}$$

where $N_K = KG_1$.

In order to deal with the nonlinear term LC_2G_2, we rewrite (4.80) as follows:

$$\Xi + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ -I \\ 0 \end{bmatrix} L\,[\,0 \quad C_2G_2 \quad 0 \quad 0 \quad 0 \quad 0\,] + [\,0 \quad C_2G_2 \quad 0 \quad 0 \quad 0 \quad 0\,]^T L^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ -I \\ 0 \end{bmatrix}^T < 0, \tag{4.81}$$

with $\Xi = \begin{bmatrix} -G_1 - G_1^T + P_1 & * & * & * & * & * \\ 0 & -G_2 - G_2^T + P_2 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ AG_1 + BN_K & 0 & 0 & -P_1 & * & * \\ \Delta AG_1 & AG_2 + \Delta AG_2 & E & 0 & -P_2 & * \\ C_1G_1 + DN_K & C_1G_2 & F & 0 & 0 & -I \end{bmatrix}$.

Then, following the derivation of Theorem 2.9, an H_∞ performance analysis criterion for the closed-loop system (4.78) is given.

Theorem 4.9 *Consider the closed-loop system* (4.78) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* M, N, U, $\tilde{L}$, N_K, P_1, P_2, G_1, G_2, *and* J, *scalar* β *such that the following matrix inequality holds:*

$$\begin{bmatrix} -G_1 - G_1^T + P_1 & * & * & * & * & * & * & * \\ 0 & -G_2 - G_2^T + P_2 + J & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * \\ AG_1 + BN_K & \tilde{L}M & 0 & -P_1 & * & * & * & * \\ \Delta AG_1 & AG_2 + \Delta AG_2 - \tilde{L}M & E & 0 & -P_2 & * & * & * \\ C_1G_1 + DN_K & C_1G_2 & F & 0 & 0 & -I & * & * \\ 0 & 0 & 0 & N^T\tilde{L}^T & -N^T\tilde{L}^T & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} \end{bmatrix} < 0, \tag{4.82}$$

where

$$\Sigma_1 = -\beta UN - \beta N^T U^T,$$

$$\Sigma_2 = (C_2G_2 - UM)^T.$$

Remark 4.8 The above-mentioned LMI decoupling approach to observer-based output feedback H_∞ control design is also applicable to the general case, that is, in (4.62) all system matrices have uncertainties.

Remark 4.9 The LMI decoupling approach is also feasible for observer-based output feedback H_∞ control design of the system (4.62) with $D \neq 0$ and $H \neq 0$. A related discussion can be known by Sect. 6.2, in which a whole design strategy of the observer and controller is considered.

4.3 Conclusion

In this chapter, the problem of observer-based output feedback H_∞ control design for discrete-time uncertain linear systems is investigated. Several different design conditions that can guarantee the H_∞ performance of the closed-loop systems with polytopic uncertainties and norm bounded uncertainties are proposed. A descriptor representation approach is exploited to derive sufficient conditions to design the observer and controller for the polytopic uncertainties case. For the norm bounded uncertainties case, the LMI decoupling approach is also mentioned mainly for observer-based output feedback H_∞ control design. The corresponding design conditions are given in the form of LMIs.

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Chapter 5
Robust H_∞ Filtering

Abstract This chapter deals with H_∞ filtering of both discrete-time systems and continuous-time systems with polytopic uncertainties. The uncertain parameters are supposed to reside in a polytope. By using the parameter-dependent Lyapunov function approach and introducing some auxiliary matrix variables, sufficient conditions for the H_∞ filter design are presented in terms of solutions to a set of linear matrix inequalities (LMIs). In contrast to the existing results of H_∞ filter design, the main advantage of the proposed design methods is the reduced conservativeness. In addition, the LMI decoupling approach is also considered for designing H_∞ filters. An example is provided to demonstrate the effectiveness of the proposed methods.

Keywords Discrete-time systems · H_∞ filtering · Parameter-dependent Lyapunov function · Linear matrix inequalities (LMIs)

5.1 Discrete-Time System

Consider a discrete-time linear system with polytopic uncertainties described by state-space equations

$$\begin{aligned} x(k+1) &= A(\theta)x(k) + B(\theta)w(k), \\ y(k) &= C(\theta)x(k) + D(\theta)w(k), \\ z(k) &= L(\theta)x(k) + H(\theta)w(k), \end{aligned} \tag{5.1}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $w(k) \in \mathscr{R}^m$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \ \infty)$, $z(k) \in \mathscr{R}^q$ is the signal to be estimated, $y(k) \in \mathscr{R}^f$ is the measurement output. The matrices $A(\theta)$, $B(\theta)$, $C(\theta)$, $D(\theta)$, $L(\theta)$, and $H(\theta)$ are constant matrices of appropriate dimensions and belong to the following uncertainty polytope [6]:

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_5,

$$\begin{aligned}\Omega &= \{[\,A(\theta),\ B(\theta),\ C(\theta),\ D(\theta),\ L(\theta),\ H(\theta)\,] \\ &= \sum_{i=1}^{r}\theta_i[\,A_i,\ B_i,\ C_i,\ D_i,\ L_i,\ H_i],\quad \sum_{i=1}^{r}\theta_i = 1,\ \theta_i \geq 0\}.\end{aligned} \tag{5.2}$$

The robust H_∞ filtering problem is to estimate the signal $z(k)$ by using the following filter:

$$\begin{aligned}x_F(k+1) &= A_F x_F(k) + B_F y(k), \\ z_F(k) &= C_F x_F(k) + D_F y(k),\end{aligned} \tag{5.3}$$

where $x_F(k) \in \mathscr{R}^n$ and $z_F(k) \in \mathscr{R}^q$ are the state and output of the filter, respectively. A_F, B_F, C_F, and D_F are filter matrices with appropriate dimensions to be determined.

Defining the augmented state vector $\psi(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}$ and $e(k) = z(k) - z_F(k)$, the following filtering error system can be obtained:

$$\begin{aligned}\psi(k+1) &= \tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k), \\ e(k) &= \tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k),\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\tilde{A}(\theta) &= \begin{bmatrix} A(\theta) & 0 \\ B_F C(\theta) & A_F \end{bmatrix}, & \tilde{B}(\theta) &= \begin{bmatrix} B(\theta) \\ B_F D(\theta) \end{bmatrix}, \\ \tilde{C}(\theta) &= [\,L(\theta)\ \ -D_F C(\theta)\ \ -C_F\,], & \tilde{D}(h) &= H(\theta) - D_F D(\theta).\end{aligned}$$

5.1.1 H_∞ Performance Analysis

In this section, we first establish a useful H_∞ filtering analysis criterion. For convenience of comparison between the proposed results and the existing ones, an existing criterion for H_∞ filtering performance analysis is given as follows.

Lemma 5.1 [3]: *Consider the filtering error system* (5.4) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* $P(\theta) > 0$, G, *and* S *such that the following matrix inequality holds:*

$$\begin{bmatrix} G\tilde{A}(\theta) + \tilde{A}^T(\theta)G^T - P(\theta) & * & * & * \\ \tilde{B}^T(\theta)G^T & -\gamma^2 I & * & * \\ -G^T + S\tilde{A}(\theta) & S\tilde{B}(\theta) & -S - S^T + P(\theta) & * \\ \tilde{C}(\theta) & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0. \tag{5.5}$$

The following theorem gives out result for H_∞ filtering analysis.

Theorem 5.1 *Consider the filtering error system (5.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta) > 0,\ G(\theta),\ N(\theta)$, and $S(\theta)$ such that the following matrix inequality holds:*

$$\begin{bmatrix} G(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)G^T(\theta) - P(\theta) & * & * & * \\ \tilde{B}^T(\theta)G^T(\theta) + N(\theta)\tilde{A}(\theta) & \Xi_{22} & * & * \\ -G^T(\theta) + S(\theta)\tilde{A}(\theta) & -N^T(\theta) + S(\theta)\tilde{B}(\theta) & \Xi_{33} & * \\ \tilde{C}(\theta) & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0, \tag{5.6}$$

where

$$\begin{aligned} \Xi_{22} &= -\gamma^2 I + N(\theta)\tilde{B}(\theta) + \tilde{B}^T(\theta)N^T(\theta), \\ \Xi_{33} &= -S(\theta) - S^T(\theta) + P(\theta). \end{aligned}$$

Proof The proof can completed by considering three different approaches.

Approach 1:
Construct a parameter-dependent Lyapunov function as

$$V(k) = \psi^T(k)P(\theta)\psi(k),\ \ P(\theta) > 0. \tag{5.7}$$

From (5.7) and recalling (5.1), it can be verified that

$$\begin{aligned} &V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) \\ &= \psi^T(k+1)P(\theta)\psi(k+1) - \psi^T(k)P(\theta)\psi(k) \\ &\quad + \left(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\right)^T\left(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\right) - \gamma^2 w^T(k)w(k) \\ &= \zeta^T(k)\left(\begin{bmatrix} -P(\theta) & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & P(\theta) \end{bmatrix} + [\,\tilde{C}(\theta) \quad \tilde{D}(\theta) \quad 0\,]^T[\,\tilde{C}(\theta) \quad \tilde{D}(\theta) \quad 0\,]\right)\zeta(k), \end{aligned} \tag{5.8}$$

where $\zeta(k) = \begin{bmatrix} \psi(k) \\ w(k) \\ \psi(k+1) \end{bmatrix}$.

By (5.4), one has

$$[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]\zeta(k) = [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]\begin{bmatrix} \psi(k) \\ w(k) \\ \psi(k+1) \end{bmatrix} = 0. \tag{5.9}$$

Applying Lemma 1.14 with

$$\Theta = \begin{bmatrix} -P(\theta) & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & P(\theta) \end{bmatrix} + [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,],$$
$$N = [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,],$$
$$\nu = \zeta(k).$$

and choose $L = \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix}$ in (1.20), if the following inequality

$$\begin{aligned} &\begin{bmatrix} -P(\theta) & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & P(\theta) \end{bmatrix} + [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,] \\ &+ \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix} [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,] + [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]^T \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix}^T < 0, \end{aligned} \tag{5.10}$$

holds, we have $V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) < 0$. Applying the Schur complement to (5.10) leads to (5.6). The equation $V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) < 0$ implies that

$$V(\infty) - V(0) + \sum_{k=0}^{\infty} e^T(k)e(k) - \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) < 0.$$

With zero initial condition $\psi(0) = 0$ and $V(\infty) > 0$, we obtain $\sum\limits_{k=0}^{\infty} e^T(k)e(k) < \gamma^2 \sum\limits_{k=0}^{\infty} w^T(k)w(k)$ for any nonzero $w(k) \in l_2[0,\ \infty)$.

Approach 2:
From (5.4), one gives

$$\tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k) - \psi(k+1) = 0. \tag{5.11}$$

According to (5.11), for any appropriately dimensioned parameter-dependent matrices $G(\theta)$, $N(\theta)$, and $S(\theta)$, we have

$$\begin{aligned} &2\big(\psi^T(k)G(\theta) + w^T(k)N(\theta) + \psi^T(k+1)S(\theta)\big) \\ &\times\big(\tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k) - \psi(k+1)\big) = 0. \end{aligned} \tag{5.12}$$

Then, combining (5.7) and (5.12), it follows that

$$
\begin{aligned}
&V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) \\
&= \psi^T(k+1)P(\theta)\psi(k+1) - \psi^T(k)P(\theta)\psi(k) \\
&\quad + \big(\psi^T(k)G(\theta) + w^T(k)N(\theta) + \psi^T(k+1)S(\theta)\big) \\
&\quad \times \big(\tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k) - \psi(k+1)\big) \\
&\quad + \big(\psi^T(k)\tilde{A}^T(\theta) + w^T(k)\tilde{B}^T(\theta) - \psi^T(k+1)\big) \\
&\quad \times \big(G^T(\theta)\psi(k) + N^T(\theta)w(k) + S^T(\theta)\psi(k+1)\big) \\
&\quad + \big(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\big)^T\big(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\big) - \gamma^2 w^T(k)w(k) \\
&= \upsilon^T(k)\left(\begin{bmatrix} -P(\theta) & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & P(\theta) \end{bmatrix} + \big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,\big]^T [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]\right. \\
&\quad \left. + \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix} [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,] + [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]^T \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix}^T\right)\upsilon(k),
\end{aligned}
\tag{5.13}
$$

where $\upsilon(k) = \begin{bmatrix} \psi(k) \\ w(k) \\ \psi(k+1) \end{bmatrix}$.

Obviously, it is observed that the matrix inequality (5.6) can guarantee the negative-definiteness of (5.13).

Approach 3:
From (5.7) and recalling (5.4), we have

$$
\begin{aligned}
&V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) \\
&= \psi^T(k+1)P(\theta)\psi(k+1) - \psi^T(k)P(\theta)\psi(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) \\
&= \big(\tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k)\big)^T P(\theta)\big(\tilde{A}(\theta)\psi(k) + \tilde{B}(\theta)w(k)\big) - \psi^T(k)P(\theta)\psi(k) \\
&\quad + \big(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\big)^T\big(\tilde{C}(\theta)\psi(k) + \tilde{D}(\theta)w(k)\big) - \gamma^2 w^T(k)w(k) \\
&= \zeta^T(k)\Big([\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,]^T P(\theta)\big[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,] \\
&\quad + \big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\big]^T\big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\big] + \begin{bmatrix} -P(\theta) & 0 \\ 0 & -\gamma^2 I \end{bmatrix}\Big)\zeta(k),
\end{aligned}
\tag{5.14}
$$

where $\zeta(k) = \begin{bmatrix} \psi(k) \\ w(k) \end{bmatrix}$.

Thus, $V(k+1) - V(k) + e^T(k)e(k) - \gamma^2 w^T(k)w(k) < 0$ for any $\zeta(k) \neq 0$ if

$$\begin{aligned}&\left[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,\right]^T P(\theta)\left[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,\right]\\&+\left[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\right]^T\left[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\right]+\begin{bmatrix}-P(\theta) & 0\\ 0 & -\gamma^2 I\end{bmatrix}<0.\end{aligned}\tag{5.15}$$

For (5.15), by using Lemma 1.6 with

$$\begin{aligned}T&=\left[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\right]^T\left[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\right]+\begin{bmatrix}-P(\theta) & 0\\ 0 & -\gamma^2 I\end{bmatrix},\\M&=\begin{bmatrix}G(\theta)\\ N(\theta)\end{bmatrix},\\G&=S(\theta),\\A&=\left[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,\right],\\P&=P(\theta),\end{aligned}$$

and Schur complement, if the inequality (5.6) is satisfied, (5.15) holds. □

Remark 5.1 Compared with (5.5) in Lemma 5.1, (5.6) in Theorem 5.1 adds a slack matrix variable $N(\theta)$ and replaces single matrix variables to be determined by parameter-dependent matrix variables. In other words, when $N(\theta)=0$, $G(\theta)$ and $S(\theta)$ are parameter-independent matrices, the H_∞ performance analysis criterion (5.6) in Theorem 5.1 reduces (5.5) in Lemma 5.1. Thus it can be easily seen that (5.5) is a special case of (5.6) and the condition in Theorem 5.1 is less conservative than that in Lemma 5.1.

5.1.2 Robust H_∞ Filters Design

For the filtering error system (5.4), [3] has presented an LMI-based condition for designing the filter in (5.3).

Lemma 5.2 [3]: *Consider the filtering error system* (5.4) *with* $H(\theta)=0$ *and give a scalar* $\gamma>0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalars* b_1 *and* b_2, *exist matrices* P_{1i}, P_{2i}, *and* P_{3i}, $i=1,\ 2,\dots,r$, F_1, F_2, K_1, K_2, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, *and* $\bar{D}_F$ *such that the following matrix inequalities hold*

$$
\begin{bmatrix}
-P_{1i} + F_1 A_i + A_i^T F_1^T + b_1 \bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T & * & * \\
-P_{2i} + F_2 A_i + b_2 \bar{B}_F C_i + b_1 \bar{A}_F^T & -P_{3i} + b_2 \bar{A}_F + b_2 \bar{A}_F^T & * \\
B_i^T F_1^T + b_1 D_i^T \bar{B}_F^T & B_i^T F_2^T + b_2 D_i^T \bar{B}_F^T & -\gamma^2 I \\
-F_1^T + K_1 A_i + \bar{B}_F C_i & -F_2^T + \bar{A}_F & K_1 B_i + \bar{B}_F D_i \\
-b_1 K^T + K_2 A_i + \bar{B}_F C_i & -b_2 K^T + \bar{A}_F & K_2 B_i + \bar{B}_F D_i \\
L_i - \bar{D}_F C_i & -\bar{C}_F & -\bar{D}_F D_i
\end{bmatrix}
$$

$$
\left.\begin{matrix}
* & * & * \\
* & * & * \\
* & * & * \\
-K_1 - K_1^T + P_{1i} & * & * \\
-K_2 - K^T + P_{2i} & -K - K^T + P_{3i} & * \\
0 & 0 & -I
\end{matrix}\right] < 0, \quad i = 1, 2, \ldots, r, \tag{5.16}
$$

$$
\begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} > 0, i = 1, 2, \ldots, r. \tag{5.17}
$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$
A_F = K^{-1} \bar{A}_F, \; B_F = K^{-1} \bar{B}_F, \; C_F = \bar{C}_F, \; D_F = \bar{D}_F.
$$

With Theorem 5.1 in hand, we are now in a position to present our filter design results. We now look at the application of Lemma 1.15. First, the inequality (5.6) can be rewritten as

$$
\begin{bmatrix}
I & \tilde{A}^T(\theta) & 0 & 0 & 0 \\
0 & \tilde{B}^T(\theta) & I & 0 & 0 \\
0 & -I & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
-P(\theta) & * & * & * & * \\
G^T(\theta) & 0 & * & * & * \\
0 & N(\theta) & -\gamma^2 I & * & * \\
0 & S(\theta) & 0 & P(\theta) & * \\
\tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 & 0 \\
\tilde{A}(\theta) & \tilde{B}(\theta) & -I & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix} < 0. \tag{5.18}
$$

Obviously, the matrix inequality (5.18) corresponds to the first equation in (1.22) with

$$
\Psi = \begin{bmatrix}
-P(\theta) & * & * & * & * \\
G^T(\theta) & 0 & * & * & * \\
0 & N(\theta) & -\gamma^2 I & * & * \\
0 & S(\theta) & 0 & P(\theta) & * \\
\tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I
\end{bmatrix}, \tag{5.19}
$$

and

$$
P^{\perp} = \begin{bmatrix}
I & 0 & 0 & 0 \\
\tilde{A}(\theta) & \tilde{B}(\theta) & -I & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}. \tag{5.20}
$$

From (5.20), we can choose

$$P = [\,\tilde{A}(\theta)\ -I\ \ \tilde{B}(\theta)\ -I\ \ 0\,]. \tag{5.21}$$

On the other hand, for another matrix Q, we divide into three cases, under which three LMI-based conditions for designing the robust H_∞ filter in (5.3) are given, respectively.

Case A:
Due to $-I < 0$, we have the following inequality:

$$[\,0\ 0\ 0\ 0\ I\,]\begin{bmatrix} -P(\theta) & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ 0 & S(\theta) & 0 & P(\theta) & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} < 0, \tag{5.22}$$

the matrix inequality corresponds to the second equation in (1.22) with $Q^\perp = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$.

Then, we can choose the matrix Q as

$$Q = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}. \tag{5.23}$$

Combining (5.18) and (5.22), based on Lemma 1.15, (5.6) holds if the following inequality holds

$$\begin{bmatrix} -P(\theta) & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ 0 & S(\theta) & 0 & P(\theta) & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix} + P^T X^T(\theta) Q + Q^T X(\theta) P < 0. \tag{5.24}$$

Assume

$$X(\theta) = \begin{bmatrix} F(\theta) \\ M(\theta) \\ E(\theta) \\ W(\theta) \end{bmatrix}. \tag{5.25}$$

Then, the inequality (5.24) is equivalent to

$$\begin{bmatrix} F(\theta)\tilde{A}(\theta)+\tilde{A}^T(\theta)F^T(\theta)-P(\theta) & * & * & * & * \\ G^T(\theta)+M(\theta)\tilde{A}(\theta)-F^T(\theta) & -M(\theta)-M^T(\theta) & * & * & * \\ \tilde{B}^T(\theta)F^T(\theta)+E(\theta)\tilde{A}(\theta) & N(\theta)-E(\theta)+\tilde{B}^T(\theta)M^T & \Upsilon_{33} & * & * \\ -F^T(\theta)+W(\theta)\tilde{A}(\theta) & S(\theta)-W(\theta)-M^T(\theta) & \Upsilon_{43} & \Upsilon_{44} & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0, \tag{5.26}$$

where

$$\begin{aligned} \Upsilon_{33} &= -\gamma^2 I + E(\theta)\tilde{B}(\theta) + \tilde{B}^T(\theta)E^T(\theta), \\ \Upsilon_{43} &= -E^T(\theta) + W\tilde{B}(\theta), \\ \Upsilon_{44} &= -W(\theta) - W^T(\theta) + P(\theta). \end{aligned}$$

We partition these designed matrix variables as

$$P(\theta) = \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j \begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0,$$

$$F(\theta) = \begin{bmatrix} F_1(\theta) & b_1K \\ F_2(\theta) & b_2K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} F_{1j} & b_1K \\ F_{2j} & b_2K \end{bmatrix},$$

$$W(\theta) = \begin{bmatrix} W_1(\theta) & K \\ W_2(\theta) & K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} W_{1j} & K \\ W_{2j} & K \end{bmatrix},$$

$$E(\theta) = [\, E_1(\theta) \;\; b_3K \,] = \sum_{j=1}^{r}\theta_j [\, E_{1j} \;\; b_3K \,],$$

$$G(\theta) = \begin{bmatrix} G_1(\theta) & G_2(\theta) \\ G_3(\theta) & G_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} G_{1j} & G_{2j} \\ G_{3j} & G_{4j} \end{bmatrix},$$

$$S(\theta) = \begin{bmatrix} S_1(\theta) & S_2(\theta) \\ S_3(\theta) & S_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} S_{1j} & S_{2j} \\ S_{3j} & S_{4j} \end{bmatrix},$$

$$N(\theta) = [\, N_1(\theta) \;\; N_2(\theta) \,] = \sum_{j=1}^{r}\theta_j [\, N_{1j} \;\; N_{2j} \,],$$

$$M(\theta) = \begin{bmatrix} M_1(\theta) & b_4K \\ M_2(\theta) & b_5K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} M_{1j} & b_4K \\ M_{2j} & b_5K \end{bmatrix}.$$

Let $\bar{A}_F = KA_F$, $\bar{B}_F = KB_F$, $\bar{C}_F = C_F$, and $\bar{D}_F = D_F$, by substituting the above matrices $P(\theta)$, $F(\theta)$, $W(\theta)$, $E(\theta)$, $G(\theta)$, $S(\theta)$, $N(\theta)$, and $M(\theta)$ into (5.26) and by considering the uncertainty set (5.2), we can obtain the following design condition for the robust H_∞ filter in (5.3).

Theorem 5.2 *Consider the filtering error system (5.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalars b_1, b_2, b_3, b_4, and b_5, there exist matrices $P_{1ij}, P_{2ij}, P_{3ij}, G_{1j}, G_{2j}, G_{3j}, G_{4j}, S_{1j}, S_{2j}, S_{3j}, S_{4j}, N_{1j}, N_{2j}, F_{1j}, F_{2j}, M_{1j}, M_{2j}, W_{1j}, W_{2j}$, and E_{1j}, $i, j = 1, 2, \ldots, r$, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$ such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0, \quad i, j = 1, 2, \ldots, r, \tag{5.27}$$

$$\Pi_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.28}$$

$$\Pi_{ij} + \Pi_{ji} < 0, \quad i < j, \ i, j = 1, 2, \ldots, r, \tag{5.29}$$

with

$$\Pi_{ij} = \begin{bmatrix}
\Pi_{11} & * & * & * & * & * & * & * \\
F_{2j}A_i + b_2\bar{B}_F C_i + b_1\bar{A}_F^T - P_{2ij} & b_2\bar{A}_F + b_2\bar{A}_F^T - P_{3ij} & * & * & * & * & * & * \\
\Pi_{31} & \Pi_{32} & -M_{1j} - M_{1j}^T & * & * & * & * & * \\
\Pi_{41} & \Pi_{42} & -M_{2j} - b_4K^T & -b_5K - b_5K^T & * & * & * & * \\
\Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55} & * & * & * \\
-F_{1j}^T + W_{1j}A_i + \bar{B}_F C_i & -F_{2j}^T + \bar{A}_F & S_{1j} - W_{1j} - M_{1j}^T & S_{2j} - K - M_{2j}^T & \Pi_{65} & -W_{1j} - W_{1j}^T + P_{1ij} & * & * \\
-b_1K^T + W_{2j}A_i + \bar{B}_F C_i & -b_2K^T + \bar{A}_F & S_{3j} - W_{2j} - b_4K^T & S_{4j} - K - b_5K^T & \Pi_{75} & -W_{2j} - K^T + P_{2ij} & \Pi_{77} & * \\
L_i - \bar{D}_F C_i & -\bar{C}_F & 0 & 0 & H_i - \bar{D}_F D_i & 0 & 0 & -I
\end{bmatrix},$$

where

$$\begin{aligned}
\Pi_{11} &= F_{1j}A_i + A_i^T F_{1j}^T + b_1\bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T - P_{1ij}, \\
\Pi_{31} &= G_{1j}^T + M_{1j}A_i + b_4\bar{B}_F C_i - F_{1j}^T, \\
\Pi_{32} &= G_{3j}^T + b_4\bar{A}_F - F_{2j}^T, \\
\Pi_{41} &= G_{2j}^T + M_{2j}A_i + b_5\bar{B}_F C_i - b_1K^T, \\
\Pi_{42} &= G_{4j}^T + b_5\bar{A}_F - b_2K^T, \\
\Pi_{51} &= E_{1j}A_i + b_3\bar{B}_F C_i + B_i^T F_{1j}^T + b_1 D_i^T \bar{B}_F^T, \\
\Pi_{52} &= b_3\bar{A}_F + B_i^T F_{2j}^T + b_2 D_i^T \bar{B}_F^T,
\end{aligned}$$

$$\begin{aligned}
\Pi_{53} &= N_{1j} - E_{1j} + B_i^T M_{1j}^T + b_4 D_i^T \bar{B}_F^T,\\
\Pi_{54} &= N_{2j} - b_3 K + B_i^T M_{2j}^T + b_5 D_i^T \bar{B}_F^T,\\
\Pi_{55} &= -\gamma^2 I + E_{1j} B_i + B_i^T E_{1j}^T + b_3 \bar{B}_F D_i + b_3 D_i^T \bar{B}_F^T,\\
\Pi_{65} &= W_{1j} B_i + \bar{B}_F D_i - E_{1j}^T,\\
\Pi_{75} &= W_{2j} B_i + \bar{B}_F D_i - b_3 K^T,\\
\Pi_{77} &= -K - K^T + P_{3ij}.
\end{aligned}$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$A_F = K^{-1}\bar{A}_F,\ B_F = K^{-1}\bar{B}_F,\ C_F = \bar{C}_F,\ D_F = \bar{D}_F. \tag{5.30}$$

Case B:
Due to $-P(\theta) < 0$, we have the following equation:

$$[\,I\ 0\ 0\ 0\ 0\,]\begin{bmatrix} -P(\theta) & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ 0 & S(\theta) & 0 & P(\theta) & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} < 0, \tag{5.31}$$

the matrix inequality corresponds to the second equation in (1.22) with $Q^\perp = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Then, we can choose the matrix Q as

$$Q = \begin{bmatrix} 0\ I\ 0\ 0\ 0 \\ 0\ 0\ I\ 0\ 0 \\ 0\ 0\ 0\ I\ 0 \\ 0\ 0\ 0\ 0\ I \end{bmatrix}. \tag{5.32}$$

By considering the definition in (5.25), for Case B, (5.24) becomes

$$\begin{bmatrix} -P(\theta) & * & * & * & * \\ G^T(\theta) + F(\theta)A(\theta) & -F(\theta) - F^T(\theta) & * & * & * \\ M(\theta)A(\theta) & \Upsilon_{32} & \Upsilon_{33} & * & * \\ E(\theta)A(\theta) & \Upsilon_{42} & E(\theta)B(\theta) - M^T(\theta) & \Upsilon_{44} & * \\ \tilde{C}(\theta) + W(\theta)\tilde{A}(\theta) & -W(\theta) & \tilde{D}(\theta) + W(\theta)B(\theta) & -W(\theta) & -I \end{bmatrix} < 0, \tag{5.33}$$

where

$$\begin{aligned}
\Upsilon_{32} &= N(\theta) - M(\theta) + B^T(\theta)F^T(\theta),\\
\Upsilon_{33} &= -\gamma^2 I + M(\theta)B(\theta) + B^T(\theta)M^T(\theta),\\
\Upsilon_{42} &= S(\theta) - E(\theta) - F^T(\theta),\\
\Upsilon_{44} &= -E(\theta) - E^T(\theta) + P(\theta).
\end{aligned}$$

We partition these designed matrix variables as

$$P(\theta) = \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j \begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0,$$

$$F(\theta) = \begin{bmatrix} F_1(\theta) & b_1 K \\ F_2(\theta) & b_2 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} F_{1j} & b_1 K \\ F_{2j} & b_2 K \end{bmatrix},$$

$$W(\theta) = [\, W_1(\theta) \;\; 0 \,] = \sum_{j=1}^{r}\theta_j [\, W_{1j} \;\; 0 \,],$$

$$E(\theta) = \begin{bmatrix} E_1(\theta) & b_4 K \\ E_2(\theta) & b_5 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} E_{1j} & b_4 K \\ E_{2j} & b_5 K \end{bmatrix},$$

$$G(\theta) = \begin{bmatrix} G_1(\theta) & G_2(\theta) \\ G_3(\theta) & G_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} G_{1j} & G_{2j} \\ G_{3j} & G_{4j} \end{bmatrix},$$

$$S(\theta) = \begin{bmatrix} S_1(\theta) & S_2(\theta) \\ S_3(\theta) & S_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} S_{1j} & S_{2j} \\ S_{3j} & S_{4j} \end{bmatrix},$$

$$N(\theta) = [\, N_1(\theta) \;\; N_2(\theta) \,] = \sum_{j=1}^{r}\theta_j [\, N_{1j} \;\; N_{2j} \,],$$

$$M(\theta) = [\, M_1(\theta) \;\; b_3 K \,] = \sum_{j=1}^{r}\theta_j [\, M_{1j} \;\; b_3 K \,].$$

Let $\bar{A}_F = KA_F$, $\bar{B}_F = KB_F$, $\bar{C}_F = C_F$, and $\bar{D}_F = D_F$, by substituting the above matrices $P(\theta)$, $F(\theta)$, $W(\theta)$, $E(\theta)$, $G(\theta)$, $S(\theta)$, $N(\theta)$, and $M(\theta)$ into (5.33) and by considering the uncertainty set (5.2), we can obtain the following design conditions for the robust H_∞ filter in (5.3).

Theorem 5.3 *Consider the filtering error system (5.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalars b_1, b_2, b_3, b_4, and b_5, there exist matrices P_{1ij}, P_{2ij}, P_{3ij}, G_{1j}, G_{2j}, G_{3j}, G_{4j}, S_{1j}, S_{2j}, S_{3j}, S_{4j}, N_{1j}, N_{2j}, F_{1j}, F_{2j}, M_{1j}, W_{1j}, E_{1j}, and E_{2j}, $i, j = 1, 2, \ldots, r$, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$ such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0, \quad i, j = 1, 2, \ldots, r, \tag{5.34}$$

$$\Xi_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.35}$$

$$\Xi_{ij} + \Xi_{ji} < 0, \quad i < j, \; i, j = 1, 2, \ldots, r, \tag{5.36}$$

with

$$\Xi_{ij} = \begin{bmatrix} -P_{1ij} & * & * & * & * & * & * & * \\ -P_{2ij} & -P_{3ij} & * & * & * & * & * & * \\ G_{1j}^T + F_{1j}A_i + b_1\bar{B}_F C_i & G_{3j}^T + b_1\bar{A}_F & -F_{1j} - F_{1j}^T & * & * & * & * & * \\ G_{2j}^T + F_{2j}A_i + b_2\bar{B}_F C_i & G_{4j}^T + b_2\bar{A}_F & -F_{2j} - b_1 K^T & -b_2K - b_2K^T & * & * & * & * \\ M_{1j}A_i + b_3\bar{B}_F C_i & b_3\bar{A}_F & \Xi_{53} & \Xi_{54} & \Xi_{55} & * & * & * \\ E_{1j}A_i + b_4\bar{B}_F C_i & b_4\bar{A}_F & S_{1j} - E_{1j} - F_{1j}^T & S_{2j} - b_4K - F_{2j}^T & \Xi_{65} & -E_{1j} - E_{1j}^T + P_{1ij} & * & * \\ E_{2j}A_i + b_5\bar{B}_F C_i & b_5\bar{A}_F & S_{3j} - E_{2j} - b_1K^T & S_{4j} - b_5K - b_2K^T & \Xi_{75} & -E_{2j} - b_4K^T + P_{2ij} & \Xi_{77} & * \\ L_i - \bar{D}_F C_i + W_{1j}A_i & -\bar{C}_F & -W_{1j} & 0 & \Xi_{85} & -W_{1j} & 0 & -I \end{bmatrix},$$

where

$$\begin{aligned} \Xi_{53} &= N_{1j} - M_{1j} + B_i^T F_{1j}^T + b_1 D_i^T \bar{B}_F^T, \\ \Xi_{54} &= N_{2j} - b_3K + B_i^T F_{2j}^T + b_2 D_i^T \bar{B}_F^T, \\ \Xi_{55} &= -\gamma^2 I + M_{1j}B_i + B_i^T M_{1j}^T + b_3\bar{B}_F D_i + b_3 D_i^T \bar{B}_F^T, \\ \Xi_{65} &= E_{1j}B_i + b_4\bar{B}_F D_i - M_{1j}^T \\ \Xi_{75} &= E_{2j}B_i + b_5\bar{B}_F D_i - b_3K^T, \\ \Xi_{77} &= -b_5K - b_5K^T + P_{3ij}, \\ \Xi_{85} &= H_i - \bar{D}_F D_i + W_{1j}B_i. \end{aligned}$$

Moreover, the gain matrices of the H_∞ filter (5.3) *are given by*

$$A_F = K^{-1}\bar{A}_F, B_F = K^{-1}\bar{B}_F, C_F = \bar{C}_F, D_F = \bar{D}_F. \tag{5.37}$$

Case C:
Due to $-\gamma^2 I < 0$, we have the following equation:

$$[0\ 0\ I\ 0\ 0] \begin{bmatrix} -P(\theta) & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ 0 & S(\theta) & 0 & P(\theta) & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix} < 0, \tag{5.38}$$

the matrix inequality corresponds to the second equation in (1.22) with $Q^{\perp} = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}$.

Then, we can choose the matrix Q as

$$Q = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{5.39}$$

By considering the definition in (5.25), for Case C, (5.24) becomes

$$\begin{bmatrix} \Upsilon_{11} & * & * & * & * \\ \Upsilon_{21} & -M(\theta) - M^T(\theta) & * & * & * \\ \tilde{B}^T(\theta)F^T(\theta) & N(\theta) + \tilde{B}^T(\theta)M^T(\theta) & -\gamma^2 I & * & * \\ E(\theta)A(\theta) - F^T(\theta) & S(\theta) - E(\theta) - M^T(\theta) & E(\theta)B(\theta) & \Upsilon_{44} & * \\ \tilde{C}(\theta) + W(\theta)\tilde{A}(\theta) & -W(\theta) & \Upsilon_{53} & -W(\theta) & -I \end{bmatrix} < 0, \tag{5.40}$$

where

$$\begin{aligned} \Upsilon_{11} &= F(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)F^T(\theta) - P(\theta), \\ \Upsilon_{21} &= G^T(\theta) + M(\theta)\tilde{A}(\theta) - F^T(\theta), \\ \Upsilon_{44} &= -E(\theta) - E^T(\theta) + P(\theta), \\ \Upsilon_{53} &= \tilde{D}(\theta) + W(\theta)B(\theta). \end{aligned}$$

Now, assume that these designed matrix variables are of the following form

$$P(\theta) = \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} \theta_i\theta_j \begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0,$$

$$F(\theta) = \begin{bmatrix} F_1(\theta) & b_1 K \\ F_2(\theta) & b_2 K \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} F_{1j} & b_1 K \\ F_{2j} & b_2 K \end{bmatrix},$$

$$W(\theta) = [\, W_1(\theta) \; 0\,] = \sum_{j=1}^{r} \theta_j [\, W_{1j} \; 0\,],$$

$$E(\theta) = \begin{bmatrix} E_1(\theta) & b_3 K \\ E_2(\theta) & b_4 K \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} E_{1j} & b_3 K \\ E_{2j} & b_4 K \end{bmatrix},$$

$$G(\theta) = \begin{bmatrix} G_1(\theta) & G_2(\theta) \\ G_3(\theta) & G_4(\theta) \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} G_{1j} & G_{2j} \\ G_{3j} & G_{4j} \end{bmatrix},$$

$$S(\theta) = \begin{bmatrix} S_1(\theta) & S_2(\theta) \\ S_3(\theta) & S_4(\theta) \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} S_{1j} & S_{2j} \\ S_{3j} & S_{4j} \end{bmatrix},$$

$$N(\theta) = [\, N_1(\theta) \;\; N_2(\theta) \,] = \sum_{j=1}^{r} \theta_j [\, N_{1j} \;\; N_{2j} \,],$$

$$M(\theta) = \begin{bmatrix} M_1(\theta) & b_5 K \\ M_2(\theta) & b_6 K \end{bmatrix} = \sum_{j=1}^{r} \theta_j \begin{bmatrix} M_{1j} & b_5 K \\ M_{2j} & b_6 K \end{bmatrix}.$$

Let $\bar{A}_F = KA_F$, $\bar{B}_F = KB_F$, $\bar{C}_F = C_F$, and $\bar{D}_F = D_F$, by substituting the above matrices $P(\theta)$, $F(\theta)$, $W(\theta)$, $E(\theta)$, $G(\theta)$, $S(\theta)$, $N(\theta)$, and $M(\theta)$ into (5.40) and by considering the uncertainty set (5.2), we can obtain the following design conditions for the robust H_∞ filter in (5.3).

Theorem 5.4 *Consider the filtering error system* (5.4) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalars* b_1, b_2, b_3, b_4, b_5, *and* b_6, *there exist matrices* P_{1ij}, P_{2ij}, P_{3ij}, G_{1j}, G_{2j}, G_{3j}, G_{4j}, S_{1j}, S_{2j}, S_{3j}, S_{4j}, N_{1j}, N_{2j}, F_{1j}, F_{2j}, M_{1j}, M_{2j}, W_{1j}, E_{1j}, *and* E_{2j}, $i, j = 1, 2, \ldots, r$, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, *and* $\bar{D}_F$ *such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0, \quad i, j = 1, 2, \ldots, r, \tag{5.41}$$

$$\Gamma_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.42}$$

$$\Gamma_{ij} + \Gamma_{ji} < 0, \quad i < j, \; i, j = 1, 2, \ldots, r, \tag{5.43}$$

with

$$\Gamma_{ij} = \begin{bmatrix}
\Gamma_{11} & * & * & * & * & * & * & * \\
\Gamma_{21} & -P_{3ij} + b_2\bar{A}_F + b_2\bar{A}_F^T & * & * & * & * & * & * \\
\Gamma_{31} & G_{3j}^T + b_5\bar{A}_F - F_{2j}^T & -M_{1j} - M_{1j}^T & * & * & * & * & * \\
\Gamma_{41} & G_{4j}^T + b_6\bar{A}_F - b_2K^T & -M_{2j} - b_5K^T & -b_6K - b_6K^T & * & * & * & * \\
B_i^T F_{1j}^T + b_1 D_i^T \bar{B}_F^T & B_i^T F_{2j}^T + b_2 D_i^T \bar{B}_F^T & N_{1j} + B_i^T M_{1j}^T + b_5 D_i^T \bar{B}_F^T & \Gamma_{54} & -\gamma^2 I & * & * & * \\
-F_{1j}^T + E_{1j}A_i + b_3\bar{B}_F C_i & -F_{2j}^T + b_3\bar{A}_F & S_{1j} - E_{1j} - M_{1j}^T & S_{2j} - b_3K - M_{2j}^T & \Gamma_{65} & -E_{1j} - E_{1j}^T + P_{1ij} & * & * \\
-b_1K^T + E_{2j}A_i + b_4\bar{B}_F C_i & -b_2K^T + b_4\bar{A}_F & S_{3j} - E_{2j} - b_5K^T & S_{4j} - b_4K - b_6K^T & \Gamma_{75} & -E_{2j} - b_3K^T + P_{2ij} & \Gamma_{77} & * \\
L_i - \bar{D}_F C_i + W_{1j}A_i & -\bar{C}_F & -W_{1j} & 0 & \Gamma_{85} & -W_{1j} & 0 & -I
\end{bmatrix},$$

where

$$
\begin{aligned}
\Gamma_{11} &= -P_{1ij} + F_{1j}A_i + b_1\bar{B}_F C_i + A_i^T F_{1j}^T + b_1 C_i^T \bar{B}_F^T,\\
\Gamma_{21} &= -P_{2ij} + F_{2j}A_i + b_2\bar{B}_F C_i + b_1\bar{A}_F^T,\\
\Gamma_{31} &= G_{1j}^T + M_{1j}A_i + b_5\bar{B}_F C_i - F_{1j}^T,\\
\Gamma_{41} &= G_{2j}^T + M_{2j}A_i + b_6\bar{B}_F C_i - b_1 K^T,\\
\Gamma_{54} &= N_{2j} + B_i^T M_{2j}^T + b_6 D_i^T \bar{B}_F^T,\\
\Gamma_{65} &= E_{1j}B_i + b_3\bar{B}_F D_i,\\
\Gamma_{75} &= E_{2j}B_i + b_4\bar{B}_F D_i,\\
\Gamma_{77} &= -b_4 K - b_4 K^T + P_{3ij},\\
\Gamma_{85} &= H_i - \bar{D}_F D_i + W_{1j}B_i.
\end{aligned}
$$

Moreover, the gain matrices of the H_∞ filter (5.3) *are given by*

$$
A_F = K^{-1}\bar{A}_F,\ B_F = K^{-1}\bar{B}_F,\ C_F = \bar{C}_F,\ D_F = \bar{D}_F. \tag{5.44}
$$

Remark 5.2 It should be noted that in order to obtain LMI-based design conditions for Case B and Case C, one is necessary to assume that $W(\theta) = [\,W_1(\theta)\ 0\,] = \sum_{j=1}^{r}\theta_j[\,W_{1j}\ \ 0\,]$ or

$$
W(\theta) = [\,W_1(\theta)\ b_7 K\,] = \sum_{j=1}^{r}\theta_j[\,W_{1j}\ b_7 K\,], \tag{5.45}
$$

where b_7 is a scalar. However, the use of (5.45) might lead to incompatible matrices dimensions. For example, we rewrite the term (6, 2) in Σ_{ij} as $-\bar{C}_F + b_7\bar{A}_F$. Since $\bar{C}_F \in R^{q\times n}$ and $\bar{A}_F \in R^{n\times n}$, they cannot be added due to incompatible dimensions. Thus, we choose $b_7 = 0$ in this study.

Theorems 5.2–5.4 present LMI-based conditions for H_∞ filter design, which introduces more auxiliary matrix variables than Lemma 5.2. Compared with Lemma 5.2, the design condition in Theorem 5.2 is less conservative. The following theorem is introduced to describe the relationship between Lemma 5.2 and Theorem 5.2.

Theorem 5.5 *If the condition in Lemma 5.2 is satisfied, the condition in Theorem 5.2 is also feasible.*

Proof Suppose P_{1i}, P_{2i}, and P_{3i}, $i = 1, 2, \ldots, r$, F_1, F_2, K_1, K_2, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$, scalars b_1 and b_2 are solutions of Lemma 5.2. Since the LMI conditions (5.16) and (5.17) of Lemma 5.2 are satisfied, which imply that the matrix K is nonsingular and $K + K^T > 0$. Then there always exists a small enough b_5 ($b_5 > 0$) such that

$$
\begin{bmatrix}
-P_{1i}+F_1A_i+A_i^TF_1^T+b_1\bar{B}_FC_i+b_1C_i^T\bar{B}_F^T & * & * & * & * & * \\
-P_{2i}+F_2A_i+b_2\bar{B}_FC_i+b_1\bar{A}_F^T & -P_{3i}+b_2\bar{A}_F+b_2\bar{A}_F^T & * & * & * & * \\
B_i^TF_1^T+b_1D_i^T\bar{B}_F^T & B_i^TF_2^T+b_2D_i^T\bar{B}_F^T & -\gamma^2 I & * & * & * \\
-F_1^T+K_1A_i+\bar{B}_FC_i & -F_2^T+\bar{A}_F & K_1B_i+\bar{B}_FD_i & -K_1-K_1^T+P_{1i} & * & * \\
-b_1K^T+K_2A_i+\bar{B}_FC_i & -b_2K^T+\bar{A}_F & K_2B_i+\bar{B}_FD_i & -K_2-K^T+P_{2i} & -K-K^T+P_{3i} & * \\
L_i-\bar{D}_FC_i & -\bar{C}_F & -\bar{D}_FD_i & 0 & 0 & -I
\end{bmatrix}
+b_5\begin{bmatrix} A_i^T & C_i^TB_F^T \\ 0 & A_F^T \\ B_i^T & D_i^TB_F^T \\ -I & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix}
$$

$$
\times\begin{bmatrix} K^T(K+K^T)^{-1}K & 0 \\ 0 & K^T(K+K^T)^{-1}K \end{bmatrix}\begin{bmatrix} A_i^T & C_i^TB_F^T \\ 0 & A_F^T \\ B_i^T & D_i^TB_F^T \\ -I & 0 \\ 0 & -I \\ 0 & 0 \end{bmatrix}^T < 0, \quad i=1,2,\ldots,r, \tag{5.46}
$$

hold.

By the Schur complement, (5.46) is equivalent to

$$
\begin{bmatrix}
-P_{1i}+F_1A_i+A_i^TF_1^T+b_1\bar{B}_FC_i+b_1C_i^T\bar{B}_F^T & * & * & * & * & * & * & * \\
-P_{2i}+F_2A_i+b_2\bar{B}_FC_i+b_1\bar{A}_F^T & -P_{3i}+b_2\bar{A}_F+b_2\bar{A}_F^T & * & * & * & * & * & * \\
B_i^TF_1^T+b_1D_i^T\bar{B}_F^T & B_i^TF_2^T+b_2D_i^T\bar{B}_F^T & -\gamma^2 I & * & * & * & * & * \\
-F_1^T+K_1A_i+\bar{B}_FC_i & -F_2^T+\bar{A}_F & K_1B_i+\bar{B}_FD_i & -K_1-K_1^T+P_{1i} & * & * & * & * \\
-b_1K^T+K_2A_i+\bar{B}_FC_i & -b_2K^T+\bar{A}_F & K_2B_i+\bar{B}_FD_i & -K_2-K^T+P_{2i} & -K-K^T+P_{3i} & * & * & * \\
L_i-\bar{D}_FC_i & -\bar{C}_F & -\bar{D}_FD_i & 0 & 0 & -I & * & * \\
A_i & 0 & B_i & -I & 0 & 0 & \Theta & * \\
B_FC_i & A_F & B_FD_i & 0 & -I & 0 & 0 & \Theta
\end{bmatrix} < 0, \quad i=1,2,\ldots,r, \tag{5.47}
$$

where $\Theta=-\frac{1}{b_5}K^{-1}(K+K^T)K^{-T}$.

Pre and postmultiplying (5.47) by T and its transpose, respectively, one has

$$
\begin{bmatrix}
-P_{1i} + F_1 A_i + A_i^T F_1^T + b_1 \bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T & * & * & * & * & * & * & * \\
-P_{2i} + F_2 A_i + b_2 \bar{B}_F C_i + b_1 \bar{A}_F^T & -P_{3i} + b_2 \bar{A}_F + b_2 \bar{A}_F^T & * & * & * & * & * & * \\
b_5 K A_i & 0 & -b_5(K + K^T) & * & * & * & * & * \\
b_5 K B_F C_i & b_5 K A_F & 0 & -b_5(K + K^T) & * & * & * & * \\
B_i^T F_1^T + b_1 D_i^T \bar{B}_F^T & B_i^T F_2^T + b_2 D_i^T \bar{B}_F^T & b_5 B_i^T K^T & b_5 D_i^T B_F^T K^T & -\gamma^2 I & * & * & * \\
-F_1^T + K_1 A_i + \bar{B}_F C_i & -F_2^T + \bar{A}_F & -b_5 K^T & 0 & K_1 B_i + \bar{B}_F D_i & -K_1 - K_1^T + P_{1i} & * & * \\
-b_1 K^T + K_2 A_i + \bar{B}_F C_i & -b_2 K^T + \bar{A}_F & 0 & -b_5 K^T & K_2 B_i + \bar{B}_F D_i & -K_2 - K^T + P_{2i} & -K - K^T + P_{3i} & * \\
L_i - \bar{D}_F C_i & -\bar{C}_F & 0 & 0 & -\bar{D}_F D_i & 0 & 0 & -I
\end{bmatrix} < 0,
$$

$$
i = 1,\ 2,\ \ldots,\ r, \tag{5.48}
$$

where

$$
T = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b_5 K & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5 K \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0
\end{bmatrix}.
$$

Consider that $P_{1ij} = P_{1i}$, $P_{2ij} = P_{2i}$, $P_{3ij} = P_{3i}$, $G_{1j} = F_1$, $G_{3j} = F_2$, $G_{2j} = b_1 K$, $G_{4j} = b_2 K$, $S_{1j} = W_{1j} = K_1$, $S_{3j} = W_{2j} = K_2$, $S_{2j} = S_{4j} = K$, $N_{1j} = N_{2j} = E_{1j} = 0$, $M_{1j} = b_5 K$, and $M_{2j} = 0$, $i, j = 1, 2, \ldots, r$, and $b_3 = b_4 = 0$, the LMIs (5.28) and (5.29) with $H_i = 0, i = 1, 2, \ldots, r$ in Theorem 5.2 can be obtained. □

Remark 5.3 It is widely accepted that there is a tradeoff between the conservatism of a given robust system design approach and the computational complexity of designing a robust system via such approach [1, 7]. In Theorems 5.2–5.4, the variables to be determined are more than the ones in Lemma 5.2, it seems that the proposed conditions are more complex than the one in Lemma 5.2. However, when those scalars are set to be fixed parameters, the design conditions in Theorems 5.2–5.4 are strictly LMIs that can be easily and effectively solved via LMI control toolbox [4]. The problem is then how to find the optimal values of those scalars in order to minimize the filtering error variance bound. In addition to this method discussed in Remark 2.9, another scalar to address the tuning issue has been used in our earliest work [2], which first sets γ to a big value and searches those scalars such that LMIs in Theorems 5.2–5.4 hold by using the function **random** in MATLAB. Then, decrease γ till the search is infeasible.

Remark 5.4 In the above derivation for obtaining LMI-based design conditions, if we define several matrix variables are of independent on the parameter θ, another design conditions with trifling conservativeness and less number of LMIs can be given. Take (5.26) for example, we define

$$
\begin{aligned}
F(\theta) &= \begin{bmatrix} F_1 & b_1K \\ F_2 & b_2K \end{bmatrix}, \\
W(\theta) &= \begin{bmatrix} W_1 & K \\ W_2 & K \end{bmatrix}, \\
E(\theta) &= [\, E_1 \;\; b_3K \,], \\
M(\theta) &= \begin{bmatrix} M_1 & b_4K \\ M_2 & b_5K \end{bmatrix},
\end{aligned}
$$

and $P(\theta) = \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r} \theta_i \begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} > 0$, the corresponding design condition is given in the following corollary. The design result has been mentioned in our earliest work [2].

Corollary 5.1 *Consider the filtering error system (5.4) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalars b_1, b_2, b_3, b_4, and b_5, there exist matrices $P_{1i}, P_{2i}, P_{3i}, G_{1i}, G_{2i}, G_{3i}, G_{4i}, S_{1i}, S_{2i}, S_{3i}, S_{4i}, N_{1i}$, and N_{2i}, $i = 1, 2, \ldots, r$, F_1, F_2, M_1, M_2, W_1, W_2, E_1, $K, \bar{A}_F, \bar{B}_F, \bar{C}_F$, and $\bar{D}_F$ such that the following matrix inequalities hold*

$$
\begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} > 0, \quad i = 1, 2, \ldots, r, \tag{5.49}
$$

$$
\begin{bmatrix}
\Pi_{11} & * & * & * & * & * & * & * \\
F_2A_i + b_2\bar{B}_FC_i + b_1\bar{A}_F^T - P_{2i} & b_2\bar{A}_F + b_2\bar{A}_F^T - P_{3i} & * & * & * & * & * & * \\
\Pi_{31} & \Pi_{32} & -M_1 - M_1^T & * & * & * & * & * \\
\Pi_{41} & \Pi_{42} & -M_2 - b_4K^T & -b_5K - b_5K^T & * & * & * & * \\
\Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55} & * & * & * \\
-F_1^T + W_1A_i + \bar{B}_FC_i & -F_2^T + \bar{A}_F & S_{1i} - W_1 - M_1^T & S_{2i} - K - M_2^T & \Pi_{65} & -W_1 - W_1^T + P_{1i} & * & * \\
-b_1K^T + W_2A_i + \bar{B}_FC_i & -b_2K^T + \bar{A}_F & S_{3i} - W_2 - b_4K^T & S_{4i} - K - b_5K^T & \Pi_{75} & -W_2 - K^T + P_{2i} & \Pi_{77} & * \\
L_i - \bar{D}_FC_i & -\bar{C}_F & 0 & 0 & H_i - \bar{D}_FD_i & 0 & 0 & -I
\end{bmatrix} < 0,
$$

$$
i = 1, 2, \ldots, r, \tag{5.50}
$$

where

$$
\begin{aligned}
\Pi_{11} &= F_1 A_i + A_i^T F_1^T + b_1 \bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T - P_{1i},\\
\Pi_{31} &= G_{1i}^T + M_1 A_i + b_4 \bar{B}_F C_i - F_1^T,\\
\Pi_{32} &= G_{3i}^T + b_4 \bar{A}_F - F_2^T,\\
\Pi_{41} &= G_{2i}^T + M_2 A_i + b_5 \bar{B}_F C_i - b_1 K^T,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{42} &= G_{4i}^T + b_5 \bar{A}_F - b_2 K^T,\\
\Pi_{51} &= E_1 A_i + b_3 \bar{B}_F C_i + B_i^T F_1^T + b_1 D_i^T \bar{B}_F^T,\\
\Pi_{52} &= b_3 \bar{A}_F + B_i^T F_2^T + b_2 D_i^T \bar{B}_F^T,\\
\Pi_{53} &= N_{1i} - E_1 + B_i^T M_1^T + b_4 D_i^T \bar{B}_F^T\\
\Pi_{54} &= N_{2i} - b_3 K + B_i^T M_2^T + b_5 D_i^T \bar{B}_F^T,\\
\Pi_{55} &= -\gamma^2 I + E_1 B_i + B_i^T E_1^T + b_3 \bar{B}_F D_i + b_3 D_i^T \bar{B}_F^T,\\
\Pi_{65} &= W_1 B_i + \bar{B}_F D_i - E_1^T,\\
\Pi_{75} &= W_2 B_i + \bar{B}_F D_i - b_3 K^T,\\
\Pi_{77} &= -K - K^T + P_{3i}.
\end{aligned}
$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$
A_F = K^{-1}\bar{A}_F,\ B_F = K^{-1}\bar{B}_F,\ C_F = \bar{C}_F,\quad D_F = \bar{D}_F. \tag{5.51}
$$

5.1.3 LMI Decoupling Approach

In this section, based on the LMI decoupling approach presented by Chap. 2, we propose other results for designing the filter in (5.3) for the filtering error system (5.4).

Let us rewrite (5.4) as follows:

$$
\begin{aligned}
\psi(k+1) &= \left(\begin{bmatrix} A(\theta) & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0\\ I\end{bmatrix}[\, A_F \;\; B_F\,]\begin{bmatrix} 0 & I\\ C(\theta) & 0\end{bmatrix}\right)\psi(k)\\
&\quad + \left(\begin{bmatrix} B(\theta)\\ 0\end{bmatrix} + \begin{bmatrix} 0\\ I\end{bmatrix}[\, A_F \;\; B_F\,]\begin{bmatrix} 0\\ D(\theta)\end{bmatrix}\right)w(k),
\end{aligned} \tag{5.52}
$$

$$
e(k) = \left[\, L(\theta) - D_F C(\theta) \;\; -C_F\,\right]\psi(k) + \big(H(\theta) - D_F D(\theta)\big)w(k).
$$

For the filtering error system (5.52), we choose a Lyapunov function as $V(k) = \psi^T(k)P(\theta)\psi(k)$, $P(\theta) = \sum_{j=1}^{r}\theta_j P_j$, $P_j > 0$.

By adding an auxiliary matrix variable $G(\theta) = \sum_{j=1}^{r}\theta_j G_j$, we give immediately the following condition to ensure the H_∞ performance γ of the filtering error system (5.52)

$$
\begin{bmatrix}
-P(\theta) & * & * & * \\
0 & -\gamma^2 I & * & * \\
G(\theta)\Delta_A & G(\theta)\Delta_E & -G(\theta) - G^T(\theta) + P(\theta) & * \\
\left[\, L(\theta) - D_F C(\theta) \;\; -C_F \,\right] & H(\theta) - D_F D(\theta) & 0 & -I
\end{bmatrix}
$$
$$
= \begin{bmatrix}
-P(\theta) & * & * & * \\
0 & -\gamma^2 I & * & * \\
G(\theta)\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} & G(\theta)\begin{bmatrix} B(\theta) \\ 0 \end{bmatrix} & -G(\theta) - G^T(\theta) + P(\theta) & * \\
\left[\, L(\theta) - D_F C(\theta) \;\; -C_F \,\right] & H(\theta) - D_F D(\theta) & 0 & -I
\end{bmatrix}
$$
$$
+ \begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 \\ I \end{bmatrix} \\ 0 \end{bmatrix} [\, A_F \;\; B_F \,] \left(\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D(\theta) \end{bmatrix} 0 \; 0 \right)
$$
$$
+ \left(\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D(\theta) \end{bmatrix} 0 \; 0 \right)^T [\, A_F \;\; B_F \,]^T \begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 \\ I \end{bmatrix} \\ 0 \end{bmatrix}^T < 0, \tag{5.53}
$$

where

$$
\Delta_A = \begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} [\, A_F \;\; B_F \,] \begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix},
$$
$$
\Delta_E = \begin{bmatrix} B(\theta) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} [\, A_F \;\; B_F \,] \begin{bmatrix} 0 \\ D(\theta) \end{bmatrix}.
$$

Define $U^{-1}[\, V_A \;\; V_B \,] = [\, A_F \;\; B_F \,]$, from (5.53), we have

$$
\begin{bmatrix}
-P(\theta) & * & * & * \\
0 & -\gamma^2 I & * & * \\
\Phi_1 & \Phi_2 & -G(\theta) - G^T(\theta) + P(\theta) & * \\
\left[\, L(\theta) - D_F C(\theta) \;\; -C_F \,\right] & H(\theta) - D_F D(\theta) & 0 & -I
\end{bmatrix}
$$
$$
+ \begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 \\ I \end{bmatrix} - MU \\ 0 \end{bmatrix} U^{-1} N^{-1} N [\, V_A \;\; V_B \,] \left(\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D(\theta) \end{bmatrix} 0 \; 0 \right)
$$
$$
+ \left(\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ D(\theta) \end{bmatrix} 0 \; 0 \right)^T [\, V_A \;\; V_B \,]^T N^T N^{-T} U^{-T} \begin{bmatrix} 0 \\ 0 \\ G(\theta)\begin{bmatrix} 0 \\ I \end{bmatrix} - MU \\ 0 \end{bmatrix}^T
$$
$$
< 0, \tag{5.54}
$$

where

$$\Phi_1 = G(\theta)\begin{bmatrix} A(\theta) & 0 \\ 0 & 0 \end{bmatrix} + M[\, V_A \ V_B \,]\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix},$$
$$\Phi_2 = G(\theta)\begin{bmatrix} B(\theta) \\ 0 \end{bmatrix} + M[\, V_A \ V_B \,]\begin{bmatrix} 0 \\ D(\theta) \end{bmatrix}.$$

Following the proof of Theorem 2.1, we obtain the following matrix inequality to verify (5.54):

$$\begin{bmatrix} -P(\theta) & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ \Phi_1 & \Phi_2 & \Phi_3 & * & * & * \\ [\, L(\theta) - D_F C(\theta) \quad -C_F \,] & H(\theta) - D_F D(\theta) & 0 & -I & * & * \\ N\,[\, V_A \quad V_B \,]\begin{bmatrix} 0 & I \\ C(\theta) & 0 \end{bmatrix} & N\,[\, V_A \quad V_B \,]\begin{bmatrix} 0 \\ D(\theta) \end{bmatrix} & 0 & 0 & \Upsilon_1 & * \\ 0 & 0 & 0 & 0 & \Upsilon_2 & -\frac{J(\theta)}{\beta^2} \end{bmatrix} < 0, \tag{5.55}$$

where

$$\Phi_3 = -G(\theta) - G^T(\theta) + P(\theta) + J(\theta),$$
$$\Upsilon_1 = -\beta N U - \beta U^T N^T,$$
$$\Upsilon_2 = G(\theta)\begin{bmatrix} 0 \\ I \end{bmatrix} - MU.$$

Based on the H_∞ performance analysis criterion in (5.55), we give a new design condition of the filter (5.3), which is demonstrated in the following theorem.

Theorem 5.6 *Consider the filtering error system* (5.52) *and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known matrices M, N and scalar β, exist matrices $U, V_A, V_B, V_C, V_D, P_j, J_j$, and G_j, $j = 1, 2, \ldots, r$ such that the following matrix inequalities hold*

$$\Delta_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.56}$$

$$\Delta_{ij} + \Delta_{ji} < 0, \quad i < j,\ i, j = 1, 2, \ldots, r, \tag{5.57}$$

with

$$\Delta_{ij} = \begin{bmatrix} -P_j & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ \Pi_1 & \Pi_2 & \Pi_3 & * & * & * \\ [\, L_i - V_D C_i \quad -V_C \,] & H_i - V_D D_i & 0 & -I & * & * \\ N\,[\, V_A \quad V_B \,]\begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix} & N\,[\, V_A \quad V_B \,]\begin{bmatrix} 0 \\ D_i \end{bmatrix} & 0 & 0 & \Pi_4 & * \\ 0 & 0 & 0 & 0 & \Pi_5 & -\frac{J_j}{\beta^2} \end{bmatrix},$$

and

$$\Pi_1 = G_j \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix} + M \begin{bmatrix} V_A & V_B \end{bmatrix} \begin{bmatrix} 0 & I \\ C_i & 0 \end{bmatrix},$$
$$\Pi_2 = G_j \begin{bmatrix} B_i \\ 0 \end{bmatrix} + M \begin{bmatrix} V_A & V_B \end{bmatrix} \begin{bmatrix} 0 \\ D_i \end{bmatrix},$$
$$\Pi_3 = -G_j - G_j^T + P_j + J_j,$$
$$\Pi_4 = -\beta N U - \beta U^T N^T,$$
$$\Pi_5 = G_j \begin{bmatrix} 0 \\ I \end{bmatrix} - MU.$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$A_F = U^{-1} V_A, B_F = U^{-1} V_B, C_F = V_C, D_F = V_D. \tag{5.58}$$

5.2 Continuous-Time Systems

Consider the following continuous-time filtering error system:

$$\begin{aligned} \dot{\psi}(t) &= \tilde{A}(\theta)\psi(t) + \tilde{B}(\theta)w(t), \\ e(t) &= \tilde{C}(\theta)\psi(t) + \tilde{D}(\theta)w(t), \end{aligned} \tag{5.59}$$

where $\tilde{A}(\theta)$, $\tilde{B}(\theta)$, $\tilde{C}(\theta)$, and $\tilde{D}(\theta)$ are the same as that in (5.4).

5.2.1 H_∞ Performance Analysis

In [3], an H_∞ performance analysis criterion has been presented for the filtering error system (5.59), and it is recalled by the following lemma.

Lemma 5.3 *Consider the filtering error system (5.59) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if there exist matrices $P(\theta) > 0$, G, and S such that the following matrix inequality holds:*

$$\begin{bmatrix} G\tilde{A}(\theta) + \tilde{A}^T(\theta)G^T & * & * & * \\ \tilde{B}^T(\theta)G^T & -\gamma^2 I & * & * \\ P(\theta) - G^T + S\tilde{A}(\theta) & S\tilde{B}(\theta) & -S - S^T & * \\ \tilde{C}(\theta) & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0. \tag{5.60}$$

In this section, we first establish a new criterion for H_∞ filtering analysis.

Theorem 5.7 *Consider the filtering error system (5.59) and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, there exist* *matrices* $P(\theta) > 0$, $G(\theta)$, $N(\theta)$, *and* $S(\theta)$ *such that the following matrix inequality holds:*

$$\begin{bmatrix} G(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)G^T(\theta) & * & * & * \\ \tilde{B}^T(\theta)G^T(\theta) + N(\theta)\tilde{A}(\theta) & -\gamma^2 I + N(\theta)\tilde{B}(\theta) + \tilde{B}^T(\theta)N^T(\theta) & * & * \\ -G^T(\theta) + S(\theta)\tilde{A}(\theta) + P(\theta) & -N^T(\theta) + S(\theta)\tilde{B}(\theta) & \Xi_{33} & * \\ \tilde{C}(\theta) & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0, \tag{5.61}$$

where $\Xi_{33} = -S(\theta) - S^T(\theta)$.

Proof Consider the following Lyapunov function:

$$V(t) = \psi^T(t)P(\theta)\psi(t),\ P(\theta) > 0. \tag{5.62}$$

Then, the time-derivative of $V(t)$ is

$$\dot{V}(t) = \dot{\psi}^T(t)P(\theta)\psi(t) + \psi^T(t)P(\theta)\dot{\psi}(t). \tag{5.63}$$

Similar to the discrete-time case, the proof also considers three different approaches.

Approach 1:
From (5.59) and (5.63), it can be verified that

$$\begin{aligned} &\dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) \\ &= \dot{\psi}^T(t)P(\theta)\psi(t) + \psi^T(t)P(\theta)\dot{\psi}(t) \\ &\quad + (\tilde{C}(\theta)\psi(t) + \tilde{D}(\theta)w(t))^T(\tilde{C}(\theta)\psi(t) + \tilde{D}(\theta)w(t)) - \gamma^2 w^T(t)w(t) \\ &= \zeta^T(k)\left(\begin{bmatrix} 0 & 0 & P(\theta) \\ 0 & -\gamma^2 I & 0 \\ P(\theta) & 0 & 0 \end{bmatrix} + [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]\right)\zeta(k), \end{aligned} \tag{5.64}$$

where $\zeta(t) = \begin{bmatrix} \psi(t) \\ w(t) \\ \dot{\psi}(t) \end{bmatrix}$.

From (5.59), one has

$$[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]\zeta(t) = [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]\begin{bmatrix} \psi(t) \\ w(t) \\ \dot{\psi}(t) \end{bmatrix} = 0. \tag{5.65}$$

By Lemma 1.14 with

$$\Theta = \begin{bmatrix} 0 & 0 & P(\theta) \\ 0 & -\gamma^2 I & 0 \\ P(\theta) & 0 & 0 \end{bmatrix} + [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,],$$
$$N = [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,],$$
$$\nu = \zeta(t),$$

and choose $L = \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix}$ in (1.20), if the following inequality

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & P(\theta) \\ 0 & -\gamma^2 I & 0 \\ P(\theta) & 0 & 0 \end{bmatrix} + [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,] \\ &+ \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix} [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,] + [\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]^T \begin{bmatrix} G(\theta) \\ N(\theta) \\ S(\theta) \end{bmatrix}^T < 0, \end{aligned} \tag{5.66}$$

i.e.,

$$\begin{aligned} &\begin{bmatrix} G(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)G^T(\theta) & * & * \\ \tilde{B}^T(\theta)G^T(\theta) + N(\theta)\tilde{A}(\theta) & -\gamma^2 I + N(\theta)\tilde{B}(\theta) + \tilde{B}^T(\theta)N^T(\theta) & * \\ -G^T(\theta) + S(\theta)\tilde{A}(\theta) + P(\theta) & -N^T(\theta) + S(\theta)\tilde{B}(\theta) & -S(\theta) - S^T(\theta) \end{bmatrix} \\ &+ [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T [\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,] < 0, \end{aligned} \tag{5.67}$$

holds, we have $\dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0$. Applying the Schur complement to (5.67) leads to (5.61). The equation $\dot{V}(t) + e^T(t)e(t) - \gamma^2 w^T(t)w(t) < 0$ which implies that

$$V(\infty) - V(0) + \int_0^\infty e^T(t)e(t)dt - \gamma^2 \int_0^\infty w^T(t)w(t)dt < 0.$$

With zero initial condition $\psi(0) = 0$ and $V(\infty) > 0$, we obtain $\int_0^\infty e^T(t)e(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt$, for any nonzero $w(t) \in L_2[0, \infty)$.

Approach 2:
From (5.59), we have

$$\tilde{A}(\theta)\psi(t) + \tilde{B}(\theta)w(t) - \dot{\psi}(t) = 0. \tag{5.68}$$

Then, for any appropriately dimensioned parameter-dependent matrices $G(\theta)$, $N(\theta)$, and $S(\theta)$, it can be given

$$\begin{aligned} &2\big(\psi^T(t)G(\theta) + w^T(k)N(\theta) + \dot{\psi}^T(t)S(\theta)\big) \\ &\times\big(\tilde{A}(\theta)\psi(t) + \tilde{B}(\theta)w(t) - \dot{\psi}(t)\big) = 0. \end{aligned} \tag{5.69}$$

By combining (5.59), (5.62), and (5.69), one has

$$\begin{aligned}
&\dot{V}(t)+e^T(t)e(t)-\gamma^2 w^T(t)w(t)\\
&=\dot{\psi}^T(t)P(\theta)\psi(t)+\psi^T(t)P(\theta)\dot{\psi}(t)\\
&\quad+\big(\psi^T(t)G(\theta)+w^T(t)N(\theta)+\dot{\psi}^T(t)S(\theta)\big)\\
&\quad\times\big(\tilde{A}(\theta)\psi(t)+\tilde{B}(\theta)w(t)-\dot{\psi}(t)\big)\\
&\quad+\big(\tilde{A}(\theta)\psi(t)+\tilde{B}(\theta)w(t)-\dot{\psi}(t)\big)^T\\
&\quad\times\big(\psi^T(t)G(\theta)+w^T(t)N(\theta)+\dot{\psi}^T(t)S(\theta)\big)^T\\
&\quad+\big(\tilde{C}(\theta)\psi(t)+\tilde{D}(\theta)w(t)\big)^T\big(\tilde{C}(\theta)\psi(t)+\tilde{D}(\theta)w(t)\big)-\gamma^2 w^T(t)w(t)\\
&=\upsilon^T(k)\left(\begin{bmatrix}0 & 0 & P(\theta)\\ 0 & -\gamma^2 I & 0\\ P(\theta) & 0 & 0\end{bmatrix}+[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]^T[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\ 0\,]\right.\\
&\quad\left.+\begin{bmatrix}G(\theta)\\ N(\theta)\\ S(\theta)\end{bmatrix}[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]+[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\ -I\,]^T\begin{bmatrix}G(\theta)\\ N(\theta)\\ S(\theta)\end{bmatrix}^T\right)\upsilon(k),
\end{aligned}\tag{5.70}$$

where $\upsilon(t)=\begin{bmatrix}\psi(t)\\ w(t)\\ \dot{\psi}(t)\end{bmatrix}$.

Obviously, it is observed that the matrix inequality (5.61) can guarantee the negative-definiteness of (5.70).

Approach 3:
From (5.59) and (5.62), we have

$$\begin{aligned}
&\dot{V}(t)+e^T(t)e(t)-\gamma^2 w^T(t)w(t)\\
&=\dot{\psi}^T(t)P(\theta)\psi(t)+\psi^T(t)P(\theta)\dot{\psi}(t)+e^T(t)e(t)-\gamma^2 w^T(t)w(t)\\
&=\big(\tilde{A}(\theta)\psi(t)+\tilde{B}(\theta)w(t)\big)^T P(\theta)\psi(t)+\psi^T(t)P(\theta)\big(\tilde{A}(\theta)\psi(t)+\tilde{B}(\theta)w(t)\big)\\
&\quad+\big(\tilde{C}(\theta)\psi(t)+\tilde{D}(\theta)w(t)\big)^T\big(\tilde{C}(\theta)\psi(t)+\tilde{D}(\theta)w(t)\big)-\gamma^2 w^T(t)w(t)\\
&=\big[\,\psi^T(t)\ w^T(t)\,\big]\begin{bmatrix}P(\theta)\tilde{A}(\theta)+\tilde{A}^T(\theta)P(\theta) & *\\ \tilde{B}^T(\theta)P(\theta) & -\gamma^2 I\end{bmatrix}\begin{bmatrix}\psi(t)\\ w(t)\end{bmatrix}\\
&\quad+\big[\,\psi^T(t)\ w^T(t)\,\big]\big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\big]^T\big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\big]\begin{bmatrix}\psi(t)\\ w(t)\end{bmatrix}.
\end{aligned}\tag{5.71}$$

Thus, $\dot{V}(t)+e^T(t)e(t)-\gamma^2 w^T(t)w(t)<0$ for any $\begin{bmatrix}\psi(t)\\ w(t)\end{bmatrix}\neq 0$ if

$$\begin{aligned}
&\begin{bmatrix}P(\theta)\\ 0\end{bmatrix}[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,]+[\,\tilde{A}(\theta)\ \tilde{B}(\theta)\,]^T\begin{bmatrix}P(\theta)\\ 0\end{bmatrix}^T\\
&+\big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\big]^T\big[\,\tilde{C}(\theta)\ \tilde{D}(\theta)\,\big]+\begin{bmatrix}0 & 0\\ 0 & -\gamma^2 I\end{bmatrix}<0.
\end{aligned}\tag{5.72}$$

Then, by using Lemma 1.9 with

$$
\begin{aligned}
T &= \begin{bmatrix} \tilde{C}(\theta) & \tilde{D}(\theta) \end{bmatrix}^T \begin{bmatrix} \tilde{C}(\theta) & \tilde{D}(\theta) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \\
M &= \begin{bmatrix} G(\theta) \\ N(\theta) \end{bmatrix}, \\
G &= S(\theta), \\
A &= [\, \tilde{A}(\theta) \ \tilde{B}(\theta) \,], \\
P &= \begin{bmatrix} P(\theta) \\ 0 \end{bmatrix}.
\end{aligned}
$$

and Schur complement, if the inequality (5.61) is satisfied, (5.72) holds. □

Remark 5.5 Compared with (5.60) in Lemma 5.3, (5.61) in Theorem 5.7 adds a slack matrix variable $N(\theta)$ and replaces single matrix variables to be determined by parameter-dependent matrix variables. In other words, when $N(\theta) = 0$, $G(\theta)$ and $S(\theta)$ are parameter-independent matrices, (5.61) in Theorem 5.7 reduces (5.60) in Lemma 5.3. Thus it can be easily seen that (5.60) is a special case of (5.61) and the condition in Theorem 5.7 is less conservative than that in Lemma 5.3.

5.2.2 Robust H_∞ Filters Design

The following lemma gives an LMI-based design condition for H_∞ filter design of the filtering error system (5.59), which has been presented in [3].

Lemma 5.4 [3]: *Consider the filtering error system* (5.59) *with* $H(\theta) = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalars* b_1 *and* b_2, *exist matrices* P_{1i}, P_{2i}, *and* P_{3i}, $i = 1, 2, \ldots, r$, F_1, F_2, K_1, K_2, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, *and* $\bar{D}_F$ *such that the following matrix inequalities hold*

$$
\begin{bmatrix}
F_1 A_i + A_i^T F_1^T + b_1 \bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T & * & * & * & * & * \\
F_2 A_i + b_2 \bar{B}_F C_i + b_1 \bar{A}_F^T & b_2 \bar{A}_F + b_2 \bar{A}_F^T & * & * & * & * \\
B_i^T F_1^T + b_1 D_i^T \bar{B}_F^T & B_i^T F_2^T + b_2 D_i^T \bar{B}_F^T & -\gamma^2 I & * & * & * \\
P_{1i} - F_1^T + K_1 A_i + \bar{B}_F C_i & P_{2i}^T - F_2^T + \bar{A}_F & K_1 B_i + \bar{B}_F D_i & -K_1 - K_1^T & * & * \\
P_{2i} - b_1 K^T + K_2 A_i + \bar{B}_F C_i & P_{3i} - b_2 K^T + \bar{A}_F & K_2 B_i + \bar{B}_F D_i & -K_2 - K^T & -K - K^T & * \\
L_i - \bar{D}_F C_i & -\bar{C}_F & -\bar{D}_F D_i & 0 & 0 & -I
\end{bmatrix} < 0, \quad i = 1, 2, \ldots, r,
\tag{5.73}
$$

$$\begin{bmatrix} P_{1i} & * \\ P_{2i} & P_{3i} \end{bmatrix} > 0, \quad i = 1, 2, \ldots, r. \tag{5.74}$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$A_F = K^{-1}\bar{A}_F, B_F = K^{-1}\bar{B}_F, C_F = \bar{C}_F, D_F = \bar{D}_F.$$

With Theorem 5.7 in hand, we will develop another result for H_∞ filtering which improves that in Lemma 5.3. First, the inequality (5.61) can be rewritten as

$$\begin{bmatrix} I & \tilde{A}^T(\theta) & 0 & 0 & 0 \\ 0 & \tilde{B}^T(\theta) & I & 0 & 0 \\ 0 & -I & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \Delta \begin{bmatrix} I & 0 & 0 & 0 \\ \tilde{A}(\theta) & \tilde{B}(\theta) & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0, \tag{5.75}$$

where $\Delta = \begin{bmatrix} 0 & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ P(\theta) & S(\theta) & 0 & 0 & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}$.

Obviously, the matrix inequality (5.75) corresponds to the first equation in (1.22) of Lemma 1.15 with

$$\Psi = \begin{bmatrix} 0 & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ P(\theta) & S(\theta) & 0 & 0 & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}, \tag{5.76}$$

and

$$P^\perp = \begin{bmatrix} I & 0 & 0 & 0 \\ \tilde{A}(\theta) & \tilde{B}(\theta) & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{5.77}$$

From (5.77), we can choose

$$P = [\, \tilde{A}(\theta) \;\; -I \;\; \tilde{B}(\theta) \;\; -I \;\; 0\,]. \tag{5.78}$$

On the other hand, for another matrix Q, we divide into two cases, under which three LMI-based conditions for designing the robust H_∞ filter in (5.3) are given.

Case A:
Due to $-I < 0$, we have the following inequality:

$$[\,0\;0\;0\;0\;I\,]\begin{bmatrix} 0 & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ P(\theta) & S(\theta) & 0 & 0 & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} < 0, \tag{5.79}$$

the matrix inequality corresponds to the second equation in (1.22) with $Q^{\perp} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$.

We can choose the matrix Q as

$$Q = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}. \tag{5.80}$$

Then, based on Lemma 1.15, (5.75) and (5.79) hold if the following inequality holds

$$\begin{bmatrix} 0 & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ P(\theta) & S(\theta) & 0 & 0 & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix} + P^T X^T(\theta) Q + Q^T X(\theta) P < 0. \tag{5.81}$$

Assume

$$X(\theta) = \begin{bmatrix} F(\theta) \\ M(\theta) \\ E(\theta) \\ W(\theta) \end{bmatrix}. \tag{5.82}$$

Then, the inequality (5.81) is equivalent to

$$\begin{bmatrix} F(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)F^T(\theta) & * & * & * & * \\ G^T(\theta) + M\tilde{A}(\theta) - F^T(\theta) & -M(\theta) - M^T(\theta) & * & * & * \\ \tilde{B}^T(\theta)F^T(\theta) + E(\theta)\tilde{A}(\theta) & \Upsilon_{32} & \Upsilon_{33} & * & * \\ P(\theta) - F^T(\theta) + W(\theta)\tilde{A}(\theta) & \Upsilon_{42} & \Upsilon_{43} & \Upsilon_{44} & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix} < 0, \tag{5.83}$$

where

$$
\begin{aligned}
\Upsilon_{32} &= N(\theta) - E(\theta) + \tilde{B}^T(\theta)M^T(\theta),\\
\Upsilon_{33} &= -\gamma^2 I + E(\theta)\tilde{B}(\theta) + \tilde{B}^T(\theta)E^T(\theta),\\
\Upsilon_{42} &= S(\theta) - W(\theta) - M^T(\theta),\\
\Upsilon_{43} &= -E^T(\theta) + W(\theta)\tilde{B}(\theta),\\
\Upsilon_{44} &= -W(\theta) - W^T(\theta).
\end{aligned}
$$

We partition these matrices as

$$
\begin{aligned}
P(\theta) &= \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j \begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0,\\
F(\theta) &= \begin{bmatrix} F_1(\theta) & b_1 K \\ F_2(\theta) & b_2 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} F_{1j} & b_1 K \\ F_{2j} & b_2 K \end{bmatrix},\\
W(\theta) &= \begin{bmatrix} W_1(\theta) & K \\ W_2(\theta) & K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} W_{1j} & K \\ W_{2j} & K \end{bmatrix},\\
E(\theta) &= [\, E_1(\theta) \;\; b_3 K \,] = \sum_{j=1}^{r}\theta_j [\, E_{1j} \;\; b_3 K \,],\\
G(\theta) &= \begin{bmatrix} G_1(\theta) & G_2(\theta) \\ G_3(\theta) & G_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} G_{1j} & G_{2j} \\ G_{3j} & G_{4j} \end{bmatrix},\\
S(\theta) &= \begin{bmatrix} S_1(\theta) & S_2(\theta) \\ S_3(\theta) & S_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} S_{1j} & S_{2j} \\ S_{3j} & S_{4j} \end{bmatrix},\\
N(\theta) &= [\, N_1(\theta) \;\; N_2(\theta) \,] = \sum_{j=1}^{r}\theta_j [\, N_{1j} \;\; N_{2j} \,],\\
M(\theta) &= \begin{bmatrix} M_1(\theta) & b_4 K \\ M_2(\theta) & b_5 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} M_{1j} & b_4 K \\ M_{2j} & b_5 K \end{bmatrix}.
\end{aligned}
$$

Let $\bar{A}_F = KA_F$, $\bar{B}_F = KB_F$, $\bar{C}_F = C_F$, and $\bar{D}_F = D_F$, by substituting the above matrices $P(\theta)$, $F(\theta)$, $W(\theta)$, $E(\theta)$, $G(\theta)$, $S(\theta)$, $N(\theta)$, and $M(\theta)$ into (5.83) and by considering the uncertainty set (5.2), we can obtain the following design conditions for the robust H_∞ filter in (5.3).

Theorem 5.8 *Consider the filtering error system* (5.59) *and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if, for known scalars b_1, b_2, b_3, b_4, and b_5, there exist matrices P_{1ij}, P_{2ij}, P_{3ij}, G_{1j}, G_{2j}, G_{3j}, G_{4j}, S_{1j}, S_{2j}, S_{3j}, S_{4j}, N_{1j}, N_{2j}, F_{1j}, F_{2j}, M_{1j}, M_{2j}, W_{1j}, W_{2j}, and E_{1j}, $i, j = 1, 2, \ldots, r$, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$ such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0, \quad i, j = 1, 2, \ldots, r, \tag{5.84}$$

$$\Pi_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.85}$$

$$\Pi_{ij} + \Pi_{ji} < 0, \quad i < j, \; i, j = 1, 2, \ldots, r, \tag{5.86}$$

with

$$\Pi_{ij} = \begin{bmatrix} \Pi_{11} & * & * & * & * & * & * & * \\ F_{2j}A_i + b_2\bar{B}_F C_i + b_1\bar{A}_F^T & b_2\bar{A}_F + b_2\bar{A}_F^T & * & * & * & * & * & * \\ \Pi_{31} & \Pi_{32} & -M_{1j} - M_{1j}^T & * & * & * & * & * \\ \Pi_{41} & \Pi_{42} & -M_{2j} - b_4K^T & -b_5K - b_5K^T & * & * & * & * \\ \Pi_{51} & \Pi_{52} & \Pi_{53} & \Pi_{54} & \Pi_{55} & * & * & * \\ P_{1ij} - F_{1j}^T + W_{1j}A_i + \bar{B}_F C_i & P_{2ij}^T - F_{2j}^T + \bar{A}_F & S_{1j} - W_{1j} - M_{1j}^T & S_{2j} - K - M_{2j}^T & \Pi_{65} & -W_{1j} - W_{1j}^T & * & * \\ P_{2ij} - b_1K^T + W_{2j}A_i + \bar{B}_F C_i & P_{3ij} - b_2K^T + \bar{A}_F & S_{3j} - W_{2j} - b_4K^T & S_{4j} - K - b_5K^T & \Pi_{75} & -W_{2j} - K^T & \Pi_{77} & * \\ L_i - \bar{D}_F C_i & -\bar{C}_F & 0 & 0 & H_i - \bar{D}_F D_i & 0 & 0 & -I \end{bmatrix},$$

where

$$\begin{aligned}
\Pi_{11} &= F_{1j}A_i + A_i^T F_{1j}^T + b_1\bar{B}_F C_i + b_1 C_i^T \bar{B}_F^T, \\
\Pi_{31} &= G_{1j}^T + M_{1j}A_i + b_4\bar{B}_F C_i - F_{1j}^T, \\
\Pi_{32} &= G_{3j}^T + b_4\bar{A}_F - F_{2j}^T, \\
\Pi_{41} &= G_{2j}^T + M_{2j}A_i + b_5\bar{B}_F C_i - b_1K^T, \\
\Pi_{42} &= G_{4j}^T + b_5\bar{A}_F - b_2K^T, \\
\Pi_{51} &= E_{1j}A_i + b_3\bar{B}_F C_i + B_i^T F_{1j}^T + b_1 D_i^T \bar{B}_F^T, \\
\Pi_{52} &= b_3\bar{A}_F + B_i^T F_{2j}^T + b_2 D_i^T \bar{B}_F^T, \\
\Pi_{53} &= N_{1j} - E_{1j} + B_i^T M_{1j}^T + b_4 D_i^T \bar{B}_F^T, \\
\Pi_{54} &= N_{2j} - b_3K + B_i^T M_{2j}^T + b_5 D_i^T \bar{B}_F^T, \\
\Pi_{55} &= -\gamma^2 I + E_{1j}B_i + B_i^T E_{1j}^T + b_3\bar{B}_F D_i + b_3 D_i^T \bar{B}_F^T, \\
\Pi_{65} &= W_{1j}B_i + \bar{B}_F D_i - E_{1j}^T, \\
\Pi_{75} &= W_{2j}B_i + \bar{B}_F D_i - b_3K^T, \\
\Pi_{77} &= -K - K^T.
\end{aligned}$$

Moreover, the gain matrices of the H_∞ filter (5.3) are given by

$$A_F = K^{-1}\bar{A}_F,\, B_F = K^{-1}\bar{B}_F,\, C_F = \bar{C}_F,\, D_F = \bar{D}_F. \tag{5.87}$$

Case B:
Due to $-\gamma^2 I < 0$, we have the following equation:

$$[0\ 0\ I\ 0\ 0]\begin{bmatrix} 0 & * & * & * & * \\ G^T(\theta) & 0 & * & * & * \\ 0 & N(\theta) & -\gamma^2 I & * & * \\ P(\theta) & S(\theta) & 0 & 0 & * \\ \tilde{C}(\theta) & 0 & \tilde{D}(\theta) & 0 & -I \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix} < 0, \tag{5.88}$$

the matrix inequality corresponds to the second equation in (1.22) with $Q^\perp = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}$.

Then, we can choose the matrix Q as

$$Q = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{5.89}$$

In this case, (5.81) is equivalent to

$$\begin{bmatrix} \Upsilon_{11} & * & * & * & * \\ \Upsilon_{21} & -M(\theta) - M^T(\theta) & * & * & * \\ \tilde{B}^T(\theta)F^T(\theta) & N(\theta) + \tilde{B}^T(\theta)M^T(\theta) & -\gamma^2 I & * & * \\ \Upsilon_{41} & S(\theta) - E(\theta) - M^T(\theta) & E(\theta)B(\theta) & \Upsilon_{44} & * \\ \tilde{C}(\theta) + W(\theta)A(\theta) & -W(\theta) & \Upsilon_{53} & -W(\theta) & -I \end{bmatrix} < 0, \tag{5.90}$$

where

$$\begin{aligned} \Upsilon_{11} &= F(\theta)\tilde{A}(\theta) + \tilde{A}^T(\theta)F^T(\theta), \\ \Upsilon_{21} &= G^T(\theta) + M(\theta)\tilde{A}(\theta) - F^T(\theta), \\ \Upsilon_{41} &= P(\theta) + E(\theta)A(\theta) - F^T(\theta), \\ \Upsilon_{44} &= -E(\theta) - E^T(\theta), \\ \Upsilon_{53} &= \tilde{D}(\theta) + W(\theta)B(\theta). \end{aligned}$$

We partition these matrices as

$$P(\theta) = \begin{bmatrix} P_1(\theta) & * \\ P_2(\theta) & P_3(\theta) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\theta_j \begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0,$$

$$F(\theta) = \begin{bmatrix} F_1(\theta) & b_1 K \\ F_2(\theta) & b_2 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} F_{1j} & b_1 K \\ F_{2j} & b_2 K \end{bmatrix},$$

$$W(\theta) = [\, W_1(\theta)\ 0 \,] = \sum_{j=1}^{r}\theta_j [\, W_{1j}\ 0 \,],$$

$$E(\theta) = \begin{bmatrix} E_1(\theta) & b_3 K \\ E_2(\theta) & b_4 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} E_{1j} & b_3 K \\ E_{2j} & b_4 K \end{bmatrix},$$

$$G(\theta) = \begin{bmatrix} G_1(\theta) & G_2(\theta) \\ G_3(\theta) & G_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} G_{1j} & G_{2j} \\ G_{3j} & G_{4j} \end{bmatrix},$$

$$S(\theta) = \begin{bmatrix} S_1(\theta) & S_2(\theta) \\ S_3(\theta) & S_4(\theta) \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} S_{1j} & S_{2j} \\ S_{3j} & S_{4j} \end{bmatrix},$$

$$N(\theta) = [\, N_1(\theta)\ N_2(\theta) \,] = \sum_{j=1}^{r}\theta_j [\, N_{1j}\ N_{2j} \,],$$

$$M(\theta) = \begin{bmatrix} M_1(\theta) & b_5 K \\ M_2(\theta) & b_6 K \end{bmatrix} = \sum_{j=1}^{r}\theta_j \begin{bmatrix} M_{1j} & b_5 K \\ M_{2j} & b_6 K \end{bmatrix}.$$

Let $\bar{A}_F = KA_F$, $\bar{B}_F = KB_F$, $\bar{C}_F = C_F$, and $\bar{D}_F = D_F$, by substituting the above matrices $P(\theta)$, $F(\theta)$, $W(\theta)$, $E(\theta)$, $G(\theta)$, $S(\theta)$, $N(\theta)$, and $M(\theta)$ into (5.90) and by considering the uncertainty set (5.2), we can obtain the following design conditions for the robust H_∞ filter in (5.3).

Theorem 5.9 *Consider the filtering error system* (5.59) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known scalars* b_1, b_2, b_3, b_4, b_5, *and* b_6 *there exist matrices* P_{1ij}, P_{2ij}, P_{3ij}, G_{1j}, G_{2j}, G_{3j}, G_{4j}, S_{1j}, S_{2j}, S_{3j}, S_{4j}, N_{1j}, N_{2j}, F_{1j}, F_{2j}, M_{1j}, M_{2j}, W_{1j}, *and* E_{1j}, $i, j = 1, 2, \ldots, r$, K, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, *and* $\bar{D}_F$ *such that the following matrix inequalities hold*

$$\begin{bmatrix} P_{1ij} & * \\ P_{2ij} & P_{3ij} \end{bmatrix} > 0, \quad i, j = 1, 2, \ldots, r, \tag{5.91}$$

$$\Gamma_{ii} < 0, \quad i = 1, 2, \ldots, r, \tag{5.92}$$

$$\Gamma_{ij} + \Gamma_{ji} < 0, \quad i < j,\ i, j = 1, 2, \ldots, r, \tag{5.93}$$

with

$$
\Gamma_{ij} = \begin{bmatrix}
\Gamma_{11} & * & * & * & * & * & * & * \\
\Gamma_{21} & b_2\bar{A}_F + b_2\bar{A}_F^T & * & * & * & * & * & * \\
\Gamma_{31} & G_{3j}^T + b_5\bar{A}_F - F_{2j}^T & -M_{1j} - M_{1j}^T & * & * & * & * & * \\
\Gamma_{41} & G_{4j}^T + b_6\bar{A}_F - b_2K^T & -M_{2j} - b_5K^T & -b_6K - b_6K^T & * & * & * & * \\
B_i^T F_{1j}^T + b_1 D_i^T \bar{B}_F^T & B_i^T F_{2j}^T + b_2 D_i^T \bar{B}_F^T & \Gamma_{53} & \Gamma_{54} & -\gamma^2 I & * & * & * \\
P_{1ij} - F_{1j}^T + E_{1j}A_i + b_3\bar{B}_F C_i & P_{2ij}^T - F_{2j}^T + b_3\bar{A}_F & S_{1j} - E_{1j} - M_{1j}^T & S_{2j} - b_3K - M_{2j}^T & \Gamma_{65} & -E_{1j} - E_{1j}^T & * & * \\
P_{2ij} - b_1K^T + E_{2j}A_i + b_4\bar{B}_F C_i & P_{3ij} - b_2K^T + b_4\bar{A}_F & S_{3j} - E_{2j} - b_5K^T & S_{4j} - b_4K - b_6K^T & \Gamma_{75} & -E_{2j} - b_3K^T & \Gamma_{77} & * \\
L_i - \bar{D}_F C_i + W_{1j}A_i & -\bar{C}_F & -W_{1j} & 0 & \Gamma_{85} & -W_{1j} & 0 & -I
\end{bmatrix},
$$

where

$$
\begin{aligned}
\Gamma_{11} &= F_{1j}A_i + b_1\bar{B}_F C_i + A_i^T F_{1j}^T + b_1 C_i^T \bar{B}_F^T, \\
\Gamma_{21} &= F_{2j}A_i + b_2\bar{B}_F C_i + b_1\bar{A}_F^T, \\
\Gamma_{31} &= G_{1j}^T + M_{1j}A_i + b_5\bar{B}_F C_i - F_{1j}^T, \\
\Gamma_{41} &= G_{2j}^T + M_{2j}A_i + b_6\bar{B}_F C_i - b_1K^T, \\
\Gamma_{53} &= N_{1j} + B_i^T M_{1j}^T + b_5 D_i^T \bar{B}_F^T, \\
\Gamma_{54} &= N_{2j} + B_i^T M_{2j}^T + b_6 D_i^T \bar{B}_F^T, \\
\Gamma_{65} &= E_{1j}B_i + b_3\bar{B}_F D_i, \\
\Gamma_{75} &= E_{2j}B_i + b_4\bar{B}_F D_i, \\
\Gamma_{77} &= -b_4K - b_4K^T, \\
\Gamma_{85} &= H_i - \bar{D}_F D_i + W_{1j}B_i.
\end{aligned}
$$

Moreover, the gain matrices of the H_∞ filter (5.3) *are given by*

$$
A_F = K^{-1}\bar{A}_F,\ B_F = K^{-1}\bar{B}_F,\ C_F = \bar{C}_F,\ D_F = \bar{D}_F. \tag{5.94}
$$

Theorems 5.8 and 5.9 present new conditions for designing the H_∞ filter (5.3), which introduces more auxiliary matrix variables than Lemma 5.4. Compared with Lemma 5.4, the proposed design conditions in Theorems 5.8 and 5.9 are less conservative. The following theorem is introduced to describe the relationship between Lemma 5.4 and Theorems 5.8 and 5.9.

Theorem 5.10 *If the condition in Lemma 5.4 is satisfied, the conditions in Theorems 5.8 and 5.9 are also feasible.*

Proof The proof from Lemma 5.4 to Theorem 5.8 directly follows the proof of Theorem 5.5. In the following, we give the proof for Lemma 5.4 to Theorem 5.9.

For simplicity, we first consider the H_∞ performance analysis criterions (5.60) and (5.90). Note that when $G(\theta) = F(\theta) = G$, $S(\theta) = E(\theta) = S$, $W(\theta) = N(\theta) = 0$, the inequality (5.90) recedes to (5.60). Defining $M(\theta) = M = \begin{bmatrix} b_6 K & 0 \\ 0 & b_6 K \end{bmatrix}$ and follows the proof of Theorem 5.5, it completes the proof. □

5.3 Numerical Example

To illustrate the effectiveness of the proposed filter design methods, we consider a numerical example. The example has been discussed in our earliest work [2]. Consider the discrete-time system (5.1) with the following parameters [3, 6]

$$
\begin{aligned}
A(\theta) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\delta \end{bmatrix}, & B(\theta) &= \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix}, \\
C(\theta) &= \begin{bmatrix} -100 & 10 \end{bmatrix}, & D(\theta) &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
L(\theta) &= [\,1\ 0\,], & H(\theta) &= 0,
\end{aligned}
$$

where δ is the uncertain parameter satisfying $|\delta| \leq 0.45$. This uncertain system can be modeled with a two-vertex polytope. Now, we consider the design problem of the H_∞ filter (5.3) for this system.

By using Lemma 5.2, [5, 6], the minimum H_∞ performances $\gamma_{\min} = 1.7030$, $\gamma_{\min} = 2.1558$, and $\gamma_{\min} = 3.2065$ are obtained, respectively. However, applying Corollary 5.1 with $b_1 = -1.37, b_2 = -0.20, b_3 = -4.40, b_4 = 1.25$, and $b_5 = 0.28$, we can find the minimum H_∞ performance $\gamma_{\min} = 1.6355$. From this comparison, it can be seen that the filter design condition in Corollary 5.1 is much less conservative than the existing results. Of course, the effect of Theorem 5.2 is better from theoretical point of view.

5.4 Conclusion

In this chapter, the problem of robust H_∞ filtering has been studied for both discrete-time systems and continuous-time uncertain systems based on the parameter-dependent Lyapunov function approach. Sufficient design conditions for the H_∞ filter have been proposed in an LMI framework, which guarantees the filtering error system to be asymptotically stable and has a prescribed H_∞ performance. Compared with the existing results concerning H_∞ filter design, the proposed conditions are less conservative. Besides, these are proposed design conditions which are obtained by introducing more auxiliary matrix variables, the LMI decoupling approach has also been applied to design robust H_∞ filters. One should note that in the new design conditions which are given by the LMI decoupling approach, the structures of added auxiliary matrix variables are free. It seems that the LMI decoupling approach might produce less conservative design results.

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Chapter 6
With Other Types of Uncertainties

Abstract This chapter studies the problems of output feedback H_∞ control and filtering for discrete-time linear systems with other types of uncertainties. Unlike usual polytopic uncertainties and norm-bounded uncertainties, this chapter is toward systems with feedback uncertainties and Frobenius norm-bounded uncertainties. Attention is focused on the design of an output feedback controller (filter) such that the closed-loop system (filtering error system) preserves a prescribed H_∞ performance, where the system matrices or the controller (observer, filter) to be designed are assumed to have gained variations. Sufficient conditions for the H_∞ controller (filter) design are proposed in terms of LMIs. When these LMIs are feasible, an explicit expression of the desired controller (filter) is given. Numerical examples will be given to show the efficiency of the proposed design methods.

Keywords Discrete-time systems · Feedback uncertainties · Frobenius norm-bounded uncertainties · Linear matrix inequalities (LMIs)

6.1 With Feedback Uncertainties

6.1.1 Robust Output Feedback H_∞ Control

6.1.1.1 For Type I

Consider the following linear discrete-time system:

$$
\begin{aligned}
x(k+1) &= A\big(I - X_A\Delta_A(k)Y_A\big)^{-1}x(k) + B\big(I - X_B\Delta_B(k)Y_B\big)^{-1}u(k) \\
&\quad + E(I - X_E\Delta_B(k)Y_B)^{-1}w(k), \\
z(k) &= C_1\big(I - X_{C1}\Delta_{C1}(k)Y_{C1}\big)^{-1}x(k) + D\big(I - X_D\Delta_D(k)Y_D\big)^{-1}u(k)
\end{aligned}
$$

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9_6,

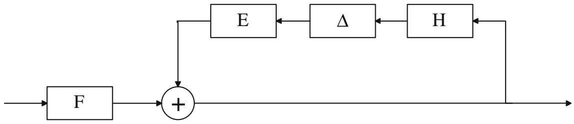

Fig. 6.1 Feedback uncertainty (Type I)

$$
\begin{aligned}
&+ F\big(I - X_F\Delta_F(k)Y_F\big)^{-1}w(k),\\
y(k) =& C_2\big(I - X_{C2}\Delta_{C2}(k)Y_{C2}\big)^{-1}x(k) + H\big(I - X_H\Delta_H(k)Y_H\big)^{-1}w(k).
\end{aligned}
\tag{6.1}
$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, and $y(k) \in \mathscr{R}^p$ is the measurement output. For system matrices $\beta = A, B, E, C_1, D, F, C_2, H$, X_β, and Y_β are constant matrices with appropriate dimensions, $\Delta_\beta(k)$ are uncertain matrices bounded such as $\Delta_\beta^T(k)\Delta_\beta(k) \le I$.

Remark 6.1 In the uncertain system (6.1), the uncertainties is referred to as a type of feedback uncertainties. This type of uncertainties can represent low frequency errors produced by parameter variations with operating conditions and aging [16]. The block diagram for representation of the type of feedback uncertainties is given in Fig 6.1, in which the symbols E, H, and F are nothing to do with the system matrices.

For simplicity, we can rewrite (6.1) as follows:

$$
\begin{aligned}
x(k+1) &= A\delta_A x(k) + B\delta_B u(k) + E\delta_E w(k),\\
z(k) &= C_1\delta_{C1}x(k) + D\delta_D u(k) + F\delta_F w(k),\\
y(k) &= C_2\delta_{C2}x(k) + H\delta_H w(k),
\end{aligned}
\tag{6.2}
$$

with $\delta_\beta = \big(I - X_\beta\Delta_\beta(k)Y_\beta\big)^{-1}$, $\beta = A, B, E, C_1, D, F, C_2, H$.

For the output feedback H_∞ control of the system (6.1), we only study the static output feedback control in this chapter. Let us consider a static output feedback controller

$$
u(k) = Ky(k) = K\big(C_2\delta_{C2}x(k) + H\delta_H w(k)\big),
\tag{6.3}
$$

then, the closed-loop system is given as

$$
\begin{aligned}
x(k+1) &= A\delta_A x(k) + B\delta_B K\big(C_2\delta_{C2}x(k) + H\delta_H w(k)\big) + E\delta_E w(k),\\
z(k) &= C_1\delta_{C1}x(k) + D\delta_D K\big(C_2\delta_{C2}x(k) + H\delta_H w(k)\big) + F\delta_F w(k).
\end{aligned}
\tag{6.4}
$$

In this chapter, for the closed-loop system (6.4), we only consider the case $D = 0$ to design the output feedback controller (6.3)

First, by following the known conclusion given in Theorem 2.1 for designing the static output feedback controllers, we can see that the closed-loop system (6.4) with $D = 0$ is asymptotically stable with the H_∞ performance γ if there exist matrices P, G, J, M, N, V, and U, scalar β such that

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA\delta_A + MVC_2\delta_{C2} & GE\delta_E + MVH\delta_H & \mathscr{G} + J & * & * & * \\ SC_1\delta_{C1} & SF\delta_F & 0 & -S - S^T + I & * & * \\ NVC_2\delta_{C2} & NVH\delta_H & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} \end{bmatrix} < 0, \tag{6.5}$$

holds, where

$$\begin{aligned} \mathscr{G} &= -G - G^T + P, \\ \Sigma_1 &= -\beta NU - \beta U^T N^T, \\ \Sigma_2 &= GB\delta_B - MU. \end{aligned}$$

On the other hand, by Lemma 1.16 with

$$\bar{A} = I, \quad \bar{B} = -X_\beta \Delta_\beta(k), \quad \bar{C} = I, \quad \bar{D} = Y_\beta,$$

we have

$$\begin{aligned} \delta_\beta &= \big(I - X_\beta \Delta_\beta(k) Y_\beta\big)^{-1} \\ &= I + X_\beta \Delta_\beta(k)\big(I - Y_\beta X_\beta \Delta_\beta(k)\big)^{-1} Y_\beta \\ &= I + \nu_\beta, \ \ \beta = A,\ B,\ E,\ C_1,\ F,\ C_2,\ H. \end{aligned} \tag{6.6}$$

By considering the description in (6.6), (6.5) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA + MVC_2 & GE + MVH & \mathscr{G} + J & * & * & * \\ SC_1 & SF & 0 & -S - S^T + I & * & * \\ NVC_2 & NVH & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ GA\nu_A + MVC_2\nu_{C2} & GE\nu_E + MVH\nu_H & 0 & * & * & * \\ SC_1\nu_{C1} & SF\nu_F & 0 & 0 & * & * \\ NVC_2\nu_{C2} & NVH\nu_H & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & GB\nu_B & 0 \end{bmatrix} \\ &= \Theta + X\Delta(k)\Lambda(k)Y + Y^T\Lambda^T(k)\Delta(k)X^T < 0, \end{aligned} \tag{6.7}$$

where

$$\Theta = \begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA + MVC_2 & GE + MVH & \mathscr{G} + J & * & * & * \\ SC_1 & SF & 0 & -S - S^T + I & * & * \\ NVC_2 & NVH & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix},$$

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ GAX_A & 0 & GEX_E & 0 & 0 & MVC_2X_{C2} & MVHX_H \\ 0 & 0 & 0 & SC_1X_{C1} & SFX_F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & NVC_2X_{C2} & NVHX_H \\ 0 & GBX_B & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Delta(k) = \begin{bmatrix} \Delta_A(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_B(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_E(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{C1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_F(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{C2}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_H(k) \end{bmatrix},$$

$$\Lambda(k) = \begin{bmatrix} \Lambda_A(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_B(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_E(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{C1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_F(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{C2}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_H(k) \end{bmatrix},$$

$$\Lambda_\beta(k) = \left(I - Y_\beta X_\beta \Delta_\beta(k)\right)^{-1}, \beta = A, B, E, C_1, F, C_2, H,$$

$$Y = \begin{bmatrix} Y_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_B & 0 \\ 0 & Y_E & 0 & 0 & 0 & 0 \\ Y_{C1} & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_F & 0 & 0 & 0 & 0 \\ Y_{C2} & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_H & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Lemma 1.11 for a scalar $\varepsilon > 0$, it follows that:

$$X\Delta(k)\Lambda(k)Y + Y^T\Lambda^T(k)\Delta(k)X^T \le \frac{1}{\varepsilon}XX^T + \varepsilon Y^T\Lambda^T(k)\Lambda(k)Y. \tag{6.8}$$

By Schur complement, the matrix inequality (6.7) can be guaranteed by

$$\begin{bmatrix} \Theta & * & * \\ X^T & -\varepsilon I & * \\ Y & 0 & -\left(\varepsilon\Lambda^T(k)\Lambda(k)\right)^{-1} \end{bmatrix} < 0. \tag{6.9}$$

It is worth noting that

$$\begin{aligned}
&-\left(\varepsilon\Lambda^T(k)\Lambda(k)\right)^{-1} \\
&\quad = -\Lambda^{-1}(k)\varepsilon^{-1}I\Lambda^{-T}(k) \\
&\quad \le -\Lambda^{-1}(k) - \Lambda^{-T}(k) + \varepsilon I \\
&\quad = -\begin{bmatrix} \Lambda_A^{-1}(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_B^{-1}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_E^{-1}(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{C1}^{-1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_F^{-1}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{C2}^{-1}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_H^{-1}(k) \end{bmatrix} \\
&\qquad - \begin{bmatrix} \Lambda_A^{-1}(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_B^{-1}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_E^{-1}(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_{C1}^{-1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_F^{-1}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_{C2}^{-1}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_H^{-1}(k) \end{bmatrix}^T + \varepsilon I \\
&\quad = \begin{bmatrix} \Phi_A(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi_B(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi_E(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi_{C1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi_F(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{C2}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi_H(k) \end{bmatrix} \\
&\qquad + \begin{bmatrix} \Phi_A(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi_B(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi_E(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi_{C1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Phi_F(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Phi_{C2}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi_H(k) \end{bmatrix}^T - 2I + \varepsilon I,
\end{aligned} \tag{6.10}$$

where $\Phi_\beta(k) = Y_\beta X_\beta \Delta_\beta(k)$, $\beta = A, B, E, C_1, F, C_2, H$.

Moreover, one can be known that

$$\bar{\Delta}(k) = \begin{bmatrix} \Delta_A(k) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_B(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_E(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_{C1}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_F(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_{C2}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_H(k) \end{bmatrix}, \ \bar{\Delta}(k)\bar{\Delta}^T(k) \le I. \tag{6.11}$$

Then, based on Lemma 1.11 with a scalar $\bar{\varepsilon} > 0$, from the end of (6.10), we have

$$\begin{aligned} &\begin{bmatrix} Y_A X_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_B X_B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_E X_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{C1} X_{C1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_F X_F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_{C2} X_{C2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_H X_H \end{bmatrix} \bar{\Delta}(k) I \\ &+ I\bar{\Delta}^T(k) \begin{bmatrix} Y_A X_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_B X_B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_E X_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{C1} X_{C1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_F X_F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_{C2} X_{C2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_H X_H \end{bmatrix}^T - 2I + \varepsilon I \\ &\le -2I + \varepsilon I + \bar{\varepsilon}^{-1} \bar{X} \bar{X}^T + \bar{\varepsilon} I, \end{aligned} \tag{6.12}$$

where

$$\bar{X} = \begin{bmatrix} Y_A X_A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_B X_B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_E X_E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_{C1} X_{C1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_F X_F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_{C2} X_{C2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_H X_H \end{bmatrix}.$$

Obviously, (6.9) is satisfied if the following matrix inequality holds

$$\begin{bmatrix} \Theta & * & * \\ X^T & -\varepsilon I & * \\ Y & 0 & -2I + \varepsilon I + \bar{\varepsilon}^{-1} \bar{X} \bar{X}^T + \bar{\varepsilon} I \end{bmatrix} < 0. \tag{6.13}$$

Using Schur complement to (6.13), it yields

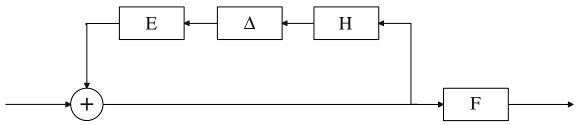

Fig. 6.2 Feedback uncertainty (Type II)

$$\begin{bmatrix} \Theta & * & * & * \\ X^T & -\varepsilon I & * & * \\ Y & 0 & -2I+\varepsilon I+\bar{\varepsilon} I & * \\ 0 & 0 & \bar{X}^T & \bar{\varepsilon} I \end{bmatrix} < 0. \tag{6.14}$$

So far, the design condition for the static output feedback H_∞ controller (6.3) is generalized in the following theorem.

Theorem 6.1 *Consider the closed-loop system* (6.4) *with* $D = 0$ *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β, *exist matrices* U, V, P, J, *and* G, *scalars* ε *and* $\bar{\varepsilon}$ *such that LMI* (6.14) *holds. Furthermore, the static output feedback* H_∞ *controller gain matrix in* (6.3) *can be given by*

$$K = U^{-1}V. \tag{6.15}$$

6.1.1.2 For Type II

Consider the following discrete-time linear system

$$\begin{aligned} x(k+1) &= \big(I - X_A\Delta_A(k)Y_A\big)^{-1}Ax(k) + \big(I - X_B\Delta_B(k)Y_B\big)^{-1}Bu(k) \\ &\quad + \big(I - X_E\Delta_E(k)Y_E\big)^{-1}Ew(k), \\ z(k) &= \big(I - X_{C1}\Delta_{C1}(k)Y_{C1}\big)^{-1}C_1x(k) + \big(I - X_D\Delta_D(k)Y_D\big)^{-1}Du(k) \\ &\quad + \big(I - X_F\Delta_F(k)Y_F\big)^{-1}Fw(k), \\ y(k) &= \big(I - X_{C2}\Delta_{C2}(k)Y_{C2}\big)^{-1}C_2x(k) + \big(I - X_H\Delta_H(k)Y_H\big)^{-1}Hw(k). \end{aligned} \tag{6.16}$$

Remark 6.2 In the uncertain system (6.16), the uncertainties are referred to as a type of feedback uncertainties. This type of uncertainties can represent low frequency errors produced by parameter variations with operating conditions and aging [16]. The block diagram for representation of the type of feedback uncertainties is given in Fig 6.2, in which the symbols E, H, and F are nothing to do with the system matrices.

Remark 6.3 In the uncertain system (6.16), the type of feedback uncertainties can be apply to handle a class of nonlinear systems by using an uncertain linear model.

The following modeling procedure is given to clarify this point.

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= \frac{-9.8x_1(t)+6x_2(t)}{4-3\cos^2\big(x_1(t)\big)}+w(t)\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}&\dot{x}_1(t) = x_2(t)\\ &\Big(4-3\cos^2\big(x_1(t)\big)\Big)\dot{x}_2(t) = -9.8x_1(t)+6x_2(t)+\Big(4-3\cos^2\big(x_1(t)\big)\Big)w(t)\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}&\dot{x}_1(t) = x_2(t)\\ &\Big(1-\tfrac{3}{4}\cos^2\big(x_1(t)\big)\Big)\dot{x}_2(t) = -\tfrac{9.8}{4}x_1(t)+\tfrac{6}{4}x_2(t)+\Big(1-\tfrac{3}{4}\cos^2\big(x_1(t)\big)\Big)w(t)\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\begin{bmatrix}1 & 0\\ 0 & 1-\frac{3}{4}\cos^2\big(x_1(t)\big)\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -\frac{9.8}{4} & \frac{6}{4}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}0\\ 1-\frac{3}{4}\cos^2\big(x_1(t)\big)\end{bmatrix}w(t)\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1-\frac{3}{4}\cos^2\big(x_1(t)\big)\end{bmatrix}^{-1}\begin{bmatrix}0 & 1\\ -\frac{9.8}{4} & \frac{6}{4}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ &\quad+\begin{bmatrix}1 & 0\\ 0 & 1-\frac{3}{4}\cos^2\big(x_1(t)\big)\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1-\frac{3}{4}\cos^2\big(x_1(t)\big)\end{bmatrix}w(t)\end{aligned}$$

$$\Downarrow$$

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \left(I-\begin{bmatrix}0\\ \frac{3}{4}\end{bmatrix}\Delta(t)\begin{bmatrix}0 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}0 & 1\\ -\frac{9.8}{4} & \frac{6}{4}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}w(t).$$

Similar to the system (6.1) with a static output feedback controller, we give the following closed-loop system for the system (6.16)

$$\begin{aligned}x(k+1) &= \delta_A Ax(k)+\delta_B BK\big(\delta_{C2}C_2x(k)+\delta_H Hw(k)\big)+\delta_E Ew(k),\\ z(k) &= \delta_{C1}C_1x(k)+\delta_D DK\big(C_2\delta_{C2}x(k)+\delta_H Hw(k)\big)+\delta_F Fw(k).\end{aligned}\tag{6.17}$$

where δ_β, $\beta = A, B, E, C_1, F, C_2, H$ are defined in (6.2).

For the the closed-loop system (6.17), it is asymptotically stable with the H_∞ performance γ if there exist matrices P, G, J, M, N, S, V, and U, scalar β such that

$$\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ G\delta_A A + MV\delta_{C2}C_2 & G\delta_E E + MV\delta_H H & \mathscr{G} + J & * & * & * \\ S\delta_{C1}C_1 & S\delta_F F & 0 & -S - S^T + I & * & * \\ NV\delta_{C2}C_2 & NV\delta_H H & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & \Sigma_2 & -\frac{J}{\beta^2} \end{bmatrix} < 0, \tag{6.18}$$

holds, where

$$\mathscr{G} = -G - G^T + P,$$
$$\Sigma_1 = -\beta NU - \beta U^T N^T,$$
$$\Sigma_2 = G\delta_B B - MU.$$

Combining (6.6), the matrix equality (6.18) becomes

$$\begin{aligned} &\begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA + MVC_2 & GE + MVH & \mathscr{G} + J & * & * & * \\ SC_1 & SF & 0 & -S - S^T + I & * & * \\ NVC_2 & NVH & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ G\nu_A A + MV\nu_{C2}C_2 & G\nu_E E + MV\nu_H H & 0 & * & * & * \\ S\nu_{C1}C_1 & S\nu_F F & 0 & 0 & * & * \\ NV\nu_{C2}C_2 & NV\nu_H H & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & G\nu_B B & 0 \end{bmatrix} \\ &= \Theta + X\Delta(k)\Lambda(k)Y + Y^T\Lambda^T(k)\Delta(k)X^T < 0, \end{aligned} \tag{6.19}$$

where $\Delta(k)$, $\Lambda(k)$ are the same as (6.7), and

$$\Theta = \begin{bmatrix} -P & * & * & * & * & * \\ 0 & -\gamma^2 I & * & * & * & * \\ GA + MVC_2 & GE + MVH & \mathscr{G} + J & * & * & * \\ SC_1 & SF & 0 & -S - S^T + I & * & * \\ NVC_2 & NVH & 0 & 0 & \Sigma_1 & * \\ 0 & 0 & 0 & 0 & GB - MU & -\frac{J}{\beta^2} \end{bmatrix},$$

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ GX_A & 0 & GX_E & 0 & 0 & MVX_{C2} & MVX_H \\ 0 & 0 & 0 & SX_{C1} & SX_F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & NVX_{C2} & NVX_H \\ 0 & GX_B & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} Y_A A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Y_B B & 0 \\ 0 & Y_E E & 0 & 0 & 0 & 0 \\ Y_{C1} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_F F & 0 & 0 & 0 & 0 \\ Y_{C2} C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_H H & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Following the same derivation of Theorem 6.1, we can establish the following robust output feedback H_∞ control design result for the the closed-loop system (6.17).

Theorem 6.2 *Consider the closed-loop system* (6.17) *with* $D = 0$ *and give a scalar* $\gamma > 0$*. Then the system is asymptotically stable with the* H_∞ *performance* γ *if, for known matrices* M, N *and scalar* β*, exist matrices* U, V, P, J, *and* G, *scalars* ε *and* $\bar{\varepsilon}$ *such that LMI* (6.14) *holds. Furthermore, the static output feedback* H_∞ *controller gain matrix in* (6.3) *can be given by* (6.15).

6.1.2 Robust H_∞ Filtering

Consider the following discrete-time linear system

For Type I

$$\begin{aligned} x(k+1) &= A\big(I - H_A\Delta_A(k)E_A\big)^{-1}x(k) + B\big(I - H_B\Delta_B(k)E_B\big)^{-1}w(k), \\ y(k) &= C\big(I - H_C\Delta_C(k)E_C\big)^{-1}x(k) + D\big(I - H_D\Delta_D(k)E_D\big)^{-1}w(k), \\ z(k) &= L\big(I - H_L\Delta_L(k)E_L\big)^{-1}x(k), \end{aligned} \tag{6.20}$$

For Type II

$$\begin{aligned} x(k+1) &= \big(I - H_A\Delta_A(k)E_A\big)^{-1}Ax(k) + \big(I - H_B\Delta_B(k)E_B\big)^{-1}Bw(k), \\ y(k) &= \big(I - H_C\Delta_C(k)E_C\big)^{-1}Cx(k) + \big(I - H_D\Delta_D(k)E_D\big)^{-1}Dw(k), \\ z(k) &= \big(I - H_L\Delta_L(k)E_L\big)^{-1}Lx(k), \end{aligned} \tag{6.21}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $w(k) \in \mathscr{R}^m$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0,\ \infty)$, $z(k) \in \mathscr{R}^q$ is the signal to be estimated, and $y(k) \in \mathscr{R}^f$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times v}$, $C \in \mathscr{R}^{f\times n}$,

$D \in \mathscr{R}^{f\times v}$, and $L \in \mathscr{R}^{q\times n}$ are system matrices. For $\beta = A, B, C, D, L$, H_β, and E_β are constant matrices with appropriate dimensions, $\Delta_\beta(k)$ are uncertain matrices bounded such as $\Delta_\beta^T(k)\Delta_\beta(k) \le I$.

In this section, we consider the following filter to estimate $z(k)$:

$$\begin{aligned} x_F(k+1) &= A_F x_F(k) + B_F y(k), \\ z_F(k) &= C_F x_F(k) + D_F y(k), \end{aligned} \tag{6.22}$$

where $x_F(k) \in \mathscr{R}^n$ and $z_F(k) \in \mathscr{R}^q$ are the state and output of the filter, respectively. $A_F \in \mathscr{R}^{n\times n}$, $B_F \in \mathscr{R}^{n\times f}$, $C_F \in \mathscr{R}^{q\times n}$, and $D_F \in \mathscr{R}^{q\times f}$ are to be determined filter gain matrices.

By defining the augmented state vector $\psi(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}$ and $e(k) = z(k) - z_F(k)$, we can obtain the following filtering error system:

$$\begin{aligned} \psi(k+1) &= \tilde{A}\psi(k) + \tilde{B}w(k), \\ e(k) &= \tilde{C}\psi(k) + \tilde{D}w(k), \end{aligned} \tag{6.23}$$

where

For Type I

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A\delta_A & 0 \\ B_F C\delta_C & A_F \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} B\delta_B \\ B_F D\delta_D \end{bmatrix}, \\ \tilde{C} &= [\, L\delta_L - D_F C\delta_C \quad -C_F \,], & \tilde{D} &= -D_F D\delta_D, \end{aligned}$$

For Type II

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} \delta_A A & 0 \\ B_F \delta_C C & A_F \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} \delta_B B \\ B_F \delta_D D \end{bmatrix}, \\ \tilde{C} &= [\, \delta_L L - D_F \delta_C C \quad -C_F \,], & \tilde{D} &= -\, D_F \delta_D D, \end{aligned}$$

with $\delta_\beta = \left(I - H_\beta \Delta_\beta(k) E_\beta\right)^{-1}$, $\beta = A, B, C, D, L$.

For the filtering error system (6.23) and a given scalar $\gamma > 0$, if there exists a matrix $P > 0$ satisfying

$$\begin{bmatrix} -P & * & * & * \\ 0 & -\gamma^2 I & * & * \\ G\tilde{A} & G\tilde{B} & -G - G^T + P & * \\ M\tilde{C} & M\tilde{D} & 0 & -M - M^T + I \end{bmatrix} < 0, \tag{6.24}$$

then the prescribed H_∞ performance $\gamma > 0$ is guaranteed. The auxiliary matrix variables G and M provide extra free dimensions in the solution space for the H_∞ filtering problem. To facilitate the design of H_∞ filters, we partition P and G in the following blocked matrices:

$$P = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix},$$
$$G = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_2 \end{bmatrix}, \tag{6.25}$$

and G_2 is nonsingular without loss of generality.

In sequel, with (6.24) in hands, we will derive conditions for designing the filter (6.22) such that the filtering error system (6.23) is asymptotically stable with H_∞ performance γ. In the following, we will treat Type I and Type II separately.

6.1.2.1 Type I

Combining (6.23), (6.24) for Type I and defining $\bar{A}_F = G_2 A_F$, $\bar{B}_F = G_2 B_F$, $\bar{C}_F = MC_F$, and $\bar{D}_F = MD_F$, we obtain

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A\delta_A + \bar{B}_F C\delta_C & \bar{A}_F & G_1 B\delta_B + \bar{B}_F D\delta_D & -G_1 - G_1^T + P_1 & * & * \\ G_3 A\delta_A + \bar{B}_F C\delta_C & \bar{A}_F & G_3 B\delta_B + \bar{B}_F D\delta_D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ ML\delta_L - \bar{D}_F C\delta_C & -\bar{C}_F & -\bar{D}_F D\delta_D & 0 & 0 & \Xi_{66} \end{bmatrix} < 0, \tag{6.26}$$

where

$$\Xi_{55} = -G_2 - G_2^T + P_3,$$
$$\Xi_{66} = -M - M^T + I.$$

As (6.6), we can know that

$$\begin{aligned} \delta_\beta &= \left(I - H_\beta \Delta_\beta(k) E_\beta\right)^{-1} \\ &= I + H_\beta \Delta_\beta(k)\left(I - E_\beta H_\beta \Delta_\beta(k)\right)^{-1} E_\beta = I + \nu_\beta,\ \ \beta = A, B, C, D, L. \end{aligned} \tag{6.27}$$

Then, it knows that the inequality (6.26) is equivalent to

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A + \bar{B}_F C & \bar{A}_F & G_1 B + \bar{B}_F D & -G_1 - G_1^T + P_1 & * & * \\ G_3 A + \bar{B}_F C & \bar{A}_F & G_3 B + \bar{B}_F D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ ML - \bar{D}_F C & -\bar{C}_F & -\bar{D}_F D & 0 & 0 & \Xi_{66} \end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix}
0 & * & * & * & * & * \\
0 & 0 & * & * & * & * \\
0 & 0 & 0 & * & * & * \\
G_1 A \nu_A + \bar{B}_F C \nu_C & 0 & G_1 B \nu_B + \bar{B}_F D \nu_D & 0 & * & * \\
G_3 A \nu_A + \bar{B}_F C \nu_C & 0 & G_3 B \nu_B + \bar{B}_F D \nu_D & 0 & 0 & * \\
M L \nu_L - \bar{D}_F C \nu_C & 0 & -\bar{D}_F D \nu_D & 0 & 0 & 0
\end{bmatrix} \\
&= \begin{bmatrix}
-P_1 & * & * & * & * & * \\
-P_2 & -P_3 & * & * & * & * \\
0 & 0 & -\gamma^2 I & * & * & * \\
G_1 A + \bar{B}_F C & \bar{A}_F & G_1 B + \bar{B}_F D & -G_1 - G_1^T + P_1 & * & * \\
G_3 A + \bar{B}_F C & \bar{A}_F & G_3 B + \bar{B}_F D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\
ML - \bar{D}_F C & -\bar{C}_F & -\bar{D}_F D & 0 & 0 & \Xi_{66}
\end{bmatrix} \\
&\quad + X_A \Delta_A(k)\big(I - E_A H_A \Delta_A(k)\big)^{-1} Y_A + Y_A^T \big(I - E_A H_A \Delta_A(k)\big)^{-T} \Delta_A^T(k) X_A^T \\
&\quad + X_B \Delta_B(k)\big(I - E_B H_B \Delta_B(k)\big)^{-1} Y_B + Y_B^T \big(I - E_B H_B \Delta_B(k)\big)^{-T} \Delta_B^T(k) X_B^T \\
&\quad + X_C \Delta_C(k)\big(I - E_C H_C \Delta_C(k)\big)^{-1} Y_C + Y_C^T \big(I - E_C H_C \Delta_C(k)\big)^{-T} \Delta_C^T(k) X_C^T \\
&\quad + X_D \Delta_D(k)\big(I - E_D H_D \Delta_D(k)\big)^{-1} Y_D + Y_D^T \big(I - E_D H_D \Delta_D(k)\big)^{-T} \Delta_D^T(k) X_D^T \\
&\quad + X_L \Delta_L(k)\big(I - E_L H_L \Delta_L(k)\big)^{-1} Y_L + Y_L^T \big(I - E_L H_L \Delta_L(k)\big)^{-T} \Delta_L^T(k) X_L^T,
\end{aligned}
\tag{6.28}
$$

where

$$
X_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 A H_A \\ G_3 A H_A \\ 0 \end{bmatrix}, \quad Y_A = [\, E_A \; 0 \; 0 \; 0 \; 0 \; 0 \,],
$$

$$
X_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 B H_B \\ G_3 B H_B \\ 0 \end{bmatrix}, \quad Y_B = [\, 0 \; 0 \; E_B \; 0 \; 0 \; 0 \,],
$$

$$
X_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F C H_C \\ \bar{B}_F C H_C \\ -\bar{D}_F C H_C \end{bmatrix}, \quad Y_C = [\, E_C \; 0 \; 0 \; 0 \; 0 \; 0 \,],
$$

$$X_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F D H_D \\ \bar{B}_F D H_D \\ -\bar{D}_F D H_D \end{bmatrix}, \quad Y_D = [0\ 0\ E_D\ 0\ 0\ 0],$$

$$X_L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ MLH_L \end{bmatrix}, \quad Y_L = [\,E_L\ 0\ 0\ 0\ 0\ 0].$$

It should be noted that different from the robust output feedback H_∞ control design presented in Sect. 6.1.1, for the of partition of matrix inequality (6.26), we use an independent partition law for each uncertain term. Obviously, such partition law may bring less-conservative design results.

By Lemma 1.11 for positive scalars $\varepsilon_A,\ \varepsilon_B,\ \varepsilon_C, \varepsilon_D$, and ε_L, we have

$$\begin{aligned} & X_\beta \Delta_\beta(k)\big(I - E_\beta H_\beta \Delta_\beta(k)\big)^{-1} Y_\beta + Y_\beta^T \big(I - E_\beta H_\beta \Delta_\beta(k)\big)^{-T} \Delta_\beta^T(k) X_\beta^T \\ & \quad \le \frac{1}{\varepsilon_\beta} X_\beta X_\beta^T + \varepsilon_\beta Y_\beta^T \big(I - E_\beta H_\beta \Delta_\beta(k)\big)^{-T} \big(I - E_\beta H_\beta \Delta_\beta(k)\big)^{-1} Y_\beta, \\ & \quad \beta = A, B, C, D, L. \end{aligned} \tag{6.29}$$

Then, (6.28) holds if the following condition is satisfied:

$$\begin{aligned} & \begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A + \bar{B}_F C & \bar{A}_F & G_1 B + \bar{B}_F D & -G_1 - G_1^T + P_1 & * & * \\ G_3 A + \bar{B}_F C & \bar{A}_F & G_3 B + \bar{B}_F D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ ML - \bar{D}_F C & -\bar{C}_F & -\bar{D}_F D & 0 & 0 & \Xi_{66} \end{bmatrix} \\ & + \tfrac{1}{\varepsilon_A} X_A X_A^T + \varepsilon_A Y_A^T \big(I - E_A H_A \Delta_A(k)\big)^{-T} \big(I - E_A H_A \Delta_A(k)\big)^{-1} Y_A, \\ & + \tfrac{1}{\varepsilon_B} X_B X_B^T + \varepsilon_B Y_B^T \big(I - E_B H_B \Delta_B(k)\big)^{-T} \big(I - E_B H_B \Delta_B(k)\big)^{-1} Y_B, \\ & + \tfrac{1}{\varepsilon_C} X_C X_C^T + \varepsilon_C Y_C^T \big(I - E_C H_C \Delta_C(k)\big)^{-T} \big(I - E_C H_C \Delta_C(k)\big)^{-1} Y_C, \\ & + \tfrac{1}{\varepsilon_D} X_D X_D^T + \varepsilon_D Y_D^T \big(I - E_D H_D \Delta_D(k)\big)^{-T} \big(I - E_D H_D \Delta_D(k)\big)^{-1} Y_D, \\ & + \tfrac{1}{\varepsilon_L} X_L X_L^T + \varepsilon_L Y_L^T \big(I - E_L H_L \Delta_L(k)\big)^{-T} \big(I - E_L H_L \Delta_L(k)\big)^{-1} Y_L < 0. \end{aligned} \tag{6.30}$$

By Schur complement to (6.30), which leads to

$$\begin{bmatrix} \Theta_{11} & * & * \\ \Theta_{21} & \Theta_{22} & * \\ \Theta_{31} & 0 & \Theta_{33} \end{bmatrix} < 0, \tag{6.31}$$

where

$$\Theta_{11} = \begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A + \bar{B}_F C & \bar{A}_F & G_1 B + \bar{B}_F D & -G_1 - G_1^T + P_1 & * & * \\ G_3 A + \bar{B}_F C & \bar{A}_F & G_3 B + \bar{B}_F D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ ML - \bar{D}_F C & -\bar{C}_F & -\bar{D}_F D & 0 & 0 & \Xi_{66} \end{bmatrix},$$

$$\Theta_{21} = \begin{bmatrix} 0 & 0 & 0 & H_A^T A^T G_1^T & H_A^T A^T G_3^T & 0 \\ 0 & 0 & 0 & H_B^T B^T G_1^T & H_B^T B^T G_3^T & 0 \\ 0 & 0 & 0 & H_C^T C^T \bar{B}_F^T & H_C^T C^T \bar{B}_F^T & -H_C^T C^T \bar{D}_F^T \\ 0 & 0 & 0 & H_D^T D^T \bar{B}_F^T & H_D^T D^T \bar{B}_F^T & -H_D^T D^T \bar{D}_F^T \\ 0 & 0 & 0 & 0 & 0 & H_L^T L^T M^T \end{bmatrix},$$

$$\Theta_{22} = \begin{bmatrix} -\varepsilon_A I & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon_B I & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_C I & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_D I & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon_L I \end{bmatrix},$$

$$\Theta_{31} = \begin{bmatrix} E_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_B & 0 & 0 & 0 \\ E_C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_D & 0 & 0 & 0 \\ E_L & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta_{33} = \begin{bmatrix} \mathscr{F}_A & 0 & 0 & 0 & 0 \\ 0 & \mathscr{F}_B & 0 & 0 & 0 \\ 0 & 0 & \mathscr{F}_C & 0 & 0 \\ 0 & 0 & 0 & \mathscr{F}_D & 0 \\ 0 & 0 & 0 & 0 & \mathscr{F}_L \end{bmatrix},$$

and

$$\mathscr{F}_\beta = -\frac{1}{\varepsilon_\beta}\big(I - E_\beta H_\beta \Delta_\beta(k)\big)\big(I - E_\beta H_\beta \Delta_\beta(k)\big)^T, \ \beta = A, B, C, D, L. \quad (6.32)$$

By Lemma 1.17, it is worth noting that

$$\begin{aligned} \mathscr{F}_\beta &= -\big(I - E_A H_A \Delta_\beta(k)\big)\varepsilon_\beta^{-1} I \big(I - E_\beta H_\beta \Delta_\beta(k)\big)^T \\ &\le \varepsilon_\beta I - 2I + \frac{1}{\varepsilon_{\beta\beta}} E_\beta H_\beta H_\beta^T E_\beta^T + \varepsilon_{\beta\beta} I, \beta = A, B, C, D, L. \end{aligned} \quad (6.33)$$

Then, (6.31) can be guaranteed by

$$\begin{bmatrix} \Theta_{11} & * & * \\ \Theta_{21} & \Theta_{22} & * \\ \Theta_{31} & 0 & \Xi + \Xi_{\Delta} \end{bmatrix} < 0, \tag{6.34}$$

where

$$\Xi = \begin{bmatrix} (\varepsilon_A - 2 + \varepsilon_{AA})I & 0 & 0 & 0 & 0 \\ 0 & (\varepsilon_B - 2 + \varepsilon_{BB})I & 0 & 0 & 0 \\ 0 & 0 & (\varepsilon_C - 2 + \epsilon_{CC})I & 0 & 0 \\ 0 & 0 & 0 & (\varepsilon_D - 2 + \varepsilon_{DD})I & 0 \\ 0 & 0 & 0 & 0 & (\varepsilon_L - 2 + \varepsilon_{LL})I \end{bmatrix},$$

$$\begin{aligned} \Xi_{\Delta} &= \begin{bmatrix} \frac{1}{\varepsilon_{AA}} E_A H_A H_A^T E_A^T & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\varepsilon_{BB}} E_B H_B H_B^T E_B^T & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\varepsilon_{CC}} E_C H_C H_C^T E_C^T & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\varepsilon_{DD}} E_D H_D H_D^T E_D^T & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\varepsilon_{LL}} E_L H_L H_L^T E_L^T \end{bmatrix} \\ &= \begin{bmatrix} E_A H_A & 0 & 0 & 0 & 0 \\ 0 & E_B H_B & 0 & 0 & 0 \\ 0 & 0 & E_C H_C & 0 & 0 \\ 0 & 0 & 0 & E_D H_D & 0 \\ 0 & 0 & 0 & 0 & E_L H_L \end{bmatrix} \begin{bmatrix} \varepsilon_{AA} I & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{BB} I & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{CC} I & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{DD} I & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{LL} I \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} E_A H_A & 0 & 0 & 0 & 0 \\ 0 & E_B H_B & 0 & 0 & 0 \\ 0 & 0 & E_C H_C & 0 & 0 \\ 0 & 0 & 0 & E_D H_D & 0 \\ 0 & 0 & 0 & 0 & E_L H_L \end{bmatrix}^T . \end{aligned}$$

Applying Schur complement to (6.34) yields

$$\begin{bmatrix} \Theta_{11} & * & * & * \\ \Theta_{21} & \Theta_{22} & * & * \\ \Theta_{31} & 0 & \Xi & * \\ 0 & 0 & \Pi & \Omega \end{bmatrix} < 0, \tag{6.35}$$

where

$$\Omega = \begin{bmatrix} -\varepsilon_{AA} I & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon_{BB} I & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_{CC} I & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon_{DD} I & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon_{LL} I \end{bmatrix},$$

$$\Pi = \begin{bmatrix} E_A H_A & 0 & 0 & 0 & 0 \\ 0 & E_B H_B & 0 & 0 & 0 \\ 0 & 0 & E_C H_C & 0 & 0 \\ 0 & 0 & 0 & E_D H_D & 0 \\ 0 & 0 & 0 & 0 & E_L H_L \end{bmatrix}.$$

At this point, the H_∞ filter design result is summarized in the following theorem.

Theorem 6.3 *Consider the filtering error system (6.23) for Type I and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, P_3, G_1, G_2, G_3, M, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$, scalars ε_A, ε_B, ε_C, ε_D, ε_L, ε_{AA}, ε_{BB}, ε_{CC}, ε_{DD}, and ε_{LL} such that LMI (6.35) holds. Furthermore, the H_∞ filter gain matrices in (6.22) can be given by*

$$A_F = G_2^{-1}\bar{A}_F, \quad B_F = G_2^{-1}\bar{B}_F, \quad C_F = M^{-1}\bar{C}_F, \quad D_F = M^{-1}\bar{D}_F. \tag{6.36}$$

Remark 6.4 It should be noted that the condition (6.35) implies $\mathscr{E}_\beta = \varepsilon_\beta I - 2I + \varepsilon_{\beta\beta} I < 0, \ \beta = A, \ B, \ C, \ D$, the term $2I$ will lead to high conservatism. In the following, we will develop another design method which improves that in Theorem 6.3.

By introducing five invertible slack matrix variables N_A, N_B, N_C, N_D, and N_L, we rewrite the inequality (6.30) as follows:

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1 A + \bar{B}_F C & \bar{A}_F & G_1 B + \bar{B}_F D & -G_1 - G_1^T + P_1 & * & * \\ G_3 A + \bar{B}_F C & \bar{A}_F & G_3 B + \bar{B}_F D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ ML - \bar{D}_F C & -\bar{C}_F & -\bar{D}_F D & 0 & 0 & \Xi_{66} \end{bmatrix}$$

$$+ \frac{1}{\varepsilon_A}\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 A H_A \\ G_3 A H_A \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 A H_A \\ G_3 A H_A \\ 0 \end{bmatrix}^T + \varepsilon_A [\, N_A E_A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T$$

$$\times \left(N_A - N_A E_A H_A \Delta_A(k)\right)^{-T}\left(N_A - N_A E_A H_A \Delta_A(k)\right)^{-1}[\, N_A E_A \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]$$

$$+ \frac{1}{\varepsilon_B}\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 B H_B \\ G_3 B H_B \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 B H_B \\ G_3 B H_B \\ 0 \end{bmatrix}^T + \varepsilon_B [\, 0 \quad 0 \quad N_B E_B \quad 0 \quad 0 \quad 0\,]^T$$

$$\times \left(N_B - N_B E_B H_B \Delta_B(k)\right)^{-T}\left(N_B - N_B E_B H_B \Delta_B(k)\right)^{-1}[\, 0 \quad 0 \quad N_B E_B \quad 0 \quad 0 \quad 0\,]$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon_C}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F C H_C \\ \bar{B}_F C H_C \\ -\bar{D}_F C H_C \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F C H_C \\ \bar{B}_F C H_C \\ -\bar{D}_F C H_C \end{bmatrix}^T + \varepsilon_C [\, N_C E_C \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,]^T \\
&\times \big(N_C - N_C E_C H_C \Delta_C(k)\big)^{-T}\big(N_C - N_C E_C H_C \Delta_C(k)\big)^{-1}[\, N_C E_C \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,] \\
&+\frac{1}{\varepsilon_D}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F D H_D \\ \bar{B}_F D H_D \\ -\bar{D}_F D H_D \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{B}_F D H_D \\ \bar{B}_F D H_D \\ -\bar{D}_F D H_D \end{bmatrix}^T + \varepsilon_D [\, 0 \quad 0 \quad N_D E_D \quad 0 \quad 0 \quad 0 \,]^T \\
&\times \big(N_D - N_D E_D H_D \Delta_D(k)\big)^{-T}\big(N_D - N_D E_D H_D \Delta_D(k)\big)^{-1}[\, 0 \quad 0 \quad N_D E_D \quad 0 \quad 0 \quad 0 \,] \\
&+\frac{1}{\varepsilon_L}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M L H_L \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ M L H_L \end{bmatrix}^T + \varepsilon_L [\, N_L E_L \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,]^T \\
&\times \big(N_L - N_L E_L H_L \Delta_L(k)\big)^{-T}\big(N_L - N_L E_L H_L \Delta_L(k)\big)^{-1}[\, N_L E_L \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,] < 0.
\end{aligned} \tag{6.37}
$$

Similar to (6.31), (6.37) is satisfied if

$$
\begin{bmatrix} \Theta_{11} & * & * \\ \Theta_{21} & \Theta_{22} & * \\ \Upsilon_{31} & 0 & \Upsilon_{33} \end{bmatrix} < 0, \tag{6.38}
$$

holds, where Θ_{11}, Θ_{21}, and Θ_{22} are the same as (6.31), and

$$
\Upsilon_{31} = \begin{bmatrix} N_A E_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_B E_B & 0 & 0 & 0 \\ N_C E_C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_D E_D & 0 & 0 & 0 \\ N_L E_L & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
$$

$$
\Upsilon_{33} = \begin{bmatrix} \mathscr{W}_A & 0 & 0 & 0 & 0 \\ 0 & \mathscr{W}_B & 0 & 0 & 0 \\ 0 & 0 & \mathscr{W}_C & 0 & 0 \\ 0 & 0 & 0 & \mathscr{W}_D & 0 \\ 0 & 0 & 0 & 0 & \mathscr{W}_L \end{bmatrix},
$$

$$
\mathscr{W}_\beta = -\frac{1}{\varepsilon_\beta}\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)^T, \ \beta = A, B, C, D, L.
$$

By Lemma 1.17, it is worth noting that

$$
\begin{aligned}
\mathscr{W}_\beta &= -\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)\epsilon_\beta^{-1} I \big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)^T \\
&\leq \epsilon_\beta I - N_\beta - N_\beta^T + \frac{1}{\epsilon_{\beta\beta}} N_\beta E_\beta H_\beta H_\beta^T E_\beta^T N_\beta^T + \epsilon_{\beta\beta} I,\ \beta = A, B, C, D, L.
\end{aligned}
\tag{6.39}
$$

Then, (6.38) can be verified by the following condition

$$
\begin{bmatrix}
\Theta_{11} & * & * & * \\
\Theta_{21} & \Theta_{22} & * & * \\
\Upsilon_{31} & 0 & \Theta_{33} & * \\
0 & 0 & \Pi_N & \Omega
\end{bmatrix} < 0,
\tag{6.40}
$$

where

$$
\Theta_{33} = \begin{bmatrix}
\rho_A & 0 & 0 & 0 & 0 \\
0 & \rho_B & 0 & 0 & 0 \\
0 & 0 & \rho_C & 0 & 0 \\
0 & 0 & 0 & \rho_D & 0 \\
0 & 0 & 0 & 0 & \rho_L
\end{bmatrix},\ \rho_\beta = -N_\beta - N_\beta^T + \epsilon_\beta I + \epsilon_{\beta\beta} I,\ A, B, C, D, L,
$$

$$
\Pi_N = \begin{bmatrix}
N_A E_A H_A & 0 & 0 & 0 & 0 \\
0 & N_B E_B H_B & 0 & 0 & 0 \\
0 & 0 & N_C E_C H_C & 0 & 0 \\
0 & 0 & 0 & N_D E_D H_D & 0 \\
0 & 0 & 0 & 0 & N_L E_L H_L
\end{bmatrix},
$$

$$
\Omega = \begin{bmatrix}
-\epsilon_{AA} I & 0 & 0 & 0 & 0 \\
0 & -\epsilon_{BB} I & 0 & 0 & 0 \\
0 & 0 & -\epsilon_{CC} I & 0 & 0 \\
0 & 0 & 0 & -\epsilon_{DD} I & 0 \\
0 & 0 & 0 & 0 & -\epsilon_{LL} I
\end{bmatrix}.
$$

The improved design condition is given by the following corollary.

Corollary 6.1 *Consider the filtering error system (6.23) for Type I and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, P_3, G_1, G_2, G_3, M, N_A, N_B, N_C, N_D, N_L, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$, scalars ε_A, ε_B, ε_C, ε_D, ε_L, ε_{AA}, ε_{BB}, ε_{CC}, ε_{DD}, and ε_{LL} such that LMI (6.40) holds. Furthermore, the H_∞ filter gain matrices in (6.22) can be given by (6.36).*

Remark 6.5 Compared with LMI condition (6.35), (6.40) replaces the identity matrix I by matrix variables N_A, N_B, N_C, N_D, and N_L. In other words, when $N_A = N_B = N_C = N_D = I$, (6.40) reduces (6.35). Thus it can be easily seen that (6.35) is a special case of (6.40) and the condition (6.40) is more relaxed than that (6.35).

6.1.2.2 Type II

For Type II, the matrix inequality (6.26) becomes

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1\delta_A A + \bar{B}_F\delta_C C & \bar{A}_F & G_1\delta_B B + \bar{B}_F\delta_D D & -G_1 - G_1^T + P_1 & * & * \\ G_3\delta_A A + \bar{B}_F\delta_C C & \bar{A}_F & G_3\delta_B B + \bar{B}_F\delta_D D & -G_3 - G_2^T + P_2 & \Xi_{55} & * \\ M\delta_L L - \bar{D}_F\delta_C C & -\bar{C}_F & -\bar{D}_F\delta_D D & 0 & 0 & \Xi_{66} \end{bmatrix} < 0, \tag{6.41}$$

where Ξ_{55} and Ξ_{66} are the same as (6.26).

Following the derivation Theorem 6.3, we do not provide the proof to give the following design result for robust H_∞ filtering with the Type II.

Theorem 6.4 *Consider the filtering error system (6.23) for Type II and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, P_3, G_1, G_2, G_3, M, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, and $\bar{D}_F$, scalars ε_A, ε_B, ε_C, ε_D, ε_L, ε_{AA}, ε_{BB}, ε_{CC}, ε_{DD}, and ε_{LL} such that the following matrix inequality holds*

$$\begin{bmatrix} \Theta_{11} & * & * & * \\ \Theta_{21} & \Theta_{22} & * & * \\ \Theta_{31} & 0 & \Xi & * \\ 0 & 0 & \Pi & \Omega \end{bmatrix} < 0, \tag{6.42}$$

where Θ_{11}, Θ_{22}, Ξ, and Ω are the same as (6.35), and

$$\Theta_{21} = \begin{bmatrix} 0 & 0 & 0 & H_A^T G_1^T & H_A^T G_3^T & 0 \\ 0 & 0 & 0 & H_B^T G_1^T & H_B^T G_3^T & 0 \\ 0 & 0 & 0 & H_C^T \bar{B}_F^T & H_C^T \bar{B}_F^T & -H_C^T \bar{D}_F^T \\ 0 & 0 & 0 & H_D^T \bar{B}_F^T & H_D^T \bar{B}_F^T & -H_D^T \bar{D}_F^T \\ 0 & 0 & 0 & 0 & 0 & H_L^T M^T \end{bmatrix},$$

$$\Theta_{31} = \begin{bmatrix} E_A A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_B B & 0 & 0 & 0 \\ E_C C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_D D & 0 & 0 & 0 \\ E_L L & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Furthermore the filter gain matrices are given by

$$A_F = G_2^{-1}\bar{A}_F, \quad B_F = G_2^{-1}\bar{B}_F, \quad C_F = M^{-1}\bar{C}_F, \text{and} \quad D_F = M^{-1}\bar{D}_F. \tag{6.43}$$

6.1.3 Output Feedback Non-fragile H_∞ Control with Type III

In practical applications and realizations, controllers (filters) do have a certain degree of errors due to finite word length in any digital systems; the imprecision inherent in analog systems and the need for additional tuning of parameters in the final controller (filter) implementation [14], brings a new issue: how to design a controller (filter) such that which is insensitive to some amount of error with respect to its gains, i.e., the controller (filter) is resilient or non-fragile [10]. Recently, non-fragile H_∞ control for systems has been investigated by many researchers. Ren and Zhang [7] was concerned with the problem of non-fragile proportional-plus-derivative (PD) state H_∞ control for a class of uncertain descriptor systems, where the parameter uncertainties are assumed to be time-varying norm-bounded appearing not only in the state matrix but also in the derivative matrix. In [15], the robust H_∞ control and non-fragile control problems for Takagi-Sugeno (T-S) fuzzy systems with linear fractional parametric uncertainties were investigated. Lien [11] investigated the H_∞ non-fragile observer-based controls for continuous dynamical systems, in which types of uncertainties which perturb the gains of control and observer were studied. Li and Jia [6] was devoted to the problems of non-fragile H_∞ and $L_2 - L_\infty$ control for a class of linear systems with time-varying state delay, in which the purpose is to design a dynamic output feedback controller with additive gain variations such that the closed-loop system is asymptotically stable while satisfying a prescribed H_∞ (or $L_2 - L_\infty$) performance level. It should be noted that the above researches on non-fragile control were taken into account the additive uncertainties, multiplicative uncertainties. However, up to now, the non-fragile control problem with feedback uncertainties has not been fully investigated and the relevant results have been very few.

This section studies observer-based non-fragile H_∞ control for discrete-time linear systems. Different from existing results for non-fragile control problems, the proposed ones are toward systems with a class of feedback uncertainties (Type III). Some auxiliary matrix variables are introduced to design the non-fragile controller and observer. Strict LMI conditions guaranteeing the system H_∞ performance are proposed.

Consider the following linear discrete-time dynamical system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ew(k), \\ z(k) &= C_1x(k) + Du(k) + Fw(k), \\ y(k) &= C_2x(k), \end{aligned} \tag{6.44}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^f$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times m}$, $E \in \mathscr{R}^{n\times v}$, $C_1 \in \mathscr{R}^{q\times n}$, $D \in \mathscr{R}^{q\times m}$, $F \in \mathscr{R}^{q\times v}$, and $C_2 \in \mathscr{R}^{p\times n}$ are system matrices.

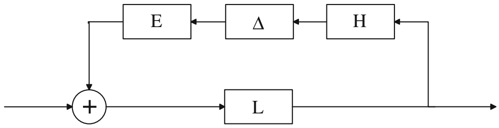

Fig. 6.3 Feedback uncertainty (Type III)

The following observer is proposed to deal with the state estimation of system (6.44):

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L\big(I - H_L\Delta_L(k)E_L L\big)^{-1}\big(y(k) - \hat{y}(k)\big),\\ \hat{y}(k) &= C_2\hat{x}(k),\end{aligned} \tag{6.45}$$

where $\hat{x}(k) \in \mathscr{R}^n$ and $\hat{y}(k) \in \mathscr{R}^p$ are the estimated state and estimated output, respectively. $L \in \mathscr{R}^{n\times c}$ is the observer gain. H_L and E_L are constant matrices with appropriate dimensions, $\Delta_L(k)$ is uncertain matrix bounded such as $\Delta_L^T(k)\Delta_L(k) \leq I$.

Remark 6.6 In the non-fragile observer (6.45), the uncertainties are referred to as feedback uncertainties. Different from additive uncertainties and multiplicative uncertainties, this class of uncertainties can represent low frequency errors produced by parameter variations with operating conditions and aging [16]. The block diagram for the representation of feedback uncertainty is given in Fig. 6.3.

Let us denote the estimation error as

$$e(k) = x(k) - \hat{x}(k). \tag{6.46}$$

From (6.44)–(6.46), we get

$$e(k+1) = x(k+1) - \hat{x}(k+1) = \Big(A - L\big(I - H_L\Delta_L(k)E_L L\big)^{-1}C_2\Big)e(k) + Ew(k). \tag{6.47}$$

Assume the following non-fragile controller is employed to deal with the design of system (6.44):

$$u(k) = K\big(I - H_K\Delta_K(k)E_K K\big)^{-1}\hat{x}(k), \tag{6.48}$$

where K is the controller gain. H_K and E_K are known constant matrices of appropriate dimensions, $\Delta_K(k)$ is uncertain matrix and satisfies $\Delta_K^T(k)\Delta_K(k) \leq I$.

By substituting (6.48) into (6.44), the closed-loop system becomes

$$\begin{aligned}\bar{x}(k+1) &= \mathbf{A}\bar{x}(k) + \mathbf{B}w(k),\\ z(k) &= \mathbf{C}\bar{x}(k) + \mathbf{D}w(k),\end{aligned} \tag{6.49}$$

where $\bar{x}(k) = \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$ and

$$\mathbf{A} = \begin{bmatrix} A + BK\phi_K & -BK\phi_K \\ 0 & A - L\phi_L C_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} E \\ E \end{bmatrix},$$
$$\mathbf{C} = [\, C_1 + DK\phi_K \quad -DK\phi_K \,], \quad \mathbf{D} = F,$$
$$\phi_K = \big(I - H_K \Delta_K(k) E_K K\big)^{-1}, \qquad \phi_L = \big(I - H_L \Delta_L(k) E_L L\big)^{-1}.$$

Without considering the controller and observer gain uncertainties, Lemma 4.2 presents a simple performance analysis result for observer-based output feedback H_∞ control for linear systems. In this study, we extend the analysis result in Lemma 4.2 into the non-fragile H_∞ control. Obviously, the non-fragile control studied in this section will lead to more complicated manipulations for using the LMI technique. Next, we seek methods to obtain strict LMI conditions for designing the non-fragile controller and observer based on the inequality condition Lemma 4.2.

From Lemma 4.2, we can see that the closed-loop system (6.49) is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, G_1, G_2, K, and N_L such that

$$\begin{bmatrix} -G_K - G_K^T + P_K & * & * & * \\ AG_K + BK\phi_K G_K & -P_K & * & * \\ C_1 G_K + DK\phi_K G_K & 0 & -I & * \\ 0 & E^T & F^T & -\gamma^2 I \end{bmatrix} < 0, \tag{6.50}$$

and

$$\begin{bmatrix} -P_L & * \\ G_L A - N_L \phi_L C_2 & -G_L - G_L^T + P_L \end{bmatrix} < 0, \tag{6.51}$$

hold.

Based on the matrix inequalities (6.50) and (6.51), in the following, we will present sufficient conditions for designing the non-fragile H_∞ observer (6.45) and controller (6.48), that is, we will determine the gain matrices L and K such that the closed-loop system (6.49) is asymptotically stable with H_∞ performance γ based on the analysis results obtained in Lemma 4.2.

Theorem 6.5 *Consider the closed-loop system* (6.49) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if there exist matrices* P_K, P_L, G_K, G_L, N_K, *and* N_L, *scalars* σ_K, σ_L, ν_K, *and* ν_L *such that the following matrix inequality holds*

$$\begin{bmatrix} \mathscr{T}_K & * & * & * & * \\ X_K^T & S_K & * & * & * \\ Y_K & 0 & U_K & * & * \\ 0 & 0 & N_K^T & J_K & * \\ 0 & 0 & \sigma_K E_K & 0 & -\nu_K I \end{bmatrix} < 0, \tag{6.52}$$

$$\begin{bmatrix} \mathscr{T}_L & * & * & * \\ X_L^T & -\sigma_L I & * & * \\ Y_L & 0 & V_L & * \\ 0 & 0 & H_L^T N_L^T & -\nu_L I \end{bmatrix} < 0, \tag{6.53}$$

where

$$\begin{aligned}
\mathscr{T}_K &= \begin{bmatrix} -G_K - G_K^T + P_K & * & * & * \\ AG_K + B_i N_K & -P_K & * & * \\ C_1 G_K + DN_K & 0 & -I & * \\ 0 & E^T & F^T & -\gamma^2 I \end{bmatrix}, \\
X_K &= \begin{bmatrix} 0 \\ BN_K \\ DN_K \\ 0 \end{bmatrix}, \\
Y_K &= [\, N_K \quad 0 \quad 0 \quad 0 \,], \\
S_K &= -G_K - G_K^T + \sigma_K H_K H_K^T, \\
U_K &= -2\sigma_K I + \sigma_K E_K^T E_K, \\
J_K &= -G_K - G_K^T + \nu_K H_K H_K^T, \\
\mathscr{T}_L &= \begin{bmatrix} -P_L & * \\ G_L A - N_L C_2 & -G_L - G_L^T + P_L \end{bmatrix}, \\
X_L &= \begin{bmatrix} 0 \\ -N_L H_L \end{bmatrix}, \\
Y_L &= [\, N_L C_2 \quad 0 \,], \\
V_L &= -G_L - G_L^T + \sigma_L E_L^T E_L + \nu_L E_L^T E_L.
\end{aligned}$$

Furthermore, the observer (6.45) *and the controller* (6.48) *gain matrices are given by*

$$L = G_L^{-1} N_L, \quad K = N_K G_K^{-1}. \tag{6.54}$$

Proof First, by Lemma 1.16, we have

$$\begin{aligned}
& BK\phi_K G_K \\
&= BK\big(I - H_K \Delta_K(k) E_K K\big)^{-1} G_K \\
&= BK\big(\underbrace{I}_{\bar{A}} \underbrace{-H_K \Delta_K(k) E_K}_{\bar{B}} \underbrace{I}_{\bar{C}} \underbrace{K}_{\bar{D}}\big)^{-1} G_K \\
&= BK\Big(I + H_K \Delta_K(k) E_K \big(I - K H_K \Delta_K(k) E_K\big)^{-1} K\Big) G_K \\
&= BN_K + BN_K G_K^{-1} H_K \Delta_K(k) E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} N_K,
\end{aligned} \tag{6.55}$$

and

$$
\begin{aligned}
&DK\phi_K G_K \\
&= DK\big(I - H_K \Delta_K(k) E_K K\big)^{-1} G_K \\
&= DK\big(\underbrace{I}_{\bar{A}}\,\underbrace{-H_K \Delta_K(k) E_K}_{\bar{B}}\,\underbrace{I}_{\bar{C}}\,\underbrace{K}_{\bar{D}}\big)^{-1} G_K \\
&= DK\Big(I + H_K \Delta_K(k) E_K \big(I - K H_K \Delta_K(k) E_K\big)^{-1} K\Big) G_K \\
&= DN_K + DN_K G_K^{-1} H_K \Delta_K(k) E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} N_K,
\end{aligned}
\tag{6.56}
$$

with $N_K = K G_K$.

Then, (6.50) can be rewritten as follows:

$$
\begin{aligned}
&\mathscr{T}_K + X_K G_K^{-1} H_K \Delta_K(k) E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} Y_K \\
&\qquad + Y_K^T \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-T} E_K^T \Delta_K^T(k) H_K^T G_K^{-T} X_K^T < 0,
\end{aligned}
\tag{6.57}
$$

where $\mathscr{T}_K$, X_K, and Y_K are defined in (6.52).

Applying Lemma 1.12, it gives

$$
\begin{aligned}
&\mathscr{T}_K + X_K G_K^{-1} H_K \Delta_K(k) E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} Y_K \\
&\quad + Y_K^T \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-T} E_K^T \Delta_K^T(k) H_K^T G_K^{-T} X_K^T \\
&\le \mathscr{T}_K + \sigma_K X_K G_K^{-1} H_K H_K^T G_K^{-T} X_K^T \\
&\quad + \frac{1}{\sigma_K} Y_K^T \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-T} E_K^T E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} Y_K.
\end{aligned}
$$

Then, (6.57) can be verified by

$$
\begin{aligned}
&\mathscr{T}_K + \sigma_K X_K G_K^{-1} H_K H_K^T G_K^{-T} X_K^T \\
&\quad + \frac{1}{\sigma_K} Y_K^T \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-T} E_K^T E_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^{-1} Y_K < 0.
\end{aligned}
\tag{6.58}
$$

By Schur complement to (6.58), we obtain

$$
\begin{bmatrix}
\mathscr{T}_K & * & * \\
X_K^T & -G_K^T(\sigma_K H_K H_K^T)^{-1} G_K & * \\
Y_K & 0 & \Xi_K
\end{bmatrix} < 0,
\tag{6.59}
$$

where $\Xi_K = -\sigma_K \big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)(E_K^T E_K)^{-1}\big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^T$.

Consider a fact

$$
-G_K^T(\sigma_K H_K H_K^T)^{-1} G_K \le -G_K - G_K^T + \sigma_K H_K H_K^T = S_K.
\tag{6.60}
$$

Using Lemma 1.12 with a scalar $\nu_K > 0$, it follows that:

$$
\begin{aligned}
\Xi_K &= -\sigma_K\big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)(E_K^T E_K)^{-1}\big(I - N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^T \\
&= -\big(\sigma_K I - \sigma_K N_K G_K^{-1} H_K \Delta_K(k) E_K\big)(\sigma_K E_K^T E_K)^{-1}\big(\sigma_K I - \sigma_K N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^T \\
&\le -\big(\sigma_K I - \sigma_K N_K G_K^{-1} H_K \Delta_K(k) E_K\big) - \big(\sigma_K I - \sigma_K N_K G_K^{-1} H_K \Delta_K(k) E_K\big)^T + \sigma_K E_K^T E_K \\
&= -2\sigma_K I + \sigma_K E_K^T E_K + \sigma_K N_K G_K^{-1} H_K \Delta_K(k) E_K + \sigma_K E_K^T \Delta_K^T(k) H_K^T G_K^{-T} N_K^T \\
&\le -2\sigma_K I + \sigma_K E_K^T E_K + \nu_K N_K G_K^{-1} H_K H_K^T G_K^{-T} N_K^T + \nu_K^{-1} \sigma_K E_K^T \sigma_K E_K \\
&= U_K + \nu_K N_K G_K^{-1} H_K H_K^T G_K^{-T} N_K^T + \nu_K^{-1} \sigma_K E_K^T \sigma_K E_K.
\end{aligned} \tag{6.61}
$$

Then, (6.59) holds, if

$$
\begin{bmatrix}
\mathscr{T}_K & * & * \\
X_K^T & S_K & * \\
Y_K & 0 & U_K + \nu_K N_K G_K^{-1} H_K H_K^T G_K^{-T} N_K^T + \nu_K^{-1} \sigma_K E_K^T \sigma_K E_K
\end{bmatrix} < 0. \tag{6.62}
$$

Once again, applying Schur complement to (6.62), it leads to

$$
\begin{bmatrix}
\mathscr{T}_K & * & * & * & * \\
X_K^T & S_K & * & * & * \\
Y_K & 0 & U_K & * & * \\
0 & 0 & N_K^T & -G_K^T(\nu_K H_K H_K^T)^{-1} G_K & * \\
0 & 0 & \sigma_K E_K & 0 & -\nu_K I
\end{bmatrix} < 0. \tag{6.63}
$$

Similar to (6.60), one has

$$
-G_K^T(\nu_K H_K H_K^T)^{-1} G_K \le -G_K - G_K^T + \nu_K H_K H_K^T = J_K. \tag{6.64}
$$

Then, if the LMI condition (6.52) is satisfied, (6.63) holds.

On the other hand, based on Lemma 1.16, we have

$$
\begin{aligned}
& N_L \phi_L C_2 \\
&= N_L\big(I - H_L \Delta_L(k) E_L L\big)^{-1} C_2 \\
&= N_L\big(\underbrace{I}_{\bar{A}} \underbrace{-H_L \Delta_L(k) E_L G_L^{-1}}_{\bar{B}} \underbrace{I}_{\bar{C}} \underbrace{N_L}_{\bar{D}}\big)^{-1} C_2 \\
&= N_L\Big(I + H_L \Delta_L(k) E_L G_L^{-1}\big(I - N_L H_L \Delta_L(k) E_L G_L^{-1}\big)^{-1} N_L\Big) C_2 \\
&= N_L C_2 + N_L H_L \Delta_L(k) E_L \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-1} N_L C_2.
\end{aligned} \tag{6.65}
$$

Then, (6.51) becomes

$$
\begin{aligned}
\mathscr{T}_L &+ X_L \Delta_L(k) E_L \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-1} Y_L \\
&+ Y_L^T \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-T} E_L^T \Delta_L^T(k) X_L^T < 0,
\end{aligned} \tag{6.66}
$$

where $\mathscr{T}_L$, X_L, and Y_L are defined in (6.53).

Applying Lemma 1.11, it follows that:

$$
\begin{aligned}
&\mathscr{T}_L + X_L \Delta_L(k) E_L \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-1} Y_L \\
&\quad + Y_L^T \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-T} E_L^T \Delta_L^T(k) X_L^T \\
&\le \mathscr{T}_L + \sigma_L^{-1} X_L X_L^T + \sigma_L Y_L^T \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-T} E_L^T \\
&\quad \times E_L \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-1} Y_L .
\end{aligned}
$$

Then, (6.66) is satisfied if matrix inequality (6.67) holds.

$$
\begin{aligned}
&\mathscr{T}_L + \sigma_L^{-1} X_L X_L^T + \sigma_L Y_L^T \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-T} E_L^T \\
&\quad \times E_L \big(G_L - N_L H_L \Delta_L(k) E_L\big)^{-1} Y_L < 0.
\end{aligned} \tag{6.67}
$$

Applying Schur complement to (6.67), then, yields

$$
\begin{bmatrix} \mathscr{T}_L & * & * \\ X_L^T & -\sigma_L I & * \\ Y_L & 0 & \Upsilon_L \end{bmatrix} < 0, \tag{6.68}
$$

where $\Upsilon_L = -\big(G_L - N_L H_L \Delta_L(k) E_L\big)\big(\sigma_L E_L^T E_L\big)^{-1}\big(G_L - N_L H_L \Delta_L(k) E_L\big)^T$.

By using Lemma 1.11, we have

$$
\begin{aligned}
\Upsilon_L &= -\big(G_L - N_L H_L \Delta_L(k) E_L\big)\big(\sigma_L E_L^T E_L\big)^{-1}\big(G_L - N_L H_L \Delta_L(k) E_L\big)^T \\
&\le -\big(G_L - N_L H_L \Delta_L(k) E_L\big) - \big(G_L - N_L H_L \Delta_L(k) E_L\big)^T + \sigma_L E_L^T E_L \\
&= -G_L - G_L^T + \sigma_L E_L^T E_L + N_L H_L \Delta_L(k) E_L + E_L^T \Delta_L^T(k) H_L^T N_L^T \\
&\le -G_L - G_L^T + \sigma_L E_L^T E_L + \nu_L E_L^T E_L + \nu_L^{-1} N_L H_L H_L^T N_L^T \\
&= V_L + \nu_L^{-1} N_L H_L H_L^T N_L^T \\
&= \Phi_L .
\end{aligned} \tag{6.69}
$$

Obviously, (6.68) can be guaranteed by

$$
\begin{bmatrix} \mathscr{T}_L & * & * \\ X_L^T & -\sigma_L I & * \\ Y_L & 0 & \Phi_L \end{bmatrix} < 0. \tag{6.70}
$$

Finally, by Schur complement, the LMI conditions (6.53) are obtained.

So far, the proof is completed. □

6.1.4 Non-fragile H_∞ Filtering with Type I and Type II

At present, there are a lot of research results on non-fragile H_∞ filtering. The problem of a non-fragile H_∞ filter design for a class of linear systems described by delta operator with circular pole constraints was investigated in [9]. Chang and Yang [3] investigated the problem of non-fragile filter designs for continuous-time fuzzy systems with additive uncertainties. An approach of designing the optimal filter transfer function and its realization was developed in [5]. In [4], the non-fragile H_∞ filtering problem for linear continuous-time systems was addressed, where the filter to be designed is assumed to have additive gain variations of interval type. Chang [2] was concerned with the H_∞ filtering problem for continuous-time T-S fuzzy systems, in which uncertain fuzzy systems with linear fractional parametric uncertainties are considered. Mahmoud [13] investigated the problem of resilient linear filtering for a class of linear continuous-time systems with norm-bounded uncertainties, in which additive gain variations were considered. It should be noted that the above researches on non-fragile filtering take into account the additive uncertainties, multiplicative uncertainties, and linear fractional parametric uncertainties. However, a few results deal with feedback uncertainties.

Motivated by the aforementioned observations, this section is concerned with the non-fragile H_∞ filter design problem for discrete-time linear systems with feedback uncertainties. The focus is on designing a filter with two types of feedback uncertainties such that the filtering error system guarantees a prescribed H_∞ performance level. The H_∞ filter design conditions are derived based on LMI techniques. An example is provided to illustrate the feasibility of the proposed design methods.

Consider the following discrete-time linear system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bw(k), \\ y(k) &= Cx(k) + Dw(k), \\ z(k) &= Lx(k), \end{aligned} \tag{6.71}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $w(k) \in \mathscr{R}^v$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the signal to be estimated, $y(k) \in R^f$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times v}$, $C \in \mathscr{R}^{f\times n}$, $D \in \mathscr{R}^{f\times v}$, and $L \in \mathscr{R}^{q\times n}$ are system matrices.

In this section, we consider the following non-fragile filter to estimate $z(k)$

For Type I

$$\begin{aligned} x_F(k+1) &= A_F\big(I - H_A\Delta_A(k)E_A\big)^{-1}x_F(k) + B_F\big(I - H_B\Delta_B(k)E_B\big)^{-1}y(k), \\ z_F(k) &= C_F\big(I - H_C\Delta_C(k)E_C\big)^{-1}x_F(k) + D_F\big(I - H_D\Delta_D(k)E_D\big)^{-1}y(k), \end{aligned} \tag{6.72}$$

For Type II

$$
\begin{aligned}
x_F(k+1) &= \big(I - H_A\Delta_A(k)E_A\big)^{-1}A_F x_F(k) + \big(I - H_B\Delta_B(k)E_B\big)^{-1}B_F y(k),\\
z_F(k) &= \big(I - H_C\Delta_C(k)E_C\big)^{-1}C_F x_F(k) + \big(I - H_D\Delta_D(k)E_D\big)^{-1}D_F y(k),
\end{aligned}
\tag{6.73}
$$

where $x_F(k) \in \mathscr{R}^n$ and $z_F(k) \in \mathscr{R}^q$ are the state and output of the filter, respectively. $A_F \in \mathscr{R}^{n\times n}$, $B_F \in \mathscr{R}^{n\times f}$, $C_F \in \mathscr{R}^{q\times n}$, and $D_F \in \mathscr{R}^{q\times f}$ are to be determined filter matrices. For $\beta = A,\ B,\ C,\ D$, H_β and E_β are constant matrices with appropriate dimensions, $\Delta_\beta(k)$ are uncertain matrices bounded such as $\Delta_\beta^T(k)\Delta_\beta(k) \le I$.

By defining the augmented state vector $\psi(k) = \begin{bmatrix} x(k) \\ x_F(k) \end{bmatrix}$ and $e(k) = z(k) - z_F(k)$, we can obtain the following filtering error system

$$
\begin{aligned}
\psi(k+1) &= \tilde{A}\psi(k) + \tilde{B}w(k),\\
e(k) &= \tilde{C}\psi(k) + \tilde{D}w(k),
\end{aligned}
\tag{6.74}
$$

where

For Type I

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} A & 0 \\ B_F\delta_B C & A_F\delta_A \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} B \\ B_F\delta_B D \end{bmatrix},\\
\tilde{C} &= [\ L - D_F\delta_D C \quad -C_F\delta_C\ ], & \tilde{D} &= -D_F\delta_D D,
\end{aligned}
$$

For Type II

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} A & 0 \\ \delta_B B_F C & \delta_A A_F \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} B \\ \delta_B B_F D \end{bmatrix},\\
\tilde{C} &= [\ L - \delta_D D_F C \quad -\delta_C C_F\ ], & \tilde{D} &= -\delta_D D_F D,
\end{aligned}
$$

with $\delta_\beta = \big(I - H_\beta\Delta_\beta(k)E_\beta\big)^{-1}$, $\beta = A, B, C, D$.

As robust H_∞ filtering in Sect. 6.2, we use also the matrix inequality (6.24) to guarantee the prescribed H_∞ performance γ of the filtering error system (6.74). To facilitate the design of H_∞ filters, we choose also the matrices P and G in the from of (6.25).

In the following, we will derive conditions for designing the filter gain matrices in (6.72) or (6.73) such that the filtering error system (6.74) is asymptotically stable with H_∞ performance γ. In the following, we will treat the Type I and Type II separately.

6.1.4.1 Type I

Combining (6.24), (6.25), and (6.74) for Type I, we obtain

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1A+\mathscr{B}_F\delta_B C & \mathscr{A}_F\delta_A & G_1B+\mathscr{B}_F\delta_B D & -G_1-G_1^T+P_1 & * & * \\ G_3A+\mathscr{B}_F\delta_B C & \mathscr{A}_F\delta_A & G_3B+\mathscr{B}_F\delta_B D & -G_3-G_2^T+P_2 & \Omega_1 & * \\ ML-\mathscr{D}_F\delta_D C & -\mathscr{C}_F\delta_C & -\mathscr{D}_F\delta_D D & 0 & 0 & \Omega_2 \end{bmatrix} < 0, \tag{6.75}$$

where

$$\begin{aligned} \mathscr{A}_F &= G_2A_F, \\ \mathscr{B}_F &= G_2B_F, \\ \mathscr{C}_F &= MC_F, \\ \mathscr{D}_F &= MD_F, \\ \Omega_1 &= -G_2-G_2^T+P_3, \\ \Omega_2 &= -M-M^T+I. \end{aligned}$$

Considering the property about δ_β in (6.27), it knows that the inequality (6.75) is equivalent to

$$\begin{aligned} &\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1A+\mathscr{B}_F C & \mathscr{A}_F & G_1B+\mathscr{B}_F D & -G_1-G_1^T+P_1 & * & * \\ G_3A+\mathscr{B}_F C & \mathscr{A}_F & G_3B+\mathscr{B}_F D & -G_3-G_2^T+P_2 & \Omega_1 & * \\ ML-\mathscr{D}_F C & -\mathscr{C}_F & -\mathscr{D}_F D & 0 & 0 & \Omega_2 \end{bmatrix} \\ &+\begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ \mathscr{B}_F\nu_B C & \mathscr{A}_F\nu_A & \mathscr{B}_F\nu_B D & 0 & * & * \\ \mathscr{B}_F\nu_B C & \mathscr{A}_F\nu_A & \mathscr{B}_F\nu_B D & 0 & 0 & * \\ -\mathscr{D}_F\nu_D C & -\mathscr{C}_F\nu_C & -\mathscr{D}_F\nu_D D & 0 & 0 & 0 \end{bmatrix} \\ &= \Theta + X_A\Delta_A(k)\big(I-E_AH_A\Delta_A(k)\big)^{-1}Y_A + Y_A^T\big(I-E_AH_A\Delta_A(k)\big)^{-T}\Delta_A^TX_A^T \\ &\quad + X_B\Delta_B(k)\big(I-E_BH_B\Delta_B(k)\big)^{-1}Y_B + Y_B^T\big(I-E_BH_B\Delta_B(k)\big)^{-T}\Delta_B^TX_B^T \\ &\quad + X_C\Delta_C(k)\big(I-E_CH_C\Delta_C(k)\big)^{-1}Y_C + Y_C^T\big(I-E_CH_C\Delta_C(k)\big)^{-T}\Delta_C^TX_C^T \\ &\quad + X_D\Delta_D(k)\big(I-E_DH_D\Delta_D(k)\big)^{-1}Y_D + Y_D^T\big(I-E_DH_D\Delta_D(k)\big)^{-T}\Delta_D^TX_D^T < 0, \end{aligned} \tag{6.76}$$

where

$$\Theta = \begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1A+\mathscr{B}_F C & \mathscr{A}_F & G_1B+\mathscr{B}_F D & -G_1-G_1^T+P_1 & * & * \\ G_3A+\mathscr{B}_F C & \mathscr{A}_F & G_3B+\mathscr{B}_F D & -G_3-G_2^T+P_2 & \Omega_1 & * \\ ML-\mathscr{D}_F C & -\mathscr{C}_F & -\mathscr{D}_F D & 0 & 0 & \Omega_2 \end{bmatrix},$$

$$X_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathscr{A}_F H_A \\ \mathscr{A}_F H_A \\ 0 \end{bmatrix}, \quad Y_A = [\,0 \; E_A \; 0 \; 0 \; 0 \; 0\,],$$

$$X_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathscr{B}_F H_B \\ \mathscr{B}_F H_B \\ 0 \end{bmatrix}, \quad Y_B = [\,E_B C \; 0 \; E_B D \; 0 \; 0 \; 0\,],$$

$$X_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\mathscr{C}_F H_C \end{bmatrix}, \quad Y_C = [\,0 \; E_C \; 0 \; 0 \; 0 \; 0\,],$$

$$X_D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\mathscr{D}_F H_D \end{bmatrix}, \quad Y_D = [\,E_D C \; 0 \; E_D D \; 0 \; 0 \; 0\,].$$

By Lemma 1.11 for positive scalars $\varepsilon_A,\ \varepsilon_B,\ \varepsilon_C$, and ε_D, it follows that

$$\begin{aligned}
&\Theta + X_A \Delta_A(k)\big(I - E_A H_A \Delta_A(k)\big)^{-1} Y_A + Y_A^T \big(I - E_A H_A \Delta_A(k)\big)^{-T} \Delta_A^T X_A^T \\
&\quad + X_B \Delta_B(k)\big(I - E_B H_B \Delta_B(k)\big)^{-1} Y_B + Y_B^T \big(I - E_B H_B \Delta_B(k)\big)^{-T} \Delta_B^T X_B^T \\
&\quad + X_C \Delta_C(k)\big(I - E_C H_C \Delta_C(k)\big)^{-1} Y_C + Y_C^T \big(I - E_C H_C \Delta_C(k)\big)^{-T} \Delta_C^T X_C^T \\
&\quad + X_D \Delta_D(k)\big(I - E_D H_D \Delta_D(k)\big)^{-1} Y_D + Y_D^T \big(I - E_D H_D \Delta_D(k)\big)^{-T} \Delta_D^T X_D^T \\
&\le \Theta + \frac{1}{\varepsilon_A} X_A X_A^T + \varepsilon_A Y_A^T \big(I - E_A H_A \Delta_A(k)\big)^{-T} \big(I - E_A H_A \Delta_A(k)\big)^{-1} Y_A \\
&\quad + \frac{1}{\varepsilon_B} X_B X_B^T + \varepsilon_B Y_B^T \big(I - E_B H_B \Delta_B(k)\big)^{-T} \big(I - E_B H_B \Delta_B(k)\big)^{-1} Y_B \\
&\quad + \frac{1}{\varepsilon_C} X_C X_C^T + \varepsilon_C Y_C^T \big(I - E_C H_C \Delta_C(k)\big)^{-T} \big(I - E_C H_C \Delta_C(k)\big)^{-1} Y_C \\
&\quad + \frac{1}{\varepsilon_D} X_D X_D^T + \varepsilon_D Y_D^T \big(I - E_D H_D \Delta_D(k)\big)^{-T} \big(I - E_D H_D \Delta_D(k)\big)^{-1} Y_D.
\end{aligned} \tag{6.77}$$

Then, (6.76) holds if the following condition is satisfied:

$$\begin{aligned}&\Theta+\frac{1}{\varepsilon_A}X_AX_A^T+\varepsilon_AY_A^T\big(I-E_AH_A\Delta_A(k)\big)^{-T}\big(I-E_AH_A\Delta_A(k)\big)^{-1}Y_A\\&+\frac{1}{\varepsilon_B}X_BX_B^T+\varepsilon_BY_B^T\big(I-E_BH_B\Delta_B(k)\big)^{-T}\big(I-E_BH_B\Delta_B(k)\big)^{-1}Y_B\\&+\frac{1}{\varepsilon_C}X_CX_C^T+\varepsilon_CY_C^T\big(I-E_CH_C\Delta_C(k)\big)^{-T}\big(I-E_CH_C\Delta_C(k)\big)^{-1}Y_C\\&+\frac{1}{\varepsilon_D}X_DX_D^T+\varepsilon_DY_D^T\big(I-E_DH_D\Delta_D(k)\big)^{-T}\big(I-E_DH_D\Delta_D(k)\big)^{-1}Y_D<0.\end{aligned}\tag{6.78}$$

By Schur complement to (6.78), which leads to

$$\begin{bmatrix}\Theta&*&*&*&*&*&*&*&*\\X_A^T&-\varepsilon_AI&*&*&*&*&*&*&*\\Y_A&0&\mathscr{F}_A&*&*&*&*&*&*\\X_B^T&0&0&-\varepsilon_BI&*&*&*&*&*\\Y_B&0&0&0&\mathscr{F}_B&*&*&*&*\\X_C^T&0&0&0&0&-\varepsilon_CI&*&*&*\\Y_C&0&0&0&0&0&\mathscr{F}_C&*&*\\X_D^T&0&0&0&0&0&0&-\varepsilon_DI&*\\Y_D&0&0&0&0&0&0&0&\mathscr{F}_D\end{bmatrix}<0,\tag{6.79}$$

where $\mathscr{F}_\beta=-\varepsilon_\beta^{-1}\big(I-E_\beta H_\beta\Delta_\beta(k)\big)\big(I-E_\beta H_\beta\Delta_\beta(k)\big)^T,\ \beta=A,\ B,\ C,\ D.$

By Lemma 1.17, it is worth noting that

$$\begin{aligned}\mathscr{F}_\beta&=-\big(I-E_AH_A\Delta_\beta(k)\big)\varepsilon_\beta^{-1}I\big(I-E_\beta H_\beta\Delta_\beta(k)\big)^T\\&\le\varepsilon_\beta I-2I+\frac{1}{\varepsilon_{\beta\beta}}E_\beta H_\beta H_\beta^TE_\beta^T+\varepsilon_{\beta\beta}I\\&=\mathscr{O}_\beta,\ \beta=A,\ B,\ C,\ D.\end{aligned}$$

Then, (6.79) can be guaranteed by

$$\begin{bmatrix}\Theta&*&*&*&*&*&*&*&*\\X_A^T&-\varepsilon_AI&*&*&*&*&*&*&*\\Y_A&0&\mathscr{O}_A&*&*&*&*&*&*\\X_B^T&0&0&-\varepsilon_BI&*&*&*&*&*\\Y_B&0&0&0&\mathscr{O}_B&*&*&*&*\\X_C^T&0&0&0&0&-\varepsilon_CI&*&*&*\\Y_C&0&0&0&0&0&\mathscr{O}_C&*&*\\X_D^T&0&0&0&0&0&0&-\varepsilon_DI&*\\Y_D&0&0&0&0&0&0&0&\mathscr{O}_D\end{bmatrix}<0,\tag{6.80}$$

Applying Schur complement to (6.80) yields

$$\begin{bmatrix} \Theta & * & * & * & * & * & * & * & * & * & * & * & * \\ X_A^T & -\varepsilon_A I & * & * & * & * & * & * & * & * & * & * & * \\ Y_A & 0 & \mathscr{E}_A & * & * & * & * & * & * & * & * & * & * \\ X_B^T & 0 & 0 & -\varepsilon_B I & * & * & * & * & * & * & * & * & * \\ Y_B & 0 & 0 & 0 & \mathscr{E}_B & * & * & * & * & * & * & * & * \\ X_C^T & 0 & 0 & 0 & 0 & -\varepsilon_C I & * & * & * & * & * & * & * \\ Y_C & 0 & 0 & 0 & 0 & 0 & \mathscr{E}_C & * & * & * & * & * & * \\ X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_D I & * & * & * & * & * \\ Y_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{E}_D & * & * & * & * \\ 0 & 0 & H_A^T E_A^T & 0 & 0 & 0 & 0 & 0 & 0 & x_A & * & * & * \\ 0 & 0 & 0 & 0 & H_B^T E_B^T & 0 & 0 & 0 & 0 & 0 & x_B & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & H_C^T E_C^T & 0 & 0 & 0 & 0 & x_C & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & H_D^T E_D^T & 0 & 0 & 0 & x_D \end{bmatrix} < 0, \tag{6.81}$$

where $\mathscr{E}_\beta = \epsilon_\beta I - 2I + \epsilon_{\beta\beta} I$, $x_\beta = -\epsilon_{\beta\beta} I$, $\beta = A, B, C, D$.

At this point, the non-fragile H_∞ filter design result is summarized in the following theorem.

Theorem 6.6 *Consider the filtering error system (6.74) for Type I and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, P_3, G_1, G_2, G_3, M, $\mathscr{A}_F$, $\mathscr{B}_F$, $\mathscr{C}_F$, and $\mathscr{D}_F$, scalars ε_A, ε_B, ε_C, ε_D, ε_L, ε_{AA}, ε_{BB}, ε_{CC}, ε_{DD}, and ε_{LL} such that LMI (6.81) holds. Furthermore the filter gain matrices in (6.72) are given by*

$$A_F = G_2^{-1}\mathscr{A}_F, \quad B_F = G_2^{-1}\mathscr{B}_F, \quad C_F = M^{-1}\mathscr{C}_F, \quad D_F = M^{-1}\mathscr{D}_F. \tag{6.82}$$

As robust H_∞ filter design result given in Theorem 6.3, the condition (6.81) implies $\mathscr{E}_\beta = \varepsilon_\beta I - 2I + \varepsilon_{\beta\beta} I < 0, \beta = A, B, C, D$, the term $2I$ will lead to high conservatism. In the following, we will develop another design method which improves that in Theorem 6.6.

By introducing four invertible slack matrix variables $N_\beta, \beta = A, B, C, D$, we rewrite the inequality (6.76) as follows:

$$\begin{aligned} \Theta &+ X_A\Delta_A(k)\big(N_A - N_A E_A H_A \Delta_A(k)\big)^{-1}\mathscr{Y}_A + \mathscr{Y}_A^T\big(N_A - N_A E_A H_A \Delta_A(k)\big)^{-T}\Delta_A^T X_A^T \\ &+ X_B\Delta_B(k)\big(N_B - N_B E_B H_B \Delta_B(k)\big)^{-1}\mathscr{Y}_B + \mathscr{Y}_B^T\big(N_B - N_B E_B H_B \Delta_B(k)\big)^{-T}\Delta_B^T X_B^T \\ &+ X_C\Delta_C(k)\big(N_C - N_C E_C H_C \Delta_C(k)\big)^{-1}\mathscr{Y}_C + \mathscr{Y}_C^T\big(N_C - N_C E_C H_C \Delta_C(k)\big)^{-T}\Delta_C^T X_C^T \\ &+ X_D\Delta_D(k)\big(N_D - N_D E_D H_D \Delta_D(k)\big)^{-1}\mathscr{Y}_D + \mathscr{Y}_D^T\big(N_D - N_D E_D H_D \Delta_D(k)\big)^{-T}\Delta_D^T X_D^T \\ &< 0, \end{aligned} \tag{6.83}$$

where $X_\beta,\ \beta = A,\ B,\ C,\ D$ are the same as that in (6.76) and

$$
\begin{aligned}
\mathscr{Y}_A &= [0 \quad N_A E_A \quad 0 \quad 0 \quad 0 \quad 0],\\
\mathscr{Y}_B &= [N_B E_B C \quad 0 \quad N_B E_B D \quad 0 \quad 0 \quad 0],\\
\mathscr{Y}_C &= [0 \quad N_C E_C \quad 0 \quad 0 \quad 0 \quad 0],\\
\mathscr{Y}_D &= [N_D E_D C \quad 0 \quad N_D E_D D \quad 0 \quad 0 \quad 0].
\end{aligned}
$$

Similar to (6.79), (6.83) is satisfied if

$$
\begin{bmatrix}
\Theta & * & * & * & * & * & * & * & * \\
X_A^T & -\varepsilon_A I & * & * & * & * & * & * & * \\
\mathscr{Y}_A & 0 & \mathscr{M}_A & * & * & * & * & * & * \\
X_B^T & 0 & 0 & -\varepsilon_B I & * & * & * & * & * \\
\mathscr{Y}_B & 0 & 0 & 0 & \mathscr{M}_B & * & * & * & * \\
X_C^T & 0 & 0 & 0 & 0 & -\varepsilon_C I & * & * & * \\
\mathscr{Y}_C & 0 & 0 & 0 & 0 & 0 & \mathscr{M}_C & * & * \\
X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_D I & * \\
\mathscr{Y}_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{M}_D
\end{bmatrix} < 0, \tag{6.84}
$$

holds, where

$$
\mathscr{M}_\beta = -\varepsilon_\beta^{-1}\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)^T,\ \beta = A,\ B,\ C,\ D.
$$

By Lemma 1.17, it is worth noting that

$$
\begin{aligned}
\mathscr{M}_\beta &= -\big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)\varepsilon_\beta^{-1} I \big(N_\beta - N_\beta E_\beta H_\beta \Delta_\beta(k)\big)^T\\
&\le \varepsilon_\beta I - N_\beta - N_\beta^T + \frac{1}{\varepsilon_{\beta\beta}} N_\beta E_\beta H_\beta H_\beta^T E_\beta^T N_\beta^T + \varepsilon_{\beta\beta} I,\ \beta = A,\ B,\ C,\ D.
\end{aligned}
$$

Then, (6.84) can be verified by the following condition:

$$
\begin{bmatrix}
\Theta & * & * & * & * & * & * & * & * & * & * & * & * \\
X_A^T & -\varepsilon_A I & * & * & * & * & * & * & * & * & * & * & * \\
\mathscr{Y}_A & 0 & \mathscr{K}_A & * & * & * & * & * & * & * & * & * & * \\
X_B^T & 0 & 0 & -\varepsilon_B I & * & * & * & * & * & * & * & * & * \\
\mathscr{Y}_B & 0 & 0 & 0 & \mathscr{K}_B & * & * & * & * & * & * & * & * \\
X_C^T & 0 & 0 & 0 & 0 & -\varepsilon_C I & * & * & * & * & * & * & * \\
\mathscr{Y}_C & 0 & 0 & 0 & 0 & 0 & \mathscr{K}_C & * & * & * & * & * & * \\
X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_D I & * & * & * & * & * \\
\mathscr{Y}_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{K}_D & * & * & * & * \\
0 & 0 & Q_A & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{AA} I & * & * & * \\
0 & 0 & 0 & 0 & Q_B & 0 & 0 & 0 & 0 & 0 & -\varepsilon_{BB} I & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & Q_C & 0 & 0 & 0 & 0 & -\varepsilon_{CC} I & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q_D & 0 & 0 & 0 & -\varepsilon_{DD} I
\end{bmatrix} < 0, \tag{6.85}
$$

where

$$
\mathscr{K}_\beta = \varepsilon_\beta I - N_\beta - N_\beta^T + \varepsilon_{\beta\beta} I, \beta = A, B, C, D,
$$

$$Q_\beta = H_\beta^T E_\beta^T N_\beta^T, \beta = A, B, C, D.$$

Remark 6.7 Compared with the LMI condition (6.81), (6.85) replaces the identity matrix I by matrix variables $N_\beta, \beta = A, B, C, D$. In other words, when $N_A = N_B = N_C = N_D = I$, (6.85) reduces (6.81). Thus it can be easily seen that (6.81) is a special case of (6.85) and the condition (6.85) is more relaxed than that (6.81). The improved design condition is given by the following corollary.

Corollary 6.2 *Consider the filtering error system* (6.74) *for Type I and give a scalar* $\gamma > 0$*. Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P_1, P_2, P_3, G_1, G_2, G_3, M, N_A, N_B, N_C, N_D, N_L, \mathscr{A}_F, \mathscr{B}_F, \mathscr{C}_F$*, and* $\mathscr{D}_F$*, scalars* $\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D, \varepsilon_L, \varepsilon_{AA}, \varepsilon_{BB}, \varepsilon_{CC}, \varepsilon_{DD}$*, and* ε_{LL} *such that LMI* (6.85) *holds. Furthermore, the* H_∞ *filter gain matrices in* (6.72) *can be given by* (6.82).

6.1.4.2 Type II

By substituting (6.25) and (6.74) for Type II into (6.24), one gives

$$\begin{bmatrix} -P_1 & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * \\ G_1A + G_2\delta_B B_F C & G_2\delta_A A_F & G_1 B + G_2\delta_B B_F D & -G_1 - G_1^T + P_1 & * & * \\ G_3A + G_2\delta_B B_F C & G_2\delta_A A_F & G_3 B + G_2\delta_B B_F D & -G_3 - G_2^T + P_2 & \Omega_1 & * \\ ML - M\delta_D D_F C & -M\delta_C C_F & -M\delta_D D_F D & 0 & 0 & \Omega_2 \end{bmatrix} < 0, \tag{6.86}$$

where Ω_1 and Ω_2 are the same as that in (6.75).

By Lemma 1.16, we have

$$\begin{aligned} \delta_\beta &= (I - H_\beta \Delta_\beta(k) E_\beta)^{-1} \\ &= (\underbrace{I}_{\bar{A}} \underbrace{-H_\beta}_{\bar{B}} \underbrace{I}_{\bar{C}} \underbrace{\Delta_\beta(k) E_\beta}_{\bar{D}})^{-1} \\ &= I + H_\beta (I - \Delta_\beta(k) E_\beta H_\beta)^{-1} \Delta_\beta(k) E_\beta \\ &= I + \mu_\beta, \beta = A, B, C, D. \end{aligned} \tag{6.87}$$

Obviously, by considering (6.87) and defining $\mathscr{A}_F = G_2 A_F$, $\mathscr{B}_F = G_2 B_F$, $\mathscr{C}_F = M C_F$, $\mathscr{D}_F = M D_F$, the inequality (6.86) can be rewritten as

$$\Theta + \begin{bmatrix} 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ G_2\mu_B B_F C & G_2\mu_A A_F & G_2\mu_B B_F D & 0 & * & * \\ G_2\mu_B B_F C & G_2\mu_A A_F & G_2\mu_B B_F D & 0 & 0 & * \\ -M\mu_D D_F C & -M\mu_C C_F & -M\mu_D D_F D & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&= \Theta + X_A\big(I - \Delta_A(k)E_AH_A\big)^{-1}\Delta_A(k)E_AG_2^{-1}Y_A \\
&\quad + Y_A^TG_2^{-T}E_A^T\Delta_A^T(k)\big(I - \Delta_A(k)E_AH_A\big)^{-T}X_A^T \\
&\quad + X_B\big(I - \Delta_B(k)E_BH_B\big)^{-1}\Delta_B(k)E_BG_2^{-1}Y_B \\
&\quad + Y_B^TG_2^{-T}E_B^T\Delta_B^T(k)\big(I - \Delta_B(k)E_BH_B\big)^{-T}X_B^T \\
&\quad + X_C\big(I - \Delta_C(k)E_CH_C\big)^{-1}\Delta_C(k)E_CM^{-1}Y_C \\
&\quad + Y_C^TM^{-T}E_C^T\Delta_C^T(k)\big(I - \Delta_C(k)E_CH_C\big)^{-T}X_C^T \\
&\quad + X_D\big(I - \Delta_D(k)E_DH_D\big)^{-1}\Delta_D(k)E_DM^{-1}Y_D \\
&\quad + Y_D^TM^{-T}E_D^T\Delta_D^T(k)\big(I - \Delta_D(k)E_DH_D\big)^{-T}X_D^T < 0,
\end{aligned}
\tag{6.88}
$$

where Θ is the same as that in (6.76) and

$$
\begin{aligned}
X_A &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_2H_A \\ G_2H_A \\ 0 \end{bmatrix}, \quad Y_A = [0 \quad \mathscr{A}_F \quad 0 \quad 0 \quad 0 \quad 0], \\
X_B &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_2H_B \\ G_2H_B \\ 0 \end{bmatrix}, \quad Y_B = [\mathscr{B}_FC \quad 0 \quad \mathscr{B}_FD \quad 0 \quad 0 \quad 0], \\
X_C &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -MH_C \end{bmatrix}, \quad Y_C = [0 \quad \mathscr{C}_F \quad 0 \quad 0 \quad 0 \quad 0], \\
X_D &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -MH_D \end{bmatrix}, \quad Y_D = [\mathscr{D}_FC \quad 0 \quad \mathscr{D}_FD \quad 0 \quad 0 \quad 0].
\end{aligned}
$$

Based on Lemma 1.11, we have

$$
\Theta + X_A\big(I - \Delta_A(k)E_AH_A\big)^{-1}\Delta_A(k)E_AG_2^{-1}Y_A
$$

$$
\begin{aligned}
&+ Y_A^T G_2^{-T} E_A^T \Delta_A^T(k)\big(I - \Delta_A(k) E_A H_A\big)^{-T} X_A^T \\
&+ X_B\big(I - \Delta_B(k) E_B H_B\big)^{-1} \Delta_B(k) E_B G_2^{-1} Y_B \\
&+ Y_B^T G_2^{-T} E_B^T \Delta_B^T(k)\big(I - \Delta_B(k) E_B H_B\big)^{-T} X_B^T \\
&+ X_C\big(I - \Delta_C(k) E_C H_C\big)^{-1} \Delta_C(k) E_C M^{-1} Y_C \\
&+ Y_C^T M^{-T} E_C^T \Delta_C^T(k)\big(I - \Delta_C(k) E_C H_C\big)^{-T} X_C^T \\
&+ X_D\big(I - \Delta_D(k) E_D H_D\big)^{-1} \Delta_D(k) E_D M^{-1} Y_D \\
&+ Y_D^T M^{-T} E_D^T \Delta_D^T(k)\big(I - \Delta_D(k) E_D H_D\big)^{-T} X_D^T \\
\le{}& \Theta + \varepsilon_A Y_A^T G_2^{-T} E_A^T E_A G_2^{-1} Y_A \\
&+ \frac{1}{\varepsilon_A} X_A\big(I - \Delta_A(k) E_A H_A\big)^{-1}\big(I - \Delta_A(k) E_A H_A\big)^{-T} X_A^T \\
&+ \varepsilon_B Y_B^T G_2^{-T} E_B^T E_B G_2^{-1} Y_B \\
&+ \frac{1}{\varepsilon_B} X_B\big(I - \Delta_B(k) E_B H_B\big)^{-1}\big(I - \Delta_B(k) E_B H_B\big)^{-T} X_B^T \\
&+ \varepsilon_C Y_C^T M^{-T} E_C^T E_C M^{-1} Y_C \\
&+ \frac{1}{\varepsilon_C} X_C\big(I - \Delta_C(k) E_C H_C\big)^{-1}\big(I - \Delta_C(k) E_C H_C\big)^{-T} X_C^T \\
&+ \varepsilon_D Y_D^T M^{-T} E_D^T E_D M^{-1} Y_D \\
&+ \frac{1}{\varepsilon_D} X_D\big(I - \Delta_D(k) E_D H_D\big)^{-1}\big(I - \Delta_D(k) E_D H_D\big)^{-T} X_D^T.
\end{aligned} \tag{6.89}
$$

Similar to the derivation of (6.79), (6.88) can be verified by

$$
\begin{bmatrix}
\Theta & * & * & * & * & * & * & * & * \\
X_A^T & \mathscr{M}_A & * & * & * & * & * & * & * \\
Y_A & 0 & \mathscr{P}_A & * & * & * & * & * & * \\
X_B^T & 0 & 0 & \mathscr{M}_B & * & * & * & * & * \\
Y_B & 0 & 0 & 0 & \mathscr{P}_B & * & * & * & * \\
X_C^T & 0 & 0 & 0 & 0 & \mathscr{M}_C & * & * & * \\
Y_C & 0 & 0 & 0 & 0 & 0 & \mathscr{W}_C & * & * \\
X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{M}_D & * \\
Y_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{W}_D
\end{bmatrix} < 0, \tag{6.90}
$$

where

$$
\begin{aligned}
\mathscr{M}_\beta &= -\varepsilon_\beta\big(I - \Delta_\beta(k) E_\beta H_\beta\big)^T\big(I - \Delta_\beta(k) E_\beta H_\beta\big), \quad \beta = A, B, C, D, \\
\mathscr{P}_\lambda &= -G_2(\varepsilon_\lambda E_\lambda^T E_\lambda)^{-1} G_2^T, \lambda = A, B, \\
\mathscr{W}_\chi &= -M(\varepsilon_\chi E_\chi^T E_\chi)^{-1} M^T, \chi = C, D.
\end{aligned}
$$

From Lemma 1.18, we can know that

$$\begin{aligned}\mathscr{M}_\beta &= -\varepsilon_\beta \big(I - \Delta_\beta(k) E_\beta H_\beta\big)^T \big(I - \Delta_\beta(k) E_\beta H_\beta\big)\\ &= -\varepsilon_\beta I \big(I - \Delta_\beta(k) E_\beta H_\beta\big)^T \varepsilon_\beta^{-1} I \big(I - \Delta_\beta(k) E_\beta H_\beta\big) \varepsilon_\beta I\\ &\le -\varepsilon_\beta I + \frac{1}{\varepsilon_{\beta\beta}} \varepsilon_\beta I H_\beta^T E_\beta^T E_\beta H_\beta \varepsilon_\beta I + \varepsilon_{\beta\beta} I\\ &= \mathscr{L}_\beta,\ \beta = A,\ B,\ C,\ D,\end{aligned}$$

and

$$\begin{aligned}\mathscr{P}_\lambda &= -G_2(\varepsilon_\lambda E_\lambda^T E_\lambda)^{-1} G_2^T \le \varepsilon_\lambda E_\lambda^T E_\lambda - G_2 - G_2^T = \mathscr{N}_\lambda, \lambda = A, B,\\ \mathscr{W}_\chi &= -M(\varepsilon_\chi E_\chi^T E_\chi)^{-1} M^T \le \varepsilon_\chi E_\chi^T E_\chi - M - M^T = \mathscr{S}_\chi, \chi = C, D.\end{aligned}$$

Then, (6.90) holds if the following condition is satisfied:

$$\begin{bmatrix} \Theta & * & * & * & * & * & * & * & * \\ X_A^T & \mathscr{L}_A & * & * & * & * & * & * & * \\ Y_A & 0 & \mathscr{N}_A & * & * & * & * & * & * \\ X_B^T & 0 & 0 & \mathscr{L}_B & * & * & * & * & * \\ Y_B & 0 & 0 & 0 & \mathscr{N}_B & * & * & * & * \\ X_C^T & 0 & 0 & 0 & 0 & \mathscr{L}_C & * & * & * \\ Y_C & 0 & 0 & 0 & 0 & 0 & \mathscr{S}_C & * & * \\ X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{L}_D & * \\ Y_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{S}_D \end{bmatrix} < 0. \tag{6.91}$$

By Schur complement to (6.91), which leads to

$$\begin{bmatrix} \Theta & * & * & * & * & * & * & * & * & * & * & * & * \\ X_A^T & \mathscr{J}_A & * & * & * & * & * & * & * & * & * & * & * \\ Y_A & 0 & \mathscr{N}_A & * & * & * & * & * & * & * & * & * & * \\ X_B^T & 0 & 0 & \mathscr{J}_B & * & * & * & * & * & * & * & * & * \\ Y_B & 0 & 0 & 0 & \mathscr{N}_B & * & * & * & * & * & * & * & * \\ X_C^T & 0 & 0 & 0 & 0 & \mathscr{J}_C & * & * & * & * & * & * & * \\ Y_C & 0 & 0 & 0 & 0 & 0 & \mathscr{S}_C & * & * & * & * & * & * \\ X_D^T & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{J}_D & * & * & * & * & * \\ Y_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{S}_D & * & * & * & * \\ 0 & \varepsilon_A E_A H_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_A & * & * & * \\ 0 & 0 & 0 & \varepsilon_B E_B H_B & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_B & * & * \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_C E_C H_C & 0 & 0 & 0 & 0 & 0 & \Lambda_C & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_D E_D H_D & 0 & 0 & 0 & 0 & \Lambda_D \end{bmatrix} < 0, \tag{6.92}$$

where

$$\Lambda_\beta = -\varepsilon_{\beta\beta} I, \quad \beta = A, B, C, D,$$
$$\mathscr{J}_\beta = -\varepsilon_\beta I + \varepsilon_{\beta\beta} I, \quad \beta = A, B, C, D.$$

Next, we give the following H_∞ filter design result for Type II.

Theorem 6.7 *Consider the filtering error system (6.74) with Type II and give a scalar $\gamma > 0$. Then the system is asymptotically stable with the H_∞ performance γ if exist matrices P_1, P_2, P_3, G_1, G_2, G_3, M, $\mathscr{A}_F$, $\mathscr{B}_F$, $\mathscr{C}_F$, and $\mathscr{D}_F$, scalars ε_A, ε_B, ε_C, ε_D, ε_L, ε_{AA}, ε_{BB}, ε_{CC}, ε_{DD}, and ε_{LL} such that LMI (6.92) holds. Furthermore, the filter gain matrices in (6.73) are given by (6.82).*

6.1.4.3 Example

It is worth pointing out that the design conditions in Theorem 6.6, Corollary 6.2, and Theorem 6.7 are LMIs over the Lyapunov matrix, auxiliary matrix variables, and the scalar γ. This implies that the scalar γ can be included as an optimization variable to obtain a reduction of the attenuation level bound. Then the minimum H_∞ performance γ_{min} can be readily found by using the MATLAB toolbox to solve the corresponding LMIs. In this following, we use an example to illustrate the effectiveness of the theoretical results developed before for non-fragile H_∞ filters design.

Consider the system (6.71) with

$$A = \begin{bmatrix} -0.30 & 0.17 & -0.34 & 0.18 \\ 0.28 & -0.08 & -0.53 & -1.21 \\ -0.98 & -0.67 & 0.14 & 0.31 \\ -0.86 & -0.17 & -0.89 & 0.20 \end{bmatrix}, \quad B = \begin{bmatrix} 0.45 \\ -0.41 \\ 0.40 \\ -0.64 \end{bmatrix},$$

$$C = [0.17 \quad -0.16 \quad 0.14 \quad 0.09], \qquad D = 0.22,$$
$$L = [0.24 \quad 0.22 \quad 0.16 \quad -0.01].$$

We assume the known parameters in (6.72) and (6.73) as

For Type I

$$H_A = \begin{bmatrix} -0.22 \\ -0.11 \\ 0.13 \\ -0.21 \end{bmatrix}, \qquad E_A = [\, 0.14 \quad -0.08 \quad 0.16 \quad 0.23 \,],$$

$$H_B = [-0.17 \quad -0.13 \quad 0.10 \quad 0.33], \quad E_B = \begin{bmatrix} 0.21 \\ -0.11 \\ -0.10 \\ 0.16 \end{bmatrix},$$

$$
\begin{aligned}
H_C &= \begin{bmatrix} -0.23 \\ 0.17 \\ 0.08 \\ 0.03 \end{bmatrix}, \quad E_C = [0.31 \quad 0.27 \quad 0.14 \quad -0.07], \\
H_D &= 0.10, \qquad\qquad E_D = -0.21,
\end{aligned} \tag{6.93}
$$

For Type II

$$
\begin{aligned}
H_A &= \begin{bmatrix} -0.19 \\ -0.38 \\ 0.94 \\ 0.07 \end{bmatrix}, \quad E_A = [0.14 \quad -0.12 \quad -0.18 \quad -0.92], \\
H_B &= \begin{bmatrix} -0.08 \\ 0.06 \\ -0.40 \\ -0.42 \end{bmatrix}, \quad E_B = [-0.37 \quad -0.12 \quad 0.64 \quad -0.10], \\
H_C &= 0.13, \qquad\qquad E_C = -0.48, \\
H_D &= -0.27, \qquad\qquad E_D = 0.01.
\end{aligned} \tag{6.94}
$$

For type I, by using the MATLAB toolbox to solve the LMI (6.81) in Theorem 6.6, the minimum H_∞ performance $\gamma_{min} = 0.8258$ is obtained, and

$$
\begin{aligned}
\mathscr{A}_F &= \begin{bmatrix} 0.0279 & 0.1260 & -0.2109 & -0.0983 \\ 0.0630 & 0.0615 & -0.0006 & -0.1100 \\ -0.1902 & -0.0888 & 0.0531 & 0.0886 \\ 0.0258 & -0.0548 & -0.0364 & 0.1367 \end{bmatrix}, \\
\mathscr{B}_F &= \begin{bmatrix} 0.1695 \\ 0.0529 \\ -0.5241 \\ 0.3603 \end{bmatrix}, \\
\mathscr{C}_F &= [-0.1239 \quad -0.0046 \quad -0.3173 \quad -0.0778], \\
\mathscr{D}_F &= -1.1606, \\
G_2 &= \begin{bmatrix} 0.4539 & 0.1082 & 0.0044 & -0.0766 \\ 0.1099 & 0.1173 & -0.0686 & -0.0951 \\ -0.1511 & -0.0847 & 0.3387 & -0.0337 \\ -0.0339 & -0.0910 & -0.0619 & 0.2111 \end{bmatrix}, \\
M &= 1.0000.
\end{aligned} \tag{6.95}
$$

Substituting (6.95) into (6.82), the filter matrices can be obtained as follows:

$$
\begin{aligned}
A_F &= \begin{bmatrix} -0.0071 & 0.2455 & -0.6010 & -0.0524 \\ 0.4318 & -0.0800 & 0.5552 & -0.2861 \\ -0.4389 & -0.2041 & 0.0254 & 0.2244 \\ 0.1785 & -0.3145 & -0.0225 & 0.5818 \end{bmatrix}, \\
B_F &= \begin{bmatrix} 0.4407 \\ 1.2010 \\ -0.8468 \\ 2.0466 \end{bmatrix}, \\
C_F &= [-0.1239 \quad -0.0046 \quad -0.3173 \quad -0.0778], \\
D_F &= -1.1606.
\end{aligned} \tag{6.96}
$$

By the LMI (6.85) in Corollary 6.2, we have the minimum H_∞ performance $\gamma_{min} = 0.7397$, which is smaller than Theorem 6.6 verifies the above discussion.

For type II, solving LMI (6.85), the minimum H_∞ performance is 0.7768, and the corresponding filter matrices are

$$
\begin{aligned}
A_F &= \begin{bmatrix} -0.7822 & -0.0014 & -0.2749 & 0.0379 \\ 1.2223 & 0.2624 & 0.3985 & -0.5194 \\ 0.1988 & -0.4010 & 0.1182 & 0.6844 \\ 0.3337 & 0.1621 & 0.0959 & -0.2066 \end{bmatrix}, \\
B_F &= \begin{bmatrix} 0.0154 \\ -2.4992 \\ -1.5461 \\ 0.2856 \end{bmatrix}, \\
C_F &= [-0.2467 \quad -0.0976 \quad -0.1037 \quad 0.0973], \\
D_F &= -0.9454,
\end{aligned} \tag{6.97}
$$

with

$$
\begin{aligned}
\mathscr{A}_F &= \begin{bmatrix} -0.0970 & -0.0261 & -0.0308 & 0.0400 \\ -0.0256 & -0.0081 & -0.0079 & 0.0118 \\ -0.0265 & -0.0114 & -0.0079 & 0.0179 \\ 0.0274 & 0.0098 & 0.0084 & -0.0142 \end{bmatrix}, \\
\mathscr{B}_F &= \begin{bmatrix} -0.3266 \\ -0.1126 \\ -0.1193 \\ 0.1084 \end{bmatrix}, \\
\mathscr{C}_F &= [-0.2392 \quad -0.0946 \quad -0.1006 \quad 0.0943], \\
\mathscr{D}_F &= -0.9167,
\end{aligned}
$$

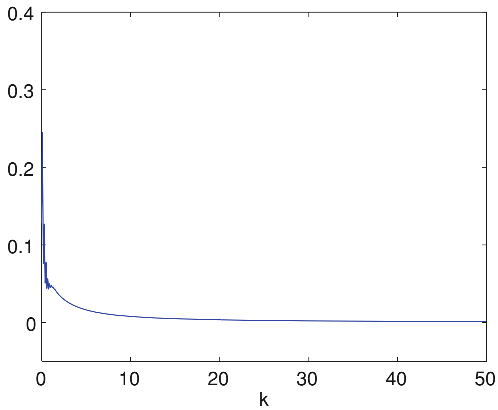

Fig. 6.4 Error response of $e(k)$ for the type I

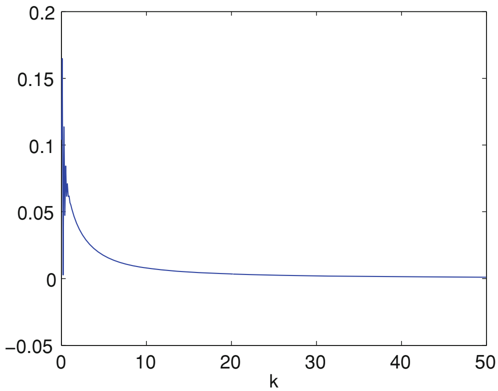

Fig. 6.5 Error response of $e(k)$ for the type II

$$G_2 = \begin{bmatrix} 0.2183 & 0.0752 & 0.0737 & -0.0983 \\ 0.0657 & 0.0270 & 0.0234 & -0.0355 \\ 0.0672 & 0.0249 & 0.0316 & -0.0320 \\ -0.0619 & -0.0245 & -0.0236 & 0.0408 \end{bmatrix},$$
$$M = 0.9696.$$

The filters consisting of (6.96) and (6.97) are non-fragile, that is, when the filters have gain variations, the H_∞ performance $\gamma = 0.8258$ (for the type I) and $\gamma = 0.7768$ (for the type II) are always guaranteed for any uncertainties satisfying $\Delta_\beta^T(k)\Delta_\beta(k) \leq I,\ \beta = A,\ B,\ C,\ D$ for this example. Based on the filters, the simulation results of the filtering error are given in Fig. 6.4 for the type I and in Fig. 6.5 for the type II, under the initial conditions $x(0) = \hat{x}_F(0) = [\,0 \quad 0 \quad 0 \quad 0\,]^T$, with $\Delta_\rho = sin(0.1k),\ \rho = A,\ B,\ C,\ D$, and the noise signal is chosen as $w(k) = (2+k^{1.3})^{-1},\ k = 1,\ 2,\ .....,$ which belongs to $l_2[\,0,\ \infty)$. Figures 6.6 and 6.7 show the simulated values of γ,

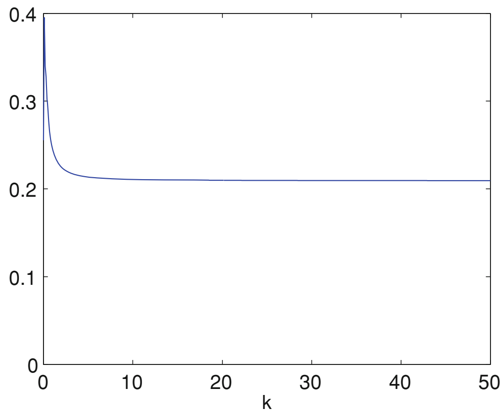

Fig. 6.6 The value of $\sqrt{\sum_{k=0}^{\infty} e^T(k)e(k)/\sum_{k=0}^{\infty} w^T(k)w(k)}$ for the type I

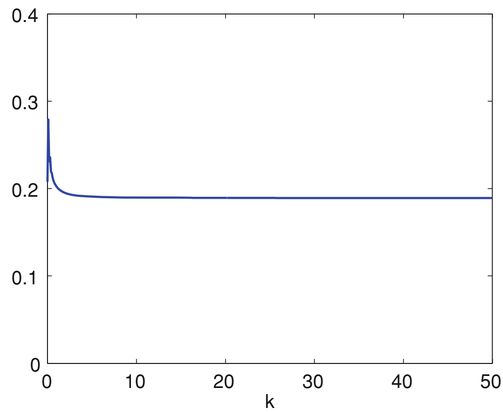

Fig. 6.7 The value of $\sqrt{\sum_{k=0}^{\infty} e^T(k)e(k)/\sum_{k=0}^{\infty} w^T(k)w(k)}$ for the type II

i.e., $\sqrt{\sum_{k=0}^{\infty} e^T(k)e(k)/\sum_{k=0}^{\infty} w^T(k)w(k)}$ for the type I and the type II, respectively. It is observed that the ratio tends to a constant value 0.2094 for the type I (0.1893 for the type II), which is less than the prescribed $\gamma = 0.8258$ ($\gamma = 0.7768$). From this simulation, we can see that the proposed non-fragile filter design methods are effective.

6.2 Frobenius Norm Uncertainties

In this section, we study the Frobenius norm uncertainties. The Frobenius norm is better than 2-norm as a measure of uncertainties and the 2-norm structure is a special case of the Frobenius norm [12]. In the study, we consider only the observer-based non-fragile H_∞ controller design problem.

6.2.1 Observer-Based Output Feedback Non-fragile H_∞ Control

Consider the following discrete-time linear dynamical system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + Ew(k), \\ z(k) &= C_1 x(k) + Du(k) + Fw(k), \\ y(k) &= C_2 x(k) + Hw(k), \end{aligned} \tag{6.98}$$

where $x(k) \in \mathscr{R}^n$ is the state variable, $u(k) \in \mathscr{R}^m$ is the control input, $w(k) \in \mathscr{R}^v$ is the noise signal that is assumed to be the arbitrary signal in $l_2[0, \infty)$, $z(k) \in \mathscr{R}^q$ is the controlled output variable, $y(k) \in \mathscr{R}^p$ is the measurement output. $A \in \mathscr{R}^{n\times n}$, $B \in \mathscr{R}^{n\times m}$, $E \in \mathscr{R}^{n\times v}$, $C_1 \in \mathscr{R}^{q\times n}$, $D \in \mathscr{R}^{q\times m}$, $F \in \mathscr{R}^{q\times v}$, $C_2 \in \mathscr{R}^{p\times n}$, and $H \in \mathscr{R}^{p\times v}$ are system matrices.

Remark 6.8 In the existing researches on designing observer-based output feedback H_∞ controllers and observers, some result have been given based the LMI technique. However, the result should also meet the other constraint on the system structure that is $H = 0$. It implies that in system equation the measured output is free of disturbances. However, in most practical applications, the measurements made in physical systems are not free of errors caused by the presence of disturbance. Thus, our design method is applicable to the more general case.

The following observer with gain variations is proposed to deal with the state estimation of system (6.98)

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + Bu(k) + \big(L + \Delta L(k)\big)\big(y(k) - \tilde{y}(k)\big), \\ \tilde{y}(k) &= C_2\tilde{x}(k), \end{aligned} \tag{6.99}$$

where $\tilde{x}(k) \in \mathscr{R}^n$ and $\tilde{y}(k) \in \mathscr{R}^f$ are the estimated state and estimated output, respectively. $L \in \mathscr{R}^{n\times f}$ is to be determined observer gain matrix, $\Delta L(k)$ is uncertain matrix formulated as

$$\Delta L(k) = \sum_{q=1}^{m_L}\sum_{s=1}^{n_L} M_{Lq}\Delta_{Lqs}(k)N_{Ls}, \tag{6.100}$$

M_{Lq} and N_{Ls} are constant matrices with appropriate dimensions and

$$\Delta_L(k) = \begin{bmatrix} \Delta_{L11}(k) & \Delta_{L12}(k) & \dots & \Delta_{L1n_L}(k) \\ \Delta_{L21}(k) & \Delta_{L22}(k) & \dots & \Delta_{L2n_L}(k) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{Lm_L1}(k) & \Delta_{Lm_L2}(k) & \dots & \Delta_{Lm_Ln_L}(k) \end{bmatrix},$$

is an unknown real time-varying matrix satisfying

$$\sum_{q=1}^{m_L}\sum_{s=1}^{n_L} \| \Delta_{Lqs}(k) \| \leq 1, \ k > 0. \tag{6.101}$$

Remark 6.9 The definition of (6.100) indicates that, the uncertain parameter $\Delta L(k)$ is decomposed into a set of norm-bounded uncertainty matrix accompanied by some known real matrices $(M_{Lq}, \ N_{Ls})$ characterizing the structure of the uncertainty [12].

By virtue of the Frobenius norm, (6.101) can be denoted by $\| \Delta_{LN}(k) \|_F \leq 1$ where the matrix is given as

$$\| \Delta_{LN}(k) \|_F = \begin{bmatrix} \| \Delta_{L11}(k) \| & \| \Delta_{L12}(k) \| & \dots & \| \Delta_{L1n_L}(k) \| \\ \| \Delta_{L21}(k) \| & \| \Delta_{L22}(k) \| & \dots & \| \Delta_{L2n_L}(k) \| \\ \vdots & \vdots & \ddots & \vdots \\ \| \Delta_{Lm_L1}(k) \| & \| \Delta_{Lm_L2}(k) \| & \dots & \| \Delta_{Lm_Ln_L}(k) \| \end{bmatrix}.$$

Remark 6.10 To draw connections with the 2-norm uncertainty structures, we specialize $m_L = n_L = 1$ to recover the 2-norm uncertainty model. Thus it can be easily seen that 2-norm uncertainty is a special case of the more general Frobenius norm.

Let us denote the estimation error as $e(k) = x(k) - \tilde{x}(k)$, then, it gets

$$e(k+1) = x(k+1) - \tilde{x}(k+1) = \big(A - (L + \Delta L)C_2\big)e(k) + \big(E - (L + \Delta L)H\big)w(k). \tag{6.102}$$

The following controller with gain variations is considered:

$$u(k) = \big(K + \Delta K(k)\big)\tilde{x}(k), \tag{6.103}$$

where $K \in \mathscr{R}^{m \times n}$ is to be determined controller gain matrix

$$\Delta K(k) = \sum_{q=1}^{m_K}\sum_{s=1}^{n_K} M_{Kq}\Delta_{Kqs}(k)N_{Ks},$$

M_{Kq} and N_{Ks} are known constant matrices of appropriate dimensions, $\Delta_{Kqs}(k)$ is uncertain matrix and satisfies (6.101).

Next, we will develop an efficient result to achieve observer-based non-fragile H_∞ controller design with uncertainties $\Delta_{Lqs}(k)$ and $\Delta_{Kqs}(k)$ satisfying (6.101). From (6.98), (6.99), and (6.103), we can obtain the closed-loop system as follows,

$$\begin{aligned} \hat{x}(k+1) &= \mathscr{A}\hat{x}(k) + \mathscr{B}w(k), \\ z(k) &= \mathscr{C}\hat{x}(k) + \mathscr{D}w(k), \end{aligned} \tag{6.104}$$

where $\hat{x}(k) = \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$ and

$$\mathscr{A} = \begin{bmatrix} A + B\big(K + \Delta K(k)\big) & -B\big(K + \Delta K(k)\big) \\ 0 & A - \big(L + \Delta L(k)\big)C_2 \end{bmatrix},$$
$$\mathscr{B} = \begin{bmatrix} E \\ E - \big(L + \Delta L(k)\big)H \end{bmatrix},$$

$$\mathscr{C} = [C_1 + D\big(K + \Delta K(k)\big) \quad - D\big(K + \Delta K(k)\big)],$$
$$\mathscr{D} = F.$$

In that following, the H_∞ performance analysis problem is concerned. From [1], we know easily that the H_∞ performance $\gamma > 0$ of the closed-loop system (6.104) can be guaranteed by

$$\begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix} & -G - G^T + \begin{bmatrix} P & * \\ 0 & I \end{bmatrix} \end{bmatrix} < 0, \tag{6.105}$$

where P is Lyapunov matrix and G is auxiliary matrix variable.

It is noted that if the controller gain matrices are given, the matrix inequality condition (6.105) is an LMI over the decision variables P and G for fixed γ. However, since our purpose is to determine the controller gain matrices, the condition (6.105) is a nonlinear matrix inequality due to the coupling between G and $\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix}$. Our main objective hereafter is to transform (6.105) into an LMI condition. In order to solve this problem, the decoupling approach presented in Chapter 2 will be considered. Our result depends on the following process.

Let us rewrite Ξ_{21} in (6.105) as follows:

$$\begin{aligned} &G\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix} \\ &= G\underbrace{\left(\begin{bmatrix} A & 0 & E \\ 0 & A & E \\ C_1 & 0 & F \end{bmatrix} + \begin{bmatrix} B\Delta K(k) & -B\Delta K(k) & 0 \\ 0 & -\Delta L(k)C_2 & -\Delta L(k)H \\ D\Delta K(k) & -D\Delta K(k) & 0 \end{bmatrix}\right)}_{\hat{A}} \\ &\quad + G\begin{bmatrix} BK & -BK & 0 \\ 0 & -LC_2 & -LH \\ DK & -DK & 0 \end{bmatrix} \\ &= G\hat{A} + G\underbrace{\begin{bmatrix} B & 0 \\ 0 & -I \\ D & 0 \end{bmatrix}}_{\hat{B}}\underbrace{\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}}_{\hat{K}}\underbrace{\begin{bmatrix} I & -I & 0 \\ 0 & C_2 & H \end{bmatrix}}_{\hat{C}} \\ &= G\hat{A} + G\hat{B}\hat{K}\hat{C}. \end{aligned} \tag{6.106}$$

Remark 6.11 In Chap. 4, based on the LMI decoupling approach, the observer-based output feedback H_∞ control design conditions have been given. Different from the results, this section considers the whole design strategy of the observer and controller.

In the LMI decoupling approach, for intuitive, we choose $M = \hat{B}$, $N = I$, and $\beta = 1$. By combining (6.105), (6.106) and defining $V = U\hat{K}$, where U is nonsingular without loss of generality, we have

$$\begin{aligned}
&\begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G\hat{A} & -G - G^T + \begin{bmatrix} P & * \\ 0 & I \end{bmatrix} \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ I \end{bmatrix} G\hat{B}\hat{K}\hat{C}[I \quad 0] + [I \quad 0]^T \hat{C}^T \hat{K}^T \hat{B}^T G^T \begin{bmatrix} 0 \\ I \end{bmatrix}^T \\
&= \begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G\hat{A} + \hat{B}V\hat{C} & -G - G^T + \begin{bmatrix} P & * \\ 0 & I \end{bmatrix} \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ I \end{bmatrix} (G\hat{B} - \hat{B}U)U^{-1}V\hat{C}[I \quad 0] + [I \quad 0]^T \hat{C}^T V^T U^{-T} (G\hat{B} - \hat{B}U)^T \begin{bmatrix} 0 \\ I \end{bmatrix}^T < 0.
\end{aligned} \tag{6.107}$$

Based on Lemma 1.3 for a positive matrix J, (6.107) holds if the following condition is satisfied,

$$\begin{aligned}
&\begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G\hat{A} + \hat{B}V\hat{C} & -G - G^T + \begin{bmatrix} P & * \\ 0 & I \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} J \begin{bmatrix} 0 \\ I \end{bmatrix}^T \\
&+ [I \quad 0]^T \hat{C}^T V^T U^{-T} (G\hat{B} - \hat{B}U)^T J^{-1} (G\hat{B} - \hat{B}U) U^{-1} V\hat{C} [I \quad 0] < 0.
\end{aligned} \tag{6.108}$$

Using Lemma 1.4 with

$$\begin{aligned}
T &= \begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * \\ G\hat{A} + \hat{B}V\hat{C} & -G - G^T + \begin{bmatrix} P & * \\ 0 & I \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} J \begin{bmatrix} 0 \\ I \end{bmatrix}^T, \\
A &= U^{-1} V\hat{C}[I \quad 0], \\
P &= (G\hat{B} - \hat{B}U)^T J^{-1} (G\hat{B} - \hat{B}U), \\
L &= U,
\end{aligned}$$

then, (6.108) can be guaranteed by

$$\begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * & * \\ G\hat{A}+\hat{B}V\hat{C} & -G-G^T+\begin{bmatrix} P & * \\ 0 & I \end{bmatrix}+J & * \\ V\hat{C} & 0 & \Pi \end{bmatrix} < 0, \tag{6.109}$$

where $= -U - U^T + (G\hat{B} - \hat{B}U)^T J^{-1}(G\hat{B} - \hat{B}U)$.

Applying Schur complement to (6.109) yields

$$\begin{bmatrix} \begin{bmatrix} -P & * \\ 0 & -\gamma^2 I \end{bmatrix} & * & * & * \\ G\hat{A}+\hat{B}V\hat{C} & -G-G^T+\begin{bmatrix} P & * \\ 0 & I \end{bmatrix}+J & * & * \\ V\hat{C} & 0 & -U-U^T & * \\ 0 & 0 & G\hat{B}-\hat{B}U & -J \end{bmatrix} < 0. \tag{6.110}$$

With the H_∞ performance analysis condition (6.110) in hands, in that following, we will present a sufficient condition for designing the non-fragile H_∞ observer and controller in the form of (6.99) and (6.103), respectively. That is, we will determine the gain matrices L in (6.99) and K in (6.103) such that the closed-loop system (6.104) is asymptotically stable with H_∞ performance γ.

Let

$$P = \begin{bmatrix} P_1 & * \\ P_2 & P_3 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2 & G_3 \\ G_4 & G_5 & G_6 \\ G_7 & G_8 & G_9 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 & * & * \\ J_2 & J_3 & * \\ J_4 & J_5 & J_6 \end{bmatrix}, \\ U = \begin{bmatrix} U_K & 0 \\ 0 & U_L \end{bmatrix}, \quad V = \begin{bmatrix} V_K & 0 \\ 0 & V_L \end{bmatrix}. \tag{6.111}$$

From (6.104), (6.110), and (6.111), one has

$$\Upsilon + \Upsilon_\Delta(k) < 0, \tag{6.112}$$

where

$$\Upsilon = \\ \begin{bmatrix} -P_1 & * & * & * & * & * & * & * & * & * & * \\ -P_2 & -P_3 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & -\gamma^2 I & * & * & * & * & * & * & * & * \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & * & * & * & * & * & * & * \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} & * & * & * & * & * & * \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & \alpha_{65} & \alpha_{66} & * & * & * & * & * \\ V_K & -V_K & 0 & 0 & 0 & 0 & -U_K-U_K^T & * & * & * & * \\ 0 & V_L C_2 & V_L H & 0 & 0 & 0 & 0 & -U_L-U_L^T & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & G_1B+G_3D-BU_K & -G_2 & -J_1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & G_4B+G_6D & -G_5+U_L & -J_2 & -J_3 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & G_7B+G_9D-DU_K & -G_8 & -J_4 & -J_5 & -J_6 \end{bmatrix}$$

$$
\begin{aligned}
\alpha_{41} &= G_1 A + G_3 C_1 + B V_K, \\
\alpha_{42} &= G_2 A - B V_K, \\
\alpha_{43} &= G_1 E + G_2 E + G_3 F, \\
\alpha_{44} &= -G_1 - G_1^T + P_1 + J_1, \\
\alpha_{51} &= G_4 A + G_6 C_1, \\
\alpha_{52} &= G_5 A - V_L C_2, \\
\alpha_{53} &= G_4 E + G_5 E + G_6 F - V_L H, \\
\alpha_{54} &= -G_4 - G_2^T + P_2 + J_2, \\
\alpha_{55} &= -G_5 - G_5^T + P_3 + J_3, \\
\alpha_{61} &= G_7 A + G_9 C_1 + D V_K, \\
\alpha_{62} &= G_8 A - D V_K, \\
\alpha_{63} &= G_7 E + G_8 E + G_9 F, \\
\alpha_{64} &= -G_7 - G_3^T + J_4, \\
\alpha_{65} &= -G_8 - G_6^T + J_5, \\
\alpha_{66} &= -G_9 - G_9^T + I + J_6,
\end{aligned}
$$

$$
\Upsilon_\Delta(k) = \begin{bmatrix}
0 & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & * & * & * & * & * & * & * & * \\
\delta_{41}(k) & \delta_{42}(k) & \delta_{43}(k) & 0 & * & * & * & * & * & * & * \\
\delta_{51}(k) & \delta_{52}(k) & \delta_{53}(k) & 0 & 0 & * & * & * & * & * & * \\
\delta_{61}(k) & \delta_{62}(k) & \delta_{63}(k) & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\begin{aligned}
\delta_{41}(k) &= G_1 B \Delta K(k) + G_3 D \Delta K(k), \\
\delta_{42}(k) &= -G_1 B \Delta K(k) - G_2 \Delta L(k) C_2 - G_3 D \Delta K(k) \\
\delta_{43}(k) &= -G_2 \Delta L(k) H, \\
\delta_{51}(k) &= G_4 B \Delta K(k) + G_6 D \Delta K(k), \\
\\
\delta_{52}(k) &= -G_4 B \Delta K(k) - G_5 \Delta L(k) C_2 - G_6 D \Delta K(k), \\
\delta_{53}(k) &= -G_5 \Delta L(k) H, \\
\\
\delta_{61}(k) &= G_7 B \Delta K(k) + G_9 D \Delta K(k), \\
\delta_{62}(k) &= -G_7 B \Delta K(k) - G_8 \Delta L(k) C_2 - G_9 D \Delta K(k), \\
\delta_{63}(k) &= -G_8 \Delta L(k) H.
\end{aligned}
$$

Obviously, $\Upsilon_\Delta(k)$ can be rewritten as

$$\begin{aligned}\Upsilon_{\Delta}(k) &= X_K \Delta K(k) Y_K + Y_K^T (\Delta K(k))^T X_K^T + X_L \Delta L(k) Y_L + Y_L^T (\Delta L(k))^T X_L^T \\ &= X_K \Big(\sum_{q=1}^{m_K} \sum_{s=1}^{n_K} M_{Kq} \Delta_{Kqs}(k) N_{Ks} \Big) Y_K + Y_K^T \Big(\sum_{q=1}^{m_K} \sum_{s=1}^{n_K} M_{Kq} \Delta_{Kqs}(k) N_{Ks} \Big)^T X_K^T \\ &\quad + X_L \Big(\sum_{q=1}^{m_L} \sum_{s=1}^{L} M_{Lq} \Delta_{Lqs}(k) N_{Ls} \Big) Y_L + Y_L^T \Big(\sum_{q=1}^{m_L} \sum_{s=1}^{n_L} M_{Lq} \Delta_{Lqs}(k) N_{Ls} \Big)^T X_L^T,\end{aligned} \tag{6.113}$$

where

$$X_K = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_1 B + G_3 D \\ G_4 B + G_6 D \\ G_7 B + G_9 D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y_K = [I \quad -I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$X_L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -G_2 \\ -G_5 \\ -G_8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y_L = [0 \quad C_2 \quad H \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

Then, by Lemma 1.20, the matrix inequality condition (6.112) holds if the following LMI condition is satisfied

$$
\begin{bmatrix}
\Upsilon & * & \cdots & & & \cdots & \cdots & * \\
M_{K1}^T X_K^T & \varepsilon_K^- I & & & & & & \\
\vdots & & \ddots & & & & & \vdots \\
M_{Km_K}^T X_K^T & 0 \;\cdots\; 0 & \varepsilon_K^- I & \ddots & & & \ddots & \\
M_{L1}^T X_L^T & 0 & \cdots \;\; 0 & \varepsilon_L^- I & & & & \\
\vdots & \vdots & & \ddots & \ddots & & \ddots & \vdots \\
M_{Lm_L}^T X_L^T & 0 & \cdots & \cdots \;\; 0 & \varepsilon_L^- I & & & \\
\varepsilon_K N_{K1} Y_K & 0 & \cdots & \cdots & 0 & \varepsilon_K^- I & & \\
\vdots & \vdots & & & \ddots & & \ddots & \\
\varepsilon_K N_{Kn_K} Y_K & 0 & \cdots & \cdots & \cdots \;\; 0 & \varepsilon_K^- I & & \vdots \\
\varepsilon_L N_{L1} Y_L & 0 & \cdots & \cdots & \cdots & 0 & \varepsilon_L^- I & \\
\vdots & \vdots & & & & & \ddots & * \\
\varepsilon_L N_{Ln_L} Y_L & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \varepsilon_L^- I
\end{bmatrix} < 0,
\tag{6.114}
$$

where $\varepsilon_\mu^- = -\varepsilon_\mu,\ \mu = K,\ L.$

Finally, the following theorem is used to observer-based non-fragile H_∞ control design for the closed-loop system (6.104).

Theorem 6.8 *Consider the closed-loop system* (6.104) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P_1,\ P_2,\ P_3,\ G_1,\ G_2,\ G_3,\ G_4,\ G_5,\ G_6,\ G_7,\ G_8,\ G_9,\ U_K,\ U_L,\ V_K,\ V_L,\ J_1,\ J_2,\ J_3,\ J_4,\ J_5,$ *and* J_6, *scalars* ϵ_K *and* ϵ_L *such that the LMI condition* (6.115) *holds. Furthermore, the observer* (6.99) *and the controller* (6.103) *gain matrices are given by*

$$
L = U_L^{-1} V_L, \quad K = U_K^{-1} V_K. \tag{6.115}
$$

Without considering the controller and observer gain uncertainties, the following corollary gives a sufficient condition for designing the standard H_∞ controller and observer.

Corollary 6.3 *Consider the closed-loop system* (6.104) *and give a scalar* $\gamma > 0$. *Then the system is asymptotically stable with the* H_∞ *performance* γ *if exist matrices* $P_1, P_2, P_3, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, U_K, U_L, V_K, V_L, J_1, J_2, J_3, J_4, J_5,$ *and* J_6 *such that* $\Upsilon < 0$ *holds, where* Υ *is defined in* (6.112). *The gain matrices for the standard controller and observer are given by* (6.115).

6.2.2 Example

In this example, we employ Theorem 6.8 to design the observer-based non-fragile H_∞ controller for the system (6.98), in which system matrices are given

$$A = \begin{bmatrix} 0.3 & 0.8 & -0.4 \\ -0.5 & 0.4 & 0.5 \\ 1.2 & 1.1 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & -1 \\ 0 & 1.3 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.1 \\ 0.4 \\ 0.1 \end{bmatrix}, \quad C_1 = [-1 \quad 0 \quad 2\,],$$
$$D = [\,1.2 \quad -1], \quad F = 0.3,$$
$$C_2 = \begin{bmatrix} -1 & 1.2 & 1 \\ 0 & -3 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}$$

For uncertainty matrices with $m_K = m_L = n_K = n_L = 2$, we give the known parameters in (6.99) and (6.103) as

$$\begin{aligned} M_{L1} &= \begin{bmatrix} 0.22 \\ -0.12 \\ 0.02 \end{bmatrix}, & M_{L2} &= \begin{bmatrix} -0.04 \\ 0.2 \\ 0.12 \end{bmatrix}, \\ N_{L1} &= [-0.31 \quad 0.13], & N_{L2} &= [0.06 \quad -0.25], \\ M_{K1} &= \begin{bmatrix} 0.28 \\ 0.12 \end{bmatrix}, & M_{K2} &= \begin{bmatrix} -0.17 \\ 0.03 \end{bmatrix}, \\ N_{K1} &= [0.23 \quad 0.02 \quad -0.13], & N_{K2} &= [0.02 \quad -0.22 \quad -0.03]. \end{aligned} \tag{6.116}$$

By using the MATLAB toolbox to solve the LMI (6.114) in Theorem 6.8, the minimum H_∞ performance $\gamma_{min} = 3.9537$ is obtained, and

$$\begin{aligned} U_K &= \begin{bmatrix} 2.3269 & 1.7494 \\ 1.6528 & 2.8928 \end{bmatrix}, \\ V_K &= \begin{bmatrix} -1.2712 & -2.7178 & -2.3672 \\ -2.0651 & -3.2440 & -2.4444 \end{bmatrix}, \\ U_L &= \begin{bmatrix} 114.6195 & 14.0316 & -2.4352 \\ 25.8412 & 61.0820 & -8.6722 \\ -21.7913 & -6.3109 & 25.0209 \end{bmatrix}, \\ V_L &= \begin{bmatrix} -2.0790 & -29.1334 \\ 20.8726 & -1.1004 \\ 17.0743 & 5.5069 \end{bmatrix}. \end{aligned} \tag{6.117}$$

Substituting (6.117) into (6.115), the controller and observer gain matrices can be given as follows:

$$\begin{aligned} K &= \begin{bmatrix} -0.0168 & -0.5696 & -0.6698 \\ -0.7043 & -0.7960 & -0.4623 \end{bmatrix}, \\ L &= \begin{bmatrix} -0.0602 & -0.2657 \\ 0.4736 & 0.0962 \\ 0.7494 & 0.0130 \end{bmatrix}. \end{aligned} \tag{6.118}$$

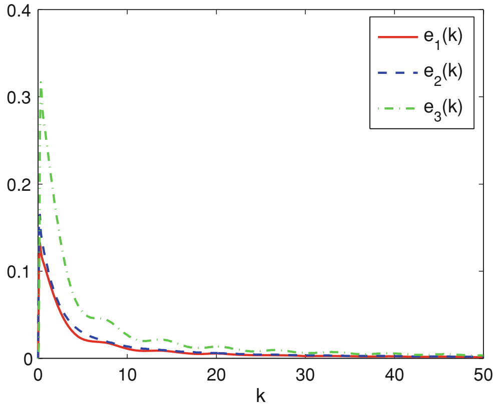

Fig. 6.8 The estimation error $e(k)$

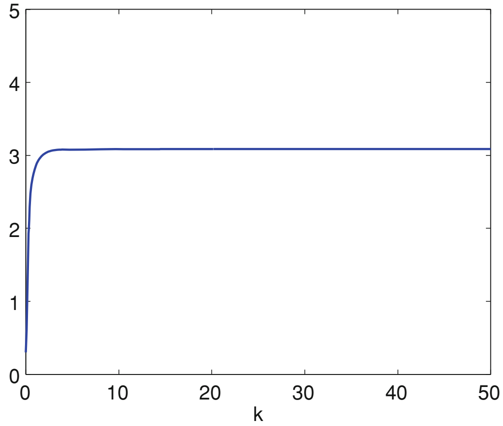

Fig. 6.9 The ratio of $\sqrt{\sum_{k=0}^{\infty} z^T(k)z(k)/\sum_{k=0}^{\infty} w^T(k)w(k)}$

The controller and observer given by (6.118) are non-fragile, that is, when the controller and observer have gain variations, the H_∞ performance $\gamma = 3.9537$ is always guaranteed for any uncertainties Δ_{Kqs} and Δ_{Lqs} satisfying (6.101) for this example. In order to verify this view, we assume that $\Delta_{K11} = \Delta_{K21} = sin(k)/\sqrt{2}$, $\Delta_{K12} = \Delta_{K22} = cos(k)/\sqrt{2}$, $\Delta_{L11} = \Delta_{L21} = sin(k)/\sqrt{2}$, and $\Delta_{L12} = \Delta_{L22} = cos(k)/\sqrt{2}$. Based on the controller and observer, the simulation result of the estimation error is given in Fig. 6.8, under the initial conditions $x(0) = \hat{x}_F(0) = [\,0\ \ 0\ \ 0\,]^T$, and the noise signal is chosen as $w(k) = (2+k^{1.3})^{-1}$. Figure 6.9 shows the simulated values of γ, i.e., $\sqrt{\sum_{k=0}^{\infty} z^T(k)z(k)/\sum_{k=0}^{\infty} w^T(k)w(k)}$. It is observed that the ratio tends to a constant value 3.0862, which is less than the prescribed $\gamma = 3.9537$. From this simulation, we can see that the proposed non-fragile control design method is effective.

6.3 Conclusion

It is well known that these results on control and filtering for systems with feedback uncertainties and Frobenius norm-bounded uncertainties have not been fully investigated and the relevant results have been very few. Motivated by the observations, this chapter has investigated the problems of output feedback H_∞ control and filtering for discrete-time linear systems with feedback uncertainties and Frobenius norm-bounded uncertainties. Three types of feedback uncertainties have been considered to design output feedback H_∞ control and filtering. And the observer-based output feedback non-fragile H_∞ control design with Frobenius norm-bounded uncertainties has also been studied. The corresponding design conditions have been derived based on linear matrix inequality (LMI) techniques. Numerical examples have been provided to illustrate the feasibility of the proposed design methods.

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Glossary

$*$	Corresponding transposed block matrix due to symmetry
A^T	Transpose of matrix A
A^{-1}	Inverse of matrix A if it exists
$A > 0 (A \geq 0)$	Matrix A is square symmetric and A is positive definite (semi-definite)
$A < 0 (A \leq 0)$	Matrix A is square symmetric and A is negative definite (semi-negative)
I	Identity matrix with appropriate dimension
$L_2[0, \infty)$	Space of square integrable vector functions on $[0, \infty)$
$l_2[0, \infty)$	Space of square summable infinite vector sequences over $[0, \infty)$
$\mathscr{R}^n$	n-dimensional Euclidean space
$\mathscr{R}^{m \times n}$	Space of $m \times n$ real matrices

X.-H. Chang, *Robust Output Feedback H-infinity Control and Filtering for Uncertain Linear Systems*, Studies in Systems, Decision and Control 7,
DOI: 10.1007/978-3-642-55107-9,